AN INFORMATION PROCESSING APPROACH TO THE INVESTIGATION OF MATHEMATICAL PROBLEM SOLVING AT SECONDARY AND UNIVERSITY LEVELS

VOLUME 2

BY

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CHAPTER EIGHT

An experiment on Paper II

The analysis of Paper II of the preliminary exam of five schools and the SCE Examination leads to the identification of some factors which may affect questions' difficulty. Some examples of these factors are the quality of diagrams, ordering of information, degree of familiarity, need for strategy, noise (in sense of irrelevant information), abstract form of mode and idea, need for complex reasoning, three dimensions. In order to test some of these factors, an experiment on Paper II was conducted among Higher-Grade pupils of five schools and first-year students of Glasgow University. The sample size was 223 and its distribution is given in Table 137.

1. The experimental material

The experimental material was produced in the light of the results obtained from the analysis of school preliminary exams and the SCE Examination. It had to be designed in a form suitable for use in a classroom situation. The material was checked by three principal teachers of mathematics. This was done to ensure that the content and language (words, graphs and symbols) were appropriate to the SCE mathematics syllabus, that the time allocation would be enough (around 80 minutes), that the material was attractively presented and that the format would be easy to administer.

2. The techniques used in the experiment

Four techniques were used in the experiment. They are:

(a) change the formulation of a question;

(b) test the subjects' understanding of the structure of a question;

(c) structural communication;

(d) test subjects' knowledge and confidence rating.

(a) Change the formulation of a question

In order to apply this technique, an original question was changed in several ways such as by:

(i) modifying and sequencing the information of the question;

(ii) converting implicit parts of the question to explicit ones;

(iii) dividing the question into subparts;

(iv) using appropriate notation;

(v) correcting or drawing a diagram;

(vi) giving the first step.

Note that in many cases more than one of the above changes was made.

(b) Test the subjects' understanding of the structure of a question

This technique was derived from three different sources.

First, according to Shuard and Rothery [135], mathematical text must have a clear **story line** or a **flow of meaning**. This leads the reader to reach some new understanding of mathematics through his reading. In order to analyse the flow of meaning in the text, they used a flow of meaning diagram (adopted from the Schools Council Project [139]). They suggested that a text should be **segmented into meaning units**. A flow diagram can then be drawn to make clear the flow of meaning through the argument. In drawing such a diagram, a distinction between three types of meaning should be made:

(i) statements in the text;

(ii) statements, not in the text, which can be discovered from questions asked and tasks provided in the text;

(iii) statements, not in the text and not discoverable from questions asked or tasks provided, which need to be inferred from the text or to be brought from a reader's background knowledge.

Secondly, in his book "Mathematical Discovery", Polya [140] suggested three different things to find in order to discover a solution of a problem or at least progress in its solution. These things are the answers to the questions:

(i) what do you want? (the aims);

(ii) what have you? (the data);

(iii) how can you get this kind of thing? (the ideas).

Thirdly, Wickelgren [141] noted that all formal problems (i.e. problems to find or to prove somethinig) were composed of three types of information concerning givens, operations, and goals. He defined givens as the set of expressions that are present in the problem, operations as the actions derived from the givens by some previous sequence of actions and the goal of a problem as a terminal expression one wishes to cause to exist in the words of the problem.

In the light of the above three approaches which I think introduce the same things with different terminology, I believe that, for understanding and then solving a question, one should identify its aims, data and develope appropriate ideas. The value of this technique may come through identification of the area of a question's difficulty in terms of its structure.

(c) Structural communication

This method came originally from Egan ([142], [143], [144]), and was developed at Glasgow University [116]. The basic element of this technique is the use a study unit which encompases intention, presentation, investigation, response, discussion and viewpoints. Information is presented to the student in the form of a grid (i.e. matrix) of knowledge elements, structured in such a way that, as the student builds his response, the various patterns and relationships of meaning emerge.

The most prominent ideas underlying Structural Communication come from Gestalt psychology rather than Stimulu-Response theory [143]. Therefore, it is more than the sum of its parts, and it makes sense only as a whole. The wholeness derives from structure.

The following example illustrates the above ideas. By using the grid below, prove that: $\cos A + \cos^2 A = 2 \cos A \cos^2 \frac{1}{2}A$.

$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (1)	Use a calculator to find an angle with a given cosine. (2)
Use a common factor to factorize an expression. (3)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (4)

The 2x2 grid contains knowledge elements which are built in such a way that, two of them are relevant to the solution (i.e. boxes 3 and

1), while the other two are not. After choosing the third box as the first step

$$\cos A + \cos^2 A = \cos A\ (1 + \cos A)$$

in the solution, the pattern of the continuation of the solution is obvious since the next step must be to use the first box:

$$\cos A\ (1 + \cos A) = \cos A\ (2 \cos^2 \tfrac{1}{2}A).$$

The solution is now practically complete:

$$\cos A\ (2 \cos^2 \tfrac{1}{2}A) = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

The size of the grid used is related to the age of the student, but a small grid size (3×4) was appropriate to the material being tested. It was found that guessing cannot interfere in this technique since student does not know how many of the boxes are required. The value of this technique is that it permits us to know what the student has correctly or wrongly chosen and determine of the correct ideas which have been omitted.

It has been suggested that the most meaningful score is obtained from:

$$\frac{\left[\begin{array}{c}\text{the number of correct}\\ \text{responses chosen}\end{array}\right]}{\left[\begin{array}{c}\text{the total number of}\\ \text{correct responses}\end{array}\right]} - \frac{\left[\begin{array}{c}\text{the number of incorrect}\\ \text{responses chosen}\end{array}\right]}{\left[\begin{array}{c}\text{the total number of}\\ \text{incorrect responses}\end{array}\right]}.$$

This gives the relevance coefficient (r) which ranges from 1 through 0 to -1.

There are many uses for grids such as [116]:

(i) testing the ability to categorise and dig into concepts;

(ii) testing the ability to sequence ideas;

(iii) testing deduction and inferences at various levels.

However, the main goals of the researcher were to use a grid as a tool to help students to organize their thinking and to test the ability to separate relevant material from irrelevant.

To sum up, the kind of intellectual activity the student is engaged in, the kind of diagnosis of responses are some the features of structural communication and in addition, its flexibility might form a basis for multi-level testing.

<u>(d) Test subjects' knowledge and confidence rating</u>

This technique is a traditional method of testing conceptual understanding (i.e. by direct questions). It was used for testing faulty recall of formulae and assessing the understanding of a specific concept. The pupil was asked to indicate his degree of confidence in each answer he gave. This was done by a five-point scale which varies from guessing to knowing the proof:

(1) I have no idea, so I have just made a guess;

(2) I am not sure, but I suspect it may be right;

(3) I think I am right;

(4) I am sure I am right;

(5) I know I am right and I can prove it.

The confidence scale may indicate that the unsuccessful answer of a pupil may be due to that pupil not having as clear an understanding of the concept being tested as a successful pupil. Therefore the main aim of confidence rating is to test how confident pupils are in their answers. I would expect a high or low confidence rating on the right or wrong answer respectively. If this is not the case, the pupils have a conflict of understanding.

3. The test

The experimental test was composed of seven questions and one standard question (three questions were taken from school exams and the remaining five from SCE exams). Each question of the seven was tested by two techniques (except one which was tested by one technique, and another by three). The role of the standard question, which was answered by all 223 pupils in the sample, was to test whether there was any difference in ability between the 7 groups into which the sample was divided.

The pupils in each group came from all five schools and the university.

Seven test papers were devised, one for the pupils in each group. The seven papers were produced in seven different colours to simplify the administration of the actual test. Each paper had 4 questions: the first question was the standard question, and the remaining three questions were selected from the pool of seven questions. The second question always appeared in its original form as a control, whereas the third and fourth questions were each modified according to one of the techniques.

Each paper was made up of two parts, A and B. Part A comprised the first three questions and an answer sheet. The technique adopted for the third question was usually a change in formulation. Part B was simply the fourth question. Since this question employed one of the three other techniques, none of which was familiar to the pupils, instructions and a sample question (and a sample grid where necessary) were provided.

I was present, as observer, during most of the actual tests in the schools and the university (although the administration of the test was carried out by the class teachers). It was clear from the way the pupils completed the test, that the written instructions were very clear. The information which was given to the teachers and a sample complete test is given in Appendix 12.

4. The experimental design

The design of the experiment is given in Table 138. Each of the seven cells presented in this design, represents one of test papers and contains the four questions (including the standard question).

The notation adopted in this design is as follows:

S refers to the standard question (common to all seven cells).

C_β refers to Control β (i.e. original Question β). Because there are seven controls in the test, $\beta = 1, 2, \ldots, 7$. For example, C_3 refers to Control 3, etc.

T^γ_β refers to Technique γ for Control β. Because there are four techniques, $\gamma = 1, \ldots, 4$. For example, T^2_5 refers to Technique 2 for the control 5. But if a technique is applied twice to the question (such as changes in formulation of the Questions 6 and 7), the addition of another number to the superscript of the technique is adopted. For example, $T^{1.1}_6$ and $T^{1.2}_6$ refer to the first and second changes in formulation of Question 6 respectively.

The links between cells refer to the relationships between controls and their techniques. There is one link for one technique. For example, there are two links between Cells 1 and 7, since the two techniques $T^{1.2}_7$ and T^1_1 are situated in the first and the last cell respectively.

school/ university	N
School 1	37
School 2	55
School 3	29
School 4	16
School 5	31
University	55
total	223

Table 137. The sample size.

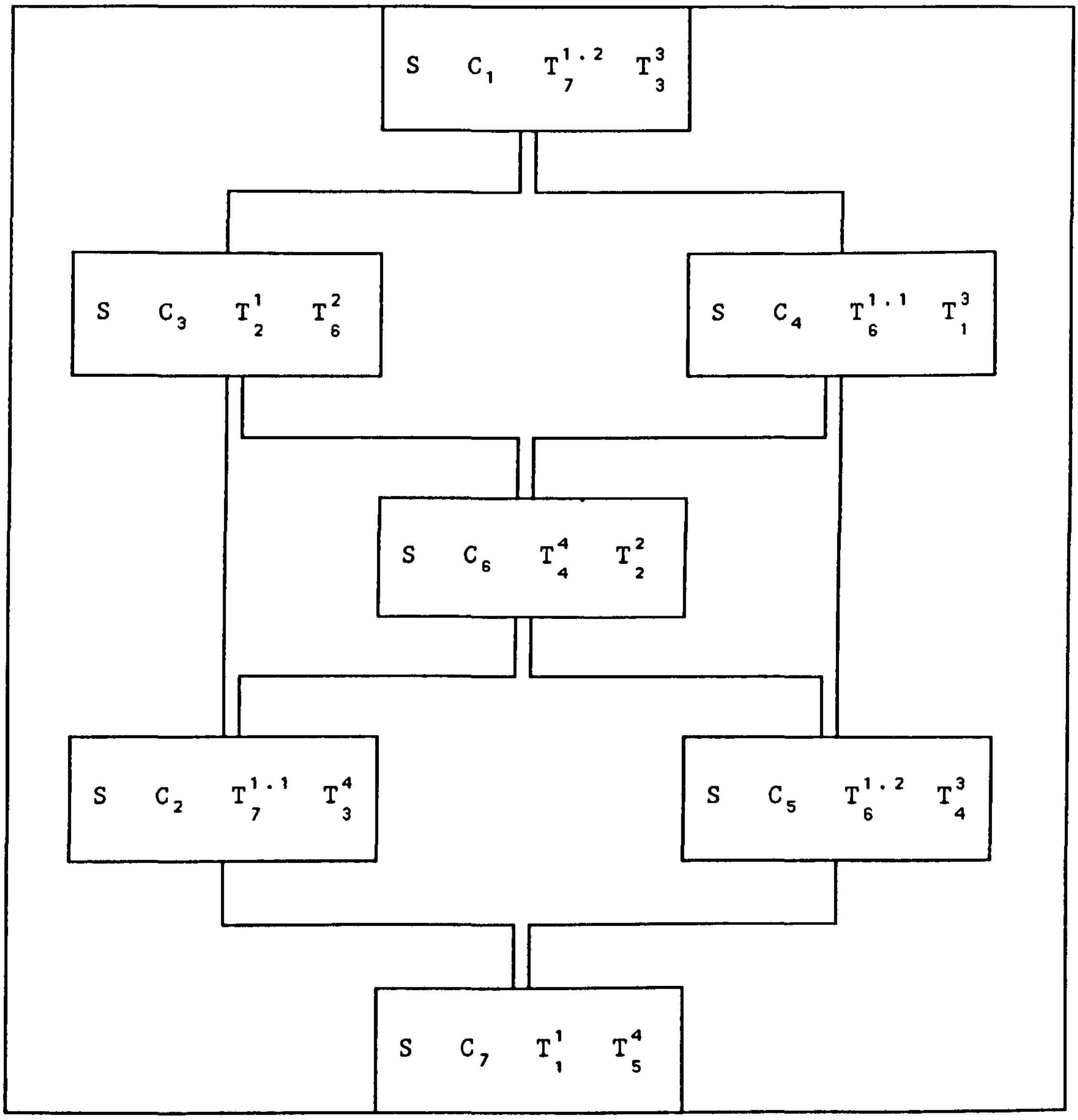


Table 138. Experimental Design.

5. The analysis of the results

I. Question 1

I.1. Rationale

The question is in Paper 1 (i.e. no. 2). Its difficulty may come from the following:

(i) the implicit division of the question;

(ii) the three dimensions factor;

(iii) the need to understand the statement "without lid";

(iv) the need for a symbol for the height of the box;

(v) the need to start by interpreting the data as an appropriate equation;

(vi) failure to realise to necessity to justify the maximum.

The two techniques used to modify this question are changeing the formulation (Paper 7, no. 3) and grid (Paper 4, Part B). Both control and its techniques are gathered in the following five pages in a small size format (the control is identified by no. 2, and its techniques by no. 3 and Part B). The real sequence of the test material can be found in Appendix 12.

The goal of the first technique is to test the first four difficulties. Therefore, the implicit division of the first part was converted to an explicit division and a diagram was drawn to overcome the other three difficulties. The goals of the second technique are to test whether a grid can improve candidates' performance and to test the pupils' ability to separate relevant information from irrelevant.

In the second technique, instructions, a sample question and a sample 2x2 grid were provided to guide the pupils. The 3x4 grid was

PAPER ONE: PART A

2. A rectangular box without a lid is made from 200 cm² of metal. Its base measures $2x$ cm by $3x$ cm.
   (a) Find the height of the box in terms of x and show that the volume V cm³ is given by

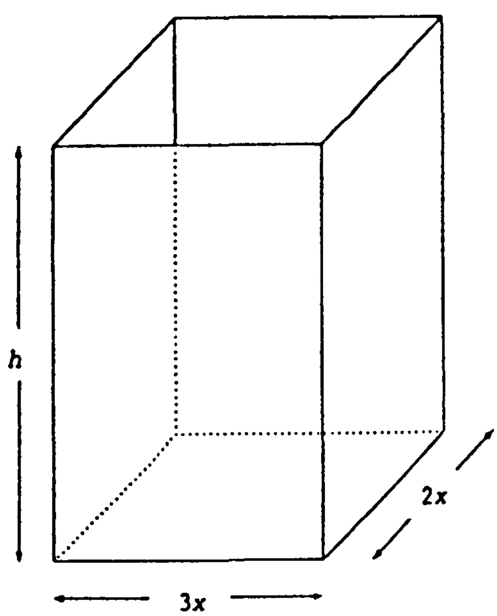
$$V = 120x - \frac{18}{5}x^3.$$

   (b) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.

3. A rectangular box without a lid is made from 200 cm² of metal. Its base measures $2x$ cm by $3x$ cm.
   (a) Find h, the height of the box, in terms of x.
   (b) Show that the volume V cm³ is given by

$$V = 120x - \frac{18}{5}x^3.$$

   (c) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.



PAPER ONE: PART B

NAME____

4. Read the INSTRUCTIONS, the SAMPLE QUESTION and the SAMPLE GRID carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

SAMPLE GRID

$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (1)	Use a calculator to find an angle with a given cosine. (2)
Use a common factor to factorize an expression. (3)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (4)

Your answer should look like this:

The numbers of the relevant steps are: (3) and (1).

The solution is:

$$\begin{aligned} \cos A + \cos^2 A &= \cos A\,(1 + \cos A) && [\text{step (3)}] \\ &= \cos A\,(2\cos^2 \tfrac{1}{2}A) && [\text{step (1)}] \\ &= 2\cos A \cos^2 \tfrac{1}{2}A. \end{aligned}$$

<u>NOW</u>

Use the <u>grid</u> on the separate sheet to answer the next question in the same way. Only some steps in the <u>grid</u> are necessary to solve the question.

<u>QUESTION</u>

A rectangular box without a lid is made from 200 cm^2 of metal. Its base measures $2x$ cm by $3x$ cm.

(a) Find the height of the box in terms of x and show that the volume V cm^3 is given by

$$V = 120x - \frac{18}{5}x^3.$$

(b) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.

The numbers of the relevant steps are:__________________________

The solution is:

Continue your solution on the next page if necessary.

GRID

Study the sign of the second derivative of V. (1)	A cuboid has six faces. (2)	$x^2 = a$, $a > 0$, has two roots. (3)
The quantity of metal is the area of the faces of the box. (4)	Calculate the minimum volume. (5)	Work out when the derivative of V is 0. (6)
Test for minimum value. (7)	The volume of the box is the product of its dimensions. (8)	Differentiate V. (9)
Calculate the maximum volume. (10)	Work out when the second derivative of V is 0. (11)	The box has five faces. (12)

supplied on a detached sheet. In this grid, eight boxes are relevant (i.e. 12, 4, 8, 9, 6, 3, 1, 10 while the remaining four are not.

I.2. The results

The mean scores on the control and its techniques and standard question for the three groups are given in Table 139. Note that S (C_1) denotes the standard question on the paper with Control 1. It was found that there was:

(i) no significant difference in ability between the three groups as measured by the means on the standard question;

(ii) no significant difference in means between the control and either of its techniques;

(iii) a significant difference in means ($p < 0.05$) between the two techniques in favour of the changes in formulation.

It was then decided to look at parts of the question rather than the whole. Table 140 gives the mean scores on parts of the control and its techniques (Parts a and b of both the control and grid (T^3) are related to a + b and c in the changed formulation T^1 respectively). It was found that there was:

(i) a significant difference in means ($p < 0.05$, $p < 0.01$) between the first part of the changed formulation and that of both the control and grid respectively (in favour of the changed formulation);

(ii) no significant difference in means between the second part of the control and its techniques.

Table 141 gives the percentage that chose each box in the grid. The relevant boxes are indicated by †. Note that box 8 is the most chosen. The existence of the word "volume" in both the control and grid may be the reason. Box 1 is the least chosen since the pupils seem not to be familiar with the offered method. The relevance coefficient is 0.4. The percentage of irrelevant and relevant boxes chosen varied from 0 to 7 and from 7 to 60 respectively (Figure 43).

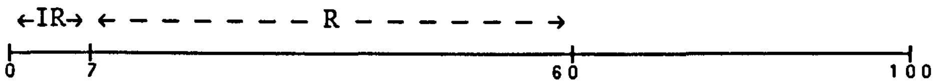


Figure 43.

The percentages of pupils who started and those who missed the justification of maximum in both the control and grid are given in Table 142. Note that, from 43% of pupils who chose the relevant boxes for starting (i.e. boxes 4 and 12), 23% of them could not start, and from 17% of pupils who missed the justification of maximum, 7% of them chose the appropriate box (i.e. box 1).

I.3. Discussion

Because the changes which have been made in the control are related to the first part only, the existence of a difference in performance in this part rather than the second is not surprise. The pupils' failure to apply the appropriate steps after they identified them may be due to the complexity of the execution of these steps.

I.4. Conclusion

The changes in formulation (with diagram) affect performance. The grid does not improve performance even though pupils, in general, succeeded in the identification of the relevant and irrelevant boxes.

	C_1	T_1^3	T_1^1	S (C_1)	S (T_1^3)	S (T_1^1)
mean* SD	2.41 2.34	1.87 2.45	3.00 2.41	3.91 1.44	4.10 1.32	4.34 1.39
size	34	30	30	34	30	30

Table 139. Mean score and standard deviation on Question 1.

* out of 7 for the control and its techniques and out of 5 for the standard question (S).

	C_1 (a)	C_1 (b)	T_1^3 (a)	T_1^3 (b)	T_1^1(a + b)	T_1^1(c)
mean* SD	0.94 1.30	1.47 1.39	0.77 1.23	1.10 1.29	1.57 1.30	1.40 1.41
size	34	34	30	30	30	30

Table 140. Mean score and standard deviation on the parts of Question 1 and its techniques.

* out of 3 and 4 for the two parts of the control and its techniques.

box	1†	2	3†	4†	5	6†	7	8†	9†	10†	11	12†	NA*
Frq	2	0	6	16	1	15	2	18	15	16	0	15	6
%	7	0	20	53	3	50	7	60	50	53	0	50	21

↑ the least chosen (box 1)　　↑ the most chosen (box 8)

Table 141. The percentage that chose each box ($N = 30$).

† refers to relevant boxes, * NA not attempted.

	C_1	T_1^3		
started	32%	23%	→ 43%	→ 23%
no justification	32%	17%	→ 7%	

Table 142. Percentages of pupils who started and those who found maximum without justification.

II. Question 2

II.1. Rationale

This question is in Paper 2 (no. 2). Its difficulty may arise from:

(i) the existence of irrelevant information;

(ii) the non-rigorous diagram;

(iii) the implicit division of the question;

(iv) the use of a letter for two meanings;

(v) the need for strategy to start;

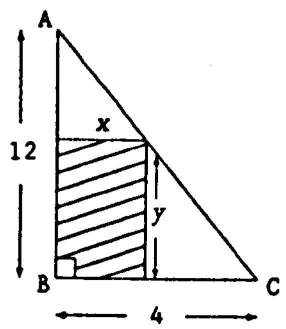
(vi) the misinterpretation of the "greatest area".

The two techniques used to modify this question are changing the formulation (Paper 3, no. 3) and testing understanding of structure (Paper 6, Part B). Both the control and its techniques are gathered in the following four pages in a small size format (the control is identified by no. 2, and its techniques by no. 3 and Part B). The real sequence of the test material can be found in Appendix 12.

The goal of the first technique is to test the first four difficulties. Therefore, the irrelevant wording was cancelled, the lengths in the diagram were corrected, the implicit division of the second part was converted to explicit division and the triangle's letters were changed (to avoid the two meanings of the letter A).

The goal of the second technique is to test the question's clarity through identification of its aims, data and ideas. It is clear that, through identification of relevant ideas, we can test the affect of the last two difficulties. In this second technique, instructions and a sample question were provided to guide the pupils to give their answers in the same way as on the sample question. In the

2. A piece of cord is in the shape of a right-angled triangle as shown with AB = 12 units, BC = 4 units, $A\hat{B}C = 90^\circ$.



A rectangle is to be cut from this triangle so that B is one of the vertices as shown.

(i) If x and y are the lengths of the sides of the rectangle as shown, show that $y = 12 - 3x$.

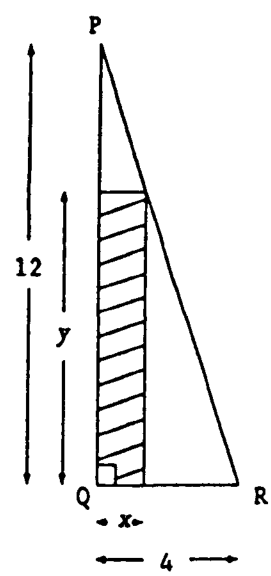
(ii) Hence express A, the area of the rectangle, in terms of x and find the dimensions of the rectangle which would have the greatest area.

3. In the diagram, PQR is a right-angled triangle with PQ = 12 units and QR = 4 units. The shaded rectangle has sides of length x and y as shown.

(i) Show that $y = 12 - 3x$.

(ii) Hence express the area A of shaded rectangle in terms of x.

(iii) Find the dimensions of the rectangle which would have the greatest area.



PAPER TWO: PART B

NAME____

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully, and then follow these instructions with the QUESTION at the end.

INSTRUCTIONS

You are NOT required to solve the question.

1. Identify the aims of the question (the things you have to find or show or prove).
2. Identify the data of the question (the information in the question which is needed in the solution).
3. Write down the ideas which are required to solve the question (these could be in the question or from your knowledge).

SAMPLE QUESTION

The positions of the points A and B on a map are given by the coordinates (4,4) and (3,5) respectively.

(a) Find the position of the point P which divides AB externally in the ratio 2:1.

(b) If the point C has the position (-3,1), prove that CP is perpendicular to AB.

Your answer should look like this:

The aims are:

1. Find the position of the point P.
2. Prove that CP is perpendicular to AB.

The data are:

1. A is the point (4,4).
2. B is the point (3,5).
3. P divides AB externally in the ratio 2:1.
4. C is the point (-3,1).

The ideas are:

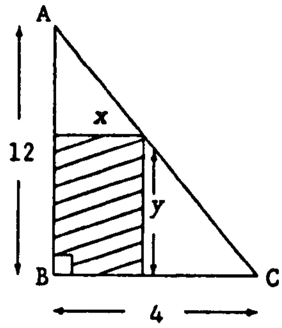
1. External division.
2. The components of a vector.
3. Scalar product.

NOW

Answer the next question in the same way.

QUESTION

A piece of cord is in the shape of a right-angled triangle as shown with AB = 12 units, BC = 4 units, $A\hat{B}C = 90^\circ$.



A rectangle is to be cut from this triangle so that B is one of the vertices as shown.

(i) If x and y are the lengths of the sides of the rectangle as shown, show that $y = 12 - 3x$.

(ii) Hence express A, the area of the rectangle, in terms of x and find the dimensions of the rectangle which would have the greatest area.

The aims are:

The data are:

The ideas are:

ns, the aims, data and ideas were introduced to the pupils in a very simple way. For example, the aim of a problem according to Polya [140] is related to the classification of problems into "to find" or "to prove", therefore, the addition of "to show" in our presentation may contribute to that simplicity. The sample question was chosen very carefully in respect of its aims, data and ideas:

(a) it has two kinds of aims (i.e. to find and to prove);

(b) it has relevant and irrelevant information to clarify the concept of data;

(c) its relevant ideas are derived from both the question and background knowledge.

II.2. The results

Tables 143 and 144 give the mean scores on the control and its first technique as complete questions and as parts. It was found that there was:

(i) no significant difference in ability between the two groups as measured by the means on the standard question;

(ii) no significant difference in means between the control and its first technique as complete questions and as parts.

The percentages of pupils who identify 3 (out of 3), 2, 1 and 0 structures (s) of the question in each of the categories aims, data and ideas are given in Table 145. While Table 146 summarises the results in terms of 3 structures against others. From this table, it was found that there was:

(i) a high significant difference ($p < 0.005$) between aims and data;

(ii) a high significant difference ($p < 0.001$) between aims and ideas;

(iii) a significant difference ($p < 0.05$) between data and ideas.

Because some expected values were less than 5, the Fisher test was applied between the two variables data (structures) and ideas (stractures) rather than the χ^2 test.

The reasons for the weakness in identification of both data and ideas may find their explanation in Table 147 which gives the percentages of each three aims, data and ideas. It is clear that the last piece of information of data and the first and - to some extent- the third ideas are the causes of such weakness. The percentages of pupils who started or recognized the maximum or missed the justification of this maximum are given in Table 148 for both the control and its first technique. Whereas, in the case of the second technique, just the percentages of those who gave the idea for starting or mentioned the maximum are given.

II.3. Discussion

It is clear that the last two difficulties (i.e. the need of a strategy to start and the interpretation of "greatest area" as the discovered maximum) are the main factors of the question's difficulty (Tables 147 and 148). This may explain why the changes in formulation do not succeed since its goals do not include overcoming these two difficulties.

The difficulty of the last piece of information of data may be due to its conditional form even though this form exists in the sample question.

Because the pupils in general are not experts, the weakness in identification of relevant ideas is expected.

II.4. Conclusion

When the strategy to start is an essential factor, any changes in a question in terms of its formulation may not guarantee an improvement in performance.

The question's structure has three different aspects: aims, data and ideas .

	C_2	T^1_2	S (C_2)	S (T^1_2)
mean*	2.78	2.81	4.00	4.09
SD	2.23	2.32	1.77	1.28
size	36	32	36	32

Table 143. Mean score and standard deviation on Question 2.

* out of 7 for the control and its technique and out of 5 for the standard question (S).

	C_2 (i)	C_2 (ii_1)	C_2 (ii_2)	T_2^1 (i)	T_2^1 (ii)	T_2^1(iii)
mean*	0.81	1.31	0.67	0.78	1.47	0.56
SD	1.33	0.92	0.72	1.29	0.88	0.76
size	36	36	36	32	32	32

Table 144. Mean score and standard deviation on the parts of Question 2 and its technique.

* out of 3, 2 and 2 for both the control and its technique respectively.

	3 (s)	2 (s)	1 (s)	0 (s)	total
aims	20 (63)	7 (22)	4 (13)	1 (13)	32
data	7 (22)	18 (56)	3 (9)	4 (13)	32
ideas	0 (0)	4 (13)	14 (44)	14 (44)	32

Table 145. Frequencies and percentages of pupils identifying the structures (s) of the question.

	3 (s)	others	total
aims	20 (62)	12 (38)	32
data	7 (22)	25 (78)	32
ideas	0 (0)	32(100)	32

Table 146. Frequencies and percentages of pupils identifying the structures (s) of the question.

	content of question's structure	Frq (%)
aims	1. show that $y = 12 - 3x$	27 (84)
	2. express the area of the rectangle	26 (81)
	3. find the dimensions of the rectangle	26 (81)
data	1. AB = 12 and BC = 4	28 (88)
	2. $A\hat{B}C = 90^\circ$	28 (88)
	3. x and y are the dimensions of the rectangle	10 (31)
ideas	1. similarity of triangles	2 (2)
	2. area of the rectangle	12 (38)
	3. maximum value of an expression	10 (31)

Table 147. Frequency and percentage of pupils identifying each piece of information of the question's structure.

	C_2	T_2^1	T_2^2
started	28%	31%	6%
maximum	36%	28%	31%
no justification	25%	16%	

Table 148. Percentage of pupils who started or found maximum or did not justify the maximum.

III. Question 3

III.1 Rationale

This question is in Paper 3 (no. 2) and its difficulty may arise from:

(i) faulty recall;

(ii) the need for complex reasoning.

The two techniques used to modify this question are grid (Paper 1, Part B) and testing knowledge and confidence rating (Paper 2, Part B). Both the control and its techniques are gathered in the following eight pages in a small size format (the control is identified by no. 2, and its techniques by Part B). The real sequence of the test material can be found in Appendix 12.

The goals of the first technique are, as we stated in the section on Question 1:

(a) to test whether a grid can improve the candidates'

PAPER THREE: PART A

2. (a) Solve, for $0 \leqslant A < 2\pi$,

$$\tan 2A + 2 \sin A = 0.$$

(b) Prove, for $0 < A < \frac{1}{2}\pi$, that

$$\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \tfrac{1}{2}A.$$

PAPER THREE: PART B

NAME____

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid on the separate sheet, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

Your answer should look like this:

The numbers of the relevant steps are: (7) and (11).

The solution is:

$$\begin{aligned} \cos A + \cos^2 A &= \cos A\,(1 + \cos A) && [\text{step } (7)] \\ &= \cos A\,(2 \cos^2 \tfrac{1}{2}A) && [\text{step } (11)] \\ &= 2 \cos A \cos^2 \tfrac{1}{2}A. \end{aligned}$$

NOW

Answer the next question in the same way.
Only some steps in the grid are necessary to solve the question.

QUESTION

(a) Solve, for $0 < A < 2\pi$,

$$\tan 2A + 2 \sin A = 0.$$

(b) Prove, for $0 < A < \frac{1}{2}\pi$, that

$$\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \tfrac{1}{2}A.$$

(a) The numbers of the relevant steps are: ____________________

The solution is:

(b) The numbers of the relevant steps are: ____________________

The solution is:

Continue your solutions on the next page if necessary.

GRID

$\cos^2 A + \sin^2 A = 1$ (1)	$\tan A = \dfrac{\sin A}{\cos A}$ (2)	Multiply by $\frac{\pi}{180}$ to convert from degrees to radians. (3)
Use the periods of sin and cos to obtain all solutions of $\sin A = 0$, $\cos A = \frac{1}{2}$, $\cos A = -1$ in specified range. (4)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (5)	$\cos 2A = 2 \cos^2 A - 1$ (6)
Use a common factor to factorize an expression. (7)	$\sin 2A = 2 \sin A \cos A$ (8)	Use a calculator to find an angle with a given cosine. (9)
The period of $\tan A$ is π. (10)	$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (11)	The equations $\sin A = 0$ and $\cos A = 0$ have two principal solutions. (12)

PAPER THREE: PART B

NAME____

4. Read both parts of the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

SAMPLE QUESTION

(i) Express $(\sin A + \cos A)^2$ in terms of $\sin 2A$.

Your answer should look like this:

$$(\sin A + \cos A)^2 = 1 + \sin 2A.$$

(ii) Indicate your degree of confidence in your answer by putting a tick in the appropriate box.

Degree of confidence	Sample Question
1. I have no idea, so I have just made a guess.	
2. I am not sure, but I suspect it may be right.	
3. I think I am right.	
4. I am sure I am right.	
5. I know I am right and I can prove it.	✓

The fifth box has been ticked because "I know I am right and I can prove it."

NOW

Answer the parts of the next question in the same way.

QUESTION

(i) Express:

(a) sin 2A in terms of sin A and cos A,

(b) cos 2A in terms of sin A,

(c) tan 2A in terms of tan A,

(d) cos 2A in terms of cos A.

Your answers are:

(a) ______________________________

(b) ______________________________

(c) ______________________________

(d) ______________________________

(ii) Indicate your degree of confidence in your answers by putting a tick in the appropriate box for each answer.
You should put ONE tick in each column.

Degree of confidence	Parts of the question			
	(a)	(b)	(c)	(d)
1. I have no idea, so I have just made a guess.				
2. I am not sure, but I suspect it may be right.				
3. I think I am right.				
4. I am sure I am right.				
5. I know I am right and I can prove it.				

performance;

(b) to test the pupils' ability to separate relevant information from irrelevant.

To achieve this, the same instructions and sample question of Paper 1 were adopted here since their aims are the same (to illustrate the grid method), but no sample grid was given since the grid itself (which was supplied on a detached sheet) was used to achieve this goal. In the grid, six boxes are relevant for the first part of the question (i.e. 2, 8, 6, 7, 12 and 4) and only five for the second part (i.e. 8, 7, 5, 11 and 2). Note that some relevant boxes are common for the two parts, but their order in the process of solution is different (i.e. 2, 7 and 8).

The goals of the second technique are:

(a) to test the faulty recall of some trigonometric formulae which are involved in the question;

(b) to test how confident pupils are in their answers.

Instructions, a sample question and a five-point scale of confidence were provided to guide pupils to give their answers in the same way as on the sample question. This (sample question) was chosen in the trigonometry field and the use of the confidence scale has been illustrated by choosing the last point of the scale to indicate the situation when the answer is right and the proof can be made.

III.2. The results

Tables 149 and 150 give the mean scores and standard deviations on the control and its first technique (grid) as complete questions and as parts. It was found that there was:

(i) no significant difference in ability between the two groups as measured by the means on the standard question;

(ii) a significant difference in means ($p < 0.05$) between the control and its first technique (in favour of the grid);

(iii) in terms of parts, no significant difference in means between the first part of both the control and its first technique, but such difference did exist ($p < 0.025$) between the second parts (in favour of the grid).

Tables 151 and 152 give the percentage that chose each box for both parts of the control (the relevant boxes are indicated by †). Note that boxes 2 & 8 and 4 & 2 are the most and the least chosen in the two parts respectively. This may explain the existence of similar patterns in both parts of the control and grid for the popular choices. While the difficulty of using period in the solution of a trigonometric equation and the different forms of tangent in both parts of the control and grid may be the reason for the unpopular choices.

The relevance coefficient was 0.2 for the first part of the question and 0.5 for the second. The percentage of irrelevant and relevant boxes chosen for both parts of question are given in Figures 44 and 45.

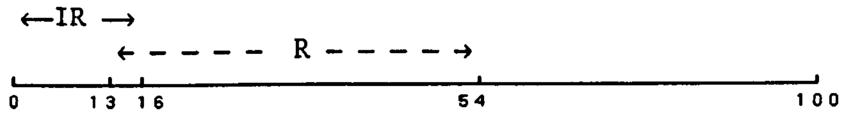


Figure 44 (for the first part).

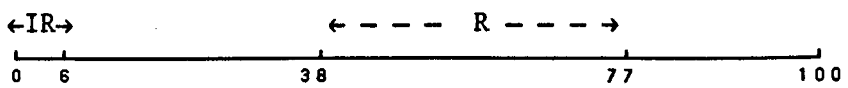


Figure 45 (for the second part).

For the recall of formulae, the percentages of pupils' answers and that of their degree of confidence is given in Table 153. Because the confidence's scale ranges from 1 to 5 and the first and the last two points of the scale show the low and high confidence respectively, it is reasonable to group them on each side to have one scale point for each of low and high confidence (while one for the neutral already exists). This is shown in Table 154 (which gives the percentages of pupils' answers and that of their degree of confidence). From this table, it was found that there was:

(i) no significant difference between the frequencies of right and wrong answers;

(ii) no significant difference in confidence of pupils with right answers (or with wrong answers) on the various parts of the question;

(iii) except in part (b) of the question, a significant difference ($p < 0.01$) between pupils' confidence in their right and wrong answers (some expected values are less than 5).

The overall measure of agreement, i.e. the difference between the number of people with high confidence (categories 4, 5) and the number with low confidence (categories 1, 2), is given in Table 155.

III.3 Discussion

There is an improvement in performance when the grid is used. This improvement certainly occurs when there is a problem with faulty recall, but not when the problem is complex reasoning. The low value of the relevance coefficient for the first part of the question compared with that for the second part may be explained by the fact that the first part needs both recall of formulae and reasoning, whereas the second needs just recall of formulae. In both parts of the question, the percentages were lower for irrelevant boxes and higher for relevant ones, but the range in both situations is in favour of the second part since no complex reasoning is required in its solution.

