André-Quillen Homology for Simplicial Algebras and Ring Spectra

by

Philipp Michael Reinhard

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Prenez un cercle, caressez-le, il deviendra vicieux.

(Eugène Ionesco, La Cantatrice Chauve)

Abstract

We discuss André-Quillen homology for simplicial algebras and algebras over simplicial algebras, extending the classical notion for rings. This extension is also discussed by Goerss and Hopkins [22], however our statements are proven in a more explicit way. We are then further able to construct spectral sequences for André-Quillen homology like the spectral sequence for the indecomposables or the Fundamental spectral sequence according to Quillen. The André-Quillen homology for algebras constructed as cellular complexes is calculated and we apply this homology theory to obtain notions of atomic and nuclear algebras, thus extending results from Baker and May. We define the notion of i-stable algebras and are able to give a comparison theorem between André-Quillen homology, stabilisation and Γ-homology for i-stable algebras up to degree i.

In the second part of the thesis we discuss topological André-Quillen homology and extend certain results by Gilmour about cellular complexes in this setting.

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Contents

Introduction

After the rise of homology theories for modules and topological spaces it was natural to look for such theories for the category of commutative rings. Barr and Beck [5] found the homology theory which is now named after them using monads. But more successful were André [2] who gave a description of a homology theory stemming from an explicit complex and Quillen [46] who gave a description of a homology theory in terms of abelianisation. The two approaches turn out to calculate the same homology theory, which is now usually called André-Quillen homology. There are various ways of generalising this notion. Goerss and Hopkins extended André-Quillen homology for algebras over a simplicial operad [22]. Basterra defined a homology theory for E_∞-ring spectra, called topological André-Quillen homology in [6] following an unpublished paper by Kříž [31]. A version for topological André-Quillen homology for non-commutative algebras was defined by Lazarev in [33]. Further there is a theory of Γ-homology for simplicial algebras and Eilenberg-MacLane spectra [48]. So, various comparison questions arise; see [9] for an overview.

In this thesis we extend the notion of André-Quillen homology from the categories of rings to the categories of algebras over a simplicial algebra. Strictly speaking, this result is also obtained by Goerss and Hopkins [22], but we are able to give more explicit proofs. We then study the notion of cellular complexes of simplicial algebras. This leads to an application of André-Quillen homology in terms of nuclear and atomic complexes, extending work of Baker and May [4]. We are able to set up a spectral sequence analogous to the fundamental spectral sequence by Quillen [46]. We define the notion of i-stable algebras with which we are able to compare André-Quillen homology for our algebras to Γ-homology using the theory of stabilisation.

The second part of the thesis deals with topological André-Quillen homology. We recall briefly the definition from Basterra and provide a proof about André-Quillen homology of free algebras, which has not appeared in this form in the literature before, although the result is well known. Finally, some results by Gilmour [20] about minimal atomic spectra are extended.

The outline of the chapters is as follows. We start by describing the categories of simplicial modules and algebras in the first chapter. The second chapter explains the notion of simplicial model categories. We define André-Quillen homology in this setting in Chapter 3. We prove that André-Quillen homology satisfies generalised Eilenberg-Steenrod axioms to form a homology theory. Chapter 4 is devoted to the definition of cellular complexes and the rôle André-Quillen homology plays with respect to these complexes. It turns out that André-Quillen homology behaves like ordinary homology for CW-complexes in topology. Furthermore, a result by Basterra for algebras with n-connected unit can be proven in our context. This allows us to define atomic and nuclear algebras in Chapter 5. In Chapter 6 we define the notion of i-stable algebras and use it to describe André-Quillen homology in terms of stabilisation, which then leads to a comparison with Γ-homology. The main result is that Γ-homology and André-Quillen homology coincides for low degrees. We end this part of the thesis with results about finiteness conditions and the construction of Quillen's fundamental spectral sequence in our context in the Chapters 7 and 8.

The second part of the thesis consists of Chapter 9 where we give a short introduction into topological André-Quillen homology and prove a result about the topological André-Quillen homology of free algebras. Finally, we complete results from Gilmour's thesis [20].

Part I

AQ

André-Quillen Homology

Chapter 1

Preliminaries

1.1 Simplicial Modules and Algebras

We summarise the notion of categories of simplicial objects. We follow [39] and [52] in exposition and notation. Let Δ be the category of ordinals $[n] = \{0, 1, \ldots, n\}$ with maps preserving the usual order $\leqslant$. Given a category $\mathfrak{C}$, we denote by $s\mathfrak{C} = [\Delta^{\mathrm{op}}, \mathfrak{C}]$ the category of (covariant) functors from Δ^{op} to $\mathfrak{C}$. Its objects are called simplicial objects in $\mathfrak{C}$. Most prominent are the categories $s\,\mathrm{Set}$ and $s\,\mathrm{Set}_*$ of simplicial sets and pointed simplicial sets. For a commutative ring R, we consider the categories $s\mathscr{M}_R$ of simplicial R-modules. Further there are the categories $s\mathscr{C}_R$ of simplicial commutative R-algebras and $s\mathscr{A}_R$ of simplicial not necessarily commutative R-algebras. All algebras are understood to be associative and unital, and algebra maps will always respect the unit. See Section 2 of this chapter for the notions involving non-unital algebras. We do not exclude the zero-ring 0 from our categories of algebras and it serves as the terminal object in these categories. But note that the category of modules over the zero-ring consists only of the zero-module.

If X is an element of $s\mathfrak{C}$, we write $X_n = X([n])$. The $n+1$ maps induced by the injective maps $d_i : [n-1] \to [n]$ are called *face maps* or *boundary maps* and denoted by

$$\delta_i : X_n \to X_{n-1}, \quad n \geqslant 1,\ 0 \leqslant i \leqslant n.$$

The $n+1$ maps induced by the surjective maps $s_i : [n+1] \to [n]$ are called *degeneracy maps* or *coboundary maps* and denoted by

$$\sigma_i : X_n \to X_{n+1}, \quad n \geqslant 0,\ 0 \leqslant i \leqslant n.$$

They satisfy the simplicial identities

$$\begin{aligned}
\delta_i\delta_j &= \delta_{j-1}\delta_i && i < j,\\
\sigma_i\sigma_j &= \sigma_{j+1}\sigma_i && i \leqslant j,\\
\delta_i\sigma_j &= \sigma_{j-1}\delta_i && i < j,\\
\delta_i\sigma_i &= \mathrm{id} && j \leqslant i \leqslant j+1,\\
\delta_i\sigma_j &= \sigma_j\delta_{i-1} && i > j+1.
\end{aligned}$$

On the other hand, given the data X_n and the boundary and the degeneracy maps for every n, then X is a simplicial object in a canonical way, because every map $[n] \to [k]$ factors as a composition of boundary and degeneracy maps. In a simplicial set, we call an element of X_n a simplex of X. Further, we call a simplex degenerate if it is in the image of a degeneracy map. Note, that applying any functor $F : \mathfrak{C} \to \mathfrak{D}$ will turn a simplicial object X in $s\mathfrak{C}$ into a simplicial object $F(X)$ in $s\mathfrak{D}$. There is always an "embedding" $\mathfrak{C} \to s\mathfrak{C}$ sending an object X to the constant simplicial object $\underline{X}$ with $\underline{X}_n = X$ and all boundary and degeneracy maps being identities.

In any category $\mathfrak{C}$ we write $\mathfrak{C}(A, B)$ for the set of morphisms from the object A to the object B. The standard n-simplex Δ^n is the simplicial set defined as

$$\Delta^n_k = \Delta([k], [n]),$$

where the degeneracies and boundary maps are induced by precomposition with the injective maps $[k-1] \to [k]$ and the surjective maps $[k+1] \to [k]$.

Geometric realisation is a functor $|-| : s\,\mathrm{Set} \to \mathrm{Top}$ from simplicial sets to the category of topological spaces, right adjoint to the functor *sing* which takes a topological space X to the simplicial set $sing(X)_n = \mathrm{Top}(\Delta^n_{\mathrm{top}}, X)$, where Δ^n_{top} is the standard topological n-simplex. Of course, there is a forgetful functor from simplicial algebras to simplicial modules and forgetful functors from these categories to simplicial sets. Thus, precomposing with the forgetful functor, we get geometric realisations of simplicial modules or algebras.

The homotopy groups π_*A of a simplicial set A are defined to be the homotopy groups of its geometric realisation. In the same way we can define the homotopy groups of any simplicial module or algebra. We set $\pi_iA = 0$ for $i < 0$ for any simplicial set A. For an alternative way of computing the homotopy groups of a simplicial module M, we can turn it into its Moore chain complex, denoted $K(M)$, which is defined by $K(M)_n = M_n$ and the differential being the alternating sum of the face-maps of M. It can be proven (e.g. see [52, 8.3.8]) that

$$\pi_*M = H_*K(M).$$

Moreover, $K(M)$ has a chain subcomplex which is acyclic. Define the normalisation functor $N : s\mathscr{M}_R \to Ch^+(R)$ from the category of simplicial R-modules to the category of bounded below R-module chain complexes by

$$N(M)_n = M_n / \bigcup_i \sigma_i(M_{n-1}), \quad n \geqslant 1$$

with differential δ_n. There is the identity

$$\pi_* M = H_* N(M).$$

Also, there is a right adjoint functor $\gamma : Ch^+(R) \to s\mathscr{M}_R$, so that

$$N : s\mathscr{M}_R \leftrightarrows Ch^+(R) : \gamma$$

is an equivalence of categories, the so called Dold-Kan correspondence [52, 8.4].

Consider the functor $\mathscr{M}_R \to s\mathscr{M}_R$ sending M to the constant simplicial object $\underline{M}$. Using the Moore complex, it is immediate that $\pi_{>0}(\underline{M}) = 0$ and $\pi_0(\underline{M}) = M$. So, these objects behave like Eilenberg-MacLane objects in $s\mathscr{M}_R$. We usually just write $0 = \underline{0}$ for the zero object in $s\mathscr{M}_R$ but hold on to the underline-notation for the non-zero constant objects in order to avoid confusion.

We say, a sequence of simplicial modules is exact, if it is exact in every degree. Using the Moore complex, it is straightforward that a short exact sequence $0 \to A \to B \to C \to 0$ of simplicial modules gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_n A \to \pi_n B \to \pi_n C \to \pi_{n-1} A \to \cdots$$

The tensor product of R-modules extends to a symmetric monoidal product on the category of simplicial R-modules, which is simply denoted by $\otimes_R$. It is defined by

$$(M \otimes_R N)_n = M_n \otimes_R N_n,$$

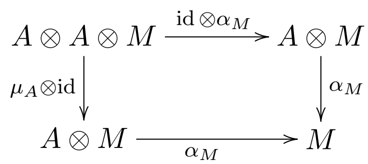
with the new face maps $\delta_i^{M \otimes N} = \delta_i^M \otimes \delta_i^N$ and the new degeneracies $\sigma_i^{M \otimes N} = \sigma_i^M \otimes \sigma_i^N$. We will write $\otimes$ for $\otimes_R$ if there is no danger of confusion.

If A is a simplicial R-algebra, then there is a multiplication

$$\mu_A : A \otimes A \to A$$

given levelwise by the usual algebra multiplications $A_n \otimes A_n \to A_n$.

In the end, we will work with algebras over a simplicial algebra. We define them by coherence diagrams and follow Quillen's original account [45, part II]. Fix a simplicial commutative R-algebra A. A module M over a simplicial R-algebra A is a simplicial R-module with a simplicial map $\alpha_M : A \otimes_R M \to M$, so that certain coherence diagrams commute. The diagrams in question are



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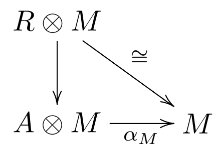
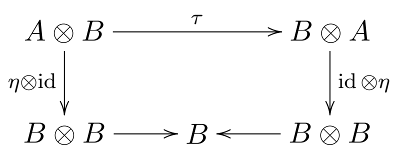


so that each M_n becomes an A_n-module. Maps of A-modules are the simplicial R-module maps that respect the α_M. We will denote the category of A-modules by $\mathscr{M}_A$ omitting R from the notation. Note that $\mathscr{M}_{\underline{R}} = s\mathscr{M}_R$.

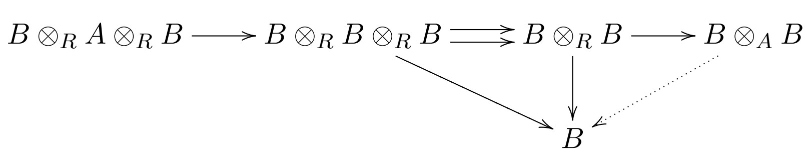
Now, an A-algebra B is a simplicial R-algebra B with a simplicial R-algebra map $\eta : A \to B$. We ask, that this map is central, if B is a non-commutative simplicial R-algebra, i.e. that the following diagram is commutative



The map η is referred to as the unit map. Together with the multiplication it turns B into an A-module via

$$\alpha : A \otimes B \xrightarrow{\eta \otimes \mathrm{id}} B \otimes B \to B.$$

The tensor product $B \otimes_A B$ is defined to be the coequaliser of the two maps $B \otimes_R A \otimes_R B \rightrightarrows B \otimes_R B$ obtained by $B \otimes_R A \otimes_R B \to B \otimes_R B \otimes_R B \rightrightarrows B \otimes_R B$ in $\mathscr{M}_{\underline{R}}$. From the diagram



we see that there is automatically a multiplication map $\mu_B : B \otimes_A B \to B$, which justifies that B is called an A-algebra. Since in a category of simplicial objects colimits are taken levelwise, we have

$$(B \otimes_A B)_n = B_n \otimes_{A_n} B_n$$

and μ_B is given levelwise by the multiplication of the A_n-algebra B_n. As usual there is the twist map $\tau : B \otimes_A B \to B \otimes_A B$, and we say that an algebra is commutative, if the obvious diagram commutes.

Given a category $\mathfrak{C}$ we can fix an object $X \in \mathfrak{C}$ and look at the categories $X \downarrow \mathfrak{C}$ and $\mathfrak{C} \downarrow X$, whose objects are objects Y in $\mathfrak{C}$ together with a specified map $X \to Y$ in the first and $Y \to X$ in the second case. Morphisms in these categories are commutative triangles. We call $X \downarrow \mathfrak{C}$ an under-category and $\mathfrak{C} \downarrow X$ an over-category. Sometimes the terminology comma-category is used in the literature. With this notation, the category of commutative A-algebras is exactly $A \downarrow s\mathscr{C}_R$ and we will denote this category simply by $\mathscr{C}_A$, again dropping R from the notation. Note that $\mathscr{C}_{\underline{R}} = s\mathscr{C}_R$.

The homotopy groups of a simplicial algebra or a simplicial module should inherit a graded algebra or module structure themselves. This is indeed the case and described in the following.

In the category $Ch^+(R)$ of bounded below chain complexes there is a symmetric monoidal product, which is denoted as well by $\otimes_R$. For two such chain complexes X and Y is given by

$$(X \otimes_R Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j$$

and the differential

$$d_n(x_i \otimes y_j) = d_n(x_i) \otimes y_j + (-1)^i x_i \otimes d_n(y_j).$$

Consider again the Moore functor

$$K : s\mathscr{M}_R \longrightarrow Ch^+(R),$$

which takes a simplicial R-module A to the chain complex $K(A)$. The Eilenberg-Zilber theorem [35, 8.1] states, that for two simplicial modules A and B there is a chain equivalence between $K(A \otimes_R B)$ and $K(A) \otimes_R K(B)$. The first tensor product is the one of simplicial modules, the second one is the one for chain complexes.

In this paragraph only we will write $\otimes^s$ for the simplicial tensor product and $\otimes^c$ for the chain complex tensor products in order to avoid confusion. Later, it will always be clear from the context, which tensor product is meant. We will use the identity $\pi_i = H_i \circ K$ and regard $\pi_*(-)$ and $H_*(-)$ as chain complexes with trivial differentials. With this notation we get for a simplicial R-algebra A the graded product

$$\begin{array}{ccc}
\pi_*A \otimes_R^c \pi_*A & \xrightarrow{=} & H_*K(A) \otimes_R^c H_*K(A) \\
 & & \downarrow \\
 & & H_*(K(A) \otimes_R^c K(A)) \\
 & & {\scriptstyle i}\downarrow{\scriptstyle \cong} \\
 & & H_*K(A \otimes_R^s A) \\
 & & \downarrow \\
 & & H_*K(A) \xrightarrow{=} \pi_*A
\end{array}$$

where the map i is the Eilenberg-Zilber map [35, 8.5].

If we have an A-module M, we get a graded map $\pi_*A \otimes_R^c \pi_*M \to \pi_*M$ in the same manner. It is straightforward to check that these maps satisfy all the coherence diagrams involved. They turn π_*A into a graded algebra and π_*M into a graded module over π_*A.

Now let us consider the case of an A-algebra B. π_*B is a graded R-algebra and comes with a module structure map $\pi_*A \otimes_R^c \pi_*B \to \pi_*B$ which makes π_*B a graded π_*A-module. Therefore we can write $\pi_*B \otimes_{\pi_*A}^c \pi_*B$ as the obvious coequaliser. Explicitly it is given as

$$\pi_*B \otimes_{\pi_*A}^c \pi_*B = \pi_*B \otimes_R^c \pi_*B / \{xa \otimes y - x \otimes ay, \quad x, y \in \pi_*B,\ a \in \pi_*A\}$$

where we use the graded module structure of π_*B.

We would like to have a map $\pi_*B \otimes_{\pi_*A}^c \pi_*B \to \pi_*B$. Using the map from the preceding diagram, we can set up the commutative diagram

$$\begin{array}{ccccc}
\pi_*B \otimes_R^c \pi_*A \otimes_R^c \pi_*B & \rightrightarrows & \pi_*B \otimes_R^c \pi_*B & \longrightarrow & \pi_*B \otimes_{\pi_*A} \pi_*B \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \vdots \\
\pi_*(B \otimes_R^s A \otimes_R^s B) & \rightrightarrows & \pi_*(B \otimes_R^s B) & \longrightarrow & \pi_*(B \otimes_A^s B) \\
 & & & & \downarrow \\
 & & & & \pi_*(B)
\end{array}$$

where the first line is a coequaliser diagram of graded modules. Hence there is a map $\pi_*B \otimes_{\pi_*A}^c \pi_*B \to \pi_*B$ of graded R-modules. The same argument works for a B-module M, where we will get a map $\pi_*B \otimes_{\pi_*A}^c \pi_*M \to \pi_*M$.

If B is a commutative A-algebra, then π_*B is a graded commutative π_*A-algebra. This means, writing μ for the multiplication, that for $x \in \pi_pB$ and $y \in \pi_qB$ we have $\mu(x \otimes y) = (-1)^{pq}\mu(y \otimes x)$. This can be seen by analysing the Eilenberg-Zilber map $i : K(B) \otimes_R^c K(B) \to K(B \otimes_R^s B)$. Let us write Σ_n for the symmetric group on n letters.

For $a \in B_p$ and $b \in B_q$, the Eilenberg-Zilber map is defined by

$$i(a \otimes b) = \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} (\sigma_{\mu(q)} \cdots \sigma_{\mu(1)} a \otimes \sigma_{\nu(p)} \cdots \sigma_{\nu(1)} b),$$

where the summation goes over all (q,p)-shuffles $(\mu,\nu) \in \Sigma_{q+p}$ and the sign $\epsilon(\mu)$ of such a shuffle is given by

$$\epsilon(\mu) = \frac{p(p-1)}{2} + \sum_{i=1}^{p} \mu(i).$$

From now on we use the standing assumption, that all our algebras are commutative.

1.2 Colimits, Limits and Adjunctions for Modules, Algebras and Nucas

Sometimes it will be necessary to drop the unit condition for algebras. Sticking to the terminology of [6], not necessarily unital commutative R-algebras shall be called R-nucas and we denote this category by $\mathscr{N}_R$. Their simplicial versions are denoted $\mathscr{N}_{\underline{R}}$ and $\mathscr{N}_A$, where A is a commutative unital simplicial R-algebra. Explicitly, an A-nuca N is an A-module N together with a multiplication

$$\mu : N \otimes_A N \to N,$$

satisfying the coherence diagrams

$$\begin{CD}
N \otimes_A N \otimes_A N @>{\mathrm{id} \otimes \mu}>> N \otimes_A N \\
@V{\mu \otimes \mathrm{id}}VV @VV{\mu}V \\
N \otimes_A N @>>{\mu}> N
\end{CD}$$

for associativity,

$$\begin{array}{ccc}
N \otimes_A N & \xrightarrow{\tau} & N \otimes_A N \\
\Big\downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mu} & \\
N & &
\end{array}$$

for commutativity and

$$\begin{CD}
A \otimes_R N \otimes_A N @>{\alpha \otimes \mathrm{id}}>> N \otimes_A N \\
@V{\mathrm{id} \otimes \mu}VV @VV{\mu}V \\
A \otimes_R N @>>{\alpha}> N
\end{CD}$$

for the compatibility of the module structure with the multiplication. So, A-nucas behave like A-algebras, but there will not in general be a map $A \to N$.

The map $N \otimes_R N \to N \otimes_A N$ from the coequaliser diagram turns an A-nuca N into an $\underline{R}$-nuca, so that there is as well a forgetful functor from A-nucas to $\underline{R}$-nucas.

On the other hand, we get a chain of forgetful functors $\mathscr{C}_A \to \mathscr{N}_A \to \mathscr{M}_A$. In the other direction are accordingly free functors, which we will describe now. First, let $\mathbb{P}_A$ denote the free functor $\mathscr{M}_A \to \mathscr{C}_A$. Explicitly, for an A-module M we set

$$M^{\otimes_A n} = \underbrace{M \otimes_A \cdots \otimes_A M}_{n \text{ times}}$$

so that we can write

$$\mathbb{P}_A M = A \ \oplus \ M \ \oplus \ M^{\otimes_A 2}/\Sigma_2 \ \oplus \ M^{\otimes_A 3}/\Sigma_3 \ \oplus \ \cdots,$$

where for all n the action of the symmetric group Σ_n, that permutes the n factors of $M^{\otimes_A n}$, is quotiented out. Everything happens in each simplicial degree separately. For a non-simplicial R-algebra S we can write $\mathbb{P}_S$ for the symmetric algebra functor $\mathscr{M}_S \to \mathscr{C}_S$. Then

$$(\mathbb{P}_A M)_n = \mathbb{P}_{A_n} M_n = A_n \ \oplus \ M_n \ \oplus \ M_n^{\otimes_{A_n} 2}/\Sigma_2 \ \oplus \ \cdots$$

We write throughout U for every instance of a forgetful functor, making sure by other means that is always clear from the context which structure it forgets. There is the adjunction

$$\mathscr{C}_A(\mathbb{P}_A M, B) \cong \mathscr{M}_A(M, U(B))$$

between A-algebras and A-modules. Let $\mathbb{A}_A M$ be the cokernel of the map $A \to \mathbb{P}_A M$, i.e.

$$\mathbb{A}_A M = M \ \oplus \ M^{\otimes_A 2}/\Sigma_2 \ \oplus \ M^{\otimes_A 3}/\Sigma_3 \ \oplus \ \cdots$$

Then, $\mathbb{A}_A M$ is an A-nuca and we get the adjunction

$$\mathscr{N}_A(\mathbb{A}_A M, N) \cong \mathscr{M}_A(M, U(N))$$

between A-nucas and A-modules. In order to pass from $\mathscr{N}_A$ to $\mathscr{C}_A$ we simply attach a unit

$$\mathscr{C}_A(A \oplus N, B) \cong \mathscr{N}_A(N, U(B))$$

where the multiplication on $A \oplus N$ is given by

$$(A \oplus N) \otimes_A (A \oplus N) \cong A \oplus N \oplus N \oplus (N \otimes_A N) \to A \oplus N$$

using the multiplication of N and folding.

The forgetful functor from $\mathscr{M}_A$ to $\mathscr{M}_{\underline{R}}$ has a left adjoint, too. More generally, for an A-algebra B, there is the following adjunction

$$\mathscr{M}_B(B \otimes_A M, M') \cong \mathscr{M}_A(M, U(M')).$$

Recall, that in the category of commutative A-algebras the following square is a pushout:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C' \otimes_B C \end{array}$$

As a special case we see the adjunction

$$\mathscr{C}_B(B \otimes_A C, C') \cong \mathscr{C}_A(C, U(C'))$$

between B-algebras and A-algebras.

There is as well an adjunction

$$\mathscr{N}_B(B \otimes_A N, N') \cong \mathscr{N}_A(N, U(N'))$$

between B-nucas and A-nucas. This can be proven by observing that for any A-module M we have

$$B \otimes_A \mathbb{A}_A U(M) = \mathbb{A}_B(B \otimes_A M)$$

and using the fact that

$$\mathbb{A}_A^2 N \rightrightarrows \mathbb{A}_A N \to N$$

is a coequaliser [52, 8.6].

All these categories have a forgetful functor to simplicial sets. In each case, there is a free functor left adjoint to it. For the category of simplicial R-modules, this functor takes a simplicial set K to $R\{K\}$, the module which in degree n is the free R-module on the set of simplices in K_n. The free functor from simplicial sets to any of the other categories above is just the free functor from simplicial R-modules to it, precomposed with the free functor from simplicial sets to simplicial R-modules. We mention in particular the free A-module functor, which sends K to $A\{K\}$ which in degree n is given by the free A_n-module on the simplices in K_n. Therefore, the free A-algebra functor, sends a simplicial set K to the simplicial A-algebra $A[K]$, which in degree n is the polynomial algebra on $|K_n|$ variables and coefficients A_n.

We summarise briefly some facts about limits and colimits. Limits and colimits of simplicial objects can be taken degreewise. We use constantly the standard fact from category theory, that left adjoints always commute with colimits and limits always commute with right adjoints. As a result, limits of A-modules, A-algebras or A-nucas are obtained as limits in $s\,\mathrm{Set}$, thus are derived from limits in Set. Colimits in general do not commute with right adjoints. But in our case, sequential colimits of A-algebras or A-nucas are obtained from the ones of A-modules.

We can deduct the form of pushouts in the category of A-nucas from the one of A-algebras. If

$$\begin{CD} L @>>> M \\ @VVV @VVV \\ M' @>>> N \end{CD}$$

is a pushout of A-nucas, then

$$\begin{CD} A \oplus L @>>> A \oplus M \\ @VVV @VVV \\ A \oplus M' @>>> A \oplus N \end{CD}$$

is one of A-algebras, hence

$$A \oplus N = (A \oplus M') \otimes_{A \oplus L} (A \oplus M),$$

which determines N. Note that for $L = 0$, i.e. for the case of the cartesian coproduct, we obtain

$$N = M' \oplus M \oplus (M' \otimes_A M).$$

Every A-module M can be seen as a trivial A-nuca with zero-multiplication. We denote M^0 for a nuca obtained this way from a module M. On the other hand, for every A-nuca, we can form the pushout along the multiplication as a module map

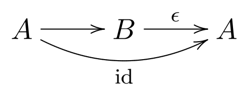
$$\begin{CD} N \otimes_A N @>>> 0 \\ @VVV @VVV \\ N @>>> Q_A(N) \end{CD}$$

i.e. $Q_A(N)$ is the cokernel of the multiplication. It is an A-module and called the *indecomposables* of N. There is the adjunction

$$\mathscr{N}_A(N, M^0) \cong \mathscr{M}_A(Q_A(N), M)$$

so that Q_A turns out to be a left adjoint.

