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Regime-Switching Option Pricing Models

This thesis submitted for the degree of Doctor of Philosophy in Quantitative Finance

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Introduction

The present thesis attempts to investigate the importance of regime-switching utilization in option pricing models. In this attempt, we construct and test a variety of regime-switching option pricing models in order to derive conclusions on the improvements that those models can consist in the option valuation process.

Many researches in the past have provided evidence that financial series can occasionally exhibit breaks in their behavior, e.g. due to a financial crisis or changes in the financial policies. For example, it is known that under financial crisis the volatilities and correlations between different assets tent to be higher compare to their long-run levels.

A common technique to represent this dynamic behaviors of data series is to employ linear models such as Autoagressive models, ARMA models or Moving Average models. However, this models fail to capture nonlinear dynamic patterns.

Markov switching models, commonly known as regime-switching models are the most popular non-linear time-series in financial modeling. These models aim to incorporate the dynamic breaks in financial series behavior into the financial modelling. By permitting the model's parameters to switch values according to the regime the economy is in, regime-switching models manage to be more flexible and thus to capture more accurately the data series. More precisely, regime-switching models consists of multiple equations each of which represents a different state, regime, in the economy. Thus, each equation captures a different behavioral pattern of the time-series and by allowing switches among the equations the models are able to capture more complex dynamic patterns. The mechanism by which these switches occur is controlled by an unobservable *Markov chain* variable, the *state variable*. This state variable follows an n –state first-order Markov chain; where n is the number of different regimes assumed to be in the economy. Thus, the value of the state variable at each time t depends only on its value at time $t-1$.

Thus, under regime-switching models the dependent variables' behavior is state-dependent since the unobservable prevailing state determines the process which generates the dependent variables' values.

In the present thesis we consider a two-state first-order Markov models. This means that we assume that the economy is divided in two states and that the prevailing state at any point of time depends only on the most resent state and not on the states prevailing before this. The unobservable state variable thus takes only two values, indicating the state the time-series is in, and its switches are controlled by the so called *transition probabilities*. The transition probabilities give the probability of being in one of the two states in the future given the present prevailing state. For example, if the state variable is s_t and can take the values $s_t = 1$ or $s_t = 2$, than

$$P(s_t = 1|s_{t-1} = 2, s_{t-2} = 1, s_{t-3} = 2, \dots) = P(s_t = 1|s_{t-1} = 2) = p_{12}$$

and $p_{11} + p_{12} = 1$.

So in contrast with single regime models where we have to estimate the parameters of a single equation, structure, in regime-switching models we have to estimate the parameters of multiple structures as well as the transition probabilities of switching between these structures. This requires a much higher computational effort which is probably the main drawback of these models. Moreover, other critics on Markov switching models relay on the fact that the regime switching is exogenous, i.e. the transition between states depends only on the current state and not on the parts of the model, the realization of the time-series and the past states. This can be considered as an unrealistic assumption since we would expect the prevailing state in the future to depend on the realization of the time series as well as the current and past states. However, besides the critics, the regime-switching models have been proven to better present the switching economic environment than single-regime models.

The present thesis tries to address the importance of regime switching into option pricing as well as to develop new option evaluation models in an attempt to contribute to the existing literature new sophisticated and accurate option pricing models. To the best of our knowledge, the models developed in this thesis are novel. Moreover, the numerical examples in the end of each chapter provide evidence that regime switching models are able to provide us with more accurate option evaluation models.

The thesis is divided into two parts, in the first part we work on district-time framework while in the second part we focus on continuous-time models. Beginning by structuring district-time regime switching option pricing models in the first part and examining how the consideration of regime-switching can improve the estimations of a lattice style option model, we continue in the second part with the development of a class of continuous time regime-switching option evaluation models. Thus, in the first part we develop an experimental option pricing model which allows us to search in deep the contribution of regime switching into option pricing. Having gained a positive feedback on the effects that regime switching consideration has on the option pricing process from the first part, we then develop in the second part of the thesis a class of sophisticated option pricing models for options written on commodities.

In Part I the thesis mainly focuses on the effect that the regime-switching in the correlation between the risk-free interest-rate and the underlying returns has on the option prices. We try to investigate whether the consideration of regime-switching correlation between the interest-rates and the underlying asset’s returns can improve the accuracy of our estimations. For this purpose we develop a *pentanomial* lattice model where the parameters are allowed to switch values over time. This part allows all the model parameters, including the mean and variance of the stock returns and risk-free interest rates, to change values over different regimes. Later in this part, aiming to isolate the effect of the regime-switching correlation on the option values from the effect of the rest regime-switching parameters, we develop and test a second model where only the correlation is allowed to follow a regime-switching process. The findings in this part suggest that the option values indeed capture the regime-switching correlation. More precisely, our findings suggest that a regime switching option pricing model in most of the cases outperform the classic Black-Scholes model while the option prices appear to be sensitive to the regime switching parameters.

In the second part of the thesis we focus on continuous-time regime-switching option evaluation models and more specifically on options written on commodities. Because of the nature of commodities, the commodities' prices are assumed to follow mean-reverting stochastic processes. In this part we first develop novel three models without assuming regime-switching and then we transform the models into Markov switching models. The models are one-, two-, and three-factor models according to the number of stochastic factors considered. In the first model only the underlying commodity price is assumed to follow a mean-reverting stochastic process. In the second model we add an extra stochastic factor, the convenience yield. Finally, in the three-factor model we assume that the risk-free interest rate also follows a stochastic process. In all the models the volatilities and correlations between the stochastic factors are assumed to follow a two-state Markov chain. The findings indicate that by assuming regime-switching and increasing the stochastic factors in the models we increase the flexibility and thus the accuracy of the models. Moreover, the findings suggest that the models can provide with accurate results for option written on a great variety of commodities.

This research provides with novel models of option evaluation and evidence that the consideration of the dynamic breaks in option pricing can improve the financial modeling. Therefore, this thesis is a proof that the observed market option values reflect the regime-switches that occur in the underlying option asset returns and the risk-free interest rate time-series as well as in the correlation between them. More precisely, in this thesis we prove that the traded option prices reflect the regime-switches that occur during the life-time of an option not only in the underlying asset driving process' parameters but also in the interest rate and convenient yield driving processes' parameters as well as in the correlations between these processes.

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Contents

Part I

A Lattice Method for Option Evaluation with Regime-Switching in the Correlation between Underlying Returns and Risk-Free Interest Rate

Abstract:

This chapter develops a lattice method for option evaluation aiming to investigate whether the option prices reflect the shifts in the distributions of the underlying asset returns and the risk-free interest rate. More precisely we try to investigate whether the option prices reflect the switches in the correlation between the underlying and risk-free bond returns that characterise different states of the economy. For this reason we develop and test two models. In the first model we allow all the parameters to follow a regime-switching process while in the second model, in order to isolate the regime-switching correlation effect on the option prices, we allow only the correlation to follow a regime-switching process. The models developed use pentanomial lattices to represent the evolution of the regime-switching underlying assets. Our findings suggest that the option prices reflect the regime-switches and that a model which considers these switches could produce more accurate results than a single-regime model.

1. Literature Review

During the two last decades regime-switching models have become accepted and widely used in finance due to their ability to reflect the random economic environment better than single-regime models. These models can capture the shifts in the distribution of economic variables by allowing the values of the parameters to change in different time periods. Many studies have found evidence that a regime-switching model can better capture the time series behaviour of a wind range of financial and economic variables. For example, it is well-known that the short-term interest rates exhibit switches in their distribution over time. Hamilton (1988) and Gray (1996) use regime-switching models to describe the dynamics of the US short-term interest rates. Other studies like Schwert (1989) and Turner, Startz, and Nelson (1989) present regime-switching models to describe the dynamics of equity returns.

In option valuation, the classic Black-Scholes formula, the most widely used model for option pricing, assumes that the underlying assets follow a geometric Brownian motion with constant mean return and volatility. The Black-Scholes formula fails to reflect the stochastic variability of the underlying asset's parameters. One of the problems caused by this is the failure of the model to capture what is known as "volatility smile". This disadvantage of the classic Black-Scholes model generates the need of a more accurate model which would allow for the parameters to change over time. Such a model could be a regime-switching model. An additional advantage of regime-switching models compare to stochastic volatility models is that the former can capture the "volatility clustering" effect that appears in the empirical data of asset returns. However, if the regimes are unobservable, the option evaluation using regime-switching models is more complicated than a single-regime model.

Naik (1993) derives option prices by developing a model in which the volatility of the returns of the underlying risky assets is subject to random shifts. Naik (1993) presents a closed form solution for the option evaluation which derives the option prices recursively based on an expectation of the usual Black-Scholes formula, where the expectations are over the future variance of the underlying asset. His model is a simplified regime-switching model which uses the expected duration of each regime over the option's life.

Bollen (1998) develops a lattice method for option evaluation in regime-switching models. In his paper he approximates the underlying stock returns by a two-regime model in which the returns are normally distributed in both regimes but with different mean and variance. For the option evaluation, instead of using the classic binomial lattice of Cox, Ross, and Rubinstein (1979) he introduces a pentanomial lattice. In his five-branch lattice the one regime is represented by a trinomial and the other by a binomial lattice in every brunch. Bollen, Gray, and Whaley (2000) examine the ability of Hamilton’s (1988-1990) Markov regime-switching model to capture the dynamics of exchange rates and exchange-traded option prices. Developing a four-regime model with independent shifts in the mean and variance they find that a regime-switching model can better fit and forecast the variance of foreign exchange rates than a single-regime model. To evaluate the options they follow the numerical method introduced by Bollen (1997). In each node the option value is the maximum between the proceeds of the early exercise and the discounted expected option value on the next node. This expectation is calculated using the regime probabilities. Bollen et al. (2000) found that the observed in the market American option prices are significantly different from those determined by their regime-switching option evaluation model. However, they show that a trading strategy based on their regime-switching option evaluation generates higher profits than the single-regime alternatives.

Yao et al. (2003) study the pricing of European options with the rate of return and volatility of the underlying depending on the state the economy is in. By assuming a finite number of regimes they formulate their regime-switching model for the underlying asset price as a geometric Brownian motion and they evaluate the options by using risk neutral evaluation. In their paper, Yao et al. (2003) present two alternative ways for the option evaluation. The first evaluation method derives a system of partial differential equations with a smooth boundary condition. This smoothness condition is to ensure the uniqueness and differentiability of the solution. The second approach is a successive approximation based on fixed points of an integral operator with a Gaussian kernel.

Duan (1995) develops an option pricing model where the underlying volatility follows a generalized GARCH process and the option evaluation is based on the locally risk-neutral valuation relationship. Duan, Popova, and Ritchken (2001) develop a class of option pricing models in which the underlying asset price is modelled by a Markov regime-switching process. Duan et al’s (2001) asset pricing model is a regime-switching model with feedback dynamic. The term feedback dynamic refers to the ability of the model to determine whether or not the process of the asset price will switch to a new volatility state not by a constant transition matrix but by an updating function together with the current volatility level. Duan et al. (2001) option evaluation models include the GARCH option evaluation method developed by Duan (1995), a lattice method, and a weighted average of Black-Scholes values which correspond to different regimes and with the weights to be determined by the transition probabilities of moving to another regime.

In the present chapter we develop an option evaluation model in which we use a regime-switching processes to drive of the option underlying parameters, i.e. the underlying asset price and the risk-free rate which under risk-neutral evaluation is used in the option evaluation. We assume that the underlying stock and bond prices are governed by two correlated geometric Brownian motions in which the drifts, volatilities, and correlation change over different regimes.

It is known that during different periods in the economic cycle the correlation between bonds and equity returns changes. We model the equity and risk-free bond returns by using a two-regime model. Our findings show that the economy can be divided into two states, a stable and an unstable state. In the stable state we have low volatilities in the stock and bond returns while in the unstable state we have higher volatilities. Alternatively, the two states can be defined by the prevailing correlation value; the numerical examples in this chapter show that by assuming two-stage regime-switching the one regime is characterized by low or negative correlation and the other by high or positive correlation between the underlying asset returns and the interest rate.

The purpose of this chapter is to investigate whether these shifts in the distributions of the underlying asset return and risk-free rates are reflected in the option prices. We try to investigate whether an option evaluation model in which the underlying option parameters are modelled by a regime-switching process provides with more accurate results than a classic option evaluation model. More precisely we try to find whether the option prices reflect these switches in the correlation between stocks and risk-free bonds that characterize the different states of the economy, i.e. whether the option prices capture the regime-switches of the economic environment. To give our analysis more focus; after we have estimated our model in which all the underlying option parameters are allowed to follow a regime switching process; we run a second model in which only the correlation is allowed to switch over different regimes. In this way we are able to investigate whether the option values are sensitive to the regime switching correlation. Finally, in the last part of this chapter we estimate our model using the Kim Filter.

The rest of the chapter is organized as follows: section 2. presents the mathematical formulation of the regime-switching model which governs the underlying asset and risk-free bond returns. In section 3. we describe the lattice method used for the option evaluation when the underlying option parameters follow a regime-switching process. In section 4. we briefly discuss the optimization techniques used for the model estimation and in section 5. we present numerical examples of the model. In section 6. we present and test an alternative method for estimating the model's parameters using the Kim Filter. Finally section 7. includes the conclusions and comments on our findings.

2. Mathematical Formulation

2.1. Continuous-time formulation

Most financial models describing stochastic variables like interest rates or asset returns assume a stationary distribution from which the changes in the variables are drawn. However, there is evidence that most financial variables cannot be accurately described by a stationary distribution because of the shifts that occur in their behavior over different time periods. Regime-switching models are able to capture this behavior of the financial variables by allowing the parameters of their data-generating processes to take different values in different time periods.

In this thesis we consider a two-regime model in which at any point in time the regime is given by an unobservable discrete Markov chain variable S_t. The evolution of the discrete variable S_t depends only upon S_{t-1} and can take only two values indicating in which of the two regimes we are in at time t, i.e. the process of S_t is a two-state first-order Markov process.

Under these assumptions we specify of the rate of return of the stock and the risk-free bond as follows:

We assume that the bond and stock prices follow two correlated geometric Brownian motions. Thus, the bond returns (risk-free interest rates) and stock returns are given as follows:

$$dr_t^b = \frac{dB_t}{B_t} = \mu_{b,S_t}\, dt + \sigma_{b,S_t}\, dW_t^1 \quad (1)$$

$$dr_t^P = \frac{dP_t}{P_t} = \mathrm{\mu}_{\mathrm{P},S_t}\, dt + \sigma_{P,S_t}\, dW_t^2 \quad (2)$$

μ_{b,S_t} and $\mathrm{\mu}_{\mathrm{P},S_t}$ are the drift of the bond and stock return processes respectively and σ_{b,S_t} and σ_{P,S_t} are the volatility of bond and stock returns, respectively.

Finally, the correlation between the two driving Brownian motions is given as:

$$dW_t^1 dW_t^2 = \rho_{S_t} dt \quad (3)$$

2.2. Discrete-time formulation

By considering that the stock and bond returns evolve according to the above model, the discrete-time approximation of the interest rate at time t is given by:

$$r_t^b = \frac{B_t - \mathrm{B}_{t-1}}{\mathrm{B}_{t-1}} = \left(\mathrm{\mu}_{b,S_t} - \frac{1}{2}\sigma_{b,S_t}{}^2\right)\Delta t + \varepsilon_t \quad (4)$$

where $\varepsilon_t \sim N\left(0, \sigma_{b,S_t}\sqrt{\Delta t}\right)$

and the stock return is given by:

$$r_t^P = \frac{P_t - P_{t-1}}{P_{t-1}} = \left(\mathrm{\mu}_{\mathrm{P},S_t} - \frac{1}{2}\sigma_{P,S_t}{}^2\right)\Delta t + \xi_t \quad (5)$$

where $\xi_t \sim N\left(0, \sigma_{P,S_t}\sqrt{\Delta t}\right)$

2.3. Matrix representation

Representing the above in a matrix format with $R_t = \left[r_t^b r_t^P\right]^T$, we have:

$$\boldsymbol{R_t} = \boldsymbol{a_{S_t}} + \boldsymbol{u_t}, \quad with\, \boldsymbol{u_t} \sim N\left(\boldsymbol{0}, \boldsymbol{V_{S_t}}\right) \quad (6)$$

where

$$\boldsymbol{a_{S_t}} = \begin{bmatrix} \left(\mu_{b,S_t} - \frac{1}{2}{\sigma_{b,S_t}}^2\right)\Delta t \\ \left(\mu_{P,S_t} - \frac{1}{2}{\sigma_{P,S_t}}^2\right)\Delta t \end{bmatrix}, \quad \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; and\; \boldsymbol{V_{S_t}} = \begin{bmatrix} {\sigma_{b,S_t}}^2\Delta t & \rho_{t,S_t}\sigma_{b,S_t}\sigma_{P,S_t}\Delta t \\ \rho_{t,S_t}\sigma_{b,S_t}\sigma_{P,S_t}\Delta t & {\sigma_{P,S_t}}^2\Delta t \end{bmatrix}$$

where $\rho_{t,S_t} = \rho_{S_t}$ is the correlation between the stock return and risk-free interest rate in the corresponding regime.

2.4. Transition Probabilities

The discretised two-state first-order Markov-switching variable S_t evolves according to the following transition probabilities:

$$\Pr[S_t = 1 | S_{t-1} = 1] = p_{11} = \frac{\exp(\mathrm{p}_0)}{1 + \exp(\mathrm{p}_0)}$$

$$\Pr[S_t = 2 | S_{t-1} = 2] = p_{22} = \frac{\exp(\mathrm{q}_0)}{1 + \exp(\mathrm{q}_0)}$$

The transition probabilities matrix is:

$$\tilde{p} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where $p_{12} = 1 - p_{11}$, $p_{21} = 1 - p_{22}$, $p_{11} + p_{12} = 1$, and $p_{22} + p_{21} = 1$.

So by letting $i_2 = [1, 1]'$, we have that $i_2'\tilde{p} = i_2'$.

If we let π_t to be the steady-state probability vector, we have:

$$\pi_t = \begin{bmatrix} \Pr[S_t = 1] \\ \Pr[S_t = 2] \end{bmatrix} = \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix}$$

3. Option Evaluation and Pentanomial Lattice

For the option evaluation we use a generalized version of the pentanomial lattice proposed by Bollen (1998) for option evaluation when the underlying returns follow a regime-switching process. In section 4.1. we briefly review the binomial lattice proposed by Cox, Ross, and Rubinstein (1979). In section 4.2. we discuss the pentanomial lattice of Bollen (1998). In section 4.3. construct two correlated lattices, one for the underlying stock price and one for the risk-free bond price. And in section 4.4. we explain how we evaluate an option in a regime-switching environment based on two correlated pentanomial lattices.