The faulty recall of trigonometric formulae occurred. Pupils' confidence does not vary when their answers are right (or when they are wrong). The pupils' confidence in the right answers is different from that in the wrong answers. It is clear that high or low degree of confidence is related to right or wrong answer respectively. In an exceptional case (such as part b), a conflict in understanding may exist.

III.4. Conclusion

Grids, in general, are helpful and are very affective in a situation when faulty recall is indicated rather than a failure of complex reasoning. Faulty recall of trigonometric formulae occurred. The confidence rating is related to the value of the answer (the high or low degree of confidence is related to right or wrong answer respectively).

	C_3	T_3^3	S (C_3)	S (T_3^3)
mean* SD	2.68 1.78	4.03 2.93	4.16 1.27	3.93 1.45
size	32	35	32	35

Table 149. Mean score and standard deviation on Question 3.

* out of 11 for the control and its technique and out of 5 for the standard question (S).

	C_3 (a)	C_3 (b)	T_3^3 (a)	T_3^3 (b)
mean* SD	1.06 1.07	1.63 1.41	1.34 1.14	2.68 2.31
size	32	32	35	35

Table 150. Mean score and standard deviation on the parts of the control and its technique.

* out of 6, 5 for the control and its technique.

box	1	2†	3	4†	5	6†	7†	8†	9	10	11	12†	NA*
Frq	5	19	4	4	0	10	7	14	2	2	0	5	12
%	16	54	13	13	0	31	22	40	6	6	0	16	34

the most and the least chosen

Table 151. The percentage that chose each box (*N* - 35).

† refers to relevant boxes, * NA not attempted.

box	1	2†	3	4	5†	6	7†	8†	9	10	11†	12	NA*
Frq	1	12	1	1	17	1	17	27	2	1	17	0	7
%	3	38	3	3	53	3	53	77	6	3	53	0	20

the least chosen

the most chosen

Table 152. The percentage that chose each box (*N* - 35).

† refers to relevant boxes, * NA not attempted.

	right and wrong answers	confidence rating				
		5	4	3	2	1
a	28 (78) 8 (22)	16 (57) 0 (0)	6 (21) 1 (13)	3 (11) 3 (38)	3 (11) 1 (13)	0 (0) 3 (38)
b	23 (64) 13 (36)	9 (39) 1 (8)	3 (13) 5 (38)	6 (26) 3 (23)	4 (18) 1 (8)	1 (4) 3 (23)
c	18 (50) 18 (50)	10 (56) 1 (6)	1 (6) 1 (6)	2 (11) 7 (39)	1 (6) 3 (17)	4 (22) 6 (33)
d	19 (53) 17 (47)	11 (58) 1 (6)	2 (11) 4 (23)	5 (26) 3 (18)	0 (0) 3 (18)	1 (5) 6 (35)

Table 153. Frequency and percentage of pupils' right and wrong answers and their confidence for each part of Question 3 ($N = 36$).

	right and wrong answers		confidence rating				
			5 & 4		3		2 & 1	
a	28	(78)	22	(78)	3	(11)	3	(11)
	8	(22)	1	(13)	3	(38)	4	(50)
b	23	(64)	12	(52)	6	(26)	5	(22)
	13	(36)	6	(46)	3	(23)	4	(31)
c	18	(50)	11	(61)	2	(11)	5	(28)
	18	(50)	2	(11)	7	(39)	9	(50)
d	19	(53)	13	(69)	5	(26)	1	(5)
	17	(47)	5	(29)	3	(18)	9	(53)

Table 154. Frequency and percentage of pupils' right and wrong answers and their confidence (on a three-point scale) for each part of Question 3 (N = 36).

	right answer	wrong answer
a	19	-3
b	7	2
c	6	-7
d	12	-4

Table 155. The overall measure of agreement.

IV. Question 4

IV.1. Rationale

This question is in Paper 4 (no. 2) and its difficulty may arise from:

(i) its abstract form;

(ii) the need of a special strategy to start it.

The techniques used to modify this question are testing knowledge (Paper 6, no. 3) and grid (Paper 5, Part B). Both the control and its techniques are gathered in the following five pages in a small size format (the control is identified by no. 2, and its techniques by no. 3 and Part B). The real sequence of the test material can be found in Appendix 12.

The goals of the first technique are:

PAPER FOUR: PART A

2. If a and b are vectors such that $|a| = |b|$ and $a \neq \pm b$, prove that the angle between the vectors u and v is a right angle, where

$$u = a + b \quad \text{and} \quad v = a - b.$$

3. (a) Given the vectors

$$a = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix},$$

find (1) $a.b$,

(2) $a + b$ and $a - b$ in component form,

(3) $|a|$.

(b) What does $c.d = 0$ tell you about the non-zero vectors c and d ?

(c) If u and v are vectors such that $|u| = |v|$, express the scalar product $(u + v).(u - v)$ in simplest form.

PAPER FOUR: PART B

NAME____

4. Read the INSTRUCTIONS, the SAMPLE QUESTION and the SAMPLE GRID carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

SAMPLE GRID

$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (1)	Use a calculator to find an angle with a given cosine. (2)
Use a common factor to factorize an expression. (3)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (4)

Your answer should look like this:

The numbers of the relevant steps are: (3) and (1).

The solution is:

$$\begin{aligned} \cos A + \cos^2 A &= \cos A\ (1 + \cos A) && [\text{step } (3)] \\ &= \cos A\ (2 \cos^2 \tfrac{1}{2}A) && [\text{step } (1)] \\ &= 2 \cos A \cos^2 \tfrac{1}{2}A. \end{aligned}$$

<u>NOW</u>

Use the <u>grid</u> on the separate sheet to answer the next question in the same way. Only some steps in the <u>grid</u> are necessary to solve the question.

<u>QUESTION</u>

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors such that $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\boldsymbol{a} \neq \pm\, \boldsymbol{b}$, prove that the angle between the vectors $\mathbf{u}$ and $\boldsymbol{v}$ is a right angle, where

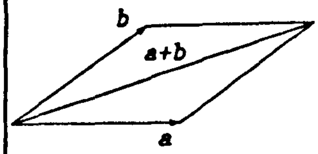
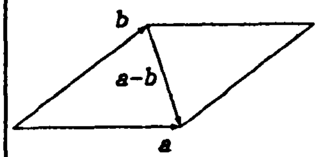
$$\mathbf{u} = \boldsymbol{a} + \boldsymbol{b} \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{a} - \boldsymbol{b}.$$

The numbers of the relevant steps are:____________________________

The solution is:

Continue your solution on the next page if necessary.

GRID

 (1)	Find the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$. (2)	Use the commutative property of scalar product. (3)
 (4)	If $u.v = 0$ then u is perpendicular to v. (5)	Find the scalar product of u and v. (6)
The diagonals of a rhombus are perpendicular. (7)	$\boldsymbol{a}.\boldsymbol{a} = \lvert\boldsymbol{a}\rvert^2$. (8)	If $\boldsymbol{a} = 0$ or $\boldsymbol{b} = 0$ then $\boldsymbol{a}.\boldsymbol{b} = 0$. (9)
Use the distributive property of scalar product. (10)	If $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\lvert\boldsymbol{a}\rvert = \sqrt{(x^2 + y^2 + z^2)}$ (11)	If $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$ then $\lvert\boldsymbol{a}\rvert^2 = \lvert\boldsymbol{b}\rvert^2$. (12)

(a) to test the knowledge required to solve such a question on scalar product;

(b) to test the understanding of the notation involved in the question such as $a \pm b$ and $|a|$;

(c) to test the affect of breaking the question into steps and at the same time to determine where the difficulty is.

To achieve these goals, a modified question was given to the pupils. It had three parts, the first was about scalar product, addition and subtraction of two vectors and the magnitude of a vector. All of these concepts were put in a concrete form. The aim was to achieve the first two goals. The second and the third parts were the result of breaking down the control into two separate bits which are in an abstract form, but direct the pupils towards the solution. This was intended to achieve the third goal.

The goals of the second technique are:

(a) to test whether a grid can improve candidates' performance;

(b) to test the pupils' ability to separate relevant information from irrelevant.

The instructions, sample question and sample grid which were used in Questions 1 and 3 were adopted here too. Because we think a strategy for solving this question is an essential factor, two methods were given in the grid as a framework for overcoming the question's difficulty. The first was an algebraic one, involving scalar product and its properties which were given in six relevant boxes (i.e. 6, 10, 3, 8, 12 and 5). The second method was geometric, using the relationship between perpendicularity of the sum and difference of two vectors with the diagonals of a rhombus. This was given in three boxes (i.e. 1, 4 and 7). Note that two of the steps were diagrams.

IV.2. The results

The mean scores of the control and its techniques are given in Table 156. It was found that there was:

(i) no significant difference in ability between the three groups as measured by the means on the standard question;

(ii) a highly significant difference in means ($p < 0.0005$) between the control and grid (in favour of the grid);

(iii) a highly significant difference in means ($p < 0.0005$) between the control and its second technique (in favour of the "break down" technique);

(iv) no significant difference in means between the two techniques.

The results of testing knowledge of some concepts and understanding of notation are given in Table 157 in terms of percentages of right answers. Tables 158 and 159 give the percentage that chose each box for the first and second method respectively (the relevant boxes in each methods are indicated by †).

In the first method, the two boxes 8 and 10 are the least chosen, while box 5 is the most chosen. It appears that the traditional definition of scalar product and the distributive property are not clear to the pupils. Whereas, the perpendicularity of two vectors when their scalar product equals zero is obvious. In the case of the second method, the three relevant boxes are very often chosen. Table 160 gives the relevant boxes (for the first method) which have very low percentages.

The relevance coefficient was 0.1 for the first method and 0.8 for the second. The percentages of irrelevant and relevant boxes chosen for both methods are given in Figures 47 and 48.

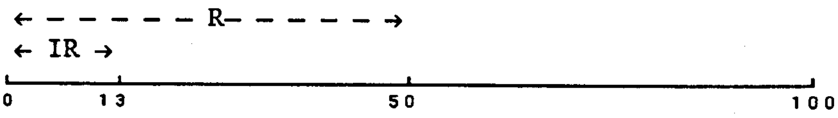


Figure 47.

Figure 48.

IV.3. Discussion

A clear improvement in performance was obtained by offering the pupils an appropriate strategy or by breaking down the question into parts. The influence of the abstract form of the question has been confirmed: pupils succeeded in concrete situations and failed in similar abstract situations even though they were directed towards solutions.

The very low value of the relevance coefficient for the first method compared with the second may be explained by the geometric method having a pictorial representation unlike the algebraic one. In addition, the relevant steps in the geometric method are easily distinguished from the irrelevant ones but this is not so in the other method.

IV.4. Conclusion

When a question is in an abstract form and finding the insight what to do is difficult, an appropriate strategy embedded in a grid is helpful, as is breaking the question into steps.

	C_4	T^3_4	T^4_4	S (C_4)	S (T^3_4)	S (T^4_4)
mean*	0.45	1.70	1.33	4.19	4.33	3.91
SD	0.91	1.51	1.08	1.41	1.27	1.51
size	29	30	33	29	30	33

Table 156. Mean score and standard deviation on Question 4.

* out of 4 for the control and its techniques and out of 5 for the standard question (S).

	percentage
scalar product	67
$\boldsymbol{a} \pm \boldsymbol{b}$	100
$\lvert\boldsymbol{a}\rvert$	67
first part	58
second part	6

Table 157. Percentages of right answers to Question 4.

box	1	2	3†	4	5†	6†	7	8†	9	10†	11	12†	NA*
Frq	1	1	1	1	8	6	0	0	0	0	2	1	5
%	6	6	6	6	50	38	0	0	0	0	13	6	31

↑ the most chosen ↑ ↑ the least chosen

Table 158. The percentage that chose each box (N = 16).

† refers to relevant boxes, * NA not attempted.

box	1†	2	3	4†	5	6	7†	8	9	10	11	12	NA*
Frq	12	1	0	13	5	2	12	0	2	0	1	3	0
%	86	7	0	93	36	14	86	0	14	0	7	21	0

Table 159. The percentage that chose each box (N = 14).

† refers to relevant boxes, * NA not attempted.

	percentage
distributive property	0
commutative property	6
$\lvert u\rvert^2 = u.u$	0
$\lvert a\rvert = \lvert b\rvert \Rightarrow \lvert a\rvert^2 = \lvert b\rvert^2$	6

Table 160. The little-chosen relevant boxes and the percentages that chose them.

V. Question 5

V.1. Rationale

The question is in Paper 5 (no. 2) and its difficulty may be the three dimensions factor. The only technique used is testing knowledge and confidence rating (Paper 7, Part B). Both control and its technique are gathered in the following four pages in a small size format (the control is identified by no. 2, and its technique by Part B). The real sequence of the test material can be found in Appendix 12.

The goals of this technique are:

(i) to test the understanding of some concepts which are involved in the question such as perpendicularity and

PAPER FIVE: PART A

2. A and B are the points (4,4,10) and (3,4,5) respectively.

   (a) Find the coordinates of the point Y which divides AB externally in the ratio 2:1.

   (b) If C is the point (-3,7,1), prove that CY is perpendicular to AB and find the image of C under reflection in AB.

PAPER FIVE: PART B

NAME____

4. Read both parts of the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

SAMPLE QUESTION

(i) Find the image of the point P(-4,11) under the translation $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Your answer should look like this:

The image of this point is (-1,11).

(ii) Indicate your degree of confidence in your answer by putting a tick in the appropriate box.

Degree of confidence	Sample Question
1. I have no idea, so I have just made a guess.	
2. I am not sure, but I suspect it may be right.	
3. I think I am right.	
4. I am sure I am right.	
5. I know I am right and I can prove it.	✓

The fifth box has been ticked because "I know I am right and I can prove it."

NOW

Answer the parts of the next question in the same way and use the last page for your rough work.

QUESTION

(i) C is the point (2,-3).

(a) Find the image of C under reflection in

(1) the x-axis,

(2) the line $x = 1$.

(b) Given the points A(4,3), B(-1,-2) and Y(0,-1) lie on the same line,

(1) show that CY is perpendicular to AB,

(2) find the image of C under reflection in AB.

Your answers are:

(a) The image of C under reflection in

(1) the x-axis is ______________________________

(2) the line $x = 1$ is ___________________________

(b) (1) CY is perpendicular to AB because ______________

(2) The image of C under reflection in AB is __________

(ii) Indicate your degree of confidence in your answers by putting a tick in the appropriate box for each answer.
You should put ONE tick in each column.

Degree of confidence	Parts of the question			
	(a)		(b)	
	(1)	(2)	(1)	(2)
1. I have no idea, so I have just made a guess.				
2. I am not sure, but I suspect it may be right.				
3. I think I am right.				
4. I am sure I am right.				
5. I know I am right and I can prove it.				

reflection;

(ii) to test the effect of the three dimensions factor;

(iii) to test how confident pupils are in their answers.

To achieve these goals, instructions, a sample question and a five-point scale of confidence were provided to guide the pupils to give their answers in the same way as on the sample question. The sample question was chosen from the area of transformations since the control belongs to this area too.

The use of the confidence scale has been illustrated by choosing the last point of the scale to indicate the situation when the answer is right and the proof can be made.

The modified question was chosen to have two dimensions rather than three such as in the control. It has two parts, each of which in turn was divided into two subparts. The first part concerns the image of a point under reflection in both the x-axis and another line. Its aim, then, is to test the first two goals of the technique. The second part is similar to that of the control except for the dimensions, which are two rather then three. This allows us to test the second goal of the technique.

V.2. The results

The mean scores and standard deviation on the control and its technique is given in Table 161. It was found that there was:

(i) no difference in ability between the two groups as measured by the means on the standard question;

(ii) even though there is a highly significant difference in means ($p < 0.0005$) between the control and its technique, the result is omitted since the two questions are different as a whole but not in the second part.

Tables 162 and 163 give the mean scores on the second part and its subparts in both the control and its technique. It was found that there was:

(i) a highly significant difference in means ($p < 0.0005$) between part (b) of the control and its technique;

(ii) a highly significant difference in means ($p < 0.0005$) between corresponding subparts of part (b) of the control and its technique

The percentages of pupils' answers and that of their degree of confidence for each part of the technique is given in Table 164. As we did in the analysis of Question 3, the first two and the last two points of the scale are grouped on each side to produce a confidence scale of three points rather than five. This is shown in Table 165 (which gives the percentages of pupils' answers and that of their degree of confidence). From this table, it was found that there was:

(i) a highly significant difference ($p < 0.0005$) between the frequencies of right and wrong answers;

(ii) no significant difference in confidence of pupils with right answers on the various parts of the question (for pupils with wrong answers, most frequencies are zero);

(iii) no significant difference between pupils' confidence in their right and wrong answers.

The overall measure of agreement is given in Table 166.

V.3. Discussion

The differences between part (b) of the control and its technique were:

(i) the dimensional factor;

(ii) the "break down" into steps;

(iii) dependence on part (a).

These factors are all in favour of the technique, but we think that the first factor was the major reason for the difficulty of the question since the "break down" into steps did not affect the difficulty and the effect of dependence on part (a) in the control is small since its first part was easy. Therefore, the dimension factor affects not only the second bit of part (b) but also the first bit, even though it is familiar.

In general, the pupils understood perpendicularity and reflection. This was confirmed by the difference between the frequencies of right and wrong answers in favour of the right answers.

Once again, pupils' confidence does not vary when their answers are right but no significant difference in confidence between pupils with right answers and those with wrong answers was found. This may be due to the nature of the question which asks for more than a yes or no response. Pupils may make errors in their calculation without being aware of it. As a result of this, high confidence can occur with a wrong answer (Table 166).

V.4. Conclusion

Reducing dimensions from three to two was very effective in improving pupils' performance.

The confidence rating is normally related to the value of the answer, but the nature of the question may reverse this.

	C_5	T_5^4	$S\ (C_5)$	$S\ (T_5^4)$
mean* SD	2.29 1.62	4.74 1.58	4.39 1.26	4.37 1.26
size	31	30	31	30

Table 161. Mean score and standard deviation on Question 5.

* out of 6 for the control and its technique and out of 5 for the standard question (S).

	C_5 (b)	T_5^4 (b)
mean* SD	1.06 1.03	3.04 1.09

Table 162. Mean score and standard deviation on Question 5(b).

* out of 4 for the control and its technique.

	C_5 (b_1)	C_5 (b_2)	T_5^4 (b_1)	T_5^4 (b_2)
mean*	1.00	0.06	1.94	1.10
SD	0.93	0.25	0.25	0.99

Table 163. Mean score and standard deviation on the Question 5(b) subparts. * out of 2 for all items.

	right and wrong answers	confidence rating				
		5	4	3	2	1
a_1	28 (93) 2 (7)	22 (79) 2 (100)	4 (14) 0 (0)	0 (0) 0 (0)	2 (7) 0 (0)	0 (0) 0 (0)
a_2	23 (77) 7 (23)	16 (70) 6 (86)	5 (22) 1 (14)	0 (0) 0 (0)	2 (8) 0 (0)	0 (0) 0 (0)
b_1	30 (100) 0 (0)	24 (80) 0 (0)	3 (10) 0 (0)	2 (7) 0 (0)	1 (3) 0 (0)	0 (0) 0 (0)
b_2	16 (53) 14 (47)	3 (19) 5 (36)	8 (50) 1 (7)	3 (19) 2 (14)	2 (12) 4 (29)	0 (0) 2 (14)

Table 164. Frequency and percentage of pupils' right and wrong answers and their confidence for each part of Question 5 (N = 30).

	right and wrong answers		confidence rating					
			5 & 4		3		2 & 1	
a_1	28	(93)	26	(93)	0	(0)	2	(7)
	2	(7)	2	(100)	0	(0)	0	(0)
a_2	23	(77)	21	(91)	0	(0)	2	(9)
	7	(23)	7	(100)	0	(0)	0	(0)
b_1	30	(100)	27	(90)	2	(7)	1	(3)
	0	(0)	0	(0)	0	(0)	0	(0)
b_2	16	(53)	11	(69)	3	(19)	2	(12)
	14	(47)	6	(43)	2	(14)	6	(43)

Table 165. Frequency and percentage of pupils' right and wrong answers and their confidence (on a three-point scale) for each part of Question 5 ($N = 30$).

	right answer	wrong answer
a_1	24	2
a_2	19	7
b_1	26	0
b_2	9	0

Table 166. The overall measure of agreement.

VI. Question 6

VI.1. Rationale

This question is in Paper 6 (no. 2). Its difficulty may arise from:

(i) the implicit division of the question;

(ii) the abstract form;

(iii) the dependent parts;

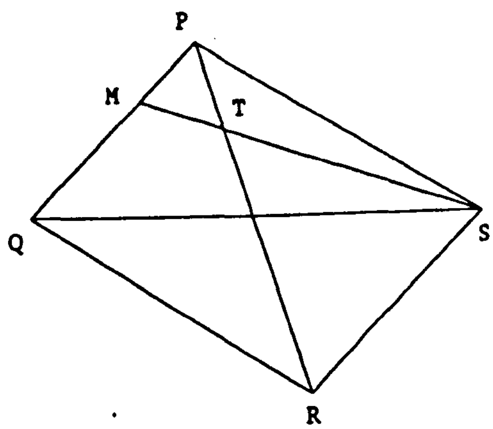
(iv) the concept of collinearity.

The techniques used to modify this question are changing the formulation (which is applied twice, Papers 4 and 5, no. 3 in each case) and testing understanding of structure (Paper 3, Part B). Both the control and its techniques are gathered in the following four pages in a small size format (the control is identified by no. 2, and

2. In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.
Given that $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ denote the position vectors of P, Q and S respectively relative to an origin O, find $\boldsymbol{m}$ and $\boldsymbol{t}$, the position vectors of M and T respectively, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.
If the parallelogram PQRS is completed, show that the points P, T and R are collinear.

3. In the triangle ABC, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ denote the position vectors of A, B and C respectively relative to an origin O.
   (a) Given that P divides BA in the ratio 2:1, find $\boldsymbol{p}$ the position vector of P in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.
   (b) Given that Q divides CP in the ratio 3:1, find $\boldsymbol{q}$ the position vector of Q in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.
   (c) If the parallelogram ABDC is completed, show that the points A, Q and D are collinear.

3. In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.
Given that $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ denote the position vectors of P, Q and S respectively relative to an origin O, find $\boldsymbol{m}$ and $\boldsymbol{t}$, the position vectors of M and T respectively, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.
If the parallelogram PQRS is completed, show that the points P, T and R are collinear.



PAPER SIX: PART B

NAME____

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully, and then follow these instructions with the QUESTION at the end.

INSTRUCTIONS

You are NOT required to solve the question.

1. Identify the aims of the question (the things you have to find or show or prove).
2. Identify the data of the question (the information in the question which is needed in the solution).
3. Write down the ideas which are required to solve the question (these could be in the question or from your knowledge).

SAMPLE QUESTION

The positions of the points A and B on a map are given by the coordinates (4,4) and (3,5) respectively.

(a) Find the position of the point P which divides AB externally in the ratio 2:1.

(b) If the point C has the position (-3,1), prove that CP is perpendicular to AB.

Your answer should look like this:

The aims are:

1. Find the position of the point P. 2. Prove that CP is perpendicular to AB.

The data are:

1. A is the point (4,4). 2. B is the point (3,5). 3. P divides AB externally in the ratio 2:1. 4. C is the point (-3,1).

The ideas are:

1. External division. 2. The components of a vector. 3. Scalar product.

NOW

Answer the next question in the same way.

<u>QUESTION</u>

In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.

Given that $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ denote the position vectors of P, Q and S respectively relative to an origin O, find $\boldsymbol{m}$ and $\boldsymbol{t}$, the position vectors of M and T respectively, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.

If the parallelogram PQRS is completed, show that the points P, T and R are collinear.

The aims are:

The data are:

The ideas are:

its techniques by no. 3 and Part B). The real sequence of the test material can be found in Appendix 12.

The goals of the first change in formulation are to test the implicit division of the control and its abstract notation. Therefore, the modified question was divided into three parts and the information given bit by bit, while the notation was changed to one which we think is more convenient for the pupils.

The aim of the second change in formulation is to test the abstract form of the question by including a diagram.

The goal of the third technique is to test the question's understandability through identification of its aims, data and ideas. It is clear that, through identification of relevant ideas, we can test the effect of the concept of collinearity. The same instructions and sample question which were in Question 2 were adopted here since their aims are the same (i.e. illustration of the testing understanding of structure method).

No technique was provided to test the effect of dependent parts since we did not wish to change the structure of the question, but a comparison between its parts was made.

VI.2. The results

The mean scores and standard deviations on the control and the two changes in formulation are given in Tables 167 and 168 as complete questions and as parts. It was found that there was:

(i) no significant difference in ability between the three groups as measured by the means on the standard question;

(ii) no significant difference in means between the control and either of the two techniques as questions or as related parts (except the mean on part b of the control differed significantly ($p < 0.05$) from that of the first change in formulation).

The percentages of pupils who identify 3 (out of 3), 2, 1 and 0 structures (s) of the question in each of the categories aims, data and ideas are given in Table 169. While Table 170 summarises the results in terms of 3 structures against others. From this table, it was found that there was:

(i) a highly significant difference ($p < 0.001$) between aims and data;

(ii) a highly significant difference ($p < 0.001$) between aims and ideas;

(iii) a significant difference ($p < 0.10$) between data and ideas.

Once again the reasons for the weakness in identification of both data and ideas may find their explanation in Table 171. It was clear that the last pieces of information in both data and ideas are the causes of such weakness.

The first part of each of the control and the two changes in formulation were compared with the first part of Question 5. Note that the latter part is in concrete form. Futhermore no significant difference in terms of pupils' ability has been found between the four groups. The result of this comparison, which is given in Table 172, is that: there was a highly significant difference in means ($p < 0.005$, $p < 0.0005$ and $p < 0.0005$) between the first part of Question 5 and that of the control, for the first and the second changes in formulation respectively. Finally, Table 173 shows the frequencies of pupils' success in parts of the control.

VI.3. Discussion

Addition of the diagram had no effect on performance. This may be due to the fact that it holds too much information since the diagram is given as a whole rather than gradually as required. However, in their study "What Makes Exam Questions Difficult?", Pollitt et al. [128] found that "the use of diagrams in the 1980 set of questions does not appear to have had an ameliorative effect on question difficulty" (page 38). The relation between illustrations (such as diagrams) and mathematical text (such as questions) has been analysed by Shuard and Rothery [135]. The following three levels of importance of illustration has been stressed:

(i) decorative;

(ii) related but non-essential;

(iii) essential.

For them, the visual material is essential if it contains ideas which cannot expressed conveniently in words. According to this definition, our diagram is certainly not essential, but it is related since it repeats and emphasises ideas stated in the question. However, our evidence (Tables 172 and 173) also suggests that the abstract notation of the question such as using letters rather than numbers and the difficulty of the concept of collinearity are the main reasons for the question's difficulty. As a result of this, the explicit division cannot help since the real reasons for the difficulty remain.

As we found in the analysis of Question 2, the conditional form indicated by if was again the reason for the difficulty of the data, and the independence of the aims, data and ideas in the structure of the question was again apparent.

It is clear that the difficulty gradually increases in a question which has dependent parts and the solution of a part is essential for that of the next.

VI.4. Conclusion

When a question is in an abstract form of **mode** and **idea**, changes in its formulation (such as converting implicit parts to explicit ones or drawing a diagram) may not be enough to get an improvement in performance.

Again the question's structure was found to have three different aspects: aims, data and ideas.

	C_6	$T_6^{1.1}$	$T_6^{1.2}$	S (C_6)	S ($T_6^{1.1}$)	S ($T_6^{1.2}$)
mean*	1.21	0.62	0.61	3.91	4.07	4.39
SD	1.89	1.35	1.20	1.51	1.41	1.58
size	33	29	31	33	29	31

Table 167. Mean score and standard deviation on Question 6.

* out of 7 for the control and its techniques and out of 5 for the standard question (S).

	C_6			$T_6^{1.1}$			$T_6^{1.2}$		
	1	2	3	1	2	3	1	2	3
mean*	0.34	0.58	0.30	0.21	0.17	0.24	0.19	0.29	0.13
SD	0.48	0.90	0.77	0.41	0.54	0.77	0.40	0.69	0.43

Table 168. Mean score and standard deviation on the parts of Question 6 and its techniques.

* out of 1, 2 and 4 for the three parts of the control and its techniques respectively.

	3 (s)	2 (s)	1 (s)	0 (s)	total
aims	26 (84)	5 (16)	0 (0)	0 (0)	31
data	10 (32)	17 (55)	4 (13)	0 (0)	31
ideas	3 (10)	8 (26)	15 (48)	5 (16)	31

Table 169. Frequencies and percentages of pupils identifying the structures (s) of the question.

	3 (s)	others	total
aims	26 (84)	5 (16)	31
data	10 (32)	21 (68)	31
ideas	3 (10)	28 (90)	31

Table 170. Frequencies and percentages of pupils identifying the structures (s) of the question.

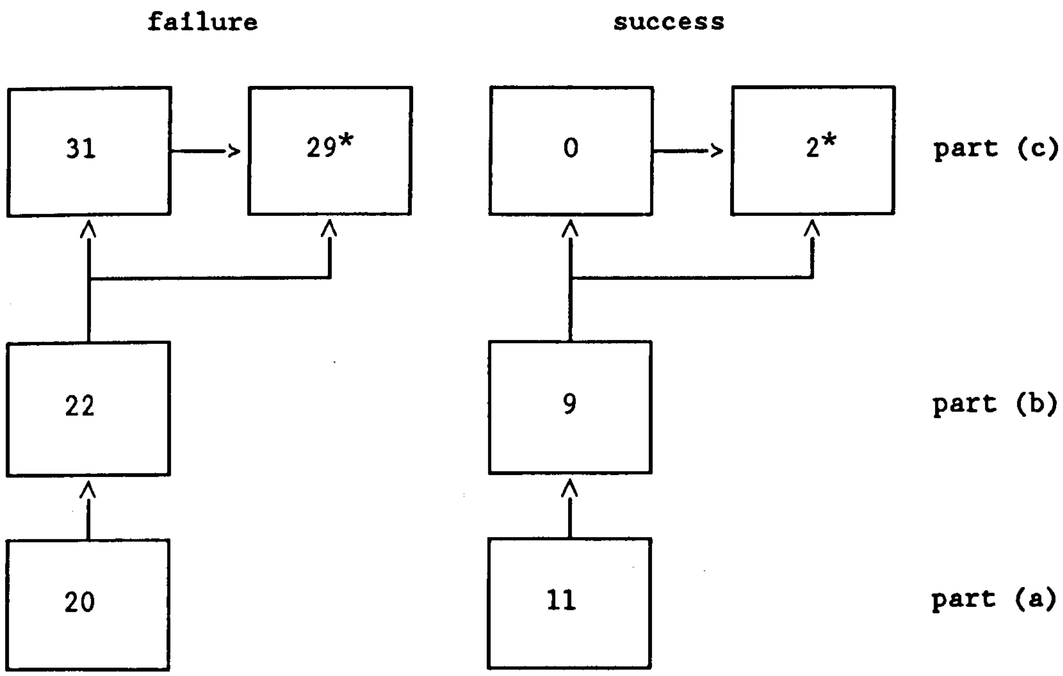
	content of question's structure	Frq (%)
aims	1. find m in terms of p, q and s	31 (100)
	2. find t in terms of p, q and s	28 (90)
	3. show that P, T and R are collinear	29 (94)
data	1. M divides QP in the ratio 2:1 T divides SM in the ratio 3:1	31 (100)
	2. p, q, s, m ,t are the position vectors of P, Q, S, M, T respectively	22 (71)
	3. PQRS is a parallelogram	10 (32)
ideas	1. internal division	17 (55)
	2. components of a vector	14 (45)
	3. collinearity of three points	9 (29)

Table 171. Frequency and percentage of pupils identifying each piece of information of the question's structure.

	C_6 1st	$T_6^{1.1}$ 1st	$T_6^{1.2}$ 1st	C_5 1st
mean*	0.55	0.41	0.39	1.26
SD	0.83	0.82	0.80	0.93

Table 172. Mean score and standard deviation on the first part of Question 6, its two techniques and Question 5.

* out of 2 for each part.



frequencies of failure and success

Table 173. The effect of dependent parts on performance.

* because no one succeeded in part (c), we consider the two pupils who got high marks (3 out of 4) as the successful ones.

VII. Question 7

VII.1. Rationale

The question is in Paper 7 (no. 2) and its difficulty may arise from:

(i) an incorrect diagram;

(ii) information not sequenced logically and an unusual choice of symbols;

(iii) the need for a strategy to start.

The technique used to modify this question is changing the formulation (which is applied twice, Papers 2 and 1, no. 3 in each case). Both the control and its techniques are gathered on the following page in a small size format (the control is identified by no. 2, and its techniques by no. 3). The real sequence of the test material can be found in Appendix 12.

The aims of the first technique are to test the first two difficulties. Therefore, the diagram was corrected (to make the triangle equilateral and AD not perpendicular to BC), the information was ordered logically and the notation was changed.

The aims of the second technique are to test all difficulties. Therefore, all the changes in the first technique were adopted here and, in addition, the first step towards a solution was given (i.e. express w in terms of u and v).

VII.2. The results

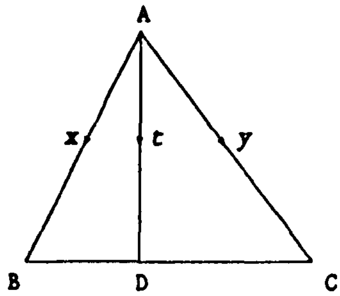
The mean scores and standard deviations on the control and its two techniques are given in Table 174. It was found that there was:

(i) no significant difference in ability between the three

PAPER SEVEN: PART A

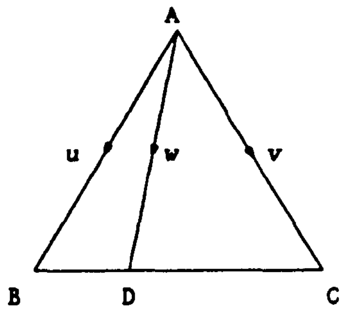
2. Triangle ABC is equilateral, with each side of length one unit. $\overrightarrow{AB}$ represents vector $\boldsymbol{x}$, $\overrightarrow{AC}$ represents $\boldsymbol{y}$ and $\overrightarrow{AD}$ represents $\boldsymbol{t}$. D lies on BC and BD $= \frac{1}{3}$BC. Show that

$$\boldsymbol{t}.\boldsymbol{x} = \frac{5}{6}.$$



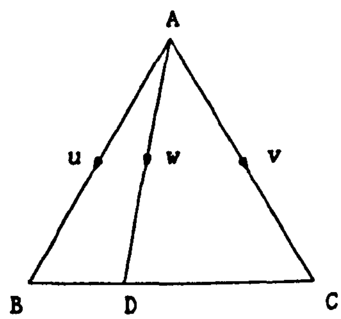
3. Triangle ABC is equilateral and the length of each side is one unit. D lies on BC and BD $= \frac{1}{3}$BC. Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the vectors represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ respectively, show that

$$\mathbf{w}.\mathbf{u} = \frac{5}{6}.$$



3. Triangle ABC is equilateral and the length of each side is one unit. D lies on BC and BD $= \frac{1}{3}$BC. Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the vectors represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ respectively, express $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathbf{v}$ and then show that

$$\mathbf{w}.\mathbf{u} = \frac{5}{6}.$$



groups as measured by the means on the standard question;

(ii) no significant difference in means between the control and the first change in formulation;

(iii) a significant difference in means ($p < 0.025$) between the control and the second change in formulation;

(iv) no significant difference in means between the two changes in formulation.

In order to find out the effect of giving the first step in improving the performance, the mean scores and standard deviations on the control and the second change in formulation have been calculated for their two parts (i.e. the first step and the rest). These can be found in Table 175. From this table, the following results were found:

(i) a highly significant difference in means ($p < 0.0005$) for the first step existed between the control and the modification in terms of the first step;

(ii) no significant difference in means on the rest existed between the control and the modification.

The effect of the wrong diagram on pupils' performance (suggesting an application of Pythagoras' theorem) and the use of a traditional geometric method (involving the cosine rule) rather than the intended vector approach was investigated. Table 176 gives the results in terms of percentages of pupils.

VII.3. Discussion

The wrong diagram caused a lot of difficulty: 27% of pupils were affected by this factor.

Correcting the diagram and sequencing the information are not enough to get a significant improvement in performance since the major difficulty is the insight required to find an appropriate strategy.

A significant number of pupils have the knowledge required to make the first step but fail to make it because they cannot think of a suitable strategy.

On the whole, it was the pupils who preferred a traditional approach to the vector method who were misled by the incorrect diagram.

VII.4. Conclusion

In a situation where a strategy is necessary, correcting a diagram and sequencing the information may not be enough to produce a significant improvement. But giving the first step is a significant factor in getting an improvement in performance.

	C_7	$T_7^{1.1}$	$T_7^{1.2}$	$S(C_7)$	$S(T_7^{1.1})$	$S(T_7^{1.2})$
mean*	0.54	0.95	1.18	4.40	4.00	3.90
SD	1.14	1.41	1.06	1.39	1.33	1.44
size	30	36	34	30	36	34

Table 174. Mean score and standard deviation on Question 7.

* out of 5 for the control, its techniques and the standard question (S).

	C_7 ($_1$)	C_7 ($_2$)	$T_7^{1.2}$ ($_1$)	$T_7^{1.2}$ ($_2$)
mean*	0.27	0.27	0.97	0.18
SD	0.69	0.64	0.87	0.46

Table 175. Mean score and standard deviation on the parts of Question 7 and the second change in formulation.

* out of 2 and 3 for the first ($_1$) and second ($_2$) parts respectively.

	C_7	$T_7^{1.1}$
affected by wrong diagram	27%	
used traditional geometrical method	3%	28%

Table 176. The effect of the incorrect diagram.

CHAPTER NINE

A "new" method for assessing the demand of a task

The following points should be made concerning the theoretical background in the field and the practical work of this research.

The demand of a question is an important factor in determining the rate of success or failure on this question. It is defined as the maximum number of thought steps and processes which had to be activated by the least able, but ultimately successful candidate in the light of what had been taught [105]. But there is no agreement about how to assess it [100]. Estimation of demand from pupils' working by the adopted method may give the relative demand rather than the absolute one, since the demand may be affected by factors not shown in the pupil's working. As a result of these factors, the facility value (FV) does not always decrease when the demand (Z) increases. This can be seen in most of the curves which illustrate the relationship between FV and Z.

