The finite dual of crossed products

by

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Abstract

In finite dimensions, Hopf algebras have a very nice duality theory, as the vector space dual of a finite-dimensional Hopf algebra is also a Hopf algebra in a canonical way. This breaks down in the infinite-dimensional setting, as here the dual need not be a Hopf algebra. Instead, one chooses a subalgebra of the vector space dual called the finite dual. This subalgebra is always canonically a Hopf algebra.

In this thesis, we aim to better understand the finite dual by trying to understand how the finite dual of a crossed product relates to the finite duals of its components.

We start by investigating what the assignment sending a Hopf algebra to its finite dual does to functions. Unlike in the finite-dimensional case, this is no longer a contravariant exact monoidal functor and might not even be a functor at all. However, many of the results true thanks to this in finite dimensions still always hold, while we can find necessary and sufficient conditions for others to hold as well as specific situations in which they are always true.

Crossed products generalise the notion of a smash product, which can be viewed as the Hopf algebra equivalent of the semidirect product. Many Hopf algebras of interest can be written as crossed products. We study the finite dual of such a product and find numerous results when assuming conditions such as one of the components being finite-dimensional or the crossed product being a smash product. These can be combined for strong statements about the finite dual under certain assumptions.

Finally, we consider Noetherian Hopf algebras which are finite modules over central Hopf subalgebras. Many of these algebras decompose as crossed products, so that we can use our previous results to study them. However, we also find results that are true without assuming such a decomposition. This allows us to calculate the finite duals of numerous examples, including a quantised enveloping algebra at a root of unity and all the prime affine regular Hopf algebras of Gelfand-Kirillov dimension one with prime PI degree.

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Declaration

The contents of this thesis are my own work except where stated otherwise. No part of this thesis has been presented elsewhere for any other degree.

Astrid Jahn

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Chapter 1

Introduction

1.1 Introduction

This thesis is about the finite duals of crossed products and of Noetherian Hopf algebras which are finite over a central Hopf subalgebra.

Let k be any field. *Hopf algebras* are a type of unital associative k-algebra with extra structure. We give the full definition in Section 1.2 along with classical examples of Hopf algebras in Section 1.2.1. Hopf algebras are widely studied for their applications to other areas as well as for their own sake.

The structure imposed on a Hopf algebra by the axioms gives rise to a number of interesting properties and symmetries. In Section 1.3 we look at one of these properties in particular, which is that when a Hopf algebra H is finite-dimensional, the vector space dual H^* consisting of the k-linear maps from H to k is also a Hopf algebra in a canonical way. Moreover, we have $(H^*)^* \cong H$ as Hopf algebras. This allows for a nice duality theory, which is very useful for proving results in the finite-dimensional setting.

However, we are primarily interested in infinite-dimensional Hopf algebras, and here this does not work out so nicely: the vector space dual H^* of an infinite-dimensional Hopf algebra H is not necessarily a Hopf algebra with respect to the structure we want. However, there exists a canonical subspace H^0 of H^*, consisting of those functions vanishing on an ideal of finite codimension in H, which is always a Hopf algebra and is moreover maximal in H^* with respect to being one. This is known as the *finite dual* of H and seems to be the obvious candidate to replace H^* in infinite dimensions.

In Section 1.4, we look at several examples of Hopf algebras H and their finite duals and what properties we can conclude the finite dual has. We find that H^0 preserves far fewer of the properties of H in comparison to the finite-dimensional case. For instance, we see that $\left(H^0\right)^0 \ncong H$ in general. We also see that H^0 need not preserve the size of H, where by *size* we refer to various different, related notions such as vector space, Gelfand-Kirillov or global dimension along with properties such as being affine or Noetherian. This can go in both directions: H^0 can be smaller than H, as seen in an example where H is infinite-dimensional but H^0 is one-dimensional. H^0 can however also be bigger than H. For instance, we see that for $H = k[x]$, H^0 is not affine, not Noetherian and has infinite global and Gelfand-Kirillov dimension as well as uncountable vector space dimension. Finally, we see that taking duals need not preserve subalgebras in any sense: when $A \subseteq H$ is a subalgebra, even Hopf subalgebra, of H, A^0 need not be either a subspace or a quotient of H^0.

All these facts are major obstructions if we want to generalise any finite-dimensional results using the dual to infinite-dimensions. We would like to look further into what is happening, what properties of H determine the size of H^0 and whether we might be able to find a subalgebra of H^0 that better preserves the properties of H in cases where H^0 is too big.

The approach we take in this thesis is one inspired by category theory. We study how functions between Hopf algebras transfer to the dual setting, among others addressing the question of what happens to subalgebras under taking finite duals. Knowing what happens to functions in the dual setting allows us to look into how the finite duals of Hopf algebras that are products of algebras relate to their components (Chapter 4) or how the finite dual of a Hopf algebra which is a finite module over some central Hopf subalgebra relates to the dual of said Hopf subalgebra and canonical quotient Hopf algebra (Chapter 5) via for instance studying what happens to canonical embedding and projection maps. The results we gain give us some feeling for what type of structure the finite dual preserves and what it does not on the one hand, while allowing us to calculate the finite duals of large classes of examples on the other.

In Chapter 2, we define the notion of a crossed product, give examples, and make note of the exact setting we will be working in. Crossed products, introduced independently by Blattner et al ([4]) and Doi and Takeuchi ([12]), are a way to combine an algebra and a Hopf algebra acting on it to form a new algebra. This can be viewed as similar to forming the semidirect product of groups, although we will see that crossed products are more general as they incorporate additional twisting.

Although the definition of a crossed product only gives us an algebra structure, and indeed we will see that not all crossed products are Hopf algebras, many Hopf algebras we are interested in arise in this way. Moreover, the crossed product structure on such a Hopf algebra H often arises from a Hopf surjection on H. From this perspective the question of when a Hopf algebra H can be written as a crossed product can be seen as a specific case of the general question of when quotient objects give rise to decompositions. In the case of Hopf algebras, there is an easy condition guaranteeing such a decomposition: the existence of a convolution invertible right comodule map, known as a *cleaving map*, from the quotient back to the original Hopf algebra.

In Chapter 3, we consider how functions transfer to the dual in the finite-dimensional world and how far this generalises to finite duals. In Proposition 3.1, we record various classical results about duality of functions in the finite-dimensional case: given a map $f : B \to C$ there is a well-defined map $f^* : C^* \to B^*$ which is injective if f is surjective and vice versa, an algebra map if f is a coalgebra map and vice versa, and so on. We will see that both the existence of such a map and it being surjective when the original map is injective cannot be immediately generalised to finite duals, with examples showing how this can fail. However, we find necessary and sufficient conditions on f which determine exactly when these implications are true. Our main result in this chapter is Theorem 3.12, which generalises Proposition 3.1 as far as possible. This means that among others we have necessary and sufficient conditions describing when a subalgebra $A \subseteq H$ gives rise to a quotient map $H^0 \to A^0$.

In Chapter 4, we use the results of Chapter 3 to study the finite dual of a crossed product where the crossed product structure arises from a Hopf surjection. Although we cannot say much in full generality, we find much stronger results when we impose various common assumptions on the crossed product. The main results of interest here are Theorems 4.9, 4.13 and 4.19. Moreover, because the assumptions we make on the Hopf algebra for each of these theorems are relatively independent, the results can combine when it satisfies several of them. We note some of these combinations in several corollaries and discuss how one of these generalises work done by Donkin in [13]. We then give a complete overview of what our results say about what type of crossed product in Table 4.1.

In Chapter 5, we look at a specific type of Hopf algebra that occurs frequently, namely a Hopf algebra H containing a central Hopf subalgebra $A \subseteq H$ such that H is finitely-generated as an A-module, where the module action is given by multiplication in H. We restrict ourselves to the case where H is Noetherian, which still includes many examples we are interested in such as quantised enveloping algebras at roots of unity.

Such algebras always have a canonical Hopf surjection from H to a finite-dimensional quotient algebra, and it is an open question when this gives rise to a crossed product decomposition of H, with a counterexample for k not algebraically closed on one hand and various affirmative results and examples on the other. This means that we can apply (and, in fact, slightly strengthen) the results of Chapter 4 in this situation. However, we also find that under certain conditions on A and H, we can describe the finite dual or certain Hopf subalgebras of the finite dual without needing to assume that H itself decomposes as a crossed product. These results let us calculate the finite dual of the quantised enveloping algebra $U_\epsilon(\mathfrak{sl}_2(k))$ for ϵ a root of unity and form a conjecture for the finite dual of $U_\epsilon(\mathfrak{g})$ for any finite-dimensional semisimple $\mathfrak{g}$ and note a promising result in this vein in the form of a canonical Hopf subalgebra of the dual.

In Chapter 6, we test our results on examples by turning our attention to the prime affine regular Hopf algebra of Gelfand-Kirillov dimension one classified by Brown and Zhang in [6]. Their work forms part of an effort to classify such Hopf algebras: Brown and Zhang give a list of examples and show it contains all such Hopf algebras satisfying another technical condition, a corollary of which says it contains all those with prime PI-degree. Recent work by Wu, Liu and Ding ([52]) completes this classification by defining another family of Hopf algebras and proving that any prime affine regular Hopf algebra with Gelfand-Kirillov dimension one is isomorphic to either one of their family or one of those listed by Brown and Zhang. Due to the fact that this part of the thesis had already been written when their work came out, we look only at those Hopf algebras listed in [6]. These consist of the polynomial and Laurent polynomial algebra in one variable, the group algebra of the dihedral group along with the two infinite families, namely the infinite-dimensional Taft algebras and the generalised Liu algebras. We use the work of Chapters 4 and 5 to calculate their finite duals.

Finally, Chapter 7 concludes the thesis by giving some additional motivation and potential application for our results by discussing the concept of a distinguished Hopf subalgebra of the finite dual as well as possible applications, work that has already been done with respect to this idea and how our results tie into this.

Chapters 1 and 2 focus on providing the definitions and explaining the setting that we work in, and thus mainly consist of providing known results in our specific context and notation with little original work. Chapters 3, 4 and 5 primarily consist of original work and contain the main results of the thesis, while 6 is partially a recap of known results but mostly original. There are also sections at the end of each chapter explaining which results stated in it are original work and which are not in more detail. When proofs are given for material that is in one of these sections stated to already be known, they are presented for clarity or because the result in question could not be located in the form needed, not because the result is original.

1.2 Hopf algebras

Throughout, let k be a field.

Throughout the thesis, when we refer to a map we always assume it is linear unless stated otherwise, and similarly all unadorned tensor products are over k. Moreover, by *algebra* we mean a unital associative k-algebra unless stated otherwise.

In order to define a Hopf algebra, we need some preliminary definitions. First, a coalgebra is the dual notion to an algebra.

Definition 1.1. A *coalgebra* C is a k-vector space along with two structure maps: a comultiplication or coproduct $\Delta : C \to C \otimes C$ and a counit $\varepsilon : C \to k$, satisfying the following axioms:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \circ \mathrm{id}) \circ \Delta, \qquad \text{(Coassociativity)}$$

and

$$\mu \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}, \qquad \text{(Counit)}$$

where μ denotes the canonical vector space isomorphism given by scalar multiplication.

Remark 1.2. We can write Δ explicitly as follows: given a coalgebra C and $c \in C$, there exists $n > 0$ and $c_1^i, c_2^i \in C$ for $1 \leq i \leq n$ such that $\Delta(c) = \sum_{i=1}^n c_1^i \otimes c_2^i$. For ease of reading we omit the summation index, leaving

$$\Delta(c) = \sum c_1 \otimes c_2.$$

This is known as *Sweedler notation* and we will use it throughout.

Under Sweedler notation, the coassociativity axiom becomes

$$\sum (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum c_1 \otimes (c_2)_1 \otimes (c_2)_2 := \sum c_1 \otimes c_2 \otimes c_3$$

and the counit axiom becomes

$$\sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2) = c$$

for $c \in C$.

Note that, similarly to the unit map in an algebra, the counit is unique.

Lemma 1.3. *Let C be a coalgebra. Then the counit $\varepsilon : C \to k$ is unique: if there exists a map $\varepsilon' : C \to k$ also satisfying the counit axiom, then $\varepsilon = \varepsilon'$.*

Proof. Suppose $\varepsilon' : C \to k$ is another map satisfying the counit axiom. Then for all $c \in C$ we have

$$\varepsilon(c) = \varepsilon\left(\sum \varepsilon'(c_1)c_2\right) = \varepsilon'\left(\sum c_1\varepsilon(c_2)\right) = \varepsilon'(c)$$

as required, using linearity of ε and ε'. □

A bialgebra is both an algebra and a coalgebra, with the two structures being compatible:

Definition 1.4. A *bialgebra* is an algebra which is also a coalgebra such that the coproduct and counit maps are algebra maps.

Remark 1.5. Any coalgebra C has a distinguished subspace of codimension one given by the kernel of the counit map. When C is a bialgebra, this subspace is in fact an ideal, called the *augmentation ideal*. We denote it by

$$C^+ := \ker \varepsilon.$$

A Hopf algebra is a bialgebra along with a specific map that translates between the structures:

Definition 1.6. A *Hopf algebra* H is a bialgebra such that there exists a map $S : H \to H$ satisfying

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon, \qquad \text{(Antipode)}$$

where $u : k \to H$ is the unit map given by $u(\lambda) = \lambda 1_H$ and $m : H \otimes H \to H$ is the multiplication map. In Sweedler notation, this means that for all $h \in H$,

$$\sum S(h_1)h_2 = \sum h_1 S(h_2) = \varepsilon_H(h)1_H.$$

We call the map S the *antipode*.

The antipode is also unique, and is always an algebra and coalgebra antihomomorphism:

Lemma 1.7. *Let H be a Hopf algebra. Then*

(i) The antipode S is unique: if $S' : H \to H$ is another map satisfying the antipode axiom, then $S = S'$.

(ii) S is an algebra and a coalgebra antihomomorphism, that is,

$$S(hk) = S(k)S(h) \qquad \sum S(h_1) \otimes S(h_2) = \sum S(h)_2 \otimes S(h)_1$$
$$S(1) = 1, \qquad \varepsilon \circ S = \varepsilon$$

for all $h, k \in H$.

Proof. (i) Suppose $S, S' : H \to H$ are two maps both satisfying the antipode axiom. Then given $h \in H$, we have

$$S(x) = \sum \varepsilon(x_1)S(x_2) = \sum S'(x_1)x_2S(x_3) = \sum S'(x_1)\varepsilon(x_2) = S'(x)$$

So $S = S'$ as required.

(ii) This is by [1, Theorem 2.1.4]. □

Note that, similarly to dualising the notion of an algebra to obtain a coalgebra, we can dualise notions such as algebra maps, ideals and modules to obtain coalgebra maps, coideals and comodules, and furthermore combine them so that we can talk about bialgebra and Hopf algebra maps or Hopf ideals. Although we do use these concepts throughout the thesis, the definitions are not given here for reasons of space. We refer the reader to [36, Chapter 1] in case they are unfamiliar.

1.2.1 Examples of Hopf algebras

The following are some classical examples of Hopf algebras we will be referring to throughout. In each case, we state what the structure maps are and give a reference to where it is verified that the axioms are satisfied.

Example 1.8. Let k be a field and G any group. The group algebra kG is the algebra with vector space basis given by the elements of G and multiplication given by the group multiplication, extended linearly. This is a Hopf algebra: given $g \in G$ we define the coproduct to be $\Delta(g) = g \otimes g$, the counit to be $\varepsilon(g) = 1$ and the antipode to be $S(g) = g^{-1}$.

The fact that the Hopf algebra axioms are satisfied follows immediately from the group axioms: for instance, the antipode axiom is simply the condition that for all $g \in G$, g^{-1} is a left and right inverse for g.

This example is reason for the following definition.

Definition 1.9. Given any Hopf algebra H, we call a nonzero element $h \in H$ *grouplike* if we have $\Delta_H(h) = h \otimes h$.

Note that the counit and antipode axioms mean that for any grouplike element $h \in H$ we must have $\varepsilon_H(h) = 1$ and h invertible with $S_H(h) = h^{-1}$.

Example 1.10. Let k be a field and $\mathfrak{g}$ a finite-dimensional Lie algebra over k with Lie bracket denoted by $[-,-]$. Suppose $\mathfrak{g}$ has basis $\{x_1, ..., x_n\}$. The *universal enveloping algebra* of $\mathfrak{g}$ is

$$U(\mathfrak{g}) := k\langle x_1, ..., x_n \mid x_j x_i - x_i x_j = [x_i, x_j]\rangle.$$

This is a Hopf algebra: given $x \in \mathfrak{g}$ we define the coproduct as $\Delta(x) = x \otimes 1 + 1 \otimes x$, counit as $\varepsilon(x) = 0$ and antipode as $S(x) = -x$, extending each algebraically or anti-algebraically as appropriate to define them on the whole of $U(\mathfrak{g})$. We refer to [1, Examples 2.5 and 2.7] for the details.

A special case of this is when $\mathfrak{g}$ is abelian, with $[x_i, x_j] = 0$ for all i and j. Here we have $U(\mathfrak{g}) = k[x_1, ..., x_n]$, and thus we find the polynomial algebra is a Hopf algebra under the coproduct, counit and antipode given above.

Definition 1.11. Given any Hopf algebra H, we call an element $h \in H$ *skew-primitive* or (g, g')*-primitive* if there exist grouplike elements $g, g' \in H$ such that

$$\Delta(h) = h \otimes g + g' \otimes h.$$

If $g = g' = 1$, we simply call h *primitive*.

Again, the counit and antipode axioms guarantee that any (g, g')-primitive element h satisfies $\varepsilon(h) = 0$ and $S(h) = -(g')^{-1} h g^{-1}$.

Both group algebras and universal enveloping algebras are *cocommutative*, meaning that the coproduct is preserved under tensor flip: in other words, given any element x of the Hopf algebra, we have

$$\sum x_1 \otimes x_2 = \sum x_2 \otimes x_1.$$

The following is an example of a family of Hopf algebras which are not cocommutative in general.

Example 1.12. Let k be a field and $G \subseteq k^n$ be an affine algebraic group. Then the *coordinate ring* of G is given by

$$\mathcal{O}(G) := k[X_1, ..., X_n]/Z(G),$$

where $Z(G)$ is the ideal given by those polynomial functions that vanish on G. We can view $\mathcal{O}(G)$ as the algebra of polynomial functions on G. This is a Hopf algebra, where given $f \in \mathcal{O}(G)$ and $x, y \in G$ we set $\varepsilon(f) = f(1_G)$, $S(f)(x) = f(x^{-1})$ and

$$\Delta(f) \in \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$$

to be such that

$$\Delta(f)(x \otimes y) = f(xy).$$

Thus the coalgebra structure comes from the group structure on G.

As in the case of the group algebra, the fact that $\mathcal{O}(G)$ satisfies the axioms of a Hopf algebra follows from the fact that G satisfies those of an affine algebraic group. For details, see for instance [21, Section 7.6] or [47, Section 2.1.2].

Remark 1.13. Note that as coordinate rings are quotients of polynomial rings, they are always commutative. In characteristic p, there exist other commutative Hopf algebras. For instance, the group algebra kC_p cannot be isomorphic to a coordinate ring as it contains nilpotent elements such as $x - 1$, where x is the generating element of C_p, while $\mathcal{O}(G)$ cannot contain nonzero nilpotent elements for any G as it is the quotient of an integral domain by a semiprime ideal.

However, we are primarily interested in Hopf algebras over fields of characteristic zero. Here, all affine commutative Hopf algebras arise through coordinate rings: there is a contravariant equivalence of categories between affine algebraic groups and commutative affine Hopf algebras given by sending an algebraic group G to $\mathcal{O}(G)$ and sending a commutative Hopf algebra H to $\mathrm{Alg}(H, k)$, the set of algebra maps from H to k. This equivalence is described in detail in for instance [23, Section 2.3].

The following is an example of a family of Hopf algebras which are neither commutative nor cocommutative:

Example 1.14. Let k be an algebraically closed field, n and t be integers with $t \geq 1$, $n > t$ and $\gcd(n, t) = 1$, and q be a primitive nth root of unity in k. Then we define the *finite-dimensional Taft algebra* on those parameters by

$$H_f(n, t, q) \cong k\langle x, g \mid xg = qgx, g^n = 1, x^n = 0\rangle.$$

This is a Hopf algebra, where the coproduct, counit and antipode are given by

$$\Delta(x) = x \otimes 1 + g^t \otimes x \qquad \varepsilon(x) = 0 \qquad S(x) = -g^{-t}x = -g^{n-t}x$$
$$\Delta(g) = g \otimes g \qquad \varepsilon(g) = 1 \qquad S(g) = g^{-1} = g^{n-1},$$

so g is grouplike and x is $(1, g^t)$-primitive. This family of Hopf algebras was first introduced by Taft in [49] for $t = 1$, and the Hopf algebra axioms are verified there.

Lu, Wu and Zhang ([32, Example 2.7]) extended this family to infinite dimensions, without the assumption that $\gcd(n, t) = 1$. The *infinite-dimensional Taft algebra* $H(n, t, q)$ is given by the same generators as $H_f(n, t, q)$, the same coproduct, counit and antipode on each generator, and the same relations except that we drop the relation that $x^n = 0$. So as an algebra, we have

$$H(n, t, q) \cong k\langle x, g \mid xg = qgx, g^n = 1\rangle.$$

We note that when $\gcd(n, t) = 1$, the finite-dimensional Taft algebra arises as the quotient of $H(n, t, q)$ by the Hopf ideal generated by x^n. This will be discussed in more detail in Example 2.12.

The Taft algebras are some of the simplest examples of noncommutative, noncocommutative algebras and also satisfy certain other properties we are interested in. We will therefore be using them as examples throughout the thesis and will learn more about their structure as we do. For instance, in Examples 2.12, we will see that infinite-dimensional Taft algebras can be described as crossed products (a notion defined in detail in Chapter 2), which among others gives us a vector space basis for them.

Remark 1.15. This last example shows us that Hopf algebras can be isomorphic as algebras but not as bialgebras: since the parameter t only affects the coalgebra structure we have $H(n, t, q) \cong H(n, t', q)$ as algebras for any $0 \leq t, t' < n$, yet these are not isomorphic as bialgebras when $t \neq t'$. This is in contrast to the antipode, which by Lemma 1.7(i) is uniquely determined by the bialgebra structure of H.

Further examples of noncommutative, noncocommutative Hopf algebras are provided by quantised enveloping algebras and quantised coordinate rings. We do not define these here, but they are studied for instance in [7] or [25] and we will look at quantised enveloping algebras again in Chapter 5.

1.3 Finite duals

Throughout, k is a field and H is a Hopf algebra.

Hopf algebras are objects which incorporate a great deal of structure and symmetry. In the finite-dimensional case, this means that they admit a very nice duality: the vector space dual H^* consisting of the k-linear maps from H to k is also a Hopf algebra in a canonical way, with the coalgebra structure of H defining the algebra structure of H^* and vice versa. Moreover, the canonical isomorphism $(H^*)^* \cong H$ is in fact an isomorphism of Hopf algebras.

However, we are primarily interested in infinite-dimensional Hopf algebras. And here this breaks down: the dual H^* of an infinite-dimensional Hopf algebra H is still an algebra but no longer a coalgebra in general. Instead, we can find a subalgebra $H^0 \subseteq H^*$ which is a Hopf algebra and is moreover maximal in H^* with respect to this. This subalgebra is called the *finite dual*, and it is our primary object of study.

1.3.1 Finite dimensional duality

First we note that the dual of any coalgebra is an algebra.

Lemma 1.16. *Let C be a coalgebra and $C^* = \mathrm{Hom}_k(C, k)$ its dual. Then C^* is an algebra in the following way: given $f, g \in C^*$ and $c \in C$,*

$$(f * g)(c) := \sum f(c_1)g(c_2).$$

This multiplication is associative with identity element ε_C.

Proof. The associativity of the multiplication follows from the coassociativity axiom: given $f, g, h \in C^*$ and $c \in C$, we have

$$\begin{aligned}((f * g) * h)(c) &= \sum (f * g)(c_1)h(c_2) \\ &= \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g * h)(c_2) = (f * (g * h))(c).\end{aligned}$$

Similarly, ε being the unit is simply the counit axiom. □

Remark 1.17. The algebra structure given in Lemma 1.16 can be defined more generally on $\mathrm{Hom}_k(C, A)$ for any coalgebra C and algebra A. This is called the *convolution algebra* and the product is called the *convolution product*. We can now say that a map $f : C \to A$

is *convolution invertible* if it is invertible in the convolution algebra, meaning that there exists some $f' : C \to A$ satisfying

$$\sum f(c_1)f'(c_2) = \sum f'(c_1)f(c_2) = \varepsilon_C(c)1_A$$

for all $c \in C$. This concept will come into play again in later chapters. For now, we simply note that the notion of the convolution algebra and convolution inverse provide an alternative proof to Lemma 1.7(i), as an antipode for a Hopf algebra H is the same thing as a convolution inverse for the identity map in $\mathrm{Hom}_k(H, H)$ and therefore must be unique if it exists.

Moreover, the dual of any finite-dimensional algebra is a coalgebra:

Lemma 1.18. *Suppose A is an algebra with $\dim_k(A) < \infty$. Then A^* is a coalgebra as follows:*

(i) Given $f \in A^*$, the coproduct is given by $\Delta_{A^*}(f) = \sum f_1 \otimes f_2$ such that

$$\sum f_1(a)f_2(b) = f(ab)$$

for all $a, b \in A$.

(ii) Given $f \in A^*$, the counit is given by evaluation at the identity element: $\varepsilon_{A^*}(f) = f(1_A)$.

Proof. First we need to show that given $f \in A^*$, $\Delta(f)$ as described in the statement of the lemma is a well-defined element of $A^* \otimes A^*$. This follows because $(A \otimes A)^* \cong A^* \otimes A^*$ for finite-dimensional A:

Let $\phi : A^* \otimes A^* \to (A \otimes A)^*$ be given by given by

$$\phi(f \otimes g)(a \otimes b) = f(a)g(b)$$

and extended linearly. First, we want to show that ϕ is injective.

Let $\alpha = \sum_{i=1}^{r} f_i \otimes g_i \in A^* \otimes A^*$ satisfying $\phi(\alpha) = 0$. We can assume that the $g_i \in A^*$ are all linearly independent. For all $a, b \in A$ we have

$$0 = \phi(\alpha)(a \otimes b) = \sum_{i=1}^{r} f_i(a)g_i(b).$$

In particular, this means that $\sum f_i(a)g_i = 0$ in A^* for all $a \in A$. Since the g_i are linearly independent, this means that $f_i(a) = 0$ for all $1 \leq i \leq r$ and $a \in A$. Hence $\alpha = 0$, and so ϕ is injective.

Now recall that $\dim_k(A) = n < \infty$ for some n. This means that $\dim_k(A^*) = n$ and so

$$\dim_k(A^* \otimes A^*) = \dim_k(A \otimes A)^* = n^2.$$

Standard linear algebra tells us that ϕ must therefore be surjective as well as injective and hence an isomorphism.

This means that we can indeed define $\Delta(f) \in A^* \otimes A^*$ by its values on $A \otimes A$.

Coassociativity and counit axioms now follow from associativity and unit axioms in A. □

Corollary 1.19. *Let H be a finite-dimensional Hopf algebra, $f, g \in H^*$ and $h, h' \in H$. Then H^* is also a Hopf algebra with respect to the following structure maps:*

(i) The multiplication is given by

$$(fg)(h) := \sum f(h_1)g(h_2),$$

(ii) The unit is given by $\varepsilon_H \in H^*$,

(iii) The comultiplication is given by $\Delta_{H^*}(f) = \sum f_1 \otimes f_2 \in H^* \otimes H^*$ such that

$$\sum f_1(h)f_2(h') = f(hh'),$$

(iv) The counit is given by

$$\varepsilon_{H^*}(f) := f(1_H),$$

and

(v) The antipode is given by

$$S_{H^*}(f) := f \circ S_H.$$

Proof. We know that H^* is both an algebra and a coalgebra by Lemmas 1.16 and 1.18 with respect to precisely those structure maps. All that remains to check is that the coproduct and counit are algebra maps and that the map S_{H^*} as defined satisfies the antipode axiom, which follows straightforwardly from the fact that the coproduct and counit in H are algebra maps and that S_H satisfies the antipode axiom. □

Example 1.20. Let k be an algebraically closed field, n be a positive integer coprime to the characteristic of k, $G = C_n$ be the cyclic group of order n and $H := kC_n$ its group algebra. Then kC_n^* is a Hopf algebra as above. In fact it is self-dual, with $kC_n^* \cong kC_n$ as Hopf algebras:

Set $g \in C_n$ to be a generating element, q a primitive nth root of unity in k, and let $f_i : kC_n \to k$ denote the algebra map given by $f_i(g) = q^i$ for $0 \leq i \leq n-1$. We see that these maps give a basis of kC_n^* as follows:

Suppose we have $\lambda_0, ..., \lambda_{n-1} \in k$ satisfying $\sum_{i=0}^{n-1} \lambda_i f_i = 0$. This means that for all $0 \leq r < n-1$, we have

$$\sum_{i=0}^{n-1} \lambda_i f_i(g^r) = \sum_{i=0}^{n-1} \lambda_i q^{ir} = 0.$$

So if we let $p(x) \in k[x]$ denote the polynomial given by $p(x) = \sum_{i=0}^{n-1} \lambda_i x^i$, this must be zero whenever $x = q^r$ for $0 \leq r < n$. However, these are n separate roots and $p(x)$ is a polynomial of degree at most $n-1$. It follows that $p(x)$ must be zero, and hence so is λ_i for each i. So the f_i are linearly independent and form a basis for kC_n^*.

Now let $\phi : kC_n \to kC_n^*$ denote the map sending g^i to f_i. This is bijective by the above. It is an algebra map: we have

$$f_i f_j(g^r) = f_i(g^r) f_j(g^r) = q^{ir} q^{jr} = q^{(i+j)r} = q^{(i+j \mod n)r} = f_{(i+j \mod n)}(g^r),$$

and ϕ sends $1 = g^0$ to $f_0 = \varepsilon_H = 1_{H^*}$. Moreover, each f_i must be grouplike because it is an algebra map and the definition of the coproduct in H^* means that any algebra map is grouplike. So ϕ is also a coalgebra map, and hence a bialgebra map. By [48, Lemma 4.0.4] any bialgebra map between Hopf algebras is a Hopf algebra map. So ϕ is an isomorphism of Hopf algebras.

In general, Hopf algebras are not self-dual. However, we do find that given a finite-dimensional Hopf algebra H, the canonical vector space isomorphism $(H^*)^* \cong H$ given by evaluation at an element of h is in fact an isomorphism of Hopf algebras:

Lemma 1.21. *Let H be a finite-dimensional Hopf algebra. Then*

$$(H^*)^* \cong H$$

as Hopf algebras.

Proof. It is a well-known fact that for finite dimensional vector spaces, the map $V \to (V^*)^*$ given by sending an element v to the map $\mathrm{ev}_v : f \mapsto f(v)$ is a linear isomorphism. So we only need to check it preserves the Hopf structure.

Let $g, h \in H$ and $f \in H^*$. Then we have

$$(\mathrm{ev}_g \, \mathrm{ev}_h)(f) = \sum \mathrm{ev}_g(f_1) \, \mathrm{ev}_h(f_2) = \sum f_1(g) f_2(h) = f(gh) = \mathrm{ev}_{gh}(f),$$

so evaluation preserves the multiplication. Similarly,

$$ev_1(f) = f(1_H) = \varepsilon_{H^*}(f),$$

so evaluation preserves the identity. Comultiplication and the counit are done analogously, and by [48, Lemma 4.0.4] any bialgebra map between Hopf algebras is a Hopf algebra map, giving us what we need. □

1.3.2 Duality in infinite dimensions

Now we suppose that H is an infinite-dimensional Hopf algebra.

By Lemma 1.16 H^* is always an algebra with respect to the same multiplication and unit as in Corollary 1.19. However, the coproduct given there relies on the isomorphism $(V \otimes V)^* \cong V^* \otimes V^*$, which holds for finite-dimensional vector spaces V but not infinite-dimensional. Instead, we consider a subspace of H, consisting of all the functions satisfying the equivalent properties we record in Proposition 1.23.

First, we give a preliminary definition we need for one of the properties:

Definition 1.22. Let H be a Hopf algebra. Then the left and right actions of H on H^* given by

$$(h \rightharpoonup f)(h') := f(h'h)$$
$$(f \leftharpoonup h)(h') := f(hh')$$

for $h, h' \in H$, $f \in H^*$ are called the *left and right hit actions* of H on H^*.

These actions are in fact module actions of H on H^* (see for instance [36, Example 1.6.6]). Moreover, the two of them together give a H-H-bimodule structure on H^*.

This gives us enough for the following statement:

Proposition 1.23. *Let H be a Hopf algebra and $f \in H^*$. The following conditions are equivalent:*

(i) The kernel of f contains a left ideal of finite codimension in H.

(ii) The kernel of f contains a right ideal of finite codimension in H.

(iii) The kernel of f contains a two-sided ideal of finite codimension in H.

(iv) The left H-submodule generated by f via the left hit action is finite-dimensional.

(v) The right H-submodule generated by f via the right hit action is finite-dimensional.

(vi) The H-subbimodule of H^* generated by f via the hit actions is finite-dimensional.

(vii) There exists an integer $n \geq 1$ and functions $g_1, ..., g_n, h_1, ..., h_n \in H^*$ such that

$$\sum_{i=1}^{n} g_i(x)h_i(y) = f(xy)$$

for all $x, y \in H$.

Proof. See [36, Lemma 9.1.1]. □

Definition 1.24. Let H be a Hopf algebra. We call the set of functions satisfying the equivalent conditions in Proposition 1.23 the *finite dual* of H, and write it as $H^0 \subseteq H^*$.

In the finite-dimensional case, this is just the vector space dual:

Example 1.25. Let k be a field and H be a finite-dimensional Hopf algebra over k. Then $\{0\}$ is an ideal with finite codimension in H. Since $\{0\} \subseteq \ker f$ for all $f \in H^*$, this means that $H^0 = H^*$.

In the infinite-dimensional case, H^0 is a subspace of H^*. In fact, H^0 is a subalgebra of H^* that is also a Hopf algebra and is maximal with respect to this property:

Proposition 1.26. *Let H be any Hopf algebra. Then*

(i) H^0 is a Hopf algebra, with the coproduct, counit and antipode as in Corollary 1.19.

(ii) H^0 is the maximal subalgebra of H^* that is a Hopf algebra with respect to this coproduct, counit and antipode.

Proof. (i) See [36, Theorem 9.1.3].

(ii) Suppose $K \subseteq H^*$ is a Hopf algebra with respect to the coproduct, counit and antipode of Corollary 1.19 and let $f \in K$. Because K is a Hopf algebra, $\Delta(f)$ is well-defined: there exists $n > 0$ and $f_1^1, ..., f_1^n, f_2^1, ..., f_2^n \in K$ satisfying

$$\sum_{i=1}^{n} f_1^i(h) f_2^i(h') = f(hh')$$

for all $h, h' \in H$.

However, this means that f satisfies Proposition 1.23(vii) and hence all the equivalent conditions, which means that $f \in H^0$. So $K \subseteq H^0$ as required. □

The following example shows that in general, H^0 is not the whole of H^*.

Example 1.27. Let k be a field of characteristic zero, $\mathfrak{g}$ be a semisimple Lie algebra over k and let $H = U(\mathfrak{g})$ be the universal enveloping algebra. By [20, Theorem 3.1] we find that $H^0 \cong \mathcal{O}(G)$ as Hopf algebras, where G is the (unique up to isomorphism) simply connected affine algebraic group satisfying $\operatorname{Lie} G = \mathfrak{g}$.

A dimension argument tells us that this need not be the whole of H^*:

Suppose $k = \mathbb{C}$. Given $x \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$, we can find $f_\lambda : H \to \mathbb{C}$ satisfying $f_\lambda(x^i) = \lambda^i$. We first want to show that these f_λ are linearly independent.

Suppose there exist $\lambda_1, ..., \lambda_n, \alpha_1, ..., \alpha_n \in \mathbb{C}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ satisfying

$$\sum_{i=1}^{n} \alpha_i f_{\lambda_i} = 0.$$

This means that $\sum \alpha_i f_{\lambda_i}(x^j) = 0$ for all j and so $\alpha := (\alpha_1, ..., \alpha_n)$ is a solution to the system of linear equations given by

$$x_1 \lambda_1^i + ... + x_n \lambda_n^i = 0 \quad \text{for } i \geq 0.$$

In particular, α is a solution to the equations for $1 \leq i \leq n$. This means that we have

$$A\alpha^T = 0,$$

where $A \in M_n(\mathbb{C})$ is the matrix whose (i, j)th entry is given by λ_i^j. However, classical results of linear algebra tell us that this matrix, also known as the Vandermonde matrix, has nonzero determinant whenever the λ_i are distinct. Thus the only vector $\mathbf{v} \in \mathbb{C}^n$ satisfying $A\mathbf{v} = 0$ is the zero vector, and hence we have

$$\alpha_1 = ... = \alpha_n = 0$$

as required.

This means that we can extend the set $\{f_\lambda \mid \lambda \in \mathbb{C}\}$ to form a basis for H^*. Thus we have

$$\dim(H^*) \geq |\{f_\lambda \mid \lambda \in \mathbb{C}\}| = |\mathbb{C}|.$$

In particular, H^* must have uncountable dimension.

However, $H^0 \cong \mathcal{O}(G)$ is the coordinate ring of an affine algebraic group and hence itself affine, which means it must have countable dimension. This means that $H^0 \neq H^*$.

Overall, H^0 seems to be the obvious replacement of H^*. However, as we will see in the next section, H^0 does not preserve the properties of H in the same way as H^* does in the finite-dimensional case.

1.4 Properties of the finite dual

Throughout, k is a field.

As discussed in Section 1.3.2, given an infinite-dimensional Hopf algebra H, we can associate with it another Hopf algebra H^0 which is a subalgebra of the dual algebra H^*. When H is finite-dimensional, H^0 is the whole of H^*, but this is not true in general.

An immediate question that arises is how far H^0 preserves the properties of H, and in particular whether it preserves the properties that H^* does. For instance: is $(H^0)^0$ isomorphic to H in the same way we have $(H^*)^* \cong H$ in the finite-dimensional case?

The following examples tell us that the answer is *no*. The problem is that taking finite duals does not preserve size.

In the finite-dimensional case, vector space dimension gives us a canonical measure of size, and the fact that $\dim_k(V) = \dim_k(V^*)$ for any finite-dimensional vector space V is well-known. In the infinite-dimensional case, vector space dimension - in particular, its cardinality - still gives us some way of comparing the size of different Hopf algebras, but it is much rougher and less useful. Instead, there are various classical algebraic notions that are used instead, such as Gelfand-Kirillov dimension and global or other homological dimensions, as well as properties such as being affine (meaning finitely generated as an algebra) or being Noetherian.

The following example shows us that H^0 can be too small.

Example 1.28. Let k be a field, $K \supsetneq k$ be a field extension such that K is infinite, and let $H := k\mathrm{PSL}_2(K)$, the k-group algebra of the projective special linear group over K. By [4, Lemma 2.7], we find that the only functions in H^* which generate a finite-dimensional left H-submodule of H^* are those functions which are scalar multiples of the counit. So we have

$$k\mathrm{PSL}_2(K)^0 = k\varepsilon_H \cong k.$$

The Hopf algebra we started with was infinite-dimensional, but its finite dual is one-dimensional. This also means that $\left(H^0\right)^0 \ncong H$, because the dual of a one-dimensional structure is itself one-dimensional.

Remark 1.29. This example also tells us that taking finite duals does not preserve subalgebras, even Hopf subalgebras, either covariantly or contravariantly:

Let $x \in \mathrm{PSL}_2(K)$ be any element with infinite order, and let A denote the subalgebra it generates. A is not just a subalgebra but a Hopf subalgebra, as $\Delta(x) = x \otimes x \in A \otimes A$ and $S(x) = x^{-1} \in A$, and we have $A \cong k[x^{\pm 1}]$ as algebras. Since H^0 has dimension one and $\varepsilon_A \in A^0$, finding any map of A^0 that is not a scalar multiple of the counit means that $\dim_k(A^0) > \dim_k(H^0)$, which in turn means that A^0 cannot be either a subspace or a quotient of H^0.

Given $\lambda \in k^*$ with $\lambda \neq 1$, we can define the map $f_\lambda : A \to k$ given by $f(x^i) = \lambda^i$. This is a member of A^0 because it is an algebra map: $\ker f_\lambda$ itself is an ideal, and since it is the kernel of a map to a one-dimensional vector space it has codimension one in A. Furthermore, f_λ is clearly not a scalar multiple of the counit, which is given by $\varepsilon(x^i) = 1$ for all i. So $\dim A^0 > 1$ and so A^0 is neither a subspace nor a quotient of H^0.

Note that there is a straightforward condition which guarantees that H^0 is at least dense in H^*, meaning that for every nonzero $h \in H$ there exists an $f \in H^0$ with $f(h) \neq 0$.

Lemma 1.30. *Let H be a Hopf algebra. The following conditions are equivalent:*

(i) The intersection of all ideals of finite codimension in H is zero.

(ii) For any nonzero $h \in H$, there exists a finite-dimensional left H-module M such that $h \notin \mathrm{Ann}_H(M)$.

(iii) H^0 is dense in H^*: for any nonzero $h \in H$, there exists an $f \in H^0$ with $f(h) \neq 0$.

Proof. (i) $\Leftrightarrow$ (iii): See [36, Proposition 9.2.10].

(i) $\Rightarrow$ (ii): Suppose that there exists some $h \in H$ such that $h \in \mathrm{Ann}_H(M)$ for all finite-dimensional modules M. This means that $h \in \mathrm{Ann}_H(H/I) = I$ for any ideal I of finite codimension. So h is in the intersection of all ideals of finite codimension: by (i), this means that $h = 0$.

(ii) $\Rightarrow$ (iii): Let $h \in H$ be nonzero. By (ii), there exists some finite-dimensional left H-module M such that $h \notin \mathrm{Ann}_H(M)$. So there must be $m \in M$ such that $h \cdot m$ is nonzero. Extend $h \cdot m$ to a basis of M, and let $f \in M^*$ be the map that sends $h \cdot m$ to 1_k and all other basis elements to 0.

We can define $\hat{f} : H \to k$ given by $\hat{f}(h') = f(h' \cdot m)$. This is nonzero on h by definition. Moreover, $\hat{f}$ is zero on $\mathrm{Ann}_H(M)$, which is a left ideal of finite codimension in H. So we have $\hat{f} \in H^0$ by Proposition 1.23. □

The condition in (i)-(iii) of the lemma is called being *residually finite-dimensional*.

Many Hopf algebras we are interested in are residually finite-dimensional. However, even when we know that H^0 is dense in H^*, we find that it can be too big.

The next example is one where H^0 is not Noetherian and has uncountable dimension, and as a consequence is also not affine. All of this stands in contrast to H.

Example 1.31. Let $\mathfrak{g}$ be a solvable Lie algebra of dimension n, and let $H := U(\mathfrak{g})$. Suppose $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ has dimension m. We have

$$U(\mathfrak{g})^0 \cong k[X_1, ...X_n] \otimes k((k, +)^{n-m})$$

as algebras (see [18, Section 6] and [19, p. 610], or [30, Proposition 1.3] for a summary). Here $(k, +)$ denotes the additive group of the underlying field, $(k, +)^{n-m}$ its $(n - m)th$ Cartesian product with itself and $k((k, +)^{n-m})$ the group algebra of this structure.

Note that $n - m > 0$ always, because $\mathfrak{g}$ is solvable and so $\mathfrak{g}'$ is a proper subspace of $\mathfrak{g}$. So when $k = \mathbb{C}$ we find that $\mathbb{C}((\mathbb{C}, +)^{n-m})$, which has basis indexed by $\mathbb{C}^{n-m}$, has uncountable dimension, which means that $U(\mathfrak{g})^0$ has uncountable dimension as well. This means that $U(\mathfrak{g})^0$ cannot be affine, since any affine algebra must be spanned by the set of words in its finitely many generators and this set is countable. Finally, $U(\mathfrak{g})^0$ is not Noetherian as follows.

First note that since $U(\mathfrak{g})^0$ is isomorphic to a tensor product of a polynomial ring with $\mathbb{C}(\mathbb{C}, +)^{n-m}$ as algebras, any infinite chain of ideals in $\mathbb{C}(\mathbb{C}, +)^{n-m}$ can be extended to one in $U(\mathfrak{g})^0$ by tensoring with the polynomial ring. Therefore, it suffices to show that $\mathbb{C}(\mathbb{C}, +)^{n-m}$ is not Noetherian.

Now note that we can define an infinite ascending chain of subgroups in $(\mathbb{C}, +)$ by setting $G_1 := \langle 1_{\mathbb{C}} \rangle = \mathbb{Z}$ and then noting that if $G_n = \langle \lambda_1, ..., \lambda_n \rangle$ for some $\lambda_1, ..., \lambda_n \in \mathbb{C}$, we have $G_n = \mathbb{Z}\lambda_1 + ... + \mathbb{Z}\lambda_n$, which is strictly contained in $\mathbb{C}$ and therefore allows us to pick $\lambda_{n+1} \notin G_n$ to construct G_{n+1}. This means we can do the same in $(\mathbb{C}, +)^{n-m}$ and that this group is not Noetherian, which means its group algebra cannot be Noetherian either (by [17, p. 421]). So $\mathbb{C}(\mathbb{C}, +)^{n-m}$ is not Noetherian as required.

