Lie–Rinehart algebras, Hopf algebroids with and without an antipode

by

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Abstract

Our main objects of study are Lie–Rinehart algebras, their enveloping algebras and their relation with other structures (Gerstenhaber algebras, Hopf algebroids, Leibniz algebras and algebroids). In particular we focus on two aspects:

1. In the same way that the universal enveloping algebra of a Lie algebra carries a Hopf algebra structure, the universal enveloping algebra of a Lie–Rinehart algebra is one of the richest class of examples of *Hopf algebroids* (a generalisation of Hopf algebras). We prove that, unlike in the classical Lie algebra case, the universal enveloping algebra of Lie–Rinehart algebras may or may not admit an antipode.

   We use the characterisation due to Kowalzig and Posthuma [KP11] of the antipode on the Hopf algebroid structure on the enveloping algebra of a Lie–Rinehart algebra in terms of left (and right) modules over its enveloping algebra [Hue98] and give examples of Lie–Rinehart algebras that do not admit these right modules structures and hence no antipode on the universal enveloping algebra of a Lie–Rinehart algebra.

   Moreover, we prove that some Lie–Rinehart algebras admit a structure weaker than right modules over its enveloping algebra which yields a generator of the corresponding Gerstenhaber algebra while not a square-zero one, hence not a differential. Our examples of these algebras arise when considering Jacobi algebras [Kir76, Lic78], a certain generalisation of Poisson algebras.

2. Following the work of Loday and Pirashvili [LP98] in which they analyse the functorial relation between Lie algebras in the category $\mathcal{LM}$ of linear maps (which they define) and Leibniz algebras, we study the relation between Lie–Rinehart algebras and Leibniz algebroids [IdLMP99]: After describing Lie–Rinehart algebra objects in the category $\mathcal{LM}$ of linear maps, we construct a functor from Lie–Rinehart algebra objects in $\mathcal{LM}$ to Leibniz algebroids .

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It is an honour to be given the chance to do maths beyond my PhD, and throughout that future, the professional standards, commitment to research and work ethics I experienced in Glasgow will be an example for me. But above all, it is the good friendships I have made during the last four years within the maths community that I consider one of the greatest achievements of my PhD.

This thesis is dedicated to my family: to my parents, to Negrito for looking after them, to my sister, and to the memory of Candelito.

Declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Contents

Introduction

Lie algebras are a fundamental structure in mathematics. However, they sometimes fail to provide an efficient framework to solve some interesting questions in non-commutative algebra or Poisson geometry which need a more general setting. This motivates the search for generalisations of Lie algebras. This generalisation has been attempted following many different approaches: for example as Lie algebroids [Pra67], Lie–Rinehart algebras [Rin63, Hue91], Leibniz algebras [LP93], Leibniz algebroids [LP93, IdLMP99], Courant algebroids [LWX97], Loday algebroids [SX08] or pre–Lie algebras [Ger63, Mat68].

Overview of results

We focus on two different problems involving Lie–Rinehart algebras:

Lie–Rinehart algebras and Hopf algebroids

This topic is developed in Chapters 1 and 2. We focus on the algebraic structure of the universal enveloping algebra of a Lie–Rinehart algebra. As in the classical case in which we study the Hopf algebra structure on the universal enveloping algebra of a Lie algebra, we ask what kind of Hopf algebra structure there exists on the universal enveloping algebra of a Lie–Rinehart algebra. It turns out that this associative algebra does not carry the structure of a Hopf algebra, but that of a Hopf algebroid, and is in fact one of the most important and richest class of examples of Hopf algebroids.

However, the term *Hopf algebroid* has been defined in different ways in the literature. See [BS04, Lu96, Sch00]. Böhm and Szlachányi [BS04] prove that their definition is inequivalent to the one given by Lu [Lu96]. Until our results in [KR15] it was an open problem whether the definitions given in [BS04] and [Sch00] were in fact equivalent. By giving a counterexample, in [KR15], we show that, unlike Hopf algebras, Hopf algebroids may or may not admit an antipode. Hopf algebroids with an antipode are called *full* Hopf algebroids, and those without an antipode are *left* Hopf algebroids following the terminology in the literature.

Furthermore, we prove that Jacobi algebras, which are a certain generalisation of Poisson algebras, give rise to Hopf algebroids (both left and full). Some of the results presented in this thesis have been published in [Rov14a].

Lie–Rinehart algebras and Leibniz algebroids

This topic is developed in Chapter 3. In this chapter we consider the relation between Lie–Rinehart algebras and other generalisations of Lie algebras given by Leibniz algebras [Lod93, LP93], Leibniz algebroids [IdLMP99] and Courant algebroids [LWX97]. Leibniz algebras were studied by Loday and Pirashvili [Lod93,LP93] (after having been introduced by Blokh [Blo65]) as a non skew-symmetric generalisation of Lie algebras. In particular, they are an R-module equipped with a bilinear form (called the Leibniz bracket) that is not necessarily skew-symmetric while satisfying a form of the Jacobi identity. Leibniz algebras appear in differential geometry as the algebraic structure on Courant algebroids [LWX97] and Leibniz algebroids [IdLMP99], and in mathematical physics, for example in Chern–Simons theory [FO09].

Loday and Pirashvili [LP93] observed that Leibniz algebras can be described as the canonical map

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{g}_{\mathrm{Lie}}$$

where $\mathfrak{g}$ is a Leibniz algebra and $\mathfrak{g}_{\mathrm{Lie}}$ is the Lie algebra which arises as the quotient of $\mathfrak{g}$ by the Leibniz ideal generated by elements $[x,x]_{\mathfrak{g}}$ for $x \in \mathfrak{g}$. This observation leads to their definition [LP98] of the monoidal category $\mathcal{LM}$ of linear maps in which a Leibniz algebra $\mathfrak{g}$ defines a Lie algebra object $\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$. Conversely, a Lie algebra object in $\mathcal{LM}$ gives rise to a Leibniz algebra (in R-mod) so that there exists a pair of adjoint functors between Lie algebras and Leibniz algebras in such a way that the functor

$$\text{Leibniz algebras} \longrightarrow \text{Lie algebras in } \mathcal{LM}$$

is left adjoint to

$$\text{Lie algebras in } \mathcal{LM} \longrightarrow \text{Leibniz algebras.}$$

Chapter 3 focuses on understanding the relation between Lie–Rinehart algebras and Leibniz algebroids, and in particular in generalising the funtorial relations between Lie algebras and Leibniz algebras. Our results [Rov14b] consist in describing Lie–Rinehart algebras in $\mathcal{LM}$ and then constructing a (right adjoint) functor

$$\text{Lie–Rinehart algebras in } \mathcal{LM} \longrightarrow \text{Leibniz algebroids.}$$

Constructing the corresponding left adjoint of this functor is part of a future research project not included in this thesis.

Structure of the Thesis

This thesis is divided into five chapters. Chapter 0 contains background material to understand the later chapters. Chapters 1 and 2 are dedicated to developing the results presented in [KR15] and [Rov14a] respectively. Chapter 3 contains some background on the monoidal category $\mathcal{LM}$ of linear maps and some results relating Lie–Rinehart algebras and Leibniz algebroids.

Chapter 0 This chapter provides some background on concepts which will be used later in the thesis.

- Section 0.1 focuses on topics in commutative algebra, which will be essential in Chapters 1 and 2.
- In Section 0.2, we explain some basic notation and results about the Lie derivative, contraction, polyvector fields and the Schouten bracket which we will use later in Chapters 2 and 3.
- Section 0.3 provides some background on Poisson geometry and Poisson algebras which will be used in Chapters 1 and 2.
- Section 0.4 introduces the main algebraic structure we aim to study: Hopf algebroids. This section is dedicated to the general description while Section 0.7 focuses on the Hopf algebroid structure on the universal enveloping algebra of a Lie–Rinehart algebra.

- Lastly, Sections 0.5 and 0.6 are dedicated to Lie–Rinehart algebras, their universal enveloping algebra and its modules.

Chapter 1 This chapter is dedicated to some fundamental constructions of Lie–Rinehart algebras and their enveloping algebras.

- We start in Section 1.1 by recalling the construction of the canonical Lie–Rinehart algebra over a Poisson algebra which we explain admits different Hopf algebroid structures on its universal enveloping algebra.
- Sections 1.2 and 1.3 are dedicated to new examples of Lie–Rinehart algebras and the construction of (A, L)-module structures on them.
- Lastly, in Section 1.4 we describe examples of Lie–Rinehart algebras whose enveloping algebra does *not* admit and antipode and is hence only a left Hopf algebroid. The first of these examples is the contents of [KR15].

Chapter 2 This chapter is broadly motivated by Poisson geometry. We study Lie–Rinehart algebras associated to Jacobi algebras, and give further examples that do not admit antipodes. These results are the contents of [Rov14a].

- In Section 2.1 we give some background and motivating examples on Jacobi algebras, and focus on the properties that we will use later.
- In Section 2.2, we present three different Lie–Rinehart algebra structures over a Jacobi algebra A and explain how one of them is related to the construction of Kerbrat and Souici-Benhammadi [KSB93], Okassa [Oka07] and Vaisman [Vai00] and for Lie algebroids.
- From the results by Huebschmann [Hue98] and in the light of [KP11], in Section 2.3 we describe the (full) Hopf algebroid structure one of the canonical Lie–Rinehart algebras over Jacobi algebras we construct in Section 2.2.
- Finally, in Section 2.4 we give further examples of Lie–Rinehart algebras that will not admit an antipode on their universal enveloping algebras. One of our examples will, unlike the example in [KR15], admit (A, L)-connections on A.

Chapter 3 This chapter is dedicated to generalising the work of Loday and Pirashvili on the functorial relation between Lie algebras and Leibniz algebras.

- In Section 3.1, we start by giving some background on Leibniz algebras, and on different generalisations of Leibniz algebras such as Leibniz algebroids, Courant algebroids and Courant–Dorfman algebras.
- In Section 3.2, we first review the category $\mathcal{LM}$ of linear maps, and then we develop the description [Rov14b] of Lie–Rinehart algebra objects in this category.
- Finally, in Section 3.3, we prove that Lie–Rinehart algebras in $\mathcal{LM}$ rise to Leibniz algebroids in R-mod, which is the main result of [Rov14b].

Chapter 0

Foundations

Throughout, let R denote a commutative ring with identity, and let A denote an associative, unital, R-algebra with multiplication map $\mu : A \otimes_R A \to A$ given by $a \otimes_R b \mapsto a \cdot b$.

0.1 Kähler differentials and 1-jet spaces

We start by studying two important A-modules which will be widely used in later sections. These A-modules arise when considering derivations of the associative R-algebra A. The main references for this section are [Har77, Mat80].

0.1.1 Kähler differentials

Definition 0.1.1. Let A be a commutative R-algebra and let $I \subset A \otimes_R A$ be the kernel of the multiplication map $\mu : A \otimes_R A \to A$. The A-module of **Kähler differentials over** A is the pair $(\Omega^1(A), d)$ where $\Omega^1(A) := I/I^2$ and d is the map

$$d : A \longrightarrow \Omega^1(A), \quad a \longmapsto 1 \otimes_R a - a \otimes_R 1 \pmod{I^2} \tag{0.1.1}$$

for all $a \in A$.

Remark 0.1.2. We will denote the Kähler differentials $\big(\Omega^1(A), d\big)$ over A as $\Omega^1(A)$.

Proposition 0.1.3. *Let* $\Omega^1(A)$ *be the* A*-module of Kähler differentials over* A*. The map* $d : A \to \Omega^1(A)$ *is a derivation of* A *with coefficients in* $\Omega^1(A)$*.*

Proof. By definition, for all $a, b \in A$, we have

$$d(a \cdot b) = 1 \otimes_R a \cdot b - a \cdot b \otimes_R 1 \pmod{I^2}. \tag{0.1.2}$$

We also have

$$a \cdot db + b \cdot da = a \cdot (1 \otimes_R b - b \otimes_R 1) + b \cdot (1 \otimes_R a - a \otimes_R 1) \pmod{I^2} \tag{0.1.3}$$
$$= a \otimes_R b - a \cdot b \otimes_R 1 + b \otimes_R a - b \cdot a \otimes_R 1 \pmod{I^2}. \tag{0.1.4}$$

Substracting Equation (0.1.4) from Equation (0.1.2), we obtain

$$\begin{aligned} d(a \cdot b) - (a \cdot db + b \cdot da) &= (1 \otimes_R a \cdot b - a \cdot b \otimes_R 1) \\ &\quad - (a \otimes_R b - a \cdot b \otimes_R 1 + b \otimes_R a - b \cdot a \otimes_R 1) \pmod{I^2} \\ &\equiv 1 \otimes_R a \cdot b - a \otimes_R b - b \otimes_R a + b \cdot a \otimes_R 1 \pmod{I^2} \\ &\equiv (1 \otimes_R a) \cdot (1 \otimes_R b) - (a \otimes_R 1) \cdot (1 \otimes_R b) \\ &\quad - (b \otimes_R 1) \cdot (1 \otimes_R a) + (b \otimes_R 1) \cdot (a \otimes_R 1) \pmod{I^2} \\ &\equiv (1 \otimes_R a - a \otimes_R 1) \cdot (1 \otimes_R b) \\ &\quad - (b \otimes_R 1) \cdot (1 \otimes_R a - a \otimes_R 1) \pmod{I^2} \\ &\equiv (1 \otimes_R a - a \otimes_R 1) \cdot (1 \otimes_R b - b \otimes_R 1) \equiv 0 \pmod{I^2}. \end{aligned}$$

Thus we find that the map $d : A \to \Omega^1(A)$ satisfies the Leibniz rule

$$d(a \cdot b) = a \cdot db + b \cdot da \tag{0.1.5}$$

and is hence a derivation of A, with coefficients in $\Omega^1(A)$. □

Proposition 0.1.4. *As an A-module, $\Omega^1(A)$ is generated by the elements $da \in \Omega^1(A)$ for $a \in A$.*

Proof. For $a \otimes_R b \in A \otimes_R A$ we have

$$\begin{aligned} \sum a_i \otimes_R b_i &= \sum a_i \cdot b_i \otimes_R 1 + \sum a_i (1 \otimes_R b_i - b_i \otimes_R 1) \\ &= \sum \mu(a_i \otimes_R b_i) \otimes_R 1 + \sum a_i \cdot (1 \otimes_R b_i - b_i \otimes_R 1) \end{aligned}$$

Therefore, if $\sum a_i \otimes_R b_i \in I$ so that $\sum \mu(a_i \otimes_R b_i) = 0$, then $\sum a_i \otimes_R b_i = \sum a_i \cdot (1 \otimes_R b_i - b_i \otimes_R 1)$. Since we have $1 \otimes_R b_i - b_i \otimes_R 1 \pmod{I^2} = db_i$, we deduce that any element of the quotient $\Omega^1(A) = I/I^2$ has the form $\sum a_i \cdot db_i$. In other words, $\Omega^1(A)$ is generated by $\{db \mid b \in A\}$ as an A-module. □

Proposition 0.1.5. *Let A be a commutative R-algebra, and $\Omega^1(A)$ the A-module of Kähler differentials of A. The differential map*

$$d : A \longrightarrow \Omega^1(A)$$

is universal in the following sense: for any A-module M, and any derivation $d' : A \to M$, there exists a unique A-module homomorphism $\psi : \Omega^1(A) \to M$ such that $d' = \psi \circ d$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega^1(A) \\ & {\scriptstyle d'}\searrow & \downarrow{\scriptstyle \psi} \\ & & M \end{array}$$

Proof. Let the A-module $A \oplus M$ be the trivial extension of A by M with ring multiplication defined by

$$(a, m) \cdot (b, n) = (a \cdot b, a \cdot n + b \cdot m)$$

for $a, b \in A, m, n \in M$ and let the map

$$f : A \otimes_R A \longrightarrow A \oplus M$$

be the A-algebra map given by

$$a \otimes_R b \longmapsto (a \cdot b, a \cdot d'(b)).$$

Since $f(I) \subseteq M$ and $M^2 = 0$, we have $f(I^2) = 0$ so that f induces a homomorphism $\tilde{f}$ of A-algebras

$$A \oplus \Omega^1(A) \longrightarrow A \oplus M$$

which maps $db \in \Omega^1(A)$ to $f(1 \otimes_R b - b \otimes_R 1) = (0, d'(b))$. Thus the restriction of $\tilde{f}$ to $\Omega^1(A)$ gives an A-linear map $\psi : \Omega^1(A) \to M$ with $\psi \circ d = d'$. □

The p-th exterior product $\Lambda^p \Omega(A)$ is denoted by $\Omega^p(A)$ and is called the module of differentials of degree p over A.

0.1.2 The 1-jet space over A

In this section we describe the A-module structure of the 1-jet space of a commutative algebra A.

Definition 0.1.6. Let A be a commutative algebra, and let $I = \mathrm{Ker}\mu \subset A \otimes_R A$. The **1-jet space of** A is the A-module given by

$$\mathcal{J}^1(A) := (A \otimes_R A)/I^2. \tag{0.1.6}$$

We now give a characterisation of the A-module $\mathcal{J}^1(A)$ as the trivial extension of A by $\Omega^1(A)$, the A-module of Kähler differentials over A, explaining its relation to Equation (0.1.6), see [Mat80, Chapter 10] for more details.

Proposition 0.1.7. *There exists a canonical isomorphism of A-modules*

$$(A \otimes_R A)/I^2 \cong A \oplus \Omega^1(A) \tag{0.1.7}$$

which identifies $a \otimes_R b \pmod{I^2}$ with $(a \cdot b, a \cdot db)$ for all $a, b \in A$.

Proof. First recall that $\Omega^1(A) = I/I^2$ and $da = 1 \otimes_R a - a \otimes_R 1 \pmod{I^2}$, so that

$$a \cdot db = a \otimes_R b - a \cdot b \otimes_R 1 \pmod{I^2}.$$

Let $\lambda : A \to A \otimes_R A$ be given by $a \mapsto a \otimes_R 1$. Then for all $a, b \in A$ we write $a \otimes_R b \in A \otimes_R A$ as

$$a \otimes_R b = a \cdot b \otimes_R 1 + (a \otimes_R b - a \cdot b \otimes_R 1)$$

where $a \cdot b \otimes_R 1 \in \lambda(A)$ and $a \otimes_R b - a \cdot b \otimes_R 1 \in I$. Since $I \cap \lambda(A) = 0$, we deduce

$$A \otimes_R A = \lambda(A) \oplus I, \quad \lambda(A)/I^2 = \lambda(A).$$

Since λ is injective, we can identify $\lambda(A)$ with A, hence

$$\begin{aligned}(A \otimes_R A)/I^2 &= (\lambda(A) \oplus I)/I^2 \\ &= \left(\lambda(A)/I^2\right) \oplus \left(I/I^2\right) \cong A \oplus \Omega^1(A).\end{aligned}$$

and $a \otimes_R b \pmod{I^2}$ is identified with $(a \cdot b, a \cdot db)$. ∎

Proposition 0.1.8. *As an A-module, $\mathcal{J}^1(A)$ is generated by the image of A under the map*

$$j^1 : A \longrightarrow \mathcal{J}^1(A), \quad a \longmapsto 1 \otimes_R a \pmod{I^2}, \tag{0.1.8}$$

*called the **1-jet map**. The elements $j^1(a) \in \mathcal{J}^1(A)$ satisfy the Leibniz rule*

$$j^1(a \cdot b) - a \cdot j^1(b) - b \cdot j^1(a) + a \cdot b \cdot j^1(1) = 0. \tag{0.1.9}$$

Proof. For $\sum a_i \otimes_R b_i \in A \otimes_R A$ we have $\sum a_i \otimes_R b_i = \sum a_i \cdot (1 \otimes_R b_i)$. Since we have $j^1(b_i) = 1 \otimes_R b_i \pmod{I^2}$ we deduce that any element in $\mathcal{J}^1(A) = (A \otimes_R A)/I^2$ is of the form $\sum a_i \cdot j^1(b_i)$. Hence we deduce that, as an A-module, $\mathcal{J}^1(A)$ is generated by $j^1(a) \in \mathcal{J}^1(A)$ for all $a \in A$. Furthermore, we have

$$\begin{aligned} j^1(a \cdot b) - a \cdot j^1(b) - b \cdot j^1(a) + a \cdot b \cdot j^1(1) &= 1 \otimes_R a \cdot b - a \otimes_R b - b \otimes_R a + a \cdot b \otimes_R 1 \pmod{I^2} \\ &\equiv (1 \otimes_R a) \cdot (1 \otimes_R b) - (a \otimes_R 1) \cdot (1 \otimes_R b) \\ &\quad - (1 \otimes_R a) \cdot (b \otimes_R 1) + (a \otimes_R 1) \cdot (b \otimes_R 1) \pmod{I^2} \\ &\equiv (1 \otimes_R a) \cdot (1 \otimes_R b - b \otimes_R 1) \\ &\quad - (a \otimes_R 1) \cdot (1 \otimes_R b - b \otimes_R 1) \pmod{I^2} \\ &= (1 \otimes_R a - a \otimes_R 1) \cdot (1 \otimes_R b - b \otimes_R 1) \pmod{I^2} \equiv 0 \end{aligned}$$

so that the Leibniz rule in Equation (0.1.9) holds. □

Remark 0.1.9. Note that the isomorphism $(A \otimes_R A)/I^2 \cong A \oplus \Omega^1(A)$ given in Equation (0.1.7) identifies $j^1(a) \in \mathcal{J}^1(A)$ with $(a, da) \in A \oplus \Omega^1(A)$ for all $a \in A$.

0.1.3 Differential operators

Definition 0.1.10. Let A be a commutative R-algebra, and M an A-module. Any R-linear map $\mathfrak{d} : A \to M$ induces a map

$$[a, \mathfrak{d}] : A \longrightarrow M, \quad b \longmapsto [a, \mathfrak{d}](b) := a \cdot \mathfrak{d}(b) - \mathfrak{d}(a \cdot b) \tag{0.1.10}$$

for $a, b \in A$. The map $\mathfrak{d} : A \to M$ **is an n^{th}-order differential operator**, if for any $n+1$ elements in A, the following identity is satisfied

$$[a_n, [a_{n-1}, [\cdots [a_2, [a_1, [a_0, \mathfrak{d}]], \cdots](b) = 0, \quad \forall a_i, b \in A. \tag{0.1.11}$$

Proposition 0.1.11. *Let A be a commutative R-algebra, and M be an A-module.*

1. *If $\xi \in \mathrm{Der}_R(A)$ and $m \in M$, then the R-linear map $f : A \to M$ given by*

$$f(a) = \xi(a) + a \cdot m \tag{0.1.12}$$

for all $a \in A$, is a first order differential operator on A.

2. *All first order differential operators $f : A \to M$ on A with coefficients in M are of the form given in (0.1.12).*

Proof. Let $\xi \in \mathrm{Der}_R(A, M)$, $m \in M$ and $f : A \to M$ be the map given by $a \mapsto \xi(a) + a \cdot m$ for all $a \in A$. For $a_1, a_2 \in A$ we have

$$\begin{aligned}
[a_1, [a_0, f]] &= [a_1, -f(a_0 \cdot \bullet) + a_0 \cdot f(\bullet)] \\
&= [a_1, -\xi(a_0 \cdot \bullet) - a_0 \cdot \bullet \cdot m + a_0 \cdot \xi(\bullet) + a_0 \cdot \bullet \cdot m] \\
&= [a_1, -\xi(a_0 \cdot \bullet) + a_0 \cdot \xi(\bullet)] \\
&= [a_1, -a_0 \cdot \xi(\bullet) - \bullet \cdot \xi(a_0) + a_0 \cdot \xi(\bullet)] \\
&= [a_1, -\bullet \cdot \xi(a_0)] = a_1 \cdot \bullet \cdot \xi(a_0) - a_1 \cdot \bullet \cdot \xi(a_0) = 0
\end{aligned}$$

so $f : A \to M$ is a first order differential operator.

Conversely, we now check that all first order differential operators are of this form. Let $\mathfrak{d} : A \to M$ be a first order differential operator, and let us define $m := \mathfrak{d}(1)$. By definition

$$\begin{aligned}
0 = [a_1, [a_0, \mathfrak{d}]](1) &= a_1 \cdot [a_0, \mathfrak{d}](1) - [a_0, \mathfrak{d}](a_1 \cdot 1) \\
&= a_1 \cdot a_0 \cdot \mathfrak{d}(1) - a_1 \cdot \mathfrak{d}(a_0 \cdot 1) - a_0 \cdot \mathfrak{d}(a_1) + \mathfrak{d}(a_0 \cdot a_1) \\
&= -a_0 \cdot \mathfrak{d}(a_1) + a_0 \cdot a_1 \cdot \mathfrak{d}(1) - a_1 \cdot \mathfrak{d}(a_0) + a_1 \cdot a_0 \cdot \mathfrak{d}(1) + \mathfrak{d}(a_0 \cdot a_1) - a_0 \cdot a_1 \cdot \mathfrak{d}(1) \\
&= -a_0 \cdot (\mathfrak{d} - m)(a_1) - a_1 \cdot (\mathfrak{d} - m)(a_0) + (\mathfrak{d} - m)(a_0 \cdot a_1)
\end{aligned}$$

Hence the operator $\zeta := \mathfrak{d} - m$ is a derivation of A with values in M. □

Corollary 0.1.12. *Let A be a commutative R-algebra, M be an A-module, and $f : A \to M$ be a first order differential operator on A with values in M. The following Leibniz rule is satisfied:*

$$f(a \cdot b) = a \cdot f(b) + b \cdot f(a) - a \cdot b \cdot f(1) \tag{0.1.13}$$

for all $a, b \in A$.

Proof. We proved in Proposition 0.1.11 that a first order differential operator on A with values in M is a map $f : A \to M$ given by $a \mapsto \xi(a) + a \cdot m$ for all $a \in A$, $\xi \in \mathrm{Der}_R(A, M)$ and $m \in M$. So we have

$$\begin{aligned} f(a \cdot b) &= \xi(a \cdot b) + a \cdot b \cdot m \\ &= a \cdot \xi(b) + b \cdot \xi(a) + a \cdot b \cdot m \\ &= a \cdot (\xi(b) + b \cdot m) + b \cdot (\xi(a) + a \cdot m) - a \cdot b \cdot (\xi(1) + 1 \cdot m) \\ &= a \cdot f(b) + b \cdot f(a) - a \cdot b \cdot f(1) \end{aligned}$$

□

Example 0.1.13. Let A be a commutative R-algebra, and $\mathcal{J}^1(A)$ its 1-jet space. From Equation (0.1.9) we deduce that the 1-jet map $j^1 : A \to \mathcal{J}^1(A)$ is a first order differential operator on A with values in $\mathcal{J}^1(A)$.

0.2 Lie derivative, Cartan formula

In this brief section, we present some basic results in differential calculus which we will use in later sections. The main reference is [Mor01].

Definition 0.2.1. The differential forms $\Omega^\bullet(M)$ on a manifold M are the sections of

$$\Lambda^\bullet T^* M := \bigoplus \Lambda^p T^* M.$$

Remark 0.2.2. By the universal property of Kähler differentials, for a given manifold M, the 1-forms $\Omega^1(M)$ are a quotient of the module of Kähler differentials of $C^\infty(M)$.

Proposition 0.2.3. *Given a vector field ξ on a manifold M, there is a $C^\infty(M)$-linear map*

$$\iota_\xi : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$$

called the interior product such that

- $\iota_\xi(df) = \xi(f)$
- $\iota_\xi(\alpha \wedge \beta) = \iota_\xi \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_\xi \beta, \quad \forall \alpha \in \Omega^p(M), \beta \in \Omega^q(M).$

Since ι_ξ is $C^\infty(M)$-linear, we have $\iota_\xi(f \cdot \alpha) = f \cdot \iota_\xi \alpha$ for $f \in C^\infty(M)$.

Example 0.2.4. Take $\alpha = dx \wedge dy$ and $\xi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$, then $\iota_\xi \alpha = xdy - ydx$.

Definition 0.2.5. Let M be a manifold, and ξ a vector field on M. The linear operator

$$\mathcal{L}_\xi : \Omega^k(M) \longrightarrow \Omega^k(M)$$

given by

$$\mathcal{L}_\xi \alpha(\xi_1 \wedge \ldots \wedge \xi_k) = \xi\alpha(\xi_1 \wedge \ldots \wedge \xi_k) - \sum_{i=1}^{k} \alpha(\xi_1 \wedge \ldots \wedge [\xi, \xi_i] \wedge \ldots \wedge \xi_k)$$

is called the **Lie derivative**.

Theorem 0.2.6 (Cartan formula)**.** *The Lie derivative $\mathcal{L}_\xi$ on a p-form α satisfies*

$$\mathcal{L}_\xi \alpha = d\left(\iota_\xi \alpha\right) + \iota_\xi d\alpha \tag{0.2.1}$$

Proof. See [Mor01, Theorem 2.11] ☐

Example 0.2.7. Let ξ be a vector field on a manifold M. Given $f \in C^\infty(M)$ we have

$$\mathcal{L}_\xi f = \xi(f).$$

Example 0.2.8. Let M be manifold. Taking $\alpha = a \cdot df \in \Omega^1(M)$ and ξ a vector field on M, we have

$$\begin{aligned}
\mathcal{L}_\xi(a \cdot df) &= d\left(\iota_\xi(a \cdot df)\right) + \iota_\xi d\left(a \cdot df\right) \\
&= d\left(a \cdot \iota_\xi(df)\right) + \iota_\xi\left(da \wedge df\right) \\
&= d\left(a \cdot \xi(f)\right) + \iota_\xi(da) \wedge df - da \wedge \iota_\xi(df) \\
&= a \cdot d\left(\xi(f)\right) + \xi(f) \cdot da + \xi(a) \cdot df - \xi(f) \cdot da \\
&= a \cdot d\left(\xi(f)\right) + \xi(a) \cdot df
\end{aligned}$$

so in particular we have that for $\alpha = df$ in Equation (0.2.1) we can write

$$\mathcal{L}_\xi(df) = d\left(\iota_\xi(df)\right) = d\left(\xi(f)\right).$$

Remark 0.2.9. Also note that $\mathcal{L}_\xi$ commutes with d so that we have

$$\mathcal{L}_\xi(df) = d\left(\mathcal{L}_\xi f\right) = d\left(\xi(f)\right), \qquad \mathcal{L}_\xi f = \iota_\xi(df) + d\left(\iota_\xi f\right).$$

Definition 0.2.10. *Let M be a manifold. There exists a wedge product on TM which is dual to that on the sections of $\Lambda^\bullet T^*M$ giving rise to $\Lambda^\bullet TM := \Lambda^k TM$. Sections of $\Lambda^k TM$ are called k-vector fields.*

Remark 0.2.11. *Let M be a manifold. k-vector fields on M are dual to differential k-forms.*

Definition 0.2.12. The Schouten–Nijenhuis bracket of k-vector fields is the bilinear map given by

$$\Lambda^\ell(TM) \otimes \Lambda^{n-\ell}(TM) \longrightarrow \Lambda^{n-1}(TM)$$

with

$$[\![u, v]\!] = (-1)^{|u|} \sum_{j \le \ell < k} (-1)^{(j+k)} \left([\xi_j, \xi_k] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n \right) \tag{0.2.2}$$

where $u = \xi_1 \wedge \ldots \wedge \xi_\ell \in \Lambda^\ell(TM)$ and $v = \xi_{\ell+1} \wedge \ldots \wedge \xi_n \in \Lambda^{n-\ell}(TM)$ and elements $\hat{\xi}_j, \hat{\xi}_k \in TM$ are omitted.

Remark 0.2.13. *Let M be a manifold, Λ be a k-vector field and $\alpha \in \Omega^k(M)$ be a differential k-form. The pairing of Λ with $\alpha \in \Omega^k(M)$, denoted by $\langle \alpha, \Lambda \rangle$, is a function in $C^\infty(M)$.*

Example 0.2.14. *Take the 2-form $\alpha = dx \wedge dy$ and the bivector field $\Lambda = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, then we have $\langle \alpha, \Lambda \rangle = x$.*

Definition 0.2.15. *Given a manifold M, a k-vector field Λ defines a skew symmetric map*

$$C^\infty(M) \otimes \ldots \otimes C^\infty(M) \longrightarrow C^\infty(M)$$

given by

$$\Lambda(df_1, \ldots, df_k) := \langle df_1 \wedge \ldots \wedge df_k, \Lambda \rangle, \quad \text{for } f_1 \ldots f_k \in C^\infty(M).$$

Proposition 0.2.16. *Given a 2-vector field Λ on a manifold M, we can associate to it a natural $C^\infty(M)$-linear homomorphism*

$$\Lambda^\# : T^*M \longrightarrow TM, \quad df \longmapsto \Lambda^\#(df)$$

such that $\langle \alpha \wedge \beta, \Lambda \rangle = \langle \beta, \Lambda^\#(\alpha) \rangle$ for $\alpha, \beta \in T^*M$.

Proof. See [DZ05]. □

0.3 Poisson geometry

In this section, we give a geometric interpretation of the Poisson bracket. See for example [DZ05] for more details.

Definition 0.3.1. A Poisson structure on a manifold M is a bivector field Π on M, such that the corresponding bracket on the space of functions on M, defined by

$$\{f,g\} := \langle df \wedge dg, \Pi\rangle$$

satisfies the Jacobi identity. (M,Π) is then called a **Poisson manifold**.

Proposition 0.3.2. *The operator $\{f,\bullet\}$ on $C^\infty(M)$ for all $f \in C^\infty(M)$ induced by a Poisson structure Π on M is a derivation.*

Proof. Note that

$$\begin{aligned}\{f,g\cdot h\} &= \langle df \wedge d(g\cdot h),\Pi\rangle = \langle df\wedge g\cdot dh,\Pi\rangle + \langle df \wedge h\cdot dg,\Pi\rangle \\ &= g\cdot\langle df\wedge dh,\Pi\rangle + h\cdot\langle df\wedge dg,\Pi\rangle \\ &= g\cdot\{f,h\} + h\cdot\{f,g\}\end{aligned}$$

□

Remark 0.3.3. The Jacobi identity for the bracket $\{f,g\} = \langle df\wedge dg,\Pi\rangle$ is equivalent to the equation $[\![\Pi,\Pi]\!] = 0$ where $[\![\bullet,\bullet]\!]$ is the Schouten bracket [Sch40].

More generally, we define a Poisson structure on an arbitrary algebra A as follows:

Definition 0.3.4. A **Poisson algebra** is a commutative R-algebra A equipped with a Lie bracket $\{\bullet,\bullet\}$, called **Poisson structure**, satisfying the Leibniz rule

$$\{a\cdot b,c\} = a\cdot\{b,c\} + b\cdot\{a,c\}, \quad \forall a,b,c\in A. \tag{0.3.1}$$

Remark 0.3.5. By Definition 0.3.4 we have $\{1,\bullet\} = 0$.

Lemma 0.3.6. *Let A be a Poisson algebra over R with bracket $\{\bullet,\bullet\}$. The Poisson structure $\{\bullet,\bullet\}$ induces an R-linear Lie algebra map from A into $\mathrm{Der}_R(A)$ defined by*

$$\xi : A \longrightarrow \mathrm{Der}_R(A), \quad a \longmapsto \xi_a := \{a,\bullet\}, \quad a\in A. \tag{0.3.2}$$

that is

$$\xi_{\{a,b\}} = [\xi_a,\xi_b]_{\mathrm{Der}(A)}. \tag{0.3.3}$$

Proof. Firstly, from Equation (0.3.1) we see that the operator $\{a, \bullet\}$ is a derivation of A. Now let $\xi_{\{a,b\}} - [\xi_a, \xi_b]_{\mathrm{Der}(A)}$ act on an arbitrary element $c \in A$, then

$$\begin{aligned}(\xi_{\{a,b\}} - [\xi_a, \xi_b]_{\mathrm{Der}_R(A)})(c) &= \xi_{\{a,b\}}(c) - [\xi_a, \xi_b]_{\mathrm{Der}(A)}(c) \\ &= \{\{a,b\},c\} - \xi_a\left(\xi_b(c)\right) + \xi_b\left(\xi_a(c)\right) \\ &= \{\{a,b\},c\} - \xi_a(\{b,c\}) + \xi_b(\{a,c\}) \\ &= \{\{a,b\},c\} - \{a,\{b,c\}\} + \{b,\{a,c\}\}.\end{aligned}$$

Thus, Equation 0.3.2 is equivalent to the Jacobi identity for the Poisson bracket. □

Proposition 0.3.7. *Let A be a Poisson algebra with bracket $\{\bullet, \bullet\}$, and let $\mathrm{Der}_R(A)$ be its Lie algebra of derivations. Elements of $\mathrm{Der}_R(A)$ that are in the image of A under the R-linear map $\xi : A \to \mathrm{Der}_R(A)$ satisfy the relation*

$$a \cdot \xi_b + b \cdot \xi_a - \xi_{a \cdot b} = 0, \quad a, b \in A \tag{0.3.4}$$

or alternatively,

$$a \cdot \{b, \bullet\} + b \cdot \{a, \bullet\} - \{a \cdot b, \bullet\} = 0. \tag{0.3.5}$$

Proof. Follows from the fact that the Poisson bracket $\{\bullet, \bullet\}$ is a derivation of A, i.e., it satisfies the Leibniz identity given in Equation (0.3.1). □

Definition 0.3.8. Let A be a Poisson algebra over R with bracket $\{\bullet, \bullet\}$. A **Poisson derivation** is an R-linear map $D : A \to A$ satisfying

$$D(a \cdot b) = a \cdot D(b) + D(a) \cdot b \tag{0.3.6}$$

$$D\left(\{a,b\}\right) = \{D(a), b\} + \{a, D(b)\}. \tag{0.3.7}$$

0.4 Basics on Hopf algebroids

Hopf algebras play an important role in non-commutative algebra, non-commutative geometry and mathematical physics. For background on Hopf algebras see for example [Mon93]. In this section we give some fundamental background on Hopf algebroids, which are a generalisation of Hopf algebras to a setting in which, for example, the underlying ground ring is allowed to be non-commutative or the antipode is twisted by a character. In some Hopf algebroids, as we will see, an antipode may not even exist.

The main references for this section are [Böh09, BS04, KK10, Lu96, Sch00]. In Chapters 1 and 2 we will focus on the Hopf algebroid structure on the universal enveloping algebra of a Lie–Rinehart algebra. Hopf algebroids in the sense of [BS04] are an algebraic structure consisting of a left bialgebroid and a right bialgebroid which are compatible with an antipode in a certain way. On their own right, bialgebroids are also the result of making two algebraic structures compatible: $A \otimes_R A^{op}$-rings and A-corings. We see that this makes Hopf algebroids a very rich structure.

0.4.1 Bialgebroids

We start by defining bialgebroids. Throughout the section, let A^{op} be the opposite algebra of the R-algebra A with multiplication $\mu^{op}(a \otimes_{R^{op}} b) := \mu(b \otimes_R a) = b \cdot a$.

$A \otimes_R A^{op}$-rings

Definition 0.4.1. Let U be an R-algebra and let $\eta : A \otimes_R A^{op} \to U$ be an R-algebra map. An $A \otimes_R A^{op}$**-ring** is a triple (U, s, t), where s, t are algebra maps given by the restrictions

$$s := \eta(\bullet \otimes_R 1) : A \longrightarrow U, \quad t := \eta(1 \otimes_R \bullet) : A^{op} \longrightarrow U$$

with commuting ranges in U.

