DUAL COVERING THEORY, CONFLUENCE STRUCTURES, AND THE LATTICE OF BICONTINUOUS FUNCTIONS

by

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To whom it may concern

I the undersigned hereby declare that the thesis

"Dual Covering Theory, Confluence Structures, and the Lattice of Bicontinuous Functions"

which I have submitted for the degree of Doctor of Philosophy in the Faculty of Science of the University of Glasgow has been composed by myself.

Yours sincerely,

[redacted]

Lawrence Michael BROWN.

To my wife Sevim

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SUMMARY.

The study of Bitopological Spaces as a distinct branch of General Topology began with the 1963 paper of J. C. Kelly, and contributions have been made by several authors since that time. Three aspects of the theory of bitopological spaces are considered in this thesis, and several new concepts introduced which seem appropriate for the future development of the subject.

Chapter One is devoted to the developement of a covering theory for the bitopological space (X, u, v) based on the notion of a dual cover, which is defined to be a binary relation on the non-empty subsets of X satisfying certain natural conditions. Firstly consideration is given to the relationships existing between the shrinkability of certain classes of dual covers and various normality conditions on X, and then using natural definitions of star refinement and locally finite refinement for dual covers such notions as full binormality, biparacompactness and strong biscreenability are defined and studied. In particular it is shown that under a suitable separation axiom a biparacompact space is fully binormal, but that the converse is false in general. Weakening the local finiteness condition also leads to the consideration of quasi-biparacompactness, etc. Following a short section on countably quasi-biparacompact spaces the notion of sequential normality is introduced as a weakening of full binormality. The class of sequentially normal spaces is important in that it contains all (pseudo-quasi) metrizable bitopological spaces, and yet is restrictive enough for its members to have such desirable properties as pairwise normality. This section contains a dual covering analogue of the Alexandroff-Urysohn Metrization Theorem, from which Salbany's Metrization Theorem is deduced, showing incidentally that the explicit assumption of pairwise normality is redundant. The treatment of dual covering properties concludes with a consideration of some weakened forms of full binormality based on such concepts as the pseudo-star refinement of a dual cover, together with weakend forms of biparacompactness and quasi-biparacompactness which are based on

the idea of a compartmental dual cover refinement. The relation between the above mentioned concepts is further clarified by the consideration of several appropriate counterexamples. The chapter ends with a brief discussion of quasi-uniform and other related structures from the point of view of dual covers, and the notion of para-quasi-uniformity is introduced and is shown to stand in the same relation to bitopological spaces as does the para-uniformity of C. I. Votaw to topological spaces.

An extension (X', u', v') of (X, u, v) which can be obtained as a (bitopological) completion of a quasi-uniform (or similar) structure on X will contain X as a $u' \vee v'$-dense subset. On the other hand there are important instances of extensions which do not satisfy this strong density condition, and the aim of Chapter Two is the development of a theory, of what may be appropriatly called Confluence Structures, designed specifically to remove this restriction in so far as is possible. Confluence quasi-uniformities (cqu) are obtained by making appropriate changes to the dual covering definition of a quasi-uniformity given in Chapter One. Their theory is more complicated than that of quasi-uniformities, but can be developed along broadly the same lines. In particular it is shown that, with respect to the appropriate definitions, every separated cqu has a completion, unique up to isomorphism, which is a separated strict extension of the corresponding bitopological space. Related to the notion of a confluence relation, which is basic to the definition of a cqu, and which is a generalisation of the relation of meeting between sets, there are defined several forms of bitopological compactness, and these are discussed in connexion with the completeness of cqu. The final section of this chapter contains an extension of the work on cqu to the case of para-quasi-uniformities. This is restricted mainly to a discussion of those bitopological extensions which can be obtained as completions of confluence para-quasi-uniformities, and partial generalisations are obtained to some of the results of Votaw on para-uniform completions.

The third chapter investigates the lattice P(X) of bicontinuous functions on (X, u, v) to the real bitopological space $(\mathbb{R}, s, t)$.

Here it is convenient to consider the notion of a bi-ideal (L, M), where L is a lattice ideal and M a lattice dual ideal in P(X), each containing O. The elementary theory of bi-ideals is first developed in the more general setting of a distributive lattice P with a real translation (T-lattice). Working in terms of a concept of ρ -regularity for bi-ideals a theory is obtained which in some respects resembles that of ring ideals. In particular the quotient P/(L, M) is defined, its order structure studied, and the notion of real bi-ideal defined. Several other aspects of the theory are also considered with a view to subsequent applications. The notion of real bi-ideal in P(X) leads to a natural definition of bireal compactness for bitopological spaces, and this is also characterized by an embedding property. The bireal compact extensions H_A are defined and studied in some detail. In particular the lattices $P(H_A)$ are considered in relation to the bi-ideal structure of P(X). It is also verified that the spaces H_A may be regarded as completions of suitable quasi-uniformities on X. The final section deals with the pair real compact spaces of Saegrove. A bi-ideal characterization is given, and the pair real compact extensions R_S defined and compaired with $H_{\langle S \rangle}$, where $\langle S \rangle$ is the smallest sub-T-lattice of P(X) containing the subset S. Finally it is shown that R_S is a strict bitopological extension if and only if it is a relatively T_0 extension, and that under this same condition R_S may be regarded as a completion of a suitable confluence quasi-uniformity on X.

For the benefit of those unfamiliar with basic definitions the following definitions are given here:

(a) A bitopological space (X, u, v) consists of a set X on which are defined two topologies u and v.

(b) If (X, u, v), (X', u', v') are bitopological spaces and f: X → X' is a function then f is said to be bicontinuous if it is continuous for the topologies u, u' and also for the topologies v, v'.

PREFACE

This thesis consists of three chapters, each prefaced by a few remarks concerning its contents. Briefly, the first chapter details a covering theory for bitopological spaces based on the notion of dual cover, the second introduces confluence structures and their completions, while the third deals with the bi-ideal structure of the lattice of bicontinuous real-valued functions and its relation to bitopological real compactness. These particular concepts have not, to the best of my knowledge, been considered before, and they constitute an original contribution to the theory of bitopological spaces. I have naturally drawn on several branches of general topology for motivation and inspiration, but the bitopological case presents many unique features, and poses questions not met with in the single topology case. Only in a limited number of areas does the theory presented here parallel, or provide an alternative approach to, known results in the theory of bitopological spaces, and I have noted in the text such instances of this as are known to me.

Apart from a few definitions, for which references are given, the thesis is self-contained. I have made an effort to maintain a consistent scheme of notation throughout, and an index of special symbols, and one of special terms, is included for the convenience of the reader.

L. M. BROWN.

CHAPTER ONE

DUAL COVERING PROPERTIES OF BITOPOLOGICAL SPACES.

The theory of covers of topological spaces has undergone a rapid developement over the past few years, following the pioneering work of A.H. STONE [33] and others. The establishment of a similar theory for bitopological spaces faces at the outset the question of deciding on a suitable counterpart to the notion of cover. Indeed it would appear that no one analogue of this notion is entirely satisfactory for all purposes. Pairwise open and weakly pairwise open covers have been the analogue most extensively considered in the literature to date, as witness for instance the papers of FLETCHER, HOYLE and PATTY [13] , RICHARDSON [29] ,CIVIC [6] and DATTA [10]. In this chapter we pursue a different line of enquiry in which our counterpart to the notion of cover is that of dual cover defined below. Dual covers correspond essentially to strong conjugate pairs of covers which were defined by GANTNER and STEINLAGE [15] , and used by them in a covering characterization of quasi-uniformities. Since this characterization generalises the covering description of a uniformity, this serves at least in part to motivate our choice of dual cover as a natural counterpart to the notion of cover. As we shall see in what follows such notions as full normality, paracompactness, etc., have various natural expressions in terms of dual covers, and our aim here is to investigate the relation between the concepts so defined. While some of the known topological results remain valid in this more general setting, this is not the case for the majority of the results of covering theory, and consequently our enquiry follows a largely independent course.

Since the basic object of study in this chapter is that of dual cover it will be convenient at this point to give some basic definitions and notation concerning these, other definitions being postponed until the appropriate point in the text.

By a <u>dual family</u> on the set X (assumed non-empty throughout) we shall mean any binary relation on the non-empty subsets of X. If d is a dual family on X we shall usually write UdV in preference to

$(U, V) \in d$.

For the dual family d let us set

$$uc(d) = \bigcup \{ U \cap V \mid UdV \},$$
$$lc(d) = \bigcup \{ U \mid \exists \ V \text{ with } UdV \} = \bigcup \text{dom } d,$$
$$rc(d) = \bigcup \{ V \mid \exists \ U \text{ with } UdV \} = \bigcup \text{ran } d$$

and call these respectively the uniform covering, the left covering and the right covering of d.

The dual family d is an ι-dual family if $U \cap V \neq \emptyset$ whenever UdV. An ι-dual family whose uniform covering is X will be called a dual cover of X.

When we are free to choose, any indexing of a dual family will be assumed to be faithful. If d is an indexed dual cover then (dom d, ran d) is a strong conjugate pair of covers in the sense of GANTNER and STEINLAGE [15], and indeed these two notions are essentially equivalent. However, working in terms of dual covers gives a certain notational economy.

If d and e are dual families we write $e \prec d$, and say that e refines d, if given ReS there exists UdV with $R \subseteq U$ and $S \subseteq V$. Unless the context makes the contrary clear, when we speak of a refinement of a dual cover we shall always mean a dual cover refinement.

If d and e are ι-dual families we set

$$d \wedge e = \{ (U \cap R, V \cap S) \mid UdV, ReS, U \cap V \cap R \cap S \neq \emptyset \}.$$

Then $d \wedge e$ is an ι-dual family which refines d and e. Indeed $d \wedge e$ is the greatest lower bound of d and e in the set of all ι-dual families partially ordered by refinement. In particular if d and e are dual covers then $d \wedge e$ is the greatest lower bound of d and e in the set of all dual covers on X.

The results of Section 1.1 were announced by the author at the 6th Balkan Mathematicians Congress held in Varna in 1977, under the title "A theory of dual covers for bitopological spaces".

Throughout this thesis a regular (normal, fully normal, compact, paracompact) topological space is not assumed to be T_1.

1.1 BINORMALITY.

The notion of pairwise normality for bitopological spaces was introduced by KELLY in [19]. A bitopological space (X, u, v) is called pairwise normal if given a v-closed set A and a u-closed set B with $A \cap B = \emptyset$ there exist $U \in u$, $V \in v$ with $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. This is a natural counterpart for bitopological spaces of the notion of normality for topological spaces, and a good many of the properties of normality carry over to pairwise normality. In particular an exact analogue of Urysohn's Lemma may be established for pairwise normality, as was shown in the above mentioned paper. In this section we are going to consider the situation with regard to the covering characterizations of normality, and dual covers. We begin by recalling the following two well known results:

A) A topological space is normal if and only if given any point finite open cover $\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$ there is an open cover $\mathcal{V} = \{ V_\alpha \mid \alpha \in A' \}$ where $A' \subseteq A$ and $\overline{V}_\alpha \subseteq U_\alpha$ for all $\alpha \in A'$.

(See, for example ([12], Theorem 6.1). Of course if we permit our open covers to contain empty sets, or if the space is R_0, then we may take A = A' in this result)

B) A topological space is normal if and only if every finite open cover has an open star refinement [24].

In order to determine if the corresponding results hold for dual covers of bitopological spaces (with "normal" replaced by "pairwise normal") we need first to give suitable counterparts for dual covers of the above mentioned properties of covers of a topological space. Let (X, u, v) be a bitopological space, and d a dual cover of X. We shall say d is open if dom $d \subseteq u$ and ran $d \subseteq v$. The property of the cover $\mathcal{U}$ described in (A) is often called shrinkability. Its analogue for dual covers is given in:

Definition 1.1.1. A dual cover $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$, faithfully indexed by A, is called shrinkable if there is an open dual cover $e = \{ (R_\alpha, S_\alpha) \mid \alpha \in A' \}$ with $A' \subseteq A$, $v\text{-cl}[R_\alpha] \subseteq U_\alpha$ and $u\text{-cl}[S_\alpha] \subseteq V_\alpha$ for each $\alpha \in A'$.

The counterpart of "point finite" for dual families is given in:

Definition 1.1.2. The dual family d is point finite if for each $x \in X$ the set

$$\{ (U, V) \mid UdV,\ x \in U \cap V \}$$

is finite.

It will transpire that if every point finite open dual cover of (X, u, v) is shrinkable then (X, u, v) is pairwise normal. However the pairwise normal space of Example 1.6.1 contains a finite open dual cover which is not shrinkable and so the converse result is false. This shows that the analogue of the condition in (A) does not characterise the pairwise normal bitopological spaces.

Now let us turn to (B). As we shall see later, the notion of "star refinement" for dual covers may be defined in several ways. However the following is by far the most useful and natural.

Definition 1.1.3 Let d be a dual family on X, and $A \subseteq X$. We set:

$$St(d, A) = \cup \{U \mid \exists\ V,\ UdV \text{ and } V \cap A \neq \emptyset \}, \text{ and}$$

$$St(A, d) = \cup \{ V \mid \exists\ U,\ UdV \text{ and } U \cap A \neq \emptyset \}.$$

If e is a second dual family on X we say d is a star refinement of e, and write $d <(*)\, e$, if given UdV $\exists$ ReS with $St(d, U) \subseteq R$ and $St(V, d) \subseteq S$.

Unless something is said to the contrary a star refinement of a dual cover will always mean a dual cover star refinement. If d is a dual cover then we have $A \subseteq St(d, A)$ and $A \subseteq St(A, d)$ for all subsets A of X. In particular a dual cover star refinement is also a refinement.

With this definition of star refinement we may now ask if it is true that a bitopological space is pairwise normal if and only if every finite open dual cover has an open star refinement. First let us note the following:

Proposition 1.1.1. Every dual cover d with an open star refinement is shrinkable.

Proof. Let $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$, and let f be an open dual cover with $f <(*)\, d$. For $\alpha \in A$ define

$$R_\alpha = \cup \{ L \mid L \in \text{dom } f,\ St(f, L) \subseteq U_\alpha \}, \text{ and}$$

$$S_\alpha = \cup \{ T \mid T \in \text{ran } f,\ St(T, f) \subseteq V_\alpha \}.$$

Then if we set $A' = \{\alpha \mid \alpha \in A,\ R_\alpha \cap S_\alpha \neq \emptyset\}$ it is easy to verify that

$$e = \{(R_\alpha, S_\alpha) \mid \alpha \in A'\}$$

has the properties required in Definition 1.1.1.

We have already mentioned that the pairwise normal space of Example 1.6.1 has a non-shrinkable finite open dual cover, and this dual cover cannot have an open star refinement by the above proposition. This shows that the above mentioned analogue of (B) for bitopological spaces is also false.

In view of these negative results two possible lines of inquiry suggest themselves. One is to determine a "reasonably large" class of dual covers of a (pairwise normal) bitopological space which do have open star refinements, and the second is to investigate the "normality" conditions imposed on a bitopological space by the requirement that certain families of dual covers should be shrinkable. Before giving one possible answer to the first question we shall need the following definition.

Definition 1.1.4. We say the dual family d is *star finite* if for each UdV the set

$$\{(U', V') \mid U'dV',\ U' \cap V \neq \emptyset \text{ or } U \cap V' \neq \emptyset\}$$

is finite.

Note that a star finite dual cover is certainly point finite.

Theorem 1.1.1. (i) Every shrinkable star finite open dual cover of a bitopological space has a star finite open star refinement.

(ii) If (X, u, v) is pairwise normal, every shrinkable star finite open dual cover has a shrinkable star finite open star refinement.

Proof. (i) Let $d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ be an open shrinkable star finite dual cover, and $e = \{(R_\alpha, S_\alpha) \mid \alpha \in A'\}$ as in Definition 1.1.1. For each $x \in X$ define

$$U_x = \bigcap \{R_\alpha \mid x \in R_\alpha\} \cap \bigcap \{U_\alpha \mid x \in U_\alpha\} \cap$$
$$\bigcap \{X - (u\text{-cl}[S_\alpha]) \mid x \notin u\text{-cl}[S_\alpha] \text{ and } \exists \beta \in A,\ x \in U_\beta \text{ and } U_\beta \cap V_\alpha \neq \emptyset\},$$

$$V_x = \bigcap \{ S_\alpha \mid x \in S_\alpha \} \cap \bigcap \{ V_\alpha \mid x \in V_\alpha \} \cap$$

$$\bigcap \{ X - (v\text{-cl}[R_\alpha]) \mid x \notin v\text{-cl}[R_\alpha] \text{ and } \exists\, \beta \in A,\ x \in V_\beta \text{ and } V_\beta \cap U_\alpha \neq \emptyset \},$$

and $f = \{ (U_x, V_x) \mid x \in X \}$.

Since d is star finite, each of the intersections involved in the definition of U_x and V_x ~~are~~ is essentially finite and so f is an open dual cover of X. Let us show that f is star finite. Take $x \in X$ and let

$$\{ \alpha \mid x \in U_\alpha \} = \{ \alpha_1, \ldots, \alpha_n \},$$

$$\{ \beta \mid V_\beta \cap U_\alpha \neq \emptyset \ \forall\ \alpha = \alpha_1, \ldots, \alpha_n \} = \{ \beta_1, \ldots, \beta_m \},$$

and for each $i = 1, 2, \ldots, m$ let

$$\{ \gamma \mid U_\gamma \cap V_{\beta_i} \neq \emptyset \} = \{ \gamma_{i1}, \ldots, \gamma_{ik_i} \}.$$

Suppose $U_x \cap V_y \neq \emptyset$, then:

(a) $(y \in S_\beta \subseteq V_\beta$ or $y \in V_\beta) \Rightarrow (V_\beta \cap U_\alpha \neq \emptyset\ \forall\ \alpha = \alpha_1, \ldots, \alpha_n) \Rightarrow (\beta = \beta_t$ for some $1 \leq t \leq m)$,

(b) $(y \notin v\text{-cl}[R_\gamma]$ and $\exists\, \beta \in A$, $y \in V_\beta$ and $V_\beta \cap U_\gamma \neq \emptyset) \Rightarrow$ $(\beta = \beta_r$ for some $1 \leq r \leq m$, and then $\gamma = \gamma_{rs}$ for some $1 \leq s \leq k_r)$

It follows that each V_y with $U_x \cap V_y \neq \emptyset$ may be defined using only $S_{\beta_1}, \ldots, S_{\beta_m}$; $V_{\beta_1}, \ldots, V_{\beta_m}$ and $R_{\gamma_{11}}, \ldots, R_{\gamma_{mk_m}}$. Hence U_x meets only a finite number of distinct V_y, and likewise V_x meets only a finite number of distinct U_y. This verifies that f is star finite.

Finally if we take $x \in X$ and $\alpha \in A'$ with $x \in R_\alpha \cap S_\alpha$ it is easy to verify that

$$St(f, U_x) \subseteq U_\alpha \text{ and } St(V_x, f) \subseteq V_\alpha$$

so $f \prec^{(*)} d$, and (i) is proved.

(ii) Take d and e as in (i). For $\alpha \in A'$ we have $v\text{-cl}[R_\alpha] \subseteq U_\alpha$ and $u\text{-cl}[S_\alpha] \subseteq V_\alpha$, so by the pairwise normality of (X, u, v) we have u-open sets P_α, M_α and H_α; and v-open sets Q_α, N_α and K_α

so that:

$v\text{-cl}[R_\alpha] \subseteq P_\alpha \subseteq v\text{-cl}[P_\alpha] \subseteq M_\alpha \subseteq v\text{-cl}[M_\alpha] \subseteq H_\alpha \subseteq v\text{-cl}[H_\alpha] \subseteq U_\alpha$,

$u\text{-cl}[S_\alpha] \subseteq Q_\alpha \subseteq u\text{-cl}[Q_\alpha] \subseteq N_\alpha \subseteq u\text{-cl}[N_\alpha] \subseteq K_\alpha \subseteq u\text{-cl}[K_\alpha] \subseteq V_\alpha$.

Now define:

$U'_x = \bigcap \{ P_\alpha \mid x \in R_\alpha \} \cap \bigcap \{ U_\alpha \mid x \in v\text{-cl}[M_\alpha] \} \cap \bigcap \{ X - (u\text{-cl}[Q_\alpha])$

$\mid x \notin u\text{-cl}[N_\alpha]$ and $\exists \beta \in A'$ with $x \in v\text{-cl}[M_\beta]$ and $U_\beta \cap V_\alpha \neq \emptyset \}$,

and

$V'_x = \bigcap \{ Q_\alpha \mid x \in S_\alpha \} \cap \bigcap \{ V_\alpha \mid x \in u\text{-cl}[N_\alpha] \} \cap \bigcap \{ X - (v\text{-cl}[P_\alpha])$

$\mid x \notin v\text{-cl}[M_\alpha]$ and $\exists \beta \in A'$ with $x \in u\text{-cl}[N_\beta]$ and $V_\beta \cap U_\alpha \neq \emptyset \}$.

Arguing as abdve $f' = \{ (U'_x, V'_x) \mid x \in X \}$ is an open dual cover of X, and with the notation used in (i) each V'_y with $U'_x \cap V'_y \neq \emptyset$ can be formed using at most $Q_{\beta_1}, \ldots, Q_{\beta_m}$; $V_{\beta_1}, \ldots, V_{\beta_m}$ and $P_{\gamma_{11}}, \ldots, P_{\gamma_{mk_m}}$ so U'_x meets only a finite number of distinct V'_y. In the same way V'_x meets only a finite number of distinct U'_y so f' is star finite.

It is easy to verify that $f' \prec(*)\, d$, so it remains only to show that f' is shrinkable. For $x \in X$ define:

$U^*_x = \bigcap \{ R_\alpha \mid x \in R_\alpha \} \cap \bigcap \{ H_\alpha \mid x \in v\text{-cl}[M_\alpha] \} \cap \bigcap \{ X - (u\text{-cl}[N_\alpha])$

$\mid x \notin u\text{-cl}[N_\alpha]$ and $\exists \beta \in A'$ with $x \in v\text{-cl}[M_\beta]$ and $U_\beta \cap V_\alpha \neq \emptyset \}$,

and

$V^*_x = \bigcap \{ S_\alpha \mid x \in S_\alpha \} \cap \bigcap \{ K_\alpha \mid x \in u\text{-cl}[N_\alpha] \} \cap \bigcap \{ X - (v\text{-cl}[M_\alpha])$

$\mid x \notin v\text{-cl}[M_\alpha]$ and $\exists \beta \in A'$ with $x \in u\text{-cl}[N_\beta]$ and $V_\beta \cap U_\alpha \neq \emptyset \}$.

Again $U^*_x \in u$, $V^*_x \in v$ and $x \in U^*_x \cap V^*_x$. Next set:

$R_x = \bigcup \{ U^*_y \mid (U'_x, V'_x) = (U'_y, V'_y) \}$, and

$S_x = \bigcup \{ V_y \mid (U'_x, V'_x) = (U'_y, V'_y) \}$.

Then $g = \{ (R_x, S_x) \mid x \in X \}$ is an open dual cover of X, and $(U'_x, V'_x) = (U'_y, V'_y)$ implies $(R_x, S_x) = (R_y, S_y)$ so it remains to show that

$$v\text{-cl}[R_x] \subseteq U'_x \quad \text{and} \quad u\text{-cl}[S_x] \subseteq V'_x \qquad \forall\ x \in X.$$

Now it is easily seen, repeating an argument used above, that the unions defining R_x and S_x are essentially finite, so it will suffice to show that

$$v\text{-cl}[U^*_x] \subseteq U'_x \quad \text{and} \quad u\text{-cl}[V^*_x] \subseteq V'_x \qquad \forall\ x \in X.$$

However this is immediate from the definitions, and f' is shrinkable as required.

This completes the proof of the theorem.

We may say that a dual cover $d = d_0$ is <u>normal</u> if there is a sequence d_n, n = 1, 2, ..., of open dual covers with $d_{n+1} <^{(*)} d_n$, n = 0, 1, 2, ... This corresponds to the terminology used for covers of topological spaces. We then have:

<u>Corollary</u>. In a pairwise normal bitopological space every open star finite shrinkable dual cover is normal.

Before going on to discuss the second question mentioned above we make the following convention of terminology which will be useful here and later. If "P" is a topological property then the term "uniformly P" applied to the bitopological space (X, u, v) will mean that P holds for the least upper bound topology $u \vee v$ (which itself will be called the uniform topology of (X, u, v)).

If $E \subseteq X$ is uniformly closed then there is an open dual family $\{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ so that

$$E = X - \bigcup\{U_\alpha \cap V_\alpha \mid \alpha \in A\}.$$

If this family may be chosen to be finite (respectively, star finite, point finite) we will say that E is <u>finitely</u> (respectively, <u>star finitely</u>, <u>point finitely</u>) <u>uniformly closed</u>.

These considerations lead us to several new forms of normality for bitopological spaces, as given below.

<u>Definition</u> 1.1.5. We say that (X, u, v) is <u>binormal</u> if given any uniformly closed set E, and any u-closed (respectively, v-closed) set F with $E \cap F = \emptyset$ there exist $U \in u$, $V \in v$ with $U \cap V = \emptyset$ and $E \subseteq U$, $F \subseteq V$ (respectively, $E \subseteq V$, $F \subseteq U$).

If this condition holds whenever E is finitely (respectively, star finitely, point finitely) uniformly closed, we will say that (X, u, v) is finitely (respectively, star finitely, point finitely) binormal.

Clearly we have:

Binormal $\Rightarrow$ point finitely binormal $\Rightarrow$ star finitely binormal $\Rightarrow$ finitely binormal $\Rightarrow$ pairwise normal.

If d is a dual cover of X we say that $(U, V) \in d$ is essential if $d - \{(U, V)\}$ is not a dual cover of X. We may now state:

Proposition 1.1.2. (X, u, v) is (finitely, star finitely, point finitely) binormal if and only if given any (finite, star finite, point finite) open dual cover d and any UdV there exist sets U_1, V_1 with $v\text{-cl}[U_1] \subseteq U$, $u\text{-cl}[V_1] \subseteq V$, $U_1 \in u$, $V_1 \in v$ and so that

$$uc[(d - \{(U, V)\}) \cup \{(U_1, V_1)\}] = X$$

Proof. $\Rightarrow$. If d is an open dual cover of the appropriate kind, and UdV is essential we need only apply the corresponding binormality property to the disjoint pairs of sets E, (X - U) and E, (X - V), where

$$E = X - uc(d - \{(U, V)\}).$$

On the other hand if UdV is not essential we may take $U_1 = V_1 = \emptyset$.

$\Leftarrow$. If $E = X - \bigcup \{U_\alpha \cap V_\alpha \mid \alpha \in A\}$ is a non-empty uniformly closed set of the appropriate kind, and F is (say) a u-closed set with $E \cap F = \emptyset$ then (X - F, X) is essential for

$$d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\} \cup \{(X - F, X)\}$$

and by the appropriate hypothesis there exist $U_1 \in u$, $V_1 \in v$ with $v\text{-cl}[U_1] \subseteq X - F$ and $uc[(d - \{(X - F, X)\}) \cup \{(U_1, V_1)\}] = X$. But then $E \subseteq U_1$ and $F \subseteq X - (v\text{-cl}\{U_1\})$, which gives the required result. A similar argument may be used when F is v-closed.

Corollary. If every (finite, star finite, point finite) open dual cover is shrinkable then (X, u, v) is (finitely, star finitely, point finitely) binormal.

The validity or otherwise of the converse result for the binormal case is an open question, but this converse result is true for the other cases, as we now show.

<u>Theorem</u> 1.1.2. If (X, u, v) is finitely binormal (respectively, star finitely, point finitely binormal) then every finite open dual cover (respectively, every star finite, point finite open dual cover) is shrinkable.

<u>Proof</u>. Let $d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ be an open dual cover of whichever kind is being considered, faithfully indexed over A. Let $\mathfrak{F}$ be the class of all functions f satisfying the following conditions:

(i) $\operatorname{dom} f = B(f) \subseteq A$,

(ii) For $\beta \in B(f)$, $f(\beta) = (R_\beta, S_\beta)$ where $R_\beta \in u$, $S_\beta \in v$,

$v\text{-cl}\{R_\beta\} \subseteq U_\beta$ and $u\text{-cl}\{S_\beta\} \subseteq V_\beta$,

(iii) $uc\left[\{(R_\beta, S_\beta) \mid \beta \in B(f)\} \cup \{(U_\alpha, V_\alpha) \mid \alpha \in A - B(f)\}\right] = X$.

For $\alpha \in A$ we may define an element f of $\mathfrak{F}$ with $B(f) = \{\alpha\}$ using Proposition 1.1.2. This shows that $\mathfrak{F} \neq \emptyset$. We may define a partial order $\leq$ on $\mathfrak{F}$ by setting $f \leq g$ if and only if $B(f) \subseteq B(g)$ and $f(\beta) = g(\beta)$ whenever $\beta \in B(f)$. Let us verify that in all cases $(\mathfrak{F}, \leq)$ is inductive. Let $\{f_\gamma \mid \gamma \in C\}$ be a chain in $\mathfrak{F}$, and define a function f by $\operatorname{dom} f = B(f) = \bigcup\{B(f_\gamma) \mid \gamma \in C\}$ and $f|B(f_\gamma) = f_\gamma$. f is clearly well defined, and if we can show that $f \in \mathfrak{F}$ it will certainly be an upper bound of $\{f_\gamma\}$. Now (i) and (ii) are clear. To see (iii) take $x \in X$ with $x \notin U_\alpha \cap V_\alpha$ for all $\alpha \in A - B(f)$. Since d is point finite in all cases there is a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ so that $x \in U_\alpha \cap V_\alpha$ if and only if $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$. By the choice of x we have $\{\alpha_1, \ldots, \alpha_n\} \subseteq B(f)$, and $\{f_\gamma\}$ is a chain so $\{\alpha_1, \ldots, \alpha_n\} \subseteq B(f_\gamma)$ for some $\gamma \in C$. It follows that for some $\beta \in B(f_\gamma) \subseteq B(f)$ we have $f(\beta) = f_\gamma(\beta) = (R_\beta, S_\beta)$ with $x \in R_\beta \cap S_\beta$. This verifies (iii) and $f \in \mathfrak{F}$ as required. Applying Zorn's Lemma to $(\mathfrak{F}, \leq)$ allows us to deduce that there is a maximal element $f \in \mathfrak{F}$. It will suffice now to show that $uc[\{(R_\beta, S_\beta) \mid \beta \in B(f)\}] = X$, for if we set $A' = \{\alpha \mid \alpha \in B(f) \text{ and } R_\alpha \cap S_\alpha \neq \emptyset\}$ and $e = \{(R_\alpha, S_\alpha) \mid \alpha \in A'\}$ then this is an open dual cover having the properties required in Definition 1.1.1. However we must in fact have $B(f) = A$ (from which the required result follows from (iii)), for if $\alpha \in A - B(f)$ we could extend f to a larger function $g \in \mathfrak{F}$ with $\alpha \in B(g)$ using Proposition 1.1.2, and this would contradict the maximality of f.

This completes the proof of the theorem.

Corollary 1. The following are equivalent for the bitopological space (X, u, v) :

(a) (X, u, v) is star finitely binormal.

(b) Every star finite open dual cover is shrinkable.

(c) Every star finite open dual cover is normal.

Since the results of Theorem 1.1.1 are clearly also true if we replace "star finite" by "finite" we may state:

Corollary 2. The following are equivalent for the bitopological space (X, u, v) :

(a) (X, u, v) is finitely binormal.

(b) Every finite open dual cover is shrinkable.

(c) Every finite open dual cover is normal.

It will be noted that, by what we have said earlier, the bitopological space of Example 1.6.1 is pairwise normal but not finitely binormal.

We end this section with the following definition and comments.

Definition 1.1.6. A bitopological space will be called:

1) Fully binormal if every open dual cover has an open star refinement, and

2) Strongly biparacompact if every open dual cover has a star finite open refinement.

These properties correspond to full normality and strong paracompactness respectively for topological spaces [25] . Note that it is equivalent to say (X, u, v) is fully binormal if and only if every open dual cover is normal. By what has been shown above it is clear that a fully binormal space is binormal, and that a star finitely binormal strongly biparacompact space is fully binormal. This latter result will be considerably improved in the next section.

1.2. LOCAL FINITENESS PROPERTIES.

In the following definition d is a dual family, and for each $x \in X$, H(x) is a nhd. of x for the topology u, and K(x) a nhd. of x for the topology v.

Definition 1.2.1. d is locally finite if we may choose H(x), K(x) in such a way that

$$d_x = \{ (U, V) \mid UdV \text{ and } U \cap K(x) \neq \emptyset \text{ or } V \cap H(x) \neq \emptyset \}$$

is finite for each $x \in X$.

d is $\leq$ -finite if we may choose H(x), K(x) in such a way that

$$\dot{\bigcup} \{ d_x \mid (U, V) \in d_x \}$$

is finite for each UdV.

If the above choice can be made so that the sets in question contain at most one element we speak of d as being discrete and $\leq$ -discrete respectively.

Finally if $d = \bigcup \{ d_n \mid n = 1, 2, \ldots \}$, and each d_n has one of the properties "L" above we say d is "σ -L".

The bitopological space (X, u, v) will be called biparacompact (respectively, strongly biscreenable) if every open dual cover has a locally finite (respectively, σ -discrete) open refinement.

The notion of a $\leq$ -finite collection on a topological space was introduced by McCANDLESS in[22]. Biparacompact and strongly biscreenable correspond respectively to paracompact and strongly screenable for topological spaces [25] .

Cleary every strongly biparacompact space is biparacompact since a star finite open dual cover is locally finite.

It is easy to see that a bitopological space is uniformly Lindelöf if and only if every open dual cover has a countable sub-cover. Hence a uniformly Lindelöf space (and more particularly a uniformly compact space) is strongly biscreenable.

It is known ([25], Corollary 2 to Theorem V.2) that for regular topological spaces the notions of paracompactness and strong screenability coincide. However the bitopological space of Example 1.6.3 is pairwise regular [19] and strongly biscreenable but not biparacompact, so the analogue of this result for bitopological spaces is false. I do not know if a (pairwise regular) biparacompact space is necessarily strongly biscreenable, but we can prove this result if we replace "biparacompact" by "strongly biparacompact". To this end we will first develop some results about Σ-finite dual families.

It is clear from the definitions that a Σ-finite dual family is both locally finite and star finite. For an open dual cover, however, we have the following improved result:

Lemma 1.2.1. An open dual cover d is Σ-finite if and only if it is star finite.

Proof. Let d be an open star finite dual cover of X, and for $x \in X$ define:

$$H(x) = \bigcap \{ U \mid \exists \ V, \ UdV \text{ and } x \in U \cap V \},$$
$$K(x) = \bigcap \{ V \mid \exists \ U, \ UdV \text{ and } x \in U \cap V \}.$$

Since d is star finite it is also point finite so $x \in H(x) \in u$ and $x \in K(x) \in v$, and it is easy to verify that with this choice of H(x), K(x) the set

$$\bigcup \{ d_x \mid (U, V) \in d_x \}$$

is finite for each UdV.

The above mentioned result will now follow if we can show that every Σ-finite dual family is σ-discrete, and this is the subject of the next theorem. This corresponds to ([22], Theorem 2), and there is only a notational difference between the proofs for the topological and bitopological cases. However the proof given in [22] contains a technical error (for the given induction hypothesis it would be quite possible for two disjoint members of $\mathcal{U}_x^{(1)}$ to be given the same index) and so we give the proof of the bitopological version of this theorem in full. We base the proof on the method of transfinite construction ([12], § 5.2) since a rigorously stated proof based on transfinite induction is somewhat cumbersome.

Theorem 1.2.1. Every $\leq$ -finite dual family is σ -discrete

Proof. Let d be a $\leq$ -finite dual family on (X, u, v), and suppose that H(x), K(x), $x \in X$, have been chosen so that

$$\bigcup \{ d_x \mid (U, V) \in d_x \}$$

is finite for each UdV. In particular each d_x is finite. For each x let F_x faithfully index d_x, and suppose the F_x are pairwise disjoint and also disjoint from X. If $\alpha \in W = \bigcup \{ F_x \mid x \in X \}$ we denote by $x(\alpha)$ the unique $x \in X$ with $\alpha \in F_x$.

Give X and each F_x a well ordering. There can be no confusion in denoting each of these orderings by $\leq$, and $<$ denotes $\leq$ and $\neq$. We may then well order W by

$$\alpha \leq \beta \text{ (in W)} \iff x(\alpha) < x(\beta) \text{ or } x(\alpha) = x(\beta) \text{ and}$$
$$\alpha \leq \beta \text{ (in } F_{x(\alpha)}).$$

We are going to show that there is a function $f : W \to \mathbb{N}$ which for each $\alpha \in W$ satisfies the conditions:

(a) $\beta < \alpha$ and $(U_\beta, V_\beta) = (U_\alpha, V_\alpha) \Rightarrow f(\beta) = f(\alpha)$, and

(b) $\beta < \alpha$, $(U_\beta, V_\beta) \neq (U_\alpha, V_\alpha)$ and $\exists$ $w \in X$ with $x(\alpha) \leq w$ and (U_β, V_β), $(U_\alpha, V_\alpha) \in d_w \Rightarrow f(\beta) \neq f(\alpha)$.

For $\alpha \in W$ we let $W(\alpha) = \{ \beta \mid \beta < \alpha \}$, and we denote by $\mathcal{F}(\alpha)$ the set of all functions $\varphi : W(\alpha) \to \mathbb{N}$ satisfying the condition :

(c) $\beta, \gamma \in W(\alpha)$, $(U_\beta, V_\beta) = (U_\gamma, V_\gamma) \Rightarrow \varphi(\beta) = \varphi(\gamma)$.

For $x \in X$ let $F_x^{(1)} = \{\alpha \mid \alpha \in F_x, \exists\ y < x \text{ with } (U_\alpha, V_\alpha) \in d_y\}$, and $F_x^{(2)} = F_x - F_x^{(1)}$. For $\gamma \in F_x^{(2)}$ let

$$S_\gamma = \{ \beta \mid x(\beta) < x \text{ and } (U_\beta, V_\beta), (U_\gamma, V_\gamma) \in d_w \text{ for some } w \in X \text{ with } x \leq w \}.$$

Since d is $\leq$ -finite S_γ is finite, and hence so too is the set

$$T(x) = \bigcup \{ S_\gamma \mid \gamma \in F_x^{(2)} \}.$$

Also for $\alpha \in W$, $T(x(\alpha)) \subseteq W(\alpha)$ so if $\varphi : W(\alpha) \longrightarrow \mathbb{N}$ is any function, $\varphi[T(x(\alpha))]$ is a finite set of natural numbers. Let

$$n(\varphi, \alpha) = \begin{cases} 0 \text{ if } T(x(\alpha)) = \emptyset \\ \max(\varphi[T(x(\alpha))]) \text{ otherwise.} \end{cases}$$

For each function $\varphi : W(\alpha) \longrightarrow \mathbb{N}$ let us define $R_\alpha[\varphi]$ as follows.

(i) If $\alpha \notin \mathfrak{F}(\alpha)$ let $R_\alpha[\varphi] = 0$

(ii) If $\alpha \in \mathfrak{F}(\alpha)$ and $\alpha \in F_{x(\alpha)}^{(1)}$ then $\exists \beta \in W$ with $x(\beta) < x(\alpha)$ and $(U_\alpha, V_\alpha) = (U_\beta, V_\beta)$; and we let $R_\alpha[\varphi] = \varphi(\beta)$.

This value is unique since φ satisfies (c).

(iii) If $\alpha \in \mathfrak{F}(\alpha)$ and $\alpha \in F_{x(\alpha)}^{(2)}$ we let $R_\alpha[\varphi] = n(\varphi, \alpha) + |K(\alpha)|$ where $|K(\alpha)|$ is the number of elements in the set $K(\alpha) = \{\beta \mid x(\beta) = x(\alpha) \text{ and } \beta \leq \alpha\}$.

By the principle of transfinite construction we have a function $f : W \longrightarrow \mathbb{N}$ such that $f(\alpha) = R_\alpha[f|W(\alpha)]$ for each $\alpha \in W$. Note that if α_0 is the least element of W then we may regard the unique function $\varphi : \emptyset \longrightarrow \mathbb{N}$ as belonging to $\mathfrak{F}(\alpha_0)$, and since $n(\varphi, \alpha_0) = 0$ this means that we are giving $f(\alpha_0)$ the unique value 1.

It remains to show that f satisfies (a) and (b) for all $\alpha \in W$. Clearly these are true for $\alpha = \alpha_0$, so it will suffice to show that if they are true for each $\alpha < \alpha'$ they are true for $\alpha = \alpha'$.

First, under this hypothesis, $f|W(\alpha') \in \mathfrak{F}(\alpha')$. For if $\beta, \gamma \in W(\alpha')$ with $\beta < \gamma$ we may apply (a) with $\alpha = \gamma$ to give $f(\beta) = f(\gamma)$, so (c) is satisfied.

<u>(a)</u> Take $\beta < \alpha'$ with $(U_\beta, V_\beta) = (U_{\alpha'}, V_{\alpha'})$. To show that $f(\alpha') = f(\beta)$. Since $\beta \neq \alpha'$ we cannot have $\beta \in F_{x(\alpha')}$ since the indexing is faithful, so $x(\beta) < x(\alpha')$. This shows that $\alpha' \in F_{x(\alpha')}^{(1)}$ and hence that $f(\alpha') = R_{\alpha'}[f|W(\alpha')] = [f|W(\alpha')](\beta) = f(\beta)$ as required.

<u>(b)</u> Take $\beta < \alpha'$ with $(U_\beta, V_\beta) \neq (U_{\alpha'}, V_{\alpha'})$, and suppose that for some $x(\alpha') \leq w$ we have $(U_\beta, V_\beta), (U_{\alpha'}, V_{\alpha'}) \in d_w$. We wish to

show that $f(\beta) \neq f(\alpha')$.

First suppose that $\alpha' \in F_{x(\alpha')}^{(1)}$, then $\exists\ \gamma \in W$ with $x(\gamma) < x(\alpha')$ and $(U_\gamma, V_\gamma) = (U_{\alpha'}, V_{\alpha'})$. Hence $f(\alpha') = R_{\alpha'}[f|W(\alpha')] = [f|W(\alpha')](\gamma) = f(\gamma)$. Now $\beta \neq \gamma$ so suppose $\gamma \prec \beta$. (b) is true for $\alpha = \beta$, and we may deduce that $f(\gamma) \neq f(\beta)$, ie. $f(\alpha') \neq f(\beta)$. The same argument applies when $\beta \prec \gamma$.

Finally suppose $\alpha' \in F_{x(\alpha')}^{(2)}$. Then $f(\alpha') = R_{\alpha'}[f|W(\alpha')] = n(f|W(\alpha'), \alpha') + |K(\alpha')|$. There are three cases to consider.

(i) $x(\beta) = x(\alpha')$ and $\beta \in F_{x(\beta)}^{(2)}$. In this case we also have $f(\beta) = n(f|W(\beta), \beta) + |K(\beta)|$. Now $T(x(\alpha')) = T(x(\beta)) \subseteq \bigcup \{F_y \mid y < x(\alpha')\} \subseteq W(\alpha') \cap W(\beta)$, and so $n(f|W(\beta), \beta) = n(f|W(\alpha'), \alpha')$. However since $\beta \prec \alpha'$ and $\beta, \alpha' \in F_{x(\alpha')}$ we have $|K(\beta)| < |K(\alpha')|$ and so $f(\beta) < f(\alpha')$.

(ii) $x(\beta) < x(\alpha')$. In this case $\beta \in S_{\alpha'} \subseteq T(x(\alpha'))$, and $\beta \in W(\alpha')$ so $n(f|W(\alpha'), \alpha') \geq f(\beta)$. Also $|K(\alpha')| \geq 1$ so again $f(\beta) < f(\alpha')$.

(iii) $x(\beta) = x(\alpha')$ and $\beta \in F_{x(\beta)}^{(1)}$. Here we have $\gamma \in W$ with $x(\gamma) < x(\beta) = x(\alpha')$ and $(U_\gamma, V_\gamma) = (U_\beta, V_\beta)$. Then as before $f(\beta) = f(\gamma)$, and we may apply case (ii) to γ in place of β and deduce that $f(\beta) = f(\gamma) < f(\alpha')$.

This verifies (b) for $\alpha = \alpha'$ in all cases, and so f has the properties (a) and (b) for all $\alpha \in W$ as stated. In particular it follows at once from (b) that $f|F_x$ is an injective function for each $x \in X$.

Now for each n = 1, 2, ... let:

$$d_n = \{ (U_\alpha, V_\alpha) \mid \alpha \in W,\ f(\alpha) = n \}.$$

Clearly $\bigcup \{d_n\} = d$, so our proof will be complete if we can show each d_n is discrete, and this will follow if no d_n can contain more than one element from any d_x, $x \in X$. Suppose, therefore, that we have (U_α, V_α), $(U_\beta, V_\beta) \in d_x$ and $f(\alpha) = f(\beta) = n$. Then $\exists\ \alpha', \beta' \in F_x$ so that $(U_\alpha, V_\alpha) = (U_{\alpha'}, V_{\alpha'})$ and $(U_\beta, V_\beta) = (U_{\beta'}, V_{\beta'})$; $f(\alpha) = f(\alpha')$ and $f(\beta) = f(\beta')$ by (a). But then $f(\alpha') = f(\beta')$ and so $\alpha' = \beta'$ since $f|F_x$ is injective. This shows $(U_\alpha, V_\alpha) = (U_\beta, V_\beta)$ and we have shown d_n is discrete as required. This completes the proof of the theorem.

Corollary. Every strongly biparacompact bitopological space is strongly biscreenable.

The other properties of Σ -finite dual families are similar to those of Σ -finite collections. If we call a dual family d closed if dom d consits of v-closed sets and ran d consists of u-closed sets then we may note in particular that a closed dual family is Σ -finite if and only if it is star finite and point finite.

We shall follow the terminology of [31] in respect to separation properties of bitopological spaces. We recall in particular that the bitopological space (X, u, v) is weakly pairwise Hausdorff if and only if given $x, y \in X$ with $x \neq y$ there exist $H \in u$, $K \in v$ with $H \cap K = \emptyset$ and $x \in H$, $y \in K$ or $x \in K$, $y \in H$. A.H. STONE [33] has shown that for Hausdorff topological spaces the notions of paracompactness and full normality coincide, but the bitopological space of Example 1.6.3 is weakly pairwise Hausdorff and fully binormal but not biparacompact, so this form of the coincidence theorem does not hold for weakly pairwise Hausdorff bitopological spaces. On the other hand, however, we are now going to show that under a suitable separation hypothesis a biparacompact bitopological space is indeed fully binormal. The required separation property is given in:

Definition 1.2.2. (X, u, v) is preseparated if given $x \notin u\text{-cl}\{y\}$ (respectively, $x \notin v\text{-cl}\{y\}$) in X there exist $U \in u$, $V \in v$ with $U \cap V = \emptyset$ and $x \in U$, $y \in V$ (respectively, $y \in U$, $x \in V$).

Clearly a preseparated bitopological space is pairwise R_0 [23], while a weakly pairwise T_0 [31] preseparated bitopological space is weakly pairwise Hausdorff.

The following definition and lemma will be useful in the proof of the above mentioned result.

Definition 1.2.3. Let d and e be dual covers of X. We say d is a delta refinement of e, and write $d \prec(\Delta) e$, if given $x \in X$ there exists ReS with $St(d, \{x\}) \subseteq R$, $St(\{x\}, d) \subseteq S$.

Lemma 1.2.2. If d, e, f are dual covers and $d \prec(\Delta)\, e \prec(\Delta)\, f$ then $d \prec(*)\, f$.

Corollary. (X, u, v) is fully binormal if and only if avery open dual cover has an open delta refinement.

We omit the proof which is straightforward. See ([25], (B), p 50) for the corresponding statement for topological spaces.

We may now give:

Theorem 1.2.2. A preseparated biparacompact bitopological space (X, u, v) is fully binormal.

Proof. (1) (X, u, v) is pairwise regular.

Let F be u-closed and $p \notin F$. For $x \in F$ we have $p \notin$ u-cl$\{x\}$ and so we have $p \in U_x \in u$, $x \in V_x \in v$ with $U_x \cap V_x = \emptyset$. The open dual cover $d = \{(X, V_x) \mid x \in F\} \cup \{(X - F, X)\}$ has a locally finite open refinement $e = \{(R_\alpha, S_\alpha) \mid \alpha \in A\}$. Now $F \subseteq St(F, e) \in v$, so it will suffice to show $p \notin$ u-cl$[St(F, e)]$. Now let $H(p) \in u$, $K(p) \in v$ be nhds. of p so that

$$\{\alpha \mid H(p) \cap S_\alpha \neq \emptyset \text{ or } K(p) \cap R_\alpha \neq \emptyset\}$$

is finite, and let

$$\{\alpha \mid H(p) \cap S_\alpha \neq \emptyset \text{ and } F \cap R \neq \emptyset\} = \{\alpha_1, \ldots, \alpha_m\}.$$

For $i = 1, 2, \ldots, m$; $R_{\alpha_i} \not\subseteq X - F$ so $\exists$ $x(i) \in F$ with $S_{\alpha_i} \subseteq V_{x(i)}$ and it follows that $M = H(p) \cap \bigcap \{U_{x(i)} \mid i = 1, 2, \ldots, m\}$ is a u-nhd. of p with $M \cap St(F, e) = \emptyset$. The case when F is v-closed is similar, and so (X, u, v) is pairwise regular.

(2) (X, u, v) is binormal.

Let F be u-closed, and T a uniformly closed set with $F \cap T = \emptyset$.

We have an open dual family $\{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$ so that

$$T = X - \bigcup \{ (U_\alpha \cap V_\alpha) \mid \alpha \in A \},$$

and we may assume without loss of generality that $U_\alpha \cap V_\alpha \neq \emptyset$ for each $\alpha \in A$. By the pairwise regularity we have for $x \in T$, $x \in U_x \in u$ and $F \in V_x \in v$ with $U_x \cap V_x = \emptyset$, and by hypothesis the open dual cover

$$d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \} \cup \{ (U_x, X) \mid x \in T \}$$

will have an open locally finite refinement $e = \{ (R_\beta, S_\beta) \mid \beta \in B \}$. Let us define:

$$W = \bigcup \{ R_\beta \mid R_\beta \cap S_\beta \cap T \neq \emptyset \};$$

then $T \subseteq W \in u$.

For $z \in F$ let $H(z) \in u$, $K(z) \in v$ be nhds. of z so that

$$\{ \beta \mid H(z) \cap S_\beta \neq \emptyset \text{ or } K(z) \cap R_\beta \neq \emptyset \}$$

is finite, and set

$$\{ \beta \mid K(z) \cap R_\beta \neq \emptyset \text{ and } R_\beta \cap S_\beta \cap T \neq \emptyset \} = \{ \beta_1, \dots, \beta_n \}.$$

For $i = 1, 2, \dots, n$ we cannot have $R_{\beta_i} \subseteq U_\alpha$ and $S_{\beta_i} \subseteq V_\alpha$ for for any $\alpha \in A$ since $U_\alpha \cap V_\alpha \cap T = \emptyset$, so there exists $x(i) \in T$ with $R_{\beta_i} \subseteq U_{x(i)}$. Then $z \in F \subseteq V_{x(i)}$ and $N(z) = K(z) \cap \bigcap \{ V_{x(i)} \mid i = 1, 2, \dots, n \}$ is a v-nhd. of z with $N(z) \cap W = \emptyset$. This shows that $F \cap v\text{-cl}[W] = \emptyset$. The case when F is v-closed is dealt with in the same way, and we deduce that (X, u, v) is binormal.

(3) (X, u, v) is fully binormal.

Let d be an open dual cover, and $e = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ an open locally finite refinement of d. In particular e is point finite, and (X, u, v) is (point finitely) binormal, so by Theorem 1.1.2 we know e is shrinkable. Hence there is an open dual cover $f = \{ (T_\alpha, Z_\alpha) \mid \alpha \in A' \}$ with $A' \subseteq A$, $v\text{-cl}[T_\alpha] \subseteq R_\alpha$ and $u\text{-cl}[Z_\alpha] \subseteq S_\alpha$ for each $\alpha \in A'$. Note that f is clearly locally finite also. For $p \in X$ define:

$$M(p) = \bigcap \{ R_\alpha \mid \alpha \in A', \ p \in v\text{-cl}[T_\alpha] \} \cap \bigcap \{ X - (u\text{-cl}[Z_\alpha]) \mid \alpha \in A', \ p \notin u\text{-cl}[Z_\alpha] \}.$$

The first intersection is finite since e is locally finite. Also, since f is locally finite and hence "closure preserving" we have:

$$\bigcap \{ X - (u\text{-cl}[Z_\alpha]) \mid \alpha \in A', \ p \notin u\text{-cl}[Z_\alpha] \}$$
$$= X - \bigcup \{ u\text{-cl}[Z_\alpha] \mid \alpha \in A', \ p \notin u\text{-cl}[Z_\alpha] \}$$
$$= X - u\text{-cl} [\bigcup \{ Z_\alpha \mid \alpha \in A', \ p \notin u\text{-cl}[Z_\alpha] \}] \in u.$$

Hence $p \in M(p) \in u$, and likewise if we set

$$N(p) = \bigcap \{ S_\alpha \mid \alpha \in A', \ p \in u\text{-cl}[Z_\alpha] \} \cap \bigcap \{ X - (v\text{-cl}[T_\alpha]) \mid \alpha \in A', \ p \notin v\text{-cl}[T_\alpha] \}$$

then $p \in N(p) \in v$.

Consider the open dual cover $g = \{ (M(p), N(p)) \mid p \in X \}$. If we take $x \in X$ and $\alpha \in A'$ with $x \in T_\alpha \cap Z_\alpha$ then it is easy to verify that

$$St(g, \{x\}) \subseteq R_\alpha \quad \text{and} \quad St(\{x\}, g) \subseteq S_\alpha.$$

Hence $g <(\Delta)\ e < d$, and (X, u, v) is fully binormal by Lemma 1.2.2. This completes the proof of the theorem.

Among other properties of paracompact spaces which carry over in a natural way to bitopological spaces we may note [3], Proposition 17, p 95 and [25], (A), p 150. Their counterparts for bitopological spaces are the subject of the next two theorems.

<u>Theorem</u> 1.2.3. The product $(X \times Y, u \times s, v \times t)$ of a biparacompact space (X, u, v) and a uniformly compact space (Y, s, t) is biparacompact.

<u>Proof</u>. Let D be an open dual cover of $X \times Y$, then for $x \in X$, $y \in Y$ we may choose nhds. $U_y(x) \in u$, $V_y(x) \in v$ of x and nhds. $S_x(y) \in s$, $T_x(y) \in t$ of y so that for some PDQ we have

$$U_y(x) \times S_x(y) \subseteq P, \ V_y(x) \times T_x(y) \subseteq Q.$$

Now for each $x \in X$, $f_x = \{ (S_x(y), T_x(y)) \mid y \in Y \}$ is an open dual cover of Y, so it has a finite refinement

$$\{ (S_x(y^x{}_1), T_x(y^x{}_1)), \ \ldots, (S_x(y^x{}_{n(x)}), T_x(y^x{}_{n(x)})) \}$$

Let $U(x) = \bigcap \{ U_{y^x_i}(x) \mid i = 1, 2, \ldots, n(x) \}$ and

$$V(x) = \bigcap \{ V_{y^x_i}(x) \mid i = 1, 2, \ldots, n(x) \}, \text{ so that}$$

$$d = \{ (U(x), V(x)) \mid x \in X \}$$

is an open dual cover of X. Let $e = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ be a locally finite open refinement of d, and for $\alpha \in A$ choose $x(\alpha) \in X$ satisfying $R_\alpha \subseteq U(x(\alpha))$, $S_\alpha \subseteq V(x(\alpha))$. Then it is easy to verify that

$$E = \{ (R_\alpha \times S_{x(\alpha)}(y^{x(\alpha)}{}_k), S_\alpha \times T_{x(\alpha)}(y^{x(\alpha)}{}_k)) \mid \alpha \in A, 1 \leq k \leq n(x(\alpha)) \}$$

is an open locally finite refinement of D.

With only minor changes the same proof also shows that the product of a strongly biscreenable space and a uniformly compact space is strongly biscreenable.

<u>Theorem</u> 1.2.4. The pairwise regular bitopological space (X, u, v) is biparacompact if and only if every open dual cover has a locally finite (not necessarily open) refinement.

<u>Proof</u>. Necessesity is clear. To show sufficiency let d be an open dual cover, and b a locally finite (not necessarily open) refinement of d. For each $x \in X$ we have nhds. $H(x) \in u$, $K(x) \in v$ of x so that

$$\{ (P, Q) \mid PbQ, P \cap K(x) \neq \emptyset \text{ or } Q \cap H(x) \neq \emptyset \}$$

is finite. Let $g = \{ (H(x), K(x)) \mid x \in X \}$. Because (X, u, v) is pairwise regular we may take $H'(x) \in u$, $K'(x) \in v$ with $x \in H'(x) \subseteq v\text{-cl}[H'(x)] \subseteq H(x)$, $x \in K'(x) \subseteq u\text{-cl}[K'(x)] \subseteq K(x)$. Let $g' = \{ (H'(x), K'(x)) \mid x \in X \}$. This is an open dual cover so there is a locally finite (not necessarily open) dual cover c with $c \prec b'$. Let

$$k = \bar{c} = \{ (v\text{-cl}[A], u\text{-cl}[B]) \mid AcB \}$$

be the <u>closure</u> of c. Then k is locally finite, and $k \prec g$.

For PbQ let

$$P' = X - \bigcup \{ N \mid N \cap P = \emptyset, N \in \text{ran } k \},$$

$$Q' = X - \bigcup \{ M \quad M \cap Q = \emptyset, M \in \text{dom } k \}.$$

Because k is locally finite, $P \subseteq P' \in u$ and $Q \subseteq Q' \in v$. Also $b \prec d$ so given PbQ we may choose $U(P, Q)$, $V(P, Q)$ with $U(P, Q)dV(P, Q)$, $P \subseteq U(P, Q)$ and $Q \subseteq V(P, Q)$. It is then easily verified that

$$e = \{ (P' \cap U(P, Q), Q' \cap V(P, Q)) \mid PbQ \}$$

is an open locally finite refinement of d, so proving the theorem.

For the strongly biscreenable case we have the following limited result.

Proposition 1.2.1. Let (X, u, v) be a pairwise regular bitopological space, and suppose that each open dual cover has a σ-discrete (not necessarily open) refinement $\cup \{ d_n \mid n = 1, 2, \ldots \}$ satisfying $\cup \{ (u\text{-int}[uc(\bar{d}_n)]) \cap (v\text{-int}[uc(\bar{d}_n)]) \mid n = 1, 2, \ldots \} = X$. Then (X, u, v) is strongly biscreenable.

We omit the proof which follows the same lines as the proof of Theorem 1.2.4.

Bitopological spaces in which one topology is paracompact with respect to the other have been considered by several authors (see, for example, [26], [27]). If (X, u, v) is biparacompact then certainly each topology is paracompact with respect to the other. For if $\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$ is (say) a u-open cover of X, and $e = \{ (R_\beta, S_\beta) \mid \beta \in B \}$ an open locally finite refinement of the open dual cover $d = \{ (U_\alpha, X) \mid \alpha \in A \}$ then $\mathcal{R} = \{ R_\beta \mid \beta \in B \}$ is a u-open v-locally finite refinement of $\mathcal{U}$. However the converse is false. Indeed the bitopological space of Example 1.6.5 has each topology compact, but it is not biparacompact.

These considerations, together with Theorem 1.2.2, show that biparacompactness is quite a ~~powerfuly~~ property. One way in which it may be weakened is to modify the notion of local finiteness. If in Definition 1.2.1 we may choose H(x), K(x) so that

$$d_x^{*} = \{ (U, V) \mid UdV, U \cap K(x) \neq \emptyset \text{ and } V \cap H(x) \neq \emptyset \}$$

is finite for each $x \in X$ we shall say that d is quasi-locally finite, and replacing d_x by d_x' in the rest of this definition gives us the terms quasi-Σ-finite, quasi-discrete, and so on. Likewise a corresponding change in Definition 1.1.4 defines the notion of quasi-star finite. The terms quasi-biparacompact, strongly quasi-biscreenable, strongly quasi-biparacompact, etc., will then have their obvious meaning.

The class of quasi-biparacompact spaces is much larger than the class of biparacompact spaces, but even so it is still not true that a bitopological space in which each topology is paracompact with respect to the other is necessarily quasi-biparacompact - as witness Example 1.6.5 again. However we can obtain this result for bitopological spaces satisfying the condition given below:

Definition 1.2.4. We shall say the dual family d is full if UdV whenever $U \in \operatorname{dom} d$, $V \in \operatorname{ran} d$ and $U \cap V \neq \emptyset$. (X, u, v) is full (respectively, σ-full) if every open dual cover has a full (respectively, σ-full) open refinement.

Clearly every uniformly Lindelöf bitopological space is σ-full, but Example 1.6.3 exhibits a uniformly Lindelöf space which is not full.

Proposition 1.2.2. Let (X, u, v) be full, and suppose that each topology is paracompact with respect to the other. Then (X, u, v) is quasi-biparacompact.

Proof. Let d be an open dual cover, e a full open refinement, $\mathfrak{m}$ a u-open v-locally finite refinement of the u-open cover dom e, and $\mathfrak{n}$ a v-open u-locally finite refinement of the v-open cover ran e. Then

$$f = \{ (M, N) \mid M \in \mathfrak{m}, N \in \mathfrak{n}, M \cap N \neq \emptyset \}$$

is clearly an open quasi-locally finite refinement of d.

Only notational changes are needed in the proofs of Lemma 1.2.1 and Theorem 1.2.1 to show that an open dual cover d is quasi-Σ-finite if and only if it is quasi-star finite, and that every quasi-Σ-finite dual family is quasi-σ-discrete. Hence we may state at once:

Theorem 1.2.5. Every strongly quasi-biparacompact bitopological space is strongly quasi-bisreenable.

In the same way the proof of Theorem 1.2.3 may be modified to give:

Theorem 1.2.6. The product $(X \times Y, u \times s, v \times t)$ of a quasi-biparacompact (respectively, strongly quasi-biscreenable) space (X, u, v), and a uniformly compact space (Y, s, t) is quasi-biparacompact (respectively, strongly quasi-biscreenable).

On the other hand, however, the proofs of Theorems 1.2.2 and 1.2.4 do not carry over to the quasi-biparacompact case, basically because a quasi-locally finite dual family may not have the "closure preserving" property. Indeed with regard to Theorem 1.2.2 we have a negative answer, for the space of Example 1.6.1 is preseparated and quasi-biparacompact but not fully binormal. I suspect that Theorem 1.2.4 may be generally false in the quasi-biparacompact case also, but we do have the following positive result:

Proposition 1.2.3. Let $(X, u\ v)$ be fully binormal, and suppose that every open dual cover has a quasi-locally finite (not necessarily open) refinement. Then (X, u, v) is quasi-biparacompact.

Proof. Let d be an open dual cover, and let d' be an open dual cover with $d' \prec_{(*)} d$. By hypothesis d' has a quasi-locally finite (not necessarily open) refinement $e' = \{ (R_\beta, S_\beta) \mid \beta \in B \}$, so there are nhds. $H(x) \in u$, $K(x) \in v$ of $x \in X$ with the property that $\{ \beta \mid R_\beta \cap K(x) \neq \emptyset \neq S_\beta \cap H(x) \}$ is finite for each x. Let $f = \{ (H(x), K(x)) \mid x \in X \}$, and take open dual covers h and g so that $g \prec_{(\Delta)} h \prec_{(*)} f$ and $g \prec d'$. Consider the open dual cover

$$e = \{ (St(g, R_\beta), St(S_\beta, g)) \mid \beta \in B \}.$$

Given $\beta \in B$ we have $U'd'V'$ with $R_\beta \subseteq U'$, $S_\beta \subseteq V'$; and UdV with $St(d', U') \subseteq U$ and $St(V', d') \subseteq V$. But then $St(g, R_\beta) \subseteq U$ and $St(S_\beta, g) \subseteq V$ so $e \prec d$. It remains to show that e is quasi-locally finite. Let us associate with $x \in X$ the nhds. $St(g, \{x\})$

and $St(\{x\}, g)$. Note that for some PhQ we have $St(g,\{x\}) \subseteq P$ and $St(\{x\}, g) \subseteq Q$, while for some $y \in X$ we have $St(h, P) \subseteq H(y)$ and $St(Q, h) \subseteq K(y)$. However if $St(g,\{x\}) \cap St(S_\beta, g) \neq \emptyset$ and $St(\{x\}, g) \cap St(g, R_\beta) \neq \emptyset$ then $H(y) \cap S_\beta \neq \emptyset$ and $K(y) \cap R_\beta \neq \emptyset$ and this is possible only for a finite number of $\beta \in B$, so e is quasi-locally finite, as required.

I do not know if a fully binormal space is necessarily quasi-biparacompact, but I would conjecture that the answer is no. Further results in this area may be found in the later sections, particularly in 1.4 and 1.5.

1.3 COUNTABLY QUASI-BIPARACOMPACT SPACES.

Countably paracompact topological spaces were introduced by C. DOWKER in [11]. In this section we consider some properties of the corresponding class of countably quasi-biparacompact bitopological spaces. We begin with:

Definition 1.3.1. (X, u, v) is countably quasi-biparacompact if every countable open dual cover has a quasi-locally finite refinement.

Our principal result is based on the following:

Lemma 1.3.1. Let (X, u, v) be a pairwise normal bitopological space, and $d = \{ (U_n, V_n) \mid n \in \mathbb{N} \}$ an open dual cover satisfying $U_n \subseteq U_{n+1}$ and $V_n \subseteq V_{n+1}$ for all $n \in \mathbb{N}$. Suppose there is a closed dual cover $c = \{ (A_n, B_n) \mid n \in \mathbb{N} \}$ with $A_n \subseteq U_n$ and $B_n \subseteq V_n$ for each n. Then d has a quasi-locally finite countable open refinement.

Proof. Since (X, u, v) is pairwise normal we have for $n \in \mathbb{N}$, $s = 1, 2, \ldots$, sets $R_{ns} \in u$ and $S_{ns} \in v$ with

$$A_n \subseteq R_{ns} \subseteq v\text{-cl}[R_{ns}] \subseteq R_{n(s+1)} \subseteq U_n, \text{ and}$$

$$B_n \subseteq S_{ns} \subseteq u\text{-cl}[S_{ns}] \subseteq S_{n(s+1)} \subseteq V_n .$$

Moreover we may suppose without loss of generality that $R_{ns} \subseteq R_{(n+1)s}$ and $S_{ns} \subseteq S_{(n+1)s}$, for if this is not so we may replace

R_{ns} and S_{ns} for $n > 0$ by $\bigcup\{R_{ks} \mid k = 0, 1, \ldots, n\}$ and $\bigcup\{S_{ks} \mid k = 0, 1, \ldots, n\}$ respectively. Let us set:

$$W_{0s} = R_{0s}\ , \quad s = 1, 2, \ldots ;$$

$$W_{ns} = R_{ns} - (u\text{-cl}[S_{(n-1)s}])\ , \ n = 1, 2, \ldots\ , \ s = 1, 2, \ldots$$

and
$$T_{0s} = S_{0s}\ , \quad s = 1, 2, \ldots ;$$

$$T_{ns} = S_{ns} - (v\text{-cl}[R_{(n-1)s}])\ , \ n = 1, 2, \ldots, \ s = 1, 2, \ldots$$

Let us also set $R_n = \bigcup\{R_{ns} \mid s = 1, 2, \ldots\}$, $S_n = \bigcup\{S_{ns} \mid s = 1, 2, \ldots\}$, $W_n = \bigcup\{W_{ns} \mid s = n, n+1, \ldots\}$ and $T_n = \bigcup\{T_{ns} \mid s = n, n+1, \ldots\}$. Then

$$e = \{(W_n, S_n) \mid n \in \mathbb{N}\} \cup \{(R_n, T_n) \mid n \in \mathbb{N}\}$$

is a countable open quasi-locally finite refinement of d. That e is countable and open is clear; and $e \prec d$ since $W_n \subseteq R_n \subseteq U_n$ and $T_n \subseteq S_n \subseteq V_n$ for each n. To see that it is a dual cover take $x \in X$ and define

$$m(x) = \min\{n \mid \exists\ s,\ x \in R_{ns}\},$$

$$n(x) = \min\{n \mid \exists\ t,\ x \in S_{nt}\}.$$

Then it is clear that if $m(x) \leq n(x)$ we have $x \in W_n \cap S_n$ for $n = n(x)$, while if $n(x) \leq m(x)$ then $x \in R_n \cap T_n$ for $n = m(x)$. Finally to show e is quasi-locally finite take $x \in X$ and suppose that, say, $m(x) \leq n(x)$. Then $x \in R_{m(x)s} \subseteq R_{n(x)s}$ for some s; while $x \in S_{n(x)t}$ for some t so we may define:

$$s(x) = \min\{s \mid x \in R_{n(x)s}\},$$

$$t(x) = \min\{t \mid x \in S_{n(x)t}\}$$

and associate with x the u-nhd. $R_{n(x)s(x)}$ and the v-nhd. $S_{n(x)t(x)}$. It is easy to verify that if $R_{n(x)s(x)} \cap T_n \neq \emptyset$ and $S_{n(x)t(x)} \cap R_n$

$\neq \emptyset$; or if $R_{n(x)s(x)} \cap S_n \neq \emptyset$ and $S_{n(x)t(x)} \cap W_n \neq \emptyset$ then $n \leq \max(n(x), s(x))$ or $n \leq \max(n(x), t(x))$ respectively. A similar appropriate assignment of nhds. to x may be made when $n(x) < m(x)$. Hence e is quasi-locally finite, and the proof is complete.

In order to state our next theorem we shall need some more terminology. We shall say the dual family d is <u>quasi-strongly point finite</u> if given $x \in X$ either $\{ U \mid x \in U \in \text{dom } d \}$ is finite or $\{ V \mid x \in V \in \text{ran } d \}$ is finite.

The dual cover d is called <u>countably medial</u> if it can be indexed over $\mathbb{N}$ in such a way that for each $x \in X$ we have $k(x) = \max(m(x), n(x))$, where $m(x) = \min\{ n \mid x \in U_n \}$, $n(x) = \min\{ n \mid x \in V_n \}$ and $k(x) = \min\{ n \mid x \in U_n \cap V_n \}$.

We may now state:

<u>Theorem</u> 1.3.1. Let (X, u, v) be a point finitely binormal space satisfying:

(a) Every countable open dual cover has a quasi-strongly point finite open refinement, and

(b) Every countable open dual cover has a countably medial open refinement.

Then (X, u, v) is countably quasi-biparacompact.

<u>Proof</u>. Let d' be a countable open cover of X. By (b) there will be no loss of generality if we assume that d' is countably medial, that is $d' = \{ (U'_n, V'_n) \mid n \in \mathbb{N} \}$, where $k'(x) = \max(m'(x), n'(x))$ for all $x \in X$, using an obvious notation. Let us set

$$U_n = \bigcup \{ U'_k \mid k = 0, 1, \ldots, n \} \text{ and } V_n = \bigcup \{ V'_k \mid k = 0, 1, \ldots n \}$$

so that $d = \{ (U_n, V_n) \mid n \in \mathbb{N} \}$ is an open dual cover satisfying $U_n \subseteq U_{n+1}$ and $V_n \subseteq V_{n+1}$. Let $e = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ be a faithfully indexed quasi-strongly point finite open refinement of d. For $s \in \mathbb{N}$ let

$$A(s) = \{ \alpha \mid \alpha \in A, R_\alpha \subseteq U_s \text{ and } S_\alpha \subseteq V_s \}.$$

Clearly $A(s) \subseteq A(s+1)$ for all s.

