SURFACE AND INTERFACIAL WAVES AND DEFORMATIONS IN PRE-STRESSED ELASTIC MATERIALS

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PREFACE

This dissertation is submitted in accordance with the regulations for the degree of Doctor of philosephy in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University

I would like here to express my deepest gratitude to Professor R. W. Ogden, Professor of Mathematics, Glasgow University for his guidence, constant interest and encouragement throughout the period of this research.

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SUMMARY

This thesis is concerned with the effect of pre-stress on the propagation of surface and interfacial waves in elastic materials. Following a review of the classical theory of Rayleigh and Stoneley waves for linear elastic materials we consider first the propagation of infinitesimal surface waves on a half-space of incompressible material subject to a general pure homogeneous pre-stress; the secular equation for propagation along a principal axis of the pre-stress is obtained for a general strain-energy function, and conditions which ensure stability of the underlying pre-stress are derived; the influence of the pre-stress on the existence of surface waves is examined, and the secular equation is analysed in detail for particular deformations and, for a number of specific forms of strain-energy function, numerical results are used to illustrated the dependence of the wave speed on the pre-stress. Necessary and sufficient conditions for the existence of a unique surface wave are obtained. Corresponding results for a compressible material are also derived.

The propagation of (Stoneley) interfacial waves along the boundary between two half-spaces of pre-stressed incompressible isotropic elastic material is then examined. The underlying deformation in each half-space corresponds to a pure homogeneous strain with one principal axis of strain normal to the interface and the others having a common orientation. The secular equation governing the wave speed for propagation along a principal axis is obtained in respect of general strain-energy functions. Detailed analysis of the secular equation reveals general sufficient

conditions for the existence of a wave and, in particular cases, necessary and sufficient conditions for the existence of a unique interfacial wave. It is also shown that when an interfacial wave exists its speed is greater than that of the least of the Rayleigh wave speeds for the separate half-spaces, paralleling a result from the linear theory. For the special case of quasi-static interfacial deformations (corresponding to vanishing wave speed) an existence criterion is found; moreover, it is shown that inequalities that exclude surface deformations in each half-space also exclude interfacial deformations. Dependence of the above results on the underlying homogeneous deformations and on material parameters is illustrated by numerical results for the neo-Hookean material.

Chapter 1

Introduction

In this chapter, we shall describe briefly the main results contained in this thesis.

In Chapter 2, we summerize the basic equations of non-linear elasticity which will be required in subsequent chapters. We introduce in Section 2.1 the notations required for the description of the deformation of an elastic body. In Section 2.2 we note the mass conservation equations and in Section 2.3 we discuss stress and the equations of motion. Constitutive laws for both compressible and incompressible elastic materials are discussed in Section 2.4, and in Section 2.5 the general forms of strain-energy functions for both compessible and incompressible elastic materials are noted. In Section 2.6 of this chapter we establish the equations of motion for both compressible and incompressible materials, which are used frequently in our discussion of surface, Love and interfacial waves. We also consider plane waves in an infinite medium in Section 2.7. In Section 2.8 we record the two-dimenstional form of the strong ellipticity condition. Finally in this chapter we specialize the equations of motion to of those the linear theory since these are used in Chapter 3. The work in this chapter is based on standard texts such as Truesdell and Noll (1965) and Ogden (1984).

In Chapter 3 we review some well-known results of the classical linear theory. In particular this chapter is concerned with Rayleigh, Stoneley and Love waves and provides the background against which to set our subsequent work on waves in pre-stressed materials. We start this chapter by an analysis for incompressible materials. In Section 3.2 we are concerned with Rayleigh waves and we derive the well-known secular equation for Rayleigh waves for an incompressible material. In Section 3.3, we discuss Stoneley waves and again we obtain the secular equation. In Section 3.4 we discuss Love waves and derive the classical dispersion relation. The corresponding analysis for compressible material can be found in Sections 3.5 - 3.8. The work in this chapter is based on, for example, Achenbach (1984), Eringen and Suhubi (1975) and Ewing, Jardetzky and Press (1957).

Chapter 4 is concerned with Rayleigh surface waves propagating in both compressible and incompressible isotropic elastic half-spaces. We start, in Section 4.1, with the analysis for an incompressible material. We suppose that the deformed half-space occupies the region $x_2 < 0$ and we consider waves propagating along the x_1-axis. For simplicity we also take the x_1-axis to correspond to a principal axis of the underlying deformation. We assume that the incremental displacement associated with the wave has no component normal to the (x_1, x_2)-plane and that the x_1 and x_2 components are independent of x_3 and we derive the secular equation for the surface wave speed in respect of a general strain-energy function. This secular equation is used in Section 4.2 to determine inequalities, involving the normal surface stress σ_2 and the strain-energy function, which ensure stability of the underlying deformation, namely

$$\gamma > 0, \tag{1.1}$$

$$\lambda_2^{-1} \lambda_1^3 \hat{W}_{11} + \lambda_1 \hat{W}_1 + 2\sigma_2(1 - \lambda_2^{-1} \lambda_1) - \sigma_2^2 /\gamma > 0, \tag{1.2}$$

where λ_1, λ_2 and λ_3 are the principal stretches of the underlying deformation (subject to the incompressibility constrain $\lambda_1\lambda_2\lambda_3 = 1$),

$$\hat{W}(\lambda_1, \lambda_3) = W(\lambda_1, \lambda_1^{-1} \lambda_3^{-1}, \lambda_3), \tag{1.3}$$

$W(\lambda_1, \lambda_2, \lambda_3)$ being the strain-energy per unit volume, $\hat{W}_1 = \partial\hat{W}/\partial\lambda_1$, $\hat{W}_{11} = \partial^2 W/\partial\lambda_1^2$ and

$$\gamma = \frac{\lambda_1 \; \lambda_2^2 \; \hat{W}_1}{\lambda_1^2 - \lambda_2^2} . \tag{1.4}$$

We note that each of (1.1) and (1.2) reduces to $\mu > 0$ in the unstressed reference configuration ($\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\sigma_2 = 0$), where μ is the shear modulus of the material in that configration.

When $\sigma_2 = \sigma$ is a hydrostatic stress, which is discussed in Section 4.2.1, (1.2) requires that $-2\mu < \sigma < 2\mu$. Within this range of values of σ the (positive) wave speed is unique and such that $c \leq c_S$, where c_S is the shear wave speed ($\rho c_S^2 = \mu$, ρ being the density of the material) in an unbounded unstrained body, with equality holding if and only if $\sigma = \mu$. At the extreme values $\sigma = \pm 2\mu$, $c = 0$ and the incremental deformation is then a quasi-static surface deformation superposed on an underlying state of hydrostatic stress. When $\sigma_2 = 0$ the inequality (1.2) provides restrictions on the domain of stability (in the (λ_1, λ_2)-plane, for example), within which surface waves exist on a free surface. Since the deformation is unaffected by a superimposed hydrostatic stress, (1.2) also provides limitions on the normal surface stress σ_2 that can be supported in any state of deformation for which surface wave exist.

For a general strain-energy function, we express the secular equation in a simple cubic form, that can be found in Section 4.2.2. The coefficients of the two highest powers of the cubic are each equal to unity, while the other two coefficients depend, in general, on λ_1, λ_2, σ_2 and the material properties. When $\sigma_2 = 0$ the equation simplifies.

In Section 4.3 we consider some particular deformations, namely plane strain and uniaxial stress, which are studied for a general strain-energy function. In Section 4.4 we specify the strain-energy function more explicitly and further results are obtained. The range of existence of surface waves is illustrated by numerical results and we observe that in each case considered a unique surface waves exists for each state of deformation in the stable regime. In Section 4.5 we note briefly how the results described above are affected by changing the incremental dead-load boundary condition to an incremental hydrostatic stress boundary condition (there is no distinction when $\sigma_2 = 0$).

In Section 4.6 we summarize the (two-dimensional) criteria appropriate to stability under all-round dead load and all-round hydrostatic stress for comparison with the stability results derived in this chapter.

In Section 4.7 we extend our discussion on surface waves to consider propagation in a general direction ($\cos\theta$, $\sin\theta$) in the (x_1, x_3)-plane. Because, in general, the equation involve cumbersome algebra, we specialize to the neo-Hookean strain-energy function in Section 4.7.1 and we obtain results equivalent to those given by Flavin (1963).

Also in this chapter we consider the corresponding results for a compressible material in Sections 4.8–4.11, we consider the surface waves propagating along a principal axis of the underlying homogeneous pure strain again in respect of the general form of strain-energy function. We derive the secular equation in Section 4.8, and, as for an incompressible material, we express the secular equation in a simple cubic form. In Section 4.9 we analyze the resulting secular equation for arbitrary configurations, including a number of special and degenerate ones. In each case necessary and sufficient conditions for the existence of a unique surface wave are found. The results obtained subsume those given by Hayes and Rivlin (1961b) and some special cases examined briefly by Willson (1972a,b; 1973). In Section 4.10 we consider some special deformations.

A general method for establishing exsistence and uniquess of surface waves in a pre-stressed compressible isotropic elastic half-space was developed by Chadwick and Jarvis (1979), and provides a different approach to that considered in this chapter. The method used by Chadwick and Jarvis applies to arbitrary strain-energy functions and arbitrary directions of propagation; in practice, it yields explicit results only for simple forms of strain-energy function. For the restricted Hadamard material they obtained an expression for the surface wave speed. For propagation along a principal axis, we examine surface wave propagation in such a material when the normal stress is non-zero. Our general results are also illustrated for another form of strain-energy function in Section 4.11.

Chapter 5 is concerned with Love waves and Rayleigh waves on a layered half-space. In Section 5.1.1 we begin the discussion by considering a pre-stressed half-space defined by $x_2 < 0$ with a layer of different pre-stressed material of thickness h on the top with boundaries $x_2 = 0$ and $x_2 = h$. We consider wave propagation along a principal axis and we derive the dispersion equation for Love waves propagating along a principal axis of the underlying deformation in respect of a general strain-energy function for an incompressible material. We also consider in section 5.1.4 the case of equibiaxial deformations and we present some numerical results. In Section 5.1.5 we recover the result for the linear theory. Also the corresponding results for a compressible material are examined briefly.

Also, in this chapter we extend the discussion of Rayleigh surface waves in which there is a layer of uniform thickness h with the boundaries $x_2 = 0$ and $x_2 = h$, but because of complicated algebra involved the secular equation is left in determinant form.

Finally in this thesis we explore a number of aspects of interfacial waves and deformations for pre-stressed incompressible isotropic elastic half-spaces. For simplicity, we consider only propagation along a common principal axis of strain of the two half-spaces, but no restriction is placed *ab initio* on the material properties; thus, we allow the materials constituting the half-spaces to have arbitrary, but different, strain-energy functions. We follow the notation used in Chapter 4 for surface waves in a single half-space.

In Section 6.1 we derive the secular equation governing the speed of interfacial waves. The secular equation is analyzed in detail in Section 6.2; firstly, the results on surfaces waves from Chapter 4 that are needed are summarized briefly; the neutral and limiting cases are discussed and the question of existence of interfacial waves is addressed. Contact is also made with the linear theory. Also, when the material of each half-space is neo-Hookean, more results are obtained; these are used as basis for numerical calculations which, for biaxial deformations, illustrate the dependence of the neutral and limiting curves on certain material parameters. Finally, the special deformation in while $\lambda_1 = \lambda_1 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$ is considered to establish the corresponding result for the classical linear theory.

In Section 6.3 we extend the discussion to consider the propagation in any direction $(\cos\theta, \sin\theta)$ in the (x_1, x_3)-plan and we derive the secular equation which corresponds to that obtained by Chadwick and Jarvis (1979) for the case $\mu = \mu^*$ and $\rho = \rho^*$. We also specialize to equibiaxial deformations and obtain the equations of neutral and limiting curves, and we present some numerical results for these cases. Finally in this chapter we consider the corresponding problem for a compressible material.

The work in Sections 4.1-4.6 forms the basis of the paper 'On surface waves and deformations in a pre-stressed incompressible elastic solid' while that in Sections 4.8-4.11 has been accepted for publication. A third paper, based on sections 6.1-6.2 is being submitted for a publication.

Chapter 2

Basic equations of non-linear elasticity

In this Chapter we summarize the basic equations of non-linear elasticity that will be used frequently in the subsequent Chapters. Detailed derivations are not given because these can be found in standard texts. We refer to Ogden (1984), for example, for full details.

2.1 Kinematics

We begin by introducing the notation required for the description of the deformation of an elastic body. We consider a continuous body which occupies the region B_r in some natural (i.e. unstressed) configuration, N_0 say. Let a typical point of B_r, P say, have position vector $\underline{X}$ relative to some (arbitrarily chosen) origin O.

The *motion* of the body, in which the body occupies the region B_t at time t, is described by the one-parameter mapping

$$\underline{\chi}_t : B_r \longrightarrow B_t \, ,$$

and we write (2.1.1)

$$\underline{x} = \underline{\chi}_t(\underline{X}) \equiv \underline{\chi}(\underline{X}, t),$$

for the position $\underline{x}$ occupied by P in the current configuraton. In cartesian components equation (2.1.1) may be expressed

$$\begin{aligned} x_1 &= \chi_1(X_1, X_2, X_3, t), \\ x_2 &= \chi_2(X_1, X_2, X_3, t), \\ x_3 &= \chi_3(X_1, X_2, X_3, t). \end{aligned} \tag{2.1.2}$$

For a given t, $\underline{\chi}_t$ is called the *deformation* of the body *relative* to B_r. We require $\underline{\chi}_t$ to have a unique inverse $\underline{\chi}_t^{-1}$ such that

$$\underline{X} = \underline{\chi}_t^{-1}(\underline{x}) \qquad \underline{x} \in B_t. \tag{2.1.3}$$

We also assume that $\underline{\chi}$ is twice-continuously differentiable when this degree of regularity is required.

The *velocity* and *acceleration* of the material particle P are given by

$$\underline{v} = \frac{\partial\underline{\chi}(\underline{X},t)}{\partial t}, \qquad \underline{a} = \frac{\partial^2\underline{\chi}(\underline{X},t)}{\partial t^2} \tag{2.1.4}$$

respectively, where $\partial/\partial t$ denotes differentiation with respect to t at fixed $\underline{X}$.

The *deformation gradient* tensor, the gradient of (2.1.1), is given by

$$\underline{A} = \text{Grad}\ \underline{\chi}(\underline{X},t) \ , \tag{2.1.5}$$

where Grad denotes the gradient operator with respect to $\underline{X}$.

It follows from (2.1.4) and (2.1.5) that

$$\dot{\underline{A}} = \underline{L}\ \underline{A} \tag{2.1.6}$$

where the superposed dot denotes $\partial/\partial t$ at fixed $\underline{X}$ and

$$\underline{L} = \text{grad}\ \underline{v} \tag{2.1.7}$$

is the *velocity gradient* tensor. Note that, in (2.1.7), grad denotes the gradient operator with respect to $\underline{x}$.

In cartesian components, we have

$$A_{ij} = \frac{\partial\chi_i(\underline{X},t)}{\partial X_j} \equiv \frac{\partial x_i}{\partial X_j}, \tag{2.1.8}$$

$$L_{ij} = \frac{\partial v_i}{\partial x_j} . \qquad (2.1.9)$$

We use the notation

$$J = \det \underline{A}, \qquad (2.1.10)$$

and impose the usual constraint

$$J > 0, \qquad (2.1.11)$$

which ensures that the deformation is locally invertible, i.e. that $\underline{A}^{-1}$ exists. For convenience we write

$$\underline{B} = (\underline{A}^{-1})^T , \qquad (2.1.12)$$

where $(\ )^T$ denotes the transpose a second-order tensor.

In components

$$B_{ij} = \frac{\partial X_j}{\partial x_i} \qquad (2.1.13)$$

We shall make use of the following polar decompositions for $\underline{A}$:

$$\underline{A} = \underline{R}\ \underline{U} = \underline{V}\ \underline{R} \qquad (2.1.14)$$

where $\underline{R}$ is proper orthogonal and $\underline{U}$ and $\underline{V}$ are positive definite and symmetric, and are called the *right* and *left stretch* tensors respectively.

Each of $\underline{U}$ and $\underline{V}$ may be expressed in spectral form

$$\begin{aligned} \underline{U} &= \sum_{i=1}^{3} \lambda_i\ \underline{u}^{(i)} \otimes \underline{u}^{(i)} \\ \underline{V} &= \sum_{i=1}^{3} \lambda_i\ \underline{v}^{(i)} \otimes \underline{v}^{(i)} \end{aligned} \qquad (2.1.15)$$

where λ_i (> 0), $i \in \{1,2,3\}$, are the *principal stretches* of the deformation, and $\underline{u}^{(i)}$ and $\underline{v}^{(i)}$ respectively are the unit eigenvectors of $\underline{U}$ and $\underline{V}$. We shall refer to $\underline{u}^{(i)}$ and $\underline{v}^{(i)}$, $i \in \{1,2,3\}$, as the *Lagrangian* and *Eulerian principal* axes respectively.

We note that

$$\underline{v}^{(i)} = \underline{R}\ \underline{u}^{(i)} \qquad i \in \{1,2,3\}, \tag{2.1.16}$$

follows from (2.1.14) and (2.1.15).

The right Cauchy-Green deformation tensor $\underline{C}$ is given by

$$\underline{C} = \underline{A}^T\underline{A} = \underline{U}^2 = \sum_{i=1}^{3} \lambda_i^2\ \underline{u}^{(i)} \otimes \underline{u}^{(i)} \tag{2.1.17}$$

For future reference we note that *principal invariants* of $\underline{C}$, denoted I_1, I_2, I_3, are given by

$$\begin{aligned} I_1 &= \mathrm{tr}(\underline{C}), \\ I_2 &= \tfrac{1}{2}\, I_1^2 - \tfrac{1}{2}\, \mathrm{tr}(\underline{C}^2) \\ I_3 &= \det \underline{C}\ . \end{aligned} \tag{2.1.18}$$

Let $\underline{N}$ denote the unit outward normal to the boundary ∂B_r of B_r and $\underline{n}$ the corresponding unit normal to the boundary ∂B_t of B_t. Then, according to Nanson's formula, area elements da_r and da of ∂B_r and ∂B_t are related by

$$\underline{n}\ da = J\ \underline{B}\ \underline{N}\ da_r. \tag{2.1.19}$$

2.2 Mass conservation

Let ρ_r denote the mass density of the material in B_r and ρ the corresponding density in B_t. *Conservation of mass* is expressed by means of the equation

$$\rho_r/\rho = J \equiv \det \underline{A}. \tag{2.2.1}$$

For an *isochoric* (volume preserving) deformation $J \equiv 1$ and $\rho = \rho_r$. An *incompressible material* is one for which *every* deformation is necessarily isochoric, i.e.

$$\rho_r/\rho = J = 1 \qquad \underline{X} \in B_r. \tag{2.2.2}$$

In view of (2.1.14) and (2.1.15) equation (2.2.1) may also be written

$$\rho_r/\rho = \det \underline{U} = \lambda_1\lambda_2\lambda_3, \tag{2.2.3}$$

with

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{2.2.4}$$

for an isochoric deformation.

We shall also require the rate form of (2.2.1), namely

$$\frac{\partial\rho}{\partial t} + \rho \operatorname{div} \underline{v} = 0. \tag{2.2.5}$$

When the motion is isochoric this simplifies to

$$\operatorname{tr}(\underline{L}) \equiv \operatorname{div} \underline{v} = 0. \tag{2.2.6}$$

2.3 Stress and the equations of motion

The traction (load) on the area element da of the deformed surface ∂B_t is expressible in the form

$$\underline{\sigma}^T \underline{n}\, da = \underline{S}^T \underline{N}\, da_r, \qquad (2.3.1)$$

where $\underline{\sigma}^T$ is the Cauchy stress tensor (independent of $\underline{n}$) and $\underline{S}$ the nominal stress tensor. In view of (2.1.9) equation (2.3.1) yields

$$\underline{S} = J\, \underline{B}^T \underline{\sigma}\ , \qquad (2.3.2)$$

and we shall use this connection later.

In terms of nominal stress the equation of motion has the form

$$\mathrm{Div}\, \underline{S} = \rho_r\, \underline{a}, \qquad (2.3.3)$$

where $\underline{a}$ is the acceleration given by (2.1.4), Div denotes the divergence operator with respect to $\underline{X}$ and body forces are assumed to be absent. Alternatively, in terms of the Cauchy stress, the equation of motion has the form

$$\mathrm{div}\, \underline{\sigma} = \rho \underline{a}, \qquad (2.3.4)$$

where div is the divergence operator in B_t.

The rate form of (2.3.3) is obtained by differentiating with respect to t at fixed $\underline{X}$ to give, using (2.1.4),

$$\mathrm{Div}\, \dot{\underline{S}} = \rho_r\, \ddot{\underline{v}}\ , \qquad (2.3.5)$$

where the dot indicates the differentiation referred to above.

Futhermore, if the reference configuration is upated from B_r to the current configuration B_t then (2.3.5) is replaced by

$$\operatorname{div} \dot{\underline{S}}_0 = \rho \,\dot{\underline{v}}, \tag{2.3.6}$$

where, again div denotes the divergence operator with respect to $\underline{x}$ and $\dot{\underline{S}}_0$ represents $\dot{\underline{S}}$ evaluated in B_t *after* differentiation with respect to t.

The equations of rotational balance are satisfied when the Cauchy stress tensor $\underline{\sigma}$ is symmetric, or, equivalently, on use of (2.3.2)

$$\underline{A}\ \underline{S} = \underline{S}^T\ \underline{A}^T. \tag{2.3.7}$$

The rate form of (2.3.7) can be obtained by differentiating with respect to t and updating the reference configuration to B_t. This yields

$$\dot{\underline{S}}_0 + \underline{L}\ \underline{\sigma} = \dot{\underline{S}}_0^{T} + \underline{\sigma}\ \underline{L}^T, \tag{2.3.8}$$

where $\underline{L}$ is defined by (2.1.7).

2.4 Constitutive laws for elastic materials

2.4.1 Compressible elastic materials

We consider an elastic material for which there is a strain energy $W(\underline{A})$ per unit reference volume, so that the nominal stress is given by

$$\underline{S} = \frac{\partial W}{\partial \underline{A}}, \tag{2.4.1}$$

or, in components,

$$S_{ji} = \frac{\partial W}{\partial A_{ij}} \tag{2.4.2}$$

It is assumed that the material is homogeneous so that W has no explicit dependence on $\underline{X}$ (i.e. it depends on $\underline{X}$ only through $\underline{A}$). For the function W to be *objective* (i.e unaffected by a superposed rigid-body rotation *after* deformation), it must depend on $\underline{A}$ only through the right stretch tensor $\underline{U}$ occurring in (2.1.14); thus

$$W(\underline{A}) \equiv W(\underline{U}). \tag{2.4.3}$$

Associated with $\underline{U}$ we define the so-called Biot stress tensor $\underline{T}$ as

$$\underline{T} = \frac{\partial W}{\partial \underline{U}}, \tag{2.4.4}$$

analogously to (2.4.1).

If the material is *isotropic* relative to B_r then W must also be unaffected by an arbitrary rigid-body rotation *before* deformation. Coupled with the objectivity requirement (2.4.3) this leads to the standard restriction on W, namely

$$W(\underline{Q}\ \underline{U}\ \underline{Q}^T) = W(\underline{U}), \tag{2.4.5}$$

which must be satisfied *for all orthogonal second-order tensors* $\underline{Q}$.

Because of (2.1.15) this ensures that W depends only on the principal stretches λ_1, λ_2, λ_3 and is indifferent to any pairwise interchange of λ_1, λ_2, λ_3. Thus, we write

$$W(\lambda_1,\lambda_2,\lambda_3) = W(\lambda_1,\lambda_3,\lambda_2) = W(\lambda_3,\lambda_1,\lambda_2). \tag{2.4.6}$$

It then follows that $\underline{T}$ is coaxial with $\underline{U}$ and hence, from (2.4.4), we obtain

$$\underline{T} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \underline{u}^{(i)} \otimes \underline{u}^{(i)}, \tag{2.4.7}$$

analogously to (2.1.15).

The principal values of $\underline{T}$ are denoted by t_i, $i \in \{1,2,3\}$. It follows from (2.4.7) that

$$t_i = \frac{\partial W}{\partial \lambda_i} \qquad i \in \{1,2,3\}, \tag{2.4.8}$$

and hence, for an isotropic elastic material,

$$\underline{T} = \sum_{i=1}^{3} t_i \, \underline{u}^{(i)} \otimes \underline{u}^{(i)}. \tag{2.4.9}$$

The corresponding expression for the Cauchy stress tensor $\underline{\sigma}$, which is coaxial with $\underline{V}$, is

$$\underline{\sigma} = \sum_{i=1}^{3} \sigma_i \, \underline{v}^{(i)} \otimes \underline{v}^{(i)}, \tag{2.4.10}$$

where σ_i and t_i are connected through

$$\sigma_i = J^{-1} \lambda_i \, t_i = J^{-1} \lambda_i \frac{\partial W}{\partial \lambda_i} \qquad i \in \{1,2,3\}. \tag{2.4.11}$$

Finally, it follows from (2.3.2) that

$$\underline{S} = \sum_{i=1}^{3} t_i \, \underline{u}^{(i)} \otimes \underline{v}^{(i)}, \tag{2.4.12}$$

which should be compared to the decomposition

$$\underline{A} = \sum_{i=1}^{3} \lambda_i \, \underline{v}^{(i)} \otimes \underline{u}^{(i)}, \tag{2.4.13}$$

for the deformation gradient, which can be obtained by using (2.1.14), (2.1.15) and (2.1.16).

2.4.2 Incompressible elastic materials

For an incompressible material it follows from (2.2.1)-(2.2.4) that the constraint

$$J \equiv \det \underline{A} \equiv \det \underline{U} \equiv \lambda_1\lambda_2\lambda_3 = 1 \tag{2.4.14}$$

must be satisfied at each point $\underline{X} \in B_r$. Equations (2.4.1) and (2.4.4) are then replaced by

$$\underline{S} = \frac{\partial W}{\partial \underline{A}} - p\underline{B}^T \tag{2.4.15}$$

and

$$\underline{T} = \frac{\partial W}{\partial \underline{U}} - p\underline{U}^{-1} \tag{2.4.16}$$

respectively, where p, which is an arbitrary function of $\underline{X}$, is the Lagrange multiplier associated with the constraint (2.4.14).

If the material is isotropic $\underline{T}$, $\underline{\sigma}$ and $\underline{S}$ are given by (2.4.9), (2.4.10) and (2.4.12) respectively, while (2.4.8) and (2.4.11) are replaced by

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\, \lambda_i^{-1} \qquad i \in \{1,2,3\} \tag{2.4.17}$$

(corresponding to the principal values of (2.4.16)) and

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \qquad i \in \{1,2,3\} \tag{2.4.18}$$

respectively.