Example 0.4.2. *Let A be an R-algebra. The vector space $A \otimes_R A^{op}$ is an $A \otimes_R A^{op}$-ring with source and target maps given respectively by*

$$s : A \longrightarrow A \otimes_R A^{op}, \quad a \longmapsto a \otimes_R 1, \quad \text{and} \quad t : A^{op} \longrightarrow A \otimes_R A^{op}, \quad a \longmapsto 1 \otimes_R a.$$

Proposition 0.4.3. *Let U be an $A \otimes_R A^{op}$-ring. There exist four different A-module structures on U given by the following actions:*

$$\begin{aligned} A \otimes_R U &\longrightarrow U, \quad a \otimes_R u \longmapsto ut(a) =: a \blacktriangleright u \\ A \otimes_R U &\longrightarrow U, \quad a \otimes_R u \longmapsto s(a)u =: a \triangleright u \\ U \otimes_R A &\longrightarrow U, \quad u \otimes_R a \longmapsto us(a) =: u \blacktriangleleft a \\ U \otimes_R A &\longrightarrow U, \quad u \otimes_R a \longmapsto t(a)u =: u \triangleleft a. \end{aligned}$$

Proof. See [BS04]. □

Definition 0.4.4. Let U be an $A \otimes_R A^{op}$-ring. We define tensor products over A and A^{op} as quotients of $U \otimes_R U$ by relations involving the left and right A-module structures on U. In particular, we have the following quotients:

$$
{}_{\blacktriangleright}U \otimes_{A^{op}} U_{\triangleleft} := U \otimes_R U/\text{span}\{a \blacktriangleright u \otimes_R v - u \otimes_R v \triangleleft a \mid u, v \in U, a \in A\} \tag{0.4.1}
$$

$$
U_{\triangleleft} \otimes_A {}_{\triangleright}U := U \otimes_R U/\text{span}\{u \triangleleft a \otimes_R v - u \otimes_R a \triangleright v \mid u, v \in U, a \in A\}. \tag{0.4.2}
$$

Example 0.4.5. *For the $A \otimes_R A^{op}$-ring $A \otimes_R A^{op}$ and $a, b, c \in A$ we have*

$$
\begin{aligned}
a \blacktriangleright (b \otimes_R c) &= (b \otimes_R c)t(a) = (b \otimes_R c)(1 \otimes_R a) = b \otimes_R a \cdot c \\
a \triangleright (b \otimes_R c) &= s(a)(b \otimes_R c) = (a \otimes_R 1)(b \otimes_R c) = a \cdot b \otimes_R c \\
(b \otimes_R c) \blacktriangleleft a &= (b \otimes_R c)s(a) = (b \otimes_R c)(a \otimes_R 1) = b \cdot a \otimes_R c \\
(b \otimes_R c) \triangleleft a &= t(a)(b \otimes_R c) = (1 \otimes_R a)(b \otimes_R c) = b \otimes_R c \cdot a.
\end{aligned}
$$

Definition 0.4.6. Let (B, μ, η) be an A-ring. A right, respectively left, character on (B, μ, η) is a R-module map $\chi : B \to A$ satisfying:

- $\chi(u \cdot \eta(a)) = \chi(u) \cdot a$ for $u \in B$ and $a \in A$
- $\chi(u \cdot u') = \chi\left((\eta \circ \chi)(u) \cdot u'\right)$, for $u, u' \in B$
- $\chi(1_B) = 1_A$

respectively

- $\chi(\eta(a) \cdot u) = a \cdot \chi(u)$ for $u \in B$ and $a \in A$
- $\chi(u \cdot u') = \chi\left((\eta \circ \chi)(u) \cdot u'\right)$, for $u, u' \in B$
- $\chi(1_B) = 1_A$

A-corings

Definition 0.4.7 (See [Swe75]). An A-**coring** is a triple (C, Δ, ϵ), where A is an R-algebra, C is an A-bimodule and

$$
\Delta : C \longrightarrow C \otimes_A C \quad \text{and} \quad \epsilon : C \longrightarrow A
$$

are A-bimodule maps, called comultiplication and counit respectively, satisfying the coassociativity and counitality conditions

$$(\Delta \otimes_A C) \circ \Delta = (C \otimes_A \Delta) \circ \Delta, \quad (\epsilon \otimes_A C) \circ \Delta = C = (C \otimes_A \epsilon) \circ \Delta. \tag{0.4.3}$$

Example 0.4.8. *Let A be an R-algebra, so that it is an A-bimodule over itself. The maps*

$$\Delta : A \longrightarrow A \otimes_A A, \quad a \longmapsto a \otimes_A 1, \quad \text{and} \quad \epsilon : A \longrightarrow A, \quad a \longmapsto a$$

are equivalent to the identity map and turn the triple (A, Δ, ϵ) into an A-coring since the counitality conditions in Equation (0.4.3) are satisfied.

Bialgebroids

Depending on the compatibility conditions between an $A \otimes_R A^{op}$-ring and an A-coring, we define *left* and *right* bialgebroids.

Definition 0.4.9 (See [Swe75])**.** Let A be an R-algebra. A **left A-bialgebroid $\mathcal{B}$** consists of an $A \otimes_R A^{op}$-ring (B, s, t) and an A-coring (B, Δ, ϵ) on the same R-module B. These two structures on B satisfy the following compatibiliby conditions:

- The bimodule structure in the A-coring (B, Δ, ϵ) is related to the $A \otimes_R A^{op}$-ring (B, s, t) via
$$a \cdot b \cdot a' := s(a)t(a')b = a \triangleright b \triangleleft a', \tag{0.4.4}$$
for $a, a' \in A, b \in B$.

- Considering B as an A-bimodule as in Equation (0.4.4), the coproduct Δ corestricts to an R-algebra map from B to
$$B_A \times B := \left\{ \sum_i b_i \otimes_A b'_i \mid \sum s(a) b_i \otimes_A b'_i = \sum_i b_i \otimes_A t(a) b'_i, \forall a \in A \right\}$$
where $B_A \times B$ is an algebra via factorwise multiplication.

- The counit ϵ is a left character on the A-ring (B, s, t). We denote elements in $B_A \times B$ by $b_{(1)} \otimes_A b_{(2)}$ for $b \in B$.

We denote left A-bialgebroids as $(B, A, \Delta_\ell, \epsilon_\ell, s_\ell, t_\ell)$.

Example 0.4.10. Let A be an R-algebra. The vector space $A \otimes_R A^{op}$ carries a left bialgebroid structure with

- Source and targets maps given respectively by

$$s_\ell : A \longrightarrow A \otimes_R A^{op}, \quad a \longmapsto a \otimes_R 1, \quad \text{and} \quad t_\ell : A \longrightarrow A \otimes_R A^{op}, \quad a \longmapsto 1 \otimes_R a$$

- Comultiplication given by

$$\Delta_\ell : A \otimes_R A^{op} \longrightarrow (A \otimes_R A^{op}) \otimes_A (A \otimes_R A^{op}), \quad a \otimes_R b \longmapsto (a \otimes_R 1) \otimes_A (1 \otimes_R b)$$

- Counit $\epsilon_\ell : A \otimes_R A^{op} \to A, \quad a \otimes b \mapsto a \cdot b$

for $a, b \in A$. We can check that the compatibility relations given in Definition 0.4.9 are satisfied.

Definition 0.4.11 (See [Tak77]). Let A be an R-algebra. A **right A-bialgebroid** $\mathcal{B}$ consists of an $A \otimes_R A^{op}$-ring (B, s, t) and an A-coring (B, Δ, ϵ) on the same R-module B. These two structures on B satisfy the following compatibiliby conditions:

- The bimodule structure in the A-coring (B, Δ, ϵ) is related to the $A \otimes_R A^{op}$-ring (B, s, t) via

$$a \cdot b \cdot a' := bs(a')t(a) = a \blacktriangleright b \blacktriangleleft a', \tag{0.4.5}$$

  for $a, a' \in A, b \in B$.

- considering B as an A-bimodule as in Equation (0.4.5), the coproduct Δ corestricts to an R-algebra map from B to

$$B \times_A B := \left\{ \sum_i b_i \otimes_A b_i' \mid \sum s(a) b_i \otimes_A b_i' = \sum_i b_i \otimes_A t(a) b_i', \forall a \in A \right\}$$

  where $B \times_A B$ is an algebra, called the Takeuchi product, via factorwise multiplication. We denote elements in $B \times_A B$ by $b^{(1)} \otimes_A b^{(2)}$ for $b \in B$.

- The counit ϵ is a right character on the A-ring (B, s, t).

We denote right A-bialgebroids as $(B, A, \Delta_{\mathrm{r}}, \epsilon_{\mathrm{r}}, s_{\mathrm{r}}, t_{\mathrm{r}})$.

Example 0.4.12. Let A be an R-algebra. The vector space $A \otimes_R A^{op}$ with opposite co-opposite structures of the ones in Example 0.4.10 carries a right bialgebroid structure with

- Source and target maps given by

$$s_{\mathrm{r}} : A \to A \otimes_R A^{op}, \quad a \mapsto 1 \otimes_R a \quad \text{and} \quad t_{\mathrm{r}} : A \to A \otimes_R A^{op}, \quad a \mapsto a \otimes_R 1$$

- Right comultiplication given by

$$\Delta_{\mathrm{r}} : A \otimes_R A^{op} \longrightarrow (A \otimes_R A^{op}) \otimes_A (A \otimes_R A^{op}), \quad a \otimes_R b \mapsto (1 \otimes_R a) \otimes_A (b \otimes_R 1);$$

- Counit $\epsilon_{\mathrm{r}} : A \otimes_R A^{op} \to A^{op}, \quad a \otimes b \mapsto b \cdot a;$

- Multiplication

$$\mu_{\mathrm{r}} ((a \otimes_R x) \otimes_R (b \otimes_R y)) = x \cdot y \otimes_R b \cdot a$$

In the next section, we will see in which sense left and right bialgebroids over the same R-module can be compatible and how this compatibility gives rise to a new structure.

0.4.2 Full Hopf algebroids

A full Hopf algebroid H consists of both a left bialgebroid $(H_\ell, \mathrm{L}, \Delta_\ell, \epsilon_\ell, s_\ell, t_\ell)$ and a right bialgebroid $(H_{\mathrm{r}}, \mathrm{R}, \Delta_{\mathrm{r}}, \epsilon_{\mathrm{r}}, s_{\mathrm{r}}, t_{\mathrm{r}})$ structure on the same underlying R-algebra H such that there exists an R-module map $S : H \to H$, called the antipode map, satisfying the following compatibility conditions with the left and right bialgebroid structures on H:

(i) $s_\ell \circ \epsilon_\ell \circ t_{\mathrm{r}} = t_{\mathrm{r}}, \quad t_\ell \circ \epsilon_\ell \circ s_{\mathrm{r}} = s_{\mathrm{r}}, \quad s_{\mathrm{r}} \circ \epsilon_{\mathrm{r}} \circ t_\ell = t_\ell, \quad t_{\mathrm{r}} \circ \epsilon_{\mathrm{r}} \circ s_\ell = s_\ell.$

(ii) $(\Delta_\ell \otimes_{\mathrm{R}} H) \circ \Delta_{\mathrm{r}} = (H \otimes_{\mathrm{L}} \Delta_{\mathrm{r}}) \circ \Delta_\ell, \quad (\Delta_{\mathrm{r}} \otimes_{\mathrm{L}} H) \circ \Delta_\ell = (H \otimes_{\mathrm{R}} \Delta_\ell) \circ \Delta_{\mathrm{r}}.$

(iii) $S(t_\ell(l) h t_{\mathrm{r}}(r)) = s_{\mathrm{r}}(r) S(h) s_\ell(l)$ for $l \in \mathrm{L}$, $r \in \mathrm{R}$ and $h \in H$.

(iv) $\mu_\ell \circ (S \otimes_{\mathrm{L}} H) \circ \Delta_\ell = s_{\mathrm{r}} \circ \epsilon_{\mathrm{r}}, \quad \mu_{\mathrm{r}} \circ (H \otimes_{\mathrm{R}} S) \circ \Delta_{\mathrm{R}} = s_\ell \circ \epsilon_\ell.$

For a full definition, see [BS04, Definition 4.1] and [Böh09].

Example 0.4.13. Let A be an R-algebra. The vector space $H = A \otimes_R A^{op}$ carries a Hopf algebroid structure with antipode

$$S : H \longrightarrow H, \quad a \otimes_R b \longmapsto b \otimes_R a.$$

We can check that the compatibility relations in [BS04, Definition 4.1] are satisfied.

Lemma 0.4.14. *Let H be either a full Hopf algebroid [BS04] or a Hopf algebroid in the sense of Lu [Lu96], with antipode $S : H \to H$, left counit $\epsilon_\ell : H \to R$, and source and target maps $s_\ell, t_\ell : A \to H$. Put $h \in H, a \in A$, then the map given by*

$$ah := \epsilon_\ell(S(h)s_\ell(a)) \tag{0.4.6}$$

yields a right H-module structure on A for which the underlying left A-action on A is given by left multiplication.

Proof. The canonical left action of a left bialgebroid H on the base algebra A is given by

$$ha := \epsilon_\ell(hs_\ell(a)) = \epsilon_\ell(ht_\ell(a)),$$

and the antipode of a Hopf algebroid is an algebra antihomomorphism (see [Böh09, Proposition 4.4] respectively [Lu96, Definition 4.1] for the two different notions). Hence (0.4.6) defines a right action of H on A. Finally, one has $S \circ t_\ell = s_\ell$ (see [Böh09, Definition 4.1 (iii)] respectively [Lu96, Definition 4.1.2.]), so

$$at_\ell(b) = \epsilon_\ell(S(t_\ell(b))s_\ell(a)) = \epsilon_\ell(s_\ell(b)s_\ell(a)) = ba, \quad \forall a, b \in A. \quad \square$$

0.4.3 Left Hopf algebroids, $\times_A$-Hopf algebras

Definition 0.4.15 (See [Sch00])**.** Let U be a left bialgebroid. The map

$$\beta : {}_\blacktriangleright U \otimes_{A^{op}} U_\triangleleft \longrightarrow U_\triangleleft \otimes_A {}_\triangleright U, \quad u \otimes_{A^{op}} v \longmapsto u_{(1)} \otimes_A u_{(2)} v$$

is called the Hopf–Galois map of U.

Definition 0.4.16. A **left Hopf algebroid** is a left bialgebroid with bijective Hopf–Galois map.

Proposition 0.4.17 (See [Böh09])**.** *A full Hopf algebroid is always a left Hopf algebroid.*

Proof. The canonical map $\beta : h \otimes h' \mapsto h_{(1)} \otimes h_{(2)} h'$ is bijective with inverse

$$h \otimes h' \longmapsto h^{(1)} \otimes S(h^{(2)})h'$$

for $h, h' \in H$. $\square$

0.5 Lie–Rinehart algebras

This algebraic structure was introduced by Herz [Her53] under the name *Lie pseudo-algebra* (also known as *Lie algebroid* [Pra67] in a differential geometric context) and has been developed and studied as a generalisation of Lie algebras. However it was not until the work of Rinehart [Rin63] that its universal enveloping algebra was defined, and its cohomology understood. The term *Lie–Rinehart algebra* was coined by Huebschmann [Hue91], a term which acknowledges Rinehart's fundamental contributions [Rin63] to the understanding of this structure. See [Hue90, Section 1] for some historical remarks on this development.

In this section, we study Lie–Rinehart algebras; their relation to Gerstenhaber algebras and BV algebras; and their universal enveloping algebras [Rin63]. In Section 0.6 we study left and right modules of Lie–Rinehart algebras over their universal enveloping algebras, and review the main tools used in the proofs of the results presented in this chapter and the following ones, namely right (A, L)-connections and right (A, L)-connection characters on A, see [Hue98, Kow09, KP11].

0.5.1 Basic definitions and examples

We now give the definition of a Lie–Rinehart algebra. Note that Lie–Rinehart algebras can be defined in any symmetric monoidal category.

Definition 0.5.1. Let A be a commutative R-algebra and L be a Lie algebra over R with bracket $[\bullet, \bullet]_L$. A pair (A, L) is called a **Lie–Rinehart algebra** over R if L is a left A-module with action $A \otimes_R L \to L$ denoted by $a \otimes_R \xi \mapsto a \cdot \xi$ for $a \in A, \xi \in L$, and there is an A-linear Lie algebra map $\rho_L : L \to \mathrm{Der}_R(A)$, called the **anchor map**, satisfying

$$[\xi, a \cdot \zeta]_L = a \cdot [\xi, \zeta]_L + \rho_L(\xi)(a) \cdot \zeta, \quad a \in A, \xi, \zeta \in L. \tag{0.5.1}$$

Remark 0.5.2. Note that, by the adjoint functor property of tensor products, we have

$$\mathrm{Hom}_A(L, \mathrm{Hom}_R(A, A)) \cong \mathrm{Hom}_A(L \otimes_R A, A).$$

Hence the action of the Lie algebra L on A by derivations can be described either by the map $\rho_L : L \to \mathrm{Der}_R(A)$ or by the equivalent formulation $\varrho : L \otimes_R A \to A$. Since a Lie algebra L is said to act by derivations on A if and only if there is a Lie algebra map from L

to $\mathrm{Der}_R(A)$, we sometimes call $\mathrm{Der}_R(A)$ the **universal** Lie algebra algebra of derivations of A.

Example 0.5.3. Let L be a Lie algebra over R with bracket $[\bullet,\bullet]_L$, i.e., L is naturally a left R-module. The R-linear Lie algebra map $\rho_L : L \to \mathrm{Der}_R(R)$ given by $\rho_L(\xi) = 0$ for all $\xi \in L$ turns the pair (R, L) into a Lie–Rinehart algebra.

Example 0.5.4. Let A be a commutative R-algebra, M be a left A-module endowed (trivially) with an abelian Lie algebra structure and let $\rho_M : M \to \mathrm{Der}_R(A)$ be the A-linear Lie algebra map given by $\rho_M(m) = 0$ for all $m \in M$. The pair (A, M) is a Lie–Rinehart algebra.

Example 0.5.5. Let A be a commutative R-algebra. The Lie algebra $\mathrm{Der}_R(A)$ of derivations of A is a left A-module with action

$$\mu : A \otimes_R \mathrm{Der}_R(A) \to \mathrm{Der}_R(A), \quad a \otimes_R \xi \longmapsto a \cdot \xi$$

given by $(a \cdot \xi)(b) = a \cdot \xi(b)$ for all $a, b \in A, \xi \in \mathrm{Der}_R(A)$. Moreover, the Lie bracket on $\mathrm{Der}_R(A)$ of derivations of A satisfies

$$\begin{aligned}
[\xi, a \cdot \zeta]_{\mathrm{Der}_R(A)} &= \xi \circ (a \cdot \zeta) - (a \cdot \zeta) \circ \xi \\
&= \xi(a) \cdot \zeta + a \cdot \xi \circ \zeta - a \cdot \zeta \circ \xi \\
&= a \cdot [\xi, \zeta]_{\mathrm{Der}_R(A)} + \xi(a) \cdot \zeta
\end{aligned}$$

for all $\xi, \zeta \in \mathrm{Der}_R(A)$. Hence the compatibility relation between the A-module structure on $\mathrm{Der}_R(A)$ and an A-linear Lie algebra map $\rho : \mathrm{Der}_R(A) \to \mathrm{Der}_R(A)$ as given by Equation (0.5.1) is satisfied for ρ the identity map. Thus, $(A, \mathrm{Der}_R(A))$ is a Lie–Rinehart algebra with anchor map ρ given by the identity.

Definition 0.5.6. Let (A, L) be a Lie–Rinehart algebra and let $(L', [\bullet,\bullet]_L)$ be a Lie subalgebra of $(L, [\bullet,\bullet]_L)$, i.e.,

$$[\xi, \zeta]_L \in L', \quad \xi, \zeta \in L', \tag{0.5.2}$$

such that L' is an A-submodule of L. The pair (A, L') is called a **Lie–Rinehart subalgebra** of (A, L).

Lastly, we define the category $\mathsf{Lie\text{–}Rinehart}$ of Lie–Rinehart algebras.

Definition 0.5.7. The objects in the category Lie–Rinehart are Lie–Rinehart algebras while the morphisms between objects are pairs of maps $\varphi := (f, g)$ such that the morphism $\varphi : (A, L) \to (A', L')$ is equivalent to an L'-module map $f : A \to A'$ and an A-linear Lie algebra map $g : L \to L'$ satisfying

$$\rho_{L'}(g(\xi))(f(a)) = f(\rho_L(\xi)(a))$$

for all $\xi \in L$ and $a \in A$.

0.5.2 The enveloping algebra of (A, L)

A fundamental milestone in the development of Lie–Rinehart algebras is Rinehart's work [Rin63] in which he gives the structure of their universal enveloping algebra (see [Rin63, Section 2]), generalising the construction of the enveloping algebra of a Lie algebra.

We first give Rinehart's original construction of the enveloping algebra of (A, L):

Proposition 0.5.8. *Let (A, L) be a Lie–Rinehart algebra. The A-module $A \oplus L$ endowed with the R-linear bracket*

$$[(a, \xi), (b, \zeta)]_{A\oplus L} := (\rho_L(\xi)(b) - \rho_L(\zeta)(a), [\xi, \zeta]_L). \tag{0.5.3}$$

is a Lie algebra which is denoted by $A \rtimes L$ and is called the semidirect product of A by L.

Proof. The bracket in Equation (0.5.3) is skew-symmetric. Furthermore, using the facts that $[\bullet, \bullet]_L$ is a Lie bracket and $\rho_L : L \to \mathrm{Der}_R(A)$ is an A-linear Lie algebra map, we have

$$\begin{aligned}
[(a, \xi), [(b, \zeta), (c, \gamma)]_{A\oplus L}]_{A\oplus L} + c.p. &= [(a, \xi), (\rho_L(\zeta)(c) - \rho_L(\gamma)(b), [\zeta, \gamma]_L)]_{A\oplus L} + c.p. \\
&= \rho_L(\xi)(\rho_L(\zeta)(c) - \rho_L(\gamma)(b)) - \rho_L([\zeta, \gamma]_L])(a) \\
&\quad - [\xi, [\zeta, \gamma]_L]_L + c.p. \\
&= 0
\end{aligned}$$

so that $[\bullet, \bullet]_{A\otimes L}$ satisfies the Jacobi identity. ☐

Definition 0.5.9. Let (A, L) be a Lie–Rinehart algebra and $U(A \rtimes L)$ be the enveloping algebra of the natural Lie algebra $A \rtimes L$ with bracket $[\bullet, \bullet]_{A\oplus L}$. The subalgebra of $U(A \rtimes L)$

generated by the canonical embedding

$$\iota : A \rtimes L \hookrightarrow U(A \rtimes L)$$

of the Lie algebra $A \rtimes L$ into its enveloping algebra is denoted by

$$U^+(A \rtimes L).$$

Remark 0.5.10. Note that $U(A \rtimes L)/U^+(A \rtimes L) \cong R$.

Definition 0.5.11. Let P be the two-sided ideal in $U^+(A \rtimes L)$ generated by the elements

$$\iota(a,0)\iota(b,\xi) - \iota(a \cdot (b,\xi)). \tag{0.5.4}$$

The **universal enveloping algebra** of (A, L) is the quotient

$$V(A,L) := U^+(A \rtimes L)/P. \tag{0.5.5}$$

Proposition 0.5.12. *Let (A, L) be a Lie–Rinehart algebra and let $V(A, L)$ be its universal enveloping algebra. The map*

$$i_A : A \longrightarrow V(A,L), \quad a \longmapsto \iota(a,0) \pmod P \tag{0.5.6}$$

is a homomorphism of R-algebras, while

$$i_L : L \longrightarrow V(A,L), \quad \xi \longmapsto \bar{\xi} := \iota(0,\xi) \pmod P \tag{0.5.7}$$

is a homomorphism of Lie algebras. Furthermore, we have

$$i_A(a) \cdot i_L(\xi) = i_L(a \cdot \xi), \quad i_L(\xi) \cdot i_A(a) - i_A(a) \cdot i_L(\xi) = i_A(\rho_L(\xi)(a)) \tag{0.5.8}$$

for all $a \in A$, $\xi \in L$.

Proof. We check that the map in Equation (0.5.6) is a homomorphism of R-algebras:

$$\begin{aligned} i_A(a\cdot b) &= \iota(a\cdot b, 0) \pmod{P} \\ &= \iota(a,0)\cdot\iota(b,0) \pmod{P} \\ &= i_A(a)\cdot i_A(b). \end{aligned}$$

Similarly, we check that the map in Equation (0.5.7) is a homomorphism of Lie algebras:

$$\begin{aligned} i_L([\xi,\zeta]) &= \iota(0,[\xi,\zeta]_L) \pmod{P} \\ &= \iota(\rho_L(\xi)(0)-\rho_L(\zeta)(0),[\xi,\zeta]_L) \pmod{P} \\ &= [\iota(0,\xi),\iota(0,\zeta)] \pmod{P} \\ &= [i_L(\xi),i_L(\zeta)]_{V(A,L)}. \end{aligned}$$

Lastly, we check

$$\begin{aligned} i_A(a)\cdot i_L(\xi) &= \iota(a,0)\iota(0,\xi) \pmod{P} \\ &= \iota(0,a\cdot\xi) \pmod{P} \\ &= i_L(a\cdot\xi) \end{aligned}$$

and

$$\begin{aligned} i_L(\xi)\cdot i_A(a) - i_A(a)\cdot i_L(\xi) &= i_A(\rho_L(\xi)(a)) \\ &= \iota(0,\xi)\iota(a,0)-\iota(a,0)\iota(0,\xi) \pmod{P} \\ &= \iota(0,\xi a)-\iota(0,a\xi) \pmod{P} \\ &= \iota(0,[\xi,a]_{A\oplus L}) \pmod{P} \\ &= i_L(\rho_L(\xi)(a)). \end{aligned}$$

□

We are now ready to describe $V(A,L)$ in terms of generators and relations:

Proposition 0.5.13 (Rinehart [Rin63])**.** *Let (A,L) be a Lie–Rinehart algebra. Its* ***universal enveloping algebra****, denoted by $V(A,L)$, is the universal associative R-algebra with*

1. *an R-algebra map $A\to V(A,L)$,*

2. *a Lie algebra map* $\iota_L : L \to V(A,L)$ *given by* $\xi \mapsto \iota_L(\xi) =: \bar{\xi}$, *where* $V(A,L)$ *is consdered with Lie bracket given by the commutator* $[\bullet,\bullet]_{V(A,L)}$

such that for all $a \in A$ *and* $\xi \in L$ *we have*

$$[\bar{\xi}, a]_{V(A,L)} = \rho_L(\xi)(a), \quad a\bar{\xi} = \overline{a \cdot \xi}$$

where the product in $V(A,L)$ *is denoted by concatenation.*

Example 0.5.14. The enveloping algebra $V(R,L)$ of the Lie–Rinehart algebra (R,L) given in Example 0.5.3 is isomorphic to the enveloping algebra of L. That is,

$$V(R,L) \cong U(L). \tag{0.5.9}$$

Example 0.5.15. Let A be the ring of smooth functions and L be the Lie algebra of smooth vector fields on a smooth manifold. Then the enveloping algebra $V(A, \mathrm{Der}_R(A))$ of the Lie–Rinehart algebra $(A, \mathrm{Der}_R(A))$ is isomorphic to the algebra of differential operators on A, that is:

$$V(A, \mathrm{Der}_R(A)) \cong \mathrm{Diff}_R(A). \tag{0.5.10}$$

See [Hue98].

0.5.3 Lie–Rinehart algebras and Gerstenhaber algebras

The aim of this section is to explain the relation between Lie–Rinehart algebras and Gerstenhaber algebras, by describing a certain pair of adjoint functors between the categories of Lie–Rinehart algebras and Gerstenhaber algebras.

We start by giving the original definition of Gerstenhaber algebras:

Definition 0.5.16 (See [Ger63])**.** A **Gerstenhaber algebra** $\mathcal{A}$ consists of

- a sequence of vector spaces $A^0, A^1, A^2, \ldots$,
- a Lie bracket $[\bullet,\bullet]_{\mathcal{A}} : A^p \otimes A^q \to A^{p+q-1}$ (of degree -1)
- an associative unital graded product $\mu : A^p \otimes A^q \to A^{p+q}$ which is graded commutative, i.e., $\alpha^p \cdot \beta^q = (-1)^{pq}\alpha \cdot \beta$.

satisfying

$$[\alpha^p, \beta^q \cdot \gamma]_{\mathcal{A}} = [\alpha,\beta]_{\mathcal{A}} \cdot \gamma + (-1)^{(p-1)q}\beta \cdot [\alpha,\gamma]_{\mathcal{A}}. \tag{0.5.11}$$

Definition 0.5.17. Gerstenhaber algebras form a category, denoted Gerstenhaber where the morphisms between objects are R-linear maps between graded components preserving the Gerstenhaber brackets and commuting with the associative graded products.

Remark 0.5.18. Since the bracket $[\bullet,\bullet]_{\mathcal{A}}$ is of degree -1, we have

$$[\bullet,\bullet]_{\mathcal{A}} : A^1 \otimes A^1 \to A^{1+1-1}$$

so that A^1 is a Lie algebra with bracket $[\bullet,\bullet]_{\mathcal{A}}$.

Proposition 0.5.19. *Let (A, L) be a Lie–Rinehart algebra. The exterior A-algebra $\Lambda_A L$ over the Lie algebra L with bracket $[\bullet,\bullet]_L$ is a Gerstenhaber algebra with bracket*

$$[u,v] = (-1)^{|u|} \sum_{j\le\ell<k} (-1)^{(j+k)} \left([\xi_j,\xi_k]_L \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n\right) \tag{0.5.12}$$

where $u = \xi_1 \wedge \ldots \wedge \xi_\ell \in \Lambda^\ell_A L$ and $v = \xi_{\ell+1} \wedge \ldots \wedge \xi_n \in \Lambda^{n-\ell}_A L$ and elements $\hat{\xi}_j, \hat{\xi}_k \in L$ are omitted. The assignment of $\Lambda_A L$ to (A, L) together with the bracket (0.5.12) on $\Lambda_A L$ yields a functor from Lie–Rinehart to Gerstenhaber which we denote by $\mathcal{G}$.

Proof. We denote elements in L by $\xi_1, \ldots, \xi_n$. Note that $\Lambda^0_A L = A$ and $\Lambda^1 L = L$. We see

$$[\xi_j,\xi_k]_L \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n \in \Lambda^{n-1}_A L$$

so that $[\bullet,\bullet]$ is of degree -1.

Since (A, L) is a Lie–Rinehart algebra, the Lie algebra L is an A-module and $[\bullet,\bullet]_L$ satisfies the Leibniz rule in Equation (0.5.1) so that

$$\begin{aligned}
[u, a\cdot v] &= (-1)^{|u|} \sum_{j\le\ell<k} (-1)^{(j+k)} \left([\xi_j, a\cdot\xi_k]_L \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n\right) \\
&= (-1)^{|u|} \sum_{j\le\ell<k} (-1)^{(j+k)} \left((a\cdot[\xi_j,\xi_k]_L + \rho_L(\xi_j)(a)\cdot\xi_k) \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n\right) \\
&= (-1)^{|u|} \sum_{j\le\ell<k} (-1)^{(j+k)} \left(a\cdot[\xi_j,\xi_k]_L \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n\right) \\
&\quad + (-1)^{|u|} \sum_{j\le\ell<k} (-1)^{(j+k)} \left(\rho_L(\xi_j)(a)\cdot\xi_k \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \hat{\xi}_k \wedge \ldots \wedge \xi_n\right) \\
&= a\cdot[u,v] + (-1)^{(\ell-1)(n-\ell)}[u,a]\cdot v
\end{aligned}$$

so that $[\bullet,\bullet]$ satisfies the rule given in Equation (0.5.11). Hence $\Lambda_A L$ is a Gerstenhaber algebra. ∎

Proposition 0.5.20. *For any Gerstenhaber algebra* $\left(\mathcal{A} = \bigoplus_{i\in\mathcal{A}} A^i, \cdot, [\bullet,\bullet]_{\mathcal{A}}\right)$*, the pair given by* (A^0, A^1) *is a Lie–Rinehart algebra with bracket* $[\bullet,\bullet]_{\mathcal{A}}$ *and anchor* $\rho : A^1 \to \mathrm{Der}_R(A^0)$ *given by* $\xi \mapsto [\xi,\bullet]_{\mathcal{A}}$ *for* $\xi \in A^1$*. The assignment to* $\mathcal{A}$ *of the pair* (A^0, A^1) *yields a functor from* **Gerstenhaber** *to* **Lie–Rinehart***, which we denote by* $\mathcal{R}$*.*

Proof. We have

- A^0-module structure on A^1 with action $\mu : A^0 \otimes A^1 \to A^1$,
- a left A^1-module action on A^0 given by $[\bullet,\bullet]_{\mathcal{A}} : A^1 \otimes A^0 \to A^0$

satisfying

1. $[\xi, a\cdot b]_{\mathcal{A}} = [\xi,a]_{\mathcal{A}}\cdot b + a\cdot[\xi,b]_{\mathcal{A}}$
2. $[b\cdot\xi, a]_{\mathcal{A}} = b\cdot[\xi,a]_{\mathcal{A}}$
3. $[\xi, a\cdot\zeta]_{\mathcal{A}} = [\xi,a]\cdot\zeta + a\cdot[\xi,\zeta]_{\mathcal{A}}$

for $a, b \in A^0$ and $\xi,\zeta \in A^1$. Then, by the adjoint functor property of tensor products, (1) means that the Lie algebra map $A^1 \to \mathrm{End}_R(A^0)$ given by $\xi \mapsto [\xi,\bullet]_{\mathcal{A}}$ factors through $\mathrm{Der}_R(A^0)$. (2) means that the Lie algebra map $A^1 \to \mathrm{Der}_R(A^0)$ map is an A^0-module map. (3) is equivalent to the Leibniz rule in Equation (0.5.1). □

We now prove that functors $\mathcal{G}$ and $\mathcal{R}$ are in adjuntion as follows:

$$\text{Lie-Rinehart algebras} \underset{\mathcal{R}}{\overset{\mathcal{G}}{\rightleftarrows}} \text{Gerstenhaber algebras.}$$

Theorem 0.5.21. *The functor* $\mathcal{G}$: **Lie–Rinehart** → **Gerstenhaber** *is left adjoint to the functor* $\mathcal{R}$: **Gerstenhaber** → **Lie–Rinehart** *since we have a natural isomophism*

$$\mathrm{Hom}_{\mathsf{Gerstenhaber}}\left(\mathcal{G}(A,L),\mathcal{A}\right) \cong \mathrm{Hom}_{\mathsf{Lie\text{–}Rinehart}}\left((A,L),\mathcal{R}(\mathcal{A})\right). \tag{0.5.13}$$

Proof. By Propositions 0.5.19 and 0.5.20 we have $\mathcal{G}(A,L) = \Lambda_A L$ and $\mathcal{R}(\mathcal{A}) = (A^0, A^1)$ respectively. Now, a map of Gerstenhaber algebras preserves brackets and multiplication on graded parts. Moreover, $\Lambda_A L$ is free so that a morphism $f : \Lambda_A L \to \mathcal{A}$ from the free exterior A-algebra $\Lambda_A L$ on L to any Gerstenhaber algebra $\mathcal{A}$ is completely determined by the maps $f : A \to A^0$ and $g : L \to A^1$ where the corresponding brackets are preserved. But this pair of maps (f, g) is equivalent to a Lie–Rinehart algebra morphism from (A, L) to (A^0, A^1) so that the isomophism in (0.5.13) holds. □

Lastly, we give the definition of a very important class of Gerstenhaber algebras:

Definition 0.5.22 (See [Ger63]). A **Batalin–Vilkovisky algebra** is a Gerstenhaber algebra $\left(\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} A^i, \cdot, [\bullet,\bullet]_{\mathcal{A}}\right)$ equipped with a square-zero operator D of degree -1 such that the bracket $[\bullet,\bullet]_{\mathcal{A}}$ is given by

$$[a,b]_{\mathcal{A}} = (-1)^{|a|}\Big(D(a\cdot b) - Da\cdot b - (-1)^{|a|}a\cdot Db\Big). \tag{0.5.14}$$

for every $a \in A^{|a|}$ and $b \in A$.

Remark 0.5.23. An operator D of degree -1, not necessarily of square zero, on a Gerstenhaber algebra $\left(\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} A^i, \cdot, [\bullet,\bullet]_{\mathcal{A}}\right)$ is said to generate the Gerstenhaber bracket $[\bullet,\bullet]_{\mathcal{A}}$ if it satisfies Equation (0.5.14).

By [Hue98, Theorem 1] we have that Lie–Rinehart algebras admitting right $V(A,L)$-module structures on A give rise to a Batalin–Vilkovisky algebra structure on $\Lambda_A L$.

0.6 $V(A,L)$-module structures

The concept of left (respectively right) (A,L)-module structures on A-modules (more generally (A,L)-connections on A-modules) was introduced in [Hue98]. There is an equivalence of categories between these module structures and left (respectively) right $V(A,L)$-module structures. On the one hand, the anchor map defines a canonical left $V(A,L)$-module structure on A itself. On the other hand, as proved in [KR15], A will not carry a right $V(A,L)$-module structure in general. Even if A admits a right $V(A,L)$-module structure, there will be no canonical choice, see [KP11, Section 3.2.2.] for discussion.

The main goal of this section is to define left and right (A,L)-modules for Lie–Rinehart algebras and provide an explicit description of right $V(A,L)$-module structures on A for several classical examples of Lie–Rinehart algebras. We begin by recalling the characterisation of (A,L)-modules given in [Hue98], and explaining how to simplify this approach by introducing an equivalent characterisation of (A,L)-modules in terms of maps $\delta : L \to A$.

0.6.1 Left (A,L)-module structures on A-modules

In this section we study left (A,L)-modules. These are L-modules satisfying certain extra conditions which make the L-module action compatible with the underlying A-module structure on L.

We start by recalling the definition of left L-modules:

Definition 0.6.1. Let L be a Lie algebra over R with bracket $[\bullet,\bullet]_L$. A left L-module is an R-module M together with a map $\varphi : L \otimes_R M \longrightarrow M$ satisfying

$$\varphi\left([\xi,\zeta]_L \otimes_R m\right) = \varphi\left(\xi \otimes_R \varphi\left(\zeta \otimes_R m\right)\right) - \varphi\left(\zeta \otimes_R \varphi\left(\xi \otimes_R m\right)\right) \tag{0.6.1}$$

for $m \in M, \xi, \zeta \in L$.

Example 0.6.2. Let (A, L) be a Lie–Rinehart algebra. It follows from Equation (0.6.1) that the Lie algebra homomorphism given by the anchor map $\rho_L : L \to \mathrm{Der}_R(A)$ defines a left L-module structure on A.

Definition 0.6.3. Let (A, L) be a Lie–Rinehart algebra with anchor $\rho_L : L \to \mathrm{Der}_R(A)$, and let M be an A-module. A **left** (A, L)**-connection on** M is an R-linear map

$$\nabla^{\ell} : L \otimes_R M \longrightarrow M, \quad \xi \otimes_R m \longmapsto \nabla^{\ell}_{\xi}(m) := \nabla^{\ell}\left(\xi \otimes_R m\right), \quad \xi \in L, m \in M \tag{0.6.2}$$

satisfying

$$\nabla^{\ell}_{\xi}(a \cdot m) = a \cdot \nabla^{\ell}_{\xi}(m) + \rho_L(\xi)(a) \cdot m \tag{0.6.3}$$

$$\nabla^{\ell}_{a\cdot\xi}(m) = a \cdot \nabla^{\ell}_{\xi}(m) \tag{0.6.4}$$

where $a \in A, m \in M, \xi \in L$.

Remark 0.6.4. Note that Equation (0.6.4) implies that $\nabla^{\ell} : L \otimes_R M \to M$ is determined when known on a set of generators of the Lie algebra L as an A-module.

Definition 0.6.5. Given a left (A, L)-connection on an A-module M we define the operator $\mathcal{C}^{\nabla}_{\ell}$ which we call **left** (A, L)**-curvature operator**, as follows,

$$\mathcal{C}^{\nabla}_{\ell} : L^{\otimes_R 2} \otimes_R M \longrightarrow M; \quad \xi \otimes_R \zeta \otimes_R m \longmapsto \mathcal{C}^{\nabla}_{\ell}(\xi,\zeta)(m) := \left([\nabla^{\ell}_{\xi}, \nabla^{\ell}_{\zeta}] - \nabla^{\ell}_{[\xi,\zeta]_L}\right)(m) \tag{0.6.5}$$

for $m \in M, \xi, \zeta \in L$. We call $\mathcal{C}^{\nabla}_{\ell}(\xi,\zeta)(m) \in M$ the **curvature** of the left (A, L)-connection $\nabla^{\ell} : M \otimes_R L \to L$ on $\xi, \zeta \in L$ evaluated at $m \in M$.

Definition 0.6.6. A left (A, L)-connection on an A-module M is called **flat** if the map $\nabla^{\ell} : L \otimes_R M \to M$ turns M into a left L-module, that is, $\mathcal{C}^{\nabla}_{\ell}(\xi,\zeta)(m) = 0$.

We are now ready to define the main object of our study:

Definition 0.6.7. Let (A, L) be a Lie–Rinehart algebra with anchor $\rho_L : L \to \mathrm{Der}_R(A)$, and let M be an A-module. A **left (A, L)-module structure on M** is a flat left (A, L)-connection on M.