As a special case, take $\mathfrak{g}$ to be the one-dimensional Lie algebra. Here $\mathfrak{g}' = 0$ and $H := U(\mathfrak{g}) \cong k[x]$. So we find that

$$k[x]^0 \cong k[X] \otimes k(k, +)$$

as algebras.

Of note in the example of $U(\mathfrak{g})^0$ for $\mathfrak{g}$ solvable is that there is a subalgebra $k[X_1, ..., X_n]$ contained in $U(\mathfrak{g})^0$ which is affine Noetherian and satisfies all the size conditions we would like. In fact, it is a Hopf subalgebra: it consists of those functions vanishing on some power of the augmentation ideal, which is always a Hopf subalgebra of H^0 by [36, Lemma 9.2.1]. This seems to indicate that under certain assumptions on H, we might be able to find a Hopf subalgebra of H^0 that preserves various properties. This and what it might be useful for is discussed further in Chapter 7.

The problem with investigating this and other questions regarding the finite dual is that we know relatively little about the finite dual and what determines its properties in general. Moreover, we do not know many examples, in particular examples of H^0 when H is a noncommutative, noncocommutative Hopf algebra. Both commutative and cocommutative Hopf algebras are special cases that satisfy properties which are not necessarily true in the general case.

As a result, we would like to know how to calculate the finite dual of a type of Hopf algebra which is very common and produces large classes of examples: a crossed product. In the next chapter, we define the crossed product and give examples.

1.5 Originality

All results stated in this chapter are known.

The definition and basic theory of coalgebras, bialgebras and Hopf algebras described in Section 1.2 is foundational and can be found in more detail in textbooks such as [1], [36] or [48]. The finite-dimensional duality theory along with the notion of a finite dual and its properties presented in Sections 1.3.1 and 1.4 are also developed there, see for instance [36, Section 1.2] for a discussion of finite-dimensional duality and [36, Chapter 9] for one of infinite-dimensional duality or [1, Section 2.3] for both. Condition (ii) of Lemma 1.30 is not discussed there, but is part of a connection between the finite dual H^0 of a Hopf algebra H and its category of finite-dimensional modules which we do not expand on further here but which is discussed in more detail in for instance [25, Section 1.4].

The examples are also known: group algebras, universal enveloping algebras of Lie algebras and coordinate rings of affine algebraic groups are classical examples of Hopf algebras, while the finite-dimensional Taft algebras were introduced by Taft in [49] and extended to infinite dimensions by Lu et al in [32]. Similarly, the finite dual of $U(\mathfrak{g})$ when $\mathfrak{g}$ is semisimple and when it is solvable is due to Hochschild ([20] and [18], [19] respectively), with the latter expanded on by Levasseur ([30]). The fact that $k\,\mathrm{PSL}_2(K)^0$ is trivial is due to Blattner, Cohen and Montgomery ([4]).

Chapter 2

Crossed products

2.1 Introduction

Throughout, k is a field.

Our eventual goal is to describe the finite dual of a crossed product. In this chapter, we define a crossed product, give examples, and describe the particular type of crossed product we will be considering in following chapters.

In general, whenever we study a specific type of structure a standard question to ask is how we can form a "product" of two given structures, incorporating an action of one on the other?

For instance, if we are working with groups, we might have a group N and another group K that acts on N. The well-known construction of the semidirect product of groups uses these to create a group $G := N \rtimes K$ such that:

A. $G \cong N \times K$ as sets,

B. Under the natural embeddings deriving from the isomorphism in A., K is a subgroup of G and N is a normal subgroup,

C. The multiplication in G encodes the action of K on N.

The analogous construction for Hopf algebras is that of a smash product. The smash product is a way of using an algebra A and a Hopf algebra T such that T acts on A (where we leave the details of what we mean by "acts on" aside for the moment) to construct an algebra $B := A\#T$ such that

A. $B \cong A \otimes T$ as vector spaces,

B. Under the natural embeddings deriving from the isomorphism in A., A and T are both subalgebras of B,

C. The multiplication in B encodes the action of T on A.

However, we look at a more general construction: that of the crossed product.

The crossed product is defined much like the smash product, except that there is also a "twisting" of the multiplication in $T \cong 1_k \otimes T \subseteq B$, given by a linear map $\sigma : T \otimes T \to A$. We write this as $B = A\#_\sigma T$, and one of the consequences is that T no longer forms a subalgebra of B in general. In the group setting, this is analogous to the case where we have a normal subgroup N of G but the quotient G/N does not "lift" to a subgroup of G. Indeed, in Example 2.10 we will see that in this case, the group algebra kG is isomorphic to a crossed product of kN by kG/N.

There are many examples of crossed products, and they generalise classic constructions such as skew group algebras (these are smash products where $T = kG$ for some group G) or differential operator rings (these are crossed products where $T = U(\mathfrak{g})$ for some finite-dimensional Lie algebra $\mathfrak{g}$ - see for instance [8, Section 2]).

In Section 2.2, we define crossed products, make note of how smash products are a special case of this construction, and give examples.

An obvious issue that arises is that the definition of a crossed product $B = A\#_\sigma T$ solely describes the algebra structure of B. It makes no reference to any coalgebra structure, and indeed Example 2.14 is an example of a crossed product which is not a Hopf algebra. Since we are solely interested in Hopf algebras, in Section 2.3 we look at a specific situation in which we are guaranteed to have a Hopf structure: namely, at crossed products which arise from surjective Hopf maps. This relates the question of when a given Hopf algebra H can be written as a crossed product $H \cong A\#_\sigma T$ to the question of when factor maps give rise to decompositions. Moreover, Proposition 2.22 tells us that there is a simple condition, involving the existence of a certain kind of map from T to H, which guarantees such a decomposition.

Finally, in Section 2.4, we look at what canonical maps we get on a crossed product of this type, what their properties are and how they connect to properties of the crossed product.

2.2 Definition

Throughout, k is a field.

Crossed products were independently introduced by Blattner, Cohen and Montgomery ([4]) and Doi and Takeuchi ([12]) in 1986 and have been much studied since. There is some ambiguity regarding the exact definition of a crossed product in the literature. Here, we follow the one given in [4]. We discuss the various definitions and our reason for choosing the one we did in Remark 2.7.

In order to define a crossed product, we need a few preliminary definitions.

Definition 2.1. Let T be a Hopf algebra and A an algebra such that T acts linearly on A, i.e. there is a linear map $T \otimes A \to A$ denoted by $t \otimes a \mapsto t \cdot a$. We say that T *measures* A if the following conditions hold.

1. For all $t \in T$ we have
$$t \cdot 1_A = \varepsilon_T(t)1_A.$$
2. Given $t \in T$, $a, b \in A$ we have
$$t \cdot (ab) = \sum (t_1 \cdot a)(t_2 \cdot b).$$

We say T *acts weakly* on A if the following condition also holds:

3. For all $a \in A$,
$$1_T \cdot a = a.$$

Definition 2.2. Let T be a Hopf algebra, A an algebra on which it acts linearly and $\sigma : T \otimes T \to A$ some linear map that is convolution invertible: that is, there is $\sigma^{-1} : T \otimes T \to A$ such that for all $s, t \in T$,

$$\sum \sigma(t_1, s_1)\sigma^{-1}(t_2, s_2) = \varepsilon_T(s)\varepsilon_T(t)1_A.$$

We say A is a *twisted T-module* with respect to σ if

3. For all $a \in A$ we have
$$1_T \cdot a = a.$$
4. Given $s, t \in T$, $a \in A$, the following equation holds:
$$s \cdot (t \cdot a) = \sum \sigma(s_1, t_1)(s_2t_2 \cdot a)\sigma^{-1}(s_3, t_3). \tag{2.1}$$

Remark 2.3. Note that condition 3. of this definition is the same as condition 3. in Definition 2.1.

Definition 2.4. Suppose T is a Hopf algebra, A an algebra, and $\sigma : T \otimes T \to A$ is some convolution invertible linear map. We say σ is a *cocycle* if we have

5. For all $t \in T$ we have
$$\sigma(t, 1_T) = \sigma(1_T, t) = \varepsilon_T(t)1_A.$$

6. For all $s, t, u \in T$ we have
$$\sum (s_1 \cdot \sigma(t_1, u_1))\sigma(s_2, t_2u_2) = \sum \sigma(s_1, t_1)\sigma(s_2t_2, u).$$

We call condition 6. on its own the *cocycle condition.*

This lets us define the crossed product of A and T:

Definition 2.5. Given an algebra A and a Hopf algebra T such that T acts weakly on A, along with a cocycle $\sigma : T \otimes T \to A$ such that A is a twisted T-module with respect to σ, we can define the *crossed product* $A\#_\sigma T$ as follows:

(a) As a vector space, $A\#_\sigma T \cong A \otimes T$, letting $a\#t$ denote the tensor of a and t in $A\#_\sigma T$.

(b) Given $a, b \in A$, $s, t \in T$ we define multiplication by
$$(a\#s)(b\#t) = \sum a(s_1 \cdot b)\sigma(s_2, t_1)\#s_3t_2. \tag{2.2}$$

Crossed products are associative algebras with identity $1_A\#1_T$.

Lemma 2.6. *Suppose A is an algebra, T a Hopf algebra acting weakly on A and σ : $T \otimes T \to A$ a cocycle such that $A\#_\sigma T$ is a crossed product. Then $A\#_\sigma T$ is an associative algebra with identity $1_A\#1_T$.*

Proof. This follows by [4, Lemma 4.4] and [4, Lemma 4.5]. □

Remark 2.7. The notion of a "crossed product", in particular the question of which conditions A, T and σ need to satisfy exactly for a space $A\#_\sigma T$ defined as in Definition 2.5 to be called one, is somewhat ambiguous in the literature. We have required all of conditions 1. through 6. to be satisfied. Doi and Takeuchi ([12]) only require that T measures A (so conditions 1. and 2. are satisfied), and Agore ([3]) as well as Agore and

Militaru ([2]) require that T acts weakly on A and that condition 5. is satisfied, in other words that

$$\sigma(t, 1_T) = \sigma(1_T, t) = \varepsilon_T(t) \qquad \text{for all } t \in T,$$

while Montgomery ([36, Chapter 7]) at first only requires that T measures A (conditions 1. and 2. are satisfied), but then restricts to the case where the multiplication is associative with identity $1_A\#1_T$. The same condition is required by Blattner et al ([4]).

In fact, it is well-known (see for instance [36, Lemma 7.1.2], [4, Lemma 4.4, Lemma 4.5] or [12, Lemma 10]) that for all these varying definitions, a necessary and sufficient condition for the multiplication to be associative with identity $1_A\#1_T$ is for all of conditions 1.-6. to hold. This means that not only is the definition given by Montgomery and Blattner et al equivalent to ours, but so are all the other varying definitions under this extra assumption. Moreover, as we are only interested in crossed products that are associative with identity $1_A\#1_T$ this definition is the obvious one to use.

Remark 2.8. Note that what we have defined are technically left crossed products: an analogous right version of this structure exists, where T acts weakly on A through a right action rather than a left action and the following definitions are adjusted correspondingly. This idea is discussed further in for instance [5, Remark 1.31]. We restrict ourselves to the left crossed products defined above, following the treatment in [36], [4] and [12].

When the cocycle σ is trivial, crossed products are simply smash products:

Definition 2.9. Let A be an algebra, T a Hopf algebra acting weakly on A and $\sigma : T \otimes T \to A$ be trivial, so

$$\sigma(s, t) = \varepsilon_T(s)\varepsilon_T(t) \qquad \text{for all } s, t \in T.$$

Then we call the resulting crossed product a *smash product* and write it as $A\#T$.

This agrees with the standard definition of a smash product as for instance given in [36, Definition 4.1.3].

There are many examples of crossed products. For instance, the following shows how any group algebra kG decomposes as a crossed product with respect to any normal subgroup $N \lhd G$. Note also that this justifies our description of a smash product as analogous to a semidirect product of groups.

Example 2.10. (See [36, Example 7.1.6]) Given a group G with some normal subgroup $N \lhd G$, we have

$$kG \cong kN \#_\sigma k(G/N),$$

where the action and cocycle are defined as follows: we define a map $\gamma : G/N \to G$ by sending a coset x to some representative $\gamma(x)$ with $\gamma(1) = 1$. Now given $x \in G/N, n \in N$ we have

$$x \cdot n = \gamma(x) n \gamma(x)^{-1} \in N$$

and given $x, y \in G/N$,

$$\sigma(x, y) = \gamma(x)\gamma(y)\gamma(xy)^{-1} \in N.$$

Here σ is trivial if and only if we can choose $\gamma : G/N \to G$ to be a group homomorphism, so if and only if G/N embeds as a subgroup in G. This happens precisely when we can write G as a semidirect product $G \cong G/N \ltimes N$.

We can decompose the universal enveloping algebras of Lie algebras in a similar way:

Example 2.11. (See [36, Corollary 7.2.8]) Given a Lie algebra $\mathfrak{g}$ with a Lie ideal $\mathfrak{h}$, then

$$U(\mathfrak{g}) \cong U(\mathfrak{h}) \#_\sigma U(\mathfrak{g}/\mathfrak{h}).$$

Again, the cocycle σ is trivial if there exists a Lie algebra embedding $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$.

Note that crossed product structures need not be unique. It is sometimes possible to describe one algebra as a crossed product in several ways.

Example 2.12. Let k be an algebraically closed field, n, t be integers with $n > 1$ and $0 \leq t \leq n - 1$ and let q be a primitive nth root of unity. Let $H = H(n, t, q)$ be the infinite-dimensional Taft algebra on these parameters, as defined in Example 1.14. So as algebras, we have

$$H(n, t, q) \cong k\langle x, g \mid xg = qgx, g^n = 1\rangle.$$

This is a Hopf algebra, with g grouplike and x $(1, g^t)$-primitive:

$$\Delta(x) = x \otimes 1 + g^t \otimes x.$$

We can find two crossed product structures that exist on H. First, we can write

$$H \cong k[x] \#_\sigma kC_n,$$

where, letting h denote the generating element of C_n, we have $h \cdot x = q^{-1}x$. Here σ is trivial: $\sigma(h^i, h^j) = 1$ for all i, j. Thus H is a smash product.

On the other hand, we can also write

$$H \cong k[x^n]\#_\tau T,$$

where T is the finite-dimensional Taft algebra on (n, t, ξ). Note that x^n commutes with all elements of H and so $k[x^n]$ is central in H. In fact, when $\gcd(n,t) = 1$ it is a Hopf subalgebra of H. This follows because by standard results on skew-commuting variables (see for instance [26, Proposition 2.2]),

$$\Delta_H(x^i) = \sum_{j=0}^{i} \binom{i}{j}_{q^{-t}} x^{i-j}g^{tj} \otimes x^j,$$

where for $\zeta \in k^*$, $n \geq 1, r \geq 0$ integers, $\binom{n}{r}_\zeta$ indicates the quantum binomial defined by

$$\binom{n}{r}_\zeta := \begin{cases} \frac{(1-\zeta^n)(1-\zeta^{n-1})\ldots(1-\zeta^{n-r+1})}{(1-\zeta)(1-\zeta^2)\ldots(1-\zeta^r)} & r \leq n \\ 0 & r > n \end{cases}.$$

By [15, 2.6 (iii)], $\binom{n}{r}_\zeta = 0$ whenever ζ is a primitive nth root of unity and $1 \leq r \leq n-1$. In particular, since q^{-t} is a primitive nth root of unity x^n is primitive.

As $k[x^n]$ is always central, the action is always trivial: we have $\overline{x}^i \cdot x^n = 0 = \varepsilon_T(\overline{x}^i)x^n$ and $\overline{g}^i \cdot x^n = x^n = \varepsilon_T(\overline{g}^i)x^n$. However, the cocycle τ is not trivial: we have

$$\tau(\overline{x}^i\overline{g}^j, \overline{x}^\ell\overline{g}^m) = \begin{cases} x^n & \text{if } i + \ell = n \\ 1 & \text{if } i = \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq i, j, \ell, m < n$.

Recall the notion of a *coordinate ring* from Example 1.12. When k is algebraically closed, we can use crossed products to describe their finite duals.

Proposition 2.13. *Suppose k is algebraically closed. Let G be an affine algebraic k-group and $\mathcal{O}(G)$ its coordinate ring. Then*

$$\mathcal{O}(G)^0 \cong U(\operatorname{Lie} G)\#kG,$$

where $\operatorname{Lie} G$ *denotes the Lie algebra of G. Here* $\operatorname{Lie} G$ *consists of the primitive elements in $\mathcal{O}(G)^0$ and G of the grouplike elements, so the algebra maps $\mathcal{O}(G) \to k$.*

Proof. See [36, Example 9.2.8]. □

This is a corollary of a general theorem regarding the structure of pointed cocommutative Hopf algebras over algebraically closed fields due to Cartier, Gabriel and Kostant, described for instance in [36, Corollary 5.6.4]. Note that the finite dual of $k[x]$, as seen in Example 1.31, is a special case of Proposition 2.13. Moreover, the group $G \subseteq \mathcal{O}(G)^0$, which is given by the algebra maps $\mathcal{O}(G) \to k$, allows us to recover the group G from its coordinate ring.

Note finally that the definition of the crossed product makes no reference to a coalgebra structure on it, and indeed the following example shows that crossed products need not be Hopf algebras.

Example 2.14. Let $\lambda \in k^*$ be such that λ is not a root of unity, and let

$$H := k\langle x^{\pm 1}, y^{\pm 1} \mid xy = \lambda yx \rangle.$$

This is not just a crossed product but a smash product: we have

$$H \cong k[x^{\pm 1}]\#k[y^{\pm 1}]$$

with $y \cdot x = \lambda^{-1}x$. However, by [24, Corollary 1.5], H is a simple ring: its only ideals are $\{0\}$ and H itself. In particular, H cannot be a Hopf algebra with respect to any coalgebra structure because it contains no ideals of codimension one, meaning that there are no algebra maps $H \to k$ and hence no counit map.

As a result, it makes sense to look at a particular situation where we have a guaranteed Hopf structure on H.

2.3 Crossed products arising from Hopf surjections

Throughout, k is a field.

In the previous section, we defined the notion of a crossed product $A\#_\sigma T$ of an algebra A and a Hopf algebra T acting on it with a twisting by a cocycle $\sigma : T \otimes T \to A$ and gave various examples. In particular, we noted that although the conditions we assumed on A, T and σ were sufficient for $A\#_\sigma T$ to be an associative algebra with identity $1_A\#1_T$, there was no need for this algebra to be a Hopf algebra - Example 2.14 being a case where a crossed product was not a Hopf algebra.

In this section, we look at a specific class of crossed products which have a guaranteed Hopf structure: those arising from Hopf surjections.

First recall that, dual to the notion of invariants for an action, given a comodule structure we can define the coinvariants of the coaction.

Definition 2.15. Let H be a Hopf algebra and M a right H-comodule, with the map $\rho : M \to M \otimes H$ describing the coaction. Then the *right coinvariants* for the coaction are given by

$$M^{co\rho} := \{m \in M \mid \rho(m) = m \otimes 1_H\}.$$

We may also write M^{coH} for this set if the coaction ρ is understood.

Analogously, given a left H-comodule N with coaction described by ν, the left coinvariants are defined by

$${}^{coH}N = {}^{co\nu}N := \{n \in N \mid \nu(n) = 1_H \otimes n\}.$$

The idea of coinvariants as dual to invariants is justified, as any H-comodule is also an H^*-module in such a way that the coinvariants of the comodule coaction are the same as the invariants of the module action:

Proposition 2.16. *Let H be a Hopf algebra, M a right H-comodule and $\rho : M \to M \otimes H$ the map describing the coaction, where for $m \in M$ we write*

$$\rho(m) = \sum m_0 \otimes m_1.$$

Define an action of H^* on M by

$$(f \cdot m) = \sum f(m_1)m_0$$

for $m \in M, f \in H^*$. Then

(i) The action defines a left H^*-module structure on M.

(ii) The coinvariants of the H-coaction are equal to the invariants of the H^*-action:

$$M^{coH} = {}^{H^*}M = \{m \in M \mid f \cdot m = \varepsilon_{H^*}(f)m \forall f \in H^*\}.$$

Proof. (i) [36, Lemma 1.6.5 1)], recalling that by Lemma 1.16 H^* is always an algebra.

(ii) [36, Lemma 1.7.2]. □

We find that, as part of a larger, well-known theory on so-called cleft extensions, there is a simple condition for a surjective Hopf map $\pi : H \to T$ to give rise to a decomposition of H as $H \cong H^{co\pi} \#_\sigma T$, where (under minor abuse of notation) we write $H^{co\pi}$ to denote the coinvariants under the canonical coaction of T on H given by π which we describe below. In this case H is always a Hopf algebra by assumption, so we always have a Hopf structure on our crossed product.

First, we record the fact that any Hopf surjection $\pi : H \to T$ determines canonical left and right T-comodule algebra structures on H, recalling that a left (right) *T-comodule algebra* is an algebra which is also a left (right) T-comodule such that the comodule coaction is an algebra map. This is a well-known result, discussed without proof in for instance [36, Section 3.4]. We give a proof here.

Lemma 2.17. *Let H and T be Hopf algebras and $\pi : H \to T$ a surjective Hopf map. Then H is both a right and left T-comodule algebra, where the right comodule coaction $\rho^r : H \to H \otimes T$ is given by $\rho^r := (\mathrm{id}_H \otimes \pi) \circ \Delta_H$, the left one by $\rho^\ell := (\pi \otimes \mathrm{id}_H) \circ \Delta_H$.*

Proof. We give a proof here for the right comodule coaction, setting $\rho := \rho^r$. The left proof is analogous.

ρ defines a comodule coaction:

Recall that $\rho : H \to H \otimes T$ is a comodule coaction if

$$(\rho \otimes \mathrm{id}_T) \circ \rho = (\mathrm{id}_H \otimes \Delta_T) \circ \rho$$

and

$$\mu \circ (\mathrm{id}_H \otimes \varepsilon_T) \circ \rho = \mathrm{id}_H,$$

where $\mu : H \otimes k \to H$ is simply the canonical isomorphism given by scalar multiplication.

We have

$$\begin{aligned}(\rho \otimes \mathrm{id}_T) \circ \rho &= ((\mathrm{id} \otimes \pi) \circ \Delta_H \otimes \mathrm{id}_T) \circ (\mathrm{id}_H \otimes \pi) \circ \Delta_H \\ &= (\mathrm{id}_H \otimes \pi \otimes \pi) \circ (\Delta_H \otimes \mathrm{id}_H) \circ \Delta_H \\ &= (\mathrm{id}_H \otimes \pi \otimes \pi) \circ (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H \\ &= (\mathrm{id}_H \otimes \Delta_T)(\mathrm{id}_H \otimes \pi) \circ \Delta_H \\ &= (\mathrm{id}_H \otimes \Delta_T) \circ \rho,\end{aligned}$$

which gives us the first condition. For the second, note that

$$\begin{aligned}\mu \circ (\mathrm{id}_H \otimes \varepsilon_T) \circ \rho &= \mu \circ (\mathrm{id}_H \otimes \varepsilon_T) \circ (\mathrm{id}_H \otimes \pi) \circ \Delta_H \\ &= \mu \circ (\mathrm{id}_H \otimes \varepsilon_H) \circ \Delta_H \\ &= \mathrm{id}_H\end{aligned}$$

by the counit axiom.

ρ is an algebra map:

This follows immediately from the definition of ρ, as it is given by composition of algebra maps. □

So given a Hopf surjection $\pi : H \to T$ we have not just a T-comodule but in fact a T-comodule algebra structure on H. This in turn means that we can say more about the coinvariants.

Proposition 2.18. *Let $\pi : H \to T$ be a Hopf surjection and let $A' := {}^{co\pi}H$ and $A := H^{co\pi}$ denote the left and right coinvariants of the corresponding coaction respectively. Then*

(i) A and A' are subalgebras of H.

(ii) $S_H(A) \subseteq A'$ and $S_H(A') \subseteq A$, with equality holding if S_H is bijective.

(iii) If $A = A'$, then A is a Hopf subalgebra of H.

Proof. (i) This follows from the fact that the comodule map $\rho := (\mathrm{id} \otimes \pi) \circ \Delta_H$ is an algebra map.

(ii), (iii) See [4, Proposition 4.19]. □

In particular, the fact that the right coinvariants form a subalgebra means that we can meaningfully ask when H decomposes as a crossed product of $H^{co\pi}$ and T. To find the answer, we need to define the notion of a cleaving map.

Definition 2.19. Suppose H and T are Hopf algebras and $\pi : H \to T$ is a surjective Hopf map. Suppose further that $\gamma : T \to H$ is a map of right T-comodules with respect to the right T-comodule structure on H defined in Lemma 2.17. We call γ a *cleaving map* if γ is convolution invertible, recalling from Remark 1.17 that this means that there exists another linear map $\gamma' : T \to H$ such that

$$\sum \gamma(t_1)\gamma'(t_2) = \sum \gamma'(t_1)\gamma(t_2) = \varepsilon_T(t)1_H$$

for all $t \in T$.

One example of a cleaving map is a coalgebra splitting. In fact, any coalgebra map $\gamma : T \to H$ will be a cleaving map precisely when it splits π.

Lemma 2.20. *Suppose H and T are Hopf algebras and $\pi : H \to T$ is a surjective map of Hopf algebras, and suppose further there is a coalgebra map $\gamma : T \to H$. Then γ is a cleaving map if and only if $\pi \circ \gamma = \mathrm{id}_T$. If this holds, the convolution inverse is given by $\gamma^{-1} := S_H \circ \gamma$.*

Proof. $\Rightarrow$: Suppose γ is a convolution invertible right T-comodule map. Then we have

$$(\gamma \otimes \pi \circ \gamma) \circ \Delta_T = (\mathrm{id}_H \otimes \pi) \circ \Delta_H \circ \gamma = (\gamma \otimes \mathrm{id}_T) \circ \Delta_T, \tag{2.3}$$

using the fact that γ preserves the coproduct and that it is a right comodule map. Now we apply $\mu \circ (\varepsilon_H \otimes \mathrm{id}_T)$ to both sides of this equation, where μ denotes the canonical isomorphism $k \otimes T \cong T$ given by scalar multiplication. The RHS of (2.3) becomes

$$\mu \circ ((\varepsilon_H \circ \gamma) \otimes \mathrm{id}_T) \circ \Delta_T = \mu \circ (\varepsilon_T \otimes \mathrm{id}_T) \circ \Delta_T = \mathrm{id}_T,$$

using the counit axiom and the fact that as γ is a coalgebra map, $\varepsilon_H \circ \gamma = \varepsilon_T$. Similarly, the LHS of (2.3) becomes

$$\mu \circ (\varepsilon_H \circ \gamma \otimes \pi \circ \gamma) \circ \Delta_T = \pi \circ \gamma \circ (\mu \circ (\varepsilon_T \otimes \mathrm{id}_T) \circ \Delta_T) = \pi \circ \gamma,$$

using the same. So we get

$$\pi \circ \gamma = \mathrm{id}_T$$

as required.

$\Leftarrow$: Suppose γ satisfies $\pi \circ \gamma = \mathrm{id}_T$.

First we note that γ is a map of right T-comodules under the right T-comodule coaction given in 2.17: we have

$$(\mathrm{id} \otimes \pi) \circ \Delta_H \circ \gamma = (\gamma \otimes (\pi \circ \gamma)) \circ \Delta_T = (\gamma \otimes \mathrm{id}_T) \circ \Delta_T$$

as required. So we only need to show that γ is convolution invertible with convolution inverse as described.

For $t \in T$, we have

$$\begin{aligned} \sum \gamma(t_1)\gamma^{-1}(t_2) &= \sum \gamma(t_1) S_H(\gamma(t_2)) \\ &= \sum \gamma(t)_1 S_H(\gamma(t)_2) \\ &= \varepsilon_H(\gamma(t))1_H = \varepsilon_T(t)1_H, \end{aligned}$$

so γ^{-1} is a right convolution inverse for γ. Showing it is a left inverse proceeds analogously. □

We can assume without loss of generality that any cleaving map fixes the identity.

Lemma 2.21. *Suppose H and T are Hopf algebras and $\pi : H \to T$ a Hopf surjection such that there exists a cleaving map $\gamma : T \to H$. Then there exists a cleaving map $\hat{\gamma} : T \to H$ such that*

$$\hat{\gamma}(1_T) = 1_H.$$

Proof. First note that γ is convolution invertible with inverse γ^{-1} and 1_T is a grouplike element of T, and so we have

$$\gamma(1_T)\gamma^{-1}(1_T) = 1_H$$

from the axioms for a convolution inverse. In particular, $\gamma(1_T)$ is a unit in H with inverse $\gamma(1_T)^{-1} = \gamma^{-1}(1_T)$.

Now define $\hat{\gamma} : T \to H$ by $\hat{\gamma}(t) := \gamma(1_T)^{-1}\gamma(t)$. This satisfies $\hat{\gamma}(1_T) = 1_H$ because $\gamma^{-1}(1_T)\gamma(1_T) = 1_H$ by the convolution invertibility axioms. We need to show that $\hat{\gamma}$ is a cleaving map, so a convolution invertible right T-comodule map.

Step 1: $\gamma(1_T)^{-1} \in H^{coT}$.

Recall from Definition 2.15 that we need to show that

$$((\mathrm{id} \otimes \pi) \circ \Delta_H) \left(\gamma(1_T)^{-1}\right) = \gamma(1_T)^{-1} \otimes 1_H.$$

We know that γ is a right T-comodule map and so

$$\gamma(1_T) \otimes 1_T = ((\gamma \otimes \mathrm{id}_T) \circ \Delta_T)(1_T) = \rho(\gamma(1_T)).$$

This means that $\gamma(1_T) \in H^{coT}$.

Now, since ρ is an algebra map we have

$$(\gamma(1_T) \otimes 1_T)\rho(\gamma(1_T)^{-1}) = \rho(\gamma(1_T))\rho(\gamma(1_T)^{-1}) = \rho(\gamma(1_T)\gamma(1_T)^{-1}) = 1_H \otimes 1_T.$$

Since multiplication in $H \otimes T$ is pointwise on simple tensors, this means that

$$\rho(\gamma(1_T)^{-1}) = \gamma(1_T)^{-1} \otimes 1_T.$$

Therefore $\gamma(1_T)^{-1} \in H^{coT}$ as well.

Step 2: $\hat{\gamma}$ is a right T-comodule map:

Let $t \in T$. We have

$$\begin{aligned}\rho(\hat{\gamma}(t)) &= \rho(\gamma(1_T)^{-1}\gamma(t)) \\ &= \rho(\gamma(1_T)^{-1})\rho(\gamma(t)) \\ &= (\gamma(1_T)^{-1} \otimes 1_T)\left(\sum \gamma(t_1) \otimes t_2\right) \\ &= \sum \gamma(1_T)^{-1}\gamma(t_1) \otimes t_2 \\ &= \sum \hat{\gamma}(t_1) \otimes t_2\end{aligned}$$

as required.

Step 3: $\hat{\gamma}$ is convolution invertible:

Let $\gamma' : T \to H$ be defined by $\gamma'(t) := \gamma^{-1}(t)\gamma(1_T)$ for $t \in T$. So we have

$$\begin{aligned}\sum \hat{\gamma}(t_1)\gamma'(t_2) &= \sum \gamma^{-1}(1_T)\gamma(t_1)\gamma^{-1}(t_2)\gamma(1_T) \\ &= \gamma^{-1}(1_T)\varepsilon_T(t)1_H\gamma(1_T) \\ &= \varepsilon_T(t)1_H.\end{aligned}$$

So γ' is the convolution inverse of $\hat{\gamma}$ as required. □

From now on, whenever we talk about a cleaving map, we will assume it fixes the identity.

Proposition 2.22. *Suppose H and T are Hopf algebras and $\pi : H \to T$ is a Hopf surjection. Suppose further that there exists a cleaving map $\gamma : T \to H$, with γ^{-1} denoting the convolution inverse. Then the map*

$$\phi : A\#_\sigma T \to H \tag{2.4}$$

given by $\phi(a\#t) = a\gamma(t)$ is an algebra, left A-module and right T-comodule isomorphism. Here $A := H^{coT}$, so is given by the coinvariants as defined in Definition 2.15, the weak action of T on A is given by

$$t \cdot a = \sum \gamma(t_1)a\gamma^{-1}(t_2) \quad \textit{for } a \in A, t \in T,$$

the cocycle σ is given by

$$\sigma(s,t) = \sum \gamma(s_1)\gamma(t_1)\gamma^{-1}(s_2t_2) \quad \textit{for } s,t \in T,$$

and the right T-comodule structure on $A\#_\sigma T$ is given by $a\#t \mapsto a\#t_1 \otimes t_2$ for $a \in A$, $t \in T$.

Proof. First note that all statements of the result follow from [12, Theorem 11] as well as [36, Proposition 7.2.3] if H is a T-comodule algebra such that A is given by the coinvariants of the comodule coaction. Now by Lemma 2.17, H is a T-comodule algebra with comodule coaction given by $\rho := (\mathrm{id}_H \otimes \pi) \circ \Delta_H : H \to H \otimes T$. The fact that A consists of the coinvariants of this action is obvious from the definition. □

The converse of this result is true as well, in that any crossed product has a cleaving map.

Proposition 2.23. *Suppose that A is an algebra and T a Hopf algebra such that there exists some cocycle σ and action of T on A giving us a crossed product $A\#_\sigma T$. Then the map $\gamma : T \to A\#_\sigma T$ given by $t \mapsto 1\#t$ for $t \in T$ is a convolution invertible right T-comodule map with respect to the comodule structure given by $a\#t \mapsto a\#t_1 \otimes t_2$ for $a \in A, t \in T$. The convolution inverse for γ is given by*

$$\gamma^{-1}(t) = \sum \sigma^{-1}(S_T(t_2), t_3)\#S_T(t_1).$$

Proof. See for instance [5, Lemma 1.5, Proposition 1.8] or [36, Proposition 7.5.7]. □

Remark 2.24. We frequently write $a\gamma(t)$ for elements of $A\#_\sigma T$ rather than $a\#t$, a notation which is justified as this is simply the image of $a\#t$ in H under the algebra isomorphism described in Proposition 2.22. We do not write at when T is not necessarily a subalgebra of H as this notation would be misleading.

Note that not all Hopf surjections give rise to a crossed product decomposition or equivalently a cleaving map.

Example 2.25. Let k be an algebraically closed field of characteristic zero, and let

$$H := \mathcal{O}(\mathrm{SL}_2(k)) \cong k[a, b, c, d \mid ad - bc = 1].$$

There is a subgroup Z of $SL_2(k)$ given by the diagonal matrices:

$$Z = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in k^* \right\},$$

so Z is isomorphic to k^* as an algebraic group and we have $T := \mathcal{O}(Z) \cong k[t^{\pm 1}]$. So there is an algebraic group embedding $k^* \to \mathrm{SL}_2(k)$ given by sending $\lambda \in k^*$ to the diagonal matrix with diagonal entries (λ, λ^{-1}). By the equivalence of categories between algebraic groups and commutative Hopf algebras discussed in Remark 1.13, this gives rise to a Hopf surjection $\pi : H \to k[t^{\pm 1}]$. This map is given by $\pi(b) = \pi(c) = 0$, $\pi(a) = t$ and $\pi(d) = t^{-1}$.

However, π does not give rise to a crossed product structure on H. We can see this via looking at the units of H:

Suppose for contradiction that $H \cong A\#_\sigma k[t^{\pm 1}]$ for some $A \subseteq H$ and cocycle σ. By Proposition 2.22, this means the natural inclusion map $\gamma : k[t^{\pm 1}] \to H$ is a cleaving map.

Since σ is convolution invertible by Definitions 2.5 and 2.2, there exists a map $\sigma^{-1} : T \otimes T \to A$ such that for all $s, u \in T$ we have

$$\sum \sigma(s_1, u_1)\sigma^{-1}(s_2, u_2) = \varepsilon_T(s)\varepsilon_T(u)1_A.$$

Since T is isomorphic to a group algebra with each t^i being grouplike, this equation becomes

$$\sigma(t^i, t^j)\sigma^{-1}(t^i, t^j) = 1_A \quad \text{for any } i, j \in \mathbb{Z}. \tag{2.5}$$

Let $i = 1, j = -1$. Equation (2.5) tells us that $\sigma(t, t^{-1})$ is a unit in H. Now there are two possibilities: either $\sigma(t, t^{-1}) = \lambda 1_H$ for some $\lambda \in k^*$ or no such λ exists. If the former holds, then by the crossed product axioms, we have

$$\gamma(t)\gamma(t^{-1}) = \sigma(t, t^{-1})\gamma(tt^{-1}) = \lambda 1_A \gamma(1_T) = \lambda 1_H.$$

In particular, this means that $\gamma(t) = \left(\lambda^{-1}\gamma(t^{-1})\right)^{-1}$, and thus $\gamma(t)$ is also a unit in H. It is nontrivial because γ is the inclusion map and hence injective, and so $\gamma(t)$ is not a scalar multiple of $1_H = \gamma(1_T)$. In either case, H contains nontrivial units, meaning units which are not scalar multiples of 1_H.

Now note that by [27, Proposition 1.2], all the units in H are given by scalar multiples of characters on $\mathrm{SL}_2(k)$. Since the group multiplication is reflected in the coalgebra structure, this means the only units in H are scalar multiples of grouplike elements. Identifying H with $U(\mathfrak{sl}_2(k))^0$, as in Example 1.27, this means any unit in H corresponds to an algebra map $U(\mathfrak{sl}_2(k)) \to k$. However, any algebra map must be zero on the ideal generated by the derived subalgebra of $\mathfrak{sl}_2(k)$, which is simply the augmentation ideal $U(\mathfrak{sl}_2(k))^+$ as $\mathfrak{sl}_2(k)$ is semisimple. So the only such map is the counit $\varepsilon_U = 1_H$, the only units in H are scalar multiples of the identity, and the above calculation is a contradiction.

We conclude that there is no cleaving map $\gamma : T \to H$ and hence no crossed product decomposition of H coming from π.

There has been work done on what conditions on the Hopf algebras H, T or factor map $\pi : H \to T$ guarantee the existence of a cleaving map and hence a crossed product decomposition of H. One result in particular we will make use of later follows.

Recall that the *coradical* H_0 of a Hopf algebra H is the sum of its simple subcoalgebras.

Proposition 2.26. *Let H and T be Hopf algebras and $\pi : H \to T$ a surjective map of Hopf algebras. Suppose also that*

(i) H is an injective right T-comodule under the canonical coaction, and

(ii) There is a coalgebra map $f : T_0 \to H$ such that $\pi \circ f : T_0 \to T$ is simply the inclusion map.

Then there exists a cleaving map $\gamma : T \to H$.

Proof. This is due to [42, Theorem 4.2]. □

This result has a lot of useful corollaries in the case where the factor map π is determined by a central Hopf subalgebra A of H. As this is the situation we study in Chapter 5, we will return to it there.

We note here that in other work looking at Hopf algebras which are crossed products as algebras, it is frequently assumed that their coalgebra structure is trivial. By this we mean that both A and T are subcoalgebras of H, giving us $H \cong A \otimes T$ as coalgebras. This is something we do not assume, nor does it follow from the crossed product structure coming from a factor Hopf map. Indeed, the following example shows that neither A nor T need be subcoalgebras of H in general.

Example 2.27. Let k be a field of characteristic zero and $G = T(4, k)$ be the second Heisenberg group. This means that G consists of the subgroup of $\mathrm{SL}_4(k)$ given by upper triangular matrices with all diagonal entries being 1_k. As algebraic varieties, $G \cong k^6$, so we have

$$H := \mathcal{O}(G) \cong k[X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}]$$

as algebras, where each X_{ij} denotes the function sending a matrix to its (i, j)th entry. Since the coproduct comes from matrix multiplication, we find that X_{12}, X_{23} and X_{34} are primitive and

$$\Delta(X_{13}) = X_{13} \otimes 1_H + 1_H \otimes X_{13} + X_{12} \otimes X_{23},$$
$$\Delta(X_{24}) = X_{24} \otimes 1_H + 1_H \otimes X_{24} + X_{23} \otimes X_{34} \quad \text{and}$$
$$\Delta(X_{14}) = X_{14} \otimes 1_H + 1_H \otimes X_{14} + X_{12} \otimes X_{24} + X_{13} \otimes X_{34}.$$

Consider the subset $D \subseteq G$ given by

$$D := \{(a_{ij}) \in G \mid a_{14} = a_{23} = a_{24} = a_{34} = 0\} = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a_{12}, a_{13} \in k \right\}.$$

This is in fact a subgroup of $T(4,k)$ and satisfies $D \cong (k,+)^2$ as algebraic groups, so by the equivalence of categories between affine algebraic groups and commutative Hopf algebras discussed in Remark 1.13 it gives rise to a Hopf surjection $\pi : H \to \mathcal{O}(D) \cong k[\overline{X_{12}}, \overline{X_{13}}]$. The map π is given by factoring along the Hopf ideal $\langle X_{14}, X_{23}, X_{24}, X_{34} \rangle$. We want to show that this map gives rise to a crossed product decomposition of H.

Let $\gamma : T \to H$ denote the algebra map given by

$$\gamma(\overline{X_{12}}^i \overline{X_{13}}^j) = X_{12}^i X_{13}^j.$$

This is a right T-comodule map: on the generators of T, we can see this because both $\overline{X_{12}}$ and X_{12} are primitive and

$$((\mathrm{id}_H \otimes \pi) \circ \Delta_H)(X_{13}) = X_{13} \otimes 1_T + 1_H \otimes \overline{X_{13}} = (\gamma \otimes \mathrm{id}_T) \circ \Delta_T(\overline{X_{13}}).$$

Now, because both $(\mathrm{id}_H \otimes \pi) \circ \Delta_H \circ \gamma$ and $(\gamma \otimes \mathrm{id}_T) \circ \Delta_T$ are compositions of algebra maps and hence themselves algebra maps, the fact that they are equal on the generators means they are equal on the whole of T.

Moreover, γ is convolution invertible, as the algebra map γ' given by $\gamma'(\overline{X_{ij}}) = -X_{ij}$ is a convolution inverse for γ. So by Proposition 2.22,

$$H \cong H^{co\pi} \#_\sigma k[\overline{X_{12}}, \overline{X_{13}}]. \tag{2.6}$$

Now we want to show that the right coinvariants are simply given by the subalgebra $A := k[X_{14}, X_{23}, X_{24}, X_{34}]$. We know that $A \subseteq H^{co\pi}$ because $X_{14}, X_{23}, X_{24}, X_{34}$ are all coinvariants and so by Proposition 2.18(i) the subalgebra they generate must be contained in H^{coT} as well.

To see that $H^{coT} \subseteq A$, recall that by Proposition 2.16, $H^{coT} = H^{T*}$, where the action of T^* on H is given by

$$f \cdot h = \sum f(\pi(h_2))h_1 \quad \text{for } f \in T^*, h \in H.$$

Assume f is grouplike. Now note that we can view $h \in H$ as a polynomial function on $T(4,k)$ whose comultiplication is defined by the group multiplication, and that there is a correspondence between grouplike elements in T^* and the algebraic group D. Taking this approach, we can view the action of T^* on H as follows: $g \in T(4,k)$ and identifying f with the element of D it corresponds to, we have

$$(f \cdot h)(g) = h(gf).$$

Assume that h is an invariant of this action. This means that $h(gf) = h(f)$ for all $g \in T(4,k)$, and particularly that $h(f) = h(1)$ for all $f \in D$. The only elements of H for which this is true are those contained in A.

Finally, the isomorphism between H and the crossed product is simply the obvious one which sends $\overline{X_{ij}}$ to X_{ij}. So we find that Equation (2.6) is simply the canonical decomposition

$$H \cong A \otimes T = k[X_{14}, X_{23}, X_{24}, X_{34}] \otimes k[X_{12}, X_{13}]$$

as algebras.

Despite the fact that the crossed product structure is effectively trivial, neither A nor T are subcoalgebras of H: we can see this because

$$\Delta(X_{13}) = X_{13} \otimes 1_H + 1_H \otimes X_{13} + X_{12} \otimes X_{23} \notin T \otimes T$$

and similarly $\Delta(X_{14}) \notin k[X_{14}, X_{23}, X_{24}, X_{34}]^{\otimes 2}$. In the world of algebraic groups, T not being a subcoalgebra corresponds to the fact that although D is a subgroup of G, it is not a normal one.

Note also that the cleaving map $\gamma : T \to H$ is a right T-comodule map but not a left one:

$$\rho^\ell(\gamma(\overline{X_{13}})) = \overline{X_{13}} \otimes 1_H + 1_T \otimes X_{13} + \overline{X_{12}} \otimes X_{23} \neq (\gamma \otimes \mathrm{id}_T)(\Delta_T(\overline{X_{13}})).$$

A consequence of this is that although all coalgebra splittings of π are cleaving maps (Lemma 2.20), not all cleaving maps are coalgebra maps - as any such coalgebra map is automatically both a left and right comodule map.

Remark 2.28. We will see another example of a crossed product with a cleaving map that is not a coalgebra map in Section 6.5. In fact, there we will see that for the family of Hopf algebras known as the generalised Liu algebras defined there, no cleaving map which is a coalgebra map can exist. Unlike the example of $\mathcal{O}(T(4,k))$, these examples are pointed, meaning that all simple subcoalgebras are one-dimensional.

2.4 Canonical maps on H, A and T

Throughout, k is a field.

Suppose that we are in the situation we described in Section 2.3: we have Hopf algebras H and T along with a factor Hopf map $\pi : H \to T$ such that there is a cleaving map $\gamma : T \to H$. In other words, by Proposition 2.22 we have

$$H \cong A\#_\sigma T$$

for some action of T on A and cocycle σ.