3.1. The Binomial Lattice

Cox, Ross, and Rubinstein (1979) developed a numerical discrete-time model for option evaluation by using binomial lattices. The so called CRR model uses a binomial tree to model the geometric Brownian motion that governs the underlying stock price process. The binomial lattice uses a two dimensional grid of nodes to illustrate the evolution of the key option's underlying variables in a finite number of time steps between the valuation time and the option expiration date.

Under the CRR model the stock price follows a geometric Brownian motion and so the stock returns are normally distributed.

$$r \sim N\left(\tilde{r}\Delta t, \sigma\sqrt{\Delta t}\right)$$

Each node in the binomial lattice approximates the normal distribution of the stock's returns using the binomial distribution. In each time step the underlying asset price will move up or down by a specific factor u or d, respectively. Thus, in each node of the tree we denote by π the probability of moving to the upper branch, making a positive upwards jump u, and by φ the continuously compounded rate of return of the stock moving to the upper branch. Likewise, in each node the probability of moving to the lower branch, by making a negative downward jump d, is $1-\pi$ and the continuously compounded rate of return $-\varphi$.

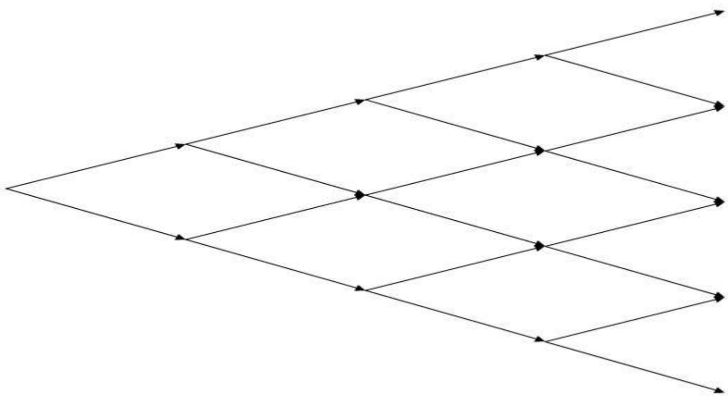
From the CRR model we have:

$$\pi = \frac{e^{\tilde{r}\Delta t} - e^{-\varphi}}{e^{\varphi} - e^{-\varphi}} \quad (11)$$

$$\varphi = \sigma\sqrt{\Delta t} \quad (12)$$

where $u = \exp(\varphi)$ and $d = 1/u$.

Figure 1: Binomial Tree



3.2. The Pentanomial Lattice

Bollen (1998) first constructs a pentanomial lattice to represent the variable's distribution when this is governed by a regime-switching process[1]. This lattice reflects the possible regime-switching of variables.

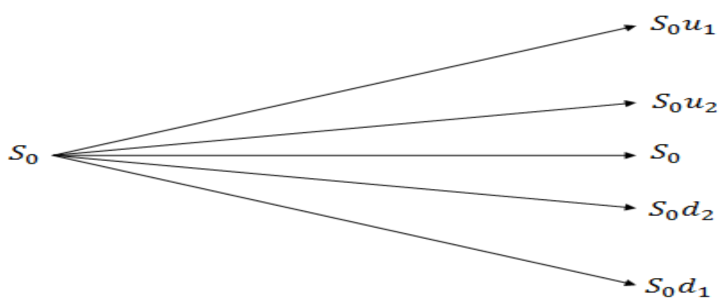
In the pentanomial lattice both regimes are represented by a trinomial instead of a binomial lattice. In every node there are two pairs of branches and one middle branch. The inner pair of branches corresponds to the regime with the lower volatility, the outer to the regime with the higher volatility, and the middle branch is shared by the two regimes. The branch probabilities and the continuously compounded rate of return are such that each trinomial

[1] In this thesis the Bollen (1998) pentanomial lattice has been adjusted for the needs of the present model. In particular in Bollen (1998) pentanomial lattice the rate of return φ in one of the regimes is adjusted so that the nodes recombine. In the present paper this would be impossible to be implemented because, as we prove later, there are additional conditions the φ has to satisfy.

lattice approximates the corresponding regime's distribution. These branch probabilities are now conditional probabilities, conditional on being in a particular regime, and so Bollen (1998) calls them *conditional branch probabilities*.

In each node of the lattice we have two conditional option values corresponding to the two different regimes.

Figure 2: Pentanomial Tree



3.3. Construction of Lattice with two Correlated Assets

In the pentanomial lattice option evaluation method of Bollen (1998) only the stock return follows a regime switching process. However, in this paper we examine the regime-switching correlation between stock returns and risk-free interest rates (government bond returns). Thus, we assume that the risk-free interest rate process is also governed by a regime-switching model. Therefore, we have to develop a second pentanomial lattice for the bond price correlated with the pentanomial lattice for the underlying stock price. To do so we first construct two correlated trinomial lattices, ignoring the regime-switching in the parameters, and later we transform these correlated trinomial lattices into two correlated pentanomial lattices, taking into consideration the regime-switching.

In section 4.3.1. we develop the trinomial model for a single underlying asset with no regime switching and in section 4.3.2. we develop two correlated trinomial lattices for the two correlated underlying assets.

3.3.1. One underlying asset

For the underlying tree we consider a three-jump process instead of two-jump process used in CRR model. Thus, starting with the initial underlying price S, the continuous distribution is approximated by a discrete distributions as follows [11]:

Table 1:

Jump	Probability	Underlying Asset Price
Up	π_u	Su
Horizontal	π_0	S
Down	π_d	Sd

To obtain the suitable values for the probabilities p_u, p_0, and p_d there are three conditions:

1. The sum of the probabilities is one:

$$\pi_u + \pi_0 + \pi_d = 1 \quad (18)$$

2. The mean of the discrete distribution is equal with the mean of the mean of the continuous lognormal distribution:

$$\pi_u S e^{\varphi} + \pi_0 S e^{0} + \pi S e^{-\varphi} = S e^{\tilde{r}\Delta t} \quad (19)$$

3. The variance of the discrete distribution is equal with the variance of the continuous lognormal distribution:

$$\pi_u \left((Se^{\varphi})^2 - \left(Se^{\tilde{r}\Delta t}\right)^2 \right) + \pi_m \left((Se^{0})^2 - \left(Se^{\tilde{r}\Delta t}\right)^2 \right) + \pi_d \left((Se^{-\varphi})^2 - \left(Se^{\tilde{r}\Delta t}\right)^2 \right) = \left(Se^{\tilde{r}\Delta t}\right)^2 \left(e^{\sigma^2 \Delta t} - 1\right) \quad (20)$$

By solving the above three equations we get the following expressions for π_u, π_0, and π_d[11]:

$$\pi_u = \frac{\left(e^{2\tilde{r}\Delta t + \sigma^2 \Delta t} - e^{\tilde{r}\Delta t}\right)e^{\varphi} - \left(e^{\tilde{r}\Delta t} - 1\right)}{(e^{\varphi} - 1)(e^{2\varphi} - 1)} \quad (21)$$

$$\pi_d = \frac{\left(e^{2\tilde{r}\Delta t+\sigma^2\Delta t} - e^{\tilde{r}\Delta t}\right)e^{2\varphi} - \left(e^{\tilde{r}\Delta t} - 1\right)e^{3\varphi}}{(e^{\varphi} - 1)(e^{2\varphi} - 1)} \quad (22)$$

$$\pi_0 = 1 - p_u - p_d \quad (23)$$

For convenience we will use the following notifications:

$$\pi_u = f(\varphi, \tilde{r}, \sigma)$$

$$\pi_d = g(\varphi, \tilde{r}, \sigma)$$

We denote the price of the underlying stock S_P and the price of the risk-free bond with S_b. Where the stock returns are normally distributed with $r_t^P \sim N\left(\tilde{r}\Delta t, \sigma_P\sqrt{\Delta t}\right)$, $\tilde{r} = \mu_P - \frac{1}{2}\sigma_P{}^2$; and the bond returns (risk free interest rate) $r_t^b \sim N\left(\widetilde{r_f}, \sigma_b\sqrt{\Delta t}\right)$, $\widetilde{r_f} = \mu_b - \frac{1}{2}\sigma_b{}^2$.

3.3.2. Two underlying assets

Now, in the case of two correlated underlying assets, based on Boyle (1988), we consider a five-jump process:

Table 2:

Event	**Probability**	**Underlying Assets Value Given Event**	
		Asset 1	**Asset 2**
E1	π_1	$S_P u_P$	$S_b u_b$
E2	π_2	$S_P u_P$	$S_b d_b$
E3	π_3	$S_P d_P$	$S_b d_b$
E4	π_4	$S_P d_P$	$S_b u_b$
E5	π_5	S_P	S_b

From the properties of the joint lognormal distribution we have that [11]:

$$E(S_P S_b) = S_P S_b e^{\tilde{r}\Delta t} e^{\widetilde{r_f}\Delta t} e^{\rho\sigma_b\sigma_P\Delta t} \quad (24)$$

This gives:

$$(\pi_1 e^{\varphi_P} e^{\varphi_b} + \pi_2 e^{\varphi_P} e^{-\varphi_b} + \pi_3 e^{-\varphi_P} e^{-\varphi_b} + \pi_4 e^{-\varphi_P} e^{\varphi_b} + \pi_5) S_P S_b$$
$$= S_P S_b e^{\tilde{r}\Delta t} e^{\widetilde{r_f}\Delta t} e^{\rho\sigma_b\sigma_P\Delta t} \quad (25)$$

By eliminating π_5 this gives:

$$\pi_1(e^{\varphi_P}e^{\varphi_b}-1)+\pi_2(e^{\varphi_P}e^{-\varphi_b}-1)+\pi_3(e^{-\varphi_P}e^{-\varphi_b}-1)+\pi_4(e^{-\varphi_P}e^{\varphi_b}-1)$$
$$=e^{\tilde{r}\Delta t}e^{\widetilde{r_f}\Delta t}e^{\rho\sigma_b\sigma_P\Delta t}-1 \quad (26)$$

To obtain suitable values for the probabilities there are three conditions:

1. The sum of the probabilities is one:

$\pi_1+\pi_2+\pi_3+\pi_4+\pi_5=1$ (27)

2. The mean of the discrete distributions is equal with the mean of the mean of the continuous lognormal distributions:

$(\pi_1+\pi_2)S_Pe^{\varphi_P}+\pi_5S_Pe^0+(\pi_3+\pi_4)S_Pe^{-\varphi_P}=S_Pe^{\tilde{r}\Delta t}$ (28)

$(\pi_1+\pi_4)S_be^{\varphi_b}+\pi_5S_b+(\pi_2+\pi_3)S_be^{-\varphi_b}=S_be^{\widetilde{r_f}\Delta t}$ (29)

3. The variance of the discrete distribution is equal with the variance of the continuous lognormal distribution:

$(\pi_1+\pi_2)((Se^{\varphi_P})^2-(Se^{\tilde{r}\Delta t})^2)+\pi_5((Se^0)^2-(Se^{\tilde{r}\Delta t})^2)+(\pi_3+\pi_4)((Se^{-\varphi_P})^2-(Se^{\tilde{r}\Delta t})^2)=$
$(Se^{\tilde{r}\Delta t})^2\left(e^{{\sigma_p}^2\Delta t}-1\right)$ (30)

$(\pi_1+\pi_4)\left((Se^{\varphi_b})^2-\left(Se^{\widetilde{r_f}\Delta t}\right)^2\right)+\pi_5\left((Se^0)^2-\left(Se^{\widetilde{r_f}\Delta t}\right)^2\right)+(\pi_3+\pi_2)\left((Se^{-\varphi_b})^2-\left(Se^{\widetilde{r_f}\Delta t}\right)^2\right)=$
$\left(Se^{\widetilde{r_f}\Delta t}\right)^2\left(e^{{\sigma_b}^2\Delta t}-1\right)$ (31)

We can note that there is a connection between equations (18), (19), and (20) and equations (27), (28), and (30). In fact the probabilities $(\pi_1+\pi_2)$, π_5, and $(\pi_3+\pi_4)$ are the new probabilities π_u, π_0, and π_d for the underlying stock price, respectively. In the same manner, $(\pi_1+\pi_4)$, π_5, and $(\pi_2+\pi_3)$ are the new probabilities π_u, π_0, and π_d for the underlying bond price, respectively. Thus, we can obtain:

$$(\pi_1+\pi_2)=f_P \quad (32)$$

$$(\pi_3+\pi_4)=g_P \quad (33)$$

$$(\pi_1 + \pi_4) = f_b \quad (34)$$

$$(\pi_2 + \pi_3) = g_b \quad (35)$$

From the above we have two expressions for π_5: $\pi_5 = 1 - f_P - g_P$, and $\pi_5 = 1 - f_b - g_b$. Thus, we have that:

$$f_P + g_P = f_b + g_b \quad (36)$$

This gives a relationship between φ_P and φ_b that must be satisfied.

Now, from equation (26) and equations (32) to (35) we can get the following expression for π_1:

$$\pi_1 = \frac{e^{\varphi_P} e^{\varphi_b} \left(e^{\tilde{r}\Delta t} e^{\widetilde{r_f}\Delta t} e^{\rho\sigma_b\sigma_P\Delta t} - 1\right) - f_P(e^{2\varphi_P} - 1) - f_b(e^{2\varphi_b} - 1) + (g_b + f_b)(e^{\varphi_P} e^{\varphi_b} - 1)}{(e^{2\varphi_P} - 1)(e^{2\varphi_b} - 1)} \quad (37)$$

Having solved for π_1, we can solve for $\pi_2, \pi_3,$ and π_4 using equations (32) to (35):

$$\pi_2 = f_P - \pi_1 \quad (38)$$

$$\pi_3 = g_b - \pi_2 \quad (39)$$

$$\pi_4 = f_b - \pi_1 \quad (40)$$

Thus, given φ_p, φ_b we have explicit expressions for the probabilities $\pi_1, \pi_2, \pi_3,$and π_4.

Note here that the two trees provide the possible stock values and bond values in any time step. However, the risk-free rate, i.e. the return of the bond, in any time step can take only three possible values, $\varphi_b, 0,$ or $-\varphi_b$, with the relative probabilities as calculated above.

Since in the present paper we consider a two regime model, the analysis in section 4.3.2. has to be repeated for both regimes, i.e. the conditional branch probabilities $(\pi_1, \pi_2, \dots, \pi_5)$ and rates of return (φ_b and φ_P) must be calculated separately for both regimes. In this way we can develop two correlated pentanomial lattices.

3.4. Option Evaluation

In a lattice the present value of an option is given as its discounted expected payoff which is determined by the expected value of its underlying asset. In risk-neutral valuation this discounting is made by using the risk-free interest-rate. However, by assuming a random switching in the regimes we introduce an additional risk in the evaluation procedure. Following Hull and White (1987) and Bollen (1998), we assume that the market does not price this additional regime-risk.

In the terminal nodes of the lattice the option value is calculated as the maximum between zero and the payoff if the option is exercised. The option is then valued iterating backwards. In the earlier node of the lattice the option values are calculated conditionally on the prevailing regime. Thus, in any earlier node we have two conditional option values; one for each regime. Here we notate by $C_{t,i}$ the option price at time t and in regime i.

This value is calculated conditional on being in regime i time t and it is equal to the weighted average of the discounted expected values of the option at time $t+1$ in each regime with the weights to be the transition probabilities. So the conditional on regime i option price at time t ($C_{t,i}$) will be equal to the discounted conditional on regime i option price at time $t+1$ multiplied by the probability of staying at $t+1$ at regime i (p_{ii}) plus the discounted conditional on regime j option price at time $t+1$ multiplied by the probability of switching at $t+1$ to regime i (p_{ij}).

For example the conditional option value of a European call option at time t, conditional on regime 1, is given as:

$$C_{t,1} = p_{11} DE\left(C_{t+1,1}\right) + p_{12} DE\left(C_{t+1,2}\right) \quad (41)$$

where DE here denotes the discounted expected values.

The transition probabilities of staying in regime 1 in time-step $t+1$ or switching to regime 2 determine the conditional option value by weighting the discounted expected future conditional option prices. The discounted expected future option values are calculated by using the appropriate regime's branches and the relative conditional branch probabilities.

$$\mathrm{DE}\left(C_{t+1,1}\right) = \pi_{1,1}\left(e^{-r_{f1,u}\Delta t} C_{t+1,1}^{u}\right) + \pi_{1,2}\left(e^{-r_{f1,d}\Delta t} C_{t+1,1}^{u}\right) + \pi_{1,3}\left(e^{-r_{f1,d}\Delta t} C_{t+1,1}^{d}\right) + \pi_{1,4}\left(e^{-r_{f1,u}\Delta t} C_{t+1,1}^{d}\right) + \pi_{1,5}\left(e^{-r_{f1,m}\Delta t} C_{t+1,1}^{m}\right) \quad (42)$$

$$\mathrm{DE}\left(C_{t+1,2}\right) = \pi_{2,1}\left(e^{-r_{f2,u}\Delta t} C_{t+1,2}^{u}\right) + \pi_{2,2}\left(e^{-r_{f2,d}\Delta t} C_{t+1,2}^{u}\right) + \pi_{2,3}\left(e^{-r_{f2,d}\Delta t} C_{t+1,2}^{d}\right) + \pi_{2,4}\left(e^{-r_{f2,u}\Delta t} C_{t+1,2}^{d}\right) + \pi_{2,5}\left(e^{-r_{f2,m}\Delta t} C_{t+1,2}^{m}\right) \quad (43)$$

where $r_{f1,u} = \varphi_{1,b}$, $r_{f1,m} = 0$, $r_{f1,d} = -\varphi_{1,b}$, $r_{f2,u} = \varphi_{2,b}$, $r_{f2,m} = 0$, $r_{f2,d} = -\varphi_{2,b}$.

In the same manner, the conditional option value of a European call option at time t, conditional on regime 2, is given as:

$$C_{t,2} = p_{22} DE\left(C_{t+1,2}\right) + p_{21} DE\left(C_{t+1,1}\right) \quad (44)$$

Thus, in the initial node two conditional option values are recorded, one for each regime. When the current regime is known, we know which of the two conditional option values in the seed node is correct. However, when regimes are not known with certainty, the European option value is the weighted average of the two conditional values where the weights are the initial regime probabilities.

4. Optimization techniques and model estimation

To estimate the model parameters we calibrate the model to the observed market option values. The two optimization techniques used in this paper are *grid search*, and *pattern search*.

Starting with the grid search we obtain a set of 10,000 plausible combinations for the model parameters. The best combinations of model parameters produced by grid search feed the pattern search which produces our final estimations for the parameters.

5. Numerical Example

To test our model we calculate the prices of European Call options on the iPath Goldman Sachs Crude Oil return Index ETN, which is a crude oil based exchange traded fund at NYSE Arca and compare our results with the actual option prices observed in the market. Since the lattice model developed is a general option pricing model, there is no particular reason here for choosing the specific option to test our model.

