Finiteness Conditions for Monoids and Small Categories

by

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Contents

Statement

Chapter 1 covers some basic notions and results from Algebraic Topology such as CW-complexes, homotopy and homology groups of a space in general and cellular homology for CW-complexes in particular. Also we give some basic ideas from abstract reduction systems and some supporting material such as several order relations on a set and the Knuth-Bendix completion procedure. There are only two original results of the author in this chapter, Theorem 1.4.5 and Theorem 1.7.3. The material related to Topology and Homological Algebra can be found in [12], [21], [40], [62], [82], [91] and [92]. The material related to reduction systems can be found in [5] and [11].

The original work of the author is included in Chapters 2, 3 and 4 apart from Section 3.2 which contains general notions from Category Theory, Section 3.5.2 which contains an account of the work in [67] and Section 4.1 which contains some basics from Combinatorial Semigroup Theory. The results of Section 4.2 are part of [83] which is accepted for publication in the International Journal of Algebra and Computation. The material related to Category Theory can be found in [59], [64], [66], [67], [74], [75], [76], [82] and [93]. The material related to Semigroup Theory is in [24] and [34].

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Abstract

In Chapter 2 we show that for every monoid S which is given by a finite and complete presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$, and for every $n \geq 2$, there is a chain of CW-complexes

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \Delta_3 \subset \ldots \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) \subset \Delta_n,$$

such that Δ_n has dimension n, for every $2 \leq s \leq n$ the s-skeleton of Δ_n is Δ_s and F acts on Δ_n. This action is called translation. Also we show that, for $2 \leq s \leq n$, the open s-cells of Δ_n are in a 1-1 correspondence with the s-tuples of positive edges of $\mathcal{D}$ with the same initial. For the critical s-tuples, the corresponding open s-cells are denoted by $\mathbf{p}_{s-1}$ and the set of their open translates by $F.\mathbf{p}_{s-1}.F$. The following holds true.

$$\Delta_s = \begin{cases} (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) \sqcup F.\mathbf{p}_{s-1}.F & \text{if} \quad s \geq 3 \\ \mathcal{D} \sqcup F.\mathbf{p}_1.F & \text{if} \quad s = 2, \end{cases}$$

where $\sqcup$ stands for the disjoint union. Also, for every $2 \leq s \leq n-1$, there exists a cellular equivalence $\sim_s$ on $K_s = (\Delta_s \times \Delta_s)^{(s+1)}$ such that $K_s / \sim_s = (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-1})$ and the following is an exact sequence of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$0 \longrightarrow H_s(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{s-1}.\mathbb{Z}S \xrightarrow{\nu} H_{s-1}(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) \longrightarrow 0,$$

where $(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) = \mathcal{D}$ if $s = 2$. Using the above short exact sequences, we deduce that S is of type bi-FP$_n$ and that the free finite resolution of $\mathbb{Z}S$ is S-graded.

In Chapter 3 we generalize the notions left-(respectively right)-FP$_n$ and bi-FP$_n$ for small categories and show that bi-FP$_n$ implies left-(respectively right)-FP$_n$. Also we show that another condition, which was introduced by Malbos and called FP$_n$, implies bi-FP$_n$. Since the name FP$_n$ is confusing, we call it here f-FP$_n$ for a reason which will be made clear in Section 3.1. Restricting to monoids, we show that, if a monoid is given by a finite and complete presentation, then it is of type f-FP$_n$. Lastly, for every small category $\mathbb{C}$, we show how to construct free resolutions of $\mathbb{Z}\mathbb{C}$, at least up to dimension 3, using some geometrical ideas which can be generalized to construct free resolutions of $\mathbb{Z}\mathbb{C}$ of any length.

In Chapter 4 we study finiteness conditions of monoids of a combinatorial nature. We show that there are semigroups S in which $\min_{\mathcal{R}}$, is independent of other conditions which S may satisfy such as being finitely generated, periodic, inverse, E-unitary and even from the finiteness of the maximal subgroups of S. We also relate the congruences of a monoid with the finiteness condition $\min_{Q}$, and show that, if S is a monoid which satisfies $\min_{Q}$, then every congruence $\mathcal{K}$ on S which contains Q is of finite index in S. If a semigroup satisfies $\min_{Q}$ and has all its maximal subgroups locally finite, then we show that it is finite. Lastly, we show that, for trees of completely 0-simple semigroups, the local finiteness of its maximal subgroups implies the local finiteness of the semigroups.

Introduction

In the mid 80's Squier initiated a program whose main purpose was to find homological and homotopical invariants for rewriting systems and, in particular, to characterize algebraically those monoids which are given by a finite complete presentation (FCP). This class of monoids is of interest since they have solvable word problem. In [95] Squier showed that, if a monoid S is given by some presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$, then there is always a free acyclic resolution of the trivial left-$\mathbb{Z}S$-module $\mathbb{Z}$:

$$\mathbb{Z}S.\mathbf{r} \xrightarrow{\partial_2} \mathbb{Z}S.\mathbf{x} \xrightarrow{\partial_1} \mathbb{Z}S \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{1}$$

and, if we assume that $\mathcal{P}$ is in addition FCP, then the resolution (1) can be prolonged with another term as follows

$$\mathbb{Z}S.\mathbf{p} \xrightarrow{\partial_3} \mathbb{Z}S.\mathbf{r} \xrightarrow{\partial_2} \mathbb{Z}S.\mathbf{x} \xrightarrow{\partial_1} \mathbb{Z}S \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{2}$$

where $\mathbf{p}$ is the set of critical pairs of $\mathbf{r}$. This in particular means that S is of type left-FP_3. In the same way one can show that, if the system $\mathcal{P}$ giving S is finite and complete, then S is of type right-FP_3. Later Kobayashi [50], Groves [36] and Brown [13] extended the result by showing that such a monoid should necessarily satisfy the conditions left/right-FP_n for all n. Unfortunately the properties left/right-FP_n together are not equivalent to FCP. To separate between these two, Squier introduced in [96] another finiteness condition, invariant of the presentation, called finite derivation type (FDT). He showed that, if S is FCP, then S is FDT and exhibited an example of a monoid satisfying the condition left-FP_∞ but not being FDT and therefore not FCP. In fact this is not enough to divide FP_∞ from FCP since the monoid of that example was left-FP_∞ but not even right-FP_3 as was pointed out by Pride and Wang in [89]. In the mean time Wang and Pride [99] introduced another finiteness condition called FHT which later was proved to be strictly implied by FDT [87]. On the other hand Otto and Kobayashi showed in [53] that FHT is equivalent to bi-FP_3. One can use this and the example given in [54] of a monoid which is left-FP_n and right-FP_n for every n but is not FHT, to separate between bi-FP_n

and left-FP$_n$ and right-FP$_n$. This in particular implies that left and right-FP$_\infty$ taken together do not guarantee that the monoid is given by a finite complete presentation. On the other hand, FDT and FCP are not equivalent. Indeed, for groups the properties FDT and bi-FP$_3$ coincide (see [53] and [86]) and bi-FP$_\infty$ is equivalent to left (right)-FP$_\infty$ as well (see [10]). On the other hand there are examples of groups which are bi-FP$_3$ but not bi-FP$_4$ as shown in [9] or [10] and therefore these groups can not be given by a finite complete presentation. Thus FDT draws a line between what we know as homological properties of a monoid and FCP itself.

Our attention is immediately drawn to the fact that the resolution (1) is in a certain sense an invariant of $\mathcal{P}$, since, as we mentioned before, one can construct such a resolution whenever a presentation for S is given. There is another similar to it found in [53]:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S \otimes \mathbb{Z}S \longrightarrow \mathbb{Z}S \longrightarrow 0, \tag{3}$$

where $\mathcal{M} = J/J^2$ is the relation $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with J being the kernel of the natural morphism $\rho : \mathbb{Z}F \longrightarrow \mathbb{Z}S$.

To define FDT for a monoid given by a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$, Squier [96] constructs the so called reduction graph $\Gamma = \Gamma(\mathbf{x}, \mathbf{r})$ whose vertices are the words of the free monoid $F = \mathbf{x}^*$ and whose edges correspond to one single step reductions on words. Then he introduces certain equivalence relations called homotopy relations, which in particular identify any two parallel paths in Γ arising from disjoint reductions on the same word. It was pointed out by Pride [85] that, instead of studying Γ together with the homotopy relations, one can construct a 2-dimensional CW-complex $\mathcal{D}$ whose 1-skeleton is Γ and 2-cells arising from the same pair of paths defining the homotopy relations in the sense of Squier. Also Pride noted that there is a bi-action of F on $\mathcal{D}$ which turns out to have homotopical and homological consequences. For example, expressed in this topological setting, FDT can now be defined as follows: there is a finite set of closed paths X in Γ such that, if we attach 2-cells for each closed path from $F.X.F$ then the new 2-complex obtained thus has fundamental homotopy groups trivial. The advantage of this approach is that we can now associate to $\mathcal{D}$ the respective cellular chain complex

$$C_2(\mathcal{D}) \xrightarrow{\partial_2} C_1(\mathcal{D}) \xrightarrow{\partial_1} C_0(\mathcal{D}) \longrightarrow 0$$

and study $H_1(\mathcal{D})$. Pride [85] proved that $H_1(\mathcal{D})$ has a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure and that there is an exact sequence of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$H_1(\mathcal{D}) \xrightarrow{\eta} \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathcal{M} \longrightarrow 0.$$

Later Guba and Sapir [39] using ideas of diagram groups, or Otto and Kobayashi [52] in an alternative way, showed that η is injective giving thus the short exact sequence

$$0 \longrightarrow H_1(\mathcal{D}) \xrightarrow{\eta} \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathcal{M} \longrightarrow 0. \tag{4}$$

Splicing together (3) and (4) one gets

$$H_1(\mathcal{D}) \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S \otimes \mathbb{Z}S \longrightarrow \mathbb{Z}S \longrightarrow 0, \tag{5}$$

which in contrast with (1) involves not only the data giving the presentation $\mathcal{P}$, namely $\mathbf{x}$ and $\mathbf{r}$, but also the first homology group $H_1(\mathcal{D})$ of the Squier complex $\mathcal{D}$ associated with that presentation. If the presentation $\mathcal{P}$ is finite and complete, then $H_1(\mathcal{D})$ is finitely generated which together with (5) shows that S is of type bi-FP$_3$. Also we mention here that the map $H_1(\mathcal{D}) \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S$ is injective. It seems that the philosophy of obtaining long exact sequences, which can then be used to obtain long free resolutions of S, is to introduce first short exact sequences as in (4), and then to splice them with long exact sequences constructed from a previous step, as we did with (3) and (4) before.

In his thesis [71] (see also [72]) S. McGlashan extended the Squier complex $\mathcal{D}$ associated with a presentation $\mathcal{P}$ by adding to it 2-cells $\mathbf{p}$ and their translates $F.\mathbf{p}.F$ such that the homology classes of the 1-cycles arising from the boundaries of cells from $\mathbf{p}$ are bi-module generators of $H_1(\mathcal{D})$. That complex was denoted by $\mathcal{D}^{\mathbf{p}}$. Then it was shown how to add 3-cells to $\mathcal{D}^{\mathbf{p}}$ in a 1-1 correspondence with 3-tuples of positive edges with the same initial which are non-critical, to obtain a 3-complex denoted there by $(\mathcal{D}, \mathbf{p})$. As before we can associate with $(\mathcal{D}, \mathbf{p})$ the cellular chain complex

$$C_3(\mathcal{D}, \mathbf{p}) \xrightarrow{\partial_3} C_2(\mathcal{D}, \mathbf{p}) \xrightarrow{\partial_2} C_1(\mathcal{D}, \mathbf{p}) \xrightarrow{\partial_1} C_0(\mathcal{D}, \mathbf{p}) \longrightarrow 0$$

and study $H_2(\mathcal{D}, \mathbf{p})$. It was proved in [71] and [72] that there is a short exact sequence

$$0 \longrightarrow H_2(\mathcal{D}, \mathbf{p}) \longrightarrow \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \longrightarrow H_1(\mathcal{D}) \longrightarrow 0 \tag{6}$$

of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules. If we splice (6) with (5), which we know from the recursive step, then we obtain the long sequence

$$0 \longrightarrow H_2(\mathcal{D}, \mathbf{p}) \longrightarrow \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S \otimes \mathbb{Z}S \longrightarrow \mathbb{Z}S \longrightarrow 0. \tag{7}$$

It was also shown that, under the assumption that $\mathcal{P}$ is finite and complete, if we choose cells from $\mathbf{p}$ to have their own boundaries arising from resolutions of the critical pairs of positive edges with the same initial, then $H_2(\mathcal{D}, \mathbf{p})$ is finitely generated and therefore one can deduce easily from (7) that S is of type bi-FP$_4$.

In Chapter 2 of this thesis, assuming that the presentation $\mathcal{P}$ is finite and complete, we keep on doing the above process in all dimensions. Roughly speaking, suppose that recursively we have constructed a sequence of CW-complexes

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \Delta_3 \subset \ldots \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) \subset \Delta_n, \tag{8}$$

such that Δ_n has dimension n, for every $2 \leq s \leq n$ the s-skeleton of Δ_n is Δ_s and F acts on Δ_n. This action is called translation. Also, we suppose that, for $2 \leq s \leq n$, the open s-cells of Δ_n are in a 1-1 correspondence with the s-tuples of positive edges of $\mathcal{D}$ with the same initial. For the critical s-tuples, the corresponding open s-cells are denoted by $\mathbf{p}_{s-1}$ and the set of their open translates by $F.\mathbf{p}_{s-1}.F$. The following holds true.

$$\Delta_s = \begin{cases} (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) \sqcup F.\mathbf{p}_{s-1}.F & \text{if} \quad s \geq 3 \\ \mathcal{D} \sqcup F.\mathbf{p}_1.F & \text{if} \quad s = 2, \end{cases}$$

where $\sqcup$ stands for the disjoint union. Also, for every $2 \leq s \leq n-1$, there exists a cellular equivalence $\sim_s$ on $K_s = (\Delta_s \times \Delta_s)^{(s+1)}$ such that $K_s / \sim_s = (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-1})$ and the following is an exact sequence of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$0 \longrightarrow H_s(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{s-1}.\mathbb{Z}S \xrightarrow{\nu} H_{s-1}(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) \longrightarrow 0,$$

where $(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{s-2}) = \mathcal{D}$ if $s = 2$.

We construct inductively an $(n+1)$-dimensional CW-complex Δ_{n+1}, having Δ_n as its n-skeleton, whose open $(n+1)$-cells are of two kinds: those which are in a 1-1 correspondence with the non-critical $(n+1)$-tuples of positive edges with the same initial, and open $(n+1)$-cells $\mathbf{p}_n$ in a 1-1 correspondence with critical $(n+1)$-tuples of positive edges with the same initial, together with their open translates $F.\mathbf{p}_n.F$. The construction is carried out in two stages. In the first stage we construct an $(n+1)$-complex $(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ whose open $(n+1)$-cells are in a 1-1 correspondence with the non-critical $(n+1)$-tuples of positive edges with the same initial. In the second stage we attach open $(n+1)$-cells $\mathbf{p}_n$ in a 1-1 correspondence with critical $(n+1)$-tuples of positive edges with the same initial, together with their open translates $F.\mathbf{p}_n.F$. As before we have

$$\Delta_{n+1} = (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \sqcup F.\mathbf{p}_n.F,$$

and then (8) extends to the sequence

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \ldots \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) \subset \Delta_n \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) \subset \Delta_{n+1}, \tag{9}$$

with the property that the following sequence of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$0 \longrightarrow H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S \xrightarrow{\nu} H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2}) \longrightarrow 0 \qquad (10)$$

is exact.

The general picture of the construction is given in our main **Theorem 2.1.1** which roughly states that *associated with a finite and complete presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$ giving a monoid S, and for every $n \geq 2$, there is a chain of CW-complexes*

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \Delta_3 \subset ... \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2}) \subset \Delta_n,$$

such that Δ_n has dimension n and for every $2 \leq m \leq n$, the m-skeleton of Δ_n is Δ_m. The complex satisfies certain properties among which is the exactness of the sequences of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$0 \rightarrow H_m(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{m-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{m-1}.\mathbb{Z}S \xrightarrow{\nu} H_{m-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{m-2}) \rightarrow 0,$$

where $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{m-2}) = \mathcal{D}$ if $m = 2$.

We use these sequences to give another proof of Corollary 7.2 of [55] for the integral monoid ring $\mathbb{Z}S$, which is stated in the following.

Theorem 2.1.2 *If a monoid S is given by some finite complete presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$, then it is of type bi-$FP_n$.*

We also reprove in **Theorem 2.4.3** the fact that *properties FDT and FHT for groups are equivalent*, by using the machinery developed earlier in Chapter 2. Other proofs can be found in [33] and also in [20], [86].

All left $\mathbb{Z}S$-modules, right $\mathbb{Z}S$-modules and $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules involved in the above mentioned result can be seen as objects from the functor categories $\mathbf{Ab}^{\mathbb{Z}S}$, $\mathbf{Ab}^{\mathbb{Z}S^{opp}}$ and $\mathbf{Ab}^{\mathbb{Z}S^{opp} \otimes \mathbb{Z}S}$ respectively. In fact all these categories are special cases of functor categories of the form $\mathbf{Ab}^{\mathbb{C}}$ with $\mathbb{C}$ a small additive category, since every ring with a unit element, in particular $\mathbb{Z}S$, $\mathbb{Z}S^{opp}$ and $\mathbb{Z}S^{opp} \otimes \mathbb{Z}S$, is a small additive category with a single object its unit element. It is then natural to ask whether it is possible to look for finiteness conditions of a homological nature for small categories which would generalize some of the results above. There is also another good reason for studying small categories as generalizations of monoids as we will explain below. In [28] (see also [7]) Dwyer and Kan introduced the notion of the category of factorizations $F\mathbb{C}$ of a small category $\mathbb{C}$. Its objects are the morphisms of $\mathbb{C}$ and a morphism $\omega \longrightarrow \omega'$ is a pair (u, v) of morphisms in $\mathbb{C}$ such that $\omega' = v\omega u$. Composition is defined by $(u', v')(u, v) = (u'u, vv')$.

One can study what are called in [7] natural systems of abelian groups on $\mathbb{C}$ which are functors $D : F\mathbb{C} \longrightarrow \mathbf{Ab}$. Every such functor extends to an additive functor $D' : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ where $\mathbb{Z}F\mathbb{C}$ is the additive category arising from $F\mathbb{C}$, or, more explicitly, it is $F\mathbb{C}$ enriched in $\mathbf{Ab}$. Thus, for a given small (non-additive) category $\mathbb{C}$, one can study two functor categories, $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}}$ and $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$. In contrast with $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}}$, whose object are functors associating with each object of $\mathbb{C}$ an abelian group, the functors of the category $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$ associate with each morphism in $\mathbb{C}$ an abelian group. In the case of monoids, the difference between these two categories is clear and one can expect to have finiteness conditions of a new nature if working with the second category. It is fruitful and more illuminating to work with small categories rather than with the special case of monoids when studying the category $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$, and then apply the results to monoids. There is yet another reason why presentations of categories are interesting to study. It appears that the homological properties of a monoid S which is given by some presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$ are "governed" by the reduction graph $\Gamma(\mathbf{x}, \mathbf{r})$. The vertices of this graph can be seen as paths of the free category $\mathbf{x}^*$ with a single object, the empty word λ, and with generating morphisms one for each generator $x \in \mathbf{x}$, and the edges of Γ are path-rewritings on $\mathbf{x}^*$ corresponding to $\mathbf{r}$-reductions. It is not essential that the graph whose paths will be rewritten has a single vertex; hence we can expect that most of the properties which are discussed in the above mentioned papers, will hold true if we try to generalize the results to monoids with several objects, known as small categories. Why generalize? FCP is not a notion borrowed from the theory of monoids, nor that of categories but from the theory of Term Rewriting Systems and occurs in many fields of Algebra whenever one speaks of presentations of algebras in general. So generalizing the existing theory to more general structures like small categories, at the very least would allow us to understand more in depth the relation between FCP and the homological properties of the algebraic structures.

In [67] Malbos defines a functor B in $\mathbf{Ab}^{\mathbb{A}}$ ($\mathbb{A}$ additive) to be of type FP_n if there is a projective resolution in $\mathbf{Ab}^{\mathbb{A}}$

$$P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

such that P_i is finitely generated for $0 \leq i \leq n$. In this context, a small category $\mathbb{C}$ is called of type f-FP_n if the constant functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$ is of type FP_n. It is called of type bi-FP_n if the functor $\mathbb{Z}\mathbb{C}$ which sends $(c, d) \in \mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{C}$ to the free abelian group with bases $\mathbb{C}(c, d)$, is of type FP_n in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}}$. It is called of type left-FP_n (respectively right-FP_n) if the trivial left (respectively right) $\mathbb{Z}\mathbb{C}$-module $\mathbb{Z}$, sending each object of $\mathbb{Z}\mathbb{C}$ on the group $\mathbb{Z}$ and each morphism of $\mathbb{C}$ on $1_{\mathbb{Z}}$, is of type FP_n in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}}$ (respectively $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$). The property f-$\mathrm{FP}_n$ was introduced by Malbos in [67] but he calls it just FP_n which is confusing with left or right FP_n. Since it is a property of the constant functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$ and since $F\mathbb{C}$ is the category of factorizations of $\mathbb{C}$, we renamed it by calling it f-FP_n.

The main result of [67] is that, for any presentation $[\mathbf{x}, \mathbf{r}]$ of a small category $\mathbb{C}$, there is an exact sequence similar to (1):

$$\mathbb{Z}\mathbb{C}[\mathbf{r}] \xrightarrow{\delta_2} \mathbb{Z}\mathbb{C}[\mathbf{x}] \xrightarrow{\delta_1} B_0(\mathbb{C}) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

which is in fact a projective resolution of $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$. He also shows that, if the presentation of $\mathbb{C}$ is finite and complete, then $\mathbb{C}$ is of type f-FP_3, by giving an exact sequence similar to (2).

In Chapter 3 we deal with finiteness conditions f-FP_n, bi-FP_n and left, right-FP_n for small categories. Being unable to verify the proofs for the implications i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) of Lemma 3.3 of [67], we give our own proofs in Theorem 3.4.5 and Theorem 3.4.10 stated below.

Theorem 3.4.5 *For every small category $\mathbb{C}$ the following implication holds true*:

$$\text{bi-FP}_n \Longrightarrow \text{left (right)-FP}_n.$$

Theorem 3.4.10 *If a small category $\mathbb{C}$ is of type f-FP_n, then it is of type bi-FP_n.*

Regarding monoids seen as categories, we prove the following.

Theorem 3.4.12 *If the monoid S is of type bi-FP_n and the corresponding free partial resolution is S-graded, then S is of type f-FP_n. In particular, monoids which are given by a finite complete presentation are of type f-FP_n.*

In Section 3.5, we look for ways to build partial resolutions for the trivial functor $\mathbb{Z} \in \mathit{Add}(\mathbb{Z}F\mathbb{C}, \mathbf{Ab})$. **Theorem 3.5.2** gives a resolution of length 3 and implicitly a condition for a category to be of type f-FP_3. The finiteness of that resolution is related to a property which we call FDT for small categories and is defined in a similar fashion to FDT for monoids (see [85] or [96]). More precisely, we prove the following.

Theorem 3.5.3 *If $\mathbb{C}$ is of type FDT, then $\mathbb{C}$ is of type f-FP_3.*

In Chapter 4 we study finiteness conditions of monoids of a combinatorial nature. Several authors have considered two approaches to studying finiteness of finitely generated semigroups. The first is to assume conditions such as permutation properties, iteration conditions or repetitivity, and combine either one of them with periodicity or with conditions imposed on the growth function of the semigroup, to obtain the finiteness of the semigroup. The second approach is to replace the first mentioned group of conditions by the minimal conditions on right/left ideals, quasi-ideals or bi-ideals, and look for similar results as in the first case. In **Theorem 4.3.1** we show that there are semigroups S in which $\min_{\mathcal{R}}$, is independent of other "good" conditions which S may satisfy such as being finitely generated, periodic, inverse, E-unitary and even from the finiteness of the maximal subgroups of S. On the one hand this reveals a rather strange nature of $\min_{\mathcal{R}}$ (and other minimal conditions similar to it), but on the other hand it justifies their consideration as candidates to obtain finiteness of the semigroups besides other conditions. We list below some of the results of this chapter.

Proposition 4.2.3 *Let S be a finitely generated monoid which satisfies* $\min_{\mathcal{Q}}$. *Every congruence $\mathcal{K}$ on S which contains $\mathcal{Q}$ is of finite index in S.*

Theorem 4.2.7 *A finitely generated semigroup S is finite if and only if it satisfies* $\min_{\mathcal{Q}}$ *and all its maximal subgroups are locally finite.*

Theorem 4.2.15 *Let $(S, *)$ be a tree of completely 0-simple semigroups $(S_t, \cdot)$ where $t \in T$ and T is a tree. If the maximal subgroups of S are locally finite, then S is locally finite.*

Notations

Topology and Homology

$X \coprod Y$	the coproduct of spaces X and Y
$X \coprod_f Y$	the attaching of X to Y via f
Φ_e	the characteristic map of a cell e
φ_e	the attaching map of a cell e
$\overline{e}$	the closure of the cell e
e°	the boundary of a cell e
∂e	the set of closed (dim$e$-1)-cells meeting $\overline{e}$
D^n	the n-dimensional disk
S^n	the n-dimensional sphere
$[f]$	the homotopy class of a map $f : S^n \longrightarrow X$
$H_n(X)$	the n^{th} homology group of a space X
$\widetilde{H}_n(X)$	the reduced n^{th} homology group of a space X
$H_n(X, A)$	the relative homology group
$\mathrm{cls}\xi$	the homology class of an n-cycle ξ
$\pi_n(X)$	the n^{th} homotopy group of a path-connected space X
Δ^n	the standard n-simplex
$\simeq$	homotopy equivalence
$\backsimeq$	homeomorphism
$[\sigma^n : \sigma^{n-1}]$	the incidence number of σ^n and σ^{n-1}
$E(\Delta)$	the cell decomposition of the complex Δ
$c_1 \parallel c_2$	cells c_1 and c_2 are parallel
$\bigsqcup$	disjoint union
$\sigma_1 \otimes \sigma_2$	the tensor product of σ_1 with σ_2

Rewriting Systems

$\mathcal{P} = [\mathbf{x}, \mathbf{r}]$	presentation of a small category
$\xrightarrow{\alpha}$	the single-step reduction relation corresponding to $\alpha \in \mathbf{r}$
λ	the empty word
$\emptyset$	the empty set
$\Gamma(\mathbf{x}, \mathbf{r})$	the reduction graph associated with the presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$
$\mathbf{e}^+$	the set of positive edges of $\Gamma(\mathbf{x}, \mathbf{r})$
$\mathcal{D}(\mathcal{P})$	the Squire complex associated with the presentation $\mathcal{P}$
$\mathbb{Z}$	the set of integers
$>_{mul}$	multiset order
$>_{lex}$	lexicographical order
$[A]$	the multiset associated with A

Categories

$\mathbb{C}$	category
$a \in \mathbb{C}$	a is an object of the category $\mathbb{C}$
$\mathbb{C}^{opp}$	the opposite of the category $\mathbb{C}$
$\hom_{\mathbb{C}}(a, b)$	the set of morphisms from a to b in a category $\mathbb{C}$
1_a	the identity morphism at some object a
$\approx$	natural bijection
$\cong$	isomorphism
$F(X)$	the free category generated by a graph X
$R^{\#}$	the congruence generated by R
$Nat(G, F)$	the hom-set $\mathbb{B}^{\mathbb{C}}(F, G) = \{\tau \mid \tau : F \longrightarrow G \text{ natural}\}$
$\bigoplus\limits_{i \in I} c_i$	the coproduct of the family $\{c_i\}$
$\times_{i \in I} c_i$	the product of the family $\{c_i\}$
$\hookrightarrow$	monomorphism
$\twoheadrightarrow$	epimorphism
$Add(\mathbb{A}, \mathbb{B})$	the category of abelian functors from $\mathbb{A}$ to $\mathbb{B}$
$Ker f$	the kernel of a morphisms $f : a \to b$
$Im f$	the image of a morphism $f : a \to b$
$Nat(S, T)_{Add(\mathbb{A}, \mathbb{B})}$	the hom-set $Add(\mathbb{A}, \mathbb{B})(S, T) = \{\tau \mid \tau : S \longrightarrow T \text{ natural}\}$
$\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{B}$	the tensor product of two additive categories $\mathbb{A}$ and $\mathbb{B}$

$F\mathbb{C}$	the category of factorizations of the category $\mathbb{C}$
$\mathbb{Z}\mathbb{C}$	the enrichment of the category $\mathbb{C}$ in **Grp**
$F(a,\_)$ and $F(\_,b)$	partial functors of the bifunctor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$
$F \otimes_{\mathbb{C}} G$	the tensor product of $F \in \mathbf{Ab}^{\mathbb{C}}$ with $G \in \mathbf{Ab}^{\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}}$
$f\,|_A$	the restriction of $f : X \to Y$ on $A \subseteq X$

Semigroups

$\mathbb{N}$	the set of natural numbers
A^*	the free monoid generated by the set A
A^+	the free semigroup generated by the set A
$\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$	Green's relations
$\mathcal{B}$	the analogue of Green's relation $\mathcal{R}$ for bi-ideals
$\mathcal{Q}$	the analogue of Green's relation $\mathcal{R}$ for quasi-ideals
FI_X	the free inverse monoid on X
FG_X	the free group on X
$F(L)$	the set of factors of a language L
$A^{\pm\omega}$	the set of bi-infinite words with letter from A
A^{ω}	the set of right-infinite words with letter from A
$A^{-\omega}$	the set of left-infinite words with letter from A
$\leq_{\mathcal{G}}$	the order induced by the relation $\mathcal{G} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}, \mathcal{B}, \mathcal{Q}\}$
$min_{\mathcal{G}}$	the minimal condition for $\mathcal{G} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}, \mathcal{B}, \mathcal{Q}\}$
$\mathrm{suff}S$	the set of suffixes of a subset $S \subseteq FG_X$
$E(S)$	the set of idempotents of a semigroup S

Chapter 1

Preliminaries

1.1 CW Complexes

A part of this thesis deals with the way we construct complex spaces starting with simple ones and then how we can study the homotopy and homology groups of these spaces. We start with a number of basic notions and concepts from Algebraic Topology which can be found in several books such as [21], [40], [62], [91] and [92].

A *coproduct* $X_1 \coprod X_2$ of two spaces X_1 and X_2 is just the disjoint union of them in which both are open subsets of the union. If $f_i : X_i \longrightarrow Y$ for $i = 1, 2$ are two continuous maps, then the continuous map $f_1 \coprod f_2 : X_1 \coprod X_2 \longrightarrow Y$ is defined by setting $(f_1 \coprod f_2)(x) = f_i(x)$, where $x \in X_i$.

Definition 1.1.1 Let X and Y be spaces, let A be a closed subspace of X, and let $f : A \longrightarrow Y$ be continuous. The space obtained from Y by *attaching* X *via* f is $(X \coprod Y)/\sim$, where $\sim$ is the equivalence relation on $X \coprod Y$ generated by $\{(a, f(a)) \in (X \coprod Y) \times (X \coprod Y) \mid a \in A\}$. This space is denoted by $X \coprod_f Y$. The map f is called the *attaching map.*

Definition 1.1.2 The map $\Phi : X \longrightarrow X \coprod_f Y$ (which is the composite $X \hookrightarrow X \coprod Y \longrightarrow X \coprod_f Y$) is called the *characteristic map.*

Definition 1.1.3 An *n-cell* e^n (or simply e) is a homeomorphic copy of the open n-disk $D^n - S^{n-1}$. Its closure will be denoted by $\overline{e}$.

Definition 1.1.4 Let Y be a Hausdorff space and let $f : S^{n-1} \longrightarrow Y$ be continuous. Then $D^n \coprod_f Y$ is called the space obtained from Y by *attaching an n-cell via* f, and is denoted by Y_f.

Definition 1.1.5 A continuous map $g : (X, A) \longrightarrow (Y, B)$ is a *relative homeomorphism* if $g|(X - A) : X - A \longrightarrow Y - B$ is a homeomorphism.

Definition 1.1.6 If a topological space X is a disjoint union of cells: $X = \cup\{e \mid e \in E\}$, then define, for each $k \geq 0$, the *k-skeleton* $X^{(k)}$ of X by

$$X^{(k)} = \cup\{e \in E \mid \dim(e) \leq k\}.$$

We then have, $X^{(k)} \subset X^{(k+1)}$ for every $k \geq 0$, and $X = \cup_{k \geq 0} X^{(k)}$.

Definition 1.1.7 A *CW-complex* is an ordered triple (X, E, Φ), where X is a Hausdorff space, E is a family of cells in X, and $\Phi = \{\Phi_e \mid e \in E\}$ is a family of characteristic maps such that

1. $X = \bigcup\{e \mid e \in E\}$ (disjoint union);

2. for each k-cell $e \in E$, the map $\Phi_e : (D^k, S^{k-1}) \longrightarrow (e \cup X^{(k-1)}, X^{(k-1)})$ is a relative homeomorphism;

3. if $e \in E$, then its closure is contained in a finite union of cells in E,

4. X has the weak topology determined by $\{\overline{e} \mid e \in E\}$: A set $A \subset X$ is open (or closed) if and only if $A \cap \overline{e}$ is open (or closed) in $\overline{e}$ for every $e \in E$.

It is proved in Lemma 8.15, p.200 of [91] or in p.193 of [92] that, for every k-cell $e \in E$, $\Phi_e(D^k) = \overline{e}$. We call $\Phi_e(D^k)$ a *closed* k-cell. The restriction of Φ_e on the boundary S^{k-1} of D^k is called the *attaching map* of e and $e^\circ = \Phi_e(S^{k-1})$ is called the *boundary* of e. We denote by ∂e the set of all closed $(k-1)$-cells which meet $\overline{e}$. In future, in order to simplify the notation, we will write $\sigma \in \partial e$ to mean that $\overline{\sigma}$ meets $\overline{e}$. Note that it is not always true that $e = \Phi_e(D^k - S^{k-1})$ is open as a subset of X even though sometimes it is referred to by several authors as the open k-cell e.

A *subcomplex* of a CW-complex X is a subspace $A \subset X$ which is a union of cells of X, such that the closure of each cell in A is contained in A.

In practice the construction of CW-complexes is done in an inductive way as follows:

(1) Start with a discrete set X^0, the 0-cells of X.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_n^α via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$. This means that X^n is the quotient space $X^{n-1} \coprod_{\varphi_\alpha} D_\alpha^n$.

(3) $X = \cup_{n\geq 0} X^n$.

According to this procedure, to attach cells we need to specify the corresponding attaching map. Actually this is not the only way to "produce" CW-complexes. Sometimes we can use a tricky way to construct an $(n+1)$-complex having as its own n-skeleton a given CW-complex. We will show this in detail in a concrete situation in Chapter 2, but before that we need some other notions and results listed below.

Theorem 1.1.8 *If K and L are CW-complexes, so is the topological product $K \times L$ provided that*

(a) one of K, L is locally compact, or

(b) both K and L have a countable number of cells.

For the proof one may see Theorem 7.3.16 of [70].

There is also a more general notion than that of a CW-complex, the one of a *cell complex.*

Definition 1.1.9 Let X be a set. A *cell structure* on X is a pair $(X, \mathbf{\Phi})$ where $\mathbf{\Phi}$ is a collection of maps of closed disks into X satisfying the following conditions.

(i) If $\Phi \in \mathbf{\Phi}$ and Φ has domain D^n, then Φ is injective on $D^n - S^{n-1}$.

(ii) The images $\{\Phi(D^n - S^{n-1}) \mid \Phi \in \mathbf{\Phi}\}$ form a partition of X, i.e. they are disjoint and have union X.

(iii) If $\Phi \in \mathbf{\Phi}$ and Φ has domain D^n, then

$$\Phi(S^{n-1}) \subset \bigcup \{\Psi(D^k - S^{k-1}) \mid \Psi \in \mathbf{\Phi} \text{ has domain } D^k \text{ and } k \leq n-1\}.$$

♦

Note that in this definition we do not have any topology specified on X. If $\Phi \in \mathbf{\Phi}$ and Φ has domain D^n, we call $\sigma^n = \Phi(D^n)$ a closed n-cell and Φ its characteristic function. As before, we denote $\Phi(S^{n-1}) = (\sigma^n)^\circ$ and by abuse of language we call $\Phi(D^n - S^{n-1})$ an *open n-cell.*

Two cell structures $(X, \mathbf{\Phi})$ and $(X, \mathbf{\Phi}')$ are *strictly equivalent* if there is a one-to-one correspondence between $\mathbf{\Phi}$ and $\mathbf{\Phi}'$ such that for a characteristic function $\Phi \in \mathbf{\Phi}$ with domain D^n there corresponds a characteristic function $\Phi' \in \mathbf{\Phi}'$ with domain D^n again, and there is a homeomorphism of pairs $h : (D^n, S^{n-1}) \longrightarrow (D^n, S^{n-1})$ such that $\Phi' = \Phi \circ h$. Strict equivalence is an equivalence relation and for some cell structure $(X, \mathbf{\Phi})$ we denote by $\mathbb{S}_{\mathbf{\Phi}}$ the set of all pairs $(\sigma^n, [\Phi])$, where $\sigma^n = \Phi(D^n)$ and $[\Phi]$ is the strict equivalence class of $\Phi \in \mathbf{\Phi}$. Obviously, if $(X, \mathbf{\Phi})$ and $(X, \mathbf{\Phi}')$ are strictly equivalent, then $\mathbb{S}_{\mathbf{\Phi}} = \mathbb{S}_{\mathbf{\Phi}'}$.

Definition 1.1.10 A *cell complex* on a set X is an equivalence class of cell structures $(X, \mathbf{\Phi})$ under the equivalence relation of strict equivalence. A cell complex on X will be denoted by a pair $(X, \mathbb{S})$ where $\mathbb{S} = \mathbb{S}_{\mathbf{\Phi}}$ for some representative cell structure $(X, \mathbf{\Phi})$. The set $\mathbb{S}$ is called the set of *closed cells* of $(X, \mathbb{S})$.

Definition 1.1.11 A *subcomplex* $(A, \mathbb{J})$ of a cell complex $(X, \mathbb{S})$ is a cell complex such that $A \subseteq X$ and $\mathbb{J} \subseteq \mathbb{S}$.

It is easy to see that every CW-complex is a cell complex and that every CW-subcomplex of a CW-complex is a subcomplex of it seen as a cell complex. We call (X, A) a *CW-pair* if X is a CW-complex and A a subcomplex of X.

Example 1.1.12 For any $n \geq 0$, the n-sphere $S^n = \{x = (x_0, x_1, ..., x_n) \mid \langle x, x \rangle = \sum_{i=0}^{n} x_i^2 = 1\}$ has a CW-complex structure (S^n, Φ), where Φ consists of two functions, $\varphi^0 : D^0 \longrightarrow S^n$ and $\varphi^n : D^n \longrightarrow S^n$. We define

$$\varphi^0(x) = (1, 0, ..., 0)$$

and

$$\varphi^n(x) = (2 \langle x, x \rangle - 1, 2x_1\sqrt{1 - \langle x, x \rangle}, ..., 2x_n\sqrt{1 - \langle x, x \rangle}).$$

In this case we have only two cells $\sigma^0 = (1, 0, ..., 0)$ and $\sigma^n = S^n$.
We can get a cell structure on the $(n+1)$-disk by taking the two characteristic maps defined above together with the identity map $\varphi^{n+1} : D^{n+1} \longrightarrow D^{n+1}$. This gives a cell complex with exactly three cells containing S^n as a subcomplex.

Definition 1.1.13, Propositions 1.1.18 and 1.1.19, and Theorem 1.1.15, which will follow below, give the outline of the procedure we use in Chapter 2 to produce CW-complexes.

Definition 1.1.13 Let $(X, \mathbb{S})$ be a cell complex and $\mathbb{R}$ an equivalence relation on X. Denote by p the quotient map. Then $\mathbb{R}$ is a *cellular equivalence* relation provided the following conditions are satisfied.

(1) If $\sigma \in \mathbb{S}$, then $p^{-1}p(\sigma - \sigma^\circ)$ is a union of open cells $\sigma_i - \sigma_i^\circ$ of the cellular partition of X.

(2) If $\sigma_0 - \sigma_0^\circ \in p^{-1}p(\sigma - \sigma^\circ)$ is of minimal dimension among all such open cells in the union, then $p \mid (\sigma_0 - \sigma_0^\circ)$ is a bijection onto $p(\sigma - \sigma^\circ)$ and $p(\sigma) = p(\sigma_0)$. Such a cell σ_0 is called $\mathbb{R}$-*minimal* for the cell σ.

(3) If σ' and σ'' are both $\mathbb{R}$-minimal for the cell σ and if Φ' and Φ'' are the respective characteristic functions, then there is a homeomorphism $h : D_{\sigma'} \longrightarrow D_{\sigma''}$ such that $p\Phi' = p\Phi'' h$.

Remark 1.1.14 We draw the attention of the reader to the difference between the notations used in [62] and those of other sources mentioned here, including the rest of this thesis. In our terminology cells are *open*, unless otherwise stated and denoted by σ, while their closure is denoted by $\overline{\sigma}$. In the terminology of [62], the closed cells $\overline{\sigma}$ are denoted simply by σ, as in the above definition, and what is an open cell for us, is denoted by $\sigma - \sigma^\circ$ in [62].

Theorem 1.1.15 *Let $(X, \mathbb{S})$ be a cell complex and $\mathbb{R}$ a cellular equivalence relation on X. Define $\mathbb{S}/\mathbb{R} = \{p(\sigma) \mid \sigma \in \mathbb{S}$ and σ is $\mathbb{R}$-minimal$\}$. Then $(X/\mathbb{R}, \mathbb{S}/\mathbb{R})$ is a cell complex.*

The proof is given in Theorem 6.2, I of [62].

Definition 1.1.16 If $(X, \mathbb{S})$ is a cell complex and $\mathbb{R}$ a cellular equivalence relation on X, the complex $(X/\mathbb{R}, \mathbb{S}/\mathbb{R})$ is the *quotient* or *identification* complex of $(X, \mathbb{S})$ with respect to $\mathbb{R}$.

We give below a few examples of cellular equivalence relations which are based on Proposition 6.8, I of [62] given below.

Proposition 1.1.17 *Let $(X, \mathbb{S})$ be a cell complex and $(A_\gamma, \mathbb{J}_\gamma)$ a family of disjoint subcomplexes. If Δ is the diagonal of $X \times X$ and $\mathbb{R} = \Delta \cup \bigcup_\gamma (A_\gamma \times A_\gamma)$, then $\mathbb{R}$ is a cellular equivalence relation.*

We say that the quotient complex $(X/\mathbb{R}, \mathbb{S}/\mathbb{R})$ is obtained from $(X, \mathbb{S})$ by *shrinking* or *collapsing the subcomplexes $(A_\gamma, \mathbb{J}_\gamma)$ to vertices of* $(X, \mathbb{S})$. In the following examples $X = (X, \mathbb{S})$ will be a cell complex and I the unit interval seen as a cell complex with one 1-cell, the open unit interval, and two 0-cell, $\{0\}$ and $\{1\}$. Recall also that the product $X \times Y$ of two cell complexes X and Y is again a cell complex with cells pairs (σ, δ) with σ and δ cells from respectively X and Y.

1. The *cone over* X, $c(X)$, is obtained from $X \times I$ by collapsing the subcomplex $X \times \{1\}$ to a vertex.

2. The *suspension* of X, $S(X)$, is obtained from $X \times I$ by collapsing the subcomplexes $X \times \{0\}$ and $X \times \{1\}$ to distinct vertices.

3. If $Y = (Y, \mathbb{J})$ is another cell complex, the *smash product* $X \wedge Y$ is obtained from $X \times Y$ by collapsing $(X \times \{y\}) \cup (\{x\} \times Y)$ to a point. Here x and y are basepoints of X and Y respectively.

Suppose that X is a space and for each $\lambda \in \Lambda$ there is given an attaching map $f_\lambda : \partial D_\lambda \longrightarrow X$, with ∂D_λ the boundary of D_λ. We let

$$\mathcal{B} = \bigcup\{D_\lambda \mid \lambda \in \Lambda\}, \ \partial\mathcal{B} = \bigcup\{\partial D_\lambda \mid \lambda \in \Lambda\}$$

and

$$F = \bigcup_\lambda f_\lambda : \partial\mathcal{B} \longrightarrow X$$

be the union map. With these notations we have the following two propositions from [62], respectively, Proposition 2.1, p. 45 and Proposition 5.7, p. 59.

Proposition 1.1.18 *Let X be a Hausdorff space, and suppose that $Y = \mathcal{B} \coprod_F X$ is obtained by attaching the cells $\{D_\lambda \mid \lambda \in \Lambda\}$ to X. Then Y is a Hausdorff space.*

Proposition 1.1.19 *Let X be a CW-complex and $\mathbb{R}$ a cellular equivalence relation such that the space $X/\mathbb{R}$ is Hausdorff. Then with the quotient structure on $X/\mathbb{R}$, the quotient space is a CW-complex.*

Remark 1.1.20 The idea of the construction of the CW-complex in Chapter 2 will be the following. Suppose we have a CW-complex Δ of finite dimension $n \geq 1$ with a countable number of cells in each dimension. The topological product $\Delta \times \Delta$ is again a CW-complex from Theorem 1.1.8, and has dimension $2n$, therefore its $(n+1)$-skeleton $\Delta^{(n+1)}$ is again a CW-complex. Suppose that $\sim$ is a cellular equivalence on $\Delta^{(n+1)}$ such that the n-skeleton of $\Delta^{(n+1)}/\sim$ is Δ and the rest of the cells are of dimension $n+1$. The last two propositions imply that $\Delta^{(n+1)}/\sim$ has a CW-structure; hence we have obtained a CW-complex $\Delta^{(n+1)}/\sim$ of dimension $n+1$, having as its own n-skeleton, the complex Δ we started with.

1.2 Homology and Homotopy Groups of a Space

1.2.1 Singular Homology

In this section we include a few basic notions of Homology Theory with a topological emphasis which can be found in books like [21] and [40].

Before we define what singular homology groups of a space X are, we give the notion of the *standard n-simplex* Δ^n for every $n \geq 0$, which by definition is

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

It is usually denoted by $[v_0, ..., v_n]$ where $v_i = (0, ..., 1, ..., 0)$ and 1 is at the i-th coordinate. To realize it geometrically, at least in low dimensions, we mention here that Δ^0 is a point, Δ^1 is a closed interval, Δ^2 is a triangle with its interior and Δ^3 is a solid tetrahedron.
A *singular n-simplex* in a space X is by definition a continuous map $\sigma : \Delta^n \longrightarrow X$. Denote by $C_n(X)$ the free abelian group with bases the set of all singular n-simplices in X. The elements of $C_n(X)$ are called n-chains. We define the boundary maps $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ by the formula:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \mid [v_0, ..., \hat{v_i}, ..., v_n]$$

where $[v_0, ..., \hat{v_i}, ..., v_n]$ is identified with $[v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$ preserving the order of vertices and $\sigma \mid [v_0, ..., \hat{v_i}, ..., v_n]$ is regarded as a singular n-simplex. For a more accurate definition of $[v_0, ..., \hat{v_i}, ..., v_n]$ the reader can see [21].

One can easily show that the boundary maps satisfy the formula $\partial_n \partial_{n+1} = 0$ and then we can define for each $n \geq 0$ the *singular homology group*

$$H_n(X) = Ker\partial_n / Im\partial_{n+1}.$$

We call $Ker\partial_n$ the group of *cycles* and $Im\partial_{n+1}$ the group of *boundaries.*

There is a nice splitting of $H_n(X)$ as the direct sum $\bigoplus_\alpha H_n(X_\alpha)$ where X_α are the path connected components of X.

There is also the notion of the *reduced homology groups* $\widetilde{H}_n(X)$ of a space X, which are by definition the homology groups of the *augmented* chain complex

$$... \rightarrow C_n(X) \xrightarrow{\partial_n} ... \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

where $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.
It is clear that $\widetilde{H}_n(X) \cong H_n(X)$ for $n > 0$ and $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Given two chain complexes in general

$$C: \ \ldots \to C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$
$$D: \ \ldots \to D_n \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_2} D_1 \xrightarrow{\delta_1} D_0 \longrightarrow 0$$

we say that $\varphi = \{\varphi_n : C_n \longrightarrow D_n \mid n \geq 0\}$ is a *chain map* from C to D, if for every $n \geq 1$ we have $\varphi_{n-1} \circ \partial_n = \delta_n \circ \varphi_n$. Such a chain map induces a group morphism $\varphi_n : H_n(C) \longrightarrow H_n(D)$ for all $n \geq 0$ (see [44]).

In particular, if X and Y are spaces and $C(X)$, $C(Y)$ their respective singular chain complexes, then every continuous map $f : X \longrightarrow Y$ induces a chain map $f_\# : C(X) \longrightarrow C(Y)$ and therefore a morphism $f_* : H_n(X) \longrightarrow H_n(Y)$ for every n.

There is an important notion of relative homology groups. Given a space X and a subspace $A \subset X$, denote by $C_n(X, A)$ the quotient group $C_n(X)/C_n(A)$. Since the boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, we have an induced quotient map $\overline{\partial} : C_n(X, A) \longrightarrow C_{n-1}(X, A)$ and as a result we have the chain complex

$$\ldots \longrightarrow C_n(X, A) \xrightarrow{\overline{\partial}} C_{n-1}(X, A) \longrightarrow \ldots$$

whose homology groups are called by definition the *relative homology groups* and denoted by $H_n(X, A)$. These groups fit into a long exact sequence

$$\begin{aligned} \ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \\ \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \ldots \longrightarrow H_0(X, A) \to 0 \end{aligned} \tag{1.1}$$

where i_* is induced by the inclusion $i : C_n(A) \hookrightarrow C_n(X)$, j_* is induced by the surjection $j : C_n(X) \longrightarrow C_n(X, A)$ and $\partial_* : H_n(X, A) \longrightarrow H_{n-1}(A)$, called the *connecting homomorphism* (see [82]), is defined as follows. Let c be some cycle from $C_n(X, A)$. Since j is onto, we find $b \in C_n(X)$ such that $j(b) = c$. The element ∂b is in $Ker j$ since $j(\partial b) = \partial j(b) = \overline{\partial} c = 0$. But $Ker j = Im i$ and therefore we find some $a \in C_{n-1}(X)$ such that $\partial b = i(a)$. We define $\partial_*(\text{cls}c)$ to be $\text{cls}a$ where $\text{cls}c$ and $\text{cls}a$ are the respective homology classes of c and a. For a detailed proof that ∂_* is indeed a well-defined group morphism, the reader may see [40], pp. 116-117.

Remark 1.2.1 The existence of (1.1) is crucial in the definition of the cellular chain complex in Section 1.3.

1.2.2 Higher Homotopy Groups

A *homotopy* from a topological space X to a topological space Y is a family of continuous maps $f_t : X \longrightarrow Y$, $t \in [0, 1]$ such that the associated map $F : X \times [0, 1] \longrightarrow Y$ given by

$F(x,t) = f_t(x)$ is continuous. One says that two maps $f_0, f_1 : X \longrightarrow Y$ are *homotopic* if there exists a homotopy f_t connecting them, and we write $f_0 \simeq f_1$.

If $A \subset X$ and f_0, $f_1 : X \longrightarrow Y$ are continuous maps such that $f_0 \mid A = f_1 \mid A$, then we write

$$f_0 \simeq f_1 \text{ rel } A$$

if there is a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ for all $x \in X$, and $F(a,t) = f_0(a) = f_1(a)$ for all $a \in A$ and $t \in [0,1]$. We say that f_0 and f_1 are *homotopic relative* to A. They are homotopic in the usual sense if $A = \emptyset$.

We say that two spaces X and Y are of the same *homotopy type* or are *homotopy equivalent*, if there exists a map $f : X \longrightarrow Y$ and a map $g : Y \longrightarrow X$ such that $f \circ g \simeq id_X$ and $g \circ f \simeq id_Y$. In such a case we say that f and g are *homotopy equivalences.* The relation of homotopy equivalence is proved to be an equivalence relation.

In what follows we denote by I^n the n-dimensional unit cube, that is the topological product of n copies of the unit interval $[0,1]$, and by ∂I^n its boundary which consists of all the points with at least one of the coordinates 1. For a space X with basepoint x_0 we define $\pi_n(X,x_0)$ to be the set of homotopy classes of continuous maps $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$, where the homotopy f_t is required to satisfy $f_t(\partial I^n) = x_0$ for all t. In the case when $n = 0$, we take I^0 to be a point and ∂I^0 to be empty, and then $\pi_0(X,x_0)$ is just the set of all path components in X.

For $n \geq 1$, a sum operation in $\pi_n(X,x_0)$ is defined by

$$(f+g)(s_1, s_2, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n), & s_1 \in [0,1/2] \\ g(2s_1 - 1, s_2, ..., s_n), & s_1 \in [1/2,1]. \end{cases}$$

It turns out that, for $n \geq 1$, $\pi_n(X,x_0)$ is a group with the operation $[f]+[g] = [f+g]$, where $[f]$ is the homotopy class of f, and for $n \geq 2$ this group is abelian (see [40]).

Every base point preserving map $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ induces a map
$\varphi^* : \pi_n(X,x_0) \longrightarrow \pi_n(Y,y_0)$ defined by $\varphi^*([f]) = [\varphi f]$. It is easy to see that φ^* is well-defined and a homomorphism for $n \geq 1$.

We mention below a number of useful results which we use later in Chapter 2.

Lemma 1.2.2 *Given a CW-pair (X,A) and a continuous map $f : A \longrightarrow Y$ with Y path-connected, then f can be extended to a map $X \longrightarrow Y$ if $\pi_{n-1}(Y) = 0$ for all n for which $X - A$ has cells of dimension n.*

For the proof of it one may see Lemma 4.7 of [40].

Recall from p. 27 of [62] that, if $(X, \mathbb{S})$ and $(Y, \mathbb{K})$ are CW-complexes, then a map $f : X \longrightarrow Y$ is said to be *cellular* provided that, for each n, $f(X^n) \subset Y^n$. If in addition it satisfies the property that, for each $\sigma \in \mathbb{S}$, $f(\sigma) = \tau \in \mathbb{K}$ and $f(\sigma - \sigma^\circ) = \tau - \tau^\circ$, then it is called *regular.*

The following is Lemma 2.3, p. 46, [62] and will be useful in Chapter 2.

Lemma 1.2.3 *Let X and Y be CW-complexes and $f : X \longrightarrow Y$ a continuous map which is regular and cellular. Then f is a homeomorphism.*

The following is Theorem 1, p. 199 of [92].

Theorem 1.2.4 *Given a CW-pair (X, A) and a continuous map $f : X \longrightarrow Y$ such that $f \mid A$ is cellular, then f is homotopic relative to A to a cellular map.*

Remark 1.2.5 Lemma 1.2.2 is called the *Extension Lemma* and, together with Theorem 1.2.4, will be crucial in the proof of Theorem 2.2.12, where we take for (X, A) the pair (D^n, S^{n-1}), which is indeed a CW-pair from our Example 1.1.12.

Theorem 1.2.6 *Let X and Y have the homotopy type of CW-complexes, and let f be a map from X to Y. The map f is a homotopy equivalence if and only if it induces isomorphisms of homotopy groups in each dimension.*

For the proof of the above one may see Theorem 3.3, IV of [62].

The following two theorems from [92], respectively, Theorem 1, p. 223 and Theorem 2, p. 225, are very useful in computing the first homotopy groups of CW-complexes.

Theorem 1.2.7 *Let K be a CW-complex, x_0 be a 0-cell and K^1 and K^2 be the 1 and 2-skeleta of K respectively. The inclusions $K^1 \subset K^2 \subset K$ induce an epimorphism $i_{1*} : \pi_1(K^1, x_0) \longrightarrow \pi_1(K^2, x_0)$ and an isomorphism $i_{2*} : \pi_1(K^2, x_0) \longrightarrow \pi_1(K, x_0)$.*

We will use the second isomorphism when we prove Theorem 2.2.12 in Chapter 2.

An *edge path* in a CW-complex is a path in its underlying 1-skeleton [92]. In the set of all closed edge paths with initial some base point x_0 one considers the following operations:

1. Allowable insertion of an edge path ee^{-1} or $e^{-1}e$, or cancellation of such a path if possible.
2. Allowable insertion of an edge path $p(\sigma)$ or $p^{-1}(\sigma)$, or deletion of such a path, if possible, where $p(\sigma)$ is the edge path which runs once round the 2-cell σ.

To every closed path like above we can now associate its corresponding equivalence class and the set of such classes $\mathcal{F}$ can be equipped with a multiplication in the same way we did in the case of homotopy groups. With this multiplication $\mathcal{F}$ forms a group.

Theorem 1.2.8 *If one associates with every closed edge path at the base point x_0 with its corresponding homotopy class, then there is defined an isomorphism $\Theta : \mathcal{F} \longrightarrow \pi_1(K, x_0)$.*

An interesting question is, how the homology and homotopy groups of a space X are related to each other. An elementary tool in the study of the relation between $\pi_n(X, x_0)$ and $H_n(X)$, is the so called *Hurewicz homomorphism* $h_n : \pi_n(X, x_0) \longrightarrow H_n(X)$ defined as follows. Recall first that $H_n(S^n) = \mathbb{Z}$ (see for example pp. 34-35 of [21]) and let $\sigma_n \in H_n(S^n)$ be the standard generator: then, if $[f] \in \pi_n(X, x_0)$ is represented by a map $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$, define $h_n[f] = f_*(\sigma_n)$. This is independent of the chosen representative map f.

For $n = 1$ there is a handy description of the Hurewicz morphism as Lemma 4.26 of [91] shows. We give it below as we will make use of it later.

Lemma 1.2.9 *Let $\eta : \Delta^1 \longrightarrow I$ be the homeomorphism $(1-t)e_0 + te_1 \longmapsto t$. The Hurewicz morphism*

$$h_1 : \pi_1(X, x_0) \longrightarrow H_1(X)$$

is given by

$$[f] \longmapsto cls f\eta$$

where $f : I \longrightarrow X$ is a closed path in X at x_0. $\bullet$

We say that a space X is *m-connected* $(m \geq 1)$ if $\pi_s(X) = 0$ for every $1 \leq s \leq m$. We state now part of the Hurewicz Theorem which we use in Chapter 2 to define the attaching mappings of critical $(n+1)$-cells.

Theorem 1.2.10 *If a space X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.*

If X is path-connected, then the Hurewicz morphism $h_1 : \pi_1(X) \longrightarrow H_1(X)$ is a surjection with kernel $[\pi_1(X), \pi_1(X)]$, the commutator subgroup of $\pi_1(X)$; hence

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X),$$

or in other words, $H_1(X)$ is the abelianization of $\pi_1(X)$. This result is originally due to Poincaré.

1.3 Cellular Homology

In the case of a CW-complex X it is possible to compute the homology groups in a rather nicer way than one does in general. We will make use of the fact that the space in this case is split into cells. The following lemma relates the cell structure of X with the singular homology of the space.

Lemma 1.3.1 *If X is a CW-complex, then:*

(a) $H_k(X^n, X^{n-1})$ is 0 for $k \neq n$ and is free abelian for $k = n$ with bases in one-to-one correspondence with n-cells of X.

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > dimX$.

(c) The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \longrightarrow H_k(X)$ if $k < n$.

See for the proof Lemma 2.3.4 of [40].

If X is a CW-complex, then parts of the long exact sequences corresponding to the pairs (X^{n+1}, X^n), (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}), fit into a diagram

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"] & H_n(X^n) \arrow[r, "i_n"] \arrow[d, "j_n"] & H_n(X^{n+1}) & \\
& H_n(X^n, X^{n-1}) \arrow[d, "\partial_n"'] \arrow[dr, "d_n"] & & \\
& H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\partial_{n-1}"'] & H_{n-2}(X_{n-2})
\end{tikzcd}$$

where $d_n = j_{n-1}\partial_n$ and $d_{n+1} = j_n\partial_{n+1}$. It follows that the composition $d_n d_{n+1} = j_{n-1}\partial_n j_n \partial_{n+1}$ equals 0 since $\partial_n j_n = 0$ in the sequence corresponding to the pair (X^n, X^{n-1}). Thus we have the chain complex of abelian groups

$$\ldots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \ldots \xrightarrow{d_1} C_0 \longrightarrow 0$$

where from Lemma 1.3.1 each $C_k = H_k(X^k, X^{k-1})$ is free abelian with bases the set of all k-cells and d_k is given by the above composition.

We call this chain, the *cellular chain complex.* The homology groups of this chain are called the *cellular homology groups.* It is proved in Theorem 2.35 of [40] that cellular homology groups

$H_n^{CW}(X)$ and singular homology groups $H_n(X)$ are isomorphic.

In applications we need to know what the cellular boundary formula is. For $n = 1$, the boundary map $d_1 : H_1(X^1, X^0) \longrightarrow H_0(X^0)$ is the same as the simplicial boundary map $\partial_1 : \Delta_1(X) \longrightarrow \Delta_0(X)$ [40]. For $n > 1$, d_n is given by the following.

Cellular Boundary Formula. $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$ where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \longrightarrow X^{n-1} \longrightarrow S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point. The summation in this formula is finite since the attaching map of the cell e_α^n has compact image and therefore it meets only finitely many cells e_β^{n-1}.

The cellular boundary formula can be derived from the following diagram

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha*}"] & \widetilde{H}_{n-1}(\partial D_\alpha^n) \arrow[rr, "\Delta_{\alpha\beta*}"] \arrow[d, "\varphi_{\alpha*}"] & & \widetilde{H}_{n-1}(S_\beta^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[rr, "q_*"] \arrow[d, "j_{n-1}"] & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta*}"'] \arrow[d, "\cong"] \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\cong"] & & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd} \tag{1.2}$$

where:

- Φ_α and φ_α are the respective characteristic and attaching maps of e_α^n.
- $q : X^{n-1} \longrightarrow X^{n-1}/X^{n-2}$ is the quotient map.
- $q_\beta : X^{n-1}/X^{n-2} \longrightarrow S_\beta^{n-1}$ collapses the complement of each cell e_β^{n-1} to a point, and the resulting quotient sphere is identified with $S_\beta^{n-1} = D_\beta^{n-1}/\partial D_\beta^{n-1}$ via the characteristic map Φ_β.
- $\Delta_{\alpha\beta} : \partial D_\alpha^n \longrightarrow S_\beta^{n-1}$ is the composition $q_\beta q \varphi_\alpha$.

To compute $d_n(e_\alpha^n)$, we choose some generator $[D_\alpha^n]$ of $\mathbb{Z} = H_n(D_\alpha^n, \partial D_\alpha^n)$, which will be referred to as the *orientation* of e_α^n, and then apply $\Phi_{\alpha*}$ which takes this element to a generator of the $\mathbb{Z}$ summand of $H_n(X^n, X^{n-1})$ corresponding to e_α^n. If we denote that generator by e_α^n, then the commutativity of the left-hand side of the diagram implies that $d_n(e_\alpha^n) = j_{n-1}\varphi_{\alpha*}\partial[D_\alpha^n]$. To see that the coefficients $d_{\alpha\beta}$ are those stated in the formula above, we use the commutativity of the other half of the diagram and the fact that $q_{\beta *}$ maps the $\mathbb{Z}$ summand of $\widetilde{H}_{n-1}(X^{n-1}, X^{n-2})$ corresponding to e_β^{n-1} to the degree of $\Delta_{\alpha\beta} : S_\alpha^{n-1} \longrightarrow X^{n-1} \longrightarrow S_\beta^{n-1}$.

In other literature, such as [62], the coefficients $d_{\alpha\beta}$ of the cellular boundary formula are called *incidence numbers* and the formula is written in the form

$$d_n \sigma_\alpha^n = \sum_{\sigma_\beta^{n-1} \in C_{n-1}} [\sigma_\alpha^n : \sigma_\beta^{n-1}] \sigma_\beta^{n-1}$$

where $[\sigma_\alpha^n : \sigma_\beta^{n-1}]$ are the incidence numbers. In particular, the above results are also obtained in [62], but using an arbitrary commutative ring R with unit $1 \neq 0$ instead of the ring of integers $\mathbb{Z}$. We will quote some further results from [62] below, but just state them for the ring $\mathbb{Z}$.

In practice it is difficult to compute the cellular boundary map coefficients, but in some cases, as shown in the following result, we can compute them provided that the attaching map of the cell satisfies a nice property. More precisely, we have from [62] the following.

Corollary 1.3.2 *If $\overline{\sigma}^n$ and $\overline{\tau}^{n-1}$ are closed cells of the CW-complex X, if $\overline{E}$ is a closed $(n-1)$-disk in ∂D^n whose interior is an open $(n-1)$-cell E, if $(\varphi_{\overline{\sigma}} \mid \partial D^n)^{-1}(\tau) = E$ and if $\varphi_{\overline{\sigma}}$ maps E homeomorphically onto τ, then $[\sigma : \tau]$ is a unit in $\mathbb{Z}$.*

Recall from [62] that, if X and Y are CW-complexes and $f : X \longrightarrow Y$ is a cellular map, then there is an induced map $f_\# : C_n(X) \longrightarrow C_n(Y)$ which is in addition a chain map (see Proposition 2.3, V of [62]).

Definition 1.3.3 Let $(X, \mathbb{S})$ and $(Y, \mathbb{K})$ be oriented CW-complexes and let $f : X \longrightarrow Y$ be a cellular map. The cells of $\mathbb{S}$ are denoted by σ_λ^n and those of $\mathbb{K}$ by τ_μ^n. If $f_\# : C_n(X) \longrightarrow C_n(Y)$ is the map induced by f, we write

$$f_\#(\sigma^n) = \sum_\mu [f : \sigma^n : \tau_\mu^n] \tau_\mu^n,$$

where $[f : \sigma^n : \tau_\mu^n] \in \mathbb{Z}$ and $[f : \sigma^n : \tau_\mu^n] = 0$ for all but finitely many μ. The integer $[f : \sigma^n : \tau_\mu^n]$ is called the *degree* with which σ^n is mapped on τ_μ^n by f.