Very closely related to A-nucas is $\mathscr{C}_A \downarrow A$, the category of A-algebras over A. Its objects are A-algebras B together with an augmentation $\epsilon : B \to A$, so that



is a commutative triangle. Let $I_A(B)$ denote the kernel of ϵ. Then $I_A(B)$ is equipped with the structure of a B-nuca and therefore with the structure of an A-nuca. On the other hand, an A-nuca N can be turned into an A-algebra over A by attaching a unit as before and we obtain an adjunction

$$\mathscr{C}_A \downarrow A(A \oplus N, B) = \mathscr{N}_A(N, I_A(B)). \tag{1.1}$$

Obviously

$$I_A(A \oplus N) = N,$$

but it is as well straightforward to prove that

$$A \oplus I_A(B) \cong B$$

as algebras, so that the adjunction is an equivalence of categories. The functor $\mathbb{P}_A$ takes an A-module to an A-algebra over A. It figures in the adjunction

$$\mathscr{C}_A \downarrow A(\mathbb{P}_A M, B) \cong \mathscr{M}_A(M, I_A(B))$$

which can be derived from (1.1).

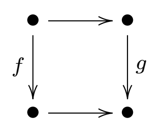
Chapter 2

Simplicial Model Structures

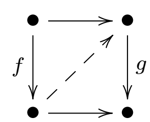
Before describing the model structures for our modules and algebras we present all the category-theoretical background in the first two sections.

2.1 Model Categories

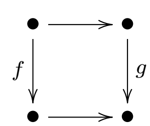
We recall the basic definitions following [16]. In a category $\mathfrak{C}$, we say that a class of morphisms I_1 has the *left lifting property (LLP)* with respect to another class of morphisms I_2 if every commutative square



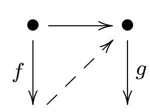
with $f \in I_1$ and $g \in I_2$ has a lift



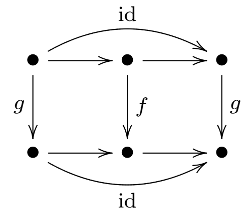
so that both triangles commute. Similarly, we say that a class of morphisms J_1 has the *right lifting property (RLP)* with respect to another class J_2, if every commutative square



with $g \in J_1$ and $f \in J_2$ has a lift



so that both triangles commute. By a *retract* of a map f we understand a map g together with the following diagram



Definition 2.1.1. A *model category* $\mathfrak{C}$ is a category with three distinguished classes of morphisms, each containing the identity maps and being closed under composition, subject to various axioms. The three classes are called *fibrations*, *cofibrations* and *weak equivalences*, denoted $\twoheadrightarrow, \rightarrowtail, \xrightarrow{\sim}$ respectively. A fibration or cofibration, which is simultaneously a weak equivalence is called *acyclic* or synonymously *trivial*. The axioms are the following:

(M1) $\mathfrak{C}$ is complete and cocomplete (i.e. bicomplete).

(M2) Given two composable functions f and g, if two out of the functions $f, g, g \circ f$ are weak equivalences, so is the third.

(M3) All three classes are closed under retracts.

(M4) The fibrations have the RLP with respect to the acyclic cofibrations and the cofibrations have the LLP with respect to the acyclic fibrations.

(M5) Every map factors as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration.

Let us remark the following: Due to bicompleteness, there is an initial object $\emptyset$ and a terminal object $*$. An object X so that $\emptyset \to X$ is a cofibration is called *cofibrant*. An object X so that $X \to *$ is a fibration is called *fibrant*. (M5) allows us to write for every object Y

$$\emptyset \rightarrowtail Y^c \overset{\sim}{\twoheadrightarrow} Y$$

and

$$Y \overset{\sim}{\rightarrowtail} Y^f \twoheadrightarrow *$$

with new objects Y^c and Y^f, which are weakly equivalent to Y. They are called *cofibrant replacement* and *fibrant replacement* respectively.

Another remark concerns isomorphisms in $\mathfrak{C}$. An isomorphism $f : A \to B$ is automatically in all three classes. The diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f^{-1}} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\mathrm{id}]{} & B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

shows that f is always a retract of the identity id_B. As a corollary we state: If $\mathfrak{C}$ is a model category and $\mathfrak{D}$ another category equivalent to $\mathfrak{C}$, then the model structure from $\mathfrak{C}$ is carried over so that $\mathfrak{D}$ is a model category,

The following proposition is straightforward from the definition. We state it for further reference

Proposition 2.1.2. *If one fixes an object X in a model category $\mathfrak{C}$, then the categories $X \downarrow \mathfrak{C}$ and $\mathfrak{C} \downarrow X$ are automatically model categories. A map is a cofibration, fibration or weak equivalences if and only if this map is such in $\mathfrak{C}$.*

Note however that cofibrant (fibrant) objects in $\mathfrak{C}$ do not need to be cofibrant (fibrant) objects in $X \downarrow \mathfrak{C}$ ($\mathfrak{C} \downarrow X$) and vice versa.

We will use the following basic proposition about model categories, which for example can be found in [16].

Proposition 2.1.3. *In a model category, two classes always determine the third. Explicitly:*

- *The cofibrations are exactly the maps which satisfy the LLP with respect to acyclic fibrations.*
- *The acyclic cofibrations are exactly the maps which satisfy the LLP with respect to fibrations.*
- *The fibrations are exactly the maps which satisfy the RLP with respect to acyclic cofibrations.*
- *The acyclic fibrations are exactly the maps which satisfy the RLP with respect to cofibrations.*
- *The weak equivalences are exactly the maps which factor as an acyclic cofibration followed by an acyclic fibration.*

Every model category $\mathfrak{C}$ comes equipped with a notion of homotopy, which we will discuss briefly. Fix an object $A \in \mathfrak{C}$. A cylinder object $C(A)$ for A is an object, so that the fold map $A \amalg A \to A$ from the cartesian coproduct into A factors as

$$A \amalg A \to C(A) \xrightarrow{\sim} A.$$

Two maps $f, g : A \to B$ are called left homotopic, if there exists a cylinder object $C(A)$ so that the sum map $f + g : A \amalg A \to B$ factors as

$$A \amalg A \to C(A) \to B.$$

When A is cofibrant and B fibrant, the notion of left homotopy turns out to be an equivalence relation, just called homotopy. The homotopy category $\operatorname{ho}\mathfrak{C}$ of the model category $\mathfrak{C}$ can be defined as follows:

- objects of $\operatorname{ho}\mathfrak{C}$ are the objects of $\mathfrak{C}$,
- morphisms $\operatorname{ho}\mathfrak{C}(A, B)$ are homotopy classes of morphisms from A^c to B^f.

We get a functor $\mathfrak{C} \to \operatorname{ho}\mathfrak{C}$. If a morphism in $\mathfrak{C}$ is a weak equivalence, then it passes to an isomorphism in the homotopy category.

We use the convention to write

$$F : \mathfrak{C} \rightleftarrows \mathfrak{D} : G$$

for an adjoint pair of functors between categories $\mathfrak{C}$ and $\mathfrak{D}$. It is implicitly understood, that F is the left adjoint and G the right adjoint.

Definition 2.1.4. An adjunction between model categories

$$F : \mathfrak{C} \rightleftarrows \mathfrak{D} : G$$

is called a *Quillen adjunction*, if the following equivalent conditions hold:

(C1) F preserves cofibrations and acyclic cofibrations,

(C2) G preserves fibrations and acyclic fibrations,

(C3) F preserves cofibrations and G preserves fibrations.

Such a pair is called a *Quillen equivalence* if given a cofibrant object X and a fibrant object Y, then $X \to G(Y)$ is a weak equivalence whenever $F(X) \to Y$ is.

The most prominent example of a Quillen equivalence are the functors *sing* and the geometric realisation between simplicial sets and topological spaces [16]. We will see more examples of Quillen adjunctions in the next section.

If F and G form a Quillen adjunction, then the functors pass down to an adjunction in the homotopy categories

$$\mathrm{L}F : \mathrm{ho}(\mathfrak{C}) \rightleftarrows \mathrm{ho}(\mathfrak{D}) : \mathrm{R}G.$$

This means that there are functors $\mathrm{L}F$, called total left derived functor of F, and $\mathrm{R}G$, called total right derived functor of G, so that for cofibrant $X \in \mathfrak{C}$ we have $F(X) \simeq \mathrm{L}F(X)$ and for fibrant $Y \in \mathfrak{D}$ we have $G(Y) \simeq \mathrm{R}G(Y)$. The adjunction of derived functors above is an equivalence of the homotopy categories if and only if the pair (F, G) is a Quillen equivalence.

For further reference we state

Lemma 2.1.5 (Ken Brown's lemma). *Whenever a functor F between model categories carries acyclic cofibrations between cofibrant objects to weak equivalences, then it respects weak equivalences between cofibrant objects. Dually, if F carries acyclic fibrations between fibrant objects to weak equivalences, then it respects weak equivalences between fibrant objects.*

Proof. See Lemma 9.9 in [16] for a proof. □

2.2 Simplicial Model Categories

We now turn to simplicial model categories. In the category of simplicial sets we can look at the set of maps $s\,\mathrm{Set}(\Delta^n \times K, L)$. They assemble to a simplicial set $\underline{Hom}_{s\,\mathrm{Set}}(K, L)$, so that $\underline{Hom}_{s\,\mathrm{Set}}(K, L)_0 = s\,\mathrm{Set}(K, L)$. That is to say, the category of simplicial sets is enriched over itself. We need to discuss categories which admit simplicial hom-sets. They are part of the next definition, which can be found in either [23], [45] or [50].

Definition 2.2.1. A category $\mathfrak{C}$ is a *simplicial category* if it is equipped with bifunctors

$$\underline{Hom}(-,-) : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \longrightarrow s\,\mathrm{Set},$$

$$- \otimes - : s\,\mathrm{Set} \times \mathfrak{C} \longrightarrow \mathfrak{C},$$

$$F(-,-) : s\,\mathrm{Set}^{\mathrm{op}} \times \mathfrak{C} \longrightarrow \mathfrak{C},$$

subject to the following conditions, where X, Y and Z denote objects in $\mathfrak{C}$ and K is a simplicial set:

- There is a natural map

$$\underline{Hom}(X,Y) \times \underline{Hom}(Y,Z) \longrightarrow \underline{Hom}(X,Z)$$

called composition, which satisfies an associativity coherence-diagram.

- There is a natural isomorphism

$$\underline{Hom}(X,Y)_0 \cong \mathfrak{C}(X,Y).$$

- There are natural adjunctions

$$\underline{Hom}(K \otimes X, Y) \cong \underline{Hom}_{s\,\mathrm{Set}}(K, \underline{Hom}(X,Y)) \cong \underline{Hom}(X, F(K,Y)).$$

The simplicial set $\underline{Hom}(X,Y)$ is called the *simplicial hom-sets.* $\otimes$ is called the *tensor structure* and F the *cotensor structure*.

We usually write X^K for $F(K,X)$. If necessary we add a subscript to $\underline{Hom}$, $\otimes$ or F in order to state explicitly to which category they belong to. Note that taking zero simplices of the last line in the definition yields the adjunctions

$$\mathfrak{C}(K \otimes X, Y) \cong s\,\mathrm{Set}(K, \underline{Hom}(X,Y)) \cong \mathfrak{C}(X, F(K,Y)),$$

so that for a fixed simplicial set K the functor $K \otimes -$ is a left adjoint and the functor $F(K,-)$ is a right adjoint. The category $s\,\mathrm{Set}$ is obviously an example of a simplicial category with

$$K \otimes L = K \times L,$$
$$\underline{Hom}(K,L) = F(K,L) = \underline{Hom}_{s\,\mathrm{Set}}(K,L).$$

If the category $\mathfrak{C}$ is pointed, i.e. there exists a zero object 0 which serves as the initial and the terminal object, then we can formulate the same conditions as above, replacing $s\,\mathrm{Set}$ with $s\,\mathrm{Set}_*$. For any two objects A and B in $\mathfrak{C}$, there is a unique zero map, namely the composition $A \to 0 \to B$ which is the basepoint in $\mathfrak{C}(A,B)$. We say that $\mathfrak{C}$ is a *pointed simplicial category* if it satisfies the modified conditions. We agree to write $\hat{\otimes}$ for the pointed tensor. Many categories thus admit simultaneously a pointed and an unpointed simplicial structure, where the functor $\underline{Hom}$ is the same. For a simplicial set K we write K_+ for the same simplicial set with a disjoint basepoint added to it.

We have the following basic facts about tensors.

Proposition 2.2.2. *If $\mathfrak{C}$ is a simplicial category then for every object A*

$$* \otimes A \cong A,$$

where $*$ is the simplicial set consisting of one single point in every degree. If there is an initial object $\emptyset$ then for all simplicial sets K

$$K \otimes \emptyset = \emptyset.$$

If $\mathfrak{C}$ is a pointed simplicial category, then

$$S^0 \hat{\otimes} A \cong A$$

and again

$$K \hat{\otimes} 0 = 0$$

for every pointed simplicial set K, where we write 0 for the zero object in $\mathfrak{C}$. Moreover, if $\mathfrak{C}$ is pointed and unpointed with the same simplicial hom-sets, then

$$K_+ \hat{\otimes} A \cong K \otimes A.$$

Proof. This follows all from Yoneda's lemma. Taking a test-object $B \in \mathfrak{C}$ it follows that

$$\mathfrak{C}(* \otimes A, B) \cong s\,\mathrm{Set}(*, \underline{Hom}(A, B)) \cong \underline{Hom}(A, B)_0 \cong \mathfrak{C}(A, B)$$

so that $* \otimes A$ and A are isomorphic. For the initial object $\emptyset$ the set $\mathfrak{C}(\emptyset, B)$ consists of exactly one point. So

$$\mathfrak{C}(K \otimes \emptyset, B) \cong * \cong \mathfrak{C}(\emptyset, B).$$

The proof for $S^0 \hat{\otimes} -$ is exactly the same. The last statement follows from

$$\mathfrak{C}(K_+ \hat{\otimes} A, B) \cong s\,\mathrm{Set}_*(K_+, \underline{Hom}(A, B)) = s\,\mathrm{Set}(K, \underline{Hom}(A, B)) \cong \mathfrak{C}(K \otimes A, B)$$

and Yoneda's lemma again. □

The next definition links simplicial structure with model structure.

Definition 2.2.3. A category $\mathfrak{C}$ is a *simplicial model category* if it is simultaneously a model category and a simplicial category and either of the following equivalent axioms (traditionally called SM7 or corner axiom) is satisfied:

(SM7) If $\alpha : A \rightarrowtail B$ is a cofibration in $\mathfrak{C}$ and $\beta : X \twoheadrightarrow Y$ a fibration in $\mathfrak{C}$, then the map

$$\underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)$$

is a fibration of simplicial sets which is a weak equivalence if α or β is.

(SM7′) If $\alpha : X \twoheadrightarrow Y$ is a fibration in $\mathfrak{C}$ and $\beta : K \rightarrowtail L$ a cofibration in $s\,\mathrm{Set}$, then the map

$$X^L \longrightarrow X^K \times_{Y^K} Y^L$$

is a fibration in $\mathfrak{C}$, which is a weak equivalence if α or β is.

(SM7″) If $\alpha : A \rightarrowtail B$ is a cofibration in $\mathfrak{C}$ and $\beta : K \rightarrowtail L$ a cofibration in $s\,\mathrm{Set}$, then the map

$$(L \otimes A) \amalg_{K \otimes A} (K \otimes B) \longrightarrow L \otimes B$$

is a cofibration in $\mathfrak{C}$ which is a weak equivalence if α or β is.

Proposition 2.2.4. *Let $\mathfrak{C}$ be a simplicial model category. If we fix a cofibrant object $A \in \mathfrak{C}$, then the adjunction*

$$- \otimes A : s\,\mathrm{Set} \leftrightarrows \mathfrak{C} : \underline{Hom}(A, -)$$

is a Quillen adjunction. Moreover, if $\mathfrak{C}$ admits a pointed simplicial structure, then the adjunction

$$- \hat{\otimes} A : s\,\mathrm{Set}_* \leftrightarrows \mathfrak{C} : \underline{Hom}(A, -)$$

is a Quillen adjunction, too.

Proof. We show that the functor $\underline{Hom}(A, -)$ respects fibrations and trivial fibrations. Let $X \twoheadrightarrow Y$ be a fibration in $\mathfrak{C}$. Together with the map $\emptyset \rightarrowtail A$ we can form the axiom (SM7) and get a fibration

$$\underline{Hom}(A, X) \twoheadrightarrow \underline{Hom}(\emptyset, X) \times_{\underline{Hom}(\emptyset,Y)} \underline{Hom}(A, Y) = \underline{Hom}(A, Y)$$

using Proposition 2.2.2. The same happens for acyclic fibrations, so that both adjunctions are Quillen adjunctions. □

This has the following corollary.

Corollary 2.2.5. *If A is a cofibrant object in $\mathfrak{C}$ and K any simplicial set, then $K \otimes A$ is cofibrant. If furthermore $\mathfrak{C}$ is pointed and admits a pointed simplicial structure, then also $K \hat{\otimes} A$ is cofibrant for any pointed simplicial set K.*

Proof. We use again Proposition 2.2.2. In 2.3.1 we state that the cofibrations in the category of simplicial sets are the injections. Therefore every simplicial set is cofibrant. The map $\emptyset \to K$ is always a cofibration in $s\,\mathrm{Set}$, hence $\emptyset = \emptyset \otimes A \to K \otimes A$ is a cofibration. The map $* \to K$ is always a cofibration in $s\,\mathrm{Set}_*$, hence $0 = * \hat{\otimes} A \to K \hat{\otimes} A$ is a cofibration. □

In order to lift model category structures from one category to another one, we need the technical notion of a cofibrantly generated model category. For our purposes the following definition suffices, which we quote from [23].

Definition 2.2.6. Take a class of maps C in a category $\mathfrak{C}$. Then an object $A \in \mathfrak{C}$ is said to be *small* with respect to C, if for any sequence $X_1 \to X_2 \to X_3 \to \cdots$ of morphisms in C the map

$$\operatorname*{colim}_n \mathfrak{C}(A, X_n) \longrightarrow \mathfrak{C}(A, \operatorname*{colim}_n X_n)$$

is an isomorphism.

With this notion of small, we can define cofibrantly generated model categories.

Definition 2.2.7. A model category $\mathfrak{C}$ is *cofibrantly generated* if there are sets of morphism I and J such that the following hold.

- The source of every morphism in I is small with respect to the class of cofibrations, and a map in $\mathfrak{C}$ is an acyclic fibration if it has the RLP with respect to every map in I.
- The source of every morphism in J is small with respect to the class of acyclic cofibrations, and a map in $\mathfrak{C}$ is a fibration if it has the RLP with respect to every map in J.

We say that I generates the cofibrations and J generates the acyclic cofibrations. One benefit of cofibrantly generated model categories is the fact that, due to the small object argument, cofibrant and fibrant replacements can be chosen functorially. The following theorem can be found in [24] or [23].

Theorem 2.2.8. *Let*

$$F : \mathfrak{C} \rightleftarrows \mathfrak{D} : G$$

be adjoint functors, where $\mathfrak{C}$ is a cofibrantly generated model category. A map α in $\mathfrak{D}$ is called a fibration or a weak equivalence, if $G(\alpha)$ is one in $\mathfrak{C}$. A map in $\mathfrak{D}$ is called *a cofibration, if it satisfies the LLP with respect to all acyclic fibrations. If furthermore G commutes with sequential colimits and every cofibration in $\mathfrak{D}$, which satisfies the LLP with respect to all fibrations is a weak equivalence, then $\mathfrak{D}$ with these notions of fibrations, cofibrations and weak equivalences becomes a cofibrantly generated model category. If the generating sets for $\mathfrak{C}$ are given by I and J, then $F(I)$ and $F(J)$ generate the cofibrations and the acyclic cofibrations in $\mathfrak{D}$. If there is a fibrant replacement functor in $\mathfrak{D}$, then the LLP-condition for cofibrations holds automatically.*

2.3 Simplicial Model Structures for Modules and Algebras

There is a standard way of producing tensors and cotensors in a category of simplicial objects. Assume a category $\mathfrak{C}$ is complete and cocomplete. We can construct tensors in $s\mathfrak{C}$ in the following way. If $\underline{X}$ is a constant simplicial object in $s\mathfrak{C}$ and $K \in s\,\mathrm{Set}$, then

$$(K \otimes \underline{X})_n = \coprod_{x \in K_n} X,$$

where the boundary and degeneracy maps are induced by the ones of K. If Y is any object in $s\mathfrak{C}$, we can look at the construction above, which yields a bisimplicial object, and take the diagonal. Explicitly

$$(K \otimes Y)_n = \coprod_{x \in K_n} Y_n.$$

Theorem 2.5. in [23] asserts that $s\mathfrak{C}$ becomes a simplicial category with the simplicial hom-sets given as

$$\underline{Hom}(A, B)_n = s\mathfrak{C}(\Delta^n \otimes A, B).$$

Before we are going to describe the simplicial model structures for modules, algebras and nucas over a simplicial R-algebra, we first have to deal with the standard simplicial model structure on $s\,\mathrm{Set}$, whose description can for example be found in [23].

Proposition 2.3.1. *The category of simplicial sets is a simplicial model category. The model structure is given by*

- *the map f is a weak equivalence if $|f|$ is a weak equivalence in topological spaces,*
- *the map f is a fibration if it is a Kan-fibration, i.e. f is a map with the RLP with respect to inclusion of horns $\Lambda^n_k \to \Delta^n$,*
- *the map f is a cofibration if it is an injection.*

For a simplicial R-algebra A and a simplicial set K recall the notation $A\{K\}$ for the free functor from simplicial sets to A-modules. We extend this notation slightly. For an A-module M let $M\{K\}$ denote the A-module $M \otimes_R R\{K\}$.

Theorem 2.3.2. *Over a simplicial R-algebra A, the categories of A-modules, A-algebras and A-nucas are all cofibrantly generated simplicial model categories. The model structure is in all three cases given by*

- *f is a weak equivalence if $\pi_i(f)$ is an isomorphism for all i,*
- *f is a fibration if f is a fibration as a function in $s\,\mathrm{Set}$,*
- *f is a cofibration if it has the left lifting property with respect to acyclic fibrations.*

Moreover, in any of these categories, all objects are fibrant.

Proof. It is a bootstrap-argument, starting with simplicial R-modules. A proof for simplicial R-modules and R-algebras can be found in various sources, for example see [45], [23] or [24]. Basically, the Theorem 2.2.8 is applied to the adjunctions between simplicial sets, simplicial abelian groups and simplicial R-modules or R-algebras. The tensors are given by the method above. Explicitly, in the category of simplicial R-modules, we write $\times$ for the tensors, and we have

$$K \times M = M\{K\}.$$

The cotensors are given as

$$F(K, M) = \underline{Hom}_{s\,\mathrm{Set}}(K, U(M)),$$

where U is the forgetful functor to simplicial sets and we use the $\underline{R}$-module structure on M to obtain one on $F(K, M)$. The simplicial hom-sets are

$$\underline{Hom}_{\mathscr{M}_{\underline{R}}}(M, N)_n = \mathscr{M}_{\underline{R}}(\Delta^n \times M, N).$$

For an R-algebra A, the tensors $K \otimes_{\mathscr{C}_{\underline{R}}} A$ are given by the general construction above, i.e.

$$(K \otimes_{\mathscr{C}_{\underline{R}}} A)_n = \bigotimes_{x \in K_n} A_n.$$

However, they do not admit anymore such an easy description as the tensors of $\underline{R}$-modules. The cotensors $F(K, A)$ are the same as in the case of modules. We simply use the algebra structure on A to obtain such a structure one on $F(K,A)$. The simplicial hom-sets are given as

$$\underline{Hom}_{\mathscr{C}_{\underline{R}}}(A,B)_n = \mathscr{C}_{\underline{R}}(\Delta^n \otimes A, B).$$

In the category of simplicial abelian groups, every object is fibrant (for example see [52]). Therefore all objects in any of the categories $\mathscr{M}_A$, $\mathscr{N}_A$ or $\mathscr{C}_A$ are fibrant. In particular, the identity functor is a fibrant replacement functor so that we do not need to check the LLP-condition for cofibrations in Theorem 2.2.8.

If we fix a simplicial R-algebra A, we can now lift this structure to a simplicial model structure on A-modules. Theorem 2.2.8 applied to the adjunction

$$A \otimes_R - : \mathscr{M}_{\underline{R}} \rightleftarrows \mathscr{M}_A : U$$

yields a model structure on $\mathscr{M}_A$. In order to show that it is a simplicial model category, we have to construct tensors and cotensors and prove *(SM7)*. The tensors and cotensors and are the same as for $\underline{R}$-modules. Explicitly, for any A-module M and M' and a simplicial set K

$$K \times M = A\{K\} \otimes_A M = M\{K\},$$
$$\underline{Hom}_{\mathscr{M}_A}(M,M')_n = \mathscr{M}_A(\Delta^n \times M, M').$$

The simplicial R-module

$$F_{\mathscr{M}_A}(K,M) = F_{\mathscr{M}_{\underline{R}}}(K,M)$$

obtains an A-module structure from M. Note that the simplicial hom sets for A-modules differ from the ones for $\underline{R}$-modules. It is straightforward to check that $\mathscr{M}_A$ is a simplicial category. Because the cotensors are exactly the same as above, the formulation *(SM7')* which only involves cotensors and fibrations, therefore holds trivially. Hence $\mathscr{M}_A$ is a simplicial model category.

In order to pass to A-algebras, we can simply note that $\mathscr{C}_A = A \downarrow \mathscr{C}_R$. It follows that $\mathscr{C}_A$ is a model category, with a map being a cofibration, a fibration or a weak equivalences if it is one when seen as a map in $\mathscr{C}_R$. This model structure on $\mathscr{C}_A$ is the same that could be obtained by lifting the model structure using the adjunction $(\mathbb{P}_A, U)$ between A-modules and A-algebras or the adjunction $(A\otimes_R, U)$ between A-algebras and R-algebras. Tensors are defined by the general construction above. Again, cotensors and fibrations are exactly as in $\mathscr{M}_{\underline{R}}$, so *(SM7)* holds as well.

The statement for simplicial nucas over a ring can be derived from the theorems [23, Theorem 4.4. and Example 5.2.]. The general case for nucas over a simplicial algebra A follows as above, using the adjunction

$$A \otimes_R - : \mathscr{N}_{\underline{R}} \rightleftarrows \mathscr{N}_A : U,$$

where U is the forgetful functor.

□

Note that in the category of A-algebras, the object A is initial, so that for any simplicial set K we have

$$K \otimes_{\mathscr{C}_A} A = A.$$

Some of our categories admit pointed simplicial structures, which we are going to describe now. We treat the case of A-modules first. Here it is enough to simply "quotient out" the basepoint. Define the bifunctor

$$\wedge : s\,\mathrm{Set}_* \times \mathscr{M}_A \longrightarrow \mathscr{M}_A$$

by forming the pushout of A-modules

$$\begin{array}{ccc} M & \longrightarrow & K \times M \\ \downarrow & & \downarrow \\ \underline{0} & \longrightarrow & K \wedge M \end{array}$$

using the inclusion of the basepoint $* \to K$ for the first horizontal map. This gives the required adjunction

$$\underline{Hom}(K \wedge M, Y) \cong s\,\mathrm{Set}_*(K, \mathscr{M}_A(M, Y)),$$

where the basepoint on the left is the zero map.