2.5 Strain-energy functions for isotropic materials

In (2.4.6) it was noted that for an isotropic elastic material the strain energy may be regarded as a symmetric function of λ_1, λ_2, λ_3. Equivalently, it may be considered as a function of the principal invariants I_1, I_2, I_3 defined in (2.1.18); in terms of λ_1, λ_2, λ_3 these are

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \\ I_3 &= \lambda_1^2\,\lambda_2^2\,\lambda_3^2\,, \end{aligned} \tag{2.5.1}$$

and when the material is incomresssible $I_3 = 1$, and the remaining independent invariants are

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \\ I_2 &= \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}. \end{aligned} \tag{2.5.2}$$

We now consider some specific forms of W for both incompressible and compressible isotropic elastic materials, which will be used in later chapters.

2.5.1 Incompressible materials

With W now regarded as a function of I_1 and I_2, as given in (2.5.2), equations (2.4.17) and (2.4.18) give

$$\lambda_i t_i = \sigma_i = 2\lambda_i^2 \frac{\partial W}{\partial I_1} - 2\lambda_i^{-2} \frac{\partial W}{\partial I_2} - p, \quad i \in \{1,2,3\} \tag{2.5.3}$$

and, on elimination of p, we obtain

$$\lambda_i t_i - \lambda_j t_j = \sigma_i - \sigma_j = 2(\lambda_i^2 - \lambda_j^2)\left(\frac{\partial W}{\partial I_1} + \lambda_i^{-2} \lambda_j^{-2} \frac{\partial W}{\partial I_2}\right). \tag{2.5.4}$$

The *Mooney (or Mooney-Rivlin)* strain-energy function is defined as

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{2.5.5}$$

where C_1 and C_2 are physical constants. The special case of this corresponding to $C_2 = 0$ yields the neo-Hookean form of strain energy, namely

$$W = C_1(I_1 - 3). \tag{2.5.6}$$

The strain-energy function (2.5.5) is a particular member of the class of strain-energy functions proposed by Ogden (1972a). For these

$$W = \sum_{n=1}^{N} \mu_n (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3)/\alpha_n, \tag{2.5.7}$$

where μ_n and α_n, $n \in \{1,2,\ldots,N\}$, are material constants, and (2.4.17) and (2.4.18) yield

$$\lambda_i t_i = \sigma_i = \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n} - p \quad i \in \{1,2,3\}. \tag{2.5.8}$$

Comparison of (2.5.5) and (2.5.7) shows that for the Mooney strain-energy function

$$\alpha_1 = 2, \quad \alpha_2 = -2, \qquad\qquad (2.5.9)$$
$$\mu_1 = 2C_1, \; \mu_2 = -2C_2, \; \mu_n = 0 \quad n \in \{3,4,\ldots,N\}.$$

A useful generalization of (2.5.7) is the Valanis-Landel strain energy, for which

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3) \qquad (2.5.10)$$

and hence

$$\lambda_i t_i = \sigma_i = \lambda_i \, w'(\lambda_i) - p, \qquad (2.5.11)$$

where w is any suitably well-behaved function.

2.5.2 Compressible materials

For a compressible material use of the invariants (2.5.1) in (2.4.11) yields

$$J\sigma_i = \lambda_i t_i = 2\lambda_i^2 \frac{\partial W}{\partial I_1} + 2\lambda_i^2 \, (I_1 - \lambda_i^2) \frac{\partial W}{\partial I_2} + 2I_3 \frac{\partial W}{\partial I_3}, \qquad (2.5.12)$$

and hence

$$J(\sigma_i - \sigma_j) = \lambda_i t_i - \lambda_j t_j = 2(\lambda_i^2 - \lambda_j^2)\left(\frac{\partial W}{\partial I_1} + \lambda_k^2 \frac{\partial W}{\partial I_2}\right), \qquad (2.5.13)$$

where (i,j,k) is a permutation of (1,2,3).

For the strain-energy function

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + F(I_3),$$

a modification of (2.5.5), where $F(I_3)$ is a suitably well-behaved function, equation (2.5.13) simplifies to

$$J(\sigma_i - \sigma_j) = \lambda_i t_i - \lambda_j t_j = 2(\lambda_i^2 - \lambda_j^2)(C_1 + C_2\lambda_k^2). \quad (2.5.14)$$

Finally, we consider a similar modification of (2.5.7) namely

$$W = \sum_{n=1}^{N} \mu_n (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) + g(\lambda_1\lambda_2\lambda_3), \quad (2.5.15)$$

where g is a function of $J = \lambda_1\lambda_2\lambda_3$. From (2.4.11) we obtain

$$J\sigma_i = \lambda_i t_i = \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n} + Jg'(J). \quad (2.5.16)$$

2.5.3 Linear isotropic materials

For infinitesimal strains we use the variables

$$e_i = \lambda_i - 1 \qquad i \in \{1,2,3\} \quad (2.5.17)$$

and linearize the stress-strain equations to obtain

$$t_i = \sigma_i = \lambda(e_1 + e_2 + e_3) + 2\mu e_i, \quad i \in \{1,2,3\} \quad (2.5.18)$$

where λ, μ are the Lamé elastic moduli, correct to first order in e_1, e_2, e_3. The bulk modulus κ is defined by

$$\kappa = \lambda + (2/3)\mu. \quad (2.5.19)$$

For incompressible materials (2.5.18) is replaced by

$$t_i - \sigma_i = 2\mu e_i - p \qquad i \in \{1,2,3\} \tag{2.5.20}$$

subject to

$$e_1 + e_2 + e_3 = 0, \tag{2.5.21}$$

with p having the same interpretaion as in (2.4.16).

Comparison of (2.5.20) with linearized form of (2.5.8) shows that

$$\sum_{n=1}^{N} \mu_n \alpha_n = 2\mu. \tag{2.5.22}$$

In particular, for the Mooney strain-energy function (2.5.5) we have

$$\mu_1 - \mu_2 = 2(C_1 + C_2) = \mu. \tag{2.5.23}$$

The corresponding linearization of (2.5.16) again yields (2.5.22), and, in addition

$$g'(1) + \sum_{n=1}^{N} \mu_n = 0, \qquad g''(1) = \lambda. \tag{2.5.24}$$

<u>2.6 Elastic moduli</u>

For use in the rate forms of the equations of motion (2.3.5) or (2.3.6) we shall require rate forms of the equations of the constitutive laws. First, for *compressible materials*, differentiation of (2.4.1) with respect to t at fixed $\underline{X}$ yields

$$\dot{\underline{S}} = \underline{A}\,\dot{\underline{A}}, \tag{2.6.1}$$

where $\underline{A}$ is the fourth-tensor given by

$$\underline{A} = \frac{\partial^2 W}{\partial \underline{A} \partial \underline{A}} , \tag{2.6.2}$$

or, in components,

$$\dot{S}_{ji} = A_{jilk} \dot{A}_{kl} \tag{2.6.3}$$

with

$$A_{jilk} = \frac{\partial^2 W}{\partial A_{ij} \partial A_{kl}} . \tag{2.6.4}$$

We refer to $\underline{A}$ as the tensor of *first-order elastic moduli* associated with the variables ($\underline{S}$, $\underline{A}$) relative to B_r.

If the reference configuration is now updated to coincide with the current configuration B_t, (2.6.1) becomes

$$\dot{\underline{S}}_0 = \underline{A}_0 \dot{\underline{A}}_0 \tag{2.6.5}$$

where the subscript zero indicates evaluation in B_t. From (2.1.6) we deduce that $\dot{\underline{A}}_0 = \underline{L}$. The tensor $\underline{A}_0$ is called the tensor of first-order *instantaneous* elastic moduli associated with ($\underline{S}$, $\underline{A}$).

For *compressible isotropic materials* the components of $\underline{A}_0$ referred to Eulerian principal axes of the underlying deformation are derived in Ogden (1984), for example, and we refer to this book for full details. Here it suffices to state that the only non-zero components of $\underline{A}_0$ are

$$
\begin{aligned}
A_{0iiii} &= \lambda_i \frac{\partial \sigma_i}{\partial \lambda_i}, \\
A_{0iijj} &= \lambda_j \frac{\partial \sigma_i}{\partial \lambda_j} + \sigma_i \quad i \neq j, \\
A_{0ijij} &= \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 \quad i \neq j, \\
A_{0ijji} &= A_{0jiij} = \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 - \sigma_i \quad i \neq j,
\end{aligned}
\tag{2.6.6}
$$

where $i,j \in \{1,2,3\}$, and

$$
J \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} . \tag{2.6.7}
$$

In components equation (2.6.5) has the form

$$
\dot{\underline{S}}_{0ji} = A_{0jilk} L_{kl} \equiv A_{0jilk} \frac{\partial v_k}{\partial x_l} \tag{2.6.8}
$$

on noting (2.1.9).

For *incompressible materials*, differentiation of (2.4.15) with respect to t at fixed $\underline{X}$ and use of (2.1.6) and (2.1.12), followed by an update of reference configuration to B_t, yields the counterpart of (2.6.5), namely

$$
\dot{\underline{S}}_0 = \underline{A}_0 \, \underline{L} + p\underline{L} - \dot{p}\underline{I}, \tag{2.6.9}
$$

where $\underline{I}$ is the identity tensor. This is coupled with the rate form of the incompressibility condition:

$$\operatorname{tr}(\underline{L}) \equiv \operatorname{div} \underline{v} = 0, \tag{2.6.10}$$

as in (2.2.6).

In components

$$\dot{S}_{0ji} = A_{0jilk} \frac{\partial v_k}{\partial x_l} + p \frac{\partial v_j}{\partial x_i} - \dot{p}\, \delta_{ij}, \tag{2.6.11}$$

with

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{2.6.12}$$

For incompressible isotropic materials the components of $\underline{A}_0$ differ slightly from (2.6.6), and are given by

$$A_{0iijj} = \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j},$$

$$A_{0ijij} = \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 \qquad i \neq j, \tag{2.6.13}$$

$$A_{0ijji} = A_{0jiij} = A_{0ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} \quad i \neq j,$$

where

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \tag{2.6.14}$$

and $i,j \in \{1,2,3\}$.

For the special case in which $\lambda_i = \lambda_j$ for $i \neq j$ the formula (2.6.6) and (2.6.13) still hold except that in the limit $\lambda_i \longrightarrow \lambda_j$ A_{0ijij} is replaced by

$$A_{0ijij} = \tfrac{1}{2}\,(A_{0iiii} - A_{0iijj} + \sigma_i) \qquad (2.6.15)$$

for *compressible materials*, and

$$A_{0ijij} = \tfrac{1}{2}\,(A_{0iiii} - A_{0iijj} + \lambda_i \frac{\partial W}{\partial \lambda_i}) \qquad (2.6.16)$$

for *incompressible materials*.

The equations of motion (2.3.6) have component form

$$\frac{\partial}{\partial x_j} \dot{S}_{0ji} = \rho \ddot{v}_i \qquad i \in \{1,2,3\}$$

so, for compressible and incompressible materials respectively, equations (2.6.8), (2.6.11) and (2.6.12) yield

$$\frac{\partial}{\partial x_j}(A_{0jilk} \frac{\partial v_k}{\partial x_l}) = \rho \ddot{v}_i \qquad (2.6.17)$$

and

$$\frac{\partial}{\partial x_j}(A_{0jilk} \frac{\partial v_k}{\partial x_l}) + \frac{\partial v_j}{\partial x_i}\frac{\partial p}{\partial x_j} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{v}_i, \qquad (2.6.18)$$

the latter being coupled with (2.6.12).

When the underlying deformation from $B_r \longrightarrow B_t$ is homogeneous $\underline{A}_0$ and p are independent of $\underline{x}$ and (2.6.17) and (2.6.18) simplify to

$$A_{0jilk} \frac{\partial^2 v_k}{\partial x_j \partial x_l} = \rho \ddot{v}_i, \qquad (2.6.19)$$

$$A_{0jilk} \frac{\partial^2 v_k}{\partial x_j \partial x_l} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{v}_i \qquad (2.6.20)$$

respectively.

Finally in this section we note that the traction rate $\dot{\underline{S}}_0^T\, \underline{n}$ on a surface with unit normal $\underline{n}$ in the current configration B_t has components

$$\dot{S}_{0ji}\, n_j \equiv (A_{0jilk} \frac{\partial v_k}{\partial x_l})n_j \tag{2.6.21}$$

and

$$\dot{S}_{0ji}\, n_j \equiv (A_{0jilk} + p\, \delta_{jk}\, \delta_{il}) \frac{\partial v_k}{\partial x_l} n_j - \dot{p}\, n_i \tag{2.6.22}$$

for compressible and incompressible materials respectively.

2.7 Plane waves in an infinite medium

As a prelude to our discussion of surface waves we now consider the propagation of plane waves in an unbounded medium. For a plane wave propagating in the direction of the unit vector $\underline{n}$ with speed c we may write

$$\underline{v} = \underline{m}\, f\, (t - \frac{\underline{n}.\underline{x}}{c}) \tag{2.7.1}$$

and, additionally, for an incompressible material,

$$\dot{p} = \frac{q}{c} f'(t - \frac{\underline{n}.\underline{x}}{c}), \tag{2.7.2}$$

where q is a constant and $\underline{m}$ a constant unit vector. We refer to $\underline{m}$ as the unit *amplitude vector.*

For an incompressible material substitution of (2.7.1) into (2.6.10) yields the constraint

$$\underline{m} \,.\, \underline{n} = 0. \tag{2.7.3}$$

Substitution of (2.7.1) and (2.7.2) into (2.6.19) and (2.6.20) yields

$$A_{0jilk}\, n_j n_l\, f''(t - \frac{\underline{n}.\underline{x}}{c})\, m_k = \rho c^2\, m_i\, f''(t - \frac{\underline{n}.\underline{x}}{c})$$

and

$$A_{0jilk}\, n_j\, n_l\, f''(t - \frac{\underline{n}.\underline{x}}{c})\, m_k + q\, n_i\, f''(t - \frac{\underline{n}.\underline{x}}{c})$$

$$= \rho c^2\, m_i\, f''(t - \frac{\underline{n}.\underline{x}}{c})$$

respectively. On the assumption that f is a twice continuously differentiable function we deduce that

$$A_{0jilk}\, n_j\, n_l\, m_k = \rho c^2\, m_i \tag{2.7.4}$$

for a compressible material, and

$$A_{0jilk}\, n_j\, n_l\, m_k + q\, n_i = \rho c^2\, m_i, \quad m_i\, n_i = 0 \tag{2.7.5}$$

for an incompressible material.

We now introduce the notation $\underline{Q}(\underline{n})$ for the second-order tensor (dependent on $\underline{n}$) with components defined by

$$Q_{ik}(\underline{n}) = A_{0jilk}\, n_j\, n_l. \tag{2.7.6}$$

Then (2.7.4) may be written compactly in the form

$$\underline{Q}(\underline{n})\underline{m} = \rho c^2\, \underline{m}, \tag{2.7.7}$$

where, in view of definitions (2.6.4) and (2.7.6), $\underline{Q}(\underline{n})$ is *symmetric* for each $\underline{n}$.

This guarantees that the secular equation

$$\det[\ \underline{Q}(\underline{n}) - \rho c^2 \underline{I}\ ] = 0 \tag{2.7.8}$$

yields real eigenvalues ρc^2 for (2.7.7). However, for the existence of plane waves ρc^2 must be positive. This follows if the *strong ellipticity condition*

$$\mathrm{tr}\left[\ [\ \underline{A}_0\ (\underline{m} \otimes \underline{n})]\ (\underline{m} \otimes \underline{n})\right] \equiv [\ \underline{Q}(\underline{n})\underline{m}\ ]\ .\ \underline{m} > 0 \quad \text{all } \underline{m} \otimes \underline{n} \neq 0 \tag{2.7.9}$$

holds.

From (2.7.7) the wave speed c associated with direction of propagation $\underline{n}$ and the amplitude $\underline{m}$ is given by

$$\rho c^2 = [\ \underline{Q}(\underline{n})\underline{m}\ ].\ \underline{m} = A_{0jilk}\ n_j\ n_l\ m_i\ m_k. \tag{2.7.10}$$

Equation (2.7.10) applies for compressible materials. For incompressible materials, using the notation (2.7.6), equation (2.7.5), yields

$$\underline{Q}(\underline{n})\underline{m} + q\underline{n} = \rho c^2\ \underline{m}, \qquad \underline{m}.\underline{n} = 0. \tag{2.7.11}$$

Taking the dot product of this with $\underline{n}$ we deduce that

$$q = -[\ \underline{Q}(\underline{n})\underline{m}\ ].\underline{n},$$

so that (2.7.11) can be rewritten, analogously to (2.7.7), in the form

$$\underline{Q}^*(\underline{n})\underline{m} = \rho c^2 \underline{m}, \qquad \underline{m}.\underline{n} = 0, \tag{2.7.12}$$

where $\underline{Q}^*(\underline{n})$ is defined by

$$\underset{\sim}{Q}^*(\underset{\sim}{n}) = \underset{\sim}{Q}(\underset{\sim}{n}) - \underset{\sim}{n} \otimes [\ \underset{\sim}{Q}^T(\underset{\sim}{n})\underset{\sim}{n}\ ]. \tag{2.7.13}$$

In this case the wave speed is given by

$$\rho c^2 = [\ \underset{\sim}{Q}^*(\underset{\sim}{n})\underset{\sim}{m}\ ].\ \underset{\sim}{m} = [\ \underset{\sim}{Q}(\underset{\sim}{n})\underset{\sim}{m}\ ].\underset{\sim}{m}, \tag{2.7.14}$$

which is the same expression as (2.7.10) except that the constraint $\underset{\sim}{m}.\underset{\sim}{n} = 0$ must be satisfied.

An important distinction between $\underset{\sim}{Q}(\underset{\sim}{n})$ and $\underset{\sim}{Q}^*(\underset{\sim}{n})$ is that, whereas $\underset{\sim}{Q}(\underset{\sim}{n})$ is symmetric, $\underset{\sim}{Q}^*(\underset{\sim}{n})$ is not in general symmetric.

Plane waves for which $\underset{\sim}{m}.\underset{\sim}{n} = 0$ are said to be *transverse waves*, and the unit amplitude vector is then referred to as the *polarization vector*.

Plane waves for which $\underset{\sim}{m} = \underset{\sim}{n}$ (in a compressible material) are called *longitudinal waves*. In general, there is no guarantee that either longitudinal or transverse waves will exist for particular choices of the direction of propagation. However, if $\underset{\sim}{n}$ is along a principal axis of the underlying deformation then some simple results follow if $\underset{\sim}{m}$ is also along a principal direction. For future reference we now record these results.

First, for a compressible material, if $\underset{\sim}{n} = \underset{\sim}{v}^{(i)}$ and $\underset{\sim}{m} = \underset{\sim}{v}^{(i)}$, where $\underset{\sim}{v}^{(1)}$, $\underset{\sim}{v}^{(2)}$, $\underset{\sim}{v}^{(3)}$ denote the Eulerian principal axes, and c_{ij} denotes the associated wave speed, then from (2.6.6), (2.6.7), and (2.7.10) we obtain

$$\rho c_{ii}^2 = A_{0iiii} = \lambda_i \frac{\partial \sigma_i}{\partial \lambda_i} \qquad i \in \{1,2,3\} \tag{2.7.15}$$

or, equivalently,

$$\rho_r\, c_{ii}^2 = \lambda_i^2 \frac{\partial^2 W}{\partial \lambda_i^2} \qquad i \in \{1,2,3\} \tag{2.7.16}$$

and also

$$\rho c_{ij}^2 = A_{0ijij} = \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2}\, \lambda_i^2 \qquad i \neq j \tag{2.7.17}$$

Equation (2.7.17) is also valid for incompressible materials. We shall make use of the notation defined in (2.7.15)– (2.7.17) in later sections of this thesis.

Finally, for waves propagating in an unstrained material we note that longitudinal and transverse waves exist for *every* direction of propagation. This follows from the fact that the components of $\underline{A}_0$ reduce to

$$A_{0ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) \tag{2.7.18}$$

in B_r, where λ and μ are the Lamé moduli introduced in (2.5.18), and, for an incompressible material, to

$$A_{0ijkl} = \mu(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}). \tag{2.7.19}$$

If c_L and c_T denote the speeds of the propagation of longitudinal and transverse waves respectively in this special case then

$$\rho c_{ii}^2 = \rho c_L^2 = \lambda + 2\mu \qquad i \in \{1,2,3\}, \tag{2.7.20}$$

$$\rho c_{ij}^2 = \rho c_T^2 = \mu \qquad i \neq j \in \{1,2,3\}. \tag{2.7.21}$$

Basic references to work on plane waves in deformed elastic materials are the paper by Hayes and Rivlin (1961), which is concerned with isotropic materials possessing a strain-energy function, and the monograph by Truesdell and Noll (1965), which generalizes this to the case where the existence of a strain-energy function is not required.

2.8 Strong ellipticity condition

At this point we consider the strong ellipticity condition, given by (2.7.9), in more detail. We write it, first of all, in the form

$$\mathrm{tr}\left\{ \left[\underline{\mathcal{A}}_0(\underline{m}\otimes\underline{n})\right] (\underline{m}\otimes\underline{n}) \right\} \equiv \left[\underline{Q}(\underline{n})\underline{m} \right].\underline{m} > 0 \quad \text{for all } \underline{m}\otimes\underline{n} \neq 0 \tag{2.8.1}$$

or, in cartesian components,

$$\mathcal{A}_{0jilk}\, n_j\, n_l\, m_i\, m_k \equiv Q_{ik}(\underline{n})\, m_i\, m_k > 0, \tag{2.8.2}$$

recalling from (2.7.6) that

$$Q_{ik}(\underline{n}) = \mathcal{A}_{0jilk}\, n_j\, n_l. \tag{2.8.3}$$

For a compressible material the components of $\underline{\mathcal{A}}_0$ are given by (2.6.6); for an incompressible material the strong ellipticity condition also takes the form (2.8.1), but with the components of $\underline{\mathcal{A}}_0$ given by (2.6.13) and the constraint

$$\underline{m}.\underline{n} = 0 \tag{2.8.4}$$

imposed.

For a compressible material the components of $\underline{Q}(\underline{n})$ are given by

$$JQ_{ii} = \frac{\partial t_i}{\partial \lambda_i} N_i^2 + \left(\frac{\lambda_i t_i - \lambda_j t_j}{\lambda_i^2 - \lambda_j^2}\right)N_j^2 + \left(\frac{\lambda_i t_i - \lambda_k t_k}{\lambda_i^2 - \lambda_j^2}\right)N_k^2, \qquad (2.8.5)$$

where (i,j,k) is a cyclic permulation of (1,2,3), and

$$JQ_{ij} = \left(\frac{\partial t_i}{\partial \lambda_j} + \frac{\lambda_j t_i - \lambda_i t_j}{\lambda_i^2 - \lambda_j^2}\right) N_i N_j \qquad i \neq j \qquad (2.8.6)$$

where $N_i = \lambda_i n_i$ and n_1, n_2 and n_3 are components of $\underline{n}$ referred to the Eulerian principal axes.

From (2.8.2), it is easy to see that

$$J\left[\underline{Q}(\underline{n})\underline{m}\right].\ \underline{m} \equiv \sum_{i,j=1}^{3} \frac{\partial t_i}{\partial \lambda_j} N_i N_j m_i m_j$$

$$+ \frac{1}{4} \sum_{i \neq j} \left\{ \left[\frac{t_i + t_j}{\lambda_i + \lambda_j}\right](N_i m_j - N_j m_i)^2 \right.$$

$$\left. + \left[\frac{t_i - t_j}{\lambda_i - \lambda_j}\right](N_i m_j + N_j m_i)^2 \right\} > 0 \qquad (2.8.7)$$

Similarly, for an incompressible material the corresponding expression is

$$[\ \underline{Q}(\underline{n})\underline{m}\ ].\underline{m} \equiv \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial\lambda_i \partial\lambda_j} N_i\, N_j\, m_i\, m_j$$

$$+ \frac{1}{4} \sum_{i \neq j} \Bigg\{ \frac{\dfrac{\partial W}{\partial\lambda_i} + \dfrac{\partial W}{\partial\lambda_j}}{\lambda_i + \lambda_j} (N_i m_j - N_j m_i)^2$$

$$+ \frac{\dfrac{\partial W}{\partial\lambda_i} - \dfrac{\partial W}{\partial\lambda_j}}{\lambda_i - \lambda_j} (N_i m_j + N_j m_i)^2 \Bigg\} > 0 \qquad (2.8.8)$$

In general, necessary and sufficient condition for the strong ellipticity condition to hold are difficult to express independently of $\underline{m}$ and $\underline{n}$.

We therefore specialize the above to the two-dimensional situation to one in which $m_3 = n_3 = 0$, so that, for compressible materials equation (2.8.7) reduces to

$$\frac{\partial t_1}{\partial\lambda_1} N_1^2 m_1^2 + 2\, \frac{\partial t_1}{\partial\lambda_2} N_1 N_2 m_1 m_2 + \frac{\partial t_2}{\partial\lambda_2} N_2^2\, m_2^2$$

$$+ \frac{1}{2} \left\{ \frac{t_1 + t_2}{\lambda_1 + \lambda_2} (N_1 m_2 - N_2 m_1)^2 + \frac{t_1 - t_2}{\lambda_1 - \lambda_2} (N_1 m_2 + N_2 m_1)^2 \right\} > 0 \qquad (2.8.9)$$

Also, (2.8.8) reduces to

$$\frac{\partial^2 W}{\partial \lambda_1^2} N_1^2 m_1^2 + 2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} N_1 N_2 m_1 m_2 + \frac{\partial^2 W}{\partial \lambda_2^2} N_2^2 m_2^2$$

$$+ \tfrac{1}{2} \left\{ \frac{\dfrac{\partial W}{\partial \lambda_1} + \dfrac{\partial W}{\partial \lambda_2}}{\lambda_1 + \lambda_2} (N_1 m_2 - N_2 m_1)^2 + \frac{\dfrac{\partial W}{\partial \lambda_1} - \dfrac{\partial W}{\partial \lambda_2}}{\lambda_1 - \lambda_2} (N_1 m_2 + N_2 m_1)^2 \right\} > 0 \tag{2.8.10}$$

Equation (2.8.9) leads to necessary and sufficient conditions for the (two-dimensional) strong-ellipticity condition for a compressible material to hold: these are

$$\frac{\partial t_1}{\partial \lambda_1} > 0, \qquad \frac{\partial t_2}{\partial \lambda_2} > 0, \qquad \frac{\lambda_1 t_1 - \lambda_2 t_2}{\lambda_1^2 - \lambda_2^2} > 0,$$

$$\frac{\partial t_1}{\partial \lambda_1} \frac{\partial t_2}{\partial \lambda_2} + \left[\frac{\lambda_1 t_1 - \lambda_2 t_2}{\lambda_1^2 - \lambda_2^2} \right]^2 - \left[\frac{\partial t_1}{\partial \lambda_2} + \frac{\lambda_2 t_1 - \lambda_1 t_2}{\lambda_1^2 - \lambda_2^2} \right]^2$$

$$+ 2 \left[\frac{\lambda_1 t_1 - \lambda_2 t_2}{\lambda_1^2 - \lambda_2^2} \right] \left[\frac{\partial t_1}{\partial \lambda_1} \frac{\partial t_2}{\partial \lambda_2} \right]^{\frac{1}{2}} > 0 \tag{2.8.11}$$

or, equivalently,

$$A_{01111} > 0, \qquad A_{02222} > 0, \qquad A_{01212} > 0,$$

$$A_{01111}\, A_{02222} + (A_{01212})^2 - (A_{01122} + A_{02121})^2$$

$$+ 2A_{01212}\, (A_{01111}\, A_{02222})^{\frac{1}{2}} > 0, \qquad (2.8.12)$$

see, for example, Knowles and Sternberg (1977), Hill (1979).