Let $r = \min\{s \mid A(s) \neq \emptyset\}$, $R_r^* = \bigcup\{R_\alpha \mid \alpha \in A(k)\}$ and $S_r^* = \bigcup\{S_\alpha \mid \alpha \in A(k)\}$. Generally for $s = 1, 2, \ldots$, let

$$R^*_{r+s} = \begin{cases} R^*_{r+s-1} \text{ if } A(r+s-1) = A(r+s) \\ \bigcup\{R_\alpha \mid \alpha \in A(r+s) - A(r+s-1)\} \text{ otherwise,} \end{cases}$$

$$S^*_{r+s} = \begin{cases} S^*_{r+s-1} \text{ if } A(r+s-1) = A(r+s) \\ \bigcup\{S_\alpha \mid \alpha \in A(r+s) - A(r+s-1)\} \text{ otherwise.} \end{cases}$$

For $x \in X$ let $s(x) = \min\{s \mid \exists\, \alpha \in A(r+s) \text{ with } x \in R_\alpha \cap S_\alpha\}$. Then clearly $x \in R^*_{r+s(x)} \cap S^*_{r+s(x)}$ and so

$$f = \{(R^*_{r+s}, S^*_{r+s}) \mid s \in \mathbb{N}\}$$

is an open dual cover refinement of d. Let us show it is point finite. For $x \in X$ let $\{\alpha_1, \ldots, \alpha_m\}$ denote the set $\{\alpha \mid x \in R_\alpha\}$ whenever this set is finite, and otherwise let it denote the set $\{\alpha \mid x \in S_\alpha\}$. Define

$$p(x) = \begin{cases} 0 \text{ if } \{\alpha_1, \ldots, \alpha_m\} \subseteq A(r), \text{ and otherwise,} \\ \max\{p \mid \exists\, i,\ 1 \leq i \leq m \text{ with } \alpha_i \in A(r+p) - A(r+p-1)\}. \end{cases}$$

If, from some point onwards, the sets $A(s)$ are equal then f is finite and hence point finite. In the contrary case, for each $x \in X$,

$q(x) = \max\{q \mid A(r+p(x)) = A(r+q)\}$

is a well defined natural number, and it is clear from the definitions that $x \in R^*_{r+s} \cap S^*_{r+s}$ implies $s \leq q(x)$. Thus f is point finite as stated. Since (X, u, v) is point finitely binormal it follows from Theorem 1.1.2 that f is shrinkable. Hence there is an open dual cover $g = \{(M_s, N_s) \mid s \in \mathbb{N}'\}$ where $\mathbb{N}' \subseteq \mathbb{N}$, $v\text{-cl}[M_s] \subseteq R^*_{r+s} \subseteq U_{r+s}$ and $u\text{-cl}[N_s] \subseteq S^*_{r+s} \subseteq V_{r+s}$ for all $s \in \mathbb{N}'$. Let $r' = \min\{s \mid s \in \mathbb{N}'\}$, and $t = r + r'$. Put $A_t = v\text{-cl}[M_{r'}]$, $B_t = u\text{-cl}[N_{r'}]$, and generally for $s = 1, 2, \ldots$,

$$A_{t+s} = \begin{cases} v\text{-cl}[M_{r'+s}] \text{ if } r' + s \in \mathbb{N}', \\ A_{t+s-1} \text{ otherwise,} \end{cases}$$

$$B_{t+s} = \begin{cases} u\text{-cl}[N_{r'+s}] \text{ if } r' + s \in \mathbb{N}', \\ B_{t+s-1} \text{ otherwise.} \end{cases}$$

Then $c = \{ (A_n, B_n) \mid n = t, t+1, \ldots \}$ is a closed dual cover, $A_n \subseteq U_n$ and $B_n \subseteq V_n$. It follows that the conditions of Lemma 1.3.1 are satisfied for the open dual cover $d_t = \{ (U_n, V_n) \mid n = t, t+1, \ldots \}$, and so we have an open quasi-locally finite refinement

$e_t = \{ (W_n, S_n) \mid n = t, t+1, \ldots \} \cup \{ (R_n, T_n) \mid n = t, t+1, \ldots \}$.

For $n \in \mathbb{N}$ set $W'_n = (\bigcup \{ W_k \mid k = n \vee t, n \vee t + L, \ldots \}) \cap U'_n$, $S'_n = (\bigcup \{ S_k \mid k = n \vee t, n \vee t + 1, \ldots \}) \cap V'_n$, $R'_n = (\bigcup \{ R_k \mid k = n \vee t, n \vee t + 1, \ldots \}) \cap U'_n$ and $T'_n = (\bigcup \{ T_k \mid k = n \vee t, n \vee t + 1, \ldots \}) \cap V'_n$.

If $x \in W_n \cap S_n$ or $x \in R_n \cap T_n$ then $n \geq k'(x) = \max(m'(x), n'(x))$, and so $x \in W'_{k'(x)} \cap S'_{k'(x)}$ or $x \in R'_{k'(x)} \cap T'_{k'(x)}$ respectively. This shows that

$e' = \{ (W'_n, S'_n) \mid W'_n \cap S'_n \neq \emptyset \} \cup \{ (R'_n, T'_n) \mid R'_n \cap T'_n \neq \emptyset \}$

is an open dual cover refiniment of d'. Finally the argument used in the proof of Lemma 1.3.1 to show e_t is quasi-locally finite will also show that e' is quasi-locally finite, and the proof is complete.

The next result is also a consequence of Lemma 1.3.1.

<u>Proposition</u> 1.3.1. Let (X, u, v) be a pairwise perfectly normal space [20], and suppose that each countable open dual cover has a countably medial open refinement. Then (X, u, v) is countably quasi-biparacompact.

Proof. Let $d' = \{(U'_n, V'_n) \mid n \in \mathbb{N}\}$ be a countably medial open dual cover, and form $d = \{(U_n, V_n) \mid n \in \mathbb{N}\}$ with $U_n \subseteq U_{n+1}$, $V_n \subseteq V_{n+1}$ as in the proof of Theorem 1.3.1. Now we have v-closed sets P_{ns}, $s \in \mathbb{N}$, and u-closed sets Q_{ns}, $s \in \mathbb{N}$, so that $P_{ns} \subseteq P_{n(s+1)}$, $Q_{ns} \subseteq Q_{n(s+1)}$, $U_n = \bigcup\{P_{ns} \mid s \in \mathbb{N}\}$, and $V_n = \bigcup\{Q_{ns} \mid s \in \mathbb{N}\}$.

For $n \in \mathbb{N}$ define $A_n = \bigcup\{P_{tn} \mid t = 1, \ldots, n\} \subseteq U_n$ and

$B_n = \bigcup\{Q_{tn} \mid t = 1, \ldots, n\} \subseteq V_n$. Then

$$c = \{(A_n, B_n) \mid n \in \mathbb{N}\}$$

is a closed dual cover, and the conditions of Lemma 1.3.1 are satisfied. The remainder of the proof is similar to the last part of the proof of Theorem 1.3.1, and is ommited.

The final lemma of this section deals with a situation at the opposite extreme from that of Lemma 1.3.1. This result can also be useful in establishing (countable) quasi-biparacompactness in some cases (See, for instance, Example 1.6.3).

Lemma 1.3.2. Let (X, u, v) be a pairwise normal bitopological space. If $d = \{(U_k, V_k) \mid k \in \mathbb{Z}\}$ is a countable open dual cover satisfying $\bigcap\{U_k\} = \bigcap\{V_k\} = \emptyset$, $U_k \subseteq U_{k+1}$ and $V_{k+1} \subseteq V_k$ for all $k \in \mathbb{Z}$, and if there exists a closed dual cover $c = \{(A_k, B_k) \mid k \in \mathbb{Z}\}$ with $A_k \subseteq A_{k+1}$, $B_{k+1} \subseteq B_k$, $A_k \subseteq U_k$ and $B_k \subseteq V_k$ for all $k \in \mathbb{Z}$, then d has a quasi-locally finite countable open refinement.

Proof. Since (X, u, v) is pairwise normal we have u-open sets R_k with $A_k \subseteq R_k \subseteq \text{v-cl}[R_k] \subseteq U_k$. Without loss of generality we may also suppose that $R_k \subseteq R_{k+1}$ for each $k \in \mathbb{Z}$, for if this is not so we may replace R_k by $\bigcup\{R_i \mid i = 0, \ldots, k\}$ for $k > 0$, and by $\bigcap\{R_i \mid i = k, \ldots, 0\}$ for $k < 0$. In just the same way we have v-open sets S_k with $B_k \subseteq S_k \subseteq \text{u-cl}[S_k] \subseteq V_k$, and we may suppose $S_{k+1} \subseteq S_k$ for each $k \in \mathbb{Z}$.

Clearly $e = \{ (R_k, S_k) \mid k \in \mathbb{Z} \}$ is an open refinement of d. We show it is quasi-locally finite. For $x \in X$ the numbers

$$m(x) = \min \{ k \mid x \in v\text{-cl}[R_k] \},$$

$$n(x) = \max \{ k \mid x \in u\text{-cl}[S_k] \}$$

both exist in Z. Also, for some k', $x \in R_{k'} \cap S_{k'}$, and so $m(x) \leq k' \leq n(x)$ for each $x \in X$. Now

$$M(x) = R_{n(x)} - (u\text{-cl}[S_{n(x)+1}])$$

is a u-nhd of x, and

$$N(x) = S_{m(x)} - (v\text{-cl}[R_{m(x)-1}])$$

is a v-nhd of x. Also if $M(x) \cap S_k \neq \emptyset$ and $N(x) \cap R_k \neq \emptyset$ then $m(x) \leq k \leq n(x)$. Hence e is quasi-locally finite, and the proof is complete.

1.4 METRIZABLE AND SEQUENTIALLY NORMAL BITOPOLOGICAL SPACES.

One of the important properties of the class of paracompact topological spacese and of the class of fully normal topological spaces is that they include the class of metrizable spaces. Let us recall that a non-negative real-valued function $p(x,y)$ on $X \times X$ satisfying the triangle inequality is called a pseudo-quasi-metric if $p(x,x) = 0$ for all $x \in X$. Corresponding to the p-q-metric p is the p-q-metric p^* defined by

$$p^*(x,y) = p(y,x), \qquad x, y \in X$$

and called the conjugate of p. Each p-q-metric p defines a topology t(p) on X in the same way that a metric does (see [19]), and the bitopological space (X, u, v) is metrizable (or, more correctly, p-q-metrizable) if there is a p-q-metric p on X satisfying $t(p) = u$ and $t(p^*) = v$. It is known [19] that a metrizable bitopological space is pairwise regular and pairwise normal, and it is also clearly pairwise R_0 and preseparated. In particular if p is a quasi-metric (that is $p(x,y) = 0 \Leftrightarrow x = y$) then $(X, t(p), t(p^*))$ is weakly pairwise Hausdorff. It is natural to ask if all metrizable bitopological spaces are biparacompact or fully binormal, and the answer in no. For the spaces of Examples 1.6.1, 1.6.2 and 1.6.3 are all metrizable while the first of these spaces is neither biparacompact nor fully binormal, and the second two are not biparacompact. We may also note in passing that the first two of these spaces are not pairwise paracompact in the sense of DATTA [10] either. It is true that all these spaces are quasi-biparacompact, but I strongly suspect that this will not be true of all metrizable bitopological spaces. This poses the problem of defining a suitable class of bitopological spaces which does include all metrizable spaces. The class of sequentially normal spaces defined in this section is obtained by weakening the condition of full binormality. I do not have a "local finiteness" characterization of these spaces, although if such a description could be obtained it would undoubtably be invaluable.

If d and e are dual families let us set

$$e * d = \{(St(e, U), St(V, e)) \mid UdV\}.$$

We may now give:

Definition 1.4.1. The dual cover d is sequentially normal if thene exist open dual families d_n, and open normal dual covers e_n so that

(i) $e_n * d_n \prec d$, $n = 1, 2, 3, \ldots$,

(ii) $\bigcup\{d_n\}$ is a dual cover of X.

(X, u, v) will be called sequentially normal if every open dual cover of X is sequentially normal.

Clearly every fully binormal space is sequentially normal.

Proposition 1.4.1. Every sequentially normal bitopological space is pairwise normal.

Proof. Let P be a u-closed set, Q a v-closed set and $P \cap Q = \emptyset$. Consider the open dual cover $d = \{(X - P, X), (X, X - Q)\}$, and let d_n, e_n have the properties (i) and (ii) above. Let $U_n =$ $St(d_n, Q)$ and $V_n = St(P, d_n)$, and set

$$W_n = U_n - \bigcup \{ u\text{-cl}[V_k] \mid k = 1, \ldots, n \},$$

$$T_n = V_n - \bigcup \{ v\text{-cl}[U_k] \mid k = 1, \ldots, n \}$$

for $n = 1, 2, \ldots$. Then if $W = \bigcup \{ W_n \mid n = 1, 2, \ldots \}$ and $T = \bigcup \{ T_n \mid n = 1, 2, \ldots \}$ then $W \in u$, $T \in v$, and it is clear that $W \cap T = \emptyset$.

Let us show that for each n we have

$$P \cap (v\text{-cl}[U_n]) = Q \cap (u\text{-cl}[V_n]) = \emptyset.$$

If $p \in P \cap (v\text{-cl}[U_n])$ then $\exists$ $R e_n S$ with $p \in R \cap S$, and then $S \cap U_n \neq \emptyset$ so $\exists$ $U d_n V$ with $S \cap U \neq \emptyset$ and $V \cap Q \neq \emptyset$. Hence $p \in R \subseteq St(e_n, U)$. But since $e_n * d_n \prec d$ we have

$$St(e_n, U) \subseteq X - P \quad \text{or} \quad St(V, e_n) \subseteq X - Q,$$

and $p \in St(e_n, U) \cap P$ contradicts the first possibility, while $V \cap Q \neq \emptyset$ contradicts the second.

It follows that $P \cap (v\text{-cl}[U_n]) = \emptyset$, and the second result is proved likewise.

From these results we deduce at once that $P \subseteq T$ and $Q \subseteq W$, so (X, u, v) is pairwise normal as required.

As promised above we are going to show that every metrizable bitopological space is sequentially normal. To this end we are going to need some terminology and results concerning p-q-metrics and equibicontinuous families of real valued functions.

Let p be a p-q-metric on X. For $x \in X$ and $\varepsilon > 0$ we set

$$H(x, \varepsilon) = \{ y \mid p(x,y) < \varepsilon \},$$

$$K(x, \varepsilon) = \{ y \mid p(y,x) < \varepsilon \}.$$

In this context $H_n(x)$ and $K_n(x)$ will denote $H(x, 2^{-n})$ and $K(x, 2^{-n})$ respectively, unless stated otherwise.

We denote by o_n the open dual cover $\{ (H_n(x), K_n(x)) \mid x \in X \}$.

It will be noted that $\{ H_n(x) \mid n \in \mathbb{N} \}$ (respectively, $\{K_n(x) \mid n \in \mathbb{N} \}$) is a base of nhds. of x for the topology t(p) (respectively, for the conjugate topology $t(p^*)$). We will say that the p-q-metric p is <u>admissible</u> for the bitopological space (X, u, v) if $t(p) \subseteq u$ and $t(p^*) \subseteq v$.

If d is a dual cover of X we will say that the p-q-metric p is <u>subordinate</u> to d if given $x \in X$ $\exists$ UdV and $n \in \mathbb{N}$ with $H_n(x) \subseteq U$ and $K_n(x) \subseteq V$. We will say that p is <u>evenly subordinate</u> to d if we have $o_n \prec d$ for some $n \in \mathbb{N}$.

Clearly (X, u, v) is metrizable if and only if there is an admissible p-q-metric p subordinate to every open dual cover of X.

Let us recall that a function $f : X \to X'$ is <u>bicontinuous</u> with respect to the bitopological spaces (X, u, v) and (X', u', v') if it is continuous for the topologies u, u'; and for the topologies v, v'. We will always consider $\mathbb{R}$ with the topologies

$$s = \{\{ x \mid x < a \} \mid a \in \mathbb{R} \} \cup \{\mathbb{R}, \emptyset \}, \text{ and}$$

$$t = \{\{x \mid a < x \} \mid a \in \mathbb{R} \} \cup \{\mathbb{R}, \emptyset \},$$

and if (X, u, v) is a bitopological space then to say that a real valued function on X is bicontinuous will mean that it is bicontinuous with respect to (X, u, v) and (R, s, t). Hence $f : X \to \mathbb{R}$ is bicontinuous if given $x \in X$ and $\varepsilon > 0$ there is a u-nhd. M(x) of x and a v-nhd. N(x) of x so that

$$y \in M(x) \Rightarrow f(y) < f(x) + \varepsilon \quad , \text{ and}$$

$$y \in N(x) \Rightarrow f(x) < f(y) + \varepsilon \quad .$$

If F is a family of real valued functions, and if for each $x \in X$ and $\varepsilon > 0$ we may find M(x), N(x) satisfying the above conditions for all $f \in F$ then we shall say that F is <u>equibicontinuous</u>.

For each $\alpha \in A$ let h_α and k_α be real valued functions on X. Then we shall say that

$$E = \{ (h_\alpha, k_\alpha) \mid \alpha \in A \}$$

is an <u>equibinormal family</u> for the bitopological space (X, u, v) if:

(a) $0 \leq h_\alpha \leq 1$ and $0 \leq k_\alpha \leq 1$, and

(b) The families $\{ h_\alpha \mid \alpha \in A \}$ and $\{ -k_\alpha \mid \alpha \in A \}$ are equibicontinuous.

If $0 \leq f \leq 1$ we set $s(f) = \{ x \mid f(x) = 0 \}$ and $e(f) = \{ x \mid f(x) < 1 \}$.

The <u>support</u> of the equibinormal family E is the dual family

$$s(E) = \{ (s(h_\alpha), s(k_\alpha)) \mid \alpha \in A,\ s(h_\alpha) \cap s(k_\alpha) \neq \emptyset \},$$

and the <u>envelope</u> of E is the dual family

$$e(E) = \{ (e(h_\alpha), e(k_\alpha)) \mid \alpha \in A,\ s(h_\alpha) \cap s(k_\alpha) \neq \emptyset \}.$$

The equibinormal family E will be called an <u>equibinormal cover</u> if s(E) is a dual cover of X.

The following lemma will play an essential role in what follows.

<u>Lemma</u> 1.4.1. Let d_n, $n = 1, 2, \ldots$, be a sequence of open dual covers of X satisfying $d_{n+1} <(*) d_n$ for each n. Then:

(1) There exists an admissible p-q-metric p so that $d_{n+2} <(*) o_n$ and $o_{n+1} < d_n^\Delta = \{ (St(d_n, \{x\}), St(\{x\}, d_n)) \mid x \in X \}$ for each n.

(2) If $d_n = \{ (U_\alpha, V_\alpha) \mid \alpha \in A_n \}$, and the index sets A_n are pairwise disjoint, then for each n there is an equibinormal cover $E_n = \{ (h_\alpha, k_\alpha) \mid \alpha \in A'_n \}$ with $A'_n \subseteq A_n$ and satisfying $d_{n+5} <(\Delta) s(E_n)$ for each n, and $e(h_\alpha) \subseteq U_\alpha$, $e(k_\alpha) \subseteq V_\alpha$ for each α.

<u>Proof.</u> (1) This is essentially a variant of ([18], Lemma 6.12). For a verification directly in terms of dual covers we may use the method of the proof of ([25], Theorem VII). Thus, let us set

$$d(1/2^n) = d_n,\ n = 1, 2, \ldots,$$

and inductively for $1 < k < 2^{n+1}$,

$$d(k/2^{n+1}) = d(2k'/2^{n+1}) = d(k'/2^n) \text{ if } k = 2k', \text{ and}$$

$$d(k/2^{n+1}) = d((2k'+1)/2^{n+1}) = d_{n+1} * d(k/2^n) \text{ if } k = 2k' + 1.$$

If finally we set $d(1) = \{(X, X)\}$ we have open dual covers $d(\nu)$ for each diadic number $\nu = k/2^n$, $1 \leq k \leq 2^n$. Now let us define

$$\varphi(x, y) = \inf\{\nu \mid y \in St(d(\nu), \{x\})\}, \text{ and}$$

$$p(x,y) = \sup\{(\varphi(x, z) - \varphi(y, z)) \vee 0 \mid z \in X\}$$

for $x, y \in X$. Then it may be verified that p is an admissible p-q-metric with the required properties.

(2) Let p be an admissible p-q-metric with the properties given in (1). For $Y \subseteq X$ we will set

$$L_n(Y) = \{x \mid H_n(x) \subseteq Y\}, \text{ and}$$

$$M_n(Y) = \{x \mid K_n(x) \subseteq Y\}.$$

Now let $A'_n = \{\alpha \mid \alpha \in A_n,\ L_{n+3}(U_\alpha) \neq \emptyset \neq M_{n+3}(V_\alpha)\}$, and for $\alpha \in A'_n$ and $x \in X$ let:

$$h_\alpha(x) = 2^{n+3}([\inf\{p(z,x) \mid z \in L_{n+3}(U_\alpha)\}] \wedge 2^{-n-3}),$$

$$k_\alpha(x) = 2^{n+3}([\inf\{p(x,z) \mid z \in M_{n+3}(V_\alpha)\}] \wedge 2^{-n-3}),$$

and $E_n = \{(h_\alpha, k_\alpha) \mid \alpha \in A'_n\}$.

That $e(h_\alpha) \subseteq U_\alpha$ and $e(k_\alpha) \subseteq V_\alpha$ is clear. To see that $\{h_\alpha \mid \alpha \in A'_n\}$ is equibicontinuous, given $\varepsilon > 0$ take $m \in \mathbb{N}$ with $2^{-m} < \varepsilon/2^{n+4}$, and $y \in H_m(x)$.

If $h_\alpha(x) = 1$ then certainly $h_\alpha(y) < h_\alpha(x) + \varepsilon$.
If $h_\alpha(x) \neq 1$, then $h_\alpha(x) = 2^{n+3}(\inf\{p(z,x) \mid z \in L_{n+3}(U_\alpha)\})$ so $\exists\ z \in L_{n+3}(U_\alpha)$ with $h_\alpha(x) \geq 2^{n+3}p(z,x) - \varepsilon/2$. Hence $h_\alpha(y) \leq 2^{n+3}p(z,y) \leq 2^{n+3}p(z,x) + 2^{n+3}p(x,y) < h_\alpha(x) + \varepsilon$ since $p(x,y) < 2^{-m} < \varepsilon/2^{n+4}$.

This proves the stated result, and $\{-k_\alpha \mid \alpha \in A'_n\}$ can be shown to be equibicontinuous in the same way and we have established that E_n is an equibinormal family. Finally $d_{n+5} \prec (*)\, o_{n+3}$ so it will suffice to show that $o_{n+3} \prec s(E_n)$. Now $o_{n+2} \prec d^{\Delta}_{n+1}$ and $d_{n+1} \prec(*)\, d_n$ so $o_{n+2} \prec d_n$. Hence if $q \in X$ we have $\alpha \in A'_n$ with $H_{n+2}(q) \subseteq U_\alpha$ and $K_{n+2}(q) \subseteq V_\alpha$. But then $x \in H_{n+3}(q) \Rightarrow$ $H_{n+3}(x) \subseteq H_{n+2}(q) \subseteq U_\alpha \Rightarrow x \in L_{n+3}(U_\alpha) \subseteq s(h_\alpha) \Rightarrow$ $H_{n+3}(q) \subseteq s(h_\alpha)$. In the same way $K_{n+3}(q) \subseteq s(k_\alpha)$. Hence $o_{n+3} \prec s(E_n)$, and the proof is complete.

Corollary 1. The following are equivalent for the dual cover d.

(a) d is normal.

(b) There is an admissible p-q-metric evenly subordinate to d.

(c) There is an equibinormal cover whose envelope refines d.

Proof. (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow directly from the lemma, and (b) $\Rightarrow$ (a) is obvious. It remains, therefore, to prove (c) $\Rightarrow$ (a). Let $E = \{(h_\alpha, k_\alpha) \mid \alpha \in A\}$, and set

$$U_m(x) = u\text{-int}\Big[\bigcap_\alpha \{\{y \mid h_\alpha(y) < h_\alpha(x) + 3^{-m}\}\} \cap \bigcap_\alpha \{\{y \mid k_\alpha(x) < k_\alpha(y) + 3^{-m}\}\}\Big],$$

$$V_m(x) = v\text{-int}\Big[\bigcap_\alpha \{\{y \mid h_\alpha(x) < h_\alpha(y) + 3^{-m}\}\} \cap \bigcap_\alpha \{\{y \mid k_\alpha(y) < k_\alpha(x) + 3^{-m}\}\}\Big].$$

Under the given hypothesis $x \in U_m(x) \in u$ and $x \in V_m(x) \in v$. Hence $d_m = \{(U_m(x), V_m(x)) \mid x \in X\}$ is an open dual cover for $m = 1, 2, \ldots$. It is easy to verify that $d_{m+1} \prec(*)\, d_m$; and that $d_1 \prec d$, so d is normal as required.

Corollary 2. (X, u, v) is metrizable if and only if there is a sequence d_n of open dual covers such that

(1) $d_{n+1} \prec(*)\, d_n$, $n = 1, 2, \ldots$,

(2) $\{St(d_n, \{x\}) \mid n = 1, 2, \ldots\}$ is a base of u-nhds. of x,

(3) $\{St(\{x\}, d_n) \mid n = 1, 2, \ldots\}$ is a base of v-nhds. of x,

for each $x \in X$.

The above corollary is the exact counterpart for bitopological spaces of the Alexandroff-Urysohn metrization theorem (See, for example, [25], Theorem VII).

Sufficient conditions for the metrizability of bitopological spaces have been given by J.C.KELLY [19], E.P.LANE [20] and S.SALBANY [32]. Let us indicate how the metrization theorem of Salbany, which includes the results of Kelly and Lane as special cases, may be deduced from Corollary 2.

Salbany defines, in effect, an <u>open pair base</u> for the bitopological space (X, u, v) to be an open dual family $b = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ satisfying:

(a) $R_\alpha \cup S_\alpha = X$ for all $\alpha \in A$,

(b) $x \in U \in u \Rightarrow \exists\ \alpha \in A$ with $x \in X - S_\alpha \subseteq R_\alpha \subseteq U$, and

(c) $x \in V \in v \Rightarrow \exists\, \beta \in A$ with $x \in X - R_\beta \subseteq S_\beta \subseteq V$.

With our terminology Salbany's theorem ([32], Theorem 2.4) now states that a pairwise normal bitopological space (X, u, v) is metrizable if there is a sequence $\{b_n\}$ of quasi-locally finite dual families so that $\cup \{b_n\}$ is an open pair base. Actually we can show that the explicit assumption of pairwise normality is unnecessary, and so we have:

<u>Theorem</u> 1.4.1. Let (X, u, v) be a bitopological space which has a sequence $\{b_n\}$ of quasi-locally finite dual families so that $\cup \{b_n\}$ is an open pair base. Then (X, u, v) is metrizable.

<u>Proof</u>. Let us first verify that (X, u, v) is pairwise normal. To this end let d be an open dual cover of X. For $n = 1, 2, \ldots,$ we may set $b_n = \{ (R_\alpha, S_\alpha) \mid \alpha \in A_n \}$ where the sets A_n are pairwise disjoint. Given $x \in X$ and n we have $H_n(x) \in u$ and $K_n(x) \in v$ so that

$$\{ \alpha \mid \alpha \in A_n,\ H_n(x) \cap S_\alpha \neq \emptyset \neq K_n(x) \cap R_\alpha \}$$

is finite. Note that

$$U_n(x) = H_n(x) \cap \bigcap \{ R_\alpha \mid \alpha \in A_n,\ H_n(x) \cap S_\alpha \neq \emptyset \neq K_n(x) \cap R_\alpha \} \in u,$$

$$V_n(x) = K_n(x) \cap \bigcap \{ S_\alpha \mid \alpha \in A_n,\ H_n(x) \cap S_\alpha \neq \emptyset \neq K_n(x) \cap R_\alpha \} \in v;$$

and that,

$U_n(x) \subseteq \bigcap\{R_\alpha \mid \alpha \in A_n,\ x \in R_\alpha\} \subseteq \bigcap\{R_\alpha \mid \alpha \in A_n,\ x \notin S_\alpha\}$,

$V_n(x) \subseteq \bigcap\{S_\alpha \mid \alpha \in A_n,\ x \in S_\alpha\} \subseteq \bigcap\{S_\alpha \mid \alpha \in A_n,\ x \notin R_\alpha\}$.

Let us define:

$d_n = \{ (U_n(x),\ V_n(x)) \mid \exists\ UdV,\ \alpha, \beta \in A_n$ with $x \in X - S_\alpha \subseteq R_\alpha \subseteq U$ and $x \in X - R_\beta \subseteq S_\beta \subseteq V \}$, and

$e_n = \{ (U_n(x),\ V_n(x)) \mid x \in X \}$.

Then for each n, d_n is an open dual family, e_n is an open dual cover of X, and it is easy to verify that

(i) $e_n * d_n < d$, $n = 1, 2, \ldots$, and

(ii) $\bigcup \{d_n\}$ is a dual cover of X.

Now the proof of Proposition 1.4.1 depends only on the properties (i) and (ii) of e_n and d_n, and not on the normality of the e_n, and so we may deduce that (X, u, v) is pairwise normal as stated.

Now let $b = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ be a quasi-locally finite open dual family satisfying $R_\alpha \cup S_\alpha = X$ for all $\alpha \in A$. By the pairwise normality we have u-open sets $M_\alpha(\nu)$ for each diadic number $\nu = k/2^n$, $1 \leq k < 2^n$, satisfying

$$X - S_\alpha \subseteq M_\alpha(\nu) \subseteq v\text{-cl}[M_\alpha(\nu)] \subseteq M\ (\nu') \subseteq R_\alpha \text{ whenever } \nu < \nu'.$$

If we set $N_\alpha(\nu) = X - (v\text{-cl}[M_\alpha(\nu)])$ for $\nu = k/2^n$, $1 \leq k < 2^n$, we have

$$X - R_\alpha \subseteq N_\alpha(\nu) \subseteq u\text{-cl}[N_\alpha(\nu)] \subseteq N_\alpha(\nu') \subseteq S_\alpha \text{ whenever } \nu' < \nu .$$

Finally we set $M_\alpha(1) = R_\alpha$ and $N_\alpha(0) = S_\alpha$.

Now define:

$R_n(x) = \bigcap\{M_\alpha((k+1)/2^n) \mid \alpha \in A,\ 1 \leq k < 2^n,\ x \in v\text{-cl}[M_\alpha(k/2^n)]\}$,

$S_n(x) = \bigcap\{N_\alpha((k-1)/2^n) \mid \alpha \in A,\ 1 \leq k < 2^n,\ x \in u\text{-cl}[N_\alpha(k/2^n)]\}$.

Since b is quasi-locally finite it is clear that $R_n(x)$ is a u-nhd. of x, and $S_n(x)$ is a v-nhd. of x. Hence

$$f_n = \{ (u\text{-int } R_n(x), v\text{-int } S_n(x)) \mid x \in X \}$$

is an open dual cover of X, and it is a straightforward matter to verify that

$$f_{n+2} \prec (*)\ f_n,\ n = 1, 2, \ldots .$$

Moreover we have

$x \in X - S_\alpha \subseteq R_\alpha \Rightarrow \mathrm{St}(f_1, \{x\}) \subseteq R_\alpha$ and

$x \in X - R_\alpha \subseteq S_\alpha \Rightarrow \mathrm{St}(\{x\}, f_1) \subseteq S_\alpha$.

It follows that if we construct f^m_n for each of the dual families b_m, m = 1, 2, ..., given in the statement of the theorem, then the sequence

$$f^1_1,\ f^1_3 \wedge f^2_1,\ f^1_5 \wedge f^2_3 \wedge f^3_1, \ldots..$$

of open dual covers has all the properties required by Corollary 2 to Lemma 1.4.1. Hence (X, u, v) is metrizable, and the proof is complete.

Let us now return to our consideration of sequential normality.

<u>Theorem</u> 1.4.2. The following are equivalent for the open dual cover d of (X, u, v).

(a) d is sequentially normal.

(b) There is an admissible p-q-metric subordinate to d.

(c) There is a sequence $\{E_n\}$ of equibinormal families so that

 (i) $e(E_n) \prec d$, n = 1, 2, ... , and

 (ii) $\bigcup \{ s(E_n) \mid n = 1, 2, \ldots \}$ is a dual cover of X.

<u>Proof</u>. (a) $\Rightarrow$ (b) Let d_n and e_n be as in Definition 1.4.1. Since each e_n is normal there is, by Lemma 1.4.1, an admissible p-q-metric p_n (evenly) subordinate to e_n. Without loss of generality we may assume $0 \leq p_n \leq 1$ so

$$p(x,y) = \sum_{n=1}^{\infty} 2^{-n} p_n(x,y)$$

is an admissible p-q-metric on X. Take $x \in X$, then for some n and $U_n d_n V_n$ we have $x \in U_n \cap V_n$. Also, since $e_n * d_n \prec d$ there exists UdV with $St(e_n, U_n) \subseteq U$ and $St(V_n, e_n) \subseteq V$. Finally p_n is subordinate to e_n so for some Re_nS with $x \in R \cap S$ and some m we have $H^n_m(x) \subseteq R$ and $K^n_m(x) \subseteq V$. But then

$$H_{m+n}(x) \subseteq U \quad \text{and} \quad K_{m+n}(x) \subseteq V$$

so p is subordinate to d.

(b) $\Rightarrow$ (c) Let p be an admissible p-q-metric subordinate to $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$. With the notation as in the proof of Lemma 1.4.1 define

$$h^n(x) = 2^n([\inf\{ p(z,x) \mid z \in L_n(U_\alpha)\}] \wedge 2^{-n}), \text{ and}$$

$$k^n(x) = 2^n([\inf\{ p(x,z) \mid z \in M_n(V_\alpha)\}] \wedge 2^{-n}).$$

Arguing as in the proof of Lemma 1.4.1 we see that

$$E_n = \{ (h^n_\alpha, k^n_\alpha) \mid \alpha \in A^n \}, \text{ where } A^n = \{\alpha \mid L_n(U_\alpha) \neq \emptyset \neq M_n(V_\alpha) \},$$

is an equibinormal family. Moreover it is clear that $e(h^n_\alpha) \subseteq U_\alpha$ and $e(k^n_\alpha) \subseteq V_\alpha$ so

$$e(E_n) \prec d, \; n = 1, 2, \ldots .$$

Finally p is subordinate to d, so given $x \in X$ there exist $\alpha \in A$ and n with

$$x \in H_n(x) \subseteq U \quad \text{and } x \in K_n(x) \subseteq V .$$

Hence $\alpha \in A^n$ and $x \in L_n(U_\alpha) \cap M_n(V_\alpha) \subseteq s(h^n_\alpha) \cap s(k^n_\alpha)$ which shows $\bigcup \{ s(E_n) \mid n = 1, 2, \ldots \}$ is a dual cover of X and so verifies (c).

(c) $\Rightarrow$ (a). Let $E_n = \{ (h_\beta, k_\beta) \mid \beta \in B_n \}$ be a sequence of equibinormal families as specified under (c). Without loss of generality we are supposing that the index sets B_n are pairwise disjoint. Form the sets $R^n_m(x)$ and $S^n_m(x)$ from the E_n in the same way that we defined the sets $R_m(x)$ and $S_m(x)$ from E in the proof of Lemma 1.4.1, Corollary 1. It follows that for each n and m,

$$e^m_n = \{ (R^n_m(x),\ S^n_m(x)) \mid x \in X \}$$

is an open dual cover of X. Also it is easy to verify that

$$e^{m+1}_n \prec (*)\ e^m_n$$

so e^m_n is a normal dual cover for each m, n.

For $\beta \in B_n$ let $U_\beta = \{ x \mid h_\beta(x) < 1/3 \}$ and $V_\beta = \{ x \mid k_\beta(x) < 1/3 \}$, and put

$$d_n = \{ (U_\beta,\ V_\beta) \mid \beta \in B_n \} .$$

Then each d_n is an open dual family, and their union is a dual cover of X by property (i) in (c). Finally let $e_n = e^1_n$. Then for $\beta \in B_n$ we have

$$St(e_n,\ U_\beta) \subseteq e(h_\beta), \text{ and}$$
$$St(V_\beta,\ e_n) \subseteq e(k_\beta)$$

from which $e_n * d_n \prec d$ follows using property (i) in (c). Hence d is sequentially normal.

Corollary. Every metrizable bitopological space is sequentially normal.

If one makes a corresponding definition of "sequentially normal" for covers of a topological space it is not difficult to verify that a topological space is fully normal if and only if every open cover is sequentially normal. Hence for topological spaces the notions of full normality and "sequential normality" coincide. However this is very far from being the case with bitopological spaces, for indeed the space of Example 1.6.1 is metrizable and therefore sequentially normal, but it is not even finitely binormal. In order to obtain a positive result here we need to weaken the condition of "full binormality", and this is the subject of the next definition.

Definition 1.4.2. Let d be a dual family, P and Q subsets of X. By the pseudo-stars of (P, Q) with respect to d we mean the sets

$PSt(d, (P, Q)) = \bigcup \{ U \mid \exists\ V \text{ with } UdV,\ U \cap Q \neq \emptyset \neq V \cap P \}$, and

$PSt((P, Q), d) = \bigcup \{ V \mid \exists\ U \text{ with } UdV,\ U \cap Q \neq \emptyset \neq V \cap P \}$.

If e is a second dual family we shall say that d is a pseudo-star refinement of e, and write $d \prec (p*)\ e$, if given UdV there exists ReS with $PSt(d, (U, V)) \subseteq R$ and $PSt((U, V), d) \subseteq S$.

(X, u, v) will be called fully pseudonormal if every open dual cover has an open pseudo-star refinement.

Theorem 1.4.3. Every sequentially normal bitopological space is fully pseudonormal.

Proof. Let d be an open dual cover of X. By theorem 1.4.2 there is an admissible p-q-metric p subordinate to d. Hence for each $x \in X$ we have $r(x)$, $0 < r(x) < 1$, so that

$$H(x, r(x)) \subseteq U \text{ and } K(x, r(x)) \subseteq V$$

for some UdV. If we put

$$d' = \{ (H(x, r(x)), K(x, r(x))) \mid x \in X \}$$

then d' is an open dual cover refinement of d.

Let $r'(x) = r(x)/6$, and consider the open dual cover

$$e = \{ (H(x, r'(x)), K(x, r'(x))) \mid x \in X \}.$$

We will show that $e \prec (p*)\ d'$.

Let $r = \sup \{ r(x') \mid H(x', r'(x')) \cap K(x, r'(x)) \neq \emptyset \neq K(x', r'(x')) \cap H(x, r'(x)) \}$.

Note in particular that $r(x) \leq r$. Now take $x_0 \in X$ such that $H(x_0, r'(x_0)) \cap K(x, r'(x)) \neq \emptyset \neq K(x_0, r'(x_0)) \cap H(x, r'(x))$ and $r(x_0) > 4r/5$.

If now $H(y, r'(y)) \cap K(x, r'(x)) \neq \emptyset \neq K(y, r'(y)) \cap H(x, r'(x))$

then $r(y) \leq r$, and so for $z \in H(y, r'(y)$ we have:

$$\begin{aligned} p(x_0,z) &\leq p(x_0,x) + p(x,y) + p(y,z) \\ &< r'(x_0) + r'(x) + r'(x) + r'(y) + r'(y) \\ &\leq (r(x_0) + 4r)/6 \\ &< r(x_0). \end{aligned}$$

This shows that $H(y, r'(y)) \subseteq H(x_0, r(x_0))$, and in the same way we have $K(y, r'(y)) \subseteq K(x_0, r(x_0))$. Hence $e <(p*)\ d'$, and the proof is complete.

The converse of this result is false. Indeed the bitopological space of Example 1.6.5 is fully pseudonormal, but it is not even pairwise normal and so by Proposition 1.4.1 it cannot be sequentially normal.

This example shows that full pseudonormality is a relatively weak condition. None the less we do have:

Proposition 1.4.2. A fully pseudonormal bitopological space is uniformly fully normal.

Proof. Let $\mathcal{U} = \{ G_\alpha \mid \alpha \in A \}$ be a uniform open cover of X. Then for $x \in G_\alpha$ we have $U_{x\alpha} \in u$, $V_{x\alpha} \in v$ with $x \in U_{x\alpha} \cap V_{x\alpha} \subseteq G_\alpha$. The open dual cover

$$d = \{ (U_{x\alpha}, V_{x\alpha}) \mid \alpha \in A,\ x \in G_\alpha \}$$

has an open pseudo-star refinement $e = \{ (R_\beta, S_\beta) \mid \beta \in B \}$, and if we put $\mathcal{V} = \{ R_\beta \cap S_\beta \mid \beta \in B \}$ it is easy to verify that $\mathcal{V}$ is a uniform open cover of X and that $\mathcal{V}^* < \mathcal{U}$. Hence (X, u, v) is uniformly fully normal.

One can easily show that a "sequentially normal" open cover of a topological space has a σ-discrete open refinement. However a corresponding result in the bitopological case would seem to require some additional restrictions, and these are detailed in the next definition.

Definition 1.4.3. The dual family d is <u>medial</u> if it can be indexed by a well ordered index set $(A, \leq)$ in such a way that for all

$x \in uc(d)$ we have $\varkappa(x) = \max(\mu(x), \nu(x))$, where

$$\varkappa(x) = \min\{\alpha \mid \alpha \in A,\ x \in U_\alpha \cap V_\alpha\},$$

$$\mu(x) = \min\{\alpha \mid \alpha \in A,\ x \in U_\alpha\}, \text{ and}$$

$$\nu(x) = \min\{\alpha \mid \alpha \in A,\ x \in V_\alpha\}.$$

We shall say that (X, u, v) is medial (respectively, σ-medial) if every open dual cover has a medial (respectively, σ-medial) open refinement.

We may now state:

Theorem 1.4.4. Every σ-medial sequentially normal open dual cover of a bitopological space has a σ-quasi-discrete open refinement.

Proof. Let the open dual cover d be the union of the medial dual families d_n. Let $(A_n, \leq)$ be a faithful indexing of d_n with the properties mentioned in Definition 1.4.3, where without loss of generality we may take the sets A_n to be pairwise disjoint. By Theorem 1.4.2 there is an admissible p-q-metric p subordinate to d, and for $B \subseteq X$ let us set

$$H_m(B) = \bigcup\{H_m(x) \mid x \in B\},$$

$$K_m(B) = \bigcup\{K_m(x) \mid x \in B\},$$

and define $L_m(B)$, $M_m(B)$ as in the proof of Lemma 1.4.1.

For $\alpha \in A_n$, $m, n = 1, 2, \ldots$, define:

$$U^m_\alpha = H_{m+3}(L_m(U_\alpha)), \quad V^m_\alpha = K_{m+3}(M_m(V_\alpha)),$$

$$R^m_\alpha = H_{m+3}(L_m(U_\alpha) - \bigcup\{M_{m+1}(V_\beta) \mid \beta \in A_n, \beta < \alpha\}), \text{ and}$$

$$S^m_\alpha = K_{m+3}(M_m(V_\alpha) - \bigcup\{L_{m+1}(U_\beta) \mid \beta \in A_n, \beta < \alpha\}).$$

Finally let $e^{nm}{}_1 = \{(U^m_\alpha, S^m_\alpha) \mid \alpha \in A_n,\ U^m_\alpha \cap S^m_\alpha \neq \emptyset\}$, and

$$e^{nm}{}_2 = \{(R^m_\alpha, V^m_\alpha) \mid \alpha \in A_n,\ R^m_\alpha \cap V^m_\alpha \neq \emptyset\},$$

for $n, m = 1, 2, 3, \ldots$.

For each n and m; $e^{nm}{}_1$ and $e^{nm}{}_2$ are open dual families refining d.

Take $x \in X$. Since p is subordinate to d we have n, $\alpha \in A_n$ and m so that $H_m(x) \subseteq U_\alpha$ and $K_m(x) \subseteq V_\alpha$. In particular $x \in uc(d_n)$ and so, using an obvious notation, we have $\kappa_n(x) = \max(\mu_n(x), \nu_n(x))$. But then it is easy to see that

$$x \in U^m_\alpha \cap S^m_\alpha \text{ if } \alpha = \kappa_n(x) = \mu_n(x), \text{ while}$$

$$x \in R^m_\alpha \cap V^m_\alpha \text{ if } \alpha = \kappa_n(x) = \nu_n(x).$$

It follows that

$$\{ e^{nm}{}_1 \mid n, m = 1, 2, \ldots \} \cup \{ e^{nm}{}_2 \mid n, m = 1, 2, \ldots \}$$

is an open dual cover of X.

It remains only to show that $e^{nm}{}_1$ and $e^{nm}{}_2$ are quasi-discrete for each n and m. To each $x \in X$ associate the nhds. $H_{n+3}(x)$ and $K_{m+3}(x)$. To show that $e^{nm}{}_1$ is quasi-discrete suppose that for some $\alpha, \beta \in A_n$ we have $H_{m+3}(x) \cap S^m_\alpha \neq \emptyset \neq K_{m+3}(x) \cap U^m_\alpha$ and $H_{m+3}(x) \cap S^m_\beta \neq \emptyset \neq K_{m+3}(x) \cap U^m_\beta$. Suppose, without loss of generality that $\beta \leq \alpha$. Then if $\beta \neq \alpha$ we have from $H_{m+3}(x) \cap S^m_\alpha \neq \emptyset$ the existance of $z \in H_{m+3}(x)$ and $t \in M_m(V_\alpha) - \bigcup \{ L_{m+1}(U_\gamma) \mid \gamma \in A_n, \gamma \prec \alpha \}$ with $z \in K_{m+3}(t)$, and so in particular,

$$t \notin L_{m+1}(U_\beta) \quad \ldots\ldots\ldots \; (1).$$

On the other hand from $K_{m+3}(x) \cap U^m_\beta \neq \emptyset$ we have $a \in K_{m+3}(x)$ and $b \in L_m(U_\beta)$ with $a \in H_{m+3}(b)$. But then

$$\begin{aligned} p(t,b) &\leq p(t,z) + p(z,x) + p(x,a) + p(a,b) \\ &< 4/2^{m+3} \\ &= 1/2^{m+1} \end{aligned}$$

and so $H_{m+1}(t) \subseteq H_m(b) \subseteq U_\beta$ which gives $t \in L_{m+1}(U_\beta)$, so contradicting (1).

This proves $e^{nm}{}_1$ is quasi-discrete. In the same way $e^{nm}{}_2$ is quasi-discrete, and the proof is complete.

Corollary. Every σ-medial sequentially normal bitopological space is strongly quasi-biscreenable.

In particular every σ-medial metrizable, and every σ-medial fully binormal bitopological space is strongly quasi-biscreenable.

We may improve this result in the fully binormal case by strengthening the "σ-medial" condition. We make the following definitions.

Definition 1.4.4. We say the dual cover $d = \bigcup\{d_n\}$ is a conservative σ-medial dual cover if there exist disjoint subsets X_1, X_2 of X (one of which could be empty) with $X_1 \cup X_2 = X$ and so that

(a) $x \in X_1 \cap uc(d_n) \Rightarrow \kappa_n(x) = \mu_n(x)$

$x \in X_2 \cap uc(d_n) \Rightarrow \kappa_n(x) = \nu_n(x)$, and

(b) $X_1 \cap uc(d_n) = X_1 \cap lc(d_n)$

$X_2 \cap uc(d_n) = X_2 \cap rc(d_n)$,

for all $n = 1, 2, \ldots$.

If $d = \bigcup\{d_n\}$ is σ-medial and sequentially normal, and p is an admissible p-q-metric subordinate to d we set

$$n(x) = \min\{n \mid x \in uc(d_n)\}, \text{ and}$$

$$m(x) = \min\{m \mid x \in L_m(U_\alpha) \cap M_m(V_\alpha) \text{ for } \alpha = \kappa_{n(x)}(x)\}.$$

With this notation we may state:

Definition 1.4.5. The σ-medial sequentially normal dual cover $d = \bigcup\{d_n\}$ is of finite type if there is an admissible p-q-metric p subordinate to d such that, for each $n = 1, 2, \ldots$, and each $\alpha \in A_n$ the set

$$\{m(z) \mid \kappa_n(z) \leq \alpha\}$$

is finite.

We may now give:

Theorem 1.4.5. Let (X, u, v) be fully binormal, and suppose that every open dual cover has a conservative σ-medial open refinement of finite type. Then (X, u, v) is quasi-biparacompact.

Proof. Let d be an open dual cover which, without loss of generality we may assume to be a conservative σ-medial dual cover $d = \bigcup\{d_n\}$ of finite type. Let X_1, X_2 be subsets of X as in Definition 1.4.4, and p a p-q-metric as in Definition 1.4.5. For n, m = 1, 2, ... , let us set

$$P_1(n, m) = \{ z \mid z \in X_1,\ n(z) = n \text{ and } m(z) = m \}, \text{ and}$$

$$P_2(n, m) = \{ z \mid z \in X_2,\ n(z) = n \text{ and } m(z) = m \}.$$

As in the proof of Theorem 1.4.4 we have quasi-discrete open dual families

$$e^{nm}{}_1 = \{ (U^m_\alpha, S^m_\alpha) \mid \alpha \in A_n,\ U^m_\alpha \cap S^m_\alpha \neq \emptyset \}, \text{ and}$$

$$e^{nm}{}_2 = \{ (R^m_\alpha, V^m_\alpha) \mid \alpha \in A_n,\ R^m_\alpha \cap V^m_\alpha \neq \emptyset \}$$

which refine d, and which together form a dual cover of X. Consider the (not necessarily open) dual family

$$e = \{ (U^m_\alpha, S^m_\alpha \cap P_1(n, m)) \mid n, m = 1, 2, \dots, \alpha \in A_n, U^m_\alpha \cap S^m_\alpha \cap P_1(n, m) \neq \emptyset \} \cup \{ (R^m_\alpha \cap P_2(n, m), V^m) \mid n, m = 1, 2, \dots, \alpha \in A_n, R^m_\alpha \cap V^m_\alpha \cap P_2(n, m) \neq \emptyset \}.$$

e is a dual cover of X. For if $x \in X_1$ then

$$x \in U^{m(x)}_\alpha \cap S^{m(x)}_\alpha \cap P_1(n(x), m(x)) \text{ for } \alpha = \kappa_{n(x)}(x) =$$

$\mu_{n(x)}(x) \in A_{n(x)}$, while if $x \in X_2$ then

$$x \in R^{m(x)}_\alpha \cap V^{m(x)}_\alpha \cap P_2(n(x), m(x)) \text{ for } \alpha = \kappa_{n(x)}(x) =$$

$\nu_{n(x)}(x) \in A_{n(x)}$.

Now let us show that e is quasi-locally finite. For $x \in X$ the set $\{ m(z) \mid \kappa_{n(x)}(z) \leq \kappa_{n(x)}(x) \}$ is finite, so we may set $M(x) = \max \{ m(z) \mid \kappa_{n(x)}(z) \leq \kappa_{n(x)}(x) \}$, and associate with x the nhds. $H_{M(x)+3}(x)$, $K_{M(x)+3}(x)$. Suppose that $H_{M(x)+3}(x) \cap P_1(n, m) \neq \emptyset$. Now

$$H_{M(x)+3}(x) \subseteq H_{m(x)}(x) \subseteq U_{\mathcal{K}_{n(x)}}(x)$$

so we may take $z \in U_{\mathcal{K}_{n(x)}}(x) \cap P_1(n, m)$. Then $z \in X_1$, $n(z) = n$ and $m(z) = m$ so $z \in X_1 \cap lc(d_{n(x)}) = X_1 \cap uc(d_{n(x)})$ by (b) of Definition 1.4.4. It follows that

$$n = n(z) \leq n(x) \quad .... \ (1)$$

Also $z \in X_1 \cap uç(d_{n(x)})$ implies $\mathcal{K}_{n(x)}(z) = \mu_{n(x)}(z)$ by (a) of Definition 1.4.4, so $\mathcal{K}_{n(x)}(z) = \mu_{n(x)}(z) \leq \mathcal{K}_{n(x)}(x)$ which gives

$$m = m(z) \leq M(x) \quad .... \ (2).$$

(1) and (2) also follow if $K_{M(x)+3}(x) \cap P_2(n, m) \neq \emptyset$, and we deduce at once that e is quasi-locally finite.

We have thus shown that every open dual cover has a quasi-locally finite refinement, and so (X, u, v) is quasi-biparacompact by Proposition 1.2.3.

The above results illustrate some of the difficulties involved in establishing even quasi-local finiteness properties of bitopological spaces. The notion of "mediality" introduced here, while providing a partial solution to some of these problems, is less than satisfactory in its present form because of its somewhat abstract nature. In particular it seems quite difficult to determine just how restrictive the conditions imposed in Theorems 1.4.4 and 1.4.5 really are.

1.5 COMPARTMENTAL DUAL COVER REFINEMENTS.

As we have noted in the previous sections, the local finiteness conditions we have imposed so far on a bitopological space are, with the possible but unlikely exception of strong quasi-biscreenability and quasi-biparacompactness, relatively stronger than the notion of paracompactness for topological spaces. In this section we discuss a much weaker form of local finiteness condition. This is based on the notion of a "compartmental dual cover", defined below.

Definition 1.5.1. If, for each $\gamma \in C$, d_γ is a dual family we say

$$d/C = \{ d_\gamma \mid \gamma \in C \}$$

is a compartmental dual family. d/C is a compartmental dual cover if

$$\bigcup \{ uc(d_\gamma) \mid \gamma \in C \} = X.$$

If e is a dual cover we say d/C refines e, and write $d/C \prec e$, if given $\gamma \in C$ there exists ReS with $d \prec \{(R, S)\}$.

Such terms as point finite, locally finite, quasi-locally finite, etc., may be defined for compartmental dual families in the obvious way. Thus, for example, d/C will be called quasi-locally finite if for each $x \in X$ there are nhds. $H(x) \in u$ and $K(x) \in v$ of x so that

$$\{ \gamma \mid \exists \; U d_\gamma V \text{ with } U \cap K(x) \neq \emptyset \neq V \cap H(x) \}$$

is finite.

A statement such as "(X, u, v) is compartmentally quasi-biparacompact" will mean that every open dual cover has a quasi-locally finite open compartmental dual cover refinement, and corresponding meanings may be given to such terms as "strongly compartmentally quasi-biscreenable", etc.

The notion of compartmental dual covers may be used to characterise uniformly paracompact bitopological spaces, as follows:

Proposition 1.5.1. Let (X, u, v) be uniformly regular. Then the following are equivalent:

(a) (X, u, v) is uniformly paracompact.

(b) Every open dual cover of X has a quasi-locally finite (not necessarily open) compartmental refinement.

Proof. (a) $\Rightarrow$ (b). If d is an open dual cover then $\mathcal{U} = \{U \cap V \mid UdV\}$ is a uniform open cover of X. If $\mathcal{V} = \{P_\alpha \mid \alpha \in A\}$ is a uniformly locally finite refinement of $\mathcal{U}$, and if for $\alpha \in A$ we set $e_\alpha = \{(\{z\},\{z\}) \mid z \in P_\alpha\}$ then $e/A = \{e_\alpha \mid \alpha \in A\}$ is the required quasi-locally finite compartmental refinement of d.

(b) $\Rightarrow$ (a). Let $\mathcal{U}$ be a uniform open cover of X, and for each $x \in X$ take $U(x) \in u$, $V(x) \in v$ with $x \in U(x) \cap V(x) \subseteq P$ for some $P \in \mathcal{U}$. If e/L is a quasi-locally finite compartmental refinement of $d = \{(U(x), V(x)) \mid x \in X\}$, and we set $Q_\lambda = uc(e_\lambda)$, $\lambda \in L$, then $\{Q_\lambda \mid \lambda \in L\}$ is a uniformly locally finite (not necessarily uniformly open) refinement of $\mathcal{U}$, and the required result now follows from a standard theorem on paracompactness (see, for example, [25]).

Corollary. A uniformly regular compartmentally quasi-biparacompact space is uniformly paracompact.

It would be tempting to conjecture from the above proposition that a uniformly regular uniformly paracompact bitopological space is necessarily compartmentally quasi-biparacompact. That such a conjecture would be false is shown by Example 1.6.8.

We may improve the above corollary with the aid of the next proposition.

Proposition 1.5.2. Let (X, u, v) be strongly compartmentally quasi-biscreenable. Then every open dual cover of X has a quasi-locally finite (not necessarily open) compartmental refinement.

Proof. Let d be an open dual cover, and let $d_n/L_n = \{d_n^\lambda \mid \lambda \in L_n\}$, n = 1, 2, ..., be quasi-discrete with respect to the nhds. $H_n(x) \in u$, $K_n(x) \in v$ of x, and such that $\bigcup\{d_n^\lambda \mid n = 1, 2, \ldots, \lambda \in L_n\}$ is an open dual cover refinement of d. Without loss of generality we may suppose that the index sets L_n are pairwise disjoint. For $x \in X$ let

$$r(x) = \min\{n \mid \exists\ \lambda \in L_n \text{ with } x \in uc(d_n^\lambda)\},$$

and denote by $\lambda(x)$ the unique λ for which $x \in uc(d_{r(x)}^{\lambda})$.

Choose a fixed $U(x)d_{r(x)}^{\lambda(x)}V(x)$ with $x \in U(x) \cap V(x)$, and

define:

$H(x) = U(x) \cap \bigcap \{ H_i(x) \mid 1 \leq i \leq r(x) \}$, and

$K(x) = V(x) \cap \bigcap \{ K_i(x) \mid 1 \leq i \leq r(x) \}$.

Let $L'_n = \{ \lambda \mid \lambda \in L_n, \exists\ x \in uc(d_n^{\lambda})$ with $r(x) = n \}$, and

for $\lambda \in L'_n$ let $d' = \{ (\{z\},\{z\}) \mid z \in uc(d_n^{\lambda})$ and $r(z) = n \}$.

Finally let $L' = \bigcup\{L'_n\}$ and $d'/L' = \{d'_\lambda \mid \lambda \in L'\}$. Let us show that d'/L' has the required properties.

If $x \in X$ then $x \in U(x) \cap V(x) \subseteq uc(d_{r(x)}^{\lambda(x)})$ so $\{x\}d'_{\lambda(x)}\{x\}$ and $\lambda(x) \in L'$. Hence d'/L' is a compartmental dual cover, and it is clearly a refinement of d. Finally suppose $z \in uc(d_n^{\lambda})$ with $r(z) = n$, and that $z \in H(x) \cap K(x)$. Then

$$z \in U(x) \cap V(x) \subseteq uc(d_{r(x)}^{\lambda(x)}),$$

and so $n = r(z) \leq r(x)$. Also if we take $Ud_n^{\lambda}V$ with $z \in U \cap V$ then $z \in U \cap K_n(x) \neq \emptyset$ and $z \in V \cap H_n(x) \neq \emptyset$ so λ is unique for this n. Hence d'/L' is quasi-locally finite as required.

<u>Corollary</u>. A uniformly regular strongly compartmentally quasi-biscreenable bitopological space is uniformly paracompact.

This last result may also be obtained by showing first that a strongly compartmentally quasi-biscreenable space is uniformly strongly biscreenable, and then using a standard theorem [25].

We are now going to show that every fully pseudonormal bitopological space is strongly compartmentally quasi-biscreenable. An apparently stronger result may be proved just as easily, however, and to state this we shall need some more notation.

Let d be a dual family, and $A \subseteq X$. By the <u>weak stars</u> of A with respect to d we shall mean the sets:

$$WSt(d, A) = \bigcup\{ U \mid \exists\ V, UdV \text{ and } U \cap V \cap A \neq \emptyset \},$$

$$WSt(A, d) = \bigcup\{ V \mid \exists\ U, UdV \text{ and } U \cap V \cap A \neq \emptyset \}.$$

By the uniform star of A with respect to d we shall mean the set:

$$USt(d, A) = USt(A, d) = \bigcup\{ U \cap V \mid UdV \text{ and } U \cap V \cap A \neq \emptyset \}.$$

The statement $e(w*)d \prec f$ between dual families will mean that given UdV there exists LfT so that $WSt(e, U \cap V) \subseteq L$ and $WSt(U \cap V, e) \subseteq T$.

We shall say that the dual cover $d = d_0$ is pseudonormal if there is a sequence d_n, $n = 1, 2, \ldots$, of open dual covers so that $d_{n+1} \prec (p*)\ d_n$, $n \in \mathbb{N}$. We will then say that the dual cover d is sequentially pseudonormal if there is a sequence d_n of open dual families, and a sequence e_n of pseudonormal dual covers so that

(i) $e_n(w*)d_n \prec d$, $n = 0, 1, \ldots$, and

(ii) $\bigcup\{d_n\}$ is a dual cover of X.

Clearly a sequentially normal dual cover is sequentially pseuonormal. We may now give:

Theorem 1.5.1. Every sequentially pseudonormal open dual cover of a bitopological space (X, u, v) has a σ-quasi-discrete open compartmental refinement.

Proof. Let $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$ be a sequentially pseudonormal open dual cover, and d_n, e_n, $n \in \mathbb{N}$, as above. Define:

$$A_\alpha^{n0} = \bigcup\{ U \cap V \mid Ud_nV, WSt(e_n, U \cap V) \subseteq U_\alpha \text{ and } WSt(U \cap V, e_n) \subseteq V_\alpha \}.$$

Since $e_n = e^0_n$ is pseudonormal, there are open dual covers e^m_n so that $e^{m+1}_n \prec (p*)\ e^m_n$, $n, m \in \mathbb{N}$.

Now define by induction on m,

$$A_\alpha^{nm} = USt(A_\alpha^{n(m-1)}, e^m_n)$$

for each $\alpha \in A$, $n \in \mathbb{N}$ and $m = 1, 2, \ldots$.

Let $\leq$ be a well ordering of the index set A, and for each $\alpha \in A$, $n, m \in \mathbb{N}$, let

$$C_\alpha^{nm} = A_\alpha^{nm} - \bigcup \{ A_\beta^{n(m+1)} \mid \beta < \alpha \} .$$

Let us show that the sets C_α^{nm} cover X. Now if $x \in X$ we have $U d_n V$ with $x \in U \cap V$ for some $n \in \mathbb{N}$, and hence we have $\alpha \in A$ with $WSt(e_n, U \cap V) \subseteq U_\alpha$ and $WSt(U \cap V, e_n) \subseteq V_\alpha$. Thus $x \in A_\alpha^{n0}$. Let

$$\gamma = \min \{ \alpha \mid \exists\ m,\ x \in A_\alpha^{nm} \} ,$$

and take an m with $x \in A_\gamma^{nm}$. Then for $\beta < \gamma$, we have $x \notin A_\beta^{nk}$ for all k, and so $x \in C_\gamma^{nm}$ from which the required result follows.

Now set $f_\alpha^{nm} = \{ (R, S) \mid R e^{m+2}{}_n S,\ R \cap S \cap C_\alpha^{nm} \neq \emptyset \}$, and $f_{nm} = \{ f_\alpha^{nm} \mid \alpha \in A \}$. This is a countable collection of open compartmental dual families, and their union is a compartmental dual cover of X by the above result.

It remains to show that each f_{nm} is quasi-discrete and refines d. First, it is easy to show by induction on m that the proposition

P(m) : If $R e^m{}_n S$ and $R \cap S \cap A_\alpha^{nm} \neq \emptyset$ then $R \subseteq U_\alpha$ and $S \subseteq V_\alpha$

is true for all $m \in \mathbb{N}$. Hence if $R f_\alpha^{nm} S$ we have $R \subseteq U_\alpha$ and $S \subseteq V_\alpha$ (since $e^{m+2}{}_n \prec e^m{}_n$), and so $f^{nm} \prec \{(U_\alpha, V_\alpha)\}$ which means that $f_{nm} \prec d$. Now for each $x \in X$ choose $R_x e^{m+2}{}_n S_x$ with $x \in R_x \cap S_x$, and suppose that for $\alpha, \beta \in A$ we have $R f^{nm} S$ and $R' f^{nm} S'$ with $R_x \cap S \neq \emptyset \neq S_x \cap R$ and $R_x \cap S' \neq \emptyset \neq S_x \cap R'$. Now $R e^{m+2}{}_n S$, $R' e^{m+2}{}_n S'$ and $e^{m+2}{}_n \prec (p*)\ e^{m+1}{}_n$ so we have $R_0 e^{m+1}{}_n S_0$ with $PSt(e^{m+2}{}_n, (R_x, S_x)) \subseteq R_0$ and $PSt((R_x, S_x), e^{m+2}{}_n) \subseteq S_0$. Suppose $\beta < \alpha$, then $R' \cap S' \cap C_\beta^{nm} \neq \emptyset \Rightarrow R_0 \cap S_0 \cap A_\beta^{nm} \neq \emptyset$

$\Rightarrow R_0 \cap S_0 \subseteq A_{\beta}^{n(m+1)} \Rightarrow R_0 \cap S_0 \cap C_{\alpha}^{nm} = \emptyset \Rightarrow R \cap S \cap C_{\alpha}^{nm}$ $= \emptyset$ which contradicts $R f_{\alpha}^{nm} S$. In the same way $\alpha < \beta$ leads to a contradiction so $\alpha = \beta$ and f_{nm} is quasi-discrete. This completes the proof of the theorem.

It follows, in particular, that every fully pseudonormal bitopological space is strongly compartmentally quasi-biscreenable. However a better result may be obtained, as follows.

<u>Theorem</u> 1.5.2. A fully pseudonormal bitopological space is compartmentally quasi-biparacompact.

<u>Proof</u>. Let d be an open dual cover of X, and let d' be an open dual cover with $d' \prec (p*)\ d$. By Theorem 1.5.1 and Proposition 1.5.2 there is a quasi-locally finite (not necessarily open) compartmental refinement $e'/L = \{ e'_\lambda \mid \lambda \in L \}$ of d'. Let $H(x) \in u$ and $K(x) \in v$ be nhds. of x so that the set

$$\{ \lambda \mid \lambda \in L, \exists\ R' e'_\lambda S' \text{ with } H(x) \cap S' \neq \emptyset \neq K(x) \cap R' \}$$

is finite, and let $f = \{ (H(x), K(x)) \mid x \in X \}$. Finally let g and h be open dual covers with $g \prec (p*)\ h \prec (p*)\ f$ and $g \prec d'$. For $\lambda \in L$ define:

$$e_\lambda = \{ (M, N) \mid MgN, \exists\ R' e'_\lambda S' \text{ with } R' \cap N \neq \emptyset \neq S' \cap M \},$$

and set $e/L = \{ e_\lambda \mid \lambda \in L \}$. Clearly e/L is an open compartmental dual cover, and $e/L \prec d$. To see that e/L is quasi-locally finite associate with $x \in X$ the nhds. $M(x) = \mathrm{PSt}(g, (\{x\}, \{x\}))$ $= \mathrm{WSt}(g, \{x\}) \in u$ and $N(x) = \mathrm{PSt}((\{x\}, \{x\}), g) = \mathrm{WSt}(\{x\}, g) \in v$. Now we have PhQ with $M(x) \subseteq P$, $N(x) \subseteq Q$; and $y \in X$ with $\mathrm{PSt}(h, (P, Q)) \subseteq H(y)$, $\mathrm{PSt}((P, Q), h) \subseteq K(y)$. Hence, if for some $\lambda \in L$ we have $M e_\lambda N$ with $M(x) \cap N \neq \emptyset\ N(x) \cap M$, then for some $R' e' S'$ we have $R' \cap N \neq \emptyset \neq S' \cap M$ and it follows that $R' \cap K(y) \neq \emptyset \neq S' \cap H(y)$. This is possible for only a finite number of λ , and the result follows.

The above theorems establish the essentially reasonable nature of the notions of compartmental strong quasi-biscreenability and compartmental quasi-biparacompactness. While these properties are somewhat on the weak side, it seems likely that they will play an important role in future developments, possibly in combination with other restrictions on the space.

Our final result of this section gives an interesting compartmental refinement property of strongly biscreenable bitopological spaces. To describe this we shall need the following terminology. We will say that the compartmental dual family $e/L = \{ e_\lambda \mid \lambda \in L \}$ is a <u>compartmental pre-dual cover</u> if $\bar{e}/L = \{ \bar{e}_\lambda \mid \lambda \in L \}$ is a compartmental dual cover. Also we shall say that e/L is <u>point singular</u> if for each $x \in X$ the set

$$\{ \lambda \mid x \in R \cap S \text{ for some } R e_\lambda S \}$$

contains at most one element.

We may now state:

<u>Theorem</u> 1.5.3. Let (X, u, v) be strongly biscreenable. Then every open dual cover d has a point singular open compartmental pre-dual cover refinement e/L.

<u>Proof</u>. Let $\bigcup \{d_n\}$ be a σ -discrete open refinement of d. For $x \in X$ define

$$k(x) = \min \{ n \mid x \in uc(\bar{d}_n) \} .$$

Let us set

$$R(x) = X - \bigcup \{ u\text{-cl}\left[\bigcup \{ V' \mid V' \in \operatorname{ran} d_i,\ x \notin u\text{-cl}[V'] \} \right] \mid 1 \le i \le k(x) - 1 \},$$

$$S(x) = X - \bigcup \{ v\text{-cl}\left[\bigcup \{ U' \mid U' \in \operatorname{dom} d_i,\ x \notin v\text{-cl}[U'] \} \right] \mid 1 \le i \le k(x) - 1 \}.$$

Then, since d_i is discrete for each i, we have $R(x) \in u$, $S(x) \in v$; and of course $x \in R(x) \cap S(x)$.

Let $L = \{ (U, V) \mid \exists\ n \text{ with } U d_n V, \text{ and } \exists\ z \in v\text{-cl}[U] \cap u\text{-cl}[V] \text{ with } k(z) = n \}$.

For $(U, V) \in L$ let $e_{(U,V)} = \{ (U \cap R(z), V \cap S(z)) \mid U d_{k(z)} V$

and $z \in v\text{-cl}[U] \cap u\text{-cl}[V]$.

$e_{(U,V)} \neq \emptyset$ by hypothesis. Also, given $x \in X$ we have $Ud_{k(x)}V$ with $x \in v\text{-cl}[U] \cap u\text{-cl}[V]$, and by definition $(U, V) \in L$ and $x \in v\text{-cl}[U \cap R(x)] \cap u\text{-cl}[V \cap S(x)]$ so

$$e/L = \{ e_{(U,V)} \mid (U, V) \in L \}$$

is an open compartmental pre-dual cover. To show it is point singular take $x \in U \cap R(z) \cap V \cap S(z)$, where $Ud_{k(z)}V$ and $z \in v\text{-cl}[U] \cap u\text{-cl}[V]$. We know $k(x) \leq k(z)$, so suppose $k(x) < k(z)$. Now we have $U'd_{k(x)}V'$ with $x \in v\text{-cl}[U'] \cap u\text{-cl}[V']$, and $z \notin v\text{-cl}[U']$ or $z \notin u\text{-cl}[V']$. Hence $(v\text{-cl}[U']) \cap S(z) = \emptyset$ or $(u\text{-cl}[V']) \cap R(z) = \emptyset$. However this means $x \notin S(z)$ or $x \notin R(z)$, which is a contradiction. Hence $k(z) = k(x)$, and since $d_{k(x)}$ is discrete this means that $Ud_{k(x)}V$ is determined uniquely by x. Hence e/L is point singular, and the proof is complete.

1.6 SOME COUNTER-EXAMPLES.

In this section we describe the examples mentioned in the previous sections.

Example 1.6.1. Let X be the closed first quadrant of the Euclidean plane, that is $X = \{ (x, y) \mid x \geq 0, y \geq 0 \}$. Let u consist of $\emptyset$ and all subsets G of X satisfying:

(i) $(x, y) \in G$, $0 < x' \leq x \Rightarrow (x', y) \in G$,

(ii) $(x, y) \in G$, $0 < y \leq y' \Rightarrow (x, y') \in G$, and

(iii) $\exists\, y > 0$ with $(0, y) \in G$.

Clearly u is a topology on X, and so is $v = \{ G^{-1} \mid G \in u \}$. We consider the bitopological space (X, u, v).

(A) (X, u, v) is metrizable.

We use Corollary 2 to Lemma 1.4.1. For $(a, b) \in X$ we define sets $R(a, b)$ as follows.

$$R(a, b) = \begin{cases} \{(x, y) \mid 0 < x \leq a \text{ and } 0 < b \leq y\} \text{ if } a \neq 0 \neq b, \\ \{(x, 0) \mid 0 < x \leq a\} \text{ if } a \neq 0,\ b = 0, \\ \{(0, y) \mid 0 < b \leq y\} \text{ if } a = 0,\ b \neq 0, \\ \{(0, 0)\} \text{ if } a = b = 0. \end{cases}$$

Then, for $n = 1, 2, 3, \ldots$, let $R_n(a, b) = R(a, b) \cup R(0, n)$. $R_n(a, b)$ is a u-open nhd. of (a, b). Finally if $S(a, b) = R(b, a)^{-1}$ and $S_n(a, b) = S(a, b) \cup S(n, 0) = R_n(b, a)^{-1}$ then $S_n(a, b)$ is a v-open nhd. of (a, b).

We now define the open dual covers

$$d_n = \{(U_n(a, b), V_n(a, b)) \mid (a, b) \in X\}$$

as follows:

$$U_n(a, b) = \begin{cases} X \text{ if } b = 0 \text{ and } a \geqslant n, \\ R_n(a, b) \text{ otherwise, and} \end{cases}$$

$$V_n(a, b) = \begin{cases} X \text{ if } a = 0 \text{ and } b \geqslant n, \\ S_n(a, b) \text{ otherwise.} \end{cases}$$

It is clear from the definitions that $d_{n+1} \prec d_n$ for each n. Hence if we can show that $d_n \prec(*)\, d_n$ it will follow that $d_{n+1} \prec(*)$ d_n for each n. Suppose that $U_n(a, b) \cap V_n(c, d) \neq \emptyset$; we wish to show that $U_n(c, d) \subseteq U_n(a, b)$. Consider the following cases:

(a) $U_n(a, b) = X$. The result is then trivial for any $(c, d) \in X$.

(b) $V_n(c, d) = X$. In this case $c = 0$ and $d \geqslant n$ so $U_n(c, d) = R_n(0, d) = R(0, n) \subseteq U_n(a, b)$ for any $(a, b) \in X$.

(c) $U_n(a, b) \neq X \neq V_n(c, d)$. In these circumstances it is easy to verify that $U_n(a, b) \cap V_n(c, d) \neq \emptyset \implies R_n(a, b) \cap S_n(c, d) \neq \emptyset \implies R_n(c, d) \subseteq R_n(a, b)$.

This shows that $St(d_n, U_n(a, b)) = U_n(a, b)$; and in the same way we have $St(V_n(a, b), d_n) = V_n(a, b)$ so $d_n \prec(*)\, d_n$ as required. This verifies condition (1) of Lemma 1.4.1, Corollary 2. To establish conditions (2) and (3) it will suffice to show that if $G \in u$, $H \in v$ and $(a, b) \in G \cap H$ then for some n we have $U_n(a, b) \subseteq G$ and $V_n(a, b) \subseteq H$. However we know that we have $(0, y) \in G$ for some $y > 0$, and $(x, 0) \in H$ for some $x > 0$; and clearly any n with $n > \max\{x, y\}$ will have the required properties.

This shows (X, u, v) is metrizable. In particular (X, u, v) is pairwise normal and preseparated.

(B) <u>(X, u, v) is uniformly discrete</u>.

