SOME APPLICATIONS OF GEOMETRIC TECHNIQUES IN COMBINATORIAL GROUP THEORY

by

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CONTENTS

STATEMENT

Chapter 1 covers basic material in combinatorial group theory and is based to some extent on notes of S.J. Pride. The modifications of his work to cover involutary complexes and the proof of Proposition 1.1 are mine.

Chapters 2,4 and Appendix B are my own work. Chapter 3 was joint work with S.J. Pride. (To be more precise the concept of NEC-complex is due to Pride. Lemmata 3.1-3.3 were obtained in collaboration with Pride, and the rest of the chapter was done by myself, at Pride's suggestion.) Chapter 3 together with some of chapter 1 has appeared in [15].

[redacted]

ACKNOWLEDGEMENTS

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Finally, I would like to thank the S.E.R.C. for awarding me a postgraduate studentship from 1985 to 1988.

ABSTRACT

Combinatorial group theory abounds with geometrical techniques. In this thesis we apply some of them to three distinct areas.

In Chapter 1 we present all of the techniques and background material neccessary to read chapters 2,3,4. We begin by defining complexes with involutary edges and define coverings of these. We then discuss equivalences between complexes and use these in §§1.3 and 1.4 to give a way (the level method) of simplifying complexes and an application of this method (Theorem 1.3). We then discuss star-complexes of complexes. Next we present background material on diagrams and pictures. The final section in the chapter deals with SQ-universality. The basic discussion of complexes is taken from notes, by Pride, on complexes without involutary edges, and modified by myself to cover complexes with involution.

Chapters 2,3, and 4 are presented in the order that the work for them was done. Chapters 2,3, and 4 are intended (given the material in chapter 1) to be self contained, and

each has a full introduction.

In Chapter 2 we use diagrams and pictures to study groups with the following structure.

(a) Let Γ be a graph with vertex set V and edge set E. We assume that no vertex of Γ is isolated.

(b) For each vertex $v \in V$ there is a non-trivial group G_v.

(c) For each edge $e=(u,v) \in E$ there is a set S_e of cyclically reduced elements of $G_u * G_v$, each of length at least two.

We define G_e to be the quotient of $G_u * G_v$ by the normal closure of S_e.

We let G be the quotient of $\underset{v \in V}{*} G_v$ by the normal closure of $S = \bigcup_{e \in E} S_e$. For convenience, we write

$$G = \langle\, G_v \ (v \in V);\ S_e \ (e \in E) \,\rangle$$

The above is a generalization of a situation studied by Pride [35], where each G_v was infinite cyclic.

Let $e=(u,v)$ be an edge of Γ. We will say that G_e has *property*-W_k if no non-trivial element of $G_u * G_v$ of free product length less than or equal to 2k is in the kernel of the natural epimorphism

$$G_u * G_v \longrightarrow G_e$$

We will work with one of the following:

(I) Each G_e has property-W_2

(II) Γ is triangle-free and each G_e has property-W_1.

Assuming that (I) or (II) holds we: (i) prove a Freihietssatz for these groups; (ii) give sufficient conditions for the groups to be SQ-universal; (iii) prove a result which allows us to give long exact sequences relating the (co)-homology G to the (co)-homology of the groups G_v $(v \in V)$, G_e $(e \in E)$.

The work in Chapter 2 is in some senses the least original. The proofs are extensions of proofs given in [35] and [39] for the case when each G_v is infinite cyclic. However, there are some technical difficulties which we had to overcome.

In chapter 3 we use the two ideas of star-complexes and coverings to look at NEC-groups.

An NEC (Non-Euclidean Crystallographic) group is a discontinuous group of isometries (some of which may be orientation reversing) of the Non-Euclidean plane. According to Wilkie [46], a finitely generated NEC-group with compact orbit space has a presentation as follows:

Involutary generators: y_{ij} $(i,j)\in\Xi_0$

Non-involutary generators: e_i $(i\in I_f)$, t_k $(1\leq k\leq r)$

a_k $(1\leq k\leq g)$, b_k $(1\leq k\leq h,\ h=0 \text{ or } g)$

(*) Defining paths: $(y_{ij}y_{ij+1})^{m_{ij}}$ $(i\in I_f,\ 1\leq j\leq n(i)-1)$

$(y_{in(i)}e_i y_{i1} e_i^{-1})^{m_{in(i)}}$ $(i\in I_f)$

$t_k^{p_k}$ $(1\leq k\leq r,\ p_k\geq 2)$

$\prod_i(e_i^{-1})(\prod_k t_k^{-1})\alpha$

where

$$\alpha = \begin{cases} \prod_k a_k^2 & \text{if } h=0 \\ \prod_k a_k b_k a_k^{-1} b_k^{-1} & \text{if } h=g, \end{cases}$$

In Hoare, Karrass and Solitar [22] it is shown that a subgroup of finite index in a group with a presentation of the form (*), has itself a presentation of the form (*). In [22] the same authors show that a subgroup of infinite index in a group with a presentation of the form (*) is a free product of groups of the following types:

(A) Cyclic groups.

(B) Groups with presentations of the form

$$<x_1,\ldots,x_n,e\ ;\ (x_1x_2)^{m_1},\ldots,(x_nex_1e^{-1})^{m_n}>$$

$x_1,\ldots,x_n$ involutary.

(C) Groups with presentations of the form

$$< x_i\ (i\epsilon Z)\ ;\ (x_ix_{i+1})^{m_i}\ (i\epsilon Z) >$$

x_i $(i\epsilon Z)$ involutary.

We define what we mean by an *NEC-complex*. (This involves a structural restriction on the form of the star-complex of the complex.) It is obvious from the definition that this class of complexes is closed under coverings, so that the class of fundamental groups of NEC-complexes is trivially closed under taking subgroups. We then obtain structure theorems for both finite and infinite NEC-complexes.

We show that the fundamental group of a finite NEC-complex has a presentation of the form (*) and that the fundamental group of an infinite NEC-complex is a free product of groups of the forms (A), (B) and (C) above.

We then use coverings to derive some of the results on normal subgroups of NEC-groups given in [5] and [6].

In chapter 4 we use the techniques of coverings and diagrams to study the SQ-universality of Coxeter groups. This is a problem due to B.H. Neumann (unpublished), see [40].

A *Coxeter pair* is a 2-tuple (Γ,φ) where Γ is a graph (with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$) and φ is a map from $E(\Gamma)$ to $\{2,3,4,\ldots\}$. We associate with (Γ,φ) the *Coxeter group* $C(\Gamma,\varphi)$ defined by the presentation

$$\mathcal{C}(\Gamma,\varphi)=\langle V(\Gamma);(xy)^{\varphi(\{x,y\})}\ (\{x,y\}\in E(\Gamma))\rangle,$$

where each generator is involutary.

Following Appel and Schupp [1] we say that a Coxeter pair is of *large type* if $2\notin \operatorname{Im}\varphi$. I conjecture that if (Γ,φ) is of large type with $|V(\Gamma)|\geq 3$ and Γ not a triangle with all edges mapped to 3 by φ, then $C(\Gamma,\varphi)$ is SQ-universal. In connection with this conjecture we firstly prove (Theorem 4.1).

Let (Γ,φ) be a Coxeter pair of large type. Suppose

(A) *Γ is incomplete on at least three vertices, or*

(B) *Γ is complete on at least five vertices and for any triangle e_1,e_2,e_3 in Γ*

$$\frac{1}{\varphi(e_1)+1}+\frac{1}{\varphi(e_2)+1}+\frac{1}{\varphi(e_3)+1}<\frac{1}{2}$$

Then $C(\Gamma,\varphi)$ *is SQ-universal.*

Secondly we prove a result (Theorem 4.2) which shows: *If* (Γ,φ) *is a Coxeter pair with* $|V(\Gamma)|\geq 4$ *and* $\mathrm{hcf}[\varphi(E(\Gamma))] > 1$, *then* $C(\Gamma,\varphi)$ *is either SQ-universal or is soluble of length at most three.*

Moreover our Theorem allows us to tell the two possibilities apart.

The proof of this result leads to consideration of the following question: If a direct sum of groups is SQ-universal, does this imply that one of the summands is itself SQ-universal?

We show (in appendix B) that the answer is "yes" for countable direct sums.

We consider the results in chapter 4 and its appendix to be the most significant part of this thesis.

NOTATIONS

Let G, H and K_i ($i\in I$) be groups.

$G\times H$ is the direct product.

$G*H$ is the free product.

$\sum_{i\in I} H_i$ is the direct sum.

$G \hookrightarrow H$ G embeds in H.

We adopt the usual notation in set theory.

$R\cup S$ is the union of sets R and S.

$R\cap S$ is the intersection of sets R and S.

$R\subseteq S$ means R is a subset of S.

$r\in R$ neans r is an element of R.

$|R|$ denotes the cardinality of R.

Z_n is the cyclic group of order n.

F_n is the free group of rank n.

Z is the integers.

The following notations are introduced in the text.

Let $\mathfrak{X}$ be a 1-complex.

$V(\mathfrak{X})$ set of vertices of $\mathfrak{X}$.

$E(\mathcal{X})$ set of edges of $\mathcal{X}$.

$\iota(e)$ initial vertex of the edge e.

$\tau(e)$ terminal vertex of the edge e.

α^{-1} inverse of the path α.

$L(\alpha)$ length of the path α.

1_v the empty path associated with the vertex v.

$L_e(\alpha)$ number of times e, e^{-1} appear in a path α.

$\mathrm{star}(\varphi)=\{e\colon e\in E(\mathcal{X}),\ \iota(e)=v\}$.

$\alpha \overset{(1)}{\sim}_{\mathcal{X}} \beta$ α is freely equal to β in $\mathcal{X}$.

Let $\mathcal{A}=\langle \mathcal{X};\rho_\lambda\ (\lambda\in\Lambda)\rangle$ be a 2-complex.

$\mathcal{A}$ is the 1-skeleton $\mathcal{X}$.

ρ_λ is a non-empty closed path in $\mathcal{A}$, called a defining path.

Λ is the set of elements called indices.

$\pi_1(\mathcal{A}, v)$ is the fundamental group of $\mathcal{A}$ at v.

Λ_m an element of Λ_m is said to be of level m.

$R(\mathcal{A})$ is the set of cyclic permutations of defining paths and there inverses

$\alpha \sim_{\mathcal{A}} \beta$ α is equivalent to β in $\mathcal{A}$.

$[\alpha]_{\mathcal{A}}$ the equivalence class containing α with respect to $\sim_{\mathcal{A}}$.

$\mathcal{A}^{st}$ the star-complex of $\mathcal{A}$.

$\iota^{st}(\gamma)$ the first edge of γ.

$\tau^{st}(\gamma)$ the inverse of the last edge of γ.

$\gamma^{-1\ st}$ the inverse of γ.

$\mathcal{A}^{st}(v)$ the full subcomplex of $\mathcal{A}^{st}$ on star(v).

CG($\mathcal{A}$) the connectivity graph of $\mathcal{A}$.

$\mathrm{star}_{\varphi}(v)=\{e\colon e\in\mathrm{star}(v),\ \varphi(e)\in E(\mathcal{B})\}$ (where $\varphi\colon \mathcal{A}\longrightarrow\mathcal{B}$).

Let $\mathcal{P}=\langle X_1,X_2,r\rangle$ be a presentation.

X_1 the set of non-involutary generators of $\mathcal{P}$.

X_2 the set of involutary generators of $\mathcal{P}$.

$=_{\mathcal{P}}$ means $\sim_{\mathcal{P}}$.

Let $\mathcal{A}$ be a diagram.

$\angle K$ the angle at the corner K.

$K(\Delta)$ the curvature of a region Δ.

$K(a)$ the curvature of a vertex a.

Let $\mathbb{P}$ be a picture.

$\overset{0}{\mathbb{P}}$ a mirror-picture.

$\vec{\gamma}$ a spray.

$\sigma_{\mathbb{P}}(\vec{\gamma})$ the sequence associated with $\vec{\gamma}$.

N the class of NEC-complexes.

F the class of Fuchsian-complexes.

S the class of Surface-complexes.

CHAPTER 1

BACKGROUND MATERIAL

1.1 COMPLEXES WITH INVOLUTION

A *1-complex*, $\mathfrak{X}$ consists of two disjoint sets $V(\mathfrak{X})$ (*vertices*) and $E(\mathfrak{X})$ (*edges*) and three maps,

$$\iota : E(\mathfrak{X}) \longrightarrow V(\mathfrak{X}),\ \tau : E(\mathfrak{X}) \longrightarrow V(\mathfrak{X}) \text{ and } {}^{-1} : E(\mathfrak{X}) \longrightarrow E(\mathfrak{X}),$$

satisfying:

(i) $(e^{-1})^{-1}=e$ $(e \in E(\mathfrak{X}))$, and

(ii) $\iota(e)=\tau(e^{-1})$ $(e \in E(\mathfrak{X}))$.

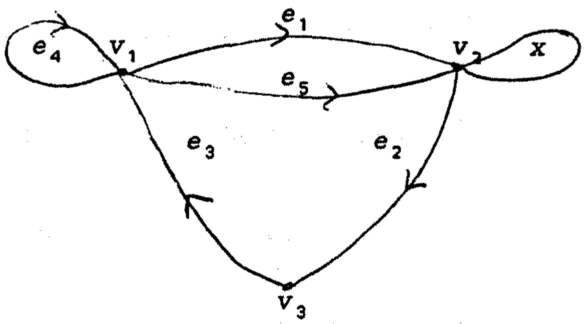
An edge of $\mathfrak{X}$ is said to be *involutary* if $e=e^{-1}$. Note that, for such an edge $\iota(e)=\tau(e)$. When $\mathfrak{X}$ has no involutary edges, the notion of 1-complex coincides with the concept of "graph" as considered by Serre, [42], and others.

Remark: In this thesis we will only use the term graph to refer to a set and a collection of two element subsets of it.

A 1-complex can be represented diagramatically as follows. A vertex is represented by a point. For each involutary edge, x say, we draw a loop (labelled x) at $\iota(x)$. The remaining edges can be divided into two element sets of the form

$\{e,e^{-1}\}$. For each of these sets we select one of the pair, e say, and draw a directed segment (labelled e) joining the point corresponding to $\iota(e)$ to the point corresponding to $\tau(e)$.

Example



This represents a 1-complex with three vertices v_1, v_2, v_3 and eleven edges; e_i, e_i^{-1} $(1 \leq i \leq 5)$ non-involutary and x involutary. We have $\iota(e_1)=\iota(e_5)=\tau(e_3)=\iota(e_4)=\tau(e_4)=v_1$, $\tau(e_1)=\tau(e_5)=\iota(x)=\iota(e_2)=v_2$, and $\tau(e_2)=\iota(e_3)=v_3$.

A *non-empty path* α in $\mathfrak{X}$ is a sequence $e_1,\ldots,e_n$ (usually written without the commas) of edges of $\mathfrak{X}$ with $\tau(e_i)=\iota(e_{i+1})$ $1 \leq i < n$. Define $\iota(\alpha)=\iota(e_1)$ and $\tau(\alpha)=\tau(e_n)$. The path α is said to be *closed* if $\iota(\alpha)=\tau(\alpha)$. The *length* of α, $L(\alpha)$, is n. The *inverse* of α , α^{-1}, is the path $e_n^{-1}\ldots e_1^{-1}$. For an edge e of we define $L_e(\alpha)$ to be the number of times e and e^{-1} occur in α. If α is closed then we can write $\alpha=\overset{0}{\alpha}{}^{p(\alpha)}$ where $\overset{0}{\alpha}$ is not a proper power and $p(\alpha)$ is a positive integer. We call $\overset{0}{\alpha}$ the *root* of α, and $p(\alpha)$ the *period*.

With each vertex, v, of $\mathfrak{X}$ we associate an *empty path*, 1_v. It has no edges and we define $L(1_v)=0$, $\iota(1_v)=\tau(1_v)=v$ and $1_v^{-1}=1_v$. If it is clear which vertex is intended then we will denote the empty path at v simply by 1.

If $|V(\mathfrak{X})|=1$, we call $\mathfrak{X}$ a *bouquet*.

A *free reduction* on a path α consists of deleting an adjacent pair of edges of the form ee^{-1}. A path α is said to be *reduced* if no free reduction can be applied to it and is *cyclically reduced* if for every cyclic permutation α^* of α, the first edge of α^* is not the inverse of the last edge.

Two paths α,β are said to be *freely equal* if there exists a sequence

$$\alpha=\alpha_0,\alpha_1,\ldots,\alpha_k=\beta$$

where in each pair (α_i,α_{i+1}) $0\leq i<k$ one path is obtained from the other by a free reduction. We write this as $\alpha\overset{(1)}{\underset{\mathfrak{X}}{\sim}}\beta$, or just $\alpha\overset{(1)}{\sim}\beta$ if no confusion can arise. A path α is said to be *freely contractible* if $\alpha\overset{(1)}{\sim}1$.

If α and β are paths in $\mathfrak{X}$ we say that the *product*, $\alpha\beta$, of

α and β is defined if $\tau(\alpha)=\iota(\beta)$. Then $\alpha\beta$ is the path consisting of the edges of α followed by the edges of β.

A 1-complex $\mathcal{X}$ is said to be *connected* if given any two vertices u,v then there is a path α in $\mathcal{X}$ with $\iota(\alpha)=u$ and $\tau(\alpha)=v$. A *subcomplex* of a 1-complex $\mathcal{X}$ is a subset of $V(\mathcal{X}) \cup E(\mathcal{X})$ which is closed under ι, τ and ${}^{-1}$. If $V \subseteq V(\mathcal{X})$ then the *full subcomplex on V* consists of V together with all edges e of $\mathcal{X}$ where both $\iota(e)$ and $\tau(e)$ lie in V. A maximal connected subcomplex of a 1-complex is called a *component*.

A *tree* is a connected 1-complex in which no non-empty closed path is reduced.

We now define *mappings (of 1-complexes)*. Let $\mathcal{X}$ and $\mathcal{Y}$ be 1-complexes.

$$\varphi: \mathcal{X} \longrightarrow \mathcal{Y}$$

is called a mapping (of 1-complexes) if it is a function sending vertices of $\mathcal{X}$ to vertices of $\mathcal{Y}$ and paths in $\mathcal{X}$ to paths in $\mathcal{Y}$, and satisfying:

(i) $\varphi(1_v)=1_{\varphi(v)}$ for all $v \in V(\mathcal{X})$.

(ii) $\varphi(\alpha^{-1}) \overset{(i)}{=} \varphi(\alpha)^{-1}$ for all paths α in $\mathcal{X}$.

(iii) Whenever $\alpha_1\alpha_2$ is defined, $\varphi(\alpha_1)\varphi(\alpha_2)$ is defined, and

$\varphi(\alpha_1\alpha_2)=\varphi(\alpha_1)\varphi(\alpha_2)$, α_1 and α_2 paths in $\mathcal{X}$.

We call φ *rigid* if $L(\varphi(\alpha))=L(\alpha)$ for all paths in $\mathcal{X}$ (that is φ maps edges to edges). We call φ *pure* if $\varphi(\alpha^{-1})=\varphi(\alpha)^{-1}$ for all paths in $\mathcal{X}$.

A 2-complex, $\mathcal{A}$, is an object

$$<\mathcal{X};\ \rho_\lambda\ (\lambda\epsilon\Lambda)>$$

where $\mathcal{X}$ is a 1-complex (called the *1-skeleton* of $\mathcal{A}$ and denoted by $\mathcal{A}^{(1)}$ where neccessary) and each ρ_λ is a closed non-empty path in $\mathcal{X}$. The ρ_λ's are called *defining paths* (for $\mathcal{A}$). The elements of Λ are called *indices*. A 2-complex is said to be *finite* if $V(\mathcal{X})\cup E(\mathcal{X})\cup\Lambda$ is a finite set. A path in $\mathcal{A}$ is a path in its 1-skeleton. The *vertices* (respectively, *edges*) *of* $\mathcal{A}$ are the vertices (respectively, edges) of its 1-skeleton, we define $V(\mathcal{A})=V(\mathcal{X})$, $E(\mathcal{A})=E(\mathcal{X})$.

If the 1-skeleton of $\mathcal{A}$ is a bouquet, we say that $\mathcal{A}$ is a *presentation*.

There are four ways that we will descibe a presentation. The first is in its form as a 2-complex.

$< p_i \; (i \in I)$  $q_j \; (j \in J) \; ; \quad \rho_\lambda \; (\lambda \in \Lambda)>$

The second is by listing its edges and defining paths

Non-involutary edges: $p_i^{\pm 1}$ $(i \in I)$

Involutary edges : q_j $(j \in J)$

Defining paths : ρ_λ $(\lambda \in \Lambda)$

The third is in the form

$< p_i \; (i \in I) \; q_j \; (j \in J) \; ; \; \rho_\lambda \; (\lambda \in \Lambda)>$ (p_i $(i \in I)$ non-involutary, q_j $(j \in J)$ involutary).

The fourth is in the form

$$< X_1, X_2 \; ; \; r>$$

Where $X_1 = \{p_i : i \in I\}$, $X_2 = \{q_j : j \in J\}$ and $r = \{\rho_\lambda : \lambda \in \Lambda\}$. In the third and fourth cases the p_i's (respectively, q_i's) are called the non-involutary (respectively, involutary) generators, and the ρ_λ's the relators.

The third and fourth forms correspond to the usual forms for a presentation, as found in Magnus, Karrass and Solitar [30] and extended, by Pride [38], to incorporate the notion of involutary generators.

Let $R(\mathcal{A})$ be the set of those paths γ, in $\mathcal{A}$ for which some cyclic permutation of γ or γ^{-1} is a defining path of $\mathcal{A}$.

Let $\mathcal{A}$ be a 2-complex. We define an equivalence relation $\sim_{\mathcal{A}}$ on paths in $\mathcal{A}$ as follows.

An *elementary reduction* of a path α in $\mathcal{A}$ is a free reduction on α or the deletion of some subpath $\gamma \epsilon R(\mathcal{A})$ from α. For two paths α and β we say $\alpha \sim_{\mathcal{A}} \beta$ if there exists a sequence

$$\alpha=\alpha_0, \alpha_1, \ldots, \alpha_n=\beta$$

where in each pair (α_i, α_{i+1}) $0 \leq i < n$ one path is obtained from the other by an elementary reduction.

If $\mathcal{A}$ is a presentation and for two paths α, β we have $\alpha \sim_{\mathcal{A}} \beta$ we will sometimes write

$$\alpha =_{\mathcal{A}} \beta.$$

The $\sim_{\mathcal{A}}$ -equivalence class containing α is denoted $[\alpha]_{\mathcal{A}}$ or $[\alpha]$ if no confusion can arise. If $\alpha \sim_{\mathcal{A}} 1_v$ for some $v \epsilon V(\mathcal{A})$ we say that α is *contractible (in* $\mathcal{A}$*)*. We note that every element of $R(\mathcal{A})$ is contractible.

If α and β are two paths in $\mathcal{A}$ such that $\alpha\beta$ is defined, we define $[\alpha]_{\mathcal{A}} [\beta]_{\mathcal{A}} =[\alpha\beta]_{\mathcal{A}}$ (this is easily seen to be well defined).

Let $v\epsilon V(\mathcal{A})$. We define the *fundamental group of* $\mathcal{A}$ *(at v)*, $\pi_1(\mathcal{A},v)$, to be the group with

$$\{[\alpha]_{\mathcal{A}} : \alpha \text{ a path in } \mathcal{A} \text{ with } \iota(\alpha)=\tau(\alpha)=v\}$$

as underlying set and with the above multiplication. The identity element is $[1_v]_{\mathcal{A}}$ and $[\alpha]_{\mathcal{A}}^{-1}=[\alpha^{-1}]_{\mathcal{A}}$

Let $\mathcal{P}=< X_1,X_2;\ r>$ be a presentation. The *group defined by* $\mathcal{P}$ is the fundamental group of the complex with a single vertex, non-involutary edges $p^{\pm1}$ $(p\epsilon X_1)$, involutary edges q $(q\epsilon X_2)$ and defining paths the elements of r.

We now define *mappings (of 2-complexes)*. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-complexes, say $\mathcal{A}=<\mathcal{X}\ ;\ \rho_\lambda\ (\lambda\epsilon\Lambda)>$ and $\mathcal{B}=<\mathcal{Y}\ ;\ \mu_\gamma\ (\gamma\epsilon\Gamma)>$. Then $\varphi: \mathcal{A}\longrightarrow\mathcal{B}$ is called a mapping of 2-complexes if it is a function sending vertices of $\mathcal{A}$ to vertices of $\mathcal{B}$ and paths in $\mathcal{A}$ to paths in $\mathcal{B}$, satisfying:

(i) $\varphi(1_v)=1_{\varphi(v)}$ for all $v\epsilon V(\mathcal{A})$

(ii) $\varphi(\alpha^{-1})\sim_{\mathcal{B}} \varphi(\alpha)^{-1}$ for all paths α in $\mathcal{A}$.

(iii) Whenever $\alpha_1\alpha_2$ is defined, $\varphi(\alpha_1)\varphi(\alpha_2)$ is defined, and $\varphi(\alpha_1\alpha_2)=\varphi(\alpha_1)\varphi(\alpha_2)$, α_1 and α_2 paths in $\mathcal{A}$.

(iv) $\varphi(\rho_\lambda)$ is contractible in $\mathcal{B}$ for all $\lambda\epsilon\Lambda$.

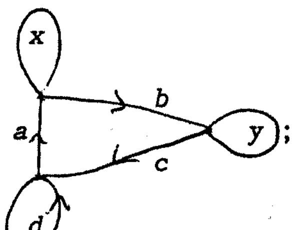
(N.b. φ need not induce a mapping of 1-complexes between $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ unless Λ and Γ are empty.)

<u>*Remark*</u>: (1) It is sufficient to define φ on the edges of $\mathcal{A}$, provided $\iota(\varphi(e))=\varphi(\iota(e))$ for all $e\epsilon E(\mathcal{A})$.

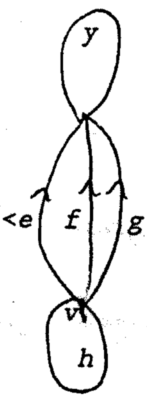
(2) (iv) guarantees that the image of any contractible path is itself contractible.

<u>Example</u>

Let

$\mathcal{A}$ = < 

; *axbc* >

and $\mathcal{B}$ = < 

; *fg*, $eyg^{-1}h^2$, h^2>

We define a mapping from $\mathcal{A}$ to $\mathcal{B}$ by

$x \mapsto y$, $a^{\pm 1} \mapsto e^{\pm 1}$, $b \mapsto f^{-1}$, $b^{-1} \mapsto g$, $c^{\pm 1}, y, d^{\pm 1} \mapsto 1_v$.

We call φ *rigid* if $L(\varphi(\alpha))=L(\alpha)$ for all paths in $\mathcal{A}$; *pure* if $\varphi(\alpha^{-1})=\varphi(\alpha)^{-1}$ for all paths in $\mathcal{A}$; and *incompressable* if no edge is mapped to an empty path.

A *based* mapping of 2-complexes

$$\varphi:(\mathcal{A},u) \longrightarrow (\mathcal{B},v)$$

is a mapping of 2-complexes from $\mathcal{A}$ to $\mathcal{B}$ which sends u to v ($u \in V(\mathcal{A})$, $v \in V(\mathcal{B})$).

Let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping of 2-complexes. Then for every vertex v of $\mathcal{A}$ we have an induced homomorphism

$$\varphi_*:\pi_1(\mathcal{A},v) \longrightarrow \pi_1(\mathcal{B},\varphi(v))$$

given by $\varphi_*([\alpha]_{\mathcal{A}})=[\varphi(\alpha)]_{\mathcal{B}}$.

Let $\mathcal{A}=<\mathcal{X}\ ;\ \rho_\lambda\ (\lambda \in \Lambda)>$ be a 2-complex. If $u \in V(\mathcal{A})$ define

$$\mathrm{star}(u)=\{e:\ e \in E(\mathcal{A}),\ \iota(e)=u\}.$$

Let $\mathcal{B}=<\mathcal{Y}\ ;\ \mu_\gamma\ (\gamma \in \Gamma)>$ and let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping of 2-complexes. Define

$$\mathrm{star}_\varphi(u)=\{e:\ e \in \mathrm{star}(u) \text{ and } \varphi(e) \in E(\mathcal{B})\}.$$

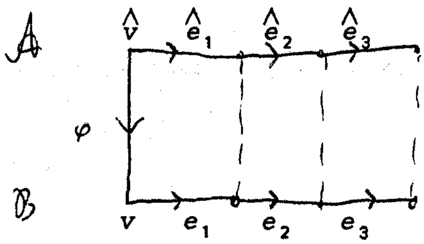
Clearly $\varphi(\mathrm{star}_\varphi(u)) \subseteq \mathrm{star}(\varphi(u))$ for all $u \in V(\mathcal{A})$. We say φ is *locally injective/surjective/bijective* if

$\varphi|\mathrm{star}_{\varphi}(u):\mathrm{star}_{\varphi}(u) \longrightarrow \mathrm{star}(\varphi(u))$.

is injective/surjective/bijective for all $u \in V(\mathcal{A})$.

If $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ is a mapping of 2-complexes and $\varphi(\hat{v})=v$ we say $\hat{v}$ *lies over* v. If α is a path in $\mathcal{B}$ and if $\hat{v}$ lies over $\iota(\alpha)$, then a path $\hat{\alpha}$ in $\mathcal{A}$ such that $\iota(\hat{\alpha})=\hat{v}$ and $\varphi(\hat{\alpha})=\alpha$ is called a *lift of* α *at* $\hat{v}$.

Example



where $\varphi(\hat{e}_i^{\pm 1})=e_i^{\pm 1}$ $(1 \leq i \leq 3)$. Then $\hat{e}_1\hat{e}_2\hat{e}_3$ is a lift of $e_1e_2e_3$ at $\hat{v}$.

LEMMA 1.1

Let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ *be an incompressable mapping of 2-complexes. The following are equivalent:*

(I) *For any vertex* $\hat{v}$ *of* $\mathcal{A}$ *and path* α *in* $\mathcal{B}$ *with* $\iota(\alpha)=\varphi(\hat{v})$, *there exists a lift of* α *at* $\hat{v}$.

(II) φ *is locally surjective.*

Proof

(I) $\Rightarrow$ (II). Let $e \in \mathrm{star}(\varphi(\hat{v}))$. Then there exists a lift of e

at $\hat{v}$ (of length one, by incompressability), namely an element of $\mathrm{star}_\varphi(\hat{v})$. Thus φ is locally surjective.

(II) $\Rightarrow$ (I). We argue by induction. If $L(\alpha)=0$ then the result is clearly true. So suppose $L(\alpha)\geq 1$ and write $\alpha=\beta e$ $(e\in E(\mathcal{B}))$. Then by the induction hypothesis there is a path $\hat{\beta}$ in $\mathcal{A}$ such that $\iota(\hat{\beta})=\hat{v}$ and $\varphi(\hat{\beta})=\beta$. Now $\varphi(\tau(\hat{\beta}))=\tau(\beta)$ so by local surjectivity there is an edge $\hat{e}$ in $\mathrm{star}_\varphi(\tau(\hat{\beta}))$ with $\varphi(\hat{e})=e$. So $\hat{\beta}\hat{e}$ is a lift of α at $\hat{v}$, the result follows by induction.□

<u>LEMMA 1.2</u>

Let $\varphi:\mathcal{A}\rightarrow\mathcal{B}$ be a mapping of 2-complexes. Suppose that for any vertex $\hat{v}$ of $\mathcal{A}$ and any path α in $\mathcal{B}$ with $\iota(\alpha)=\varphi(\hat{v})$, there exists at most one lift of α at $\hat{v}$. Then φ is locally injective.

<u>Proof</u>

Let $\hat{e}_1,\hat{e}_2\in\mathrm{star}_\varphi(\hat{v})$ and suppose that $\varphi(\hat{e}_1)=\varphi(\hat{e}_2)=e$, say. Then $\hat{e}_1$ and $\hat{e}_2$ are both lifts of e at $\hat{v}$ so $\hat{e}_1=\hat{e}_2$. Thus φ is locally injective.□

LEMMA 1.3

Let $\varphi: \mathcal{A} \to \mathcal{B}$ be a rigid, locally injective mapping of 2-complexes. Let $\hat{v}$ be a vertex of $\mathcal{A}$ and α a path in $\mathcal{B}$ with $\iota(\alpha)=\varphi(\hat{v})$. Then there exists at most one lift of α at $\hat{v}$.

Proof

We argue by induction on $L(\alpha)$. If $L(\alpha)=0$ the result is obvious. So suppose that $L(\alpha)\geq 1$ and that α has a lift at $\hat{v}$. Write $\alpha=\beta e$ ($e\in E(\mathcal{B})$). Let $\hat{\alpha}_1$ and $\hat{\alpha}_2$ be lifts of α at $\hat{v}$. Then $\hat{\alpha}_1=\hat{\beta}_1\hat{e}_1$ and $\hat{\alpha}_2=\hat{\beta}_2\hat{e}_2$ where $\varphi(\hat{\beta}_1)=\varphi(\hat{\beta}_2)=\beta$ and $\varphi(\hat{e}_1)=\varphi(\hat{e}_2)=e$. Since $\hat{\beta}_1$ and $\hat{\beta}_2$ are both lifts of β at $\hat{v}$, $\hat{\beta}_1=\hat{\beta}_2$ by the induction hypothesis. Let $\hat{u}=\tau(\hat{\beta}_1)$. Then $\hat{e}_1,\hat{e}_2\in\mathrm{star}(\hat{u})$ and thus by rigidity (to guarantee $\hat{e}_1,\hat{e}_2\in\mathrm{star}\varphi(\hat{u})$) and local injectivity we have that $\hat{e}_1=\hat{e}_2$. Thus $\hat{\alpha}_1=\hat{\alpha}_2$ as required. The result now follows by induction.□

Combining the above we have

LEMMA 1.4

Let $\varphi: \mathcal{A} \to \mathcal{B}$ be a rigid, locally bijective mapping of 2-complexes. For any vertex $\hat{v}$ of $\mathcal{A}$ and any path α in $\mathcal{B}$ with $\iota(\alpha)=\varphi(\hat{v})$ there exists a unique lift of α at $\hat{v}$.□

Let $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping of 2-complexes. We say φ is *equivalence preserving* if it satisfies

(1.1) φ is rigid and locally injective.

(1.2) For all $e \in E(\mathcal{B})$, if $\nu \in \varphi^{-1}(ee^{-1})$ then ν is contractible in $\mathcal{A}$.

(1.3) $\varphi^{-1}R(\mathcal{B}) = R(\mathcal{A})$.

<u>Example</u>

Consider

$\mathcal{A} = \langle\, a$  $b\ ;\ ab \rangle$, and

$\mathcal{B} = \langle e$  $;\ ee^{-1} \rangle$.

Define a mapping from $\mathcal{A}$ to $\mathcal{B}$ by $a^{\pm 1} \mapsto e^{\pm 1}$, $b^{\pm 1} \mapsto e^{\mp 1}$. This is equivalence preserving. It also illustrates the fact that (1.2) is not vacuous since $ab \in \varphi^{-1}(ee^{-1})$.

<u>LEMMA 1.5</u>

Let $\varphi\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an equivalence preserving mapping of 2-complexes. Let $\hat{v}$ be a vertex of $\mathcal{A}$ and let α, β be paths in $\mathcal{B}$ with $\iota(\alpha) = \iota(\beta) = \varphi(\hat{v})$. Suppose $\hat{\alpha}$ and $\hat{\beta}$ are lifts of α and β respectively at $\hat{v}$. Then $\hat{\alpha} \sim_{\mathcal{A}} \hat{\beta}$ if and only if $\alpha \sim_{\mathcal{B}} \beta$.

Proof

$\Rightarrow$. To prove this it suffices to deal with the case when $\hat{\beta}$ is obtained from $\hat{\alpha}$ by an elementary reduction. The general case then follows by induction.

Case i $\hat{\alpha}=\hat{\alpha}_1\hat{e}\hat{e}^{-1}\hat{\alpha}_2$ and $\hat{\beta}=\hat{\alpha}_1\hat{\alpha}_2$

$$\begin{aligned} \text{Now } \alpha=\varphi(\hat{\alpha})=\varphi(\hat{\alpha}_1\hat{e}\hat{e}^{-1}\hat{\alpha}_2)&=\varphi(\hat{\alpha}_1)\varphi(\hat{e})\varphi(\hat{e}^{-1})\varphi(\hat{\alpha}_2) \\ &\sim_{\mathcal{B}} \varphi(\hat{\alpha}_1)\varphi(\hat{e})\varphi(\hat{e})^{-1}\varphi(\hat{\alpha}_2) \\ &\overset{(1)}{\sim} \varphi(\hat{\alpha}_1\hat{\alpha}_2)=\varphi(\hat{\beta})=\beta \end{aligned}$$

Case ii $\hat{\alpha}=\hat{\alpha}_1\hat{\gamma}\hat{\alpha}_2$ $(\hat{\gamma}\in R(\hat{\mathcal{A}}))$ and $\hat{\beta}=\hat{\alpha}_1\hat{\alpha}_2$

$$\alpha=\varphi(\hat{\alpha})=\varphi(\hat{\alpha}_1\hat{\gamma}\hat{\alpha}_2)=\varphi(\hat{\alpha}_1)\varphi(\hat{\gamma})\varphi(\hat{\alpha}_2)\sim_{\mathcal{B}} \varphi(\hat{\alpha}_1)\varphi(\hat{\alpha}_2)=\varphi(\hat{\alpha}_1\hat{\alpha}_2)=\varphi(\hat{\beta})=\beta$$

(since $\varphi(\hat{\gamma})$ is contractible in $\hat{\mathcal{A}}$).