Proposition 1.3.4 *Let $(X, \mathbb{S})$ and $(Y, \mathbb{K})$ be oriented CW-complexes and let $f : X \longrightarrow Y$ be a cellular map. The degrees satisfy the following conditions.*

(1) If τ_μ^n is not a subset of $f(\sigma^n)$, then $[f : \sigma^n : \tau_\mu^n] = 0$.

(2) If $f(\sigma^0) = \tau^0$, then $[f : \sigma^0 : \tau^0] = 1$.

(3) For $n \geq 1$ and $\sigma^n \in \mathbb{S}$ and $\tau^{n-1} \in \mathbb{K}$,

$$\sum_{\mu} [f : \sigma^n : \tau^n_\mu][\tau^n_\mu : \tau^{n-1}] = \sum_{\lambda} [\sigma^n : \sigma^{n-1}_\lambda][f : \sigma^{n-1}_\lambda : \tau^{n-1}].$$

For the proof one may see Proposition 3.12, V of [62].

Example 1.3.5 The closed 2-disk D^2 has a CW-structure with three cells: a 0-cell $\sigma^0 \in \partial D^2$, $\sigma^1 = \partial D^2 = S^1$ and $\sigma^2 = D^2 \backslash \partial D^2$. From the cellular boundary formula, the incidence numbers are: $[\sigma^1 : \sigma^0] = 0$ and $[\sigma^2 : \sigma^1] = 1$. One can now calculate easily the homology groups of D^2 by writing down the cellular chain complex for the above cell structure. We have that $H_0(D^2) = \mathbb{Z}$, $H_i(D^2) = 0$ for $i = 1, 2$ and $H_k(D^2) = 0$ for $k > 2$ from Lemma 1.3.1 (b).

Example 1.3.6 Let X be the space obtained from the 2-sphere S^2 by identifying two antipodal points. To describe the CW-structure of it, we give the 2-sphere S^2 a CW-structure with two 0-cells σ^0_1, σ^0_2; a 1-cell σ^1 with boundary cells σ^0_1 and σ^0_2; a 2-cell attached by projecting S^1 onto D^1 by $(x, y) \to x$ and then using the characteristic map of σ^1; and finally identifying σ^0_1 with σ^0_2. The incidence numbers are: $[\sigma^1 : \sigma^0] = [\sigma^2 : \sigma^1] = 0$, therefore, similar to the previous example, we have that $H_0(X) = H_1(X) = H_2(X) = \mathbb{Z}$ and $H_k(X) = 0$ for $k > 2$.

1.4 Abstract Reduction Systems

An *abstract reduction system* is a pair $(A, \to)$, where the *reduction* $\to$ is a binary relation on the set A. We write $a \to b$ instead of $(a, b) \in \to$. In what follows we denote by $\xrightarrow{+}$ the transitive closure of $\to$, by $\xrightarrow{*}$ the reflexive transitive closure of $\to$ and by $\xleftrightarrow{*}$ the equivalence relation generated by $\to$. We call $a \in A$ *reducible* if and only if there is a $b \in A$ such that $a \xrightarrow{+} b$, otherwise we call it *irreducible* or in *normal form*. We call b a *normal form of* a if and only if $a \xrightarrow{*} b$ and b is irreducible. If it happens that b is unique, then we denote b by $a\downarrow$. We call a and a' *joinable* (or *resolvable*) if and only if there is c such that $a \xrightarrow{*} c \xleftarrow{*} a'$, in which case we write $a \downarrow a'$.
A reduction $\to$ is called

- *Church-Rosser* if and only if $a \xleftrightarrow{*} b \implies a \downarrow b$.
- *Confluent* if and only if $a \xleftarrow{*} c \xrightarrow{*} b \implies a \downarrow b$.
- *Semi-Confluent* if and only if $a \leftarrow c \xrightarrow{*} b \implies a \downarrow b$.

- *Locally-Confluent* if and only if $a \leftarrow c \rightarrow b \implies a \downarrow b$.
- *Terminating* (or *Noetherian*) if and only if there is no infinite descending chain $a_0 \rightarrow a_1 \rightarrow \ldots$
- *Normalizing* if and only if every element has a normal form.
- *Convergent* if and only if it is both confluent and terminating.

The first three concepts coincide as Theorem 2.1.5, [5] given below, shows.

Theorem 1.4.1 *The following conditions are equivalent:*

1. $\rightarrow$ *has the Church-Rosser property.*
2. $\rightarrow$ *is confluent.*
3. $\rightarrow$ *is semi-confluent.*

Suppose that $(A, \rightarrow)$ is a reduction system such that $\rightarrow$ is Noetherian and let P be some property on the elements of A. The following inference rule gives what we call *well founded induction* or simply *Noetherian induction.*

$$\frac{\forall a \in A.\ (\forall b \in A.\ a \xrightarrow{+} b \implies P(b)) \implies P(a)}{\forall a \in A.\ P(a)}$$

So to prove that the property P is a property of all the elements of A we *must* show the implication $(\forall b \in A.\ a \xrightarrow{+} b \implies P(b)) \implies P(a)$ for every $a \in A$. This in particular means that we show P for irreducible elements.

It turns out that the Noetherian induction always holds on any reduction system $(A, \rightarrow)$ which is Noetherian, and conversely, if the Noetherian induction holds on $(A, \rightarrow)$, then $\rightarrow$ is Noetherian. We will use Noetherian induction in many proofs in Chapter 2.
The following is known as the Newman's Lemma. We give its proof in full as an example of Noetherian induction techniques.

Lemma 1.4.2 *A Noetherian system is confluent if it is locally confluent.*

Proof. The confluence can be written in the form of a predicate as follows

$$P(x) :\ \forall y, z.\ y \xleftarrow{*} x \xrightarrow{*} z \implies y \downarrow z.$$

We must show that $P(x)$ holds under the assumption that $P(t)$ holds for all t such that $x \xrightarrow{+} t$. If in the "fork" $y \xleftarrow{*} x \xrightarrow{*} z$ either $x = y$ or $x = z$, then the result follows. Otherwise we have $x \rightarrow y_1 \xrightarrow{*} y$ and $x \rightarrow z_1 \xrightarrow{*} z$ as shown in the following diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "l.c."] & z_1 \arrow[r, "*"] \arrow[d, dashed, "*"] \arrow[ddr, phantom, "ind"] & z \arrow[dd, dashed, "*"] \\
y_1 \arrow[r, "*"] \arrow[d, "*"] \arrow[dr, phantom, "ind"] & u \arrow[d, "*"] & \\
y \arrow[r, "*"] & v \arrow[r, "*"] & w
\end{tikzcd}$$

The confluence of the pair $y_1 \leftarrow x \rightarrow z_1$ follows from the local confluence, the confluence of $y \xleftarrow{*} y_1 \xrightarrow{*} u$ and $v \xleftarrow{*} z_1 \xrightarrow{*} z$ follow from induction hypothesis since $x \xrightarrow{+} y_1$ and $x \xrightarrow{+} z_1$. ■

An important notion is that of a complete reduction system. A reduction system $(A, \rightarrow)$ is called *complete* if and only if every element has a unique normal form. The following characterization of complete systems, due to Newman [80], is important because it translates the completeness in terms of confluence and termination.

Lemma 1.4.3 *A reduction system is complete if and only if it is Noetherian and confluent.*

This lemma is the reason why sometimes complete systems are called convergent. Combining Lemma 1.4.2 and Lemma 1.4.3, we get the following characterization.

Lemma 1.4.4 *A reduction system is complete if and only if it is Noetherian and locally confluent.*

1.4.1 An Algebraic Characterization for a Complete Reduction Systems

In this section, we give a new algebraic characterization for a Noetherian reduction systems $(A, \rightarrow)$ to be complete. First, for every reduction systems $(A, \rightarrow)$, we construct a submonoid P of the full transformation monoid $\mathcal{T}(A)$ on the set A as follows:

$$P = \{\tau \in \mathcal{T}(A) \mid \tau(u) = v \text{ only if } v \text{ is a descendant of } u \text{ or } u = v\}.$$

It is clear that, under the usual composition of transformations, P forms a submonoid of $\mathcal{T}(A)$. Before we give the announced characterization, we recall that a Noetherian reduction systems $(A, \rightarrow)$ is complete if and only if every element from A has a unique irreducible descendant.

Theorem 1.4.5 *Let $(A, \rightarrow)$ be a Noetherian reduction systems. Then, $(A, \rightarrow)$ is complete if and only if the monoid P has a zero element.*

Proof. If $\mathcal{P}$ is complete, then, for every $\omega \in A$, the respective congruence class $[\omega]$ has a unique irreducible element, say $irr([\omega])$. Let $\theta \in P$ be the element which sends every $\omega \in A$ to its corresponding $irr([\omega])$. It is easy to show that θ is the zero of P.

Conversely, suppose that P has a zero element θ. Denote by $Irr(\omega)$ the set of irreducibles which are descendants of ω, and write $Irr = \cup_{\omega \in A} Irr(\omega)$. If we think of θ as a $2 \times \infty$ matrix, then we first show that the second row of θ consists only of elements from Irr. Indeed, if there is $u \in A$ such that $\theta(u) = v$ and $v \notin Irr$, then for τ which sends v to some corresponding descendant v', we would have $\tau\theta(u) = v'$, which means that $\tau\theta \neq \theta$. Note also that in the second row we always have represented all the elements from Irr because they are not transformed under any element of P. Hence the second row of θ consists only of all the elements of Irr. Next we show that every $\omega \in A$ has a *unique* irreducible descendant. Suppose by way of contradiction that there is some $u \in A$ which has $n > 1$ distinct irreducible descendants, say $i_1, ..., i_n$. Let $K_1, ..., K_n$ be respectively $\theta^{-1}(i_1), ..., \theta^{-1}(i_n)$. Suppose that $u \in K_1$. Since i_s with $s \neq 1$ is a descendant of u too, then there will be some v such that v is a descendant of u and i_s is a descendant of v or $i_s = v$. Distinguish between two cases.

1. $v \notin K_1$. Let $\tau \in P$ be such that it sends u to v. Then $\theta\tau(u) = \theta(v) \neq i_1$ which contradicts the fact that θ is the zero.

2. $v \in K_1$. Let $\tau \in P$ be such that it sends v to i_s. Then $\theta\tau(v) = \theta(i_s) = i_s \neq i_1$ which again contradicts the fact that θ is the zero.

So it remains that u can not have more than one irreducible descendant and hence the system is complete. ■

Corollary 1.4.6 *A Noetherian reduction system $(A, \rightarrow)$ is complete if and only if the monoid P constructed as above, has cohomological dimension* 0.

Proof. This follows immediately from Theorem 1.4.5 and from [38]. ■

1.5 Monoid Presentations and Finiteness Conditions

1.5.1 Monoid Presentations Via Reduction Systems

If $\mathbf{x}$ is a non empty alphabet, then the set of all words with letters from $\mathbf{x}$ together with the empty word λ is denoted by $\mathbf{x}^*$ and forms the *free monoid* $F(\mathbf{x})$ on $\mathbf{x}$ under the concatenation of words. We denote $F(\mathbf{x})$ for simplicity by F. The unit element of F is the empty word λ.

A *string rewriting system* (also called a *monoid presentation*) is a pair $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ where $\mathbf{r}$ is a subset of $\mathbf{x}^* \times \mathbf{x}^*$ whose elements are called *rewrite rules.* The reflexive and transitive closure induced from $\mathbf{r}$ on $\mathbf{x}^*$, is called the *reduction relation* induced from $\mathbf{r}$ and denoted by $\longrightarrow^*_{\mathbf{r}}$. The congruence closure $\longleftrightarrow^*_{\mathbf{r}}$ of $\mathbf{r}$ is a congruence relation on $\mathbf{x}^*$, called the *Thue congruence*, and therefore every rewrite system defines a monoid, namely the quotient $S = \mathbf{x}^* / \longleftrightarrow^*_{\mathbf{r}}$. For every word $u \in F$, we denote by $\overline{u}$ the element of S represented by u. We say that a monoid S *admits a presentation* $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ if and only if $S \cong \mathbf{x}^* / \longleftrightarrow^*_{\mathbf{r}}$. If S admits a finite presentation, then we say that the monoid S defined by this presentation, is *finitely presented.*

Along with a rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ giving a monoid S, there is the reduction system $(\mathbf{x}^*, \to)$ with $\to := \{(u\omega v, u\omega' v) \mid (\omega, \omega') \in \mathbf{r}\}$ which obviously contains $\mathbf{r}$, and $\overset{*}{\longleftrightarrow} = \mathbf{r}^{\#}$. This connection between presentation of monoids and reduction systems allows us to use concepts like termination or local confluence to study the syntactic properties, and most importantly, homotopical and homological properties, of monoids. We say that a rewriting system is *terminating* (*locally confluent, confluent, complete*) if its underlying reduction system is such.

1.5.2 Complete Monoid Presentations

The test for local confluence seems at a first sight to be difficult, but it can be simplified to a certain degree for reduction systems arising from presentations of monoids.

Definition 1.5.1 Suppose that $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ is a presentation of a monoid and $(\mathbf{x}^*, \to)$ where

$$\to := \{(u\rho_1 v, u\rho_2 v) \mid (\rho_1, \rho_2) \in \mathbf{r}\}$$

is the *underlying reduction system.*

If $\alpha = (\rho_1, \rho_2) \in \mathbf{r}$ then we write $\rho_1 \xrightarrow{\alpha} \rho_2$. A *critical pair* is a pair $(\alpha, \beta) \in \mathbf{r} \times \mathbf{r}$ as in one of the following situations:

i) *inclusion ambiguities*: $u\omega v \xrightarrow{\alpha} \omega'$ and $\omega \xrightarrow{\beta} \omega''$, $\omega \neq \lambda$;

ii) *overlap ambiguities*: $u\omega \xrightarrow{\alpha} \omega'$ and $\omega v \xrightarrow{\beta} \omega''$, $\omega \neq \lambda$.

Lemma 1.5.2 *The reduction system $(\mathbf{x}^*, \to)$ of Definition 1.5.1 is locally confluent if and only if all the critical pairs resolve.*

Proof. The "only if" part is trivial. For the converse, suppose that we have the "fork" $\omega_1 \overset{\alpha_1}{\leftarrow} \omega \overset{\alpha_2}{\to} \omega_2$ where $(\alpha_1, \alpha_2) \in \to \times \to$. There are three possible cases.

1) $\omega = aubvc$ and $\alpha_1 = (aubvc, au_1bvc)$ with $(u, u_1) \in \mathbf{r}$, and $\alpha_2 = (aubvc, aubv_1c)$ with $(v, v_1) \in \mathbf{r}$. In this case there is always a resolution of the (α_1, α_2). We can transform au_1bvc to au_1bv_1c by applying (v, v_1) on it and similarly we transform $aubv_1c$ to au_1bv_1c by applying (u, u_1) on it. Both the applications are in $\rightarrow$.

2) $\omega = \rho aub\rho'$, $\alpha_1 = (\rho aub\rho', \rho\omega_1'\rho')$ with $\alpha_1' = (aub, \omega_1') \in \mathbf{r}$, and $\alpha_2 = (\rho aub\rho', \rho au'b\rho')$ with $\alpha_2' = (u, u') \in \mathbf{r}$. We see that the pair (α_1', α_2') is an inclusion ambiguity and, if it resolves, then one obtains the resolution of our fork by simply acting on the left and on the right of the assumed resolution by respectively ρ and ρ'.

3) $\omega = \rho aub\rho'$, $\alpha_1 = (\rho aub\rho', \rho u'b\rho')$ with $\alpha_1' = (au, u') \in \mathbf{r}$, and $\alpha_2 = (\rho aub\rho', \rho au''\rho')$ with $\alpha_2' = (ub, u'') \in \mathbf{r}$. In this case the pair (α_1', α_2') is an overlap ambiguity and, if we suppose that the fork $u'b \overset{\alpha_1'}{\leftarrow} aub \overset{\alpha_2'}{\rightarrow} au''$ resolves, we act on that resolution with ρ and ρ' respectively on the left and on the right, to obtain the resolution of our fork $\omega_1 \overset{\alpha_1}{\leftarrow} \omega \overset{\alpha_2}{\rightarrow} \omega_2$. ∎

An immediate consequence of this is

Lemma 1.5.3 *If the reduction system of Definition 1.5.1 is Noetherian, then it is complete if and only if all the critical pairs resolve.*

1.5.3 Geometrical Constructions Associated with a Monoid Presentation

Associated with every rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ presenting a monoid, say S, there is the *reduction graph* $\Gamma = \Gamma(V, E, \iota, \tau, ^{-1})$ (see [96] or [85]) with

a) $V = F$ the set of vertices;

b) $E = \{(u, (\alpha, \beta), v, \varepsilon) \mid u, v \in F, (\alpha, \beta) \in \mathbf{r} \text{ and } \varepsilon = \pm 1\}$ the set of edges;

c) the *initial* and *terminal* maps $\iota, \tau : E \longrightarrow V$ defined by

$$\iota(u, (\alpha, \beta), v, \varepsilon) = \begin{cases} u\alpha v & \text{if} \quad \varepsilon = 1 \\ u\beta v & \text{if} \quad \varepsilon = -1 \end{cases}$$

and

$$\tau(u, (\alpha, \beta), v, \varepsilon) = \begin{cases} u\beta v & \text{if} \quad \varepsilon = 1 \\ u\alpha v & \text{if} \quad \varepsilon = -1 \end{cases}$$

d) the *inverse* map $^{-1} : E \longrightarrow E$ sending $(u, (\alpha, \beta), v, \varepsilon) \longmapsto (u, (\alpha, \beta), v, -\varepsilon)$.

An edge $(u, (\alpha, \beta), v, \varepsilon)$ is called *positive* if $\varepsilon = 1$ and *negative* otherwise. We will denote the set of positive edges by $\mathbf{e}^+$.

There is a bi-action of F on Γ:

$$\xi.\omega.\eta = \xi\omega\eta$$

and

$$\xi.(u,(\alpha,\beta),v,\varepsilon).\eta = (\xi u,(\alpha,\beta),v\eta,\varepsilon)$$

which can be extended to paths of Γ in the obvious way. This graph can be extended to a 2-complex by adding 2-cells $[e,f]$, where e and f are edges, through closed paths of the form $(e.\iota(f))\circ(\tau(e).f)\circ(e.\tau(f))^{-1}\circ(\iota(e).f)^{-1}$. This means that we identify the 1-sphere S^1 with the above closed path to make the attachment of 2-disk D^2. This is a CW-complex which is called the *Squier complex* of the presentation and is denoted by $\mathcal{D}(\mathcal{P})$ or simply by $\mathcal{D}$ if no confusion arises. The complex $\mathcal{D}$ is not path-connected, but it splits as a disjoint union of the form $\mathcal{D} = \bigsqcup_{s\in S}\mathcal{D}_s$ where $\mathcal{D}_s$ is a path-connected component of $\mathcal{D}$ whose 0-skeleton consists of the words $\omega \in F$ representing s.

The bi-action of F on Γ now induces a bi-action on $\mathcal{D}$ by simply acting on the boundaries of 2-cells. Sometimes we call these actions, simply *translations*.
Actually the original definition of the Squier complex includes only cells $[e^{\varepsilon},f^{\delta}]$ with $\varepsilon=\delta=1$. The other cells which are introduced in [85] give an "oriented" version of $\mathcal{D}$. For example, $[e,f^{-1}]$ and $[e^{-1},f]$ have opposite orientations, and $[e^{-1},f^{-1}]$ have opposite orientation with $[e,f]$. Positive edges $e.\iota(f)$ and $\iota(e).f$ both acting on $\iota(e)\iota(f)$ will be called *disjoint.*

There is also a geometric interpretation of critical pairs in this dimension. We say that an edge e is *left-reduced* (respectively, *right-reduced*) if it cannot be written in the form $u.f$ (respectively, $f.u$) for some non-empty word $u\in F$ and edge f. A pair of positive edges with the same initial vertex form a critical pair if either:

1. One of the pair is both left- and right-reduced (a critical pair of inclusion type); or

2. One of the pair is left-reduced but not right-reduced, the other is right-reduced but not left-reduced, and they are not disjoint (a critical pair of overlapping type).

Then any pair of edges in $star^{+}(w)$ $(w\in F)$ $(star^{+}(w)$ denotes the set of edges starting at $w)$ are either disjoint or are a translate of a critical pair by the two-sided action of F (see [72]).

The above interpretation of critical pairs can be taken as the definition of them.
Indeed, every critical pair as defined in Definition 1.5.1, gives rise to a pair of positive edges

with the same initial $u\omega v$, of types 1 or 2 as above. The converse is obvious.

There is also a 3-dimensional CW-complex associated with any monoid presentation $[\mathbf{x}, \mathbf{r}]$ as shown in [72]. We recall briefly here how it is constructed. First we attach 2-cells $\mathbf{p}$ together with their translates $u.\mathbf{p}.v$ with $u, v \in F$, to kill off the first homotopy groups $\pi_1(\mathcal{D}_\omega)$ of the connected components $\mathcal{D}_\omega$ of $\mathcal{D}$, thus forming a 2-dimensional CW-complex $(\mathcal{D}, \mathbf{p})$. Then we add 3-cells $[f, \sigma]$ and $[\sigma, f]$ for every positive edge f and every $\sigma \in F.\mathbf{p}.F$. The boundary of $[f, \sigma]$ is made of 2-cells $\iota f.\sigma$, $\tau f.\sigma$ and $[f, e_i^{\varepsilon_i}]$ for every $e_i^{\varepsilon_i} \in \partial\sigma$. Likewise, the boundary of $[\sigma, f]$ is made of cells $\sigma.\iota f$, $\sigma.\tau f$ and $[e_i^{\varepsilon_i}, f]$ for every $e_i^{\varepsilon_i} \in \partial\sigma$. The action of F on the 3-cells is defined as follows: let $u.[f, \sigma].v = [u.f, \sigma.v]$ and $u.[\sigma, f].v = [u.\sigma, f.v]$ for all $u, v \in F$. The complex is denoted there by $\mathcal{D}(\mathcal{P})^{\mathbf{P}}$ or simply $\mathcal{D}^{\mathbf{P}}$. In the next chapter we will show another way of obtaining this and its n-dimensional analogue. If the system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ is complete then there is a simple way of attaching 2-cells to trivialize the first homotopy group of the complex. Simply take the set $\mathbf{p}$ to be made of cells with boundary $e \circ e_* \circ f_*^{-1} \circ f^{-1}$ with (e, f) a critical pair and (e_*, f_*) a resolution of that pair. (See [96] for more details.) In [72] this is done one dimension higher, resolving what we define there *critical triples*, but we will not stop here to explain the construction because this will be done for every $n \geq 3$ in the next chapter in a more general way.

1.5.4 Finiteness Conditions for Monoids

Let $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ be a finite presentation of a monoid S and as before denote by $\mathcal{D}$ the Squier complex associated with $\mathcal{P}$.

Definition 1.5.4 The presentation $\mathcal{P}$ is said to have *finite derivation type* (FDT) if, by adding to $\mathcal{D}$ a *finite* set of 2-cells X together with their translates $F.X.F$, we obtain a 2-complex $\mathcal{D}^X$ with trivial first homotopy groups.

The property FDT is proved in [96] to be independent of the presentation and therefore it is a structural property of S. Also in [96] it is shown that, if the presentation $\mathcal{P}$ is finite and complete, then S is FDT.

One can associate with the Squier complex $\mathcal{D}$ its cellular chain complex, whose chain groups turn out to have a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure, and then study the first cellular homology $H_1(\mathcal{D})$ which as is shown in [85] has an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure.

Definition 1.5.5 A finite presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ giving a monoid S, is said to have *finite homological type* (FHT) if the $H_1(\mathcal{D})$ is finitely generated as a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule.

It turns out that FHT is an invariant of the presentation. Also the property FDT implies FHT. Indeed, since every closed path f in $\mathcal{D}$ is homotopic with a closed path φ in the underlying graph Γ, Lemma 1.2.9 shows that the Hurewicz homomorphism $h_1 : \pi_1(\mathcal{D}) \longrightarrow H_1(\mathcal{D})$ sends the homotopy class $[f]$ to the homology class of φ. The fact that φ can be "filled in" using standard cells of $\mathcal{D}$ and the translates of finitely many new 2-cells $\mathbf{p}$, implies that the homology generators of $H_1(\mathcal{D})$ are the cycles corresponding to the boundaries of cells from $\mathbf{p}$.

The converse is not true in general as it is shown in [87].

In [72] there are introduced the analogues of FDT and FHT, one dimension higher.

Definition 1.5.6 We say that a finite rewriting system $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is of *second order finite derivation type* FDT_2 if:

1. is of type FDT,
2. for some finite homotopy trivializer $\mathbf{p}$ of $\mathcal{D}$, the 3-complex $(\mathcal{D}, \mathbf{p})$ has a finite set $\mathbf{X}$ of sphere tessellations such that attaching 3-cells to the set $F.\mathbf{X}.F$ gives a new 3-complex with trivial second homotopy groups.

Definition 1.5.7 We say that a finite rewriting system $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is of *second order finite homological type* FHT_2 if:

1. is of type FHT,
2. for some finite homology trivializer $\mathbf{p}$ of $\mathcal{D}$, there is a finite set of 2-cycles $\mathbf{Y}$ whose homology classes generate the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $H_2(\mathcal{D}, \mathbf{p})$.

Remark 19 of [72] implies in particular that FDT_2 implies FHT_2. On the other hand, it follows that $H_2(\mathcal{D}, \mathbf{p})$ being finitely generated (equivalent with FHT_2 from the above definition) implies the property bi-FP_4. The property bi-FP_n is introduced in [53] as a generalization of the property FHT in all dimensions.
A monoid S is called *bi-FP_n* for some $n \geq 1$ if there is a partial free finitely generated resolution of $\mathbb{Z}S$-bimodules of the $\mathbb{Z}S$-bimodule $\mathbb{Z}S$:

$$C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} \mathbb{Z}S \rightarrow 0.$$

If it is bi-FP$_n$ for every n, then it is called *bi-FP$_\infty$*.

Finally, if the system is finite and complete, then it is FDT$_2$ (see Theorem 7 of [72]).

1.6 Useful Orders

- **Multiset Orders**

A *multiset* M over a set A is a function $M : A \longrightarrow \mathbb{N} \cup \{0\}$. Intuitively, $M(x)$ is the number of copies of $x \in A$ in M. For example, if $A = \{a, b, c\}$, then a multiset over A would look like, $M = \{a, a, b, b, b, b, c\}$, which means that M as a map sends $a \to 2$, $b \to 4$ and $c \to 1$. A multiset M is finite if there are only finitely many x such that $M(x) > 0$.

Denote by $\mathcal{M}(A)$ the set of all finite multisets over A. Below are some basic operations on $\mathcal{M}(A)$.

Element : $x \in M :\Longleftrightarrow M(x) > 0$.

Inclusion : $M \subseteq N :\Longleftrightarrow \forall x \in A.\ M(x) \leq N(x)$.

Union : $(M \cup N)(x) := M(x) + N(x), \forall x \in A$.

Difference : $(M - N)(x) := M(x) \div N(x)$, where $m \div n$ is $m - n$ if $m \geq n$ and 0 otherwise.

We say that $(A, >)$ is a *strict order* if and only if $>$ is an irreflexive and transitive relation on A.

Definition 1.6.1 Given a strict order $>$ on a set A, we define the corresponding *multiset order* $>_{mul}$ on $\mathcal{M}(A)$ as follows:

$M >_{mul} N$ if and only if there exist X, $Y \in \mathcal{M}(A)$ such that

$X \subseteq M$ and

$N = (M - X) \cup Y$ and

$\forall y \in Y.\ \exists x \in X.\ x > y$

It is not difficult to show that $>_{mul}$ is strict if $>$ is so, and that $>_{mul}$ is Noetherian if $>$ is Noetherian too. The following characterization (see Lemma 2.5.6 of [5]) will be useful in the next chapter.

Lemma 1.6.2 *If $>$ is a strict order on A and $M, N \in \mathcal{M}(A)$, then*

$$M >_{mul} N \iff M \neq N \wedge \forall n \in N - M.\ \exists m \in M - N.\ m > n.$$

We mention that, in the case of a *partial* order $\geq$, the multiset order $\geq_{mul}$ can be defined as follows:

$$M \geq_{mul} N \iff (M = N) \vee (M >_{mul} N).$$

Note that any set A can be considered as a multiset by simply taking the constant map

$$[A] : A \longrightarrow \mathbb{N} \cup \{0\}$$

sending every element of A to 1. If we apply Lemma 1.6.2 for sets, it is translated as follows:

$$[M] >_{mul} [N] \iff M \neq N \wedge \forall n \in N - M.\ \exists m \in M - N.\ m > n,$$

where now $-$ is the usual difference of sets.

Multiset ordering for sets does not allow comparing between a set and its subsets. We can extend the above order to make possible that comparison, at least in the case of finite sets.

Let $>$ be a strict order on a set A and let M be a finite subset of A. We let

$$\overline{M} = \{m \in M \mid m \text{ is maximal in } M \text{ with respect to } <\}.$$

We say for two finite subsets M and N of A that $[M] \prec_{mul} [N]$ if and only if

1. either $\overline{M} \subset \overline{N}$, or
2. $\overline{M}$ and $\overline{N}$ are incomparable and $[\overline{M}] <_{mul} [\overline{N}]$.

It is obvious that $\prec_{mul}$ is again Noetherian.

- **Lexicographic Orders**

Given two strict orders $(A, >_A)$ and $(B, >_B)$, the (*left*) *lexicographic product* $>_{A\times B}$ on $A \times B$ is defined by

$$(x, y) >_{A\times B} (x', y') \;:\iff\; (x >_A x') \vee (x = x' \wedge y >_B y').$$

The lexicographic product of two strict (respectively Noetherian) orders is again a strict (respectively Noetherian) order.

One can extend the above for every $n \in \mathbb{N}$. Given strict orders $(A_i, >_{A_i})$, $i = 1, ..., n \geq 2$, the lexicographic product $>_{\Pi A_i}$, written shortly by $>$, is defined as

$$(x_1, ..., x_n) > (y_1, ..., y_n) \;:\iff\; \exists\, k \leq n.\ (\forall\, i < k.\ x_i = y_i) \wedge (x_k >_{A_k} y_k) \tag{1.3}$$

If $(A_i, >_{A_i}) = (A, >_A)$ for all $n \geq 2$, then one can define the *lexicographic order* $>^*_{lex}$ on A^* as follows. One can see A^* as the disjoint union $\cup_{n \geq 0} A^n$ where each A^n contains words of length n with letters from A. Then for every $m > n$ take any word from A^n to be less than any word from A^m, and if $u, v \in A^n$, then compare them as in (1.3). Again, $>^*_{lex}$ is strict (respectively Noetherian) if $>$ is strict (respectively Noetherian). If $>$ is a total order, then $>^*_{lex}$ is total too.

- **Reduction Orders**

Let the monoid S be given by the rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. As before, we let $\rightarrow := \{(u\omega v, u\omega' v) \mid (\omega, \omega') \in \mathbf{r}\}$.

Definition 1.6.3 A strict order $>$ in $\mathbf{x}^*$ is called a *reduction order* if

1. $u > v$ implies $\omega_1 u \omega_2 > \omega_1 v \omega_2$ for every ω_1, $\omega_2 \in \mathbf{x}^*$,

2. $>$ is Noetherian.

We have the following result:

Proposition 1.6.4 *With the above notations, $\rightarrow$ is Noetherian if and only if there exists a reduction order $>$ compatible with $\mathbf{r}$, that is, $u > v$ for all $(u, v) \in \mathbf{r}$.*

Proof. Assume that $\rightarrow$ is Noetherian. It follows that $\xrightarrow{+}$ is itself a reduction order compatible with $\mathbf{r}$ from the definition of $\rightarrow$.

Conversely, if we assume by way of contradiction that there is an infinite chain $u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_n \rightarrow \ldots$, then this would imply that $u_1 > u_2 > \ldots > u_n > \ldots$ is infinite. ∎

Definition 1.6.5 If $\rightarrow$ is Noetherian, then the partial order $\succ_{\mathbf{r}}$ on $F(\mathbf{x})$ given by writing

$$\omega_1 u \omega_2 \succ_{\mathbf{r}} \omega_1 v \omega_2$$

for each $(u, v) \in \mathbf{r}$ and ω_1, $\omega_2 \in \mathbf{x}^*$ such that composition is possible, is a reduction order, the *reduction order induced* by $\mathbf{r}$.

1.7 Completion of a Rewriting System

We will describe in this section the Knuth-Bendix completion procedure for a finite rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ giving a monoid and find a sufficient condition for a finite rewriting system in such a case to be equivalent with a finite complete one.

The Knuth-Bendix completion procedure is as follows.

Input: A finite Noetherian rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ and a reduction order $>$ on $\mathbf{x}^*$.

Output: A finite complete rewriting system $\hat{\mathcal{P}} = [\mathbf{x}, \hat{\mathbf{r}}]$ equivalent with $\mathcal{P}$, if the procedure terminates successfully; "Fail" if the procedure terminates unsuccessfully; runs forever without terminating.

begin: If there exists $(u, v) \in \mathbf{r}$ such that $u \neq v$, but incomparable, then terminate with output "Fail". Otherwise, $i := -1$ and $\mathbf{r}_0 = \{u \to v \mid (u, v) \in \mathbf{r} \cup \mathbf{r}^{-1} \wedge u > v\}$.

repeat: $i \leftarrow i + 1$;

    $\mathbf{r}_{i+1} \leftarrow \emptyset$;

    $CP \leftarrow$ the set of critical pairs of $\mathbf{r}_i$;

while $CP \neq \emptyset$ do

begin choose $\langle z_1, z_2 \rangle \in CP$; compute normal forms $\hat{z}_1$ and $\hat{z}_2$ of z_1 and z_2, respectively;

    if $\hat{z}_1 > \hat{z}_2$ then $\mathbf{r}_{i+1} :\leftarrow \mathbf{r}_{i+1} \cup \{(\hat{z}_1, \hat{z}_2)\}$;

    if $\hat{z}_2 > \hat{z}_1$ then $\mathbf{r}_{i+1} :\leftarrow \mathbf{r}_{i+1} \cup \{(\hat{z}_2, \hat{z}_1)\}$;

    $CP \leftarrow CP - \{(z_1, z_2)\}$

end;

(Comment: all the critical pairs of $\mathbf{r}_i$ have been resolved.)

if $\mathbf{r}_{i+1} \neq \emptyset$ then $\mathbf{r}_{i+1} \leftarrow \mathbf{r}_i \cup \mathbf{r}_{i+1}$

until $\mathbf{r}_{i+1} = \emptyset$;

    $\mathbf{r}^* \leftarrow \cup_{i \geq 0} \mathbf{r}_i$;

end.

Example 1.7.1 Let $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ be a rewriting system with $\mathbf{x} = \{x, x^{-1}\}$ and

$$\mathbf{r} = \{xx^{-1} = \lambda, x^{-1}x = \lambda, xx = \lambda\}.$$

The reduction order is induced from the *lex* order based on $x^{-1} \rhd x$. There are four critical pairs, two arising from the rules $xx^{-1} = \lambda$ and $x^{-1}x = \lambda$ which are both resolvable, and two others as follows

$$x^{-1} \longleftarrow x^{-1}\underline{xx} = \underline{x^{-1}x}x \longrightarrow x \text{ and } x \longleftarrow x\underline{xx^{-1}} = \underline{xx}x^{-1} \longrightarrow x^{-1}.$$

After the first loop of Knuth-Bendix, we obtain the set of rules $\mathbf{r}_1 = \mathbf{r} \cup \{x = x^{-1}\}$. This gives rise to two new critical pairs

$$\lambda \longleftarrow \underline{xx^{-1}} = x\underline{x^{-1}} \longrightarrow xx \text{ and } \lambda \longleftarrow \underline{x^{-1}x} \longrightarrow xx$$

which are resolved by the existing rule $xx = \lambda$. The output is thus, the system $\mathcal{P} = [\mathbf{x}, \mathbf{r}_1]$.

Example 1.7.2 Let $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ be a rewriting system with $\mathbf{x} = \{a, \overline{a}, b, \overline{b}\}$ and

$$\mathbf{r} = \{a\overline{a} \to \lambda, \overline{a}a \to \lambda, b\overline{b} \to \lambda, \overline{b}b \to \lambda, ba \to ab, b\overline{a} \to \overline{a}b, \overline{b}a \to a\overline{b}, \overline{b}\overline{a} \to \overline{a}\overline{b}\}.$$

(i) Let $>$ be the lexicographical order induced by the linear order $a < \overline{a} < b < \overline{b}$. Then $\mathbf{r}_0 = \mathbf{r}$, $\mathbf{r}$ is confluent and hence, $\mathbf{r}^* = \mathbf{r}$.

(ii) Let $>$ be the lexicographical order induced by the linear order $\overline{b} < a < \overline{a} < b$. The Knuth-Bendix procedure runs as follows.

$\mathbf{r}_0 = \mathbf{r}$;
$CP_0 = \{(a, \overline{b}ab),\ (\overline{a}, \overline{b}\overline{a}b)\}$
$\mathbf{r}_1 = \mathbf{r}_0 \cup \{\overline{b}ab \to a,\ \overline{b}\overline{a}b \to \overline{a}\}$;
$CP_1 = \{(\overline{b}a^2b, a^2),\ (\overline{b}\overline{a}^2b, \overline{a}^2)\}$;
$\mathbf{r}_2 = \mathbf{r}_1 \cup \{\overline{b}a^2b \to a^2,\ \overline{b}\overline{a}^2b \to \overline{a}^2\}$.

Repeating this process, we obtain $\mathbf{r}_i = \mathbf{r}_0 \cup \{\overline{b}a^jb \to a^j,\ \overline{b}\overline{a}^jb \to \overline{a}^j \mid j = 1, ..., i\}$.
Hence, $\mathbf{r}^* = \mathbf{r}_0 \cup \{\overline{b}a^jb \to a^j,\ \overline{b}\overline{a}^jb \to \overline{a}^j \mid j \geq 1\}$.
This example shows that the termination of Knuth-Bendix depends on the well-order chosen.

Suppose we are given a finite Noetherian rewriting system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. List all the possible critical pairs as follows, (e_i, f_i) with $i = 1, ..., s$. Let u_i and v_i be irreducibles such that $\tau e_i \xrightarrow{*} u_i$ and $\tau f_i \xrightarrow{*} v_i$. Denote by U the union of all irreducibles chosen as above. Suppose that U satisfies the following four properties.

1. None of $u \in U$ is a proper factor of any of the remaining,
2. none of $u \in U$ overlaps with any of the $u' \in U$,
3. none of $u \in U$ is a proper factor of some ω such that $(\omega, \omega') \in \mathbf{r}$,

4. none of $u \in U$ overlaps with any of the ω such that $(\omega, \omega') \in \mathbf{r}$.

The following gives new conditions under which a rewriting system is equivalent with a complete one.

Theorem 1.7.3 *If the system $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ is finite and Noetherian and for some choice of irreducibles U as above, we have 1, 2, 3 and 4, then the system is equivalent with a finite complete one.*

Proof. Suppose that U is included in classes $C_1,...,C_t$. Add new letters $*_1,..., *_t$ to the alphabet, one for each C_i, and edges from each $u \in U$ to $*_j$, if $u \in C_j$. The new system is equivalent with the previous one since all the above transformations are of Tietze type (see Definition 3.6.4) and is obviously Noetherian. It remains to check whether this procedure produces critical pairs. If such a pair exists, then it should have at least one of its edges a new one. They can not be both new edges, from Conditions 1 and 2. Also we can not have a critical pair with one new edge and the other an old one since this would contradict Conditions 3 and 4. So it follows that the new system is complete. ■

Systems satisfying Conditions 1-4 exist as the following shows.

Example 1.7.4 Take $\mathbf{x} = \{a, b, c, d, e, f\}$ and $\mathbf{r} = \{(abc, acb), (bcd, bdc), (acbd, e), (abdc, f)\}$. There is only one critical pair of overlapping type $acbd \longleftarrow abcd \longrightarrow abdc$ which does not resolve. The only irreducible descendent of $acbd$ is e and the only irreducible descendent of $abdc$ is f. In this case the set U of the theorem is $\{e, f\}$ and it obviously satisfies Conditions 1-4.

1.8 Gröbner Bases for Algebras

In this section we will introduce some basic notions about Gröbner bases which can be found in several sources such as [8], [55], [68], [77] or [78].

Let $\mathbf{x}$ be a finite alphabet and $\mathbf{x}^*$ the free monoid with bases $\mathbf{x}$. For a commutative ring K with unit 1, denote by $P = K \cdot \mathbf{x}^*$ the free left K-module generated by $\mathbf{x}^*$. The elements of P have the form of noncommutative polynomials

$$f = \sum_{i=1}^{n} k_i \omega_i$$

with $k_i \in K \backslash \{0\}$ and $\omega_i \in \mathbf{x}^*$.

Suppose we have defined some *lex* order $\succ$ on $\mathbf{x}^*$. We can write f above in such a way that the terms are in descending order. In particular $k_1\omega_1$ is the biggest term, which we call the *leading term* of f and denote by $\mathrm{lt}(f)$. Let $\mathrm{rt}(f) = f - \mathrm{lt}(f)$. The order $\succ$ extends to an order on P in the following way. First set $f \succ 0$ if $f \neq 0$. Now, if f and g are non zero polynomials with $\mathrm{lt}(f) = k \cdot u$ and $\mathrm{lt}(g) = s \cdot v$, then define $f \succ g$ if and only if either $u \succ v$ or $u = v$ and $\mathrm{rt}(f) \succ \mathrm{rt}(g)$. This order is of course partial and Noetherian. A *monic rewriting rule* is a pair $(u, \rho) \in \mathbf{x}^* \times P$ such that $u \succ \rho$ and will be written as $u \to \rho$. A *monic rewriting system* $\mathbf{r}$ is a set of rewriting rules in P; in fact the pair $(P, \mathbf{r}) = (P, \to)$ is a reduction system.

The rewriting process of polynomials works as follows. If the polynomial f contains the term $k \cdot x_1 u x_2$ and we want to apply the rule $u \to \rho$ on it, then replace $x_1 u x_2$ with $x_1 \rho x_2$ and in this way f transforms into $g = f - k \cdot x_1(u - \rho)x_2$. Denote by $\to_{\mathbf{r}}^*$ the reflexive transitive closure of $\to$ and by $\leftrightarrow_{\mathbf{r}}^*$ the equivalence generated by $\to$. Let $I(\mathbf{r})$ be the ideal generated by the set $\{u - \rho \mid u \to \rho \in \mathbf{r}\}$.

Proposition 1.8.1 *The relation $\longleftrightarrow_{\mathbf{r}}^*$ is equal to the congruence on P modulo $I(\mathbf{r})$, that is,*

$$f \leftrightarrow_{\mathbf{r}}^* g \Longleftrightarrow f \equiv g(mod\ I(\mathbf{r}))$$

for f, $g \in P$. In particular,

$$f \leftrightarrow_{\mathbf{r}}^* 0 \Longleftrightarrow f \equiv 0(mod\ I(\mathbf{r}))$$

for $f \in P$, that is,

$$I(\mathbf{r}) = \{f \in P \mid f \leftrightarrow_{\mathbf{r}}^* 0\}.$$

The above is the reason why we say that the quotient algebra $A = P/I(\mathbf{r})$ is *defined by* the rewriting system $\mathbf{r}$ and use the notation $A = P/\leftrightarrow_{\mathbf{r}}^*$. We say that a set G of P is *monic*, if every $g \in G$ is monic, that is, the leading coefficient of g is 1. Let I be an ideal of P and $G \subset I$ a set of generators. We say that G is a *Gröbner base* of I if it is monic and the system $\mathbf{r}_G = \{\mathrm{lt}(g) \to -\mathrm{rt}(g) \mid g \in G\}$ associated with G is a complete reduction system in P. We say that an algebra A over K admits a Gröbner base if it is isomorphic to the quotient P/I of some finitely generated free algebra P over K modulo an ideal I with Gröbner base.

Example 1.8.2 Let $\mathbf{x}$ be a finite alphabet and $\mathbf{x}^{(2)}$ the subset of $\mathbf{x}^*$ consisting of words of length 2. Let U be a subset of $\mathbf{x}^{(2)}$ and $\phi : U \longrightarrow K \cdot \mathbf{x} \oplus K$ be a map. Let

$$\phi' : K \cdot \mathbf{x}^{(2)} \oplus K \cdot \mathbf{x} \longrightarrow K \cdot \mathbf{x}^{(2)} \oplus K \cdot \mathbf{x} \oplus K$$

be a K-linear map defined by

$$\phi'(u) = \begin{cases} \phi(u) & \text{if} \quad u \in U \\ u & \text{if} \quad u \notin U \end{cases}$$

for $u \in \lambda \cup \mathbf{x} \cup \mathbf{x}^{(2)}$. Suppose that

$$\phi'(\phi(ab)c) = \phi'(a\phi(bc)) \tag{1.4}$$

holds for every $a, b, c \in \mathbf{x}$ such that $ab, bc \in U$. The ideal I of $P = K \cdot \mathbf{x}^*$ generated by $G = \{u - \phi(u) \mid u \in U\}$ has G as a Gröbner base since (1.4) ensures that the rewriting system $\mathbf{r}_G = \{u \longrightarrow \phi(u) \mid u \in U\}$ is confluent.

Example 1.8.3 Let again $\mathbf{x}$ be a finite alphabet but differently from the general case, take the rewriting system $\mathbf{r}$ to be made of pairs $(r_+, r_-) \in \mathbf{x}^* \times \mathbf{x}^*$. Let S be the monoid defined by the presentation $[\mathbf{x}, \mathbf{r}]$ and $K \cdot \mathbf{x}^*$, $K \cdot S$ be the respective free left K-modules generated by $\mathbf{x}^*$ and S. The canonical epimorphism $\mathbf{x}^* \longrightarrow S$ extends to a ring epimorphism $\theta : K \cdot \mathbf{x}^* \longrightarrow K \cdot S$ whose kernel J is the abelian group generated by elements $k \cdot \xi.(r_+ - r_-).\eta$, where ξ, $\eta \in \mathbf{x}^*$, $k \in K$ and $(r_+, r_-) \in \mathbf{r}$. Therefore, as an ideal of $P = K \cdot \mathbf{x}^*$, J is generated by the set $G = \{r_+ - r_- \mid (r_+, r_-) \in \mathbf{r}\}$ or, in other words, $J = I(\mathbf{r})$. This in particular means that the algebra $A = P/I(\mathbf{r})$ defined by the rewriting system $\mathbf{r}$, in our case is $K \cdot \mathbf{x}^*/J = K \cdot S$. If we take $\mathbf{r}$ to be complete, then, since $\mathbf{r}_G = \{\mathrm{lt}(g) \longrightarrow -\ \mathrm{rt}(g) \mid g \in G\}$ is in fact $\mathbf{r}$, we have that G is a Gröbner base for J.

1.9 Homological Finiteness Conditions for Monoids

As before, K will denote a commutative ring with unity 1 and S a monoid. We let KS be the monoid ring over K. One defines the standard augmentation

$$\varepsilon : KS \longrightarrow K; \quad s \mapsto 1 \quad (s \in S),$$

which allows us to regard K as a left KS-module ${}_S K$ where the KS-action is given via ε:

$$a.k = \varepsilon(a)k \quad (a \in KS,\, k \in K).$$

We say that S is of type *left-FP_n* (over K) if there is a partial resolution

$$P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow_S K \longrightarrow 0 \tag{1.5}$$

where P_k is a *free finitely generated* left KS-module for all $k = 0, \ldots, n$.

Symmetrically, one defines the property *right-FP_n* by first regarding K as a right KS-module K_S. These two properties are different from each other as shown in [17] where there is given an example of a monoid which is right-FP_n over $\mathbb{Z}$ for every n but not even left-FP_1 over $\mathbb{Z}$. In contrast with monoids in general, for groups the properties left-FP_n and right-FP_n coincide. Interestingly, for any $n \in \mathbb{N}$ there are groups which are left-FP_n over $\mathbb{Z}$ but fail to be left-FP_{n+1} over $\mathbb{Z}$ as shown in [10].

Regarding K as a (KS, KS)-bimodule ${}_SK_S$ via the 2-sided action

$$a.k.a' = \varepsilon(a).k.\varepsilon(a') \quad (a, a' \in KS,\ k \in K),$$

we can define the finiteness condition *weak bi-FP_n* [84] (see also [2]). It is shown in [2] that the implication

$$\text{left-FP}_n \ + \ \text{right-FP}_n \Longrightarrow \ \text{weak bi-FP}_n$$

holds in the case when K is a *PID*. The converse is also true in general as shown in [84].

One can regard KS as a (KS, KS)-bimodule where the action is the multiplication itself. In [53] it is defined the finiteness condition bi-FP_n for a monoid S if there exists a partial bi-resolution

$$F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow KS \longrightarrow 0 \tag{1.6}$$

where $F_0, F_1, \dots, F_n$ are free finitely generated (KS, KS)-bimodules.

In [53] the authors show that

$$\text{bi-FP}_n \Longrightarrow \text{left-FP}_n \ + \ \text{right-FP}_n$$

by tensoring on the right (respectively left) hand side (1.6) with ${}_SK$ (respectively K_S). We will use a similar technique in Chapter 3 of this thesis to show that bi-FP_n for small categories implies left-FP_n and right-FP_n.

In the case of groups properties bi-FP_n and FP_n coincide as shown in [84]. This is not the case for monoids in general. Kobayashi and Otto have given in [54] an example of a monoid which is left-FP_n and right-FP_n for every n but is not bi-FP_3.

Note that in both resolution (1.5) and (1.6) we may take the respective modules to be projective rather than free and the respective (seemingly new) property is equivalent with the original one. See for this Proposition 4.3 of [12] or its bi-module version of [53].

Lastly, one can define the property bi-FP_∞ for a monoid S by requiring the existence of a bi-resolution

$$\dots \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow KS \longrightarrow 0 \tag{1.7}$$

where $F_0, F_1, ..., F_n, ...$ are finitely generated projective (KS, KS)-bimodules. It turns out that

$$\text{bi-FP}_\infty \Longleftrightarrow \text{bi-FP}_n \text{ for every } n.$$

Also its analogue holds true for one sided versions of bi-FP_∞.

Chapter 2

Topological and Homological Aspects of Rewriting Systems

2.1 Introduction

Given a monoid presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$, one can associate with the Squier complex $\mathcal{D}(\mathcal{P}) = \mathcal{D}$ the respective cellular chain complex, whose chain groups turn out to have a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure where F is the free monoid on $\mathbf{x}$, and then study the first cellular homology $H_1(\mathcal{D})$ which as is shown in [85] has a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure induced by its $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure, where S is the monoid given by $\mathcal{P}$. There is an important short exact sequence

$$0 \to H_1(\mathcal{D}) \xrightarrow{\eta} \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathcal{M}(\mathcal{P}) \to 0, \tag{2.1}$$

where $\mathcal{M}(\mathcal{P})$ is the relation bimodule of $\mathcal{P}$ introduced by Ivanov [45]. The sequence (2.1) was obtained by Pride [85], apart from the injectivity of η which was proved by Guba and Sapir [39] using ideas of diagram groups, and in an alternative way in [52]. In [71] and [72] there is constructed a 3-dimensional CW-complex$(\mathcal{D}, \mathbf{p}_1)$ containing the Squier complex $\mathcal{D}$. There is a bi-action of F on the cells of this complex whose restriction on the 0-skeleton F coincides with the concatenation of words in F and the empty word λ acts trivially on $(\mathcal{D}, \mathbf{p}_1)$. This action makes the homology groups $H_1(\mathcal{D})$ and $H_2(\mathcal{D}, \mathbf{p}_1)$ have both a $(\mathbb{Z}F, \mathbb{Z}F)$ and an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure. Here $\mathbf{p}_1$ is a set of 2-cells whose homology classes of the corresponding 1-cycles are $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule generators of $H_1(\mathcal{D})$ and the 2-skeleton of $(\mathcal{D}, \mathbf{p}_1)$ is $\Delta_2 = \mathcal{D} \sqcup F.\mathbf{p}_1.F$. Then there are added 3-cells $[f, \sigma]$ and $[\sigma, f]$ for every positive edge f and every $\sigma \in F.\mathbf{p}_1.F$. The boundary of $[f, \sigma]$ is made of 2-cells $\iota f.\sigma$, $\tau f.\sigma$ and $[f, e_i^{\varepsilon_i}]$ for every $e_i^{\varepsilon_i} \in \partial\sigma$. Likewise, the boundary of $[\sigma, f]$ is made of cells $\sigma.\iota f$, $\sigma.\tau f$ and $[e_i^{\varepsilon_i}, f]$ for every $e_i^{\varepsilon_i} \in \partial\sigma$. The action of F on the 3-cells is defined by letting $u.[f, \sigma].v = [u.f, \sigma.v]$ and $u.[\sigma, f].v = [u.\sigma, f.v]$ for all $u, v \in F$. In this way we do not distinguish between 3-cells $[\sigma.u, f]$ and $[\sigma, u.f]$ for any $u \in F$, or between 3-cells $[f.u, \sigma]$ and $[f, u.\sigma]$ for any $u \in F$. In other words, for every $\sigma \in F.\mathbf{p}_1.F$, $f \in \mathbf{e}^+$ and $u \in F$, we identify each pair of cells from $\Delta_2 \times \Delta_2$ of the form $(\sigma.u, f)$ with $(\sigma, u.f)$, and similarly we identify $(f.u, \sigma)$ with $(f, u.\sigma)$. As we will see later, these identifications arise from a cellular equivalence $\sim_2$ on the 3-skeleton K_2 of the complex $\Delta_2 \times \Delta_2$ and that $K_2/\sim_2 = (\mathcal{D}, \mathbf{p}_1)$. Also, it turns out that cells of dimension 3 of $(\mathcal{D}, \mathbf{p}_1)$ are in a 1-1 correspondence with triples of positive edges with the same initial which are translates of non-critical triples. If the system $\mathcal{P}$ is finite and complete, there is a canonical way of constructing cells of $\mathbf{p}_1$ by firstly taking resolutions of all the critical pairs (finitely many) and then attaching the 2-cells through the corresponding closed paths in a 1-1 fashion. In this case the set $F.\mathbf{p}_1.F$ trivializes $\pi_1(\mathcal{D})$ and therefore the homology classes of cycles corresponding to elements of $\mathbf{p}_1$ will be $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule generators of $H_1(\mathcal{D})$. So, in the complete case, cells of $(\mathcal{D}, \mathbf{p}_1)$ are in a 1-1 correspondence with the translates of critical k-tuples ($k = 1, 2$), or in a 1-1 correspondence with k-tuples ($k = 2, 3$) of edges with the same initial which are not critical. Again, if $\mathcal{P}$ is finite and complete, we introduced in [72] a canonical way of attaching 3-cells $\mathbf{p}_2$ to $(\mathcal{D}, \mathbf{p}_1)$ in a 1-1 correspondence with critical triples, together with their translates $F.\mathbf{p}_2.F$, and showed that the new complex $\Delta_3 = (\mathcal{D}, \mathbf{p}_1) \sqcup F.\mathbf{p}_2.F$ has trivial second homology group. In other words, 2-cycles arising from $\mathbf{p}_2$ are $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule generators of $H_2(\mathcal{D}, \mathbf{p}_1)$. Since $\mathbf{p}_2$ is finite, then $H_2(\mathcal{D}, \mathbf{p}_1)$ is a *finitely generated* $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule.

Most importantly, it is shown that, for every system $\mathcal{P}$ and every set of 2-cells $\mathbf{p}_1$ whose homology classes of the corresponding 1-cycles are $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule generators of $H_1(\mathcal{D})$, there is a short exact sequence

$$0 \to H_2(\mathcal{D}, \mathbf{p}_1) \longrightarrow \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \longrightarrow H_1(\mathcal{D}) \to 0$$

which, if spliced with (2.1) and then with the exact sequence

$$0 \to \mathcal{M}(\mathcal{P}) \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S \otimes \mathbb{Z}S \longrightarrow \mathbb{Z}S \to 0 \tag{2.2}$$

found in [53], gives the exact sequence

$$0 \to H_2(\mathcal{D}, \mathbf{p}_1) \longrightarrow \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S \otimes \mathbb{Z}S \longrightarrow \mathbb{Z}S \to 0. \tag{2.3}$$

In the case when $\mathcal{P}$ is finite and complete, then $H_2(\mathcal{D}, \mathbf{p}_1)$ is a finitely generated $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule and then using (2.3) it is easy to deduce that S satisfies the property bi-FP_4.

In the present, focusing on finite complete presentations $\mathcal{P}$, we keep on doing the above process in all dimensions. Our main result will be the following theorem whose proof covers the whole of Section 2.2.

Theorem 2.1.1 *Associated with a finite and complete presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$ giving a monoid S, and for every $n \geq 2$, there is a chain of CW-complexes*

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \Delta_3 \subset \ldots \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) \subset \Delta_n,$$

such that Δ_n has dimension n and, for every $2 \leq m \leq n$, the m-skeleton of Δ_n is Δ_m. The complex satisfies properties $\mathbf{A}_n$-$\mathbf{F}_n$ together with properties (i)-(v).

We give below properties $\mathbf{A}_n$-$\mathbf{F}_n$ and leave for Section 2.2.1 properties (**i**)-(**v**).

$\mathbf{A}_n$ There is a bi-action of F on the cells of Δ_n with λ acting trivially and such that the restriction on the 0-cells coincides with the multiplication of F. We call this action *translation.*

$\mathbf{B}_n$ For every $2 \leq m \leq n$, the open m-cells of Δ_n are in a 1-1 correspondence with the m-tuples of positive edges of $\mathcal{D}$ with the same initial. For the critical m-tuples, the corresponding open m-cells are denoted by $\mathbf{p}_{m-1}$ (note that the index is one less than the dimension), and the set of their open translates by $F.\mathbf{p}_{m-1}.F$. The following holds true.

$$\Delta_m = \begin{cases} (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-2}) \sqcup F.\mathbf{p}_{m-1}.F & \text{if} \quad m \geq 3 \\ \mathcal{D} \sqcup F.\mathbf{p}_1.F & \text{if} \quad m = 2, \end{cases}$$

where $\sqcup$ stands for the disjoint union.

$\mathbf{C}_n$ For every $2 \leq m \leq n-1$, there exists a cellular equivalence $\sim_m$ on $K_m = (\Delta_m \times \Delta_m)^{(m+1)}$ such that $K_m / \sim_m = (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-1})$.

$\mathbf{D}_n$ For every $2 \leq m \leq n-1$, $H_m(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-1})$ has a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure and an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure.

$\mathbf{E}_n$ For every $2 \leq m \leq n-1$, $H_m(\Delta_{m+1}) = 0$.

$\mathbf{F}_n$ For every $2 \leq m \leq n-1$, the following is an exact sequence of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$0 \to H_m(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{m-1}.\mathbb{Z}S \xrightarrow{\nu} H_{m-1}(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-2}) \to 0, \qquad (2.4)$$

where $(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{m-2}) = \mathcal{D}$ if $m = 2$.

The first immediate advantage of (2.4) is that we can now see the $(\mathbb{Z}S,\mathbb{Z}S)$-bimodule $H_{m-1}(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{m-2})$ as the cokernel of the map Φ. The second advantage is that we give a shorter proof of Corollary 7.2 of [55] for the integral monoid ring $\mathbb{Z}S$. This is given in the following theorem.

Theorem 2.1.2 *If a monoid S is given by some finite complete presentation $\mathcal{P}=\mathcal{P}[\mathbf{x},\mathbf{r}]$, then it is of type bi-$FP_{n+1}$. In particular, the free finite partial resolution of $\mathbb{Z}S$ can be chosen to be S-graded.*

Proof. Using the property $\mathbf{F}_n$, one obtains the exactness of the following

$$0 \to H_{n-1}(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-2}) \longrightarrow \mathbb{Z}S.\mathbf{p}_{n-2}.\mathbb{Z}S \longrightarrow ... \longrightarrow \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \longrightarrow H_1(\mathcal{D}) \to 0,$$

and then, if we splice it with (2.1) and then with (2.2), we obtain the exact sequence

$$\begin{gathered} 0 \to H_{n-1}(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-2}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{n-2}.\mathbb{Z}S \longrightarrow ... \longrightarrow \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \longrightarrow \\ \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S\otimes\mathbb{Z}S \longrightarrow \mathbb{Z}S \to 0. \end{gathered}$$

But now, $H_{n-1}(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-2})$ is a finitely generated bimodule, as Proposition 2.2.23 shows, and then there is a finitely generated free $(\mathbb{Z}S,\mathbb{Z}S)$-bimodule P_{n-1} with bases $\mathbf{p}_{n-1}$ and a surjective bimodule morphism $\delta : P_{n-1} \longrightarrow H_{n-1}(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-2})$. As a consequence, the following

$$\begin{gathered} P_{n-1} \xrightarrow{\Phi\circ\delta} \mathbb{Z}S.\mathbf{p}_{n-2}.\mathbb{Z}S \longrightarrow ... \longrightarrow \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \\ \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \longrightarrow \mathbb{Z}S\otimes\mathbb{Z}S \longrightarrow \mathbb{Z}S \longrightarrow 0 \end{gathered} \tag{2.5}$$

is exact, which shows that S is bi-FP_{n+1}.

That the above resolution is S-graded, will be made clear in Definition 2.2.33. ∎

Throughout Section 2.2, $\mathcal{P}=[\mathbf{x},\mathbf{r}]$ will be a *finite complete* presentation.
We will assume in addition that $\mathcal{P}$ is *uniquely terminating*, that is, it is finite and complete and if (r,s_1), $(r,s_2)\in\mathbf{r}$, then $s_1=s_2$.
By a result of Squier [95], any finite complete presentation $\mathcal{P}$ is equivalent to a uniquely terminating one.

In Section 2.3 we ask the question whether we can always construct the sequence of CW-complexes of Theorem 2.1.1 and prove that for every $2\le m\le n-1$, the sequence (2.4) remains exact. In such a case we would be able to define properties FDT_n and FHT_n for $n\ge 3$ generalizing the results of [72].

In Section 2.4 we give a topological proof of the known fact that FHT and FDT for groups coincide. This proof is based on our techniques for constructing complexes by taking products and quotients.

2.2 Proof of Theorem 2.1.1

Recall from the above that we have assumed that we have constructed the chain of CW-complexes

$$\mathcal{D} \subset \Delta_2 \subset (\mathcal{D}, \mathbf{p}_1) \subset \Delta_3 \subset \ldots \subset \Delta_{n-1} \subset (\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-2}) \subset \Delta_n,$$

with properties $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{E}_n$, $\mathbf{F}_n$. We will construct inductively an $(n+1)$-dimensional CW-complex Δ_{n+1}, having Δ_n as its n-skeleton, whose open $(n+1)$-cells are of two kinds: those which are in a 1-1 correspondence with the non-critical $(n+1)$-tuples of positive edges with the same initial, and open $(n+1)$-cells $\mathbf{p}_n$ in a 1-1 correspondence with critical $(n+1)$-tuples of positive edges with the same initial, together with their open translates $F.\mathbf{p}_n.F$. The construction will be carried out in two stages. In the first stage we construct an $(n+1)$-complex $(\mathcal{D}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$ whose open $(n+1)$-cells are in a 1-1 correspondence with the non-critical $(n+1)$-tuples of positive edges with the same initial. In this stage we do not specify the attaching maps in order to attach the $(n+1)$-cells, but we obtain the complex as the quotient by a cellular equivalence $\sim_n$ on the complex $K_n = (\Delta_n \times \Delta_n)^{(n+1)}$ whose attaching maps are easily calculated in terms of the respective maps of Δ_n. In the second stage we attach open $(n+1)$-cells $\mathbf{p}_n$ in a 1-1 correspondence with critical $(n+1)$-tuples of positive edges with the same initial, together with their open translates $F.\mathbf{p}_n.F$. Note that the constructing procedure we introduce in the first stage works for all the presentations whether they are complete or not, but there is not a canonical way of constructing $(n+1)$-cells from $F.\mathbf{p}_n.F$ unless we assume that the presentation is finite and complete.

Recall from [72] (see also Section 1.5.3 of this thesis) that the 1-skeleton Γ of $(\mathcal{D}, \mathbf{p}_1)$ is oriented and the set of positive edges is denoted by $\mathbf{e}^+$. We have defined in [72] the following Noetherian strict (irreflexive and transitive) orders on the 0- and 1-skeleton, as follows.

(0) We say that $u \prec_0 v$, $(u,\, v \in F)$ if and only if $v \longrightarrow_{\mathbf{r}}^{+} u$.

(1) For $e = (u, r, +1, v)$ and $f = (u', r', +1, v')$ from $\mathbf{e}^+$, $e \prec_1 f$ if $\iota e \prec_0 \iota f$ or if they have the same initial and one of the following occurs:

(1.1) v' is a proper suffix of v or, (1.2) $v = v'$, $|r_{+1}| < |r'_{+1}|$.

The assumption that the system is uniquely terminating guarantees that it never happens that, for edges e and f as above, we have

$$v = v', \ r_{+1} = r'_{+1} \text{ and } r_{-1} \neq r'_{-1}.$$

This in particular means that we can always compare between two edges with the same initial. Note also that the above strict ordering is Noetherian.

Note that order of 0-cells and 1-cells will be the bases to define a strict Noetherian order in each skeleton of the complex Δ_n.

We call two positive edges $e = (u, r, +1, v)$ and $f = (u', r', +1, v')$ with the same initial $ur_{+1}v = u'r'_{+1}v'$ *disjoint*, if and only if the occurrences of the words r_{+1} and r'_{+1} in $ur_{+1}v = u'r'_{+1}v'$, neither overlap with each other, nor is one of them a subword of the other.

2.2.1 Properties of the Complex Δ_n

To construct $\Delta_{n+1} = (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \sqcup F.\mathbf{p}_n.F$, we suppose by induction that the complex Δ_n satisfies the following five properties which are needed to carry out the construction of Δ_{n+1}. Of course all these properties hold true for $\Delta_3 = (\mathcal{D}, \mathbf{p}_1) \sqcup F.\mathbf{p}_2.F$ of [72], and then, after we construct Δ_{n+1}, we have to show that they hold true for Δ_{n+1} as well.