This is trivial since $R_n(a, b) \cap S_n(a, b) = \{(a, b)\}$ for each n and $(a, b) \in X$.

(C) <u>(X, u, v) is not finitely binormal</u>.

Consider the sets

$G_1 = \{(x, y) \mid y > 0\}$, and $G_2 = \{(x, 0) \mid x \geq 0\} \cup \{(0, y) \mid y \geq 0\}$

Both these sets are u-open, and $G_1 \cup G_2 = X$, so

$$d = \{(G_1, X), (G_2, X)\}$$

is an open dual cover of X. If (X, u, v) were finitely binormal there would be u-open sets R_1, R_2 with $R_1 \cup R_2 = X$, $v\text{-cl}[R_1] \subseteq G_1$ and $v\text{-cl}[R_2] \subseteq G_2$. However if H is any non-empty v-open set there exists $x > 0$ with $(x, 0) \in H$, and $(x, 0) \notin G_1 \Rightarrow (x, 0) \notin R_1 \Rightarrow (x, 0) \in R_2$ so that $v\text{-cl}[R_2] = X$ which contradicts $G_2 \neq X$.

Note that we have not even had to use the fact that R_1 and R_2 are u-open in order to obtain this contradiction.

In particular it follows that (X, u, v) is a metrizable bitopological space which is neither fully binormal nor biparacompact. However:

(D) <u>(X, u, v) has an open quasi-discrete dual cover</u>. <u>In partic- (X, u, v) is quasi-biparacompact</u>.

For $(a, b) \in X$ let $H(a, b) = R(a, b) \cup R(0, a \vee b + 1)$, and $K(a, b) = H(a, b)^{-1}$. Consider:

$g = \{ (H(a, b), K(a, b)) \mid (a, b) \in X \}$.

This is an open dual cover of X, let us show it is quasi-discrete. Suppose that $H(a, b) \cap K(c, d) \neq \emptyset \neq K(a, b) \cap H(c, d)$. Then
$H(a, b) \cap K(c, d) = [R(0, a \vee b + 1) \cap S(c \vee d + 1, 0)] \cup$
$[R(0, a \vee b + 1) \cap S(c, d)] \cup [R(a, b) \cap S(c \vee d + 1, 0)] \cup$
$[R(a, b) \cap S(c, d)]$,
and under the above hypothesis the first three terms of this union are empty so we deduce $R(a, b) \cap S(c, d) \neq \emptyset$. In just the same way we have $R(c, d) \cap S(a, b) \neq \emptyset$, and it follows easily from this that $(a, b) = (c, d)$. Hence g is quasi-discrete as stated.

Now let $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$ be an open dual cover of X, and for $(a, b) \in X$ choose $\alpha(a, b)$ with $(a, b) \in U_{\alpha(a,b)} \cap V_{\alpha(a,b)}$. Then
$e = \{ (U_{\alpha(a,b)} \cap H(a, b), V_{\alpha(a,b)} \cap K(a, b)) \mid (a, b) \in X \}$

is clearly a quasi-discrete open refinement of d. Hence (X, u, v) is quasi-biparacompact and strongly quasi-biscreenable.

(E) <u>(X, u, v) is not pairwise paracompact</u> (In the sense of [10])

For $0 \leq r < 1$ let $U(r) = \{ (x, y) \mid y > 0 \} \cup \{ (x, 0) \mid 0 \leq x \leq r \} \in u$, and consider the pairwise open cover

$$\mathcal{b} = \{ S(1, 0) \} \cup \{ U(r) \mid 0 \leq r < 1 \}.$$

If this had a pairwise locally finite open refinement then there would be a v-open nhd N of (1/2, 0) meeting only finitely many u-open sets $U_{\alpha(1)}, \ldots, U_{\alpha(n)}$ in this refinement. We shall have $U_{\alpha(i)} \subseteq U(r_i)$, $i = 1, \ldots, n$; and if $\max\{r_1, \ldots, r_n\} < r < 1$, then there must be a u-open set U_α in the refinement with $(r, 0) \in U_\alpha$. However $(r, 0) \in U_\alpha \cap N$, and $\alpha \neq \alpha(i)$, $1 \leq i \leq n$, which is a contradiction.

<u>Example</u> 1.6.2. Consider the space (X, u, v) defined as in Example 1.6.1 with the exception that condition (iii) is removed.

(A) <u>(X, u, v) is metrizable and fully binormal</u>.

Note that in this space R(a, b) (respectively, S(a, b)) is the smallest u-open (respectively, v-open) set containing (a, b) for each $(a, b) \in X$. Hence the open dual cover

$$d_0 = \{ (R(a, b), S(a, b)) \mid (a, b) \in X \}$$

refines all open dual covers of X. Moreover, argueing as in the last example, we see that $d_0 \prec (*) \, d_0$ and the above mentioned properties are now immediate.

(B) <u>(X, u, v) is neither biparacompact nor strongly biscreenable</u>.

We show that there is no sequence d_n, n = 1, 2, ..., of locally finite open dual families each of which refines d_0, and whose union is a dual cover of X. Suppose that such d_n do exist. Since $d_n \prec d$ we must have $d_n = \{ (R(a, b), S(a, b)) \mid (a, b) \in X_n \}$ where $X_n \subseteq X$. Also since $R(a, b) \cap S(a, b) = \{ (a, b) \}$ we see that $\cup \{ X_n \} = X$. The set R(1, 1) is uncountable, so $X_n \cap R(1, 1)$ is infinite for at least one n, and $(c, d) \in X_n \cap R(1, 1)$ implies $S(c, d) \in \operatorname{ran} d_n$ and $S(c, d) \cap R(1, 1) \neq \emptyset$. Finally $(c, d) \neq (c', d')$ implies $S(c, d) \neq S(c', d')$ so R(1, 1) meets infinitely many different sets in ran d_n for this n. Since R(1, 1) is the smallest u-nhd. of (1, 1) this contradicts the locall finiteness of d_n.

(C) <u>(X, u, v) is quasi-biparacompact and strongly quasi-biscreenable.</u>

d_0 is easily seen to be an open quasi-discrete dual cover of X.

(D) <u>(X, u, v) is not pairwise paracompact</u>.

The proof is just as in Example 1.6.1, (E).

<u>Example</u> 1.6.3. Let $X = R$, $u = \{ \{ x \mid x < a \} \mid a \in \mathbb{R} \} \cup \{ R, \emptyset \}$ and $v = \{ \{ x \mid x > a \} \mid a \in \mathbb{R} \} \cup \{ R, \emptyset \}$.

(In the remaining sections of this thesis this space will be invariably denoted by $(\mathbb{R}, s, t)$)

(A) <u>(X, u, v) is metrizable</u>.

The required p-q-metric is $p(x,y) = (y - x) \vee 0$.

(B) <u>(X, u, v) is fully binormal</u>.

Let $d = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$ be an open dual cover of X. Since (X, u, v) is uniformly fully normal (the uniform topology being the usual topology of R) there is an open covering of X by bounded open intervals which is a star refinement of $\{ U_\alpha \cap V_\alpha \mid \alpha \in A \}$. Hence we have an open dual cover $e = \{ (R_\beta, S_\beta) \mid \beta \in B \}$ so that $\{ R_\beta \cap S_\beta \mid \beta \in B \} \prec(*) \{ U_\alpha \cap V_\alpha \mid \alpha \in A \}$. However it is easy to deduce from this that $e \prec(*)$ d, and the result follows.

(C) <u>(X, u, v) is strongly biscreenable</u>.

This is an immediate consequence of the fact that (X, u, v) is uniformly Lindelöf.

(D) <u>(X, u, v) is not biparacompact</u>.

Indeed neither topology is paracompact with respect to the other. For if we consider, for example, the u-open cover $\{ U(k) \mid k \in \mathbb{Z} \}$, where $U(k) = \{ x \mid x < k \}$ it is clear that this cannot have a v-locally finite u-open refinement.

(E) <u>(X, u, v) is pairwise paracompact but not strongly pairwise paracompact</u>.

Any pairwise open cover $\mathcal{b}$ will contain at least one v-open set $V(r) = \{ x \mid x > r \}$, and at least one u-open set $U(s) = \{ x \mid x < s \}$. Indeed there must be such sets in $\mathcal{b}$ with $r < s$, for otherwise $q \in X$ satisfying

$$\sup \{ s \mid U(s) \in \mathcal{b} \} \leq q \leq \inf \{ r \mid V(r) \in \mathcal{b} \}$$

is contained in no set of $\mathcal{b}$, and in this event $\{ U(s), V(r) \}$ is a finite sub-cover of $\mathcal{b}$. (This expresses, of course, the well known pairwise compactness property of this space).

That (X, u, v) is not strongly pairwise paracompact follows as in (D).

(F) <u>(X, u, v) is quasi-biparacompact</u>.

Let $d' = \{ (U_\alpha, V_\alpha) \mid \alpha \in A \}$ be an open dual cover of X, and

consider the uniform open cover $\{ U_\alpha \cap V_\alpha \mid \alpha \in A \} = \mathcal{b}$.
Using well known properties of the real numbers we may show the existance of real numbers a_k, b_k, p_k, q_k, $k \in \mathbb{Z}$, satisfying

$b_k < a_k$, $q_k < p_k$, $[b_k, a_k] \subseteq (q_k, p_k)$, $a_k \leq a_{k+1}$, $b_k \leq b_{k+1}$

$p_k \leq p_{k+1}$ and $q_k \leq q_{k+1}$ for each $k \in \mathbb{Z}$, and so that the closed intervals $[b_k, a_k]$ cover $\mathbb{R}$, and the open intervals (q_k, p_k) refine $\mathcal{b}$. If now we set

$$A_k = \{ x \mid x \leq a_k \}, \quad B_k = \{ x \mid x \geq b_k \},$$
$$U_k = \{ x \mid x < p_k \}, \quad V_k = \{ x \mid x > q_k \}$$

then we see that $d = \{ (U_k, V_k) \mid k \in \mathbb{Z} \}$ and $c = \{ (A_k, B_k) \mid k \in \mathbb{Z} \}$ satisfy the conditions of Lemma 1.3.2. Hence d has a quasi-locally finite open refinement, and since $d \prec d'$ the result is proved.

<u>Example</u> 1.6.4. Let $X = \mathbb{N} \cup \{w\}$, $U_n = \{ 0, 1, 2, \ldots, n \}$, $U_{\infty} = \cup \{ U_n \mid n \in \mathbb{N} \}$ and $V_n = \{ n, n+1, \ldots \} \cup \{w\}$.
Consider the bitopological space (X, u, v) where
$u = \{ U_0, U_1, \ldots ; U_\infty, X, \emptyset \}$ and $v = \{ V_0, V_1, \ldots ; \emptyset \}$.

(A) <u>(X, u, v) is metrizable and uniformly compact</u>. <u>In particular it is biparacompact and fully binormal</u>.

For $m \in \mathbb{N}$ consider the finite open dual cover

$$d_m = \{ (U_n, V_n) \mid 0 \leq n \leq m \} \cup \{ (X, V_{m+1}) \}$$

If d is any open dual cover then $d_m \prec d$ for a suitable m, so (X, u, v) is uniformly compact. Also $d_{m+1} \prec d_m$, and $d_m \prec(*) \, d_m$ for each m so (X, u, v) is metrizable.

<u>Example</u> 1.6.5. Let $(X', \mathcal{T})$ be any non-paracompact topological space, and take $p, q \notin X'$. On $X = X' \cup \{p\} \cup \{q\}$ let u be the topology with base $\mathcal{T} \cup \{\{p\}\} \cup \{X\}$, and v the topology with

base $\mathfrak{J} \cup \{\{q\}\} \cup \{X\}$ ~~, and v the topology with base $\mathfrak{J} \cup \{\{q\}\} \cup \{X\}$~~. Consider the bitopological space (X, u, v).

(A) <u>(X, u) and (X, v) are compact. In particular each topology is paracompact with respect to the other</u>.

X is the only u-open set containing q, and the only v-open set containing p.

(B) <u>(X, u, v) is not quasi-biparacompact</u>.

Let $\mathcal{G} = \{ G_\alpha \mid \alpha \in A \}$ be a $\mathfrak{J}$ -open covering of X' with no $\mathfrak{J}$ -open $\mathfrak{J}$ -locally finite refinement, and consider the open dual cover

$$d = \{(\{p\}, X)\} \cup \{(X, \{q\})\} \cup \{(G_\alpha, G_\alpha) \mid \alpha \in A\}$$

of X. If this had an open quasi-locally finite refinement $e = \{(R_\beta, S_\beta) \mid \beta \in B\}$ then

$$\{ R_\beta \cap S_\beta \mid \beta \in B,\ R_\beta \cap S_\beta \cap X' \neq \emptyset \}$$

would be a $\mathfrak{J}$ -open $\mathfrak{J}$ -locally finite refinement of $\mathcal{G}$ in X', which is impossible.

<u>Example</u> 1.6.6. Let X be the open upper half plane, that is $X = \{(x, y) \mid y > 0\}$. For $P = (p_1, p_2) \in X$ let

$$U(P) = \{ ([y(p_1 + 1) - p_2]/p_2,\ y) \mid 0 < y \leq p_2 \},$$

$$V(P) = \{ ([y(p_1 - 1) + p_2]/p_2,\ y) \mid 0 < y \leq p_2 \}.$$

Let u be the topology on X with base $\{ U(P) \mid P \in X \}$, and v the topology with base $\{ V(P) \mid P \in X \}$. Consider the bitopological space (X, u, v).

(A) <u>(X, u, v) is fully pseudonormal, quasi-biparacompact and strongly quasi-biscreenable</u>.

Note that U(P) is the smallest u-open set containing P, and V(P) is the smallest v-open set containing P. Hence

$$d_0 = \{ (U(P), V(P)) \mid P \in X \}$$

refines every open dual cover d. It is also clear that if $U(P) \cap V(Q) \neq \emptyset$ and $V(P) \cap U(Q) \neq \emptyset$ then P = Q. Hence $d_0 < (p*)\ d_0$

and d_0 is quasi-discrete, from which the above stated properties follow at once.

(B) <u>(X, u, v) is not pairwise R_0 (and hence, in particular, not pairwise regular)</u>

For $P \in X$ we have $v\text{-cl}[P] = \{ Q \mid Q = (q_1, q_2), q_1 = [q_2(p_1 - 1) + p_2]/p_2, q_2 \geqslant p_2 \} \not\subseteq U(P)$.

(C) <u>(X, u, v) is not pairwise normal</u>.

Let $P = (1, 1)$ and $Q = (-1, 1)$. Then $v\text{-cl}[P] = \{(1, y) \mid y \geqslant 1\}$ and $u\text{-cl}[Q] = \{(-1, y) \mid y \geqslant 1\}$ so these sets are disjoint. However $(0, 1/2) \in U(P) \cap V(Q) \neq \emptyset$.

<u>Example</u> 1.6.7. With the set X and the notation as in Example 1.6.6, this time let u have base

$$\{ U(P) - \{P_1, \ldots, P_n\} \mid P \in X, P_1, \ldots, P_n \in X, P_i \neq P \},$$

and make a corresponding change to the base of v.

(A) <u>(X, u, v) is fully pseudonormal, quasi-biparacompact and strongly quasi-biscreenable.</u>

We need only make rather obvious modifications to the argument used in Example 1.6.1 (A).

(B) <u>(X, u, v) is pairwise completely regular</u>.

For $Q \in X$ define the sets

$$R(Q) = \cup \{ U(T) \mid T = (t_1, t_2) \in X,\ t_1 = \frac{t_2(q_1 - 1) + q_2}{q_2},\ t_2 \geqslant q_2 \},$$

$$S(Q) = \cup \{ V(T) \mid T = (t_1, t_2) \in X,\ t_1 = \frac{t_2(q_1 + 1) - q_2}{q_2},\ t_2 \geqslant q_2 \}.$$

Clearly $R(Q) \in u$ and $S(Q) \in v$. Now let P be a fixed point of X, and let $H \in u$, $K \in v$ be nhds. of P. Then for some finite set P_1, $P_2, \ldots, P_n$ of X with $P_i \neq P$, $1 \leq i \leq n$, we have

$U'(P) = U(P) - \{P_1, \ldots, P_n\} \subseteq H$, $V'(P) = V(P) - \{P_1, \ldots, P_n\}$

$\subseteq K$.

Note that $R(Q) - V'(P) \in u$ for each $Q \in X$ since it is the union of sets of the form $U(T) - U(T) \cap V'(P)$; and $U(T) \cap V'(P)$ contains at most one element of X. Likewise $S(Q) - U'(P) \in v$. Now consider the open dual cover:

$$d = \{ (U'(P), V'(P)) \} \cup \{ (R(Q) - V'(P), S(Q) - U'(P)) \mid Q \in X, Q \notin U'(P) \cup V'(P) \} \cup \{ (R(Q), V(Q) \cap V'(P)) \mid Q \in V'(P), P \neq Q \} \cup \{ (U(Q) \cap U'(P), S(Q)) \mid Q \in U'(P), P \neq Q \}.$$

Clearly $St(d, P) = U'(P) \subseteq H$ and $St(P, d) = V'(P) \subseteq K$. Moreover d is normal, for indeed we may show by direct computation that $d \prec_{(*)} d$. It follows by Proposition 1.7.1 that (X, u, v) is pairwise completely regular.

(C) <u>(X, u, v) is pairwise Hausdorff.</u>

If $P \neq Q$ in X then $U(P) - V(Q)$ is a u-nhd. of P, and it is disjoint from the v-nhd. $V(Q)$ of Q.

(D) <u>(X, u, v) is not pairwise normal.</u>

Consider the u-closed set $F = \{ (-1, y) \mid y > 0 \}$, and the v-closed set $T = \{ (1, y) \mid y > 0 \}$. Clearly $F \cap T = \emptyset$. Let $K \in v$ contain F and $H \in u$ contain T. Without loss of generality we may suppose H and K lie in the set

$$Y = \{ (x, y) \mid -1 \leq x \leq 1, y > 0 \}.$$

Let us show that the set

$$E = \{ A \mid A \in F, |V(A) \cap H| < \infty \}$$

is at most finite. Suppose E contains an infinite sequence of distinct elements A_n; then the set $\bigcup \{ V(A_n) \cap H \mid n \in \mathbb{N} \}$ is a countable subset of Y, and since T is uncountable there exists a point $B \in T$ so that $U(B)$ contains non of the points of $\bigcup \{ V(A_n) \cap H \mid n \in \mathbb{N} \}$. However, for some $P_1, \ldots, P_m \in X$, $U'(B) = U(B) - \{ P_1, \ldots, P_m \} \subseteq H$ and so for some $n \in \mathbb{N}$, $V(A_n) \cap U'(B) \neq \emptyset$, which is a contradiction. Hence E is finite and we may choose $A \in F - E$. But then for some $Q_1, \ldots, Q_k \in X$ the set $V(A) - \{ Q_1, \ldots, Q_k \}$ is contained in K and meets H, so $H \cap K \neq \emptyset$ and (X, u, v) cannot be pairwise normal.

Example 1.6.8. Let $A = (-1, 0)$, $B = (1, 0)$ and $X = \{A, B\} \cup \{(x, y) \mid y > 0\}$. For $P \in X - \{A, B\}$ let $U(P)$, $V(P)$ be as in Example 1.6.6, and let u have base

$$\{U(P) \mid P \in X - \{A, B\}\} \cup \{\{A\}, X\},$$

and v have base

$$\{V(P) \mid P \in X - \{A, B\}\} \cup \{\{B\}, X\}.$$

Then:

(A) <u>(X, u) and (X, v) are compact</u>.

The proof is trivial.

(B) <u>(X, u, v) is fully pseudonormal, quasi-biparacompact and strongly quasi-biscreenable</u>.

We need only consider the open dual cover

$$d_0 = \{(U(P), V(P)) \mid P \in X - \{A, B\}\} \cup \{(\{A\}, X), (X, \{B\})\}.$$

(C) <u>(X, u, v) is not pairwise R_0 or pairwise normal</u>.

This may be proved as in Example 1.6.6 (B) and (C).

Example 1.6.9. With X, $U(P)$, $V(P)$ as in Example 1.6.8, let u have base

$$\{U(P) \cup \{A\} - \{P_1, \ldots, P_n\} \mid P, P_i \in X - \{A, B\}, P \neq P_i\}$$
$$\cup \{\{A\}, X\},$$

and let v have base

$$V(P) \cup \{B\} - \{P_1, \ldots, P_n\} \mid P, P_i \in X - \{A, B\}, P \neq P_i\}$$
$$\cup \{\{B\}, X\}.$$

Then:

(A) <u>(X, u) and (X, v) are compact</u>.

The proof is trivial.

(B) <u>(X, u, v) is fully pseudonormal, quasi-biparacompact and strongly quasi-biscreenable</u>.

We need only modify the proof of Example 1.6.8 (B) in the obvious way.

(C) <u>(X, u, v) is pairwise completely regular</u>.

If the given point P lies in $X - \{A, B\}$ it is easy to see how we should modify the open dual cover d of Example 1.6.7 (B). On the other hand for $P = A$ or $P = B$ we may consider in its place the dual cover

$$d = \{ (\{A\}, X), (X, \{B\}) \} \cup \{ (R(Q) \cup \{A\}, S(Q) \cup \{B\}) \mid Q \in X - \{A, B\} \} .$$

(D) <u>(X, u, v) is not pairwise R_1</u>.

$u\text{-cl}[A] = X \neq \{P, A\} = v\text{-cl}[P]$ for $P \in X - \{A, B\}$, but X is the only v-open set containing A and this meets every u-open set containing P.

(E) <u>(X, u, v) is not pairwise normal</u>.

We may use essentially the same proof as in Example 1.6.7 (D). Note that this example disproves ([23], Theorem 4.20).

<u>Example 1.6.10</u>. Let $X = \{ (x, y) \mid y > 0 \}$, and for $P \in X$ let
$M(P) = \{ ([y(p_1 + 1) - p_2]/p_2, y) \mid y > 0 \}$, and
$N(P) = \{ ([y(p_1 - 1) + p_2]/ p_2, y) \mid y > 0 \}$.
Let u have base $\{ M(P) \mid P \in X \}$ and v have base $\{ N(P) \mid P \in X \}$. Then:

(A) <u>(X, u, v) is uniformly discrete (and hence, in particular, uniformly paracompact.</u>

Trivial since $M(P) \cap N(P) = \{P\}$.

(B) <u>(X, u, v) is not fully pseudonormal</u>.

$d_0 = \{ (M(P), N(P)) \mid P \in X \}$ refines all open dual covers, but clearly $d_0 \not< (p*) \; d_0$.

(C) <u>(X, u, v) is neither compartmentally quasi-biparacompact nor compartmentally quasi-biscreenable</u>.

Suppose there is a sequence of quasi-locally finite open compartmental dual families

$$d_n/L_n = \{ d^n_\gamma \mid \gamma \in L_n \}$$

with $d_n/L_n \prec d_0$ and $\bigcup \{ d_n/L_n \mid n \in \mathbb{N} \}$ a compartmental dual cover of X.

Then it is clear that for $P \in X$ we must have $n(P) \in \mathbb{N}$ and $\gamma(P) \in L_{n(P)}$ satisfying

$$d^{n(P)}_{\gamma(P)} = \{ (M(P), N(P)) \} \quad \ldots.. (1).$$

Let $X_n = \{ P \mid n(P) = n \}$ and $L = \{ (x, 1) \mid -2 < x < 2 \}$. L is an uncountable set, and $\bigcup \{ X_n \mid n \in \mathbb{N} \} = X$, so for some $m \in \mathbb{N}$ the set $L \cap X_m$ is infinite. It follows from (1) that the subset

$$L'_m = \{ \gamma(P) \mid P \in L \cap X_m \}$$

of L_m is infinite also. However

$L'_m \subseteq \{ \gamma \mid \gamma \in L_m, \exists \; Md^m_\gamma N$ with $M \cap N((0, 1)) \neq \emptyset \neq N \cap M((0, 1))\}$

and this contradicts the quasi-locall finiteness of d_m/L_m. Hence no such sequence of compartmental dual families exists, and the stated properties follow at once.

Note that we could clearly modify Example 1.6.10 in the same way that we modified Example 1.6.6 to produce Examples 1.6.7 - 1.6.9, and with very much the same result.

1.7. QUASI-UNIFORM BITOPOLOGICAL SPACES AND GENERALIZATIONS.

In this section we are going to discuss, very briefly, some structures on a set which can be defined using dual covers, and which give rise to a bitopological space in a natural way. The first, and by far the most important, of these is the quasi-uniform structure introduced by A CSASZAR [8] in 1960. There is quite an extensive literature on this subject (see, for example, the book of MURDESHWAR and NAIMPALLY [23] for a survey of some of the earlier work in the field), and our aim here is limited to the consideration of one or two aspects of the theory where our notion of dual cover seems particularly relevant.

In terms of dual covers the definition of a quasi-uniformity may be expressed as follows:

Definition 1.7.1. Let $\mathcal{S}$ be a non-empty collection of dual covers of the set X. Then $\mathcal{S}$ is a <u>(dual covering) quasi-uniformity</u> if:

(i) Given $d \in \mathcal{S}$ there exists $e \in \mathcal{S}$ with $e \prec(*) d$.

(ii) $d, e \in \mathcal{S} \Rightarrow d \wedge e \in \mathcal{S}$.

(iii) If $d \in \mathcal{S}$ and e is a dual cover with $d \prec e$ then $e \in \mathcal{S}$.

<u>Note 1</u>. The notions of <u>base</u> and <u>subbase</u> may be defined in the obvious way.

<u>Note 2</u>. To obtain the corresponding quasi-uniformity in diagonal form we need only consider the sets $W(d) = \bigcup \{ V \times U \mid UdV \}$ for $d \in \mathcal{S}$.

<u>Note 3</u>. Since, as we have noted, a dual cover corresponds to a strong conjugate pair of covers, a dual covering quasi-uniformity as defined above actually corresponds to a base of a covering quasi-uniformity in the sense of GANTNER and STEINLAGE [15]. However these notions are essentially equivalent.

A quasi-uniformity $\mathcal{S}$ gives rise to the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, where $\{ St(d, x) \mid d \in \mathcal{S} \}$ is a base of nhds. of x for the topology $t_u(\mathcal{S})$, and $\{ St(x, d) \mid d \in \mathcal{S} \}$ is a base of nhds. of x for the topology $t_v(\mathcal{S})$.

These topologies may also be described in terms of closure, as follows:

$x \in t_u(\mathcal{S})\text{-cl}[A] \Leftrightarrow$ given $d \in \mathcal{S}$ $\exists$ UdV with $x \in V$, $U \cap A \neq \emptyset$,

$x \in t_v(\mathcal{S})\text{-cl}[A] \Leftrightarrow$ given $d \in \mathcal{S}$ $\exists$ UdV with $x \in U$, $V \cap A \neq \emptyset$.

The following results are easily verified (see also [15])

<u>Lemma 1.7.1</u>. For $d \in \mathcal{S}$ and $A \subseteq X$ we have

$t_v(\mathcal{S})\text{-cl}[A] \subseteq St(d, A)$, $\quad t_u(\mathcal{S})\text{-cl}[A] \subseteq St(A, d)$,

$A \subseteq t_u(\mathcal{S})\text{-int}[St(d, A)]$, $\quad A \subseteq t_v(\mathcal{S})\text{-int}[St(A, d)]$.

<u>Corollary</u>. A quasi-uniformity $\mathcal{S}$ has a base of open dual covers, and a base of closed dual covers.

<u>Example</u> 1.7.1. If p is a p-q-metric compatible with (X, u, v) then $d_\varepsilon = \{ (H(x, \varepsilon), K(x, \varepsilon)) \mid x \in X \}$ is an open dual cover,

and $d_{\varepsilon/3} <(*) d$ for each $\varepsilon > 0$. Hence $\{ d_\varepsilon \mid \varepsilon > 0 \}$ is a base for a quasi-uniformity on X which is clearly compatible with (X, u, v). This corresponds, of course, to the usual diagonal quasi-uniformity defined by a p-q-metric.

As mentioned we will denote the space of Example 1.6.3 by $(\mathbb{R}, s, t)$. For this space we shall write $M(x, \varepsilon)$ in place of $H(x, \varepsilon)$ and $N(x, \varepsilon)$ in place of $K(x, \varepsilon)$, so $M(x, \varepsilon) = \{ y \mid y < x + \varepsilon \}$ and $N(x, \varepsilon) = \{ y \mid y > x - \varepsilon \}$. We shall also set $m_\varepsilon = \{ (M(x, \varepsilon), N(x, \varepsilon)) \mid x \in \mathbb{R} \}$, and denote by μ the quasi-uniformity of which these dual covers are a base.

If for n = 1, 2, ..., we set $m(\alpha, n) = \{ (M(k/n, \alpha/n), N(k/n, \alpha/n)) \mid k \in Z \}$ and $\mu(\alpha) = \{ m(\alpha, n) \mid n = 1, 2, \ldots \}$ then for each fixed α with $1/2 < \alpha < 1$ it is clear that $\mu(\alpha)$ is a countable base of μ consisting of countable open dual covers.

If $\mathcal{S}$ is a quasi-uniformity compatible with (X, u, v), and $\mathcal{S}'$ a quasi-uniformity compatible with (X', u', v') then a function $f:X \to X'$ is quasi-uniformly continuous if for each $d' \in \mathcal{S}'$ we have $f^{-1}(d') \in \mathcal{S}$, where

$$f^{-1}(d') = \{ (f^{-1}(U'), f^{-1}(V')) \mid U'dV', f(X) \cap U' \cap V' \neq \emptyset \}.$$

This clearly aqrees with the usual definition of quasi-uniform continuity [23].

<u>Example</u> 1.7.2. Let (X, u, v) be a bitopological space, and S a set of real-valued pairwise continuous functions on X. By the initial quasi-uniformity generated by S we shall mean the quasi-uniformity qu(S) with subbase

$$\{ f^{-1}(m_\varepsilon) \mid f \in S, \varepsilon > 0 \}.$$

This is the smallest quasi-uniformity on X for which the functions in S are quasi-uniformly continuous with respect to μ. Clearly qu(S) will be compatible with (X, u, v) if and only if the functions in S define the topologies u and v, and such an S exists if and only if (X, u, v) is pairwise completely regular.

It will be clear from the corollary to Lemma 1.7.1 that all the dual covers belonging to a quasi-uniformity $\mathcal{S}$ are normal dual covers for the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$. Conversely for the bitopological space (X, u, v) let $\mathcal{S}_n$ denote the set of all normal dual covers of X. It is trivial to verify that $\mathcal{S}_n$ is a quasi-uniformity, and that $t_u(\mathcal{S}_n) \subseteq u$, $t_v(\mathcal{S}_n) \subseteq v$. With regard to equality we have:

<u>Proposition</u> 1.7.1. The following are equivalent for the bitopological space (X, u, v).

(i) (X, u, v) is pairwise completely regular.

(ii) (X, u, v) has a compatible quasi-uniformity.

(iii) Given $H \in u$, $K \in v$ and $x \in H \cap K$ there exists an open normal dual cover d with $St(d,\{x\}) \subseteq H$, $St(\{x\}, d) \subseteq K$.

(iv) $\mathcal{S}_n$ is compatible with (X, u, v).

Using Lemma 1.4.1 the proof is straight-forward, and will be omitted. The equivalence of (i) and (ii) is well known, see for example [20] .

It is clear that $\mathcal{S}_n$ is the largest quasi-uniformity compatible with a pairwise completely regular bitopological space (X, u, v). It will contain all the open dual covers on X if and only if (X, u, v) is fully binormal.

The following definition is useful in discussing $\mathcal{S}_n$.

<u>Definition</u> 1.7.2. The dual cover d of X is <u>divisible</u> if there is a $v \times u$-open nhd. W of the diagonal in $X \times X$ so that

$$W \circ W \subseteq W(d).$$

If we are given a $v \times u$-nhd. of the diagonal in $X \times X$ we may form open dual covers $d(W)$, $e(W)$ as follows:

$$d(W) = \{ (W_u(x), W_v(x)) \mid x \in X \},$$

where $W_u(x) = \{ y \mid (x, y) \in W \}$ and $W_v(x) = \{ y \mid (y, x) \in W \}$; and

$$e(W) = \{ (R, S) \mid R \in u, S \in v, R \cap S \neq \emptyset \text{ and } S \times R \subseteq W \}.$$

We then call the dual cover d <u>even</u> if there is a $v \times u$-nhd. W of the diagonal in $X \times X$ so that

$$d(W) \prec d.$$

These terms are the natural counterparts for dual covers of the corresponding terms as applied to covers of a topological space (see, for example, [4] or [18]).

Since

$$d(W) \prec d \implies W \circ W \subseteq W(d)$$

we see that every even dual cover is divisible. In general the converse is false, however, for consider the open dual cover $d = \{ (G_1, X), (G_2, X) \}$ of Example 1.6.1 (C). d is divisible since $W(d) = X \times X$. However, for any nhd. W of the diagonal we have, say, $((1, 1), (1, 1)) \in W$ so we have $U \in u$, $V \in v$ with $((1, 1), (1, 1)) \in V \times U \subseteq W$. Now $\exists$ $a > 0$ with $(a, 0) \in V$ and so $(1, 1) \in W_u((a, 0))$. It follows that $W_u((a, 0)) \nsubseteq G_1$ and $W_u((a, 0)) \nsubseteq G_2$, that is $d(W) \not\prec d$ so d is not even.

It is a trivial matter to verify that

$$e \prec(\Delta)\, d \iff d(W(e)) \prec d$$

and that $W(e(W)) \subseteq W$ for any $v \times u$-nhd. W of the diagonal, and we deduce that (X, u, v) is fully binormal if and only if every open dual cover is even. With regard to divisibility we have:

<u>Theorem</u> 1.7.1. Let (X, u, v) be pairwise completely regular. Then the following are equivalent.

(a) The largest compatible (diagonal) quasi-uniformity contains all the $v \times u$-nhds. of the diagonal in $X \times X$.

(b) Every open dual cover is divisible.

(c) If d is an open dual cover there exists an open dual cover e with $W(e^{\Delta}) \subseteq W(d)$.

<u>Proof</u>. (a) $\implies$ (b). Let d be an open dual cover. Then $W(d)$ is a $v \times u$-nhd. of the diagonal so there is a normal open dual cover e with $W(e) \subseteq W(d)$. If then f is an open dual cover with $f \prec(*)\, e$ then $W(f)$ is a $v \times u$-nhd. of the diagonal, and $W(f) \circ W(f) \subseteq W(d)$, so d is divisible.

(b) $\implies$ (c). Let d be an open dual cover. Since d is divisible there is a $v \times u$-nhd. W of the diagonal with $W \circ W \subseteq W(d)$. If we set $e = e(W)$ it is easy to verify that $W(e^{\Delta}) \subseteq W(d)$.

(c) $\Rightarrow$ (a). Let W be a $v \times u$-nhd. of the diagonal. Using (c) we have open dual covers d, f with $W(d^{\Delta}) \subseteq W(e(W))$ and $W(f^{\Delta}) \subseteq W(d)$. But then $W(f^{*}) \subseteq W(e(W))$, and so

$$W(f)\circ W(f) \subseteq W(e(W)) \subseteq W.$$

Hence the largest compatible (diagonal) quasi-uniformity contains all the $v \times u$-nhds. of the diagonal.

By what we have said above, any pairwise R_0 fully binormal space will satisfy the conditions of Theorem 1.7.1. However we may considerably improve this result as follows.

<u>Theorem</u> 1.7.2. Let (X, u, v) be a pairwise R_0 sequentially normal bitopological space. Then every open dual cover of X is divisible.

<u>Proof</u>. Since (X, u, v) is pairwise completely regular we need only verify (c) of Theorem 1.7.1. Let d be an open dual cover. By Theorem 1.4.2 there is an admissible p-q-metric p subordinate to d. Hence, given $x \in X$, there exists $r(x)$ with $0 < r(x) < 1$ and so that $H(x, r(x)) \subseteq U$, $K(x, r(x)) \subseteq V$ for some UdV. Let $r'(x) = r(x)/4$,

$$d' = \{ (H(x, r(x)), K(x, r(x))) \mid x \in X \} < d, \text{ and}$$

$$e = \{ (H(x, r'(x)), K(x, r'(x))) \mid x \in X \}.$$

Let us show that $W(e^{\Delta}) \subseteq W(d)$. Take $(x, y) \in W(e^{\Delta})$, then $\exists$ $z \in X$ with $x \in St(\{z\}, e)$ and $y \in St(e,\{z\})$. Hence we have a, b $\in X$ with $x \in K(a, r'(a))$, $z \in H(a, r'(a))$ and $y \in H(b, r'(b))$, $z \in K(b, r'(b))$. Let

$$s = \sup \{ r(x') \mid H(a, r'(a)) \cap K(x', r'(x')) \neq \emptyset \neq K(b, r'(b)) \cap H(x', r'(x')) \}.$$

Note that $r(a) \leq s$ and $r(b) \leq s$. Now choose $x_0 \in X$ so that $H(a, r'(a)) \cap K(x_0, r'(x_0)) \neq \emptyset \neq K(b, r'(b)) \cap H(x_0, r'(x_0))$ and $r(x_0) > 2s/3$. Then

$$p(x,x_0) \leq 2r'(a) + r'(x_0) < r(x_0), \text{ and}$$

$$p(x_0,y) \leq 2r'(b) + r'(x_0) < r(x_0). \text{ Hence}$$

$$(x, y) \in K(x_o, r(x_o)) \times H(x_o, r(x_o)) \subseteq W(d') \subseteq W(d)$$

as required.

Bitopological spaces having the properties of Theorem 1.7.1 satisfy a condition which corresponds to the property of collective normality for topological spaces (cf [4] , IX § 4 Exercise 18). Let us make the following definition.

<u>Definition</u> 1.7.3. (X, u, v) is <u>collectively binormal</u> if whenever $c = \{(P_\alpha, Q_\alpha) \mid \alpha \in A\}$ is a discrete closed dual family there is an open dual family $d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ with $P_\alpha \subseteq U_\alpha$, $Q_\alpha \subseteq V_\alpha$ for all $\alpha \in A$, and $U_\alpha \cap V_\beta = \emptyset$ for all $\alpha \neq \beta$.

It is clear that a collectively binormal space is pairwise normal.

<u>Theorem</u> 1.7.3. Let (X, u, v) be a bitopological space satisfying the conditions of Theorem 1.7.1. Then (X, u, v) is collectively binormal.

<u>Proof</u>. For $c = \{(P_\alpha, Q_\alpha) \mid \alpha \in A\}$ as in Definition 1.7.3 let us set $R_\alpha = X - \bigcup\{Q_\beta \mid \beta \neq \alpha\}$ and $S_\alpha = X - \bigcup\{P_\beta \mid \beta \neq \alpha\}$. Since c is discrete, $R_\alpha \in u$ and $S_\alpha \in v$. Also

$$e = \{(R_\alpha, S_\alpha) \mid \alpha \in A\}$$

is clearly an open dual cover of X. Now applying Theorem 1.7.1 (c) twice we have an open dual cover f with $W(f^*) \subseteq W(d)$. Let us set

$$U_\alpha = St(f, P_\alpha),\ V_\alpha = St(Q_\alpha, f).$$

Then it is easy to verify that $d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ has all the properties required by Definition 1.7.3.

In particular we see that every pairwise R_0 sequentially normal, and hence every metrizable bitopological space is collectively binormal. Example 1.6.1 therefore shows that a collectively binormal bitopological space need not be finitely binormal.

To discuss some further properties of δ_n we shall need:

Definition 1.7.4. By a bifilter on the set X we will mean a product

$$\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$$

of two filters $\mathcal{B}_u$ and $\mathcal{B}_v$ on X.

The bifilter $\mathcal{B}$ is 1-regular if $F \cap G \neq \emptyset$ whenever $(F, G) \in \mathcal{B}$.

If (X, u, v) is a bitopological space, and $x \in X$, then

$\mathcal{B}(x) = \{ (H(x), K(x)) \mid H(x)$ is a u-nhd. and $K(x)$ a v-nhd. of $x \}$

is an 1-regular bifilter which we shall call the nhd. bifilter of x.

The bifilter $\mathcal{B}$ will be said to converge to x if $\mathcal{B}(x) \subseteq \mathcal{B}$.

If $\mathcal{S}$ is a (dual covering) quasi-uniformity compatible with (X, u, v) then the bifilter $\mathcal{B}$ will be called $\mathcal{S}$-Cauchy if $d \cap \mathcal{B} \neq \emptyset$ for all $d \in \mathcal{S}$. (X, u, v) is $\mathcal{S}$-complete if every $\mathcal{S}$-Cauchy 1-regular bifilter on X is convergent.

Since with each 1-regular bifilter $\mathcal{B}$ we may associate the filter $\{ F \cap G \mid (F, G) \in \mathcal{B} \}$, we may of course express the above definition of completeness in terms of filters, as is usual in the literature. However bifilters will be involved in an essential way in the next chapter, and are introduced here to maintain consistency of terminology.

Proposition 1.7.2. Let (X, u, v) be fully binormal and pairwise R_0. Then (X, u, v) is $\mathcal{S}_n$-complete.

Proof. $\mathcal{S}_n$ is compatible since (X, u, v) is pairwise completely regular. Suppose that there exists a $\mathcal{S}_n$-Cauchy 1-regular bifilter $\mathcal{B}$ which does not converge in X. Then for each $x \in X$ we have nhds. $M(x) \in u$ and $K(x) \in v$ of x so that $(M(x), N(x)) \notin \mathcal{B}$. However the open dual cover $d = \{ (M(x), N(x)) \mid x \in X \}$ belongs to $\mathcal{S}_n$ since (X, u, v) is fully binormal, and hence $d \cap \mathcal{B} \neq \emptyset$, which is a contradiction. This proves the proposition.

We note for future reference the following characterisations of uniform compactness. This notion has been considered by various authors under a variety of different names (see, for example, [9] and [34]).

Lemma 1.7.2. The following are equivalent for the bitopological space (X, u, v).

(a) (X, u, v) is uniformly compact.

(b) Every open dual cover has a finite subcover.

(c) The diagonal is a compact set in $(X \times X, v \times u)$.

(d) Every maximal 1-regular bifilter on X is convergent.

We omit the proof which is straightforward.

In terms of a dual covering quasi-uniformity the notion of total boundedness ([23], Definition 4.8(2)) takes the following form:

The dual covering quasi-uniformity $\mathcal{S}$ is totally bounded if and only if it has a base consisting of finite dual covers.

Proposition 1.7.3. Let (X, u, v) be a preseparated pairwise R_0 bitopological space. Then (X, u, v) is uniformly compact if and only if $\mathcal{S}_n$ is compatible, complete and totally bounded.

Proof. If (X, u, v) is uniformly compact then (X, u, v) is fully binormal by Theorem 1.2.2. In particular it is pairwise normal and pairwise R_0, and hence pairwise completely regular by the counterpart of Urysohn's Lemma for bitopological spaces [19]. It follows that $\mathcal{S}_n$ is compatible by Proposition 1.7.1, and complete by Proposition 1.7.2. Finally if $d \in \mathcal{S}_n$ there is an open dual cover $d' \in \mathcal{S}_n$ with $d' \prec d$, and d' has a finite subcover d''. $d'' \in \mathcal{S}_n$ since (X, u, v) is fully binormal, and so $\mathcal{S}_n$ is totally bounded.

Conversely suppose $\mathcal{S}_n$ is compatible, complete and totally bounded. If $\mathcal{B}$ is a maximal 1-regular bifilter and $d \in \mathcal{S}_n$ we have $d' \in \mathcal{S}_n$ with d' finite and $d' \prec d$, and it is clear that $\mathcal{B} \cap d' \neq \emptyset$ so $\mathcal{B} \cap d \neq \emptyset$. Hence $\mathcal{B}$ is $\mathcal{S}_n$-Cauchy, and so converges in X. Hence (X, u, v) is uniformly compact by Lemma 1.7.2.

It is well known that a quasi-uniformity compatible with a uniformly compact bitopological space must contain all the nhds.

of the diagonal in $(X \times X, v \times u)$ in its diagonal form, and so we may conclude that a uniformly compact ~~preseparated~~ pairwise R_0 bitopological space is uniquely quasi-uniformizable.

Now let us denote by β_{fn} the set of all finite open dual covers $d = d_1$ for which there exists a sequence $\{d_n\}$ of finite open dual covers with $d_{n+1} \prec(*)\ d_n$. Clearly β_{fn} is a base for a quasi-uniformity $\mathcal{S}_{fn}$, and we have $t_u(\mathcal{S}_{fn}) \subseteq u$, $t_v(\mathcal{S}_{fn}) \subseteq v$. It is immediate from the definition that if $\mathcal{S}_{fn}$ is compatible with (X, u, v) it must be the largest compatible totally bounded quasi-uniformity. To establish compatibility when (X, u, v) is pairwise completely regular, let $P^*(X)$ denote the set of all bounded real valued pairwise continuous functions on (X, u, v), and note that for $f \in P^*(X)$ and $n = 1, 2, \ldots$, $f^{-1}(m(1, n))$ is a finite open dual cover of X, and

$$f^{-1}(m(1, 6n)) \prec(*)\ f^{-1}(m(1, n))$$

so that $f^{-1}(m(1, n)) \in \mathcal{S}_{fn}$. This shows that $\mathcal{S}_{fn}$ is finer than the initial quasi-uniformity generated by the set $P^*(X)$, and therefore is compatible with (X, u, v) if (X, u, v) is pairwise completely regular. Now let $d = d_1$ be a finite open dual cover of (X, u, v), and suppose there are open dual covers d_n (not necessarily finite) so that $d_{n+1} \prec(*)\ d_n$, $n = 1, 2, \ldots$. By Lemma 1.4.1 (2) we have a finite equibinormal cover $E_1 = \{(h_i, k_i) \mid 1 \leq i \leq m\}$ so that $d_6 \prec s(E_1)$ and $e(E_1) \prec d_1$. From this we may deduce at once that

$$d_6 \prec \left(\bigwedge_{i=1}^{m} h_i^{-1}(m_o) \right) \wedge \left(\bigwedge_{i=1}^{m} (1 - k_i)^{-1}(m_o) \right) \prec d$$

where $m_o \in \mu|[0, 1]$ is given by $m_o = \{(P, [0, 1]), ([0, 1], Q)\}$, $P = \{x \mid 0 \leq x < 2/3\}$ and $Q = \{x \mid 1/3 < x \leq 1\}$.

On the one hand this result shows that $\mathcal{S}_{fn}$ contains all finite normal dual covers, and on the other hand it proves that every totally bounded quasi-uniformity is the initial quasi-uniformity generated by a set of bounded pairwise continuous real valued functions, and in particular that $\mathcal{S}_{fn}$ is the initial quasi-uniformity generated by $P^*(X)$. We summarize these properties of $\mathcal{S}_{fn}$ in the next proposition.

<u>Proposition</u> 1.7.2. Let (X, u, v) be pairwise completely regular. Then:

(a) $\mathcal{S}_{fn}$ is the largest compatible totally bounded quasi-uniformity.

(b) $\mathcal{S}_{fn}$ is the initial quasi-uniformity generated by $P^*(X)$.

(c) $\mathcal{S}_{fn}$ is generated by all the finite normal (open) dual covers of X.

In case (X, u, v) is pairwise normal and pairwise R_0 we may present some alternatives to (c).

<u>Proposition</u> 1.7.3. Let (X, u, v) be pairwise normal and pairwise R_0. Then:

(a) The set of all finite open shrinkable dual covers is a base of $\mathcal{S}_{fn}$.

(b) The open dual covers $\{(X, K), (H, X)\}$, where $H \in u$, $K \in v$ and $H \cup K = X$, form a subbase of $\mathcal{S}_{fn}$.

<u>Proof</u>. (a). On the one hand every (finite) normal open dual cover is shrinkable by Proposition 1.1.1, while by the corollary to Theorem 1.1.1 every finite shrinkable open dual cover is normal. Hence (a) is proved.

(b). Firstly, since (X, u, v) is pairwise normal, it is clear that the finite open dual cover $\{(H, X), (X, K)\}$ is shrinkable, and hence belongs to $\mathcal{S}_{fn}$ by (a). To show these sets form a subbase take $f \in P^*(X)$ and $n \in \mathbb{N}$, $n \geqslant 1$. Since f(X) is a bounded subset of $\mathbb{R}$ we may choose itegers p, q so that

$$f(X) \subseteq [p/2n, (q-1)/2n] .$$

For $p \leq k \leq q$ let

$$H_k = f^{-1}(M((2k-1)/4n, 1/4n)) \text{ and } K_k = f^{-1}(N((2k-1)/4n, 1/4n))$$

so that $H_k \in u$, $K_k \in v$ and $H_k \cup K_k = X$. Then it is easy to see that

$$\wedge\{\{(H_k, X), (X, K_k)\} \mid p \leq k \leq q\} \prec f^{-1}(m(1, 2n))$$

from which the required result follows by Proposition 1.7.2 (b).

Note that if (X, u, v) is pairwise normal and pairwise R_0 the set of all star finite shrinkable open dual covers will form a base for a quasi-uniformity $\mathcal{S}_{sfs}$ (Corollary to Theorem 1.1.1). Since $\mathcal{S}_{fn} \subseteq \mathcal{S}_{sfs} \subseteq \mathcal{S}_n$ we see that $\mathcal{S}_{sfs}$ is compatible with (X, u, v). Of course $\mathcal{S}_{fn}$ (respectively, $\mathcal{S}_{sfs}$) will contain all finite open (respectively, all star finite open) dual covers if and only if (X, u, v) is finitely binormal (respectively, star finitely binormal).

Let us call the dual family $d = \{(U_\alpha, V_\alpha) \mid \alpha \in A\}$ <u>transitive</u> if

$$St(d, U_\alpha) = U_\alpha \text{ and } St(V_\alpha, d) = V_\alpha$$

for each $\alpha \in A$. If d is transitive then in particular $d \prec (*) \, d$. Several examples of trasitive dual covers have been seen in 1.6. The quasi-uniformity $\mathcal{S}$ may be called transitive if it has a base of trasitive dual covers. This corresponds to the usual definition of transitivity for diagonal quasi-uniformities. See, for example, [14] where the discussion is based on the notion of Q-covers. If $d \in \mathcal{S}$ is a transitive dual cover then it is clear that it is both open and closed. Indeed for any $d' \subseteq d$ and $x \in X$ the set

$$U(d', x) = \bigcup\{U \mid U \in \operatorname{dom} d', x \notin U\}$$

is $t_u(\mathcal{S})$-open and $t_v(\mathcal{S})$-closed, while the set

$$V(d', x) = \bigcup\{V \mid V \in \operatorname{ran} d', x \notin V\}$$

is $t_v(\mathcal{S})$-open and $t_u(\mathcal{S})$-closed. Now suppose $\mathcal{S}$ is transitive and compatible with (X, u, v), and let

$$u' = \{U(d', x) \mid x \in X, d' \subseteq d \in \mathcal{S} \text{ and } d \text{ is transitive}\}.$$

Then u' is a base of open sets of u, and a base of closed sets for v. In the same way,

$v' = \{ V(d', x) \mid x \in X,\ d' \subseteq d \in \mathcal{S}$ and d is transitive $\}$

is a base of open sets of v, and a base of closed sets for u.

Conversely if u' is an open base of u and a closed base for v and we set

$$d(U) = \{ (U, X), (X, X - U) \}$$

then d(U) is transitive and $\{ d(U) \mid U \in u' \}$ is a subbase for a compatible totally bounded quasi-uniformity. We may also obtain such a quasi-uniformity begining with an open base of v which is a closed base for u, and we have established, (cf [17])

Proposition 1.7.4. The following are equivalent for the bitopological space (X, u, v).

(1) There is a compatible transitive quasi-uniformity.

(2) There is a base u' of open sets of u which is a base of closed sets for v.

(3) There is a base v' of open sets of v which is a base of closed sets for u.

(4) There is a compatible totally bounded transitive quasi-uniformity.

I.L. REILLY [28] has called bitopological spaces satisfying the equivalent conditions (2) and (3) zero dimensional.

One extreme case is where u is a base of closed sets for v, and v is a base of closed sets for u. In this case u (respectively, v) is the largest (quasi-uniform) conjugate of v (respectively, of u), and so we might call such bitopological spaces pairwise reflexive. A pairwise reflexive bitopological space is clearly binormal, and so $\mathcal{S}_{fn}$ coincides with the transitive quasi-uniformity constructed as above from u or v. On the other hand, a bitopological space in which $\mathcal{S}_{fn}$ is transitive is not necessarily pairwise reflexive. For consider the space of Example 1.6.4. Here $\mathcal{S}_{fn}$ is clearly transitive, and u is the largest (indeed, only) conjugate of v, but v is not the largest conjugate of u. For if it were we should have $v = (v^{-1})^{-1}$ (where v^{-1} denotes the largest conjugate of v), and it follows easily from this that v would have to be a Q-space, which it is not since $\cap \{ V \mid w \in V \in v \}$

$= \{w\} \notin v$.

We end this section by defining two new forms of structure which may be obtained by weakening Definition 1.7.1.

If in Definition 1.7.1 we replace " $<(*)$ " by " $<(p*)$ " we obtain the definition of a structure to which it would be appropriate to give the name pseudo-quasi-uniformity. Certainly any quasi-uniformity is a pseudo-quasi-uniformity, but the converse is false as we shall see in a moment.

If (X, u, v) is fully pseudonormal then the set of all open dual covers of X will form a base for a pseudo-quasi-uniformity, and a consideration of the relation between this structure and the topologies for the space of Example 1.6.6 will indicate that in place of the sets $St(d,\{x\})$ and $St(\{x\}, d)$ we should use the sets $PSt(d, (\{x\},\{x\})) = WSt(d,\{x\})$ and $PSt((\{x\},\{x\}), d) = WSt(\{x\}, d)$ in the definition of the topologies of a pseudo-quasi-uniformity. Consequently if $\mathcal{S}$ is a pseudo-quasi-uniformity, $t_u(\mathcal{S})$ will be the topology generated by the filter bases $\{WSt(d,\{x\}) \mid d \in \mathcal{S}\}$ for $x \in X$, and $t_v(\mathcal{S})$ will be the topology generated by the filter bases $\{WSt(\{x\}, d) \mid d \in \mathcal{S}\}$ for $x \in X$. It will be clear that in general the sets $WSt(d,\{x\})$ and $WSt(\{x\}, d)$ will not necessarily be nhds. of x for the appropriate topology, and that a pseudo-quasi-uniformity need not have an open base.

As mentioned above, if (X, u, v) is fully pseudonormal the set of all open dual covers is a base for a pseudo-quasi-uniformity on X. This clearly is compatible with (X, u, v), and has an open base so the above sets are all nhds. in this instance. This pseudo-quasi-uniformity for the space of Example 1.6.1 cannot be a quasi-uniformity since (X, u, v) is not pairwise completely regular. (Note that if a pseudo-quasi-uniformity is in fact a quasi-uniformity then the bitopological space it generates as a quasi-uniformity is indeed the same as it generates as a pseudo-quasi-uniformity, since if $e < (*) \, d$ we clearly have

$$St(e,\{x\}) \subseteq WSt(d,\{x\}) \subseteq St(d,\{x\}), \text{ and}$$
$$St(\{x\}, e) \subseteq WSt(\{x\}, d) \subseteq St(\{x\}, d)$$

for each $x \in X$. Hence the topologies $t_u(\mathcal{S})$ and $t_v(\mathcal{S})$ are defined unambiguously). Hence not every pseudo-quasi-uniformity is a quasi-uniformity.

Example 1.6.6 shows that a pseudo-quasi-uniformizable bitopological space is not necessarily pairwise R_0. However not every bitopological space is pseudo-quasi-uniformizable, as the next proposition shows.

Proposition 1.7.5. A pseudo-quasi-uniformisable bitopological space is uniformly completely regular.

Proof. Let $\mathcal{S}$ be a compatible pseudo-quasi-uniformity, and for $d \in \mathcal{S}$ set

$$C(d) = \{ U \cap V \mid UdV \}.$$

Then $e \prec (p*)\ d \Rightarrow C(e)^* \prec C(d)$ and so $\{C(d) \mid d \in \mathcal{S}\}$ is a base for a uniformity compatible with the uniform topology of (X, u, v), from which the result follows.

The converse seems to be an open question. Certainly the uniformly completely regular space of Example 1.6.10 has no compatible pseudo-quasi-uniformity with an open base, but I do not know if it is, none the less, pseudo-quasi-uniformizable.

A second generalisation of the notion of quasi-uniformity may be obtained by replacing "$e \prec (*)\ d$" in Definition 1.7.1 (i) by the requirement

$$WSt(e, WSt(e,\{x\})) \subseteq WSt(d,\{x\}), \text{ and}$$
$$WSt(WSt(\{x\}, e), e) \subseteq WSt(\{x\}, d)$$

for each $x \in X$. We call such a structure a weak local quasi-uniformity. We may justify this name by noting that if in place of "WSt" we were to specify "St" above we should obtain the dual covering equivalent of the notion of local quasi-uniformity [21]. In particular every local quasi-uniformity, and hence every quasi-uniformity, is a weak local quasi-uniformity. If $\mathcal{S}$ is a weak local quasi-uniformity a base of $t_u(\mathcal{S})$- (respectively, $t_v(\mathcal{S})$-) nhds. of $x \in X$ is taken to be $\{WSt(d,\{x\}) \mid d \in \mathcal{S}\}$ (respectively, $\{WSt(\{x\}, d) \mid d \in \mathcal{S}\}$). Note that a weak local quasi-uniformity always has an open base, just like a quasi-uniformity. Also if $\mathcal{S}$ is a local quasi-uniformity then the bitopological space generated by $\mathcal{S}$ as a local quasi-uniformity will be the same as that which it generates as a weak local quasi-uniformity.

Not every pseudo-quasi-uniformity, and indeed not every pseudo-quasi-uniformity with an open base, is a weak local quasi-uniformity, as the next example shows.

Example 1.7.3. In the fully pseudonormal space of Example 1.6.1 the set of all open dual covers is not a base of a weak local quasi-uniformity.

Proof. With the notation as in Example 1.6.1 let us set

$$U(a, b) = \begin{cases} R(a, b) \cup R(0, 1/b) \text{ if } a > 0 \text{ and } b > 0, \\ R(a, 0) \cup R(0, 1) \text{ if } a \geq 0 \text{ and } b = 0, \\ R(0, b) \text{ if } a = 0 \text{ and } b \geq 0; \end{cases}$$

$$V(a, b) = \begin{cases} S(a, b) \cup S(1/a, 0) \text{ if } a > 0 \text{ and } b > 0, \\ S(a, 0) \text{ if } a \geq 0 \text{ and } b = 0, \\ S(0, b) \cup S(1, 0) \text{ if } a = 0 \text{ and } b \geq 0. \end{cases}$$

and consider the open dual cover

$d = \{ (U(a, b), V(a, b)) \mid (a, b) \in X \}$.

Suppose that $e = \{ (R_\alpha, S_\alpha) \mid \alpha \in A \}$ is an open dual cover of X satisfying $WSt(e, WSt(e,\{x\})) \subseteq WSt(d,\{x\})$ for all $x \in X$, and take $\alpha \in A$ with $(1, 1) \in R_\alpha \cap S_\alpha$. Since $R_\alpha \in u$ $\exists$ $b > 0$ with $(0, b) \in R_\alpha$, and consider the point $x = (1, 1/(b+1)) \in X$. Then $WSt(d,\{x\}) = U(1, 1/(b+1))$ and so

$(0, b) \in R_\alpha \subseteq U(1, 1/(b+1)) = R(1, 1/(b+1)) \cup R(0, b+1)$

which is impossible since it implies $b + 1 \leq b$. This contradiction completes the proof.

On the other hand there are weak local quasi-uniformities which are not pseudo-quasi-uniformities. Thus, for example, we have seen that the set $\mathcal{S}$ of all open dual covers of the space of Example 1.6.10 is not a pseudo-quasi-uniformity, whereas

$WSt(d_0, WSt(d_0,\{P\})) = M(P)$ and $WSt(WSt(\{P\}, d_0), d_0) = N(P)$

for all $P \in X$, and so $\mathcal{S}$ is a, clearly compatible, weak local quasi-uniformity.

This shows that in general the notions of pseudo-quasi-uniformity and weak local quasi-uniformity are independent of one another.

1.8. PARA-QUASI-UNIFORMITIES.

In this section we present a rather different generalisation of the notion of quasi-uniformity. Here we are motivated by the definition of a para-uniformity given by C.I. VOTAW [35]. A para-uniformity maintains the symmetry of a uniformity, and so gives rise to a single topological space. Moreover any topology may be defined by a suitable para-uniformity. A para-quasi-uniformity as defined below gives rise to a bitopological space, just as a quasi-uniformity does, and moreover any bitopological space may be defined by a suitable para-quasi-uniformity. Hence para-quasi-uniformities stand in the same relation to bitopological spaces as para-uniformities stand in relation to topological spaces.

In accordance with our general approach in this chapter we will define a para-quasi-uniformity in terms of dual families. In addition to the notation and terminology used so far we shall need the following. If d is a dual family we define:

$uc_1(d) = \{ x \mid x \in uc(d) \text{ and } St(d,\{x\}) \subseteq uc(d) \}$, and

$uc_2(d) = \{ x \mid x \in uc(d) \text{ and } St(\{x\}, d) \subseteq uc(d) \}$.

We then set $\mathcal{G} = \{ d \mid d \subseteq \imath \text{ and } uc_1(d) \cup uc_2(d) \neq \emptyset \}$. Note that $\mathcal{G}$ contains all dual covers of X, and more generally all non-empty $\imath$-dual families satisfying $lc(d) \subseteq uc(d)$ or $rc(d) \subseteq uc(d)$.

For convenience we shall take $e \precsim d$ (respectively, $e \precsim^{(*)} d$) to mean $e \prec d$ (respectively, $e \prec^{(*)} d$) and $uc(e) = uc(d)$. We note without proof the following elementary facts.

Lemma 1.8.1. Let d and e be dual families on X. Then:

(a) $uc(d \wedge e) = uc(d) \cap uc(e)$

(b) $uc_j(d) \cap uc_j(e) \subseteq uc_j(d \wedge e)$, $j = 1, 2$.

(c) If $e \precsim d$ then $uc_j(d) \subseteq uc_j(e)$, $j = 1, 2$.

(d) If $e' \precsim e$, $d' \precsim d$ and $e \wedge d \in \mathcal{G}$ then $e' \wedge d' \in \mathcal{G}$.

We may now give:

Definition 1.8.1. The non-empty subset $\mathcal{S}$ of $\mathcal{C}$ is a para-quasi-uniformity (pqu) on X if it satisfies:

PQ.1 $d \in \mathcal{S} \Rightarrow \exists\ e \in \mathcal{S}$ with $e \lesssim (*)\ d$

PQ.2 $d_i \in \mathcal{S}$, $i = 1, \ldots, n$, $\wedge d_i \in \mathcal{C} \Rightarrow \wedge d_i \in \mathcal{S}$.

PQ.3 $d \in \mathcal{S}$, $e \in \mathcal{C}$ with $d \lesssim e \Rightarrow e \in \mathcal{S}$.

PQ.4 $\{(X, X)\} \in \mathcal{S}$.

It will be noted that in particular any (dual covering) quasi-uniformity is a pqu.

We may define a pqu base as follows.

Definition 1.8.2. $\beta \subseteq \mathcal{C}$ is a pqu base if it satisfies:

B.1 $d \in \beta \Rightarrow \exists\ e \in \beta$, $e \lesssim (*)\ d$.

B.2 $d_i \in \beta$, $i = 1, \ldots, n$, $\wedge d_i \in \mathcal{C} \Rightarrow \exists\ f \in \beta$, $f \lesssim \wedge d_i$.

We then have

Proposition 1.8.1. If β is a pqu base then

$\mathcal{S} = \{ d \mid d \in \mathcal{C}, \exists\ e \in \beta \cup \{(X, X)\}$ with $e \lesssim d \}$

is a pqu.

We omit the proof which is straightforward.

If we insist that all the elements d of β should satisfy the condition $rc(d) \subseteq uc(d)$, and add a symmetry condition, we may consider pqu bases β satisfying

P.1 $d \in \beta \Rightarrow rc(d) \subseteq uc(d)$

P.2 $d \in \beta \Rightarrow \exists\ e \in \beta$ with $e \lesssim (*)\ d$.

P.3 $d, e \in \beta$, $\emptyset \neq d \wedge e \Rightarrow \exists\ f \in \beta$ with $f \lesssim d \wedge e$.

P.4 $d \in \beta \Rightarrow \exists\ e \in \beta$ with $e \lesssim d \wedge d^{-1}$.

For such a β it is easy to see that $\{ W(d) \mid d \in \beta \}$ is a base for a para-uniformity on X. This verifies that the notion of para-quasi-uniformity generalises the notion of para-uniformity as well as that of quasi-uniformity.

Definition 1.8.3. The subset σ of $\mathcal{C}$ is a para-quasi-uniform subbase if it satisfies:

S.1 $d \in \sigma \Rightarrow \exists\ e \in \sigma$ with $e \lesssim (*)\ d$.

<u>Proposition</u> 1.8.2. If σ is a pqu subbase then

$\beta = \{ \wedge d_i \mid d_i \in \sigma , i = 1, \ldots, n, \wedge d_i \in \mathscr{P} \}$

is a pqu base.

We omit the proof which is straightforward.

Let us now show that a pqu defines a bitopological space.

<u>Theorem</u> 1.8.1. For $A \subseteq X$ we define

$$(x \in \overline{A}^u) \Longleftrightarrow (\text{Given } d \in \mathcal{S} \text{ with } x \in uc_1(d) \ \exists\ UdV \text{ with } x \in V, \ U \cap A \neq \emptyset).$$

Then $-u$ is a closure operation for a topology $t_u(\mathcal{S})$ on X.

<u>Proof</u>. Of the four closure axioms which must be verified, $\overline{\emptyset}^u = \emptyset$ is clear from PQ.4, $A \subseteq \overline{A}^u$ follows from $uc_1(d) \subseteq uc(d)$ and $\overline{A}^u \cup \overline{B}^u = \overline{A \cup B}^u$ follows from PQ.2 and Lemma 1.8.1 (b). Finally $\overline{A}^u \subseteq \overline{\overline{A}^u}^u$ is clear, so it remains to prove the reverse inclusion. Take $x \in \overline{\overline{A}^u}^u$ and $d \in \mathcal{S}$ with $x \in uc_1(d)$. By PQ.1 we have $e \in \mathcal{S}$ with $e \leq (*) d$, and $x \in uc_1(e)$ by Lemma 1.8.1 (c). Hence $\exists$ ReS with $x \in S$ and $R \cap \overline{A}^u \neq \emptyset$. Take $y \in R \cap \overline{A}^u$, and let us show $y \in uc_1(e)$. First $y \in R \subseteq St(e,\{x\}) \subseteq uc(e)$ since $x \in uc_1(e)$, and if UdV has $St(e, R) \subseteq U$, $St(S, e) \subseteq V$ then $x \in V$ and $St(e,\{y\}) \subseteq St(e, R) \subseteq U \subseteq St(d,\{x\}) \subseteq uc(d) = uc(e)$ since $x \in uc_1(d)$. This verifies $y \in uc_1(e)$ so $\exists$ R'eS' with $y \in S'$ and $R' \cap A \neq \emptyset$. With UdV as above, $x \in V$ and $R' \subseteq St(e, R) \subseteq U$ since $S' \cap R \neq \emptyset$, and so $U \cap A \neq \emptyset$. Hence $x \in \overline{A}^u$ and the proof is complete.

In just the same way

$$(x \in \overline{A}^v) \Longleftrightarrow (\text{Given } d \in \mathcal{S} \text{ with } x \in uc_2(d) \ \exists\ UdV \text{ with } x \in U, \ V \cap A \neq \emptyset)$$

defines a topology $t_v(\mathcal{S})$ on X. In this way a pqu $\mathcal{S}$ gives rise to the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, Note that if $\mathcal{S}$ is a quasi-uniformity, then the bitopological space generated by $\mathcal{S}$ as a quasi-uniformity is the same as that generated by $\mathcal{S}$ as a para-quasi-uniformity.

Proposition 1.8.3. (a) $\{ St(d, \{x\}) \mid d \in \mathcal{S}, x \in uc_1(d) \}$ is a base of $t_u(\mathcal{S})$-nhds. of $x \in X$.

(b) If $d \in \mathcal{S}$ and $A \subseteq uc_1(d)$ then $St(d, A)$ is a $t_u(\mathcal{S})$-nhd. of $A \subseteq X$.

(c) If $d \in \mathcal{S}$ and $A \subseteq X$ then $uc_1(d) \cap (t_u(\mathcal{S})\text{-cl}[A]) \subseteq St(A, d)$.

We omit the proof, which is straightforward. Of course corresponding results hold for the other topology.

In general a pqu need not have an open (or closed) base. However many important examples (including all quasi-uniformities and para-uniformities) do have an open base. Let us now verify that every bitopological space has a compatible pqu with an open base.

Theorem 1.8.2. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of non-empty subsets of X. Then

$$\sigma = \{ \{(A, X)\}, \{(X, B)\} \mid A \in \mathcal{A}, B \in \mathcal{B} \}$$

is a subbase for a pqu $\mathcal{S}$ on X. If u and v are topologies on X, $\mathcal{A} \subseteq u$ and $\mathcal{B} \subseteq v$ then $t_u(\mathcal{S}) \subseteq u$ and $t_v(\mathcal{S}) \subseteq v$. Moreover we have equality if and only if $\mathcal{A} \cup \{X\}$ is a subbase of u and $\mathcal{B} \cup \{X\}$ is a subbase of v.

We omit the proof which is quite trivial. Note that if we construct $\mathcal{S}$ as above for the bitopological space (X, u, v), say by taking $\mathcal{A} = u - \{\emptyset\}$ and $\mathcal{B} = v - \{\emptyset\}$, then $\mathcal{S}$ will be a compatible pqu with an open transitive base.

A second way of constructing a compatible pqu with an open base for a given bitopological space is described in the next lemma.

Lemma 1.8.2. Let $r \subseteq (u - \{\emptyset\})$ and $s \subseteq (v - \{\emptyset\})$ be such that $r \cup \{X\}$ is a subbase of u, and $s \cup \{X\}$ a subbase of v. For $R \in r$ let $d(R) = \{(R, X), (X, X - (v\text{-cl}[R]))\}$ if $v\text{-cl}[R] \neq X$, and $d(R) = \{(R, X)\}$ otherwise, and for $S \in s$ let $e(S) = \{(X, S), (X - (u\text{-cl}[S]), X)\}$ if $u\text{-cl}[S] \neq X$ and $e(S) = \{(X, S)\}$

otherwise. Then the set

$$\{ d(R), e(S) \mid R \in r, S \in s \}$$

is a subbase for a compatible pqu $\mathcal{S}$ with an open transitive base. Again the proof is immediate, and is omitted.

We now give for pqu with an open base an exact analogue of ([35], Proposition 2.11).

Theorem 1.8.3. The bitopological space (X, u, v) has a unique compatible pqu with an open base if and only if it satisfies the two conditions below.
(a) u is the only base of u, and v is the only base of v, which is closed under finite intersections and contains X and $\emptyset$.
(b) Every non-empty u-open set is dense in (X, v), and every non-empty v-open set is dense in (X, u).

Proof. First suppose that (X, u, v) satisfies (a) and (b), and that the compatible pqu $\mathcal{S}$ has an open base. Define

$$u' = \{ U \mid U \in u,\ \{(U, X)\} \in \mathcal{S} \} \cup \{ \emptyset \} .$$

Then u' contains X and $\emptyset$, and is closed under finite intersections. We show it is a base of u. Take $x \in G \in u$, and $d \in \mathcal{S}$ so that $x \in uc_1(d)$ and $St(d,\{x\}) \subseteq G$. Take $e \in \mathcal{S}$, open, with $e \leq (*)\ d$. Then $x \in uc_1(d) \subseteq uc(d) = uc(e)$ so we have ReS with $x \in R \cap S$. Take UdV with $St(e, R) \subseteq U$, $St(S, e) \subseteq V$. Then using (b) and the fact that $x \in V$ and $x \in uc_1(d)$ we have

$$uc(e) \subseteq lc(e) = St(e, R) \subseteq U \subseteq St(d,\{x\}) \subseteq uc(d) = uc(e)$$

and so all these sets are equal. Hence if we set $f = \{(U, X)\}$ we see that $f \in \mathcal{P}$ and $e \leq f$ so $f \in \mathcal{S}$ and $U \in u'$. However $x \in U \subseteq G$, and so u' is a base of u, and hence u' = u by (a). This shows $\mathcal{S}$ contains all dual families of the form $\{(U, X)\}$ with $U \in u - \{\emptyset\}$, and likewise it contains all dual families of the form $\{(X, V)\}$ with $V \in v - \{\emptyset\}$. Hence $\mathcal{S}$ is finer than the pqu $\mathcal{f}$ with subbase

$$\{ \{(U, X)\}, \{(X, V)\} \mid U \in u - \{\emptyset\},\ V \in v - \{\emptyset\} \} .$$

On the other hand take any $d \in \mathcal{S}$, and take e open with $e \in \mathcal{S}$ and $e \leq (*)\ d$. If $uc_1(d) \neq \emptyset$ we may take $x \in uc_1(d)$, ReS and UdV as above. If, in addition to the above equalities, we note that by (b) we have $rc(e) = St(S, e) \subseteq V$ then it is clear that $g = \{(U, rc(e))\} \leq d$. However $g = \{(U, X)\} \wedge \{(X, rc(e))\} \in \rho$ and so $d \in \rho$. A similar argument shows $d \in \rho$ also if $uc_2(d) \neq \emptyset$, and so $\mathcal{S}$ is coarser than ρ. Hence ρ is the only compatible pqu with an open base.