$\Leftarrow$. To prove this it suffices to deal with the case when β is obtained from α by an elementary reduction. The general case then follows by induction.

Case i $\alpha=\alpha_1ee^{-1}\alpha_2$ and $\beta=\alpha_1\alpha_2$.

Since φ is rigid we may write $\hat{\alpha}=\hat{\alpha}_1\hat{e}\hat{f}\hat{\alpha}_2$ where $\varphi(\hat{\alpha}_1)=\alpha_1$, $\varphi(\hat{e})=e$, $\varphi(\hat{f})=e^{-1}$ and $\varphi(\hat{\alpha}_2)=\alpha_2$. Now $\hat{e}\hat{f}\in\varphi^{-1}(ee^{-1})$ and so is a contractible path in $\hat{\mathcal{A}}$. Thus

$$\hat{\alpha}\sim_{\hat{\mathcal{A}}} \hat{\alpha}_1\hat{\alpha}_2.$$

Now $\varphi(\hat{\alpha}_1\hat{\alpha}_2)=\beta$, hence by Lemma 1.3 $\hat{\alpha}_1\hat{\alpha}_2=\hat{\beta}$. Thus $\hat{\alpha}\sim_{\mathcal{A}}\hat{\beta}$.

<u>Case ii</u> $\alpha=\alpha_1\gamma\alpha_2$ $(\gamma\epsilon R(\mathcal{B}))$ and $\beta=\alpha_1\alpha_2$.

Since φ is rigid we may write $\hat{\alpha}=\hat{\alpha}_1\hat{\gamma}\hat{\alpha}_2$ where $\varphi(\hat{\alpha}_1)=\alpha_1$, $\varphi(\hat{\gamma})=\gamma$ and $\varphi(\hat{\alpha}_2)=\alpha_2$. Now $\hat{\gamma}\epsilon\varphi^{-1}(\gamma)$ hence $\hat{\gamma}\epsilon R(\mathcal{A})$. Thus

$$\hat{\alpha}\sim_{\mathcal{A}}\hat{\alpha}_1\hat{\alpha}_2.$$

As above $\hat{\alpha}_1\hat{\alpha}_2=\hat{\beta}$. Thus $\hat{\alpha}\sim_{\mathcal{A}}\hat{\beta}$.□

<u>LEMMA 1.6</u>

Let $\varphi:\mathcal{A}\longrightarrow\mathcal{B}$ be an equivalence preserving mapping of 2-complexes. Let $\hat{v}$ be a vertex of $\mathcal{A}$. Then the induced homomorphism

$$\varphi_*:\pi_1(\mathcal{A},\hat{v})\longrightarrow\pi_1(\mathcal{B},\varphi(\hat{v}))$$

is injective.

<u>Proof</u>

Let $[\alpha]_{\mathcal{A}}\epsilon\ker\varphi_*$. Then $\varphi(\hat{\alpha})\sim_{\mathcal{B}}1$. Hence by Lemma 1.5, $\hat{\alpha}\sim_{\mathcal{A}}1$ i.e. $[\hat{\alpha}]_{\mathcal{A}}=[1]_{\mathcal{A}}$. So φ_* is injective.□

<u>LEMMA 1.7</u>

Let $\varphi:\mathcal{A}\longrightarrow\mathcal{B}$ be an equivalence preserving mapping of 2-complexes. Let $\hat{v}$ be a vertex of $\mathcal{A}$. Suppose α is a closed path in $\mathcal{B}$ with $\iota(\alpha)=\varphi(\hat{v})$ and that the lift, $\hat{\alpha}$, of α at $\hat{v}$

exists. Then $\hat{\alpha}$ *is closed if and only if* $[\alpha]_{\mathcal{B}} \in \varphi_* \pi_1(\mathcal{A}, \hat{v})$.

<u>Proof</u>

$\Rightarrow$. Suppose $\hat{\alpha}$ is closed. Then

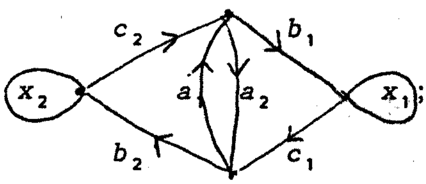
$[\alpha]_{\mathcal{B}} = [\varphi(\hat{\alpha})]_{\mathcal{B}} = \varphi_*[\hat{\alpha}]_{\mathcal{A}} \in \varphi_* \pi_1(\mathcal{A}, \hat{v})$.

$\Leftarrow$. Suppose $[\alpha]_{\mathcal{B}} \in \varphi_* \pi_1(\mathcal{A}, \hat{v})$. Then $[\alpha]_{\mathcal{B}} = [\varphi(\hat{\beta})]_{\mathcal{B}}$ for some closed path $\hat{\beta}$ at $\hat{v}$. Thus $\alpha \sim_{\mathcal{B}} \varphi(\hat{\beta})$. So $\hat{\alpha} \sim_{\mathcal{A}} \hat{\beta}$ by Lemma 1.5. Hence in particular $\tau(\hat{\alpha}) = \tau(\hat{\beta}) (= \hat{v})$ i.e. $\hat{\alpha}$ is closed.□

Suppose $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ is a locally surjective, pure, equivalence preserving mapping, between two connected 2-complexes. Then φ is called a *covering*.

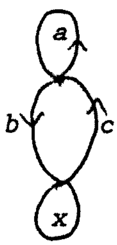
<u>Example</u>

Let

$\mathcal{A}$ = 

; $(x_1b_1c_2x_2b_2c_1)^2$, $(a_1a_2c_1x_1b_1)^2$
$(a_2a_1c_2x_2b_2)^2$

and

$\mathcal{B}$ = 

; $(xbc)^4$, $(a^2cxb)^2$

If we define a mapping φ from $\mathcal{A}$ to $\mathcal{B}$ by

$a_1^{\pm 1}, a_2^{\pm 1} \mapsto a^{\pm 1}$, $b_1^{\pm 1}, b_2^{\pm 1} \mapsto b^{\pm 1}$, $c_1^{\pm 1}, c_2^{\pm 1} \mapsto c^{\pm 1}$, $x_1, x_2 \mapsto x$.

Then φ is a covering.

Remark: (1) If $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ is a covering and $\mathcal{B}$ has no involutary edges then the same is true of $\mathcal{A}$.

(2) We emphasise again, because of its central importance, that if φ is a covering then φ_* is a monomorphism.

<u>THEOREM 1.1</u>

Let $\mathcal{A}$ be a connected 2-complex. Let v be a vertex of $\mathcal{A}$ and let H be a subgroup of $\pi_1(\mathcal{A},v)$. Then there is a covering $\varphi_H: \mathcal{A}_H \longrightarrow \mathcal{A}$ and a vertex v_H of $\mathcal{A}_H$ such that $\varphi_{H*}\pi_1(\mathcal{A}_H, v_H)=H$.

<u>Proof</u>

Let $\mathcal{A}=<\mathcal{X};\rho_\lambda\ (\lambda\epsilon\Lambda)>$ and $X=\{[\alpha];\iota(\alpha)=v\}$.

We say that two elements $[\alpha]$ and $[\beta]$ of X are *equivalent mod H* if $\tau(\alpha)=\tau(\beta)$ and $[\alpha\beta^{-1}]\epsilon H$. The equivalence class containing $[\alpha]$ is $\{[\gamma][\alpha]:[\gamma]\epsilon H\}$. We denote this by $H[\alpha]$.

Define the 1-skeleton of $\mathcal{A}_H$ as follows.

Vertices: $H[\alpha]$ $([\alpha]\epsilon X)$.

Edges : $(H[\alpha],e)$ $(e\epsilon E(\mathcal{A}),\ [\alpha]\epsilon X$ and $\tau(\alpha)=\iota(e))$.

For an edge $(H[\alpha],e)$ we set

$\iota((H[\alpha],e))=H[\alpha]$, $\tau((H[\alpha],e))=H[\alpha e]$ and $(H[\alpha],e)^{-1}=(H[\alpha e],e^{-1})$.

We take v_H to be the vertex $H[1_v]$.

For a defining path $\rho_\lambda = e_1 e_2 \ldots e_n$ of $\mathcal{A}$ and a vertex $H[\alpha]$ of $\mathcal{A}_H$ with $\tau(\alpha) = \iota(\rho_\lambda)$ let

$$\rho(\lambda, H[\alpha]) = (H[\alpha], e_1)(H[\alpha e_1], e_2) \ldots (H[\alpha e_1 e_2 \ldots e_{n-1}], e_n)$$

We note that this is a closed path in $\mathcal{A}_H$.

The defining paths of $\mathcal{A}_H$ are then all of the $\rho(\lambda, H[\alpha])$ ($\lambda \in \Lambda$, and $[\alpha] \in X$ such that $\tau(\alpha) = \iota(\rho_\lambda)$).

φ_H is defined as follows:

$$\varphi_H(H[\alpha]) = \tau(\alpha) \quad (H[\alpha] \text{ a vertex of } \mathcal{A}_H),$$

$$\varphi_H((H[\alpha], e) = e \quad ((H[\alpha], e) \text{ an edge of } \mathcal{A}_H)$$

We now show that φ_H is locally surjective. Let u be a vertex of $\mathcal{A}$ and let α be a path in $\mathcal{A}$ from v to u, so $H[\alpha]$ lies over u. Let $e \in star(u)$. Then $(H[\alpha], e) \mapsto e$ and $(H[\alpha], e) \in star(H[\alpha])$. Thus φ_H is locally surjective.

Clearly φ is pure. We now show φ_H is equivalence preserving. I.e. we verify (1.1), (1.2), and (1.3). Firstly, φ_H is clearly rigid and locally injective; secondly the elements of $\varphi^{-1}(ee^{-1})$ are of the form $(H[\alpha], e)(H[\alpha e], e^{-1}) = (H[\alpha], e)(H[\alpha], e)^{-1}$ which is (freely) contractible in $\mathcal{A}_H$; and thirdly, $\varphi^{-1} R(\mathcal{A}) = R(\mathcal{A}_H)$ by construction.

Now we show that $\mathcal{A}_H$ is connected. Let $H[\alpha]$ be a vertex of $\mathcal{A}_H$ with $\alpha = e_1 e_2 \ldots e_n$. Then

$$(H[1_v], e_1)(H[e_1], e_2)\ldots(H[e_1 \ldots e_{n-1}], e_n)$$

is a path in $\mathcal{A}_H$ from $H[1_v]$ to $H[\alpha]$. Thus $\mathcal{A}_H$ is connected.

Hence $\varphi_H : \mathcal{A}_H \longrightarrow \mathcal{A}$ is a covering.

Finally we show that $\varphi_{H*}\pi_1(\mathcal{A}_H, H[1_v]) = H$. Let α be a closed loop at v. Then by construction of $\mathcal{A}_H$ and Lemma 1.7 $[\alpha] \in H$ if and only if there exists a closed lift $\hat{\alpha}$ of α at $H[1_v]$ if and only if $[\alpha] \in \varphi_{H*}\pi_1(\mathcal{A}, \hat{v})$.$\square$

Remark: Since φ_{H*} is a monomorphism, if we are only interested in the group theoretical structure of H, we need only consider $\pi_1(\mathcal{A}_H, H[1_v])$ as this is isomorphic to H.

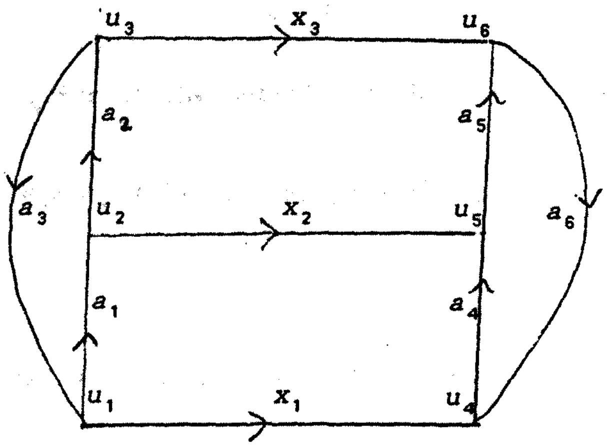
<u>Examples</u>

Let $\mathcal{A} = <$ x  a ; $a^3xa^{-3}x$ $>$

(1) Consider the homomorphism of $\pi_1(\mathcal{A}, v)$ onto $Z_3 \times Z_2$ defined by $a \mapsto (1,0)$, $x \mapsto (0,1)$. Let H be the kernel of this homomorphism. A transversal for H in $\pi_1(\mathcal{A}, v)$ is $[1], [a], [a^2], [x], [ax], [a^2x]$, thus $\mathcal{A}_H$ has vertices

u_1=H[1], u_2=H[a], u_3=H[a^2], u_4=H[x], u_5=H[ax], u_6=H[a^2x], and the edges are a_1=(H[1],a), a_2=(H[a],a), a_3=(H[a^2],a), a_4=(H[x],a) a_5=(H[ax],a), a_6=(H[a^2x],a), x_1=(H[1],x), x_2=(H[a],x), x_3=(H[a^2],x)

Then $\mathcal{A}_H$ has 1-skeleton



The lifts of the defining path $a^3xa^{-3}x$ are

$a_1a_2a_3x_1a_6^{-1}a_5^{-1}a_4^{-1}x_1^{-1}$, $a_2a_3a_1x_2a_4^{-1}a_6^{-1}a_5^{-1}x_2^{-1}$,

$a_3a_1a_2x_3a_5^{-1}a_4^{-1}a_6^{-1}x_3^{-1}$, $a_4a_5a_6x_1^{-1}a_3^{-1}a_2^{-1}a_1^{-1}x_1$,

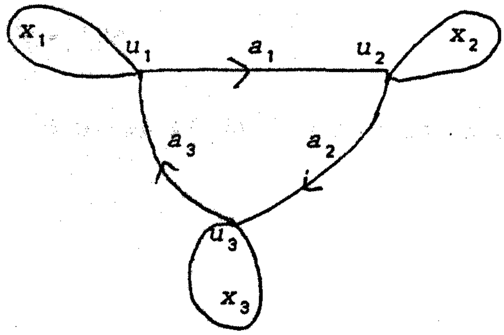
$a_5a_6a_4x_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}x_2$, $a_6a_4a_5x_3^{-1}a_2^{-1}a_1^{-1}a_3^{-1}x_3$.

(2) Let $\mathcal{A}$ be as above and consider the homomorphism of $\pi_1(\mathcal{A},v)$ onto Z_3(={0,1,2}) defined by $a \mapsto 1$, $x \mapsto 0$. Let H be the kernel of this homomorphism. A transversal for H in $\pi_1(\mathcal{A},v)$ is [1],[a],[a^2]. Thus $\mathcal{A}_H$ has vertices

u_1=H[1], u_2=H[a], u_3=H[a^2] and the edges are

a_1=(H[1],a), a_2=(H[a],a), a_3=(H[a^2],a), x_1=(H[1],x)

$x_2=(H[a],x)$, $x_3=(H[a^2],x)$. Then $\mathcal{A}_H$ has 1-skeleton



The lifts of the defining path $a^3xa^{-3}x$ are

$a_1a_2a_3x_1a_3^{-1}a_2^{-1}a_1^{-1}x_1$, $a_2a_3a_1x_2a_1^{-1}a_3^{-1}a_2^{-1}x_2$, $a_3a_1a_2x_3a_2^{-1}a_1^{-1}a_3^{-1}x_3$.

Let X be a class of connected complexes that is closed under taking coverings (i.e. if $\mathcal{A}$ is an element of X and if $\varphi: \mathcal{A}' \to \mathcal{A}$ is a covering then $\mathcal{A}'$ is an element of X). Call a group an X-*group* if it is isomorphic to the fundamental group of an element of X. Call a group an X_f-*group* (respectively, X_i-*group*) if it is isomorphic to the fundamental group of a finite (respectively, infinite) element of X

Using Theorem 1.1, we then have the following simple but useful result.

LEMMA 1.8 (THE SUBGROUP LEMMA)

Let X be as above. Then

(I) A subgroup of finite index in an X_f-group is an X_f-group.

(II) A subgroup of infinite index in an X_f-group is an X_i-group.

(III) A subgroup of an X_i-group is an X_i-group.$\square$

1.2 EQUIVALENCES AND TIETZE TRANSFORMATIONS

If $\mathcal{A}$ and $\mathcal{B}$ are 2-complexes a mapping

$$\varphi: \mathcal{A} \rightarrow \mathcal{B} ,$$

is called an *equivalence* if there is a mapping

$$\theta: \mathcal{B} \rightarrow \mathcal{A} ,$$

such that

(1.4) $$\theta\varphi(\alpha) \sim_{\mathcal{A}} \alpha$$

for each path α in $\mathcal{A}$, and ,

(1.5) $$\varphi\theta(\beta) \sim_{\mathcal{B}} \beta$$

for each path β in $\mathcal{B}$. We say that the equivalence θ is *inverse* to the equivalence φ. Two 2-complexes are said to be *equivalent* if there is an equivalence between them. It is easily checked that being equivalent is an equivalence relation.

Since $\theta\varphi(1_v) = 1_{\theta\varphi(v)} \sim_{\mathcal{A}} 1_v$ we have $\theta\varphi(v) = v$ $(v \in V(\mathcal{A}))$, similarly $\varphi\theta(u) = u$ $(u \in V(\mathcal{B}))$. Hence the restriction of φ to the vertices of $\mathcal{A}$ is a bijection from the vertices of $\mathcal{A}$ to those of $\mathcal{B}$.

The notion of equivalence is related to "Tietze transformations", as we now explain.

Let $\mathcal{A} = \langle \mathcal{X}; \rho_\lambda\ (\lambda\in\Lambda)\rangle$.

Suppose V' is a set in 1:1 correspondance with $V(\mathcal{A})$, and let $\sigma: V(\mathcal{A}) \to V'$ be a specific bijection. Let $\mathcal{X}'$ be the 1-complex with vertex set V', edge set $E(\mathcal{A})$, and functions ι', τ' and $^{-1'}$ define by

$$\iota'(e) = \sigma\iota(e),\ \tau'(e) = \sigma\tau(e) \text{ and } e^{-1'} = e^{-1}\ (e\in E(\mathcal{A})).$$

Let $\mathcal{A}' = \langle \mathcal{X}'; \rho_\lambda\ (\lambda\in\Lambda)\rangle$. We have an equivalence from $\mathcal{A}$ to $\mathcal{A}'$ given by

$$v \mapsto \sigma(v)\ (v\in V(\mathcal{A})) \text{ and } e \mapsto e\ (e\in E(\mathcal{A})).$$

We say that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a *Tietze transformation* (T0).

Next, let ξ_i $(i\in I)$ be a collection of contractible paths in $\mathcal{A}$, and let $\mathcal{B} = \langle \mathcal{X};\ \rho_\lambda\ (\lambda\in\Lambda),\ \xi_i\ (i\in I)\ \rangle$. The identity on $\mathcal{X}$ induces an equivalence from $\mathcal{A}$ to $\mathcal{B}$. We say that $\mathcal{B}$ is obtained from $\mathcal{A}$ by a *Tietze transformation* (T1). The transformation is said to be *elementary* if $|I| = 1$.

Finally, suppose $\mathcal{Y}$ is a 1-complex obtained from $\mathcal{X}$ by adjoining additional edges f_j, f_j^{-1} $(j\in J)$. Suppose that for $j\in J$

there is a path γ_j <u>in</u> $\mathcal{A}$ from $\iota(f_j)$ to $\tau(f_j)$ with γ_j^2 contractible in $\mathcal{A}$ if $f_j = f_j^{-1}$. Let

$$\mathcal{B} = \langle \mathcal{Y}; \rho_\lambda \ (\lambda \in \Lambda),\ f_j^{-1}\gamma_j \ (j \in J) \rangle.$$

The inclusion of $\mathcal{X}$ in $\mathcal{Y}$ induces an equivalence from $\mathcal{A}$ to $\mathcal{B}$ (with inverse equivalence given by

$$v \mapsto v \ (v \in V(\mathcal{A})),\ e \mapsto e \ (e \in E(\mathcal{A})) \text{ and } f_j \mapsto \gamma_j \ (j \in J)).$$

We say that $\mathcal{B}$ is obtained from $\mathcal{A}$ by a *Tietze transformation* (T2). The transformation is said to be elementary if $|J|=1$.

<u>THEOREM 1.2</u>

Two 2-complexes $\mathcal{A}$, $\mathcal{A}'$ *are equivalent if and only if there is a finite sequence of 2-complexes*

$$\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n = \mathcal{A}'$$

where for $i=0,1,\ldots,n-1$, *one of* $\mathcal{A}_i, \mathcal{A}_{i+1}$ *is obtained from the other by a Tietze transformation* (T0),(T1) *or* (T2). *If* $\mathcal{A}$, $\mathcal{A}'$ *are finite then all* (T1) *and* (T2) *transformations can be taken to be elementary.*

<u>Proof</u>

The "if" part follows from the above discussion. We now prove the "only if" part. Let

$$\mathcal{A}=\langle\mathcal{X};\ \alpha_\lambda\ (\lambda\epsilon\Lambda)\rangle \text{ and}$$

$$\mathcal{A}'=\langle\mathcal{Y};\ \beta_\mu\ (\mu\epsilon M)\rangle,$$

and $\varphi:\mathcal{A}\rightarrow\mathcal{A}'$ be an equivalence with inverse θ.

Firstly we show that we may assume that

(1.6) $E(\mathcal{A})\cap E(\mathcal{A}')=\emptyset$

Let $e_i^{\pm1}$ $(i\epsilon I)$ be the non-involutary edges of $\mathcal{A}$ and e_j $(j\epsilon J,\ J\cap I=\emptyset)$ be the involutary edges of $\mathcal{A}$. Let $\mathcal{X}_1$ be obtained from $\mathcal{X}$ by adjoining new non-involutary edges

$$f_i^{\pm1}\ \ (\iota(f_i)=\iota(e_i),\tau(f_i)=\tau(e_i)\ \ (i\epsilon I))$$

and new involutary edges

$$f_j\ \ (\ \iota(f_j)=\iota(e_j)\ \ (j\epsilon J))$$

where the f's are chosen so that

$$\{f_k^{\pm1}:\ k\epsilon I\cup J\}\cap E(\mathcal{A}')=\emptyset.$$

Let $\mathcal{A}_1=\langle\mathcal{X}_1;\ \alpha_\lambda\ (\lambda\epsilon\Lambda),\ f_k^{-1}e_k\ (k\epsilon I\cup J)\rangle$. So $\mathcal{A}_1$ is obtained from $\mathcal{A}$ by a (T2) transformation. Clearly $e_k^{-1}f_k$ $(k\epsilon I\cup J)$ is contractible in $\mathcal{A}_1$. Also, if we let α_λ' be that word obtained from α_λ by replacing any occurence of e_k^{ϵ} $(k\epsilon I\cup J)$ by f_k^{ϵ} in it, the collection α_λ' $(\lambda\epsilon\Lambda)$ is also contractible in $\mathcal{A}_1$.

Let $\mathcal{A}_2=\langle\mathcal{X}_1;\ \alpha_\lambda,\alpha_\lambda'\ (\lambda\epsilon\Lambda),\ f_k^{-1}e_k,\ e_k^{-1}f_k\ (k\epsilon I\cup J)\rangle$. Then $\mathcal{A}_2$ is obtained from $\mathcal{A}_1$ by a (T1) transformation. By symmetry if

$$\mathcal{A}_3 = \langle \mathfrak{X}_1;\ \alpha'_\lambda\ (\lambda\epsilon\Lambda),\ e_k^{-1} f_k\ (k\epsilon(I\cup J)\rangle,$$

then $\mathcal{A}_3$ is obtained from $\mathcal{A}_2$ by a transformation inverse to (T1). If we let $\mathfrak{X}_2$ be obtained from $\mathfrak{X}_1$ by deleting the e_k's and their inverses and if we let

$$\mathcal{A}_4 = \langle \mathfrak{X}_2;\ \alpha'_\lambda\ (\lambda\epsilon\Lambda)\rangle,$$

we find (again by symmetry) that $\mathcal{A}_4$ is obtained from $\mathcal{A}_3$ by a transformation inverse to (T2). Thus we may assume, without loss of generality that (1.6) holds.

Secondly we show that we may assume that

(1.7) $V(\mathcal{A}) = V(\mathcal{A}')$ and φ maps $V(\mathcal{A})$ identically onto $V(\mathcal{A}')$

We take the restriction of φ to $V(\mathcal{A})$ as the σ in the definition of the Tietze transformation (T0), to obtain a new 2-complex $\mathcal{A}_1$ equivalent to $\mathcal{A}$ with vertex set $V(\mathcal{A}')$.

So we now assume that (1.6) and (1.7) hold.

Let $e_i^{\pm 1}$ $(i\epsilon I)$ (respectively $f_\rho^{\pm 1}$ $(\rho\epsilon P)$) be the non-involutary edges of $\mathcal{A}$ (respectively $\mathcal{A}'$) and e_j $(j\epsilon J)$ (respectively f_η $(\eta\epsilon H)$) be the involutary edges of $\mathcal{A}$ (respectively $\mathcal{A}'$).

Let $\mathcal{Y}'$ be obtained from $\mathcal{Y}$ by adjoining $e_k^{\pm 1}$ $(k\epsilon IUJ)$ to it. Let $\mathcal{A}_1' = \langle \mathcal{Y}';\ \beta_\mu\ (\mu\epsilon M),\ e_k^{-1}\varphi(e_k)\ (k\epsilon IUJ)\rangle$. (We note that $\varphi(e_k)^2 = \varphi(e_k^2)$, so if e_k is involutary $\varphi(e_k)^2$ is contractible in $\mathcal{A}'$).

Now $\mathcal{A}_1'$ is obtained from $\mathcal{A}'$ by a (T2) transformation. Now since $e_k \sim_{\mathcal{A}_1'} \varphi(e_k)$ $(k\epsilon IUJ)$ we have $W \sim_{\mathcal{A}_1'} \varphi(W)$ for all paths W in $\mathcal{A}_1'$, in particular

$$\alpha_\lambda \sim_{\mathcal{A}_1'} \varphi(\alpha_\lambda) \sim_{\mathcal{A}_1'} 1\ (\lambda\epsilon\Lambda).$$

Also we have $\theta(f_\sigma) \sim_{\mathcal{A}_1'} \varphi(\theta(f_\sigma)) \sim_{\mathcal{A}_1'} f_\sigma$ $(\sigma\epsilon PUH)$. So $f_\sigma^{-1}\theta(f_\sigma)$ is contractible in $\mathcal{A}_1'$ $(\sigma\epsilon(PUH))$. Let

$\mathcal{A}_2' = \langle \mathcal{Y}';\alpha_\lambda\ (\lambda\epsilon\Lambda), \beta_\mu\ (\mu\epsilon M), e_k^{-1}\varphi(e_k)\ (k\epsilon IUJ), f_\sigma^{-1}\theta(f_\sigma)\ (\sigma\epsilon PUH)\rangle$.

Then $\mathcal{A}_2'$ is obtained from $\mathcal{A}_1'$ by a (T1) transformation.

Now, by symmetry, $\mathcal{A}_2'$ can be obtained from $\mathcal{A}$ by a similar sequence of Tietze transformations. The result follows.□

There is also the notion of based equivalences. We say that the based mapping $\varphi:(\mathcal{A},u) \longrightarrow (\mathcal{B},v)$ is a *based equivalence* if there is a based mapping $\theta:(\mathcal{B},v) \longrightarrow (\mathcal{A},u)$ such that (1.4) and (1.5) hold for all paths α and β with $\iota(\alpha) = \tau(\alpha) = u$, $\iota(\beta) = \tau(\beta) = v$. Obviously an equivalence $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$

gives rise to a based equivalence for any choice of u and $v=\varphi(u)$. If $\varphi:(\mathcal{A},u) \longrightarrow (\mathcal{B},v)$ is a based equivalence then

$$\varphi_*:\pi_1(\mathcal{A},u) \longrightarrow \pi_1(\mathcal{B},v)$$

is an isomorphism, with inverse θ_*.

Given any connected 2-complex $\mathcal{A}$ and vertex u, there is a based mapping from $(\mathcal{A},u)$ to a presentation. The method of obtaining such a presentation is called *collapsing a maximal subtree*, which we now describe.

Let $\mathcal{T}$ be a maximal subtree of $\mathcal{A}^{(1)}$, and let $f_i^{\pm 1}$ $(i\epsilon I)$ be the edges of $\mathcal{A}$ lying outside $\mathcal{T}$. Let $\mathcal{W}$ be the bouquet with vertex v and edges $g_i^{\pm 1}$ $(i\epsilon I)$. Where $g_i=g_i^{-1}$ if and only if $f_i=f_i^{-1}$ $(i\epsilon I)$. Define,

$$\varphi:\mathcal{A}^{(1)}\longrightarrow \mathcal{W} \text{ by}$$

$$\varphi(e)=\begin{cases} 1 & e\epsilon\,\mathcal{T} \\ g_i^{\epsilon} & e=f_i^{\epsilon}\ (i\epsilon I,\ \epsilon=\pm 1). \end{cases}$$

Let $\mathcal{P}=\langle\mathcal{W},\ \varphi(\alpha_\lambda)\ (\lambda\epsilon\Lambda)\rangle$. Then

$$\varphi:(\mathcal{A},u) \longrightarrow (\mathcal{P},v),$$

is a based mapping. We show that it is a based equivalence, by exhibiting an inverse, θ, for it. For $i\epsilon I$ let p_i (respectively q_i) be the geodesic in τ from u to $\iota(f_i)$ (respectively $\tau(f_i)$).

Define

$$\theta:\mathcal{P} \to \mathcal{A} \text{ by}$$

$$\theta(e)=(p_i f_i q_i^{-1})^{\epsilon} \quad e=g_i^{\epsilon} \ (i\epsilon I,\ \epsilon=\pm 1).$$

Then

$\theta:(\mathcal{P},v) \longrightarrow (\mathcal{A},u)$ is a based mapping. Clearly $\varphi\theta=\mathrm{Id}_{\mathcal{P}}$,

and for all closed paths α in $\mathcal{A}$ starting at u

$$\theta\varphi(\alpha)\sim_{\mathcal{A}}\alpha.$$

To see this let α be such a path. Write

$$\alpha=A_0 e_1 A_1 \ldots e_n A_n,$$

where e_i is an edge lying outside $\mathcal{T}$ $(1\leq i\leq n)$ and A_i is a path in $\mathcal{T}$ $(0\leq i\leq n)$. Then

$$\theta\varphi(\alpha)=p_{i_1}e_1 q_{i_1}^{-1}\cdots p_{i_n}e_n q_{i_n}^{-1}$$

Now $p_{i_1}\overset{(1)}{\sim}A_0$, $q_{i_n}^{-1}\overset{(1)}{\sim}A_n$ and $q_{i_j}^{-1}p_{i_{j+1}}\overset{(1)}{\sim}A_j$ $(1\leq j<n)$.

Hence $\theta\varphi(\alpha)\sim_{\mathcal{A}}\alpha$. Thus θ is inverse to φ, and so φ is a based equivalence.

1.3 THE LEVEL METHOD

Let

$$\mathcal{A} = \langle \mathcal{X};\ \rho_\lambda\ (\lambda \in \Lambda) \rangle.$$

Suppose that Λ is written as a disjoint union of subsets (some of which may be empty):

$$\Lambda = \bigcup_{m=0}^{\infty} \Lambda_m.$$

An element of Λ_m will be said to be of *level* m. We assume that if λ has level at least one, then some cyclic permutation of ρ_λ has the form $e_\lambda \alpha_\lambda^{-1}$, where e_λ is an edge, $L_{e_\lambda}(\alpha_\lambda)=0$, and $L_{e_\lambda}(\rho_\mu)=0$ ($\mu \neq \lambda$, with μ of level k, $0 \leq k \leq m$). We call e_λ the *edge associated with* λ.

Let

$$\mathcal{X}_0 = \mathcal{X} - \{e_\lambda, e_\lambda^{-1} : \lambda \text{ has level greater then } 0\},$$

and for $m > 0$ let

$$\mathcal{X}_m = \mathcal{X}_{m-1} \cup \{e_\lambda, e_\lambda^{-1} : \lambda \in \Lambda_m\}.$$

Note that if λ has level $m \geq 1$, then α_λ is a path in $\mathcal{X}_{m-1}$. For suppose not. Then there is some edge in α_λ that is $e_\mu^{\pm 1}$ for some $\mu \in \Lambda_k, k \geq m$. Since $\mu \neq \lambda$, this edge must contradict our assumptions about the e_λ's.

Define $\varphi: \mathfrak{X} \longrightarrow \mathfrak{X}_0$ as follows. First define φ on $\mathfrak{X}_0$ to be the identity. Suppose that φ has been defined on $\mathfrak{X}_{m-1}$ $(m\geq 0)$. Extend φ to $\mathfrak{X}_m$ by setting

$$\varphi(e_\lambda)=\varphi(\alpha_\lambda) \quad (\lambda\in\Lambda_m)$$

$$\varphi(e_\lambda^{-1})=\varphi(\alpha_\lambda^{-1}) \quad (\lambda\in\Lambda_m,\ e_\lambda^{-1}\neq e_\lambda).$$

Let $\mathcal{A}_0=\langle \mathfrak{X}_0;\ \varphi(\rho_\lambda)\ (\lambda\in\Lambda_0),\ \varphi(e_\lambda)^2\ (\lambda\in\Lambda_m,\ m\geq 0,\ e_\lambda^{-1}=e_\lambda)\rangle$.

We now show that $\mathcal{A}_0$ and $\mathcal{A}$ are equivalent. First note the following three results.

(i) For any path α in $\mathcal{A}$, $\varphi(\alpha^{-1})\sim_{\mathcal{A}_0} \varphi(\alpha)^{-1}$. We prove this inductively. Certainly this is true if α is a path in $\mathcal{A}_0$ for $\varphi(\alpha)=\alpha$ and $\varphi(\alpha^{-1})=\alpha^{-1}$. So suppose inductively that the result is true for all paths in $\mathfrak{X}_{m-1}$, and let α be a path in $\mathfrak{X}_m$. Suppose $\alpha=\alpha_0 e_1^{\epsilon_1}\alpha_1 e_2^{\epsilon_2}..e_{n-1}^{\epsilon_{n-1}}\alpha_n$, where e_i $(i=1,\dots,n)$ is associated with an index λ_i of level m, and no edge involved in any α_i is associated with an index of level m.

If e_i is non-involutary then

$$\begin{aligned}\varphi(e_i^{-1})&=\varphi(\alpha_{\lambda_i}^{-1})\\ &\sim\varphi(\alpha_{\lambda_i})^{-1} \text{ (by inductive hypothesis)}\\ &=\varphi(e_i)^{-1}.\end{aligned}$$

If e_i is involutary then since $\varphi(e_i)^2$ is a defining path of $\mathcal{A}_0$ we have

$$\varphi(e_i^{-1})\varphi(e_i)\sim 1$$

so $\varphi(e_i^{-1})\sim\varphi(e_i)^{-1}$. We now have that

$$\varphi(\alpha^{-1})=\varphi(\alpha_n^{-1})\varphi(e_n^{-\epsilon_n})\ldots\varphi(\alpha_1^{-1})\varphi(e_1^{-\epsilon_1})\varphi(\alpha_0^{-1})$$

$$\sim\varphi(\alpha_n)^{-1}\varphi(e_n^{\epsilon_n})^{-1}\ldots\varphi(\alpha_1)^{-1}\varphi(e_1^{\epsilon_1})^{-1}\varphi(\alpha_0)^{-1}$$

(by the above remarks, and the inductive hypothesis)

$$=\varphi(\alpha)^{-1}.$$

Thus (i) holds.

(ii) $\varphi(\rho_\mu)\sim_{\mathcal{A}_0} 1$ (μ of level greater than 0).

Let $\rho_\mu=\alpha e_\mu\beta$, then $\alpha_\mu^{-1}=\beta\alpha$.

$$\begin{aligned}\varphi(\rho_\mu)=\varphi(\alpha e_\mu\beta)&=\varphi(\alpha)\varphi(e_\mu)\varphi(\beta)\\&=\varphi(\alpha)\varphi(\alpha_\mu)\varphi(\beta)\\&\sim\varphi(\alpha)\varphi(\alpha_\mu)\varphi(\beta)\varphi(\alpha)\varphi(\alpha)^{-1}\\&=\varphi(\alpha)\varphi(\alpha_\mu)\varphi(\alpha_\mu^{-1})\varphi(\alpha)^{-1}\\&\sim\varphi(\alpha)\varphi(\alpha_\mu)\varphi(\alpha_\mu)^{-1}\varphi(\alpha)^{-1}\\&\sim 1.\end{aligned}$$

(iii) For any path α in $\mathcal{A}$, $\varphi(\alpha)\sim_{\mathcal{A}}\alpha$.