Our approach in the next several chapters is to use canonical maps on $H \cong A\#_\sigma T$ and their behaviour in the dual setting to investigate the structure of H^0. In this section, we specify which maps we are interested in, and also introduce an extra assumption we will need to make.

We already know that $\pi : H \to T$ is a Hopf surjection by assumption. We can also find a surjective left A-module map $\Pi : H \to A$ given by factoring along $A\gamma(T^+)$.

Lemma 2.29. *Suppose H and T are Hopf algebras and $\pi : H \to T$ a Hopf factor map such that there is a cleaving map $\gamma : T \to H$. Let $A := H^{coT}$. Then the map $\Pi : H \to A$ given by factoring along the left A-submodule $A\gamma(T^+)$ of H is a surjective map of left A-modules satisfying $\Pi|_A = \mathrm{id}_A$, given by*

$$\Pi(a\gamma(t)) = a\varepsilon_T(t) \qquad \text{for } a \in A, t \in T.$$

Proof. By Proposition 2.22, $H \cong A\#_\sigma T$ for some cocycle σ and action of T on A, with the isomorphism given by $a\#t \mapsto a\gamma(t)$. So since $T \cong k1_T \oplus T^+$, as left A-modules there is a decomposition

$$H \cong A\gamma(T) \cong A\gamma(1_T) \oplus A\gamma(T^+).$$

Since by Lemma 2.21 we can assume that $\gamma(1_T) = 1_H = 1_A$, this turns into

$$H \cong A \oplus A\gamma(T^+).$$

In this decomposition Π is simply the canonical projection map to A, and so is a surjective left A-module map satisfying $\operatorname{im}\Pi = A$ and $\Pi|_A = \mathrm{id}_A$.

The fact that Π is of the form described follows immediately from the definition. □

We know that the cleaving map $\gamma : T \to H$ is a right T-comodule map, although Example 2.27 tells us it need not be a left one in general and hence also need not be a coalgebra map. The map Π lets us see that γ is an algebra map if and only if H is a smash product.

Lemma 2.30. *Given a crossed product $H \cong A\#_\sigma T$, the cleaving map γ is an algebra map if and only if σ is trivial, so if and only if the product is a smash product.*

Proof. First note that by definition of multiplication in a crossed product (defined in Equation (2.2)) and using the identification of $a\#t$ with $a\gamma(t)$ described in Remark 2.24, for all $s, t \in T$ we have

$$\gamma(s)\gamma(t) = \sum \sigma(s_1, t_1)\gamma(s_2t_2). \tag{2.7}$$

$\Leftarrow$: Suppose σ is trivial. This means that (2.7) turns into

$$\gamma(s)\gamma(t) = \sum \sigma(s_1, t_1)\gamma(s_2t_2) = \sum \varepsilon_T(s_1)\varepsilon_T(t_1)\gamma(s_2t_2) = \gamma(st),$$

and so γ preserves multiplication. Since by Lemma 2.21 we can assume without loss of generality that $\gamma(1_T) = 1_H$, this means that γ is an algebra map.

$\Rightarrow$: Suppose γ is an algebra map. Now (2.7) becomes

$$\gamma(st) = \gamma(s)\gamma(t) = \sum \sigma(s_1, t_1)\gamma(s_2t_2),$$

and by the counit axiom this means

$$\sum \varepsilon_T(s_1)\varepsilon_T(t_1)\gamma(s_2t_2) = \sum \sigma(s_1, t_1)\gamma(s_2t_2). \tag{2.8}$$

By Lemma 2.29, there is a surjective map $\Pi : H \to A$ given by factoring along $A\gamma(T^+)$. Explicitly, given $a \in A$ and $t \in T$ we have $\Pi(a\gamma(t)) = \varepsilon_T(t)a$. We now apply Π to both sides of (2.8). Using the counit axiom the equation becomes

$$\sigma(s, t) = \varepsilon_T(s)\varepsilon_T(t)1_A,$$

which is exactly what we want. □

We have already seen in Example 2.27 that even when Lemma 2.30 holds, neither A nor T need be Hopf subalgebras of H.

Note that Π itself is also not an algebra map in general. In fact, a necessary and sufficient condition for it to be one is for the crossed product structure to be trivial. By this we mean that both σ and the action of T on A are trivial, and thus $H \cong A \otimes T$ as algebras:

Lemma 2.31. *Suppose $\pi : H \to T$ is some Hopf factor map and $\gamma : T \to H$ a cleaving map. Let $A := H^{coT}$ and $\Pi : H \to A$ be the map given by factoring along $A\gamma(T^+)$. Then the following are equivalent:*

(i) Π *is an algebra map.*

(ii) Both σ and the action of T on A are trivial: we have

$$\sigma(s,t) = \varepsilon_T(s)\varepsilon_T(t) \quad \textit{and} \quad t \cdot a = \varepsilon_T(t)a$$

for all $s, t \in T$, $a \in A$.

(iii) $H \cong A \otimes T$ as algebras, where the isomorphism is given by

$$a \otimes t \mapsto a\gamma(t).$$

Proof. (i) $\Rightarrow$ (ii):

Suppose Π is an algebra map, and recall that on elements we have

$$\Pi(a\gamma(t)) = \varepsilon_T(t)a.$$

First we show that σ is trivial:

Let $s, t \in T$. By definition of the multiplication in a crossed product (Equation (2.2)), we have

$$\gamma(s)\gamma(t) = \sum \sigma(s_1, t_1)\gamma(s_2t_2) \in H.$$

Now consider Π applied to this. The RHS gives us

$$\Pi\left(\sum \sigma(s_1, t_1)\gamma(s_2t_2)\right) = \sum \varepsilon_T(s_2t_2)\sigma(s_1, t_1) = \sigma(s, t).$$

On the other hand, applying Π to the LHS gives us

$$\Pi(\gamma(s)\gamma(t)) = \Pi(\gamma(s))\Pi(\gamma(t)) = 1_A\varepsilon_T(s)\varepsilon_T(t),$$

using the fact that Π is an algebra map by assumption. Putting both of these results together, we have

$$\sigma(s,t) = \varepsilon_T(s)\varepsilon_T(t)1_A$$

for any $s, t \in T$. So σ is trivial.

Now we want to show the action of T on A is trivial. Let $t \in T, a \in A$. Again by the definition of multiplication in a crossed product, we have $\gamma(t)a = (t_1 \cdot a)\gamma(t_2)$. Applying Π to the LHS just gives us

$$\Pi(\gamma(t)a) = \Pi(\gamma(t))\Pi(a) = \varepsilon_T(t)a.$$

Applying Π to the RHS gives us

$$\Pi((t_1 \cdot a)\gamma(t_2)) = \varepsilon_T(t_2)(t_1 \cdot a) = t \cdot a.$$

Combining the two tells us that $t \cdot a = \varepsilon_T(t)a$. So the action is trivial as required.

(ii) $\Rightarrow$ (iii):

By Equation (2.2), given $a, b \in A$, $s, t \in T$ we have

$$a\gamma(s)b\gamma(t) = \sum a(s_1 \cdot b)\sigma(s_2, t_1)\gamma(s_3 t_2).$$

Now, because both action and σ are trivial, this becomes

$$a\gamma(s)b\gamma(t) = \sum a\varepsilon_T(s_1)b\varepsilon_T(s_2)\varepsilon_T(t_1)\gamma(s_3 t_2),$$

which by the counit axiom simply gives us

$$a\gamma(s)b\gamma(t) = ab\gamma(st).$$

Letting ϕ denote the canonical vector space isomorphism given by $a \otimes t \mapsto a\gamma(t)$ from Proposition 2.22, this means that we have

$$\phi(a \otimes s)\phi(b \otimes t) = a\gamma(s)b\gamma(t) = ab\gamma(st) = \phi(ab \otimes st) = \phi((a \otimes s)(b \otimes t).$$

So ϕ is in fact an algebra isomorphism as required.

(iii) $\Rightarrow$ (i):

Suppose that $H \cong A \otimes T$ as algebras, where the isomorphism is given by $a \otimes t \mapsto a\gamma(t)$. This means that γ is an algebra map and that A and $\gamma(T)$ commute.

This means that

$$HA\gamma(T^+) = A\gamma(T)A\gamma(T^+) \subseteq AA\gamma(T)\gamma(T^+) \subseteq A\gamma(TT^+) = A\gamma(T^+),$$

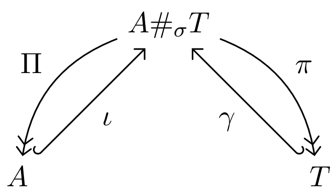
meaning that $A\gamma(T^+)$ is a left ideal. A similar argument gives $A\gamma(T^+)H \subseteq A\gamma(T^+)$, which means $A\gamma(T^+)$ is a two-sided ideal. Now since the kernel of Π is an ideal, it is an algebra map. $\square$

Similarly, Π is not a coalgebra map in general, as we can see in the following example.

Example 2.32. Let $H = \mathcal{O}(T, 3, k)$. We have $H \cong k[x, y, z]$ as algebras, where x and y are primitive and $\Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y$. This decomposes as a crossed product $H \cong k[x, y] \otimes k[\overline{z}]$ given by the Hopf surjection $\pi : H \to H/\langle x, y\rangle \cong k[\overline{z}]$. Here $k[x, y]$ is a Hopf subalgebra, so the map $\iota : k[x, y] \to H$ is a map of Hopf algebras. The cleaving map $\gamma : k[\overline{z}] \to H$ is given by $\overline{z}^i \mapsto z^i$. It is an algebra map and a map of both left and right $k[\overline{z}]$-comodules, but is not a coalgebra map. Finally, the factor map $\Pi : H \to k[x, y]$ is given by the ideal $\langle z\rangle$. It is an algebra map, but is not a coalgebra map since $\Pi(z) = 0$ but $(\Pi \otimes \Pi)(\Delta_H(z)) = x \otimes y \neq 0$.

The last map that we will be working with is the canonical embedding map $\iota : A \to H$. By Proposition 2.18(i), A is always a subalgebra of H and so ι is an algebra map. We know ι is not always a coalgebra map because it is not always the case that the coinvariants form a subbialgebra: for instance, in Example 2.27 they do not.

So in summary, given a crossed product $H \cong A\#_\sigma T$ which arises from a factor Hopf map $\pi : H \to T$, we have



Here π is a surjective Hopf map, ι is an injective algebra map, γ is an injective right T-comodule map that is an algebra map if and only if the cocycle σ is trivial, and $\Pi : H \to A$ a surjective left A-module map satisfying $\Pi \circ \iota = \mathrm{id}_A$ which is an algebra map if and only if both σ and the action of T on A are trivial.

2.5 Originality

Crossed products as defined in Section 2.2 were independently introduced in [4] and [12] and have been much studied since - an overview is available in [36, Chapter 7]. The connection between the existence of a cleaving map $\gamma : T \to H$ and a crossed product decomposition $H \cong H^{coT} \#_\sigma T$ discussed in Section 2.3 is studied in greater generality in the latter two, while [4] restricts to the case where γ is a coalgebra splitting. None of the results in these sections are original.

In Section 2.4, Lemma 2.30 is common knowledge but not generally stated in this form, while Lemmas 2.29 and 2.31 are new as the quotient map $\Pi : H \to A$ which these refer to is not generally studied.

Chapter 3

Duality of functions

3.1 Introduction

Our main purpose in this chapter is to prove Theorem 3.12, and to set it in the appropriate context. Throughout, k is a field.

In order to compute finite duals, we pursue an approach that is inspired by category theory: we study objects via studying the functions on them. If we can say something about the (Hopf) subalgebras or factor (Hopf) algebras of H^0, that tells us something about its structure. This is particularly relevant to understanding the duals of crossed products. As we saw in Section 2.4, a Hopf algebra that is a crossed product has certain canonical maps on it, and moreover the existence of certain maps can guarantee a crossed product structure on a given Hopf algebra. As a result, working out how these canonical maps translate into the dual setting brings us closer to our ultimate aim: describing the finite dual of certain crossed products as crossed products themselves.

As before, we try to generalise from the finite dimensional case. Here it is clear:

Proposition 3.1. *Let B and C be finite-dimensional algebras and $\phi : B \to C$ some linear map between them.*

(i) There is a linear map $\phi^* : C^* \to B^*$ given by composition with ϕ, and given any finite-dimensional algebra D and any linear map $\psi : C \to D$ we have

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

(ii) If ϕ is injective, ϕ^* is surjective.

(iii) If ϕ is surjective, ϕ^* is injective.

(iv) If ϕ is an algebra map, ϕ^* is a coalgebra map.

(v) Suppose B and C are both bialgebras. Then if ϕ is a coalgebra map, ϕ^* is an algebra map.

(vi) Suppose B and C are both Hopf algebras. Then if ϕ is a Hopf algebra map, so is ϕ^*.

These results are well-known. We do not give a reference here because we will see in Section 3.4 that this follows as an immediate corollary of Theorem 3.12.

Remark 3.2. Looking at this in category theoretic terms, what (i)-(iii) are saying is that $(-)^*$ is an exact contravariant functor from the category of finite-dimensional k-algebras to that of finite-dimensional k-coalgebras, both with linear maps as morphisms, and (iv)-(vi) follow from the functor preserving the monoidal structure. Although this is a useful perspective, we do not approach the problem this way. The reason is because we find on trying to generalise to the infinite-dimensional case that the question of what category we should be working in, and in particular of what the morphisms should be, becomes rather complicated.

We want to know how Proposition 3.1 generalises to the infinite-dimensional case. It is tempting to simply generalise this result via replacing vector space duals by finite duals and restricting maps appropriately:

Naive Conjecture. *Let B and C be algebras and $\phi : B \to C$ some linear map between them. Then*

(i) There is a linear map $\phi^0 : C^0 \to B^0$ given by composition, and given an algebra D and a linear map $\psi : C \to D$ we have

$$(\psi \circ \phi)^0 = \phi^0 \circ \psi^0.$$

(ii) If ϕ is injective, ϕ^0 is surjective.

(iii) If ϕ is surjective, ϕ^0 is injective.

(iv) If ϕ is an algebra map, ϕ^0 is a coalgebra map.

(v) If B and C are both bialgebras and ϕ is a coalgebra map, then ϕ^0 is an algebra map.

(vi) If B and C are both Hopf algebras and ϕ is a Hopf algebra map, so is ϕ^0.

The next two sections show how this fails in two ways. In Section 3.2, we show that (i) does not hold in general via looking at an example of a given B, C and map $\phi : B \to C$ where the image of the resulting map ϕ^0 cannot lie in B^0. Thus, in this case ϕ cannot give rise to a well-defined map $\phi^0 : C^0 \to B^0$. We then introduce a technical condition we call being a *finite overlay*. In Lemma 3.5, we prove that ϕ being a finite overlay is a necessary and sufficient condition for $\operatorname{im}\phi^0$ to be contained in B^0. Section 3.3 does the same with condition (ii): we show that given a finite overlay $\phi : B \to C$ which is injective, the dual function ϕ^0 is not always surjective by giving an example where it is not. We then define a condition we call being *reciprocal*, and prove in Lemma 3.10 that this is necessary and sufficient for ϕ^0 to be surjective if ϕ is injective.

In Section 3.4, we show that these are the only two problems that occur. In particular, Theorem 3.12, the main result of this chapter, fixes the Naive Conjecture and generalises Proposition 3.1 as far as possible. Finally, Section 3.5 and in particular Lemma 3.16 look at a specific situation in which the conditions of being a finite overlay and being reciprocal hold.

3.2 The first problem: well-definedness

Throughout, k is a field.

As soon as we start looking into the Naive Conjecture in detail, we run into problems. In particular, the first one we encounter is the question of whether ϕ^0 is well-defined. Does a map $\phi : B \to C$ give rise to a map $\phi^0 : C^0 \to B^0$?

In the finite-dimensional case, this is easy to see: any linear map composed with ϕ is again a linear map, so if we set $\phi^*(f) := f \circ \phi$ for all $f \in C^*$, we have $\operatorname{im}\phi^* \subseteq B^*$. In the infinite-dimensional case, things become more complicated because we are no longer working with C^* and B^* but instead C^0 and B^0, the spaces consisting of those maps vanishing on some ideal of finite codimension. The same argument as before tells us that if we start with $f \in C^0$ and take $\phi^0(f) := f \circ \phi$, this will be an element of B^* - but in order to get $\phi^0(f) \in B^0$ we also need it to vanish on some ideal J of finite codimension in B.

The following example, an expansion on Remark 1.29, makes it clear this is not always true.

Example 3.3. Let $k = \mathcal{A}$ be the algebraic numbers, $K = \mathbb{C}$ and $B = k\,\mathrm{PSL}_2(K)$. We have seen in Example 1.28 that $B^0 = k\varepsilon_B$, as the only proper ideals of B are $\{0\}$ and the augmentation ideal B^+.

Let $x \in \mathrm{PSL}_2(K)$ be any nonidentity element, and let $C := k\langle x \rangle$. We can assume that x has infinite order and thus $C \cong k[x]$. As vector spaces, we have $B \cong C \oplus X$ for some vector space complement X of C. Now let $\phi : B \to C$ be the projection along X.

If the Naive Conjecture held, we would have an injective linear map $\phi^0 : C^0 \to B^0$. However, this is impossible, because by Example 1.31, we have $k[x]^0 \cong k[y] \otimes k(k, +)$. Hence $\dim C^0 = \infty > 1 = \dim B^0$.

So what goes wrong? We might assume composition with ϕ does still give us a linear map $\phi^0 : C^0 \to B^0$, simply one that is not injective, but by checking what the map does on elements it becomes clear the problem is that $\mathrm{im}\, \phi^0$ is not contained in B^0:

Let $f \in C^0$ be the algebra map given by $f(x^i) = \lambda^i$ for $\lambda \in k^*$ with $\lambda \neq 1$. Then $f \circ \phi : B \to k$ satisfies $(f \circ \phi)(x) = \lambda$ and $(f \circ \phi)(1) = 1$. So in particular, $f \circ \phi$ is not a scalar multiple of ε_B as this satisfies $\varepsilon_B(x) = \varepsilon_B(1) = 1$. Since $B^0 = k\varepsilon_B$, this means that $f \circ \phi$ does not lie in B^0.

Our choice of base fields shows that this is the case even when both fields in question are algebraically closed of characteristic zero.

The problem lies in the fact that although we can always define a map $\phi^* : C^* \to B^*$ even in the infinite-dimensional case and define a map ϕ^0 by restricting to C^0, as this example shows there is no guarantee that the image of this map lies in B^0. In order to guarantee this, we need a technical condition.

Definition 3.4. Suppose $\phi : B \to C$ is a linear map such that for every ideal J of finite codimension in C there exists some ideal I of finite codimension in B with $\phi(I) \subseteq J$. In this case we say ϕ is a *finite overlay*.

The next result tells us that this is a necessary and sufficient condition for $\phi^0 : C^0 \to B^*$ to have image in B^0.

Lemma 3.5. *Suppose B and C are algebras and $\phi : B \to C$ is a linear map. Let $\phi^0 : C^0 \to B^*$ denote the map on duals given by composition:*

$$\phi^0(f) := f \circ \phi.$$

Then $\mathrm{im}(\phi^0) \subseteq B^0$ if and only ϕ is a finite overlay.

Proof. $\Leftarrow$: Suppose that ϕ is a finite overlay, and let $f \in C^0$. Then there exists an ideal $J \subseteq \ker f$ with finite codimension in C. By assumption, there exists some ideal I of finite codimension in B with $\phi(I) \subseteq J$. This means that $\phi^0(f)(I) = f(\phi(I)) = 0$. This gives us $\phi^0(f) \in B^0$ as required.

$\Rightarrow$: Suppose $\operatorname{im} \phi^0 \subseteq B^0$, and let $J \subseteq C$ be an ideal of finite codimension. Then we can find maps $f_1, ..., f_n \in C^0$ such that

$$\bigcap_{i=1}^{n} \ker f_i = J.$$

Since $\phi^0(f_i) \in B^0$ for each i, this means we can find ideals $I_1, .., I_n$ of finite codimension in B such that $I_i \subseteq \ker \phi^0(f_i)$. Set

$$I := \bigcap_{i=1}^{n} I_i.$$

This is also an ideal of finite codimension in B, and we have

$$f_i(\phi(I)) = \phi^0(f_i)(I) = 0$$

for all i, giving us

$$\phi(I) \subseteq \bigcap_{i=1}^{n} \ker f_i = J$$

as required. □

Note that by Proposition 1.23, the proof of Lemma 3.5 works equally if we assume I is a two-sided ideal, left ideal, or right ideal - hence all of these conditions are mutually equivalent and equivalent to the notion of ϕ being a finite overlay.

One situation where a map is always a finite overlay is when it is an algebra map.

Corollary 3.6. *Let B and C be algebras and $\phi : B \to C$ an algebra map. Then ϕ is a finite overlay.*

Proof. Suppose $J \subseteq C$ is an ideal of finite codimension. We want to show that there exists some ideal of finite codimension $I \subseteq B$ with $\phi(I) \subseteq J$.

Let $I := \phi^{-1}(J) \subseteq B$. This is an ideal because ϕ is an algebra map, we have $\phi(I) \subseteq J$ by definition and since ϕ induces an embedding of B/I into C/J it follows that I must have finite codimension in B. □

However, this is not the only case in which a finite overlay occurs. For example, it is trivial to see that ϕ always gives a finite overlay when B is finite-dimensional - in this case, we can always take $I := \{0\}$, which satisfies $\phi(I) \subseteq J$ for any ideal J in C. In Section

3.5 we will discuss other situations which force the finite overlay condition to hold, with particular focus on ones relevant to the crossed product setting.

Remark 3.7. We can put the notion of a map being a finite overlay into a wider context as follows: any k-algebra can be given the structure of a topological ring by defining the open neighbourhoods of zero to be the ideals of finite codimension (where we can consider left, right, or two-sided ideals). This topology, known as the fc-topology, is studied for instance in [40]. Viewed in this setting, saying a map is a finite overlay is simply saying it is continuous with respect to this topology.

3.3 The second problem: surjectivity in the dual

Throughout, k is a field.

From now on, we suppose $\phi : B \to C$ is a linear map which is a finite overlay, which by Lemma 3.5 means we have a linear map $\phi^0 : C^0 \to B^0$ given by composition with ϕ. We want to see what happens to the Naive Conjecture we proposed in Section 3.1 once we add in this extra assumption.

The next problem that arises is in part (ii) of the Naive Conjecture: we would like ϕ being injective to guarantee that ϕ^0 is surjective. The following example shows that this is not always true.

Example 3.8. Suppose k an algebraically closed field that is not locally finite and set $C := kG$, where G is the Heisenberg group given by

$$G = \langle x, y, z \mid [x, y] = z, z \text{ central}\rangle.$$

Now set $B := k[z^{\pm 1}]$, and let $\phi : B \to C$ denote the natural inclusion map. Throughout, we identify B with its image $\phi(B)$ in C. Since ϕ is an algebra embedding, these two spaces are isomorphic as algebras. Also, ϕ being an algebra map means that by Corollary 3.6 it is a finite overlay, and so we get a map $\phi^0 : C^0 \to B^0$ given by restriction to B. We want to show this map is not surjective. This follows from the next claim.

Claim. *Suppose $I \subseteq kG$ is an ideal of finite codimension. Then $I \cap B = fB$ for some $f \in B$ such that every irreducible factor of f has the form $(z - \lambda)$ for some root of unity $\lambda \in k^*$.*

Proof. We know that some $f \in B$ with $I \cap B = fB$ must exist because B is a principal ideal domain, and we know that the irreducible elements of B look like $(z - \lambda)$ for $\lambda \in k^*$

because k is algebraically closed. So the only thing we need to show is that f contains no irreducible factors of the form $(z - \mu)$ where μ is not a root of unity.

Suppose to the contrary that $f = (z - \mu)g$ for some such μ, where $g \in B$. Then we have

$$((z - \mu)kG)(gkG) = fkG \subseteq I$$

because $(z - \mu)$ is central in kG. However, we know that $g \notin I$ because $I \cap B = fB$ and g does not contain f as a factor. So $g + I$ is nonzero in the finite dimensional algebra kG/I and has annihilator ideal $P \subsetneq kG$ such that $(z - \mu)kG \subseteq P$. This means we have a canonical isomorphism of kG-modules

$$kG/P \cong kG(g + I)/I.$$

Since $kG(g + I)/I \subseteq kG/I$ is finite dimensional, showing that kG/P is not finite dimensional suffices to give us a contradiction. We do this by showing that $P = (z - \mu)kG$.

We have

$$R_\mu := \frac{kG}{(z - \mu)kG} = k\langle \overline{x}^{\pm 1}, \overline{y}^{\pm 1} \mid \overline{xy} = \mu\overline{yx}\rangle.$$

This is infinite dimensional and arises by localising the quantum plane

$$Q_\mu = k\langle \overline{x}, \overline{y} \mid \overline{xy} = \mu\overline{yx}\rangle$$

at the multiplicatively closed set consisting of all monomials in $\overline{x}$ and $\overline{y}$. However, because μ is not a root of unity this construction gives a simple ring (see Example 2.14.) The only proper ideal of R_μ is the zero ideal. So $(z - \mu)kG$ is maximal in kG.

Since we know that $(z - \mu)kG \subseteq P$ by the above, this means that $P = (z - \mu)kG$. However, this does not have finite codimension in kG, which is a contradiction by the above. Such μ cannot exist.

This proves the claim. □

From the claim we can show that $\phi^0 : C^0 \to B^0$ is not surjective as follows:

Let $\mu \in k^*$ be given such that μ is not a root of unity. Define $f : B \to k$ to be the algebra homomorphism given by $f(z) = \mu$. We have $f \in B^0$ because $\ker f = (z - \mu)B$, which is an ideal of codimension 1 in B. Now suppose we have $\hat{f} \in C^0$ such that $\phi^0(\hat{f}) = f$. By definition any $\hat{f} \in C^0$ contains an ideal $I \subseteq C$ of finite codimension in its kernel, and because $\hat{f}|_B = f$ we have $I \cap B \subseteq \ker f = (z - \mu)B$. Since B is a principal ideal domain, we have $I \cap B = gB$ for some $g \in B$, where $g \neq 0$ since $I \cap B$ has finite codimension in

B. Moreover, $gB \subseteq (z-\mu)B$ means that g has an irreducible factor of the form $(z-\mu)$. This is a contradiction by the above claim, and so no such $\hat{f} \in C^0$ can exist. So the map $\phi^0 : C^0 \to B^0$ is not surjective.

We need another technical condition.

Definition 3.9. Let $\phi : B \to C$ be a finite overlay. We say that ϕ is *reciprocal* if whenever I is an ideal of finite codimension in B, we can find an ideal J of finite codimension in C with

$$J \cap \operatorname{im}\phi \subseteq \phi(I).$$

If $\phi : B \to C$ is an injective finite overlay, it being reciprocal is equivalent to the dual map $\phi^0 : C^0 \to B^0$ being surjective:

Lemma 3.10. *Suppose B and C are algebras and $\phi : B \to C$ is an injective linear map which is a finite overlay. Then $\phi^0 : C^0 \to B^0$ is surjective if and only if ϕ is reciprocal.*

Proof. $\Leftarrow$: Suppose the stated condition holds, and suppose $f \in B^0$. We want to show there exists $g \in C^0$ such that $\phi^0(g) = f$.

Since $f \in B^0$, there exists some ideal $I \subseteq \ker f$ with finite codimension in B. By assumption, we can find an ideal $J \subseteq C$ of finite codimension such that $J \cap \operatorname{im}\phi \subseteq \phi(I)$.

We want $g(J) = 0$, so we can define g on C/J. Now, we have

$$C/J \cong \frac{(\operatorname{im}\phi + J)}{J} \oplus X$$

as vector spaces, where X is some vector space complement of $(\operatorname{im}\phi + J)/J$. Set $g(X) = 0$.

It remains to define g on $(\operatorname{im}\phi + J)/J \cong \operatorname{im}\phi/(J \cap \operatorname{im}\phi)$ - or, alternatively, to define g on $\operatorname{im}\phi$ in such a way that $g(J \cap \operatorname{im}\phi) = 0$.

To do this, we note that since ϕ is injective, it is an isomorphism on its image: we have an inverse map $\phi^{-1} : \operatorname{im}\phi \to B$. We define $g|_{\operatorname{im}\phi} := f \circ \phi^{-1}$. This satisfies

$$g(J \cap \operatorname{im}\phi) = f(\phi^{-1}(J \cap \operatorname{im}\phi)) \subseteq f(\phi^{-1}(\phi(I))) = f(I) = 0,$$

so it is well-defined. It is also obvious that this means we have $\phi^0(g) = f$.

$\Rightarrow$: Suppose we know that ϕ^0 is surjective. Let I be an ideal of finite codimension in B. Then we can find $f_1, ..., f_n \in B^0$ such that

$$\bigcap_{i=1}^{n} \ker f_i = I.$$

By the surjectivity of ϕ^0, there exist $g_1, ..., g_n \in C^0$ with $\phi^0(g_i) = f_i$. For each $1 \leq i \leq n$, since $g_i \in C^0$ there exists an ideal $J_i \subseteq \ker g_i$ such that J_i has finite codimension in C. Set

$$J := \bigcap_{i=1}^{n} J_i.$$

We want to show that $J \cap \operatorname{im} \phi \subseteq \phi(I)$.

Let $h \in B$. Since $g_i \circ \phi = f_i$, we have $g_i(\phi(h)) = f_i(h)$. In particular, $g_i(\phi(h)) = 0$ if and only if $f_i(h) = 0$, and so

$$\ker g_i \cap \operatorname{im} \phi = \phi(\ker f_i).$$

This means that

$$\begin{aligned} J \cap \operatorname{im} \phi &\subseteq \left(\bigcap_{i=1}^{n} \ker g_i \right) \cap \operatorname{im} \phi \\ &= \bigcap_{i=1}^{n} (\ker g_i \cap \operatorname{im} \phi) \\ &= \bigcap_{i=1}^{n} \phi(\ker f_i) \\ &= \phi \left(\bigcap_{i=1}^{n} \ker f_i \right) \qquad \text{because } \phi \text{ is injective} \\ &= \phi(I) \end{aligned}$$

as required. □

This time, we find that any finite overlay $\phi : B \to C$ is trivially reciprocal whenever C is finite dimensional - in this case, we can just take $J = \{0\}$, which has finite codimension in C by finite-dimensionality and obviously satisfies $\{0\} = J \cap \operatorname{im} \phi \subseteq \phi(I)$ for any ideal I in B. However, unlike in Section 3.2, we do not immediately see any "nice" properties such as being an algebra map which guarantee that a map is reciprocal. In Example 3.8, we saw that even an injective map which is the embedding of a central Hopf subalgebra is not necessarily reciprocal. Again, in Section 3.5 we will look at a specific situation relevant to the crossed product situation which guarantees reciprocity.

Remark 3.11. Just as in the case of finite overlays in Remark 3.7, we can also view the notion of a reciprocal map in terms of the fc-topology on a k-algebra given by the ideals of finite codimension. In this case, a map being reciprocal is equivalent to it being a topological embedding: a continuous map that is a homeomorphism onto its image.

3.4 The main result

Throughout, k is a field.

In Section 3.2, we encountered the first problem with the Naive Conjecture: given a linear map $\phi : B \to C$, we needed to impose a technical condition which we called being a *finite overlay* in order to be able to talk about a canonical map $\phi^0 : C^0 \to B^0$ at all. In Section 3.3, we saw the second problem: given an injective finite overlay $\phi : B \to C$, we needed to impose another technical condition which we called being *reciprocal* in order to have ϕ^0 surjective. Now, we prove that these are the only two obstructions to the truth of the Naive Conjecture.

Theorem 3.12. *Suppose B and C are both algebras and $\phi : B \to C$ is some linear map between them which is a finite overlay. Then*

(i) There is a well-defined map $\phi^0 : C^0 \to B^0$ given by composition with ϕ, and given any algebra D and finite overlay $\psi : C \to D$, $\psi \circ \phi$ is a finite overlay such that

$$(\psi \circ \phi)^0 = \phi^0 \circ \psi^0.$$

(ii) Suppose ϕ is injective. Then ϕ^0 is surjective if and only if ϕ is reciprocal.

(iii) If ϕ is surjective, ϕ^0 is injective.

(iv) If ϕ is an algebra map, ϕ^0 is a coalgebra map.

(v) Suppose B and C are bialgebras. Then if ϕ is a coalgebra map, ϕ^0 is an algebra map.

(vi) Suppose B and C are Hopf algebras. Then if ϕ is a Hopf algebra map, so is ϕ^0.

Proof. (i) For the first part, see Lemma 3.5. Now let D be an algebra and $\psi : C \to D$ be some linear map which is a finite overlay. Let $f \in D^0$. Then

$$(\psi \circ \phi)^0(f) = f \circ \psi \circ \phi = (\phi^0(f \circ \psi)) = \phi^0 \circ \psi^0 \circ f$$

and thus we have

$$(\psi \circ \phi)^0 = \phi^0 \circ \psi^0.$$

The fact that $\psi \circ \phi$ is a finite overlay follows from this: we have

$$(\psi \circ \phi)^0(D^0) = (\phi^0 \circ \psi^0)(D^0) = \phi^0(\psi^0(D^0)) \subseteq \phi^0(C^0) \subseteq B^0$$

because both ϕ and ψ are finite overlays. So $\operatorname{im}(\psi \circ \phi)^0 \subseteq B^0$, which by Lemma 3.5 means that $\psi \circ \phi$ is a finite overlay.

(ii) This is simply Lemma 3.10.

(iii) Suppose ϕ is surjective and $\phi^0(f) = 0$ for some $f \in C^0$. Then $f(\phi(h)) = 0$ for all $h \in B$. Since ϕ is surjective, this means that $f(h') = 0$ for all $h' \in C$, and so $f = 0$. So ϕ^0 is injective.

(iv) Suppose ϕ is an algebra map and let $\mu : k \otimes k \to k$ denote the scalar multiplication map. The fact that the coalgebra structure on B^0 is induced by the algebra structure on B gives us what we need. In particular, assume $f \in C^0$ and $g, h \in B$. Then we have

$$\begin{aligned}
\Delta_{B^0}(\phi^0(f))(g \otimes h) &= \left(\sum \phi^0(f)_1 \otimes \phi^0(f)_2\right)(g \otimes h) \\
&= \phi^0(f)(gh) \\
&= f(\phi(gh)) \\
&= f(\phi(g)\phi(h)) \\
&= \sum f_1(\phi(g)) f_2(\phi(h)) \\
&= \mu\left(\left(\sum \phi^0(f_1) \otimes \phi^0(f_2)\right)(g \otimes h)\right) \\
&= \mu\left((\phi^0 \otimes \phi^0)(\Delta_{C^0}(f))(g \otimes h)\right).
\end{aligned}$$

so ϕ^0 preserves the coproduct. Similarly, we find that

$$\varepsilon_{B^0}(\phi^0(f)) = \phi^0(f)(1_B) = f(\phi(1_B)) = f(1_C) = \varepsilon_{C^0}(f)$$

and hence ϕ^0 preserves the counit: it is a coalgebra map as required.

(v) Suppose ϕ is a coalgebra map. The algebra structure on B^0 is induced by the coalgebra structure on B, so if ϕ respects the coalgebra structure then for all $f, g \in C^0$ and $h \in B$ we have

$$\begin{aligned}
\left(\phi^0(f)\phi^0(g)\right)(h) &= \sum (f \circ \phi)(h_1)(g \circ \phi)(h_2) \\
&= \sum f(\phi(h)_1) g(\phi(h)_2) \\
&= (fg)(\phi(h)) = \phi^0(fg)(h).
\end{aligned}$$

Similarly, $\phi^0(\varepsilon_C) = \varepsilon_C \circ \phi = \varepsilon_B$, so ϕ^0 preserves the identity. This means ϕ^0 is an algebra map.

(vi) Suppose ϕ is a Hopf algebra map. By (iv), this means that ϕ^0 is a coalgebra map, and by (v) it is also an algebra map. So we already know that $\phi^0 : C^0 \to B^0$ is a bialgebra map. Now note that both C^0 and B^0 are Hopf algebras and by [48, Lemma 4.0.4], any bialgebra map between Hopf algebras is a Hopf algebra map. Hence ϕ^0 is a Hopf algebra map as required. □

Remark 3.13. Since the embedding $B^0 \to B^*$ is an algebra embedding, parts (iii) and (v) of Theorem 3.12 also hold for the vector space dual: given any linear map $\phi : B \to C$, the dual map $\phi^* : C^* \to B^*$ is still injective whenever ϕ surjective, and is still an algebra map whenever B and C are bialgebras and ϕ is a coalgebra map.

Remark 3.14. Note that if we assume B and C finite-dimensional, $B^0 = B^*$ and $C^0 = C^*$, and moreover ϕ is always trivially a reciprocal finite overlay. Thus under these assumptions Theorem 3.12 gives us exactly Proposition 3.1, which we can therefore view as a corollary.

3.5 Subalgebras and projection maps arising from a direct sum decomposition

Throughout, k is a field.

Recall the notion of a finite overlay from Definition 3.4 and that of it being reciprocal from Definition 3.9.

These two conditions are quite difficult to check in practice. We want to find conditions that imply them which are easier to check and tell us more about the structure of B or C. In particular, we are interested in crossed products as defined in Chapter 2 and the maps arising on those and so would like to find conditions relevant to that situation.

One situation that arises frequently is that we have some Hopf algebra H and subalgebra $A \subseteq H$ along with a linear quotient map $\psi : H \to A$ with $\psi|_A = \mathrm{id}_A$. For instance, when $H \cong A\#_\sigma T$, we have a canonical subalgebra $A \subseteq H$ along with a quotient map $\Pi : H \to A$ given by factoring along $A\#T^+$. In this case, Π is not just a linear map but also one of left A-modules.

Given such a subalgebra A and map ψ, we know that the canonical inclusion map $\iota : A \to H$ is a finite overlay by Corollary 3.6, but as we saw in Example 3.8 it need not be reciprocal. Similarly, $\psi : H \to A$ need not be a finite overlay at all.

It turns out that these two conditions are related.

Lemma 3.15. *Suppose H is an algebra and $A \subseteq H$ is a subalgebra of H, with ι denoting the inclusion map. Suppose also that we have a linear map $\psi : H \to A$ such that $\iota \circ \psi = \mathrm{id}_A$.*

If ψ is a finite overlay, then ι is reciprocal.

Proof. Suppose we are given A, H, ι and ψ as in the statement and suppose ψ is a finite overlay. So for any ideal I of finite codimension in A there exists an ideal J with finite codimension in H such that $\psi(J) \subseteq I$. By Corollary 3.6, ι being an algebra map means it is a finite overlay. Now we want to show that ι is reciprocal. Since I was an arbitrary ideal of finite codimension in A, this means we need to find some ideal K of finite codimension in H with $K \cap \operatorname{im} \iota \subseteq \iota(I)$. However, we can just take $K := J$, since we have

$$J \cap \operatorname{im} \iota = J \cap A = \iota \circ \psi(J \cap A) \subseteq \iota \circ \psi(J) \subseteq \iota(I).$$

This means ι is reciprocal. □

As mentioned above, one situation where Lemma 3.15 applies is when H is a crossed product, and in that case the projection map Π arises from a left A-module decomposition

$$H \cong A \oplus A\#T^+.$$

In that case, we can say something more:

Lemma 3.16. *Suppose A and H are algebras, and suppose further that we have*

$$H \cong A \oplus X$$

as A-modules, where the first summand A is a subalgebra of H and X is some left A-module. If X is finitely-generated as an A-module, then the canonical projection map $\psi : H \to A$ along X is a finite overlay and the inclusion map $\iota : A \to H$ is reciprocal.

Proof. Let ψ be as stated. In order to show that ψ is a finite overlay, we want to show that given any ideal I of finite codimension in A there exists some left, right or two-sided ideal $J \subseteq H$ of finite codimension such that $\psi(J) \subseteq I$.

Define $J := I \oplus IX = IH$. This is a right ideal in H, and we have $\psi(J) = \psi(I) + \psi(IX) = I$ since $IX \subseteq X \subseteq \ker \psi$. So we only need to check that J has finite codimension.

Note that $\dim(X/IX) < \infty$ because X/IX is a finitely-generated left A/I-module and A/I is finite-dimensional. So J has finite codimension because

$$\dim(H/J) = \dim(A/I) + \dim(X/IX) < \infty,$$

and so ψ is a finite overlay.

The fact that ι is then reciprocal follows from Lemma 3.15. □

This result is particularly useful for the crossed product case.

Corollary 3.17. *Suppose H and T are Hopf algebras and A is an algebra such that $H \cong A\#_\sigma T$ for some cocycle σ and action of T on H. Suppose further that T is finite-dimensional. Then the canonical projection map $\Pi : H \to A$ given by $\Pi(a\#t) = \varepsilon_T(t)a$ for $a \in A, t \in T$ is a finite overlay, and the inclusion map $\iota : A \to H$ is reciprocal.*

Proof. We can write H as

$$H \cong A\#1_T \oplus A\#T^+.$$

This is a decomposition of left A-modules, and because T and hence T^+ are finite-dimensional, $A\#T^+$ is finitely generated as a left A-module. So Lemma 3.16 applies: the projection map given by factoring along $A\#T^+$, which is precisely the one sending $a\#t$ to $\varepsilon_T(t)a$, is a finite overlay and the inclusion map $\iota : A \to H$ is reciprocal. □

An obvious question to ask is whether we can weaken the hypotheses of Lemma 3.16 at all. In particular, we would like to do away with the assumption that X is finitely generated, thus allowing us to extend these results to all crossed products rather than just those where T is finite-dimensional. The following example shows us that this is not possible. Lemma 3.16 does not necessarily hold without that assumption, even when we are working in "nice" situations such as one in which H is Noetherian and A is a central Hopf subalgebra.

Example 3.18. Let G be the Heisenberg group defined in Example 3.8 and H its group algebra. This means that

$$H = kG \cong k\langle x^{\pm 1}, y^{\pm 1}, z^{\pm 1} \mid [x, y] = z, z \text{ central}\rangle.$$

Let $A = k[z^{\pm 1}] \subseteq H$. We know that $H \cong A \oplus X$ as left A-modules, where X is given by

$$X = \bigoplus_{\substack{i,j\in\mathbb{Z} \\ \text{one of } i,j\neq 0}} Ax^i y^j.$$

However, X is not finitely generated as a left A-module.

Recall that in Example 3.8 we showed the inclusion map $A \to H$ is not reciprocal. By Lemma 3.15, this implies that the canonical projection from H to A along X cannot be a finite overlay. Thus neither of the statements of Lemma 3.16 holds, although all of its conditions hold save for X being finitely generated as a left A-module.

Moreover, in this case, A is a central Hopf subalgebra, and since G is polycyclic we know by [17, Theorem 1 and following remarks] or [38, Corollary 10.2.8] that H is Noetherian.

However, although we cannot drop X being finitely-generated from our conditions for Lemma 3.16, at the same time it is not always a necessary condition. This follows from Corollary 3.6, which says that in any case where $\psi : H \to A$ is an algebra map it is always a finite overlay. One of the cases where this holds is when H is isomorphic to a tensor product.

Example 3.19. Suppose that $H \cong A \otimes T$ as Hopf algebras and suppose T is not finite-dimensional. We have $H \cong A \oplus AT^+$ as left A-modules, and $X := AT^+$ is not finitely generated as a left A-module. However, the map $\psi : H \to A$ given by factoring along X is an algebra map and hence by Corollary 3.6 a finite overlay. By Lemma 3.15, this also implies that the inclusion map $\iota : A \to H$ is reciprocal.

In the next chapter, we will use the results of this chapter to work out results about the finite duals of crossed products.

3.6 Originality

Proposition 3.1 is classical knowledge, as is the extension to infinite dimensions in Theorem 3.12 (iii)-(v). However, the notions of finite overlay and reciprocity and their properties introduced in Sections 3.2 and 3.3 are new in the context of Hopf algebras, as are Theorem 3.12(i)-(ii) and Section 3.5. Although there is some work on similar questions in [1, Theorem 2.3.16], the approach taken there is asking what properties two algebras must satisfy so that every injective algebra map between them becomes surjective in the dual and thus different from ours.

Chapter 4

Duals of crossed products

4.1 Introduction

In Chapter 2, we looked at how functions between Hopf algebras translate to the world of finite duals. This chapter, we endeavour to use these results to describe the finite dual of a crossed product, in particular one arising from a Hopf surjection.

Throughout, k is a field, H and T are Hopf algebras over k and $\pi : H \to T$ is a surjective Hopf map. We recall various results from Chapter 2:

H is always a right T-comodule algebra with comodule map given by $(\mathrm{id}_H \otimes \pi) \circ \Delta_H$ (Lemma 2.17). Suppose there exists a convolution invertible right T-comodule map $\gamma : T \to H$. We call γ a cleaving map (Definition 2.19) and by Proposition 2.22 its existence means that we have

$$H \cong A \#_\sigma T$$

for a given weak action of T on A and cocycle $\sigma : T \otimes T \to A$, where $A := H^{coT}$ is given by the coinvariants of the T-comodule action.

Recall from Definition 2.5 and Lemma 2.6 that this notation means that $H \cong A \otimes T$ as vector spaces, and moreover that H is an associative algebra with multiplication given by

$$(a\#t)(b\#s) = \sum a(t_1 \cdot b)\sigma(t_2, s_1)\#t_3 s_2,$$

where $a, b \in A, s, t \in T$ and we write $a\#t$ for the element corresponding to $a \otimes t$ in the description of H as a tensor product. In this notation, the multiplication also has idenity $1_A\#1_T$.