Lemma 0.6.8. *Let (A, L) be a Lie–Rinehart algebra and let $\nabla^\ell : L \otimes_R M \to M$ be a left (A, L)-connection on an A-module M. The (A, L)-curvature operator $\mathcal{C}_\ell^\nabla$ is A-trilinear, that is, it satisfies the following identity:*

$$\mathcal{C}_\ell^\nabla(a \cdot \xi, b \cdot \zeta)(c \cdot m) = a \cdot b \cdot c \cdot \mathcal{C}_\ell^\nabla(\xi, \zeta)(m), \quad a, b, c \in A, m \in M. \tag{0.6.6}$$

Proof. Using Definition 0.6.3 we argue as follows

$$\begin{aligned}
\mathcal{C}_\ell^\nabla(a\cdot\xi, b\cdot\zeta)(c\cdot m) &= \nabla^\ell_{a\cdot\xi}\left(\nabla^\ell_{b\cdot\zeta}(c\cdot m)\right) - \nabla^\ell_{b\cdot\zeta}\left(\nabla^\ell_{a\cdot\xi}(c\cdot m)\right) - \nabla^\ell_{[a\cdot\xi, b\cdot\zeta]_L}(c\cdot m)\\
&= a\cdot\nabla^\ell_\xi\left(b\cdot\nabla^\ell_\zeta(c\cdot m)\right) - b\cdot\nabla^\ell_\zeta\left(a\cdot\nabla^\ell_\xi(c\cdot m)\right)\\
&\quad - a\cdot b\cdot\nabla^\ell_{[\xi,\zeta]_L}(c\cdot m) - a\cdot\rho_L(\xi)(b)\cdot\nabla^\ell_\zeta(c\cdot m) + b\cdot\rho_L(\zeta)(a)\cdot\nabla^\ell_\zeta(c\cdot m)\\
&= a\cdot\nabla^\ell_\xi\left(b\cdot c\cdot\nabla^\ell_\zeta(m) + b\cdot\rho_L(\zeta)(c)\cdot m\right) - b\cdot\nabla^\ell_\zeta\left(a\cdot c\cdot\nabla^\ell_\xi(m) + a\cdot\rho_L(\xi)(c)\cdot m\right)\\
&\quad - a\cdot b\cdot\nabla^\ell_{[\xi,\zeta]_L}(c\cdot m) - a\cdot c\cdot\rho_L(\xi)(b)\cdot\nabla^\ell_\zeta(m) - a\cdot\rho_L(\xi)(b)\cdot\zeta(c)\cdot m\\
&\quad + b\cdot c\cdot\rho_L(\zeta)(a)\cdot\nabla^\ell_\zeta(m) + b\cdot\rho_L(\zeta)(a)\cdot\zeta(c)\cdot m\\
&= a\cdot b\cdot c\cdot\nabla^\ell_\xi\left(\nabla^\ell_\zeta(m)\right) + a\cdot\rho_L(\xi)(b\cdot c)\cdot\nabla^\ell_\zeta(m) + a\cdot b\cdot\rho_L(\zeta)(c)\cdot\nabla^\ell_\xi(m) + a\cdot\rho_L(\xi)(b\cdot\rho(\zeta)(c))\cdot m\\
&\quad - b\cdot a\cdot c\cdot\nabla^\ell_\zeta\left(\nabla^\ell_\xi(m)\right) - b\cdot\rho_L(\zeta)(a\cdot c)\cdot\nabla^\ell_\xi(m) - a\cdot b\cdot\rho_L(\xi)(c)\cdot\nabla^\ell_\zeta(m) - b\cdot\rho_L(\zeta)(a\cdot\rho_L(\xi)(c))\cdot m\\
&\quad - a\cdot b\cdot c\cdot\nabla^\ell_{[\xi,\zeta]_L}(m) - a\cdot b\cdot\rho_L([\xi,\zeta])(c)\cdot m\\
&\quad - a\cdot c\cdot\rho_L(\xi)(b)\cdot\nabla^\ell_\zeta(m) - a\cdot\rho_L(\xi)(b)\cdot\zeta(c)\cdot m + b\cdot c\cdot\rho_L(\zeta)(a)\cdot\nabla^\ell_\zeta(m) + b\cdot\rho_L(\zeta)(a)\cdot\xi(c)\cdot m\\
&= a\cdot b\cdot c\cdot\mathcal{C}_\ell^\nabla(\xi,\zeta)(m).
\end{aligned}$$

□

Example 0.6.9. Let (A, L) be a Lie–Rinehart algebra over a commutative R-algebra A. The anchor map $\varrho : L \otimes_R A \to A$ is a flat left (A, L)-connection on A since it is an L-module on $(A, \cdot)$ satisfying

$$\begin{aligned}
\varrho(\xi \otimes_R a \cdot b) &= a \cdot \varrho(\xi \otimes_R b) + \varrho(\xi \otimes_R a) \cdot b\\
\varrho(a \cdot \xi \otimes_R b) &= a \cdot \varrho(\xi \otimes_R b).
\end{aligned}$$

0.6.2 Right (A, L)-module structures

In this section we define right (A, L)-module structures on A-modules [Hue98] and provide a way to describe them in similar terms as used for left (A, L)-module structures. We start by recalling the definition of right L-modules:

Definition 0.6.10. Let L be a Lie algebra over R with bracket $[\bullet,\bullet]_L$. A right L-module is an R-module N together with a map $\varphi : N \otimes_R L \longrightarrow N$ satisfying

$$\varphi(\xi \otimes_R \varphi(\zeta \otimes_R n)) - \varphi(\zeta \otimes_R \varphi(\xi \otimes_R n)) + \varphi([\xi,\zeta]_L \otimes_R n) = 0, \tag{0.6.7}$$

for $n \in N, \xi, \zeta \in L$.

When the right L-module N also carries a left A-module structure, the compatibility between both actions on N gives rise to the following structure:

Definition 0.6.11. Let (A, L) be a Lie–Rinehart algebra with anchor $\rho_L : L \to \mathrm{Der}_R(A)$, and let N be an A-module. A **right (A,L)-connection** on N is an R-linear map

$$N \otimes_R L \longrightarrow N, \quad n \otimes_R \xi \longmapsto \nabla^{\mathrm{r}}_{\xi}(n), \quad \xi \in L, n \in N \tag{0.6.8}$$

satisfying

$$\nabla^{\mathrm{r}}_{\xi}(a \cdot n) = \nabla^{\mathrm{r}}_{a\cdot\xi}(n) = a \cdot \nabla^{\mathrm{r}}_{\xi}(n) - \rho_L(\xi)(a) \cdot n, \tag{0.6.9}$$

where $a \in A, n \in N, \xi \in L$.

Remark 0.6.12. From Equation (0.6.9) it follows that

$$\nabla^{\mathrm{r}}(a \cdot n \otimes_R \xi) = \nabla^{\mathrm{r}}(n \otimes_R a \cdot \xi) \tag{0.6.10}$$

so that a right (A, L)-connection is in fact a map

$$\nabla^{\mathrm{r}} : N \otimes_A L \longrightarrow N \tag{0.6.11}$$

satisfying Equation (0.6.9).

Remark 0.6.13. Note that Equation (0.6.9) implies that $\nabla^{\mathrm{r}} : N \otimes_A L \to N$ is determined when known on a set of generators of L as an A-module.

Definition 0.6.14. Given a (A, L)-connection on a A-module N, we define the operator $\mathcal{C}^{\nabla}_{\mathrm{r}}$ which we call **right (A,L)-curvature operator**, as follows,

$$\mathcal{C}^{\nabla}_{\mathrm{r}} : L^{\otimes_A 2} \otimes_A N \longrightarrow N; \quad n \longmapsto \mathcal{C}^{\nabla}_{\mathrm{r}}(\xi,\zeta)(n) := ([\nabla^{\mathrm{r}}_{\xi}, \nabla^{\mathrm{r}}_{\zeta}] + \nabla^{\mathrm{r}}_{[\xi,\zeta]_L})(n) \tag{0.6.12}$$

for $n \in N, \xi, \zeta \in L$. We call $\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)(n) \in N$ the **curvature** of the right (A, L)-connection $\nabla^{\mathrm{r}} : N \otimes_A L \to N$ on $\xi, \zeta \in N$ evaluated at $n \in N$.

Definition 0.6.15. A right (A, L)-connection on an A-module N is called **flat** if the map $\nabla^{\mathrm{r}} : N \otimes_A L \to N$ turns N into a right L-module, that is, $\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)(n) = 0$.

We are now ready to define right (A, L)-modules:

Definition 0.6.16. Let (A, L) be a Lie–Rinehart algebra with anchor $\rho_L : L \to \mathrm{Der}_R(A)$, and let N be an A-module. A **right (A, L)-module structure on** N is a flat right (A, L)-connection on N.

Lemma 0.6.17. *Let (A, L) be a Lie–Rinehart algebra and let $\nabla^{\mathrm{r}} : N \otimes_A L \to N$ be a right (A, L)-connection on an A-module N. The (A, L)-curvature operator $\mathcal{C}_{\mathrm{r}}^{\nabla}$ is A-trilinear, so*

$$\mathcal{C}_{\mathrm{r}}^{\nabla}\left(a\cdot\xi, b\cdot\zeta\right)\left(c\cdot n\right) = a\cdot b\cdot c\cdot \mathcal{C}_{\mathrm{r}}^{\nabla}\left(\xi,\zeta\right)\left(n\right), \quad a, b, c \in A, n \in N. \tag{0.6.13}$$

Proof. Using Definition 0.6.11 we argue as follows

$$\begin{aligned}
\mathcal{C}_{\mathrm{r}}^{\nabla}\left(a\cdot\xi, b\cdot\zeta\right)\left(c\cdot n\right) &= \nabla^{\mathrm{r}}_{a\cdot\xi}\left(\nabla^{\mathrm{r}}_{b\cdot\zeta}(c\cdot n)\right) - \nabla^{\mathrm{r}}_{b\cdot\zeta}\left(\nabla^{\mathrm{r}}_{a\cdot\xi}\left(c\cdot n\right)\right) + \nabla^{\mathrm{r}}_{[a\cdot\xi, b\cdot\zeta]_L}\left(c\cdot n\right)\\
&= \nabla^{\mathrm{r}}_{\xi}\left(a\cdot\nabla^{\mathrm{r}}_{\zeta}\left(b\cdot c\cdot n\right)\right) - \nabla^{\mathrm{r}}_{\zeta}\left(b\cdot\nabla^{\mathrm{r}}_{\xi}\left(a\cdot c\cdot n\right)\right)\\
&\quad + \nabla^{\mathrm{r}}_{[\xi,\zeta]_L}\left(a\cdot b\cdot c\cdot n\right) + \nabla^{\mathrm{r}}_{\zeta}\left(a\cdot\rho_L\left(\xi\right)\left(b\right)\cdot c\cdot n\right) - \nabla^{\mathrm{r}}_{\xi}\left(b\cdot\rho_L\left(\zeta\right)\left(a\right)\cdot c\cdot n\right)\\
&= \nabla^{\mathrm{r}}_{\xi}\left(a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{\zeta}\left(n\right) - a\cdot\rho_L\left(\zeta\right)\left(b\cdot c\right)\cdot n\right) - \nabla^{\mathrm{r}}_{\zeta}\left(a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{\xi}\left(n\right) - b\cdot\rho_L\left(\xi\right)\left(a\cdot c\right)\cdot n\right)\\
&\quad + a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{[\xi,\zeta]_L}\left(n\right) - \rho_L\left([\xi,\zeta]_L\right)\left(a\cdot b\cdot c\right)\cdot n\\
&\quad + a\cdot\rho_L\left(\xi\right)\left(b\right)\cdot c\cdot\nabla^{\mathrm{r}}_{\zeta}\left(n\right) - \rho_L\left(\zeta\right)\left(a\cdot\rho_L\left(\xi\right)\left(b\right)\cdot c\right)\cdot n\\
&\quad - b\cdot\rho_L\left(\zeta\right)\left(a\right)\cdot c\cdot\nabla^{\mathrm{r}}_{\xi}\left(n\right) + \rho_L\left(\xi\right)\left(b\cdot\rho_L\left(\zeta\right)\left(a\right)\cdot c\right)\cdot n\\
&= a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{\xi}\left(\nabla^{\mathrm{r}}_{\zeta}\left(n\right)\right) - \rho_L\left(\xi\right)\left(a\cdot b\cdot c\right)\cdot\nabla^{\mathrm{r}}_{\zeta}\left(n\right)\\
&\quad - a\cdot\rho_L\left(\zeta\right)\left(b\cdot c\right)\cdot\nabla^{\mathrm{r}}_{\xi}\left(n\right) + \rho_L\left(\xi\right)\left(a\cdot\rho_L\left(\zeta\right)\left(b\cdot c\right)\right)\cdot n\\
&\quad - a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{\zeta}\left(\nabla^{\mathrm{r}}_{\xi}\left(n\right)\right) - \rho_L\left(\zeta\right)\left(a\cdot b\cdot c\right)\cdot\nabla^{\mathrm{r}}_{\xi}\left(n\right)\\
&\quad + b\cdot\rho_L\left(\xi\right)\left(a\cdot c\right)\cdot\nabla^{\mathrm{r}}_{\zeta}\left(n\right) + \rho_L\left(\zeta\right)\left(b\cdot\rho_L\left(\xi\right)\left(a\cdot c\right)\right)\cdot n\\
&\quad + a\cdot b\cdot c\cdot\nabla^{\mathrm{r}}_{[\xi,\zeta]_L}(n) - \rho_L\left([\xi,\zeta]_L\right)\left(a\cdot b\cdot c\right)\cdot n\\
&\quad + a\cdot\rho_L\left(\xi\right)\left(b\right)\cdot c\cdot\nabla^{\mathrm{r}}_{\zeta}\left(n\right) - \rho_L\left(\zeta\right)\left(a\cdot\rho_L\left(\xi\right)\left(b\right)\cdot c\right)\cdot n\\
&\quad - b\cdot\rho_L\left(\zeta\right)\left(a\right)\cdot c\cdot\nabla^{\mathrm{r}}_{\xi}\left(n\right) + \rho_L\left(\xi\right)\left(b\cdot\rho_L\left(\zeta\right)\left(a\right)\cdot c\right)\cdot n\\
&= a\cdot b\cdot c\cdot\mathcal{C}_{\mathrm{r}}^{\nabla}\left(\xi,\zeta\right)\left(n\right).
\end{aligned}$$

□

0.6.3 (A, L)-module structures on A

In this section we will consider (A, L)-module structures on the algebra A, viewed as an A-module over itself. We first see that in this situation, left (A, L)-connection maps $\nabla^{\ell} : L \otimes_R A \to A$ can be simplified to an A-linear map $\psi : L \to A$. Furthermore, we see that, since $A \otimes_A L \cong L$, right (A, L)-connection maps $\nabla^{\mathrm{r}} : A \otimes_A L \to A$ are equivalent to maps $\nabla^{\mathrm{r}} : L \to A$, which are not A-linear as we will see. These facts will be crucial throughout the proofs of our results in Chapters 1 and 2.

We begin with an observation:

Proposition 0.6.18. *Let (A, L) be a Lie–Rinehart algebra, and let $\nabla^{\ell} : L \otimes_R A \to A$ and $\nabla^{\mathrm{r}} : A \otimes_A L \to A$ be left, respectively right (A, L)-connections on A viewed as an A-module over itself. Then ∇^{ℓ} and ∇^{r} satisfy:*

$$\nabla^{\ell}_{a\cdot\xi}(b) = a \cdot \nabla^{\ell}_{\xi}(b) = a \cdot b \cdot \nabla^{\ell}_{\xi}(1) \tag{0.6.14}$$

and

$$\nabla^{\mathrm{r}}_{a\cdot\xi}(b) = \nabla^{\mathrm{r}}_{\xi}(a \cdot b) = a \cdot b \cdot \nabla^{\mathrm{r}}_{\xi}(1) - \rho_L(\xi)(a \cdot b) \cdot 1 \tag{0.6.15}$$

respectively.

Proof. Put $M = A$ in Definition 0.6.3, then the condition in Equation (0.6.4) implies Equation (0.6.14). Similarly, from Definition 0.6.11 we deduce that the relation given in Equation (0.6.9) implies Equation (0.6.15). □

Remark 0.6.19. From Equations (0.6.14) and (0.6.15) we see that left, respectively right (A, L)-connections on $(A, \cdot)$ are completely determined by their values on the generators $\xi \in L$ at $1 \in A$, i.e., by the expressions $\nabla^{\mathrm{r}}_{\xi}(1)$ and $\nabla^{\ell}_{\xi}(1)$.

The above obsevation leads us to define the following maps.

Definition 0.6.20. Given left, respectively right (A, L)-connections on $(A, \cdot)$, viewed as a module over itself, we define the operators δ^{∇}_{ℓ} and $\delta^{\nabla}_{\mathrm{r}}$ called **left (A, L)-connection character** and **right (A, L)-connection character**, as follows

$$\delta^{\nabla}_{\ell} : L \longrightarrow A; \quad \xi \longmapsto \nabla^{\ell}_{\xi}(1), \quad \xi \in L, \tag{0.6.16}$$

$$\delta^{\nabla}_{\mathrm{r}} : L \longrightarrow A, \quad \xi \longmapsto \nabla^{\mathrm{r}}_{\xi}(1), \quad \xi \in L. \tag{0.6.17}$$

Proposition 0.6.21. *Let (A, L) be a Lie–Rinehart algebra. Given a left, respectively right (A, L)-connection on A, viewed as a module over itself, left respectively right (A, L)-connection character operators δ_ℓ^∇ and $\delta_{\mathrm{r}}^\nabla$ satisfy the following equations respectively:*

$$\delta_\ell^\nabla (a \cdot \xi) = a \cdot \delta_\ell^\nabla (\xi), \tag{0.6.18}$$

$$\delta_{\mathrm{r}}^\nabla (a \cdot \xi) = a \cdot \delta_{\mathrm{r}}^\nabla (\xi) - \rho_L (\xi) (a). \tag{0.6.19}$$

Proof. From Equation (0.6.16), we have

$$\delta_\ell^\nabla (a \cdot \xi) = \nabla^\ell_{a \cdot \xi} (1) = a \cdot \nabla^\ell_\xi (1) = a \cdot \delta_\ell^\nabla (\xi).$$

Similarly, from Equation (0.6.17), we find

$$\delta_{\mathrm{r}}^\nabla (a \cdot \xi) = \nabla^{\mathrm{r}}_{a \cdot \xi} (1) = a \cdot \nabla^{\mathrm{r}}_\xi (1) - \rho_L(\xi)(a) \cdot 1 = a \cdot \delta_{\mathrm{r}}^\nabla (\xi) - \rho_L(\xi)(a).$$ □

Lemma 0.6.22. *Let (A, L) be a Lie–Rinehart algebra and let $\ell : L \to A$ be an A-linear map, i.e., $\ell(a \cdot \xi) = a \cdot \ell(\xi)$. The A-linear map*

$$\varphi : L \otimes_R A \longrightarrow A, \quad \xi \otimes_R a \longmapsto \varphi_\xi(a) := a \cdot \ell(\xi) + \rho_L(\xi)(a) \tag{0.6.20}$$

is a left (A, L)-connection on A.

Proof. For $a, b \in A$, we have

$$\begin{aligned}\varphi(a \cdot \xi \otimes_R b) &= \varphi_{a \cdot \xi}(b) = b \cdot \ell(a \cdot \xi) + \rho_L(a \cdot \xi)(b) \\ &= a \cdot b \cdot \ell(\xi) + a \cdot \rho_L(\xi)(b) \\ &= a \cdot (b \cdot \ell(\xi) + \rho_L(\xi)(b)) = a \cdot \varphi_\xi(b) = a \cdot \varphi(\xi \otimes_R b)\end{aligned}$$

so that $\varphi : L \otimes_R A \to A$ is A-linear and the condition given in Equation (0.6.4) holds. Now,

$$\begin{aligned}\varphi_\xi (a \cdot b) &= a \cdot b \cdot \ell(\xi) + \rho_L(\xi)(a \cdot b) \\ &= a \cdot b \cdot \ell(\xi) + a \cdot \rho_L(\xi)(b) + b \cdot \rho_L(\xi)(a) = a \cdot (b \cdot \ell(\xi) + \rho_L(\xi)(b)) + b \cdot \rho_L(\xi)(a) \\ &= a \cdot \varphi_\xi(b) + b \cdot \rho_L(\xi)(a)\end{aligned}$$

so Equation (0.6.3) holds. □

Lemma 0.6.23. *Let* (A, L) *be a Lie–Rinehart algebra and let the map* $\mathrm{r} : L \to A$ *satisfy the relation*

$$\mathrm{r}\,(a \cdot \xi) = a \cdot \mathrm{r}\,(\xi) - \mathrm{r}\,(\xi)\,(a)\,. \tag{0.6.21}$$

Then the map

$$\psi : A \otimes_A L \longrightarrow A, \quad a \otimes_A \xi \longmapsto \psi_\xi(a) := a \cdot \mathrm{r}\,(\xi) - \rho_L(\xi)(a) \tag{0.6.22}$$

is a right (A, L)*-connection on* A.

Proof. We now check that the map $\psi : A \otimes_A L \to A$ as defined in Equation (0.6.22) is a right (A, L)-connection on A.

$$\begin{aligned}\psi_\xi\,(a \cdot b) &= a \cdot b \cdot \mathrm{r}\,(\xi) - \rho_L\,(\xi)\,(a \cdot b)\\ &= a \cdot b \cdot \mathrm{r}\,(\xi) - a \cdot \rho_L\,(\xi)\,(b) - b \cdot \rho_L\,(\xi)\,(a)\\ &= a \cdot (b \cdot \mathrm{r}\,(\xi) - \rho_L\,(\xi)\,(b)) - (\rho_L\,(\xi)\,(a)) \cdot b\\ &= a \cdot \psi_\xi\,(b) - (\rho_L\,(\xi)\,(a)) \cdot b,\end{aligned}$$

so the condition given in Equation (0.6.9) holds. Similarly, we have

$$\begin{aligned}\psi_{a\cdot\xi}(b) &= b \cdot \mathrm{r}\,(a \cdot \xi) - \rho_L\,(a \cdot \xi)\,(b)\\ &= b \cdot (a \cdot \mathrm{r}(\xi) - \rho_L\,(\xi)\,(a)) - a \cdot \rho_L\,(\xi)\,(b)\\ &= b \cdot a \cdot \mathrm{r}\,(\xi) - b \cdot \rho_L\,(\xi)\,(a) - a \cdot \rho_L\,(\xi)\,(b)\\ &= a \cdot (b \cdot \mathrm{r}\,(\xi) - \rho_L\,(\xi)\,(b)) - b \cdot \rho_L\,(\xi)\,(a)\\ &= a \cdot \psi_\xi(b) - (\rho_L(\xi)(a)) \cdot b,\end{aligned}$$

so the condition given in Equation (0.6.9) holds. □

Theorem 0.6.24. *Let* (A, L) *be a Lie–Rinehart algebra. There exists a one to one correspondence between right* (A, L)*-connections* $\nabla^{\mathrm{r}} : L \otimes_A A \to A$ *on* A *and right connection characters* $\delta_{\mathrm{r}}^{\nabla} : L \to A$ *on* A.

Proof. Follows from Lemma 0.6.23 and from the fact $A \otimes_A L \cong L$. □

Remark 0.6.25. Proposition 0.6.21 and Lemma 0.6.23 imply that (A, L)-connections on A are determined by maps satisfying the conditions for them to be connection characters.

Remark 0.6.26. Equations (0.6.18) and (0.6.19) imply that when L is free as an A-module, we can choose arbitrary elements in A as the values of the connection characters $\delta_{\mathrm{r}}^{\nabla}$ and δ_{ℓ}^{∇} evaluated on the generators of L.

Example 0.6.27. Let k be a field, $A := k[t]$ so that $\mathrm{Der}_k(A)$ is generated as an A-module by $\frac{d}{dt}$. The Lie–Rinehart algebra $(A, \mathrm{Der}_k(A))$ admits a right $(A, \mathrm{Der}_k(A))$-connection character on A given by the map

$$\varphi : \mathrm{Der}_k(A) \longrightarrow A, \quad f \cdot \frac{d}{dt} \longmapsto -\frac{d}{dt}(f), \quad \forall f \in A \tag{0.6.23}$$

since we have

$$\varphi\left(f \cdot g \cdot \frac{d}{dt}\right) = -\frac{d}{dt}(f \cdot g) = -f \cdot \frac{d}{dt}(g) - g \cdot \frac{d}{dt}(f) = f \cdot \varphi\left(g \frac{d}{dt}\right) - g \cdot \frac{d}{dt}(f)$$

and hence Equation (0.6.19) is satisfied.

Let us now express the curvature of the left, respectively right (A, L)-connections on A only in terms of the operators δ_{ℓ}^{∇} and $\delta_{\mathrm{r}}^{\nabla}$ respectively.

Theorem 0.6.28. *Let (A, L) be a Lie–Rinehart algebra. Let the maps $\nabla^{\ell} : L \otimes_R A \to L$ and $\nabla^{\mathrm{r}} : A \otimes_R L \to A$ be left, respectively right (A, L)-connections on A, and let $\delta_{\ell}^{\nabla} : L \to A$ and $\delta_{\mathrm{r}}^{\nabla} : L \to A$ be the corresponding left, respectively right (A, L)-connection characters on A. The A-linear operators $\mathcal{C}_{\ell}^{\nabla}$ and $\mathcal{C}_{\mathrm{r}}^{\nabla}$ are given at the identity $1 \in A$ by the expressions*

$$\mathcal{C}_{\ell}^{\nabla}(\xi, \zeta)(1) = \rho_L(\xi)\left(\delta_{\ell}^{\nabla}(\zeta)\right) - \rho_L(\zeta)\left(\delta_{\ell}^{\nabla}(\xi)\right) - \delta_{\ell}^{\nabla}([\xi, \zeta]_L) \tag{0.6.24}$$

$$\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi, \zeta)(1) = -\rho_L(\xi)\left(\delta_{\mathrm{r}}^{\nabla}(\zeta)\right) + \rho_L(\zeta)\left(\delta_{\mathrm{r}}^{\nabla}(\xi)\right) + \delta_{\mathrm{r}}^{\nabla}([\xi, \zeta]_L) \tag{0.6.25}$$

where $\xi, \zeta \in L$ are generators of L.

Proof. Since by Lemma 0.6.17, the (A, L)-curvature operators $\mathcal{C}_{\ell}^{\nabla}$ and $\mathcal{C}_{\mathrm{r}}^{\nabla}$ are A-linear, we only need to evaluate them at the identity 1 where we obtain the following,

$$\begin{aligned}
\mathcal{C}_{\ell}^{\nabla}(\xi, \zeta)(1) &= \left([\nabla_{\xi}^{\ell}, \nabla_{\zeta}^{\ell}] - \nabla_{[\xi,\zeta]_L}^{\ell}\right)(1) \\
&= \nabla_{\xi}^{\ell}\left(\nabla_{\zeta}^{\ell}(1)\right) - \nabla_{\zeta}^{\ell}\left(\nabla_{\xi}^{\ell}(1)\right) - \delta_{\ell}^{\nabla}([\xi, \zeta]_L) \\
&= \nabla_{\xi}^{\ell}\left(\delta_{\ell}^{\nabla}(\zeta)\right) - \nabla_{\zeta}^{\ell}\left(\delta_{\ell}^{\nabla}(\xi)\right) - \delta_{\ell}^{\nabla}([\xi, \zeta]_L) \\
&= \delta_{\ell}^{\nabla}(\zeta) \cdot \delta_{\ell}^{\nabla}(\xi) + \rho_L(\xi)\left(\delta_{\ell}^{\nabla}(\zeta)\right) - \delta_{\ell}^{\nabla}(\xi) \cdot \delta_{\ell}^{\nabla}(\zeta) - \rho_L(\zeta)\left(\delta_{\ell}^{\nabla}(\xi)\right) - \delta_{\ell}^{\nabla}([\xi, \zeta]_L) \\
&= \rho_L(\xi)\left(\delta_{\ell}^{\nabla}(\zeta)\right) - \rho_L(\zeta)\left(\delta_{\ell}^{\nabla}(\xi)\right) - \delta_{\ell}^{\nabla}([\xi, \zeta]_L).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)(1) &= \left([\nabla_\xi^{\mathrm{r}},\nabla_\zeta^{\mathrm{r}}]+\nabla_{[\xi,\zeta]_L}^{\mathrm{r}}\right)(1)\\
&= \nabla_\xi^{\mathrm{r}}\left(\nabla_\zeta^{\mathrm{r}}(1)\right)-\nabla_\zeta^{\mathrm{r}}\left(\nabla_\xi^{\mathrm{r}}(1)\right)+\delta_{\mathrm{r}}^{\nabla}\left([\xi,\zeta]_L\right)\\
&= \nabla_\xi^{\mathrm{r}}(\delta_{\mathrm{r}}^{\nabla}(\zeta))-\nabla_\zeta^{\mathrm{r}}\left(\delta_{\mathrm{r}}^{\nabla}(\xi)\right)+\delta_{\mathrm{r}}^{\nabla}\left([\xi,\zeta]_L\right)\\
&= \delta_{\mathrm{r}}^{\nabla}(\zeta)\cdot\delta_{\mathrm{r}}^{\nabla}(\xi)-\rho_L(\xi)\left(\delta_{\mathrm{r}}^{\nabla}(\zeta)\right)-\delta_{\mathrm{r}}^{\nabla}(\xi)\cdot\delta_{\mathrm{r}}^{\nabla}(\zeta)+\rho_L(\zeta)\left(\delta_{\mathrm{r}}^{\nabla}(\xi)\right)+\delta_{\mathrm{r}}^{\nabla}\left([\xi,\zeta]_L\right)\\
&= -\rho_L(\xi)\left(\delta_{\mathrm{r}}^{\nabla}(\zeta)\right)+\rho_L(\zeta)\left(\delta_{\mathrm{r}}^{\nabla}(\xi)\right)+\delta_{\mathrm{r}}^{\nabla}\left([\xi,\zeta]_L\right).
\end{aligned}$$

□

We write

$$\mathcal{C}_\ell^{\nabla}(\xi,\zeta)(1) \coloneqq \mathcal{C}_\ell^{\nabla}(\xi,\zeta),\quad \mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)(1)\coloneqq \mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)$$

for $\xi,\zeta\in L$.

Example 0.6.29. Recall the Lie–Rinehart algebra in Example 0.6.27, given by $(A,\mathrm{Der}_k(A))$ for $A=k[t]$. The right $(A,\mathrm{Der}_k(A))$-connection character on A given by the map

$$\varphi:\mathrm{Der}_k(A)\longrightarrow A,\quad f\cdot\frac{d}{dt}\longmapsto -\frac{d}{dt}(f),\quad \forall f\in A \tag{0.6.26}$$

gives rise to a flat right $(A,\mathrm{Der}_k(A))$-connection on A since we have

$$\mathcal{C}_{\mathrm{r}}^{\nabla}\left(f\cdot\frac{d}{dt},g\cdot\frac{d}{dt}\right)=f\cdot g\cdot\mathcal{C}_{\mathrm{r}}^{\nabla}\left(\frac{d}{dt},\frac{d}{dt}\right)=0.$$

Let us now see an example where L is free as an A-module and of rank 2:

Proposition 0.6.30. *Let (A,L) be a Lie–Rinehart algebra, let L be generated as a free A-module by $\xi,\zeta\in L$, and let $m,n\in A$ satisfy*

$$[\xi,\zeta]_L=m\cdot\xi+n\cdot\zeta. \tag{0.6.27}$$

The map

$$\psi:L\longrightarrow A;\quad a\cdot\xi+b\cdot\zeta\longmapsto a\cdot n-b\cdot m-\rho_L(\xi)(a)-\rho_L(\zeta)(b),\quad a,b\in A, \tag{0.6.28}$$

is a right (A,L)-connection character on A which give rise to a flat right (A,L)-connection on A.

Proof. We first check that $\psi:L\to A$ is a right (A,L)-connection character on A. Let

$\nu = a \cdot \xi + b \cdot \zeta$ for some $a, b \in A$. Then

$$\begin{aligned}\psi(c \cdot \nu) &= \psi(c \cdot a \cdot \xi + c \cdot b \cdot \zeta) \\ &= c \cdot a \cdot n - c \cdot b \cdot m - \rho_L(\xi)(c \cdot a) - \rho_L(\zeta)(c \cdot b) \\ &= c(a \cdot n - \rho_L(\xi)(a) - b \cdot m - \rho_L(\zeta)(b)) - a \cdot \rho_L(\xi)(c) - b \cdot \rho_L(\zeta)(c) \\ &= c \cdot \psi(\nu) - \rho_L(\nu)(c).\end{aligned}$$

Since $\psi : L \to A$ satisfies Equation (0.6.19) we deduce that it is a right connection character on A. We now check that $\psi : L \to A$ induces a flat right (A, L)-connection on A. Following Theorem 0.6.28 and using the expression from Equation (0.6.27) we obtain

$$\begin{aligned}\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi, \zeta) &= -\rho_L(\xi)(\psi(\zeta)) + \rho_L(\zeta)(\psi(\xi)) + \psi([\xi, \zeta]_L) \\ &= -\rho_L(\xi)(-m) + \rho_L(\zeta)(n) + \psi(m \cdot \xi + n \cdot \zeta) \\ &= \rho_L(\xi)(m) + \rho_L(\zeta)(n) + m \cdot n - \rho_L(\xi)(m) - n \cdot m - \rho_L(\zeta)(n) = 0\end{aligned}$$ □

We have considered examples of Lie–Rinehart algebras where L is free as an A-module and described how under these assumptions right (A, L)-module structures exist on A. Furthermore, by [Hue98, Theorem 3], it is well-known that L only needs to be finitely generated projective over A to ensure that (A, L)-modules on A exist.

Proposition 0.6.31. *Let L be projective as an A-module and of constant finite rank n. There is a bijective correspondence between (A, L)-connections on $\Lambda_A^n L$ and right (A, L)-connections on A. Moreover, left (A, L)-modules on $\Lambda_A^n L$ correspond to right (A, L)-modules on A.*

Proof. See [Hue98]. □

Lastly, we describe the relation between (A, L)-modules on A and $V(A, L)$-modules on A. In the next section, we describe results by Kowalzig and Posthuma [KP11] explaining how these module structures play a fundamental role in the existence (or not) of an antipode on $V(A, L)$.

Proposition 0.6.32 (See [Hue90])**.** *Let (A, L) be a Lie–Rinehart algebra and $V(A, L)$ its universal enveloping algebra. There is an equivalence of categories between left, respectively right (A, L)-module structures on A and left, respectively right $V(A, L)$-modules on A.*

Proof. First assume that A is a right $V(A,L)$-module with right action $A \otimes_R V(A,L) \to A$ given by $a \otimes_R \bar{\xi} \mapsto a \prec \bar{\xi}$ for $a \in A$ and $\bar{\xi} \in V(A,L)$ and define an R-linear map $\partial : L \to A$ by $\partial\xi := 1 \prec \bar{\xi} \in A$. Then

$$\begin{aligned}\partial(a \cdot \xi) &= 1 \prec a\bar{\xi} = 1 \prec \bar{\xi}a - 1 \prec \bar{\xi}a + 1 \prec a\bar{\xi} = 1 \prec \bar{\xi}a - 1 \prec [\bar{\xi}, a]_{V(A,L)} \\ &= a \cdot \left(1 \prec \bar{\xi}\right) - \rho_L(\xi)(a) = a \cdot \partial\xi - \rho_L(\xi)(a) \\ &= a \cdot \partial\xi - \rho_L(\xi)(a)\end{aligned}$$

so $\partial : L \to A$ satisfies Equation (0.6.9) and is hence a right (A,L)-connection on A. Furthermore,

$$\begin{aligned}0 &= 1 \prec \bar{\xi} \prec \bar{\zeta} - 1 \prec \bar{\zeta} \prec \bar{\xi} + 1 \prec [\bar{\xi}, \bar{\zeta}]_{V(A,L)} \\ &= \partial(\xi) \prec \zeta - \partial(\zeta) \prec \bar{\xi} + \partial\left([\xi,\zeta]_L\right) \\ &= \partial(\xi) \cdot \partial(\zeta) - \rho_L(\zeta)\left(\partial(\xi)\right) - \partial(\zeta) \cdot \partial(\xi) - \rho_L(\xi)\left(\partial(\zeta)\right) + \partial\left([\xi,\zeta]_L\right) \\ &= \mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta)\end{aligned}$$

so that $\partial : L \to A$ is a *flat* right (A,L)-connection on A, hence a right (A,L)-module on A.

Conversely, we assume that A is a right (A,L)-module. Then there exists a map $\varphi : L \to A$ satisfying Equation (0.6.9). We see that putting $\partial\xi =: 1 \prec \bar{\xi}$ induces a right $V(A,L)$-module structure on A given by $a \otimes_R \bar{\xi} \mapsto a \prec \bar{\xi}$.

The equivalence between left (A,L)-module structures on A and left $V(A,L)$-modules on A is proved following an identical argument. □

0.7 The left Hopf algebroid $V(A,L)$

This section focuses on understanding the universal enveloping algebra of a Lie algebroid [Pra67] or more generally of a Lie–Rinehart algebra [Her53, Hue91, Hue90, Rin63] and its Hopf algebroid structure, see Section 0.4. These enveloping algebras are one of the richest class of examples of Hopf algebroids.

It is known that these universal enveloping algebras are always *left bialgebroids* (introduced under the name $\times_R$-bialgebras by Takeuchi [Tak77]), see [Xu01], and in fact *left Hopf algebroids* (introduced under the name $\times_R$-Hopf algebras by Schauenburg [Sch00]), see [KK10, Example 2]; see also [Hue08, MM10]. In the light of [KP11, Proposition 3.11], it is known that the universal enveloping algebras of Lie algebroids [ELW99] and of the Lie–Rinehart algebras associated to Poisson algebras [Hue98, Section (3.2)] are full Hopf algebroids (Hopf algebroids in the sense of [BS04]).

However, until our work [KR15] it was an open question whether the enveloping algebra of a Lie–Rinehart algebra was *always* a full Hopf algebroid (with an antipode). It turns out that there exist examples that admit no antipode and hence are only *left* Hopf algebroids. The aim of Chapter 1 is to develop the proof of this result.

In this section we describe the left Hopf algebroid structure on $V(A, L)$. We start by describing the left bialgebroid structure on $V(A, L)$.

Proposition 0.7.1. *Let (A, L) be a Lie–Rinehart algebra over R. Its universal enveloping algebra $V(A, L)$ is a left bialgebroid with*

1. $s_\ell \equiv t_\ell \equiv i_A : A \to V(A, L)$
2. $\epsilon_\ell : V(A, L) \to A, \quad a + \bar{\xi} \mapsto a$
3. $\Delta_\ell : V(A, L) \to V(A, L) \times_A V(A, L), \quad a + \bar{\xi} \mapsto a \otimes_A 1 + 1 \otimes_A \bar{\xi} + \bar{\xi} \otimes_A 1$

where $\times_A$ denotes the Takeuchi product.

Proof. It is straightforward to see that the compatibility conditions given in [BS04] are satisfied. □

Moreover, since the Hopf–Galois map

$$\beta : {}_\blacktriangleright V(A, L) \otimes_{A^{op}} V(A, L)_\triangleleft \longrightarrow V(A, L)_\triangleleft \otimes_A {}_\triangleright V(A, L)$$

has an inverse [KK10], we deduce from [KK10, Sch00] that $V(A, L)$ is a left Hopf algebroid.

Theorem 0.7.2. *There exists an antipode on $V(A, L)$ if and only if there exist right (A, L)-modules on A.*

Proof. See [KP11] □

Proposition 0.7.3. *Let (A, L) be a Lie–Rinehart algebra. If there exists a right (A, L)-module structure on A given by $\delta_{\mathrm{r}}^{\nabla} : L \to A$, the left Hopf algebroid $V(A, L)$ admits a right bialgebroid structure given by*

1. $s_\mathrm{r} \equiv t_\mathrm{r} \equiv i_A : A \to V(A,L)$

2. $\epsilon_\mathrm{r} : V(A,L) \to A, \quad a + \bar{\xi} \mapsto a + \delta_\mathrm{r}^\nabla(\xi)$

3. $\Delta_\mathrm{r} : V(A,L) \to V(A,L) \times^A V(A,L), \quad a + \bar{\xi} \mapsto a \otimes_A 1 + 1 \otimes_A \bar{\xi} + \bar{\xi} \otimes_A 1 - \delta_\mathrm{r}^\nabla(\xi) \otimes_A 1$

turning $V(A,L)$ *into a full Hopf algebroid with antipode* $S : V(A,L) \to V(A,L)$ *given by* $a + \bar{\xi} \mapsto a - \bar{\xi} + \delta_\mathrm{r}^\nabla(\xi)$.