Certainly this is true if α is a path in $\mathscr{X}_0$, for $\varphi(\alpha)=\alpha$ there. So suppose inductively that the result is true for all paths in $\mathcal{X}_{m-1}$ ($m\geq 0$), and let α be a path in $\mathcal{X}_m$, but not in $\mathcal{X}_{m-1}$. Let α' be the path obtained from α by replacing any edge e_λ by α_λ, for λ of level m (and also replacing e_λ^{-1} by α_λ^{-1} if $e_\lambda^{-1}\neq e_\lambda$). Since $e_\lambda \sim_{\mathcal{A}} \alpha_\lambda$, $\alpha' \sim_{\mathcal{A}} \alpha$. By definition of φ, $\varphi(\alpha')=\varphi(\alpha)$, and by induction $\varphi(\alpha')\sim_{\mathcal{A}} \alpha'$. Hence $\varphi(\alpha)=\varphi(\alpha')\sim_{\mathcal{A}} \alpha$, as required.

By (i) and (ii) above, φ induces a mapping of 2-complexes (also denoted by φ) from $\mathcal{A}$ to $\mathcal{A}_0$. By (iii) the inclusion of $\mathcal{X}_0$ in $\mathcal{X}$ induces a mapping θ from $\mathcal{A}_0$ to $\mathcal{A}$. Clearly $\varphi\theta=\mathrm{Id}_{\mathcal{A}_0}$. By (iii) $\theta\varphi(\alpha)\sim_{\mathcal{A}} \alpha$ for every path α in $\mathcal{A}$. Thus φ is an equivalence, with inverse θ.

1.4 AN APPLICATION OF THE LEVEL METHOD TO "QUADRATIC-LIKE" COMPLEXES

We begin with some terminology. If $\mathfrak{X}$ is a 1-complex and β is a path in $\mathfrak{X}$ then we let $E(\beta)$ denote the set of edges occuring in β and β^{-1}.

Suppose we have a collection β_i $(i\epsilon I)$ of closed paths in $\mathfrak{X}$. We define the *connectivity graph* of this collection as follows: the vertex set is I; and $\{i,j\}$ is an edge if and only if $E(\beta_i)\cap E(\beta_j)$ is non-empty. A *label* of an edge $\{i,j\}$ is a choice of element $e\epsilon E(\beta_i)\cap E(\beta_j)$. A label, e, is said to be *quadratic* if e is non-involutary, $L_e(\overset{0}{\beta}_i)=L_e(\overset{0}{\beta}_j)=1$, and $L_e(\beta_k)=0$ for $k\neq i,j$. A subgraph in which each edge has a quadratic label is said to be *quadratically labelled*.

The connectivity graph, $CG(\mathcal{A})$, of a 2-complex $\mathcal{A}$ is defined to be the connectivity graph of its collection of defining paths.

Throughout the remainder of this section, let

$$\mathcal{A}=<\mathfrak{X};\ \rho_\lambda\ (\lambda\epsilon\Lambda)>.$$

For convenience we will denote the period of ρ_λ by $p(\lambda)$

(rather than $p(\rho_\lambda)$).

<u>THEOREM 1.3</u>

Let Λ^* *be a subset of* Λ, *and suppose that the full subgraph of* $CG(\mathcal{A})$ *on* Λ^* *has a spanning subforest* F *which is quadratically labelled. Assume that the following condition holds:*

If T *is a connected component of* F *which is finite,*

(1.8) *then there is a vertex* o *of* T, *and an edge* e *of* $\mathcal{A}$,

such that $L_e(\mathring{\rho}_o)=1$ *and* $L_e(\rho_\lambda)=0$ $(\lambda\in\Lambda-\{o\})$.

Then $\mathcal{A}$ *is equivalent to a 2-complex*

(1.9) $$\langle \mathcal{Y};\ t_\lambda^{p(\lambda)}\ (\lambda\in\Lambda^*),\ \rho_\lambda\ (\lambda\in\Lambda-\Lambda^*)\rangle$$

where the t's are non-involutary edges of $\mathcal{Y}-\mathcal{X}$ *and* $t_\lambda\neq t_\mu^{\pm1}$ *for* $\lambda\neq\mu$.

<u>Proof</u>

For notational convenience we will carry out the proof for the case when F consists of a single tree T. If T is finite then we take o to be a vertex as in (1.8). If T is infinite we take o to be any vertex of T.

Let $\mathring{\mathcal{X}}$ be the 1-complex obtained from $\mathcal{X}$ by adding new non-involutary edges $t_\lambda^{\pm 1}$ $(\lambda \epsilon \Lambda^*)$, where $\iota(t_\lambda) = \tau(t_\lambda) = \iota(\rho_\lambda)$. Let

$$\overset{0}{\mathcal{G}} = \langle \overset{0}{\mathcal{X}} \, ; \; t_\lambda \overset{0}{\rho}_\lambda, \; t_\lambda^{p(\lambda)} (\lambda \epsilon \Lambda^*), \; \rho_\lambda \; (\lambda \epsilon \Lambda - \Lambda^*) \rangle$$

For the purposes of constructing CG($\overset{0}{\mathcal{G}}$), the subscript of $t_\lambda \overset{0}{\rho}_\lambda$ will be taken to be λ and that of $t_\lambda^{p(\lambda)}$ will be taken to be λ'. We then consider CG($\mathcal{G}$) to be a subgraph of CG($\overset{0}{\mathcal{G}}$) in the obvious way.

Now there is a subtree T_∞ of T such that T_∞ either has no extremal vertices, or has just one extremal vertex, namely o, and such that removing the edges of T_∞ from T gives a forest of finite trees T_i $(i \epsilon I)$. If T is finite then T_∞ consists of the single vertex o, and I is a singleton. To see that T_∞ exists when T is infinite, note that since each vertex of T has finite valence, by Konigs' infinity lemma, [47,p.79], there is an infinite reduced path in T begining at o. Let T_∞ be the union of all such infinite paths. Clearly at most o is an extremal vertex of T_∞. Suppose if possible that removing the edges of T_∞ from did not leave a forest of finite trees. Then there would be an infinite tree, T′ say , joined to T_∞ at a vertex v of T , containing no edges of T_∞. Again by Konigs' infinity lemma there is an infinite reduced path γ in T' begining at v, but now the concatenation of the geodesic from o to v with γ is an infinite reduced path in T not in T_∞ - a contradiction. Hence all the trees are finite.

Each of the finite trees T_i has a unique vertex λ_i in T_∞. We let d_i be the maximum of the lengths of the geodesics in T_i starting at λ_i.

We now partition the set

$$\Theta=\{\lambda,\lambda': \lambda\epsilon\Lambda^*\}\cup(\Lambda-\Lambda^*)$$

of subscripts of defining paths of $\overset{0}{\mathcal{A}}$ so that we can apply the level method. Let

$$\Theta_0=\{\lambda_i:i\epsilon I\}\cup\{\lambda': \lambda\epsilon\Lambda^*\}\cup(\Lambda-\Lambda^*),$$

and for m≥0, let

$$\Theta_m=\{\lambda:\lambda\epsilon T_i,\lambda\neq\lambda_i,d_i-d(\lambda,\lambda_i)=m-1,i\epsilon I\}.$$

(Here $d(\lambda,\lambda_i)$ is the length of the geodesic from λ_i to λ.) For $\lambda\epsilon\Theta_m$ (m≥0) the edge e_λ associated with λ is obtained as follows. Suppose $\lambda\epsilon T_i$. There is a unique edge in T_i joining λ to a vertex of distance $d(\lambda,\lambda_i)-1$ from λ_i. We take e_λ to be the label on this edge.

Using the above partition of Θ and applying the level method we obtain an equivalence from $\mathcal{A}$ to a 2-complex

$$\mathcal{A}' = \langle \mathfrak{X}';\ t_{\lambda_i}\beta_{\lambda_i}\ (i \in I),\ tR_{\lambda}^{(\lambda)}\ (\lambda \in \Lambda^*),\ \rho_\lambda\ (\lambda \in \Lambda - \Lambda^*) \rangle$$

The tree T_∞ is a spanning subtree of the connectivity graph of the collection

$$t_{\lambda_i}\beta_{\lambda_i} (i \in I)$$

and retains its original labelling.

We now partition the set

$$J = I \cup \{\lambda' :\ \lambda \in \Lambda^*\} \cup (\Lambda - \Lambda^*)$$

of subscripts of defining paths of $\mathcal{A}'$.

Let

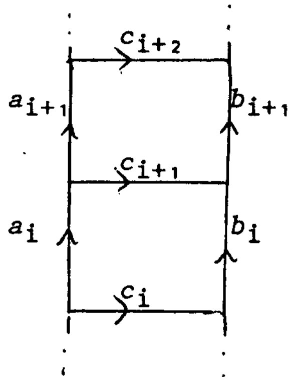
$$J_0 = \{\lambda' :\ \lambda \in \Lambda^*\} \cup (\Lambda - \Lambda^*).$$

For $m=1,2,\ldots$ let J_m be the set of $i \in I$ such that the distance from o to $\bar{\lambda}_i$ in T_∞ is $m-1$. The edge f_i associated with $i \in J_m$ $(m \geq 0)$ is obtained as follows. First consider the case when T_∞ is infinite. Choose an edge of T_∞ joining i to a vertex in J_{m+1}, and take f_i to be the label on this edge. Next consider the case when T_∞ consists of the single vertex o. Then $J_1 = \{o\}$ and $J_m = \emptyset$ for $m \geq 1$. We take the edge f_o associated with $o \in J_1$ to be the edge e as in (1.8).

Using the level method, we then obtain an equivalence from $\mathcal{A}'$ to a 2-complex as in (1.9).□

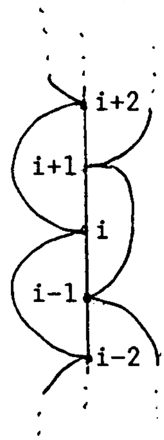
<u>Example</u>

$\mathcal{A}$ = <



; $(a_i a_{i+1} c_{i+2} b_{i+1}^{-1} b_i^{-1} c_i^{-1})^{m_i}$ $(i \epsilon Z)$>

CG($\mathcal{A}$)=



Each edge $\{i,i+1\}$ can be labelled b_{i+1}. This labelling is clearly quadratic. Thus CG($\mathcal{A}$) has a spanning subtree consisting of $\{i,i+1\}$ $(i \epsilon Z)$ which can be quadratically labelled. Hence by Theorem 1.3, $\mathcal{A}$ is equivalent to a 2-complex

$$< \mathcal{Y} ; t_i^{m_i} \ (i \epsilon Z)>.$$

1.5 STAR-COMPLEXES OF 2-COMPLEXES

Let $\mathcal{A}$ be a 2-complex. We can associate with $\mathcal{A}$ a 1-complex $\mathcal{A}^{st}$, called the *star-complex* of $\mathcal{A}$, as follows:

Vertex set of $\mathcal{A}^{st}$: $E(\mathcal{A})$,

Edge set of $\mathcal{A}^{st}$: $R(\mathcal{A})$,

with maps ι^{st}, τ^{st} and ${}^{-1\ st}$. Given by

$\iota^{st}(\gamma)$=first edge of γ $(\gamma \epsilon R(\mathcal{A}))$,

$\tau^{st}(\gamma)$=inverse of the last edge of γ $(\gamma \epsilon R(\mathcal{A}))$

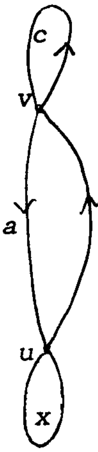
and $\gamma^{-1\ st} = \gamma^{-1}$ $(\gamma \epsilon R(\mathcal{A}))$.

Let γ be an edge of $\mathcal{A}^{st}$, with $\iota^{st}(\gamma)=e$ and $\tau^{st}(\gamma)=f$. Then since γ is a closed path in $\mathcal{A}$ it is easy to see $\iota(e)=\iota(f)$. Hence if g,h are two vertices of $\mathcal{A}^{st}$ in the same component of $\mathcal{A}^{st}$ then $\iota(g)=\iota(h)$.

For a vertex v of $\mathcal{A}$ we denote the full subcomplex of $\mathcal{A}^{st}$ on star(v) by $\mathcal{A}^{st}(v)$. We say that a 2-complex, $\mathcal{A}$, is *star-connected* if $\mathcal{A}^{st}(v)$ is connected for each vertex, v, of $\mathcal{A}$.

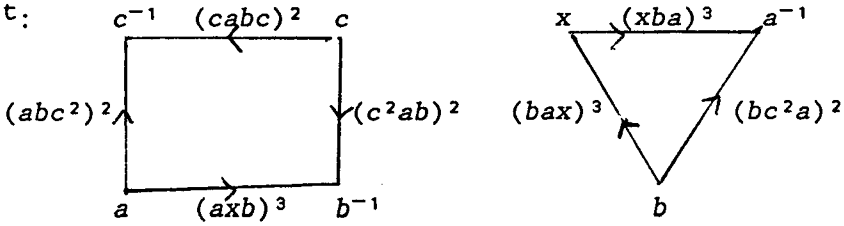
Examples

(1)



$\mathcal{A} = < a \quad b \; ; \; (c^2ab)^2, \; (xba)^3 >$

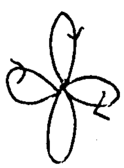
$\mathcal{A}^{st}$:



$(= \mathcal{A}^{st}(v))$ $\qquad$ $(= \mathcal{A}^{st}(u))$

(2)

$\mathcal{B}$



$c; \; (c^2ab)^2, \; (xba)^3$

$\mathcal{B}^{st}$ is identical with $\mathcal{A}^{st}$, but note that $\mathcal{A}$ is star-connected whilst $\mathcal{B}$ is not.

The following will prove crucial in chapter 3.

PROPOSITION 1.1

If $\mathcal{A}$ *is connected and star-connected then* CG($\mathcal{A}$) *is connected.*

Proof

Let ρ_λ $(\lambda\epsilon\Lambda)$ be the collection of defining paths of $\mathcal{A}$. Let $\lambda,\lambda'\epsilon\Lambda$, and let $e\epsilon E(\rho_\lambda)$ and $f\epsilon E(\rho_{\lambda'})$. Since $\mathcal{A}$ is connected there is a path $e_1e_2\ldots e_{n+1}$ in $\mathcal{A}$, where $e_1=e$ and $e_{n+1}=f$. Since $\mathcal{A}$ is star-connected, for i=1,...,n there is a path $\beta_{i1}\beta_{i2}\ldots\beta_{ir(i)}$ in $\mathcal{A}^{st}$ starting at the vertex e_i^{-1} and ending at the vertex e_{i+1}. Let $d(\beta_{ij})$ be an element λ of Λ such that β_{ij} is a cyclic permutation of $\rho_\lambda^{\pm 1}$. Then the following are elements of CG($\mathcal{A}$) (where a singleton is to be regarded as a vertex).

$$\{\lambda,d(\beta_{11})\},\ \{d(\beta_{nr(n)}),\lambda'\},$$

$$\{d(\beta_{ij}),d(\beta_{i,j+1})\} \qquad 1\leq i\leq n,\ 1\leq j\leq r(i)-1,$$

$$\text{and } \{d(\beta_{ir(i)}),d(\beta_{i+1,1})\} \qquad 1\leq i\leq n.$$

Thus we obtain a path in CG($\mathcal{A}$) from λ to λ'.□

INDUCED MAPPINGS OF STAR-COMPLEXES

Let φ: $\mathcal{A}\longrightarrow\mathcal{B}$ be a pure, incompressible mapping, with

$\varphi(R(\mathcal{A}))\subseteq R(\mathcal{B})$. We then have an induced (rigid and pure) mapping

$$\varphi^{st}: \mathcal{A}^{st} \longrightarrow \mathcal{B}^{st}$$

defined by

$$\varphi^{st}(e)= \text{first edge of } \varphi(e) \quad (e \text{ a vertex of } \mathcal{A}^{st})$$

$$\varphi^{st}(\gamma)=\varphi(\gamma) \qquad\qquad (\gamma \text{ an edge of } \mathcal{A}^{st})$$

It is easily seen that if v is a vertex of $\mathcal{A}$ then φ^{st} maps $\mathcal{A}^{st}(v)$ into $\mathcal{B}^{st}(\varphi(v))$.

<u>THEOREM 1.4</u>

Let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ be a locally bijective, rigid, pure mapping, with $\varphi(R(\mathcal{A}))\subseteq R(\mathcal{B})$. Then the following are equivalent:

(A) φ^{st} *is locally bijective.*

(B) $\varphi^{-1}R(\mathcal{B})=R(\mathcal{A})$.

(C) *For each vertex v of $\mathcal{A}$, φ^{st} maps $\mathcal{A}^{st}(v)$ isomorphically onto $\mathcal{B}^{st}(\varphi(v))$.*

<u>Proof</u>

See [37] Theorem 1.□

1.6 DIAGRAMS

DIAGRAMS OVER PRESENTATIONS

Let $\mathcal{P}=<X_1,X_2;r>$ be a presentation. Planar (Van Kampen) and conjugacy diagrams over $\mathcal{P}$ (at least when $X_2=\emptyset$) are discussed at length in [27,Chp. V]. Spherical diagrams are discussed in [7] and elsewhere. Here we give a general treatment of diagrams which includes all of the above, and more. The treatment follows

[36],[37] and the reader is refered there for further information.

A $\mathcal{P}$*-spine* is a finite combinatorial subdivision of a closed interval, where the oriented edges are labelled by elements of $X_1 \cup X_1^{-1} \cup X_2$ (with the understanding that if an oriented edge is labelled by $z \epsilon X_1 \cup X_1^{-1} \cup X_2$, then if we traverse the edge against the orientation we read z^{-1}). A $\mathcal{P}$*-sphere* is a tesselated sphere, whose oriented edges are labelled by elements of $X_1 \cup X_1^{-1} \cup X_2$ and for which there is a subset of regions (called *non-distinguished* regions, possibly consisting of all of the regions on the sphere) each of which has a boundary cycle labelled by a element of $r \cup r^{-1}$. A *label* on a region Δ is $\varphi(e_1)\ldots\varphi(e_n)$ for any anti-clockwise boundary cycle $e_1 \ldots e_n$ of Δ.The remaining regions are called *distinguished* regions. A *diagram* , $\mathcal{D}$, *over* $\mathcal{P}$ is an ascending union

$$\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \ldots \subseteq \mathcal{D}_n = \mathcal{D}$$

where $\mathcal{D}_0$ is a single vertex, and $\mathcal{D}_{i+1}$ is obtained from $\mathcal{D}_i$ either by attaching a $\mathcal{P}$-spine to $\mathcal{D}_i$ by one of its endpoints to a vertex of $\mathcal{D}_i$, or by attatching a $\mathcal{P}$-sphere by one of its vertices to a vertex of $\mathcal{D}_i$.

If $\mathcal{D}$ consists of a single sphere we will wish, in chapter 4, to assign numbers, called *angles*, to the corners of the regions of $\mathcal{D}$. We denote the angle at a corner K by $\angle K$. For a region Δ of $\mathcal{D}$ we define the *curvature*, $K(\Delta)$, of Δ to be

$$h-(s-2)\pi,$$

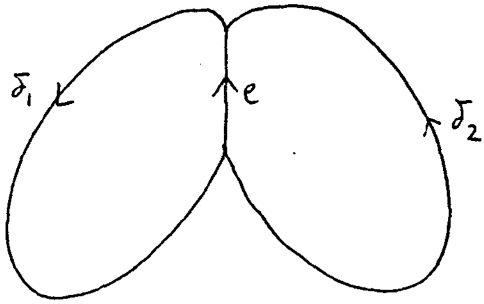
where h is the sum of the angles at the corners of Δ and s is the number of corners of Δ. For a vertex, *a*, we define the *curvature*, $K(a)$, at *a* to be

$$2\pi-g,$$

where g is the sum of the angles of the corners incident at *a*. Using the Euler characterisic of the sphere it is easily shown that

$$\sum K(a) + \sum K(\Delta) = 4\pi.$$

Let $\mathcal{D}$ be a diagram over $\mathcal{P}$. Let Δ_1 and Δ_2 be $\mathcal{D}$ non-distinguished regions (not necessarily distinct) of $\mathcal{D}$ with an edge $e \subseteq \partial\Delta_1 \cap \partial\Delta_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of Δ_1 and Δ_2 respectively. Let U_1, U_2 be the labels on δ_1 and δ_2 respectively. $\mathcal{D}$ will be called *reduced* if one never has $U_1 = U_2^{-1}$.



The following two Lemmata are adaptions of results in [7].

<u>LEMMA 1.9</u>

Let $\mathcal{P} = \langle X_1, X_2\ ;\ r\rangle$ *be a presentation. Let* $\mathcal{D}$ *be a diagram over* $\mathcal{P}$ *with* k *distinguished regions labelled by words* $W_1, \dots, W_k$. *Then there exist words* $U_1, \dots, U_k$ *on* $X_1 \cup X_2$ *such that*

$$U_1 W_1 U_1^{-1} \dots U_k W_k U_k^{-1} =_{\mathcal{P}} 1. \square$$

LEMMA 1.10

Let $\mathcal{P} = \langle X_1, X_2;\ r\rangle$ *be a presentation. Suppose that there exist words* $U_1,\ldots,U_k,W_1,\ldots W_k$ *on* $X_1 \cup X_2$ *such that*

$$U_1W_1U_1^{-1}\ldots U_kW_kU_k^{-1} =_{\mathcal{P}} 1.$$

Then there exists a reduced diagram over $\mathcal{P}$ *with distinguished regions* $\Delta_1,\ldots,\Delta_k$ *such that for some boundary cycle of* Δ_i *the label on* Δ_i *is* W_i $(1 \leq i \leq k)$.□

We note that Van Kampen diagrams correspond to the case of one distinguished region, conjugacy diagrams to two distinguished regions, and spherical diagrams to no distinguished regions. In general, more distinguished regions relate to general dependence problems, see [37].

PLANAR DIAGRAMS OVER QUOTIENTS OF FREE PRODUCTS

It would be possible to give a general treatment of diagrams over quotients of free products along the lines of that described above for presentations. However, we will only require the concept of planar (Van Kampen) diagrams, so we will content ourselves with describing these. The following is a variation on the discussion in Lyndon and Schupp [27, Chp.V].

Let $H = \underset{i\in I}{*}H_i$, a *word* on H is a finite sequence (usually written without the commas) of elements of $\underset{i\in I}{\cup}H_i$. The *length* of a word $a_1\ldots a_n$ is n, and its *inverse* $(a_1\ldots a_n)^{-1}$ is $a_n^{-1}\ldots a_1^{-1}$. Clearly we may talk about the element of H (or of any quotient of H) that any word on H defines. A word on H is said to be *trivial* if it defines 1 in H, and *non-trivial* otherwise. Let **r** be a set of words on H. An **r**-diagram is a finite oriented planar map M and a function φ from the edges of M to $\underset{i\in I}{\cup}H_i$ satisfying

(i) If e is an edge of M then $\varphi(e)^{-1}=\varphi(e^{-1})$.

(ii) M is connected and its complement in the plane has precisely one component.

(iii) If Δ is any region of M there is a boundary cycle $e_1,e_2,\ldots,e_n$ of Δ such that $\varphi(e_1)\varphi(e_2)\ldots\varphi(e_n)$ is equal (in $\underset{i\in I}{*}H_i$) to an element of **r**.

A *label* on a region Δ is $\varphi(e_1)\varphi(e_2)\ldots\varphi(e_n)$ for any anti-clockwise boundary cycle of Δ. An **r**-*trivial* word on H is a word which defines 1 in <H;**r**> (the quotient of H by the normal closure of the elements defined by the elements of **r**).

LEMMA 1.11

Let $a_1a_2\ldots a_n$ be an r*-trivial word on* H. *Then there exists an* r*-diagram* M *over* <H;r> *and a vertex* v *on* ∂M *such that if* $e_1,\ldots,e_t$ *is the boundary cycle of* M *beginning at* v, *then* t=n *and,*

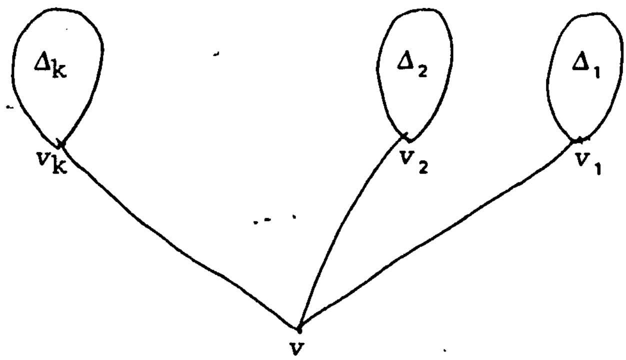
$$\varphi(e_1)\ldots\varphi(e_n) = a_1\ldots a_n.$$

Proof

To begin note that $a_1a_2\ldots a_n$ defines the same element in H as some product $\prod_{j=1}^{k} U_jR_j^{\epsilon_j}U_j^{-1}$ (U_j a word on H, $R_j \epsilon$ r, and $\epsilon_j=\pm1$ $1\leq j\leq k$).

STEP 1

Draw a "bunch of k lollipops", as follows.



Now subdivide the "stalk" from v to v_j into a number of segments equal to the length of U_j. For j=1,...,k label the segment from v to v_j so that, reading from v to v_j we read U_j.

Next subdivide $\partial\Delta_j$ into a number of segments equal to the length of R_j. Label these segments so that the label on Δ_j, reading once anti-clockwise around Δ_j from v_j, is $R_j^{\epsilon_j}$. Then the label on the above "bunch of lollipops", reading once anti-clockwise around its boundary from v, is

$\prod_{j=1}^{k} U_j R_j^{\epsilon_j} U_j^{-1}$. The result is an r-diagram, M′ say. We know that we can turn $\prod_{j=1}^{k} U_j R_j^{\epsilon_j} U_j^{-1}$ into $a_1 \ldots a_n$ by a series of following operations:

(1) *Insertion*. Replace a word $b_1 \ldots b_j b_{j+1} \ldots b_n$ by

$b_1 \ldots b_j h h^{-1} b_{j+1} \ldots b_n$ $(h \in \bigcup_{i \in I} H_i)$.

(2) *Deletion*. The inverse of insertion.

(3) *Splitting*. Replace a word $b_1 \ldots g \ldots b_n$ by a word

$b_1 \ldots hk \ldots b_n$ where $g,h,k \in H_i$ and $g=hk$ $(i \in I)$.

(4) *Coalition*. The inverse of splitting.

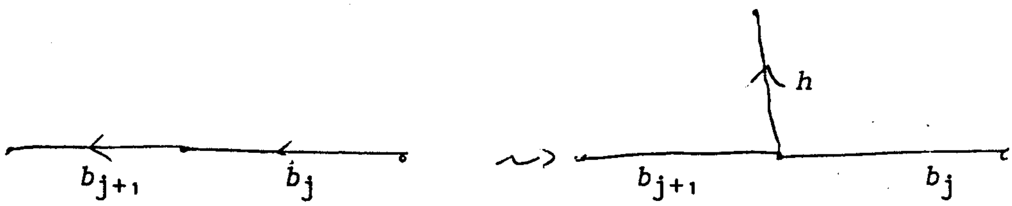
Remark: (1) An operation (4) can also be achieved by an operation (3) followed by an operation (2).

(2) We may remove a term b_j if $b_j=1$. Since, by splitting we may replace $\ldots, b_j, \ldots$ by $\ldots, b_j, b_j, \ldots$ which is equal to $\ldots, b_j, b_j^{-1}, \ldots$ and so both terms can deleted.

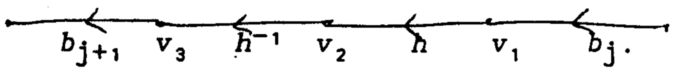
<u>STEP 2</u>

We show that we can mimic the above operations on the boundary of M′.

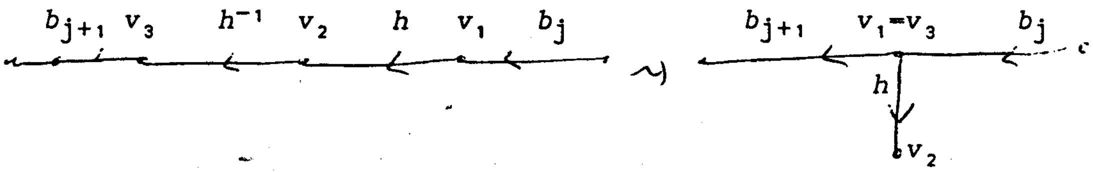
(1). This is mimicked an M′ as follows.



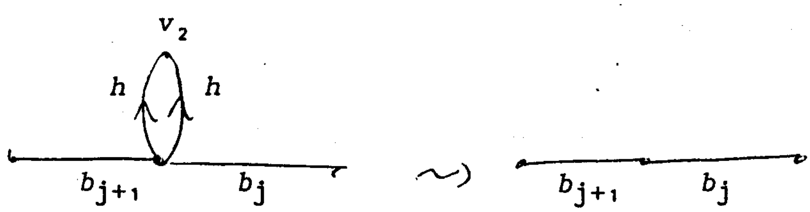
(2). We have,



There are two cases. Firstly suppose $v_1 \neq v_3$. Then



Now suppose that $v_1 = v_3$. Then



(3). We have,

$$v_2 \xleftarrow{\;g\;} v_1 \quad \rightsquigarrow \quad v_1 \xleftarrow{\;k\;} \bullet \xleftarrow{\;h\;} v_2$$

(4). This is dealt with by the above Remark.

Thus by iteration of these operations we obtain a diagram of the required form.□

LEMMA 1.12 (NORMAL SUBGROUP LEMMA)

Let M *be an* r*-diagram with regions* $\Delta_1,\ldots,\Delta_m$. *Let* $\alpha = e_1\ldots e_n$ *be a boundary cycle of* M *beginning at a vertex* $v_0 \in \partial M$. *Let* $W = \varphi(e_1)\ldots\varphi(e_n)$. *Then there exist labels* R_i *of* Δ_i *and words* U_i *on* H, $1 \leq i \leq m$, *such that* W *defines the same element of* H *as* $(U_1R_1U_1^{-1})\ldots(U_mR_mU_m^{-1})$.

Proof

The proof is identical to the analogous result for presentations in [27,Chp. V].□

Let M be a diagram over <H ; r>. Let Δ be a region of M with $e_1\ldots e_n$ a boundary cycle of Δ. We define

$$t(\Delta) = \{\ i:\ i \in I \text{ and for some } 1 \leq j \leq n\ \varphi(e_j) \in H_i\}.$$

A diagram M is *minimal* if there is no diagram with fewer regions and the same boundary label.

Let W be a word on H which defines 1 in $\langle H ; r \rangle$. Then we know (Lemma 1.11) that there exists an r-diagram with boundary label W. We define deg(W) to be the number of regions in a minimal diagram with boundary label W.

1.7 SEQUENCES AND PICTURES

SEQUENCES AND PICTURES OVER PRESENTATIONS WITHOUT INVOLUTARY GENERATORS

Let

$$\langle X;r\rangle$$

be a presentation without involutary generators (i.e $X=X_1$), and let W be the set of words on X (reduced or not). For $t\subseteq r$ we let

$$t^W=\{Wt^{\epsilon}W^{-1}:\ W\epsilon W,\ t\epsilon t,\ \epsilon=\pm1\}.$$

Two elements $W_1R_1^{\epsilon_1}W_1^{-1}$ and $W_2R_2^{\epsilon_2}W_2^{-1}$ of r^W will be said to be *G-equivalent* if $R_1=R_2$, $\epsilon_1=\epsilon_2$ and $W_1N=W_2N$ (N being the normal closure of r in the free group on X). Two finite sequences $(C_1,\ldots,C_m)$, $(C_1',\ldots,C_n')$ of elements of r^W will be said to be *G-equivalent* if m=n, there is a permutation σ of $\{1,\ldots,m\}$ such that C_λ' is G-equivalent to $C_{\sigma(\lambda)}$ and $C_1C_2\ldots C_m$ is freely equal to $C_1'C_2'\ldots C_n'$.

Two finite sequences of elements of r^W will be said to be *equivalent* if one can be obtained from the other by a finite number of operations of the following form.

(1.10) Replace a sequence by a G-equivalent sequence.

(1.11) Delete two successive terms

$$\ldots, WR^{\epsilon}W^{-1}, WR^{-\epsilon}W^{-1}, \ldots$$

from a sequence, or insert two such terms into a sequence.

A sequence $(C_1, C_2, \ldots C_m)$ of elements of r^W is called an *identity sequence* if $C_1C_2\ldots C_m$ is freely equal to 1.

We now describe pictures over $<X ; r>$. The following basic exposition is taken from [41]. For further information see [4],[7], and [8].

A *picture* $\mathbb{P}$ *(over* $<X:r>$*)* consists of the following.

(a) A disk D with a basepoint o on ∂D.

(b) Disjoint disks $\Delta_1, \ldots, \Delta_n$ in the interior of D with basepoints $o_1, \ldots, o_n$ on $\partial\Delta_1, \ldots, \partial\Delta_n$, respectively.

(c) A finite number of disjoint arcs. Each arc lies in the closure of

$$D - \bigcup_{\lambda=1}^{n} \Delta_\lambda$$

and is either a simple closed curve having empty intersection with $\partial D \cup \partial\Delta_1 \cup \partial\Delta_2 \ldots \cup \partial\Delta_n$, or it is a simple non-closed curve which joins two distinct points of $\partial D \cup \partial\Delta_1 \cup \ldots \cup \partial\Delta_n$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting the edge transversaly and is labelled by an element of $\mathbf{X}$.

(d) If we travel around $\partial\Delta_\lambda$ once in the anti-clockwise direction starting at o_λ and reading off the labels on the arcs encountered then we obtain a word $R_\lambda^{\epsilon_\lambda}$ where $R_\lambda \in \mathbf{r}$ and $\epsilon_\lambda = \pm 1$. The word is called the *label on* Δ_λ.

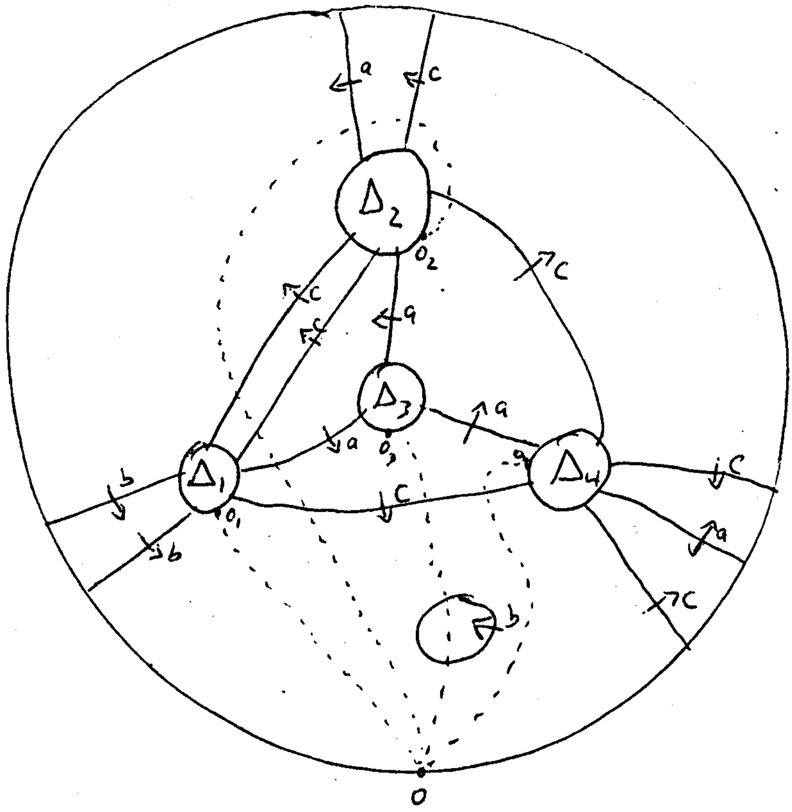
The *label on* $\mathbb{P}$ is the word one reads off by travelling around ∂D once in the anti-clockwise direction, starting at o.

The *discs of* $\mathbb{P}$ are the discs $\Delta_1, \ldots, \Delta_n$ (but not the ambient disc D).