The day the data were collected the share value of the fund had been at $19.96. The options evaluated are options with different expiration date and strike prices. Since the interest rate distribution changes for each maturity, the models have been estimated separately for different maturities. Moreover, since the underlying asset distribution is influenced through correlation by the risk-free interest rate distribution, the parameters of the process driving the underlying asset price also change in different maturities.

We consider options with three different maturity dates and for each maturity we evaluate ten options; options in-the-money, approximately at-the-money and out-of-the money. In this way we manage to test the model's performance on options with different life expectancy and distance of the underlying asset price to the strike price. More precisely, we evaluate options with maturity in 54,144 and 235 days which were the options' maturities on the data collection date.

Note here that because of limitation in computer memory we cannot construct lattices with as many steps as the days to expiration. Thus, for each maturity we assume different number of steps and length of time steps. However, in order the results to be comparable we present the parameters of the underlying distributions in daily basis.

5.1. Model parameters

5.1.1. Maturity in 54 days

We first calculate the optimum parameters for the model. The parameters above are presented in a daily basis. The best fit parameters based on our data are:

		Regime 1	Regime 2
mean (μ)	Stock	0.1336%	0.1073%
	Bond	0.0341%	0.2476%
volatility (σ)	Stock	3.3926%	0.6621%
	Bond	0.9251%	0.4373%
correlation (ρ)		1	0.30493

And the transition probabilities:

p11	98.20%	p12	1.80%
p22	99.38%	p21	0.62%

We can see that in Regime 2 we have lower volatility than in Regime 1 in both stock and bond returns. We can also notice that the correlation between stock and bonds is higher in the high volatile regime than it is in the low volatile regime.

Now we have to calculate the φ for the bond and the stock in both regimes so that equation (36) is true by assuming three time steps of length 18 days. In Regime 1 we have that for $\varphi_{P,1} = 15.492\%$ and for $\varphi_{b,1} = 4.070\%$ the condition is satisfied. In particular:

f_P	0.54573
g_P	0.54563
f_b	0.41336
g_b	0.41346

In Regime 2 we have that for $\varphi_{P,2} = 3.443\%$ and for $\varphi_{b,2} = 4.942\%$ the condition is satisfied. In particular:

f_P	0.77647
g_P	0.22257
f_b	0.95996
g_b	0.03907

And the conditional branch probabilities are:

	$\pi 1$	$\pi 2$	$\pi 3$	$\pi 4$	$\pi 5$
Regime 1	0.24267	0.30306	0.11040	0.30296	0.04092
Regime 2	0.74242	0.03405	0.00503	0.21754	0.00097

5.1.2. Maturity in 144 days

The best fit parameters based on our data are:

		Regime 1	**Regime 2**
mean (μ)	**Stock**	0.0489%	0.0036%
	Bond	0.0015%	0.0028%
volatility (σ)	**Stock**	2.0335%	0.7879%
	Bond	0.0004%	0.000001%
correlation (ρ)		0.032848	0.9999

And the transition probabilities:

p11	59.63%	**p12**	40.37%
p22	71.53%	**p21**	28.47%

We can see that in this maturity the volatilities in both assets' returns are higher in Regime 1 than in Regime 2. Moreover, in Regime 1 we have low and positive correlation while in Regime 2 we have correlation almost one.

Now we have to calculate the φ for the bond and the stock in both regimes so that equation (36) is true by assuming three time steps of length 48 days. In Regime 1 we have that for $\varphi_{P,1} = 14.524\%$ and for $\varphi_{b,1} = 0.071\%$ the condition is satisfied. In particular:

$\boldsymbol{f_P}$	0.54525
$\boldsymbol{g_P}$	0.45280
$\boldsymbol{f_b}$	0.98268
$\boldsymbol{g_b}$	0.01537

In Regime 2 we have that for $\varphi_{P,2} = 5.525\%$ and for $\varphi_{b,2} = 0.136\%$ the condition is satisfied. In particular:

f_P	0.49471
g_P	0.48782
f_b	0.93390
g_b	0.04863

The conditional branch probabilities are:

	$\pi 1$	$\pi 2$	$\pi 3$	$\pi 4$	$\pi 5$
Regime 1	0.53656	0.00869	0.00667	0.44612	0.00195
Regime 2	0.46630	0.02841	0.02022	0.46760	0.01747

5.1.3. Maturity in 235 days

The best fit parameters based on our data are:

		Regime 1	Regime 2
mean (μ)	**Stock**	0.11130%	-0.14230%
	Bond	0.10695%	0.00025%
volatility (σ)	**Stock**	1.46930%	0.83009%
	Bond	0.09795%	0.00000%
correlation (ρ)		0.00037	-0.25952

And the transition probabilities:

p11	73.24%	**p12**	26.76%
p22	64.20%	**p21**	35.80%

We can see that in this maturity in Regime 1 we have higher volatilities while the correlation is positive. In contrast, in Regime 2, where the volatilities are lower, the correlation is negative.

Now we have to calculate the φ for the bond and the stock in both regimes so that equation (36) is true by assuming five time steps of 47 days. In Regime 1 we have that for $\varphi_{P,1} = 11.674\%$ and for $\varphi_{b,1} = 5.199\%$ the condition is satisfied. In particular:

f_P	0.11035
g_P	0.88956
f_b	0.99971
g_b	0.00020

In Regime 2 we have that for $\varphi_{P,2} = 8.499\%$ and for $\varphi_{b,2} = 0.012\%$ the condition is satisfied. In particular:

f_P	0.70645
g_P	0.29346
f_b	0.99528
g_b	0.00463

And the conditional branch probabilities are:

	$\pi 1$	$\pi 2$	$\pi 3$	$\pi 4$	$\pi 5$
Regime 1	0.70306	0.00339	0.00124	0.29222	0.00009
Regime 2	0.11026	0.00009	0.00011	0.88945	0.00008

5.2. Presentation and Analysis of the Results

The tables 3 to 5 present the results for the maturities 54, 144, and 235 days respectively. We compare our results to those obtained by the Black-Scholes model. For each maturity Bloomberg provides with the appropriate interest-rate to be used for the relevant options calculations. Thus, the interest rate used in Black-Scholes model for maturity in 54 days is 0.19%, in 114 days is 0.29% and in 235 days is 0.39%. The volatility used in Black-Scholes model was estimated by the same optimization process used for the lattice model parameters by calibrating the model to the option data. The Black-Scholes daily volatilities for each maturity, as these obtained by the optimization, are 1.6523%, 1.5656% and 1.4428% for the maturities 54, 144, and 235 days respectively.

The first column of the tables has the strike price, the second column contains the observed option market prices, the third column contains the option prices as there were calculated by the model and the last column contains the option prices by Black-Scholes model. In the end of the predicted values it is presented the sum of absolute percentage differences between the predicted option values and the observed ones.

Table 3

Strike Price	Market Price	Predicted by Lattice	Predicted by BS
16	4.9	3.881616	3.9936
17	2.85	3.127282	3.0613
18	2.35	2.373772	2.2162
19	1.65	1.629733	1.5029
20	0.95	0.949999	0.9500
21	0.4	0.399999	0.5587
22	0.2	0.200003	0.3059
23	0.2	0.136097	0.1565
24	0.1	0.105053	0.0750
25	0.15	0.088113	0.0339
Sum of absolute differences		1.110163	2.573241

Table 4

Strike Price	Market Price	Predicted by Lattice	Predicted by BS
16	4.5	4.34865943	4.1725616
17	3.6	3.5642556	3.35169784
18	2.75	2.57296396	2.62325745
19	2	1.93626598	1.99996353
20	1.45	1.46448434	1.48590397
21	1.05	1.01660883	1.07684673
22	0.61	0.73864212	0.76221078
23	0.35	0.47745972	0.52773376
24	0.5	0.4942085	0.35800034
25	0.25	0.22325247	0.23834467
Sum of absolute differences		0.85790036	1.30846523

Table 5

Strike Price	Market Price	Predicted by Lattice	Predicted by BS
16	5	4.4153631	4.3208036
17	3.8	3.7998973	3.5464351
18	2.4	2.3996588	2.8585724
19	2.25	1.9270998	2.2636265
20	1.6	1.5997621	1.7622281
21	1.35	1.3087541	1.3499542
22	0.95	1.0205032	1.0186664
23	1.05	1.1713058	0.7580323
24	0.6	0.6000609	0.5569055
25	0.5	0.4960518	0.4043912
Sum of absolute differences		0.4890502	1.1145081

The results indicate that the model works accurately. In particular it seems to outperform the Black-Scholes model in most of the cases especially in the case of Out-of-the-Money options. However, it seems that there are few cases when the options are in-the-money that Black-Scholes model provide option prices slightly closer to the real prices that the lattice regime-switching model. Nevertheless, the sum of absolute differences between the estimated option values and the real observed ones is in all the cases significantly lower in the case of the regime switching model. This could be an indication that the consideration of regime-switching in the parameters' values during the option evaluation can improve our results.

We can also observe here that the regime switching model works more accurately as the time to maturity increases. The reason for this can be that the probability of switches in the underlying parameters' value to occur increases with the time expectancy. In other words, we could conclude here that the consideration of regime switching makes more sense or that it is more important as the time horizon under consideration expands.

5.3. Sensitivity of Option Values to the Regime Parameters

After having proved that the current model can accurately calculate the option prices, we need to test how the regime-switching affects the option prices, i.e. how important is the consideration of regime-switching in the option prices calculation. To test this we need to test the sensitivity of the option prices to the regime parameters. To do so we consider an option on US Oil with expiration in 235 days and strike price $19.

The option value increases as the volatility increases. Thus, when the regime resistance of the regime with the high volatility increases, i.e. when the p_{11} increases, the option value must increase as well. On the other hand when the regime resistance of the regime with the low volatility increases, i.e. when p_{22} increases, the option value should decrease. We test this hypothesis by evaluating a European Call option for different values of p_{11} and p_{22}, setting the initial regime probabilities to 50%. The results are displayed on Figure 3.

Since the option value increases when the volatility is higher, the option value should increase as the initial regime probability of being in Regime 1 increases. This relationship should be weaker as the regime resistance decreases. We test this hypothesis by evaluating a European Call option for different values of regime resistance and initial regime probabilities, setting $p11 = p22$. The results are displayed on Figure 4.

5.4. Sensitivity of Option Prices to the Regime Correlation

The results above indicate that the option prices are sensitive to the regime parameters of the model. However, the main purpose of this paper is to investigate whether the regime switching in the correlation between the riskless interest rate and the return of the underlying asset affect the option price, and if the consideration of this regime switching can improve our estimation on the option prices. Thus we have to isolate this influence of the regime switching correlation in the option prices in order to test whether this on its own affects the option prices. To do so we estimate the model for options with maturity in 54 days again considering regime switching only in the correlation between the returns of the underlying index and riskless bond. The model we have to estimate now is:

$$\boldsymbol{R_t} = \boldsymbol{\alpha} + \boldsymbol{u_t}, \quad with\ \boldsymbol{u_t} \sim N\left(\boldsymbol{0}, \boldsymbol{V_{S_t}}\right) \quad (45)$$

where

$$\boldsymbol{\alpha} = \begin{bmatrix} \left(\mu_b - \frac{1}{2}\sigma_b{}^2\right)\Delta t \\ \left(\mu_P - \frac{1}{2}\sigma_P{}^2\right)\Delta t \end{bmatrix}, \quad \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ and\ \boldsymbol{V_{S_t}} = \begin{bmatrix} \sigma_b{}^2\Delta t & \rho_{S_t}\sigma_b\sigma_P\Delta t \\ \rho_{S_t}\sigma_b\sigma_P\Delta t & \sigma_P{}^2\Delta t \end{bmatrix}$$

The underlying asset has mean return 0.110% and volatility 10.081%. The risk-free interest rate has mean return 0.019% and volatility 5.544%. We first calculate the optimum parameters for the model, i.e. the parameters that give the maximum likelihood function. The best fit parameters based on our data are:

	Regime 1	**Regime 2**
Correlation	0.0444	0.800
Regime Resistance	99.75%	91.35%

So in Regime 1 the correlation between the underlying stock and riskless bond is positive and low while in Regime 2 is positive and high.

Now we have to calculate the φ for the bond and the stock for which equation (36) is true. Since φ is independent of the correlation between the bond and the stock, it remains the same in both regimes. For $\varphi_b = 4.500\%$ and for $\varphi_P = 7.00\%$ the condition is satisfied. In particular:

$\boldsymbol{f_P}$	0.542452
$\boldsymbol{g_P}$	0.145266
$\boldsymbol{f_b}$	0.328477
$\boldsymbol{g_b}$	0.359241

For these values the conditional branch probabilities are:

	π_1	π_2	π_3	π_4	π_5
Regime 1	0.212032	0.33042	0.028821	0.116446	0.312281
Regime 2	0.910673	0.01986	0.015211	0.01714	0.037115

To test the sensitivity of option prices to the regime parameters, i.e. the correlation between the underlying stock and riskless bond we evaluate a European Call option for different values of p_{11} and p_{22}, setting the initial regime probabilities of both regimes to 50%. The results indicate that as the regime resistance of Regime 1 (the regime with the lower correlation) increases the option price decreases and as the regime resistance of Regime 2 (the regime with the higher correlation) increases the option price increases. The results are displayed on Figure 5.

To further test the sensitivity of the option price to the regime switching correlation between the underlying stock and riskless bond, we evaluate the option for different values of regime resistance (setting $p11 = p22$) and initial regime probability of being in Regime 1. The results indicate that as the initial regime probability of being in Regime 1 increases, the option value decreases, i.e. as the correlation between the stock and the bond returns decreases, the option value decreases as well; and as the initial probability of being in regime 2 increases the option price increases. This relationship becomes stronger as the regime resistance increases. The results are displayed on Figure 6.

Figure 3: Sensitivity of Call Option value to Transition Probabilities

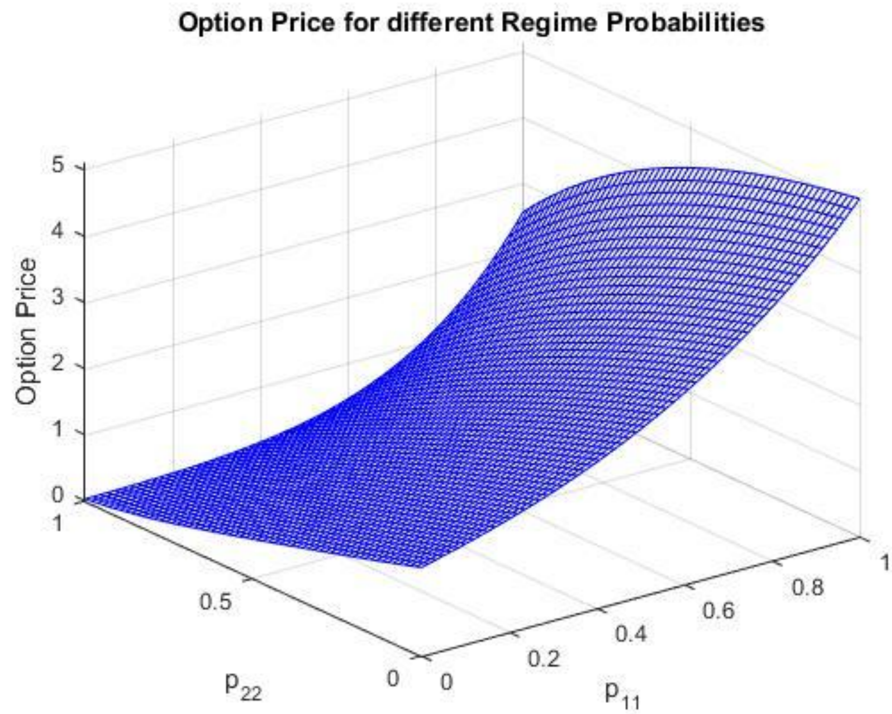


Figure 4: Sensitivity of Call Option Value to the Regime Parameters

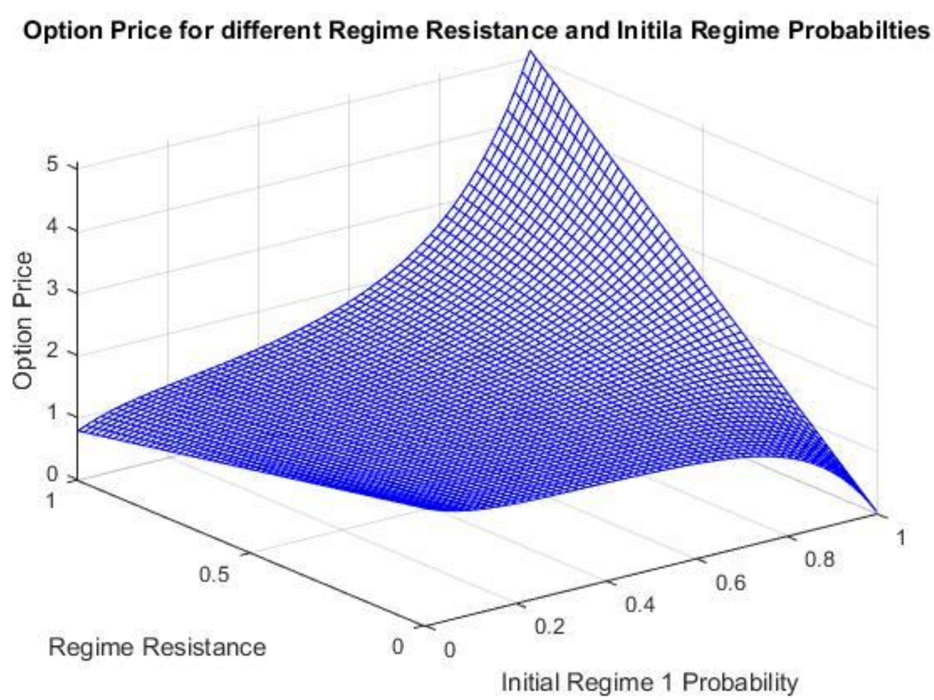


Figure 5: Sensitivity of the option price to the regime resistance of the two regimes