It was found [5] that overload (i.e. demand exceeding a candidate's capacity) could be reached by:

(i) what is presented in the question itself;

(ii) what has to be recalled;

(iii) the complexity of the executed steps.

The familiarity of the language, the use of negative forms, the way the data is arranged, the existence of much irrelevant information, etc., illustrate the first factor. Recall of formulae, definitions and theories, etc., exemplify the second. However, failure to organise the key ideas into an appropriate strategy characterises the third one.

This leads us to the research programme. In order to identify how these factors affect the demand of a task, two experiments were carried out with school and SCE examination questions. The findings may be summarised as follows.

(1) Changes made to the language of questions which increase the amount of positive form and familiarity or replace a multiple-completion style by a multiple-choice one are, in general, positive.

(2) (a) An improved performance can be obtained by adding a diagram and changing the formulation of a question.

(b) A diagram carrying a lot of information is less helpful.

(c) A misleading diagram can cause problems, but this depends on the method of solution selected by pupils.

(3) When finding a strategy to start is an crucial factor of the difficulties, changes in a question's formulation may not be helpful, but giving a first step may improve performance.

(4) When a question is in an abstract form and there is a problem finding the insight to start, breaking it into steps or reducing abstract notation may help.

(5) An improved performance can be obtained by reducing dimensions from three to two.

(6) Grids, in general, are helpful and very effective in situations when:

(a) a question's difficulty is caused by faulty recall of formulae;

(b) a question is in an abstract form.

A grid which involves the appropriate formulae or offers the construction of a suitable diagram in stages is certainly very efficient. But, when a question requires complex reasoning (i.e. the executed steps are expected to be hard), the grid may not be helpful even though pupils, in general, may succeed in the identification of the relevant steps which are involved in the process of solution.

(7) The confidence rating is generally related to the value of the answer. It is high or low according to whether the answer is right or wrong, but few pupils are highly confident about incorrect responses. This may be due to the nature of the question or to the existence of a conflict in understanding.

(8) A question's structure has three aspects: aims, data and ideas. Pupils understand what they want (i.e. the aims), they have difficulty in identifying all the data (in particular, data in a conditional form indicated by "if") and they cannot find relevant ideas.

The question now is how we use the above findings to assess the demand of a task. Such an estimation - if it exists - should ensure that "pupils' performance decreases when the demand of the task increases and increases when it decreases. In order to achieve this goal, looking at the construction of a question is a crucial since it gives insight about the kind of links between the question's parts. The questions which are used in secondary schools and at the tertiary level are characterised by:

(i) indivisible questions;

(ii) divisible questions in which their parts are dependent or independent or a mixture (very few).

Therefore the estimation of the **relative demand** (as it is established in the literature), the question's division and the factors which are found to have a significant affect on performance are all involved in the "new" assessment of the demand.

In order to apply the "new" approach, follow the instructions below:

(i) No theoretical questions are involved in this method.

(ii) Questions which have dependent parts are taken as a whole rather than in parts.

(iii) Because questions which have a mixture of parts are rare, they are also taken as a whole but they are treated differently from dependent ones.

(iv) Questions which have independent parts are taken as separate parts.

(v) Indivisible questions are taken entire of course.

(vi) The symbols s, f and n denote the number of <u>thought steps</u>, <u>factors</u> and <u>dependent parts</u> involved in a question respectively, while d refers to the <u>demand</u> of the question.

(vii) Each factor is coded by <u>plus or minus one step</u> depending on how it affects a question by increasing or reducing the load. For example, if the <u>method of solution is mentioned</u>, the <u>first step is given</u> or a <u>question is "broken down" into two parts</u>, etc., one step is subtracted from the demand in each case. Where there is an <u>abstract form</u>, the <u>need for a strategy</u>, <u>the need for complex reasoning</u> or the <u>need to recall formulae</u>, etc., one step is added to the demand in each situation.

A question which may arise here is why the factor is coded by ±1 step and not by ±2 steps (or whatever). The answer is that the aim of this method is to ensure that the new estimation of demand does not fall below the "actual" demand.

According to the above instructions, the estimation of the demand of a question can be found by application of the following two rules.

RULE 1

(indivisible questions or questions which have independent parts)

$$\boxed{d = s \pm f}$$

RULE 2

(questions which have dependent parts or a mixture of parts*)

$$\boxed{d = s \pm f + (n - 1)}$$

(*) In the case of a mixture of parts, an alternative way of assessing the demand is separate the independent parts from the question and then apply Rule 1 to each of them and Rule 2 to the reminder of question (whose parts are dependent). But because the parts of a question in this category are (to some extent) related to each other, application of Rule 2 may be more convenient.

Application of the method to university and school samples

When the estimation of demand is revised according to the above rules, the curves of FV against demand appear to be more realistic. Some cases are discussed in detail below.

"A" class

In the revision of the April 1988 exam (see Table 8 and Figure 7 in Chapter 3), there was no change in the estimation of demand of items 1(i), 3(iii), 4(i) and 4(ii), while the demand of other items changed. Table 177 gives the new estimation of the demand of these items and the reason for the change. On the following page, the original curve (Figure 7) and the modified version (Figure 49) are shown in a small size format.

"B" class

The changes to the demand in the revision of the November 1987 exam (see Table 13 and Figure 9 in Chapter 3) are given in Table 178. After the table, both the original curve (Figure 9) and the modified version (Figure 50) are shown.

Table 179 gives the changes to the demand in the revision of the April 1988 exam (see Table 17 and Figure 11 in Chapter 3). Both the original curve (Figure 11) and the modified version (Figure 51) are shown after Table 179.

item	change in demand	reason
1ii	8 — 9	strategy
1iii	7 — 8	strategy
2	17 - 15	mixture, broken twice
3ii	9 - 10	collinearity concept
5a	8 — 7	familiarity
5b	9 — 8	familiarity
6i	8 — 9	strategy
6iia	4 — 7	understanding, recall, abstract
6iib	8 — 9	abstract

Table 177. The revised estimation of the demand of items in the second exam of the "A" class.

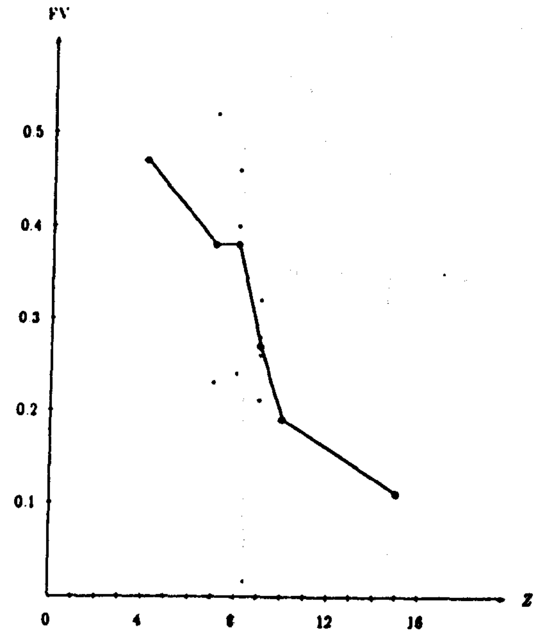


Figure 49

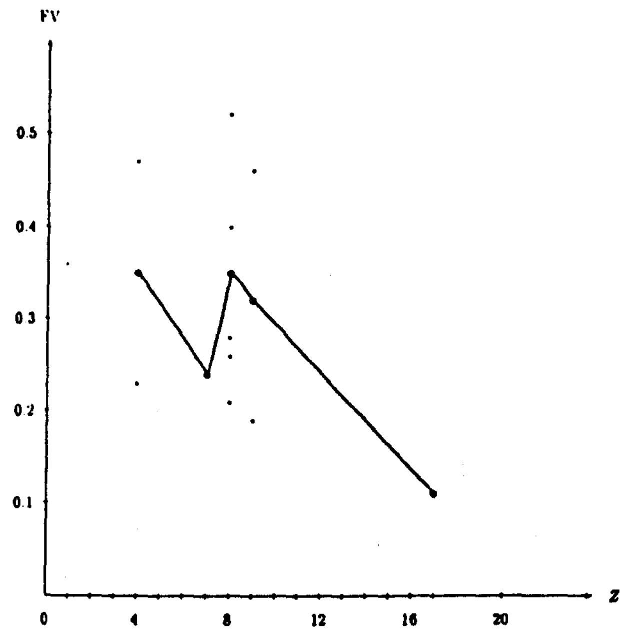


Figure 7

item	change in demand	reason
1i	3 — 2	help
1b	2 — 3	recall
1iiia	5 — 6	strategy
1iiib	3 — 4	recall
1iiib'	5 — 6	strategy
2ii	5 — 6	reasoning
2iiib	7 — 6	method given
2iv	8 — 7	help
3i	5 — 6	mixture, broken once, recall and reasoning
3ii	6 — 8	reasoning and recall
3iv	4 — 6	reasoning and recall
4ii	6 — 7	recall

Table 178. The revised estimation of the demand of items in the first exam of the "B" class.

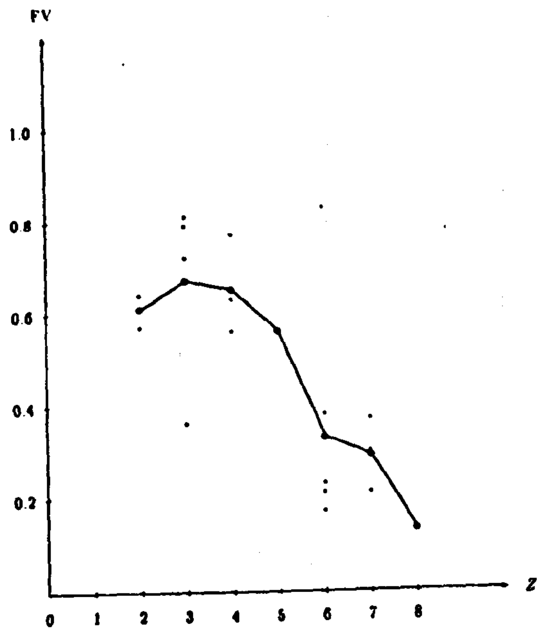


Figure 50

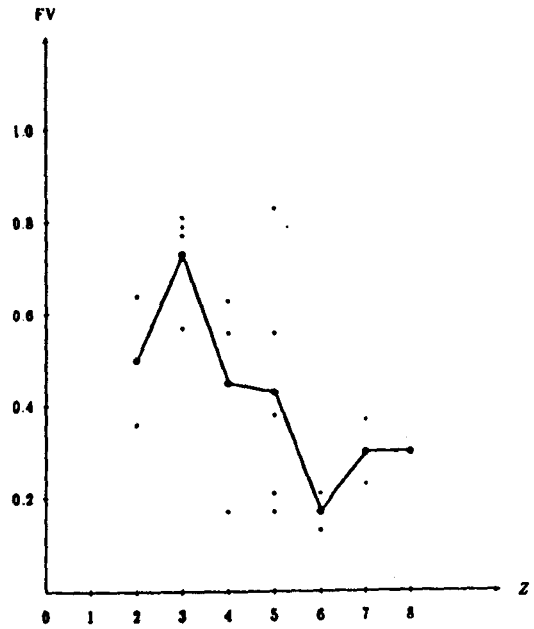


Figure 9

item	change in demand	reason
1i	14 - 10	mixture, broken four times
2i	6 — 5	method is given
2ii	4 — 3	method is given
2iii	7 — 6	method is given
3i	7 — 6	method is given
3iia	5 — 6	data
3iib	8 — 9	data
4i	7 — 8	recall
4ii	5 — 6	data
4iii	6 — 5	method is given
5iiia	4 — 5	data
5iiib	5 — 6	data
5iiic	4 — 5	data
6i	9 — 8	method is given
6ii	4 — 5	abstract form
6iii	5 — 6	abstract form

Table 179. The revised estimation of the demand of items in the second exam of the "B" class.

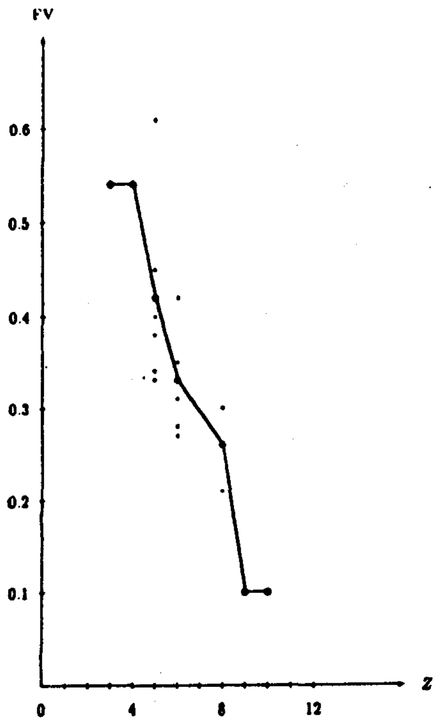


Figure 51

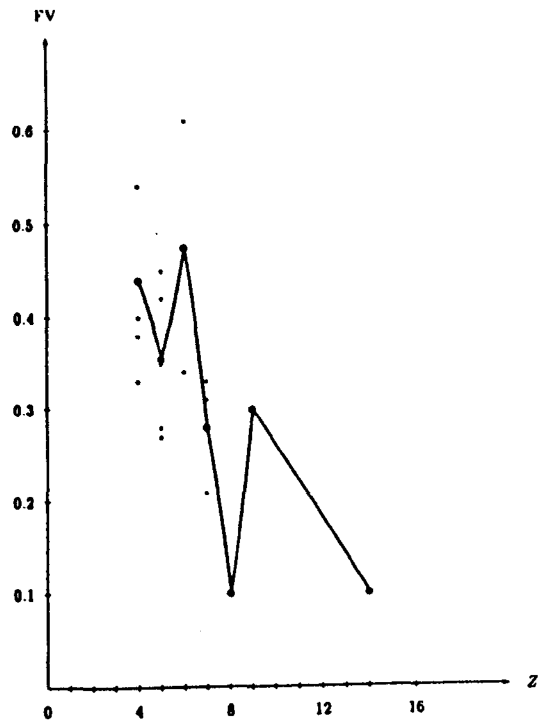


Figure 11

Scottish schools and the SCE

Table 89 (Chapter 7) gives the results of School 1. The changes in demand of items are given in Table 180. Figures 32 and 52 display the original curve and the modified one respectively. They are shown after the table.

The results of School 2 are given in Table 97 (Chapter 7). The changes are given in Table 181, followed by the original (Figure 34) and modified curve (Figure 53).

Table 105 (Chapter 7) gives the results of School 3. Table 182 shows the changes. Both the original curve (Figure 36) and the modified one (Figure 54) are shown after the table.

The results of School 4 are shown in Table 113 (Chapter 7). Table 183 gives the changes. The Figures 38 and 55 display the original curve and the modified one.

Table 121 (Chapter 7) gives the results of School 5. Table 184 shows the changes. Both the original curve (Figure 40) and the modified one (Figure 56) are shown after the table.

The results from the SCE are shown in Table 129 (Chapter 7). Table 185 gives the changes. Figures 42 and 57 display the original curve and the modified one.

Algerian school

Demand was estimated for the parts of the Algerian questions, even though the parts were dependent. This was done to show the difficulty of obtaining a realistic demand for such questions. Hence the original curves were of no relevance to the present discussion.

item	change in demand	reason
2	11 — 9	broken twice
3b	4 — 5	strategy
4c	5 — 6	recall
5	9 — 8	help
6	12 - 10	broken twice
7a	5 — 5	recall and given method
7b	7 — 8	reasoning
8c	5 — 6	reasoning
9b	7 — 8	strategy
10	11 - 10	broken once

Table 180. The revised estimation of the demand of items from School 1.

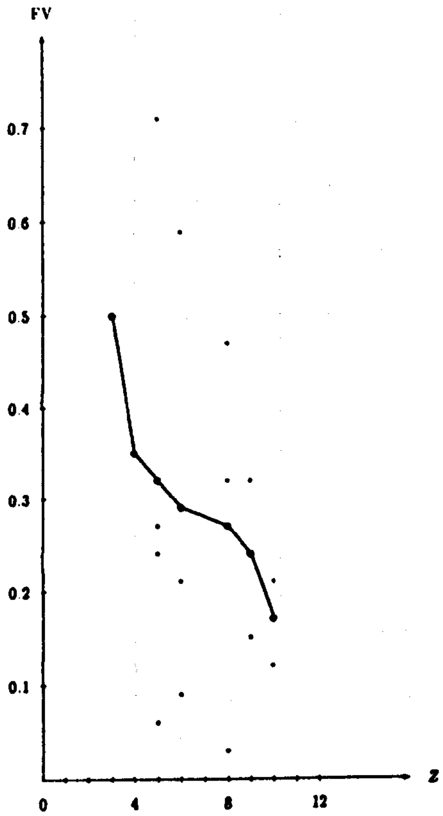


Figure 52

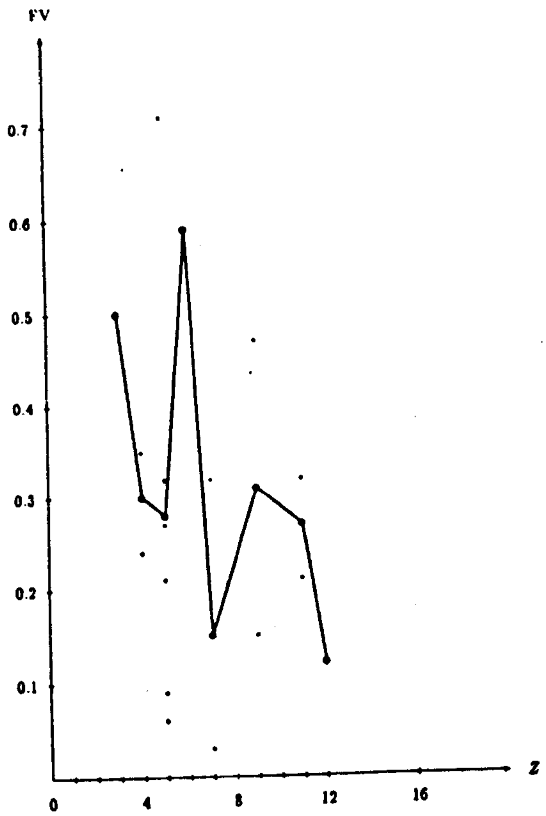


Figure 32

item	change in demand	reason
4	8 — 7	broken once
5	7 — 8	reasoning
6	17 - 14	broken three times
7	10 — 8	broken twice
8iii	4 — 6	recall and dimensions
9	11 - 10	broken once
10	7 — 8	abstract form

Table 181. The revised estimation of the demand of items from School 2.

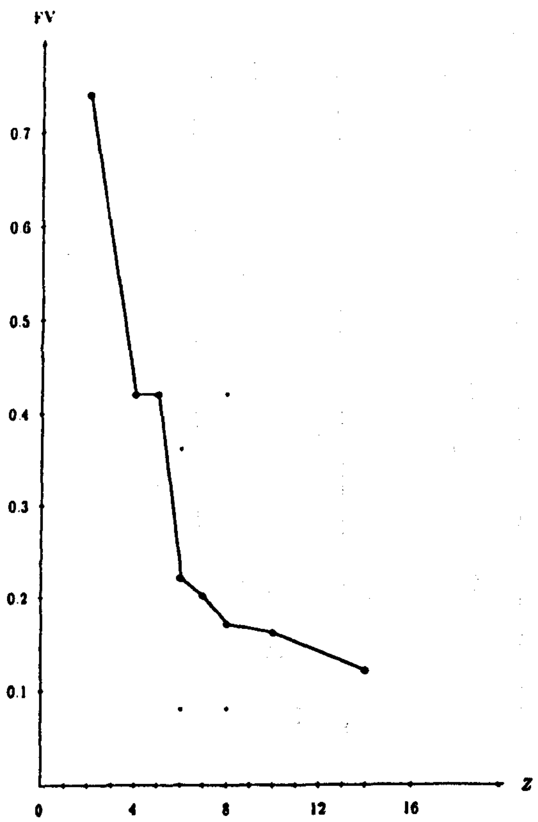


Figure 53

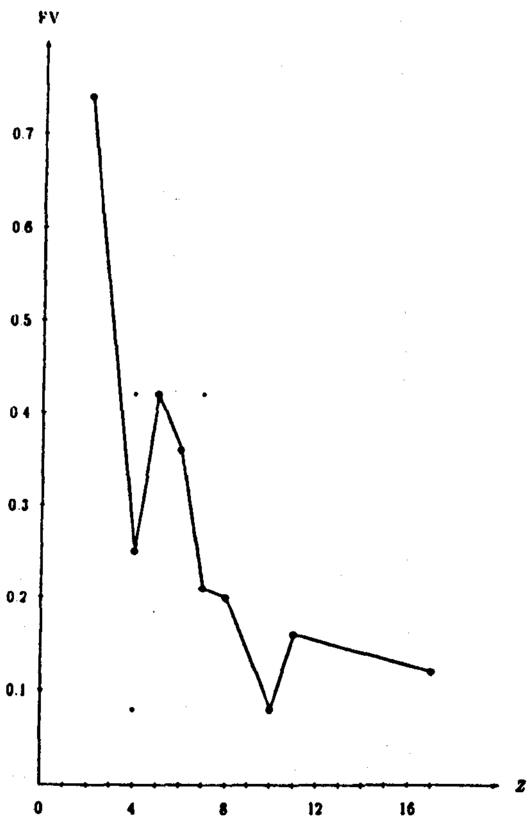


Figure 34

item	change in demand	reason
3a	6 — 7	reasoning
3b	5 — 6	recall
5	12 – 10	broken once and dimensions
6	12 – 10	broken twice
8	9 — 8	help
9a	9 – 10	recall, reasoning and broken once
9b	7 – 10	recall, reasoning, abstract
10i	4 — 5	strategy
10ii	6 — 7	data

Table 182. The revised estimation of the demand of items from School 3.

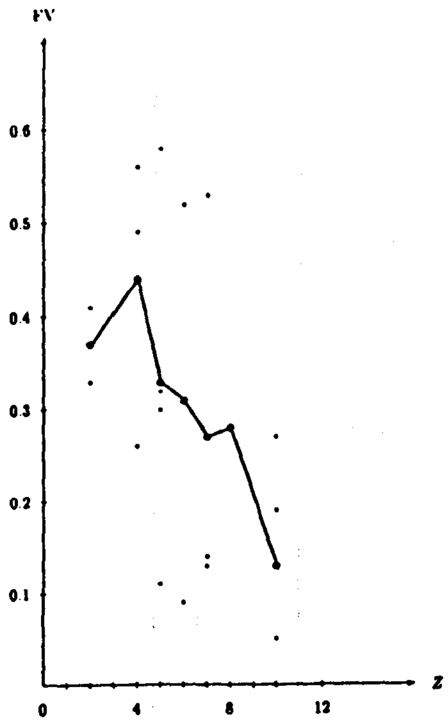


Figure 34

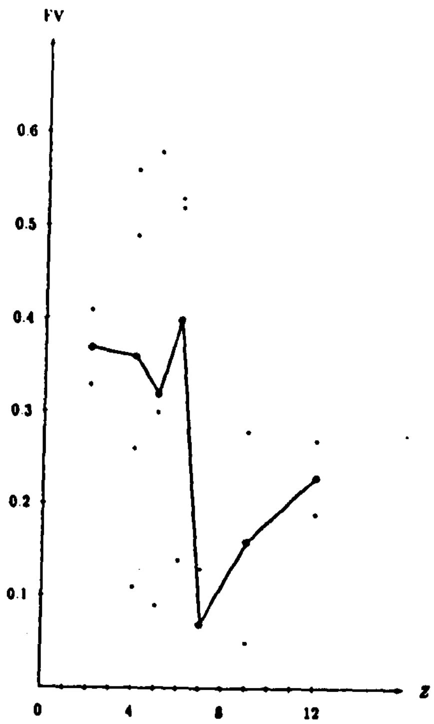


Figure 36

item	change in demand	reason
2a	5 — 6	recall, reasoning, broken once
2b**	4 — 5	strategy
3	11 — 9	broken twice
4	8 — 6	broken twice
5b	4 — 5	recall
6	10 — 8	broken twice
7	7 — 6	broken once
8a*	12 - 10	broken twice
8a**	3 — 4	abstract form
9	10 — 9	broken once
10	13 - 11	broken twice
12a*	7 — 8	recall
12b*	9 - 10	reasoning

Table 183. The revised estimation of the demand of items from School 4.

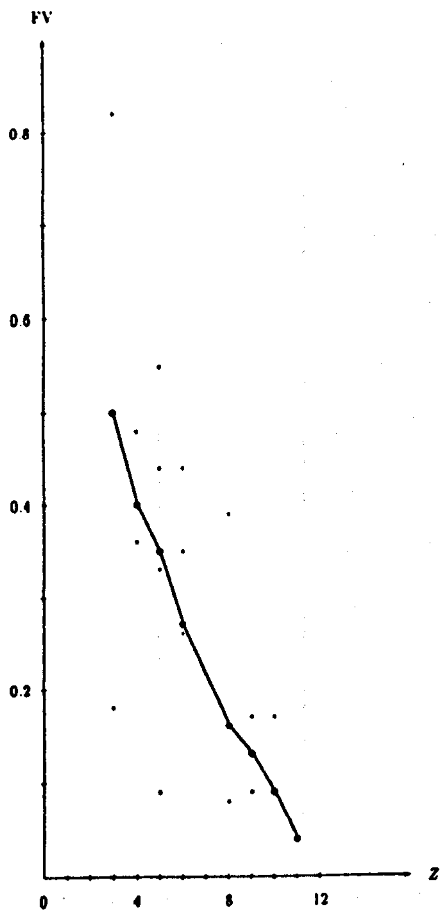


Figure 55

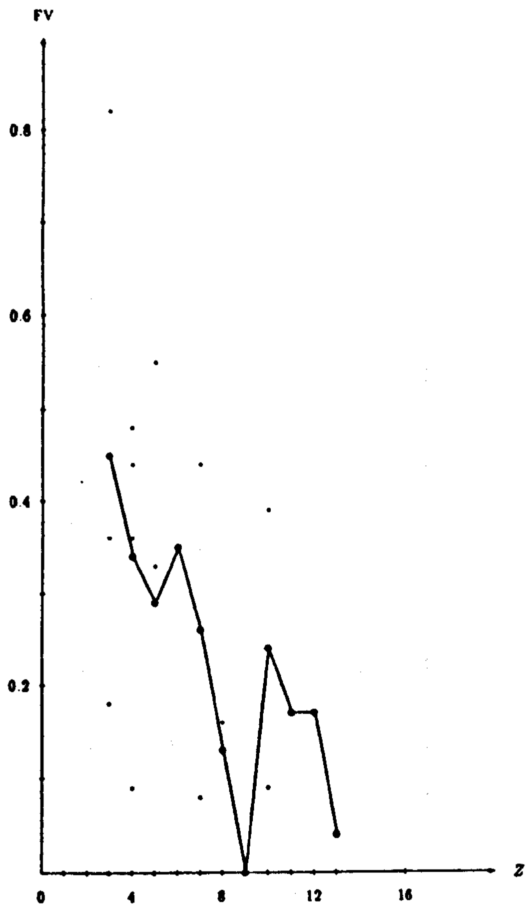


Figure 38

item	change in demand	reason
2	10 — 8	broken once and dimensions
3a	6 — 7	reasoning
5	13 - 11	mixture, broken twice
6	7 — 8	recall
7_1	4 — 5	concept of collinearity
8	12 - 13	strategy
9b	9 — 8	broken three times, abstract form and collinearity

Table 184. The revised estimation of the demand of items from School 5.

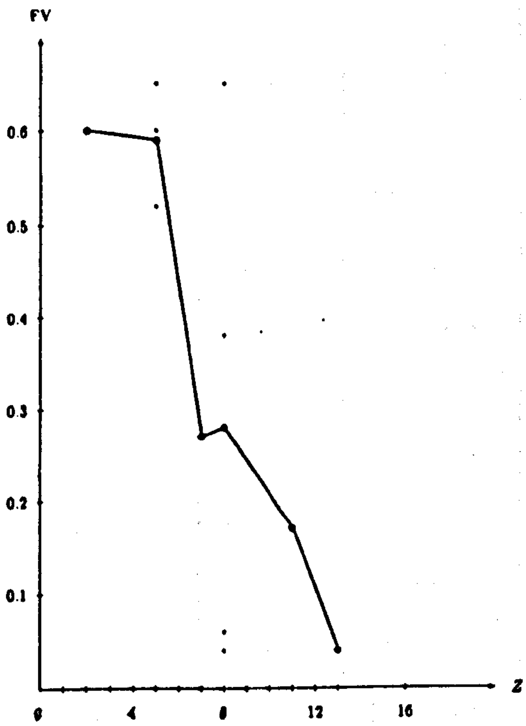


Figure 56

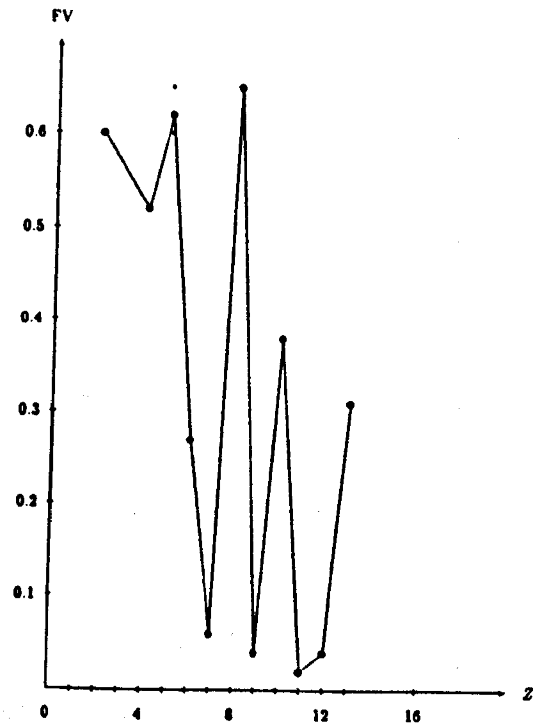


Figure 40

item	change in demand	reason
3	11 — 9	broken twice
4	12 — 9	broken three times
6	11 — 9	broken twice
7	9 — 9	broken once and dimensions
8	11 — 9	broken twice
9c	5 — 6	abstract form
10a	5 — 6	strategy
10b	6 — 7	concept of maximum
11a	7 — 8	reasoning
11b	6 — 7	recall
12	10 - 12	abstract form and collinearity
13	8 — 7	broken once

Table 185. The revised estimation of the demand of items from the SCE.

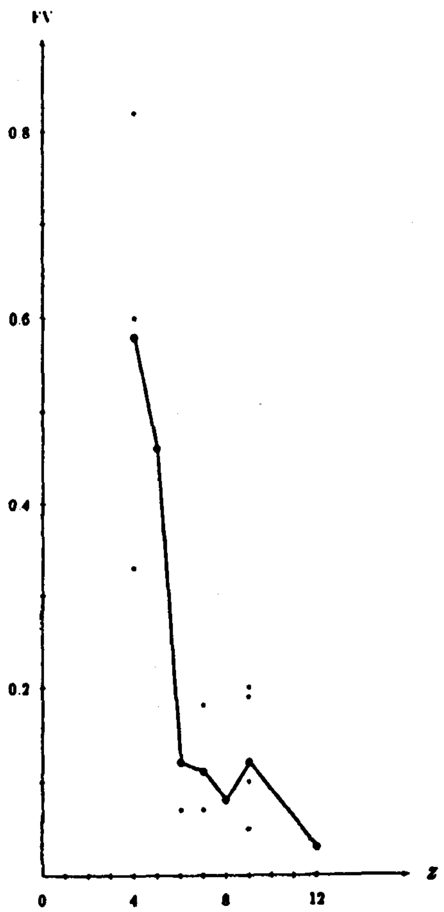


Figure 57

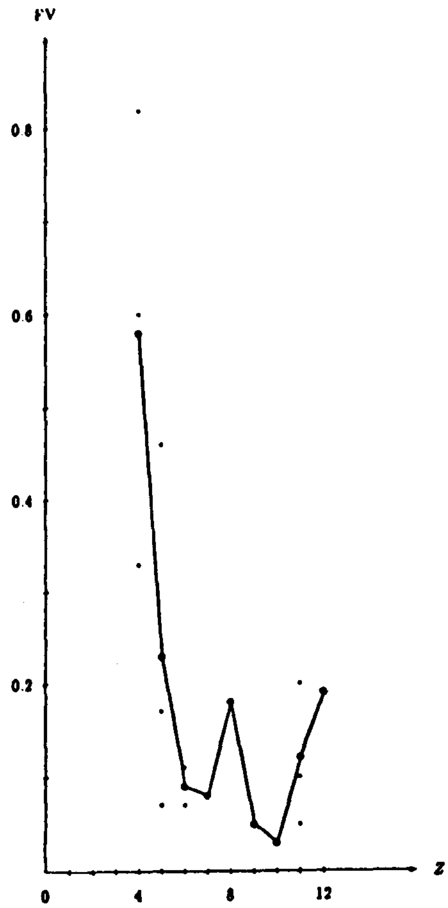


Figure 42

Conclusion

In the early part of this thesis, generally disappointing results were obtained where the demand of a question was determined solely by trying to total the number of thought steps taken by the least able, but successful candidate. Clearly this concept of demand was too naive.

As a result of studying of other factors affecting difficulty, a more complete picture of demand has emerged. When these new insights are brought to bear upon the analysis of a question's demand, a new set of curves are obtained which more nearly conform to the shape expected from Information Processing Theory. In most curves there is a sharp drop in performance indicating a point of overload. Questions whose demand exceeds the demand at this point of overload may test both mathematics, and the size and use of working memory space.

The modifications of our ideas of demand are clearly incomplete but we have moved to a more realistic position than before.

CHAPTER TEN

CONCLUSIONS AND RECOMMENDATIONS

In this final chapter, we focus on the following points:

I. summary of the findings from the research,

II. suggestions for further research,

III. applications for teaching and learning strategies.

I. Summary of the findings from the research

1. (i) Subjects' performance generally decreases (or increases) as the demand of a task increases (or decreases).

   (ii) A task with a high demand may test both knowledge and understanding plus psychological factors simultaneously.

   (iii) The analysis of a solution of a task into its thought steps has been made at both tertiary and secondary levels. But the method adopted can only give a relative demand rather than an absolute one which may involve factors not shown in subjects' overt working. Therefore, it is necessary to identify other factors in many situations.

   (iv) A new method for assessing the demand of a question has been devised and applied. The factors which were found in the experimental work, the structure of the question and the relative demand are involved in this method. The new estimation of the demand is more realistic than the relative one and it has potential for playing a role in the design of questions of moderate demand.

2. Field dependent-independent people differ in their performance. This may be due to the small or large working memory space available for handling irrelevant information. It was found that the influence of cognitive style on subjects' performance was generally as expected.

3. A positive improvement in performance occurred with changes from negative items to positive ones, and from multiple-completion items to multiple-choice ones, and for certain wording changes.

4. The difficulty of a question may be caused by the subject matter in respect of its nature, thought processes demanded and symbolism, or by other factors such as familiarity of language, structure (aims, data and ideas), appearance, etc.

The findings can be listed under four headings.

(i) An improved performance can be obtained by:

1. changing the formulation of a question and adding a diagram,
2. giving the first step,
3. breaking a question into steps,
4. simplifying the notation,
5. reducing the dimensions from three to two.

But changes in formulation of a question may not be helpful in situations such as when:

1. a strategy for starting is an crucial factor,
2. a question is in an abstract form.

(ii) We can use grids for diagnosis by noting certain omissions or particular inclusions. They can also be used for organising a subject's thinking and planning before doing a task. It was found that a grid is helpful when there is:

1. a faulty recall of formulae,
2. an abstract form of a question.

But when the question requires complex reasoning it may not be helpful.

(iii) The confidence rating, in general, is high in the right answers and low in the wrong answers.

(iv) The question's structure has three aspects: aims, data and ideas. Subjects can identify the aims, they have difficulties in finding all the data and they cannot find the relevant ideas.

II. Suggestions for further research

This research generated the following questions which need to be investigated.

1. Why do certain factors affect subjects' performance in a specific task rather than in general?

2. How do field dependence-independence and working memory capacity interact in the new method for the estimation of the demand of a task?

3. In the question structure, how can we classify the following in terms of their difficulties:

(i) indivisible questions,

(ii) questions which have dependent parts,

(iii) questions which have independent parts,

(iv) questions which have a mixture of parts?

4. Does the order of independent parts in a question affect its demand?

5. How do the following types of questions differ in their demand:

(i) questions which have dependent parts,

(ii) questions which have dependent parts and each part (except the first) needs the result of the solution of the previous one?

6. How can we assess the demand of a diagram?

7. Are there other factors which affect a task's demand?

III. Applications for teaching and learning strategies

Most of these applications are deduced from the findings of the research.

1. Learning should be regarded as an active process that occurs within the learner and can be influenced by the learner. Its outcome depends on what information is presented and how the learner processes it.

2. Better problem solvers in mathematics focus on the mathematical structure of the problem rather than superfluous information in its formulation. Therefore opportunity should be given to subjects to organize their knowledge into large chunks based on fundamental mathematical principles.

3. Chunking devices (graphs, tables, concept maps, etc.) should be used in presenting information in a meaningful way.

4. Techniques such as grids, investigating a question's structure, confidence rating, etc., could be used in the classroom situation.

5. The modification of instruction in the light of the limitation of working memory space is necessary and care should be given to any factor which could influence the demand of a task.

6. Familiarity with the wording of a task is essential before we deal with its solution. Rational thinking by subjects certainly will occur if we provide appropriate material in a right way and a right language.

7. The amount of irrelevant information should be minimised especially at the early stages of learning a new concept.

Finally, in the light of the above findings and recommendations, it is suggested that examiners should concentrate on questions of a demand which remains within the limitation of working memory space so that mathematical skill and competence is being measured without the interference of other factors. Teachers are recommended to devise means of helping their pupils to operate efficiently within the limitation of working memory space by adopting strategies to prevent overload. Such strategies are:

(i) breaking questions into workable parts,

(ii) using algorithms as working memory saves,

(iii) converting ideas and problems into visualisable form and using interconected knowledge.