(i) For every pair of cells (σ_1, σ_2) such that $\dim \sigma_1 + \dim \sigma_2 = m \leq n$, Δ_n contains m-cells $\sigma_1 \otimes \sigma_2$ such that the $(m-1)$-boundary $\Phi_{\sigma_1 \otimes \sigma_2}(S^{m-1}_{\sigma_1 \otimes \sigma_2})$ is equal to the union of the closed cells of the form $\overline{\sigma_{1i} \otimes \sigma_2}$ and those $\overline{\sigma_1 \otimes \sigma_{2j}}$, for all $\overline{\sigma_{1i}} \in \partial\sigma_1$ and all $\overline{\sigma_{2j}} \in \partial\sigma_2$.

For every $\sigma_1, \sigma_2, \sigma_3 \in \Delta_n$ such that the sum of their dimensions is at most n, we have $\sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3$.

If any of the cells σ_1 or σ_2 are from F, then $\sigma_1 \otimes \sigma_2 = \sigma_1.\sigma_2$, where . is the action of F on Δ_n.

(ii) For every pair of words $u, v \in F$ and every cell $\sigma \in \Delta_n$ there is a homeomorphism

$$h_{(u,\sigma,v)} : \overline{\sigma} \longrightarrow \overline{u.\sigma.v}$$

such that for every $\sigma' \in \partial\sigma$,

$$h_{(u,\sigma,v)} \mid_{\overline{\sigma'}} = h_{(u,\sigma',v)}.$$

Notation 2.2.1 For each $1 \leq m \leq n$, we identify D^m with the m-cube $\underbrace{I \times \ldots \times I}_{m}$, and denote it by $[I_1, ..., I_m]$. Denote by $[I_1, ..., \overset{\wedge}{I_i}, ..., I_m]$ the boundary cell of D^m obtained by replacing the i-th factor I_i with $\{1\}$. In other words, the i-th coordinate of the elements of $[I_1, ..., \overset{\wedge}{I_i}, ..., I_m]$ will be 1. We let $A_0 = (0, 1, ..., 1)$, $A_1 = (1, 0, 1, ..., 1)$,..., $A_{m-1} = (1, ..., 1, 0)$, $A_m = (1, ..., 1)$. Also we make the notation

$$d_i = A_m A_{i-1} = \{(1, ..., 1, x, 1, ...1) \mid x \in I_i\} \text{ for every } i = 1, ..., m.$$

The following reveals some similarities of the cells of Δ_n with simplicies in a simplicial complex.

(iii) Every cell σ of dimension m with $2 \leq m \leq n$ the following hold true:

1. The characteristic map Φ of σ sends A_m to a 0-cell ω_σ which is the biggest vertex of the 0-skeleton of σ, and there are positive edges $e_1, ..., e_m$ from the 1-skeleton of σ coming out of ω_σ, (this set of edges will be later referred to as *starσ*) such that $\Phi(d_1) = \overline{e_1}, ..., \Phi(d_m) = \overline{e_m}$.

2. These edges determine σ in a unique way. We say that they *generate* σ and write $\sigma = [\omega; (e_1, e_2, ..., e_m)]$.

3. For every $m-1$ of these edges $e_{i_1}, ..., e_{i_{m-1}}$ there is a boundary cell of σ generated by them meeting ω_σ, and conversely every boundary cell meeting ω_σ is generated by such $m-1$ edges of $e_1, ..., e_m$. We denote by $[\omega; (e_1, ..., \overset{\wedge}{e_i}, ..., e_m)]$, with $i = 1, ..., m$, the boundary cell of σ generated by the edges $\{e_1, e_2, ..., e_m\} \backslash \{e_i\}$.

4. The restriction of Φ on $[I_1, ..., \overset{\wedge}{I_i}, ..., I_m]$ agrees with the characteristic map of the cell generated by $\{e_1, ..., e_m\} \backslash \{e_i\}$. The restriction of Φ on the union of the boundary cells of $[I_1, ..., I_m]$ that do not meet A_m is a union ζ of closed $(m-1)$-cells whose maximal boundary cells are less than ω_σ.

Definition 2.2.2 An m-tuple of positive edges e_i, $i = 1, ...m$ coming out of the same vertex ω will be called *critical* if the following hold:

(1) there are no k edges $(0 < k < m)$ disjoint from the remaining $m - k$;

(2) ω cannot be written in the form $\omega = u\omega' v$ $(u, v \neq \lambda)$ such that there are positive edges e'_i, $i = 1, ...m$ coming out of ω' and for each i, $e_i = u \otimes e'_i \otimes v$.

We call a cell $[\omega; (e_1, e_2, ..., e_m)]$ *critical* if $(e_1, e_2, ..., e_m)$ is a critical m-tuple.

(iv) If σ is critical then $\sigma = \sigma_1 \otimes \sigma_2$ implies that either $\sigma_1 = \lambda$ or $\sigma_2 = \lambda$.

Critical cells turns out to be the building blocks of the complex.

(v) **(Unique Factorization Property)** Every cell σ from Δ_n of dimension at least 1, is expressed uniquely as

$$\sigma = u_1 \otimes \sigma_1 \otimes u_2 \otimes ... \otimes u_k \otimes \sigma_k \otimes u_{k+1},$$

with $u_i \in F$ and σ_i critical cells of dimensions at least 1.

If $u_1 = \lambda$ in the above decomposition, then we call σ *left-reduced*, and if $u_{k+1} = \lambda$, then we call it *right-reduced.*

2.2.2 The Construction of the Complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$

Now we start constructing the $(n+1)$-CW complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ we mentioned before. Property **(iii)** implies that Δ_n is countable, therefore we have that the topological product $\Delta_n \times \Delta_n$ is again a CW-complex with countably many cells (see Theorem 7.3.16 of [70]). The cells of $\Delta_n \times \Delta_n$ are pairs (σ, σ') with σ and σ' cells from Δ_n.

Let K_n be the $(n+1)$-skeleton of $\Delta_n \times \Delta_n$. Construct a mapping $j : E(K_n^{(i)}) \longrightarrow E(\Delta_n)$, where $0 \leq i \leq n$ and E stands for the cell decomposition of the complex, such that

$$j(\sigma, \sigma') = \sigma \otimes \sigma', \ (\sigma, \sigma') \in E(K_n^{(i)}).$$

Observe that $\dim c = \dim j(c)$, for every $c \in E(K_n^{(i)})$, $0 \leq i \leq n$.

We say that two open cells c_1 and c_2 from K_n are *parallel*, denoted by $c_1 \parallel c_2$, if either they are both of the same dimension which is at most n and $j(c_1) = j(c_2)$, or else if c_1, c_2 are both of dimension $n+1$ and there are $\sigma_1, \sigma_2, \sigma_3$ such that $c_1 = (\sigma_1 \otimes \sigma_2, \sigma_3)$ and $c_2 = (\sigma_1, \sigma_2 \otimes \sigma_3)$. So in general we have that two open cells c_1 and c_2 from K_n of dimension $0 \leq m \leq n+1$ are parallel if and only if

$$c_1 = (\sigma_1 \otimes \sigma_2, \sigma_3) \text{ and } c_2 = (\sigma_1, \sigma_2 \otimes \sigma_3) \text{ for some } \sigma_1, \sigma_2, \sigma_3 \in \Delta_n.$$

This is due to the Unique Factorization Property (v) and from the definition of j. We identify as before the closed ball D^m for $m > 1$ with $I^m = \underbrace{I \times ... \times I}_{m}$, where I is the unit segment, and for $m = 0$ we take $D^0 = I^0$ to be a singleton set; then we can write

$$D^m_{c_1} = (I^{\dim \sigma_1} \times I^{\dim \sigma_2}) \times I^{\dim \sigma_3} \text{ and } D^m_{c_2} = I^{\dim \sigma_1} \times (I^{\dim \sigma_2} \times I^{\dim \sigma_3})$$

and therefore there is an obvious homeomorphism $h_{12} : D^m_{c_1} \longrightarrow D^m_{c_2}$.
Define the following binary relation in K_n.

$x_1 \sim_n x_2$ if and only if $x_1 \in c_1$, $x_2 \in c_2$, with c_1 and c_2 open cells such that

1) $c_1 \parallel c_2$,

2) $h_{12}(\Phi^{-1}_{c_1}(x_1)) = \Phi^{-1}_{c_2}(x_2)$, where Φ_{c_i} $(i = 1, 2)$ is the corresponding characteristic map and h_{12} is the above homeomorphism.

The relation $\sim_n$ is an equivalence relation as may easily be checked.

Definition 2.2.3 Denote by $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ the quotient space $K_n / \sim_n$ and let

$$p : K_n \longrightarrow (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$$

be the quotient map. For every cell $c \in E(K_n)$, define $p(c) = \{p(x) \mid x \in c\}$.

Lemma 2.2.4 *For every two open cells c_1 and c_2 of K_n, $p(c_1) = p(c_2)$ if and only if $c_1 \parallel c_2$.*

Proof. The direct implication follows from the definition of $\sim_n$.
Conversely, let $c_1 \parallel c_2$ and $D^i_{c_1}$, $D^i_{c_2}$ be the balls corresponding to $\overline{c_1}$ and $\overline{c_2}$ respectively. If we take $x_1 \in c_1$, then there is $\alpha \in D^i_{c_1} - S^{i-1}_{c_1}$ such that $\Phi_{c_1}(\alpha) = x_1$. Then $x_2 = \Phi_{c_2}(h_{12}(\alpha)) \in c_2$. From the definition of $\sim_n$ we have that $x_1 \sim_n x_2$, or in other words $p(x_1) = p(x_2) \in p(c_2)$. This shows that $p(c_1) \subseteq p(c_2)$. By symmetry one shows that $p(c_2) \subseteq p(c_1)$, obtaining the equality.
■

This lemma shows that parallel open cells are identified under the map p.

2.2.3 $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ is a CW-complex

We proceed by showing that $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ is a CW-complex. First we show that $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ is a cell complex by using Definition 1.1.13 and Theorem 1.1.15, and then using Propositions 1.1.18 and 1.1.19, we derive that $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ is a CW-complex.

We are now going to show that all the Conditions (1), (2) and (3) of Definition 1.1.13 are satisfied for the equivalence relation $\sim_n$.

(1) Indeed, for every open cell $c \in E(K_n)$, Lemma 2.2.4 implies that $p^{-1}p(c) = \cup c'$ where $c' \parallel c$.

(2) From above, every two open cells $c_1, c_2 \in p^{-1}p(c)$ have the same dimension since $c_1 \parallel c_2$. Also, from the definition of $\sim_n$, the restriction $p\mid_{c_1}$ is a bijection onto $p(c_2)$. It remains to show that $p(\overline{c_1}) = p(\overline{c_2})$, where again by $\overline{c}$ we denote the closure of the cell c. For this we make some preparatory work. First we make a remark.

Remark 2.2.5 If $(\sigma_1, \sigma_2) \in E(K_n)$, then the characteristic function $\Phi_{(\sigma_1,\sigma_2)}$ is just $\Phi_{\sigma_1} \times \Phi_{\sigma_2}$. If $\dim \sigma = k$ and $\dim \sigma' = l$, then $(\Phi_{\sigma_1} \times \Phi_{\sigma_2})(S^{k+l-1}_{(\sigma_1,\sigma_2)}) = (\Phi_{\sigma_1} \times \Phi_{\sigma_2})(S^{k-1}_{\sigma_1} \times D^l_{\sigma_2} \cup D^k_{\sigma_1} \times S^{l-1}_{\sigma_2}) = \Phi_{\sigma_1}(S^{k-1}_{\sigma_1}) \times \overline{\sigma_2} \cup \overline{\sigma_1} \times \Phi_{\sigma_2}(S^{l-1}_{\sigma_2})$ and, since the restriction of the characteristic map on the boundary of the ball acts as the attaching map, we derive from Property (**i**) that the boundary of the cell (σ_1, σ_2) consists of all the cells $(\overline{\sigma_{1i}}, \overline{\sigma_2})$, with $\overline{\sigma_{1i}}$ from the $(k-1)$-boundary of σ_1 and those of the form $(\overline{\sigma_1}, \overline{\sigma_{2j}})$, with $\overline{\sigma_{2j}}$ from the $(l-1)$-boundary of σ_2. In the case, when either σ_1 or σ_2 is a zero cell, we have, respectively, that the boundary of (σ_1, σ_2) is either $\overline{\sigma_1} \times \Phi_{\sigma_2}(S^{l-1}_{\sigma_2})$ or $\Phi_{\sigma_1}(S^{k-1}_{\sigma_1}) \times \overline{\sigma_2}$.

We prove the following.

Lemma 2.2.6 *For every two open cells σ and δ of Δ_n, we have $\overline{\sigma \times \delta} = \overline{\sigma} \times \overline{\delta}$.*

Proof. We have $\overline{\sigma} \times \overline{\delta} = (\sigma \cup \sigma^\circ) \times (\delta \cup \delta^\circ) = \sigma \times \delta \cup \sigma \times \delta^\circ \cup \sigma^\circ \times \delta \cup \sigma^\circ \times \delta^\circ$. On the other hand since $(\sigma \times \delta)^\circ = (\Phi_\sigma \times \Phi_\delta)(S^{\dim\sigma+\dim\delta-1}_{(\sigma,\delta)}) = \overline{\sigma} \times \delta^\circ \cup \sigma^\circ \times \overline{\delta}$, then $\overline{\sigma \times \delta} = \sigma \times \delta \cup \overline{\sigma} \times \delta^\circ \cup \sigma^\circ \times \overline{\delta} = \sigma \times \delta \cup \sigma \times \delta^\circ \cup \sigma^\circ \times \delta \cup \sigma^\circ \times \delta^\circ$ and then we obtain $\overline{\sigma \times \delta} = \overline{\sigma} \times \overline{\delta}$. ■

Remark 2.2.7 Remark 2.2.5 and the proof of Lemma 2.2.6 imply that the boundary c° of any k-dimensional cell $c \in E(K_n)$ is a finite union of closed $(k-1)$-cells as below:

$$c^\circ = \Phi_c(S^{k-1}_c) = \bigcup_{i=1}^{m} \overline{c_i}.$$

Lemma 2.2.8 *If the open cells c and d from $K_n^{(i)}$ are such that $c \parallel d$, then*

$$\{j(\mu) \mid \overline{\mu} \in \Phi_c(S^{i-1}_c)\} = \{j(\delta) \mid \overline{\delta} \in \Phi_d(S^{i-1}_d)\}.$$

Proof. The following cases are possible.

(a) *Both c and d are of dimension $n+1$.* Then from the definition of $\|$ we must have that $c = (\sigma_1 \otimes \sigma_2, \sigma_3)$ and $d = (\sigma_1, \sigma_2 \otimes \sigma_3)$. From Remark 2.2.5 and from Property (**i**), we see that the cells μ of the boundary of c are of one of the following forms, $(\sigma_1' \otimes \sigma_2, \sigma_3)$, $(\sigma_1 \otimes \sigma_2', \sigma_3)$ or $(\sigma_1 \otimes \sigma_2, \sigma_3')$ with σ_1', σ_2' and σ_3' from the respective boundaries of $\sigma_1, \sigma_2, \sigma_3$. Similarly for d they are $(\sigma_1', \sigma_2 \otimes \sigma_3)$, $(\sigma_1, \sigma_2' \otimes \sigma_3)$ or $(\sigma_1, \sigma_2 \otimes \sigma_3')$. The equality of the sets now follows from the definition of j.

(b) *Both c and d have dimension at most n where $c = (\sigma, \sigma')$ has both coordinates of dimension at least 1 and either $d = (u, \sigma_1 \otimes \sigma_2)$ with $u.(\sigma_1 \otimes \sigma_2) = \sigma \otimes \sigma'$ and $u \in F$, or $d = (\sigma_1 \otimes \sigma_2, u)$ with $(\sigma_1 \otimes \sigma_2).u = \sigma \otimes \sigma'$ and $u \in F$.* In the first case we may assume that $u.\sigma_1 = \sigma$. This can be always achieved by taking in the unique decomposition (Property (**v**)) of $u.(\sigma_1 \otimes \sigma_2)$ that "prefix" which equals σ. On the other hand $\Phi_{(\sigma,\sigma')}(S^{\dim\sigma+\dim\sigma'-1}_{(\sigma,\sigma')})$ consists of the union of cells $\overline{\sigma_i} \times \overline{\sigma'}$, $\overline{\sigma} \times \overline{\sigma_j'}$ where σ_i and σ_j' are from the respective boundaries of σ and σ', and $\Phi_{(u,\sigma_1\otimes\sigma_2)}(S^{\dim\sigma+\dim\sigma'-1}_{(u,\sigma_1\otimes\sigma_2)})$ consists of the union of cells $\overline{u} \times \overline{(\sigma_{1i} \otimes \sigma_2)} = u \times \overline{(\sigma_{1i} \otimes \sigma_2)}$ or $\overline{u} \times \overline{(\sigma_1 \otimes \sigma_{2j})} = u \times \overline{(\sigma_1 \otimes \sigma_{2j})}$, with σ_{1i} from the boundary of σ_1, and σ_{2j} from that of σ_2. Using Lemma 2.2.6, the definition of j and Property (**i**), one can easily see that $\{j(\mu) \mid \overline{\mu} \in \Phi_{(\sigma,\sigma')}(S^{\dim\sigma+\dim\sigma'-1}_{(\sigma,\sigma')})\} = \{j(\delta) \mid \overline{\delta} \in \Phi_{(u,\sigma_1\otimes\sigma_2)}(S^{\dim\sigma+\dim\sigma'-1}_{(u,\sigma_1\otimes\sigma_2)})\}$. The second case is proved similarly.

(c) *Both c and d are of dimension at most n and each has one of the coordinates from F.* Again this is proved similarly to the previous case.

(d) *Both $c = (\sigma_1, \sigma_2)$ and $d = (\delta_1, \delta_2)$ are of dimension at most n and all $\sigma_1, \sigma_2, \delta_1, \delta_2$ are of dimension at least 1.* In this case we proceed similarly as in the previous cases. ∎

Now we are ready to show that for c_1, $c_2 \in p^{-1}p(c)$ we have $p(\overline{c_1}) = p(\overline{c_2})$. Since $c_1 \parallel c_2$, from Lemma 2.2.8 we have that there is a 1-1 correspondence between the boundary cells of c_1 and c_2 such that the corresponding boundary cells μ and η are parallel, and then, applying an inductive argument on dimension for μ and η, we have that $p(\overline{\mu}) = p(\overline{\eta})$. Finally, Remark 2.2.7 implies that

$$p(\overline{c_1}) = p(c_1) \bigcup_{\overline{\mu}\in\Phi_{c_1}(S^{i-1}_{c_1})} p(\overline{\mu}) = p(c_2) \bigcup_{\overline{\mu}\in\Phi_{c_2}(S^{i-1}_{c_2})} p(\overline{\eta}) = p(\overline{c_2}).$$

(3) Now let cells c_1 and c_2 be parallel cells, say $c_1 = (\alpha \otimes \beta, \gamma)$ and $c_2 = (\alpha, \beta \otimes \gamma)$. We

have on the one hand that

$$p\Phi_{(\alpha\otimes\beta,\gamma)}((I^{\dim\alpha}\times I^{\dim\beta})\times I^{\dim\gamma})=p(\Phi_{\alpha\otimes\beta}(I^{\dim\alpha}\times I^{\dim\beta})\times\Phi_{\gamma}(I^{\dim\gamma}))=p(\overline{\alpha\otimes\beta},\overline{\gamma}),$$

and on the other hand

$$p\Phi_{(\alpha,\beta\otimes\gamma)}h_{12}((I^{\dim\alpha}\times I^{\dim\beta})\times I^{\dim\gamma})=p\Phi_{(\alpha,\beta\otimes\gamma)}(I^{\dim\alpha}\times(I^{\dim\beta}\times I^{\dim\gamma}))=$$
$$p(\Phi_{\alpha}(I^{\dim\alpha})\times\Phi_{\beta\otimes\gamma}(I^{\dim\beta}\times I^{\dim\gamma}))=p(\overline{\alpha},\overline{\beta\otimes\gamma}).$$

But now $p(\overline{\alpha\otimes\beta},\overline{\gamma})=\overline{p(\alpha\otimes\beta,\gamma)}=\overline{p(\alpha,\beta\otimes\gamma)}=p(\overline{\alpha},\overline{\beta\otimes\gamma})$ from Lemma 2.2.6 and the second part of (2), and as a result $p\Phi_{(\alpha\otimes\beta,\gamma)}=p\Phi_{(\alpha,\beta\otimes\gamma)}h_{12}$.

Proposition 2.2.9 $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ *is an* $(n+1)$*-dimensional CW-complex whose* n*-skeleton is the complex* Δ_n.

Proof. That $\sim_n$ is a cellular equivalence is immediate from (1), (2), (3) above, and therefore by Theorem 1.1.15, $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ is a cell complex with cells $p((\sigma_1,\sigma_2))$ and characteristic maps $p(\Phi_{\sigma_1}\times\Phi_{\sigma_2})$. To see that $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ is Hausdorff, we proceed as follows.

We see first that Δ_n is embedded into $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ via the following injection map on cells

$$\iota: E(\Delta_n)\longrightarrow E((\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1}));\ \sigma\longmapsto p((\sigma,\lambda))=p((\lambda,\sigma)).$$

Secondly, let us denote by $\Upsilon=\{p((\sigma_1,\sigma_2))\mid(\sigma_1,\sigma_2)\in K_n^{(n+1)}\}$, $\mathcal{B}=\cup\{D_\tau^{n+1}\mid\tau\in\Upsilon\}$, $\partial\mathcal{B}=\cup\{\partial D_\tau^{n+1}\mid\tau\in\Upsilon\}$ and $\mathcal{F}=\cup_\tau f_\tau:\partial\mathcal{B}\longrightarrow\Delta_n$ where $f_\tau=p(\varphi_{\sigma_1}\times\varphi_{\sigma_2})$ does not depend on the choice of the representative (σ_1,σ_2) of $\tau=p((\sigma_1,\sigma_2))$ since, as we saw in Lemma 2.2.4, parallel open cells are identified under $\sim_n$. Since the closure of each $(n+1)$-cell $\tau=p(\sigma_1,\sigma_2)$ splits as $D_\tau^{n+1}\backslash\partial D_\tau^{n+1}\cup p\{(\sigma_1,\sigma_2)^\circ\}$ and ∂D_τ^{n+1} is identified with $p\{(\sigma_1,\sigma_2)^\circ\}\subset\Delta_n$ via the attaching map $f_\tau=p(\varphi_{\sigma_1}\times\varphi_{\sigma_2})$, then it follows that $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})=\Delta_n\coprod_{\mathcal{F}}\mathcal{B}$, or in other words that $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ is the adjunction space of $\mathcal{B}$ to Δ_n via the attaching map $\mathcal{F}$. From II, Proposition 2.1 of [62], since Δ_n is Hausdorff, we derive that $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ is Hausdorff too and then, from II, Proposition 5.7 of [62], we get that $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$ is a CW-complex, as required. ■

Cells of Δ_n, seen as a subcomplex of $(\mathcal{D},\mathbf{p}_1,...,\mathbf{p}_{n-1})$, are those of the form $p((\sigma_1,\sigma_2))$ for some $\sigma_1,\sigma_2\in\Delta_n$, with $\dim p((\sigma_1,\sigma_2))\leq n$. We denote $(n+1)$-cells again by $\sigma_1\otimes\sigma_2$ instead of $p((\sigma_1,\sigma_2))$.

2.2.4 Construction of the Critical $(n+1)$-cells

Before we attach critical $(n+1)$-cells and their translates to $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ to form Δ_{n+1}, we prove two properties of the complex Δ_n. First we need to order the skeleta of Δ_n and then obtain an induced order on the n-chains.

- The order of the skeleta of Δ_n.

Recall from Section 1.6 that for any set A, we have denoted by $[A]$ the corresponding multiset.

For every two cells of the same dimension $2 \le m \le n$, $\sigma = [\omega_\sigma; (e_1, e_2, ..., e_m)]$ and $\delta = [\omega_\delta; (f_1, f_2, ..., f_m)]$ we define

$$[\omega_\sigma; (e_1, e_2, ..., e_m)] \prec_m [\omega_\delta; (f_1, f_2, ..., f_m)]$$

if and only if

$$\text{either } \omega_\sigma \prec \omega_\delta \text{ or } \omega_\sigma = \omega_\delta \text{ and } [\{e_1, e_2, ..., e_m\}] \prec_{mul} [\{f_1, f_2, ..., f_m\}],$$

where $\prec_{mul}$ is the multiset ordering induced by $\prec_1$. The ordering $\prec_m$ is a Noetherian irreflexive and transitive order since $\prec_1$ is so.

From now on, we use the same symbol $\prec$ for the order of cells of Δ_n.

- The induced order on the m-chains.

We write an m-chain in the form $\xi = \sum_{i=1}^{s} n_i\sigma_i$, where $s \ge 0$, $\sigma_1, ..., \sigma_s$ are all distinct m-cells and $n_1, ..., n_s$ are non-zero integers. When $s = 0$, the sum is empty and $\xi = 0$.

We define an *elementary reduction* on an m-chain

$$\xi = \sum_{i=1}^{s} n_i\sigma_i$$

to be the replacement of a non-empty subchain $\xi' = n_{i_1}\sigma_{i_1} + ... + n_{i_q}\sigma_{i_q}$ of ξ by a chain $\varepsilon_1\delta_{i_1} + ... + \varepsilon_k\delta_{i_k}$, with $k \ge 0$ such that $[\{\delta_{i_1}, ..., \delta_{i_k}\}] \prec_{mul} [\{\sigma_{i_1}, ..., \sigma_{i_q}\}]$, where $\prec_{mul}$ is the multiset ordering induced by the order $\prec$ of the m-cells.

For every non-zero m-chain $\xi = \sum_{i=1}^{s} n_i\sigma_i$, we denote

$$\xi^{(0)} = \bigcup_{i=1}^{s} \overline{\sigma}_i^{(0)}$$

where $\overline{\sigma}_i^{(0)}$ is the 0-skeleton of $\overline{\sigma}_i$.

We define $\zeta < \xi$ if:

1. ζ is obtained by ξ by a finite positive number of elementary reductions, and

2. $[\zeta^{(0)}] \prec_{mul} [\xi^{(0)}]$, where $\prec_{mul}$ is induced by $\prec$.

Since $\prec$ is Noetherian, we have that $<$ is Noetherian too.

Lemma 2.2.10 *For every $(n-1)$-cycle ξ, the n-chain ζ such that $\partial_n(\zeta) = \xi$ which exists from $\mathbf{E}_n$, has the property that for every n-cell δ represented in ζ, the maximal 0-cell of δ is less than or equal to some 0-cell from the 0-skeleton of some $(n-1)$-cell σ represented in ξ.*

Proof. Let $\xi = \sum_j n_j \sigma_j$, an $(n-1)$-cycle, and let u be some maximal 0-cell of $\xi^{(0)}$. Suppose $\sigma_1 = [u; (e_1, e_2, ..., e_{n-1})]$ is a maximal cell of ξ. Since ξ is a cycle, there will be a cell say $\sigma_i = [u; (e'_1, e_2, ..., e_{n-1})]$ represented in ξ with $e'_1 \prec e_1$. That $e'_1 \prec e_1$ follows from the fact that $\sigma_i \prec \sigma_1$, which from the definition of $\prec$, is equivalent to $[\{e'_1, e_2, ..., e_{n-1}\}] \prec_{mul} [\{e_1, e_2, ..., e_{n-1}\}]$. It follows that there is the n-cell

$$\varsigma_1 = [u; (e'_1, e_1, e_2, ..., e_{n-1})] \in \Delta_n.$$

Let ε'_1 be the incidence number of $[u; (e_1, e_2, ..., e_{n-1})]$ in ς_1. The new cycle $\xi' = \xi - \varepsilon'_1 n_1 \partial_n \varsigma_1$ obtained from ξ by replacing $n_1\sigma_1$ by the chain

$$-n_1 \left(\sum_{i>1} \varepsilon_i [e'_1, ..., \hat{e}_i, ..., e_{n-1})] + \mu \right)$$

where $\varepsilon_i = \pm 1$, $[u; (e'_1, ..., \hat{e}_i, ..., e_{n-1})] \prec \sigma_1$ for $i > 1$, and the cells represented in μ have maximal 0-cells strictly less than u.

If after all the possible cancellations in $\xi - \varepsilon'_1 n_1 \partial_{n+1} \varsigma$ there are still cells represented there which meet u, we repeat the above process finitely many times until we obtain a cycle

$$\xi' = \xi - \partial_{n+1}(\varepsilon'_1 n_1 \varsigma_1 + m_2 \varsigma_2 + ... + m_t \varsigma_t)$$

without cells meeting u. That this process terminates in finitely many steps, follows from the fact that $\prec$ is Noetherian.

From the definition of $<$ we have that $\xi' < \xi$, therefore, by Noetherian induction, there is an n-chain ζ' such that $\xi' = \partial_n \zeta'$ and the 0-cells of the cells represented in ζ' are not bigger than those of the cells from ξ'. It follows that

$$\xi = \partial_n(\zeta' + \varepsilon'_1 n_1 \varsigma_1 + m_2 \varsigma_2 + ... + m_t \varsigma_t)$$

and then the induction hypothesis for ζ' complete the proof. ■

We make the following.

Definition 2.2.11 We call a subcomplex K of Δ_n *lower complete,* if $\sigma \in \Delta_n$ and $\delta \in K$ are such that $[\sigma^0] \prec_{mul} [\delta^0]$, then $\sigma \in K$. In other words K has the property that, if some cell $\sigma \in \Delta_n$ has its own 0-skeleton lower in the multiset ordering induced by $\prec_0$ than the 0-skeleton of some $\delta \in K$, then $\sigma \in K$.

The order of edges extends naturally to the paths of the 1-skeleton Γ of Δ_n. We say that $p_1 \prec p_2$ if and only if $[p_1] \prec_{mul} [p_2]$ where by $[p]$ we denote the multiset of edges represented in the path p and $\prec_{mul}$ is the multiset ordering induced by $\prec_1$.

Theorem 2.2.12 *If K is a path-connected n-dimensional lower complete subcomplex of Δ_n, then every continuous map $f : S^{n-1} \longrightarrow K$ extends to a continuous map $F : D^n \longrightarrow K$. Furthermore, if f is cellular, then F is homotopic relative to S^{n-1} to a cellular map.*

Proof. From the Extension Lemma (Lemma 1.2.2), it suffices to show that $\pi_{n-1}(K) = 0$. Since every $(n-1)$-cycle from K is an $(n-1)$-cycle in Δ_n, and since K is lower complete, we have from Lemma 2.2.10 that $H_{n-1}(K) = 0$. To complete the proof we need to show that $\pi_{n-1}(K) \cong H_{n-1}(K)$. For this we will make use of the Hurewicz Theorem by showing first that $\pi_1(K) = 0$. Since K is path-connected and lower complete, it contains the unique irreducible word of the corresponding class and therefore its 1-skeleton $K^{(1)}$ is path-connected. From Theorem 1.2.7 we have that $\pi_1(K) \cong \pi_1(K^{(2)})$; hence to prove our claim we need to show that $\pi_1(K^{(2)}) = 0$. Since from Theorem 1.2.8 every continuous map $f : S^1 \longrightarrow K^{(2)}$ deforms homotopically to a closed path in the underlying graph $K^{(1)}$, we need only look at closed paths in $K^{(1)}$. So let p be such a closed path and u be some maximal vertex represented in p. Since u is maximal, there are two positive edges coming out of u, say e_1 and e_2. If they are disjoint we can add to our picture a standard 2-cell σ of the Squier complex generated by e_1 and e_2 (this belongs to K due to the fact that it is lower complete) and then replace the part $e_1^{-1} \circ e_2$ of p by the rest of the boundary of σ to obtain a lower (in the multiset ordering) closed path p' and then apply Noetherian induction on it. The same argument applies if the pair (e_1, e_2) is a translate of a critical one since the set of critical 2-cells $\mathbf{p}_1$ is contained in Δ_n.

The second part of the theorem follows immediately from Theorem 1.2.4. ■

Remark 2.2.13 The idea of the above proof is a generalization of the proof that $\pi_1(\mathcal{D}^{\mathbf{p}_1}) = 0$ with $\mathcal{D}^{\mathbf{p}_1} = \mathcal{D} \sqcup F.\mathbf{p}_1.F$, found in [89].

For some path-connected subcomplex K of Δ_n, denote by $\overline{K}$ the subcomplex of Δ_n of maximal dimension which is spanned from the following set of 0-cells

$$\{\omega \in \Delta_n \mid \omega \preceq_0 \omega' \text{ for some } \omega' \in K^0\}.$$

It is easy to see that $\overline{K}$ is lower complete and we will call it in future the *lower completion* of K. It is obvious that $\overline{K}$ is path-connected and, if K is finite, since the system $\mathcal{P}$ is finite and Noetherian, so will be $\overline{K}$.

Let $(e_1, e_2, ..., e_{n+1})$ be a critical $(n+1)$-tuple of positive edges coming out of some ω. Consider the cells $[\omega; (e_1, ..., \hat{e_i}, ..., e_{n+1})]$, where $i = 1, ..., (n+1)$, whose respective boundaries are as follows

$$\partial[\omega; (e_2, ..., e_{n+1})] = \cup_{1 \le j \le n}[\omega; (e_2, ..., \hat{e_j}, ..., e_{n+1})] \cup \zeta_1,$$
$$\partial[\omega; (e_1, e_3 ..., e_{n+1})] = \cup_{1 \le j \le n}[\omega; (e_1, e_3, ..., \hat{e_j}, ..., e_{n+1})] \cup \zeta_2,$$
$$\cdots$$
$$\partial[\omega; (e_1, ..., e_{n-1}, e_{n+1})] = \cup_{1 \le j \le n}[\omega; (e_1, ..., \hat{e_j}, ..., e_{n-1}, e_{n+1})] \cup \zeta_n,$$
$$\partial_n[\omega; (e_1, ..., e_{n-1}, e_n)] = \cup_{1 \le j \le n}[\omega; (e_1, ..., \hat{e_j}, ..., e_{n-1}, e_n)] \cup \zeta_{n+1},$$

Here we have that all ζ_i, $i = 1, ..., n+1$, are made of closed cells with respective maximal 0-cells less than ω.

Make the following notations. Denote by K_η the subcomplex of $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ generated by the closed cells represented in some m-chain η and as before by $\overline{K_\eta}$ the lower completion of K_η. Recall that, for each $1 \le m \le n+1$, we have identified D^m with the m-cube $\underbrace{I \times ... \times I}_{m}$, which we denote in another form as $[I_1, ..., I_m]$, and let $A_0 = (0, 1, ..., 1)$, $A_1 = (1, 0, 1, ..., 1)$,..., $A_{m-1} = (1, ..., 1, 0)$, $A_m = (1, ..., 1)$. Also we made the notation $[I_1, ..., \hat{I_i}, ..., I_m]$ for the boundary cell of D^m obtained by replacing the i-th factor I_i with $\{1\}$. Let

$$d_i = A_m A_{i-1} = \{(1, ..., 1, x, 1, ...1) \mid x \in I_i\} \text{ for every } i = 1, ..., m.$$

In order to attach an $(n+1)$-cell to the complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$, we need to introduce the attaching map of that cell. To do this, we proceed as follows. Instead of taking S^n as the boundary of D^{n+1}, we take the boundary of the $(n+1)$-cube $[I_1, ..., I_{n+1}]$. We have that, for $i \ge 1$, the characteristic map Φ_i from the cell $[I_1, ..., \hat{I_i}, ..., I_{n+1}] \in \partial[I_1, ..., I_{n+1}]$ to $[\omega; (e_1, ..., \hat{e_i}, ..., e_{n+1})]$, has the property that

$$\Phi_i (A_{n+1}) = \omega \text{ and } \Phi_i (d_k) = \overline{e_k} \text{ for } k \ne i,$$

and for the union $\cup\sigma'$ of $\sigma' \in \partial[I_1, ..., \overset{\wedge}{I_i}, ..., I_{n+1}]$ not meeting A_{n+1} we have

$$\Phi_i(\cup\sigma') = K_{\zeta_i}. \tag{2.6}$$

The two above claims are consequences of Property **(iii)**.

Now denote by φ' the map

$$\varphi' : \cup\{\delta \in \partial[I_1, ..., I_{n+1}] \mid \delta \text{ meeting } A_{n+1}\} \longrightarrow \bigcup_{i=1}^{n+1}[\omega; (e_1, ..., \overset{\wedge}{e_i}, ..., e_{n+1})]$$

whose restriction on each $[I_1, ..., \overset{\wedge}{I_i}, ..., I_{n+1}]$ for $i = 1, ..., n+1$, coincides with Φ_i. This is well defined because of the assumption for the maps Φ_i, and is continuous from the construction. From (2.6) the restriction f of φ' on

$$S^{n-1} \backsimeq \cup_i\{\sigma' \in \partial[I_1, ..., \overset{\wedge}{I_i}, ..., I_{n+1}] \mid \sigma' \text{ not meeting } A_{n+1}\}$$

sends the latter to $K_\zeta = \bigcup_{i=1}^{n+1} K_{\zeta_i}$ and therefore is cellular. Next we show how to construct a map φ'' from $\{\delta \in \partial[I_1, ..., I_{n+1}] \mid \delta \text{ not meeting } A_{n+1}\} \backsimeq D^n$ to $\overline{K_\zeta}$ whose restriction on S^{n-1} equals f. In other words, we want to prove that f extends to a map $\varphi'' : D^n \longrightarrow \overline{K_\zeta}$. Since $\overline{K_\zeta}$ is path-connected, lower complete containing K_ζ and there is a map $f : S^{n-1} \longrightarrow K_\zeta$, we have from Theorem 2.2.12 that f extends to a cellular map φ'' from D^n to $\overline{K_\zeta}$. Then we can "glue" φ' with φ'' since they coincide on the boundary of D^n, to obtain the attaching map for our cell which we denote by $[\omega; (e_1, ..., e_{n+1})]$. Note that the finiteness of K_ζ and hence that of $\overline{K_\zeta}$, together with the fact that φ'' is cellular, imply that the "bottom" part $\varphi''(D^n)$ of the boundary of $[\omega; (e_1, ..., e_{n+1})]$, is made of finitely many closed cells whose maximal 0-cells do not exceed the 0-cells of the complex K_ζ and therefore are less than ω. On the other hand the "top" cells of the boundary of $[\omega; (e_1, ..., e_{n+1})]$ are those of the form $[\omega; (e_1, ..., \overset{\wedge}{e_i}, ..., e_{n+1})]$, all meeting ω. However, since we do not know exactly what the bottom part $\varphi''(D^n)$ of a critical cell σ is, we can not have an explicit form for the cellular boundary formula of σ.

We attach translates $u \otimes \sigma \otimes v$ of critical cells σ, in the same way as above, by letting the boundary of such cells to be made of cells $u \otimes \sigma' \otimes v$, with σ' from the boundary of σ. Explicitly, we can construct as before the top of the cell, this time to be made of cells $u.[\omega; (e_1, ..., \overset{\wedge}{e_i}, ..., e_{n+1})].v$. The analogue of the map f above, here denoted by $\widetilde{f}$, sends S^{n-1} to $u.K_\zeta.v$. We define the analogue $\widetilde{\varphi''}$ of φ'' by the following commutative diagram:

$$\begin{tikzcd} D^n \arrow[r, equal] \arrow[d, "\varphi''"'] & D^n \arrow[d, "\widetilde{\varphi''}"] \\ \overline{K_\zeta} \arrow[r, "\kappa"'] & \overline{u.K_\zeta.v} \end{tikzcd}$$

with κ the homeomorphism which sends every $\overline{\sigma} \in \varphi''(D^n)$ to $\overline{u.\sigma.v} \in \overline{u.K_\zeta.v}$. The map κ can be constructed using Property (**ii**). It follows that there is a continuous bijection $h : \overline{\sigma} \longrightarrow \overline{u.\sigma.v}$ which from Lemma 1.2.3 satisfies Property (**ii**). The set of all critical $(n+1)$-cells is denoted by $\mathbf{p}_n$ and the set of their translates by $F.\mathbf{p}_n.F$. The resulting complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \sqcup F.\mathbf{p}_n.F$ is denoted by Δ_{n+1}. The properties $\mathbf{A}_{n+1}$ and $\mathbf{C}_{n+1}$ for Δ_{n+1} obviously hold true from the construction of it. Also the second half of Property $\mathbf{B}_{n+1}$ holds true. The first half will be proved in Lemma 2.2.18.

2.2.5 Properties (i)-(v) of the Complex Δ_{n+1}

In this section we show that properties (**i**)-(**v**) hold true for the complex Δ_{n+1}. Note that for $(n+1)$-cells from $F.\mathbf{p}_n.F$, Property (**i**) is an obvious consequence of the way the action of F on critical $(n+1)$-cells is defined, while Property (**ii**) follows from the construction.

Lemma 2.2.14 *For every open cell $c \in E(K_n)$ we have $p(c^\circ) = p(c)^\circ$ and $p(\overline{c}) = \overline{p(c)}$.*

Proof. From Remark 2.2.7 we have that $c^\circ = \bigcup_{i=1}^{m} \overline{c_i}$ and then $p(c^\circ) = \bigcup_{i=1}^{m} p(\overline{c_i})$. On the other hand $p(c)^\circ = \Phi_{p(c)}(S_c^{k-1}) = p\Phi_c(S_c^{k-1}) = \bigcup_{i=1}^{m} p(\overline{c_i})$, which shows that $p(c^\circ) = p(c)^\circ$. To show the second equality, it is enough to split $\overline{c}$ as the disjoint union $c \cup c^\circ$ and then take $p(\overline{c}) = p(c \cup c^\circ) = p(c) \cup p(c^\circ) = p(c) \cup p(c)^\circ = \overline{p(c)}$. ■

We prove now properties (**i**)-(**iii**) for $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$.

(**i**) For cells of dimension $\leq n$ the statement follows by induction since we have $\Delta_n \subset (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$. If $\sigma \otimes \delta = p((\sigma, \delta))$ is a non-critical $(n+1)$-cell, from Remark 2.2.5 and Lemmas 2.2.6 and 2.2.14 we have that, $\Phi_{\sigma\otimes\delta}(S^n_{\sigma\otimes\delta}) = \Phi_{p((\sigma,\delta))}(S^n_{p((\sigma,\delta))}) = p\Phi_{(\sigma,\delta)}(S^n_{(\sigma,\delta)})$ is made of cells

$$p((\overline{\sigma'}, \overline{\delta})) = p(\overline{(\sigma', \delta)}) = \overline{p((\sigma', \delta))} = \overline{\sigma' \otimes \delta}$$

and

$$p((\overline{\sigma}, \overline{\delta'})) = p(\overline{(\sigma, \delta')}) = \overline{p((\sigma, \delta'))} = \overline{\sigma \otimes \delta'},$$

where $\overline{\sigma'}$ and $\overline{\delta'}$ are from the respective boundaries of σ and δ.

From the definition of p, for every $\sigma_1, \sigma_2, \sigma_3 \in (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ whose sum of dimensions is at most $n+1$, we have $\sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3$.

The bi-action of F on Δ_n now extends to $(n+1)$-cells $\sigma_1 \otimes \sigma_2 \in (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ as follows. For $\omega \in F$ we let

$$\omega.(\sigma_1 \otimes \sigma_2) = \omega \otimes (\sigma_1 \otimes \sigma_2) = (\omega \otimes \sigma_1) \otimes \sigma_2 = (\omega.\sigma_1) \otimes \sigma_2$$

by induction, and similarly $(\sigma_1 \otimes \sigma_2).\omega = \sigma_1 \otimes (\sigma_2.\omega)$.

Lemma 2.2.15 *For every two cells σ and δ from Δ_n such that $dim\sigma + dim\delta \leq n+1$, p maps homeomorphically $\overline{\sigma \times \delta}$ onto $\overline{\sigma \otimes \delta}$ with the property that for every $\sigma' \in \partial\sigma$ and $\delta' \in \partial\delta$, $p(\overline{\sigma' \times \delta}) \backsimeq \overline{\sigma' \otimes \delta}$ and $p(\overline{\sigma \times \delta'}) \backsimeq \overline{\sigma \otimes \delta'}$.*

Proof. The claim is obviously true if both the cells have dimension 0. Let now $\sigma \times \delta$ some cell of dimension m. From the definition of p, we have that p maps $\sigma \times \delta$ homeomorphically onto $\sigma \otimes \delta$. On the other hand, by our assumption, we have for every $\sigma' \in \partial\sigma$ and $\delta' \in \partial\delta$ the homeomorphisms $p(\overline{\sigma' \times \delta}) \backsimeq \overline{\sigma' \otimes \delta}$ and $p(\overline{\sigma \times \delta'}) \backsimeq \overline{\sigma \otimes \delta'}$. The claim follows. ∎

We continue with the proof of Property **(ii)**.

(ii) If σ is a translate of a critical $(n+1)$-cell, then the result follows from the construction of such cells. Let $\sigma \otimes \delta$ an $(n+1)$-cell. By assumption, we have that for every $u, v \in F$, there are homeomorphisms $\overline{\sigma} \backsimeq \overline{u.\sigma}$ and $\overline{\delta} \backsimeq \overline{\delta.v}$ such that $\overline{\sigma'} \backsimeq \overline{u.\sigma'}$ and $\overline{\delta'} \backsimeq \overline{\delta'.v}$ for every $\sigma' \in \partial\sigma$ and $\delta' \in \partial\delta$. It follows that there is a homeomorphism $\overline{\sigma \times \delta} \backsimeq \overline{u.\sigma \times \delta.v}$ with the property: $\overline{\sigma' \times \delta} \backsimeq \overline{u.\sigma' \times \delta.v}$ and $\overline{\sigma \times \delta'} \backsimeq \overline{u.\sigma \times \delta'.v}$ for every $\sigma' \in \partial\sigma$ and $\delta' \in \partial\delta$. Lemma 2.2.15 implies that there is a homeomorphism $\overline{\sigma \otimes \delta} \backsimeq \overline{u.\sigma \otimes \delta.v}$ such that $\overline{\sigma' \otimes \delta} \backsimeq \overline{u.\sigma' \otimes \delta.v}$ and $\overline{\sigma \otimes \delta'} \backsimeq \overline{u.\sigma \otimes \delta'.v}$ for every $\sigma' \in \partial\sigma$ and $\delta' \in \partial\delta$ proving thus the claim.

Before we prove Property **(iii)** we prove some preparatory results.

Lemma 2.2.16 *For every $(n+1)$-cell of the form $\sigma \otimes \delta$, its 0-skeleton is made of cells uv, with u and v being respectively in the zero skeleton of σ and δ.*

Proof. The claim follows easily from Property **(i)**. ∎

Lemma 2.2.17 *For every $(n+1)$-cell σ, there is ω from the 0-skeleton σ^0 of σ which is bigger than any other 0-cell of σ.*

Proof. We distinguish the following two cases.

1) $\sigma = \alpha \otimes \beta \in (\mathcal{D}, \mathrm{p}_1, \ldots, \mathrm{p}_{n-1})$. Property **(iii)** for α whose dimension is less than $(n+1)$, implies that there is ω_α from α^0 which is bigger than any other 0-cell of α. Similarly, there is ω_β from β^0 which is bigger than any other 0-cell of β. From the above lemma, $\omega_\alpha\omega_\beta$ is in σ^0 and then the compatibility of multiplication in F with $\prec$ implies that $\omega_\alpha\omega_\beta$ is the biggest 0-cell of σ.

2) If σ is a translate of a critical $(n+1)$-cell, then the construction of σ shows the result. ∎

The following lemma proves the Property (**iii**) for the complex Δ_{n+1}. Similarly as in the construction of the critical $(n+1)$-cells, we take D^{n+1} to be $[I_1, ..., I_{n+1}]$.

Lemma 2.2.18 *The following hold true for every $(n+1)$-cell σ.*

1. *The characteristic map Φ of σ sends A_{n+1} to a 0-cell ω_σ which is the biggest vertex of the 0-skeleton of σ, and there are positive edges $e_1, ..., e_{n+1}$ from the 1-skeleton of σ coming out of ω_σ, which we call starσ, such that $\Phi(d_1) = \overline{e_1}, ..., \Phi(d_{n+1}) = \overline{e_{n+1}}$.*

2. *These edges determine σ in a unique way. We say that they generate σ.*

3. *For every n of these edges $e_{i_1}, ..., e_{i_n}$ there is a boundary cell of σ generated by them meeting ω_σ, and conversely every boundary cell meeting ω_σ is generated by such n edges of $e_1, ..., e_{n+1}$.*

4. *The restriction of Φ on $[I_1, ..., \hat{I_i}, ..., I_{n+1}]$ agrees with the characteristic map of the cell generated by $\{e_1, ..., e_{n+1}\} \backslash \{e_i\}$. The restriction of Φ on the union of the boundary cells of $[I_1, ..., I_{n+1}]$ that do not meet A_{n+1} is a union ζ of closed n-cells whose maximal boundary cells are less than ω_σ.*

Proof. 1) The claims hold true for $(n+1)$-cells belonging to $F.\mathbf{p}_n.F$ from the construction of these cells.

Let $\sigma \otimes \delta$ be some $(n+1)$-cell from $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ with σ a k-cell and δ an l-cell. We can take $S^n = S^{k+l-1}$ to be the boundary of $[I_1, ..., I_{n+1}] = [I_1, ..., I_k] \times [I_1, ..., I_l]$ and then from Proposition 2.2.9 we can write the boundary of $\sigma \otimes \delta$ in the form

$$p(\Phi_\sigma(S_\sigma^{k-1}) \times \Phi_\delta(D_\delta^l)) \cup p(\Phi_\sigma(D_\sigma^k) \times \Phi_\delta(S_\delta^{l-1})).$$

Then replacing S_σ^{k-1} and S_δ^{l-1} by the boundary of the respective cubes we get the form

$$p(\Phi_\sigma(\partial([I_1, ..., I_k])) \times \Phi_\delta([I_1, ..., I_l])) \cup p(\Phi_\sigma([I_1, ..., I_k]) \times \Phi_\delta(\partial[I_1, ..., I_l])).$$

From Property (**iii**) for σ we have that Φ_σ maps the edges $d_1, ..., d_k$ of $[I_1, ..., I_k]$ to the closures of the positive edges $e'_1, ..., e'_k$ of the 1-skeleton of σ with common initial ω_σ. Similarly Φ_δ maps the edges (renumbered) $d_{k+1}, ..., d_{n+1}$ of $[I_1, ..., I_l]$ to respectively the closures of the positive edges $e'_{k+1}, ..., e'_{n+1}$ coming out of ω_δ. Therefore $p(\Phi_\sigma \times \Phi_\delta)$ will map $d_1, ..., d_k, d_{k+1}, ..., d_{n+1}$, seen as edges of $[I_1, ..., I_{n+1}]$, to respectively the closures of the positive edges

$$e'_1 \otimes \omega_\delta, ..., e'_k \otimes \omega_\delta, \omega_\sigma \otimes e'_{k+1}, ..., \omega_\sigma \otimes e'_{n+1},$$

which we denote by $e_1, ..., e_k, e_{k+1}, ..., e_{n+1}$ and are ordered in the ascending order. Lastly, from Lemma 2.2.17 we have that $\omega_\sigma \omega_\delta$ is the biggest 0-cell of $\sigma \otimes \delta$.

2) From 1 we have that

$$star(\sigma \otimes \delta) = star(\sigma) \otimes \omega_\delta \cup \omega_\sigma \otimes star(\delta).$$

Taking the unique decompositions of σ and δ, we have that

$$star(\sigma \otimes \delta) = \bigcup_{i=1}^{t} u_1 \otimes \omega_{\alpha_1} \otimes ... \otimes u_i \otimes star(\alpha_i) \otimes u_{i+1} \otimes ... \otimes u_t \otimes \omega_{\alpha_t} \otimes u_{t+1},$$

where $star(\alpha_i)$ is a critical tuple of edges for every $i = 1, ..., t$. Suppose that there is another cell γ such that $star(\gamma) = star(\sigma \otimes \delta)$ and let

$$star(\gamma) = \bigcup_{i=1}^{s} v_1 \otimes \omega_{\beta_1} \otimes ... \otimes v_i \otimes star(\beta_i) \otimes v_{i+1} \otimes ... \otimes v_t \otimes \omega_{\beta_s} \otimes v_{s+1}.$$

Note that γ can not be a translate of a critical $(n+1)$-cell since otherwise its star would be a translate of a critical $(n+1)$-tuple of edges and therefore not equal with $star(\sigma \otimes \delta)$. Since, for every $1 \leq i \leq t$, $star(\alpha_i)$ is a critical tuple of edges and so is $star(\beta_j)$ for $1 \leq j \leq s$, the equality $star(\gamma) = star(\sigma \otimes \delta)$ implies that $s = t$ and for every $1 \leq i \leq t$,

$$u_1 \otimes ... \otimes u_i \otimes star(\alpha_i) \otimes u_{i+1} \otimes ... \otimes u_{t+1} = v_1 \otimes ... \otimes v_i \otimes star(\beta_i) \otimes v_{i+1} \otimes ... \otimes v_{t+1}. \quad (2.7)$$

It remains to show that, for every $1 \leq i \leq t$, $star(\alpha_i) = star(\beta_i)$.

Since the number of edges in each side of (2.7) is less than $n+1$, we have from Property **(iii)** that

$$u_1 \otimes ... \otimes u_i \otimes \alpha_i \otimes u_{i+1} \otimes ... \otimes u_{t+1} = v_1 \otimes ... \otimes v_i \otimes \beta_i \otimes v_{i+1} \otimes ... \otimes v_{t+1}.$$

and then the unique factorization property implies that $\alpha_i = \beta_i$.

3) Suppose that the cell is $\sigma \otimes \delta$, where $\sigma = [\omega_\sigma, (e'_1, ..., e'_k)]$ and $\delta = [\omega_\delta, (e'_{k+1}, ..., e'_{n+1})]$. From 1 we have that

$$star(\sigma \otimes \delta) = \{e'_1.\omega_\delta, ..., e'_k.\omega_\delta, \omega_\sigma.e'_{k+1}, ..., \omega_\sigma.e'_{n+1}\};$$

therefore every n of those edges will contain either $e'_1.\omega_\delta, ..., e'_k.\omega_\delta$ together with $n-k$ of $\omega_\sigma.e'_{k+1}, ..., \omega_\sigma.e'_{n+1}$, or $\omega_\sigma.e'_{k+1}, ..., \omega_\sigma.e'_{n+1}$ together with $k-1$ of $e'_1.\omega_\delta, ..., e'_k.\omega_\delta$. In the first case $e'_1, ..., e'_k$ generate σ and each $n-k$ of $e'_{k+1}, ..., e'_{n+1}$ generate a boundary cell δ' of δ. Therefore we have the boundary cell $\sigma \otimes \delta'$. Similarly we prove 3 in the second case.

To prove the converse let $\sigma \otimes \delta' \in \partial(\sigma \otimes \delta)$. By Property (**iii**), δ' is generated by $n-k$ of $e'_{k+1}, ..., e'_{n+1} \in star(\delta)$, say $e'_{i,k+1}, ..., e'_{i,n}$, and as a result

$$star(\sigma \otimes \delta') = \{e'_1.\omega_\delta, ..., e'_k.\omega_\delta, \omega_\sigma.e'_{i,k+1}, ..., \omega_\sigma.e'_{i,n}\}.$$

But *star* determines the cell uniquely, hence $\sigma \otimes \delta'$ is generated by

$$\{e'_1.\omega_\delta, ..., e'_k.\omega_\delta, \omega_\sigma.e'_{i,k+1}, ..., \omega_\sigma.e'_{i,n}\}.$$

Similarly one can prove the case when $\sigma' \otimes \delta \in \partial(\sigma \otimes \delta)$.

4) Let σ be a k-cell and δ a $(n+1-k)$-cell. From Property (**iii**), the restriction of $\Phi_{\sigma\otimes\delta} = p(\Phi_\sigma \times \Phi_\delta)$ in $[I_1, ..., \hat{I_i}, ..., I_{n+1}]$ is of the form $p(\Phi_\sigma \times \Phi_{\delta'})$ if $i \geq k+1$ or $p(\Phi_{\sigma'} \times \Phi_\delta)$ if $i \leq k$. In the first case it coincides with $\Phi_{\sigma\otimes\delta'}$, and in the second case with $\Phi_{\sigma'\otimes\delta}$.

Splitting I^{n+1} as $I^{\dim\sigma} \times I^{\dim\delta}$ and applying induction, we have that

$$p(\Phi_\sigma \times \Phi_\delta)(I^{\dim\sigma-1} \times I^{\dim\delta}) = \cup\overline{\sigma' \otimes \delta}$$

where $I^{\dim\sigma-1}$ is the union of the boundary cells of $I^{\dim\sigma}$ not meeting $A_{\dim\sigma}$, and $\sigma' \in \partial\sigma$ not meeting ω_σ. Similarly,

$$p(\Phi_\sigma \times \Phi_\delta)(I^{\dim\sigma} \times I^{\dim\delta-1}) = \cup\overline{\sigma \otimes \delta'}$$

with $\delta' \in \partial\delta$ not meeting ω_δ. ∎

Lemma 2.2.19 *Let* $\sigma_1 = [\omega_1; (f_1, ..., f_s)]$ *and* $\sigma_2 = [\omega_2; (g_1, ..., g_t)]$ *be cells of dimensions at least one. Then* $\sigma_1 \otimes \sigma_2 = [\omega_1\omega_2; (f_1.\omega_2, ..., f_s.\omega_2, \omega_1.g_1, ..., \omega_1.g_t)]$.

Proof. This follows from 1 and 2 of Lemma 2.2.18. ∎

Now we prove Properties (**iv**) and (**v**).

(**iv**) If a critical cell σ is decomposed as $\sigma_1 \otimes \sigma_2$ with $\sigma_1 = [\omega_1; (f_1, ..., f_s)]$ and $\sigma_2 = [\omega_2; (g_1, ..., g_t)]$, then from Lemma 2.2.19 we have $star(\sigma) = \{f_1.\omega_2, ..., f_s.\omega_2, \omega_1.g_1, ..., \omega_1.g_t\}$, a contradiction.

(**v**) (**Unique Factorization Property**) If the $(n+1)$-cell is a translate of a critical $(n+1)$-cell, then Property (**iv**) applies. Let now $\sigma \otimes \delta$ be an $(n+1)$-cell from $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$. If

$$\sigma = u_1 \otimes \sigma_1 \otimes u_2 \otimes ... \otimes u_k \otimes \sigma_k \otimes u_{k+1} \text{ and } \delta = v_1 \otimes \delta_1 \otimes v_2 \otimes ... \otimes v_s \otimes \delta_s \otimes v_{s+1},$$

then from Property (**v**) we have

$$\sigma \otimes \delta = u_1 \otimes \sigma_1 \otimes u_2 \otimes ... \otimes u_k \otimes \sigma_k \otimes (u_{k+1}v_1) \otimes \delta_1 \otimes v_2 \otimes ... \otimes v_s \otimes \delta_s \otimes v_{s+1}$$

with all σ_i's and δ_j's critical. To show the uniqueness, suppose that $\omega_1 \otimes \kappa_1 \otimes \omega_2 \otimes ... \otimes \kappa_i \otimes \omega_i = v_1 \otimes \rho_1 \otimes v_2 \otimes ... \otimes \rho_j \otimes v_j$ are two different expressions of $\sigma \underset{\prime}{\otimes} \delta$. From the definition of $\sim$ we must have $(\omega_1 \otimes \kappa_1) \times (\omega_2 \otimes ... \otimes \kappa_i \otimes \omega_i) \parallel (v_1 \otimes \rho_1) \times (v_2 \otimes ... \otimes \rho_j \otimes v_j)$; hence we either have $\omega_1 \otimes \kappa_1 = v_1 \otimes \rho_1 \otimes \alpha$ for some cell α, or $v_1 \otimes \rho_1 = \omega_1 \otimes \kappa_1 \otimes \beta$ for some cell β. In the first case, Property (**v**) for Δ_n implies that $\omega_1 = v_1$ and then the indecomposability of κ_1 (Property (**iv**)) implies that $\kappa_1 = \rho_1$ and that $\alpha = \lambda$. As a consequence we have that $\omega_2 \otimes ... \otimes \kappa_i \otimes \omega_i = v_2 \otimes ... \otimes \rho_j \otimes v_j$. Since the cell $\omega_2 \otimes ... \otimes \kappa_i \otimes \omega_i$ has dimension less than $n + 1$, Property (**v**) for Δ_n implies that the above expression is unique. The same argument applies in the second case.

2.2.6 The Cellular Boundary Maps for Δ_{n+1}

In order to prove Properties $\mathbf{D}_{n+1}$, $\mathbf{E}_{n+1}$ and $\mathbf{F}_{n+1}$ for Δ_{n+1}, we need to compute the cellular boundary maps for the cellular chain complex associated with Δ_{n+1}. Let this chain be

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} ... \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \tag{2.8}$$

where C_k for $k = n+1, ..., 0$, are free abelian with bases the respective k-cells and ∂_k are calculated from the cellular boundary formula. Here C_k for $k \leq n$, are the same as those represented in the cellular chain complex associated with Δ_n. To compute ∂_1, we recall that the presentation $\mathcal{P}$ is terminating, hence the 1-skeleton Γ of $\mathcal{D}$ does not contain edges with the same initial and terminal, therefore we can think of Γ as a simplicial complex. Now Example 1, p.222 of [79] implies that the map $\partial_1 : C_1 \longrightarrow C_0$ is the same as the simplicial boundary map $\partial_1 : \Delta_1(\Gamma) \longrightarrow \Delta_0(\Gamma)$ and then we have

$$\partial_1(e) = \iota e - \tau e. \tag{2.9}$$

Property (**iii**) and Corollary V, 3.6 of [62] imply that for every $2 \leq k \leq n$

$$\partial_k[\omega, (e_1, ..., e_k)] = \sum_{i=1}^{k} \varepsilon_i[\omega, (e_1, ..., \hat{e_i}, ..., e_k)] + \zeta, \tag{2.10}$$

where for every $i = 1, ..., k$, $\varepsilon_i = \pm 1$ and ζ is a chain made of $(k-1)$-cells whose maximal 0-cells are less than ω.

It remains to find an explicit form for ∂_{n+1}. We will split the work for this in two parts. First we compute the restriction $\widetilde{\partial}_{n+1}$ of ∂_{n+1} on $C_{n+1}(\mathrm{p}_1, ..., \mathrm{p}_{n-1})$, the free abelian group generated by $(n+1)$-cells of $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$, and then we compute $\partial_{n+1}(\sigma)$ for $\sigma \in F.\mathbf{p}_n.F$. To compute $\widetilde{\partial}_{n+1}$ we consider the cellular chain complex associated with $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$:

$$C_{n+1}(\mathbf{p}_1, ..., \mathbf{p}_{n-1}) \xrightarrow{\widetilde{\partial}_{n+1}} C_n \xrightarrow{\partial_n} ... \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0. \tag{2.11}$$

On the other hand, the cellular chain complex associated with K_n is

$$D_{n+1} \xrightarrow{d_{n+1}} ... \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0, \tag{2.12}$$

where D_s is free abelian with bases the set of s-cell $\sigma \times \delta \in E(K_n^{(s)})$, with $0 \leq s \leq n+1$, and then the map $p : K_n \longrightarrow (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$, which is obviously cellular, induces a chain map $p_\#$ from (2.12) to (2.11) such that $p_\#(\sigma \times \delta) = \sigma \otimes \delta = p(\sigma \times \delta)$, which means that $[p : \sigma \times \delta : p(\sigma \times \delta)] = 1$.