In this situation, we want to use our results from Chapter 3 to describe H^0.

Naively, we might think that

$$H^0 \cong A^0 \#_\tau T^0 \tag{4.1}$$

as algebras for some cocycle τ. We will see that this fails to hold in a number of situations: the main issues are that there need not be a canonical embedding $A^0 \subseteq H^0$ and that even when we do have such an embedding, there may not exist an isomorphism such as the above. However, we show that there are multiple similar, even some stronger results that hold with some classical constraints on the structure, such as H being a smash product.

Remark 4.1. Another possible decomposition of H^0 is

$$H^0 \cong T^0 \#_{\tau'} A^0 \tag{4.2}$$

for some cocycle τ'. Indeed, this might even seem more natural than (4.1) at first glance as we know the translation to the dual setting is in some sense contravariant and so we would expect this kind of reversal. We do not consider the possibility here because in order for (4.2) to hold, we would need both a Hopf algebra structure on A^0 and for it to satisfy various compatibility conditions regarding the structure of H^0, and moreover the obvious way to get such a structure would be through a Hopf algebra structure on A. However, neither the definition of a crossed product nor the conditions we have assumed to get a crossed product structure on H make any reference to a coalgebra structure on A, much less a Hopf algebra structure compatible with H. Instead, we look at the possibility of (4.2) in Chapter 5, where we study Hopf algebras that are finitely generated over central Hopf subalgebras.

In Section 4.1.1, we record the precise assumptions we will be working with throughout this chapter and the notation we will be using.

In Section 4.2, we recall the canonical left A-module map $\Pi : H \to A$ from Section 2.4 and define a subspace A^0_Π of A^0 which is the maximal subspace that gets mapped into H^0 by the canonical map Π^*. We then define a linear embedding

$$\xi : A^0_\Pi \otimes T^0 \to H^0.$$

We do not know whether this embedding is always bijective, but in any case Example 4.8 shows it does not always give rise to an algebra isomorphism $A^0_\Pi \#_\sigma T^0 \cong H^0$. Lemma 4.4 gives a necessary and sufficient condition for A^0_Π to be the whole of A^0.

The map ξ does not tell us very much in the general case. However, in the next three sections we look at what ξ tells us under certain extra conditions which arise frequently.

In Section 4.3 we ask when ξ is bijective. Theorem 4.9 gives two conditions which, together, are necessary and sufficient for ξ to be bijective: the conditions are that γ is a finite overlay (as in Definition 3.4) and that $\operatorname{im}\iota^0 \subseteq A^0_\Pi$. Since these two conditions are rather technical, two corollaries look at specific situations of interest in which they both hold. In particular, Corollary 4.11 tells us that they always hold when T is finite-dimensional, and Corollary 4.12 that they also hold when the action on A and cocycle are trivial, which means that $H \cong A \otimes T$ as algebras.

In the next two sections, we impose a stronger hypothesis on γ, namely that it is a coalgebra map.

Section 4.4 is concerned with smash products: what happens when the cocycle σ is trivial? Theorem 4.13 tells us that here, ξ gives rise to an algebra isomorphism between $A^0_\Pi \# T^0$ and H^0. Then Section 4.5 considers the case where A is a Hopf subalgebra of H. Here, Theorem 4.19 tells us that ξ is an algebra map.

Since all of these results concern the same function and are about relatively independent conditions on H, they can combine. In Section 4.6, we summarise all our results in Table 4.1, which tells us what we can say about H^0 under what assumptions on H.

4.1.1 Setting and notation

We recall the results of Section 2.4.

Throughout, k is a field and H and T are Hopf algebras, with $\pi : H \to T$ being some Hopf surjection and $\gamma : T \to H$ a convolution invertible right T-comodule map. By Proposition 2.22, this means that

$$H \cong A \#_\sigma T.$$

Here $A := H^{coT}$ are the right T-coinvariants, forming a subalgebra of H by Proposition 2.18(i), while the action of T on A is given by

$$t \cdot a = \sum \gamma(t_1) a \gamma^{-1}(t_2) \qquad \text{for } t \in T, a \in A$$

and σ is given by

$$\sigma(s,t) = \sum \gamma(s_1)\gamma(t_1)\gamma^{-1}(s_2 t_2) \qquad \text{for } s,t \in T.$$

The isomorphism is given by $a \# t \mapsto a\gamma(t)$.

In this setting, by Section 2.4, we have four canonical maps on H, A and T. There is the canonical Hopf surjection $\pi : H \to T$ along with the cleaving map $\gamma : T \to H$, which is a convolution invertible right T-comodule map where we can assume without loss of generality that

$$\gamma(1_T) = 1_H.$$

We also have the canonical algebra map which gives the embedding of A into H as a subalgebra, which we call ι. Finally, there is a canonical left A-module map $\Pi : H \to A$ given by factoring along the left A-module $A\gamma(T^+)$. That is, for $a \in A$ and $t \in T$,

$$\Pi(a\gamma(t)) = a\varepsilon_T(t). \tag{4.3}$$

As we saw in Section 2.4, γ is neither an algebra nor coalgebra map in general, and in fact a necessary and sufficient condition for it to be an algebra map is for the cocycle σ to be trivial (this is by Lemma 2.30). Similarly, Π is neither an algebra nor coalgebra map in general.

As discussed in Remark 2.24, we write $a\gamma(t)$ for the element of H given by $m_H \circ (\iota \otimes \gamma)(a \otimes t)$, which corresponds to $a\#t$ in the crossed product description of H. In particular, we usually suppress ι in notation but do not suppress γ. The reason for this is that γ is not generally an algebra map, so $\gamma(s)\gamma(t) \neq \gamma(st)$ in general for $s, t \in T$. As a result, suppressing γ would lead to our notation being ambiguous and misleading.

In Sections 4.4 and 4.5, we impose a stronger assumption on γ: namely, that it is a coalgebra map. As we saw in Example 2.27, this condition need not always be true. We will see in this chapter that γ is a coalgebra map in many examples of interest to us, such as when H is a Taft algebra (Examples 4.16, 4.22). By Lemma 2.20, in this case we always have $\pi \circ \gamma = \mathrm{id}_T$.

4.2 General results

Throughout, k is a field.

Let H and T be Hopf algebras with a Hopf surjection $\pi : H \to T$ and a convolution invertible right T-comodule map $\gamma : T \to H$, so that by Proposition 2.22 we have

$$H \cong A\#_\sigma T$$

for some weak action of T on A and cocycle $\sigma : T \otimes T \to A$ (recalling these definitions from Chapter 2).

As discussed in the introduction, we want to find a way to express H^0 in terms of A^0 and T^0. In particular, our naive assumption might be that $H^0 \cong A^0\#_\tau T^0$ for some action of T^0 on A^0 and cocycle $\tau : T^0 \otimes T^0 \to A^0$. We begin this chapter by attempting to construct a map that could provide such an isomorphism. However, in Example 4.2, we find that there does not necessarily exist an embedding $A^0 \to H^0$ satisfying the properties we would like. So instead we switch to a subspace $A^0_\Pi \subseteq A^0$ which we know always embeds into H^0 and define a map $\xi : A^0_\Pi \otimes T^0 \to H^0$ which may give us a crossed product structure. This map ξ will be our object of study in the rest of the chapter.

Retaining the notation and hypotheses of Section 4.1.1, we start off looking for a way to define a canonical map $A^0 \otimes T^0 \to H^0$.

The logical approach seems to be to find two canonical embeddings $A^0 \to H^0$ and $T^0 \to H^0$, tensor these together and then compose with the multiplication on H^0. Indeed, for T^0 the first part of this works, as we now show.

First, recall the condition of a map $\phi : B \to C$ being a *finite overlay* from Definition 3.4, in particular that by Lemma 3.5 this condition is necessary and sufficient for the image of the map $\phi^0 : C^0 \to B^*$ given by composition with ϕ to be contained in B^0. By Corollary 3.6, all algebra maps are finite overlays so the Hopf surjection $\pi : H \to T$ is certainly one. So we have a well-defined map $\pi^0 : T^0 \to H^0$, and moreover Theorem 3.12 (iii) and (vi) tells us this map is an embedding of Hopf algebras.

However, we run into problems looking for a canonical embedding $\psi : A^0 \to H^0$. We would like it to be natural in some sense, for instance by having maps in A^0 and their images in H^0 agree on A, so that given $f \in A^0$ and $a \in A$ we have

$$\psi(f)(a) = f(a). \tag{4.4}$$

An obvious candidate here might be the map Π^0: since $\Pi|_A = \mathrm{id}_A$, the map on duals given by composition with Π will satisfy (4.4). However, there seems to be no reason for Π to be a finite overlay. Indeed, in the following example it is not. In fact, in the example no $\psi : A^0 \to H^0$ satisfying Equation (4.4) can exist.

Example 4.2. Let k be an algebraically closed field of characteristic zero, $\mathfrak{g} := k\{x, y \mid [y, x] = x\}$ be the nonabelian 2-dimensional Lie algebra, and $H := U(\mathfrak{g})$ its universal enveloping algebra. We find that H is a smash product:

$$H \cong k[x]\#k[y],$$

where the action of $k[y]$ on $k[x]$ is given by $y \cdot x = x$. Here, $\Pi : H \to k[x]$ is the map given by factoring along the left $k[x]$-module $\bigoplus_{i\geq 1} k[x]y^i$, so that we have $\Pi(x^i y^j) = \delta_{j0} x^i$.

Recall from Example 1.31 that we have

$$H^0 \cong k[X, Y] \otimes k(k, +).$$

By [30, 1.6], viewed as linear maps from H to k we have $X(x^i y^j) = \delta_{i1}\delta_{j0}$, $Y(x^i y^j) = \delta_{i0}\delta_{j1}$ and the map $f_\lambda \in (k, +)$ corresponding to a scalar $\lambda \in k$ is given by $f_\lambda(x^i y^j) = \delta_{i0}\lambda^j$. In other words, the f_λ are the characters arising from $k[y]$ viewed as the abelianisation of $U(\mathfrak{g})$.

This means we are missing the characters coming from $k[x]$: by Example 1.31, we have

$$k[x]^0 \cong k[z] \otimes k(k, +),$$

where $g_\lambda \in k(k, +)$ is given by $g_\lambda(x^i) = \lambda^i$. When we go from $k[x]^0$ to H^0, although the maps in $k[z]$ lift to those in $k[X]$, the characters in $k(k, +)$ get "lost".

We can see what is happening here by looking at $U(\mathfrak{g})$-modules.

First, we note that every finite dimensional irreducible $U(\mathfrak{g})$-module is one-dimensional as a vector space (see for instance [10, Corollary 1.3.13]). Since $[y, x] = x$, this means that x acts as zero on every finite dimensional irreducible $U(\mathfrak{g})$-module. As any finite dimensional $U(\mathfrak{g})$-module has a composition series of finite length, this means that any such module must be killed by a power of x. Since every ideal of finite codimension is the annihilator of some finite-dimensional module, this means every such ideal contains some power of x, and so given $f \in U(\mathfrak{g})^0$ we must have $f(x^n) = 0$ for some n. In particular, this means that

$$f(x^n) = 0 \neq \lambda^n = g_\lambda(x^n)$$

for all $\lambda \in k^*$.

Since $\Pi^0(g_\lambda)(x^i) = \lambda^i$, this means that $\Pi^0(g_\lambda) \notin U(\mathfrak{g})^0$, and hence by Lemma 3.5 Π is not a finite overlay. Moreover, the same applies to any map $\psi : A^0 \to H^*$ which satisfies Equation (4.4).

This example shows us that the map $\Pi^0 : A^0 \to H^*$ given by composition with the canonical quotient map $\Pi : H \to A$ does not necessarily give us a well-defined map $A^0 \to H^0$. In fact, it shows that the same is true for any map $A^0 \to H^*$ that preserves evaluation on A, even when (as in the example) A and H satisfy properties such as H being a smash product or A being a Hopf subalgebra.

However, the example also shows that even when there is no such embedding, we might be able to find an embedding of a subspace of A^0. In the case of $U(\mathfrak{g}) \cong k[x]\#k[y]$, we have $H^0 \cong k[X] \otimes k[y]^0$. Here we can identify $k[X]$ with $\Pi^0(k[z])$, where

$$k[z] \subseteq k[z] \otimes k(k,+) \cong k[x]^0.$$

We can do something similar in the general case:

Definition 4.3. Under the notation and assumptions of Section 4.1.1, we define a subspace A^0_Π of A^0 as follows:

$$A^0_\Pi := \left(\Pi^0\right)^{-1}(\operatorname{im}\Pi^0 \cap H^0) = \{f \in A^0 \mid f \circ \Pi \in H^0\}.$$

Thus A^0_Π is just the subspace of A^0 consisting of those functions which get mapped into H^0 by Π^0.

A^0_Π is always a subspace of A^0. It is easy to see the necessary and sufficient condition for it being all of A^0:

Lemma 4.4. $A^0_\Pi = A^0$ *if and only if* Π *is a finite overlay.*

Proof. This follows straight from the definition and from Lemma 3.5:

We know that A^0_Π consists exactly of those functions in A^0 which are mapped to H^0 by Π^0. This is going to be all of A^0 if and only if $\operatorname{im}\Pi^0 \subseteq H^0$, which by Lemma 3.5 is true if and only if Π is a finite overlay. □

In the case of Example 4.2, we have $A^0_\Pi = k[z]$.

In general, under the hypotheses of Section 4.1.1 we have a well-defined linear map

$$\Pi^0|_{A^0_\Pi} : A^0_\Pi \to H^0,$$

since we know that $\Pi^0(A^0_\Pi) \subseteq H^0$ by definition. For $a \in A, t \in T$ and $f \in A^0_\Pi$ it satisfies

$$\Pi^0(f)(a\gamma(t)) = (f \circ \Pi)(a\gamma(t)) = \varepsilon_T(t)f(a).$$

Moreover, it is an embedding: if $\Pi^0(f) = f \circ \Pi = 0$, the fact that Π is surjective implies that $f = 0$.

Thus we define a linear map $\xi : A^0_\Pi \otimes T^0 \to H^0$ by

$$\xi := m_{H^0} \circ (\Pi^0|_{A^0_\Pi} \otimes \pi^0). \tag{4.5}$$

It turns out that viewed on elements, this map has a nice form:

Lemma 4.5. *Under the notation and assumptions of Section 4.1.1 and with A^0_Π and ξ as in Definition 4.3 and (4.5) respectively, take $f \in A^0_\Pi$, $g \in T^0$, $a \in A$ and $t \in T$. Then we have*

$$\xi(f \otimes g)(a\gamma(t)) = f(a)g(t).$$

Proof. By (4.5) we have

$$\xi(f \otimes g) = (f \otimes g) \circ (\Pi \otimes \pi) \circ \Delta_H,$$

so a sufficient condition to prove the lemma is to show that

$$((\Pi \otimes \pi) \circ \Delta_H)(a\gamma(t)) = a \otimes t. \tag{4.6}$$

That this is sufficient follows because what we want to show is just $(f \otimes g)$ applied to both sides of this equation.

Now note that

$$(\Pi \otimes \pi) \circ \Delta_H = (\Pi \otimes \mathrm{id}_T) \circ (\mathrm{id}_H \otimes \pi) \circ \Delta_H = (\Pi \otimes \mathrm{id}_T) \circ \rho_T,$$

where ρ_T is the canonical right T-comodule structure of H. We know that ρ_T is an algebra map and that $A = H^{coT}$ with respect to ρ_T, and since $a \in A$ this means that $\rho_T(a) = a \otimes 1_T$. Finally, we have

$$\rho_T \circ \gamma = (\gamma \otimes \mathrm{id}_T) \circ \Delta_T$$

because γ is a right T-comodule map.

Combining all of this we get

$$\begin{aligned}
((\Pi \otimes \pi) \circ \Delta_H)(a\gamma(t)) &= (\Pi \otimes \mathrm{id}_T)(\rho_T(a\gamma(t))) \\
&= (\Pi \otimes \mathrm{id}_T)\rho_T(a)\rho_T(\gamma(t)) \\
&= (\Pi \otimes \mathrm{id}_T)\left((a \otimes 1_T)\sum(\gamma(t_1) \otimes t_2)\right) \\
&= (\Pi \otimes \mathrm{id}_T)\left(\sum a\gamma(t_1) \otimes t_2\right) \\
&= \sum a \otimes \varepsilon_T(t_1)t_2 = a \otimes t,
\end{aligned}$$

where we use the characterisation of Π given in (4.3) for the last equality. This gives us Equation (4.6) as required. □

Corollary 4.6. *Under the notation and hypotheses of Section 4.1.1, with A^0_Π and ξ as in Definition 4.3 and Equation (4.5) respectively, the map ξ is injective.*

Proof. Suppose $\alpha = \sum f_i \otimes g_i \in \ker \xi$ for some $f_i \in A_\Pi^0, g_i \in T^0$. We can assume without loss of generality that the f_i are linearly independent.

We have

$$0 = \xi \left(\sum (f_i \otimes g_i) \right) (a\gamma(t)) = \sum f_i(a) g_i(t) = \left(\sum g_i(t) f_i \right) (a)$$

by the linearity of ξ and by Lemma 4.5. As this is true for all $a \in A$, it means that

$$\sum g_i(t) f_i = 0 \quad \text{for all } t \in T.$$

By the linear independence of the f_is, this means that for every i we have $g_i(t) = 0$ for all $t \in T$ and hence $g_i = 0$. So $\alpha = 0$ as required. □

We find that ξ always gives rise to a Hopf algebra embedding on T^0.

Lemma 4.7. *Under the notation and hypotheses of Section 4.1.1 and with ξ as in (4.5), the map $\xi|_{k \otimes T^0} : T^0 \to H^0$ is an embedding of Hopf algebras.*

Proof. By the definition of ξ, we have $\xi|_{k \otimes T^0} = \pi^0$. We know that π is a surjective map of Hopf algebras, so by Corollary 3.6 it is a finite overlay. So by Theorem 3.12 (iii) and (vi), π^0 is an injective map of Hopf algebras as required. □

In general, however, we cannot say much more about ξ. Although we know $\xi|_{T^0}$ is always a Hopf embedding, the whole of ξ is neither an algebra map (Example 4.8) nor a coalgebra map (see Example 4.22) in general, and it need not give rise to an isomorphism $H^0 \cong A_\Pi^0 \#_\tau T^0$. The problem is that $\xi|_{A_\Pi^0}$ need not be an algebra map and the coinvariants are always a subalgebra of H^0. The following example shows how this can fail.

Example 4.8. Let k be an algebraically closed field of characteristic zero and $G = T(3, k)$ be the three-dimensional Heisenberg group, so the subgroup of $M_3(k)$ given by those upper triangular matrices with all diagonal entries equal to 1_k. Set $H := \mathcal{O}(G)$.

Of course, the finite duals of coordinate rings of affine algebraic groups are well-understood and we discussed the general case in Proposition 2.13. In particular, we know that

$$\mathcal{O}(G)^0 \cong U(\operatorname{Lie} G) \# kG,$$

where $G \subseteq \mathcal{O}(G)^0$ consists of the grouplike elements and $\operatorname{Lie} G$ of the primitive elements. Here, we put this aside and instead look at a particular decomposition of $\mathcal{O}(T(3, k))$ to show how ξ does not always give us a crossed product structure on H^0.

Since, as a variety, G is affine 3-space, we have $H \cong k[x, y, z]$ as algebras, setting $x = X_{12}$, $y = X_{23}$ and $z = X_{13}$, where X_{ij} is the map that picks out the $(i, j)th$ entry of a matrix. Moreover, by the definition of the coproduct on coordinate rings we calculate easily that x and y are primitive and

$$\Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y.$$

So we can write

$$H \cong k[\tilde{x}, \tilde{y}] \otimes k[\tilde{z}],$$

where the isomorphism from left to right sends x to $\tilde{x}$, y to $\tilde{y}$ and z to $\tilde{z}$.

This is a crossed product of Hopf algebras, arising from the factor map $\pi : H \to k[\tilde{z}]$ given by factoring out by the Hopf ideal $\langle x, y\rangle$. The cleaving map is $\gamma : k[\tilde{z}] \to H$ given by $\gamma(\tilde{z}^i) = z^i$. This satisfies $\pi \circ \gamma = \mathrm{id}_{k[\tilde{z}]}$. Let $A = k[\tilde{x}, \tilde{y}]$ and $T = k[\tilde{z}]$.

In this case, $\Pi : H \to A$ is the map given by factoring out by the ideal generated by z. This map is an algebra map and hence a finite overlay by Corollary 3.6. Hence, by Theorem 3.12 (iii) and (iv) we have a coalgebra embedding $\Pi^0 : A^0 \to H^0$. This in turn means, by the definition of A^0_Π in Definition 4.3, that we have

$$A^0_\Pi = A^0.$$

So we have an injective linear map

$$\xi : A^0 \otimes T^0 \to H^0.$$

By Lemma 4.5, it is given by

$$\xi(f \otimes g)(x^i y^j z^k) = f(x^i y^j)g(z^k).$$

We want to show that ξ does not give rise to any algebra isomorphism $H^0 \cong A^0 \#_\sigma T^0$. We do this by showing that $\mathrm{im}\, \Pi^0 = \mathrm{im}\, \xi|_{A^0 \otimes k\varepsilon_T}$ is not a subalgebra of H^0.

First note that it is clear from the definition of Π that

$$\mathrm{im}\, \Pi^0 = \{f \in H^0 \mid f(x^i y^j z^k) = 0 \text{ whenever } k > 0\}.$$

Let f be the character of H given by $f(x) = 1, f(y) = 0, f(z) = 0$ and g be the one given by $g(x) = 0, g(y) = 1, g(z) = 0$. Because these are algebra maps, they are contained in H^0, and further are elements of $\mathrm{im}\, \Pi^0$ by the above. Now we have

$$\begin{aligned} fg(z) &= \sum f(z_1)g(z_2) \\ &= f(z)g(1) + f(1)g(z) + f(x)g(y) \\ &= 0 + 0 + 1 = 1 \end{aligned}$$

So $fg(z) \neq 0$, meaning that $fg \notin \operatorname{im} \Pi^0$. Hence $\operatorname{im} \Pi^0$ is not closed under multiplication and is not a subalgebra.

4.3 A necessary and sufficient condition for ξ to be bijective

Throughout, we retain the notations and hypotheses of Sections 4.1.1, and recall the definition of A^0_Π from Definition 4.3 and that of ξ from Equation (4.5).

In the previous section, we defined a map $\xi : A^0_\Pi \otimes T^0 \to H^0$ which we aim to use to describe H^0 in terms of A^0_Π and H^0. We know by Corollary 4.6 that ξ is always injective. In order to use ξ to determine the structure of H^0, we need it to be an isomorphism. In this section, we find a necessary and sufficient condition for ξ to be surjective in Theorem 4.9. The condition is quite technical, but Corollaries 4.11 and 4.12 show it holds in two situations of interest: when T is finite-dimensional and when $H \cong A \otimes T$ as algebras.

Theorem 4.9. *Recall the notation and assumptions of Section 4.1.1 and let A^0_Π be as in Definition 4.3 and ξ be as in (4.5). Then the following are equivalent:*

(i) ξ is bijective.

(ii) γ is a finite overlay and $\operatorname{im} \iota^0 \subseteq A^0_\Pi$.

(iii) γ is a finite overlay and $\operatorname{im} \iota^0 = A^0_\Pi$.

Proof. (ii) $\Leftrightarrow$ (iii):

The direction (iii) $\Rightarrow$ (ii) is obvious. As for (ii) $\Rightarrow$ (iii), all we need to do to show this is that $A^0_\Pi \subseteq \operatorname{im} \iota^0$ always.

Let $f \in A^0_\Pi$. This means that $f \in A^0$ is a map such that $f \circ \Pi \in H^0$. Since $\Pi|_A = \mathrm{id}_A$, or equivalently $\Pi \circ \iota = \mathrm{id}_A$, we have

$$\iota^0(f \circ \Pi) = f \circ \Pi \circ \iota = f,$$

and so $f \in \operatorname{im} \iota^0$. This gives us what we need.

(i) $\Rightarrow$ (ii):

Assume ξ is bijective. We want to show that γ is a finite overlay and $\operatorname{im} \iota^0 \subseteq A^0_\Pi$.

Choose $f \in H^0$. Since ξ is bijective, there exists $n \geq 1$ along with $g_i \in A^0_\Pi$ and $h_i \in T^0$ for each $1 \leq i \leq n$ such that

$$f = \xi\left(\sum_{i=1}^{n} g_i \otimes h_i\right) = \sum_{i=1}^{n} (g_i \circ \Pi)(h_i \circ \pi).$$

By Lemma 4.5, this means that given $a \in A, t \in T$ we have

$$f(a\gamma(t)) = \sum g_i(a)h_i(t). \tag{4.7}$$

Now we want to show that γ is a finite overlay. By Lemma 3.5, this is equivalent to showing $\operatorname{im}\gamma^0 \subseteq T^0$. So it sufficies to show that $\gamma^0(f) \in T^0$.

Using Equation (4.7), we find that

$$\gamma^0(f)(t) = f(\gamma(t)) = \sum g_i(1_A)h_i(t),$$

which means that $\gamma^0(f)$ is a linear combination of the h_i. Since $h_i \in T^0$, it follows that $\gamma^0(f) \in T^0$. So γ is a finite overlay as required.

Showing that $\operatorname{im}\iota^0 \subseteq A^0_\Pi$ is similar: it suffices to show that given $f \in H^0$ we have $\iota^0(f) \in A^0_\Pi$. Equation (4.7) tells us that given $a \in A$, we have

$$\iota^0(f)(a) = f(\iota(a)) = f(a\gamma(1_T)) = \sum g_i(a)h_i(1_T).$$

Again, this means that $\iota^0(f)$ is a linear combination of the g_i, which are all in A^0_Π by assumption. Since A^0_Π is a subspace of A^0, $\iota^0(f) \in A^0_\Pi$ as required.

(ii) ⇒ (i):

(This proof is inspired by [13, Lemma 1.5.2].)

Assume γ is a finite overlay and $\operatorname{im}\iota^0 \subseteq A^0_\Pi$. We want to show that ξ is bijective. By Corollary 4.6, ξ is injective. So all that remains to be shown is that ξ is surjective.

Given $g \in H^0$, define $g^A := \iota^0(g) = g|_A \in A^*$ and $g^T := \gamma^0(g) \in T^*$, so we have $g^T(t) = g(\gamma(t))$ for $t \in T$. We know that γ is a finite overlay by assumption, so by Lemma 3.5 $g^T \in T^0$. Moreover, $\operatorname{im}\iota^0 \subseteq A^0_\Pi$ by assumption, so $g^A \in A^0_\Pi$.

Now let $f \in H^0$ be any map and consider the element

$$\widehat{f} := \sum (f_1)^A \otimes (f_2)^T.$$

This is in $A^0_\Pi \otimes T^0$ because H^0 is a coalgebra and so f being in H^0 implies that each $f_1, f_2 \in H^0$. So $\widehat{f}$ is in the domain of ξ. We want to show that $\xi(\widehat{f}) = f$.

By Lemma 4.5, given any $a \in A$ and $t \in T$, we have

$$\xi(\widehat{f})(a\gamma(t)) = \sum m_{H^0}\left((f_1)^A(a) \otimes (f_2)^T(t)\right) = \sum f_1(a)f_2(\gamma(t)) = f(a\gamma(t)).$$

Thus $\xi(\widehat{f}) = f$, and so ξ is surjective as required. □

Corollary 4.10. *Keep the notation and assumptions of Section 4.1.1, and let ξ be as in (4.5) and A^0_Π be as in Definition 4.3. If ξ is bijective, then A^0_Π is a subcoalgebra of A^0.*

Proof. By Theorem 4.9 (iii), if ξ is bijective we have $A^0_\Pi = \operatorname{im} \iota^0$. We know that ι is an algebra map, so by Corollary 3.6 it is a finite overlay. This means that by Theorem 3.12, $\iota^0 : H^0 \to A^0$ is a coalgebra map and hence $\operatorname{im} \iota^0 = A^0_\Pi$ is a subcoalgebra of A^0. □

Although we have found necessary and sufficient conditions for ξ to be bijective, we do not know if they are meaningful: we know of no example where they do not hold, so it is unclear whether they are always satisfied. We record this in the following question.

Question 4.A. Is ξ always bijective?

Note that the condition that $\operatorname{im} \iota^0 \subseteq A^0_\Pi$ is trivially true whenever Π is a finite overlay, as in this case Lemma 4.4 tells us that $A^0_\Pi = A^0$. This gives rise to the following two corollaries. We shall give a further case where Question 4.A has a positive answer in Section 4.4.

Corollary 4.11. *Using the notation and assumptions of Section 4.1.1, assume T is finite-dimensional. Then ξ is bijective and $A^0_\Pi = A^0$.*

Proof. By Lemma 4.4, $A^0_\Pi = A^0$ if and only if Π is a finite overlay, and in this case we immediately have $\operatorname{im} \iota^0 \subseteq A^0_\Pi$. Thus Theorem 4.9 tells us that it suffices to show that Π and γ are finite overlays.

Recalling that we have $\gamma(1_T) = 1_H$, we have

$$H \cong A\gamma(1_T) \oplus A\gamma(T^+) = A \oplus A\gamma(T^+)$$

as left A-modules. Since T is finite-dimensional $A\gamma(T^+)$ is finitely-generated as an A-module. So Lemma 3.16 tells us that the projection from H to A given by sending $A\gamma(T^+)$ to zero gives a finite overlay of A by H. This map is precisely Π by definition.

Now consider $\gamma : T \to H$. This is a finite overlay of H because T is finite-dimensional: given any ideal $I \subseteq H$ of finite codimension, we have $\gamma(\{0\}) = \{0\} \subseteq I$, and $\{0\}$ is an ideal of finite codimension in T. □

Corollary 4.12. *Using the notation and assumptions of Section 4.1.1, assume both σ and the action of T on A are trivial, i.e.*

$$H \cong A \otimes T \qquad \textit{as algebras.}$$

Then ξ is bijective and $A^0_\Pi = A^0$.

Proof. By Lemma 4.4, $A^0_\Pi = A^0$ if and only if the map $\Pi : H \to A$ given by factoring along $A\gamma(T^+)$ is a finite overlay, and in this case we immediately have $\operatorname{im} \iota^0 \subseteq A^0_\Pi$. Thus Theorem 4.9 tells us that it suffices to show that Π and γ are finite overlays.

We know that γ is an algebra map because it is just the canonical inclusion map and T is a subalgebra of H. Furthermore, we know that Π is an algebra map by Lemma 2.31, which states that this is a necessary and sufficient condition for σ and the action of T on A to be trivial. So by Corollary 3.6, both γ and Π are finite overlays as required. □

This is a less trivial situation than it might seem at first glance, because although the algebra structure is trivial, we have made no extra assumptions about the coalgebra structure on H. For instance, Example 4.8 falls into this category and we have seen that its coalgebra structure is nontrivial.

We generalise the first part of this result - that ξ is bijective - to smash products in the next section, working under the additional assumption that γ is a coalgebra map.

4.4 The case where σ is trivial

We retain the notation and hypotheses of Section 4.1.1, and let A^0_Π be as in Definition 4.3 and ξ be as in (4.5). We also assume throughout that the cleaving map $\gamma : T \to H$ is a coalgebra map. As seen in Example 2.32 this is a nontrivial assumption, and we will see in Remark 4.14 that the results of this section do not hold without it.

In Section 4.3 we investigated the question of when ξ was bijective. What we have not done so far is look at when ξ tells us something about the algebra structure of H^0, which is really what we are interested in.

In this section, we investigate the situation where H is a smash product. In other words, σ is trivial, which by Lemma 2.30 is equivalent to γ being an algebra map. It turns out that here ξ is always bijective, and further that the isomorphism it gives rise to determines the algebra structure of H^0. In particular, ξ tells us that H^0 is itself not just a crossed product but a smash product.

This result is somewhat surprising because as we will see, in the case where A^0 has a canonical algebra structure coming from A being a Hopf algebra, Π^* and so $\xi|_{A^0_\Pi}$ need not be an algebra map with respect to it. We will discuss this further in Remark 4.15 and look at an example in Example 4.16. For now, this issue means we write $\xi(A^0_\Pi)$ instead of A^0_Π for the copy of A^0_Π mapped to H^0 by ξ in the theorem:

Theorem 4.13. *Keep the notation and assumptions of Section 4.1.1 with A^0_Π be as in Definition 4.3 and ξ as in Equation (4.5). Assume further that H is a smash product and T is a Hopf subalgebra of H, meaning that*

(1) γ is a coalgebra map.

(2) σ is trivial:

$$\sigma(s,t) = \varepsilon_T(s)\varepsilon_T(t) \qquad \text{for all } s,t \in T.$$

Then H^0 is a smash product:

$$H^0 \cong \xi(A^0_\Pi)\#T^0,$$

where the isomorphism is one of algebras and given by ξ.

Proof. First, we show that there is a Hopf surjection $H^0 \to T^0$ given by γ^0, and that there is also a cleaving map $T^0 \to H^0$. By Proposition 2.22, this means we have an algebra isomorphism

$$H^0 \cong \left(H^0\right)^{coT^0} \#_\tau T^0$$

for a given cocycle τ and action of T^0 on $\left(H^0\right)^{coT^0}$. Then we show that τ is trivial, that the coinvariants of H^0 are just $\xi(A^0_\Pi)$ and that the isomorphism thus constructed is just ξ.

Throughout, when we derive a fact we will use later in the proof, we set it apart and label it for easier reading and referring back.

Step 1: $\gamma^0 : H^0 \to T^0$ is a surjective Hopf map:

First note that under the standard hypotheses of Section 4.1.1, π^0 is always an embedding of Hopf algebras. We can see this fact follows from the results in Chapter 3:

By Corollary 3.6,

$$\pi \text{ is a finite overlay.} \tag{A}$$

This means that by Theorem 3.12 (iii) and (vi), π^0 is injective and a Hopf algebra map:

$$\pi^0 : T^0 \to H^0 \text{ is an injective Hopf algebra map.} \tag{B}$$

The extra assumptions we have made tell us that γ and γ^0 are both Hopf algebra maps. To see this, we note that by assumption (1) γ is a coalgebra map and by assumption (2) σ is trivial and so H is a smash product. By Lemma 2.30, the latter is equivalent to γ being an algebra map. Since T is a Hopf algebra, this means that $\gamma : T \to H$ is a bialgebra map between Hopf algebras, and by [48, Lemma 4.0.4]

$$\gamma \text{ is a map of Hopf algebras.} \tag{C}$$

So again we see by Corollary 3.6 that

$$\gamma \text{ is a finite overlay.} \tag{D}$$

So by Theorem 3.12 (vi)

$$\gamma^0 : H^0 \to T^0 \text{ is a Hopf algebra map.} \tag{E}$$

Finally, we want to show that γ^0 is surjective. First, recall that γ is injective because it is a cleaving map. Since by (C) γ is an algebra map, this means the hypotheses of Lemma 3.15 hold, and this along with π being a finite overlay ((A)) implies that γ is reciprocal. As γ is a finite overlay ((D)), by Lemma 3.10 this means we have

$$\gamma^0 : H^0 \to T^0 \text{ is surjective.} \tag{F}$$

Step 2: H^0 is isomorphic to a crossed product.

By the above, we have a surjective Hopf epimorphism $\gamma^0 : H^0 \to T^0$. By Proposition 2.22, this means that we only need to find a cleaving map, in other words a convolution invertible right T^0-comodule map, from T^0 to H^0 in order to gain a crossed product structure on H^0.

We see that π^0 is a cleaving map as follows:

We know that π^0 is an injective Hopf algebra map by (B), and moreover by Lemma 2.20 the fact that the cleaving map γ is a coalgebra map (assumption (1)) means that we have $\pi \circ \gamma = \mathrm{id}_T$, which in turn gives us

$$\gamma^0 \circ \pi^0 = \mathrm{id}_{T^0} .$$

Also by Lemma 2.20, any coalgebra splitting of γ^0 provides a cleaving map, so

$$\pi^0 : T^0 \to H^0 \text{ is a cleaving map for the extension } \left(H^0\right)^{co\gamma^0} \subseteq H^0. \tag{G}$$

By Proposition 2.22, this means we have

$$H^0 \cong \left(H^0\right)^{co\gamma^0} \#_\tau T^0.$$

In fact, since by (B) the cleaving map π^0 is an algebra map, by Lemma 2.30 τ must be trivial. So as algebras, we have

$$H^0 \cong \left(H^0\right)^{co\gamma^0} \# T^0, \tag{4.8}$$

a smash product of $\left(H^0\right)^{co\gamma^0}$ by the Hopf subalgebra T^0.

Step 3: $\left(H^0\right)^{co\gamma^0} = \xi(A^0_\Pi)$

Let $f \in H^0$. Then $f \in \left(H^0\right)^{co\gamma^0}$ if and only if

$$f \otimes \varepsilon_T = \rho(f) := (\mathrm{id}_{H^0} \otimes \gamma^0) \circ \Delta_{H^0}(f) = \sum f_1 \otimes f_2 \circ \gamma \tag{4.9}$$

In particular, given $h = a\gamma(s) \in H$ for some $a \in A$ and $s \in T$, and also given $t \in T$, the RHS of (4.9) evaluated at $h \otimes t$ becomes

$$\rho(f)(h \otimes t) = \sum f_1(h) f_2(\gamma(t)) = f(h\gamma(t)) = f(a\gamma(s)\gamma(t)) = f(a\gamma(st)),$$

because γ is an algebra map by (C).

Moreover, the LHS of (4.9) evaluated at $h \otimes t$ becomes

$$(f \otimes \varepsilon_T)(h \otimes t) = f(h)\varepsilon_T(t) = f(a\gamma(s))\varepsilon_T(t).$$

So $\rho(f) = f \otimes \varepsilon_T$ if and only if for all $a \in A$ and $s, t \in T$ we have

$$f(a\gamma(st)) = f(a\gamma(s))\varepsilon_T(t).$$

This is true if and only if it is true when $s = 1$, so when for all $a \in A, t \in T$ we have

$$f(a\gamma(t)) = f(a)\varepsilon_T(t) = f|_A(\Pi(a\gamma(t))).$$

Equivalently, $f \in \left(H^0\right)^{co\gamma^0}$ if and only if

$$f = \left(\Pi^0 \circ \iota^0\right)(f).$$

Since $\iota^0(f) \in A^0$, every coinvariant of T^0 must be of the form $\Pi^0(g)$ for some $g \in A^0$. Moreover, any $g \in A^0$ satisfies $\iota^0(\Pi^0(g)) = g$ and hence

$$\left(\Pi^0 \circ \iota^0\right)(\Pi^0(g)) = \Pi^0(g),$$

which by the above tells us that as long as $\Pi^0(g)$ is a map in H^0 it will be a coinvariant. This means that the coinvariants of T^0 are precisely the elements in the image of Π^0 which are contained in H^0. By definition, this is just $\Pi^0(A^0_\Pi) = \xi(A^0_\Pi)$.

This means that Equation (4.8) becomes

$$H^0 \cong \xi(A^0_\Pi)\#T^0 \tag{4.10}$$

Step 4: The isomorphism in Equation (4.10) is given by ξ:

Since ξ is injective, we can identify $\xi(A^0_\Pi)$ with A^0_Π. Moreover, since the coinvariants are always a subalgebra, this means we can always put an algebra structure on A^0_Π such that $\xi|_{A^0_\Pi} = \Pi^0|_{A^0_\Pi}$ is an algebra map with respect to it. (See Remark 4.15 for a discussion on how this may compare to a canonical algebra structure on A^0.)

Let $\phi : A^0_\Pi\#T^0 \to H^0$ denote the algebra isomorphism we get via Equation (4.10) in this way. By Proposition 2.22 and the fact that π^0 is the cleaving map, this map is given by $\phi(f\#g) = \psi(f)\pi^0(g)$, where $\psi : A^0_\Pi \to H^0$ is the canonical embedding. This means ψ is just $\Pi^0|_{A^0_\Pi}$. So for all $f \in A^0_\Pi, g \in T^0$ we have

$$\phi(f\#g) = m_{H^0} \circ \left(\Pi^0|_{A^0_\Pi} \otimes \pi^0\right)(f \otimes g) = \xi(f \otimes g).$$

□

Remark 4.14. If we remove the assumption that γ is a coalgebra map, the theorem does not hold in general. We have already seen an example where all other hypotheses hold but the statement of the theorem is false: Example 4.8 shows that if $H = \mathcal{O}(T(3, k))$, we have $H \cong k[x, y] \otimes k[z]$, which is not just a smash product but actually has trivial action as well. However, the subspace

$$\xi(k[x, y]^0_\Pi) = \xi(k[x, y]^0) = \{f \in H^0 \mid f(x^i y^j z^\ell) = 0 \text{ when } \ell > 0\}$$

is not a subalgebra of H^0 and so ξ cannot give rise to any algebra isomorphism between $\xi(k[x, y]^0_\Pi)\#k[z]^0$ and H^0.

Remark 4.15. The algebra structure on A^0_Π induced by ξ does not have to be compatible with any canonical algebra structure on A^0. In particular, we frequently have a Hopf algebra structure on A, in which case A^0 has a Hopf algebra structure as its finite dual. The next example shows that the algebra structure on A^0_Π induced by ξ does not have to agree with this canonical algebra structure on A^0, and this is true even if $A^0_\Pi = A^0$.

Example 4.16. Let k be an algebraically closed field of characteristic zero, n and t be coprime integers with $n > 1$ and $1 \leq t \leq n - 1$, and let q be a primitive nth root of unity in k. We take $H := H(n,t,q)$ to be the infinite-dimensional Taft algebra on these parameters as introduced in Example 1.14: that is, we have

$$H = k\langle x, g \mid xg = qgx, g^n = 1\rangle$$

as algebras, with g grouplike and $\Delta(x) = x \otimes 1 + g^t \otimes x$. This has a basis consisting of monomials $\{x^i g^j \mid i \geq 0,\ 0 \leq j \leq n-1\}$.

As discussed in Example 2.12, we have $H \cong k[x]\#kC_n$. This smash product decomposition arises from the factor Hopf map $H \to kC_n$ given by factoring out the Hopf ideal $\langle x\rangle$. The cleaving map γ is just the natural embedding of the Hopf subalgebra of H generated by g.

Since kC_n is finite dimensional, Lemma 3.16 tells us that the canonical quotient map $\Pi : H \to k[x]$, which is given by factoring out the $k[x]$-module $\sum_{i=1}^{n-1} k[x](g^i - 1)$, is a finite overlay. Thus by Lemma 4.4 we have $A^0_\Pi = A^0$, and so by Theorem 4.13 we have

$$H^0 \cong \xi\left(k[x]^0\right)\#kC_n^0 \cong \xi\left(k[x]^0\right)\#kC_n,$$

using the fact that by Example 1.20, $kC_n^0 \cong kC_n^* \cong kC_n$. We also know that $k[x]^0 \cong k[z] \otimes k(k,+)$ as Hopf algebras by Example 1.31. The question now is how the algebra structure of $k[x]^0$, viewed as a subalgebra of H^0 via ξ, compares to this canonical one.

Consider the element $z \in k[x]^0$. This is the map given by $z(x^i) = \delta_{i1}$. Thus the map $\overline{z} := \Pi^0(z) \in H^0$ is given by

$$\overline{z}(x^i g^j) = z(\Pi(x^i g^j)) = z(x^i) = \delta_{i1}.$$

We want to show that this is nilpotent: in particular, that we have $\overline{z}^n = 0$.

Let $H(i) := k\{x^i g^j \mid 0 \leq j \leq n-1\}$, so with $H(i)$ consisting of those terms with x-degree i. This is a coalgebra grading as follows.

Let $A(i) := \bigoplus_{j=0}^{i} H(j)$. Now $A(0) = k\{g^j \mid 0 \leq j \leq n-1\}$ is a Hopf subalgebra of H, $A(1)$ generates H as an algebra and $A(i) = A(1)^i$. Moreover, $A(0)A(1) = A(1)A(0) = A(1)$ and

$$\Delta(A(1)) \subseteq A(1) \otimes A(0) + A(0) \otimes A(1).$$

This means that by [36, Lemma 5.5.1], $\{A(i)\}_{i\geq 0}$ is a coalgebra filtration. The associated graded coalgebra is just H with grading $H(i)$.

This means that because $\overline{z}$ is only nonzero on $H(1)$, $\overline{z}^n$ can only be nonzero on $H(n)$, in particular on those $h \in H(n)$ such that $\Delta^n_H(h)$ has a summand in $H(1)^{\otimes n}$, where Δ^i_H is recursively defined by $\Delta^0_H = \Delta_H$, $\Delta^i_H = (\mathrm{id} \circ \Delta_H) \circ \Delta^{i-1}_H$.

However, as we saw in Example 2.12, x^n is primitive. Therefore, for all $0 \leq j \leq n-1$ we have

$$\Delta(x^n g^j) \in H(n) = x^n g^j \otimes g^j + g^j \otimes x^n g^j \in H(n) \otimes H(0) + H(0) \otimes H(n),$$

which means that

$$\Delta^n(x^n g^j) \in \bigoplus_{i=0}^{n-1} H(0)^{\otimes i} \otimes H(n) \otimes H(0)^{\otimes(n-i-1)}.$$

Since the monomials $x^n g^j$ give a basis for $H(n)$, it follows that there exists no $h \in H(n)$ such that $\Delta^n(h)$ has a summand in $H(1)^{\otimes n}$. Thus we have $\overline{z}^n = 0$.

So although there is an embedding $\Pi^0 : k[x]^0 \to H^0$, and although the image of this embedding is a subalgebra in H^0, the resulting algebra structure is not the same as the canonical one on $k[x]^0$. The element z which generates a polynomial ring in $k[x]^0$ gets mapped to a nilpotent element by Π^0.