Proof. A straightforward computation yields the compatibility conditions given in [BS04, Definition 4.1]. □

Chapter 1

Lie–Rinehart algebras with and without antipode

In this chapter we focus on presenting examples of Lie–Rinehart algebras beyond the basic examples in Section 0.5.1 and describing how some of these Lie–Rinehart algebra will not admit an antipode on its universal enveloping algebra, which as we saw in Section 0.5.2, is a left Hopf algebroid. Some of these results, in particular of Section 1.4, are the contents of our paper [KR15].

1.1 The canonical Lie–Rinehart algebra over a Poisson algebra

In this section we start by describing the canonical Lie–Rinehart algebra over a Poisson algebra. We then describe the different (full) Hopf algebroid structures on the universal enveloping algebra of this canonical Lie–Rinehart algebra. This description extends some results by Huebschmann [Hue98].

1.1.1 The canonical Lie–Rinehart algebra $(A, \Omega^1(A))$

To construct the canonical Lie–Rinehart algebra over a Poisson algebra A with bracket $\{\bullet, \bullet\}$, we first endow the A-module $A \otimes_R A$ with a Lie bracket, denoted by $[\bullet, \bullet]_I$ and induced by the Poisson structure $\{\bullet, \bullet\}$ on A. We then prove that the Lie bracket $[\bullet, \bullet]_I$ on $A \otimes_R A$ restricts to a Lie bracket on I, the kernel of the multiplication map $\mu : A \otimes_R A \to A$ and furthermore, that it descends to a Lie bracket, denoted $[\bullet, \bullet]_{\Omega^1(A)}$, on the quotient

I/I^2, defining a Lie algebra structure on the A-module $\Omega^1(A)$ of Kähler differentials of A. We next define an A-linear Lie algebra map from $\Omega^1(A)$ to $\mathrm{Der}_R(A)$ compatible with $[\bullet,\bullet]_{\Omega^1(A)}$ in such a way that the pair $(A,\Omega^1(A))$ becomes a Lie–Rinehart algebra over the Poisson algebra A.

Lemma 1.1.1. *Let A be Poisson algebra over R with bracket $\{\bullet,\bullet\}$. The bilinear form on the A-module $A\otimes_R A$ given by*

$$[a\otimes_R f, b\otimes_R g]_I = a\cdot b\otimes_R \{f,g\} - a\cdot b\cdot\{f,g\}\otimes_R 1 + a\cdot\{f,b\}\otimes_R g + b\cdot\{a,g\}\otimes_R f \tag{1.1.1}$$

is a Lie bracket. Furthermore, $[\bullet,\bullet]_I$ restricts to the kernel I of the multiplication map $\mu: A\otimes_R A\to A$ endowing I with a a Lie algebra structure.

Proof. We see that the bracket $[\bullet,\bullet]_I$ in Equation (1.1.1) is skew–symmetric. We now check that $[\bullet,\bullet]_I$ satisfies the Jacobi identity:

$$\begin{aligned}
&[a\otimes_R f,[b\otimes_R g,c\otimes_R h]_I]_I + c.p. = [a\otimes_R f, b\cdot c\otimes_R\{g,h\}]_I - [a\otimes_R f, b\cdot c\cdot\{g,h\}\otimes_R 1]_I\\
&\quad + [a\otimes_R f, b\cdot\{g,c\}\otimes_R h]_I + [a\otimes_R f, c\cdot\{b,h\}\otimes_R g]_I + c.p.\\
&= a\cdot b\cdot c\otimes_R\{f,\{g,h\}\} - a\cdot b\cdot c\cdot\{f,\{g,h\}\}\otimes_R 1 + a\cdot\{f,b\cdot c\}\otimes_R\{g,h\} + b\cdot c\cdot\{a,\{g,h\}\}\otimes_R f\\
&\quad - a\cdot\{f,b\cdot c\cdot\{g,h\}\}\otimes_R 1\\
&\quad + a\cdot b\cdot\{g,c\}\otimes_R\{f,h\} - a\cdot b\cdot\{g,c\}\cdot\{f,h\}\otimes_R 1 + a\cdot\{f,b\cdot\{g,c\}\}\otimes_R h + b\cdot\{g,c\}\cdot\{a,h\}\otimes f\\
&\quad + a\cdot c\cdot\{b,h\}\otimes_R\{f,g\} - a\cdot c\cdot\{b,h\}\cdot\{f,g\}\otimes_R 1 + a\cdot\{f,c\cdot\{b,h\}\}\otimes_R g + c\cdot\{b,h\}\cdot\{a,g\}\otimes f\\
&\quad + c.p.\\
&= -a\cdot\{f,b\cdot c\cdot\{g,h\}\}\otimes_R 1 - a\cdot b\cdot\{g,c\}\cdot\{f,h\}\otimes_R 1 - a\cdot c\cdot\{b,h\}\cdot\{f,g\}\otimes_R 1\\
&\quad + b\cdot c\cdot\{a,\{g,h\}\}\otimes_R f + a\cdot\{f,b\cdot\{g,c\}\}\otimes_R h + b\cdot\{g,c\}\cdot\{a,h\}\otimes f\\
&\quad + a\cdot\{f,c\cdot\{b,h\}\}\otimes_R g + c\cdot\{b,h\}\cdot\{a,g\}\otimes f\\
&\quad + a\cdot\{f,b\cdot c\}\otimes_R\{g,h\} + a\cdot b\cdot\{g,c\}\otimes_R\{f,h\} + a\cdot c\cdot\{b,h\}\otimes_R\{f,g\}\\
&\quad + c.p.,
\end{aligned}$$

so that $[\bullet,\bullet]_I$ is a Lie bracket on $A\otimes_R A$. We now check that $[\bullet,\bullet]_I$ restricts to a Lie bracket on I. Let

$$\sum a_i\otimes_R f_i\in I,\quad \sum b_j\otimes_R g_j\in I \tag{1.1.2}$$

so that $\sum a_i\cdot f_i = \sum b_j\cdot g_j = 0$ and by the Leibniz rule for $\{\bullet,\bullet\}$ given in Equation (0.3.1) we have

$$0 = \sum a_i \cdot \{f_i, b_j \cdot g_j\} = a_i \cdot b_j \cdot \{f_i, g_j\} + a_i \cdot g_j \cdot \{f_i, b_j\}$$
$$0 = \sum b_j \cdot \{g_j, a_i \cdot f_i\} = a_i \cdot b_j \cdot \{g_j, f_i\} + b_j \cdot f_i \cdot \{g_j, a_i\}.$$

Then we have

$$\begin{aligned}\mu\left([\sum a_i \otimes_R f_i, \sum b_j \otimes_R g_j]_I\right) &= \sum (a_i \cdot b_j \cdot \{f_i, g_j\} - a_i \cdot b_j \cdot \{f_i, g_j\} + a_i \cdot \{f_i, b_j\} \cdot g_j + b_j \cdot \{a_i, g_j\} \cdot f_i) \\ &= \sum (a_i \cdot g_j \cdot \{f_i, b_j\} + b_j \cdot f_i \cdot \{a_i, g_j\}) \\ &= \sum (-a_i \cdot b_j \cdot \{f_i, g_j\} - a_i \cdot b_j \cdot \{g_j, f_i\})\end{aligned}$$

which vanishes so that $\sum[a_i \otimes_R f_i, b_j \otimes_R g_j]_I \in I$. □

Theorem 1.1.2. *Let A be a Poisson algebra over R with bracket $\{\bullet, \bullet\}$ and let $\Omega^1(A)$ be the A-module of Kähler differentials of A. The pair $(A, \Omega^1(A))$ is a Lie–Rinehart algebra with anchor*

$$\rho_{\Omega^1(A)} : \Omega^1(A) \longrightarrow \mathrm{Der}_R(A), \quad a \cdot df \longmapsto a \cdot \xi_f := a \cdot \{f, \bullet\} \tag{1.1.3}$$

and Lie bracket $[\bullet, \bullet]_{\Omega^1(A)} : \Omega^1(A) \otimes_R \Omega^1(A) \to \Omega^1(A)$ on $\Omega^1(A)$ given by

$$[df, dg]_{\Omega^1(A)} = d\{f, g\}. \tag{1.1.4}$$

Proof. First recall that $\Omega^1(A) \cong I/I^2$. The Lie bracket $[\bullet, \bullet]_I$ on I given in Equation (1.1.1) induces the map given in Equation (1.1.4) if it maps $I^2 \otimes_R I + I \otimes_R I^2$ to I^2. Since $[\bullet, \bullet]_I$ is skew-symmetric it is enough to check that $I \otimes_R I^2$ is mapped to I^2 under $[\bullet, \bullet]_I : I \otimes_R I \to I$, i.e.,

$$[\alpha, \beta]_I \in I^2, \quad \text{for all } \alpha \in I, \beta \in I^2. \tag{1.1.5}$$

We now check that Equation (1.1.5) holds. Let

$$\sum a_i \otimes_R b_i \in I, \quad \sum a_j \otimes_R b_j \in I, \quad \sum f_k \otimes_R g_k \in I \tag{1.1.6}$$

so that

$$\left(\sum a_i \otimes_R b_i\right) \otimes_R \left(\sum a_j \otimes_R b_j\right) \cdot \left(\sum f_k \otimes_R g_k\right) \in I \otimes_R I^2 \tag{1.1.7}$$

which under the map $[\bullet,\bullet]_I : I \otimes_R I \to I$ becomes

$$
\begin{aligned}
&[\sum a_i \otimes_R b_i, (\sum a_j \otimes_R b_j)\cdot(\sum f_k \otimes_R g_k)]_I = [\sum a_i \otimes_R b_i, \sum a_j \cdot f_k \otimes_R b_j \cdot g_k]_I \\
&\quad = \sum a_i \cdot a_j \cdot f_k \otimes_R \{b_1, b_2 \cdot g_k\} - \sum a_i \cdot a_j \cdot f_k \cdot \{b_i, b_j \cdot g_k\} \otimes_R 1 \\
&\qquad + \sum a_i \cdot \{b_i, a_j \cdot f_k\} \otimes_R (b_j \cdot g_k) - \sum a_j \cdot f_k \cdot \{b_j \cdot g_k, a_i\} \otimes_R b_i \\
&\quad = \sum a_i \cdot a_j \cdot f_k \otimes_R (b_j \cdot \{b_i, g_k\} + g_k \cdot \{b_i, b_j\}) + \sum a_i \cdot (a_j \cdot \{b_i, f_k\} + f_k \cdot \{b_i, a_j\}) \otimes_R (b_j \cdot g_k) \\
&\quad = (\sum a_i \cdot a_j \otimes_R b_j)\cdot(\sum f_k \otimes_R \{b_i, g_k\}) + (\sum a_i \cdot f_k \otimes_R g_k)\cdot(\sum a_j \otimes_R \{b_i, b_j\}) \\
&\qquad + (\sum a_i \cdot a_j \otimes_R b_j)\cdot(\sum \{b_i, f_k\} \otimes g_k) + (\sum a_i \cdot f_k \otimes_R g_k)\cdot(\sum \{b_i, a_j\} \otimes_R b_j) \\
&\quad = (\sum a_i \cdot a_j \otimes_R b_j)\cdot(\sum f_k \otimes_R \{b_i, g_k\} + \sum \{b_i, f_k\} \otimes_R g_k) \\
&\qquad + (\sum a_i \cdot f_k \otimes_R g_k)\cdot(\sum a_j \otimes_R \{b_i, b_j\} + \sum \{b_i, a_j\} \otimes_R b_j)
\end{aligned}
$$

which we deduce is in I^2 since by the Leibniz rule for $\{\bullet,\bullet\}$ given in Equation (0.3.1) we have we have

$$
f_2 \otimes_R \{b_1, g_2\} + \{b_1, f_2\} \otimes_R g_2 \in I, \quad a_2 \otimes_R \{b_1, b_2\} + \{b_1, a_2\} \otimes_R b_2 \in I. \tag{1.1.8}
$$

Hence we deduce that $I^2 \otimes_R I + I \otimes_R I^2$ is mapped to I^2 under $[\bullet,\bullet]_I : I \otimes_R I \to I$, and consequently, the map given in Equation (1.1.1) descends to the map in Equation (1.1.4). Since $[\bullet,\bullet]_I$ satisfies the Jacobi identify, $[\bullet,\bullet]_{\Omega^1(A)}$ also does. Hence we deduce that the A-module $\Omega^1(A)$ is a Lie algebra with bracket $[\bullet,\bullet]_{\Omega^1(A)}$. Moreover, the map $\rho_{\Omega^1(A)} : \Omega^1(A) \to \mathrm{Der}_R(A)$ is well-defined if and only if the map

$$
\rho_I : I \longrightarrow \mathrm{Der}_R(A), \quad a \cdot f \longmapsto a \cdot \xi_f = a \cdot \{f, \bullet\} \quad \text{for } a \otimes_R f \in I
$$

maps I^2 to 0. We check that given $a \otimes_R f \in I$ and $b \otimes_R g \in I$, we have

$$
\begin{aligned}
\rho_I((a \otimes_R f)\cdot(b \otimes_R g)) &= \rho_I(a \cdot b \otimes_R f \cdot g) \\
&= a \cdot b \cdot \xi_{f\cdot g} \\
&= a \cdot b \cdot f \cdot \xi_g + a \cdot b \cdot g \cdot \xi_f
\end{aligned}
$$

which vanishes. We now check that the A-linear map $\rho_{\Omega^1(A)} : \Omega^1(A) \to \mathrm{Der}_R(A)$ is a Lie algebra homomorphism. By Lemma 0.3.6 and Proposition 1.1.2, we have

$$\begin{aligned}\rho_{\Omega^1(A)}\left([a\cdot df, b\cdot dg]_{\Omega^1(A)}\right) &= \rho_{\Omega^1(A)}\left(a\cdot b\cdot d\{f,g\} + a\cdot\{f,b\}\cdot dg + b\cdot\{a,g\}\cdot df\right)\\ &= a\cdot b\cdot \xi_{\{f,g\}} + a\cdot\{f,b\}\cdot\xi_g + b\cdot\{a,g\}\cdot\xi_f\\ &= a\cdot b\cdot[\xi_f,\xi_g]_{\mathrm{Der}_R(A)} + a\cdot\xi_f(b)\cdot\xi_g - b\cdot\xi_g(a)\cdot\xi_f\\ &= a\cdot b\cdot\left(\xi_f\left(\xi_g\right) - \xi_g\left(\xi_f\right)\right) + a\cdot\xi_f(b)\cdot\xi_g - b\cdot\xi_g(a)\cdot\xi_f\\ &= a\cdot\left(b\cdot\xi_f(\xi_g) + \xi_f(b)\cdot\xi_g\right) - b\cdot\left(a\cdot\xi_g(\xi_f) + \xi_g(a)\cdot\xi_f\right)\\ &= a\cdot\left(\xi_f\left(b\cdot\xi_g\right)\right) - b\cdot\left(\xi_g\left(a\cdot\xi_f\right)\right)\\ &= [a\cdot\xi_f, b\cdot\xi_g]_{\mathrm{Der}_R(A)}\\ &= [\rho_{\Omega^1(A)}(a\cdot df), \rho_{\Omega^1(A)}(b\cdot dg)]_{\mathrm{Der}_R(A)}.\end{aligned}$$

Furthermore, the commutative algebra structure $\mu : A\otimes_R A \to A$ underlying the Poisson algebra $(A,\{\bullet,\bullet\})$ and the Lie bracket $[\bullet,\bullet]_{\Omega^1(A)}$ on $\Omega^1(A)$ induced by the Poisson structure $\{\bullet,\bullet\}$ satisfy the compatibility condition in Equation (0.5.1): by Proposition 1.1.2 we can write,

$$\begin{aligned}[df, h\cdot dg]_{\Omega^1(A)} &= h\cdot d\{f,g\} + \{f,h\}\cdot dg\\ &= h\cdot[df,dg]_{\Omega^1(A)} + \rho_{\Omega^1(A)}\left(df\right)(h)\cdot(dg).\end{aligned}$$

Hence $\left(A,\Omega^1(A)\right)$ is a Lie–Rinehart algebra. □

Remark 1.1.3. Let A be a Poisson algebra over R with bracket $\{\bullet,\bullet\}$ and let $\Omega^1(A)$ be the A-module of Kähler differentials over A equipped with the Lie algebra structure $[\bullet,\bullet]_{\Omega^1(A)}$. Equation (1.1.4) implies that the Lie bracket on the generators of $\Omega^1(A)$ is

$$[df,dg]_{\Omega^1(A)} = d\{f,g\},\quad f,g\in A,$$

so that the differential map $d : A \longrightarrow \Omega^1(A)$ is a Lie algebra homomorphism.

1.1.2 $(A,\Omega^1(A))$-connections on a Poisson algebra A

We now focus on $(A,\Omega^1(A))$-connections on a Poisson algebra $(A,\{\bullet,\bullet\})$. First recall from Remark 0.6.13 that left, respectively right (A,L)-connections $\nabla^\ell : L\otimes_R A\to A$ and $\nabla^{\mathrm{r}} : A\otimes_A L\to A$ are completely determined when known on the generators of L as an A-module. Since we consider $L=\Omega^1(A)$, we only need to specify ∇^ℓ and ∇^{r} on the elements

$da \in \Omega^1(A)$ for all $a \in A$.

Recall also that a right (A, L)-connection on A is determined by the non-linear right connection character operator $\delta_{\mathrm{r}}^{\nabla}$ as given in Proposition 0.6.21, while a left (A, L)-connection on A is determined by the A-linear left connection character operator δ_{ℓ}^{∇}.

Proposition 1.1.4. *Let A be a Poisson algebra with bracket $\{\bullet, \bullet\}$, $\Omega^1(A)$ be the A-module of Kähler differentials and $D : A \to A$ be an R-linear map. The map*

$$\psi : \Omega^1(A) \longrightarrow A, \quad a \cdot db \longmapsto a \cdot D(b) + \{a, b\} \tag{1.1.9}$$

is a right $\big(A, \Omega^1(A)\big)$-connection character on A if and only if $D : A \to A$ is a derivation of A.

Proof. First we check the conditions under which the map given in Equation (1.1.9) is well-defined. The map

$$\gamma_\psi : I \longrightarrow A, \quad a \otimes_R b \longmapsto a \cdot D(b) + \{a, b\} \tag{1.1.10}$$

descends to the map given in Equation (1.1.9) if and only if $\gamma_\psi\left(I^2\right) = 0$. We now check this. Let $\sum a_i \otimes_R b_i, \sum f_j \otimes_R g_j \in I$, that is, $\sum a_i \cdot b_i = \sum f_j \cdot g_j = 0$, so that we have

$$\left(\sum a_i \otimes_R b_i\right) \cdot \left(\sum f_j \otimes_R g_j\right) \in I^2 \tag{1.1.11}$$

which under the map $\gamma_\psi : I \to A$ becomes

$$\begin{aligned}
\gamma_\psi&\left(\left(\sum a_i \otimes_R b_i\right) \cdot \left(\sum f_j \otimes_R g_j\right)\right) = \gamma_\psi\left(\sum a_i \cdot f_j \otimes b_i \cdot g_j\right)\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum \{a_i \cdot f_j, b_i \cdot g_j\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum a_i \cdot b_i \cdot \{f_j, g_j\} + \sum a_i \cdot g_j \cdot \{f_j, b_i\}\\
&\quad + \sum b_i \cdot f_j \cdot \{a_i, g_j\} + \sum f_j \cdot g_j \cdot \{a_i, b_i\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum a_i \cdot g_j \cdot \{f_j, b_i\} + \sum b_i \cdot f_j \cdot \{a_i, g_j\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum a_i \cdot g_j \cdot \{f_j, b_i\} + \sum b_i \cdot f_i \cdot \{a_i, g_j\} - \sum g_j \cdot \{f_j, a_i \cdot b_i\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) - \sum g_j \cdot b_i \cdot \{f_j, a_i\} + \sum b_i \cdot f_j \cdot \{a_i, g_j\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum b_i \cdot g_j \cdot \{a_i, f_j\} + \sum b_i \cdot f_j \cdot \{a_i, g_j\}\\
&= \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right) + \sum b_i \cdot \{a_i, f_j \cdot g_j\} = \sum a_i \cdot f_j \cdot D\left(b_i \cdot g_j\right).
\end{aligned}$$

which vanishes if and only if

$$\sum a_i \cdot f_i \cdot D\left(b_j \cdot g_j\right) = 0 \quad \text{for } a_i \cdot b_j = f_i \cdot g_j = 0. \tag{1.1.12}$$

Note that if $D : A \to A$ is a first order differential operator, this condition is satisfied. We now prove that $\psi : \Omega^1(A) \to A$ is a right connection character on A. We have

$$\begin{aligned}\psi\left(a \cdot b \cdot dc\right) &= a \cdot b \cdot D\left(c\right) + \{a \cdot b, c\} \\ &= a \cdot b \cdot D\left(c\right) + a \cdot \{b, c\} + b \cdot \{a, c\} \\ &= a\left(b \cdot D\left(c\right) + \{b, c\}\right) + b \cdot \{a, c\} \\ &= a \cdot \psi\left(b \cdot dc\right) + b \cdot \{a, c\} \\ &= a \cdot \psi\left(b \cdot dc\right) - \rho_{\Omega^1(A)}\left(b \cdot dc\right)\left(a\right)\end{aligned}$$

so the map $\psi : \Omega^1(A) \to A$ satisfies Equation (0.6.19) and is hence a right connection character on A. Conversely, let the map given in Equation (1.1.9) be a connection character on A. Then we can write:

$$\begin{aligned}D(b \cdot c) &= \psi(d(b \cdot c)) \\ &= \psi(b \cdot dc + c \cdot db) \\ &= b \cdot D(c) + \{b, c\} + c \cdot D(b) + \{c, b\} \\ &= b \cdot \psi(dc) + c \cdot \psi(db) \\ &= b \cdot D(c) + c \cdot D(b).\end{aligned}$$

So $D : A \to A$ satisfies the Leibniz rule in Equation (0.1.5), and is hence a derivation which satisfies the condition on the map $D : A \to A$ given in Equation (1.1.12). □

Proposition 1.1.5. *Let $(A, \Omega^1(A))$ be the canonical Lie–Rinehart algebra over a Poisson algebra A with bracket $\{\bullet, \bullet\}$, and let $D : A \to A$ be a derivation of A. The $(A, \Omega^1(A))$-connection character*

$$\psi : \Omega^1(A) \longrightarrow A; \quad a \cdot db \longmapsto a \cdot D\left(b\right) + \{a, b\} \tag{1.1.13}$$

induces a flat right $\left(A, \Omega^1(A)\right)$-connection on A if and only if $D : A \to A$ is a Poisson derivation of A.

Proof. Following Theorem 0.6.28 we obtain the curvature of the right $\big(A, \Omega^1(A)\big)$-connection $\psi : \Omega^1(A) \to A$ on A:

$$\begin{aligned}\mathcal{C}^{\nabla}_{\mathrm{r}}(df, dg) &= -\rho_{\Omega^1(A)}(df)(\psi(dg)) + \rho_{\Omega^1(A)}(dg)(\psi(df)) + \psi([df, dg]_{\Omega^1(A)}) \\ &= \{f, D(g)\} - \{D(f), g\} + D(\{f, g\})\end{aligned}$$

for all $f, g \in A$. Let us assume that $D : A \to A$ is a Poisson derivation of A, then $\mathcal{C}^{\nabla}_{\mathrm{r}}(df, dg)$ vanishes. Conversely, let us assume $\mathcal{C}^{\nabla}_{\mathrm{r}}(df, dg)$ vanishes for all $f, g \in A$: then we have $\{f, D(g)\} - \{D(f), g\} + D(\{f, g\}) = 0$ so we find that $D : A \to A$ is a Poisson derivation. □

1.2 New constructions of Lie–Rinehart algebras

The following three results (Propositions 1.2.1, 1.2.2, 1.2.4) present constructions of Lie–Rinehart algebras which play a fundamental role in Section 1.4 and Chapter 2.

We first consider the structure of the pair (A, A). Since A is a commutative algebra and can be seen as a Lie algebra, we deduce that the pair (A, A) always admits a Lie–Rinehart algebra structure with trivial Lie bracket on A and anchor map given by the zero map. Moreover, if A admits non-zero derivations, then A can also be endowed with a certain (non-trivial) Lie bracket which turns A into a Jacobi algebra. Then the pair (A, A) admits a non-trivial Lie–Rinehart algebra structure. This structure will be discussed in the next chapter. Note that this provides a one to one correspondence between derivations of A and Lie–Rinehart algebra structures on (A, L) where $L \cong A$ as an A-module.

Proposition 1.2.1. *Let A be a commutative R-algebra, and let $E : A \to A$ be an R-linear map.*

1. *If $E \in \mathrm{Der}_R(A)$, then the skew-symmetric bilinear form*

$$[\bullet, \bullet] : A \otimes_R A \longrightarrow A, \quad (a, b) \longmapsto [a, b] := a \cdot E(b) - b \cdot E(a) \tag{1.2.1}$$

and the A-linear map

$$\rho_A : A \longrightarrow \mathrm{Hom}_R(A, A), \quad a \longmapsto a \cdot E \tag{1.2.2}$$

define a Lie–Rinehart algebra structure on (A, A).

2. *Moreover, all Lie–Rinehart algebra structures on (A, L), where $L \cong A$ as an A-module, arise in this way.*

Proof. Assume $E \in \mathrm{Der}_R(A)$ and set $E(a) =: a'$. First note $[1, a] = a'$ for all $a \in A$. By Equation (1.2.1), the bracket $[\bullet, \bullet]$ is skew symmetric, so we only need to check that it satisfies the Jacobi identity:

$$\begin{aligned}[a,[b,c]] + [b,[c,a]] + [c,[a,b]] &= [a, b\cdot c' - c\cdot b'] + c.p.\\ &= a\cdot\left(b\cdot c' - c\cdot b'\right)' - \left(b\cdot c' - c\cdot b'\right)\cdot a' + c.p.\\ &= a\left(b\cdot c'' - b''\cdot c\right) - a'\cdot b\cdot c' + a'\cdot b'\cdot c + c.p. = 0\end{aligned}$$

so $[\bullet, \bullet]$ is a Lie bracket on A. We next check that the A-linear map in Equation (1.2.2) is a Lie algebra map:

$$\rho_A([a,b]) = \rho_A\left(a\cdot b' - a'\cdot b\right) = \left(a\cdot b' - a'\cdot b\right)\cdot E = [a\cdot E, b\cdot E] = [\rho_A(a), \rho_A(b)].$$

Also, we check that the compatibility condition between the A-module structure on $\mathrm{Der}_R(A)$ and the anchor map $\rho_A : A \to \mathrm{Der}_R(A)$ given in Equation (0.5.1) is satisfied:

$$\begin{aligned}[a, b\cdot c] &= a\cdot(b\cdot c)' - a'\cdot b\cdot c\\ &= a\cdot b\cdot c' + a\cdot b'\cdot c - a'\cdot b\cdot c\\ &= b\cdot\left(a\cdot c' - a'\cdot c\right) + a\cdot b'\cdot c\\ &= b\cdot[a,c] + \rho_A(a)(b)\cdot c.\end{aligned}$$

Hence (A, A) is a Lie–Rinehart algebra. Conversely, let us assume (A, A) is a Lie–Rinehart algebra. Since the anchor map $\rho : A \to \mathrm{Der}_R(A)$ is A-linear, we have

$$\rho(a) = a\cdot\rho(1), \quad a \in A \tag{1.2.3}$$

so the anchor map is completely determined by its image on the identity element $1 \in A$, i.e., the image of A under the anchor map is generated, as an A-module, by a single derivation of A, namely $\rho(1)$, which we denote by $E \in \mathrm{Der}_R(A)$. So we write the anchor map as

$$\rho : A \longrightarrow \mathrm{Der}_R(A), \quad a \longmapsto a\cdot E. \tag{1.2.4}$$

Now, since (A, A) has a Lie–Rinehart algebra structure, the Lie bracket $[\bullet, \bullet]$ on A satisfies the Leibniz rule given in Equation (0.5.1), so

$$[a, b] = [a \cdot 1, b \cdot 1] = a \cdot b \cdot [1, 1] + a \cdot \rho(1) \cdot (b) - b \cdot \rho(1) \cdot (a) = a \cdot E(b) - b \cdot E(a)$$

so a Lie–Rinehart algebra structure on (A, A) induces a Lie bracket on A given by the map in Equation (1.2.1). □

We now consider Lie–Rinehart algebras (A, L) over a field k such that the A-module structure on the Lie algebra L is given by a character $\chi : A \to k$, that is, the action $A \otimes L \to L$ is given by $a \otimes \xi \mapsto \chi(a)\xi$. This construction is fundamental in some of the counterexamples in Section 1.4.

Proposition 1.2.2. *Let A be a commutative k-algebra, L be a Lie algebra over k with bracket $[\bullet, \bullet]_L$ and $\rho_L : L \to \mathrm{Der}_k(A)$ be a Lie algebra map. Define an A-module structure on L by*

$$\mu : A \otimes L \longrightarrow L, \quad a \otimes \xi \longmapsto a \cdot \xi := \chi(a)\xi,$$

where $\chi : A \to k$ is a character on A. Then (A, L) is a Lie–Rinehart algebra if and only if ρ_L is A-linear and $\rho_L(\xi)(a) \in \ker\chi$ for all $a \in A, \xi \in L$.

Proof. First note that the A-linearity of ρ_L is equivalent to

$$0 = \rho_L(a \cdot \xi) - a \cdot \rho_L(\xi) = \rho_L(\chi(a)\xi) - a \cdot \rho_L(\xi) = \chi(a)\rho_L(\xi) - a \cdot \rho_L(\xi)$$

so that the A-module structure on $\mathrm{Der}_k(A)$ satisfies

$$a \cdot \rho_L(\xi) = \chi(a)\rho_L(\xi). \tag{1.2.5}$$

The proof is divided into two sections:

First assume (A, L) is a Lie–Rinehart algebra. Then, since $[\bullet, \bullet]_L$ satisfies the Leibniz rule in Equation (0.5.1), we have:

$$\begin{aligned} 0 &= [\xi, a \cdot \zeta]_L - a \cdot [\xi, \zeta]_L - \rho_L(\xi)(a) \cdot \zeta \\ &= [\xi, \chi(a)\zeta]_L - \chi(a)[\xi, \zeta]_L - \chi(\rho_L(\xi)(a))\zeta \\ &= -\chi(\rho_L(\xi)(a))\zeta \end{aligned}$$

so we see that the k-linearity of $[\bullet,\bullet]_L$ becomes equivalent to $\rho_L(\xi)(a) \in \ker(\chi)$ for all $a \in A$ and $\xi \in L$.

Conversely, let us assume $\rho_L(\xi)(a) \in \ker(\chi)$ for all $a \in A$ and $\xi \in L$. Then we have $\rho_L(\xi)(a)\cdot\zeta = \chi(\rho_L(\xi)(a))\,\zeta = 0$, so that

$$\begin{aligned} 0 &= [\xi,\chi(a)\zeta]_L - \chi(a)[\xi,\zeta]_L \\ &= [\xi,\chi(a)\zeta]_L - \chi(a)[\xi,\zeta]_L - \chi(\rho_L(\xi)(a))\zeta \\ &= [\xi, a\cdot\zeta]_L - a\cdot[\xi,\zeta]_L - \rho_L(\xi)(a)\cdot\zeta \end{aligned}$$

which is equivalent to Equation (0.5.1). □

Note that Equation (1.2.5) is a very strong condition. This is satisfied for example when $A = k$ and for examples involving zero-divisors as in Section 1.4. Note also that for examples of Lie–Rinehart algebras constructed following Lemma 1.2.2, the Lie bracket $[\bullet,\bullet]_L$ is even A-linear, so L is a Lie algebra over A. However, in general we have $\rho_L \neq 0$.

In the next two propositions we discuss how given a Lie–Rinehart algebra (A, L), we can define two different Lie–Rinehart algebra structures on a pair (A, M) where M is a certain A-submodule of L.

Proposition 1.2.3. *Let (A, L) be a Lie–Rinehart algebra with Lie bracket on L denoted by $[\bullet,\bullet]_L$ and anchor ρ_L. For $h \in A$, define $\mu_h : L \to L$ by $\zeta \mapsto h\cdot\zeta$ for $\zeta \in L$ and put $M := \mathrm{Im}(\mu_h) = hL$. The pair (A, M) is a Lie–Rinehart subalgebra of (A, L).*

Proof. First we check that M is closed under $[\bullet,\bullet]_L$:

$$[h\cdot\zeta, h\cdot\gamma]_L = h^2\cdot[\zeta,\gamma]_L + h\cdot\rho_L(\zeta)(h)\cdot\gamma - h\cdot\rho_L(\gamma)(h)\cdot\zeta \in M.$$

Since M is an A-submodule of L, by Definition 0.5.6, we have that (A, M) is a Lie–Rinehart subalgebra of (A, L). □

We now define a new Lie bracket on M and subsequently a new Lie–Rinehart algebra structure on (A, M):

Proposition 1.2.4. *Let (A, L) be a Lie–Rinehart algebra with Lie bracket on L denoted by $[\bullet,\bullet]_L$ and anchor ρ_L. For $h \in A$ and $\mu_h : L \to L$ the A-linear map $\zeta \mapsto h\cdot\zeta$, put*

$M = \mathrm{Im}(\mu_h) = hL$ and define $K := \mathrm{Ker}(\mu_h) = \{\xi \in L \mid h \cdot \xi = 0\}$. The pair (A, M) admits a Lie–Rinehart algebra structure with Lie bracket on M given by

$$[h \cdot \zeta, h \cdot \gamma]_M := h \cdot [\zeta, \gamma]_L, \quad \zeta, \gamma \in L, \tag{1.2.6}$$

and anchor

$$\rho_M : M \longrightarrow \mathrm{Der}_k(A), \quad \rho_M(h \cdot \zeta) := \rho_L(\zeta), \quad \zeta \in L \tag{1.2.7}$$

turning the map

$$\mu_h : L \longrightarrow M, \quad \zeta \longmapsto h \cdot \zeta \tag{1.2.8}$$

into a Lie–Rinehart algebra homomorphism if and only if

1. *$\rho_L(\xi) = 0$, for all $\xi \in K$,*

2. *K is a Lie ideal in L, i.e., $h \cdot [\xi, \bullet]_L = 0$, for all $\xi \in K$.*

Proof. Assume first that (A, L) and $h \in A$ satisfy conditions (1) and (2) above. The bracket $[\bullet, \bullet]_M : M \otimes M \to M$, defined by

$$[\bullet, \bullet]_M \circ (\mu_h \otimes \mu_h) = \mu_h \circ [\bullet, \bullet]_L,$$

is well-defined since $K = \mathrm{Ker}(\mu_h)$ and $[K, M]_L \subset K$ by condition (2). Moreover, $[\bullet, \bullet]_M$ is skew-symmetric and satisfies the Jacobi identity since $[\bullet, \bullet]_L$ does. Furthermore, we have $\rho_M(h \cdot \xi) = \rho_L(\xi)$ which vanishes by our assumptions, so the map $\rho_M : M \to \mathrm{Der}_k(A)$ is well-defined. We next check that ρ_M is a Lie algebra map:

$$\begin{aligned} \rho_M([h \cdot \zeta, h \cdot \gamma]_M) &= \rho_M(h \cdot [\zeta, \gamma]_L) \\ &= \rho_L([\zeta, \gamma]_L)) \\ &= [\rho_L(\zeta), \rho_L(\gamma)]_{\mathrm{Der}_k(A)} \\ &= [\rho_M(h \cdot \zeta), \rho_M(h \cdot \gamma)]_{\mathrm{Der}_k(A)}. \end{aligned}$$

We finally check that $[\bullet, \bullet]_M$ is compatible with ρ_M since the Leibniz rule in Equation (0.5.1) is satisfied:

$$\begin{aligned} [h \cdot \zeta, a \cdot h \cdot \gamma]_M &= h \cdot [\zeta, a \cdot \gamma]_L = h \cdot a \cdot [\zeta, \gamma]_L + h \cdot \rho_L(\zeta)(a) \cdot \gamma \\ &= a \cdot [h \cdot \zeta, h \cdot \gamma]_M + \rho_M(h \cdot \zeta)(a) \cdot \gamma. \end{aligned}$$

Conversely, assume M admits the Lie–Rinehart algebra structure with bracket on M given by Equation (1.2.6) and anchor given by Equation (1.2.7). Since for all $\xi \in K$ we have $\rho_M(h \cdot \xi) = 0$, by Equation (1.2.7) we deduce $\rho_L(\xi) = 0$. Moreover, for elements $\xi \in K$ and $\gamma \in L$ we have

$$[h \cdot \xi, h \cdot \gamma]_M = 0 = h \cdot [\xi, \gamma]_L$$

so $[\xi, \gamma]_L \in K$ for all $\gamma \in L$, so that K is a Lie ideal in L. □

Remark 1.2.5. Note that since $[h \cdot \xi, \gamma_1]_L = 0$ for all $\xi \in K$, by the Leibniz rule in (0.5.1) we deduce that $\rho_L(\gamma_1)(h) \cdot \xi = 0$ for all $\gamma_1 \in L$ so that $[\rho_L(\gamma_1)(h) \cdot \xi, \gamma_2]_L = 0$ and $\rho_L(\gamma_2) \circ \rho_L(\gamma_1)(h) \cdot \xi = 0$. Repeating this iteration process we deduce $\rho_L(\gamma_i) \circ \cdots \circ \rho_L(\gamma_1)(h) \cdot \xi = 0$.

1.3 Some new Lie–Rinehart algebras with an antipode

In this section we present some examples of Lie–Rinehart algebras (A, L) where L is not finitely generated and projective over A and whose universal enveloping algebras $V(A, L)$ still admit an antipode. These examples go beyond the situation explained by [Hue98] in the light of [KP11].

Example 1.3.1. Let A be a commutative R-algebra, and let $E \in \mathrm{Der}_R(A)$. The Lie–Rinehart algebra (A, A) with bracket on A given by

$$[\bullet, \bullet] : A \otimes_R A \longrightarrow A, \quad (a, b) = a \cdot E(b) - b \cdot E(a) \tag{1.3.1}$$

and anchor map

$$\rho : A \longrightarrow \mathrm{Der}_R(A), \quad a \longmapsto a \cdot E$$

admits a right (A, A)-connection character on A given by the map

$$\partial : A \longrightarrow A, \quad a \longmapsto a \cdot \mathfrak{d} - E(a)$$

for all $\mathfrak{d} \in A$. Hence its enveloping algebra $V(A, A)$ is a full Hopf algebroid. Since $\partial : A \to A$ satisfies

$$\begin{aligned} \partial(a \cdot b) &= a \cdot b \cdot \mathfrak{d} - E(a \cdot b) = a \cdot b \cdot \mathfrak{d} - a \cdot E(b) - b \cdot E(a) \\ &= a(b \cdot \mathfrak{d} - E(b)) - b \cdot E(a) = a \cdot \partial(b) - b \cdot E(a) \end{aligned}$$

we deduce that the condition given in Equation (0.6.19) is satisfied. A short computation provides the curvature:

$$\mathcal{C}_{\mathrm{r}}^{\nabla}(a,b) = a \cdot b \cdot \mathcal{C}_{\mathrm{r}}^{\nabla}(1,1) = 0.$$

From Section 0.7, we see that while there is a single canonical left Hopf algebroid structure on $V(A,A)$, for each element $\mathfrak{d} \in A$, there exists a right bialgebroid structure on $V(A,A)$, and hence a full Hopf algebroid on $V(A,A)$.

Theorem 1.3.2. *Let k be a field with $\mathrm{Char}(k) \neq 2$ and let $A := k[t]/\langle t^2 \rangle$ so that we have $\mathrm{Der}_k(A) = \mathrm{Span}_A \left\{ E := t \cdot \frac{d}{dt} \right\}$. The map*

$$\varphi : \mathrm{Der}_k(A) \longrightarrow A; \quad a \cdot E \longmapsto a \cdot c - E(a) \tag{1.3.2}$$

where $c = (1 - b \cdot t)$ for $b \in A$ is a flat right $(A, \mathrm{Der}_k(A))$-connection character on A.