Let us now consider the category of A-nucas. Given a simplicial set K and an A-nuca N, we can form the pushout

$$\begin{array}{ccc} N & \longrightarrow & K \otimes N \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K \hat{\otimes} N \end{array}$$

using the unpointed tensor structure to define the pointed one. The pushout property gives that $\mathscr{N}_A(K \hat{\otimes} N, N')$ is isomorphic to the set of maps $\mathscr{N}_A(K \otimes N, N')$ that precomposed with $N \to K \otimes N$ are the zero maps. But using the adjunction $\mathscr{N}_A(K \otimes N, M) \cong s\,\mathrm{Set}(K, \underline{Hom}_{\mathscr{N}_A}(N.N'))$, these are exactly the basepoint-preserving maps, hence

$$\mathscr{N}_A(K \hat{\otimes} N, N') \cong s\,\mathrm{Set}_*(K, \underline{Hom}_{\mathscr{N}_A}(N, N')).$$

Since the category of A-algebras is not pointed (there is no "zero map" between two algebras), this category does not admit a pointed simplicial structure. However, the category $\mathscr{C}_A \downarrow A$ is pointed with zero object A. For any two algebras B and C over A there is now a distinguished map $B \to A \to C$, which serves as the basepoint in simplicial hom-sets. And indeed we obtain a pointed simplicial structure on this category.

The tensors $\hat{\otimes}$ in $\mathscr{C}_A \downarrow A$ can be defined via the pointed tensor structure of nucas. Writing I_A for the augmentation functor we have that every A-algebra B over A is isomorphic to $A \oplus I_A(B)$ and we set

$$K \hat{\otimes}_{\mathscr{C}_A} B = A \oplus (K \hat{\otimes}_{\mathscr{N}_A} I_A(B)).$$

Using that the zero map of nucas $N \to N'$ corresponds to $A \oplus N \to A \to A \oplus N'$ when passing to algebras, the following chain of adjunctions

$$\begin{aligned}
\mathscr{C}_A \downarrow A(K \hat{\otimes} B, B') &\cong \mathscr{C}_A \downarrow A \left(A \oplus (K \hat{\otimes} I_A(B)), B'\right) \\
&\cong \mathscr{N}_A \left(K \hat{\otimes} I_A(B), I_A(B')\right) \\
&\cong s\,\mathrm{Set}_* \left(K, \underline{Hom}_{\mathscr{N}_A}(I_A(B), I_A(B'))\right) \\
&\cong s\,\mathrm{Set}_* \left(K, \underline{Hom}_{\mathscr{C}_A}(B, B')\right)
\end{aligned}$$

shows that $\hat{\otimes}$ defined this way is indeed a pointed tensor structure.

The tensors of the categories above are linked in various ways.

Proposition 2.3.3. *Suppose there is an adjunction of simplicial categories*

$$F : \mathfrak{C} \leftrightarrows \mathfrak{D} : G,$$

that is an adjunction that satisfies

$$\underline{Hom}_{\mathfrak{D}}(F(X), Y) \cong \underline{Hom}_{\mathfrak{C}}(X, G(Y))$$

for all $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$. Then the left adjoint F commutes with the tensors. In particular, if we fix a simplicial R-algebra A, A-algebras B and C and an A-module M, *then*

$$\mathbb{P}_A(K \wedge M) \cong K \hat{\otimes} \mathbb{P}_A(M),$$

$$\mathbb{P}_A(K \times M) \cong K \otimes \mathbb{P}_A(M),$$

$$\mathbb{A}_A(K \wedge M) \cong K \hat{\otimes}_{\mathscr{N}_A} \mathbb{A}_A(M),$$

$$C \otimes_A (K \wedge_{\mathscr{M}_A} M) \cong K \wedge_{\mathscr{M}_C} (C \otimes_A M),$$

$$C \otimes_A (K \hat{\otimes}_{\mathscr{C}_A} B) \cong K \hat{\otimes}_{\mathscr{C}_C} (C \otimes_A B).$$

Proof. All this follows from Yoneda's lemma. Fixing a simplicial set K and an object $X \in \mathfrak{C}$ we can take a test-object $Y \in \mathfrak{D}$ and have

$$\begin{aligned} \mathfrak{D}(F(K \otimes_{\mathfrak{C}} X), Y) &\cong \mathfrak{C}(K \otimes_{\mathfrak{C}} X, G(Y)) \\ &\cong s\,\mathrm{Set}(K, \underline{Hom}_{\mathfrak{C}}(X, G(Y))) \\ &\cong s\,\mathrm{Set}(K, \underline{Hom}_{\mathfrak{D}}(F(X), Y)) \cong \mathfrak{D}(K \otimes_{\mathfrak{D}} F(X), Y), \end{aligned}$$

hence left adjoints always commute with tensors. The same proof goes through for pointed structures. It is left to show that all the adjunctions in question are actually adjunctions of simplicial categories. But this follows from the degreewise description of our functors and the description of $(K \otimes X)_n$ as a pushout for all n. □

There is an alternative description of tensors in terms of coequalisers. If given an adjunction

$$F : \mathfrak{C} \rightleftarrows \mathfrak{D} : G$$

we write $T = F \circ G : \mathfrak{D} \to \mathfrak{D}$ and there is a coequaliser for every object X in $\mathfrak{D}$

$$T^2 X \rightrightarrows TX \to X.$$

For example see [52, 8.6] for this fact. From this, the next proposition follows immediately.

Proposition 2.3.4. *If $\mathfrak{C}$ and $\mathfrak{D}$ are simplicial categories then $K \otimes_{\mathfrak{D}} X$ in $\mathfrak{D}$ is given as the coequaliser*

$$F(K \otimes_{\mathfrak{C}} (G \circ TX)) \rightrightarrows F(K \otimes_{\mathfrak{C}} G(X)) \to K \otimes_{\mathfrak{D}} X.$$

This description of tensors in $\mathfrak{D}$ can be very helpful, if the tensors are known explicitly in $\mathfrak{C}$.

Recall the adjunctions

$$\mathscr{C}_A {\downarrow} A(A \oplus N, B) \cong \mathscr{N}_A(N, I_A(B)),$$

$$\mathscr{N}_A(N, M^0) \cong \mathscr{M}_A(Q_A(N), M).$$

Let us write $V(N) = A \oplus N$ and $Z(M) = M^0$ for the functors involved.

Proposition 2.3.5. *The first equation above is a Quillen equivalence, whereas the second is a Quillen adjunction.*

Proof. The first adjunction is a Quillen adjunction by definition. Because $\pi_* V(N) = \pi_* A \oplus \pi_* N$, the condition for Quillen equivalences is satisfied. A diagram chase shows that the functor Z respects fibrations and acyclic fibrations. $\square$

It follows that the functors above give rise to derived functors

$$\begin{aligned} \operatorname{ho}\mathscr{C}_A \downarrow A(\mathrm{L}V(N), B) &\cong \operatorname{ho}\mathscr{N}_A(N, \mathrm{R}I_A(B)), \\ \operatorname{ho}\mathscr{N}_A(N, \mathrm{R}Z(M)) &\cong \operatorname{ho}\mathscr{M}_A(\mathrm{L}Q_A(N), M). \end{aligned}$$

2.3.1 Description of the Cofibrations

The cofibrations in $\mathscr{C}_A$ are not so easy to describe. There are various ways of characterising cofibrations as retracts of "free" or "almost free" maps, for example see [45], [21] or [24]. For our purposes, it is enough to follow the characterisation for the cofibrations of algebras, which is given in [41] and [30].

Definition 2.3.6. An *almost free* A-module map $f : M \to N$ is one, where there is a sequence of sets $X_n \in N_n$, so that the X_n are closed under degeneracies, i.e. $\sigma_i(X_n) \subseteq X_{n+1}$ for all i and n, and the natural map

$$M_n \oplus A_n\{X_n\} \to N_n$$

is an isomorphism for all n.

An *almost free* A-algebra map $g : B \to C$ is one, where there is a sequence of free A_n-submodules $V_n \subseteq C_n$ for all n, so that the V_n are closed under degeneracies and the natural map

$$B_n \otimes_{A_n} \mathbb{P}_{A_n} V_n \longrightarrow C_n$$

is an isomorphism for all n.

Proposition 2.3.7 (Miller [41], Quillen [45]). *There is the following formal link between almost free maps and cofibrations in the categories $\mathscr{M}_A$ and $\mathscr{C}_A$:*

- *Almost free maps are cofibrations.*
- *Any map factors as $\bullet \to \bullet \overset{\sim}{\twoheadrightarrow} \bullet$, where the first map is an almost free one.*
- *A cofibration is a retract of an almost free map.*

2.4 The Derived Tensor Product

Fix a simplicial R-algebra A. For an A-module M, the tensor product $M \otimes_A -$ does not respect weak equivalences in general. However, it gives rise to a derived functor $M \otimes^L - : \mathrm{ho}\,\mathscr{M}_A \to \mathrm{ho}\,\mathscr{M}_A$, which we are going to describe now.

Definition 2.4.1. For two A-modules M and N, the *derived tensor product* $M \otimes^L N$ is defined as the A-module $M^c \otimes N$, where the cofibrant replacement takes place in $\mathscr{M}_A$.

Of course we could as well fix an A-module M and consider the functor $- \otimes_A M$. However, the next proposition shows among other things, that we wouldn't gain anything from that.

Proposition 2.4.2 (Quillen). *The derived tensor product enjoys the following properties:*

- *For two different projective replacements P and Q of M we have $P \otimes_A N \simeq Q \otimes_A N$.*
- *It is balanced in the sense that $M^c \otimes_A N \simeq M \otimes_A N^c$.*
- *There is a first quadrant spectral sequence referred to as Künneth spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^{\pi_* A}_{p,q}(\pi_* M, \pi_* N) \Longrightarrow \pi_{p+q}(M \otimes^L_A N),$$

  where the Tor-functor is the one of graded modules over graded rings.

From this it follows that if M is a cofibrant module the tensor product $M \otimes_A -$ coincides with the derived tensor product and hence respects weak equivalences.

Chapter 3

Introduction to André-Quillen Homology

We define André-Quillen homology for algebras over a simplicial algebra. The main results of this chapter are in Section 3.3 and 3.4. These results can as well be obtained by methods from Goerss and Hopkins [22], where they derive the existence of general André-Quillen homology theories for categories of algebras over a simplicial operad. The proofs presented in this thesis are more explicit. The exposition here is a generalisation of the André-Quillen homology for rings, which was originally described by André [2] and Quillen [46] and [45, part 1]. The André-Quillen homology for rings is treated in detail for example in the books [34] and [52]. Our version of André-Quillen homology coincides with the traditional one if one uses constant simplicial algebras. In Section 3.5 we derive the existence of a spectral sequence to compute the indecomposables. This is similar to the case of commutative ring spectra, which was treated in [6].

3.1 Abelian Objects, Square-Zero Extensions and Derivations

Let A be a simplicial R-algebra. We work in the category $\mathscr{C}_A \downarrow B$ of A-algebras over a fixed A-algebra B. Take a B-module N which is simultaneously an A-nuca, so that

$$\begin{array}{ccc} B \otimes_A N \otimes_A N & \xrightarrow{\mathrm{id} \otimes \nu} & B \otimes_A N \\ {\scriptstyle \alpha \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \alpha} \\ B \otimes_A N & \xrightarrow[\alpha]{} & N \end{array}$$

where α is the B-module structure map and ν is the multiplication of N as a nuca. We can form an A-algebra $B \oplus N$ with multiplication

$$\mu : (B \oplus N) \otimes_A (B \oplus N) \cong (B \otimes_A B) \ \oplus \ (B \otimes_A N) \ \oplus \ (N \otimes_A B) \ \oplus \ (N \otimes_A N) \longrightarrow B \oplus N$$

given by $\mu = \mu_B \oplus (\alpha + \alpha \circ \tau + \nu)$, where μ_B is the algebra multiplication of B. We can certainly perform this construction with such N, where the multiplication is trivial. That way we get a functor

$$sq : \mathscr{M}_B \longrightarrow \mathscr{C}_A \downarrow B.$$

Definition 3.1.1. An A-algebra over B isomorphic to one of the form $sq(M)$ with a B-module M is called a *square-zero extension* of B with M.

The importance of square-zero extensions will become clear in this and the next section.

Definition 3.1.2. A *group object* in a category $\mathfrak{C}$ with finite products is an object A with a "multiplication" $m : A \times A \to A$, a "unit" $u : * \to A$ from the terminal object into A and an "inverse" $i : A \to A$ satisfying various coherence diagrams [36, p 75]. If furthermore the multiplication m commutes with the twist map $A \times A \to A \times A$, then A is called an *abelian group object*.

The multiplication map thus turns the set $\mathfrak{C}(X, A)$ naturally in X into an abelian group via

$$\mathfrak{C}(X, A) \times \mathfrak{C}(X, A) \cong \mathfrak{C}(X, A \times A) \longrightarrow \mathfrak{C}(X, A).$$

We write $\mathfrak{C}_{\mathrm{Ab}}$ for the subcategory of abelian group objects in a category $\mathfrak{C}$. There is always the question, whether the forgetful functor $U : \mathfrak{C}_{\mathrm{Ab}} \to \mathfrak{C}$ admits a left adjoint, a so called abelianisation $\mathrm{Ab} : \mathfrak{C} \to \mathfrak{C}_{\mathrm{Ab}}$. In the following we construct the functor Ab if $\mathfrak{C} = \mathscr{C}_A \downarrow B$. This functor will give rise to André-Quillen homology in the end. Crucial for this is the next proposition. We provide an easy proof for it as we were not able to find a complete proof in the literature.

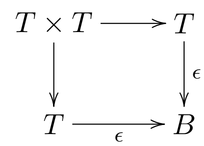
Proposition 3.1.3. *In the category $\mathscr{C}_A \downarrow B$ the abelian group objects are exactly the square-zero extensions.*

Proof. Let T be an abelian group object in $\mathscr{C}_A \downarrow B$. We write $\mu_T : T \otimes_A T \to T$ for the algebra multiplication of T. The terminal object in $\mathscr{C}_A \downarrow B$ is B so that we get a map

$u : B \to T$. Denoting ϵ for the augmentation map $\epsilon : T \to B$, we get $\epsilon \circ u = \mathrm{id}_B$. This means we can write $T \cong B \oplus N$, where $N \cong \ker(\epsilon)$ and multiplication

$$\mu_T = \mu_B \oplus (\alpha + \alpha \circ \tau + \nu) : (B \otimes_A B) \ \oplus \ (B \otimes_A N) \ \oplus \ (N \otimes_A B) \ \oplus \ (N \otimes_A N) \longrightarrow B \oplus N$$

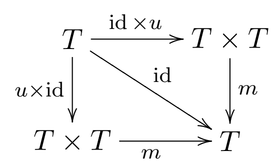
where α is the B-module structure map for N and ν is an A-nuca multiplication map for N. We have to prove that ν is the zero map. The product $T \times T$ is given as the pullback



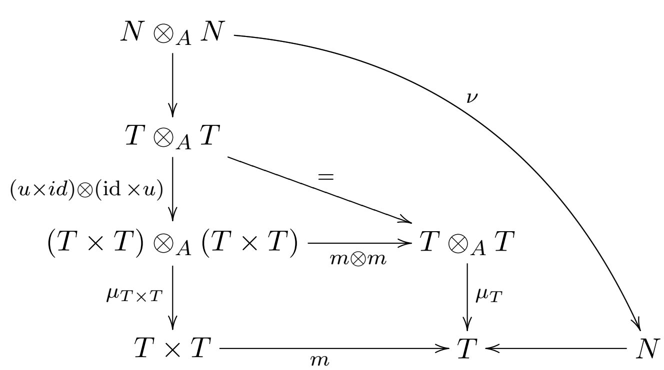
in $\mathscr{C}_A \downarrow B$, hence in the category $\mathscr{C}_A$. But this is the same as the pullback in the category of A-modules, hence

$$T \times T = \{(a, b) \in T \oplus T \mid \epsilon(a) = \epsilon(b)\} \cong B \oplus N \oplus N.$$

Denoting by $m : T \times T \to T$ the multiplication map of T as an abelian group object, we get the following commutative diagram



linking the map m with u. The multiplication μ_T of the algebra T determines the multiplication $\mu_{T\times T}$ of the A-algebra $T \times T$. We obtain the following commutative diagram



where the composition of the left vertical arrows is the zero map, which shows that ν is the zero map.

Conversely, if we have a square-zero extension $C \cong B \oplus N$, then we can set $m : C \times C \cong B \oplus N \oplus N \to B \oplus N$ to be the map which is the identity on B and adds the elements of N. Furthermore we can set $u : B \to B \oplus N$ to be the inclusion of the first summand and

$i : B \oplus N \to B \oplus N$ to be the map which is the identity on B and sends $x \in N_n$ to $-x$ for all n. This data turns C into an abelian group object. □

If $C \cong B \oplus M$ is an abelian group object in $\mathscr{C}_A \downarrow B$, then for any other A-algebra D over B, the set $\mathscr{C}_A \downarrow B(D, B \oplus M)$ is an abelian group. Given two functions f and g in this set, we denote their projection onto M by f_M and g_M, so that we can calculate their addition as

$$f + g = \epsilon \oplus (f_M + g_M).$$

We could have dropped the abelian condition from the definition of abelian group objects and so define the group objects in a category. The proof above never uses this abelian condition. Therefore every group object in $\mathscr{C}_A \downarrow C$ is automatically an abelian group object.

Definition 3.1.4. Given an A-algebra B and a B-module M. A *derivation* is a map $d : B \to M$ of A-modules that satisfies degreewise

$$d(b_1 b_2) = b_1 d(b_2) + b_2 d(b_1).$$

There is the following straightforward observation about derivations and square-zero extensions:

Proposition 3.1.5. *For fixed B and M as above, the set of derivations $d : B \to M$ is in correspondence with $\mathscr{C}_A \downarrow B(B, B \oplus M)$, where we regard $B \oplus M$ as a square-zero extension. In particular, the set of derivations is always an abelian group.*

3.2 Kähler-differentials

We repeat the description of the *Kähler-differentials* given in the book by Matsumura [37, §25]. Let S be a (non-simplicial) R-algebra. The Kähler-differential $\Omega_{S/R}$ is defined to be the S-module given by generators ds, $s \in S$ and relations $\mathfrak{R}$:

$$\mathfrak{R} = \begin{cases} d(s+t) = ds + dt & s, t \in S, \\ d(rs) = rds & r \in R,\ s \in S, \\ d(st) = sdt + tds & s, t \in S. \end{cases}$$

The Kähler differentials $\Omega_{S/R}$ always come with an S-module map

$$\delta : S \to \Omega_{S/R}$$
$$s \mapsto ds$$

called the *universal derivation*. It is characterised by the following proposition, which for example can be found in [37].

Proposition 3.2.1. *The universal derivation δ is a derivation and if M is an S-module, then any derivation $d : S \to M$ factors uniquely as $S \xrightarrow{\delta} \Omega_{S/R} \to M$.*

Setting $I = \ker(S \otimes S \to S)$, there is the identity (see [37])

$$\Omega_{S/R} \cong I/I^2.$$

Furthermore, there is an adjunction of the following form. Let S be an R-algebra and consider the category $\mathscr{C}_R \downarrow S$ of algebras over S. Let T be an R-algebra over S and M an S-module, then

$$\mathscr{M}_S(\Omega_{T/R} \otimes_T S, M) \cong \mathscr{C}_R \downarrow S(T, S \oplus M), \tag{3.1}$$

so that the functor $\Omega_{(-/R)} \otimes_{-} S : \mathscr{C}_R \downarrow S \to \mathscr{M}_S$ is the left adjoint to the square-zero extension functor.

The notion of Kähler-differentials can easily be adapted to the simplicial context. Let A be a simplicial R-algebra and B an A-algebra. Define $\Omega_{B/A} \in \mathscr{M}_B$ by simply setting $(\Omega_{B/A})_n = \Omega_{B_n/A_n}$. The universal derivations $B_n \to \Omega_{B_n/A_n}$ assemble to a universal derivation $\delta : B \to \Omega_{B/A}$.

Let E be an A-algebra.

Proposition 3.2.2. *If we take an A-algebra B over E and a E-module M, then the adjunction (3.1) above generalises to*

$$\mathscr{M}_E(\Omega_{B/A} \otimes_B E, M) \cong \mathscr{C}_A \downarrow E(B, E \oplus M).$$

Proof. The isomorphism is given by turning a E-module map $\phi : \Omega_{B/A} \otimes_B E \to M$ into an A-algebra map ψ defined by

$$\psi : B \xrightarrow{\mathrm{id} \oplus \delta} B \oplus \Omega_{B/A} \cong B \oplus (\Omega_{B/A} \otimes_B B) \xrightarrow{\eta \oplus (\mathrm{id} \otimes \eta)} E \oplus (\Omega_{B/A} \otimes_B E) \xrightarrow{\mathrm{id} \oplus \phi} E \oplus M,$$

where $\eta : B \to E$ is the unit of E as a B-algebra. If we have an algebra map $\psi : B \to E \oplus M$, the projection to M yields a derivation of E-modules, hence it corresponds to a map $\Omega_{B/A} \otimes_B E \to M$. This is the inverse of the isomorphism. $\square$

The next proposition summarises standard identities concerning Kähler differentials and can for example be found in Matsumura's book [37, §25].

Proposition 3.2.3. *If S and T are two (non-simplicial) R-algebras then*

$$\Omega_{S\otimes_R T/T} \cong \Omega_{S/R} \otimes_R T,$$
$$\Omega_{S\otimes_R T/R} \cong (\Omega_{S/R} \otimes_R T) \ \oplus \ (S \otimes_R \Omega_{T/R}).$$

If $R \to S \to T$ is a sequence of algebras then there is an exact sequence (though not short exact)

$$\Omega_{S/R} \otimes_S T \longrightarrow \Omega_{T/R} \longrightarrow \Omega_{T/S} \longrightarrow 0.$$

In the following we will set up Kähler-differentials in a homotopy invariant way.

3.3 André-Quillen (Co-)homology for Algebras over Simplicial Algebras

We start with a remark about the adjunction in the last section:

Proposition 3.3.1. *Let E be an A-algebra. The adjunction*

$$\Omega_{-/A} \otimes_{-} E : \mathscr{C}_A \!\downarrow\! E \leftrightarrows \mathscr{M}_E : sq$$

is a Quillen adjunction.

Proof. We simply have to check that the functor sq preserves fibrations and acyclic fibrations. Because fibrations and acyclic fibrations in algebras coincides with the ones in modules, we will prove this for the functor $U \circ sq : \mathscr{M}_E \to \mathscr{M}_E$. Take a fibration $M \twoheadrightarrow N$ in $\mathscr{M}_E$ and an acyclic cofibration $X \overset{\sim}{\rightarrowtail} Y$ in $\mathscr{M}_E$ so that there is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & E \oplus M & \longrightarrow & M \\ {\scriptstyle\wr}\downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & E \oplus N & \longrightarrow & N \end{array}$$

We have to show that there is a lift $Y \to E \oplus M$ making everything commute. From the given data, there is a lift $Y \to M$. Note that the righthand square is a pullback, so that we get a lift $Y \to E \oplus M$ in the lefthand square and the only thing to check is whether the upper left triangle commutes. Because the middle vertical arrow is the identity on E, this is indeed the case. So, $U \circ sq$ respects fibrations. By exactly the same argument one shows that $U \circ sq$ preserves acyclic fibrations. □

This means that there is a left derived functor of the abelianisation. If $B \in \mathscr{C}_A \downarrow E$ is any A-algebra over E, then we can factor the unit into

$$A \rightarrowtail P \overset{\sim}{\twoheadrightarrow} B$$

where now P is a cofibrant A-algebra. That the following definition is well defined up to homotopy equivalence shows the Proposition 3.3.3.

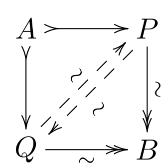
Definition 3.3.2. By the *cotangent complex* of B over A we understand the B-module

$$\mathfrak{L}_{B/A} = \Omega_{P/A} \otimes_P B.$$

for a cofibrant replacement P of B.

Proposition 3.3.3. *The cotangent complex $\mathfrak{L}_{B/A}$ is well defined up to homotopy equivalence of B-modules.*

Proof. If we choose two cofibrant replacements P and Q for B, then there are weak equivalences between them, say $f : P \xrightarrow{\sim} Q$ and $g : Q \xrightarrow{\sim} P$. This follows from the diagram



and the lifting and 2 out of 3-axioms. Since id_P and gf are both liftings over id_A they are homotopic (see [30, 4.6.]). The same is of course true for id_Q and fg. The abelianisation functor preserves cofibrations and acyclic cofibrations, in particular it preserves acyclic cofibrations between cofibrant objects. But then Ken Brown's Lemma 2.1.5 states, that it preserves any weak equivalences between cofibrant objects. In particular

$$\Omega_{P/A} \otimes_P B \simeq \Omega_{Q/A} \otimes_Q B,$$

where the homotopy equivalence is induced by f and g. □

Definition 3.3.4. The *André-Quillen homology groups* of an A-algebra B with coefficients in a B-module M are defined to be

$$\mathrm{AQ}_i(B, A; M) = \pi_i(\mathfrak{L}_{B/A} \otimes_B M)$$

for $i \in \mathbb{N}$.

It follows, that $\mathrm{AQ}_*(B, A; M)$ has a graded π_*B-module structure.

The next proposition relaxes the condition imposed on the cofibrant replacement.

Proposition 3.3.5. *In order to get the correct André-Quillen groups, it is actually enough to factor the unit into $A \rightarrowtail P' \xrightarrow{\sim} B$ without demanding that the second arrow be a fibration.*

Proof. To see this, simply consider a further factorisation of $P' \xrightarrow{\sim} B$ as

$$P' \overset{\sim}{\rightarrowtail} Q \overset{\sim}{\twoheadrightarrow} B.$$

Since $B \otimes_- \Omega_{-/A}$ respects weak equivalences between cofibrant objects, the homotopy groups we get by applying that functor to P' or Q are isomorphic, though not in a canonical way. ☐

The universal derivation $B \to \Omega_{B/A}$ induces a map

$$B \otimes_A M \to B \otimes_B M \to \Omega_{B/A} \otimes_B M,$$

so that there is a comparison

$$\pi_*(B \otimes_A^L M) \to \mathrm{AQ}_*(B, A; M)$$

between the derived tensor product and André-Quillen homology. Furthermore, if C is any B-algebra, the universal derivation together with the unit $B \to C$ induces a map

$$\theta : \pi_* B \to \pi_* \mathfrak{L}_{B/A} \to \pi_* \mathfrak{L}_{B/A} \otimes_B C \cong \mathrm{AQ}_*(B, A; C) \tag{3.2}$$

called the *Hurewicz map*.

We describe now an alternative way to calculate André-Quillen homology. If $A \to B$ is a map of simplicial R-algebras, we can factor it as before to get

$$A \rightarrowtail P \overset{\sim}{\twoheadrightarrow} B.$$

Let I be the kernel of the multiplication map μ_P

$$0 \to I \to P \otimes_A P \xrightarrow{\mu_P} P \to 0.$$

Because $\mathrm{Tor}_1^{P_k}(P_k, B_k) = 0$ for all k we can tensor this sequence with $- \otimes_P B$ to get a short exact sequence

$$0 \to I \otimes_P B \to P \otimes_A B \to B \to 0.$$

We set $J = I \otimes_P B$. Taking indecomposables shows

$$\Omega_{P/A} \otimes_P B \cong I/I^2 \otimes_P B \cong J/J^2.$$

So,

$$\mathrm{AQ}_*(B, A; M) \cong \pi_*(I/I^2 \otimes_P B) \cong \pi_*(J/J^2 \otimes_B M)$$

which is an alternative way to compute André-Quillen homology.