Example

$<a,b,c;\ c^2ac^{-2}a^{-1},\ c^{-1}a^{-1}c^2b^2,\ a^3>$

ℙ



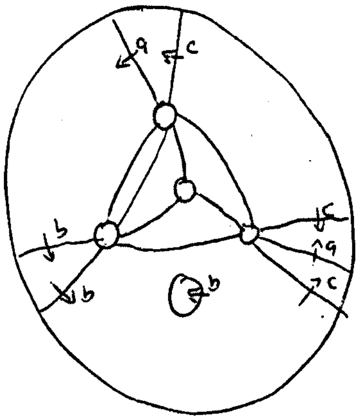
The labels on $\Delta_1, \Delta_2, \Delta_3$, and Δ_4 are $c^{-1}a^{-1}c^2b^2$, $c^2ac^{-2}a^{-1}$, a^3, $c^2ac^{-2}a^{-1}$ respectively. (The dotted lines in this picture represent a *spray*, as defined below.)

A *spherical picture* is a picture in which no arcs meet ∂D.

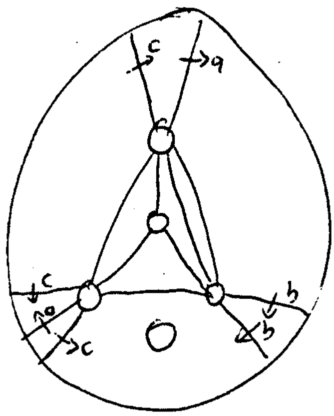
A *spherical subpicture* of a picture is obtained by considering a subset ξ of the picture homeomorphic to a closed disc such that $\partial\xi$ does not intersect any arcs or discs of the picture. The ξ with the arcs and discs of the picture on it is a spherical subpicture of the picture.

If $\mathbb{P}$ is a picture with label $x_1,\ldots x_n$ then the *mirror-picture* $\overset{0}{\mathbb{P}}$ is the picture obtained by "glueing" $\mathbb{P}$ to its mirror-image $\mathbb{P}'$. We illustrate what we mean with an example.

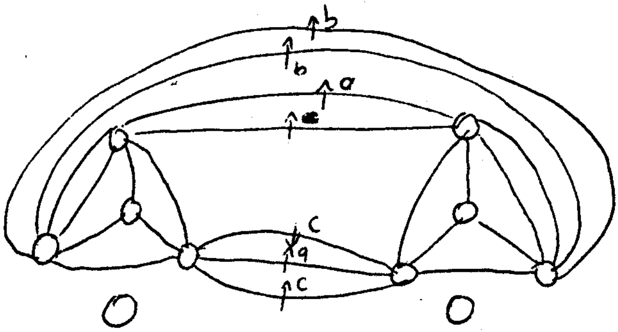
$\mathbb{P}$:



$\mathbb{P}'$:



$\overset{0}{\mathbb{P}}$:



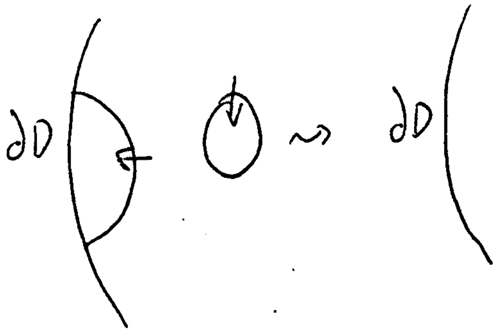
Note that $\overset{0}{\mathbb{P}}$ is a spherical picture, and so according to our definition, its label is the empty word. However, it will be convenient to define the *label on the mirror-picture* $\overset{0}{\mathbb{P}}$ to be the label on $\mathbb{P}$.

We define three operations on pictures:

(1.12) *Bridge moves.*

where the above are arcs in a picture.

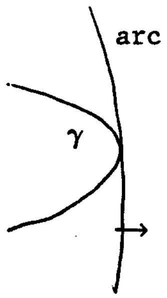
(1.13) *Deletion of floating arcs.*



(1.14) *Insertion and deletion of mirror-pictures.*

A *transverse path* in $\mathbb{P}$ is a path γ in Δ with the following properties

(a) The intersection of γ and the union of all the arcs in $\mathbb{P}$ is finite, moreover, if γ intersects an arc then it does not just touch it but crosses it.



not allowed.

(b) If γ intersects $\partial D \cup \partial \Delta_1 \cup \ldots \cup \partial \Delta_n$ then it does so in a subset of $\{o, o_1, \ldots o_n\}$.

Since we will only ever consider transverse paths we will from now on drop the adjective "transverse", and simply refer to paths.

If we travel along a path γ from its initial point to its terminal point then we will cross various arcs, and we read off the labels on these arcs, giving a word $w(\gamma)$.

A *spray* in $\mathbb{P}$ is a sequence $\vec{\gamma}=(\gamma_1, \ldots, \gamma_n)$ of simple paths satisfying the following:

(a) There exists a permutation θ of $\{1, \ldots, n\}$ (depending on $\vec{\gamma}$) such that for $\lambda=1, \ldots, n$, γ_λ starts at o and ends at $o_{\theta(\lambda)}$.

(b) For $1 \leq \lambda, \mu \leq n$ distinct, γ_λ and γ_μ intersect only at o

(c) Travelling around o anti-clockwise we encounter the paths in the order $\gamma_1, \gamma_2, \ldots, \gamma_n$.

The *sequence* $\sigma_{\mathbb{P}}(\vec{\gamma})$ *associated with* $\vec{\gamma}$ is

$$(w(\gamma_1) r_{\theta(1)}^{\epsilon(1)} w(\gamma_1)^{-1}, \ldots, w(\gamma_n) r_{\theta(n)}^{\epsilon\theta(n)} w(\gamma_n)^{-1}).$$

LEMMA 1.13

If $\vec{\gamma}, \vec{\gamma}'$ are any two sprays in $\mathbb{P}$ *then* $\sigma_{\mathbb{P}}(\vec{\gamma})$ *and* $\sigma_{\mathbb{P}}(\vec{\gamma}')$ *are* G-*equivalent.*

Proof

See [41].□

We say that two pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are *equivalent* if for a spray $\vec{\gamma}_1$ over $\mathbb{P}_1$ and a spray $\vec{\gamma}_2$ over $\mathbb{P}_2$, $\sigma_{\mathbb{P}_1}(\vec{\gamma}_1)$ is equivalent to $\sigma_{\mathbb{P}_2}(\vec{\gamma}_2)$. This is well defined by Lemma 1.13.

Clearly, if $\mathbb{P}'$ is a picture obtained from $\mathbb{P}$ by a series of operations of types (1.12), (1.13) and (1.14) then $\mathbb{P}$ and $\mathbb{P}'$ are equivalent.

PICTURES OVER QUOTIENTS OF FREE PRODUCTS

A picture over a quotient of a free product

$$\langle G_v\ (v \in V);\ S \rangle$$

is identical to a picture over a presentation, except in the following ways.

(a) Each arc is labelled by an element of some G_v.

(b) If we travel around $\partial\Delta_k$ once in the anti-clockwise direction starting at o_k and read off the labels on the arcs encountered we obtain a word $S_\lambda^{\epsilon_\lambda}$ where S_λ is equal (in $\underset{v\epsilon V}{*} G_v$) to an element of S and $\epsilon_\lambda = \pm 1$.

We note that the definitions of spherical picture and spherical subpicture carry over into this new situation.

1.8 SQ-UNIVERSALITY

A group, G, is said to be *SQ-universal* if every countable group can be embedded in some quotient group of G, see [27,p.282]

Example: If A and B are non-trivial groups, not both of order two then $A*B$ is SQ-universal, see [33]. Hence, in particular, the free group of rank two, F_2, is SQ-universal.

The following two facts will be very important. See [33].

(1.15) *Suppose* $\varphi:G \longrightarrow H$ *is an epimorphism and that* H *is SQ-universal. Then* G *is SQ-universal.*

(1.16) *If* H *and* G *are groups with* H *of finite index in* G *then* H *is SQ-universal if and only if* G *is SQ-universal.*

We remark that being SQ-universal is a measure of the "largeness" of a group. A more general discussion of "largeness" in group theory is given in [9], [12] and [34]. Following [9], [12] and [34] we say that a group G is *as large as* F_2 (written $G \succcurlyeq F_2$) if G has a subgroup of finite index which can be mapped homomorphically onto F_2. Note that, by (1.15) and (1.16) above, if $G \succcurlyeq F_2$ then G is SQ-universal.

CHAPTER 2

ON SOME QUOTIENTS OF FREE PRODUCTS

2.1 INTRODUCTION

In this chapter we will consider groups with the following structure.

(a) Let Γ be a graph with vertex set V and edge set E. We assume that no vertex of Γ is isolated.

(b) For each vertex $v \in V$ there is a non-trivial group G_v.

(c) For each edge $e=\{u,v\} \in E$ there is a set S_e of cyclically reduced elements of $G_u * G_v$, each of length at least two.

We define G_e to be the quotient of $G_u * G_v$ by the normal closure of S_e.

We let G be the quotient of $\underset{v\in V}{*} G_v$ by the normal closure of $S = \bigcup_{e \in E} S_e$. For convenience, we write

$$G = \langle G_v\ (v \in V);\ S_e\ (e \in E) \rangle$$

The above is a generalization of a situation studied by Pride [35], where each G_v was infinite cyclic.

Let $e=\{u,v\}$ be an edge of Γ. We will say that G_e has *property*-W_k if no non-trivial element of $G_u * G_v$ of free product length less than or equal to 2k is in the kernel of the natural epimorphism

$$G_u * G_v \longrightarrow G_e$$

We will work with one of the following:

(I) Each G_e has property-W_2

(II) Γ is triangle-free and each G_e has property-W_1.

Our results will concern a Freiheitssatz, SQ-universality, and (co)-homology. Our results will be discussed shortly, but first we give some examples of situations when conditions (I)/(II) hold.

Example 1

For an edge $e=\{u,v\}$ of Γ, let D_e denote the Cartesian subgroup of $G_u * G_v$ (i.e. D_e is the kernel of the natural epimorphism $G_u * G_v \longrightarrow G_u \times G_v$). Then G_e clearly has property-W_1 if $S_e \subseteq D_e$ and G_e has property-W_2 if $S_e \subseteq D_e^{p(e)} D_e'$ for some prime $p(e)$ since

$$\frac{D_e}{D_e^{p(e)} D_e'}$$

is an elementary abelian $p(e)$-group with basis $[x,y] D_e^{p(e)} D_e'$ ($x \in G_u$, $y \in G_v$, neither x nor y equal to 1). See [30].

Example 2

For an edge $e=\{u,v\}$ of Γ, let $S_e=\{(xy)^r\}$ ($x\epsilon G_v$,$y\epsilon G_u$, neither x nor y is trivial). Then G_e has property-W_k if $r\geq k+2$. This is easily verified using small cancellation theory.

FREIHEITSSATZ

Let Φ be a full subgraph of Γ with vertex set V' and edge set E' say. Then we have the group

$$G_\Phi = \langle G_v \ (v\epsilon V') \ ; \ S_e \ (e\epsilon E') \rangle$$

and there is a natural homomorphism $G_\Phi \longrightarrow G$.

THEOREM 2.1 (FREIHEITSSATZ)

Suppose (I) *or* (II) *holds. For every full subgraph* Φ *of* Γ *the natural map*

$$G_\Phi \longrightarrow G$$

is an injection.

SQ-UNIVERSALITY

We prove

THEOREM 2.2

Suppose (I) *or* (II) *holds. Assume that there are vertices* u,v *of* Γ *satisfying the following: not both* G_u,G_v *have order*

2; $\{u,v\}$ is not an edge of Γ and if (II) *holds (but* (I) *does not), then adjoining $\{u,v\}$ to Γ does not create a triangle.*

Then G is SQ-universal.

<u>(CO)-HOMOLOGY</u>

The following is adapted from [39].

For each vertex $v\epsilon V$ let $G_v = \langle X_v; r_v\rangle$ (a presentation with no involutary edges) and for each $g\epsilon G_v$ let $w(g)$ be a word on X_v representing g. If $e=\{u,v\}$ is an edge of Γ and h is an element of S_e, say $h=x_1y_1\ldots x_ny_n\epsilon S_e$ $(x_1,\ldots,x_n\epsilon G_u,\ y_1,\ldots,y_n\epsilon G_v)$, let $\hat{h}=w(x_1)w(y_1)\ldots w(x_n)w(y_n)$. Let $\hat{r}_e=\{\hat{s}:s\epsilon S_e\}$ and $r_e=\hat{r}_e \cup r_u \cup r_v$, so

$$G=\langle X;r\rangle$$

where $X=\bigcup_{v\epsilon V} X_v$, $r=\bigcup_{e\epsilon E} r_e$.

Let N be the normal closure of r in F, the free group on X. We let M denote the relation module for the given presentation of G. Thus M is the left G-module with underlying abelian group

$$N^{ab}=N/N'$$

and G-action

$$wN \cdot uN' = wuw^{-1}N' \quad (w \in F,\ u \in N)$$

We have the submodule M_e of M generated by

$$\{RN' : R \in r_e\}.$$

For $e \in E$, let P_e be the free left $\mathbb{Z}G$-module with basis $\{t_R^e : R \in r_e\}$, and let K_e be the kernel of the epimorphism

$$P_e \longrightarrow M_e,\ t_R^e \mapsto RN' \quad (R \in r_e).$$

Let P be the free left $\mathbb{Z}G$-module with basis $\{t_R : R \in r\}$ and let K be the kernel of the epimorphism

$$P \longrightarrow M,\ t_R \mapsto RN' \quad (R \in r).$$

Now we have an epimorphism

$$\alpha : \bigoplus_{e \in E} P_e \longrightarrow P,\ t_R^e \mapsto t_R \quad (e \in E,\ R \in r_e)$$

which clearly carries $\bigoplus_{e \in E} K_e$ into K.

Pride works with two assumptions:

(A) The natural maps $G_v \longrightarrow G$ $(v \in V)$, $G_e \longrightarrow G$ $(e \in E)$ are injective.

(B) α carries $\bigoplus_{e \in E} K_e$ onto K.

Under these assumptions he proves the following result.

For v a vertex of Γ let $n_v = |\mathrm{Adj}(v)| - 1$, where $\mathrm{Adj}(v)$ is the set of vertices of Γ adjacent to v.

THEOREM (PRIDE)

Let A be any right G-module, and B be any left G-module.

(i) *There is a long exact sequence*

$$\ldots \longrightarrow H_{n+1}(G,A) \longrightarrow \bigoplus_{v\epsilon V} H_n(G_v,A)^{n_v} \longrightarrow \bigoplus_{e\epsilon E} H_n(G_e,A) \longrightarrow H_n(G,A) \longrightarrow \ldots$$

terminating in

$$\ldots \longrightarrow H_2(G,A) \longrightarrow \bigoplus_{v\epsilon V}(A\otimes_{G_v} IG_v)^{n_v} \longrightarrow \bigoplus_{e\epsilon E} A\otimes_{G_e} IG_e \longrightarrow A\otimes_G IG \longrightarrow 0.$$

(ii) *There is a long exact sequence*

$$\ldots H^n(G,B) \longrightarrow \prod_{e\epsilon E} H^n(G_e,B) \longrightarrow \prod_{v\epsilon V} H^n(G_v,B)^{n_v} \longrightarrow H^{n+1}(G,B) \longrightarrow \ldots$$

starting with

$$0 \longrightarrow \mathrm{Hom}_{ZG}(IG,B) \longrightarrow \prod_{e\epsilon E}\mathrm{Hom}_{ZG_e}(IG_e,B) \longrightarrow \prod_{v\epsilon V}\mathrm{Hom}(IG_v,B)^{n_v}$$

$$\longrightarrow H^2(G,B) \longrightarrow \ldots \square$$

From this and a theorem due to Serre (see [24]) we have

COROLLARY (PRIDE)

Suppose that there is a global bound on the cohomological dimension of all of the G_v's. Then any finite subgroup of G is contained in a conjugate of some subgroup G_e ($e\epsilon E$).

Clearly if (I) or (II) holds then Pride's assumption (A) holds (by the Freiheitssatz).

<u>THEOREM 2.3</u>

If (I) *or* (II) *holds then Pride's assumption* (B) *holds.*

2.2 PROOF OF THEOREM 2.1

The proof is very similar to the proof of Theorem 4 of Pride [35] (which considers the special case when each G_v is infinite cyclic). However, for the readers convenience we describe the main points of the proof.

We ask the reader to begin by recalling the definitions and terminology of diagrams over free products (see section 1.6).

Consider an S-diagram M. We define an equivalence relation on the regions of M as follows:

$D \sim D'$ if and only if there exist regions $D=D_0, D_1, \ldots, D_n=D'$ with $t(D_0)=t(D_1)=\ldots=t(D_n)$ and where D_i, D_{i+1} have an edge in common for $i=1,\ldots,n-1$. The regions in a $\sim$-equivalence class give rise to a connected subdiagram of M, which we call a *federation*.

Let $e=\{u,v\}$ be an edge of Γ. Define $\hat{S}_e$ to be the set of all non-trivial words on $G_u * G_v$ which define 1 in G_e. Let $\hat{S} = \bigcup_{e \in E} \hat{S}_e$.

Suppose M satisfies

$$(2.1)\qquad \begin{array}{l}\textit{Each federation is simply connected and no}\\ \textit{federation has boundary label defining } 1 \textit{ in } \underset{v\in V}{*}G_v.\end{array}$$

We may then obtain from M an $\hat{S}$-diagram by removing the interior edges and vertices of each federation. This diagram satisfies (a) and (b) below. By performing slight modifications we can obtain an $\hat{S}$-diagram $\hat{M}$ which additionally satisfies (c). For details of this construction see Pride [35].

(a) Each internal edge of $\hat{M}$ has a label from some G_v.

(b) If each G_e has property-W_k then each almost interior region of $\hat{M}$ has at least $2(k+1)$ sides.

(c) Every internal vertex of $\hat{M}$ has valence at least three, and if Γ has no triangles then every internal vertex of $\hat{M}$ has valence at least four.

We now deduce that if M satisfies (2.1) then it has a boundary region D with

$$t(D)\subseteq t(\partial M).$$

We show this for the case where hypothesis (I) holds. (The case where hypothesis (II) holds is similar.) Since every internal vertex has valence at least three and every almost interior region has at least six sides, $\hat{M}$ has a simple boundary region $\hat{D}$ with at most three interior edges (see Lyndon and Schupp [27,Chp. V]). Now $\hat{D}$ arises from a federation L in M, which has a region D, which is a boundary region of M. Suppose $t(L)=\{u,v\}$. Hypothesis (I) together with (a) implies that the label on $\partial\hat{D}\cap\partial\hat{M}$ involves elements from both G_u and G_v. Thus

$$t(D)=t(L)=t(\hat{D})\subseteq t(\partial\hat{M})=t(\partial M).$$

Next we deduce that any minimal S-diagram satisfies (2.1) above.

To show this we argue by contradicion. Let K be a counterexample with as few regions as possible. Let L be a federation in K which is not simply connected, and let M be a bounded component of K-L. Then since all federations in M are simply connected, no federation in M can have boundary label defining 1 in $\underset{v\epsilon V}{*}G_v$, else K is not minimal. Hence, by the above, M has a boundary region D with $t(D)\subseteq t(\partial M)=t(L)$, contradicting the fact that L is a federation.

We can now outline the proof of Theorem 2.1

Let Φ be a full subgraph of Γ with vertex set V'. Let Z be a word in $\underset{v\in V'}{*}G_v$ defining 1 in G_Γ. We argue by induction on deg(Z). If deg(Z)=0 the result is clearly true, so suppose deg(Z)>0. Let M be a connected, simply-connected S-diagram with deg(Z) regions (which guarantees that M is minimal) and boundary label Z. By the above M has a boundary region D with $t(D)\subseteq V'$.

Let M' be obtained from M by removing the interior of D and one edge of $\partial D\cap\partial M$. Let Z' be the boundary label of M'. Then Z' is equal to 1 in G and deg(Z')<deg(Z) so Z' equals 1 in G_Φ. Now Z equals Z' in G_Φ. Hence Z equals 1 in G_Φ.$\square$

We note that Edjvet [10] has also obtained this result by different methods, as a consequence of his work on "filtered presentations".

2.3 PROOF OF THEOREM 2.2

Let $A=\langle a,b;T\rangle$ be any two generator group.

Suppose $|G_u|\geq 2$ and $|G_v|\geq 3$, let k be a non-trivial element of G_u and g,h distinct non-trivial elements of G_v. Consider the following situation.

Let Γ' be the graph obtained from Γ by adjoining a new edge $\{u,v\}$. For x a vertex of Γ', let $H_x=G_x$ if $x\neq u$, and let $H_u=A*G_u$. For e an edge of Γ', let $S_e'=S_e$ if $e\neq\{u,v\}$ and let $S'_{\{u,v\}}=\{akg(kh)kg(kh)^2kg(kh)^3\ldots kg(kh)^{40},bkg(kh)^{41}\ldots kg(kh)^{80}\}$

If $\{x,y\}$ is an edge of Γ' let $H_{\{x,y\}}$ be the quotient of H_x*H_y by the normal closure of $S_{\{x,y\}}$. Let

$$H=\langle H_v\ (v\in V(\Gamma'))\ ;\ S_e'\ (e\in E(\Gamma'))\rangle$$

We show (I) or (II) holds for H. The Theorem will then follow because firstly, A embeds into H (by the Freiheitssatz); secondly, by Tietze transformations that eliminate a and b, we can show that H is a quotient of G; and thirdly, any countable group can be embedded in some two generator group (see Lyndon and Schupp [27,p.188]).

For an edge $e=\{x,y\}$ of Γ' we let H_e be the quotient of

H_x*H_y by the normal closure of S'_e.

We show first that $H_{\{u,v\}}$ has property-W_2. Consider any word on H_u*H_v that defines 1 in $H_{\{u,v\}}$ but not in H_u*H_v. Then there is a reduced $S_{\{u,v\}}$-diagram representing this. Eliminate all of the vertices of this diagram of valence two, in the standard way, to obtain a diagram M. It is easily seen that any almost interior region of M has at least six sides. Thus M has a simple boundary region D with at most three internal edges (see Lyndon and Schupp [27,Chp.V]). Thus we find that the label on $\partial M \cap \partial D$ has free product length at least 1200. Thus $G_{\{u,v\}}$ has property-W_{599}! Hence $G_{\{u,v\}}$ certainly has property-W_2.

We now show that if $\{x,y\}$ is an edge of Γ' distinct from $\{u,v\}$ then $H_{\{x,y\}}$ has property-W_i if G_e has property-W_i.

Clearly if $\{x,y\}$ is an edge of Γ' with neither endpoint equal to u then the above assertion holds.

Suppose $\{u,y\}$ is an edge of Γ' distinct from $\{u,v\}$ and suppose that the assertion is false. Then there exists a word $g_1h_1\ldots g_mh_m$ $(g_1,\ldots,g_m \in H_u,\ h_1,\ldots,h_m \in H_y)$ on H_u*H_y which defines 1 in $H_{\{u,y\}}$ but not $H_u * H_y$ and for which $m < i$. Choose such a word with m as small as possible. Now, write each g_i, as an element of $A * G_u$, in normal form. Next consider the subwords of $g_1 h_1 \ldots g_m h_m$ that lie between the elements of A. At least one of these, W say, must define 1 in $G_{\{u,y\}}$. Since $G_{\{u,y\}}$ has property-W_i, W defines 1 in $G_u * G_y$. Now since we wrote the terms from $A * G_u$ in normal form no term in W from G_u is 1, hence some h_i is equal to 1 and we can create a shorter counterexample - a contradiction.

It follows that H satisfies (I) or (II).□

2.4 PROOF OF THEOREM 2.3

To prove the result we first need the following:

PROPOSITION[39]

Assumption (B) *holds if the following condition is satisfied: Every identity sequence* σ *is equivalent to a product* $\sigma_1\sigma_2\ldots\sigma_k$ *of identity sequences* σ_i $(1 \leq i \leq k)$ *such that for* $i=1,\ldots,k$ *there is an edge* $e(i)$ *of* Γ *such that all of the terms of* σ_i *belong to* $r^W_{e(i)}$.

Proof of Theorem 2.3

We prove that the condition in the above proposition is satisfied.

For $e=(u,v)$ an edge of Γ let $r'_e=r_e-(r_u \cup r_v)$ and let $r'=\bigcup_{e\in E} r'_e$.

Let σ be an identity sequence over <X ; R>. The proof is by induction on the number, $m(\sigma)$, of terms of σ in r'^W.

We proceed geometrically, using pictures. We will always assume that our pictures have no floating arcs. This can always be achieved by elimination.

Consider first the case when $m(\sigma)=0$. Let $\mathbb{P}$ be a spherical picture representing σ. Let $\mathbb{P}'$ be a spherical subpicture of

$\mathbb{P}$ containing at least one disc and which is minimal with this property. Clearly all of the discs in $\mathbb{P}'$ are labelled by elements from some $r_u^{\pm 1}$ (otherwise we would have to have a disc labelled by an element of $r_u^{\pm 1}$ joined by an arc to a disc labelled by an element of $r_v^{\pm 1}$, $u \neq v$, which is impossible). Now put a spray over the picture, the first arcs of which go to the discs in $\mathbb{P}'$. This gives us that σ is equivalent to a product $\sigma_1\sigma_2$ of identity sequences where σ_1 consists of terms from r_u^W, and the number of terms in $\sigma_1\sigma_2$ is the same as in σ. A simple induction finishes this case.

Now suppose $m(\sigma)\geq 0$. We then prove that σ is equivalent to a product $\sigma_1\sigma_2$ of identity sequences, where there exists an edge $e=\{u,v\}$ of Γ such that all of the terms of σ_1 lie in r_e^W and at least one term lies in $r_e'^W$ and $m(\sigma_1\sigma_2)=m(\sigma)$. A simple then induction completes the proof.

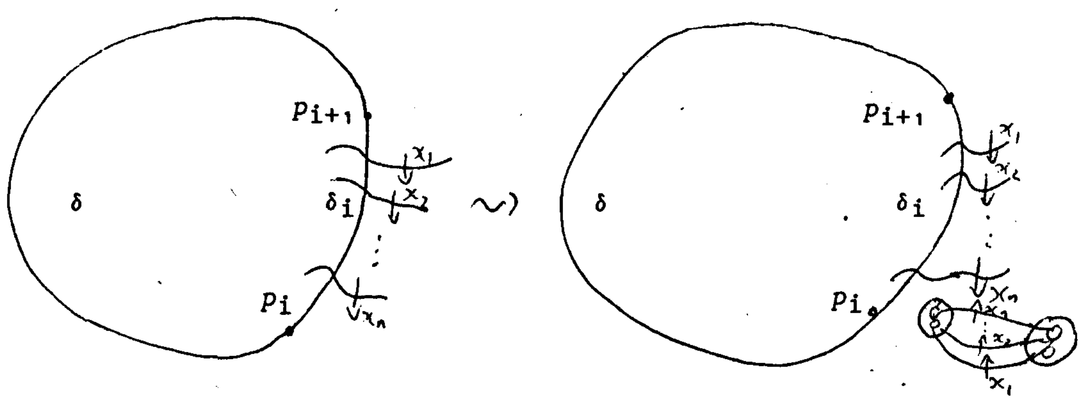
To begin, take a spherical picture $\mathbb{P}$ representing σ. Then, it turns out (see pp.83-88 below) that we can alter this picture to an equivalent picture for which there exists a simple closed path δ satisfying the following conditions.

(i) There exists an edge e of Γ such that each disc inside δ is labelled by an element of $r_e^{\pm 1}$; moreover, at least one disc is labelled by an element of $r_e'^{\pm 1}$.

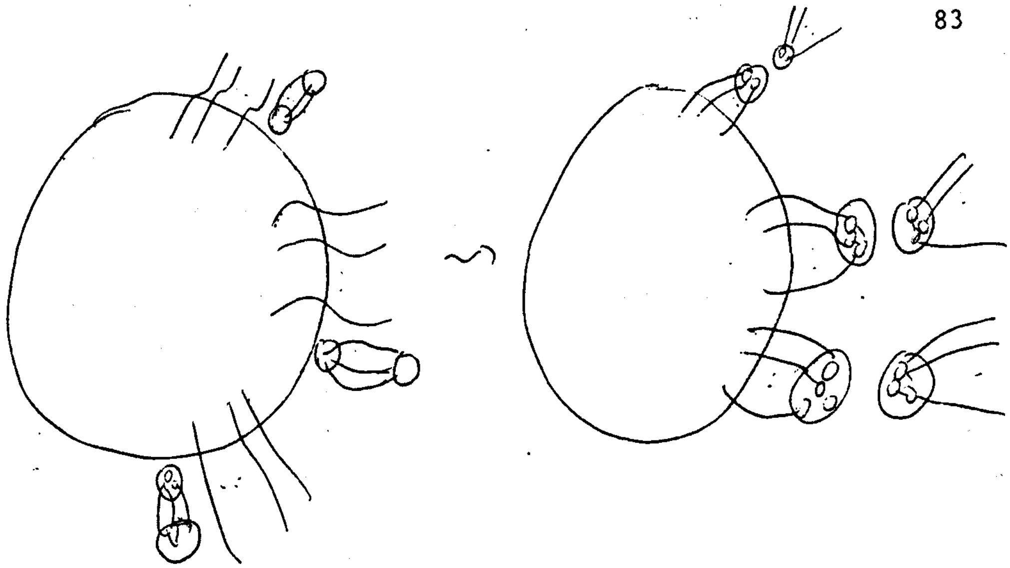
(ii) There exist n distinct points $p_1,\ldots,p_n$ on δ (none of which lies over an arc) such that if we read around δ anti-clockwise the label on the segment δ_i of δ from p_i to p_{i+1} ($i=1,\ldots,n$, subscripts computed mod n) is a word on $X_{v(i)}$ that defines the identity in

$$\langle X_{v(i)} \ ; \ r_{v(i)} \rangle.$$

Now for each segment δ_i there exists a mirror-picture $\overset{0}{\mathbb{D}}_i$ over $\langle X_{v(i)} \ ; \ r_{v(i)} \rangle$, formed from a picture $\mathbb{D}_i$ who's label is the same as the label on δ_i. We insert these into the picture in the following way.



Next we use bridge moves as follows



Now put a spray over the resulting picture, the first arcs of which go to the spherical subpicture containing $|D_i$. The result follows.

We now show how to obtain a picture equivalent to $|P$, and a path δ as above. Colour each disc labelled by an element of $r'^{\pm 1}$ red, each disc labelled by an element of $(r-r')^{\pm 1}$, blue. Colour an arc between two red discs, red; between two blue discs, blue; and all others green.

If C is a blue component of the arcs and discs of $|P$, a *C-region* is a subset Δ_C of $|P$ satisfying the following conditions

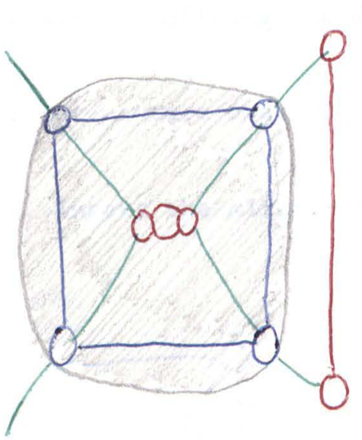
(i) Δ_C is homeomorphic to a closed disc.

(ii) Δ_C contains C.

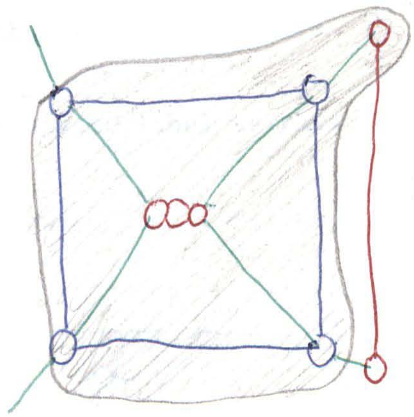
(iii) Subject to (i) and (ii), Δ_C contains as few discs as possible.

(iv) Subject to (i),(ii) and (iii), Δ_C contains as few segments of arcs as possible.

e.g.



but not



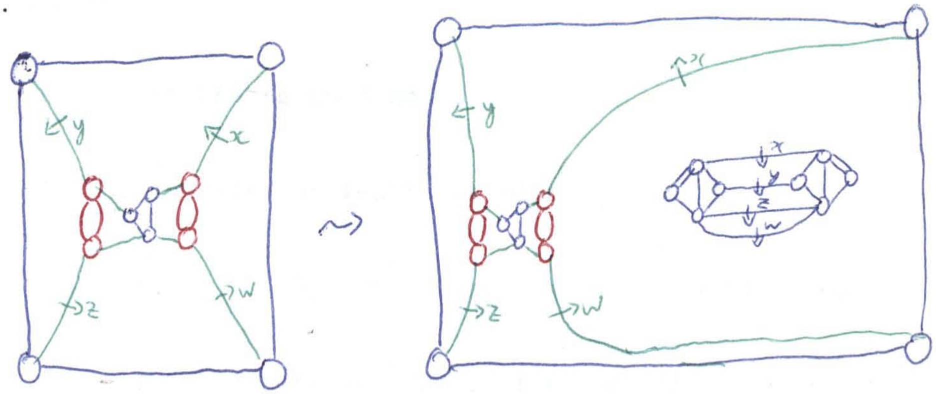
For each blue component, C, of the arcs and discs of $\mathbb{P}$ fix a C-region.

We say that C is *simply connected* if there exists a C-region that contains no red discs.

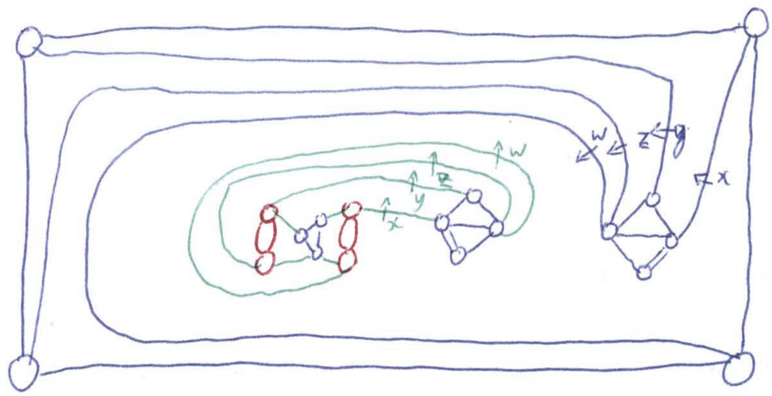
Suppose that not every blue component of $\mathbb{P}$ is simply connected. Pick a blue component C such that: (i) Δ_C contains a minimal number of red discs; (ii) subject to (i) Δ_C contains as few discs as possible. Consider a particular red disc contained in Δ_C. Clearly this red disc lies in some bounded component of the complement of C. We consider this component as a picture over <X ; R>. Clearly by minimality every blue component of this picture is simply connected.

Now the label on the boundary of this picture is a word on some X_V, defining the identity in G, hence by the Freiheitssatz, defining the identity in $<X_V ; r_V>$. Thus there is a mirror-picture over $<X_V ; r_V>$ with the same label as this picture. Insert the mirror-picture as follows.

e.g.



and then perform bridge moves to obtain



Hence $\mathbb{P}$ is equivalent to a picture with a spherical subpicture $\mathbb{P}'$ containing at least one red disc and for which every blue component is simply connected.

We now show how to put δ over $\mathbb{P}'$. To do this we need the following Lemma, the proof of which is identical to that for the analogous result in [41].

CONTRACTIBLE LOOP LEMMA

Let $\mathbb{U}$ be a spherical picture over $<G_v\ (v\epsilon V);\ S>$. Then there exists a simple closed path δ over $\mathbb{U}$ with the following properties:

(I) δ intersects no discs

(II) δ contains at least one disc.

(III) If $\Delta_1,\ldots,\Delta_n$ are all of the discs inside δ then

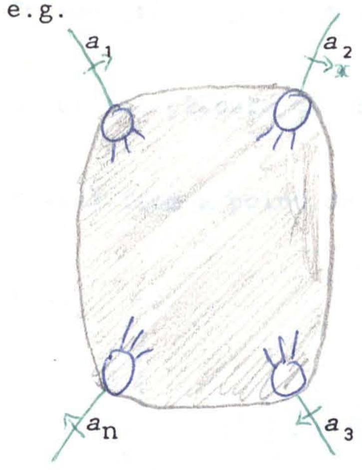
- there exists an edge e of Γ such that the labels on

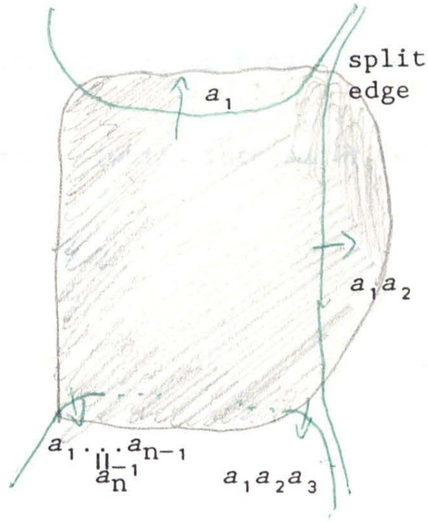
$\Delta_1,\ldots,\Delta_n$ are equal in $\underset{v\epsilon V}{*}G_v$ to elements of $S_e^{\pm 1}$.