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Appendix 1

Saturday, 28th November, 1987 9.30 *a.m.* to 11.30 *a.m.*

UNIVERSITY OF GLASGOW

MATHEMATICS

CLASS IA - ORDINARY A

First Examination

Use separate answer books for Course 1 *and Course* 2.

Course 1

1. (i) Simplify $\dfrac{1}{\sqrt{(1 - \frac{1}{1 - x^2})}} \cdot \dfrac{1}{(1 + x^2)^{3/2}}$.

(ii) Prove that, if $a > b > -\frac{4}{3}$, then

$$\frac{a + 1}{3a + 4} > \frac{b + 1}{3b + 4}.$$

(iii) Show that, provided $\theta \neq k\pi \pm \frac{\pi}{3}$ $(k \in \mathbb{Z})$,

$$\sin\theta = \frac{\sin 3\theta}{2\cos 2\theta + 1}.$$

Deduce the values of $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

(iv) Prove that $\sqrt{2}$ is irrational.

2. (i) Verify that -2 and 3 are roots of the polynomial

$$f(x) = 2x^4 - 5x^3 - 7x^2 + 16x - 12$$

and *deduce* that f has no further real roots.

(ii) Find partial fraction for the rational function

$$u(x) = \frac{x^2 - x}{(x + 1)(x^3 + 1)}.$$

(iii) Prove by induction that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n + 1)^2.$$

Hence (or otherwise) find a formula for the sum of the cubes of the first n odd positive integers.

[OVER

<u>Course 2</u>

3. (i) Given functions $f : A \to B$, $g : B \to C$ and $h : C \to D$, prove that $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) Find the maximal domain of definition of the real function g defined by

$$g(x) = \sqrt{(x^2 - 6x + 8)}.$$

Solve the equation

$$(h \circ g)(2 - x) = 2x$$

where h is the real function defined by $h(x) = x^2 - 1$.

(iii) Prove that the function $f : [1, 2] \to [2, 3]$, where $f(x) = 1 + \frac{2}{x}$, is a bijection and find its inverse.

4. (i) Given that $\sin^{-1} x = -\frac{1}{\sqrt{3}}$, evaluate $\cos^{-1} x$ and $\tan^{-1} x$.

(ii) By arguing from first principles, find $f'(x)$ when $f(x) = \sqrt{(2x + 3)}$.

(iii) Let f and g be real functions, differentiable at x. Prove that $f + g$ is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.

(iv) Express $\frac{dy}{dx}$ in terms of x and y for the curve defined by

$$x^2 + y^2 = xy + 1.$$

Find the points on the curve where $\frac{dy}{dx} = 0$.

END]

The analysis of some items into their demand

Item 2(i)

step 1 check that $f(-2) = 0$ and $f(3) = 0$,

step 2 notice that both $x + 2$ and $x - 3$ are factors of $f(x)$,

step 3 deduce that $x^2 - x - 6$ is a factor of $f(x)$,

step 4 determine the other factor,

step 5 calculate the discriminant of the other factor,

step 6 deduce there are no further real roots.

Therefore the Z-demand of this item is equal to 6 (thought steps).

Item 2(ii)

step 1 factorise $x^3 + 1$,

step 2 express the fraction as the sum of its partial fractions involving constants to be determined,

step 3 multiply by the denominator of the given fraction,

step 4 put $x = -1$ to determine one of the constants,

step 5 equate coefficients of three powers of x,

steps 6, 7 and 8 solve the resulting system of equations
(eliminate one of the constants from two equations,
solve these two equations,
determine the remaining constants),

step 9 substitute the values of constants in the general form.

Then the Z-demand of this item is equal to 9 (thought steps).

Item 2(iii)

step 1 establish the $n = 1$ case,

step 2 make the induction hyphothesis,

step 3 obtain the relationship between the formulae in the cases $n = k + 1$ and $n = k$,

step 4 take out the common factor $\frac{1}{4}(k + 1)^2$,

step 5 simplify the other factor,

step 6 observe that the $n = k + 1$ case is true and refer to induction to complete the proof,

step 7 use the given formula to find an expression for $\sum_{r=1}^{r=2n} r^3$,

step 8 use the given formula to obtain $\sum_{\substack{r=1 \\ r \text{ even}}}^{r=2n} r^3$,

step 9 substract the last result from the preceding one,

step 10 simplify the expression·

Therefore the Z-demand of this item is equal to 10 (thought steps).

Item 3(ii)

step 1 note that the quadratic expression must be non-negative,

step 2 factorise the quadratic,

step 3 construct the table of sign for the quadratic,

step 4 deduce the domain,

step 5 find an expression for $g(2 - x)$,

step 6 find an expression for $(h \circ g)(2 - x)$,

step 7 simplify the expression,

step 8 solve the equation,

step 9 check whether the solutions lie in the domain of function defined by $g(2 - x)$,

step 10 state the conclusion.

So the Z-demand of this item is equal to 10 (thought steps).

Appendix 2

Saturday, 23rd April, 1988 9.30 *a.m. to* 11.30 *a.m.*

UNIVERSITY OF GLASGOW

MATHEMATICS

CLASS IA - ORDINARY A (Including IAS)

Second Class Examination

Use separate answer books for Course 1 *and Course* 2.

Course 1

1. (i) Find the real and imaginary parts of $\frac{3+i}{1-i}$.

(ii) Express $\cos^4 x$ in the form $a \cos 4x + b \cos 2x + c$, with $a, b, c \in \mathbb{R}$.

(iii) Determine the polar form of $\surd 3 - i$, and hence find the real and imaginary parts of $(\surd 3 - i)^{13}$.

2. Sketch the curve

$$y = \frac{x^3}{(x-1)^2},$$

showing the behaviour at the origin, approaches to the asymptotes, and any critical points. Find where the curve crosses its non-vertical asymptote.

3. (i) *State* the section formula.

(ii) Let PQR be a triangle, and let S (respectively T, U) lie on PQ (respectively QR, RP) such that

$$PS:SQ = 2:1, \quad QT:TR = 2:3, \quad RU:UP = -3:4.$$

Prove that S, T and U are collinear, and find $ST:TU$.

(iii) Sppose that L_1 and L_2 are the lines

$$L_1: \ \frac{x}{1} = \frac{y-1}{2} = \frac{z-3}{-1},$$
$$L_2: \ \frac{x+1}{-2} = \frac{y-4}{1} = \frac{z}{-2}.$$

Show that L_1 and L_2 intersect, and find the point of intersection.

[OVER

4. (i) A particle moves on the x-axis so that its x-coordinate at time t is

$$x = \log(t^2 - 2t + 2).$$

Find the x-coordinate and the acceleration of the particle when its velocity is 0.

(ii) For the curve with parametric equations

$$x = t^2 + 2t, \quad y = \frac{1}{3} t^3 - t \quad (t \in \mathrm{R}),$$

Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t when $t \neq -1$.

5. Evaluate

(a) $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^2 x \cos^3 x \, dx$, (b) $\displaystyle\int_{-1}^{1} \frac{2x + 3}{x^2 + 2x + 5} \, dx$.

6. (i) Find the general solution of the differential equation

$$x \frac{dy}{dx} + 2y = e^x.$$

(ii) Consider the differential equation

$$ay'' + by' + cy = 0, \qquad (*)$$

where $a, b, c \in \mathrm{R}$ and $a \neq 0$. Given that r is the only root of the quadratic equation,

$$am^2 + bm + c = 0,$$

Show that $2ar + b = 0$, and that $y = xe^{rx}$ is a solution of (*).

END]

The analysis of the items into their demand

Item 1(i)

step 1 recall the complex conjugate method,

step 2 multiply numerator and denominator by complex conjugate,

step 3 simplify,

step 4 identify the real and imaginary parts of the complex number.

So the demand of this item is equal to 4 (thought steps).

Item 1(ii)

step 1 recall the formula cos $\frac{1}{2}(e^{ix} + e^{-ix})$,

step 2 express it as 2 cos $x = \frac{1}{z}$, $z = e^{ix}$,

step 3 take 4th power of both sides,

step 4 use the binominal theorem,

step 5 simplify,

step 6 collect together terms of the form $z^n + \frac{1}{z^n}$,

step 7 substitute 2 cos nx for $z^n + \frac{1}{z^n}$,

step 8 obtain the expression for $\cos^4 x$.

Therefore the demand of this item is equal to 8 (thought steps).

Item 1(iii)

step 1 determine the modulus of $z = \sqrt{3} - i$,

steps 2 and 3 determine the argumant (write down equations, and solve them),

step 4 write down the polar form,

step 5 deduce the polar form of z^{13}, step 6 find the trigonometric form of z^{13},

step 7 determine the real and imaginary parts of z^{13}.

The demand of this item is equal to 7 (thought steps).

Item 2

step 1 find the zero of the function,

step 2 deduce the horizontal point of inflexion at the origin,

step 3 identify the vertical asymptote,

step 4 construct a table of sign for y,

step 5 find the nature of the approaches to the asymptote,

step 6 divide the numerator by the denominator (long division),

step 7 obtain the approximation $\frac{3}{x}$ to the remainder fraction,

step 8 identify the non vertical asymptote,

step 9 find the x-coordinate of the point where the curve crosses the asymptote,

step 10 find the y-coordinate of the intersection point,

step 11 differentiate the expression,

step 12 simplify the derivative,

steps 13 and 14 find the x-coordinate of the critical point and the corresponding y-value,

step 15 construct a table of sign for y',

step 16 classify the critical point,

step 17 sketch the curve.

So the demand of this question is equal to 17 (thought steps).

Item 3(ii)

step 1 recall the section formula,

steps 2,3 find the position vectors $\boldsymbol{s}$, $\boldsymbol{t}$ and $\boldsymbol{u}$ of S, T and U

and 4 respectively,

steps 5,6 find the vectors $\overrightarrow{ST}$ and $\overrightarrow{TU}$ in terms of r, q and p,

step 7 obtain the relationship between $\overrightarrow{ST}$ and $\overrightarrow{TU}$,

step 8 deduce collinearity,

step 9 find the ratio.

The demand of this item is equal to 9 (thought steps).

<u>Item 3(iii)</u>

steps 1,2 obtain the parametric equations for the two lines,

step 3 equate the two expressions for x, for y and for z,

step 4 simplify the equations,

steps 5,6 solve the system of equations (find the values for parameters),

step 7 check the solution,

step 8 substitute the parameter to find the coordinates of the intersection point.

The demand of this item is equal to 8 (thought steps).

<u>Item 4(i)</u>

step 1 recall the velocity is $\dot{x}$,

step 2 differentiate the function,

step 3 recall the acceleration is $\ddot{x}$,

step 4 differentiate again,

step 5 find when $\dot{x}$ is 0,

steps 6,7 find the corresponding value of x (using $\log 1 = 0$),

step 8 find the value of $\ddot{x}$,

Therefore the demand of this item is equal to 8 (thought steps), Item 4(ii)

step 1 recall $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$,

step 2 find $\dot{x}$,

step 3 find $\dot{y}$,

step 4 factorize the numerator and denominator of $\frac{dy}{dx}$,

step 5 simplify $\frac{dy}{dx}$,

step 6 recall $\frac{d^2y}{dx^2} = \frac{1}{\dot{x}} \frac{d}{dt}(\frac{dy}{dx})$,

step 7 substitute for $\dot{x}$ and $\frac{dy}{dx}$,

step 8 differentiate $\frac{dy}{dx}$ with respect to t,

step 9 simplify $\frac{d^2y}{dx^2}$.

The demand of this item is equal to 9 (thought steps).

Item 5(a)

step 1 pick the substitution,

step 2 differentiate the substitution function,

step 3 calculate the new limits of integration,

step 4 recall the formula $\cos^2 x + \sin^2 x = 1$,

step 5 make the substitution,

step 6 multiply out the integral,

step 7 integrate,

step 8 substitute the limits of integration.

Therefore the demand of this item is equal to 8 (thought steps).

item 5(b)

step 1 split the fraction into two fractions,

step 2 notice the relation between the numerator and denominator,

step 3 recall the integral $\int \frac{f'(x)}{f(x)}\, dx$, step 4 complete the square in denominator of the second fraction,

step 5 recall the integral $\int \frac{dt}{t^2 + \lambda^2}$,

step 6 obtain the integrals,

step 7 substitute the limits of integration,

steps 8,9 simplify, using properties of log and $\tan^{-1}$.

The demand of this item is equal to 9 (thought steps).

Item 6(i)

step 1 identify the equation as a linear differential equation,

step 2 recall the formula for the integrating factor,

step 3 integrate $\frac{1}{x}$,

step 4 simplify the integrating factor,

step 5 multiply the differential equation by the integrating factor,

step 6 recall the form of the left hand side,

steps 7,8 integrate the right-hand side by parts.

The demand of this item is equal 8 (thought steps).

Item 6(ii)$_1$

step 1 recall the formula for the roots of a quadratic,

step 2 note that $\Delta = 0$ when there is one root,

step 3 substitute the value of Δ,

step 4 obtain the final result.

The demand of this item is equal to 4 (thought steps).

Item 6 (ii)$_2$

steps 1,2 find the derivative of the product xe^{rx} (using the derivative of e^{rx}), step 3 find the derivative of y,

step 4 substitute y' and y'' in the left-hand side of the equation,

step 5 multiply out and collect similar terms,

step 6 take out the common factor,

step 7 use the conditions for r being the only root,

step 8 confirm the final result.

The demand of this item is equal to 8 (thought steps).

Appendix 3

Saturday, 28th November, 1987 9.30 *to* 11.30 *a.m.*

UNIVERSITY OF GLASGOW

MATHEMATICS

CLASS IB - ORDINARY CLASS B

First Examination

1. (i) Sketch the curve $y = a^{-x}$, where a is a constant greater than 1. Your sketch should show clearly how the curve lies relative to each coordinate axis.

To what simpler expression is

$$e^{2x} + 10e^{-3x}$$

approximately equal

(a) when x is large and positive,

(b) when $|x|$ is large and x is negative?

(ii) Simplify

(a) $\frac{(2x^2)^3}{4x^5}$, (b) $\frac{3x - 2}{x^2 - 4} + \frac{1}{2 - x}$,

(c) $\exp[\frac{1}{2}\ln(1 + \sin^2 \theta)]$, (d) $\frac{4a^2b - 4ab^2}{a^2 - b^2}$.

(iii) (a) A quantity y is decaying exponentially in accordance with the equation $y = Ae^{-kt}$ (A, k being positive constants and t denoting time). Prove that the half-life of y is $\frac{\ln 2}{k}$.

(b) Two radioactive sources S_1 and S_2 are decaying exponentially.

(1) In the case of S_1 it is observed that one eighth of the original is left after 27 days. Without doing any non-trivial calculations, state which of the following is the half-life:

3 days; $6\frac{3}{4}$ days; 9 days.

(2) In the case of S_2 the half-life is known to be 12 days. Do a

[OVER

calculation to discover when just 10% of the original will be left. [If not using a calculator, you may take $\ln 2 = 0.69$, $\ln 10 = 2.3$.]

2. (i) Indicate in a sketch the set of points in the plane whose polar coordinates satisfy both the conditions

$$r > 1 \quad \text{and} \quad 0 < \theta < \tfrac{1}{4}\pi .$$

(ii) Find all roots in $\mathbb{C}$ (including any real roots) of the equation

$$x^3 - 5x^2 + 4x + 10 = 0.$$

(iii) Find the real and imaginary parts of

$$\text{(a)} \quad \frac{9 + 7i}{1 + 3i}, \qquad \text{(b)} \quad (-\sqrt{3} + i)^7,$$

using de Moivre's theorem in the case of (b).

(iv) Show that if $z = e^{i\theta} + 2$ (where $\theta \in \mathbb{R}$), then

$$z\bar{z} = 5 + 4 \cos \theta.$$

Hence find the real part of

$$\frac{e^{i\theta}}{e^{i\theta} + 2}$$

3. (i) Sketch the curve

$$y = x^2 - 6x + 5,$$

showing clearly the point where y reaches its minimum value, and also the points where the curve cuts the x-axis. Express as a union of intervals

$$\{x \in \mathbb{R} : x^2 - 6x + 5 > 0\}.$$

(ii) The height y of a particular water wave passing a stationary marker buoy varies with time t according to the equation

$$y = 3 \cos(2t + \tfrac{2\pi}{3}).$$

[OVER

Sketch the graph of this function, and state the period, amplitude and phase shift of the function.

(iii) Write down the formulae for $\sin(x + y)$ and $\sin(x - y)$ and hence prove that

$$\sin(x + y)\sin(x - y) = (\sin x + \sin y)(\sin x - \sin y).$$

(iv) Solve the equation

$$\cos(3x + \tfrac{\pi}{6}) = \tfrac{1}{2}.$$

4. (i) Differentiate each of the following with respect to x, simplifying your answers where possible:

(a) $(2x + 1)^3 (3x + 5)^4$; (b) $\tan[(3x - 1)^{\frac{1}{2}}]$;

(c) $\dfrac{x^2 - 3x + 6}{x + 2}$; (d) $\dfrac{\sin^2 x}{\cos 2x}$.

(ii) Find the tangent to the curve

$$y = x \ln x$$

at the point with x-coordinate e^2.

END]

The analysis of some items into their demand

Item 2(i)

step 1 identify the region determined by the first condition,

step 2 identify the region determined by the second condition,

step 3 sketch the set.

The Z-demand of this item is equal to 3 (thought steps).

Item 2(ii)

step 1 obtain the root -1 by inspection,

step 2 use synthetic division,

step 3 write the cubic as a product of a linear and a quadratic polynominal,

step 4 apply the formula for the roots of a quadratic,

step 5 simplify the roots.

The Z-demand of this item is equal to 5 (thought steps).

Item 2(iv)

It is possible to consider this item as two parts since the second part could be solved without solving the first, but it was taken as a whole since the students' solutions did not separate them.

step 1 find the complex conjugate of z,

step 2 multiply z by its conjugate,

step 3 convert the polar forms of the complex number to trigonometric forms,

step 4 simplify,

step 5 multiply the numerator and denominator by the complex conjugate of the denominator,

step 6 simplify (using the result of the previous part),

step 7 replace the polar form by a trigonometric form,

step 8 obtain the real part.

So the Z-demand of this example is equal to 8 (thought steps).

Item 3(i)

step 1 complete the square of the quadratic,

step 2 identify the minimum value,

step 3 factorize the quadratic (to find the intersections with the x-axis),

step 4 sketch the curve,

step 5 express the set as a union of intervals.

Therefore the Z-demand of this item is equal to 5 (thought steps).

Item 3(ii)

step 1 take out the factor 2 from the argument of the cosine,

step 2 determine the period,

step 3 determine the amplitude,

step 4 determine the phase-shift,

step 5 find some points of intersection with the axes,

step 6 sketch the curve, using the above information.

Therefore, the Z-demand of this item is equal to 6 (thought steps).

Appendix 4

Saturday, 23rd April, 1988 *9.30 to 11.30 a.m.*

UNIVERSITY OF GLASGOW

MATHEMATICS

CLASS IB - ORDINARY CLASS B

Second Examination

1. (i) For the curve C defined by

$$y = \frac{x^2 - 9x + 18}{x^2},$$

Find $\frac{dy}{dx}$ and show that

$$\frac{d^2y}{dx^2} = \frac{18(6 - x)}{x^4}.$$

Show that the curve C has one critical point and one point of inflection, determining the coordinates of each of these points and the nature of the critical point. Find the vertical asymptote of C, the intersections of C with the x-axis, and the asymptotic behaviour of C as $x \to \pm\infty$. Hence sketch the curve.

(ii) Show that

$$y = x\, e^{3x}$$

satisfies the differential equation

$$x \frac{d^2y}{dx^2} = 3x \frac{dy}{dx} + 3y.$$

2. (i) By making a suitable trigonometric substitution, find

$$\int \sin^4 x \cos^3 x \, dx.$$

(ii) Use integration by parts to find

$$\int x^2 \ln x \, dx.$$

(iii) By first completing to square, find

$$\int \frac{dx}{\sqrt{(5 + 4x - x^2)}}$$

[OVER

3. (i) Use partial fractions to evaluate the definite integral

$$\int_2^4 \frac{x + 5}{(x - 1)(x + 2)}\, dx.$$

(ii) Let R denote the region in the x,y-plane bounded by the the curve $y = x^3/3$, the x-axis, and the lines $x = 1$ and $x = 2$. A solid body is formed by rotating the region R through one revolution about the x-axis. Find the volume V and the surface area S of the curved surface of the body so formed.

4. (i) State the formulae for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$, and hence find a formula for the sum to n terms of the series

$$1.3 + 2.4 + 3.5 + 4.6 + 5.7 + \ldots .$$

Give your answer in fully factorized form.

(ii) At the end of each year a librarian records the value of s, the length of shelf-space occupied by the new books acquired by his library during the year. He finds a pattern: each year the value of s exceeds its value in the previous year by 10%.

(a) State what kind of sequence (arithmetic or geometric) the successive values of s form.

(b) If s was 50 metres in 1987 and if the pattern noted by the librarian continues, what will be the total length of shelf-space needed for books acquired by the library in the 15-year period 1987 to 2001, inclusive?

(iii) Use complex numbers to express $\cos^5 \theta$ in the form

$$a \cos 5\theta + b \cos 3\theta + c \cos \theta$$

with a, b, c constants.

[OVER

5. (i) A doctor has 5 patients to visit one morning. In how many different orders might he make the 5 visits?

(ii) A box contains 10 light bulbs, of which 6 work and 4 do not. A man chooses 3 bulbs at random from the box (without replacement). What is the probability that the 3 bulbs which he chooses all work?

(iii) Three students A, B, and C form a team to take part in a television quiz show. Extensive testing during auditions indicates that, on average, they can answer correctly 75%, 85%, and 50%, respectively, of the questions of the type used in the show.

(a) If 5 questions are put to A, what is the probability that he will answer correctly exactly 3 of them?

(b) If 5 questions are put to C, what is the probability that he will answer correctly more than one of them?

(c) If the question-master is about to select one member of the team at random and put a question to him, what is the probability that the outcome will be a correct answer?

6. (i) Find the general solution of the system of equations

$$\left\{\begin{aligned} x_1 + 2x_2 + 2x_3 + 5x_4 &= 3 \\ x_1 + 3x_2 + 2x_3 + 2x_4 &= 1 \\ x_1 + 4x_2 - 2x_3 - 3x_4 &= 3 \end{aligned}\right\}$$

by first transforming the augmented matrix of the system to reduced echelon form.

(ii) A square matrix A satisfies

$$A^2 - 3A + 3I = O.$$

Prove that A is nonsingular, and express A^{-1} in terms of A.

(iii) It is given that P and B are $n \times n$ matrices, that P is nonsingular, and that $(P^{-1}BP)^2 = I$. Prove that $B^2 = I$. END]

Appendix 5

Amara Rachid School   Time: 2½ hours

The first term examination in mathematics

1. (a) Find the set of remainders of (i) $(4)^n$, (ii) $(3)^n$ upon division by 7 for integers $n > 0$.

   (b) Prove that $106(32)^{3n+1} + (17)^{6n-5}$ is divisible by 7 for all integers $n > 0$.

   (c) Find the remainder of $(851)^{1989} + (4343)^{1409}$ upon division by 7.

2. Let a and b be positive integers and let $\alpha = \gcd(a,b)$ and $\beta = \text{lcm}(a,b)$. Dtermine all pairs(a,b) which satisfy

$$\beta = \alpha^2, \quad \alpha + \beta = 156 \quad \text{and } a > b.$$

3. Let p be a prime number and a, b and c the integers which are represented by 7, 238 and 1541 in base p respectively.

   (a) Find p such that $c = a.b$.

   (b) Represent these integers in base 10.

   (c) Solve the equation $ax + by = 1$, where x and y are integers.

[OVER

4\. A. Let $(S_n)_{n\epsilon\mathbb{N}}$ be the sequence of numbers defined recursively by

$$S_1 = 2, \quad S_{n+1} = \frac{1}{3}(S_n + 5).$$

(a) Evaluate S_2 and S_3.

(b) Prove by induction that $(S_n)_{n\epsilon\mathbb{N}}$ is an increasing sequence.

(c) Prove that $(S_n)_{n\epsilon\mathbb{N}}$ is bounded above by $\frac{5}{2}$.

B. Let b be a real number and $(S'_n)_{n\epsilon\mathbb{N}}$ the sequence of numbers defined by

$$S'_n = S_n + b \qquad (n\epsilon\mathbb{N}).$$

(a) Find b such that $(S'_n)_{n\epsilon\mathbb{N}}$ is a geometric sequence. Find the common ratio and the first term of this sequence. Determine whether $(S'_n)_{n\epsilon\mathbb{N}}$ is convergent.

(b) Express S'_n and S_n in term of n.

(c) Determine $\lim\limits_{n\to\infty} \left(\sum\limits_{r=1}^{r=n} S'_r \right)$ and $\lim\limits_{n\to\infty} \left(\sum\limits_{r=1}^{r=n} S_r \right)$.

(d) Find a simple expression, in terms of n, for the product P defined by

$$P = S'_1 \, . \, S'_2 \, . \, \ldots . S'_n \, .$$

Hence determine $\lim\limits_{n\to\infty} P$.

Appendix 6

Amara Rachid School　　　　Time: 3 hours

The second term examination in mathematics

Question 1

I. Consider the equation

$z^3 - 2(2 + 3i)z^2 - 4(1 - 5i)z + 16(1 - i) = 0$..(*)

in the complex number z.

(a) Prove that equation (*) has a real solution $z = \alpha$ and find this solution.

(b) Determine the other two solutions of equation (*).

(c) Plot all the solutions in the complex plane.

II.(a) Show that the solutions are the first three terms of a geometric sequence which has first term $S_0 = \alpha$. Find the common ratio and the 17th term.

(b) Find an expression in n for the general term S_n. Determine the numbers n such S_n belongs to the set of integers $\mathbb{Z}$.

Question 2 In the triangle ABC, A is a right angle, AB = 3 and AC = 4. Associated with A, B and C are the coefficients $\alpha - 3$, α and $1 - \alpha$, respectively, where α is a real number.

[OVER

(1) For which values of α do $A(\alpha - 3)$, $B(\alpha)$ and $C(1 - \alpha)$ have a barycentre O_α in the plane of triangle ABC?

(2) Determine the point O_4.

(3) Determine the set of points P in the plane which satisfy the equation

$$(\overrightarrow{PA})^2 + 4(\overrightarrow{PB})^2 - 3(\overrightarrow{PC})^2 = \lambda,$$

where λ is a given real number, and discuss the geometrical nature of this set.

(4) Let T_α ($\alpha \neq 2$) be the transformation which maps any point P of the plane to the point P' such that

$$\overrightarrow{PP'} = (\alpha - 3)\overrightarrow{PA} + \alpha\overrightarrow{PB} + (1 - \alpha)\overrightarrow{PC}.$$

Find the real number k such that

$$\overrightarrow{O_\alpha P'} = k\overrightarrow{O_\alpha P}.$$

What happens if $\alpha = 2$?

<u>Problem</u> Let f be the real function defined by

$$f(x) = \frac{2x}{x^2 + 1}.$$

(1) Investigate the behaviour of f and sketch its curve (α) in a coordinate plane in which the lengths of the orthogonal unit vectors $\imath$ and $\jmath$ are given by

$|\imath| = |\jmath| = 2$ (centimetres).

(2) Let g be the real function defined by

$$g(x) = \frac{2|x|}{x^2 + 1}.$$

(a) Investigate the continuity and differentiability of this function at $x = 0$.

(b) Show how to derive a sketch of the curve (β) of the function g from the sketch of the curve (α).

[OVER

(3) Let h be the mapping of the interval $[-1, 1]$ to itself defined by

$$h(x) = f(x),$$

for every x belonging to the interval $[-1, 1]$. Show that h is bijective. Sketch the curve (γ) of the inverse function h^{-1} of h in the same diagram as (β).

(4)(a) Find the area $A(\lambda)$ of the region bounded by the curve (α) and the lines $y = 0$, $x = 0$ and $x = \lambda$, where λ is a positive real number.

(b) Evaluate $\lim_{\lambda\to\infty} \dfrac{A(\lambda)}{\lambda^2}$ and $\lim_{\lambda\to 0} \dfrac{A(\lambda)}{\lambda^2}$.

(You may find it helpful to put $k = \lambda^2 + 1$.)

(c) Find the area of the region consisting of the points $P(x,y)$ defined by the inqualities

$$\begin{cases} 0 \leqslant x \leqslant 1, \\ x \leqslant y \leqslant f(x). \end{cases}$$

Deduce the area in cm^2 of the region situated between the curves (α) and (γ) and the lines $x = -1$ and $x = 1$. (Take $\ln 2 = 0.69$.)

END]

Appendix 7

Amara Rachid School Time: 4 hours

The third term examination in mathematics

Question 1

(1) Find the product $(a + ib)(a - ib)$ in $\mathbb{C}$, the set of complex numbers.

(2) Let f be the polynomial function defined over the complex numbers by

$$f(z) = (z^2 + 3z)^2 + (3z + 5)^2.$$

(a) Factorize $f(z)$ as a product of two polynomials of second degree with complex coefficients.

(b) Solve the equation

$$z^2 + 3(1 + i)\, z + 5i = 0, \quad z \in \mathbb{C}.$$

(c) Deduce the solutions of the equation $f(z) = 0$ in the set of complex numbers $\mathbb{C}$, and prove that $f(z)$ is a product of two polynomials of second degree with real coefficients.

(3) Application

Let α be a natural number which is greater than or equal to 6 and a, b the numbers represented by 130, 35 in base α respectively. Prove that $a^2 + b^2$ is a product of two natural numbers whose representations in base α are independent of α.

[OVER

Question 2

A curve (Γ) has equation

$$x^2 - y^2 - 2xy - x + y = 0,$$

in a coordinate plane in which i and j are the unit vectors.

(1) Let f be the transformation of the plane which maps a point $P(x, y)$ to the point $P'(x', y')$ given by

$$\begin{cases} x' = 5x + 2y - 2, \\ y' = 2x + y - 1. \end{cases}$$

Prove that f is bijective and the curve (Γ) is invariant under the transformation f.

(2) Let S_0 be the point $(3, 1)$ and $(S_k)_{k\in\mathbb{N}}$ the sequence defined by

$$S_{k+1} = f(S_k), \quad k\in\mathbb{N},$$

where $\mathbb{N}$ is the set of non-negative integers. Prove that $S_k \in (\Gamma)$ for all $k \in \mathbb{N}$.

Let (x_k, y_k) be the coordinates of S_k, $k \in \mathbb{N}$. Prove that $(x_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ are both strictly increasing sequences and

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = \infty.$$

Problem

I. Let f_λ be the real function defined by

$$f_\lambda(x) = \frac{\lambda}{x} + \log x^2, \quad \lambda \in \mathbb{R},$$

and (C_λ) its curve in a coordinate plane in which i and j are the orthogonal unit vectors.

[OVER

(1) Investigate the behaviour of f_0 (i.e. f_λ in the case $\lambda = 0$) and sketch its curve (C_0).

(2) Let (γ) be the curve with equation $y = \log x$. What are the two transformations which enable you to sketch (C_0) from (γ)?

(3)(a) Given that Ω_λ is the point on (C_λ) at which the tangent is parallel to the x-axis, determine the set E of points Ω_λ, where λ varies over $\mathbb{R} - \{0\}$.

(b) Prove that the set E is the image of (C_0) under a simple transformation and find this transformation.

(4) Given that $\lambda \neq 0$, deduce the geometrical relationship between the curves (C_λ) and $(C_{-\lambda})$ by comparing $f_\lambda(x)$ and $f_{-\lambda}(x)$.

II.(1) Given that

$$g(x) = (2x^2 - 3x)\, e^x, \; x \in \mathbb{R},$$

verify that the function g is a solution of the differential equation

$$y'' - 2y' + y = 4e^x,$$

and deduce a primitive function for g.

(2) Evaluate

$$A(\lambda) = \int_\lambda^0 g(x)\, dx,$$

and find $\lim\limits_{\lambda \to -\infty} A(\lambda)$.

(3) Recall that $g^{(n)}$ is the nth derivative of g. Prove that, for all positive integers n,

$$g^{(n)}(x) = (2x^2 + a_n x + b_n)e^x,$$

[OVER

for some integers a_n and b_n, finding expressions for a_n and b_n in terms of n.

(4) What is the nature of the sequence $(a_n)_{n\in\mathbb{N}-\{0\}}$?

III. Let T be the transformation of the affine plane (Π) which maps a point $P(x, y)$ to the point $P'(x', y')$ given by

$$\begin{cases} x' = \dfrac{\lambda x}{2x - \lambda}, \\ y' = \dfrac{\lambda y}{2x - \lambda}, \end{cases}$$

where λ is a positive real number.

(1) Determine F, the set of points P which do not have an image P' under the transformation T.

(2) Suppose that P is not in the set F.

(a) Determine the set of invariant points under the transformation T.

(b) Determine the inverse transformation T^{-1}; what can you deduce?

(c) Prove that the points O, P and P' lie in the same line.

(d) (Γ) is the circle which passes through the origin O and has centre $O_1(-\frac{1}{4}\lambda, 0)$. Given that $(\Gamma') = T(\Gamma)$, determine the equation of (Γ') and state the characteristics of (Γ').

IV. Throughout this part, (Π) is the Euclidean affine plane oriented directly.

(1) D is the dilatation with centre $O'(1, 0)$ and scale factor 2.

[OVER

Find the algebra formulae which specify this dilatation. (Δ_1) is a line through the point H(1, 1) with direction vector $v = 2i + 3j$. Determine $D\ (\Delta_1)$.

(2) R is the transformation of the plane (Π) which maps a point P (x, y) to the point $P'(x', y')$ such that

$$\begin{cases} x' = \frac{x}{2} - \frac{\sqrt{3}}{2} y, \\ y' = \frac{\sqrt{3}}{2} x + \frac{y}{2}. \end{cases}$$

Prove that R is a retation and determine the centre and the angle of the retation.

(3) Let $K = R \circ D$. Determine the set of invariant points under the transformation K. Prove that K is bijective and determine its inverse K^{-1}.

(4) In the complex plane associated with (Π), L is a transformation wich maps a point P (z) to the point $P'(z')$ given by

$$z' = \sqrt{3}\, iz + 2$$

(a) Express L as composition of three transformation $(L = T_3 \circ T_2 \circ T_1)$, and state the formulae which define these transformations.

(b) Determine the nature of the transformation L and its characteristic properties.

V. The variable point $P(x, y)$ in the plane (Π) is a given by

$$x = \varphi(t), \quad y = \psi(t),$$

where,

[OVER

$$\begin{cases} \varphi(t) = \lambda^2 \dfrac{\cos t}{1 - \cos t}, \\ \psi(t) = \dfrac{\sin t}{1 - \cos t}, \end{cases}$$

the constant $\lambda \neq 0$ and $t \in (0, 2\pi)$.

(1) Find the equation of the trajectory of P.

(2) The point $P_1(x_1, y_1)$ on the plane (Π) is given by

$$x_1 = \varphi(t + \pi), \quad y_1 = \psi(t + \pi).$$

Prove that O, P and P_1 lie on the same line.

(3) Determine the trajectory of the mid-point N_0 of PP_1 as t varies over $(0, 2\pi)$.

<u>Note</u>

Parts I, II, III, IV and V are independent.

END]

Appendix 8

Hidden Figures Test (HFT)

The aim of this test is to measure the students' degree of FD/FI. The classification of these students into FD/FI groups has been made according to their ability to discriminate the required item from its context.

The HFT contains 18 complex figures and 2 examples. In this test, there are 6 simple geometric and non-geometric shapes which are embedded in the complex figures and the subject should isolate these shapes. He is required to locate a hidden simple shape in each complex figure and then trace it in pen against the lines of the complex figure. The following conditions are given to the subject:

(a) the simple shape has to be found with the same size, same proportions, and the same orientation within the complex figure;

(b) the subject is not allowed to use a ruler or any other means to measure the size of the simple shape in the complex figure;

(c) there is more than one simple shape embedded in some complex figures but the subject is required to locate only the ones which satisfy the (a) condition and trace only one;

(d) the time allowed to the subject to do this test is 20 minutes.

Because the test comprises 18 complex figures, the maximum mark that can be obtained is 18 (i.e. one mark per correct figure).

A sample of the HFT

NAME......................   SEX.................

SCHOOL....................   DATE OF BIRTH.......

CLASS.....................

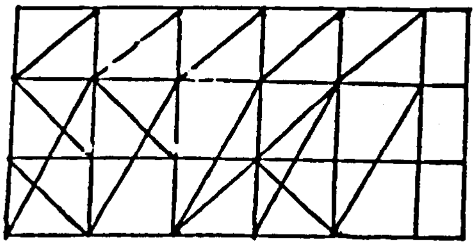
This is a test of your ability to find a simple shape when it is hidden within a complex pattern. The results will not affect your school work in any way.

Example 1

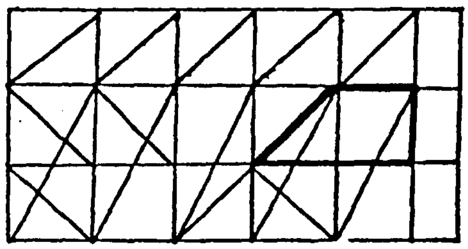
Here is a simple shape which we have labelled (*X*)

(*X*)

This simple shape is hidden within the more complex figure below



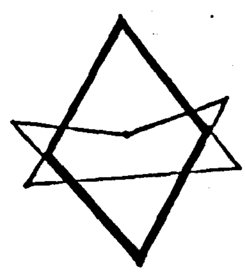
Try to find the simple shape in the complex figure and trace it in pen directly over the lines of the complex figure. It is the same size, in the same proportions, and faces in the same direction within the complex figure as when it appeared alone. (When you finish, turn the page to check your answer).



Example 2

Find and trace the sample shape (Y) in the complex figure beside it.

The answer is



In the following pages, problems like the ones above will appear. On each page you will see a complex shape, and beside it will be an indication of the sample shape which is hidden in it. For each problem, try to trace the simple shape in pen over the lines of the complex shape.

Note these points:

(1) rub out all mistakes;

(2) do the problems in order; do not skip a problem unless you are absolutely stuck on it;

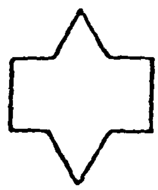
(3) trace only one simple shape in each problem, you may see more than one, but just trace one of them;

(4) the sample shape is always present in the complex figure with the same size, same proportions, and facing in the same direction as it appears alone;

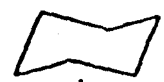
(5) look back at the simple forms as often as necessary.