The fact that this problem arises is not that surprising, because the definition of a crossed product we are using does not guarantee a coalgebra structure on A at all, and certainly not one that is compatible with both the algebra structure of A and the coalgebra structure of H. In the dual setting, this means there is not necessarily a canonical algebra structure on A^0, and that when we do have one it need not be compatible with both the coalgebra structure on A^0 and the multiplication in H^0. This means that the algebra isomorphism guaranteed by Theorem 4.13 is less useful than we would like it to be.

As a result, it makes sense to look at a specific case where we have a given canonical coalgebra - in fact, a Hopf algebra - structure on A compatible with that of H, namely when A is a Hopf subalgebra.

4.5 The case where A is a Hopf subalgebra

Throughout this section, we keep the notation of assumptions of Section 4.1.1, with A^0_Π as in Definition 4.3 and ξ as in (4.5). We also assume that γ is a coalgebra map.

In the last section, we saw that even when ξ gives rise to a description of the algebra structure of H^0 as a crossed product of A^0_Π and T^0, this does not always preserve the algebra structure of A^0_Π as we might like - in particular, the algebra structure of $\xi(A^0_\Pi)$ viewed as a subalgebra of H^0 does not have to coincide with that of A^0 when such exists. The reason for this is because the definition of the crossed product makes no reference to any coalgebra structure on A.

Therefore, it makes sense to look at a situation in which we are guaranteed a coalgebra structure on A compatible with its algebra structure and the coalgebra structure of H. One such situation which arises frequently is that where A is not just a subalgebra but in fact a Hopf subalgebra of H. We look at this in this section, under the additional assumption that the cleaving map γ is a coalgebra map and hence T is a subcoalgebra of H. Our main result is Theorem 4.19, which tells us that in this case $\xi : A^0_\Pi \otimes T^0 \to H^0$ is an algebra map. Hence, when ξ is known to be bijective (discussed in Section 4.3), it follows that $H^0 \cong A^0_\Pi \otimes T^0$ as algebras.

First, we note that under our new assumptions Π is always a coalgebra map with respect to the Hopf algebra structures of H and A, something that Example 4.16 and Example 4.8 made clear is not true in general.

Lemma 4.17. *Under the same assumptions and notations as in Section 4.1.1, assume γ is a coalgebra map and suppose A is a Hopf subalgebra of H. Then $\Pi : H \to A$ is a coalgebra map with respect to the canonical coalgebra structures on H and A.*

Proof. Step 1: Π *preserves the coproduct*

Given $a \in A, t \in T$ we have

$$\begin{aligned}
((\Pi \otimes \Pi) \circ \Delta_H)(a\gamma(t)) &= (\Pi \otimes \Pi)\left(\sum a_1\gamma(t)_1 \otimes a_2\gamma(t)_2\right) \\
&= (\Pi \otimes \Pi)\left(\sum a_1\gamma(t_1) \otimes a_2\gamma(t_2)\right) \\
&= \sum a_1\varepsilon_T(t_1) \otimes a_2\varepsilon_T(t_2) \\
&= \Delta_A(a)\varepsilon_T(t) \\
&= (\Delta_A \circ \Pi)(a\gamma(t))
\end{aligned}$$

as required.

Step 2: Π *preserves the counit*

Given $a \in A, t \in T$ we have

$$(\varepsilon_A \circ \Pi)(a\gamma(t)) = \varepsilon_A(a)\varepsilon_T(t) = \varepsilon_H(a)\varepsilon_H(\gamma(t)) = \varepsilon_H(a\gamma(t))$$

as required.

□

Remark 4.18. Note that this result does not hold true in general if γ is not a coalgebra map. For instance, in Example 4.8 we saw that for

$$H := \mathcal{O}(T(3,k)) \cong k[x,y] \otimes k[z]$$

the corresponding map $\Pi : H \to k[x, y]$ is given by factoring by the ideal $\langle z\rangle$. This is not a coideal and Π is not a map of coalgebras. Here $k[x, y]$ is a Hopf subalgebra of H, but the cleaving map $\gamma : k[z] \to H$ is not a map of coalgebras. However, γ being a coalgebra map is not a necessary condition and there are cases where it is not and the result of the lemma still holds. This is discussed further in Chapter 5.

In fact, when A is a Hopf subalgebra and γ is a coalgebra map, we can say more about the algebra structure of H^0. Since both A and $\gamma(T)$ are subcoalgebras of H, we have

$$H \cong A \otimes T \quad \text{as coalgebras.}$$

This is regardless of whether H is a crossed or smash product, and indeed we do not assume that the cocycle σ is trivial. Since the coalgebra structure determines the algebra structure of the dual, the following result seems natural:

Theorem 4.19. *Using the notation and assumptions of Section 4.1.1, let A^0_Π be as in Definition 4.3 and ξ as in (4.5). Suppose γ is a coalgebra map and ι is a map of Hopf algebras. Then*

(i) $A^0_\Pi \subseteq A^0$ is a subalgebra of A^0 with respect to the canonical multiplication coming from the coproduct in A.

(ii) ξ is an algebra homomorphism.

(iii) $A^0_\Pi \otimes T^0 \subseteq H^0$ as algebras.

(iv) If ξ is surjective, there is a Hopf surjection $\phi : H^0 \to A^0_\Pi$ given by

$$\phi(\xi(f \otimes g)) = g(1_H)f$$

for $f \in A^0_\Pi$, $g \in T^0$.

Proof. (i) *A^0_Π is a subalgebra of A^0:*

We know A is a Hopf subalgebra of H by assumption. So by Lemma 4.17, Π is a coalgebra map.

Now let $f, g \in A^0_\Pi$, so we have $f, g \in A^0$ such that $f \circ \Pi, g \circ \Pi \in H^0$. Then

$$(fg) \circ \Pi = \mu \circ (f \otimes g) \circ \Delta_A \circ \Pi = \mu \circ (f \otimes g) \circ (\Pi \otimes \Pi) \circ \Delta_H = (f \circ \Pi)(g \circ \Pi).$$

Since H^0 is a subalgebra, this means that $(fg) \circ \Pi \in H^0$ and so $fg \in A^0_\Pi$.

Moreoever, we have $\varepsilon_A \circ \Pi = \varepsilon_H \in H^0$, so $\varepsilon_A \in A^0_\Pi$. Thus A^0_Π is closed under multiplication and contains the identity element: it is a subalgebra as required.

(ii) ξ *is an algebra map:*

First we note that $\Pi^0|_{A^0_\Pi}$ and π^0 are both algebra maps. Π^* is an algebra map because by Lemma 4.17, Π is a coalgebra map, and by Remark 3.13 this means Π^* is an algebra map. As a result, $\Pi^0|_{A^0_\Pi}$, which is simply the restriction of Π^* to a subalgebra of A^0 and therefore a subalgebra of A^*, is also an algebra map. On the other hand, π is a map of Hopf algebras, so a finite overlay by Corollary 3.6 and so by Theorem 3.12(vi) π^0 is also a map of Hopf algebras.

We need to show that given $f, \tilde{f} \in A^0_\Pi$ and $g, \tilde{g} \in T^0$ we have

$$\xi((f \otimes g)(\tilde{f} \otimes \tilde{g})) = \xi(f \otimes g)\xi(\tilde{f} \otimes \tilde{g}). \tag{4.11}$$

Note that the LHS of this is given by

$$\xi((f \otimes g)(\tilde{f} \otimes \tilde{g})) = \xi(f\tilde{f} \otimes g\tilde{g}) = \Pi^0(f\tilde{f})\pi^0(g\tilde{g}) = \Pi^0(f)\Pi^0(\tilde{f})\pi^0(g)\pi^0(\tilde{g})$$

by definition of ξ and because, as discussed above, π^0 and $\Pi^0|_{A^0_\Pi}$ are both algebra maps. The definition of ξ means the RHS of (4.11) becomes

$$\xi(f \otimes g)\xi(\tilde{f} \otimes \tilde{g}) = \Pi^0(f)\pi^0(g)\Pi^0(\tilde{f})\pi^0(\tilde{g}).$$

In particular, ξ is a homomorphism if and only if

$$\pi^0(g)\Pi^0(\tilde{f}) = \Pi^0(\tilde{f})\pi^0(g),$$

for all $\tilde{f} \in A^0_\Pi$ and $g \in T^0$, that is, if and only if $\Pi^0(A^0_\Pi)$ and $\pi^0(T^0)$ commute.

Let $g \in T^0$, $f \in A^0_\Pi$, $a \in A$ and $t \in T$. Because we have assumed A a Hopf subalgebra and γ a coalgebra map we know that both A and $\gamma(T)$ are subcoalgebras. So we have

$$\Delta_H(a\gamma(t)) = \sum a_1\gamma(t)_1 \otimes a_2\gamma(t)_2 = \sum a_1\gamma(t_1) \otimes a_2\gamma(t_2)$$

with $a_1, a_2 \in A$. This means that

$$\begin{aligned}\pi^0(g)\Pi^0(f)(a\gamma(t)) &= \sum (g \circ \pi)(a_1\gamma(t_1))(f \circ \Pi)(a_2\gamma(t_2))\\ &= \sum \varepsilon_A(a_1)g(t_1)\varepsilon_T(t_2)f(a_2)\\ &= f(a)g(t).\end{aligned}$$

Now we know that

$$\Pi^0(f)\pi^0(g)(a\gamma(t)) = \xi(f \otimes g)(a\gamma(t)) = f(a)g(t)$$

by Lemma 4.5. So $\Pi^0(f)$ and $\pi^0(g)$ commute, and so ξ respects multiplication.

Furthermore, we have

$$\xi(\varepsilon_A \otimes \varepsilon_T)(a\gamma(t)) = \varepsilon_A(a)\varepsilon_T(t) = \varepsilon_H(a)\varepsilon_H(\gamma(t)) = \varepsilon_H(a\gamma(t))$$

by Lemma 4.5 and the fact that A is a subcoalgebra and γ is a coalgebra map. So ξ respects the identity element: it is an algebra map.

(iii) The fact that $A^0_\Pi \otimes T^0 \subseteq H^0$ as algebras follows immediately from (i), (ii) and the fact that by Corollary 4.6, ξ is injective.

(iv) By (ii) and the assumption that ξ is surjective, we have

$$H^0 \cong A^0_\Pi \otimes T^0 \tag{4.12}$$

as algebras, where T^0 is a Hopf subalgebra. Now consider the right ideal $(T^0)^+H^0$. Identifying H^0 with $A^0_\Pi \otimes T^0$ and T^0 with $k \otimes T^0$ through the isomorphism in (4.12), we get

$$\begin{aligned}(T^0)^+H^0 &= (k \otimes (T^0)^+)(A^0_\Pi \otimes T^0)\\ &= A^0_\Pi \otimes (T^0)^+ \\ &= (A^0_\Pi \otimes T^0)(k \otimes (T^0)^+) = H^0(T^0)^+,\end{aligned} \tag{4.13}$$

since A^0_Π and T^0 commute. In particular, $(T^0)^+H^0$ is both a left and a right ideal and factoring along it simply gives us the map $\phi : H^0 \to A^0_\Pi$. All we have to show is that it is a Hopf ideal.

By [36, Lemma 3.4.2], it suffices to show that T^0 is normal in H^0, meaning that the left and right adjoint actions of H^0 on itself leave T^0 fixed. By [36, Proposition 3.4.3], this is true whenever H^0 is faithfully flat as a T^0-module and $(T^0)^+H = H(T^0)^+$. The former follows because by (4.12), H^0 is in fact a free T^0-module. The latter follows by (4.13).

□

Remark 4.20. Theorem 4.19 is not valid without the assumption that γ is a map of coalgebras. Again, Example 4.8 provides a counterexample. There we saw that for $H := \mathcal{O}(T(3,k))$, $H \cong k[x,y] \otimes k[z]$, where $k[x,y]$ is a Hopf subalgebra but the cleaving map γ is not a coalgebra map, and moreover that in this example $\xi(k[x,y]^0_\Pi) = \xi(k[x,y]^0)$ does not form a subalgebra of H^0. This of course means that $\xi : k[x,y]^0 \otimes k[z]^0 \to H^0$ cannot be an algebra map.

We will discuss the situation where A is a Hopf subalgebra but γ is not necessarily a coalgebra map further in Chapter 5.

Remark 4.21. We know of no examples where the inclusion in Theorem 4.19 (iii) is strict, or in other words where the inclusion map $\xi : A^0_\Pi \otimes T^0 \to H^0$ is not surjective and so the equivalent conditions guaranteeing its bijectivity given in 4.9 are not satisfied. It is possible they always are, but there is nothing in the theorem that guarantees this.

Theorem 4.19 provides an alternative way of computing the dual of the Taft algebra, using a different crossed product decomposition than the one used in Example 4.16.

Example 4.22. Let k be an algebraically closed field of characteristic zero, n, t be coprime integers with $n > 1$ and $1 \leq t \leq n-1$, and let $q \in k^*$ be a primitive nth root of unity. Take $H := H(n,t,q)$ be the infinite-dimensional Taft algebra on those parameters as in Example 2.12 or 4.16. So we have

$$H \cong k\langle x, g \mid xg = qgx, g^n = 1\rangle$$

as algebras with g grouplike and

$$\Delta_H(x) = x \otimes 1 + g^t \otimes x.$$

Recall from Example 2.12 that we can express H as

$$H \cong k[y]\#_\sigma T = A\#_\sigma T, \tag{4.14}$$

where $y = x^n$ and $T \cong H/\langle x^n\rangle$ is the finite-dimensional Taft algebra on the same parameters:

$$T \cong k\langle \overline{x}, \overline{g} \mid \overline{xg} = q\overline{gx}, \overline{g}^n = 1, \overline{x}^n = 0\rangle.$$

Here the action of T on $k[y]$ is trivial, so we have $t \cdot y = \varepsilon_T(t)y$ for all $t \in T$. The map γ is the inclusion given by $\gamma(\overline{x}^i\overline{g}^j) = x^i g^j$ for $0 \leq i, j \leq n-1$. Standard computation shows that γ is a coalgebra map.

However, γ is not a map of Hopf algebras, because $\gamma(\overline{x})^n \neq 0 = \gamma(\overline{x}^n)$. Thus by Lemma 2.30, σ is not trivial and the crossed product decomposition in Equation (4.14) is not a smash product.

Since T is finite-dimensional, Corollary 4.11 applies: ξ is bijective, and $A^0_\Pi = A^0$.

Recall from Example 4.16 that $y = x^n$ is primitive, so $k[y]$ is a Hopf subalgebra of H. Since γ is a coalgebra map, Theorem 4.19 applies. Its results can be combined with those of Corollary 4.11 to give us

$$H^0 \cong k[y]^0 \otimes T^0 \tag{4.15}$$

as algebras.

Now we find that $T^0 \cong T^* \cong T$ as Hopf algebras. The fact that finite-dimensional Taft algebras are self-dual is well-known in the $t = 1$ case - see for instance [39, Section 1.1]. It holds true when t is coprime to n as well: this is a standard calculation with the isomorphism given by the map $\phi : T \to T^0$ defined by

$$\phi(\overline{x})(\overline{x}^i\overline{g}^j) = \delta_{i1} \quad \text{and} \quad \phi(\overline{g})(\overline{x}^i\overline{g}^j) = \delta_{i0}q^{-t^{-1}j}.$$

Here t^{-1} denotes the inverse of t modulo n. Moreover, by Example 1.31, we have $k[y]^0 \cong k[v] \otimes k(k, +)$ as algebras, with

$$v(y^i) = \delta_{i1}. \tag{4.16}$$

So Equation (4.15) becomes

$$H^0 \cong k[v] \otimes k(k, +) \otimes T \quad \text{as algebras.} \tag{4.17}$$

The map $\widehat{x} \in H^0$ corresponding to $\overline{x} \in T$ is the map given by

$$\widehat{x}(x^i g^j) = \delta_{i1}.$$

In other words, it is precisely the map z we found in Example 4.16, which we expected to generate a polynomial ring and were surprised to find was nilpotent. No such surprises await us now: the fact that $\widehat{x}$ is nilpotent is obvious from the relations in T and the fact that T is isomorphic to a subalgebra of H^0, and the canonical embedding map $\Pi^0 : k[y]^0 \to H^0$ is an algebra map. So Equation (4.17) tells us everything about the algebra structure of H^0.

We note at this point that it does not tell us everything about the coalgebra structure. We know that T^0 is not just a subalgebra but a Hopf subalgebra of H^0 by Lemma 4.7, and part (iv) of Theorem 4.19 tells us that $k[y]^0$ is a factor Hopf algebra of H^0. However, $k[y]^0$ does not form a subcoalgebra of H^0:

The embedding of $k[y]^0$ into H^0 as a subalgebra is given by composition with Π, meaning that all maps in $k[y]^0$ are zero on $\ker \Pi$. Similarly, we find that any element of $k[y]^0 \otimes k[y]^0$ will be zero on $\ker \Pi \otimes \ker \Pi$. However, we have

$$\Delta(v)(x \otimes x^{n-1}) = v(x^n) = 1 \neq 0$$

by (4.16). Since $\ker \Pi = k\{x^{in+j}g^\ell \mid 1 \leq j \leq n-1\}$, we have $x, x^{n-1} \in \ker \Pi$ and thus

$$\Delta(v) \notin k[y]^0 \otimes k[y]^0.$$

We capture this fact in the following remark.

Remark 4.23. In the statement of Theorem 4.19, ξ need not be a map of coalgebras. In particular, although $\xi|_{T^0}$ always gives us a map of Hopf algebras by Lemma 4.7, A^0_Π need not be a subcoalgebra of H^0. This is despite the fact that it is always a quotient Hopf algebra by Theorem 4.19(iv).

However, overall we are more interested in working out the algebra structure of H^0, and when ξ is bijective and its assumptions hold Theorem 4.19 gives us a complete description.

4.6 Summary

Throughout, we retain the hypotheses of Section 4.1.1, and let A^0_Π be as in Definition 4.3 and ξ be as in (4.5).

Since all our statements concern the same map ξ and all our assumptions are relatively independent of one another, as in Example 4.22 we can combine them when several hold. For instance, we immediately get the following two corollaries to our previous results:

Corollary 4.24. *Keeping the notation and assumptions of Section 4.1.1, let ξ be as in (4.5) and A^0_Π as in Definition 4.3. Suppose further that ι is a Hopf map, γ is a coalgebra map and T is finite dimensional. Then*

$$H^0 \cong A^0 \otimes T^0$$

as algebras, where the isomorphism is given by ξ.

Proof. Again, Theorem 4.19 tells us that $\xi : A^0_\Pi \otimes T^0 \to H^0$ is an algebra homomorphism. Now Corollary 4.11 tells us that $A^0_\Pi = A^0$ and ξ is bijective. This means we have

$$H^0 \cong A^0 \otimes T^0$$

as algebras via ξ as required. □

Corollary 4.25. *Keeping the notation and assumptions of Section 4.1.1, let ξ be as in (4.5) and A^0_Π be as in Definition 4.3. Suppose further that γ and ι are both Hopf maps, or in other words that $H \cong A\#T$ is a smash product with both T and A Hopf subalgebras. Then*

$$H^0 \cong A^0_\Pi \otimes T^0$$

as algebras, where the isomorphism is given by ξ and T^0 is a Hopf subalgebra of H^0.

Proof. By Theorem 4.19, γ being a coalgebra map and ι being a Hopf map imply that $\xi : A^0_\Pi \otimes T^0 \to H^0$ is an algebra homomorphism. It is bijective by Theorem 4.13, which means we have

$$H^0 \cong A^0_\Pi \otimes T^0$$

as algebras via ξ, as required. □

Remark 4.26. This generalises earlier work: Donkin shows in [13, Proposition 1.5.3] that under these assumptions and also assuming that H is cocommutative, $H^0 \cong A^0_\Pi \otimes T^0$ as algebras. Note that while we show Donkin's assumption that H is cocommutative to be unnecessary, Donkin also shows that $H^0 \cong A^0_\Pi \otimes T^0$ as right H^0-comodules without assuming H^0 cocommutative, something we have not looked into at all, and does not assume the existence of a Hopf factor map $\pi : H \to T$.

Furthermore, our work and Donkin's also build on Hochschild's work regarding the finite dual of universal enveloping algebras of semisimple Lie algebras in [18], where he used a similar approach to work out the finite dual of $U(\mathfrak{g})$, both for $\mathfrak{g}$ solvable (a result we recorded in Example 1.31) and the general case. We make note of this in the following corollary, a reproduction of [18, Theorem 5] which largely uses Corollary 4.25 but refers to [18] for the calculation of A^0_Π in this case.

Corollary 4.27. *Let k be a field of characteristic zero, $\mathfrak{g}$ a finite-dimensional Lie algebra over k, $\mathfrak{s} := \operatorname{rad}\mathfrak{g}$ its radical and and $\mathfrak{t} := [\mathfrak{g}, \mathfrak{s}]$. Write $n := \dim_k(\mathfrak{s}), m := \dim_k(\mathfrak{t})$ and let L be the unique simply connected algebraic group with Lie algebra $\mathfrak{g}/\mathfrak{s}$. Then*

$$U(\mathfrak{g})^0 \cong k[x_1, ..., x_n] \otimes k(k,+)^{n-m} \otimes \mathcal{O}(L).$$

Here $\mathcal{O}(L)$ is a Hopf subalgebra of $U(\mathfrak{g})^0$.

Proof. First note that there exists a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{l}, \tag{4.18}$$

and since $\mathfrak{l} \cong \mathfrak{g}/\mathfrak{s}$ it is clear that $\mathfrak{l}$ is semisimple. This is the Levi decomposition and follows by [51, Theorem 20.3.5]. Now note that by Example 2.11, (4.18) means that

$$U(\mathfrak{g}) \cong U(\mathfrak{s}) \# U(\mathfrak{l}),$$

with trivial cocycle because $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$. Moreover, both $U(\mathfrak{l})$ and $U(\mathfrak{s})$ are Hopf subalgebras of $U(\mathfrak{g})$ and the factor map $\mathfrak{g} \to \mathfrak{l}$ turns into a Hopf factor map $U(\mathfrak{g}) \to U(\mathfrak{l})$.

So Corollary 4.25 applies and we have

$$U(\mathfrak{g})^0 \cong U(\mathfrak{s})^0_\Pi \otimes U(\mathfrak{l})^0$$

as algebras with $U(\mathfrak{l})^0$ being a Hopf subalgebra. Since by Example 1.27 (also due to Hochschild) we have $U(\mathfrak{l})^0 \cong \mathcal{O}(L)$, the equation becomes

$$U(\mathfrak{g})^0 \cong U(\mathfrak{s})^0_\Pi \otimes \mathcal{O}(L).$$

So all that needs to be done is to show that

$$U(\mathfrak{s})^0_\Pi \cong k[x_1, ..., x_n] \otimes k(k, +)^{n-m}$$

as algebras. This is where we can no longer rely on our results and instead refer to Hochschild, who shows this in [18, pages 516-519] and writes $\mathbf{R}(L)_A$ for what we denote as $U(\mathfrak{s})^0_\Pi$. □

A complete overview of how the results of this chapter combine is provided in Table 4.1 on the following page. In the table, we keep the notation and assumptions of Section 4.1.1 throughout and let ξ be as in (4.5) and A^0_Π as in Definition 4.3.

Overview

$\dim_k(T) < \infty$?	γ coalgebra map?	A Hopf subalgebra?	σ trivial?	$\#$ trivial?	What we know about H^0	References
yes	yes	yes	y/n	y/n	$H^0 \cong A^0 \otimes T^0$ as algs	Cor 4.11, Thm 4.19
y/n	yes	yes	yes	yes		Cor 4.12, Thm 4.19
yes	yes	no	yes	y/n	$H^0 \cong \xi(A^0)\#T^0$ as algs	Cor 4.11, Thm 4.13
y/n	yes	no	yes	yes		Cor 4.12, Thm 4.13
no	yes	yes	yes	no	$H^0 \cong A^0_\Pi \otimes T^0$ as algs	Thm 4.13, Thm 4.19
no	yes	y/n	yes	no	$H^0 \cong \xi(A^0_\Pi)\#T^0$ as algs	Thm 4.13
no	yes	yes	no	y/n	$A^0_\Pi \otimes T^0 \subseteq H^0$ as algs	Thm 4.19
yes	y/n	y/n	y/n	y/n	$\xi : A^0 \otimes T^0 \to H^0$ bijective	Cor 4.11
y/n	no	y/n	yes	yes		Cor 4.12
no	y/n	y/n	y/n	y/n	$\xi : A^0_\Pi \otimes T^0 \to H^0$ injective	Cor 4.6

Table 4.1: An overview of the results in Chapter 4 and how they combine.

A cell marked 'yes' means we need the condition to be true for the result to hold, a cell marked 'no' means we do not need the condition to be true and there is a stronger result available if it is, and a cell marked 'y/n' means we do not need the condition to be true and cannot say any more when it is. When we write $\xi(A^0)$ or $\xi(A^0_\Pi)$ instead of A^0 or A^0_Π, we do so in order to emphasise that the isomorphism in column 6 need not respect any canonical algebra structure on A^0 or A^0_Π when such exists.

4.7 Originality

The results of this chapter are original. The proof of Theorem 4.9 is partially inspired by earlier work of Donkin (see [13]), and Corollary 4.25 duplicates and generalises part of one of the results of that paper. This is further discussed in Remark 4.26. Corollary 4.27 then uses this result further to duplicate part of the work done by Hochschild to compute the universal enveloping algebra of a general finite-dimensional Lie algebra (see [18]).

Chapter 5

Hopf algebras which are finite over central Hopf subalgebras

5.1 Introduction

In this chapter, we use the results of the previous chapters to study a specific class of Hopf algebras which is of interest to us and gives rise to some important and large families of examples. In particular, we are interested in Hopf algebras H which are finitely-generated as modules over some central Hopf subalgebra A.

Examples of Hopf algebras H that contain such a central Hopf subalgebra include the Taft algebras studied in Examples 2.12, 4.16 and 4.22, all other prime Hopf algebras of GK-dimension one described in [6] such as the generalised Liu algebras and the group algebra of the dihedral group, as well as quantised enveloping algebras and coordinate rings at roots of unity. If we also consider Hopf algebras over a field with positive characteristic, then this class includes all enveloping algebras of finite-dimensional Lie algebras (see for instance [22, Proposition 2]).

We are specifically interested in Noetherian Hopf algebras, which covers all the cases mentioned above. So, assuming a given base field k, we say that $A \subseteq H$ *satisfy* (F) if we have

$$\begin{gathered} H \text{ is a Noetherian Hopf } k\text{-algebra and } A \subseteq Z(H) \text{ is a Hopf sub-} \\ \text{algebra of } H \text{ such that } H \text{ is a finitely-generated left } A\text{-module.} \end{gathered} \tag{F}$$

We also assume that k is algebraically closed of characteristic zero throughout.

In Section 5.2 we discuss the implications of $A \subseteq H$ satisfying these conditions and record various facts that always hold about the structure of H and H^0, such as the existence of a canonical quotient Hopf algebra $\overline{H} := H/A^+H$. Using results from Chapter 3, we also find that there is always a canonical Hopf surjection $\iota^0 : H^0 \to A^0$ given by restriction. This and other results lead us to Corollary 5.5, which tells us that when we have a convolution invertible right A^0-comodule map $\phi : A^0 \to H^0$, then

$$H^0 \cong \overline{H}^0 \#_\tau A^0 \tag{5.1}$$

for some action and cocycle τ. In Section 5.2.1, we then investigate the question of when such a map ϕ exists. The main result of this section is Theorem 5.8, which tells us that when we have an A-module decomposition $H \cong A \oplus X$ such that A is a Hopf subalgebra and X is a coideal of H, then such a map always exists and in fact H^0 decomposes as

$$H^0 \cong \overline{H}^0 \# A^0,$$

a smash product with no twisting given by any cocycle.

In Section 5.3, we turn our attention to subalgebras of H^0. We consider two canonical Hopf subalgebras - W, consisting of those functions vanishing on some power of A^+H, and $k\widehat{G}$, consisting of those functions extending characters on A to the whole of H. The main result of this section regarding W is Theorem 5.21, which tells us that whenever there exists an A-module projection $\Pi : H \to A$, then

$$W \cong \overline{H}^0 \#_\sigma U(\mathrm{Lie}\, G),$$

where G is the affine algebraic group satisfying $A \cong \mathcal{O}(G)$. The main results regarding $k\widehat{G}$ are Theorems 5.19 and 5.20, which put together tell us that given some right comodule map $\psi : A^0 \to H^0$,

$$k\widehat{G} \cong \overline{H}^0 \#_\tau kG$$

with cleaving map $\psi|_{A^0}$ if and only if (5.1) holds with cleaving map ψ.

We look at how our results work on an example in Section 5.4, where we use Theorem 5.8 to compute the finite dual of $U_\epsilon(\mathfrak{sl}_2(k))$ for $\epsilon \in k^*$ a root of unity. We then use the results of Section 5.3 to talk about its Hopf subalgebras. We also formulate a conjecture regarding $U_\epsilon(\mathfrak{g})^0$ for any finite-dimensional semisimple $\mathfrak{g}$ and make note of a partial result in this vein regarding the Hopf subalgebra W.

In Section 5.5, we restrict ourselves to the case where H itself decomposes as a crossed product. As, under hypothesis (F), we always have a canonical quotient Hopf algebra $\overline{H}$, the results of Chapter 2 tell us that such a crossed product decomposition of H exists whenever there exists a cleaving map $\gamma : \overline{H} \to H$. In Section 5.5.1, we ask when this happens. It turns out that it is an open question whether all Hopf algebras $A \subseteq H$ satisfying (F) over an algebraically closed field k decompose in this way, and we make note of some positive results from the literature. In Section 5.5.2, we then look at what we can say about H^0 when such a decomposition exists. In this case, we can use the results of Chapter 4. We record these in Theorem 5.33. Finally, we note that our results so far give us several potential crossed product decompositions of H^0. We define the notion of two such decompositions being equivalent and note that any two decompositions gained from our results must be equivalent, and in fact under certain conditions coincide.

Section 5.6 concludes our investigation into H^0 by noting as a conjecture what we (optimistically) expect to hold true generally, or perhaps (more realistically) under one of two given assumptions on H. We summarise how the results of the previous sections tie into the three variations of the conjecture and what sort of partial answers they give us.

5.2 A potential crossed product decomposition of H^0

Throughout, let k be an algebraically closed field of characteristic zero and let $A \subseteq H$ be Hopf algebras satisfying (F).

In this section, we record some of the results that always hold about the structure of A and H as well as how the maps between them translate to the dual setting. We find that there is always a canonical Hopf surjection $H^0 \to A^0$ given by restriction, which means we can apply the results recorded in Chapter 2 to work out whether this gives rise to a crossed product decomposition. We apply the results recorded in Chapter 2 in Corollary 5.5, which tells us that whenever there is a cleaving map $\phi : A^0 \to H^0$, we have $H^0 \cong \overline{H}^0 \#_\sigma A^0$ for some cocycle σ and action of A^0 on H^0. The immediate question is when such a map exists. Theorem 5.8 gives us two cases in which it does.

A number of conditions about the structure of A and H follow from (F). Before listing them, we note the following definition.

Definition 5.1. Let R be a ring. Then we say that R *satisfies a polynomial identity* or that R is a *polynomial identity ring* or *PI ring* if there is some $n \geq 1$ and some nonzero polynomial $p(x_1, ..., x_n) \in \mathbb{Z}\langle x_1, ..., x_n\rangle$ such that

$$p(r_1, ..., r_n) = 0 \quad \text{for all } r_1, ..., r_n \in R.$$

We can assume that p has minimal degree with respect to this property. We then call the degree of p the *PI-degree* of the ring R.

Proposition 5.2. *Suppose $A \subseteq H$ are Hopf algebras satisfying* (F)*. Then*

(i) A is Noetherian,

(ii) A and H are both affine,

(iii) H satisfies a polynomial identity,

(iv) The antipode S_H of H is bijective, and

(v) H is a faithfully flat and projective left A-module.

Proof. (i) This follows by [34, Corollary 10.1.10]

(ii) Since A is commutative and Noetherian by (i), it is affine (by [35]). Now H is a finitely-generated module over a finitely-generated algebra, and hence itself finitely generated as an algebra: given a generating set $S = \{a_1, ..., a_n\}$ for A as an algebra and $S' = \{h_1, ..., h_m\}$ for H as an A-module, $S \cup S'$ provides a generating set for H as an algebra.

(iii) This follows because any ring which is a finite module over its centre is PI ([34, Corollary 13.1.13(iii)]).

(iv) This follows immediately from (ii) and (iii), as by [44, Corollary 2] any Noetherian affine Hopf algebra satisfying a polynomial identity has bijective antipode.

(v) This follows by [43, Theorem 3.3], which states that any Noetherian Hopf algebra is faithfully flat over its central Hopf subalgebras. Now by [41, Corollary 3.57], any finitely-generated flat module over a Noetherian ring is projective. Since A is Noetherian by (i), it follows that H is projective. □

Moreover, we know there is always a canonical finite-dimensional factor Hopf algebra of H coming from the augmentation ideal of H.

Proposition 5.3. *Suppose $A \subseteq H$ are Hopf algebras satisfying* (F)*. Then the subspace A^+H is a Hopf ideal, and the resulting factor Hopf algebra H/A^+H is finite-dimensional.*

Proof. The fact that A^+H is a Hopf ideal is easy to see: it is an ideal thanks to the centrality of A, as this means that

$$HA^+H = A^+HH = A^+H.$$

Moreover, it is a coideal because

$$\begin{aligned}\Delta_H(A^+H) &= \Delta_H(A^+)\Delta_H(H) = \Delta_A(A^+)\Delta_H(H)\\ &\subseteq (A^+ \otimes A + A \otimes A^+)(H \otimes H)\\ &\subseteq A^+H \otimes H + H \otimes A^+H\end{aligned}$$

and

$$\varepsilon_H(A^+H) = \varepsilon_H(A^+)\varepsilon_H(H) = \varepsilon_A(A^+)\varepsilon_H(H) = 0,$$

using the fact that A^+ is a coideal of A and A is a Hopf subalgebra and hence subcoalgebra of H. Finally, note that

$$S_H(A^+H) = S_H(H)S_H(A^+) = S_H(H)S_A(A^+) \subseteq HA^+ = A^+H$$

since A is a Hopf subalgebra and central in H.

The fact that H/A^+H is finite-dimensional follows immediately from the fact that H is finitely-generated as a left A-module, as this means there is a finite set of elements generating H as an A-module. The images of these elements under the factor map span H/A^+H as a vector space. □

So we always have a canonical Hopf surjection $\pi : H \to \overline{H}$, where we write $\overline{H} := H/A^+H$.

Now we wish to work out how these maps translate to the dual setting.

Recall some notation from Chapter 3: given a map $f : A \to B$, where B is an algebra, we write f^0 for the map from B^0 to A^* given by composition with f.

Lemma 5.4. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F). Let $\overline{H} := H/A^+H$ and $\pi : H \to \overline{H}$ denote the canonical factor map. Then*

(i) The map $\iota^0 : H^0 \to A^0$ given by restriction to A is a surjective map of Hopf algebras.

(ii) The map $\pi^0 : \overline{H}^0 \to H^0$ given by composition with π is an embedding of Hopf algebras.

(iii) The left and right coinvariants of the A^0-coaction $\rho := (\mathrm{id} \otimes \iota^0) \circ \Delta_{H^0}$ on H^0 are given by $\overline{H}^0$:

$$\left(H^0\right)^{co\iota^0} = {}^{co\iota^0}\left(H^0\right) = \overline{H}^0,$$

where we view $\overline{H}^0$ as a Hopf subalgebra of H^0 through the embedding in (ii).

Proof. (i) Let $\iota : A \to H$ denote the natural Hopf algebra embedding map. By Corollary 3.6, it is a finite overlay (recalling the definition of *finite overlay* from Definition 3.4, although we do not use the details here.) This means that by Theorem 3.12 (ii) and (vi), $\iota^0 : H^0 \to A^0$ is a map of Hopf algebras that is surjective if and only if ι is reciprocal, where we recall from Definition 3.9 that $\iota : A \to H$ being reciprocal means that for any ideal of finite codimension $I \subseteq A$ there is an ideal of finite codimension $J \subseteq H$ with

$$J \cap A \subseteq I.$$

To see that this holds, we recall that an ideal $K \subseteq A$ is said to satisfy the *right Artin-Rees property* if, for all finitely-generated right A-modules M and submodules $N \subseteq M$ there exists some $n \geq 1$ with

$$N \cap MK^n \subseteq NK$$

(see [16, Lemma 13.1] for this and other equivalent conditions.) Left Artin-Rees is defined analogously.

Now note that because A is commutative (as it is central in H) and also Noetherian by Proposition 5.2(i), all its ideals satisfy both left and right Artin-Rees properties (see for instance [16, Theorem 13.3]). This means that given an ideal $I \subseteq A$ of finite codimension, there exists some $n \geq 1$ such that

$$A \cap HI^n \subseteq AI = I.$$

Since HI^n is an ideal of H, all that remains to show is that this also has finite codimension in H, or equivalently that I^n has finite codimension in A as H is a finitely generated A-module.

It suffices to show this for $n = 2$, as the general result can then be obtained by induction.

First note that because A is Noetherian, both I and I^2 are finitely generated, and we can choose a generating set $\{a_1, ..., a_s, a_{s+1}, ..., a_t\}$ of I such that $\{a_1, ..., a_s\}$ forms a generating set for I^2.

Now consider the quotient space I/I^2. This is a canonical A/I-module, and it is generated by the elements $\{a_{s+1} + I^2, ..., a_t + I^2\}$. Since A/I is finite-dimensional by assumption, this means that I/I^2 is a finitely-generated module over a finite-dimensional algebra and therefore itself finite-dimensional. By a standard isomorphism theorem we know that

$$\dim(A/I^2) = \dim(A/I) + \dim(I/I^2) < \infty.$$

So I^2 is an ideal of finite codimension as required.

(ii) By Proposition 5.3, there is a surjective Hopf algebra map $\pi : H \to \overline{H}$. By Corollary 3.6, it is a finite overlay. So by Theorem 3.12 (iii) and (vi), the map $\pi^0 : \overline{H}^0 \to H^0$ given by composition with π is a well-defined injective map of Hopf algebras as required.

(iii) *Step 1: Identifying $\pi^0(\overline{H}^0)$ with a specific subspace of H^0*

We want to show that $\pi^0(\overline{H}^0)$ consists precisely of those maps in H^0 which are zero on A^+H.

To see this, note that whenever $A^+H \subseteq \ker f$ we can construct a map $\hat{f} \in \overline{H}^*$ with $\pi^*(\hat{f}) = f$ by setting $\hat{f}(h + A^+H) := f(h)$. This is well-defined because f is zero on A^+H. Since $\overline{H}$ is finite-dimensional by Proposition 5.3, we have $\hat{f} \in \overline{H}^0$ and thus $f = \pi^0(\hat{f}) \in \pi^0(\overline{H}^0)$ as required.

For the other direction, note that any map in H^0 that arises by composition with π will be zero on $\ker \pi = A^+H$.

Step 2: Showing that for any coinvariant $f \in \left(H^0\right)^{co\iota^0}$ we have $A^+H \subseteq \ker f$

Step 1 means that in order to show the coinvariants are just those maps in the image of π^0, it suffices to show that the left and right coinvariants of the $\operatorname{im} \iota^0$-action are exactly those maps which are zero on A^+H.

Suppose $f \in \left(H^0\right)^{co\iota^0}$, so $f \in H^0$ is such that

$$\rho(f) = (\operatorname{id} \otimes \iota^0) \circ \Delta_{H^0}(f) = f \otimes \varepsilon_A.$$

On elements, this means that for $h \in H, a \in A$ we need to have

$$f(ah) = f(ha) = \left(\sum f_1 \otimes f_2 \circ \iota\right)(h \otimes a) = \rho(f)(h \otimes a) = f(h)\varepsilon_A(a)$$

(recalling that by assumption, A is central in H). In particular, whenever $a \in A^+$ we have $f(ah) = \varepsilon_A(a)h = 0$. So f is zero on A^+H. By Step 1 this gives us

$$\left(H^0\right)^{co\iota^0} \subseteq \pi^0(\overline{H}^0).$$

Step 3: Showing that any map that is zero on A^+H is a coinvariant

Suppose that $g(A^+H) = 0$. So given $h \in H$ and $a = \varepsilon_A(a)1_A + a'$ for some $a' \in A^+$, we have

$$g(ah) = g(\varepsilon_A(a)h + a'h) = \varepsilon_A(a)g(h) + g(a'h) = \varepsilon_A(a)g(h)$$

and thus find that g is a right coinvariant. Again, by Step 1 this means we have

$$\pi^0(\overline{H}^0) \subseteq \left(H^0\right)^{co\iota^0},$$

which combined with Step 2 gives us equality.

The proof for $\pi^0(\overline{H}^0) = {}^{co\iota^0}(H^0)$ is analogous. □

So given $A \subseteq H$ satisfying (F), we always have a Hopf epimorphism $H^0 \to A^0$ such that the coinvariants are given by a Hopf embedding of $\overline{H}^0$ into H^0, where $\overline{H}$ is the canonical factor Hopf algebra.

Recall from Definition 2.19 that given a Hopf surjection $\psi : H \to T$, a *cleaving map* is a convolution invertible right T-comodule map $\gamma : T \to H$. Recall further from Proposition 2.22 that the existence of a cleaving map guarantees that H decomposes as a crossed product

$$H \cong H^{co\psi} \#_\sigma T$$

for some cocycle and action. Using these results, we find a natural corollary to Lemma 5.4:

Corollary 5.5. *Suppose $A \subseteq H$ are Hopf algebras satisfying* (F)*, and let $\overline{H} := H/A^+H$ and $\pi : H \to \overline{H}$ be the canonical factor map. Suppose further that there exists a cleaving map $\phi : A^0 \to H^0$. Then*

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

as algebras for some cocycle σ and action of A^0 on $\overline{H}^0$. Moreover, the isomorphism is given by $\zeta := m_{H^0} \circ (\pi^0 \otimes \phi)$, and ζ is also a map of left $\overline{H}^0$-modules and right A^0-comodules. Finally, the restricted map $\zeta|_{\overline{H}^0} = \pi^0 : \overline{H}^0 \subseteq H^0$ is a Hopf algebra embedding.

Proof. This follows immediately by Proposition 2.22, noting that by Lemma 5.4 (iii) the subalgebra of coinvariants of the action of A^0 on H^0 is $\overline{H}^0$, viewed as a subalgebra of H^0 under the embedding given by π^0. □

We discuss when such a map exists in the next section.

5.2.1 On the existence of a cleaving map $A^0 \to H^0$

A cleaving map is simply a convolution invertible right comodule map. We find that right comodule maps $A^0 \to H^0$ can be retrieved from left A-module projections $H \to A$:

Lemma 5.6. *Suppose $A \subseteq H$ are Hopf algebras satisfying* (F) *and suppose that $\phi : H \to A$ is an A-module map such that $\phi|_A = \mathrm{id}_A$. Then there exists a well-defined injective map $\phi^0 : A^0 \to H^0$ given by composition with ϕ. Moreover, ϕ^0 is a map of right A^0-comodules.*

Proof. By a standard result about split short exact sequences (see for instance [41, Theorem 2.7]), the existence of an A-module projection $\phi : H \to A$ satisfying $\phi \circ \iota = \mathrm{id}_A$ is equivalent to saying that H decomposes as a direct sum:

$$H \cong A \oplus X$$

as left A-modules, where $X = \ker \phi$. Note that X must be finitely-generated because H is a finitely-generated module over the Noetherian ring A and hence all its submodules are finitely-generated. So by Lemma 3.16, ϕ is a finite overlay, and the map $\phi^0 : A^0 \to H^0$ is well-defined. It is injective by Theorem 3.12(iii), using the fact that ϕ is surjective.

Now note that the right A^0-comodule structure on H^0 is given by

$$\rho := (\mathrm{id} \otimes \iota^0) \circ \Delta_{H^0} : H^0 \to H^0 \otimes A^0.$$

To show that ϕ^0 is a right comodule map, we need to show that

$$\rho \circ \phi^0 = (\phi^0 \otimes \mathrm{id}_{A^0}) \circ \Delta_{A^0}. \tag{5.2}$$

First note that given $f \in H^0$, $h \in H$ and $a \in A$ and letting μ denote the canonical isomorphism $k \otimes k \cong k$, we have

$$\mu \circ \rho(f)(h \otimes a) = \sum f_1(h) f_2(\iota(a)) = f(ha) = f(ah),$$

using the fact that A is central. Now we can use this to show that (5.2) holds: we have

$$\begin{aligned}
\mu \circ (\rho \circ \phi^0)(f)(h \otimes a) &= \rho(f \circ \phi)(h \otimes a) \\
&= (f \circ \phi)(ah) \\
&= f(a\phi(h)) \\
&= f(\phi(h)a) \\
&= \mu \circ (\sum f_1(\phi(h)) \otimes f_2(a)) \\
&= \mu \circ \left(\sum \phi^0(f_1) \otimes f_2\right)(h \otimes a) \\
&= \mu \circ ((\phi^0 \otimes \mathrm{id}_{A^0}) \circ \Delta_{A^0})(f)(h \otimes a).
\end{aligned}$$

Since μ is an isomorphism and this is true for all $f \in H^0$, $h \in H$, $a \in A$, this gives us exactly (5.2). □

Remark 5.7. Note that when $H \cong A\#_\tau \overline{H}$ is itself a crossed product, the assumptions of this lemma always hold by taking $X := A\left(\gamma(\overline{H}^+)\right)$, where $\gamma : \overline{H} \to H$ is the cleaving map. We will discuss this situation further in Section 5.5.