Recall the functors $I_B : \mathscr{C}_A \downarrow B \to \mathscr{N}_B$ and $Q_B : \mathscr{N}_B \to \mathscr{M}_B$. They allow the following description of the cotangent complex.

Theorem 3.3.6. *If B is an A-algebra then*

$$\mathfrak{L}_{B/A} \simeq \mathrm{L}Q_B \, \mathrm{R}I_B(B \otimes^L_A B).$$

Proof. This is the same argument as in [6, Proposition 3.2]. Recall the Quillen adjunctions

$$\mathscr{C}_B \downarrow B(V(N), C) \cong \mathscr{N}_B(N, I_B(C)),$$
$$\mathscr{N}_B(N, Z(M)) \cong \mathscr{M}_B(Q_B(N), M)$$

from Proposition 2.3.5. Fix an A-algebra C over B. There is the following chain of adjunctions

$$\begin{aligned} \mathrm{ho}\, \mathscr{C}_A \downarrow B(C, B \oplus M) &\cong \mathrm{ho}\, \mathscr{C}_B \downarrow B(B \otimes^L_A C, B \oplus M) \\ &\cong \mathrm{ho}\, \mathscr{C}_B \downarrow B(\mathrm{L}V \, \mathrm{R}I_B(B \otimes^L_A C), \mathrm{L}V \, \mathrm{R}Z(M)) \\ &\cong \mathrm{ho}\, \mathscr{N}_B(\mathrm{R}I_B(B \otimes^L_A C), \mathrm{R}Z(M)) \\ &\cong \mathrm{ho}\, \mathscr{M}_B(\mathrm{L}Q_B \, \mathrm{R}I_B(B \otimes^L_A C), M). \end{aligned}$$

Therefore, by Yoneda's lemma we obtain

$$\mathfrak{L}_{C/A} \otimes_C B \simeq \mathrm{L}Q_B \, \mathrm{R}I_B(B \otimes^L_A C)$$

and setting $B = C$ yields the result. □

Dual to André-Quillen homology we can define André-Quillen cohomology by setting

$$\mathrm{AQ}^n(B, A; M) = \pi_{-n} F_{\mathscr{M}_B}(\Omega_{P/A} \otimes_P B, M),$$

where P is a cofibrant replacement for the A-algebra B. As before, it can be proven, that this definition is independent of the cofibrant replacement chosen.

In the case of rings, i.e. constant simplicial objects we obtain the definition of Quillen given in [45].

3.4 Eilenberg-Steenrod Axioms

The Eilenberg-Steenrod axioms were introduced in [18]. Given a model category $\mathfrak{C}$ they can be reformulated in the following form [7].

We consider the category $ar(\mathfrak{C})$ of arrows $A \to B$ in $\mathfrak{C}$. Morphisms in $ar(\mathfrak{C})$ are commutative squares in $\mathfrak{C}$. A homology theory h_* on $\mathfrak{C}$ is a family of functors h_n with $n \in \mathbb{Z}$ taking arrows $A \to B$ to an abelian category $\mathfrak{A}$. We write $h_n(A \to B) = h_n(B, A)$ reminiscent of ordinary homology for topological spaces. The axioms are the following. For all $n \in \mathbb{Z}$:

- Homotopy:

  Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\sim} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & B' \end{array}$$

  then

$$h_n(B, A) \cong h_n(B', A').$$

- Long exact sequence:

  Given two arrows $A \to B \to C$ then there is a map $h_n(C, B) \to h_{n-1}(B, A)$ for all n so that

$$\cdots \to h_n(B, A) \to h_n(C, A) \to h_n(C, B) \to h_{n-1}(B, A) \to \cdots$$

  is a natural long exact sequence.

- Base change:

  Given cofibrations $A \rightarrowtail B$ and $A \rightarrowtail C$ then

$$h_n(B \amalg_A C, C) \cong h_n(B, A).$$

- Coproduct:

  Given a set $A \rightarrowtail A_i$ of cofibrations indexed by $i \in I$, then

$$h_n(\coprod_{i \in I} A_i, A) \cong \coprod_{i \in I} h_n(A_i, A).$$

A cohomology theory h^* is defined dually. In the following we will prove that André-Quillen homology satisfies the Eilenberg-Steenrod axioms in the category $\mathscr{C}_A \downarrow X$, where X is a fixed A-algebra.

The following proposition is stated in [7] for topological S-algebras. However, the same proof applies in the case of simplicial algebras.

Proposition 3.4.1. *There is no homology or cohomology theory in $\mathscr{C}_A$ except for the one which is zero everywhere.*

This is why we have to insist to work in the category $\mathscr{C}_A \downarrow E$ for an A-algebra E, although the E is only important in the way it restricts the choice of coefficients. In practice, when calculating the André-Quillen homology of an A-algebra B with coefficients in a B-module M one sets simply $E = B$.

Theorem 3.4.2. *For an E-module M, the families of functors $h_* = \mathrm{AQ}_*(-,-;M)$ and $h^* = \mathrm{AQ}^*(-,-;M)$ form a homology and cohomology theory on $\mathscr{C}_A \downarrow E$ respectively.*

The proof is at the end of this section. We prove first some intermediate results. Recall Proposition 3.2.3 on which the following propositions are based. Consider the pushout diagram of simplicial A-algebras

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A C \end{array}$$

Proposition 3.4.3. *Let B and C be cofibrant A-algebras and let M be a $B \otimes_A C$-module. Then we have for all $n \in \mathbb{N}$*

$$\mathrm{AQ}_n(B \otimes_A C, A; M) \cong \mathrm{AQ}_n(B, A; M) \oplus \mathrm{AQ}_n(C, A; M),$$
$$\mathrm{AQ}_n(B \otimes_A C, B; M) \cong \mathrm{AQ}_n(C, A; M).$$

Furthermore, if $A \rightarrowtail A^i, i \in I$ is a family of cofibrations and M a $(\bigotimes A^i)$-module, then

$$\mathrm{AQ}_n(\bigotimes_i A^i, A; M) \cong \bigoplus_i \mathrm{AQ}_n(A^i, A; M).$$

Proof. Since the units into B and C are cofibrations, $B \otimes_A C$ is a cofibrant algebra and

$$\mathfrak{L}_{B/A} = \Omega_{B/A},$$
$$\mathfrak{L}_{C/A} = \Omega_{C/A},$$
$$\mathfrak{L}_{B \otimes_A C/A} = \Omega_{B \otimes_A C/A}.$$

Therefore, for a $B \otimes_A C$-module M, we have

$$\begin{aligned}\mathfrak{L}_{B\otimes_A C/A} \otimes_{B\otimes_A C} M &= \Omega_{B\otimes_A C/A} \otimes_{B\otimes_A C} M \\ &\cong \big((\Omega_{B/A} \otimes_A C) \ \oplus\ (B \otimes_A \Omega_{C/A})\big) \otimes_{B\otimes_A C} M \\ &\cong (\Omega_{B/A} \otimes_B M) \ \oplus\ (\Omega_{C/A} \otimes_C M) \\ &= (\mathfrak{L}_{B/A} \otimes_B M) \ \oplus\ (\mathfrak{L}_{C/A} \otimes_C M),\end{aligned}$$

where the second isomorphism is a simplicial extension of the last proposition. This yields

$$\mathrm{AQ}_*(B \otimes_A C, A; M) \cong \mathrm{AQ}_*(B, A; M) \oplus \mathrm{AQ}_*(C, A; M).$$

In the same manner we get

$$\begin{aligned}\mathfrak{L}_{B\otimes_A C/C} \otimes_{B\otimes_A C} M &= \Omega_{B\otimes_A C/C} \otimes_{B\otimes_A C} M \\ &\cong (\Omega_{B/A} \otimes_A C) \otimes_{B\otimes_A C} M \\ &\cong \mathfrak{L}_{B/A} \otimes_B M,\end{aligned}$$

which yields

$$\mathrm{AQ}_*(B \otimes_A C, C; M) = \mathrm{AQ}_*(B, A; M).$$

Because $\Omega_{-/A} \otimes_A -$ is a left adjoint, it commutes with colimits. This proves the last point of the proposition. □

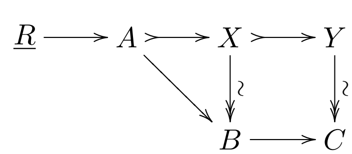
It is slightly trickier to establish the transitivity sequence. This is where the notion of an almost free map is convenient.

Proposition 3.4.4 (Transitivity long exact sequence). *For three simplicial R algebras $A \to B \to C$ and a C-module M there is a long exact sequence*

$$\cdots \to \mathrm{AQ}_n(B, A; M) \to \mathrm{AQ}_n(C, A; M) \to \mathrm{AQ}_n(C, B; M) \to \mathrm{AQ}_{n-1}(B, A; M) \to \cdots$$

Proof. We will construct a short exact sequence of simplicial R-algebras, whose homology long exact sequence will be the transitivity long exact sequence.

As usual, factor the map $A \to B$ into $A \rightarrowtail X \twoheadrightarrow B$. Then factorise further in the model category of A-algebras the composite $X \to B \to C$ into $X \rightarrowtail Y \xrightarrow{\sim} C$, so that the map $\psi : X \rightarrowtail Y$ is an almost free map. This means that for each n we have an isomorphism $Y_n \cong X_n \otimes_{A_n} \mathbb{P}_{A_n} V_n$ for some A_n-module V_n. We obtain the diagram



Since everything happens levelwise, there is an exact sequence of Y-modules

$$\Omega_{X/A} \otimes_X Y \to \Omega_{Y/A} \to \Omega_{Y/X} \to 0.$$

For each n, the map $\psi_n : X_n \to Y_n$ has a splitting of A_n-algebras. So, after applying in each degree n the functor $\Omega_{-/A_n} \otimes_- Y_n$ to the injection ψ_n, we still have an injection. Therefore the sequence above is actually short exact. Since Ω_{Y_n/X_n} is a projective Y_n-module for every n, tensoring the sequence with any Y-module yields another short exact sequence. We end up with the short exact sequence

$$0 \to C \otimes_X \Omega_{X/A} \to C \otimes_Y \Omega_{Y/A} \to C \otimes_Y \Omega_{Y/X} \to 0$$

and finally

$$0 \to M \otimes_X \Omega_{X/A} \to M \otimes_Y \Omega_{Y/A} \to M \otimes_Y \Omega_{Y/X} \to 0. \tag{3.3}$$

We now have to identify these terms. $C \otimes_X \Omega_{X/A}$ is $C \otimes_B \mathfrak{L}_{B/A}$ and $C \otimes_Y \Omega_{Y/A}$ is $\mathfrak{L}_{C/A}$. It is left to show that $C \otimes_Y \Omega_{Y/X} \simeq \mathfrak{L}_{C/B}$. There is a map from the pushout into C

$$B \otimes_X Y \to C.$$

This is a weak equivalence because $Y \cong X \otimes_X Y \to B \otimes_X Y$ is one by the 2 out of 3 axiom. Furthermore, $B \otimes_X Y$ is a cofibrant B-algebra. Therefore

$$\begin{aligned}
\mathfrak{L}_{C/B} &\simeq \mathfrak{L}_{B \otimes_X Y/B} \\
&= \Omega_{B \otimes_X Y/B} \\
&= B \otimes_X \Omega_{Y/X} \\
&\simeq B \otimes_X Y \otimes_Y \Omega_{Y/X} \\
&\simeq C \otimes_Y \Omega_{Y/X}.
\end{aligned}$$

With this, the short exact sequence 3.3 turns into the long exact sequence

$$\cdots \to \mathrm{AQ}_n(B, A; M) \to \mathrm{AQ}_n(C, A; M) \to \mathrm{AQ}_n(C, B; M) \to \mathrm{AQ}_{n-1}(B, A; M) \to \cdots$$

This concludes the proof. □

We finish this section with the proof of Theorem 3.4.2.

Proof. We show that André-Quillen homology with the E-module M as coefficients is a homology theory on $\mathscr{C}_A \downarrow E$. Note that if B is an A-algebra, then a cofibration of B-algebras is as well one of A-algebras, so that we do not need to specify in which category we take the cofibrant replacements. The homotopy axiom is obvious from Proposition 3.3.3. Given $B \to C \to D$ in $\mathscr{C}_A \downarrow E$, the transitivity exact sequence serves as the long exact sequence needed. We only have to note that M is by restriction a D-module. Given two cofibrations $B \to C$ and $B \to D$, the base change axiom is satisfied because of Proposition 3.4.3. Again, M is a $C \otimes_B D$-module by restriction. As a last point, the same proposition asserts that the sum axiom holds. □

3.5 Spectral Sequence for the Indecomposables

We go back to the description of $\Omega_{B/A} \simeq J/J^2$, where J is the kernel of the map $P \otimes_A B \to B$. Calculating $\pi_* J$ is not too much of a problem because of the Künneth spectral sequence and the long exact sequence associated to a homotopy fibre. However, calculating $\pi_* Q_B(J)$ from $\pi_* J$ is much harder. There is a spectral sequence, which in some cases can be put to use.

Set $X_{ji} = (\mathbb{A}_B^j J)_i$ for $i, j \geqslant 0$. X is a bisimplicial complex, say j is in the horizontal axis, i in the vertical. The fact that it is simplicial in i is clear, for j it follows from the property, that $\mathbb{A}_B \circ U : \mathscr{N}_B \to \mathscr{N}_B$ is a monad [52, 8.6].

So, X_{**} is in $s\mathscr{N}_B$. But $s\mathscr{N}_A$ inherits a Reedy model structure from $\mathscr{N}_B$, so that if $J \in \mathscr{N}_B$ is cofibrant, then $\underline{J} \in s\mathscr{N}_B$ is cofibrant [28, 15.10.1]. Moreover, $s\mathscr{M}_B$ obtains such a model structure and the Quillen adjunction (Q, Z) extends to a Quillen adjunction between $s\mathscr{N}_B$ and $s\mathscr{M}_B$.

For every B-module M we have that $Q(\mathbb{A}_B M) = M$. We can therefore compute $Q_B(X)$ as

$$Q_B(X_{ji}) = \begin{cases} (\mathbb{A}_A^{j-1} J)_i & j \geqslant 1, \\ Q_B(J)_i & j = 0. \end{cases}$$

There is a double complex associated to it, denoted $K(Q_B(X))$ when passing to alternative sums of boundary maps.

Given a cofibrant B-nuca J, we can look at $\underline{J} \in s\mathscr{M}_B$ and there is a weak equivalence $\mathbb{A}_B^* J \to \underline{J}$. Therefore $Q_B(X_{**}) \to Q_B(J)$ is a weak equivalence and this leaves us with the spectral sequence for a double complex, which is

$$E^1_{ji} = \pi_i \mathbb{A}_B^j J \Longrightarrow \pi_{j+i} Q_B(J).$$

Unfortunately, $\pi_i \mathbb{A}_B^j J$ is usually very hard to compute if $j \geqslant 1$.

Chapter 4

Cellular Complexes

In this chapter we define the notion of cellular complexes in the categories $\mathscr{M}_A$ and $\mathscr{C}_A$ and show that every module and algebra can be approximated by cellular complexes. We show that André-Quillen homology behaves like ordinary homology theory with respect to cellular complexes. This is similar to the case of commutative ring spectra, cf. [3]. The sphere objects itselves have been studied before, see [21] and [15]. Proposition 4.4.2 is based on a result by Basterra for commutative ring spectra.

4.1 Sphere and Disc-Objects

First we will describe the basic building blocks of cellular complexes in the categories $\mathscr{M}_A$ and $\mathscr{C}_A$. They behave analogously to spheres and discs in topology when building CW-complexes.

Before we start, we have to describe our model of the n-sphere S^n as a simplicial set. We regard it with a basepoint $*$ and a unique non-degenerate simplex e_n in degree n. In order to give a combinatorial description, we are taking refuge to the subcategory $\Delta' \subset \Delta$ of ordered ordinals $[n]$ with maps $\alpha : [k] \twoheadrightarrow [n]$ which are order preserving and surjective. With this notation we get

$$(S^n)_k = \begin{cases} \{*\} & k < n, \\ \{*, e_n\} & k = n, \\ \{*, x_\alpha \mid \alpha \in \Delta'([k],[n]) \} & k > n. \end{cases}$$

For convenience we write $x_{\text{id}} = e_n$. Then the boundary maps are given by $\delta_i(x_\alpha) = x_{\alpha \circ d_i}$, if $\alpha \circ d_i$ is surjective and $\delta_i(x_\alpha)) = *$ otherwise. The degeneracies are given by $\sigma_i(x_\alpha) = x_{\alpha \circ s_i}$. Note that $\alpha \circ s_i$ is always surjective. The so defined boundary and degeneracy maps satisfy the simplicial identities. For $\alpha \neq \mathrm{id}$, the x_α are all degenerate. It follows that the geometric realisation $|S^n| = S^n_{\mathrm{top}}$ is the topological n-sphere.

Recall the construction $S^n \wedge M = (S^n \times M)/(* \times M)$ for a simplicial R-module M from Section 2.3. We write $\Sigma^n M = S^n \wedge M$ and call it the n-fold suspension of M. We now describe $\Sigma^n \underline{R}$ in the category $\mathscr{M}_{\underline{R}}$ for various n.

$S^n \times \underline{R}$ is the simplicial R-module obtained by applying the free R-module functor degreewise on the sets S^n_k. So, $\Sigma^n \underline{R}$ consists of the simplices

$$(\Sigma^n \underline{R})_k = \begin{cases} 0 & k < n, \\ R\{x_{\mathrm{id}}\} & k = n, \\ R\{x_\alpha \mid \alpha \in \Delta'([k],[n])\ \} & k > n. \end{cases}$$

It is enough to characterise the boundary and degeneracy maps on the generators x_α. But there they coincide with the boundary and degeneracy maps of the simplicial set S^n when identifying 0 with the basepoint $*$. It follows that $\Sigma^n \underline{R}$ is an almost free object in $\mathscr{M}_{\underline{R}}$ hence a cofibrant $\underline{R}$-module. In order to calculate the homotopy groups, we compare it to $R\{S^n\} = S^n \times \underline{R}$. We have that $\pi_*(R\{S^n\}) \cong H_*(S^n_{\mathrm{top}}, R)$, for example see [52, 8.3.8]. Using the short exact sequence of simplicial modules

$$0 \to \underline{R} \to \underline{R}\{S^n\} \to \Sigma^n \underline{R} \to 0,$$

we have

$$\pi_k \Sigma^n \underline{R} = \begin{cases} R & k = n, \\ 0 & \text{otherwise}. \end{cases}$$

Another method to obtain $\Sigma^n \underline{R}$ is to use the Dold-Kan correspondence. Let $R[n] \in Ch^+(R)$ denote the chain complex which is the module R concentrated in degree n. Then $\Sigma^n \underline{R} = \gamma(R[n])$. Since $A \otimes_R -$ is a left adjoint $\mathscr{M}_{\underline{R}} \to \mathscr{M}_A$, it commutes with the functor $S^n \wedge -$, see Proposition 2.3.3. Therefore $\Sigma^n A = A \otimes_R \Sigma^n \underline{R}$, where the first Σ^n takes place in the category of A-modules, whereas the second Σ^n is in $\mathscr{M}_{\underline{R}}$. Explicitly:

$$(\Sigma^n A)_k = \begin{cases} 0 & k < n, \\ A_n\{x_{\mathrm{id}}\} & k = n, \\ A_k\{x_\alpha \mid \alpha : [k] \twoheadrightarrow [n]\ \} & k > n. \end{cases}$$

The boundary and degeneracy maps adjust accordingly.

Since $R\{S^n\}$ is cofibrant as a $\underline{R}$-module we can use the Künneth spectral sequence from 2.4.2

$$E^2_{p,q} = \mathrm{Tor}^R_{p,q}(\pi_* A, \pi_* R\{S^n\}) \Longrightarrow \pi_{p+q} A\{S^n\}$$

to calculate the homotopy of $A\{S^n\} = A \otimes_R \underline{R}\{S^n\}$.

$$\begin{aligned}
\operatorname{Tor}^R_{p,*}(\pi_* A, \pi_* R\{S^n\}) &\cong \operatorname{Tor}^R_{p,*}(\pi_* A, H_*(S^n_{\text{top}}, R)) \\
&\cong \operatorname{Tor}^R_{p,*}(\pi_* A, R\epsilon_0) \oplus \operatorname{Tor}^R_{p,*}(\pi_* A, R\epsilon_n) \\
&\cong \begin{cases} \pi_* A \oplus \pi_{*-n} A & p = 0, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

where ϵ_0 and ϵ_n are generators for copies of R in degree 0 and n. So, $\pi_* A\{S^n\} \cong \pi_* A \oplus \pi_{*-n} A$. Examining the pushout

$$\begin{array}{ccc}
A & \longrightarrow & A\{S^n\} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma^n A
\end{array}$$

we see that

$$\pi_k \Sigma^n A = \begin{cases} 0 & k < n, \\ \pi_{k-n} A & k \geqslant n \end{cases}$$

as expected.

In order to pass to discs, one kills this homotopy group in degree n by attaching cells. We call this process forming the cone and denote the $(n+1)$-disc in the category of A-modules by $C\Sigma^n A$. Its simplices are given by

$$(C\Sigma^n A)_k = \begin{cases} 0 & k < n, \\ A_n\{x_{\text{id}}\} & k = n, \\ A_{n+1}\{x_\alpha, y_{\text{id}} \mid \alpha \in \Delta'([n+1],[n])\ \} & k = n+1, \\ A_k\{x_\alpha, y_\beta \mid \alpha \in \Delta'([k],[n]),\ \beta \in \Delta'([k],[n+1])\ \} & k > n. \end{cases}$$

The boundary and degeneracy maps for the x_α stay exactly the same as before. For the y_β they are defined the same way with the only exception being

$$\delta_i y_{\text{id}} = \begin{cases} x_{\text{id}} & i = 0, \\ 0 & i > 0. \end{cases}$$

Again, the maps assemble to satisfy the simplicial identities. By direct computation using the Moore chain complex one gets that $\pi_*(C\Sigma^n A) = 0$ and it is clear that $\Sigma^n A \to C\Sigma^n A$ is an almost free map of $\underline{R}$-modules as defined in 2.3.6.

In order to pass from modules to algebras, we simply apply the functor $\mathbb{P}_A$. We call the resulting object $\mathbb{P}_A \Sigma^n A$ the n-sphere in $\mathscr{C}_A$. Explicitly, its simplices are

$$(\mathbb{P}_A \Sigma^n A)_k = \begin{cases} A_k & k < n, \\ A_n[x_{\text{id}}] & k = n, \\ A_k[x_\alpha \mid \alpha \in \Delta'([k],[n])\ ] & k > n, \end{cases}$$

so that we have now polynomial algebras in each degree $\geqslant n$. Its homotopy groups are not known for general simplicial algebras A. However they can be calculated in some special cases. We will do this in the next section. Because $\mathbb{P}_A\Sigma^n A$ is an A-algebra over A, we can write

$$\mathbb{P}_A\Sigma^n A \cong S^n \hat{\otimes} \mathbb{P}_A A.$$

The rôle of the discs is played by $\mathbb{P}_A C\Sigma^n A$. Its simplices are explicitly

$$(\mathbb{P}_A C\Sigma^n A)_k = \begin{cases} A_k & k < n, \\ A_n[x_{\mathrm{id}}] & k = n, \\ A_{n+1}[x_\alpha, y_{\mathrm{id}} \mid \alpha \in \Delta'([n+1],[n])\ ] & k = n+1, \\ A_k[x_\alpha, y_\beta \mid \alpha \in \Delta'([k],[n]),\ \beta \in \Delta'([k],[n+1])\ ] & k > n. \end{cases}$$

For the homotopy groups we have

$$\pi_*\mathbb{P}_A C\Sigma^n A = \pi_* A$$

for all n. A proof is given as Lemma 4.3.4. Because the functor $\mathbb{P}_A$ respects cofibrations, the map $\mathbb{P}_A\Sigma^n A \rightarrowtail \mathbb{P}_A C\Sigma^n A$ is one.

One justification why $\mathbb{P}_A\Sigma^n A$ can be called the n-sphere in $\mathscr{C}_A$ is the following proposition. The forgetful functor U from algebras to modules and the free functor $\mathbb{P}_A$ from modules to algebra

$$\mathbb{P}_A : \mathscr{M}_A \rightleftarrows \mathscr{C}_A : U$$

form a Quillen adjoint pair. Therefore the functors pass down to the homotopy category. This proves the following statement.

Proposition 4.1.1. *Let B be an A-algebra, then for all $n \in \mathbb{N}$*

$$\mathrm{ho}\,\mathscr{C}_A(\mathbb{P}_A\Sigma^n A, B) \cong \mathrm{ho}\,\mathscr{M}_A(\Sigma^n A, B) \cong \mathrm{ho}\, s\,\mathrm{Set}_*(S^n, B) \cong \pi_n B.$$

Proof. The only thing left to show is the last isomorphism sign which comes from the fact that *sing* and geometric realisation is a Quillen equivalence. □

4.2 The Homotopy Groups of the Spheres

This section is for later reference. We will not need it in this chapter, but it will provide us with examples. We rely on work by Dold [14] and Puppe [44] about symmetric powers of simplicial sets.

For a simplicial set X with basepoint $*$ define $SP^n(X)$ to be the simplicial set with

$$SP^n(X)_k = (X_k \times \cdots \times X_k)/\Sigma^n,$$

the n-fold cartesian product, where we divide out by the action of the symmetric group which permutes the factors. By including an extra copy of the basepoint $*$ we find a map $SP^n(X) \to SP^{n+1}(X)$ and we define the symmetric power of X as

$$SP(X) = \operatorname*{colim}_n SP^n(X).$$

The k-simplices of $SP(X)$ are given as unordered monomials $x_1^{n_1} \cdots x_m^{n_m}$ with $x_i \in X_k$ respecting the relation $* = 1$. Similarly, the symmetric power of pointed topological spaces can be defined, using the cartesian product of topological spaces. The formalism is exactly the same (e.g. [26], 3.C). Moreover, there is a homeomorphism

$$|SP(X)| \cong SP(|X|),$$

see [44, page 403]. For the topological sphere S^n_{top}, we have that $SP(S^n_{\text{top}})$ is an Eilenberg MacLane space $K(\mathbb{Z}, n)$. A proof of this fact can be found in [26, Section 4.K].

We turn back to the simplicial symmetric powers and the simplicial sphere S^n. We see now by inspection that

$$\underline{R}\{SP(S^n)\} = \mathbb{P}_{\underline{R}}\Sigma^n \underline{R}.$$

By the Dold-Thom theorem, we are left with

$$\pi_*\mathbb{P}_{\underline{R}}\Sigma^n \underline{R} = H_*(|SP(S^n)|; R) = H_*(K(\mathbb{Z}, n); R)$$

The latter was first calculated for $R = \mathbb{F}_p$ in the éxposé 7 of the séminaire Cartan [12]. Although $\pi_*\mathbb{P}_{\underline{R}}\Sigma^n \underline{R}$ carries a graded ring structure, we deal only with the additive structure.

We describe in detail the homotopy groups of $\mathbb{P}_{\underline{R}}\Sigma^n \underline{R}$ for $R = \mathbb{F}_p$ with p a prime and R an algebra over the rationals.