Now attempt each of the items in the following sheets.

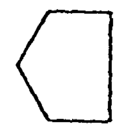
SIMPLE FORMS



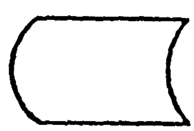
A



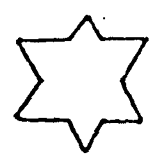
B



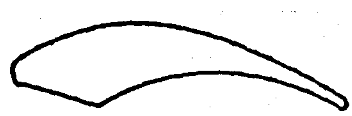
C



D

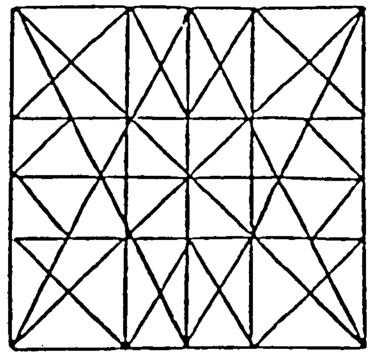


E



G

FIND SIMPLE FORM 'C'



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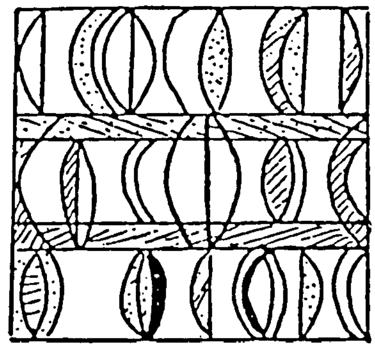

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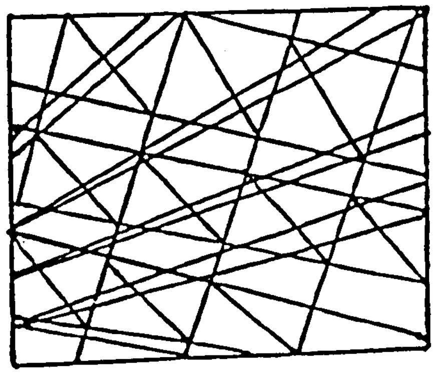

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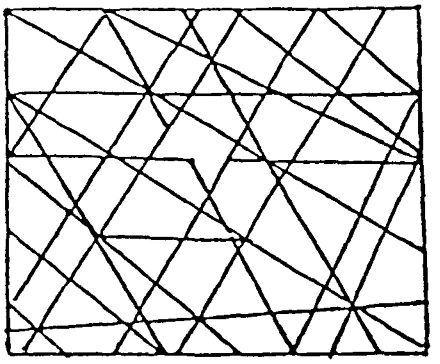

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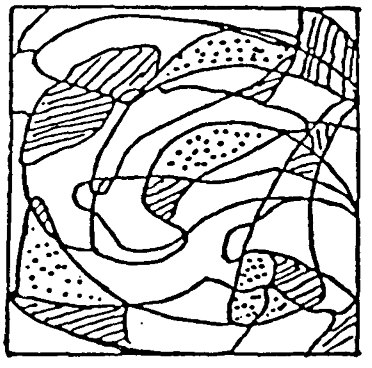

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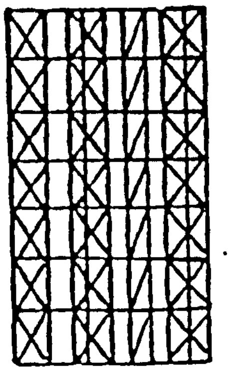

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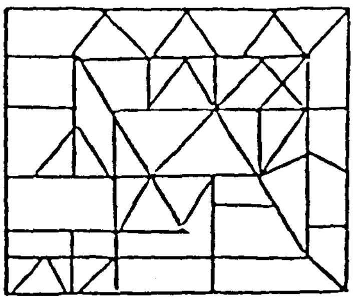

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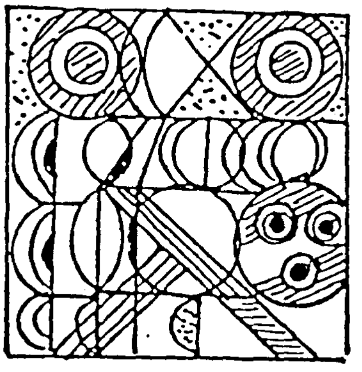

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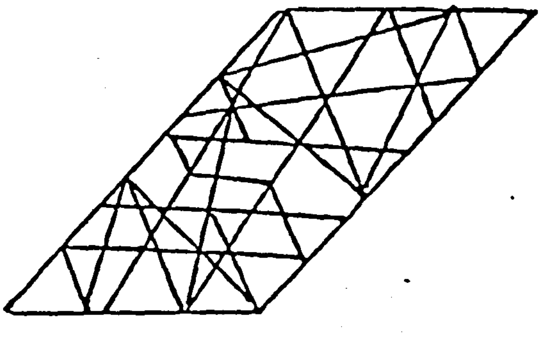

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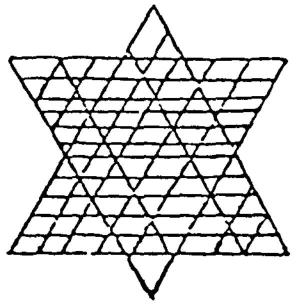

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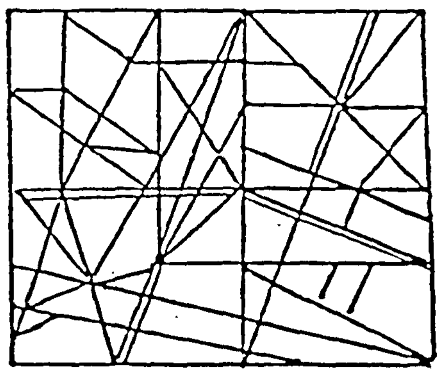

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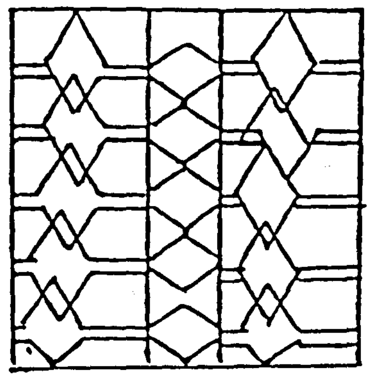

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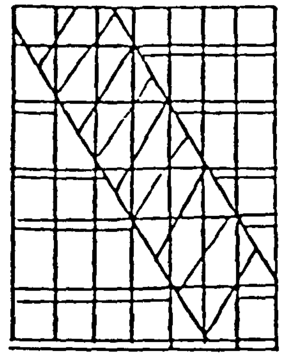

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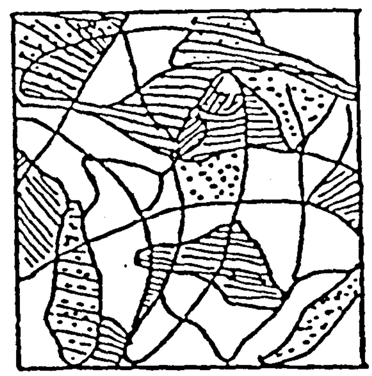

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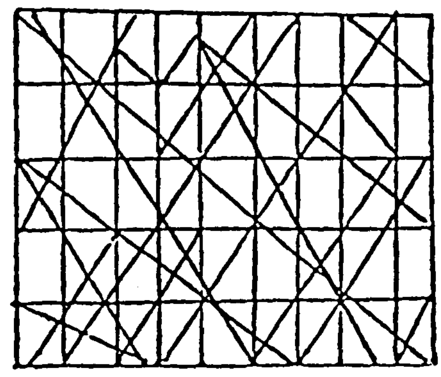

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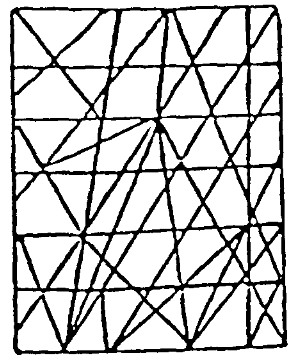

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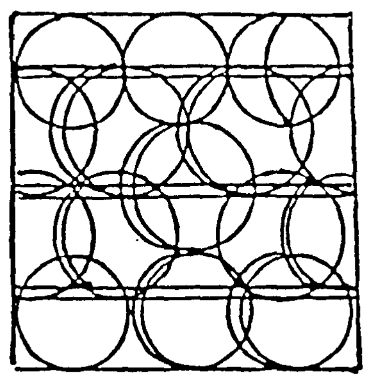

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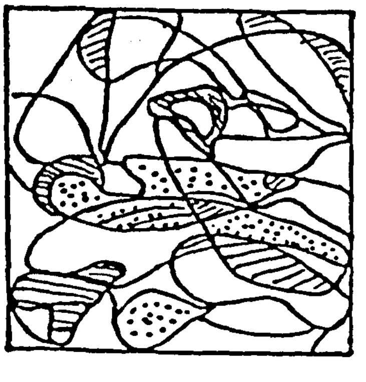
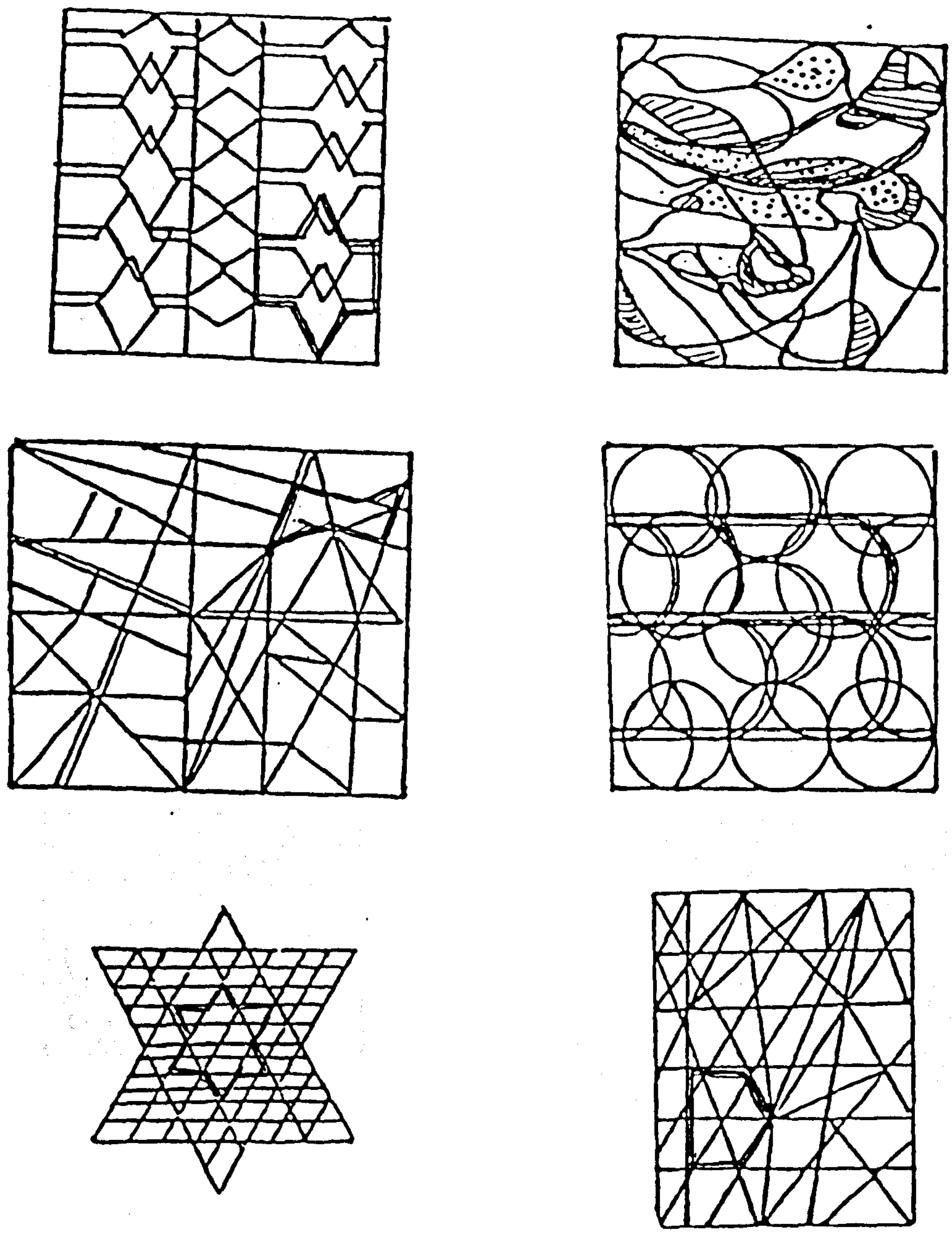
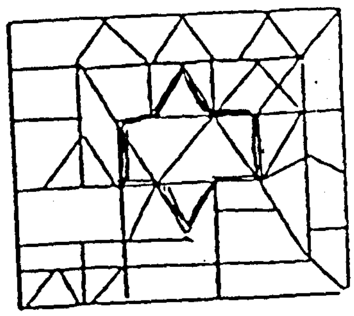
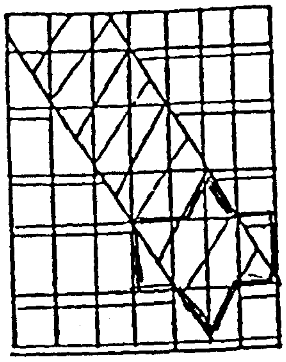
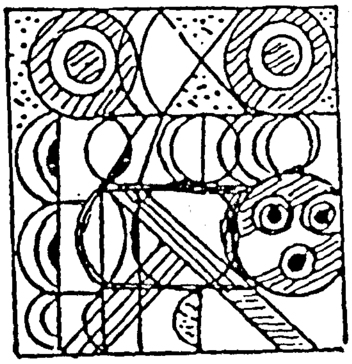
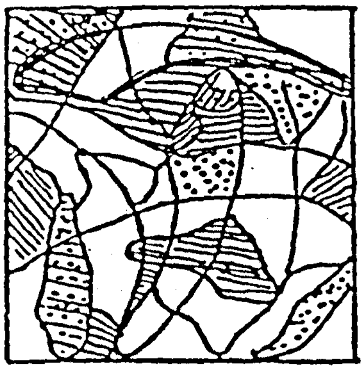
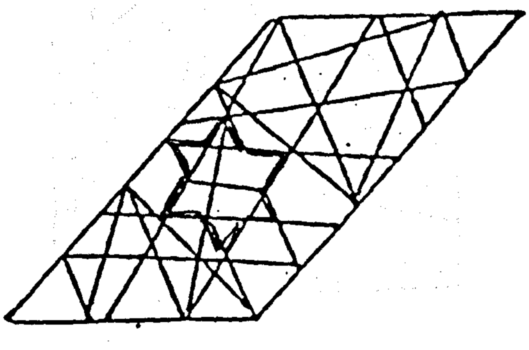
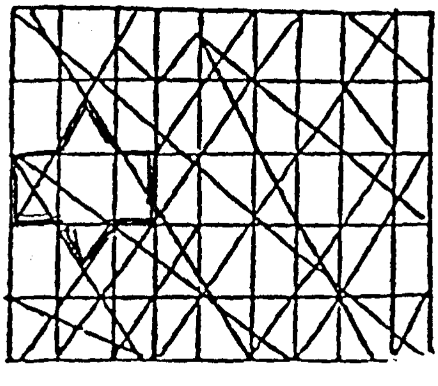
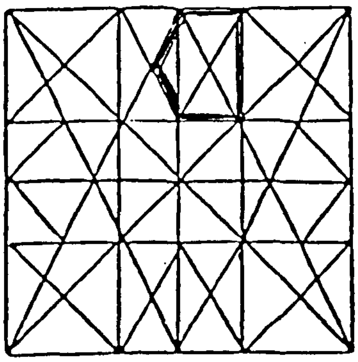
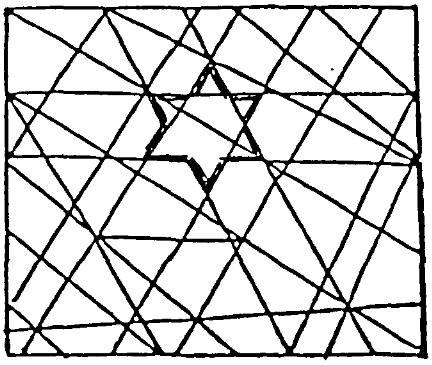


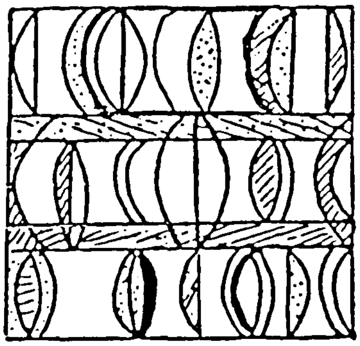
The scoring key of the HFT

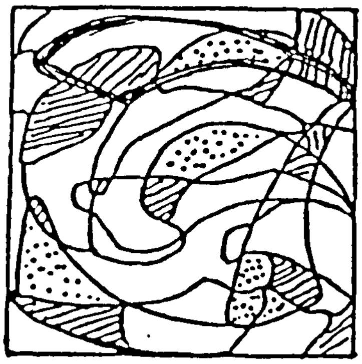


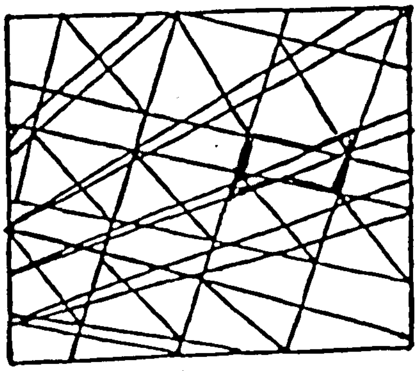







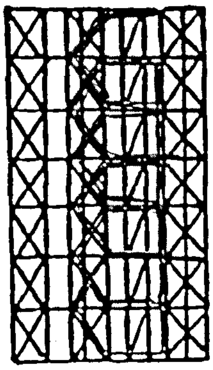












Appendix 9

Preliminary examination 1988-89

School 1

(a) Paper I

1. In the arithmetic sequence $-\frac{2}{3}, 0, \frac{2}{3}, \ldots$ the 13th term is

   A. $6\frac{2}{3}$ B. $7\frac{1}{3}$ C. 8 D. $8\frac{2}{3}$ E. $12\frac{2}{3}$.

2. $f(x) = 3x + 2$ and $g(x) = \frac{1}{x-1}$, $x \neq 1$, are functions from $\mathbb{R}$ to $\mathbb{R}$. $(f \circ g)(x) =$

   A. $\frac{3}{x-1} + 2$ B. $\frac{1}{3(x-1)} + 2$ C. $\frac{3x+2}{x-1}$

   D. $\frac{1}{3x+1}$ E. $\frac{1}{3x-1}$.

3. If $\sin A = -\frac{4}{5}$ then cos 2A is

   A. $-\frac{3}{5}$ B. $-\frac{7}{25}$ C. $\frac{7}{25}$ D. $\frac{2}{5}$ E. $\frac{3}{5}$.

4. Given that k is a constant of integration, $\int x^{1/3}\,dx$ equals

   A. $\frac{1}{3x^{2/3}} + k$ B. $\frac{3}{x^{2/3}} + k$ C. $\frac{3}{4}x^{4/3} + k$

   D. $\frac{4}{3}x^{4/3} + k$ E. $3x^{4/3} + k$.

5. The equation $2x^2 + (k-1)x + 8$ has equal roots in x. The value(s) of k are

   A. 1 B. 0 and 1 C. 4 and -4 D. 3 and -3

   E. some other value or values.

6. The equation of the straight line through the points (-1, 3) and (3, 0) is

A. $3x - 4y = 9$ B. $3x - 4y = -15$ C. $3x + 4y = 9$
D. $4x + 3y = 5$ E. $4x - 3y = 12$.

7. $f(x) = \frac{2}{x^3}$; $f'(x)$ equals

A. $\frac{2}{3x^2}$ B. $-\frac{2}{3x^2}$ C. $-\frac{6}{x^2}$ D. $-\frac{2}{3x^4}$ E. $-\frac{6}{x^4}$.

8. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$ is an infinite series. Its sum to infinity is

A. $\frac{2}{3}$ B. $\frac{3}{4}$ C. $\frac{4}{3}$ D. $\frac{3}{2}$ E. non-existent.

9. $\lim\limits_{h \to 0} \frac{(3 + h)^3 - 27}{h}$ is

A. -3 B. 0 C. 3 D. 27 E. undefined.

10. Which of the following is/are zero(s) of the function $f(x) = 3\cos(\theta + \frac{\pi}{6})$?

(1) 0 (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$.

A. (1) only B. (2) only C. (3) only D. (4) only

E. some other combination of (1), (2), (3) and (4).

11. The gradient of a straight line perpendicular to the line $x + 5y + 3 = 0$ is

A. -5 B. -3 C. $-\frac{1}{5}$ D. $\frac{1}{5}$ E. 5

12. The equation of the circle which passes through the origin and the points (0, 4) and (-6, 0) is

A. $(x - 3)^2 + (y + 2)^2 = 13$ B. $(x + 3)^2 + (y - 2)^2 = 13$

C. $(x - 3)^2 + (y + 2)^2 = 52$ D. $(x + 3)^2 + (y + 2)^2 = 52$

E. $(x - 6)^2 + (y + 4)^2 = 52$.

13. Which of the following has/have a period of 180°

(1) $2 \sin x^\circ$ (2) $\sin 2x^\circ$ (3) $2 \tan x^\circ$ (4) $\tan 2x^\circ$

A. (1) and (3) only B. (2) and (4) only

C. (1) and (4) only D. (2) and (3) only

E. some other combination of (1), (2), (3) and (4).

14. O is the origin, P the point (1, -2, 0) and Q the point (0, -1, 2). The cosine of angle POQ is

A. -1 B. -0.4 C. 0 D. 0.4 E. 1.

15. Given that $\cos \theta^\circ = a$, $0 < \theta < 90$, which of the following is/are true?

(1) $\sin \theta^\circ = \surd(1 - a)$ (2) $\sin(90 + \theta)^\circ = a$

(3) $\sin(90 - \theta)^\circ = a$

A. (1) and (2) only B. (2) and (3) only C. (1) and (3) only

D. (1), (2) and (3) E. None of (1), (2) and (3).

16. If $g(x) = x^2 + 1$, $x \in \mathrm{R}$, $g^{-1}(x)$ equals

A. $(x - 1)^2$ B. $\surd(x - 1)$ C. $\surd(x + 1)$ D. $\surd x - 1$

E. no inverse exists.

17. Chord PQ is 1 unit from O, the centre of a circle with radius 3 units. cos POQ equals

A. $-\frac{7}{9}$ B. $-\frac{2}{3}$ C. $-\frac{1}{3}$ D. $\frac{2}{3}$ E. $\frac{7}{9}$.

18. The line $y = mx$ passes through the point of intersection of the lines $x = p$ and $y = q$ if and only if

A. $m = \frac{p}{q}$ B. $m = -\frac{p}{q}$ C. $m = \frac{q}{p}$ D. $m = -\frac{q}{p}$

E. m has some other value involving p and q.

19. The length of the arc of a circle of radius r, subtended by an angle of x° at the centre of the circle, is a. x equals

A. $\frac{a}{r}$ B. $\frac{r}{a}$ C. $\frac{180a}{\pi r}$ D. $\frac{\pi a}{180r}$ E. $\frac{180r}{\pi a}$.

20. The centre of the circle $2x^2 + 2y^2 - 8x + 6y - 3 = 0$ is

A. $(4, -3)$ B. $(-4, 3)$ C. $(-2, \frac{3}{2})$ D. $(2, -3)$ E. $(2, -\frac{3}{2})$.

21. T is the point (1, 2, 3) relative to rectangular axes OX, OY and OZ. The cosine of angle TOZ is

A. $\frac{1}{\sqrt{14}}$ B. $\frac{2}{\sqrt{14}}$ C. $\frac{3}{\sqrt{14}}$ D. $\frac{3}{\sqrt{13}}$ E. $\frac{2}{\sqrt{13}}$.

22. The line $\frac{x}{7} - \frac{y}{2} = 1$ cuts the x-axis at X and the y-axis at Y. The coordinates of X and Y are

A. (2, 0) and (7, 0) B. (−2, 0) and (0, 7)

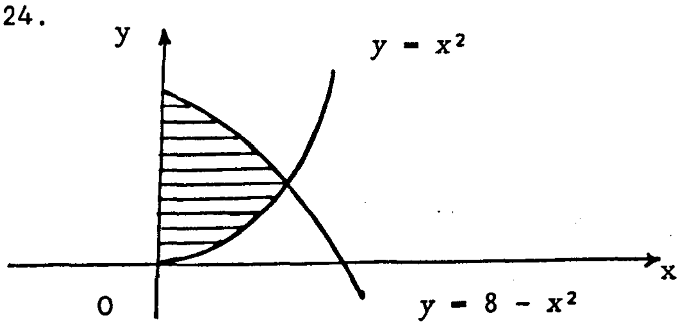
C. (7, 0) and (0, −2) D. (1/7, 0) and $(0, -\frac{1}{2})$

E. $(-\frac{1}{2}, 0)$ and (0, 1/7).

23. $f(x) = qx^2 - 3x - 7$ has a stationary value where $x = -2$. The value of q is

A. $-\frac{3}{2}$ B. $-\frac{3}{4}$ C. $\frac{3}{4}$ D. $\frac{3}{2}$ E. 3.

24.



The shaded area in the diagram is given by

A. $\int_0^8 (8 - 2x^2)\,dx$ B. $\int_0^8 (2x^2 - 8)\,dx$ C. $\int_0^4 (8 - 2x^2)\,dx$

D. $\int_0^2 (2x^2 - 8)\,dx$ E. $\int_0^2 (8 - 2x^2)\,dx$.

25. If $\tan(\alpha + \beta) = 2$ and $\tan \alpha = 3 \tan \beta$, then $\tan \beta$ has the values

(1) -1 (2) $\frac{1}{3}$ (3) 3

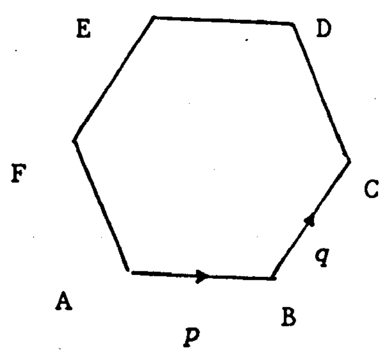
A. (1) only B. (2) only C. (3) only

D. (1) and (2) only E. (1), (2) and (3) only.

26. $f(x) = 3x(9 + x^2)(1 - x^2)$, $x \in \mathbb{R}$. The number of values of x for which $f(x) = 0$ is

A. zero B. 1 C. 2 D. 3 E. 4 or more.

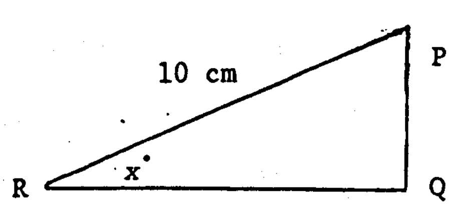
27.



ABCDEF is a regular hexagon. $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent vectors p and q respectively. $\overrightarrow{BF}$ represents the vector

A. $-p - q$ B. $-p + q$ C. $-2p + q$ D. $2p - q$ E. $-p + 2q$.

28.



Given that $\tan x° = \frac{1}{3}$, then the length of PQ, in cm is

A. $\frac{3}{10}$ B. $\frac{3\sqrt{10}}{10}$ C. $\frac{\sqrt{10}}{3}$ D. $\sqrt{10}$ E. $\frac{4}{3}$.

29. For which of the following values of x is the vector

$u = \begin{bmatrix} x \\ 5 \\ -3 \end{bmatrix}$ perpendicular to the vector $v = \begin{bmatrix} 3x \\ x \\ 4 \end{bmatrix}$?

(1) -3 (2) 0 (3) $\frac{4}{3}$ (4) 3

A. (1) only B. (2) only C. (1) and (2) only

D. (3) and (4) only E. some other one or combination of responses

30. If $L = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ then $(LM)^{-1}$ is

A. $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ B. $\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ 6/4 & \frac{1}{4} \end{bmatrix}$ C. $\begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}$

D. $\begin{bmatrix} 6/4 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ E. $\begin{bmatrix} 6/4 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

31. For all suitable α, β, $\dfrac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} =$

A. $\tan \alpha + \tan \beta$ B. $\tan \alpha - \tan \beta$ C. $1 + \tan \alpha \tan \beta$

D. $1 - \tan \alpha \tan \beta$ E. $\sin \alpha + \sin \beta$.

32. The gradient of the tangent to the curve $y = \cos x$ at $x = \frac{\pi}{3}$ is

A. $-\frac{\sqrt{3}}{2}$ B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. $\frac{\sqrt{3}}{2}$ E. $\sqrt{3}$.

33. The image of $(x - 2)^2 + (y - 3)^2 = 9$ under the dilatation with centre the origin and scale factor -2 is

A. $(x - 2)^2 + (y - 3)^2 = 18$ B. $(x - 2)^2 + (y - 3)^2 = 36$

C. $(x + 4)^2 + (y + 6)^2 = 9$ D. $(x + 4)^2 + (y + 6)^2 = 18$

E. $(x + 4)^2 + (y + 6)^2 = 36$.

34. A quadratic equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers and $a \neq 0$, has one root $3 - \sqrt{2}$. Which of the following must be true?

(1) the other root is $3 + \sqrt{2}$ (2) $\frac{b}{a} = 6$ (3) $\frac{c}{a} = 5$

A. none of (1), (2) and (3) B. (1) only C. (1) and (2) only

D. (1) and (3) only E. (1), (2) and (3).

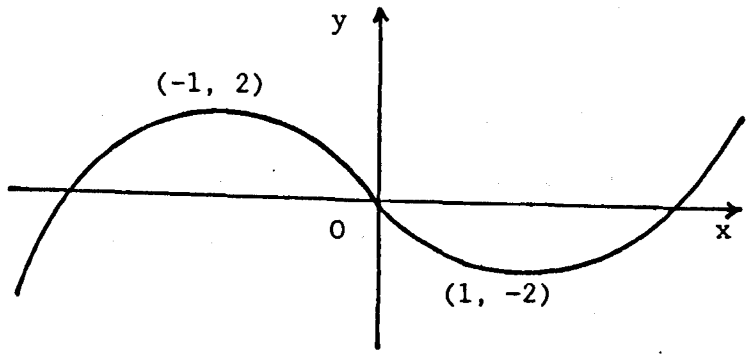
35. For which of the following is $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} f(x)\, dx = 2 \int_{0}^{\frac{1}{4}\pi} f(x)\, dx$?

(1) $f(x) = 2x$ (2) $f(x) = \sin 2x$ (3) $f(x) = \cos 2x$

A. (1) only B. (2) only C. (3) only

D. (1) and (3) E. (2) and (3)

36. The diagram shows the sketch of a cubic function f.



Which of the following is most likely to be $f(x)$?

A. $-x^3 - x$ B. $-x^3 + 3x$ C. $-x^3 - x + 3$

D. $x^3 - 3x$ E. $x^3 - 2x^2 - 3x + 2$.

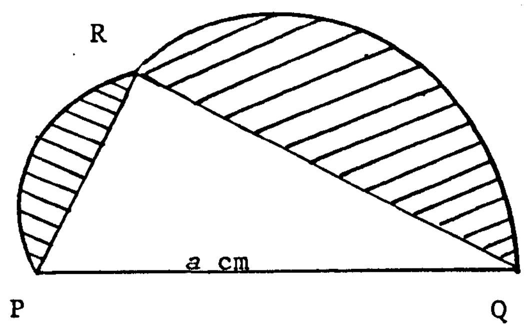
37. The maximum value of $f(x) = 4 \sin(3x - \frac{1}{2}\pi)$ occurs when x equals

A. $\frac{\pi}{6}$ B. $\frac{\pi}{3}$ C. $\frac{1}{2}\pi$ D. $\frac{2\pi}{3}$ E. $\frac{7\pi}{6}$.

38. The curve with equation $y = 4 \sin 3x$ is mapped on to itself when reflected in the line with equation

A. $x = 0$ B. $x = \frac{\pi}{6}$ C. $x = \frac{1}{4}\pi$ D. $x = \frac{\pi}{3}$ E. $x = \frac{2\pi}{3}$.

39.



The diagram shows three semi-circles with diameters PQ, PR and RQ. R is a point on the semi-circle with diameter PQ = a cm. The area of the shaded portion in cm² is

A. $\frac{1}{6}\pi a^2$ B. $\frac{1}{4}\pi a^2$ C. $\frac{1}{2}\pi a^2$ D. πa^2

E. indeterminable without knowing the position of R.

40. P, Q, R have position vectors $u - v$, $2u - 4v$ and $3u - 7v$ respectively. Which of the following statements is/are true?

(1) $\overrightarrow{PQ}$ represents $2u - 6v$ (2) P, Q and R are collinear

(3) Q is the mid-point of PR.

A. (1) only B. (2) only C. (3) only D. (1), (2) and (3)

E. some other combination of responses.

Paper II

1. (a) Prove that the sum of the first n terms of the arithmetic sequence a, $a + d$, $a + 2d$, ... is

$$\tfrac{1}{2}n[2a + (n - 1)d].$$

(b) Prove that if two straight lines of gradient m_1 and m_2 are perpendicular to each other, then $m_1 m_2 = -1$.

2. The line joining A(-2, 6) and B(4, 0) is a diameter of a circle, centre C.

(a) State the coordinates of C and find the equation of the circle.

(b) The circle crosses the x-axis at a second point D. Find the coordinates of D.

(C) Hence find the equation of the tangent to the circle which passes through D.

3. (a) Solve the equations $7x^2 + 13xy - 2y^2 = 0$ and $2x - y = 5$, where $x, y \in \mathbb{R}$.

(b) f is a function given by the formula $f(x) = \frac{x}{x-2}$, $x \neq 2$, $x \in \mathbb{R}$. Find the formula for the inverse function $f^{-1}(x)$ and state clearly the domain of f^{-1}.

(c) The functions f and g are defined by

$$f: x \to 2x^2 \quad \text{and} \quad g: x \to x^2 - 2.$$

Find in simplest form $(f \circ g')(x) - (g \circ f')(x)$.

4. (a) Differentiate

$$f(x) = \frac{(x^2 - 3)(x + 1)}{\sqrt{x}}$$

with respect to x.

(b) Evaluate the definite integral $\int_1^4 (3\sqrt{x} + \frac{1}{3\sqrt{x}})\, dx$.

(c) Find the equation of the tangent to the curve $y = \sin x + \cos x$ at the point where $x = \pi$.

5. $A = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}$. Show that $A^2 = 2A - 5I$ where I is the 2x2 unit matrix. Hence or otherwise, find p and q such that

$$A^2 = pA + qI.$$

6. (a) Find the equation of the straight line which passes through the point P(4, 2) and is perpendicular to the line with equation $x + 2y - 5 = 0$.

(b) This line, whose equation you have found, cuts the x-axis at A and y-axis at B. Find the coordinates of A and B and the area of triangle AOB.

(c) State the coordinates of O', A' and B', the images of O, A and B under the dilatation $[p, 2]$. What is the area of triangle O'A'B'?

7. (a) By expressing 15° as $(45 - 30)^\circ$, show that $\tan 15^\circ = 2 - \sqrt{3}$

(b) Solve for $0 \leq x \leq 360$, the equation

$$3 \cos 2x^\circ - \sin x^\circ - 2 = 0.$$

8. (a) The first three terms of a geometric sequence are $3(x + 2)$, $3(x - 2)$ and $(x + 6)$ respectively. Show that there are two possible values of x and hence find the two possible sequences. Show that only one of these sequences can be summed to infinity and find this sum.

(b) If $x = 49$ and $y = 27$ calculate the value of

$$(x^{1/2} + y^{2/3})^{-3/4}.$$

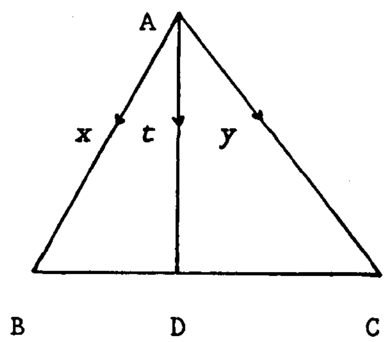
(c) For what range of values of k will the equation

$$k\,(2 - x) = x^2 - x - 1$$

have no real roots?

9. (a) A and B are the points (2, 1, -1) and (8, 4, 8) respectively. C divides AB externally in the ratio 2:1. Find the coordinates of C.

(b)



Triangle ABC is equilateral, with each side of length one unit. $\overrightarrow{AB}$ represents vector $\boldsymbol{x}$, $\overrightarrow{AC}$ represents $\boldsymbol{y}$ and $\overrightarrow{AD}$ represents $\boldsymbol{t}$. D lies on BC and BD = $\frac{1}{3}$ BC.

Show that $\boldsymbol{t}.\boldsymbol{x} = \frac{5}{6}$.

10. (a) Calculate the area enclosed between the curves

$$y = x^2 - 2x + 6 \quad \text{and} \quad y = 6 + 4x - x^2.$$

(b) Show that the line $x = 1$ divides this area in the ratio 7:20.