Lemma 2.2.20 *If* $(\sigma \times \delta)_{\mu'}$, $(\sigma \times \delta)_{\mu''} \in \partial(\sigma \times \delta)$ *then*

$$[p : (\sigma \times \delta)_{\mu'} : p((\sigma \times \delta)_{\mu''})] = \begin{cases} 1 & \textit{if} \quad \mu' = \mu'' \\ 0 & \textit{if} \quad \mu' \neq \mu'' \end{cases}$$

Proof. When $\mu' = \mu''$ the assumption is obvious. In the case when $\mu' \neq \mu''$ we need to show that there are no different parallel cells in the boundary of $\sigma \times \delta$. Suppose that $\sigma = u_1 \otimes \sigma_1 \otimes ... \otimes u_k \otimes \sigma_k \otimes u_{k+1}$ and $\delta = v_1 \otimes \delta_1 \otimes ... \otimes v_s \otimes \delta_s \otimes v_{s+1}$. The boundary cells of $\sigma \times \delta$ are of two kinds; $\sigma' \times \delta$ with $\sigma' \in \partial\sigma$ and $\sigma \times \delta'$ with $\delta' \in \partial\delta$. By an inductive argument on dimension one can show that $\sigma' = u_1 \otimes \sigma_1 \otimes ... \otimes u_i \otimes \sigma_i' \otimes u_{i+1} \otimes ... \otimes u_{k+1}$ with $\sigma_i' \in \partial\sigma_i$ and similarly $\delta' = v_1 \otimes \delta_1 \otimes ... \otimes v_j \otimes \delta_j' \otimes v_{j+1} \otimes ... \otimes v_{s+1}$ with $\delta_j' \in \partial\delta_j$. Using Property (**v**), one can easily show that $\sigma' \otimes \delta \neq \sigma \otimes \delta'$ and therefore $\sigma' \times \delta \nparallel \sigma \times \delta'$. The same holds true for the cells $\sigma \times \delta'$ and $\sigma \times \delta''$ where $\delta', \delta'' \in \partial\delta$ and $\delta' \neq \delta''$, and similarly $\sigma' \times \delta \nparallel \sigma'' \times \delta$ with $\sigma', \sigma'' \in \partial\sigma$ and $\sigma' \neq \sigma''$. ∎

Now let $\sigma \times \delta \in E(K_n^{(n+1)})$ and $\sigma' \in \partial\sigma$. We take $p(\sigma' \times \delta)$ to be in the role of τ of Proposition 1.3.4. We have that

$$[p : \sigma \times \delta : p(\sigma \times \delta)][p(\sigma \times \delta) : p(\sigma' \times \delta)] = \sum_{\mu \in \Lambda_n} [\sigma \times \delta : (\sigma \times \delta)_\mu][p : (\sigma \times \delta)_\mu : p(\sigma' \times \delta)].$$

From Lemma 2.2.20, in the sum of the right hand side, only one of the factors $[p : (\sigma \times \delta)_\mu : p(\sigma' \times \delta)]$ is non zero, the one with $(\sigma \times \delta)_\mu = (\sigma' \times \delta)$ and then the equality above changes to

$$[p : \sigma \times \delta : p(\sigma \times \delta)][p(\sigma \times \delta) : p(\sigma' \times \delta)] = [\sigma \times \delta : (\sigma' \times \delta)][p : (\sigma' \times \delta) : p(\sigma' \times \delta)],$$

which immediately implies that $[p(\sigma \times \delta) : p(\sigma' \times \delta)] = [\sigma \times \delta : (\sigma' \times \delta)]$. In the same way we prove that for $\delta' \in \partial\delta$ we have that $[p(\sigma \times \delta) : p(\sigma \times \delta')] = [\sigma \times \delta : (\sigma \times \delta')]$. So the coefficients in which a boundary cell $p(\sigma' \times \delta)$ or $p(\sigma \times \delta')$ is represented in the cellular boundary formula for $p(\sigma \times \delta)$ in (2.11) are the same as the coefficients of the respective boundary cells $\sigma' \times \delta$ and $\sigma \times \delta'$ for $\sigma \times \delta$ in (2.12). But on the other hand from [40] we have that the cellular boundary formula for (2.12) is given by

$$d_{n+1}(\sigma^i \times \sigma^j) = \partial_i(\sigma^i) \times \sigma^j + (-1)^i \sigma^i \times \partial_j(\sigma^j)$$

which if written explicitly gives

$$d_{n+1}(\sigma^i \times \sigma^j) = \sum_s n_s \sigma^i_s \times \sigma^j + (-1)^i \sum_t n_t \sigma^i \times \sigma^j_t,$$

where n_{is} and $(-1)^i n_{jt}$ are the incidence numbers of $\sigma^i_s \times \sigma^j$ and $\sigma^i \times \sigma^j_t$ respectively. So we finally have for (2.11) the formula

$$\widetilde{\partial}_{n+1}(\sigma^i \otimes \sigma^j) = \sum_s n_s(\sigma^i_s \otimes \sigma^j) + (-1)^i \sum_t n_{jt}(\sigma^i \otimes \sigma^j_t). \tag{2.13}$$

Before we describe ∂_{n+1} for cells from $F.\mathbf{p}_n.F$, we will show that ∂_{n+1} is a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule homomorphism.

Property (**v**) shows that the chain group C_{n+1} defined in (2.8) is in fact free $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule with bases the set of right and left-reduced cells and the F-action is inherited from the two-sided action $\otimes$ of F on Δ_{n+1}. To show that ∂_{n+1} is a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule homomorphism, we need firstly to show that $\forall\ \sigma, \sigma' \in \Delta^{(n+1)}_{n+1}$ and $u, v \in F$,

$$u \otimes \sigma \otimes v = u \otimes \sigma' \otimes v \textit{ if and only if } \sigma = \sigma'$$

and then we show that $\forall\ \sigma \in \Delta^{(n+1)}_{n+1}$, $\forall\ \sigma_i \in \partial\sigma$ and $u, v \in F$, we have

$$[\sigma : \sigma_i] = [u \otimes \sigma \otimes v : u \otimes \sigma_i \otimes v].$$

To show the first claim we take the respective decompositions of both σ and σ'. We then have

$$uu_1 \otimes \sigma_1 \otimes u_2 ... u_k \otimes \sigma_k \otimes u_{k+1}v = u \otimes (u_1 \otimes \sigma_1 \otimes u_2 ... u_k \otimes \sigma_k \otimes u_{k+1}) \otimes v = u \otimes \sigma \otimes v =$$
$$u \otimes \sigma' \otimes v = u \otimes (u'_1 \otimes \sigma'_1 \otimes u'_2 ... u'_s \otimes \sigma'_s \otimes u'_{s+1}) \otimes v = uu'_1 \otimes \sigma'_1 \otimes u'_2 ... u'_s \otimes \sigma'_s \otimes u'_{s+1}v.$$

The uniqueness of such decompositions and the fact that multiplication in F is cancellative, imply that $\sigma = \sigma'$. To show the second claim, we define a chain map f from the chain complex in (2.8) to itself that sends each σ to $u \otimes \sigma \otimes v$. We have that $[f : \sigma : u \otimes \sigma \otimes v] = 1$ and $[f : \sigma : u \otimes \sigma' \otimes v] = 0$ for $\sigma' \neq \sigma$. Now, fixing some $\sigma_i \in \partial\sigma$, by Proposition 1.3.4 we may write

$$[f : \sigma : u \otimes \sigma \otimes v][u \otimes \sigma \otimes v : u \otimes \sigma_i \otimes v] = \sum_{\mu \in \Lambda_n} [\sigma : \sigma_\mu][f : \sigma_\mu : u \otimes \sigma_i \otimes v],$$

which yields $[u \otimes \sigma \otimes v : u \otimes \sigma_i \otimes v] = [\sigma : \sigma_i]$.

To show that the boundary map ∂_{n+1} is a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule homomorphism, one can see from Property (i) that the boundary of every k-cell with $1 \leq k \leq n+1$, $u.\sigma.v = u \otimes \sigma \otimes v$ is made of cells $u.\sigma_i.v = u \otimes \sigma_i \otimes v$ with $\sigma_i \in \partial\sigma$; hence

$$\partial_{n+1}(u.\sigma.v) = \textstyle\sum_i [u \otimes \sigma \otimes v : u \otimes \sigma_i \otimes v](u \otimes \sigma_i \otimes v) =$$
$$u. \left(\textstyle\sum_i [u \otimes \sigma \otimes v : u \otimes \sigma_i \otimes v]\sigma_i\right).v = u. \left(\textstyle\sum_i [\sigma : \sigma_i]\sigma_i\right).v = u. \left(\partial_{n+1}\sigma\right).v.$$

The following completes the description of the cellular boundary maps for all the $(n+1)$-cells.

Lemma 2.2.21 *For every $(n+1)$-cell $\sigma = [\omega; (e_1, e_2, ..., e_{n+1})]$, we have*

$$\partial_{n+1}\sigma = \sum_i \varepsilon_i[\omega; (e_1, ..., \hat{e}_i, ..., e_{n+1})] + \zeta,$$

where $\varepsilon_i = \pm 1$ for every $i = 1, ..., n+1$ and ζ is a chain made of n-cells whose respective maximal 0-cells are below ω.

Proof. If σ is a critical $(n+1)$-cell, then the formula holds true because of Corollary V, 3.6 of [62] and from the construction of such cells. If σ is a translate of a critical $(n+1)$-cell, then the above and the fact that ∂_{n+1} is a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule morphism imply the result. Now suppose that $\sigma = \sigma_1 \otimes \sigma_2$, where $\dim \sigma_1 = s \geq 1$ and $\dim \sigma_2 = t \geq 1$ and that $\sigma_1 = [\omega_1; (f_1, ..., f_s)]$ and $\sigma_2 = [\omega_2; (g_1, ..., g_t)]$. From the cellular boundary formula for σ, from (2.10) and (2.9), we have that

$$\partial_{s+t}\sigma = \sum_i \varepsilon_i[\omega_1; (f_1, ..., \hat{f}_i, ..., f_s)] \otimes \sigma_2 + (-1)^s \sum_j \varepsilon_j \sigma_1 \otimes [\omega_2; (g_1, ..., \hat{g}_j, ..., g_t)] + \zeta,$$

where $\varepsilon_i = \pm 1$ for every $i = 1, ..., s$, $\varepsilon_j = \pm 1$ for every $j = 1, ..., t$ and ζ is made of n-cells whose respective maximal 0-cells are below $\omega = \omega_1\omega_2$. Now from Lemma 2.2.19 we can write $\sigma = [\omega_1\omega_2; (f_1.\omega_2, ..., f_s.\omega_2, \omega_1.g_1, ..., \omega_1.g_t)]$, and, since again from that lemma

$$[\omega_1; (f_1, ..., \hat{f}_i, ..., f_s)] \otimes \sigma_2 = [\omega_1\omega_2; (f_1.\omega_2, ..., \widehat{f_i.\omega_2}, ..., f_s.\omega_2, \omega_1.g_1, ..., \omega_1.g_t)]$$

and

$$\sigma_1 \otimes [\omega_2; (g_1, ..., \hat{g}_j, ..., g_t)] = [\omega_1\omega_2; (f_1.\omega_2, ..., f_s.\omega_2, \omega_1.g_1, ..., \widehat{\omega_1.g_j}, ..., \omega_1.g_t)],$$

we can finally write

$$\partial_{n+1}\sigma = \sum_i \varepsilon_i[\omega; (e_1, ..., \overset{\wedge}{e_i}, ..., e_{n+1})] + \zeta,$$

where ζ is made of n-cells whose respective maximal 0-cells are below $\omega = \omega_1\omega_2$ and $\varepsilon_i = \pm 1$ for every $i = 1, ..., n+1$. ■

2.2.7 The Homology of the Complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$

In this section we study the homology group $H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ and prove Properties $\mathbf{D}_{n+1}$ and $\mathbf{E}_{n+1}$ which are related to this group.

We introduce the following notations:

$$Im\widetilde{\partial}_{n+1} = B_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \subseteq Z_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) = Ker\partial_n;$$

hence the n-homology group now is

$$H_n = H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) = \frac{Z_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})}{B_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})}.$$

Note that, since both $Ker\partial_n$ and $Im\widetilde{\partial}_{n+1}$ have a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure, H_n has an induced $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure too.

Before we prove the next lemma, we observe that if in (2.13) we take $i = 1$, and hence $\sigma^i = e \in \mathbf{e}^+$, we obtain that $(\iota e - \tau e).\sigma^j$ is homologous with $\sum_t n_t(e \otimes \sigma_t^j)$.

Lemma 2.2.22 $J.H_n = 0 = H_n.J$.

Proof. We only show that $J.H_n = 0$; the other equality is obtained similarly. Suppose that $\xi = \sum_{i=1}^{t} \lambda_i \varsigma_i \in Z_n$, $\lambda_i \in \{\pm 1\}$ for all $i = 1, ..., t$, ς_i is an n-cell. Suppose that $\partial_n \varsigma_i = \sum_{j=1}^{m_i} \delta_{ij}\sigma_{ij}$. Now from the condition we have

$$0 = \partial_n \xi = \sum_{i=1}^{t} \lambda_i \left(\sum_{j=1}^{m_i} \delta_{ij}\sigma_{ij} \right). \tag{2.14}$$

This shows that each element in (2.14) has its opposite again in this sum. As we saw earlier, for every $e \in \mathbf{e}^+$, $(\iota e - \tau e).\varsigma_i$ is homologous to $\sum_{j=1}^{m_i} \delta_{ij}(e \otimes \sigma_{ij})$ for $1 \le i \le t$. Taking the sums of all $\lambda_i(\iota e - \tau e).\varsigma_i$, we obtain that $(\iota e - \tau e).\xi$ is homologous to the following sum

$$\sum_{i=1}^{t} \lambda_i \left(\sum_{j=1}^{m_i} \delta_{ij}(e \otimes \sigma_{ij}) \right).$$

Comparing it with (2.14), one can see easily that this sum is 0, and hence $(\iota e - \tau e).\xi$ is null homologous. ∎

Lemma 2.2.22 implies that there is an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure on the n-th homology, with the action

$$\overline{u}.(\xi + B_n).\overline{v} = u.\xi.v + B_n \quad (\xi \in Z_n, u, v \in F).$$

In this way we have proved Property $\mathbf{D}_{n+1}$ for the complex Δ_{n+1}.

Proposition 2.2.23 *The complex Δ_{n+1} satisfies Property $\mathbf{E}_{n+1}$.*

Proof. To show that Property $\mathbf{E}_{n+1}$ holds true for the complex Δ_{n+1}, one can apply the argument of the proof of Lemma 2.2.10 and see that the bi-module generators of $H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ are the homology classes of cycles arising from $\mathbf{p}_n$, therefore by adding cells from $F.\mathbf{p}_n.F$ to $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1})$ we obtain the complex

$$\Delta_{n+1} = (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \sqcup F.\mathbf{p}_n.F$$

with $H_n(\Delta_{n+1}) = 0$. ∎

2.2.8 Proof of the Property $\mathbf{F}_{n+1}$ for Δ_{n+1}

The strategy of proving that there are morphisms Φ and ν such that the sequence

$$0 \longrightarrow H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S \xrightarrow{\nu} H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2}) \longrightarrow 0 \qquad (2.15)$$

is exact, will be the following. First we introduce a short exact sequence of $(\mathbb{Z}F, \mathbb{Z}F)$-bimodules

$$0 \longrightarrow Im\widetilde{\partial}_{n+1} \overset{incl}{\hookrightarrow} K^{\mathbf{p}_{n-1}} \xrightarrow{\partial_n} Im\widetilde{\partial}_n \longrightarrow 0 \qquad (2.16)$$

which fits into a commutative ladder with another short exact sequence of $(\mathbb{Z}F, \mathbb{Z}F)$-bimodules and then we use the Snake Lemma to prove the exactness of (2.15).

Let us now define the modules represented in (2.16). Define first the map

$$\varphi : C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2}) \oplus \mathbb{Z}F.\mathbf{p}_{n-1}.\mathbb{Z}F \longrightarrow \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S, \qquad (2.17)$$

where $C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2})$ is the free abelian group with bases the n-cells of $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2})$ and

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x \in C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2}) \\ \overline{u}.\sigma.\overline{v} & \text{if} \quad x = u.\sigma.v \in F.\mathbf{p}_{n-1}.F \end{cases}$$

and secondly the map

$$\nu : \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S \longrightarrow H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2})$$

such that

$$\nu(\sigma) = \partial_n(\sigma) + B'_{n-1},\ \sigma \in \mathbf{p}_{n-1} \text{ and } B'_{n-1} = \partial_n(C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2})).$$

From the induction hypothesis, $H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2})$ is generated (as a bimodule) by the elements $\partial_n(\sigma) + B'_{n-1}$, $\sigma \in \mathbf{p}_{n-1}$ and therefore ν is surjective. Letting the kernel of φ be $K^{\mathbf{p}_{n-1}}$ we have the following.

Lemma 2.2.24 *We can write* $K^{\mathbf{p}_{n-1}} = C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2}) + J.\mathbf{p}_{n-1}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{p}_{n-1}.J$.

Proof. This follows from the general fact that for every set $\mathbf{X}$ the natural homomorphism mapping $\mathbb{Z}F.\mathbf{X}.\mathbb{Z}F$ to $\mathbb{Z}S.\mathbf{X}.\mathbb{Z}S$ has the kernel $J.\mathbf{X}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{X}.J$. ∎

Beside the chain complex (2.8) we consider the chain complex

$$C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2}) \xrightarrow{\widetilde{\partial}_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0 \tag{2.18}$$

associated with the complex $(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2})$. The difference between (2.18) and (2.8) is that C_n in (2.8) is the direct sum of $C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2})$ with the free $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule with bases $\mathbf{p}_{n-1}$, while the other bimodules of both the chains, indexed $n-1$ or less, coincide and therefore the respective boundary maps coincide as well. On the other hand the restriction of ∂_n on $C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2})$ coincides with $\widetilde{\partial}_n$. It is clear from the formulas (2.13) for $\widetilde{\partial}_{n+1}$ that $Im\widetilde{\partial}_{n+1} \subseteq K^{\mathbf{p}_{n-1}}$. Now let $(\iota f - \tau f).\varsigma \in J.\mathbf{p}_{n-1}.\mathbb{Z}F$. Again from formulas (2.13), taking $\sigma_1 = f$ we get that $(\iota f - \tau f).\varsigma = \widetilde{\partial}_{n+1}(f \otimes \varsigma) + \eta$ where $\eta \in C_n(\mathbf{p}_1, ..., \mathbf{p}_{n-2})$. Taking ∂_n of both sides of the above equality, we have that

$$\partial_n((\iota f - \tau f).\varsigma) = \partial_n \eta = \widetilde{\partial}_n \eta \in Im\widetilde{\partial}_n.$$

Likewise $\partial_n(\varsigma.(\iota f - \tau f)) \in Im\widetilde{\partial}_n$ and hence from Lemma 2.2.24 we have that $\partial_n(K^{\mathbf{p}_{n-1}}) = Im\widetilde{\partial}_n$. Lastly, the above and the fact that $\partial_n \partial_{n+1} = 0$, imply that (2.16) is a chain complex. Our intention is to show the following:

Proposition 2.2.25 *The sequence (2.16) is exact.*

We first prove some technical lemmas.

Lemma 2.2.26 *For every $(n+1)$-cell $\sigma = [\omega; (e_1, e_2, ..., e_{n+1})]$, the boundary cells of σ which have ω_σ in their own respective 0-boundaries, are comparable. The maximal boundary cell is $[\omega; (\overset{\wedge}{e_1}, e_2, ..., e_{n+1})]$.*

Proof. From Lemma 2.2.18 (3), the n-cells of σ which have ω_σ in their own respective 0-boundaries are in a 1-1 correspondence with subsets of $star(\sigma)$ with n elements. Let σ_1 and σ_2 be two of them with respective $star$'s, $\{e_{i_i}, e_{i_2}, ..., e_{i_n}\}$ and $\{e_{j_i}, e_{j_2}, ..., e_{j_n}\}$. We have either $[\{e_{i_i}, e_{i_2}, ..., e_{i_n}\}] \prec_{mul} [\{e_{j_i}, e_{j_2}, ..., e_{j_n}\}]$ or $[\{e_{j_i}, e_{j_2}, ..., e_{j_n}\}] \prec_{mul} [\{e_{i_i}, e_{i_2}, ..., e_{i_n}\}]$ and hence, from the definition of $\prec$ we have either $\sigma_1 \prec \sigma_2$ or $\sigma_2 \prec \sigma_1$. The maximality of $[\omega; (\overset{\wedge}{e_1}, e_2, ..., e_{n+1})]$ follows from the fact that $\{e_2, ..., e_{n+1}\}$ is maximal in $star(\sigma)$. ■

Lemma 2.2.27 *If the cells σ_1 and σ_2 of the same dimension are such that $\sigma_1 \prec \sigma_2$, then for every cell δ such that $dim(\sigma_1 \otimes \delta) \leq n$ we have $\sigma_1 \otimes \delta \prec \sigma_2 \otimes \delta$.*

Proof. Let $\sigma_1 = [\omega_{\sigma_1}, (e_1, ..., e_s)]$ and $\sigma_2 = [\omega_{\sigma_2}, (f_1, ..., f_s)]$.
If $\omega_{\sigma_1} \prec \omega_{\sigma_2}$, then $\omega_{\sigma_1}\omega_\delta \prec \omega_{\sigma_2}\omega_\delta$ and then from the definition of $\prec$ we have $\sigma_1 \otimes \delta \prec \sigma_2 \otimes \delta$.
If $\omega_{\sigma_1} = \omega_{\sigma_2} = \omega$ and $[\{e_1, ..., e_s\}] \prec_{mul} [\{f_1, ..., f_s\}]$, then it is easy to see that

$$[\{e_1.\omega_\delta, ..., e_s.\omega_\delta, \omega.g_1, ..., \omega.g_t\}] \prec_{mul} [\{f_1.\omega_\delta, ..., f_s.\omega_\delta, \omega.g_1, ..., \omega.g_t\}],$$

where $\{g_1, ..., g_t\} = star(\delta)$. This again implies that $\sigma_1 \otimes \delta \prec \sigma_2 \otimes \delta$. ■

Lemma 2.2.28 *If the cell δ_1 and δ_2 of the same dimension, are such that $\delta_1 \prec \delta_2$, then for every cell σ such that $dim(\sigma \otimes \delta_1) \leq n$ we have $\sigma \otimes \delta_1 \prec \sigma \otimes \delta_2$.*

Proof. The proof of this runs similarly to that of the previous lemma. ■

Lemma 2.2.29 *Let $\sigma \otimes \delta$ be a non-critical cell such that $2 \leq dim(\sigma \otimes \delta) \leq n$. Let $\sigma_1 \in \partial\sigma$ and $\delta_1 \in \partial\delta$ be the respective maximal boundary cells. We have $\sigma_1 \otimes \delta \succ \sigma \otimes \delta_1$.*

Proof. Let $\sigma = [\omega_\sigma, (e_1, ..., e_s)]$ and $\delta = [\omega_\delta, (f_1, ..., f_t)]$. From Lemma 2.2.26 we have that $\sigma_1 = [\omega_\sigma, (e_2, ..., e_s)]$ and $\delta_1 = [\omega_\delta, (f_2, ..., f_t)]$. It follows that

$$\sigma_1 \otimes \delta = [\omega_\sigma\omega_\delta, (e_2.\omega_\delta, ..., e_s.\omega_\delta, \omega_\sigma.f_1, ..., \omega_\sigma.f_t)]$$

and

$$\sigma \otimes \delta_1 = [\omega_\sigma\omega_\delta, (e_1.\omega_\delta, ..., e_s.\omega_\delta, \omega_\sigma.f_2, ..., \omega_\sigma.f_t)].$$

Since $e_1.\omega_\delta \prec \omega_\sigma.f_1$, we have that

$$[\{e_2.\omega_\delta, ..., e_s.\omega_\delta, \omega_\sigma.f_1, ..., \omega_\sigma.f_t\}] \succ [\{e_1.\omega_\delta, ..., e_s.\omega_\delta, \omega_\sigma.f_2, ..., \omega_\sigma.f_t\}]$$

which proves the lemma. ∎

Lemma 2.2.30 *If $\sigma \in F.\mathbf{p}_{n-1}.F$ is a maximal cell from an n-cycle ξ in $K^{\mathbf{p}_{n-1}}$, then σ can not be written in the form $\sigma = u.[\omega; (e_1, e_2, ..., e_n)].v$ where u and v are both irreducible and $[\omega; (e_1, e_2, ..., e_n)] \in \mathbf{p}_{n-1}$.*

Proof. Suppose by the way of contradiction that we can write $\sigma = u.[\omega; (e_1, e_2, ..., e_n)].v$ where u and v are both irreducible and $[\omega; (e_1, e_2, ..., e_n)] \in \mathbf{p}_{n-1}$. Since $\varphi\xi = 0$, there must exists some $\sigma_i = u_i.[\omega; (e_1, e_2, ..., e_n)].v_i$ from ξ, such that $\overline{u_i} = \overline{u}$ and $\overline{v_i} = \overline{v}$. The choice of u and v implies that $\sigma_i \succ \sigma$, which contradicts the maximality of σ. ∎

Lemma 2.2.31 *Every n-cycle $\xi = \sum_{i=1}^{m} n_i\sigma_i$ in $K^{\mathbf{p}_{n-1}}$ is homologous to some n-cycle ξ' in $K^{\mathbf{p}_{n-1}}$ such that ξ' is obtained from ξ by a positive number of elementary transitions.*

Proof. Let ω be some maximal 0-cell from $\xi^{(0)}$ and σ_1 some maximal cell represented in ξ whose maximal 0-cell is ω. We distinguish the following three possibilities.

1) Suppose that σ_1 is represented in ξ as $m\sigma_1$, with $m \in \mathbb{Z}$ and $\sigma_1 = \alpha \otimes \beta$ with α and β cells of dimension at least 1. Letting $\alpha = [\omega_\alpha; (e_1, ..., e_k)]$ and $\beta = [\omega_\beta; (e_{k+1}, ..., e_n)]$, from Lemma 2.2.19 we have that

$$\sigma_1 = [\omega_\alpha\omega_\beta; (e_1.\omega_\beta, ..., e_k.\omega_\beta, \omega_\alpha.e_{k+1}, ..., \omega_\alpha.e_n)].$$

From Lemma 2.2.26 we have that the maximal boundary cell of σ_1 is

$$\sigma_{11} = [\omega_\alpha\omega_\beta; (e_2.\omega_\beta, ..., e_k.\omega_\beta, \omega_\alpha.e_{k+1}, ..., \omega_\alpha.e_n)].$$

Since σ_1 must cancel under the boundary map, there is some other cell σ_1' represented in ξ with σ_{11} in its own boundary. This cell must be of the form

$$\sigma_1' = [\omega_\alpha\omega_\beta; (f, e_2.\omega_\beta, ..., e_k.\omega_\beta, \omega_\alpha.e_{k+1}, ..., \omega_\alpha.e_n)],$$

where $f \prec e_1.\omega_\beta$. It follows that f does not act on ω_β and hence we can write $f = f'.\omega_\beta$ and f' is disjoint from $e_{k+1}, ..., e_n$. Consequently we have the $(n+1)$-cell $\sigma_1^* = \alpha^* \otimes \beta$ with $\alpha^* = [\omega_\alpha; (f', e_1, e_2, ..., e_k)]$. We can perform an elementary transition on ξ by replacing $m\sigma_1$ by the chain $-\varepsilon m\zeta$ with $\varepsilon = \pm 1$ being the incidence number of σ_1 in σ_1^*, to obtain thus the n-cycle $\xi' = \xi - \varepsilon m(\varepsilon\sigma_1 + \zeta) \in Z_n \cap K^{\mathbf{p}_{n-1}}$, where ζ is given by the formula

$$\partial_{n+1}\sigma_1^* = \varepsilon\sigma_1 + \zeta$$

and hence is made of cells which are strictly less than σ_1.

2) Suppose that σ_1 is represented in ξ as $m\sigma_1$, with $m \in \mathbb{Z}$ and $\sigma_1 = \iota e.\sigma_1'$ for some positive edge e and σ_1' arising from a critical n-tuple. The boundary of the $(n+1)$-cell $e \otimes \sigma_1'$ is given by the formula

$$\partial_{n+1}(e \otimes \sigma_1') = (\iota e - \tau e).\sigma_1' + (-1)^1 \sum_j (e \otimes \sigma_{1j}'),$$

where $\sigma_{1j}' \in \partial\sigma_1'$. From Lemma 2.2.29, $\iota e.\sigma_1' \succ (e \otimes \sigma_{11}')$, with σ_{11}' the biggest of all $\sigma_{1j}' \in \partial\sigma_1'$ and, from Lemma 2.2.27, $\iota e.\sigma_1' \succ \tau e.\sigma_1'$. So we can now perform the elementary transition of replacing $m\sigma_1 = m(\iota e.\sigma_1')$ in ξ by the chain $m\zeta$, and obtain thus the n-cycle $\xi' = \xi - m(\sigma_1 - \zeta) \in Z_n \cap K^{\mathrm{P}_{n-1}}$. Here

$$\zeta = \tau e.\sigma_1' + \sum_j (e \otimes \sigma_{1j}')$$

is made of cells lesser than σ_1.

3) Suppose that $\sigma_1 = \sigma_1'.\iota e$ for some positive edge e and σ_1' arising from a critical n-tuple. Let $\sigma_2, ..., \sigma_k$ be all the other cells (re-indexed) represented in ξ meeting ω. For $i = 1, ..., k$, we let $\sigma_i = [\omega; (e_{i1}, e_{i2}, ..., e_{in})]$. From the condition, we can write $\sigma_1 = [\omega'\iota e; (e_{11}'.\iota e, e_{12}'.\iota e, ..., e_{1n}'.\iota e)]$ for some positive edges e_{1j}', $j = 1, ..., n$. For every $2 \leq i \leq k$ and $1 \leq s \leq n$, since $\sigma_i \prec \sigma_1$, it follows from the definition of $\prec$ that $e_{is} = e_{is}'.\iota e$ for some positive edge e_{is}'. As a consequence there are $(n+1)$-cells

$$[\omega; (e_{i1}, e_{i2}, ..., e_{in}, \omega'.e)] = [\omega'; (e_{i1}', e_{i2}', ..., e_{in}')] \otimes e \tag{2.19}$$

for every $i = 1, ..., k$. On the other hand we have that

$$\sum_{i=1}^{k} n_i \sum_{s_i=1}^{n} \varepsilon_{s_i} [\omega; (e_{i1}, ..., \overset{\wedge}{e_{is_i}}, ..., e_{in})] = 0, \tag{2.20}$$

where $\varepsilon_{s_i} = \pm 1$ are the incidence numbers. Indeed,

$$0 = \partial_n \xi = \sum_{i=1}^{k} n_i(\partial_n \sigma_i) + \sum_{i=k+1}^{m} n_i(\partial_n \sigma_i) =$$

$$\sum_{i=1}^{k} n_i \sum_{s_i=1}^{n} \varepsilon_{s_i} [\omega; (e_{i1}, ..., \overset{\wedge}{e_{is_i}}, ..., e_{in})] + \left(\eta + \sum_{i=k+1}^{m} n_i(\partial_n \sigma_i)\right),$$

where the bracket is a chain made of cells whose maximal boundary cell is different than ω.

It follows from (2.19) and (2.20) that

$$\sum_{i=1}^{k} n_i \sum_{s_i=1}^{n} \varepsilon_{s_i} [\omega; (e_{i1}, ..., \overset{\wedge}{e_{is_i}}, ..., e_{in}, \omega'.e)] = 0. \tag{2.21}$$

Applying formula (2.13) on the cell in (2.19) we see that the incidence number of $[\omega; (e_{i1}, e_{i2}, ..., e_{in})]$ in $[\omega; (e_{i1}, e_{i2}, ..., e_{in}, \omega'.e)]$ is $(-1)^n$. This fact together with (2.21) imply:

$$\xi - (-1)^n \sum_{i=1}^{k} n_i \partial_{n+1} [\omega; (e_{i1}, e_{i2}, ..., e_{in}, \omega'.e)] = \sum_{i=k+1}^{m} n_i \sigma_i + \zeta, \tag{2.22}$$

where the chain ζ is either zero or is made of cells whose maximal 0-cell is less than ω. Now the cycle $\xi' = \sum_{i=k+1}^{m} n_i\sigma_i + \zeta$ is homologous with ξ, it is from $Z_n \cap K^{\mathbf{p}_{n-1}}$ and is obtained from ξ by replacing $\sum_{i=1}^{k} n_i\sigma_i$ by the chain ζ. ■

The following proof will be the combination of the last two lemmas.

Proof of Proposition 2.2.25. Suppose that $\xi \in Z_n \cap K^{\mathbf{p}_{n-1}}$ and want to show that it is null-homologous. Let ω be some maximal 0-cell from $\xi^{(0)}$ and σ some maximal cell represented in ξ whose maximal 0-cell is ω. From Lemma 2.2.30 we have that σ can only be as in one of the three cases of the proof of Lemma 2.2.31. We can then apply Lemma 2.2.31 to obtain an homologous cycle ξ'. If ξ' does not satisfy the condition $\xi' < \xi$, we chose ω' to be some maximal 0-cell from $\xi'^{(0)}$ and σ' some maximal cell represented in ξ' whose maximal 0-cell is ω', and repeat the above procedure. After finitely many steps we obtain ξ'' homologous with ξ and such that $[\xi''^{(0)}] \prec_{mul} [\xi^{(0)}]$. Therefore we have $\xi'' < \xi$ and then we can apply Noetherian induction on ξ''. ■

Now we prove the property $\mathbf{F}_{n+1}$ for Δ_{n+1}. From Proposition 2.2.25 we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Im\widetilde{\partial}_{n+1} & \xrightarrow{incl.} & K^{\mathbf{p}_{n-1}} & \xrightarrow{\partial_n} & Im\widetilde{\partial}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & \\
0 & \longrightarrow & Z_n(\Delta_n) & \xrightarrow{incl.} & C_n(\Delta_n) & \xrightarrow{\partial_n} & Z_{n-1}(\Delta_n) & \longrightarrow & 0
\end{array}$$

where all the vertical maps are inclusions, both rows are exact and $C_n(\Delta_n) = C_n$ in (2.8), $Z_n(\Delta_n)$ is the kernel of ∂_n in (2.8), and $Z_{n-1}(\Delta_n)$ is the kernel of ∂_{n-1} in (2.8). Applying the "Snake Lemma" [90] to it, we obtain the short exact sequence

$$0 \longrightarrow H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S \xrightarrow{\nu} H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2}) \longrightarrow 0,$$

where

$$\Phi : H_n(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-1}) = \mathrm{coker}(\iota_1) \quad \longrightarrow \quad \mathrm{coker}(\iota_2) = \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S$$

is given by

$$\xi_n + B_n \quad \longmapsto \quad \varphi(\xi_n) \quad (\xi_n \in Z_n),$$

with φ defined as in (2.17), and

$$\nu : \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S = \mathrm{coker}(\iota_2) \longrightarrow \mathrm{coker}(\iota_3) = H_{n-1}(\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2})$$

is the homomorphism taking each $\varsigma \in \mathbf{p}_{n-1}$ to the homology class of the corresponding $(n-1)$-cycle.

This completes the proof of Theorem 2.1.1.

2.2.9 The S-graded Resolutions

The following remark will be useful in Chapter 3.

Remark 2.2.32 Associated with a finite and complete presentation $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$, we constructed in Theorem 2.1.2 the resolution:

$$\mathbb{Z}S.\mathbf{p}_n.\mathbb{Z}S \xrightarrow{\delta_{n+2}} \ldots \xrightarrow{\delta_4} \mathbb{Z}S.\mathbf{p}_1.\mathbb{Z}S \xrightarrow{\delta_3} \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \xrightarrow{\delta_2} \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S \xrightarrow{\delta_1} \mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{Z}S \xrightarrow{\delta_0} \mathbb{Z}S \to 0.$$

We can think of each set $\mathbf{p}_k$ with $k = 1, ..., n$, as the subset of S whose elements are represented by the 0-cells occurring in the cells of $\mathbf{p}_k$. Of course this is a 1-1 correspondence. Similarly, $\mathbf{r}$ (respectively $\mathbf{x}$) is the subset of S whose elements are represented by the coordinates of the elements of $\mathbf{r}$ (respectively by the generators in $\mathbf{x}$). Now it is more illuminating to write the above resolution in the form:

$$\bigoplus_{\mathbf{p}_n}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_{n+2}} \ldots \xrightarrow{\delta_3} \bigoplus_{\mathbf{r}}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_2} \bigoplus_{\mathbf{x}}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_1} \mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S \xrightarrow{\delta_0} \mathbb{Z}S \to 0.$$

We identify here the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{Z}S$ with the free left $\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$ module $\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$ via the isomorphism

$$u^{opp} \otimes v \longrightarrow v \otimes u.$$

From [53], [72], [85] and our definitions of mappings δ_k with $k \geq 3$, we see that, for every $1 \leq k \leq n+2$ and every $s \otimes t \in \mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$,

$$\delta_k(s \otimes t) = \sum_{i \in I} n_i(s_i \otimes t_i),$$

where $n_i \in \mathbb{Z}$ and for every $i \in I$, $s_i w_i t_i = swt$ if $s \otimes t$ is taken from that direct sum component related to $w \in S$ and $s_i \otimes t_i$ is in the direct sum component related to $w_i \in S$.

This is a motivation to make the general definition below. Before that we introduce some notation. If in the coproduct $\bigoplus_{\mathbf{p}}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S)$ we think of $\mathbf{p}$ as a subset of S, then any element $s \otimes t$ belonging to that $\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$ which is the u^{th} component of the coproduct will be denoted by $(s \otimes t)_u$.

Definition 2.2.33 A free resolution

$$\bigoplus_{\mathbf{p}_n}(\mathbb{Z}S^{opp}\otimes_{\mathbb{Z}}\mathbb{Z}S) \xrightarrow{\delta_n} \dots \xrightarrow{\delta_1} \bigoplus_{\mathbf{p}_0}(\mathbb{Z}S^{opp}\otimes_{\mathbb{Z}}\mathbb{Z}S) \xrightarrow{\delta_0} \mathbb{Z}S \to 0,$$

of $\mathbb{Z}S$ is called *S-graded* if for each $1 \leq k \leq n$ and each $(s\otimes t)_u \in \bigoplus_{\mathbf{p}_k}(\mathbb{Z}S^{opp}\otimes_{\mathbb{Z}}\mathbb{Z}S)$ we have

$$\delta_k(s\otimes t)_u = \sum_{i\in I} n_i(s_i\otimes t_i)_{u_i},$$

such that

$$sut = s_i u_i t_i$$

for every $i \in I$, and

$$\delta_0(s\otimes t)_u = sut.$$

2.3 A Remark and an Open Problem

The result of Theorem 2.1.2 is a special case of that of Corollary 7.2 of [55] which states that *if an algebra A over a commutative ring K with a unit element admits a finite Grobner base G, then a finitely generated A-bimodule with finite Grobner base modulo G has type* bi-FP$_\infty$.

Indeed, as we saw in Example 1.8.3, $\mathbb{Z}S$ admits a finite Grobner base if S is given by a finite complete presentation; therefore considering $\mathbb{Z}S$ as a $(\mathbb{Z}S,\mathbb{Z}S)$-bimodule, we obtain straightaway from the above result that $\mathbb{Z}S$ has type bi-FP$_\infty$.

The advantage of our topological approach restricted in the case of integral monoid rings, is that there is the possibility that one can define finiteness conditions FDT$_n$ and FHT$_n$ for monoids with $n \geq 3$ in a similar way with that of [72], generalizing McGlashan's results. A first step to achieve this is to solve the following.

Problem 2.3.1 If $\mathcal{P} = \mathcal{P}[\mathbf{x},\mathbf{r}]$ is a finite presentation for a monoid S, then there is a CW-complex Δ_n of dimension $n > 3$ containing the 3-complex $(\mathcal{D},\mathbf{p}_1)$ of [72] and such that Δ_n is expressed as a disjoint union $(\mathcal{D},\mathbf{p}_1,\dots,\mathbf{p}_{n-2}) \sqcup F.\mathbf{p}_{n-1}.F$ where $(\mathcal{D},\mathbf{p}_1,\dots,\mathbf{p}_{n-2})$ is a n-subcomplex of Δ_n and $\mathbf{p}_1,\dots,\mathbf{p}_{n-1}$ are finite sets of cells which give rise to $(\mathbb{Z}S,\mathbb{Z}S)$-bimodule generators of respectively $H_1(\mathcal{D})$,..., $H_1(\mathcal{D},\mathbf{p}_1,\dots,\mathbf{p}_{n-2})$. Secondly, there is a short exact sequence

$$0 \longrightarrow H_{n-1}(\mathcal{D},\mathbf{p}_1,\dots,\mathbf{p}_{n-2}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}_{n-1}.\mathbb{Z}S \xrightarrow{\nu} H_{n-2}(\mathcal{D},\mathbf{p}_1,\dots,\mathbf{p}_{n-3}) \longrightarrow 0$$

of $(\mathbb{Z}S,\mathbb{Z}S)$-bimodules.

As we saw in the previous sections, we can construct the complex with the above properties if the system is finite and complete. In [72] the above problem is solved for $n = 3$ by first making $\mathcal{P} = \mathcal{P}[\mathbf{x}, \mathbf{r}]$ compatible with some $<_{lex}$ and completing the resulting system using Knuth-Bendix. The output system $\mathcal{P}^{\infty} = \mathcal{P}[\mathbf{x}, \mathbf{r}^{\infty}]$ gives rise to a new Squier complex $\mathcal{D}(\mathcal{P}^{\infty})$ with homology trivializers $\mathbf{p}^{\infty}$ obtained by choosing resolutions of all the critical pairs of $\mathbf{r}^{\infty}$. Then, similarly as we did in Proposition 2.2.25, one can obtain the short exact sequence

$$0 \to B_2(\mathcal{D}(\mathcal{P}^{\infty}), \mathbf{p}^{\infty}) \longrightarrow K^{\infty} \longrightarrow B_1(\mathcal{D}(\mathcal{P}^{\infty})) \to 0$$

where $K^{\infty} = C_2(\mathcal{D}(\mathcal{P}^{\infty})) + J.\mathbf{p}^{\infty}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{p}^{\infty}.J$, and then using Lemma 13 of [72], we get the other short exact sequence

$$0 \to B_2(\mathcal{D}(\mathcal{P}), \mathbf{p}) \longrightarrow K^{\mathbf{p}} \longrightarrow B_1(\mathcal{D}(\mathcal{P})) \to 0 \tag{2.23}$$

corresponding to $(\mathcal{D}(\mathcal{P}), \mathbf{p})$, where $\mathbf{p}$ is a set of bimodule generators of $H_1(\mathcal{D})$.
Finally using the "Snake Lemma", in an identical fashion as in Theorem 2.1.2, one gets the basic short exact sequence

$$0 \longrightarrow H_2(\mathcal{D}, \mathbf{p}) \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \xrightarrow{\nu} H_1(\mathcal{D}) \longrightarrow 0$$

which can then be used to define FDT_2, FHT_2 and prove the independence of them from the presentation.

Despite the confusing notation, $\mathbf{p}^{\infty}$ and $\mathbf{p}$ are not related with each other, but Lemma 13 of [72] states that, if (2.23) is exact for some homology trivializer $\mathbf{q}$ of $H_1(\mathcal{D})$, then it stays exact for any other, say $\mathbf{p}$. The key to proving that lemma is that every 2-cycle from $K^{\mathbf{p}}$ is homologous to some 2-cycle in $C_2(\mathcal{D})$ and it is here that we use the other trivializer $\mathbf{q}$ to express this cycle as the boundary of some 3-chain in $C_3(\mathcal{D}, \mathbf{q})$. Then it is not difficult to see that the cycle we started with is a boundary of a 3-chain in $C_3(\mathcal{D}(\mathcal{P}), \mathbf{p})$, proving the exactness of (2.23).

In the higher dimensional case, two different choices of the set of the trivializers for the first homology groups, say $\mathbf{p}_1$ and $\mathbf{q}_1$, give rise to two different 3-complexes and therefore, in each case, the corresponding $\mathbf{p}_2$ and $\mathbf{q}_2$ will be different. Continuing the construction of complexes up to dimension n, we end up with two different complexes $\Delta_n(\mathbf{p}) = (\mathcal{D}, \mathbf{p}_1, ..., \mathbf{p}_{n-2}) \sqcup F.\mathbf{p}_{n-1}.F$ and $\Delta_n(\mathbf{q}) = (\mathcal{D}, \mathbf{q}_1, ..., \mathbf{q}_{n-2}) \sqcup F.\mathbf{q}_{n-1}.F$. The use of the techniques of Lemma 13 of [72] becomes useless in this case, but we believe that it is possible to find a relation between these complexes which would then enable us to show the equivalence of the respective short exact sequences analogues of (2.23).

2.4 FDT and FHT for Groups

It is already known that FDT and FHT are equivalent for groups. Different proofs can be found for example in [33] and also in [20], [86]. We show this equivalence in a different way, using the topological settings established so far.

Let G be a *group* given by a *monoid* presentation

$$\mathcal{P} = [\mathbf{x} \cup \mathbf{x}^{-1} : \mathbf{r}, (xx^{-1} = \lambda = x^{-1}x)] \tag{2.24}$$

and let $\mathcal{C}$ be the *cubical* ∞-complex constructed as follows. Let $\mathcal{C}_1 = \Gamma$ be the usual graph associated with $\mathcal{P}$. Take $\Gamma \times \Gamma$ and quotient it by $\sim_2 = \sim$ introduced in Section 2.2.2 obtaining a 2-complex $\mathcal{C}_2$. In fact $\mathcal{C}_2$ is just the Squier complex $\mathcal{D} = \mathcal{D}(\mathcal{P})$. Then take $(\mathcal{D} \times \mathcal{D})^{(3)}$ and quotient it by the appropriate $\sim_3$ obtaining a 3-complex $\mathcal{C}_3$ which contains $\mathcal{D}$. Recursively we obtain an increasing sequence of complexes

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset ... \subset \mathcal{C}_n \subset \mathcal{C}_{n+1} \subset ...$$

of respective dimensions $1, 2, 3, ..., n, n+1, ...$ where

$$\mathcal{C}_{n+1} = (\mathcal{C}_n \times \mathcal{C}_n)^{(n+1)} / \sim_{n+1}$$

and let

$$\mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}_n.$$

Note that for every $n \geq 1$ there is always a quotient map

$$p_{n+1} : (\mathcal{C}_n \times \mathcal{C}_n)^{(n+1)} \longrightarrow \mathcal{C}_{n+1}$$

arising from $\sim_{n+1}$. The n-cells of $\mathcal{C}$ with $n \geq 2$, are those of the form $e_1 \otimes ... \otimes e_n$ with e_i positive edges and can be thought of as cubes with either top cell $(e_1 \otimes ... \otimes e_{n-1}).\iota e_n$ and bottom cell $(e_1 \otimes ... \otimes e_{n-1}).\tau e_n$, or with top cell $\iota e_1.(e_2 \otimes ... \otimes e_n)$ and bottom cell $\tau e_1.(e_2 \otimes ... \otimes e_n)$.

Proposition 2.4.1 *For any other component $\mathcal{D}_\omega$ of the Squier complex of $\mathcal{P}$, we have $\pi_1(\mathcal{D}_\omega) \cong \pi_1(\mathcal{D}_\lambda)$.*

Proof. Since the existence of the n-cells with $n \geq 3$, do not influence the homotopy type of the complex, we need only to show that $\pi_1(\mathcal{C}_\omega) \simeq \pi_1(\mathcal{C}_\lambda)$. We will make use of Proposition 1.18 of [40] and for this we need a homotopy equivalence $f : \mathcal{C}_\omega \longrightarrow \mathcal{C}_\lambda$. We define f to be the map whose restriction on each cell coincides with the map of Property (ii) which in this case sends each cell $\sigma \in \mathcal{C}_\omega$ to $\omega^{-1}.\sigma \in \mathcal{C}_\lambda$. Similarly, define $g : \mathcal{C}_\lambda \longrightarrow \mathcal{C}_\omega$ by sending every cell $\sigma \in \mathcal{C}_\lambda$ to $\omega.\sigma \in \mathcal{C}_\omega$. We will show that $gf \simeq id_{\mathcal{C}_\omega}$ and $fg \simeq id_{\mathcal{C}_\lambda}$. For simplicity we will suppose that ω is a single letter. The proof in general does not differ in essence from the above special case. To show the first homotopy we need to introduce a continuous map $F : I \times \mathcal{C}_\omega \longrightarrow \mathcal{C}_\omega$ such that $F(\{1\} \times \mathcal{C}_\omega) = gf$ and $F(\{0\} \times \mathcal{C}_\omega) = id_{\mathcal{C}_\omega}$. Recall that for every cell $\sigma \in \mathcal{C}_\omega$, $gf(\sigma) = \omega\omega^{-1}.\sigma$ and that from the choice of $\omega \in \mathbf{x}$, we have that $\omega\omega^{-1} = \iota e$ and $\lambda = \tau e$ where e is the edge transforming $\omega\omega^{-1}$ to λ. There is a continuous map F :

$$p(\Phi_e \times id_{\mathcal{C}_\omega}) : I \times \mathcal{C}_\omega \longrightarrow \cup_\sigma e \otimes \sigma \tag{2.25}$$

where Φ_e is the characteristic map of e and p is the map whose restriction on $\mathcal{C}_1 \times \mathcal{C}_n$ for every $n \geq 1$ is p_{n+1}. It is easy to see that F maps $\{1\} \times \sigma \to \iota e.\sigma \in \partial(e \otimes \sigma)$ and $\{0\} \times \sigma \to \tau e.\sigma \in \partial(e \otimes \sigma)$, and when composed with the inclusion $\iota : \cup_\sigma e \otimes \sigma \hookrightarrow \mathcal{C}_\omega$ it gives the desired homotopy F. The second homotopy $fg \simeq id_{\mathcal{C}_\lambda}$ is shown in a similar fashion. ∎

Corollary 2.4.2 *For every ω, we have an isomorphism of groups $\pi_1(\mathcal{D}_\omega) \cong H_1(\mathcal{D}_\omega)$.*

Proof. Indeed, from Lemma 7.4 of [39] we have that $\pi_1(\mathcal{D}_\lambda)$ is abelian and then from Proposition 2.4.1 we obtain that $\pi_1(\mathcal{D}_\omega)$ is abelian for every ω. Since $H_1(\mathcal{D}_\omega)$ is the abelianization of $\pi_1(\mathcal{D}_\omega)$, it follows that $\pi_1(\mathcal{D}_\omega) \cong H_1(\mathcal{D}_\omega)$. ∎

Note that the isomorphism of the above corollary is the Hurewicz morphism $h_1 : \pi_1(\mathcal{D}_\omega) \longrightarrow H_1(\mathcal{D}_\omega)$ as described in Lemma 1.2.9. Since changing the base point of $\mathcal{D}_\omega$ does not alter $\pi_1(\mathcal{D}_\omega)$, we will take the base point to be a vertex of $\mathcal{D}_\omega$. Also, since every closed path in $\mathcal{D}_\omega$ is homotopic with a closed path in the underlying 1-skeleton Γ_ω, then the morphism of Lemma 1.2.9 has a simpler form:

$$[f] \longmapsto \mathrm{cls}\xi_f \tag{2.26}$$

where f is a closed path in Γ_ω with initial and terminal the base point chosen and $\mathrm{cls}\xi_f$ is the homology class of the cycle ξ_f corresponding to f.

Theorem 2.4.3 *For groups, FDT and FHT are equivalent.*

Proof. As we have seen before, FDT implies FHT in general so it remains to show the converse. First we define a bi-action of F on $\oplus_{\omega \in F'}\pi_1(\mathcal{D}_\omega)$ as follows

$$u.[f].v = [u.f.v]$$

where $u, v \in F$ and f is a closed path in Γ_ω. It is easy to see that this action is well defined. Suppose now that $\oplus_{\omega \in F} H_1(\mathcal{D}_\omega)$ is a finitely generated $(\mathbb{Z}G, \mathbb{Z}G)$ bi-module and let $\{\xi_i \mid i = 1, ..., n\}$ be representative cycles of these generators which we may take without restriction to be *polygons* in the sense that the edges represented in each one ξ_i, form a closed path f'_{ξ_i} in Γ. Denote by f_{ξ_i} the path $\gamma f'_{\xi_i} \gamma^{-1}$ where γ is a path in Γ from the base point to any of the vertices represented in f'_{ξ_i}.

Let now f be some closed path in Γ_ω and $\mathrm{cls}\xi_f$ be the corresponding element of $H_1(\mathcal{D}_\omega)$ which from the assumption can be written in the form

$$\mathrm{cls}\xi_f = \sum_j \varepsilon_j \overline{u_j}.\mathrm{cls}\xi_j.\overline{v_j} = \sum_j \varepsilon_j \mathrm{cls}(u_j.\xi_j.v_j)$$

where $\varepsilon_j = \pm 1$, $\mathrm{cls}\xi_j$ are generators and $u_j, v_j \in F$. From (2.26), we have that

$$[f] = \prod_j [f_{u_j.\xi_j.v_j}]^{\varepsilon_j}.$$

But

$$[f_{u_j.\xi_j.v_j}] = [u_j.f_{\xi_j}.v_j] = u_j.[f_{\xi_j}].v_j$$

and then we get

$$[f] = \prod_j (u_j.[f_{\xi_j}].v_j)^{\varepsilon_j}.$$

This means that, if the classes $[f_{\xi_j}]$ are all 0, then every $[f] = 0$ and therefore $\pi_1(\mathcal{D}_\omega) = 0$. In topological terms this means that, if we add 2-cells $\mathbf{p}$ to the complex $\mathcal{D}$ with boundaries the closed paths f_{ξ_j} together with their translates, then we obtain a complex $\mathcal{D}^{\mathbf{P}}$ with $\pi_1(\mathcal{D}^{\mathbf{P}}) = 0$, or in other words, the group is FDT. ■

Chapter 3

Finiteness Conditions for Small Categories

3.1 Introduction

Small categories are sometimes called monoids with several objects. This is due to the simple fact that, if the category has a single object, then it is a monoid. Similarly additive categories generalize unitary rings and are sometimes called rings with several objects. The analogy goes further. For every fixed unitary ring $\mathbf{R}$, the category $\mathbf{R}$-$\mathbf{Mod}$ of left $\mathbf{R}$-modules and $\mathbf{R}$-module morphisms is a special case of the category $Add(\mathbb{A}, \mathbf{Ab})$ of additive functors and natural transformations between them. Here $\mathbb{A}$ is an additive category. In other words, if we replace a ring $\mathbf{R}$ by an additive category $\mathbb{A}$, all the left-$\mathbf{R}$-modules by additive functors from $\mathbb{A}$ to $\mathbf{Ab}$ and all the $\mathbf{R}$-module morphisms by natural transformations between additive functors, then we get the category $Add(\mathbb{A}, \mathbf{Ab})$. Also notions such as free, projective and finitely generated in the category $\mathbf{R}$-$\mathbf{Mod}$ have their natural analogues in the class of functor categories $Add(\mathbb{A}, \mathbf{Ab})$ where again $\mathbb{A}$ is additive.

The main scope of this chapter is to introduce finiteness conditions of a homological nature for small categories which would generalize finiteness conditions for monoids such as bi-FP_n and left (respectively right)-FP_n, and find the relations between them. We make those definitions in Section 3.4.1 but first we recall from [67] the definition of functors $B \in Add(\mathbb{A}, \mathbf{Ab})$ of type FP_n and then we say that a small category $\mathbb{C}$ is of *type bi-FP_n* if a certain functor $\mathbb{Z}\mathbb{C} \in Add(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}, \mathbf{Ab})$ is of type FP_n in $Add(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}, \mathbf{Ab})$. Similarly, we say that a small category $\mathbb{C}$ is of type *left* (respectively *right*)-*FP_n* if a certain functor $\mathbb{Z} \in Add(\mathbb{Z}\mathbb{C}, \mathbf{Ab})$ (respectively

$\mathbb{Z} \in Add(\mathbb{Z}\mathbb{C}^{opp}, \mathbf{Ab})$) is of type FP_n in $Add(\mathbb{Z}\mathbb{C}, \mathbf{Ab})$ (respectively in $Add(\mathbb{Z}\mathbb{C}^{opp}, \mathbf{Ab})$). As it is expected, we show that the implication bi-$\mathrm{FP}_n \Longrightarrow$ left (right)-FP_n holds true. More precisely we prove the following.

Theorem 3.4.5 *For every small category $\mathbb{C}$ the following implication holds true:*

$$\text{bi-FP}_n \Longrightarrow \text{left (right)-FP}_n.$$

In [67] Malbos claims that one can obtain a projective finitely generated resolution of the trivial left $\mathbb{Z}\mathbb{C}$ -module $\mathbb{Z}$ in $Add(\mathbb{Z}\mathbb{C}, \mathbf{Ab})$ by applying the left additive Kan extension functor $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}}$ _ to a projective finitely generated resolution of the $(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})$-bimodule $\mathbb{Z}\mathbb{C}$ in $Add(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}, \mathbf{Ab})$. In fact he does not give a proof for this. It seems that he is referring to the Corollary 10.5 of [74] which states that, if *$\mathcal{C}$ is an additive category and*

$$\ldots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow G \longrightarrow 0$$

is an exact sequence of projectives in $Add(\mathcal{C}^{opp}, \mathbf{Ab})$, and if $F \in Add(\mathcal{C}, \mathbf{A})$ where $\mathbf{A}$ is an abelian category with coproducts, then

$$\ldots \longrightarrow F \otimes_{\mathcal{C}} X_n \longrightarrow F \otimes_{\mathcal{C}} X_{n-1} \longrightarrow \ldots \longrightarrow F \otimes_{\mathcal{C}} X_1 \longrightarrow F \otimes_{\mathcal{C}} X_0 \longrightarrow F \otimes_{\mathcal{C}} G \longrightarrow 0$$

is exact in $\mathbf{A}$.

This can be adapted to work for the category $Add(\mathcal{C}^{opp} \otimes_{\mathbb{Z}} \mathcal{D}, \mathbf{Ab})$ instead of $Add(\mathcal{C}^{opp}, \mathbf{Ab})$ with $\mathcal{D}$ additive and for $Add(\mathcal{C}, \mathbf{Ab})$ instead of $Add(\mathcal{C}, \mathbf{A})$, but still the condition that $G \in Add(\mathcal{C}^{opp} \otimes_{\mathbb{Z}} \mathcal{D}, \mathbf{Ab})$ is projective will not be satisfied in our case, because in that case we have that $G = \mathbb{Z}\mathbb{C} \in Add(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}, \mathbf{Ab})$ and $\mathbb{Z}\mathbb{C}$ is not projective in general. For example, if $\mathbb{C}$ is a group $\mathbb{G}$ such that $\mathbb{G} \neq [\mathbb{G}, \mathbb{G}]$, then we know that $\mathbb{Z}\mathbb{G}$ is not a projective $(\mathbb{Z}\mathbb{G}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{G})$-bimodule.

Besides that, one still need to compute $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}} \mathbb{Z}\mathbb{C}$ and $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}} X_k$ for every $k \geq 0$ and show that indeed $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}} \mathbb{Z}\mathbb{C} \cong \mathbb{Z}$ and $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}} X_k$ are projective and finitely generated if X_k are such.

There is also a possibility that one may use Proposition 11.8 of [74] which states that, if *$\mathcal{C}$ and $\mathcal{D}$ are K-projective K-categories, and X be a projective resolution for F in $Add(\mathcal{C}, \mathbf{A})$, and Y be a projective resolution for G in $Add(\mathcal{C}^{opp} \otimes_{\mathbb{Z}} \mathcal{D}, \mathbf{Ab})$, then $X \otimes_{\mathcal{C}} Y$ is a projective resolution for $F \otimes_{\mathcal{C}} G$ in $Add(\mathcal{D}, \mathbf{Ab})$ provided that $Tor_n^{\mathcal{C}}(F, G) = 0$ for all $n > 0$.*

Again, this approach would need to compute explicitly the tensor $X \otimes_{\mathcal{C}} Y$ to check for finite generation and for the condition $Tor_n^{\mathcal{C}}(F, G) = 0$.

For this reason we decided to give here our own proof. To prove our Theorem 3.4.5, we generalize the techniques introduced in [53] to show that for monoids the condition bi-FP_n

implies left (right)-FP_n. Since this techniques requires the tensor product of modules, we give in Section 3.3 the notion of the tensor product of functors which, as can be easily verified, coincides with the tensor product of modules if the functors happen to be modules.

In [28] Dwyer and Kan introduced the notion of the category of factorizations $F\mathbb{C}$ of a small category $\mathbb{C}$. Its objects are the morphisms of $\mathbb{C}$ and a morphism $\omega \longrightarrow \omega'$ is a pair (u, v) of morphisms in $\mathbb{C}$ such that $\omega' = v\omega u$. One can study what are called in [7] *natural systems of abelian groups* on $\mathbb{C}$ which are functors $D : F\mathbb{C} \longrightarrow \mathbf{Ab}$. Every such functor extends to an additive functor $D' : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ where $\mathbb{Z}F\mathbb{C}$ is the free additive category on $F\mathbb{C}$. In contrast with $Add(\mathbb{Z}\mathbb{C}, \mathbf{Ab})$ whose object are functors associating with each object of $\mathbb{C}$ an abelian group, the functors of the category $Add(\mathbb{Z}F\mathbb{C}, \mathbf{Ab})$ associate with each morphism in $\mathbb{C}$ to an abelian group. In the case of monoids, the difference between these two categories is apparent and one can expect to have finiteness conditions of a new nature if working with the second category.

In Section 3.4.2 we deal with small categories of type f-FP_n defined as those small categories $\mathbb{C}$ with the property that a certain functor $\mathbb{Z} \in Add(\mathbb{Z}F\mathbb{C}, \mathbf{Ab})$ called there the trivial natural system, is of type FP_n in $Add(\mathbb{Z}F\mathbb{C}, \mathbf{Ab})$. In fact what we call here type f-FP_n, is introduced from Malbos in [67] and called there type FP_n. This is slightly confusing with properties bi-FP_n or left and right-FP_n, so we decided to change its name to f-FP_n with f- standing for factorization.

To relate properties bi-FP_n and left (respectively right)-FP_n with FP_n, we prove the following.

Theorem 3.4.10 *If a small category $\mathbb{C}$ is of type f-FP_n, then it is of type bi-FP_n.*

Regarding to monoids seen as categories, we prove the following.

Theorem 3.4.12 *If the monoid S is of type bi-FP_n and the corresponding free partial resolution is S-graded, then S is of type f-FP_n. In particular, monoids which are given by a finite complete presentation are of type f-FP_n.*

In Section 3.5, we look for ways to build partial resolutions for the trivial functor $\mathbb{Z} \in Add(\mathbb{Z}F\mathbb{C}, \mathbf{Ab})$. **Theorem 3.5.2** gives a resolution of length 3 and implicitly a condition for a category to be of type f-FP_3. The finiteness of that resolution is related with a property which we call FDT for small categories and is defined in a similar fashion to FDT for monoids (see [85] or [96]). More precisely, we prove the following.

Theorem 3.5.3 *If $\mathbb{C}$ is of type FDT, then $\mathbb{C}$ is of type f-FP_3.*

The proof of Theorem 3.5.2, which is the category version of that of Theorem 3.2 of [19], deserves a little bit more attention. The exact sequence of functors we construct

$$\mathbb{Z}\mathbb{C}[B] \xrightarrow{\delta_3} \mathbb{Z}\mathbb{C}[\mathbf{r}] \xrightarrow{\delta_2} \mathbb{Z}\mathbb{C}[\mathbf{x}] \xrightarrow{\delta_1} B_0(\mathbb{C}) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{3.1}$$

is related to three graphs. The first one is $UF(\mathbf{x})$, the underlaying graph of the free category $F(\mathbf{x})$. The second is the Squier graph $\Gamma(\mathbf{x}, \mathbf{r})$ which rewrites the paths of $UF(\mathbf{x})$ by using rules from $\mathbf{r}$ seen as parallel paths in $UF(\mathbf{x})$. The third one is $\Delta(\mathbf{x}, \mathbf{r}, B)$ which, similarly with the first two, rewrites the paths of $\Gamma(\mathbf{x}, \mathbf{r})$ by using rules from B consisting of parallel paths from $\Gamma(\mathbf{x}, \mathbf{r})$. The expectation is that, if we want to extend the sequence (3.1) further, we have to extend the above sequence of graphs further by introducing rewrite rules of paths of the current graph at each stage, but the notations become complicated and the boundary transformations δ_n are difficult to compute for $n > 3$.

3.2 Basic Notions from Category Theory

3.2.1 Categories and Functors

We will give in this section a few basic notions from Category Theory which are the essential ingredients to understand the work in the two last sections. One can find the relevant material in books like [64], [75] and [93] or in [66], [67], [74] and [76]. There is also a very helpful treatment of additive and abelian categories in Chapter 7 of [82].

Definition 3.2.1 A *category* $\mathbb{C}$ is a class $\mathcal{O}$, together with a class $\mathcal{M}$ which is a disjoint union of the form

$$\mathcal{M} = \bigcup_{(a,b)\in\mathcal{O}\times\mathcal{O}} \hom_{\mathbb{C}}(a, b).$$

We call the members of $\mathcal{M}$ *morphisms* or *arrows* and those of $\mathcal{O}$ *objects*. For each triple of objects $(a, b, c) \in \mathcal{O} \times \mathcal{O} \times \mathcal{O}$, there is a function $\hom_{\mathbb{C}}(b, c) \times \hom_{\mathbb{C}}(a, b) \longrightarrow \hom_{\mathbb{C}}(a, c)$. The image of the pair (β, α) under this function will be called the *composition* of β and α, and will be denoted by $\beta\alpha$. The composition satisfies the following two axioms.

(i) *Associativity*: Whenever the compositions are defined, we have $(\gamma\beta)\alpha = \gamma(\beta\alpha)$.

(ii) *Existence of identity*: For each $a \in \mathcal{O}$ we have an element $1_a \in \hom_{\mathbb{C}}(a, a)$ such that $1_a\alpha = \alpha$ and $\beta 1_a = \beta$ whenever the composition is defined.

If there is no danger of confusion, we write $\mathbb{C}(a, b)$ instead of $\hom_{\mathbb{C}}(a, b)$ or sometimes $\hom(a, b)$. Note that $\hom(a, b)$ may be empty for certain pairs (a, b). For each morphism

$\alpha \in \hom_{\mathbb{C}}(a,b)$, we say that a is the *domain* of α and b is the *codomain* of α. Any two morphisms $\alpha, \beta \in \mathbb{C}(a,b)$ are called *parallel*. If only the morphism α is given and we want to indicate the domain and the codomain of it, we use $\iota\alpha$ for the domain and $\tau\alpha$ for the codomain.

If $\mathcal{O}$ and $\mathcal{M}$ are sets, then we call the category *small*.

We will write $c \in \mathbb{C}$ to mean that c is an *object* of $\mathbb{C}$ and the objects of a category will be denoted by the first letters of the Latin alphabet a, b, c, $d, \ldots$. For the morphisms, we use letters e, f, g, h, or Greek letters α, β, γ, δ.

Definition 3.2.2 We say that a category $\mathbb{C}'$ is a *subcategory* of $\mathbb{C}$ if

(i) $\mathbb{C}' \subseteq \mathbb{C}$.

(ii) $\hom_{\mathbb{C}'}(a,b) \subseteq \hom_{\mathbb{C}}(a,b)$ for all $(a,b) \in \mathbb{C}' \times \mathbb{C}'$.

(iii) The composition of two arrows in $\mathbb{C}'$ is the same as their composition in $\mathbb{C}$.

(iv) 1_a is the same in $\mathbb{C}'$ as in $\mathbb{C}$ for every $a \in \mathbb{C}'$.

If furthermore $\hom_{\mathbb{C}'}(a,b) = \hom_{\mathbb{C}}(a,b)$ for all $(a,b) \in \mathbb{C}' \times \mathbb{C}'$ we say that $\mathbb{C}'$ is a *full* subcategory of $\mathbb{C}$.

Below is a list of some well known examples of categories.

Set: the category of all sets and functions between them.

Grp: the category of all groups and group homomorphisms between them.

Ab: the category of all abelian groups and group homomorphisms between them.

Rings: the category of all rings and ring homomorphisms between them.

R-Mod: the category of all left **R**-modules and module homomorphisms between them.