Since $N_i = \lambda_i n_i$, equation (2.8.10) becomes

$$\lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} n_1^2\, m_1^2 + 2\,\lambda_1\,\lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} n_1 n_2 m_1 m_2 + 2\lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} n_1^2\, m_2^2$$

$$+ \lambda_1^2 \frac{\left[\lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2}\right]}{\left[\lambda_1^2 - \lambda_2^2\right]} n_1^2\, m_2^2 + 2\lambda_1\lambda_2 \frac{\left[\lambda_2 \frac{\partial W}{\partial \lambda_1} - \lambda_1 \frac{\partial W}{\partial \lambda_2}\right]}{\left[\lambda_1^2 - \lambda_2^2\right]} n_1 n_2 m_1 m_2$$

$$+ \lambda_2^2 \frac{\left[\lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_1 \frac{\partial W}{\partial \lambda_1}\right]}{\left[\lambda_1^2 - \lambda_2^2\right]} n_2^2\, m_1^2 > 0 \qquad (2.8.13)$$

On taking $m_1 = n_2 = \cos\theta$, $m_2 = -\, n_1 = \sin\theta$, equation (2.8.13) can be written as

$$A_{01212} \sin^4\theta + (A_{01111} + A_{02222} + 2A_{01122} - 2A_{01221}) \sin^2\theta\cos^2\theta$$

$$+ A_{02121}\cos^4\theta > 0, \qquad (2.8.14)$$

for all $\theta \in [\theta,\ 2\pi]$.

It follows that necessary and sufficient conditions for the (two-dimensional) strong-ellipticity condition for an incompressible material to hold are

$$A_{01212} > 0, \qquad A_{02121} > 0,$$

(2.8.15)

$$A_{01111} + A_{02222} - 2A_{01122} - 2A_{01221} > -2\sqrt{A_{01212}A_{02121}}.$$

2.9 Equations of linear elasticity

In terms of the infinitesimal strain tensor e_{ij} and the corresponding stress σ_{ij}, the constitutive law for a linear isotropic elastic material can be written

$$\sigma_{ij} = 2\mu\, e_{ij} + \lambda\, e_{kk}\, \delta_{ij}, \tag{2.9.1}$$

for a compressible material and

$$\sigma_{ij} = 2\mu\, e_{ij} - p\, \delta_{ij}, \tag{2.9.2}$$

for an incompressible material.

With the small-strain tensor components

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.9.3}$$

The equation of motion is

$$\sigma_{ij,j} = \rho\ddot{u}_i, \tag{2.9.4}$$

and the traction $\underline{t}$ per unit area on a surface with normal $\underline{n}$ is given by

$$t_i = \sigma_{ij}\, n_j. \tag{2.9.5}$$

On substituting for σ_{ij} from (2.9.1) and (2.9.2), the equations of motion in terms of displacement u_i are

$$\mu\, u_{i,jj} + (\lambda + \mu)\, u_{j,ij} = \rho \ddot{u}_i \tag{2.9.6}$$

for a compressible material, and

$$\mu(u_{i,jj} + u_{j,ij}) + p_{,i} = \rho \ddot{u}_i$$

with (2.9.7)

$$u_{j,j} = 0$$

for an incompressible material.

Chapter 3

Rayleigh, Stoneley and Love waves in the classical linear theory

In this Chapter we shall use the equations given in Section 2.9 to establish the well-known results on Rayleigh surface waves, Stoneley waves and Love waves for compressible materials in the classical linear theory. Also we obtain corresponding results for incompressible materials. This provides some background against which to set our subsequent work on waves in pre-stressed materials.

3.1 Analysis for an incompressible material

Consider the current position $\underline{x}$ of the particle $\underline{X}$, with a small displacement $\underline{u}$, such that

$$\underline{u} = \underline{x} - \underline{X}. \tag{3.1.1}$$

Then

$$\underline{x} = \underline{X} + \underline{u}, \tag{3.1.2}$$

or, in cartesian components,

$$\begin{aligned} x_1 &= X_1 + u_1, \\ x_2 &= X_2 + u_2, \\ x_3 &= X_3 + u_3, \end{aligned} \tag{3.1.3}$$

where in general u_1, u_2 and u_3 depend on x_1, x_2, x_3 and t.

Recall from Section 2.9 the stress tensor σ_{ij}, for an incompressible material

$$\sigma_{ij} = 2\mu\, e_{ij} - p\delta_{ij}, \tag{3.1.4}$$

with the small-strain tensor

$$e_{ij} = \tfrac{1}{2}\,(u_{i,j} + u_{j,i}). \tag{3.1.5}$$

On use of equations (3.1.4) and (3.1.5), we have, in full

$$
\begin{aligned}
\sigma_{11} &= 2\mu\, u_{1,1} - p, \\
\sigma_{22} &= 2\mu\, u_{2,2} - p, \\
\sigma_{33} &= 2\mu\, u_{3,3} - p, \\
\sigma_{12} &= \sigma_{21} = \mu(u_{1,2} + u_{2,1}), \\
\sigma_{13} &= \sigma_{31} = \mu(u_{1,3} + u_{3,1}), \\
\sigma_{23} &= \sigma_{32} = \mu(u_{2,3} + u_{3,2}).
\end{aligned}
\tag{3.1.6}
$$

With

$$u_{1,1} + u_{2,2} + u_{3,3} = 0. \tag{3.1.7}$$

Next, we take $u_3 \equiv 0$ and assume u_1, u_2 are independent of x_3. Then equation (3.1.7) becomes

$$u_{1,1} + u_{2,2} = 0, \tag{3.1.8}$$

and hence there exists a function $\psi(x_1,x_2,t)$ such that

$$u_1 = \frac{\partial\psi}{\partial x_2} = \psi_{,2}, \qquad u_2 = -\frac{\partial\psi}{\partial x_1} = -\psi_{,1} \tag{3.1.9}$$

Also, equations (3.1.6) become

$$
\begin{aligned}
\sigma_{11} &= 2\mu\, u_{1,1} - p, \\
\sigma_{22} &= 2\mu\, u_{2,2} - p, \\
\sigma_{33} &= -p, \\
\sigma_{12} &= \sigma_{21} = \mu(u_{1,2} + u_{2,1}),
\end{aligned}
\tag{3.1.10}
$$

where p is independent of x_3.

From (2.9.3), the equations of motion then simplify to

$$\begin{aligned} \sigma_{11,1} + \sigma_{21,2} &= \rho \ddot{u}_1, \\ \sigma_{12,1} + \sigma_{22,2} &= \rho \ddot{u}_2. \end{aligned} \tag{3.1.11}$$

From equations (3.1.10), we have

$$\begin{aligned} \sigma_{11,1} &= 2\mu\, u_{1,11} - p_{,1}, \\ \sigma_{21,2} &= \mu(u_{1,22} + u_{2,12}), \\ \sigma_{12,1} &= \mu(u_{1,12} + u_{2,11}), \\ \sigma_{22,2} &= 2\mu\, u_{2,22} - p_{,2}, \end{aligned} \tag{3.1.12}$$

or, in terms of ψ,

$$\begin{aligned} \sigma_{11,1} &= 2\mu\, \psi_{,112} - p_{,1}, \\ \sigma_{21,2} &= \mu(\psi_{,222} - \psi_{,112}), \\ \sigma_{12,1} &= \mu(\psi_{,122} - \psi_{,111}), \\ \sigma_{22,2} &= -\,(2\mu\, \psi_{,122} + p_{,2}). \end{aligned} \tag{3.1.13}$$

On use of (3.1.12) in (3.1.11), we have

$$\begin{aligned} 2\mu\, u_{1,11} + \mu(u_{1,22} + u_{2,12}) - p_{,1} &= \rho \ddot{u}_1, \\ 2\mu\, u_{2,22} + \mu(u_{1,12} + u_{2,11}) - p_{,2} &= \rho \ddot{u}_2, \end{aligned} \tag{3.1.14}$$

or, in terms of ψ,

$$\begin{aligned} \mu\, \psi_{,112} + \mu\, \psi_{,222} - p_{,1} &= \rho \ddot{\psi}_{,2}, \\ \mu\, \psi_{,122} - \mu\, \psi_{,111} - p_{,2} &= -\, \rho \ddot{\psi}_{,1}. \end{aligned} \tag{3.1.15}$$

To eliminate p we must differentiate equations (3.1.15) with respect to x_2 and x_1 respectively and the subtraction of the two equations yields

$$\mu(\psi_{,1111} + \psi_{,2222}) + 2\mu\, \psi_{,1122} - \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) = 0. \qquad (3.1.16)$$

We now choose axes so that $\underline{u}$ corresponds to a wave propagating along the x_1-axis, and we take $\psi(x_1, x_2, t)$ to have the form

$$\psi = f(x_2) e^{i\omega(t - x_1/c)}. \qquad (3.1.17)$$

This represents a wave propagating with (a constant) wave speed c in the x_1-direction. ω is the angular frequency.

We also assume that the x_2 variation of ψ is of the form e^{-skx_2}, where $k = {}^{\omega}/_{c}$ is the wave number. Then (3.1.16) gives

$$\mu\, s^4 - (2\mu - \rho c^2)s^2 + \mu - \rho c^2 = 0. \qquad (3.1.18)$$

Equation (3.1.18) is a quadratic equation for s^2. Suppose it has roots s_1^2 and s_2^2. Then

$$s_1^2 + s_2^2 = 2 - \frac{\rho c^2}{\mu}, \qquad s_1^2 s_2^2 = 1 - \frac{\rho c^2}{\mu}. \qquad (3.1.19)$$

In fact, the roots s_1^2 , s_2^2 are 1 and $1 - \dfrac{\rho c^2}{\mu}$.

We also assume that p has the same time and spatial dependence as ψ. It follows from (3.1.15) that

$$ikp = k^2(\mu - \rho c^2)\, \psi_{,2} - \mu\, \psi_{,222}. \qquad (3.1.20)$$

3.2 Rayleigh waves

Here we consider the half-space $x_2 > 0$ occupied by the material with traction-free boundary $x_2 = 0$. We therefore wish to solve the equations of motion for the given boundary conditions

$$\sigma_{2i} = 0, \qquad \text{on } x_2 = 0,$$

that is

$$\left.\begin{aligned} &\mu(u_{1,2} + u_{2,1}) = 0, \\ &2\mu\, u_{2,2} - p = 0, \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (3.2.1)$$

or, in terms of ψ,

$$\left.\begin{aligned} &\mu(\psi_{,22} - \psi_{,11}) = 0, \\ &2\mu\, \psi_{,12} + p = 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (3.2.2)$$

On use of equation (3.1.20), equations (3.2.2) become

$$\left.\begin{aligned} &\mu(\psi_{,22} - \psi_{,11}) = 0, \\ &(3\mu - \rho c^2)k^2\, \psi_{,2} - \mu\, \psi_{,222} = 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (3.2.3)$$

For surface waves we must have a solution for ψ in equation (3.1.16) which decays when $x_2 \longrightarrow \infty$ and which satisfies the boundary conditions (3.2.3) at the surface $x_2 = 0$. Hence, in (3.1.17) and (3.1.18), if a solution of this type is to exist, we should be able to find numbers s_1 and s_2, with positive real parts, and the solution for ψ may be written as

$$\psi = (A_1 e^{-s_2 k x_2} + A_2\, e^{-s_2 k x_2})\, e^{i\omega(t - x_1/c)}. \qquad (3.3.4)$$

On substitution of this expression into (3.2.3) we have

$$(s_1{}^2 + 1)A_1 + (s_2{}^2 + 1)A_2 = 0,$$

$$s_1(\mu s_1{}^2 - 3\mu + \rho c^2)A_1 + s_2(\mu s_2{}^2 - 3\mu + \rho c^2)A_2 = 0. \tag{3.2.5}$$

For these equations to have non-trivial solutions we must have

$$\begin{vmatrix} s_1{}^2 + 1 & s_2{}^2 + 1 \\ s_1(\mu s_1{}^2 - 3\mu + \rho c^2) & s_2(\mu s_2{}^2 - 3\mu + \rho c^2) \end{vmatrix} = 0,$$

i.e.

$$s_2(s_1{}^2 + 1)(\mu s_2{}^2 - 3\mu + \rho c^2) - s_1(s_2{}^2 + 1)(\mu s_1{}^2 - 3\mu + \rho c^2) = 0.$$

On rearrangement this becomes

$$(s_1 - s_2)\left[s_1 s_2(\rho c^2 - 4\mu) - \mu(s_1^2 s_2^2 + s_1^2 + s_2^2) + 3\mu - \rho c^2\right] = 0. \tag{3.2.6}$$

Assuming $s_1 \neq s_2$ this reduces to

$$s_1 s_2(\rho c^2 - 4\mu) - \mu(s_1^2\, s_2^2 + s_1^2 + s_2^2) + 3\mu - \rho c^2 = 0. \tag{3.2.7}$$

Substituting (3.1.19) into (3.2.7), the secular equation becomes

$$(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}}(\rho c^2 - 4\mu) - \mu(3 - 2\frac{\rho c^2}{\mu}) + 3\mu - \rho c^2 = 0,$$

i.e.

$$\mu(1- \frac{\rho c^2}{\mu})^{\frac{1}{2}} (\frac{\rho c^2}{\mu} - 4)- \mu(3 -2\frac{\rho c^2}{\mu})+ \mu(3 - \frac{\rho c^2}{\mu}) = 0,$$

on setting $x = \frac{\rho c^2}{\mu}$, this becomes

$$(1 - x)^{\frac{1}{2}}(4 - x) - x = 0,$$

which, on squaring, gives the well-known result for Rayleigh waves, namely

$$x^2 = (1 - x)(4 - x)^2.$$

i.e.

$$x^3 - 8x^2 + 24x - 16 = 0. \tag{3.2.8}$$

The only positive real solution of (3.2.8) is $x = x_0 \cong 0.912622$. Thus there exists a Rayleigh wave with speed c given by

$$\rho c^2 = x_0\mu.$$

In equation (3.2.7), we have assumed that $s_1 \neq s_2$. We now consider the special case in which $s_1=s_2=s$, so the general solution (3.2.3) becomes

$$\psi = (A + Bx_2)\, e^{skx_2-i\omega(t - x_1/c)}. \tag{3.2.9}$$

Using this in the boundary condition (3.2.4), we have

$$\begin{aligned} &(s^2 + 1)A - 2sB = 0, \\ &s(\mu s^2 - 3\mu + \rho c^2)A + \{3\mu(1 - s^2) - \rho c^2\}B =0. \end{aligned} \tag{3.2.10}$$

For these equations to have non-trivial solutions we have

$$\begin{vmatrix} s^2 + 1 & -2s \\ s(\mu s^2 - 3\mu + \rho c^2) & 3\mu(1 - s^2) - \rho c^2 \end{vmatrix} = 0$$

i.e.

$$(s^2 + 1)\left[3\mu\left[1 - s^2\right] - \rho c^2\right] + 2s^2\left[\mu s^2 - 3\mu + \rho c^2\right] = 0,$$

which gives

$$\mu s^4 + (6\mu - \rho c^2)s^2 - 3\mu + \rho c^2 = 0. \qquad (3.2.11)$$

Equation (3.2.11) is a quadratic for s^2. Therefore, the sum of roots for this case is

$$2s^2 = -\frac{(6\mu - \rho c^2)}{\mu} = -\left(6 - \frac{\rho c^2}{\mu}\right), \qquad (3.2.12)$$

since we assumed $s_1 = s_2 = s$, equation $(3.1.19)_1$ becomes

$$2s^2 = 2 - \frac{\rho c^2}{\mu} \qquad (3.2.13)$$

From (3.2.12) and (3.2.13), we see s is pure imaginary, that means there is no decay when $x_2 \longrightarrow \infty$, and therefore this case cannot arise and we conclude that A = B = 0.

3.3 Stoneley waves

Here we consider waves propagating along the interface $x_2 = 0$ between two half-spaces of different materials with moduli of rigidity μ, μ^* in the $x_2 < 0$ and $x_2 > 0$ half-spaces respectively. Let σ_{ij} and σ^*_{ij} be the stress components in the two half-spaces and p, p^*, u, u^* are the corresponding hydrostatic pressures and displacements. Then

$$\begin{aligned}
\sigma_{11} &= 2\mu\, u_{1,1} - p, \\
\sigma_{22} &= 2\mu\, u_{2,2} - p, \\
\sigma_{12} &= \sigma_{21} = \mu(u_{1,2} + u_{2,1}), \\
\sigma^*_{11} &= 2\mu^*\, u^*_{1,1} - p^*, \\
\sigma^*_{22} &= 2\mu^*\, u^*_{2,2} - p^*, \\
\sigma^*_{12} &= \sigma^*_{21} = \mu^*(u^*_{1,2} + u^*_{2,1}).
\end{aligned} \tag{3.3.1}$$

From (3.1.18), we have

$$\mu\, s^4 - (2\mu - \rho c^2)\, s^2 + \mu - \rho c^2 = 0,$$

for the half-space $x_2 < 0$.

Similarly

$$\mu^*\, s^{*4} - (2\mu^* - \rho^* c^2)\, s^{*2} + \mu^* - \rho^* c^2 = 0, \tag{3.3.2}$$

for the half-space $x_2 > 0$.

Also, from (3.1.19), we have

$$s_1^2 + s_2^2 = 2 - \frac{\rho c^2}{\mu}, \qquad s_1^2\, s_2^2 = 1 - \frac{\rho c^2}{\mu}.$$

Similarly (3.3.3)

$$s_1^{*2} + s_2^{*2} = 2 - \frac{\rho^* c^2}{\mu^*}, \qquad s_1^{*2}\, s_2^{*2} = 1 - \frac{\rho^* c^2}{\mu^*}$$

Also, from (3.1.20), we get

$$ikp = k^2(\mu - \rho c^2)\, \psi_{,2} - \mu\, \psi_{,222},$$

and similarly (3.3.4)

$$ikp^* = k^2(\mu^* - \rho^* c^2)\, \psi^*_{,2} - \mu^*\, \psi^*_{,222},$$

the superscript * in all cases indicating quantities in the region $x_2 > 0$. The half-spaces are bonded together and the boundary conditions are

$$\underline{u} = \underline{u}^*, \qquad \sigma_{2i} = \sigma^*_{2i} \qquad \text{on } x_2 = 0,$$

that is

$$\begin{aligned} u_1 - u^*_1 &= 0, \\ u_2 - u^*_2 &= 0, \\ \sigma_{21} - \sigma^*_{21} &= 0, \\ \sigma_{22} - \sigma^*_{22} &= 0. \end{aligned} \tag{3.3.5}$$

On use of equations (3.3.1), we have

$$\begin{aligned} u_1 - u^*_1 &= 0, \\ u_2 - u^*_2 &= 0, \\ \mu(u_{1,2} + u_{2,1}) - \mu^*(u^*_{1,2} + u^*_{2,1}) &= 0, \\ 2\mu\, u_{2,2} - p - 2\mu^*\, u^*_{2,2} + p^* &= 0. \end{aligned} \qquad \text{on } x_2 = 0 \tag{3.3.6}$$

From equations (3.1.9) and the corresponding equations for the starred quantities, the above condition can be written in terms of ψ as

$$
\left.
\begin{aligned}
&\psi_{,2} - \psi^*_{,2} = 0,\\
&\psi^*_{,1} - \psi_{,1} = 0,\\
&\mu(\psi_{,22} - \psi_{,11}) - \mu^*(\psi^*_{,22} - \psi^*_{,11}) = 0,\\
&2\mu^* \psi^*_{,12} + p^* - (2\mu \psi_{,12} + p) = 0.
\end{aligned}
\right\} \text{ on } x_2 = 0 \qquad (3.3.7)
$$

Substituting (3.3.4) in (3.3.7), the boundary conditions become

$$
\left.
\begin{aligned}
&\psi_{,2} - \psi^*_{,2} = 0,\\
&\psi^*_{,1} - \psi_{,1} = 0,\\
&\mu(\psi_{,22} - \psi_{,11}) - \mu^*(\psi^*_{,22} - \psi^*_{,11}) = 0,
\end{aligned}
\right\} \text{ on } x_2 = 0 \qquad (3.3.8)
$$

$$
\begin{aligned}
2ik\mu^* \psi^*_{,12} + k^2(\mu^* - \rho^* c^2)\, \psi^*_{,2} - \mu^* \psi^*_{,222} - 2ik\, \mu\, \psi_{,12}&\\
- k^2(\mu k^2 - \rho c^2)\, \psi_{,2} + \mu\, \psi_{,222} &= 0.
\end{aligned}
$$

Next, suppose that the general solutions are given by

$$
\begin{aligned}
\psi &= (A\, e^{s_1 k x_2} + B\, e^{s_2 k x_2})\, e^{i\omega(t - x_1/c)} && \text{for } x_2 < 0,\\
& && (3.3.9)\\
\psi^* &= (A^*\, e^{-s_1^* k x_2} + B^*\, e^{-s_1^* k x_2})\, e^{i\omega(t - x_1/c)} && \text{for } x_2 > 0,
\end{aligned}
$$

where s_1, s_2 are the solutions of (3.1.18) with positive real part and s_1^*, s_2^* the solutions of (3.3.2) with positive real part, this ensuring that the solutions (3.3.9) decay as $x_2 \longrightarrow \pm \infty$.

Substituting (3.3.9) into (3.3.8), we have

$$
\begin{aligned}
& s_1\, A + s_2\, B + s_1^*\, A^* + s_2^*\, B^* = 0, \\
& A + B - A^* - B^* = 0, \\
& \mu(s_1^2 + 1)A + \mu(s_2^2 + 1)B - \mu(s_1^{*2} + 1)A^* - \mu^*(s_2^{*2} + 1)B^* = 0, \\
& s_1(\mu s_1^2 - 3\mu + \rho c^2)A + s_2(\mu s_2^2 - 3\mu + \rho c^2)B + s_1^*(\mu^* s_1^{*2} - 3\mu^* + \rho^* c^2)A^* \\
& \qquad + s_2^*(\mu^* s_2^{*2} - 3\mu^* + \rho^* c^2)B^* = 0.
\end{aligned}
\tag{3.3.10}
$$

For these equations to have non-trivial solutions, we must have

$$
\begin{vmatrix}
s_1 & s_2 & s_1^* & s_2^* \\
1 & 1 & -1 & -1 \\
\mu(s_1^2 + 1) & \mu(s_2^2 + 1) & -\mu^*(s_1^{*2} + 1) & -\mu^*(s_2^{*2} + 1) \\
\begin{matrix} s_1(\mu s_1^2 - 3\mu \\ + \rho c^2) \end{matrix} & \begin{matrix} s_2(\mu s_2^2 - 3\mu \\ + \rho c^2) \end{matrix} & \begin{matrix} s_1^*(\mu^* s_1^{*2} - 3\mu^* \\ + \rho^* c^2) \end{matrix} & \begin{matrix} s_2^*(\mu^* s_2^{*2} - 3\mu^* \\ + \rho^* c^2) \end{matrix}
\end{vmatrix} = 0
$$

On rearrangement this becomes

$$(s_1 - s_2)(s_1^* - s_2^*)\Bigg[\mu^{*2}\Big\{s_1^{*2}s_2^{*2} + s_1^{*2} + s_2^{*2}\Big\} + s_1^* s_2^*\ \mu^*\big(4\mu^* - \rho^* c^2\big)$$

$$- 3\mu^{*2} + \mu^*\rho^* c^2$$

$$-\mu^2\Big\{s_1^2 s_2^2 + s_1^2 + s_2^2\Big\} - s_1 s_2\ \mu\big(4\mu - \rho c^2\big) + 3\mu^2 - \mu\ \rho c^2$$

$$+ \mu\mu^*\ s_1^* s_2^*\ (s_1 + s_2)(s_1^* + s_2^*) + \mu\mu^* s_1 s_2 (s_1 + s_2)(s_1^* + s_2^*)$$

$$+ \mu^*\ (s_1^* s_2^* - 1)\ \Big\{3\mu - \mu\ \big(s_1^2 + s_2^2 + s_1 s_2\ \big) - \rho c^2\Big\}$$

$$+ \mu\ (1 - s_1 s_2\ )\ \Big\{\mu^*\big(s_1^{*2} + s_2^{*2} + s_1^*\ s_2^*\big) - 3\mu^* + \rho^* c^2\ \Big\}\Bigg] = 0$$

(3.3.11)

As for Rayleigh wave the case of $s_1 = s_2$ does not lead to the existence of waves. We therefore assume $s_1 \neq s_2$, $s_1^* \neq s_2^*$, and the above equation reduces to

$$\mu^{*2}\ \Big\{\ s_1^{*2}s_2^{*2} + s_1^{*2} + s_2^{*2} + s_1^* s_2^*\ (4 - \frac{\rho^* c^2}{\mu^*}) - (3 - \frac{\rho^* c^2}{\mu^*})\ \Big\}$$

$$+ \mu^2\Big\{s_1^2\ s_2^2 + s_1^2 + s_2^2 + s_1^2 s_2^2\ (4 - \frac{\rho c^2}{\mu}) - (3 - \frac{\rho c^2}{\mu})\Big\}$$

$$+ \mu\ \mu^*\ \Big[\ (s_1 + s_2)\ (s_1^* + s_2^*)\ (s_1 s_2 + s_1^* s_2^*)$$

$$+ (s_1^* s_2^* - 1)\Big\{(3 - \frac{\rho c^2}{\mu}) - (s_1^2 + s_2^2 + s_1 s_2)\Big\}$$

$$+ (1 - s_1 s_2)\Big\{\ (s_1^{*2} + s_2^{*2} + s_1 s_2) - (3 - \frac{\rho^* c^2}{\mu^*})\Big\}\Big] = 0. \quad (3.3.12)$$

On use of equations (3.3.3), the secular equation becomes

$$\mu 2\left[(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} (4 - \frac{\rho c^2}{\mu}) - \frac{\rho c 2}{\mu}\right]$$

$$+ \mu \mu^*\Bigg[\left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} - 1\right\}\left\{(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - 1\right\}$$

$$+ \left\{1 - (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}}\right\}\left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} - 1\right\} - \left\{2 - \frac{\rho c^2}{\mu} + 2(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

$$\left\{2 - \frac{\rho\ c^2}{\mu^*} + 2(1 - \frac{\rho\ c^2}{\mu^*})^{\frac{1}{2}}\right\} \left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} + (1 - \frac{\rho c^2}{\mu^*})^{\frac{1}{2}}\right\}\Bigg]$$

$$+ \mu^{*2} \left[(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} (4 - \frac{\rho^* c^2}{\mu^*}) - \frac{\rho^* c^2}{\mu^*} \right] = 0. \qquad (3.3.13)$$

This equation (3.3.13) can be written as

$$(4 - \frac{\rho c^2}{\mu})(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} - \frac{\rho c^2}{\mu}$$

$$- \frac{\mu^*}{\mu} \Bigg[2 \left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} + (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - (1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}}(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - 1\right\}$$

$$- \left\{ 2 - \frac{\rho c^2}{\mu} + 2 (1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} \right\}^{\frac{1}{2}}\left\{ 2 - \frac{\rho^* c^2}{\mu^*} + 2 (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\left\{ (1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}}\right\}\Bigg]$$

$$+ (\frac{\mu^*}{\mu})^2 \left[(4 - \frac{\rho^* c^2}{\mu^*})(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - \frac{\rho^* c^2}{\mu^*}\right] = 0. \qquad (3.3.14)$$

This is the secular equation for the propagation of Stoneley waves at the plane interface between two incompressible linearly isotropic elastic materials.