4.2.1 The Case $R = \mathbb{F}_2$

Over the field of two elements $\mathbb{F}_2$ we are able to make some explicit computations. Set $A = \underline{\mathbb{F}_2}$. It is possible to compute $\pi_*(\mathbb{P}_A \Sigma^n A)$. Of course, if $n = 0$ we simply get $\mathbb{P}_A A = \underline{\mathbb{F}_2[\iota_0]}$, a constant simplicial algebra on a polynomial ring on one generator. For $n > 0$ we need to calculate the homology of $K(\mathbb{Z}, n)$ with $\mathbb{F}_2$-coefficients. The cohomology can be found in many places, e.g. [12] [42] [27] or [38]. The cohomology groups are finitely generated, so the homology groups are isomorphic to the cohomology groups. Their description uses the combinatorics of admissible sequences. A sequence of natural numbers

$$I = i_1 i_2 \cdots i_r$$

is said to be *admissible* if $i_j \geqslant 2i_{j+1}$ for $j < r$ and $i_r \geqslant 2$. The *excess* of an admissible sequence is defined as

$$e(I) = i_1 - \sum_{j=2}^{r} i_j,$$

whereas its degree is defined as

$$d(I) = \sum_{j+1}^{r} i_j.$$

We regard the empty sequence to be admissible with excess and degree zero. With this notation, the cohomology of $K(\mathbb{Z}, n)$ with $\mathbb{F}_2$-coefficients is given as a polynomial algebra, with generators $\delta_I(\iota_n)$ in degree $n + d(I)$, where the I are of excess less than n.

$$H^*(K(\mathbb{Z}, n); \mathbb{F}_2) = \mathbb{F}_2[\delta_I(\iota_n) \mid I \text{ admissible and } e(I) < n].$$

Additively, that is now the same as $\pi_* \mathbb{P}_A \Sigma^n A$. However, the ring structure is quite different. Its description can be found in [21].

$$\pi_*(\mathbb{P}_A \Sigma^n A) = \Lambda[\delta_I(\iota_n) \mid I \text{ admissible and } e(I) \leqslant n].$$

Because of $i_r \geqslant 2$ we have that $d(I) \geqslant 2$ for $I \neq \emptyset$, so that $\pi_{n+1}(\mathbb{P}_A \Sigma^n A) = 0$.

4.2.2 The Case $R = \mathbb{F}_p$ for $p > 2$

Set $A = \underline{\mathbb{F}_p}$, where p is now an odd prime number. We follow the Séminaire Cartan [12] to give a description of the algebras $\pi_* \mathbb{P}_A \Sigma^n A$. There is as well a description analogous of the case $p = 2$, see [38, Theorem 10.4.], but we will only use Cartan's version. Let us consider words consisting of the letters σ, ϕ and γ. The *length* n of a word α is the number of letters in α. The *degree* d of a word is given inductively. The degree of the empty word is 0, and then we use

$$d(\sigma\alpha) = 1 + d(\alpha),$$

$$d(\phi\alpha) = p \cdot d(\alpha),$$

$$d(\gamma\alpha) = 2 + p \cdot d(\alpha)$$

for any word of positive length. A word is *admissible* if it starts with either σ or ϕ, it ends with σ and the total number of σ's to the right of a ϕ or γ is even.

For every admissible word α of height n and degree d, there is a map

$$\mathbb{F}_p \to H_d(K(\mathbb{Z}, n); \mathbb{F}_p). \tag{4.1}$$

Let $M_n = \bigoplus_\alpha \mathbb{F}_p$ be the free graded $\mathbb{F}_p$-module which is generated by the admissible words of length n. Furthermore, consider the free noncommutative tensor algebra

$$\mathbb{T}(M_n) = \mathbb{F}_p \;\oplus\; M_n \;\oplus\; M_n^{\otimes_{\mathbb{F}_p} 2} \;\oplus\; M_n^{\otimes_{\mathbb{F}_p} 3} \;\oplus\; \cdots$$

and its subalgebra of symmetric tensors, which we denote by $\mathrm{Sym}(M)$. There is now the following theorem.

Theorem 4.2.1 (Cartan). *The map (4.1) extends to an isomorphism of graded $\mathbb{F}_p$-algebras*

$$\mathrm{Sym}(M_n) \xrightarrow{\cong} H_*(K(\mathbb{Z}, n); \mathbb{F}_p).$$

One can read off from this isomorphism that the word σ^n is in correspondance to $H_n(K(\mathbb{Z}, n); \mathbb{F}_p)$. If $n < 3$, there is no other word with length n. If $n \geqslant 3$, the word with the lowest degree greater than n is $\sigma^{n-3}\phi\sigma^2$ with $d(\sigma^{n-3}\phi\sigma^2) = n + 2p - 3$.

Therefore, $\pi_k \mathbb{P}_A \Sigma^n A = 0$ for $n < k < 2n$ if $n + 2p - 3 \geqslant 2n$ or for $n < k < n + 2p - 3$ if $n + 2p - 3 < 2n$. It follows that the map $A \to \mathbb{A}_A \Sigma^n A$ is $(2n-1)$-connected if $n \leqslant 2p - 3$ and $(n + 2p - 3)$-connected if $n > 2p - 3$.

4.2.3 The Case when R is a $\mathbb{Q}$-algebra

We only discuss the homotopy groups in low degrees. We can use the fact that $\Sigma^n \underline{R}$ is a $\mathbb{Q}$-vector space in every simplicial degree. Setting $M = \Sigma^n \underline{R}$, we have that $M^{\otimes k}$ is $(nk - 1)$-connected. Furthermore, the map

$$\begin{aligned} \alpha : M^{\otimes k}/\Sigma_k &\to M^{\otimes k} \\ m_1 \otimes \cdots \otimes m_k &\mapsto \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma(m_1 \otimes \cdots \otimes m_k) \end{aligned}$$

turns $M^{\otimes k}/\Sigma_k$ into a retract of $M^{\otimes k}$, hence $M^{\otimes k}/\Sigma_k$ is as well $(nk-1)$-connected. It follows that the map $M \to \mathbb{A}_{\underline{R}}M$ is $(2n-1)$-connected. So

$$\pi_i \mathbb{P}_{\underline{R}} \Sigma^n \underline{R} = \begin{cases} R & i = 0, \\ 0 & 0 < i < n, \\ R & i = n, \\ 0 & n < i < 2n. \end{cases}$$

4.3 André-Quillen Homology for Cellular Complexes

We work over a fixed simplicial R-algebra A and start with the definition of the mapping cone.

Definition 4.3.1. Let $\phi : M \to N$ be a map of A-modules. Factor it as $M \rightarrowtail Q \overset{\sim}{\twoheadrightarrow} N$. The *mapping cone* of ϕ is the A-module C_ϕ obtained by the pushout

$$\begin{array}{ccc} M & \rightarrowtail & Q \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{C}_\phi \end{array}$$

The mapping cone is well-defined up to homotopy. If the map ϕ is already a cofibration, no replacements need to be made.

We now describe how to build cellular complexes. By attaching an $(n+1)$-cell to an A-algebra B we mean the following process: Take $K_n = \bigoplus_i \Sigma^n A$ a sum of (possibly infinite) $\Sigma^n A$'s together with an attaching map of A-modules

$$k_n : K_n \to B$$

Defining $CK_n = \bigoplus C\Sigma^n A$, we get a cofibration of modules

$$K_n \rightarrowtail CK_n,$$

where CK_n is contractible. By definition, attaching a sum of cells as a module to B is the same as taking the pushout

$$\begin{array}{ccc} K_n & \longrightarrow & B \\ \downarrow & & \downarrow \\ CK_n & \longrightarrow & CK_n \amalg_{K_n} B \end{array}$$

in the category of A-modules. So the resulting module coincides with the mapping cone C_{k_n} of the attaching map $k_n : K_n \to B$

$$CK_n \amalg_{K_n} B \simeq \mathrm{C}_{k_n}\,.$$

In the category of algebras we can proceed in the following way: The attaching map k_n induces an algebra map $\mathbb{P}_A K_n \to B$ and we can form the pushout in the category of A-algebras.

$$\begin{array}{ccc} \mathbb{P}_A K_n & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{P}_A C K_n & \longrightarrow & B' \end{array}$$

with

$$B' = B \otimes_{\mathbb{P}_A K_n} \mathbb{P}_A C K_n.$$

Because the map $K_n \to CK_n$ was a cofibration of modules, $\mathbb{P}_A K_n \to \mathbb{P}_A C K_n$ is a cofibration of algebras. Pushouts respect cofibrations, so that $B \to B'$ is a cofibration.

In order to compute the homotopy groups of B' we have to understand the homotopy groups of $\mathbb{P}_A K_n$. Unfortunately, they are rather complicated. However, we have the following statements:

Lemma 4.3.2. *If A is a simplicial R-algebra, then for all $m > 1$, $n > 0$ and $k \leqslant n$.*

$$\pi_k((\Sigma^n A)^{\otimes_A m}/\Sigma_m) = 0.$$

Proof. If $k < n$, then the statement is true trivially. So assume $k = n$. The boundary maps $\delta_i : ((\Sigma^n A)^{\otimes_A m})_{n+1} \to ((\Sigma^n A)^{\otimes_A m})_n$ are Σ_m-invariant. Hence the differential $d = \sum (-1)^i \delta_i$ is Σ_m-invariant, when passing to the Moore chain complex $K((\Sigma^n A)^{\otimes_A m})$. Because $\Sigma^n A$ is a cofibrant A-module, we can apply the Künneth spectral sequence to compute $\pi_n(\Sigma^n A)^{\otimes_A m} = 0$, and hence $d : ((\Sigma^n A)^{\otimes_A m})_{n+1} \to ((\Sigma^n A)^{\otimes_A m})_n$ is surjective. Therefore it is still surjective when passing to orbits, which means $\pi_n((\Sigma^n A)^{\otimes_A m}/\Sigma_m) = 0$. □

Note, that there is no need to assume cofibrancy conditions on A as an $\underline{R}$-algebra. This lemma immediately yields the following result.

Corollary 4.3.3. *If $K_n = \bigoplus \Sigma^n A$ is a sum of n-spheres, then*

$$\pi_i \mathbb{P}_A K_n = \begin{cases} \pi_i A & i < n, \\ \pi_n A \oplus \pi_n K_n & i = n. \end{cases}$$

The homotopy groups of $\mathbb{P}_A C K_n$ are indeed what one might expect.

Lemma 4.3.4. *For any simplicial R-algebra A we have*

$$\pi_* \mathbb{P}_A C K_n = \pi_* A$$

Proof. Since $0 \to CK_n$ is an acyclic cofibration of cofibrant modules, it is respected by $\mathbb{P}_A$, hence $A = \mathbb{P}_A 0 \to \mathbb{P}_A CK_n$ is a weak equivalence. □

When attaching a sum of $(n+1)$-cells to B as an algebra, we can compute the homotopy groups of $B' = \mathbb{P}_A CK_n \otimes_{\mathbb{P}_A K_n} B$ up to n. Take the homotopy cofibre of the attaching map $K_n \xrightarrow{k_n} B \to \mathrm{C}_{k_n}$.

Proposition 4.3.5. *With this notations we have*

$$\pi_i B' = \begin{cases} \pi_i B & i < n, \\ \pi_n \mathrm{C}_{k_n} & i = n. \end{cases}$$

Proof. We have that $B_i = B'_i$ for $i \leqslant n$, so that $\pi_i B = \pi_i B'$ for $i < n$. Because $B \to B'$ is a cofibration, we can use the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\pi_* \mathbb{P}_A K_n}_{p,q}(\pi_* B, \pi_* A) \Longrightarrow \pi_{p+q} B'$$

All the terms $\mathrm{Tor}_{p,q}$ are 0 for the pairs $\{p, q\}$ with $p \geqslant 1$ and $q < n$. So

$$\pi_n B' = (\pi_* B \otimes_{\pi_* \mathbb{P}_A K_n} \pi_* A)_n \cong \pi_n B / \operatorname{im} \pi_n(k_n) \cong \pi_n \mathrm{C}_{k_n}\,.$$ □

Using a repeated n-cell gluing process, we can define cellular complexes in the following way. Fix $B^0 = A$ and define inductively B^{n+1} by attaching an n-cell to B^n for all n. We get a tower of cofibrations

$$A = B^0 \rightarrowtail B^1 \rightarrowtail B^2 \rightarrowtail \cdots \rightarrowtail B^n \rightarrowtail \cdots \rightarrowtail \operatorname*{colim}_n B^n =: B$$

We call such a B a cellular complex with skeleta B^n. A *finite* cellular complex B is one where only a finite number of cells is attached.

Calculating André-Quillen homology of cellular complexes is rather easy.

Proposition 4.3.6. *Let X be a cofibrant A-module. Then*

$$\mathrm{AQ}_*(\mathbb{P}_A X, A; M) = \pi_*(X \otimes_A M)$$

Proof. We shall prove that $\Omega_{\mathbb{P}_A X/A} \cong \mathbb{P}_A X \otimes_A X$, from which the proposition follows. We start with the Kähler-differentials for rings. Let N be a (non-simplicial) R-module and consider the algebra map $R \to \mathbb{P}_R N$. Let N' be any R-module, then an R-derivation

$$d : R \;\oplus\; N \;\oplus\; N^2/\Sigma_2 \;\oplus\; \cdots \longrightarrow N'$$

is zero on R and an R-module map on N. Moreover, an R-module map $N \to N'$ extends uniquely to a derivation $d : \mathbb{P}_R N \to N'$, hence

$$\mathscr{C}_R \downarrow \mathbb{P}_R N(\mathbb{P}_R N, R \oplus N') \cong \mathscr{M}_R(N, N') \cong \mathscr{M}_{\mathbb{P}_R N}(\mathbb{P}_R N \otimes_R N, N').$$

So, the Kähler-differentials are

$$\Omega_{\mathbb{P}_R N/N} \cong \mathbb{P}_R N \otimes N.$$

It follows that in the simplicial context

$$(\Omega_{\mathbb{P}_A X/A})_n = \Omega_{\mathbb{P}_{A_n} X_n/A_n} \cong \mathbb{P}_{A_n} X_n \otimes_{A_n} X_n = (\mathbb{P}_A X \otimes_A X)_n,$$

hence

$$\Omega_{\mathbb{P}_A X/A} \cong \mathbb{P}_A X \otimes_A X.$$

Now, tensoring with $- \otimes_{\mathbb{P}_A X} M$ and taking homotopy groups yields the result. □

We want to compute the groups $\mathrm{AQ}_*(\mathbb{P}_A CX, \mathbb{P}_A X; M) \cong \mathrm{AQ}_*(A, \mathbb{P}_A X; M)$ for cofibrant A-modules X. This can be done via the transitivity long exact sequence.

Corollary 4.3.7. *With the notation as above:*

$$\mathrm{AQ}_*(A, \mathbb{P}_A X; M) \cong \pi_*(\Sigma X \otimes_A M),$$

where the suspension Σ is in the category of A-modules.

Proof. The sequence $A \to \mathbb{P}_A X \to A$ yields a transitivity exact sequence of the form

$$\cdots \to 0 \to \mathrm{AQ}_n(A, \mathbb{P}_A X; M) \to \mathrm{AQ}_{n-1}(\mathbb{P}_A X, A; M) \to 0 \to \cdots$$

hence

$$\mathrm{AQ}_n(A, \mathbb{P}_A X; M) \cong \pi_{n-1}(X \otimes_A M).$$ □

This last corollary makes the computation of the André-Quillen homology of $B' = B \otimes_{\mathbb{P}_A K_n} \mathbb{P}_A CK_n$ accessible.

Proposition 4.3.8. *Denoting $B' = B \otimes_{\mathbb{P}_A K_n} \mathbb{P}_A CK_n$, we have for a B'-module M*

$$\mathrm{AQ}_*(B', B; M) \cong \pi_*(\Sigma K_n \otimes_A M) \cong \pi_{*-1}(K_n \otimes_A M).$$

Proof. Since $B \rightarrowtail B'$ is a cofibration, the pushout axiom holds. This together with the last corollary yields

$$\mathrm{AQ}_*(B', B; M) = \mathrm{AQ}_*(\mathbb{P}_A CK_n, \mathbb{P}_A K_n; M) \cong \pi_{*-1}(K_n \otimes_A M).$$ □

Let us restrict to the case, where $\pi_0 A = \pi_0 B = R$, so that we can use "trivial" coefficients $\underline{R}$ which become a B-module via the map $B \to \underline{\pi_0 B}$.

Proposition 4.3.9. *Subject to above restrictions, let B be an n-dimensional cell complex and let B' be obtained from B via attaching an n-cell $K_n = \bigoplus_{i\in I} \Sigma^n A$, so that B' is an $(n+1)$-dimensional cell complex.*

$$\mathrm{AQ}_k(B', A; \underline{R}) = \begin{cases} 0 & k > n+1, \\ \mathrm{AQ}_k(B, A; \underline{R}) & k < n. \end{cases}$$

In the middle we get the exact sequence

$$0 \to \mathrm{AQ}_{n+1}(B', A; \underline{R}) \to \bigoplus_{i\in I} R \to \mathrm{AQ}_n(B, A; \underline{R}) \to \mathrm{AQ}_n(B', A; \underline{R}) \to 0.$$

Proof. Setting $M = \underline{R}$ in the last proposition yields

$$\mathrm{AQ}_k(B', B; \underline{R}) = \pi_k \bigoplus_{i\in I} \Sigma^{n+1} A \otimes_A \underline{R} = \pi_k \bigoplus_{i\in I} \Sigma^{n+1} \underline{R} = \begin{cases} \bigoplus_{i\in I} R & k = n+1, \\ 0 & k \neq n+1. \end{cases}$$

We put this into the transitivity exact sequence for $A \to B \to B'$, which gives us

$$\mathrm{AQ}_k(B'A; \underline{R}) = \mathrm{AQ}_k(B, A; \underline{R})$$

for $k < n$ and $k > n+1$. By induction we know that $\mathrm{AQ}_k(B, A; \underline{R}) = 0$ for $k > n$. The middle part of the transitivity sequence is therefore

$$0 = \mathrm{AQ}_{n+1}(B, A; \underline{R}) \to \mathrm{AQ}_{n+1}(B', A; \underline{R}) \to \bigoplus_{i\in I} R \to \mathrm{AQ}_n(B, A; \underline{R}) \to \mathrm{AQ}_n(B', A; \underline{R}) \to 0$$

as required. □

As a last remark about these kinds of results, we mention the following corollary. Assume that R is a principal ideal domain, so that we have a notion of rank for modules. We define for an R-module M, that $\operatorname{rank} M = \dim_K M \otimes_R K$, where K is the fraction field of R.

Corollary 4.3.10. *If B is a finite cellular complex, with cells $K_n = \bigoplus_{i\leqslant m_n} \Sigma^n A$, then*

$$\sum_k \operatorname{rank} \mathrm{AQ}_k(B, A; \underline{R}) \leqslant \sum_k m_k,$$

where $m_k = 0$, if no cell of dimension k is attached.

Informally, this means that $\sum_k \operatorname{rank} \mathrm{AQ}_k(B, A; \underline{R})$ does not exceed the number of cells.

Proof. This follows inductively from the last proposition, using the fact that submodules of free modules are free. □

We can define the Euler-characteristic $\chi(B)$ in the usual manner as

$$\chi(B) = \sum_{i\geqslant 0}(-1)^i \operatorname{rank} \mathrm{AQ}_i(B, A; \underline{R}),$$

if this sum is finite.

Assume that there is an algebra map $R \to \Bbbk$ from R into a field $\Bbbk$. We then can consider

$$\chi_{\Bbbk}(B) = \sum_{i\geqslant 0}(-1)^i \dim_{\Bbbk} \mathrm{AQ}_i(B, A; \underline{\Bbbk}).$$

As in the ordinary homology theory of finite CW-complexes we have the following proposition.

Proposition 4.3.11. *For any such field* $\Bbbk$*, there is the identity*

$$\chi_{\Bbbk}(B) = \chi(B).$$

Proof. This follows immediately from the fact that

$$\mathfrak{L}_{B/A} \otimes_B \underline{\Bbbk} \cong \mathfrak{L}_{B/A} \otimes_B \underline{R} \otimes_{\underline{R}} \underline{\Bbbk}.$$

Passing to the Moore-complex and using the general coefficient theorem for modules yields the result. □

4.4 Cellular Approximation

We use the following conventions. We say that a simplicial module A is *n-connected*, if $\pi_i A = 0$ for $i \leqslant n$. A map $f : A \to B$ is said to be *n-connected* or an *n-equivalence* if the homotopy cofibre C_f is n-connected, i.e. $\pi_i(f)$ is an isomorphism for $i < n$ and $\pi_n(f)$ is surjective. In contrast, a *degree n equivalence* $f : A \to B$ is a map so that $\pi_i(f)$ is an isomorphism for $i \leqslant n$.

Let $\phi : A \to B$ be a map of simplicial A-algebras. We will exploit the consequences of n-equivalence to the André-Quillen homology. The next result is the analogue of attaching cells to kill homotopy groups.

Proposition 4.4.1. *If* $f : A \to B$ *is* n*-connected for an* $n \geqslant 0$*, then there is a factorisation* $A \rightarrowtail P \xrightarrow{\sim} B$ *in the category of* A*-algebras, where* $A_i \cong P_i$ *for* $i \leqslant n$*.*

Proof. Step by step we will construct a factorisation $A \rightarrowtail P \xrightarrow{\sim} B$, such that $A_i = P_i$ for $i \leqslant n$.

The map $\pi_n f : \pi_n A \to \pi_n B$ is a surjection. We use the identity $\pi_n A \cong \operatorname{ho} \mathscr{M}_A(\Sigma^n A, A)$. Suppose that the kernel is generated by homotopy classes of maps which are represented by $u_i : \Sigma^n A \to A$. We get an attaching map $K_n := \bigoplus_i \Sigma^n A \xrightarrow{k_n} A$, so that the composition $K_n \to A \to B$ is nullhomotopic and $\pi_n B \cong \pi_n A / \operatorname{im} \pi_n(k_n)$. Denoting $CK_n = \bigoplus C\Sigma^n A$, the composition factors as $K_n \to CK_n \to B$. Therefore we get a diagram of algebras

$$\begin{array}{ccc} \mathbb{P}_R K_n & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{P}_R C K_n & \longrightarrow & Q^n \end{array}$$

where $Q^n = \mathbb{P}_A CK_n \otimes_{\mathbb{P}_A K_n} A$ is the pushout.

We claim that $Q^n \to B$ is a degree n equivalence. Because $\pi_0 A \cong \pi_0 B$, we can assume that $n > 0$. By Lemma 4.3.2, one can see that $\pi_n \mathbb{P}_A K_n = \pi_n A \oplus \pi_n K_n$, the terms $K_n^{\otimes m}/\Sigma_m$ being at least n-connected.

For $i \leqslant n$, we have $Q_i^n = A_i$. Since $\mathbb{P}_A CK_n$ is cofibrant over $\mathbb{P}_A K_n$, there is the spectral sequence

$$\operatorname{Tor}_{p,q}^{\pi_* \mathbb{P}_A K}(\pi_* A, \pi_* A) \Longrightarrow \pi_{p+q} Q^n.$$

Because the higher Tor-terms are only nontrivial in degrees $\geqslant n$, we have

$$\pi_n Q^n = (\pi_* A \otimes_{\pi_* \mathbb{P}_A K_n} \pi_* A)_n = \pi_n A / \operatorname{im} \pi_n(k_n) \cong \pi_n B.$$

The map $g : Q^n \to B$ is therefore a degree n equivalence.

However, the map $\pi_{n+1} Q^n \to \pi_{n+1} B$ is not necessarily surjective. This can be adjusted like this: Take representants in B_{n+1} of the generators of the cokernel of $\pi_{n+1}(g)$. As before, there is a map $K'_{n+1} := \bigoplus \Sigma^{n+1} A \to B$ hitting these generators in π_{n+1}. This time, consider the pushout

$$\begin{array}{ccc} A & \rightarrowtail & \mathbb{P}_A K'_{n+1} \\ \downarrow & & \downarrow \\ Q^n & \rightarrowtail & P^n \end{array}$$

With this notation, $P_i^n = Q_i^n$ for $i \leqslant n$. Noting, that $\mathbb{P}_A K'_{n+1}$ is cofibrant over A, there is the spectral sequence

$$\operatorname{Tor}_{p,q}^{\pi_* A}(\pi_* Q^n, \pi_* \mathbb{P}_A K'_{n+1}) \Longrightarrow \pi_{p+q} P^n$$

from which follows that $\pi_n P^n \cong \pi_n Q^n$ and

$$\pi_{n+1} P^n = \pi_{n+1} Q^n \oplus \pi_0 Q^n \otimes_A \bigoplus A,$$

where the map $P^n \to B$ now induces a surjection on π_{n+1}.

The step from Q^n to P^n is attaching cells trivially. This can be seen like this: There is a pushout of modules for some (additive) cells $\widetilde{K}_n$

$$\begin{array}{ccc} \widetilde{K}_n & \longrightarrow & C\widetilde{K}_n \\ \downarrow & & \downarrow \\ \underline{0} & \longrightarrow & K'_{n+1} \end{array}$$

Since $\mathbb{P}_A$ is a left adjoint, this gives rise to the pushout diagram

$$\begin{array}{ccc} \mathbb{P}_A\widetilde{K}_n & \longrightarrow & \mathbb{P}_A C\widetilde{K}_n \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathbb{P}_A K'_{n+1}, \end{array}$$

which yields

$$\mathbb{P}_A K'_{n+1} = A \otimes_{\mathbb{P}_A \widetilde{K}_n} \mathbb{P}_A C\widetilde{K}_n.$$

Taking the trivial map $\mathbb{P}_A\widetilde{K}_n \to A \to Q^n$ and the map $\mathbb{P}_A C\widetilde{K}_n \to \mathbb{P}_A K'_{n+1}$, we can form the diagram

$$\begin{array}{ccccc} \mathbb{P}_A\widetilde{K}_n & \longrightarrow & \mathbb{P}_A C\widetilde{K}_n & & \\ \downarrow & & \downarrow & \searrow & \\ Q^n & \longrightarrow & Q^n \otimes_{\mathbb{P}_A\widetilde{K}_n} \mathbb{P}_A C\widetilde{K}_n & & \\ & \searrow & & \searrow & \\ & & & & B \end{array}$$

We get back

$$P^n = Q^n \otimes_A \mathbb{P}_A K'_{n+1} = Q^n \otimes_A A \otimes_{\mathbb{P}_A\widetilde{K}_n} \mathbb{P}_R C\widetilde{K}_n = Q^n \otimes_{S_R\widetilde{K}_n} \mathbb{P}_A C\widetilde{K}_n$$

with its map $P^n \to B$ as before.

We can iterate this construction to get an algebra $P = \operatorname{colim}_n P^n$, such that

$$A \rightarrowtail P \xrightarrow{\sim} B,$$

with $P_i = A_i$ for $i \leqslant n$. $\square$

By the same construction, a cellular approximation of an A-algebra B can be obtained even when there are no connectivity-conditions. Write $\phi : A \to B$ for the unit and take generators $u_i \in \pi_0 B / \operatorname{im} \pi_0(\phi)$ to form the A-algebra $A[u_1, \ldots]$, which in degree n is the polynomial algebra $A_n[u_1, \ldots]$. The boundary and degeneracy maps of A are extended to identities on the variables u_i, so that $A[u_1 \ldots]$ becomes an A-algebra. There is an induced map $A[u_1, \ldots] \to B$ which is now surjective on π_0, i.e. the map is 0-connected. We can set $P_0 = A[u_1, \ldots]$ and continue as in the last proposition.