Top: the category of all topological spaces and continuous functions between them.

Top$_*$: the category of all topological spaces with a base point and continuous functions between them which are base point preserving.

Toph: the category of all topological spaces and homotopy classes of functions between them.

Definition 3.2.3 If $\mathbb{B}$ and $\mathbb{C}$ are two categories, we can construct the *product category* $\mathbb{B} \times \mathbb{C}$ having objects pairs $\langle b, c \rangle \in \mathbb{B} \times \mathbb{C}$ with $b \in \mathbb{B}$, $c \in \mathbb{C}$, and hom-sets consisting of arrows

$$\langle b, c \rangle \xrightarrow{\langle f, g \rangle} \langle b', c' \rangle,$$

where $f \in \mathbb{B}(b, b')$, $g \in \mathbb{C}(c, c')$. The composition of two arrows

$$\langle b, c\rangle \xrightarrow{\langle f,g\rangle} \langle b', c'\rangle \xrightarrow{\langle f',g'\rangle} \langle b'', c''\rangle$$

is defined to be $\langle f', g'\rangle \langle f, g\rangle = \langle f'f, g'g\rangle$.

Definition 3.2.4 To each category $\mathbb{C}$ we assign a category denoted by $\mathbb{C}^{opp}$ whose objects and morphisms are in a 1-1 correspondence with those of $\mathbb{C}$ in such a way that to each morphism $\alpha : a \longrightarrow b$ in $\mathbb{C}$ the corresponding morphism in $\mathbb{C}^{opp}$ is $\alpha^{opp} : b \longrightarrow a$. The composition is defined by $\alpha^{opp}\beta^{opp} = (\beta\alpha)^{opp}$ whenever $\beta\alpha$ is defined. The category $\mathbb{C}^{opp}$ is called the *opposite* category of $\mathbb{C}$.

Definition 3.2.5 Let $\mathbb{C}$ and $\mathbb{B}$ be categories. (i) A *covariant functor* $T : \mathbb{C} \longrightarrow \mathbb{B}$ with domain $\mathbb{C}$ and codomain $\mathbb{B}$, consists of two functions: The *object function* T, which assigns to each object $c \in \mathbb{C}$ an object $T(c) \in \mathbb{B}$ and the *arrow function* written again by T, which assigns to each arrow $\alpha : c \longrightarrow c'$ of $\mathbb{C}$ an arrow $T(\alpha) : T(c) \longrightarrow T(c')$ of $\mathbb{B}$. We require that

$$T(1_c) = 1_{T(c)}, \text{ for all } c \in \mathbb{C}$$

and $T(\beta\alpha) = T(\beta)T(\alpha)$ whenever $\beta\alpha$ is defined.

(ii) A *contravariant functor* $T : \mathbb{C} \longrightarrow \mathbb{B}$ again consists of two functions: The *object function* T, which assigns to each object $c \in \mathbb{C}$ an object $T(c) \in \mathbb{B}$ and the *arrow function* written again by T, which assigns to each arrow $\alpha : c \longrightarrow c'$ of $\mathbb{C}$ an arrow $T(\alpha) : T(c') \longrightarrow T(c)$ of $\mathbb{B}$. We require that

$$T(1_c) = 1_{T(c)}, \text{ for all } c \in \mathbb{C}$$

and $T(\beta\alpha) = T(\alpha)T(\beta)$ whenever $\beta\alpha$ is defined.

Remark 3.2.6 In general, if we refer to a functor, we will mean a covariant functor, unless otherwise stated.

Remark 3.2.7 It is easy to see that a functor $T : \mathbb{C} \longrightarrow \mathbb{B}$ is *contravariant* if the functor $\overline{T} : \mathbb{C}^{opp} \longrightarrow \mathbb{B}$, which sends $c \longmapsto T(c)$ and $\alpha^{opp} : b \longrightarrow a$ to $T(\alpha)$, is covariant.

Example 3.2.8 Let $\mathbb{C}$ be a small category. For each fixed object $a \in \mathbb{C}$, the *covariant hom-functor*

$$\mathbb{C}(a, \_) = \hom(a, \_) : \mathbb{C} \longrightarrow \mathbf{Set}$$

sends each object $b \in \mathbb{C}$, to $\hom(a,b) \in \mathbf{Set}$ and each arrow $\beta : b \longrightarrow b'$ to the map

$$\hom(a,\beta) : \hom(a,b) \longrightarrow \hom(a,b')$$

defined by the assignment $f \longmapsto \beta f$ for each $f : a \longrightarrow b$.

Similarly one defines the *contravariant hom-functor*

$$\mathbb{C}(\_,b) = \hom(\_,b) : \mathbb{C} \longrightarrow \mathbf{Set}$$

which sends each object $a \in \mathbb{C}$, to $\hom(a,b) \in \mathbf{Set}$ and each arrow $\alpha : a \longrightarrow a'$ to the map

$$\hom(\alpha,b) : \hom(a',b) \longrightarrow \hom(a,b)$$

defined by $f \longmapsto f\alpha$ for each $f : a' \longrightarrow b$.

Example 3.2.9 (Homology as a Functor) As we mentioned in Section 1.2.1, every continuous map $f : X \longrightarrow Y$ between two topological spaces X and Y induces a morphism $f_* : H_n(X) \longrightarrow H_n(Y)$ between the respective n^{th} homology groups of X and Y, which we now denote by $H_n(f)$. It is easy to see that $H_n : \mathbf{Top} \longrightarrow \mathbf{Ab}$ is a functor, which we call the n^{th} *homology functor.*

Example 3.2.10 (π_1 as a Functor) Also $\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Grp}$ is a functor from based spaces $\mathbf{Top}_*$ to $\mathbf{Grp}$ as can be easily checked.

Example 3.2.11 A functor $U : \mathbb{C} \longrightarrow \mathbb{B}$ which forgets some of the structure of the domain category $\mathbb{C}$, is called *forgetful.* The functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ which sends each group G to its underlying set G and each morphism $f : G \longrightarrow G'$ to the underlying set function $f : G \longrightarrow G'$, is an example of a forgetful functor.

♦

Definition 3.2.12 A functor $T : \mathbb{C} \longrightarrow \mathbb{B}$ is called *faithful (respectively full)* if the function

$$\mathbb{C}(c_1,c_2) \longrightarrow \mathbb{B}(T(c_1),T(c_2))$$

is injective (respectively surjective) for every $c_1, c_2 \in \mathbb{C}$. A full and faithful functor T which is bijective on objects will be called an *isomorphism.* We say that two categories $\mathbb{C}$ and $\mathbb{B}$ are *isomorphic*, denoted by $\mathbb{C} \cong \mathbb{B}$, if there is an isomorphism $T : \mathbb{C} \longrightarrow \mathbb{B}$.

Example 3.2.13 For any two categories $\mathbb{B}$ and $\mathbb{C}$, $\mathbb{B} \times \mathbb{C} \cong \mathbb{C} \times \mathbb{B}$ via the isomorphism

$$\tau : \mathbb{B} \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{B} \tag{3.2}$$

sending $\langle b, c\rangle \longmapsto \langle c, b\rangle$ and $\langle \beta, \gamma\rangle \longmapsto \langle \gamma, \beta\rangle$.

Any functor $S : \mathbb{B} \times \mathbb{C} \longrightarrow \mathbb{D}$ is called a *bifunctor.* If $S : \mathbb{B} \times \mathbb{C} \longrightarrow \mathbb{D}$ is a bifunctor and $b \in \mathbb{B}$ (respectively $c \in \mathbb{C}$) are fixed objects, then we have the "obvious" induced *partial* functors $S(b, \_) : \mathbb{C} \longrightarrow \mathbb{D}$ (respectively $S(\_, c) : \mathbb{B} \longrightarrow \mathbb{D}$).

Definition 3.2.14 An arrow $\alpha : a \longrightarrow b$ in a category $\mathbb{C}$ is called *invertible* if there is $\beta : b \longrightarrow a$ such that $\alpha\beta = 1_b$ and $\beta\alpha = 1_a$. If such an arrow exists, then we call objects a and b *isomorphic* and denote this fact by $a \cong b$.

Definition 3.2.15 Let F_1, $F_2 : \mathbb{C} \longrightarrow \mathbb{B}$ be two functors. A *natural transformation* from F_1 to F_2, is a family of morphisms $\partial_c : F_1(c) \longrightarrow F_2(c)$, $c \in \mathbb{C}$ such that, for every $c, c' \in \mathbb{C}$ and every morphism $e : c \longrightarrow c'$, the diagram

$$\begin{tikzcd}
F_1(c) \arrow[r, "\partial_c"] \arrow[d, "F_1(e)"'] & F_2(c) \arrow[d, "F_2(e)"] \\
F_1(c') \arrow[r, "\partial_{c'}"'] & F_2(c')
\end{tikzcd}$$

commutes.

Whenever the above commutativity occurs, we say that $\partial_c : F_1(c) \longrightarrow F_2(c)$ is *natural* in c. A natural transformation with every ∂_c invertible, is called a *natural isomorphism.*

Example 3.2.16 In Example 3.2.9 take $\mathbf{Top}_*$ instead of $\mathbf{Top}$. The family

$$h_1(X, x) : \pi_1(X, x) \longrightarrow H_1(X, x)$$

of Hurewicz morphisms for every $X \in \mathbf{Top}_*$ is a natural transformation from π_1 to H_1.

♦

Definition 3.2.17 Let $\mathbb{B}$ and $\mathbb{C}$ be categories. The *functor category* $\mathbb{C}^{\mathbb{B}}$ has objects all functors from $\mathbb{B}$ to $\mathbb{C}$ and each hom-set $\mathbb{C}^{\mathbb{B}}(S, T)$ consists of all natural transformations from S to T. The composition of natural transformations is defined as follows. If $\sigma : R \longrightarrow S$ and $\tau : S \longrightarrow T$ are natural transformations, then their components for each b define arrows $(\tau \cdot \sigma)_b = \tau_b \circ \sigma_b$ which are the components of a natural transformation $\tau \cdot \sigma : R \longrightarrow T$. Sometimes when we need to simplify the notations and, if there is no confusion, we denote the hom-set $\mathbb{C}^{\mathbb{B}}(S, T)$ by $Nat(S, T)$.

We usually denote the objects of a functor category $\mathbb{C}^{\mathbb{B}}$ by Latin upper case letters.

One can define the *evaluation (bi)functor*

$$Eval : \mathbb{C}^{\mathbb{B}} \times \mathbb{B} \longrightarrow \mathbb{C} \tag{3.3}$$

which is defined on objects by the assignment: $(F, b) \in \mathbb{C}^{\mathbb{B}} \times \mathbb{B} \longmapsto F(b) \in \mathbb{C}$ and on morphisms as the diagonal of the commutative diagram

$$\begin{tikzcd} F(b) \arrow[r, "\alpha_b"] \arrow[d, "T(f)"'] \arrow[dr] & G(b) \arrow[d, "G(f)"] \\ F(b') \arrow[r, "\alpha_{b'}"] & G(b') \end{tikzcd}$$

An important notion in Category Theory is that of representable functors.

Definition 3.2.18 Let $\mathbb{D}$ be a category with small hom-sets. A *representation* of a functor $K : \mathbb{D} \longrightarrow \mathbf{Set}$ is a pair $\langle r, \psi \rangle$, with r an object of $\mathbb{D}$ and

$$\psi : \mathbb{D}(r, \_) \longrightarrow K$$

a natural bijection. The object r is called the *representing object*. The functor K is said to be *representable* if such representation exists. We will denote a natural bijection by $\approx$.

Example 3.2.19 Let $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ be the forgetful functor. The free cyclic group $\mathbb{Z}$ is a representing object of U where the natural transformation $\psi : \mathbf{Grp}(\mathbb{Z}, \_) \longrightarrow U$ is the family of mappings ψ_G for every $G \in \mathbf{Grp}$, sending each morphism $f : \mathbb{Z} \longrightarrow G$ to $f(1) \in U(G)$.

There is an elegant way of realizing in the general situation the natural transformations $\psi : \mathbb{D}(r, \_) \longrightarrow K$, given by the Yoneda Lemma below.

Lemma 3.2.20 (Yoneda) *If $K : \mathbb{D} \longrightarrow \mathbf{Set}$ is a functor from $\mathbb{D}$ and $r \in \mathbb{D}$ (for $\mathbb{D}$ a category with small hom-sets), there is a bijection*

$$Y : Nat(\mathbb{D}(r, \_), K) \approx K(r)$$

which sends each natural transformation $\alpha : \mathbb{D}(r, \_) \longrightarrow K$ to $\alpha_r(1_r)$, the image of the identity $r \longrightarrow r$. Furthermore, Y is natural in both r and K.

By a careful inspection of the following commutative diagram, one can see that every natural transformation from $\mathbb{D}(r, \_)$ to K is uniquely determined by the image under the arrow

$$\alpha_r : \mathbb{D}(r, r) \longrightarrow K(r)$$

of the identity $1_r : r \longrightarrow r$.

$$\begin{tikzcd} D(r,r) \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"'] & K(r) \arrow[d, "K(f)"] \\ D(r,d) \arrow[r, "\alpha_d"] & K(d) \end{tikzcd}$$

We call the map $Y : Nat(\mathbb{D}(r,\_), K) \longrightarrow K(r)$ of the lemma, the *Yoneda map.*

Theorem 3.2.21 *The Yoneda map is a bifunctor isomorphism*

$$Y : Nat(\mathbb{D}(\_,\_),\_) \longrightarrow E\tau(\_,\_)$$

where E is the evaluation functor and τ is the isomorphism (3.2).

For the proof see Theorem 4.2.2 of [93]. We will return to the Yoneda Lemma later when we study additive categories.

3.2.2 Special Objects and Special Morphisms

We will devote this section to some special types of objects and morphisms in general categories which will be of particular interest in functor categories $Add(\mathbb{C}, \mathbf{Ab})$ in later sections.

Definition 3.2.22 An arrow $\alpha : a \longrightarrow b$ in a category $\mathbb{C}$ is *epi* if for any two arrows $\beta_1, \beta_2 : b \longrightarrow c$, the equality $\beta_1\alpha = \beta_2\alpha$ implies that $\beta_1 = \beta_2$. We will denote an epi in the future by $\twoheadrightarrow$. In this case b is called a *quotient* object of a.

Example 3.2.23 In **Set** epi arrows coincide with surjections. In **Grp** epi arrows coincide with surjective group homomorphisms. See Exercise 5, p. 21, [64].

Definition 3.2.24 A functor $T : \mathbb{C} \longrightarrow \mathbb{B}$ will be called an *epifunctor*, if $T(\gamma)$ is epi whenever γ is epi.

Definition 3.2.25 An arrow $\alpha : a \longrightarrow b$ in a category $\mathbb{C}$ is *mono* if for any two arrows $\beta_1, \beta_2 : c \longrightarrow a$, the equality $\alpha\beta_1 = \alpha\beta_2$ implies that $\beta_1 = \beta_2$. We will denote a monic in the future by $\rightarrowtail$. In this case a will be called a *subobject* of b.

Example 3.2.26 In **Set** and **Grp**, monomorphisms are precisely the injections.

Definition 3.2.27 An object $t \in \mathbb{C}$ is called *terminal* in $\mathbb{C}$ if for each $a \in \mathbb{C}$ there is exactly one arrow $a \longrightarrow t$. An object $s \in \mathbb{C}$ is called *initial* in $\mathbb{C}$ if for each $a \in \mathbb{C}$ there is exactly one arrow $s \longrightarrow a$. A *null* or *zero* object $z \in \mathbb{C}$ is an object which is at the same time initial and terminal.

Example 3.2.28 In **Set** the empty set is an initial object and every one-point set is a terminal object. In **Grp** the trivial group $\{1\}$ is a null object.

Definition 3.2.29 If $\mathbb{C}$ has a zero object 0, then for any two objects $a, b \in \mathbb{C}$ there is exactly one morphism $a \longrightarrow b$ which factors through 0; that is, it can be represented in the form $a \longrightarrow 0 \longrightarrow b$. We call this the *zero morphism* and denote it by $0_{a,b}$. It does not depend on the choice of the zero.

Example 3.2.30 In **Grp** the one element group $\{1\}$ is a null object and for every two groups A and B, the morphism $\theta : A \longrightarrow B$ which maps every element of A to the identity of B is a zero morphism.

Definition 3.2.31 Let $\mathbb{C}$ be a category. We call an object $c \in \mathbb{C}$ *projective* if, for every morphism $f : c \longrightarrow b$ and every epimorphism $\mu : a \twoheadrightarrow b$, there is a morphism $g : c \longrightarrow a$ such that the following diagram commutes.

$$\begin{tikzcd} & c \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ a \arrow[r, two heads, "\mu"'] & b \end{tikzcd}$$

This is equivalent to saying that $\mathbb{C}(c, \_)$ is an epifunctor.

Example 3.2.32 In **Set** every object is projective and in **Grp** (respectively **Ab**) projectives coincide with free groups (respectively free abelian groups) ([63], p. 2).

Definition 3.2.33 If every object in a category $\mathbb{C}$ is a quotient object of a projective, then we say that $\mathbb{C}$ has *enough* projectives.

Example 3.2.34 The categories **Ab**, **Grp** and **R-Mod** have enough projectives.

Definition 3.2.35 Let $\mathbb{C}$ be a category and $\{c_i\}_{i \in I}$ be a family of objects in $\mathbb{C}$. A *coproduct* of this family is a family of morphisms $\{u_i : c_i \longrightarrow c\}$, called *injections*, such that for each family of morphisms $\{\alpha_i : c_i \longrightarrow c'\}$ there is a unique morphism $\alpha : c \longrightarrow c'$ with $\alpha u_i = \alpha_i$ for all $i \in I$. The object c is unique up to isomorphism and will be denoted by $\oplus_{i \in I} c_i$.

Dually one can define the *product* of a family $\{c_i\}_{i \in I}$ of objects of $\mathbb{C}$ as follows:

Definition 3.2.36 It is a family of morphisms $\{p_i : c \longrightarrow c_i\}$, called *projections*, such that for any family of morphisms $\{\alpha_i : c' \longrightarrow c_i\}$ there is a unique morphism $\alpha : c' \longrightarrow c$ with $p_i\alpha = \alpha_i$ for all $i \in I$. The object c is unique up to isomorphism and will be denoted by $\times_{i\in I}c_i$.

Example 3.2.37 The coproduct of a family of modules (respectively abelian groups) $\{A_i\}_{i\in I}$ exists in **R-Mod** (respectively **Ab**) and is equal with $\oplus_{i\in I}A_i$. Products also exist in the respective categories and are just direct products denoted usually by $\prod_{i\in I} A_i$.

Definition 3.2.38 A set $\mathcal{G}$ of objects in a category $\mathbb{C}$ is called a *generating set* if, for every pair of different parallel morphisms $\alpha, \beta : a \longrightarrow b$, there is a morphism $\gamma : g \longrightarrow a$ with $g \in \mathcal{G}$ such that $\alpha\gamma \neq \beta\gamma$. An object g is called a *generator* if $\{g\}$ is a generating set.

Example 3.2.39 Every one point set generates **Set**, $\mathbb{Z}$ generates **Ab** and **Grp**, and **R** generates **R-Mod**. As we will see later in Theorem 3.2.67, for any small category $\mathbb{C}$, the functor category $\mathbf{Ab}^{\mathbb{C}}$ is generated by the set $\{\mathbb{C}(c, \_) \mid c \in \mathbb{C}\}$.

Definition 3.2.40 We call an object of a category *finitely generated* with respect to a family of generators $\{g_i\}_{i\in I}$ if it is a quotient object of a finite coproduct of the form $\overset{n}{\underset{k=1}{\oplus}} g_{i_k}$ where $i_k \in I$ for $k = 1, ..., n$. We call the object *free* with respect to the above family if it is of the form $\oplus_{k\in K}g_{i_k}$ where $i_k \in I$ for $k \in K$.

Remark 3.2.41 The existence of a family of generators $\mathcal{G}$ for a category $\mathbb{C}$ which has coproducts does not imply that every element of that category is a coproduct of elements from $\mathcal{G}$. For example, the empty set $\emptyset$ is a generating objects in the poset $\mathcal{P}(S)$ where S is a non-empty set, but it is not true that for every $a \in S$, $a \neq \emptyset$ we have $a = \oplus_{k\in K}\emptyset$ with $K \neq \emptyset$.

There is also a more algebraic definition of free objects in a category satisfying a specific condition. Before that we need the following.

Definition 3.2.42 Let $\mathbb{A}$ and $\mathbb{X}$ be categories. An *adjunction* from $\mathbb{X}$ to $\mathbb{A}$ is a triple $\langle F, G, \varphi\rangle : \mathbb{X} \longrightarrow \mathbb{A}$, where F and G are functors

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{A}\,,$$

while φ is a function which assigns to each pair of objects $x \in \mathbb{X}$ and $a \in \mathbb{A}$ a bijection of sets

$$\varphi = \varphi_{x,a} : \mathbb{A}(Fx, a) \approx \mathbb{X}(x, Ga)$$

which is natural in x and a. We say that F is a *left-adjoint* for G and G is a *right-adjoint* for F. This will be denoted for short by $F \dashv G$.

Definition 3.2.43 Let $\mathbb{C}$ be a category and let $U : \mathbb{C} \longrightarrow \mathbf{Set}$ be a faithful functor. Suppose that there exists a functor $Fr : \mathbf{Set} \longrightarrow \mathbb{C}$ such that $Fr \dashv U$. Then, for every set S, $Fr(S)$ is called the *free object* on S (relative to U).

For such a category $\mathbb{C}$ as above, one can easily show that the set

$$\{Fr(S) \mid \text{with } S \text{ non-empty}\},$$

is a set of generators of $\mathbb{C}$. This follows easily from the existence of the natural bijection $\mathbb{C}(Fr(S), c) \approx \mathbf{Set}(S, U(c))$ for any $c \in \mathbb{C}$, $S \in \mathbf{Set}$, and from the fact that together with one-point sets of $\mathbf{Set}$, all non-empty sets generate $\mathbf{Set}$. On the other hand, Proposition 10.6 of [41] shows that coproducts of free objects are free objects. Therefore any coproduct of generators $Fr(S)$ yields a free object in our new sense as well. As for the connection between free and projective objects in a category we give the following.

Lemma 3.2.44 *If the functor $U : \mathbb{C} \longrightarrow \mathbf{Set}$ sends epimorphisms to surjections, then every free object in $\mathbb{C}$ is projective.*

This is proved in Corollary 10.3 of [41].

The following lemma will be useful later.

Lemma 3.2.45 *For every small category $\mathbb{C}$, the category $\mathbf{Ab}^{\mathbb{C}}$ contains the coproducts and the products of every family of objects $\{T_i\}_{i \in I}$.*

Proof. We give the proof for coproducts only because the proof for products is the dual. For the family of objects $\{T_i\}_{i \in I} \in \mathbf{Ab}^{\mathbb{C}}$ we define for every $c \in \mathbb{C}$, $T(c) = \oplus_{i \in I} T_i(c)$ and, if $f : c \longrightarrow c'$, then we define $T(f) = \oplus_{i \in I} T_i(f)$. In the following prism

$$\begin{tikzcd}
T_i(c) \arrow[rr, "\alpha_{i,c}"] \arrow[dr, "u_{i,c}"'] \arrow[dd, "T_i(f)"'] & & T'(c) \arrow[dd, "T'(f)"] \\
& T(c) \arrow[ur, "\alpha_c"'] \arrow[dd, "T(f)"] & \\
T_i(c') \arrow[rr, "\alpha_{i,c'}"] \arrow[dr, "u_{i,c'}"'] & & T'(c') \\
& T(c') \arrow[ur, "\alpha_{c'}"'] &
\end{tikzcd}$$

we have by definition

$$\alpha_c(\sum_i x_i) = \sum_i \alpha_{i,c}(x_i),$$

for $x_i \in T_i(c)$ and $i \in J$ where J is a finite subset of I, and therefore the top and the bottom of the diagram are commutative. Also the back and the left hand side squares are commutative since α_i and u_i are natural transformations. The only thing we have to show is the commutativity of the right hand side square, thus proving the naturality of α. Indeed,

$(T'(f) \circ \alpha_c)(\sum_i x_i) = T'(f)(\sum_i \alpha_{i,c}(x_i))$ (from above)

$= \sum_i (T'(f) \circ \alpha_{i,c})(x_i) = \sum_i (\alpha_{i,c'} \circ T_i(f))(x_i)$ (the front square commutes)

$= (\alpha_{c'} \circ T'(f))(\sum_i x_i)$ (from the definition of the maps $T'(f)$ and $\alpha_{c'}$). ■

Definition 3.2.46 Let $\mathbb{A}$ be a category with a null object 0, and let $\alpha : a \longrightarrow b$. We will say that $u : k \longrightarrow a$ is the *kernel* of α, denoted by $Ker\alpha$, if $\alpha u = 0$, and if for every morphism $u' : k' \longrightarrow a$ such that $\alpha u' = 0$ we have a unique morphism $\gamma : k' \longrightarrow k$ such that $u\gamma = u'$. Equivalently, the kernel of α is given by the following pullback diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d, "u"'] & 0 \arrow[d] \\ a \arrow[r, "\alpha"'] & b \end{tikzcd}$$

Definition 3.2.47 Let $\mathbb{A}$ be a category with a null object 0, and let $\alpha : a \longrightarrow b$. Define the *cokernel* of α, denoted by $coker\alpha$, to be the opposite of the kernel of α^{opp} in $\mathbb{A}^{opp}$. In other words, it is given by the following pushout diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\ b \arrow[r, "p"'] & c \end{tikzcd}$$

Example 3.2.48 We can employ the above definition to describe kernels in **Grp** since it has a null object which is the one element group $\{1\}$. It turns out that the kernel of an arbitrary morphism $f : G \longrightarrow H$ is the inclusion $N \hookrightarrow G$ where $N = \{x \in G \mid f(x) = 1_H\}$. In **R-Mod** the null object is the trivial module $\{0\}$ and then we have that the kernel of a module morphism $f : A \longrightarrow B$ is the inclusion $K \hookrightarrow A$ where $K = \{x \in A \mid f(x) = 0_B\}$. In **Ab** and **R-Mod** the cokernel of a morphism $f : A \longrightarrow B$ exists and is given by the arrow $B \longrightarrow B/f(A)$.

Definition 3.2.49 Let $\mathbb{A}$ be a category which has a null object and contains kernels and cokernels. The *image* $Im(\alpha)$ of a morphism $\alpha : a \longrightarrow b$ is defined as $Im(\alpha) = ker(coker\alpha)$.

Example 3.2.50 In **Grp** the image of $f : G \longrightarrow H$ is just

$$Im(f) = \{h \in H \mid \exists g \in G : h = f(g)\}.$$

Definition 3.2.51 In a category with a null object and kernels, a *sequence* of two morphisms

$$a \xrightarrow{f} b \xrightarrow{g} c$$

is called *exact* if:

(i) $gf = 0$;

(ii) in the factorization $f = (Kerg)f'$ guaranteed by (i) (see Definition 3.2.46), f' is an epimorphism.

A sequence of morphisms

$$\dots \longrightarrow a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \xrightarrow{f_{n-1}} a_{n-2} \dots$$

is said to be *exact at* a_n if f_{n+1} and f_n satisfy (i) and (ii). It is *exact* if it is exact at every a_n. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow a \xrightarrow{m} b \xrightarrow{p} c \longrightarrow 0.$$

We denote it succinctly by

$$a \overset{m}{\rightarrowtail} b \overset{p}{\twoheadrightarrow} c.$$

Lemma 13.1.4 of [93] shows that the exactness of the sequence $a \xrightarrow{f} b \xrightarrow{g} c$ at b is equivalent to the condition that $Im(f)$ and $Ker(g)$ are equivalent subobjects of b.

Definition 3.2.52 A morphism $p : b \longrightarrow c$ is called a *retraction* if there is $q : c \longrightarrow b$ such that $p \circ q = id_c$. A short exact sequence

$$a \overset{m}{\rightarrowtail} b \overset{p}{\twoheadrightarrow} c$$

is called *split* if p is a retraction.

Definition 3.2.53 Let $\mathbb{C}$ be a category which has a null object and let

$$\begin{array}{rl} (a, \partial) : & \xrightarrow{\partial_{n+2}} a_{n+1} \xrightarrow{\partial_{n+1}} a_n \xrightarrow{\partial_n} a_{n-1} \xrightarrow{\partial_{n-1}} \dots \\ (b, \delta) : & \xrightarrow{\delta_{n+2}} b_{n+1} \xrightarrow{\delta_{n+1}} b_n \xrightarrow{\delta_n} b_{n-1} \xrightarrow{\delta_{n-1}} \dots \end{array}$$

be two sequences of morphisms such that $\delta^2 = 0$ and $\partial^2 = 0$. A *chain transformation* $f : (a, \partial) \longrightarrow (b, \delta)$ is a family of morphisms

$$f = \{f_n : a_n \longrightarrow b_n \mid n \in \mathbb{Z}\}$$

such that

$$\delta_{n-1} \circ f_n = f_{n-1} \circ \partial_n.$$

3.2.3 Presentations of Small Categories

We will give in this section the definition of a presentation of a small category which mimics that of a monoid. First recall that a *directed graph* $\mathbf{x}$ is a pair (O, A) with O the set of vertices and A the set of edges e, together with a pair of functions

$$A \underset{\tau}{\overset{\iota}{\rightrightarrows}} O$$

We call ιe the *domain* of e and τe the *codomain* of e. A *morphism* $D : \mathbf{x} \longrightarrow \mathbf{x}'$ of graphs is a pair of functions $D_O : O \longrightarrow O'$ and $D_A : A \longrightarrow A'$ such that

$$D_O \iota e = \iota D_A e \text{ and } D_O \tau e = \tau D_A e$$

for every $e \in A$. It is easy to see that graphs together with graph morphisms form a category, which we denote by **Grph**.

Every category $\mathbb{C}$ determines a graph $U\mathbb{C}$ with the same set of objects, and the set of arrows coincides with the set of arrows of $\mathbb{C}$. Thus every path in $U\mathbb{C}$ gives rise to a graph arrow. Also every functor between categories $F : \mathbb{C} \longrightarrow \mathbb{B}$ can be seen as a graph morphism $UF : U\mathbb{C} \longrightarrow U\mathbb{B}$ and as a result we have the forgetful functor U : **Cat** $\longrightarrow$ **Grph**. We call $U\mathbb{C}$ the *underlying graph* of $\mathbb{C}$. This is not the only relation between these two categories. Indeed, any graph $\mathbf{x}$ "generates" a category $F(\mathbf{x})$ with the same set of vertices and with arrows, paths of $\mathbf{x}$. We call $F(\mathbf{x})$ the *free category* generated by the graph $\mathbf{x}$. The following theorem (see Theorem 1, pp. 49 of [64]) certifies the above chosen term.

Theorem 3.2.54 *Let $\mathbf{x}$ be a graph. There is a morphism $P : \mathbf{x} \longrightarrow UF(\mathbf{x})$ of graphs from $\mathbf{x}$ to the underlying graph $UF(\mathbf{x})$ of $F(\mathbf{x})$ with the following universal property. Given any category $\mathbb{B}$ and any morphism $D : \mathbf{x} \longrightarrow U\mathbb{B}$ of graphs, there is a unique functor $D' : F(\mathbf{x}) \dashrightarrow \mathbb{B}$ with $(UD') \circ P = D$ such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{x} \arrow[r, "P"] \arrow[dr, "D"'] & UF(\mathbf{x}) \arrow[d, dashed, "UD'"] \\ & U\mathbb{B} \end{tikzcd}$$

There is even more in this theorem. Graph morphisms $D : \mathbf{x} \longrightarrow U\mathbb{B}$ are in a 1-1 correspondence with functors $D' : F(\mathbf{x}) \longrightarrow \mathbb{B}$ via the bijection $D' \longmapsto D = (UD') \circ P$. This bijection

$$\mathbf{Cat}(F(\mathbf{x}), B) \approx \mathbf{Grph}(\mathbf{x}, U\mathbb{B})$$

is natural in $\mathbf{x}$ and $\mathbb{B}$. In fact this is an example of an adjunction as described in Definition 3.2.42.

Once we have assigned to every graph a free category generated by this graph, assignment which generalizes the construction of the free monoid generated by some set, we can attempt to extend this analogy further by expressing categories through generators and relations. The Proposition 1, p. 51 of [64] gives the general idea of taking the *quotient* of a category by a congruence relation. We state it below and make a comment afterwards.

Proposition 3.2.55 *For a given category $\mathbb{C}$, let $\mathbf{r}$ be a function which assigns to each pair of objects $a, b \in \mathbb{C}$ a binary relation $\mathbf{r}_{a,b}$ on the hom-set $\mathbb{C}(a,b)$. Then there exist a category $\mathbb{C}/\mathbf{r}$ and a functor $Q = Q_{\mathbf{r}} : \mathbb{C} \longrightarrow \mathbb{C}/\mathbf{r}$ such that (i) If $f\mathbf{r}_{a,b}f'$ in $\mathbb{C}$, then $Qf = Qf'$; (ii) If $H : \mathbb{C} \longrightarrow \mathbb{D}$ is any other functor for which $f\mathbf{r}_{a,b}f'$ implies $Hf = Hf'$ for all f and f', then there is a unique functor $H' : \mathbb{C}/\mathbf{r} \longrightarrow \mathbb{D}$ with $H' \circ Q_{\mathbf{r}} = H$. Moreover, the functor $Q_{\mathbf{r}}$ is a bijection on objects.*

The notation $\mathbb{C}/\mathbf{r}$ used in this proposition is a bit misleading. Actually we do not take the quotient of $\mathbb{C}$ by $\mathbf{r}$ but we first define a new relation $\mathbf{r}^{\#}$, the *congruence generated* by $\mathbf{r}$, as the smallest relation containing $\mathbf{r}_{a,b}$ for any $a, b \in \mathbb{C}$, which is reflexive, symmetric and transitive, and satisfies the property: if $f, f' : a \longrightarrow b$ such that $f\mathbf{r}_{a,b}f'$ and if $g : a' \longrightarrow a$ and $h : b \longrightarrow b'$, then $(hfg)\mathbf{r}_{a,b}(hf'g)$. Then we define $\mathbb{C}/\mathbf{r}$ to be the category with the same objects as $\mathbb{C}$ and with hom-sets $(\mathbb{C}/\mathbf{r})(a,b) = \mathbb{C}(a,b)/\mathbf{r}^{\#}_{a,b}$. In the special case when $\mathbb{C} = F(\mathbf{x})$ is the free category generated by some graph $\mathbf{x}$, we call $F(\mathbf{x})/\mathbf{r}^{\#}$ the *category with generators* $\mathbf{x}$ and *relations* $\mathbf{r}$. In contrast to categories, we will agree to use the notation $x \in \mathbf{x}$ to mean that x is an edge of $\mathbf{x}$.

3.2.4 Additive Categories

In the rest of this chapter we deal with additive categories.

Definition 3.2.56 An *additive*[1] category is a category $\mathbb{A}$ together with an abelian group structure on each of its hom-sets, subject to the following condition:
The composition functions $\hom(b,c) \times \hom(a,b) \longrightarrow \hom(a,c)$ are *bilinear.* That is, if $\alpha, \beta \in \hom(a,b)$ and $\gamma \in \hom(b,c)$, then $\gamma(\alpha+\beta) = \gamma\alpha + \gamma\beta$, and if $\gamma \in \hom(a,b)$ and $\alpha, \beta \in \hom(b,c)$, then $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$.

[1]In [64] these categories are called *Ab*-categories and the name additive is reserved for those satisfying two extra conditions: having a zero element and biproducts.

Example 3.2.57 Any ring $\mathbf{R}$ with unit element can be thought of as an additive category with a single object $*$ corresponding to the unit element of $\mathbf{R}$, and morphisms $r : * \longrightarrow *$ for each $r \in \mathbf{R}$. In fact in the above definition, every $\hom(a, a)$ has a ring structure. Also the category of left $\mathbf{R}$-modules **R-Mod** (respectively right $\mathbf{R}$-modules **Mod-R**) for some ring $\mathbf{R}$, form additive categories.

Example 3.2.58 For every non additive category $\mathbb{C}$ one can construct the category $\mathbb{ZC}$ with objects those of $\mathbb{C}$ and hom-sets $\mathbb{ZC}(a, b)$ the free abelian group generated by $\mathbb{C}(a, b)$. If we require $\mathbb{ZC}$ to satisfy the condition of Definition 3.2.56, then $\mathbb{ZC}$ becomes an additive category. We say that $\mathbb{ZC}$ is the *free additive category on* $\mathbb{C}$. If $\mathbb{C}$ is the trivial category with a single object and a single morphism, then we denote $\mathbb{ZC}$ by simply $\mathbb{Z}$ and call it the *trivial additive category*.

Example 3.2.59 For every additive category $\mathbb{A}$, its opposite $\mathbb{A}^{opp}$ is again additive where $\alpha^{opp} + \beta^{opp} = (\alpha + \beta)^{opp}$.

A functor $T : \mathbb{A} \longrightarrow \mathbb{B}$ with $\mathbb{A}$ and $\mathbb{B}$ both additive, will be called *additive* if it satisfies the condition $T(\alpha + \beta) = T(\alpha) + T(\beta)$ whenever $\alpha + \beta$ is defined.

If $S : \mathbb{B} \times \mathbb{C} \longrightarrow \mathbb{D}$ is a bifunctor where $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are additive and for all $b \in \mathbb{B}$ and $c \in \mathbb{C}$, the respective partial functors are additive, then we call S *biadditive.*

Definition 3.2.60 If $\mathbb{A}$ and $\mathbb{B}$ are both additive categories, then we can consider the full subcategory $Add(\mathbb{A}, \mathbb{B})$ of $\mathbb{B}^{\mathbb{A}}$ with objects all the additive functors from $\mathbb{A}$ to $\mathbb{B}$ and with hom-sets equipped with an additive operation: if $\alpha, \beta : S \longrightarrow T$ are natural transformations then, for any object $a \in \mathbb{A}$, we define $(\alpha + \beta)_a = \alpha_a + \beta_a$. For any two additive functors $S, T : \mathbb{A} \longrightarrow \mathbb{B}$, we denote by $Nat(S, T)_{Add(\mathbb{A},\mathbb{B})}$ the hom-set in $Add(\mathbb{A}, \mathbb{B})$ with domain S and codomain T.

Example 3.2.61 (Left R-modules) An element of $Add(\mathbf{R}, \mathbf{Ab})$ is an additive functor $\mu : \mathbf{R} \longrightarrow \mathbf{Ab}$ sending $* \longrightarrow A$ and each morphisms r as explained in Example 3.2.57, to some endomorphism $\mu(r)$ of A satisfying the following conditions:

1. $\mu(*)(x) = x$ for all $x \in A$,
2. $\mu(r)(x_1 + x_2) = \mu(r)(x_1) + \mu(r)(x_2)$,
3. $\mu(r_1 r_2)(x) = \mu(r_1)(\mu(r_2)(x))$,
4. $\mu(r_1 + r_2)(x) = \mu(r_1)(x) + \mu(r_2)(x)$.

Thus μ can be identified with the left $\mathbf{R}$-module with underlying abelian group A. Motivated by this example we sometimes call $Add(\mathbb{C}, \mathbf{Ab})$, where $\mathbb{C}$ is additive, the category of *left* $\mathbb{C}$-*modules*.

If $\mathbb{B}$ and $\mathbb{C}$ are additive then one can define in a similar fashion with (3.3) the additive version of the evaluation functor

$$E : Add(\mathbb{B}, \mathbb{C}) \times \mathbb{B} \longrightarrow \mathbb{C} \tag{3.4}$$

The following mimics the definition of the tensor product of abelian groups.

Definition 3.2.62 If $\mathbb{A}$ and $\mathbb{B}$ are two additive categories, then their *tensor* product $\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{B}$ is the additive category with object set all the pairs (a, b) with $a \in \mathbb{A}$ and $b \in \mathbb{B}$, and the abelian group of morphisms from (a_1, b_1) to (a_2, b_2) is the tensor product

$$\mathbb{A}(a_1, a_2) \otimes_{\mathbb{Z}} \mathbb{B}(b_1, b_2).$$

The bilinear composition in $\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{B}$ is defined by

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = (\alpha_1 \alpha_2) \otimes (\beta_1 \beta_2).$$

Example 3.2.63 We can use the tensor product of categories to define bimodules. If $\mathbf{R}$ and $\mathbf{S}$ are rings with unit elements, then, as we saw in Example 3.2.57, they are both additive categories and from Example 3.2.59 and Definition 3.2.62 we have that $\mathbf{S}^{opp} \otimes_{\mathbb{Z}} \mathbf{R}$ is an additive category. As in Example 3.2.61 one can define the category of left-$(\mathbf{S}^{opp} \otimes_{\mathbb{Z}} \mathbf{R})$ modules $Add(\mathbf{S}^{opp} \otimes_{\mathbb{Z}} \mathbf{R}, \mathbf{Ab})$ whose functors satisfy all the properties of the $(\mathbf{R}, \mathbf{S})$-bimodules. This is why we call its objects, $(\mathbf{R}, \mathbf{S})$-*bimodules.*

Example 3.2.64 If $\mathbb{A}$ is additive and $\mathbb{Z}$ is the free additive category generated by the trivial category, then it is easy to show that $\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{Z}$ is isomorphic with $\mathbb{A}$.

♦

The analogues of Lemma 3.2.20 and Theorem 3.2.21 in the additive case also hold true. We include them below for the convenience of the reader.

Lemma 3.2.65 *If $\mathbb{D}$ is additive and $K : \mathbb{D} \longrightarrow \mathbf{Ab}$ is an additive functor, then the Yoneda map*

$$Y : Nat(\mathbb{D}(r, \_), K)_{Add(\mathbb{D}, \mathbf{Ab})} \longrightarrow K(r),$$

which sends each natural transformation $\alpha : \mathbb{D}(r, \_) \longrightarrow K$ to $\alpha_r(1_r)$, is an isomorphism of additive groups. Furthermore, Y is natural in both r and K.

For the proof see Lemma 4.3.1 of [93]. We remark here that the functor $\mathbb{D}(r, \_)$ is additive if $\mathbb{D}$ is such.

Proposition 3.2.66 *If $\mathbb{D}$ is additive, then for $\alpha : \mathbb{D}(r, \_) \longrightarrow K$ the Yoneda map $Y(\alpha) = \alpha_r(1_r)$ determines an isomorphism*

$$Y : Nat(\mathbb{D}(\_, \_), \_)_{Add(\mathbb{D}, \mathbf{Ab})} \longrightarrow E\tau(\_, \_)$$

of biadditive functors $\mathbb{D} \times Add(\mathbb{D}, \mathbf{Ab}) \longrightarrow \mathbf{Ab}$.

The proof of the above is given in Proposition 4.3.3 of [93].

Theorem 3.2.67 *If $\mathbb{D}$ is an additive category, then the set of all representables $\mathbb{D}(r, \_)$ is a set of generators for $Add(\mathbb{D}, \mathbf{Ab})$.*

Proof. Let F and G be two objects from $Add(\mathbb{D}, \mathbf{Ab})$ and τ, τ' two different natural transformations from F to G. We must find $d \in \mathbb{D}$ and $\gamma : \mathbb{D}(d, \_) \longrightarrow F$ such that $\tau\gamma \neq \tau'\gamma$. The fact that $\tau \neq \tau'$ implies that there is some $c \in \mathbb{D}$ such that the group morphisms τ_c, τ'_c : $F(c) \longrightarrow G(c)$ are different; hence there is some $x \in F(c)$ such that $\tau_c(x) \neq \tau'_c(x)$. Since, from Lemma 3.2.65, γ is uniquely determined from the value $\gamma_c(1_c)$, we take $\gamma : \mathbb{D}(c, \_) \longrightarrow F$ to satisfy the condition $\gamma_c(1_c) = x$. It follows that $\tau_c\gamma_c(1_c) \neq \tau'_c\gamma_c(1_c)$ since $\tau_c(x) \neq \tau'_c(x)$. ∎

There is also an alternative proof of the above based on Proposition 3.2.66 as is shown in Example 10.5.2 of [93].

We prove below two interesting properties of representables $\mathbb{C}(c, \_) \in Add(\mathbb{C}, \mathbf{Ab})$. First we have this.

Proposition 3.2.68 *Let $\mathbb{C}$ be a small category. The functor $\bigoplus_{c \in \mathbb{C}} \mathbb{C}(c, \_) \in Add(\mathbb{C}, \mathbf{Ab})$ is free (in the sense of Definition 3.2.43) relative to the functor $U : Add(\mathbb{C}, \mathbf{Ab}) \longrightarrow$ **Set** which is defined on objects by*

$$U(G) = \prod_{c \in \mathbb{C}} G(c),$$

and on morphisms $\mu : G \longrightarrow H$ by

$$U(\mu) : \prod_{c \in \mathbb{C}} G(c) \longrightarrow \prod_{c \in \mathbb{C}} H(c),$$

where

$$U(\mu) \mid_{G(c)} = \mu_c \text{ for every } c \in \mathbb{C}.$$

Proof. Let $Fr : \mathbf{Set} \longrightarrow Add(\mathbb{C}, \mathbf{Ab})$ be the functor defined on objects by

$$Fr(S) = \bigoplus_{s \in S} \bigoplus_{c \in \mathbb{C}} \mathbb{C}(c, \_).$$

Denote $\bigoplus_{c \in \mathbb{C}} \mathbb{C}(c, \_)$ for the moment by A. If $h : S \longrightarrow S'$ it can be seen as the map

$$\{A_s \mid s \in S\} \longrightarrow \{A_{s'} \mid s' \in S'\}$$

defined by

$$h(A_s) = A_{h(s)} \text{ for every } s \in S,$$

which, when composed with the injections $u_{s'} : A_{s'} \longrightarrow \bigoplus_{s' \in S'} A_{s'}$, induces a unique morphism

$$Fr(h) : \bigoplus_{s \in S} A_s \longrightarrow \bigoplus_{s' \in S'} A_{s'}.$$

It is easy now to check that Fr is a functor and it sends every set to an additive functor. It remains to show that Fr is a left adjoint of U and for this we have to show, that for each functor $G \in Add(\mathbb{C}, \mathbf{Ab})$ and each non-empty set S, there is a natural bijection

$$Add(\mathbb{C}, \mathbf{Ab})(Fr(S), G) \approx \mathbf{Set}(S, U(G)).$$

First observe that each $\tau \in Add(\mathbb{C}, \mathbf{Ab})(Fr(S), G)$ induces for every $s \in S$ and every $c \in \mathbb{C}$ a natural transformation $\tau^{(s,c)} : \mathbb{C}(c, \_) \longrightarrow G$ such that for every $d \in \mathbb{C}$, $\tau_d^{(s,c)} = \tau_d \mid_{\mathbb{C}(c,d)}$. From the definition of the coproduct, the family

$$\{\tau^{(s,c)} \mid s \in S, c \in \mathbb{C}\}$$

determines τ uniquely and therefore can be identified with τ. On the other hand, from the Yoneda Lemma, for every $c \in \mathbb{C}$, $\tau^{(s,c)}$ is uniquely determined by $\tau_c^{(s,c)}(1_c)$. Now we can construct the map

$$\Phi : Add(\mathbb{C}, \mathbf{Ab})(\bigoplus_{s \in S} \bigoplus_{c \in \mathbb{C}} \mathbb{C}(c, \_), G) \longrightarrow \mathbf{Set}(S, \prod_{c \in \mathbb{C}} G(c)),$$

by

$$\tau \longmapsto (\varphi_\tau : S \longrightarrow \prod_{c \in \mathbb{C}} G(c)),$$

where

$$\varphi_\tau(s) = \prod_{c \in \mathbb{C}} \tau_c^{(s,c)}(1_c), \forall s \in S.$$

We will show that Φ is bijective.

Indeed, it is injective since, if $\tau, t : Fr(S) \longrightarrow G$ are two different natural transformations, then there is $d \in \mathbb{C}$ such that

$$\tau_d : \underset{s\in S}{\oplus} \underset{c\in \mathbb{C}}{\oplus} \mathbb{C}(c,d) \longrightarrow G(d)$$

is different from

$$t_d : \underset{s\in S}{\oplus} \underset{c\in \mathbb{C}}{\oplus} \mathbb{C}(c,d) \longrightarrow G(d).$$

This implies that there is $s \in S$ and $c \in \mathbb{C}$ such that $\tau_d^{(s,c)} \neq t_d^{(s,c)}$ and, from the Yoneda Lemma, this is equivalent to $\tau_c^{(s,c)}(1_c) \neq t_c^{(s,c)}(1_c)$, which proves the injectivity of Φ.

For the surjectivity, for any

$$\varphi : S \longrightarrow \prod_{c\in\mathbb{C}} G(c)$$

such that

$$\varphi(s) = \prod_{c\in\mathbb{C}} g_c^{(s)},\ \forall s \in S,$$

we can define $\tau^{(s,c)}$ for any $s \in S$ and $c \in \mathbb{C}$, by letting $\tau_c^{(s,c)}(1_c) = g_c^{(s)}$. In this way we have defined a natural transformation $\tau : \underset{s\in S}{\oplus} \underset{c\in \mathbb{C}}{\oplus} \mathbb{C}(c,\_) \longrightarrow G$ since, as we saw earlier,

$$\tau = \{\tau^{(s,c)} \mid s \in S, c \in \mathbb{C}\}.$$

From the definition of Φ we have that $\Phi(\tau) = \varphi$.

Lastly, we have to check the naturality of Φ in both S and G.

Let $h : S \longrightarrow S'$. For any $\tau \in Add(\mathbb{C}, \mathbf{Ab})(\underset{s'\in S'}{\oplus} \underset{c\in \mathbb{C}}{\oplus} \mathbb{C}(c,\_), G)$, we have

$$(\varphi_\tau \circ h)(s) = \prod_{c\in\mathbb{C}} \tau_c^{(h(s),c)}(1_c),\ \forall s \in S.$$

On the other hand we have,

$$\varphi_{\tau\circ Fr(h)}(s) = \prod_{c\in\mathbb{C}} \tau_c^{(h(s),c)}(1_c),\ \forall s \in S,$$

which proves that Φ is natural in S.

For the naturality of Φ in G, let $\mu : G \longrightarrow G'$ be some natural transformation. For every $\tau \in Add(\mathbb{C}, \mathbf{Ab})(\underset{s\in S}{\oplus} \underset{c\in \mathbb{C}}{\oplus} \mathbb{C}(c,\_), G)$, we have

$$\varphi_{\mu\circ\tau}(s) = \prod_{c\in\mathbb{C}} (\mu \circ \tau)_c^{(s,c)}(1_c) = \prod_{c\in\mathbb{C}} \mu_c((\tau)_c^{(s,c)}(1_c)),\ \forall s \in S.$$

On the other hand we have

$$(U(\mu) \circ \varphi_\tau)(s) = \prod_{c\in\mathbb{C}} \mu_c((\tau)_c^{(s,c)}(1_c)),\ \forall s \in S,$$

which proves the naturality in G.

Finally, if in our definition of $Fr(S)$ we take S to be a singleton, then we get $\underset{c\in\mathbb{C}}{\oplus}\mathbb{C}(c,\_)$ to be free. ■

Remark 3.2.69 The fact that U, as defined in the above proposition, is injective on morphisms, can be proved in another way using the fact that $\underset{c\in\mathbb{C}}{\oplus}\mathbb{C}(c,\_)$ is a generator in $Add(\mathbb{C},\mathbf{Ab})$ as shown in Proposition 15.4.1 of [93].

We could have defined for every fixed $c \in \mathbb{C}$ the functor U by

$$U = Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),\_)$$

and the functor Fr by letting $Fr(S) = \underset{s\in S}{\oplus}\mathbb{C}(c,\_)$ for every set S. Again Fr is a left adjoint of U (see [94], p. 378) but U may fail to be injective on morphisms. If every $\mathbb{C}(c,\_)$ is a generator, then of course U as defined above will be injective on morphisms. We state below a condition under which $\mathbb{C}(c,\_)$ is a generator in $Add(\mathbb{C},\mathbf{Ab})$.

Lemma 3.2.70 *If the small category $\mathbb{C}$ has the property that, for every $c,d \in \mathbb{C}$ the identity morphism 1_c can be expressed as $1_c = \beta_{c,d}\alpha_{c,d}$ where $\alpha_{c,d} : c \longrightarrow d$ and $\beta_{c,d} : d \longrightarrow c$, then every representable $\mathbb{C}(c,\_)$ is a generator in $\mathbf{Ab}^{\mathbb{C}}$.*

Proof. One can show easily as in the proof of Yoneda Lemma that for every $F \in \mathbf{Ab}^{\mathbb{C}}$, every $d \in \mathbb{C}$ and every natural transformation $\tau : \mathbb{C}(c,\_) \longrightarrow F$, τ is uniquely determined by the value $\tau(\alpha_{c,d})$. Then the lemma follows. ■

The second property of the representables $\mathbb{C}(c,\_) \in Add(\mathbb{C},\mathbf{Ab})$ is proved in the following.

Proposition 3.2.71 *Representable functors $\mathbb{C}(c,\_) \in Add(\mathbb{C},\mathbf{Ab})$ are projective.*

Proof. In fact, as we have observed in Definition 3.2.31, this is equivalent to showing that the hom functor $Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),\_)$ is an epifunctor. So let $\tau : G \longrightarrow H$ be an epi in $Add(\mathbb{C},\mathbf{Ab})$. From Lemma 3.2.65, $Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),F) \cong F(c)$ and $Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),G) \cong G(c)$, and the isomorphism Y is natural in F. This implies that the induced morphism

$$Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),\tau) : Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),F) \longrightarrow Add(\mathbb{C},\mathbf{Ab})(\mathbb{C}(c,\_),G)$$

is an epimorphism. ■

3.2.5 Abelian Categories

A special type of additive categories are Abelian categories. A consequence of Proposition 3.2.76 below is that, for every additive category $\mathbb{A}$, the category of additive functors $Add(\mathbb{A}, \mathbf{Ab})$ is Abelian. This type of category will be the focus of our study in the rest of this chapter.

Definition 3.2.72 Let $\mathbb{A}$ be an additive category. A *biproduct* diagram for the objects $a, b \in \mathbb{A}$ is a diagram

$$a \underset{i_1}{\overset{p_1}{\rightleftarrows}} c \underset{i_2}{\overset{p_2}{\rightleftarrows}} b \tag{3.5}$$

with morphisms p_1, p_2, i_1, i_2 satisfying the identities

$$p_1 i_1 = 1_a, \quad p_2 i_2 = 1_b, \quad i_1 p_1 + i_2 p_2 = 1_c.$$

The following important result (Theorem 2, p. 194 of [64]) relates the biproduct of two elements with their product and coproduct.

Theorem 3.2.73 *Let $\mathbb{A}$ be an additive category. Two objects $a, b \in \mathbb{A}$ have a product in $\mathbb{A}$ if and only if they have a biproduct in $\mathbb{A}$. Specifically, given a biproduct diagram (3.5), the object c with projections p_1 and p_2 is a product of a and b, while, dually, c with injections i_1 and i_2 is a coproduct of a and b. In particular, two objects a and b have a product in $\mathbb{A}$ if and only if they have a coproduct in $\mathbb{A}$.*

Definition 3.2.74 An additive category $\mathbb{A}$ with a null object is called *pre-Abelian* if it contains the biproducts of any two objects, and if any morphism has both kernel and cokernel. It is called *Abelian* if it is pre-Abelian and satisfies two further conditions: every monic is a kernel and every epi is a cokernel.

Example 3.2.75 **Ab** is obviously Abelian and for every ring **R** with unit element, the category of **R-Mod** is also Abelian. ♦

The following result of Grothendieck [35] (see also Proposition 3.1, p. 258 of [65]) shows that when $\mathbb{A}$ is a small category, not necessarily additive, the functor category $\mathbb{B}^{\mathbb{A}}$ is Abelian.

Proposition 3.2.76 *If the category $\mathbb{A}$ is small and $\mathbb{B}$ is Abelian, then $\mathbb{B}^{\mathbb{A}}$ is Abelian. A sequence*

$$0 \longrightarrow F \xrightarrow{\tau_1} G \xrightarrow{\tau_2} H \longrightarrow 0$$

is exact in $\mathbb{B}^{\mathbb{A}}$ if and only if, for each $a \in \mathbb{A}$, the sequence

$$0 \longrightarrow F(a) \xrightarrow{\tau_1(a)} G(a) \xrightarrow{\tau_2(a)} H(a) \longrightarrow 0$$

is exact in $\mathbb{B}$.

Staying with exact sequences in abelian categories, we make the following remark.

Remark 3.2.77 Firstly, if $0 \longrightarrow a \xrightarrow{m} b \xrightarrow{p} c \longrightarrow 0$ is exact which splits, then $b \cong a \oplus c$ (see [93], p. 126). Secondly, if $0 \longrightarrow a \xrightarrow{m} b \xrightarrow{p} c \longrightarrow 0$ is exact, then m is the kernel of p and p is the cokernel of m. Also $b \xrightarrow{p} c \longrightarrow 0$ is exact if and only if p is epi, and $0 \longrightarrow a \xrightarrow{m} b$ is exact if and only if m is mono (see [93], p. 124).

In the case when $\mathbb{B} = \mathbf{Ab}$, there is a nice description of the kernel, cokernel and the image of a morphism as the following shows.

Example 3.2.78 For any category $\mathbb{C}$, since $\mathbf{Ab}^{\mathbb{C}}$ is Abelian, it contains a null object, namely the functor $0 \in \mathbf{Ab}^{\mathbb{C}}$ which sends every object to the trivial group $\{0\}$, and in addition it contains all pullbacks; therefore Definition 3.2.46 and Example 3.2.48 imply that for every $\tau : F \longrightarrow G$ in $\mathbf{Ab}^{\mathbb{C}}$, $Ker\tau$ can be given functorially on objects by

$$(Ker\tau)(c) = Ker\tau_c,$$

where $Ker\tau_c$ is the kernel of the group morphism $\tau_c : F(c) \longrightarrow G(c)$, and on morphisms by

$$(Ker\tau)(\delta) = F(\delta) \mid_{Ker\tau_c} ,$$

for every $\delta : c \longrightarrow c'$. Similarly, for every $\tau : F \longrightarrow G$ in $\mathbf{Ab}^{\mathbb{C}}$, $coker\tau$ is given functorially by

$$(coker\tau)(c) = coker\tau_c,$$

where $coker\tau_c$ is the cokernel in $\mathbf{Ab}$ of $\tau_c : F(c) \longrightarrow G(c)$, and on morphisms by

$$(coker\tau)(\delta) = F(\delta) \mid_{coker\tau_c} ,$$

for every $\delta : c \longrightarrow c'$. Lastly, combining the two above results, one gets that the image of $\tau : F \longrightarrow G$ in $\mathbf{Ab}^{\mathbb{C}}$ can be given functorially by

$$(Im\tau)(c) = Im(\tau_c),$$

where $\tau_c : F(c) \longrightarrow G(c)$, and on morphisms by

$$(Im\tau)(\delta) = F(\delta) \mid_{Im(\tau_c)} ,$$

for every $\delta : c \longrightarrow c'$.

The following is useful in proving the Schanuel's Lemma for abelian categories.

Lemma 3.2.79 *In an abelian category, an exact sequence $k \rightarrowtail b \twoheadrightarrow c$ and a morphism $\gamma : c' \longrightarrow c$ can be put into a commutative diagram*

$$\begin{tikzcd} k \arrow[r, tail] \arrow[d, equal] & b' \arrow[r, two heads] \arrow[d] & c' \arrow[d, "\gamma"] \\ k \arrow[r, tail] & b \arrow[r, two heads] & c \end{tikzcd}$$

where the top row is exact and the right square is a pullback.

Proof. The full proof is given in Corollary 20.3 of [75]. We give here only a sketch of it. First one shows that, if in the following commutative diagram

$$\begin{tikzcd} k \arrow[r, "\gamma"] \arrow[d, equal] & b' \arrow[r, "\beta_2"] \arrow[d, "\beta_1"] & c' \arrow[d, "\alpha_2"] \\ k \arrow[r, "u"] & b \arrow[r, "\alpha_1"] & c \end{tikzcd} \tag{3.6}$$

the right-hand side square is pullback, u is the kernel of α_1 and γ is the morphism into the pullback induced by the morphisms $u : k \longrightarrow b$ and $0 : k \longrightarrow c'$, then γ is the kernel of β_2. Secondly one shows that, if in the pullback square

$$\begin{tikzcd} b' \arrow[r, "\beta_2"] \arrow[d, "\beta_1"] & c' \arrow[d, "\alpha_2"] \\ b \arrow[r, "\alpha_1"] & c \end{tikzcd} \tag{3.7}$$

α_1 is epi (respectively mono), then β_2 has to be epi (respectively mono). Combining (3.6) with (3.7) for the epi case, gives the result. ∎

Lemma 3.2.80 (Schanuel's Lemma) *Let $k \rightarrowtail p \twoheadrightarrow m$ and $k' \rightarrowtail p' \twoheadrightarrow m$ be two short exact sequences in an abelian category and let p and p' be projective objects. Then $p \oplus k' \cong p' \oplus k$.*

Proof. Using Lemma 3.2.79 twice one can construct the following commutative diagram

$$\begin{tikzcd} & k' \arrow[r, equal] \arrow[d, tail] & k' \arrow[d, tail] \\ k \arrow[r, tail] \arrow[d, equal] & q \arrow[r, two heads] \arrow[d, two heads] & p' \arrow[d, two heads] \\ k \arrow[r, tail] & p \arrow[r, two heads] & m \end{tikzcd}$$

with exact rows and columns. Since p and p' are projective, then, using a similar argument as in Homological Algebra, one can see that the exact sequences involving q must split and then from Remark 3.2.77 we have that $p \oplus k' \cong q \cong p' \oplus k$. ∎

Before we prove the generalized Schanuel's Lemma for abelian categories, we give a number of preliminary results. First we give below Proposition 13.3.4 of [93].

Proposition 3.2.81 *In abelian categories finite coproducts of exact sequences are exact. Here we define the coproduct of two sequences*

$$\ldots \longrightarrow a_{n+1} \longrightarrow a_n \longrightarrow a_{n-1} \longrightarrow \ldots$$

and

$$\ldots \longrightarrow b_{n+1} \longrightarrow b_n \longrightarrow b_{n-1} \longrightarrow \ldots$$

to be the sequence

$$\ldots \longrightarrow a_{n+1} \oplus b_{n+1} \longrightarrow a_n \oplus b_n \longrightarrow a_{n-1} \oplus b_{n-1} \longrightarrow \ldots .$$

Corollary 3.2.82 *The coproduct of finitely many finitely generated objects in an abelian category is finitely generated.*

Proof. It is enough to prove the claim for any two objects. Let a and b be two finitely generated objects and

$$\bigoplus_{i=1}^{n} g_i \twoheadrightarrow a$$

and

$$\bigoplus_{j=1}^{m} g_j' \twoheadrightarrow b.$$

Remark 3.2.77 implies that we have the following exact sequences

$$\bigoplus_{i=1}^{n} g_i \rightarrow a \rightarrow 0$$

and

$$\bigoplus_{j=1}^{m} g_j' \rightarrow b \rightarrow 0.$$

Their coproduct

$$\left(\bigoplus_{i=1}^{n} g_i\right) \oplus \left(\bigoplus_{j=1}^{m} g_j'\right) \longrightarrow a \oplus b \rightarrow 0$$

is still exact and then again from Remark 3.2.77 we get that

$$\left(\bigoplus_{i=1}^{n} g_i\right) \oplus \left(\bigoplus_{j=1}^{m} g_j'\right) \longrightarrow a \oplus b$$

is epi and therefore $a \oplus b$ is finitely generated. ■

Lemma 3.2.83 *Let* 0 *be a null object for a category* $\mathbb{A}$ *and let* a_1 *and* a_2 *be two objects. Then the diagram*

$$\begin{tikzcd} p \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & a_2 \arrow[d] \\ a_1 \arrow[r] & 0 \end{tikzcd}$$

is a pullback if and only if p *is the product of* a_1 *and* a_2.

For the proof see Lemma 17.6 of [75]. Note here that, if the category $\mathbb{A}$ is in addition additive, then from Theorem 3.2.73 we have that p is also a coproduct of a_1 and a_2.

Corollary 3.2.84 *In an abelian category* $\mathbb{A}$ *with a null object* 0*, for any biproduct diagram*

$$a \underset{i_1}{\overset{p_1}{\leftrightarrows}} a \oplus b \underset{i_2}{\overset{p_2}{\rightleftarrows}} b$$

the projection $p_2 : a \oplus b \longrightarrow b$ *is an epimorphism. In particular,*

$$0 \oplus b \cong b.$$

Proof. From Lemma 3.2.83 we can see $a \oplus b$ as part of the pullback diagram

$$\begin{tikzcd} a \oplus b \arrow[r, "p_2"] \arrow[d, "p_1"'] & b \arrow[d] \\ a \arrow[r] & 0 \end{tikzcd}$$

But now $a \longrightarrow 0$ is epi as there is exactly one arrow from 0 to any other object of $\mathbb{A}$; therefore from the second part of the proof of Lemma 3.2.79 we have that p_2 is epi. To see the second claim, put in the above diagram $a = 0$ and then again from Lemma 3.2.79 the projection $p_2 : 0 \oplus b \longrightarrow b$ is mono. On the other hand from the definition of a biproduct, p_2 is a retraction, hence there is $\beta : b \longrightarrow 0 \oplus b$ such that $p_2\beta = 1_b$. It follows that $p_2\beta p_2 = 1_b p_2 = p_2 = p_2 1_{0\oplus b}$. Since p_2 is mono, we have that $\beta p_2 = 1_{0\oplus b}$, therefore p_2 is an isomorphism. ■

Lemma 3.2.85 (Generalized Schanuel's Lemma) *Let*

$$0 \to p_n \longrightarrow p_{n-1} \longrightarrow \dots \longrightarrow p_0 \longrightarrow m \to 0$$

and

$$0 \to p'_n \longrightarrow p'_{n-1} \longrightarrow \dots \longrightarrow p'_0 \longrightarrow m \to 0$$

be exact sequences in an abelian category $\mathbb{A}$ *and* p_i, p'_i *are projective for* $i \leq n-1$. *Then*

$$p_0 \oplus p'_1 \oplus p_2 \oplus p'_3 \oplus \dots \cong p'_0 \oplus p_1 \oplus p'_2 \oplus p_3 \oplus \dots .$$

Consequently, if for $i \leq n-1$, p_i and p'_i are finitely generated with respect to some set of generators of $\mathbb{A}$, then p_n is finitely generated if and only if p'_n is such.