The Rayleigh secular equation which was given in the last section can be obtained from this equation by taking $\mu^* \longrightarrow 0$, namely

$$(4 - \frac{\rho c^2}{\mu})^2 (1 - \frac{\rho c^2}{\mu}) = (\frac{\rho c^2}{\mu})^2.$$

i.e.

$$(\frac{\rho c^2}{\mu})^3 - 8\,(\frac{\rho c^2}{\mu})^2 + 24\,(\frac{\rho c^2}{\mu}) - 16 = 0.$$

Finally in this section we present some numerical results based on the solution of (3.3.14). First of all set $\alpha = \mu^*/\mu$, $\beta = \rho^*/\rho$, $\eta = \rho c^2/\mu$. Then (3.3.14) can be written as

$$\begin{aligned}
&(4 - \eta)(1 - \eta)^{\frac{1}{2}} - \eta \\
&- \alpha \Big[\, 2 \Big\{ (1 - \eta)^{\frac{1}{2}} + (1 - \frac{\beta\eta}{\alpha})^{\frac{1}{2}} - (1 - \eta)^{\frac{1}{2}} (1 - \frac{\beta\eta}{\alpha})^{\frac{1}{2}} - 1 \Big\} \\
&\quad - \Big\{ 2 - \eta + 2(1 - \eta)^{\frac{1}{2}} \Big\}^{\frac{1}{2}} \Big\{ 2 - \frac{\beta\eta}{\alpha} + 2(1 - \frac{\beta\eta}{\alpha})^{\frac{1}{2}} \Big\}^{\frac{1}{2}} \\
&\qquad \Big\{ (1 - \eta)^{\frac{1}{2}} + (1 - \frac{\beta\eta}{\alpha})^{\frac{1}{2}} \Big\} \Big] \\
&+ \alpha^2 \Big[(4 - \frac{\beta\eta}{\alpha})(1 - \frac{\beta\eta}{\alpha})^{\frac{1}{2}} - \frac{\beta\eta}{\alpha} \Big] = 0. \qquad (3.3.15)
\end{aligned}$$

We solve equation (3.3.15) for η for given values of the material constants α and β. The results are plotted for η as a function of $1/\alpha$ in Figure 3.1 for a number of different values of β. Note, in particular, that $\eta \longrightarrow 0$ (i.e. $c \longrightarrow 0$) as $\alpha \longrightarrow 1$ (i.e. $\mu^* \longrightarrow \mu$) in which case the two materials are indistinguishable when $\beta = 1$. On the other hand when $\mu^*/\mu \longrightarrow 0$ the result (3.2.8) for Rayleigh waves is recovered, as we mentioned above.

Note that the inequalities

$$\eta < 1, \qquad \frac{\beta\eta}{\alpha} < 1,$$

must be satisfied for the equation to have real solutions. For the range of values of β and α considered we have $\beta/\alpha \leq 1$ so the latter inequality above follows from the former.

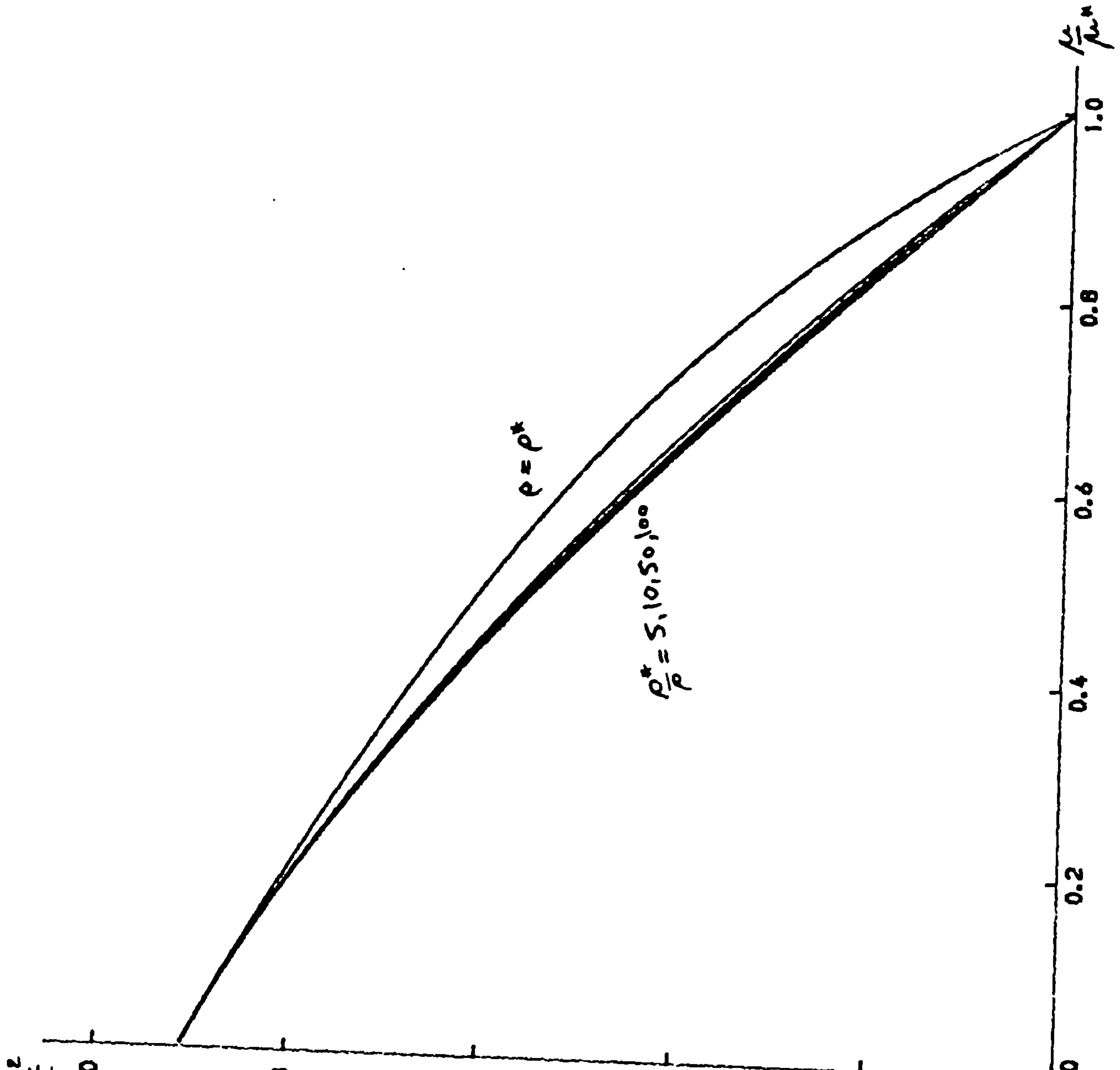


Figure 3.1

3.4 Love waves

We consider a half-space defined by $x_2 < 0$ on which there is a layer of different material of thickness h with boundaries $x_2 = 0$ and $x_2 = h$. Let μ and μ^* be the modulus of rigidity in the half-space and layer respectively.

Now suppose the boundary conditions are given by

$$\begin{aligned} \sigma_{2i} &= 0 \qquad && \text{on } x_2 = h, \\ \underline{u} = \underline{u}^*, \quad \sigma^*_{2i} &= \sigma_{2i} \qquad && \text{on } x_2 = 0. \end{aligned} \tag{3.4.1}$$

We assume that

$$\begin{aligned} \underline{u} &= (0,\ 0,\ u_3) = (0,\ 0,\ A\ e^{-skx_2 + i\omega(t - x_1/c)}), \\ \underline{u}^* &= (0,\ 0,\ u^*_3) = (0,\ 0,\ f(x_2)\ e^{i\omega(t - x_1/c)}), \end{aligned} \tag{3.4.2}$$

where

$$f(x_2) = (A'\cos s^*kx_2 + A''\sin s^*kx_2). \tag{3.4.3}$$

From the equation of motion (2.9.3), we have

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= \rho\ddot{u}_1, \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} &= \rho\ddot{u}_2, \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} &= \rho\ddot{u}_3, \end{aligned} \tag{3.4.4}$$

On use of equations (3.1.6) and (3.4.2) equations (3.4.4) lead to

$$p_{,1} = 0,$$

$$p_{,2} = 0, \tag{3.4.5}$$

$$\mu(u_{3,11} + u_{3,22}) = \rho\ddot{u}_3,$$

for the half-space, and

$$p^*_{,1} = 0,$$

$$p^*_{,2} = 0, \tag{3.4.6}$$

$$\mu^*(u^*_{3,11} + u^*_{3,22}) = \rho\ddot{u}^*_2,$$

for the layer.

Substituting (3.4.2) into $(3.4.5)_3$, we have

$$\mu(s^2 - 1) = -\rho c^2, \tag{3.4.7}$$

and hence

$$s^2 = (1 - \frac{\rho c^2}{\mu}). \tag{3.4.8}$$

Similarly

$$s^{*2} = (1 - \frac{\rho^* c^2}{\mu^*}) \tag{3.4.9}$$

From equations (3.1.6), we have

$$\sigma_{32} = \mu(u_{2,3} + u_{3,2}) = \mu\ u_{3,2}. \tag{3.4.10}$$

The boundary conditions (3.4.1), give $u^*_{3,2} = 0$ on $x_2 = h$ and then $(3.4.1)_2$ leads to

$$A' \sin s^* h k - A'' \cos s^* h k = 0. \tag{3.4.11}$$

Also, $(3.4.1)_2$ leads to

$$A = A', \tag{3.4.12}$$

and hence

$$s\ \mu\ A = s^*\ \mu^*\ A''. \tag{3.4.13}$$

Next, on use of equations (3.4.11), (3.4.12) and (3.4.13), we get the secular equation

$$\frac{s^*\mu^*}{s\ \mu} = \cot\ (s^*\ h\ k),$$

i.e.

$$\tan\ (s^*\ h\ k) = \frac{s\ \mu}{s^*\mu^*}\ . \tag{3.4.14}$$

Recall equation (2.7.21),

$$\mu = \rho c_T^2.$$

The secular equation (3.4.14) becomes

$$\tan s^*hk = \frac{s\rho c_T^2}{s^*\rho^* c_T^{*2}}, \tag{3.4.15}$$

where

$$\rho^* c^*{}_T{}^2 < \rho c^2 < \rho c_T{}^2. \tag{3.4.16}$$

Also, equations (3.4.8) and (3.4.9) become

$$s^2 = (1 - \frac{c^2}{c_T^2}),$$

(3.4.17)

$$s^{*2} = (\frac{c^2}{c_T^{*2}} - 1).$$

Substituting (3.4.17) into (3.4.15), we have

$$\tan\left[kh \left(\frac{c^2}{c_T^{*2}} - 1\right)^{\frac{1}{2}} \right] - \frac{\mu}{\mu^*} \frac{(1 - c^2/c_T^2)}{(c^2/c^{*2} - 1)} = 0. \qquad (3.4.18)$$

This equation is the well-known dispersion relation for Love waves in classical linear theory . See, for example, Achenbach (1984).

3.5 Analysis for a compressible materials

For a compressible material, the stress tensor σ_{ij} is given by (2.9.1), namely

$$\begin{aligned} \sigma_{ij} &= \lambda\, e_{kk}\, \delta_{ij} + 2\mu e_{ij}, \\ e_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}). \end{aligned} \qquad (3.5.1)$$

If i = j, $(3.5.1)_1$ gives

$$\sigma_{ii} = (3\lambda + 2\mu)e_{ii}, \qquad (3.5.2)$$

with summation over i.

On use of equations(3.5.1), we get

$$\begin{aligned} \sigma_{11} &= \lambda(u_{1,1} + u_{2,2} + u_{3,3}) + 2\mu u_{1,1}, \\ \sigma_{22} &= \lambda(u_{1,1} + u_{2,2} + u_{3,3}) + 2\mu u_{2,2}, \\ \sigma_{33} &= \lambda(u_{1,1} + u_{2,2} + u_{3,3}) + 2\mu u_{3,3}, \\ \sigma_{12} &= \sigma_{21} = \mu(u_{1,2} + u_{2,1}), \\ \sigma_{31} &= \sigma_{31} = \mu(u_{1,3} + u_{3,1}), \\ \sigma_{23} &= \sigma_{32} = \mu(u_{2,3} + u_{3,2}). \end{aligned} \qquad (3.5.3)$$

Next, if we take $u_3 \equiv 0$ and assume u_1 and u_2 are independent of x_3, the above equations reduce to

$$
\begin{aligned}
\sigma_{11} &= \lambda(u_{1,1} + u_{2,2}) + 2\mu u_{1,1}, \\
\sigma_{22} &= \lambda(u_{1,1} + u_{2,2}) + 2\mu u_{2,2}, \\
\sigma_{33} &= \lambda(u_{1,1} + u_{2,2}), \\
\sigma_{12} &= \sigma_{21} = \mu(u_{1,2} + u_{2,1}).
\end{aligned} \tag{3.5.4}
$$

The equation of motion is given by

$$\sigma_{ij,j} = \rho \ddot{u}_i,$$

from this equation, when $i = 1$ and $j = 2$, we have

$$
\begin{aligned}
\sigma_{11,1} + \sigma_{12,2} &= \rho \ddot{u}_1, \\
\sigma_{21,1} + \sigma_{22,2} &= \rho \ddot{u}_2
\end{aligned} \tag{3.5.5}
$$

Equations (3.5.4) give

$$
\begin{aligned}
\sigma_{11,1} &= \lambda(u_{1,11} + u_{2,12}) + 2\mu\, u_{11,1}, \\
\sigma_{22,2} &= \lambda(u_{1,12} + u_{2,22}) + 2\mu\, u_{22,2}, \\
\sigma_{12,2} &= \mu(u_{1,22} + u_{2,12}) \\
\sigma_{21,1} &= \mu(u_{1,12} + u_{2,11}).
\end{aligned} \tag{3.5.6}
$$

On use of equations (3.5.6) in (3.5.5), we get

$$
\begin{aligned}
\lambda(u_{1,11} + u_{2,12}) + 2\mu\, u_{1,11} + \mu(u_{1,22} + u_{2,12}) &= \rho \ddot{u}_1, \\
\lambda(u_{1,12} + u_{2,22}) + 2\mu\, u_{2,22} + \mu(u_{1,12} + u_{2,22}) &= \rho \ddot{u}_2.
\end{aligned} \tag{3.5.7}
$$

Assume u_1 and u_2 are given by

$$u_1 = A_1 \, e^{skx_2 + ix_1k - i\omega t},$$
$$\tag{3.5.8}$$
$$u_2 = A_2 \, e^{skx_2 + ix_1k - i\omega t}.$$

Substituting (3.5.8) into (3.5.7), we have

$$\{\mu s^2 + \rho c^2 - (\lambda + 2\mu)\}A_1 - is(\lambda + \mu)A_2 = 0,$$
$$\tag{3.5.9}$$
$$is(\lambda + \mu)A_1 - \{(\lambda + 2\mu)s^2 - \mu + \rho c^2\} \, A_2 = 0.$$

For these equations to have non-trivial solution for A_1 and A_2, we must have

$$\begin{vmatrix} \mu s^2 + \rho c^2 - (\lambda + 2\mu) & - is \, (\lambda + \mu) \\ - is \, (\lambda + \mu) & (\lambda + 2\mu)s^2 + \rho c^2 - \mu \end{vmatrix} = 0,$$

which gives

$$\mu(\lambda + 2\mu)s^4 - \{ \, 2\mu(\lambda + 2\mu) - (\lambda + 3\mu)\rho c^2\}s^2 + \mu(\lambda + 2\mu)$$
$$- \rho c^2(\lambda + 3\mu - \rho c^2) = 0. \tag{3.5.10}$$

Recalling (2.7.20) and (2.7.21) that

$$\rho c_L^2 = \lambda + 2\mu, \qquad \rho c_T^2 = \mu,$$

and hence (3.5.11)

$$\lambda = \rho c_L^2 - 2\rho c_T^2.$$

Equation (3.5.10) becomes

$$c_L^2\, c_T^2\, s^4 - \{\, 2c_L^2\, c_T^2 - c^2 c_L^2 + c_T^2\,)\}\, s^2 + c_L^2\, c_T^2 - c^2(c_L^2 + c_T^2 - c^2) = 0.$$

This is a quadratic equation for s^2, with roots given by

$$s_1^2 = 1 - \frac{c^2}{c_L^2}\,, \qquad\qquad s_2^2 = 1 - \frac{c^2}{c_T^2}\,. \tag{3.5.12}$$

From (3.5.8) we can now write the general solution for u_1 and u_2 as

$$u_1 = (A_1\, e^{s_1 kx_2} + B_1\, e^{s_2 kx_2})\, e^{ikx_1 - i\omega t},$$

$$u_2 = (A_2\, e^{s_1 kx_2} + B_2\, e^{s_2 kx_2})\, e^{ikx_1 - i\omega t},$$

where, from (3.5.9), we have

$$\frac{iA_2}{A_1} = \frac{s_1^2\mu - (\lambda + 2\mu) + \rho c^2}{s_1(\lambda + \mu)}\,,$$

and similarly,

$$\frac{iB_2}{B_1} = \frac{\mu s_2^2 - (\lambda + 2\mu) + \rho c^2}{s_2(\lambda + \mu)}\,.$$

On use of equations (3.5.11), these become

$$\frac{iA_2}{A_1} = \frac{s_1^2 c_T^2 - c_L^2 + c^2}{s_2^2(c_L^2 - c^2)} = \frac{(c_L^2 - c^2)(c^2/\, c_L^2 - 1)}{s_1^2(c_L^2 - c_T^2\,)} = -\, s_1\,,$$

$$\frac{iB_2}{B_1} = \frac{s_2^2 c_T^2 - c_L^2 + c^2}{s_2^2(c_L^2 - c_T^2)} = \frac{(c_L^2 - c_T^2\,)}{s_2^2(c_L^2 - c_T^2)} = -\,\frac{1}{s_2} \tag{3.5.13}$$

3.6 Rayleigh waves

As in Section 3.2 we consider the propagation of surface waves on a half-space $x_2 \geqslant 0$. The boundary conditions are given by

$$\sigma_{2i} = 0, \qquad \text{on } x_2 = 0,$$

that is

$$\left.\begin{aligned} &\mu(u_{1,2} + u_{2,1}) = 0, \\ &(\lambda + 2\mu)u_{2,2} + \lambda\, u_{1,1} = 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \tag{3.6.1}$$

We seek solutions for u_1 and u_2 of the form

$$\begin{aligned} u_1 &= (A_1\, e^{s_1 kx_2} + B_1\, e^{s_2 kx_2})e^{ikx_1 - i\omega t}, \\ u_2 &= (A_2\, e^{s_2 kx_2} + B_2\, e^{s_2 kx_2})e^{ikx_1 - i\omega t}. \end{aligned} \tag{3.6.2}$$

On use of equations (3.6.2) in (3.6.1), we have

$$\begin{aligned} &\mu\, s_1\, A_1 + i\mu\, A_2 + s_2\, \mu\, B_1 + i\mu\, B_2 = 0, \\ &i\lambda\, A_1 + (\lambda + 2\mu)s_1\, A_2 + i\lambda\, B_1 + (\lambda + 2\mu)\, B_2 = 0. \end{aligned} \tag{3.6.3}$$

Substituting (3.5.11) into (3.6.3), we have

$$\begin{aligned} &c_T^2\, s_1\, A_1 + ic_T^2\, A_2 + c_T^2\, s_2\, B_1 + ic_T^2\, B_2 = 0, \\ &i(c_L^2 - 2c_T^2)\, A_1 + c_L^2\, s_1\, A_2 + i(c_L^2 - c_T^2)\, B_1 + c_L^2\, B_2 = 0. \end{aligned} \tag{3.6.4}$$

On use of (3.5.13) in (3.6.4), we have

$$2c_T^2\, s_1\, A_1 + c_T^2\, (s_2 + \frac{B_1}{s_2}) = 0,$$

$$\{c_L^2\, s_1 - (c_L^2 - 2c_T^2)\}\, A_1 + 2c_T^2\, B_1 = 0, \tag{3.6.5}$$

For these equations to have non-trivial solutions we must have

$$\begin{vmatrix} 2s_1 s_2 & 1 + s_2 \\ c_L^2\, s_1^2 - c_L^2 + 2c_T^2 & 2c_T^2 \end{vmatrix} = 0,$$

which gives

$$4c_T^2\, s_1 s_2 + (1 + s_2^2)(c_L^2 - 2c_T^2 - c_L^2\, s_1^2) = 0, \tag{3.6.6}$$

Substituting the expression s_1^2, which given in (3.5.12) into (3.6.6), the secular equation becomes

$$4c_T^2\, s_1 s_2 - c_T^2\, (1 + s_2^2)(2 - c^2/\, c_T^2) = 0,$$

i.e.

$$(1 + s_2^2)^2 - 4s_1 s_2 = 0. \tag{3.6.7}$$

This is an equivalent result to that given by Eringen and Suhubi (1975), which also can be written, by using (3.5.12) in (3.6.7), as

$$(2 - \frac{c^2}{c_T^2}) - 4(1 - \frac{c^2}{c_L^2})\, (1 - \frac{c^2}{c_T^2}) = 0. \tag{3.6.8}$$

By squaring and rearrangment, this becomes

$$\left[\frac{c}{c_T}\right]^2\left[\left[\frac{c}{c_L}\right]^6 - 8\left[\frac{c}{c_T}\right]^4 + \left[24 - 16\frac{c^2}{c_L^2}\right]\left[\frac{c}{c_T}\right]^2 - 16\left[1 - \frac{c^2}{c_L^2}\right]\right] = 0$$

This equation can be written as an equivalent for $\frac{\rho c^2}{\mu}$ with parameters λ and μ:

$$\left[\frac{\rho c^2}{\mu}\right]\left[\left[\frac{\rho c^2}{\mu}\right]^3 - 8\left[\frac{\rho c^2}{\mu}\right]^2 + \left[24 - 16\frac{\mu}{\lambda+2\mu}\right]\left[\frac{\rho c^2}{\mu}\right] - 16\left[1 - \frac{\rho c^2}{\mu}\right]\right] = 0 \quad (3.6.9)$$

This is the secular equation of Rayleigh waves in the classical theory. Numerical solutions of this equation can be found in Ewing, Jardetzky and Press (1957), for example.

3.7 Stoneley waves

As for an incompressible material, we consider propagation along the interface $x_2 = 0$ between two half-spaces of different materials with Lamé moduli μ, λ and μ^*, λ^* in $x_2 < 0$ and $x_2 > 0$ respectively. Again we let σ_{ij}, σ^*_{ij} be the stress components in the two half-spaces and $\underline{u}$ and $\underline{u}^*$ be the corresponding displacements. Then, restricting attention to two-dimensional motion, as before, we obtain

$$\sigma_{11} = \lambda(u_{1,1} + u_{2,2}) + 2\mu u_{1,1},$$

$$\sigma_{22} = \lambda(u_{1,1} + u_{2,2}) + 2\mu u_{2,2},$$

$$\sigma_{12} = \sigma_{21} = \mu(u_{1,2} + u_{2,1}), \tag{3.7.1}$$

$$\sigma^*_{11} = \lambda^*(u^*_{1,1} + u^*_{2,2}) + 2\mu^* u^*_{1,1},$$

$$\sigma^*_{22} = \lambda^*(u^*_{1,1} + u^*_{2,2}) + 2\mu^* u^*_{2,2},$$

$$\sigma^*_{12} = \sigma^*_{21} = \mu^*(u^*_{1,2} + u^*_{2,1}).$$

The equation of motion in $x_2 < 0$ is (3.5.7). Similarly, in $x_2 > 0$, we have

$$\lambda^*(u^*_{1,11} + u^*_{2,12}) + 2\mu^* u^*_{1,11} + \mu^*(u^*_{1,22} + u^*_{2,12}) = \rho^* \ddot{u}^*_1,$$
$$(3.7.2)$$
$$\lambda^*(u^*_{1,12} + u^*_{2,22}) + 2\mu^* u^*_{2,22} + \mu^*(u^*_{1,12} + u^*_{2,11}) = \rho^* \ddot{u}^*_2.$$

Assume that u^*_1 and u^*_2 are given by

$$u^*_1 = A^*_1 \, e^{-s^* kx_2 - i\omega t + ix_1 k},$$
$$(3.7.3)$$
$$u^*_2 = A^*_2 \, e^{-s^* kx_2 - i\omega t + ix_1 k},$$

and n_1 and n_2 by (3.5.8).

On use of (3.7.3) in (3.7.2), we have

$$\{ \mu^* s^* - (\lambda^* + 2\mu^*) + \rho^* c^2 \} A^*_1 + is^*(\lambda^* + \mu^*) A^*_2 = 0,$$
$$(3.7.4)$$
$$is^*(\lambda^* + \mu^*) A^*_1 + \{(\lambda^* + 2\mu^*) s^* - \mu^* + \rho^* c^2\} A^*_2 = 0.$$

From (3.7.4), we have

$$\frac{iA_2^*}{A_1^*} = - \frac{\{\mu^* s_1^* - (\lambda^* + 2\mu^*) + \rho^* c^2\}}{s_1^* (\lambda^* + \mu^*)},$$

and similarly (3.7.5)

$$\frac{iB_2^*}{B_1^*} = - \frac{\{\mu^* s_2^* - (\lambda^* + 2\mu^*) + \rho^* c^2\}}{s_2^* (\lambda^* + \mu^*)},$$

for the half-space $x_2 > 0$.