Conversely suppose ρ is the only compatible pqu with an open base. To verify (a) suppose that $\mathcal{A}$ is a base of u containing X and $\emptyset$, and closed under finite intersections. By Theorem 1.8.2 $\sigma = \{ \{(A, X)\}, \{(X, B)\} \mid A \in \mathcal{A} - \{\emptyset\}, B \in v - \{\emptyset\} \}$ is a subbase for a compatible pqu with an open base, and this must therefore be ρ. Hence if $G \in u - \{X, \emptyset\}$ $\exists$ $A \in \mathcal{A} - \{\emptyset\}$, $B \in v - \{\emptyset\}$ with $\{(A, B)\} \leq \{(G, X)\}$, since $\mathcal{A}$ is closed under finite intersections. Hence $A \subseteq G$ and $A \cap B = G$ which gives $G = A \in \mathcal{A}$, and therefore $\mathcal{A} = u$. The second part of (a) is proved likewise.

Finally let us verify (b). Suppose that, for instance, we have $G \in u - \{\emptyset\}$ with $v\text{-cl}[G] \neq X$, and construct the compatible pqu with an open base as in Lemma 1.8.2 taking $r = u - \{\emptyset\}$ and $s = v - \{\emptyset\}$. This pqu is equal to ρ, and contains the dual family $\{(G, X), (X, X - (v\text{-cl}[G]))\}$ so we have $U \in u - \{\emptyset\}$, $V \in v - \{\emptyset\}$ with

$$\{(U, V)\} \leq \{(G, X), (X, X - (v\text{-cl}[G]))\} .$$

However this is clearly impossible, and the proof is complete.

Other aspects of the theory of para-uniformities may be generalized to include para-quasi-uniformities in the obvious way. Some of these will be considered in a more general setting in the next chapter.

CHAPTER TWO

BITOPOLOGICAL EXTENSIONS WHICH ARE COMPLETIONS OF CONFLUENCE STRUCTURES

Let (X, u, v) be a bitopological space and (X', u', v') a bitopological subspace for which X' is bidense in X, that is dense for each of the topologies u and v. We may express this by saying that (X, u, v) is a bitopological extension of (X', u', v'). Suppose $\mathcal{S}$ is a (dual covering) quasi-uniformity compatible with (X, u, v), and that d is an element of the open base of $\mathcal{S}$. Then if UdV we certainly have $U \cap X' \neq \emptyset \neq V \cap X'$, but there will be no guarantee that $U \cap V \cap X' \neq \emptyset$, unless of course X' is actually uniformly dense in X. It follows that $\{(U \cap X', V \cap X') \mid UdV\}$ will not be an element of the induced structure on X' in general; and while we may obtain from it an element of this induced structure by removing all pairs $(U \cap X', V \cap X')$ with $U \cap V \cap X' = \emptyset$, it is possible that in so doing we may, in certain circumstances, be lo~~o~~sing information which could enable us to characterize (X, u, v). This suggests that in order to widen the class of bitopological extensions which can be obtained with the aid of quasi-uniform-like structures it would be an advantage to consider the enlargement of such structures to include elements which are not ι-dual families. Of course this must be done in an organised and well defined way in order to produce a workable theory, and this is the object of the present chapter. Briefly the idea is to consider more general "confluence relations" (see Definition 2.1.1) in place of the relation ι of "meeting" between sets. Clearly any quasi-uniform-like structure is amenable to such a generalization; however to keep our discussion as concrete as possible we will confine ourselves to the cases of quasi-uniformities and para-quasi-uniformities.

The princip~~le~~al results of Sections 2.1 - 2.7 were presented by the author at the Bolyai Janos Colloquium on Topology, held in Budapest in August 1978, and these are to be published in the Proceedings under the title "On Extensions of Bitopological Spaces".

2.1 CONFLUENCE QUASI-UNIFORM STRUCTURES.

Definition 2.1.1. The binary relation c on the non-empty subsets of X will be called a confluence relation on X if it satisfies:

(a) $P \cap Q \neq \emptyset \Rightarrow PcQ$, and

(b) $PcQ,\ P \subseteq P' \subseteq X,\ Q \subseteq Q' \subseteq X \Rightarrow P'cQ'$.

Denoting by ι the relation $P \iota Q \Leftrightarrow P \cap Q \neq \emptyset$ of having a non-empty intersection, we see that ι is a confluence relation, and indeed it is the smallest conflunce relation on X. Clearly a confluence relation has some of the basic properties of the relation ι.

Definition 2.1.2. Let c be a confluence relation on X. The dual family d on X is a c-dual family if $c \subseteq d$. If in addition $uc(d) = X$ we say d is a c-dual cover.

This agrees with our earlier use of the term ι-dual family. Note that an ι-dual cover is a dual cover.

Now let b, c be confluence relations, d a c-dual cover and e a b-dual cover. Consider the pairs (d, c) and (e, b). We shall say that (d, c) refines (e, b), and write $(d, c) \prec (e, b)$, if $d \prec e$ and $c \subseteq b$.

For $A \subseteq X$ let us set

$St_c(d, A) = \bigcup \{ U \mid \exists\ V,\ UdV \text{ and } AcV \}$,

$St_c(A, d) = \bigcup \{ V \mid \exists\ U,\ UdV \text{ and } UcA \}$.

Then we shall say that (d, c) is a star refinement of (e, b), and write $(d, c) \prec^{(*)} (e, b)$, if $c \subseteq b$ and given UdV there exists ReS with $St_c(d, U) \subseteq R$ and $St_c(V, d) \subseteq S$.

Since $A \subseteq St_c(d, A)$ and $A \subseteq St_c(A, d)$ we see that a star refinement is also a refinement.

With this notation we may now define a confluence quasi-uniformity as follows:

Definition 2.1.3. Let $\mathcal{S}$ be a non-empty collection of pairs (d, c). Then $\mathcal{S}$ is a confluence quasi-uniformity (cqu) if it satisfies.

(i) If $(d, c) \in \mathcal{S}$ then c is a confluence relation on X and d is a c-dual cover.

(ii) If $(d, c), (e, b) \in \mathcal{S}$ there exists $(f, a) \in \mathcal{S}$ so that $(f, a) \prec^{(*)} (d, c)$ and $(f, a) \prec^{(*)} (e, b)$.

(iii) If $(d, c) \in \mathcal{S}$, b is a confluence relation and e a b-dual cover of X with $(d, c) \prec (e, b)$ then $(e, b) \in \mathcal{S}$.

The notion of a base for a confluence quasi-uniformity may be defined in the obvious way.

Example 2.1.1. Let $\mathcal{S}$ be a dual covering quasi-uniformity on X. Then $\{(d, 1) \mid d \in \mathcal{S}\}$ is a base for a confluence quasi-uniformity $\mathcal{S}'$ on X.

Note that dom $\mathcal{S}'$ corresponds to the covering quasi-uniformity of which $\mathcal{S}$ is the base, and in this way we have made exact the relation between these two definitions.

Just as for a quasi-uniformity, a confluence quasi-uniformity gives rise to a bitopological space.

Proposition 2.1.1. $\{St(d,\{x\}) \mid d \in \text{dom } \mathcal{S}\}$ is a base of nhds. of $x \in X$ for a topology $t_u(\mathcal{S})$, and $\{St(\{x\}, d) \mid d \in \text{dom } \mathcal{S}\}$ is a base of nhds. of $x \in X$ for a topology $t_v(\mathcal{S})$.

We omit the proof which is straightforward. Note that for the dual covering quasi-uniformity $\mathcal{S}$ we have $t_u(\mathcal{S}) = t_u(\mathcal{S}')$ and $t_v(\mathcal{S}) = t_v(\mathcal{S}')$, so no confusion can arise here. Let us note also the following elementary results which we present without proof. In each $\mathcal{S}$ is a cqu.

Proposition 2.1.2. If $A \subseteq X$ and $(d, c) \in \mathcal{S}$ then

$A \subseteq t_u(\mathcal{S})\text{-int}[St_c(d, A)]$ and $A \subseteq t_v(\mathcal{S})\text{-int}[St_c(A, d)]$.

Proposition 2.1.3. If $A \subseteq X$ and $(d, c) \in \mathcal{S}$ then

$t_u(\mathcal{S})\text{-cl}[A] \subseteq St_c(A, d)$ and $t_v(\mathcal{S})\text{-cl}[A] \subseteq St_c(d, A)$.

For convenience we shall call $(d, c) \in \mathcal{S}$ open (respectively, closed) if d is open (respectively, closed) for $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$. We may deduce at once from the above results that a cqu has a base of open and a base of closed elements. In particular the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is pairwise regular.

We shall find the following notion of extreme importance later.

Definition 2.1.4. The confluence relation c is interior for the bitopological space (X, u, v) if whenever PcQ and $P \cap Q = \emptyset$ we have (u-int[P])c(v-int[Q]). We shall say that (d, c) $\in \mathcal{S}$ is interior if c is interior for (X, $t_u(\mathcal{S})$, $t_v(\mathcal{S})$). We denote the set of interior elements of $\mathcal{S}$ by $\mathcal{S}_i$, and say $\mathcal{S}$ is an interior cqu if $\mathcal{S}_i$ is a base of $\mathcal{S}$.

In particular $\imath$ is interior for any bitopological space, and if $\mathcal{S}$ is a dual covering quasi-uniformity then the cqu $\mathcal{S}'$ of Example 2.1.1 is interior.

It is easily verified that an interior cqu has a base of open interior elements, and a base of closed interior elements.

In all that follows all cqu considered will be assumed to be interior. We will denote the base of open interior elements of $\mathcal{S}$ by $\mathcal{S}_o$.

Example 2.1.2. For the space ($\mathbb{R}$, s, t) considered earlier, let K be the relation

$$PKQ \Leftrightarrow P\imath Q \text{ or } (t\text{-cl}(s\text{-int}[P]))\imath(s\text{-cl}(t\text{-int}[Q])).$$

Clearly K is an interior confluence relation. For any $\alpha > 0$ let $k(\alpha) = \{(M(x, \varepsilon), N(x, \varepsilon)) \mid x \in X, 0 \leq \varepsilon \leq \alpha\}$, so that $k(\alpha)$ is a K-dual cover of $\mathbb{R}$. Since

$$(k(\alpha/3), K) \prec^{(*)} (k(\alpha), K)$$

for all $\alpha > 0$ we see that $\{(k(\alpha), K) \mid \alpha > 0\}$ is a base for an interior cqu compatible with ($\mathbb{R}$, s, t). Note that this cqu contains no dual covers.

2.2 SEPARATION PROPERTIES.

We begin by generalizing the notion of preseparated given in Definition 1.2.2.

Definition 2.2.1. Let c be a confluence relation on X. The bitopological space (X, u, v) will be called c-preseparated if given x, y $\in$ X with $x \notin$ u-cl{y} (respectively, $x \notin$ v-cl{y}) there exist $G \in u$, $H \in v$ with $G \not c H$ and $x \in G$, $y \in H$ (respectively,

$y \in G$, $x \in H$).

A c-preseparated weakly pairwise T_o bitopological space will be called <u>c-separated.</u> Note that a c-preseparated space is preseparated, and that a c-separated space is weakly pairwise Hausdorff.

For the cqu $\mathcal{S}$ (assumed interior as mentioned above) we denote by D the interior confluence relation

$$\bigcap \{ c \mid c \in \operatorname{ran} \mathcal{S} \} = \bigcap \{ c \mid c \in \operatorname{ran} \mathcal{S}_o \} .$$

<u>Proposition</u> 2.2.1. The bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-preseparated.

The proof is trivial and is omitted. Note that if $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is weakly pairwise T_o it will, therefore, be D-separated. We shall indicate this more shortly by saying that $(X, \mathcal{S})$ is <u>separated.</u>

<u>Definition</u> 2.2.2. Let $\mathcal{B}$ be a bifilter, $\mathcal{S}$ a cqu on X and $\beta \subseteq \mathcal{S}_o$ a base of $\mathcal{S}$.

(a) $\mathcal{B}$ is <u>β-regular</u> if whenever $(d, c) \in \beta$ and $(U, V) \in \mathcal{B} \cap ((\operatorname{dom} d) \times (\operatorname{ran} d))$ we have UcV.

(b) $\mathcal{B}$ is <u>D-regular</u> if $\mathcal{B} \subseteq D$.

In the absence of a cqu, Definition 2.2.2 (b) may be applied to any (interior) confluence relation on X. This notation agrees with our use of the term "ı-regular bifilter" in the last chapter. Note that a D-regular bifilter is β-regular for all bases β .

$\mathcal{B}(x)$ will denote, as usual, the nhd. bifilter of x in $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$. The following proposition gives, without proof, some elementary facts relating bifilters and the separatedness of $(X, \mathcal{S})$.

<u>Proposition</u> 2.2.2. The following are equivalent for the cqu $\mathcal{S}$ with base $\beta \subseteq \mathcal{S}_o$.

(a) $(X, \mathcal{S})$ is separated.

(b) The map $x \to \mathcal{B}(x)$ is one to one.

(c) Given $x \neq y$ in X $\exists$ $(d, c) \in \beta$ and UdV', $U'dV$ so that

$St_c(d, U) \not\subseteq St_c(V, d)$ and $x \in U \cap V'$, $y \in U' \cap V$, or $x \in U' \cap V$, $y \in U \cap V'$.

(d) Every convergent β-regular bifilter has a unique limit.

(e) Every convergent D-regular bifilter has a unique limit.

If $\mathcal{S}$ is a cqu on X let us define the relation $\sim$ on X by $(x \sim y) \iff (St(d,\{x\})cSt(\{y\}, d)$ and $St(d,\{y\})cSt(\{x\}, d)$ for all $(d, c) \in \mathcal{S}$). Then:

Lemma 2.2.1. $\sim$ is an equivalence relation on X.

The proof is a matter of straightforward verification. We denote by $\dot{x}$ the equivalence class containing x, by $\dot{X}$ the quotient set $X/\sim$, and by φ the canonical mapping of X onto $\dot{X}$. For $A \subseteq X$ we shall often write $\dot{A}$ in place of $\varphi(A)$.

If c is a confluence relation on X we define a confluence relation $\dot{c}$ on X by:

$P\dot{c}Q \iff P\imath Q$ or $(t_u(\mathcal{S})\text{-int}[\varphi^{-1}(P)])c(t_v(\mathcal{S})\text{-int}[\varphi^{-1}(Q)])$.

Also for $d \in \operatorname{dom} \mathcal{S}_o$ we define $\dot{d}$ on $\dot{X}$ by

$P\dot{d}Q \iff \exists$ UdV with $P = \dot{U}$ and $Q = \dot{V}$.

Then:

Lemma 2.2.2. $\dot{d}$ is a $\dot{c}$-dual cover of $\dot{X}$ for each $(d, c) \in \mathcal{S}_o$.

Proof. For UdV it is clear that $\dot{U}\dot{c}\dot{V}$, and so $\dot{d} \subseteq \dot{c}$. Also if UdV and $x \in U \cap V$ then $\dot{U}\dot{d}\dot{V}$ and $\dot{x} \in \dot{U} \cap \dot{V}$ so $uc(\dot{d}) = \dot{X}$, which completes the proof.

Lemma 2.2.3. If $G \subseteq X$ is $t_u(\mathcal{S})$ or $t_v(\mathcal{S})$ - open and $x \in G$ then $\dot{x} \subseteq G$.

Proof. Take $x \in G \in t_u(\mathcal{S})$; then we have $(d, c) \in \mathcal{S}_o$ with $St(d,\{x\}) \subseteq G$. Take (e, b), $(f, a) \in \mathcal{S}_o$ with $(f, a) \prec^{(*)} (e, b) \prec^{(*)} (d, c)$, and $y \in \dot{x}$. Since $y \sim x$ we have $St(f,\{x\})aSt(\{y\}, f)$. Take LfT with $x \in L \cap T$, L'fT' with $y \in L' \cap T'$, and take ReS, R'eS' with $St_a(f, L) \subseteq R$, $St_a(T, f) \subseteq S$, $St_a(f, L') \subseteq R'$ and $St_a(T', f) \subseteq S'$. Then $St(f,\{x\}) \subseteq St_a(f, L) \subseteq R$ and $St(\{y\},f) \subseteq St_a(T', f)$ so RaS'. Hence RbS' since $a \subseteq b$. Now take UdV with $St_b(e, R) \subseteq U$

and $St_b(S, e) \subseteq V$. Then $x \in T \subseteq St_a(T, f) \subseteq S \subseteq St_b(S, e) \subseteq V$ and so $y \in L' \subseteq St_a(f, L') \subseteq R' \subseteq St_b(e, R) \subseteq U \subseteq St(d, \{x\}) \subseteq G$ as required. The proof of the other case is similar.

<u>Corollary</u> 1. If G is $t_u(\mathcal{S})$ or $t_v(\mathcal{S})$ - open then $\varphi^{-1}(\varphi(G)) = G$

<u>Corollary</u> 2. If $G \in t_v(\mathcal{S})$ and $H \in t_v(\mathcal{S})$ then $\dot{G}\dot{c}\dot{H} \Leftrightarrow GcH$ for all $c \in \operatorname{ran} \mathcal{S}_o$.

<u>Corollary</u> 3. For $(d, c) \in \mathcal{S}_o$ and $x \in X$ we have $\varphi^{-1}(St(\dot{d}, \{\dot{x}\})) = St(d, \{x\})$ and $\varphi^{-1}(St(\{\dot{x}\}, \dot{d})) = St(\{x\}, d)$.

We may now give:

<u>Theorem</u> 2.2.1. $\{(\dot{d}, \dot{c}) \mid (d, c) \in \mathcal{S}_o\}$ is an open interior base for a separated equ on $\dot{X}$. The canonical mapping φ is bi-open and bicontinuous.

<u>Proof</u>. If $(e, b) \in \mathcal{S}_o$ and $(e, b) \prec(*) (d, c)$ it follows easily from Lemma 2.2.3, Corollary 2, that $(\dot{e}, \dot{b}) \prec(*) (\dot{d}, \dot{c})$. Hence $\{(\dot{d}, \dot{c}) \mid (d, c) \in \mathcal{S}_o\}$ is a base for a equ $\dot{\mathcal{S}}$ on $\dot{X}$. That φ is bi-open follows at once from Lemma 2.2.3, and in particular this means that $\dot{d}$ is open for each $d \in \operatorname{dom} \mathcal{S}_o$. To see that $\dot{c}$ is interior for each $c \in \operatorname{ran} \mathcal{S}_o$ take $P\dot{c}Q$ with $P \cap Q = \emptyset$. To show $(t_u(\dot{\mathcal{S}})\text{-int}\{P\})\dot{c}(t_v(\dot{\mathcal{S}})\text{-int}\{Q\})$ it will suffice to show that $t_u(\mathcal{S})\text{-int}[\varphi^{-1}(P)] \subseteq t_u(\mathcal{S})\text{-int}[\varphi^{-1}(t_u(\dot{\mathcal{S}})\text{-int}[P])]$, and a corresponding result for Q. However these results follow easily from Lemma 2.2.3, and the fact that φ is bi-open. Hence $\{(\dot{d}, \dot{c}) \mid (d, c) \in \mathcal{S}_o\}$ is an open interior base for $\dot{\mathcal{S}}$ as stated.

That φ is bicontinuous is an immediate consequence of Lemma 2.2.3, Corollary 3.

Finally to show $(\dot{X}, \dot{\mathcal{S}})$ is separated take $\dot{x}, \dot{y} \in \dot{X}$ with $\dot{x} \neq \dot{y}$. Then for some $(d, c) \in \mathcal{S}_o$ we have, say, $St(d, \{x\}) \not\subseteq St(\{y\}, d)$, from which we deduce $St(\dot{d},\{\dot{x}\}) \not\subseteq St(\{\dot{y}\}, \dot{d})$. This completes the proof of the theorem.

Definition 2.2.3. We call $(\dot{X}, \dot{\mathcal{S}})$ the associated separated cqu for $(X, \mathcal{S})$.

We have noted above that if $\mathcal{S}$ is a cqu then $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is pairwise regular. With an additional assumption on the confluence relation D we may give a stronger result. First we shall need the following definitions.

Definition 2.2.4. Let (X, u, v) be a bitopological space, c a confluence relation on X and $A \subseteq X$. We set

${}^cA = \{ x \mid x \in X \text{ and } x \in H \in v \Rightarrow AcH \}$, and

$A^c = \{ x \mid x \in X \text{ and } x \in G \in u \Rightarrow GcA \}$.

We say that c is bicompatible with (X, u, v) if ${}^cA = v\text{-cl}[A]$ and $A^c = u\text{-cl}[A]$ for all $A \subseteq X$.

Definition 2.2.5. Let (X, u, v) be a bitopological space and c a confluence relation on X. Then (X, u, v) is c-regular if for $x \in X$ and $A \subseteq X$ we have

(a) $x \notin {}^cA \Rightarrow \exists\ G \in u$ and $H \in v$ with $x \in H$, $A \subseteq G$ and $G \not{c} H$, and

(b) $x \notin A^c \Rightarrow \exists\ G \in u$ and $H \in v$ with $x \in G$, $A \subseteq H$ and $G \not{c} H$.

Note that the confluence relation ι is bicompatible with any bitopological space, and that ι-regular means the same as pairwise regular.

Definition 2.2.6. Let (X, u, v) be a bitopological space and c a confluence relation on X. We say c has the open union property (oup) if whenever $G_\alpha \in u$, $H_\beta \in v$ and $(\bigcup_\alpha \{G_\alpha\}) c (\bigcup_\beta \{H_\beta\})$ then $G_\alpha c H_\beta$ for some α, β.

ι, of course, has the oup for any bitopological space.

Proposition 2.2.3. Let $(X, \mathcal{S})$ be a cqu for which D has the oup. Then D is bicompatible with $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, and this bitopological space is D-regular.

Proof. Take $x \notin u\text{-cl}[A]$; then for some $(d, c) \in \mathcal{S}_0$ we have $St(d,\{x\}) \cap A = \emptyset$. Take $(e, b) \in \mathcal{S}_0$ with $(e, b) \prec(*) (d, c)$, and consider $x \in G = St(e,\{x\}) \in u$ and $A \subseteq H = \bigcup\{ St(\{y\}, e) \mid y \in A\} \in v$. Suppose GDH, then, since D has the oup and e is open, we have ReS with $x \in S$ and R'eS' with $R' \cap A \neq \emptyset$ satisfying RDS'. Hence RbS', and for some UdV we have $x \in S \subseteq St_b(S, e) \subseteq V$ and hence $R' \subseteq St_b(e, R) \subseteq U \subseteq St(d,\{x\})$ and this gives the contradiction $St(d,\{x\}) \cap A \neq \emptyset$. Thus $G\not{D}H$. Since clearly $x \notin A^D$ implies $x \notin u\text{-cl}[A]$ this establishes Definition 2.2.5 (b). (a) may be proved likewise, so $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-regular. On the other hand $G\not{D}H$ above implies $G\not{D}A$, so $x \notin u\text{-cl}[A]$ implies $x \notin A^D$. Hence $A^D = u\text{-cl}[A]$. In the same way ${}^DA = v\text{-cl}[A]$, and so D is bicompatible.

Note that if D has the oup for $\mathcal{S}$ then $\dot{D}$ has the oup for $\dot{\mathcal{S}}$ and so in this event the conclusions of Proposition 2.2.3 will apply to the bitopological space $(\dot{X}, t_u(\dot{\mathcal{S}}), t_v(\dot{\mathcal{S}}))$.

2.3 INDUCED STRUCTURES.

Let (X, u, v) be a bitopological space, c a confluence relation on X and $A \subseteq X$. We define the <u>induced confluence relation</u> c_A on A by

$$Pc_AQ \iff P\imath Q \text{ or } \exists\ P' \in u,\ Q' \in v \text{ with } P'cQ',\ \emptyset \neq P' \cap A \subseteq P \text{ and } \emptyset \neq Q' \cap A \subseteq Q.$$

We may express this definition in another way. For $B \subseteq A$ let us define

$$B^*_u = \bigcup\{ G \mid G \in u,\ \emptyset \neq G \cap A \subseteq B\}, \text{ and}$$

$$B^*_v = \bigcup\{ H \mid H \in v,\ \emptyset \neq H \cap A \subseteq B\}.$$

Then it is immediate that $Pc_AQ \iff P\imath Q$ or $(P^*_u)c(Q^*_v)$.

Note that c_A is necessarily an interior confluence relation for the induced bitopological space on A, even if c is not interior on X. This is one of the main reasons why we have found it convenient to restrict ourselves to the study of interior cqu.

If $\mathcal{S}$ is a cqu on X consider the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, and take $(d, c) \in \mathcal{S}_o$. We define the relation

d_A on the non-empty subsets of A by

$$P d_A Q \iff \exists\ UdV \text{ with } P = U \cap A \neq \emptyset,\ Q = V \cap A \neq \emptyset.$$

Because $(d, c) \in \mathcal{S}_o$ is open for $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, we see that d_A is an open c_A-dual cover of A relative to the induced bitopological space. We are therefore lead to the following definition.

<u>Definition</u> 2.3.1. $\mathcal{S}_A = \{(d, c) \mid d$ is a c-dual cover of A and there exists $(e, b) \in \mathcal{S}_o$ with $(e_A, b_A) \prec (d, c)\}$ is called the <u>induced structure</u> of $\mathcal{S}$ on A.

In general $\mathcal{S}_A$ need not be a cqu on A. In order to ensure that it is we need to impose some additional embedding conditions on A. The following seem to be appropriate.

<u>Definition</u> 2.3.2. Let $(X, \mathcal{S})$ be a cqu and $A \subseteq X$.

(a) We say A is <u>$\mathcal{S}$-embedded</u> in X if for some base $\beta \subseteq \mathcal{S}_o$ of $\mathcal{S}$ we have UcV whenever $(d, c) \in \beta$, $U \in \text{dom } d$, $V \in \text{ran } d$ and $(U \cap A) c_A (V \cap A)$.

(b) We say A is <u>strictly $\mathcal{S}$-embedded</u> in X if for some base $\beta \subseteq \mathcal{S}_o$ we have

(i) $(U \cap A)^*_u \subseteq U$ and $(V \cap A)^*_v \subseteq V$ whenever $d \in \text{dom } \beta$, $U \in \text{dom } d$ and $V \in \text{ran } d$; and

(ii) Given $c \in \text{ran } \beta$, PcQ and $P \cap Q = \emptyset$ we have $P' \in t_u(\mathcal{S})$, $Q' \in t_v(\mathcal{S})$ with $P'cQ'$, $(P' \cap A)^*_u \subseteq P$ and $(Q' \cap A)^*_v \subseteq Q$.

If A is strictly $\mathcal{S}$-embedded in X it is clearly also $\mathcal{S}$-embedded.

<u>Lemma</u> 2.3.1. Let A be strictly $\mathcal{S}$-embedded and bidense in X, and let $\beta \subseteq \mathcal{S}_o$ be a base of $\mathcal{S}$ as in Definition 2.3.2 (b). Then if $c \in \text{ran } \beta$ and b is a confluence relation on X with $c_A \subseteq b_A$, we have $c \subseteq b$.

<u>Proof</u>. Take PcQ with $P \cap Q = \emptyset$. Then $\exists\ P' \in t_u(\mathcal{S})$, $Q' \in t_v(\mathcal{S})$ with $P'cQ'$ and $(P' \cap A)^*_u \subseteq P$, $(Q' \cap A)^*_v \subseteq Q$. Now $P' \cap A c_A Q' \cap A$ so $P' \cap A b_A Q' \cap A$, and hence $\exists\ P'' \in t_u(\mathcal{S})$, $Q'' \in t_v(\mathcal{S})$ with

$\emptyset \neq P'' \cap A \subseteq P' \cap A$, $\emptyset \neq Q'' \cap A \subseteq Q' \cap A$ and $P''bQ''$. But then $P'' \subseteq (P' \cap A)^*_u \subseteq P$ and $Q'' \subseteq (Q' \cap A)^*_v \subseteq Q$, and so PbQ as required.

Proposition 2.3.1. Let $(X, \mathcal{S})$ be a cqu and $A \subseteq X$. If A is $\mathcal{S}$-embedded in X then $(A, \mathcal{S}_A)$ is an interior cqu compatible with the induced bitopological space on A.

Proof. For (d, c), (e, b) belonging to the base $\beta \subseteq \mathcal{S}_0$ having the properties mentioned in Definition 2.3.2 (a) it is easy to verify that $(d, c) \prec^{(*)} (e, b)$ implies $(d_A, c_A) \prec^{(*)} (e_A, b_A)$ so $\mathcal{S}_A$ is a cqu. Also $St(d_A, \{x\}) = St(d, \{x\}) \cap A$ and $St(\{x\}, d_A) = St(\{x\}, d) \cap A$ for all $x \in A$, so $\mathcal{S}_A$ defines the induced topologies, and the fact that $\{(d_A, c_A) \mid (d, c) \in \mathcal{S}_0\}$ is an open interior base of $\mathcal{S}_A$ follows at once from this.

It will be noted that if A is a bidense and strictly $\mathcal{S}$-embedded subset of X then $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is a strict bitopological extension of $(A, t_u(\mathcal{S}_A), t_v(\mathcal{S}_A))$ in the sense that $(X, t_u(\mathcal{S}))$ is a strict topological extension of $(A, t_u(\mathcal{S}_A))$, and $(X, t_v(\mathcal{S}))$ is a strict topological extension of $(A, t_v(\mathcal{S}_A))$. The reader is refered to [11] for a general discussion of strict topological extensions.

Definition 2.3.3. Let (X, u, v) be a bitopological space, $x \in X$ and $A \subseteq X$. We set

$$\mathcal{B}^A(x) = \{(P \cap A, Q \cap A) \mid (P, Q) \in \mathcal{B}(x),\ (u\text{-int}[P]) \cap A \neq \emptyset \neq (v\text{-int}[Q]) \cap A\}.$$

Note that if A is bidense in X then $\mathcal{B}^A(x)$ will be a bifilter on A. Also $B^*_u = \{x \mid B \in \mathcal{B}^A_u(x)\}$, and $B^*_v = \{x \mid B \in \mathcal{B}^A_v(x)\}$ for all $B \subseteq A$.

Definition 2.2.4. Let $(X, \mathcal{S})$ be a cqu. We say the bifilter $\mathcal{B}$ on X is $\mathcal{S}$-Cauchy if $d \cap \mathcal{B} \neq \emptyset$ for all $d \in \operatorname{dom} \mathcal{S}$.

Clearly it is sufficient for this condition to hold for any base of $\mathcal{S}$.

Proposition 2.3.2. Let $(X, \mathcal{S})$ be a cqu, $A \subseteq X$ bidense for $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, and $x \in X$. Then $\mathcal{B}^A(x)$ is a D_A-regular $\mathcal{S}_A$-Cauchy bifilter on A.

We omit the proof which is straightforward.

Corollary 1. If A is bidense and $\mathcal{S}$-embedded in X then $\mathcal{B}^A(x)$ is a minimal D_A-regular $\mathcal{S}_A$-Cauchy bifilter on A.

Proof. It remains only to show the minimality. Let $\mathcal{B}$ be a $\mathcal{S}_A$-Cauchy (necessarily D_A-regular) bifilter on A with $\mathcal{B} \subseteq \mathcal{B}^A(x)$. Take $(P, Q) \in \mathcal{B}^A(x)$, and let $\beta \subseteq \mathcal{S}_0$ be a base of $\mathcal{S}$ as in Definition 2.3.2 (a). Then we have $(d, c) \in \mathcal{S}$ with $St(d, \{x\}) \cap A \subseteq P$, $St(\{x\}, d) \cap A \subseteq Q$ and $(e, b) \in \beta$ with $(e, b) \prec(\omega) (d, c)$. Since $\mathcal{B}$ is $\mathcal{S}_A$-Cauchy $\exists$ ReS with $(R \cap A, S \cap A) \in \mathcal{B} \subseteq \mathcal{B}^A(x)$. Take R'eS' with $x \in R' \cap S'$. Since e is open we also have $(R' \cap A, S' \cap A) \in \mathcal{B}^A(x)$. It follows by Proposition 2.3.2 that $R' \cap A b_A S \cap A$ and $R \cap A b_A S' \cap A$ so that R'bS and RbS'. Now take UdV with $St_b(e, R') \subseteq U$ and $St_b(S', e) \subseteq V$. Then $R \subseteq U \subseteq St(d, \{x\})$ and $S \subseteq V \subseteq St(\{x\}, d)$ so $R \cap A \subseteq P$ and $S \cap A \subseteq Q$ which proves $(P, Q) \in \mathcal{B}$. Hence $\mathcal{B} = \mathcal{B}^A(x)$, and $\mathcal{B}^A(x)$ is minimal.

Since X is bidense and $\mathcal{S}$-embedded in itself we have

Corollary 2. The nhd. bifilters $\mathcal{B}(x)$ of $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ are minimal D-regular $\mathcal{S}$-Cauchy bifilters on X.

With regard to the existence of minimal $\mathcal{S}$-Cauchy bifilters in general we have:

Proposition 2.3.3. Let $(X, \mathcal{S})$ be a cqu with base $\beta \subseteq \mathcal{S}_0$ and $\mathcal{B}$ a bifilter on X. Denote by $\mathcal{B}^*$ the bifilter with subbase

$$\{(St_c(d, U), St_c(V, d)) \mid (d, c) \in \beta, U \in \text{dom } d, V \in \text{ran } d \text{ and } (U, V) \in \mathcal{B}\}.$$

Then if $\mathcal{B}$ is β-regular and $\mathcal{S}$-Cauchy, $\mathcal{B}^*$ is minimal D-regular $\mathcal{S}$-Cauchy and is contained in $\mathcal{B}$.

Proof. That $\mathcal{B}^*$ is a bifilter with $\mathcal{B}^* \subseteq \mathcal{B}$ is clear. Also if

$\mathcal{B}$ is $\mathcal{S}$-Cauchy it is easy to see that $\mathcal{B}^*$ is also. Now let $\mathcal{B}$ be β-regular and $\mathcal{S}$-Cauchy, and let $\mathfrak{b}$ be a $\mathcal{S}$-Cauchy bifilter with $\mathfrak{b} \subseteq \mathcal{B}^*$. If $(P, Q) \in \mathcal{B}^*$ $\exists$ $(d_i, c_i) \in \beta$, $i = 1, 2, \ldots, n$, $U_i \in \operatorname{dom} d_i$ and $V_i \in \operatorname{ran} d_i$ with $(U_i, V_i) \in \mathcal{B}$, $\bigcap \{ St_{c_i}(d_i, U_i)\} \subseteq P$ and $\bigcap \{ St_{c_i}(V_i, d_i)\} \subseteq Q$. Take any $(e, b) \in \mathcal{S}$, and take $(d, c) \in \mathcal{S}$ with $(d, c) \prec (d_i, c_i)$, $1 \leq i \leq n$, and $(d, c) \prec (e, b)$. Since $\mathfrak{b}$ is $\mathcal{S}$-Cauchy we have UdV with $(U, V) \in \mathfrak{b} \subseteq \mathcal{B}^* \subseteq \mathcal{B}$. If we take $U'_i d_i V'_i$ with $U \subseteq U'_i$, $V \subseteq V'_i$ we have $(U'_i, V_i) \in \mathcal{B}$ and $(U_i, V'_i) \in \mathcal{B}$ and so $U'_i c_i V_i$ and $U_i c_i V'_i$ since $\mathcal{B}$ is β-regular. Hence

$$U'_i \subseteq St_{c_i}(d_i, U_i) \text{ and } V'_i \subseteq St_{c_i}(V_i, d_i)$$

and so $U \subseteq P$ and $V \subseteq Q$. It follows that $(P, Q) \in \mathfrak{b}$ and so $\mathfrak{b} = \mathcal{B}^*$ which proves that $\mathcal{B}^*$ is minimal $\mathcal{S}$-Cauchy. Also $UdV \Rightarrow UcV \Rightarrow UbV \Rightarrow PbQ \Rightarrow PDQ$ since $b \in \operatorname{ran} \mathcal{S}$ was arbitrary. Hence $\mathcal{B}^*$ is D-regular, and the proof is complete.

This result enables us to work almost exclusively in terms of D-regular bifilters.

2.4 CONFLUENCE QUASI-UNIFORM CONTINUITY.

In this section we define the notion of confluence quasi-uniform continuity. As with the induced structure this involves in an essential way our assumption that the cqu under consideration should be interior.

Definition 2.4.1. Let X and Y be non-empty sets, $f : X \to Y$ a function and μ a cqu on Y. Let c be a confluence relation on Y, and $P, Q \subseteq X$. We define the confluence relation $f^{-1}(c)$ on X by:

$$Pf^{-1}(c)Q \Leftrightarrow P\iota Q \text{ or } \exists \; P' \in t_u(\mathcal{S}),\; Q' \in t_v(\mathcal{S}) \text{ with } P'cQ',$$
$$\emptyset \neq f^{-1}(P') \subseteq P \text{ and } \emptyset \neq f^{-1}(Q') \subseteq Q.$$

If d is a c-dual cover of Y then $f^{-1}(d)$ is defined by

$$Pf^{-1}(d)Q \Leftrightarrow \exists \; UdV \text{ with } \emptyset \neq f^{-1}(U) = P,\; \emptyset \neq f^{-1}(V) = Q.$$

Note that if d is an open c-dual cover of Y then $f^{-1}(d)$ is a $f^{-1}(c)$-dual cover of X.

If δ is a cqu on X we shall say that f is <u>$(\delta - \mu)$ confluence quasi-uniformly continuous</u> (cquc) if for each open $(d, c) \in \mu$ we have $(f^{-1}(d), f^{-1}(c)) \in \delta$.

Note that it will be sufficient for this condition to hold for all $(d, c) \in \gamma$, where γ is a base of μ with $\gamma \subseteq \mu_0$.

We may note the following examples of cquc functions. The verification is trivial and is omitted.

(i) The identity mapping $i : (X, \delta) \to (X, \delta)$.

(ii) The injection mapping $j : (A, \delta_A) \to (X, \delta)$, where $A \subseteq X$ is δ-embedded.

(iii) The canonical mapping $\varphi : (X, \delta) \to (\dot{X}, \dot{\delta})$, where $(\dot{X}, \dot{\delta})$ is the associated separated cqu of (X, δ).

Indeed in (i) and (ii) the cqu on the left is the coarsest for which the corresponding function is cquc. (Initial cqu).

The following result is a trivial consequence of the definitions.

<u>Proposition</u> 2.4.1. If $f : X \to Y$ is $(\delta - \mu)$ cquc it is $(X, t_u(\delta), t_v(\delta)) - (Y, t_u(\mu), t_v(\mu))$ bicontinuous.

<u>Definition</u> 2.4.2. Let (X, δ) and (Y, μ) be cqu. (X, δ) and (Y, μ) are <u>isomorphic</u> if there is a bijective map $f : X \to Y$ so that f is $(\delta - \mu)$ cquc and f^{-1} is $(\mu - \delta)$ cquc.

<u>Definition</u> 2.4.3. Let (X, δ) and (Y, μ) be cqu, and $g : X \to Y$ a mapping. We shall say that (Y, μ) is an <u>extension</u> of (X, δ) under g if

(a) (X, δ) and $(g(X), \mu_{g(X)})$ are isomorphic under g, and

(b) g(X) is bidense and μ-embedded in (Y, μ).

If g(X) is in fact strictly μ-embedded in Y we shall say (Y, μ) is a <u>strict extension</u> of (X, δ). Note that if (Y, μ) is a (strict) extension of (X, δ), then certainly $(Y, t_u(\mu), t_v(\mu))$ will be a (strict) bitopological extension of $(X, t_u(\delta), t_v(\delta))$.

We note the following for future use.

Lemma 2.4.1. Let $f:X \to Y$ be a function and μ a cqu on Y.
(a) If $f(X)$ is μ-embedded in Y and $\gamma \subseteq \mu_0$ is a base of μ as in Definition 2.3.2 (a) then $(d, c) \in \gamma$, $U \in \operatorname{dom} d$, $V \in \operatorname{ran} d$ and $f^{-1}(U) f^{-1}(c) f^{-1}(V)$ imply UcV.
(b) If $f(X)$ is strictly μ-embedded in Y and $\gamma \subseteq \mu_0$ is a base of μ as in Definition 2.3.2 (b) then $c \in \operatorname{ran} \gamma$ and $f^{-1}((P \cap f(X))^*_u) f^{-1}(c) f^{-1}((Q \cap f(X))^*_v)$ imply $(P \cap f(X))^*_u c (Q \cap f(X))^*_v$.

We omit the proof which is straightforward.

If we have an extension of a separated cqu we may form a separated extension by forming the associated separated cqu. The details are given below.

Proposition 2.4.2. Let (X, δ) be a separated cqu, and (Y, μ) an extension (respectively, strict extension) of (X, δ) for the mapping g. If $(\dot{Y}, \dot{\mu})$ is the associated separated cqu of (Y, μ), and $\varphi : Y \to \dot{Y}$ is the canonical mapping then $(\dot{Y}, \dot{\mu})$ is a separated extension (respectively, separated strict extension) of (X, δ) for the mapping $\dot{g} = \varphi \circ g$.

Proof. Since $(g(X), \mu_{g(X)})$ is isomorphic to the separated space (X, δ) it follows that the relation $\sim$ on Y reduces to the identity on $g(X)$, that is $\dot{g(x)} = \dot{g}(x) = \{g(x)\}$ for all $x \in X$. Hence $\dot{g}$ is a set isomorphism of X and $\dot{g}(X)$. Also $\dot{g}$, being the composition of the cquc maps g and φ is also cquc. Let us verify $\dot{g}^{-1} : \dot{g}(X) \to X$ is $(\dot{\mu}_{\dot{g}(X)} - \delta)$ cquc. Now since $g^{-1} : g(X) \to X$ is $(\mu_{g(X)} - \delta)$ cquc, given $(d, c) \in \delta_0$ $\exists$ $(e, b) \in \mu_0$ with $(e_{g(X)}, b_{g(X)}) \prec ((g^{-1})^{-1}(d), (g^{-1})^{-1}(c))$. We may deduce from this that $(\dot{e}_{\dot{g}(X)}, \dot{b}_{\dot{g}(X)}) \prec ((\dot{g}^{-1})^{-1}(d), (\dot{g}^{-1})^{-1}(c))$, and the result follows at once. Hence $\dot{g}$ is an isomorphism of X and $\dot{g}(X)$. That $\dot{g}(X)$ is bidense in Y is clear.

Now suppose $g(X)$ is μ-embedded in Y, and let $\gamma \subseteq \mu_0$ be a base of μ as in Definition 2.3.2 (a). Consider the base $\dot{\gamma}$ =

$\{(\dot{d}, \dot{c}) \mid (d, c) \in \gamma\} \subseteq \dot{\mu}_0$. Take $(d, c) \in \gamma$, $U \in \operatorname{dom} d$, $V \in \operatorname{ran} d$ with $(\dot{U} \cap \dot{g}(X)) c_{\dot{g}(X)} (\dot{V} \cap \dot{g}(X))$. Then either $\dot{U} \cap \dot{g}(X) \cap \dot{V} \cap \dot{g}(X) \neq \emptyset$, in which event $\dot{U} \dot{c} \dot{V}$ follows at once, or we have $P \in t_u(\dot{\mu})$, $Q \in t_v(\dot{\mu})$ with $P \dot{c} Q$, $\emptyset \neq P \cap \dot{g}(X) \subseteq \dot{U} \cap \dot{g}(X)$ and $\emptyset \neq Q \cap \dot{g}(X) \subseteq \dot{V} \cap \dot{g}(X)$. Then certainly $\varphi^{-1}(P) c\, \varphi^{-1}(Q)$, while $\varphi^{-1}(P) \in t_u(\mu)$ and $\varphi^{-1}(Q) \in t_v(\mu)$, and so $(\varphi^{-1}(P) \cap g(X)) c_{g(X)} (\varphi^{-1}(Q) \cap g(X))$. However $\varphi^{-1}(P) \cap g(X) \subseteq \varphi^{-1}(\dot{U}) \cap g(X) = U \cap g(X)$ by Lemma 2.2.3, Corollary 1, since $U \in t_u(\mu)$; and likewise $\varphi^{-1}(Q) \cap g(X) \subseteq V \cap g(X)$. Thus $(U \cap g(X)) c_{g(X)} (V \cap g(X))$, which implies UcV since $g(X)$ is μ-embedded in Y. Hence we again have $\dot{U} \dot{c} \dot{V}$, and so $\dot{g}(X)$ is $\dot{\mu}$-embedded in $\dot{Y}$ as required.

Finally suppose $g(X)$ is strictly μ-embedded in Y. For $A \in t_u(\mu)$ and $B \in t_v(\mu)$ it is easy to verify that

$$(\dot{A} \cap \dot{g}(X))^*_u = \varphi((A \cap g(X))^*_u), \text{ and}$$
$$(\dot{B} \cap \dot{g}(X))^*_v = \varphi((B \cap g(X))^*_v).$$

It follows easily from these results that if $\gamma \subseteq \mu_0$ is a base of μ as in Definition 2.3.2 (b), then the base $\dot{\gamma}$ defined as above for this γ also satisfies the conditions of Definition 2.3.2 (b). Hence $\dot{g}(X)$ is strictly $\dot{\mu}$-embedded in $\dot{Y}$, and the proof is complete.

2.5 COMPLETENESS AND COMPLETIONS OF CONFLUENCE QUASI-UNIFORMITIES.

Definition 2.5.1. Let (X, δ) be a cqu. We shall say that (X, δ) is complete if each D-regular δ-Cauchy bifilter on X is convergent in $(X, t_u(\delta), t_v(\delta))$.

Proposition 2.5.1. Let (X, δ) be a complete cqu, and $\beta \subseteq \delta_0$ a base of δ. Then every β-regular δ-Cauchy bifilter on X is convergent.

Proof. If $\mathcal{B}$ is β-regular and δ-Cauchy, and we construct $\mathcal{B}^*$ as in Proposition 2.3.3, then $\mathcal{B}^*$ is (minimal) D-regular and

$\mathcal{S}$-Cauchy, and hence convergent. However $\mathcal{B}^* \subseteq \mathcal{B}$ so $\mathcal{B}$ is convergent also.

Let $\mathcal{S}$ be an interior separated cqu on X, and let $\Gamma(X)$ be any set of $\mathcal{S}_o$-regular $\mathcal{S}$-Cauchy bifilters on X which includes the set $\{\mathcal{B}(x) \mid x \in X\}$ of nhd. bifilters on X. We denote by j the one to one map $j(x) = \mathcal{B}(x)$ of X into $\Gamma(X)$. We are going to show how we may give $\Gamma(X)$ an interior confluence quasi-uniformity $\tilde{\mathcal{S}}$ with the property that $(\Gamma(X), \tilde{\mathcal{S}})$ is a strict extension of $(X, \mathcal{S})$ for the map j. For $A \subseteq X$ define

$$A_u = \{\mathcal{B} \mid \mathcal{B} \in \Gamma(X),\ A \in \mathcal{B}_u\}, \text{ and}$$

$$A^o_u = \{\mathcal{B} \mid \mathcal{B} \in \Gamma(X) \text{ and } \exists\ (d, c) \in \mathcal{S}_o \text{ such that } UdV \text{ and } V \in \mathcal{B}_v \Rightarrow U \subseteq A\}.$$

For $B \subseteq X$ we define B_v and B^o_v in a similar way. Some important properties of these sets are set out in the next lemma.

<u>Lemma</u> 2.5.1. For each $A, B \subseteq X$:

(a) $A^o_u \subseteq A_u$ and $B^o_v \subseteq B_v$

(b) $j^{-1}(A^o_u) = j^{-1}(A_u) = t_u(\mathcal{S})\text{-int}[A]$, and

$j^{-1}(B^o_v) = j^{-1}(B_v) = t_v(\mathcal{S})\text{-int}[B]$.

We omit the proof, which is straightforward.

If c is a confluence relation on X we may define the confluence relation $\tilde{c}$ on $\Gamma(X)$ by

$$P\,\tilde{c}\,Q \iff P\iota Q \text{ or } \exists\ A \in t_u(\mathcal{S}),\ B \in t_v(\mathcal{S}) \text{ with } AcB,\ A^o_u \subseteq P \text{ and } B^o_v \subseteq Q.$$

If d is a c-dual cover of X we define $\tilde{d}$ on $\Gamma(X)$ by

$$P\,\tilde{d}\,Q \iff \exists\ UdV \text{ with } P = U^o_u \text{ and } Q = V^o_v.$$

If $(d, c) \in \mathcal{S}$ is open then $\tilde{d}$ is a $\tilde{c}$-dual cover of $\Gamma(X)$. That $\tilde{d} \subseteq \tilde{c}$ is clear. To show the uniform covering of $\tilde{d}$ is $\Gamma(X)$ take $\mathcal{B} \in \Gamma(X)$ and $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec(*) (d, c)$. Since $\mathcal{B}$ is $\mathcal{S}$-Cauchy we have ReS with $(R, S) \in \mathcal{B}$. Also we have UdV with

$St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$, and it is easy to verify that $\mathcal{B} \in U^o{}_u \cap V^o{}_v$; while by definition $U^o{}_u \; \tilde{d} \; V^o{}_v$.

We may now give:

Theorem 2.5.1. With the notation as above, $\{ (\tilde{d}, \tilde{c}) \mid (d, c) \in \mathcal{S}_o \}$ is an open interior base for a confluence quasi-uniformity $\tilde{\mathcal{S}}$ on $\Gamma(X)$. $(\Gamma(X), \tilde{\mathcal{S}})$ is a strict extension of $(X, \mathcal{S})$ under the map j.

Proof. For (d, c), $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec (*) (d, c)$ take ReS and UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$. If $\mathcal{B} \in St_{\tilde{b}}(\tilde{e}, R^o{}_u)$ we have $R'eS'$ with $\mathcal{B} \in R'^o{}_u$ and $R^o{}_u \; \tilde{b} \; S'^o{}_v$.
There are two cases:

(i) $R^o{}_u \cap S'^o{}_v \neq \emptyset$. Then $\exists \; \mathcal{L} \in R_u \cap S'_v$ by Lemma 2.5.1 (a), and $(R, S') \in \mathcal{L}$ implies RbS' since $\mathcal{L}$ is $\mathcal{S}_o$-regular.

(ii) $R^o{}_u \cap S'^o{}_v = \emptyset$. Then $\exists \; A \in t_u(\mathcal{S})$, $B \in t_v(\mathcal{S})$ with AbB, $A^o{}_u \subseteq R^o{}_u$ and $B^o{}_v \subseteq S'^o{}_v$. But then $A = t_u(\mathcal{S})\text{-int}[A] = j^{-1}(A^o{}_u) \subseteq j^{-1}(R^o{}_u) = t_u(\mathcal{S})\text{-int}[R] = R$ by Lemma 2.5.1 (b). In the same way $B \subseteq S'$, and so we again have RbS'.

In either case we therefore have $R' \subseteq St_b(e, R) \subseteq U$, and hence $St_{\tilde{b}}(\tilde{e}, R^o{}_u) \subseteq U^o{}_u$. In just the same way $St_{\tilde{b}}(S^o{}_v, \tilde{e}) \subseteq V^o{}_v$. Finally we clearly have $\tilde{b} \subseteq \tilde{c}$, and hence $(\tilde{e}, \tilde{b}) \prec (*) (\tilde{d}, \tilde{c})$. This proves that $\{ (\tilde{d}, \tilde{c}) \mid (d, c) \in \mathcal{S}_o \}$ is a base for a cqu $\tilde{\mathcal{S}}$ on $\Gamma(X)$.

Now let us verify that for $A, B \subseteq X$ we have:

$$[j(A)]^*_u = A^o{}_u = t_u(\tilde{\mathcal{S}})\text{-int}[A_u] \text{ , and}$$
$$[j(B)]^*_v = B^o{}_v = t_v(\tilde{\mathcal{S}})\text{-int}[B_v] \text{ ,}$$

where the sets on the left are formed for the subset $j(X)$ of the bitopological space $(\Gamma(X), t_u(\tilde{\mathcal{S}}), t_v(\tilde{\mathcal{S}}))$.
First take $\mathcal{B} \in [j(A)]^*_u$, then $\mathcal{B} \in G \in t_u(\tilde{\mathcal{S}})$ with $G \cap j(X) \subseteq j(A)$.

Take $(d, c) \in \delta_o$ with $St(\tilde{d}, \{\mathcal{B}\}) \subseteq G$, and $(e, b) \in \delta_o$ with $(e, b) \prec(\omega) (d, c)$. Since $\mathcal{B}$ is δ-Cauchy we have ReS with $(R, S) \in \mathcal{B}$, and we have UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$. Then $\mathcal{B} \in V^o{}_v$ and so $U^o{}_u \subseteq St(\tilde{d}, \{\mathcal{B}\}) \subseteq G$. Hence $U \subseteq A$ since j is one to one. To show that $\mathcal{B} \in A^o{}_u$ take $R'eS'$ with $S' \in \mathcal{B}_v$. Then $(R, S') \in \mathcal{B}$ so RbS' since $\mathcal{B}$ is δ_o-regular. Hence $R' \subseteq St_b(e, R) \subseteq U \subseteq A$, and $\mathcal{B} \in A^o{}_u$ as required. This verifies $[j(A)]^*_u \subseteq A^o{}_u$.

If now $\mathcal{B} \in A^o{}_u$ and $(d, c) \in \delta_o$ satisfies UdV, $V \in \mathcal{B}_v \Rightarrow U \subseteq A$, then it is easy to verify $St(\tilde{d}, \{\mathcal{B}\}) \subseteq A_u$, and so $t_u(\tilde{\delta})\text{-int}[A_u]$. Hence $A^o{}_u \subseteq t_u(\tilde{\delta})\text{-int}[A_u]$.

Finally it is trivial to verify $t_u(\tilde{\delta})\text{-int}[A_u] \subseteq [j(A)]^*_u$, and so the firt set of equalities is proved. The proof of the other equalities is similar.

It follows at once that $\{(\tilde{d}, \tilde{c}) \mid (d, c) \in \delta_o\}$ is an open interior base for $\tilde{\delta}$.

It is trivial to verify that $j(X)$ is bidense in $\Gamma(X)$. Also, using the above equalities together with Lemma 2.5.1 we may easily show that the conditions of Definition 2.3.2 (b) are satisfied for the base $\{(\tilde{d}, \tilde{c}) \mid (d, c) \in \delta_o\}$ of $\tilde{\delta}$, and so $j(X)$ is strictly $\tilde{\delta}$-embedded in $\Gamma(X)$.

It remains to show that j is an isomorphism of (X, δ) and $(j(X), \tilde{\delta}_{j(X)})$. However this follows at once from the relations

$$(d, c) \prec (j^{-1}(\tilde{d}_{j(X)}, j^{-1}(\tilde{c}_{j(X)}), \text{ and}$$

$$(\tilde{d}_{j(X)}, \tilde{c}_{j(X)}) \prec ((j^{-1})^{-1}(d), (j^{-1})^{-1}(c))$$

which are easily verified for any $(d, c) \in \delta_o$. This completes the proof of the theorem.

The strict extension $(\Gamma(X), \tilde{\delta})$ constructed above will not in general be separated. As mentioned above (Proposition 2.4.2) we may obtain a separated strict extension by taking the associated separated cqu. A second way in which we may obtain a separated extension is to require that $\Gamma(X)$ contain only <u>minimal</u> $\mathcal{S}$-Cauchy bifilters. We know in any event by the corollary 2 to Proposition 2.3.2 that the elements of $j(X)$ do have this property. We use this second method in the following theorem.

<u>Theorem</u> 2.5.2. Let $\Gamma_o(X)$ denote the set of all D-regular minimal $\mathcal{S}$-Cauchy bifilters on the separated cqu $(X, \mathcal{S})$. Then $(\Gamma_o(X), \tilde{\mathcal{S}})$ is a complete separated strict extension of $(X, \mathcal{S})$.

<u>Proof</u>. By Theorem 2.5.1 we know that $(\Gamma_o(X), \tilde{\mathcal{S}})$ is a strict extension of $(X, \mathcal{S})$, so it remains only to show it is complete and separated. Let B be a $\tilde{D}$-regular $\tilde{\mathcal{S}}$-Cauchy bifilter on $\Gamma_o(X)$, and set

$$\mathcal{B} = \{(P, Q) \mid P, Q \subseteq X, (P^o{}_u, Q^o{}_v) \in B\}.$$

It is easy to see that $\mathcal{B}$ is a D-regular $\mathcal{S}$-Cauchy bifilter on X. Construct $\mathcal{B}^*$ as in Proposition 2.3.3 (for the base $\mathcal{S}_o$, say); then $\mathcal{B}^* \in \Gamma_o(X)$. Let us show B converges to $\mathcal{B}^*$ in $\Gamma_o(X)$. Take $(d, c) \in \mathcal{S}_o$. Since $\tilde{d}$ is a $\tilde{c}$-dual cover of $\Gamma_o(X)$ we have UdV with $\mathcal{B}^* \in U^o{}_u \cap V^o{}_v$. Hence $U^o{}_u \subseteq St(\tilde{d}, \{\mathcal{B}^*\})$ and $V^o{}_v \subseteq St(\{\mathcal{B}^*\}, \tilde{d})$. On the other hand $\mathcal{B}^* \in U_u \cap V_v$ implies $(U, V) \in \mathcal{B}^* \subseteq \mathcal{B}$ and so $(U^o{}_u, V^o{}_v) \in B$. Hence $(St(\tilde{d}, \{\mathcal{B}^*\}), St(\{\mathcal{B}^*\}, \tilde{d}) \in B$ for all $(d, c) \in \mathcal{S}_o$, and $B \rightarrow \mathcal{B}^*$ as required. This proves that $(\Gamma_o(X), \tilde{\mathcal{S}})$ is complete.

To show $(\Gamma_o(X), \tilde{\mathcal{S}})$ is separated take $\mathcal{B}, \mathcal{L} \in \Gamma_o(X)$ with $\mathcal{B} \neq \mathcal{L}$. Say, for example, that $\mathcal{B}_u \nsubseteq \mathcal{L}_u$. Since $\mathcal{B}$ is minimal $\mathcal{S}$-Cauchy and $\mathcal{B}^* \subseteq \mathcal{B}$ we have $\mathcal{B} = \mathcal{B}^*$. Hence $\exists$ $(d, c) \in \mathcal{S}_o$ and $U \in \text{dom } d$ with $U \in \mathcal{B}_u$ and $St_c(d, U) \notin \mathcal{L}_u$. Since $\mathcal{B}$ is D-regular we may deduce from this that $\mathcal{L} \notin St(\tilde{d}, \{\mathcal{B}\})$. The other cases may be dealt with in the same way, and we deduce that $(\Gamma_o(X), t_u(\tilde{\mathcal{S}}), t_v(\tilde{\mathcal{S}}))$ is weakly pairwise T_0. Hence $(\Gamma_o(X), \tilde{\mathcal{S}})$ is separated, and the proof is complete.

We now give a theorem on the extension of cquc functions. This is a basic step in proving that separated strict completions of separated cqu are unique up to isomorphism.

Theorem 2.5.3. Let (X, μ) be a cqu, (Y, δ) a complete separated cqu and A a bidense and strictly μ-embedded subset of X. If the function $f : A \to Y$ is $(\mu_A - \delta)$ cquc, and $f(A)$ is δ-embedded in Y, then f has a unique $(\mu - \delta)$ cquc extension $\overline{f} : X \to Y$.

Proof. Take $x \in X$. Since A is bidense in X, the nhd. bifilter trace $\mathcal{B}^A(x)$ is a bifilter on A. Hence

$$\mathcal{b}(x) = \{ (P, Q) \mid (f^{-1}(P), f^{-1}(Q)) \in \mathcal{B}^A(x) \}$$

is a bifilter on Y. Let $\beta \subseteq \delta_o$ be a base of δ with the properties of Definition 2.3.2 (a) for the δ-embedded subset $f(A)$ of Y. We will show that $\mathcal{b}(x)$ is β-regular and δ-Cauchy. Take any $(d, c) \in \beta$. Then $(f^{-1}(d), f^{-1}(c)) \in \mu_A$. Hence if $U \in$ dom d, $V \in$ ran d and $(U, V) \in \mathcal{b}(x)$ then $(f^{-1}(U), f^{-1}(V)) \in \mathcal{B}^A(x)$, and so $f^{-1}(U) f^{-1}(c) f^{-1}(V)$ since $\mathcal{B}^A(x)$ is D_A-regular by Proposition 2.3.2. Hence UcV by Lemma 2.4.1, and so $\mathcal{b}(x)$ is β-regular. Next take $(e, b) \in \mu_o$ with $(e_A, b_A) \prec (f^{-1}(d), f^{-1}(c))$, and ReS with $x \in R \cap S$. Since A is bidense $(R \cap A) e_A (S \cap A)$ so we have U'dV' with $R \cap A \subseteq f^{-1}(U')$ and $S \cap A \subseteq f^{-1}(V')$. However we also have $(R \cap A, S \cap A) \in \mathcal{B}^A(x)$ and so $(U', V') \in \mathcal{b}(x)$. Thus $\mathcal{b}(x)$ is δ-Cauchy.

It follows by Proposition 2.5.1 that $\mathcal{b}(x)$ converges in Y, and the limit is unique since Y is separated. We denote this limit by $\overline{f}(x)$, and in this way we have defined a function $\overline{f}: X \to Y$.

It is clear that if $x \in A$ then $\mathcal{b}(x) \to f(x)$ so in this case $\overline{f}(x) = f(x)$. Hence $\overline{f}$ is an extension of f.

We now show that $\overline{f} : X \to Y$ is $(\mu - \delta)$ cquc. Take $(d, c) \in \delta_o$, and $(e, b) \in \beta$ with $(e, b) \prec(*) (d, c)$. Since A is strictly μ-embedded in X we also have a base $\gamma \subseteq \mu_o$ with the properties of Definition 2.3.2 (b), and since $(f^{-1}(e), f^{-1}(b)) \in \mu_A$ we have $(g, a) \in \gamma$ with $(g_A, a_A) \prec (f^{-1}(e), f^{-1}(b))$. Let us verify $(g, a) \prec (\bar{f}^{-1}(d), \bar{f}^{-1}(c))$, from which the required result follows.

First take LgK. Since g is open we have $(L \cap A)g_A(K \cap A)$ so we have ReS with $L \cap A \subseteq f^{-1}(R)$ and $K \cap A \subseteq f^{-1}(S)$. Take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$. If $x \in L$ then $L \cap A \in \mathcal{B}_u^A(x)$ and so $f(L \cap A) \in \mathcal{b}_u(x)$. But $f(L \cap A) \subseteq R$, so we also have $R \in \mathcal{b}_u(x)$. On the other hand there exists $R'eS'$ with $\bar{f}(x) \in R' \cap S'$. In particular $S' \in \mathcal{b}_v(x)$ since $\mathcal{b}(x) \rightarrow \bar{f}(x)$. Hence $(R, S') \in \mathcal{b}(x)$, and so RbS' since $\mathcal{b}(x)$ is β-regular. Hence $\bar{f}(x) \in R' \subseteq St_b(e, R) \subseteq U$, and we have shown $L \subseteq \bar{f}^{-1}(U)$. In just the same way we have $K \subseteq \bar{f}^{-1}(V)$, so it remains only to show $a \subseteq \bar{f}^{-1}(c)$. By Lemma 2.3.1 it will suffice for us to verify that $a_A \subseteq [\bar{f}^{-1}(c)]_A$. To this end take Pa_AQ with $P \cap Q = \emptyset$. Then $Pf^{-1}(b)Q$ as $a_A \subseteq f^{-1}(b)$ so $\exists$ $P' \in t_u(\delta)$, $Q' \in t_v(\delta)$ with $P'bQ'$, $\emptyset \neq f^{-1}(P') \subseteq P$ and $\emptyset \neq f^{-1}(Q') \subseteq Q$. Since $\bar{f}$ is the same as f on A we deduce that $\bar{f}^{-1}(P') \cap A \subseteq P$ and $\bar{f}^{-1}(Q') \cap A \subseteq Q$. On the other hand $P'cQ'$ as $b \subseteq c$, and so $\bar{f}^{-1}(P')\bar{f}^{-1}(c)\bar{f}^{-1}(Q')$. Hence if we can show $\bar{f}^{-1}(P') \in t_u(\mu)$ and $\bar{f}^{-1}(Q') \in t_v(\mu)$ we shall have

$$(\bar{f}^{-1}(P') \cap A)(\bar{f}^{-1}(c))_A(\bar{f}^{-1}(Q') \cap A),$$

that is $P(\bar{f}^{-1}(c))_AQ$ as required. To show $\bar{f}^{-1}(P') \in t_u(\mu)$ take $z \in \bar{f}^{-1}(P')$, that is $\bar{f}(z) \in P'$. Now take $(h, q) \in \beta$ with $St(h, \bar{f}(x)) \subseteq P'$, $(k, p) \in \beta$ with $(k, p) \prec(*) (h, q)$, and $(m, s) \in \gamma$ with $(m_A, s_A) \prec (f^{-1}(k), f^{-1}(p))$. Then repeating the argument used above we see that $St(m, \{z\}) \subseteq \bar{f}^{-1}(P')$, and hence $\bar{f}^{-1}(P') \in t_u(\mu)$. Likewise $\bar{f}^{-1}(Q') \in t_v(\mu)$, and we have completed the proof that $a \subseteq \bar{f}^{-1}(c)$. Hence $\bar{f}$ is $(\mu - \delta)$ cquc.

Let us prove finally that the extension $\bar{f}$ is unique. Suppose that $\check{f} : X \to Y$ is also cquc, and that $\check{f}(x) = \bar{f}(x) = f(x)$ for all $x \in A$. Suppose that for some $x \in X$ we have $\check{f}(x) \neq \bar{f}(x)$. Since $(Y, \mathcal{S})$ is separated we see from Proposition 2.2.2 (c) that there exists $(d, c) \in \beta$, $U \in \text{dom } d$, $V \in \text{ran } d$ with $U \not\subseteq V$ and, say, $U \in \mathcal{B}_u(\bar{f}(x))$, $V \in \mathcal{B}_v(f(x))$. Since cquc functions are bicontinuous we have $(\bar{f}^{-1}(U) \cap A,\ f^{-1}(V) \cap A) \in \mathcal{B}^A(x)$. It follows at once that $(U, V) \in \mathcal{b}(x)$, and so $U \subseteq V$ since $\mathcal{b}(x)$ is β-regular. This contradiction shows $\bar{f} = \check{f}$, and completes the proof of the theorem.

We may now state our uniqueness theorem.

<u>Theorem</u> 2.5.4. Let $(X, \mathcal{S})$ be a separated cqu. Let (Y, μ) and (Z, τ) be separated strict completions of $(X, \mathcal{S})$ with respect to the maps j and k respectively. Then (Y, μ) and (Z, τ) are cqu isomorphic.

<u>Proof</u>. Let $h : j(X) \to Z$ be the map $h = k \circ j^{-1}$, and $t : k(X) \to Y$ the map $t = j \circ k^{-1}$. Since the conditions of Theorem 2.5.3 are satisfied for these maps we have cquc extensions $\bar{h} : Y \to Z$ and $\bar{t} : Z \to Y$. We complete the proof by showing that $\bar{t} = \bar{h}^{-1}$.

Take $y \in Y$, and let $\bar{h}(y) = z \in Z$. Then with the notation as in the proof of Theorem 2.5.3 we have $\mathcal{b}(y) \to z$, and so $\mathcal{B}(z) \subseteq \mathcal{b}(y)$. If we set $\bar{t}(z) = y'$, then $\mathcal{b}(z) \to y'$ and so $\mathcal{B}(y') \subseteq \mathcal{b}(z)$. We wish to show that $y = y'$.

Suppose that $y \neq y'$, and suppose $j(X) = t(k(X))$ is (strictly) μ-embedded in Y relative to the base $\beta \subseteq \mu_0$. Since (Y, μ) is separated we have by Proposition 2.2.2 (c) that there exists $(d, c) \in \beta$, $U \in \text{dom } d$ and $V \in \text{ran } d$ with $St_c(d, U) \not\subseteq St_c(V, d)$ and, say, $U \in \mathcal{B}_u(y')$ and $V \in \mathcal{B}_v(y)$. In particular we have $G \in t_u(\tau)$ with

$$z \in G \text{ and } t(G \cap k(X)) \subseteq U \quad \ldots\ldots. \ (1).$$

Now suppose $k(X) = h(j(X))$ is (strictly) τ-embedded in Z relative to the base $\gamma \subseteq \tau_0$. We have $(j^{-1}(d), j^{-1}(c)) \in \mathcal{S}$ so there exists $(e, a) \in \gamma$ with

$(k^{-1}(e), k^{-1}(a)) \prec (j^{-1}(d), j^{-1}(c))$ .... (2),

and $St(e,\{z\}) \subseteq G$ .... (3).

Finally $(k^{-1}(e), k^{-1}(a)) \in \delta$ so there exists $(g, b) \in \beta$ with

$(j^{-1}(g), j^{-1}(b)) \prec (k^{-1}(e), k^{-1}(a))$ .... (4),

and $St(\{y\}, g) \subseteq V$ .... (5).

Take ReS and LgK with $z \in R \cap S$ and $y \in L \cap K$. From (1) we have

$t(R \cap k(X)) \subseteq U$ .... (6),

and from (5) we have

$K \subseteq V$ .... (7).

Also $K \cap j(X) \in \mathcal{B}_v^{j(X)}(y)$ so

$h(K \cap j(X)) \in \mathcal{b}_v(y)$ .... (8).

Now $j^{-1}(L)j^{-1}(g)j^{-1}(K)$ so by (4) we have $R'eS'$ with

$j^{-1}(L) \subseteq k^{-1}(R')$ and $j^{-1}(K) \subseteq k^{-1}(S')$ .... (9).

From (9) we have $h(K \cap j(X)) \subseteq S' \cap k(X)$ and so $S' \in \mathcal{b}_v(y)$ from (8). On the other hand $R \in \mathcal{b}_u(y)$ and so $(R, S') \in \mathcal{b}(y)$. However $\mathcal{b}(y)$ is γ-regular and so

RaS' .... (10).

Now $k^{-1}(R')k^{-1}(e)k^{-1}(S')$ so by (2) we have $U'dV'$ with

$k^{-1}(R') \subseteq j^{-1}(U')$ and $k^{-1}(S') \subseteq j^{-1}(V')$ ... (11).

Since $R \in t_u(\tau)$ and $S' \in t_v(\tau)$ we have $k^{-1}(R)k^{-1}(a)k^{-1}(S')$ by (10), so using (11), (6) and the fact (from (2)) that $k^{-1}(a) \subseteq j^{-1}(c)$, we deduce that $j^{-1}(U)j^{-1}(c)j^{-1}(V')$. However $j(X)$ is, in particular, μ-embedded in Y so from Lemma 2.4.1 (a) we see:

UcV' ...... (12).

On the other hand from (9), (11) and (7) we have $L \cap j(X) \subseteq U' \cap j(X)$ and $K \cap j(X) \subseteq V \cap j(X)$. We deduce that $j^{-1}(U')j^{-1}(b)j^{-1}(V)$ and hence from (2) and (4) we have $j^{-1}(U')j^{-1}(c)j^{-1}(V)$. Using Lemma 2.4.1 (a) again gives $U'cV$, and so $V' \subseteq St_c(V, d)$. But also $U \subseteq St_c(d, U)$ and we have the contradiction $St_c(d, U) c\, St(V, d)$. Hence $y = y'$ and $\overline{h^{-1}} = \overline{t}$, which completes the proof.

Corollary. The complete separated strict extension of a separated cqu is unique up to isomorphism.

2.6 COMPACTNESS.

The notion of compactness for a bitopological space can be defined in very many different ways. We have already mentioned some properties of uniform compactness, and the reader may consult [7] for a discussion of several forms of bitopological compactness. In this section we define compactness, modulo a confluence relation, by using bifilters, and relate this to the completeness of cqu.

Throughout this section D will denote $\bigcap \{c \mid c \in \operatorname{ran} \mathcal{S}_0\}$ if we are discussing the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ defined by a cqu $\mathcal{S}$, but in the absence of a cqu, D may be regarded as denoting any fixed confluence relation on X which is interior for the bitopological space under consideration.

We will say the bifilter $\mathcal{B}$ on (X, u, v) is open if $\mathcal{B} \cap (u \times v)$ is a base of $\mathcal{B}$. We will say that $x \in X$ is a D-cluster point (or just cluster point if there can be no confusion) of the bifilter $\mathcal{B}$ if $x \in {}^D P \cap Q^D$ for all $(P, Q) \in \mathcal{B}$.

We may now give:

Definition 2.6.1. The bitopological space (X, u, v) will be called D-compact (respectively, almost D-compact) if every D-regular (respectively, every open D-regular) bifilter on X has a D-cluster point.

Clearly a uniformly compact bitopological space is ı-compact. With regard to the properties of a D-compact space when D is bicompatible we have:

Proposition 2.6.1. If D is bicompatible and (X, u, v) is D-compact, then every u-closed subset of X is v-compact, and every v-closed subset of X is u-compact.

Proof. Let F be a u-closed subset of X, and $\mathcal{K}$ a filter of v_F-closed subsets of F. Let $\mathcal{F}$ be the filter on X generated by $\mathcal{K}$, and put $\mathcal{B} = \mathcal{F} \times \mathcal{F}$. Then $\mathcal{B}$ is an ı-bifilter and hence a D-bifilter so it has a D-cluster point x. Hence $x \in {}^{D}K \cap K^{D} = (v\text{-cl}[K]) \cap (u\text{-cl}[K])$ for all $K \in \mathcal{K}$ since D is bicompatible. Now $x \in u\text{-cl}[K] \Rightarrow x \in F$ so $x \in (v\text{-cl}[K]) \cap F = K$, and F is v-compact as required. The proof of the other result is similar.