(IV) The label on δ is equal to 1 in $\underset{v\epsilon V}{*}G_v$.□

We turn $\mathbb{P}'$ into a spherical picture over $<G_v\ (v\epsilon V);\ S>$ in the following way. Firstly relable each edge. Do this by replacing any label by the group element it represents. Then replace the blue components as follows (where the shaded area represents the chosen C-region for the component).

e.g.

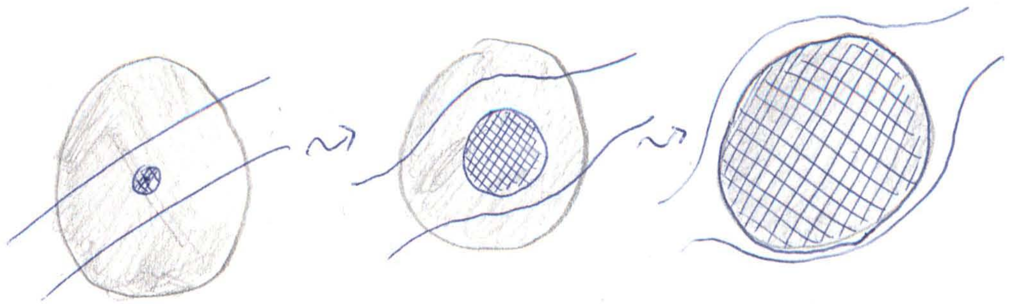




Let δ be given by the Contactible Loop Lemma. By suitable alterations we may assume that (i) δ does not intersect any C-region and (ii) δ contains no C-regions arising from blue spherical subpictures of $\mathbb{P}'$.

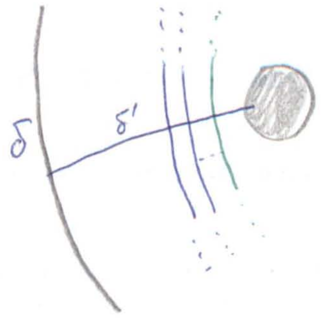
To see (i) suppose that δ does intersect some C-region Δ_C. Pick a point in Δ_C not on δ and draw a small disc around it, again not intersecting δ. Expand this disc and continuously deform the arcs of δ so that they never intersect it. In this way we may "push" all of the arcs of δ off the C-disc.

e.g.

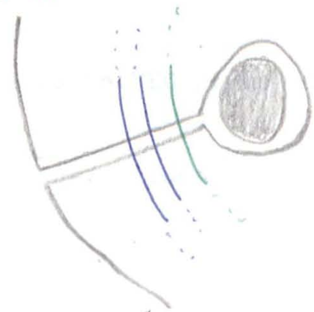


To see (ii) suppose δ contains a C-region arising from a blue spherical picture of $\mathbb{P}'$. Draw a path δ' (which intersects no discs) from a point on δ to a point on $\partial\Delta_C$.

I.e.



Then alter δ to



where the alterations are carried out "local" to δ' and Δ_C.

Considering δ as over $\mathbb{P}'$ we obtain a path as required.□

CHAPTER 3

SUBGROUPS OF NEC-GROUPS

3.1 INTRODUCTION

BACKGROUND

An NEC (Non-Euclidean Crystallographic) group is a discontinuous group of isometries (some of which may be orientation reversing) of the Non-Euclidean plane. For further information on this see Appendix A. According to Wilkie [46], a finitely generated NEC-group with compact orbit space has a presentation $\mathcal{P}$ as follows:

Involutary generators: y_{ij} $(i,j)\in\Xi_0$

Non-involutary generators: e_i $(i\in I_f)$, t_k $(1\leq k\leq r)$

a_k $(1\leq k\leq g)$, b_k $(1\leq k\leq h$, $h=0$ or $g)$

(3.1) Defining paths: $(y_{ij}y_{ij+1})^{m_{ij}}$ $(i\in I_f,\ 1\leq j\leq n(i)-1)$

$(y_{in(i)}e_iy_{i1}e_i^{-1})^{m_{in(i)}}$ $(i\in I_f)$

$t_k^{p_k}$ $(1\leq k\leq r,\ p_k\geq 2)$

$\prod_i(e_i^{-1})(\prod_k t_k^{-1})\alpha$

where

$$\alpha = \begin{cases} \prod_k a_k^2 & \text{if } h=0 \\ \prod_k a_kb_ka_k^{-1}b_k^{-1} & \text{if } h=g, \end{cases}$$

In Hoare, Karrass and Solitar [22] it is shown that a subgroup of finite index in a group with a presentation of the form (3.1), has itself a presentation of the form (3.1). In [22] the same authors show that a subgroup of infinite index in a group with a presentation of the form (3.1) is a free product of groups of the following types:

(A) Cyclic groups.

(B) Groups with presentations of the form

$$< x_1,\ldots,x_n,e\ ;\ (x_1x_2)^{m_1},\ldots,(x_nex_1e^{-1})^{m_n} >$$

$x_1,\ldots,x_n$ involutary.

(C) Groups with presentations of the form

$$< x_i\ (i\epsilon Z)\ ;\ (x_ix_{i+1})^{m_i}\ (i\epsilon Z) >$$

x_i $(i\epsilon Z)$ involutary.

In this chapter we are going to define what we mean by an *NEC-complex*. It will be obvious from the definition that this class of complexes is closed under coverings, so that the class of fundamental groups of NEC-complexes is trivially closed under taking subgroups. Our aim is then to obtain structure theorems for both finite and infinite NEC-complexes.

We show that the fundamental group of a finite NEC-complex has a presentation of the form (3.1) and that the fundamental group of an infinite NEC-complex is a free product of groups of the forms (A), (B) and (C) above.

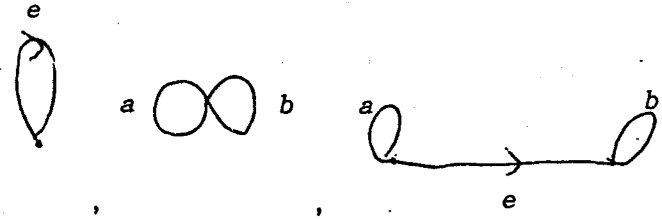
The usual approach to subgroup theorems for NEC-groups is to specify the groups by means of presentations and then try and show that every subgroup can be specified by a similar presentation. The approach here is different and has several advantages: (i) By using complexes, rather than presentations, we avoid a lot of technicalities involving the Reidermeister-Schreier rewriting process; (ii) by allowing involutary edges we get a more streamlined use of the star-complex (= coinitial graph), and avoid having to consider coinitial graphs of presentations with "identifying relators" (as defined in [22]); (iii) The results of Hoare, Karrass and Solitar [20], [21] and [22] are unified, and the proofs considerably shortened; (iv) modulo an understanding of the basic theory of complexes, the arguments are straight forward and quite transparent.

The approach is analogous to the geometric proof of the Neilsen-Schreier Theorem [27,p.119]. There one looks at the class of graphs. This is clearly closed under taking coverings. One then shows that the fundamental group of a graph is free.

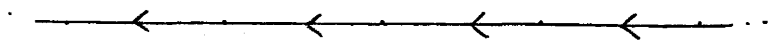
This work can also be viewed in a wider context as part of a general program to study groups through properties of star-complexes, i.e specifying some structural restriction on the star-complex of a complex and seeing what this tells us about the fundamental group of the complex. See [11], [13], [14], [16], [18], [37], [38].

NEC-COMPLEXES

A *circle* is a connected 1-complex such that $|star(x)|=2$ for each vertex x. We also require that there are no loops in the 1-complex, in order to avoid pathologies like the following:



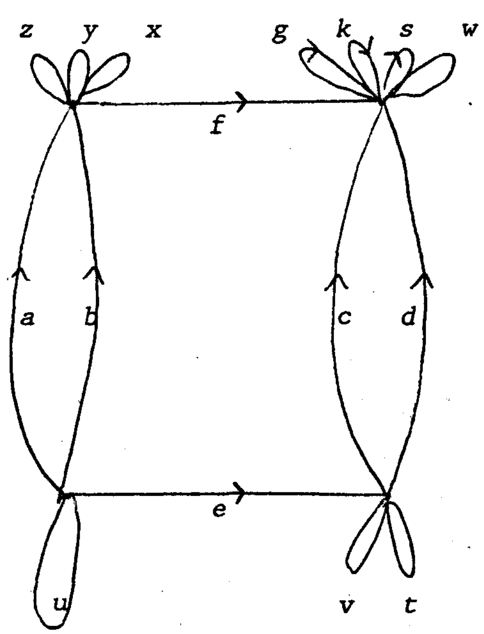
Note that we allow a circle to be infinite, so that it may, in fact, be a "line" stretching off to infinity in both directions



We define an *NEC-complex* to be a connected, slender complex $\mathcal{K}$ such that $\mathcal{K}^{st}(v)$ is a circle for each vertex v of $\mathcal{K}$. A *Fuchsian-complex* is an NEC-complex with no involutary edges, and a *surface-complex* is a Fuchsian-complex with all defining paths of period one. We use N,F and S to denote the classes of NEC-complexes, Fuchsian-complexes and surface-complexes respectively.

<u>EXAMPLE</u>

$\mathcal{K}$ =



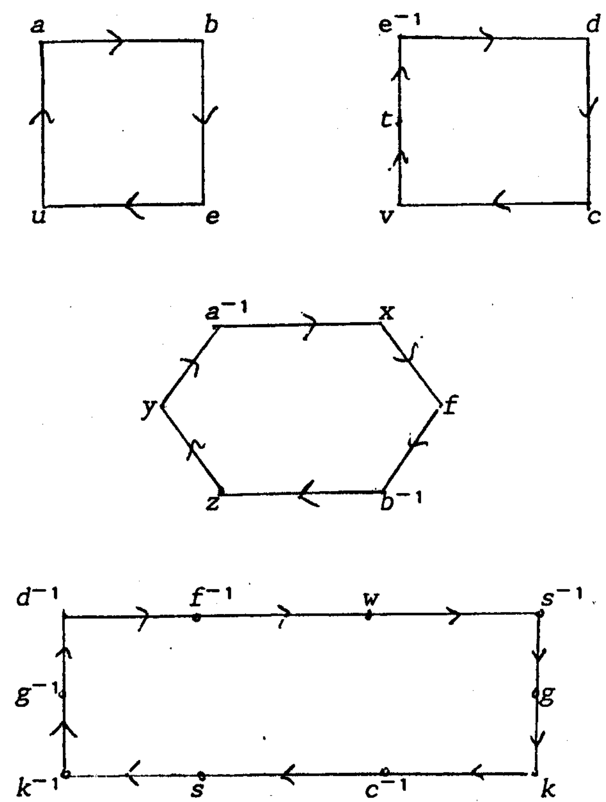
$(ayzb^{-1})^3, (csgk^{-1}gd^{-1})^3$

$(bfd^{-1}e^{-1})^4, (etvte^{-1}u)^5$

; $(vcksws^{-1}k^{-1}c^{-1})^4$

$(fwf^{-1}x)^2, (uaxa^{-1})^3$

$\mathcal{N}^{st}$



It follows from Theorem 1.4 that N is closed under coverings; also F and S are closed under coverings (F by a remark in §1.1, and S since if $\varphi: \mathcal{A} \to \mathcal{B}$ is a covering and all defining paths of $\mathcal{B}$ have period one, then the same is true for $\mathcal{A}$). We thus see that the Subgroup Lemma (see §1.1) applies for the classes N, F and S.

It should, however, be noted that our use of the term "Fuchsian" is not strictly correct. In the finite case, for example, the term should really only apply to complexes for which the path α in Theorem 3.1 terminates in a product of commutators rather than a product of squares. However, we will use the term in this wider sense (cf. [27,p126]). We also note that the Fuchsian-complexes defined here are not the same as those defined in [27, section III.7].

DEFINING PATHS AND CHAINS IN NEC-COMPLEXES

Let $\mathcal{K}$ be an NEC-complex.

We will say that a defining path ρ of $\mathcal{K}$ is of *type* I if, whenever we have a cyclic permutation of $\overset{0}{\rho}$ of the form $a\gamma$ with a involutary then $\gamma \neq \gamma^{-1}$. We will say that a defining path ρ of $\mathcal{K}$ is of *type* II (respectively *type* III) if some cyclic permutation of $\overset{0}{\rho}$ has the form $a\alpha a\alpha^{-1}$ with a involutary (respectively $a\alpha b\alpha^{-1}$, with $a \neq b$ and a, b involutary).

Remark: It will be seen from Lemmata 3.1, 3.2 and 3.3 that no path can be of two different types.

LEMMA 3.1

If ρ is of type I, *and if e is an involutary edge occuring in ρ then $L_e(\overset{0}{\rho})=1$, and e does not occur in any other defining path.*

Proof

Some cyclic permutation of $\overset{0}{\rho}$ will have the form $e\alpha$. By assumption $\alpha \neq \alpha^{-1}$, so $(e\alpha)P(\rho)$, $(e\alpha^{-1})P(\rho)$ are distinct edges of $\mathcal{K}^{st}$ starting at e. It follows immediately that no other defining path of $\mathcal{K}$ can contain e.

Suppose now that $L_e(\alpha) \neq 0$, so that $\alpha = \alpha_1 e \alpha_2$, ($\alpha_1, \alpha_2$ non-empty and reduced since ρ is cyclically reduced). Then $(e\alpha_2 e\alpha_1)P(\rho)$ is an edge of $\mathcal{K}^{st}$ starting at e, and so must be one of $(e\alpha)P(\rho)$, $(e\alpha^{-1})P(\rho)$. However it cannot be the former since $\overset{0}{\rho}$ is not a proper power. But neither can it be the latter, for otherwise we would have $\alpha_1 = \alpha_1^{-1}$ and $\alpha_2 = \alpha_2^{-1}$. Then $\alpha_1 = \beta c \beta^{-1}$, $\alpha_2 = \gamma d \gamma^{-1}$ where c and d are involutary edges, and hence $c\beta^{-1} e \gamma d \gamma^{-1} e \beta$ is a cyclic permutation of $\overset{0}{\rho}$ with $\beta^{-1} e \gamma d \gamma^{-1} e \beta$ equal to its own inverse - a contradiction.□

LEMMA 3.2

If ρ is of type II, and if e is an involutary edge occuring in ρ then $L_e(\overset{0}{\rho}) = 2$, and e does not occur in any other defining path. If $a\alpha a\alpha^{-1}$ and $a_1\alpha_1 a_1 \alpha_1^{-1}$ are two cyclic permutations of $\overset{0}{\rho}$ with a, a_1 involutary then $a = a_1$ and $\alpha = \alpha_1^{\pm 1}$.

Proof

It is clear that $L_e(\overset{0}{\rho})$ is even, and since the valence of e in $\mathcal{K}^{st}$ is at least $L_e(\overset{0}{\rho})$ it must be precisely two. Obviously, then, no other defining path can contain e.

To prove the second part, suppose, by way of a contradiction that $a \neq a_1$. Then a_1 must occur in α, so $\alpha = \beta a_1 \gamma$ say, and we get that $a_1 \alpha_1 a_1 \alpha_1^{-1}$ must be one of $a_1 \gamma a \gamma^{-1} a_1 \beta^{-1} a \beta$, $a_1 \beta^{-1} a \beta a_1 \gamma a \gamma^{-1}$. In either case we deduce that $\beta = \gamma^{-1}$, so that $\overset{0}{\rho}$ is a cyclic permutation of $(a\beta a_1 \beta^{-1})^2$, contradicting the fact that $\overset{0}{\rho}$ is not a proper power.□

LEMMA 3.3

If $\bar{\rho}$ *is of type* III *then there are distinct involutary edges* a *and* b *with* $L_a(\overset{0}{\rho}) = L_b(\overset{0}{\rho}) = 1$. *There are unique defining paths* ρ_1 *and* ρ_2 *(both of type* III*) different from* ρ *with* $L_a(\overset{0}{\rho}_1) = L_b(\overset{0}{\rho}_2) = 1$. *If* e *is an involutary edge different from* a *or* b *occuring in* ρ, *then* $L_e(\overset{0}{\rho}) = 2$ *and* e *does not occur in any other defining path.*

Proof

By assumption, $\overset{0}{\rho}$ has a cyclic permutation $a\alpha b \alpha^{-1}$ with a

and b distinct and involutary. Thus $L_a(\overset{0}{\rho})$ is odd and since the valence of a in $\mathcal{K}^{st}$ is at least $L_a(\overset{0}{\rho})$ we must have $L_a(\overset{0}{\rho})=1$. Similarly $L_b(\overset{0}{\rho})=1$. Clearly if e is an involutary edge different from a and b occuring in ρ then e occurs in α, so $L_e(\overset{0}{\rho})$ is even and hence must be two, and e cannot occur in any other defining path.

Now ρ contributes only one edge to $\mathcal{K}^{st}$ starting at a, namely the edge $(a\alpha b\alpha^{-1})^{P(\rho)}$. Hence a must occur in some other defining path ρ_1, which by Lemmata 3.1 and 3.2 must be of type III. Clearly $L_a(\overset{0}{\rho}_1)=1$. Similarly for b.□

We let Ξ_0 denote the set of $\xi\epsilon\Xi$ such that ρ_ξ is of type II or III. It follows from Lemmata 3.2 and 3.3 that we can arrange the defining paths ρ_ξ $(\xi\epsilon\Xi_0)$ into *chains*, which we now describe.

If ρ is a path of type III let $j(\rho)$ be the two element set containing the edges a,b given by Lemma 3.3. Define two type III paths ρ and ρ' to be equivalent if and only if there is a finite sequence

$$\rho=\rho_1,\rho_2,\ldots,\rho_n=\rho'$$

of defining paths of type III where $j(\rho_i) \cap j(\rho_{i+1}) \neq \emptyset$ $(i=1,\ldots,n-1)$.

A *finite chain* is either a path of type II or consists of the elements of a finite equivalence class. An *infinite chain* consists of the elements of an infinite equivalence class.

It is convenient to take the elements of Ξ_0 to be ordered pairs which reflect this arrangement. There will be elements

$$(i,j) \quad i \in I_f \quad 1 \leq j \leq n(i), \text{ j computed mod } n(i),$$

coming from *finite chains*, and elements

$$(i,j) \quad i \in I_\infty \quad j \in \mathbf{Z},$$

coming from *infinite chains*. By cyclically permuting, if necessary, we may write

$$\rho(i,j) = (x_{ij} A_{ij} x_{ij+1} A_{ij}^{-1})^{m_{ij}}$$

where x_{ij}, x_{ij+1} are involutary edges, and m_{ij} is the period of $\rho_{(i,j)}$. The x_{ij}'s are called the *chain edges*.

The *period cycles* are the sequences

$$(m_{i1}, m_{i2}, \ldots, m_{in(i)}) \quad (i \in I_f)$$

$$(\ldots, m_{i,-1}, m_{i0}, m_{i1}, \ldots) \quad (i \in I_\infty)$$

The *proper periods* are the periods $p(\rho_\xi)$ $(\xi\epsilon\Xi-\Xi_0,\ p(\rho_\xi)\geq 2)$, together with a list of twos, one for each involutary edge which is not a chain edge.

EXAMPLE (CONTINUED)

There is one (finite) chain

$(uetvte^{-1})^5, (vcksws^{-1}k^{-1}c^{-1})^4, (wf^{-1}xf)^2, (xa^{-1}ua)^3.$

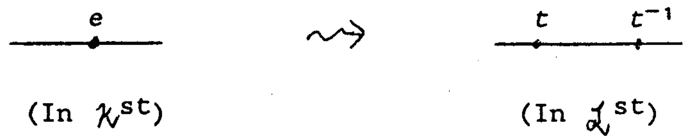
The corresponding period cycle is (5,4,2,3). The proper periods are 3,3,4,2,2,2 since there are three involutary edges which are not chain edges in $\mathcal{K}$, namely y,z and t.

It will be convenient later to assume that $\mathcal{K}$ has no involutary edges except the chain edges. This can always be achieved by modifying $\mathcal{K}$ as follows.

Let e_s $(s\epsilon S)$ be the collection of all involutary edges which are not chain edges. Introduce new non-involutary edges $t_s^{\pm 1}$ $(s\epsilon S)$ where $\iota(t_s)=\tau(t_s)=\iota(e_s)$. Now e_s will appear once in the root of some defining path of type I (and nowhere else), or will appear twice in the root of some defining path $\rho_{(i,j)}$ of type II or III (and nowhere else). In the former case replace the occurrence of e_s by t_s; in the latter case replace the occurrence of e_s in A_{ij} by t_s and replace the occurrence of e_s in A_{ij}^{-1} by t_s^{-1}. Delete the edge e_s from $\mathcal{K}$, and add a new defining path t_s^2 for each $s \in S$, to obtain a complex $\mathcal{L}$. Then $\mathcal{K}$ is equivalent to $\mathcal{L}$.

We show in general that $\mathcal{L}$ is an NEC-complex. If r is a non-involutary edge or a chain edge of $\mathcal{K}$ then since in changing from $\mathcal{K}$ to $\mathcal{L}$ we do not alter the number of edges in the star-complex beginning with r, r has valence two in $\mathcal{L}^{st}$. We now look at the vertices $t_s^{\pm 1}$ of $\mathcal{L}^{st}$. Each of these has precisely one edge incident to it arising from t_s^2, and by construction of $\mathcal{L}$ precisely one edge of $\mathcal{L}^{st}$ arising from the modified defining paths of $\mathcal{K}$ is incident to t_s and precisely one to t_s^{-1}. Thus every vertex of $\mathcal{L}^{st}$ has valence two.

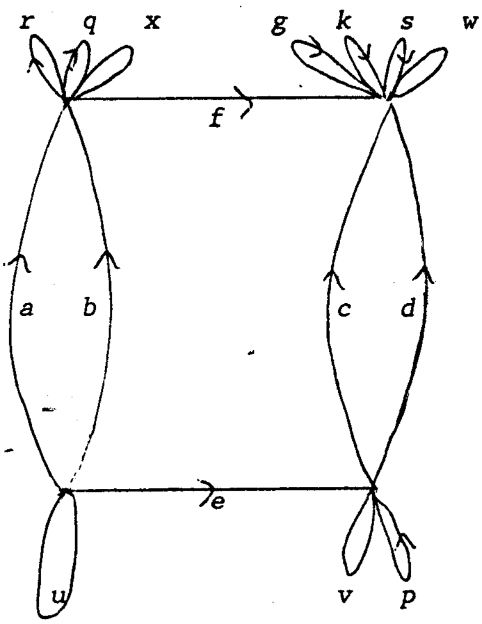
We now show that $\mathcal{L}^{st}(v)$ is connected for all vertices of $\mathcal{L}$. Let x,y be vertices of $\mathcal{L}^{st}(v)$. Then these arise from two vertices $\hat{x},\hat{y}$ in $\mathcal{K}^{st}(v)$. Now there is a path from $\hat{x}$ to $\hat{y}$ in $\mathcal{K}^{st}(v)$. This path will involve only finitely many vertices from the set $\{e_s\colon s \in S\}$. In passing from $\mathcal{K}$ to $\mathcal{L}$ each such vertex e is "expanded"



Thus it is easily seen that x and y are connected in $\mathcal{L}^{st}(v)$ and hence $\mathcal{L}$ is an NEC-complex.

EXAMPLE (CONTINUED)

Making the above modifications we obtain the complex $\mathcal{L}$.



; $(aqrb^{-1})^3, (csgk^{-1}gd^{-1})^3$

$(bfd^{-1}e^{-1})^4, (uepvp^{-1}e^{-1})^5$

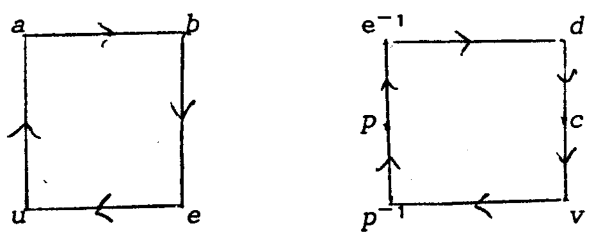
$(vcksws^{-1}k^{-1}c^{-1})^4$

$(wf^{-1}xf)^2, (xa^{-1}ua)^3$

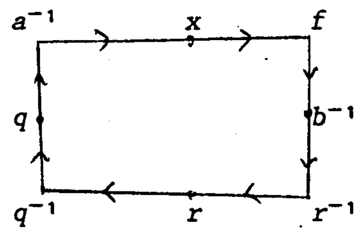
p^2, q^2, r^2

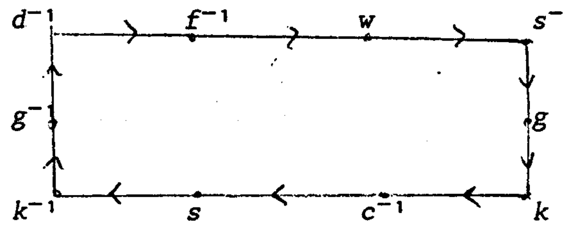
where t, y, z have been replaced by $p^{\pm 1}, q^{\pm 1}, r^{\pm 1}$ respectively.

$\mathcal{L}^{st}$



$\mathcal{L}^{st}$ (continued)





3.2 FINITE NEC-COMPLEXES

Let $\mathcal{K}$ be an NEC-complex.

THEOREM 3.1

If $\mathcal{K}$ is finite then there is a based equivalence from $\mathcal{K}$ to a presentation $\mathcal{P}$ of the form

Involutary edges: y_{ij} $(i,j)\in\Xi_0$

Non-involutary edges: $e_i^{\pm1}$ $(i\in I_f)$, $t_k^{\pm1}$ $(1\leq k\leq r)$

$a_k^{\pm1}$ $(1\leq k\leq g)$, $b_k^{\pm1}$ $(1\leq k\leq h,\ h=0 \text{ or } g)$

Defining paths: $(y_{ij}y_{ij+1})^{m_{ij}}$ $(i\in I_f,\ 1\leq j\leq n(i)-1)$

$(y_{in(i)}e_iy_{i1}e_i^{-1})^{m_{in(i)}}$ $(i\in I_f)$

$t_k^{p_k}$ $(1\leq k\leq r,\ p_k\geq 2)$

(3.2) $$\prod_i(e_i^{\epsilon_i})(\prod_k t_k^{-1})\alpha$$

where

$$\alpha = \begin{cases} \prod_k a_k^2 & \text{if } h=0 \\ \prod_k a_kb_ka_k^{-1}b_k^{-1} & \text{if } h=g, \end{cases}$$

and where $\epsilon_i=\pm1$ $(i\in I_f)$. If $\mathcal{K}\in F$ then there are no y's or e's, and if $\mathcal{K}\in S$ then there are no y's, e's or t's.

Remarks: (i) The period cycles and proper periods of $\mathcal{P}$ are the same as those of $\mathcal{K}$.

(ii) We may make $\epsilon_i=-1$ if for those i for which $\epsilon_i=1$ we replace $(y_{in(i)}e_iy_{i1}e_i^{-1})^{m_{in(i)}}$ by $(y_{in(i)}e_i^{-1}y_{i1}e_i)^{m_{in(i)}}$.

Proof

The proof consists of six reductions:

FIRST REDUCTION: *Modify* $\mathcal{K}$ *so that there are no involutary edges except the chain edges.*

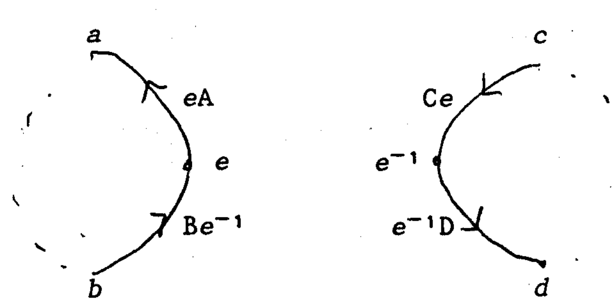
This has already been dealt with, and we note that the resulting complex, $\mathcal{L}$, is an NEC-complex.

SECOND REDUCTION: *Collapse a maximal subtree of* $\mathcal{L}$ *to obtain a presentation* $\mathcal{P}_1$.

We show that $\mathcal{P}_1$ is an NEC-complex. This can be seen by examining the effect on the star-complex of $\mathcal{L}$ of collapsing a single edge pair $\{e,e^{-1}\}$ with $\iota(e)\neq\tau(e)$, and then iterating the process. Let $\mathcal{M}$ be obtained from $\mathcal{L}$ by collapsing $\{e,e^{-1}\}$. If γ is a path in $\mathcal{L}$ denote by $\hat{\gamma}$ the path in $\mathcal{M}$ obtained from γ by removing every occurrence of $e^{\pm 1}$.

The situation in $\mathcal{L}^{st}$ is as follows.

(3.3)

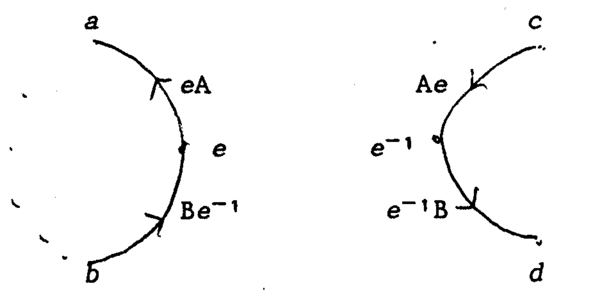


Now since eA and eB^{-1} are distinct, $e^{-1}A^{-1}$ and $e^{-1}B$ are distinct. Thus

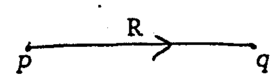
$$\{e^{-1}A^{-1}, e^{-1}B\} = \{e^{-1}C^{-1}, e^{-1}D\}.$$

Without loss of generality we take A=C and B=D, thus (3.3) becomes

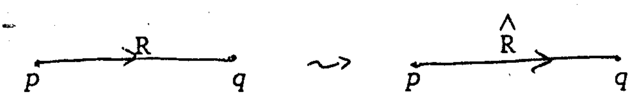
(3.4)



Let

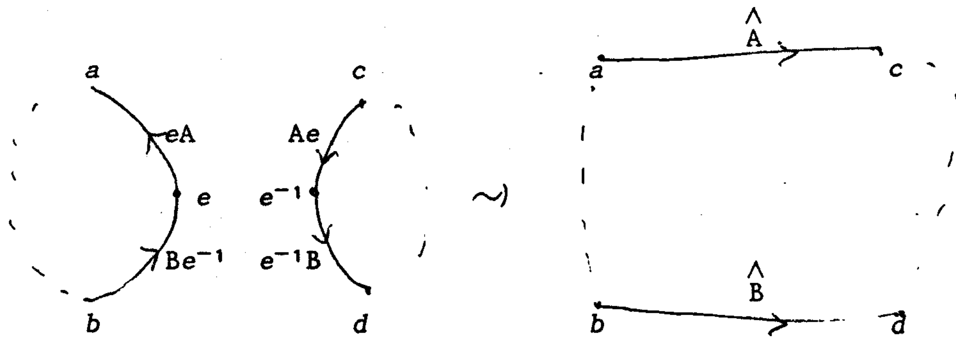


be an edge of $\mathcal{L}^{st}$ with neither endpoint e or e^{-1}. Then on collapsing $e^{\pm 1}$ we have



since R does not begin or end with e or e^{-1}.

Thus with the exception of (3.4) the star-complex of $\mathcal{M}$ is isomorphic to that of $\mathcal{L}$. Now A does not begin or end with e or e^{-1}, and similarly for B, so we have



So $\mathcal{M}^{st}(v)$ is a circle for all vertices v. Hence by induction, $\mathcal{P}_1$ is an NEC-complex.

<u>THIRD REDUCTION</u>: *Modify the chains of $\mathcal{P}_1$ as follows, to obtain a presentation $\mathcal{P}_2$.*

Clearly chains in $\mathcal{L}$ go to chains in $\mathcal{P}_1$. Now suppose

$$(x_1A_1x_2A_1^{-1})^*,\ldots,(x_{n-1}A_{n-1}x_nA_{n-1}^{-1})^*,(x_nA_nx_1A_n^{-1})^*$$

is a chain in $\mathcal{P}_1$. Replace it by

$$(y_1y_2)^*,\ldots,(y_{n-1}y_n)^*,(y_ney_1e^{-1})^*,e^{-1}A_1\ldots A_n,$$

$y_1,\ldots,y_n$ involutary, $e^{\pm 1}$ non-involutary. (The resulting complex is equivalent to $\mathcal{P}_1$ under the mapping defined by

$$y_i \mapsto A_1\ldots A_{i-1}x_iA_{i-1}^{-1}\ldots A_1^{-1} \qquad 1\leq i\leq n$$

$$e_i^{\pm 1} \mapsto (A_1\ldots A_n)^{\pm 1}.)$$

We now consider how the above operation affects the star-complex:

(i) $$\underset{}{Yx_{i+1}A_i^{-1}x_iX} \rightsquigarrow YA_{i+1}\ldots A_n e^{-1}A_1\ldots A_{i-1}X$$

($XY=A_i$, X, Y non-empty).

(ii) $$\underbrace{(A_jx_{j+1}A_j^{-1}x_j)^*\ (x_jA_{j-1}^{-1}x_{j-1}A_{j-1})^*}_{x_j} \rightsquigarrow A_j\ldots A_n e^{-1}A_1\ldots A_{j-1}$$

($1<j\leq n$).

(iii) $$\underbrace{(A_n^{-1}x_nA_nx_1)^*\ (x_1A_1x_2A_1^{-1})^*}_{x_1} \rightsquigarrow$$

$$A_n^{-1}\ldots A_1^{-1}e \underset{e^{-1}}{\ } (e^{-1}y_ney^1)^* \underset{y_1}{\ } (y_1y_2)^* \underset{y_2}{\ } \ldots$$

$$\ldots \underset{y_{n-1}}{\ } (y_{n-1}y_n)^* \underset{y_n}{\ } (y_ney_1e^{-1})^* \underset{e}{\ } eA_n^{-1}\ldots A_1^{-1}$$

It now follows that $\mathcal{P}_2$ is an NEC-complex.

<u>FOURTH REDUCTION</u>: *Modify the defining paths of $\mathcal{P}_2$ which have period at least two and which do <u>not</u> belong to any chain, as follows, to obtain a presentation $\mathcal{P}_3$.*

Let ρ be such a defining path. Add new non-involutary edges $t^{\pm 1}$, add new defining paths

$$t^{-1}\overset{0}{\rho}, tP(\rho),$$

then delete ρ. (The resulting complex is equivalent to $\mathcal{P}_2$ under the mapping which sends $t^{\pm 1}$ to $\overset{0}{\rho}{}^{\pm 1}$ and is the identity on all other edges.)

The effect of this on the star-complex is as follows

(i) $\xrightarrow{(\gamma_2\gamma_1)^{P(\rho)}} \quad \rightsquigarrow \quad \xrightarrow{\gamma_2 t^{-1}\gamma_1}$

($\gamma_1\gamma_2 = \overset{0}{\rho}$, γ_1,γ_2 non-empty).

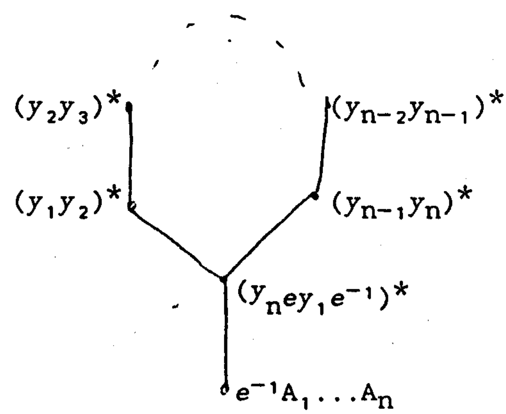
(ii) $\xrightarrow{\rho} \quad \rightsquigarrow \quad \xrightarrow{\overset{0}{\rho}t^{-1}} \underset{t}{\bullet} \xrightarrow{t^{P(\rho)}} \underset{t^{-1}}{\bullet} \xrightarrow{t^{-1}\overset{0}{\rho}}$

It now follows that $\mathcal{G}_3$ is an NEC-complex.

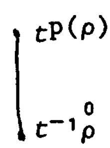
We describe the form of CG($\mathcal{G}_3$). Each chain

$$(y_1y_2)^*,\ldots,(y_{n-1}y_n)^*,(y_ney_1e^{-1})^*$$

arising from the third reduction gives rise to a "hoop"

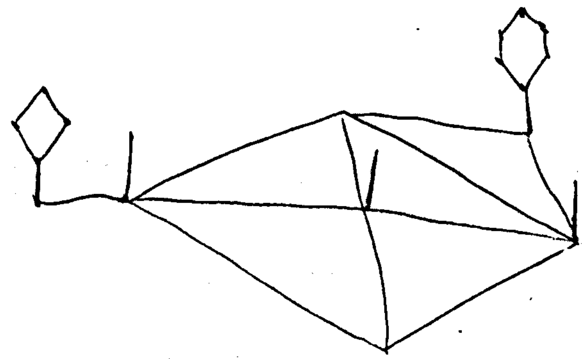


and each pair $t^{P(\rho)}, t^{-1}\overset{0}{\rho}$ arising from the fourth reduction gives rise to a "stalk"



These "hoops" and "stalks" are each attached by a single vertex to the full subgraph on Q, where Q is the set of defining paths of $\mathcal{G}_3$ minus the chains and powers of t's.