We present now the proposition, which in the case for spectra is due to Basterra [6, Lemma 8.2].

Proposition 4.4.2. *If the unit $\phi : A \to B$ is an n-connected map for some $n \geqslant 1$ so that we can set $\pi_0 A = \pi_0 B$, then $\mathfrak{L}_{B/A}$ is n-connected and there is a map of A-modules $\mathrm{C}_\phi \to \mathfrak{L}_{B/A}$ which induces*

$$\pi_{n+1} \mathrm{C}_\phi \xrightarrow{\cong} \pi_{n+1}(\mathfrak{L}_{B/A}) \cong \mathrm{AQ}_{n+1}(B, A; \underline{\pi_0 B}).$$

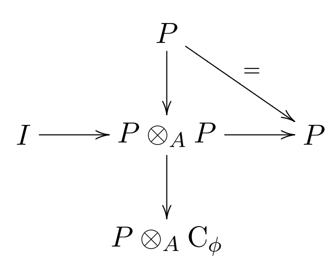
If $\phi : A \to B$ is 0-connected, then $\mathfrak{L}_{B/A}$ is 0-connected and there is a map of A-modules $\mathrm{C}_\phi \to \mathfrak{L}_{B/A}$ which induces

$$\pi_1(\mathrm{C}_\phi \otimes_A \underline{\pi_0 B}) \xrightarrow{\cong} \mathrm{AQ}_1(B, A; \underline{\pi_0 B}).$$

Proof. Let us first factor the unit into $A \rightarrowtail P \xrightarrow{\sim} B$, where P is as in the preceding proposition. Factoring further we obtain

$$A \rightarrowtail P \overset{\sim}{\rightarrowtail} P' \overset{\sim}{\twoheadrightarrow} B.$$

So the cofibre of ϕ is given by P'/A. But a diagram chase shows, that $P/A \xrightarrow{\sim} P'/A$, so that we can set $\mathrm{C}_\phi = P/A$. This has the advantage, that $(\mathrm{C}_\phi)_k = 0$ for $k \leqslant n$. The composition $A \to P \to \Omega_{P/A}$ is the zero map, hence there is a map $\tau : \mathrm{C}_\phi \to \Omega_{P/A}$ which extends by the unit $P \to B$ to a map $\mathrm{C}_\phi \to B \otimes_P \Omega_{P/A}$. Let I denote the kernel of the multiplication map $P \otimes_A P \to P$. Tensoring the short exact sequence $A \to P \to \mathrm{C}_\phi$ with $P \otimes_A -$ yields another short exact sequence, because the induced map $P \to P \otimes_A P$ splits the multiplication map μ_P, so that it is injective. The diagram of levelwise short exact sequences



shows that as A-modules $I \cong P \otimes_A \mathrm{C}_\phi$. Hence $I_k = 0$ for $k \leqslant n$ and I is n-connected. Since I is a retract of $P \otimes_A P$, it is cofibrant as $P \otimes_A P$-module. Hence the Künneth spectral sequence proves that $I \otimes_P I$ is at least $(2n+1)$-connected. We now have to calculate $\pi_k(Q_P(I)) = \pi_k(I/I^2)$ for $k \leqslant n+1$.

By [49, 11.27] there is a short exact sequence for every k of the form

$$\mathrm{Tor}_1^{P_k \otimes P_k}(I_k, P_k) \to I_k \otimes_{P_k} I_k \to I_k^2.$$

They assemble to a short exact sequence of A-modules. From $I_k = 0$ for $k \leqslant n$ follows that we have an exact sequence

$$\pi_{n+1}(I \otimes_P I) \to \pi_{n+1}(I^2) \to \pi_n(\mathrm{Tor}_1^{P \otimes P}(I, P)) \to 0.$$

The first and the last term are zero, hence $\pi_{n+1}(I^2) = 0$. The long exact sequence

$$\pi_{n+1}(I^2) \to \pi_{n+1} I \to \pi_{n+1}(I/I^2) \to \pi_n(I^2)$$

yields that $\pi_{n+1}(I/I^2) \cong \pi_{n+1} \mathrm{C}_\phi$. In total, $\left(\Omega_{P/A}\right)_k = 0$ for $k \leqslant n$, that means $\Omega_{P/A}$ is n-connected and

$$\pi_{n+1}\Omega_{P/A} \cong \pi_{n+1} I \cong \pi_{n+1}(P \otimes_A \mathrm{C}_\phi).$$

The spectral sequence

$$\mathrm{Tor}_{p,q}^{\pi_* P}(\pi_* \Omega_{P/A}, \underline{\pi_0 B}) \Longrightarrow \pi_{p+q}(\Omega_{P/A} \otimes_P \underline{\pi_0 B})$$

finally shows that

$$\mathrm{AQ}_r(B, A; \underline{\pi_0 B}) = \mathrm{AQ}_r(P, A; \underline{\pi_0 B}) = \begin{cases} 0 & r \leqslant n, \\ \pi_{n+1}(\mathrm{C}_\phi \otimes_A \underline{\pi_0 B}) & r = n+1. \end{cases}$$

If now $n > 0$, the hypothesis $\pi_0 A = \pi_0 B$ together with the Künneth spectral sequence shows that

$$\mathrm{AQ}_{n+1}(B, A; \underline{\pi_0 B}) \cong \pi_{n+1} \mathrm{C}_\phi$$

as required. □

This proposition has the following corollary.

Corollary 4.4.3. *If $\phi : A \to B$ is an algebra map which induces an isomorphism on π_0, then ϕ is a weak equivalence if and only if* $\mathrm{AQ}_*(B, A, \underline{\pi_0 B}) = 0$.

Proof. Only the "if" is to be shown. Assume $0 = \pi_*(\mathfrak{L}_{B/A} \otimes_B \underline{k})$. We prove that $A \to B$ is a weak equivalence by induction. It is a 0-equivalence by hypothesis. Now assume it is an n-equivalence. Then $\pi_{n+1}\mathrm{C}_\phi \cong \pi_{n+1}(\mathfrak{L}_{B/A} \otimes_B \underline{k}) = 0$. So $A \to B$ is an $(n+1)$-equivalence. □

Chapter 5

Minimal Atomic Complexes

In this chapter we follow closely the thesis by Gilmour [20], where the case for ring spectra has been treated. Most ideas originate in the paper [4] by Baker and May.

All our algebras in this chapter are algebras over a noetherian local ground ring R. Its maximal ideal is denoted by $\mathfrak{m}$ so that $R/\mathfrak{m}$ is the residue field. We have mostly $R = \mathbb{Z}_{(p)}$ with $R/\mathfrak{m} = \mathbb{F}_p$ in mind, but the results are more general. We will only deal with cellular simplicial A-algebras B, so that $\pi_i(B)$ are finitely generated R-modules for all i. Furthermore we will assume in this chapter that the unit $A \to B$ induces an epimorphism on $\pi_0 B$. In particular, the 0-skeleton of B can be chosen to be A. We will constantly be using the fact that finitely generated modules over a noetherian ring are noetherian.

5.1 Minimal Atomic Algebras

We start with the definition of a nuclear complex and a core.

Definition 5.1.1. A cellular commutative A-algebra B is called *nuclear*, if the attaching maps $k_n : K_n \to B_n$ satisfy

$$\ker \pi_n(k_n) \subseteq \mathfrak{m}\pi_n(K_n)$$

for all $n \in \mathbb{N}$.

As an example take $A = \underline{\mathbb{F}}_p \oplus \Sigma^n \underline{\mathbb{F}}_p$ and use the attaching map $k_n : \Sigma^n A \to A$ which is the identity on $\Sigma^n \underline{\mathbb{F}}_p$. This produces a nuclear algebra.

Definition 5.1.2. A *core* of a simplicial A-algebra B is a nuclear A-algebra Q with an algebra map $g : Q \to B$, so that the induced map $\pi_*(g)$ is injective.

Note that the standing assumption that $\pi_0 A \to \pi_0 B$ is an epimorphism forces $\pi_0(g)$ to be an isomorphism.

Proposition 5.1.3. *For every algebra there exists a core.*

Proof. We construct a core Q by killing homotopy groups, proceeding inductively. Set $Q^0 = A$ and set g^0 to be the unit of B. Assume we have constructed Q^n and g^n, so that g^n induces an injection on homotopy groups in dimensions less than n. We set $L = \ker \pi_n(g^n)$. The R-module L is finitely generated over R. Moreover, $L/\mathfrak{m}L$ is a vector space over the residue field $R/\mathfrak{m}R$, so that we can find a finitely generated free R-module F with $F/\mathfrak{m}F \cong L/\mathfrak{m}L$. Because F is free there is a lift

$$\begin{array}{ccccc} & & & & L \\ & & \overset{f}{\nearrow} & & \downarrow \\ F & \longrightarrow & F/\mathfrak{m}F & \longrightarrow & L/\mathfrak{m}L \end{array}$$

which we denote by f. It follows that $L = \operatorname{im}(f) + \mathfrak{m}L$, so that by Nakayama's lemma we have that f is surjective. Furthermore, $\ker(f) \subseteq \mathfrak{m}F$ by construction.

Now, take a coproduct of copies of $\Sigma^n A$ corresponding to the generators of F. We get an attaching map $k_n : K_n = \bigoplus \Sigma^n A \to Q^n$ so that $\ker \pi_n(k_n) \cong \ker(f) \subseteq \mathfrak{m}\pi_n K_n$. We further get a map $\mathbb{P}_A(\bigoplus \Sigma^n A) \to Q^n$ so that we can form the pushout

$$\begin{array}{ccc} \mathbb{P}_A(\bigoplus \Sigma^n A) & \longrightarrow & Q^n \\ \downarrow & & \downarrow \\ \mathbb{P}_A(C \bigoplus \Sigma^n A) & \longrightarrow & Q^{n+1} \end{array}$$

to obtain Q^{n+1} and the map $g^{n+1} : Q^{n+1} \to B$. By construction $\pi_i(g^{n+1})$ is the same as $\pi_i(g^n)$ for $i < n$ and $\pi_n(g^{n+1})$ is an injection because of 4.3.2. □

Remark that cores do not need to be unique, not even up to homotopy. If the unit $A \to B$ of an A-algebra is an injection, then A serves trivially as a core.

Definition 5.1.4. An A-algebra B is called *atomic*, if any A-algebra self map of B is a weak equivalence. It is *minimal atomic*, if it is atomic and any map $C \to B$ from an atomic A-algebra C to B that induces an injection on homotopy groups is a weak equivalence.

The main theorem is that nuclear implies minimal atomic. The proof goes over several intermediate results.

Proposition 5.1.5. *Every nuclear A-algebra B is atomic.*

Proof. The unit of B induces an isomorphism on π_0. Take a self map f of B, which we can assume to be cellular. We will prove that $f^n : B^n \to B^n$ is an n-equivalence for all n. We proceed by induction. Since $B^0 = A$, the statement is true for $n = 0$. Assume f^n is a n-equivalence. Then $B^{n+1} = B^n \otimes_{\mathbb{P}K_n} \mathbb{P}CK_n$ is formed as a pushout, where K_n is a sum of n-spheres.

Using that $\mathrm{AQ}_n(B^{n+1}, B^n; \underline{R}) = \pi_n K_n$, we get the following diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{AQ}_{n+1}(B^{n+1}, A; \underline{R}) & \longrightarrow & \pi_n K_n & \longrightarrow & \mathrm{AQ}_n(B^n, A; \underline{R}) & \longrightarrow & \mathrm{AQ}_n(B^{n+1}, A; \underline{R}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow \alpha & & \downarrow \cong & & \downarrow & \\
0 \longrightarrow & \mathrm{AQ}_{n+1}(B^{n+1}, A; \underline{R}) & \longrightarrow & \pi_n K_n & \longrightarrow & \mathrm{AQ}_n(B^n, A; \underline{R}) & \longrightarrow & \mathrm{AQ}_n(B^{n+1}, A; \underline{R}) & \longrightarrow 0
\end{array}$$

with an induced map α. If we can prove that α is an isomorphism, then from the 5-lemma follows that the first and the last vertical arrow are isomorphisms. The homotopy cofibre sequence $K_n \to B^n \to \mathrm{C}_{k_n}$ yields a long exact sequence in homotopy. We get the diagram

$$\begin{array}{ccccccc}
\pi_n K_n & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \mathrm{C}_{k_n} & \longrightarrow & 0 \\
\alpha \downarrow & & \downarrow \beta & & \gamma \downarrow & & \\
\pi_n K_n & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \mathrm{C}_{k_n} & \longrightarrow & 0
\end{array}$$

with induced maps β and γ. Since β is an isomorphism, it follows that γ is an epimorphism. However, a surjective endomorphism of a noetherian module is always an isomorphism.

The diagram above induces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}\, \pi_n(k_n) & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \mathrm{C}_{k_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \beta & & \gamma \downarrow & & \\
0 & \longrightarrow & \mathrm{im}\, \pi_n(k_n) & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \mathrm{C}_{k_n} & \longrightarrow & 0
\end{array}$$

hence f induces an isomorphism on $\mathrm{im}\, \pi_n(k_n)$. Furthermore we obtain the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \pi_n(k_n) & \longrightarrow & \pi_n K_n & \longrightarrow & \mathrm{im}\, \pi_n(k_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow & & \\
0 & \longrightarrow & \ker \pi_n(k_n) & \longrightarrow & \pi_n K_n & \longrightarrow & \mathrm{im}\, \pi_n(k_n) & \longrightarrow & 0
\end{array}$$

By a diagram chase we find that

$$\pi_n K_n = \mathrm{im}\, \alpha + \ker \pi_n(k_n).$$

By construction we have $\ker \pi_n(k_n) \subseteq \mathfrak{m}\pi_n K_n$ so that

$$\pi_n K_n = \mathrm{im}\, \alpha + \mathfrak{m}\pi_n K_n.$$

From Nakayama's lemma follows that $\operatorname{im}\alpha = \pi_n K_n$. This shows that α is an epimorphism and hence an isomorphism by the same argument as above. So the induced maps

$$\mathrm{AQ}_{n+1}(B^{n+1}, A; \underline{R}) \to \mathrm{AQ}_{n+1}(B^{n+1}, A; \underline{R}),$$
$$\mathrm{AQ}_{n}(B^{n+1}, A; \underline{R}) \to \mathrm{AQ}_{n}(B^{n+1}, A; \underline{R})$$

are isomorphisms. By Proposition 4.3.9 and the induction hypothesis we have that the self map f induces an isomorphism on $\mathrm{AQ}_*(B, A; \underline{R})$. Taking the transitivity exact sequence for $A \to B \xrightarrow{f} B$ we get that the André-Quillen homology of the pair $B \xrightarrow{f} B$ vanishes. From Corollary 4.4.3 it follows that f induces an isomorphism on homotopy. □

Corollary 5.1.6. *Any minimal atomic A-algebra B is weakly equivalent to a nuclear A-algebra.*

Proof. Let B be a minimal atomic A-algebra. Consider any core $Q \to B$. By Proposition 5.1.5, Q is atomic. By minimality of B we conclude that $Q \to B$ is a weak equivalence. □

Theorem 5.1.7. *If B is a nuclear A-algebra, then every core Q of B is weakly equivalent to B.*

Proof. Let $f : Q \to B$ be a core of B. We assume again that f is cellular and proceed using induction on the skeleta. Assume that $f : Q^n \to B^n$ is an n-equivalence. Let the $(n+1)$-skeleta be given by

$$Q^{n+1} = Q^n \otimes_{\mathbb{P}_A K_n} \mathbb{P}_A C K_n,$$
$$B^{n+1} = B^n \otimes_{\mathbb{P}_A J_n} \mathbb{P}_A C J_n,$$

where the K_n and J_n are coproducts of n-spheres attached by the maps $k_n : K_n \to Q^n$ and $j_n : K_n \to B^n$.

By the transitivity exact sequence, we get the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{AQ}_{n+1}(Q^{n+1}, A, \underline{R}) & \longrightarrow & \pi_n K_n & \longrightarrow & \mathrm{AQ}_n(Q^n, A; \underline{R}) & \longrightarrow & \mathrm{AQ}_n(Q^{n+1}, A; \underline{R}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \cong} & & \downarrow & \\
0 \longrightarrow & \mathrm{AQ}_{n+1}(B^{n+1}, A, \underline{R}) & \longrightarrow & \pi_n J_n & \longrightarrow & \mathrm{AQ}_n(B^n, A; \underline{R}) & \longrightarrow & \mathrm{AQ}_n(B^{n+1}, A; \underline{R}) & \longrightarrow 0
\end{array}$$

If we can show that α is an isomorphism it follows that all vertical maps in the diagram above are isomorphisms.

The cofibre sequences

$$\begin{array}{ccccc} K_n & \longrightarrow & Q^n & \longrightarrow & \mathrm{C}_{k_n} \\ \downarrow & & \downarrow & & \downarrow \\ J_n & \longrightarrow & B^n & \longrightarrow & \mathrm{C}_{j_n} \end{array}$$

lead to the diagram

$$\begin{array}{ccccccc} \pi_n K_n & \longrightarrow & \pi_n Q^n & \longrightarrow & \pi_n \mathrm{C}_{k_n} & \longrightarrow & 0 \\ \downarrow & & \downarrow \cong & & \downarrow & & \\ \pi_n J_n & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \mathrm{C}_{j_n} & \longrightarrow & 0 \end{array}$$

where the rows are exact. By a diagram chase, the last vertical arrow is an epimorphism. There is a canonical map $\mathrm{C}_{k_n} \to Q^{n+1} \to Q$. By construction of Q^{n+1} we have an isomorphism

$$\pi_n \mathrm{C}_{k_n} \cong \pi_n Q^{n+1} \cong \pi_n Q$$

and similarly for B. Put in a diagram

$$\begin{array}{ccccc} \pi_n \mathrm{C}_{k_n} & \xrightarrow{\cong} & \pi_n Q^{n+1} & \xrightarrow{\cong} & \pi_n Q \\ \downarrow & & & & \downarrow \\ \pi_n \mathrm{C}_{j_n} & \xrightarrow{\cong} & \pi_n B^{n+1} & \xrightarrow{\cong} & \pi_n B \end{array}$$

we see that the first vertical arrow is a monomorphism, hence it is an isomorphism.

As in the proof of Proposition 5.1.5, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi_n(k_n) & \xrightarrow{i} & \pi_n K_n & \longrightarrow & \operatorname{im} \pi_n(k_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & \ker \pi_n(j_n) & \xrightarrow{i} & \pi_n J_n & \longrightarrow & \operatorname{im} \pi_n(j_n) & \longrightarrow & 0 \end{array}$$

with exact rows, where the last vertical arrow is an isomorphism.

Tensoring with the residue field $R/\mathfrak{m}$, both maps i become zero due to the gluing conditions for nuclear algebras. Hence $\alpha \otimes R/\mathfrak{m}$ is an isomorphism. Since $\pi_n K_n$ and $\pi_n J_n$ are both free R-modules, we get the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\pi_n K_n) \otimes_R \mathfrak{m} & \longrightarrow & \pi_n K_n & \longrightarrow & (\pi_n K_n) \otimes_R R/\mathfrak{m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \cong & & \\ 0 & \longrightarrow & (\pi_n J_n) \otimes_R \mathfrak{m} & \longrightarrow & \pi_n J_n & \longrightarrow & (\pi_n J_n) \otimes_R R/\mathfrak{m} & \longrightarrow & 0 \end{array}$$

We see that

$$\pi_n J_n = \operatorname{im} \alpha + \mathfrak{m} \pi_n J_n,$$

hence α is surjective by Nakayama's lemma. It follows from the Snake lemma and the freeness of $\pi_n J_n$ that $\ker \alpha \cong \mathfrak{m} \ker \alpha$, so that $\ker \alpha = 0$ by Nakayama's lemma again. $\square$

Corollary 5.1.8. *Every nuclear A-algebra B is minimal atomic.*

Proof. Let B be a nuclear A-algebra and let $f : Q \to B$ be a map from an atomic algebra Q which induces a monomorphism in homotopy. We show that f is a weak equivalence. Take a core $g : W \to Q$. The composition $f \circ g$ induces a monomorphism in homotopy, hence it is a weak equivalence by the preceding theorem. It follows that f induces also an epimorphism in homotopy, hence it is a weak equivalence. □

One question is, how nuclear algebras can be detected. One easy description is in terms of no mod $\mathfrak{m}$ detectable homotopy and minimality which we will define now. We abbreviate $\Bbbk = R/\mathfrak{m}$ for the residue field.

Definition 5.1.9. A cellular A-algebra $B = \operatorname{colim}_n B^n$ with $\pi_0 B \cong \pi_0 A$ is said to *have no mod* $\mathfrak{m}$ *detectable homotopy* if the Hurewicz maps (3.2)

$$\theta_n : \pi_n B^n \to \mathrm{AQ}_n(B^n, A; \underline{\Bbbk})$$

are zero for all n.

Definition 5.1.10. A cellular A-algebra $B = \operatorname{colim}_n B^n$ is said to be *minimal*, if the induced epimorphism

$$\mathrm{AQ}_n(B^n, A; \underline{\Bbbk}) \to \mathrm{AQ}_n(B, A; \underline{\Bbbk})$$

is an isomorphism for all n.

This definition of minimal originates in the theory of minimal complexes (see for example [17]). Be aware that this is a different kind of minimal than the one used in the term minimal atomic. A minimal atomic algebra does not need to be minimal in the sense above.

Theorem 5.1.11. *An A-algebra B is nuclear if and only if it is minimal and has no mod* $\mathfrak{m}$ *detectable homotopy.*

Proof. We use the identity

$$\mathrm{AQ}_{n+1}(B^{n+1}, B^n; \underline{\Bbbk}) = \pi_n(K_n \otimes_R \underline{\Bbbk}).$$

There is a commutative diagram

$$\begin{array}{ccccccc} & & \pi_{n+1} B^{n+1} & \xrightarrow{\gamma} & \pi_n K_n & \longrightarrow & \pi_n B^n \\ & & \downarrow{\scriptstyle \theta_{n+1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \theta_n} \\ 0 & \longrightarrow & \mathrm{AQ}_{n+1}(B^{n+1}, A; \underline{\Bbbk}) & \longrightarrow & \pi_n(K_n \otimes_R \underline{\Bbbk}) & \longrightarrow & \mathrm{AQ}_n(B^n, A; \underline{\Bbbk}). \end{array} \tag{5.1}$$

Assume that B is nuclear. Then the map γ has its image contained in $\mathfrak{m}\pi_n K_n$, hence the composition of γ with α is zero. Therefore θ_{n+1} is the zero map. Furthermore, θ_0 is the zero map because $B^0 = A$, and hence all θ_n are just zero maps. Because the map $\pi_n K_n \to \pi_n(K_n \otimes_R \underline{\Bbbk})$ is an epimorphism, it follows that the map

$$\pi_n(K_n \otimes_R \underline{\Bbbk}) \to \mathrm{AQ}_n(B^n, A; \underline{\Bbbk})$$

is the zero map, therefore the surjective map

$$\mathrm{AQ}_n(B^n, A; \underline{\Bbbk}) \to \mathrm{AQ}_n(B^{n+1}, A; \underline{\Bbbk}) \cong \mathrm{AQ}_n(B, A; \underline{\Bbbk})$$

is an isomorphism. Hence B is minimal. Moreover, the map

$$\pi_n B^n \to \mathrm{AQ}_n(B^n, A; \underline{\Bbbk}) \cong \mathrm{AQ}_n(B, A; \underline{\Bbbk})$$

is the zero map, hence B has no mod $\mathfrak{m}$ detectable homotopy.

On the other hand, minimal and no mod $\mathfrak{m}$ detectable homotopy means that

$$\mathrm{AQ}_n(B^n, A; \underline{\Bbbk}) \cong \mathrm{AQ}_n(B^{n+1}, A; \underline{\Bbbk})$$

and the map

$$\pi_n B^n \to \mathrm{AQ}_n(B, A; \underline{\Bbbk}) \cong \mathrm{AQ}_n(B^{n+1}, A; \underline{\Bbbk})$$

is the zero map. Therefore

$$\theta_n : \pi_n B^n \to \mathrm{AQ}_n(B^n, A; \underline{\Bbbk})$$

is the zero map for all n. From the diagram (5.1) above it follows that B is nuclear. □

Further we can define irreducible algebras in the following way:

Definition 5.1.12. An A-algebra B is called *irreducible* if every injection $C \to B$ of A-algebras is a weak equivalence.

As analogue of Theorem 1.8 by [4] we can state the next theorem.

Theorem 5.1.13. *The following conditions on an A-algebra B are equivalent*

- *B is irreducible,*
- *B is minimal atomic.*

Proof.

- (i) $\Longrightarrow$ (ii)

  Let B be an irreducible algebra and let f be self-map. There exists a core $Q \to B$ and B is weakly equivalent to B by definition. Furthermore, f induces a map on Q, but Q is atomic. Therefore f is a weak equivalence on B, i.e. B is atomic. That B is minimal atomic follows directly from the definition of irreducible.

- (ii) $\Longrightarrow$ (i)

  Let B be minimal atomic and take any injection $C \to B$ of A-algebras. Take further a core $Q \to C \to B$ and note that the composition $Q \to B$ is a weak equivalence. Hence $C \to B$ is already a weak equivalence.

□

5.2 Minimal Atomic Modules

We digress to introduce the concepts above also for modules. We do this in order to detect commutative minimal atomic algebras.

Again, let R be a local ring with maximal ideal $\mathfrak{m}$ and let A denote a simplicial R-algebra. Let X be a cellular complex of A-modules. We say that n_0 is the Hurewicz-dimension of X, if $\pi_i(X) = 0$ for $i < n_0$ but $\pi_{n_0}(X) \neq 0$. We say that X is a Hurewicz-complex, if $\pi_{n_0}(X)$ is a quotient of R. In this section, we say that a map $X \to Y$ of A-modules with Hurewicz-dimension n_0 is a monomorphism, if $\pi_n(f) : \pi_n X \to \pi_n Y$ is a monomorphism for all n and the induced map $\pi_{n_0}(X) \otimes R/\mathfrak{m} \to \pi_{n_0}(Y) \otimes R/\mathfrak{m}$ is a monomorphism, too. This last condition is necessary to get the behaviour at the bottom dimension n_0 under control.

Definition 5.2.1. We use the following definition from [4].

- Y is *irreducible*, if every monomorphism $X \to Y$ is an equivalence.
- Y is *atomic*, if it is a Hurewicz-complex and any self map $f : Y \to Y$ that induces an isomorphism in the Hurewicz-dimension is an equivalence.
- Y is *minimal atomic*, if it is atomic and any monomorphism $f : X \to Y$ from an atomic module X into Y is an equivalence.
- Y has *no homotopy detected by mod* $\mathfrak{m}$, if it is a Hurewicz-complex and the map $\pi_n Y \to \pi_n(Y \otimes_R \underline{R/\mathfrak{m}})$ is zero for all $n > n_0$.

We summarise Theorems 1.3, and 1.8 of [4].

Theorem 5.2.2. *The following conditions are equivalent*

- Y *is irreducible,*
- Y *has no homotopy detected by mod* $\mathfrak{m}$ *homology,*
- Y *is minimal atomic.*

We omit the proof, because it is word by word identical to the one given by Baker and May.