Proof. We use induction on n. When $n = 2$, the result follows from Schanuel's Lemma. Let k (respectively k') be the kernel of $p_{n-2} \longrightarrow p_{n-3}$ (respectively $p'_{n-2} \longrightarrow p'_{n-3}$). By the induction hypothesis we have

$$k \oplus p'_{n-2} \oplus p_{n-3} \cdots \cong k' \oplus p_{n-2} \oplus p'_{n-3} \cdots .$$

Let

$$q = p'_{n-2} \oplus p_{n-3} \cdots$$

and

$$q' = p_{n-2} \oplus p'_{n-3} \cdots .$$

The following two exact sequences

$$0 \longrightarrow p_n \longrightarrow p_{n-1} \oplus q \longrightarrow k \oplus q \longrightarrow 0$$

and

$$0 \longrightarrow p'_n \longrightarrow p'_{n-1} \oplus q' \longrightarrow k' \oplus q' \longrightarrow 0$$

are obtained from

$$0 \longrightarrow p_n \longrightarrow p_{n-1} \longrightarrow k \longrightarrow 0$$

and

$$0 \longrightarrow p'_n \longrightarrow p'_{n-1} \longrightarrow k' \longrightarrow 0$$

respectively, by taking the coproduct of each with the exact sequences

$$0 \longrightarrow 0 \longrightarrow q \xrightarrow{1_q} q \longrightarrow 0 \text{ and } 0 \longrightarrow 0 \longrightarrow q' \xrightarrow{1_{q'}} q' \longrightarrow 0$$

and then applying Proposition 3.2.81. We can then apply Schanuel's Lemma to obtain

$$p_n \oplus p'_{n-1} \oplus q' \cong p'_n \oplus p_{n-1} \oplus q.$$

For the last part, if for example p'_n, p_{n-1} and q are finitely generated, then from Corollary 3.2.82 $p'_n \oplus p_{n-1} \oplus q$ is finitely generated. Since the isomorphisms are epi (see Exercise 5.1.5 of [93]) then it follows that $p_n \oplus p'_{n-1} \oplus q'$ is finitely generated. On the other hand, Corollary 3.2.84 implies the existence of an epi $p_n \oplus p'_{n-1} \oplus q' \twoheadrightarrow p_n$ which finally proves that p_n is finitely generated. ∎

The following definition is the restriction of Definition 7.7 of [82] to the category $Add(\mathbb{A}, \mathbf{Ab})$ which is Abelian from Proposition 3.2.76.

Definition 3.2.86 Let $\mathbb{A}$ be an additive category and $B \in Add(\mathbb{A}, \mathbf{Ab})$. A *projective resolution* of B is an exact sequence

$$\ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

with all P_j projective objects.

The following theorem will play an important role in the next sections.

Theorem 3.2.87 *A sequence*

$$Q_{n+1} \xrightarrow{\partial_{n+1}} Q_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\partial_0} B \longrightarrow 0, \tag{3.8}$$

in $Add(\mathbb{A}, \mathbf{Ab})$ is exact if and only if the sequence of abelian groups

$$Q_{n+1}(a) \xrightarrow{\partial_{n+1,a}} Q_n(a) \xrightarrow{\partial_{n,a}} \ldots \xrightarrow{\partial_{2,a}} Q_1(a) \xrightarrow{\partial_{1,a}} Q_0(a) \xrightarrow{\partial_{0,a}} B(a) \longrightarrow 0,$$

is exact for every object $a \in \mathbb{A}$.

Proof. We argue by induction on the length of the sequence. If the length is 3, then the claim is true from Proposition 3.2.76. If $K \longrightarrow Q_0$ is the kernel of $Q_0 \xrightarrow{\partial_0} B$, then we have the short exact sequence

$$0 \longrightarrow K \xrightarrow{\kappa} Q_0 \xrightarrow{\partial_0} B \longrightarrow 0. \tag{3.9}$$

The exactness of (3.8) implies that there is an epi $Q_1 \xrightarrow{\varepsilon} K$ such that

$$\partial_1 = \kappa\varepsilon. \tag{3.10}$$

Therefore the exactness of (3.8) is equivalent to the exactness of (3.9) together with the exactness of the following

$$Q_{n+1} \xrightarrow{\partial_{n+1}} Q_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} Q_1 \xrightarrow{\varepsilon} K \longrightarrow 0.$$

By induction hypothesis we have that they are equivalent to the exactness of

$$0 \longrightarrow K(a) \xrightarrow{\kappa_a} Q_0(a) \xrightarrow{\partial_{0,a}} B(a) \longrightarrow 0$$

and

$$Q_{n+1}(a) \xrightarrow{\partial_{n+1,a}} Q_n(a) \xrightarrow{\partial_{n,a}} \ldots \xrightarrow{\partial_{2,a}} Q_1(a) \xrightarrow{\varepsilon_a} K(a) \longrightarrow 0,$$

for every $a \in \mathbb{A}$. Using (3.10), we can now splice the two last sequences to obtain the exactness of

$$Q_{n+1}(a) \xrightarrow{\partial_{n+1,a}} Q_n(a) \xrightarrow{\partial_{n,a}} \ldots \xrightarrow{\partial_{2,a}} Q_1(a) \xrightarrow{\partial_{1,a}} Q_0(a) \xrightarrow{\partial_{0,a}} B(a) \longrightarrow 0$$

as desired. ∎

Definition 3.2.88 Let $\mathbb{C}$ be an additive category which has a null object. With the notations of Definition 3.2.53, two chain transformations $f, g : (a, \partial) \longrightarrow (b, \delta)$ are said to be *homotopic*, denoted by $f \simeq g$, if there exists a family of morphisms

$$h = \{h_n : a_n \longrightarrow b_{n+1} \mid n \in \mathbb{Z}\}$$

such that

$$\delta_{n+1}h_n + h_{n-1}\partial_n = f_n - g_n.$$

The following can be found in [82] and will play an important role in the next sections.

Proposition 3.2.89 *Let $\mathbb{C}$ be a pre-Abelian category which has enough projectives. Then any object in $\mathbb{C}$ has a projective resolution which can be chosen by a choice function. If b, $b' \in \mathbb{C}$ and $h : b \longrightarrow b'$, and if*

$$\begin{array}{rl} (p, \partial) : & \xrightarrow{\partial_{n+1}} p_n \xrightarrow{\partial_n} p_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} p_0 \xrightarrow{\partial_0} b \longrightarrow 0 \\ (p', \delta) : & \xrightarrow{\delta_{n+1}} p'_n \xrightarrow{\delta_n} p'_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} p'_0 \xrightarrow{\delta_0} b' \longrightarrow 0 \end{array}$$

are projective resolutions, then there is a chain transformation $f : (p, \partial) \longrightarrow (p', \delta)$ such that $f_{-1} = h$. Any two such chain transformations are homotopic.

Let $\mathbb{C}$ be a pre-Abelian category with enough projectives and let

$$\begin{array}{rl} (a, \partial) : & \xrightarrow{\partial_{n+1}} a_{n+1} \xrightarrow{\partial_n} a_n \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} a_0 \xrightarrow{\partial_0} x \longrightarrow 0 \\ (b, \delta) : & \xrightarrow{\delta_{n+1}} b_{n+1} \xrightarrow{\delta_n} b_n \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} b_0 \xrightarrow{\delta_0} x \longrightarrow 0 \end{array}$$

be two projective resolutions of $x \in \mathbb{C}$. Denote by $\iota : x \longrightarrow x$ be the identity morphism on x. Proposition 3.2.89 tells us that ι can be extended into a chain transformation $f : (a, \partial) \longrightarrow (b, \delta)$ such that $f_{-1} = \iota$. In a similar fashion, one can construct $g : (b, \delta) \longrightarrow (a, \partial)$ such that $g_{-1} = \iota$. It is easy to see that

$$gf = \{g_n f_n : a_n \longrightarrow a_n \mid n \in \mathbb{Z}\}$$

and

$$fg = \{f_n g_n : b_n \longrightarrow b_n \mid n \in \mathbb{Z}\}$$

are both chain transformations of (a, ∂) and (b, δ) respectively such that $g_{-1}f_{-1} = \iota = f_{-1}g_{-1}$. Proposition 3.2.89 implies that gf and fg are homotopic with the identity chain transformations of (a, ∂) and (b, δ) respectively. We call such chain transformations, *chain equivalences.* Whenever there are two projective resolutions (a, ∂) and (b, δ) of x and chain equivalences $f : (a, \partial) \longrightarrow (b, \delta)$ and $g : (b, \delta) \longrightarrow (a, \partial)$, we call (a, ∂) and (b, δ) *homotopically equivalent.* The following is immediate.

Lemma 3.2.90 *Any two projective resolutions of some $x \in \mathbb{C}$ are homotopically equivalent.*

Proposition 3.2.91 *If $j : a \longrightarrow p$ in $\mathbb{C}$ is an epimorphism and p projective, then j is a retraction. If $\mathbb{C}$ has a null object, then a coproduct $\oplus_{i \in I} p_i$ is projective if and only if each p_i is projective.*

See for the proof Proposition 10.4.6 of [93]. As a consequence of this we have that any coproduct of representables $\mathbb{C}(c, \_) \in Add(\mathbb{C}, \mathbf{Ab})$ is projective. Our intention is to find conditions under which a category of additive functors has enough projectives. Corollary 10.5.5 of [93] gives such conditions and can be obtained immediately from Proposition 10.5.4 of [93]. We give below both of them.

Proposition 3.2.92 *Let $\mathbb{C}$ be a category with coproducts (and thus an initial object). A set $\mathcal{G}$ of objects is a generating set if and only if for every $c \in \mathbb{C}$, the following holds true: For*

$$g_c = \bigoplus_{e \in \bigcup_{g \in \mathcal{G}} \mathbb{C}(g,c)} g_e, \quad g_e \text{ is the domain of } e,$$

the morphism $\pi_c : g_c \longrightarrow c$ defined by $\pi_c u_e = e$ is an epi.

Corollary 3.2.93 *A category with coproducts and a generating set of projectives has enough projectives.*

Proposition 3.2.94 *For every additive category $\mathbb{C}$, the category of additive functors $Add(\mathbb{C}, \mathbf{Ab})$ has enough projectives.*

Proof. Lemma 3.2.45 implies that $Add(\mathbb{C}, \mathbf{Ab})$ has coproducts. Theorem 3.2.67, Proposition 3.2.71 and Corollary 3.2.93 imply the result. ■

The consequence of this proposition is that we can apply Proposition 3.2.89 to compare between resolutions of functors from $Add(\mathbb{C}, \mathbf{Ab})$.

3.3 Tensor Product of Functors

In this section we will define the tensor product $\otimes_{\mathbb{C}}$ of two functors $F \in Add(\mathbb{C}, \mathbf{Ab})$ and $G \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and prove a number of properties.

3.3.1 The Definition of the Tensor Product

Definition 3.3.1 Let $\mathbb{C}$ and $\mathbb{D}$ be two additive categories and let $F \in Add(\mathbb{C}, \mathbf{Ab})$ and $G \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$. We define the tensor product of F and G, $F \otimes_{\mathbb{C}} G$ as a covariant functor from $Add(\mathbb{D}, \mathbf{Ab})$ defined on objects by

$$(F \otimes_{\mathbb{C}} G)(d) = \left(\underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d)) \right) / M,$$

where M is the subgroup generated by elements $x \otimes G(\gamma^{opp}, d)(y) - F(\gamma)(x) \otimes y$ for every $\gamma \in \mathbb{C}(c_1, c_2)$, $x \in F(c_1)$ and $y \in G(c_2, d)$, and on morphisms by the group morphism

$$(F \otimes_{\mathbb{C}} G)(\delta) : \left(\underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_1)) \right) / M_1 \longrightarrow \left(\underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_2)) \right) / M_2$$

for every $\delta : d_1 \longrightarrow d_2$, which sends the class of each element $x \otimes y \in F(c) \otimes_{\mathbb{Z}} G(c, d_1)$ to the class of $x \otimes G(c, \delta)(y) \in F(c) \otimes_{\mathbb{Z}} G(c, d_2)$.

In the future, the elements of the quotient group $\left(\underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_1)) \right) / M$ will be denoted either by $x + M$ or by $\overline{x}$ where $x \in \underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_1))$.

To make sure that $(F \otimes_{\mathbb{C}} G)(\delta)$ is well-defined and is a homomorphism, it is sufficient to show that it is induced from the homomorphism

$$\underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_1)) \longrightarrow \underset{c \in \mathbb{C}}{\oplus} (F(c) \otimes_{\mathbb{Z}} G(c, d_2))$$

arising from $G(c, \delta)$ with $\delta : d_1 \longrightarrow d_2$. For this, we need to show that $(F \otimes_{\mathbb{C}} G)(\delta)(M_1) \subseteq M_2$. Let $x \otimes G(\gamma^{opp}, d_1)(y) - F(\gamma)(x) \otimes y$ be a generator from M_1, where $x \in F(c)$, $y \in G(\gamma(c), d_1)$ for some morphism $\gamma : c \longrightarrow \gamma(c)$ in $\mathbb{C}$. Now $x \otimes G(\gamma^{opp}, d_1)(y)$ will be mapped to $x \otimes (G(c, \delta) \circ G(\gamma^{opp}, d_1))(y)$ and $F(\gamma)(x) \otimes y$ to $F(\gamma)(x) \otimes G(\gamma(c), \delta)(y)$. But

$$\begin{aligned} & x \otimes (G(c, \delta) \circ G(\gamma^{opp}, d_1))(y) - F(\gamma)(x) \otimes G(\gamma(c), \delta)(y) = \\ & x \otimes (G(\gamma^{opp}, d_2) \circ G(\gamma(c), \delta))(y) - F(\gamma)(x) \otimes G(\gamma(c), \delta)(y) \end{aligned}$$

from the commutativity of $G(\_, \delta)$ with $G(\gamma^{opp}, \_)$. On the other hand,

$$x \otimes (G(\gamma^{opp}, d_2) \circ G(\gamma(c), \delta))(y) - F(\gamma)(x) \otimes G(\gamma(c), \delta)(y)$$

is equivalent mod(M_2) to $x \otimes (G(\gamma^{opp}, d_2) \circ G(\gamma(c), \delta))(y) - x \otimes (G(\gamma^{opp}, d_2) \circ G(\gamma(c), \delta))(y) = 0$.

Lastly, using the fact that G is a functor, one can check easily that $F \otimes_{\mathbb{C}} G$ is a functor too. Since

$$G(c, \delta_1 + \delta_2) : G(c, d) \longrightarrow G(c, d')$$

is defined by

$$y \longmapsto G(c,\delta_1)(y) + G(c,\delta_2)(y),$$

(because G is additive), then passing to quotients, we get

$$(F \otimes_{\mathbb{C}} G)(\delta_1 + \delta_2)(\overline{x \otimes_{\mathbb{Z}} y}) = (F \otimes_{\mathbb{C}} G)(\delta_1)(\overline{x \otimes_{\mathbb{Z}} y}) + (F \otimes_{\mathbb{C}} G)(\delta_2)(\overline{x \otimes_{\mathbb{Z}} y}),$$

thus proving the additivity of $F \otimes_{\mathbb{C}} G$.

The following will be needed to prove Theorem 3.4.5.

Lemma 3.3.2 *Let $\mathbb{C}$ and $\mathbb{D}$ be two additive categories and let $F \in Add(\mathbb{C}, \mathbf{Ab})$ and $G_i \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ with $i \in I$. Then,*

$$(F \otimes_{\mathbb{C}} (\bigoplus_{i \in I} G_i)) \cong \bigoplus_{i \in I} (F \otimes_{\mathbb{C}} G_i).$$

Proof. From Lemma 3.2.45 and Definition 3.3.1 we have on the one hand that

$$\left(F \otimes_{\mathbb{C}} (\bigoplus_{i \in I} G_i)\right)(d) = \left(\bigoplus_{c \in \mathbb{C}} \left(F(c) \otimes_{\mathbb{Z}} (\bigoplus_{i \in I} G_i(c,d))\right)\right) / M^{\oplus},$$

where $M^{\oplus}$ is the subgroup generated by elements $x \otimes (\bigoplus_{i \in I} G_i)(\gamma^{opp}, d))(y) - F(\gamma)(x) \otimes y$ for every $\gamma \in \mathbb{C}(c_1, c_2)$, $x \in F(c_1)$ and $y \in (\bigoplus_{i \in I} G_i(c_2, d))$. By definition (see Lemma 3.2.45)

$$(\bigoplus_{i \in I} G_i)(\gamma^{opp}, d) = (\bigoplus_{i \in I} G_i(\gamma^{opp}, d)).$$

On the other hand, again from Definition 3.3.1, we have

$$\bigoplus_{i \in I} (F \otimes_{\mathbb{C}} G_i)(d) = \bigoplus_{i \in I} \left(\left(\bigoplus_{c \in \mathbb{C}} (F(c) \otimes_{\mathbb{Z}} G_i(c,d))\right) / M_i\right),$$

where M_i is the subgroup generated by elements $x \otimes G_i(\gamma^{opp}, d)(y) - F(\gamma)(x) \otimes y$ for every $\gamma \in \mathbb{C}(c_1, c_2)$, $x \in F(c_1)$ and $y \in G_i(c_2, d)$.

The isomorphism

$$h : \bigoplus_{c \in \mathbb{C}} \left(F(c) \otimes_{\mathbb{Z}} (\bigoplus_{i \in I} G_i(c,d))\right) \longrightarrow \bigoplus_{i \in I} \left(\bigoplus_{c \in \mathbb{C}} F(c) \otimes_{\mathbb{Z}} (G_i(c,d))\right),$$

defined on finite sums by

$$h\left(\sum_c \left(x_c \otimes_{\mathbb{Z}} \sum_i G_i(c,d)\right)\right) = \sum_i \sum_c (x_c \otimes_{\mathbb{Z}} G_i(c,d)),$$

induces a morphism

$$h^* : \left(F \otimes_{\mathbb{C}} (\bigoplus_{i \in I} G_i)\right)(d) \longrightarrow \bigoplus_{i \in I} (F \otimes_{\mathbb{C}} G_i)(d)$$

defined by

$$\left(x \otimes \sum_{j\in J} \alpha_j\right) + M^{\oplus} \longmapsto \sum_{j\in J} (x \otimes \alpha_j + M_j)\,,$$

where $J \subseteq I$ is finite and $\alpha_j \in G_j(c,d)$. One can show easily that h^* is an isomorphism, proving that for every $d \in \mathbb{D}$,

$$(F \otimes_{\mathbb{C}} (\underset{i\in I}{\oplus} G_i))(d) \cong \underset{i\in I}{\oplus} (F \otimes_{\mathbb{C}} G_i)(d).$$

Also h^* is natural in each $d \in \mathbb{D}$ which proves the claim. ■

3.3.2 A Universal Property for the Tensor Product

Given three additive functors $F \in Add(\mathbb{C}, \mathbf{Ab})$, $G \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and $H \in Add(\mathbb{D}, \mathbf{Ab})$ and let $d \in \mathbb{D}$ be a fixed object. We say that a map $\psi_d : \underset{c\in\mathbb{C}}{\cup} F(c) \times G(c,d) \longrightarrow H(d)$ is *bilinear* if for every $c \in \mathbb{C}$,

$$\psi_d((x, y' + y'')) = \psi_d((x,y')) + \psi_d((x,y'')), \forall x \in F(c) \text{ and } \forall y',\ y'' \in G(c,d),$$

$$\psi_d((x' + x'', y)) = \psi_d((x',y)) + \psi_d((x'',y)),\ \forall x',\ x'' \in F(c) \text{ and } \forall y \in G(c,d),$$

and

$$\psi_d((x, G(\gamma^{opp}, d)(y))) = \psi_d((F(\gamma)(x), y)),$$

for every c_1, c_2 objects in $\mathbb{C}$ and every $x \in F(c_1)$, $\gamma \in \mathbb{C}(c_1, c_2)$ and $y \in G(c_2, d)$.

Let $F \in Add(\mathbb{C}, \mathbf{Ab})$ and $G \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and $d \in \mathbb{D}$ a fixed object. Let

$$\mathcal{F} = \underset{c\in\mathbb{C}}{\oplus} \mathbb{Z}(F(c) \times G(c,d)),$$

where $\mathbb{Z}(F(c) \times G(c,d))$ is the free abelian group generated by $F(c) \times G(c,d)$, and

$$i : \underset{c\in\mathbb{C}}{\cup} F(c) \times G(c,d) \longrightarrow \mathcal{F}$$

be the inclusion map. From Definition 3.3.1 we have that $(F \otimes_{\mathbb{C}} G)(d)$ is obtained from $\mathcal{F}$ by factoring it by the subgroup B generated by the elements of the sets

$$\cup_{c\in\mathbb{C}}\{i((x, y' + y'')) - i((x,y')) - i((x,y''))\mid\ \forall x \in F(c) \text{ and } \forall y',\ y'' \in G(c,d)\}$$

and

$$\cup_{c\in\mathbb{C}}\{i((x' + x'', y)) - i((x',y)) - i((x'',y))\ \mid \forall x',\ x'' \in F(c) \text{ and } \forall y \in G(c,d)\},$$

together with elements of the form:

$$i((x, G(\gamma^{opp}, d)(y))) - i((F(\gamma)(x), y)),$$

for every c_1, c_2 objects in $\mathbb{C}$ and every $x \in F(c_1)$, $\gamma \in \mathbb{C}(c_1, c_2)$ and $y \in G(c_2, d)$.

We let $\mu_d = p \circ i$ where $p : \mathcal{F} \longrightarrow \mathcal{F}/B$ is the canonical epimorphism.

Lemma 3.3.3 *Given three additive functors $F \in Add(\mathbb{C}, \mathbf{Ab})$, $G \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and $H \in Add(\mathbb{D}, \mathbf{Ab})$, and let $d \in \mathbb{D}$ be a fixed object. The pair $\left(\bigcup_{c\in\mathbb{C}} F(c) \times G(c,d), \mu_d\right)$ has the universal property: for every bilinear map $\psi_d : \bigcup_{c\in\mathbb{C}} F(c) \times G(c,d) \longrightarrow H(d)$, there is a unique homomorphism θ_d making the diagram*

$$\begin{tikzcd} \bigcup_{c\in\mathbb{C}} (F(c) \times G(c,d)) \arrow[rr, "\mu_d"] \arrow[dr, "\forall\psi_d"'] & & \left(\bigoplus_{c\in\mathbb{C}} (F(c) \otimes_{\mathbb{Z}} G(c,d))\right)/M \arrow[dl, dashed, "\exists!\theta_d"] \\ & H(d) & \end{tikzcd} \tag{3.11}$$

commutative.

Proof. We consider the commutative diagram

$$\begin{tikzcd} \bigcup_{c\in\mathbb{C}} (F(c) \times G(c,d)) \arrow[rr, "i"] \arrow[dr, "\psi_d"'] & & \mathcal{F} \arrow[dl, dashed, "j"] \\ & H(d) & \end{tikzcd}$$

where j is a morphism such that $j \circ i = \psi$. The existence of j comes from the freeness of $\mathcal{F}$. Using the fact that ψ_d is bilinear, one can easily see that $B \subseteq Kerj$ and hence j induces a morphism $\theta : \mathcal{F}/B \longrightarrow H(d)$ such that $\theta \circ p = j$. It follows that the triangle (3.11) commutes since $\theta \circ \mu = \theta \circ p \circ i = j \circ i = \psi$. To show the uniqueness of θ, we suppose by the way of contradiction that there is another θ' making (3.11) commute. Every generator $t \in (F \otimes_{\mathbb{C}} G)(d)$ is expressed in the form

$$t = \mu_d((x, y)),$$

where $u = (x, y) \in F(c) \times G(c, d)$ for some $c \in \mathbb{C}$. Therefore, $\theta_d(t) = \theta_d\mu_d(u) = \psi_d(u) = \theta'_d\mu_d(u) = \theta'_d(t)$. ■

3.3.3 Functorial Properties of the Tensor Product

We prove in this section that the tensor product can be regarded as an additive functor

$$\otimes_{\mathbb{C}} : Add(\mathbb{C}, \mathbf{Ab}) \otimes_{\mathbb{Z}} Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab}) \longrightarrow Add(\mathbb{D}, \mathbf{Ab})$$

defined on objects by

$$(F, G) \longrightarrow F \otimes_{\mathbb{C}} G.$$

It remains to define it on morphisms and then check for the functorial properties of it. Since every morphisms $\alpha \otimes_{\mathbb{Z}} \beta : (F_1, G_1) \longrightarrow (F_2, G_2)$ equals with the composition $(\alpha \otimes_{\mathbb{Z}} 1_{G_2})(1_{F_1} \otimes_{\mathbb{Z}} \beta)$, or with $(1_{F_2} \otimes_{\mathbb{Z}} \beta)(\alpha \otimes_{\mathbb{Z}} 1_{G_1})$, then it is sufficient to make the definition for morphisms of the form $1_F \otimes_{\mathbb{Z}} \beta$ and $\alpha \otimes_{\mathbb{Z}} 1_G$.

Using the universal property of Lemma 3.3.3, we show that, if $\partial : G_1 \longrightarrow G_2$ is a morphism in $Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and $F \in Add(\mathbb{C}, \mathbf{Ab})$, then there is an induced morphism

$$\theta : F \otimes_{\mathbb{C}} G_1 \longrightarrow F \otimes_{\mathbb{C}} G_2$$

in $Add(\mathbb{D}, \mathbf{Ab})$.

Let now G_1 and G_2 be two functors in $Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ and

$$\partial = \{\partial_{(c,d)} \mid c \in \mathbb{C} and d \in \mathbb{D}\}$$

a natural transformation from G_1 to G_2. If $F \in Add(\mathbb{C}, \mathbf{Ab})$ then for every $c \in \mathbb{C}$ and $d \in \mathbb{D}$ one has an induced map

$$F(c) \times G_1(c, d) \overset{F \times \partial_{(c,d)}}{\longrightarrow} F(c) \times G_2(c, d)$$

and therefore by extension the map

$$\bigcup_{c \in \mathbb{C}} (F(c) \times G_1(c, d)) \overset{\cup F \times \partial_{(c,d)}}{\longrightarrow} \bigcup_{c \in \mathbb{C}} (F(c) \times G_2(c, d)).$$

For $j = 1, 2$ we denote the elements of $\bigoplus_{c \in \mathbb{C}} (F(c) \otimes_{\mathbb{Z}} G_j(c, d))$ by

$$((z_1 \otimes_{\mathbb{Z}} \alpha_1), ..., (z_n \otimes_{\mathbb{Z}} \alpha_n)) \text{ where } (z_i \otimes_{\mathbb{Z}} \alpha_i) \in F(c_i) \otimes_{\mathbb{Z}} G_j(c_i, d) \text{ for } i = 1, ..., n,$$

and the elements of $(F \otimes_{\mathbb{C}} G_j)(d)$ by

$$((z_1 \otimes_{\mathbb{Z}} \alpha_1), ..., (z_n \otimes_{\mathbb{Z}} \alpha_n)) + M_j$$

where M_j is the subgroup of $\bigoplus_{c \in \mathbb{C}} (F(c) \otimes_{\mathbb{Z}} G_j(c, d))$ defined as in Definition 3.3.1.

Using the universal property of the tensor $F \otimes_{\mathbb{C}} G_1$ depicted in the diagram (3.11), we obtain the following commutative diagram

$$\begin{tikzcd}
\bigcup_{c \in \mathbb{C}} (F(c) \times G_1(c, d)) \arrow[rr, "\mu_1"] \arrow[dr, "\cup F \times \partial_{(c,d)}"'] & & (F \otimes_{\mathbb{C}} G_1)(d) \arrow[dd, dashed, "\exists! \theta_d"] \\
& \bigcup_{c \in \mathbb{C}} (F(c) \times G_2(c, d)) \arrow[dr, "\mu_2"'] & \\
& & (F \otimes_{\mathbb{C}} G_2)(d)
\end{tikzcd} \tag{3.12}$$

with

$$\theta_d : (F \otimes_{\mathbb{C}} G_1)(d) \longrightarrow (F \otimes_{\mathbb{C}} G_2)(d)$$

defined by

$$((z_1 \otimes_{\mathbb{Z}} \alpha_1), ..., (z_n \otimes_{\mathbb{Z}} \alpha_n)) + M_1 \longmapsto ((z_1 \otimes_{\mathbb{Z}} \partial_{c,d}(\alpha_1)), ..., (z_n \otimes_{\mathbb{Z}} \partial_{c,d}(\alpha_n)) + M_2. \quad (3.13)$$

Also we have that both μ_1 and $\mu_2 \circ (\oplus F \times \partial_{(c,d)})$ are bilinear. That μ_1 is bilinear follows from the way it is defined. Let use check for convenience that the 3^{rd} condition of bilinearity for $\psi_d = \mu_2 \circ (\oplus F \times \partial_{(c,d)})$ is satisfied. Let $z \in F(c_1)$, $\gamma \in \mathbb{C}(c_1, c_2)$ and $y \in G_1(c_2, d)$. Then,

$$\psi_d(z, G_1(\gamma^{opp}, d)(y)) = z \otimes_{\mathbb{Z}} \partial_{(c_1,d)}(G_1(\gamma^{opp}, d)(y)) + M_2;$$

but the naturality of ∂ implies that $\partial_{(c_1,d)}(G_1(\gamma^{opp}, d)(y)) = G_2(\gamma^{opp}, d)(\partial_{(c_2,d)}(y))$ and therefore

$$\psi_d(z, G_1(\gamma^{opp}, d)(y)) = z \otimes_{\mathbb{Z}} G_2(\gamma^{opp}, d)(\partial_{(c_2,d)}(y)) + M_2.$$

On the other hand

$$z \otimes_{\mathbb{Z}} G_2(\gamma^{opp}, d)(\partial_{(c_2,d)}(y)) + M_2 = F(\gamma)(z) \otimes_{\mathbb{Z}} \partial_{(c_2,d)}(y) + M_2 = \psi_d(F(\gamma)(z), y),$$

which proves the condition.

Next we prove that the family

$$\{\theta_d \mid d \in \mathbb{D}\}$$

is a natural transformation. For this we examine the following diagram with $\delta : d \longrightarrow d'$

$$\begin{tikzcd}
(z,\alpha) \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{z \otimes \alpha} \arrow[dl, dashed] \arrow[dd] \\
& \overline{z \otimes \partial_{(c,d)}(\alpha)} \arrow[dd] & \\
(z, G_1(c,\delta)(\alpha)) \arrow[rr] \arrow[dr] & & \overline{z \otimes G_1(c,\delta)(\alpha)} \arrow[dl, dashed] \\
& \overline{z \otimes \partial_{(c,d')}(G_1(c,\delta)(\alpha))} &
\end{tikzcd}$$

and (z, α) being for simplicity a vector with a single coordinate. The commutativity of the right-hand side square comes from the fact that the family $\partial_{(c,d)} : G_1(c,d) \longrightarrow G_2(c,d)$ is a natural transformation and therefore, for every $\alpha \in G_1(c,d)$, $G_2(c,\delta)(\partial_{(c,d)}(\alpha)) = \partial_{(c,d')}(G_1(c,\delta)(\alpha))$.

In the future, we denote the induced morphism θ by $F \otimes_{\mathbb{C}} \partial$.

Finally, by using twice the diagram (3.12), one can show that for every $\partial_2 : G_2 \longrightarrow G_3$, $\partial_1 : G_1 \longrightarrow G_2$ and $F \in Add(\mathbb{C}, \mathbf{Ab})$ we have that

$$F \otimes_{\mathbb{C}} \partial_2\partial_1 = (F \otimes_{\mathbb{C}} \partial_2)(F \otimes_{\mathbb{C}} \partial_1),$$

which proves the functoriality of $\otimes_{\mathbb{C}}$ in the second variable. Also (3.13) implies that for every $\partial, \partial' : G_1 \longrightarrow G_2$ we have that

$$F \otimes_{\mathbb{C}} (\partial + \partial') = F \otimes_{\mathbb{C}} \partial + F \otimes_{\mathbb{C}} \partial',$$

which proves the additivity.

One can show in exactly the same way as above that $\otimes_{\mathbb{C}}$ is an additive covariant functor in the first variable, making thus $\otimes_{\mathbb{C}}$ a covariant additive functor of two variables as claimed.

3.3.4 Tensoring Over Sequences

Let

$$(A, \partial) : \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} A_{-1} \longrightarrow 0$$

be a sequence of functors in $Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$ such that $\partial^2 = 0$. For every $Y \in Add(\mathbb{C}, \mathbf{Ab})$ we will denote by $Y \otimes_{\mathbb{C}} A$ the sequence

$$\longrightarrow Y \otimes_{\mathbb{C}} A_{n+1} \xrightarrow{Y \otimes_{\mathbb{C}} \partial_{n+1}} Y \otimes_{\mathbb{C}} A_n \xrightarrow{Y \otimes_{\mathbb{C}} \partial_n} \dots \xrightarrow{Y \otimes_{\mathbb{C}} \partial_1} Y \otimes_{\mathbb{C}} A_0 \xrightarrow{Y \otimes_{\mathbb{C}} \partial_0} Y \otimes_{\mathbb{C}} A_{-1} \longrightarrow 0,$$

in $Add(\mathbb{D}, \mathbf{Ab})$ where θ_n is induced from ∂_n for every $n \geq 0$.

As the following lemma shows, $Y \otimes_{\mathbb{C}} A$ has the property that $\theta^2 = 0$.

Lemma 3.3.4 *Let for $i = 1, 2, 3$, $G_i \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$. Suppose that in the diagram*

$$G_3 \xrightarrow{\partial'} G_2 \xrightarrow{\partial''} G_1$$

we have $\partial''\partial' = 0$; then for any $F \in Add(\mathbb{C}, \mathbf{Ab})$ in the induced diagram

$$(F \otimes_{\mathbb{C}} G_3) \xrightarrow{F \otimes_{\mathbb{C}} \partial'} (F \otimes_{\mathbb{C}} G_2) \xrightarrow{F \otimes_{\mathbb{C}} \partial''} (F \otimes_{\mathbb{C}} G_1),$$

we have $(F \otimes_{\mathbb{C}} \partial'')(F \otimes_{\mathbb{C}} \partial') = 0$.

Proof. From the condition we have that $\partial''\partial' = 0$ therefore, $F \otimes_{\mathbb{C}} \partial''\partial' = 0$ since $\otimes_{\mathbb{C}}$ is additive. On the other hand $F \otimes_{\mathbb{C}} \partial''\partial' = (F \otimes_{\mathbb{C}} \partial'')(F \otimes_{\mathbb{C}} \partial')$ which implies that $(F \otimes_{\mathbb{C}} \partial'')(F \otimes_{\mathbb{C}} \partial') = 0$. ■

We prove now the analogue of III, Lemma 2.1 of [44] for the tensor product $\otimes_{\mathbb{C}}$.

Lemma 3.3.5 *Let*

$$(A,\partial): \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_0} X \longrightarrow 0$$

and

$$(B,\delta): \xrightarrow{\delta_{n+2}} B_{n+1} \xrightarrow{\delta_{n+1}} B_n \xrightarrow{\delta_n} B_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_0} X \longrightarrow 0$$

two projective resolutions of the functor $X \in Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$. *Then, for every* $Y \in Add(\mathbb{C}, \mathbf{Ab})$ *and every* $d \in \mathbb{D}$, *we have that* $H_n((Y \otimes_{\mathbb{C}} A)(d)) \cong H_n((Y \otimes_{\mathbb{C}} B)(d))$ *for every* n.

Proof. From Lemma 3.2.90 there exist chain transformations

$$f = \{f_n : A_n \longrightarrow B_n \mid n \geq -1\}$$
$$g = \{g_n : B_n \longrightarrow A_n \mid n \geq -1\}$$

such that the compositions

$$gf = \{g_n f_n : A_n \longrightarrow A_n \mid n \geq -1\}$$
$$fg = \{f_n g_n : B_n \longrightarrow B_n \mid n \geq -1\}$$

are homotopic to the identity chain transformations of A and B respectively.

It is easy to see that

$$\begin{aligned} \varphi &= \{\varphi_n : Y \otimes_{\mathbb{C}} A_n \longrightarrow Y \otimes_{\mathbb{C}} B_n \mid n \geq -1\} \\ \psi &= \{\psi_n : Y \otimes_{\mathbb{C}} B_n \longrightarrow Y \otimes_{\mathbb{C}} A_n \mid n \geq -1\} \end{aligned} \tag{3.14}$$

are chain transformations, where $\varphi_n = \{\varphi_{d,n} \mid d \in \mathbb{D}\}$ and $\psi_n = \{\psi_{d,n} \mid d \in \mathbb{D}\}$ are the natural transformations induced by respectively f_n and g_n. In the case of φ for example, since for every $n \geq 0$ we have that $\partial_n f_{n-1} = \delta_n f_n$, then $(Y \otimes_{\mathbb{C}} \partial_n)\varphi_{n-1} = (Y \otimes_{\mathbb{C}} \partial_n)(Y \otimes_{\mathbb{C}} f_{n-1}) = (Y \otimes_{\mathbb{C}} \delta_n)(Y \otimes_{\mathbb{C}} f_n) = (Y \otimes_{\mathbb{C}} \delta_n)\varphi_n$.

Since φ and ψ are chain transformations, it follows that $\forall d \in \mathbb{D}$,

$$\begin{aligned} \varphi_d &= \{\varphi_{d,n} : (Y \otimes_{\mathbb{C}} A_n)(d) \longrightarrow (Y \otimes_{\mathbb{C}} B_n)(d) \mid n \geq -1\} \\ \psi_d &= \{\psi_{d,n} : (Y \otimes_{\mathbb{C}} B_n)(d) \longrightarrow (Y \otimes_{\mathbb{C}} A_n)(d) \mid n \geq -1\} \end{aligned} \tag{3.15}$$

are chain transformations, therefore there are induced homomorphisms

$$\varphi^*_{d,n} : H_n((Y \otimes_{\mathbb{C}} A)(d)) \longrightarrow H_n((Y \otimes_{\mathbb{C}} B)(d))$$
$$\psi^*_{d,n} : H_n((Y \otimes_{\mathbb{C}} B)(d)) \longrightarrow H_n((Y \otimes_{\mathbb{C}} A)(d))$$

for every $n \geq 0$ and $d \in \mathbb{D}$. Using the fact that gf and fg are homotopic to the identity chain transformations of (A,∂) and (B,δ) respectively, one can easily show that $\varphi\psi$ and $\psi\varphi$ are respectively homotopic to the identity of $Y \otimes_{\mathbb{C}} A_n$ and $Y \otimes_{\mathbb{C}} B_n$ and therefore for every $d \in \mathbb{D}$,

$\varphi_d \circ \psi_d$ and $\psi_d \circ \varphi_d$ are respectively homotopic to the identity of $(Y \otimes_{\mathbb{C}} A_n)(d)$ and $(Y \otimes_{\mathbb{C}} B_n)(d)$. Now as in III, Lemma 2.1 of [44] we can show that $\psi^*_{d,n} \circ \varphi^*_{d,n}$ and $\varphi^*_{d,n} \circ \psi^*_{d,n}$ are the identity automorphisms of the groups $H_n((Y \otimes_{\mathbb{C}} A)(d))$ and $H_n((Y \otimes_{\mathbb{C}} B)(d))$ which means that these groups are isomorphic. ■

Proposition 3.3.6 *If X is a projective functor in $Add(\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{D}, \mathbf{Ab})$, then for every $Y \in Add(\mathbb{C}, \mathbf{Ab})$ and $d \in \mathbb{D}$ we have that $H_n((Y \otimes_{\mathbb{C}} A)(d)) = 0$ where $n \in \mathbb{N}$ and A is any projective resolution of X.*

Proof. From Lemma 3.3.5 we are free to chose the projective resolution of X. Let just take it to be

$$\ldots 0 \longrightarrow 0 \longrightarrow X \xrightarrow{1_X} X \longrightarrow 0$$

where ι is the identity, and then the following is exact

$$\ldots 0 \longrightarrow 0 \longrightarrow Y \otimes_{\mathbb{C}} X \xrightarrow{Y \otimes_{\mathbb{C}} 1_X} Y \otimes_{\mathbb{C}} X \longrightarrow 0$$

because the induced $Y \otimes_{\mathbb{C}} 1_X$ is again the identity, which proves the claim. ■

3.4 Homological Finiteness Conditions for Small Categories

In this section we will give the homological finiteness conditions left (respectively right)-FP$_n$ and bi-FP$_n$ for small categories as natural generalizations of their counterparts for monoids and relate them with a new finiteness condition called f-FP$_n$ which is introduced by Malbos in [67] but called there FP$_n$.

In what follows we will deal with categories $Add(\mathbb{A}, \mathbf{Ab})$ which we denote for short by $\mathbf{Ab}^{\mathbb{A}}$. In the future we take the generators of $\mathbf{Ab}^{\mathbb{A}}$ to be the representables $\mathbb{A}(a, \_)$ with $a \in \mathbb{A}$.

Definition 3.4.1 Let $n \geq 0$ be an integer and $\mathbb{A}$ an additive category. An object B in $\mathbf{Ab}^{\mathbb{A}}$ is said to be of *type FP_n* if there is a partial projective resolution in $\mathbf{Ab}^{\mathbb{A}}$

$$P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

such that P_i is finitely generated for $0 \leq i \leq n$.

The following from [67] holds true.

Lemma 3.4.2 *Let $\mathbb{A}$ an additive category. For each B in $\mathbf{Ab}^{\mathbb{A}}$ and $n \geq 0$, the following are equivalent:*

1. *there is a partial resolution $F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$, where each object F_i is free (in the sense of Definition 3.2.40) and finitely generated,*

2. *B is of type FP_n,*

3. *B is finitely generated and for every partial projective resolution*

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} \ldots \longrightarrow P_0 \xrightarrow{d_0} B \longrightarrow 0,$$

with $k < n$ and each P_i finitely generated, then $Kerd_k$ is finitely generated.

Proof. The proof runs the same as the proof of Proposition 4.3 of [12] for modules and uses Lemma 3.2.85. ∎

3.4.1 Small Categories of Type bi-FP$_n$ and left (right)-FP$_n$

We will generalize in this section the notions of bi-FP$_n$ and left (right)-FP$_n$ monoids for small categories.

Define $\mathbb{ZC}$ to be the functor $\mathbb{ZC} : \mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC} \longrightarrow \mathbf{Ab}$ where $\mathbb{ZC}(p, q)$ is the free abelian group generated by $\mathbb{C}(p, q)$ and if $(\alpha \otimes \beta) : (p, q) \longrightarrow (p', q')$ is an arrow in $\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC}$, where $\alpha \in \mathbb{C}^{opp}(p, p')$ and $\beta \in \mathbb{C}(q, q')$, then $\mathbb{ZC}(\alpha \otimes \beta) : \mathbb{ZC}(p, q) \longrightarrow \mathbb{ZC}(p', q')$ is defined by sending every arrow $\gamma \in \mathbb{ZC}(p, q)$ to $\beta\gamma\alpha^{opp} \in \mathbb{ZC}(p', q')$.

Define the trivial left (respectively right) $\mathbb{ZC}$-module $\mathbb{Z}$, as the additive functor $\mathbb{Z} : \mathbb{ZC} \longrightarrow \mathbf{Ab}$ (respectively $\mathbb{Z} : \mathbb{ZC}^{opp} \longrightarrow \mathbf{Ab}$) by sending each object of $\mathbb{ZC}$ to the group $\mathbb{Z}$ and each morphism of $\mathbb{C}$ to $1_{\mathbb{Z}}$.

Definition 3.4.3 A small category $\mathbb{C}$ is said to be of *type*

1. *bi-FP_n* if the functor $\mathbb{ZC}$ is of type FP$_n$ in $\mathbf{Ab}^{\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC}}$,

2. of *type left-FP_n* (respectively *right-FP_n*) if the trivial left (respectively right) $\mathbb{ZC}$-module $\mathbb{Z}$ is of type FP$_n$ in $\mathbf{Ab}^{\mathbb{ZC}}$ (respectively $\mathbf{Ab}^{\mathbb{ZC}^{opp}}$).

Remark 3.4.4 In the case of monoids, conditions *bi-FP_n* and *left-FP_n* (respectively *right-FP_n*) just defined coincide with those defined in Section 1.9 for monoids. Indeed, since a monoid S is a category with a single object $*$, then the functor $\mathbb{Z}S$ is just the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\mathbb{Z}S$. The value at $(*,*)$ of the only representable $(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S)((*,*),(\_,\_))$ is the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$; therefore every free resolution in $\mathbf{Ab}^{\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S}$ of the functor $\mathbb{Z}S$ can be seen as a free resolution of the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\mathbb{Z}S$. Similarly one can discuss the left or right case.

Theorem 3.4.5 *For every small category $\mathbb{C}$ the following implication holds true:*

$$\textit{bi-}FP_n \Longrightarrow \textit{left (right)-}FP_n.$$

Before we prove the theorem, we prove the following proposition which will be useful in the proof of the theorem.

Proposition 3.4.6 $\mathbb{Z} \otimes_{\mathbb{Z}\mathbb{C}} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a,b),(\_,\_)) \cong \mathbb{Z}\mathbb{C}(b,\_)$ *where $\mathbb{Z}$ is the trivial left $\mathbb{Z}\mathbb{C}$-module $\mathbb{Z}$.*

Proof. First we prove that, for every $d \in \mathbb{C}$, there is an isomorphism

$$f_d : \left(\bigoplus_{c \in \mathbb{C}} \mathbb{Z}(c) \otimes_{\mathbb{Z}} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a,b),(c,d)) \right) / M \longrightarrow \mathbb{Z}\mathbb{C}(b,d), \tag{3.16}$$

where M is the subgroup generated by the elements

$$x \otimes (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a,b),(\gamma^{opp},d))(y) - \mathbb{Z}(\gamma)(x) \otimes y$$

for every $x \in \mathbb{Z}(c)$, $\gamma \in \mathbb{C}(c,c')$ and $y \in (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a,b),(c',d))$.

For every fixed $c \in \mathbb{C}$ and $d \in \mathbb{C}$, we can write $(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a,b),(c,d))$ in the form $\mathbb{Z}\mathbb{C}^{opp}(a,c) \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}(b,d)$ and use the universal property of $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}^{opp}(a,c) \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}(b,d)$ presented by the following diagram

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z}\mathbb{C}^{opp}(a,c) \times \mathbb{Z}\mathbb{C}(b,d) \arrow[rr, "\mu"] \arrow[dr, "\psi"'] & & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}^{opp}(a,c) \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}(b,d) \arrow[dl, "\theta"] \\ & \mathbb{Z}\mathbb{C}(b,d) & \end{tikzcd}$$

The map ψ is defined by

$$\psi\left(x, \sum_i n_i \alpha_i, \beta\right) = \left(x \sum_i n_i\right).\beta$$

for every $x \in \mathbb{Z}$, $\alpha_i \in \mathbb{C}^{opp}(a,c)$, $n_i \in \mathbb{Z}$ and $\beta \in \mathbb{Z}\mathbb{C}(b,d)$. It is obviously a trilinear map, therefore there is θ defined by

$$\theta\left(x \otimes \left(\sum_i n_i \alpha_i\right) \otimes \beta\right) = \left(x \sum_i n_i\right).\beta$$

which makes the diagram commutative. This diagram induces the diagram

$$\begin{tikzcd}
\underset{c\in\mathbb{C}}{\oplus}(\mathbb{Z}\times\mathbb{Z}\mathbb{C}^{opp}(a,c)\times\mathbb{Z}\mathbb{C}(b,d)) \arrow[rr, "\oplus\mu"] \arrow[dr, "\oplus\psi"'] & & \underset{c\in\mathbb{C}}{\oplus}(\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,c)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)) \arrow[dl, "\oplus\theta"] \\
& \mathbb{Z}\mathbb{C}(b,d) &
\end{tikzcd}$$

where $\oplus\mu$, $\oplus\theta$ and $\oplus\psi$ are linear extensions of respectively μ, θ and ψ.

We claim that $M \subseteq Ker(\oplus\theta)$. For this we need to check whether $\oplus\theta$ vanishes on the generators of M. Since $\mathbb{Z}$ is the trivial functor and since $(\mathbb{Z}\mathbb{C}^{opp}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C})((a,b),(\gamma^{opp},d))$ acts only on $\mathbb{Z}\mathbb{C}^{opp}(a,c')$, we can write the generators of M in the form

$$x\otimes\gamma^{opp}\alpha\otimes\beta - x\otimes\alpha\otimes\beta, \tag{3.17}$$

where $\alpha\in\mathbb{Z}\mathbb{C}^{opp}(a,c')$, $\beta\in\mathbb{Z}\mathbb{C}(b,d)$ and $\gamma\in\mathbb{C}(c,c')$.

Now it is straightforward that $(\oplus\theta)(x\otimes\gamma^{opp}\alpha\otimes\beta - x\otimes\alpha\otimes\beta)=0$. The fact that $M\subseteq Ker(\oplus\theta)$ implies that $\oplus\theta$ induces a morphism

$$f_d : \left(\underset{c\in\mathbb{C}}{\oplus}\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,c)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)\right)/M \longrightarrow \mathbb{Z}\mathbb{C}(b,d)$$

such that for the generators $1\otimes\alpha\otimes\beta+M$ with $\alpha\in\mathbb{C}^{opp}(a,c)$ and $\beta\in\mathbb{C}(b,d)$ we have:

$$f_d(1\otimes\alpha\otimes\beta+M)=\beta.$$

We need the following claim.

Claim 1 *For every $x\in\mathbb{Z}$, $\alpha\in\mathbb{C}^{opp}(a,c)$ and $\beta\in\mathbb{Z}\mathbb{C}(b,d)$, the element $x\otimes\alpha\otimes\beta\in\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,c)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)$ is equivalent mod(M) with $x\otimes id_a\otimes\beta\in\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,a)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)$.*

Proof. We can write $x\otimes\alpha\otimes\beta = x\otimes\gamma^{opp}id_a\otimes\beta$ where $\gamma\in\mathbb{C}(c,a)$ is such that $\gamma^{opp}=\alpha$. From (3.17) $x\otimes\gamma^{opp}id_a\otimes\beta$ is equivalent mod(M) with $x\otimes id_a\otimes\beta$ proving the claim. ∎

The claim and the fact that

$$x\otimes\alpha\otimes\beta = x.(1\otimes\alpha\otimes\beta)$$

for every $x\in\mathbb{Z}$, implies that every generator of $\left(\underset{c\in\mathbb{C}}{\oplus}\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,c)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)\right)/M$ has the form $1\otimes id_a\otimes\beta+M$ for some $\beta\in\mathbb{C}(b,d)$.

We also define a morphism

$$g_d : \mathbb{Z}\mathbb{C}(b,d) \longrightarrow \left(\underset{c\in\mathbb{C}}{\oplus}\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}^{opp}(a,c)\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}(b,d)\right)/M$$

by

$$\beta \longmapsto 1 \otimes id_a \otimes \beta + M$$

and see that f_d and g_d are mutually inverse of each other which finally proves the isomorphism (3.16).

As we saw in Definition 3.3.1, the morphism $\delta : d_1 \longrightarrow d_2$ induces the group morphism

$$(\mathbb{Z} \otimes_{\mathbb{ZC}} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a,b),(\_,\_)))(d_1) \longrightarrow (\mathbb{Z} \otimes_{\mathbb{ZC}} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a,b),(\_,\_)))(d_2)$$

defined on generators by

$$1 \otimes id_a \otimes \beta + M_1 \longmapsto 1 \otimes id_a \otimes \delta\beta + M_2.$$

On the other hand,

$$f_{d_1}(1 \otimes id_a \otimes \beta + M_1) = \beta$$

and

$$f_{d_2}(1 \otimes id_a \otimes \delta\beta + M_2) = \delta\beta,$$

which imply that

$$\mathbb{ZC}(b,\delta) \circ f_{d_1} = f_{d_2} \circ (\mathbb{Z} \otimes_{\mathbb{ZC}} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a,b),(\_,\_)))(\delta).$$

This means that f_d in natural in every $d \in \mathbb{D}$. Therefore we have the isomorphism $\mathbb{Z} \otimes_{\mathbb{ZC}} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a,b),(\_,\_)) \cong \mathbb{ZC}(b,\_)$. ■

Proof of Theorem 3.4.5. Since $\mathbb{C}$ is bi-FP$_n$, then by Lemma 3.4.2 there exists a free finitely generated partial resolution of the functor $\mathbb{ZC} \in \mathbf{Ab}^{\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC}}$:

$$\bigoplus_{i \in I_n} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a_{ni}, b_{ni}),(\_,\_)) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \bigoplus_{i \in I_0} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a_{0i}, b_{0i}),(\_,\_))$$

$$\xrightarrow{\partial_0} \mathbb{ZC}(\_,\_) \longrightarrow 0; \tag{3.18}$$

we want to construct a projective finitely generated resolution of the trivial functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{ZC}}$.

By repeating the argument of the proof of Proposition 3.4.6, one can see that tensoring $\mathbb{ZC} \in \mathbf{Ab}^{\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC}}$ on the left by the trivial module $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{ZC}}$ yields $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{ZC}}$. Again from Proposition 3.4.6 if we tensor on the left by the trivial functor $\mathbb{Z}$ each of $(\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a_{si}, b_{si}),(\_,\_))$ for some fixed $0 \le s \le n$ obtain $\mathbb{ZC}(b_{si},\_)$. Lemma 3.3.2 implies that for every $d \in \mathbb{C}$

$$(\mathbb{Z} \otimes_{\mathbb{ZC}} \bigoplus_{i \in I_k} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a_{ki}, b_{ki}),(\_,\_)))(d) \cong (\bigoplus_{i \in I_k} \mathbb{Z} \otimes_{\mathbb{ZC}} (\mathbb{ZC}^{opp} \otimes_{\mathbb{Z}} \mathbb{ZC})((a_{ki}, b_{ki}),(\_,\_)))(d),$$

for all $0 \leq k \leq n$. Therefore if we tensor through on the left of (3.18) by $\mathbb{Z}$ and evaluate all the functors in some d, we obtain from Lemma 3.3.4 and above, the chain complex of abelian groups

$$\underset{i\in I_n}{\oplus} \mathbb{Z}\mathbb{C}(b_{ni}, d) \xrightarrow{\mathbb{Z}\otimes_{\mathbb{C}}\partial_n} \dots \xrightarrow{\mathbb{Z}\otimes_{\mathbb{C}}\partial_1} \underset{i\in I_0}{\oplus} \mathbb{Z}\mathbb{C}(b_{0i}, d) \xrightarrow{\mathbb{Z}\otimes_{\mathbb{C}}\partial_0} \mathbb{Z} \longrightarrow 0 \tag{3.19}$$

The last thing is to show that (3.19) is exact. To do this, we will obtain (3.19) is another way which will allow us to use Proposition 3.3.6.

Firstly, for every fixed representable $(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a, b), (\_, \_)) \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}\otimes_{\mathbb{Z}}\mathbb{Z}\mathbb{C}}$ and a fixed $d \in \mathbb{C}$, we let the functor $R_{(a,b,d)} \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$ be

$$R_{(a,b,d)} = \underset{\mathbb{C}(b,d)}{\oplus} \mathbb{Z}\mathbb{C}^{opp}(a, \_)$$

which is projective since it is a coproduct of projectives in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$. Note here that for every $c \in \mathbb{C}$,

$$R_{(a,b,d)}(c) = (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((a, b), (c, d)).$$

Secondly, for every fixed $d \in \mathbb{C}$, we let $S_d \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$ be

$$S_d = \mathbb{Z}\mathbb{C}^{opp}(d, \_),$$

which is again projective in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$. Note again that for every $c \in \mathbb{C}$,

$$S_d(c) = \mathbb{Z}\mathbb{C}(c, d).$$

The resolution (3.18) induces a projective resolution of S_d in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp}}$ as follows:

$$\underset{i\in I_n}{\oplus} R_{(a_{ni}, b_{ni}, d)} \xrightarrow{\partial_{n,d}} \dots \xrightarrow{\partial_{1,d}} \underset{i\in I_0}{\oplus} R_{(a_{0i}, b_{0i}, d)} \xrightarrow{\partial_{0,d}} S_d \longrightarrow 0 \tag{3.20}$$

where for every $0 \leq i \leq n$, $\partial_{i,d}$ is the natural transformation induced by ∂_i.

If we tensor (3.20) on the left by the trivial module $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}}$ we obtain (3.19) and then we apply Proposition 3.3.6 by taking $Y = \mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}}$, $\mathbb{D} = \mathbb{Z}$, $X = S_d$ and (3.20) as the resolution of X, obtaining that (3.19) is exact. ∎

3.4.2 Small Categories of Type f-FP$_n$

In [67] Malbos has defined a new finiteness condition for small categories called FP$_n$. We will give the definition for it but, as we explained in Section 3.1, we will change its name to f-FP$_n$. Also we will relate it with the other finiteness conditions studied in the previous section: bi-FP$_n$ and left (right)-FP$_n$.

> In what follows we will write the composition of two arrows $u : a \longrightarrow b$ with $v : b \longrightarrow c$ as uv instead of the standard notation vu.

To define f-FP_n, we need the following.

Definition 3.4.7 Let $\mathbb{C}$ be a small category. The *category of factorizations* $F\mathbb{C}$ in $\mathbb{C}$, has objects the set of morphisms in $\mathbb{C}$ and a morphism in $F\mathbb{C}$ from ω to ω' is a pair (u, v) of morphisms in $\mathbb{C}$ such that the following diagram commute in $\mathbb{C}$:

$$\begin{tikzcd} q \arrow[r, "u"] \arrow[d, "\omega'"] & p \arrow[d, "\omega"] \\ q' & p' \arrow[l, "v"] \end{tikzcd}$$

The composition is defined by pasting such squares: if $(u, v) : \omega \longrightarrow \omega'$ and $(u', v') : \omega' \longrightarrow \omega''$ are morphisms in $F\mathbb{C}$ then $(u', v')(u, v) = (u'u, vv')$. The triple (u, ω, v) is called a *factorization of* ω'.

Definition 3.4.8 An *abelian natural system* on $\mathbb{C}$ is a functor $D : F\mathbb{C} \longrightarrow \mathbf{Ab}$. If $(u, v) : \omega \longrightarrow \omega'$ is a morphism in $F\mathbb{C}$, then its image $D(u, v) : D(\omega) \longrightarrow D(\omega')$ will be denoted shortly by u_*v^*. It extends uniquely to an additive functor $D : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$.

Recall from Example 3.2.58 that $\mathbb{Z}F\mathbb{C}$ is the free additive category on $F\mathbb{C}$.

The *trivial natural system* $\mathbb{Z} : F\mathbb{C} \longrightarrow \mathbf{Ab}$ is the functor, defined on objects by $\mathbb{Z}(\omega) = \mathbb{Z}$, and for each morphism (u, v) we let $u_*v^* = 1_{\mathbb{Z}}$. It extends uniquely to an additive functor from $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$ which we denote again by $\mathbb{Z}$ and call the *trivial natural system.*

Definition 3.4.9 A small category $\mathbb{C}$ is said to be of *type f-FP_n* if the trivial natural system $\mathbb{Z}$ is of type FP_n in $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$.

Theorem 3.4.10 *If a small category $\mathbb{C}$ is of type f-FP_n, then it is of type bi-FP_n.*

Proof. From Lemma 3.4.2 we may suppose that

$$\bigoplus_{u \in U_n} \mathbb{Z}F\mathbb{C}(u, \_) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \bigoplus_{u \in U_0} \mathbb{Z}F\mathbb{C}(u, \_) \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

is a finitely generated free partial resolution of the trivial natural system $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$. From Theorem 3.2.87 this is equivalent with the exactness of the following sequence of abelian groups

$$\bigoplus_{u \in U_n} \mathbb{Z}F\mathbb{C}(u, \omega) \xrightarrow{\partial_{n,\omega}} \dots \xrightarrow{\partial_{1,\omega}} \bigoplus_{u \in U_0} \mathbb{Z}F\mathbb{C}(u, \omega) \xrightarrow{\partial_{0,\omega}} \mathbb{Z}(\omega) \longrightarrow 0,$$

for any $\omega \in F\mathbb{C}$. It follows that for every $(c_1, c_2) \in \mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$, the sequence of abelian groups

$$\begin{gathered} \bigoplus_{\omega \in \mathbb{C}(c_1, c_2)} \bigoplus_{u \in U_n} \mathbb{Z}F\mathbb{C}(u, \omega) \xrightarrow{\partial_{n,(c_1,c_2)}} \dots \xrightarrow{\partial_{1,(c_1,c_2)}} \bigoplus_{\omega \in \mathbb{C}(c_1, c_2)} \bigoplus_{u \in U_0} \mathbb{Z}F\mathbb{C}(u, \omega) \\ \xrightarrow{\partial_{0,(c_1,c_2)}} \bigoplus_{\omega \in \mathbb{C}(c_1, c_2)} \mathbb{Z}(\omega) \longrightarrow 0 \end{gathered} \tag{3.21}$$

is exact, where for every $s = 0, 1, ..., n$ and $(c_1, c_2) \in \mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$, we have defined

$$\partial_{s,(c_1,c_2)} = \bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \partial_{s,\omega}$$

to be the extension of the maps $\partial_{s,\omega}$ with $\omega \in \mathbb{C}(c_1, c_2)$ on the direct sum

$$\bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega).$$

We claim that $\partial_{s,(c_1,c_2)}$ is natural in (c_1, c_2). Indeed, if $\gamma \otimes \delta : (c_1, c_2) \longrightarrow (c_1', c_2')$ is a morphism in $\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$, then it induces a map

$$\bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega) \longrightarrow \bigoplus_{\omega' \in \mathbb{C}(c_1',c_2')} \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega')$$

by

$$(\alpha, \beta) \longmapsto (\gamma^{opp}\alpha, \beta\delta)$$

for every $\omega \in \mathbb{C}(c_1, c_2)$ and every generator $(\alpha, \beta) \in \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega)$. But the restriction of this map on $\bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega)$ for some fixed $\omega \in \mathbb{C}(c_1, c_2)$ is the same as the map

$$\bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega) \longrightarrow \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega')$$

induced by $(\gamma^{opp}, \delta) : \omega \longrightarrow \omega' = \gamma^{opp}\omega\delta$. Since from the definition of $\partial_{s,(c_1,c_2)}$, for every $\omega \in \mathbb{C}(c_1, c_2)$, $\partial_{s,(c_1,c_2)} \mid_{\bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u,\omega)} = \partial_{s,\omega}$ and since $\partial_{s,\omega}$ is natural in each $\omega \in \mathbb{C}(c_1, c_2)$, we obtain the naturality of $\partial_{s,(c_1,c_2)}$ as claimed.

For every $s = 0, 1, ..., n$ we will construct finitely generated projective functors $P_s \in \mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}}$ and then introduce a resolution

$$P_n \xrightarrow{\delta_n} \dots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{Z}\mathbb{C} \longrightarrow 0$$

of $\mathbb{Z}\mathbb{C}$ in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}}$, proving the theorem.

Define, for every $s = 0, 1, ..., n$,

$$P_s = \bigoplus_{u \in U_s} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (\_, \_)),$$

which is projective being a coproduct of representables in $\mathbf{Ab}^{\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}}$, and is obviously finitely generated.

Before we start defining transformations δ_s for every $s = 0, 1, ..., n$, we note that for every $(c_1, c_2) \in \mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$ and $u \in F\mathbb{C}$ there is an isomorphism

$$\mu_{(c_1,c_2)} : \bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \mathbb{Z}F\mathbb{C}(u, \omega) \longrightarrow (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2))$$

defined by

$$(\alpha, \beta) \longmapsto \alpha^{opp} \otimes \beta$$

for every $\omega \in \mathbb{C}(c_1, c_2)$ and every $(\alpha, \beta) : u \longrightarrow \omega = \alpha u \beta$. This is natural in (c_1, c_2) too. Indeed, if $\gamma \otimes \delta : (c_1, c_2) \longrightarrow (c_1', c_2')$ is a morphism in $\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$, then it induces a map

$$\bigoplus_{\omega \in \mathbb{C}(c_1, c_2)} \mathbb{Z}F\mathbb{C}(u, \omega) \longrightarrow \bigoplus_{\omega' \in \mathbb{C}(c_1', c_2')} \mathbb{Z}F\mathbb{C}(u, \omega')$$

by

$$(\alpha, \beta) \longmapsto (\gamma^{opp}\alpha, \beta\delta)$$

for every $\omega \in \mathbb{C}(c_1, c_2)$ and every generator $(\alpha, \beta) \in \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega)$. It induces also a map

$$(\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2)) \longrightarrow (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1', c_2'))$$

by

$$\alpha \otimes \beta \longmapsto \alpha\gamma \otimes \beta\delta$$

for every $\alpha \otimes \beta : (\iota u, \tau u) \longrightarrow (c_1, c_2)$. Now the naturality of $\mu_{(c_1, c_2)}$ in (c_1, c_2) follows easily.

For every $s = 1, ..., n$ and $(c_1, c_2) \in \mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$ we define

$$\delta_{s,(c_1,c_2)} : \bigoplus_{u \in U_s} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2)) \longrightarrow \bigoplus_{v \in U_{s-1}} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota v, \tau v), (c_1, c_2))$$

to be

$$\mu_{s-1,(c_1,c_2)} \circ \partial_{s,(c_1,c_2)} \circ \mu^{-1}_{s,(c_1,c_2)},$$

where

$$\mu_{s,(c_1,c_2)} : \bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \bigoplus_{u \in U_s} \mathbb{Z}F\mathbb{C}(u, \omega) \longrightarrow \bigoplus_{u \in U_s} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2))$$

is the extension of $\mu_{(c_1,c_2)}$. The morphism $\delta_{s,(c_1,c_2)}$ is natural in (c_1, c_2) since it factors through naturals.