From (3.5.11), we have

$$\rho c_L^2 = \lambda + 2\mu , \qquad \rho c_T^2 = \mu , \qquad \lambda = \rho c_L^2 - 2\rho c_T^2 ,$$

and similarly (3.7.6)

$$\rho^* c_L^{*2} = \lambda^* + 2\mu^*; \qquad \rho^* c_T^{*2} = \mu^*, \qquad \lambda^* = \rho^* c_L^{*2} - 2\rho^* c_T^{*2}.$$

On use of (3.7.6), (3.7.5) become

$$\frac{iA_2^*}{A_1^*} = - \frac{s_1^{*2} c_T^{*2} - c_L^{*2} + c^2}{s_1^*(c_L^{*2} - c_T^{*2})} = - \frac{\{c_T^{*2}(1 - c^2/c_L^{*2}) - c_L^{*2} + c^2\}}{s_1^*(c_L^{*2} - c_T^{*2})} = - s_1^*,$$

(3.7.7)

$$\frac{iB_2^*}{B_1^*} = - \frac{(s_2^{*2} c_T^{*2} - c_L^{*2} + c^2)}{s_2^* (c_L^{*2} - c_T)} = - \frac{\{c_T^{*2} (1 - c^2/c_T^{*2}) - c_L^{*2} + c^2\}}{s_2^*(c_L^{*2} - c_T^{*2})} = - \frac{1}{s_2^*}.$$

Also, from equation (3.5.12), we have

$$s_1^2 = 1 - c^2/c_L^2, \qquad s_2^2 = 1 - c^2/c_T^2, \qquad \text{for } x_2 < 0.$$

Similarly, (3.7.8)

$$s_1^{*2} = 1 - c^2/c_L^{*2}, \qquad s_2^{*2} = 1 - c^2/c_T^{*2}, \qquad \text{for } x_2 > 0.$$

From (3.5.13) and (3.7.5), we get

$$iA_2 = -s_1\ A_1, \qquad iB_2 = -\frac{B_1}{s_2}, \qquad \text{for } x_2 < 0,$$

$$(3.7.9)$$

$$iA_2^* = s_1^*\ A_1^*, \qquad iB_2^* = \frac{B_1^*}{s_2^*}, \qquad \text{for } x_2 > 0.$$

Let now consider the boundary conditions, which are given by

$$\underline{u} = \underline{u}^*, \qquad \sigma_{2i} = \sigma_{2i}^* \qquad \text{on } x_2 = 0.$$

i.e

$$\left.\begin{aligned} u_1 - u_1^* &= 0, \\ u_2 - u_2^* &= 0, \\ \sigma_{21} - \sigma_{21}^* &= 0, \\ \sigma_{22} - \sigma_{22}^* &= 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \quad (3.7.10)$$

Substituting (3.7.1) in (3.7.4), we have

$$\left.\begin{aligned} u_1 - u_1^* &= 0, \\ u_2 - u_2^* &= 0, \\ \mu(u_{1,2} + u_{2,1}) - \mu^*(u_{1,2}^* + u_{2,1}^*) &= 0, \\ (\lambda + 2\mu)u_{2,2} + \lambda u_{1,1} - (\lambda^* + 2\mu^*)u_{2,2}^* - \lambda^* u_{1,1}^* &= 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \quad (3.7.11)$$

We seek solutions for u_i and u_i^*, $i \in \{1,2\}$, of the form

$$u_1 = (A_1\, e^{s_1 k x_2} + B_1\, e^{s_2 k x_2}) e^{i(\omega t - k x_1)},$$

$$u_2 = (s_1 A_2 e^{s_1 k x_2} + s_2\, B_2\, e^{s_2 k x_2}) e^{i(\omega t - k x_1)}, \tag{3.7.12}$$

$$u_1^* = (A_1^*\, e^{-s_1^* k x_2} + B_1^*\, e^{-s_2^* k x_2})\, e^{i(\omega t - k x_1)},$$

$$u_2^* = (s_1^*\, A_2^*\, e^{-s_1^* k x_2} + s_2^*\, B_2^*\, e^{-s_2^* k x_2})\, e^{i(\omega t - k x_1)},$$

On use of these equations (3.7.9) and (3.7.12) in (3.7.11), we obtain

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_1 A_1 + \frac{B_1}{s_2} - s_1^* A_1^* - \frac{B_1^*}{s^*_2} = 0,$$

$$\text{on } x_2 = 0 \tag{3.7.13}$$

$$2\mu s_1 A_1 + \mu(s_2 + \frac{1}{s_2})\, B_1 + 2\mu^* s_1^*\, A_1^* + \mu^*(s_2^* + \frac{1}{s_2{}^*})B_1^* = 0,$$

$$\{\lambda - (\lambda + 2\mu)s_1^2\}A_1 - 2\mu B_1 + \{(\lambda^* + 2\mu^*)s_1^{*2} - \lambda^*\}\, A_1^* + 2\mu^* B_1^* = 0.$$

On use of equations (3.7.6) and (3.7.8) in $(3.7.13)_{3,4}$, we get

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_1 A_1 + \frac{B_1}{s_2} - s_1^*\, A_1^* - \frac{B_1^*}{s_2^*} = 0, \qquad \text{on } x_2 = 0 \tag{3.7.14}$$

$$2\mu s_1 A_1 + \mu(2 - c^2/c_T^2)\, \frac{B_1}{s_2} + 2\mu^* s_1^*\, A_1^* + \mu^*(2 - c^2/c_T^{*2})\, \frac{B_1^*}{s_2^*} = 0,$$

$$\rho c_T^2\, (2 - c^2/c_T^2)A_1 + 2\mu B_1 - \rho^* c_T^{*2}(2 - c^2/c_T^{*2})A_1^* - 2\mu^* B_1^* = 0,$$

i.e.

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_1 A_1 + \frac{B_1}{s_2} - s_1^* A_1^* - \frac{B_1^*}{s_2^*} = 0,$$

$$(3.7.15)$$

$$2s_1 A_1 + (2 - c^2/c_T^2)\frac{B_1}{s_2} + 2\frac{\mu^*}{\mu} s_1^* A_1^* + \frac{\mu^*}{\mu}(2 - c^2/c_T^{*2})\frac{B_1^*}{s_2^*} = 0,$$

$$(2 - c^2/c_T^2)A_1 + 2B_1 - \frac{\mu^*}{\mu}(2 - c^2/c_T^{*2})A_1^* - 2\frac{\mu^*}{\mu}B_1^* = 0.$$

For these equations to have non-trivial solutions for A_i, B_i^*, A_i^*, B_i, $i \in \{1,2\}$ we must have

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ s_1 & \frac{1}{s_2} & -s_1^* & -\frac{1}{s_2^*} \\ 2s_1 & (2 - \frac{c^2}{c_T^2})\frac{1}{s_2} & 2\frac{\mu^*}{\mu}s_1 & \frac{\mu^*}{\mu}(2 - \frac{c^2}{c_T^{*2}})\frac{1}{s_2^*} \\ 2 - c^2/c_T^2 & 2 & \frac{\mu^*}{\mu^{*2}}(2 - \frac{c^2}{c_T^{*2}}) & -2\frac{\mu^*}{\mu} \end{vmatrix} = 0$$

This is the secular equation for Stoneley waves, in the matrix form, which is the same form given by Achenbach (1984). The analysis of this secular equation can be found in Cagniard (1962).

3.8 Love waves

Let us assume, as in the incompressible material, that the half-space is defined by $x_2 \leqslant 0$ on which there is a layer of different material of thickness h with boundaries $x_2 = 0$ and $x_2 = h$. Also we assume μ, λ and μ^*, λ^* the modulus of rigidity in the half-space and the layer respectively.

Next, let the boundary conditions be given by

$$\begin{aligned} &\sigma_{2i} = 0 && \text{on } x_2 = h, \\ &\underline{u} = \underline{u}^*, \quad \sigma_{2i} = \sigma^*_{2i} && \text{on } x_2 = 0, \end{aligned} \tag{3.8.1}$$

where $\underline{u}$ and $\underline{u}^*$ are given by (3.4.2).

For a compressible elastic material, the equations of motion (3.4.4) are

$$\begin{aligned} &\mu(u_{2,11} + u_{2,22}) = \rho \ddot{u}_2, \\ &\mu^*(u^*_{2,11} + u^*_{2,22}) = \rho^* \ddot{u}^*_2. \end{aligned} \tag{3.8.2}$$

for the half-space and the layer respectively.

Substituting (3.4.2) into $(3.8.2)_1$, we get

$$\mu(s^2 - 1) = -\rho c^2, \tag{3.8.3}$$

and hence

$$s^2 = 1 - \frac{\rho c^2}{\mu}. \tag{3.8.4}$$

Similarly

$$s^{*2} = 1 - \frac{\rho^* c^2}{\mu^*}. \tag{3.8.5}$$

On use of (2.7.21), equations (3.8.4) and (3.8.5) become

$$s^2 = 1 - \frac{c^2}{c_T^2}, \qquad s^{*2} = \frac{c^2}{c_T^{*2}} - 1. \tag{3.8.6}$$

From equations (3.5.3), we have

$$\sigma_{32} = \mu(u_{2,2} + u_{3,2}) = \mu u_{3,2}. \tag{3.8.7}$$

The boundary conditions (3.8.1) also give $\overset{*}{u}_{2,2} = 0$ on $x_2 = h$ and from (3.8.3) and (3.8.7), we get the same results given in an incompressible case.

That is, the secular equation (3.4.18) is also the secular equation for a compressible material, namely

$$\tan\left[kh\left[\frac{c^2}{c_T^{*2}} - 1 \right]^{\frac{1}{2}} \right] - \frac{\mu}{\mu^*} \frac{(1 - c^2/c_T^2)}{(c^2/c_T^{*2} - 1)} = 0. \tag{3.8.8}$$

Chapter 4

Rayleigh waves in a pre-stressed elastic material

In this Chapter we extend the discussion of Rayleigh waves in the linear theory covered in Section 3.2 to the situation in which there is an underlying pre-stress. In particular, we recover certain results obtained by Hayes and Rivlin (1961a), who used a different notation, and we generalize other results given by Flavin (1963) and Willson (1973a, 1974a,b) for an incompressible material and Willson (1972, 1973b) for a compressible material.

4.1 Analysis for an incompressible material

Consider the large homogeneous pure strain defined by

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3. \tag{4.1.1}$$

Upon this deformation we superpose a small displacment $\underline{u}$, such that

$$\begin{aligned} x_1 &= \lambda_1 X_1 + u_1, \\ x_2 &= \lambda_2 X_2 + u_2, \\ x_3 &= \lambda_3 X_3 + u_3, \end{aligned} \tag{4.1.2}$$

where u_1, u_2, u_3 are the components of $\underline{u}$. The velocity components are given by

$$v_i = \left.\frac{\partial u_i}{\partial t}\right|_{\underline{X}}. \tag{4.1.3}$$

From (2.4.18), we have the principal components of the Cauchy stress tensor associated with the homogeneous deformation

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \qquad i \in \{1, 2, 3\}.$$

On use of the incremental equations (2.6.11), we deduce that

$$\begin{aligned}
\dot{s}_{011} &= A_{01111}\, v_{1,1} + A_{01122}\, v_{2,2} + A_{01133}\, v_{3,3} + pv_{1,1} - \dot{p}\,, \\
\dot{s}_{022} &= A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2} + A_{02233}\, v_{3,3} + pv_{2,2} - \dot{p}\,, \\
\dot{s}_{033} &= A_{03311}\, v_{1,1} + A_{02233}\, v_{2,2} + A_{03333}\, v_{3,3} + pv_{3,3} - \dot{p}\,, \\
\dot{s}_{012} &= A_{01212}\, v_{2,1} + A_{01221}\, v_{1,2} + pv_{1,2}, \\
\dot{s}_{013} &= A_{01313}\, v_{3,1} + A_{01331}\, v_{1,3} + pv_{1,3}, \\
\dot{s}_{021} &= A_{02121}\, v_{1,2} + A_{02112}\, v_{2,1} + pv_{2,1}, \\
\dot{s}_{023} &= A_{02323}\, v_{3,2} + A_{02332}\, v_{2,3} + pv_{2,3}, \\
\dot{s}_{031} &= A_{01313}\, v_{1,3} + A_{03113}\, v_{3,1} + pv_{3,1}, \\
\dot{s}_{032} &= A_{03232}\, v_{2,3} + A_{03223}\, v_{3,2} + pv_{3,2},
\end{aligned} \tag{4.1.4}$$

subject to the incompressibility condition (2.6.12).

4.1.1 Plane incremental motion

In order to keep the algebra as simple as possible we take $v_3 \equiv 0$ and assume v_1, v_2 are independent of x_3.
Then (2.6.12) becomes

$$v_{1,1} + v_{2,2} = 0. \tag{4.1.5}$$

Hence there exists a function $\psi(x_1, x_2, t)$ such that

$$v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1}. \tag{4.1.6}$$

Also, (4.1.4) reduce to

$$\begin{aligned}
\dot{s}_{011} &= A_{01111}\, v_{1,1} + A_{01122}\, v_{2,2} + p\, v_{1,1} - \dot{p},\\
\dot{s}_{022} &= A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2} + p\, v_{2,2} - \dot{p},\\
\dot{s}_{033} &= A_{03311}\, v_{1,1} + A_{02233}\, v_{2,2} - \dot{p},\\
\dot{s}_{012} &= A_{01212}\, v_{2,1} + A_{01221}\, v_{1,2} + p\, v_{1,2},\\
\dot{s}_{021} &= A_{01212}\, v_{1,2} + A_{02112}\, v_{2,1} + p\, v_{2,1}.
\end{aligned} \tag{4.1.7}$$

From (2.3.5), the incremental motion is governed by

$$\dot{s}_{0ji,j} = \rho \dot{v}_i. \tag{4.1.8}$$

Thus, from the incremental equations (2.6.11), we get

$$\dot{s}_{0ji,j} \equiv A_{0jilk}\, v_{k,lj} - p_{,i} = \rho \dot{v}_i.$$

from this equation, if we take i = 1,2 we have

$$\begin{aligned}
\dot{s}_{011,1} + s_{021,2} &= \rho \dot{v}_1,\\
\dot{s}_{012,2} + s_{022,2} &= \rho \dot{v}_2.
\end{aligned} \tag{4.1.9}$$

From (4.1.6) and (4.1.7) we get

$$\begin{aligned}
\dot{s}_{011,1} &= (A_{01111} - A_{01122} + p)\psi_{,112} - \dot{p}_{,1}\ ,\\
\dot{s}_{012,1} &= (A_{01221} + p)\psi_{,221} - A_{01212}\, \psi_{,111}\ ,\\
\dot{s}_{021,2} &= A_{02121}\psi_{,222} - (A_{02112} + p)\psi_{,112}\ ,\\
\dot{s}_{022,2} &= (A_{02211} - A_{02222})\psi_{,122} - \dot{p}_{,3}\ .
\end{aligned} \tag{4.1.10}$$

Substituting (4.1.10) into (4.1.9), we get

$$\rho\ddot{\psi}_{,2} = -\dot{p}_{,1} + (A_{01111} - A_{01122} - A_{02112})\psi_{,112} + A_{02121}\psi_{,222}$$

$$\text{on } x_2 = 0 \quad (4.1.11)$$

$$-\rho\ddot{\psi}_{,1} = -\dot{p}_{,2} + (A_{01221} + A_{02211} - A_{02112})\psi_{,122} - A_{01212}\psi_{,111}.$$

To eliminate $\dot{p}$ we must differentiate equation (4.1.11) with respect to x_2 and x_1 then we obtain

$$-\rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) + A_{01212}\psi_{,1111} + (A_{01111} + A_{02222} - 2A_{01221}$$

$$- 2A_{02211})\psi_{,1122} + A_{02121}\psi_{,2222} = 0.$$

On setting

$$\alpha = A_{01212}, \quad \gamma = A_{02121},$$

$$(4.1.12)$$

$$2\beta = A_{01111} + A_{02222} - 2A_{01221} - 2A_{02211},$$

the above equation becomes

$$\alpha\, \psi_{,1111} + 2\beta\, \psi_{,1122} + \gamma\, \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}). \qquad (4.1.13)$$

Also, necessary and sufficient conditions for the strong-ellipticity condition $(2.8.15)_{1,3}$ simplify to

$$\alpha > 0, \qquad \beta > -\sqrt{\alpha\gamma}\,. \qquad (4.1.14)$$

Suppose the elastic medium occupies the half-space defined by $x_2 < 0$. In this basic homogeneous configuration the normal stress on the surface $x_2 = 0$ is σ_2. We assume this is unaffected by the perturbed deformation, so the incremental boundary conditions vanish. From (2.6.22) we therefore obtain

$$\dot{s}_{021} = \dot{s}_{022} = 0 \qquad \text{on } x_2 = 0.$$

i.e.

$$\left.\begin{aligned} A_{02121}\psi_{,22} - (A_{01221} + p)\psi_{,11} &= 0, \\ (A_{01122} - A_{02222} - p)\psi_{,12} - \dot{p} &= 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (4.1.15)$$

From $(4.1.11)_1$, we have

$$\dot{p}_{,1} = (A_{01111} - A_{01122} - A_{02112})\psi_{,112} + A_{02121}\psi_{,222} - \rho\ddot{\psi}. \quad (4.1.16)$$

To eliminate $\dot{p}$ we must differentiate equations (4.1.15) with respect to x_1 and using equation (4.1.16) and the notation (4.1.12), the boundary conditions can be written as

$$\left.\begin{aligned} \gamma\,(\psi_{,22} - \psi_{,11}) + \sigma_2\,\psi_{,11} &= 0, \\ (2\beta + \gamma - \sigma_2)\,\psi_{,112} + \gamma\,\psi_{,222} - \rho\ddot{\psi}_{,2} &= 0, \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (4.1.17)$$

where from equation (2.6.13), we have $A_{01221} = A_{02121} - \lambda_2 \dfrac{\partial W}{\partial \lambda_2}$ and hence $A_{01221} + p = A_{02121} - \sigma_2$.

The equibilrum counterpart of equation (4.1.13) was derived by Nowinski (1969a) in the context of quasi-static surface instabilities, and also by Hill and Hutchinson (1975) for a class of incrementally-linear materials. For the dynamic case (4.1.13) appear to be new although, in different notation, Willson (1973a) obtained an equivalent equation for the time-harmonic case. Willson also obtained boundary conditions equivalent to (4.1.17) for the time-harmonic case with the restriction $\sigma_2 = 0$ from the outset and also considered an equibiaxial underlying deformation $\lambda_1 = \lambda_2$.

4.1.2 Propagation along a principal axis

Let us assume $\psi(x_1,x_2,t)$ has the form

$$\psi = f(x_2)e^{i\omega t - ikx_1}, \qquad (4.1.18)$$

which represents a wave propagating with (constant) wave speed c in the x_1-direction. Also, the frequency of the wave ω is constant.

Also, we assume the spatial variation of ψ of the form $e^{(skx_2 - ikx_1 + i\omega t)}$, where $k = \omega/c$ is the wave number. Then equation (4.1.13) leads to

$$\gamma\, s^4 - (2\beta - \rho c^2)\, s^2 + \alpha - \rho c^2 = 0, \qquad (4.1.19)$$

which is a quadratic equation for s^2. Suppose it has root s_1^2 and s_2^2. Then

$$s_1^2 + s_2^2 = \frac{2\beta - \rho c^2}{\gamma}, \quad s_1^2\, s_2^2 = \frac{\alpha - \rho c^2}{\gamma}. \qquad (4.1.20)$$

For surface waves we must have a solution for ψ in (4.1.13) which decays when $x_2 \longrightarrow -\infty$ so we shall require s to have positive real part. In either case $s_1{}^2 s_2{}^2 > 0$. Since γ is required to be positive by the strong-ellipticity condition and $\gamma = \mu > 0$ in the unstressed configuration, we get from (4.1.20) that the wave speed c is restricted according to the inequality

$$\rho c^2 < \alpha. \qquad (4.1.21)$$

On the other hand, $2\beta - \rho c^2$ may be positive or negative.

We note here that if c_S denotes the speed of a plane (shear) wave propagating in the x_1-direction with displacement in the x_2-direction in an unbounded body subject to the same homogeneous pure strain then

$$\rho c_S^2 = \alpha. \tag{4.1.22}$$

In the unstressed configuration $\alpha = \mu$ and (4.1.22) is expressible as

$$c < c_S. \tag{4.1.23}$$

We now write the general solution for ψ in the form

$$\psi = (A\ e^{s_1 k x_2} + B\ e^{s_2 k x_2})\ e^{i\omega t - ikx_1}, \tag{4.1.24}$$

where A and B are constants.

On use of equation (4.1.24) into (4.1.17), we obtain

$$(\gamma\ s_1^2 + \gamma - \sigma_2)A + (\gamma\ s_2^2 + \gamma - \sigma_2)B = 0 \tag{4.1.25}$$

$$s_1(2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma s_1^2)\ A + s_2(2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma\ s_2)^2 B = 0. \tag{4.1.26}$$

For these equations to have a non-trivial solution, we must have

$$\begin{vmatrix} \gamma\ s_1^2 + \gamma - \sigma_2 & \gamma\ s_2^2 + \gamma - \sigma_2 \\ s_1(2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma\ s_1^2) & s_2(2\beta + \gamma - \sigma_2 - \rho c^2 - \gamma\ s_2^2) \end{vmatrix} = 0,$$

which gives

$$(s_1 - s_2)[\gamma(s_1^2 + s_2^2)(\gamma - \sigma_2) + \gamma^2 s_1^2 s_2^2 + \gamma s_1 s_2(2\beta + 2\gamma - \rho c^2 - 2\sigma_2)$$

$$- (\gamma - \sigma_2)^2 + (\rho c^2 - 2\beta)(\gamma - \sigma_2)] = 0. \qquad (4.1.27)$$

Assuming $s_1 \neq s_2$, this equation reduces to

$$\gamma(s_1^2 + s_2^2)(\gamma - \sigma_2) + \gamma^2 s_1^2 s_2^2 + \gamma s_1 s_2(2\beta + 2\gamma - 2\sigma_2 - \rho c^2)$$

$$- (\gamma - \sigma_2)^2 + (\gamma - \sigma_2)(\rho c^2 - 2\beta) = 0. \qquad (4.1.28)$$

Substituting (4.1.18) into (4.1.28), we get

$$\gamma(\alpha - \rho c^2) + (2\beta + 2\gamma - 2\sigma_2 - \rho c^2)\sqrt{\gamma(\alpha - \rho c^2)} = (\gamma - \sigma_2)^2. \qquad (4.1.29)$$

Equation (4.1.29) is *the secular equation for Rayleigh surface waves in a pre-stressed incompressible elastic material.*

On squaring and rearrangement of (4.1.29), we obtain

$$(\rho c^2)^3 - p(\rho c^2)^2 + q(\rho c^2) - r = 0, \qquad (4.1.30)$$

where

$$p = 4\beta + 3\gamma + \alpha - 4\sigma_2, \qquad (4.1.31)$$

$$q = (2\beta + 2\gamma - 2\sigma_2)^2 + 2\alpha(2\beta + 2\gamma - \sigma_2) + 2(\gamma - \sigma_2)^2 - 2\gamma\alpha, \qquad (4.1.32)$$

$$r = \{\sqrt{\gamma\alpha}\,(2\beta + 2\gamma - 2\sigma_2)^2 + (\gamma - \sigma_2) - \gamma\alpha\}$$

$$\{\sqrt{\gamma\alpha}\,(2\beta + 2\gamma - 2\sigma_2) - (\gamma - \sigma_2)^2 + \gamma\alpha\}/\gamma \qquad (4.1.33)$$

In different notation equation (4.1.30) generalizes the results given by Willson (1973a), who took $\sigma_2 = 0$ and considered only equibiaxial underlying deformations $\lambda_2 = \lambda_1$ ($\lambda_3 = \lambda_1$ in our notation) throughout his calculation. The squaring process may give solutions of (4.1.30), which are not solutions of (4.1.29). For instance, when $c = 0$, $r = 0$ and either of the factor in braces in (4.1.33) may vanish, but only the second of these corresponds to a solution of (4.1.29).

To avoid this problem we work directly with (4.1.29) and introduce the notation

$$\eta = \sqrt{(\alpha - \rho c^2)/\gamma}\ , \tag{4.1.34}$$

so that

$$\rho c^2 = \alpha - \gamma\eta^2. \tag{4.1.35}$$

From (2.6.13), we have

$$\alpha\lambda_2^2 = \gamma\ \lambda_1^2, \tag{4.1.36}$$

and from (4.1.21), we must have

$$0 \leq \eta \leq \lambda_1\lambda_2^{-1}. \tag{4.1.37}$$

Now, equation (4.1.29) becomes

$$\eta^3 + \eta^2 + (2\beta + 2\gamma - \alpha - 2\sigma_2)\eta/\gamma - (\gamma - \sigma_2)^2/\gamma^2 = 0. \tag{4.1.38}$$

For the special case in which $\sigma_2 = 0$ this simplifies to give

$$\eta^3 + \eta^2 + (2\beta + 2\gamma - \alpha)\eta/\alpha - 1 = 0. \tag{4.1.39}$$

In equation (4.1.28) we assumed $s_1 \neq s_2$. We now consider the special case in which $s_1 = s_2 = s$, say. Equation (4.1.24) becomes

$$\psi = (A + Bx_2)e^{skx_2 + i\omega t - ikx_1}.$$

On use of this equation in (4.1.16) and (4.1.16), for $\sigma_2 = 0$ we deduce

$$\begin{aligned}&(s^2 + 1)k^2 A + 2sk B = 0,\\ &(\gamma s^2 - 2\beta - \gamma - \rho c^2)sk^3 A + (3\gamma s^2 - 2\beta - \gamma - \rho c^2)k^2 B = 0,\end{aligned} \qquad (4.1.40)$$

For equations (4.1.40) to have non-trivial solution for A and B we must have

$$\begin{vmatrix} (s^2 + 1)k^2 & 2sk \\ (\gamma s^2 - 2\beta - \gamma - \rho c^2)sk^3 & (3\gamma s^2 - 2\beta - \gamma - \rho c^2)k^2 \end{vmatrix} = 0,$$

which gives

$$\gamma s^4 + (2\beta + 4\gamma + \rho c^2)s^2 - 2\beta - \gamma - \rho c^2 = 0, \qquad (4.1.41)$$

which is a quadratic equation for s^2. Therefore the sum of roots of this equation is

$$2s^2 = -\frac{(2\beta + 4\gamma + \rho c^2)}{\gamma}, \qquad (4.1.42)$$

this equation can be written as

$$2s^2 = -\frac{A_{01111} + A_{02222} - 2A_{01122} + 2A_{02121} + 2p}{A_{02121}}. \qquad (4.1.43)$$

Since it is assumed $s_1 = s_2 = s$, equation $(4.1.20)_2$, becomes

$$2s^2 = \frac{A_{01111} + A_{02222} - 2A_{01221} - 2A_{01122} - \rho c^2}{A_{02121}} . \qquad (4.1.44)$$

From equations (4.1.43) and (4.1.44), we get

$$s^2 = -1. \qquad (4.1.45)$$

That is, s is pure imaginary, there is no decay when $x_2 \longrightarrow -\infty$. So this case cannot arise and we conclude that A = B = 0. This result appears to be new, although a corresponding result for the compressible case has been found by Hayes and Rivlin (1961b).