Proof. First note that $t \cdot \frac{d}{dt}\left(t^2\right) = 2 \cdot t^2 = 0$ so that $t \cdot \frac{d}{dt}$ is a derivation of A. Note also $t \cdot E = 0$. We now check that the map $\varphi : \mathrm{Der}_k(A) \to A$ is well-defined:

$$0 = \varphi(t \cdot E) = t \cdot c - E(t) = t \cdot (1 - b \cdot t) - t = t - b \cdot t^2 - t$$

which holds since $t^2 = 0$. We now check that $\varphi : \mathrm{Der}_k(A) \to A$ is a right $(A, \mathrm{Der}_k(A))$-connection on A:

$$\begin{aligned}
\varphi(f \cdot g \cdot E) &= f \cdot g \cdot c - E(f \cdot g) \\
&= f \cdot g \cdot c - f \cdot E(g) - g \cdot E(f) \\
&= f \cdot (g \cdot c - E(g)) - g \cdot E(f) \\
&= f \cdot \varphi(g \cdot E) - E(f) \cdot g
\end{aligned}$$

so that Equation (0.6.19) is satisfied. Moreover, note

$$\mathcal{C}_{\mathrm{r}}^{\nabla}(f \cdot E, g \cdot E) = f \cdot g \cdot \mathcal{C}_{\mathrm{r}}^{\nabla}(E,E) = 0$$

so that $\varphi : \mathrm{Der}_k(A) \to A$ induces a flat $(A, \mathrm{Der}_k(A))$-connection on A. □

Theorem 1.3.3. *Let k be a field with $\mathrm{Char}(k) \neq 2$ and let $A := k[x,y]/\langle x \cdot y \rangle$ so that*

$$\mathrm{Der}_k(A) = \mathrm{Span}_A \left\{ \xi := x \cdot \frac{\partial}{\partial x}, \zeta := y \cdot \frac{\partial}{\partial y} \right\}.$$

For all $f, g, h \in A$, the map

$$\psi : \mathrm{Der}_k(A) \longrightarrow A; \quad f \cdot \xi + g \cdot \zeta \longmapsto x \cdot f \cdot h + y \cdot g \cdot h - x \cdot \frac{\partial}{\partial x}(f) - y \cdot \frac{\partial}{\partial y}(g) \tag{1.3.3}$$

is a flat right $(A, \mathrm{Der}_k(A))$-connection character on A. Hence the Lie–Rinehart algebra $(A, \mathrm{Der}_k(A))$ admits a right $V(A, \mathrm{Der}_k(A))$-module structure on A that extends multiplication in A.

Proof. First note that $\xi(x \cdot y) = \zeta(x \cdot y) = 0$ and $y \cdot \xi = x \cdot \zeta = 0$. We now check that the map $\psi : \mathrm{Der}_k(A) \to A$ is well-defined:

$$\psi(y \cdot \xi) = x \cdot y \cdot h - x \cdot \frac{\partial}{\partial x}(y) = 0, \qquad \psi(x \cdot \zeta) = y \cdot x \cdot h - y \cdot \frac{\partial}{\partial y}(x) = 0$$

Following an identical argument as in the proof of Theorem 1.3.2, a straightforward computation shows that ψ satisfies Equation (0.6.19) so that $\psi : \mathrm{Der}_k(A) \to A$ is a right $(A, \mathrm{Der}_k(A))$-connection on A. Moreover, note

$$\begin{aligned} \mathcal{C}_{\mathrm{r}}^{\nabla}(f \cdot \xi, g \cdot \zeta) &= f \cdot g \cdot \mathcal{C}_{\mathrm{r}}^{\nabla}(\xi, \zeta) \\ &= f \cdot g \cdot (-\xi(\psi(\zeta)) + \zeta(\psi(\xi)) + \psi([\xi, \zeta])) \\ &= f \cdot g \cdot (-\xi(y \cdot h) + \zeta(x \cdot h)) \\ &= f \cdot g \cdot (-y \cdot \xi(h) + x \cdot \zeta(h)) \\ &= f \cdot g \cdot \left(-y \cdot x \frac{\partial}{\partial x}(h) + x \cdot y \cdot \frac{\partial}{\partial y}(h)\right) = 0 \end{aligned}$$

so $\psi : \mathrm{Der}_k(A) \to A$ induces a flat right $(A, \mathrm{Der}_k(A))$-connection on A. □

We now consider an example of a Lie–Rinehart algebra (A, L) where L is generated, as an A-module, by two elements, i.e., L is of rank 2. Moreover, L is not free as an A-module.

Theorem 1.3.4. *Let (A, L) be a Lie–Rinehart algebra where A is an integral domain over R, and L is a Lie algebra generated as an A-module by $\xi, \zeta \in L$ satisfying the relation $f \cdot \xi = g \cdot \zeta$ for $f, g \in A$, and let $m, n \in A$ satisfy $[\xi, \zeta]_L = m \cdot \xi + n \cdot \zeta$. The map*

$$\delta : L \longrightarrow A, \quad a \cdot \xi + b \cdot \zeta \longmapsto a \cdot (h \cdot g + n) - b \cdot (h \cdot f - m) - \rho_L(\xi)(a) - \rho_L(\zeta)(b), \tag{1.3.4}$$

for $a, b, h \in A$, is a flat (A, L)-connection on A.

Proof. First we note the following: since $f\cdot\xi = g\cdot\zeta$, we have

$$\begin{aligned}0 = [f\cdot\xi, g\cdot\zeta]_L &= f\cdot g\cdot[\xi,\zeta]_L + f\cdot\rho_L(\xi)(g)\cdot\zeta - g\cdot\rho_L(\zeta)(f)\cdot\xi\\ &= f\cdot g\cdot(m\cdot\xi + n\cdot\zeta) + f\cdot\rho_L(\xi)(g)\cdot\zeta - g\cdot\rho_L(\zeta)(f)\cdot\xi\\ &= g\cdot\rho_L(\zeta)(g)\cdot\zeta - g\cdot\rho_L(\zeta)(f)\cdot\xi + f\cdot g\cdot(m\cdot\xi+n\cdot\zeta)\\ &= f\cdot\rho_L(\zeta)(g)\cdot\xi - f\cdot\rho_L(\xi)(f)\cdot\xi + f\cdot g\cdot m\cdot\xi + f\cdot n\cdot f\cdot\xi\end{aligned}$$

so, we deduce

$$g\cdot m + f\cdot n = \rho_L(\xi)(f) - \rho_L(\zeta)(g). \tag{1.3.5}$$

We first check that the map $\delta : L \to A$ is well-defined.

$$\begin{aligned}\delta(f\cdot\xi - g\cdot\zeta) &= f\cdot(h\cdot g + n) - g\cdot(h\cdot f - m) - \rho_L(\xi)(f) - \rho_L(\zeta)(g)\\ &= f\cdot h\cdot g + f\cdot n - g\cdot h\cdot f + g\cdot m - \rho_L(\xi)(f) - \rho_L(\zeta)(g)\\ &= f\cdot n + g\cdot m - \rho_L(\xi)(f) - \rho_L(\zeta)(g) = 0\end{aligned}$$

We now check that the map $\delta : L \to A$ is a right (A, L)-connection character on A. Let $\nu = a\cdot\xi + b\cdot\zeta$ for all $a, b \in A$. Then

$$\begin{aligned}\delta(c\cdot\nu) = \delta(c\cdot a\cdot\xi + c\cdot b\cdot\zeta) &= c\cdot a\cdot(h\cdot g + n) - c\cdot b\cdot(h\cdot f - m) - \rho_L(\xi)(c\cdot a) - \rho_L(\zeta)(c\cdot b)\\ &= c\cdot(a\cdot h\cdot g + a\cdot n - \rho_L(\xi)(a) - b\cdot h\cdot f + b\cdot m - \rho_L(\zeta)(b))\\ &\quad - a\cdot\rho_L(\xi)(c) - b\cdot\rho_L(\zeta)(c) = c\cdot\delta(\nu) - \rho_L(\nu)(c).\end{aligned}$$

Since $\delta : L \to A$ satisfies Equation (0.6.19) we deduce that it is a right (A, L)-connection character on A. We now want to check that the right (A, L)-connection it induces on A is flat. Following Lemma 0.6.28 and using Equations (0.6.27) and (1.3.5), we obtain

$$\begin{aligned}\mathcal{C}_{\mathrm{r}}^{\nabla}(\xi,\zeta) &= -\rho_L(\xi)(\delta(\zeta)) + \rho_L(\zeta)(\delta(\xi)) + \delta([\xi,\zeta]_L)\\ &= -\rho_L(\xi)(h\cdot f - m) + \rho_L(\zeta)(h\cdot g + n) + \delta(m\cdot\xi + n\cdot\zeta)\\ &= -\rho_L(\xi)(h\cdot f - m) + \rho_L(\zeta)(h\cdot g + n)\\ &\quad + m\cdot(h\cdot g + n) - \rho_L(\xi)(m) + n\cdot(h\cdot f - m) - \rho_L(\zeta)(n)\\ &= -\rho_L(\xi)(h\cdot f) + \rho_L(\zeta)(h\cdot g) + h\cdot(m\cdot g + n\cdot f)\\ &= -h\cdot\rho_L(\xi)(f) - f\cdot\rho_L(\xi)(h) + h\cdot\rho_L(\zeta)(g) + g\cdot\rho_L(\zeta)(h) + h\cdot(m\cdot g + n\cdot f)\\ &= h\cdot(-\rho_L(\xi)(f) + \rho_L(\zeta)(g)) + h\cdot(m\cdot g + n\cdot f) = 0\end{aligned}$$

Note that the value of $\delta : L \to A$ on the generators of L is

$$\delta(\xi) = h \cdot g + n, \quad \delta(\zeta) = h \cdot f - m, \quad \forall h \in A. \qquad \square$$

1.4 Lie–Rinehart algebras with no antipode

As indicated in the introduction and developed in Section 0.7, the universal enveloping algebra $V(A,L)$ of a Lie–Rinehart algebra (A,L) has the structure of a left Hopf algebroid. Its counit endows A with the structure of a left $V(A,L)$-module, in such a way that the induced action of $a \in A$ is given by multiplication, and the induced action of $\xi \in L$ is given by the anchor map.

In this section we give explicit examples of Lie–Rinehart algebras whose enveloping algebra is not a Hopf algebroid, either in the sense of [BS04] or in the sense of [Lu96]. We provide several examples of Lie–Rinehart algebras (A,L) such that A does not admit a right module structure over the enveloping algebra $V(A,L)$ extending multiplication in A, i.e., there exists no antipode on $V(A,L)$. Since for the Lie–Rinehart algebras presented in this section, $V(A,L)$ admits no antipode, they are neither full Hopf algebroids (in the sense of [BS04] nor Hopf algebroids in the sense of Lu [Lu96]. This establishes that full Hopf algebroids are a more restrictive structure than left Hopf algebroids. The results we present in this section go beyond the work in [KR15].

Theorem 1.4.1. *Let (A,L) be a Lie–Rinehart algebra with anchor ρ_L, and let $a \cdot \xi = 0$ for some non-zero element $a \in A$ and $\xi \in L$. If there exists no $b \in A$ satisfying $a \cdot b = \rho_L(\xi)(a)$, then (A,L) admits no right (A,L)-connection on A, i.e., there exists no right $V(A,L)$-module structure on A that extends multiplication in A. Hence $V(A,L)$ is neither a full Hopf algebroid (in the sense of [BS04]), nor a Hopf algebroid in the sense of [Lu96].*

Proof. Assume there exists a right (A,L)-connection $\delta_{\mathrm{r}}^{\nabla} : L \to A$ on A. Then we have

$$0 = \delta_{\mathrm{r}}^{\nabla}(a \cdot \xi) = a \cdot \delta_{\mathrm{r}}^{\nabla}(\xi) - \rho_L(\xi)(a)$$

which is a contradiction if there exists no $b \in A$ satisfying $a \cdot b = \rho_L(\xi)(a)$. $\square$

We now present three examples of Lie–Rinehart algebras without an antipode. Example 1.4.4 is contained in [KR15].

Example 1.4.2. *Let k be a field and let $A \coloneqq k[x,y]/\langle x \cdot y, y^2 \rangle$ so that*

$$\mathrm{Der}_k(A) = \mathrm{Span}_A \left\{ \xi \coloneqq x\frac{\partial}{\partial x}, \zeta \coloneqq y\frac{\partial}{\partial x}, \gamma \coloneqq y\frac{\partial}{\partial y} \right\}.$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ span $\mathrm{Der}_k(k[x,y])$ as an $k[x,y]$-module. Assume that the Lie–Rinehart algebra $(A, \mathrm{Der}_k(A))$ admits a right $(A, \mathrm{Der}_k(A))$-connection on A, as a module over itself, given by $\delta_{\mathrm{r}}^{\nabla} : \mathrm{Der}_k(A) \to A$, then we have

$$0 = \delta_{\mathrm{r}}^{\nabla}\left(x \cdot y\frac{\partial}{\partial x}\right) = x \cdot \delta_{\mathrm{r}}^{\nabla}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}(x) = x \cdot \delta_{\mathrm{r}}^{\nabla}\left(y\frac{\partial}{\partial x}\right) - y$$

which is a contradiction since there exists no $z \in A$ satisfying $x \cdot z = y$. Hence the pair $(A, \mathrm{Der}_k(A))$ admits no right $(A, \mathrm{Der}_k(A))$-connection on A, so we deduce that $V(A, \mathrm{Der}_k(A))$ is not a full Hopf algebroid.

Example 1.4.3. Let k be a field and let $A \coloneqq k[x,y]/\langle x \cdot y, x^2, y^2 \rangle$ so that

$$\mathrm{Der}_k(A) = \mathrm{Span}_A \left\{ x\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y} \right\}.$$

As in Example 1.4.2, we assume that the Lie–Rinehart algebra $(A, \mathrm{Der}_k(A))$ admits a right $(A, \mathrm{Der}_k(A))$-connection given by $\delta_{\mathrm{r}}^{\nabla} : \mathrm{Der}_k(A) \to A$ on A, then we have

$$0 = \delta_{\mathrm{r}}^{\nabla}\left(x \cdot y\frac{\partial}{\partial x}\right) = x \cdot \delta_{\mathrm{r}}^{\nabla}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}(x) = x \cdot \delta_{\mathrm{r}}^{\nabla}\left(y\frac{\partial}{\partial x}\right) - y$$

which is a contradiction since there exists no $z \in A$ satisfying $x \cdot z = y$. Hence the pair $(A, \mathrm{Der}_k(A))$ admits no right $(A, \mathrm{Der}_k(A))$-connection on A, hence there is no right $V(A, \mathrm{Der}_k(A))$-module structure on $(A, \cdot)$ extending multiplication in A.

The last example of a Lie–Rinehart algebra without an antipode we present in this section is contained in [KR15]. The proof is based on the results in Lemma 1.2.2. For this Lie–Rinehart algebra, we will give an explicit description of the left Hopf algebroid structure on its enveloping algebra.

Example 1.4.4 (See [KR15])**.** Let k be a field, $A \coloneqq k[x,y]/\langle x \cdot y, x^2, y^2 \rangle$, L be the 1-dimensional Lie algebra with basis $\{\xi\}$ and $E \in \mathrm{Der}_k(A)$ be the derivation with

$$E(x) = y, \quad E(y) = 0.$$

1. There is a Lie–Rinehart algebra structure on (A, L) with A-module structure on L given by $x \cdot \xi = y \cdot \xi = 0$ and anchor map given by $\rho(\xi) = E$.

2. There is no right $V(A, L)$-module structure on A that extends multiplication in A.

We first check that the Lie–Rinehart algebra (A, L) is an example of the construction in Lemma 1.2.2. Note that as a vector space, A is three-dimensional with basis $\{1, x, y\}$. We define a character $\chi : A \to k$ by

$$\chi(1) \longmapsto 1, \quad \chi(x) \longmapsto 0, \quad \chi(y) \longmapsto 0$$

so that the A-module structure $\mu : A \otimes L \to L$ is given by

$$1 \otimes \xi \longmapsto \xi, \quad x \otimes \xi \longmapsto 0, \quad y \otimes \xi \longmapsto 0.$$

The A-linearity of the anchor is equivalent to $a \cdot E = \chi(a)E$ for $a \in A$ which holds since

$$x \cdot E = 0, \quad y \cdot E = 0.$$

A straightforward computation shows that the Leibniz rule in Equation (0.5.1) holds. As in Example 1.4.2, we now assume that (A, L) admits a right (A, L)-connection on A given by $\delta_{\mathrm{r}}^{\nabla} : L \to A$, then we have

$$\begin{aligned} 0 &= \delta_{\mathrm{r}}^{\nabla}(x \cdot \xi) \\ &= x \cdot \delta_{\mathrm{r}}^{\nabla}(\xi) - \rho_L(\xi)(x) \\ &= x \cdot \delta_{\mathrm{r}}^{\nabla}(\xi) - E(x) \\ &= x \cdot \delta_{\mathrm{r}}^{\nabla}(\xi) - y \end{aligned}$$

which is a contradiction since there exists no $z \in A$ satisfying $x \cdot z = y$. Hence the pair (A, L) admits no right (A, L)-connection on A, hence there is no right $V(A, L)$-module structure on A extending multiplication in A.

For the Lie–Rinehart algebra presented in Example 1.4.4, we carry out Rinehart's construction explicitly which yields a presentation of the associative k-algebra $V(A, L)$ in terms of generators $x, y, \bar{\xi}$ satisfying the relations

$$\bar{\xi}x = y, \quad \bar{\xi}y = x\bar{\xi} = y\bar{\xi} = x^2 = y^2 = xy = yx = 0.$$

Hence $V(A,L)$ has a k-linear basis given by $\{\bar{\xi}^n, x, y\}_{n\in\mathbb{N}}$. The source and target maps are both the inclusion of A into $V(A,L)$. Hence one can also see directly that $V(A,L)$ admits no antipode: S would satisfy $S(x) = x$, $S(y) = y$ and one would have

$$y = S(y) = S(\bar{\xi}x) = S(x)S(\bar{\xi}) = xS(\bar{\xi}),$$

but there is no element $z \in V(A,L)$ such that $y = xz$.

Chapter 2

Lie–Rinehart algebras over Jacobi algebras

Jacobi algebras are a generalisation of Poisson algebras first introduced in a differential geometric context in [Kir76, Lic78]. Both Poisson and Jacobi algebras are commutative algebras endowed with a Lie bracket. However, while the Poisson bracket is a *derivation* of the underlying commutative algebra, the Jacobi bracket is a *first order differential operator* on the commutative algebra, see Proposition 0.1.11. In this sense, Poisson algebras are a particular case of Jacobi algebras.

In this chapter, we study examples of Lie–Rinehart algebras arising from Jacobi algebras, and consider the Hopf algebroid structure on their universal enveloping algebras. The main results presented here have been published in [Rov14a], in which we give examples of Hopf algebroids without an antipode, i.e., *left* Hopf algebroids which are neither *full* nor satisfy the axioms of [Lu96].

More precisely, the results in this Chapter are the following:

1. In Section 2.2, we describe Lie–Rinehart algebras over Jacobi algebras in particular a canonical construction given by the pair $(A, A \otimes_R A)$ and two different quotients given by $(A, \mathcal{J}^1(A))$ and $(A, Ah \otimes_R A)$, this last one following the general method given in Proposition 1.2.4. Our construction of $(A, \mathcal{J}^1(A))$ generalises the construction of Kerbrat and Souici-Benhammadi [KSB93] (see also [Vai00]) and Okassa [Oka07] for Lie algebroids.

2. In Section 2.3, we prove the following:

- The universal enveloping algebra $V(A, A \otimes_R A)$ of the canonical Lie–Rinehart algebra $(A, A \otimes_R A)$ over a Jacobi algebra A admits an antipode,
- The universal enveloping algebra of the Lie–Rinehart algebra $(A, \mathcal{J}^1(A))$ over a Jacobi algebra admits an antipode.

To our knowledge, these facts had not been stated in the literature until [Rov14a].

3. Lastly, Section 2.4 is dedicated to examples of Lie–Rinehart algebras over Jacobi algebras that do not admit an antipode on its universal enveloping algebra. A new feature of these examples is that some of them will admit right (A, L)-connections on A although not flat, i.e., A is not a right $V(A, L)$-module. These examples constitute the main result in [Rov14a].

2.1 Jacobi algebras

Jacobi algebras were first introduced by Kirillov [Kir76] under the name *local Lie algebras* and independently by Lichnerowicz [Lic78] as the algebraic structure on the ring of C^∞-functions on a certain kind of smooth manifolds, called Jacobi manifolds, see Section 2.2.2 below. (See [Mar91, Section 2.2] for some remarks comparing both definitions). Here we give a purely algebraic definition, see [GM03] for a graded version and [AM14] for results on Frobenius Jacobi algebras, representations of Jacobi algebras, and classification.

Definition 2.1.1. *A **Jacobi algebra** is a commutative R-algebra A endowed with a Lie bracket $\{\bullet, \bullet\}_J$, called the Jacobi bracket, satisfying the Leibniz rule*

$$\{a \cdot b, c\}_J = a \cdot \{b, c\}_J + b \cdot \{a, c\}_J - a \cdot b \cdot \{1, c\}_J \quad \forall a, b, c \in A. \tag{2.1.1}$$

Remark 2.1.2. *The Leibniz rule given in Equation (2.1.1) can be written as*

$$\{a, b \cdot c\}_J = b \cdot \{a, c\}_J + c \cdot \{a, b\}_J + b \cdot c \cdot \{1, a\}_J, \quad \forall a, b, c, \in A. \tag{2.1.2}$$

Remark 2.1.3. *From (2.1.1) and Proposition 0.1.11, we deduce that $\{a, \bullet\}_J$ is a first order differential operator on A for all $a \in A$, so that $\{a, \bullet\}_J + \{1, a\} \cdot \bullet \in \mathrm{Der}_R(A)$. Hence we deduce that the operator $\{a, \bullet\}_J$ is a derivation on A if and only if $\{1, a\}_J = 0$, so that Poisson algebras are Jacobi algebras where $\{1, a\}_J = 0$ for all $a \in A$.*

See also [Kir76, Proof of Lemma 4].

Theorem 2.1.4. *Let A be a commutative R-algebra and $E \in \mathrm{Der}_R(A)$ a derivation of A. The pair (A, A) is a Lie–Rinehart algebra with anchor $\rho_A : A \to \mathrm{Der}_R(A)$ given by $a \mapsto a \cdot E$ and bracket $[\bullet, \bullet]_A$ on A if and only if A is a Jacobi algebra with bracket*

$$[a, b]_A := a \cdot E(b) - E(a) \cdot b.$$

Proof. Assume that (A, A) is a Lie–Rinehart algebra. Then, by Proposition 1.2.1 we deduce that A admits a derivation $E \in \mathrm{Der}_R(A)$. Hence, by Proposition 0.1.11 we deduce that A admits a first order differential operator so that, by Equation (2.1.2) and Remark 2.1.3 we deduce that A admits a Jacobi algebra structure. Conversely, let us assume that A is a Jacobi algebra. Hence A admits a first order differential operator, in particular, it admits a derivation so by Proposition 1.2.1, (A, A) is a Lie–Rinehart algebra. □

Proposition 2.1.5. *The Jacobi bracket $\{\bullet, \bullet\}_J$ induces a Lie algebra map defined by*

$$\Phi : A \longrightarrow \mathrm{Der}_R(A), \quad a \longmapsto \Phi_a := \{a, \bullet\}_J + \bullet \cdot \{1, a\}_J, \quad a \in A, \tag{2.1.3}$$

that is, we have $\Phi_{\{a,b\}_J} = [\Phi_a, \Phi_b]_{\mathrm{Der}_R(A)}$.

Proof. Since $\{a, \bullet\}_J$ is a first order differential operator on A, it follows from Proposition 0.1.11 that Φ_a is a derivation on A for all $a \in A$. Furthermore, since $\{\bullet, \bullet\}_J$ satisfies the Jacobi identity, and letting $\Phi_{\{a,b\}_J} - [\Phi_a, \Phi_b]_{\mathrm{Der}_R(A)}$ act on an arbitrary element $c \in A$, we have the following:

$$\begin{aligned}
\left(\Phi_{\{a,b\}_J} - [\Phi_a, \Phi_b]_{\mathrm{Der}_R(A)}\right)(c) &= \Phi_{\{a,b\}_J}(c) - \Phi_a\left(\Phi_b(c)\right) + \Phi_b\left(\Phi_a(c)\right) \\
&= \{\{a, b\}_J, c\}_J + c \cdot \{1, \{a, b\}_J\}_J \\
&\quad - \Phi_a\left(\{b, c\}_J + c \cdot \{1, b\}_J\right) + \Phi_b\left(\{a, c\}_J + c \cdot \{1, a\}_J\right) \\
&= \{\{a, b\}_J, c\}_J + c \cdot \{1, \{a, b\}_J\}_J \\
&\quad - \{a, \{b, c\}_J\}_J - \{b, c\}_J \cdot \{1, a\}_J - \{1, b\}_J \cdot \Phi_a(c) - c \cdot \Phi_a\left(\{1, b\}_J\right) \\
&\quad + \{b, \{a, c\}_J\}_J + \{a, c\}_J \cdot \{1, b\}_J + \{1, a\}_J \cdot \Phi_b(c) + c \cdot \Phi_b\left(\{1, a\}_J\right) \\
&= c \cdot \{1, \{a, b\}_J\}_J - \{b, c\}_J \cdot \{1, a\}_J - \{1, b\}_J \cdot \{a, c\}_J \\
&\quad - \{1, b\}_J \cdot c \cdot \{1, a\}_J - c \cdot \{a, \{1, b\}_J\}_J - c \cdot \{1, b\}_J \cdot \{1, a\}_J \\
&\quad + \{a, c\}_J \cdot \{1, b\}_J + \{1, a\}_J \cdot \{b, c\}_J + \{1, a\}_J \cdot c \cdot \{1, b\}_J \\
&\quad + c \cdot \{b, \{1, a\}_J\}_J + c \cdot \{1, a\}_J \cdot \{1, b\}_J \\
&= c \cdot \left(\{1, \{a, b\}_J\}_J - \{a, \{1, b\}_J\}_J - \{\{1, a\}_J, b\}_J\right).
\end{aligned}$$

Since the derivation $\Phi_{\{a,b\}_J} - [\Phi_a, \Phi_b]_{\mathrm{Der}_R(A)}$ vanishes on all $c \in A$, we deduce that Φ is a Lie algebra homomorphism. □

Remark 2.1.6. *From Equation (2.1.3) we deduce that* $\Phi_1 = \{1, \bullet\}_J$.

Example 2.1.7. *Let* $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ *be vector fields on a manifold* X. *The bilinear map*

$$\{\bullet, \bullet\}_J : C^\infty(X) \otimes C^\infty(X) \longrightarrow C^\infty(X)$$

given by

$$f \otimes_R g \longmapsto \{f, g\}_J := \frac{\partial}{\partial x}(f) \cdot \frac{\partial}{\partial y}(g) - \frac{\partial}{\partial y}(f) \cdot \frac{\partial}{\partial x}(g) + f \cdot \frac{\partial}{\partial z}(g) - \frac{\partial}{\partial z}(f) \cdot g$$

is a Jacobi structure on X. *Note* $\{1, \bullet\}_J = \frac{\partial}{\partial z}$. *We now show explicitly that the Leibniz rule in (2.1.1) holds:*

$$\begin{aligned}
\{f, g \cdot h\}_J &= \frac{\partial}{\partial x}(f) \cdot \frac{\partial}{\partial y}(g \cdot h) - \frac{\partial}{\partial y}(f) \cdot \frac{\partial}{\partial x}(g \cdot h) + f \cdot \frac{\partial}{\partial z}(g \cdot h) - \frac{\partial}{\partial z}(f) \cdot g \cdot h \\
&= \frac{\partial}{\partial x}(f) \cdot \left(\frac{\partial}{\partial y}(g) \cdot h + g \cdot \frac{\partial}{\partial y}(h)\right) - \frac{\partial}{\partial y}(f) \cdot \left(\frac{\partial}{\partial x}(g) \cdot h + g \cdot \frac{\partial}{\partial x}(h)\right) \\
&\quad + f \cdot \left(\frac{\partial}{\partial z}(g) \cdot h + g \cdot \frac{\partial}{\partial z}(h)\right) - \frac{\partial}{\partial z}(f) \cdot g \cdot h \\
&= g \cdot \left(\frac{\partial}{\partial x}(f) \cdot \frac{\partial}{\partial y}(h) - \frac{\partial}{\partial y}(f) \cdot \frac{\partial}{\partial x}(h) + f \cdot \frac{\partial}{\partial z}(h) - \frac{\partial}{\partial z}(f) \cdot h\right) \\
&\quad + h \cdot \left(\frac{\partial}{\partial x}(f) \cdot \frac{\partial}{\partial y}(g) - \frac{\partial}{\partial y}(f) \cdot \frac{\partial}{\partial x}(g) + f \cdot \frac{\partial}{\partial z}(g) - \frac{\partial}{\partial z}(f) \cdot g\right) - g \cdot h \cdot \frac{\partial}{\partial z}(f) \\
&= g \cdot \{f, h\}_J + h \cdot \{f, g\}_J - g \cdot h \cdot \{1, f\}_J.
\end{aligned}$$

Similarly, a straightforward computation using the Leibniz rule for partial derivatives shows the Jacobi identity is satisfied.

Proposition 2.1.8. *Let* A *be a Jacobi algebra with bracket* $\{\bullet, \bullet\}_J$, *then the elements* $\Phi_a \in \mathrm{Der}_R(A)$ *for all* $a \in A$ *satisfy the relation*

$$a \cdot \Phi_b + b \cdot \Phi_a + a \cdot b \cdot \Phi_1 - \Phi_{(a \cdot b)} = 0. \tag{2.1.4}$$

Proof. By the Leibniz identity given in Equation (2.1.1),

$$\begin{aligned}
0 &= a\cdot\{b,\bullet\}_J + b\cdot\{a,\bullet\}_J - a\cdot b\cdot\{1,\bullet\} - \{a\cdot b,\bullet\}_J \\
&= a\cdot\{b,\bullet\}_J + b\cdot\{a,\bullet\}_J - a\cdot b\cdot\{1,\bullet\}_J - \{a\cdot b,\bullet\}_J + b\cdot\bullet\cdot\{1,a\}_J + a\cdot\bullet\cdot\{1,b\}_J - \bullet\cdot\{1,a\cdot b\}_J \\
&= a\cdot(\{b,\bullet\}_J + \bullet\cdot\{1,b\}_J) + b\cdot(\{a,\bullet\}_J + \bullet\cdot\{1,a\}_J) - a\cdot b\cdot\{1,\bullet\}_J - \{a\cdot b,\bullet\}_J - \bullet\cdot\{1,a\cdot b\}_J \\
&= a\cdot\Phi_b + b\cdot\Phi_a + a\cdot b\cdot\Phi_1 - \Phi_{(a\cdot b)}.
\end{aligned}$$

□

In the following sections, we construct Lie–Rinehart algebra over both Poisson and Jacobi algebras. We first construct the canonical Lie–Rinehart algebra over a Poisson algebra. Secondly, we construct a canonical Lie–Rinehart algebra, given by $(A, A\otimes A)$, over a Jacobi algebra A, and consider two quotients: $(A, \mathcal{J}^1(A))$ and $(A, Ah\otimes A)$.

2.2 Lie–Rinehart algebras over Jacobi algebras

In this section, we firstly construct the canonical Lie–Rinehart algebra $(A, A\otimes_R A)$ over a Jacobi algebra A with bracket $\{\bullet,\bullet\}_J$. We then consider two different quotient Lie–Rinehart algebras of $(A, A\otimes_R A)$:

- In our first quotient construction, one associates a Jacobi algebra with its 1-jet space $\mathcal{J}^1(A)$ which we recall from Section 0.1.2 is isomorphic to the quotient $(A\otimes_R A)/I^2$.
- The second quotient is based on Proposition 1.2.2 and will be used used in Section 2.4 as a source of examples of Hopf algebroids without an antipode.

2.2.1 The canonical Lie–Rinehart algebra $(A, A\otimes_R A)$

Our first aim is to endow the A-module $A\otimes_R A$, where A is a Jacobi algebra with bracket $\{\bullet,\bullet\}_J$, with a Lie bracket, denoted $[\bullet,\bullet]_{A\otimes_R A}$, and an A-linear Lie algebra map from $A\otimes_R A$ to $\mathrm{Der}_R(A)$ compatible with $[\bullet,\bullet]_{A\otimes_R A}$ so that the pair $(A, A\otimes_R A)$ is a canonical Lie–Rinehart algebra over $(A, \{\bullet,\bullet\}_J)$.

Lemma 2.2.1. *Let A be a Jacobi algebra over R with bracket $\{\bullet,\bullet\}_J$. The pair $(A, A\otimes_R A)$ is a Lie–Rinehart algebra with anchor*

$$\rho_{A\otimes_R A} : A\otimes_R A \longrightarrow \mathrm{Der}_R(A), \quad a\otimes_R b \longmapsto a\cdot\Phi_b \tag{2.2.1}$$

and Lie bracket on $A \otimes_R A$ *given by*

$$[a \otimes_R f, b \otimes_R g]_{A\otimes_R A} = a \cdot b \otimes_R \{f, g\}_J + a \cdot \Phi_f(b) \otimes_R g - b \cdot \Phi_g(a) \otimes_R f. \tag{2.2.2}$$

Proof. The bracket $[\bullet, \bullet]_{A\otimes_R A}$ in Equation (2.2.2) is skew-symmetric and by a brief computation we see that it satisfies the Jacobi identity. We now check that the A-linear map in Equation (2.2.1) is a Lie algebra map:

$$\begin{aligned}\rho_{A\otimes_R A}([a \otimes_R f, b \otimes_R g]_{A\otimes_R A}) &= \rho_{A\otimes_R A}\big(a \cdot b \otimes_R \{f, g\}_J + a \cdot \Phi_f(b) \otimes_R g - b \cdot \Phi_g(a) \otimes_R f\big)\\ &= a \cdot b \cdot \Phi_{\{f,g\}_J} + a \cdot \Phi_f(b) \cdot \Phi_g - b \cdot \Phi_g(a) \cdot \Phi_f\\ &= a \cdot b \cdot [\Phi_f, \Phi_g]_{\mathrm{Der}_R(A)} + a \cdot \Phi_f(b) \cdot \Phi_g - b \cdot \Phi_g(a) \cdot \Phi_f\\ &= [a \cdot \Phi_f, b \cdot \Phi_g]_{\mathrm{Der}_R(A)}\\ &= [\rho_{A\otimes_R A}(a \otimes_R f), \rho_{A\otimes_R A}(b \otimes_R g)]_{\mathrm{Der}_R(A)}.\end{aligned}$$

Lastly, we show that the bracket in Equation (2.2.2) is compatible with the A-module structure on $A \otimes_R A$ since the Leibniz rule in Equation (0.5.1) is satisfied:

$$\begin{aligned}[a \otimes_R f, b \cdot c \otimes_R g]_{A\otimes_R A} &= a \cdot b \cdot c \otimes_R \{f, g\}_J + a \cdot \Phi_f(b \cdot c) \otimes_R g - b \cdot c \cdot \Phi_g(a) \otimes_R f\\ &= a \cdot b \cdot c \otimes_R \{f, g\}_J + a \cdot b \cdot \Phi_f(c) \otimes_R g + a \cdot c \cdot \Phi_f(b) \otimes_R g - b \cdot c \cdot \Phi_g(a) \otimes_R f\\ &= b \cdot (a \cdot c \otimes_R \{f, g\}_J + a \cdot \Phi_f(c) \otimes_R g - c \cdot \Phi_g(a) \otimes_R f) + a \cdot c \cdot \Phi_f(b) \otimes_R g\\ &= b \cdot [a \otimes_R f, c \otimes_R g]_{A\otimes_R A} + \rho_{A\otimes_R A}(a \otimes_R f)(b) \cdot (c \otimes_R g).\end{aligned}$$

□

2.2.2 The canonical 1-jet algebroid on a Jacobi algebra

In this Section we focus on the algebraic characterisation (as a Lie–Rinehart algebra) of a Lie algebroid associated to Jacobi manifolds proposed in [KSB93, Vai00]. Our description is equivalent to the one given by Okassa [Oka07], although we use a different approach: while [Oka07] uses so-called Jacobi 1- and 2-forms on the trivial extension of a Jacobi algebra A by its module of Kähler differentials $\Omega^1(A)$, we consider the A-module $A \oplus \Omega^1(A)$ in its characterisation as the quotient $\mathcal{J}^1(A) = (A \otimes_R A)/I^2$, as explained in Section 0.1.2.

We prove that $(A, \mathcal{J}^1(A))$ is a Lie–Rinehart algebra: the bracket $[\bullet, \bullet]_{A\otimes_R A}$ on $A \otimes_R A$ given in (2.2.2) and the anchor map $\rho_{A\otimes_R A} : A \otimes_R A \to \mathrm{Der}_R(A)$ in (2.2.1) descend respectively to a Lie bracket on $\mathcal{J}^1(A)$ and to an A-linear Lie algebra map from $\mathcal{J}^1(A)$ to $\mathrm{Der}_R(A)$ compatible with each other since they satisfy the Leibniz rule in (0.5.1).