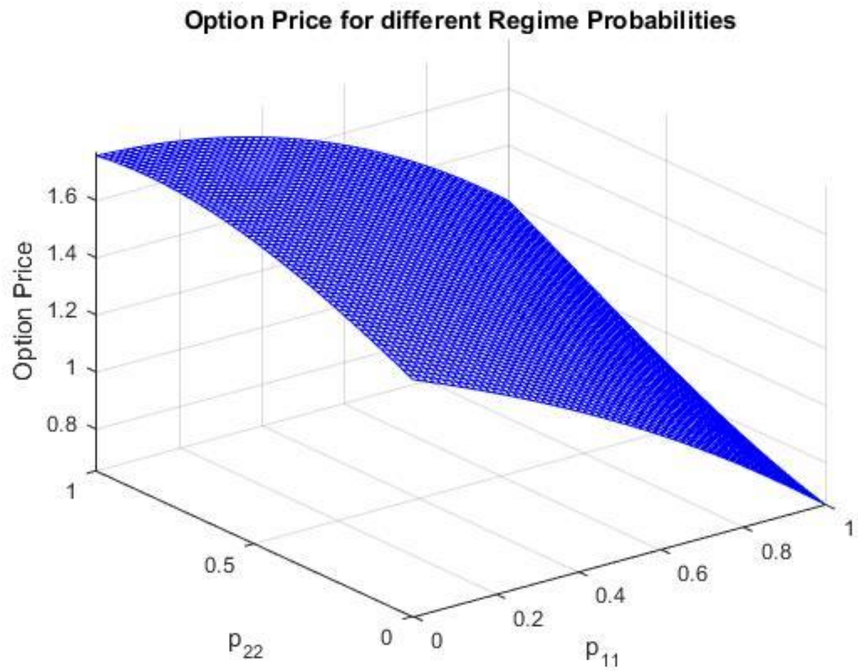
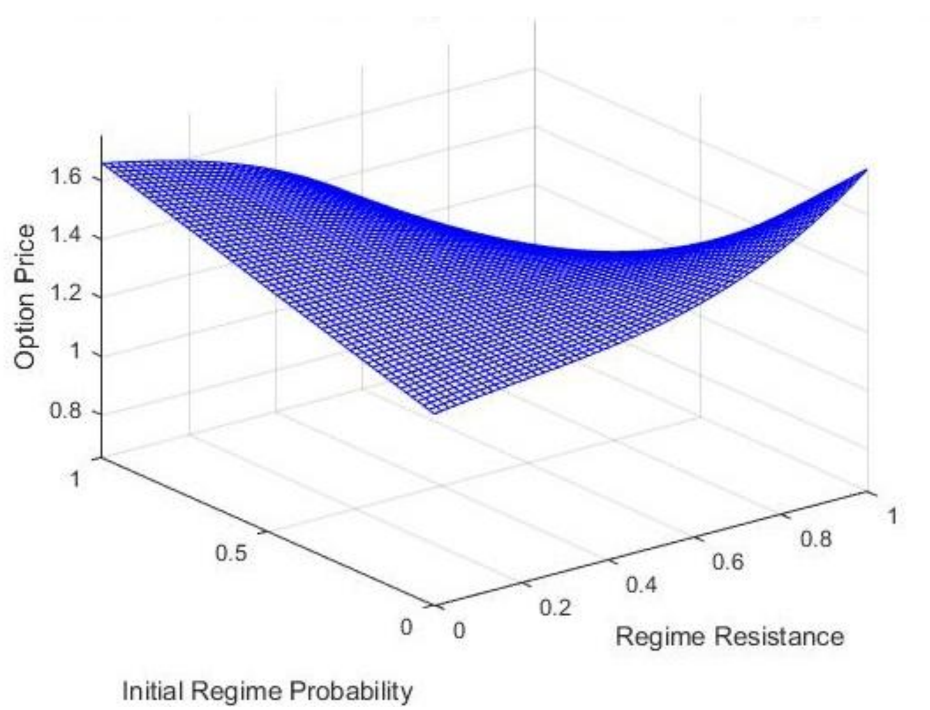


Figure 6: Sensitivity of Option Price to the regime switching correlation



6. Further Discussion: Kim Filter

As discussed before, the model has been calibrated according to the observed option values. By doing this we can say that the option values are used to forecast the distributions of the underlying asset return and the interest-rate. However, the parameters can be estimated by historical data, this can be done by employing state space models and utilizing Kim Filter.

6.1. State Space Representation

We first consider our model as a state-space model. State-space models deal with dynamic time series and are employed to make inferences about unobserved variables. In our case the unobserved variable, $\boldsymbol{\beta_t}$, is normally distributed and its dynamics are described by the "Transition equation". Then, the returns of the stock and the bond, i.e. the matrix $\boldsymbol{R_t}$, is given by the "Measurement equation" which describes the relationship between the observed variable $\boldsymbol{R_t}$, and the unobserved state-variable $\boldsymbol{\beta_t}$.

We consider the following representation for our state-space model with switching in both measurement and transition equations:

Measurement Equation: $\boldsymbol{R_t = H_{S_t}\beta_t + e_t}$

Transition Equation: $\boldsymbol{\beta_t = a_{S_t} + K_{S_t}\beta_{t-1} + G_{S_t}u_t}$

$$\begin{pmatrix} e_t \\ u_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} W_{S_t} & 0 \\ 0 & Q_{S_t} \end{pmatrix}\right)$$

In our specific case $\boldsymbol{H_{S_t}}$ is a (2×2) identity matrix, $\boldsymbol{K_{S_t}}$ is a (2×2) matrix of zeros, and $\boldsymbol{R_t}, \boldsymbol{a_{S_t}}$ are as determined in §1.3.

Moreover, we set $\boldsymbol{G_1}$ equal to a (2×2) matrix of zeros and G_2 equal to a (2×2) identity matrix. Thus, for $S_t = 1$ the variance of the process comes only from the measurement equation and it is equal to R_1 while for $S_t = 2$ the variance of the process comes from both state and measurement equations. Then, by setting $\boldsymbol{W_{S_t}} = \boldsymbol{V_1}$ and $\boldsymbol{Q_{S_t}} = \boldsymbol{V_2} - \boldsymbol{V_1}$ for both states, the variance in state 1 is $\boldsymbol{V_1}$ and the variance in state 2 is $\boldsymbol{V_2}$.

6.2. Transition Probabilities

The discretised two-state first-order Markov-switching variable S_t evolves according to the following transition probabilities:

$$\Pr[S_t = 1 | S_{t-1} = 1] = p_{11} = \frac{\exp(\mathrm{p}_0)}{1 + \exp(\mathrm{p}_0)}$$

$$\Pr[S_t = 2 | S_{t-1} = 2] = p_{22} = \frac{\exp(\mathrm{q}_0)}{1 + \exp(\mathrm{q}_0)}$$

The transition probabilities matrix is:

$$\tilde{p} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where $p_{12} = 1 - p_{11}$, $p_{21} = 1 - p_{22}$, $p_{11} + p_{12} = 1$, and $p_{22} + p_{21} = 1$.

So by letting $i_2 = [1, 1]'$, we have that $i_2' \tilde{p} = i_2'$.

If we let π_t to be the steady-state probability vector, we have:

$$\pi_t = \begin{bmatrix} \Pr[S_t = 1] \\ \Pr[S_t = 2] \end{bmatrix} = \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix}$$

Then, by definition of the steady-state probabilities we have $\pi_{t+1} = \tilde{p}\pi_t$ and $\pi_{t+1} = \pi_t$. And as Kim and Nelson (1999)[2] show $\pi_t = \tilde{p}\pi_t \Rightarrow (I_2 - \tilde{p})\pi_t = 0_2$, where 0_2 is a 2×1 vector of zeros. Thus,

$$\begin{bmatrix} I_2 - \tilde{p} \\ i_2' \end{bmatrix} \pi_t = \begin{bmatrix} 0_2 \\ 1 \end{bmatrix}$$

and by setting $A = \begin{bmatrix} I_2 - \tilde{p} \\ i_2' \end{bmatrix}$, we have that the $\pi_t = (A'A)^{-1}A' \begin{bmatrix} 0_2 \\ 1 \end{bmatrix}$. In other words, π_t equals the last column of $(A'A)^{-1}A'$.

At any point in time t the probability that each regime will govern the next observation based on the information available, $\tilde{y}_t$, is called regime probability $\Pr[\, S_t = j | \, \tilde{y}_t], j = 1, 2$. Gray (1996) shows that the regime probabilities at any point in time are related to the prior regime probabilities as follows:

$$\Pr[\, S_t = j | \, \tilde{y}_t] = \sum_{i=1}^{2} \Pr[\, S_t = j,\, S_{t-1} = i | \, \tilde{y}_t] \quad (j = 1, 2) \quad (7)$$

where

$$\Pr[\, S_t = j,\, S_{t-1} = i | \, \tilde{y}_t] = p_{ij} \frac{f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) \Pr[\, S_{t-1} = i | \, \tilde{y}_{t-1}]}{f(y_t | \, \tilde{y}_{t-1})} \quad (8)$$

and

$$f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) = (2\pi)^{-\frac{N}{2}} |f_{t|t-1}^{(i,j)}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \eta_{t|t-1}^{(i,j)}{}' f_{t|t-1}^{(i,j)}{}^{-1} \eta_{t|t-1}^{(i,j)} \right\} \quad (i, j = 1, 2) \quad (9)$$

$$f(y_t | \, \tilde{y}_{t-1}) = \sum_{i=1}^{2} \sum_{j=1}^{2} f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) \Pr[\, S_t = j,\, S_{t-1} = i | \, \tilde{y}_{t-1}] \quad (10)$$

6.3. Kim's Filter and Model Estimation

To estimate the parameters of the model we have to maximise the likelihood function with respect to the model's parameters. The basic tool to estimate the parameters of a state space model is the Kalman filter. The Kalman filter is a recursive procedure for estimating the unobserved state vector at time t, based on all the available information at that time. Moreover, via Kalman filter we estimate the likelihood function which we have to maximize. Kim (1994)[1] combines the Hamilton (1989) filter for regime-switching models estimation with the Kalman filter and develops a method to estimate the parameters of the state-space models with regime-switching. We follow the Kim's filter to estimate the parameters of our model; its steps are presented in the next section.

We represent with $\boldsymbol{\theta}$ the (12×1) vector with the parameters of the model we want to estimate, i.e. $\boldsymbol{\theta} = \left[\mathrm{\mu_{b,1}}, \mathrm{\mu_{b,2}}, \mathrm{\mu_{P,1}}, \mathrm{\mu_{P,2}}, \sigma_{b,1}, \sigma_{b,2}, \sigma_{P,1}, \sigma_{P,2}, \rho_1, \rho_2, \mathrm{p_0}, \mathrm{q_0}\right]'$.

Setting $l(\theta) = 0$ at the beginning of the iteration, the following steps are repeated for $t = 1,2, \dots, T$:

Kalman Filter

Step 1: Prediction

$\beta_{t|t-1}^{(i,j)} = a_j + K_j \beta_{t-1|t-1}^{i}$: *prior estimation of state vector*

$P_{t|t-1}^{(i,j)} = K_j P_{t-1|t-1}^{i} K_j' + G_j Q_j G_j{}'$: *prior estimation of state vector covariance*

Step 2: Prediction Error

$\eta_{t|t-1}^{(i,j)} = R_t - H_j \beta_{t|t-1}^{(i,j)}$: *prediction error*

$f_{t|t-1}^{(i,j)} = H_j P_{t|t-1}^{(i,j)} H_j' + R_j$: *conditional variance of prediction error*

Step 3: Updating

$\beta_{t|t}^{(i,j)} = \beta_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} H_j' \left[f_{t|t-1}^{(i,j)}\right]^{-1} \eta_{t|t-1}^{(i,j)}$: *posterior estimation of state vector*

$P_{t|t}^{(i,j)} = \left(I - P_{t|t-1}^{(i,j)} H_j' \left[f_{t|t-1}^{(i,j)}\right]^{-1} H_j\right) P_{t|t-1}^{(i,j)}$: *posterior estimation of state vector covariance*

Hamilton Filter

Step 4

$$\Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_{t-1}] = \Pr[\, S_t = j |\, S_{t-1} = i] \Pr[\, S_{t-1} = i | \tilde{y}_{t-1}] \quad (i, j = 1, 2)$$

where $\Pr[\, S_t = j |\, S_{t-1} = i]$ is the transition probability p_{ij}.

Step 5: Conditional and marginal density of y_t

$$f(y_t |\, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) = (2\pi)^{-\frac{N}{2}} |f_{t|t-1}^{(i,j)}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \eta_{t|t-1}^{(i,j)}{}' f_{t|t-1}^{(i,j)}{}^{-1} \eta_{t|t-1}^{(i,j)}\right\} \quad (i, j = 1, 2)$$

$$f(y_t | \tilde{y}_{t-1}) = \sum_{i=1}^{2} \sum_{j=1}^{2} f(y_t |\, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_{t-1}]$$

Step 6: Update the probability term

$$\Pr[\,S_t = j,\ S_{t-1} = i|\,\tilde{y}_t] = \frac{f(y_t|\,S_t = j,\ S_{t-1} = i,\tilde{y}_{t-1})\Pr[\,S_t = j,\ S_{t-1} = i|\,\tilde{y}_{t-1}]}{f(y_t|\,\tilde{y}_{t-1})}\quad (i,j = 1,2)$$

$$\Pr[\,S_t = j|\,\tilde{y}_t] = \sum_{i=1}^{2}\Pr[\,S_t = j,\ S_{t-1} = i|\,\tilde{y}_t]$$

Approximation

Step 7: Collapse terms to make filter operable

$$\beta_{t|t}^{j} = \frac{\sum_{i=1}^{2}\Pr[\,S_t = j,\ S_{t-1} = i|\,\tilde{y}_t]\,\beta_{t|t}^{(i,j)}}{\Pr[\,S_t = j|\,\tilde{y}_t]}$$

$$P_{t|t}^{j} = \frac{\sum_{i=1}^{2}\Pr[\,S_t = j,\ S_{t-1} = i|\,\tilde{y}_t]\left\{P_{t|t}^{(i,j)} + \left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)\left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)'\right\}}{\Pr[\,S_t = j|\,\tilde{y}_t]}$$

Log Likelihood Function

$$LL = \sum_{t=1}^{T}\ln\big(f(y_t|\,\tilde{y}_{t-1})\big)$$

6.4. Numerical Example

In order to examine how this method works we estimated the optimal parameters by maximizing the likelihood function and then estimated the option values with maturity in 54 days. The data used for the parameters estimation are the daily US Oil prices and the US 10-year Treasury Bond yield for the time period 26/10 /2010 to 24/10/2014, when the option prices were observed. The best fit parameters of the model are:

		Regime 1	Regime 2
mean (μ)	Stock	0.1131%	0.0424%
	Bond	0.0545%	0.0648%
volatility (σ)	Stock	0.6744%	1.6390%
	Bond	0.2630%	0.2216%
correlation (ρ)		-0.40045	-0.26764

And the transition probabilities:

p11	93.60%	p12	6.40%
p22	43.41%	p21	56.59%

Now we have to calculate the φ for the bond and the stock in both regimes so that equation (36) is true by assuming three time steps of length 18 days. In Regime 1 we have that for $\varphi_{P,1} = 5.18\%$ and for $\varphi_{b,1} = 2.09\%$ the condition is satisfied. In particular:

f_P	0.68280
g_P	0.30739
f_b	0.72707
g_b	0.26313

In Regime 2 we have that for $\varphi_{P,2} = 13.58\%$ and for $\varphi_{b,2} = 2.30\%$ the condition is satisfied. In particular:

f_P	0.35893
g_P	0.35051
f_b	0.60743
g_b	0.102025

And the conditional branch probabilities are:

	$\pi 1$	$\pi 2$	$\pi 3$	$\pi 4$	$\pi 5$
Regime 1	0.41988	0.26293	0.00020	0.30719	0.00981
Regime 2	0.26634	0.09259	0.00944	0.34108	0.29055

According to the above values, the predicted option prices for options with expiration in 54 days are presented in table 6.

Table 6

Strike	Price	Predicted
16	4.9	1.625144372
17	2.85	1.269327353
18	2.35	0.940833993
19	1.65	0.626509398
20	0.95	0.420399871
21	0.4	0.245524212
22	0.2	0.158808617
23	0.2	0.079890983
24	0.1	0.045583587
25	0.15	0.023566284

As it can be seen the results here are significantly less accurate than those estimated above. This is because the parameters of the model now are calibrated based on the historical underlying data rather than the option data.

7. Conclusion

In this chapter we showed how to evaluate options considering the shifts in the distribution of the returns of the underlying stock and the risk-free interest rate and in the correlation between them. The numerical examples proved that the consideration of these switches in our calculations improves the accuracy of the results. More precisely, we proved that the option prices are indeed sensitive to the regime switches in the correlation between the returns of the underlying stock and riskless bond. The numerical example also saw that the present model may provide with more accurate results that the classic Black-Scholes model especially when the options are out-of-the-money.

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Appendix

Kim's Filter

$$l(\theta) = 0, \beta_{0|0}^{j}, P_{0|0}^{j},$$

$$\Pr(\, S_0 = \pi_i) \quad (Steady - state\ prob.)$$

$$\Downarrow$$

Kalman Filter

$$\beta_{t|t-1}^{(i,j)}, P_{t|t-1}^{(i,j)}, \eta_{t|t-1}^{(i,j)}, f_{t|t-1}^{(i,j)}, \beta_{t|t}^{(i,j)}, P_{t|t}^{(i,j)}$$

Hamilton Filter

$$\Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_{t-1}] = \Pr[\, S_t = j | \, S_{t-1} = i] \Pr[\, S_{t-1} = i | \tilde{y}_{t-1}]$$

$$f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) = (2\pi)^{-\frac{N}{2}} |f_{t|t-1}^{(i,j)}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \eta_{t|t-1}^{(i,j)\,\prime} f_{t|t-1}^{(i,j)\ -1} \eta_{t|t-1}^{(i,j)}\right\}$$

$$f(y_t | \tilde{y}_{t-1}) = \sum_{i=1}^{2} \sum_{j=1}^{2} f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_{t-1}]$$

$$\Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_t] = \frac{f(y_t | \, S_t = j,\, S_{t-1} = i,\, \tilde{y}_{t-1}) \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_{t-1}]}{f(y_t | \tilde{y}_{t-1})}$$

$$\Pr[\, S_t = j | \tilde{y}_t] = \sum_{i=1}^{2} \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_t]$$

$$\Downarrow$$

Approximation

$$\beta_{t|t}^{j} = \frac{\sum_{i=1}^{2} \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_t] \, \beta_{t|t}^{(i,j)}}{\Pr[\, S_t = j | \tilde{y}_t]}$$

$$P_{t|t}^{j} = \frac{\sum_{i=1}^{2} \Pr[\, S_t = j,\, S_{t-1} = i | \tilde{y}_t] \left\{P_{t|t}^{(i,j)} + \left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)\left(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)}\right)'\right\}}{\Pr[\, S_t = j | \tilde{y}_t]}$$

$$\Downarrow$$

$t - 1 = t$

Log Likelihood Function

$$LL = \sum_{t=1}^{T} \ln\left(f(y_t | \tilde{y}_{t-1})\right)$$

Part II

Closed-form Option pricing formulas for Mean-Reverting Commodity Prices with Regime-Switching

Abstract:

This part develops a class of closed-form models for options on commodities evaluation under the assumptions of mean-reversion in the commodity prices and factors' values and regime-switching in the volatilities and correlations. At first we develop novel closed-form solutions of the 1-, 2- and 3-factors models and later in the paper these three models are transformed into regime switching models. The six models (three with and three without regime-switching) are then tested and compared on real market data. Our findings suggest that the by increasing the stochastic factors and assuming regime-switching in the models their flexibility and thus their accuracy increases.