So whenever we have a decomposition of H as in Lemma 5.6, we have a right A^0-comodule map $\phi^0 : A^0 \to H^0$. All we need for Corollary 5.5 to apply is for it to be convolution invertible.

Two types of maps which we know are always convolution invertible are algebra and coalgebra maps. In particular, if X is either an ideal or a coideal, Corollary 5.5 applies:

Theorem 5.8. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F), and let $\overline{H} := H/A^+H$. Suppose also that we have $H \cong A \oplus X$ as left A-modules, where the first summand A is the embedding of A into H as a Hopf subalgebra. Then*

(i) If X is a coideal, we have

$$H^0 \cong \overline{H}^0 \# A^0$$

as left $\overline{H}^0$-modules, right A^0-comodules and algebras for some action of A^0 on $\overline{H}^0$.

(ii) If X is an ideal, we have

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

as left $\overline{H}^0$-modules, right A^0-comodules and algebras for some action of A^0 on $\overline{H}^0$ and cocycle σ. Moreover, this isomorphism induces a coalgebra isomorphism

$$H^0 \cong \overline{H}^0 \otimes A^0$$

through the canonical identification of a crossed product with its underlying tensor product.

In both cases the isomorphism is given by $m_{H^0} \circ (\pi^0 \otimes \Pi^0)$, where $\pi : H \to \overline{H}$ denotes the canonical Hopf surjection and $\Pi : H \to A$ the quotient map along X. Moreover, this map restricts to a Hopf algebra embedding on $\overline{H}^0$.

Proof. Let $\phi : H \to A$ denote the map given by factoring along X. By Lemma 5.6, it is a finite overlay and the map $\phi^0 : A^0 \to H^0$ given by composition is a right A^0-comodule map.

(i) Because $X := \ker \phi$ is a coideal, ϕ is a coalgebra map. So by Theorem 3.12(v), ϕ^0 is an algebra map. This means it is convolution invertible, with inverse $\phi^0 \circ S_{A^0}$. That is, given $f \in A^0$, we have

$$\sum (\phi^0 \circ S_{A^0})(f_1)\phi^0(f_2) = \phi(S_{H^0}(f_1)f_2)) = \phi(1_{H^0})\varepsilon_{H^0}(f)$$

by the antipode axiom, and showing $\phi^0 \circ S_{A^0}$ is a right inverse is similar. So ϕ^0 is a cleaving map: by Corollary 5.5, we have

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

for some action of A^0 on $\overline{H}^0$ and cocycle σ, and $\overline{H}^0$ is a Hopf subalgebra in this decomposition.

Now note that by Lemma 2.30, the cleaving map ϕ^0 being an algebra map tells us that σ is trivial. This means that the crossed product is in fact a smash product as required.

(ii) Since $X := \ker \phi$ is an ideal, ϕ is an injective algebra map. So by Theorem 3.12(ii) and (iv), ϕ^0 is a coalgebra embedding, and moreover one that satisfies $\phi^0 \circ \iota^0 = \mathrm{id}_{A^0}$ since $\phi|_A = \mathrm{id}_A$. So by Lemma 2.20, ϕ^0 is convolution invertible. By Corollary 5.5, we have

$$H^0 \cong \overline{H}^0 \#_\sigma A^0 \tag{5.3}$$

for some action and cocycle σ, with $\overline{H}^0$ a Hopf subalgebra. Furthermore, we know that the isomorphism in (5.3) is given by $\zeta := m_{H^0} \circ (\pi^0 \otimes \phi^0)$. So the corresponding bijection $\zeta' : A^0 \otimes \overline{H}^0 \to H^0$ on tensor products is given by $\zeta' = m_{H^0} \circ (\pi^0 \otimes \phi^0)$.

Now note that the cleaving map $\phi^0 : A^0 \to H^0$ is a coalgebra map. Similarly, π^0 is a Hopf algebra and hence coalgebra map by Lemma 5.4 (ii). So $\pi^0 \otimes \phi^0 : \overline{H}^0 \otimes A^0 \to H^0 \otimes H^0$ is a coalgebra map. Since H^0 is a Hopf algebra, m_{H^0} is also a coalgebra map. So ζ' is the composition of coalgebra maps and hence itself a coalgebra map. Since it is bijective, this means that

$$H^0 \cong \overline{H}^0 \otimes A^0$$

as coalgebras as required. □

We will see applications of Theorem 5.8 in Section 5.4, which uses part (i) to calculate the finite dual of the quantised enveloping algebra $U_\epsilon(\mathfrak{sl}_2(k))$ for ϵ a root of unity, and in Section 6.5, which uses part (i) to calculate the finite dual of the generalised Liu algebras introduced by Liu ([31]) and Brown and Zhang ([6]).

Remark 5.9. It is unclear whether the assumptions made in Theorem 5.8(i) and (ii) are necessary for the dual of the projection map to be convolution invertible. In particular, we know of no examples of Hopf algebras $A \subseteq H$ satisfying (F) such that there is a left A-module projection $\Pi : H \to A$ where H^0 does not decompose as a crossed product. We record this in the following question.

Question 5.A. Are there any Hopf algebras $A \subseteq H$ satisfying (F) such that $H \cong A \oplus X$ as left A-modules for some left A-module X and such that H^0 does not decompose as a crossed product of $\overline{H}^0$ and A^0, meaning that

$$H^0 \ncong \overline{H}^0 \#_\sigma A^0$$

for any action of A^0 on $\overline{H}^0$ or cocycle σ?

What about if $H \cong A \#_\tau \overline{H}$, so that H itself decomposes as a crossed product?

We will discuss this in more detail in Section 5.6.

Remark 5.10. In the case where H itself decomposes as a crossed product and $X := A\gamma(\overline{H}^+)$, we can look at previous results to see when Theorem 5.8 will apply. In particular, the assumption that X is an ideal made in part (ii) of the theorem is then true if and only if both the action and cocycle are trivial and $H \cong A \otimes \overline{H}$ as algebras, by Lemma 2.31. The assumption that X is a coideal made in part (i) is then true if γ is a coalgebra map by Lemma 4.17, but we will see in Section 6.5 that this is not a necessary condition.

5.3 Canonical Hopf subalgebras of H^0

Throughout, k is an algebraically closed field of characteristic zero.

In the previous section, we looked at Hopf algebras $A \subseteq H$ satisfying (F) and tried to understand the structure of H^0. In this section, we study two canonical Hopf subalgebras of H^0 instead. It turns out that we can say quite a bit about them. We see that the Hopf subalgebra $k\widehat{G}$, given by those functions in H^0 that restrict to characters on A^0, is a G-graded algebra, where G denotes the affine algebraic group satisfying $A \cong \mathcal{O}(G)$ (Lemma 5.15). Furthermore, Theorems 5.19 and 5.20 together tell us that $k\widehat{G}$ decomposes as a crossed product if and only if the whole of H^0 does.

The Hopf subalgebra W, given by those functions in H^0 vanishing on some power of A^+H, is isomorphic to $(H/A^+H)^0\#_\sigma U(\operatorname{Lie} G)$ whenever we have an A-module projection $\Pi : H \to A$ (Theorem 5.21). This is for instance always true when H is a free A-module with A-basis containing 1_H, such as when H is a crossed product of A and H/A^+H.

To define $k\widehat{G}$ and W, we first recall the following facts about coordinate rings. Recall also that $G(K)$ denotes the group of grouplike elements of a Hopf algebra K.

Proposition 5.11. *Suppose A is a commutative affine Hopf k-algebra where k is an algebraically closed field of characteristic zero. Then there exists an affine algebraic group*

G such that $A \cong \mathcal{O}(G)$ as Hopf algebras. Moreover, there exists an isomorphism of groups given by

$$G \cong G(A^0) = \mathrm{Alg}(A, k).$$

Proof. These are both standard results of algebraic geometry. The first part has been discussed in Remark 1.13, the second we have seen in the characterisation of $\mathcal{O}(G)^0$ given in Proposition 2.13, □

Remark 5.12. Note that, as discussed in Remark 1.13, this result does not hold in positive characteristic. For instance, if $\mathrm{char}\, k = p$, the Hopf algebra kC_p is a commutative affine Hopf algebra but not isomorphic to the coordinate ring of any affine algebraic group. Further, the characterisation of the dual of $\mathcal{O}(G)^0$ coming from Proposition 2.13 also requires k to be algebraically closed.

We write α_g for the map in A^0 corresponding to $g \in G$, and set $\mathfrak{m}_g := \ker \alpha_g$. Since α_g is an algebra map, $\mathfrak{m}_g$ is a maximal ideal of codimension one in A for each $g \in G$.

This gives us all we need to define several canonical subspaces of H^0.

Definition 5.13. Let $A \subseteq H$ be Hopf algebras satisfying (F) and let G be the affine algebraic group such that $A \cong \mathcal{O}(G)$.

(i) Let $W := \{f \in H^0 \mid f\left((A^+H)^n\right) = 0 \text{ for some } n > 0\}$.

(ii) Given $g \in G$, set

$$\widehat{g} := \{f \in H^0 \mid f(\mathfrak{m}_g H) = 0\} \cong (H/\mathfrak{m}_g H)^*$$

and

$$k\widehat{G} := \bigoplus_{g \in G} \widehat{g}.$$

In fact, the following lemmas show that both W and $k\widehat{G}$ are Hopf subalgebras of H^0.

First, the fact that W is a Hopf subalgebra follows from standard results regarding subspaces of H^0 consisting of functions vanishing on some power of a Hopf ideal.

Lemma 5.14. *Retain the notation and assumptions of Definition 5.13 let $\iota : A \to H$ denote the canonical embedding map. Then W is a Hopf subalgebra of H^0 such that $\iota^0(W) \subseteq U(\mathrm{Lie}\, G) \subseteq A^0$.*

Proof. This is due to standard results: by [36, Lemma 9.2.1], given any Hopf ideal $I \subseteq H$ the set

$$H_I^0 := \{f \in H^0 \mid f(I^n) = 0 \text{ for some } n > 0\}$$

is a Hopf subalgebra. Since A^+H is a Hopf ideal by Proposition 5.3, this means that $W = H^0_{A^+H}$ is a Hopf subalgebra of H^0.

Now note that following [36, Proposition 9.2.5], we can identify $U(\operatorname{Lie} G) \subseteq A^0$ with

$$U(\operatorname{Lie} G) = \{f \in A^0 \mid f((A^+)^n) = 0 \text{ for some } n \geq 1\}.$$

In particular, if $f \in W$, then $f((A^+H)^n) = 0$ for some $n \geq 1$ and $\iota^0(f) = f|_A$ is zero on $(A^+)^n$, meaning that $\iota^0(f) \in U(\operatorname{Lie} G)$. □

Showing that $k\widehat{G}$ is a Hopf subalgebra is slightly more complicated. The result relies on the fact that we can show that $k\widehat{G}$ is a G-graded subalgebra where each component of the grading is a subcoalgebra.

Lemma 5.15. *Retain the notation and assumptions of Definition 5.13. Then*

(i) $k\widehat{G}$ is a G-graded subalgebra of H^0, with $\widehat{g}\widehat{h} \subseteq \widehat{gh}$ for $g, h \in G$.

(ii) $k\widehat{G}$ is a Hopf subalgebra of H^0, with each $\widehat{g}$ being a subcoalgebra and $S_{H^0}(\widehat{g}) \subseteq \widehat{g^{-1}}$.

Proof. (i) Let $\alpha_g \in A^0 = \mathcal{O}(G)^0$ be the algebra map associated to $g \in G$, and let $\mathfrak{m}_g := \ker \alpha_g$ be the corresponding maximal ideal in A.

Let $g, h \in G$. We start by showing that

$$\Delta_A(\mathfrak{m}_{gh}) \subseteq \mathfrak{m}_g \otimes A + A \otimes \mathfrak{m}_h.$$

We have $A = \mathfrak{m}_g \oplus k1_A = \mathfrak{m}_h \oplus k1_A$, so we can write

$$\Delta_A(\mathfrak{m}_{gh}) \subseteq (\mathfrak{m}_g \oplus k1_A) \otimes (\mathfrak{m}_h \oplus k1_A) = \mathfrak{m}_g \otimes A + A \otimes \mathfrak{m}_h \oplus k1_A \otimes k1_A.$$

We write $\lambda 1_A \otimes 1_A$ for the summand belonging to $k1_A \otimes k1_A$ in this decomposition and $\sum x \otimes y$ for the remainder of the sum contained in $\mathfrak{m}_g \otimes A + A \otimes \mathfrak{m}_h$.

Now by Proposition 5.11, $\alpha_g \alpha_h = \alpha_{gh}$. This means that, letting μ denote the canonical isomorphism $k \otimes k \cong k$ given by scalar multiplication,

$$\begin{aligned} 0 &= \alpha_g \alpha_h(\mathfrak{m}_{gh}) \\ &= \mu \circ (\alpha_g \otimes \alpha_h)(\Delta(\mathfrak{m}_{gh})) \\ &= \mu \circ (\alpha_g \otimes \alpha_h)\left(\sum x \otimes y + \lambda 1_A \otimes 1_A\right) \\ &= \sum \alpha_g(x) \otimes \alpha_h(y) + \lambda \alpha_g(1_A)\alpha_h(1_A) \\ &= \lambda, \end{aligned}$$

noting that α_g, α_h are algebra maps and hence send 1_A to 1_k. So $\lambda = 0$ and so

$$\Delta_A(\mathfrak{m}_{gh}) \subseteq \mathfrak{m}_g \otimes A + A \otimes \mathfrak{m}_h.$$

Now let $f \in \widehat{g}, f' \in \widehat{h}$. We have

$$\begin{aligned} ff'(\mathfrak{m}_{gh}H) &= \mu \circ (f \otimes f')(\Delta_H(\mathfrak{m}_{gh}H)) \\ &= \mu \circ (f \otimes f')(\Delta_A(\mathfrak{m}_{gh})\Delta_H(H)) \\ &\subseteq \mu \circ (f \otimes f')((\mathfrak{m}_g \otimes A + A \otimes \mathfrak{m}_h)(H \otimes H)) \\ &\subseteq \mu \circ (f \otimes f')(\mathfrak{m}_g H \otimes H + H \otimes \mathfrak{m}_h H) \\ &= f(\mathfrak{m}_g H)f'(H) + f(H)f'(\mathfrak{m}_h H) = 0 \end{aligned}$$

by assumption on f and f'. In particular, $ff' \in \widehat{gh}$, so $k\widehat{G}$ is closed under multiplication and is G-graded.

Finally, note that $1_{H^0} = \varepsilon_H$ is zero on $H^+ \supseteq A^+H = \mathfrak{m}_1 H$, so $1_{H^0} \in \widehat{1_G} \subseteq k\widehat{G}$. This together with the above means that $k\widehat{G}$ is a subalgebra of H^0.

(ii) Let $g \in G$ and $f \in \widehat{g}$, and consider $\Delta(f) := \sum f_1 \otimes f_2$. We want to show that $\Delta(f) \in \widehat{g} \otimes \widehat{g}$.

We can choose this expression for $\Delta(f)$ such that the f_2 are linearly independent. Now note that for all $h \in H, m \in \mathfrak{m}_g H$ we have $mh \in \mathfrak{m}_g H$ and so

$$\sum f_1(m)f_2(h) = f(mh) = 0.$$

In particular, $\sum f_1(m)f_2 \equiv 0$ on the whole of H, and since the f_2 are linearly independent this means that $f_1(m) = 0$ for each f_1 in the sum and all $m \in \mathfrak{m}_g H$. So $\Delta(f) \subseteq \widehat{g} \otimes H^0$. The exact same argument works for the right hand side as well, giving us

$$\Delta(f) \subseteq (\widehat{g} \otimes H^0 \cap H^0 \otimes \widehat{g}) = \widehat{g} \otimes \widehat{g}.$$

Now we want to show that $S_{H^0}(\widehat{g}) = \widehat{g^{-1}}$.

$\subseteq$:

First recall that by Proposition 5.2, S_H is bijective. This means that given $f \in \widehat{g}$ for $g \in G$, $S_{H^0}(f) = f \circ S_H$ is zero on $S_H^{-1}(\mathfrak{m}_g H)$. Note that

$$S_H^{-1}(\mathfrak{m}_g H) = S_H^{-1}(H) S_H^{-1}(\mathfrak{m}_g) = H S_A^{-1}(\mathfrak{m}_g) = S_A^{-1}(\mathfrak{m}_g) H, \tag{5.4}$$

using the fact that A is a central Hopf subalgebra and that S_H and hence S_H^{-1} are algebra antihomomorphisms.

Now we recall that there is a group homomorphism $G \cong G(A^0)$ given by $g \mapsto \alpha_g$. In particular, $S_{A^0}(\alpha_g) = \alpha_g^{-1} = \alpha_{g^{-1}}$ and so $S_A^{-1}(\mathfrak{m}_g) = \mathfrak{m}_{g^{-1}}$. So Equation (5.4) becomes

$$S_H^{-1}(\mathfrak{m}_g H) = \mathfrak{m}_{g^{-1}} H$$

and so $S_{H^0}(f) \in \widehat{g^{-1}} \subseteq k\widehat{G}$ as required.

$\supseteq$:

By the above, S_H is invertible. Since S_H is an algebra antihomomorphism, so is S_H^{-1}, meaning that it sends two-sided ideals to two-sided ideals. Since S_H^{-1} is also bijective and hence preserves the notion of finite codimension, this means that S_H^{-1} is a finite overlay and gives rise to a well-defined map $\left(S_H^{-1}\right)^0 : H^0 \to H^0$ given by composition. The same argument as above performed for S_H^{-1} tells us that $\left(S_H^{-1}\right)^0 \left(\widehat{g^{-1}}\right) \subseteq \widehat{g}$. Moreover, $\left(S_H^{-1}\right)^0$ and S_{H^0} are mutually inverse, and therefore mutually inverse on restriction to $\widehat{g}$ and $\widehat{g^{-1}}$. In particular, this means that $S_{H^0}|_{\widehat{g}}$ must be surjective. □

Remark 5.16. In fact, Lemma 5.15 tells us that not only is $k\widehat{G}$ a Hopf subalgebra of H^0, so is $\bigoplus_{g \in K} \widehat{g}$ for any subgroup $K < G$.

We will see later that $k\widehat{G}$ is only a crossed product if the whole of H^0 is. However, Lemma 5.15(i) tells us that it is always G-graded and hence a kG-comodule, which we can view as a weaker version of that statement.

Both W and $k\widehat{G}$ contain $\overline{H}^0$ as a Hopf subalgebra, and in fact this is all of their intersection.

Lemma 5.17. *Retain the notation and assumptions of Definition 5.13. Then*

(i) $\overline{H}^0 \subseteq W$ *and* $\overline{H}^0 = \widehat{1_G} \subseteq k\widehat{G}$.

(ii) $W \cap k\widehat{G} = \overline{H}^0$.

Proof. (i) We can identify $\overline{H}^0$ in H^0 as

$$\overline{H}^0 \cong \{f \in H^0 \mid f(A^+H) = 0\}.$$

On the one hand, since W consists of those functions vanishing on some power of A^+H, it certainly contains those functions that vanish on A^+H itself and hence $\overline{H}^0$. On the other, $A^+ = \ker \varepsilon = \mathfrak{m}_{1_G}$ and so $\overline{H}^0 = \widehat{1_G}$ by definition.

(ii) From (i), we know that $\overline{H}^0 \subseteq W \cap k\widehat{G}$, so we only need to show the other inclusion.

Let $f \in W \cap k\widehat{G}$. So there exists some $n \geq 1$ such that $f((A^+H)^n) = 0$ and some $g_1, ..., g_n \in G$ with $g_i \neq g_j$ for $i \neq j$ such that $f\left(\bigcap_{i=1}^n \mathfrak{m}_{g_i} H\right) = 0$. We want to show that $f(A^+H) = 0$.

Write $I := \bigcap_{i=1}^n \mathfrak{m}_{g_i} H$. We have

$$\left(\bigcap_{i=1}^n \mathfrak{m}_{g_i}\right) H \subseteq I$$

and

$$(A^+H)^n = (A^+)^n H^n = (A^+)^n H,$$

using the fact that A is central in H.

Suppose first that $g_i \neq 1_G$ for all i. Then $\bigcap_{i=1}^n \mathfrak{m}_{g_i}$ and $(A^+)^n$ are comaximal as there is no maximal ideal of A containing both of them, so

$$\bigcap_{i=1}^n \mathfrak{m}_{g_i} + (A^+)^n = A.$$

This means that

$$\begin{aligned} I + (A^+H)^n &= \bigcap_{i=1}^n (\mathfrak{m}_{g_i} H) + (A^+)^n H \\ &\supseteq \left(\bigcap_{i=1}^n \mathfrak{m}_{g_i}\right) H + (A^+)^n H \\ &= \left(\bigcap_{i=1}^n \mathfrak{m}_{g_i} + (A^+)^n\right) H = AH = H. \end{aligned}$$

So $I + (A^+H)^n = H$. Since $f(I) = 0$ and $f((A^+H)^n) = 0$, it follows that f is also zero on their sum and so the whole of H: $f = 0$.

So we can assume that $g_1 = 1_G$, which means that $\bigcap_{i=1}^n \mathfrak{m}_{g_i} H \subseteq A^+H$.

Since A^+H and $I + (A^+H)^n$ are both ideals in H, we can view them as left A-modules and consider their quotient module $B := (A^+H)/(I + (A^+H)^n)$. B is finite-dimensional because I and $(A^+H)^n$ have finite codimension in H. We find that $\left(\bigcap_{i=2}^n \mathfrak{m}_{g_i}\right) B = 0$, because

$$\left(\bigcap_{i=2}^{n} \mathfrak{m}_{g_i}\right) A^+H \subseteq \left(\bigcap_{i=1}^{n} \mathfrak{m}_{g_i}\right) H \subseteq I.$$

However, we also have $(A^+)^{n-1}B = 0$. So in particular, $(A^+)^{n-1} + \bigcap_{i=2}^n \mathfrak{m}_{g_i}$ acts as zero on B. By the same argument as above, these two ideals are comaximal and this sum is the whole of A. So $B = 0$, meaning that $I + (A^+H)^n = A^+H$ and so $f(A^+H) = 0$ as required. □

Our aim is to express W in terms of a crossed product of $\overline{H}^0$ and $U(\operatorname{Lie} G)$ and similarly express $k\widehat{G}$ in terms of $\overline{H}^0$ and kG.

We start with $k\widehat{G}$. The following lemma, regarding the structure of the subcoalgebras $\widehat{g} \subseteq k\widehat{G}$, records some preliminary facts which we will need for the proof of the main results.

Lemma 5.18. *Retain the notation and assumptions of Definition 5.13, let $\iota : A \to H$ denote the canonical embedding map, ρ_{A^0} the canonical A^0-comodule structure on H^0 given by ι^0 as in Lemma 5.4(i) and (iii), $g \in G$ and $\alpha_g \in A^0$ be the algebra map corresponding to g. Then*

(i) The space $\widehat{g}$ is a finite-dimensional nonzero left $\overline{H}^0$-module. If H is free as an A-module, then $\dim_k(\widehat{g}) = \dim_k(\overline{H})$.

(ii) $\iota^0(\widehat{g}) = k\alpha_g$ and so $\iota^0(k\widehat{G}) = kG$.

(iii) $\widehat{g} = \{f \in H^0 \mid \rho_{A^0}(f) = f \otimes \alpha_g\}$.

Proof. (i) The fact that $\widehat{g}$ is a left $\overline{H}^0$-module follows immediately from the fact that by Lemma 5.15(i) and Lemma 5.17(i), $k\widehat{G} := \bigoplus_{h\in G} \widehat{h}$ is a G-graded algebra and $\widehat{1_G} = \overline{H}^0$. So $\overline{H}^0\widehat{g} \subseteq \widehat{g}$ by definition.

$\widehat{g}$ is finite dimensional because we can identify $\widehat{g} = (H/\mathfrak{m}_g H)^*$, where $H/\mathfrak{m}_g H$ is a finite-dimensional module. The only way for it to be zero is for $\mathfrak{m}_g H$ to be the whole of H. However, in this case, by Nakayama's Lemma (in the form stated for instance in [33, Theorem 2.2]) there would have to exist an element $a \in A$ satisfying $aH = 0$ and $a \equiv 1_A \mod \mathfrak{m}_g$. In particular, the latter statement means that a would have to be nonzero. This is impossible and so $\mathfrak{m}_g H \neq H$ and $\widehat{g} \neq 0$.

Finally, if H is free as an A-module, then $H/\mathfrak{m}_g H \cong (A/\mathfrak{m}_g)^n$ for $n = \dim_k(\overline{H})$, and so by the above $\dim_k(\widehat{g}) = \dim_k(\overline{H})$.

(ii) α_g is the algebra map $A \to k$ given by factoring along $\mathfrak{m}_g$ by definition. This means that in A^0, any map which is zero on $\mathfrak{m}_g$ is a scalar multiple of α_g.

Let $f \in \widehat{g}$. This means that f is zero on $\mathfrak{m}_g H$ and so $\iota^0(f) = f|_A$ is zero on $\mathfrak{m}_g$, meaning that $\iota^0(f)$ is a scalar multiple of α_g. Since $k\alpha_g$ is a one-dimensional vector space, in order to show that $\iota^0(\widehat{g}) = k\alpha_g$ we only need to show that we can choose f such that $\iota^0(f)$ is nonzero.

Since by (i) $\widehat{g} \neq 0$ and so $\mathfrak{m}_g H \neq H$, we know that $1_H \notin \mathfrak{m}_g H$. This means that we can choose f such that $f(1_H) \neq 0$. Then $\iota^0(f)(1_A) = f(\iota(1_A)) = f(1_H) \neq 0$, giving us what we want.

So

$$\iota^0(k\widehat{G}) = \iota^0\left(\bigoplus_{g\in G}\widehat{g}\right) = \sum_{g\in G}\iota^0(\widehat{g}) = \sum_{g\in G}k\alpha_g = kG$$

as required.

(iii) We want to show that

$$\widehat{g} = \{f \in H^0 \mid \rho(f) = f \otimes \alpha_g\}.$$

$\subseteq$: Suppose $f \in \widehat{g}$. By Lemma 5.15(ii), $\widehat{g}$ is a subcoalgebra of H^0. Since by (ii) $\iota^0(\widehat{g}) \subseteq k\alpha_g$, we have $\rho(f) = \sum \lambda_1 f_1 \otimes \alpha_g$ for some $\lambda_1 \in k$. Now the counit part of the comodule axiom means that

$$(\mathrm{id} \otimes \varepsilon_{A^0}) \circ \rho(f) = f \otimes 1$$

and so $\sum \lambda_1 f_1 = f$ as required.

$\supseteq$: Suppose $f \in H^0$, and f is such that $\rho(f) = f \otimes \alpha_g$. We need to show that $f(\mathfrak{m}_g H) = 0$.

Using the fact that A is central in H and letting $\mu : k \otimes k \to k$ denote the canonical isomorphism given by scalar multiplication, we find that

$$\begin{aligned}
f(\mathfrak{m}_g H) &= f(H\mathfrak{m}_g)\\
&= f(H\iota(\mathfrak{m}_g))\\
&= \sum f_1(H) f_2(\iota(\mathfrak{m}_g))\\
&= \mu \circ (\mathrm{id} \otimes \iota^0) \circ \Delta_{H^0}(f)(H \otimes \mathfrak{m}_g)\\
&= \mu \circ \rho(f)(H \otimes \mathfrak{m}_g)\\
&= f(H)\alpha_g(\mathfrak{m}_g) = 0,
\end{aligned}$$

giving us what we want.

□

With these results, we can show that whenever H^0 decomposes as a crossed product, so does $k\widehat{G}$ with respect to the same action and cocycle.

Theorem 5.19. *Retain the notation and assumptions of Definition 5.13. Suppose that H^0 decomposes as a crossed product:*

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

as left $\overline{H}^0$-modules, algebras and right A^0-comodules for some cocycle σ and action of A^0. Then $k\widehat{G}$ also decomposes as a crossed product, namely

$$k\widehat{G} \cong \overline{H}^0 \#_{\sigma|_{kG\otimes kG}} kG.$$

Here the action and cocycle both arise from those in the decomposition of H^0 as a crossed product under restriction to kG or $kG \otimes kG$ respectively . Moreover, for all $g \in G$,

$$\widehat{g} = \overline{H}^0 \# \alpha_g,$$

where $\alpha_g : A \to k$ denotes the algebra map corresponding to g.

Proof. Step 1: Showing that $\widehat{g} = \overline{H}^0 \# \alpha_g$.

$\supseteq$: Let $f \in \overline{H}^0$. We want to show that $f \# \alpha_g \in \widehat{g}$.

By Proposition 2.23, there exists a convolution invertible right A^0-comodule map $\phi : A^0 \to H^0$ such that we can identify $f \# \alpha_g$ with $f\phi(\alpha_g)$. Moreover, $\phi(\alpha_g) \in \widehat{g}$: this follows as ϕ is a right comodule map, meaning that, since $\alpha_g \in A^0$ is grouplike,

$$\rho(\phi(\alpha_g)) = (\phi \otimes \mathrm{id}) \circ \Delta_{A^0}(\alpha_g) = \phi(\alpha_g) \otimes \alpha_g,$$

and so by Lemma 5.18 $\phi(\alpha_g) \in \widehat{g}$. Now we can use this and the fact that

$$\Delta_H(\mathfrak{m}_g) \subseteq A^+H \otimes H + H \otimes \mathfrak{m}_g H$$

by the proof of Lemma 5.15(i) to get

$$\begin{aligned} f\phi(\alpha_g)(\mathfrak{m}_g H) &\subseteq \mu \circ (f \otimes \phi(\alpha_g)) \left(A^+H \otimes H + H \otimes \mathfrak{m}_g H\right) \\ &= f(A^+H)\phi(\alpha_g)(H) + f(H)\phi(\alpha_g)(\mathfrak{m}_g H) \\ &= 0 + 0 = 0. \end{aligned}$$

So $f\#\alpha_g(\mathfrak{m}_g H) = 0$, meaning that $f\#\alpha_g \in \widehat{g}$ by definition.

$\subseteq$: Let $\rho := (\mathrm{id} \otimes \iota^0) \circ \Delta_{H^0} : H^0 \to H^0 \otimes A^0$ denote the map giving the right A^0-comodule structure on H^0. By Lemma 5.18(iii), it suffices to show that the only $f \in H^0$ satisfying $\rho(f) = f \otimes \alpha_g$ are those such that $f \in \overline{H}^0\#\alpha_g$.

Suppose $f \in H^0$ is such that $\rho(f) = f \otimes \alpha_g$. Recall from Proposition 5.11 that $A^0 \cong U(\mathrm{Lie}\, G)\#kG$. This and the assumed crossed product decomposition of H^0 means we have

$$f = \sum_{i=1}^{n} f_i \phi(u_i\#\alpha_{g_i})$$

for some $n \geq 1$, $f_i \in \overline{H}^0$, $u_i \in U(\mathrm{Lie}\, G)$ and $g_i \in G$. Then

$$\begin{aligned}
\rho(f) &= \rho\left(\sum_{i=1}^{n} f_i \phi(u_i\#\alpha_{g_i})\right) \\
&= \sum_{i=1}^{n} \rho(f_i)\rho(\phi(u_i\#\alpha_{g_i})) \\
&= \sum_{i=1}^{n} (f_i \otimes 1_{A^0})(\phi \otimes \mathrm{id}) \circ \Delta_{A^0}(u_i\#\alpha_{g_i}) \\
&= \sum_{i=1}^{n} (f_i \otimes 1_{A^0})(\phi \otimes \mathrm{id})(\Delta_{A^0}(u_i)(\alpha_{g_i} \otimes \alpha_{g_i})) \\
&= \sum_{i=1}^{n} \sum f_i \phi((u_i)_1 \alpha_{g_i}) \otimes (u_i)_2 \alpha_{g_i}.
\end{aligned}$$

The map ϕ is injective and since $u_i \in U(\mathrm{Lie}\, G)$, u_i is a polynomial in primitive elements. In particular, the only way for $\rho(f) = f \otimes \alpha_g$ is for $n = 1$, $g_1 = g$ and $u_1 = 1_{H^0}$.

Step 2: $k\widehat{G} \cong \overline{H}^0\#_\tau kG$.

By the above, $k\widehat{G} = \bigoplus_{g \in G} \overline{H}^0\#\alpha_g$. So in particular, we can find a linear isomorphism $\psi : k\widehat{G} \to \overline{H}^0 \otimes kG$ given by $\psi(f\#\alpha_g) := f \otimes g$ and extending linearly. It follows immediately from this definition that ψ is a left $\overline{H}^0$-module map as well as a right kG-comodule map.

So by [36, Theorem 8.2.4], to show that $k\widehat{G}$ decomposes as

$$k\widehat{G} \cong \overline{H}^0\#_\tau kG$$

for some action of kG on $\overline{H}^0$ and cocycle τ it suffices to show that $\overline{H}^0 \subseteq k\widehat{G}$ is Galois, meaning that the map

$$\beta : k\widehat{G} \otimes_{\overline{H}^0} k\widehat{G} \to k\widehat{G} \otimes_k kG$$

given by $\beta(a \otimes b) := (a \otimes 1)\rho(b)$ is bijective.

We have $k\widehat{G} = \bigoplus_{g\in G} \overline{H}^0 \# \alpha_g$ and

$$\overline{H}^0 \phi(\alpha_g) = \overline{H}^0(1_H \# \alpha_g) = \overline{H}^0 \# \alpha_g.$$

So $k\widehat{G}$ is a free $\overline{H}^0$-module with basis $\{\phi(\alpha_g) \mid g \in G\}$ and so we can identify $k\widehat{G} \otimes_{\overline{H}^0} k\widehat{G}$ with $\bigoplus_{g\in G} k\widehat{G} \otimes_k \alpha_g$. Viewed like this, the map β sends $\sum a_i \otimes \alpha_{g_i}$ to $\sum a_i\phi(\alpha_{g_i}) \otimes \alpha_{g_i}$.

Define β^{-1} by $\beta^{-1} : f \otimes \alpha_g \mapsto f\phi^{-1}(\alpha_g) \otimes \alpha_g$. We know that α_g is grouplike and ϕ^{-1} is the convolution inverse for ϕ. This means that $\phi(\alpha_g)\phi^{-1}(\alpha_g) = 1_H$ and so for all $a_i \in A, g_i \in G$,

$$\begin{aligned}(\beta^{-1} \circ \beta)\left(\sum a_i \otimes \alpha_{g_i}\right) &= \beta^{-1}\left(\sum a_i\phi(\alpha_{g_i}) \otimes \alpha_{g_i}\right)\\ &= \sum a_i\phi(\alpha_{g_i})\phi^{-1}(\alpha_{g_i}) \otimes \alpha_{g_i}\\ &= \sum a_i \otimes \alpha_{g_i}\end{aligned}$$

The same argument shows that $\beta \circ \beta^{-1} = \mathrm{id}$. So β is invertible and hence bijective as required.

Step 3: Action and cocycle agree

This simply follows from the fact that $k\widehat{G}$ is a Hopf subalgebra of H^0 such that $\overline{H}^0 \#_\tau \alpha_g \subseteq k\widehat{G}$ corresponds to $\overline{H}^0 \#_\sigma \alpha_g \subseteq H^0$. So the crossed product multiplication from the decomposition

$$k\widehat{G} \cong \overline{H}^0 \#_\tau kG$$

gained in part (ii) has to agree with that of H^0, forcing the action and cocycle to coincide.

□

Thanks to a result of Takeuchi, we find that as long as we can extend the cleaving map $kG \to k\widehat{G}$ to the whole of A^0 and H^0, the converse of Theorem 5.19 also holds: in this case, if $k\widehat{G}$ decomposes as a crossed product so does the whole of H^0.

Theorem 5.20. *Retaining the notation and assumptions of Definition 5.13, suppose that*

$$k\widehat{G} \cong \overline{H}^0 \#_\sigma kG$$

with cleaving map given by ψ and suppose there exists some right A^0-comodule map ϕ : $A^0 \to H^0$ satisfying $\phi|_{kG} = \psi$. Then ϕ is convolution invertible and

$$H^0 \cong \overline{H}^0 \#_\tau A^0$$

with cleaving map ϕ, and $\tau_{kG\otimes kG} = \sigma$ and the two actions agree on kG.

Proof. Suppose $\psi : kG \to k\widehat{G}$ is a cleaving map for $k\widehat{G}$. So $\psi : kG \to k\widehat{G}$ is convolution invertible, which means that $\phi|_{kG} : kG \to H^0$ is so as well. Now we note that the coradical of A^0 is given by kG as by Proposition 2.13, $A^0 \cong U(\operatorname{Lie} G)\#kG$ as Hopf algebras, where $U(\operatorname{Lie} G)$ is connected and G consists of grouplike elements. So by [50, Lemma 14], since the restriction of ϕ to the coradical is convolution invertible, so is the whole map $\phi : A^0 \to H^0$. Corollary 5.5 gives us that

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

in the ways required, and the actions and cocycles agreeing follows by the fact that they are defined via the cleaving maps and one is a restriction of the other. □

So whenever we have some right A^0-comodule map $\phi : A^0 \to H^0$ (such as, by Lemma 5.6, whenever there exists an A-module projection $H \to A$), then $k\widehat{G}$ decomposes as a crossed product with respect to it if and only if the whole of H^0 does so.

We now return to looking at W. In contrast to $k\widehat{G}$, we discover that W decomposes as a crossed product with only very minor assumptions on H.

Theorem 5.21. *Retain the notation and assumptions of Definition 5.13. Suppose that there exists a left A-module projection $\Pi : H \to A$ satisfying $\Pi|_A = \mathrm{id}_A$. Then W decomposes as a crossed product, with*

$$W \cong \overline{H}^0 \#_\sigma U(\operatorname{Lie} G)$$

for some action and cocycle σ.

Proof. We first note that by Lemma 5.14, W is a Hopf subalgebra of H^0 satisfying $\iota^0(W) \subseteq U(\operatorname{Lie} G)$. In particular,

$$\rho(W) = (\mathrm{id} \otimes \iota^0) \circ \Delta_{H^0}(W) \subseteq W \otimes U(\operatorname{Lie} G)$$

and so the canonical A^0-comodule structure on H^0 restricts to a canonical $U(\operatorname{Lie} G)$-comodule structure on W. Further, the map $\Pi^0 : A^0 \to H^0$, which by Lemma 5.6 is a right A^0-comodule map, satisfies $\Pi^0(U(\operatorname{Lie} G)) \subseteq W$. This follows because Π^0 extends maps in A^0 to maps in H^0 by setting them to be zero on X: any map $u \in U(\operatorname{Lie} G)$ is zero on $(A^+)^n$ for some $n \geq 1$, and it follows from this that

$$\Pi^0(u)((A^+H)^n) \subseteq \Pi^0(u)((A^+ \oplus X)^n) \subseteq \Pi^0(u)((A^+)^n \oplus X) = 0.$$

So $u \in W$ as required.

So $\Pi^0|_{U(\mathrm{Lie}\,G)} : U(\mathrm{Lie}\,G) \to W$ is a right $U(\mathrm{Lie}\,G)$-comodule map. By [50, Lemma 14], it is convolution invertible if and only if its restriction to the coradical is convolution invertible. However, $U(\mathrm{Lie}\,G)$ is connected, meaning that its coradical is simply k, and of course any map $k \to W$ is convolution invertible. So $\Pi^0|_{U(\mathrm{Lie}\,G)}$ is convolution invertible and so a cleaving map, meaning that by Proposition 2.22 we have

$$W \cong (W)^{coU(\mathrm{Lie}\,G)} \#_\sigma U(\mathrm{Lie}\,G),$$

and all that remains to be shown is that the coinvariants are given by $\overline{H}^0$.

For this, first note that since the $U(\mathrm{Lie}\,G)$-comodule structure on W is the restriction of the A^0-comodule structure on H^0,

$$(W)^{coU(\mathrm{Lie}\,G)} = W \cap \left(H^0\right)^{coA^0}. \tag{5.5}$$

Now by Lemma 5.4, $\left(H^0\right)^{coA^0} = \overline{H}^0$, and by Lemma 5.17 we have $\overline{H}^0 \subseteq W$. This means that (5.5) turns into

$$(W)^{coU(\mathrm{Lie}\,G)} = \overline{H}^0$$

as required. □

The condition that there is a left A-module projection $\Pi : H \to A$ is frequently fulfilled and may easily always be true - as we will see in Section 5.5.1, since we are assuming that k is algebraically closed it is an open question whether a far stronger condition on H is always true. Among others, Theorem 5.21 gives rise to the following corollary.

Corollary 5.22. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F), with G denoting the affine algebraic group such that $A \cong \mathcal{O}(G)$ and $\overline{H} := H/A^+H$ denoting the canonical quotient Hopf algebra, and suppose that H is free as an A-module with an A-basis that includes 1_H. Then there exists a Hopf subalgebra W of H^0 that decomposes as a crossed product, with*

$$W \cong \overline{H}^0 \#_\sigma U(\mathrm{Lie}\,G)$$

for some cocycle σ and action of $U(\mathrm{Lie}\,G)$ on $\overline{H}^0$.

This is always satisfied when $H \cong A\#_\tau \overline{H}$ is itself a crossed product, a case we will look at in more detail in Section 5.5.

5.4 The finite dual of $U_\epsilon(\mathfrak{sl}_2)$

Throughout, k is an algebraically closed field of characteristic zero.

In this section, we use Theorem 5.8 to work out the finite dual of the quantised enveloping algebra $U_\epsilon(\mathfrak{sl}_2(k))$.

First, recall that given a semisimple Lie algebra $\mathfrak{g}$ and a nonzero scalar $q \in k^*$ with $q \neq 1$, we can define the *quantised enveloping algebra* $U_q(\mathfrak{g})$, which can be viewed as a noncocommutative deformation of the usual enveloping algebra on the parameter q. This construction and its properties are discussed in detail in [7, Section I.6].

The theory of quantised enveloping algebras divides into two different cases. In the case where q is *generic*, one studies quantised enveloping algebras for $q \in k^*$ not a root of unity. In this case, the finite dual $U_q(\mathfrak{g})^0$ is known: it is isomorphic to $\mathcal{O}_q(G)\#k\mathbb{Z}_2^{\mathrm{rk}(\mathfrak{g})}$, where G is the unique simply connected affine algebraic group satisfying $\operatorname{Lie} G = \mathfrak{g}$ (see [25, Section 9.1.1] for details).

When q is a root of unity, the finite dual of $U_q(\mathfrak{g})$ is less well understood. However, $U_q(\mathfrak{g})$ always has a central Hopf subalgebra satisfying (F) (see [7, Theorem III.6.2]) and so provides an example of the sort of Hopf algebra we are studying in this chapter.

In this section, we look at the specific case where $\mathfrak{g} := \mathfrak{sl}_2(k)$ and show how our results apply, in particular that we can describe $U_\varepsilon(\mathfrak{sl}_2(k))^0$ as a crossed product. We then form a conjecture about the general case $U_\varepsilon(\mathfrak{g})^0$ and note a partial positive result in this area.

Definition 5.23. Let k be an algebraically closed field of characteristic zero, $n \geq 3$ be odd and $\epsilon \in k^*$ be a primitive nth root of unity. The *quantised enveloping algebra* $U_\epsilon(\mathfrak{sl}_2(k))$ is defined as follows:

As an algebra, $U_\epsilon(\mathfrak{sl}_2(k))$ is generated by $E, F, K^{\pm 1}$ under the relations

$$KE = \epsilon^2 EK \qquad KF = \epsilon^{-2} FK$$
$$EF - FE = \frac{K - K^{-1}}{\epsilon - \epsilon^{-1}} \qquad KK^{-1} = 1 = K^{-1}K.$$

The coalgebra structure is given by K grouplike and

$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

We write U for $U_\epsilon(\mathfrak{sl}_2(k))$.

This algebra has a nice PBW basis, along with a central Hopf subalgebra satisfying (F):

Proposition 5.24. *Let $\epsilon \in k^*$ be a primitive nth root of unity for some odd $n > 1$, and let $U := U_\epsilon(\mathfrak{sl}_2(k))$ denote the quantised universal enveloping algebra. Then*

(i) U has a PBW basis given by $\{E^i K^j F^t \mid i, j, t \in \mathbb{Z}, i, t \geq 0\}$.

(ii) The subalgebra $A := k\langle E^n, F^n, K^{\pm n}\rangle$ is a central Hopf subalgebra of U, satisfying

$$A \cong k[E^n, F^n, K^{\pm n}].$$

(iii) U is Noetherian as an algebra and is finitely-generated as an A-module.

Proof. (i) This is by [7, Corollary I.3].

(ii) This follows by [7, Theorem III.6.2] (see also [7, Section III.2.1] for further discussion).

(iii) The fact that U is Noetherian follows as it can be written as an iterated skew polynomial algebra. To see that U is finitely-generated, note that the finite set $\{E^i K^j F^t \mid 0 \leq i, j, t < n\}$ provides an A-generating set for U. □

The proposition means that we have an A-module decomposition of U as follows:

$$U \cong A \oplus \bigoplus_{\substack{0 \leq i,j,t < n \\ i+t>0}} AE^i K^j F^t \oplus \bigoplus_{s=1}^{n-1} A(K^s - 1),$$

where the summand A is the embedding of A into U as a Hopf subalgebra. As such, if we can show that the complement of A under this decomposition is a coideal, then Theorem 5.8(i) applies. We choose $K^s - 1$ rather than K^s in the latter summand in order to ensure that the complement will lie inside U^+, one of the requirements to be a coideal.