Theorem 2.2.2. *Let A be a Jacobi algebra over R with bracket $\{\bullet,\bullet\}_J$, $\mathcal{J}^1(A)$ be the 1-jet space of A and $j^1 : A \to \mathcal{J}^1(A)$ be the 1-jet map $a \mapsto 1 \otimes_R a \pmod{I^2}$, for all $a \in A$. The pair $(A, \mathcal{J}^1(A))$ is a Lie–Rinehart algebra with anchor*

$$\rho_{\mathcal{J}^1} : \mathcal{J}^1(A) \longrightarrow \mathrm{Der}_R(A), \quad j^1(a) \longmapsto \Phi_a := \{a,\bullet\}_J + \bullet \cdot \{1,a\}_J \tag{2.2.3}$$

and Lie bracket on $\mathcal{J}^1(A)$ given by

$$[j^1(f), j^1(g)]_{\mathcal{J}^1(A)} = j^1(\{f,g\}_J). \tag{2.2.4}$$

Proof. The Lie bracket $[\bullet,\bullet]_{A\otimes_R A}$ on $A \otimes_R A$ given in (2.2.2) descends to the bracket $[\bullet,\bullet]_{\mathcal{J}^1(A)}$ given in Equation (2.2.4) if it maps the A-module $A \otimes_R A \otimes_R I^2 + I^2 \otimes_R A \otimes_R A$ to I^2. Since $[\bullet,\bullet]_{A\otimes_R A}$ is skew-symmetric, it is enough to check that it maps $A \otimes_R A \otimes_R I^2$ to I^2. Moreover, since the expressions we will consider below are additive, we can assume without loss of generality that all tensors are elementary. Let $a_1 \otimes_R b_1 \in A \otimes_R A$ and $a_2 \otimes_R b_2, f_2 \otimes_R g_2 \in I$ so that $a_2 \cdot b_2 = f_2 \cdot g_2 = 0$. Then we have

$$(a_1 \otimes_R b_1) \otimes_R ((a_2 \otimes_R b_2) \cdot (f_2 \otimes_R g_2)) \in A \otimes_R A \otimes_R I^2$$

which under $[\bullet,\bullet]_{A\otimes_R A}$ becomes

$$\begin{aligned}
&[a_1 \otimes_R b_1, (a_2 \otimes_R b_2) \cdot (f_2 \otimes_R g_2)]_{A\otimes_R A} = [a_1 \otimes_R b_1, a_2 \cdot f_2 \otimes_R b_2 \cdot g_2]_{A\otimes_R A} \\
&= a_1 \cdot a_2 \cdot f_2 \otimes_R \{b_1, b_2 \cdot g_2\}_J + a_1 \cdot \{b_1, a_2 \cdot f_2\}_J \otimes_R (b_2 \cdot g_2) - a_2 \cdot f_2 \cdot \{b_2 \cdot g_2, a_1\} \otimes_R b_1 \\
&\quad + a_1 \cdot a_2 \cdot f_2 \cdot (\{1, b_1\}_J \otimes_R b_2 \cdot g_2 - \{1, b_2 \cdot g_2\}_J \otimes_R b_1) \\
&= a_1 \cdot a_2 \cdot f_2 \otimes_R (b_2 \cdot \{b_1, g_2\}_J + g_2 \cdot \{b_1, b_2\}_J + b_2 \cdot g_2 \cdot \{1, b_1\}_J) \\
&\quad + a_1 \cdot (a_2 \cdot \{b_1, f_2\}_J + f_2 \cdot \{b_1, a_2\}_J + a_2 \cdot f_2 \cdot \{1, b_1\}_J) \otimes_R (b_2 \cdot g_2) \\
&\quad + a_1 \cdot a_2 \cdot f_2 \cdot \{1, b_1\}_J \otimes_R b_2 \cdot g_2 \\
&= (a_1 \cdot a_2 \otimes_R b_2) \cdot (f_2 \otimes_R \{b_1, g_2\}_J) + (a_1 \cdot f_2 \otimes_R g_2) \cdot (a_2 \otimes_R \{b_1, b_2\}_J) \\
&\quad + (a_1 \cdot a_2 \otimes_R b_2)(\{b_1, f_2\}_J \otimes g_2) + (a_1 \cdot f_2 \otimes_R g_2) \cdot (\{b_1, a_2\}_J \otimes_R b_2) \\
&\quad + (a_1 \cdot a_2 \otimes_R b_2) \cdot (f_2 \cdot \{1, b_1\}_J \otimes_R g_2) + (a_1 \cdot a_2 \otimes_R b_2) \cdot (f_2 \cdot \{1, b_1\}_J \otimes_R g_2) \\
&= a_1 \cdot (a_2 \otimes_R b_2) \cdot (f_2 \otimes_R \{b_1, g_2\}_J) + a_1 \cdot (f_2 \otimes_R g_2) \cdot (a_2 \otimes_R \{b_1, b_2\}_J) \\
&\quad + a_1 \cdot (a_2 \otimes_R b_2)(\{b_1, f_2\}_J \otimes_R g_2) + a_1 \cdot (f_2 \otimes_R g_2) \cdot (\{b_1, a_2\}_J \otimes_R b_2) \\
&\quad + a_1 \cdot (a_2 \otimes_R b_2) \cdot (f_2 \otimes_R g_2) \cdot (1 \otimes_R \{1, b_1\}_J + 2\{1, b_1\}_J \otimes_R 1) \in I^2
\end{aligned}$$

which we deduce is in I^2 since by the Leibniz rule we have

$$\begin{aligned} f_2 \otimes_R \{b_1, g_2\}_J + \{b_1, f_2\}_J \otimes_R g_2 + f_2 \cdot g_2 \otimes_R \{1, b_1\}_J \in I \\ a_2 \otimes_R \{b_1, b_2\}_J + \{b_1, a_2\}_J \otimes_R b_2 + a_2 \cdot b_2 \otimes_R \{1, b_1\}_J \in I. \end{aligned} \tag{2.2.5}$$

and we have

$$a_2 \otimes b_2 \in I, \quad f_2 \otimes g_2 \in I, \quad f_2 \otimes \{b_1, g_2\}_J + \{b_1, f_2\} \otimes g_2 \in I, \quad a_2 \otimes \{b_1, b_2\}_J + \{b_1, a_2\}_J \otimes b_2 \in I.$$

Hence we deduce that $A \otimes A \otimes I^2 + I^2 \otimes A \otimes A$ is mapped to I^2 under $[\bullet, \bullet]_{A\otimes A}$ and consequently that the map $[\bullet, \bullet]_{A\otimes A}$ given in Equation (2.2.2) descends to the map $[\bullet, \bullet]_{\mathcal{J}^1(A)}$ given in Equation (2.2.4). Moreover, by a similar argument, we show that the map $\rho_{A\otimes_R A}$ in Equation (2.2.1) descends to the map given in Equation (2.2.3). Let

$$a \otimes_R b \in I, \quad f \otimes_R g \in I$$

so that $a \cdot b = f \cdot g = 0$ and $a \cdot f \otimes_R b \cdot g \in I^2$, which under $\rho_{A\otimes_R A}$ becomes

$$\rho_{A\otimes_R A}(a \cdot f \otimes_R b \cdot g) = a \cdot f \cdot \Phi_{b\cdot g} = a \cdot f \cdot \{b \cdot g, \bullet\}_J + a \cdot f \cdot \{1, b \cdot g\}_J \cdot \bullet = 0.$$

Since $\rho_{A\otimes_R A}$ maps I^2 to 0, we deduce that it descends to the map $\rho_{\mathcal{J}^1}$. Following a similar argument as in the proof of Theorem 1.1.2, we now check that the A-linear map $\rho_{\mathcal{J}^1} : \mathcal{J}^1(A) \to \mathrm{Der}_R(A)$ is a Lie algebra homomorphism:

$$\begin{aligned} \rho_{\mathcal{J}^1}\left([a \cdot j^1(f), b \cdot j^1(g)]_{\mathcal{J}^1(A)}\right) &= \rho_{\mathcal{J}^1}\left(a \cdot b \cdot j^1(\{f, g\}_J) + (a \cdot \{f, b\}_J + a \cdot b \cdot \{1, f\}_J) \cdot j^1(g)\right) \\ &\quad + \rho_{\mathcal{J}^1}\left(-(b \cdot \{g, a\}_J + b \cdot a \cdot \{1, g\}_J) \cdot j^1(f)\right) \\ &= a \cdot b \cdot \Phi_{\{f,g\}_J} + (a \cdot \{f, b\}_J + a \cdot b \cdot \{1, f\}_J) \cdot \Phi_g \\ &\quad - (b \cdot \{g, a\}_J + b \cdot a \cdot \{1, b\}_J) \cdot \Phi_f \\ &= a \cdot b \cdot [\Phi_f, \Phi_g]_{\mathrm{Der}_R(A)} + a \cdot \Phi_f(b) \cdot \Phi_g - b \cdot \Phi_g(a) \cdot \Phi_f \\ &= [a \cdot \Phi_f, b \cdot \Phi_g]_{\mathrm{Der}_R(A)} \\ &= [\rho_{\mathcal{J}^1}(a \cdot j^1(f)), \rho_{\mathcal{J}^1}(b \cdot j^1(g))]_{\mathrm{Der}_R(A)}. \end{aligned}$$

Lastly, the following short computation similar to one performed in the proof of Lemma

2.2.1 shows that the Leibniz rule given in Equation (0.5.1) holds:

$$\begin{aligned}[j^1(f), h\cdot j^1(g)]_{\mathcal{J}^1(A)} &= h\cdot j^1(\{f,g\}_J) + (\{f,h\}_J + h\cdot\{1,f\}_J)\cdot j^1(g) \\ &\quad - (h\cdot\{g,1\}_J + \{1,g\}_J\cdot h)\cdot j^1(f) \\ &= h\cdot[j^1(f), j^1(g)]_{\mathcal{J}^1(A)} + (\{f,h\}_J + h\cdot\{1,f\}_J)\cdot j^1(g) \\ &= h\cdot[j^1(f), j^1(g)]_{\mathcal{J}^1(A)} + \rho_{\mathcal{J}^1(A)}\left(j^1(f)\right)(h)\cdot j^1(g).\end{aligned}$$

Thus, the Lie bracket $[\bullet,\bullet]_{\mathcal{J}^1(A)}$ given in Equation (2.2.4) and the anchor $\rho_{\mathcal{J}^1}$ in Equation (2.2.3) turn $(A, \mathcal{J}^1(A))$ into a Lie–Rinehart algebra. □

Relation of $(A, \mathcal{J}^1(A))$ to the Lie algebroid over a Jacobi manifold

In this brief section we explain how our algebraic description, given in Theorem 2.2.2, of the canonical Lie–Rinehart algebra $(A, \mathcal{J}^1(A))$ associated to a Jacobi algebra A with bracket $\{\bullet,\bullet\}_J$ is related to the geometric description of the Lie algebroid over a Jacobi manifold M as given in [KSB93].

Definition 2.2.3. *Let M be a smooth manifold equipped with a bivector field Λ and a vector field E. The ring $C^\infty(M)$ admits a Lie bracket, called a* ***Jacobi structure****, given by*

$$\{f,g\}_J = \Lambda(df,dg) + f\cdot E(g) - g\cdot E(f), \quad f,g\in C^\infty(M) \tag{2.2.6}$$

if and only if Λ and E satisfy

$$[\![\Lambda,\Lambda]\!] = 2E\wedge\Lambda, \quad [\![\Lambda,E]\!] = 0,$$

where $[\![\bullet,\bullet]\!]$ is the Schouten bracket. Then (M,Λ,E) is called a ***Jacobi manifold****.*

To construct the canonical Lie algebroid associated to a Jacobi manifold (M,Λ,E), Kerbrat and Souici-Benhammadi [KSB93] endow the bundle $J^1(M,\mathbb{R})$ of 1-jets of smooth functions on M, which is isomorphic as a $C^\infty(M)$-module to the direct sum $C^\infty(M)\oplus\Omega^1(M)$ where $\Omega^1(M)$ are the smooth differential 1-forms, with a Lie bracket given by

$[(f_1, a_1 \cdot db_1), (f_2, a_2 \cdot db_2)] = (f, a \cdot db)$ where

$$\begin{aligned} f &= -\Lambda(a_1 \cdot db_1, a_2 \cdot db_2) + \iota_{(\Lambda^{\#}(a_1 \cdot db_1) + f_1 \cdot E)}(df_2) - \iota_{(\Lambda^{\#}(a_2 \cdot db_2) + f_2 \cdot E)}(df_1) \\ a \cdot db &= \mathcal{L}_{(\Lambda^{\#}(a_1 \cdot db_1) + f_1 \cdot E)}(a_2 \cdot db_2) - \mathcal{L}_{(\Lambda^{\#}(a_2 \cdot db_2) + f_2 \cdot E)}(a_1 \cdot db_1) \\ &\quad - \langle a_1 \cdot db_1, E \rangle (a_2 \cdot db_2 - df_2) + \langle a_2 \cdot db_2, E \rangle (a_1 \cdot db_1 - df_1) \\ &\quad - d(\Lambda(a_1 \cdot db_1, a_2 \cdot db_2)) \end{aligned}$$

for $f, f_1, f_2 \in C^\infty(M)$ and $a \cdot db, a_1 \cdot db_1, a_2 \cdot db_2 \in \Omega^1(M)$, which we can write in terms of the Jacobi structure given in Equation (2.2.6) as

$$\begin{aligned} f &= -a_1 \cdot a_2 \cdot \{b_1, b_2\}_J + a_1 \cdot a_2 \cdot b_1 \cdot \{1, b_2\}_J - a_1 \cdot a_2 \cdot b_2 \cdot \{1, b_1\}_J \\ &\quad + a_1 \cdot \{b_1, f_2\}_J - a_1 \cdot b_1 \cdot \{1, f_2\}_J + a_1 \cdot f_2 \cdot \{1, b_1\}_J + f_1 \cdot \{1, f_2\}_J \\ &\quad - a_2 \cdot \{b_2, f_1\}_J + a_2 \cdot b_2 \cdot \{1, f_1\}_J - a_2 \cdot f_1 \cdot \{1, b_2\}_J - f_2 \cdot \{1, f_1\}_J \\ a \cdot db &= a_1 \cdot a_2 \cdot d\{b_1, b_2\}_J \\ &\quad + (a_2 \cdot \{a_1, b_2\}_J - f_2 \cdot \{1, a_1\}_J + a_2 \cdot b_2 \cdot \{1, a_1\}_J - a_1 \cdot a_2 \cdot \{1, b_2\}_J) \cdot db_1 \\ &\quad - (a_1 \cdot \{a_2, b_1\}_J - f_1 \cdot \{1, a_2\}_J + a_1 \cdot b_1 \cdot \{1, a_2\}_J - a_1 \cdot a_2 \cdot \{1, b_1\}_J) \cdot db_2 \\ &\quad - (a_1 \cdot f_2 - a_1 \cdot a_2 \cdot b_2) \cdot d\{1, b_1\}_J + (a_2 \cdot f_1 - a_1 \cdot a_2 \cdot b_1) \cdot d\{1, b_2\}_J. \end{aligned}$$

Now, take $A = C^\infty(M)$ in Equation (2.2.4). By the isomorphism given in Equation (0.1.7) we can identify

$$a \cdot j^1(b) + (f - a \cdot b) \cdot j^1(1) \in \mathcal{J}^1(C^\infty(M))$$

with

$$(f, a \cdot db) \in C^\infty(M) \oplus \Omega^1(C^\infty(M)).$$

By the universal property of Kähler differentials, a straightforward computation shows that the bracket (on the algebraic Kähler differentials) given in Equation (2.2.4) *descends* to the bracket (on differential forms) defined in [KSB93], so that both constructions are in fact compatible.

Furthermore, since elements $a \cdot j^1(b) \in \mathcal{J}^1(C^\infty(M))$ are identified, as above, with elements $(a \cdot b, a \cdot db) \in C^\infty(M) \oplus \Omega^1(M)$, we see that the anchor map we defined in Equation (2.2.3) yields the anchor defined in [KSB93], that is a map $\rho : J^1(M, \mathbb{R}) \to TM$ given by $(f, a \cdot db) \mapsto \Lambda^{\#}(a \cdot db) + f \cdot E$.

1-jet algebroid of a Poisson algebra

Since Poisson algebras can be seen as a special case of Jacobi algebras where $\{1, \bullet\}_J$ vanishes, the pair $\left(A, \mathcal{J}^1(A)\right)$ is also a Lie–Rinehart algebra when A is a Poisson algebra. We now clarify the relation between this Lie–Rinehart algebra over a Poisson algebra A and the canonical construction $(A, \Omega^1(A))$ given in Section 1.1.

Proposition 2.2.4. *Let A be a Poisson algebra over R with non-zero bracket $\{\bullet, \bullet\}$. The canonical Lie–Rinehart algebra $(A, \Omega^1(A))$ is not a subalgebra of $(A, \mathcal{J}^1(A))$.*

Proof. From Equations (0.1.1) and (0.1.8) we deduce $da = j^1(a) - a \cdot j^1(1)$. Hence

$$[da, db]_{\mathcal{J}^1(A)} = [j^1(a) - a \cdot j^1(1), j^1(b) - b \cdot j^1(1)]_{\mathcal{J}^1(A)} = j^1\{a, b\} = d\{a, b\} - \{a, b\} \cdot j^1(1)$$

so we see that the Lie bracket $[\bullet, \bullet]_{\mathcal{J}^1(A)}$ will not restrict to the A-module $\Omega^1(A)$ in general so that $\Omega^1(A)$ it is not a Lie subalgebra of $\mathcal{J}^1(A)$. □

2.2.3 Other quotient Lie–Rinehart algebras of $(A, A \otimes_R A)$

In the previous section we considered $(A, \mathcal{J}^1(A))$ as a quotient of the Lie–Rinehart algebra $(A, A \otimes_R A)$ associated to a Jacobi algebra $(A, \{\bullet, \bullet\}_J)$ over R. In this section we construct new quotient Lie–Rinehart algebras of $(A, A \otimes_R A)$.

Following the construction in Proposition 1.2.2, our aim in this section is to endow the A-module $Ah \otimes A$, where A is a Jacobi algebra over a field k with bracket $\{\bullet, \bullet\}_J$ and $h \in A$, with a Lie bracket that is compatible with the bracket on $\mathcal{J}^1(A)$ given in Equation (2.2.4).

Throughout this section, we denote $\otimes_k$ by an unadorned tensor product $\otimes$.

Lemma 2.2.5. *Let A be a Jacobi algebra over a field k with bracket $\{\bullet, \bullet\}_J$, and let $h \in A$ be such that $\{\bullet, \bullet\}_J$ satisfies $r \cdot \{\bullet, \bullet\}_J = 0$ for all $r \in \mathrm{Ann}_A(\{h\})$. Then $(A, Ah \otimes A)$ admits a Lie–Rinehart algebra structure with Lie bracket on $Ah \otimes A$ given by*

$$[h \cdot a \otimes f, h \cdot b \otimes g]_{Ah \otimes A} := h \cdot [a \otimes f, b \otimes g]_{A \otimes A} \tag{2.2.7}$$

and anchor

$$\rho_{Ah \otimes A}(h \cdot a \otimes f) = \rho_{A \otimes A}(a \otimes f) = a \cdot \Phi_f. \tag{2.2.8}$$

Proof. First recall that $(A, A\otimes A)$ is a Lie–Rinehart algebra with Lie bracket $[\bullet,\bullet]_{A\otimes A}$ given in Equation (2.2.2) and anchor $\rho_{A\otimes A} : A\otimes A \to \mathrm{Der}_k(A)$ as given in Equation (2.2.1). Note also that there exists a map of A-modules given by

$$\mu_h : A\otimes A \longrightarrow Ah\otimes A, \quad 1\otimes a \longmapsto h\otimes a.$$

We now prove that the conditions in Lemma 1.2.4 are satisfied.

Let $\{e_i\}$ be a basis of the Jacobi algebra $(A, \{\bullet,\bullet\}_J)$ as a k-vector space and let

$$K = \left\{\sum a_i\otimes e_i \in A\otimes A \mid \sum a_i\cdot h\otimes e_i = 0\right\}.$$

Now, let $\sum a_i\otimes e_i \in K$. Then $a_i\cdot h = 0$ so $a_i \in \mathrm{Ann}_A(\{h\})$ for all i.

1. Since $r\cdot\{\bullet,\bullet\}_J = 0$ for all $r\in \mathrm{Ann}_A(\{h\})$, we have $r\cdot\Phi_\bullet = 0$. Then, for elements $\sum a_i\otimes_R e_i \in K$ we have

$$\rho_{A\otimes A}\left(\sum a_i\otimes e_i\right) = \sum a_i\cdot\Phi_{e_i} = 0$$

since $a_i\in \mathrm{Ann}_A(\{h\})$.

2. Since $\{h\cdot r,\bullet\}_J = r\cdot\{\bullet,\bullet\}_J = 0$, the Leibniz rule given in Equation (2.1.1) yields $h\cdot\{r,\bullet\}_J = 0$ for all $r\in\mathrm{Ann}_A(\{h\})$. Furthermore, by computing the Lie bracket on $A\otimes A$ in (2.2.2) we obtain, for $\sum a_i\otimes e_i\in K$, $\sum b_j\otimes e_j\in A\otimes A$:

$$\begin{aligned}\left[\sum a_i\otimes e_i, \sum b_j\otimes e_j\right]_{A\otimes A} &= \sum\sum a_i\cdot b_j\otimes\{e_i,e_j\}_J + \sum\sum a_i\cdot\Phi_{e_i}(b_j)\otimes e_j\\ &\quad - \sum\sum b_j\cdot\Phi_{e_j}(a_i)\otimes e_i\end{aligned}$$

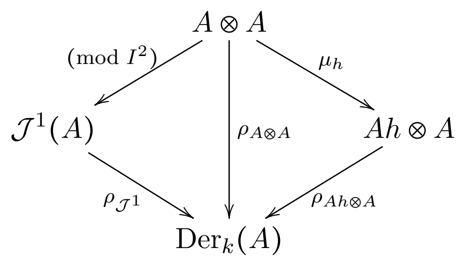
hence we deduce

$$\left[\sum a_i\otimes e_i, \sum b_j\otimes e_j\right]_{A\otimes A}\in K$$

so that K is a Lie ideal in $(A\otimes A, [\bullet,\bullet]_{A\otimes A})$.

Hence, by Lemma 1.2.4, the pair $(A, Ah\otimes A)$ can be endowed with a Lie–Rinehart algebra structure with anchor $\rho_{Ah\otimes A}(h\cdot a\otimes f) = \rho_{A\otimes A}(a\otimes b) = a\cdot\Phi_b$ and Lie bracket on $Ah\otimes A$ given by $[h\cdot a\otimes f, h\cdot b\otimes g]_{Ah\otimes A} = h\cdot[a\otimes f, b\otimes g]_{A\otimes A}$. □

The following commutative diagram describes the compatibility and relations between the Lie–Rinehart algebra structures in $(A, A \otimes A)$, $(A, \mathcal{J}^1(A))$ and $(A, Ah \otimes A)$:



2.3 The full Hopf algebroid $V(A, \mathcal{J}^1(A))$ over a Jacobi algebra A

In this section we analyse the right (A, L)-module structures on the Lie–Rinehart algebra $(A, \mathcal{J}^1(A))$ of a Jacobi algebra $(A, \{\bullet, \bullet\}_J)$. As in Section 1.1, we will see that there exist different full Hopf algebroid structures on $V(A, \mathcal{J}^1(A))$.

Proposition 2.3.1. *Let A be a Jacobi algebra with bracket $\{\bullet, \bullet\}_J$, $(A, \mathcal{J}^1(A))$ be the Lie–Rinehart algebra on the 1-jet space of A and $D_J : A \to A$ be an R-linear map. The map*

$$\varphi : \mathcal{J}^1(A) \longrightarrow A, \quad a \cdot j^1(b) \longmapsto a \cdot D_J(b) + \{a, b\}_J - a \cdot \{1, b\}_J \tag{2.3.1}$$

is a right $\left(A, \mathcal{J}^1(A)\right)$-connection character on A if and only if $D_J : A \to A$ is a first order differential operator on A.

Proof. First we check the conditions under which this map is well-defined. Again, recall that $\mathcal{J}^1(A) = (A \otimes_R A)/I^2$. The map

$$\gamma_\varphi : A \otimes_R A \longrightarrow A, \quad a \otimes_R b \longmapsto a \cdot D_J(b) + \{a, b\}_J - a \cdot \{1, b\}_J \tag{2.3.2}$$

induces the map given in Equation (2.3.1) if and only if $\gamma_\varphi\left(I^2\right) = 0$. We check this. Let $\sum a_i \otimes_R b_j, \sum f_i \otimes_R g_j \in I$, that is, $\sum a_i \cdot b_j = 0, \quad \sum f_i \cdot g_j = 0$, so that we have

$$\left(\sum a_i \otimes_R b_j\right) \cdot \left(\sum f_i \otimes_R g_j\right) \in I^2 \tag{2.3.3}$$

which under the map $\gamma_\varphi : A \otimes_R A \longrightarrow A$ becomes

$$\begin{aligned}
\gamma_\varphi\left(\left(\sum a_i \otimes_R b_j\right)\cdot\left(\sum f_i \otimes_R g_j\right)\right) &= \gamma_\varphi\left(\sum a_i \cdot f_i \otimes_R b_j \cdot g_j\right) \\
&= \sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) + \sum\{a_i \cdot f_i, b_j \cdot g_j\}_J - a_i \cdot f_i \cdot \{1, b_j \cdot g_j\}_J \\
&= \sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) - a_i \cdot f_i \cdot g_j \cdot \{1, b_j\}_J - a_i \cdot f_i \cdot b_j \cdot \{1, g_j\}_J \\
&\quad + a \cdot b \cdot \{f, g\}_J + b \cdot f \cdot \{a, g\}_J - a \cdot b \cdot \{1, g\}_J + a \cdot g \cdot \{f, b\}_J \\
&\quad + f \cdot g \cdot \{a, b\}_J - a \cdot f \cdot g \cdot \{1, b\}_J + a \cdot b \cdot g \cdot \{1, f\} + b \cdot f \cdot g \cdot \{1, a\}_J \\
&= \sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) + b \cdot f \cdot \{a, g\}_J + a \cdot g \cdot \{f, b\}_J \\
&= \sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) + b \cdot f \cdot \{a, g\}_J + a \cdot g \cdot \{f, b\}_J - b \cdot \{a, f \cdot g\}_J \\
&= \sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) + g \cdot \{f, a \cdot b\}_J
\end{aligned}$$

which vanishes if and only if $\sum a_i \cdot f_i \cdot D_J\left(b_j \cdot g_j\right) = 0$. Note that if $D_J : A \to A$ is a first order differential operator, this condition is satisfied. We now prove that $\varphi : \mathcal{J}^1(A) \to A$ is a right connection character on A. We have

$$\begin{aligned}
\varphi\left(h \cdot a \cdot db + h \cdot f \cdot dt\right) &= \varphi\left(h \cdot a \cdot j^1(b) + (h \cdot f - h \cdot a \cdot b) \cdot j^1(1)\right) \\
&= h \cdot a \cdot D_J(b) + \{h \cdot a, b\}_J - h \cdot a \cdot \{1, b\}_J + (h \cdot f - h \cdot a \cdot b) \cdot D_J(1) - \{1, h \cdot f - h \cdot a \cdot b\}_J \\
&= h \cdot a \cdot D_J(b) + h \cdot \{a, b\}_J + a \cdot \{h, b\}_J - h \cdot a \cdot \{1, b\}_J - h \cdot a \cdot \{1, b\}_J \\
&\quad + (h \cdot f - h \cdot a \cdot b) \cdot D_J(1) - h \cdot \{1, f\}_J - f \cdot \{1, h\}_J + h \cdot a \cdot \{1, b\}_J + h \cdot b \cdot \{1, a\}_J + a \cdot b \cdot \{1, h\}_J \\
&= h\left(a \cdot D_J(b) + \{a, b\}_J - a \cdot \{1, b\}_J\right) + h \cdot \left((f - a \cdot b) \cdot D_J(1) - \{1, f - a \cdot b\}_J\right) \\
&\quad - a \cdot \{b, h\}_J - a \cdot h \cdot \{1, b\}_J - (f - a \cdot b) \cdot \{1, h\}_J \\
&= h\left(\varphi\left(a \cdot db + f \cdot dt\right)\right) - \rho\left(a \cdot db + f \cdot dt\right)(h)
\end{aligned}$$

so the map $\varphi : \mathcal{J}^1(A) \to A$ satisfies Equation (0.6.19) and is hence a right connection character on A. Conversely, let $\varphi : \mathcal{J}^1(A) \to A$ be a right connection character on A. By Equation (2.3.1) we have

$$\begin{aligned}
a \cdot D_J(b \cdot c) &= \varphi\left(a \cdot j^1(b \cdot c)\right) - \{a, b \cdot c\}_J + a \cdot \{1, b \cdot c\}_J \\
&= \varphi\left(a \cdot b \cdot j^1(c) + a \cdot c \cdot j^1(b) - a \cdot b \cdot c \cdot j^1(1)\right) - \{a, b \cdot c\}_J + a \cdot \{1, b \cdot c\}_J \\
&= a \cdot b \cdot D_J(c) + \{a \cdot b, c\} - a \cdot b \cdot \{1, c\} + a \cdot c \cdot D_J(b) + \{a \cdot c, b\} - a \cdot c \cdot \{1, b\} \\
&\quad - a \cdot b \cdot c \cdot D_J(1) + \{1, a \cdot b \cdot c\} - \{a, b \cdot c\}_J + a \cdot \{1, b \cdot c\}_J \\
&= a \cdot b \cdot D_J(c) + a \cdot c \cdot D_J(b) - a \cdot b \cdot c \cdot D_J(1)
\end{aligned}$$

since $D_J : A \to A$ satisfies the Leibniz rule given in Equation (0.1.13), we deduce that it is a first order differential operator on A. □

Proposition 2.3.2. *Let A be a Jacobi algebra with bracket $\{\bullet,\bullet\}_J$, $(A,\mathcal{J}^1(A))$ be the Lie–Rinehart algebra on the 1-jet space of A, and $D_J : A \to A$ a first order differential operator on A. The connection character*

$$\varphi : \mathcal{J}^1(A) \longrightarrow A; \quad a \cdot j^1(b) \longrightarrow a \cdot D_J(a) + \{a,b\}_J - a \cdot \{1,b\}_J \tag{2.3.4}$$

gives rise to a flat $\left(A,\mathcal{J}^1(A)\right)$-connection on A if and only $D_J : A \to A$ satisfies

$$D_J(\{a,b\}_J) - \{a, D_J(b)\}_J - \{D_J(a), b\}_J = D_J(b) \cdot \{1,a\}_J - D_J(a) \cdot \{1,b\}_J. \tag{2.3.5}$$

Proof. Following a similar argument as in the proof of Proposition 1.1.5, we first compute the curvature of the right $(A,\mathcal{J}^1(A))$-connection $\varphi : \mathcal{J}^1(A) \to A$:

$$\begin{aligned}
\mathcal{C}^{\nabla}_{\mathrm{r}}\left(j^1(a), j^1(b)\right) &= -\rho_{\mathcal{J}^1(A)}\left(j^1(a)\right)\left(\varphi\left(j^1(b)\right)\right) + \rho_{\mathcal{J}^1(A)}\left(j^1(b)\right)\left(\varphi\left(j^1(a)\right)\right) \\
&\quad + \varphi\left([j^1(a), j^1(b)]_{\mathcal{J}^1(A)}\right) \\
&= -\rho_{\mathcal{J}^1(A)}\left(j^1(a)\right)(D_J(b)) + \rho_{\mathcal{J}^1(A)}\left(j^1(b)\right)(D_J(a)) + D_J(\{a,b\}_J) \\
&= -\{a, D_J(b)\}_J - D_J(b) \cdot \{1,a\}_J + \{b, D_J(a)\}_J + D_J(a) \cdot \{1,b\} + D_J(\{a,b\}_J) \\
&= D_J(\{a,b\}_J) - \{a, D_J(b)\}_J - \{D_J(a), b\}_J - D_J(b) \cdot \{1,a\}_J + D_J(a) \cdot \{1,b\}_J
\end{aligned}$$

We see that $\mathcal{C}^{\nabla}_{\mathrm{r}}(j^1(a), j^1(b))$ vanishes if and only if D_J satisfies the condition given in Equation (2.3.5). □

Example 2.3.3. *The derivation $D_J = \{1,\bullet\}_J$ satisfies the condition given in Equation (2.3.5).*

$\left(A,\mathcal{J}^1(A)\right)$-connections on a Poisson algebra $(A,\{\bullet,\bullet\})$

From Propositions 2.3.1 and 2.3.2 we obtain the following description:

Proposition 2.3.4. *Let A be a Poisson algebra over R with bracket $\{\bullet,\bullet\}$, let $\left(A,\mathcal{J}^1(A)\right)$ be the Lie–Rinehart algebra on the 1-jet space of A and let $D : A \to A$ be an R-linear map. The map*

$$\Psi : \mathcal{J}^1(A) \longrightarrow A; \quad a \cdot j^1(b) \longmapsto a \cdot D(b) + \{a,b\} \tag{2.3.6}$$

is a right connection character on A if and only if $D : A \to A$ is a first order differential operator on A. Moreover, $\Psi : \mathcal{J}^1(A) \to A$ gives rise to a flat $\left(A, \mathcal{J}^1(A)\right)$-connection on A if $D : A \to A$ satisfies

$$D(\{a,b\}) - \{a, D(b)\} - \{a, D(b)\} = 0. \tag{2.3.7}$$

Proof. Follows directly from Proposition 2.3.2 when setting $\{1, \bullet\}_J = 0$ so that $\{\bullet,\bullet\}_J$ becomes a Poisson bracket. □

Example 2.3.5. *Let $D : A \to A$ be a Poisson derivation. The map $\Phi : \mathcal{J}^1(A) \to A$ given by $a \cdot j^1(b) \mapsto a \cdot D(b) + \{a,b\}$ is a flat right $(A, \mathcal{J}^1(A))$-connection on A.*

2.4 Lie–Rinehart algebras $(A, Ah \otimes A)$ with no antipode

2.4.1 Right $(A, Ah \otimes A)$-connections on $(A, \{\bullet,\bullet\}_J)$

Throughout this section we assume A is a Jacobi algebra over a field k with bracket $\{\bullet,\bullet\}_J$, $h \in A$ and $r \cdot \{\bullet,\bullet\}_J = 0$ for all $r \in \mathrm{Ann}_A(\{h\})$. Again, an unadorned $\otimes$ denotes $\otimes_k$.

Lemma 2.4.1. *A k-linear map $\varphi_h : Ah \otimes A \to A$ is a right $(A, Ah \otimes A)$-connection on A if and only if it is of the form*

$$\varphi_h : Ah \otimes A \longrightarrow A, \quad a \cdot h \otimes b \longmapsto a \cdot \mathfrak{D}(b) + \{a,b\}_J - a \cdot \{1,b\}_J \tag{2.4.1}$$

where $\mathfrak{D} : A \to A$ satisfies:

$$r \cdot \mathfrak{D}(a) - \{a, r\}_J = 0, \quad \forall r \in \mathrm{Ann}_A(\{h\}). \tag{2.4.2}$$

In terms of $\mathfrak{D}$, the curvature of a right $(A, Ah \otimes A)$-connection on A is

$$\mathcal{C}_{\mathrm{r}}^{\nabla}(h \otimes a, h \otimes b) = \mathfrak{D}\{a,b\}_J - \{a, \mathfrak{D}(b)\}_J - \{\mathfrak{D}(a), b\}_J - \{1,a\}_J \cdot \mathfrak{D}(b) + \mathfrak{D}(a) \cdot \{1,b\}_J. \tag{2.4.3}$$

Proof. We start by recalling that $Ah \otimes A$ is generated as an A-module by the elements $h \otimes a \in Ah \otimes A$, $a \in A$. Let φ_h be a right $(A, Ah \otimes A)$-connection character on A, then it satisfies

$$\varphi_h : Ah \otimes A \longrightarrow A, \quad a \cdot h \otimes b \longmapsto a \cdot \varphi_h(h \otimes b) - \rho_{Ah \otimes A}(h \otimes b)(a).$$

Let $\mathfrak{D}(a) \coloneqq \varphi_h\left(h \otimes a\right), \forall a \in A$. Then we have

$$0 = \varphi_h\left(r \cdot h \otimes a\right) = r \cdot \mathfrak{D}(a) - \{a, r\}_J - \{1, a\}_J \cdot r = r \cdot \mathfrak{D}(a) - \{a, r\}_J. \tag{2.4.4}$$

We now prove the converse statement: a k-linear map $\varphi_h : Ah \otimes A \to A$ given by

$$a \cdot h \otimes b \longmapsto a \cdot \mathfrak{D}(b) + \{a, b\}_J - a \cdot \{1, b\}_J,$$

where $\mathfrak{D}$ satisfies Equation (2.4.2), is a right $(A, Ah \otimes A)$-connection on A. Note that the map $\varphi_h : Ah \otimes A \to A$ is well-defined by Equation (2.4.4). Let $\sum a_i \cdot h \otimes b_i \in Ah \otimes A$, then

$$\begin{aligned}
\varphi_h\left(\sum c \cdot a_i \cdot h \otimes b_i\right) &= \sum c \cdot a_i \cdot \mathfrak{D}(b_i) + \sum \{c \cdot a_i, b_i\}_J - \sum c \cdot a_i \cdot \{1, b_i\}_J \\
&= c \cdot \sum a_i \cdot \mathfrak{D}(b_i) + c \cdot \sum \{a_i, b_i\}_J - \sum a_i \cdot \{b_i, c\}_J \\
&\quad - \sum c \cdot a_i \cdot \{1, b_i\}_J - \sum c \cdot a_i \cdot \{1, b_i\}_J \\
&= c \cdot \varphi_h\left(\sum a_i \cdot h \otimes b_i\right) - \rho_{Ah \otimes A}\left(\sum a_i \cdot h \otimes b_i\right)(c)
\end{aligned}$$

so φ_h satisfies (0.6.19). Lastly, the expression for the curvature in (2.4.3) follows directly from Lemma 0.6.28. □

Remark 2.4.2. *Assume right $(A, Ah \otimes A)$-connections exist on A, and let $\mathcal{D}$ be the (non-empty) set of maps $\mathfrak{D} : A \to A$ satisfying (2.4.2) for all $a \in A$. Since maps $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{D}$ satisfy*

$$r \cdot \mathfrak{D}_1(a) - r \cdot \mathfrak{D}_2(a) = \{a, r\}_J - \{a, r\}_J = 0, \quad \text{for all } a \in A,$$

we deduce

$$\mathfrak{D}_1(a) - \mathfrak{D}_2(a) \in \bigcap_{r \in \mathrm{Ann}_A(\{h\})} \mathrm{Ann}_A(r) =: H,$$

hence it follows that the set $\mathcal{D}$ is an affine space over $\mathrm{Lin}_k(A, H)$.

Furthermore, from Equation (2.4.1) we see that right $(Ah \otimes A, A)$-connections on A are determined by maps $\mathfrak{D} \in \mathcal{D}$ so that given two connections φ_h and φ'_h we have

$$r \cdot (\varphi_h - \varphi'_h)(a) = 0$$

for all $a \in A$. As before, we deduce that the set of right $(A, Ah \otimes A)$-connections on A is an affine space over $\mathrm{Lin}_k(A, H)$.

Theorem 2.4.3. *Let A be a Jacobi algebra over a field k with bracket $\{\bullet,\bullet\}_J$, let $h \in A$, and $r \cdot \{\bullet,\bullet\}_J = 0$ for all $r \in \operatorname{Ann}_A(\{h\})$.*

1. *The pair $(A, Ah\otimes A)$ is a Lie–Rinehart algebra with anchor $\rho_{Ah\otimes A} : Ah\otimes A \to \operatorname{Der}_k(A)$ given by $h \otimes a \mapsto \Phi_a$, and Lie bracket on $Ah \otimes A$ given by*

$$[h \otimes f, h \otimes g]_{Ah\otimes A} = h \otimes \{f, g\}_J. \tag{2.4.5}$$

2. *Assume there exists a right $(A, Ah \otimes A)$-connection ∇^{r} on A.*
   (a) *Then there exists some $a \in A$ satisfying $a \cdot r = \{1, r\}_J$ for all $r \in \operatorname{Ann}_A(\{h\})$.*
   (b) *Moreover, if ∇^{r} is flat, so that A is a right $(A, Ah \otimes A)$-module extending multiplication in A, then for all $b \in A$ satisfying $\{1, b\}_J = 0$, there exists an element $a \in A$ satisfying $a \cdot r = \{1, r\}_J$ and such that the following compatibility condition holds:*

$$\{b, a\}_J = \{1, c\}_J, \quad \text{for some } c \in A.$$

Proof. To prove Part 2 (a), take $\xi = r \otimes 1$ where $r \in \operatorname{Ann}_A(\{h\})$, $\zeta = 1 \otimes 1$ and $a = r$ in Lemma 1.4.1, so we deduce that if there exists a right $(A, Ah \otimes A)$-connection on A, not necessarily flat, there must exist some $b \in A$ such that $r \cdot b = \{1, r\}_J$. Note that taking $a = 1$ in (2.4.2) yields

$$0 = r \cdot \mathfrak{D}(1) - \{1, r\}_J$$

for all $r \in \operatorname{Ann}_A(\{h\})$.

For all $a \in A$, we denote by S_a the set $\{s \in A \mid r \cdot s = \{a, r\}_J, \forall r \in \operatorname{Ann}_A(\{h\})\}$ of solutions of (2.4.2).

To prove Part 2 (b), assume there exist right $(A, Ah \otimes A)$-connections on A which are flat, i.e., $\mathcal{D}$ is non-empty, so that (2.4.2) has solutions $\mathfrak{D}(a) \in S_a \subset A$ for all $a \in A$ and furthermore, $\mathcal{C}^{\nabla}_{\mathrm{r}}(h \otimes f, h \otimes g) = 0$ for all $f, g \in A$. Let $b \in A$ satisfy $\{1, b\}_J = 0$, then by (2.4.3) we have

$$0 = \mathcal{C}^{\nabla}_{\mathrm{r}}(h \otimes 1, h \otimes b) = -\{1, \mathfrak{D}(b)\}_J - \{\mathfrak{D}(1), b\}_J. \tag{2.4.6}$$

Hence if there exists no $c \in A$ satisfying $\{1, c\}_J = \{b, s\}_J$ for some $s \in S_1$, then there exists no such map $\mathfrak{D} : A \to A$, and hence there exists no flat right $(A, Ah \otimes A)$-connection on A. ∎

We see that if a Jacobi algebra A with bracket $\{\bullet,\bullet\}_J$ satisfying $r\cdot\{\bullet,\bullet\}_J = 0$ for all $r \in \mathrm{Ann}_A(\{h\})$ for some fixed $h \in A$ does not satisfy conditions (a) or (b) in Theorem 2.4.3 Part 2, then there is an obstruction to the existence of flat right $(A, Ah\otimes A)$-connections on $(A,\cdot)$. By a result of Kowalzig and Posthuma [KP11], see Section 0.6.3 below, the universal enveloping algebra of the Lie–Rinehart algebra $(A, Ah\otimes A)$ associated to these Jacobi algebras will provide new examples of left Hopf algebroids without antipode. Section 2.4 is dedicated to examples of this construction.