1. Introduction and Previous Bibliography

In order to evaluate claims on commodities we have to consider the stochastic behavior of commodity prices. Because of the nature of commodity market, the commodity prices tend to exhibit mean reversion. As Schwartz (1997) explains, we would expect that if the commodity prices are relatively high, more producers would enter the market increasing the supply and putting a downward pressure on the price. On the other hand, if the commodity prices are at some point relatively low, the higher cost producers would exit the market decreasing the supply and causing an upward lift on the prices. Thus, while Geometric Brownian motion is an appropriate model to formulate stock prices, in the case of commodity prices we need a mean-reverting stochastic process. Evidence on the mean-reverting nature of commodity prices has been provided in a number of papers; see Gibson and Schwartz (1990), Cortazar and Schwartz (1994), Bessembinder et al. (1995), Baker et al. (1998).

An important aspect when evaluating claims on commodities is the *convenience yield*. The convenience yield on the commodity can be defined as the profit that occurs by the ownership of the physical commodity (see Brennan and Schwartz (1985)). This profit may arise by the volatility of the market price or by the maintenance of a production process due to the ownership of the physical commodity. The convenience yield can be thought as the premium of holding the underlying commodity (physical good) rather than holding a contract or derivative contract on the commodity. This is because the users of consumption assets are able to obtain benefits by physically holding the asset as inventory from temporary shortages and the ability to keep a production process running; clearly these benefits cannot be obtained by holding a contract.

Gibson and Schwartz (1990) develop a two-factor model for pricing contingent claims on oil. In their model the two factors are the spot oil price and the instantaneous convenience yield. They assume that the spot price and the convenience yield follow two joint diffusion processes. More precisely, they formulate the spot price by a Geometric Brownian motion and the convenience yield by an Ornstein-Uhlenbeck process and assume that the two stochastic processes have correlated increments. Using It$\hat{o}$'s Lemma and under the standard perfect market assumptions they derive the partial deferential equation that the claim should satisfy and apply their model to determine the present value of a barrel of oil delivered in the future. By adopting the Gibson and Schwartz (1990) assumptions of the economy, Bjerksund (1991) provides an analytical solution for pricing European call options. The author considers the two-factor model described by Gibson and Schwartz (1990) and derives a Black-Scholes option pricing formula.

Schwartz (1997) extends the previous models by introducing an third stochastic factor, the instantaneous interest rate. In particular he develops and compares a class of mean-reverse models to describe the stochastic behaviour of commodity prices and to price future contracts on commodities. Schwartz (1997) compares three models; one-, two-, and three-factor models. The first model is a one-factor model in which the spot commodity price is assumed to follow a mean-reverting process. The second model is similar to the Gibson and Schwartz (1990) model and has as additional stochastic factor the convenience yield of the commodity which is assumed to follow a mean-reverting process. Finally, the third model is a three-factor model in which the stochastic factors are the spot commodity price, the convenience yield, and the instantaneous interest rate.

Based on the previous works, Miltersen and Schwartz (1998) develop a closed-form Black-Scholes/Merton pricing formula for options written on commodity future contracts. Their model is developed in the presence of stochastic interest rate and convenience yield.

Swishchuk (2008) produces a closed-form option pricing formula for mean-reverting assets in the energy market. The author considers the one factor model of Schwartz (1997) and by employing the *change of time approach* (see Ikeda and Watanabe (1981) and Swishchuk (2007)) he derives an explicit expression for European option prices both in real and in risk-neutral world.

1.1. Regime-Switching in Commodity prices

The last decade, many researchers have employed regime-switching models to describe the stochastic behaviour of commodity prices and to investigate whether such models could perform better than classic single-regime models. Chen and Insley (2010) provide evidence that a regime-switching model can more closely matches the future prices of lumber prices than a single-regime model. In their paper they use prices of derivatives on lumber to calibrate their model and their analysis shows that there are significant differences in the optimal tree harvest thresholds between single and regime switching models.

Alizadeh et al. (2008) employ Markov regime-switching to determine the time-varying minimum variance hedge ration in energy future market. The results indicate that the hedge ratios provided by the regime-switching model outperform the hedge ratios by alternative single-regime methods. Thus, their study provides evidence that by using Markov regime-switching models investors may be able to increase the performance of their hedges measured in terms of variance reduction and increase in utility.

Almounsour (2012) develops an one-factor regime-switching model in order to evaluate future contracts on crude oil market and they compare their model's performance with the Gibson and Schwartz (1990) two-factor model (G&S model). Their findings suggest that even if G&S model outperforms the regime-switching model for short-term maturities, the regime-switching model outperforms the G&S model for long-term maturities. This may suggest that the assumption of a unique equilibrium level, which implies that the future structure should revert to one slope at all times, is not correct.

In this part we develop a class of closed-form option pricing models under regime switching for pricing options on commodities. In the first chapters of this part we develop three novel closed-form option pricing models for options written on mean-reverting commodities without considering regime-switching, while later we turn these models into regime-switching models. More precisely, in chapters 2 to 4 we produce three option pricing models; the first model is an one-factor model in which the only stochastic factor is the underlying commodity spot price, the second model is a two-factor model in which the stochastic factors (or state variables) are the spot commodity price and the spot convenience yield, and the third model is a three-factor model which has as an additional stochastic factor the instantaneous interest rate. Thus, in the first two models we assume that the interest rate is constant at all time (from the time of the option evaluation till the time of the option expiration) while in the third model we drop this assumption by introducing a third stochastic factor (or state variable) in the model, the instantaneous interest rate. Later in section 5 we transform our option pricing models into discrete-time regime-switching models. Finally, in section 6 we compare and test the six option pricing models developed in the previous sections by evaluating options on different commodities and comparing the calculated values with the real market option values.

In contrast with most of the bibliography till now which develops models for options on commodity future contracts; here we produce models on options on commodities. However, assuming the spot commodity price is a reasonable fit for the closest future price, the models developed here can be also used to evaluate options on commodity future contracts.

1.2. Assumptions

Before moving to the mathematical part we first have to state the assumptions under which the models are developed. The approach to evaluate the options is based on the following assumptions:

1. The trading takes place continuously.
2. The commodity is tradable at the spot price continuously.
3. There is no transaction cost, fees, or taxes.
4. There are no arbitrage opportunities in the market
5. It is possible to borrow and lend any amount of cash (even fractional) at the risk-free interest rate.
6. It is possible to buy and sell any amount of the commodity (even fractional).
7. Short selling is permitted.

2. One-Factor Model

2.1. The model

Following Schwartz (1997) we assume that the commodity spot prices follow a mean-reverting stochastic process given as:

$$dS_t = \alpha(\mu - \ln S_t)S_t dt + \sigma S_t dW_t \quad (1)$$

Defining $X_t = \ln S_t$ and applying Ito's lemma, this implies that the log price follows an Ornstein–Uhlenbeck process:

$$dX_t = a(L - X_t)dt + \sigma dW_t \quad (2)$$

$$L = \mu - \frac{\sigma^2}{2\alpha} \quad (3)$$

2.2. Risk Adjustment

Let P be the physical measure and Q be the equivalent martingale measure. Under standard assumptions, the dynamics of Ornstein–Uhlenbeck process under the equivalent martingale measure can be written as:

$$dX_t = [a(L - X_t) - k]dt + \sigma d{W_t}^* \quad (4)$$

where k is the market price of risk. Equivalently, (4) can be written as:

$$dX_t = a(L^* - X_t)dt + \sigma d{W_t}^* \quad (5)$$

where $L^* = L - \frac{k}{a}$ and ${W_t}^*$ is a Brownian motion under the equivalent martingale measure. The term $\frac{k}{a}$ is the normalized risk-premium subtracted from the long-run mean.

In equation (5) the conditional expectation of X at time T under the equivalent martingale measure is normal with mean and variance [16]:

$$E_0[X_T] = e^{-aT}X_0 + (1 - e^{-aT})L^* \ (6)$$

$$Var_0[X_T] = (1 - e^{-aT})\frac{\sigma^2}{2a} \quad (7)$$

Since $X = \ln S_t$, the spot price of the commodity at time T is log-normally distributed under the martingale measure with the above parameters.

2.3. Closed-form solution for 1-factor model

The SDE in equation (5) is known to have the solution:

$$X_t = X_0 e^{-at} + L^*(1 - e^{-at}) + \int_0^t \sigma e^{-\alpha(s-t)} dW_s^* \quad (8)$$

which has the same distributions as:

$$X_t = X_0 e^{-at} + L^*(1 - e^{-at}) + \sigma \sqrt{\left(\frac{1 - e^{-2at}}{2a}\right)} W_t^* \quad (9)$$

For convenience, we will set $\tilde{\sigma}_t^2 = \sigma^2 \left(\frac{1-e^{-2at}}{2a}\right)$.

Thus the call option value at time $t = 0$ is given as:

$$C_0 = e^{-Tr} E[(S_T - K)^+ | S_t]$$

$$= e^{-Tr} E[\exp(\ln(S_0) e^{-aT} + L^*(1 - e^{-aT}) + \tilde{\sigma}_T W_T^*) - K]^+$$

$$= e^{-Tr} \int_{-\infty}^{+\infty} \left[\exp\left(\ln(S_0) e^{-aT} + L^*(1 - e^{-aT}) + \tilde{\sigma}_T \sqrt{T} z\right) - K\right]^+ \varphi(z) dz \quad (10)$$

where $\varphi(z)$ is the standard normal distribution probability density:

$$\varphi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \quad (11)$$

It is clear that the function under the integral is non-negative if and only if the folllowing inequality holds:

$$\exp\left(\ln(S_0)e^{-aT} + L^*(1-e^{-aT}) + \tilde{\sigma}_T\sqrt{T}z\right) - K \geq 0$$

Thus,

$$z \geq \frac{\ln K - \ln(S_0)e^{-aT} - L^*(1-e^{-aT})}{\tilde{\sigma}_T\sqrt{T}} = -d(S_0, T) \ (12)$$

So (10) becomes:

$$C_t = e^{-Tr}\int_{-d}^{+\infty}\left(\exp\left(\ln(S_0)e^{-aT} + L^*(1-e^{-aT}) + \tilde{\sigma}_T\sqrt{T}z\right) - K\right)\varphi(z)dz =$$

$$= e^{-Tr}e^{\ln(S_0)e^{-aT}+L^*\left(1-e^{-aT}\right)}\int_{-d}^{+\infty}\left(e^{\tilde{\sigma}_T\sqrt{T}z}\right)\varphi(z)dz - e^{Tr}K\int_{-d}^{+\infty}\varphi(z)dz$$

$$= e^{-Tr}e^{\ln(S_0)e^{-aT}+L^*\left(1-e^{-aT}\right)}\int_{-d}^{+\infty}\frac{e^{\tilde{\sigma}_T\sqrt{T}z-\frac{z^2}{2}}}{\sqrt{2\pi}}dz - e^{Tr}KN(d)$$

$$= e^{-Tr+\ln(S_0)e^{-aT}+L^*\left(1-e^{-aT}\right)+\frac{1}{2}\tilde{\sigma}_T^2T}\int_{-d}^{+\infty}\frac{e^{-\frac{1}{2}\left(z-\tilde{\sigma}_T\sqrt{T}\right)^2}}{\sqrt{2\pi}}dz - e^{-Tr}KN(d)$$

$$= S_0\int_{-d-\tilde{\sigma}_T\sqrt{T}}^{+\infty}\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}du - e^{-Tr}KN(d)$$

$$= S_0N\left(d + \tilde{\sigma}_T\sqrt{T}\right) - e^{-Tr}KN(d)$$

$$= S_0N(d_1) - e^{-Tr}KN(d_2) \quad (13)$$

where we define $d_1 = d + \tilde{\sigma}_T\sqrt{T}$, $d_{2=}d_1 - \tilde{\sigma}_T\sqrt{T}$, and $\tilde{\sigma}_T^2 = \sigma^2\left(\frac{1-e^{-2aT}}{2a}\right)$.

In the above equation we have substituted $e^{-Tr+\ln(S_0)e^{-aT}+L^*(1-e^{-aT})+\frac{1}{2}\tilde{\sigma}_T^2 T}$ with S_0. This is because $e^{\ln(S_0)e^{-aT}+L^*(1-e^{-aT})+\frac{1}{2}\tilde{\sigma}_T^2 T} = E_0(S_T)$ and under the martingale measure $e^{-Tr}E_0(S_T) = S_0$

Theorem 1:

Under the equivalent martingale measure Q, if we assume that the log price of the underlying asset follows a Ornstein–Uhlenbeck process:

$$dX_t = \alpha(L^* - X_t)dt + \sigma dW_t{}^*$$

the price of an European call option on the asset is given as:

$$C_0 = S_0 N(d_1) - e^{-Tr} K N(d_2)$$

$$d_1 = \frac{X_0 e^{-aT} - \ln K + L^*(1 - e^{-aT}) + \tilde{\sigma}_T^2 T}{\tilde{\sigma}_T \sqrt{T}}$$

$$d_2 = d_1 - \tilde{\sigma}_T \sqrt{T}$$

$$\tilde{\sigma}_T^2 = \sigma^2 \left(\frac{1 - e^{-2aT}}{2a} \right)$$

3. Two-Factor Model

3.1. The model

The two-factor model is based on the ones developed by Gibson and Schwartz (1990) and Schwartz (1997). In this model the first factor is the spot commodity price and the second is the convenience yield, δ. These factors are assumed to follow two correlated stochastic processes given by:

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_1 S_t dW_{1,t} \quad (14)$$

$$d\delta_t = \alpha(L - \delta_t)dt + \sigma_2 dW_{2,t} \quad (15)$$

where:

$$dW_{1,t}dW_{2,t} = \rho dt \quad (16)$$

Thus, in the two-factor model the spot commodity price follows a standard process allowing for stochastic convenience yield, which is assumed to follow an Ornstein-Uhlenbeck stochastic process. Moreover, we can note here that if we set the convenience yield, instead of being stochastic, to be a function of the spot price: $\delta_t = \alpha \ln S_t$, the two-factor model is identical with the one-factor model.

3.2. Risk Adjustment

As we did in the one-factor model, we want to find a probability Q equivalent to the physical measure P. We can note here that by (14) we can regard the commodity as an asset that pays a stochastic dividend yield δ_t. Thus, the risk adjusted drift in the commodity spot price process will be $r - \delta_t$. And because the convenience yield cannot be hedged, we have to attribute a market price of risk to it[2]. Thus, under the equivalent martingale measure, equations (14) and (16) become:

$$dS_t = (r - \delta_t)S_t dt + \sigma_1 S_t dW_{1,t}^* \quad (17)$$

[2] See Bjerksund (1991) and Schwartz (1997)

$$d\delta_t = [a(L-\delta_t)-\lambda]dt + \sigma_2 dW_{2,t}^* \quad (18)$$

and:

$$dW_{1,t}^* dW_{2,t}^* = \rho dt \quad (19)$$

where λ is the market price of convenience yield risk.

By setting $L^* = L - \frac{k}{a}$ we can rewrite (18) as:

$$d\delta_t = a(L^*-\delta_t)dt + \sigma_2 dW_{2,t}^* \quad (20)$$

3.3. Closed-form solution for 2-factor model

Bjerksund (1991) has derived a close form solution for the two-factor model. With some modifications[3], we follow Bjerksund (1991) methodology to derive the option price.

From the SDE in equation (20) we have that:

$$\delta_t = \delta_0 e^{-at} + L^*(1-e^{-at}) + \sigma_2 \sqrt{\left(\frac{1-e^{-2at}}{2a}\right)} W_{2,t}^*$$

So we have that:

$$\delta_T = \delta_t e^{-a(T-t)} + L^*\left(1-e^{-a(T-t)}\right) + \sigma_2 \int_t^T e^{-a(T-s)}\, dW_{2,s}^* \quad (21)$$

And from (17) we have that:

$$S_T = S_t \exp\left[\left(r-\frac{1}{2}\sigma_1^2\right)(T-t) - \int_t^T \delta_s\, ds + \sigma_1 \int_t^T dW_{1,t}^*\right] \quad (22)$$

By following Bjerksund (1991), we define the cumulative convenience yield rate from time 0 to time t:

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[3] Bjerksund (1991) based on Gibson and Schwartz (1990) assumes that $dS_t = \mu S_t dt + \sigma_1 S_t dW_{1,t}$ while in this paper based on Schwartz (1997) we assume that $dS_t = (\mu-\delta_t)S_t dt + \sigma_1 S_t dW_{1,t}$.

$$\mathcal{D}_t = \int_0^t \delta_s \, ds \quad (23)$$

By integrating equation (20) we have that:

$$\int_t^T d\delta_s = \int_t^T a(L^* - \delta_s)ds + \int_t^T \sigma_2 \, dW_{2,s}^*$$

$$= aL^*(T-t) - a\int_t^T \delta_s ds + \sigma_2 \int_t^T dW_{2,s}^* \quad (24)$$

Note that:

$$\int_t^T d\delta_s = \delta_T - \delta_t \quad (25)$$

And from (23) we have that:

$$\int_t^T \delta_s \, ds = \mathcal{D}_T - \mathcal{D}_t \quad (26)$$

By substituting (25) and (26) into (24) we have that:

$$\delta_T - \delta_t = aL^*(T-t) - a(\mathcal{D}_T - \mathcal{D}_t) + \sigma_2 \int_t^T dW_{2,s}^* \quad (27)$$

Substituting (21) into (27) and solving for $\mathcal{D}_T$ results:

$$\mathcal{D}_T = \mathcal{D}_t + L^*(T-t) + \frac{1}{a}(\delta_t - L^*)\left(1 - e^{-a(T-t)}\right) - \frac{\sigma_2}{a}\int_t^T e^{-a(T-s)} \, dW_{2,s}^* + \frac{\sigma_2}{a}\int_t^T dW_{2,s}^* \quad (28)$$

If we consider now a *self-financing portfolio* with initial value $P_0 = S_0$ where the convenience yield is continuously re-invested, its value at time T will be:

$$P_T = \exp\left(\int_0^T \delta_s \, ds\right) S_T = e^{\mathcal{D}_T} S_T \quad (29)$$

And by assuming no arbitrage opportunities in the market, at time t the value of such a self-financing portfolio will be:

$$V_t(P_T) = P_t \quad (30)$$

Now, substituting (22) into (29) we have that:

$$P_T = S_t \exp\left[\mathcal{D}_T + \left(r - \frac{1}{2}\sigma_1^2\right)(T-t) - \int_t^T \delta_s \, ds + \sigma_1 \int_t^T dW_{1,t}^*\right] \quad (31)$$