<u>Definition</u> 2.6.2. The bitopological space (X, u, v) is <u>D-normal</u> if whenever ${}^{D}A \cap B^{D} = \emptyset$ there exist $G \in u$, $H \in v$ with $A \subseteq G$, $B \subseteq H$ and $G \not{D} H$.

We now have:

<u>Proposition</u> 2.6.2. Let (X, u, v) be D-compact and D-preseparated, and suppose that D is bicompatible and has the oup. Then (X, u, v) is D-regular and D-normal.

<u>Proof</u>. Let F be u-closed and $x \notin F$. For $y \in F$ we have $x \notin u\text{-cl}\{y\}$ and so there exist $x \in U(y) \in u$, $y \in V(y) \in v$ with $U(y) \not{D} V(y)$. Now by Proposition 2.6.1 we know F is v-compact so for some $y_1, \ldots y_n \in F$ we have $F \subseteq V = \bigcup \{ V(y_i) \mid 1 \leq i \leq n \}$. Also $x \in U = \bigcap \{ U(y_i) \mid 1 \leq i \leq n \}$ and $U \in u$, $V \in v$, $U \not{D} V$ since D has the oup. The case when F is v-closed is dealt with in the same way, and we see (X, u, v) is D-regular since D is bicompatible.

Now take A, B with ${}^{D}A \cap B^{D} = \emptyset$, and suppose that $A \subseteq U \in u$, $B \subseteq V \in v$ implies UDV. Then $\{ (U, V) \mid A \subseteq U \in u, B \subseteq V \in v \}$ is a base for an (open) D-regular bifilter $\mathcal{B}$ on X. Let x be a cluster point of $\mathcal{B}$. If $x \notin {}^{D}A$ we have $x \in H \in v$, $A \subseteq G \in u$ with $G \not{D} H$ since (X, u, v) is D-regular. However $G \in \mathcal{B}_u$ implies $x \in {}^{D}G$ which contradicts $G \not{D} H$. Hence $x \in {}^{D}A$, and in just the same way we can show $x \in B^{D}$ and we have a contradiction to ${}^{D}A \cap B^{D} = \emptyset$. This proves that (X, u, v) is D-normal.

Note that this last argument has actually established the following.

<u>Corollary</u>. A D-regular almost D-compact bitopological space is D-normal.

It will be noted that if D is bicompatible then a D-normal bitopological space is necessarily pairwise normal.

A D-compact space is certainly almost D-compact, but the converse is not true in general. However we do have the following result.

Proposition 2.6.3. A D-regular almost D-compact bitopological space is D-compact.

Proof. Let $\mathcal{B}$ be a D-regular bifilter, and define

$\mathcal{B}' = \{(P', Q') \mid \exists\ (P, Q) \in \mathcal{B},\ P \subseteq u\text{-int}[P'],\ Q \subseteq v\text{-int}[Q']\}$.

Then $\mathcal{B}'$ is an open D-regular bifilter, and so has a D-cluster point x. However, because (X, u, v) is D-regular, it is easy to see that x is a cluster point of $\mathcal{B}$ also, and so (X, u, v) is D-compact.

Corollary. If $(X, \mathcal{S})$ is a cqu for which D has the oup, and if $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is almost D-compact then $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact.

The assumption that a bitopological space be almost D-compact imposes a restriction on D, as described below.

Definition 2.6.3. The confluence relation D on (X, u, v) is conjunctive if $PDQ \Rightarrow {}^{D}P \text{ı} Q^{D}$.

Note that, as we are assuming D to be interior, it will be sufficient for the above to hold for $P \in u$ and $Q \in v$. Verification of the following result is trivial and is omitted.

Proposition 2.6.4. If (X, u, v) is almost D-compact then D is conjunctive.

Let us now note:

Proposition 2.6.5. Let $(X, \mathcal{S})$ be a cqu for which D is conjunctive. Then if $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is almost ı-compact it is D-compact.

Proof. Let $\mathcal{B}$ be a D-regular bifilter, and $(P, Q) \in \mathcal{B}$. For $(d, c) \in \mathcal{S}_o$ it is easy to verify ${}^{D}P \subseteq St_c(d, P)$ and $Q^{D} \subseteq St_c(Q, d)$ and so, since D is conjunctive, $\mathcal{B}^*$ with base $\{(St_c(d, P), St_c(Q, d)) \mid (P, Q) \in \mathcal{B},\ (d, c) \in \mathcal{S}_o\}$ is an open $\imath$-regular bifilter on X, and so has an $\imath$-cluster point x. Take $c \in \operatorname{ran} \mathcal{S}_o$, $x \in H \in v$ and $d \in \operatorname{dom} \mathcal{S}_o$ with $St(\{x\}, d) \subseteq H$. Then we have $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec^{(*)} (d, c)$, and $St(\{x\}, e)$ is a v-nhd. of x so $St(\{x\}, e) \cap St_b(e, P) \neq \emptyset$. Hence we have ReS and R'eS' with $x \in R$, PbS' and $S \cap R' \neq \emptyset$. If now we take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$ then $x \in R \in St_b(e, R)$ $\subseteq U$ and so $S' \subseteq St_b(S, e) \subseteq V \subseteq St(d, \{x\}) \subseteq H$. Hence PbH. However $b \subseteq c$, and $c \in \operatorname{ran} \mathcal{S}_o$ was arbitrary, so PDH and we have shown $x \in {}^{D}P$. In just the same way $x \in Q^{D}$; hence x is a D-cluster point of $\mathcal{B}$ and $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact as required.

With regard to the converse, we have noted above that if $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact then D is conjunctive. If D is bicompatible, and in particular, therefore, if D has the oup, then it is clear that the D-compactness of $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ will imply that this space is also $\imath$-compact.

Basic to the relation between these compactness notions and completeness is the following.

<u>Lemma</u> 2.6.1. A D-cluster point of a D-regular $\mathcal{S}$-Cauchy bifilter is a limit point.

We omit the proof which is straightforward.

<u>Proposition</u> 2.6.6. If $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is almost D-compact then $(X, \mathcal{S})$ is complete.

<u>Proof</u>. Let $\mathcal{B}$ be a D-regular $\mathcal{S}$-Cauchy bifilter on X, and form $\mathcal{B}^*$ as in Proposition 2.3.3 for the base $\mathcal{S}_o$. Then $\mathcal{B}^*$ is an open D-regular bifilter so it has a cluster point x, and this point is a limit point of $\mathcal{B}^*$ by Lemma 2.6.1 since $\mathcal{B}^*$ is $\mathcal{S}$-Cauchy. But then x is also a limit point of $\mathcal{B}$, and so $(X, \mathcal{S})$ is complete.

With regard to the definition of a suitable "total boundedness" property we consider the following.

<u>TB.1</u>. For each D-regular bifilter $\mathcal{B}$ there exists a D-regular bifilter $\mathcal{L}$ such that given $(P, Q) \in \mathcal{B}$ and $d \in \operatorname{dom} \mathcal{S}_o$

there exists UdV with $(U, V) \in \mathcal{L}$, PDV and UDQ.

TB.2. For every open D-regular bifilter $\mathcal{B}$ there exists $\mathcal{L}$ as above.

TB.3. For every D-regular bifilter $\mathcal{B}$, $(P, Q) \in \mathcal{B}$ and $d \in \operatorname{dom} \mathcal{S}_o$ there exists UdV with PDV and UDQ.

TB.4. For every open D-regular bifilter $\mathcal{B}$, $(P, Q) \in \mathcal{B}$ and $d \in \operatorname{dom} \mathcal{S}_o$ there exists UdV with PDV and UDQ.

TB.5. Every maximal D-regular bifilter is $\mathcal{S}$-Cauchy.

TB.6. Every maximal open D-regular bifilter is $\mathcal{S}$-Cauchy.

We have:

Proposition 2.6.7. $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact (respectively, almost D-compact) if and only if $(X, \mathcal{S})$ satisfies TB.1 (respectively, TB.2) and is complete.

We omit the proof which is straightforward.

The following relations between the above properties follow at once from the definitions.

$$\begin{array}{ccccc}
TB.5 & \Rightarrow & TB.1 & \Rightarrow & TB.3 \\
\Downarrow & & \Downarrow & & \Downarrow \\
TB.6 & \Rightarrow & TB.2 & \Rightarrow & TB.4
\end{array}$$

In order to obtain conditions under which these properties are equivalent we make the following definition.

Definition 2.6.4. Let $(X, \mathcal{S})$ be a cqu. We say D is $\mathcal{S}$-compatible if for $G \in t_u(\mathcal{S})$, $H \in t_v(\mathcal{S})$ with GDH, and $d \in \operatorname{dom} \mathcal{S}_o$, we have $e \in \operatorname{dom} \mathcal{S}_o$ with $e \prec d$ and ReS with RDS, $R \subseteq G$ and $S \subseteq H$.

Note in particular that if for $\mathcal{S}$ we have $D = \imath$ then certainly D is $\mathcal{S}$-compatible.

Proposition 2.6.8. Let $(X, \mathcal{S})$ be a cqu for which D is $\mathcal{S}$-compatible and has the oup. Then all the conditions TB.i, $i = 1, \ldots, 6$, are equivalent.

Proof. It will suffice to show TB.4 $\Rightarrow$ TB.6 $\Rightarrow$ TB.5.

TB.4 $\Rightarrow$ TB.6. Let $\mathcal{B}$ be a maximal open D-regular bifilter. Take

$(d, c) \in \delta_o$ and $(e, b) \in \delta_o$ with $(e, b) \prec(*) (d, c)$. For $(G, H) \in \mathcal{B} \cap (u \times v)$ let

$$U(G, H) = \bigcup \{ L \mid L \subseteq G \text{ and } \exists \; T \subseteq H \text{ with LDT and LfT for some } f \in \operatorname{dom} \delta_o \text{ with } f \prec e \},$$

$$V(G, H) = \bigcup \{ T \mid T \subseteq H \text{ and } \exists \; L \subseteq G \text{ with LDT and LfT for some } f \in \operatorname{dom} \delta_o \text{ with } f \prec e \}.$$

Because D is δ -compatible it is easy to verify that

$$\{ (U(G, H), V(G, H)) \mid (G, H) \in \mathcal{B} \cap (u \times v) \}$$

is a base for an open D-regular refinement of $\mathcal{B}$. Hence $(U(G, H), V(G, H)) \in \mathcal{B}$ for all $(G, H) \in \mathcal{B} \cap (u \times v)$ since $\mathcal{B}$ is a maximal open D-regular bifilter. By TB.4 we have ReS with $U(G, H)DS$ and $RDV(G, H)$. Take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$. Since D has the oup $\exists \; L \subseteq G, T \subseteq H$ with LDT, LDS and LfT for some $f \in \operatorname{dom} \delta_o$ with $f \prec e$. Take R'eS' with $L \subseteq R'$ and $T \subseteq S'$. Then R'DS and so R'bS. Hence $T \subseteq S' \subseteq St_b(S, e) \subseteq V$, and it follows that $GD(H \cap V)$ for all $(G, H) \in \mathcal{B} \cap (u \times v)$. This means that

$$\{ (G, H \cap V) \mid (G, H) \in \mathcal{B} \cap (u \times v) \}$$

is a base for an open D-regular refinement of $\mathcal{B}$, and so $V \in \mathcal{B}_v$ by the maximality of $\mathcal{B}$. In the same way $RDV(G, H)$ leads to $U \in \mathcal{B}_u$, and so $(U, V) \in \mathcal{B}$ which proves that $\mathcal{B}$ is δ -Cauchy.

<u>TB.6 $\Rightarrow$ TB.5</u>. Let $\mathcal{B}$ be a maximal D-regular bifilter, and set

$$\mathcal{B}_u^{(1)} = \{ P \mid P \in \mathcal{B}_u, \exists \; (P', Q') \in \mathcal{B} \text{ with } P \cap P' \cap Q' = \emptyset \},$$

$\mathcal{B}_u^{(2)} = \mathcal{B}_u - \mathcal{B}_u^{(1)}$; and make a corresponding definition of $\mathcal{B}_v^{(1)}$ and $\mathcal{B}_v^{(2)}$. Because of the maximality of $\mathcal{B}$, and the fact that D is interior, it is easy to verify that $P \in \mathcal{B}_u^{(1)} \Rightarrow t_u(\delta)\text{-int}[P] \in \mathcal{B}_u^{(1)}$, with a corresponding result for $\mathcal{B}_v^{(1)}$. It follows that

$$(\{ t_u(\delta)\text{-int}[P] \mid P \in \mathcal{B}_u^{(1)} \} \cup \{ St(d, P) \mid P \in \mathcal{B}_u^{(2)}, d \in \operatorname{dom} \delta_o \}) \times$$

$(\{t_v(\mathcal{S})\text{-int}\{Q\} \mid Q \in \mathcal{B}_v^{(1)}\} \cup \{St(Q, d) \mid Q \in \mathcal{B}_v^{(2)}, d \in \operatorname{dom} \mathcal{S}_o\})$

is a base of an open D-regular bifilter $\mathcal{B}'$ contained in $\mathcal{B}$. Let $\mathcal{L}$ be a maximal open D-regular refinement of $\mathcal{B}'$. By TB.6 $\mathcal{L}$ is $\mathcal{S}$-Cauchy. Hence for (d, c), $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec^{(*)}$ (d, c) we have ReS with $(R, S) \in \mathcal{L}$. Take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$, and $(P, Q) \in \mathcal{B}$. If $P \in \mathcal{B}_u^{(1)}$ or $Q \in \mathcal{B}_v^{(1)}$ we see at once that $(P \cap U)D(Q \cap V)$, so suppose $P \in \mathcal{B}_u^{(2)}$ and $Q \in \mathcal{B}_v^{(2)}$. Then we have $P \cap Q \in \mathcal{B}_u^{(2)} \cap \mathcal{B}_v^{(2)}$ because of the maximality of $\mathcal{B}$, and hence

$$(R \cap St(e, P \cap Q))D(S \cap St(P \cap Q, e)).$$

Since D has the oup we then have $R'eS'$ with $R'DS$ and $S' \cap P \cap Q \neq \emptyset$. In particular $R'bS$ and so $S' \subseteq St_b(S, e) \subseteq V$ which gives $V \cap P \cap Q \neq \emptyset$, that is $(P)D(Q \cap V)$. Likewise $(P \cap U)D(Q)$, and we deduce from the maximality of $\mathcal{B}$ that $(U, V) \in \mathcal{B}$. Hence $\mathcal{B}$ is $\mathcal{S}$-Cauchy, and the proof is complete.

Corollary. A pairwise completely regular $\imath$-compact space is uniformly compact.

Proof. By Proposition 1.7.1 we have a compatible quasi-uniformity $\mathcal{S}$, and we may form the compatible cqu $\mathcal{S}'$ with $D' = \imath$ as in Example 2.1.1. By Proposition 2.6.7, $(X, \mathcal{S}')$ is complete and satisfies TB.1; hence it satisfies TB.5 by the above result. Hence every maximal $\imath$-regular bifilter is convergent, and so X is uniformly compact by Lemma 1.7.2.

It is a well known, and entirely trivial fact, that in a topological space a cluster point of a maximal filter is a limit point. The next proposition gives a partial generalisation of this result for bifilters.

Proposition 2.6.9. Let $(X, \mathcal{S})$ be a cqu for which D is $\mathcal{S}$-compatible and has the oup. Then a D-cluster point of a maximal D-regular bifilter, or of a maximal open D-regular bifilter, is a limit point.

Proof. The proof of the case where $\mathcal{B}$ is a maximal open D-regular bifilter follows exactly the same steps as the proof of the implication TB.4 $\Rightarrow$ TB.6 above. Moreover if $\mathcal{B}$ is a maximal D-regular bifilter we may consider $\mathcal{B}'$ and $\mathcal{L}$ as in the proof of TB.6 $\Rightarrow$ TB.5. Because of the maximality of $\mathcal{B}$ it is not difficult to verify that the bifilter with base

$$\{ (St_D(d, G), St_D(H, d)) \mid (G, H) \in \mathcal{L} \cap (u \times v), d \in \operatorname{dom} \mathcal{S}_0 \}$$

is contained in $\mathcal{B}$. We may deduce from this that if x is a D-cluster point of $\mathcal{B}$ it is a D-cluster point of $\mathcal{L}$ also. However $\mathcal{L}$ is a maximal open D-regular bifilter so by what we have noted above x is a limit point of $\mathcal{L}$. However it is not difficult to deduce from this that x is a limit of $\mathcal{B}$, and the proof is complete.

It is clearly of vital interest to know when a total boundedness condition will carry over to a extension. Let us make the following definition.

Definition 2.6.5. Let (X, u, v) be a bitopological space, D an interior confluence relation and $A \subseteq X$. Then A is D-embedded in X if GDH whenever $G \in u$, $H \in v$ and $(G \cap A)D_A(H \cap A)$.

Verification of the following result is trivial, and is omitted.

Proposition 2.6.10. Let $A \subseteq X$ be bidense and D-embedded in the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$. Then if $(A, \mathcal{S}_A)$ is a cqu which satisfies TB.2, TB.4 or TB.6 so does $(X, \mathcal{S})$.

In general strictly $\mathcal{S}$-embedded would not seem to imply D-embedded, but we do have the following result.

Proposition 2.6.11. Let $A \subseteq X$ be bidense and strictly $\mathcal{S}$-embedded in X for the bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, and suppose that D_A has the oup. Then A is D-embedded in X, and D has the oup.

We omit the proof, which is straightforward. We may state the following as an immediate corollary to the above results.

Theorem 2.6.1. Let $(X, \mathcal{S})$ be a separated cqu which satisfies

TB.2, and for which D has the oup. Then the separated strict extension $(\Gamma_o(X), \tilde{\zeta})$ is D-compact.

2.7. D-HYPERFILTERS.

In this section we examine bitopological compactness from a different viewpoint, by introducing the notion of D-hyperfilter.

Definition 2.7.1. Let D be an interior confluence relation on the bitopological space (X, u, v), regarded as a subset of $\mathcal{P}(X) \times \mathcal{P}(X)$. By a D-hyperfilter on X we shall mean any filter on D. We say the D-hyperfilter $\mathfrak{Z}$ is open if it has a base whose elements F satisfy $F \subseteq u \times v$.

Clearly any D-hyperfilter finer than an open D-hyperfilter is open, so the terms "maximal open" and "open maximal" have the same meaning when applied to D-hyperfilters.

We shall find the following notation and terminology useful. If $\mathfrak{Z}$ is a D-hyperfilter, $F \in \mathfrak{Z}$ and $P, Q \subseteq X$, we say that (P, Q) dominates F, and write $F \lhd (P, Q)$, if $(L, K) \in F \Rightarrow L \subseteq P$ and $K \subseteq Q$. We will say that (P, Q) weakly dominates F, and write $F \lhd\!\rhd (P, Q)$, if $(L, K) \in F \Rightarrow LDQ$ and PDK.

There is a natural link between D-hyperfilters and D-regular bifilters, as follows. Let $\mathcal{B}$ be a D-regular bifilter on X, and for $(P, Q) \in \mathcal{B}$ let $F(P, Q) = \{(P \cap P', Q \cap Q') \mid (P', Q') \in \mathcal{B}\}$. Then $\{F(P, Q) \mid (P, Q) \in \mathcal{B}\}$ is a base for a D-hyperfilter $h(\mathcal{B})$ on X. Conversely, if $\mathfrak{Z}$ is a D-hyperfilter on X, then $b(\mathfrak{Z}) = \{(P, Q) \mid \exists\, F \in \mathfrak{Z},\ F \lhd (P, Q)\}$ is a D-regular bifilter on X. To describe the set of D-hyperfilters which have the form $h(\mathcal{B})$ we make the following definition.

Definition 2.7.2. The D-hyperfilter $\mathfrak{Z}$ is dominated if it has a base $\mathfrak{Z}'$ satisfying

(a) $(L, K) \in F \in \mathfrak{Z}' \Rightarrow \exists\ F' \in \mathfrak{Z}'$ with $F' \lhd (L, K)$,

(b) $F \in \mathfrak{Z}' \Rightarrow \exists\ (L, K) \in F$ with $F \lhd (L, K)$.

(c) $(L, K) \in F \in \mathfrak{Z}'$ and $(L', K') \in F' \in \mathfrak{Z}' \Rightarrow (L \cap L', K \cap K') \in F \cap F'$

(d) $F \in \mathfrak{Z}'$, $F \lhd (U, V) \Rightarrow \exists\ F' \in \mathfrak{Z}'$ with $(U, V) \in F'$.

Now we may state:

Proposition 2.7.1. (i) $h(\mathcal{B})$ is dominated for all D-regular bifilters $\mathcal{B}$, and $\mathcal{B} = b(h(\mathcal{B}))$.

(ii) Let $\mathcal{F}$ be a D-hyperfilter. Then $h(b(\mathcal{F})) = \mathcal{F}$ if and only if $\mathcal{F}$ is dominated.

Corollary. h is a one to one mapping of the set of D-regular bifilters onto the set of all dominated D-hyperfilters, and the restriction of b to this set is the inverse of h.

We omit the proof of the above statements, since they are a matter of straightforward verification.

Note also that the maps h and b preserve the property of being open, which we have defined for bifilters and D-hyperfilters.

We may define the notions of "limit" and "cluster point" for a D-hyperfilter in several ways, as detailed below.

Definition 2.7.3. Let $\mathcal{F}$ be a D-hyperfilter on (X, u, v). We say that $x \in X$ is

(a) A weak cluster point of $\mathcal{F}$ if $x \in {}^{D}P \cap Q^{D}$ whenever $F \in \mathcal{F}$ and $F \lhd (P, Q)$.

(b) A cluster point of $\mathcal{F}$ if whenever $(M, N) \in \mathcal{B}(x)$ and $F \in \mathcal{F}$ we have $(P, Q) \in F$ with PDN and MDQ.

(c) A weak limit point of $\mathcal{F}$ if for $(M, N) \in \mathcal{B}(x)$ there exists $F \in \mathcal{F}$ with $F \lhd\!\rhd (M, N)$.

(d) A limit point of $\mathcal{F}$ if for $(M, N) \in \mathcal{B}(x)$ there exists $F \in \mathcal{F}$ with $F \lhd (M, N)$.

If $\mathcal{F}$ is a D-hyperfilter for the cqu $(X, \mathcal{S})$ we say $\mathcal{F}$ is $\mathcal{S}$-Cauchy (respectively, weakly $\mathcal{S}$-Cauchy) if given $d \in \mathcal{S}_0$ there exists UdV and $F \in \mathcal{F}$ with $F \lhd (U, V)$ (respectively, $F \lhd\!\rhd (U, V)$).

We list below some easy consequences of these definitions.

Lemma 2.7.1. Let $\mathcal{F}$ be a D-hyperfilter on X. Then:

(i) x is a weak cluster point of $\mathcal{F}$ if and only if x is a cluster point of $b(\mathcal{F})$.

(ii) x is a limit point of $\mathcal{F}$ if and only if x is a limit point of $b(\mathcal{F})$.

(iii) $\mathcal{H}$ is $\mathcal{S}$ -Cauchy if and only if $b(\mathcal{H})$ is $\mathcal{S}$ -Cauchy.

Corresponding results may, of course, be stated for the mapping h. It is immediate from the above that:

Proposition 2.7.2 The bitopological space (X, u, v) is D-compact (respectively, almost D-compact) if and only if every D-hyperfilter (respectively, every open D-hyperfilter) on X has a weak cluster point.

Before going on to characterize completeness in terms of D-hyperfilters we shall find it convenient to make the following definition.

Definition 2.7.4. Let $\mathcal{H}$ be a D-hyperfilter on the cqu (X, $\mathcal{S}$). We will say $\mathcal{H}$ is $\mathcal{S}$ -refined if given $d \in \operatorname{dom} \mathcal{S}_0$ there exists $F \in \mathcal{H}$ with $F \prec d$ and $F \subseteq u \times v$.

In particular it will be noted that a $\mathcal{S}$ -refined D-hyperfilter is open. We now have:

Proposition 2.7.3. The following conditions on the cqu (X, $\mathcal{S}$) are equivalent.

(i) (X, $\mathcal{S}$) is complete.

(ii) Every $\mathcal{S}$ -Cauchy D-hyperfilter on X is convergent.

(iii) Every $\mathcal{S}$ -refined $\mathcal{S}$ -Cauchy D-hyperfilter on X is convergent.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, so it remains to prove (iii) $\Rightarrow$ (i). Let $\mathcal{B}$ be a $\mathcal{S}$ -Cauchy D-regular bifilter on X, and consider the D-hyperfilter $\mathcal{K}$ with base $\{ K(d) \mid d \in \operatorname{dom} \mathcal{S}_0 \}$, where

$$K(d) = \{ (R, S) \mid RDS,\ F \lhd (R, S) \text{ for some } F \in h(\mathcal{B}) \text{ and } ReS \text{ for some } e \in \operatorname{dom} \mathcal{S}_0 \text{ with } e \prec d \}.$$

It is clear from the definition that $\mathcal{K}$ is $\mathcal{S}$ -refined, let us show that it is also $\mathcal{S}$ -Cauchy. Take $(d, c) \in \mathcal{S}_0$ and $(e, b) \in \mathcal{S}_0$ with $(e, b) \prec^{(*)} (d, c)$. Since $\mathcal{B}$ is $\mathcal{S}$ -Cauchy so to is $h(\mathcal{B})$, and hence we have ReS and $F \in h(\mathcal{B})$ with $F \lhd (R, S)$. Take UdV with $St_b(e, R) \subseteq U$ and $St_b(S, e) \subseteq V$, and take $(L, K) \in K(e)$. Then we have $f \in \operatorname{dom} \mathcal{S}_0$ with $f \prec e$ and LfK, and also

$F' \in h(\mathcal{B})$ with $F' \lhd (L, K)$. Take $R'eS'$ with $L \subseteq R'$ and $K \subseteq S'$, then we have $F' \lhd (R', S')$. Now $F \wedge F' \in h(\mathcal{B})$ so we have $(G, H) \in F \cap F'$, and GDH since $h(\mathcal{B})$ is a D-hyperfilter. We have $G \subseteq R \cap R'$ and $H \subseteq S \cap S'$, so RbS' and $R'bS$. Hence $L \subseteq R' \subseteq St_b(e, R) \subseteq U$, and likewise $K \subseteq V$. This shows $K(e) \lhd (U, V)$, and so $\mathcal{K}$ is $\mathcal{S}$-Cauchy as stated. Let x be a limit point of $\mathcal{K}$, and take $(M, N) \in \mathcal{B}(x)$. Then for some $d \in \mathcal{S}_0$ we have $K(d) \lhd (M, N)$. Take $(R, S) \in K(d)$, and $F \in h(\mathcal{B})$ with $F \lhd (R, S)$. Then $F \lhd (M, N)$ so x is a limit point of $h(\mathcal{B})$, and hence of $\mathcal{B}$, which completes the proof.

Below we give without proof some useful relationships which exist between the concepts introduced in Definition 2.7.3.

Lemma 2.7.2. (a) A weak limit point of a $\mathcal{S}$-refined D-hyperfilter is a limit point.

(b) A cluster point (and, in particular, a weak limit point) of a $\mathcal{S}$-Cauchy D-hyperfilter is a limit point.

(c) A cluster point of a maximal D-hyperfilter is a weak limit point.

(d) Every weakly $\mathcal{S}$-Cauchy $\mathcal{S}$-refined D-hyperfilter is $\mathcal{S}$-Cauchy.

(e) A convergent (respectively, weakly convergent) D-hyperfilter is $\mathcal{S}$-Cauchy (respectively, weakly $\mathcal{S}$-Cauchy).

The following bitopological compactness and completeness properties may be defined quite naturally in terms of D-hyperfilters. We will see later to what extent they coincide with previously discussed properties.

Definition 2.7.5. The bitopological space (X, u, v) will be called D-hypercompact (respectively, almost D-hypercompact) if every D-hyperfilter (respectively, every open D-hyperfilter) on X has a cluster point.

The cqu $(X, \mathcal{S})$ will be called hypercomplete if every weakly $\mathcal{S}$-Cauchy D-hyperfilter on X is weakly convergent.

It is clear from the results mentioned above that an (almost)

D-hypercompact bitopological space is (almost) D-compact, while a hypercomplete cqu is complete. To examine the relation between these concepts in more detail we consider the following conditions of "hyper-total boundedness".

HTB.1. Every maximal D-hyperfilter is weakly $\mathcal{S}$ -Cauchy.

HTB.2. Every maximal open D-hyperfilter is weakly $\mathcal{S}$ -Cauchy.

HTB.3. Every maximal $\mathcal{S}$ -refined D-hyperfilter is $\mathcal{S}$ -Cauchy.

HTB.4. Given $d \in \operatorname{dom} \mathcal{S}_o$ there exists $U_i d_i V_i$, $i = 1, 2, \ldots, n$, with $X = \bigcup \{ U_i \cap V_i \mid 1 \leq i \leq n \}$.

First let us note the following:

<u>Lemma</u> 2.7.3. Each of the following conditions on the cqu $(X, \mathcal{S})$ are equivalent.

(i) HTB.3.

(ii) Given any base $\beta \subseteq \mathcal{S}_o$ of $\mathcal{S}$ and any $(d, c) \in \mathcal{S}_o$ there exists $U_i d\, V_i$, $i = 1, \ldots, n$ with $U_i D V_i$, and so that for some $(e, b) \in \beta$ with $(e, b) \prec (d, c)$ we have ReS and RDS imply $U_k bS$ and RbV_k for some k, $1 \leq k \leq n$.

(iii) Given any $(d, c) \in \mathcal{S}_o$ there exist $U_i dV_i$, $i = 1, \ldots, n$, with $U_i DV_i$ and so that for some $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec (d, c)$ we have ReS and RDS imply $U_k bS$ and RbV_k for some k, $1 \leq k \leq n$.

<u>Proof</u>. (i) $\Rightarrow$ (ii). Suppose (ii) is false, then we have $(d, c) \in \mathcal{S}_o$ and a base $\beta \subseteq \mathcal{S}_o$ of $\mathcal{S}$ so that, if a is any finite subset of d and $(e, b) \in \beta$ satisfies $(e, b) \prec (d, c)$, then there exists ReS with RDS so that $U \not b S$ or $R \not b V$ for all $(U, V) \in a$. It follows that, if for each $(d', c') \in \mathcal{S}_o$ we set

$F_a(d', c') = \{ (R, S) \mid RDS$ and $\exists\ (e, b) \in \beta$ with $(e, b) \prec (d', c')$ so that ReS and $R \not b V$ or $U \not b S$ for all $(U, V) \in a \}$,

then $\{ F_a(d', c') \mid (d', c') \in \mathcal{S}_o$, a is a finite subset of $d \}$ is a base for a D-hyperfilter $\mathcal{F}$ on X. It is clear from the definition that $\mathcal{F}$ is $\mathcal{S}$ -refined, and so by (i) it has a $\mathcal{S}$ - Cauchy refinement $\mathcal{H}$. Take $U_o dV_o$ and $H \in \mathcal{H}$ with $H \lhd (U_o, V_o)$, and let $a_0 = \{(U_0, V_0)\}$. Then $F_{a_0}(d, c) \in \mathcal{F} \subseteq \mathcal{H}$. Take $(R, S) \in H \cap F_{a_0}(d, c)$. Then RDS, and for some $(e, b) \in \beta$, $(e, b) \prec (d, c)$ we have ReS and $R \not\delta V_0$ or $U_0 \not\delta S$. But $R \subseteq U_0$, $S \subseteq V_0$ so $R \not\delta S$ which contradicts RDS. Hence (ii) is satisfied.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let $\mathcal{F}$ be a maximal δ-refined D-hyperfilter, and take $(d, c) \in \delta_0$. Take $(e, b) \in \delta_0$ with $(e, b) \prec (d, c)$. By (iii) we have $R_i e S_i$, $i = 1, 2, \ldots, n$, with $R_i D S_i$, and $(f, a) \in \delta_0$ with $(f, a) \prec (e, b)$, so that LfK with LDK implies LaS_k and $S_k aK$ for some k, $1 \leq k \leq n$. For $G \in \mathcal{F}$ and k, $1 \leq k \leq n$, let us set

$$G^k = \{ (L', K') \mid (L', K') \in G \text{ and } \exists\ LfK,\ L' \subseteq L,\ K' \subseteq K, \text{ so that } LaS_k \text{ and } R_k aK \}.$$

Now it is not difficult to show that for some k, $1 \leq k \leq n$, $\{ G^k \mid G \in \mathcal{F} \}$ is a base for a D-hyperfilter $\mathcal{G}$. Moreover, since $G^k \subseteq G$ and $\mathcal{F}$ is maximal we see that $\mathcal{F} = \mathcal{G}$, and so $G^k \in \mathcal{F}$ for this value of k and all $G \in \mathcal{F}$. However if we take UdV with $St_b(e, R_k) \subseteq U$ and $St_b(S_k, e) \subseteq v$ it is easy to see that $G^k \lhd (U, V)$ for any G, and so $\mathcal{F}$ is δ-Cauchy as required.

Corollary. HTB.i $\Rightarrow$ HTB.(i + 1), i = 1, 2, 3.

Under certain conditions on the interior confluence relation D some of these properties are equivalent. We may note the following:

Proposition 2.7.4. (a) If D is conjunctive then HTB.4 $\Rightarrow$ HTB.1
(b) If D is δ-compatible then HTB.3 $\Rightarrow$ HTB.1
(c) If D has the oup then HTB.2 $\Rightarrow$ HTB.1.

Proof. (a) Let $\mathcal{F}$ be a maximal D-hyperfilter, and take $(d, c) \in \delta_0$. By HTB.4 we have $U_i d V_i$, $i = 1, 2, \ldots, n$, with $\bigcup \{ U_i \cap V_i \mid 1 \leq i \leq n \} = X$, and without loss of generality we may suppose

$U_i \cap V_i \neq \emptyset$ for each i. For $1 \leq k \leq n$ let us set

$F^k = \{(G, H) \mid (G, H) \in F,\ GDV_k \text{ and } U_kDH\}$,

where $F \in \mathcal{F}$. Using the fact that D is conjunctive we may easily see that for some k, $1 \leq k \leq n$, the set $\{F^k \mid F \in \mathcal{F}\}$ is a base of a D-hyperfilter refinement of $\mathcal{F}$. Hence, by the maximality of $\mathcal{F}$, for this k and any $F \in \mathcal{F}$ we have $F^k \in \mathcal{F}$. However $F^k \triangleleft\!\triangleright (U_k, V_k)$, and so $\mathcal{F}$ is weakly $\mathcal{S}$-Cauchy.

(b) Let $\mathcal{F}$ be a maximal D-hyperfilter, and take $(d, c) \in \mathcal{S}_o$. Take $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec(*) (d, c)$. By Lemma 2.7.3 (iii) we have R_ieS_i, $i = 1, 2, \ldots, n$, with R_iDS_i; and $(f, a) \in \mathcal{S}_o$ with $(f, a) \prec (e, b)$, so that LfK with LDK $\Rightarrow \exists$ k, $1 \leq k \leq n$, with LaS_k and R_kaK. If we take U_idV_i with $St_b(e, R_i) \subseteq U_i$ and $St_b(S_i, e) \subseteq V_i$, then the remaining steps are as in (a) above, except that we use the $\mathcal{S}$-compatibility of D in place of the conjunctivity.

(c) Let $\mathcal{F}$ be a maximal D-hyperfilter, take $F \in \mathcal{F}$ and $d \in \operatorname{dom} \mathcal{S}_o$, and let $F_d = \{(St_D(d', L), St_D(K, d')) \mid (L, K) \in F,\ d' \in \operatorname{dom} \mathcal{S}_o,\ d' \prec d\}$. It is clear that

$$\{F_d \mid F \in \mathcal{F},\ d \in \operatorname{dom} \mathcal{S}_o\}$$

is a base for an open D-hyperfilter. Let $\mathcal{H}$ be a weakly $\mathcal{S}$-Cauchy D-hyperfilter refinement. For $(d, c) \in \mathcal{S}_o$ and $(e, b) \in \mathcal{S}_o$ with $(e, b) \prec(*) (d, c)$ take R_oeS_o and $K \in \mathcal{H}$ with $K \triangleleft\!\triangleright (R_o, S_o)$, and take U_odV_o with $St_b(e, R_o) \subseteq U_o$ and $St_b(S_o, e) \subseteq V_o$. Then using the fact that D has the oup it is easy to see that each $F \in \mathcal{F}$ contains some (G, H) with GDV_o and U_oDH, and the proof may then be completed as in (a) and (b).

The next proposition tells us something about the relation between completeness and hypercompleteness under rather restrictive conditions.

Proposition 2.7.5. If (X, δ) is complete and satisfies HTB.3 then it is hypercomplete.

Proof. Let $\mathcal{F}$ be a weakly δ-Cauchy D-hyperfilter. For $d \in \operatorname{dom} \delta_o$ define

$$H(d) = \{(R, S) \mid RDS,\ \exists\ F \in \mathcal{F} \text{ with } F \lhd (R, S) \text{ and } \exists\ e \in \operatorname{dom} \delta_o \text{ with } e \prec d \text{ and } ReS\}.$$

Then it is easy to verify that

$$\{ H(d) \mid d \in \operatorname{dom} \delta_o \}$$

is a base for a δ-refined D-hyperfilter $\mathcal{H}$ on X. Let $\mathcal{H}'$ be a maximal refinement of $\mathcal{H}$; then $\mathcal{H}'$ is δ-Cauchy by HTB.3, and so has a limit point $x \in X$. However it is easy to verify that x is in fact a weak limit point of $\mathcal{F}$, and the result is proved.

We may now give:

Theorem 2.7.1. The following are equivalent for the cqu (X, δ).

(i) $(X, t_u(\delta), t_v(\delta))$ is almost D-hypercompact.

(ii) (X, δ) is complete and satisfies HTB.2.

(iii) (X, δ) is hypercomplete and satisfies HTB.2.

(iv) $(X, t_u(\delta), t_v(\delta))$ is D-hypercompact.

Proof. (i) $\Rightarrow$ (ii). By Proposition 2.7.3, completeness will follow if we can show that every δ-refined δ-Cauchy D-hyperfilter $\mathcal{F}$ is convergent. However such an $\mathcal{F}$ is open and so has a cluster point x, while $\mathcal{F}$ converges to x by Lemma 2.7.2 (b). To verify HTB.2 let $\mathcal{H}$ be a maximal open D-hyperfilter. $\mathcal{H}$ has a cluster point x, x is a weak limit point by Lemma 2.7.2 (c), and hence $\mathcal{H}$ is weakly δ-Cauchy by Lemma 2.7.2 (e).

(ii) $\Rightarrow$ (iii). This follows from Proposition 2.7.5 and the Corollary to Lemma 2.7.3.

(iii) $\Rightarrow$ (iv). Let $\mathcal{F}$ be a D-hyperfilter on X. For each $F \in \mathcal{F}$ and $e \in \operatorname{dom} \delta_o$ define

$$F(e) = \{ (P', Q') \mid \exists\ (P, Q) \in F \text{ with } P \cap Q = \emptyset,\ P' = t_u(\delta)\text{-int}[P]$$

and $Q' = t_v(\mathcal{S})\text{-int}[Q]$, or $\exists$ $(P, Q) \in F$ with $P \cap Q \neq \emptyset$ and $e' \in \text{dom}\, \mathcal{S}_0$ with $e' \prec e$ so that $P'e'Q'$ and $P' \cap Q' \cap P \cap Q \neq \emptyset \}$.

Since D is an interior confluence relation on $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ it is immediate that

$$\{ F(e) \mid F \in \mathcal{F},\ e \in \text{dom}\, \mathcal{S}_0 \}$$

is a base for an open D-hyperfilter $\mathcal{F}'$ on X. Then if $\mathcal{H}$ is a maximal open refinement of $\mathcal{F}'$, $\mathcal{H}$ is weakly $\mathcal{S}$-Cauchy by HTB.2, and hence $\mathcal{H}$ has a weak limit point x. Let us show that x is a cluster point of $\mathcal{F}$. To this end take $(M, N) \in \mathcal{B}(x)$ and $(d, c) \in \mathcal{S}_0$ with $St(d, \{x\}) \subseteq M$ and $St(\{x\}, d) \subseteq N$. Now take $(e, b) \in \mathcal{S}_0$ with $(e, b) \prec(*) (d, c)$, and ReS with $x \in R \cap S$. Then $(R, S) \in \mathcal{B}(x)$ so $\exists$ $H \in \mathcal{H}$ with $H \triangleleft\!\triangleright (R, S)$. Now for $F \in \mathcal{F}$ we have $F(e) \in \mathcal{F}' \subseteq \mathcal{H}$ so $\exists$ $(P', Q') \in F(e) \cap H$. If $P' = t_u(\mathcal{S})\text{-int}[P]$, $Q' = t_v(\mathcal{S})\text{-int}[Q]$ for some $(P, Q) \in F$ then clearly PDN and MDQ. On the other hand if $P'e'Q'$ for some $e' \in \text{dom}\, \mathcal{S}_0$ with $e' \prec e$ and $P' \cap Q' \cap P \cap Q \neq \emptyset$ for some $(P, Q) \in F$ then it is easily verified that $M \cap N \cap P \cap Q \neq \emptyset$, and so again PDN and MDQ. Hence x is a cluster point of $\mathcal{F}$ as required.

(iv) $\Rightarrow$ (i). Immediate.

Corollary. For a pairwise completely regular space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, where $(X, \mathcal{S})$ is a cqu for which D has the oup, the notions of D-compact, almost D-compact, D-hypercompact and almost D-hypercompact are all equivalent to the requirement that $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ should be uniformly compact and D conjunctive.

Proof. In view of previously established results it remains only to verify that under the given conditions D-compactness implies D-hypercompactness. Suppose that $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact, then in particular $(X, \mathcal{S})$ is complete and D is conjunctive. Also, for the conditions stated, $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is uniformly compact (see the comment after Proposition 2.6.5 and the corollary to Proposition 2.6.8), and so $(X, \mathcal{S})$ clearly satisfies HTB.4. But then it satisfies HTB.2 by Proposition 2.7.4

and the corollary to Lemma 2.7.3, and the result now follows from the above theorem.

Note. Since the conditions of the above theorem imply that D is conjunctive we could of course replace HTB.2 by HTB.1 in (ii) and (iii). However HTB.2 is the weakest form of this axiom for which I have been able to establish (iii) $\Rightarrow$ (iv) in general.

In the absence of completeness the relation between the "TB" and "HTB" conditions does not seem very clear in general. However we may note the following results.

Proposition 2.7.6. (a) If $(X, \mathcal{S})$ satisfies HTB.4 and D is conjunctive then $(X, \mathcal{S})$ satisfies TB.1.

(b) If $(X, \mathcal{S})$ satisfies HTB.3 and D is $\mathcal{S}$-compatible then $(X, \mathcal{S})$ satisfies TB.5.

Proof. (a) Let $\mathcal{B}$ be a D-regular bifilter on X, and consider the D-hyperfilter $h(\mathcal{B})$. For $d \in \operatorname{dom} \mathcal{S}_o$ define

$$T(d) = \{ (R, S) \mid RDS,\ ReS \text{ for some } e \in \operatorname{dom} \mathcal{S}_o \text{ with } e \prec d, \text{ and } F \in h(\mathcal{B}) \Rightarrow \exists\ (G, H) \in F \text{ with } GDS \text{ and } RDH \}.$$

Using the conjunctiveness of D and the fact that $(X, \mathcal{S})$ satisfies HTB.4 it is not difficult to verify that

$$\{ T(d) \mid d \in \operatorname{dom} \mathcal{S}_o \}$$

is a base for a $\mathcal{S}$-refined D-hyperfilter $\mathfrak{F}$. Let $\mathfrak{H}$ be a maximal, necessarily $\mathcal{S}$-refined, D-hyperfilter with $\mathfrak{F} \subseteq \mathfrak{H}$. $\mathfrak{H}$ is $\mathcal{S}$-Cauchy since HTB.3 is satisfied (Proposition 2.7.4 (a) and the corollary to Lemma 2.7.3). Let $\mathfrak{b} = b(\mathfrak{H})$. Now take $(P, Q) \in \mathcal{B}$ and $d \in \operatorname{dom} \mathcal{S}_o$, then $T(d) \in \mathfrak{F} \subseteq \mathfrak{H}$ and $F(P, Q) \in h(\mathcal{B})$. Also take UdV and $H' \in \mathfrak{H}$ with $H' \lhd (U, V)$. Then $H' \cap T(d) \in \mathfrak{H}$ so $\exists$ $(R, S) \in H' \cap T(d)$. Hence $R \subseteq U$, $S \subseteq V$, and $\exists$ $(G, H) \in F(P, Q)$ with GDS and RDH. However $F(P, Q) \lhd (P, Q)$ and so PDV and so PDV and UDQ. Finally $(U, V) \in b(\mathfrak{H}) = \mathfrak{b}$, and TB.1 is verified.

(b) Let $\mathcal{B}$ be a maximal D-regular bifilter and consider the D-hyperfilter $h(\mathcal{B})$. For $F \in h(\mathcal{B})$ and $d \in \operatorname{dom} \mathcal{S}_o$ define

$F^d = \{ (R, S) \mid RDS,\ ReS$ for some $e \in \operatorname{dom} \mathcal{S}_o$ with $e \prec d$, and

$\exists\ (P, Q) \in F$ so that either $R \subseteq P$, $S \subseteq Q$ or $R \cap S \cap P \cap Q \neq \emptyset \}$.

Since D is $\mathcal{S}$-compatible it is clear that

$$\{ F^d \mid F \in h(\mathcal{B}),\ d \in \operatorname{dom} \mathcal{S}_o \}$$

is a base of a $\mathcal{S}$-refined D-hyperfilter $\mathcal{F}$. Let $\mathcal{H}$ be a maximal refinement of $\mathcal{F}$, so that $\mathcal{H}$ is $\mathcal{S}$-Cauchy by HTB.3. Hence we have UdV and $H \in \mathcal{H}$ with $H \lhd (U, V)$. However it is trivial to verify that if $(P, Q) \in \mathcal{B}$ then $(P \cap U)D(Q \cap V)$, and so $\mathcal{B} \cup \{(U, V)\}$ is a base for a D-regular refinement of $\mathcal{B}$. Since $\mathcal{B}$ is maximal this implies $(U, V) \in \mathcal{B}$, and we have shown that $\mathcal{B}$ is $\mathcal{S}$-Cauchy as required.

Let us also note the following result:

<u>Proposition</u> 2.7.7. If $(X, \mathcal{S})$ satisfies HTB.1 (respectively, HTB.2) then every maximal (respectively, maximal open) dominated D-hyperfilter is weakly $\mathcal{S}$-Cauchy. Moreover, under these latter conditions, if $(X, \mathcal{S})$ is hypercomplete then $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact (respectively, almost D-compact).

<u>Proof</u>. Let $(X, \mathcal{S})$ satisfy HTB.1, and let $\mathcal{F}$ be a dominated D-hyperfilter with base $\mathcal{F}'$ satisfying (a) - (d) of Definition 2.7.2. If $\mathcal{H}$ is a (not necessarily dominated) maximal refinement of $\mathcal{F}$ then $\mathcal{H}$ is weakly $\mathcal{S}$-Cauchy. Take $d \in \operatorname{dom} \mathcal{S}_o$, UdV and $H \in \mathcal{H}$ with $H \lhd\!\rhd (U, V)$. If $F' \in \mathcal{F}'$ and $F' \lhd (L, K)$ then $F' \cap H \neq \emptyset$ so LDV and UDK. Now take $F \in \mathcal{F}'$, then by (a) if $(L, K) \in F$ we have $F' \in \mathcal{F}'$ with $F' \lhd (L, K)$, and so LDV, UDK by the above. Hence $F \lhd\!\rhd (U, V)$, that is $\mathcal{F}$ is weakly $\mathcal{S}$-Cauchy.

Now assume that all the maximal dominated D-hyperfilters are weakly $\mathcal{S}$-Cauchy, and let $\mathcal{B}$ be a maximal D-regular bifilter. Using Proposition 2.7.1 it is easy to verify that $h(\mathcal{B})$ is a maximal dominated D-hyperfilter, and hence weakly $\mathcal{S}$-Cauchy. Thus, if $(X, \mathcal{S})$ is hypercomplete, $h(\mathcal{B})$ has a weak limit point x. In particular x is a weak cluster point of $h(\mathcal{B})$, so $\mathcal{B}$ has a cluster point by Lemma 2.7.1. It follows at once that $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is D-compact.

The remaining cases are dealt with in just the same way.

We now examine some conditions under which the "HTB" axioms carry over to an extension.

Proposition 2.7.8. Let $(X, \mathcal{S})$ be a cqu, $A \subseteq X$ bidense and D-embedded. Then if $(A, \mathcal{S}_A)$ satisfies HTB.2 so does $(X, \mathcal{S})$.

Proof. Let $\mathcal{F}$ be a maximal open D-hyperfilter on X, and $\mathcal{F}'$ an open base of $\mathcal{F}$. For $F \in \mathcal{F}'$ let

$$F_A = \{ (L \cap A, K \cap A) \mid (L, K) \in F \}.$$

Then $\{ F_A \mid F \in \mathcal{F}' \}$ is a base for an open D_A-hyperfilter $\mathcal{F}_A$ on A. Let $\mathcal{G}$ be a maximal open D_A-hyperfilter refinement of $\mathcal{F}_A$ on A. Then $\mathcal{G}$ is weakly $\mathcal{S}_A$-Cauchy so given $d \in \operatorname{dom} \mathcal{S}_0$ we have UdV and $G \in \mathcal{G}$ so that $G \triangleleft\triangleright (U \cap A, V \cap A)$. Now let

$$G^* = \{ (P', Q') \mid P'DQ', P' \in t_u(\mathcal{S}), Q' \in t_v(\mathcal{S}) \text{ and } P \subseteq P' \cap A,$$
$$Q \subseteq Q' \cap A \text{ for some } (P, Q) \in G \}.$$

It is easy to verify that $F \cap G^* \neq \emptyset$ for each $F \in \mathcal{F}'$, and so $G^* \in \mathcal{F}$ as $\mathcal{F}$ is maximal. However since A is D-embedded in X we see that $G^* \triangleleft\triangleright (U, V)$, and so $\mathcal{F}$ is weakly $\mathcal{S}$-Cauchy as required.

Corollary. Let $(X, \mathcal{S})$ be a separated cqu satisfying HTB.2, and suppose that D has the oup. Then the separated strict extension $(\Gamma_0(X), \tilde{\mathcal{S}})$ is $\tilde{D}$-hypercompact.

Proposition 2.7.9. Let $(X, \mathcal{S})$ be a cqu, $A \subseteq X$ bidense and strictly $\mathcal{S}$-embedded. Then if $(A, \mathcal{S}_A)$ satisfies HTB.3 so does $(X, \mathcal{S})$.

Proof. This result follows trivially from the characterizations of HTB.3 given in Lemma 2.7.3.

Corollary. If $(X, \mathcal{S})$ is a separated cqu which satisfies HTB.3 then the separated strict extension $(\Gamma_0(X), \tilde{\mathcal{S}})$ is hypercomplete.

Finally in this connection let us note:

Proposition 2.7.10. Let $(X, \mathcal{S})$ be a cqu and $A \subseteq X$ uniformly dense in $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$. Then if $(A, \mathcal{S}_A)$ satisfies HTB.4 so does $(X, \mathcal{S})$.

We omit the proof, which is straightforward.

D-hypercompactness and almost D-hypercompactness may also be described in terms of dual covering properties. In particular this gives us a simple characterization of all non-pathological D-hypercompact spaces.

Theorem 2.7.2. Let D be an internal confluence relation on the bitopological space (X, u, v). Then (X, u, v) is D-hypercompact (respectively, almost D-hypercompact) if and only if given any open D-dual cover d of X there is a finite subfamily d_o of d so that given any PDQ (respectively, given any PDQ with $P \in u$ and $Q \in v$) there exists Ud_oV with PDV and UDQ.

Proof. First suppose that (X, u, v) is D-hypercompact, but that there exists an open D-dual cover d not satisfying the conditions mentioned in the theorem. Then for any finite subset d_o of d the set

$$F(d_o) = \{ (P, Q) \mid PDQ \text{ and } Ud_oV \Rightarrow P\not{D}V \text{ or } U\not{D}Q \}$$

is non-empty, and so $\{ F(d_o) \mid d_o \subseteq d \text{ is finite} \}$ is a base for a D-hyperfilter $\mathfrak{F}$ on X. Let x be a cluster point of $\mathfrak{F}$, and take $U_o d\, V_o$ with $x \in U_o \cap V_o$. We may take $d_o = \{ (U_o, V_o) \}$, which means $F(d_o) \in \mathfrak{F}$, while $(U_o, V_o) \in \mathcal{B}(x)$ implies that there exists $(P, Q) \in F(d_o)$ with PDV_o and U_oDQ, which contradicts the definition of $F(d_o)$.

Conversely suppose the condition is satisfied, and let $\mathfrak{F}$ be a D-hyperfilter. If $\mathfrak{F}$ has no cluster point then for each $x \in X$ we have $x \in U(x) \in u$, $x \in V(x) \in v$ and $F_x \in \mathfrak{F}$ so that $(P, Q) \in F_x \Rightarrow P\not{D}V(x)$ or $U(x)\not{D}Q$. Then

$$d = \{ (U(x), V(x)) \mid x \in X \}$$

is an open dual cover (and hence an open D-dual cover) of X, and hence we have $x_1, \ldots, x_n \in X$ so that given PDQ there exists i, $1 \leq i \leq n$, with $PDV(x_i)$ and $U(x_i)DQ$. However if we take $(P, Q) \in \bigcap \{ F_{x_i} \mid 1 \leq i \leq n \} \in \mathfrak{F}$ we obtain an immediate contradicion, and so $\mathcal{F}$ has a cluster point as required. Note that this argument actually shows that the stated condition need only be assumed to apply to all open dual covers of X.

The necessary changes to be made for the almost D-hypercompact case are obvious, and so the proof is complete.

Corollary 1. Let d be an open dual cover of the almost D-hypercompact space (X, u, v). Then there is a finite subfamily d_o of d so that $\bigcup \{ {}^{D}U \cap V^{D} \mid (U, V) \in d_o \} = X$.

Corollary 2. Let D be an interior confluence relation on (X, u, v), and if $D \neq 1$ suppose that no single point set is open in either topology. Then (X, u, v) is D-hypercompact if and only if it is uniformly compact and D is conjunctive.

In particular it follows from Corollary 2 that 1-hypercompact and uniformly compact are identical for all bitopological spaces.

We end this section with a generalisation of the result, established in Chapter One, that every preseparated uniformly compact space is fully binormal. We assume that D satisfies the condition of Corollary 2 above so that we may say that D-hypercompactness implies uniform compactness.

Theorem 2.7.3. Let D be an interior confluence relation on (X, u, v), and suppose that (X, u, v) is D-separated and D-hypercompact. Then if d is any open D-dual cover of X there is an open D-dual cover e of X with $(e, D) \prec(*) (d, D)$.

Proof. Suppose that there is an open D-dual cover d which does not have this property. Then for every open D-dual cover e,

$$F(e) = \{ (L, K) \mid LDK, \exists \ ReS \text{ with } L \subseteq R, K \subseteq S, \text{ and for each } UdV \text{ we have } St_D(e, R) \nsubseteq U \text{ or } St_D(S, e) \nsubseteq V \}$$

is non-empty. It follows at once that $\{ F(e) \mid e$ is an open D-dual cover of X $\}$ is a base for a D-hyperfilter $\mathcal{F}$ on X. $\mathcal{F}$ has a cluster point x, and we may take UdV with $x \in U \cap V$.

Let $Y = X - \{x\}$. We divide Y into three mutually disjoint subsets as follows:

$Y_1 = \{ y \mid y \notin u\text{-cl}\{x\}$ and $y \notin v\text{-cl}\{x\} \}$,

$Y_2 = \{ y \mid y \notin u\text{-cl}\{x\}$ and $y \in v\text{-cl}\{x\} \}$,

$Y_3 = \{ y \mid y \in u\text{-cl}\{x\}$ and $y \notin v\text{-cl}\{x\} \}$.

For $y \in Y_1$ $\exists$ $R(y) \not\!\! \emptyset V(y)$ with $y \in R(y) \in u$, $x \in V(y) \in v$, and

$\exists$ $U(y) \not\!\! \emptyset S(y)$ with $y \in S(y) \in v$, $x \in U(y) \in u$.

For $y \in Y_2$ $\exists$ $R(y) \not\!\! \emptyset V(y)$ with $y \in R(y) \in u$, $x \in V(y) \in v$, $R(y) \subseteq U$.

For $y \in Y_3$ $\exists$ $U(y) \not\!\! \emptyset S(y)$ with $y \in S(y) \in v$, $x \in U(y) \in u$, $S(y) \subseteq V$.

Consider $e = \{ (R(y), S(y)), (U(y), V(y)) \mid y \in Y_1 \} \cup \{ (R(y), X), (U, V(y)) \mid y \in Y_2 \} \cup \{ (X, S(y)), (U(y), V) \mid y \in Y_3 \}$.

e is an open dual cover of X, and (X, u, v) is uniformly compact, so there is a finite sub-dual cover e_o which we may take in the form:

$$e_o = \{ (R(y_i), S(y_i)), (U(y_i), V(y_i)) \mid 1 \le i \le n \} \cup \{ (R(y_i), X), (U, V(y_i)) \mid n+1 \le i \le m \} \cup \{ (X, S(y_i)), (U(y_i), V) \mid m+1 \le i \le k \},$$

where $y_1, \ldots, y_n \in Y_1$; $y_{n+1}, \ldots, y_m \in Y_2$ and $y_{m+1}, \ldots, y_k \in Y_3$.

Now let $U' = U \cap \bigcap \{ U(y_i) \mid 1 \le i \le n \text{ or } m+1 \le i \le k \}$, and

$$V' = V \cap \bigcap \{ V(y_i) \mid 1 \le i \le m \}.$$

Then $U' \in u$, $V' \in v$ and $x \in U' \cap V'$. Define:

$$Z = X - \left[\{x\} \cup \bigcup \{ R(y_i) \cap S(y_i) \mid 1 \le i \le n \} \cup \bigcup \{ R(y_i) \mid n+1 \le i \le m \} \cup \bigcup \{ S(y_i) \mid m+1 \le i \le k \} \right] \subseteq U \cap V.$$

We suppose $Z \ne \emptyset$, omitting the case $Z = \emptyset$ which is somewhat simpler. We divide Z into two mutually disjoint subsets as follows.

$Z_1 = \{ z \mid z \in Z \text{ and } z \notin u\text{-cl}\{x\} \}$,

$Z_2 = \{ z \mid z \in Z,\ z \in u\text{-cl}\{x\} \text{ and } z \notin v\text{-cl}\{x\} \}$.

For $z \in Z_1$ $\exists$ $R'(z) \not\!\! \emptyset V'(z)$ with $z \in R'(z) \in u$, $x \in V'(z) \in v$ and $V'(z) \subseteq V'$.

For $z \in Z_2$ $\exists$ $U'(z) \not{D} S'(z)$ with $z \in S'(z) \in \boldsymbol{v}$, $x \in U'(z) \in \boldsymbol{u}$,
$S'(z) \subseteq V$ and $U'(z) \subseteq U'$.

Consider the following open dual cover of X:

$$f = \{ (R(y_i), S(y_i)) \mid 1 \leq i \leq n \} \cup \{ (R(y_i), X) \mid n+1 \leq i \leq m \} \cup \{ (X, S(y_i)) \mid m+1 \leq i \leq k \} \cup \{ (R'(z), V), (U', V'(z)) \mid z \in Z_1 \} \cup \{ (U, S'(z)), (U'(z), V') \mid z \in Z_2 \}.$$

Again f will have a finite sub-dual cover f_o which we may take in the form:

$$f_o = \{ (R(y_i), S(y_i)) \mid 1 \leq i \leq n \} \cup \{ R(y_i), X) \mid n+1 \leq i \leq m \} \cup \{ (X, S(y_i)) \mid n+1 \leq i \leq k \} \cup \{ (R'(z_i), V), (U', V'(z_i)) \mid 1 \leq i \leq s \} \cup \{ (U, S'(z_i)), (U'(z_i), V') \mid s+1 \leq i \leq t \},$$

where $z_1, \ldots, z_s \in Z_1$ and $z_{s+1}, \ldots, z_t \in Z_2$.

Let $M = U' \cap \bigcap \{ U'(z_i) \mid s+1 \leq i \leq t \}$, and
$N = V' \cap \bigcap \{ V'(z_i) \mid 1 \leq i \leq s \}$. Note that $(M, N) \in \mathcal{B}(x)$.

Finally let us define:

$$g = \{ (R(y_i), S(y_i)) \mid 1 \leq i \leq n \} \cup \{ (R(y_i), X) \mid n+1 \leq i \leq m \} \cup \{ (X, S(y_i)) \mid m+1 \leq i \leq k \} \cup \{ (R'(z_i), V) \mid 1 \leq i \leq s \} \cup \{ (U, S'(z_i)) \mid s+1 \leq i \leq t \} \cup \{ (U', V') \}.$$

g is an open dual cover of X, and so in particular an open D-dual cover. Hence $F(g) \in \mathcal{F}$, and so we have $(L, K) \in F(g)$ with LDN and MDK. Also there exists GgH with $L \subseteq G$, $K \subseteq H$ and $St_D(g, G) \nsubseteq U$ or $St_D(H, g) \nsubseteq V$. However by considering all the possible choices of (G, H) it is not difficult to verify that we must have $G = U'$ and $H = V'$. On the other hand we may easily show that $St_D(g, U') \subseteq U$ and $St_D(V', g) \subseteq V$, and this contradiction proves the theorem.

<u>Corollary</u>. If (X, u, v) is D-separated and D-hypercompact then $\{ (d, D) \mid d$ is an open D-dual cover of $X \}$ is a base for a cqu $\mathcal{S}$ on X which is compatible with (X, u, v).

If we call a cqu $\mathcal{S}$ basic if it has a base β with ran β = $\{D\}$ then this corollary says that a (non-pathological) D-separated D-hypercompact space always has a compatible basic cqu. Moreover this cqu is unique, for it is easy to verify that a basic cqu compatible with a D-hypercompact space must contain (d, D) for every open D-dual cover d.

2.8. CONFLUENCE PARA-QUASI-UNIFORMITIES.

In this section we extend our work on confluence quasi-uniformities to para-quasi-uniformities.

If c is a confluence relation on X, and d is a dual family with $d \subseteq c$, we shall say that d is a c-dual family. If d is a c-dual family and e is a b-dual family, then the meaning of such expressions as

$$(e, b) \prec (d, c), \quad (e, b) \prec^{(*)} (d, c)$$

is clear. We will write

$$(e, b) \preceq (d, c), \quad (e, b) \preceq^{(*)} (d, c)$$

respectively if in addition we have uc(d) = uc(e).

For c_i-dual families d_i, i = 1, ..., n, we define

$$\bigwedge\{(d_i, c_i)\} = (d, c)$$

where $c = \bigcap\{c_i\}$ and $PdQ \iff PcQ$ and $\exists\ U_i d_i V_i$, $1 \leq i \leq n$, so that $P = \bigcap\{U_i\}$ and $Q = \bigcap\{V_i\}$.

(Note that, for ease of writing, we shall omit the range $1 \leq i \leq n$ of i for the operations $\bigwedge$, $\bigcap$ etc. where no confusion can arise).

Note that $\bigwedge\{(d_i, c_i)\}$ is a (possibly empty) $\bigcap\{c_i\}$-dual family. It is worthwhile noting that if $(e_i, b_i) \preceq (d_i, c_i)$ for each i then $\bigwedge\{(e_i, b_i)\} \preceq \bigwedge\{(d_i, c_i)\}$.

Finally let us denote by $\mathcal{C}^c$ (or, more precisely, $\mathcal{C}^c{}_X$) the set of pairs (e, b) where b is a confluence relation on X and e is a b-dual family belonging to $\mathcal{C}$.

On analogy with a pqu we may now give:

Definition 2.8.1. The subset $\mathcal{S}$ of $\wp^c$ is a confluence para-quasi-uniformity (cpqu) if it satisfies.

CPQ.1. $(d, c) \in \mathcal{S} \Rightarrow \exists\ (e, b) \in \mathcal{S}$ with $(e, b) \preceq (*)\ (d, c)$.

CPQ.2. $(d_i, c_i) \in \mathcal{S}$, $i = 1, \ldots, n$, with $\bigwedge\{(d_i, c_i)\} \in \wp^c$
$\Rightarrow \bigwedge\{(d_i, c_i)\} \in \mathcal{S}$.

CPQ.3. $(d, c) \in \mathcal{S}$, $(e, b) \in \wp^c$ with $(d, c) \preceq (e, b)$
$\Rightarrow (e, b) \in \mathcal{S}$.

CPQ.4. $\{(X, X)\} \in \operatorname{dom} \mathcal{S}$.

Note that every cqu is also a cpqu.

cpqu bases and subbases may be defined in the obvious way, and we omit the details. Exactly as for a pqu, a cpqu defines a bitopological space $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$, and we note in particular that a base of $t_u(\mathcal{S})$- (respectively, $t_v(\mathcal{S})$-) nhds. of $x \in X$ is given by $\{ St(d, \{x\}) \mid d \in \operatorname{dom} \mathcal{S},\ x \in uc_1(d) \}$ (respectively, $\{ St(\{x\}, d) \mid d \in \operatorname{dom} \mathcal{S},\ x \in uc_2(d) \}$).

We denote by $\mathcal{S}^*$ the base of $\mathcal{S}$ given by

$$\mathcal{S}^* = \{ (d, c) \mid (d, c) \in \mathcal{S} \text{ and } \exists\ (d', c') \in \mathcal{S} \text{ with } (d, c) \preceq (*)\ (d', c') \}.$$

Lemma 2.8.1. For $d \in \operatorname{dom} \mathcal{S}^*$ we have

$$(t_u(\mathcal{S})\text{-int}[uc_1(d)]) \cup (t_v(\mathcal{S})\text{-int}[uc_2(d)]) \neq \emptyset.$$

We omit the proof, which is straightforward.

In general a cpqu need not have an open base; however for convenience in all that follows we will assume that all cpqu under consideration are such that

$$\mathcal{S}_0 = \{ (d, c) \mid (d, c) \in \mathcal{S}^*,\ c \text{ is interior, and } UdV \Rightarrow (t_u(\mathcal{S})\text{-int}[U])c(t_v(\mathcal{S})\text{-int}[V]) \}$$

is a base of $\mathcal{S}$. Of course this is automatically true for a cqu, and $\mathcal{S}_0$ then has the same meaning as it did in the earlier sections.

For a dual family d on $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ we shall denote by $\dot{d}$ the dual family $\{(t_u(\mathcal{S})\text{-int}[U], t_v(\mathcal{S})\text{-int}[V]) \mid UdV$ and $(t_u(\mathcal{S})\text{-int}[U]) \neq \emptyset \neq (t_v(\mathcal{S})\text{-int}[V])\}$. Note that for $d \in \text{dom}\, \mathcal{S}_o$ we have $\dot{d} = \{(t_u(\mathcal{S})\text{-int}[U], t_v(\mathcal{S})\text{-int}[V]) \mid UdV\}$, and $\dot{d}$ is a c-dual family. However $(\dot{d}, c)$ need not belong to $\mathcal{S}$ in general, and we may well have $uc(\dot{d}) \neq uc(d)$.

Some of the results given earlier for cqu will carry over basicly unchanged to the cpqu case, but the majority will need at least some modifications to the definitions involved, while others will not hold at all for general cpqu. Our aim in this section is to concentrate mainly on those results which have a direct bearing on the question of induced cpqu structures and of extensions.

Let $(X, \mathcal{S})$ be a cpqu, and $A \subseteq X$. If c is a confluence relation on X and d a dual family we may define c_A and d_A as previously. Note that if $(d, c) \in \mathcal{S}_o$ then d_A is a c_A-dual family. The induced structure $\mathcal{S}_A$ on A may be defined by

$$\mathcal{S}_A = \{(d, c) \mid (d, c) \in \mathcal{B}^c{}_A \text{ and } \exists\ (e, b) \in \mathcal{S}_o \text{ with } (e_A, b_A) \leq (d, c)\}.$$

The conditions (a) or (b) of Definition 2.3.2 will no longer suffice, in general, to ensure that $(A, \mathcal{S}_A)$ is a cpqu, and in order to describe the additional conditions required we shall need some more definitions and notation. Let d be a c-dual family on X and define

$$(d, c)_A = (d', c_A)$$

where $U'd'V' \iff U'c_AV'$ and $U' = U \cap A$, $V' = V \cap A$ for some UdV. Clearly if $(d, c) \in \mathcal{S}_o$ we shall have $(d, c)_A = (d_A, c_A)$, but this equality need not hold in general. Now for $(d^i, c^i) \in \mathcal{S}$, $i = 1, \ldots, n$, let us define

$$\overset{A}{\wedge}\{(d^i, c^i)\} = (d, c)$$

where $c = \bigcap\{c^i\}$ and $PdQ \Leftrightarrow PcQ$, $(P \cap A)c_A(Q \cap A)$ and $\exists\ U_i d^i V_i$ with $P = \bigcap\{U_i\}$ and $Q = \bigcap\{V_i\}$.

Clearly $\overset{A}{\wedge}\{(d^i, c^i)\} \leq \wedge\{(d^i, c^i)\}$, and if $\overset{A}{\wedge}\{(d^i, c^i)\}$ $= (d, c)$ then

$$[\overset{A}{\wedge}\{(d^i, c^i)\}]_A = (d_A, c_A) = [\wedge\{(d^i, c^i)\}]_A \leq \wedge\{(d^i{}_A, c^i{}_A)\}.$$

We will now say that $A \subseteq X$ is $\mathcal{S}$ -embedded (respectively, strictly $\mathcal{S}$ -embedded) in X if there is a base $\beta \subseteq \mathcal{S}_o$ of $\mathcal{S}$ satisfying condition (a) (respectively, condition (b)) of Definition 2.3.2, and in addition the condition

(c) For $(e, b) = \overset{A}{\wedge}\{(e^i, b^i)\}$ where $(e^i, b^i) \in \beta$, $i = 1, \ldots,$ n we have:

 (α) $(e, b) \in \mathcal{C}^c{}_X \Rightarrow (e, b) \in \mathcal{S}$, and

 (β) $uc_j(e_A) \subseteq uc_j(e)$, $j = 1, 2$.

Note that this extra condition is trivially satisfied for a cqu and so our terminology remains consistent.

We may now give:

<u>Proposition</u> 2.8.1. Let $(X, \mathcal{S})$ be a cpqu, and $A \subseteq X$ bidense and $\mathcal{S}$ -embedded. Then $(A, \mathcal{S}_A)$ is a cpqu, $t_u(\mathcal{S}_A) = t_u(\mathcal{S})_A$ and $t_v(\mathcal{S}_A) = t_v(\mathcal{S})_A$.

<u>Proof</u>. Let $\beta \subseteq \mathcal{S}_o$ be a base of $\mathcal{S}$ satisfying (a) and (c). First let us note that for a dual family d on X we have $uc(d) \cap A$ $= uc(d_A)$ and $uc_j(d) \cap A \subseteq uc_j(d_A)$, $j = 1, 2$. Hence, since A is bidense in X, we see that $(d_A, c_A) \in \mathcal{C}^c{}_A$ for each $(d, c) \in \mathcal{S}_o$ and so it will suffice to show that $\{(e_A, b_A) \mid (e, b) \in \beta\}$ is cpqu base on A. Now it is easy to verify that if $(d, c), (e, b) \in \beta$ and $(e, b) \leq (*) (d, c)$ then $(e_A, b_A) \leq (*) (d_A, c_A)$, so let us take $(e^i, b^i) \in \beta$, $i = 1, \ldots, n$, with $\wedge\{(e^i{}_A, b^i{}_A)\} \in \mathcal{C}^c{}_A$. As noted above, if we set $\overset{A}{\wedge}\{(e^i, b^i)\} = (e, b)$, then $(e_A, b_A) \leq \wedge\{(e^i{}_A, b^i{}_A)\}$ and so we also have $(e_A, b_A) \in \mathcal{C}^c{}_A$

by Lemma 1.8.1 (c). But then $(e, b) \in \mathcal{C}^c_X$ by (c) (β), and so $(e, b) \in \mathcal{S}$ by (c) (α). Hence we have $(g, a) \in \beta$ with $(g, a) \preceq (e, b)$, and then $(g_A, a_A) \preceq \bigwedge\{(e^i_A, b^i_A)\}$ which completes the proof that $\{(e_A, b_A) \mid (e, b) \in \beta\}$ is a base for a cpqu on A. Finally, using (c) (β) we may verify that for $(e, b) \in \beta$ we have $uc_j(e_A) = uc_j(e) \cap A$, $j = 1, 2$, and the topological identities follow from this.

For the cpqu $\mathcal{S}$ we define the interior confluence relation D by

$$D = \bigcap\{c \mid c \in \operatorname{ran} \mathcal{S}_o\} = \bigcap\{c \mid c \in \operatorname{ran} \mathcal{S}\}$$

as before. If β is a base of $\mathcal{S}$ and $\mathcal{B}$ is a bifilter on X then the notions of β-regularity and D-regularity for $\mathcal{B}$ will be as given in Definition 2.2.2. On the other hand we shall say that $\mathcal{B}$ is $\mathcal{S}$-Cauchy if it satisfies

$$(d_i, c_i) \in \mathcal{S}_o,\ i = 1, \ldots, n,\ \bigwedge\{(d_i, c_i)\} = (d, c) \Rightarrow \dot{d} \cap \mathcal{B} \neq \emptyset.$$

For a cqu this condition is, of course, equivalent to that given in Definition 2.2.4, and so no confusion can arise here.