Appendix 9 (ctd)

Preliminary examination 1988-89

School 2

(a) Paper I

1. P(−2, −5) lies on the circumference of a circle whose centre is Q(1, −3). The coordinates of R, the other extremity of the diameter through P, are

   A. (−5, −7) B. (−2, −5) C. (1, −3) D. (4, −1) E (−4, 1).

2. The equation of the circle, centre (1, 3), radius 4 is

   A. $x^2 + y^2 = 16$ B. $(x + 1)^2 + (y + 3)^2 = 16$

   C. $(x - 1)^2 + (y - 3)^2 = 16$ D. $(x + 1)^2 + (y + 3)^2 = 4$

   E. $(x - 1)^2 + (y - 3)^2 = 4$.

3. Which of the following points lie(s) inside the circle whose equation is $x^2 + y^2 = 81$?

   1. (6, 6) 2. (6, 7) 3. (4, 8)

   A. 1, 2 and 3 B. 1 and 2 only C. 1 and 3 only

   D. 2 and 3 only E. none of 1, 2 and 3.

4. P is the point (5, 5) and Q is the point (1, 1). R is the point on the x-axis such that PR = QR. R is the point

   A. (3, 0) B. (5, 0) C. (6, 0) D. (0, 3) E. (0, 5)

5. If $x + y = a$ and $xy = b$, then x must satisfy

A. $x(a - x) = b$ B. $x(a + x) = b$ C. $x(x - a) = b$

D. $x^2 - a = b$ E. $a - x^2 = b$.

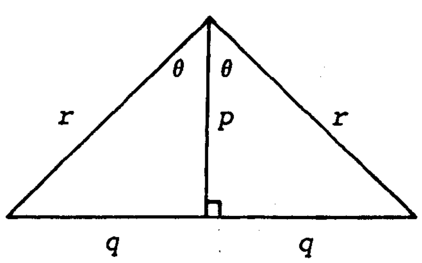
6. P and Q are the points (2, 3) and (-1, 4). A line perpendicular to PQ will have gradient

A. -3 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. $\frac{4}{3}$ E. 3.

7. If $f(x) = \frac{1}{x^2}$ and $x \neq 0$, then $f'(x)$ is

A. $\frac{1}{2\sqrt{x}}$ B. $\frac{-1}{x}$ C. $\frac{-2}{x}$ D. $\frac{-2}{x^2}$ E. $\frac{1}{-2x^3}$.

8.



sin 2θ equals

A. $\frac{pq}{r}$ B. $\frac{pq}{r^2}$ C. $\frac{2q}{r}$ D. $\frac{2pq}{r}$ E. $\frac{2pq}{r^2}$.

9. The least period of $\sin(x + \frac{1}{2}\pi)$ is

A. $\frac{1}{2}\pi$ B. π C. $3\pi/2$ D. 2π E. $5\pi/2$.

10 The x-coordinate of the point at which the curve $y = 6 - 3x^2$ has gradient 12 is

A. -6 B. -2 C. $-\sqrt{2}$ D. -1 E. $\sqrt{2}$.

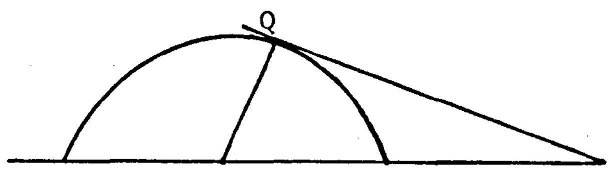
11 The equation of a circle, centre (-5, 6), with the x-axis as a tangent is

A. $(x + 5)^2 + (y - 6)^2 = 1$ B. $(x + 5)^2 + (y - 6)^2 = 25$

C. $(x - 5)^2 + (y + 6)^2 = 25$ D. $(x + 5)^2 + (y - 6)^2 = 36$

E. $(x - 5)^2 + (y + 6)^2 = 36$.

12.



PQ is a tangent to the semi-circle, centre r. If PS = 7, and RQ = 5, then PQ equals

A. $\sqrt{24}$ B. $\sqrt{84}$ C. $\sqrt{119}$ D. 12 E. 13.

13. If k is a constant of integration, $\int x^{3/2}\, dx$ equals

A. $\frac{2}{5} x^{5/2} + k$ B. $\frac{5}{2} x^{5/2} + k$ C. $\frac{3}{2} x^{1/2} + k$

D. $\frac{3}{2} x^{5/2} + k$ E. $\frac{2}{3} x^{1/2} + k$.

14. If the straight line joining the points (0, 8) and (-4, 0) passes through the point $(p, -4)$, then p is equal to

A. -8 B. -6 C. -2 D. 2 E. 6.

15. $(\sqrt{5} - 2\sqrt{3})^2$ equals

A. 17 B. $11 - 4\sqrt{15}$ C. $17 - 2\sqrt{15}$ D. $17 - 4\sqrt{15}$

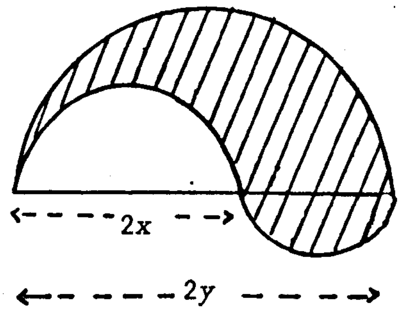
E. $61 - 4\sqrt{15}$

16. If $(3x - a)(4x - 3) = 12x^2 - 17x + b$ for all real values of x then

A. $a = 2,\ b = 6$ B. $a = -2,\ b = 6$ C. $a = -2,\ b = -6$

D. $a = 8,\ b = -24$ E. $a = 8,\ b = 24$

17.



The shaded area is bounded by three semi-circles. Its area is

A. $\pi y(y - x)$ B. $\pi y(y + x)$ C. $\pi y(y + 2x)$ D. πy^2

E. none of these.

18. If k is a constant of integration, $\int 2x^{2/3}\ dx$ equals

A. $\frac{4}{3}x^{-2/3} + k$ B. $-6x^{-2/3} + k$ C. $\frac{3}{5}x^{5/3} + k$

D. $\frac{10}{3}x^{5/3} + k$ E. $\frac{6}{5}x^{5/3} + k$.

19. $\lim\limits_{h\to 0} \dfrac{(x + h)^2 - x^2}{h}$ equals

A. x^2 B. $2x$ C. $2x + h$ D. 0 E. none of these.

20. The image of the circle $(x - a)^2 + (y - b)^2 = r^2$ under reflection in the line $y = x$ is

A. $(x + a)^2 + (y + b)^2 = r^2$ B. $(x + a)^2 + (y - b)^2 = r^2$

C. $(x - a)^2 + (y + b)^2 = r^2$ D. $(x + b)^2 + (y + a)^2 = r^2$

E. $(x - b)^2 + (y - a)^2 = r^2$

21. If $x + 5$ is a factor of the polynomial $x^3 + 4x^2 + kx - 10$ then k equals

A. 7 B. -3 C. -7 D. -35 E. -43.

22. The side PQ of triangle PQR has gradient 1/3 while R is the point (-4, 7). The equation of the altitude from R to PQ is

A. $x + 3y - 25 = 0$ B. $x + 3y - 17 = 0$ C. $3x + y - 19 = 0$

D. $3x + y + 5 = 0$ E. $3x + y + 19 = 0$.

23. State which one of the following is false. The function $\sin x$ has the same period as the function

A. $\cos x$ B. $\sin 2x$ C. $2 \sin x$ D. $\sin (\frac{1}{2}\pi + x)$

E. $2 + \sin x$.

24. The equation of the tangent to the curve $y = 5x^3 - 7x + 3$ at the point (-1, 5) is

A. $y = -22x - 17$ B. $y = 3x + 8$ C. $y = 5x + 10$

D. $y = 8x + 13$ E. $y = 22x + 13$.

25. $f(x) = ax^2 - 2x - 5$ has a stationary value where $x = 3$. The value of a is

A. -3 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. $\frac{11}{9}$ E. 3.

26. The units digit in the answer to $1234^2 + 6543^4$ is

A. 3 B. 4 C. 5 D. 6 E. 7.

27. If $f(x) = \dfrac{3}{5x^{3/2}}$ then $f'(x)$ equals

A. $\dfrac{2}{5x^{1/3}}$ B. $\dfrac{2}{5x^{1/2}}$ C. $-\dfrac{45}{2x^{5/2}}$ D. $-\dfrac{9}{10x^{1/2}}$

E. $-\dfrac{9}{10x^{5/2}}$.

28 The equation of the circle having the line joining P(3, 2) to Q(7, 2) as a diameter is

A. $(x - 7)^2 + (y - 3)^2 = 4$ B. $(x - 4)^2 + y^2 = 4$

C. $(x - 5)^2 + (y - 2)^2 = 2$ D. $(x - 4)^2 + y^2 = 2$

E. $(x - 5)^2 + (y - 2)^2 = 4$.

29. $3 \cos(\theta - \frac{1}{4}\pi)$ has a maximum value when θ is

A. $\frac{1}{4}\pi$ B. $\frac{1}{2}\pi$ C. $\frac{1}{4}3\pi$ D. π E. $\frac{1}{4}\ \pi$.

30. $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h}$, $x \neq 0$, is

A. 0 B. 1 C. $\dfrac{1}{x^2}$ D. $\dfrac{-1}{x^2}$ E. undefined.

31. The gradient of the curve $y = x^2$ at the point (2, 8) is

A. 1 B. 4 C. 12 D. 64 E. 192.

32. For $0 \leqslant \theta \leqslant \pi$, the maximum value of $\sin(2\theta - \frac{1}{4}\pi)$ occurs when θ equals

A. $\frac{1}{8}\pi$ B. $3\pi/8$ C. $\frac{1}{2}\pi$ D. $5\pi/8$ E. $3\pi/4$.

33. $f(x) = (3x + 1)^2$; $f'(1)$ equals

A. 0 B. 8 C. 15 D. 24 E. 48.

34. $f(x) = \dfrac{x^2 + 1}{x}$, $x \neq 0$ then $f'(x)$ equals

A. $2x$ B. $2x + 1$ C. $\dfrac{2x + 1}{x}$ D. $\dfrac{x^2 - 1}{x^2}$ E. $\dfrac{x^2 + 1}{x}$.

35. If $x \neq 0$, $\lim\limits_{h \to 0} \dfrac{\dfrac{1}{x + h} + \dfrac{1}{x}}{h}$ is

A. $-\frac{1}{x^2}$ B. 0 C. $\frac{1}{2x}$ D. $\frac{1}{x^2}$ E. non-existent.

36. Given that $\sin\theta^\circ = a$, $a \neq 0$, $\sin(90 - \theta)^\circ$ equals

A. $5a$ B. $\frac{1}{a}$ C. $1 - a$ D. $\sqrt{(1 - a^2)}$ E. $\frac{\sqrt{3}}{2} + a$.

37. The radius of the circle $x^2 + y^2 = 4 - 4x + 2y$ is

A. 1 B. 2 C. 3 D. 4 E. $\sqrt{24}$.

38. Under the translation $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, the image of the line $y = 2x + 4$ is

A. $y = 2x$ B. $y = 2x + 2$ C. $y = 2x + 4$

D. $y = 2x + 8$ E. $y = 2x - 4$.

39. The solution set of the system

$$y = (x - 1)(x - 4)$$
$$y = x - 1$$

is

A. $\{(4, 3)\}$ B. $\{(5, 4)\}$ C. $\{(1, 0), (4, 3)\}$

D. $\{(1, 0), (4, 0)\}$ E. $\{(1, 0), (5, 4)\}$.

40. If the exat value of $\cos x$ is $\frac{1}{\sqrt{5}}$, then the exact value of $\cos 2x$ is

A. $-\frac{3}{5}$ B. $-\frac{2}{\sqrt{5}}$ C. $\frac{2}{\sqrt{5}}$ D. $\frac{3}{5}$ E. $\frac{4}{5}$.

(b) PaperII

1. (a) Prove that, if $g^2 + f^2 - c > 0$, the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

represents a circle with centre $(-g, -f)$ and radius $\sqrt{(g^2 + f^2 - c)}$.

(b) Obtain, from first principles, the derivative of $-\frac{5}{x}$, with respect to x.

2. Solve the system of equations

$$\left.\begin{aligned} 2x - 3y + 1 &= 0 \\ 2x^2 + 3y^2 + 3x + y &= 4 \end{aligned}\right\}.$$

3. (a) Given that $f(x) = \dfrac{(x^2 + 3)^2}{3x^{3/2}}$, $x \neq 0$, find $f'(x)$.

(b) Evaluate $\int_1^4 (3x^2 - \frac{1}{2\sqrt{x}})\, dx$.

4. $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

(a) Show that $A^2 = 5A - 7I$ where I is the 2×2 unit matrix.

(b) Find integers p and q such that $A^3 = pA + qI$.

5. Solve, for $0 < x < 360$ the equation

$$3 \cos 2x + 2 \cos x = 1.$$

6. (a) Write down the coordinates of the centre and the radius of the circle $x^2 + y^2 - 8x + 4y - 5 = 0$ and verify that B(7, 2) lies on the circle.

(b) Find the equation of the tangent at B to the circle.

(c) P is the point (-5, -14). Find the equation of the image of the circle under the dilatation [P, 2].

(d) Show that the original circle and its image in part (c) touch externally.

7. If $f(x) = (x - 2)(x + 3)(x - 5) = x^3 - 4x^2 - 11x + 30$.

(a) Find where the graph of $y = f(x)$ meets the coordinate axes.

(b) Find the stationary points of $f(x)$ and determine thier nature, justify your answers.

(c) Make a rough sketch of the graph.

8. A, B, C and D are the points (1, 2, 3), (7, -1, 5), (5, -7, 2) and (-1, -4, 0) respectively.

Prove that

(a) $\overrightarrow{AD} = \overrightarrow{BC}$,

(b) AC is perpendicular to BD,

(c) AC = BD. What kind of figure is ABCD?

9. A, B and C are the points (5, 8), (-2, 1) and (6, -1) respectively and M is the mid-point of AC.

(a) Calculate the coordinates of M and the equation of BM.

(b) If the perpendicular from C to AB meets BM in Q, find the coordinates of Q.

10. Show that the roots of the equation $(x - k)(x - l) = m^2$ are real for all values of k, l and m. State the nature of the roots of the equation if $k = l$.

Appendix 9 (ctd)

Preliminary examination 1988-89

School 3

(a) Paper I

1. $\sin(90 - x)$ equals

   A. $\cos x$ B. $-\cos x$ C. $\sin x$ D. $-\sin x$ E. $1 - \sin x$.

2. The line joining the points (-2, 3) and (6, k) has gradient 2/3. The value of k is

   A. 15 B. 17/3 C. 25/3 D. 9 E. 7/3.

3. P(1, -2, 5) Q(2, -4, 4) and R(-1, 2, 7) are three collinear points. The ratio PQ:QR equals

   A. 3:1 B. -1:3 C. 2:3 D. 1:3 E. -3:1

4. Which of the following pairs consist (s) of a matrix and its inverse?

   (1) $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

   (2) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

   (3) $\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

   A. none B. (1) only C. (3) only D. (1) and (2) only E. (1) and (3) only.

5. If $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ then $x + y$ equals

A. $-3/2$ B. 3 C. 6 D. 20 E. none of these.

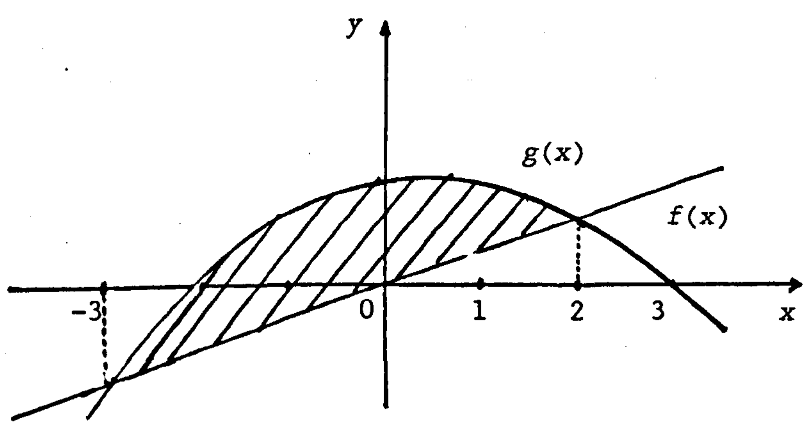
6. Given that $\cos 2x = p$, then $\cos^2 x$ equals

A. $\frac{1}{4}p^2$ B. $\frac{1}{2}(1 - p)$ C. $\frac{1}{2}(p - 1)$ D. $\frac{1}{2}p$ E. $\frac{1}{2}(p + 1)$.

7. The gradient of the tangent to the curve $y = \frac{1}{3}x^3 + \frac{1}{2}x^2$ at the point where $x = -1$ is

A. 0 B. -1 C. -2 D. 1 E. 2.

8.



The shaded area is

A. $\int_{-3}^{2} (f(x) - g(x))\, dx$ B. $\int_{-2}^{2} (f(x) - g(x))\, dx$

C. $\int_{-3}^{2} (g(x) - f(x))\, dx$ D. $\int_{-2}^{3} (g(x) - f(x))\, dx$

E. $\int_{-3}^{0} (g(x) - f(x))\, dx + \int_{0}^{3} (g(x) - f(x))\, dx$.

9. f and g are functions on the set of real numbers such that $f(x) = 2x - 1$ and $(f \circ g)(x) = 4x + 1$. $g(x)$ equals

A. $8x + 1$ B. $2x + 1$ C. $8x - 3$ D. $2x + 3$

E. $2x + 2$.

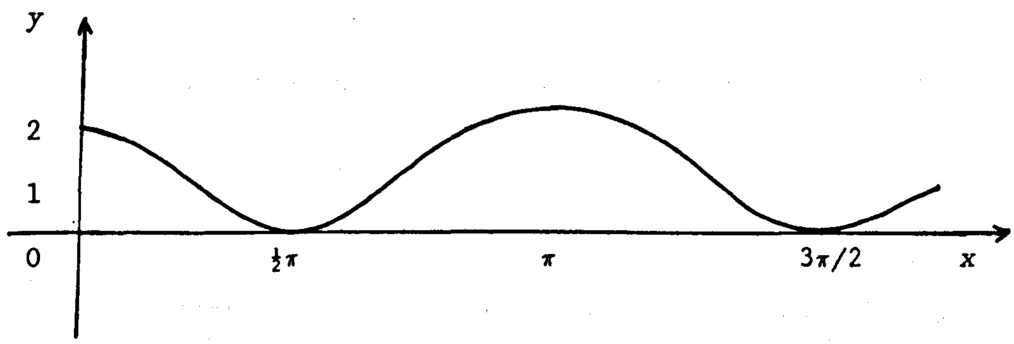
10 $\int_{-1}^{1} x(x^2 - 2)\, dx$ is

A. $-\frac{5}{3}$ B. $-\frac{3}{2}$ C. 0 D. $\frac{3}{2}$ E. $\frac{5}{3}$.

11. $f(x) = 2x^{3/2}$; $f'(4)$ equals

A. $\frac{3}{4}$ B. 2 C. 3 D. 4 E. 6.

12.

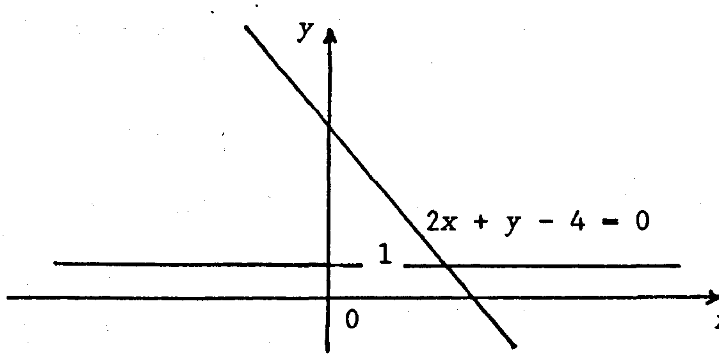


The graph is most likely to be that of $g(x) =$

A. $1 + \cos x$ B. $2 + \sin x$ C. $2 \cos x$ D. $1 + \cos 2x$

E. $2 + \sin 2x$.

13.



The image of the line $2x + y - 4 = 0$ under reflection in the line $y = 1$ is

A. $x - 2y - 6 = 0$ B. $x - 2y - 4 = 0$ C. $2x - y - 3 = 0$

D. $2x - y - 2 = 0$ E. $x + 2y + 4 = 0$.

14. $\cos P \sin Q - \sin P \cos Q$ is equal to

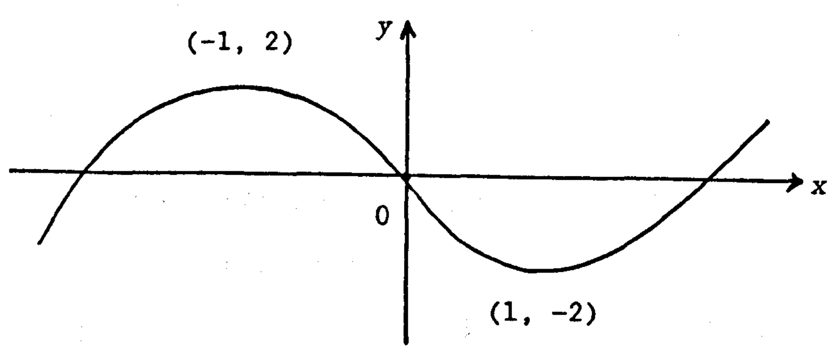
A. $\sin(P + Q)$ B. $\sin(P - Q)$ C. $- \sin(P + Q)$

D. $\cos(P + Q)$ E. $- \sin(P - Q)$.

15. If $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, $u, f, v \neq 0$, then f equals

A. $\frac{u + v}{uv}$ B. $u + v$ C. uv D. $\frac{1}{u + v}$ E. $\frac{uv}{u + v}$.

16. The diagram shows the sketch of a cubic function f.



Which of the following is most likely to be f (x)?

A. $-x^2 - x$ B. $-x^3 + 3x$ C. $-x^3 + 3x^2 - x + 3$

D. $x^3 - 3x$ E. $x^3 - 2x^2 - 3x + 2$.

17. LMN is a triangle such that $\overrightarrow{LM}$ and $\overrightarrow{NM}$ have components

$\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$ respectively.

If parallelogram LMNP is completed then $\overrightarrow{PM}$ has components

A. $\begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$ B. $\begin{bmatrix} 0 \\ -10 \\ -1 \end{bmatrix}$ C. $\begin{bmatrix} 0 \\ 10 \\ -13 \end{bmatrix}$ D. $\begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$ E. $\begin{bmatrix} 8 \\ 0 \\ -13 \end{bmatrix}$

18. If $f'(x) = \lim\limits_{h \to 0} \dfrac{\dfrac{3}{x+h} - \dfrac{3}{h}}{h}$ then $f'(x)$ is

A. $-\dfrac{3}{x^2}$ B. $\dfrac{3}{x}$ C. $\dfrac{3}{2x}$ D. $\dfrac{3}{x^2}$ E. non-existent

19. If i, j and k are unit vectors along mutually perpendicular axes then $(i + 2j).(j + 2k)$ equals

A. 2 B. 1 C. 0 D. 7 E. 9.

20. If k is a constant of integration, $\int x^{2/3}\, dx$ equals

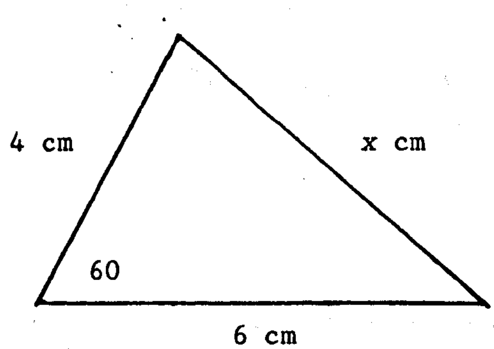
A. $\frac{4}{3} x^{-1/3} + k$ B. $-6x^{-1/3} + k$ C. $\frac{3}{5} x^{5/3} + k$

D. $\frac{10}{3} x^{5/3} + k$ E. $\frac{6}{5} x^{5/3} + k$.

21. The sum to infinity of a geometric series with common ratio $-2/3$ is 9. Its first term is

A. 15 B. 27/5 C. 3 D. 27 E. non of these

22.



x equal

A. $2\sqrt{7}$ B. $2\sqrt{3}$ C. $\sqrt{34}$ D. $2\sqrt{10}$ E. $2\sqrt{19}$.

23. $(\sqrt{5} - 2\sqrt{3})^2$ equals

A. 17 B. $11 - 4\sqrt{15}$ C. $17 - 2\sqrt{15}$ D. $17 - 4\sqrt{15}$

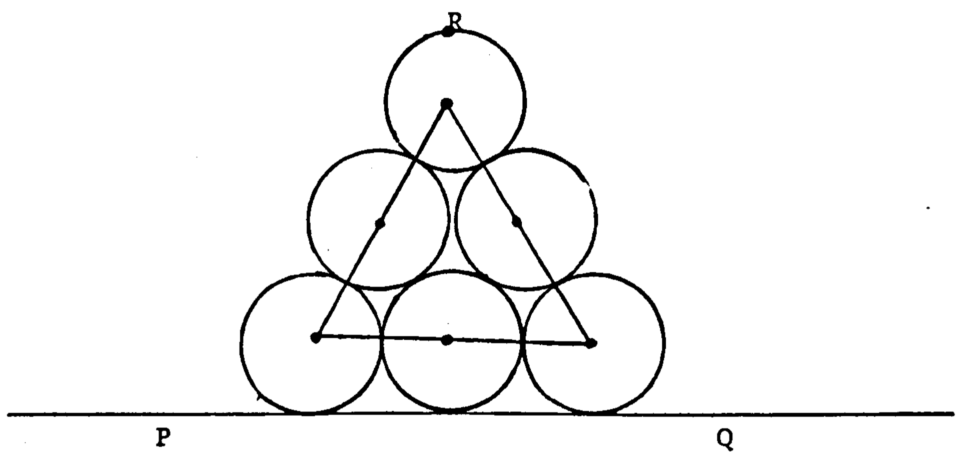
E. $61 - 4\sqrt{15}$.

24. $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. If $pu + qv = 7v$, then

A. $p = q$ B. $p = 2q$ C. $p + q = 7$ D. $2p + q = 7$

E. $p + 2q = 7$.

25.



Six cylinders of unit radius are stacked as shown with R the uppermost point of the top cylinder. The perpendicular distance of R from PQ is

A. $2 + \sqrt{3}$ B. $6 - \sqrt{3}$ C. $1 + 2\sqrt{3}$ D. $2 + 2\sqrt{3}$

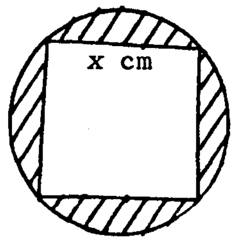
E. $4 + \sqrt{3}$.

26. $f'(x) = x^2 - 2x$ and $f(2) = \frac{8}{3}$ then $f(x)$ equals

A. $\frac{x^3}{3} - x^2 + 4$ B. $\frac{x^3}{3} - x^2 - 2$ C. $\frac{x^3}{3} - x^2 - 4$

D. $\frac{x^3}{3} - x^2$ E. $\frac{x^3}{3} - x^2 + 2$.

27. ABCD is a square with length x cm, inscribed in a circle.



An expression for the shaded area is

A. $x^2(\pi - 1)$ B. $x^2(\frac{1}{2}\pi - 1)$ C. $x^2\left(\frac{\pi - 1}{2}\right)$

D. $x^2(\pi - \frac{1}{2})$ E. $x^2(\pi - \frac{1}{4})$

28. $5^{1/3}$, $5^{1/2}$ are the first two terms of a geometric sequence. The third term is 5^n where n equals

A. 2/3 B. 1/6 C. 2/9 D. 1/4 E. 5/6.

29. If 2 is added to the numerator of the fraction $\frac{a}{b}$, the quantity which must be added to the denominator so that the value of the fraction is unaltered is

A. 2 B. -2 C. $\frac{2b}{a}$ D. $-\frac{2b}{a}$ E. $2b - a$.

30. The angle between the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ \sqrt{6} \end{bmatrix}$ is

A. $\frac{1}{2}\pi$ B. $\pi/6$ C. $\pi/3$ D. $\frac{1}{4}\pi$ E. 0.

31. For the system of equations

$$x + y - z = 4,$$
$$x - y + z = 2,$$
$$x + y + z = 6, \qquad x, y, z \in \mathrm{R}$$

Which of the following is true?

A. $x = 2y$ B. $z = 2y$ C. $y = x = z$ D. $x < y < z$

E. $y + z > x$.

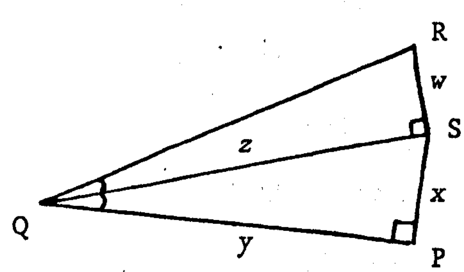
32. The length of vector a, where $a = 5i + 2\surd 2j - 4\surd 3k$ is

A. 9 B. 15 C. $5 + 2\surd 2 - 4\surd 3$ D. 21 E. 81.

33. If the exact value of tan A is $\frac{1}{7}\surd 5$, then the exact value of cos A is

A. $\frac{\surd 54}{7}$ B. $\frac{7}{3\surd 6}$ C. $\frac{7}{5}$ D. $\frac{7}{8}$ E. $\frac{5}{\surd 54}$.

34. In the diagram angle PQS = angle SQR and angle RSQ = angle SPQ = 90 degrees, tan PQR can be written as



A. $\frac{2x}{y}$ B. $\frac{\frac{2x}{y}}{1 + \frac{x^2}{y^2}}$ C. $\frac{\frac{2x}{y}}{1 - \frac{x^2}{y^2}}$ D. $\frac{x}{y} + \frac{w}{z}$ E. $\frac{x + w}{y}$

35. The equation of the straight line through the points (1, -2) and (-3, 4) is

A. $3x + 2y = -1$ B. $3x - 2y = 7$ C. $2x + 3y = -4$

D. $2x - 3y = 8$ E. none of these.

36. The solution set of the system of equations $xy = 6$ and $y = x - 1$ where x, y real, contains which of the following values of x?

A. -3 B. 1 C. 2 D. 6 E. -2.

37. $k - 1$, $k + 3$, $3k - 1$, in that order are the first three terms of an arithmetic sequence. Which of the following is/are true about k?

1. k is even 2. k is odd 3. k is a perfect square.

A. 1 and 3 only B. 1 only C. 2 only D. 2 and 3 only

E. none of 1, 2 and 3.

In questions 38 and 39 one or more of the four responses may be correct.

Answer A. if only (1), (2) and (3) are correct.

B. if only (1) and (3) are correct.

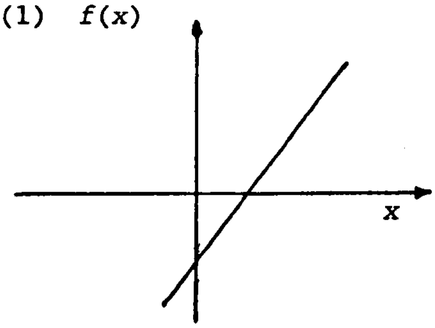
C. if only (2) and (4) are correct.

D. if only (4) is correct.

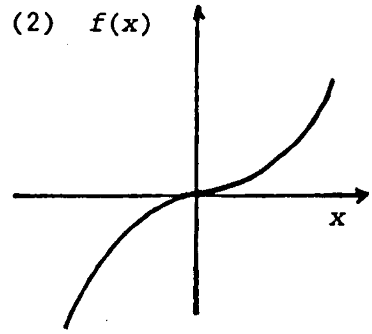
E. if some other response or combination of responses is correct.

38. Which of the following graphs show/s a function which has no inverse?

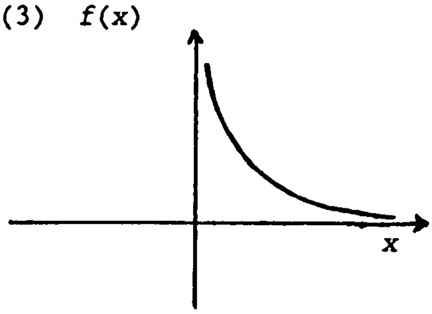
(1) $f(x)$



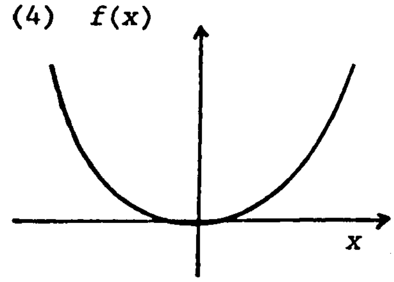
(2) $f(x)$



(3) $f(x)$



(4) $f(x)$



39. The vectors with components $\begin{bmatrix} k \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} k \\ 1 \\ -1 \end{bmatrix}$ will be perpendicular if k has the value

(1) $-\sqrt{2}$ (2) 0 (3) $\sqrt{2}$ (4) $\sqrt{3}$

40. An indicator line of length 10 cm rotates about the centre of a circular radar screen at 2 radians per second. The area (in square centimetres) swept out per second is

A. 20 B. 40 C. 100 D. 200 E. 400.

(b) Paper II

1. (a) The points A and B have position vectors a and b respectively, relative to an origin O and the point P with position vector p divides AB internally in the ratio $m:n$. Prove that

$$p = \frac{na + mb}{m + n}$$

(b) Assuming the expansions for cos(A − B) and sin(A − B) prove that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(c) Find from first principles the derivative of $f(x) = 2x^2$

2. (a) Solve the system of equations

$$2x + y = 2,$$

$$x^2 + y^2 + 3x - 2y - 4 = 0, \qquad x, y \in \mathbb{R}$$

(b) f and g are functions given by the formulae

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^2 + 1, \quad x \in \mathbb{R}$$

Express $(f \circ g)(x) - (g \circ f)(x)$ in its simplest form. Sate a suitable domain for f (x) such that f^{-1} (x) would exist.

3. (a) Solve the equation

$$3 \cos 2\theta^{\circ} + \cos \theta^{\circ} + 2 = 0$$

(b) Prove that

$$\sin A \cos^3 A - \sin^3 A \cos A = \tfrac{1}{4} \sin 4A.$$

4. (a) Differentiate $f(x) = \dfrac{(x^2 - 3)(x + 1)}{\sqrt{x}}$ with respect to x

(b) Show that $y - 2ax + a^2 + 4 = 0$ is the equation of the tangent at the point on the curve $y = x^2 - 4$ where $x = a$ and find the values of a for which the tangent would pass through the point (3, 1).

(c) Evaluate $\int_1^8 (\sqrt[3]{x} + \dfrac{1}{2(\sqrt[3]{x})})\, dx$.

5. The points A(1, -9), B(11, 4) and C(-3, 11) are the vertices of a triangle.

(a) Find the equation of the altitude AP and the equation of the median BQ of this triangle.

(b) Find also the equation of the line through K parallel to BC where K is the point of intersection of AP and BQ.

6. $f(x) = 8 + 2x^2 - x^4$, $x \in \mathbb{R}$. Find

(a) $f'(x)$

(b) where the graph of the function cuts the x- and y- axes.

(c) the stationary values of $f(x)$ and their nature. Hence sketch the graph of $y = 8 + 2x^2 - x^4$ marking clearly the intercepts with the X and Y axes and the turning points.

7. (a) If $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 3 \end{bmatrix}$

find (i) $A^{-1}B^{-1}$, (ii) $(BA)^{-1}$.

(b) Prove that

$$\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}^n = \begin{bmatrix} \cos nA & -\sin nA \\ \sin nA & \cos nA \end{bmatrix} \text{ where } n = 2.$$

(c) If $M = \begin{bmatrix} 4 & 3 \\ 3 & 10 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

express M^2 in the form $pM + qI$, where $p, q \in \mathbb{Z}$.

8. A is the point (1, 8, −1) and B is the point (5, 4, 7). Find the coordinates of C if C divides AB in the ratio 3:1.

Find also the coordinates of D if D divides AB externally in the ratio 3:1.

Hence calculate the size of $\angle$ COD where O is the origin.

9. (a) (i) State the common ratio r for the geometric sequence

$\cos 2A,\ 2 \cos 2A \sin^2 A,\ 4 \cos 2A \sin^4 A,\ \ldots$

(ii) Find the sum to infinity of the series

$\cos 2A + 2 \cos 2A \sin^2 A + 4 \cos 2A \sin^4 A + \ldots$

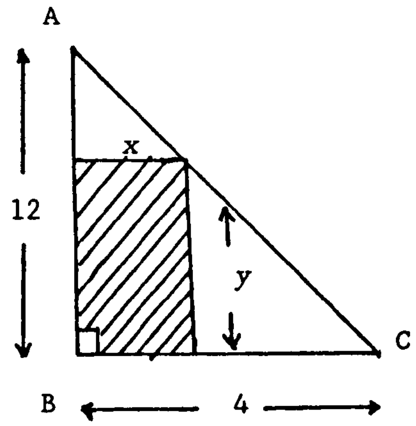
and find the range of values of A between 0 and $\frac{1}{2}\pi$ for which the sum to infinity exists.

(b) Show that the sum of all the numbers in the square array of numbers below which contains n rows of numbers with n numbers in each row is

$$\frac{n^2(n + 1)}{4}$$

$$\begin{array}{llll}
1 & 2 & 3 & 4 \ldots\ldots\ldots\ldots\ n \\
2 & 4 & 6 & 8 \ldots\ldots\ldots\ldots\ 2n \\
3 & 6 & 9 & \ldots\ldots\ldots\ldots\ldots\ 3n \\
\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots \\
\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots \\
n & 2n & 3n & \ldots\ldots\ldots\ldots\ldots nn
\end{array}$$

10. A piece of cord is in the shape of a right-angled triangle as shown with AB = 12 units, BC = 4 units and $\angle ABC = 90^\circ$.



A rectangle is to be cut from this triangle so that B is one of the vertices as shown.

(a) If x and y are the lengths of the sides of the rectangle as shown, show that $y = 12 - 3x$.

(b) Hence express A, the area of the rectangle, in terms of x and find the dimensions of the rectangle which would have the greatest area.

11. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors such that $|\boldsymbol{a}| = |\boldsymbol{b}|$, prove that the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is a right angle where

$$\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{a} - \boldsymbol{b}.$$

Appendix 9 (ctd)

Preliminary examination 1988-89

School 4

(a) Paper I

1. The gradient of a line perpendicular to $2x + 3y = 12$ is

A. $-\frac{3}{2}$ B. $-\frac{2}{3}$ C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. $\frac{3}{2}$.

2. Given that P and Q are the points (-2, 1, -5) and (3, -4, 4) respectevely, then the vector $\overrightarrow{QP}$ is equal to

A. $\begin{bmatrix} -5 \\ 5 \\ -9 \end{bmatrix}$ B. $\begin{bmatrix} -5 \\ 5 \\ -1 \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ D. $\begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$ E. $\begin{bmatrix} 5 \\ -5 \\ 9 \end{bmatrix}$

3. The inverse of the matrix $Q = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ is

A. $\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$ B. $\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$ C. $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ E. $\begin{bmatrix} -3 & 2 \\ -4 & 3 \end{bmatrix}$.