And from (26) this is:

$$P_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma_1^2\right)(T-t) + \mathcal{D}_t + \sigma_1 \int_t^T dW_{1,t}^*\right]$$

$$= P_t \exp\left[\left(r - \frac{1}{2}\sigma_1^2\right)(T-t) + \sigma_1 \int_t^T dW_{1,t}^*\right] \quad (32)$$

Under the equivalent martingale measure Q we know that:

$$V_t(P_T) = e^{-r(T-t)} E(P_T) = e^{-r(T-t)} \, P_t \, E\left[\exp\left[\left(r - \frac{1}{2}\sigma_1^2\right)(T-t) + \sigma_1 \int_t^T dW_{1,t}^*\right]\right] = P_t \quad (33)$$

From (22) we have that the discounted future price of the commodity can be expressed as:

$$e^{-r(T-t)} S_T = S_t \exp\left[\left(-\frac{1}{2}\sigma_1^2\right)(T-t) - \int_t^T \delta_s \, ds + \sigma_1 \int_t^T dW_{1,t}^*\right]$$

And from (26) this is:

$$e^{-r(T-t)}S_T = S_t \exp\left[\left(-\frac{1}{2}\sigma_1^2\right)(T-t) - (\mathcal{D}_T - \mathcal{D}_t) + \sigma_1 \int_t^T dW_{1,t}^*\right]$$

Substituting (28) into the above equation we get:

$$e^{-r(T-t)}S_T = S_t \exp\left[-\left(\frac{1}{2}\sigma_1^2 + L^*\right)(T-t) + \frac{1}{a}(L^* - \delta_t)\left(1 - e^{-a(T-t)}\right) + \frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\, dW_{2,s}^*\right.$$

$$\left. - \frac{\sigma_2}{a}\int_t^T dW_{2,s}^* + \sigma_1 \int_t^T dW_{1,t}^*\right] = S_t \exp(z^*) \quad (34)$$

We can see that under the equivalent martingale measure[4]:

$$E(z^*) = -\left(\frac{1}{2}\sigma_1^2 + L^*\right)(T-t) + \frac{1}{a}(L^* - \delta_t)\left(1 - e^{-a(T-t)}\right) = \tilde{\mu} \quad (35)$$

and

$$E\left[\left(\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\, dW_{2,s}^*\right)^2\right] = \frac{\sigma_2^2}{a^2}\left(\frac{1 - e^{-2a(T-t)}}{2a}\right) \quad (36)$$

$$E\left[\left(\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*\right)^2\right] = \frac{\sigma_2^2}{a^2}(T-t) \quad (37)$$

$$E\left[\left(\sigma_1 \int_t^T dW_{1,t}^*\right)^2\right] = \sigma_1^2(T-t) \quad (38)$$

$$E\left[\left(\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\, dW_{2,s}^*\right)\left(\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*\right)\right] = \frac{\sigma_2^2}{a^3}\left(1 - e^{-a(T-t)}\right) \quad (39)$$

[4] See Bjerksund (1991)

$$E\left[\left(\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\,dW_{2,s}^*\right)\left(\sigma_1\int_t^T dW_{1,t}^*\right)\right]=\frac{1}{a^2}\sigma_2\sigma_1\rho\left(1-e^{-a(T-t)}\right)\quad(40)$$

$$E\left[\left(\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*\right)\left(\sigma_1\int_t^T dW_{1,t}^*\right)\right]=\frac{1}{a}\sigma_2\sigma_1\rho(T-t)\quad(41)$$

Equations (36) to (41) imply that the variance of z^* is:

$$\tilde{\sigma}^2=E[(z^*)^2]-[E(z^*)]^2$$

$$=\frac{\sigma_2^2}{a^2}\left(\frac{1-e^{-2a(T-t)}}{2a}\right)+\frac{\sigma_2^2}{a^2}(T-t)+\sigma_1^2(T-t)-2\frac{\sigma_2^2}{a^3}\left(1-e^{-a(T-t)}\right)$$

$$+2\frac{1}{a^2}\sigma_2\sigma_1\rho\left(1-e^{-a(T-t)}\right)-2\frac{1}{a}\sigma_2\sigma_1\rho(T-t)$$

$$=\frac{\sigma_2^2}{a^2}\left(\frac{1-e^{-2a(T-t)}}{2a}\right)+\left(\frac{\sigma_2^2}{a^2}+\sigma_1^2-2\frac{1}{a}\sigma_2\sigma_1\rho\right)(T-t)+2\left(\frac{1}{a^2}\sigma_2\sigma_1\rho-\frac{\sigma_2^2}{a^3}\right)\left(1-e^{-a(T-t)}\right)\quad(42)$$

Thus, the variable $z^*\sim N(\tilde{\mu},\tilde{\sigma}^2)$ with $\tilde{\mu}$ and $\tilde{\sigma}$ given by (35) and (42) respectively. Therefore, the right-hand side of (34) is log-normally distributed and so:

$$E\left(e^{-r(T-t)}S_T\right)=E\left(S_te^{z^*}\right)=S_t\exp\left(\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2\right)\quad(43)$$

It is known that if x is a standard normal deviate, then $z=\mu+\sigma x$ will have a normal distribution with mean μ and standard deviation σ. So the option price will be given as:

$$C_t=e^{-(T-t)r}E[(S_T-K)^+|S_t]$$

$$=E\left[\left(S_te^{z^*}-e^{-(T-t)r}K\right)^+\middle|S_t\right]$$

$$=\int_{-\infty}^{+\infty}\left[S_te^{\tilde{\mu}+\tilde{\sigma}z}-e^{-(T-t)r}K\right]^+\varphi(z)dz\quad(44)$$

It is clear that the function under the integral is non-negative if and only if the following inequality holds:

$$S_t e^{\tilde{\mu}+\tilde{\sigma}z} - e^{-(T-t)r}K \geq 0$$

Thus,

$$z \geq \frac{\ln(K/S_t) - \tilde{\mu} - (T-t)r}{\tilde{\sigma}} = -d(S_t, T-t) \quad (45)$$

Therefore, (44) becomes:

$$C_t = \int_{-d_-}^{+\infty} \left(S_t e^{\tilde{\mu}+\tilde{\sigma}z} - e^{-(T-t)r}K\right) \varphi(z)dz$$

$$= \int_{-d}^{+\infty} S_t e^{\tilde{\mu}+\tilde{\sigma}z}\, \varphi(z)dz - e^{-(T-t)r}K \int_{-d}^{+\infty} \varphi(z)dz$$

$$= S_t e^{\tilde{\mu}} \int_{-d}^{+\infty} e^{\tilde{\sigma}z} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz - e^{-(T-t)r}KN(d)$$

$$= S_t e^{\tilde{\mu}} \int_{-d}^{+\infty} \frac{e^{\tilde{\sigma}z-\frac{z^2}{2}}}{\sqrt{2\pi}} dz - e^{-(T-t)r}KN(d)$$

$$= S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} \int_{-d}^{+\infty} \frac{e^{-\frac{1}{2}\left(\tilde{\sigma}^2-2\tilde{\sigma}z+z^2\right)}}{\sqrt{2\pi}} dz - e^{-(T-t)r}KN(d)$$

$$= S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} \int_{-d}^{+\infty} \frac{e^{-\frac{1}{2}(z-\tilde{\sigma})^2}}{\sqrt{2\pi}} dz - e^{-(T-t)r}KN(d)$$

$$= S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} \int_{-d-\tilde{\sigma}}^{+\infty} \frac{e^{-\frac{1}{2}(u)^2}}{\sqrt{2\pi}} du - e^{-(T-t)r}KN(d)$$

$$= S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} \int_{-d-\tilde{\sigma}}^{+\infty} \frac{e^{-\frac{1}{2}(u)^2}}{\sqrt{2\pi}} du - e^{-(T-t)r} K N(d)$$

$$= S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} N(d+\tilde{\sigma}) - e^{-(T-t)r} K N(d) \quad (46)$$

Theorem 2:

Under the equivalent martingale measure, if we consider a two-factor model, in which the factors are the commodity spot price and the convenience yield which are given by:

$$dS_t = (r - \delta_t) S_t dt + \sigma_1 S_t dW_{1,t}^*$$

$$d\delta_t = a(L^* - \delta_t) dt + \sigma_2 dW_{2,t}^*$$

and:

$$dW_{1,t}^* dW_{2,t}^* = \rho dt$$

the price of a European Call option on the commodity is:

$$C_t = S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2} N(d_1) - e^{-(T-t)r} K N(d_2)$$

where:

$$d_1 = \frac{\ln\left(S_t e^{\tilde{\mu}+\frac{1}{2}\tilde{\sigma}^2}/K\right) + (T-t)r + \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}$$

$$d_2 = d_1 - \tilde{\sigma}$$

$$\tilde{\mu} = -\left(\frac{1}{2}\sigma_1^2 + L^*\right)(T-t) + \frac{1}{a}(L^* - \delta_t)\left(1 - e^{-a(T-t)}\right)$$

$$\tilde{\sigma}^2 = \frac{\sigma_2^2}{a^2}\left(\frac{1 - e^{-2a(T-t)}}{2a}\right) + \left(\frac{\sigma_2^2}{a^2} + \sigma_1^2 - 2\frac{1}{a}\sigma_2\sigma_1\rho\right)(T-t) + 2\left(\frac{1}{a^2}\sigma_2\sigma_1\rho - \frac{\sigma_2^2}{a^3}\right)\left(1 - e^{-a(T-t)}\right)$$

4. Three-Factor Model

4.1. The model

The three factor model is an extension of the two-factor model above. In the three-factor model the stochastic factors (state variables) are the spot commodity price, the instantaneous convenience yield and the instantaneous interest rate. The first two factors, the spot price and convenience yield, follow the stochastic processes described in the two-factor model while the additional factor, the instantaneous interest rate, is formulated according to the Vasicek model (Vasicek 1977).

The joint stochastic processes of the factors can be presented as:

$$dS_t = (\mu - \delta_t)S_t dt + \sigma_1 S_t dW_{1,t} \quad (47)$$

$$d\delta_t = a(L - \delta_t)dt + \sigma_2 dW_{2,t} \quad (48)$$

$$dr_t = k(m - r_t)dt + \sigma_3 dW_{3,t} \quad (49)$$

and:

$$dW_{1,t}dW_{2,t} = \rho_{12}dt \ , dW_{1,t}dW_{3,t} = \rho_{13}dt \ , dW_{2,t}dW_{3,t} = \rho_{23}dt \quad (50)$$

4.2. Risk Adjustments

Under the equivalent martingale measure, the joint stochastic processes of the factors can be expressed as:

$$dS_t = (r_t - \delta_t)S_t dt + \sigma_1 S_t dW^*_{1,t} \quad (51)$$

$$d\delta_t = a(L^* - \delta_t)dt + \sigma_2 dW^*_{2,t} \quad (52)$$

$$dr_t = k(m^* - r_t)dt + \sigma_3 dW^*_{3,t} \quad (53)$$

and:

$$dW^*_{1,t}dW^*_{2,t} = \rho_{12}dt \ , dW^*_{1,t}dW^*_{3,t} = \rho_{13}dt \ , dW^*_{2,t}dW^*_{3,t} = \rho_{23}dt \quad (54)$$

Where k and m^* are respectively the speed of mean reversion and the risk-adjusted long-term mean of the instantaneous interest rate.

So we have that:

$$S_T = S_t \exp\left[\left(-\frac{1}{2}\sigma_1^2\right)(T-t) + \int_t^T r_s\,ds - \int_t^T \delta_s\,ds + \sigma_1 \int_t^T dW_{1,t}^*\right] \quad (55)$$

$$\delta_T = \delta_t e^{-a(T-t)} + L^*\left(1 - e^{-a(T-t)}\right) + \sigma_2 \int_t^T e^{-a(T-s)}\,W_{2,s}^* \quad (56)$$

$$r_T = r_t e^{-k(T-t)} + m^*\left(1 - e^{-k(T-t)}\right) + \sigma_3 \int_t^T e^{k(T-s)}\,W_{3,s}^* \quad (57)$$

4.3. Closed-form solution for 3-factor model

From the two-factor model we know that:

$$\int_t^T \delta_s\,ds = \mathcal{D}_T - \mathcal{D}_t = L^*(T-t) + \frac{1}{a}(\delta_t - L^*)\left(1 - e^{-a(T-t)}\right) - \frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\,dW_{2,s}^* + \frac{\sigma_2}{a}\int_t^T dW_{2,s}^*$$

Similarly, in the case of the spot interest rate, we have that:

$$\int_t^T r_s\,ds = m^*(T-t) + \frac{1}{k}(r_t - m^*)\left(1 - e^{-k(T-t)}\right) - \frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^* + \frac{\sigma_3}{k}\int_t^T dW_{3,s}^* \quad (58)$$

So we have that:

$$\int_t^T r_s\,ds = B(T,t) + Z(T,t) \quad (59)$$

where:

$$B(T,t) = m^*(T-t) + \frac{1}{k}(r_t - m^*)\left(1 - e^{-k(T-t)}\right)$$

$$Z(T,t) = -\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^* + \frac{\sigma_3}{k}\int_t^T dW_{3,s}^*$$

The option value is then given as:

$$C_t = E\left[\exp\left(-\int_t^T r_s\,ds\right)(S_T - K)^+\right]$$

$$= E\left[\left(S_t \mathrm{e}^{z^*} - Ke^{\left(-B(T,t)-Z(T,t)\right)}\right)^+\right] \quad (60)$$

We can see that $\left[S_t \mathrm{e}^{z^*} - Ke^{\left(-B(T,t)-Z(T,t)\right)}\right]$ is positive for

$$z^* + Z(T,t) \geq \ln\left(\frac{K}{S_t}\right) - B(T,t)$$

where

$$\begin{pmatrix} z^* \\ Z(T,t) \end{pmatrix} \sim N_2\left(\begin{pmatrix} \tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} \widetilde{\sigma_1} & \widetilde{\sigma_{12}} \\ \widetilde{\sigma_{12}} & \widetilde{\sigma_2} \end{pmatrix}\right)$$

Where N_2 here denotes a 2-dimensional Gaussian distribution function.

Here, $\widetilde{\sigma_1}$ is the $\tilde{\sigma}$ from two-factor model. So we have to calculate the values $\widetilde{\sigma_2}$ (which is the variance of $Z(T,t)$) and $\widetilde{\sigma_{12}}$ (which is the covariance of $z^*$and $Z(T,t)$).

We have that:

$$E\left[\left(\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*\right)^2\right] = \frac{\sigma_3^2}{k^2}\left(\frac{1-e^{-2k(T-t)}}{2k}\right) \quad (61)$$

$$E\left[\left(\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)^2\right] = \frac{\sigma_3^2}{k^2}(T-t) \quad (62)$$

$$E\left[\left(\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*\right)\left(\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]=\frac{\sigma_3^2}{k^3}\left(1-e^{-k(T-t)}\right)\quad(63)$$

So equations (61) to (63) imply that the variance of $Z(T,t)$ is:

$$\widetilde{\sigma_2}^{\,2}=E\left[\left(Z(T,t)\right)^2\right]-\left[E\left(Z(T,t)\right)\right]^2=\frac{\sigma_3^2}{k^2}\left(\frac{2k(T-t)+1-e^{-2k(T-t)}}{2k}\right)-2\frac{\sigma_3^2}{k^3}\left(1-e^{-k(T-t)}\right)\quad(64)$$

The covariance between z^* and $Z(t,t)$ is calculated as:

$$\widetilde{\sigma_{12}}=E(z^*Z)-E(z^*)E(Z)=E(z^*Z)=$$

$$=E\left[\left(\tilde{\mu}+\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\,dW_{2,s}^*-\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*+\sigma_1\int_t^T dW_{1,t}^*\right)\left(-\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*+\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]$$

$$=\tilde{\mu}E\left[\left(-\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*+\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]$$

$$+E\left[\left(\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\,dW_{2,s}^*\right)\left(-\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*\right)\right]$$

$$+E\left[\left(\frac{\sigma_2}{a}\int_t^T e^{-a(T-s)}\,dW_{2,s}^*\right)\left(\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]$$

$$+E\left[\left(-\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*\right)\left(-\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*\right)\right]+E\left[\left(-\frac{\sigma_2}{a}\int_t^T dW_{2,s}^*\right)\left(\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]$$

$$+E\left[\left(\sigma_1\int_t^T dW_{1,t}^*\right)\left(-\frac{\sigma_3}{k}\int_t^T e^{-k(T-s)}\,dW_{3,s}^*\right)\right]+E\left[\left(\sigma_1\int_t^T dW_{1,t}^*\right)\left(\frac{\sigma_3}{k}\int_t^T dW_{3,s}^*\right)\right]$$

$$=0-\frac{\sigma_2\sigma_3}{a^2k^2}\rho_{23}\left(1-e^{-a(T-t)}\right)\left(1-e^{-k(T-t)}\right)+\frac{\sigma_2\sigma_3}{a^2k}\rho_{23}\left(1-e^{-a(T-t)}\right)+\frac{\sigma_2\sigma_3}{k^2a}\rho_{23}\left(1-e^{-k(T-t)}\right)$$

$$-\frac{\sigma_2\sigma_3}{ak}\rho_{23}(T-t)-\frac{\sigma_1\sigma_3}{k}\rho_{13}(T-t)$$

$$= \frac{\sigma_2 \sigma_3}{ak} \rho_{23} \left(\left(1 - \frac{\left(1 - e^{-a(T-t)}\right)}{a} \right) \left(\frac{\left(1 - e^{-k(T-t)}\right)}{k} - 1 \right) - a(T-t) + 1 \right) - \frac{\sigma_1 \sigma_3}{k} \rho_{13} (T-t) \; (65)$$

So the option price is:

$$C_t = E\left[S_t \mathrm{e}^{z^*} I\left(z^* + Z(T,t) \geq \ln\left(\frac{K}{S_t}\right) - B(T,t) \right) \right]$$

$$- E\left[K e^{(-B(T,t) - Z(T,t))} I\left(z^* + Z(T,t) \geq \ln\left(\frac{K}{S_t}\right) - B(T,t) \right) \right] \quad (66)$$

where $I(.)$ is 1 if $(.)$ is true and 0 otherwise.