As previously $(X, \mathcal{S})$ will be called complete if every D-regular $\mathcal{S}$-Cauchy bifilter on X is convergent on $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$.

Convergence of a bifilter is, of course, a purely bitopological notion. For the cpqu $\mathcal{S}$ let us say that $\mathcal{B}$ is <u>$\mathcal{S}$-convergent to $x \in X$</u> if $(St(d,\{x\}), St(\{x\}, d)) \in \mathcal{B}$ for all $d \in \operatorname{dom} \mathcal{S}$. $\mathcal{S}$-convergence certainly implies convergence, and these notions are equivalent for a cqu, but they will not be equivalent in general since some of the sets $St(d,\{x\})$, $St(\{x\}, d)$ need not be nhds. of x.

Note that the existence of a $\mathcal{S}$-Cauchy bifilter implies, in particular, that $\bigwedge\{(d_i, c_i)\} \neq \emptyset$ for any $(d_i, c_i) \in \mathcal{S}_o$, $i = 1, \ldots, n$, and so any cpqu not satisfying this condition must, of necessity, be complete.

If $\beta \subseteq \mathcal{S}_o$ is a base of $\mathcal{S}$ and $\mathcal{B}$ is a $\mathcal{S}$-Cauchy β-regular bifilter, then arguing as in the proof of Proposition 2.3.3 it is easy to verify that the bifilter $\mathcal{B}^*$ with subbase

$\{(St_c(d, U), St_c(V, d)) \mid (d, c) \in \beta$, $U \in \text{dom } d$, $V \in \text{ran } d$ and $(t_u(\mathcal{S})\text{-int}[U], t_v(\mathcal{S})\text{-int}[V]) \in \mathcal{B}\}$

is a minimal D-regular $\mathcal{S}$ -Cauchy bifilter contained in $\mathcal{B}$. It is also clearly open. In particular it follows ~~that~~, as for cqu, that a cpqu $(X, \mathcal{S})$ is complete if and only if every β -regular $\mathcal{S}$ -Cauchy bifilter is convergent.

Nhd. bifilters $\mathcal{B}(x)$ and nhd. bifilter traces $\mathcal{B}_A(x)$ maintain their regularity properties, but they need not be Cauchy when we are dealing with a cpqu. This represents an important difference between cpqu and cqu. If in constructing a bifilter extension of a cpqu space we were to include the elements of X in the form $\{\mathcal{B}(x) \mid x \in X\}$ we should, in any case, have to apply different arguments to the elements $\mathcal{B}(x)$ from those used for the remaining $\mathcal{S}$ -Cauchy elements of the extension, and this suggests that we might just as well include X in the form $\{x \mid x \in X\}$. This, of course, also has the added advantage that we can then deal equally well with the case when $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is not weakly pairwise T_o. Bearing these comments in mind let us now show how we may construct a strict completion of a (non-complete) cpqu $(X, \mathcal{S})$.

Denote by $\mathcal{N}_o(X)$ the set of all non-convergent D -regular minimal $\mathcal{S}$ -Cauchy bifilters on X, and set

$$\mathcal{N}_1(X) = X \cup \mathcal{N}_o(X).$$

In place of the sets A^o_u of § 2.5 we consider the following:

$$A^o_u = [t_u(\mathcal{S})\text{-int}[A]] \cup \{\mathcal{B} \mid \mathcal{B} \in \mathcal{N}_o(X) \text{ and } \exists\ d \in \text{dom } \mathcal{S}_o \text{ with } UdV,\ V \in \mathcal{B}_v \Rightarrow U \subseteq A \cap uc(d)\}, \text{ and}$$

$$A^1_u = A \cup \{\mathcal{B} \mid \mathcal{B} \in \mathcal{N}_o(X) \text{ and } \exists\ d \in \text{dom } \mathcal{S}_o \text{ with } UdV,\ V \in \mathcal{B}_v \Rightarrow U \subseteq A\}.$$

Note that if $\mathcal{S}$ is a cqu then $A^o_u = A^1_u$ for all $A \in t_u(\mathcal{S})$, but in general we can only claim that $A^o{}_u \subseteq A^1{}_u$.

We may define $B^o{}_v$ and $B^1{}_v$ in an analogous way.

Note that for any subsets $A_1, \ldots, A_n$; $B_1, \ldots, B_m$ of X with $(\bigcap\{(A_i)^1{}_u\}) \cap (\bigcap\{(B_j)^1{}_v\}) \neq \emptyset$ we clearly have $(\bigcap\{A_i\}) D (\bigcap\{B_j\})$.

If c is a confluence relation on X we may define the confluence relation $\hat{c}$ on $\mathcal{N}_1(X)$ by

$$P\hat{c}Q \iff P \iota Q \text{ or } \exists\ A \in t_u(\mathcal{S}),\ B \in t_v(\mathcal{S}) \text{ with } AcB,\ A^o{}_u \subseteq P \text{ and } B^o{}_v \subseteq Q.$$

If $(d, c) \in \mathcal{S}_o$ we define $\hat{d}$ on $\mathcal{N}_1(X)$ by

$$P\hat{d}Q \iff \exists\ UdV \text{ with } P = U^1{}_u \text{ and } Q = V^1{}_v.$$

Since c is interior it is clear that $\hat{d}$ is a $\hat{c}$-dual family. Note also that

$uc(\hat{d}) = uc(d) \cup \mathcal{N}_o(X)$ and $uc_j(\hat{d}) \cap X = uc_j(d)$, $j = 1, 2$,

so that for $(d, c) \in \mathcal{S}_o$ we have $(\hat{d}, \hat{c}) \in \mathcal{C}^c{}_{\mathcal{N}_1(X)}$.

Now let us verify that $\{(\hat{d}, \hat{c}) \mid (d, c) \in \mathcal{S}_o\}$ is a base for a cpqu $\hat{\mathcal{S}}$ on $\mathcal{N}_1(X)$. Firstly for (d, c), $(e, b) \in \mathcal{S}_o$ with $(e, b) \preceq_{(*)} (d, c)$, the verification that $(\hat{e}, \hat{b}) \preceq_{(*)} (\hat{d}, \hat{c})$ is essentially the same as for the corresponding result for cqu. Secondly let us take $(d_i, c_i) \in \mathcal{S}_o$ with $\bigwedge\{(\hat{d}_i, \hat{c}_i)\} = (F, \bigcap\{\hat{c}_i\}) \in \mathcal{C}^c{}_{\mathcal{N}_1(X)}$, and suppose, for example, that $uc_1(F) \neq \emptyset$. Now take $(e_i, b_i) \in \mathcal{S}_o$ with $(e_i, b_i) \preceq_{(*)} (d_i, c_i)$, and let $\bigwedge\{(e_i, b_i)\} = (e, \bigcap\{b_i\})$. Now if $x \in uc_1(F) \cap X \neq \emptyset$ then it is immediate that $x \in uc_1(e)$, so let us consider the case $\mathcal{B} \in uc_1(F) \cap \mathcal{N}_o(X)$. Since $\mathcal{B}$ is $\mathcal{S}$-Cauchy $\exists\ R_i e_i S_i$ with $(\bigcap\{R_i\}, \bigcap\{S_i\}) \in \mathcal{B}$. But if we then take $z \in \bigcap\{R_i\}$ it may easily be verified that $z \in uc_1(e)$, and so in either event $uc_1(e) \neq \emptyset$. In the same way $uc_2(F) \neq \emptyset \Rightarrow uc_2(e) \neq \emptyset$, so $e \in \mathcal{C}_X$

and $(e, \cap\{b_i\}) \in \mathcal{S}$. Hence if we take $(g, a) \in \mathcal{S}_o$ with $(g, a) \preccurlyeq (e, \cap\{b_i\})$ it is immediate that

$$(\hat{g}, \hat{a}) \preccurlyeq \wedge\{(\hat{d}_i, \hat{c}_i)\}$$

which completes the proof that $\{(\hat{d}, \hat{c}) \mid (d, c) \in \mathcal{S}_o\}$ is a base for a cpqu $\hat{\mathcal{S}}$ on $\Omega_1(X)$.

It is not difficult to verify that for all $A \subseteq X$ we have

$$A^*_u = A^o_u = t_u(\hat{\mathcal{S}})\text{-int}[A^1_u] \qquad \text{............ (1)}$$

where A^*_u is formed with respect to the subset X of $(\Omega_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$. Also these sets are clearly unchanged if we replace A by $t_u(\mathcal{S})$-int[A]. Similar statements hold for the other topologies, and we deduce in particular that for $(d, c) \in \mathcal{S}_o$ and UdV we have $[t_u(\hat{\mathcal{S}})\text{-int}[U^1_u]]\hat{c}[t_v(\hat{\mathcal{S}})\text{-int}[V^1_v]]$. Hence $(\hat{\mathcal{S}})_o$ is a base for $\hat{\mathcal{S}}$, that is $(\Omega_1(X), \hat{\mathcal{S}})$ satisfies our general hypothesis.

Next let us note that X is strictly $\hat{\mathcal{S}}$-embedded in $\Omega_1(X)$ with respect to the base $\beta = \{(\hat{d}, \hat{c}) \mid (d, c) \in \mathcal{S}_o$ and $\exists$ $(e, b) \in \mathcal{S}_o$ with $(d, c) \preccurlyeq(\omega) (e, b)\} \subseteq (\hat{\mathcal{S}})_o$. Certainly (b) (i) and (ii) of Definition 2.3.2 follow at once from the equalities (1) above. Finally for (c) (α) and (β) take $(\hat{e}^i, \hat{b}^i) \in \beta$ and set $\overset{X}{\wedge}\{(\hat{e}^i, \hat{b}^i)\} = (E, \cap\{\hat{b}^i\})$. That $(E, \cap\{\hat{b}^i\}) \in \mathcal{P}^c_{\Omega_1(X)}$ $\Rightarrow$ $(E, \cap\{\hat{b}^i\}) \in \hat{\mathcal{S}}$ follows exactly as in the proof of the corresponding result with " $\overset{X}{\wedge}$ " replaced by " $\wedge$ ", outlined above, and so (α) is proved. (β) follows from the evident fact that $\Omega_o(X) \subseteq uc(E)$.

It is not difficult to verify that for $(d, c) \in \mathcal{S}_o$ we have $(\hat{d}_X, \hat{c}_X) = (d, c)$, and so $\hat{\mathcal{S}}_X = \mathcal{S}$. Moreover X is clearly bidense in $\Omega_1(X)$ and so we have verified that $(\Omega_1(X), \hat{\mathcal{S}})$ is a strict extension of (X, δ).

Now let B be a $\hat{D}$-regular $\hat{\delta}$-Cauchy bifilter on $\Lambda_1(X)$, and let

$$\mathcal{B} = \{ (P, Q) \mid (P^*_u, Q^*_v) \in B \}$$

It is clear that $\mathcal{B}$ is a D-regular δ-Cauchy bifilter on X, so there are two possibilities. Either $\mathcal{B}$ converges in X to an element $x \in X$, or it is non-convergent. In the first case we may easily see that B converges to x in $\Lambda_1(X)$, while in the second, if we form $\mathcal{B}^*$ for the base δ_o then $\mathcal{B}^* \in \Lambda_1(X)$ and B is then $\hat{\delta}$-convergent (and hence convergent) to $\mathcal{B}^*$ in $\Lambda_1(X)$.

This completes the proof that $(\Lambda_1(X), \hat{\delta})$ is a strict completion of (X, δ).

It is of some interest to try to characterize the cpqu $(\Lambda_1(X), \hat{\delta})$, and this is the aim of the next theorem. For this purpose it is first necessary to make definite the notion of cpqu isomorphism. If (X, δ) and (Y, μ) are cpqu, and $f : X \to Y$ is a function we may define $(f^{-1}(d), f^{-1}(c))$ for $(d, c) \in \mu_o$ as in § 2.4. In general there will be no guarantee that $(f^{-1}(d), f^{-1}(c))$ should belong to $\mathcal{P}^c_X$, and so we shall say that f is $(\delta - \mu)$ cpqu continuous if $(f^{-1}(d), f^{-1}(c)) \in \delta$ whenever $(d, c) \in \mu_o$ and $(f^{-1}(d), f^{-1}(c)) \in \mathcal{P}^c_X$. f will then be a cpqu isomorphism if it is bijective, f is $(\delta - \mu)$ cpquc and f^{-1} is $(\mu - \delta)$ cpquc.

We may now state:

<u>Theorem</u> 2.8.1. Let (X, δ) and (Y, μ) be cpqu with $X \subseteq Y$ bidense for $(Y, t_u(\mu), t_v(\mu))$ and having $\delta = \mu_X$. Then (Y, μ) and $(\Lambda_1(X), \hat{\delta})$ are cpqu isomorphic if and only if there is a subbase $\sigma \subseteq \mu_o$ of μ so that

1) For $d \in \operatorname{dom} \sigma$ we have $uc_j(d_X) \subseteq uc_j(d) \cap X$, $j = 1, 2$.

2) For $d \in \operatorname{dom} \sigma$, $U \in \operatorname{dom} d$, $V \in \operatorname{ran} d$ and $y \in Y - X$ we have

(a) $y \in U \Leftrightarrow \exists\ e \in \operatorname{dom} \mu_o$ with $St(e,\{y\}) \cap X \subseteq U \cap X$,

(b) $y \in V \Leftrightarrow \exists\ e \in \operatorname{dom} \mu_o$ with $St(\{y\}, e) \cap X \subseteq V \cap X$.

3) For $c \in \operatorname{ran} \mu_o$ and PcQ with $P \cap Q = \emptyset$ $\exists$ $P' \in t_u(\mu)$, $Q' \in t_v(\mu)$ with $P'cQ'$, $(P' \cap X)^*_u \subseteq P$, $(Q' \cap X)^*_v \subseteq Q$.

4) Given $y \in Y - X$ and $(d_i, c_i) \in \sigma$, $i = 1, \dots, n$, $\exists$ $U_i d_i V_i$ with $y \in (\bigcap\{U_i\}) \cap (\bigcap\{V_i\})$ and $(t_u(\mu)\text{-int}[\bigcap\{U_i\}])(\bigcap\{c_i\})(t_v(\mu)\text{-int}[\bigcap\{V_i\}])$.

5) For $y \in Y - X$ and $z \in Y$ $\exists$ $(d, c) \in \sigma$ so that

(a) $St(d,\{z\}) \not\subseteq St(\{y\}, d)$, or (b) $St(d,\{y\}) \not\subseteq St(\{z\}, d)$;

where if $z \in X$ we can take $z \in uc_1(d)$ in case (a), and $z \in uc_2(d)$ in case (b)

6) Each M-regular μ-Cauchy bifilter on Y either converges in Y to $x \in X$, or μ-converges in Y to $y \in Y - X$.

<u>Proof</u>. We have already noted above that $(\mathcal{N}_1(X), \hat{\mathcal{S}})$ satisfies various of the properties listed above, and the remainder are easily verified, again for the base $\beta \subseteq (\hat{\mathcal{S}})_o$, so we will concentrate on the proof of the sufficiency of (1) - (6).

First let us note from (1) that $t_u(\mathcal{S}) = t_u(\mu)_X$ and $t_v(\mathcal{S}) = t_v(\mu)_X$, facts that will be used below without specific mention.

Secondly from (2) we have in particular that

7) Given $(d_i, c_i) \in \sigma$, $U_i \in \operatorname{dom} d_i$ and $V_i \in \operatorname{ran} d_i$, $i = 1, \dots, n$ then $(\bigcap\{(U_i \cap X)\})(\bigcap\{c_i\})_X(\bigcap\{(V_i \cap X)\})$ implies $(\bigcap\{U_i\})(\bigcap\{c_i\})(\bigcap\{V_i\})$.

Finally, from (4), we see that for each $d \in \operatorname{dom} \mu_o$ we have $Y - X \subseteq uc(d)$.

Now take $y \in Y - X$, and consider the bifilter $\mathcal{K}(y)$ with subbase

$$\{(t_u(\mu)\text{-int}[St(d,\{y\})],\ t_v(\mu)\text{-int}[St(\{y\}, d)]) \mid d \in \operatorname{dom} \sigma\}.$$

It follows easily from (4) that $\mathcal{K}(y)$ is a M-regular μ-Cauchy bifilter on Y, and in particular it contains $(St(d,\{y\}), St(\{y\},d))$ for all $d \in \operatorname{dom}\mu_o$. Hence by (7),

$$\mathcal{K}^X(y) = \{ (P \cap X, Q \cap Y) \mid (P, Q) \in \mathcal{K}(y) \}$$

is a M_X = D-regular minimal $\mu_X = \delta$-Cauchy bifilter on X. It follows from (4) and (5) that $\mathcal{K}^X(y)$ is not convergent in X, and so $\mathcal{K}^X(y) \in \mathcal{N}_o(X)$. In this way we have a map $\varphi(y) = \mathcal{K}^X(y)$ of Y - X to $\mathcal{N}_o(X)$, which can be extended to a map $\varphi : Y \to \mathcal{N}(X)$ by setting $\varphi(x) = x$ for $x \in X$. Moreover φ is injective, as follows easily from (4), (5), and the fact that $\mathcal{K}^X(y)$ is D-regular. To see that φ is surjective take $\mathcal{B} \in \mathcal{N}_o(X)$ and set

$$B = \{ (P, Q) \mid P, Q \subseteq Y, ((t_u(\mu)\text{-int}[P]) \cap X, (t_v(\mu)\text{-int}[Q]) \cap X) \in \mathcal{B} \} .$$

Then if γ is the base of μ defined by the subbase σ it is clear that B is a γ-regular μ-Cauchy bifilter on Y by (7), and it follows from (6) that either B converges to some $x \in X$ or is μ-convergent to some $y \in Y - X$. However in the first instance we could then deduce from (1) and (4) that $\mathcal{B}$ must also converge to x in X, and this is contrary to the choice of $\mathcal{B}$ so B is μ-convergent to $y \in Y - X$. It follows at once from (4) that $\mathcal{K}^X(y) \subseteq \mathcal{B}$, and so $\mathcal{K}^X(y) = \mathcal{B}$ since $\mathcal{B}$ is minimal δ-Cauchy. This verifies that φ is surjective, and so φ is a bijection of Y with $\mathcal{N}_1(X)$. Note that this result has not used (3) or the full force of (2); that is, loosely speaking, it depends on the " μ-embedding" of X in Y, and not on the "strict μ-embedding".

Now let us verify that $\varphi : Y \to \mathcal{N}_1(X)$ is $(\mu - \hat{\delta})$ cpquc. Corresponding to the subbase σ of μ we have the subbase

$$\sigma_X = \{ (d'_X, c'_X) \mid (d', c') \in \sigma \} ,$$

and clearly $\sigma_X \subseteq \delta_o$. Take $(d', c') \in \sigma$ and $(e', b') \in \sigma$ with $(e', b') \preceq (\omega) (d', c')$. Then it will suffice to show that

$$(e', b') \preccurlyeq (\varphi^{-1}(\hat{d}), \varphi^{-1}(\hat{c})),$$

where, for convenience, we have written $d = d'_X$, $c = c'_X$. Take $R'e'S'$; $U'd'V'$ with $St_{b'}(e', R') \subseteq U'$, $St_{b'}(S', e') \subseteq V'$; and set $U' \cap X = U$, $V' \cap X = V$ so that we have UdV. Let us verify

$$R' \subseteq \varphi^{-1}(U^1{}_u).$$

Now within X this is clear, so take $y \in R'$ with $y \in Y - X$. Then by 2(a) we have $(f, a) \in \gamma$ with $St(f, \{y\}) \subseteq R'$. Take $(g, k) \in \gamma$ with $(g, k) \preccurlyeq_{(*)} (f, a)$; then as noted above $Y - X \subseteq uc(g)$ and so we have MgN with $y \in M \cap N$. Take LfT with $St(g, \{y\}) \subseteq St_k(g, M) \subseteq L$, $St_k(N, g) \subseteq T$; and take $L'fT'$ with $T' \cap X \in \mathcal{K}^X{}_v(y)$. By the comment after the definition of $\mathcal{K}(y)$ we have $L \cap X \in \mathcal{K}^X{}_u(y)$, and so $(L \cap X)a_X(T' \cap X)$ since $\mathcal{K}^X(y)$ is D-regular. Hence LaT' by (7), and so we have ReS with $L' \subseteq St_a(f, L) \subseteq R$, $y \in N \subseteq T \subseteq St_a(T, f) \subseteq S$. Hence $L' \subseteq R \subseteq St(e', \{y\}) \subseteq St_{b'}(e', R') \subseteq U'$, and so $L' \cap X \subseteq U' \cap X = U$. Since $f_X \in \delta_o$ this shows that $\varphi(y) = \mathcal{K}^X(y) \in U^1{}_u$, and completes the proof that $R' \subseteq \varphi^{-1}(U^1{}_u)$. Likewise $S' \subseteq \varphi^{-1}(V^1{}_v)$, and so $e' \angle \varphi^{-1}(\hat{d})$. In fact we also have $uc(e') = uc(\varphi^{-1}(\hat{d}))$, as is easily verified, so it remains to show that $b' \subseteq \varphi^{-1}(\hat{c})$. Take $Pb'Q$ with $P \cap Q = \emptyset$. By (3) we have $P' \in t_u(\mu)$, $Q' \in t_v(\mu)$ with $P'b'Q'$, $(P' \cap X)^*{}_u \subseteq P$ and $(Q' \cap X)^*{}_v \subseteq Q$. If $A = P' \cap X$, $B = Q' \cap X$ then $\emptyset \neq A \in t_u(\delta)$, $\emptyset \neq B \in t_v(\delta)$ and $Ab'_X B$. Hence AcB since $b'_X \subseteq c'_X = c$, and so $A^o{}_u \hat{c}\, B^o{}_v$. Clearly $\emptyset \neq \varphi^{-1}(A^o{}_u)$ and $\emptyset \neq \varphi^{-1}(B^o{}_v)$ so $P\, \varphi^{-1}(\hat{c})Q$ will follow if we can show $\varphi^{-1}(A^o{}_u) \subseteq P$ and $\varphi^{-1}(B^o{}_v) \subseteq Q$. This is clear within X, so take $y \in Y - X$ with $y \in \varphi^{-1}(A^o{}_u)$, that is $\mathcal{K}^X(y) \in A^o{}_u$. Now we have $(f, a) \in \delta_o$ satisfying LfT, $T \in \mathcal{K}^X_v(y) \Rightarrow L \subseteq A \cap uc(f)$, and we may take $(f', a') \in \gamma$ with $(f'_X, a'_X) \preccurlyeq (f, a)$. Take $(g', k') \in \gamma$ with $(g', k') \preccurlyeq_{(*)} (f', a')$, and $M'g'N'$ with

$y \in M' \cap N'$. Then we have $L'f'T'$ with $y \in M' \subseteq St(g', \{y\}) \subseteq St_{b'}(g', M') \subseteq L'$ and $y \in N' \subseteq St(\{y\}, g') \subseteq St_{b'}(N', g') \subseteq T'$; and LfT with $L' \cap X \subseteq L$, $T' \cap X \subseteq T$. Then $St(g, f') \cap X \in \mathcal{K}_v^X(y)$ $\Rightarrow$ $T \in \mathcal{K}_v^X(y) \Rightarrow L \subseteq A \cap uc(f)$. Hence $St(g', \{y\}) \cap X \subseteq uc(f) = uc(g') \cap X$, while on the other hand $Y - X \subseteq uc(g')$ by (4), so $y \in St(g', \{y\}) \subseteq uc(g')$. This shows $y \in uc_1(g')$, and so $St(g', \{y\})$ is a $t_u(\mu)$-nhd. of y in Y. However we also have $St(g', \{y\}) \cap X \subseteq A = P' \cap X$, and so $y \in (P' \cap X)^*_u \subseteq P$. This verifies that $\varphi^{-1}(A^o_u) \subseteq P$, and likewise we have $\varphi^{-1}(B^o_v) \subseteq Q$. Hence $b' \subseteq \varphi^{-1}(\hat{c})$, and we have shown that φ is $(\mu - \hat{S})$ cpquc.

Finally consider $\psi = \varphi^{-1} : \mathcal{N}_1(X) \rightarrow Y$. If we take $(d', c') \in \sigma$ and set $(d, c) = (d'_X, c'_X)$, then an argument exactly similar to that used above enables us to show that

$$(\hat{d}, \hat{c}) \preccurlyeq (\psi^{-1}(d'), \psi^{-1}(c')),$$

and so ψ is $(\hat{S} - \mu)$ cpquc. This completes the proof of the theorem.

We may make the following definitions for bitopological extensions in general.

Definition 2.8.2. Let M be an interior confluence relation on the bitopological space (Y, u, v), and let $X \subseteq Y$ be bidense. Then the extension (Y, u, v) of (X, u_X, v_X) will be said to:

(i) be M-separated except for X if given $y \in Y - X$ and $z \in Y$ there exists $G \in u$, $H \in v$ with $G \not M H$ and either $y \in G$, $z \in H$ or $z \in G$, $y \in H$.

(ii) have pairwise relatively zero-dimensional outgrowth if u has a base u^* and v has a base v^* so that

$$G \in u^* \Rightarrow v\text{-}cl[G] - G \subseteq X \text{ and } (G \cap X)^*_u = G,$$

$$H \in v^* \Rightarrow u\text{-}cl[H] - H \subseteq X \text{ and } (H \cap X)^*_v = H.$$

Note that in (ii) it is sufficient for the stated conditions to hold for subbases u^*, v^* of u, v respectively. Also the conditions $(G \cap X)^*_u = G$ and $(H \cap X)^*_v = H$ are redundant if X is uniformly dense in Y.

Clearly if (Y, u, v) has pairwise relatively zero-dimensional outgrowth then it is, in particular, a strict extension of (X, u_X, v_X).

The following theorem gives sufficient conditions for $(\mathcal{M}_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ to be $\mathcal{D}$-separated except for X, and to have pairwise relatively zero-dimensional outgrowth.

<u>Theorem</u> 2.8.2. Let $(X, \mathcal{S})$ be a cpqu, and suppose there is a subbase $\sigma \subseteq \mathcal{S}_0$ of $\mathcal{S}$ satisfying

(a) Given $\mathcal{B} \in \mathcal{M}_0(X)$ and $(d, c) \in \sigma$ we have UdV with $St_c(d, U) \notin \mathcal{B}_u$ (respectively, $St_c(V, d) \notin \mathcal{B}_v$) implies $St_c(d, U) \subseteq uc(d)$ (respectively, $St_c(V, d) \subseteq uc(d)$) and there exists U'dV' with $(U', V') \in \mathcal{B}$ and $St_c(V', d) \subseteq uc(d)$ (respectively, $St_c(d, U') \subseteq uc(d)$).

Then $(\mathcal{M}_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is $\mathcal{D}$-separated except for X. If in addition σ satisfies

(b) Each $(d, c) \in \sigma$ is transitive

then $(\mathcal{M}_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ has pairwise relatively zero-dimensional outgrowth.

<u>Proof</u>. First take $\mathcal{B}, \mathcal{A} \in \mathcal{M}_0(X)$ with $\mathcal{B} \neq \mathcal{A}$. Since $\mathcal{B}$ is minimal $\mathcal{S}$-Cauchy we have $\mathcal{A}_u \not\subseteq \mathcal{B}_u$ or $\mathcal{A}_v \not\subseteq \mathcal{B}_v$. Suppose $\mathcal{A}_u \not\subseteq \mathcal{B}_u$, then since $\mathcal{A}$ is minimal $\mathcal{S}$-Cauchy we have $(f, a) \in \sigma$ and LfT with $(L, T) \in \mathcal{A}$ and $St_a(f, L) \notin \mathcal{B}_u$. Take (e, b), $(d, c) \in \sigma$ with

$$(d, c) \preceq^{(*)} (e, b) \preceq^{(*)} (f, a).$$

Since $\mathcal{A}$ is $\mathcal{S}$-Cauchy $\exists$ UdV with $(U, V) \in \mathcal{A}$. Take ReS and L'fT' with $St_c(d, U) \subseteq R$, $St_c(V, d) \subseteq S$, and $St_b(e, R) \subseteq L'$, $St_b(S, e) \subseteq T'$. Now $(L', T') \in \mathcal{A}$ so LaT' and so

$St_c(d, U) \subseteq L' \subseteq St_a(f, L)$ which gives $St_c(d, U) \notin \mathcal{B}_u$. Hence by (a) we have $St_c(d, U) \subseteq uc(d)$ and $\exists$ $U'dV'$ with $(U', V') \in \mathcal{B}$ and $St_c(V', d) \subseteq uc(d)$. If now we take $R'eS'$ with $St_c(d, U') \subseteq R'$ and $St_c(V', d) \subseteq S'$ it is easy to verify that

$$\mathcal{b} \in R^o{}_u \in t_u(\hat{\mathcal{S}}), \mathcal{B} \in (S')^o{}_v \in t_v(\hat{\mathcal{S}}) \text{ and } (R^o{}_u) \not\delta ((S')^o{}_v).$$

A similar result may be obtained if $\mathcal{b}_v \notin \mathcal{B}_v$. Secondly if we take $\mathcal{B} \in \mathcal{N}_o(X)$ and $x \in X$ then $\mathcal{B} \not\to x$, and a similar argument to that used above may be employed to complete the proof that $(\mathcal{N}_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is $\mathfrak{D}$-separated except for X.

Now suppose that in addition σ is transitive. Then it is clear that

$$\{ U^o{}_u \mid U \in \operatorname{dom} d \in \operatorname{dom} \sigma, U \subseteq uc(d) \}$$

is a subbase of $t_u(\hat{\mathcal{S}})$. Also if $(d, c) \in \sigma$ and $U \in \operatorname{dom} d$ has $U \subseteq uc(d)$ then $(U^o{}_u \cap X)^*{}_u = U^o{}_u$, so let us verify that

$$t_v(\hat{\mathcal{S}})\text{-cl}[U^o{}_u] - U^o{}_u \subseteq X.$$

Suppose on the contrary that for some $\mathcal{B} \in \mathcal{N}_o(X)$ we have $\mathcal{B} \in t_v(\hat{\mathcal{S}})\text{-cl}[U^o{}_u]$ but $\mathcal{B} \notin U^o{}_u$. Then $\exists$ $U'dV'$ with $V \in \mathcal{B}_v$ and $U' \notin U \cap uc(d) = U$. There are two cases to consider:

(i) $U \in \mathcal{B}_u$. In this event UcV' and we have the immediate contradiction $U' \subseteq St_c(d, U) = U$.

(ii) $U \notin \mathcal{B}_u$. Then, since $U = St_c(d, U)$, we have by (a) that $\exists$ $U''dV''$ with $(U'', V'') \in \mathcal{B}$ and $V'' \subseteq uc(d)$. But now $\mathcal{B} \in (V'')^o{}_v \in t_v(\hat{\mathcal{S}})$ and so $U^o{}_u \cap (V'')^o{}_v \neq \emptyset$, which implies UcV''. Hence $U'' \subseteq St_c(d, U) = U$, which gives the contradiction $U \in \mathcal{B}_u$. In just the same way

$$\{ V^o{}_v \mid V \in \operatorname{ran} d \in \operatorname{dom} \sigma, V \subseteq uc(d) \}$$

is a subbase of $t_v(\hat{\mathcal{S}})$ satisfying $(V^o{}_v \cap X)^*{}_v = V^o{}_v$ and

$t_u(\hat{\mathcal{S}})\text{-cl}[V^o{}_v] - V^o{}_v \subseteq X$, and the proof is complete.

Clearly condition (a) of this theorem is satisfied for any cqu, and it is also satisfied in the symmetric situation afforded by a cpu. The following example gives another case in which both (a) and (b) are satisfied.

Example 2.8.1. Let (X, u, v) be a bitopological space, and D an interior confluence relation satisfying $G \not{D} (X - v\text{-cl}[G])$ and $(X - u\text{-cl}[H]) \not{D} H$ for all $G \in u$ and $H \in v$ (this is true, in particular, if $D = 1$). Define d(G) and e(H) as in Lemma 1.8.2. Then

$$\sigma = \{ (d(G), D), (e(H), D) \mid G \in u - \{\emptyset\}, H \in v - \{\emptyset\} \}$$

is a transitive open subbase for a basic cpqu $\mathcal{S}$ on X which is compatible with (X, u, v).

That conditions (a) and (b) of Theorem 2.8.2 are satisfied for this subbase is clear, and so $(\mathcal{N}_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is a strict completion of $(X, \mathcal{S})$ which is $\hat{D}$-separated except for X and has pairwise relatively zero-dimensional outgrowth.

We will now give a result in the opposite direction to Theorem 2.8.2. First let us make the following definition:

Definition 2.8.3. Let (X, u, v) be a bitopological space, and D an interior confluence relation. We will say that the cpqu $\mathcal{S}$ is compatible with (X, u, v, D) if $t_u(\mathcal{S}) = u$, $t_v(\mathcal{S}) = v$ and $D = \bigcap \{ c \mid c \in \operatorname{ran} \mathcal{S} \}$.

We will say that (X, u, v) is quasi-D-biclosed if every cpqu $\mathcal{S}$ which is compatible with (X, u, v, D) is complete. We may now give:

Theorem 2.8.3. Let (X', u', v') be a quasi-D'-biclosed bitopological space, and (X, u, v) a bidense subspace. Suppose that D' has the oup, and that $D = D'_X$ satisfies $G \not{D} (X - v\text{-cl}[G])$ and $(X - u\text{-cl}[H]) \not{D} H$ for all $G \in u$, $H \in v$. Finally suppose that (X', u', v') is D'-separated except for X, and that it has pairwise relatively zero-dimensional outgrowth. Then there exists a cpqu $\mathcal{S}$ on X with an open transitive base which is compatible with (X, u, v, D), and such that (X', u', v') is bitopologically homeomorphic with $(\Lambda_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$.

<u>Proof</u>. Let

$$u^* = \{ G' \mid G' \in u' - \{\emptyset\},\ (G' \cap X)^*_u = G' \text{ and } v'\text{-cl}[G'] - G' \subseteq X \}$$

and make a corresponding definition for v^*. Under the given conditions, u^* is a base of u' and v^* is a base of v'. For $G' \in u^*$ let us set

$$d'(G') = \begin{cases} \{ (G', \mathbf{X}'), (\mathbf{X}', [(\mathbf{X}' - v'\text{-cl}[G']) \cap X]^*_v) \} & \text{if } v'\text{-cl}[G'] \neq \mathbf{X}' \\ \{ (G', \mathbf{X}') \} & \text{otherwise,} \end{cases}$$

and for $H' \in v^*$ let us make an analogous definition of $e'(H')$. We may note that $(\mathbf{X}' - v'\text{-cl}[G']) \cap X \subseteq X - v\text{-cl}[G' \cap X]$, and that $(G' \cap X)\not{D}(X - v\text{-cl}[G' \cap X])$, so that $G' \not{D}' [(\mathbf{X}' - v'\text{-cl}[G']) \cap X]^*_v$. Hence $(d'(G'), D') \preceq^{(*)} (d'(G'), D')$, and a similar result holds for $(e'(H'), D')$. It follows that

$$\sigma' = \{ (d'(G'), D'), (e'(H'), D') \mid G' \in u^*, H' \in v^* \}$$

is an open transitive subbase for a cpqu $\mathcal{S}'$ compatible with (X', u', v', D'). Moreover it is clear that $\mathcal{S} = \mathcal{S}'_X$ is a cpqu on X with an open transitive base, which is compatible with (X, u, v, D). To show that (X', u', v') and $(\Lambda_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ are bitopologically homeomorphic it will suffice to verify the conditions (1) - (6) of Theorem 2.8.1 for the subbase σ'. Condition (1) is clear from the definition. To show (4) take $y \in \mathbf{X}' - X$ and $G' \in u^*$. If $y \in G'$ then of course $y \in G' \cap \mathbf{X}'$. On the other hand if $y \notin G'$ then $y \notin v'\text{-cl}[G']$ and so $y \in [(\mathbf{X}' - v'\text{-cl}[G']) \cap X]^*_v \cap \mathbf{X}'$. A similar result holds for $H' \in v^*$, and (4) now follows since the elements of σ' are open. For (2) take $d' \in \operatorname{dom} \sigma'$, $U' \in \operatorname{dom} d'$ and $y \in \mathbf{X}' - X$. As the elements of σ' are transitive we have

$$y \in U' \Rightarrow St(d', \{y\}) \cap X = St_{D'}(d', U') \cap X = U' \cap X.$$

On the other hand let $\mathcal{V}'$ be the base generated by σ', and suppose $\exists$ $e' \in \operatorname{dom} \mathcal{V}'$ with $St(e',\{y\}) \cap X \subseteq U' \cap X$. Now e' is open and by (4) we have $y \in uc(e')$ so $St(e',\{y\})$ is a $t_u(\mathcal{S}')$-open nhd. of y. Hence $y \in (U' \cap X)^*_u = U'$ by the definition of the elements of σ'. A similar result holds for $V' \in \operatorname{ran} d'$.

To establish (3) take $PD'Q$ with $P \cap Q = \emptyset$. Since D' is interior and u^*, v^* are bases we have

$$(\bigcup\{G' \mid G' \in u^*, G' \subseteq P\})D'(\bigcup\{H' \mid H' \in v^*, H' \subseteq Q\})$$

and since D' has the oup $\exists$ $G' \in u^*$, $G' \subseteq P$ and $H' \in v^*$, $H' \subseteq Q$ with $G'D'H'$. But then $(G' \cap X)^*_u = G' \subseteq P$ and $(H' \cap X)^*_v = H' \subseteq Q$ as required.

For (5) take $y \in \mathfrak{X}' - X$, $z \in \mathfrak{X}' - X$ and $x \in X$. Suppose, for example, that we have $y \in G' \in u^*$ with $z \notin G'$, and $x \in H' \in v^*$ with $y \notin H'$. Then $St(d'(G'),\{y\}) = G'$ and $St(\{z\},d'(G')) = ((\mathfrak{X}' - v'\text{-}cl[G']) \cap X)^*_v$, while $G'\not{D}'((\mathfrak{X}' - v'\text{-}cl[G']) \cap X)^*_v$. On the other hand $St(\{x\},e'(H')) = H'$, $x \in uc_2(e'(H'))$, $St(e'(H'),\{y\}) = ((\mathfrak{X}' - u'\text{-}cl[H']) \cap X)^*_u$ and $((\mathfrak{X}' - u'\text{-}cl[H']) \cap X)^*_u \not{D}'H'$. The other cases are similar.

Finally for (6) we note that by hypothesis $(Y, \mathcal{S}')$ is complete, while by (4) and the fact that $\mathcal{V}'$ is an open base, we see that " $\mathcal{S}'$-convergent to $y \in \mathfrak{X}' - X$" is the same as " convergent to $y \in \mathfrak{X}' - X$".

This completes the proof of the theorem.

The above theorem is a natural generalisation of the characterization, given by VOTAW [35] ,of those quasi H-closed topological extensions which are Hausdorff except for X and which have relatively zero-dimensional outgrowth. This would encourage one to believe that it might be possible to characterize those quasi-D'-biclosed extensions of a bitopological space which are D'-separated except for X, and which are bitopologically homeomorphic to some $(\Lambda_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$, in terms of a notion of "relatively pairwise completely regular outgrowth". Now it is certainly true that, under some fairly mild restrictions on $\mathcal{S}$, the extension $(\Lambda_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ does indeed satisfy a natural "pairwise" analogue of the notion of relatively completely regular outgrowth, but this seems insufficient, in general, to ensure the converse result. In principle a characterization in terms of the existence of certain "normal sequences" of dual families would be quite feasible, however this would amount to a virtual restatement of a special case of Theorem 2.8.1, and we omit the details.

We end this section by considering the relation between quasi-D-biclosedness and almost D-compactness, each of which is an analogue of a characteristic property of quasi H-closed [35]. If (X, u, v) is almost D-compact and $\mathcal{S}$ is a cpqu compatible with (X, u, v, D) then $\mathcal{S}$ is complete, the argument being the same as in the case of cqu. Hence an almost D-compact space is quasi-D-biclosed. Let us consider the converse. For the cpqu $(X, \mathcal{S})$ we may define TB.1 and TB.2 as in § 2.6, except that in place of "$(U, V) \in \mathcal{G}$" we require "$(t_u(\mathcal{S})\text{-int}\{U\}, t_v(\mathcal{S})\text{-int}\{V\}) \in \mathcal{G}$". Of course this is the same if $\mathcal{S}_o$ is an open base. We may verify at once that if $(X, \mathcal{S})$ is complete and satisfies TB.2 then $(X, t_u(\mathcal{S}), t_v(\mathcal{S}))$ is almost D-compact. Hence a quasi-D-biclosed space which has a compatible cpqu satisfying TB.2 must be almost D-compact. However it is conceivable that there might be bitopological spaces for which no compatible cpqu has a Cauchy bifilter, and such a bitopological space would be quasi D-biclosed for any D but no compatible cpqu could satisfy TB.2 and consequently it might not be almost D-compact. Below we give sufficient conditions for a quasi D-biclosed space to be almost D-compact. First let us make one or two comments about D-hyperfilters on a cpqu space. If $\mathcal{F}$ is a D-hyperfilter we make the same changes to the definition of "$\mathcal{S}$-Cauchy" and "weakly $\mathcal{S}$-Cauchy" as we made in the case of bifilters. Note in particular that a $\mathcal{S}$-refined D-hyperfilter $\mathcal{F}$ will be $\mathcal{S}$-Cauchy, exactly as before, if for a subbase σ of $\mathcal{S}$ and any $d \in \operatorname{dom} \sigma$ we have UdV and $F \in \mathcal{F}$ with $F \lhd (U, V)$. The conditions of hyper-total boundedness have the same definitions as before, but the relations which hold between them for a cqu will not hold, in general, for a cpqu.

We may now give:

<u>Proposition</u> 2.8.1. (a) Suppose the cpqu $(X, \mathcal{S})$ has a subbase σ so that d is finite for each $d \in \operatorname{dom} \sigma$. Then $(X, \mathcal{S})$ satisfies HTB.3.

(b) Suppose that $(X, \mathcal{S})$ has a subbase satisfying the condition above, and in addition:

Given $d_1, \ldots, d_n \in \operatorname{dom} \sigma$, and $G \in t_u(\mathcal{S})$, $H \in t_v(\mathcal{S})$ with GDH there exists $U_i d_i V_i$, $1 \leq i \leq n$, with $GD(\bigcap\{V_i\})$, $(\bigcap\{U_i\})DH$, and $(t_u(\mathcal{S})\text{-int}[\bigcap\{U_i\}])D(t_v(\mathcal{S})\text{-int}[\bigcap\{V_i\}])$. Then $(X, \mathcal{S})$ satisfies TB.2.

<u>Proof</u>. (a) Let $\mathcal{F}$ be a maximal $\mathcal{S}$-refined D-hyperfilter, and take $d = \{(U_1, V_1), \ldots, (U_n, V_n)\} \in \operatorname{dom} \sigma$. Take $F' \in \mathcal{F}$ with $F' \subseteq u \times v$ and $F' \prec d$. Now if for each $1 \leq i \leq n$ we had $F_i \in \mathcal{F}$ satisfying

$$(G, H) \in F_i \Rightarrow G \nsubseteq U_i \text{ or } H \nsubseteq V_i$$

we should obtain an immediate contradiction from $F' \cap \bigcap\{F_i\} \neq \emptyset$; hence for some k, $1 \leq k \leq n$, we have

$$F^* = \{(G, H) \mid (G, H) \in F,\ G \subseteq U_k \text{ and } H \subseteq V_k\} \neq \emptyset$$

for each $F \in \mathcal{F}$. Since $\mathcal{F}$ is maximal we deduce that $F^* \in \mathcal{F}$, while clearly $F \lhd (U_k, V_k)$, which completes the proof.

(b) Let $\mathcal{G}$ be an open D-regular bifilter, and set $\mathcal{F} = h(\mathcal{G})$. Take $d_1, \ldots, d_n \in \operatorname{dom} \sigma$, and suppose that for each selection $S(\alpha) = \{(U_i, V_i) \mid 1 \leq i \leq n\}$, where $U_i d_i V_i$ and $(t_u(\mathcal{S})\text{-int}[\bigcap\{U_i\}])D(t_v(\mathcal{S})\text{-int}[\bigcap\{V_i\}])$, there exists $F(\alpha) \in \mathcal{F}$ so that $G\not{D}(\bigcap\{V_i\})$ or $(\bigcap\{U_i\})\not{D}H$ for all $(G, H) \in F(\alpha) \cap (u \times v)$. Now each d_i is finite, so there are only a finite number of possible selections $S(\alpha_1), \ldots, S(\alpha_m)$, and if we take $(G, H) \in (\bigcap\{F(\alpha_i)\}) \cap (u \times v)$, then GDH and so by hypothesis we have a selection $S(\alpha_j)$, $1 \leq j \leq m$, with $GD(\bigcap\{V_i\})$ and

$(\bigcap\{U_i\})DH$ which contradicts $(G, H) \in F(\alpha_j) \cap (u \times v)$. Hence

$T(d_1,\ldots,d_n) = \{(P, Q) \mid P \in t_u(\mathcal{S}), Q \in t_v(\mathcal{S}), PDQ, \exists U_i d_i V_i$ with $P \subseteq \bigcap\{U_i\}$ and $Q \subseteq \bigcap\{V_i\}$; and $F \in \mathcal{F} \Rightarrow \exists (G, H) \in F \cap (u \times v)$ with GDQ and PDH $\}$

is non-empty, and so $\{T(d_1,\ldots,d_n) \mid d_1, \ldots, d_n \in \operatorname{dom} \sigma\}$ is a base for a D-hyperfilter $\mathcal{T}$, which is clearly $\mathcal{S}$-refined. Let $\mathcal{H}$ be a maximal D-hyperfilter refinement of $\mathcal{T}$. Then $\mathcal{H}$ is $\mathcal{S}$-refined and hence $\mathcal{S}$-Cauchy by (a). Hence if we set $\mathcal{b}$ = $b(\mathcal{H})$ then it is clear that $\mathcal{b}$ is a D-regular bifilter with the properties required in the definition of TB.2, and the proof is complete.

Corollary 1. Suppose that the bitopological space (X, u, v) satisfies the following conditions for the interior confluence relation D:

(i) $G \not{D} (X - v\text{-cl}[G])$ and $(X - u\text{-cl}[H]) \not{D} H$ $\forall$ $G \in u$ and $H \in v$,

(ii) Given $G_1, \ldots, G_n$, $G \in u$ and $H_1, \ldots, H_m$, $H \in v$ with GDH there exist (possibly improper) partitions (p_1, p_2) and (q_1, q_2) of the sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ respectively, so that if $U = \bigcap\{G_i \mid i \in p_1\} \cap \bigcap\{(X - u\text{-cl}[H_j]) \mid j \in q_2\}$ and $V = \bigcap\{(X - v\text{-cl}[G_i]) \mid i \in p_2\} \cap \bigcap\{H_j \mid j \in q_1\}$ then UDV, GDV and UDH.

Then (X, u, v) is almost D-compact if and only if it is quasi-D-biclosed.

Proof. We need only consider the cpqu of Example 2.8.1.

Corollary 2. Let (X, u, v, D) have the properties (i) and (ii) of Corollary 1, and suppose in addition that D has the oup. Then if $\mathcal{S}$ is the cpqu of Example 2.8.1, $(\Lambda_1(X), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is almost $\hat{D}$-compact, and hence, in particular, quasi-$\hat{D}$-biclosed.

We now give a simple example which serves to illustrate this last result.

Example 2.8.2. Consider the bitopological space (R, s, t) mentioned earler, and take $D = \imath$. If for $r \in \mathbb{R}$ we set $H(r) = \{ x \mid x < r \}$ and $K(r) = \{ x \mid x > r \}$ then the subbase σ of the cpqu $\mathcal{S}$ of Example 2.8.1 takes the form

$$\{ (d_r, \imath) \mid r \in \mathbb{R} \} \cup \{ (\{(\mathbb{R}, \mathbb{R})\}, \imath) \}$$

where $d_r = \{ (H(r), \mathbb{R}), (\mathbb{R}, K(r)) \}$.

Clearly $(\mathbb{R}, s, t, \imath)$ satisfies the conditions of Corollary 2 above, and so $(\mathcal{N}_1(\mathbb{R}), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is almost $\imath$-compact. Let us identify this space. If $\mathcal{B}$ is an $\imath$-regular bifilter which contains both $(H(r), \mathbb{R})$ and $(\mathbb{R}, K(s))$ for some $r, s \in \mathbb{R}$, then clearly $\mathcal{B}$ converges to r_o where

$$\sup\{ s \mid (\mathbb{R}, K(s)) \in \mathcal{B} \} = r_o = \inf\{ r \mid (H(r), \mathbb{R}) \in \mathcal{B} \} .$$

Hence the only non-convergent $\imath$-regular minimal $\mathcal{S}$-Cauchy bifilters are $\mathcal{H}$ and $\mathcal{K}$, where $\mathcal{H}$ has base $\{ (H(r), \mathbb{R}) \mid r \in \mathbb{R} \}$, and $\mathcal{K}$ has base $\{ (\mathbb{R}, K(s)) \mid s \in \mathbb{R} \}$. Then

$$\mathcal{N}_1(\mathbb{R}) = \mathbb{R} \cup \{\mathcal{H}, \mathcal{K}\} ,$$

and clearly $H(r)^1{}_u = H(r)^o{}_u = H(r) \cup \{\mathcal{H}\}$, $K(s)^1{}_v = K(s)^o{}_v = K(s) \cup \{\mathcal{K}\}$; these sets, together with $\mathcal{N}_1(R)$ and $\emptyset$, being the open sets of $t_u(\hat{\mathcal{S}})$, $t_v(\hat{\mathcal{S}})$ respectively. Note that $(\mathcal{N}_1(R), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is actually uniformly compact, and $\mathbb{R}$ is uniformly dense. Indeed $(\mathcal{N}_1(\mathbb{R}), t_u(\hat{\mathcal{S}}) \vee t_v(\hat{\mathcal{S}}))$ is the usual two-point compactification of the real line.

If in our construction of the completion $(\mathcal{N}_1(X), \hat{\mathcal{S}})$ we are prepared to forgo the separation properties (5) of Theorem 2.8.1 we may include in $\mathcal{N}_o(X)$, D-regular $\mathcal{S}$-Cauchy bifilters which are convergent in X, or which are not minimal $\mathcal{S}$-Cauchy. Our final example of this section is an illustration of this.

Example 2.8.3. Let us again consider the space $(\mathbb{R}, s, t)$, but this time let D be the relation

$$PDQ \Leftrightarrow P\imath Q \text{ or } s\text{-int}[P] \neq \emptyset \neq t\text{-int}[Q] .$$

Clearly D is the largest interior confluence relation on $\mathbb{R}$. Also if $\mathcal{B}$ is any open D-regular bifilter then every point of $\mathbb{R}$ is a D-cluster point, and so $(\mathbb{R}, s, t)$ is almost D-compact. In fact this space is almost D-hypercompact, but it is not D-compact as consideration of the bifilter with base

$$\{ (\mathbb{R}, \{n, n+1, \ldots\}) \mid n \in \mathbb{N} \}$$

will show.

Now consider the cpqu $\mathcal{S}$ with subbase $\sigma = \{ (e_r, D), (f_r, D) \mid r \in \mathbb{R} \}$, where $e_r = \{ (H(r), \mathbb{R}) \}$ and $f_r = \{ (\mathbb{R}, K(r)) \}$. Clearly $\mathcal{S}$ is compatible with $(\mathbb{R}, s, t, D)$, and has an open transitive base. There is just one minimal D-regular $\mathcal{S}$-Cauchy bifilter, and that is the bifilter $\mathcal{E}$ with base

$$\{ (H(r), K(s)) \mid r, s \in \mathbb{R} \}.$$

Of course $\mathcal{E}$ converges to all the points of $\mathbb{R}$, but none the less we may consider the completion $(\mathcal{N}_1(\mathbb{R}), \hat{\mathcal{S}})$, where

$$\mathcal{N}_1(\mathbb{R}) = \mathbb{R} \cup \{\mathcal{E}\} .$$

By the above general discussion $(\mathcal{N}_1(\mathbb{R}), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ will be almost $\hat{D}$-compact, and have pairwise relatively zero-dimensional outgrowth.

Now for $r \in \mathbb{R}$ we have $H(r)^1{}_u = H(r)^\circ{}_u = H(r) \cup \{\mathcal{E}\}$ and $K(r)^1{}_v = K(r)^\circ{}_v = K(r) \cup \{\mathcal{E}\}$; and these are the non-trivial open sets for $t_u(\hat{\mathcal{S}})$ and $t_v(\hat{\mathcal{S}})$ respectively. It follows at once that $P\hat{D}Q \Leftrightarrow P \cap Q \neq \emptyset$, that is $\hat{D} = \imath$ on $\mathcal{N}_1(\mathbb{R})$. This illustrates the extreme difference which can exist between a confluence relation and its restriction or extension.

Note that $(\mathcal{N}_1(\mathbb{R}), t_u(\hat{\mathcal{S}}), t_v(\hat{\mathcal{S}}))$ is not $\imath$-compact or uniformly compact, and $\mathbb{R}$ is not uniformly dense in this space since $\mathcal{E}$ is an isolated point for the uniform topology.

CHAPTER THREE

THE LATTICE OF BICONTINUOUS REAL-VALUED FUNCTIONS.

In this chapter we consider the relation between certain bitopological properties and properties of the lattice of bicontinuous real-valued functions. The set of bicontinuous real-valued functions is a lattice, and it is closed under addition and multiplication by functions taking only non-negative values, but in general it is not a ring. In place of the notion of ring ideal, which plays such a central role in the study of the ring of continuous functions on a topological space (see, for example, [16]), we shall consider the notion of bi-ideal, to be defined below. The elementary theory of bi-ideals resembles somewhat that of ring ideals, and can be developed in a more general setting than that of the lattice of bicontinuous real-valued functions on a bitopological space. These considerations occupy the first two sections of this chapter. Following this we relate the theory so developed to the study of bitopological real compactness, and consider the connection with completeness.

For established terminology and notation concerning lattices the reader is refered to any standard text, for example ([2], [30]). Throughout the first two sections P denotes a distributive lattice with a distinguished element O.

3.1 ELEMENTARY THEORY OF BI-IDEALS.

Let us first recall that $L \subseteq P$ is a (lattice) <u>ideal</u> in P if it satisfies:

(i) $a, b \in L \Rightarrow a \vee b \in L$, and

(ii) $a \in L$, $b \in P$ with $b \leq a \Rightarrow b \in L$.

An ideal L is called <u>prime</u> if it satisfies

$$a \wedge b \in L \Rightarrow a \in L \text{ or } b \in L.$$

Likewise $M \subseteq P$ is a (lattice) <u>dual ideal</u> in P if it satisfies:

(i) $a, b \in M \Rightarrow a \wedge b \in M$, and

(ii) $a \in M$, $b \in P$ with $a \leqslant b \Rightarrow b \in M$;

while M is <u>prime</u> if it satisfies

$$a \vee b \in M \Rightarrow a \in M \text{ or } b \in M.$$

We may now give:

<u>Definition</u> 3.1.1. The pair (L, M) is a <u>bi-ideal</u> in P if L is an ideal, M is a dual ideal and $0 \in L \cap M$.

The bi-ideal (L, M) will be called <u>prime</u> if L and M are prime, and it will be called <u>total</u> if $L \cup M = P$.

In order to obtain useful properties of bi-ideals we need to impose suitable "regularity" conditions. Accordingly we make the following definition.

<u>Definition</u> 3.1.2. The binary relation ϱ on P is a <u>dispersion</u> if it satisfies:

(i) $a \varrho b$, $a \leqslant a' \in P$ and $b \geqslant b' \in P \Rightarrow a' \varrho b'$, and

(ii) $a \varrho b$, $a \varrho b'$, $a' \varrho b$ and $a' \varrho b' \Rightarrow (a \wedge a') \varrho (b \vee b')$.

For convenience of writing we denote the negation of $a \varrho b$ by $a \bar{\varrho} b$. If (L, M) is a bi-ideal we shall now say that (L, M) is <u>ϱ-regular</u> if it satisfies

$$a \in L \text{ and } b \in M \Rightarrow a \bar{\varrho} b.$$

We may partially order the bi-ideals by

$$(L, M) \prec (L', M') \Leftrightarrow L \subseteq L' \text{ and } M \subseteq M'.$$

If (L, M) is a ϱ-regular bi-ideal then the set of all ϱ-regular bi-ideals greater that (L, M) is clearly inductive, and so by Zorn's Lemma each ϱ-regular bi-ideal has a maximal ϱ-regular bi-ideal refinement.

<u>Definition</u> 3.1.3. Let ϱ be a dispersion on P. We say the bi-ideal (L, M) is <u>ϱ-outer prime</u> if it satisfies:

$$(a \vee b) \varrho (a' \wedge b') \Rightarrow \exists \; p \in L \text{ and } q \in M \text{ with } (a \vee p) \varrho q \text{ or } (b \vee p) \varrho q$$
$$\text{or } p \varrho (a' \wedge q) \text{ or } p \varrho (b' \wedge q).$$

We now have:

<u>Proposition</u> 3.1.1. The ϱ-regular bi-ideal (L, M) has a unique maximal ϱ-regular refinement if and only if it is ϱ-outer prime.

<u>Proof</u>. Suppose (L, M) has a unique maximal ϱ-regular refinement.

Then if (L, M) is not ρ-outer prime $\exists$ $a, b, a', b' \in P$ with $(a \vee b)\rho(a' \wedge b')$ and $(a \vee p)\bar{\rho} q$, $(b \vee p)\bar{\rho} q$, $p\bar{\rho}(a' \wedge q)$ and $p\bar{\rho}(b' \wedge q)$ for all $p \in L$ and $q \in M$. If for $u \in P$ we define

$L^u = \{ v \mid v \in P, \exists \ p \in L \text{ with } v \leq p \vee a \}$, and

$M^u = \{ v \mid v \in P, \exists \ q \in M \text{ with } u \wedge q \leq v \}$

then (L^a, M), (L^b, M), $(L, M^{a'})$ and $(L, M^{b'})$ are ρ-regular refinements of (L, M), and so have a common ρ-regular refinement (L', M'). However we now have $a \vee b \in L'$ and $a' \wedge b' \in M'$, which gives the immediate contradiction $(a \vee b)\bar{\rho}(a' \wedge b')$.

To prove the converse, let (L, M) be a ρ-regular bi-ideal and define:

$L(\rho) = \{ a \mid a \in P, \exists \ p \in L, q \in M \text{ with } (a \vee p)\rho q \}$, and

$M(\rho) = \{ b \mid b \in P, \exists \ p \in L, q \in M \text{ with } p\rho(b \wedge q) \}$.

Clearly $L(\rho)$ is a dual ideal, and $M(\rho)$ is an ideal. Moreover, if (L, M) is ρ-outer prime, then $L(\rho)$ and $M(\rho)$ are prime, and so $P - L(\rho)$ is an ideal and $P - M(\rho)$ is a dual ideal. Also $L \subseteq P - L(\rho)$ and $M \subseteq P - M(\rho)$ so $(P - L(\rho), P - M(\rho))$ is a bi-ideal refinement of (L, M), and it is clearly ρ-regular. Finally let (L', M') be any ρ-regular refinement of (L, M), and suppose $\exists$ $a \in L' \cap L(\rho) \neq \emptyset$. Then for some $p \in L$, $q \in M$ we have $(a \vee p)\rho q$. However this is impossible since $a, p \in L'$ and $q \in M'$; and we deduce that $L' \subseteq P - L(\rho)$. Likewise $M' \subseteq P - M(\rho)$, and we have shown that $(P - L(\rho), P - M(\rho))$ is the unique maximal ρ-regular refinement of (L, M).

<u>Corollary</u> 1. A maximal ρ-regular bi-ideal is prime and ρ-outer prime.

<u>Corollary</u> 2. (L, M) is maximal ρ-regular if and only if it is ρ-regular and $L \cup L(\rho) = M \cup M(\rho) = P$.

<u>Corollary</u> 3. If (L, M) is ρ-regular and total then $L(\rho) \cap M(\rho) = \emptyset$. If (L, M) is maximal ρ-regular and $L(\rho) \cap M(\rho) = \emptyset$ then (L, M) is total.

We now introduce some additional structure into P. Let $T : \mathbb{R} \times P \rightarrow P$ be a mapping, and for $r \in \mathbb{R}$ and $a \in P$ let us set $T_r(a) = T(r, a)$. We make the following definition:

<u>Definition</u> 3.1.4. We say P is a <u>T-lattice</u> if T satisfies:

(i) $T_r : P \longrightarrow P$ is a lattice homomorphism for each $r \in \mathbb{R}$.

(ii) $T_r o T_s = T_s o T_r = T_{r+s}$ for all $r, s \in \mathbb{R}$.

(iii) $T_r(a) = a \iff r = 0$, for all $a \in P$.

(iv) $T_r(a) \leqslant a$ for all $a \in P$ and $r > 0$.

Note in particular that the map $r \rightarrow T_{-r}(0)$ is an injection of $\mathbb{R}$ into P, and it takes $0 \in \mathbb{R}$ to the distinguished element $0 \in P$. In general we shall denote the element $T_{-r}(0)$ of P by r. Clearly we have $T_s(r) = r - s$ for all r and s.

This added structure on P enables us to define various special dispersions. Initially let us note the following:

<u>Definition</u> 3.1.5. On the T-lattice P the dispersions $\mathcal{f}_e$ and $\mathcal{f}_b$ are given by

$\mathcal{f}_e = \{(a, b) \mid a, b \in P, \exists \; r \in R \text{ with } b \leqslant r < 0 \text{ or } 0 < r \leqslant a\}$,

$\mathcal{f}_b = \{(a, b) \mid a, b \in P, \exists \; r > 0 \text{ with } T_r(a \vee 0) \geqslant b \wedge 0\}$.

Note that, since $\mathcal{f}_e \leq \mathcal{f}_b$, a $\mathcal{f}_b$-regular bi-ideal is also $\mathcal{f}_e$-regular.

<u>Lemma</u> 3.1.1. If (L, M) is $\mathcal{f}_e$-regular and $r \in \mathbb{R}$, then $r \in L \iff r \leq 0$ and $r \in M \iff r \geqslant 0$.

<u>Lemma</u> 3.1.2. If (L, M) is $\mathcal{f}_b$-regular it satisfies

(a) $a \in L \Rightarrow T_r(a) \notin M \;\forall\; r > 0$, and

(b) $b \in M \Rightarrow T_{-r}(b) \notin L \;\forall\; r > 0$.

Conversely if the bi-ideal (L, M) satisfies (a) or (b) it is $\mathcal{f}_b$-regular.

We omit the proofs, which are straightforward. Note in particular that if (L, M) is $\mathcal{f}_b$-regular and $T_r(a) \in L$, $T_s(a) \in M$ then we must have $s \leq r$. Hence we can have $T_r(a) \in L \cap M$ for at most one $r \in \mathbb{R}$.

The following is a further corollary to Proposition 3.1.1.

<u>Proposition</u> 3.1.2. If (L, M) is ℓ_b-regular and prime then $L(\ell_b) \cap M(\ell_b) = \emptyset$. In particular every maximal ℓ_b-regular bi-ideal is total.

<u>Proof</u>. Suppose we had some $a \in L(\ell_b) \cap M(\ell_b)$. Since $a \in L(\ell_b)$ we have $p \in L$, $q \in M$ with $(a \vee p)(\ell_b)q$. Also $a \notin M$ since $M(\ell_b) \cap M = \emptyset$. Now $\exists\ t > 0$ with $T_r(a \vee p \vee 0) \geq q \wedge 0$, and $q \wedge 0 \in M$, so $T_t(a \vee p \vee 0) = T_t(a) \vee T_t(p \vee 0) \in M$. But M is prime, so $T_t(a) \in M$ or $T_t(p \vee 0) \in M$. However the first is impossible since $T_t(a) \leq a$, and the second is impossible since $p(\ell_b)T_t(p \vee 0)$. This contradiction shows that $L(\ell_b) \cap M(\ell_b) = \emptyset$. Note that we have only used the fact that M is prime, and likewise it would be sufficient to assume only that L is prime.

Finally, if (L, M) is maximal ℓ_b-regular then (L, M) is prime by Corollary 1 to Proposition 3.1.1. Hence $L(\ell_b) \cap M(\ell_b) = \emptyset$, and so (L, M) is total by Corollary 3 to Proposition 3.1.1.

<u>Definition</u> 3.1.6. Let (L, M) be a bi-ideal in the T-lattice P. The bi-ideal refinement (L^+, M^+) of (L, M) is defined by

$$L^+ = \{ a \mid a \in P,\ T_r(a) \in L\ \ \forall\ r > 0 \}, \text{ and}$$

$$M^+ = \{ a \mid a \in P,\ T_{-r}(a) \in M\ \ \forall\ r > 0 \}.$$

(L, M) is <u>nearly total</u> (respectively, <u>nearly prime</u>) if (L^+, M^+) is total (respectively, prime).

<u>Note</u>. (1) (L, M) is ℓ_e-regular or ℓ_b-regular if and only if the same is true of (L^+, M^+).

(2) (L, M) is nearly total if and only if it satisfies either of the equivalent conditions:

(a) $L \cup M^+ = P$, or (b) $L^+ \cup M = P$.

(3) (L, M) is nearly prime if and only if it satisfies:

(a) $a \wedge b \in L \Rightarrow a \in L^+$ or $b \in L^+$; and

(b) $a \vee b \in M \Rightarrow a \in M^+$ or $b \in M^+$.

In particular a prime bi-ideal is nearly prime.

We omit the proofs of the above results since they are all elementary.

Proposition 3.1.3. Let (L, M) be a ρ_b-regular bi-ideal. Then (L, M) is nearly total if and only if it is ρ_b-outer prime and nearly prime.

Proof. First let us suppose that (L, M) is nearly total. Suppose that for some $a \wedge b \in L$ we have $a \notin L^+$ and $b \notin L^+$. Then we clearly also have some $t > 0$ with $T_t(a) \notin L^+$ and $T_t(b) \notin L^+$. However $L^+ \cup M = P$, and so $T_t(a \wedge b) = T_t(a) \wedge T_t(b) \in M$ which contradicts the fact that (L, M) is ρ_b-regular (Lemma 3.1.2). In the same way we cannot have $a \vee b \in M$ with $a \notin M^+$ and $b \notin M^+$, and it follows by Note (3) above that (L, M) is nearly prime. To show that (L, M) is ρ_b-outer prime, let us assume the contrary. Then we shall have $a, b, a', b' \in P$ so that $(a \vee b)(\rho_b)(a' \wedge b')$ and $(a \vee p)(\bar{\rho}_b)q$, $(b \vee p)(\bar{\rho}_b)q$, $p(\bar{\rho}_b)(a' \wedge q)$ and $p(\bar{\rho}_b)(b' \wedge q)$ for all $p \in L$ and $q \in M$. Now for any $t > 0$ we have $(a \vee 0)(\rho_b)T_t(a)$, and since $0 \in L$ this implies $T_t(a) \notin M$. Since (L, M) is nearly total this shows that $a \in L^+$. In just the same way we can show that $b \in L^+$, $a' \in M^+$ and $b' \in M^+$. Now for some $s > 0$ we have $T_s(a \vee b \vee 0) \succcurlyeq a' \wedge b' \wedge 0$, and we may deduce that for all r with $0 < 2r \leq s$ we have $T_r(a \vee b \vee 0) \in L \cap M$. This, by the remark made after Lemma 3.1.2, contradicts the fact that (L, M) is ρ_b-regular.

For the converse, suppose that (L, M) is ρ_b-regular, ρ_b-outer prime and nearly prime; and that for some $a \in P$ and $t > 0$ we have $a \in L$ and $T_{-t}(a) \notin M$. Now $T_{-t}(a)(\rho_b)a$, so $\exists\ p \in L$, $q \in M$ with $(T_{-t}(a) \vee p)(\rho_b)q$ or $p(\rho_b)(q \wedge a)$. In the first case there exists $s > 0$ with $T_{2s}(T_{-t}(a) \vee p \vee 0) \geq q \wedge 0$, and so $T_{2s-t}(a) \vee T_{2s}(p \vee 0) \in M$. Since (L, M) is nearly prime we deduce that either $T_{s-t}(a) \in M$ or $T_s(p \vee 0) \in M$. Now $T_s(p \vee 0) \in M$ is impossible since (L, M) is $\mathcal{f}_b$-regular, and so $T_{s-t}(a) \in M$. However $T_{s-t}(a) \leq T_{-t}(a)$ and so $T_{-t}(a) \in M$, which contradicts the above hypothesis. The second case leads to a contradiction in the same way, and the proof is complete.

For a fixed bi-ideal (L, M) let us consider the following binary relation $\sim$ on P.

$$a \sim b \iff (T_r(a) \in L \iff T_r(b) \in L \text{ and } T_r(a) \in M \iff T_r(b) \in M)$$

Clearly $\sim$ is an equivalence relation on P. We denote the equivalence classes by $[a]$, the quotient set by $P/(L, M) = \{ [a] \mid a \in P \}$. ψ will denote the canonical map $\psi(a) = [a]$ of P onto $P/(L, M)$.

We may partially order $P/(L, M)$ by setting:

$$[a] \leq [b] \iff (T_r(b) \in L \Rightarrow T_r(a) \in L \text{ and } T_r(a) \in M \Rightarrow T_r(b) \in M)$$

It is clear that ψ is order preserving, and so $[a \vee b]$ is an upper bound and $[a \wedge b]$ a lower bound of the set $\{ [a], [b] \}$, for all $a, b \in P$. Moreover if (L, M) is prime then it is easy to see that

$$[a] \vee [b] = [a \vee b] \quad \text{and} \quad [a] \wedge [b] = [a \wedge b]$$

so that $P/(L, M)$ is then a lattice, and ψ a lattice homomorphism. Also $T_r([a]) = [T_r(a)]$ is well defined, and so when (L, M) is prime we may make $P/(L, M)$ into a T-lattice in such a way that ψ is a T-lattice homomorphism. Finally let us note that if (L, M) is $\mathcal{f}_e$-regular then

$$r \to [r]$$

is an order preserving injection of $\mathbb{R}$ into $P/(L, M)$.

The proofs of the following lemmas are elementary, and are omitted.

<u>Lemma</u> 3.1.3. Let (L, M) be $\mathcal{f}_b$-regular. Then:

(a) $L = \{ a \mid [a] \leq [0] \}$

(b) $M = \{ b \mid [0] \leq [b] \}$

(c) $L \cap M = [0]$

Lemma 3.1.4. Let (L, M) be $\mathscr{f}_b$-regular. Then P/(L, M) is totally ordered if and only if each element of P/(L, M) is compatible with $[0]$.

We may now give:

Proposition 3.1.4. Let (L, M) be a $\mathscr{f}_e$-regular bi-ideal. Then P/(L, M) is totally ordered if and only if (L, M) is $\mathscr{f}_b$-regular and total.

Proof. Sufficiency follows at once from Lemma 3.1.3 and Lemma 3.1.4. To prove necessity suppose P/(L, M) is totally ordered. If (L, M) were not $\mathscr{f}_b$-regular we should have $p \in L$, $q \in M$ with $p(\mathscr{f}_b)q$, and hence some $t > 0$ with $T_{2t}(p \vee 0) \geq q \wedge 0$. In that case $T_t(T_t(p \vee 0)) \in M$ but $T_t(0) \notin M$ so $[T_t(p \vee 0)] \not\leq [0]$; while $T_{-t}(T_t(p \vee 0)) = p \vee 0 \in L$ but $T_{-t}(0) \notin L$ and so $[0] \not\leq [T_t(p \vee 0)]$. This would contradict the total orderedness of P/(L, M), and we deduce that (L, M) is $\mathscr{f}_b$-regular. The fact that (L, M) is total now follows at once from Lemma 3.1.3.

Definition 3.1.7. We say $[a] \in P/(L, M)$ is infinitesimal if $[-t] \leq [a] \leq [t]$ for all $t > 0$, and in that case we say that a is infinitesimal at (L, M). We denote by $\imath_{(L,\ M)}(0)$ (or just by $\imath(0)$) the set of all elements of P which are infinitesimal at (L, M).

First let us note the following elementary consequences of our definition.

Lemma 3.1.5. If (L, M) is $\mathscr{f}_e$-regular then

$\imath(0) = \{ a \mid T_t(a) \in L - M \text{ and } T_{-t}(a) \in M - L \ \forall \ t > 0 \}$.

Corollary 1. If (L, M) is $\mathscr{f}_e$-regular and nearly prime then $\imath(0)$ is a sub-lattice of P.

Corollary 2. If (L, M) is $\mathcal{f}_b$-regular then $\imath(0) = L^+ \cap M^+$.

We may now give:

Proposition 3.1.5. The $\mathcal{f}_b$-regular bi-ideal (L, M) is maximal $\mathcal{f}_b$-regular if and only if it is nearly total and P/(L, M) contains no non-zero infintesimal elements.

Proof. If (L, M) is maximal $\mathcal{f}_b$-regular then it is total by Proposition 3.1.1, and hence nearly total. Also, since (L^+, M^+) is a $\mathcal{f}_b$-regular refinement of (L, M) we have

$$\imath(0) = L^+ \cap M^+ = L \cap M = [0]$$

by Lemma 3.1.3 and Lemma 3.1.5, Corollary 2.