Thus, since by Proposition 1.1 it is connected, $CG(\mathcal{P}_3)$ "looks like"



where the part of $CG(\mathcal{P}_3)$ "lying in the plane" is the full subgraph on Q.

<u>FIFTH REDUCTION</u>: *Modify* $\mathcal{P}_3$ *to obtain a presentation* $\mathcal{P}_4$ *in which* Q *is replaced by a single defining path* β*, and all other paths are unaltered.*

To see how to do this, observe first that CG(Q) is connected by the above discussion. Moreover it can be quadratically labelled by virtue of the following three observations, where $\{\rho,\sigma\}$ is an edge of CG(Q).

(i) The label on $\{\rho,\sigma\}$ must be non-involutary.

(ii) If $e\epsilon E(\rho)\cap E(\sigma)$ then $L_e(\mu)=0$ for $\mu\neq\rho,\sigma$ (otherwise e would have valence at least three in $\mathcal{P}_3{}^{st}$.)

(iii) $L_e(\rho)=L_e(\sigma)=1$. Suppose, by way of a contradiction, that this was not true. Say $L_e(\rho)\geq 2$. Then since $\mathcal{P}_3$ is an NEC-complex ρ must give rise to precisely one edge in $\wp_3{}^{st}$ beginning at e. Let $e\alpha$ be a cyclic permutation of ρ or ρ^{-1}. If $\alpha=\alpha_1 e\alpha_1$, ρ gives rise to two edges in $\wp_3{}^{st}$ beginning at e since it is not a proper power. So $\alpha=\alpha_1 e^{-1}\alpha_2$ (α_1,α_2 non-empty) Hence $\alpha_1=\alpha_1^{-1}$ - a contradiction (α_1 contains no involutary edges).

Hence by an application of the level method (see §1.3) we may replace Q by a single defining path and leave all other defining paths unaltered.

<u>SIXTH REDUCTION</u>: *Modify β to be of the form* (3.2).

The procedure for doing this is well known (see Henle [17,§21] and [20]) and will not be given here in detail (see the example below for an illustration). The strategy is roughly as follows. Note that in β each e arising in the third reduction and each t arising in the fourth reduction is involved precicely once, and all other edges are involved precisely twice. We first bring the e's to the front, and then bring the t's to the front, inverting as neccessary. Next we turn the remainder of the defining path into a product of squares followed by a product of commutators. Finally, if there are any squares we turn the product of squares and commutators into a product of squares.□

EXAMPLE (CONTINUED)

We illustrate the above steps for our example.

First reduction: Already done (see p.102).

Second reduction: Collapsing the maximal subtree of consisting of the edges $a^{\pm 1}, f^{\pm 1}$ and $e^{\pm 1}$ gives the following presentation $\mathcal{P}_1$.

Involutary edges : u, v, w, x

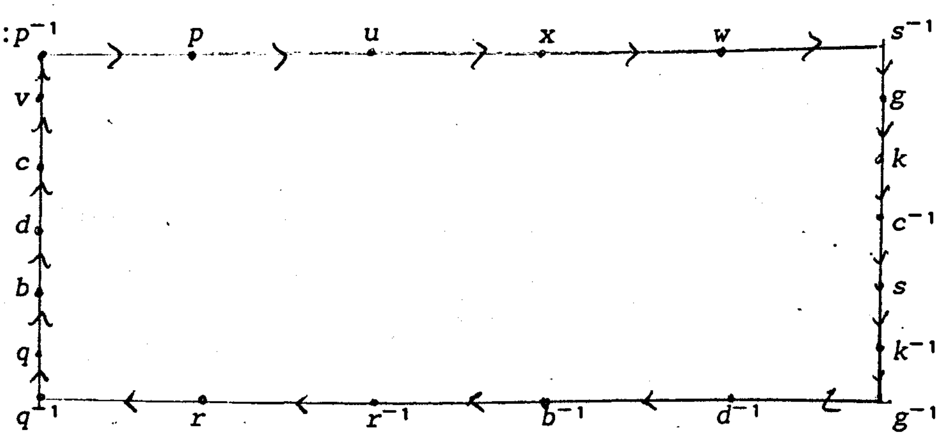
Non-involutary edges: $b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, g^{\pm 1}, k^{\pm 1}, s^{\pm 1}, p^{\pm 1}, q^{\pm 1}, r^{\pm 1}$

Defining paths : $(upvp^{-1})^5$, $(vcksws^{-1}k^{-1}c^{-1})^4$, $(wx)^2$, $(xu)^3$

$(qrb^{-1})^3$, $(csgk^{-1}gd^{-1})^3$, $(bd^{-1})^4$,

p^2, q^2, r^2

$\mathcal{P}_1^{st}$:



<u>Third reduction</u>: Modifying the chain $(upvp^{-1})^5$, $(vcksws^{-1}k^{-1}c^{-1})^4$, $(wx)^2, (xu)^3$ of $\mathcal{P}_1$ gives the following presentation $\mathcal{P}_2$.

Involutary edges : y_1, y_2, y_3, y_4

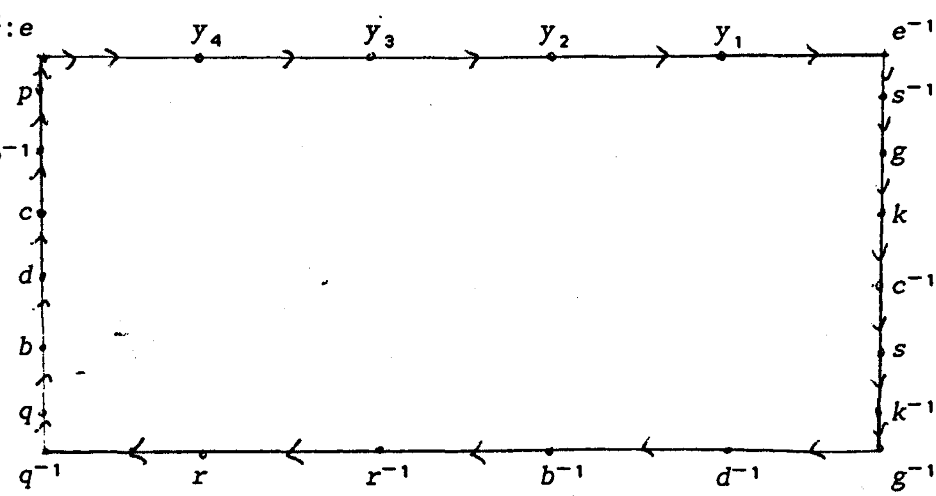
Non-involutary edges: $e^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, g^{\pm 1}, k^{\pm 1}, s^{\pm 1}, p^{\pm 1}, q^{\pm 1}, r^{\pm 1}$

Defining paths : $(y_1y_2)^5, (y_2y_3)^4, (y_3y_4)^2, (y_4ey_1e^{-1})^3$

$e^{-1}pcks$

$(qrb^{-1})^3, (csgk^{-1}gd^{-1})^3, (bd^{-1})^4, p^2, q^2, r^2$

$\mathcal{P}_2^{st}$:



<u>Fourth reduction</u>: Modifying the defining paths $(qrb^{-1})^3, (csgk^{-1}gd^{-1})^3, (bd^{-1})^4, p^2, q^2, r^2$ of $\mathcal{P}_2$ gives the following presentation $\mathcal{P}_3$.

Involutary edges : y_1, y_2, y_3, y_4

Non-involutary edges: $t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}$

$e^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, g^{\pm 1}, k^{\pm 1}, s^{\pm 1}, p^{\pm 1}, q^{\pm 1}, r^{\pm 1}$

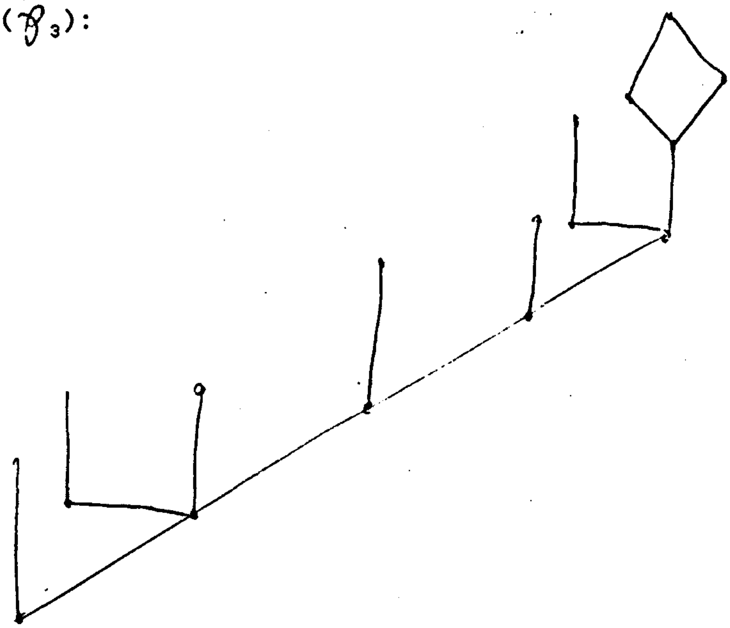
Defining paths : $(y_1y_2)^5$, $(y_2y_3)^4$, $(y_3y_4)^2$, $(y_4ey_1e^{-1})^3$

$e^{-1}pcks$

$t_1^{-1}qrb^{-1}$, $t_2^{-1}csgk^{-1}gd^{-1}$, $t_3^{-1}bd^{-1}$

$t_4^{-1}p$, $t_5^{-1}q$, $t_6^{-1}r$, t_1^3, t_2^3, t_3^4, t_4^2, t_5^2, t_6^2

$CG(\mathcal{P}_3)$:



$Q=\{e^{-1}pcks,\ t_1^{-1}qrb^{-1},\ t_2^{-1}csgk^{-1}gd^{-1},\ t_3^{-1}bd^{-1},\ t_4^{-1}p,\ t_5^{-1}q,\ t_6^{-1}r\}$

<u>Fifth reduction:</u> Replacing Q by a single defining path gives the following presentation $\mathcal{P}_4$.

Involutary edges : y_1, y_2, y_3, y_4

Non-involutary edges: $t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}$

$e^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, g^{\pm 1}, k^{\pm 1}, s^{\pm 1}, p^{\pm 1}, q^{\pm 1}, r^{\pm 1}$

Defining paths : $(y_1y_2)^5$, $(y_2y_3)^4$, $(y_3y_4)^2$, $(y_4ey_1e^{-1})^3$

t_1^3, t_2^3, t_3^4, t_4^2, t_5^2, t_6^2

$e^{-1}t_4t_2t_3^{-1}t_1^{-1}t_5t_6g^{-1}kg^{-1}s^{-1}ks$.

We do this in the following way. Eliminating p via $p \mapsto t_4$; q via $q \mapsto t_5$; r via $r \mapsto t_6$; d via $d \mapsto t_3^{-1}b$; b via $b \mapsto t_1^{-1}t_5t_6$ and c via $c \mapsto t_4^{-1}es^{-1}k^{-1}$, and then inverting and cyclically permuting.

th reduction:

The e's are already at the front.

Bringing the t's to the front (see [20]) gives a path

$$e^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}t_5^{-1}t_6^{-1}g^{-1}kg^{-1}s^{-1}k^{-1}$$

The next step (see Henle [17,p.125]) turns the word into

$$e^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}t_5^{-1}t_6^{-1}g^2[k^{-1},s^{-1}],$$

and the last step (see Henle [17,p.127]) turns the word into

$$e^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}t_5^{-1}t_6^{-1}s^2t^2u^2.$$

I.e. a word of the required form.

3.3 INFINITE NEC-COMPLEXES

Let $\mathcal{K}$ be an NEC-complex.

THEOREM 3.2

If $\mathcal{K}$ is infinite then there is a based equivalence from $\mathcal{K}$ to a presentation $\mathcal{P}$ of the following form:

Involutary edges: y_{ij} $(i,j)\in\Xi_0$

Non-involutary edges: $e_i^{\pm1}$ $(i\in I_f)$, $t_j^{\pm1}$ $(j\in J)$, $s_k^{\pm1}$ $(k\in K)$

Defining paths: $(y_{ij}y_{ij+1})^{m_{ij}}$ $((i,j)\in\Xi_0-\{(i,n(i)):i\in I_f\}$

$(y_{in(i)}e_i y_{i1}e_i^{-1})^{m_{in(i)}}$ $(i\in I_f)$

$t_j^{p_j}$ $(j\in J$ $p_j\geq2)$

If $\mathcal{K}\in$F then there are no y's or e's, and if $\mathcal{K}\in$S then there are no y's, e's or t's.

Remarks: (i) If $\mathcal{K}$ belongs to F (or S) then the Theorem follows immediately from Theorem 1.3 since CG($\mathcal{K}$) may be quadratically labelled, in this case.

(ii) The Theorem provides an alternative proof of the main result of Macbeath and Hoare [29].

(iii) Although in general, $\mathcal{P}$ is not an NEC-presentation, it is still clear what one means by the period cycles and proper periods of $\mathcal{P}$. We then have that the period cycles and proper periods of $\mathcal{P}$ are the same as those of $\mathcal{K}_V$.

Proof

The proof consists of four reductions:

FIRST REDUCTION: *Modify $\mathcal{K}_V$ so that there are no involutary edges except chain edges.*

This has already been dealt with, and we note that the resulting complex, $\mathcal{L}_V$, is an NEC-complex.

SECOND REDUCTION: *Modify the chains of $\mathcal{L}_V$, as follows, to obtain a new complex $\mathcal{M}$.*

A finite chain, say

(3.5) $$\rho_1=(x_1A_1x_2A_1^{-1})^*,\ldots,\rho_n=(x_nA_nx_1A_n^{-1})^*$$

is replaced by

$$\rho_1'=(x_1'x_2')^*,\ldots,\rho_{n-1}'=(x_{n-1}'x_n')^*,\rho_n'=(x_n'ex_1'e^{-1})^*,\rho'=e^{-1}A_1\ldots A_n$$

$x_1',\ldots,x_n'$ involutary, $e^{\pm 1}$ non-involutary.

An infinite chain, say

(3.6) $$\theta_i=(y_iB_iy_{i+1}B_i^{-1})^* \quad (i\epsilon Z),$$

is replaced by

$$\theta'_i=(y'_i y'_{i+1})^* \qquad (i\epsilon Z),$$

y'_i $(i\epsilon Z)$ involutary.

The mapping which sends

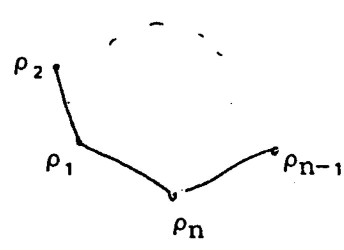
$$x'_i \mapsto A_1\ldots A_{i-1}x_iA_{i-1}^{-1}\ldots A_1^{-1},\ e^{\pm1} \mapsto (A_1\ldots A_n)^{\pm1}$$

$$y'_i \mapsto \begin{cases} A_0\ldots A_{i-1}y_iA_{i-1}^{-1}\ldots A_0^{-1} & i\geq 0 \\ A_{-1}^{-1}\ldots A_i^{-1}y_iA_i\ldots A_{-1} & i<0, \end{cases}$$

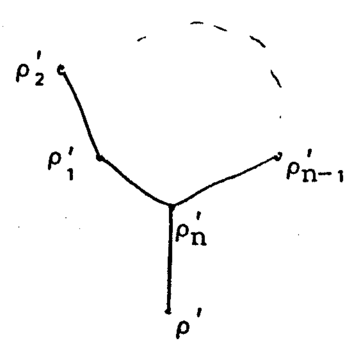
and is the identity on all other edges, defines an equivalence from $\mathcal{M}$ to $\mathcal{L}$.

We note the effect of the above operations on the connectivity graph. A chain as in (3.5) gives rise to a "circle" in the connectivity graph.

(3.7)



On passing from $\mathcal{L}$ to $\mathcal{M}$ this "circle" becomes a "hoop"



All edges incident to one of the vertices $\rho_1,\rho_2,\ldots,\rho_n$ of the circle (3.7) are reattached to the vertex ρ' (and retain their original labelling.)

A chain as in (3.6) gives rise to an "infinite line" in the connectivity graph.

(3.8) $$\theta_{-2} \text{---} \theta_{-1} \text{---} \theta_{0} \text{---} \theta_{1} \text{---} \theta_{2} \cdots$$

On passing from $\mathcal{L}$ to $\mathcal{M}$ we get an infinite line

$$\cdots \theta'_{-2} \text{---} \theta'_{-1} \text{---} \theta'_{0} \text{---} \theta'_{1} \text{---} \theta'_{2} \cdots$$

and all edges incident to one of the vertices θ_i ($i \in Z$) of (3.8) are removed.

<u>THIRD REDUCTION</u>: *Modify* $\mathcal{M}$ *so that any defining path not involving an involutary edge has the form* t^k, *as follows, to obtain a new complex* $\mathcal{N}$.

Let γ_i ($i \in I$) be the defining paths of $\mathcal{M}$ not involving any involutary edges. Then we modify $\mathcal{M}$ by first adjoining for each $i \in I$ a defining path of the form $t_i p(\gamma_i)$ (t_i non-involutary, $\iota(t_i) = \iota(\gamma_i)$), and then eliminating those defining paths which are not of the form t^k and which do not involve an involutary edge. More formally we apply Theorem 1.3 to $\mathcal{M}$, as follows.

By an argument similar to that in the proof of Theorem 3.1 it is seen that A, the full subgraph of the connectivity graph of $\mathcal{M}$ on the defining paths of $\mathcal{M}$ minus the chains, is quadratically labelled.

Let T be a finite, connected component of A. Pick a vertex of the connectivity graph of $\mathcal{L}$ that corresponds to a vertex of T. Now, over all paths in the connectivity graph of $\mathcal{L}$ from this vertex to a vertex of an infinite chain, pick a path of minimal length. Let a be the label on the final edge of this path. Note that it must be non-involutary. Let ρ_0 be the vertex of T to which the penultimate vertex corresponds. Clearly then $L_a(\rho_0)=1$ and $L_a(\rho_\lambda)=0$ for $\rho_\lambda \neq \rho_0$.

Hence we may apply Theorem 1.3.

<u>FOURTH REDUCTION</u>: *Collapse a maximal subtree of $\mathcal{N}$ and eliminate those $t_i p(\gamma i)$ for which* $p(\gamma_i)=1$.□

3.4 NORMAL SUBGROUPS OF NEC-GROUPS

Let $\mathcal{P}$ be an NEC-presentation as in Theorem 3.1, and let H be a normal subgroup of $\pi_1(\mathcal{P})$. E. Bujalance [5] and J. A. Bujalance [6] have obtained results relating the period cycles and proper periods of $\mathcal{P}$ to those of H. Their proofs use an analysis of fundamental region. Most of their results can be proved more directly and quickly using standard results about coverings. We will give short proofs of Propositions 2.2 and 2.3 of [5], and Theorems 3.1 and 4.1 of [6].

By the proper periods and period cycles of H we mean those of the covering $\varphi_H:\mathcal{P}_H \longrightarrow \mathcal{P}$ corresponding to H.

Throughout the following H is a normal subgroup of $\pi_1(\mathcal{P})$ of finite index n.

(A) n is odd

In this case we describe the proper periods and period cycles of H.

We will need the following concept (this will also be needed in (C) below).

For an edge d of $\mathcal{P}$, we define two vertices u,v of $\mathcal{P}_H$ to be *d-equivalent* if there is a path β from u to v such that $\varphi_H(\beta)$ is a power of d. This is clearly an equivalence relation on the vertices of $\mathcal{P}_H$. Let $o(d)$ denote the order of the element $H[d]_{\mathcal{P}}$ of $\pi_1(\mathcal{P})/H$. Then there are $n/o(d)$ equivalence classes each having $o(d)$ elements. For let $H[\beta]_{\mathcal{P}}$ be a representative of some d-equivalence class, then since H is normal in $\pi_1(\mathcal{P})$,

$$H[\beta]_{\mathcal{P}}, H[\beta d]_{\mathcal{P}}, \ldots, H[\beta d^{o(d)-1}]_{\mathcal{P}},$$

are distinct.

Consider the defining paths $t_k^{p_k}$ $(1 \leq k \leq l)$ of $\mathcal{P}$. Let t^p denote a typical one of these. There will be $n/o(t)$ defining paths of $\mathcal{P}_H$ mapping to t^p in 1:1 correspondance with the t-equivalence classes, and each of these defining paths will have period $p/o(t)$. Thus the proper periods of H are $p_k/o(t_k)$ repeated $n/o(t_k)$ times for each $1 \leq k \leq l$ where $o(t_k) \neq p_k$.

Now consider the chains of $\mathcal{P}$, and let

$$(3.9) \qquad (xy)^{m_1}, \ldots, (zfxf^{-1})^{m_r}$$

denote a typical chain. (We note that since n is odd the lift of an involutary edge is itself involutary.) This chain will give rise to $n/o(f)$ chains in $\mathcal{Y}_H$ in 1:1 correspondance with the f-equivalence classes. For any given f-equivalence class the corresponding chain will have the form

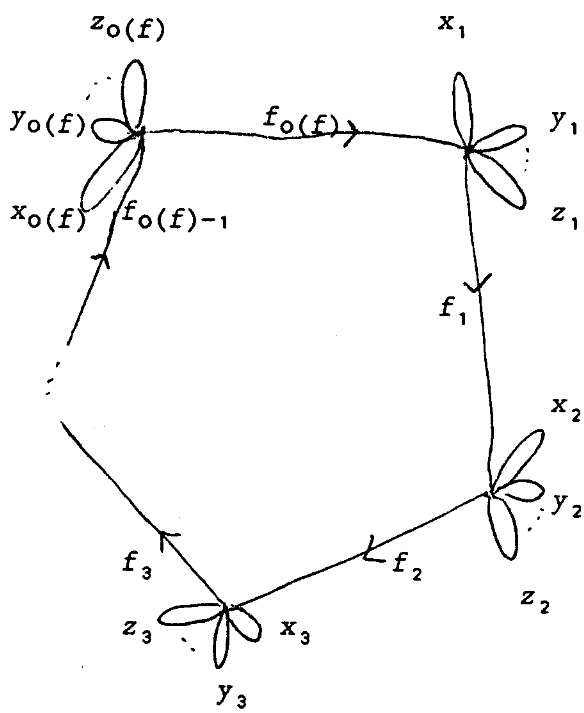
$$(x_1y_1)^{m_1},\ldots,(z_1f_1x_2f_1^{-1})^{m_r}$$

$$(x_2y_2)^{m_1},\ldots,(z_2f_2x_3f_2^{-1})^{m_r}$$

$$\vdots$$

$$(x_{o(f)}y_{o(f)})^{m_1},\ldots,(z_{o(f)}f_{o(f)}x_1f_{o(f)}^{-1})^{m_r}$$

obtained by lifting (3.9) at the vertices of



Thus the chain (3.9) gives rise to $n/o(f)$ period cycles of H, each of the form $(m_1,\ldots,m_r)$ concatenated with itself $o(f)$ times.

i.e. $\underbrace{(m_1,\ldots,m_r,m_1,\ldots,m_r,\ldots,m_1,\ldots,m_r)}_{o(f)\text{ times}}$

Remark: This combines Propositions 2.2 and 2.3 of [5].

<u>(B) n is even</u>

We obtain some of the period cycles of H. Suppose $(m_1,\ldots,m_s)$ is a period cycle of $\mathcal{G}$ with associated chain edges $x_1,\ldots,x_s$. Suppose

(I) $x_i,x_{i+1},\ldots,x_j \in H$

(II) $x_{i-1},x_{j+1} \notin H$

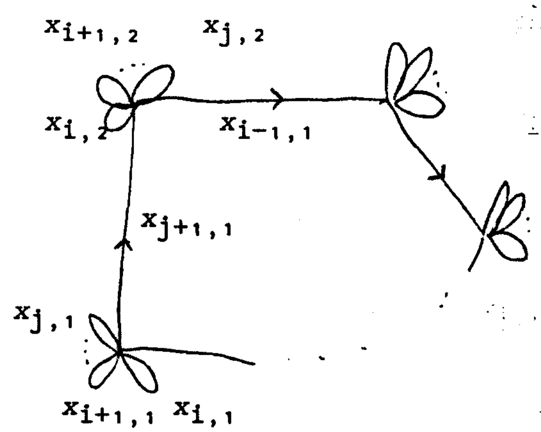
(III) q is the order of $H[x_{i-1}x_{j+1}]_{\mathcal{G}}$ in $\pi_1(\mathcal{G})/H$

Say two vertices $H[\alpha]_{\mathcal{G}},H[\beta]_{\mathcal{G}}$ are equivalent if there exists $c \geq 0$ for which

$$H[\alpha x_{i-1}(x_{j+1}x_{i-1})^c]_{\mathcal{G}}=H[\beta]_{\mathcal{G}}\text{, or}$$

$$H[\alpha(x_{i-1}x_{j+1})^c]_{\mathcal{G}}=H[\beta]_{\mathcal{G}}.$$

This equivalence relation partitions the vertices of $\mathcal{G}_H$ into n/2q equivalence classes, each of which gives rise to a period cycle as illustrated by the following:



Taking lifts, we find that we get the following chain

$(x_{i,1}x_{i+1,1})^{m_{i+1}},\ldots,(x_{j-1,1}x_{j,1})^{m_j},(x_{j,1}x_{j+1,1}x_{j,2}x_{j-1,1}^{-1})^{\frac{1}{2}m_{j+1}}$

$(x_{j,2}x_{j-1,2})^{m_j},\ldots,(x_{i+1}x_{i,2})^{m_{i+1}},(x_{i,2}x_{i-1,2}x_{i,3}x_{i-1,2}^{-1})^{\frac{1}{2}m_j}$

$\vdots$

$\cdot \quad \cdot \quad \cdot \quad (x_{i,\frac{n}{2q}}x_{i-1,\frac{n}{2q}}x_{i,1}x_{i-1,\frac{n}{2q}}^{-1})^{\frac{1}{2}m_i}$

Thus H has n/2q period cycles of the form

$(m_{i+1},\ldots,m_j,\frac{1}{2}m_{j+1},m_j,\ldots,m_{i+1},\frac{1}{2}m_i)$ concatenated with itself q times.

Remark: This is Theorem 3.1 of [6].

(C) $\pi_1(\mathcal{P})/H$ has precisely one element of order two, $H[\gamma]_{\mathcal{P}}$; $\mathcal{P}H$ has no proper periods, and all of its period cycles are of the form $(1,\ldots,1)$

We determine the period cycles and proper periods of $\mathcal{P}$.

We will need the concept of *d*-equivalence introduced in (A) above.

(similarly for y_r and y_1). Now $H[y_i y_{i+1}]_{\mathscr{F}} = H[\gamma]_{\mathscr{F}}$. Thus the lift of $(y_i y_{i+1})^{m_i}$ (respectively $(y_r e y_1 e^{-1})^{m_r}$) is closed if and only if $m_i = 2k_i$ $(i=1,\ldots,r)$. Now the lift of $(y_i y_{i+1})^{2k_i}$ (respectively $(y_r e y_1 e^{-1})^{2k_r}$) has period k_i, hence $k_i = 1$ $(i=1,\ldots,r)$ else $\mathscr{F}_H$ has a period cycle not of the form $(1,\ldots,1)$.

Thus the period cycles of $\mathscr{F}$ consist of

$$\sigma_i = (1) \qquad i=1,\ldots,s$$

$$\text{and} \qquad \mu_j = (2,\ldots,2) \quad j=1,\ldots,k$$

where the number of terms in μ_j is r_j, an even number.

We now prove

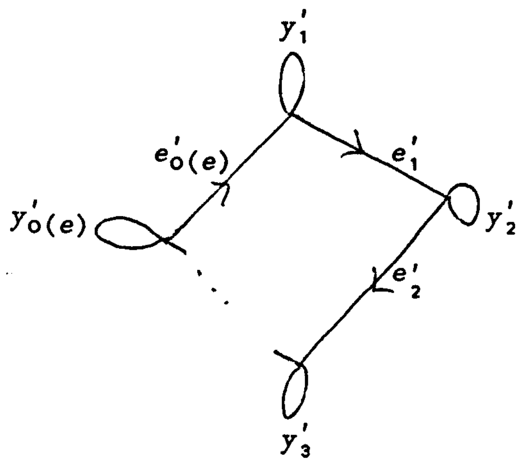
(3.11) $$\text{The number of period cycles of } \mathscr{F}_H = \sum_{i=1}^{p} \frac{n}{n_i} + \frac{n}{2} \sum_{i=1}^{k} \frac{r_i}{2}$$

where $(y_i e_i y_i e_i^{-1})$ $(i=1,\ldots,p)$ are all of the chains of $\mathscr{F}$ with period cycle (1) for which y_i lies in H, and where $n_1,\ldots,n_p$ are the orders of $e_1,\ldots,e_p$ respectively in $\pi_1(\mathscr{F})/H$.

Let

$$(yeye^{-1})$$

be one of the above chains. This gives rise to $n/o(e)$ period cycles of the form (1,...,1) in $\wp_H$ in 1:1 correspondance with the e_i-equivalence classes, in the following way. Lifting $yeye^{-1}$ at each vertex of the following



gives rise to a chain

$$(y_1'e_1'y_2'e_1'^{-1}),\ldots,(y'_{o(e)}e'_{o(e)}y_1'e'^{-1}_{o(e)})$$

in $\wp_H$ which has period cycle (1,...1)

Thus we obtain

$$\sum_{i=1}^{p}\frac{n}{n_i}$$

period cycles of the form (1,...,1) in $\wp_H$ in this manner. Discussion (B) gives that each period cycle

$$(\overset{r_i}{2,\ldots,2})$$

gives $\frac{n}{2}.\frac{r_i}{2}$ period cycles of the form (1,...1) for $\wp_H$, since there are $\frac{r_i}{2}$ involutaries in the chain, and the q of (III) in

(B) above is one. Hence (3.11) holds.

Remark: This is Theorem 4.1 of [6].

APPENDIX A

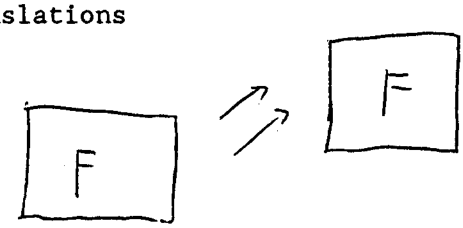
GENERAL INFORMATION ON NEC-GROUPS

We begin by describing what the NEC–groups are. We do this by analogy with the Euclidean crystallographic groups.

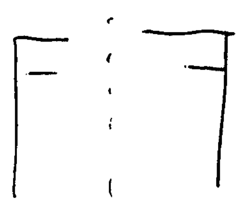
(1) Euclidean preliminaries

Let $I(\mathbb{E})^2$ be the group of isometries of the Euclidean plane. There are four types (see [26,Chp.2]).

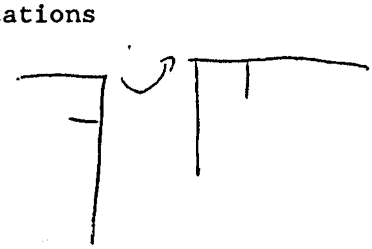
(i) Translations



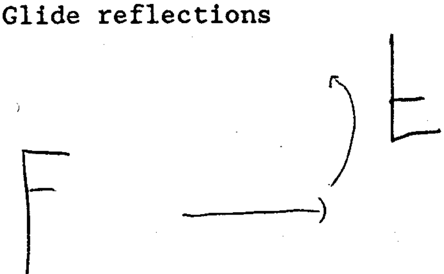
(ii) Reflections



(iii) Rotations



(iv) Glide reflections



Now since:

(i) any translation can be effected by a product of two reflections with parallel axes, [26,Chp.2];

(ii) any rotation can be effected by a product of two reflections with non-parallel axes, [26,Chp.2];

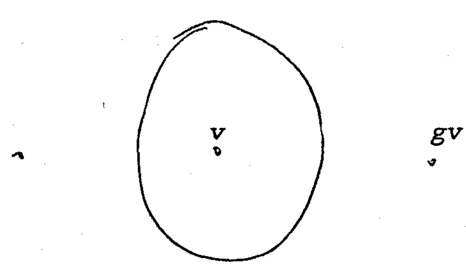
(iii) a glide reflection is a product of a translation and a rotation, [26,Chp.2];

we have that $I(\mathbb{E}^2)$ is generated by reflections.

A subgroup, G, of $I(\mathbb{E}^2)$ is *discontinuous* if for every point v of $\mathbb{E}^2$ there is a neighbourhood U of v such that

$$\mathrm{Orb}_G(v) \cap U = \{v\}$$

i.e.



e.g. Let G consist of the translations

$$\tau_{m,n}(x,y) = (x+m, y+n) \quad (m,n \in \mathbb{Z})$$

Clearly a disc of radius half about any point (x,y) contains no point of the orbit of (x,y) except (x,y).

Corresponding to G we have a tesselation of $\mathbb{E}^2$:

Now since:

(i) any translation can be effected by a product of two reflections with parallel axes, [26,Chp.2];

(ii) any rotation can be effected by a product of two reflections with non-parallel axes, [26,Chp.2];

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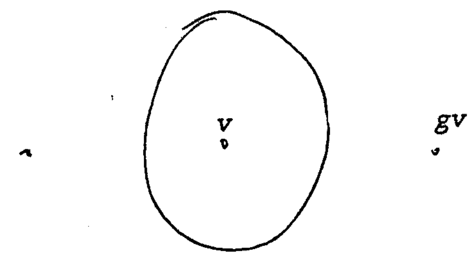

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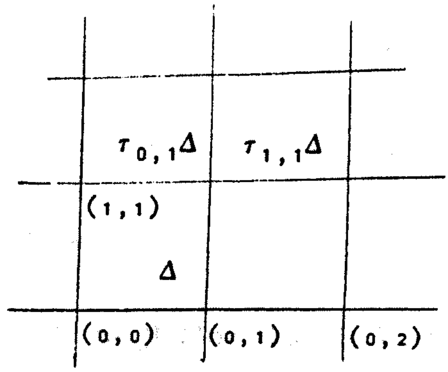
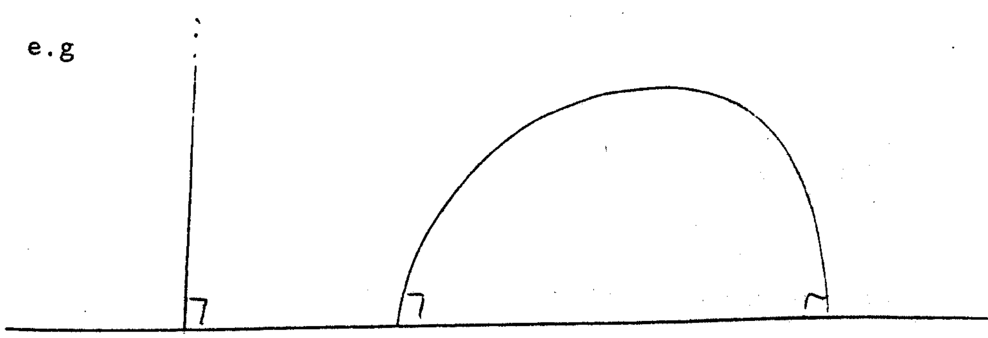


Δ is called a *fundamental region* (no two points in Int(Δ) lie in the same orbit, and the translates of Δ under the elements of G tesselate the plane).

(2) The geometry of the hyperbolic plane $\mathbb{H}^2$

This is represented by the upper half of the complex plane. Lines in $\mathbb{H}^2$, *$\mathbb{H}$-lines*, are Euclidean lines perpendicular to the x-axis and semi-circles with their origins on the x-axis.

e.g



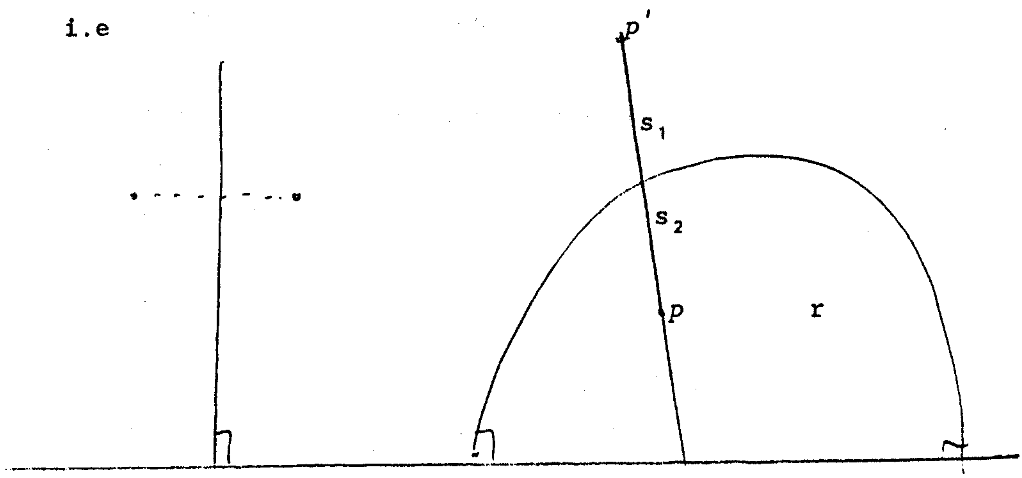
There is a metric one can put on $\mathbb{H}^2$, (the *hyperbolic* metric), given by

$$\rho(z,w) = \ln \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$$

$I(\mathbb{H}^2)$ is then the group of isometries of $\mathbb{H}^2$. This is generated by reflections in $\mathbb{H}$-lines, [2,p.137], which are defined as follows.