4.2 Analysis of the secular equation

Before considering the general form (4.1.34) of the secular equation it is instructive to examine the special case in which the material is undeformed but subject to a uniform hydrostatic stress.

4.2.1 The case of $\lambda_3 = \lambda_2 = \lambda_1 = 1$ in the presence of hydrostatic pre-stress

If the undeformed configuration is subject to a hydrostatic pre-stress $\sigma_1 = \sigma_2 = \sigma_3$ then we have from (2.7.19)

$$\begin{gathered} A_{01111} = A_{02222} = 2\mu, \\ A_{01212} = A_{02121} = A_{02112} = A_{01221} = \mu, \\ A_{01122} = A_{02211} = 0. \end{gathered} \qquad (4.2.1)$$

From (4.2.1) and (4.1.12), it follows that in a state of hydrostatic stress $\alpha = \beta = \gamma = \mu$ and (4.1.20) become

$$s_1^2 + s_2^2 = 2 - \frac{\rho c^2}{\mu}, \qquad s_1^2 s_2^2 = 1 - \frac{\rho c^2}{\mu}. \tag{4.2.2}$$

Also, the secular equation in the form (4.1.38) for this case becomes

$$f(\eta) = \eta 3 + \eta 2 + (3 - 2\bar{\sigma})\eta - (1 - \bar{\sigma})^2 = 0, \tag{4.2.3}$$

subject to

$$0 < \eta < 1, \tag{4.2.4}$$

where $\bar{\sigma} = \sigma_2/\mu$.

From (4.2.3), it follows that

$$f(0) = -(1 - \bar{\sigma})^2, \quad f(1) = 4 - \bar{\sigma}^2. \tag{4.2.5}$$

Also, from (4.2.3), we deduce it has a solution in interval (4.2.4) provided

$$-2 < \bar{\sigma} < 2. \tag{4.2.6}$$

The extreme values $\bar{\sigma} = \pm 2$ in (4.2.6) yield the solution $\eta = 1$, by (4.1.35), yields $c = 0$, while $\bar{\sigma} = 1$ corresponds to the solution $\eta = 0$ or $c = c_s$ where $\rho {c_s}^2 = \mu$. It is easy to show that $f(\eta)$ is monotonic increasing for $\bar{\sigma} < {}^4/_3$ and monotonic increasing for $\eta > 0$ if $\bar{\sigma} < {}^3/_2$. For ${}^3/_2 < \bar{\sigma} < 2$ the product of the roots of $f'(\eta) = 0$ is negative so the maximum of $f(\eta)$ occurs in $\eta < 0$ and the minimum in $\eta > 0$. Since $f(0) < 0$ it follows that (4.2.3) has a *unique* positive solution in the interval (4.2.4) if and only if (4.2.6) holds. Thus, a unique wave speed exists for hydrostatic stress satisfying (4.2.6). When $\bar{\sigma} = 0$, equation (4.2.3) reduces to

$$\eta^3 + \eta^2 + 3\eta - 1 = 0, \tag{4.2.7}$$

which has a unique (positive) real solution, say η_0. The approximate value of η_0 is 0.2956, which by (4.1.35), leads to an approximate value of 0.9126 for $\rho c^2/\mu$. This agrees with the classical results for an incompressible linear theory, which is given in (3.2.8); see, for example Ewing, Jardetzky and Press (1957).

Next, on setting $\xi = \dfrac{\rho c^2}{\mu}$, equation (4.1.30) for hydrostatic stress can be written

$$g(\xi) \equiv \xi^3 + 4(\bar{\sigma} - 2)\xi^2 + 6(\bar{\sigma} - 2)^2\xi + (\bar{\sigma} + 2)(\bar{\sigma} - 2)^3 = 0. \tag{4.2.8}$$

Then, at the end-points of the interval $0 \leqslant \xi \leqslant 1$, we have

$$g(0) = (\bar{\sigma} + 2)(\bar{\sigma} - 2)^3, \qquad g(1) = (\bar{\sigma} - 1)^4.$$

Clearly $\quad g(0) < 0 \quad$ for $-2 < \bar{\sigma} < 2$,

while $\quad g(1) > 0 \quad$ for $-2 < \bar{\sigma} < 2 \quad$ except at $\xi = 1$.

Also, we have

$$g'(\xi) = 3\xi^2 + 8(\bar{\sigma} - 2)\xi + 6(\bar{\sigma} - 2)^2,$$

and this is strictly positive except for $\bar{\sigma} = 2$, $\xi = 0$. Thus, $g(\xi)$ is monotonically increasing for $0 \leqslant \xi \leqslant 1$ and it has a unique solution $\xi \in [0,1]$ if and only if $\bar{\sigma}$ satisfies (4.2.6). Furthermore, the solutions of (4.2.3) and (4.2.8) are such that $\xi = 1 - \eta^2$. In particular, for $\bar{\sigma} = 0$, $\xi_0 = 1 - \eta_0{}^2$ is the (unique positive) solution of

$$\xi^3 - 8\xi^2 + 24\xi - 16 = 0, \tag{4.2.9}$$

and has the approximate value 0.9126, which has been mentioned above.

Equation (4.2.9) is equivalent to an equation given by Willson (1973a), and is the same as that obtained in Ewing, Jardetzky and Press (1957).

A Rayleigh wave will therefore propagate in a hydrostatically pre-stressed half-space provided the pre-stress satisfies $-2 < (\bar{\sigma} = \sigma/\mu) < 2$, except $\bar{\sigma} = 1$. The limiting case $\bar{\sigma} = \pm 2$ corresponding to situations in which the underlying homogeneous deformation becomes neutrally stable. (The stability of such configurations has been discussed in detail in the book by Ogden (1984), for example).

The form of $g(\xi)$ is illustrated in Figure 4.1 for different values of $\bar{\sigma}$. Also the results of (4.2.8) are illustrated in Figure 4.2, where we plot ξ as function of $\bar{\sigma} \in [-2,2]$. The value ξ_0 is marked in the figure. We note here that $\xi = 0$ when $\bar{\sigma} = \pm 2$. Also $\xi \leq 1$, with equality holding only for $\bar{\sigma} = 1$; for $\bar{\sigma} = 1$ the solutions s^2 of (4.1.18) are 0 and 3, and $B = 0$ in the solution (4.1.27). In this case the wave is a plane shear wave with speed given by $\xi = 1$; it is not a surface wave.

When $\xi = 0$ $(c = 0)$ $g(1)$ is positive provided $-2 < \bar{\sigma} < 2$. This indicates that, on a quasi-static hydrostatic path of loading from the stress-free configuration no incremental quasi-static surface deformations can appear. In other words, the configuration of the body in question is incrementally stable. Stability fails in either tension or compression when $\bar{\sigma}$ reaches 2 or -2 respectively and the body is then in a neutrally stable configuration.

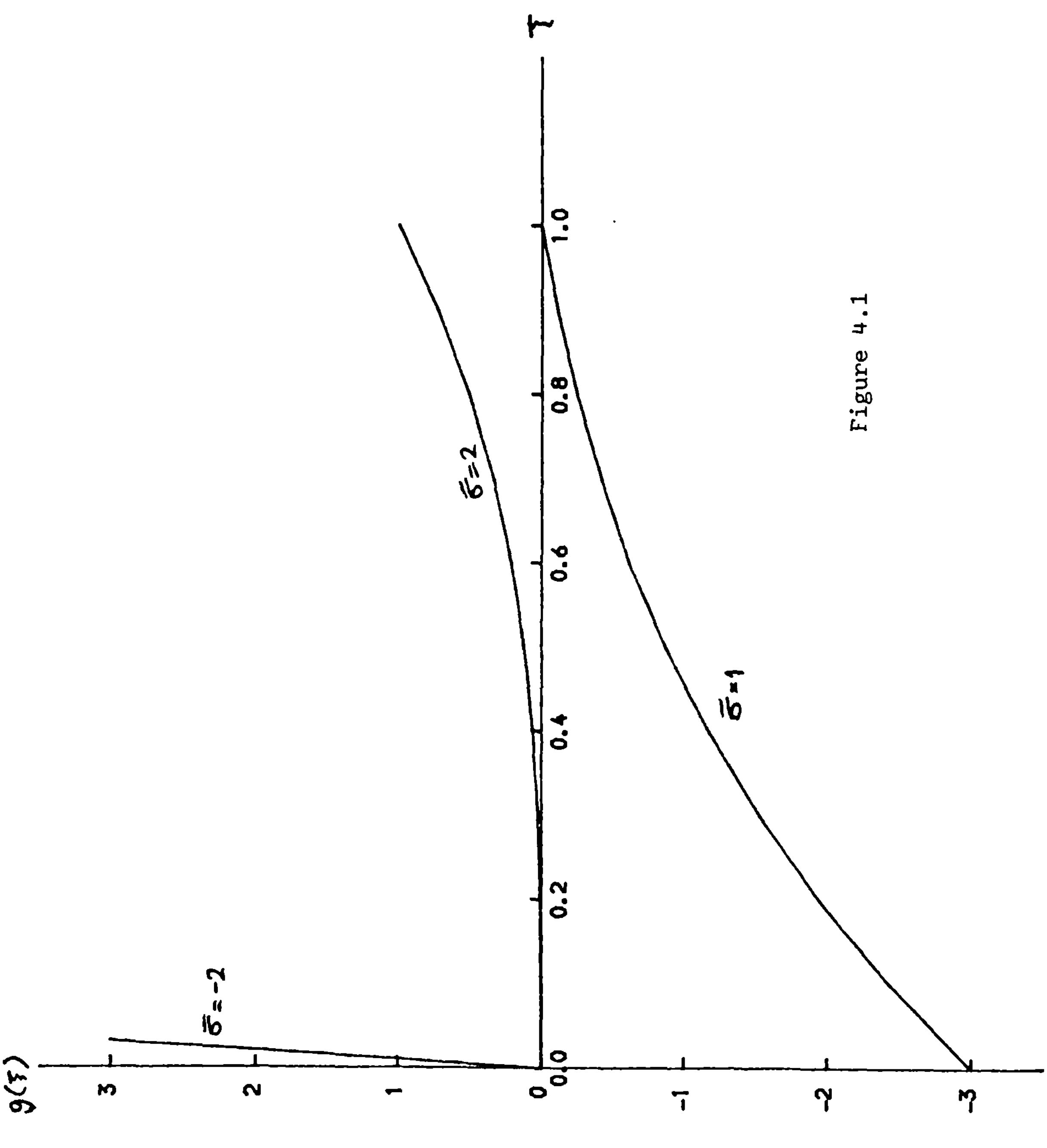


Figure 4.1

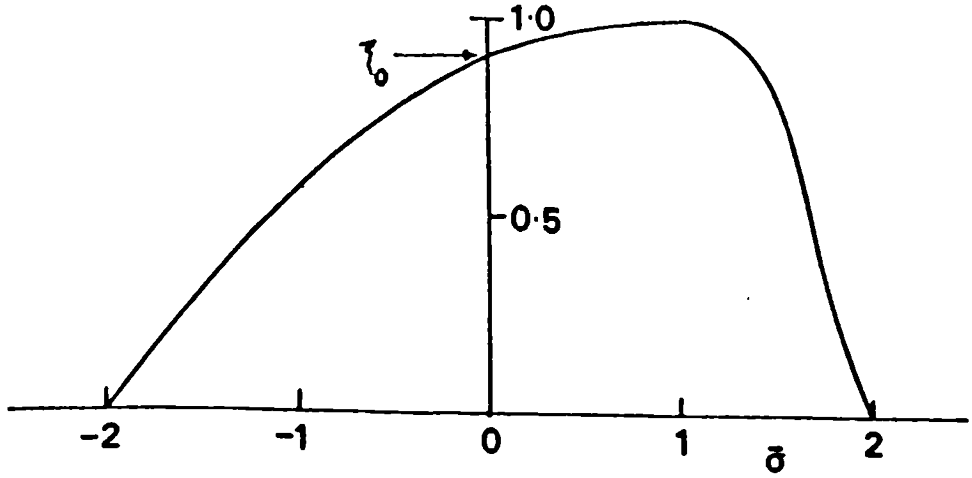


Figure 4.2

4.2.2 The general case

We now write the secular equation (4.1.38) in the form

$$f(\eta) \equiv \eta^3 + \eta^2 + d\eta - e = 0, \tag{4.2.10}$$

where

$$d = (2\beta + 2\gamma - \alpha - 2\sigma_2)/\gamma, \tag{4.2.11}$$

$$e = (\gamma - \sigma_2)^2/\gamma^2 \geqslant 0. \tag{4.2.12}$$

We consider $e > 0$ and $e = 0$ separately.

a) $e > 0$

Here when $\eta = 0$ $f(\eta) < 0$. If $d \geqslant 0$ then $f'(\eta) > 0$, for $\eta > 0$. If $d < 0$ then $f(\eta)$ has a minimum for $\eta < 0$ and a maximum for $\eta > 0$, and $f'(0) < 0$. In each case, equation (4.2.10) has at most one root in the interval (4.1.37). To ensure that a root corresponding to a non-zero wave speed ($\xi \equiv \rho c^2/\mu \neq 0$) exists we require $f(\sqrt{\alpha/\gamma}\,) > 0$, which, after rearrangement, yields

$$(\alpha - \gamma)\gamma + \sqrt{\alpha\,\gamma}\,(2\beta + 2\gamma) + 2\sigma_2(\gamma - \sqrt{\alpha\,\gamma}) - \sigma_2^2 > 0. \tag{4.2.13}$$

We note that $f(\sqrt{\alpha/\gamma}\,) = 0$ ensures (4.2.10) is satisfied by $\xi = 0$.

For the case $\sigma_2 = 0$ (4.2.13) becomes

$$\gamma\{\alpha - \gamma + \sqrt{\alpha/\gamma}\,(2\beta + 2\gamma)\} > 0. \tag{4.2.14}$$

In the undeformed configuration $\gamma = \mu$, and the second factor in (4.2.14) reduces to 4μ. Since $\mu > 0$ then, by continuity, (4.2.10) requires that

$$\gamma > 0, \tag{4.2.15}$$

$$\alpha - \gamma + \sqrt{\alpha/\gamma}\ (2\beta + 2\gamma) > 0. \tag{4.2.16}$$

We note that (4.2.15) is one of the inequalities required by the (two-dimensional) strong-ellipticity condition (see (4.1.14)). Next, since

$$2\beta + 2\sqrt{\alpha}\,\gamma - \{2\beta + 2\gamma + \sqrt{\gamma/\alpha}\ (\alpha - \gamma)\} = \sqrt{\gamma/\alpha}\ (\sqrt{\alpha} - \sqrt{\gamma}\ )^2 \geq 0,$$

it follows that (4.2.16) entails $(4.1.14)_2$. Thus, (4.2.15) and (4.2.16) together imply that the strong-ellipticity condition holds.

The inequalities (4.2.15), (4.2.16) ensure the existence of a unique surface waves when the surface $x_2 = 0$ is free of traction ($\sigma_2 = 0$). They may also be interpreted as ensuring that the underlying deformation is stable under the given boundary conditions. Put otherwise, (4.2.15) and (4.2.16) exclude the possibility of 'positions of adjacent equilibrium.'

For an incompressible material the state of deformation $(\lambda_1, \lambda_2, \lambda_3)$ is unaffeceted by a change in the hydrostatic stress. If we superimpose a hydrostatic stress of magnitude σ_2 then α, β and γ are unaffected and (4.2.13) serves to place restrictions on the range of values of the surface stress σ_2 that can be supported in the given state of deformation. It follows from (4.2.13) that

$$\gamma - \sqrt{\alpha\gamma} - [2\sqrt{\alpha\gamma}\ (\ \beta + \sqrt{\alpha\ \gamma})\,]^{\frac{1}{2}} < \sigma_2 < \gamma - \sqrt{\alpha\gamma} + [\ 2\sqrt{\alpha\gamma}\ (\beta + \sqrt{\alpha\gamma}\ ]^{\frac{1}{2}}. \tag{4.2.17}$$

We note here that, for the same underlying state of deformation the corresponding (two-dimensional) inequalities for stability under all-round dead load are (see section 4.6)

$$(\beta + \gamma - \sigma_2) > 0,$$
$$\alpha\,\gamma > (\gamma - \sigma_2)^2, \tag{4.2.18}$$

along with $\gamma > 0$. Clearly, (4.2.18) imply (4.1.36), but the converse does not hold in general since the sign of $\alpha\,\gamma - (\gamma - \sigma_2)^2$ may be either positive or negative. In particular, in a state of hydrostatic stress σ, the latter simplifies to $\sigma(2\mu - \sigma)$, which is positive for $0 < \sigma < 2\mu$ but negative for $-2\mu < \sigma < 0$.

When $\sigma_2 = 0$, equation (4.2.18) reduces to

$$\beta + \gamma > 0, \qquad \alpha > \gamma, \tag{4.2.19}$$

which imply $(4.1.14)_2$

<u>b) e = 0</u>

For this case we get from (4.2.12) that $\sigma_2 = \gamma$ and from (4.2.10) we get

$$\eta = 0 \quad \text{or} \quad \eta^2 + \eta + d = 0,$$

where, for this case,

$$d = (2\beta - \alpha)/\gamma. \tag{4.2.20}$$

The solution $\eta = 0$ corresponds to a plane shear wave with speed c_s given by $\rho c_s^2 = \alpha$ and does not correspond to a surface wave. A different solution exists provided

$$-\sqrt{\gamma\alpha} \leq 2\beta \leq \alpha, \tag{4.2.21}$$

the left-hand equality corresponding to zero wave speed.

If $\eta = 0$ then $\rho c^2 = \alpha$ and from (4.1.18) we can take $s_2 = 0$ and $s_1{}^2 = (2\beta - \gamma)\gamma$.

At this point it is instructive to express certain of the inequalities in this section in terms of the strain-energy function

$$\hat{W}(\lambda_1, \lambda_2) = \hat{W}(\lambda_2, \lambda_1), \tag{4.2.22}$$

which is defined in (1.3). It follows from (2.4.18), (2.6.13) and (4.1.12) that

$$\sigma_1 - \sigma_2 = \alpha - \gamma = \lambda_1 \hat{W}_1, \tag{4.2.23}$$

where $\hat{W}_1 = \partial\hat{W}/\partial\lambda_1$, γ is given by (1.4),

$$\alpha = \lambda_1^3 \hat{W}_1/(\lambda_1^2 - \lambda_2^2) \tag{4.2.24}$$

and

$$2\beta + 2\gamma = \lambda_1^2 \hat{W}_{11}. \tag{4.2.25}$$

The inequality (4.2.13) becomes

$$\lambda_2^{-1} \lambda_1^3 \hat{W}_{11} + \lambda_1\hat{W}_1 + 2\sigma_2(1 - \lambda_2^{-1}\lambda_1) - \sigma_2^2/\gamma > 0, \tag{4.2.26}$$

as (1.2), and this reduces to

$$\lambda_1^2 \hat{W}_{11} + \lambda_2\hat{W}_1 > 0 \tag{4.2.27}$$

when $\sigma_2 = 0$. From (4.2.17) the upper and lower bounds on σ_2 are

$$- \frac{\lambda_1\lambda_2 \hat{W}_1}{\lambda_1 + \lambda_2} \pm \left[\frac{\lambda_1^3 \lambda_2 \hat{W}_1}{\lambda_1 - \lambda_2} \{\lambda_1\hat{W}_{11} + \frac{2 \lambda_2 \hat{W}_1}{\lambda_1 + \lambda_2} \}\right]^{\frac{1}{2}} \qquad (4.2.28)$$

which reduce to $\pm 2\mu$ in the case of hydrostatic stress.

4.2.3 Surface deformations

On a path of deformation and (hydrostatic) stress from the origin (1,1,0) in $(\lambda_1,\lambda_2,\sigma_2)$-space, the inequality (4.2.26) just fails at points satisfying

$$\lambda_2^{-1} \lambda_1^3 \hat{W}_{11} + \lambda_1\hat{W}_1 + 2\sigma_2(1 - \lambda_2^{-1} \lambda_1) - \sigma_2 /\gamma = 0, \qquad (4.2.29)$$

with γ given by (1.4). Equation (4.2.29) defines the boundary of the domain of stability in $(\lambda_1,\lambda_2,\sigma_2)$-space. At points of this boundary the secular equation (4.2.10) has solution $\eta = \sqrt{\alpha/\gamma}$ or, equivalently, $\xi = 0$ (i.e. c = 0).

The general solution (4.1.24) with $\omega = kc = 0$ then represents a quasi-static incremental surface deformation, or standing wave with wave number k. In other words, bifurcation from a state of pure homogeneous strain into an inhomogeneous mode of deformation can occur at points $(\lambda_1,\lambda_2,\sigma_2)$ satisfying (4.2.29).

For the case $\sigma_2 = 0$, equation (4.2.29) reduces to

$$\lambda_1^2 \hat{W}_{11} + \lambda_2\hat{W}_1 = 0, \qquad (4.2.30)$$

which describes a curve in the (λ_1,λ_2)-space.

Incremental surface deformations, or, 'instabilities', in an incompressible elastic material have been considered previously by Nowinski (1969a,b), Usmani and Beatty (1974), Reddy (1982,1983), Ogden (1984) and Biot (1965) for $\sigma_2 = 0$ either in respect of a specific strain-energy function or for a particular class of pre-strains (or both). For the neo-Hookean strain-energy function, surface deformations in which the displacement $\underline{u}$ depends on x_3 as well as on x_1 and x_2, and with $u_2 \neq 0$, were examined by Nowinski (1969b). Nowinski's results were recovered in the incompressible limit by Usmani and Beatty (1974), who used a compressible counterpart of the neo-Hookean strain-energy function. With the exception of the results relating to the latter type of incremental deformation the results given in the papers mentioned above are embraced by equation (4.2.30).

For a neo-Hookean strain-energy function, we have

$$\hat{W}(\lambda_1,\lambda_3) = \tfrac{1}{2}\,\mu\,(\lambda_1^2 + \lambda_1^{-2}\,\lambda_3^{-2} + \lambda_3^2 - 3), \tag{4.2.31}$$

and equation (4.2.3) becomes

$$\lambda_1^3 + \lambda_1^2\,\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_2^3 = 0. \tag{4.2.32}$$

This is the same equation as (4.2.7) and therefore has a unique positive solution $\lambda_1 = \eta_0\,\lambda_2$, which may also be written as $\lambda_1^2\,\lambda_3 = \eta_0$, by using the incompressibility conditions (2.2.4). In different notation the latter result is contained in Usmani and Beatty (1974). In plane strain $\lambda_2 = \lambda_1^{-1}$ ($\lambda_3 = 1$) equation (4.2.32) reduces to

$$\lambda_1^6 + \lambda_1^4 + 3\lambda_1^2 - 1 = 0, \tag{4.2.33}$$

which was obtained by Nowinski (1969 a,b). The solution of this equation (4.2.33) is $\lambda_1 = \sqrt{\eta_0}$.

4.3 Results for particular deformations

4.3.1 Plane strain

When the underlying deformation of the half-space corresponds to plane strain with $\lambda_3 = 1$ we write $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$ and

$$\tilde{W}(\lambda) = \hat{W}(\lambda, 1). \tag{4.3.1}$$

It follows from (4.2.23)-(4.2.25) that

$$\sigma_1 - \sigma_2 = \alpha - \gamma = \lambda \tilde{W}' , \tag{4.3.2}$$

$$\alpha = \lambda^4 \gamma = \frac{\lambda^5 \tilde{W}'}{\lambda^4 - 1} \tag{4.3.3}$$

and

$$2\beta + 2\gamma = \lambda^2 \tilde{W}'' , \tag{4.3.4}$$

with $\tilde{W}''(1) = 4\mu$, where the prime indicates differentiation with respect to λ.

The stability inquality (4.2.26) becomes

$$\lambda^4 \tilde{W}'' + \lambda \tilde{W}' + 2\sigma_2 (1 - \lambda^2) - \sigma_2^2/\gamma > 0 \tag{4.3.5}$$

and, when $\sigma_2 = 0$, this simplifies to

$$\lambda^3 \tilde{W}'' + \tilde{W}' > 0. \tag{4.3.6}$$

We note later, in Section 4.6, that the corresponding dead-load stability inequalities are $\tilde{W}' > 0$, $\tilde{W}'' > 0$.

The bifurcation criterion

$$\lambda^3 \tilde{W}'' + \tilde{W}' = 0 \tag{4.3.7}$$

for the case $\sigma_2 = 0$ was obtained by Reddy (1983); see also Ogden (1984, p.448). In different notation an equivalent result was given by Nowinski (1969a).

For values of the stretch λ satisfying (4.3.6) the bounds on σ_2 given by (4.2.28) are specialized similarly, and the secular equation (4.2.10) becomes

$$\eta^3 + \eta^2 + (\lambda^2 \tilde{W}'' - \frac{\lambda^5 \tilde{W}'}{\lambda^4 - 1} - 2\sigma_2)\ \eta/\gamma - (\gamma - \sigma_2)^2/\gamma^2 = 0, \tag{4.3.8}$$

with γ given by (4.3.3). When $\sigma_2 = 0$ this simplifies to

$$\eta^3 + \eta^2 + (\lambda^2 \tilde{W}'' - \frac{\lambda^5 \tilde{W}'}{\lambda^4 - 1})\eta/\gamma - 1 = 0. \tag{4.3.9}$$

In respect of the special case e = 0 discussed in Section 4.2 the right-hand inequality in (4.2.21) now requires

$$\lambda^2 \tilde{W}'' < \frac{\lambda \hat{W}'}{\lambda^4 - 1} (\lambda^4 + 2). \tag{4.3.10}$$

As we shall see in the next section this can be satisfied for particular strain-energy functions.