Following Kim (2001), we apply:

Lemma (Kunitomo and Takahashi (1992)): Let $\boldsymbol{x} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where N_2 is a 2-dimensional Gaussian distribution function. For arbitrary 2-dimentional vector $\boldsymbol{\alpha}$ and scalars b and c the following is true:

$$\iint_{(1,-b)\boldsymbol{x} \geq c} \exp(\boldsymbol{\alpha}' \boldsymbol{x})\, n_2(\boldsymbol{x} |\ \boldsymbol{\mu}, \boldsymbol{\Sigma}) dx = \exp\left(\boldsymbol{\alpha}' \boldsymbol{\mu} + \frac{1}{2} \boldsymbol{\alpha}' \boldsymbol{\Sigma} \boldsymbol{\alpha} \right) N\left[\frac{(1,-b)(\boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{\alpha}) - c}{\sqrt{(1,-b)\boldsymbol{\Sigma}(1,-b)'}} \right]$$

where n_2 is a 2-dimensional Gaussian density function and N is a standard Gaussian distribution function.

If we apply the above lemma on the first part of (66) for $b = -1$ and $\boldsymbol{a} = (1{,}0)'$ we have that:

$$E\left[S_t \mathrm{e}^{z^*} I\left(z^* + Z(T,t) \geq \ln\left(\frac{K}{S_t}\right) - B(T,t) \right) \right]$$

$$= S_t e^{\left(\tilde{\mu} + \frac{1}{2}\widetilde{\sigma_1}^2\right)} N\left[\frac{\widetilde{\sigma_1}^2 + \widetilde{\sigma_{12}} + \tilde{\mu} - \ln\left(\frac{K}{S_t}\right) + B(T,t)}{\sqrt{\widetilde{\sigma_1}^2 + 2\widetilde{\sigma_{12}} + \widetilde{\sigma_2}^2}} \right] \quad (67)$$

And if we apply the lemma on the 2nd part of (66) for $b=-1$ and $\boldsymbol{a}=(0,-1)'$ we have that:

$$E\left[Ke^{(-B(T,t)-Z(T,t))}I\left(z^{*}+\ Z(T,t)\geq\ln\left(\frac{K}{S_t}\right)-\ B(T,t)\right)\right]$$

$$=Ke^{\left(\frac{1}{2}\widetilde{\sigma_2}^2-B(T,t)\right)}N\left[\frac{\tilde{\mu}-\widetilde{\sigma_1}^2-\widetilde{\sigma_{12}}-\ln\left(\frac{K}{S_t}\right)+\ B(T,t)}{\sqrt{\widetilde{\sigma_1}^2+2\widetilde{\sigma_{12}}+\widetilde{\sigma_2}^2}}\right]\quad(68)$$

Thus, the option price is:

$$C_t=S_te^{\left(\tilde{\mu}+\frac{1}{2}\widetilde{\sigma_1}^2\right)}N(d_1)-Ke^{\left(\frac{1}{2}\widetilde{\sigma_2}^2-B(T,t)\right)}N(d_2)\quad(69)$$

where

$$d_1=\frac{\ln\left(\frac{S_t}{K}\right)+\widetilde{\sigma_1}^2+\widetilde{\sigma_{12}}+\tilde{\mu}+\ B(T,t)}{\sqrt{D}},\qquad d_2=d_1-\sqrt{D},$$

and $D=\widetilde{\sigma_1}^2+2\widetilde{\sigma_{12}}+\widetilde{\sigma_2}^2$

Theorem 3:

Under the equivalent martingale measure, if we consider a three-factor model, in which the factors are the commodity spot price, the convenience yield, and the interest rate which are given by:

$$dS_t=(r_t-\delta_t)S_tdt+\sigma_1S_tdW_{1,t}^{*}$$

$$d\delta_t=a(L^{*}-\delta_t)dt+\sigma_2dW_{2,t}^{*}$$

$$dr_t=k(m^{*}-r_t)dt+\sigma_3dW_{3,t}^{*}$$

and:

$dW_{1,t}^{*}dW_{2,t}^{*}=\rho_{12}dt$, $dW_{1,t}^{*}dW_{3,t}^{*}=\rho_{13}dt$, $dW_{2,t}^{*}dW_{3,t}^{*}=\rho_{23}dt$

the price of a European Call option on the commodity is:

$$C_t = S_t e^{\left(\tilde{\mu}+\frac{1}{2}\widetilde{\sigma_1}^2\right)} N(d_1) - K e^{\left(\frac{1}{2}\widetilde{\sigma_2}^2 - B(T,t)\right)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \widetilde{\sigma_1}^2 + \widetilde{\sigma_{12}} + \tilde{\mu} + B(T,t)}{\sqrt{D}}$$

$$d_2 = d_1 - \sqrt{D}$$

$$B(T,t) = m^*(T-t) + \frac{1}{k}(r_t - m^*)\left(1 - e^{-k(T-t)}\right)$$

$$D = \widetilde{\sigma_1}^2 + 2\widetilde{\sigma_{12}} + \widetilde{\sigma_2}^2$$

$$\tilde{\mu} = -\left(\frac{1}{2}\sigma_1^2 + L^*\right)(T-t) + \frac{1}{a}(L^* - \delta_t)\left(1 - e^{-a(T-t)}\right)$$

$$\widetilde{\sigma_1}^2 = \frac{\sigma_2^2}{a^2}\left(\frac{1 - e^{-2a(T-t)}}{2a}\right) + \left(\frac{\sigma_2^2}{a^2} + \sigma_1^2 - 2\frac{1}{a}\sigma_2\sigma_1\rho_{12}\right)(T-t) + 2\left(\frac{1}{a^2}\sigma_2\sigma_1\rho_{12} - \frac{\sigma_2^2}{a^3}\right)\left(1 - e^{-a(T-t)}\right)$$

$$\widetilde{\sigma_2}^{\,2} = \frac{\sigma_3^2}{k^2}\left(\frac{2k(T-t) + 1 - e^{-2k(T-t)}}{2k}\right) - 2\frac{\sigma_3^2}{k^3}\left(1 - e^{-k(T-t)}\right)$$

$$\widetilde{\sigma_{12}} = \frac{\sigma_2\sigma_3}{ak}\rho_{23}\left(\left(1 - \frac{\left(1 - e^{-a(T-t)}\right)}{a}\right)\left(\frac{\left(1 - e^{-k(T-t)}\right)}{k} - 1\right) - a(T-t) + 1\right) - \frac{\sigma_1\sigma_3}{k}\rho_{13}(T-t)$$

5. Option Evaluation under Regime-Switching

To evaluate the options under the regime switching assumption we work based on Duan et al. (2002). Thus, the option value results as a weighted average of the BS formula values corresponding to different regimes with weights determined by the probabilities of the regimes.

Note here that by assuming a random switching in the regimes we introduce an additional risk in the market. Following Hull and White (1987) and Bollen (1998), we assume that the market does not price the additional regime-risk (or stochastic volatility risk) and so the risk-neutral valuation is proper to be used.

We consider a 2-state model, where ξ_t is a Markov process representing the state in the business cycle. Let:

$$\xi_t = \begin{cases} \xi_t = 0, & \textit{whene the cycle is in contraction} \\ \xi_t = 1, & \textit{whene the cycle is in expansion} \end{cases}$$

The two-state first-order Markov-switching variable ξ_t evolves according to the following transition probabilities:

$$\Pr[\xi_t = 0 | \xi_{t-1} = 0] = p_{00} = \frac{\exp(\mathrm{p}_0)}{1 + \exp(\mathrm{p}_0)}$$

$$\Pr[\xi_t = 1 | \xi_{t-1} = 1] = p_{11} = \frac{\exp(\mathrm{q}_0)}{1 + \exp(\mathrm{q}_0)}$$

So the transition probabilities matrix is:

$$\tilde{p} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

where $p_{01} = 1 - p_{00}$, $p_{10} = 1 - p_{11}$, $p_{00} + p_{01} = 1$, and $p_{11} + p_{10} = 1$.

Now let N_n^i be the number of visits to state 0 in n trials ($n = T/\Delta t$), given that at time $t = 0$ the prevailing state is state i (i.e. $\xi_0 = i, i = 0{,}1$). Also, assume that $\gamma_{n,j}^i$ is the probability under the equivalent martingale measure Q that in n periods starting from state i the number of visits to state 0 and 1 are j and $n - j$ respectively. Moreover, we denote by $f^n(y|\xi_0 = i)$ the density function under Q of the total log return after n periods starting from state i and by $\varphi_j\left(y; \eta_j, \theta_j^2\right)$ we denote the normal density function with mean η and variance θ_j^2. Thus, we have that[5]:

$$f^n(y|\xi_0 = i) = \sum_{j=0}^{n} \gamma_{n,j}^i \varphi_j\left(y; \eta, \theta_j^2\right) \ for \ i = 0{,}1 \quad (70)$$

where,

$$\theta_j^2 n = j * (variance\ of\ \log return\ in\ state\ 0) + (n - j) * (var.of\ \log return\ in\ state\ 1)$$

$$\gamma_{n,j}^i = \Pr^{Q}\left(N_n^i = j\right) \quad for\ j = 0{,}1{,}2, \dots, n\ and\ i = 0{,}1 \quad (71)$$

The Option price can now be written as the weighted average of the BS formula values corresponding to different regimes:

$$C_0^i(X, n) = \sum_{j=0}^{n} \gamma_{n,j}^i C_{j.0}^i \quad (72)$$

So we need to derive expressions for $\gamma_{n,j}^i$. We start by developing equations for $\gamma_{n,j}^0$. For $m = 1, 2, \dots, n$ we have:

$$\gamma_{1,1}^0 = p_{00}$$

$$\gamma_{m,0}^0 = p_{01} p_{11}^{m-1} \quad for\ m = 1, 2, \dots, n$$

[5] See Appendix 1

$$\gamma_{m,1}^{0} = p_{00}p_{01}p_{11}^{m-2} + (m-2)p_{01}^{2}p_{10}p_{11}^{m-3} + p_{01}p_{10}p_{11}^{m-2} \quad for\; m = 2,3,\ldots,n$$

To compute the remaining probabilities, we first consider the probabilities of the first passage of state 0. Let $F^{0}(k)$ be the probability that the 1st visit to state 0 occurs after k periods, given that the initial state is 0:

$$F^{0}(1) = p_{00}$$

$$F^{0}(m) = p_{01}p_{11}^{m-2}p_{10} \quad for\; m = 2,3,\ldots,n$$

Therefore, for $k = 2,3,\ldots,m$, we have that:

$$\gamma_{n,k}^{0} = \sum_{j=0}^{n-k+1} F^{0}(j)\gamma_{n-j,k-1} \quad (72) \quad for\; k = 2,3,\ldots,n$$

Similar expressions we can be computed for $\gamma_{n,k}^{1}$.

5.1. One-factor model

Under the regime-switching assumption, the log underlying commodity price follows a discrete-time stochastic differential equation with regime switching[6] in the volatility:

$$X_t = X_{t-1}e^{-a} + L^{*}(1 - e^{-a}) + \sigma_{\xi_t}\sqrt{\left(\frac{1 - e^{-2a}}{2a}\right)}\varepsilon_t \quad (73)$$

where $\xi_t = 0{,}1$ is the indicator variable of the regime that we are in at each time t and it is independent of $\varepsilon_t \sim N(0,1)$.

When volatility σ remains unchanged between different states, it is difficult to observe ξ_t. If we assume that the volatility in different states is distinct, then without loss of generality we can assume that ξ_t is observable. Thus, the filtration $\mathcal{F}^{X}$ generated by $\{X_t\}$ contains the filtration $\mathcal{F}^{\xi}$ generated by $\{\xi_t\}$.

[6] The proof for the discretization of the OU process can be found in Appendix 2.

So the option price is given as:

$$C_0^i(X,n) = \sum_{j=0}^{n} \gamma_{n,j}^i C_{j.0}^i$$

where

$$C_{j.0}^i = S_0 N\left(d_{j,1}\right) - e^{-nr} K N\left(d_{j,2}\right)$$

$$d_{j,1} = \frac{X_0 e^{-an} - \ln K + L^*(1 - e^{-an}) + \tilde{\sigma}_j^2 n}{\tilde{\sigma}_j \sqrt{n}}$$

$$d_{j,2} = d_{j,1} - \tilde{\sigma}_j$$

$$\tilde{\sigma}_j^2 n = \tilde{\sigma}_0^2 j + \tilde{\sigma}_1^2 (n-j)$$

5.2. Two-factor model

Under the regime-switching assumption, the underlying commodity price and the convenient yield follow the discrete-time stochastic differential equations[7]:

$$S_{t+\Delta t} = S_t + (r - \delta_t) S_t \Delta t + \sigma_{1,\xi_t} S_t \sqrt{\Delta t} \varepsilon_{1,t} \quad (74)$$

$$\delta_{t+\Delta t} = \delta_t + a(L^* - \delta_t)\Delta t + \sigma_{2,\xi_t} \sqrt{\Delta t} \varepsilon_{2,t} \quad (75)$$

where $\xi_t = 0{,}1$ is the indicator variable of the regime that we are in at each time t and it is independent of $\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \sim N\left(0, \begin{matrix} 1 & \rho_{\xi_t} \\ \rho_{\xi_t} & 1 \end{matrix}\right)$. To generate $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ with correlation ρ_{ξ_t} we take two independent standard normal variables Z_1 and Z_2 and we set $\varepsilon_{1,t} = Z_1$ and $\varepsilon_{2,t} = \rho_{\xi_t} * Z_1 + \sqrt{1 - \rho_{\xi_t}} Z_2$.

Hence, the option price is given as:

[7] The discretisation here has been done by employing the Euler method. For the proof see Appendix 3.

$$C_0^i(X,n) = \sum_{j=0}^{n} \gamma_{n,j}^i C_{j.0}^i$$

where

$$C_{j.0}^i = S_0 e^{\tilde{\mu}_j + \frac{1}{2}\tilde{\sigma}_j^2} N(d_{j,1}) - e^{-nr} K N(d_{j,2})$$

and:

$$d_{1,j} = \frac{\ln\left(S_0 e^{\tilde{\mu}_j + \frac{1}{2}\tilde{\sigma}_j^2} / K\right) + nr + \frac{1}{2}\tilde{\sigma}_j^2}{\tilde{\sigma}_j}$$

$$d_{2,j} = d_{1,j} - \tilde{\sigma}_j$$

$$\tilde{\mu}_j n = j * \tilde{\mu}_0 + (n-j) * \tilde{\mu}_1$$

$$\tilde{\sigma}_j^2 n = \tilde{\sigma}_0^2 j + \tilde{\sigma}_1^2 (n-j)$$

5.3. Three-Factor model

Under the regime-switching assumption, the underlying commodity price, the convenient yield, and the interest rate follow the discrete-time stochastic differential equations:

$$S_{t+\Delta t} = S_t + (r_t - \delta_t) S_t \Delta t + \sigma_1 S_t \sqrt{\Delta t} \varepsilon_{1,t}$$

$$\delta_{t+\Delta t} = \delta_t + a(L^* - \delta_t)\Delta t + \sigma_2 \sqrt{\Delta t} \varepsilon_{2,t}$$

$$r_{t+\Delta t} = r_t + k(m^* - r_t)\Delta t + \sigma_3 \sqrt{\Delta t} \varepsilon_{3,t}$$

where $\xi_t = 0{,}1$ is the indicator variable of the regime that we are in at each time t and it is independent of $\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{2,t} \end{bmatrix} \sim N\begin{pmatrix} & 1 & \rho_{12,\xi_t} & \rho_{13,\xi_t} \\ 0\ , & \rho_{12,\xi_t} & 1 & \rho_{23,\xi_t} \\ & \rho_{13,\xi_t} & \rho_{23,\xi_t} & 1 \end{pmatrix} = (0, \Sigma)$. To generate $\varepsilon_{1,t}$, $\varepsilon_{2,t}$ and $\varepsilon_{3,t}$ with pre-specified correlation between them, we take a vector of uncorrelated Gaussian variables, Z.

Then, by using the Cholesky decomposition, we find the square root of Σ, i.e. a matrix C such as $CC^T = \Sigma$. Finally, we create the target vector as: $\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{2,t} \end{bmatrix} = 0 + CZ$.

The option price is given as:

$$C_0^i(X,n) = \sum_{j=0}^{n} \gamma_{n,j}^i C_{j.0}^i$$

where

$$C_{j.0}^i = S_0 e^{\left(\widetilde{\mu_j} + \frac{1}{2}\widetilde{\sigma_{1,J}}^2\right)} N(d_1) - K e^{\left(\frac{1}{2}\widetilde{\sigma_{2,J}}^2 - B_j(n)\right)} N(d_2)$$

and:

$$d_{j,1} = \frac{\ln(S_t/K) + \widetilde{\sigma_{1,J}}^2 + \widetilde{\sigma_{12,J}} + \tilde{\mu} + B(T,t)}{\sqrt{D_j}}$$

$$d_{j,2} = d_1 - \sqrt{D_j}$$

$$D_j = \widetilde{\sigma_{1,J}}^2 + 2\widetilde{\sigma_{12,J}} + \widetilde{\sigma_{2,J}}^2$$

$$B(n) = m^* n + \frac{1}{k}(r_t - m^*)\left(1 - e^{-kn}\right)$$

$$\tilde{\mu}_j n = j * \tilde{\mu}_0 + (n-j) * \tilde{\mu}_1$$

$$\tilde{\sigma}_{1,j}^2 n = \tilde{\sigma}_{1,0}^2 j + \tilde{\sigma}_{1,1}^2 (n-j)$$

$$\tilde{\sigma}_{2,j}^2 n = \tilde{\sigma}_{2,0}^2 j + \tilde{\sigma}_{2,1}^2 (n-j)$$

$$\tilde{\sigma}_{12,j}^2 n = \tilde{\sigma}_{12,0}^2 j + \tilde{\sigma}_{12,1}^2 (n-j)$$

6. Numerical Example

To test the models developed we evaluate options on three different types of commodities; one agricultural, one precious metal and one industrial metal. Namely, we evaluate options on corn, gold, and copper brass. The reason we have chosen these types of commodities is to test how the models perform on a range of different kind of commodities. The results above indicate that by increasing the stochastic factors of the models and assuming regime-switching in the volatilities and correlations the model accuracy increases. The results also illustrate that the models developed above can provide with accurate results on options written on a wide range of commodities.

The options considered are options on Teucrium Corn ETF (CORN US), traded in NYSE Arca, on Randgold Resources Ltd (GOLD US equity) traded in NASDAQ, and Global Brass & Copper Holdings Inc. (BRASS US equity) traded in NYSE Arca. The data have been collected from Bloomberg on the 24th of October 2014. The options evaluated are options with different expiration date and strike prices while the interest rate for each maturity used in 1- and 2- factor models were provided by Bloomberg for the evaluation of the options depending on their specific expiration date. For the 3-factor models the stochastic differential equations which drive the interest rates have been calculated for each maturity. For each commodity we consider options with different maturity dates and for each maturity we evaluate one option in-the-money, one approximately at-the-money and one out-of-the money. At the end of each table there is the sum of absolute differences between the market price and the models' predicted prices.

The best fit parameters of the models can be seen in Appendix 4.

6.1. Options on Corn

Table 6.1.