Reflection in an $\mathbb{H}$-line that is a Euclidean line perpendicular to the x-axis is exactly the same as Euclidean reflection. For any other $\mathbb{H}$-line, the reflection of any point is obtained by thinking of $\mathbb{H}^2$ as $\mathbb{E}^2$ and then inverting the point in the circle.

i.e



$s_1 s_2 = r^2$ (s_1, s_2, r Euclidean distances between the points)

The discontinuous subgroups of $I(\mathbb{H}^2)$ are the NEC-groups. Wilkie [46] showed that finitely generated NEC-groups with compact orbit space have presentations of the form (3.1) and Singerman [44] showed that the area of a fundamental region of such a group is

$$2\pi\mu > 0$$

where $\mu = g + h + s + \sum_{1}^{r} \frac{1}{p_k} + \frac{1}{2}\sum_{i,j} \left[1 - \frac{1}{m_{ij}}\right]$

Thus in our work we work with a slightly wider class of groups, than just the NEC-groups, as for particular choices of $\mathcal{P}$, μ may be negative, and thus $\pi_1(\mathcal{P})$ is not an NEC-group. In this regard see [48].

It is interesting to note that the class of presentations of the form (3.1) for which $\mu > 0$ form one of the few classes for which all three of Dehn's classical problems are solvable.

Macbeath [28] solved the isomorphism problem. The word and conjugacy problems are solved in [38].

CHAPTER 4

ON THE SQ-UNIVERSALITY OF COXETER GROUPS

4.1 INTRODUCTION

A *Coxeter pair* is a 2-tuple (Γ,φ) where Γ is a graph and φ is a map from $E(\Gamma)$ to $\{2,3,4,\ldots\}$. With each Coxeter pair we associate a presentation

$$\mathcal{P}(\Gamma,\varphi)=\langle V(\Gamma);(xy)^{\varphi(\{x,y\})}\ (\{x,y\}\in E(\Gamma))\rangle$$

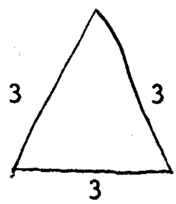
where each generator is involutary. $\mathcal{P}(\Gamma,\varphi)$ is called a *Coxeter presentation* and the associated group $C(\Gamma,\varphi)$ is called a *Coxeter group*.

We will often represent a Coxeter pair (Γ,φ) by drawing the graph Γ and writing numbers on the edges to represent the values of φ. Sometimes, if no confusion can arise, we will use such a diagram to represent the group $C(\Gamma,\varphi)$.

Let (Γ,φ) be a Coxeter pair. When discussing the SQ-universality of $C(\Gamma,\varphi)$ it suffices to deal with the case when $|V(\Gamma)|\geq 4$ and Γ is connected. For if $|V(\Gamma)|\leq 2$ then $C(\Gamma,\varphi)$ is either cyclic or dihedral (finite or infinite) and so is not SQ-universal. If $|V(\Gamma)|\geq 3$ and Γ is not connected then we can express $C(\Gamma,\varphi)$ as a free product of two non-trivial groups not both of order two, and so $C(\Gamma,\varphi)$ is again SQ-universal (see [33]). *If* $|V(\Gamma)|=3$ *and* Γ *is connected then* $C(\Gamma,\varphi)$ *is SQ-universal if and only* $\displaystyle\sum_{\{x,y\}\in E(\Gamma)} \frac{1}{\varphi(\{x,y\})} < 1$, by Neumann [33].

Following Appel and Schupp [1] we will say that a Coxeter pair is of *large type* if $2\nmid \operatorname{Im} \varphi$.

I conjecture that if (Γ,φ) is a Coxeter pair of large type with $|V(\Gamma)|\geq 3$, then $C(\Gamma,\varphi)$ is SQ-universal except when (Γ,φ) is



In connection with this conjecture we prove the following.

<u>THEOREM 4.1</u>

Let (Γ,φ) *be a Coxeter pair of large type. Suppose*

(A) Γ *is incomplete on at least three vertices, or*

(B) Γ *is complete on at least five vertices and for any triangle* e_1, e_2, e_3 *in* Γ

(4.1) $$\frac{1}{\varphi(e_1)+1} + \frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1} < \frac{1}{2}$$

Then $C(\Gamma,\varphi)$ *is SQ-universal.*

can express $C(\Gamma,\varphi)$ as a free product of two non-trivial groups not both of order two, and so $C(\Gamma,\varphi)$ is again SQ-universal (see [33]). *If* $|V(\Gamma)|=3$ *and* Γ *is connected then* $C(\Gamma,\varphi)$ *is SQ-universal if and only* $\displaystyle\sum_{\{x,y\}\in E(\Gamma)} \frac{1}{\varphi(\{x,y\})} < 1$, by Neumann [33].

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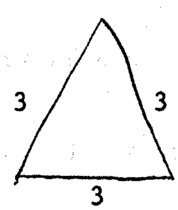

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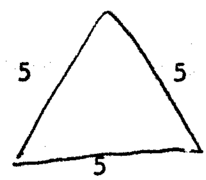
(B) Γ *is complete on at least five vertices and for any triangle* e_1, e_2, e_3 *in* Γ

$$\frac{1}{\varphi(e_1)+1} + \frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1} < \frac{1}{2} \qquad (4.1)$$

Then $C(\Gamma,\varphi)$ *is SQ-universal.*

This result is in fact a corollary of more general results stated and proved in §§4.2, 4.3.

Note that (4.1) always holds if $2,3,4 \notin \mathrm{Im}\ \varphi$ and there is no triangle in Γ of the form



Before stating our second result we need the following definition. Let (Γ,φ) be a Coxeter pair. We define an equivalence relation $\sim$ on $V(\Gamma)$ as follows:

$x \sim y$ if and only if there exist vertices $x = x_1, x_2, \ldots, x_n = y$ satisfying $x_i \neq x_{i+1}$ and, if $\{x_i, x_{i+1}\} \in E(\Gamma)$ then $\varphi(\{x_i, x_{i+1}\}) \geq 3$, $(i=1,\ldots,n-1)$.

An *island* in (Γ,φ) is a Coxeter pair (Γ',φ') where Γ' is the full subgraph of Γ on some $\sim$-equivalance class and φ' is the restriction of φ to $E(\Gamma')$.

Using the terminology of §1.8, our second result is

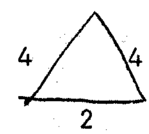
<u>THEOREM 4.2</u>

Let (Γ,φ) *be a Coxeter pair with* $|V(\Gamma)|\geq 4$ *and*

$\mathrm{hcf}[\varphi(E(\Gamma))] > 1$.

(I) *If* $\mathrm{hcf}[\varphi(E(\Gamma))] \geq 3$ *then* $C(\Gamma,\varphi) \geqslant F_2$.

(II) *If* $\mathrm{hcf}[\varphi(E(\Gamma))] = 2$ *and not all islands have the form*

(4.2) $\qquad \cdot\ ,\ \cdot\ \ \cdot\ ,$ ——— , 

then $C(\Gamma,\varphi) \geqslant F_2$.

(III) *If* $\mathrm{hcf}[\varphi(E(\Gamma))] = 2$ *and all islands have the form* (4.2) *then* $C(\Gamma,\varphi)$ *is soluble of length at most three.*

<u>COROLLARY 4.1</u>

Let (Γ,φ) *be a Coxeter pair as in the statement of the above Theorem. Then* $C(\Gamma,\varphi)$ *is SQ-universal if and only if* (I) *or* (II) *holds.*

Remarks: (1) If (Γ_1,φ_1) and (Γ_2,φ_2) are distinct islands in (Γ,φ) and $x\in V(\Gamma_1)$ and $y\in V(\Gamma_2)$, then clearly $\{x,y\}\in E(\Gamma)$ and $\varphi\{x,y\}=2$, giving the following observation:

If (Γ,φ) *is a Coxeter pair and* (Γ_i,φ_i) $(i\in I)$ *are all of the islands in* (Γ,φ) *then*

$$C(\Gamma,\varphi)\cong\sum_{i\in I} C(\Gamma_i,\varphi_i)$$

(2) The proof of (II) above proceeds by picking an island not of the form (4.2) and showing that it is equally as large as F_2, whence $C(\Gamma,\varphi)$ is equally as large as F_2.

(3) Bearing in mind remark (1) above it is interesting to ask the following question: If a direct sum of groups is SQ-universal, does this imply that one of the summands is itself SQ-universal? We will show (in an appendix to this chapter) that the answer is "yes" for countable direct sums.

At various points in the chapter we will need to use

THE SOLUTION TO THE WORD PROBLEM FOR COXETER GROUPS.

This is effectively the algorithmn given in Tits [45]. Let

$$\mathcal{P}(\Gamma,\varphi)=\langle V(\Gamma);(xy)^{\varphi(\{x,y\})}\ \ (\{x,y\}\in E(\Gamma))\rangle$$

be a Coxeter presentation. Let A be a word on $V(\Gamma)$. We define two operations on words:

(4.3) If B is a subword of A of length k that is also a subword of a relator $(xy)^k$, replace B by the word obtained by interchanging x and y in it.

(4.4) Delete any subword of the form x^2.

The *derived set* of A is the set of all words obtainable from A by a finite number of operations of types (4.3) and (4.4). We have: *a word* A *on* $V(\Gamma)$ *is equal to 1 in* $C(\Gamma,\varphi)$ *if and only if the empty word is in the derived set of* A. We will say that a word is *minimal* if there is no shorter word in its derived set.

THE FREIHEITSSATZ FOR COXETER GROUPS.

This says that if Γ' is a full subgraph of Γ and φ' the restriction of φ to $E(\Gamma')$ then the natural mapping

$$C(\Gamma',\varphi') \longrightarrow C(\Gamma,\varphi)$$

given by $v \mapsto v$ $(v \in V(\Gamma'))$ is injective. See [3] for details.

4.2 THEOREM 4.3

In this section we prove the following result:

THEOREM 4.3

Let (Γ,φ) *be a Coxeter pair with* $|v(\Gamma)|\geq 3$. *Suppose there exists a vertex* v *of* Γ *not joined to every other vertex, satisfying: If* $\{u,v\}$ *is an edge of* Γ *then* $\varphi(\{u,v\})\geq 3$. *Then* $C(\Gamma,\varphi)$ *is SQ-universal.*

Remark: Part (A) of Theorem 4.1 is a special case of this result.

We delay the proof until after the following discussion, taken from Shelah [43]. If A and B are any groups with A a subgroup of B we say an element x of B is *malnormal* over B (relative to A) if

$$A^x\cap A=\{1\},$$

where $A^x=\{xax^{-1}:a\in A\}$. Now suppose

$$D=A\ast_C B\quad (A\neq C,\ B\neq C)$$

is a free product with amalgamated subgroup C.

THEOREM (SHELAH)

Let A,B,C *and* D *be as above. If there exists a malnormal element* x *in either* A *or* B *(relative to* C*) then* D *is* the empty word is not in the derived set of either of these.

Case 2 L(X)=2

Z has one of the following forms (where p,q,r,s are distinct elements of $V(\Gamma_3)$):

1) $vpqvr$

2) $vpqvp$

3) $vpqvq$

4) $vpqvrs$

5) $vpqvps$

6) $vpqvqs$

7) $vpqvrp$

8) $vpqvrq$

9) $vpqvpq$

10) $vpqvqp$

We now show that the empty word is not in the derived set of any of the above. We only give subcase 9) a fuller treatment, as this is most complicated. The other subcases are obtained similarly.

Subcases 1), 2), and 3)

This is obvious as no word of odd length ever defines 1 in a Coxeter group.

Subcase 4).

The derived set is a subset of

$$\{vpqvrs,\ vqpvrs,\ vpqvsr,\ vqpvsr\}.$$

Subcase 5).

The derived set is a subset of

$$\{vpqvps,\ vpqvsp,\ vqpvsp,\ vqpvps,\ vqvpvs,\ qvqpvs,\ qvpqvs\}.$$

Subcase 6).

The derived set is a subset of

$$\{vpqvqs,\ vpqvsq,\ vqpvqs,\ vqpvsq,\ vpvqvs,\ pvpqvs,\ pvqpvs\}.$$

Subcase 7).

This reduces to subcase 5) if $(rp)^2$ is a relator, so suppose it is not. Then the derived set is a subset of

$$\{vpqvrp,\ vqpvrp\}.$$

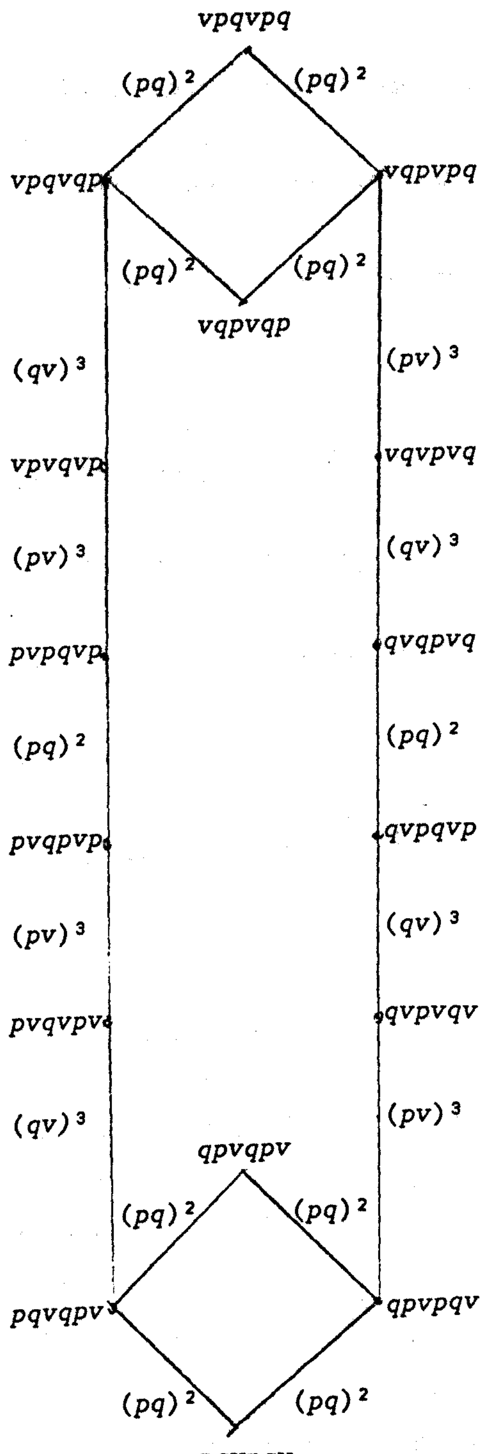
Subcase 8).

This reduces to Subcase 6) if $(rq)^2$ is a relator, so suppose it is not. Then the derived set is a subset of

$$\{vpqvrq,\ vqpvrq\}.$$

Subcase 9).

If $(pq)^2$, $(qv)^3$, and $(pv)^3$ are all relators we find that the derived set has the following structure.



So suppose that they are not all relators. Then the derived set is a subset of

$$\{vpqvpq,\ vpqvqp,\ vqpvpq,\ vqpvqp,\ vpvqvp,\ vqvpvq\}.$$

<u>Subcase 10).</u>

If $(pq)^2$ is a relator this reduces to subcase 9), so suppose it is not. Then the derived set is a subset of

$$\{vpqvqp,\ vpvqvp,\ pvpqvp\}.$$

<u>Case 3 $L(X)\geq 3$</u>

Let $\{X_i : i\epsilon I\}$, $\{Y_j : j\epsilon J\}$ be the derived sets of X and Y respectively. For $i\epsilon I$ write $X_i = X_i' p_i$ where p_i is an element of $V(\Gamma_3)$, and for $j\epsilon J$ write $Y_j = q_j Y_j'$, where q_j is an element of $V(\Gamma_3)$. We let

$$K = \{(i,j) : p_i = q_j,\ \{v,p_i\}\epsilon E(\Gamma),\ \varphi(\{v,p_i\}) = 3\}.$$

Note that if $(i,j)\epsilon K$ then $vX_i vY_j$ $(= vX_i' p_i v p_i Y_j')$ can be changed to $vX_i' v p_i v Y_j'$, and since $L(X_i')\geq 2$ and X_i, Y_j are minimal, no further type (4.3) or type (4.4) operations involving a relator containing v, can be applied to this word, apart from changing the word back to $vX_i vY_j$. It now easily follows that

$$vX_i vY_j \qquad i\epsilon I,\ j\epsilon J$$

$$vX_i'vp_ivY_j' \quad (i,j)\epsilon K$$

are all of the words in the derived set of Z. Hence $vXvY\neq 1$ in $C(\Gamma_1,\varphi_1)$ - a contradiction. The result follows.□

§4.3 THEOREM 4.4

THEOREM 4.4

Let (Γ,φ) be a Coxeter pair. Suppose that the following hold:

(1) *There exist five distinct vertices v,u,w,x,y of Γ such that*

(i) *the full subgraph of Γ on $\{v,u,w,x,y\}$ is complete, and*

(ii) *the image under φ of any edge of Γ with at least one endpoint in $\{v,u,w,x,y\}$ is at least 3.*

(2) *If $e_1,e_2,e_3 \in E(\Gamma)$ form a triangle in Γ then*

$$\frac{1}{\varphi(e_1)+1} + \frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1} < \frac{1}{2},$$

unless some of e_1,e_2,e_3 are mapped to 2 by φ, in which instance we may replace $\frac{1}{2}$ by $^7/_{12}$.

(3) *If $e_1,e_2,e_3,e_4 \in E(\Gamma)$ form a square in Γ then $\{\varphi(e_1),\varphi(e_2),\varphi(e_3),\varphi(e_4)\}$ is not equal to $\{3\}$ or $\{2,3\}$.*

Then $C(\Gamma,\varphi)$ is SQ-universal.

Remark: Part (B) of Theorem 4.1 is a special case of this result

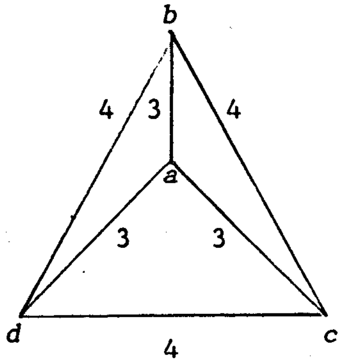
Recall that if H and G are groups with H a subgroup of G, then H is said to be *normal-convex* in G if for every normal subgroup N of H the intersection of H with the normal closure of N in G is N (see [23]). Thus if H is normal-convex in G, and is SQ-universal, then G is SQ-universal. For let X be a countable group and let N be a normal subgroup of H such that

$$X \hookrightarrow H/N$$

Then $X \hookrightarrow H/N = H/H \cap N^G \cong HN^G/N^G < G/N^G$

Our strategy is thus the following: we will find a free subgroup H of rank two in $C=C(\Gamma,\varphi)$ and we will show that it is normal convex in C. We begin with a technical lemma, the proof of which uses some of the ideas in Howie [23] and is quite long. For that reason it is presented in two halves; the first, the outline, is the proof omitting technical details, the second, the details, contains the technical details missing from the outline.

Remark: Originally we had hoped to prove that for any three vertices x,y,z of Γ the subgroup generated by them was normal-convex in $C(\Gamma,\varphi)$. This has turned out to be false as the following example shows



For b and c are not conjugate in the subgroup generated by b,c,d and yet are conjugate in the whole group.

LEMMA 4.1

Let (Γ,φ) *be as in the statement of Theorem* 4.4 *with* $\mathcal{P}(\Gamma,\varphi)=\langle V(\Gamma);R\rangle$, *and suppose*

$$\mu=(uvw)x(uvw)^2x\ldots x(uvw)^{20,001}$$

$$\eta=(xvy)u(xvy)^2u\ldots u(xvy)^{20,001}.$$

If $\mathcal{D}$ *is a reduced diagram over* $\langle V(\Gamma);R\rangle$ *with* n *distinguished regions labelled by words* $V_1,\ldots,V_n$ *in* μ *and* η, *then there exist words* $S_1,\ldots,S_n$ *in* μ *and* η *such that*

(4.5) $\qquad (S_1V_1S_1^{-1})(S_2V_2S_2^{-1})\ldots(S_nV_nS_n^{-1}) = 1$ in C.

Remarks: (i) Note that we can change the order of the terms in (4.5) by "Peiffer-type" transformations (at the expense of altering the S's). For example we could alter the first two terms to

$$(S_2V_2S_2^{-1})(S_1'V_1S_1'^{-1})\ldots$$

where $S_1'=S_2V_2^{-1}S_2^{-1}S_1$.

(ii) In (4.5) we could, of course, take S_1 to be empty (by conjugating), but it is convenient to allow S_1 to be non-empty for symmetry.

We indicate straight away how Theorem 4.4 follows from the Lemma 4.1.

Let H be the subgroup of C generated by μ and η. It is easily seen, by using the solution to the word problem, that H is free of rank two and hence is SQ-universal. We now show that H is normal-convex in C. If W_0 is a word on μ and η that represents an element of the normal closure of some normal subgroup N of H in C, we must show that W_0 represents an element of N itself.

We in fact have the following: *Suppose*

$$W_0U_1W_1U_1^{-1}\ldots U_nW_nU_n^{-1}=1 \textit{ in } C$$

where $W_0,\ldots,W_n$ *are words on* μ *and* η, *and* $U_1,\ldots,U_n$ *are words* on $V(\Gamma)$. *Then there exist words* $T_1,\dots,T_n$ *on* μ *and* η *such that*

$$W_0T_1W_1T_1^{-1}\dots T_nW_nT_n^{-1} = 1 \text{ in } C. \tag{4.6}$$

This is proved by appealing to Lemma 1.12 by which we may assume that there exists a reduced diagram with $n+1$ distinguished regions labelled by $W_0,\dots,W_n$. Then Lemma 4.1 and the remarks following it give us (4.6) as required.

Thus H is normal-convex in C.

<u>Proof of Lemma 4.1</u>

<u>The outline</u>

It suffices to prove the result when $\mathcal{D}$ consists of a single sphere. The proof is by induction on n. Clearly the result holds if $n=0$ or 1, so suppose $n>1$.

The idea now is to assign angles to the corners of the regions of $\mathcal{D}$ in such a way that for every <u>non-distinguished region</u> Δ, $K(\Delta)\leq 0$, and for <u>every</u> vertex a of $\mathcal{D}$, $K(a)\leq 0$. Since

$$\sum_{a \text{ a vertex}} K(a) + \sum_{\Delta \text{ a region}} K(\Delta) = 4\pi$$

there then exists some distinguished region Δ_1 with $K(\Delta_1)>0$.

In order to explain the next step of the proof we need some terminology. If Δ is a distinguished region then an edge of $\partial\Delta$ is called a *distinguished edge* if it occurs twice in a boundary cycle of Δ, or if it separates Δ from another distinguished region. A *non-distinguished edge* of $\partial\Delta$ is one which separates Δ from a non-distinguished region. A subpath of $\partial\Delta$ is a *distinguished segment* if each of its edges is a distinguished edge and each of its intermediate vertices has valence two. It is a *non-distinguished segment* if each of its edges is a non-distinguished edge and each intermediate vertex has precisely one corner from Δ incident at it and corners from no other distinguished regions are incident at it. Then $\partial\Delta$ splits up uniquely into a collection of maximal distinguished segments and maximal non-distinguished segments. Our aim is to show that Δ_1 has a "very long" distinguished segment. To do this we consider the angles at the various corners of Δ_1. What we show is that for a suitable small $\epsilon>0$, if K is a corner of Δ_1 then

$$\angle K \begin{cases} =\pi & \text{if K is incident to an intermediate vertex of a distinguished segment} \\ \leq\pi-\epsilon & \text{otherwise.} \end{cases} \tag{4.7}$$

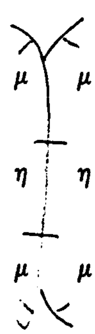
Now let q be the number of corners of Δ_1 with angle π and let p be the number with angle at most $\pi-\epsilon$. Then

$$0 < K(\Delta_1) \leq p(\pi-\epsilon) + q\pi - (p+q-2)\pi,$$

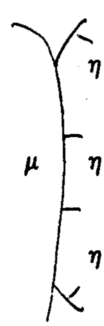
whence $p<2\pi/\epsilon$. Since the length of a boundary cycle of Δ_1 is at least the length of μ, we deduce that there exists a distinguished segment ξ of $\partial\Delta_1$ of length at least $|\mu|\epsilon/2\pi$.

For each distinguished region Δ of $\mathcal{D}$, the boundary cycle of Δ can be broken uniquely into segments labelled by either $\mu^{\mp 1}$ or $\eta^{\mp 1}$. We may show that these factorizations coincide exactly on ξ (see pp.168-169 below).

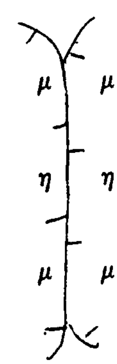
e.g.



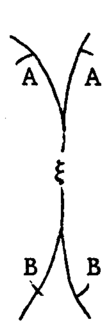
but not



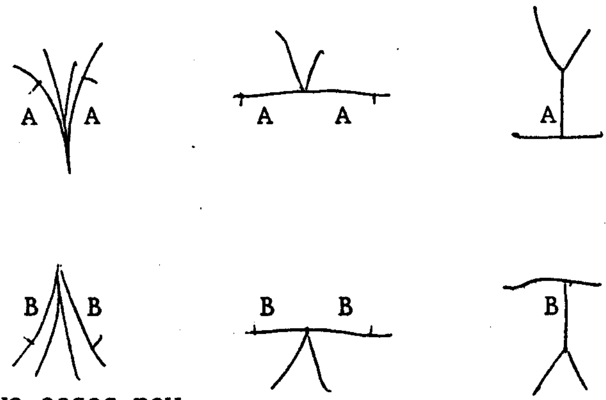
or



We remove ξ to obtain a new diagram;

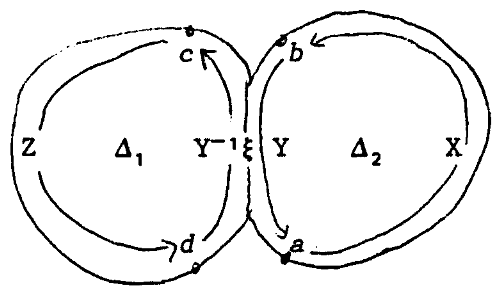


where A and B are words, possibly empty, on $V(\Gamma)$. We then "fold" these segments out, as follows



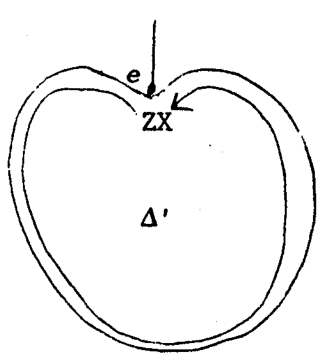
There are two cases now.

(i) The regions on either side of ξ were distinct.



Here X (respectively Y) is the label on the segment of $\partial\Delta_2$ reading anti-clockwise from a to b (respectively b to a), and where Z (respectively Y^{-1}) is the label on the segment of $\partial\Delta_1$ reading anti-clockwise from c to d (respectively d to c).

Performing the above modifications gives a new region Δ'



Where ZX is the label on $\partial\Delta'$ reading anticlockwise from e.

Note that ZX is a word on μ and η. Using the inductive hypothesis on this new diagram, which has only n-1 distinguished regions, and Remark (i), we find there exist words $S, S_3, \ldots, S_n$ on μ and η such that

$$(SZXS^{-1})(S_3V_3S_3^{-1})\ldots(S_nV_nS_n^{-1}) = 1 \text{ in } C$$

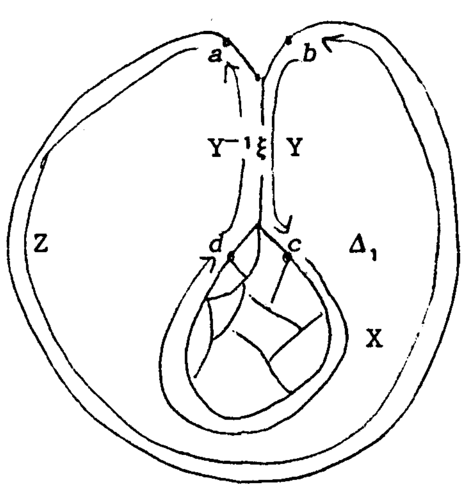
where $V_3, \ldots, V_n$ are the labels on the distinguished regions excluding Δ'.

Now there exist words P,Q on μ and η such that $PXYP^{-1}$ is equal (in C) to the label V_2 on Δ_2, and $QY^{-1}ZQ^{-1}$ is equal (in C) to the label V_1 on Δ_1. Replacing $SZXS^{-1}$ by

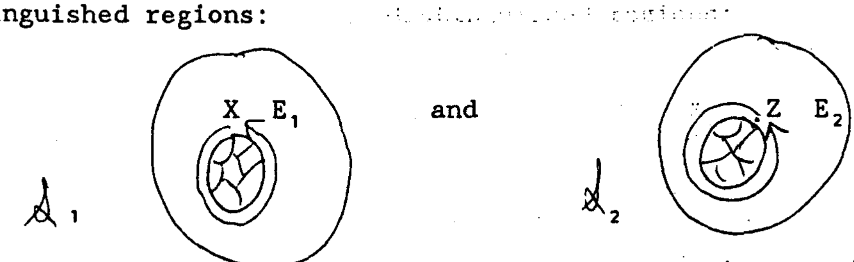
$$(SYQ^{-1}V_1QY^{-1}S^{-1})(SYP^{-1}V_2PY^{-1}S^{-1})$$

completes the proof.

(ii) The regions on either side of ξ coincide.



Here Z (respectively Y,X, and Y^{-1}) is the label on the segment of $\partial\Delta_1$ reading anti-clockwise from *a* to *b* (respectively *b* to *c*, *c* to *d*, and *d* to *a*). Eliminating ξ and performing the above modifications and then removing the interior of the resulting annular region gives rise to two planar diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$, one with boundary label X and the other with boundary label Z. From this we create two new spherical diagrams $\mathcal{S}_1$ and $\mathcal{S}_2$ in the following way. We "glue" $\mathcal{D}_1$ to an untesselated sphere to obtain $\mathcal{S}_1$ and "glue" $\mathcal{D}_1$ to a second untesselated sphere to obtain $\mathcal{S}_2$. Now each of $\mathcal{S}_1$ and $\mathcal{S}_2$ has n-1 or fewer distinguished regions:



Then by applying the inductive hypothesis to $\mathcal{S}_1$ and $\mathcal{S}_2$ and making use of Remark (i) after Lemma 4.1, we find that

(4.8) $$XT_1L_1T_1^{-1}\ldots T_aL_aT_a^{-1} = 1 \text{ in } C$$

(4.9) $$ZU_1M_1U_1^{-1}\ldots U_bM_bU_b^{-1} = 1 \text{ in } C$$

where $L_1,\ldots,L_a$ are the labels on the distinguished regions of $\mathcal{S}_1$ other than E_1, in some order; $M_1,\ldots,M_b$ are the labels on the distinguished regions of $\mathcal{S}_2$ other than E_2, in some order, and $T_1,\ldots,T_a,U_1,\ldots,U_b$ are words on μ and η.

Combining (4.8) and (4.9) we find

$$XY^{-1}ZY(Y^{-1}U_1M_1U_1^{-1}Y)(Y^{-1}U_2M_2U_2^{-1}Y)\dots(Y^{-1}U_bM_bU_b^{-1}Y)(T_1L_1T_1^{-1})\dots(T_aL_aT_a^{-1})$$

$=1$ in C.

Conjugating each term by a suitable word on μ and η and then performing free reductions on the first term, allows us to replace it by V_1, where V_1 is the label on Δ_1. This completes the outline of the proof.

<u>The details</u>

Consider the following two assertions:

(4.10) *If ρ is a freely reduced word on $\{\mu,\eta\}$, no cyclic permutation of ρ begins with sts for s,t distinct elements of $\{u,v,w,x,y\}$.*

(4.11) *Any subword of an element of the symmetrized closure of $\{\mu,\eta\}$ of length at least* 120,008 *is not a piece.*

(4.10) is easily verified by inspection. We verify (4.11).

Let γ be a maximal piece. We suppose without loss of generality that it is a subword of a cyclic permutation of μ. Since

$$x(uvw)^k x$$

is never a piece, γ must be contained in

$$(uvw)^{20000}x(uvw)^{20002}$$

and hence has length at most 120,007.

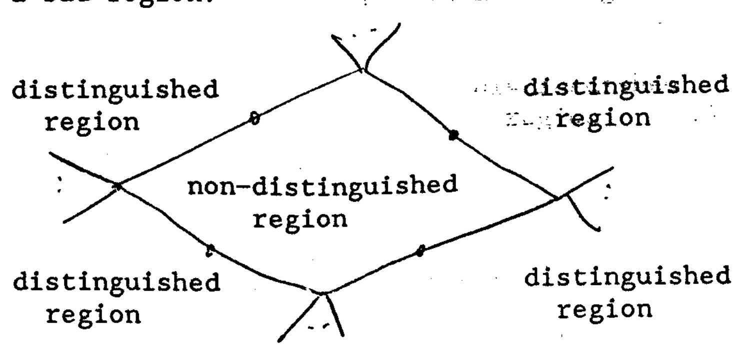
(4.10) and (4.11) are both crucial to our proof, and will be refered back to.

We now begin the details proper by describing how to assign the angles to the corners of the regions of $\mathcal{D}$ and showing that they have the required properties.

Let Δ be a non-distinguished region. Then Δ has $2r$ corners for some $r \geq 2$. A corner of Δ is *bad* if the vertex it is incident to has valence two, and *good* otherwise. Now two successive corners cannot both be bad since this would imply that Δ had a common boundary of length at least three with a distinguished region, which would violate (4.10) above.

Thus at most r corners can be bad. Call Δ *bad* if exactly r corners are bad, and *good* otherwise.

Example of a bad region:



We observe that if two non-distinguished regions have an edge in common then both regions must be good.

We first assign angles to the corners of the non-distinguished regions. Let $\epsilon = \pi/421$.

Firstly suppose that Δ has four sides. Assign the angle $\pi/2$ to each of its corners. Then

$$K(\Delta) = \frac{4\pi}{2} - (4-2)\pi = 0.$$

So now suppose that Δ has at least $k \geq 6$ sides. Assign the angle $\pi+\epsilon$ to each of its bad corners and, $\left[1 - \frac{4}{k}\right]\pi - \epsilon$ to each of its good corners if Δ is bad, or $\frac{(k-2)}{(k+2)}(\pi-\epsilon)$ to each of its good corners if Δ is good.

If Δ is bad we have

$$\begin{aligned} K(\Delta) &= \frac{k}{2}\left[\left[1 - \frac{4}{k}\right]\pi - \epsilon\right] + \frac{k}{2}(\pi+\epsilon) - (k-2)\pi \\ &= 0. \end{aligned}$$

If Δ is good we have, where b is the number of bad corners of Δ,

$$\begin{aligned} K(\Delta) &= (k-b)\left[\frac{k-2}{k+2}\right](\pi-\epsilon) + b(\pi+\epsilon) - (k-2)\pi \\ &= k\left[\frac{k-2}{k+2}\right](\pi-\epsilon) + b\left[\pi + \epsilon - \left[\frac{k-2}{k+2}\right](\pi-\epsilon)\right] - (k-2)\pi \end{aligned}$$

$$= k\left[\frac{k-2}{k+2}\right](\pi-\epsilon) + b\left[\frac{4\pi+2k\epsilon}{k+2}\right] - (k-2)\pi$$

$$\leq k\left[\frac{k-2}{k+2}\right](\pi-\epsilon) + \left[\frac{k-2}{2}\right]\left[\frac{4\pi+2k\epsilon}{k+2}\right] - (k-2)\pi$$

$$= \left[\frac{k-2}{k+2}\right](k+2)\pi - (k-2)\pi$$

$$= 0.$$

Thus for all non-distinguished regions Δ, $K(\Delta) \leq 0$ as required.

Let a be a vertex of $\mathcal{A}$. A corner incident with a will be said to be *distinguished* (respectively, *non-distinguished*) if it arises from a distinguished (respectively, non-distinguished) region.