4.3.2 Equibiaxial deformation $\lambda_2 = \lambda_1$

For this deformation we take $\lambda_3 = \lambda$ and $\lambda_1 = \lambda_2 = \lambda^{-\frac{1}{2}}$ so that $\alpha = \beta = \gamma = {}^1/_4\ \lambda^{-1}\ \bar{W}_{11}(\lambda^{-\frac{1}{2}}\ ,\ \lambda)$ and the secular equation (4.1.38) for this case becomes

$$\eta^3 + \eta^2 + (3 - 2\sigma_2/\gamma)\eta - (1 - \sigma_2/\gamma)^2 = 0. \tag{4.3.11}$$

This equation is the same as (4.2.3), so we deduce from (4.2.6) that the bounds on σ_2 are $\pm 2\gamma$. When $\sigma_2 = 0$ equation (4.2.7) yields once more the surface wave speed is therefore given by $\rho c^2 = (1 - \eta_0{}^2)\alpha$.

4.3.3 Equibiaxial deformation $\lambda_1 = \lambda_3$

In this case we write $\lambda_1 = \lambda_3 = \lambda$, $\lambda_2 = \lambda^{-2}$, and the secular equation (4.1.38) is specialized accordingly. For the case $\sigma_2 = 0$ the secular equation in different notation was given by Willson (1973a) for this deformation in the form (4.1.30). Numerical results are similar to those given for plane strain so we omit further details here.

For the case $\sigma_2 = 0$, the boundary of the stability regime is given by

$$\lambda^4 \hat{W}_{11}(\lambda,\lambda) + \hat{W}_1(\lambda,\lambda) = 0. \tag{4.3.12}$$

This is equivalent to a formula of Reddy (1982), who considered its implications in respect of a number of strain-energy functions. We shall discuss it in the next section.

4.3.4 Equibiaxial deformation $\lambda_2 = \lambda_3$

For this case we write $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-\frac{1}{2}}$ and the counterpart of (4.3.12) is

$$\lambda^{5/2} \hat{W}_{11}(\lambda,\lambda^{-\frac{1}{2}}) + \hat{W}_1(\lambda,\lambda^{-\frac{1}{2}}) = 0 \tag{4.3.13}$$

Again numerical calculations give broadly similar results for the wave speed to those obtained for plane strain.

4.4 Results for some special strain-energy functions

4.4.1 The neo-Hookean material

For the neo-Hookean material, the strain-energy function is given by (4.2.31), namely

$$W = \tfrac{1}{2}\,\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3). \tag{4.4.1}$$

In plane strain (4.3.1) and (4.4.1) yield

$$\tilde{W}(\lambda) = \tfrac{1}{2}\,\mu(\lambda^2 + \lambda^{-2} - 2)$$

and the secular equation of the form (4.3.9) reduces to

$$\eta^3 + \eta^2 + 3\eta - 1 = 0.$$

Again this equation is the same as (4.2.7). Note that η is independent of λ. It follows from (4.1.35) that the wave speed c is given by

$$\xi \equiv \frac{\rho c^2}{\mu} = \lambda^2 - \lambda^{-2}\,\eta_0^2. \tag{4.4.2}$$

The more general secular equation is obtained by using (4.4.1) in the secular equation (4.1.39), giving

$$\left[\frac{\mu\,\lambda_1^2 - \rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}\left[\frac{\mu\,\lambda_1^2 + 3\mu\,\lambda_2^2 - \rho c^2}{\mu\,\lambda_2^2}\right] = \frac{\mu\,\lambda_2^2 - \mu\,\lambda_1^2 + \rho c^2}{\mu\,\lambda_2^2} \tag{4.4.3}$$

On squaring and rearranging this equation can be written as

$$\xi^3 - (3\,\lambda_1^2 + 5\,\lambda_2^2)\xi^2 + (3\lambda_1^4 + 10\,\lambda_1^2\lambda_2^2 + 11\lambda_2^4)\xi - \lambda_1^6 + \lambda_2^6 - 11\,\lambda_1^2\lambda_2^4 - 5\,\lambda_1^4\lambda_2^2 = 0. \tag{4.4.4}$$

In Fig. 4.3 we plot $\xi/\lambda^2 \equiv c^2/c_s^2$ as a function of λ, where $\rho c_s^2 = \mu\lambda^2$ (recalling (4.1.22)). The underlying deformation is stable for $\lambda > \sqrt{\eta_0}$, and we see that c rapidly approaches c_s as λ increases from unity.

Turning next to the bounds on σ_2, given by (4.2.28) appropriately specialized, we find

$$-\lambda - 1 - \lambda^{-1} - \lambda^{-2} < \bar{\sigma}_2 < \lambda - 1 + \lambda^{-1} + \lambda^{-2}, \quad (4.4.5)$$

where $\bar{\sigma}_2 = \sigma_2/\mu$. The range of $\bar{\sigma}_2$ and λ for which the underlying state of deformation and stress is stable (and hence admits surface waves) is shown in Fig. 4.4. In view of the factorization

$$\lambda^6 + \lambda^4 + 3\lambda^2 - 1 = (\lambda^3 + \lambda^2 + \lambda - 1)(\lambda^3 + \lambda^2 + \lambda + 1)$$

we see that the lower bound in (4.4.5) vanishes where $\lambda = \sqrt{\eta_0}$, and this is reflected in Figure 4.4 .

In Figures 4.5 – 4.7 we solve equation (4.4.3) to plot ξ as a function of λ_1 for a series of different values of λ_3.

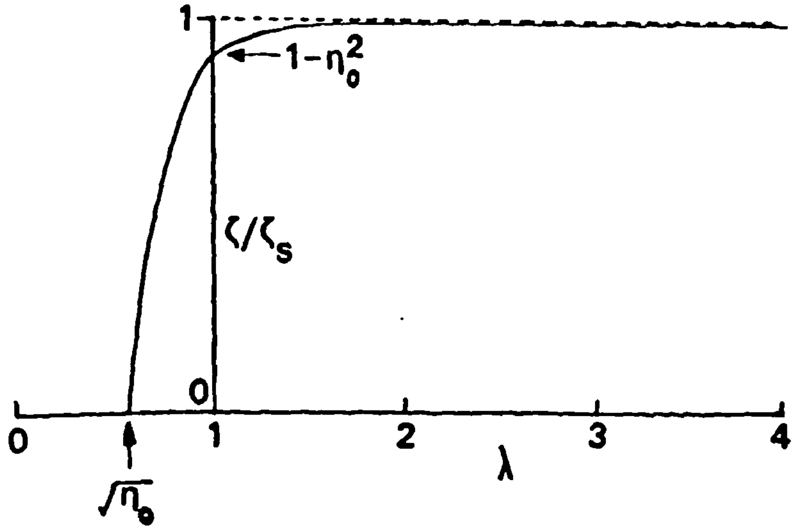


Figure 4.3

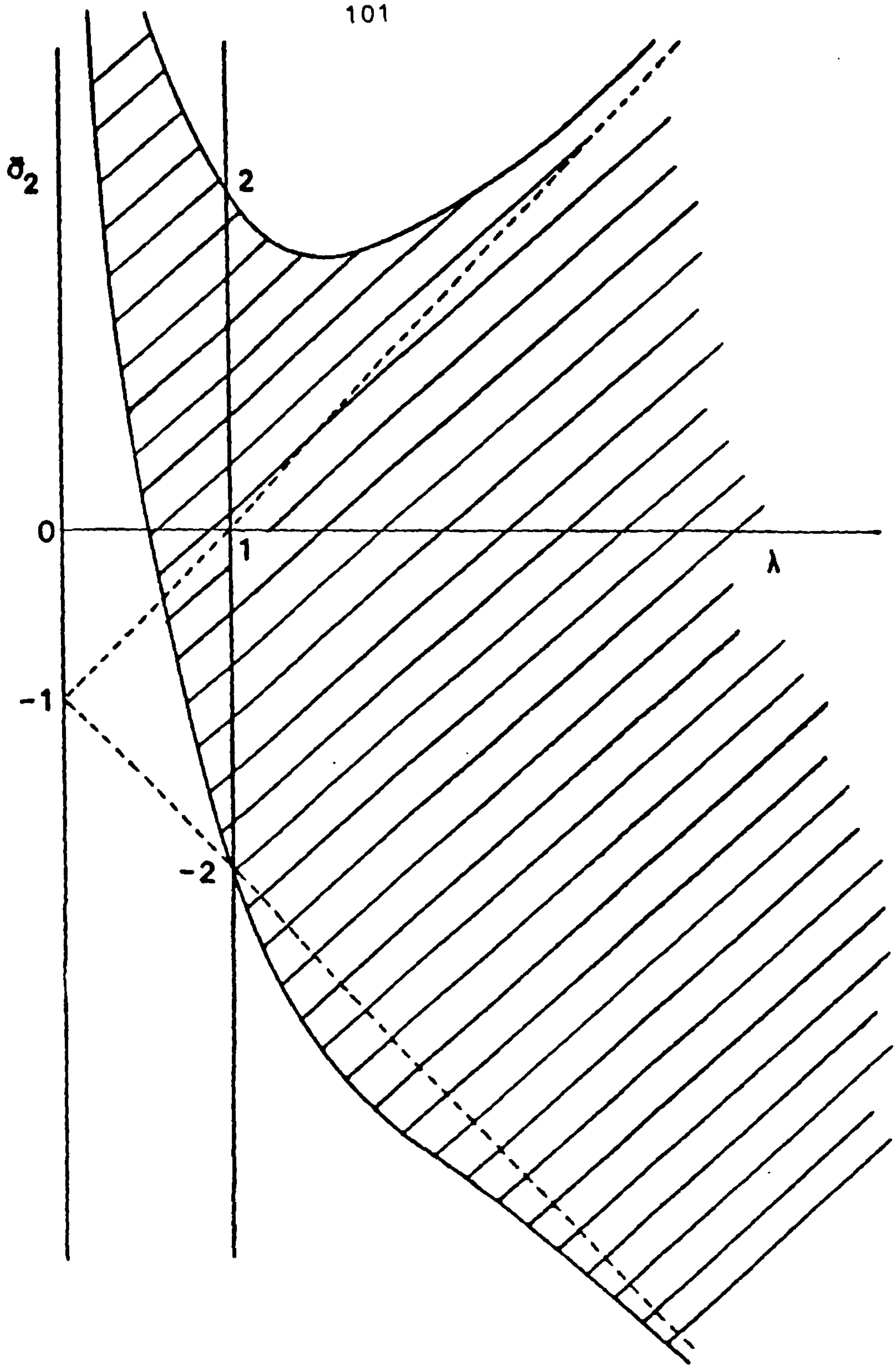
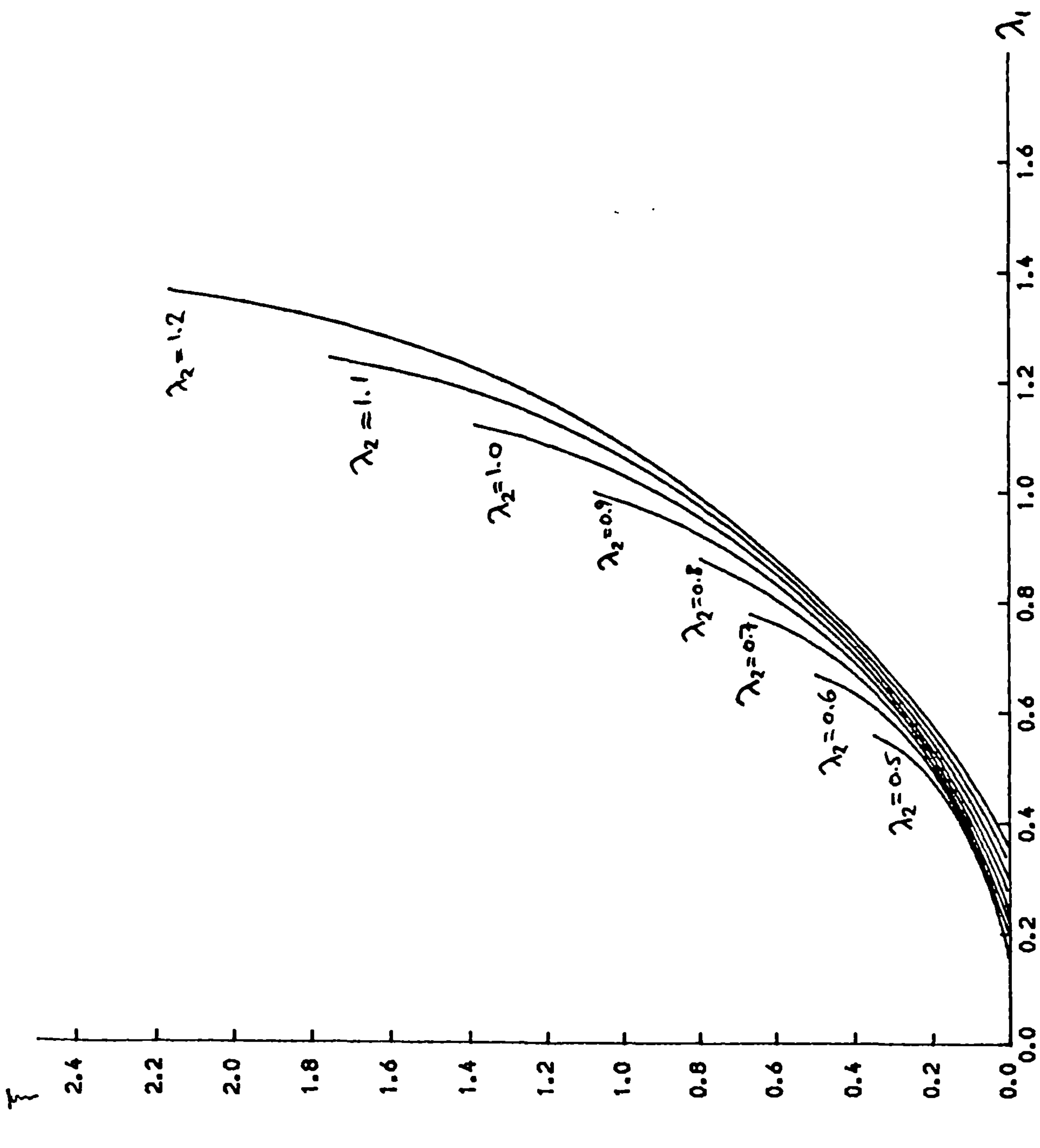


Figure 4.4



ξ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
λ_1
$\lambda_2 = 1.2$
$\lambda_2 = 1.1$
$\lambda_2 = 1.0$
$\lambda_2 = 0.9$
$\lambda_2 = 0.8$
$\lambda_2 = 0.7$
$\lambda_2 = 0.6$
$\lambda_2 = 0.5$

Figure 4.5

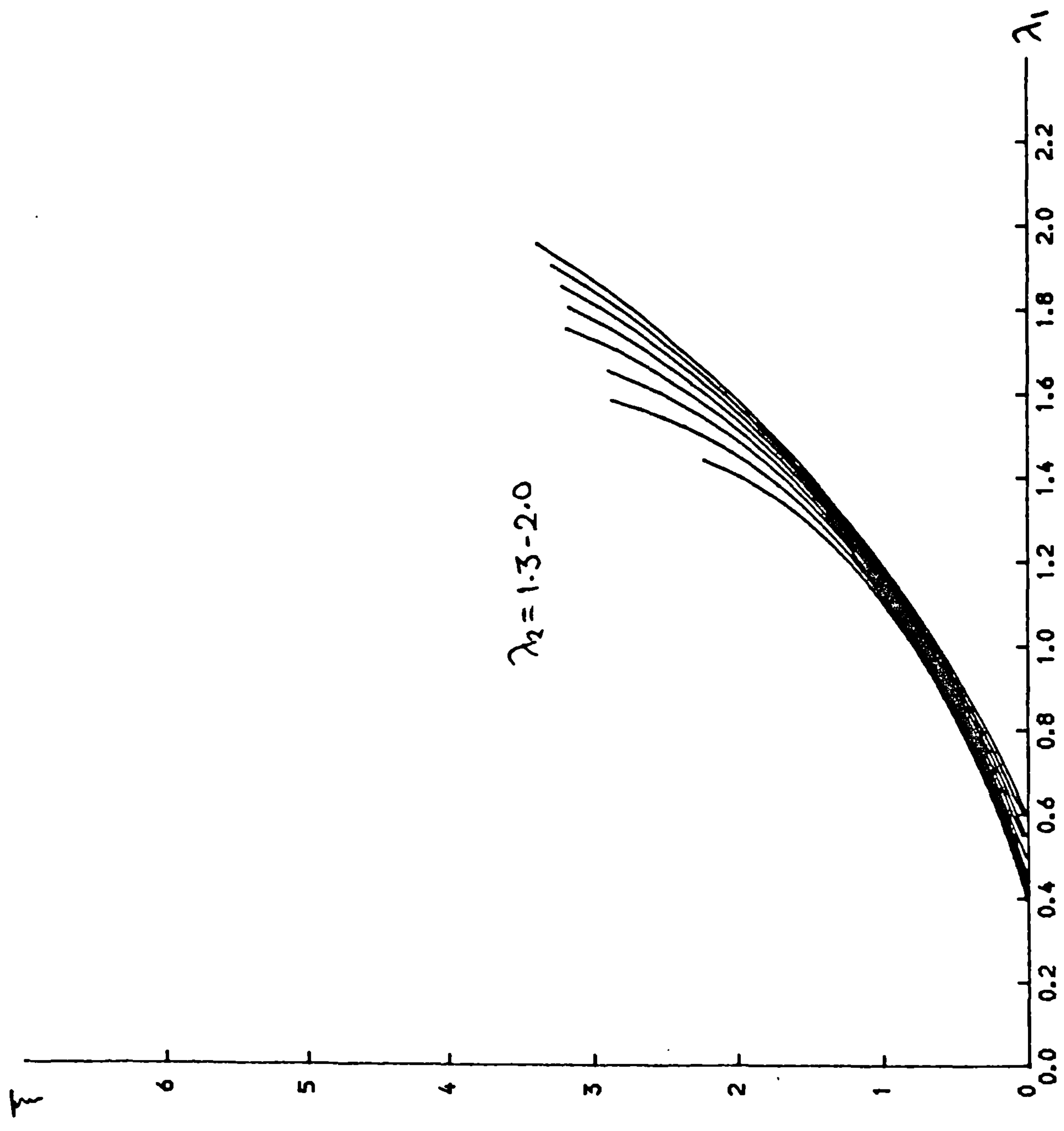


Figure 4.6

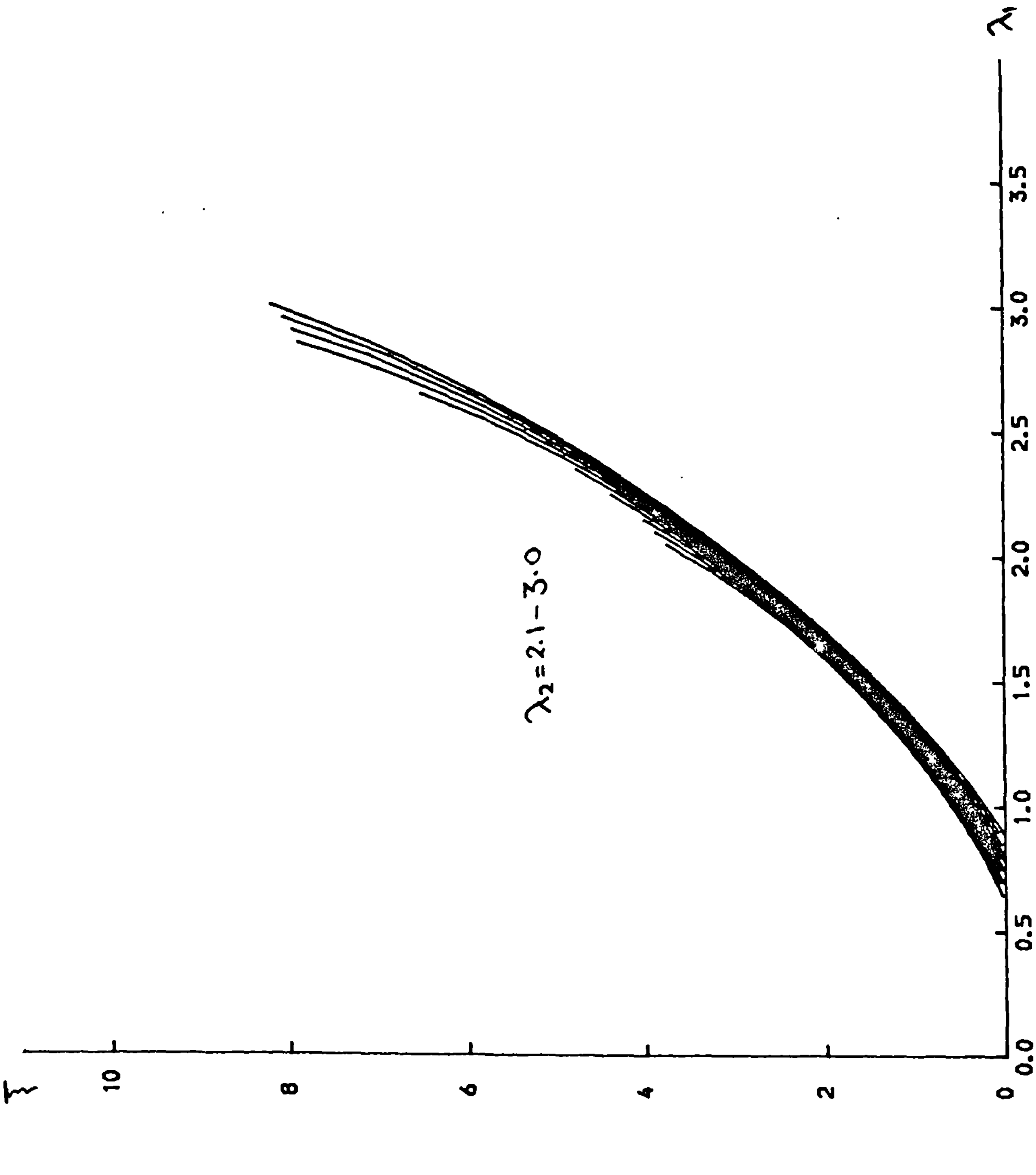


Figure 4.7

4.4.2 The Varga material

For the Varga material, the strain-energy function is given by

$$W = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{4.4.6}$$

so that

$$\tilde{W}(\lambda) = 2\mu(\lambda + \lambda^{-1} - 2). \tag{4.4.7}$$

When $\sigma_2 = 0$ the secular equation (4.3.9) reduces to

$$\eta^3 + \eta^2 + (2\lambda^2 + 2 - \lambda^4)\eta - 1 = 0, \tag{4.4.8}$$

which has a unique positive solution for η in the stable regime. From (4.3.6) it follows that the latter is defined by $\lambda^2 > {}^1/_3$, and σ_2 is bounded according to

$$-2\lambda^{-1}(1 - 3\lambda^2)/(\lambda^2 + 1) \quad < \bar{\sigma}_2 < \quad 2\lambda^{-1}.$$

From the solution of (4.4.7) we get

$$\xi \equiv \frac{\rho c^2}{\mu} = \frac{2(\lambda^2 - \lambda^{-2}\eta^2)}{\lambda + \lambda^{-1}}, \tag{4.4.9}$$

while the shear wave speed c_s is given by

$$\xi_s \equiv \frac{\rho c_s^2}{\mu} = \frac{2\lambda^2}{\lambda + \lambda^{-1}}. \tag{4.4.10}$$

For the case of $e = 0$ ($\sigma_2 = \gamma$) we have

$$\bar{\sigma}_2 = \frac{2\lambda^{-2}}{\lambda + \lambda^{-1}} \tag{4.4.11}$$

and from section 4.2.2(b) the secular equation yields

$$\eta = 0 \quad \text{or} \quad \eta^2 + \eta + \lambda^2(2 - \lambda^2) = 0. \tag{4.4.12}$$

A non-zero positive solution of the latter equation exists if $\lambda^2 > 2$, and, using (4.4.9) and (4.4.10), this can be shown to give

$$\xi/\xi_s = 2\lambda^{-2} + \tfrac{1}{2}\lambda^{-4}\{[1 + 4\lambda^2(\lambda^2-2)]^{\frac{1}{2}} - 1\}. \tag{4.4.13}$$

Thus, when $\lambda^2 > 2$, two waves can propagate, with speeds given by (4.4.10) and (4.4.13), provided the surface stress has the value (4.4.11) but only one of these is a surface wave.

Figure 4.8 shows the stable region in the $(\lambda,\bar{\sigma}_2)$-plane; it includes a plot of $\bar{\gamma} = \gamma/\mu$ as a function of λ, and we note that this lies entirely within the stable region. Figure 4.9 shows ξ_s and ξ as functions of λ for the case $\sigma_2 = \gamma$.

Also, the more general secular equation is obtained by using (4.4.6) in (4.1.39), giveing

$$\left[\frac{2\mu\lambda_1^2 - \rho c^2(\lambda_1 + \lambda_2)}{2\mu\lambda_2^2}\right]^{\frac{1}{2}}\left[\frac{2\mu\lambda_1^2 - \rho c^2(\lambda_1 + \lambda_2) + 4\mu\lambda_1\lambda_2 + 4\mu\lambda_2^2 - 2\mu\lambda_1^2}{2\mu\lambda_2^2}\right]$$

$$= \frac{\rho c^2(\lambda_1^2 + \lambda_2^2) + 2\mu\lambda_2^2 - 2\mu\lambda_2^2}{2\mu\lambda_2^2}$$

On squaring and rearranging this becomes

$$(\lambda_1 + \lambda_2)\xi^3 - 2(\lambda_1^2 + 4\lambda_1\lambda_2 + 3\lambda_2^2)\xi^2 + 8\lambda_2^2(2\lambda_1 + 2\lambda_1\lambda_2 - 3\lambda_2^2)\xi$$

$$- 8\lambda_2^2(3\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2) = 0. \tag{4.4.14}$$

In Figure 4.10 we solve equation (4.4.14) to plot ξ as a function of λ_1 for a series of different values of λ_2.