Lemma 5.25. *Let $\epsilon \in k$ be a primitive nth root of unity for some $n > 1$, n odd, and let $U := U_\epsilon(\mathfrak{sl}_2(k))$ and $A := k[E^n, F^n, K^{\pm n}] \subseteq U$ be the canonical central Hopf subalgebra described in Proposition 5.24. Then there is an A-module decomposition of U given by*

$$U \cong A \oplus \bigoplus_{\substack{0 \leq i,j,t < n \\ i+t>0}} AE^i K^j F^t \oplus \bigoplus_{s=1}^{n-1} A(K^s - 1).$$

Moreover, the complement of A under this decomposition is a coideal.

Proof. The fact that the decomposition given is an A-module decomposition with all sums direct follows straightforwardly from Proposition 5.24 (i) and (ii). Define X by

$$X := \bigoplus_{\substack{0 \leq i,j,t < n \\ i+t>0}} AE^i K^j F^t \oplus \bigoplus_{s=1}^{n-1} A(K^s - 1),$$

the complement of A. We need to show X is a coideal.

First note that as an A-module, X has generating set $Y \cup P$, where $Y = \{E^i K^j F^t \mid 0 \leq i, j, t < n, i + t > 0\}$ and $P = \{K^s - 1 \mid 1 \leq s < n\}$. It suffices to check that $\varepsilon(Y) = \varepsilon(P) = 0$ and also that both $\Delta(Y)$ and $\Delta(P)$ are contained in $U \otimes X + X \otimes U$. As A is a coalgebra, this will give us the result for the whole of X.

Step 1: $\varepsilon(Y) = \varepsilon(P) = 0$

Given $E^i K^j F^t \in Y$, we have

$$\varepsilon(E^i K^j F^t) = \varepsilon(E)^i \varepsilon(K)^j \varepsilon(F)^t = 0^{i+t} = 0$$

as $i + t > 0$.

Similarly, given $(K^s - 1) \in P$, we have

$$\varepsilon(K^s - 1) = \varepsilon(K^s) - \varepsilon(1) = 1 - 1 = 0.$$

Step 2: $\Delta(P) \subseteq U \otimes X + X \otimes U$

Let $(K^s - 1) \in P$. Because K^s is grouplike, $(K^s - 1)$ is primitive, and so

$$\Delta(K^s - 1) = (K^s - 1) \otimes 1 + 1 \otimes (K^s - 1) \in X \otimes U + U \otimes X$$

as required.

Step 3: $\Delta(Y) \subseteq U \otimes X + X \otimes U$

Let $E^i K^j F^t \in Y$, i.e. we have $0 \leq i, j, t < n$ and at least one of i, t nonzero.

First, note that

$$\Delta(E^i) = \sum_{\ell=0}^{i} \binom{i}{\ell}_{\epsilon^2} E^{i-\ell} K^\ell \otimes E^\ell,$$

using standard facts about skew-commuting variables and recalling the notation $\binom{r}{s}_\tau$ for $r \geq 1$, $s \geq 0$ and $\tau \in k^*$ from Example 2.12.

Similarly,

$$\Delta(F^t) = \sum_{r=0}^{t} \binom{t}{r}_{\epsilon^2} F^t \otimes K^{-t} F^{r-t}.$$

This means that

$$\begin{aligned} \Delta(E^i K^j F^t) &= \left(\sum_{\ell=0}^{i} \binom{i}{\ell}_{\epsilon^2} E^{i-\ell} K^\ell \otimes E^\ell \right) \Delta(K^j) \left(\sum_{r=0}^{t} \binom{t}{r}_{\epsilon^2} F^t \otimes K^{-t} F^{r-t} \right) \\ &= \sum_{\ell=0}^{i} \sum_{r=0}^{t} \binom{i}{\ell}_{\epsilon^2} \binom{t}{r}_{\epsilon^2} E^{i-\ell} K^{\ell+j} F^t \otimes E^\ell K^{j-t} F^{r-t}. \end{aligned}$$

Consider the normed summand $E^{i-\ell} K^{\ell+j} F^t \otimes E^\ell K^{j-t} F^{r-t}$ and assume for a contradiction that it is not an element of $X \otimes H + H \otimes X$.

This means that neither of the tensorands are elements of X, meaning that both the degrees of E and F must be divisible by n in both of the tensorands. This in turns means that $\ell + (i - \ell) = i \equiv 0 \mod n$, and similarly $r \equiv 0 \mod n$. Since $0 \leq i, r < n$, this means that $i = r = 0$. However, we assumed that $i + r > 0$. This is a contradiction: at least one of the tensorands must be in X.

So for each summand we have

$$\binom{i}{\ell}_{q^2} \binom{t}{r}_{q^2} E^{i-\ell} K^{\ell+j} F^t \otimes E^\ell K^{j-t} F^{r-t} \in H \otimes X + X \otimes H.$$

As a result,

$$\Delta(E^i K^j F^t) \in H \otimes X + X \otimes H$$

as required. □

This allows us to fully describe the algebra structure of U^0, recalling the definition of a quantised coordinate ring from [7, Chapters I.7 and III.7] and using it to define the restricted quantised coordinate ring as in [7, Section III.4.4].

Corollary 5.26. *Suppose that k is an algebraically closed field of characteristic zero, $n \geq 3$ is odd and $\epsilon \in k^*$ a primitive nth root of unity.*

(i) The finite dual of $U_\epsilon(\mathfrak{sl}_2(k))$ is a smash product, with

$$U_\epsilon(\mathfrak{sl}_2(k))^0 \cong \overline{\mathcal{O}_\epsilon(SL_2(k))} \# \left(U(\mathfrak{g}) \# k((k^2, +) \rtimes (k^*, *))\right),$$

where $\mathfrak{g} := k\langle a, b, c \mid [a, b] = 0, [a, c] = a, [b, c] = b\rangle$ and $\overline{\mathcal{O}_\epsilon(SL_2(k))}$ denotes the restricted quantised coordinate ring of $SL_2(k)$.

(ii) Neither action in the two smash products in (i) is trivial.

(iii) There are two canonical Hopf subalgebras of $U_\epsilon(\mathfrak{sl}_2(k))^0$, given by $\overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))} \# U(\mathfrak{g})$ and $\overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))} \# k((k^2, +) \rtimes (k^*, *))$ respectively.

Proof. (i) Let $U := U_\varepsilon(\mathfrak{sl}_2(k))$.

By Lemma 5.25, we have $U \cong A \oplus X$ as A-modules, where X is a coideal and the direct summand A corresponds to the embedding of A into U as a central Hopf subalgebra. By Theorem 5.8, this means that we have

$$U^0 \cong \left(H/A^+H\right)^* \# A^0.$$

Because A is an affine commutative Hopf algebra and k has characteristic 0, as discussed in Remark 1.13 it is isomorphic to $\mathcal{O}(G)$ for some affine algebraic group G. Following the discussion in [7, Section III.6.5], we have $G \cong (k^2, +) \rtimes (k^*, *)$ with action given by $\lambda \cdot (f, g) = (\lambda^{-2} f, \lambda^{-2} g)$ for $\lambda \in k^*$, $f, g \in k$, using the maximal torus T in $\mathrm{SL}_2(k)$ given by

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in k^* \right\} \cong (k^*, *).$$

Now we want to work out $\operatorname{Lie} G$. As discussed in [21, Section 5.1], one of the ways it arises is via

$$\operatorname{Lie} G \cong \left(A^+/(A^+)^2\right)^*.$$

We first note that $A \cong k[E^n, F^n, K^{\pm n}]$, where E^n and F^n are skew-primitive with

$$\Delta(E^n) = E^n \otimes 1 + K^n \otimes E^n \qquad \Delta(F^n) = F^n \otimes K^{-n} + 1 \otimes F^n$$

and K^n is grouplike - this simply follows by standard results on skew-commuting variables and q-binomials. This means that $A^+ = \langle E^n, F^n, K^n - 1 \rangle \subseteq A$.

We have

$$\dim_k(A^+/(A^+)^2) = \dim_k(\operatorname{Lie} G) = \dim(G) = 3$$

by [1, Theorem 4.3.11]. Since $E^n, F^n, K^n - 1 \notin (A^+)^2$ and their images are linearly independent in $A^+/(A^+)^2$, this means said images form a basis for that vector space. Under minor abuse of notation via identifying E^n, F^n and $K^n - 1$ with their images, this gives us a resulting dual basis for $\operatorname{Lie} G \cong \left(A^+/(A^+)^2\right)^*$:

$$\begin{array}{lll} a(E^n) = 1 & a(F^n) = 0 & a(K^n - 1) = 0 \\ b(E^n) = 0 & b(F^n) = 1 & b(K^n - 1) = 0 \\ c(E^n) = 0 & c(F^n) = 0 & c(K^n - 1) = 1. \end{array}$$

Setting $a(1) = b(1) = c(1) = 0$ allows us to extend these maps to $\left(A/(A^+)^2\right)^*$ and from there to A^0 by setting them to be zero on $(A^+)^2$ (noting that $(A^+)^2$ is an ideal of finite codimension and so the resulting maps are indeed elements of A^0). This means that on an element $\alpha := E^{kn} F^{rn} K^{\ell n} \in A$ with $k, r \geq 0, \ell \in \mathbb{Z}$, a, b and c become

$$a(\alpha) = \delta_{k1}\delta_{r0}, \qquad b(\alpha) = \delta_{k0}\delta_{r1}, \qquad c(\alpha) = \ell\delta_{k0}\delta_{r0}.$$

As discussed in [21, Section 9.3], in this embedding $\operatorname{Lie} G \to A^0$, the Lie bracket of $\operatorname{Lie} G$ agrees with the commutator in A^0. This lets us compute the Lie bracket of a, b, c using the multiplication in A^0. Some calculation shows that this gives us precisely $\operatorname{Lie} G = \mathfrak{g}$ as in the statement of the theorem.

Finally, by Proposition 2.13, all this means that we have

$$A^0 \cong U(\operatorname{Lie} G)\# kG \cong U(\mathfrak{g})\# k((k^2,+) \rtimes (k^*,*)).$$

Now H/A^+H is the restricted quantised enveloping algebra $\overline{U_\epsilon(\mathfrak{sl}_2(k))}$. By [7, Theorem III.7.10], this satisfies

$$\overline{U_\epsilon(\mathfrak{sl}_2(k))}^* \cong \overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}.$$

Combining these, we find that

$$U^0 \cong \overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}\# \left(U(\mathfrak{g})\# kG\right) \tag{5.6}$$

as required.

(ii) We want to see that neither smash product is trivial. To do this we find elements in $\overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}$, $U(\mathfrak{g})$ and kG that do not commute.

First, note that the elements of $\overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}$ are those functions in $\operatorname{Hom}_k(U,k)$ that are zero on $A^+U = \langle E^n, F^n, K^{\pm n}-1\rangle$. So we can find a map $f \in \overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}$ satisfying $f(EK^{n-1}) = 1$. Now, the elements of $U(\operatorname{Lie} G)\# kG$ consist of functions in $k[E^n, F^n, K^{\pm n}]^0$ extended to the whole of U by being set to be zero on X where we recall that

$$X := \bigoplus_{\substack{0\le i,j,t<n \\ i+t>0}} AE^iK^jF^t \oplus \bigoplus_{s=1}^{n-1} A(K^s-1).$$

We can set $g \in U(\mathfrak{g}) \subseteq U^0$ to be $g = a+b$ where a and b are as in (i), so a map satisfying $g(E^n) = g(F^n) = 1$, $g(K^{\pm n}) = 0$. Since kG corresponds to the grouplike elements in U^0, for $\lambda \in k^*$ such that $\lambda \neq 1$ we can set $h_\lambda \in kG$ to be the algebra map such that $h_\lambda(E^n) = h_\lambda(F^n) = h_\lambda(K^n) = \lambda$.

Step 1: f and h_λ do not commute

Consider EK^{n-1}. We have

$$\begin{aligned} fh_\lambda(EK^{n-1}) &= f(EK^{n-1})h_\lambda(K^{n-1}) + f(K^n)h_\lambda(EK^{n-1}) \\ &= h_\lambda(1) + 0 = 1, \end{aligned}$$

using the fact that $f(EK^{n-1}) = 1$ by assumption and that because h_λ is zero on X we have $h_\lambda(K^{n-1}) = h_\lambda(1)$ and $h_\lambda(EK^{n-1}) = 0$. However,

$$\begin{aligned} h_\lambda f(EK^{n-1}) &= h_\lambda(EK^{n-1})f(K^{n-1}) + h_\lambda(K^n)f(EK^{n-1}) \\ &= 0 + \lambda = \lambda, \end{aligned}$$

again using the definition of h_λ and f. Since $\lambda \neq 1$ by assumption, we are left with

$$fh_\lambda(EK^{n-1}) \neq h_\lambda f(EK^{n-1}),$$

meaning that $\overline{\mathcal{O}_\epsilon(\mathrm{SL}_2(k))}$ and $U(\mathrm{Lie}\, G)\#kG$ do not commute and the smash product between them is nontrivial.

Step 2: g and h_λ do not commute

Consider the element $E^n \in U$. Previous results on the coproduct of skew-primitive elements with respect to a grouplike element with which they skew-commute (see e.g. the discussion in Example 2.12) tell us that we have

$$\Delta_U(E^n) = E^n \otimes 1 + K^n \otimes E^n.$$

This means that

$$gh_\lambda(E^n) = g(E^n)h_\lambda(1) + g(K^n)h_\lambda(E^n) = 1 + 0 = 1.$$

However,

$$h_\lambda g(E^n) = h_\lambda(E^n)g(1) + h_\lambda(K^n)g(E^n) = 0 + \lambda = \lambda.$$

Again, this means that g and h_λ and hence $U(\mathrm{Lie}\, G)$ and kG do not commute, and so the smash product between them is nontrivial.

(iii) By Lemmas 5.14 and 5.15, we have Hopf subalgebras W and $k\widehat{G}$ of H^0. Since U^0 decomposes as a smash product with respect to the its central Hopf subalgebra $A \cong \mathcal{O}(G)$ by part (i), Theorem 5.19 and Corollary 5.22 apply. So the Hopf subalgebras W and $k\widehat{G}$ become

$$W := \{f \in U^0 \mid f((A^+U)^n) = 0 \text{ for some } n > 0\} \cong \overline{\mathcal{O}_\varepsilon(SL_2(k))}\#U(\mathrm{Lie}\, G)$$
$$k\widehat{G} := \{f \in U^0 \mid f(\mathfrak{m}_g U) = 0 \text{ for some } g \in G\} \cong \overline{\mathcal{O}_\varepsilon(SL_2(k))}\#kG,$$

with trivial cocycle because the cocycle in U^0 is trivial. This gives us exactly what we want. □

The relations between E and F did not come into play at all in this proof. Our result relies on the fact that $U_\epsilon(\mathfrak{sl}_2(k))$ has a PBW-basis given by $\{E^iK^jF^r \mid i, r \geq 0, j \in \mathbb{Z}\}$ such that K is grouplike, E and F skew-primitive involving K, and E and K as well as F and K skew-commute.

The reason this cannot be immediately extended to further $U_\epsilon(\mathfrak{g})$ is due to the PBW basis of $U_\varepsilon(\mathfrak{sl}_2(k))$ being formed by grouplike and skew-primitive elements. Although

$U_\epsilon(\mathfrak{g})$ for a finite-dimensional semisimple $\mathfrak{g}$ is generated by grouplike elements K_λ along with skew-primitive generators $E_1, ..., E_n$ and $F_1, ..., F_n$ such that each E_i and K_λ skew-commute, these do not generally form a PBW basis. Instead, one typically defines additional generators $E_{\alpha_i}, F_{\alpha_i}$, corresponding to non-simple roots of $\mathfrak{g}$ where the $E_1, ..., E_n$, and $F_1, ..., F_n$ correspond to simple roots. These extra generators give us a PBW basis $\{\mathbf{E}^\mathbf{i}\mathbf{K}^\mathbf{j}\mathbf{F}^\mathbf{r}\}$. The coproduct of these new generators is not immediately obvious - a particular problem for us is that they need not be skew-primitive.

However, it still seems reasonable to suggest that even if Corollary 5.26 cannot immediately be extended to further Lie algebras, there still exists some cleaving map so that Corollary 5.5 applies. We record this in the following conjecture.

Conjecture 5.A. *Let k be an algebraically closed field of characteristic zero, $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra, $n \geq 3$ odd and $\epsilon \in k^*$ a primitive nth root of unity. Let $Z \subseteq U_\epsilon(\mathfrak{g})$ denote the maximal central Hopf subalgebra in $U_\epsilon(\mathfrak{g})$, K denote the affine algebraic group such that $\mathcal{O}(K) \cong Z$ and G be the unique simply connected affine algebraic group satisfying* $\operatorname{Lie} G = \mathfrak{g}$. *Then*

$$U_\epsilon(\mathfrak{g})^0 \cong \overline{\mathcal{O}_\epsilon(G)}\#_\sigma(U(\operatorname{Lie} K)\#kK)$$

as algebras, left $\overline{\mathcal{O}_\epsilon(G)}$-modules and right $U(\operatorname{Lie} K)\#kK$-comodules for some action of $U(\operatorname{Lie} K)\#kK$ on $\overline{\mathcal{O}_\epsilon(G)}$ and cocycle σ.

The results of Section 5.3 give us a partial result in this vein regarding a Hopf subalgebra of $U_\epsilon(\mathfrak{g})^0$.

Theorem 5.27. *Let k be an algebraically closed field of characteristic zero, $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra, $n \geq 3$ odd and $\epsilon \in k^*$ a primitive nth root of unity. Let $Z \subseteq U_\epsilon(\mathfrak{g})$ denote the maximal central Hopf subalgebra in $U_\epsilon(\mathfrak{g})$, G the unique simply connected affine algebraic group satisfying* $\operatorname{Lie} G = \mathfrak{g}$ *and K the affine algebraic group such that $\mathcal{O}(K) \cong Z$. Then there exists a Hopf subalgebra W of $U_\epsilon(\mathfrak{g})^0$ such that*

$$W \cong \overline{\mathcal{O}_\epsilon(G)}\#_\sigma U(\operatorname{Lie} K)$$

as algebras, left $\overline{\mathcal{O}_\epsilon(G)}$-modules and right $U(\operatorname{Lie} K)$-comodules.

Proof. By [7, Theorem III.6.2], the central Hopf subalgebra Z noted in the statement of the theorem exists and $U_\epsilon(\mathfrak{g})$ is free over it with a basis that includes 1. So Corollary 5.22 applies: there exists a canonical Hopf subalgebra W of $U_\epsilon(\mathfrak{g})^0$ such that

$$W \cong \overline{U_\epsilon(\mathfrak{g})}^0\#_\sigma U(\operatorname{Lie} K)$$

in the required ways. Now by [7, Theorem III.7.10],

$$\overline{U_\epsilon(\mathfrak{g})}^* = \overline{\mathcal{O}_\epsilon(G)},$$

giving us the result. □

We will see another partial positive result for Conjecture 5.A later, as Example 5.34 shows that there always exists at least a right $U(\operatorname{Lie} K)\#kK$-comodule isomorphism between the desired objects.

5.5 The case when H decomposes as a crossed product

Throughout, we let k be an algebraically closed field of characteristic zero, let $A \subseteq H$ be Hopf algebras satisfying (F), and let $\overline{H} := H/A^+H$.

In the previous sections, we looked at the structure of Hopf algebras $A \subseteq H$ satisfying (F), discovered a canonical Hopf surjection $H^0 \to A^0$ and discussed when this surjection produces a crossed product decomposition of H^0. In this section, we look into the possibility of H itself decomposing as a crossed product. In Section 5.5.1, we ask when this happens and make note of relevant results in the literature. In Section 5.5.2, we look at the consequences for H^0 if H does decompose as a crossed product. In this case, we can use the results of Chapter 4, which we record in Theorem 5.33. Of particular note is that this along with the results of the previous section potentially gives us different decompositions of H^0 as a crossed product if certain conditions on H are met. We note that the resulting crossed product structures are always equivalent (defined in Definition 5.36) and in certain cases coincide.

5.5.1 The existence of a cleaving map

By Proposition 5.3, the canonical quotient map $\pi : H \to \overline{H}$ is in fact a Hopf surjection. So the results in Chapter 2 tell us when H decomposes as a crossed product:

Lemma 5.28. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F), and let $\overline{H} := H/A^+H$. Suppose we have an invertible right $\overline{H}$-comodule map $\gamma : \overline{H} \to H$ with respect to the canonical $\overline{H}$-comodule structure on H coming from the quotient map. Then there is a cocycle σ and action of $\overline{H}$ on A such that $A\#_\sigma\overline{H}$ is a crossed product, and the map*

$$\phi : A\#_\sigma\overline{H} \to H$$

given by $\phi(a\#h) = a\gamma(h)$ for $a \in A, h \in \overline{H}$ is an algebra, left A-module and right $\overline{H}$-comodule isomorphism, where the right $\overline{H}$-comodule structure on $A\#_\sigma\overline{H}$ is given by $a\#h \mapsto a\#h_1 \otimes h_2$.

Moreover, the converse holds: if there is an algebra isomorphism

$$H \cong A\#_\sigma\overline{H}$$

for some cocycle σ and action of $\overline{H}$ on H, then the canonical embedding $\overline{H} \to A\#_\sigma\overline{H}$ is a convolution invertible right $\overline{H}$-comodule map with respect to the structure defined above.

Proof. We know that the existence of a cleaving map γ means we have

$$H \cong H^{co\overline{H}}\#_\sigma\overline{H}$$

as algebras for some cocycle σ and relevant action by Proposition 2.22. Moreover, by Proposition 5.2(v), H is a faithfully flat A-module and so by [36, Proposition 3.4.3] we find that $H^{co\overline{H}} = A$. So all we need to show is that the action of $\overline{H}$ on A is trivial. We see as follows that this is a consequence of the fact that A is central.

Recall that given $a \in A$, $t \in \overline{H}$, the crossed product multiplication rules in $H \cong A\#_\sigma\overline{H}$ mean we have

$$\begin{aligned}\gamma(t)a &= \sum(t_1 \cdot a)\sigma(t_2, 1)\gamma(t_3)\\ &= \sum(t_1 \cdot a)\varepsilon_T(t_2)\gamma(t_3)\\ &= \sum(t_1 \cdot a)\gamma(t_2).\end{aligned}$$

So by centrality of A, we find that for any $t \in \overline{H}$ we have

$$a\gamma(t) = \sum(t_1 \cdot a)\gamma(t_2) \tag{5.7}$$

for all $a \in A$.

Now consider the map $\Pi : H \to A$ given by factoring along $A\gamma(\overline{H}^+)$. Given $s \in \overline{H}$ we can write $s = \varepsilon_{\overline{H}}(s)1 + s'$, with $s' \in \overline{H}^+$. So for $b \in A$ we have

$$\begin{aligned}\Pi(b\gamma(s)) &= \Pi(\varepsilon_{\overline{H}}(s)b\gamma(1_{\overline{H}}) + b\gamma(s'))\\ &= \varepsilon_{\overline{H}}(s)\Pi(b\gamma(1_{\overline{H}})) + \Pi(b\gamma(s'))\\ &= \varepsilon_{\overline{H}}(s)\Pi(b1_H)\\ &= \varepsilon_{\overline{H}}(s)\Pi(b) = \varepsilon_{\overline{H}}(s)b,\end{aligned}$$

recalling from Lemma 2.21 that we can assume without loss of generality that $\gamma(1_{\overline{H}}) = 1_H$.

So if we apply Π to both sides of (5.7), we get

$$\varepsilon_{\overline{H}}(t)a = \Pi(a\gamma(t)) = \sum \Pi((t_1 \cdot a)\gamma(t_2)) = \sum \varepsilon_T(t_2)(t_1 \cdot a) = t \cdot a.$$

This means that the action of T on A is trivial, giving us what we want.

Finally, note that the converse statement holds by Proposition 2.23. □

The immediate question that arises is when such a map exists. In fact, since we are assuming that k is algebraically closed, it appears to be an open question whether such a map always exists.

Question 5.B. Suppose k is an algebraically closed field and $A \subseteq H$ are Hopf algebras satisfying (F). Let $\overline{H} := H/A^+H$. Is there a cleaving map $\gamma : \overline{H} \to H$, and therefore a crossed product decomposition of H as in Lemma 5.28?

The following example, due to Oberst and Schneider ([37]), shows that the assumption that k is algebraically closed is definitely necessary.

Example 5.29. Let k be a field of characteristic zero and $k \subset K$ a field extension of degree 2, and let $G = \{1_G, \nu\}$ denote the Galois group of the extension. We can extend G to act on the Hopf algebra $R := K[x^{\pm 1}]$ by 1_G acting as the identity and $\nu(x^i) = x^{-i}$. Let $H := \left(K[x^{\pm 1}]\right)^G$ be the invariants under the action and $A := \left(K[x^{\pm 2}]\right)^G$.

Step 1: $A \subseteq H$ satisfy (F):

Both R and $K[x^{\pm 2}] \subseteq R$ are Hopf K-algebras, viewed as the group algebra $K\mathbb{Z}$ and its subalgebra $K2\mathbb{Z}$ respectively. Since G acts k-linearly on both of them, by [36, Lemma 3.5.1] A and H are both k-Hopf algebras with multiplication and comultiplication induced from R and $K[x^{\pm 2}]$. Therefore A is a Hopf subalgebra of H, which is obviously central because both A and H are commutative. It is left to show that H is Noetherian and a finitely-generated A-module.

Since R is a finitely-generated commutative k-algebra and G is a finite group acting on it by k-linear automorphisms, by [46, Theorem 2.3.1] the resulting ring of coinvariants H is Noetherian. The same result tells us that A is Noetherian, and furthermore that $K[x^{\pm 2}]$ is a finitely-generated A-module. Since R is a finitely-generated $K[x^{\pm 2}]$-module with generating elements 1 and x, this means that R is a finitely-generated A-module as well. Now $H \subseteq R$ is an A-submodule of a finitely-generated A-module. Since A is Noetherian, this means that H is also finitely-generated.

So $A \subseteq H$ are Hopf algebras satisfying (F).

Step 2: $A \subseteq H$ does not give rise to a crossed product structure

By [37, 10. Proposition], H is not free as an A-module. This means it cannot be a crossed product as in Question (5.B): any decomposition $H \cong A\#_\sigma T$ would give a free basis for H as an A-module through the embedding of a vector space basis of T into H.

Note that our assumption that k has a field extension of degree 2 means that in this example, k is not algebraically closed.

On the other hand, we know the answer to Question 5.B is yes if we impose certain extra conditions on H. In particular, recall from Chapter 2 that we made note of some conditions which force any Hopf surjection to give rise to a cleaving map and hence a crossed product structure in Proposition 2.26. We restate this result for our new situation.

First, recall that the *coradical* H_0 of a Hopf algebra H is simply the sum of its simple subcoalgebras, and so a subcoalgebra of H.

Proposition 5.30. *Suppose $A \subseteq H$ are Hopf algebras satisfying* (F)*. Let $\overline{H} := H/A^+H$ with $\pi : H \to \overline{H}$ denoting the canonical factor map. Suppose that the following two conditions hold:*

A. H is an injective $\overline{H}$-comodule under the canonical coaction.

B. There is a coalgebra map $f : (\overline{H})_0 \to H$ such that $\pi \circ f : (\overline{H})_0 \to \overline{H}$ is simply the inclusion map.

Then there is a cleaving map $\gamma : \overline{H} \to H$ and so

$$H \cong A\#_\sigma \overline{H}$$

for some cocycle σ and trivial action of $\overline{H}$ on A.

Proof. Recall that by Proposition 5.3, $\overline{H}$ is a factor Hopf algebra and $\pi : H \to \overline{H}$ a surjective map of Hopf algebras. So we are in the situation described by Proposition 2.26, and the result that there exists a cleaving map γ if conditions A. and B. are satisfied is simply a restatement of it. Now the fact that in this case, $H \cong A\#_\sigma \overline{H}$ with trivial action of $\overline{H}$ on A follows immediately from 5.28. □

In [42, Corollary 4.3], we see that working with Hopf surjections arising from central Hopf subalgebras gives us some immediate corollaries covering a number of cases we are interested in. We state these here.

Recall that we say a Hopf algebra H is *pointed* if all its simple subcoalgebras are one-dimensional, and *connected* if the only simple subcoalgebra is $k1_H$.

Proposition 5.31. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F) and $\overline{H} := H/A^+H$ with $\pi : H \to \overline{H}$ denoting the canonical factor map. Suppose further that any one of the following conditions holds:*

(i) $H_0 \subseteq AG(H)$, where $G(H)$ denotes the grouplike elements of H, or

(ii) H is pointed.

Then there is a cleaving map $\gamma : \overline{H} \to H$ and so

$$H \cong A\#_\sigma \overline{H}$$

for some cocycle σ and trivial action of $\overline{H}$ on A.

Proof. This is due to [42, Corollary 4.3]. Note that this result also requires in (i) that the antipode S_H of H is bijective. However, we know by Proposition 5.2(iv) that this is always true under the conditions we assume in (F). □

Remark 5.32. Note that this result implies that any quantised enveloping algebra $U_\varepsilon(\mathfrak{g})$ for $\mathfrak{g}$ a finite-dimensional Lie algebra and ε a root of unity decomposes as a crossed product in the way we want. This is because $U_\varepsilon(\mathfrak{g})$ is pointed as it is generated by grouplike and skew-primitive elements, and it has a central Hopf subalgebra satisfying (F) by [7, Theorem III.6.2], so satisfies the conditions for Proposition 5.31(ii).

5.5.2 Consequences for the finite dual

When H does decompose as a crossed product as discussed in the previous section, we are in the situation studied in Chapter 4. The following theorem is a restatement of the results we obtained in that chapter for the central Hopf subalgebra case, along with some extra information we gain through this specific situation.

Theorem 5.33. *Suppose k is an algebraically closed field of characteristic zero, $A \subseteq H$ are Hopf k-algebras satisfying (F) and let $\overline{H} := H/A^+H$ denote the canonical factor Hopf algebra with π the factor map. Suppose $\gamma : \overline{H} \to H$ is a cleaving map. Then there is a isomorphism of right A^0-comodules*

$$\overline{H}^0 \otimes A^0 \cong H^0.$$

The isomorphism is given by the map

$$m_{H^0} \circ \left(\Pi^0 \otimes \pi^0\right) \circ \tau : \overline{H}^0 \otimes A^0 \to H^0,$$

where Π *denotes the canonical quotient map* $H \to A$ *given by factoring along* $A\gamma(\overline{H}^+)$ *and* τ *denotes the tensor flip map, and restricts to a Hopf algebra embedding on* $\overline{H}^0$*. Moreover, if* γ *is a map of coalgebras, this map is not just a right* A^0*-comodule map but also a left* $\overline{H}^0$*-module map and an algebra map, so that the isomorphism is one of crossed products.*

Proof. First note that by Lemma 5.28, we have

$$H \cong A\#_\sigma \overline{H}$$

as algebras for some cocycle σ and trivial action of $\overline{H}$ on A. Moreover, by Proposition 5.3, $\overline{H}$ is finite-dimensional.

By Corollary 4.11, we have a vector space isomorphism

$$\xi := m_{H^0} \circ \left(\Pi^0 \otimes \pi^0\right) : A^0 \otimes \overline{H}^0 \to H^0.$$

Since the tensor flip τ is also a vector space isomorphism, so is $\xi \circ \tau : \overline{H}^0 \otimes A^0 \to H^0$, and it restricts to π^0 on $\overline{H}^0$ which is a Hopf embedding by Lemma 5.4(ii). So for the first part, we only need to show that $\xi \circ \tau$ is a map of A^0-comodules. We see this as follows.

Given $f \in A^0, g \in \overline{H}^0$ we have

$$\begin{aligned} \rho_{H^0} \circ \xi \circ \tau(g \otimes f) &= (\mathrm{id} \otimes \iota) \circ \Delta_{H^0} \circ \xi(f \otimes g) \\ &= (\mathrm{id} \otimes \iota) \circ \Delta_{H^0}(\Pi^0(f)\pi^0(g)) \\ &= \rho_{H^0}(\Pi^0(f))\rho_{H^0}(\pi^0(g)), \end{aligned} \tag{5.8}$$

using the fact that the map $\rho_{H^0} = (\mathrm{id} \otimes \iota) \circ \Delta_{H^0}$ giving the A^0-comodule structure on H^0 is an algebra map by Lemma 2.17. Now note that $\pi^0(g)$ is a coinvariant of the comodule coaction by Lemma 5.4 and Π^0 is a right A^0-comodule map by Lemma 5.6. So (5.8) becomes

$$\rho_{H^0} \circ \xi \circ \tau(g \otimes f) = (\Pi^0(f_1)\pi^0(g) \otimes f_2 = (\xi \circ \tau) \circ (\mathrm{id}_{\overline{H}^0} \otimes \Delta_{A^0})(g \otimes f)$$

as required.

Now assume that γ is a coalgebra map.

This means that by Corollary 4.24, the map ξ is an algebra isomorphism. The tensor flip map $\tau : \overline{H}^0 \otimes A^0 \to A^0 \otimes \overline{H}^0$ is also an algebra isomorphism, so $\xi \circ \tau$ is an algebra map.

It is a left $\overline{H}^0$-module map because it is an algebra map and the $\overline{H}^0$-module structure on H^0 comes from the multiplication in H^0.

□

The following example shows that Theorem 5.33 gives us a partial positive answer to Conjecture 5.A.

Example 5.34. Let k be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a finite-dimensional semisimple Lie algebra, $n \geq 3$ and $\varepsilon \in k^*$ a primitive nth root of unity. We know that the quantised universal enveloping algebra $U := U_\varepsilon(\mathfrak{g})$ has a central Hopf subalgebra $Z \subseteq U$ satisfying (F) by [7, Theorem III.6.2], and that it is pointed as it is generated by primitive and skew-primitive elements. So Proposition 5.31 applies and $U_\varepsilon(\mathfrak{g})$ decomposes as a crossed product, with

$$U \cong Z\#_\sigma \overline{U_\varepsilon(\mathfrak{g})}.$$

This means that we can apply Theorem 5.33 to get

$$U_\varepsilon(\mathfrak{g})^0 \cong \overline{\mathcal{O}_\varepsilon(G)} \otimes Z^0 \tag{5.9}$$

as right Z^0-comodules, where G is the unique simply connected affine algebraic group such that $\operatorname{Lie} G = \mathfrak{g}$, $\overline{\mathcal{O}_\varepsilon(G)}$ is the restricted quantised coordinate ring defined via [7, Theorem III.7.2] and we use [7, Theorem III.7.10] to get $\overline{\mathcal{O}_\varepsilon(G)} \cong \overline{U_\varepsilon(\mathfrak{g})}^*$.

Note that the condition needed for the remainder of Theorem 5.33 to apply (in particular, that there exists a cleaving map which is a coalgebra map) cannot possibly be satisfied for general $\mathfrak{g}$. If this were the case, (5.9) would be not just a right A^0-comodule isomorphism but also an algebra isomorphism. However, we have seen in Corollary 5.26 that when $\mathfrak{g} = \mathfrak{sl}_2(k)$, U^0 decomposes as a smash product with nontrivial action. As we will see below this means that in this case, no such cleaving map can exist.

In general, our results so far give us what are potentially multiple different crossed product decompositions of H^0 in terms of $\overline{H}^0$ and A^0. However, any such decompositions have to agree in some sense. To clarify what we mean by this, we make note of the following result, due to Doi ([11]) although we follow the presentation given in [36].

Proposition 5.35. *Suppose B is an algebra and T a Hopf algebra with two actions of T on B given by $t \otimes b \mapsto t \cdot b$ and $t \otimes b \mapsto t \cdot' b$ for $t \in T, b \in B$ and two cocycles $\sigma, \sigma' : T \otimes T \to B$. Suppose further that $B\#_\sigma T$ and $B\#'_{\sigma'} T$ are crossed products with respect to these actions. Then the following are equivalent:*

(i) There exists an algebra, left B-module, right T-comodule isomorphism

$$\phi : B\#_\sigma T \to B\#'_{\sigma'}T.$$

(ii) There exists a convolution invertible linear map $u : T \to B$ such that for $b \in B$, $s, t \in T$

$$t \cdot' b = \sum u^{-1}(t_1)(t_2 \cdot b)u(t_3),$$
$$\sigma'(s,t) = \sum u^{-1}(s_1)(s_2 \cdot u^{-1}(t_1))\sigma(s_3, t_2)u(s_4 t_3).$$

If this holds, then ϕ and u can be chosen such that

$$\phi(b\#t) = \sum bu(t_1)\#' t_2.$$

Proof. This is [36, Theorem 7.3.4]. □

Definition 5.36. Suppose B is an algebra, T a Hopf algebra and we have two crossed product decompositions $B\#_\sigma T$, $B\#'_{\sigma'}T$ such that the equivalent conditions in Proposition 5.35 hold. Then we call these crossed products *equivalent.*

Since the isomorphisms $H^0 \cong \overline{H}^0\#_\sigma A^0$ found in Corollary 5.5, Theorem 5.8 as well as Theorem 5.33 (under the assumption that γ is a coideal map) are all algebra, left module and right comodule maps, any decompositions of H^0 for given $A \subseteq H$ satisfying (F) we find through these results must be equivalent.

Remark 5.37. In fact, when $A \subseteq H$ satisfying (F) are such that $H \cong A\#_\sigma\overline{H}$ and the cleaving map $\gamma : \overline{H} \to H$ is a coalgebra map, we can apply not just Theorem 5.33 immediately but also Theorem 5.8(i) as we have a canonical A-module decomposition $H \cong A \oplus A\gamma(\overline{H}^+)$ where $A\gamma(\overline{H}^+)$ is a coideal. In this case, the resulting decompositions of H^0 are not just equivalent but in fact equal.

This follows because the isomorphism in Theorem 5.8 is given by $\zeta := m_{H^0} \circ (\pi^0 \otimes \Pi^0)$, since $\Pi : H \to A$ is the canonical projection in the decomposition. Moreover, the isomorphism in Theorem 5.33 is given by $\xi' := \xi \circ \tau = m_{H^0} \circ (\Pi^0 \otimes \pi^0) \circ \tau$, and ξ' is an algebra map. Let $f \in A^0$, $g \in \overline{H}^0$. This means we have

$$\begin{aligned}\xi'(g \otimes f) &= \xi'(g \otimes \varepsilon_A)\xi'(\varepsilon_{\overline{H}} \otimes f)\\ &= \Pi^0(\varepsilon_A)\pi^0(g)\Pi^0(f)\pi^0(\varepsilon_{\overline{H}})\\ &= \pi^0(g)\Pi^0(f) = \zeta(g \otimes f).\end{aligned}$$

So the two isomorphisms are in fact the same.

5.6 Conjectures and results

Throughout, k is an algebraically closed field of characteristic zero and $A \subseteq H$ are Hopf algebras satisfying (F).

We now have numerous partial results regarding H^0 under numerous assumptions. In this section, we give an overview by conjecturing what we believe is always true for H^0 (possibly under one of two assumptions) and explaining how the results we have obtained so far fit into that picture.

What we have seen leads us to the following, rather optimistic conjecture.

Conjecture 5.B. *Suppose $A \subseteq H$ are Hopf algebras satisfying (F) and let $\overline{H}$ denote the canonical factor Hopf algebra. Then H^0 is a crossed product, with*

$$H^0 \cong \overline{H}^0 \#_\sigma A^0$$

for some action and cocycle σ.

It is quite possible that this conjecture does not hold true at such a level of generality. However, even if this is the case we can still hope for it to hold if we have an A-module projection $H \to A$:

$$\begin{gathered} A \subseteq H \text{ are Hopf algebras satisfying (F) and there exists} \\ \text{a left } A\text{-module map } \Pi : H \to A \text{ such that } \Pi|_A = \mathrm{id}_A. \end{gathered} \tag{Mod}$$

or if H decomposes as a crossed product:

$$\begin{gathered} A \subseteq H \text{ are Hopf algebras satisfying (F) and there exists} \\ \text{a cleaving map } \gamma : H/A^+H \to H, \text{ so } H \cong A\#_\sigma H/A^+H. \end{gathered} \tag{CP}$$

Note that by Lemma 2.29, (CP) holding is a sufficient condition for (Mod) to hold. Moreover, as discussed in Section 5.2.1, it is an open question whether (CP) (and hence certainly (Mod)) is always satisfied for k algebraically closed and there are numerous positive examples, such as pointed Hopf algebras.

Although none of our results prove the conjecture either on its own or assuming (CP) or (Mod), we do have a number of partial results.

Section 5.2 gives us some positive results regarding (Mod) with extra assumptions. Corollary 5.5 tells us that the conjecture holds whenever (Mod) is satisfied and the map $\Pi^0 : A^0 \to H^0$ is convolution invertible, and the related result Theorem 5.8(i) and (ii) tells us that this is true when (Mod) is satisfied and the map Π is also either a coalgebra or an algebra map.

Section 5.3 gives us a partial result in a different vein by looking at Hopf subalgebras in place of the whole of H^0. Of particular interest to the conjecture is Theorem 5.21, which tells us that when (Mod) is satisfied, there always exists a Hopf subalgebra W of H^0 such that W is a crossed product with

$$W \cong \overline{H}^0 \#_\sigma U(\operatorname{Lie} G),$$

where G is such that $\mathcal{O}(G) \cong A$. Here $U(\operatorname{Lie} G) \subseteq A^0$ in a canonical way.

Moreover, Theorems 5.19 and 5.20 give a necessary and sufficient condition for the conjecture to hold: namely, that the canonical Hopf subalgebra $k\widehat{G}$ defined in Definition 5.13 decomposes as a crossed product, with cleaving map given by the restriction of some right A^0-comodule map $\phi : A^0 \to H^0$. So when such as a map exists (such as when (Mod) holds), it suffices to look at $k\widehat{G}$ rather than the whole of H^0.

Finally, Section 5.5 looks at the case when (CP) is satisfied. Theorem 5.33 tells us that in this situation, part of the statement of the conjecture always holds because then we have a right A^0-comodule isomorphism

$$H^0 \cong \overline{H}^0 \otimes A^0.$$

We also note that although the conjecture is framed in terms of when H^0 decomposes as a crossed product, all examples we know of and all those we will see in Chapter 6 are in fact smash products, with trivial cocycle. We record this in the following question.

Question 5.C. Are there any $A \subseteq H$ satisfying (F) such that H^0 decomposes as a crossed product with

$$H^0 \cong (H/A^+H)^0 \#_\sigma A^0$$

where the cocycle σ is nontrivial? What if H satisfies (Mod) or (CP)?

Recall that when H satisfies (Mod), Question 5.A asks whether the projection map can be convolution invertible (and hence give rise to a crossed product decomposition) without being either an algebra or a coalgebra map. If the projection is a coalgebra map, then the crossed product decomposition is always a smash product, so these two questions are related.

5.7 Originality

In Section 5.2, Propositions 5.2 and 5.3 are well-known, while Lemma 5.4 and Corollary 5.5 are original. Section 5.2.1 is original, as are Section 5.3 and Section 5.4 apart from the known results regarding the structure of $U_q(\mathfrak{sl}_2(k))$ at a root of unity recorded in Proposition 5.24.

In Section 5.5, Section 5.5.1 is known. Example 5.29 is due to Oberst and Schneider ([37]), while Proposition 5.30 and its corollary Proposition 5.31 are due to Schneider ([42]). Section 5.5.2 is original save for Proposition 5.35, which gives the notion of two crossed products being equivalent and is due to Doi ([11]).

Chapter 6

Prime affine regular Hopf algebras of Gelfand-Kirillov dimension one

We conclude by showing how the results of the previous two chapters let us calculate the finite duals of the prime affine regular Hopf algebras of Gelfand-Kirillov dimension one described by Brown and Zhang in [6].

There has been a great deal of interest in the classification of infinite-dimensional Hopf algebras in recent years. One of the approaches taken is classifying Hopf algebras with specific Gelfand-Kirillov dimension. In [6], as a corollary of their main result Brown and Zhang classified all prime affine regular Hopf algebras of Gelfand-Kirillov dimension one with prime PI-degree over some algebraically closed field of characteristic zero. They showed that any such Hopf algebra is one of a given list, and conjectured that this list along with those Taft algebras and generalised Liu algebras (defined in Section 6.5) with non-prime PI degree should give a classification of all prime affine regular Hopf algebras of Gelfand-Kirillov dimension one.

Recent work by Wu, Liu and Ding ([52]) has shown that this conjecture is not true. They define another family of Hopf algebras $D(m, d, q)$, show that these are prime affine regular with Gelfand-Kirillov dimension one and not isomorphic to any of the Hopf algebras listed in [6], and then prove that these Hopf algebras complete the classification: any prime affine regular Hopf algebra of GK-dimension one is either $D(m, d, q)$ for some m, d, q or one of the Hopf algebras in the conjectured classification of [6].

In this chapter, we compute the finite duals of all Hopf algebras listed in [6]. As this part of the thesis had already been written when Wu, Liu and Ding's work came out, we do not consider the family of Hopf algebras $D(m, d, q)$ they introduced. It is not clear whether we can calculate their finite dual using the results obtained in Chapters 4 and 5.

In Section 6.1, we give the preliminary definitions needed to understand the classification of those Hopf algebras with prime PI-degree in [6], recording said classification in Proposition 6.4. We state the main result of this chapter - the finite duals of all Hopf algebras listed in Proposition 6.4 - without proof in Theorem 6.5.