Definition 5.2.3. We say that a Hurewicz-complex X with attaching maps $k_n : K_n \to X^n$ is *nuclear*, if

$$\ker \pi_n(k_n) \subseteq \mathfrak{m}\pi_n K_n.$$

Given any complex X, we say that a *core* of X is a nuclear complex Y together with a monomorphism $Y \to X$.

Without a proof we give the following proposition from Baker and May.

Proposition 5.2.4. *For every A-module X there exists a core. Any core Y of a minimal atomic module X is equivalent to X.*

Proposition 5.2.5. *For any core $Q \to B$ of A-algebras there exists a core $X \to B$ of modules together with an A-module map $\alpha : X \to Q$, so that the core $X \to B$ factors through α. Moreover, α is an injection on homotopy groups.*

Proof. Take X to be a core of the A-module Q. So the composition $X \to Q \to B$ is a core of B as an A-module. □

This leads to the following corollary.

Corollary 5.2.6. *If B is an A-algebra, which is minimal atomic as an A-module, then it is weakly equivalent to a minimal atomic A-algebra.*

Proof. Given such an algebra B, then the Proposition 5.2.5 shows that we can take a core $Q \to B$ as A-algebras and a core $X \to Q$ as A-modules so that $X \to B$ is a core of A-modules. Because B is minimal atomic as a module, it is a weak equivalence. Therefore the map $Q \to B$ is a weak equivalence. So, Q is nuclear as an algebra, hence minimal atomic by Corollary 5.1.8. □

Chapter 6

Stabilisation and Connections to Γ-Homology

In this chapter we introduce the concept of stabilisation according to Goodwillie. In [8] a link between Γ-homology and topological André-Quillen homology is established using stabilisation. In general this does not work for André-Quillen homology of simplicial algebras. However, we are able to prove a similar result for small degrees. We use the computation of the homotopy groups of spheres to obtain an explicit bound up to what degree Γ-homology and André-Quillen homology coincide.

6.1 i-stable Algebras

We fix a simplicial R-algebra A for the whole chapter. The results on stabilisation highly depend on the connectivity of the maps $\Sigma^n A \to \mathbb{A}_A \Sigma^n A$ for all n. The best one can hope for is that this map is $(2n-1)$-connected as is the case if A is a simplicial $\mathbb{Q}$-algebra (see Section 4.2.3).

Definition 6.1.1. We say that A is *i-stable* if the map $\Sigma^n A \to \mathbb{A}_A \Sigma^n A$ is $(2n-1)$-connected if $n \leqslant i$ and at least $(n+i-1)$-connected if $n > i$.

Note that, if A is a simplicial $\mathbb{Q}$-algebra, then A is i-stable for every i. The same phenomenon happens in the case of ring spectra: The map $\Sigma^n A \to \mathbb{A}_A \Sigma^n A$ is always $(2n-1)$-connected [19, III.5], if A is a strictly commutative R-ring spectrum. Due to Lemma 4.3.2 every simplicial algebra A is 1-stable. As described in the Section 4.2.2, the algebras $\underline{\mathbb{F}_p}$ are $(2p-3)$-stable. We present more examples of i-stable algebras for $i > 1$

as 6.1.4 later.

If an algebra A is i-stable, its André-Quillen homology can be described up to degree i in terms of stabilisation, so mimicking the behaviour of strictly commutative ring spectra [32] in low degrees. This is the main result of this chapter and presented as Theorem 6.5.1.

Lemma 6.1.2. *Let A be i-stable. For a sum $K_n = \bigoplus \Sigma^n A$ we have that the map*

$$K_n \to \mathbb{A}_A K_n$$

is $(2n-1)$-connected if $n \leqslant i$ and at least $(n+i-1)$-connected if $n > i$.

Proof. It suffices to note that $\mathbb{A}_A$ is a left adjoint. Using the description of coproducts of nucas from Section 1.2 we obtain therefore

$$\mathbb{A}_A K_n = \coprod \mathbb{A}_A \Sigma^n A = \bigoplus \mathbb{A}_A \Sigma^n A \oplus L,$$

where L is a $(2n-1)$-connected A-module. Therefore by the definition of i-stable, the map $K_n \to \mathbb{A}_A K_n$ is $(2n-1)$-connected if $n \leqslant i$ and $(n+i-1)$-connected if $n > i$. □

We use this lemma to prove the next proposition about n-connected modules over i-stable algebras. Note that the lemma is a special case of the proposition since K_n is $(n-1)$-connected.

Proposition 6.1.3. *Let M be a n-connected cofibrant module over an i-stable algebra A. Then the map $M \to \mathbb{A}_A M$ is $(2n+1)$-connected if $n \leqslant i+1$ and at least $(n+i)$-connected if $n > i+1$.*

Proof. Approximate M by a cell complex of modules $N = \operatorname{colim}_k N^k \xrightarrow{\sim} M$. The bottom cell can be chosen to be in dimension $n+1$, i.e. $N^k = 0$ for $k \leqslant n$ and $N^{n+1} = \bigoplus \Sigma^{n+1} A$. Thus, by the last lemma we have that the map $N^{n+1} \to \mathbb{A}_A N^{n+1}$ satisfies the connectivity conditions and we can proceed by induction.

Assume that $N^k \to \mathbb{A}_A N^k$ for some $k > n+1$ is $(2n+1)$-connected if $n \leqslant i+1$ and at least $(n+i)$-connected if $n > i+1$. We will show that the same holds for N^{k+1}. By gluing cells we obtain a pushout of modules of the form

$$\begin{array}{ccc} K_k & \longrightarrow & N^k \\ \downarrow & & \downarrow \\ CK_k & \longrightarrow & N^{k+1} \end{array}$$

so that N^{k+1} is the mapping cone of the attaching map. Applying $\mathbb{P}_A$ respects pushouts, hence

$$\mathbb{P}_A N^{k+1} = \mathbb{P}_A C K_k \otimes_{\mathbb{P}_A K_k} \mathbb{P}_A N^k.$$

Using the Künneth spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\pi_* \mathbb{P}_A K_k}_{p,q}(\pi_* A, \pi_* \mathbb{P}_A N^k) \Longrightarrow \pi_{p+q} \mathbb{P}_A N^{k+1}$$

we can do the following calculation. Let $j = \min(2n+1, n+i)$ denote the smaller of the two degrees. Because we are only interested in the degrees $\leqslant j$ we can, using again the last lemma, replace $\mathbb{P}_A K_k$ by the square-zero extension $A \oplus K_k$ and we have

$$\mathrm{Tor}^{\pi_* \mathbb{P}_A K_k}_{p,q}(\pi_* A, \pi_* \mathbb{P}_A N^k) \cong \mathrm{Tor}^{\pi_* A \oplus \pi_* K_k}_{p,q}(\pi_* A, \pi_* A \oplus \pi_* N^k)$$

for p and q smaller than j.

The right hand side is a spectral sequence converging to $\pi_*(A \otimes^L_{A \oplus K_k} (A \oplus N^k))$ which is therefore isomorphic to $\pi_* \mathbb{P}_A N^{k+1}$ in the desired degrees. Forming a resolution of $\pi_* A$ by the square-zero extension $\pi_* A \oplus \pi_* K_k$, one sees that all the $\mathrm{Tor}_{p,q}$-terms vanish for $p > 1$. So we get for the abutment that there are short exact sequences for $m <\leqslant j$ of the form

$$0 \longrightarrow \pi_m A \oplus \mathrm{im}(\pi_m K_k \to \pi_m N^k) \longrightarrow \pi_m \mathbb{P}_A N^{k+1} \longrightarrow \ker(\pi_{m-1} K_k \to \pi_{m-1} N^k) \longrightarrow 0$$

which turn into the short exact sequences

$$0 \longrightarrow \mathrm{im}(\pi_m K_k \to \pi_m N^k) \longrightarrow \pi_m \mathbb{A}_A N^{k+1} \longrightarrow \ker(\pi_{m-1} K_k \to \pi_{m-1} N^k) \longrightarrow 0.$$

So, the short exact sequence can be spliced into a long exact sequence of the form

$$\cdots \pi_m K_k \to \pi_m N^k \to \pi_m \mathbb{A}_A N^{k+1} \to \pi_{m-1} K_k \to \cdots$$

where $m \leqslant j$. But this defines the mapping cone of the attaching map $K_k \to N^k$ in degrees $\leqslant j$ which is N^{k+1}. □

The following corollary yields a lot more examples of i-stable algebras.

Corollary 6.1.4. *If B is a cofibrant algebra over an i-stable algebra A, then B is i-stable itself.*

Proof. Let B be a cofibrant A-algebra. Because the left adjoint $B \otimes_A -$ commutes with tensor structures we have that

$$\Sigma^n B = B \otimes_A \Sigma^n A$$

and

$$\mathbb{A}_B \Sigma^n B = B \otimes_A \mathbb{A}_A \Sigma^n A.$$

Since B is cofibrant, we get a map of Künneth spectral sequences induced by $\Sigma^n A \to \mathbb{A}_A \Sigma^n A$, which induces an isomorphism up to degree $2n+1$ if $n \leqslant i+1$ and $n+i$ if $n > i+1$. Hence the map $\Sigma^n B \to \mathbb{A}_A \Sigma^n B$ satisfies the desired connectivity conditions. □

Corollary 6.1.5. *Let A be ani-stable algebra and N a cofibrant n-connected A-nuca. Then the map $N \to Q_A(N)$ is $(2n+1)$-connected if $n \leqslant i+1$ and $(n+i)$-connected if $n > i+1$.*

Proof. We use the spectral sequence

$$E^1_{p,q} = \pi_q \mathbb{A}^p_A N \Longrightarrow \pi_{p+q} Q_A(N)$$

from Section 3.5. Its E^1-page has horizontal differentials and for $q \leqslant 2n+1$ or $q \leqslant n+i+1$, depending on n, it looks like

$$\pi_q N \xleftarrow{0} \pi_q N \xleftarrow{\mathrm{id}} \pi_q N \xleftarrow{0} \cdots .$$

So in the degrees above, the E^2-page consists of a single column $\pi_* N$ concentrated in degree 0. This column stabilises and yields the result. □

Recall the pointed tensor structure of the category of A-nucas from Section 2.3. We will need the following result.

Proposition 6.1.6. *If N is an n-connected cofibrant A-nuca, then $S^m \hat{\otimes} N$ is a cofibrant $(m+n)$-connected A-nuca.*

Proof. It is enough to prove the proposition for $m = 1$. We use the equivalence of categories between $\mathscr{C}_A \downarrow A$ and $\mathscr{N}_A$. If B is such an algebra, then the induced map $\pi_* A \to \pi_* B$ is injective and the induced map $\pi_* B \to \pi_* A$ is surjective. Assume that the map $A \to B$ is n-connected, i.e. $\pi_i A \to \pi_i B$ is an isomorphism for $i < n$ and surjective for $i = n$. But because of the injectivity above $\pi_n A \cong \pi_n B$. Recall the suspension in the category of A-algebras over A. $S^1 \hat{\otimes} B$ is equivalent to the homotopy pushout of algebras

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & S^1 \hat{\otimes} B \end{array}$$

So we are left to analyse the spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*B}_{p,q}(\pi_*A, \pi_*A) \Longrightarrow \pi_{p+q}(S^1 \hat{\otimes} B)$$

in order to prove that the map $\pi_{n+1}A \to \pi_{n+1}(S^1 \hat{\otimes} B)$ is surjective. The spectral sequence has $E^2_{p,q} = 0$ for $p > 0$ and $q < n+1$. So $\pi_{n+1}(S^1 \hat{\otimes} B) = (\pi_*A \otimes_{\pi_*B} \pi_*A)_{n+1}$. But because the map $\pi_*B \to \pi_*A$ is surjective anyway, we have

$$E^2_{0,q} \cong \pi_q A, \quad q \leqslant n+1.$$

Thus we end up with

$$\pi_q(S^1 \hat{\otimes} B) \cong \pi_q A, \quad q \leqslant n+1.$$

If N is an n-connected A-nuca, it is now easy to prove that $S^1 \hat{\otimes} N$ is $(n+1)$-connected. The A-algebra $B = A \oplus N$ is augmented over A and the map $A \to B$ is n-connected. From $A \oplus (S^1 \hat{\otimes} N) \cong S^1 \hat{\otimes} B$ follows that for $q \leqslant n+1$

$$\pi_q A \oplus \pi_q(S^1 \hat{\otimes} N) \cong \pi_q A$$

so that $\pi_q(S^1 \hat{\otimes} N) = 0$.

As a last point, the A-nuca $S^1 \hat{\otimes} N$ is always a cofibrant A-nuca due to 2.2.5. □

6.2 Stabilisation

For an A-module M we write ΩM for M^{S^1}. The functor Ω is a left inverse of the suspension, meaning that for any A-module M

$$\Omega\Sigma M \xrightarrow{\sim} M.$$

This is Proposition II.6.1 in [45].

We give the definition of the stabilisation of a functor according to Goodwillie [25]. Fix a functor $F : \mathfrak{C} \to \mathfrak{D}$ between pointed simplicial model categories, which respects the basepoint.

Definition 6.2.1. The *stabilisation* of F is the functor $F^{\mathrm{st}} : \mathfrak{C} \to \mathrm{ho}\,\mathfrak{D}$ defined by

$$F^{\mathrm{st}}(-) = \operatorname*{hocolim}_n \Omega^n F(S^n \hat{\otimes} -).$$

Under certain conditions there is a link between stabilisation and André-Quillen homology. A prominent rôle will be played by the forgetful functor U from A-nucas to A-modules.

Theorem 6.2.2. *Let A be an i-stable simplicial R-algebra. If N is a cofibrant A-nuca, then the map $U^{\mathrm{st}}(N) \to Q_A(N)$ is an i-equivalence.*

Proof. $S^n \hat{\otimes} N$ is n-connected. Let $j = \min(2n+1, n+i)$ denote the smaller of the two degrees. Corollary 6.1.5 says that the map $S^n \hat{\otimes} N \to Q_A(S^n \hat{\otimes} N)$ is j-connected. It follows that $\Omega^n(S^n \hat{\otimes} N) \to \Omega^n Q_A(S^n \hat{\otimes} N)$ is still $(j-n)$-connected. Taking homotopy colimits over n and passing to homotopy groups, we get an isomorphism up to degree i:

$$\pi_k(U^{\mathrm{st}}(N)) \cong \pi_k Q_A^{\mathrm{st}}(N) \quad k \leqslant i.$$

On the other hand, the functor Q_A is a left adjoint, hence it commutes with the tensored structure:

$$\Omega^n Q_A(S^n \hat{\otimes} N) \simeq \Omega^n \Sigma^n Q_A(N) \simeq Q_A(N),$$

so that $Q_A^{\mathrm{st}}(N) \simeq Q_A(N)$ which concludes the proof. □

6.3 Γ-homology

We discuss the definition of Γ-homology for simplicial algebras which goes back to Robinson and Whitehouse [48]. We follow Basterra and McCarthy [8] in this section.

Let $\underline{n}$ denote the n-element set $\{1, 2 \dots, n\}$. We work in the category with objects the $\underline{n}$ for $n \geqslant 1$ and surjections. With $N_q(\underline{n}, \underline{m})$ we denote the set of q composable functions $(f_q, \dots, f_1)$ in the category such that $f_q \circ f_{q-1} \circ \dots \circ f_1 : \underline{m} \to \underline{n}$, and by $R\{N_q(\underline{n}, \underline{m})\}$ we denote the free R-module on that set.

If g is in $N_1(\underline{n}, \underline{m})$ and $i \in \underline{m}$ then we can look at g^i, the component of g at i, defined by restricting g to the preimage of i. So, $g^i \in N_1(\underline{|g^{-1}(i)|}, \underline{1})$. Similarly, for a chain of functions $(f_q, \dots, f_1) \in N_q(\underline{n}, \underline{m})$ we can look at the component at $i \in \underline{m}$ taking successive preimages to get a chain, denoted $(f_q, \dots, f_1)^i$, ending in $\underline{1}$. With this notation we can set up the Γ-complex or Robinson-Whitehouse complex $\Gamma C_* = \Gamma C_*(S, R; M)$ for a (so far non-simplicial) R-algebra S and an S-module M.

$$\Gamma C_0 = M \otimes S,$$
$$\Gamma C_q = \bigoplus_{n \geqslant 1} R\{N_q(\underline{n}, \underline{1})\} \otimes M \otimes S^{\otimes n} \quad \text{for } q \geqslant 1.$$

The definition of its differential takes various steps. Given a surjective pointed function $f : [n] \to [m]$, we can associate to it a homomorphism $f_* : M \otimes S^{\otimes n} \to M \otimes S^{\otimes m}$. For an element $x = s_0 \otimes s_1 \otimes \cdots s_n \in M \otimes S^{\otimes n}$ we denote

$$f_*(x) = t_0 \otimes t_1 \otimes \cdots \otimes t_m,$$

where

$$t_i = \prod_{j \in f^{-1}(i)} s_j$$

If g is a surjective function $g : \underline{n} \to \underline{m}$, then there is a unique pointed function $g' : [n] \to [m]$, which extends g. Taking x as before, we write $g_*(x) = g'_*(x)$. Note, that in this case $t_0 = s_0$, so the module structure of M is not needed. Take a chain of functions $(f_q, \dots, f_1) \in N_q(\underline{n}, \underline{1})$ and assume that $\underline{r}$ is the domain of f_q, so that $(f_{q-1}, \dots, f_1) \in N_{q-1}(\underline{n}, \underline{r})$. Set $r_j = |(f_{q-1} \circ \cdots \circ f_1)^{-1}(j)|$. For each $j \in \underline{r}$ the preimage of this function is a subset of $\underline{n}$. Define the function $l : [n] \to [r_j]$ by mapping $k \in [n]$ to 0 except if k lies in the preimage, where the function is an order preserving bijection.

The differential $d : \Gamma C_q \to \Gamma C_{q-1}$ can be written as $d = \sum_{i=0}^{q} (-1)^i \delta_i$. Take an $x \in M \otimes S^{\otimes^n}$ and $(f_q, \dots f_1) \in N_q(\underline{n}, \underline{1})$ as before. For $q > 1$ we have

$$\begin{aligned}
\delta_0((f_q, \dots, f_1) \otimes x) &= (f_q, \dots, f_2) \otimes f_{1*}(x), \\
\delta_i((f_q, \dots, f_1) \otimes x) &= (f_q, \dots, f_i \circ f_{i-1}, \dots, f_1) \otimes x \quad \text{for } 0 < i < q, \\
\delta_q((f_q, \dots, f_1) \otimes x) &= \sum_{j=1}^{r} (f_{q-1}, \dots, f_1)^j \otimes l_{j*}(x).
\end{aligned}$$

If $q = 1$, then $N_1(\underline{n}, \underline{1})$ is just a singleton. For $j \in \underline{n}$ define $g_j : [n] \to [1]$ to be the map which is zero everywhere except that j is mapped to 1. For an x as above, we set

$$\begin{aligned}
\delta_0([\underline{n} \to \underline{1}] \otimes x) &= \prod_i s_i, \\
\delta_1([\underline{n} \to \underline{1}] \otimes x) &= \sum_{j=1}^{n} g_{j*}(x).
\end{aligned}$$

It is now straightforward to prove that $\delta_i \delta_j = \delta_{j-1} \delta_i$ for $i \leqslant j$, hence $d \circ d = 0$.

If B is a simplicial A-algebra, then the Robinson-Whitehouse complex can be defined the same way, forming it in each degree. We get thus a double complex $\Gamma C_* * (B, A; M)$. We denote the total complex again by $\Gamma C_*(B, A; M)$. In order to define a homotopy invariant notion we have to pass to cofibrant replacements. If B is a simplicial A-algebra, take a cofibrant replacement $A \rightarrowtail P \overset{\sim}{\twoheadrightarrow} B$ in the category of A-algebras. Γ-homology of B over A with respect to any B-module M is defined as

$$H_*^{\Gamma}(B, A; M) = H_* \Gamma C_*(P, A; M)$$

where we take the homology of the total complex on the right hand side. By definition, this is a homotopy invariant notion. Because it is calculated as the homology of a double complex there is always a spectral sequence

$$E^2_{pq}\pi_p H^\Gamma_q(P_*, A_*; M_*) \Longrightarrow H^\Gamma(B, A; M)$$

where we first compute degreewise $H^\Gamma_*(P_i, A_i, M_i)$ to obtain a simplicial object of which we then can take the homotopy groups.

In [8] is proven that for an R-algebra S, which is flat as a module over R, we do not need to pass to cofibrant replacements:

$$H^\Gamma_*(S, R; M) = H_*\Gamma C_*(S, R; M).$$

6.4 Comparison for Rational Algebras

Let us assume in this chapter that the groundring R is a $\mathbb{Q}$-algebra. We prove that in this case Γ-homology coincides with André-Quillen homology. This is an easy extension of the known result for constant simplicial algebras, which can be found in [43] or [9].

Theorem 6.4.1. *If A is a simplicial $\mathbb{Q}$-algebra, and B any A-algebra and M a B-module, then*

$$\mathrm{AQ}_*(B, A; M) \cong H^\Gamma_*(B, A; M).$$

Proof. By definition we can replace B with an A-algebra P, such that in each simplicial degree $P_i \cong A_i[X_i]$ is a polynomial algebra. First, let us compute Γ-homology. There is a spectral sequence

$$E^2_{p,q} = \pi_p H^\Gamma_q(P_*, A_*; M_*) \Longrightarrow H^\Gamma_{p+q}(P, R; M).$$

We use the fact that $\underline{P_i}$ is cofibrant over $\underline{A_i}$ for all i and the fact that $H^\Gamma(P_i, A_i; M_i) \cong \mathrm{AQ}(\underline{P_i}, \underline{A_i}; \underline{M_i})$, so that we have the following chain of isomorphisms

$$E^2_{p,q} \cong \pi_p\,\mathrm{AQ}_q(P_*, A_*; M_*) \cong \begin{cases} \pi_p(\Omega_{P/A} \otimes_P M) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the spectral sequence collapses to give

$$H^\Gamma_p(P, A; M) \cong \pi_p(\Omega_{P/A} \otimes_P M).$$

On the other hand, $\mathrm{AQ}_p(P, A; M) = \pi_p(\Omega_{P/A} \otimes_P M)$ by definition. □

6.5 Comparison for i-stable Algebras

We work now over an i-stable algebra A. For example assume that $A = \underline{\mathbb{F}_p}$ and $i = 2p-3$, or take any simplicial algebra A and assume that $i = 1$.

Theorem 6.5.1. *Let B be an algebra over an i-stable algebra A. Then we have an isomorphism*

$$\mathrm{AQ}_k(B, A; B) \cong H_k^\Gamma(B, A; B)$$

for $k \leqslant i$.

Proof. Let C be in $\mathscr{C}_B \downarrow B$. In [8] it is proven that $\Gamma C_*(C, B; B) \simeq I_B^{\mathrm{st}}(C)$. Write U for the forgetful functor from B-nucas to B-modules. Moreover, we can write $C \cong B \oplus I_B(C)$ and because the functor $V : N \mapsto B \oplus N$ is a left adjoint we have

$$\begin{aligned} U \circ I_B^{\mathrm{st}}(C) &= U(\operatorname*{colim}_n \Omega^n I_B(S^n \hat{\otimes} C)) \\ &= U(\operatorname*{colim}_n \Omega^n I_B(B \oplus S^n \hat{\otimes}_{\mathscr{N}_B} I_B(C))) \\ &\simeq \operatorname*{colim}_n \Omega^n U(S^n \hat{\otimes}_{\mathscr{N}_B} I_B(C)) \\ &= U^{\mathrm{st}}(I_B(C)). \end{aligned}$$

Taking a cofibrant replacement P for B and setting $C = P \otimes_A B$, it follows that

$$H_*^\Gamma(C, B; B) \cong \pi_* U^{\mathrm{st}}(I_B(P \otimes_A B)) \cong \pi_* U^{\mathrm{st}}(\mathrm{R}I_B(P \otimes_A B))$$

where the last term can be interpreted up to degree i as the cotangent complex according to Theorem 6.2.2. So there are isomorphisms $\pi_m \mathfrak{L}_{B/A} \cong \pi_m Q_B I_B(C) \xrightarrow{\cong} \pi_m \Gamma_* C(C, B; B)$ for $m \leqslant i$. The base-change property for Γ-homology [8, page 3] finally shows that

$$H_*^\Gamma(C, B; B) \cong H_*^\Gamma(B, A; B). \qquad \square$$

Chapter 7

Finiteness Conditions for André-Quillen Homology

We consider a simplicial ring extension $A \to B$. Due to the homotopy invariance of André-Quillen homology, it is reasonable to ask, what kind of finiteness conditions of $\pi_*(B)$ over $\pi_*(A)$ are inherited by $\mathrm{AQ}_*(B, A; B)$. In this short chapter we study noetherian finiteness conditions for this purpose.

7.1 Noetherian Finiteness Conditions

Interesting are noetherian finiteness conditions, formulated in the graded context by Eilenberg [17]. We give the definitions in the category of non-negatively graded modules.

Definition 7.1.1. A graded module $M_* = \bigoplus_{n\in\mathbb{N}} M_n$ over a graded ring $L_* = \bigoplus_{n\in\mathbb{N}} L_n$ is said to be *locally finitely generated* if M_* is generated by a set of homogenous elements X, such that $X \cap M_n$ is finite for all n. We denote the full subcategory of locally finitely generated modules by $\mathfrak{L}_{L_*}$.

Let $L_* = \bigoplus_{n\in\mathbb{N}} L_n$ be a graded ring, satisfying

$$\begin{aligned}
&K = L^0/J \text{ is semisimple, where } J \text{ is the Jacobson radical of } L_0,\\
&\text{each idempotent in } K \text{ is the image of an idempotent in } L_0,\\
&L_0 \text{ is Noetherian,}\\
&L_n \text{ is finitely generated over } L_0. \qquad (7.1)
\end{aligned}$$

Following the terminology of Eilenberg, the category $\mathfrak{L}_{L_*}$ is perfect for such an L_*. In particular we have the following two lemmata.

Lemma 7.1.2 (Eilenberg). *$\mathfrak{L}_{L_*}$ is closed under quotients and submodules. Every element of $\mathfrak{L}_{L_*}$ has a projective resolution, where the projectives also belong to $\mathfrak{L}_{L_*}$.*

Lemma 7.1.3. *If $M^1_*, M^2_* \in \mathfrak{L}_{L_*}$ are bounded below, then $M_* = M^1_* \otimes_{L_*} M^2_*$ is in $\mathfrak{L}_{L_*}$.*

Proof. If we denote the generating sets by X_1 and X_2 respectively, then the set $X = \{x_1 \otimes x_2 \mid x_i \in X_i\}$, which is finite in every degree, generates M_*. □

As a consequence we have the following corollary.

Corollary 7.1.4. *For $M^1_*, M^2_* \in \mathfrak{L}_{L_*}$, $\operatorname{Tor}^{L_*}_n(M^1_*, M^2_*)$ is in $\mathfrak{L}_{L_*}$ for all n.*

In addition we get 2 out of 3 theorems:

Lemma 7.1.5. *If $M^1_* \to M^2_* \to M^3_*$ is a short exact sequence of graded L_*-modules and two out of the M^i_* are in $\mathfrak{L}_{L_*}$, then so is the third. Alternatively, if $M^1_* \to M^2_* \to M^3_* \xrightarrow{\circ} M^1_*$ is an exact triangle, all the M^i_* are bounded below and two out of them are in $\mathfrak{L}_{L_*}$, then so is the third.*

Proof. The first statement follows easily from Schanuel's Lemma [52]. The second statement can be proven by induction, unwinding the triangle into a long exact sequence of L_0-modules. □

7.2 Application to André-Quillen Homology

We now turn our attention back to simplicial modules. Let A be a simplicial ring, so that $\pi_*(A)$ satisfies conditions (7.1). Let B be an A-algebra, so that $\pi_* B$ is an element of $\mathfrak{L}_{\pi_* A}$.