Corollary 2.4.4. *If* $\mathrm{Ann}_A(h) = 0$*, the map* $\mathfrak{D} : A \to A$ *given by* $a \mapsto \{1,a\}_J$ *satisfies* (2.4.2) *and hence induces the right* $(A, Ah\otimes A)$*-connection* $\varphi : Ah\otimes A \to A$ *on* A *given by* $a\cdot h\otimes b \mapsto \{a,b\}_J$ *which is shown to be flat, by a straightforward computation using* (2.4.3).

This section is dedicated to provide examples of Lie–Rinehart algebras $(A, Ah\otimes A)$, constructed as in Section 2.2.3. The first of our examples admits no right $(A, Ah\otimes A)$-connections on A while the second one does admit them, although none of them can be flat.

2.4.2 Example with no $(A, Ah\otimes A)$-connections on A

Let $A = k[x,y]/\langle x\cdot y, x^2, y^2\rangle$, let $E \in \mathrm{Der}_k(A)$ be a derivation with $E(x) = y$, $E(y) = 0$, and let A be endowed with the Jacobi bracket $\{a,b\}_J = a\cdot E(b) - E(a)\cdot b$.

Take $h = y$, then $\mathrm{Ann}_A(\{y\}) = \mathrm{Span}_k\{x,y\}$ and $r\cdot\{\bullet,\bullet\}_J = 0$ for all $r \in \mathrm{Ann}_A(\{y\})$. Then, the pair $(A, Ay\otimes A)$ is a Lie–Rinehart algebra with Lie bracket on $Ay\otimes A$ given by

$$[y\otimes f, y\otimes g]_{Ay\otimes A} = y\otimes\{f,g\}_J$$

and anchor

$$\rho_{Ay\otimes A} : Ay\otimes A \longrightarrow \mathrm{Der}_k(A), \quad y\otimes a \longmapsto \Phi_a = a\cdot E(\bullet).$$

Since $x \in \mathrm{Ann}_A(\{y\})$ and there exists no $a \in A$ satisfying

$$a\cdot x - \{1,x\}_J = a\cdot x - y = 0$$

we deduce by Theorem 2.4.3 Part 2 (a) that the Lie–Rinehart algebra $(A, Ay\otimes A)$ does not admit right $(A, Ay\otimes A)$-connections on A. Hence its universal enveloping algebra does not admit an antipode.

Alternatively, we can prove this result by noting that $(Ay\otimes A, [\bullet,\bullet]_{Ay\otimes A})$ is isomorphic to the Heisenberg Lie algebra $H_3(k)$ of dimension 3, with central element $y\otimes y$. Let $\{\alpha_1,\alpha_2,\alpha_3\}$ be a basis for $H_3(k)$ with central element α_3. Following [KR15, Proposition 3.1], we define an A-module structure on $H_3(k)$ by

$$a\cdot\alpha_i := \chi(a)\alpha_i$$

for $1\le i\le 3$ where $\chi : A\to k$ is a character on A given by $\chi(x)=\chi(y)=0$. Furthermore, we define an anchor map $\rho : H_3(k)\to \mathrm{Der}_k(A)$ given by $\rho(\alpha_1)=E$ and $\rho(\alpha_2)=\rho(\alpha_3)=0$. Then, $(A,H_3(k))$ is a Lie–Rinehart algebra isomorphic to $(A, Ay\otimes A)$. A similar argument as in the proof of [KR15, Theorem 1.1] yields that there exist no right $(A,H_3(k))$-connections on A.

2.4.3 Example with non-flat $(A, Ah\otimes A)$-connections on A

Let $k=\mathbb{Z}_2$ and let $A=\mathbb{Z}_2[x,y,z]/\langle x^4,y^6,z^2,x\cdot y^4,x^3\cdot y,x^3\cdot z\rangle$. A basis for A as $\mathbb{Z}_2$-module is:

$$\begin{aligned}1,x,x^2,x^3,y,y^2,y^3,y^4,y^5,z,x\cdot y,x^2\cdot y,x\cdot y^2,x\cdot y^3,x^2\cdot y^2,x^2\cdot y^3,x\cdot z,\\ x^2\cdot z,y\cdot z,y^2\cdot z,y^3\cdot z,y^4\cdot z,y^5\cdot z,x\cdot y\cdot z,x^2\cdot y\cdot z,x\cdot y^2\cdot z,x\cdot y^3\cdot z,\\ x^2\cdot y^2\cdot z,x^2\cdot y^3\cdot z.\end{aligned} \tag{2.4.7}$$

Let $E,F\in \mathrm{Der}_R(A)$ be derivations with

$$E(x)=E(z)=x^2,\quad E(y)=0,\quad \text{and}\quad F(x)=F(z)=0,\quad F(y)=z.$$

Then the images of $E,F\in\mathrm{Der}_R(A)$ characterised in terms of the basis for A in (2.4.7) are:

$$\begin{aligned}\mathrm{Im}(E)=\mathrm{Span}_{\mathbb{Z}_2}\{&x^2,x^2\cdot y,x^2\cdot y^2,x^2\cdot y^3,x^2\cdot z+x^3,\\ &x^2\cdot y\cdot z,x^2\cdot y^2\cdot z,x^2\cdot y^3\cdot z\}\subset Ax^2\end{aligned} \tag{2.4.8}$$

and

$$\mathrm{Im}(F)=\mathrm{Span}_{\mathbb{Z}_2}\{z,y^2\cdot z,y^4\cdot z,x\cdot z,x^2\cdot z,x\cdot y^2\cdot z,x^2\cdot y^2\cdot z\} \tag{2.4.9}$$

Hence we deduce that

$$\mathrm{Im}(F\circ E)=\mathrm{Span}_{\mathbb{Z}_2}\{x^2\cdot z,x^2\cdot y^2\cdot z\}\subset Ax^2\cdot z$$

and

$$\operatorname{Im}(E \circ F) = \operatorname{Span}_{\mathbb{Z}_2}\{x^2, x^2 \cdot y^2, x^2 \cdot z + x^3, x^2 \cdot y^2 \cdot z\} \subset Ax^2$$

so $E(a) \cdot F(E(b)) \in A$, $E(a) \cdot E(F(b)) \in A$ vanish for all $a, b, c \in A$. So A admits the Jacobi bracket

$$\{a, b\}_J = E(a) \cdot F(b) - F(a) \cdot E(b) + a \cdot E(b) - E(a) \cdot b.$$

To see this, we note $\{1, \bullet\}_J = E$; next we check that $\{\bullet, \bullet\}_J$ satisfies the Leibniz rule in (2.1.1):

$$\begin{aligned}\{a, b \cdot c\}_J &= E(a) \cdot F(b \cdot c) - F(a) \cdot E(b \cdot c) + a \cdot E(b \cdot c) - E(a) \cdot b \cdot c \\ &= b \cdot \{a, c\}_J + c \cdot \{a, b\}_J + \{1, a\}_J \cdot b \cdot c;\end{aligned}$$

and finally we check that $\{\bullet, \bullet\}_J$ satisfies the Jacobi identity:

$$\begin{aligned}\{a, \{b, c\}_J\}_J + c.p. &= E(a) \cdot F(E(b) \cdot F(c) - F(b) \cdot E(c) + b \cdot E(c) - E(b) \cdot c) \\ &\quad - F(a) \cdot E(E(b) \cdot F(c) - F(b) \cdot E(c) + b \cdot E(c) - E(b) \cdot c) \\ &\quad + a \cdot E(E(b) \cdot F(c) - F(b) \cdot E(c) + b \cdot E(c) - E(b) \cdot c) \\ &\quad - E(a) \cdot (E(b) \cdot F(c) - F(b) \cdot E(c) + b \cdot E(c) - E(b) \cdot c) + c.p. \\ &= E(a) \cdot F(E(b)) \cdot F(c) - E(a) \cdot F(b) \cdot F(E(c)) \\ &\quad + E(a) \cdot b \cdot F(E(c)) - E(a) \cdot c \cdot F(E(b)) \\ &\quad - F(a) \cdot E(b) \cdot E(F(c)) + F(a) \cdot E(F(b)) \cdot E(c) \\ &\quad + a \cdot E(b) \cdot E(F(c)) - a \cdot E(F(b)) \cdot E(c) + c.p. = 0\end{aligned}$$

Now, taking $h = y^2$, we characterise $\operatorname{Ann}_A(\{y^2\})$ in terms of the basis for A given in Equation (2.4.7) as

$$\begin{aligned}\operatorname{Ann}_A(\{y^2\}) = \operatorname{Span}_{\mathbb{Z}_2} \big\{&x^3, y^4, y^5, x \cdot y^2, x \cdot y^3, x^2 \cdot y^2, x^2 \cdot y^3, \\ &y^4 \cdot z, y^5 \cdot z, x \cdot y^2 \cdot z, x \cdot y^3 \cdot z, x^2 \cdot y^2 \cdot z, x^2 \cdot y^3 \cdot z\big\}\end{aligned} \tag{2.4.10}$$

so that $r \cdot \{\bullet, \bullet\}_J = 0$ for all $r \in \operatorname{Ann}_A(\{y^2\})$. Hence, we deduce that $(A, Ay^2 \otimes_{\mathbb{Z}_2} A)$ is a Lie–Rinehart algebra with anchor map

$$\rho_{Ay^2 \otimes_{\mathbb{Z}_2} A} : Ay^2 \otimes_{\mathbb{Z}_2} A \longrightarrow \operatorname{Der}_R(A), \quad y^2 \otimes_{\mathbb{Z}_2} a \longmapsto \Phi_a$$

and Lie bracket on the A-module $Ay^2 \otimes_{\mathbb{Z}_2} A$ given by

$$[y^2 \otimes_{\mathbb{Z}_2} f, y^2 \otimes_{\mathbb{Z}_2} g]_{Ay^2 \otimes_{\mathbb{Z}_2} A} = y^2 \otimes_{\mathbb{Z}_2} \{f, g\}_J.$$

We now show that right $(A, Ay^2 \otimes_{\mathbb{Z}_2} A)$-connections do exist on $(A, \{\bullet, \bullet\}_J)$. A straightforward computation using the characterisation of $\mathrm{Ann}_A(\{y^2\})$ given in Equation (2.4.10) shows that $x \in A$ satisfies

$$r \cdot x = \{1, r\}_J \tag{2.4.11}$$

for all $r \in \mathrm{Ann}_A(\{y^2\})$. It is now easy to check that, for all $a \in A$, the map

$$\mathfrak{D} : A \longrightarrow A, \quad a \longmapsto -x \cdot F(a) + x \cdot a$$

satisfies the condition given in Equation (2.4.2) since

$$\begin{aligned} r \cdot \mathfrak{D}(a) &= r \cdot (-x \cdot F(a) + x \cdot a) \\ &= -E(r) \cdot F(a) + E(r) \cdot a \\ &= E(a) \cdot F(r) - E(r) \cdot F(a) + a \cdot E(r) - r \cdot E(a) \\ &= \{a, r\}_J \end{aligned}$$

for all $r \in \mathrm{Ann}_A(\{y^2\})$. Hence, by Lemma 2.4.1, the map

$$\varphi : Ay^2 \otimes_{\mathbb{Z}_2} A \longrightarrow A, \quad a \cdot y^2 \otimes_{\mathbb{Z}_2} b \longrightarrow -a \cdot x \cdot F(b) + a \cdot b \cdot x + \{a, b\}_J - a \cdot \{1, b\}_J \tag{2.4.12}$$

is a right $(A, Ay^2 \otimes_{\mathbb{Z}_2} A)$-connection character on A.

We now prove that none of the right $(A, Ay^2 \otimes_{\mathbb{Z}_2} A)$-connections on A is flat. First note

$$H = \bigcap_{r \in \mathrm{Ann}_A(\{y^2\})} \mathrm{Ann}_A(r) = x^2 A + y^2 A$$

that is

$$\begin{aligned} H = \mathrm{Span}_{\mathbb{Z}_2}(&x^2, x^3, x^2 \cdot y, x^2 \cdot y^2, x^2 \cdot y^3, x^2 \cdot z, x^2 \cdot y \cdot z, x^2 \cdot y^2 \cdot z, \\ &x^2 \cdot y^3 \cdot z, y^2, x \cdot y^2, y^3, y^4, y^5, y^2 \cdot z, x \cdot y^3, x \cdot y^2 \cdot z, \\ &y^3 \cdot z, y^4 \cdot z, y^5 \cdot z, x \cdot y^3 \cdot z). \end{aligned}$$

Then, from Remark 2.4.2 and Equation (2.4.11) we deduce that the only solutions of the equation

$$a \cdot r = \{1, r\}_J$$

for all $r \in \mathrm{Ann}_A(\{y^2\})$, are elements

$$a = x + \alpha$$

for all $\alpha \in H$. Since $\{1, y\}_J = E(y) = 0$, we can take $a = x + \alpha$ and $b = y$ in Theorem 2.4.3, and compute

$$\begin{aligned}\{a, b\}_J &= \{x + \alpha, y\}_J \\ &= x^2 \cdot z - x^2 \cdot y + E(\alpha) \cdot F(y) - y \cdot E(\alpha) \\ &= x^2 \cdot z - x^2 \cdot y + \lambda_1 \cdot x^2 \cdot y^2 \cdot z + \lambda_2 \cdot x^2 \cdot y^3 + \lambda_3 \cdot x^2 \cdot y^3 \cdot z\end{aligned}$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_2$, which by (2.4.8) is not in the image of E. Hence there exists no $c \in A$ satisfying $\{a, b\}_J = \{1, c\}_J$ and by Theorem 2.4.3 Part 2 (b), we find that A is not a right $(A, Ay^2 \otimes_{\mathbb{Z}_2} A)$-module.

Chapter 3

Lie–Rinehart algebras and Leibniz algebroids

Our goal is to understand the relation between Lie–Rinehart algebras and Leibniz algebras (in particular Leibniz algebroids and Courant algebroids). In this chapter we start by giving some background on Leibniz algebras, providing an algebraic description of Leibniz algebroids and giving some examples. We then describe Lie–Rinehart algebras in $\mathcal{LM}$ and finally construct a (right adjoint) functor from Lie–Rinehart algebras in $\mathcal{LM}$ to Leibniz algebroids.

Throughout this chapter let R be a unital commutative ring and let an unadorned $\otimes$ denote $\otimes_R$.

Theorem A. *Let $(M \xrightarrow{g} A)$ be a commutative R-algebra object and $(N \xrightarrow{f} L)$ be a Lie algebra object in $\mathcal{LM}$. The pair $\Big((M \xrightarrow{g} A), (N \xrightarrow{f} L)\Big)$ is a Lie–Rinehart algebra in $\mathcal{LM}$ if*

- *(A, L) is a Lie–Rinehart algebra (in R-mod) with anchor $\rho_0 : L \to \mathrm{Der}_R(A)$,*
- *the A-bimodule M is a left (A, L)-module with action given by $\rho_2 : L \to \mathrm{Hom}_R(M, M)$,*
- *the right L-module N with action $N \otimes L \to N$ given by $n \otimes \xi \mapsto [n, \xi]$ for all $n \in N$, $\xi \in L$ is also a left A-module with $[\bullet, \bullet]$ satisfying*

$$[a \cdot n, \xi] = a \cdot [n, \xi] - \rho_0(\xi)(a) \cdot a \quad \forall a \in A,$$

- *both f and g are L-equivariant and A-linear*

and there exist

- *an A-module map* $\lambda : M \otimes_A L \to N$,
- *an A-module map* $\rho_1 : N \to \mathrm{Der}_R(A, M)$ *satisfying*

$$g(\rho_1(n)(a)) = \rho_0(f(n))(a), \quad \rho_1([n,\xi]) = [\rho_1(n), (\rho_0 + \rho_2)(\xi)]$$

for all $a \in A$, $m \in M$, $n \in N$, $\xi \in L$.

Our second result consists in the construction of a functor from the category of Lie–Rinehart algebra objects in $\mathcal{LM}$ to the category of Leibniz algebroids.

Theorem B. *For any Lie–Rinehart algebra* $\Big((M \overset{g}{\to} A), (N \overset{f}{\to} L)\Big)$ *in the category* $\mathcal{LM}$ *of linear maps, the pair* $(A, M \oplus N)$ *is a Leibniz algebroid with anchor*

$$\rho_N := -\rho_0 \circ f : M \oplus N \longrightarrow \mathrm{Der}_R(A) \tag{B.1}$$

and Leibniz bracket on the A-module $M \oplus N$ *given by*

$$[m_1 + n_1, m_2 + n_2]_{M \oplus N} := -\rho_2(f(n_2))(m_1) + [n_1, f(n_2)]. \tag{B.2}$$

for all $m_1, m_2 \in M$ *and* $n_1, n_2 \in N$.

3.1 Leibniz algebroids

In this section we first recall the definitions of Leibniz algebras as defined by Loday and Pirashvili [Lod93, LP93]. Secondly, we discuss Leibniz algebroids, see [IdLMP99] for a differential geometric description, and give some motivating examples.

3.1.1 Leibniz algebras

Leibniz algebras are a generalisation of Lie algebras. In particular, they are a vector space equipped with a bilinear form (called a Leibniz bracket) that is not necessarily skew-symmetric while satisfying the Jacobi identity (in a form not requiring skew symmetry). Leibniz algebras were first defined by Blokh [Blo65], later rediscovered and more intensively studied since [Cuv91, LP93]. For motivation, definitions and basic examples see [LP93, Lod93].

Definition 3.1.1. A right Leibniz algebra $\mathfrak{g}$ is an R-module equipped with a bilinear map, called the right Leibniz bracket, given by $[\bullet,\bullet]_{\mathfrak{g}} : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the identity

$$[x,[y,z]_{\mathfrak{g}}]_{\mathfrak{g}} - [[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,z]_{\mathfrak{g}},y]_{\mathfrak{g}} = 0, \quad \text{for all } x,y,z\in\mathfrak{g}. \tag{3.1.1}$$

Correspondingly, a *left* Leibniz algebra structure $[\bullet,\bullet]$ on an R-module V is defined by $[x,y] := [y,x]_{\mathfrak{g}}$ for $x,y\in V$ where $[\bullet,\bullet]_{\mathfrak{g}}$ satisfies (3.1.1), see [LP93, Lod93].

Since Loday and Pirashvili use right Leibniz algebra structures in their work [Lod93, LP93, LP98], we choose right Leibniz algebras as well, which we call from now on Leibniz algebras.

Remark 3.1.2. For a Leibniz algebra $\mathfrak{g}$ there exists a corresponding Lie algebra, denoted by $\mathfrak{g}_{\text{Lie}}$ and called the reduced Lie algebra of $\mathfrak{g}$, arising as the quotient of $\mathfrak{g}$ by the Leibniz ideal generated by elements $[x,x]_{\mathfrak{g}}\in\mathfrak{g}$ for $x\in\mathfrak{g}$. Hence there exists a surjective map

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{g}_{\text{Lie}}.$$

Example 3.1.3 (See [LP93])**.** All Lie algebras are Leibniz algebras.

Example 3.1.4 (See [KP02])**.** Let L be a Lie algebra over R with bracket $[\bullet,\bullet]_L$. The bracket on the second tensor power of L given by

$$[x_1\otimes y_1, x_2\otimes y_2]_{L\otimes L} = [x_1,[x_2,y_2]_L]_L\otimes y_1 + x_1\otimes[y_1,[x_2,y_2]_L]_L \tag{3.1.2}$$

for all $x_1,x_2,y_1,y_2\in L$ endows $L\otimes L$ with a Leibniz algebra structure.

In the following example we reformulate the construction of the *hemi–semi–direct product* for left Leibniz algebras introduced by Kinyon and Weinstein in [KW01, Example 2.2] and endow the direct sum of a Lie algebra L and a (left) L-module V with a (right) Leibniz algebra structure.

Example 3.1.5. Let L be a Lie algebra over R and V be a L-module with left action $L\otimes V\to V$ given by $\xi\otimes a\mapsto\xi(a)$ for all $a\in V$ and $\xi\in L$. The direct sum (of R-modules) $V\oplus L$ together with the bracket

$$[a+\xi, b+\zeta]_{V\oplus L} := \zeta(a) - [\xi,\zeta]_L, \quad a,b\in V, \xi,\zeta\in L \tag{3.1.3}$$

becomes a (right) Leibniz algebra since the identity (3.1.1) is satisfied:

$$\begin{aligned}
&[a+\xi,[b+\zeta,c+\gamma]_{V\oplus L}]_{V\oplus L} - [[a+\xi,b+\zeta]_{V\oplus L},c+\gamma]_{V\oplus L} + [[a+\xi,c+\gamma]_{V\oplus L},b+\zeta]_{V\oplus L}\\
&= [a+\xi,\gamma(b)-[\zeta,\gamma]_L]_{V\oplus L} - [\zeta(a)-[\xi,\zeta]_L,c+\gamma]_{V\oplus L} + [\gamma(a)-[\xi,\gamma]_L,b+\zeta]_{V\oplus L}\\
&= -[\zeta,\gamma]_L(a) + [\xi,[\zeta,\gamma]_L]_L - \gamma(\zeta(a)) - [[\xi,\zeta]_L,\gamma]_L + \zeta(\gamma(a)) + [[\xi,\gamma]_L,\zeta]_L = 0.
\end{aligned}$$

Definition 3.1.6 (See [Lod93])**.** Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Leibniz algebras. A map of Leibniz algebras $\varphi:\mathfrak{g}\to\mathfrak{g}'$ is a homomorphism of R-modules satisfying $\varphi([x,y]_{\mathfrak{g}}) = [\varphi(x),\varphi(y)]_{\mathfrak{g}'}$ for all $x,y\in\mathfrak{g}$.

Proposition 3.1.7. *Let $\mathfrak{g}$ be a Leibniz algebra over R and let M be a left module over its reduced Lie algebra $\mathfrak{g}_{\text{Lie}}$ with left action $\mathfrak{g}_{\text{Lie}}\otimes M\to M$ given by $\pi(g)\otimes m\mapsto\pi(g)(m)$ for all $m\in M$ and $g\in\mathfrak{g}$. The direct sum (of R-modules) $M\oplus\mathfrak{g}$ together with the bracket*

$$[m_1+g_1,m_2+g_2]_{M\oplus\mathfrak{g}} := -\pi(g_2)(m_1) + [g_1,g_2]_{\mathfrak{g}}, \quad m_1,m_2\in M, g_1,g_2\in\mathfrak{g} \tag{3.1.4}$$

is a Leibniz algebra.

Proof. Since $\pi:\mathfrak{g}\to\mathfrak{g}_{\text{Lie}}$ is a map of Leibniz algebras, a straightforward computation identical to the one carried out in Example 3.1.5 yields that $[\bullet,\bullet]_{M\oplus\mathfrak{g}}$ satisfies (3.1.1) and is hence a Leibniz bracket. □

Lemma 3.1.8. *Let $\mathfrak{g}_\ell$ be a left Leibniz algebra with bracket $[\bullet,\bullet]_{\mathfrak{g}_\ell}$, $(\mathfrak{g},[\bullet,\bullet]_{\mathfrak{g}})$ be the corresponding (right) Leibniz algebra and L be a Lie algebra with bracket $[\bullet,\bullet]_L$. A homomorphism $\varphi:\mathfrak{g}_\ell\to L$ induces an antihomomorphism $\varphi:\mathfrak{g}\to L$.*

Proof. We have $\varphi([x,y]_{\mathfrak{g}_\ell}) = \varphi([y,x]_{\mathfrak{g}}) = [\varphi(x),\varphi(y)]_L$. □

3.1.2 Leibniz algebroids and related structures

While Lie algebras can be generalised to Lie–Rinehart algebras [Her53, Hue91, Rin63] (Lie algebroids [Pra67] in differential geometric context), Leibniz algebras [Blo65, LP93] give rise to different algebraic objects: Leibniz algebroids, first defined in a differential geometric context in [IdLMP99]; Loday algebroids [SX08]; Courant algebroids [LWX97]; Courant-Dorfman algebras, a term coined by Roytenberg [Roy09] to denote a structure encompassing both Courant algebroids [LWX97] and Dorfman algebras. See [KS13] for a description of the historic development of these structures.

Leibniz algebroids

We propose a definition of Leibniz algebroids in purely algebraic terms, following the definitions given by Rinehart [Rin63] and later by Huebschmann [Hue91] for Lie–Rinehart algebras as an algebraic description of Lie algebroids.

Definition 3.1.9. Let A be a commutative R-algebra and $\mathcal{E}$ be a Leibniz algebra over R with bracket $[\bullet,\bullet]_{\mathcal{E}}$. The pair $(A,\mathcal{E})$ is called a **Leibniz algebroid** if the Leibniz algebra $\mathcal{E}$ has a left A-module structure $\mu : A\otimes\mathcal{E}\to\mathcal{E}$ given by $a\otimes e\mapsto a\cdot e$ for all $a\in A$ and $e\in\mathcal{E}$, and there exists an A-linear Leibniz algebra antihomomorphism $\rho_{\mathcal{E}} : \mathcal{E}\to \mathrm{Der}_R(A)$, called the anchor, satisfying

$$[a\cdot e_1, e_2]_{\mathcal{E}} = a\cdot[e_1,e_2]_{\mathcal{E}} + \rho_{\mathcal{E}}(e_2)(a)\cdot e_1, \quad \text{for } e_1,e_2\in\mathcal{E}, a\in A. \tag{3.1.5}$$

Example 3.1.10. A Lie–Rinehart algebra (A,L), with anchor ρ_L, is a Leibniz algebroid with anchor $-\rho_L$.

Proposition 3.1.11. *Let (A,L) be a Lie–Rinehart algebra, with anchor ρ_L, and let M be a left (A,L)-module with action $L\otimes M\to M$ given by $\xi\otimes m\mapsto\nabla^\ell_\xi(m)$. The pair $(A, M\oplus L)$ is a Leibniz algebroid with anchor*

$$\rho_{M\oplus L}(m+\xi) := -\rho_L(\xi), \quad \text{for } m\in M,\ \xi\in L \tag{3.1.6}$$

and bracket on the A-module $M\oplus L$ given by the negative hemi-semi-direct product $[\bullet,\bullet]_{M\oplus L}$ of M by L with action $L\otimes M\to M$ (or equivalently $L\to\mathrm{Hom}_R(M,M)$) given by ∇^ℓ.

Proof. First note that $[m_1+\xi, m_2]_{M\oplus L}=0$ for all $m_1,m_2\in M$, $\xi\in L$. Now, since M is an (A,L)-module with action ∇^ℓ, as in Example 3.1.5, we endow the direct sum $M\oplus L$ with the Leibniz bracket given in (3.1.3), that is:

$$[m_1+\xi, m_2+\zeta]_{M\oplus L} = -\nabla^\ell_\zeta(m_1) + [\xi,\zeta]_L.$$

We now check that the map in (3.1.6) is an antihomomorphism:

$$\begin{aligned}\rho_{M\oplus L}([m_1+\xi, m_2+\zeta]_{M\oplus L}) &= \rho_{M\oplus L}\left(-\nabla^\ell_\zeta(m_1)+[\xi,\zeta]_L\right) = -\rho_L([\xi,\zeta]_L)\\ &= -[\rho_L(\xi),\rho_L(\zeta)]_{\mathrm{Der}_R(A)} = [\rho_L(\zeta),\rho_L(\xi)]_{\mathrm{Der}_R(A)}\\ &= [\rho_{M\oplus L}(\zeta),\rho_{M\oplus L}(\xi)]_{\mathrm{Der}_R(A)}.\end{aligned}$$

Lastly, we check that the compatibility condition beween $[\bullet,\bullet]_{M\oplus L}$ and the A-module structure on $M \oplus L$ given in (3.1.5) is satisfied:

$$\begin{aligned}
[a\cdot(m+\xi),\zeta]_{M\oplus L} &= -\nabla^{\ell}_{\zeta}(a\cdot m) + [a\cdot\xi,\zeta]_L \\
&= -a\cdot\nabla^{\ell}_{\zeta}(m) - \rho_L(\zeta)(a)\cdot m + a\cdot[\xi,\zeta]_L - \rho_L(\zeta)(a)\cdot\xi \\
&= a\cdot(-\nabla^{\ell}_{\zeta}(m) + [\xi,\zeta]_L) - \rho_L(\zeta)(a)\cdot(m+\xi) \\
&= a\cdot[m+\xi,\zeta]_{M\oplus L} + \rho_{M\oplus L}(\zeta)(a)\cdot(m+\xi).
\end{aligned}$$
□

Example 3.1.12. Let (A, L) be a Lie–Rinehart algebra, with anchor ρ_L. The pair $(A, L\otimes L)$ where $L \otimes L$ is the Leibniz algebra with bracket given by (3.1.2) will not be a Leibniz algebroid in general. Since $[\bullet,\bullet]_L$ satisfies the Leibniz rule (3.1.5), we have

$$\begin{aligned}
[a\cdot x_1\otimes y_1, x_2\otimes y_2]_{L\otimes L} &= [a\cdot x_1,[x_2,y_2]_L]_L\otimes y_1 + a\cdot x_1\otimes[y_1,[x_2,y_2]_L]_L \\
&= a\cdot[x_1,[x_2,y_2]_L]_L\otimes y_1 - \rho_L\left([x_2,y_2]_L\right)(a)\cdot x_1\otimes y_1 \\
&\quad + a\cdot x_1\otimes[y_1,[x_2,y_2]_L]_L \\
&= a\cdot\left([x_1,[x_2,y_2]_L]_L\otimes y_1 + x_1\otimes[y_1,[x_2,y_2]_L]_L\right) \\
&\quad - \rho_L\left([x_2,y_2]_L\right)(a)\cdot x_1\otimes y_1 \\
&= a\cdot[x_1\otimes y_1, x_2\otimes y_2]_{L\otimes L} - \rho_L([x_2,y_2])(a)\cdot x_1\otimes y_1
\end{aligned}$$

but the map $\gamma(x_2\otimes y_2) \coloneqq \rho_L([x_2,y_2]_L)$ is not A-linear since $[\bullet,\bullet]_L$ is not, hence the pair $(A, L\otimes L)$ does not admit an anchor map induced by ρ_L.

We now see that given a Leibniz algebroid $(A,\mathcal{E})$, the pair $(A,\mathcal{E}_{\mathrm{Lie}})$ will not be a Lie–Rinehart algebra in general.

Proposition 3.1.13. *Let $(A,\mathcal{E})$ be a Leibniz algebroid with anchor $\rho_{\mathcal{E}}$. If $\mathrm{Ker}(\pi)$ is an A-submodule of $\mathcal{E}$, then the pair $(A,\mathcal{E}_{\mathrm{Lie}})$ is a Lie–Rinehart algebra with anchor denoted by $\rho_{\mathcal{E}_{\mathrm{Lie}}}$ and given by $-\rho_{\mathcal{E}}$.*

Proof. First note that the anchor $\rho_{\mathcal{E}}$ descends to an R-linear map $\gamma_{\mathcal{E}_{\mathrm{Lie}}}:\mathcal{E}_{\mathrm{Lie}}\to\mathrm{Der}_R(A)$ since $\rho_{\mathcal{E}}\left([e,e]_{\mathcal{E}}\right) = [\rho_{\mathcal{E}}(e),\rho_{\mathcal{E}}(e)]_{\mathrm{Der}_R(A)} = 0$ for all $e\in\mathcal{E}$. Let us assume that $\pi:\mathcal{E}\to\mathcal{E}_{\mathrm{Lie}}$ is A-linear. Then we have

$$\gamma_{\mathcal{E}_{\mathrm{Lie}}}(a\cdot\pi(e)) = \gamma_{\mathcal{E}_{\mathrm{Lie}}}\left(\pi(a\cdot e)\right) = \rho_{\mathcal{E}}(a\cdot e) = a\cdot\rho_{\mathcal{E}}(e) = a\cdot\gamma_{\mathcal{E}_{\mathrm{Lie}}}(\pi(e))$$

so that $\gamma_{\mathcal{E}_{\mathrm{Lie}}} : \mathcal{E}_{\mathrm{Lie}} \to \mathrm{Der}_R(A)$ is A-linear. Since $\rho_{\mathcal{E}}$ is a antihomomorphism while the anchor of a Lie–Rinehart algebra is a homomorphism, we set $\rho_{\mathcal{E}_{\mathrm{Lie}}} := -\gamma_{\mathcal{E}_{\mathrm{Lie}}}$ so that $(A, \mathcal{E}_{\mathrm{Lie}})$ is a Lie–Rinehart algebra and the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho_{\mathcal{E}}} & \mathrm{Der} A \\ {\scriptstyle \pi}\downarrow & \nearrow_{-\rho_{\mathcal{E}_{\mathrm{Lie}}}} & \\ \mathcal{E}_{\mathrm{Lie}} & & \end{array}$$

commutes. □

Courant algebroids, Courant–Dorfman algebras

The original definition of Courant–Dorfman algebras by Roytenberg [Roy09] involves a left Leibniz algebra structure, while Loday and Pirashvili's work [LP93] on the enveloping algebra of a Leibniz algebra uses a right Leibniz algebra structure.

Since for our purposes, we work with right Leibniz algebras, we now give the corresponding version of Roytenberg's definition reformulated for right Leibniz algebras. See Definition 2.1 [Roy09] for the original definition.

Definition 3.1.14. A (right) **Courant–Dorfman algebra** over a field k of characteristic zero consists of the following data:

- a commutative k-algebra A, an A-module $\mathcal{E}$ with action $A \otimes \mathcal{E} \to \mathcal{E}$ given by $a \otimes e \mapsto a \cdot e$
- a symmetric bilinear form $\langle \bullet, \bullet \rangle : \mathcal{E} \otimes_A \mathcal{E} \to A$,
- a derivation $\partial : A \to \mathcal{E}$,
- a right Leibniz bracket $[\bullet, \bullet]_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$

These data are required to satisfy the following conditions:

1. $[a \cdot e_1, e_2]_{\mathcal{E}} = a \cdot [e_1, e_2]_{\mathcal{E}} + \langle e_2, \partial a \rangle \cdot e_1$
2. $\langle e_1, \partial \langle e_2, e_3 \rangle \rangle = \langle [e_2, e_1]_{\mathcal{E}}, e_3 \rangle + \langle e_2, [e_3, e_1]_{\mathcal{E}} \rangle$
3. $[e_1, e_2]_{\mathcal{E}} + [e_2, e_1]_{\mathcal{E}} = \partial \langle e_1, e_2 \rangle$
4. $[e_1, [e_2, e_3]_{\mathcal{E}}]_{\mathcal{E}} - [[e_1, e_2]_{\mathcal{E}}, e_3]_{\mathcal{E}} + [[e_1, e_3]_{\mathcal{E}}, e_2]_{\mathcal{E}} = 0$
5. $[e, \partial a]_{\mathcal{E}} = 0$

6. $\langle \partial a, \partial b \rangle = 0$

for all $e, e_1, e_2, e_3 \in \mathcal{E}$ and $a, b \in A$.

Note that from (1) and (3) we deduce

$$[e_1, b \cdot e_2]_{\mathcal{E}} = b \cdot [e_1, e_2]_{\mathcal{E}} - \langle e_1, \partial b \rangle \cdot e_2 + \langle e_1, e_2 \rangle \cdot \partial b$$

so that the Leibniz-type rule in (1) can be rewritten as

$$[a \cdot e_1, b \cdot e_2]_{\mathcal{E}} = ab \cdot [e_1, e_2]_{\mathcal{E}} - a\langle e_1, \partial b \rangle \cdot e_2 + b\langle e_2, \partial a \rangle \cdot e_1 + a\langle e_1, e_2 \rangle \cdot \partial b \tag{3.1.7}$$

Example 3.1.15. On a manifold M, the bundle $E = TM \oplus T^*M$ has a natural Courant–Dorfman algebra structure with:

- a bilinear form given by $\langle X + \xi, Y + \zeta \rangle = \iota_X \zeta + \iota_Y \zeta$, where $X, Y \in TM$, $\xi, \zeta \in T^*M$ and $\iota_X \xi$ is the contraction of X with ξ,
- the derivation $d : C^\infty(M) \to T^*X$ given by the differential of a function,
- a right Leibniz bracket given by $[X + \xi, Y + \zeta]_E = [X, Y] - \mathcal{L}_X \zeta + \mathcal{L}_Y \xi + d(\iota_X \zeta)$ where $[\bullet, \bullet]$ is the commutator of vector fields and $\mathcal{L}$ is the Lie derivative.

Note that we have $[X + \xi, X + \xi]_E = d(\iota_X \xi)$ so that the reduced Lie algebra E_{Lie} corresponding to $E = TM \oplus T^*M$ is

$$E_{\text{Lie}} = TM \oplus T^*M / \langle \text{exact forms} \rangle$$

with Lie bracket given by

$$[X + \xi, Y + \zeta]_E = [X, Y] - \mathcal{L}_X \zeta + \mathcal{L}_Y \xi.$$

In particular, we see that the map $\pi : E \to E_{\text{Lie}}$ is not $C^\infty(M)$-linear. Note also that while $[X + \xi, df] = 0$, we have $[X + \xi, a \cdot df]_E = -\langle X, da \rangle \cdot df + \langle X, df \rangle \cdot da$. Note that the reduced Lie algebra $\mathcal{E}_{\text{Lie}}$ is not an $C^\infty(M)$-module so that $(C^\infty(M), E_{\text{Lie}})$ is not a Lie–Rinehart algebra.

3.2 Lie–Rinehart algebras in the category $\mathcal{LM}$ of linear maps

Lie–Rinehart algebras [Rin63,Her53,Hue91] (Lie algebroids [Pra67] in differential geometric context) were introduced by Herz [Her53] under the name *Lie pseudo–algebra* (also known as *Lie algebroid* [Pra67] in a differential geometric context) and has been developed and studied as a generalisation of Lie algebras. The term *Lie–Rinehart algebra* was coined by Huebschmann [Hue91], a term which acknowledges Rinehart's fundamental contributions [Rin63] to the understanding of this structure. See [Hue90, Section 1] for some historical remarks on this development.

We start by giving an overview of the category $\mathcal{LM}$ as defined by Loday and Pirashvili [LP98]. In Section 3.2.2 we give the necessary tools and background to describe the universal algebra of derivations of an algebra in $\mathcal{LM}$ (see Proposition 3.2.16). Lastly, in Section 3.2.3, we describe Lie–Rinehart algebras in the category $\mathcal{LM}$ of linear maps.

3.2.1 The category $\mathcal{LM}$ of linear maps

We start by recalling some fundamental concepts and definitions about the category $\mathcal{LM}$ of linear maps, introduced by Loday and Pirashvili in [LP98], that are relevant for our main constructions later. We refer to [LP98] for further details. See [KW14] for results on Hopf algebras in $\mathcal{LM}$.

Definition 3.2.1. The objects in the category $\mathcal{LM}$ are R-module maps $(V \xrightarrow{u} W)$, where u is called the vertical map. The morphisms between objects in $\mathcal{LM}$ are pairs of maps $\mathsf{h} := (h_1, h_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{h_1} & V' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ W & \xrightarrow{h_0} & W' \end{array}$$

Given two morphisms $\mathsf{g} := (g_1, g_0)$ and $\mathsf{h} := (h_1, h_0)$ in $\mathcal{LM}$, their composition $\mathsf{h} \circ \mathsf{g}$ is

$$\mathsf{h} \circ \mathsf{g} = (h_1, h_0) \circ (g_1, g_0) := (h_1 \circ g_1, h_0 \circ g_0). \tag{3.2.1}$$

Remark 3.2.2. *The category $\mathcal{LM}$ of linear maps can be understood as the category of truncated chain complexes of length one.*

A morphism $\phi := (\phi_1, \phi_0)$ is an isomorphism between objects $(V \overset{u}{\to} W)$ and $(V' \overset{u'}{\to} W')$ if and only if ϕ_1 and ϕ_2 are isomorphisms of R-modules.