4. The exact value of $\sin \frac{4\pi}{3}$ is

A. $-\frac{\surd 3}{2}$ B. $-\frac{1}{2}$ C. $-\frac{1}{\surd 3}$ D. $\frac{1}{2}$ E. $\frac{\surd 3}{2}$.

5. The point O is the origin and the point P has coordinates (4, 7). After an anticlockwise rotation of 90° about O followed by reflection in the y-axis, the image of P is

A. (-7, -4) B. (7, -4) C. (0, -6) D. (2, 6) E. (7, 4).

6. The tangent to the curve $y = x^2 + 6x + 2$ has gradient 2 where x is equal to

A. -6 B. -2 C. 0 D. 2 E. 4.

7. The value of $(27^{-1/3} + 16^{1/2})^{-1}$ is

A. $\frac{1}{7}$ B. 7 C. $\frac{13}{4}$ D. $\frac{3}{13}$ E. $\frac{4}{13}$.

8. The exact value of $\sin 75^\circ$ is

A. $\frac{\sqrt{3} + 1}{\sqrt{3}}$ B. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ C. $\frac{\sqrt{3} + 1}{1 - \sqrt{3}}$ D. $\frac{\sqrt{3} - 1}{1 + \sqrt{3}}$ E. $\frac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

9. Given that $f(x) = \frac{x + 1}{\sqrt{x}}$, then $f'(4)$ is equal to

A. $-\frac{1}{4}$ B. $\frac{3}{16}$ C. $\frac{1}{4}$ D. $\frac{5}{16}$ E. 4.

10. Given that the lines $2x + ay + 3 = 0$ and $3x - 4y + 12 = 0$ are parallel, then the value of a is

A. $-\frac{8}{3}$ B. $-\frac{3}{2}$ C. $\frac{3}{8}$ D. $\frac{3}{2}$ E. $\frac{8}{3}$.

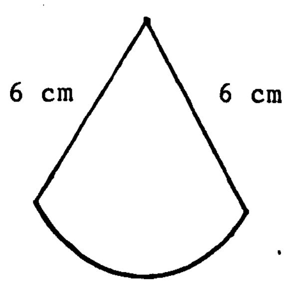
11. Given that $f(x) = \frac{x^4 + 4x^2 + 1}{2x^2}$, then $f'(x)$ is equal to

A. $x^2 + 2$ B. $x^2 + 2 - \frac{1}{2x^3}$ C. $x - \frac{1}{2x^3}$ D. $x - \frac{1}{x^3}$ E. $x - \frac{4}{x^3}$

12. Given that the points A, B, C have coordinates A(-3, 4, 5), B(5, -4, 5) and C(-5, 6, 0), then $\overrightarrow{AC}$ equals

A. $-\frac{4}{5}\overrightarrow{BC}$ B. $-\frac{1}{5}\overrightarrow{BC}$ C. $-5\overrightarrow{BC}$ D. $\frac{1}{5}\overrightarrow{BC}$ E. $5\overrightarrow{BC}$.

13.



Given that the perimeter of this sector is 24 cm, then its area in cm² is

A. 18 B. 36 C. 72 D. 144

E. 288.

14. The line through (3, 1) perpendicular to the line $x + 2y = 5$ has equation

A. $x - 2y = 1$ B. $2x + y = 5$ C. $2x - y = 1$

D. $2x + y = 1$ E. $2x + y = 7$.

15. $\cos 75^\circ - \cos 15^\circ$ equals

A. $-\sqrt{3}$ B. $-\frac{1}{\sqrt{2}}$ C. $\frac{1}{\sqrt{2}}$ D. $\frac{\sqrt{3}}{\sqrt{2}}$ E. $\sqrt{3}$.

16. Given that the line joining the points (0, 6) and (4,0) passes through the point $(k, -9)$, then the value of k is

A. -10 B. -2 C. 2 D. 10 E. 14.

17. Given that $f'(x) = x^2 - 4x$ and $f(3) = 1$, then $f(x)$ equals

A. $\frac{1}{3}x^3 - 2x^2$ B. $\frac{1}{3}x^3 - 2x^2 + 1$ C. $\frac{1}{3}x^3 - 2x^2 - 4$

D. $\frac{1}{3}x^3 - 2x^2 + 4$ E. $\frac{1}{3}x^3 - 2x^2 + 10$.

18. Given that the third and fourth terms of a geometric series are 12 and -6, respectively, then the sum to infinity is

A. 72 B. 32 C. 16 D. 8 E. -48.

19. The lines $px - 4y + 2 = 0$ and $px - ax - 8 = 0$ are perpendicular. Which of the following must be true?

A. $a = \frac{1}{4}$ B. $a = -4$ C. $p^2 = 4a$ D. $p^2 = -4a$ E. $a = -\frac{1}{4}$.

20. $2 \sin 50^\circ \sin 40^\circ$ equals

A. $1 + \sin 10^\circ$ B. $1 - \sin 10^\circ$ C. $\sin 10^\circ - 1$

D. $-\cos 10^\circ$ E. $\cos 10^\circ$.

21. Given that $f(x) = \frac{2}{\sqrt{x}}$, then $f'(4)$ equals

A. 4 B. 2 C. 1 D. $\frac{1}{8}$ E. $-\frac{1}{8}$.

22. Given the functions $f: x \to \frac{1}{2}(x + 2)$ and $g: x \to 2x - 1$, $x \in R$ then $(g \circ f)^{-1}: x \to$

A. $x + 1$ B. $x - 1$ C. $1 - x$ D. $\frac{1}{2}(2x + 1)$ E. $\frac{1}{2}(2x - 1)$

23. The factors of $x^3 - 12x - 16$ include which of the following

I $(x - 2)$ II $(x + 2)$ III $(x - 4)$ IV $(x + 4)$

A. I and III only B. I, II and III C. I and IV only

D. II and III only E. I, II and IV.

24. The vectors with $\begin{bmatrix} -2 \\ 6 \\ x \end{bmatrix}$ and $\begin{bmatrix} y \\ z \\ -2 \end{bmatrix}$ are parallel. Which of the following must be true?

A. $y = x$ B. $x + y - 3z = 0$ C. $z = 3y$

D. $xy = 4$ E. $xz = 12$.

25. $\int \frac{x + 1}{\sqrt{x}}\, dx$ equals

A. $2x^{5/2} + 2x^{3/2} + c$ B. $-\frac{1}{2}x^{5/2} - \frac{1}{2}x^{3/2} + c$ C. $2x^{3/2} - 2x^{\frac{1}{2}} + c$

D. $\frac{2}{3}x^{3/2} + 2x^{\frac{1}{2}} + c$ E. $\frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$.

26. C_1 is the circle with equation $x^2 + y^2 = 9$ and C_2 the circle with equation $(x - 3)^2 + (y - 4)^2 = 9$. Which of the following statements is false?

A The common tangents are parallel.

B The circles cut in two points.

C C_2 lies completely in the first quadrant.

D The circles meet on an axis.

E The distance between the centres is 5 units.

27. Given that $\frac{\tan 2x}{\tan x} = p$, then $\tan^2 x$ in terms of p is

A. $p(p + 1)$ B. $2 + \frac{1}{p}$ C. $\frac{1}{2 + p}$ D. $1 - \frac{2}{p}$

E. none of these.

28. $\overrightarrow{RS}$ represents the vector $\begin{bmatrix} 3 \\ -7 \end{bmatrix}$ and $\overrightarrow{ST}$ represents $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$

Given that R is the point (-1, 5), then T is

A. (2, −16) B. (−2, 16) C. (0, −6) D. (2, 6) E. (−2, 6).

29. The tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$ has equation

A. $y = 4x$ B. $y = 4x + 1$ C. $y = 8x$ D. $y = 8x - 7$

E. $y = 8x - 16$.

30. Given that $\tan\theta = -\frac{3}{4}$ and $90^\circ < \theta < 180^\circ$, then the exact value of $\dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$ is

A. −7 B. $-\frac{1}{7}$ C. $\frac{1}{7}$ D. 7 E. none of these values.

31. An arithmetic sequence has first term 4 and ninth term −20. The common difference is

A. −3 B. −2 C. −2 D. −2 E. −2.

32. The equation $3x^2 + px + 3 = 0$ has real roots. All possible values of p satisfy

A. $-6 < p < 6$ B. $0 < p < 3$ C. $-3 < p < 0$

D. $p > 6$ or $p < -6$ E. $p > 3$ or $p < -3$.

33 The angle between the lines $y = x$ and $y -3x$ has tangent equal to

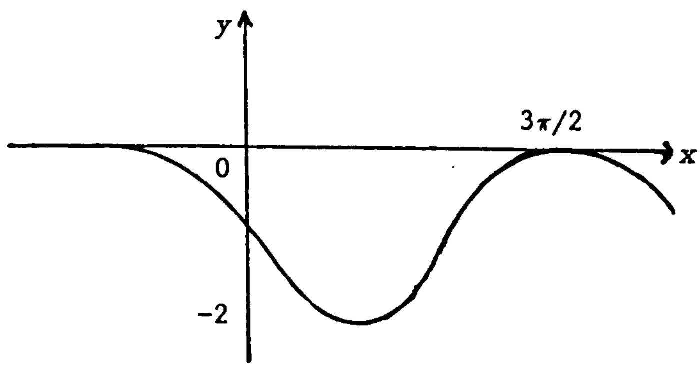
A. −4 B. −2 C. −1 D. $-\frac{1}{2}$ E. 2.

34. If $x + y = a$ and $xy = b$, then x must satisfy

A. $x(a - x) = b$ B. $x(a + x) = b$ C. $x(x - a) = b$

D. $x^2 - a = b$ E. $a - x^2 = b$.

35. The graph in the diagram is most likely to represent



A. cos x − 1

B. 1 − cos x

C. −2 sin x

D. −1 − sin x

E. 1 − sin x.

36. p is The vector with $\begin{bmatrix} a \\ -2a \\ 2b \end{bmatrix}$ and q is the vector $\begin{bmatrix} -4b \\ a \\ -b \end{bmatrix}$.

If p and q are perpendicular, which of the following must be true?

A. $a = -2b$ B. $a = -b$ C. $a = \frac{1}{2}b$ D. $a = b$ E. $a = 2b$.

37. Given that $0 < x < 2\pi$, the solution set of the inequation $\sin x > \cos x$ is

A. $\{x: \frac{1}{4}\pi < x < 5\pi/4\}$ B. $\{x: x < \frac{1}{4}\pi\} \cup \{x: x > 5\pi/4\}$

C. $\{x: \frac{1}{4}\pi < x < \frac{1}{2}\pi\} \cup \{x: 5\pi/4 < x < 3\pi/2\}$

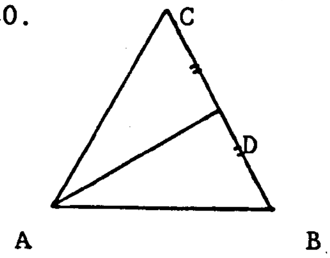
D. $\{x: x > \frac{1}{4}\pi\}$ E. $\{x: x > 5\pi/4\}$.

38. A is the point (3, −2) and B is (−3, 4). Given that P divides BA externally in the ratio 2:1, then P is the point

A. (−6, 10) B. (−1, 2) C. (1, 0) D. (6, −8) E. (15, −14)

39. $\lim_{h \to 0} \dfrac{(5 + h)^3 - 125}{h}$ is

A. −125 B. −120 C. 0 D. 75 E. undefined.

40. 

ABC is an equilateral triangle of side 1 unit, and D is the mid-point of BC. Given that AB = p and AD = q, then $p \cdot q$ equals

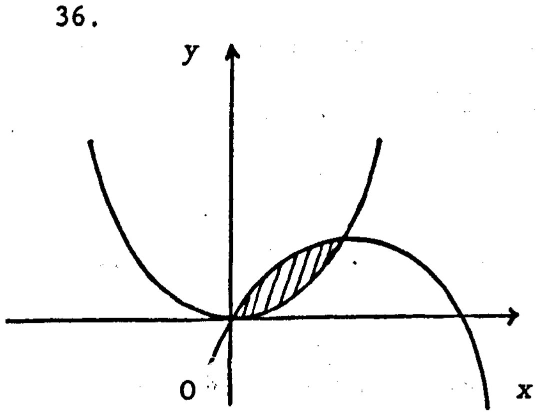
A. $\frac{1}{4}\sqrt{3}$ B. $3/4$ C. $\frac{1}{2}\sqrt{3}$ D. 1 E. $3/2$.

<u>The following five items were given to the second class instead of the earlier ones</u>

33. Given that $f(x) = \cos(5x^2 + 3)$, then $f'(x)$ is equal to

A. $10x \sin(5x^2 + 3)$ B. $-10x \sin(5x^2 + 3)$ C. $\sin(10x)$

D. $-\sin(10x)$ E. $-5 \sin(5x^2 + 3)$.

36. 

The diagram shows the graphs $y = x^2$ and $y = x(4 - x)$. The shaded area is given by

A. $\int_0^4 \{x^2 - x(4 - x)\}\, dx$ B. $\int_0^4 \{x(4 - x) - x^2\}\, dx$

C. $\int_0^2 \{x^2 - x(4 - x)\}\, dx$ D. $\int_0^2 \{x(4 - x) - x^2\}\, dx$

E. $\int_2^4 \{x^2 - x(4 - x)\}\, dx$.

38. $\int_0^1 (2x + 3)\, dx$ is equal to

A. $\frac{1}{2}\, 5^4$ B. $\frac{1}{4}\, 5^4$ C. $\frac{1}{8}\, 5^4$ D. $\frac{1}{4}\,(5^4 - 3^4)$ E. $\frac{1}{8}\,(5^4 - 3^4)$

40. Given that $\log a = 2 \log b - \log 3$, then a is equal to

A. $b^2 - 3$ B. $2b - 3$ C. $3b^2$ D. $\frac{b^2}{3}$ E. $\frac{2b}{3}$.

(b) Paper II

1. (a) Find, from first principles, the derivative with respect to x of $2x^2 - 3x$.

(b) Prove that if P divides AB in the ratio $m{:}n$, then

$$p = \frac{mb + na}{m + n}.$$

2. (a) Given that $f(x) = x^3 - 3x^2 + 6x + 5$,

(i) find $f'(x)$, and

(ii) deduce that $f(x)$ is always increasing.

(b) Evaluate $\int_{\frac{1}{2}}^{1} \frac{x + 1}{x^3}\, dx$.

3. The first three terms of a geometric sequence are $3(2 + x)$, $3(x - 2)$ and $6 - x$, respectively.

(a) Show that this can be true for two values of x.

(b) Write down the values of the first three terms in each case and explain why only one of these cases gives a geometric series with a sum to infinity.

(c) Calculate the value of this sum to infinity.

4. (a) Write down the coordinates of the centre, C, of the

circle $(x + 2)^2 + (y - 1)^2 = 13$.

(b) Find the equation of the diameter of the circle through the point P(1, 3).

(c) Find the coordinates of the other end, Q, of the diameter through P.

5. (a) Solve, for $0 \leqslant x \leqslant 360$, the equation

$$\cos 2x^\circ + 3 \sin x^\circ - 2 = 0.$$

(b) Prove that

$$\cos 2\theta + \sin 2\theta \tan \theta = 1.$$

6. Find the stationary points of the curve $y = x^4 - 2x^2$ and determine their nature.

Sketch the curve $y = x^4 - 2x^2$, indicating clearly where it cuts the axes.

7. Functions $h(x) = 3x - 2$ and $k(x) = \frac{x + 2}{3}$, are defined on the set of real numbers.

(a) Find expressions for $(h \circ k)(x)$ and $(k \circ h)(x)$.

(b) Deduce a relationship between h^{-1} and k.

8. A, B and C are the points (7, 2, 5), (1, -1, -1) and (3, -3, 1), respectively. The point D divides BC externally in the ratio 3:1.

(a) Find the coordinates of D.

(b) Prove that triangle ABC is obtuse-angled at C.

(c) Prove that AD is perpendicular to BC.

9. (a) Find the values of a and b if

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix}.$$

Hence calculate the matrix product BA.

(b) If X is another 2×2 matrix such that $XAB = BA$, find the matrix X and state the geometrical transformation with which it is associated.

10. The curve $y = x(x - 2)(x - 4)$ cuts the x-axis in three points.

(a) Prove that the tangents at two of these points are parallel.

(b) Find the equation of the tangent at the third point.

(c) Sketch the curve and calculate the area enclosed between the curve and the x-axis.

11. If α and β are the roots of the equation $2x^2 - 4x + 1 = 0$, find the values of

(i) $\alpha + \beta$ (ii) $\alpha\beta$ (iii) $\alpha^2 + \alpha\beta + \beta^2$.

12. (a) Prove that

$$\frac{\sin 5\theta + 2 \sin 3\theta + \sin \theta}{\cos \theta - \cos 5\theta} = \frac{1}{\tan \theta}.$$

(b) Solve, for $0 \leq \theta \leq 2\pi$

$$\tan 2\theta + 2 \sin \theta = 0.$$

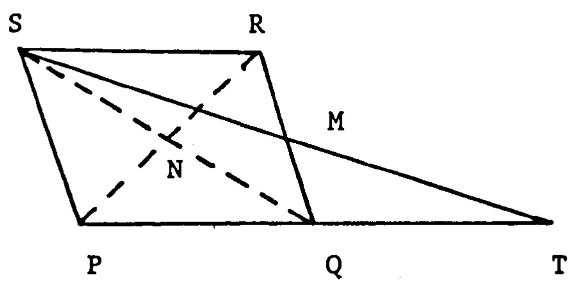
<u>The following items were given to the second class instead of the earlier ones</u>

1. (b) Prove that the equation of the straight line with gradient m which passes through the point (a, b) is

$$y - b = m(x - a).$$

2. (b) Evaluate $\int_0^2 \frac{1}{(4x + 1)^2}\, dx$.

8.



In the diagram, PQRS is a parallelogram and PQ = QT. The lines ST and QR intersect at M and the lines PR and SQ intersect at N. $\overrightarrow{PS}$ and $\overrightarrow{PQ}$ represent the vectors **u** and **v**.

(a) Find, in terms of **u** and **v**, the vectors represented by $\overrightarrow{QM}$ and $\overrightarrow{NR}$.

(b) If the coordinates of P, Q and S are (3, -3, 1), (7, 2, 5) and (1, -1, -1), respectively, find the vectors **u** and **v**, the coordinates of T and the length SQ.

11. (b) A function is defined by $g(x) = x^3 + k$, where k is constant. When $g(x)$ is divided by $x - 3$, the reminder is 36. Find the value of k.

(c) Find the value of x in the following equations.

(i) $(x^{\frac{1}{2}} + 1)^{1/3} = 2$

(ii) $\frac{1}{2} \log 25 + \frac{1}{3} \log 1000 = \log x + \log 4$.

12. If $f(x) = \cos^2 x + \cos 2x$, show that

$$f'(x) = -3 \sin 2x.$$

Appendix 9 (ctd)

Preliminary examination 1988-89

School 5

(a) Paper I

1. Write down the equation of the line pasing through the points (-5, -2) and (-2, -1).

2. P has position vector $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and Q $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Calculate the length of PQ.

3. Given $\tan A = \frac{1}{5}$ and $\tan B = \frac{2}{3}$, calculate the exact value of $\tan(A + B)$.

4. Given $f(x) = \frac{x + 1}{x^{\frac{1}{2}}}$, $x > 0$, find $f'(x)$.

5. Integrate: $2x^{2/3}$

6. Find, from first principles, the derivative with respect to x of x^2.

7. Given the sum to infinity of the geometric series with common ratio $-2/3$ is 9, calculate the first term.

8. P, Q, R are points such that $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\overrightarrow{PR} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and R is (0, 2, 1). Find the coordinates of Q.

9. In the triangle PQR, PQ has gradient 1/3 and R has coordinates (-4, 7). Write down the equation of the altitude from R to PQ.

10. Find the gradient of the curve $y = x^3$ at the point (2, 8).

11. Write down the x coordinates of the stationary points of the curve with equation $y = x^3 - 4x^2 - 3x$.

12. The functions f and g are defined for $0 \leqslant x \leqslant 2$ by

$$f(x) = x \quad , \quad 0 \leqslant x \leqslant 1$$
$$f(x) = 2 - x \ , \quad 1 \leqslant x \leqslant 2$$
$$g(x) = 2x \quad , \quad 0 \leqslant x \leqslant 1$$
$$g(x) = 3 - x \ , \quad 1 \leqslant x \leqslant 2$$

Calculate $f \circ g\ (3/2)$.

13. Given $x = 2\ (t^2 + 1)$ and $y = 4t$, find an equation expressing y in terms of x.

14. If i, j and k are unit vectors along mutually perpendicular axes, calculate $(i + 2j).(j + 2k)$.

15. Find the equation of the tangent to the curve

$y = 5x^3 - 7x + 3$ at the point $(-1, 5)$.

16. Simplify $\sin 4x^\circ \sin 3x^\circ - \cos 4x^\circ \cos 3x^\circ$.

17. Given $M = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$, find M^{-1}.

18. Find the sum of the following sequence to 40 terms.

$$2 + 5 + 8 + 11 + \ldots.$$

19. Given $f(x) = x + 1$ and $g(x) = x^2$, $x \in \mathbb{R}$. Find formulae for $f \circ g(x)$ and $(f \circ g)^{-1}(x)$.

20. Calculate the angle between the two vectors $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

21. Find the coordinates of the point dividing the line joining E(5, 2, 1) and F(9, 10, 13) externally in the ratio 1:3.

22. Solve, for x, the equation

$$2 \cos 2x + \cos x - 1 = 0, \; 0 \leq x \leq 360.$$

23. Calculate the area bounded by the curve $y = x^2 - 3x$ and the x-axis.

24. Find the intervals in which the function

$$f(x) = x^3 - 6x^2 + 5$$

is increasing and those in which it is decreasing.

(b) Paper II

1. Solve the system of equations

$$2x + y = 1$$
$$x^2 - xy - y^2 = -11.$$

2. The points A(0, -10), B(10, 3) and C(-4, 10) are the vertices of a triangle ABC.

Find: (a) the equation of the altitude AP and of the median BQ of this triangle;

(b) The coordinates of the point of intersection of AP and BQ.

3. Let $f: x \to \sin x$, $g: x \to x^2$ and $h: x \to 1 - 2x$ be mappings on the set of rea numbers.

(a) Find a formula in its simplest form for the function k such that $k(x) = (h \circ g \circ f)(x)$. Hence, or otherwise, find the image of $\frac{\pi}{6}$ under the mapping k.

(b) Give the formula for $h^{-1}(x)$, the inverse of h.

4. Calculate the area enclosed between the two curves

$$y = x(x - 3) \quad \text{and} \quad y = 2x(3 - x).$$

5. A curve has equation $y = x^3(3 - x)$.

(a) For this curve find

(i) the coordinates of the points at which it meets the coordinate axes;

(ii) the coordinates and nature of the stationary points on it.

(b) Make a rough sketch of the curve.

(c) Obtain the equation of the tangent at the point on the curve whose x-coordinate is 1.

6. Find the common ratio of the geometric sequence

$$9,\ 6\cos^2\theta,\ 4\cos^4\theta,\ \frac{8}{3}\cos^6\theta,\ \ldots,$$

and deduce that the series

$$9 + 6\cos^2\theta + 4\cos^4\theta + \frac{8}{3}\cos^6\theta + \ldots$$

has a sum to infinity. Show that this sum is $\dfrac{27}{2 - \cos 2\theta}$

7. With reference to mutually perpendicular axes, two sets of points A, B and C are as follows:-

First set —> A (3, -1, 0) B (7, 5, 6) C (17, 20, 21)

Second set -> A (-2, 1, 2) B (0, 3, 3) C (1, 7, 11).

For each set give a reason to show whether the points are collinear or non-collinear. Where A, B and C are collinear give the ratio AB:BC, and where non-collinear find the area of the triangle ABC as a surd.

8. A rectangular box without a lid is made of cardboard of negligible thickness. The side of the base are $2x$ cm and $3x$ cm, and the height is y cm. If the total area of the cardboard is 200 cm^2,

prove that

$$y = \frac{20}{x} - \frac{3x}{5}.$$

Find the dimensions of the box when its <u>volume</u> is a maximum.

9. (a) A is the point $(4, -1, 1)$ and F is the point $(-1, 4, -4)$. Find the coordinates of the point G dividing EF in the ratio 3:2 and of the point H dividing EF in the ratio $-2:3$.

(b) The vertices P, Q and S of a triangle PQS have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ respectively.

(i) Find $\boldsymbol{m}$, the position vector of M, the midpoint of PQ, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$.

(ii) Find $\boldsymbol{t}$, the position vector of T, on SM such that ST:TM = 2:1, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.

(iii) If the parellelogram PQRS is now completed, express $\boldsymbol{r}$, the position vector of R, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.

(iv) Prove that P, T and R are collinear.

The SCE examination 1988

Paper II

1. (a) Prove that if two straight lines with gradients m_1 and m_2 are perpendicular to each other, then

$$m_1 m_2 = -1.$$

(b) Find from first principles the derivative with respect to x of $\frac{1}{x^2}$, $x \neq 0$.

2. Solve, for $x, y \in \mathbb{R}$,

$$x - y - 5 = 2$$

$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0 \qquad x, y \in \mathbb{R}$$

3. (a) Given that A and B are the points (4, -1) and (-2, 7) respectively, state the centre and the radius of the circle on AB as diameter and write down its equation.

(b) Verify that P(-4, 3) and Q($-\frac{2}{5}$, $-\frac{9}{5}$) are the ends of a chord of the circle parallel to AB.

(c) Write down the coordinates of the mid-point of PQ and hence, or otherwise, calculate the distance between the chord PQ and the diameter AB.

4. (a) Solve, for $0 < x < 360$, the equation

$$4 \sin(x - 50)^\circ = 0.$$

(b) Write down the maximum and minimum values of $4 \sin(x - 50)^\circ$.

(c) Calculate the values of x at which the maximum and minimum values of $4 \sin(x - 50)^\circ$ occur in the interval $0 < x < 360$.

(d) Use your previous answers to sketch the graph of $y = 4 \sin(x - 50)^\circ$ in the interval $0 < x < 360$, showing clearly the points of intersection with the axes.

5. (a) Given that $y = \dfrac{x + 4x^{\frac{7}{3}}}{x^2}$, find $\dfrac{dy}{dx}$.

(b) Evaluate $\int_1^2 (3x - 5)^7 \, dx$.

6. A function $f(x)$ is defined by the formula

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12, \; x \in \mathbb{R}.$$

where $\mathbb{R}$ is the set of real numbers.

(a) Find the stationary points of $f(x)$ and determine their nature, justifing your answer.

(b) Sketch the graph of $y = f(x)$ showing the stationary points and the position of the curve without calculating the points of intersction with the x-axis.

(c) State the number of roots of $f(x) = 0$.

7. A and B are the points (4, 4, 10) and (3, 4, 5) respectively.

(a) Find the coordinates of the point Y which divides AB externally in the ratio 2:1.

(b) If C is the point (-3, 7, 1), prove that CY is perpendicular to AB and find the image of C under reflection in AB.

8. The transformation M is represented by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix},$$

(a) Write down the coordinates of two points on the line $y = \frac{1}{2}x$ and, hence or otherwise, find the equation of the image of this line under M.

(b) Find the equation of the image of the line $y = 3x + 1$ under M.

(c) Find the image of the point $(x, mx + c)$ under M and hence write down the equation of the image of the line $y = mx + c$.

(d)*Describe the effect of the transformation M on every line making particular reference to its effect on the line $y = x$.

9. Functions $f(x) = x + a$ and $g(x) = bx^2$, where a and b are rational constants, are defined on the set of real numbers.

(a) Find expression for $(f \circ g)(x)$ and $(g \circ f)(x)$.

(b) Show that

$$(f \circ g)(x) + (g \circ f)(x) = 2bx^2 + 2abx + (a^2b + a).$$

(c) Find a pair of non-zero values for a, b such that $(f \circ g)(x) + (g \circ f)(x)$ has equal roots.

10. A rectangular box without a lid is made from 200 cm^2 of metal. Its base measures $2x$ cm by $3x$ cm.

(a) Find the height of the box in terms of x and show that the volume V cm^3 is given by

$$V = 120x - \frac{18}{5}x^3.$$

(b) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.

11. (a) Solve the equation

$$3 \sin 7x^\circ + 3 \sin x^\circ = 2 \sin 4x^\circ,$$

where $0 \leq x < 180$.

(b) Prove, for $0 < A < \frac{1}{2}\pi$, that

$$\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \tfrac{1}{2}A.$$

12. In Δ PQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.

Given that $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ denote the position vectors of P, Q and S respectively relative to an origin O, find $\boldsymbol{m}$ and $\boldsymbol{t}$, the position vectors of M and T respectively, in terms of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$.

If the parallelogram PQRS is completed, show that the points P, T and R are collinear.

13. A series of squares is cut from a piece of card. The first square is 8 cm by 8 cm and subsequent squares are constructed with **diagonal** equal to the side of the previous square.

(a) Find the first three terms of the sequence of areas of squares and show that it is a geometric sequence, identifying the common ratio **r**.

(b) Show that a piece of card of area 128 cm^2 will be sufficient to construct all such squares.

(*) This item is omitted for a lack of precision.

Appendix 10

A "negative" questions changed to a "positive" questions

Original question

State which one of the following is fase.

The function sin x has the same period as the function

A $\cos x$

B $\sin 2x$

C $2 \sin x$

D $\sin(\frac{1}{2}\pi + x)$

E $2 + \sin x$

Modified question (no. 6)

One of the following functions has a different period to the function sin x; the function with a different period is

A. $\cos x$ B. $\sin x$ C. $2 \sin x$ D. $\sin(\frac{1}{2}\pi + x)$ E. $2 + \sin x$.

Original question

C_1 is the circle with equation $x^2 + y^2 = 9$ and C_2 the circle with equation $(x - 3)^2 + (y - 4)^2 = 9$. Which of the following statements is false?

A The common tangents are parallel.

B The circles cut in two points.

C C_2 lies completely in the first quadrant.

D The circles meet on an axis.

E The distance between the centres is 5 units.

Modified question (no. 11)

C_1 is the circle with equation $x^2 + y^2 = 9$ and C_2 the circle with equation $(x - 3)^2 + (y - 4)^2 = 9$. One of the following statements is false; the false statement is:

A. The common tangents are parallel.

B. The circles cut in two points.

C. C_2 lies completely in the first quadrant.

D. The circles meet on an axis.

E. The distance between the centres is 5 units.

Original question

In this question ONE or MORE of the four responses may be correct.

Answer A. if only (1), (2) and (3) are correct

B. if only (1) and (3) are correct

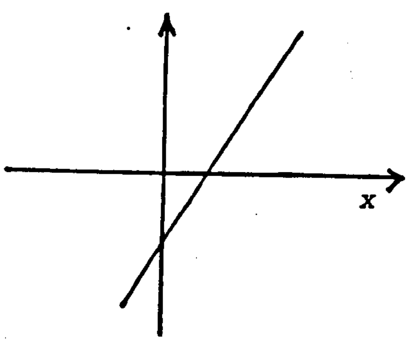
C. if only (2) and (4) are correct

D. if only (4) is correct.

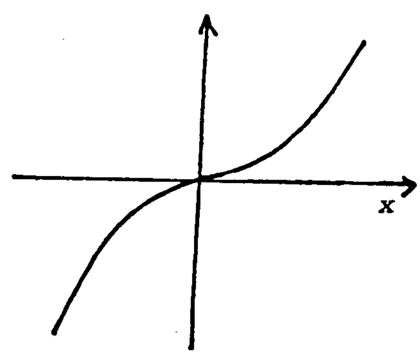
E. if some other response or combination of responses is correct.

Which of the following graphs show/s a function which has no inverse?

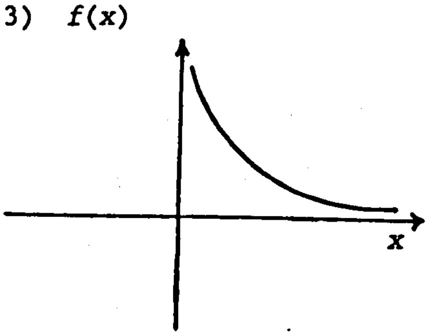
(1) $f(x)$



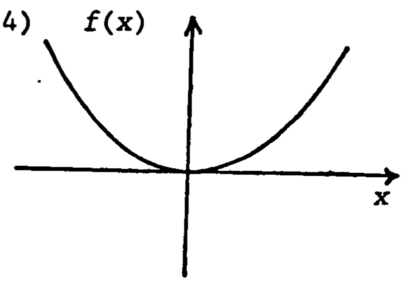
(2) $f(x)$



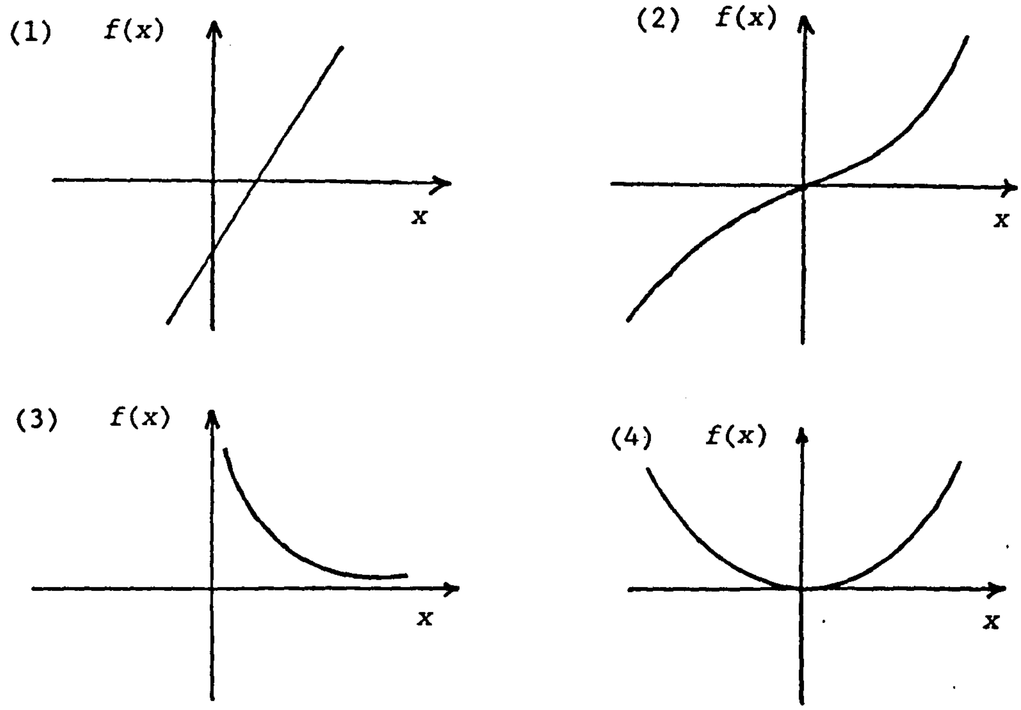
(3) $f(x)$



(4) $f(x)$



Modified question (no. 19)



Which of the following is true about the functions shown in the graphs?

A. Only (1), (2) and (3) have no inverse.

B. Only (1) and (3) have no inverse.

C. Only (2) and (4) have no inverse.

D. Only (4) has no inverse.

E. Some other combination of the functions have/has no inverse.

A change of wording

Original question

The length of the arc of a circle of radius r, subtended by an angle of x° at the centre of the circle, is a. x equals

A. a/r

B. r/a

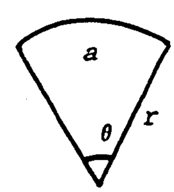
C. $\dfrac{180a}{\pi r}$

D. $\dfrac{\pi r}{180r}$

E. $\dfrac{180r}{\pi r}$.

Modified question (no. 2)

The length of the arc of a sector is a and its radius is r. Then the angle θ° of this radius is



A. $\dfrac{a}{r}$ B. $\dfrac{r}{a}$ C. $\dfrac{180a}{\pi r}$ D. $\dfrac{\pi a}{180r}$ E. $\dfrac{180r}{\pi a}$.

Original question

$f(x) = 3x\ (9 + x^2)(1 - 3x^2)$, $x \in \mathbb{R}$. The number of values of x for which $f(x) = 0$ is

A. zero

B. 1

C. 2

D. 3

E. 4 or more.

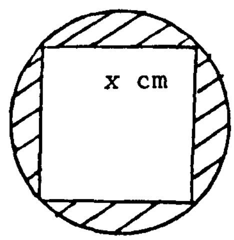
Modified question (no. 4)

The number of roots in $\mathbb{R}$ of the equation $3x(9 + x^2)(1 - 3x^2) = 0$ is

A. zero B. 1 C. 2 D. 3 E. 4 or more.

Original question

ABCD is a square with length x cm, inscribed in a circle.



An expression for the shaded area is

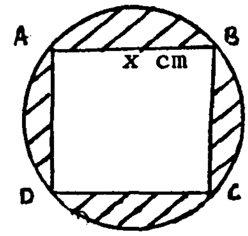
A. $x^2(\pi - 1)$ B. $x^2(\frac{1}{2}\pi - 1)$

C. $x^2(\frac{\pi - 1}{2})$ D. $x^2(\pi - \frac{1}{2})$

E. $x^2(\pi - \frac{1}{4})$

Modified question (no. 5)

The length of a side of the square ABCD is x cm. This square is inscribed in a circle. The shaded region has area



A. $x^2(\pi - 1)$ B. $x^2(\frac{1}{2}\pi - 1)$

C. $x^2(\frac{\pi - 1}{2})$ D. $x^2(\pi - \frac{1}{2})$ E. $x^2(\pi - \frac{1}{4})$.

Original question

Given the functions $f: x \to \frac{x+2}{2}$ and $g: x \to 2x - 1$, $x \in \mathbb{R}$.

then $(g \circ f)^{-1}: x \to$

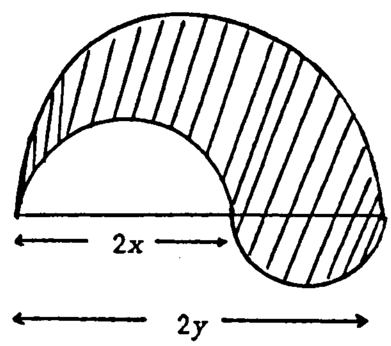
A $x + 1$ B $x - 1$ C $1 - x$ D $\frac{1}{2}(2x + 1)$ E $\frac{1}{2}(2x - 1)$.