Proposition 7.2.1. *Let P a cofibrant replacement for the A-algebra B. Denote by J the kernel of the map $P \otimes_A B \to B$. Then $\pi_* J$ is in $\mathfrak{L}_{\pi_* B}$.*

Proof. Factor the unit $A \to B$ into $A \overset{\sim}{\rightarrowtail} P \twoheadrightarrow B$ and consider the short exact sequence

$$J \to P \otimes_A B \to B$$

The E^2-page of the spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^{\pi_* A}_{p,q}(\pi_* B, \pi_* B) \Longrightarrow \pi_*(P \otimes_A B)$$

has every column in $\mathfrak{L}_{\pi_*A}$. Since $\mathfrak{L}_{\pi_*A}$ is closed under subquotients, the E^r-page enjoys the same property for all r. By the passage to the E^∞-page we only have to consider quotients, hence the E^∞-page has again all columns in $\mathfrak{L}_{\pi_*A}$. Since everything is connective, we can break down the extension problem for each degree, where it only involves a finite extension. As a result, $\pi_n(P \otimes_A B)$ is finitely generated over $\pi_0 A$. The two out of three conditions yields that $\pi_* J$ is in $\mathfrak{L}_{\pi_*A}$. □

Theorem 7.2.2. *Let $\eta : A \to B$ as in the preceding proposition so that $\pi_0(\eta)$ is an isomorphism. Furthermore, let the B-module M be such $\pi_* M$ is in $\mathfrak{L}_{\pi_*B}$. Then $\mathrm{AQ}_*(B, A; M)$ is in $\mathfrak{L}_{\pi_*B}$.*

Proof. The last proposition asserts that $\pi_* J$ is in $\mathfrak{L}_{\pi_*A}$, but therefore it is as well in $\mathfrak{L}_{\pi_*B}$. We now use the spectral sequence

$$E^1_{i,j} = \pi_i \mathbb{A}^j_B J \Longrightarrow \pi_* Q_B(J).$$

Because J is 0-connected, the $\pi_* J^{\otimes n}$ are at least n-connected. It follows that $\pi_i \mathbb{A}_B J = \bigoplus_{n \geqslant 1} \pi_i J^{\otimes n}/\Sigma_n$ is actually a finite sum. Therefore $\pi_* \mathbb{A}_B J$ is in $\mathfrak{L}_{\pi_*B}$. Furthermore $\mathbb{A}_B J$ is 0-connected so that the argument can be repeated in order to get that $\mathbb{A}^j_B J \in \mathfrak{L}_{\pi_*B}$. By the same argument as above we get that $\pi_* Q_B(J)$ is in $\mathfrak{L}\pi_* B$.

For the last step we have

$$\mathrm{AQ}_*(B, A; M) = \pi_*(Q_B(J) \otimes_B M)$$

so that the result follows from the Künneth spectral sequence. □

Chapter 8

Quillen's Fundamental Spectral Sequence

In this chapter we generalise Quillen's Fundamental spectral sequence [46] from rings to simplicial algebras. Our approach follows closely Quillen's original account.

8.1 Preliminaries

First a remark about derived tensor products. We quote the following lemma from Quillen [45, 6.10].

Lemma 8.1.1. *Let A be a simplicial R-algebra and $M, N \in \mathscr{M}_A$. Then $M \otimes_A N$ is weakly equivalent to the derived tensor product $M \otimes_A^L N$, if*

$$\mathrm{Tor}_q^{A_n}(M_n, N_n) = 0 \qquad q \geqslant 1, n \geqslant 0.$$

This has the nice effect, that cofibrant replacements in the category of algebras are good enough to obtain the derived tensor product.

Corollary 8.1.2. *If $A \rightarrowtail P \overset{\sim}{\twoheadrightarrow} B$ is a cofibrant replacement in the category of A-algebras which is almost free, and if M is any A-module, then*

$$P \otimes_A M \simeq P \otimes_A^L M.$$

Proof. The last lemma proves it, since $\mathrm{Tor}_q^{A_n}(P_n, M_n) = 0$ due to 2.3.6 and $B \otimes_A^L M \simeq P \otimes_A^L M$. □

The following discussion is an extension to the simplicial context of the arguments given in Quillen's paper [46, Section 6].

Lemma 8.1.3. *For a surjective ring extension $\phi : R \to S$, write $I = \ker \phi$. Then for any S-module N we have $\mathrm{Tor}_1^R(S, N) \cong I/I^2 \otimes_S N$.*

Proof. From the short exact sequence $I \to R \to S$, we get after applying the functor $- \otimes_R N$ the long exact sequence

$$0 \to \mathrm{Tor}_1^R(S, N) \to I \otimes_R N \to N \to S \otimes_R N \to 0$$

Since the map $I \otimes_R N \to N$ is the zero map, it is left to prove that $I \otimes_R N \cong I/I^2 \otimes_S N$. Define maps $\epsilon : i \otimes n \mapsto (i + I^2) \otimes n$ and $\eta : (i + I^2) \otimes n \mapsto i \otimes n$, so that $\epsilon\eta = \mathrm{id}_{I/I^2 \otimes_S N}$ and $\eta\epsilon = \mathrm{id}_{I \otimes N}$. The only thing to show, is that η is well-defined. This follows from

$$i + i_1 i_2 \otimes_S n \mapsto i \otimes n + i_1 \otimes i_2 n = i \otimes n$$

and the fact that ϕ is surjective. □

Lemma 8.1.4. *Let $\phi : A \to B$ be a morphism of simplicial R-algebras such that $\pi_0(\phi)$ is surjective and let M be a B-module, then*

$$\mathrm{AQ}_0(B, A; M) = 0.$$

Proof. This is just a special case of 4.4.2 □

8.2 The Fundamental Spectral Sequence

Lemma 8.2.1. *Define J to be the kernel of the map $P \otimes_A B \to B$. If J is 0-connected, then J^n is at least n-connected.*

Proof. Assume that P was an almost free resolution of B. Degreewise the sequence has the form

$$J_k \to B_k[X] \to B_k$$

For any A_k-module M we have

$$\mathrm{AQ}_q(B_k, B_k[X]; M) = \mathrm{AQ}_{q-1}(B_k[X], A; M) = 0 \text{ for } q \geqslant 1.$$

By Theorem 6.13 in [46], we get that J_k is quasiregular, i.e. J_k/J_k^2 is flat over B_k and $\Lambda_*^{B_k}(J_k/J_k^2) \cong \mathrm{Tor}_*^{B_k[X]}(B_k, B_k)$. By Theorem 6.12 in [46] the statement follows directly. □

Let us assume from now on that N is a constant simplicial B-module. For example set $N = \underline{\pi_0 B}$.

Lemma 8.2.2.

$$\pi_n(J \otimes_B N) = \pi_n(B \otimes^L_R N).$$

Proof. The short exact sequence

$$0 \to J \to P \otimes_A B \to B \to 0$$

stays exact after tensoring with N over B. The resulting long exact sequence gives isomorphisms

$$\pi_n(J \otimes_B N) \cong \pi_n(P \otimes_A N) = \pi_n(B \otimes^L_A N) \quad n \geqslant 1.$$

□

Theorem 8.2.3. *For an algebra map $\phi : A \to B$ so that $\pi_0(\phi)$ is surjective we obtain*

$$\mathrm{AQ}_1(B, A; N) = \pi_1(B \otimes^L_R N).$$

Proof. Assume that P was chosen to be almost free, so that $P_k = A_k[X_k]$ for a set X_k for all k. Then $(P \otimes_A B)_k \cong B_k[X_k]$ and J_k is some ideal in $B_k[X_k]$.

Consider the short exact sequence

$$J^2 \to J \to J/J^2.$$

On the i-th simplicial level J_i is the kernel of the map $f_i : B_i[X_i] \to B_i$. Hence, J_i is generated as a $B_i[X_i]$-module by the elements $x - f_i(x)$ for $x \in X_i$. It follows that

$$(J/J^2)_i = \bigoplus_{x \in X_i} (x - f_i(x))B_i,$$

i.e. J_i/J_i^2 is free as a B_i-module. We know that $\pi_1(J^2) = 0$. Tensoring the short exact sequence above with a B-module N yields another short exact sequence

$$0 \to J^2 \otimes_B N \to J \otimes_B N \to J/J^2 \otimes_B N \to 0.$$

The long exact sequence attached to it produces

$$\mathrm{AQ}_1(B, A; N) \cong \pi_1(J \otimes_B N) \cong \pi_1(B \otimes^L_A N),$$

where the last isomorphism comes from the precedent lemma. □

Lemma 8.2.4. *With the notation from above, there is an isomorphism*

$$J^n/J^{n+1} \cong (J/J^2)^{\otimes_B n}/\Sigma_n.$$

Proof. It is straightforward to check that the map $(j_1 \cdots j_n) \mapsto (j_1 \otimes \cdots \otimes j_n)$ induces an isomorphism. □

For an A-module M we write

$$S_0^A(M) = A$$

and

$$S_q^A(M) = M^{\otimes_A q}/\Sigma_q, \quad q \geqslant 1,$$

so that

$$\mathbb{P}_A M = \bigoplus_{q \geqslant 0} S_q^A(M).$$

Quillen's fundamental spectral sequence now generalises as in [46], using arguments from [13].

Theorem 8.2.5. *If $A \to B$ is a map of simplicial R-algebras that induces an epimorphism on π_0, then there is a convergent spectral sequence*

$$E^2_{p,q} = \pi_{p+q}(S_q^B \mathfrak{L}_{B/A}) \Longrightarrow \pi_{p+q}(B \otimes_A^L B).$$

Proof. There is a filtration

$$\cdots \hookrightarrow J^q \hookrightarrow J^{q-1} \hookrightarrow \cdots \hookrightarrow J \hookrightarrow Q := P \otimes_A B$$

of the B-module Q. Note, that the grading in q is not the standard one for a homological spectral sequence. The first differentials go from $\pi_{p+q} J^q/J^{q+1}$ to $\pi_{p+q-1} J^{q+1}/J^{q+2}$. So, there is a upper-halfplane spectral sequence of the following form attached to the filtration:

$$E^2_{p,q} = \pi_{p+q}(J^q/J^{q+1}) \Longrightarrow \pi_{p+q} Q.$$

By the preceding lemma, we have the isomorphism $E^2_{p,q} \cong \pi_{p+q} S_q^B \mathfrak{L}_{B/A}$, while the abutment is $\pi_*(Q) = \pi_*(B \otimes_A^L B)$. The convergence is assured because $\pi_k(J^n) = 0$ if $n > k$, which makes the terms $E^r_{p,q}$ for fixed p and q stabilise with $r \to \infty$. □

Corollary 8.2.6. *The five-term exact sequence of this spectral sequence is*

$$\pi_3(B \otimes_A^L B) \to \mathrm{AQ}_3(B, A; B) \to \Lambda^2_B \pi_1(B \otimes_A^L B) \to \pi_2(B \otimes_A^L B) \to \mathrm{AQ}_2(B, A; B) \to 0$$

Theorem 8.2.7. *The above results generalise to constant simplicial coefficients. If N is a constant simplicial B-module, then the five term sequence is*

$$\pi_3(B \otimes_A^L N) \to \mathrm{AQ}_3(B, A; N) \to \Lambda_B^2 \pi_1(B \otimes_A^L N \to \pi_2(B \otimes_A^L N) \to \mathrm{AQ}_2(B, A; N) \to 0.$$

We can reparametrise the spectral sequence in the following way. Taking refuge in [52, chapter 5.4], we define the filtration

$$F_p Q = J^{-p}.$$

Note that $F_p Q = 0$ for positive p. We get a spectral sequence

$$E_{p,q}^1 = \pi_{p+q}(J^{-p}/J^{-p+1}) \Longrightarrow \pi_*(Q).$$

This spectral sequence is now a 2nd quadrant spectral sequence.

Part II

TAQ

Topological André-Quillen Homology

Chapter 9

Topological André-Quillen Homology

In the second part of this thesis we give a short account of topological André-Quillen homology following Basterra [6]. The main result in this chapter is Theorem 9.2.2, which was stated as conjecture in Gimour's thesis. On the way we provide a complete proof of the well known Theorem 9.1.1 because we were unable to find a correct published proof for this fact in the literature.

9.1 Introduction to Topological André-Quillen Homology

We give a short introduction to André-Quillen homology in the topological context of ring spectra. Our category of spectra is the category of S-modules following [19] seen as a model category with the standard model structure defined in [19, chapter VII]. S stands for the sphere spectrum and we denote the category of S-modules as usual by $\mathscr{M}_S$. In this chapter we will not deal with simplicial algebras, so that this notation cannot cause confusion. The category of S-modules comes equipped with a smash product, denoted $\wedge$. As in the first chapter of the first part of this thesis we can define commutative S-algebras by asking that the appropriate coherence diagrams commute. For such an S-algebra R we can further construct the category $\mathscr{M}_R$ of R-modules and $\mathscr{C}_R$ of commutative R-algebras. Dropping the unit condition, we end up with the category $\mathscr{N}_R$ of R-nucas. By the same coequaliser construction as for simplicial algebras and the tensor product, the category of R-modules comes equipped with a smash product, which we denote by $\wedge_R$. Smashing with a fixed R-module is a left adjoint. Explicitly, for every two R-modules M and N, there is an internal function object $F_R(M, N) \in \mathscr{M}_R$, so that

$$\mathscr{M}_R(A \wedge_R M, N) \cong \mathscr{M}_R(A, F_R(M, N)).$$

Every S-module M comes with its homotopy groups $\pi_n(M)$, where $n \in \mathbb{Z}$. A spectrum M such that $\pi_n(M) = 0$ for $n < 0$ is called *connective.* As with simplicial modules and algebras, $\pi_*(M)$ inherits a graded module structure over $\pi_*(S)$. As before, for a commutative R-algebra A, the homotopy groups $\pi_*(A)$ form a commutative graded $\pi_*(R)$-algebra.

The weak equivalences in any of these categories are those maps which induce an isomorphism on π_*. When passing to the homotopy category one obtains from $\mathscr{M}_S$ the stable homotopy category first described by Boardman [10]. This category is triangulated with an endofunctor called suspension Σ which is an equivalence of categories. There are more model categories with a symmetric monoidal smash product, such that the homotopy category is triangulated, notably the category of symmetric spectra [29]. But all such categories are Quillen equivalent by the rigidity theorem [51]. That is why we can restrict to the case of S-modules. However, the categories of Bousfield-Friedländer spectra [11] or the category of spectra according to Adams [1] fail to be symmetric monoidal, which is crucial for our applications.

There is a suspension spectrum functor

$$\Sigma^\infty : \mathrm{Top} \longrightarrow \mathscr{M}_S$$

which embeds topological spaces into spectra. The homotopy groups $\pi_*(\Sigma^\infty X)$ are the stable homotopy groups

$$\pi_n(\Sigma^\infty X) = \pi_n^{\mathrm{st}}(X) = \operatorname*{colim}_k \pi_{n+k}\Sigma^k X,$$

where Σ^k is the usual k-fold suspension of topological spaces. The Freudenthal Suspension Theorem says that this colimit is obtained after finitely many steps. Explicitly

$$\pi_n^{\mathrm{st}}(X) = \pi_{n+k}\Sigma^k X$$

for $k > 2n + 1$.

Although there is no topological space, apart from contractible ones, of which we know the stable homotopy groups, they are in some sense better behaved than the actual homotopy groups of the space and carry lots of information.

Let $\mathscr{R}$ be the category of commutative rings. There is an Eilenberg-MacLane functor

$$H : \mathscr{R} \longrightarrow \mathscr{C}_S$$

which assigns to a ring Z the spectrum HZ. The homotopy groups of an Eilenberg-MacLane spectrum give back the original ring

$$\pi_n(HZ) = \begin{cases} Z & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, an Eilenberg-MacLane functor $H : \mathscr{M}_Z \to \mathscr{M}_{HZ}$ can be arranged, so that $\pi_*(HM)$ is again M concentrated in degree 0, if M is a Z-module. If A is a Z-algebra, then HA is an HZ-algebra.

This allows us to see the category of ring spectra as an extension of the category of rings, which led to the expression "brave new algebra" for the studies of the categories of S-modules and S-algebras motivated by algebra. Moreover, the Eilenberg-MacLane functor extends to a functor

$$s\mathscr{R} \longrightarrow \mathscr{M}_S$$

sending the simplicial ring A to the simplicial spectrum obtained by applying H degreewise to the A_n and then taking geometric realisation [19, chapter X]. By a spectral sequence argument one can show that

$$\pi_* A = \pi_* HA.$$

If A is a constant simplicial ring, HA coincides with the Eilenberg-MacLane functor above.

André-Quillen homology and cohomology for S-algebras has first been introduced by Basterra in [6] following an unpublished version by Kříž [31]. A completely different description in terms of stabilisation was later given by Kuhn [32]. We summarise here the approach by Basterra.

We write again $\mathbb{P}_R$ for the free functor between R-modules and R-algebras and $\mathbb{A}_R$ for the non-unital version. There is the category of R-algebras over R, which we will denote by $\mathscr{C}_R{\downarrow}R$. The coproduct of two R-modules is usually denoted by $\vee$ in reminiscence of the wedge of two topological spaces. This operation allows us to add a unit to a non-unital algebra to obtain a unital one. We write in this context $V(N) = R \vee N$. The augmentation functor

$$I_R : \mathscr{C}_R{\downarrow}R \longrightarrow \mathscr{N}_R$$

is defined as the pullback

$$\begin{array}{ccc} I_R(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & R \end{array}$$

in the category of A-modules, where $*$ is the trivial spectrum. Because of the diagram

$$\xymatrix{ I_R(A) \wedge_R I_R(A) \ar[rr] \ar@{.>}[dr] \ar[ddr] & & A \wedge_R A \ar[d] \\ & I_R(A) \ar[r] \ar[d] & A \ar[d] \\ & * \ar[r] & R }$$

there is a unique map $I_R(A) \wedge_R I_R(A) \to I_R(A)$ which turns $I_R(A)$ into an R-nuca. There is the adjunction

$$\mathscr{C}_R{\downarrow}R(V(N), A) \cong \mathscr{N}_R(N, I_R(A)),$$

which can be proven (see [6, Proposition 2.1 and Proposition 2.2]) to be a Quillen equivalence. In particular, the homotopy categories are equivalent and the equivalence is given as right and left derived functors

$$\mathrm{ho}\,\mathscr{C}_R{\downarrow}R(\mathbb{L}V(N), A) \cong \mathrm{ho}\,\mathscr{N}_R(N, \mathbb{R}I_R(A)).$$

The indecomposable functor

$$Q_R : \mathscr{N}_R \to \mathscr{M}_R$$

is defined to be the pushout

$$\begin{array}{ccc} N \wedge_R N & \longrightarrow & N \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q_R(N) \end{array}$$

in the category of R-modules. It is part of a Quillen adjunction

$$\mathscr{N}_R(N, Z(M)) \cong \mathscr{M}_R(Q_R(N), M)$$

where $Z(M)$ denotes the nuca obtained from the module M by setting the multiplication map to be the trivial map. This is not a Quillen equivalence, but the adjunction passes down to the homotopy categories and we obtain the left derived functor LQ_R.

By essentially the same argument as for simplicial algebras, the abelian group objects in $\mathscr{C}_R$ are the square-zero extensions. The functor sending an R-module M to $V(Z(M))$

is an equivalence between the category of R-modules and the category of square-zero extensions. In contrast to the simplicial context there is no left adjoint to this functor. However there is one, when one passes to homotopy. Define the abelianisation

$$\mathrm{Ab}_R^A : \mathrm{ho}\,\mathscr{C}_R \downarrow A \longrightarrow \mathrm{ho}\,\mathscr{M}_A$$

by

$$\mathrm{Ab}_R^A(B) = LQ_B RI_B(B^c \wedge_R A)$$

where we took a cofibrant replacement of B in the model category of R-algebras. Basterra proves that indeed

$$\mathrm{ho}\,\mathscr{C}_R \downarrow A(B, A \vee M) = \mathrm{ho}\,\mathscr{M}_R(\mathrm{Ab}_R^A(B), M). \tag{9.1}$$

Note that in contrast to simplicial algebras, this adjunction cannot be lifted to an adjunction between the categories before passing to homotopy.

We set

$$\Omega_{A/R} = \mathrm{Ab}_R^A(A)$$

and for an A-module M we define topological André-Quillen homology and cohomology as

$$\mathrm{TAQ}_*(A, R; M) = \pi_*(\Omega_{A/R} \wedge_A M),$$
$$\mathrm{TAQ}^*(A, R; M) = \pi_{-*}F_A(\Omega_{A/R}, M).$$

Applying this construction to Eilenberg-MacLane spectra does not give the same groups as applying André-Quillen homology to the rings in question. This has to do with the fact that the adjunction (9.1) seen as an adjunction of spectra or of simplicial modules is quite different in nature. There are many comparison results of topological André-Quillen homology of Eilenberg-MacLane spectra with other notions of homology. There is a comparison of André-Quillen homology of rings with its topological counterpart given in [47]. It is stated that for an algebra A over a field k, there is a spectral sequence

$$E^2_{p,q} = \mathrm{AQ}_p(A, k; M) \otimes \mathrm{TAQ}_q(Hk[x], Hk; HM) \Longrightarrow \mathrm{TAQ}_{p+q}(HA, Hk; HM).$$

Another result is that topological André-Quillen homology and Γ-homology coincide for Eilenberg-MacLane spectra for flat ring extensions [8], [9] and [43].

The following theorem is “folklore” and used in many publications. However, we were not able to find a complete proof in the literature. We present a proof that McCarthy and

Minasian probably had in mind [40, page 256], but which unfortunately is not correct as stated.

Theorem 9.1.1. *Let X be a cofibrant A-module. Then*

$$\Omega_{\mathbb{P}_A X/A} \simeq \mathbb{P}_A X \wedge_A X$$

as $\mathbb{P}_A X$-modules.

Proof. For every $M \in \mathscr{M}_{\mathbb{P}_A X}$ there is the following adjunction

$$\mathscr{C}_A{\downarrow}\mathbb{P}_A X(\mathbb{P}_A X, \mathbb{P}_A X \vee M) \cong \mathscr{M}_A{\downarrow}\mathbb{P}_A X(X, \mathbb{P}_A X \vee M).$$

Because the forgetful functor: $\mathscr{C}_A \downarrow \mathbb{P}_A X \to \mathscr{M}_A \downarrow \mathbb{P}_A X$ respects fibrations and acyclic fibrations the adjunction passes down to homotopy categories. It is obvious that

$$\mathscr{M}_A{\downarrow}\mathbb{P}_A X(X, \mathbb{P}_A X \vee M) \cong \mathscr{M}_A{\downarrow}X(X, X \vee M)$$

and that this adjunction is as well valid in the homotopy categories. In the homotopy category of A-modules, the pushout of two modules coincides with the pullback, so that we can write

$$\mathrm{ho}\,\mathscr{M}_A{\downarrow}X(X, X \vee M) \cong \mathrm{ho}\,\mathscr{M}_A(X, M).$$

We use the free functor from A-modules to $\mathbb{P}_A X$-modules to get

$$\mathrm{ho}\,\mathscr{M}_A{\downarrow}X(X, X \vee M) \cong \mathrm{ho}\,\mathscr{M}_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A X, M).$$

In total, we have proven that

$$\mathrm{ho}\,\mathscr{M}_{\mathbb{P}_A X}(\Omega_{\mathbb{P}_A X/A}, M) \cong \mathrm{ho}\,\mathscr{C}_A{\downarrow}\mathbb{P}_A X(\mathbb{P}_A X, \mathbb{P}_A X \vee M) \cong \mathrm{ho}\,\mathscr{M}_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A X, M)$$

so that by Yoneda's lemma we end up with

$$\Omega_{\mathbb{P}_A X/A} \simeq \mathbb{P}_A X \wedge_A X$$

as required. □

9.2 Minimal Atomic Spectra

We recall the definitions from [3] and [20], which are analogous to the ones in chapter 5. Let R be a connective p-local commutative S-algebra so that $\pi_0 R$ is a noetherian local ring with maximal ideal $p\pi_0 R$.

We work in the category of commutative p-local R-algebras and we assume that for all such algebras A all homotopy groups are noetherian modules over $\pi_0 R$ and $\pi_0 R \cong \pi_0 A$.

The definition of cellular complexes is exactly the same as for simplicial algebras. We take for the R-sphere the p-local spectrum $R \wedge_S S^n_{(p)}$, where $S_{(p)}$ is the p-local sphere spectrum.

Definition 9.2.1.

- A cellular R-algebra A is *nuclear*, if the attaching maps $k_n : K_n \to B_n$ satisfy

$$\ker \pi_m(k_n) \subset p \cdot \pi_n(K_n)$$

for all $n \in \mathbb{N}$.

- A *core* of an R-algebra B is a nuclear R-algebra A with a map $g : A \to B$ so that the induced map $\pi_*(g)$ is injective.

- An R-algebra A whose unit induces an isomorphism on π_0 is *atomic*, if any self map is a weak equivalence.

- A is said to be *minimal atomic* it is atomic and any map $A' \to A$ from an atomic algebra A' to A that induces an injection on homotopy groups is a weak equivalence.

The same theorems as in chapter 5 hold. For analogues of Propositions 5.1.3, 5.1.5 and Corollary 5.1.6 see [20]. However, the analogues of Theorem 5.1.7 and Corollary 5.1.8 in [20] are only stated as conjectures, so that we supply a full proof, the missing ingredient being basically our last diagram in the proof.

Theorem 9.2.2. *Every core Q of a nuclear A-algebra B is an equivalence. As a consequence, every nuclear algebra is minimal atomic.*

Proof. We use the same notation as in the proof of Theorem 5.1.7.

$$k_n : K_n \to Q^n,$$

$$j_n : J_n \to B^n$$

are the attaching maps respectively. By [20, 5.7] we only have to prove that the induced map $\pi_n \, \mathrm{C}_{k_n} \to \pi_n \, \mathrm{C}_{j_n}$ on the mapping cones is an isomorphism. By the diagram

$$\begin{array}{ccccc} \pi_n K_n & \longrightarrow & \pi_n Q^n & \longrightarrow & \pi_n \, \mathrm{C}_{k_n} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_n J_n & \longrightarrow & \pi_n B^n & \longrightarrow & \pi_n \, \mathrm{C}_{j_n} \end{array}$$

the right vertical arrow is an epimorphism. It follows straight from the definitions of cellular complexes that $\pi_n \mathrm{C}_{k_n} \cong \pi_n Q^{n+1} \cong \pi_n Q$ and $\pi_n \mathrm{C}_{j_n} \cong \pi_n B^{n+1} \cong \pi_n B$. The diagram

$$\begin{array}{ccc} \pi_n \mathrm{C}_{k_n} & \xrightarrow{\cong} & \pi_n Q \\ \downarrow & & \downarrow \\ \pi_n \mathrm{C}_{j_n} & \xrightarrow{\cong} & \pi_n B \end{array}$$

shows that the first vertical map is a monomorphism, because the second vertical map is a monomorphism by definition. □

Index

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