So=£24.9

Strike	Price	1-factor Reg.-Swit.	1-factor Non-R.S	2-factor Reg.-Swit.	2-factor Non-R.S	3-factor Reg.-Swit.	3-factor Non-R.S	T	r
24	1.300	1.1796	1.0613	1.2532	1.2192	1.2653	1.2509	26	0.15
25	0.650	0.5859	0.4552	0.6771	0.6478	0.6818	0.6799		
26	0.320	0.2635	0.1450	0.3192	0.2945	0.3200	0.3200		
24	1.450	1.4500	1.3903	1.4581	1.4659	1.4500	1.4499	54	0.19
25	0.850	0.8980	0.8391	0.9069	0.9305	0.8728	0.8927		
26	0.550	0.5323	0.4629	0.5224	0.5500	0.4808	0.5043		
20	4.950	4.9611	4.9329	4.9476	4.8600	4.9500	4.9148	82	0.22
25	1.020	1.0994	1.0911	1.0936	1.1410	1.0200	1.0200		
30	0.100	0.1114	0.0702	0.0737	0.0923	0.0631	0.0488		
24	1.800	1.8234	1.8516	1.8121	1.8512	1.8098	1.8000	116	0.26
25	1.350	1.2990	1.3316	1.2861	1.3500	1.2831	1.2981		
26	0.880	0.9021	0.9260	0.8800	0.9557	0.8800	0.9065		
24	2.280	2.2114	2.3014	2.1850	2.2307	2.2800	2.2800	200	0.35
25	1.800	1.7044	1.8000	1.6756	1.7534	1.7313	1.7405		
26	1.250	1.2928	1.3844	1.2595	1.3575	1.2826	1.2978		
sum of abs. difference		**0.71222352**	**1.156012**	**0.5745461**	**0.753827**	**0.3735554**	**0.439678**		

6.2. Options on Gold

Table 6.2.

So=£64.69

Strike	Price	1-factor Reg.-Swit.	1-factor Non-R.S	2-factor Reg.-Swit.	2-factor Non-R.S	3-factor Reg.-Swit.	3-factor Non-R.S	T	r
62.5	3.800	3.5490	3.2899	3.7764	3.7131	3.7994	3.7950	26	0.15
65	2.450	2.1761	1.8943	2.4387	2.3664	2.4500	2.4500		
67.5	1.450	1.2335	0.9731	1.4746	1.4051	1.4777	1.4802		
62.5	4.800	4.6179	4.4646	4.7047	4.7012	4.7915	4.7020	54	0.19
65	3.400	3.3214	3.1618	3.4357	3.4260	3.4000	3.4000		
67.5	2.250	2.3131	2.1554	2.4346	2.4212	2.4465	2.3781		
62.5	5.400	5.3176	5.2426	5.3971	5.4230	5.1905	5.2169	82	0.22
65	3.900	4.0540	3.9771	4.1701	4.1899	3.9544	3.9493		
67.5	2.950	3.0277	2.9500	3.1629	3.1773	2.9500	2.9216		
62.5	6.700	6.5131	6.5450	6.5505	6.5685	6.7000	6.7000	144	0.29
65	5.400	5.2922	5.3259	5.3885	5.4000	5.2477	5.2477		
67.5	4.000	4.2540	4.2868	4.3928	4.3989	4.0227	4.0227		
62.5	7.900	7.9000	8.0263	7.7934	7.6755	7.9000	7.9000	235	0.39
65	6.700	6.7179	6.8482	6.7000	6.5772	6.6929	6.6929		
67.5	5.600	5.6801	5.8109	5.7352	5.6100	5.6327	5.6327		
sum of abs. difference		**2.025813**	**3.446769**	**1.656649**	**1.939336**	**0.712097**	**0.736817**		

6.3. Options on Copper Brass

Table 6.3.

So=£13.9

Strike	Price	1-factor Reg.-Swit.	1-factor Non-R.S	2-factor Reg.-Swit.	2-factor Non-R.S	3-factor Reg.-Swit.	3-factor Non-R.S	T	r
12.5	2.000	1.8784	1.5698	1.9305	1.8418	1.9403	1.9523	26	0.15
15	0.650	0.6500	0.2771	0.7302	0.6496	0.7357	0.7687		
17.5	0.250	0.2298	0.0176	0.2472	0.1779	0.2500	0.2500		
12.5	2.100	2.1071	1.9083	2.0023	2.1026	2.1000	2.1000	54	0.19
15	0.400	0.9039	0.6888	0.8190	0.9890	0.9143	0.9704		
17.5	0.600	0.4010	0.1896	0.3021	0.4204	0.3623	0.4014		
12.5	2.350	2.2624	2.2789	2.2419	2.2953	2.3500	2.1971	100	0.22
15	1.050	1.0791	1.1133	1.0777	1.2733	1.0484	1.0512		
17.5	0.350	0.5309	0.4897	0.4793	0.6771	0.4044	0.4522		
12.5	2.750	2.4659	2.7088	2.7493	2.3905	2.7500	2.4785	172	0.32
15	1.200	1.3085	1.5943	1.6022	1.4772	1.6051	1.5098		
17.5	0.900	0.7102	0.9000	0.9000	0.9000	0.9000	0.8998		
sum of abs. difference		**1.731876**	**2.636123**	**1.635193**	**2.243782**	**1.358483**	**1.773185**		

The above tables indicate that the Regime-switching models work significantly better than the single-regime ones. While the single regime models asign one value at the volatilities and correlation parameters so that the model values will best fit the market values, the regime-switching model can asign two values in the volatilities and correlations parameters, depending on the regime the economy is in. This allows the regime-switching models to be more flexible and so to better fit the data. We can also observe that as the number of stochatsic parameters increases in the models, whether we assume regime-switching or not, the accuracy of the models increases. We can also colclude here that the number of stochastic factors considered in most of the cases influencies more the accuracy of the model than the consideration of regime-switching. Moreover, the results indicate that as the number of stochastic factors increases the influence of regime-switching in the accuracy of the results decreases. Indeed we can observe that the reduction on the sum of absolut differences occured by adding regime-switching in the modelling is greatest in the 1-factor model, less in the 2-factor model and least in the 3-factor model. In other words, it is more vital to consider regime-switching in the 1-factor model rather than in the 3-factor model.

We can also see that even if the models fit in general well the data, the models' predictability seems to be better in the cases of options on Gold and Corn while both regime-switching and single-regime models seem to perform the least in the case of options on Copper. However, even if the models seem to perform relatively worse in the case of Copper compare to the cases of Gold and Corn, we can still conclude that the models can be used and provide accurate prices for options written on a wide range of commodities. In fact, based on the results we can argue that the models work significantly well considering that they perform accurately pricing options on various types of commodities. Moreover, in contrast with traditional models like Black-Scholes option evaluation model, we can notice hare that the model does not seem to perform significantly different when the options are ITM or OTM.

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Appendix 1

Let N_n^i be the number of visits to state 0 in n trials ($n = T/\Delta t$), given that at time $t = 0$ the prevailing state is state i (i.e. $\xi_0 = i, i = 0,1$). Also, assume that $\gamma_{n,j}^i$ is the probability under the equivalent martingale measure Q that in n periods starting from state i the number of visits to state 0 and 1 are j and $n-j$ respectively. The probability density at time period n of the log return will be:

$$f^n(y|\xi_0 = i) = \gamma_{n,j}^i \varphi_j\left(y; \eta_j, \theta_j^2\right)$$

where,

$$\varphi_j\left(y; \eta_j, \theta_j^2\right) = \frac{1}{\theta_j\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(y-\eta_j\right)^2}{\theta_j^2}}$$

$$\eta_j n = j * (mean \log return\ in\ state\ 0) + (n-j) * (mean \log return\ in\ state\ 1)$$

$$\theta_j^2 n = j * (variance\ of\ \log return\ in\ state\ 0) + (n-j) * (var.of\ \log return\ in\ state\ 1)$$

$$\gamma_{n,j}^i = \Pr^{\mathrm{Q}}\left(N_n^i = j\right)\ \ for\ j = 0,1,2,\ldots,n\ and\ i = 0,1 \quad (2)$$

Thus, the density function under Q of the total log return after n periods starting from state i is:

$$f^n(y|\xi_0 = i) = \sum_{j=0}^{n} \gamma_{n,j}^i \varphi_j\left(y; \eta_j, \theta_j^2\right)\ for\ i = 0,1$$

which is the same with the density function of the total log return in Dual et al. 2002.

Appendix 2

If X is an Ornstein-Uhlenbeck process

$$dX_t = \alpha(L^* - X_t)dt + \sigma dW_t^{\ *} \quad (2.1)$$

where a, L^*, σ are positive parameters, its solution in $[t, t+1]$is:

$$X_{t+1} = L^* - (L^* - X_t)e^{-a(t+1-t)} + \sigma \int_t^{t+1} e^{-a(t+1-t)} dW_u \quad (2.2)$$

Since $u \to e^{-a(t+1-t)}$ is a deterministic function, the stochastic integral appearing in the solution is a normal random variable with zero mean and variance:

$$\int_t^{t+1} \left(e^{-a(t+1-t)}\right)^2 du = \int_t^{t+1} e^{-2a(t+1-t)} du = \frac{1 - e^{-2a(t+1-t)}}{2a} \quad (2.3)$$

Therefore,

$$X_t = X_{t-1}e^{-a_{\xi_t}} + L_{\xi_t}^{\ *}(1 - e^{-a_{\xi_t}}) + \sigma_{\xi_t}\sqrt{\left(\frac{1 - e^{-2a_{\xi_t}}}{2a_{\xi_t}}\right)}\varepsilon_t \quad (2.4)$$

where $\varepsilon_t \sim N(0,1)\ i.i.d.$

Appendix 3

The system of PDF's we have to discretise is given by:

$$dS_t = (r - \delta_t)S_t dt + \sigma_1 S_t dW_{1,t}^* \quad (3.1)$$

$$d\delta_t = a(L^* - \delta_t)dt + \sigma_2 dW_{2,t}^* \quad (3.2)$$

and:

$$dW_{1,t}^* dW_{2,t}^* = \rho dt \quad (3.3)$$

<u>Discritisation of S_t (using Euler method)</u>

The SDE of S_t in (3.1) in integral form is:

$$S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} (r - \delta_u)S_u du + \int_t^{t+\Delta t} \sigma_1 S_u dW_{1,u} \quad (3.4)$$

The Euler discretization approximates the integrals using the left-point rule. Hence:

$$\int_t^{t+\Delta t} (r - \delta_u)S_u du \approx (r - \delta_t)S_t \Delta t \quad (3.5)$$

$$\int_t^{t+\Delta t} \sigma_1 S_u dW_{1,u} \approx \sigma_1 S_t \left(W_{1,t+\Delta t} - W_{1,t}\right) = \sigma_1 S_t \sqrt{\Delta t}\varepsilon_{1,t} \quad (3.6)$$

where $\varepsilon_{1,t} \sim N(0,1)\ i.i.d.$

The right hand side involves with $(L^* - X_t)$ instead of $(L^* - X_{t+\delta t})$, since at time t we do not know $X_{t+\delta t}$. So the discretised form of (3.1) is:

$$S_{t+\Delta t} = S_t + (r - \delta_t)S_t \Delta t + \sigma_1 S_t \sqrt{\Delta t}\varepsilon_{1,t} \quad (3.7)$$

Discritisation of δ_t (using Euler method)

The SDE of δ_t in (3.2) in integral form is:

$$\delta_{t+\Delta t} = \delta_t + \int_t^{t+\Delta t} a(L^* - \delta_u)du + \int_t^{t+\Delta t} \sigma_2 dW_{2,u} \quad (3.8)$$

As before, the Euler discretization approximates the integrals using the left-point rule. Thus:

$$\int_t^{t+\Delta t} a(L^* - \delta_u)du \approx a(L^* - \delta_t)\Delta t \quad (3.9)$$

$$\int_t^{t+\Delta t} \sigma_2 dW_{2,u} \approx \sigma_2\left(W_{2,t+\Delta t} - W_{2,t}\right) = \sigma_2\sqrt{\Delta t}\varepsilon_{2,t} \quad (3.10)$$

where $\varepsilon_{2,t} \sim N(0,1)\ i.i.d.$

So the discretised form of (3.2) is:

$$\delta_{t+\Delta t} = \delta_t + a(L^* - \delta_t)\Delta t + \sigma_2\sqrt{\Delta t}\varepsilon_{2,t} \quad (3.11)$$

To generate $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ with correlation ρ_{ξ_t} we take two independent standard normal variables Z_1 and Z_2 and we set $\varepsilon_{1,t} = Z_1$ and $\varepsilon_{2,t} = \rho_{\xi_t} * Z_1 + \sqrt{1 - \rho_{\xi_t}} Z_2$.

Appendix 4

The tables below contain the best fit parameters of the model which have been fitted to the observed option values.

The table $A.4.1.$ contains the parameters of the stochastic differential equation of the interest rate which has been estimated for each maturity. The parameters k, m* and r remain the same for both regime-switching and single-regime models and only the volatility changes from the single-regime to the regime-switching model.

Table A.4.1.

				Regime-Switching		**Non-R.S.**
Maturity	**k**	**m***	**r**	$\boldsymbol{\sigma_{31}}$	$\boldsymbol{\sigma_{32}}$	$\boldsymbol{\sigma_3}$
26 Days	7.98034	0.062805	0.390168	1.49E-08	0.061768	3.747225
54 Days	3.85498	0.104736	0.599158	0.063815	0.174805	0.187073
82 Days	6.005459	0.048513	0.509789	0.749549	2.516045	0.581979
100 Days	4.863281	0.330078	0.585316	1.570313	1.585938	0.5336
116 Days	7.762987	0.005371	0.385645	2.869141	0.826172	0.814862
144 Days	6.304932	0.218384	0.153259	1.95752	0.928223	0.613251
172 Days	9.850769	0.102753	0.220548	6.046722	4.677254	0.247192
200 Days	5.495605	0.081299	0.021826	1.742142	1.581055	0.582001
235 Days	8.692017	0.071472	0.105011	3.034424	0.145264	0.582001

The tables below contain the best fit parameters of the models. Note here that even if for convenience we have used the same notations a and L^* in the 1-, 2-, and 3-factors models; the a and L^* in the 1-factor model are not the same with those in the 2-, and 3-factors models.

Table A.4.2./1-factor Model on Options on Corn

a	11.99979			
L^*	3.173157			
	σ			
	σ_{11}	σ_{12}	π_{11}	π_{22}
Reg.Swit.	4.399719	0.85873	0.593567	0.965027
Non.-R.S.	2.453932613			

Table A.4.3./2- and 3-factor Models on Options on Corn

	a	31.030735						
	L^*	0.0131989						
	δ	5.00E-16						
	σ_1		σ_2		ρ_{12}			
	σ_{11}	σ_{12}	σ_{21}	σ_{22}	$\rho_{12,1}$	$\rho_{12,2}$	π_{11}	π_{22}
Reg.Swit.	0.818726	0.1170654	6.149292	3.785034	0.997984	-0.38745	0.356323	0.992798
Non.-R.S.	0.282791138		2.744613647		0.413421631			

Regime-Switching Model

Maturity (In Days)	ρ_{13}		ρ_{23}	
	$\rho_{13,1}$	$\rho_{13,2}$	$\rho_{23,1}$	$\rho_{23,2}$
26	0.95638	-0.87396	-0.75348	0.19257
54	0.195004	0.614136	-0.938354	0.035034
82	-0.411457	0.831234	-0.472561	-0.011097
116	-0.860352	-0.52832	-0.647461	0.049805
200	0.091309	0.7182617	-0.120605	-0.003418

Non-Regime-Switching Model

Maturity	ρ_{13}	ρ_{23}
26	0.570399	-0.09949
54	0.018677	5.21E-02
82	-0.30862	0.080627
116	-0.54718	0.080627
200	-0.28212	0.080627

Table A.4.4./1-factor Model on Options on Gold

a	14.7160			
L^*	4.1749			
	σ			
	σ_{11}	σ_{12}	π_{11}	σ_{11}
Reg.Swit.	5.0915	2.7499	0.4733	0.9689
Non.-R.S.	3.156219482			

Table A.4.5./2- and 3-factor Models on Options on Gold

	a	2.063093						
	L^*	0.053386						
	δ	0.001415						
	σ_1		σ_2		ρ_{12}			
	σ_{11}	σ_{12}	σ_{21}	σ_{22}	$\rho_{12,1}$	σ_{11}	σ_{12}	σ_{21}
Reg.Swit.	0.574066	0.376801	4.705982	0.500000	1.000000	0.351471	0.388214	0.998138
Non.-R.S.	0.369644165		0.076675415		0.920240834			

Regime-Switching Model

Maturity (In Days)	ρ_{13}		ρ_{23}	
	$\rho_{13,1}$	$\rho_{13,2}$	$\rho_{23,1}$	$\rho_{23,2}$
26	0.47081	-0.03055	0.76340	-0.08597
54	0.28513	-0.77502	0.130493	-0.5719
82	0.374749	0.231812	0.305298	0.016968
116	0.311768	0.940186	-0.4	0.741943
200	0.942742	0.99231	-0.41609	0.421021

Non-Regime-Switching Model

Maturity (In Days)	ρ_{13}	ρ_{23}
26	-0.89598	-0.76822
54	0.690491	0.89679
82	0.215942	-0.48499
144	0.995789	-0.67
235	0.691101	-0.97737

Table A.4.6./1-factor Model on Options on Copper Brass

a	13.0727			
L^*	2.6788			
	σ			
	σ_{11}	σ_{12}	π_{11}	σ_{11}
Reg.Swit.	7.2427	0.000521	0.814575	0.992798
Non.-R.S.	3.822180708			

Table A.4.5./2- and 3-factor Models on Options on Gold

	a	6.712534						
	L^*	0.003859						
	δ	1.82E-12						
	σ_1		σ_2		ρ_{12}			
	σ_{11}	σ_{12}	σ_{21}	σ_{22}	$\rho_{12,1}$	σ_{11}	σ_{12}	σ_{21}
Reg.Swit.	3.016663	0.280457	2.932787	3.856812	0.232239	0.356262	0.411926	0.996887
Non.-R.S.	0.822773695		2.254467991		0.918650628			

Regime-Switching Model

Maturity (In Days)	ρ_{13}		ρ_{23}	
	ρ131	**ρ132**	**ρ231**	**ρ232**
26	-0.88392	-0.79996	0.880951	-0.21524
54	-0.97516	-0.78668	0.051436	-0.60712
100	0.925781	0.582031	-0.27734	-0.07422
172	-0.534	0.99469	-0.69452	-0.0943

Non-Regime-Switching Model

Maturity (In Days)	ρ_{13}	ρ_{23}
26	-0.31528	-0.07132
54	0.519714	-0.89923
100	0.434387	0.457703
172	0.848877	0.144775