Conversely suppose (L, M) is $\mathcal{f}_b$-regular and nearly total, and that P/(L, M) has no non-zero infintesimals. By Proposition 3.1.1, Corollary 2, it will suffice to show that $L \cup L(\mathcal{f}_b) = M \cup M(\mathcal{f}_b) = P$. Take $a \in P - L$. Since $L \cup M^+ = P$ we have $a \in M^+$. On the other hand $a \notin L \cap M = [0] = L^+ \cap M^+$ and so $a \notin L^+$. Hence for some $s > 0$ we have $T_{2s}(a) \notin L$, and so $T_{2s}(a) \in M^+$, which implies that $T_s(a) \in M$. Hence $0 \in L$ and $(a \vee 0)(\mathcal{f}_b)T_s(a)$ which gives us $a \in L(\mathcal{f}_b)$ as required. $M \cup M(\mathcal{f}_b) = P$ may be shown in the same way, and the proof is complete.

On P/(L, M) let us define the relation $\ll$ by

$$[a] \ll [b] \Leftrightarrow ([T_t(a)] \leq [T_{-t}(b)] \;\forall\; t > 0).$$

We have at once:

Lemma 3.1.6. The following are equivalent.

(a) $[a] \ll [b]$,

(b) $[T_t(a)] \leq [b] \;\forall\; t > 0$,

(c) $[a] \leq [T_{-t}(b)] \;\forall\; t > 0$.

Corollary 1. $\ll$ is a pseudo-order on P/(L, M). (That is, it is reflexive and transitive)

Corollary 2. The sets $I([a]) = \{ [b] \mid [a] \ll [b]$ and $[b] \ll [a] \} = \{ [b] \mid [T_t(a)] \leq [b] \leq [T_{-t}(a)] \ \forall \ t > 0 \}$, $a \in P$, form a partition of $P/(L, M)$.

We may regard the elements of $I([a])$ as being "infinitely close" to $[a]$. Note in particular that $I([0]) = \psi(\iota(0))$.

Definition 3.1.8. We say $P/(L, M)$ is <u>nearly totally ordered</u> if $\ll$ is a total pseudo-order.

Now consider the set $I(P) = \{ I([a]) \mid a \in P \}$. We may partially order $I(P)$ by setting

$$I([a]) \leq I([b]) \Leftrightarrow [a] \ll [b].$$

On the other hand let us denote the elements of $P/(L^+, M^+)$ by $[a]^+$, and the cannonical mapping by ψ^+. If we note that

$$[a] \ll [b] \Leftrightarrow [a]^+ \leq [b]^+$$

then it is immediate that $\kappa(I([a])) = [a]^+$ is an order preserving isomorphism of $I(P)$ with $P/(L^+, M^+)$. In particular if (L, M) is nearly prime then $P/(L^+, M^+)$ is a lattice, and hence so too is $I(P)$. Moreover, since $I([a]) = \kappa^{-1}(\psi^+(a))$ is a composition of lattice homomorphisms we have

$$I([a]) \vee I([b]) = I([a \vee b]), \text{ and}$$
$$I([a]) \wedge I([b]) = I([a \wedge b])$$

for all $a, b \in P$. We summarize these results in the next theorem.

Theorem 3.1.1. Let (L, M) be nearly prime. Then $I(P)$ is a lattice and $a \rightarrow I([a])$ is a lattice homomorphism. Moreover $I(P)$ is lattice isomorphic with $P/(L^+, M^+)$ under the mapping $\kappa(I([a])) = [a]^+$, $a \in P$.

Next let us note:

Theorem 3.1.2. Let (L, M) be $\mathcal{f}_b$-regular. Then the following are equivalent.

(i) $P/(L, M)$ is nearly totally ordered.

(ii) (L, M) is nearly total.

(iii) (L^+, M^+) is maximal $\mathcal{f}_b$-regular.

Proof. (i) $\Rightarrow$ (ii). If P/(L, M) is nearly totally ordered then $I(P) \cong P/(L^+, M^+)$ is totally ordered, and hence (L^+, M^+) is total by Proposition 3.1.4. This shows that (L, M) is nearly total.

(ii) $\Rightarrow$ (iii). Note first that $P/(L^+, M^+)$ can contain no non-zero infintesimals, for by Lemma 3.1.5, Corollary 2, and Lemma 3.1.3, we have

$$\imath_{(L^+, M^+)}(0) = (L^+)^+ \cap (M^+)^+ = L^+ \cap M^+ = [0]^+$$

Hence if (L, M) is nearly total then (L^+, M^+) is total and hence maximal $\mathcal{f}_b$-regular by Proposition 3.1.5.

(iii) $\Rightarrow$ (i) If (L^+, M^+) is maximal $\mathcal{f}_b$-regular then it is total by Proposition 3.1.2, and hence $P/(L^+, M^+) \cong I(P)$ is totally ordered by Proposition 3.1.4. Hence P/(L, M) is nearly totally ordered.

We are now going to consider the situation with regard to the existence of infinite elements of P/(L, M).

Definition 3.1.9. The element $[a] \in P/(L, M)$ is finite if $[s] \leq [a] \leq [t]$ for some $s, t \in \mathbb{R}$, and it is infinite if $[s] \leq [a]$ $\forall$ $s \in \mathbb{R}$ or $[a] \leq [t]$ $\forall$ $t \in \mathbb{R}$. $a \in P$ is finite or infinite at (L, M) if [a] has the corresponding property in P/(L, M). Finally (L, M) is finite if all the elements of P are finite at (L, M).

Lemma 3.1.7. If (L, M) is $\mathcal{f}_e$-regular then:

(a) $[s] \leq [a]$ $\forall$ $s \in \mathbb{R} \Leftrightarrow T_r(a) \in M - L$ $\forall$ $r \in \mathbb{R}$.

(b) $[a] \leq [t]$ $\forall$ $t \in R \Leftrightarrow T_r(a) \in L - M$ $\forall$ $r \in \mathbb{R}$.

Moreover, if (L, M) is $\mathcal{f}_b$-regular we may replace M - L by M and L - M by L in (a) and (b) above.

Corollary. If (L, M) is $\mathcal{f}_e$-regular then the only infinite elements of P/(L, M) are the greatest and least elements, when these exist.

We omit the proof, which is trivial.

Proposition 3.1.6. Let (L, M) be a finite $\mathcal{f}_b$-regular bi-ideal.

(i) If (L', M') is $\mathcal{f}_b$-regular and (L, M) $\prec$ (L', M') then (L', M') is finite.

(ii) If (L', M') is nearly total and (L', M') $\prec$ (L, M) then (L', M') is finite.

Proof. (i) is trivial, and (ii) follows from Lemma 3.1.7 if we note that when (L', M') is a nearly total $\mathcal{f}_b$-regular bi-ideal each element of P/(L', M') is either finite or infinite.

We have already noted that when (L, M) is $\mathcal{f}_e$-regular $r \to [r]$ is an order preserving injection of $\mathbb{R}$ into P/(L, M). If P/(L, M) has a greatest and/or least element this mapping may be extended in the obvious way to a mapping defined on $\mathbb{R} \cup \{\infty\}$, $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{\infty, -\infty\}$, as the case may be.

Definition 3.1.10. The $\mathcal{f}_e$-regular bi-ideal (L, M) is real if P/(L, M) is the image of $\mathbb{R}$ under the mapping $r \to [r]$. (L, M) is extended real if it is the image of any one of the sets $\mathbb{R}$, $\mathbb{R} \cup \{\infty\}$, $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{\infty, -\infty\}$.

Lemma 3.1.8. Let (L, M) be $\mathcal{f}_e$-regular. Then (L, M) is real if and only if given $a \in P$ $\exists\, \alpha \in \mathbb{R}$ with $T_r(a) \in L \Leftrightarrow r \geq \alpha$ and $T_r(a) \in M \Leftrightarrow r \leq \alpha$.

Corollary 1. Let (L, M) be $\mathcal{f}_e$-regular. Then (L, M) is real if and only if given $a \in P$ $\exists !\, \alpha \in \mathbb{R}$ with $T_\alpha(a) \in L \cap M$.

Corollary 2. Let (L, M) be $\mathcal{f}_e$-regular. Then (L, M) is real if and only if it is $\mathcal{f}_b$-regular and given $a \in P$ $\exists\, \alpha \in \mathbb{R}$ with $T_\alpha(a) \in L \cap M$.

We omit the proof, which is straightforward. Corresponding characterizations of extended real may be obtained by replacing "$a \in P$" by "$a \in P$ which is finite at (L, M)". In particular we note that an extended real bi-ideal is always $\mathcal{f}_b$-regular.

Corollary 1 of the next proposition gives a considerably improved result.

<u>Proposition</u> 3.1.7. Let (L, M) be $\mathcal{f}_b$-regular and nearly prime. Then (L, M) is nearly total if and only if $\bigcup\{ I([r]) \mid r \in \mathbb{R}\}$ is the set of all finite elements of P/(L, M).

<u>Proof</u>. Suppose (L, M) is nearly total, and let $a \in P$ be finite at (L, M). Then $s = \sup\{ r \mid [r] \leq [a]\} \in \mathbb{R}$ and $t = \inf\{ r \mid [a] \leq [r]\} \in \mathbb{R}$, and clearly $s \leq t$. If $s < t$ take w with $s < w < t$, and $0 < h < \max\{t - w, w - s\}$. By Proposition 3.1.2, P/(L, M) is nearly totally ordered so $[a] \ll [w]$ or $[w] \ll [a]$. In the first case $[a] \leq [T_{-h}(w)] = [w + h]$ and so $t \leq w + h$, which contradicts the choice of h. Likewise the second case leads to a contradiction, and we deduce $s = t$. It then follows easily that $[a] \in I([s])$.

The converse result is clear.

<u>Corollary</u> 1. The bi-ideal (L, M) is extended real if and only if it is maximal $\mathcal{f}_b$-regular.

<u>Proof</u>. First suppose that (L, M) is maximal $\mathcal{f}_b$-regular. Then (L, M) is prime and total, so by Proposition 3.1.7 $\bigcup\{ I([r]) \mid r \in \mathbb{R}\}$ is the set of all finite elements of P/(L, M). However by Proposition 3.1.5 P/(L, M) has no non-zero infintesimal elements, so if $a \in P$ is finite at (L, M) we have $[a] \in I([\alpha])$ for some $\alpha \in \mathbb{R}$, while $[a] \in I([\alpha]) \Leftrightarrow [T_\alpha(a)] \in I([0]) = \{[0]\}$. Hence $T_\alpha(a) \in L \cap M$, and (L, M) is extended real.

Conversely suppose (L, M) is extended real. Then by definition (L, M) is $\mathcal{f}_e$-regular, and also P/(L, M) is totally ordered so by Proposition 3.1.4 (L, M) is $\mathcal{f}_b$-regular and total. Finally P/(L, M) contains no non-zero infintesimals so (L, M) is maximal $\mathcal{f}_b$-regular by Proposition 3.1.5.

As an immediate consequence of this result and Proposition 3.1.6 we have:

<u>Corollary</u> 2. Any maximal $\mathcal{f}_b$-regular refinement of a finite bi-ideal is real.

Corollary 3. Let (L, M) be a nearly prime $\mathscr{f}_b$-regular bi-ideal. Then the following are equivalent:

(a) $P/(L, M) = \bigcup \{ I([r]) \mid r \in \mathbb{R} \}$.

(b) (L^+, M^+) is real.

(c) (L, M) is $\mathscr{f}_b$-outer prime and finite.

Proof. (a) $\Rightarrow$ (b). By the proposition (L, M) is nearly total so by Theorem 3.1.2, (L^+, M^+) is maximal $\mathscr{f}_b$-regular. Also (L, M) is finite so (L^+, M^+) is real by Corollary 2.

(c) $\Rightarrow$ (a). This follows at once from the above proposition and Proposition 3.1.3.

(b) $\Rightarrow$ (c). (L^+, M^+) is maximal $\mathscr{f}_b$-regular by Corollary 1, and so (L, M) is nearly total by Theorem 3.1.2. Hence (L, M) is $\mathscr{f}_b$-outer prime by Proposition 3.1.3, and (L, M) is finite since (L^+, M^+) is, using Proposition 3.1.6(ii).

Now let A and B be sub-T-lattices of P with $A \subseteq B$. A dispersion $\mathscr{f}$ on P induces a dispersion on A and B which we continue to denote by $\mathscr{f}$, and a statement that a bi-ideal in A or B is $\mathscr{f}$-regular, $\mathscr{f}$-outer prime, etc., will mean that it has the stated property for the induced dispersion.

If (L, M) is a $\mathscr{f}$-regular bi-ideal in B then $(L \cap A, M \cap A)$ is a $\mathscr{f}$-regular bi-ideal in A, and moreover if (L, M) is maximal $\mathscr{f}$-regular then the same is true of $(L \cap A, M \cap A)$ in A.

On the other hand suppose (L, M) is a $\mathscr{f}$-regular bi-ideal in A, and assume that $\mathscr{f}_e \subseteq \mathscr{f}$. Let

$L_B = \{ b \mid b \in B, \exists \; a \in L \text{ and } t > 0 \text{ with } b \wedge t \preccurlyeq a \}$, and

$M_B = \{ b \mid b \in B, \exists \; a \in M \text{ and } t > 0 \text{ with } a \preccurlyeq b \vee (-t) \}$.

Then (L_B, M_B) is a $\mathscr{f}$-regular bi-ideal in B which is contained in every prime $\mathscr{f}$-regular bi-ideal in B whose restriction to A is (L, M). Hence if (L, M) is maximal $\mathscr{f}$-regular in A we have at least one maximal $\mathscr{f}$-regular bi-ideal in B whose restriction to A is (L, M). In particular this maximal $\mathscr{f}$-regular extension of (L, M) to B will be unique if and only if (L_B, M_B) is $\mathcal{f}$-outer prime. This leads us to the following:

Definition 3.1.11. Let A, B be sub-T-lattices of P with $A \subseteq B$. Then B is a $\mathcal{f}$-refinement of A if every maximal $\mathcal{f}$-regular bi-ideal in A has a unique extension to a maximal $\mathcal{f}$-regular bi-ideal in B.

A is $\mathcal{f}$-complete in P if it has no proper $\mathcal{f}$-refinement in P. A $\mathcal{f}$-complete $\mathcal{f}$-refinement of A will be called a $\mathcal{f}$-completion of A.

In the same way if every finite maximal $\mathcal{f}$-regular bi-ideal in A has a unique finite maximal $\mathcal{f}$-regular extension to B we may speak of B as a finite $\mathcal{f}$-refinement of A, and give obvious meanings to the terms finitely $\mathcal{f}$-complete and finite $\mathcal{f}$-completion.

Proposition 3.1.8. Every sub-T-lattice A of P has a $\mathcal{f}$-completion in P.

Proof. If A is $\mathcal{f}$-complete there is nothing to prove so assume the contrary. Then by Zorn's Lemma it will be sufficient to show that the set of $\mathcal{f}$-refinements of A is inductive when ordered by set inclusion. Let $\{B_\alpha\}$ be a chain of $\mathcal{f}$-refinements of A, and set $B = \cup\{B_\alpha\}$. Clearly B is a sub-T-lattice of P with $A \subseteq B$. Let (L, M) be a maximal $\mathcal{f}$-regular bi-ideal in A, and note that

$$L_B = \cup\{L_{B_\alpha}\} \quad \text{and } M_B = \cup\{M_{B_\alpha}\}$$

It may be verified at once that (L_B, M_B) is $\mathcal{f}$-outer prime, so (L, M) has a unique maximal $\mathcal{f}$-regular extension to B. Hence B is an upper bound of the $\mathcal{f}$-refinements $\{B_\alpha\}$, and the proof is complete.

For the case $\mathcal{f} = \mathcal{f}_b$ we are going to show that, more particularly, every sub-T-lattice A has a unique $\mathcal{f}_b$-completion and a unique finite $\mathcal{f}_b$-completion. To this end we shall need the following lemma, which is true for an arbitrary dispersion.

Lemma 3.1.9. Let A, B be sub-T-lattices of P with $A \subseteq B$, and (L, M) a maximal $\mathcal{f}$-regular bi-ideal in A. Define:

$L(B) = \{ b \mid b \in B$ and $\exists\ a \in L$ such that $\forall\ r \in \mathbb{R}$, $T_r(a) \in L'$ $\Rightarrow T_r(b) \in L'$ $\forall$ maximal $\mathcal{f}$-regular bi-ideals (L', M') in P with $L' \cap A = L$ and $M' \cap A = M \}$,

$M(B) = \{ b \mid b \in B$ and $\exists\ a \in M$ such that $\forall\ r \in \mathbb{R}$, $T_r(a) \in M'$ $\Rightarrow T_r(b) \in M'$ $\forall$ maximal $\mathcal{f}$-regular bi-ideals (L', M') in P with $L' \cap A = L$ and $M' \cap A = M \}$.

Then:

(i) (L(B), M(B)) is a $\mathcal{f}$-regular bi-ideal in B with $L(B) \cap A = L$ and $M(B) \cap A = M$,

(ii) If (L_o, M_o) is any maximal $\mathcal{f}$-regular bi-ideal in B with $L_o \cap A = L$ and $M_o \cap A = M$ then $L(B) \subseteq L_o$ and $M(B) \subseteq M_o$,

(iii) If (L, M) has a unique maximal $\mathcal{f}$-regular extension to B then (L(B), M(B)) is prime.

Proof. (i) and (ii) are clear, so let us prove (iii). Take b, b' $\in$ B with $b \wedge b' \in L(B)$, and let (L', M') be a maximal $\mathcal{f}$-regular bi-ideal in P with $L' \cap A = L$ and $M' \cap A = M$. Then for some $a \in L$ we have $T_r(a) \in L' \Rightarrow T_r(b \wedge b') \in L'$, and since L' is prime it follows at once that $T_r(a) \in L' \Rightarrow T_r(b) \in L'$ $\forall\ r \in \mathbb{R}$ or $T_r(a) \in L' \Rightarrow T_r(b') \in L'$ $\forall\ r \in \mathbb{R}$. Since we are assuming that (L, M) has a unique maximal $\mathcal{f}$-regular extension to B this is sufficient to prove $b \in L(B)$ or $b' \in L(B)$, that is L(B) is prime. M(B) may be shown to be prime in the same way.

The results of this lemma, together with Propositions 3.1.1, 3.1.3 and 3.1.6, and Corollories 1 and 2 of Proposition 3.1.7, give us the following result:

Corollary. B is a $\mathcal{f}_b$-refinement (respectively, a finite $\mathcal{f}_b$-refinement) of A if and only if for each extended real (respectively, real) bi-ideal (L, M) in A the bi-ideal (L(B), M(B)) is nearly total (respectively, nearly total and finite).

We may now give:

Theorem 3.1.3. Every sub-T-lattice A of P has a unique $\mathcal{f}_b$-completion and a unique finite $\mathcal{f}_b$-completion in P.

Proof. If A is $\mathcal{f}_b$-complete (respectively, finitely $\mathcal{f}_b$-complete) there is nothing to prove, so assume the contrary and let $\{ B_\alpha \}$ be the class of all $\mathcal{f}_b$-refinements (respectively, finite $\mathcal{f}_b$-refinements) of A in P. Let $B = \langle B_\alpha \rangle$ denote the smallest sub-T-lattice of P containing all the B_α; then the proof will be complete if we can show that B is a $\mathcal{f}_b$-refinement (respectively, finite $\mathcal{f}_b$-refinement) of A. Let (L, M) be an extended real (respectively, real) bi-ideal in A. Now the elements of B may be obtained from the elements of $\bigcup \{ B_\alpha \}$ by a finite number of applications of the operations $\vee$, $\wedge$, T_r, $r \in \mathbb{R}$, and let us denote by B_n the set of elements of B which may be obtaind using n such applications. In particular, therefore, $B_0 = \bigcup \{ B_\alpha \}$ and $B = \bigcup \{ B_n \mid n = 0, 1, 2, \ldots \}$.

Make the following induction hypothesis:

P(n) : For all $b \in B_n$, $r \in \mathbb{R}$, we have $T_r(b) \notin L(B) \Rightarrow T_r(b) \in M(B)^+$.

P(0) is certainly valid, for $L(B) \cap B_\alpha = L(B_\alpha)$, $M(B) \cap B_\alpha = M(B_\alpha)$, and $(L(B_\alpha), M(B_\alpha))$ is nearly total for each α by the corollary to Lemma 3.1.9. Hence suppose P(m) for all $m < n$, and take $b \in B_n$. There are three cases to consider:

(i) $b = b' \vee b''$, (ii) $b = b' \wedge b''$, (iii) $b = T_s(b')$,

where $b' \in B_{n'}$, $b'' \in B_{n''}$ with $n' \vee n'' < n$, and $s \in \mathbb{R}$.

Consider (i), and suppose $T_r(b) = T_r(b' \vee b'') = T_r(b') \vee T_r(b'') \notin L(B)$. Then $T_r(b') \notin L(B)$ or $T_r(b'') \notin L(B)$, so by the induction hypothesis $T_r(b') \in M(B)^+$ or $T_r(b'') \in M(B)^+$, and in either event $T_r(b) = T_r(b') \vee T_r(b'') \in M(B)^+$.

Case (ii) is dealt with in the same way, and (iii) is trivial. so We have established P(n). This shows P(n) is true for all n, and hence (L(B), M(B)) is nearly total. If (L, M) is real and each B_α is a finite $\mathcal{f}_b$-refinement then a similar induction argument can be used to show that (L(B), M(B)) is also finite, and the result now follows from the Corollary to Lemma 3.1.9.

An element $a \in P$ is <u>bounded</u> if $s \leq a \leq t$ for some $s, t \in \mathbb{R}$, and we denote the set of bounded elements of P by P^*. Of course P^* is a sub-T-lattice of P. If $a \in P^*$ then a is finite at every $\mathcal{f}_e$-regular bi-ideal in P. Hence if we set

$P' = \{ a \mid a \in P$, a is finite at every extended real bi-ideal in P $\}$

then P' is a sub-T-lattice of P, and $P^* \subseteq P'$. Note that every extended real bi-ideal in P^* or P' is real.

<u>Proposition</u> 3.1.9. (i) The $\mathcal{f}_b$-completion of P^* is P.

(ii) If A is a sub-T-lattice of P^* then its finite $\mathcal{f}_b$-completion in P is a subset of P'. In particular the finite $\mathcal{f}_b$-completion of P^* is P'.

<u>Proof</u>. Let (L, M) be a real bi-ideal in P, and define

$L' = \{ a \mid a \in P, (a \wedge t) \vee s \in L \ \forall \ s < 0 < t \}$,

$M' = \{ b \mid b \in P, (b \wedge t) \vee s \in M \ \forall \ s < 0 < t \}$.

It is a straightforward matter to verify that (L', M') is the unique maximal $\mathcal{f}_b$-regular extension of (L, M) to P, and so P is the $\mathcal{f}_b$-completion of P^*, and P' is a finite $\mathcal{f}_b$-refinement of P^*. Finally if $A \leq P^*$ is a sub-T-lattice, B a finite $\mathcal{f}_b$-refinement of A and (L', M') an extended real bi-ideal in P, then $(L' \wedge B, M' \cap B)$ is finite and hence real in B, since it is the extension to B of the real bi-ideal $(L' \cap A, M' \cap A)$ in A, and so every element of B is finite at (L', M'); that is $B \subseteq P'$. This completes the proof.

3.2. S-RESOLUTIONS AND DERIVATIVES.

Let S be a subset of the T-lattice P. In general S will not be a sub-T-lattice, but it will be convenient to assume throughout that S contains the distinguished element O. We will denote by $\langle S \rangle$ the smallest sub-T-lattice of P which contains S. The elements of $\langle S \rangle$ are obtained from those of S by a finite number of applications of the operations $\vee$, $\wedge$ and T_r.

Let $p : S \to \mathbb{R}$ be a function and define

$$L^p = \{ a \mid a \in \langle S \rangle,\ \exists\ a_1, \ldots, a_n \in S \text{ and } t > 0 \text{ with}$$
$$a \wedge t \leq \vee \{ T_{p(a_i)}(a_i) \} \vee 0 \}, \text{ and}$$

$$M^p = \{ a \mid a \in \langle S \rangle,\ \exists\ a_1, \ldots, a_n \in S \text{ and } t > 0 \text{ with}$$
$$a \vee (-t) \geq \wedge \{ T_{p(a_i)}(a_i) \} \wedge 0 \}.$$

Clearly L^p is an ideal in $\langle S \rangle$, and M^p a dual ideal. Also if $p(0) = 0$ we have $0 \in L^p \cap M^p$ and so (L^p, M^p) is a bi-ideal. We make the following definition:

<u>Definition</u> 3.2.1. The function $p : S \to \mathbb{R}$ is a <u>S-resolution</u> if $p(0) = 0$ and the bi-ideal (L^p, M^p) is $\mathcal{f}_e$-regular. (L^p, M^p) is then called the <u>S-derivative</u> corresponding to p.

We denote by R_S the set of all S-resolutions.

If (L, M) is a real bi-ideal in $\langle S \rangle$ then for each $a \in \langle S \rangle$ we have a unique real number, which we may denote by p(a), satisfying $T_r(a) \in L \Leftrightarrow r \geq p(a)$ and $T_r(a) \in M \Leftrightarrow r \leq p(a)$. p is clearly a $\langle S \rangle$-resolution, for $p(0) = 0$ and $(L^p, M^p) = (L, M)$ is $\mathcal{f}_b$-regular and hence certainly $\mathcal{f}_e$-regular. Let us characterize those $\langle S \rangle$-resolutions p whose derivatives are real in $\langle S \rangle$. Noting that $\mathbb{R}$ is a T-lattice for the translation $T_r(s) = s - r$ we may state:

Proposition 3.2.1. The $<S>$-resolution p has a real derivative if and only if p is a T-lattice homomorphism of $<S>$ into $\mathbb{R}$. We omit the proof, which is straightforward. We denote by $H_{<S>}$ the set of all T-lattice homomorphisms p of $<S>$ into $\mathbb{R}$ such that $p(0) = 0$. For $p \in H_{<S>}$ the derivative is specified more simply by

$$L^p = \{ a \mid a \in <S>, \ p(a) \leq 0 \}, \text{ and}$$

$$M^p = \{ a \mid a \in <S>, \ p(a) \geq 0 \}.$$

Note that $H_{<S>}$ is in one to one correspondence with the set of all real bi-ideals in $<S>$.

There is a natural relation between R_S and $H_{<S>}$, as follows. For $p \in H_{<S>}$ we may consider $p|S : S \to \mathbb{R}$. Clearly $p|S(0) = 0$ and $(L^{p|S}, M^{p|S}) \prec (L^p, M^p)$ so $(L^{p|S}, M^{p|S})$ is $\mathcal{f}_e$-regular, and hence $p|S \in R_S$. In this way we have a map $\vartheta : H_{<S>} \to R_S$ defined by $\vartheta(p) = p|S$, $p \in H_{<S>}$. Moreover:

Proposition 3.2.2. For each $p \in H_{<S>}$ we have $(L^{p|S}, M^{p|S}) = (L^p, M^p)$. In particular the mapping ϑ is injective.

Proof. Here and later let us denote by S_n the set of all elements of $<S>$ which may be expressed in terms of the elements of S using not more than n applications of the operations $\vee$, $\wedge$, T_r. For $A \subseteq <S>$ let us also set

$$T[A] = \{ T_r(a) \mid a \in A, \ r \in \mathbb{R} \}.$$

Consider the induction hypothesis

$P(n) : L^p \cap T[S_n] \subseteq L^{p|S}$.

Since $S_0 = S$ it is easy to see that $P(0)$ is true. On the other hand $P(n) \Rightarrow P(n+1)$ follows easily from the fact that p is a T-lattice homomorphism, and so $P(n)$ is true for each n. However

$\bigcup\{S_n\} = \langle S \rangle$, and we have shown that $L^p = L^{p|S}$. Likewise $M^p = M^{p|S}$, as required. Since a real bi-ideal clearly has a unique $\langle S \rangle$-resolution it follows at once that θ is injective.

Corollary. θ is a bijection if and only if every $\langle S \rangle$-resolution has a real derivative.

The following notion is useful in our study of S-derivatives.

Definition 3.2.2. Let L be an ideal in $\langle S \rangle$ and $F \subseteq \langle S \rangle$. Then L is F-prime if whenever $a_1, \ldots, a_n \in F$, $r_1, \ldots, r_n \in \mathbb{R}$ and $\bigwedge\{T_{r_i}(a_i)\} \in L$ then $T_{r_i}(a_i) \in L$ for some i, $1 \leq i \leq n$.

The notion of a F-prime dual ideal M is defined in a corresponding way, and (L, M) will be called F-prime if L and M are.

Proposition 3.2.3. The ideal L in $\langle S \rangle$ is prime if and only if it is S-prime.

Proof. Necessity is clear so let us show sufficiency. Let L be S-prime and make the following induction hypothesis.

P(n) : L is S_n-prime.

Since $S_0 = S$ we see that P(0) is valid. Assume P(n), and take $a_1, \ldots, a_m \in S_{n+1}$, $r_1, \ldots, r_m \in \mathbb{R}$ with $\bigwedge\{T_{r_i}(a_i)\} \in L$.

We may partition the set $I = \{1, 2, \ldots, m\}$ into three sets I_1, I_2, I_3 (one or more of which may be empty) so that

$$i \in I_1 \Rightarrow a_i = b_i \vee c_i \ ; \ b_i, c_i \in S_n,$$

$$i \in I_2 \Rightarrow a_i = b_i \wedge c_i \ ; \ b_i, c_i \in S_n, \text{ and}$$

$$i \in I_3 \Rightarrow a_i = T_{s_i}(b_i) \ ; \ b_i \in S_n, s_i \in \mathbb{R}.$$

With the understanding below that the term in question is to be removed if the index set involved is empty, we may write:

$$\bigwedge\{T_{r_i}(a_i)\} = \bigwedge\{T_{r_i}(b_i) \vee T_{r_i}(c_i) \mid i \in I_1\} \wedge a,$$

where

$$a = \bigwedge\{T_{r_i}(b_i) \wedge T_{r_i}(c_i) \mid i \in I_2\} \wedge \bigwedge\{T_{r_i+s_i}(b_i) \mid i \in I_3\}.$$

Consider a general element of $\langle S \rangle$ of the form

$$\bigwedge \{ T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \mid i \in J \} \wedge a',$$

where J is finite, b'_i, $c'_i \in S_n$, $r'_i \in R$ and a', if it is present, is a finite infimum of elements of $T[S_n]$. Make the following induction hypothesis on the finite number k of elements in J.

Q(k) : If $\bigwedge \{ T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \mid i \in J \} \wedge a' \in L$ and $|J| = k$ then $a' \in L$ or $T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \in L$ for some $i \in J$.

Q(0) is clear, so assume Q(k) and take $|J| = k + 1$. Choose $j \in J$, then:

$\bigwedge \{ T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \mid i \in J \} \wedge a' = e \vee f$, where

$$e = \bigwedge \{ T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \mid i \in J - \{j\}\} \wedge (T_{r'_j}(b'_j) \wedge a') \quad ,$$

$$f = \bigwedge \{ T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \mid i \in J - \{j\}\} \wedge (T_{r'_j}(c'_j) \wedge a') \quad .$$

Hence, if the above element belongs to L, we have $e \in L$ and $f \in L$, and so by Q(k) either $T_{r'_i}(b'_i) \vee T_{r'_i}(c'_i) \in L$ for some $i \in J - \{j\}$, or $T_{r'_j}(b'_j) \wedge a' \in L$, or $T_{r'_j}(c'_j) \wedge a' \in L$. In the last two cases we may apply P(n) and deduce that $T_{r'_j}(b'_j) \vee T_{r'_j}(c'_j) \in L$ or $a' \in L$, and we have verified Q(k+1). It follows that Q(k) is true for all k, and applying this result for $k = |I_1|$ to $\bigwedge \{ T_{r_i}(b_i) \vee T_{r_i}(c_i) \mid i \in I_1 \} \wedge a \in L$ gives $T_{r_i}(b_i) \vee T_{r_i}(c_i) \in L$ for some $i \in I_1$, or $a \in L$. In the first case we have $T_{r_i}(a_i) \in L$ directly, and in the second we may deduce from P(n) that $T_{r_i}(a_i) \in L$ for some $i \in I_2 \vee I_3$. This verifies P(n+1), and so L is S_n-prime for each n.

If now we take $a, b \in \langle S \rangle$ with $a \wedge b \in L$, then for sufficiently large n we have $a, b \in S_n$ and so $a \in L$ or $b \in L$, since L is S_n-prime. Hence L is prime in $\langle S \rangle$ as required.

A corresponding result holds for dual ideals. In particular we have:

Corollary. If p is a S-resolution then (L^p, M^p) is prime if and only if it is S-prime.

We now give some properties of S-derivatives.

Proposition 3.2.4. Every S-derivative is total.

Proof. Let p be a S-resolution, and make the induction hypothesis

$P(n) : T[S_n] \subseteq L^p \cup M^p$.

For $a = T_s(b) \in T[S_0] = T[S]$, $b \in S$, we have $a \in L^p$ if $s \geq p(b)$ and $a \in M^p$ if $s \leq p(b)$. Hence P(0) is valid.

Now assume P(n) is valid, and take $a \in T[S_{n+1} - S_n]$. If $a = T_s(b)$ and $b \in S_{n+1} - S_n$ has the form $b = b' \vee b''$ with $b', b'' \in S_n$ we have by hypothesis that $T_s(b') \in L^p$ or $T_s(b') \in M^p$ and $T_s(b'') \in L^p$ or $T_s(b'') \in M^p$. Now if $T_s(b') \in M^p$ or $T_s(b'') \in M^p$ then $a = T_s(b') \vee T_s(b'') \in M^p$, while if $T_s(b') \notin M^p$ and $T_s(b'') \notin M^p$ then $T_s(b') \in L^p$ and $T_s(b'') \in L^p$ so $a \in L^p$ as required. The other possible forms of b are dealt with in the same way, and the inductive proof of P(n) is complete. Since $\bigcup\{S_n\} = \langle S \rangle$ it follows at once that (L^p, M^p) is total.

Proposition 3.2.5. No element of $\langle S \rangle$ is infinite at a S-derivative.

Proof. It is a straightforward matter to verify the proposition

$P(n) : a \in S_n \Rightarrow \exists\ s, t \in \mathbb{R}$ with $T_s(a) \in L^p$ and $T_t(a) \in M^p$

for each n by induction. The result now follows from $\bigcup\{S_n\} = \langle S \rangle$, and Lemma 3.1.7.

Lemma 3.2.1. Take $p \in \mathbb{R}_S$ and let $q : S \to \mathbb{R}$ be a function so that $p(b) \neq q(b)$ for some $b \in S$, and $p(a) = q(a)$ for all $a \in S - \{b\}$.

Then:

(i) If $q(b) < p(b)$ and $T_{q(b)}(b) \in L^p$ then $q \in R_S$ and $L^p = L^q$.

(ii) If $q(b) > p(b)$ and $T_{q(b)}(b) \in M^p$ then $q \in R_S$ and $M^p = M^q$.

We omit the proof, which is straightforward.

Proposition 3.2.6. (a) The maps $p \longrightarrow L^p$ and $p \longrightarrow M^p$, $p \in R_s$, are both one to one if and only if for each $a \in S$ we have $T_r(a) \in L^p \Leftrightarrow r \geq p(a)$ and $T_r(a) \in M^p \Leftrightarrow r \leq p(a)$.

(b) The map $p \longrightarrow (L^p, M^p)$, $p \in R_s$, is one to one; that is each S-derivative is determined by a unique S-resolution, if and only if, given $a \in S$, we have $T_r(a) \in L^p \Leftrightarrow r \geq p(a)$ or $T_r(a) \in M^p \Leftrightarrow r \leq p(a)$.

Proof. (a) We show $p \longrightarrow L^p$ is one to one on R_S if and only if $T_r(a) \in L^p \Leftrightarrow r \geq p(a)$, the remaining case being similar. Now for $a \in S$ we have $T_{p(a)}(a) \in L^p$ so certainly $r \geq p(a) \Rightarrow T_r(a) \in L^p$. Hence we need only consider the reverse implication.

(i) Suppose that $p \longrightarrow L^p$ is I - I but that for some $b \in S$ we have $T_s(b) \in L^p$ for some $s < p(b)$. Define $q : S \longrightarrow \mathbb{R}$ by

$$q(a) = \begin{cases} p(a) \text{ if } a \neq b \\ s \text{ if } a = b. \end{cases}$$

Then by Lemma 3.2.1 (i) we have $q \in R_S$ and $L^p = L^q$. However $p \neq q$, and this contradicts our hypothesis. Hence $T_r(a) \in L^p \Rightarrow r \geq p(a)$ as required.

(ii) Suppose $T_r(a) \in L^p \Leftrightarrow r \geq p(a)$ for $p \in R_S$ and $a \in S$, and let $L^p = L^q$. Then for $a \in S$, $T_{p(a)}(a) \in L^p \Rightarrow T_{p(a)}(a) \in L^q \Rightarrow p(a) \geq q(a)$. Likewise $q(a) \leq p(a)$ so $p = q$, and $p \longrightarrow L^p$ is one to one as required.

(b) The proof is similar, and is omitted.

The property mentioned in (a) above is of importance in future applications, and we make it the subject of the next definition.

Definition 3.2.3. The S-derivative (L^p, M^p) is S-real if it satisfies $T_r(a) \in L^p \Leftrightarrow r \geq p(a)$ and $T_r(a) \in M^p \Leftrightarrow r \leq p(a)$ for each $a \in S$.

Lemma 3.2.2. Let L be a nearly prime ideal in $\langle S \rangle$ which satisfies the condition

$$T_r(a) \in L \quad \forall \quad r > k \quad \Rightarrow \quad T_k(a) \in L$$

for all $a \in S$. Then L is prime.

Proof. By Proposition 3.2.3 we need only show that L is S-prime. But if $a_1, \ldots, a_n \in S$; $r_1, \ldots, r_n \in \mathbb{R}$ and $\bigwedge\{T_{r_i}(a_i)\}$ $\in L$ then $T_{r_i}(a_i) \in L^+$ for some i, $1 \leq i \leq n$, and so $T_r(a_i) \in$ L for all $r > r_i$. Hence $T_{r_i}(a_i) \in L$ by the above hypothesis, and L is S-prime as required.

A corresponding result holds for dual ideals. In particular we have:

Corollary. Let the S-derivative (L^p, M^p) be S-real. Then L^p (respectively, M^p) is prime if and only if L^p (respectively, M^p) is nearly prime.

Finally let us note:

Proposition 3.2.7. Take $p \in R_S$. Then the following are equivalent.

(i) $\exists\ p' \in H_S$ with $p = p'|S$.

(ii) (L^p, M^p) is real.

(iii) (L^p, M^p) is S-real and prime.

(iv) (L^p, M^p) is S-real, and L^p is nearly prime or M^p is nearly prime.

(v) (L^p, M^p) is $\mathcal{f}_b$-regular.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.2.2 we have $(L^p, M^p) = (L^{p'}, M^{p'})$, and hence it is real.

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (iv). Immediate.

(iv) $\Rightarrow$ (v). Let (L^p, M^p) be S-real and L^p nearly prime. Then L^p is prime by the Corollary to Lemma 3.2.2. Now if (L^p, M^p) is not $\mathcal{f}_b$-regular we have $a \in \langle S \rangle$ and $s > 0$ with $a \in L^p$ and $T_s(a) \in M^p$, by Lemma 3.1.2. In particular we have $b_1, \ldots, b_m \in S$ and $t > 0$ so that

$$0 \wedge \bigwedge \{ T_{p(b_i)}(b_i) \} \leq T_s(a) \vee (-t).$$

Without loss of generality we may assume $t < s$, and so

$$t \wedge \bigwedge \{ T_{p(b_i)-t}(b_i) \} \leq a \vee 0.$$

It follows at once that $\bigwedge \{ T_{p(b_i)-t}(b_i) \} \in L^p$, and since L^p is prime this means $T_{p(b_i)-t}(b_i) \in L^p$ for some i, $1 \leq i \leq m$. However this contradicts the fact that (L^p, M^p) is S-real, and we deduce that (L^p, M^p) is $\mathcal{f}_b$-regular.

(v) $\Rightarrow$ (i). Let (L^p, M^p) be $\mathcal{f}_b$-regular. Since (L^p, M^p) is total by Proposition 3.2.4 it follows from Proposition 3.1.4 that $\langle S \rangle/(L^p, M^p)$ is totally ordered. Hence we may deduce from Proposition 3.2.5 that (L^p, M^p) is finite. If (L, M) is a maximal $\mathcal{f}_b$-regular refinement of (L^p, M^p) in $\langle S \rangle$, then (L, M) is real by Corollary 2 to Proposition 3.1.7, and hence $(L, M) = (L^{p'}, M^{p'})$ for some $p' \in H_{\langle S \rangle}$. However it is trivial to verify that $p = p' | S$, and the proof is complete.

3.3 BIREAL COMPACTNESS.

Throughout this section and the next (X, u, v) will denote a pairwise completely regular weakly pairwise T_o bitopological space, and we will denote by P(X) the set of all bicontinuous functions $f : (X, u, v) \rightarrow (R, s, t)$.

Note that the sets

$f^{-1}(-\infty, f(x) + r)$, $f \in P(X)$, $r > 0$

form a base of u-open nhds. of x, while the sets

$f^{-1}(f(x) - r, \infty)$, $f \in P(X)$, $r > 0$

form a base of v-open nhds. of x. Likewise the sets

$Z^+(f, r) = Z^+(f - r) = \{ x \mid f(x) \leq r \}$, $f \in P(X)$, $r > 0$

form a base of closed sets for the topology v, while the sets

$Z^-(f, r) = Z^-(f + r) = \{ x \mid f(x) \geq -r \}$, $f \in P(X)$, $r > 0$

form a base of closed sets for the topology u.

We note also that the functions in P(X) separate the points of X.

We shall say that $S \subseteq P(X)$ is <u>bigenerating</u> if $0 \in S$ and for each $x \in X$, $\{ f^{-1}(-\infty, f(x) + r) \mid f \in S, r > 0 \}$ is a subbase of the u-nhd. filter of x, and $\{ f^{-1}(f(x) - r, \infty) \mid f \in S, r > 0 \}$ is a subbase of the v-nhd. filter of x.

P(X) is a T-lattice under the translation $T_r(f) = f - r$, and so we may apply the notation and results of the last two sections. Let us first note:

<u>Proposition</u> 3.3.1. In the notation of Proposition 3.1.9 we have $P'(X) = P^*(X)$.

<u>Proof</u>. Take $f \in P'(X)$ and suppose, say, that f is not bounded above. Then if we set

$$L = \{ g \mid g \in P(X), g \leq 0 \},$$

$$M = \{ h \mid h \in P(X), \exists \ r \in \mathbb{R} \text{ with } T_r(f) \wedge 0 \leq h \}$$

it is easy to see that (L, M) is a $\mathcal{f}_b$-regular bi-ideal in P(X). If (L', M') is a maximal $\mathcal{f}_b$-regular refinement of (L, M) then $T_s(f) \in L' \wedge M'$ for some $s \in \mathbb{R}$. However $T_{s+1}(f) \in M \subseteq M'$, and

$(T_s(f)\vee 0)(\mathcal{f}_b)(T_{s+1}(f)\wedge 0)$, which is a contradiction. Likewise each element of $P'(X)$ is bounded below.

There is a natural link between the $\mathcal{f}_e$-regular bi-ideals in $P(X)$ and a certain family of bifilters on X, which we now describe. If (L, M) is a $\mathcal{f}_e$-regular bi-ideal in $P(X)$ (or, more generally, in a bigenerating sub-T-lattice of $P(X)$) we denote by $Z_b(L, M)$ the bifilter with base

$$\{ (Z^+(f, r), Z^-(g, r)) \mid f \in L,\ g \in M,\ r > 0 \}$$

It is clear that $Z_b(L, M)$ is ı-regular if and only if (L, M) is $\mathcal{f}_b$-regular.

In the opposite direction let $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_v$ be a bifilter on X and define

$$Z_b^{-1}(\mathcal{B}_u) = \{ f \mid f \in P(X),\ Z^+(f, r) \in \mathcal{B}_u \quad \forall\ r > 0 \},$$

$$Z_b^{-1}(\mathcal{B}_v) = \{ g \mid g \in P(X),\ Z^-(g, r) \in \mathcal{B}_v \quad \forall\ r > 0 \}.$$

It is immediate that $(Z_b^{-1}(\mathcal{B}_u), Z_b^{-1}(\mathcal{B}_v))$ is a $\mathcal{f}_e$-regular bi-ideal in $P(X)$, and we have the relations

$$Z_b(Z_b^{-1}(\mathcal{B}_u), Z_b^{-1}(\mathcal{B}_v)) \subseteq \mathcal{B} \qquad \ldots\ldots (1)$$

and $$(L, M) \prec (Z_b^{-1}(Z_b(L, M)_u), Z_b^{-1}(Z_b(L, M)_v)) \quad \ldots (2)$$

for all bifilters $\mathcal{B}$ and $\mathcal{f}_e$-regular bi-ideals (L, M).

If A is a bigenerating sub-T-lattice of $P(X)$ and the bifilter $\mathcal{B}$ satisfies

$$Z_b(Z_b^{-1}(\mathcal{B}_u)\cap A, Z_b^{-1}(\mathcal{B}_v)\cap A) = \mathcal{B}$$

we shall say that $\mathcal{B}$ is a $\underline{Z_b^A\text{ -bifilter}}$. The Z_b^A -bifilters are exactly those which have the form $\mathcal{B} = Z_b(L, M)$ for some $\mathcal{f}_e$-regular bi-ideal in A. Note in particular that the relations (1) and (2) imply that the maximal ı-regular Z_b^A -bifilters are in one to one correspondence with the maximal $\mathcal{f}_b$-regular bi-ideals in A.

On the other hand if the $\mathcal{f}_e$-regular bi-ideal (L, M) in A satisfies

$$(L, M) = (Z_b^{-1}(Z_b(L, M)_u) \cap A,\ Z_b^{-1}(Z_b(L, M)_v) \cap A)$$

we say it is a $\underline{Z_b^A}$ -bi-ideal. These are exactly the bi-ideals in A which have the form $(Z_b^{-1}(\mathcal{B}_u) \cap A,\ Z_b^{-1}(\mathcal{B}_v) \cap A)$ for some bifilter $\mathcal{B}$. It is not difficult to verify that the Z_b^A - bi-ideals are characterized among the $\mathcal{f}_e$-regular bi-ideals by the condition

$$(L, M) = (L^+, M^+).$$

In particular every maximal $\mathcal{f}_b$-regular bi-ideal in A is a Z_b^A -bi-ideal.

For $x \in X$ let us set

$$L(x) = \{ f \mid f \in P(X),\ f(x) \leq 0 \},$$

$$M(x) = \{ f \mid f \in P(X),\ f(x) \geq 0 \}.$$

It is clear that (L(x), M(x)) is a real bi-ideal in P(X), and that $Z_b(L(x), M(x))$ is the nhd. bifilter of x. Consequently if (L, M) is a Z_b^A -bi-ideal in A, $Z_b(L, M)$ will converge to $x \in X$ if and only if $L(x) \cap A \subseteq L$ and $M(x) \cap A \subseteq M$. We express this by saying that <u>(L, M) is fixed by x</u>. Since we shall, in practice, usually apply this condition to $\mathcal{f}_b$-regular bi-ideals the above condition will then be equivalent to $L(x) \cap A = L$ and $M(x) \cap A = M$.

For $x \in X$ we define the function

$$\hat{x} : P(X) \longrightarrow \mathbb{R}$$

by $\hat{x}(f) = f(x)\ \ \forall\ \ f \in P(X)$. Clearly $\hat{x} \in H_{P(X)}$ for all $x \in X$. We note the following result.

<u>Lemma</u> 3.3.1. For each $x \in X$ and $S \subseteq P(X)$ we have

$$(L^{\hat{x}|S}, M^{\hat{x}|S}) = (L^{\hat{x}|\langle S \rangle}, M^{\hat{x}|\langle S \rangle}) = (L(x) \cap \langle S \rangle, M(x) \cap \langle S \rangle).$$

<u>Proof</u>. This is immediate from the definitions and Proposition

3.2.2.

We now give a characterization of the bigenerating subsets of $P(X)$.

Proposition 3.3.2. $S \subseteq P(X)$ is bigenerating if and only if $0 \in S$ and for each $x \in X$ the bi-ideal

$$([(L^{\hat{x}|S})_{P(X)}]^+, [(M^{\hat{x}|S})_{P(X)}]^+)$$

is fixed by x.

Proof. Suppose S is bigenerating, take $f \in L(x)$ and $s > 0$. Since $f^{-1}(-\infty, f(x) + s)$ is a u-nhd. of x we have $f_1, \ldots, f_n \in S$ and $t > 0$ so that

$$\bigcap\{f_i^{-1}(-\infty, f_i(x) + 2t)\} \subseteq f^{-1}(-\infty, f(x) + s).$$

It follows from this that

$$T_s(f) \wedge t \leq \bigvee\{T_{f_i(x)}(f_i) \vee 0\},$$

and since $\hat{x}|S(f_i) = f_i(x)$ this means that

$$T_s(f) \wedge t \leq \bigvee\{T_{\hat{x}|S(f_i)}(f_i) \vee 0\}.$$

Hence $T_s(f) \in (L^{\hat{x}|S})_{P(X)}$, and since $s > 0$ was arbitrary we have established

$$L(x) \subseteq [(L^{\hat{x}|S})_{P(X)}]^+.$$

The other part of the result is proved likewise.

Conversely suppose that $([(L^{\hat{x}|S})_{P(X)}]^+, [(M^{\hat{x}|S})_{P(X)}]^+)$ is fixed by x for each $x \in X$. To show that S generates the topology u it will suffice to show that for $x \in X$, $f \in P(X)$ and $s > 0$ we have $f_1, \ldots, f_n \in S$ and $t > 0$ so that

$$\bigcap\{f_i^{-1}(-\infty, f_i(x) + t)\} \subseteq f^{-1}(-\infty, f(x) + s) \quad \ldots\ldots (3).$$

Now $T_{f(x)}(f) \in L(x) \subseteq [(L^{\hat{x}|S})_{P(X)}]^+$, and so $T_{s/2}\, T_{f(x)}(f) \in (L^{\hat{x}|S})_{P(X)}$. Hence we have $g \in L^{\hat{x}|S}$ and $t' > 0$ with

$$T_{f(x)+s/2}(f) \wedge t' \leq g.$$

Also we have $f_1, \ldots, f_n \in S$ and $t'' > 0$ with

$$g \wedge t'' \leq \bigvee \{ T_{\hat{x}|S(f_i)}(f_i) \vee 0 \}.$$

It is now easy to verify (3) for this $f_1, \ldots, f_n$ and $t = t' \wedge t'' \wedge (s/2)$, and so S generates the topology u. Likewise S generates v, and hence if $0 \in S$ it is bigenerating.

On analogy with the definition of real compactness for topological spaces (see, for example, [16]) we now give:

Definition 3.3.1. Let $S \subseteq P(X)$ be bigenerating. Then (X, u, v) is S-bireal compact if every real bi-ideal in $\langle S \rangle$ is fixed. A P(X)-bireal compact space will be called bireal compact for short.

Proposition 3.3.2. Let S, W be bigenerating subsets of P(X) with $\langle S \rangle \subseteq \langle W \rangle$. Then if (X, u, v) is S-bireal compact it is W-bireal compact.

Proof. Let (L, M) be a real bi-ideal in $\langle W \rangle$. Then $(L \cap \langle S \rangle, M \cap \langle S \rangle)$ is a real bi-ideal in $\langle S \rangle$, and hence fixed by some $x \in X$. Hence

$$(L^{\hat{x}|S}, M^{\hat{x}|S}) = (L(x) \cap \langle S \rangle, M(x) \cap \langle S \rangle) = (L \cap \langle S \rangle, M \cap \langle S \rangle)$$

by Lemma 3.3.1. Since (L, M) is a maximal $\mathcal{f}_b$-regular extension of the above bi-ideal to $\langle W \rangle$ we have $(L^{\hat{x}|S})_{\langle W \rangle} \subseteq L$ and $(M^{\hat{x}|S})_{\langle W \rangle} \subseteq M$. Finally $L = L^+$ and $M = M^+$ and so

$$\left(\left[(L^{\hat{x}|S})_{\langle W \rangle} \right]^+, \left[(M^{\hat{x}|S})_{\langle W \rangle} \right]^+ \right) \prec (L, M).$$

However W is bigenerating and hence it follows by Proposition 3.3.2 that (L, M) is fixed by x.

In particular a S-bireal compact space is bireal compact. In the opposite direction we have:

Theorem 3.3.1. Let S and W be bigenerating subsets of P(X) with $\langle S \rangle \subseteq \langle W \rangle$, and suppose that (X, u, v) is W-bireal compact.

Then the following are equivalent:

(i) (X, u, v) is S-bireal compact.

(ii) $(L_{\langle W\rangle}, M_{\langle W\rangle})$ is finite for all real bi-ideals (L, M) in $\langle S\rangle$.

(iii) $\langle W\rangle$ is a finite $\mathcal{f}_b$-refinement of $\langle S\rangle$ in $P(X)$.

Proof. (i) $\Rightarrow$ (ii). Let (L, M) be a real bi-ideal in $\langle S\rangle$. Then (L, M) is fixed by some $x \in X$, and so by Lemma 2.3.1 we have

$$(L_{\langle W\rangle}, M_{\langle W\rangle}) = ((L^{\hat{x}|S})_{\langle W\rangle}, (M^{\hat{x}|S})_{\langle W\rangle}) \quad \ldots. (4).$$

On the other hand $S \subseteq P(X)$ is bigenerating, and we may deduce at once from Proposition 3.3.2 that

$$([(L^{\hat{x}|S})_{\langle W\rangle}]^+, [(M^{\hat{x}|S})_{\langle W\rangle}]^+) = (L(x) \cap \langle W\rangle, M(x) \cap \langle W\rangle).$$

Comparing this with (4) gives

$$(L_{\langle W\rangle}{}^+, M_{\langle W\rangle}{}^+) = (L(x) \cap \langle W\rangle, M(x) \cap \langle W\rangle),$$

and hence $(L_{\langle W\rangle}{}^+, M_{\langle W\rangle}{}^+)$ is real in $\langle W\rangle$, and so, in particular, finite. Hence $(L_{\langle W\rangle}, M_{\langle W\rangle})$ is finite also.

(ii) $\Rightarrow$ (iii). Let (L, M) be a real bi-ideal in $\langle S\rangle$, and (L', M') any maximal $\mathcal{f}_b$-regular extension of (L, M) to $\langle W\rangle$. Since $(L_{\langle W\rangle}, M_{\langle W\rangle}) \prec (L', M')$ we see that (L', M') is real by Corollary 2 to Proposition 3.1.7. However (X, u, v) is W-bireal compact so (L', M') is fixed by some $x \in X$, that is $(L', M') = (L(x) \cap \langle W\rangle, M(x) \cap \langle W\rangle)$. In particular $(L, M) = (L(x) \cap \langle S\rangle, M(x) \cap \langle S\rangle)$. Now since S is bigenerating the functions in S separate the points of X and it follows at once that $(L(x) \cap \langle S\rangle, M(x) \cap \langle S\rangle) = (L(y) \cap \langle S\rangle, M(y) \cap \langle S\rangle) \Rightarrow x = y$. Hence $(L(x) \cap \langle W\rangle, M(x) \cap \langle W\rangle)$ is the unique real extension of (L, M) to $\langle W\rangle$, and it follows that $\langle W\rangle$ is a finite $\mathcal{f}_b$-refinement of $\langle S\rangle$ in $P(X)$.

(iii) $\Rightarrow$ (i). Let (L, M) be a real bi-ideal in $\langle S\rangle$ and (L', M') its unique real extension to $\langle W\rangle$. By hypothesis (L', M') is fixed by some $x \in X$, and clearly (L, M) is fixed by the same x. Hence (X, u, v) is S-bireal compact.

We shall find it convenient to study the relation between (X, u, v) and the bitopological subspace $(H_{\langle S\rangle}, u_S, v_S)$ of $(\prod_{f\in\langle S\rangle} \mathbb{R}_f, \prod_{f\in\langle S\rangle} s_f, \prod_{f\in\langle S\rangle} t_f)$ where for each $f\in\langle S\rangle$, $\mathbb{R}_f = \mathbb{R}$, $s_f = s$ and $t_f = t$; S being a bigenerating subset of P(X). We denote by π_f the projection

$$\pi_f : \prod_{f\in\langle S\rangle} \mathbb{R}_f \longrightarrow \mathbb{R}_f = \mathbb{R}$$

defined by $\pi_f(g) = g(f)$, and also the restriction of this projection to the subset $H_{\langle S\rangle}$. First let us note the following:

<u>Lemma</u> 3.3.2. (a) The map $\sigma_{\langle S\rangle} : X \longrightarrow H_{\langle S\rangle}$ defined by $\sigma_{\langle S\rangle}(x) = \hat{x}|\langle S\rangle$ is a bitopological homeomorphism of (X, u, v) onto a uniformly dense subset of $(H_{\langle S\rangle}, u_S, v_S)$.

(b) $H_{\langle S\rangle}$ is a uniformly closed subset of $\prod_{f\in\langle S\rangle} \mathbb{R}_f$.

<u>Proof</u>. (a) It is a straightforward matter to verify that $\sigma_{\langle S\rangle}$ is a bitopological homeomorphism into $(H_{\langle S\rangle}, u_S, v_S)$, and the uniform density of $\sigma_{\langle S\rangle}(X)$ in $H_{\langle S\rangle}$ is an easy consequence of the fact that the bifilter $Z_b(L^p, M^p)$ is 1-regular for each $p \in H_{\langle S\rangle}$.

(b) Let $g \in \prod_{f\in\langle S\rangle} \mathbb{R}_f$ be in the uniform closure of $H_{\langle S\rangle}$. If $g(0) = t > 0$ then $\pi_0^{-1}(\pi_0(g) - t, \infty)$ meets $H_{\langle S\rangle}$, say in a point p, and we obtain the contradiction $g(0) - t = \pi_0(g) - t < \pi_0(p) = p(0) = 0$. Likewise $g(0) < 0$ is impossible and so $g(0) = 0$. Now let us establish that

$$g(f \vee f') = g(f) \vee g(f')$$

for all $f, f' \in \langle S\rangle$. Without loss of generality we may suppose $g(f') \leq g(f)$, that is $g(f) \vee g(f') = g(f)$. Suppose that $g(f\vee f') < g(f)$, then we may set $g(f) = g(f\vee f') + 2t$, where $t > 0$. But then the uniform nhd.

$$\pi^{-1}_{f\vee f'}(-\infty,\ \pi_{f\vee f'}(g) + t) \cap \pi^{-1}_{f}(\pi_f(g) - t,\ \infty)$$

of g meets $H_{<S>}$, say in a point p, and we then have the contradiction

$g(f) - t < p(f) \leq p(f) \vee p(f') = p(f \vee f') < g(f \vee f') + t.$

In the same way we obtain a contradiction if $g(f \vee f') > g(f)$, and the result is established. In the same way we may show that

$$g(f \wedge f') = g(f) \wedge g(f') \text{ and } g(T_r(f)) = T_r(g(f))$$

for all f, $f' \in <S>$, $r \in \mathbb{R}$. Hence $g \in H_{<S>}$, and $H_{<S>}$ is uniformly closed as stated.

If (X, u, v) is S-bireal compact then of course $\mathcal{O}_{<S>}(X) = H_{<S>}$, and we have shown in particular that a bireal compact space may always be embedded as a uniformly closed subset of a product of copies of the space $(\mathbb{R}, s, t)$. In fact this property is characteristic of the bireal compact spaces, as the next theorem shows.

<u>Theorem</u> 3.3.2. (X, u, v) is bireal compact if and only if it is bitopologically homeamorphic to a uniformly closed subspace of a product of copies of $(\mathbb{R}, s, t)$.

<u>Proof</u>. Necessity has been established above, so it remains only to show the sufficiency. Without loss of generality we may suppose X is itself a uniformly closed subspace of $(\prod R_\alpha,\ \prod s_\alpha,\ \prod t_\alpha)$. If we denote the projections by π_α then $S = \{\pi_\alpha | X\} \cup \{0\}$ is a bigenerating subset of P(X). By Proposition 3.3.2 it will be sufficient to show that (X, u, v) is S-bireal compact. Let (L, M) be a real bi-ideal in $<S>$, and let $p \in H_{<S>}$ be the $<S>$-resolution of (L, M). For each α let $x_\alpha = p(\pi_\alpha | X)$, and consider the element $x = (x_\alpha)$ of $\prod R_\alpha$. We show first that $x \in X$. Suppose this is not so, then we have α_i, $1 \leq i \leq n$ and α_j, $n + 1 \leq j \leq m$, and $t > 0$ so that

$$X \cap \bigcap \{\pi_{\alpha_i}^{-1}(-\infty,\ \pi_{\alpha_i}(x) + t)\} \cap \bigcap \{\pi_{\alpha_j}^{-1}(\pi_{\alpha_j}(x) - t,\ \infty)\} = \emptyset \quad .. (4).$$

Now $[\bigvee\{(\pi_{\alpha_i} - x_{\alpha_i})\}]\vee 0 \in L$, and (L, M) is $\mathcal{f}_b$-regular so $T_t([\bigvee\{(\pi_{\alpha_i} - x_{\alpha_i})\}]\vee 0) \notin M$ by Lemma 3.1.2. However $[\bigwedge\{(\pi_{\alpha_j} - x_{\alpha_j})\}]\wedge 0 \in M$ and so

$$[\bigwedge\{(\pi_{\alpha_j} - x_{\alpha_j})\}]\wedge 0 \nleq T_t([\bigvee\{(\pi_{\alpha_i} - x_{\alpha_i})\}]\vee 0).$$

Hence we have $y \in X$ so that for all i, j we have

$$(\pi_{\alpha_j}(y) - x_{\alpha_j})\wedge 0 > (\pi_{\alpha_i}(y) - x_{\alpha_i})\vee 0 - t.$$

But then y belongs to the set on the left hand side of (4), and this contradiction shows $x \in X$.

For $f \in S$ it is immediate from the definition that $f(x) = p(f)$, and the same equality may be deduced for all $f \in \langle S\rangle$ by using a simple induction argument on the form of the elements in $\langle S\rangle$. Hence

$$(L(x)\cap\langle S\rangle,\ M(x)\cap\langle S\rangle) = (L^p, M^p) = (L, M)$$

and (L, M) is fixed in X as required.

Corollary 1. If (X, u, v) is bireal compact it is uniformly real compact.

Corollary 2. Let $S \subseteq P(X)$ be bigenerating and set $\pi_{\langle S\rangle} = \{\pi_f \mid f \in \langle S\rangle\}$. Then $(H_{\langle S\rangle}, u_S, v_S)$ is $\pi_{\langle S\rangle}$-bireal compact.

Corollary 3. Any bireal extension of (X, u, v) in which X is uniformly dense is bitopologically homeomorphic to $(H_{\langle S\rangle}, u_S, v_S)$ for some bigenerating subset of $P(X)$.

Proof. Let (X', u', v') be a bireal compact extension of (X, u, v) in which X is uniformly dense. Without loss of generality (up to a bitopological homeomorphism) we may assume $X \subseteq X'$. By the theorem we know that (X', u', v') is bitopologically homeomorphic to $(H_{P(X')}, u_{P(X')}, v_{P(X')})$. On the other hand $S = \{f'|X \mid f' \in P(X')\}$ is a bigenerating subset (actually, sub-T-lattice) of $P(X)$, and it is a trivial matter to verify that $(H_{P(X')}, u_{P(X')}, v_{P(X')})$ is bitopologically homeomorphic with (H_S, u_S, v_S) under the correspondence $p' \rightarrow p$ defined by $p(f'|X) = p'(f')\ \forall\ f' \in P(X')$. This establishes the result.

Definitions of bitopological real compactness have been given by SAEGROVE [31], and by BRUMMER and SALBANY [5]. Saegrove's pair real compactness is considered in more detail in the next section. At the time of writing only the review of [5] in Zbl. für Math, 371, is available to the author, but there are clear parallels with some of the results obtained here, as the reader will observe.

An important special case of the above notion of S-bireal compactness is covered by the following:

<u>Proposition</u> 3.3.3. The following are equivalent for (X, u, v):

(i) $P^*(X)$-bireal compact.

(ii) S-bireal compact for all bigenerating $S \subseteq P^*(X)$.

(iii) Uniformly compact.

<u>Proof</u>. (i) $\Rightarrow$ (ii) follows from Theorem 3.3.1 since all $\mathcal{f}_e$-regular bi-ideals in $P^*(X)$ are finite, and (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (iii). If (X, u, v) is $P^*(X)$-bireal compact then (X, u, v) is bitopologically homeomorphic to $(H_{P^*(X)}, u_{P^*(X)}, v_{P^*(X)})$ by Lemma 3.3.2(a). On the other hand the proof of Lemma 3.3.2 (b) makes it clear that $H_{P^*(X)}$ is in fact a uniformly closed subset of a product of bounded closed intervals in $\mathbb{R}$, and hence is uniformly compact.

(iii) $\Rightarrow$ (i). Suppose (X, u, v) is uniformly compact, and let (L, M) be a real bi-ideal in $P^*(X)$. Then $Z_b(L, M)$ is an $\imath$-regular bifilter, and if $\mathcal{B}$ is a maximal $\imath$-regular bifilter refinement of $Z_b(L, M)$ we have $\mathcal{B} \rightarrow x$ for some $x \in X$ by Lemma 1.7.2 (c). Hence $Z_b(L(x), M(x)) \subseteq \mathcal{B}$, and it is easy to deduce that (L, M) is fixed by x. Thus (X, u, v) is $P^*(X)$-bireal compact.

For a bigenerating sub-T-lattice $A \subseteq P(X)$ define $\nu : P(H_A) \rightarrow P(X)$ by $\nu(g) = \check{g}$, $g \in P(H_A)$, where $\check{g}(x) = g(\hat{x}|A)$, $\forall\ x \in X$.

<u>Lemma</u> 3.3.3. (i) ν is an injective T-lattice homomorphism.

(ii) $\nu(P(H_A))$ is a finite $\mathcal{f}_b$-refinement of A.

<u>Proof</u>. (i). ν is clearly a T-lattice homomorphism, and the fact

that it is one to one follows at once from the uniform density of $\sigma_A(X)$ in H_A.

(ii) First $A \subseteq \nu(P(H_A))$ since if $f \in A$ then $\pi_f \in P(H_A)$ and $f = \nu(\pi_f)$. Now let (L, M) be a real bi-ideal in A; then $(L, M) = (L^p, M^p)$ where $p \in H_A$ is the A-resolution of (L, M). Corresponding to p we have $\hat{p} \in H_{P(H_A)}$ defined by $\hat{p}(g) = g(p)$ for all $g \in P(H_A)$. $(L^{\hat{p}}, M^{\hat{p}})$ is a real bi-ideal in $P(H_A)$, and therefore $(\nu(L^{\hat{p}}), \nu(M^{\hat{p}}))$ is a real bi-ideal in $\nu(P(H_A))$ by (i). However it is easy to verify that

$$\nu(L^{\hat{p}}) \cap A = L \quad \text{and} \quad \nu(M^{\hat{p}}) \cap A = M$$

so $(\nu(L^{\hat{p}}), \nu(M^{\hat{p}}))$ is a real extension of (L, M) to $\nu(P(H_A))$. On the other hand if (L', M') is any real extension of (L, M) to $\nu(P(H_A))$ then $(\nu^{-1}(L'), \nu^{-1}(M'))$ is a real bi-ideal in $P(H_A)$, and $H_{P(H_A)}$ is bireal compact, so it is fixed by some $q \in H_A$. It follows that $(L', M') = (\nu(L^{\hat{q}}), \nu(M^{\hat{q}}))$, while $(L^p, M^p) = (L' \cap A, M' \cap A) = (\nu(L^{\hat{q}}) \cap A, \nu(M^{\hat{q}}) \cap A) = (L^q, M^q)$ implies $p = q$ and hence $\hat{p} = \hat{q}$. Thus (L', M') is unique, and $\nu(P(H_A))$ is a finite $\mathcal{f}_b$-refinement of A as required.

<u>Corollary</u>. If (X, u, v) is uniformly compact then $P(X) = P^*(X)$.

<u>Proof</u>. By the lemma $\nu(P(H_{P^*(X)}))$ is a finite $\mathcal{f}_b$-refinement of $P^*(X)$ and so $\nu(P(H_{P^*(X)})) = P^*(X)$ by Propositions 3.1.9 and 3.3.1. It follows at once that $P(H_{P^*(X)}) = P^*(H_{P^*(X)})$. However if (X, u, v) is $P^*(X)$-bireal compact then $H_{P^*(X)}$ and X are bitopologically homeomorphic so $P(X) = P^*(X)$ also.

Of course the above proof of the pseudo compactness of a uniformly compact bitopological space is of technical interest only, and an elementary topological proof of this result is easily given.

Note that the above results imply in particular that the separated uniform compactifications of (X, u, v) in which X is uniformly dense are, up to bitopological homeomorphism, exactly the spaces (H_A, u_A, v_A), where A is any bigenerating sub-T-lattice of $P^*(X)$.

Proposition 3.3.4. Let A be a bigenerating sub-T-lattice of $P(X)$ and A' a finite $\mathscr{f}_b$-refinement of A in $P(X)$. Then the following are equivalent:

(i) $A' \subseteq \nu(P(H_A))$,

(ii) $(L_{A'}, M_{A'})$ is almost prime for every real bi-ideal (L, M) in A.

Proof. (i) $\Rightarrow$ (ii). Suppose $A \subseteq A' \subseteq \nu(P(H_A))$ and let $(L, M) = (L^p, M^p)$, $p \in H_A$, be a real bi-ideal in A. Let $\hat{p} \in H_{P(H_A)}$ be defined as in the proof of Lemma 3.3.3. By definition $\pi_A = \{\pi_f \mid f \in A\}$ is bigenerating for the space H_A and so by Proposition 3.3.2 we have

$$\left(\left[(L^{\hat{p}}|\pi_A)_{P(H_A)}\right]^+, \left[(M^{\hat{p}}|\pi_A)_{P(H_A)}\right]^+\right) = (L^{\hat{p}}, M^{\hat{p}}).$$

It follows by Theorem 3.1.2 that

$$((L^{\hat{p}}|\pi_A)_{P(H_A)}, (M^{\hat{p}}|\pi_A)_{P(H_A)})$$

is nearly total. Take $g \in A' \subseteq \nu(P(H_A))$; then $g = \nu(h) = \check{h}$ for some $h \in P(H_A)$. Suppose, for instance, that

$$T_s(h) \in (L^{\hat{p}}|\pi_A)_{P(H_A)} \quad \forall \quad s > 0.$$

Then for each $s > 0$ we have $t > 0$ and $f \in A$ with $\pi_f \in L^{\hat{p}}|\pi_A$ so that $T_s(h) \wedge t \preceq \pi_f$. But then $T_s(g) \wedge t \preccurlyeq f$, and $f \in L^p$, so $T_s(g) \in (L^p)_{A'} = L_{A'}$ for all $s > 0$. We deduce that $(L_{A'}, M_{A'})$ is nearly total, and hence nearly prime by Proposition 3.1.3.

(ii) $\Rightarrow$ (i) For $f' \in A'$ define $f : H_A \longrightarrow \mathbb{R}$ by

$$f(p'|A) = p'(f') \quad \text{for all } p' \in H_{A'}.$$

It is clear from the definition that if $f \in P(H_A)$ then $\nu(f) = f'$, so it remains only to show the former property. Take $p = p'|A \in H_A$, $p' \in H_{A'}$, and $s > 0$. Since $(L^p{}_{A'}, M^p{}_{A'})$ is nearly prime we have by Corollary 3 to Proposition 3.1.7 that $T_{p'(f')+s/2}(f') \in L^p{}_{A'}$. Hence for some $0 < t < s/2$ and $g \in L^p$ we have

$$T_{p'(f')+s/2}(f') \wedge t \leq g.$$

We deduce at once that

$$\pi_g^{-1}(-\infty, \pi_g(p) + t) \subseteq f^{-1}(-\infty, f(p) + s)$$

so f is continuous for the first topologies. Likewise it is continuous for the second topologies, that is $f \in P(H_A)$ as required.

The above result suggests the following definition.

<u>Definition</u> 3.3.2. A' is a <u>finite $\mathcal{f}_b$-prime-refinement</u> of A if it is a finite $\mathcal{f}_b$-refinement and $(L_{A'}, M_{A'})$ is nearly prime in A' for all real bi-ideals (L, M) in A.

The relation of being a finite $\mathcal{f}_b$-prime-refinement is easily seen to be transitive, and so we may state the following corollary to Proposition 3.3.4.

<u>Corollary</u> 1. If $A \subseteq P(X)$ is a bigenerating sub-T-lattice then $\nu(P(H_A))$ is the finite $\mathcal{f}_b$-prime-completion af A.

We also have:

<u>Corollary</u> 2. Properties (i) and (ii) of Proposition 3.3.4 are each equivalent to

(iii) (H_A, u_A, v_A) is bitopologically homeomorphic to $(H_{A'}, u_{A'}, v_{A'})$ under the correspondence $p = p'|A \leftrightarrow p'$.

Proof. If $p \leftrightarrow p'$ is a bitopological homeomorphism then $P(H_A)$ and $P(H_{A'})$ are isomorphic T-lattices under the correspondence $f \leftrightarrow f'$ defined by $f(p) = f'(p')$. It follows at once that $A' \subseteq \nu'(P(H_{A'})) = \nu(P(H_A))$, using an obvious notation. To establish the converse it will clearly suffice to show (H_A, u_A, v_A) is bitopologically homeomorphic to $(H_{A''}, u_{A''}, v_{A''})$ for the case $A'' = \nu(P(H_A))$. Now by Lemma 3.3.3 (ii), and Proposition 3.3.4, applied to A'' in place of A we know $\nu''(P(H_{A''}))$ is a finite $\mathscr{F}_b$-prime-refinement of A''. However $\nu(P(H_A))$ is finitely $\mathscr{F}_b$-prime-complete so $\nu(P(H_A)) = \nu''(P(H_{A''}))$, and $P(H_A)$ and $P(H_{A''})$ are isomorphic T-lattices under the correspondence $f \leftrightarrow f''$ defined by $f(p) = f''(p'')$. It follows that $p \leftrightarrow p''$ is a bitopological homeomorphism as required.