Define

$$\delta_{0,(c_1,c_2)} : \bigoplus_{u \in U_0} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2)) \longrightarrow \mathbb{Z}\mathbb{C}(c_1, c_2)$$

by

$$\mu_{-1,(c_1,c_2)} \circ \partial_{0,(c_1,c_2)} \circ \mu^{-1}_{0,(c_1,c_2)},$$

where

$$\mu_{-1,(c_1,c_2)} : \bigoplus_{\omega \in \mathbb{C}(c_1,c_2)} \mathbb{Z}(\omega) \longrightarrow \mathbb{Z}\mathbb{C}(c_1, c_2)$$

is the natural isomorphism in (c_1, c_2) which sends the generator of $\mathbb{Z}(\omega)$ to $\omega \in \mathbb{C}(c_1, c_2)$. Again $\delta_{0,(c_1,c_2)}$ is natural in (c_1, c_2).

Since, for every $s = 1, ..., n$, we have $\partial_{s-1,(c_1,c_2)} \circ \partial_{s,(c_1,c_2)} = 0$, then it follows that

$$\delta_{s-1,(c_1,c_2)} \circ \delta_{s,(c_1,c_2)} = 0.$$

It remains to show that for every $(c_1, c_2) \in \mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C}$ the sequence of abelian groups

$$\bigoplus_{u \in U_n} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota u, \tau u), (c_1, c_2)) \xrightarrow{\delta_{n,(c_1,c_2)}} \cdots \xrightarrow{\delta_{1,(c_1,c_2)}} \bigoplus_{v \in U_0} (\mathbb{Z}\mathbb{C}^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{C})((\iota v, \tau v), (c_1, c_2))$$
$$\xrightarrow{\delta_{0,(c_1,c_2)}} \mathbb{Z}\mathbb{C}(c_1, c_2) \longrightarrow 0$$

is exact. But this is straightforward from the definition of morphisms $\delta_{s,(c_1,c_2)}$ and from the exactness of (3.21). ■

Combining Theorem 3.4.5 with Theorem 3.4.10, we obtain immediately the following.

Corollary 3.4.11 *If a small category $\mathbb{C}$, then the following implications hold true:*

$$\mathbb{C} \text{ is of type f-}FP_n \Longrightarrow \mathbb{C} \text{ is of type bi-}FP_n \Longrightarrow \mathbb{C} \text{ is of type left (right)-}FP_n.$$

3.4.3 Monoids of Type f-FP$_n$

As we explained in the introduction, we can consider every monoid S as a category with a single object; hence all the notions and results of the previous sections apply for monoids. The following reveals an interesting property of the S-graded resolutions of $\mathbb{Z}S$.

Theorem 3.4.12 *If the monoid S is of type bi-FP_n and the corresponding free resolution is S-graded, then S is of type f-FP_n. In particular, monoids which are given by a finite complete presentation are of type f-FP_n.*

Proof. Suppose that there is a free finitely generated S-graded resolution of $\mathbb{Z}S$

$$\bigoplus_{\mathbf{q}_n}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_4} \bigoplus_{\mathbf{q}_3}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_3} \bigoplus_{\mathbf{q}_2}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S)$$
$$\xrightarrow{\delta_2} \bigoplus_{\mathbf{q}_1}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S) \xrightarrow{\delta_1} \mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S \xrightarrow{\delta_0} \mathbb{Z}S \to 0$$

as explained in Definition 2.2.33.

As discussed in Remark 3.4.4, the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$ is nothing but the value of the functor $(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S)((*,*), \_) \in \mathbf{Ab}^{\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S}$ at $(*,*)$, hence, from the proof of Theorem 3.4.10, for every $0 \leq t \leq n$, there is an isomorphism

$$\mu_t = \mu_{t,(*,*)} : \bigoplus_{s \in S} \bigoplus_{u \in \mathbf{q}_t} \mathbb{Z}FS(u, s) \longrightarrow \bigoplus_{\mathbf{q}_t}(\mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S)$$

which is natural in $(*,*)$.

There is also a natural isomorphism

$$\mu_{-1} : \underset{s\in S}{\oplus} \mathbb{Z} \longrightarrow \mathbb{Z}S.$$

Similarly with Theorem 3.4.10, one can construct the exact sequence of abelian groups

$$\underset{s\in S}{\oplus} \underset{u\in \mathbf{q}_n}{\oplus} \mathbb{Z}FS(u,s) \xrightarrow{d_n} \dots \xrightarrow{d_2} \underset{s\in S}{\oplus} \underset{v\in \mathbf{q}_1}{\oplus} \mathbb{Z}FS(v,s) \xrightarrow{d_1} \underset{s\in S}{\oplus} \mathbb{Z}FS(1,s) \xrightarrow{d_0} \underset{s\in S}{\oplus} \mathbb{Z} \to 0. \quad (3.22)$$

Note that d_t is S-graded for every $0 \le t \le n$. Indeed, for any fixed $s \in S$, we denote $(\alpha,\beta) \in \underset{u\in \mathbf{q}_t}{\oplus} \mathbb{Z}FS(u,s)$ by $(\alpha,\beta)_u$ if it is in the u^{th} component of the direct sum. Similarly we denote its μ-image $\alpha^{opp} \otimes \beta \in \underset{u\in \mathbf{q}_t}{\oplus} \mathbb{Z}S^{opp} \otimes_{\mathbb{Z}} \mathbb{Z}S$ by $(\alpha^{opp} \otimes \beta)_u$. Then we have

$$d_t((\alpha,\beta)_u) = (\mu_{t-1}^{-1} \circ \delta_t \circ \mu_t)((\alpha,\beta)_u) = (\mu_{t-1}^{-1} \circ \delta_t)((\alpha^{opp} \otimes \beta)_u) =$$
$$\mu_{t-1}^{-1}(\textstyle\sum_{i\in I} n_i(\alpha_i^{opp} \otimes \beta_i)_{v_i}) = \sum_{i\in I} n_i(\alpha_i,\beta_i)_{v_i}.$$

But now since, for every $i \in I$, we have from Definition 2.2.33 that $\alpha_i v_i \beta_i = s = \alpha u \beta$, it follows that

$$d_t\left(\underset{u\in \mathbf{q}_t}{\oplus} \mathbb{Z}FS(u,s)\right) \subseteq \underset{v\in \mathbf{q}_{t-1}}{\oplus} \mathbb{Z}FS(v,s).$$

We claim that

$$\underset{u\in \mathbf{q}_n}{\oplus} \mathbb{Z}FS(u,\_) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} \underset{v\in \mathbf{q}_1}{\oplus} \mathbb{Z}FS(v,\_) \xrightarrow{\partial_1} \mathbb{Z}FS(1,\_) \xrightarrow{\partial_0} \mathbb{Z} \to 0,$$

is a resolution of $\mathbb{Z}$, where $\partial_{t,s}$ is the restriction of d_t in the s^{th} component of the direct sum $\underset{s\in S}{\oplus} \underset{u\in \mathbf{q}_t}{\oplus} \mathbb{Z}FS(u,s)$.

Since d_t is S-graded for every $0 \le t \le n$, and since (3.22) is exact, then it follows that, for every $s \in S$,

$$\underset{u\in \mathbf{q}_n}{\oplus} \mathbb{Z}FS(u,s) \xrightarrow{\partial_{n,s}} \dots \xrightarrow{\partial_{2,s}} \underset{v\in \mathbf{q}_1}{\oplus} \mathbb{Z}FS(v,s) \xrightarrow{\partial_{1,s}} \mathbb{Z}FS(1,s) \xrightarrow{\partial_{0,s}} \mathbb{Z} \to 0,$$

is exact. ♦

Lastly, for every $0 \le t \le n$, the family

$$\{\partial_{t,s} \mid s \in S\}$$

is a natural transformation. This follows from the definition of μ_t and from the fact that δ_t is a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule morphism.

The second part of the theorem follows immediately from the above and from Remark 2.2.32.

■

3.5 Partial Resolutions of the Trivial Functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$

3.5.1 Introduction

The main scope of this section is to introduce partial resolutions of the trivial functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$ which involve the data giving a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of $\mathbb{C}$, namely $\mathbf{x}$ and $\mathbf{r}$, and some other data arising from $\mathcal{P}$ which we will introduce in the next sections.

Before doing that, let us recall how we construct partial resolutions of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ where M is a monoid given by a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. It is shown in [95] that in such a case there is always the partial free resolution

$$\mathbb{Z}M[\mathbf{r}] \xrightarrow{\delta_2} \mathbb{Z}M[\mathbf{x}] \xrightarrow{\delta_1} \mathbb{Z}M \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \tag{3.23}$$

of the trivial functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}M}$ where

$$\mathbb{Z}M[\mathbf{r}] = \underset{\mathbf{r}}{\oplus}\mathbb{Z}M \text{ and } \mathbb{Z}M[\mathbf{x}] = \underset{\mathbf{x}}{\oplus}\mathbb{Z}M.$$

In order to extend (3.23) with another term, Cremanns and Otto in [19] (see also [57] and [85]) introduced other data arising from $\mathcal{P}$: a homotopy base B for the relation

$$P^2(\Gamma) = \{(p, q) \mid \iota p = \iota q \text{ and } \tau p = \tau q\}.$$

They showed that there is an exact sequence

$$\mathbb{Z}M[B] \xrightarrow{\delta_3} \mathbb{Z}M[\mathbf{r}] \xrightarrow{\delta_2} \mathbb{Z}M[\mathbf{x}] \xrightarrow{\delta_1} \mathbb{Z}M \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where

$$\mathbb{Z}M[B] = \underset{B}{\oplus}\mathbb{Z}M.$$

This was achieved by a heavy use of the graph $\Delta = \Delta(\mathbf{x}, \mathbf{r}, B)$ which rewrites the paths of the graph $\Gamma(\mathbf{x}, \mathbf{r})$ by using as rewriting rules the set $B \subseteq P^2(\Gamma)$.

Returning to the category $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$, we recall from [67] the following important theorem.

Theorem 3.5.1 *If the small category $\mathbb{C}$ is given by the presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$, then there is a partial free resolution of the trivial functor $\mathbb{Z} \in \mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$*

$$\underset{(l,r)\in\mathbf{r}}{\oplus} \mathbb{Z}F\mathbb{C}(l, \_) \xrightarrow{\delta_2} \underset{x\in\mathbf{x}}{\oplus}\mathbb{Z}F\mathbb{C}(x, \_) \xrightarrow{\delta_1} \underset{c\in\mathbb{C}}{\oplus}\mathbb{Z}F\mathbb{C}(1_c, \_) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{3.24}$$

where ϵ is the augmentation defined functorially by $\epsilon_{\overline{\omega}}(\overline{u}, \overline{v}) = \{\overline{uv}\}$ for any morphism $\overline{\omega} \in \mathbb{C}$.

To extend this partial resolution with another term, we will follow here the approach of [19]. For this we will define in Section 3.5.3 two graphs associated with a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of a small category $\mathbb{C}$, the Squier graph $\Gamma(\mathbf{x}, \mathbf{r})$ and the graph $\Delta = \Delta(\mathbf{x}, \mathbf{r}, B)$ which is the analogue of Δ explained above. Then using similar techniques to those used in Theorem 3.2 of [19], we prove the following.

Theorem 3.5.2 *Let the small category $\mathbb{C}$ be given by the presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. For every $B \subseteq P^2(\Gamma(\mathbf{x}, \mathbf{r}))$ such that $\simeq_B = P^2(\Gamma(\mathbf{x}, \mathbf{r}))$, the sequence*

$$\underset{(p,q)\in B}{\oplus} \mathbb{Z}F\mathbb{C}(\iota(\iota p), \_) \xrightarrow{\delta_3} \underset{(l,r)\in \mathbf{r}}{\oplus} \mathbb{Z}F\mathbb{C}(l, \_) \xrightarrow{\delta_2} \underset{x\in \mathbf{x}}{\oplus} \mathbb{Z}F\mathbb{C}(x, \_) \xrightarrow{\delta_1} \underset{c\in \mathbb{C}}{\oplus} \mathbb{Z}F\mathbb{C}(1_c, \_) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact.

The following follows immediately from the above and the definition of FDT.

Theorem 3.5.3 *If $\mathbb{C}$ is of type FDT, then $\mathbb{C}$ is of type f-FP_3.*

3.5.2 A Basic Exact Sequence Associated With a Presentation $\mathcal{P}$

We will give in this section a short account of the notions involved in [67] which are used to prove Theorem 3.5.1 (Lemma 4.2 of [67]).

Let $\mathbb{C}$ be a small category presented by the presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. The free abelian natural system $\mathbb{Z}\mathbb{C}[\mathbf{x}] : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ generated by the set of all edges x in the graph $\mathbf{x}$ viewed as morphisms in $\mathbb{C}$ is defined by

$$\mathbb{Z}\mathbb{C}[\mathbf{x}] = \underset{x\in \mathbf{x}}{\oplus} \mathbb{Z}F\mathbb{C}(x, \_)$$

For each morphism $\overline{\omega}$ in $\mathbb{C}$, the elements of $\mathbb{Z}F\mathbb{C}(x, \overline{\omega})$ are denoted by $(\overline{u}, [x], \overline{v})$, where $\overline{u}, \overline{v}$ are morphisms in $\mathbb{C}$ such that $\overline{\omega} = \overline{uxv}$ and the actions are given by the abelian group morphisms

$$\overline{u'}_* \overline{v'}^* : \mathbb{Z}\mathbb{C}[\mathbf{x}](\overline{\omega}) \longrightarrow \mathbb{Z}\mathbb{C}[\mathbf{x}](\overline{u'\omega v'})$$

defined by

$$\overline{u'}_* \overline{v'}^* (\overline{u}, [x], \overline{v}) = (\overline{u'u}, [x], \overline{vv'}).$$

Similarly, the free abelian natural system $\mathbb{Z}\mathbb{C}[\mathbf{r}] : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ generated by the set of all rules of R, is the free abelian group

$$\mathbb{Z}\mathbb{C}[\mathbf{r}] = \underset{(l,r)\in \mathbf{r}}{\oplus} \mathbb{Z}F\mathbb{C}(l, \_).$$

For each morphism $\overline{\omega}$ in $\mathbb{C}$, the elements of $\mathbb{ZC}[\mathbf{r}](\overline{\omega})$ are denoted by $(\overline{u}, [l,r], \overline{v})$, where $\overline{u}, \overline{v}$ are morphisms in $\mathbb{C}$ such that $\overline{\omega} = \overline{ul\overline{v}}$ and the actions on $\mathbb{ZC}[\mathbf{r}]$ are given by morphisms

$$\overline{u'}_*\overline{v'}^* : \mathbb{ZC}[\mathbf{r}](\overline{\omega}) \longrightarrow \mathbb{ZC}[\mathbf{r}](\overline{u'\omega v'})$$

defined by

$$\overline{u'}_*\overline{v'}^*(\overline{u}, [l,r], \overline{v}) = (\overline{u'u}, [l,r], \overline{vv'}).$$

Lastly for a set $B \subseteq P^2(\Gamma(\mathbf{x},\mathbf{r}))$, the free abelian natural system $\mathbb{ZC}[B] : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ generated by B, is the free abelian group

$$\mathbb{ZC}[B] = \bigoplus_{(p,q)\in B} \mathbb{Z}F\mathbb{C}(\iota(\iota p), \_).$$

Similarly to the above, for morphisms $\overline{\omega}$ in $\mathbb{C}$, the elements of $\mathbb{ZC}[B](\overline{\omega})$ are denoted by $(\overline{u}, [p,q], \overline{v})$ and the actions on $\mathbb{ZC}[B]$ are given by morphisms

$$\overline{u'}_*\overline{v'}^* : \mathbb{ZC}[B](\overline{\omega}) \longrightarrow \mathbb{ZC}[B](\overline{u'\omega v'})$$

defined by

$$\overline{u'}_*\overline{v'}^*(\overline{u}, [l,r], \overline{v}) = (\overline{u'u}, [p,q], \overline{vv'}).$$

Denote $(1_{\iota(x)}, [x], 1_{\tau(x)})$ by $[x]$, $(1_{\iota(l)}, [l,r], 1_{\tau(l)})$ by $[l,r]$ and $(1_{\iota(\iota(p))}, [p,q], 1_{\tau(\iota(p))})$ by $[p,q]$.

Call a *natural system of sets* a functor $S : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Set}$. Associated with S there is the so called *abelian natural system over* $\mathbb{C}$, $\mathbb{ZC}[S] : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Ab}$ defined as the composition $\mathbb{ZC}[S] = \mathbb{Z}[\_] \circ S$, where $\mathbb{Z}[\_] : \mathbf{Set} \longrightarrow \mathbf{Ab}$ denotes the *free abelian functor* which sends every non-empty set to the free abelian group generated by that set, the empty set to the group $\{0\}$, and is defined on morphisms by sending every map of sets to the unique group morphisms induced by that map. Thus, for every morphism $\overline{\omega}$ in $\mathbb{C}$, $\mathbb{ZC}[S](\overline{\omega})$ is the free abelian group generated by $S(\omega)$. For every factorization $\overline{\omega'} = \overline{u\omega v}$, the action $\overline{u'}_*\overline{v'}^* : \mathbb{Z}[S(\overline{\omega})] \longrightarrow \mathbb{Z}[S(\overline{\omega'})]$ is the morphism of abelian groups induced by the map $S(u,v)$.

Let $N_n(\mathbb{C}) : \mathbb{Z}F\mathbb{C} \longrightarrow \mathbf{Set}$ be the natural system of sets such that

$$N_n(\mathbb{C})(\overline{\omega}) = \{(\overline{u_1}, ..., \overline{u_n}) \mid \overline{\omega} = \overline{u_1 ... u_n}\}$$

for $n > 0$ and $N_0(\mathbb{C})(\overline{\omega}) = \phi$ if $\omega \neq \lambda$ and $N_0(\mathbb{C})(\lambda) = \{1\}$.

For any factorization $\overline{\omega'} = \overline{u\omega v}$ in $\mathbb{C}$, the action

$$\overline{u}_*\overline{v}^* : N_n(\mathbb{C})(\overline{\omega}) \longrightarrow N_n(\mathbb{C})(\overline{\omega'})$$

is defined by

$$\overline{u}_*\overline{v}^*(\overline{u_1}, ..., \overline{u_n}) = (\overline{uu_1}, ..., \overline{u_n v}).$$

For each $n \geq 0$, denote by

$$B_n(\mathbb{C}) = \mathbb{ZC}[N_{n+2}(\mathbb{C})]$$

the free abelian natural system generated by $N_{n+2}(\mathbb{C})$.

A *derivation* from a small category $\mathbb{C}$ with values into an abelian natural system D over $\mathbb{C}$ is a function

$$d : \mathcal{O}(F\mathbb{C}) \longrightarrow \bigcup_{\omega \in F\mathbb{C}} D(\omega)$$

where $d(\overline{\omega}) \in D(\overline{\omega})$ for every $\overline{\omega} \in F\mathbb{C}$ and such that

$$d(\overline{uv}) = \overline{u}_* d(\overline{v}) + \overline{v}^* d(\overline{u}),$$

for every $\overline{u}, \overline{v} \in \mathbb{C}$.

Lemma 3.5.4 *Let* $\mathbf{x}$ *be a graph,* $F(\mathbf{x})$ *the free category generated by* $\mathbf{x}$ *and* D *be an abelian natural system on* $F(\mathbf{x})$. *Any family* $([x])_{x \in \mathbf{x}}$, *with* $[x] \in D(x)$, *has a unique extension into derivation* $[\,] : \mathcal{O}(FF(\mathbf{x})) \longrightarrow D$ *by setting for every* n *composable morphisms* $x_1, ..., x_n \in F(\mathbf{x})$

$$[x_1 ... x_n] = \sum_{i=1}^{n} (x_1 ... x_{i-1})_* (x_{i+1} ... x_n)^* [x_i].$$

The proof of the above is given in Lemma 4.1 of [67].

For a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of a small category $\mathbb{C}$, denote by $\pi : F(\mathbf{x}) \longrightarrow \mathbb{C}$ the canonical morphism sending u to its corresponding class $\overline{u}$. For every morphism u in $F(\mathbf{x})$, π induces a homomorphism

$$\mathbb{Z}\pi[\mathbf{x}](u) : \bigoplus_{x \in \mathbf{x}} \mathbb{Z}F(F(\mathbf{x}))(x, u) \longrightarrow \bigoplus_{x \in \mathbf{x}} \mathbb{Z}F\mathbb{C}(x, \overline{u})$$

by

$$\mathbb{Z}\pi[\mathbf{x}](u)((\alpha, [x], \beta)) = (\overline{\alpha}, [x], \overline{\beta}).$$

Now for a fixed $x \in \mathbf{x}$, define the map

$$\frac{\partial}{\partial x} : \mathbf{x} \longrightarrow \bigcup_{u \in F(\mathbf{x})} \mathbb{Z}F(\mathbf{x})[\mathbf{x}](u)$$

by

$$\frac{\partial x'}{\partial x} = \begin{cases} [x] & \text{if} \quad x' = x \\ 0 & \text{otherwise} \end{cases}$$

Again from Lemma 4.1 of [67] we can extend it uniquely to a derivation

$$\frac{\partial}{\partial x} : F(\mathbf{x}) \longrightarrow \bigcup_{u \in F(\mathbf{x})} \mathbb{Z}F(\mathbf{x})[\mathbf{x}](u)$$

which composed with

$$\mathbb{Z}\pi[\mathbf{x}] : \bigcup_{u \in F(\mathbf{x})} \mathbb{Z}F(\mathbf{x})[\mathbf{x}](u) \longrightarrow \bigcup_{\overline{u} \in \mathbb{C}} \mathbb{Z}\mathbb{C}[\mathbf{x}](\overline{u})$$

gives a derivation

$$\frac{\overline{\partial}}{\partial x} : F(\mathbf{x}) \longrightarrow \bigcup_{\overline{u} \in \mathbb{C}} \mathbb{Z}\mathbb{C}[\mathbf{x}](\overline{u})$$

by the formula

$$\frac{\overline{\partial} uv}{\partial x} = \overline{u}_* \frac{\overline{\partial} v}{\partial x} + \overline{v}^* \frac{\overline{\partial} u}{\partial x}$$

for every morphisms u, v in $F(\mathbf{x})$.

Let now

$$\delta_1 : \mathbb{Z}\mathbb{C}[\mathbf{x}] \longrightarrow B_0(\mathbb{C})$$

be the natural transformation defined as follows. For each morphisms $\overline{u}, \overline{v}$ in $\mathbb{C}$ and $x \in \mathbf{x}$ such that $\overline{\omega} = \overline{uxv}$,

$$\delta_{1,\overline{\omega}}(\overline{u}, [x], \overline{v}) = (\overline{ux}, \overline{v}) - (\overline{u}, \overline{xv}).$$

Definition 3.5.5 Let $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ be a presentation for the small category $\mathbb{C}$. The *Reidemeister-Fox Jacobian* of $\mathcal{P}$ is the morphism of abelian natural systems

$$\delta_2 : \mathbb{Z}\mathbb{C}[\mathbf{r}] \longrightarrow \mathbb{Z}\mathbb{C}[\mathbf{x}]$$

defined functorially, for each morphism $\overline{\omega}$ in $\mathbb{C}$ by

$$\delta_{2,\overline{\omega}}(\overline{u}, [l, r], \overline{v}) = \overline{u}_* \, \overline{v}^* \sum_{x \in \mathbf{x}} \left(\frac{\overline{\partial} l}{\partial x} - \frac{\overline{\partial} r}{\partial x} \right),$$

for morphisms $\overline{u}, \overline{v}$ in $\mathbb{C}$ and $(l, r) \in \mathbf{r}$ such that $\overline{\omega} = \overline{ulv}$.

Finally we make the following.

Remark 3.5.6 In the original statement of Lemma 2.4 of [67] (here Theorem 3.5.1), the last term of the resolution of $\mathbb{Z}$ is $B_0(\mathbb{C})$ which is nothing but $\bigoplus_{c \in \mathbb{C}} \mathbb{Z}F\mathbb{C}(1_c, \_)$ and therefore this is a partial projective resolution in $\mathbf{Ab}^{\mathbb{Z}F\mathbb{C}}$. If $\mathbb{C}$ is finitely generated, then $B_0(\mathbb{C})$ is finitely generated too. Furthermore, if $\mathbb{C}$ is finitely presented, the sequence of the above lemma gives a finitely generated partial projective resolution of $\mathbb{Z}$ of length 2.

3.5.3 Geometrical Constructions Associated with a Presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$

We will construct a graph $\Gamma(\mathbf{x}, \mathbf{r})$ associated with a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of a small category $\mathbb{C}$, which will contain information how the morphisms of $\mathbb{C}$ are presented by paths from $UF(\mathbf{x})$. We denote by $PUF(\mathbf{x})$ the set of paths of $\mathbf{x}$ and by $\circ$ their composition.

Definition 3.5.7 Let $\Gamma(\mathbf{x}, \mathbf{r}) = (V, E, \iota, \tau, ^{-1})$ be as follows:

1. The set of vertices is $V = PUF(\mathbf{x})$.

2. The set of edges is $E = \{(u, (l, r), v, \varepsilon) \mid u, v \in PUF(\mathbf{x}),\ (l, r) \in \mathbf{r} \text{ and } \varepsilon = \pm 1\}$

3. the maps $\iota, \tau : E \longrightarrow V$ which associate with each edge $e \in E$ its initial vertex ιe and its terminal τe, are defined by

$$\iota(u, (l, r), v, \varepsilon) = \begin{cases} u \circ l \circ v & \text{if} \quad \varepsilon = 1 \\ u \circ r \circ v & \text{if} \quad \varepsilon = -1 \end{cases}$$

   and

$$\tau(u, (l, r), v, \varepsilon) = \begin{cases} u \circ r \circ v & \text{if} \quad \varepsilon = 1 \\ u \circ l \circ v & \text{if} \quad \varepsilon = -1 \end{cases}$$

4. and the map $^{-1} : E \longrightarrow E$, which associates with each edge $e \in E$, its inverse edge e^{-1} defined by

$$(u, (l, r), v, \varepsilon)^{-1} = (u, (l, r), v, -\varepsilon).$$

Below is how an edge of $\Gamma(\mathbf{x}, \mathbf{r})$ looks like:

$$\begin{tikzcd} \iota u \arrow[r, "u"] & a \arrow[r, bend left, "l"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow\alpha'"] & b \arrow[r, "v"] & \tau v \end{tikzcd}$$

An edge $(u, (l, r), v, \varepsilon)$ is called positive if $\varepsilon = 1$ and negative if $\varepsilon = -1$. There is a *partial action* of vertices of $\Gamma(\mathbf{x}, \mathbf{r})$ on the set of edges defined as follows:

$$\omega.(u, (l, r), v, \varepsilon).\omega' = (\omega \circ u, (l, r), v \circ \omega', \varepsilon)$$

where $\omega, \omega' \in PUF(\mathbf{x})$ and the compositions $\omega \circ u$ and $v \circ \omega'$ are defined. This action extends in the obvious fashion to paths of $\Gamma(\mathbf{x}, \mathbf{r})$.

We call $\Gamma(\mathbf{x}, \mathbf{r})$ the *Squier graph* of the presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$. We define the composition $e \cdot f$ of two edges e and f in $\Gamma(\mathbf{x}, \mathbf{r})$ whenever $\iota f = \tau f$. Inductively we can define *paths* $\alpha_1 \cdot \ldots \cdot \alpha_n$ provided that for every $i = 1, \ldots, n-1$ we have $\tau\alpha_i = \iota\alpha_{i+1}$. We say that the path

$p = \alpha_1 \cdot ... \cdot \alpha_n \in \Gamma(\mathbf{x}, \mathbf{r})$ has *length* n. A path $p = \alpha_1 \cdot ... \cdot \alpha_n$ will be called *positive* if $\alpha_k = 1$ is a positive for every $k = 1, ..., n$. The inverse of a path $p = \alpha_1 \cdot ... \cdot \alpha_n$ is by definition the path $p = \alpha_1^{-1} \cdot ... \cdot \alpha_n^{-1}$.

Denote by $P(\Gamma(\mathbf{x}, \mathbf{r}))$, or simply by $P(\Gamma)$, the set of paths of $\Gamma(\mathbf{x}, \mathbf{r})$ and by $P^2(\Gamma(\mathbf{x}, \mathbf{r}))$, or simply by $P^2(\Gamma)$, the set of paths in $\Gamma(\mathbf{x}, \mathbf{r})$ which have the same initial and terminal. In the future we call $P^2(\Gamma)$ the set of *parallel paths* of $\Gamma(\mathbf{x}, \mathbf{r})$.

Define this set of relations in $P^2(\Gamma(\mathbf{x}, \mathbf{r}))$:

1. Relations $I : (e \cdot e^{-1}, id_{\iota e}), (e^{-1} \cdot e, id_{\tau e})$ for every edge e of $\Gamma(\mathbf{x}, \mathbf{r})$,

2. Relations D : For every two edges $e, f \in \Gamma(\mathbf{x}, \mathbf{r})$ we take

$$((e \circ \iota f) \cdot (\tau e \circ f), (\iota e \circ f) \cdot (e \circ \tau f)).$$

In the sequel we call $e \circ \iota f$ and $\iota e \circ f$ *disjoint edges.* For any $(p_1, p_2) \in P^2(\Gamma)$, we define its *whisker* (*translate*) by $\rho', \rho'' \in UF(\mathbf{x})$ to be the pair $(\rho' \circ p_1 \circ \rho'', \rho' \circ p_2 \circ \rho'')$, whenever this is defined. We say that some set $B \subseteq P^2(\Gamma)$ is *whisker closed* if it contains the set of all the possible translates of its elements by paths from $F(\mathbf{x})$. We denote the whisker closure of some set B by $w(B)$.

Remark 3.5.8 We could have used 2-categories to define $\Gamma(\mathbf{x}, \mathbf{r})$ by considering the 2-category arising from $F(\mathbf{x})$ by adding 2-cells in a 1-1 correspondence to $\mathbf{r}^{\#}$. But this approach does not give a clear description of the rewriting process.

Definition 3.5.9 An equivalence relation $\simeq \subseteq P^2(\Gamma(\mathbf{x}, \mathbf{r}))$ is called a *homotopy relation* if it satisfies the following:

1. $I \cup D \subseteq \simeq$,

2. it is whisker closed, and

3. for every $(\rho_1, \rho_2) \in \simeq$, and every $p, q \in \Gamma(\mathbf{x}, \mathbf{r})$ such that $\tau p = \iota \rho_1 (= \iota \rho_2)$ and $\iota q = \tau \rho_1 (= \tau \rho_2)$, $(p \cdot \rho_1 \cdot q, p \cdot \rho_2 \cdot q) \in \simeq$.

   We denote by $\simeq_B$ the smallest homotopy relation containing a set $B \subseteq P^2(\Gamma(\mathbf{x}, \mathbf{r}))$.

Lemma 3.5.10 *Let $(\alpha \cdot \beta, \gamma) \in P^2(\Gamma)$. Then, $\alpha \cdot \beta \simeq \gamma$ if and only if $\alpha \simeq \gamma \cdot \beta^{-1}$.*

Proof. If $\alpha \cdot \beta \simeq \gamma$, then from the above definition $\alpha \cdot \beta \cdot \beta^{-1} \simeq \gamma \cdot \beta^{-1}$ and again from the definition, $\beta \cdot \beta^{-1} \simeq id_{dom\beta}$ which implies that $\alpha \simeq \gamma \cdot \beta^{-1}$. The converse is proved similarly. ■

Definition 3.5.11 We say that a presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of some small category $\mathbb{C}$ has *finite derivation type* (FDT) if it is finite and if there is a finite set $B \subset P^2(\Gamma(\mathbf{x}, \mathbf{r}))$ such that $\simeq_B = P^2(\Gamma(\mathbf{x}, \mathbf{r}))$.

Definition 3.5.12 Let $B \subseteq P^2(\Gamma(\mathbf{x}, \mathbf{r}))$. Define the graph $\Delta(\mathbf{x}, \mathbf{r}, B) = (V, E, \iota, \tau, ^{-1})$ as follows:

1. $V = P(\Gamma(\mathbf{x}, \mathbf{r}))$ is the set of vertices;

2. $E = \{(r_1, u, (p, q), v, r_2, \varepsilon) \mid r_1, r_2 \in V,\ u, v \in UF(\mathbf{x}),\ (p, q) \in D \cup I \cup B,\ \varepsilon = \pm 1$ such that $\tau r_1 = \iota(u \circ p \circ v)$ and $\iota r_2 = \tau(u \circ p \circ v)\}$ is the set of edges;

3. the maps $\iota, \tau : E \longrightarrow V$ which associate with each edge $e \in E$ its initial vertex ιe and its terminal τe, are defined by
$$\iota(r_1, u, (p, q), v, r_2, \varepsilon) = \begin{cases} r_1 \cdot (u \circ p \circ v) \cdot r_2 & \text{if} \quad \varepsilon = 1 \\ r_1 \cdot (u \circ q \circ v) \cdot r_2 & \text{if} \quad \varepsilon = -1 \end{cases}$$
and
$$\tau(r_1, u, (p, q), v, r_2, \varepsilon) = \begin{cases} r_1 \cdot (u \circ q \circ v) \cdot r_2 & \text{if} \quad \varepsilon = 1 \\ r_1 \cdot (u \circ p \circ v) \cdot r_2 & \text{if} \quad \varepsilon = -1 \end{cases}$$

4. and the map $^{-1} : E \longrightarrow E$, which associates with each edge $e \in E$, its inverse edge e^{-1} defined by
$$(r_1, u, (p, q), v, r_2, \varepsilon)^{-1} = (r_1, u, (p, q), v, r_2, -\varepsilon).$$

The following is an immediate consequence of Definition 3.5.9.

Lemma 3.5.13 *Let Δ be the graph defined above and let p and q be two morphisms in $P(\Gamma(\mathbf{x}, \mathbf{r}))$. The following are equivalent.*

1. *there is a path in Δ from p to q,*

2. *$p \simeq_B q$.*

3.5.4 Proof of the Theorem 3.5.2

In what follows we denote by $P(G)$ the set of paths of a given graph G. Again suppose that $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ gives a small category $\mathbb{C}$.

Define the map

$$\gamma_1 : P(UF(\mathbf{x})) \longrightarrow \bigoplus_{\omega \in F\mathbb{C}} \bigoplus_{x \in \mathbf{x}} \mathbb{Z}F\mathbb{C}(x, \omega)$$

by

$$\gamma_1(u) = \begin{cases} 0 & \text{if} \quad |u| = 0 \\ \sum\limits_{x \in \mathbf{x}} \frac{\partial u}{\partial x} & \text{if} \quad |u| \neq 0, \end{cases}$$

where $|u|$ denotes the length of $u \in P(UF(\mathbf{x}))$. With this notation we can rewrite the expression for $\delta_{2,\overline{\omega}}$ as follows:

$$\delta_{2,\overline{\omega}}(\overline{u}, [l, r], \overline{v}) = \overline{u}_* \ \overline{v}^* \left(\gamma_1(l) - \gamma_1(r)\right).$$

Similarly we define the map

$$\gamma_2 : P(\Gamma(\mathbf{x}, \mathbf{r})) \longrightarrow \bigoplus_{\omega \in F\mathbb{C}} \bigoplus_{(r,l) \in \mathbf{r}} \mathbb{Z}F\mathbb{C}(l, \omega)$$

as follows:

$$\gamma_2(p) = \begin{cases} 0 & \text{if} \qquad |p| = 0 \\ \gamma_2(p') + \varepsilon \overline{u}_* \ \overline{v}^*[l, r] & \text{if} \quad p = p' \cdot (u, (l, r), v, \varepsilon) \end{cases}$$

Define the transformation

$$\delta_3 : \mathbb{Z}\mathbb{C}[B] \longrightarrow \mathbb{Z}\mathbb{C}[\mathbf{r}],$$

by

$$\delta_{3,\overline{\omega}}(\overline{u}, [p, q], \overline{v}) = \overline{u}_* \overline{v}^* (\gamma_2(p) - \gamma_2(q)).$$

It is easy to see that δ_3 is natural in every $\overline{\omega}$.

Lastly we define the map

$$\gamma_3 : P(\Delta) \longrightarrow \bigoplus_{\omega \in F\mathbb{C}} \bigoplus_{(p,q) \in B} \mathbb{Z}F\mathbb{C}(u, \omega)$$

(here u is the morphism of $\mathbb{C}$ represented by $\iota(\iota p)$) as follows:

$$\gamma_3(\alpha) = 0 \text{ if } |\alpha| = 0$$

and

$$\gamma_3(\alpha' \bullet (r_1, u, (p, q), v, r_2, \varepsilon)) = \begin{cases} \gamma_3(\alpha') & \text{if} \quad (p, q) \notin B \\ \gamma_3(\alpha') + \varepsilon \overline{u}_* \ \overline{v}^*[p, q] & \text{if} \quad (p, q) \in B \end{cases}$$

where $\bullet$ denotes the concatenation of paths in $P(\Delta)$.

In [67] for every morphism $\overline{\omega}$ in $\mathbb{C}$ there is defined a group morphism

$$\eta_{2,\overline{\omega}} : \mathbb{ZC}[\mathbf{x}](\overline{\omega}) \longrightarrow \mathbb{ZC}[\mathbf{r}](\overline{\omega})$$

by

$$(\overline{u}, [x], \overline{v}) \longmapsto \gamma_2(\Gamma(\overline{u}, x) \circ \widetilde{v}),$$

where $\Gamma(\overline{u}, x)$ is a path in $\Gamma(\mathbf{x}, \mathbf{r})$ from $\widetilde{u} \circ x$ to $\widetilde{ux}$. From here on, $\widetilde{\omega}$ will be some fixed representative of the morphism $\overline{\omega}$ in $\mathbb{C}$.

Before we define for every morphism $\overline{\omega}$ in $\mathbb{C}$ a group morphism $\eta_{3,\overline{\omega}} : \mathbb{ZC}[\mathbf{r}](\overline{\omega}) \longrightarrow \mathbb{ZC}[B](\overline{\omega})$, we need to introduce another notion.

Let $(l, r) \in \mathbf{r}$ where $l = a_1 \circ \ldots \circ a_s$ and $r = b_1 \circ \ldots \circ b_t$. For every morphisms $\overline{u}, \overline{v}$ in $\mathbb{C}$, define two paths in $\Gamma(\mathbf{x}, \mathbf{r})$ as follows:

$$\Gamma_1(\overline{u}, (l, r), \overline{v}) = (\widetilde{u}, (l, r), \widetilde{v}, 1) \cdot (\lambda, \Gamma(\overline{u}, b_1) \circ b_2 \circ \ldots \circ b_t, \widetilde{v}, 1) \cdot \ldots \cdot (\lambda, \Gamma(\overline{u \circ b_1 \circ \ldots \circ b_{t-1}}, b_t), \widetilde{v}, 1)$$

and

$$\begin{aligned}\Gamma_2(\overline{u}, (l, r), \overline{v}) = (\lambda, \Gamma(\overline{u}, a_1) \circ a_2 \circ \ldots \circ a_s, \widetilde{v}, 1) \cdot (\lambda, \Gamma(\overline{u \circ a_1}, a_2) \circ a_3 \circ \ldots \circ a_s, \widetilde{v}, 1) \cdot \ldots \\ \cdot (\lambda, \Gamma(\overline{u \circ a_1 \circ \ldots \circ a_{s-1}}, a_s), \widetilde{v}, 1).\end{aligned}$$

It is easy to see that $\Gamma_1(\overline{u}, (l, r), \overline{v})$ is a path in $\Gamma(\mathbf{x}, \mathbf{r})$ from $\widetilde{u} \circ l \circ \widetilde{v}$ to $\widetilde{urv}$ and $\Gamma_2(\overline{u}, (l, r), \overline{v})$ is a path in $\Gamma(\mathbf{x}, \mathbf{r})$ from $\widetilde{u} \circ l \circ \widetilde{v}$ to $\widetilde{ulv} = \widetilde{urv}$. Therefore, assuming that $\simeq_B = P^2(\Gamma(\mathbf{x}, \mathbf{r}))$ for a set $B \subset P^2(\Gamma(\mathbf{x}, \mathbf{r}))$, we have from Lemma 3.5.13 that there is a path $\alpha(\overline{u}, (l, r), \overline{v})$ in $\Delta(\mathbf{x}, \mathbf{r}, B)$ connecting $\Gamma_1(\overline{u}, (l, r), \overline{v})$ with $\Gamma_2(\overline{u}, (l, r), \overline{v})$.

Now we define for every morphism $\overline{\omega}$ in $\mathbb{C}$, a group morphism

$$\eta_{3,\overline{\omega}} : \mathbb{ZC}[\mathbf{r}](\overline{\omega}) \longrightarrow \mathbb{ZC}[B](\overline{\omega})$$

by

$$(\overline{u}, [l, r], \overline{v}) \longmapsto \gamma_3(\alpha(\overline{u}, (l, r), \overline{v})).$$

The following diagram gives all the maps defined here and in [67].

$$\begin{tikzcd}
P(\Delta) \arrow[d, "\gamma_3"] & P(\Gamma(\mathbf{x},\mathbf{r})) \arrow[d, "\gamma_2"] & P(UF(\mathbf{x})) \arrow[d, "\gamma_1"] & & \\
\bigoplus_{\omega \in F\mathbb{C}} \mathbb{ZC}[B](\omega) \arrow[r, shift right, "\delta_3"'] & \bigoplus_{\omega \in F\mathbb{C}} \mathbb{ZC}[\mathbf{r}](\omega) \arrow[l, shift right, "\eta_3"'] \arrow[r, shift right, "\delta_2"'] & \bigoplus_{\omega \in F\mathbb{C}} \mathbb{ZC}[\mathbf{x}](\omega) \arrow[l, shift right, "\eta_2"'] \arrow[r, shift right, "\delta_1"'] & \bigoplus_{\omega \in F\mathbb{C}} B_0(\mathbb{C})(\omega) \arrow[l, shift right, "\eta_1"'] \arrow[r, two heads, "\varepsilon"] & \bigoplus_{\omega \in F\mathbb{C}} \mathbb{Z}
\end{tikzcd}$$

Lemma 3.5.14 *$\delta_2(\gamma_2(p)) = \gamma_1(\iota p) - \gamma_1(\tau p)$ for all $p \in P(\Gamma(\mathbf{x}, \mathbf{r}))$.*

Proof. First we show that $\gamma_1(\iota e) - \gamma_1(\tau e) = \varepsilon\overline{u}_*\overline{v}^*(\gamma_1(l) - \gamma_1(r))$ for every edge $e = (u,(l,r),v,\varepsilon)$. For $\varepsilon = 1$, $\gamma_1(\iota e) = \gamma_1(u \circ l \circ v) = \overline{l \circ v}^*\gamma_1(u) + \overline{u}_*\ \overline{v}^*\gamma_1(l) + \overline{(u \circ l)}_*\gamma_1(v)$ and $\gamma_1(\tau e) = \gamma_1(u \circ r \circ v) = \overline{r \circ v}^*\gamma_1(u) + \overline{u}_*\ \overline{v}^*\gamma_1(r) + \overline{(u \circ r)}_*\gamma_1(v)$. Subtracting, we get the result. The case $\varepsilon = -1$ is the analogue of the above. To prove the lemma, we use an induction argument on the length of p. If the length of p is 0, then we certainly have the result. Now suppose that $p = p' \cdot e$ with $p' \in P(\Gamma(\mathbf{x},\mathbf{r}))$ and $e = (u,(l,r),v,\varepsilon)$, then

$$\delta_2(\gamma_2(p)) = \delta_2(\gamma_2(p') + \varepsilon\overline{u}_*\overline{v}^*[l,r]) = \delta_2(\gamma_2(p')) + \varepsilon\overline{u}_*\overline{v}^*\delta_2([l,r]) =$$
$$\gamma_1(\iota p') - \gamma_1(\tau p') + \varepsilon\overline{u}_*\overline{v}^*(\gamma_1(l) - \gamma_1(r)) = \gamma_1(\iota p') - \gamma_1(\tau p') + (\gamma_1(\iota e) - \gamma_1(\tau e)) = \gamma_1(\iota p) - \gamma_1(\tau p).$$

■

Lemma 3.5.15 *For every morphism ϖ in $\mathbb{C}$, $\delta_{2,\varpi}\delta_{3,\varpi} = 0$.*

Proof. Indeed, if $(p,q) \in B$ such that the vertices in p represent ϖ, then $\delta_2(\delta_3([p,q])) = \delta_2(\gamma_2(p) - \gamma_2(q)) = \delta_2(\gamma_2(p)) - \delta_2(\gamma_2(q)) = (\gamma_1(\iota p) - \gamma_1(\tau p)) - (\gamma_1(\iota q) - \gamma_1(\tau q)) = 0$. ■

Lemma 3.5.16 $\delta_3(\gamma_3(\alpha)) = \gamma_2(\iota\alpha) - \gamma_2(\tau\alpha)$ *for all paths* $\alpha \in P(\Delta)$.

Proof. We first show that for each edge $e = (r_1,u,(p,q),v,r_2,\varepsilon) \in P(\Delta)$,

$$\gamma_2(\iota e) - \gamma_2(\tau e) = \varepsilon\overline{u}_*\overline{v}^*(\gamma_2(p) - \gamma_2(q)).$$

Indeed, for e positive we have:

$$\gamma_2(\iota e) = \gamma_2(r_1 \cdot (u \circ p \circ v) \cdot r_2) = \gamma_2(r_1) + \overline{u}_*\overline{v}^*\gamma_2(p) + \gamma_2(r_2)$$

and

$$\gamma_2(\tau e) = \gamma_2(r_1 \cdot (u \circ q \circ v) \cdot r_2) = \gamma_2(r_1) + \overline{u}_*\overline{v}^*\gamma_2(q) + \gamma_2(r_2).$$

It follows that $\gamma_2(\iota e) - \gamma_2(\tau e) = \overline{u}_*\ \overline{v}^*(\gamma_2(p) - \gamma_2(q))$. Similarly we prove the result for $\varepsilon = -1$. We proceed with the proof of the lemma by induction on the length of α. If α is a path of length 0, then $\delta_3(\gamma_3(\alpha)) = \delta_3(0) = 0$, and $\gamma_2(\iota\alpha) - \gamma_2(\tau\alpha) = \gamma_2(0) = 0$.

If $\alpha = \beta \bullet e$ with $e = (r_1,u,(p,q),v,r_2,\varepsilon)$, then we distinguish between two cases, $(p,q) \in B$ and $(p,q) \notin B$. If $(p,q) \in B$, then

$$\begin{aligned}
\delta_3(\gamma_3(\alpha)) &= \delta_3(\gamma_3(\beta \bullet e)) = \delta_3(\gamma_3(\beta) + \varepsilon\overline{u}_*\ \overline{v}^*[p,q]) \\
&= \delta_3(\gamma_3(\beta)) + \varepsilon\overline{u}_*\ \overline{v}^*\delta_3([p,q]) \\
&= \gamma_2(\iota\beta) - \gamma_2(\tau\beta) + \varepsilon\overline{u}_*\ \overline{v}^*\delta_3([p,q]) \ \text{(by induction hypothesis)} \\
&= \gamma_2(\iota\alpha) - \gamma_2(\tau\beta) + \varepsilon\overline{u}_*\ \overline{v}^*(\gamma_2(p) - \gamma_2(q))
\end{aligned}$$

$= \gamma_2(\iota\alpha) - \gamma_2(\tau\beta) + \gamma_2(\iota e) - \gamma_2(\tau e)$ (from above)

$= \gamma_2(\iota\alpha) - \gamma_2(\tau\alpha)$, since $\tau\beta = \iota e$ and $\tau e = \tau\alpha$.

If $(p,q) \notin B$, we have

$\delta_3(\gamma_3(\alpha)) = \delta_3(\gamma_3(\beta \bullet e)) = \delta_3(\gamma_3(\beta))$

$= \gamma_2(\iota\beta) - \gamma_2(\tau\beta)$ (by induction hypothesis)

$= \gamma_2(\iota\alpha) - \gamma_2(\tau\beta) + \varepsilon\overline{u}_* \, \overline{v}^*(\gamma_2(p) - \gamma_2(q))$ since

for all $(p,q) \in D \cup I$, $\gamma_2(p) = \gamma_2(q)$ (easily checked). Therefore,

$\delta_3(\gamma_3(\alpha)) = \gamma_2(\iota\beta) - \gamma_2(\tau\beta) + \gamma_2(\iota e) - \gamma_2(\tau e)$ (from above)

$= \gamma_2(\iota\alpha) - \gamma_2(\tau\alpha)$, since $\tau(\beta) = \iota(e)$, $\iota(\beta) = \iota(\alpha)$ and $\tau(e) = \tau(\alpha)$. ■

Lemma 3.5.17 *Let $B \subset P^2(\Gamma(\mathbf{x},\mathbf{r}))$ such that $\simeq_B = P^2(\Gamma(\mathbf{x},\mathbf{r}))$. Then, for every morphism $\overline{\omega}$ in $\mathbb{C}$, $\delta_{3,\overline{\omega}}\eta_{3,\overline{\omega}} + \eta_{2,\overline{\omega}}\delta_{2,\overline{\omega}} = id_{\mathbb{Z}\mathbb{C}[\mathbf{r}](\overline{\omega})}$.*

Proof. Let $\overline{u}$ and $\overline{v}$ be morphisms in $\mathbb{C}$ and $(l,r) \in \mathbf{r}$. We have the following:

$\delta_3(\eta_3(\overline{u},[l,r],\overline{v})) = \delta_3(\gamma_3(\alpha(\overline{u},(l,r),\overline{v})))$

$= \gamma_2(\iota(\alpha(\overline{u},(l,r),\overline{v})) - \gamma_2(\tau(\alpha(\overline{u},(l,r),\overline{v}))$ (by Lemma 3.5.16)

$= \gamma_2(\Gamma_1(\overline{u},(l,r),\overline{v})) - \gamma_2(\Gamma_2(\overline{u},(l,r),\overline{v}))$

$= (\overline{u},[l,r],\overline{v}) + \overline{b_2 \circ \ldots \circ b_t \circ \overline{v}}^* \gamma_2(\Gamma(\overline{u}, b_1)) + \ldots + \overline{v}^* \gamma_2(\Gamma(\overline{u \circ b_1 \circ \ldots \circ b_{t-1}}, b_t))$

$- \overline{a_2 \circ \ldots \circ a_s \circ \overline{v}}^* \gamma_2(\Gamma(\overline{u}, a_1)) - \ldots - \overline{v}^* \gamma_2(\Gamma(\overline{u \circ a_1 \circ \ldots \circ a_{s-1}}, a_s))$

On the other hand,

$\eta_2(\delta_2(\overline{u},[l,r],\overline{v})) = \eta_2(\overline{u}_* \, \overline{v}^*(\gamma_1(l) - \gamma_1(r))) = \eta_2\{\overline{u}_* \, \overline{v}^*(\overline{a_2 \circ \ldots \circ a_s}^*[a_1] + \overline{a_1}_* \, \overline{a_3 \circ \ldots \circ a_s}^*[a_2] + \ldots + \overline{(a_1 \circ \ldots \circ a_{s-1})}_*[a_s] - \overline{b_2 \circ \ldots \circ b_t}^*[b_1] + \overline{b_1}_* \, \overline{b_3 \circ \ldots \circ b_t}^*[b_2] + \ldots + \overline{(b_1 \circ \ldots \circ b_{t-1})}_*[b_t])\}$.

After summing up we have that $\delta_3(\eta_3(\overline{u},[l,r],\overline{v})) + \eta_2(\delta_2(\overline{u},[l,r],\overline{v})) = (\overline{u},[l,r],\overline{v})$ as claimed. ■

Lemma 3.5.18 *Let $B \subset P^2(\Gamma(\mathbf{x},\mathbf{r}))$ be such that $\simeq_B = P^2(\Gamma(\mathbf{x},\mathbf{r}))$. Then $\delta_{3,\overline{\omega}}(\mathbb{Z}\mathbb{C}[B](\overline{\omega})) = Ker(\delta_{2,\overline{\omega}})$.*

Proof. From Lemma 3.5.15 we have that $\delta_{3,\overline{\omega}}(\mathbb{Z}\mathbb{C}[B](\overline{\omega})) \subseteq Ker(\delta_{2,\overline{\omega}})$. Conversely, if $x \in Ker(\delta_{2,\overline{\omega}})$, then $(\eta_{2,\overline{\omega}}\delta_{2,\overline{\omega}})(x) = 0$ and hence $x = (\delta_{3,\overline{\omega}}\eta_{3,\overline{\omega}})(x) + (\eta_{2,\overline{\omega}}\delta_{2,\overline{\omega}})(x) = \delta_{3,\overline{\omega}}(\eta_{3,\overline{\omega}}(x))$ by Lemma 3.5.17. As a result $Ker(\delta_{2,\overline{\omega}}) \subseteq \delta_{3,\overline{\omega}}(\mathbb{Z}\mathbb{C}[B](\overline{\omega}))$. ■

Theorem 3.5.2 now follows easily.

3.6 Open Problems

In this section we raise a few questions regarding the relation between conditions f-FP_n and bi-FP_n for small categories and another one for the invariance of FDT.

Problem 3.6.1 Is it true that for small categories, f-FP_n is equivalent to bi-FP_n? What about monoids?

We believe that for small categories in general f-FP_n strictly implies bi-FP_n. The second question would have a positive answer if we could solve the following.

Problem 3.6.2 If the monoid S is of type bi-FP_n, then is there a free finite partial resolution of $\mathbb{Z}S$ which is S-graded?

Indeed, if the answer is positive, then we can apply Theorem 3.4.12.

Problem 3.6.3 Is FDT an invariant of the presentation?

We will discuss this in some detail. For monoids we know that: *two finitely presented monoids S_1 and S_2 given by the respective presentations $\mathcal{P}_1 = [\mathbf{x}_1, \mathbf{r}_1]$ and $\mathcal{P}_2 = [\mathbf{x}_2, \mathbf{r}_2]$ are isomorphic if and only if $\mathcal{P}_1$ and $\mathcal{P}_2$ can be obtained from the other by applying finitely many Tietze transformations.* We can certainly extend the notion of the elementary Tietze transformations for presentations of categories.

Definition 3.6.4 Let $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ be a presentation of a small category. The following four types of transformations on $\mathcal{P}$ will be called *Tietze transformations.*

(T_1) If $u : a \longrightarrow b$ and $v : a \longrightarrow b$ are such that $(u, v) \in \mathbf{r}^{\#}$, then add to $\mathbf{r}$ the pair (u, v).

(T_2) If $(u, v) \in \mathbf{r}$ such that $(u, v) \in (\mathbf{r} \backslash (u, v))^{\#}$, then we remove (u, v) from $\mathbf{r}$.

(T_3) For some arrow $u : a \longrightarrow b$ in $UF(\mathbf{x})$, add to $\mathbf{x}$ a new edge $\alpha : a \longrightarrow b$ and add the relation (α, u) to $\mathbf{r}$.

(T_4) If $\alpha : a \longrightarrow b$ is an edge in $\mathbf{x}$ and $u : a \longrightarrow b$ is a path in $UF(\mathbf{x})$ such that u does not factor through α and (α, u) is from $\mathbf{r}$, then remove α from the set of edges of $\mathbf{x}$ together with the respective relation (α, u) or (u, α) from $\mathbf{r}$ and then in every relation $(f, g) \in \mathbf{r}$ if α is a factor of either f or g, it will be replaced by u.

One can show in a similar way as in [96] that FDT is invariant under applying Tietze transformations. The problem is that Tietze transformations seem to be not enough to transform one presentation of a category to another. It is easy to show that for some given presentation $\mathcal{P} = [\mathbf{x}, \mathbf{r}]$ of a small category $\mathbb{C}$, if we apply one of T_1-T_4 to $\mathcal{P}$, then the resulting category is isomorphic to $\mathbb{C}$, but it is not clear whether the converse holds true or not.

Chapter 4

Notes on Finitely Generated Semigroups

4.1 Results in Combinatorial Semigroup Theory

In this chapter we will present some finiteness conditions for semigroups which are of a combinatorial nature such as permutation properties, iteration conditions, repetitivity, and minimal conditions on ideals. We show that in some cases minimal conditions on ideals are not necessary to ensure the finiteness of semigroups, but on the other hand, we exhibit an example of a semigroup S in which $\min_{\mathcal{R}}$ is independent of other "good" conditions which S may satisfy such as being finitely generated, periodic, inverse, E-unitary and even from the finiteness of the maximal subgroups of S. Also we prove that if a semigroup S is finitely generated and satisfies $\min_{\mathcal{Q}}$ (respectively $\min_{\mathcal{B}}$, $\min_{\mathcal{L}}$, $\min_{\mathcal{R}}$, $\min_{\mathcal{J}}$), then every congruence on S which contains $\mathcal{Q}$ (respectively $\mathcal{B}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$), is of finite index in S.

Throughout we will denote by A^* the free monoid with letters from a finite alphabet A. We say that a word $u \in A^*$ is a *factor* of $\omega \in A^*$, if there are ξ, $\eta \in A^*$ such that $\omega = \xi u \eta$. It is called a prefix if $\xi = \lambda$ and a suffix if $\eta = \lambda$. Any subset L of A^* will be called a *language* over A. For any language L, we denote by $F(L)$, $P(L)$, $S(L)$, the sets of factors, prefixes, suffixes of all the words of L. We say that L is *closed by factors* if $F(L) = L$. A language L is called *bounded* if there exists finitely many words $u_1,\ldots,u_n \in A^*$ such that $L \subseteq u_1^* \cdots u_n^*$.

A *two-sided* infinite (or *bi-infinite*) word ω over an alphabet A is any map

$$\omega : \mathbb{Z} \longrightarrow A.$$

For every $n \in \mathbb{Z}$, we set $\omega_n = \omega(n)$ and denote ω in the form

$$\omega = \cdots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \cdots.$$

The set of bi-infinite words on A will be denoted by $A^{\pm\omega}$. A word $u \in A^*$ is a *finite factor* of $\omega \in A^{\pm\omega}$ if $u = \lambda$ or there exist $i, j \in \mathbb{Z}$ such that $i \leq j$ and $u = \omega_i \cdots \omega_j$. Also one can define *right-infinite* (respectively *left-infinite*) words ω over an alphabet A by a map

$$\omega : \mathbb{N}_+ \longrightarrow A \text{ (respectively } \omega : \mathbb{N}_- \longrightarrow A).$$

The set of right-infinite words will be denoted by A^ω and that of left-infinite words by $A^{-\omega}$. For every bi-infinite word $\omega = \cdots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \cdots$, we denote by ω_+ the word $\omega_1\omega_2 \cdots$.

For all the definitions and results given in this section we refer the reader to [26].

Definition 4.1.1 Let S be a set and $\phi : A^* \longrightarrow S$ a map. A word $\omega = \omega_1 \cdots \omega_k$ with $\omega_i \in A^*$, is called a *k-power modulo* ϕ if

$$\phi(\omega_1) = \ldots = \phi(\omega_k).$$

Definition 4.1.2 Let A^* be a free monoid, S a set and $k \in \mathbb{N}$, $k > 1$. A map $\phi : A^* \longrightarrow S$ is called *k-repetitive* if there exist a positive integer L, depending on ϕ and k, such that every word ω with $|\omega| \geq L$ has a factor which is a k-power modulo ϕ. One says that is *repetitive* if it k-repetitive for every $k > 1$.

Definition 4.1.3 A factor u of an infinite word $\omega \in A^\omega$ (respectively $\omega \in A^{\pm\omega}$) is *recurrent* if the set of all $i \in \mathbb{N}_+$ (respectively $i \in \mathbb{Z}$) such that $u = \omega[i, i + |u| - 1]$ has not an upper (respectively upper and lower) bound. The word is *recurrent* if and only if all its factors are recurrent.

Definition 4.1.4 A factor u of an infinite word t occurs *sydentically* in t if there exists an integer k such that in any factor of t of length k there is at least one occurrence of u. An infinite word is called *uniformly recurrent*, or with *bounded gaps*, if all its factors occur sydentically in t.

Theorem 4.1.5 *Let $L \subseteq A^*$ be an infinite language. There exists an infinite word $x \in A^{\pm\omega}$ such that*

(i) x is uniformly recurrent and

(ii) $F(x) \subseteq F(L)$.

Corollary 4.1.6 *Let J be a two sided ideal of A^*. If, for every uniformly recurrent word $\omega \in A^\omega$, $F(\omega) \cap J \neq \phi$, then there exists an $n > 0$ such that $A^n A^* \subseteq J$.*

Proof. Suppose that there exist infinitely many words that belong to the set $C = A^* \backslash J$. Being the complement of an ideal, C is closed by factors, then by the above theorem, there exists a uniformly recurrent word $\omega \in A^{\pm\omega}$ such that $F(\omega) \subseteq C$. Hence, $F(\omega_+) \subseteq C$ which is a contradiction since $\omega_+ \in A^\omega$ is uniformly recurrent as well. ■

If a semigroup S is generated by a set A, then we can define the canonical morphism $\phi : A^+ \longrightarrow S$ where A^+ is the free semigroup with base A and ϕ sends each word of A^+ to the element of S it represents. Suppose we have a total order $<$ on A and define the *alphabetical* order $<_a$ on A^+ as follows:

$$u <_a v \iff (|u| < |v|) \text{ or}$$
$$(|u| = |v| \text{ and } u = hx\xi,\ v = hy\eta,\ h, \xi, \eta \in A^*,\ x, y \in A \text{ and } x < y).$$

A word $\omega \in A^+$ is called *reducible*, if there exists $u \in A^+$ such that

$$u <_{lex} \omega \quad \text{and} \quad \phi(u) = \phi(\omega).$$

A word which is not reducible, is called *irreducible.* Let $s \in S$. The unique minimal element of $\phi^{-1}(s)$ will be called the *canonical representative* of s. For every subset $T \subseteq S$, we denote by C_T the set of canonical representatives of the elements of T.

A sequence $s_1, ..., s_n$ of elements of a semigroup S is called a *bi-ideal* sequence if for $i > 0$

$$s_{i+1} \in s_i S^1 s_i,$$

where $S^1 = S \cup \{1\}$ with 1 a unit element, or $S^1 = S$ if S already has such an element.

Proposition 4.1.7 *Let S be a finitely generated semigroup. If T is an infinite subset of S closed by factors, then there exists a bi-ideal sequence $(s_n)_{n>0}$ such that for all $n > 0$, $s_n \in T$ and for all positive integers i, j with $i \neq j$, one has $s_i \neq s_j$.*

Let S be a finitely generated semigroup, A its generating set and $\phi : A^+ \longrightarrow S$ be the *canonical morphism.* The *growth function* of S is defined for all $n > 0$, as

$$g_S(n) = card\{s \in S \mid \phi^{-1}(s) \cap A^{\leq n} \neq \phi\}.$$

Proposition 4.1.8 *Let S be a finitely generated semigroup such that there exists an integer $n > 0$ for which*

$$g_S(n) < \frac{n(n+3)}{2}.$$

If S is infinite, then S contains an element of infinite order.

In what follows we will define some finiteness conditions for semigroups and give a few important results related with them.

Definition 4.1.9 Let S be a semigroup and n an integer > 1. A sequence $s_1, ..., s_n$ of n elements of S is called *permutable* if the product $s_1 \cdots s_n$ remains invariant under some non-trivial permutation of its factors.

We say that S is *n-permutable* if every sequence of n elements of S is permutable, and that S is *permutable* if it is n-permutable for some $n > 0$. Obviously, permutability generalizes commutativity. There are a number of interesting results which we mention briefly below.

Proposition 4.1.10 *Let S be a finitely generated semigroup which is permutable. Then, C_S is a bounded language and the growth function of S is polynomially upper bounded.*

Theorem 4.1.11 *Let S be a finitely generated and periodic semigroup. S is finite if and only if it is permutable.*

There is a characterization of finitely generated groups.

Theorem 4.1.12 *A finitely generated group G is permutable if and only if it is abelian-by-finite, i.e., G has an abelian (normal) subgroup of finite index.*

The similarity of the following two results is not surprising if we recall that to a certain degree, completely 0-simple semigroups are similar to groups.

Theorem 4.1.13 *If the growth function of a group G is bounded by a polynomial of degree ≤ 3, then G is permutable.*

Proposition 4.1.14 *A completely 0-simple semigroup whose growth function is bounded by a polynomial of degree ≤ 3 is permutable.*

Next we give some finiteness condition of a different nature. They are related with the so called chain conditions which are conditions on the ideal structure of the semigroups. We advise the reader to read first a few basic notions from Semigroup Theory in the Appendix of this Thesis.

Consider the following relations on a semigroup S. For $s, t \in S$ we set

$$\begin{aligned} s \leq_{\mathcal{R}} t &\iff sS^1 \subseteq tS^1 \\ s \leq_{\mathcal{L}} t &\iff S^1 s \subseteq S^1 t \\ s \leq_{\mathcal{J}} t &\iff S^1 s S^1 \subseteq S^1 t S^1. \end{aligned}$$

Definition 4.1.15 A semigroup S satisfies the minimal condition on principal right (respectively left, two-sided) ideals if the quasi-order $\leq_{\mathcal{L}}$ (respectively $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$) is well-founded. We denote by $\min_{\mathcal{L}}$ (respectively $\min_{\mathcal{R}}$, $\min_{\mathcal{J}}$) these minimal conditions.

Definition 4.1.16 An element $s \in S$ is called *right-stable* (respectively *left-stable*) if for every $t \in J_s$, $tS^1 \subseteq sS^1$ (respectively $S^1 t \subseteq S^1 s$) implies $t\mathcal{R}s$ (respectively $t\mathcal{L}s$). It is called stable if it is both right and left stable. A subset $X \subseteq S$ is called stable if every element of X is stable.

Lemma 4.1.17 *Every periodic semigroup is stable.*

Before we state the $\mathcal{J}$-depth decomposition theorem, which we use in the proof of Theorem 4.2.7, we give some definitions and preliminary results.

Definition 4.1.18 Let s be an element of a semigroup S. The *$\mathcal{J}$-depth* of s is the length of the longest strictly ascending chain of two-sided principal ideals starting with s. The $\mathcal{J}$-depth of s can be infinite. A semigroup S admits a *$\mathcal{J}$-depth function $d_{\mathcal{J}}$* if for every $s \in S$ the $\mathcal{J}$-depth $d_{\mathcal{J}}(s)$ of s is finite.