In the following sections, we go through the Hopf algebras and families of Hopf algebras listed in Proposition 6.4 to compute their finite duals. Section 6.2 looks at the polynomials and Laurent polynomials, whose finite duals are classical. Section 6.3 looks at the finite dual of the group algebra of the dihedral group, finding it as a corollary of a general result about the finite duals of group algebras thanks to the results in Chapter 4. Section 6.4 looks at the finite dual of the Taft algebras, which we already computed in the $\gcd(n, t) = 1$ case in Example 4.22 but extend to $\gcd(n, t) > 1$ here in order to also find the finite dual of those with non-prime PI degrees. Finally, in Section 6.5 we define and then compute the finite dual of the generalised Liu algebras, again of any PI-degree. All these results combine to give us the proof of Theorem 6.5.

6.1 Preliminaries

First, we need some definitions.

Definition 6.1. Let R be a ring. We call R *regular* if R has finite global dimension. The global dimension is defined by

$$\text{gldim}(A) = \sup_{M \in R-Mod} \text{pdim}_R M,$$

where pdim denotes the projective dimension of M as an R-module.

Definition 6.2. Let R be a ring. We call R *prime* if for all ideals $I, J \subseteq R$, we have $IJ = 0$ if and only if one of $I, J = \{0\}$.

Definition 6.3. Let k be any field and A be an affine k-algebra. Let $\{a_1, ..., a_n\}$ be a generating set for A and set $V := \sum ka_i$ be the span of these elements. Then the *Gelfand-Kirillov dimension* or *GK-dimension* of A is defined by

$$\text{GKdim}(A) := \limsup_{n \to \infty} \left(\log_n(\dim_k(V^n))\right).$$

Note that although the value of the Gelfand-Kirillov dimension appears to depend on the choice of generating elements, it is in fact independent: any generating set for A will give the same value for GKdim(A) in the above formula (see [28, Lemma 1.1]).

By [45], any prime affine regular algebra of GK-dimension one is Noetherian and a finite module over its centre. The latter means that by [34, Corollary 13.1.13(i)] it is a PI algebra (recalling what this means from Definition 5.1). Furthermore, if the algebra in question is a Hopf algebra this means we are very close to (F) being satisfied, as all we still need is for the algebra to be not just a finite module over its centre but in fact a finite module over a central Hopf subalgebra. This is not always the case (for instance, we note in Section 6.4 that the infinite-dimensional Taft algebra $H(n, t, q)$ for $\gcd(n, t) > 1$ is finite over its centre but not finite over a central Hopf subalgebra) but will often be true.

These definitions give us everything we need to understand the classification in [6].

Let k be an algebraically closed field of characteristic zero. In [6], as a corollary of their main result Brown and Zhang classified all prime affine regular Hopf k-algebras of Gelfand-Kirillov dimension one with prime PI-degree (again recalling the definition of this from Definition 5.1), showing that any such Hopf algebra is one of a given list. They conjectured that this list along with those Taft algebras and generalised Liu algebras (defined in Section 6.5) with non-prime PI degree should give a classification of all prime affine regular Hopf algebras of Gelfand-Kirillov dimension one. However, recent work by Wu, Liu and Ding ([52]) has shown that this is not true. They define another family of Hopf algebras $D(m, d, q)$, show that these are prime affine regular with Gelfand-Kirillov dimension one and not isomorphic to any of the Hopf algebras listed in [6], and then prove that these Hopf algebras complete the classification: any prime affine regular Hopf algebra of GK-dimension one is either $D(m, d, q)$ for some m, d, q or one of the Hopf algebras in the conjectured classification of [6].

First, we record Brown and Zhang's classification in the following proposition, noting that the Taft algebras were defined in Example 1.14 and the generalised Liu algebras are defined in Section 6.5.

Proposition 6.4. *Suppose k is an algebraically closed field of characteristic zero and H is a prime affine regular Noetherian Hopf k-algebra of Gelfand-Kirillov dimension one. Suppose further that the polynomial identity degree of H is prime. Then H is isomorphic to one of the following:*

(1) The polynomial algebra $k[x]$,

(2) The Laurent polynomial algebra $k[x^{\pm 1}]$,

(3) The group algebra $k\mathbb{D}$ *of the infinite dihedral group* $\mathbb{D} := \langle x, g \mid gxg = x^{-1}, g^2 = 1\rangle$,

(4) An infinite-dimensional Taft algebra $H(p, t, q)$ *for some integers* $p > 1$ *prime,* $0 \leq t < p$ *and primitive pth root of unity* $q \in k^*$*, or*

(5) A generalised Liu algebra $B(p, w, q)$ *for some integers* $p > 1$ *prime,* $w \geq 1$ *and primitive pth root of unity* $q \in k^*$.

Proof. This result is [6, Corollary 0.3]. □

We can use our results to determine the finite dual of all Hopf algebras listed in this classification.

Theorem 6.5. *Suppose* k *is an algebraically closed field of characteristic zero and* H *is a prime affine regular Noetherian Hopf algebra of Gelfand-Kirillov dimension one. Suppose further that the PI-degree of* H *is prime. Then one of the following holds:*

(1) $H \cong k[x]$*, and*

$$H^0 \cong k[y] \otimes k(k, +).$$

(2) $H \cong k[x^{\pm 1}]$*, and*

$$H^0 \cong k[y] \otimes k(k^*, *).$$

(3) $H \cong k\mathbb{D}$*, and*

$$H^0 \cong k[y] \otimes k(k^*, *) \otimes kC_2,$$

where kC_2*,* $k[y] \otimes kC_2$ *and* $k(k^*, *) \otimes kC_2$ *are Hopf subalgebras.*

(4) $H \cong H(p, t, q)$ *for some prime* $p > 1$*, integer* $0 \leq t < p$ *and primitive pth root of unity* $q \in k^*$*. Then*

$$H^0 \cong (k[y] \otimes k(k, +)) \otimes H_f(p, t, q)$$

as crossed products, where $H_f(p, t, q)$ *denotes the finite-dimensional Taft algebra on those parameters. Here* $H_f(p, t, q)$*,* $k[y] \otimes H_f(p, t, q)$ *and* $k(k, +) \otimes H_f(p, t, q)$ *are Hopf subalgebras.*

(5) H *is a generalised Liu algebra on* (p, w, q) *for prime* $p > 1$*, integer* $w \geq 1$ *and primitive pth root of unity* q*. Then*

$$H^0 \cong H_f(p, 1, q)\#(k[z] \otimes (k^*, *))$$

as crossed products, where $H_f(p,1,q)$ *denotes the finite-dimensional Taft algebra on those parameters. Here* $H_f(p,1,q)$ *is a Hopf subalgebra, as are* $H_f(p,1,q)\#k[z]$ *and* $H_f(p,1,q)\#(k^*,*)$.

All isomorphisms on H^0 *are as algebras. They are also of left modules and right comodules with respect to the appropriate structures when explicitly stated to be a crossed product isomorphism.*

6.2 The polynomial and Laurent polynomial algebras

We do not use our results to calculate the finite dual of these algebras. Instead, they are special cases of classical results already presented previously, and we will use them as building blocks to determine the finite dual of the other algebras listed in Proposition 6.4.

First, consider the polynomial algebra $k[x]$ on one variable. This is a Hopf algebra, isomorphic to the universal enveloping algebra of the one-dimensional Lie algebra. Recall that by Example 1.31 this means we have

$$k[x]^0 \cong k[y] \otimes k(k,+)$$

as algebras, where $k(k,+)$ denotes the group algebra of the additive group of the underlying field.

Now let $H := k[x^{\pm 1}]$ be the Laurent polynomial ring, with the Hopf structure given by x^i grouplike. We find that

$$H \cong k[X,Y]/\langle XY-1\rangle \cong \mathcal{O}((k^*,*)),$$

where $(k^*,*)$ denotes the multiplicative group of the underlying field, viewed as an algebraic variety. In fact, this is an isomorphism of Hopf algebras. This means that we can use Proposition 2.13: we have

$$H^0 \cong U(\mathrm{Lie}(k^*,*))\#k(k^*,*).$$

The Lie algebra associated with $(k^*,*)$ must be the one-dimensional Lie algebra. Moreover, it is easy to see that the definition of multiplication in the finite dual means that $k[x^{\pm 1}]$ being cocommutative forces its finite dual to be commutative. It then follows from the definition of multiplication in a smash product that the action of $k(k^*,*)$ on $k[z]$ must be trivial. So we find that

$$H^0 \cong k[z] \otimes k(k^*,*)$$

as algebras.

Note that there are multiple ways to view these structures: for instance, $k[x]$ is also the coordinate ring of the algebraic group $(k,+)$, and $k[x^{\pm 1}]$ is the group algebra of the integers.

6.3 The group algebra of the dihedral group

Let $\mathbb{D}$ denote the infinite dihedral group, given by

$$\mathbb{D} := \langle x, y \mid xyx = y^{-1}, x^2 = 1 \rangle.$$

This means that the corresponding group algebra is

$$H := k\mathbb{D} = k\langle x, y \mid xyx = y^{-1}, x^2 = 1 \rangle.$$

As this is a group algebra, both x and y are grouplike.

We can use our results to calculate H^0. In fact, we see that this is a special case of a general result about group algebras:

Theorem 6.6. *Let G be any group and N a normal subgroup of finite index. Then*

$$kG^0 \cong kN^0 \otimes k\left(G/N\right)^0$$

as algebras, with $k\left(G/N\right)^0$ a Hopf subalgebra.

Proof. By [36, Example 7.1.6], we have

$$kG \cong kN \#_\sigma k\left(G/N\right)$$

for some cocycle σ, where the cleaving map $\gamma : k(G/N) \to kG$ is given by a map picking out coset representatives for elements of G/N in G and extending linearly. Moreover, the crossed product structure is induced by the map $\pi : kG \to k(G/N)$ given by extending the canonical quotient map $G \to G/N$, and γ splits π: $\pi \circ \gamma = \mathrm{id}_{k(G/N)}$. So we are in the setting of Section 4.1.1.

Now note that kN simply consists of the subalgebra spanned by elements of N. Since all these are grouplike, kN is in fact a Hopf subalgebra of kG. Moreover, since γ sends the elements of G/N which form a basis for $k(G/N)$ to elements in G, it is a coalgebra map (as it sends a basis of grouplike elements to grouplike elements). Finally, since N has finite index in G, $k(G/N)$ is finite-dimensional. So Corollary 4.24 applies: we have

$$kG^0 \cong kN^0 \otimes k(G/N)^0$$

as algebras as required. □

Corollary 6.7. *Let* $\mathbb{D} = \langle x, y \mid xyx = y^{-1}, x^2 = 1\rangle$ *be the dihedral group. Then*

$$(k\mathbb{D})^0 \cong k[z] \otimes k(k^*, *) \otimes kC_2$$

as algebras, where kC_2 *is a Hopf subalgebra.*

Proof. This follows because the cyclic subgroup N generated by y is a normal subgroup of $\mathbb{D}$, which is isomorphic to $\mathbb{Z}$ as y has infinite order, and $\mathbb{D}/N = C_2$. So the theorem says that

$$k\mathbb{D}^0 \cong k\mathbb{Z}^0 \otimes kC_2^0,$$

with kC_2^0 a Hopf subalgebra. Since $k\mathbb{Z} \cong k[x^{\pm 1}]$, by the previous section we know that $k\mathbb{Z}^0 \cong k[z] \otimes k(k^*, *)$. Moreover, group algebras of finite cyclic groups are self-dual by Example 1.20, giving us what we want. □

Remark 6.8. Although the corollary tells us what the algebra structure of $k\mathbb{D}^0$ looks like, it does not give us much information about the coalgebra structure.

Let f denote the generating element of C_2 and z_λ the element of $(k^*, *)$ corresponding to $\lambda \in k^*$. Keep z as in Corollary 6.7. We know that f is grouplike because kC_2 is a Hopf subalgebra. As for z and z_λ, some calculation gives z skew-primitive with $\Delta(z) = z \otimes 1 + f \otimes z$ along with $\varepsilon(z_\lambda) = 1$ and

$$\Delta(z_\lambda) = \frac{1}{2}\big(z_\lambda \otimes (z_\lambda + z_{\lambda^{-1}}) + (fz_\lambda) \otimes (z_\lambda - z_{\lambda^{-1}})\big).$$

We omit the details here; interested readers can check that these correspond to the definition of Δ_{H^0} as coming from the multiplication on H on the basis of H given by $\mathbb{D}$.

It is clear that $k[z] \otimes k(k^*, *)$ is not a Hopf subalgebra and that moreover the elements z_λ coming from characters on $k[y^{\pm 1}]$ do not give characters on H. However, both $k[z] \otimes kC_2$ and $k(k^*, *) \otimes kC_2$ are Hopf subalgebras.

6.4 The Taft algebras

Let k be an algebraically closed field of characteristic zero, $n > 1$ and $1 \leq t < n$ be integers and q be a primitive nth root of unity, and let H be the infinite-dimensional Taft algebra on these parameters: that is,

$$H := H(n, t, q) = k\langle x, g \mid xg = qgx, g^n = 1\rangle$$

as algebras, with $\Delta(x) = x \otimes 1 + g^t \otimes x$ and g grouplike.

As discussed in [32, Examples 2.7, 7.3] and [6, Section 3.3]), the Taft algebras form examples of prime affine regular Hopf algebras of Gelfand-Kirillov dimension one, and the PI-degree of $H(n, t, q)$ is simply n.

We have already calculated H^0 for the case where $\gcd(n, t) = 1$ in Chapter 4. We record it again here, with additional information about Hopf subalgebras.

Lemma 6.9. *Let $n > 1$, $1 \leq t < n - 1$ be integers and $q \in k^*$ a primitive nth root of unity. Let $H := H(n, t, q)$ denote the infinite-dimensional Taft algebra on those paramters and suppose that $\gcd(n, t) = 1$.*

(i) H^0 decomposes as a crossed product, with

$$H^0 \cong H_f(n, t, q) \otimes k[y] \otimes k(k, +). \tag{6.1}$$

Here action and cocycle are trivial.

(ii) $H_f(n, t, q)$, $k[y] \otimes H_f(n, t, q)$ and $k(k, +) \otimes H_f(n, t, q)$ are Hopf subalgebras of H^0.

Proof. The algebra decomposition and $H_f(n, t, q)$ being a Hopf subalgebra are simply Example 4.22, using the fact that the tensor flip map is an algebra map. We can also get the same decomposition through Theorem 5.33, which also guarantees that the decomposition is not just as algebras but also as left $H_f(n, t, q)$-modules and right $k[y] \otimes k(k, +)$-comodules.

Now note that $k[x^n]$ is a central Hopf subalgebra of H^0 satisfying (F). So by Theorem 5.21, there exists a Hopf subalgebra W of H^0 such that

$$W \cong H_f(n, t, q) \#_\sigma k[y],$$

and the fact that W is a Hopf subalgebra and $H_f(n, t, q)$ and $k[y]$ in its decomposition correspond to those in (6.1) mean that the action and cocycle must be trivial. Similarly, by Theorem 5.19, there exists a Hopf subalgebra

$$k\widehat{G} \cong H_f(n, t, q) \otimes k(k, +).$$

We know that action and cocycle are trivial here because the cocycle and action on $k\widehat{G}$ are induced from (6.1). □

Since H having prime PI-degree and hence n being prime forces $\gcd(n,t) = 1$, this result already gives us the finite duals of all Hopf algebras listed in Proposition 6.4. However, we would like to find the finite duals of all prime affine regular Hopf algebras of GK-dimension one, not just those whose PI-degree is prime.

From now on, we assume $\gcd(n,t) = d > 1$, and write $n := n'd, t := t'd$.

In this case, although H is finite over its center by the above, it is *not* finite over any central Hopf subalgebra. We make note of this fact in the following lemma.

Lemma 6.10. *Let $n > 1$, $1 \leq t < n$ be integers such that $\gcd(n,t) = d > 1$ and let $q \in k^*$ be a primitive nth root of unity. Then there does not exist any central Hopf subalgebra $A \subseteq H(n,t,q)$ such that $H(n,t,q)$ is a finite left A-module.*

Proof. Suppose for a contradiction that such an A exists. By the discussion in [6, Section 3.3], $Z(H) = k[x^n]$. Moreover, by Example 2.12 $x^{n'}$ is primitive, meaning that $k[x^{n'}]$ is a Hopf subalgebra of $H(n,t,q)$, where we write $n' := n/d$. So we have $A \subsetneq k[x^{n'}]$ as Hopf subalgebras. Moreover, $H(n,t,q)$ being a finite A-module means that $k[x^{n'}]$ is a finite A-module. Since polynomial rings are Noetherian, this means that $A \subseteq k[x^{n'}]$ satisfies (F) and so Proposition 5.3 applies: there exists a canonical finite dimensional factor Hopf algebra $T \cong k[x^{n'}]/A^+k[x^{n'}]$. Since $k[x^{n'}]$ is commutative, T is commutative.

Now recall from Remark 1.13 that there exists a contravariant equivalence of categories between commutative Hopf algebras and coordinate rings of affine algebraic groups. This means that T is the coordinate ring of some affine algebraic group which is a subgroup of $(k,+)$. Since T is finite dimensional, the group must be finite. But since we are in characteristic zero, the only finite subgroup of $(k,+)$ is $\{0\}$. This means that $T = k$ and $A = k[x^{n'}]$ itself. But this is not central in $H(n,t,q)$, giving us a contradiction. □

So the results of Chapter 5 do not apply when $\gcd(n,t) > 1$. However, we can use the results of Chapter 4 to calculate H^0.

Theorem 6.11. *Let $n > 1$, $1 \leq t < n-1$ be integers and $q \in k$ a primitive nth root of unity. Suppose $\gcd(n,t) = d$ and write $n = n'd$, $t = t'd$. Then*

$$H(n,t,q)^0 \cong k[y] \otimes k(k,+) \otimes H_f(n',t',q^d) \otimes kC_d$$

as algebras, where kC_d is a Hopf subalgebra.

Proof. Let $I := \langle x, g^d - 1\rangle$. This is an ideal of finite codimension in H which is also a Hopf ideal as $g^d - 1$ is primitive and x is skew-primitive. It satisfies $H/I \cong kC_d$. Let $\pi : H \to H/I$ denote the canonical quotient map, and let $\overline{g} := \pi(g)$.

We have a coalgebra embedding $\gamma : kC_d \to H$ given by $\gamma(\overline{g}^i) = g^i$ for $0 \leq i < d$ which satisfies $\pi \circ \gamma = \mathrm{id}_{kC_d}$. By Lemma 2.20, γ is a cleaving map and so by Proposition 2.22 we have

$$H \cong H^{co\pi} \#_\sigma kC_d \tag{6.2}$$

as algebras, left $H^{co\pi}$-modules and right kC_d-comodules for some action of kC_d on $H^{co\pi}$ and cocycle σ. We claim that $A := H^{co\pi}$ is the Hopf subalgebra of H generated by x and g^d.

For this, let $\rho := (\mathrm{id} \otimes \pi) \circ \Delta_H : H \to H \otimes kC_d$ denote the map giving the comodule algebra structure. Note that $\rho(x) = x \otimes 1 + g^t \otimes \pi(x) = x \otimes 1$, and $\rho(g) = g \otimes \overline{g}$. Because ρ is an algebra map, this means that for $i \geq 0, 0 \leq j < n$ we have

$$\rho(x^i g^j) = \rho(x)^i \rho(g)^j = x^i g^j \otimes \overline{g}^j.$$

Now let $\alpha \in H$ be any element. We can write α as $\alpha_0 + \alpha_1 + ... + \alpha_{d-1}$, where each α_r consists of the sum of those summands $x^i g^j$ of α where $j \equiv r \mod d$. Since any such summand satisfies $\rho(x^i g^j) = x^i g^j \otimes \overline{g}^r$ by the above, we have

$$\rho(\alpha) = \sum_{r=0}^{d-1} \rho(\alpha_r) = \sum_{r=0}^{d-1} \alpha_r \otimes \overline{g}^r.$$

In particular, $\alpha \in A$ if and only if $\alpha_r = 0$ for all $r \geq 1$.

This means that $A = k\langle x, g^d\rangle$. This is in fact a Hopf subalgebra of H, and setting $g' := g^d$, we find that

$$A \cong k\langle x, g' \mid xg' = q^d g' x, (g')^{n'} = 1\rangle$$

as algebras, with coproduct given by $\Delta(x) = x \otimes 1 + (g')^{t'} \otimes x$, g' grouplike. In other words, A is isomorphic to the infinite-dimensional Taft algebra on parameters (n', t', q^d).

By Corollary 4.24, we now have

$$H^0 \cong A^0 \otimes (kC_d)^0$$

as algebras. By Example 1.20, $(kC_d)^0 \cong kC_d$. Moreover, since $\gcd(n,t) = d$ by assumption, we must have $\gcd(n', t') = 1$ and so our previous results apply to A^0: we have

$$A^0 \cong k[y] \otimes k(k, +) \otimes T,$$

where T is the finite-dimensional Taft algebra on parameters (n', t', q^d).

Inserting this into (6.2), we find that

$$H^0 \cong k[y] \otimes k(k,+) \otimes T \otimes kC_d$$

as algebras as required. □

Remark 6.12. The isomorphism in Theorem 6.11 is not one of Hopf algebras. This can be seen for instance by the fact there are no characters of $H(n,t,q)$ other than those given by the n simple modules annihilated by x. In particular, the elements in $(k^+,+)$ cannot all be grouplike elements. Similarly, $y \in k[y]$ corresponds to the map satisfying $y(x^i g^j) = \delta_{in'}$, and so

$$\mu \circ \Delta(y)(x^{n'-1} \otimes x) = y(x^{n'-1}x) = y(x^{n'}) = 1$$

which means that y cannot be primitive.

In the $\gcd(n,t) = 1$ case, some calculation tells us that if $\overline{x_i}$ is defined to be the map such that

$$\overline{x_i}(x^j g^r) = \delta_{ij} \qquad \text{for } j \geq 0, 0 \leq r < n,$$

then

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{i=1}^{n-1} \overline{x_i g}^i \otimes \overline{x_{n-i}}$$

and, letting y_λ denote the element of $(k,+) \subseteq H^0$ corresponding to $\lambda \in k$,

$$\Delta(y_\lambda) = y_\lambda \otimes y_\lambda + \lambda(y_\lambda \otimes y_\lambda)\left(\sum_{i=1}^{n-1} \overline{x_i g}^i \otimes \overline{x_{n-i}}\right).$$

We note that $\overline{x_i}$ is some scalar multiple of $\overline{x}^i$: for instance,

$$\overline{x_2} = (1+q^{-t})^{-1}\overline{x}^2.$$

Here the subalgebra $H_f(n', t', q^d) = H_f(n,t,q)$ is in fact a Hopf subalgebra by Lemma 6.9, and the calculation of $\Delta(f_\lambda)$ confirms Lemma 5.15(ii) in that it shows that the $(k,+)$-indexed subspaces $H_f(n,t,q) \otimes y_\lambda$ are all subcoalgebras of H^0. This also means that $H_f(n,t,q) \otimes kG$ is a Hopf subalgebra for any subgroup $G < (k,+)$. Finally, although $k[y] \otimes k(k,+)$ is not a subcoalgebra, we do find that the map $H^0 \to k[y] \otimes k(k,+)$ given by restriction to $k[x^n]$ is a map of Hopf algebras - as stated in Lemma 5.4(i).

Note also that the fact that action and coaction are trivial in Theorem 6.11 is due to the fact that there exists a cleaving map for $H(n,t,q)$ which is a coalgebra map, and since the coinvariants form a Hopf subalgebra. This means that Theorem 4.19 applies. As we will see in the next section, this forms a contrast to the generalised Liu algebras.

6.5 The generalised Liu algebras

Let k be an algebraically closed field of characteristic zero.

Liu algebras were introduced by Liu in [31] and generalised by Brown and Zhang in [6]. In our definition, we follow the second presentation given in [6, Section 3.4].

Definition 6.13. Let k be an algebraically closed field, $n > 1$ and $1 \leq w < n$ be integers and $q \in k$ be a primitive nth root of unity. Then the *Liu algebra* on (n, w, q) is given by

$$B(n, w, q) := k\langle x^{\pm 1}, g^{\pm 1}, y \mid yg = qgy, x \text{ central}, y^n = 1 - g^n = 1 - x^w \rangle.$$

This is a Hopf algebra, with the coalgebra structure given by x, g, being grouplike and y skew-primitive with $\Delta(y) = y \otimes g + 1 \otimes y$.

By [6, Theorem 3.4], Liu algebras are prime affine regular Hopf algebras of Gelfand-Kirillov dimension one, and the PI-degree of a Liu algebra $B(n, w, q)$ is simply n. Moreover, Liu algebras are finite dimensional over central Hopf subalgebras and so satisfy (F). We note this and other results we need to compute the finite dual in the following lemma.

Lemma 6.14. *Let k be an algebraically closed field, $n > 1$ and $1 \leq w < n$ be integers and $q \in k$ be a primitive nth root of unity, and let $B := B(n, w, q)$ be the Liu algebra on those parameters. Then*

(i) $A := k[x^{\pm 1}]$ is a central Hopf subalgebra of B such that $A \subseteq B$ satisfies (F)*.*

(ii) The canonical Hopf quotient $\overline{B} := B/A^+B$ is isomorphic to the finite-dimensional Taft algebra $H_f(n, 1, q)$:

$$\overline{B} \cong k\langle \overline{y}, \overline{g} \mid \overline{yg} = q\overline{gy}, \overline{y}^n = 0, \overline{g}^n = 1 \rangle$$

as algebras, where $\overline{g}$ is grouplike and $\overline{y}$ is $(g, 1)$-primitive.

(iii) B has a PBW basis given by $\{x^r y^i g^j \mid r \in \mathbb{Z}, 0 \leq i, j < n\}$.

Proof. (i) By [6, Theorem 3.4(b)], B is a Noetherian Hopf algebra with a central subalgebra $A := k[x^{\pm 1}]$. Further, A is a Hopf subalgebra because x is grouplike. So in order to show that $A \subseteq B$ satisfy (F), we only need to show that B is finitely-generated as an A-module. We do this by showing that the set $\{y^i g^j \mid 0 \leq i, j < n\}$ is a generating set for B.

First note since B is generated by $x^{\pm 1}$, y and g it is spanned by monomials in these variables. Because x is central and y and g skew-commute, any element of B which is a monomial in x, y and g can be rearranged into the form $\lambda x^r y^s g^t$ for $r, t \in \mathbb{Z}$, $s \geq 0$, $\lambda \in k^*$. Now suppose that $t = un + i$ and $s = vn + j$ for some $0 \leq i, j < n$, $u \in \mathbb{Z}$, $v \geq 0$. Then we have

$$\begin{aligned} x^r y^s g^t &= x^r (y^n)^v y^j (g^n)^u g^i \\ &= x^r (1 - x^w)^v y^j (x^w)^u g^i \\ &= \sum_{\ell=0}^{v} \binom{v}{\ell} (-1)^\ell x^{r+(u+\ell)w} y^j g^i, \end{aligned}$$

giving us what we need.

(ii) Write $\pi : B \to \overline{B}$ for the canonical quotient map given by factoring by $\langle x - 1 \rangle$ and $\overline{y} := \pi(y), \overline{g} := \pi(g)$. We want to show that $\overline{B}$ is isomorphic to a Taft algebra.

Consider $F := k[x^{\pm 1}]$ and $E := F\langle \hat{y}, \hat{g} \rangle$ to be free on generators $\hat{y}$ and $\hat{g}$. It is clear that if we set I and J to be the ideals of E given by

$$J := \langle \hat{y}\hat{g} - q\hat{g}\hat{y}, \hat{y}^n - x^w + 1, \hat{y}^n - \hat{g}^n + 1 \rangle, \qquad I := J + \langle x - 1 \rangle,$$

then by standard isomorphism theorems,

$$E/I \cong (E/J)/(I/J) \cong B/\langle x - 1 \rangle_B \cong \overline{B}.$$

However, we can also view E/I as

$$E/I \cong (E/\langle x - 1 \rangle)/(I/\langle x - 1 \rangle) \cong k\langle \hat{y}, \hat{g} \rangle/(I/\langle x - 1 \rangle).$$

Now we have

$$I/\langle x - 1 \rangle = (J + \langle x - 1 \rangle)/\langle x - 1 \rangle = \langle \hat{y}^n, \hat{g}^n + 1, \hat{y}\hat{g} - q\hat{g}\hat{y} \rangle_{k\langle \hat{y}, \hat{g} \rangle}.$$

Since these are precisely the relations giving a Taft algebra on parameters n and q and $k\langle \hat{y}, \hat{g} \rangle$ is the free algebra on $\hat{y}$ and $\hat{g}$, it follows that

$$\overline{B} \cong E/I \cong H_f(n, 1, q)$$

as algebras as required.

The fact that the factor map π is a coalgebra map means that under this identification, $\hat{g}$ is grouplike and $\hat{y}$ is $(\hat{g}, 1)$-primitive. So in fact,

$$\overline{B} \cong H_f(n, 1, q)$$

as Hopf algebras.

(iii) We first note that B is torsion-free as an A-module, because B is a domain and the A-module structure is given by multiplication in B. This means that B must be free, following a basic result about the structure of finitely-generated modules over principal rings (see for instance [29, Theorem 7.3]), and must have finite rank $\ell > 0$. Any A-basis of B gives a k-basis of B/A^+B and vice versa, so by (ii) $\ell = \dim_k(\overline{B}) = n^2$ and the spanning set $\{y^i g^j \mid 0 \leq i,j < n\}$ given in (i) is a basis for B as a free A-module. Since $\{x^r \mid r \in \mathbb{Z}\}$ is a k-basis for A, it follows immediately that $\{x^r y^i g^j \mid r \in \mathbb{Z}, 0 \leq i,j < n\}$ is a k-basis for B as required.

□

This gives us what we need to apply Theorem 5.8.

Theorem 6.15. *Let k be an algebraically closed field of characteristic zero, $n, w > 0$ be integers and $q \in k^*$ be a primitive nth root of unity, and let $B := B(n,w,q)$ denote the generalised Liu algebra on those parameters.*

(i) B^0 decomposes as a crossed product with trivial cocycle:

$$B^0 \cong H_f(n,1,q)\#(k[z] \otimes k(k^*,*)),$$

where $H_f(n,1,q)$ denotes the finite-dimensional Taft algebra on those parameters. Here $H_f(n,1,q)$ is a Hopf subalgebra of B^0.

(ii) The action determining the smash product in (i) is nontrivial.

(iii) The crossed product in (i) is transitive, meaning that there are two subalgebras $H_f(n,1,q)\#k[z]$ and $H_f(n,1,q)\#k(k^*,*)$ of B^0. In fact, these are Hopf subalgebras.

Proof. (i) To find the stated crossed product decomposition of B^0, we use Theorem 5.8. This tells us that given $A \subseteq B$ satisfying (F), if we have a left A-module decomposition

$$B \cong A \oplus X \tag{6.3}$$

where A is a Hopf subalgebra and X is a coideal, then B^0 is a crossed product with

$$B^0 \cong \overline{B}^0 \# A^0. \tag{6.4}$$

By Lemma 6.14(i), $A := k[x^{\pm 1}] \subseteq B$ satisfies (F) and by (ii) the canonical quotient algebra $\overline{B}$ is isomorphic to the finite-dimensional Taft algebra $H_f(n,1,q)$. Since we saw in Example 4.22 that finite-dimensional Taft algebras are self-dual and in Section 6.2 that $k[x^{\pm 1}]^0 \cong k[z] \otimes k(k^*, *)$, (6.4) becomes

$$B^0 \cong H_f(n, 1, q)\#(k[z] \otimes k(k^*, *)), \tag{6.5}$$

which is exactly what we want. So all we need to do is show that (6.3) holds for some left A-module and coideal X.

Let

$$X := \bigoplus_{\substack{1 \le j < n \\ 0 \le r < n}} Ay^j g^r \oplus \bigoplus_{1 \le s < n} A(g^s - 1).$$

This is an A-module by definition, and we have

$$B \cong A \oplus X$$

as a slightly modified form of the canonical PBW-basis decomposition. So all that remains to check is that X is a coideal.

Step 1: Showing $\varepsilon(X) = 0$

This is quite straightforward: we have $\varepsilon(y^j) = \varepsilon(y)^j = 0$ whenever $1 \le j < n$, and furthermore g^s is grouplike and so $\varepsilon(g^s - 1) = 0$ for all s. Since the counit is an algebra map, this means that

$$\varepsilon(X) = \bigoplus_{\substack{1 \le j < n \\ 0 \le r < n}} \varepsilon(A)\varepsilon(y^j)\varepsilon(g^r) \oplus \bigoplus_{1 \le s < n} \varepsilon(A)\varepsilon(g^s - 1) = 0.$$

Step 2: Showing $\Delta(X) \subseteq X \otimes H + H \otimes X$

Let

$$Y := \{y^j g^r, (g^s - 1) \mid 1 \le j, s < n, 0 \le r < n\}.$$

This is a generating set for X as an A-module, and we first check that $\Delta(Y) \subseteq H \otimes X + X \otimes H$.

Let $1 \le s < n$. Since g^s is grouplike,

$$\Delta(g^s - 1) = (g^s - 1) \otimes 1 + 1 \otimes (g^s - 1) \in X \otimes H + H \otimes X$$

as required.

Now let $1 \leq j < n, 0 \leq r < n$. Then we have

$$\begin{aligned}
\Delta(y^j g^r) &= \Delta(y^j)\Delta(g^r) \\
&= \left(\sum_{s=0}^{j} \binom{j}{s}_{q^{-1}} y^s \otimes y^{j-s} g^s\right)(g^r \otimes g^r) \\
&= \sum_{s=0}^{j} \binom{j}{s}_{q^{-1}} y^s g^r \otimes y^{j-s} g^{s+r} \\
&= \sum_{s=0}^{\min(j,n-r-1)} \binom{j}{s}_{q^{-1}} y^s g^r \otimes y^{j-s} g^{s+r} \\
&\quad + \sum_{t=n-r}^{j} \binom{j}{t}_{q^{-1}} y^t g^r \otimes x^w y^{j-t} g^{t+r-n},
\end{aligned}$$

where $\binom{j}{s}_{q^{-1}}$ denotes the quantum binomial defined in Example 2.12.

Every term in both the sums is expressed in terms of the decomposition $H \cong A \oplus X$ and has at least one of the tensorands with degree of y between 1 and $n-1$, so every term is contained in $H \otimes X + X \otimes H$. This means that $\Delta(y^j g^r) \in H \otimes X + X \otimes H$ as required.

This gives us all of Y. Now we have

$$\begin{aligned}
\Delta(X) &= \Delta(AY) = \Delta(A)\Delta(Y) \\
&\subseteq (A \otimes A)(H \otimes X + X \otimes H) \\
&\subseteq AH \otimes AX + AX \otimes AH \subseteq H \otimes X + X \otimes H.
\end{aligned}$$

So X is a coideal as required: Theorem 5.8(i) applies and (6.5) holds, giving us what we want.

(ii) We want to show that the action determining the smash product in (6.5) is non-trivial. We do this by showing that there are elements in $H_f(n,1,q)$ and elements in $k[z] \otimes (k^*, *)$ that do not commute.

Let $f \in \overline{B}^0 = \overline{B}^* \cong H_f(n,1,q)$ be any map such that $f(yg^{n-1}) = 1$, and let $\alpha \in k[x^{\pm 1}]^*$ be given by $\alpha(x^i) = \lambda^i$ for some $\lambda \in k^*$ such that $\lambda \neq 1, \lambda^w \neq 1$. Since α is an algebra map, it is contained in $k[x^{\pm 1}]^0$.

Under minor abuse of notation, we also write f and α for the images of these maps in B^0, where the embedding maps are given by π^0 and the dual of the canonical map given by factoring along X respectively. We find that $f(x^i y^j g^r) = f(y^j g^r)$ for any $i \in \mathbb{Z}$ and that $\alpha(x^i g^r) = \alpha(x^i) = \lambda^i$, while $\alpha(x^i y^j g^r) = 0$ whenever $0 < j \leq n-1$.

We now show that $f\alpha(yg^{n-1}) \neq \alpha f(yg^{n-1})$, meaning that f and α do not commute.

This follows as we have

$$\begin{aligned} f\alpha(yg^{n-1}) &= \mu \circ (f \otimes \alpha)\left(yg^{n-1} \otimes g^n + g^{n-1} \otimes yg^{n-1}\right) \\ &= \mu \circ (f \otimes \alpha)\left(yg^{n-1} \otimes x^w + g^{n-1} \otimes yg^{n-1}\right) \\ &= f(yg^{n-1})\alpha(x^w) + f(g^{n-1})\alpha(yg^{n-1}) \\ &= \lambda^w + 0 = \lambda^w. \end{aligned}$$

However,

$$\begin{aligned} \alpha f(yg^{n-1}) &= \mu \circ (\alpha \otimes f)\left(yg^{n-1} \otimes g^n + g^{n-1} \otimes yg^{n-1}\right) \\ &= \alpha(yg^{n-1})f(x^w) + \alpha(g^{n-1})f(yg^{n-1}) \\ &= 0 + \lambda^0 = 1. \end{aligned}$$

Since we assumed $\lambda^w \neq 1$, this means that $f\alpha(yg^{n-1}) \neq \alpha f(yg^{n-1})$ and so the two maps do not commute: the smash product structure in (6.5) is nontrivial.

(iii) By (i), B^0 decomposes as a smash product as in (6.5) and so is also a free $\overline{H}^0$-module. So Theorems 5.19 and 5.21 apply: we have Hopf subalgebras

$$W := \{f \in B^0 \mid f(\langle x-1\rangle^n) = 0 \text{ for some } n > 0\} \cong \overline{B}^0 \#_\sigma U(\operatorname{Lie} G)$$

and

$$k\widehat{G} := \{f \in B^0 \mid f(\mathfrak{m}_g B) = 0 \text{ for some } g \in G\} \cong \overline{B}^0 \#_\tau kG,$$

where G is the affine algebraic group such that $\mathcal{O}(G) \cong k[x^{\pm 1}]$. In other words, $G = (k^*, *)$ and $\operatorname{Lie} G$ is simply the one-dimensional Lie algebra, so that $U(\operatorname{Lie} G) = k[y]$. Moreover, the cocycles σ and τ arise from restriction of the cocycle on B and are hence also trivial. Since we know that $\overline{B}^0 \cong H_f(n,1,q)$ by (i), these equations become

$$\begin{aligned} W &\cong H_f(n,1,q)\#k[y], \\ k\widehat{G} &\cong H_f(n,1,q)\#k(k^*,*), \end{aligned}$$

with $k[y]$ and $k(k^*,*)$ in these decompositions corresponding to those in (6.5). □

Remark 6.16. An immediate consequence of part (ii) of Theorem 6.15 is that although we know that the Liu algebras can be written as a crossed product with respect to $k[x^{\pm 1}]$ by Proposition 5.31(ii) (using the fact that they are generated by grouplike and skew-primitive elements and hence pointed), the resulting cleaving map cannot be a coalgebra map. We can see this as follows.

Suppose for a contradiction that there did exist a coalgebra map $\gamma : \overline{B} \to B$. Then by Theorem 5.33 we have

$$B^0 \cong \overline{B}^0 \otimes k[x^{\pm 1}]^0$$

as algebras, left $\overline{B}^0$-modules and right $k[x^{\pm 1}]^0$-comodules. Given $b \in \overline{B}$, $a \in k[x^{\pm 1}]$, we write $b \cdot a$ for the action of $\overline{B}$ on A in the crossed product decomposition in Theorem 6.15(i). Now by Proposition 5.35, there exists a convolution invertible map $u : H \to A$ satisfying

$$\begin{aligned} b \cdot a &= \sum u^{-1}(b_1)(\varepsilon_B(b_2)a)u(b_3) \\ &= \sum u^{-1}(b_1)au(b_2) \\ &= \sum u^{-1}(b_1)u(b_2)a = \varepsilon_B(b)a. \end{aligned}$$

So the action of $\overline{B}$ on A and hence the smash product structure in the decomposition of Theorem 6.15(i) is trivial - contradicting part (ii) of said theorem.

6.6 Originality

The first part of this chapter, up to Theorem 6.5, relates known results and work regarding the structure and classification of prime affine regular Hopf algebras of Gelfand-Kirillov dimension one. Section 6.2 is known. Section 6.3 is to our knowledge original, but Theorem 6.6 may reproduce results in the representation theory of finite groups. Section 6.4 is original, as is Section 6.5.

Chapter 7

Conclusion

In this thesis, we have worked out various ways of calculating the finite dual of various Hopf algebras, focusing on those which can be decomposed as crossed products (Chapter 4) or which have some distinguished central Hopf subalgebra (Chapter 5). These results let us use known classical results to compute the finite duals of numerous Hopf algebras or families of Hopf algebras of interest: we have done so with the prime regular Noetherian Hopf algebras classified by Brown and Zhang in [6] (Chapter 6) along with the quantised enveloping algebra $U_\epsilon(\mathfrak{sl}_2(k))$ at a root of unity (Section 5.4), with an eye to the possibility of extending the latter to $\mathfrak{sl}_n(k)$ or other Lie algebras.

Such examples allow us to better study how various well-known results about duality in the finite-dimensional world do or do not transfer to the infinite-dimensional world. The duality theory of finite-dimensional Hopf algebras is a powerful tool, but as we already saw in the introduction numerous properties that are preserved under taking finite-dimensional duals need not transfer from an infinite-dimensional Hopf algebra to its dual.

One of the problems we discussed in the introduction was that of the size of H^0. Most examples we have seen in the intervening chapters have been "too big": in particular, they often have uncountable dimension thanks to containing some group of characters isomorphic to an algebraic group. So for instance, in Example 4.22, we saw that given an algebraically closed field k of characteristic zero, the finite dual of a Taft algebra $H(n,t,q)$ on coprime integers $n>1, t>0$ and a primitive nth root of unity $q \in k$ is given by

$$H(n,t,q)^0 \cong k[z] \otimes H_f(n,t,q) \otimes k(k,+),$$

where $H_f(n,t,q)$ is the finite-dimensional Taft algebra with respect to the same parameters and $k(k,+)$ is the group algebra of the additive group of the underlying field. This last subalgebra has basis indexed by k, and so when $k = \mathbb{C}$ it, and hence $H(n,t,q)^0$ as a whole, has uncountable dimension. This also means the finite dual is not affine, in distinct contrast to the Hopf algebra we started with. Another example is given by the coordinate ring of an affine algebraic group. In Proposition 2.13, we saw that this satisfies

$$\mathcal{O}(G)^0 \cong U(\operatorname{Lie} G)\# kG.$$

Again, kG is the group algebra of G and hence has basis indexed by G, which as an algebraic variety is generally going to have the same cardinality as the underlying field.

However, in many cases including these we can find a smaller Hopf subalgebra inside the finite dual with size far closer to the size of H.

In the case of the Taft algebras above, this might be the subalgebra $k[z] \otimes H_f(n,t,q)$, which we know is a Hopf subalgebra by Lemma 6.9(ii), or perhaps $k[z] \otimes H_f(n,t,q) \otimes kK$ for some finitely-generated subgroup $K < (k,+)$, this also being a Hopf subalgebra by the discussion in Remark 6.12. In the case of the coordinate ring $\mathcal{O}(G)$ of an affine algebraic group G, we have $U(\operatorname{Lie} G)$ - hinting at a duality between coordinate rings and universal enveloping algebras of Lie algebras. Finally, Theorem 5.21 extends the example of coordinate rings: it says that whenever H is a Noetherian Hopf algebra which is a finite module over some central Hopf subalgebra $A \cong \mathcal{O}(G)$ such that there exists a left A-module projection $\Pi : H \to A$, there is a Hopf subalgebra $W \cong (H/A^+H)^* \# U(\operatorname{Lie} G)$.

Chin and Musson also investigate this question in [9], focusing on the injective hull $E_H(k)$ of the trivial module as a potential replacement for H^0.

One potential situation where having a distinguished subalgebra of H^0 would be useful is in a potential extension of the Drinfel'd double to infinite dimensions.

The Drinfel'd double, introduced by Drinfel'd in [14], is a Hopf algebra $D(H)$ constructed from a finite-dimensional Hopf algebra H and its dual H^*. This Hopf algebra is isomorphic to $H \otimes H^*$ as a vector space, contains both H and H^* as Hopf subalgebras, and satisfies some good structural properties such as being quasitriangular (see [14]). It is hence a useful tool in the study of finite-dimensional Hopf algebras.

In the infinite-dimensional case, it is possible to form a so-called *Hopf pairing* between a Hopf algebra H and any Hopf algebra $K \subseteq H^0$ that is dense in H^*, a construction which is analogous to the Drinfel'd double and for instance described in [25, Section 3.2]. However, such a pairing does not necessarily satisfy any of the properties it does in the finite-dimensional case. Moreover, using $K = H^0$ means that we end up with the same size problems as we have for H^0 itself: the resulting Hopf pairing may have uncountable dimension and hence not be affine and possibly not Noetherian, even when the Hopf algebra we started with satisfied those properties. This makes it of limited usefulness for many applications and means that some smaller algebra may often be more suitable instead.

In all of this, the results of this thesis are immediately useful in two ways. First, they allow us to explicitly compute the structure of H^0 for various Hopf algebras H and hence can provide a large class of examples on which we can check theories and look for patterns. Second, there is the possibility of using them to extending any results in this area about classical Hopf algebras like coordinate rings, group algebras and enveloping algebras to crossed products of these algebras or, in the case of coordinate rings, to Hopf algebras containing them as a central Hopf subalgebra. This is of particular interest because this might allow us to use results about commutative and cocommutative Hopf algebras to say things about Hopf algebras that are neither, such as quantum groups.

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