Modified question (no. 7)

Given that $f(x) = \frac{1}{2}(x + 2)$ and that $g(x) = 2x - 1$, $x \in \mathbb{R}$, the formula $(g \circ f)^{-1}(x)$ for the inverse of $g \circ f$ is

A. $x + 1$ B. $x - 1$ C. $1 - x$ D. $\frac{1}{2}(2x + 1)$ E. $\frac{1}{2}(2x - 1)$.

Original question



The shaded area is bounded by three semi-circles. Its area is

A $\pi y(y - x)$

B $\pi y(y + x)$

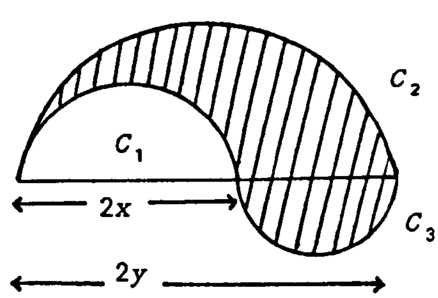
C $\pi y(y + 2x)$

D πy^2

E none of these.

Modified question (no. 9)

C_1 ana C_2 are semi-circles with diameters 2x and 2y respectively and C_3 is another semi-circle. The area of the shaded region is



A. $\pi y(y - x)$ B. $\pi y(y + x)$

C. $\pi y(y + x)$ D. $\pi\ y^2$ E. none of these.

Original question

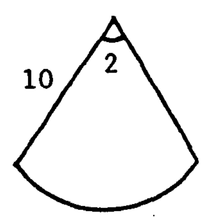
An indicator line of length 10 cm rotates about the centre of a circular radar screen at 2 radians per second. The area (in square centimetres) swept out per second is

A. 20 B. 40 C. 100 D. 200 E. 400

Modified question (no. 12)

Given that the angle of a sector is 2 radians and its radius is 10 cm, then its area is

A. 20 B. 40 C. 100 D. 200 E. 400.



Original question

If $g(x) = x^2 + 1$, $x \in \mathbb{R}$, $g^{-1}(x)$ equals

A. $(x - 1)^2$

B. $\sqrt{(x - 1)}$

C. $\sqrt{(x + 1)}$

D. $\sqrt{x} - 1$

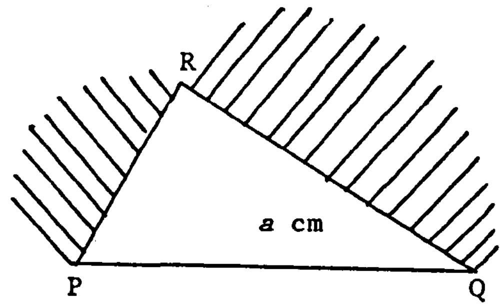
E. no inverse exists.

Modified question (no. 14)

Given that $g(x) = x^2 + 1$ for all real numbers x, then $g^{-1}(x)$ equals

A. $(x - 1)^2$ B. $\sqrt{(x - 1)}$ C. $\sqrt{(x + 1)}$ D. $\sqrt{x} - 1$ E. no inverse exists

Original question



The diagram shows three semi-circles with diameters PQ, PR and RQ. R is a point on the semi-circle with diameter PQ. PQ = a cm. The area of the shaded portion in cm² is

A. $1/8\pi a^2$

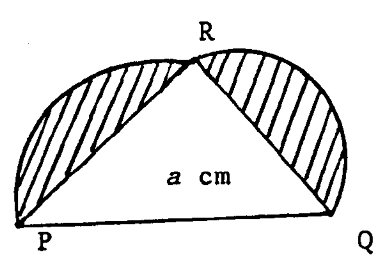
B. $\frac{1}{4}\pi a^2$

C. $\frac{1}{2}\pi a^2$

D. πa^2

E. indeterminable without knowing the position of R.

Modified question (no. 16)



PQ = a cm and R is a point on the circle with diameter PQ. The diagram shows two semi-circles with diameters PR and RQ. The area of shaded region in cm² is

A. $\frac{1}{8}\pi a^2$ B. $\frac{1}{4}\pi a^2$ C. $\frac{1}{2}\pi a^2$ D. πa^2

E. indeterminable without knowing the position of R.

Original question

The curve with equation $y = 4 \sin 3x$ is mapped on to itself when reflected in the line with equation

A. $x = 0$

B. $x = \pi/6$

C. $x = \pi/4$

D. $x = \pi/3$

E. $x = 2\pi/3$.

Modified question (no. 17)

Under a reflection in a line, the curve with equation $y = 4 \sin 3x$ is mapped into $y = 4 \sin 3x$ itself. The equation of the line is

A. $x = 0$ B. $x = \frac{\pi}{6}$ C. $x = \frac{\pi}{4}$ D. $x = \frac{\pi}{3}$ E. $x = \frac{2\pi}{3}$.

A multiple-completion question changed to a multiple-choice question

Original question

In this question one or more of the four responses may be correct.

Answer A. if only (1), (2) and (3) are correct.

B. if only (1) and (3) are correct.

C. if only (2) and (4) are correct.

D. if only (4) is correct.

E. if some other response or combination of responses is correct.

The vectors with components $\begin{bmatrix} k \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} k \\ 1 \\ -1 \end{bmatrix}$ will be perpendicular if k has the value

(1) $-\sqrt{2}$ (2) 0 (3) $\sqrt{2}$ (4) $\sqrt{3}$

Modified question (no. 1)

The vectors with components $\begin{bmatrix} k \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} k \\ 1 \\ -1 \end{bmatrix}$ will be perpendicular if k has the value

A. $-\sqrt{2}$ B. 0 C. 2 D. $\sqrt{3}$ E. -2

Original question

If $\tan(\alpha + \beta) = 2$ and $\tan \alpha = 3 \tan \beta$, then $\tan \beta$ has the values

(1) -1 (2) $1/3$ (3) 3

A. (1) only

B. (2) only

C. (3) only

D. (1) and (2) only

E. (1), (2) and (3).

Modified question (no. 3)

If $\tan(\alpha + \beta) = 2$ and $\tan \alpha = 3 \tan \beta$ then $\tan \beta$ could have the value

A. -1 B. $-\frac{1}{3}$ C. 3 D. 1 E. none of these.

<u>Original question</u>

For which of the following values of x is the vector

$$u = \begin{bmatrix} x \\ 5 \\ -3 \end{bmatrix} \text{ perpendicular to the vector } v = \begin{bmatrix} 3x \\ x \\ 4 \end{bmatrix}\ ?$$

(1) -3 (2) 0 (3) 4/3 (4) 3.

A. (1) only

B. (2) only

C. (1) and (2) only

D. (3) and (4) only

E. some other one or combination of responses.

<u>Modified question (no. 10)</u>

For which of the following values of x is the vector

$$u = \begin{bmatrix} x \\ 5 \\ -3 \end{bmatrix} \text{ perpendicular to the vector } v = \begin{bmatrix} 3x \\ x \\ 4 \end{bmatrix}?$$

A. -4 B. 0 C. $\frac{4}{3}$ D. 3 E. none of these.

Original question

In this question one or more of the four responses may be correct.

Answer A. if only (1), (2) and (3) are correct.

B. if only (1) and (3) are correct.

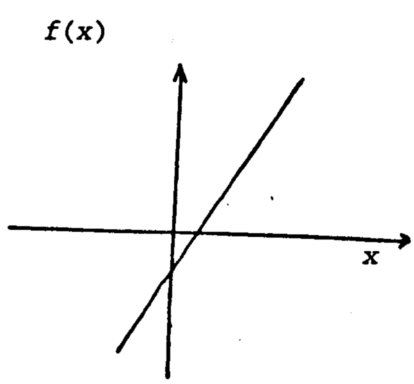
C. if only (2) and (4) are correct.

D. if only (4) is correct.

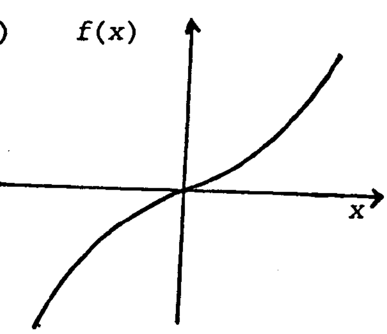
E. if some other response or combination of responses is correct.

Which of the following graphs show/s a function which has no inverse?

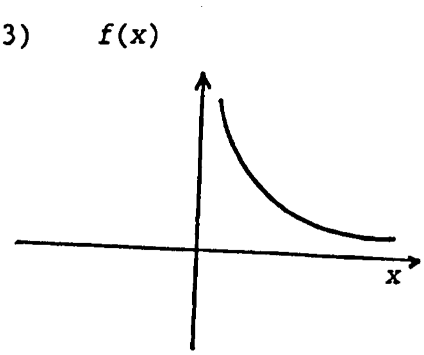
(1)



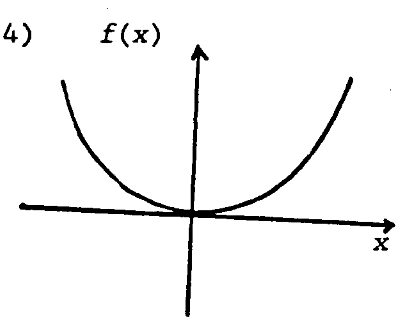
(2)



(3)



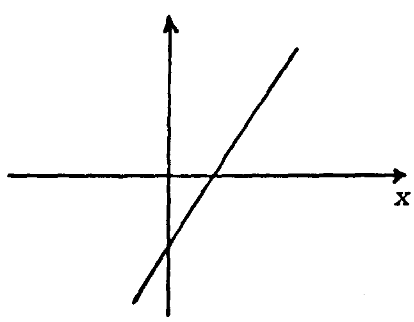
(4)



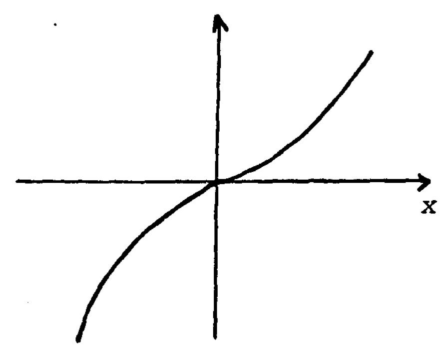
Modified question (no. 8)

Which of the following graphs shows a function wich has no inverse?

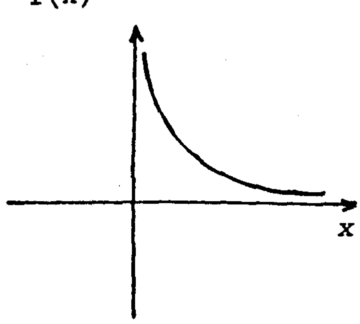
(1) $f(x)$



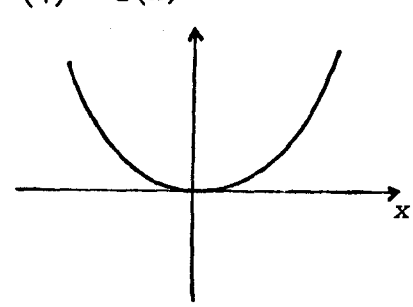
(2) $f(x)$



(3) $f(x)$



(4) $f(x)$



Original question

A quadratic equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers and $a \neq 0$, has one root $3 - \sqrt{2}$. Which of the following must be true?

(1) The other root is $3 + \sqrt{2}$

(2) $b/a = 6$

(3) $c/a = 5$

A. None of (1), (2) and (3)

B. (1) only

C. (1) and (2) only

D. (1) and (3) only

E. (1), (2) and (3).

Modified question (no. 13)

A quadratic equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers and $a \neq 0$, has one root $3 - \sqrt{2}$. Which of the following must be true?

A. The other root is $3 + \sqrt{2}$ B. $\frac{b}{a} = 6$ C. $\frac{c}{a} = 5$

D. The other root is $-3 + \sqrt{2}$ E. none of these.

Original question

For which of the folloing is $\int_{-\pi/4}^{\pi/4} f(x)\, dx = 2 \int_{0}^{\pi/4} f(x)\, dx$?

(1) $f(x) = 2x$ (2) $f(x) = \sin 2x$ (3) $f(x) = \cos 2x$

A. (1) only

B. (2) only

C. (3) only

D. (1) and (3)

E. (2) and (3).

Modified question (no. 15)

Given that $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} f(x)\, dx = 2 \int_{0}^{\frac{1}{4}\pi} f(x)\, dx$, then $f(x)$ could be

A. $2x$ B. $\sin 2x$ C. $\cos 2x$ D. $x + 2$ E. none of these.

Original question

Which of the following has a period of 180°

(1) $2 \sin x^\circ$ (2) $\sin 2x^\circ$ (3) $2 \tan x^\circ$ (4) $\tan 2x^\circ$

A. (1) and (3) only

B. (2) and (4) only

C. (1) and (4) only

D. (2) and (3) only

E. some other combination of (1), (2), (3) and (4).

Modified question (no. 18)

Which of the following has a period of 180°?

A. $2 \sin x^\circ$ B. $\sin 2x^\circ$ C. $\tan \frac{1}{2}x^\circ$ D. $\tan 2x^\circ$ E. $2 \tan \frac{1}{2}x^\circ$

Original question

P, Q, R have position vectors $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} - 4\mathbf{v}$ and $3\mathbf{u} - 7\mathbf{v}$ respectively. Which of the following statements is/are true?

(1) $\overrightarrow{PQ}$ represents $2\mathbf{u} - 6\mathbf{v}$

(2) P, Q and R are collinear

(3) Q is the mid-point of PR.

A. (1) only

B. (2) only

C. (3) only

D. (1), (2) and (3)

E. Some other combination of responses.

Modified question (no. 20)

P, Q, R are different points with position vectors $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} - 4\mathbf{v}$ and $3\mathbf{u} - 7\mathbf{v}$ respectively. Then

A. $\overrightarrow{PQ}$ represents $2\mathbf{u} - 6\mathbf{v}$,

B. P, Q and R are collinear,

C. $\overrightarrow{QR}$ represents $\mathbf{u} - 11\mathbf{v}$,

D. $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are perpendicular,

E. none of these.

Appendix 11

Classification of Division into Parts

Examples

Indivisible

If S_n denotes the sum of the first n terms of the geometric series

$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots, \; x > 1,$$

prove that $\dfrac{S_{2n}}{S_n} = 1 + \dfrac{1}{x^n}$.

Divisible

A, B and C are the points with co-ordinates (2, 7), (-2, 3) and (3, 13) respectively.

(a) Find the co-ordinates of of the point S which divides BC in the ratio 1:4.

(b) Find the equation of the line through S parallel to AC.

(C) Find the co-ordinates of the point D where ASDC is a parallelogram.

Divided Implicitly

A and B are the points with co-ordinates (2, 0, 3) and (4, -2, 1) respectively.

If the point P divides AB in the ratio 1:k, write down the co-ordinates of P in terms of k.

Given that OP is perpendicular to AB, find the value of k.

Divided Explicitly

(a) The terms $3(x + 5)$, $3(x + 3)$ and $(x + 7)$ are the first three terms of a geometric sequence.

Find the possible values of x.

(b) In each case, find the first term and the common ratio of the sequence.

(c) In the case where a sum to infinity S exists, evaluate S.

Divided Explicitly, Dependent Parts

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = 4n - n^2.$$

(a) Find (i) u_1 and u_2, the first two terms;

(ii) the common difference.

(b) Calculate the sum of the first 50 terms of the series

$$u_1 + u_3 + u_5 + \ldots$$

Divided Explicitly, Independent Parts

(a) Prove that the sum S_n of the first n terms of the geometric series

$$a + ar + ar^2 + \ldots \text{ is } \frac{a(1 - r^n)}{1 - r}, \; r \neq 1.$$

(b) Evaluate $\displaystyle\int_1^2 (3x - 5)\, dx$.

Divided Implicitly, Dependent Parts

Show that the line $x + 3y - = 0$ is a tangent to the circle $x^2 + y^2 + 10x + 6y - 56 = 0$ and find the coordinates of the point of contact.

Divided Implicitly, Dependent Parts and Independent Parts

(1) For the curve C defined by

$$y = \frac{x^2 - 9x + 18}{x^2},$$

Find $\frac{dy}{dx}$ and show that

$$\frac{d^2y}{dx^2} = \frac{18\ (6 - x)}{x^4}$$

Show that the curve C has one critical point and one point of inflection, determining the coordinates of each of these points and the nature of the critical point. Find the vertical asymptote of C, the intersections of C with the x-axis, and the asymptotic behaviour of C as $x \to \pm\infty$. Hence sketch the curve.

Appendix 12

Information for Teachers

1. There are seven different papers, each in a diffrent colour with accompanying white "answer sheets"; a pupil should work on ONE paper.
2. The papers are supplied in bundles of seven papers, one of each colour.
3. Each paper has two parts: A and B.
4. Part A comprises THREE questions to be answered on the accompanying "answer sheets".
5. Part B has an unfamiliar style. There are instructions, a sample question and ONE question for the pupil to answer in the same way as the sample question, in the space provided on the question paper.
   (This question may involve reference to a grid on an accompanying detached sheet.)

Instructions

1. Distribute the papers at random issuing all seven papers (complete with the white covers) from one bundle before starting on another bundle.
2. Tell the pupils to write their names in the spaces provided:
   (a) on the first page of the "answer sheets",
   (b) on the first page of part B of the question paper.
3. Advise the pupils to take about 50 minutes for part A and about 30 minutes for part B.
4. At the end, ask the pupils to place all their papers inside the white cover before you collect them.

Thank you for your help.

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

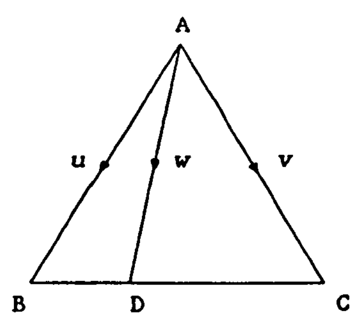
2. A rectangular box without a lid is made from 200 cm² of metal. Its base measures $2x$ cm by $3x$ cm.

   (a) Find the height of the box in terms of x and show that the volume V cm³ is given by

$$V = 120x - \frac{18}{5}x^3.$$

   (b) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.

3. Triangle ABC is equilateral and the length of each side is one unit. D lies on BC and BD = $\frac{1}{3}$BC. Given that **u**, **v** and **w** are the vectors represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ respectively, express **w** in terms of **u** and **v** and then show that



$$\mathbf{w}.\mathbf{u} = \frac{5}{6}.$$

PAPER ONE: PART B

NAME______________________________

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid on the separate sheet, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

Your answer should look like this:

The numbers of the relevant steps are: (7) and (11).

The solution is:

$$\begin{aligned} \cos A + \cos^2 A &= \cos A\,(1 + \cos A) && \text{[step (7)]} \\ &= \cos A\,(2 \cos^2 \tfrac{1}{2}A) && \text{[step (11)]} \\ &= 2 \cos A \cos^2 \tfrac{1}{2}A. \end{aligned}$$

NOW

Answer the next question in the same way.
Only some steps in the grid are necessary to solve the question.

QUESTION

(a) Solve, for $0 < A < 2\pi$,

$$\tan 2A + 2 \sin A = 0.$$

(b) Prove, for $0 < A < \frac{1}{2}\pi$, that

$$\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \tfrac{1}{2}A.$$

(a) The numbers of the relevant steps are:____________________________

The solution is:

(b) The numbers of the relevant steps are:____________________________

The solution is:

Continue your solutions on the next page if necessary.

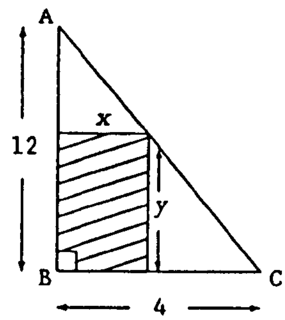
GRID

$\cos^2 A + \sin^2 A = 1$ (1)	$\tan A = \frac{\sin A}{\cos A}$ (2)	Multiply by $\frac{\pi}{180}$ to convert from degrees to radians. (3)
Use the periods of sin and cos to obtain all solutions of $\sin A = 0$, $\cos A = \frac{1}{2}$, $\cos A = -1$ in specified range. (4)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (5)	$\cos 2A = 2 \cos^2 A - 1$ (6)
Use a common factor to factorize an expression. (7)	$\sin 2A = 2 \sin A \cos A$ (8)	Use a calculator to find an angle with a given cosine. (9)
The period of $\tan A$ is π. (10)	$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (11)	The equations $\sin A = 0$ and $\cos A = 0$ have two principal solutions. (12)

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. A piece of cord is in the shape of a right-angled triangle as shown with AB = 12 units, BC = 4 units, $\hat{ABC} = 90^\circ$.



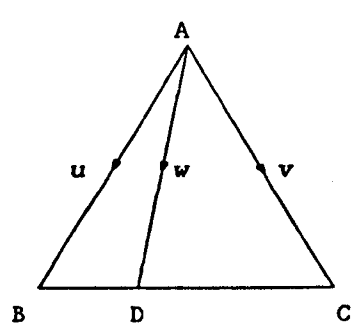
A rectangle is to be cut from this triangle so that B is one of the vertices as shown.

(i) If x and y are the lengths of the sides of the rectangle as shown, show that $y = 12 - 3x$.

(ii) Hence express A, the area of the rectangle, in terms of x and find the dimensions of the rectangle which would have the greatest area.

3. Triangle ABC is equilateral and the length of each side is one unit. D lies on BC and BD = $\frac{1}{3}$BC. Given that **u**, **v** and **w** are the vectors represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ respectively, show that

$$\mathbf{w}.\mathbf{u} = \frac{5}{6}.$$



PAPER TWO; PART B

NAME____

4. Read both parts of the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

SAMPLE QUESTION

(i) Express $(\sin A + \cos A)^2$ in terms of $\sin 2A$.

Your answer should look like this:

$$(\sin A + \cos A)^2 = 1 + \sin 2A.$$

(ii) Indicate your degree of confidence in your answer by putting a tick in the appropriate box.

Degree of confidence	Sample Question
1. I have no idea, so I have just made a guess.	
2. I am not sure, but I suspect it may be right.	
3. I think I am right.	
4. I am sure I am right.	
5. I know I am right and I can prove it.	✓

The fifth box has been ticked because "I know I am right and I can prove it."

NOW

Answer the parts of the next question in the same way.

<u>QUESTION</u>

(i) Express:

(a) sin 2A in terms of sin A and cos A,

(b) cos 2A in terms of sin A,

(c) tan 2A in terms of tan A,

(d) cos 2A in terms of cos A.

Your answers are:

(a) ______________________________

(b) ______________________________

(c) ______________________________

(d) ______________________________

(ii) Indicate your degree of confidence in your answers by putting a tick in the appropriate box for each answer.
You should put ONE tick in each column.

Degree of confidence	Parts of the question			
	(a)	(b)	(c)	(d)
1. I have no idea, so I have just made a guess.				
2. I am not sure, but I suspect it may be right.				
3. I think I am right.				
4. I am sure I am right.				
5. I know I am right and I can prove it.				

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. (a) Solve, for $0 \leqslant A < 2\pi$,

$$\tan 2A + 2 \sin A = 0.$$

(b) Prove, for $0 < A < \frac{1}{2}\pi$, that

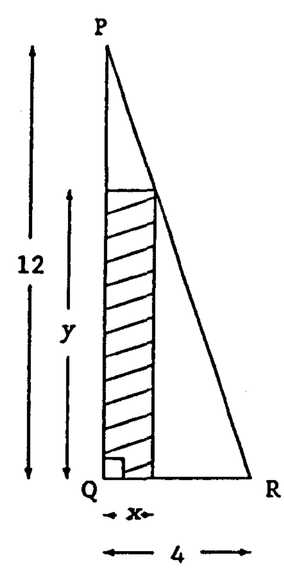
$$\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \tfrac{1}{2}A.$$

3. In the diagram, PQR is a right-angled triangle with PQ = 12 units and QR = 4 units. The shaded rectangle has sides of length x and y as shown.

(i) Show that $y = 12 - 3x$.

(ii) Hence express the area A of shaded rectangle in terms of x.

(iii) Find the dimensions of the rectangle which would have the greatest area.



PAPER THREE: PART B

NAME____

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully, and then follow these instructions with the QUESTION at the end.

INSTRUCTIONS

You are NOT required to solve the question.

1. Identify the aims of the question (the things you have to find or show or prove).
2. Identify the data of the question (the information in the question which is needed in the solution).
3. Write down the ideas which are required to solve the question (these could be in the question or from your knowledge).

SAMPLE QUESTION

The positions of the points A and B on a map are given by the coordinates (4,4) and (3,5) respectively.

(a) Find the position of the point P which divides AB externally in the ratio 2:1.

(b) If the point C has the position (-3,1), prove that CP is perpendicular to AB.

Your answer should look like this:

The aims are:

1. Find the position of the point P. 2. Prove that CP is perpendicular to AB.

The data are:

1. A is the point (4,4). 2. B is the point (3,5). 3. P divides AB externally in the ratio 2:1. 4. C is the point (-3,1).

The ideas are:

1. External division. 2. The components of a vector. 3. Scalar product.

NOW

Answer the next question in the same way.

QUESTION

In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.

Given that p, q and s denote the position vectors of P, Q and S respectively relative to an origin O, find m and t, the position vectors of M and T respectively, in terms of p, q and s.

If the parallelogram PQRS is completed, show that the points P, T and R are collinear.

The aims are:

The data are:

The ideas are:

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors such that $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\boldsymbol{a} \neq \pm \boldsymbol{b}$, prove that the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is a right angle, where

$$\mathbf{u} = \boldsymbol{a} + \boldsymbol{b} \quad \text{and} \quad \mathbf{v} = \boldsymbol{a} - \boldsymbol{b}.$$

3. In the triangle ABC, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ denote the position vectors of A, B and C respectively relative to an origin O.
   (a) Given that P divides BA in the ratio 2:1, find $\boldsymbol{p}$ the position vector of P in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.
   (b) Given that Q divides CP in the ratio 3:1, find $\boldsymbol{q}$ the position vector of Q in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.
   (c) If the parallelogram ABDC is completed, show that the points A, Q and D are collinear.

PAPER FOUR; PART B

NAME____

4. Read the INSTRUCTIONS, the SAMPLE QUESTION and the SAMPLE GRID carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

SAMPLE GRID

$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (1)	Use a calculator to find an angle with a given cosine. (2)
Use a common factor to factorize an expression. (3)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (4)

Your answer should look like this:

The numbers of the relevant steps are: (3) and (1).
The solution is:

$$\cos A + \cos^2 A = \cos A\,(1 + \cos A) \qquad [\text{step (3)}]$$
$$= \cos A\,(2\cos^2 \tfrac{1}{2}A) \qquad [\text{step (1)}]$$
$$= 2\cos A \cos^2 \tfrac{1}{2}A.$$

<u>NOW</u>

Use the <u>grid</u> on the separate sheet to answer the next question in the same way. Only some steps in the <u>grid</u> are necessary to solve the question.

<u>QUESTION</u>

A rectangular box without a lid is made from 200 cm^2 of metal. Its base measures $2x$ cm by $3x$ cm.

(a) Find the height of the box in terms of x and show that the volume V cm^3 is given by

$$V = 120x - \frac{18}{5}x^3.$$

(b) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.

The numbers of the relevant steps are: ____________________

The solution is:

Continue your solution on the next page if necessary.

<u>GRID</u>

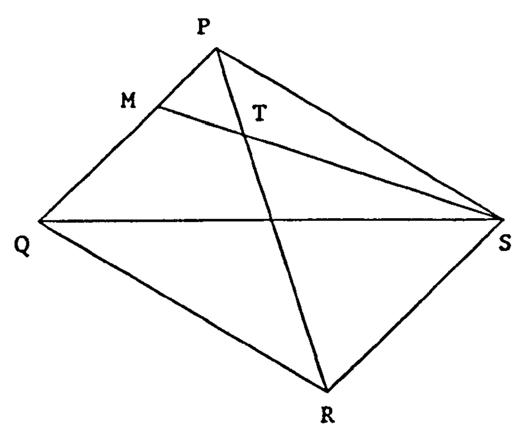
Study the sign of the second derivative of V. (1)	A cuboid has six faces. (2)	$x^2 - a$, $a > 0$, has two roots. (3)
The quantity of metal is the area of the faces of the box. (4)	Calculate the minimum volume. (5)	Work out when the derivative of V is 0. (6)
Test for minimum value. (7)	The volume of the box is the product of its dimensions. (8)	Differentiate V. (9)
Calculate the maximum volume. (10)	Work out when the second derivative of V is 0. (11)	The box has five faces. (12)

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. A and B are the points (4,4,10) and (3,4,5) respectively.
   (a) Find the coordinates of the point Y which divides AB externally in the ratio 2:1.
   (b) If C is the point (-3,7,1), prove that CY is perpendicular to AB and find the image of C under reflection in AB.

3. In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.
Given that p, q and s denote the position vectors of P, Q and S respectively relative to an origin O, find m and t, the position vectors of M and T respectively, in terms of p, q and s.
If the parallelogram PQRS is completed, show that the points P, T and R are collinear.



PAPER FIVE: PART B

NAME____

4. Read the INSTRUCTIONS, the SAMPLE QUESTION and the SAMPLE GRID carefully before you attempt the QUESTION at the end.

INSTRUCTIONS

1. Read the question carefully.
2. Think of the way you would go about answering the question.
3. From the grid, select the numbers of the relevant steps you would take to solve the question and write down these numbers.
4. Use these relevant steps to solve the question.

SAMPLE QUESTION

Prove that

$$\cos A + \cos^2 A = 2 \cos A \cos^2 \tfrac{1}{2}A.$$

SAMPLE GRID

$1 + \cos A = 2 \cos^2 \frac{1}{2}A$ (1)	Use a calculator to find an angle with a given cosine. (2)
Use a common factor to factorize an expression. (3)	$1 - \cos A = 2 \sin^2 \frac{1}{2}A$ (4)

Your answer should look like this:

The numbers of the relevant steps are: (3) and (1).
The solution is:

$$\cos A + \cos^2 A = \cos A\,(1 + \cos A) \qquad [\text{step } (3)]$$
$$= \cos A\,(2\cos^2 \tfrac{1}{2}A) \qquad [\text{step } (1)]$$
$$= 2\cos A \cos^2 \tfrac{1}{2}A.$$

NOW

Use the grid on the separate sheet to answer the next question in the same way. Only some steps in the grid are necessary to solve the question.

QUESTION

If a and b are vectors such that $|a| = |b|$ and $a \neq \pm b$, prove that the angle between the vectors u and v is a right angle, where

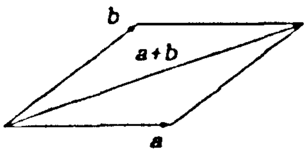
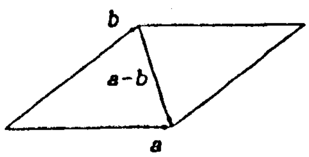
$$u = a + b \quad \text{and} \quad v = a - b.$$

The numbers of the relevant steps are: ______________________

The solution is:

Continue your solution on the next page if necessary.

GRID

 (1)	Find the scalar product of $\mathbf{a}$ and $\mathbf{b}$. (2)	Use the commutative property of scalar product. (3)
 (4)	If $\mathbf{u}.\mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular to $\mathbf{v}$. (5)	Find the scalar product of $\mathbf{u}$ and $\mathbf{v}$. (6)
The diagonals of a rhombus are perpendicular. (7)	$\mathbf{a}.\mathbf{a} = \|\mathbf{a}\|^2$. (8)	If $\mathbf{a} = 0$ or $\mathbf{b} = 0$ then $\mathbf{a}.\mathbf{b} = 0$. (9)
Use the distributive property of scalar product. (10)	If $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\|\mathbf{a}\| = \sqrt{(x^2 + y^2 + z^2)}$ (11)	If $\|\mathbf{a}\| = \|\mathbf{b}\|$ then $\|\mathbf{a}\|^2 = \|\mathbf{b}\|^2$. (12)

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. In ΔPQS, M divides QP in the ratio 2:1 and T divides SM in the ratio 3:1.
Given that p, q and s denote the position vectors of P, Q and S respectively relative to an origin O, find m and t, the position vectors of M and T respectively, in terms of p, q and s.
If the parallelogram PQRS is completed, show that the points P, T and R are collinear.

3. (a) Given the vectors

$$a = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix},$$

find (1) $a.b$,
(2) $a + b$ and $a - b$ in component form,
(3) $|a|$.

(b) What does $c.d = 0$ tell you about the non-zero vectors c and d ?

(c) If u and v are vectors such that $|u| = |v|$, express the scalar product $(u + v).(u - v)$ in simplest form.

PAPER SIX: PART B

NAME____

4. Read the INSTRUCTIONS and the SAMPLE QUESTION carefully, and then follow these instructions with the QUESTION at the end.

INSTRUCTIONS

You are NOT required to solve the question.

1. Identify the aims of the question (the things you have to find or show or prove).
2. Identify the data of the question (the information in the question which is needed in the solution).
3. Write down the ideas which are required to solve the question (these could be in the question or from your knowledge).

SAMPLE QUESTION

The positions of the points A and B on a map are given by the coordinates (4,4) and (3,5) respectively.

(a) Find the position of the point P which divides AB externally in the ratio 2:1.

(b) If the point C has the position (-3,1), prove that CP is perpendicular to AB.

Your answer should look like this:

The aims are:

1. Find the position of the point P.
2. Prove that CP is perpendicular to AB.

The data are:

1. A is the point (4,4).
2. B is the point (3,5).
3. P divides AB externally in the ratio 2:1.
4. C is the point (-3,1).

The ideas are:

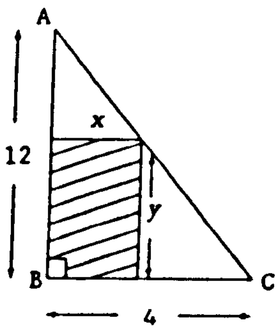
1. External division.
2. The components of a vector.
3. Scalar product.

NOW

Answer the next question in the same way.

QUESTION

A piece of cord is in the shape of a right-angled triangle as shown with AB = 12 units, BC = 4 units, $A\hat{B}C = 90^\circ$.



A rectangle is to be cut from this triangle so that B is one of the vertices as shown.

(i) If x and y are the lengths of the sides of the rectangle as shown, show that $y = 12 - 3x$.

(ii) Hence express A, the area of the rectangle, in terms of x and find the dimensions of the rectangle which would have the greatest area.

The aims are:

The data are:

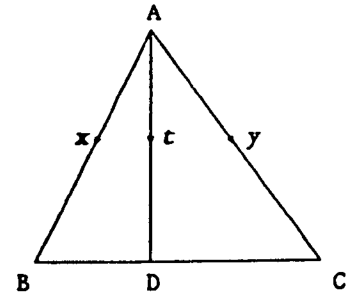
The ideas are:

1. Solve, for $x, y \in R$,

$$x - y - 5 = 0,$$
$$x^2 - xy + 2y^2 - 3x + 4y - 10 = 0.$$

2. Triangle ABC is equilateral, with each side of length one unit. $\overrightarrow{AB}$ represents vector x, $\overrightarrow{AC}$ represents y and $\overrightarrow{AD}$ represents t. D lies on BC and BD = $\frac{1}{3}$BC. Show that

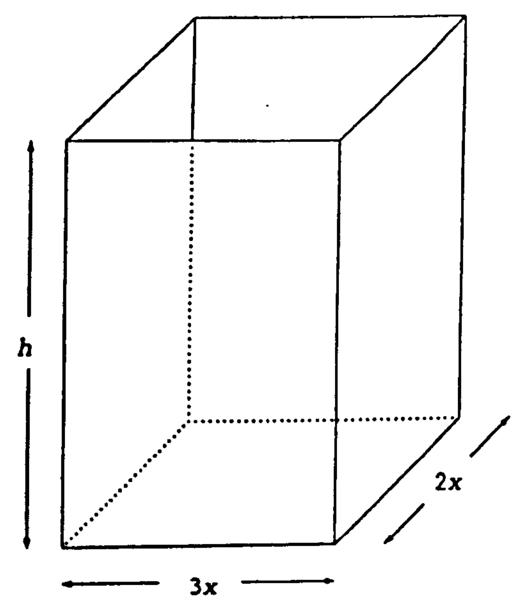
$$t.x = \frac{5}{6}.$$



3. A rectangular box without a lid is made from 200 cm² of metal. Its base measures $2x$ cm by $3x$ cm.
   (a) Find h, the height of the box, in terms of x.
   (b) Show that the volume V cm³ is given by

   $$V = 120x - \frac{18}{5}x^3.$$

   (c) Show that the maximum volume occurs when $x = \frac{10}{3}$ and find this maximum volume.



PAPER SEVEN: PART B

NAME____

4. Read both parts of the SAMPLE QUESTION carefully before you attempt the QUESTION at the end.

SAMPLE QUESTION

(i) Find the image of the point P(-4,11) under the translation $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Your answer should look like this:

The image of this point is (-1,11).

(ii) Indicate your degree of confidence in your answer by putting a tick in the appropriate box.

Degree of confidence	Sample Question
1. I have no idea, so I have just made a guess.	
2. I am not sure, but I suspect it may be right.	
3. I think I am right.	
4. I am sure I am right.	
5. I know I am right and I can prove it.	✓

The fifth box has been ticked because "I know I am right and I can prove it."

NOW

Answer the parts of the next question in the same way and use the last page for your rough work.

QUESTION

(i) C is the point (2,-3).

(a) Find the image of C under reflection in

(1) the x-axis,

(2) the line $x = 1$.

(b) Given the points A(4,3), B(-1,-2) and Y(0,-1) lie on the same line,

(1) show that CY is perpendicular to AB,

(2) find the image of C under reflection in AB.

Your answers are:

(a) The image of C under reflection in

(1) the x-axis is ____________________

(2) the line $x = 1$ is ____________________

(b) (1) CY is perpendicular to AB because ____________________

(2) The image of C under reflection in AB is ____________

(ii) Indicate your degree of confidence in your answers by putting a tick in the appropriate box for each answer.
You should put ONE tick in each column.

Degree of confidence	Parts of the question			
	(a)		(b)	
	(1)	(2)	(1)	(2)
1. I have no idea, so I have just made a guess.				
2. I am not sure, but I suspect it may be right.				
3. I think I am right.				
4. I am sure I am right.				
5. I know I am right and I can prove it.				