The above results show that the bireal compactifications of (X, u, v) in which X is uniformly dense are in one to one correspondence with the subsets $\nu(P(H_A))$ of $P(X)$ for A bigenerating, and that these are characterized internally amongst the bigenerating sub-T-lattices of $P(X)$ by the requirement that they are finitely $\mathscr{F}_b$-prime-complete. It will be noted that if (X, u, v) is bireal compact then the bigenerating sub-T-lattices of $P(X)$ may themselves be characterised in terms of the internal lattice structure of $P(X)$, one such characterization being obtained explicitly by applying Proposition 3.3.2 to the bitopologically homeomorphic space $(H_{P(X)}, u_{P(X)}, v_{P(X)})$. In this case all the bireal compact extensions of (X, u, v) in which X is uniformly dense, including the space itself, of course, can be obtained explicitly from the lattice structure of $P(X)$.

In case A is a bigenerating sub-T-lattice of $P^*(X)$ we may state an alternative form of Corollary 2. First we need the following lemma, which is of some interest in itself.

Lemma 3.3.4. Let (X, u, v) be preseparated, (X', u', v') uniformly compact and $\varphi : X' \to X$ a bijective bicontinuous mapping. Further suppose that for each $x' \in X'$ we have

$$\varphi(u'\text{-cl}\{x'\}) = u\text{-cl}\{\varphi(x')\}, \text{ and}$$

$$\varphi(v'\text{-cl}\{x'\}) = v\text{-cl}\{\varphi(x')\}.$$

Then φ is a bitopological homeomorphism.

Proof. Let F be a u'-closed subset of X'. By hypothesis F is (in particular) v'-compact, and hence $\varphi(F)$ is v-compact in X. Take $x \in X - \varphi(F)$ and $x' \in X'$ with $\varphi(x') = x$. Then $x' \notin F$ and so if $y' \in F$ we have $x' \notin u'\text{-cl}\{y'\}$ since F is u'-closed. Hence, by hypothesis, $x = \varphi(x') \notin \varphi(u'\text{-cl}\{y'\}) = u\text{-cl}\{\varphi(y')\}$, and since (X, u, v) is preseparated we have $x \in U(y') \in u$ and $\varphi(y') \in V(y') \in v$ with $U(y') \cap V(y') = \emptyset$. The sets $V(y')$, $y' \in F$, cover $\varphi(F)$, and so we have $y'_1, \ldots, y'_n \in F$ so that $\varphi(F)$ is covered by $V(y'_1), \ldots, V(y'_n)$. But then $\bigcap\{U(y'_i) \mid i = 1, \ldots, n\}$ is a u-open nhd. of x which does not meet $\varphi(F)$, and $\varphi(F)$ is u-closed. In just the same way φ maps v'-closed sets to v-closed sets, and we have shown that φ is a bitopological homeomorphism, as required.

We may now give at once:

Corollary 3. If A is a bigenerating sub-T-lattice of $P^*(X)$ then the following conditions are equivalent to each other and to (i) and (ii) of Proposition 3.3.4.

(iv) $p'|A \preceq q'|A \Leftrightarrow p' \leq q'$ for all $p', q' \in H_{A'}$.

(v) $q' \in u_{A'}\text{-cl}\{p'\} \Leftrightarrow q'|A \in u_A\text{-cl}\{p'|A\}$, and

$q' \in v_{A'}\text{-cl}\{p'\} \Leftrightarrow q'|A \in v_A\text{-cl}\{p'|A\}$

for all $p', q' \in H_{A'}$.

Let us denote by $\mathfrak{A}$ the set of all bigenerating finitely $\mathcal{f}_b$-prime-complete sub-T-lattices of P(X), and set $\mathfrak{A}^* = \{A \mid A \in \mathfrak{A}, A \subseteq P^*(X)\}$. For each $B \in \mathfrak{A}$ which is finitely $\mathcal{f}_b$-complete let $\mathfrak{A}_B = \{A \mid A \in \mathfrak{A}$, B is a finite $\mathcal{f}_b$-completion of $A\}$. By Theorem 3.1.3 the sets $\mathfrak{A}_B$ form a partition of $\mathfrak{A}$.

The set $\mathcal{A}$ is partially ordered by set inclusion, and indeed it is an upper semi-lattice, the least upper bound of A, A' $\in$ $\mathcal{A}$ being the finite $\mathcal{P}_b$-prime-completion of $\langle A \cup A' \rangle$. Likewise $\mathcal{A}^*$ and each $\mathcal{A}_{\mathcal{B}}$ is a sub upper semi lattice of $\mathcal{A}$.

Take A, A' $\in \mathcal{A}$. As usual we shall say that a mapping $\varphi : H_{A'} \to H_A$ <u>preserves X</u> if $\varphi(\sigma_{A'}(x)) = \sigma_A(x)$ for all $x \in X$. We may now note:

<u>Lemma</u> 3.3.5. Take A, A' $\in \mathcal{A}$. Then $A \subseteq A'$ if and only if there exists a bicontinuous mapping $\varphi : (H_{A'}, u_{A'}, v_{A'}) \to (H_A, u_A, v_A)$ which preserves X.

<u>Proof</u>. If $A \subseteq A'$ the required function is clearly $\varphi(p') = p'|A$. Conversely let $\varphi : (H_{A'}, u_{A'}, v_{A'}) \to (H_A, u_A, v_A)$ be bicontinuous and preserve X. For $f \in A$ define $f' : H_{A'} \to \mathbb{R}$ by

$$f'(p') = \pi_f(\varphi(p')),\ p' \in H_{A'}.$$

Since f' is the composition of the bicontinuous mappings π_f and φ it is bicontinuous, that is $f' \in P(H_{A'})$. On the other hand, since φ preserves X, it is easy to verify $f = \nu'(f') \in \nu'(P(H_{A'})) = A'$. Hence $A \subseteq A'$ as required.

When $A \subseteq A'$ the mapping $\varphi : H_{A'} \to H_A$ defined by $\varphi(p') = p'|A$ need not be onto, or in other words the extension $(H_{A'}, u_{A'}, v_{A'})$ need not be projectively larger than the extension (H_A, u_A, v_A). Let us note, however, the following special cases.

(a) A, A' $\in \mathcal{A}_{\mathcal{B}}$. In this case, of course, φ is bijective. Note that if $\mathcal{B} \subseteq P^*(X)$ and $\mathcal{A}_{\mathcal{B}}$ contains more than one element then the spaces (H_A, u_A, v_A), $A \in \mathcal{A}_{\mathcal{B}}$ cannot all be (for instance) pairwise Hausdorff because of Corollary 3 to Proposition 3.3.4. This would lead to a contradiction in the event that u = v, but in general it seems quite conceivable that $\mathcal{A}_{\mathcal{B}}$ could contain more than one element, even when $\mathcal{B} \subseteq P^*(X)$, and this represents a significant difference between the topological and bitopological cases.

(b) $A, A' \in \mathcal{A}^*$. In this case φ is onto. For if $A \subseteq A'$ and $p \in H_A$ then $(L^p_{A'}, M^p_{A'})$ has a (not necessarily unique) maximal $\mathcal{f}_b$-regular refinement in A'. Since $A' \subseteq P^*(X)$ such a refinement has the form $(L^{p'}, M^{p'})$ for some $p' \in H_{A'}$, and clearly $p'|A = p$. This shows that the ordering in $\mathcal{A}^*$ reflects the projective ordering of the corresponding uniform compactifications. In particular the largest of these is $(H_{P^*(X)}, u_{P^*(X)}, v_{P^*(X)})$, and this is the uniform compactification in which X is $P^*(X)$-embedded. Despite this, of course, $(H_{P^*(X)}, u_{P^*(X)} \vee v_{P^*(X)})$ might not be the Stone-Čech compactification of $(X, u \vee v)$.

(c) $A \in \mathcal{A}$ and $A^* = A \cap P^*(X)$. Let us first verify that $A^* \in \mathcal{A}^*$. Now $A = \nu(P(H_A))$ and so A^* is isomorphic to $P^*(H_A)$. Hence H_{A^*} is bitopologically homeomorphic with $H_{P^*(H_A)}$, and so $P(H_{A^*})$ is isomorphic with $P(H_{P^*(H_A)})$, each under the natural correspondence. However $P^*(H_A)$ is finitely $\mathcal{f}_b$-complete in $P(H_A)$, so $P(H_{P^*(H_A)})$ is isomorphic with $P^*(H_A)$, and hence with A^* also. This shows that $A^* = \nu^*(P(H_{A^*}))$, using an obvious notation, and so $A^* \in \mathcal{A}^*$. The above argument also shows that H_{A^*} is the largest of the separated uniform compactifications of H_A in which H_A is uniformly dense. Hence $\varphi : H_A \to H_{A^*}$ is a bitopological homeomorphism of H_A with a uniformly dense subset of H_{A^*} which preserves X, and this means that the extension $(H_{A^*}, u_{A^*}, v_{A^*})$ is injectively larger that the extension (H_A, u_A, v_A).

Note in particular that $H_{P^*(X)}$ is a uniform compactification of $H_{P(X)}$. $(H_{P(X)}, u_{P(X)}, v_{P(X)})$ is the bireal compactification of (X, u, v) in which X is $P(X)$-embedded. For each $A \in \mathcal{A}$, $(H_A, u_A \vee v_A)$ is a real compactification of $(X, u \vee v)$, but in general $(H_{P(X)}, u_{P(X)} \vee v_{P(X)})$ may not be the Hewitt real compactification of $(X, u \vee v)$.

In the remainder of this section we relate the notion of bireal compactness with the completeness of certain quasi-uniformities. Let us recall that if $S \subseteq P(X)$ is bigenerating then $qu(S)$ denotes the quasi-uniformity with subbase

$$\{ f^{-1}(\mathcal{M}_u) \mid f \in S, s > 0 \},$$

(see Chapter 1), and it is clearly compatible with (X, u, v). The following result is basic.

Proposition 3.3.5. Let $\mathcal{B}$ be an $\imath$-regular bifilter on X. Then $\mathcal{B}$ is $qu(S)$-Cauchy if and only if $(Z_b^{-1}(\mathcal{B}_u) \cap \langle S \rangle, Z_b^{-1}(\mathcal{B}_v) \cap \langle S \rangle)$ is real in $\langle S \rangle$.

Proof. Sufficiency is clear so we will establish the necessity. Let $\mathcal{B}$ be $qu(S)$-Cauchy. Note that since $qu(S) = qu(\langle S \rangle)$, $\mathcal{B}$ is $qu(\langle S \rangle)$-Cauchy, and it follows easily that $(Z_b^{-1}(\mathcal{B}_u) \cap \langle S \rangle, Z_b^{-1}(\mathcal{B}_v) \cap \langle S \rangle)$ is finite. Hence we have $p \in H_{\langle S \rangle}$ with

$$(Z_b^{-1}(\mathcal{B}_u) \cap \langle S \rangle, Z_b^{-1}(\mathcal{B}_v) \cap \langle S \rangle) < (L^p, M^p).$$

Take $f \in L^p$ and suppose $f \notin Z_b^{-1}(\mathcal{B}_u)$. As we have noted earlier $Z_b^{-1}(\mathcal{B}_u) = [Z_b^{-1}(\mathcal{B}_u)]^+$, and so we have $s > 0$ with $T_s(f) \notin Z_b^{-1}(\mathcal{B}_u)$. Hence we have $s' > 0$ with $Z^+(T_s(f), s') \notin \mathcal{B}_u$. Let $s'' = (s + s')/2$, then for some $r \in R$ we have $(f^{-1}(M(r, s'')), f^{-1}(N(r, s''))) \in \mathcal{B}$. It follows that $f^{-1}(M(r, s'')) \nsubseteq Z^+(T_s(f), s')$, and hence

$$s' + s < r + s'' \quad \ldots\ldots\ldots (5).$$

Also we have $T_{r-s''}(f) \in Z_b^{-1}(\mathcal{B}_v) \cap \langle S \rangle \subseteq M^p$ and so $p(f) - r + s'' \geqslant 0$. Finally $f \in L^p$ implies $p(f) \leq 0$, and so $r \leq s''$ which contradicts (5). Hence $L^p = Z_b^{-1}(\mathcal{B}_u) \cap \langle S \rangle$, and $M^p = Z_b^{-1}(\mathcal{B}_v) \cap \langle S \rangle$ is proved likewise.

We may now give:

Theorem 3.3.3. (X, u, v) is S-bireal compact if and only if it is qu(S)-complete.

Proof. Let (X, u, v) be S-bireal compact, and let $\mathcal{B}$ be a qu(S)-Cauchy 1-regular bifilter on X. By the last proposition $(Z_b^{-1}(\mathcal{B}_u)\cap \langle S\rangle,\ Z_b^{-1}(\mathcal{B}_v)\cap\langle S\rangle)$ is real in $\langle S\rangle$, and hence fixed by some $x \in X$. It follows at once that $\mathcal{B}$ converges to x, and so X is qu(S)-complete.

Conversely suppose X is qu(S),complete, and let (L, M) be a real bi-ideal in $\langle S\rangle$ with resolution $p \in H_{\langle S\rangle}$. It is clear that for $f \in \langle S\rangle$ and $s > 0$ we have

$$(f^{-1}(M(p(f), s)),\ f^{-1}(N(p(f), s))) \in Z_b(L, M),$$

and hence $Z_b(L, M)$ is a qu(S)-Cauchy 1-regular bifilter on X. By hypothesis $Z_b(L, M)$ converges to some $x \in X$, and so $Z_b(L(x)\cap\langle S\rangle,\ M(x)\cap\langle S\rangle) \subseteq Z_b(L, M)$. However the mapping Z_b is injective on the maximal $\mathscr{I}_b$-regular bi-ideals of $\langle S\rangle$ and we have $(L(x)\cap\langle S\rangle,\ M(x)\cap\langle S\rangle) = (L, M)$, that is (L, M) is fixed by x as required.

Corollary 1. For each bigenerating sub-T-lattice A of P(X), $(H_A,\ qu(\pi_A))$ is a weakly pairwise Hausdorff completion of (X, qu(A)).

Proof. The restriction of $qu(\pi_A)$ to $\sigma_A(X)$ is clearly quasi-uniformly isomorphic with qu(A), and the completeness follows from the theorem and Corollary 2 to Theorem 3.3.2.

For the quasi-uniformity $\mathcal{S}$ on X we shall denote by $Q(\mathcal{S})$ the set of all functions $f : X \to \mathbb{R}$ which are $(\mathcal{S} - \mu)$ quasi-uniformly continuous. By applying (essentially) Theorem 2.5.3 to the completion $(H_A,\ qu(\pi_A))$ we have at once:

Corollary 2. If A is a bigenerating sub-T-lattice of P(X) then

$$A \subseteq Q(qu(A)) \subseteq \nu(P(H_A)).$$

In particular for $A \in \mathfrak{A}$ we have $A = Q(qu(A))$.

If A is bigenerating sub-T-lattice of $P^{*}(X)$ then (H_A, u_A, v_A) is uniformly compact and so has a unique compatible quasi-uniformity. It follows in this case that $Q(qu(\Pi_A)) = P(H_A)$ and hence

$$Q(qu(A)) = \nu(P(H_A)) \in \mathcal{A}^{*}.$$

Consequently there is a one to one correspondence between $\mathcal{A}^{*}$ and the set of totally bounded quasi-uniformities compatible with (X, u, v). On the other hand, however, it is possible to have a bigenerating sub-T-lattice A of $P(X)$ for which $Q(qu(A)) \notin \mathcal{A}$, as the next example shows.

Example 3.3.1. Let (X, u, v) be the space $(\mathbb{R}, s, t)$, and let $S = \{ i, 0 \}$, where $i: \mathbb{R} \longrightarrow \mathbb{R}$ is the identity function. Clearly $S \subseteq P(X)$ is bigenerating. Let us first show that $(\mathbb{R}, s, t)$ is S-bireal compact. Let (L, M) be a real bi-ideal in $\langle S \rangle$. Then for some $a \in \mathbb{R}$ we have $T_a(i) \in L \cap M$, and we will show that (L, M) is fixed by a. To do this it will suffice to show $f \in L \cap M \Rightarrow f(a) = 0$. Suppose on the contrary that $f(a) = 2b > 0$. Then $a \notin Z^{+}(f, b)$, which is a closed lower set in R, and so $k = \sup Z^{+}(f, b) < a$. Take $0 < t < a - k$; then $Z^{+}(f, b) \cap Z^{-}(T_a(i), t) = \emptyset$, which contradicts the fact that $Z_b(L, M)$ is $\mathcal{f}_b$-regular.

Likewise $f(a) < 0$ is impossible, and we have shown that $f(a) = 0$ as required. Hence $(\mathbb{R}, s, t)$ is indeed S-bireal compact. It follows by Theorem 3.3.1 that $P(\mathbb{R})$ is a finite $\mathcal{f}_b$-refinement of $\langle S \rangle$, for $(\mathbb{R}, s, t)$ is bireal compact by Theorem 3.3.2. Indeed examination of the proof of Theorem 3.3.1 shows that in fact $P(\mathbb{R})$ is a finite $\mathcal{f}_b$-prime-refinement of $\langle S \rangle$, and hence of $Q(qu(S))$ also. If we can show $Q(qu(S)) \neq P(\mathbb{R})$ it will then follow that $Q(qu(S)) = Q(qu(\langle S \rangle)) \notin \mathcal{A}$. Now it is clear from the definition that $qu(S) = \mu$ so we have to show $Q(\mu) \neq P(\mathbb{R})$. Take

$$f(x) = e^{x} \in P(\mathbb{R})$$

and suppose $f \in Q(\mu)$. Then given $t > 0 \ \exists \ s > 0$ with

$$m_s < f^{-1}(m_t).$$

Hence for $k \in \mathbb{R}$ there exists $r \in \mathbb{R}$ with

$M(k, s) \subseteq f^{-1}(M(r, t))$ and $N(k, s) \subseteq f^{-1}(N(r, t))$.

Hence $e^{k+s/2} < r + t$ and $e^{k-s/2} > r - t$, and so

$$0 < e^{s/2} - e^{-s/2} < 2t/e^k.$$

However this gives a contradiction for large enough k, so $f \notin Q(\mu)$ and the result is established.

We know from the general theory above that $Q^*(\mu) \in \mathcal{A}^*$. Actually $Q^*(\mu) = P^*(R)$. To see this take $f \in P^*(R)$ and let $a = \inf[f(R)]$, $b = \sup[f(R)]$. Since the subspace $[a, b]$ of R is uniformly compact the restriction to $[a, b]$ of μ coincides with qu(P(R)) restricted to $[a, b]$. Hence given $t > 0\ \exists\ s > 0$ with

$$(\mathcal{M}_s)_{[a, b]} < (f^{-1}(\mathcal{M}_t))_{[a, b]}.$$

Hence for $a - s < k < b + s$ we have $r \in R$ with

$$M(k, s) \cap [a, b] \subseteq f^{-1}(M(r, t)) \cap [a, b],$$

$$N(k, s) \cap [a, b] \subseteq f^{-1}(N(r, t)) \cap [a, b].$$

However, using the fact that $f^{-1}(M(r, t))$ is an open lower set and $f^{-1}(N(r, t))$ is an open upper set we may deduce that

$$M(k, s) \subseteq f^{-1}(M(r, t)) \text{ and } N(k, s) \subseteq f^{-1}(N(r, t)).$$

On the other hand for $k \leq a - s$ (respectively, $k \geq b + s$) we have $M(k, s) \subseteq f^{-1}(M(a, t))$, $N(k, s) \subseteq f^{-1}(N(a, t))$, (respectively, $M(k, s) \subseteq f^{-1}(M(b, t))$, $N(k, s) \subseteq f^{-1}(N(b, t))$). This shows that

$$\mathcal{M}_s < f^{-1}(\mathcal{M}_t)$$

and so $f \in Q^*(\mu)$ as required.

We now turn our attention to the question of the existence of a minimal element of $\mathcal{A}$. The situation does not seem to be as straightforward as in the case of a single topology, and in particular there does not seem to be a simple characterization in terms of the number of elements in the outgrowth of a minimal uniform compactification. Before giving our partial solution to this question let us note some terminology. If A is a sub-T-lattice of $P(X)$ and (F, K) an ordered pair of non-empty subsets of X we shall say (F, K) is <u>A-completely separated</u> if for some $h \in A$ with $0 \leq h \leq 1$ we have $h(F) = \{0\}$ and $h(K) = \{1\}$. (F, K) will be called <u>closed</u> if F is v-closed and K is u-closed. We then have:

<u>Theorem</u> 3.3.4. (a) Let A be a minimal element of $\mathfrak{A}$. Then for every A-completely separated pair (F, K) either every filter $\mathcal{F}$ with $F \in \mathcal{F}$ has a u-cluster point in X or every filter $\mathcal{K}$ with $K \in \mathcal{K}$ has a v-cluster point in X.

(b) Let $A \in \mathfrak{A}$, and suppose that every closed A-completely separated pair (F, K) has the property that either F is u-compact or K is v-compact. Then A is the least element of $\mathfrak{A}$.

<u>Proof</u>. (a). Suppose that for some A-completely separated pair (F, K) there is a filter $\mathcal{F}$ with $F \in \mathcal{F}$ that has no u-cluster point in X, and a filter $\mathcal{K}$ with $K \in \mathcal{K}$ which has no v-cluster point in X. We will show that the bifilter $\mathcal{K} \times \mathcal{F}$ is qu(A)-Cauchy.

Take $a \in X$. Then since $\mathcal{F}$ has no u-cluster point there is a u-closed set $F(a) \in \mathcal{F}$ with $a \notin F(a)$, and likewise we have a v-closed set $K(a) \in \mathcal{K}$ with $a \notin K(a)$. Consequently we have $e(a) \in qu(A)$ with $St(e(a), \{a\}) \cap F(a) = St(\{a\}, e(a)) \cap K(a) = \emptyset$. Take any $e \in qu(A)$ and a sequence $d_n \in qu(A)$ with $d_1 \prec (\Delta)$ $e \wedge e(a)$ and $d_{n+1} \prec (*) \, d_n$, $n = 1, 2, \ldots$. Let p^e_a be the admissible p-q-metric obtained as in Lemma 1.4.1 for the sequence d_n. For $x \in X$ let

$$f^e_a(x) = p^e_a(a, x).$$

Since for $2^{-n} < t$ we have $d_{n+2} < o_n \prec f^{e\,-1}_a(m_t)$ we see that $f^e_a \in Q(qu(A)) = A$. Also

$$f^{e\,-1}_a(-\infty, f^e_a(a) + 2^{-2}) \subseteq St(e, a)$$

so the set $\{ f^e_a \mid e \in qu(A), a \in X \}$ generates the topology u. In exactly the same way if we set

$$g^e_a(x) = -p^e_a(x, a)$$

then $g^e_a \in A$ and the set $\{ g^e_a \mid e \in qu(A),\ a \in X \}$ generates the topology v. It follows that

$$S = \{ f^e_a \wedge (1/4),\ g^e_a \vee (-1/4) \mid e \in qu(A),\ a \in X \}$$

is bigenerating and that $Q(qu(S)) \subseteq A$. However $S \subseteq P^*(X)$ so $Q(qu(S)) \in \mathcal{A}$, and since A is minimal this implies

$$A = Q(qu(S)) \qquad \ldots\ldots\ldots \quad (6).$$

Now by the definition of p^e_a we know that $o_2 < d_1^A$, and so in particular $o_2 < e(a)$. It follows that for all $t > 0$ we have

$$X = (f^e_a \wedge (1/4))^{-1}(M(1/4,\ t)),$$

$$F(a) \subseteq (f^e_a \wedge (1/4))^{-1}(N(1/4,\ t)); \text{ and}$$

$$K(a) \subseteq (g^e_a \vee (-1/4))^{-1}(M(-1/4,\ t)),$$

$$X = (g^e_a \vee (-1/4))^{-1}(N(-1/4,\ t))$$

and so, using (6), $\mathcal{K} \times \mathcal{F}$ is qu(A)-Cauchy as required. However we have $h \in A$ with $h(F) = \{0\}$ and $h(K) = \{1\}$, and hence for some $r \in R$,

$$(h^{-1}(M(r,\ 1/4)),\ h^{-1}(N(r,\ 1/4))) \in \mathcal{K} \times \mathcal{F}.$$

$h^{-1}(M(r,\ 1/4)) \cap K \neq \emptyset$ and $h^{-1}(N(r,\ 1/4)) \cap F \neq \emptyset$ now gives an immediate contradiction, and (a) is proved.

(b) Suppose that $A \in \mathcal{A}$ has the properties stated in the theorem, and that A' is any other element of $\mathcal{A}$. Take $f \in A$ and set $a = \inf[f(X)]$, $b = \sup[f(X)]$. Since in any case A' contains all constant functions on X we may suppose that f is not constant, and hence that $a < b$. Take $t > 0$, and a natural number n with $1/n < t$. Let p and q be integers satisfying

$$(p - 1)/n \leq a < p/n \quad \text{and} \quad q/n < b \leq (q + 1)/n.$$

Since the pairs

$$(Z^+(T_{i/n}(f)),\ Z^-(T_{(i+1)/n}(f))), \quad p \leq i \leq q - 1$$

are closed and A-completely separated we know by hypothesis that for each pair either the first set is u-compact or the second set is v-compact. It follows by a simple compactness argument that in either case each pair is A'-completely separated, say by the functions $h_i \in A'$. Now consider

$$\bigwedge \{ h_i^{-1}(\mathcal{M}_{1/2}) \mid p \leq i \leq q-1 \} \in qu(A').$$

If for $r_p, \ldots, r_{q-1} \in R$ we have

$$\bigcap \{ h_i^{-1}(M(r_i, 1/2)) \cap h_i^{-1}(N(r_i, 1/2)) \mid p \leq i \leq q-1 \} \neq \emptyset$$

and if we define

$$k = \begin{cases} q \text{ if } r_i \geq 1/2 \ \forall \ p \leq i \leq q-1, \\ \min \{ i \mid r_i < 1/2 \} \text{ otherwise,} \end{cases}$$

then it is trivial to verify

$$\bigcap \{ h_i^{-1}(M(r_i, 1/2)) \mid p \leq i \leq q-1 \} \subseteq f^{-1}(M(k/n, t)),$$

$$\bigcap \{ h_i^{-1}(N(r_i, 1/2)) \mid p \leq i \leq q-1 \} \subseteq f^{-1}(N(k/n, t)),$$

and hence

$$\bigwedge \{ h_i^{-1}(\mathcal{M}_{1/2}) \mid p \leq i \leq q-1 \} < f^{-1}(\mathcal{M}_t).$$

This shows $f^{-1}(\mathcal{M}_t) \in qu(A')$ for all $t > 0$, and so $f \in Q(qu(A'))$ $= A'$, and we have shown $A \subseteq A'$ as required.

In the case u = v the conditions (a) and (b) coincide, and give a familiar condition for the existence of a minimal compatible uniformity, but in general they are unfortunately rather far apart.

The results given so far in this section show very clearly that the most natural dispersion to study in relation to the notion of bireal compactness is $\mathcal{f}_b$. However to close this section we will make a few comments about another dispersion which could be studied in this context. This is the dispersion $\mathcal{f}_z$ defined by

$$f(\mathcal{f}_z)g \iff Z^+(f) \cap Z^-(g) = \emptyset , \ f, g \in P(X).$$

It will be noted that the definition of $\mathcal{f}_z$ depends explicitly on the space X, and it is not in general possible to define it using the internal lattice structure of P(X). The reason why

ρ_z might be of interest is that corresponding to a ρ_z-regular bi-ideal (L, M) we have an 1-regular bifilter Z(L, M) with base

$$\{ (Z^+(f), Z^-(g)) \mid f \in L, g \in M \},$$

and this might well be considered a more natural analogue of a z-filter than is $Z_b(L, M)$. ($Z_b(L, M)$, where (L, M) is ρ_b-regular, is more exactly an analogue of an e-filter. C.f. [16], Problem 2.L). It is not our intention here to give a detailed treatment of ρ_z-regular bi-ideals, and indeed the results we do present tend to suggest that this notion might be of less value in the bitopological case than it is for topologies.

It is clear from the definition that $\rho_b \subseteq \rho_z$, and so every ρ_z-regular bi-ideal is ρ_b-regular, although the converse will be false in general. Let us first note:

Proposition 3.3.6. Let A be a bigenerating sub-T-lattice of P(X). Then every maximal ρ_z-regular bi-ideal (L, M) in A is total.

Proof. Suppose that for some $f \in A$ we have $f \notin L$ and $f \notin M$, and let

$$L' = \{ f' \mid f' \in A,\ f' \leq f \vee h \text{ for some } h \in L \},$$

$$M' = \{ g' \mid g' \in A,\ g' \geq f \wedge k \text{ for some } k \in M \}.$$

Then by the ρ_z-maximality the bi-ideals (L', M) and (L, M') are not ρ_z-regular and so we have $h \in L$, $k' \in M$ with $(f \vee h) \bar{\rho}_z k'$ and $h' \in L$, $k \in M$ with $h' \bar{\rho}_z (f \wedge k)$. Now $h \vee h' \in L$, $k \wedge k' \in M$ so we have $a \in Z^+(h \vee h') \cap Z^-(k \wedge k') = Z^+(h) \cap Z^+(h') \cap Z^-(k) \cap Z^-(k')$. However $Z^+(f \vee h) \cap Z^-(k') = Z^+(f) \cap Z^+(h) \cap Z^-(k') = \emptyset$ and $Z^+(h') \cap Z^-(f \wedge k) = Z^+(h') \cap Z^-(f) \cap Z^-(k) = \emptyset$ so we obtain the contradiction $a \notin Z^+(f) \cup Z^-(f)$. This proves the result.

Corollary. A maximal ρ_z-regular bi-ideal in A has a unique maximal ρ_b-regular refinement.

Proof. This follows at once from Propositions 3.1.3 and 3.1.1.

In particular we see that every maximal ρ_z-regular bi-ideal which is finite has a unique real refinement, and this will have the form (L^+, M^+). Hence if (L, M) is a finite maximal $\mathcal{f}_z$-regular bi-ideal we have $p \in H_A$ so that

$$\tau(L, M) = (L^+, M^+) = (L^p, M^p),$$

and in this way we have defined a mapping τ from the set Z_A of all finite maximal $\mathcal{f}_z$-regular bi-ideals on A into the set H_A. Now for each $\mathcal{f}_b$-regular bi-ideal (L, M) we may define

$$L^- = \{ f \mid f \in A, \ \exists \ h \in L, \ s > 0 \text{ with } f \preccurlyeq T_s(h) \vee 0 \},$$

$$M^- = \{ g \mid g \in A, \ \exists \ k \in M, \ s > 0 \text{ with } g \succcurlyeq T_{-s}(k) \wedge 0 \}.$$

Clearly (L^-, M^-) is a $\mathcal{f}_z$-regular bi-ideal, and we have $(L, M) \prec ((L^-)^+, (M^-)^+)$. Consequently if (L, M) is maximal $\mathcal{f}_b$-regular then (L^-, M^-) is nearly total, and (L, M) is its unique maximal $\mathcal{f}_b$-regular refinement. It is clear that all the maximal $\mathcal{f}_z$-regular finite bi-ideals (L, M) with $\tau(L, M) = p$ satisfy

$$((L^p)^-, (M^p)^-) \prec (L, M) \prec (L^p, M^p) \quad \ldots\ldots \quad (7).$$

We may now show that τ is onto H_A. For if $p \in H_A$ then $((L^p)^-, (M^p)^-)$ has a maximal $\mathcal{f}_z$-regular refinement (L, M), and (L, M) is a finite $\mathcal{f}_b$-regular bi-ideal so it has a real refinement; and since this is also a refinement of $((L^p)^-, (M^p)^-)$ it must be (L^p, M^p). Hence $\tau(L, M) = p$, and τ is onto as required.

For $f \in A$ let us set

$$\overline{Z^+(f)} = \{ (L, M) \mid (L, M) \in Z_A, \ f \in L \},$$

$$\overline{Z^-(g)} = \{ (L, M) \mid (L, M) \in Z_A, \ g \in M \}.$$

Clearly $\{ \overline{Z^+(f)} \mid f \in A \}$ may be taken as a base of closed sets for a topology v'_A on Z_A, and $\{ \overline{Z^-(g)} \mid g \in A \}$ as a base of closed sets for a topology u'_A. (Z_A, u'_A, v'_A) is an extension of

(X, u, v) with respect to the mapping $x \to (L(x) \cap A, M(x) \cap A)$, and X is uniformly dense in Z_A. It is easily seen that (Z_A, u'_A, v'_A) is weakly pairwise T_0, but I would conjecture that it need be neither pairwise completely regular nor separated in the general case. We may note:

Proposition 3.3.7. If $A \subseteq P^*(X)$ is a bigenerating sub-T-lattice then (Z_A, u'_A, v'_A) is $\imath$-compact.

Proof. Let $\mathcal{B}$ be an $\imath$-regular bifilter on Z_A, and define

$$L = \{ f \mid f \in A, \overline{Z^+(f)} \in \mathcal{B}_u \},$$

$$M = \{ g \mid g \in A, \overline{Z^-(g)} \in \mathcal{B}_v \}.$$

Clearly (L, M) is a ρ_z-regular bi-ideal in A, and because $A \subseteq P^*(X)$ it has a finite maximal ρ_z-regular refinement (L', M'). $(L', M') \in Z_A$ and is easily seen to be an $\imath$-cluster point of $\mathcal{B}$ so the proof is complete.

Corollary. If under the conditions of the proposition (Z_A, u'_A, v'_A) is pairwise completely regular then it is uniformly compact.

Proof. This follows at once from the corollary to Proposition 2.6.8.

Now let us take $(L, M) \in Z_A$, $f \in A$ and $t > 0$. Then it is easy to verify that

$$(L, M) \in Z_A - \overline{Z^-(T_{p(f)+t}(f))} \subseteq \tau^{-1}(\pi_f^{-1}(-\infty, \pi_f(p) + 2t),$$

$$(L, M) \in Z_A - \overline{Z^+(T_{p(f)-t}(f))} \subseteq \tau^{-1}(\pi_f^{-1}(\pi_f(p) - 2t, \infty),$$

where $p = \tau(L, M)$, and so

$$\tau : (Z_A, u'_A, v'_A) \longrightarrow (H_A, u_A, v_A)$$

is bicontinuous. Since τ also preserves X this means that the extension Z_A is projectively larger than H_A, and in particular X is A-embedded (in fact, $\nu(P(H_A))$-embedded) in Z_A. Recalling that $H_{P^*(X)}$ is the projectively largest separated uniformly compact extension in which X is uniformly dense we see from the

Corollary to Proposition 3.3.7 that if $Z_{P^*(X)}$ is pairwise completely regular then

$$\tau : Z_{P^*}(X) \to H_{P^*}(X)$$

is a bitopological homeomorphism. By embedding $Z_{P(X)}$ in $Z_{P^*(X)}$, or otherwise, we see that

$$\tau : Z_{P(X)} \to H_{P(X)}$$

is a bitopological homeomorphism under the same hypothesis. Before examining conditions under which τ is a bitopological homeomorphism in general we note the following result.

<u>Lemma</u> 3.3.5. Let (L, M) be a maximal $\mathcal{f}_z$-regular bi-ideal in A. If $f \in L$ and $f' \in A$ satisfies $Z^+(f) \subseteq Z^+(f')$ then $f' \in L$.

We omit the proof, which is straightforward. Of course a corresponding result holds for M.

We may now give:

<u>Theorem</u> 3.3.5. Let $A \subseteq P^*(X)$ be a bigenerating sub-T-lattice, and $\tau : Z_A \to H_A$ be defined as above. Then the following are equivalent:

(a) τ is a bitopological homeomorphism.

(b) The set $\tilde{A} = \{ \tilde{f} \mid f \in A \} \subseteq P(Z_A)$, where $\tilde{f} = \pi_f \circ \tau$, is bigenerating.

(c) The following conditions hold for all f, g, h $\in$ A,

(i) If the sets $Z^+(f)$ and $Z^+(g) \cap Z^-(h)$ are non-empty and disjoint then the pair $(Z^+(f), Z^+(g) \cap Z^-(h))$ is A-completely separated, and

(ii) If the sets $Z^+(f) \cap Z^-(g)$ and $Z^-(h)$ are non-empty and disjoint then the pair $(Z^+(f) \cap Z^-(g), Z^-(h))$ is A-completely separated.

<u>Proof</u>. (a) $\Rightarrow$ (b). This is immediate since $\pi_A = \{ \pi_f \mid f \in A \}$ is bigenerating in H_A.

(b) $\Rightarrow$ (c). We establish (i), the proof of (ii) being similar. By hypothesis Z_A is pairwise completely regular, and hence uniformly compact by the corollary to Proposition 3.3.7, so we could use a compactness argument. However the following proof is more in keeping with our general approach. Suppose that $(Z^+(f), Z^+(g) \cap Z^-(h))$ is not A-completely separated. Then given $f' \in A$ with $Z^+(f) \subseteq Z^+(f')$, and $s > 0$, we have

$$Z^+(f', s) \cap Z^+(g) \cap Z^-(h) \neq \emptyset,$$

and hence if we define

$$L' = \{ g' \mid g' \in A,\ Z^+(T_s(f') \vee g) \subseteq Z^+(g') \text{ for some } s > 0 \text{ and } f' \in A \text{ with } Z^+(f) \subseteq Z^+(f') \},$$

$$M' = \{ h' \mid h' \in A,\ Z^-(h) \subseteq Z^-(h') \},$$

we see that (L', M') is a $\mathcal{f}_Z$-regular bi-ideal in A. Let (L, M) be a maximal $\mathcal{f}_Z$-regular refinement of (L', M'); then $(L, M) \in Z_A$ since $A \subseteq P^*(X)$, and we set $\mathcal{T}(L, M) = p$. Let us show $f \in L$. Suppose the contrary, then $(L, M) \notin \overline{Z^+(f)}$, and since A is bigenerating we have $f' \in A$ with $\overline{Z^+(f)} \subseteq Z^+(\widetilde{f'})$ and $(L, M) \notin Z^+(\widetilde{f'})$. Now $\overline{Z^+(f)} \subseteq Z^+(\widetilde{f'})$ implies $Z^+(f) \subseteq Z^+(f')$ so $T_s(f') \in L' \subseteq L \ \ \forall\ s > 0$, that is $f' \in L^+ = L^p$. On the other hand $(L, M) \notin Z^+(\widetilde{f'})$ means $p(f') = \pi_{f'}(p) = \widetilde{f}(L, M) > 0$, which is a contradiction. Hence $f \in L$, and $g \in L$, $h \in M$ are immediate from the definitions, so we are lead to the contradiction $Z^+(f) \cap Z^+(g) \cap Z^-(h) \neq \emptyset$. This establishes (i).

(c) $\Rightarrow$ (a). Let us first demonstrate that $\mathcal{T}$ is injective. Suppose on the contrary that $\mathcal{T}(L, M) = \mathcal{T}(L', M')$ and that, say, we have $f \in L$ with $f \notin L'$. By the maximality of (L', M') we have $h' \in L'$ and $k' \in M'$ with

$$Z^+(f \vee h') \cap Z^-(k') = \emptyset.$$

Applying (i) we have $g_1 \in A$ with $0 \leq g_1 \leq 1$, $g_1(Z^+(f \vee h')) = \{0\}$ and $g_1(Z^-(k')) = \{1\}$. By Lemma 3.3.5 we see that $T_t(g_1) \in M' \ \ \forall\ t \leq 1$. Hence $Z^+(h') \cap Z^-(T_{1/2}(g_1)) \neq \emptyset$, while $Z^+(f \vee h') \cap Z^-(T_{1/2}(g_1)) = Z^+(f) \cap [Z^+(h') \cap Z^-(T_{1/2}(g_1))] = \emptyset$.

Hence we may again apply (i) to give us $g_2 \in A$ with $0 \leq g_2 \leq 1$, $g_2(Z^+(f)) = \{0\}$ and $g_2(Z^+(h') \cap Z^-(T_{1/2}(g_1))) = \{1\}$. By Lemma 3.3.5 we have $g_2 \in L \subseteq L^p$, and since $T_1(g_1) \in M' \subseteq M^p$ we see that $Z^+(T_{1/2}(g_2)) \cap Z^-(T_{1/2}(g_1)) \neq \emptyset$. However $Z^+(T_{1/2}(g_2)) \cap (Z^+(h') \cap Z^-(T_{1/2}(g_1))) = Z^+(h') \cap (Z^+(T_{1/2}(g_2)) \cap Z^-(T_{1/2}(g_1))) = \emptyset$, and a final application of (i) now gives us $g_3 \in A$, $0 \leq g_3 \leq 1$, with $g_3(Z^+(h')) = \{0\}$ and $g_3(Z^+(T_{1/2}(g_2)) \cap Z^-(T_{1/2}(g_1))) = \{1\}$. On the one hand $g_3 \in L' \in L^p$ by Lemma 3.3.5, and on the other we have

$$Z^+(T_{1/2}(g_3)) \cap (Z^+(T_{1/2}(g_2)) \cap Z^-(T_{1/2}(g_1))) = \emptyset \quad \ldots\ldots \quad (7).$$

However $T_1(g_1) \in M^p$, $g_2 \vee g_3 \in L^p$; and (7) implies

$$Z^+(g_2 \vee g_3, 1/2) \cap Z^-(T_1(g_1), 1/2) = \emptyset$$

which contradicts the fact that (L^p, M^p) is $\mathscr{f}_b$-regular. This shows that $\mathcal{T}$ is injective, as required.

Now let us show that for each $f \in A$ the set $\mathcal{T}\,\overline{(Z^+(f))}$ is v_A-closed in H_A. Take $p \notin \mathcal{T}\,\overline{(Z^+(f))}$ in H_A, and define

$$L' = \{ f' \mid f' \leq f \vee T_s(h) \vee 0,\ h \in L^p,\ s > 0 \}.$$

Clearly L' is an ideal containing $(L^p)^-$, and hence the bi-ideal $(L', (M^p)^-)$ cannot be $\mathscr{f}_z$-regular, for if it were it would have a (necessarily finite) maximal $\mathscr{f}_z$-regular refinement (L, M) satisfying $f \in L$ and $\mathcal{T}(L, M) = p$, and this would contradict $p \notin \mathcal{T}\,\overline{(Z^+(f))}$. Hence we have $h \in L^p$, $k \in M^p$, s, $t > 0$, so that

$$Z^+(f \vee T_s(h)) \cap Z^-(T_{-t}(k)) = \emptyset,$$

that is

$$Z^+(f) \cap (Z^+(T_s(h)) \cap Z^-(T_{-t}(k))) = \emptyset.$$

We may now apply (i) to give $g \in A$, $0 \leq g \leq 1$, with $g(Z^+(f)) = \{0\}$ and $g(Z^+(T_s(h)) \cap Z^-(T_{-t}(k))) = \{1\}$. In particular we therefore have

$$Z^+(T_{1/2}(g) \vee h,\ s \wedge (1/4)) \cap Z^-(k,\ t) = \emptyset \quad \ldots\ldots (8).$$

Now suppose that $q \in \pi_g^{-1}(\pi_g(p) - 1/2,\ \infty) \cap \tau(\overline{Z^+(f)})$. Then $q = \tau(L, M)$ with $f \in L$, and $Z^+(f) \subseteq Z^+(g)$ implies $g \in L \subseteq L^q$ by Lemma 3.3.5, so we have $q(g) \leq 0$. On the other hand $\pi_g(q) \geq \pi_g(p) - 1/2$ implies $g(p) < q(g) + 1/2 \leq 1/2$, and so $T_{1/2}(g) \in L^p$. However (8) now contradicts the fact that (L^p, M^p) is $\mathcal{f}_b$-regular, and we have established that

$$\pi_g^{-1}(\pi_g(p) - 1/2,\ \infty) \cap \tau(\overline{Z^+(f)}) = \emptyset.$$

Hence $\tau(\overline{Z^+(f)})$ is v_A-closed, and an exactly similar argument shows that $\tau(\overline{Z^-(f)})$ is u_A-closed. Thus τ is a biclosed mapping and hence a bitopological homeomorphism, as required.

The argument used in proving (c) $\Rightarrow$ (a) does not use the fact that $A \subseteq P^*(X)$. Hence we may state

Corollary 1. If $A \subseteq P(X)$ is a bigenerating sub-T-lattice which satisfies (i) and (ii) then $\tau : Z_A \to H_A$ is a bitopological homeomorphism.

Let us also note

Corollary 2. If (X, u, v) is uniformly compact then every bigenerating sub-T-lattice $A \subseteq P(X)$ satisfies (i) and (ii).

It is known ([20] , Proposition 2.8) that if $Z^+(f)$ and $Z^-(g)$ are non-empty and disjoint then the pair $(Z^+(f),\ Z^-(g))$ is $P(X)$-completely separated. This condition is, however, considerably weaker than (i) and (ii) of Theorem 3.3.5, and I would conjecture that $P(X)$ need not satisfy (i) and (ii) in the general case. That not every element of $\mathfrak{R}$ need satisfy even the weaker condition mentioned above is demonstrated by the following simple example.

Example 3.3.2. Let $X = (-1, 0) \cup (0, 1)$, and consider the induced bitopological space $(X, s|X, t|X)$. Clearly the quasi-uniformity $\mu|X$ is compatible with this space, and we may consider $A =$

$Q^*(\mu|X) \in \mathcal{A}^*$. If we set $f = [(i \vee 0) \wedge 1]|X$ and $g = [(i \wedge 0) \vee (-1)]|X$ then $f, g \in A$ and $Z^+(f) \cap Z^-(g) = (-1, 0) \cap (0, 1) = \emptyset$. Suppose we have $h \in A$ with $h(Z^+(f)) = \{0\}$ and $h(Z^-(g)) = \{1\}$. Now $h^{-1}(m_{1/2}) \in \mu|X$ so there exists $k > 0$ with $(m_k)_X \prec h^{-1}(m_{1/2})$. Since $(M(0, k) \cap X) \cap (N(0, k) \cap X) = (-k, 0) \cup (0, k) \neq \emptyset$ we have $r \in \mathbb{R}$ with

$$M(0, k) \cap X \subseteq h^{-1}(M(r, 1/2)), \text{ and}$$

$$N(0, k) \cap X \subseteq h^{-1}(N(r, 1/2)).$$

However $k/2 \in M(0, k) \cap X$ implies $1 = h(k/2) < r + 1/2$, and $-k/2 \in N(0, k) \cap X$ implies $0 = h(-k/2) > r - 1/2$, so we have an immediate contradiction. Hence the pair $(Z^+(f), Z^-(g))$ is not A-completely separated.

Note that although $Q^*(\mu) = P^*(\mathbb{R})$ we do not have $A = Q^*(\mu| X) = P^*(X)$, for of course the above pair is $P^*(X)$-completely separated.

One particular case in which we should have a unique maximal $\mathcal{f}_z$-regular bi-ideal (L, M) with $\tau(L, M) = p$ would be when $(L, M) = (L^p, M^p)$. Our final result in this section investigates this situation. Let us say that the $\mathcal{f}_z$-regular bi-ideal (L, M) has the <u>countable intersection property</u> (C.I.P) if the set $\{Z^+(f) \cap Z^-(g) \mid f \in L, g \in M\}$ has the countable intersection property. We then have:

<u>Theorem</u> 3.3.6. Take $A \in \mathcal{A}$. Then the maximal $\mathcal{f}_z$-regular bi-ideal (L, M) in A is real if and only if it has the countable intersection property.

<u>Proof</u>. Suppose that the maximal $\mathcal{f}_z$-regular bi-ideal (L, M) in A is real, but that it does not have the C.I.P. Then we have $f_n \in L$, $g_n \in M$ with $\bigcap\{Z^+(f_n) \cap Z^-(g_n)) \mid n = 1, 2, \dots\} = \emptyset$, and without loss of generality we may assume $0 \le f_n \le 1/n$ and $-1/n \le g_n \le 0$. Let us set

$$f = \bigvee\{f_n \mid n = 1, 2, \dots\} \text{ and } g = \bigwedge\{g_n \mid n = 1, 2, \dots\}.$$

Clearly f and g are well defined, $0 \leq f \leq 1$ and $-1 \leq g \leq 0$. Take $s > 0$ and choose a natural number k with $1/2k < s$. If given $r_1, \ldots, r_k \in \mathbb{R}$ we set $r = \bigvee\{ r_i \mid 1 \leq i \leq k \} \vee (1/2k)$ then it is easy to verify

$$\bigcap \{ f_i^{-1}(M(r_i, 1/2k)) \mid 1 \leq i \leq k \} \subseteq f^{-1}(M(r, s)), \text{ and}$$

$$\bigcap \{ f_i^{-1}(N(r_i, 1/2k)) \mid 1 \leq i \leq k \} \subseteq f^{-1}(N(r, s)),$$

and so

$$\bigwedge \{ f_i^{-1}(\mathcal{m}_{1/2k}) \mid 1 \leq i \leq k \} \prec f^{-1}(\mathcal{m}_s).$$

Hence $f \in Q(qu(A))$, and so $f \in A$ by Corollary 2 to Theorem 3.3.3, and likewise $g \in A$. Since (L, M) is real we have $s, t \in \mathbb{R}$ with $T_s(f) \in L \cap M$ and $T_t(g) \in L \cap M$. Suppose that $s > 0$, and take a natural number m with $1/m \leq s$. Then since (L, M) is $\mathcal{f}_z$-regular we have $x \in \bigcap\{Z^+(f_i) \mid 1 \leq i \leq m\} \cap Z^-(T_s(f))$. However $f_i(x) = 0$, $i = 1, \ldots, m$ implies $f(x) < 1/m \leq s$ which contradicts $x \in Z^-(T_s(f))$. Hence $s \leq 0$; and likewise $t \geq 0$, which shows $f \in L$ and $g \in M$. Hence there exists

$$z \in Z^+(f) \cap Z^-(g)$$

which gives the contradiction $z \in \bigcap \{ Z^+(f_n) \cap Z^-(g_n) \mid n = 1, 2, \ldots \}$. Hence (L, M) has the C.I.P.

To prove the converse let (L, M) be a maximal $\mathcal{f}_z$-regular bi-ideal in A with the countable intersection property. Suppose first that we have some $f \in A$ with $T_r(f) \in L \;\; \forall\; r \in \mathbb{R}$. Then by hypothesis we have $a \in \bigcap\{Z^+(T_{-n}(f) \mid n = 1, 2, \ldots \}$, which is clearly impossible. Likewise $T_r(f) \in M \;\; \forall\; r \in R$ is impossible so no element of A is infinite at (L, M). Since (L, M) is total and $\mathcal{f}_b$-regular this implies (L, M) is finite, and hence to show (L, M) is real it will be sufficient to verify $L^+ \subseteq L$ and $M^+ \subseteq M$. Suppose, for example, that there exists $f \in L^+$ with $f \notin L$. Then since (L, M) is maximal $\mathcal{f}_z$-regular $\exists\; h \in L$,

$k \in M$ with

$$Z^{+}(f \vee h) \cap Z^{-}(k) = \emptyset \quad \dots\dots\dots (9).$$

On the other hand $T_{1/n}(f) \in L \;\; \forall \;\; n = 1, 2, \dots$, and so by hypothesis we have $b \in X$ with

$$b \in \bigcap \{ Z^{+}(T_{1/n}(f)) \mid n = 1, 2, \dots \} \cap Z^{+}(h) \cap Z^{-}(k).$$

However $f(b) \leq 1/n \;\; \forall \;\; n = 1, 2, \dots$ implies $f(b) = 0$, that is $b \in Z^{+}(f)$, and we have a contradiction to (9). This completes the proof of the theorem.

Corollary. Every real $\mathcal{f}_z$-regular bi-ideal in A^* has a real $\mathcal{f}_z$-regular extension to A.

Proof. This follows at once from the observation that the $\mathcal{f}_z$-regular bi-ideal (L, M) in A has the C.I.P if and only if $(L \cap A^*, M \cap A^*)$ does.

In particular it will be noted that there is a one to one correspondence between the real $\mathcal{f}_z$-regular bi-ideals in A and in A^*. Consequentially if (X, u, v) is not pseudo compact there will exist real bi-ideals in $P^*(X)$ which are not $\mathcal{f}_z$-regular. This implies in particular that the conditions (i) and (ii) of Theorem 3.3.5 are in general not sufficient to ensure that all the real bi-ideals on A should be $\mathcal{f}_z$-regular. Of course it will be apparent that if (X, u, v) is A-bireal compact then certainly all the real bi-ideals in A will be $\mathcal{f}_z$-regular. It can also be shown quite easily that for the space (X, u, v) with u = v then all the real bi-ideals in P(X) are $\mathcal{f}_z$-regular, but I do not know if this is always true when $u \neq v$.

3.4 PAIR REAL COMPACTNESS.

Throughout this section, as in the last, (X, u, v) will always denote a pairwise completely regular weakly pairwise T_0 bitopological space. M.J. SAEGROVE [31] has called such a space pair real compact if it is bitopologically homeomorphic to an intersection of a $\prod s_\alpha$-closed subset and a $\prod t_\alpha$-closed subset of a product $(\prod \mathbb{R}_\alpha, \prod s_\alpha, \prod t_\alpha)$ of copies of $(\mathbb{R}, s, t)$. If in addition (X, u, v) is pseudo compact he calles it a <u>bicompact</u> space. It follows at once from Theorem 3.3.2 and Proposition 3.3.3 that a pair real compact space is bireal compact, and hence uniformly real compact, while a bicompact space is uniformly compact. However the converse is not true in general.

We shall begin by giving a characterization of pair real compactness in terms of the notions of S-resolution and S-derivative introduced in § 3.2. First we make the following definition.

<u>Definition</u> 3.4.1. Let $S \subseteq P(X)$ be a bigenerating subset. We say the S-resolution p is <u>S-fixed by x</u> in X if $f(x) = p(f)$ for all $f \in S$.

<u>Lemma</u> 3.4.1. If the S-resolution p is S-fixed by $x \in X$ then the derivative (L^p, M^p) is a real bi-ideal in $\langle S \rangle$ which is fixed by x.

<u>Proof</u>. By Proposition 3.2.7 it will be sufficient to show that (L^p, M^p) is $\mathscr{I}_b$-regular. However if we had $f_1, \dots, f_n; g_1, \dots, g_m \in S$ and $t > 0$ with

$$T_t(\vee\{T_{p(f_i)}(f_i)\}\vee 0) \geqslant \wedge\{T_{p(g_j)}(g_j)\}\wedge 0$$

we should obtain an immediate contradiction by calculating the value of each side at the point x, and we deduce that (L^p, M^p) is real. It follows that p has an extension to an element $p' \in H_{\langle S \rangle}$, and the fact that $(L^p, M^p) = (L^{p'}, M^{p'})$ is fixed by x now follows by a simple induction argument.

We may now give:

<u>Theorem</u> 3.4.1. (X, u, v) is pair real compact if and only if there exists a bigenerating subset S of P(X) so that each S-resolution is S-fixed in X.

<u>Proof</u>. First suppose (X, u, v) is pair real compact. Then without loss of generality we may suppose $X \subseteq \prod R_\alpha$, and that

$$X = (\prod t_\alpha\text{-cl}[X]) \cap (\prod s_\alpha\text{-cl}[X]).$$

Let us set $S = \{\pi_\alpha | X\} \cup \{0\}$ and suppose $p : S \to \mathbb{R}$ is a S-resolution. Then if we put $x_\alpha = p(\pi_\alpha | X)$ we have $x = (x_\alpha) \in \prod R_\alpha$, and using the fact that the S-derivative (L^p, M^p) is $\mathcal{f}_e$-regular it is not difficult to verify $x \in X$, while by definition p is S-fixed by x.

For the converse let S be a bigenerating subset of $P(X)$, and consider the set R_S of all S-resolutions. $R_S \subseteq \prod\{R_f \mid f \in S\}$, and we may make R_S into a bitopological space (R_S, u_S, v_S) by means of the projections $\pi_f : R_S \to \mathbb{R}_f = \mathbb{R}$, $f \in S$, as usual. The map $\sigma_S : X \to R_S$ given by $\sigma_S(x) = \hat{x} | S$ is a bitopological homeomorphism of (X, u, v) with the bidense subset $\sigma_S(X)$ of R_S, and it is trivial to verify that

$$R_S = (\prod t_f\text{-cl}[R_S]) \cap (\prod s_f\text{-cl}[R_S])$$

in $\prod \mathbb{R}_f$. If every S-resolution p is S-fixed in X then by Lemma 3.4.1 we have $\sigma_S(X) = R_S$, and so (X, u, v) is pair real compact as required.

<u>Corollary</u> 1. (R_S, u_S, v_S) is a pair real compact (respectively, bicompact) extension of (X, u, v) for every bigenerating set S in $P(X)$ (respectively, in $P^*(X)$).

<u>Corollary</u> 2. Every pair real compact bitopological extension of (X, u, v) is bitopologically homeomorphic to (R_S, u_S, v_S) for some bigenerating subset S of $P^*(X)$.

<u>Proof</u>. Let (X', u', v') be pair real compact, and (X, u, v) a bidense subspace. By the theorem we have a bigenerating subset $S' \subseteq P(X')$ so that every S'-resolution is S'-fixed in X'. Let

$$S = \{ f' | X \mid f' \in S' \},$$

then $S \subseteq P(X)$ is clearly bigenerating. Take $p \in R_S$, and for $f' \in S'$ let

$$p'(f') = p(f' | X).$$

Clearly p' is a S'-resolution, so the derivative $(L^{p'}, M^{p'})$ is fixed by some $x_p \in X'$. x_p is unique since (X', u', v') is weakly pairwise T_0, so we may define $\alpha : R_S \to X'$ by

$$\alpha(p) = x_p.$$

It is easy to verify that α is one to one onto X', and the fact that it is a bitopological homeomorphism then follows at once from the relations

$$\pi_f^{-1}(-\infty, r) = \alpha^{-1}(f'^{-1}(-\infty, r)),$$

$$\pi_f^{-1}(r, \infty) = \alpha^{-1}(f'^{-1}(r, \infty))$$

which hold for all $r \in \mathbb{R}$ and $f \in S$, $f' \in S'$ with $f = f'|X$.

Corollary 3. If (X, u, v) is pair real compact there exists a bigenerating set $S \subseteq P(X)$ satisfying

(a) $P(X)$ is a finite $\mathcal{f}_b$-refinement of $<S>$, and

(b) For any S-resolution p the derivative (L^p, M^p) is real.

Conversely if (X, u, v) is a space with a bigenerating set S satisfying (a) and (b) then (X, u, v) is pair real compact if and only if it is bireal compact.

Proof. If (X, u, v) is pair real compact we have a bigenerating S so that every S-resolution is S-fixed. In particular every real bi-ideal in $<S>$ is fixed so (X, u, v) is S-bireal compact and (a) follows from Theorem 3.3.1. (b) follows at once from Lemma 3.4.1. For the converse we only have to show that a bireal compact space (X, u, v) satisfying (a) and (b) is pair real compact. However by (a) and Theorem 3.3.1 we know every real bi-ideal in $<S>$ is fixed, and so by (b) every S-resolution is S-fixed. Thus (X, u, v) is pair real compact by the theorem.

For a given bigenerating $S \subseteq P(X)$ it is natural to consider the relation between the spaces R_S and $H_{<S>}$. The proof of the following result is straightforward, and is omitted.

Proposition 3.4.1. (R_S, u_S, v_S) is a pair real compact extension of $(H_{<S>}, u_S, v_S)$ for the mapping θ defined by $\theta(p) = p|S \ \forall \ p \in H_{<S>}$.

For $f \in P(R_S)$ we may define $\eta(f) = \check{f} \in P(X)$ by $\check{f}(x) = f(\hat{x}|S)$ $\forall\ x \in X$. The following result is clear.

Lemma 3.4.2. For all bigenerating sets $S \subseteq P(X)$ we have

$$<S> \subseteq \eta(P(R_S)) \subseteq \nu(P(H_{<S>})).$$

η is, like ν, a T-lattice homomorphism, but in general it is not injective. Indeed we have the following.

Theorem 3.4.2. The following ane equivalent for a given bigenerating S.

(a) η is injective.

(b) Each bicontinuous real function on $H_{<S>}$ has a unique extension to a bicontinuous real function on R_S.

(c) ϑ is surjective.

(d) $\theta(H_{<S>})$ is uniformly dense in R_S.

(e) $\sigma_S(X)$ is uniformly dense in R_S.

Proof. (a) $\Rightarrow$ (b). Since R_S is bireal compact we know that R_S and $H_{P(R_S)}$ are bitopologically homeomorphic spaces. Also if η is injective then $P(R_S)$ and $\eta(P(R_S))$ are isomorphic T-lattices, and it is easy to deduce that

$$\eta(P(R_S)) = \nu(P(H_{\eta(P(R_S))})).$$

It follows from this that $\eta(P(R_S))$ is finitely $\mathcal{f}_b$-prime-complete. However $<S> \subseteq \eta(P(R_S))$ by Lemma 3.4.2, and we know that $\nu(P(H_{<S>}))$ is the finite $\mathcal{f}_b$-prime-completion of $<S>$ and so we have

$$\eta(P(R_S)) = \nu(P(H_{<S>})) \qquad \ldots\ldots\ (1).$$

(b) now follows at once from (1), and the fact that η is one to one.

(b) $\Rightarrow$ (a). Immediate from the definitions

(a) $\Rightarrow$ (c). From (1) it is immediate that $P(R_S)$ and $P(H_{<S>})$ are isomorphic T-lattices. Since these spaces are bireal compact it follows that ϑ is a bitopological homeomorphism of $H_{<S>}$ onto R_S.

(c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (e). Immediate from the fact that $\sigma_{<S>}(X)$ is uniformly dense in $H_{<S>}$.

(e) $\Rightarrow$ (a). Straightforward.

It is clear that in general equality (1) may hold without η being injective. For example if (X, u, v) is a bireal compact space which is not pair real compact then $\vartheta : H_{P(X)} \rightarrow R_{P(X)}$ cannot be onto, and so by the theorem $\eta : P(R_{P(X)}) \rightarrow P(X)$ is not injective. However

$$P(X) \subseteq \eta(P(R_{P(X)})) \subseteq \nu(P(H_{P(X)})) \subseteq P(X)$$

and so certainly (1) holds in this case.

Let us now consider in more detail the properties of the bitopological extension (R_S, u_S, v_S) of (X, u, v). For $p \in R_S$ we denote by $\mathcal{B}(p)$, as usual, the nhd. bifilter of p, and we shall denote by $\mathcal{B}^X(p)$ the trace of $\mathcal{B}(p)$ on X, that is

$$\mathcal{B}^X(p) = \sigma_S^{-1}(\mathcal{B}(p)).$$

The proof of the following result is straightforward, and is omitted.

<u>Lemma</u> 3.4.3. $\mathcal{B}^X(p) = Z_b(L^p, M^p)$ for each $p \in R_S$. Moreover $\mathcal{B}_u^X(p) = \mathcal{B}_u^X(q) \Leftrightarrow L^p \subseteq (L^q)^+$ and $L^q \subseteq (L^p)^+$, and with a similar result for $\mathcal{B}_v^X(p)$ and $\mathcal{B}_v^X(q)$.

Now let us verify that $p \rightarrow \mathcal{B}_u^X(p)$ is one to one on R_S if and only if $p \rightarrow L^p$ is one to one. First suppose $p \rightarrow \mathcal{B}_u^X(p)$ is one to one. Then $L^p = L^q \Rightarrow L^p \subseteq (L^q)^+$ and $L^q \subseteq (L^p)^+$ $\Rightarrow \mathcal{B}_u^X(p) = \mathcal{B}_u^X(q) \Rightarrow p = q$, so $p \rightarrow L^p$ is one to one. Conversely suppose $p \rightarrow L^p$ is one to one, and let $\mathcal{B}_u^X(p) = \mathcal{B}_u^X(q)$, that is $L^p \subseteq (L^q)^+$ and $L^q \subseteq (L^p)^+$. By the proof of Proposition

3.2.6 we know $T_r(f) \in L^p$ (respectively, L^q) if and only if $r \geq p(f)$ (respectively, $r \leq q(f)$) for each $f \in S$. Hence for $f \in S$, $T_{p(f)}(f) \in L^p \Rightarrow T_{p(f)+s}(f) \in L^q \;\forall\; s > 0 \Rightarrow p(f) + s \geq q(f) \;\forall\; s > 0 \Rightarrow p(f) \geq q(f)$. Likewise $p(f) \leq q(f)$ and so $p = q$. This shows $p \to \mathfrak{B}^X_u(p)$ is one to one. and in exactly the same way $p \to \mathfrak{B}^X_v(p)$ is one to one on R_S if and only if $p \to M^p$ is one to one.

Let us say the bitopological extension (X', u', v') is a <u>relatively T_0 extension</u> of (X, u, v) if (X', u') is a relatively T_0 extension of (X, u) and (X', v') is a relatively T_0 extension of (X, v). (See [1]). Combining the above results with Proposition 3.2.6 now gives us:

<u>Proposition</u> 3.4.2. (R_S, u_S, v_S) is a relatively T_0 extension of (X, u, v) if and only if every S-derivative is S-real.

It will be noted that if we require only that the mapping $p \to \mathfrak{B}^X(p)$ should be one to one on R_S we obtain a condition which will, in general, be weaker than relatively T_0.

Let us now examine under what conditions (R_S, u_S, v_S) will be a strict extension of (X, u, v).

<u>Proposition</u> 3.4.3. (R_S, u_S, v_S) is a strict bitopological extension of (X, u, v) if and only if every S-derivative is S-real.

<u>Proof</u>. It is a straightforward matter to verify that a strict bitopological extension which is weakly pairwise T_1 is necessarily relatively T_0, so necessity follows at once from Proposition 3.4.2. Let us prove the sufficiency. For $f \in \langle S \rangle$ and $s > 0$ we will denote the u-open subset $\{ x \mid f(x) < s \}$ of X by $G^+(f, s)$, and we will denote the set $[\sigma_S(G^+(f, s))]^*_u$ by $G^+(f, s)^*_u$ for short. Take $p \in R_S$ and $f \in S$. Then for $s > 0$ we have

$$\pi_f^{-1}(-\infty, \pi_f(p) + s) \cap \sigma_S(X) = \sigma_S(G^+(T_{p(f)}(f), s))$$

and so

$$p \in G^+(T_{p(f)}(f), s)^*_u \quad \ldots\ldots\ldots (2).$$

Now take $q \in G^+(T_{p(f)}(f), s)^*_u$. Then clearly $G^+(T_{p(f)}(f), s) \in \mathcal{B}^X_u(q) = (Z_b(L^q, M^q))_u$, so for some $g \in L^q$ and $t > 0$ we have

$$Z^+(g, t) \subseteq G^+(T_{p(f)}(f), s).$$

From this we deduce at once that

$$T_{p(f)+s}(f) \wedge t \leq g \vee 0,$$

and because of the definition of L^q this implies $T_{p(f)+s}(f) \in L^q$. Now by hypothesis (L^q, M^q) is S-real and so $p(f) + s \geq q(f)$. Hence $q \in \pi_f^{-1}(-\infty, \pi_f(p) + 2s)$, and we have shown

$$G^+(T_{p(f)}(f), s)^*_u \subseteq \pi_f^{-1}(-\infty, \pi_f(p) + 2s) \quad \ldots (3).$$

That (R_S, u_S) is a strict extension of (X, u) is now clear from (2) and (3). Likewise (R_S, v_S) is a strict extension of (X, v), and the proof is complete.

Let us now note

<u>Proposition</u> 3.4.4. L^p is nearly prime for all $p \in R_S$ if and only if for each $f, g \in \langle S \rangle$ and $s > 0$ we have

$$[G^+(f, s) \cup G^+(g, s)]^*_u \subseteq G^+(f, 2s)^*_u \cup G^+(g, 2s)^*_u \quad \ldots\ldots (4).$$

<u>Proof</u>. Suppose L^p is nearly prime for all $p \in R_S$. If $p \in [G^+(f, s) \cup G^+(g, s)]^*_u$ then $G^+(f, s) \cup G^+(g, s) = G^+(f \wedge g, s) \in \mathcal{B}^X_u(p)$, and exactly as in the proof above we deduce $T_s(f \wedge g) = T_s(f) \wedge T_s(g) \in L^p$. Hence $T_{3s/2}(f) \in L^p$ or $T_{3s/2}(g) \in L^p$, and we have $p \in G^+(f, 2s)^*_u \cup G^+(g, 2s)^*_u$ as required. The proof of the converse is similar, and is omitted.

<u>Corollary</u>. Suppose that (R_S, u_S, v_S) is a strict bitopological extension of (X, u, v), and that the topology u_S satisfies (4) for all $f, g \in \langle S \rangle$, $s > 0$. Then (R_S, u_S, v_S) is bitopologically homeomorphic with $(H_{\langle S \rangle}, u_S, v_S)$ under the mapping ϑ.

<u>Proof</u>. This follows at once from Propositions 3.4.4 and 3.2.7.

Note that a result corresponding to Proposition 3.4.4 will also hold for M^p and the topology v_S. In particular if (R_S, u_S, v_S) is a strict bitopological extension then u_S will satisfy (4) if and only if the corresponding result is satisfied by v_S. This represents an interesting symmetry between these two topologies in this case.

We have seen earlier that the spaces H_A may be regarded as completions of a suitable quasi-uniformity on X, and we are now going to show that those spaces R_S which form a strict extension of X (and these indeed comprise all the strict pair real compact extensions of X) may be regarded as the completion of a suitable confluence quasi-uniformity on X. Let us denote by $cqu(\pi_S)$ the basic confluence quasi uniformity on R_S with subbase

$$\{ (\pi_f^{-1}(m_s), \imath) \mid f \in S, s > 0 \}.$$

Then we have:

<u>Lemma</u> 3.4.4. (R_S, u_S, v_S) is $cqu(\pi_S)$-complete.

<u>Proof</u>. Let $\mathcal{B}$ be an $\imath$-regular $cqu(\pi_S)$-Cauchy bifilter on R_s. For $f \in S$ it is easy to verify that

$$p(f) = \inf \{ r \mid \pi_f^{-1}(H(r)) \in \mathcal{B}_u \}$$

exists in R, and that moreover we have

$$p(f) = \sup \{ r \mid \pi_f^{-1}(K(r)) \in \mathcal{B}_v \}.$$

But then $(\pi_f^{-1}(M(p(f), s)), \pi_f^{-1}(N(p(f), s))) \in \mathcal{B}$ for all $s > 0$, and it follows easily that $p \in R_S$ and that $\mathcal{B}$ converges to p as required. Hence R_S is complete.

<u>Lemma</u> 3.4.5. If (R_S, u_S, v_S) is a strict extension of (X, u, v) then $\sigma_S(X)$ is strictly $cqu(\pi_S)$-embedded in R_S.

<u>Proof</u>. For $f \in S$ and $s > 0$ let us set

$$d_f(s) = \{ ([U \cap \sigma_S(X)]^*_u, [V \cap \sigma_S(X)]^*_v) \mid U \pi_f^{-1}(m_s) V \}.$$

Since R_S is a strict extension of X we may use Proposition 3.4.3, and in particular the relation (3) and its equivalent for the topology v_S to show

$$\pi_f^{-1}(m_s) \prec d_f(s) \prec \pi_f^{-1}(m_{2s})$$

for all $s > 0$. Hence

$$\{ (d_f(s), \imath) | f \in S, s > 0 \}$$

is a subbase of $\text{cqu}(\pi_S)$, and it is clear from the definition that the corresponding base β satisfies conditions (b) (i) and (ii) of Definition 2.3.2.

It follows from Proposition 2.3.1 that the induced structure on X is an interior confluence quasi-uniformity, which we will denote by δ_S. Now let us define on X the interior confluence relation D_S by

$$PD_SQ \Leftrightarrow P\imath Q \text{ or } \exists \ p \in R_S \text{ with } (P, Q) \in Z_b(L^p, M^p),$$

and for $f \in S$, $s > 0$ let us set

$$d_S(f, s) = \{ (f^{-1}(M(r, s)), f^{-1}(N(r, s))) \mid r \in R, \\ f^{-1}(M(r, s)) D_S f^{-1}(N(r, s)) \} .$$

Then it is a straightforward matter to verify that

$$\{ (d_S(f, s), D_S) \mid f \in S, s > 0 \}$$

is a subbase for the induced structure δ_S. Summarizing these results we have:

<u>Theorem</u> 3.4.3. Let $S \subseteq P(X)$ be bigenerating, and suppose that every S-derivative in $< S >$ is S-real. Then

$$\{ (d_S(f, s), D_S) \mid f \in S, s > 0 \}$$

is a subbase for a separated basic interior cqus δ_S on X, and the separated completion of (X, δ_S) is the space $(R_S, \text{cqu}(\pi_S))$.

<u>Corollary</u>. Each pair real compact strict extension of (X, u, v) may be obtained as the completion of a suitable confluence quasi-uniformity on X.

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INDEX OF SPECIAL SYMBOLS

This list contains, under the appropriate headings, the symbols and abbreviations used frequently in this thesis, with the number of the page on which the definition may be found. Standard symbols and abbreviations are not included.

T-lattices.

Note. The symbols $\prec$, $\leq$ and $\wedge$ listed above are also used with their standard meaning at various points in the text. It will be clear from the context which meaning is intended.

INDEX OF SPECIAL TERMS

The number denotes the page on which the definition may be found. Well established terms are not included in the list.