Proposition 3.2.3. *The category $\mathcal{LM}$ of linear maps is monoidal where the tensor product of two objects is given by*

$$(V \overset{u}{\to} W) \otimes (V' \overset{u'}{\to} W') := (V \otimes W' \oplus W \otimes V' \xrightarrow{u \otimes 1_{W'} + 1_W \otimes u'} W \otimes W'),$$

and the unit object is $(\{0\} \overset{0}{\to} R)$. *Furthermore, given two morphisms* $\mathsf{g} := (g_1, g_0)$ *and* $\mathsf{h} := (h_1, h_0)$ *in* $\mathcal{LM}$, *their tensor product* $\mathsf{g} \otimes \mathsf{h}$ *is given by*

$$\mathsf{g} \otimes \mathsf{h} = (g_1, g_0) \otimes (h_1, h_0) := (g_1 \otimes h_0 + g_0 \otimes h_1, g_0 \otimes h_0). \tag{3.2.2}$$

Proof. See [LP98]. □

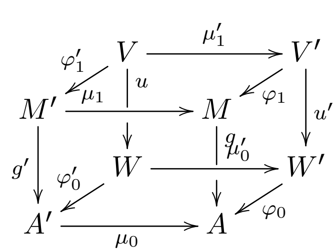
The monoidal category $\mathcal{LM}$ is symmetric, with interchange morphism $\tau := (\tau_1, \tau_0)$ denoted by $\tau : (V \overset{u}{\to} W) \otimes (V' \overset{u'}{\to} W') \to (V' \overset{u'}{\to} W') \otimes (V \overset{u}{\to} W)$ and given by the pair of maps $\tau_0 : W \otimes W' \to W' \otimes W$ and $\tau_1 : V \otimes W' \oplus W \otimes V' \to V' \otimes W \oplus W' \otimes V$ given by the standard flip maps, see [LP98]. Note that there is no change of sign prescribed in [LP98].

Commutative diagrams in $\mathcal{LM}$ can be seen as commutative "cubes" in R-mod.

Example 3.2.4. The commutative diagram in $\mathcal{LM}$ given by

$$\begin{array}{ccc} (V \overset{u}{\to} W) & \xrightarrow{\mu'} & (V' \overset{u'}{\to} W') \\ \Big\downarrow{\scriptstyle \varphi'} & & \Big\downarrow{\scriptstyle \varphi} \\ (M' \overset{g'}{\to} A') & \xrightarrow{\mu} & (M \overset{g}{\to} A) \end{array}$$

corresponds to the commuting "cube" given by



We now describe some of the fundamental algebraic structures in $\mathcal{LM}$. For further details and proofs see [LP98].

Proposition 3.2.5.

- *An* ***associative algebra object*** $(M \overset{g}{\to} A)$ *in* $\mathcal{LM}$ *is a triple consisting of an associative R-algebra A, an A-bimodule M and an A-bimodule map* $g : M \to A$*. Moreover, the algebra object* $(M \overset{g}{\to} A)$ *is commutative, if and only if the A-bimodule M is symmetric and A is commutative.*

- *A* ***Lie algebra object*** $(N \overset{f}{\to} L)$ *in* $\mathcal{LM}$ *is equivalent to a Lie algebra L, a right L-module N with right action* $N \otimes L \to N$ *given by* $n \otimes \xi \mapsto [n, \xi]$ *for all* $n \in N$ *and* $\xi \in L$*, and an R-linear L-equivariant map* $f : N \to L$*, i.e.,* $f([n, \xi]) = [f(n), \xi]_L$*.*

Example 3.2.6. We give the following examples of objects in $\mathcal{LM}$:

- The surjective map $\pi : \mathcal{E} \to \mathcal{E}_{\text{Lie}}$ is a Lie algebra object in $\mathcal{LM}$.

- Let I be a two-sided ideal in an associative algebra A. The identity map $\text{id} : I \to A$ is an associative algebra in $\mathcal{LM}$.

We now focus on the description of $(M \overset{g}{\to} A)$-modules in $\mathcal{LM}$:

Proposition 3.2.7. *Let* $(M \overset{g}{\to} A)$ *be an algebra object. A* ***left*** $(M \overset{g}{\to} A)$***-module object*** *is a map* $(V \overset{u}{\to} W)$ *of left A-modules together with an A-module map* $\mu_1 : W \otimes M \to V$ *satisfying* $g \circ \mu_1(m \otimes w) = \mu_0(g(m) \otimes w)$ *for* $w \in W$ *and* $m \in M$*, which descends to an A-module map* $\alpha_\ell^V : M \otimes_A W \to V$ *(called* ***structure map*** *of the left* $(M \overset{g}{\to} A)$*-module* $(V \overset{u}{\to} W)$*) satisfying*

$$u \circ \alpha_\ell^V (m \otimes_A w) = \mu_0 \circ (g(m) \otimes_A w). \tag{3.2.3}$$

Proof. Since $(V \overset{u}{\to} W)$ is a left $(M \overset{g}{\to} A)$-module, there exists a morphism

$$\mu : (M \overset{g}{\to} A) \otimes (V \overset{u}{\to} W) \longrightarrow (V \overset{u}{\to} W),$$

that is, a commuting square

$$\begin{array}{ccc} M \otimes W \oplus A \otimes V & \xrightarrow{\mu_1} & V \\ {\scriptstyle g \otimes 1_W + 1 \otimes u} \downarrow & & \downarrow {\scriptstyle u} \\ A \otimes W & \xrightarrow{\mu_0} & W \end{array} \tag{3.2.4}$$

where μ_1 and μ_0 satisfy some associativity conditions. From (3.2.4) we see that μ_0 and the restriction of μ_1 to $A \otimes V$ turn W and V respectively into left A-modules, so that the vertical map u becomes a map of left A-modules. Now, since M is an A-bimodule and W is a left A-module, we can construct the tensor product $M \otimes_A W$. Moreover, by the associativity of the module action μ, we deduce that μ_1 vanishes on $m \cdot a \otimes w - m \otimes a \cdot w$ where $m \in M$, $a \in A$, $w \in W$ so that the map $\mu_1 : M \otimes W \to V$ descends to a map $\alpha_\ell^V : M \otimes_A W \to V$ yielding the following diagram:

$$\begin{array}{ccc} M \otimes_A W & \xrightarrow{\alpha_\ell^V} & V \\ {\scriptstyle g \otimes_A 1_W} \big\downarrow & & \big\downarrow {\scriptstyle u} \\ A \otimes_A W & \xrightarrow{\mu_0} & W \end{array} \tag{3.2.5}$$

The commutativity of (3.2.5) ensures that the compatibility relation (3.2.3) is satisfied. □

Similarly, a **right** $(M \overset{g}{\to} A)$-module is an object $(V' \overset{u'}{\to} W')$ where $u' : V' \to W'$ is a map of right A-modules and $\alpha_r^V : W \otimes_A M \to V$ is an A-module map (called the **structure map** of the right $(M \overset{g}{\to} A)$-module $(V' \overset{u'}{\to} W')$) satisfying

$$u \circ \alpha_r^V (w' \otimes_A m) = \mu_0 \circ (w' \otimes_A g(m)). \tag{3.2.6}$$

Example 3.2.8. The associative algebra $(M \overset{g}{\to} A)$ is a bimodule over itself where the structure maps $\alpha_\ell^M : M \otimes_A A \to M$ and $\alpha_r^M : A \otimes_A M \to M$ are both induced by the identity map id_M.

Proposition 3.2.9. *An object $(V \overset{u}{\to} W)$ admits a left (respectively right) $(A \overset{id}{\to} A)$-module structure if and only if V and W are left (respectively right) A-modules and there exists an A-module map $\alpha : W \to V$ which is a section of the (A-linear) map $u : V \to W$.*

Proof. From (3.2.5) we observe that a left $(A \overset{id}{\to} A)$-module structure on $(V \overset{u}{\to} W)$ yields a diagram

$$\begin{array}{ccc} A \otimes_A W & \xrightarrow{\alpha_\ell^V} & V \\ {\scriptstyle 1 \otimes_A 1_W} \big\downarrow & & \big\downarrow {\scriptstyle u} \\ A \otimes_A W & \xrightarrow{\mu_0} & W \end{array} \tag{3.2.7}$$

which since $A \otimes_A W \cong W$ can be written as

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & V \\ & \searrow^{1_W} & \downarrow u \\ & & W \end{array} \tag{3.2.8}$$

where $\alpha : W \to V$ satisfies $u \circ \alpha = 1_W$ and is hence a section of u. □

Corollary 3.2.10. *Let $(V \overset{u}{\to} W)$ be a map of A-modules and $I \subset A$ be an ideal in A. The object $(V \overset{u}{\to} W)$ is an $(I \overset{id}{\to} A)$-module structure iff u splits as an I-module map:*

$$\begin{array}{ccc} I \otimes_A W & \xrightarrow{\alpha} & V \\ & \searrow^{1_W} & \downarrow u \\ & & W \end{array} \tag{3.2.9}$$

Proposition 3.2.11. *For any Lie algebra object $(N \overset{f}{\to} L)$ endowed with an $(M \overset{g}{\to} A)$-module structure, the Leibniz bracket on N given by $[n_1, n_2]_N := [n_1, f(n_2)]$ satisfies*

$$[n_1, a \cdot [n_2, n_2]_N]_N = 0, \quad \forall a \in A. \tag{3.2.10}$$

Proof. By definition [LP98], a Lie algebra object $(N \overset{f}{\to} L)$ carries a Leibniz algebra structure on the L-module N with bracket given by $[n_1, n_2]_N := [n_1, f(n_2)]$ so that $[n_1, n_2]_N = 0$ for all $n_2 \in \mathrm{Ker}(f)$. Now, since $(N \overset{f}{\to} L)$ is an $(M \overset{g}{\to} A)$-module, N and L are A-modules and the vertical map f is A-linear, we deduce that $\mathrm{Ker}(f)$ is an A-module. Hence $[\bullet, \bullet]_N$ satisfies condition (3.2.10). □

3.2.2 The Lie algebra of derivations in $\mathcal{LM}$

In this section we describe the Lie algebra of derivations of an associative algebra object in $\mathcal{LM}$. We first describe morphisms between Lie algebra objects and Lie algebra actions.

Proposition 3.2.12. *An algebra morphism $\phi : (M \overset{g}{\to} A) \to (M' \overset{g'}{\to} A')$ in $\mathcal{LM}$ is given by a pair of maps (ϕ_1, ϕ_0) satisfying*

$$\phi_1(a_1 m_1 + m_2 a_2) = \phi_0(a_1)\phi_1(m_1) + \phi_1(m_2)\phi_0(a_1) \tag{3.2.11}$$

$$\phi_0(a_1 \cdot a_2) = \phi_0(a_1) \cdot \phi_0(a_2) \tag{3.2.12}$$

for $a_1, a_2 \in A$ and $m_1, m_2 \in M$.

Proof. Assume $\phi := (\phi_1, \phi_0)$ is an algebra morphism (in $\mathcal{LM}$). Then using (3.2.1) and (3.2.2), a straightforward computation shows

$$0 = \phi \circ \mu - \mu \circ (\phi \otimes \phi) = (\phi_1 \circ \mu_1 - \mu_1 \circ (\phi_1 \otimes \phi_0), \phi_0 \circ \mu_0 - \mu_0 \circ (\phi_0 \otimes \phi_0)),$$

that is:

$$\phi_1 \circ \mu_1 - \mu_1 \circ (\phi_1 \otimes \phi_0) = 0, \quad \phi_0 \circ \mu_0 - \mu_0 \circ (\phi_0 \otimes \phi_0) = 0 \tag{3.2.13}$$

which yield the relations in (3.2.11) and (3.2.12). □

Proposition 3.2.13. *A Lie algebra map* $\mathrm{a} : (N \xrightarrow{f} L) \to (N' \xrightarrow{f'} L')$ *in* $\mathcal{LM}$ *is given by a pair of maps* (a_1, a_0) *satisfying*

$$a_1([n, \xi]) = [a_1(n), a_0(\xi)], \quad a_0([\xi, \zeta]_L) = [a_0(\xi), a_0(\zeta)]_{L'} \tag{3.2.14}$$

for $n \in N$ *and* $\xi, \zeta \in L$.

Proof. An identical argument as in the proof for Proposition 3.2.12 and abusing notation so that μ is the Lie bracket on $(N \xrightarrow{f} L)$ we obtain the relation in (3.2.13) which, in this case, yields the relations in (3.2.14). □

Proposition 3.2.14. *Given a Lie algebra object* $(N \xrightarrow{f} L)$*, a left* $(N \xrightarrow{f} L)$*-module (in* $\mathcal{LM}$ *) is an object* $(V \xrightarrow{u} W)$ *such that* V *and* W *are left* L*-modules with actions given by*

$$\alpha_0 : L \otimes W \longrightarrow W, \quad \alpha_2 : L \otimes V \longrightarrow V \tag{3.2.15}$$

and there exists an R*-linear map*

$$\alpha_1 : N \otimes W \longrightarrow V \tag{3.2.16}$$

satisfying the following compatibility condition:

$$\alpha_1([n, \xi] \otimes w) = \alpha_1(n \otimes \alpha_0(\xi \otimes w)) - \alpha_2(\xi \otimes \alpha_1(n \otimes w)). \tag{3.2.17}$$

Moreover, the following compatibility conditions between u*,* f*,* α_0*,* α_1 *and* α_2 *are satisfied:*

$$u \circ \alpha_1 = \alpha_0 \circ (f \otimes 1_W), \quad u \circ \alpha_2 = \alpha_0 \circ (1_L \otimes u).$$

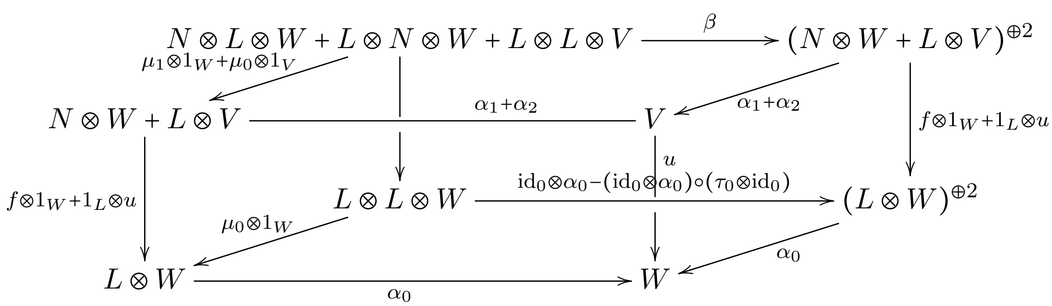
Proof. Since $(V \xrightarrow{u} W)$ is a left $(N \xrightarrow{f} L)$-module, we have a morphism

$$\alpha : (N \xrightarrow{f} L) \otimes (V \xrightarrow{u} W) \longrightarrow (V \xrightarrow{u} W)$$

that is, a pair of maps $(\alpha_1 + \alpha_2, \alpha_0)$ where $\alpha_0, \alpha_1, \alpha_2$ are given by the maps in (3.2.15) and (3.2.16), such that the diagram commutes

$$\begin{array}{ccc} (N \xrightarrow{f} L) \otimes (N \xrightarrow{f} L) \otimes (V \xrightarrow{u} W) & \xrightarrow{\mathrm{id}\otimes\alpha - (\mathrm{id}\otimes\alpha)\circ(\tau\otimes\mathrm{id})} & \Big((N \xrightarrow{f} L) \otimes (V \xrightarrow{u} W)\Big)^{\oplus 2} \\ \Big\downarrow{\scriptstyle \mu\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \alpha} \\ (N \xrightarrow{f} L) \otimes (V \xrightarrow{u} W) & \xrightarrow{\quad\alpha\quad} & (V \xrightarrow{u} W) \end{array} \tag{3.2.18}$$

which can be seen as the following diagram in cube shape:



where

$$\beta := \mathrm{id}_1 \otimes \alpha_0 - (\mathrm{id}_0 \otimes \alpha_1) \circ (\tau_1 \otimes \mathrm{id}_0) + \mathrm{id}_0 \otimes \alpha_1 - (\mathrm{id}_1 \otimes \alpha_0) \circ (\tau_1 \otimes \mathrm{id}_0) + \mathrm{id}_0 \otimes \alpha_2 - (\mathrm{id}_0 \otimes \alpha_2) \circ (\tau_0 \otimes \mathrm{id}_1)$$

Using (3.2.1) and (3.2.2), a long but straightforward computation shows that the compatibility relation making (3.2.18) commute can be expressed as

$$\begin{aligned} 0 &= \alpha \circ (\mathrm{id} \otimes \alpha - (\mathrm{id} \otimes \alpha) \circ (\tau \otimes \mathrm{id})) - \alpha \circ (\mu \otimes \mathrm{id}) \\ &= \Big(\alpha_1 \circ (\mathrm{id}_1 \otimes \alpha_0) - \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_1) \circ (\tau_1 \otimes \mathrm{id}_0) + \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_1) \\ &\qquad\qquad - \alpha_1 \circ (\mathrm{id}_1 \otimes \alpha_0) \circ (\tau_1 \otimes \mathrm{id}_0) + \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_2) - \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_2) \circ (\tau_0 \otimes \mathrm{id}_1), \\ &\qquad \alpha_0 \circ (\mathrm{id}_0 \otimes \alpha_0) - \alpha_0 \circ (\mathrm{id}_0 \otimes \alpha_0) \circ (\tau_0 \otimes \mathrm{id}_0)\Big) \\ &\quad - \Big(\alpha_1 \circ (\mu_1 \otimes \mathrm{id}_0) + \alpha_2 \circ (\mu_0 \otimes \mathrm{id}_1), \alpha_0 \circ (\mu_0 \otimes \mathrm{id}_0)\Big) \\ &= \Big(\alpha_1 \circ (\mathrm{id}_1 \otimes \alpha_0) - \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_1) \circ (\tau_1 \otimes \mathrm{id}_0) - \alpha_1 \circ (\mu_1 \otimes \mathrm{id}_0) \\ &\quad + \Big(\alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_1) - \alpha_1 \circ (\mathrm{id}_1 \otimes \alpha_0) \circ (\tau_1 \otimes \mathrm{id}_0) - \alpha_1 \circ (\mu_1 \otimes \mathrm{id}_0) \\ &\quad + \Big(\alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_2) - \alpha_2 \circ (\mathrm{id}_0 \otimes \alpha_2) \circ (\tau_0 \otimes \mathrm{id}_1) - \alpha_2 \circ (\mu_0 \otimes \mathrm{id}_1), \\ &\qquad \alpha_0 \circ (\mathrm{id}_0 \otimes \alpha_0) - \alpha_0 \circ (\mathrm{id}_0 \otimes \alpha_0) \circ (\tau_0 \otimes \mathrm{id}_0) - \alpha_0 \circ (\mu_0 \otimes \mathrm{id}_0)\Big) \end{aligned}$$

so that the following diagrams commute

- A diagram encoding the L-module action $\alpha_0 : L \otimes W \to W$:

$$\begin{array}{ccc} L \otimes L \otimes W & \xrightarrow{\mathrm{id}_0 \otimes \alpha_0 - (\mathrm{id}_0 \otimes \alpha_0) \circ (\tau_0 \otimes \mathrm{id}_0)} & L \otimes W \oplus L \otimes W \\ {\scriptstyle \mu_0 \otimes 1_W} \downarrow & & \downarrow {\scriptstyle \alpha_0} \\ L \otimes W & \xrightarrow{\alpha_0} & W \end{array}$$

- A diagram encoding the L-module action $\alpha_2 : L \otimes V \to V$:

$$\begin{array}{ccc} L \otimes L \otimes V & \xrightarrow{\mathrm{id}_0 \otimes \alpha_2 - (\mathrm{id}_0 \otimes \alpha_2) \circ (\tau_0 \otimes \mathrm{id}_1)} & L \otimes V \oplus L \otimes V \\ {\scriptstyle \mu_0 \otimes 1_V} \downarrow & & \downarrow {\scriptstyle \alpha_2} \\ L \otimes V & \xrightarrow{\alpha_2} & V \end{array}$$

- And lastly

$$\begin{array}{ccc} N \otimes L \otimes W + L \otimes N \otimes W & \xrightarrow[+\mathrm{id}_0 \otimes \alpha_1 - (\mathrm{id}_1 \otimes \alpha_0) \circ (\tau_1 \otimes \mathrm{id}_0)]{\mathrm{id}_1 \otimes \alpha_0 - (\mathrm{id}_0 \otimes \alpha_1) \circ (\tau_1 \otimes \mathrm{id}_0)} & (N \otimes W \oplus L \otimes V)^{\oplus 2} \\ {\scriptstyle \mu_1 \otimes 1_W} \downarrow & & \downarrow {\scriptstyle \alpha_1 + \alpha_2} \\ N \otimes W & \xrightarrow{\alpha_1} & V \end{array}$$

  encoding the compatibility relation in (3.2.16). □

By the adjoint functor property of tensor products, the maps in (3.2.15) correspond to the maps

$$\alpha_0 : L \to \mathrm{Hom}_R(W, W), \quad \alpha_1 : N \to \mathrm{Hom}_R(W, V), \quad \alpha_2 : L \to \mathrm{Hom}_R(V, V)$$

that we can describe as the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\alpha_1} & \mathrm{Hom}_R(W, V) \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle g \circ h + h \circ g} \\ L & \xrightarrow{\alpha_0 + \alpha_2} & \mathrm{Hom}_R(W, W) \oplus \mathrm{Hom}_R(V, V) \end{array} \tag{3.2.19}$$

where $h \in \mathrm{Hom}_R(W, V)$.

Proposition 3.2.15. *Let $(M \xrightarrow{g} A)$ be a commutative algebra object in $\mathcal{LM}$. A Lie algebra object $(N \xrightarrow{f} L)$ acts on $(M \xrightarrow{g} A)$ by derivations if there exist two Lie algebra maps*

$$\rho_0 : L \to \mathrm{Der}_R(A), \quad \rho_2 : L \to H := (\mathrm{Hom}_R(M, M), [\bullet, \bullet]_H) \tag{3.2.20}$$

satisfying the compatibility conditions

$$\rho_2(\xi)(a \cdot m) = a \cdot \rho_2(\xi)(m) + \rho_0(\xi)(a) \cdot m, \quad g\left(\rho_2(\xi)(m)\right) = \rho_0(\xi)(g(m)) \tag{3.2.21}$$

and an R-module map

$$\rho_1 : N \to \mathrm{Der}_R(A, M) \tag{3.2.22}$$

satisfying

$$\rho_1([n, \xi]) = [\rho_1(n), (\rho_0 + \rho_2)(\xi)], \quad g(\rho_1(n)(a)) = \rho_0(f(n))(a) \tag{3.2.23}$$

for all $\xi \in L$, $a \in A$ and $m \in M$.

Proof. Let a Lie algebra $(N \xrightarrow{f} L)$ act on $(M \xrightarrow{g} A)$ by derivations, then there exists a left $(N \xrightarrow{f} L)$-module structure on $(M \xrightarrow{g} A)$, denoted by $\rho : (N \xrightarrow{f} L) \otimes (M \xrightarrow{g} A) \to (M \xrightarrow{g} A)$, which by Proposition 3.2.14 endows M and A with left L-actions given by $\varrho_0 : L \otimes A \to A$ and $\varrho_2 : L \otimes M \to M$ respectively, and induces a map $\varrho_1 : N \otimes A \to M$ satisfying (3.2.17). Furthermore, the action of $(N \xrightarrow{f} L)$ on $(M \xrightarrow{g} A)$ by derivations makes the following diagram commute:

$$\begin{array}{ccc}
(N \xrightarrow{f} L) \otimes (M \xrightarrow{g} A) \otimes (M \xrightarrow{g} A) & \xrightarrow{\rho \otimes \mathrm{id} + (\mathrm{id} \otimes \rho) \circ (\tau \otimes \mathrm{id})} & (M \xrightarrow{g} A)^{\otimes 2} \oplus (M \xrightarrow{g} A)^{\otimes 2} \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \mu} & & \Big\downarrow {\scriptstyle \mu} \\
(N \xrightarrow{f} L) \otimes (M \xrightarrow{g} A) & \xrightarrow{\quad\quad \rho \quad\quad} & (M \xrightarrow{g} A)
\end{array}$$

Since $\rho = (\varrho_1 + \varrho_2, \varrho_0)$, by (3.2.1) and (3.2.2) we find

$$\begin{aligned}
0 &= \mu \circ \big(\rho \otimes \mathrm{id} + (\mathrm{id} \otimes \rho) \circ (\tau \otimes \mathrm{id})\big) - \rho \circ (\mathrm{id} \otimes \mu) \\
&= \big(\mu_1 \circ (\varrho_1 \otimes \mathrm{id}_0) + \mu_1 \circ (\mathrm{id}_0 \otimes \varrho_1) \circ (\tau_0 \otimes \mathrm{id}_1) - \rho_1 \circ (\mathrm{id}_1 \otimes \mu_0) \\
&\quad + \mu_1 \circ (\varrho_2 \otimes \mathrm{id}_0) + \mu_1 \circ (\mathrm{id}_1 \otimes \varrho_0) \circ (\tau_1 \otimes \mathrm{id}_0) + \mu_1 \circ (\varrho_0 \circ \mathrm{id}_1) + \mu_1 \circ (\mathrm{id}_0 \otimes \varrho_2) \circ (\tau_1 \otimes \mathrm{id}_0) \\
&\quad - \varrho_2 \circ (\mathrm{id}_1 \otimes \mu_0), \mu_0 \circ (\varrho_0 \otimes \mathrm{id}_0) + \mu_0 \circ (\mathrm{id}_0 \otimes \varrho_0) \circ (\tau_0 \otimes \mathrm{id}_0) - \varrho_0 \circ (\mathrm{id}_0 \otimes \mu_0)\big)
\end{aligned}$$

so the following diagrams commute:

- a diagram which encodes the universal action of a Lie algebra L on an R-algebra A by derivations:

$$\begin{array}{ccc} L\otimes A\otimes A & \xrightarrow{\varrho_0\otimes \mathrm{id}_0+(\mathrm{id}_0\otimes\varrho_0)\circ(\tau_0\otimes\mathrm{id}_0)} & (A\otimes A)^{\oplus 2} \\ \downarrow{\scriptstyle 1_L\otimes\mu_0} & & \downarrow{\scriptstyle \mu_0} \\ L\otimes A & \xrightarrow{\varrho_0} & A \end{array}$$

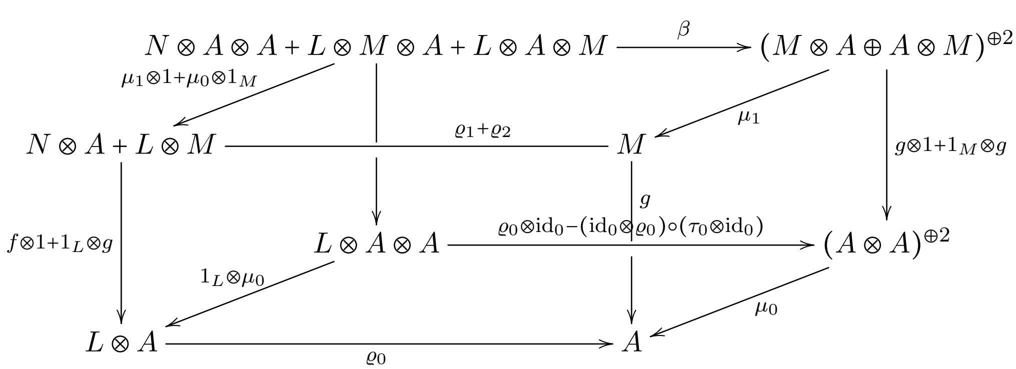
- a diagram encoding the action of N on A:

$$\begin{array}{ccc} N\otimes A\otimes A & \xrightarrow{\varrho_1\otimes \mathrm{id}_0+(\mathrm{id}_1\otimes\varrho_1)\circ(\tau_0\otimes\mathrm{id}_0)} & (M\otimes A\oplus A\otimes M)^{\oplus 2} \\ \downarrow{\scriptstyle 1_N\otimes\mu_0} & & \downarrow{\scriptstyle \mu_1} \\ N\otimes A & \xrightarrow{\varrho_1} & M \end{array}$$

- and lastly, a commutative diagram encoding the action of the Lie algebra L on both A and M:

$$\begin{array}{ccc} L\otimes M\otimes A\oplus L\otimes A\otimes M & \xrightarrow[+\varrho_0\otimes\mathrm{id}_1+(\mathrm{id}_0\otimes\varrho_2)\circ(\tau_0\otimes\mathrm{id}_1)]{\varrho_2\otimes \mathrm{id}_0+(\mathrm{id}_1\otimes\varrho_0)\circ(\tau_1\otimes\mathrm{id}_0)} & (M\otimes A\oplus A\otimes M)^{\oplus 2} \\ \downarrow{\scriptstyle 1_L\otimes\mu_1} & & \downarrow{\scriptstyle \mu_1} \\ L\otimes M & \xrightarrow{\varrho_2} & M \end{array}$$

These maps make the following cube commute:



By the adjoint functor property of tensor products, the maps $\varrho_1, \varrho_2, \varrho_3$ are equivalent to the maps in (3.2.20) and (3.2.22). $\square$

Proposition 3.2.16. *Let* $(M \xrightarrow{g} A)$ *be a commutative algebra object in the category* $\mathcal{LM}$*, and let* $H = \mathrm{Hom}_R(M, M)$*. Let the* R*-module maps* $\pi_1 : \mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A)$ *and* $\pi_2 : \mathrm{Der}_R(A, M) \to H$ *be given by* $\pi_1(\partial) := g \circ \partial$ *and* $\pi_2(\partial) := \partial \circ g$ *respectively. The universal* ***Lie algebra of derivations*** *of the algebra object* $(M \xrightarrow{g} A)$ *is given by*

$$\mathrm{Der}_{\mathcal{LM}}\Big((M \xrightarrow{g} A)\Big) = \Big(\mathrm{Der}_R(A, M) \xrightarrow{\pi_1+\pi_2} \mathrm{Der}_R(A) \oplus H\Big)$$

where the Lie bracket on $\mathrm{Der}_R(A) \oplus H$ *is*

$$[(\alpha,\beta),(\alpha',\beta')]_{\mathrm{Der}_R(A)\oplus H} := \big([\alpha,\alpha']_{\mathrm{Der}_R(A)}, -[\beta,\beta']_H\big)$$

and the right $\mathrm{Der}_R(A) \oplus H$*-module structure on* $\mathrm{Der}_R(A, M)$ *is given by*

$$\partial \otimes (\alpha,\beta) \longmapsto [\partial,(\alpha,\beta)] := \partial \circ \alpha - \beta \circ \partial. \tag{3.2.24}$$

Proof. We first check that the action $[\bullet,\bullet]$ in (3.2.24) endows $\mathrm{Der}_R(A, M)$ with a right Lie algebra module structure over (the Lie algebra) $\mathrm{Der}_R(A) \oplus H$:

On the one hand we have

$$\begin{aligned}\big[\partial, [(\alpha,\beta),(\alpha',\beta')]_{\mathrm{Der}_R(A)\oplus H}\big] &= \big[\partial, \big([\alpha,\alpha']_{\mathrm{Der}_R(A)}, [\beta,\beta']_H\big)\big] \\ &= \partial \circ [\alpha,\alpha']_{\mathrm{Der}_R(A)} - [\beta,\beta']_H \circ \partial.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\big[[\partial,(\alpha,\beta)],(\alpha',\beta')\big] &= [(\partial\circ\alpha - \beta\circ\partial),(\alpha',\beta')] = (\partial\circ\alpha - \beta\circ\partial)\circ\alpha' - \beta'\circ(\partial\circ\alpha - \beta\circ\partial) \\ &= \partial\circ\alpha\circ\alpha' - \beta\circ\partial\circ\alpha' - \beta'\circ\partial\circ\alpha + \beta'\circ\beta\circ\partial\end{aligned}$$

so that

$$\begin{aligned}\big[[\partial,(\alpha,\beta)],(\alpha',\beta')\big] - \big[[\partial,(\alpha',\beta')],(\alpha,\beta)\big] &= \partial\circ\alpha\circ\alpha' - \beta\circ\partial\circ\alpha' - \beta'\circ\partial\circ\alpha + \beta'\circ\beta\circ\partial \\ &\quad - \partial\circ\alpha'\circ\alpha + \beta'\circ\partial\circ\alpha + \beta\circ\partial\circ\alpha' - \beta\circ\beta'\circ\partial \\ &= \partial\circ[\alpha,\alpha']_{\mathrm{Der}_R(A)} - [\beta,\beta']_H\circ\partial \\ &= \big[\partial,\big([\alpha,\alpha']_{\mathrm{Der}_R(A)},[\beta,\beta']_H\big)\big].\end{aligned}$$

This shows that the right action of $\mathrm{Der}_R(A) \oplus H$ on $\mathrm{Der}_R(A, M)$ is well defined. Hence, the morphism $\rho : (N \xrightarrow{f} L) \to \mathrm{Der}_{\mathcal{LM}}\big((M \xrightarrow{g} A)\big)$ is given by the following diagram, which follows from (3.2.19)

$$\begin{array}{ccc} N & \xrightarrow{\rho_1} & \mathrm{Der}_R(A, M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_1 + \pi_2} \\ L & \xrightarrow{\rho_0 + \rho_2} & \mathrm{Der}_R(A) \oplus H. \end{array}$$

which commutes. □

Remark 3.2.17. The morphism $\rho : (N \xrightarrow{f} L) \to \mathrm{Der}_{\mathcal{LM}}\big((M \xrightarrow{g} A)\big)$ given by $(\rho_1, \rho_0 + \rho_2)$ is a morphism of Lie algebras in $\mathcal{LM}$.

Example 3.2.18. The universal Lie algebra of derivations of the commutative algebra object $(A \xrightarrow{id} A)$ is

$$\big(\mathrm{Der}_R(A) \longrightarrow \mathrm{Der}_R(A) \oplus \mathrm{Der}_R(A)\big).$$

Then, the action of a Lie algebra object $(N \xrightarrow{f} L)$ by derivations on $(A \xrightarrow{id} A)$ is given by

- a Lie algebra map $\rho_0 \equiv \rho_2 : L \to \mathrm{Der}_R(A)$,
- an A-module map $\rho_1 : N \to \mathrm{Der}_R(A)$

satisfying $\rho_1(n) = \rho_0(f(n))$.

3.2.3 Lie–Rinehart algebra objects in $\mathcal{LM}$

A Lie–Rinehart algebra [Hue91, Rin63] is an algebraic structure which encompasses a Lie algebra and a commutative algebra which act on each other in a way that both actions are compatible. This object can be described in any symmetric monoidal category.

In this Section we focus on the description of Lie–Rinehart algebra objects in the category $\mathcal{LM}$ of linear maps. Based on [LP98, Lemma 3.6] we give a proof of Theorem A.

Proof of Theorem A. Assume the pair $\Big((M \xrightarrow{g} A), (N \xrightarrow{f} L)\Big)$ is a Lie–Rinehart algebra object. Then there exist

- a left $(M \xrightarrow{g} A)$-module structure on the Lie algebra object $(N \xrightarrow{f} L)$,
- an action ρ of the Lie algebra object $(N \xrightarrow{f} L)$ on the commutative algebra object $(M \xrightarrow{g} A)$ by derivations,

and a compatibility condition between these two actions. We now describe what these structures involve.

Firstly, by Proposition 3.2.7, we deduce that a left $(M \overset{g}{\to} A)$-module structure on the Lie algebra object $(N \overset{f}{\to} L)$ turns the L-equivariant map $f : N \to L$ into an A-module map. Also, it yields an A-module map $\lambda := \alpha_\ell^N : M \otimes_A L \to N$. Moreover the Leibniz algebra structure on N given by $[\bullet, \bullet]_N$ must satisfy $[n_1, a \cdot [n_2, n_2]_N]_N = 0$ for all $a \in A$ and $n_1, n_2 \in N$.

Secondly, by Proposition 3.2.15, an action ρ of the Lie algebra object $(N \overset{f}{\to} L)$ on the commutative algebra $(M \overset{g}{\to} A)$ by derivations yields

- an A-linear Lie algebra map $\rho_0 : L \to \mathrm{Der}_R(A)$,
- an A-module map $\rho_1 : N \to \mathrm{Der}_R(A, M)$,
- an A-linear Lie algebra map $\rho_2 : L \to \mathrm{Hom}_R(M, M)$

satisfying conditions (3.2.21), (3.2.23), and turns $g : M \to A$ into an L-equivariant map. Lastly, the following compatibility conditions between the two module structures are satisfied:

- the pair (A, L) is a Lie–Rinehart algebra with anchor ρ_0,
- the right L-action on the A-module N satisfies

$$[a \cdot n, b \cdot \xi] = a \cdot [n, b \cdot \xi] - b \cdot \rho_0(\xi)(a) \cdot n \tag{3.2.25}$$

  which provides a compatibility relation between the right L-action on N, given by $[\bullet, \bullet]$ and the A-module structure on N.

Note that ρ_2 endows M with is a left (A, L)-module structure, see [Hue98] for further details. □

Example 3.2.19. Let (A, L) be a Lie–Rinehart algebra with anchor $\rho_L : L \to \mathrm{Der}_R(A)$. The pair $\left((A \overset{id}{\to} A), (L \overset{\mathrm{id}}{\to} L)\right)$ is a Lie–Rinehart algebra object in $\mathcal{LM}$ with

$$\rho_0 \equiv \rho_1 \equiv \rho_2 = \rho_L$$

and $\lambda \equiv id$.

Example 3.2.20. The pair

$$\Big((M \overset{g}{\to} A), \Big(\mathrm{Der}_R(A, M) \xrightarrow{\pi_1+\pi_2} \mathrm{Der}_R(A) \oplus H\Big)\Big)$$

is a Lie–Rinehart algebra object in $\mathcal{LM}$ with

$$\rho_0 : \mathrm{Der}_R(A) \oplus H \to \mathrm{Der}_R(A),$$
$$\rho_1 : \mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A, M),$$
$$\rho_2 : \mathrm{Der}_R(A) \oplus H \to H.$$

3.3 Leibniz algebroids and Lie–Rinehart algebras in $\mathcal{LM}$

In this short section we prove Theorem B. This result provides a functorial relation from Lie–Rinehart algebras to Leibniz algebroids.

Proof of Theorem B. First note that N is both a right L-module and a left A-module. Also, recall from [LP98] that the right L-module N becomes a Leibniz algebra by defining the bracket $[n_1, n_2]_N = [n_1, f(n_2)]$. Furthermore, since M is a left (A, L)-module, by Proposition 3.1.7 we can endow the A-module $M \oplus L$ with a Leibniz algebra structure given by the bracket (B.2). Since $f : N \to L$ and $\rho_0 : L \to \mathrm{Der}_R(A)$ are A-linear maps, the map $\rho_{M\oplus N} = -\rho_0 \circ f$ is also A-linear. We now prove that $\rho_{M\oplus N}$ is a Leibniz algebra antihomomorphism:

$$\begin{aligned}\rho_N([n_1, n_2]_N) = \rho_N([n_1, f(n_2)]) = -\rho_0(f([n_1, f(n_2)])) = -\rho_0([f(n_1), f(n_2)]_L)\\ = [\rho_0(f(n_2)), \rho_0(f(n_1))]_{\mathrm{Der}_k(A)} = [\rho_N(n_2), \rho_N(n_1)]_{\mathrm{Der}_k(A)}\end{aligned}$$

so the relations in (B.1) hold. We now prove that the Leibniz rule in (3.1.5) hold:

$$\begin{aligned}[a\cdot(m_1+n_1), m_2+n_2]_{M\oplus N} &= -\rho_2(f(n_2))(a\cdot m_1) + [a\cdot n_1, f(n_2)]\\ &= -a\cdot\rho_2(f(n_2))\cdot m_1 - \rho_0(f(n_2))(a)\cdot m_1\\ &\quad + a\cdot[n_1, f(n_2)] - \rho_0(f(n_2))(a)\cdot n_1\\ &= a\cdot(-\rho_2(f(n_2))\cdot m_1 + [n_1, f(n_2)]) + \rho_{M\oplus N}(a)\cdot(m_1+n_1)\\ &= a\cdot[m_1+n_1, m_2+n_2]_{M\oplus N} + \rho_{M\oplus N}(m_2+n_2)(a)\cdot(m_1+n_1).\ \square\end{aligned}$$

Example 3.3.1. Given the Lie–Rinehart algebra

$$\Big((M \overset{g}{\to} A), \Big(\mathrm{Der}_R(A, M) \xrightarrow{\pi_1+\pi_2} \mathrm{Der}_R(A) \oplus H\Big)\Big)$$

in $\mathcal{LM}$, the pair $(A, \mathrm{Der}_R(A, M))$ is a Leibniz algebroid.

Example 3.3.2. Given a Lie–Rinehart algebra (A, L), the pair

$$\Big(((A \overset{id}{\to} A), (A \oplus L \to L)\Big)$$

is a Lie–Rinehart algebra object in $\mathcal{LM}$ with corresponding Leibniz algebroid given by $(A, A \oplus L)$ with structure presented in Proposition 3.1.11.

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Index