For $s, t \in S$, if $J_s < J_t$, then we say that the $\mathcal{J}$-class J_t is *above* J_s.

Definition 4.1.19 A semigroup S is *weakly finite J-above* if each $\mathcal{J}$-class of S has only finitely many $\mathcal{J}$-classes above it.

Definition 4.1.20 A semigroup S is *finite J-above* if it is weakly finite $\mathcal{J}$-above and every $\mathcal{J}$-class is finite.

Let S be a semigroup. We define recursively a sequence $(K_n)_{n\geq 0}$ of sets as follows : $K_0 = \emptyset$ and for all $n > 0$,

$$K_n = \bigcup_{1 \leq j \leq n} C_j,$$

where for $j > 0$, C_j is the set of elemens of $S \backslash K_{j-1}$ which are maximal with respect to $\leq_{\mathcal{J}}$ in $S \backslash K_{j-1}$. We set $K_S = \bigcup_{j>0} K_j$.

Lemma 4.1.21 *Let S be a semigroup. For all $j > 0$, K_j is closed by factors and is a union of $\mathcal{J}$-classes.*

Definition 4.1.22 A semigroup S has a *weak $\mathcal{J}$-depth decomposition* if for all $j > 0$ the sets K_j are finite. Moreover, if S is infinite then K_S has to be infinite. A semigroup S has a *$\mathcal{J}$-depth decomposition* if it has a weak $\mathcal{J}$-depth decomposition and $S = K_S$.

Proposition 4.1.23 *Let S be a semigroup. The following are equivalent.*

(i) S has a $\mathcal{J}$-depth function and a weak $\mathcal{J}$-depth decomposition.

(ii) S has a $\mathcal{J}$-depth decomposition.

A more direct connection between the $\mathcal{J}$-depth decomposition and the ideals of a semigroup, is given in the following.

Proposition 4.1.24 *If a semigroup S has a $\mathcal{J}$-depth decomposition, then S is finite $\mathcal{J}$-above.*

The following has many applications to finiteness conditions for finitely generated semigroups with maximal subgroups locally finite.

Theorem 4.1.25 ($\mathcal{J}$-depth decomposition theorem) *Let S be a finitely generated semigroup which is right stable and whose subgroups are locally finite. Then S has a weak $\mathcal{J}$-depth decomposition.*

Returning to the chain conditions, we give a theorem found in [24] which generalizes a theorem of Hotzel [42].

Theorem 4.1.26 *Let S be a finitely generated semigroup whose subgroups are locally finite. If S satisfies $\min_{\mathcal{R}}$ (respectively$\min_{\mathcal{L}}$), then S is finite.*

Proof. If S satisfies $\min_{\mathcal{R}}$ then it is right-stable. Suppose that S is infinite, then from Theorem 4.1.25 so will be K_S. Since K_S is closed by factors, from Proposition 4.1.7 one finds a bi-ideal sequence $(f_n)_{n>0}$ of elements of K_S such that

$$f_n = f_{n-1}g_{n-1}f_{n-1},\ g_{n-1} \in S^1,\ n > 1,$$

and $f_n \neq f_m$ for $n \neq m$. Since in particular $f_nS^1 \subseteq f_mS^1$, from $\min_{\mathcal{R}}$ there exists an integer k such that for all $n \geq k$ we have $f_n\mathcal{R}f_k$. This means that the class J_{f_n} is infinite and therefore K_j which contains f_n will be so, which is a contradiction. ■

Recall from [97] the definition a bi-ideal in a semigroup S. We call $B \subseteq S$ a *bi-ideal* if

$$BSB \subseteq B.$$

It is easy to see that the principal bi-ideal generated by $s \in S$ has the form

$$B(s) = sS^1s \cup s.$$

This gives rise to another relation on S:

$$s\mathcal{B}t \iff B(s) = B(t).$$

We say that $s \leq_{\mathcal{B}} t$ if $B(s) \subseteq B(t)$. If this quasi-order is well-founded, we say that S satisfies $\min_{\mathcal{B}}$.

The following generalizes a theorem of Coudrain and Schutzenberger [18].

Theorem 4.1.27 *Let T be a semigroup which satisfies* $\min_{\mathcal{B}}$. *Let T' be a subsemigroup of T such that the subgroups of T are locally finite in T'. Then T' is locally finite.*

Corollary 4.1.28 *Let T be a semigroup satisfying* $\min_{\mathcal{B}}$. *If T' is a periodic subsemigroup whose subgroups are locally finite, then T' is finite.*

We will use Corollary 4.1.28 to prove the McNaughton and Zalcstein Theorem [24]:

Theorem 4.1.29 *A torsion semigroup of $n \times n$ matrices over a field is locally finite.*

Sketch of proof. For every field F, the semigroup $\mathcal{M}_n(F)$ of $n \times n$ matrices over F can be identifies with $\mathrm{End}_n(V, F)$ where V is a vectorial space of dimension n. In a next step, one can prove that $\mathrm{End}_n(V, F)$ satisfies $\min_{\mathcal{R}}$ and $\min_{\mathcal{L}}$ and therefore as can be easily seen, also $\min_{\mathcal{B}}$. Also it is know that all the maximal subgroups of $\mathcal{M}_n(F)$ are locally finite (see [46]). All we stated above, hold true for every subsemigroup S of $\mathcal{M}_n(F)$ and as a result Corollary 4.1.28 applies. ■

We say that a semigroup S satisfies the *iteration* property if for any product $s_1 \cdots s_m$ of a sufficiently great number m of elements of S, there exists a factor $s_i \cdots s_j$ with $1 \leq i \leq j \leq m$ which can be iterated, i.e., can be replaced by $(s_i \cdots s_j)^n$, $n \neq 1$, without changing the value of the product.

Definition 4.1.30 Let S be a semigroup and m and n integers such that $m > 0$ and $n \geq 0$. We say that the sequence $s_1, ..., s_m$ of m elements of S is *n-iterable* if there exist integers i, j such that $1 \leq i \leq j \leq m$ and

$$s_1 \cdots s_m = s_1 \cdots s_{i-1} \cdot (s_i \cdots s_j)^n \cdot s_{j+1} \cdots s_m.$$

We say that S is *(m,n)-iterable*, or satisfies the property $C(n,m)$, if all sequences of m elements are n-iterable.

The condition stated below is rather weaker.

Definition 4.1.31 Let m be a positive integer. A semigroup S satisfies the condition $C(m)$ if for any sequence $s_1, ..., s_m$ of m elements of S there exist integers i, j, n such that $1 \leq i \leq j \leq m$, $0 \leq n \neq 1$, and

$$s_1 \cdots s_m = s_1 \cdots s_{i-1} \cdot (s_i \cdots s_j)^n \cdot s_{j+1} \cdots s_m.$$

There is also a stronger version of Definition 4.1.30 called the *iteration property on the right.* A semigroup satisfies $D(n,m)$, if for every sequence $s_1, ..., s_m$ of m elements of S there exist integers i, j such that $1 \leq i \leq j \leq m$ and

$$s_1 \cdots s_j = s_1 \cdots s_{i-1} \cdot (s_i \cdots s_j)^n.$$

Theorem 4.1.32 *Let S be a finitely generated semigroup. Then S is finite if and only if S satisfies properties $D(2,m)$ (respectively $C(2,m)$) or $D(3,m)$ (respectively $C(3,m)$).*

The following shows that in the case of $C(m)$ one needs to assume the finiteness of the finitely generated subgroups of S.

Theorem 4.1.33 *Let S be a finitely generated semigroup satisfying the iteration condition $C(m)$. If the subgroups of S are locally finite, then S is finite.*

Lastly we define strong repetitivity as a candidate for a finiteness condition for semigroups. Before doing so we need the following.

Definition 4.1.34 Let S be a semigroup. We say that a morphism $\phi : A^+ \longrightarrow S$ is *strongly repetitive* if it satisfies the following condition: for any map $f : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ there exists a positive integer M, depending on f, such that for every $\omega \in A^+$ if $|\omega| \geq M$, then ω can be factorized as

$$\omega = hv_1 \cdots v_{f(p)}h',$$

where $p \in \mathbb{N}_+$, $h, h' \in A^*$, $0 < |v_i| \leq p$, $1 \leq i \leq f(p)$, and

$$\phi(v_1) = \ldots = \phi(v_{f(p)}).$$

Definition 4.1.35 A semigroup S is *strongly repetitive*, if for every finite alphabet A, every morphism $\phi : A^+ \longrightarrow S$ is strongly repetitive.

Theorem 4.1.36 *A finite semigroup is strongly repetitive.*

The following in due to Brown [14] and will be useful in the next section.

Theorem 4.1.37 *Let $\phi : S \longrightarrow T$ be a morphism of semigroups. If T is locally finite and if for each idempotent $e \in T$, $\phi^{-1}(e)$ is locally finite, then S is locally finite.*

Sketch of proof. One must show that for every finite alphabet A and every morphism $\zeta : A^+ \longrightarrow S$, $\zeta(A^+)$ is finite. Since T is locally finite, one has that $T' = \psi(A^+)$ is finite where $\psi = \zeta\phi$. There is $r \in \mathbb{N}_+$ such that for every $t \in T'$, $t^r \in E(T')$. For every idempotent $e \in T'$ and $n \in \mathbb{N}_+$ we denote by $X_{n,e}$ the set

$$X_{n,e} = \{u \in A^+ \mid \zeta(u) \in \phi^{-1}(e) \text{ and } |u| \leq n\}.$$

Since $\phi^{-1}(e)$ is locally finite and $\zeta(X_{n,e}^+)$ is a finitely generated subsemigroup of $\phi^{-1}(e)$, we have that $\zeta(X_{n,e}^+)$ is finite. There exists an integer $p(n,e)$ such that if $u \in X_{n,e}^+$ and $|u| \geq p(n,e)$, then there exists $u' \in X_{n,e}^+$ such that $|u'| < |u|$ and $\zeta(u) = \zeta(u')$. Let us set

$$f(n) = r \max\{p(rn, e) \mid e \in E(T')\}.$$

Using the fact that $\psi : A^+ \longrightarrow T'$ is strongly repetitive (Theorem 4.1.36), one can show that there exist $M \in \mathbb{N}_+$ such that all the words of A^+ of length at least M are ζ equivalent with shorter words of A^+ and as a result $\zeta(A^+)$ is finite. ■

Lastly we give the following.

Theorem 4.1.38 *Let S be a semigroup. S is locally finite if and only if it is strongly repetitive.*

4.2 Some other Finiteness Conditions

The results of this section will appear in [83].

Let $(S, \cdot)$ be a monoid generated by a finite set A ($A \subseteq S$). Denote by A^* the free monoid on A and by φ the canonical morphism $\varphi : A^* \longrightarrow S$ sending every word $\omega \in A^*$ to the element of S it is representing. For any congruence $\mathcal{K}$ on S and for any $\mathcal{K}$-class, $\mathcal{K}_x$ ($x \in S$) we define the set

$$M_x = \left\{\omega \in \varphi^{-1}(\mathcal{K}_x) \mid \forall \omega' \in \varphi^{-1}(\mathcal{K}_x),\, l(\omega) \leq l(\omega')\right\},$$

where $l(\omega)$ is the length of the word ω. Clearly M_x contains all the words of $\varphi^{-1}(\mathcal{K}_x)$ of minimal length which will be referred to later as *the set of the minimal length representatives of the $\mathcal{K}$-class $\mathcal{K}_x$*. Next we show that the set $M = \bigcup_{x \in S} M_x$ is closed by factors. Indeed, let $\omega = u_1vu_2 \in M$ and suppose $v \in A^* - M$. It follows that there is $v' \in \varphi^{-1}(\mathcal{K}_{\varphi(v)})$ such that $l(v') < l(v)$. On the other hand, since $\varphi(v')\mathcal{K}\varphi(v)$,we obtain $\varphi(u_1)\varphi(v')\varphi(u_2)\mathcal{K}\varphi(u_1)\varphi(v)\varphi(u_2)$ or equivalently $\omega' = u_1v'u_2 \in \varphi^{-1}(\mathcal{K}_{\varphi(\omega)})$. But $l(\omega') = l(u_1)+l(v')+l(u_2) < l(u_1)+l(v)+l(u_2) = l(\omega)$, which is a contradiction. Similarly it can be shown that if $\omega \in M$, then all its suffixes and prefixes are also in M, hence M is closed by factors. This fact is denoted for short by $M = F(M)$.

Suppose that $A^* - M \neq \emptyset$. Under this assumption we show that $I = A^* - M$ is an ideal of A^*, that is $IA^* \subseteq I$ and $A^*I \subseteq I$. Indeed, suppose by the way of contradiction that there is $u \in I$, $v \in A^*$ such that $uv \in M$, then since $M = F(M)$ it follows that $u \in M$, which is a contradiction. Similarly I is a right ideal. Thus we have proved the following.

Lemma 4.2.1 *If $(S, \cdot)$ is a monoid generated by a finite set A and $\mathcal{K}$ a congruence on S such that $A^* - M \neq \emptyset$ where M is the set of the minimal length representatives of the $\mathcal{K}$-classes. Then $I = A^* - M$ is an ideal of the free monoid A^*.*

In the next lemma, Δ will denote the trivial relation $\{(x,x) \mid x \in S\}$ on the set S.

Lemma 4.2.2 *Let S be a monoid generated by a finite set A, $\mathcal{K}$ a congruence on S and let M be the set of the minimal length representatives of the $\mathcal{K}$-classes. If $\mathcal{K}$ contains $\mathcal{Q}$ (respectively $\mathcal{B}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$) and either $\mathcal{Q} \neq \Delta$ (respectively $\mathcal{B} \neq \Delta$, $\mathcal{L} \neq \Delta$, $\mathcal{R} \neq \Delta$, $\mathcal{J} \neq \Delta$), or $\mathcal{Q} = \Delta$ (respectively $\mathcal{B} = \Delta$, $\mathcal{L} = \Delta$, $\mathcal{R} = \Delta$, $\mathcal{J} = \Delta$) and S satisfies* $\min_{\mathcal{Q}}$ *(respectively* $\min_{\mathcal{B}}$, $\min_{\mathcal{L}}$, $\min_{\mathcal{R}}$, $\min_{\mathcal{J}}$*), then $A^* - M \neq \phi$.*

Proof. We will prove the claim for $\mathcal{Q}$ only since the proofs for the other cases run similarly. Suppose first that $\mathcal{Q} \neq \Delta$ and let $x, y \in S$ such that $x \neq y$ and $x\mathcal{Q}y$. It follows in particular that there are $s_1, s_2 \in S\backslash\{1\}$ (1 is the unit element of S) such that $x = xs_1s_2$. This means that there are two representations of x with words from A^* of different lengths. Since $\mathcal{Q} \subseteq \mathcal{K}$, we have that $M \neq A^*$.

Suppose now that $\mathcal{Q} = \Delta$ and S satisfies $\min_{\mathcal{Q}}$. It is easy to see that in general for every $x \in S$ we have

$$x \geq_{\mathcal{Q}} x^2 \geq_{\mathcal{Q}} \dots \geq_{\mathcal{Q}} x^n \geq_{\mathcal{Q}} \dots,$$

therefore $\min_{\mathcal{Q}}$ implies that there is an $n \geq 1$ such that $x^n \mathcal{Q} x^{n+1}$. Since $\mathcal{Q} = \Delta$, we must have $x^n = x^{n+1}$ and as a consequence there are two representations of x^n with words from A^* of different lengths. As with the first case one deduces that $M \neq A^*$. ■

Proposition 4.2.3 *Let S be a finitely generated monoid which satisfies* $\min_{\mathcal{Q}}$. *Every congruence $\mathcal{K}$ on S which contains $\mathcal{Q}$ is of finite index in S.*

Proof. We use the notation of Lemma 4.2.1 and Lemma 4.2.2. Lemma 4.2.2 assures that we always have $M \neq A^*$. Since M meets every $\varphi^{-1}(\mathcal{K}_x)$, $x \in S$, then it suffices to show that $M = A^* - I$ is finite. We will make use of Corollary 2.3.2 of [26] to prove the finiteness of M. According to that result, we must show that for any uniformly recurrent word $w \in A^\omega$ we have $F(w) \cap I \neq \phi$, where $F(w)$ is the set of factors of w. Let $w = a_1 a_2 \dots$, be a uniformly recurrent word from the set of infinite words A^ω with letters from A. Denote by

$$Q_0 = (\varphi(a_1))_q = (\varphi(a_1)S \cap S\varphi(a_1)) \cup \varphi(a_1)$$

the principal quasi-ideal generated by $\varphi(a_1)$. Since w is uniformly recurrent, then there is $v_1 \in F(w)$ such that $u_1 = a_1 v_1 a_1 \in F(w)$. Observe that

$$\begin{aligned} Q_1 = (\varphi(u_1))_q &= (\varphi(a_1)\varphi(v_1)\varphi(a_1)S \cap S\varphi(a_1)\varphi(v_1)\varphi(a_1)) \\ &\cup \varphi(a_1)\varphi(v_1)\varphi(a_1) \subseteq \varphi(a_1)S \cap S\varphi(a_1) \subseteq Q_0. \end{aligned}$$

Inductively one can construct a sequence as follows

$$a_1, u_1 = a_1 v_1 a_1, u_2 = u_1 v_2 u_1, \dots, u_k = u_{k-1} v_k u_{k-1}, \quad u_{k+1} = u_k v_{k+1} u_k, \dots,$$

where $u_k \in F(w)$, $k \geq 1$ and $\varphi(a_1) \geq_{\mathcal{Q}} \varphi(u_1) \geq_{\mathcal{Q}} \varphi(u_2) \geq_{\mathcal{Q}} \dots \varphi(u_k) \geq_{\mathcal{Q}} \varphi(u_{k+1}) \geq_{\mathcal{Q}} \dots$. Recalling that S satisfies $\min_{\mathcal{Q}}$, we find $k \in \mathbb{N}$ such that $\varphi(u_k)\mathcal{Q}\varphi(u_{k+1})$ or equivalently $\varphi(u_k)$ and $\varphi(u_{k+1})$ are in the same $\mathcal{Q}$-class and hence in the same $\mathcal{K}$-class, say $K_{\varphi(u_k)}$. It follows that u_k and $u_{k+1} \in \varphi^{-1}(K_{\varphi(u_k)})$, but from the construction we have $l(u_k) < l(u_{k+1})$, hence $u_{k+1} \notin M$. So $u_{k+1} \in F(w) \cap I$ and we are done. ■

Combining Theorem 4.1.37 with Proposition 4.2.3, we obtain the finiteness condition of Corollary 4.2.4 below.

Corollary 4.2.4 *Let S be a finitely generated monoid and let $\mathcal{K}$ be a congruence containing $\mathcal{Q}$. Then S is finite if and only if it satisfies* $\min_{\mathcal{Q}}$ *and every idempotent of the factor monoid $S/\mathcal{K}$ is a locally finite subsemigroup of S.*

Similar results to Proposition 4.2.3, and consequently to Corollary 4.2.4, hold if we put respectively relations $\mathcal{B}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ instead of $\mathcal{Q} = \mathcal{H}$.

Let us now consider the semigroup S with r generators which satisfies the equation $x^n = x^{n+1}$ for a fixed $n \in N$ and every $x \in S$. We denote it by $B(r, n, n+1)$. Since the $\mathcal{Q}$-classes of $B(r, n, n+1)$ are trivial (see Lemma 4.6.1 of [26]), we have that $\mathcal{Q}$ is a congruence. The presentation giving the semigroup in this case is length reducing, therefore there are words representing the same element of S which do not have the same length. In particular this means that we automatically have the condition $M \neq A^*$ satisfied.

Proposition 4.2.5 $S = B(r, n, n+1)$ *is finite if and only if it satisfies* $\min_{\mathcal{Q}}$.

Proof. From the above comment, the set of the minimal length representatives M related with any congruence is never equal to A^*, the free monoid of rank r. On the other hand, since in this case $\mathcal{Q}$ is itself a congruence whose classes are of a single element, one can get the result by applying Corollary 4.2.4. ■

Lemma 4.2.6 *If S is a semigroup that satisfies* $\min_{\mathcal{Q}}$ *and all its maximal subgroups are locally finite, then it is periodic.*

Proof. As we mentioned earlier, for every $a \in S$,

$$a \geq_{\mathcal{Q}} a^2 \geq_{\mathcal{Q}} a^3 \geq_{\mathcal{Q}} \dots \geq_{\mathcal{Q}} a^n \geq_{\mathcal{Q}} a^{n+1} \geq_{\mathcal{Q}} \dots.$$

On the other hand $\min_{\mathcal{Q}}$ implies the existence of $n \in \mathbb{N}$ such that $a^n \mathcal{Q} a^{2n}$. It follows that the $\mathcal{Q}$-class H_{a^n} is a subgroup of S. Now denoting by $\langle a^n \rangle$ the subgroup of H_{a^n} generated by a^n, we have from the assumption that $\langle a^n \rangle$ is finite, hence there is some $k \in \mathbb{N}$ such that $a^n = a^{nk}$. This shows that a is periodic. ■

Theorem 4.2.7 *A finitely generated semigroup S is finite if and only if it satisfies* $\min_{\mathcal{Q}}$ *and all its maximal subgroups are locally finite.*

Proof. From Lemma 4.2.6, S is periodic and consequently it is stable. From the $\mathcal{J}$-depth decomposition theorem it follows that S has a weak $\mathcal{J}$-depth decomposition, consequently if we suppose S to be infinite, then so will be K_S. Under this assumption, from Proposition 3.2.2 and

Lemma 3.6.3 of [26], there is a bi-ideal sequence $(f_n)_{n\geq 0}$ of elements of K_S such that $f_n \neq f_m$ for all $n \neq m$. Since $f_n \in f_{n-1}S^1 \cap S^1 f_{n-1} \subseteq (f_{n-1})_q$, then from $\min_Q$ one has that there exists $n \in \mathbb{N}$ such that $f_n \mathcal{H} f_m$ for all $m \geq n$, hence J_{f_n} contains infinitely many elements of S. It follows that the one K_j which contains J_{f_n} is infinite, a contradiction. ■

Observe that Theorem 4.2.7 is a substantial generalization of Proposition 4.2.5.

Now we focus our study in finding finiteness conditions for some special kinds of regular semigroups. First we recall the following from [81].

Lemma 4.2.8 *Let S be a completely 0-simple semigroup. Then S is locally finite if and only if a maximal subgroup of S is locally finite.*

Proposition 4.2.9 *Let the semigroup S be a union of completely 0-simple semigroups S_i, $i \in I$ such that for any $i, j \in I$, $S_iS_j \subseteq S_i \cap S_j$. Then S is locally finite if and only if every subgroup of S is locally finite.*

Proof. Here we use an induction argument on the minimal number of completely 0-simple subsemigroups S_i, $i \in I$ needed to contain the set of generators X of a finitely generated subsemigroup of S. Assume that every subgroup of S is locally finite. If X is contained in a single S_i for a certain $i \in I$, then Lemma 4.2.8 implies that the subsemigroup $\langle X \rangle$ is finite. Let us now suppose that any finitely generated subsemigroup of S whose set of generators is contained in at most $k-1$ semigroups of the collection $\{S_i\}_{i\in I}$, is finite. Let $X \subseteq S$ be finite and the minimal number of the subsemigroups of S of the collection $\{S_i\}_{i\in I}$ which contain X is k. Denoting these semigroups by $S_1, S_2, ..., S_{k-1}, S_k$, we may write

$$X \subseteq S_1 \cup S_2 \cup ... \cup S_{k-1} \cup S_k.$$

Denote by $Y_1 = X \cap (\bigcup_{j\neq k} S_j)$ and $Y_2 = X \cap (S_k \setminus \bigcup_{j\neq k} S_j)$. Now since $Y_1 \subseteq S_1 \cup S_2 \cup ... \cup S_{k-1}$, it follows that the minimal number of S_i, $i \in I$ that contain Y_1 is at most $k-1$. From the induction hypothesis it follows that $T_1 = \langle Y_1 \rangle$ is finite. Of course $T_2 = \langle Y_2 \rangle$ is finite too. Next we show that

$$\langle X \rangle = \langle Y_1 \cup Y_2 \rangle = T_1 \cup T_2 \cup T_3 \cup T_1T_3 \cup T_3T_1 \cup T_2T_3 \cup T_3T_2,$$

where $T_3 = \langle T_1T_2 \cup T_2T_1 \rangle$ and that T_3 is finite. Indeed, any element of $\langle Y_1 \cup Y_2 \rangle$ belongs to one of the following four types of products of powers of Y_1 and Y_2:

$$(Y_1^{\alpha_{11}}Y_2^{\alpha_{21}})(Y_1^{\alpha_{12}}Y_2^{\alpha_{22}})...(Y_1^{\alpha_{1n}}Y_2^{\alpha_{2n}}),$$
$$(Y_1^{\alpha_{11}}Y_2^{\alpha_{21}})(Y_1^{\alpha_{12}}Y_2^{\alpha_{22}})...(Y_1^{\alpha_{1n-1}}Y_2^{\alpha_{2n-1}})Y_1^{\alpha_{1n}},$$
$$(Y_2^{\alpha_{21}}Y_1^{\alpha_{11}})(Y_2^{\alpha_{22}}Y_1^{\alpha_{12}})...(Y_2^{\alpha_{2n}}Y_1^{\alpha_{1n}}),$$
$$(Y_2^{\alpha_{21}}Y_1^{\alpha_{11}})(Y_2^{\alpha_{22}}Y_1^{\alpha_{12}})...(Y_2^{\alpha_{2n-1}}Y_1^{\alpha_{1n-1}})Y_2^{\alpha_{2n}},$$

where n ranges over $\mathbb{N}$ and for all $k = 1,..,n$, α_{1k} and α_{2k} are non negative integers. It is clear that products of the first type are included in $(T_1T_2)^n \subseteq T_3$, products of the second type are included in $(T_1T_2)^{n-1}T_1 \subseteq T_3T_1$, those of the third type in $(T_2T_1)^n \subseteq T_3$ and lastly products of the fourth type in $(T_2T_1)^{n-1}T_2 \subseteq T_3T_2$, hence

$$\langle Y_1 \cup Y_2\rangle = \langle T_1 \cup T_2\rangle \subseteq T_1 \cup T_2 \cup T_3 \cup T_1T_3 \cup T_3T_1 \cup T_2T_3 \cup T_3T_2,$$

while the converse is obvious. Finally to prove that $\langle X\rangle$ is finite we need only the finiteness of T_3. First we see that

$$\begin{aligned} T_1T_2 &\subseteq (S_1 \cup S_2 \cup ... \cup S_{k-1})S_k = S_1S_k \cup ... \cup S_{k-1}S_k \\ &\subseteq (S_1 \cap S_k) \cup ... \cup (S_{k-1} \cap S_k) \\ &\subseteq (S_1 \cup S_2 \cup ... \cup S_{k-1}) \cap S = S_1 \cup S_2 \cup ... \cup S_{k-1}. \end{aligned}$$

Similarly $T_2T_1 \subseteq S_1 \cup S_2 \cup ... \cup S_{k-1}$. Consequently, $T_1T_2 \cup T_2T_1 \subseteq S_1 \cup S_2 \cup ... \cup S_{k-1}$. Now since $T_1T_2 \cup T_2T_1$ is finite and since the minimal number of semigroups of $\{S_i\}_{i \in I}$ which contain $T_1T_2 \cup T_2T_1$, is at most $k-1$, it follows from induction that $T_3 = \langle T_1T_2 \cup T_2T_1\rangle$ is finite. ■

Corollary 4.2.10 *A primitive regular semigroup S is locally finite if and only if every subgroup of S is locally finite.*

Proof. From Theorem 1.9 of [34], S is a 0-direct union of completely 0-simple semigroups. The result follows from Proposition 4.2.9. ■

In what follows one needs the concept of a tree of semigroups. The description in general of such structures is given in Lemma 3.3 and Lemma 3.4 of [34]. We include below these lemmas for the convenience of the reader.

Lemma 4.2.11 *Let $A \geq B$ be $\mathcal{J}$-classes of a strict regular semigroup. Let $\varphi_B^A : A \longrightarrow B$ assign to each $x \in A$ the element $y \in B$ such that $y \leq x$. Then φ_B^A is a partial homomorphism. Furthermore φ_A^A is the identity on A; if $A \geq B \geq C$, then $\varphi_C^B \circ \varphi_B^A = \varphi_C^A$; and*

$$xy = x_1y_1 = \varphi_C^A(x)\varphi_C^B(y)$$

when $A = J_x$, $B = J_y$, $C = J_{xy}$ and $x \geq x_1 \geq \varphi_C^A(x)$, $y \geq y_1 \geq \varphi_C^B(y)$.

Lemma 4.2.12 *Let T be a tree in which every element has finite height. For each $t \in T$ let S_t be a semigroup with 0. For each $t \in T$ let φ_t be a partial homomorphism of $S_t \backslash 0$ into $S_{t'} \backslash 0$ if t is not minimal. Assume that the partial semigroups $S_t \backslash 0$ are pairwise disjoint. On the disjoint union $S = (\cup_{t \in T}(S_t \backslash 0) \cup \{0\})$ define a multiplication $*$ recursively as follows: for all $x \in S_t \backslash 0$ and $y \in S_u \backslash 0$,*

$$x * y = \begin{cases} xy & \text{if } t = u \text{ and } xy \neq 0 \text{ in } S_t, \\ \varphi_t(x)\varphi_t(y) & \text{if } t = u \text{ and } xy = 0 \text{ in } S_t, \\ \varphi_t(x)y & \text{if } t > u, \\ x\varphi_u(y) & \text{if } t < u, \\ \varphi_t(x)\varphi_u(y) & \text{if } t \not\geq u \text{ and } t \not\leq u. \end{cases}$$

Then S is a semigroup.

Next we study semigroups $(S, *)$ which are trees of completely 0-simple semigroups $(S_t, \cdot)$, $t \in T$ and T is a tree. In fact, as Theorem 3.5 of [34] shows (see also Theorem A.0.6 in Appendix), such semigroups are regular and their idempotents form a tree in which every element has finite height, or equivalently, they are strict regular and their $\mathcal{J}$-classes form a tree in which every element has finite height. As it turns out from this theorem, the $\mathcal{J}$-classes of S are the sets $J_t = S_t \backslash 0$, and $S/\mathcal{J} \simeq T$. This implies that each maximal subgroup of S is included in $S_t \backslash 0$ for some $t \in T$. If we add the condition that the maximal subgroups of S are locally finite, then from Lemma 4.2.8 we obtain that each $(S_t, \cdot)$ is locally finite. Here arises a question: Is a tree of completely 0-simple semigroups whose subgroups are locally finite, a locally finite semigroup? Before dealing with this question, let us introduce the following notations. For $X \subseteq S$, denote by $\langle X \rangle$ the subsemigroup of $(S, *)$ generated by X. For $X \subseteq S_t$, $t \in T$, denote by $\langle X \rangle_t$ the subsemigroup of $(S_t, \cdot)$ generated by X. Observe these two facts.

Fact 1. $\langle X \rangle \subseteq J_t \Longrightarrow \langle X \rangle = \langle X \rangle_t$.

Fact 2. $\langle X \rangle \subsetneq J_t \Longrightarrow \langle X \rangle_t = (\langle X \rangle \cap S_t) \cup \{0\}$.

Fact 1 is obvious.

Fact 2. Since $\langle X \rangle \subsetneq J_t$, then there are elements of $\langle X \rangle$ which equal to the zero of S_t, 0, hence $\langle X \rangle_t$ consists of $\{0\}$ and of those elements of $\langle X \rangle$ which belong to S_t.

Lemma 4.2.13 *Let $(S, *)$ be a tree of completely 0-simple semigroups and let the maximal subgroups of S be locally finite. If $X \subseteq J_t$, $t \in T$ and $|X| < \infty$, then $\langle X \rangle$ is finite.*

Proof. We use an inductive argument on the cardinality of X. Let first $X = \{x\} \subseteq J_{t_0}$, $t_0 \in T$. There are two possibilities.

1) For each $n \in N$, $x^n \in J_{t_0}$. From Fact 1 above we have that $\langle X \rangle = \langle X \rangle_{t_0}$ and since $(S_{t_0}, \cdot)$ is completely 0-simple, from Lemma 4.2.8 the finiteness of $\langle X \rangle = \langle X \rangle_{t_0}$ follows.

2) $\langle x \rangle \subsetneq J_{t_0}$, then there is $n_0 \in \mathbb{N}$: $x^{n_0} \in J_{t_0}$ and $x^{n_0+1} \in J_{t_1}$ where $t_1 < t_0$. From the definition of the product is S, we find that

$$x^{n_0+1} = \varphi_{S_{t_1}}^{S_{t_0}}(x) \cdot \varphi_{S_{t_1}}^{S_{t_0}}(x^{n_0}) = \varphi_{S_{t_1}}^{S_{t_0}}(x) \cdot (\varphi_{S_{t_1}}^{S_{t_0}}(x))^{n_0} = (\varphi_{S_{t_1}}^{S_{t_0}}(x))^{n_0+1}.$$

Now let $n_1 \in \mathbb{N}$, $n_1 \geq n_0 + 1$ be such that $x^{n_1} \in J_{t_1}$ and $x^{n_1+1} \in J_{t_2}$ where $t_2 < t_1$. Similarly with above we have

$$x^{n_1+1} = \varphi_{S_{t_2}}^{S_{t_1}}(x) \cdot \varphi_{S_{t_2}}^{S_{t_1}}(x^{n_1}) = \varphi_{S_{t_2}}^{S_{t_1}}(x) \cdot (\varphi_{S_{t_2}}^{S_{t_1}}(x))^{n_1} = (\varphi_{S_{t_2}}^{S_{t_1}}(x))^{n_1+1},$$

and so on. Since the height of t_0 is finite, there is $m \in \mathbb{N}$ such that $x^{n_m} \in J_{t_m}$ and $A = \{x^{n_m+1}, x^{n_m+2}, ...\} \subseteq J_{t_{m+1}}$where $t_{m+1} < t_m$. Now since obviously $S_{t_i} \cap \langle x \rangle$ is finite for each $i \leq m$, we need only to prove the finiteness of A. Indeed, $(A, \cdot)$ is a subsemigroup of $(S_{t_{m+1}}, \cdot)$ and furthermore it is finitely generated with $B = \{x^{n_m+1}, x^{n_m+2}, ..., x^{2n_m}, x^{2n_m+1}\}$ as its generating set. Recalling that $(S_{t_{m+1}}, \cdot)$ is locally finite, we obtain that A is finite. Suppose now that for $X' \subseteq J_t$, $t \in T$ and $|X'| = k - 1$ we have $|\langle X' \rangle| < \infty$. Let $X \subseteq J_t$, $t \in T$ and $|X| = k$, that is $X = \{x_1, x_2, ..., x_{k-1}, x_k\}$. Denote by $X_1 = \{x_1, x_2, ..., x_{k-1}\}$ and $X_2 = \{x_k\}$. From the induction hypothesis we have that $|\langle X_1 \rangle| < \infty$ and $|\langle X_2 \rangle| < \infty$. Let $S_t = S_{t_0}$, S_{t_1}, $S_{t_2}, ..., S_{t_n}$, where $t_0 = t > t_1 > ... > t_n$, be the sets which $\langle X \rangle$ intersects with. It suffices to prove that $\langle X \rangle \cap S_{t_i}$, $i = 0, 1, ..., n$ is finite. First $\langle X \rangle \cap S_{t_0} \subseteq \langle X \rangle_{t_0}$, from Fact 2, and since $|\langle X \rangle_{t_0}| < \infty$ we obtain that $|\langle X \rangle \cap S_{t_0}| < \infty$. Now for $i \geq 1$, each element of $\langle X \rangle \cap S_{t_i}$ is expressed as a product of elements taken in the set

$$A_i = \bigcup_{0 \leq j \leq i} \left\{\varphi_{S_{t_i}}^{S_{t_j}}(((\langle X_1 \rangle \cup \langle X_2 \rangle) \cap S_{t_j})\right\},$$

which implies that $\langle X \rangle \cap S_{t_i} \subseteq \langle A_i \rangle_{t_i}$. From induction hypothesis A_i is finite and hence $\langle A_i \rangle_{t_i}$ is finite, consequently $\langle X \rangle \cap S_{t_i}$ is finite too. ■

Lemma 4.2.14 *Let $(S, *)$ be a tree of completely 0-simple semigroups and let the maximal subgroups of S be locally finite. If $X \subseteq J_{t_1} \cup J_{t_2} \cup ... \cup J_{t_k}$, $|X| < \infty$ and $t_1 > t_2 > ... > t_k$, $t_i \in T$, $i = 1, 2, ..., k$, then $|\langle X \rangle| < \infty$.*

Proof. First we observe that if the lemma holds true in the special case when $t_2 = t_1 + 1$, $t_3 = t_2 + 1$, ..., $t_k = t_{k-1} + 1$, where by $t_i + 1$ we denote the predecessor of t_i, then it holds true in general. Indeed, let $X \subseteq J_{t_1} \cup ... \cup J_{t_k}$. Put $X_i = X \cap J_{t_i}$, $1 \leq i \leq k$ and let us consider the set

$$X^* = X_1 \cup \varphi^{S_{t_1}}_{S_{t_1+1}}(X_1) \cup \varphi^{S_{t_1}}_{S_{t_1+2}}(X_1) \cup ... \cup \varphi^{S_{t_1}}_{S_{t_2-1}}(X_1) \cup \varphi^{S_{t_1}}_{S_{t_2}}(X_1) \cup$$
$$X_2 \cup \varphi^{S_{t_2}}_{S_{t_2+1}}(X_2) \cup \varphi^{S_{t_2}}_{S_{t_2+2}}(X_2) \cup ... \cup \varphi^{S_{t_2}}_{S_{t_3}}(X_2) \cup X_3 \cup ...$$
$$... \cup \varphi^{S_{t_{k-1}}}_{S_{t_{k-1}+1}}(X_{k-1}) \cup \varphi^{S_{t_{k-1}}}_{S_{t_{k-1}+2}}(X_{k-1}) \cup ... \cup \varphi^{S_{t_{k-1}}}_{S_{t_k-1}}(X_{k-1}) \cup \varphi^{S_{t_{k-1}}}_{S_{t_k}}(X_{k-1}) \cup X_k.$$

Obviously, X^* is finite and $X^* \subseteq S_{t_1} \cup S_{t_1+1} \cup ... \cup S_{t_{k-1}} \cup S_{t_{k-1}+1} \cup ... \cup S_{t_k}$. Since, by our assumption, $|\langle X^* \rangle| < \infty$ and since $\langle X \rangle \subseteq \langle X^* \rangle$, we obtain that $\langle X \rangle$ is finite too. To prove the lemma in the special case, we use induction on k. In case $k = 1$, from Lemma 4.2.13, it follows that $|\langle X \rangle| < \infty$. Let us now denote $X' = X_1 \cup X_2 \cup ... \cup X_{k-1}$ and $X'' = X_k$. From the induction hypothesis we have $|\langle X' \rangle| < \infty$ and $|\langle X'' \rangle| < \infty$. Let $n \in \mathbb{N}$, $n \geq k$ be such that $\langle X \rangle \subseteq S_{t_1} \cup S_{t_2} \cup ... \cup S_{t_k} \cup ... \cup S_{t_n}$ and $\langle X \rangle \cap S_{t_n} \neq \emptyset$, $\langle X \rangle \cap S_{t_n+1} = \emptyset$. From the above we obtain that $Y'_i = \langle X' \rangle \cap S_{t_i}$, $i = 1, 2, ..., n$ is finite and $Y''_j = \langle X'' \rangle \cap S_{t_j}$, $j = k$, $k+1, ..., n$ is finite too. Now it suffices to prove that $|\langle X \rangle \cap S_{t_i}| < \infty$, for each $i \leq n$. Observe that $\langle X \rangle \cap S_{t_i} = \langle X' \rangle \cap S_{t_i} = Y'_i$ for $i = 1, 2, ..., k-1$ and as we mentioned before these are all finite. For all $t_l < t_k$, letting in general $\varphi^{S_\alpha}_{S_\beta}(\emptyset) = \emptyset$, where $\alpha \geq \beta$ and $\emptyset$ is the empty set, as in the proof of Lemma 4.2.13 we have $\langle X \rangle \cap S_{t_l} \subseteq \langle A_l \rangle_{t_l}$ where

$$A_l = \Big(\bigcup_{t_1 \geq t_i \geq t_l} \varphi^{S_{t_i}}_{S_{t_l}}(Y'_i) \Big) \cup \Big(\bigcup_{t_k \geq t_j \geq t_l} \varphi^{S_{t_j}}_{S_{t_l}}(Y''_j) \Big).$$

Since A_l is finite, it follows that $\langle X \rangle \cap S_{t_l}$ is finite too. ∎

Theorem 4.2.15 *Let $(S, *)$ be a tree of completely 0-simple semigroups $(S_t, \cdot)$ where $t \in T$ and T is a tree. If the maximal subgroups of S are locally finite, then S is locally finite.*

Proof. First for any $J_{t_1}, J_{t_2}, ..., J_{t_k}$, denote by $]J_{t_1} \cup J_{t_2} \cup ... \cup J_{t_k}]$ the set

$$\{x \in S \mid x \in S_t,\ t \leq t_i \text{ for some } i = 1, 2, ..., k\}.$$

Let $X \subseteq S$ be finite and $X \subseteq]J_{t_1} \cup J_{t_2} \cup ... \cup J_{t_k}]$ where $t_1, t_2, ..., t_k$ do not necessarily form a chain. If $k = 1$, then from Lemma 4.2.14, $|\langle X \rangle| < \infty$. Make the following notations

$$X_{k-1} = X \cap]J_{t_1} \cup J_{t_2} \cup ... \cup J_{t_{k-1}}] \text{ and } X_k = X \cap]J_{t_k}].$$

Let $Y_{k-1} = \langle X_{k-1} \rangle$ and $Y_k = \langle X_k \rangle$, both of them finite by the induction hypothesis. Let

$$T_k = \{t \in T \mid S_t \subseteq]J_{t_1} \cup J_{t_2} \cup ... \cup J_{t_k}],\ S_t \cap \langle X \rangle \neq \emptyset\}$$

and

$$T_k^{(k-1)} = \{t \in T_k \mid Y_{k-1} \cap S_t \neq \phi\},\ T_k^{(k)} = \{t \in T_k \mid Y_k \cap S_t \neq \emptyset\}.$$

It suffices to prove that $S_t \cap \langle X \rangle$ is finite for each $t \in T_k$. Clearly, $\langle X \rangle \cap S_{t_i} = Y_{k-1} \cap S_{t_i}$ for $i = 1, 2, ..., k-1$ and $\langle X \rangle \cap S_{t_k} = Y_k \cap S_{t_k}$. From the induction both intersections are finite. Now for $t \in T_k \setminus \{t_1, t_2, ..., t_{k-1}, t_k\}$ we have that

$$\langle X \rangle \cap S_t \subseteq \left\langle \Big(\bigcup_{\substack{t' \in T_k^{(k-1)} \\ t' \geq t}} \varphi_{S_t}^{S_{t'}}(Y_{k-1} \cap S_{t'}) \Big) \cup \Big(\bigcup_{\substack{t' \in T_k^{(k)} \\ t' \geq t}} \varphi_{S_t}^{S_{t'}}(Y_k \cap S_{t'}) \Big) \right\rangle_t$$

The latter semigroup is finite because it is generated by a finite subset of S_t and $(S_t, \cdot)$ is locally finite. ■

4.3 A counterexample

In Theorem 4.2.7 and Proposition 4.2.5, or in Corollary 3.1 of [16], the minimality condition on principal quasi ideals and right ideals respectively, is required besides the local finiteness of the maximal subgroups of the semigroup, as a finiteness condition for the semigroup. A natural question which arises here is whether or not there is any case when $\min_{\mathcal{R}}$ ($\min_{\mathcal{Q}}$, $\min_{\mathcal{B}}$, or other minimal conditions) follows from the rest of the conditions under which a semigroup is expected to be locally finite. In this section we find a negative answer of the above. Precisely, we find finitely generated, periodic, E-unitary inverse semigroups which have all their maximal subgroups finite, but do not satisfy $\min_{\mathcal{R}}$ and consequently are not finite.

Theorem 4.3.1 *There is a finitely generated, periodic, E-unitary inverse semigroup which has all its maximal subgroups finite, but does not satisfy* $\min_{\mathcal{R}}$.

In general we can construct, as will be shown later, an inverse semigroup which is generated by one of its subsets along with a group that is generated by one of its subsets. The semigroup has the following properties. If the group is finitely generated, we show that so is the semigroup constructed. If the group is periodic, then so will be the semigroup. Further if the group is infinite, then the semigroup does not satisfy the condition $\min_{\mathcal{R}}$. Since there are finitely generated and periodic groups which are infinite (see [1]), then we can deduce from Theorem 4.3.1 that $\min_{\mathcal{R}}$ on a finitely generated semigroup does not depend on its being inverse and periodic, E-unitary and on having all its maximal subgroups finite. The way of constructing such a semigroup is quite similar to that one of constructing free inverse semigroups presented in the Theorem of Scheiblich (see [34]).

Let G be such a group generated by one of its subsets X. Along with G, let us consider the free group FG_X on X. Denote by φ the canonical morphism $\varphi : FG_X \longrightarrow G$ defined by

$$\varphi(y_1 y_2 ... y_n) = \varphi(y_1)\varphi(y_2)...\varphi(y_n),$$

$\forall\, y_1, y_2, ... y_n \in X \cup X^{-1}$ and for every $i = 1, ..., n$, $\varphi(y_i) = y_i$, $\forall\, n \in \mathbb{N}$.

For every fixed $g \in G$ and for every $w \in \varphi^{-1}(g) \subseteq FG_X$, we take $A_w \subseteq FG_X$ such that $w \in A_w$, A_w is finite but nontrivial and $\text{suff}(A_w) = A_w$, where $\text{suff}(A_w)$ is the set of all suffixes of the words in A_w. In the sequel we consider pairs $(\varphi(w), \varphi(A_w)) = (g, \varphi(A_w))$ for $g \in G$ and A_w chosen as above. For any fixed w, denote by $\mathcal{A}_w$ the set of all possible A_w described above. Define the set

$$S = \bigcup_{g\in G} \; \bigcup_{w\in\varphi^{-1}(g)} \; \bigcup_{A_w\in\mathcal{A}_w} (g, \varphi(A_w)).$$

Define in S the following multiplication

$$(g_1, \varphi(A_{w_1})) \cdot (g_2, \varphi(A_{w_2})) = (g_1 g_2, \varphi(A_{w_1})g_2 \cup \varphi(A_{w_2})).$$

Observe that

$$\varphi(A_{w_1})g_2 \cup \varphi(A_{w_2}) = \varphi(A_{w_1}w_2 \cup A_{w_2})$$

and that

$$w_1 w_2 \in \; A_{w_1}w_2 \cup A_{w_2}$$

which is finite, nontrivial and clearly

$$\text{suff}(A_{w_1}w_2 \cup A_{w_2}) = A_{w_1}w_2 \cup A_{w_2}.$$

This shows that the pair $(g_1 g_2, \varphi(A_{w_1})g_2 \cup \varphi(A_{w_2})) = (\varphi(w_1 w_2), \varphi(A_{w_1}w_2 \cup A_{w_2}))$ belongs to S, which assures one for the correctness of the multiplication $\cdot$ as a mapping $S \times S \longrightarrow S$. Let us now show that $(S, \cdot)$ is a semigroup. Indeed,

$$(g_1, \varphi(A_{w_1}))\left[(g_2, \varphi(A_{w_2}))(g_3, \varphi(A_{w_3}))\right] = (g_1, \varphi(A_{w_1}))(g_2 g_3, \varphi(A_{w_2})g_3 \cup \varphi(A_{w_3})) =$$
$$(g_1 g_3 g_3, \varphi(A_{w_1})g_2 g_3 \cup \varphi(A_{w_2})g_3 \cup \varphi(A_{w_3})) =$$
$$(g_1 g_2, \varphi(A_{w_1})g_2 \cup \varphi(A_{w_2}))(g_3, \varphi(A_{w_3})) = \left[(g_1, \varphi(A_{w_1}))(g_2, \varphi(A_{w_2}))\right](g_3, \varphi(A_{w_3})).$$

To show that S is an inverse semigroup and at the same time to find a generating set of S, we proceed as follows. Consider the free inverse semigroup F on X constructed in the theorem of Scheiblich (Theorem 7.1 of [34]) and the mapping $f : F \longrightarrow S$ such that $f((w, A_w)) = (\varphi(w), \varphi(A_w))$. Clearly f is onto.

Let us now show that f is homomorphism. Indeed,

$$f((w_1, A_{w_1})(w_2, A_{w_2})) = f(w_1w_2, A_{w_1}w_2 \cup A_{w_2}) =$$
$$(\varphi(w_1w_2), \varphi(A_{w_1}w_2 \cup A_{w_2})) = (\varphi(w_1)\varphi(w_2), \varphi(A_{w_1})\varphi(w_2) \cup \varphi(A_{w_2})) =$$
$$(\varphi(w_1), \varphi(A_{w_1}))(\varphi(w_2), \varphi(A_{w_2})) = f((w_1, A_{w_1}))f((w_2, A_{w_2})).$$

As S is a homomorphic image of F and F is inverse, we have from Proposition 1.2 of [34] that so will be S. We already know that

$$F = \langle\{(y, \{y, 1\}) \mid y \in Y = X \cup X^{-1}\}\rangle.$$

Next we show that

$$S = \langle\{(\varphi(y), \{\varphi(y), e\}) \mid y \in Y = X \cup X^{-1}\}\rangle = \langle\{(y, \{y, e\}) \mid y \in Y = X \cup X^{-1}\}\rangle,$$

where e is the unit element of G (recall that $\varphi(y) = y, \forall y \in Y$). Indeed,

$$(\varphi(w), \varphi(A_w)) = f((w, A_w)) = f((y_1, \{y_1, 1\})(y_2, \{y_2, 1\})...(y_n, \{y_n, 1\})) =$$
$$f((y_1, \{y_1, 1\}))f((y_2, \{y_2, 1\}))...f((y_n, \{y_n, 1\})) =$$
$$(\varphi(y_1), \{\varphi(y_1), \varphi(1)\})(\varphi(y_2), \{\varphi(y_2), \varphi(1)\})...(\varphi(y_n), \{\varphi(y_n), \varphi(1)\}) =$$
$$(y_1, \{y_1, e\})(y_2, \{y_2, e\})...(y_n, \{y_n, e\}).$$

In particular if X is finite, then S is finitely generated. Now we prove that the periodicity of S follows from that of G. Indeed, in such a case $\forall g \in G$, $\exists n \in N$ such that $g^n = e$ and hence $g^{n+1} = g$. Let $(g, \varphi(A)) \in S$ and $g^n = e$ $(g^{n+1} = g)$. One can easily check that

$$(g, \varphi(A))^{n+1} = (g, \varphi(A)g^n \cup \varphi(A)g^{n-1} \cup ... \cup \varphi(A)g \cup \varphi(A)),$$

and then, since $g^n = e$, we can write

$$(g, \varphi(A))^{n+1} = (g, \varphi(A)g^{n-1} \cup ... \cup \varphi(A)g \cup \varphi(A)).$$

It is now easy to see that $(g, \varphi(A))^{2n+1} = (g, \varphi(A))^{n+1}$ which shows the periodicity of S.

Before we prove that S is in addition E-unitary, we note that $(g, \varphi(A))$ is idempotent if and only if $g = e \in G$. If we have

$$(g, \varphi(A))(e, \varphi(B)) = (ge, \varphi(A) \cup \varphi(B)) \in E(S),$$

where $E(S)$ is the set of idempotents of S, then $ge = e$, which implies $g = e$ and hence $(g, \varphi(A)) \in E(S)$.

Next we show that each maximal subgroup of S is finite. Indeed, from Proposition 5.1.2 of [43] and from the fact that the inverse of $(g, \varphi(A))$ is $(g, \varphi(A))' = (g^{-1}, \varphi(A)g^{-1})$ (this can be checked directly), we obtain that $(g_1, \varphi(A_1))\mathcal{H}(g_2, \varphi(A_2))$ if and only if $\varphi(A_1) = \varphi(A_2)$ and $\varphi(A_1)g_1^{-1} = \varphi(A_2)g_2^{-1}$. Now if $H_{(g,\varphi(A))}$ is a maximal subgroup of S, then it contains an idempotent $(e, \varphi(A))$. Hence $(e, \varphi(A))\mathcal{H}(g, \varphi(A))$ and it follows that $\varphi(A)g = \varphi(A)$. Since $e \in \varphi(A)$, then from the last equality we have that $g \in \varphi(A)$. But recalling that $\varphi(A)$ is finite, we have finitely many $g \in G$ such that $(g, \varphi(A)) \in H_{(e,\varphi(A))}$ and consequently $|H_{(e,\varphi(A))}| < \infty$.

Let us now observe a few simple facts regarding the idempotents of S. We begin with the following equivalence

$$(e, \varphi(A)) \leq_{\mathcal{R}} (e, \varphi(B)) \iff \varphi(A) \supseteq \varphi(B).$$

The following sequence of implications is easily checked:

$$\begin{aligned} &(e, \varphi(A))S \subseteq (e, \varphi(B))S \\ \Longrightarrow\ &(e, \varphi(A)) \in (e, \varphi(B))S \\ \Longrightarrow\ &(e, \varphi(A)) = (e, \varphi(B))(\varphi(u), \varphi(C)). \end{aligned}$$

Since in this case, $\varphi(u)$ must equal to e, then one has $(e, \varphi(A)) = (e, \varphi(B))(e, \varphi(C))$ which implies that $\varphi(A) = \varphi(B) \cup \varphi(C)$ and consequently $\varphi(A) \supseteq \varphi(B)$. Conversely, it suffices to show that $(e, \varphi(A)) \in (e, \varphi(B))S$. Indeed,

$$(e, \varphi(A)) = (e, \varphi(B))(e, \varphi(A)) \in (e, \varphi(B))S,$$

because $\varphi(A) = \varphi(B) \cup \varphi(A)$.

Secondly we claim that

$$(e, \varphi(A))\mathcal{R}(e, \varphi(B)) \iff \varphi(A) = \varphi(B).$$

This follows from the first claim.

And thirdly,

$$(e, \varphi(A))S \subset (e, \varphi(B))S \iff \varphi(A) \supset \varphi(B).$$

This follows from the previous two claims.

Finally we show that if G is infinite, then S does not satisfy $\min_{\mathcal{R}}$.

Let

$$(e, \varphi(A_1))S \supset (e, \varphi(A_2))S \supset ... \supset (e, \varphi(A_n))S \supset (e, \varphi(A_{n+1}))S \supset ...$$

be a descending chain of principal right ideals. From the third claim above, this is equivalent to:

$$\varphi(A_1) \subset \varphi(A_2) \subset ... \subset \varphi(A_n) \subset \varphi(A_{n+1}) \subset .... \tag{4.1}$$

If we require the chain of right ideals to be infinite, then we must construct an infinite ascending chain of subsets of G of the form (4.1) where A_i are finite, nontrivial and $\text{suff}(A_i) = A_i$ for each $i \geq 1$. Since G is infinite, one can find an infinite ascending chain of finite subsets of G like the following.

$$B_1 \subset B_2 \subset ... \subset B_n \subset ... \tag{4.2}$$

It follows that for every $i, j \in \mathbb{N}$ with $i < j$, we have

$$\overline{\varphi^{-1}(B_i)} \subset \overline{\varphi^{-1}(B_j)},$$

where for a set $C \subset G$ in general, we denote by $\overline{\varphi^{-1}(C)}$ the set of words of minimal length in FG_X representing the elements of C. Note that since X is finite, then $\overline{\varphi^{-1}(C)}$ has to be finite if C is finite.

It is clear that for every $i < j$,

$$A_i = \text{suff}(\overline{\varphi^{-1}(B_i)}) \subseteq \text{suff}(\overline{\varphi^{-1}(B_j)}) = A_j,$$

and then we obtain the chain

$$A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ..., \tag{4.3}$$

where from above A_n is finite and $\text{suff}(A_n) = A_n$ for every $n \geq 1$.

This chain can not terminate at some $n \geq 1$ since otherwise we would have infinitely many elements of the chain (4.2) being represented by finitely many words from the finite chain (4.3), namely those of $\overline{\bigcup_{i=1}^{n} \varphi^{-1}(B_i)}$. Therefore we can extract from (4.3) an infinite subchain as below.

$$A_{k_1} \subset A_{k_2} \subset ... \subset A_{k_n} \subset ...,$$

which induces the chain

$$\varphi(A_{k_1}) \subseteq \varphi(A_{k_2}) \subseteq ... \subseteq \varphi(A_{k_n}) \subseteq .... \tag{4.4}$$

Also this chain can not terminate at some $n \geq 1$, since for every $i \in \mathbb{N}$, $B_{k_i} \subseteq \varphi(A_{k_i})$ and as a result we would have

$$\cup_{i\in\mathbb{N}} B_{k_i} \subseteq \varphi(A_{k_n})$$

which is impossible as $\varphi(A_{k_n})$ is finite. Hence we can finally extract from (4.4) an infinite ascending chain of the form

$$\varphi(A_{k_{t_1}}) \subset \varphi(A_{k_{t_2}}) \subset ... \subset \varphi(A_{k_{t_n}}) \subset ...,$$

as desired.

Appendix A

Basics from Semigroup Theory

Notions from Algebraic Theory of Semigroups can be found in standard books like [34], [43], [97] or [16]. Let S be a semigroup. We define the relation $\mathcal{R}$ by

$$a\mathcal{R}b \iff aS^1 = a \cup aS = b \cup bS = bS^1,$$

that is,

$$a\mathcal{R}b \iff a \text{ and } b \text{ generate the same } \textit{principal right ideal.}$$

By symmetry one can define the relation $\mathcal{L}$. Relations $\mathcal{R}$ and $\mathcal{L}$ are respectively left and right congruences. Also one defines the relation $\mathcal{H}$ on S by

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

In [97] it is defined the relation $\mathcal{Q}$ by

$$a\mathcal{Q}b \iff a \cup (aS \cap Sa) = b \cup (bS \cap Sb),$$

where $x \cup (xS \cap Sx)$ is the *principal quasi-ideal* generated by $x \in S$. It can be shown that $\mathcal{Q} = \mathcal{H}$.

Yet another relation

$$\mathcal{D} = \mathcal{R} \vee \mathcal{L},$$

that is,

$$\mathcal{D} \text{ is the least equivalence containing both } \mathcal{R} \text{ and } \mathcal{L}.$$

Since $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ (an easy exercise), one has that

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L}.$$

Lastly we have the relation $\mathcal{J}$ defined by

$$a\mathcal{J}b \iff S^1aS^1 = a \cup aS \cup Sa \cup SaS = b \cup bS \cup Sb \cup SbS = S^1bS^1,$$

that is,

$$a\mathcal{J}b \iff a \text{ and } b \text{ generate the same } \textit{principal ideal.}$$

We say that S satisfies $\min_{\mathcal{R}}$ (reps. $\min_{\mathcal{L}}$, $\min_{\mathcal{Q}}$, $\min_{\mathcal{J}}$) if and only if every descending chain of principal right ideals (reps. left ideals, quasi ideals, ideals) terminates.

Every $\mathcal{D}$-class can be visualize like an egg-box as we always have

$$a\mathcal{D}b \iff R_a \cap L_b \neq \phi \iff L_a \cap R_b \neq \phi.$$

Lemma A.0.2 (Green's Lemma) *Let $a\mathcal{R}b$ in a semigroup S, and let $s, s' \in S^1$ be such that*

$$as = b, \;\; bs' = a.$$

Then the right translations $\rho_s \mid L_a$, $\rho_{s'} \mid L_b$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and L_b onto L_a respectively.

There is also an $\mathcal{L}$-version of this Lemma know as well as the Green's Lemma.

Theorem A.0.3 (Green's Theorem) *If H is an $\mathcal{H}$-class in a semigroup S, then either $H^2 \cap H = \phi$ or $H^2 = H$ and H is a subgroup of S.*

In fact it is easy to show that the group $\mathcal{H}$-classes of a semigroup S are the maximal subgroups of S.

We call an element $a \in S$ *regular* if there is $x \in S$ such that $a = axa$. If a $\mathcal{D}$-class D contains a regular element, then every element of D is regular. In fact, in a regular $\mathcal{D}$-class, every $\mathcal{R}$-class and every $\mathcal{L}$-class contains respectively an idempotent. Also every two group $\mathcal{H}$-classes in a $\mathcal{D}$-class are isomorphic.

A semigroup S with zero 0 is called *0-simple* if

(i) $\{0\}$ and S are the only ideals of S and

(ii) $S^2 \neq \{0\}$.

There is a partial order in the set of idempotents of a semigroup S defined as follows

$$e \leq f \iff ef = fe = e$$

and called the *Rees order.*

There is also a more general order in a semigroup S defined by

$$x \leq y \iff xS^1 \subseteq yS^1 \text{ and } x = ey \text{ for some } e \in R_x.$$

This order is called the *natural order.* It turns out that for regular semigroups the Rees order on idempotents coincides with the natural order.

We call a 0-simple semigroup S, *completely 0-simple* if for every idempotent $e \neq 0$, the only idempotent "below" e is 0. Below we give a receipt (due to Rees) how to construct completely 0-simple semigroups. Let G be a group with identity e, and let I and Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in the 0-group $G^0 = G \cup \{0\}$, and suppose that P is regular, that is, no row or column of it is entirely 0. Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define a composition on S by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } \ p_{\lambda j} \neq 0, \\ 0 & \text{if } \ p_{\lambda j} = 0, \end{cases}$$
$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0.$$

The semigroup S defined thus is completely 0-simple. In fact every completely 0-simple semigroup S arises in this way. Indeed, since S has exactly two $\mathcal{D}$-classes, $\{0\}$ and $D = S\backslash\{0\}$, we let I and Λ be the set of respectively the $\mathcal{R}$ and $\mathcal{L}$-classes of D, and denote by $H_{i\lambda} = R_i \cap L_\lambda$. Since S is regular, we can choose a group $\mathcal{H}$-class H_{11} and then using the Green's Lemma it is easy to show that there is a bijection

$$\phi : (I \times H_{11} \times \Lambda) \cup \{0\} \longrightarrow S$$

given by

$$(i, a, \lambda)\phi = r_i a q_\lambda, \; 0\phi = 0,$$

where $r_i \in H_{i1}$ and $q_\lambda \in H_{1\lambda}$ are fixed elements. Since

$$(r_i a q_\lambda)(r_j b q_\mu) = r_i(a q_\lambda r_j b) q_\mu,$$

we can define $p_{\lambda j} = q_\lambda r_j$ (which in fact is proved to be an element of $L_{q_\lambda} \cap R_{r_j} = H_{11}$) if and only if the $\mathcal{H}$-class $H_{j\lambda}$ is a group (and therefore isomorphic to H_{11}), otherwise we take $p_{\lambda j} = 0$. So $(I \times H_{11} \times \Lambda) \cup \{0\}$ is a regular Rees matrix. That the bijection ϕ is an isomorphism, this is easy to show.

The *principal factors* of a semigroups S are defined as follows. If S has a single $\mathcal{J}$-class J, then we let $P_J = J$. If there are more than one $\mathcal{J}$-classes, then for every such class J, we let

$P_J = J \cup \{0\}$ with the multiplication defined by

$$a * b = \begin{cases} ab & \text{if} \qquad\qquad ab \in J \\ 0 & \text{if} \quad ab \notin J \text{ or either } a = 0 \text{ or } b = 0. \end{cases}$$

We call a semigroup *completely semisimple* if all its principal factros are completely simple or completely 0-simple semigroups.

Proposition A.0.4 *A regular semigroup is completely semisimple if and only if $x \leq y$ and $x\mathcal{D}y$ implies $x = y$.*

Theorem A.0.5 (Lallement) *For a regular semigroup S the following conditions are equivalent:*

1. *S is a subdirect product of completely simple and completely 0-simple semigroups (this semigroups are called strict regular);*

2. *for every $\mathcal{J}$-classes $A \geq B$ and idempotent $e \in A$ there is exactly one idempotent $f \in B$ such that $e \geq f$;*

3. *for every $\mathcal{J}$-classes $A \geq B$ and $x \in A$ there is exactly one $y \in B$ such that $x \geq y$.*

   Either condition implies that S is completely semisimple.

Here $A \geq B$ menas that for every $x \in A$ and $y \in B$ we have $S^1yS^1 \subseteq S^1xS^1$.

A *tree* is a partially ordered set T in which the principal ideal $\{x \in T \mid x \leq t\}$ is a chain. In a tree T the *height* $h(t)$ of an element t is the cardinality of $\{x \in T \mid x \leq t\}$. If t has finite height, then either t is minimal ($h(t) = 0$), or there is a greatest $x < t$, the *predecessor* of t (which has height $h(t) - 1$).

Theorem A.0.6 (Lallement and Petrich) *The following conditions on a semigroup S with zero are equivalent:*

1. *S is regular and its idempotents form a tree in which every element has finite height;*

2. *S is strict regular and its $\mathcal{J}$-classes form a tree in which every element has finite height;*

3. *S is a tree of completely 0-simple semigroups.*

It turns out that the completely 0-simple semigroups of the theorem are the principal factors of S.

Either one of the following can be taken as the definition of an *inverse* semigroup S.

(1) S is regular and its idempotents commute;

(2) Every $\mathcal{L}$-class and every $\mathcal{R}$-class contains a unique idempotent;

(3) Every element of S has a unique inverse.

Of interest are free inverse semigroups. By definition, the *free inverse* semigroup on a set X, is an inverse semigroup FI_X which satisfies the universal property given by the following commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & FI_X \arrow[d, "\exists! f^*"] \\ & S \end{tikzcd}$$

The existence of FI_X is given by the following.

Theorem A.0.7 (Scheiblich) *The free inverse semigroup on X is isomorphic to the semigroup*

$$F' = \{(\omega, A) \mid A \subseteq G \textit{ is finite nontrivial closed and } \omega \in A\}$$

with multiplication given by $(u, A)(v, B) = (u \cdot v, A \cdot v \cup B)$.

Here G is the free group on X and closed means that the set contains all suffixes of its elements.

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