Suppose that there is at least one distinguished corner incident with a. Assign angles to the incident distinguished corners as follows: Suppose there are t such corners and that the sum of the angles of the non-distinguished corners incident at a is θ, then assign an angle

$$\frac{(2\pi-\theta)}{t}$$

to each incident distinguished corner. Then

$$K(a) = 0.$$

Hence we need only show $K(a) \leq 0$ for those vertices a of with all incident corners non-distinguished corners.

<u>Case 1. Five or more non-distinguished corners are incident at a.</u>

We first note (for use in this and the following case) that if K is a corner incident to a then

$$\angle K \geq \frac{\pi-\epsilon}{2}$$

For by a previous remark K must be a corner of a good region. If that region has four sides then $\angle K = \frac{\pi}{2}$, whereas if that region has $k \geq 6$ sides then

$$\begin{aligned}\angle K &= \left[\frac{k-2}{k+2}\right](\pi-\epsilon)\\ &= \left[1 - \frac{4}{k+2}\right](\pi-\epsilon)\\ &\geq \frac{1}{2}(\pi-\epsilon).\end{aligned}$$

Thus

$$K(a) \leq 2\pi - 5\left[\frac{\pi-\epsilon}{2}\right] \leq 2\pi - 2\pi = 0.$$

<u>Case 2. Precisely four non-distinguished corners are incident at a.</u>

Suppose first that all four corners come from regions with four sides. Then

$$K(a) = 2\pi - 4\frac{\pi}{2} = 0.$$

Suppose now that less than four corners come from regions with four sides. By hypothesis (3) of the Theorem at least one region incident to a has $k \geq 8$ sides so

$$K(a) \leq 2\pi - \left[3\left[\frac{\pi-\epsilon}{2}\right] + \frac{3}{5}(\pi-\epsilon)\right] \leq 0.$$

<u>Case 3. Precisely three non-distinguished corners are incident at a.</u>

Let $e_1, e_2, e_3 \epsilon E(\Gamma)$ be the three edges in Γ corresponding to the three regions. The following is crucial:

If hypothesis (2)(i) of the Theorem holds for e_1, e_2, e_3 then in fact

(4.12) $$\frac{1}{\varphi(e_1)+1} + \frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1} \leq \frac{209}{420}$$

and if (2)(ii) holds, then the sum is bounded above by 61/105.

Verification of this is given after the proof.

<u>Subcase 3.1. $\min\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\} \geq 3$.</u>

$$K(a) = 2\pi - \left[\left[\frac{2\varphi(e_1)-2}{2\varphi(e_1)+2}\right](\pi-\epsilon) + \left[\frac{2\varphi(e_2)-2}{2\varphi(e_2)+2}\right](\pi-\epsilon) + \left[\frac{2\varphi(e_3)-2}{2\varphi(e_3)+2}\right](\pi-\epsilon)\right]$$

$$= 2\pi - \left[\left[1-\frac{2}{\varphi(e_1)+1}\right](\pi-\epsilon) + \left[1-\frac{2}{\varphi(e_2)+1}\right](\pi-\epsilon) + \left[1-\frac{2}{\varphi(e_3)+1}\right](\pi-\epsilon)\right]$$

$$= 2\pi - \left[(\pi-\epsilon)\left[3 - 2\left[\frac{1}{\varphi(e_1)+1} + \frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1}\right]\right]\right]$$

$$\leq 2\pi - (\pi-\epsilon)\frac{421}{210} = 0; \text{ using } (4.12)$$

Subcase 3.2. $\min\{\varphi(e_1),\varphi(e_2),\varphi(e_3)\} = 2.$

Note that at most one of these can be equal to two, so without loss of generality we assume that it is $\varphi(e_1)$. Using (4.12) we find

$$\frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1} \leq \frac{26}{105}.$$

Now

$$K(a) = 2\pi - \left[\frac{\pi}{2} + \left[\frac{2\varphi(e_2)-2}{2\varphi(e_2)+2}\right](\pi-\epsilon) + \left[\frac{2\varphi(e_3)-2}{2\varphi(e_3)+2}\right](\pi-\epsilon)\right]$$

$$= 2\pi - \left[\frac{\pi}{2} + \left[1 - \frac{2}{\varphi(e_2)+1}\right](\pi-\epsilon) + \left[1 - \frac{2}{\varphi(e_2)+1}\right](\pi-\epsilon)\right]$$

$$= 2\pi - \left[\frac{\pi}{2} + (\pi-\epsilon)\left[2 - 2\left[\frac{1}{\varphi(e_2)+1} + \frac{1}{\varphi(e_3)+1}\right]\right]\right]$$

$$\leq 2\pi - \left[\frac{\pi}{2} + \frac{420\pi}{421}\left[2 - \frac{52}{105}\right]\right]$$

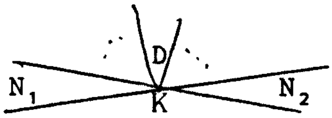
$$= 2\pi - \frac{1685}{842}\pi$$

$$= -\frac{\pi}{842} \leq 0.$$

Thus for all vertices $K(a) \leq 0$, and we may assign the angles as asserted.

We now verify (4.7). Let K be a corner of Δ_1.

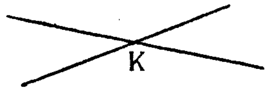
Case 1. K separates two maximal non-distinguished segments.



$$\measuredangle K \leq \frac{1}{2}\left[2\pi - 2\left[\frac{\pi}{3} - \epsilon\right]\right] = \frac{2\pi}{3} + \epsilon \leq \pi-\epsilon,$$

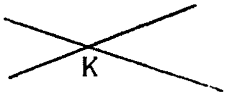
since both N_1 and N_2 have at least six sides, but may both be bad.

Case 2. K separates two maximal distinguished segments.



$$\measuredangle K \leq \frac{2}{3}\pi \leq \pi-\epsilon.$$

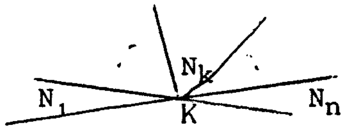
Case 3. K separates a non-distinguished segment and a distinguished segment.



$$\measuredangle K \leq \frac{1}{2}\left[2\pi - \left[\frac{\pi}{3} - \epsilon\right]\right] = \frac{5}{6}\pi + \frac{\epsilon}{2} \leq \pi-\epsilon.$$

Case 4. K is intermediate to a non-distinguished segment.

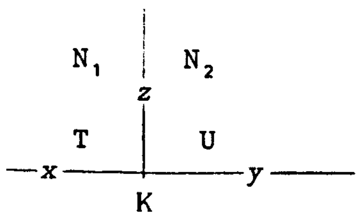
Subcase 4.1. There are at least three non-distinguished corners incident at the same vertex as K.



$$\measuredangle K \leq 2\pi - 3\left[\frac{\pi-\epsilon}{2}\right] = \frac{\pi}{2} + \frac{3}{2}\epsilon \leq \pi-\epsilon,$$

since each region $N_1,\ldots,N_n$ must be good.

Subcase 4.2. There are precisely two non-distinguished corners incident at the same vertex as K.



Firstly: by hypothesis (1) of the Theorem, neither $\varphi(\{x,z\})$ nor $\varphi(\{y,z\})$ is 2; also $\varphi(\{x,z\})$ and $\varphi(\{z,y\})$ are not both three, since, if they were, hypothesis (2) would be violated. It can now be shown that the following holds.

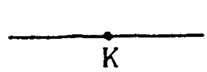
$$\text{(4.13)} \qquad \frac{1}{\varphi(\{x,z\})+1} + \frac{1}{\varphi(\{z,y\})+1} \leq \frac{9}{20}.$$

Verification of this is given after the proof. See p.173 below.

$$\text{So } \angle T + \angle U = \left[2-2\left[\frac{1}{\varphi(\{x,z\})+1} + \frac{1}{\varphi(\{z,y\})+1}\right]\right](\pi-\epsilon) \geq \frac{11}{10}(\pi-\epsilon) \geq \pi+\epsilon$$

$$\text{so } \angle K \leq 2\pi - (\pi+\epsilon) = \pi-\epsilon.$$

Subcase 4.3. There is precisely one non-distinguished corner incident at the same vertex as K.



$$\angle K = 2\pi - (\pi+\epsilon) = \pi-\epsilon.$$

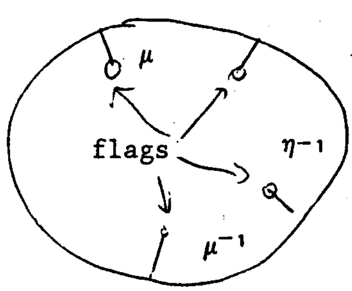
<u>Case 5. K is intermediate to a distinguished segment</u>.

$$\angle K = \frac{2}{2}\pi = \pi.$$

Thus (4.7) holds and so the number p referred to in the discussion after (4.7) is less than 842. So some distinguished segment of $\partial\Delta_1$, ξ say, has length at least 713568.

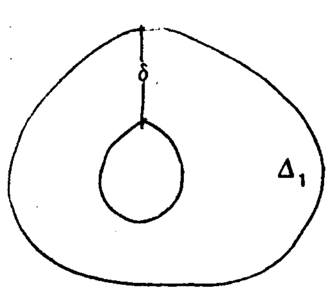
Now consider Δ_1, for some boundary cycle of Δ_1 the label on this cycle is a word on μ and η. We mark the vertices of $\partial\Delta_1$ that correspond to the endpoints of the μ's and η's with *flags* in Δ_1.

e.g.

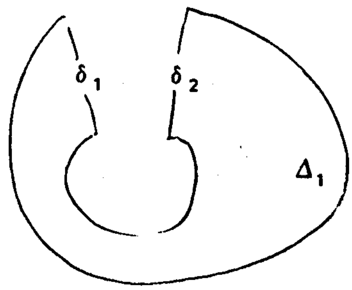


If Δ_1 has a boundary with self intersection we always draw the flags as though Δ_1 were simply connected.

e.g.



we think of it as though it were

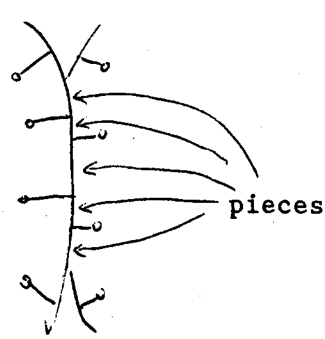


for the purposes of drawing the flags. Thus no flag on a vertex of δ_1, lies attached to a vertex of δ_2 and vice versa, and hence flags are always drawn on the "right" side of any such boundary. We may also do this for any distinguished region.

We now show that the factorization of the boundary cycles of the regions on either side of ξ, match up on ξ. Suppose by way of contradiction that it does not.

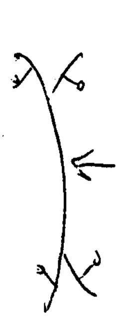
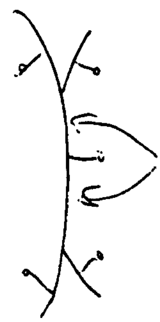
If there are two or more flags on either side of ξ then we must have a piece of length at least $|\mu|/2$, - a contradiction to (4.11).

i.e.

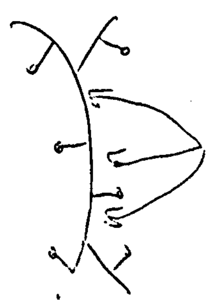


So we may suppose that there is at most one flag on either side of ξ.

Now if there is no flag on one side of ξ we find that we have a piece of length at least $|\xi|/2$, – a contradiction to (4.11).

i.e.  a piece or  pieces

So we may assume that there is precisely one flag on each side.

i.e.  pieces

Now at least one of these pieces has length at least $|\xi|/3$ i.e. length at least 237832 – a contradiction to (4.11). Hence the factorizations match up as required. This completes the details.□

We now verify (4.12). We begin by verifying the first part. Let $X=\{4,5,6,\ldots\}$ and let $\theta : X^3 \longrightarrow \mathbb{R}$ be given by $\theta(p,q,r) = 1/p + 1/q + 1/r$. The problem reduces to showing the following: *suppose* $\theta(p,q,r) < 1/2$, *then* $\theta(p,q,r) \leq 209/420 = \theta(4,5,21)$. We argue by contradiction. Suppose if possible

$(p,q,r) \in X^3$ and $209/420 < \theta(p,q,r) < 1/2$. Without loss of generality we may assume that $p \leq q \leq r$. Firstly

$1/p > 209/1260$ so $p = 4$, 5 or 6;

<u>Case 1. $p = 4$</u>.

$1/2 > 1/4 + 1/q + 1/r > 209/420$ hence

$1/4 > 1/q + 1/r > 26/105$. So $1/q > 13/105$ hence $q \leq 8$ also

$1/q < 1/4$ so $q \geq 5$

<u>subcase 1.1. $q = 5$</u>.

$1/2 > 1/4 + 1/5 + 1/r > 209/420$ hence $1/20 > 1/r > 1/21$

– a contradiction.

<u>subcase 1.2. $q = 6$</u>.

$1/2 > 1/4 + 1/6 + 1/r > 209/420$ hence $1/12 > 1/r > 17/210$

– a contradiction.

<u>subcase 1.3. $q = 7$</u>.

$1/2 > 1/4 + 1/7 + 1/r > 209/420$ hence $3/28 > 1/r > 11/105$

– a contradiction.

<u>subcase 1.4. $q = 8$</u>.

$1/2 > 1/4 + 1/8 + 1/r > 209/420$ hence $1/8 > 1/r > 103/840$

– a contradiction.

Case 2. $p = 5$.

$1/2 \geq 1/5 + 1/q + 1/r \geq 209/420$ hence

$3/10 \geq 1/q + 1/r \geq 25/84$. So $1/q \geq 25/168$ hence $q = 5$ or 6;

subcase 2.1. $q = 5$.

$1/2 \geq 1/5 + 1/5 + 1/r \geq 209/420$ hence $1/10 \geq 1/r \geq 41/420$

– a contradiction.

subcase 2.2. $q = 6$.

$1/2 \geq 1/5 + 1/6 + 1/r \geq 209/420$ hence $2/15 \geq 1/r \geq 11/84$ –

a contradiction.

Case 3. $p = 6$.

$1/2 \geq 1/6 + 1/q + 1/r \geq 209/420$ hence

$1/3 \geq 1/q + 1/r \geq 139/420$. So $1/q \geq 139/840$ hence $q = 6$. So

$1/2 \geq 1/6 + 1/6 + 1/r \geq 209/420$ hence $1/6 \geq 1/r \geq 23/140$ – a

contradiction.

This completes the proof of the first part of (4.12).

We now verify the second part. We note that precisely one

of $\varphi(e_1), \varphi(e_2), \varphi(e_3)$ is two. Hence the problem reduces to

showing the following: *Let* $\mu : X^2 \longrightarrow \mathbb{R}$ *(given by* $\mu(p,q) =$

$1/p + 1/q$) *satisfy* $\mu(p,q) \leq 1/4 = (7/12 - 1/3)$, *then*

$\mu(p,q) \le 26/105 = 61/105 - 1/3 = \mu(5,21)$.

We argue by contradiction. Suppose $26/105 < \mu(p,q) < 1/4$.

Without loss of generality we assume that $p \le q$. Firstly

$1/p \ge 13/105$ so $p \le 8$

<u>Case 1. p = 4</u>.

$1/4 \ge 1/4 + 1/q \ge 26/105$ hence $0 \ge 1/q \ge -1/420$ – a

contradiction.

<u>Case 2. p = 5</u>.

$1/4 \ge 1/5 + 1/q \ge 26/105$ hence $1/20 \ge 1/q \ge 1/21$ – a

contradiction.

<u>Case 3. p = 6</u>.

$1/4 \ge 1/6 + 1/q \ge 26/105$ hence $1/12 \ge 1/q \ge 17/210$

a contradiction.

<u>Case 4. p = 7</u>.

$1/4 \ge 1/7 + 1/q \ge 26/105$ hence $3/28 \ge 1/q \ge 77/735$

a contradiction.

<u>Case 5. p = 8</u>.

$1/4 \ge 1/8 + 1/q \ge 26/105$ hence $1/8 \ge 1/q \ge 103/840$

a contradiction.

Thus the second part of (4.12) holds.

Lastly we verify (4.13). To do this we must show that if $\mu(p,q) \leq 1/2$ then $\mu(p,q) \leq 9/20 = \mu(4,5)$. We argue by contradiction. Suppose $1/2 \geq \mu(p,q) \geq 9/20$. Without loss of generality we assume that $p \leq q$. Firstly $1/p \geq 9/40$ hence $p = 4$. Thus $1/2 \geq 1/4 + 1/q \geq 9/20$. Hence $1/4 \geq 1/q \geq 1/5$ – a contradiction.

§4.4 PROOF OF THEOREM 4.2

We ask the reader to recall what it means for a group to be as large as F_2, from §1.8. We will use the following fact throughout this section.

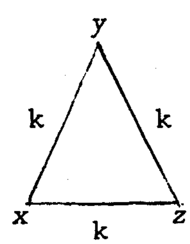
(4.14) *If* $|V(\Gamma)|=3$ *and* Γ *is connected then* $C(\Gamma,\varphi)$ *is as large as* F_2 *if and only if* $\displaystyle\sum_{\{x,y\}\in E(\Gamma)} \frac{1}{\varphi(\{x,y\})} <1$, by [34].

Let $k = \mathrm{hcf}[\varphi(E(\Gamma))]$.

Proof of (I).

Suppose first that $k\geq 4$; let u,v,w be distinct vertices of Γ. Then there is a homomorphism from $C(\Gamma,\varphi)$ onto

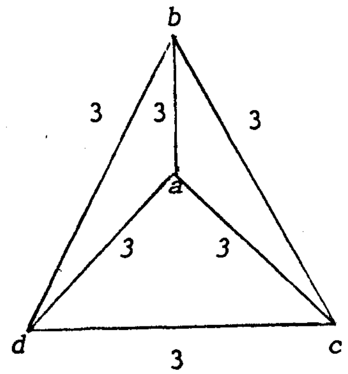
H=



given by $u \mapsto x$, $v \mapsto y$, $w \mapsto z$, and $t \mapsto x$ for $t\in V(\Gamma)-\{u,v,w\}$. By (4.14), H is as large as F_2, and hence so is $C(\Gamma,\varphi)$

Now suppose $k=3$. Let u,v,w,x be distinct elements of $V(\Gamma)$. Then there is a homomorphism from $C(\Gamma,\varphi)$ onto

H_0 =



given by $u \mapsto a$, $v \mapsto b$, $w \mapsto c$, $x \mapsto d$ and $t \mapsto a$ for $t \in V(\Gamma)-\{u,v,w,x\}$.

We now show H_0 is equally as large as F_2.

Now there is an automorphism of this group of order dividing four carrying $a \mapsto b$, $b \mapsto c$, $c \mapsto d$, $d \mapsto a$, hence

$$H_1=\langle x,\theta;\ (x\theta x\theta^{-1})^3,\ (x\theta^2x\theta^{-2})^3,\ \theta^4\rangle \text{ (x involutary)}$$

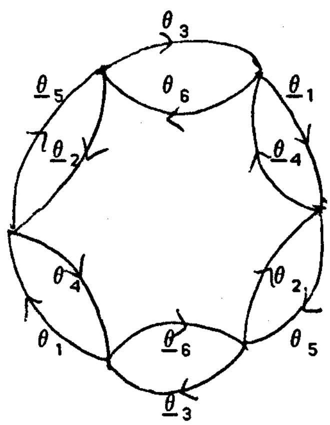
is a finite extension of H_0. Let H_2 be the kernel of the homomorphism from H_1 onto $\mathbb{Z}_2=\{0,1\}$ given by $x \mapsto 1$, $\theta \mapsto 0$. The covering corresponding to H_2 is

$$;\quad (x\theta x^{-1}\underline{\theta}^{-1})^3,\ (x^{-1}\underline{\theta}x\theta^{-1})^3,\ (x\theta^2x^{-1}\underline{\theta}^{-2})^3 \\ (x^{-1}\underline{\theta}^2x\theta^{-2})^3,\ \theta^4,\ \underline{\theta}^4$$

Collapsing the maximal subtree gives

$$H_2=\langle\theta,\underline{\theta};\ (\theta\underline{\theta}^{-1})^3,\ (\theta^2\underline{\theta}^{-2})^3,\ \theta^4,\ \underline{\theta}^4\rangle.$$

Let H_3 be the kernel of the homomorphism from H_2 onto $\langle x,y;\ x^2,\ y^2,\ (xy)^3\rangle$ given by $\theta \mapsto x$, $\underline{\theta} \mapsto y$. The covering corresponding to H_3 is



$$;\quad \begin{array}{l} \theta_1\underline{\theta}_2^{-1}\theta_3\underline{\theta}_4^{-1}\theta_5\underline{\theta}_6^{-1},\ \theta_4\underline{\theta}_3^{-1}\theta_2\underline{\theta}_1^{-1}\theta_6\underline{\theta}_5^{-1} \\ (\theta_1\theta_4\underline{\theta}_5^{-1}\underline{\theta}_6^{-1})^3,\ (\theta_4\theta_1\underline{\theta}_2^{-1}\underline{\theta}_5^{-1})^3 \\ (\theta_6\theta_3\underline{\theta}_4^{-1}\underline{\theta}_1^{-1})^3,\ (\theta_3\theta_6\underline{\theta}_5^{-1}\underline{\theta}_2^{-1})^3 \\ (\theta_5\theta_2\underline{\theta}_1^{-1}\underline{\theta}_4^{-1})^3,\ (\theta_2\theta_5\underline{\theta}_6^{-1}\underline{\theta}_3^{-1})^3 \\ (\theta_1\theta_4)^2,\ (\theta_3\theta_6)^2,\ (\theta_2\theta_5)^2 \\ (\underline{\theta}_5\underline{\theta}_2)^2,\ (\underline{\theta}_1\underline{\theta}_4)^2,\ (\underline{\theta}_3\underline{\theta}_6)^2 \end{array}$$

Collapsing the maximal subtree consisting of $\theta_4,\underline{\theta}_3,\theta_2,\underline{\theta}_1,\theta_6$ and then eliminating $\underline{\theta}_5$ by a Tietze transformation gives the following

$$\begin{aligned} H_3 = \langle \theta_1,\theta_3,\theta_5,\underline{\theta}_2,\underline{\theta}_4,\underline{\theta}_6;\ & (\theta_3\underline{\theta}_2^{-1})^3,\ (\theta_3\underline{\theta}_4^{-1})^3,\ (\theta_5\underline{\theta}_4^{-1})^3,\ (\theta_5\underline{\theta}_6^{-1})^3 \\ & \theta_1\underline{\theta}_2^{-1}\theta_3\underline{\theta}_4^{-1}\theta_5\underline{\theta}_6^{-1},\ (\theta_1\underline{\theta}_6^{-1})^3,\ (\theta_1\underline{\theta}_2^{-1})^3 \\ & \theta_1^2,\theta_3^2,\theta_5^2,\ \underline{\theta}_2^2,\ \underline{\theta}_4^2,\ \underline{\theta}_6^2\rangle. \end{aligned}$$

There is now an homomorphism from H_3 onto

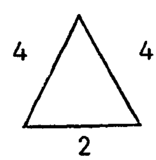
$$H_4 = y \overset{3}{\text{———}} x \overset{3}{\text{———}} z$$

given by $\theta_1,\underline{\theta}_4 \mapsto x$, $\theta_5,\underline{\theta}_6 \mapsto y$, $\theta_3,\underline{\theta}_2 \mapsto z$. By (4.14) H_4 is as large as F_2, and hence so is H_0.

<u>Proof of (II)</u>.

Let (Γ',φ') be an island not of the form (4.2). Then there is an homomorphism from $C(\Gamma,\varphi)$ onto $C(\Gamma',\varphi')$ given by $v \mapsto v$ ($v \in V(\Gamma')$) and $u \mapsto 1$ ($u \in V(\Gamma)-V(\Gamma')$). Let Γ_1 be the complete graph on $V(\Gamma')$ and let φ_1 be the extension of φ' to $E(\Gamma_1)$ for which, if Γ' is not complete, $\varphi_1(E(\Gamma_1)-E(\Gamma'))=\{6\}$. Thus there is a homomorphism from $C(\Gamma,\varphi)$ onto $C(\Gamma_1,\varphi_1)$.

If $|V(\Gamma_1)|=3$ then, since

$(\Gamma_1,\varphi_1) \neq$ 

,

it follows from (4.14) that $C(\Gamma_1,\varphi_1)$ is as large as F_2, hence so is $C(\Gamma,\varphi)$.

Suppose $|V(\Gamma_1)| \geq 4$. We begin by showing the following:

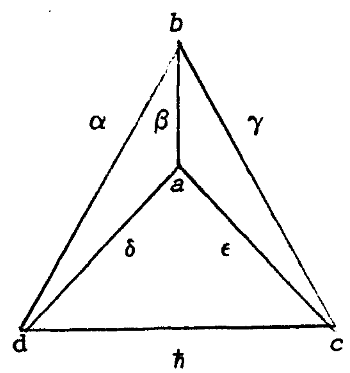
(4.15) *There is a complete subgraph Γ_2 on four vertices in Γ_1 such that any two vertices of Γ_2 are joined by a path in Γ_2 no edge of which is mapped to 2 by φ_1.*

Pick a maximal subtree T of Γ_1 consisting of edges with image at least 3. (T exists since (Γ',φ') was an island.) Pick a subtree of T containing precisely four vertices and let Γ_2 be the full subgraph of Γ_1 on these.

Let φ_2 be the restriction of φ_1 to the edge set of Γ_2. Now there is a homomorphism from $C(\Gamma_1,\varphi_1)$ onto $C(\Gamma_2,\varphi_2)$ given by $v \mapsto v$ $(v \in V(\Gamma_2))$ and $u \mapsto 1$ $(u \in V(\Gamma_1)-V(\Gamma_2))$. We show that $C(\Gamma_2,\varphi_2)$ is as large as F_2.

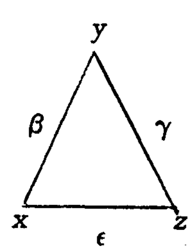
<u>Case 1. There is a triangle in Γ_2 with no edge mapped to 2 by φ_2.</u>

$(\Gamma_2,\varphi_2)=$



We suppose, without loss of generality, that no edge of the triangle with vertices a,b,c is mapped to 2. Then there is a homomorphism from $C(\Gamma_2,\varphi_2)$ onto

H=



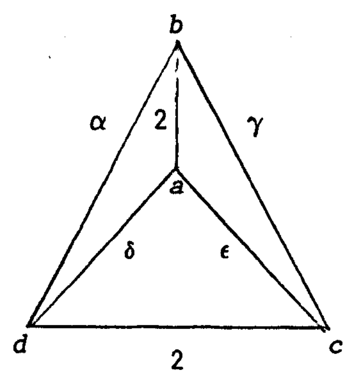
given by $a \mapsto x$, $b \mapsto y$, $c \mapsto z$, $d \mapsto 1$. By (4.14), H is as large as F_2, and hence so is $C(\Gamma_2,\varphi_2)$.

<u>Case 2. In each triangle in Γ_2 there is an edge mapped to 2 by φ_2.</u>

We note that by (4.15) there can be at most three edges mapped to 2 by φ_2.

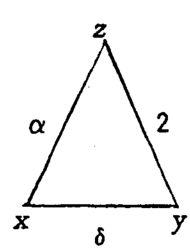
<u>Subcase 2.1. Precisely two edges of Γ_2 are mapped to 2 by φ_2.</u>

Then we must have the following situation.



If at least one of $\alpha,\gamma,\delta,\epsilon$ is at least 6 (say α), then there is a homomorphism from $C(\Gamma_2,\varphi_2)$ onto

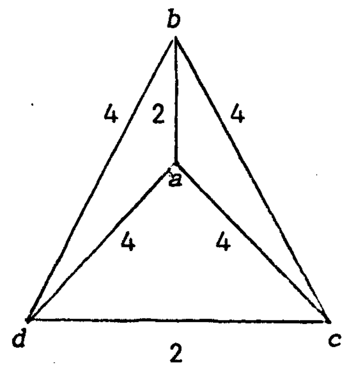
H=



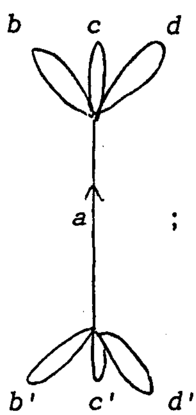
given by $a \mapsto y$, $b \mapsto z$, $c \mapsto 1$, and $d \mapsto x$. By (4.14), H is as large as F_2, and hence so is $C(\Gamma_2,\varphi_2)$.

So now assume that

$(\Gamma_2,\varphi_2)=$



Let H be the kernel of the homomorphism from $C(\Gamma_2,\varphi_2)$ onto $Z_2=\{0,1\}$ given by $a \mapsto 1$; $b,c,d \mapsto 0$. The covering corresponding to H is



; $aba^{-1}b'$, $(aca^{-1}c')^2$, $(ada^{-1}d')^2$, $(bc)^4$
$(bd)^4$, $(cd)^2$, $(b'c')^4$, $(b'd')^4$, $(c'd')^2$

Collapsing the maximal subtree and then eliminating b' gives

$H=\langle b,c,c',d,d';(cc')^2,(dd')^2,(bc)^4,(bd)^4,(cd)^2,(bc')^4,(bd')^4,(c'd')^2\rangle$

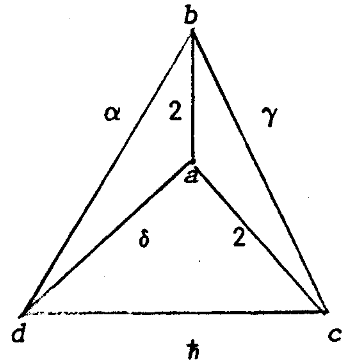
(all generators involutary). There is a homomorphism from H to

$$H' = y \;\; 4 \;\; x \;\; 4 \;\; z$$

given by $b \mapsto x$, $c' \mapsto y$, $d \mapsto z$ and $c,d' \mapsto 1$. By (4.14), H' is as large as F_2, and hence so is $C(\Gamma_2,\varphi_2)$.

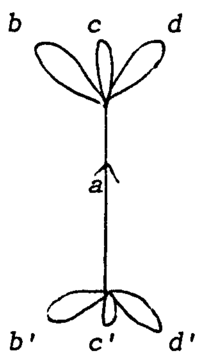
<u>Subcase 2.2. Precisely three edges are mapped to 2 by φ_2.</u>

Then we must have the following situation.



where $\delta, \hbar \neq 2$ and exactly one of α, γ is 2.

Let H be the kernel of the homomorphism from $C(\Gamma_2, \varphi_2)$ onto $Z_2 = \{0,1\}$ given by $a \mapsto 1$; $b,c,d \mapsto 0$. The covering corresponding to H is



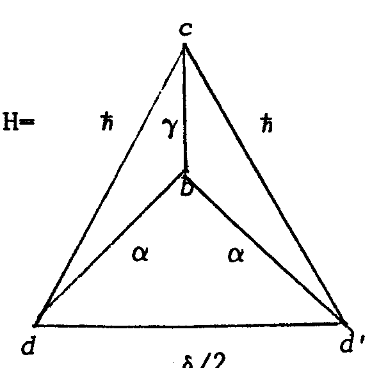
$$aba^{-1}b',\ aca^{-1}c',\ (ada^{-1}d')^{\delta/2}$$

$$;\quad (bc)^{\gamma},\ (bd)^{\alpha},\ (cd)^{\hbar},\ (b'c')^{\gamma}$$

$$(b'd')^{\alpha},\ (c'd')^{\hbar}$$

Collapsing the maximal subtree and then eliminating b' and c' gives

H=



Firstly suppose $\gamma=2$ and $\alpha\neq 2$. Then H is as large as F_2 either by Case 1 (if $\delta\geq 6$) or Subcase 2.1 (if $\delta=4$).

Suppose now that $\alpha=2$ and $\gamma\neq 2$. Then H is as large as F_2 either by Subcase 2.1 (if $\delta\geq 6$) or the above case (if $\delta=4$).

Hence $C(\Gamma_2,\varphi_2)$ is as large as F_2.

Proof of (III).

The Coxeter groups associated with the graphs in (4.2) are Z_2, $Z_2 * Z_2$, the dihedral group of order 2k, and a group which can be written as $((Z\times Z) \rtimes (Z_2\times Z_4)) \rtimes (Z_2\times Z_2)$, respectively. Each of these groups is soluble of length at most three. Hence any direct sum of such groups is soluble of length at most three [32]. Hence $C(\Gamma,\varphi)$ is soluble of length at most three by remark (1) on p.138.□

APPENDIX B

ON THE SQ-UNIVERSALITY OF A DIRECT SUM

LEMMA B.1

If A *and* B *are any groups, and* $A\times B$ *is SQ-universal then* A *or* B *is SQ-universal.*

Proof

We argue by contradiction. So suppose that we have groups A and B such that $A\times B$ is SQ-universal but that neither A nor B is.

Let H_1, H_2 be countable groups that embed into no quotient of A and B respectively. Let X be a countably infinite simple group embedding $H_1\times H_2$ (see Lyndon and Schupp [27,p.189]). Then X embeds into some quotient

$$\frac{A\times B}{N}$$

of $A\times B$. For convenience we assume that X is actually a subgroup of $\frac{A\times B}{N}$.

Let $B(N)=\{b\in B : (a,b)\in N \text{ for some } a\in A\}$. Then there is a homomorphism

$$\eta:\frac{A\times B}{N} \longrightarrow \frac{B}{B(N)}$$

given by $(a,b)N \mapsto bB(N)$. Since X is simple, the restriction of η to X is either an isomorphism or the trivial homomorphism. By construction of X it cannot be the former (or else $H_2 \hookrightarrow B/B(N)$), so must be trivial.

Let $(a,b)N\epsilon X$. Then $b\epsilon B(N)$ so there exists $a'\epsilon A$ such that $(a',b)\epsilon N$. Hence $(a,b)N=(aa'^{-1},1)N(a',b)N=(aa'^{-1},1)N$.

Thus $X\leq\{(a,1)N : a\epsilon A\}$, a homomorphic image of A under the map $a \mapsto (a,1)N$. Thus X injects into some quotient of A - a contradiction. The result follows.□

<u>COROLLARY B.1</u>

Suppose that G_i $(i\epsilon I)$ *is a collection of groups with* I *finite, and* $\sum_{i\epsilon I} G_i$ *SQ-universal. Then for some* $i\epsilon I$, G_i *is SQ-universal.*

<u>Proof</u>

By repeated application of Lemma B.1.□

<u>LEMMA B.2</u>

Let G_i $(i\epsilon I)$ *be a collection of groups. Then* $G=\sum_{i\epsilon I} G_i$ *is SQ-universal if and only if for every countable group* A *there is a* <u>*finite*</u> *subset* J *of* I *such that* A *embeds into some* *quotient of* $\sum\limits_{j\in J} G_j$.

<u>Proof</u>

Since for any subset of I the direct sum of the groups indexed by that set is a homomorphic image of G. Clearly the "if" part holds.

To show the "only if" part we show that if there exists a countable group A that embeds in no quotient of any finite subsum of the G_i's then G is not SQ-universal.

We argue by contradiction. Suppose such an A exists but that G is SQ-universal. Embed A in a two generator group B (see Lyndon and Schupp [27,p.188]). Then B embeds in some quotient G/N of G. Now since B is finitely generated and since each generator of B can be written as a finite sum of terms of the form xN, where x is in some G_i, there thus exists a finite subset J of I such that B embeds in

$$\frac{\sum\limits_{j\in J} G_i}{\sum\limits_{j\in J} G_i \cap N},$$

a contradiction. The result follows.□

THEOREM B.1

Let G_i ($i \epsilon I$) be a collection of groups with I countable. Suppose that $\sum_{i \epsilon I} G_i$ is SQ-universal. Then there exists $i \epsilon I$ such that G_i is SQ-universal.

Proof

We argue by contradiction. Suppose that no G_i is SQ-universal. By Corollary B.1 every finite sum of G_i's is not SQ-universal. Hence, for every finite subset J of I there is a countable group A_J which injects into no quotient of $\sum_{j \epsilon J} G_j$. Let $\mathcal{F}$ be the set of all non-empty finite subsets of I. Clearly $\mathcal{F}$ is countable. Hence

$$A = \sum_{J \epsilon \mathcal{F}} A_J$$

is countable. By construction A embeds in no quotient of any finite subsum of the G_i's. Hence by Lemma B.2 $\sum_{i \epsilon I} G_i$ is not SQ-universal - a contradiction. The result follows.□

We note that in Theorem B.1 the restriction that I be countable cannot be dropped. For let $X = (G_i : i \epsilon 2^{\aleph_0})$ be a collection of two generator groups such that every two generator group is isomorphic to some element of X and no two elements of X are isomorphic (see Lyndon and Schupp [27,p.188]).

Let H_i be a countable simple group into which G_i embeds ($i \in 2^{\aleph_0}$). Then

$$H = \sum_{i \in 2^{\aleph_0}} H_i$$

is SQ-universal (in fact every countable group embeds in H) but no H_i is itself SQ-universal.

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