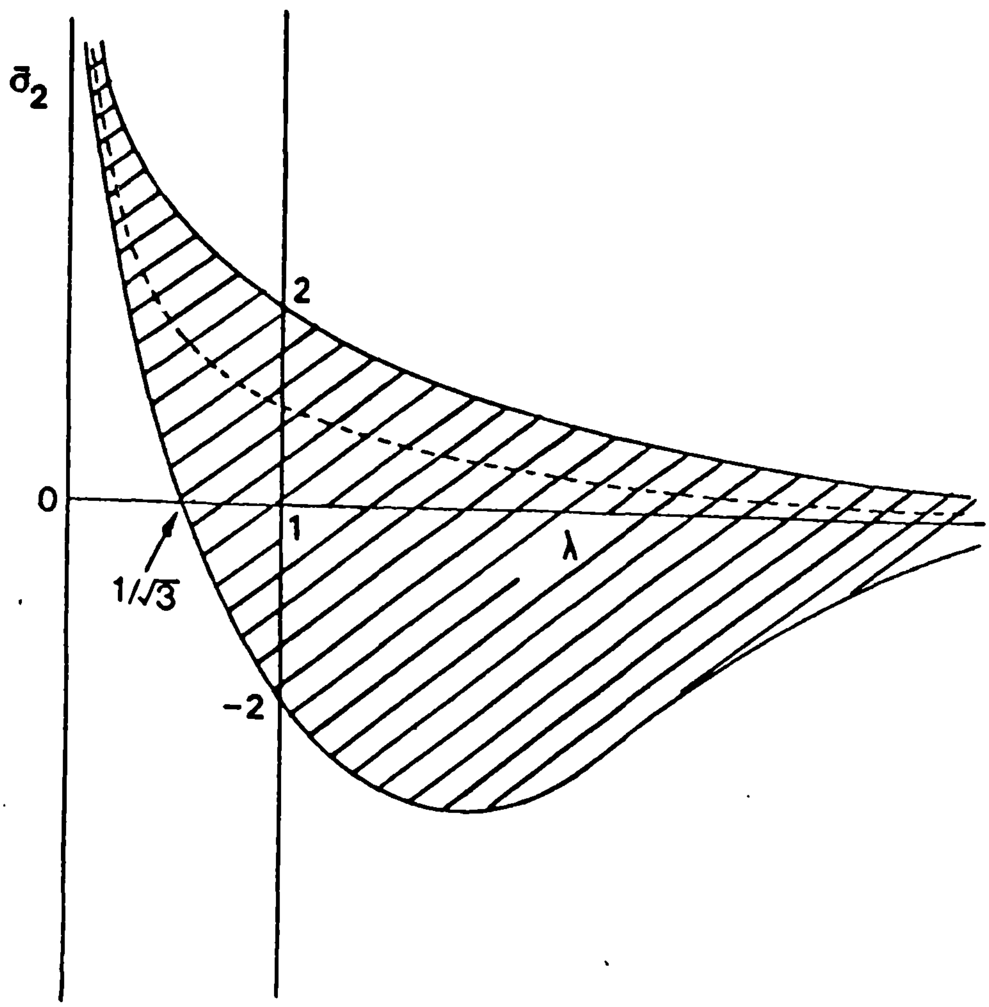


Figure 4.8

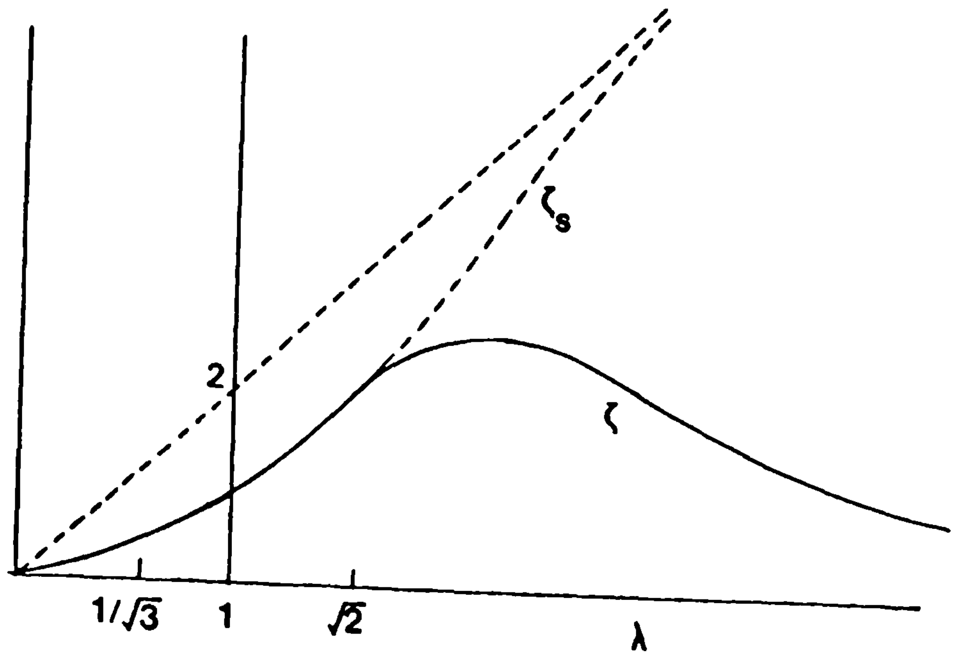


Figure 4.9

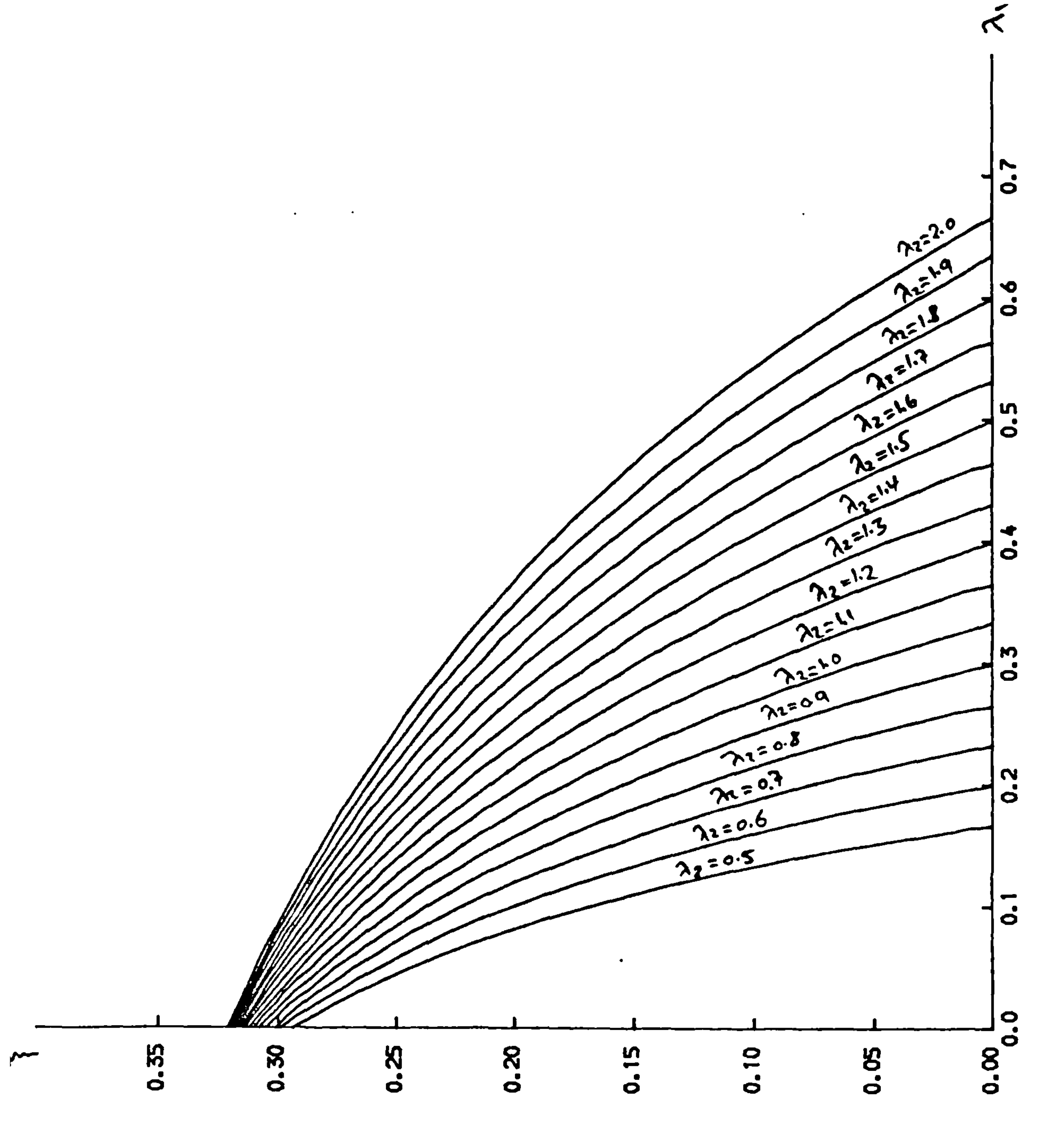

$\lambda_2=2.0$
$\lambda_2=1.9$
$\lambda_2=1.8$
$\lambda_2=1.7$
$\lambda_2=1.6$
$\lambda_2=1.5$
$\lambda_2=1.4$
$\lambda_2=1.3$
$\lambda_2=1.2$
$\lambda_2=1.1$
$\lambda_2=1.0$
$\lambda_2=0.9$
$\lambda_2=0.8$
$\lambda_2=0.7$
$\lambda_2=0.6$
$\lambda_2=0.5$
λ_1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35

Figure 4.10

The strain-energy functions considered in 4.4.1 and 4.4.2 above are members of the class of functions which, for plane strain, are defined by

$$\tilde{W}(\lambda) = 2\mu(\lambda^{m} + \lambda^{-m} - 2)/m^2 . \qquad (4.4.15)$$

Next we consider one further member of this class.

4.4.3 The $m = \frac{1}{2}$ strain-energy function

For $m = \frac{1}{2}$ in (4.4.15) the results are similar to those for $m = 1$ except that there is an upper bound to the set of λ for which the underlying deformation is stable. The stability inequality yields

$$-\lambda^2 + 3\lambda^2 + 2\lambda^2 - 2 > 0 \qquad (4.4.16)$$

and the bounds on $\bar{\sigma}_2$ are

$$-4\lambda^{-\frac{1}{2}} \frac{\lambda - 1}{\lambda^2 + 1} \pm 2\sqrt{2}\, \lambda^{\frac{1}{2}} \left[\frac{-\lambda^3 + 3\lambda^2 + 3\lambda - 1}{(\lambda^3 + \lambda^2 + \lambda + 1)(\lambda^2 + 1)}\right]^2, \qquad (4.4.17)$$

while

$$\gamma = \frac{4\mu\lambda^{-\frac{1}{2}}}{\lambda^3 + \lambda^2 + \lambda + 1} = \alpha\, \lambda^{-4}.$$

In Figure 4.11 we plot $\xi = \rho c^2/\mu$ as a function of λ for values of λ for which (4.4.16) holds, corresponding to free surface waves ($\sigma_2 = 0$). In Figure 4.12 are shown the bounds on $\bar{\sigma}_2$, given by (4.4.17), as functions of λ along with the corresponding plot of $\bar{\gamma} = \gamma/\mu$ from (4.4.18). We note that $\sigma_2 = \gamma$ is possible in the stable region for a wide range of values of λ.

In Figure 4.13 we illustrate the result for the case $\sigma_2 = \gamma$, noting that $\eta = 0$ ($\xi = \xi_s$) is the unique solution of the secular equation up to a critical value of λ, after which the positive solution of

$$\eta^2 + \eta - \tfrac{1}{2}(3\lambda^4 - 2\lambda^3 - 2\lambda^2 - 2\lambda + 1) = 0$$

becomes effective. The latter solution corresponds to a value of ξ which vanishes at a value of λ between 3 and 4. This reflects the fact that the stable region is bounded.

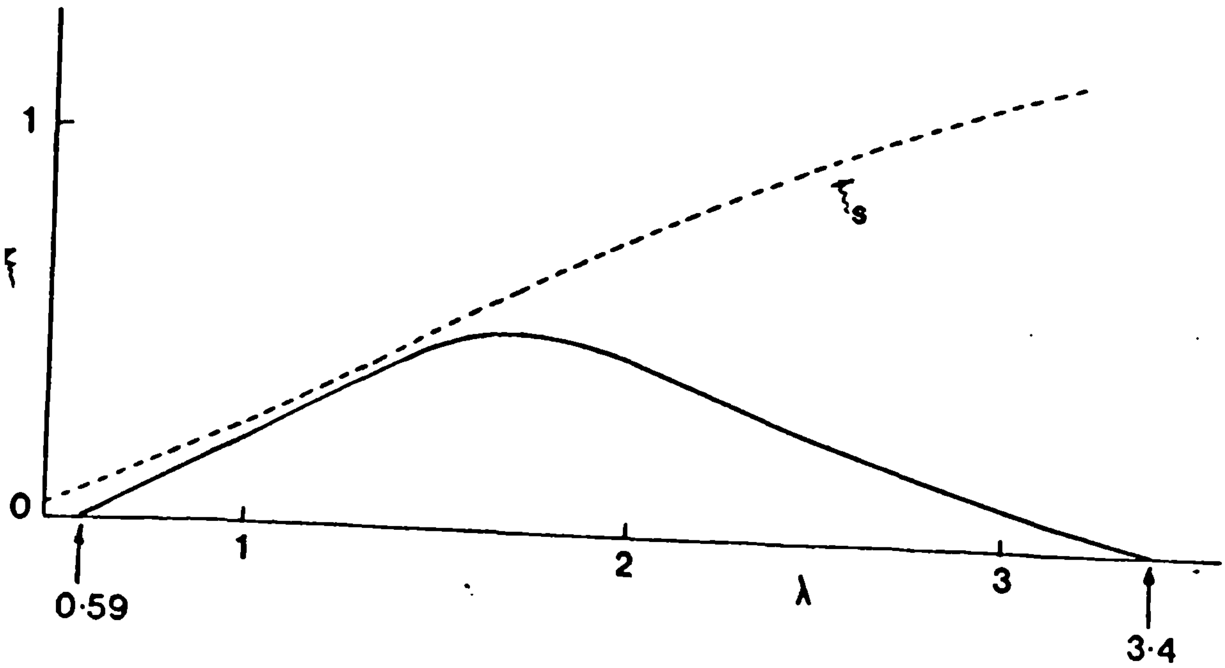


Figure 4.11

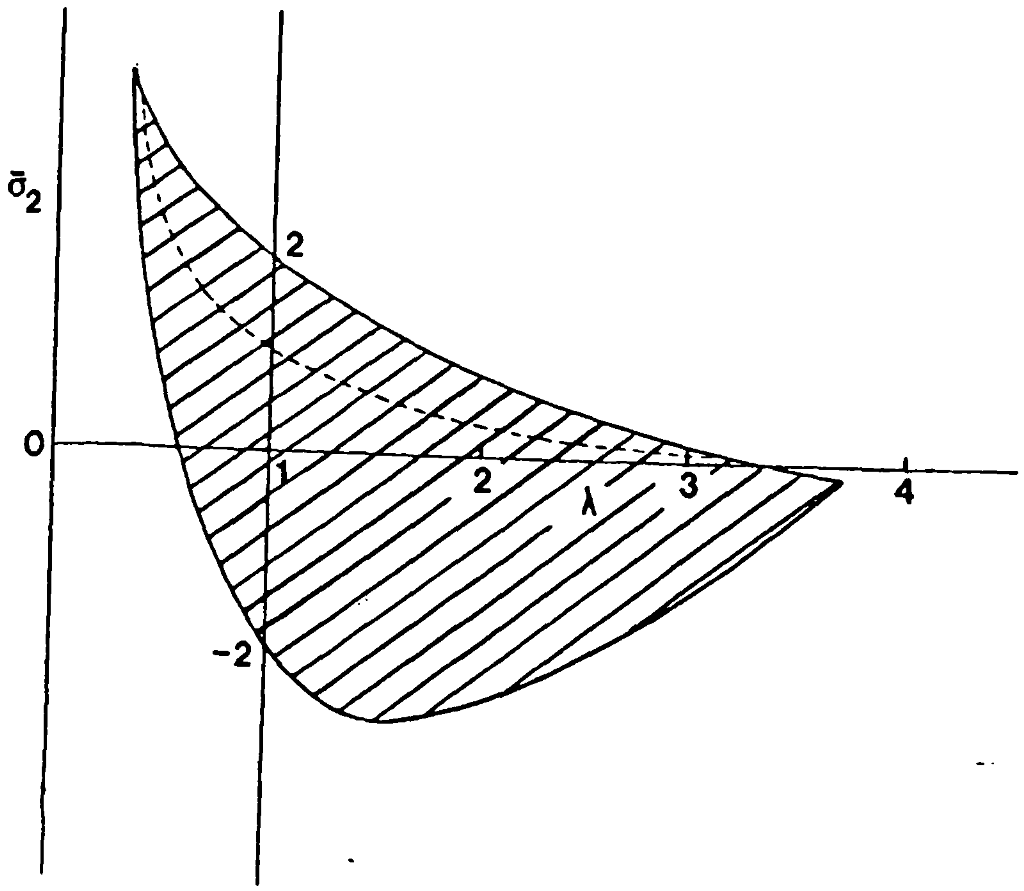


Figure 4.12

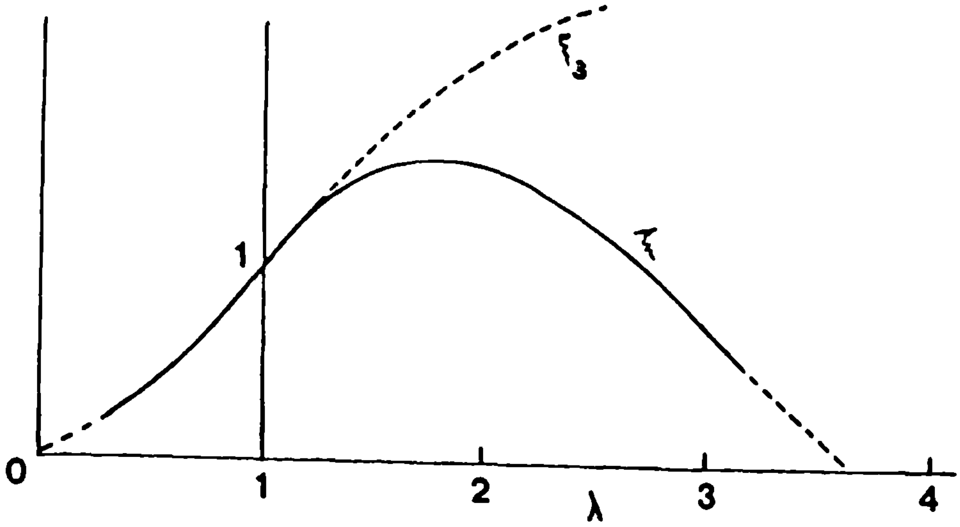


Figure 4.13

4.4.4 A three-term strain-energy function

The final case of the special strain-energy functions we consider corresponds to the model of rubber elasticity due to Ogden (1972). In plane strain the strain-energy function is a linear combination of terms of the form (4.4.15) for different values of m. We write this in the form

$$\tilde{W}(\lambda) = \sum_{n=1}^{3} \mu_n(\lambda^{\alpha_n} + \lambda^{-\alpha_n} - 2)/\alpha_n, \tag{4.4.19}$$

where

$$\sum_{n=1}^{3} \mu_n \alpha_n = 2\mu. \tag{4.4.20}$$

Specifically, the numerical values of the constants in (4.4.19) are

$$\begin{aligned} \alpha_1 &= 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0 \\ \mu_1 &= 1.491\mu, \quad \mu_2 = 0.003\mu, \quad \mu_3 = -0.0237\mu. \end{aligned} \tag{4.4.21}$$

For these values and for $\sigma_2 = 0$ the wave speed is shown in Figure 4.14 in the form ξ as a function of λ. We note that the surface wave speed is very close to the shear wave speed, which is also shown in Figure 4.14, for the range of values of λ between 1.9 and 3.4 approximately. The underlying plane strain is stable for λ greater than about 0.55. Bounds on σ_2 similar to those for the neo-Hookean strain energy are obtained for this strain-energy function, but they grow more rapidly as $\lambda \longrightarrow \infty$ in this case.

When $\sigma_2 = \gamma$ there is a range of values of λ for which d in (4.2.20) is negative and so the trend of the departure of the surface wave speed from the shear wave speed is similar to that shown in Figure 4.14.

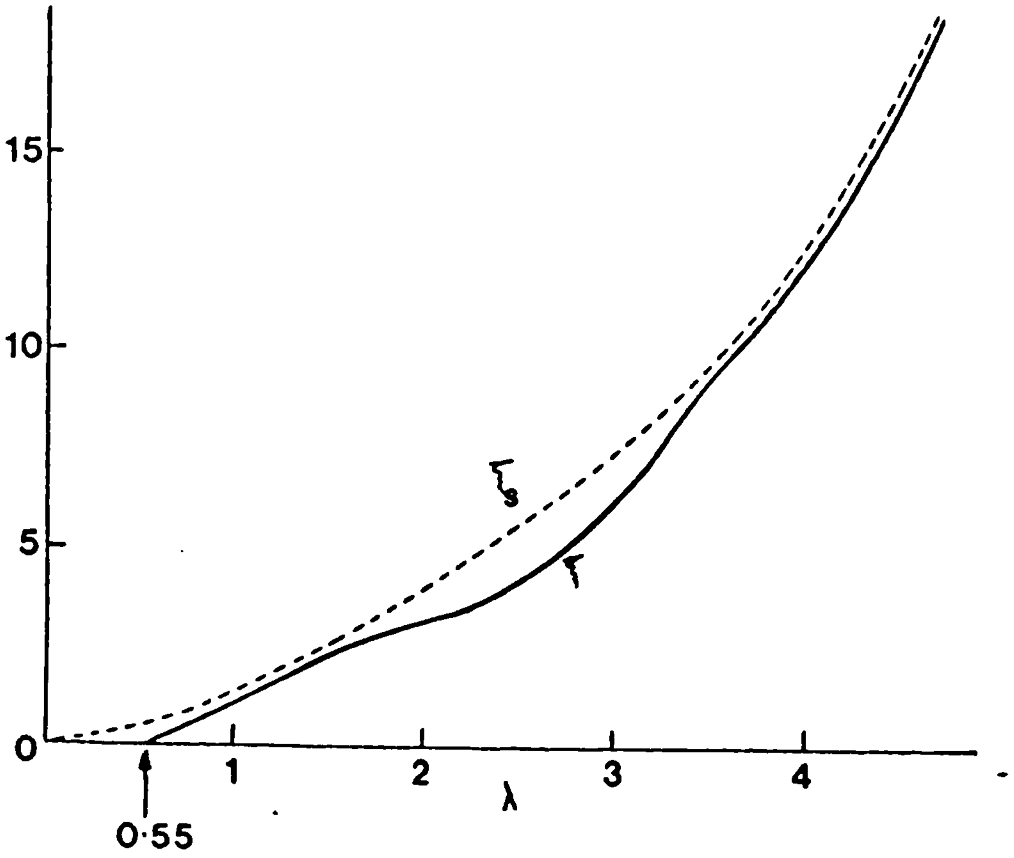


Figure 4.14

4.5 Hydrostatic stress boundary conditions

If the incremental dead-load traction boundary condition on $x_2 = 0$, which leads to (4.1.9) is replaced by a corresponding hydrostatic stress boundary condition then we have

$$\left.\begin{aligned} \dot{s}_{021} &= -\sigma_2 v_{2,1}, \\ \dot{s}_{022} &= -\sigma_2 v_{2,2}. \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (4.5.1)$$

Equations (4.1.15) are then replaced by

$$\left.\begin{aligned} \psi_{,22} - \psi_{,11} &= 0, \\ (2\beta + \gamma)\psi_{,112} + \gamma\, \psi_{,222} - \rho\ddot{\psi}_{,2} &= 0, \end{aligned}\right\} \quad \text{on } x_2 = 0 \qquad (4.5.2)$$

and the secular equation (4.1.29) by

$$(2\beta + 2\gamma - \rho c^2)\sqrt{\alpha - \rho c^2} = \sqrt{\gamma}\,(\gamma - \alpha + \rho c^2), \qquad (4.5.3)$$

or, equivalently,

$$\eta^3 + \eta^2 + (2\beta + 2\gamma - \alpha)\eta/\gamma - 1 = 0. \qquad (4.5.4)$$

This is the same as (4.1.39) for $\sigma_2 = 0$, but σ_2 is not in general zero here.

The stability regime is again determined by (4.2.15) and (4.2.16), but no restriction is placed on the hydrostatic stress. When the stress is purely hydrostatic equation (4.5.4) reduces to (4.2.7). The surface wave speed is therefore given by $\rho c^2 = \mu(1 - \eta_0^2)$ independently of the hydrostatic stress.

4.6 Note on infinitesimal stability

In this section, it worth noting that, since $\mu > 0$, the undeformed configuration is stable under arbitrary all-round hydrostatic loading.

Under all-round dead load the infinitesimal stability condition (or exculsion condition) is

$$\mathrm{tr}\ \{\ (\underline{A}_0\ \underline{L})\underline{L} + \mathrm{p}\ \underline{L}^2\} > 0, \tag{4.6.1}$$

for all $\underline{L} \neq \underline{0}$ such that $\mathrm{tr}(\underline{L})=0$; see, for example, Ogden(1984) for detailed discussion. When specialized to two-dimensional incremental deformations (4.6.1) becomes

$$(A_{01111} + A_{02222} - 2A_{01122} + 2\mathrm{p})\ L_{11}^2 + A_{01212}\ L_{21}^2 + A_{02121}\ L_{12}^2$$
$$+ 2(A_{02121} - \sigma_2)\ L_{12}\ L_{21} > 0. \tag{4.6.2}$$

In terms of the notation in (4.1.12) necessary and sufficient conditions for (4.6.2) to hold are

$$\begin{aligned} 2\beta + 2\gamma - 2\sigma_2 &> 0, \\ \alpha &> 0, \\ \gamma\ \alpha &> (\gamma - \sigma_2)^2. \end{aligned} \tag{4.6.3}$$

On the other hand, if the loading is an all-round hydrostatic stress σ then (4.6.1)is replaced by

$$\mathrm{tr}\{(\underline{A}_0\ \underline{L}) + (\sigma + \mathrm{p})\underline{L}^2\} > 0 \tag{4.6.4}$$

and (4.6.2) by

$$4\mu\ L_{11}^2 + \mu(L_{12} + L_{21})^2 > 0. \tag{4.6.5}$$

Since $\mu > 0$ the undeformed configuration is stable for all hydrostatic stress provided $L_{12} + L_{21} \neq 0$. The incremental mode of deformation corresponding to $L_{12} + L_{21} \neq 0$ represents a shear in the (1,2) principal plane.

4.7 Propagation in a general direction

In this section we shall obtain equations for Rayleigh surface waves propagating in general direction in the (x_1, x_3)-plane, in which the direction of the propagation has the direction $(\cos\theta, \sin\theta)$.

For an incompressible material the incremental equations of motion are given by

$$\begin{aligned} A_{0jilk} v_{k,jl} - \dot{p}_{,i} &= \rho \dot{v}_i, \\ v_{i,i} &= 0. \end{aligned} \tag{4.7.1}$$

Assume $\underline{v}$ and $\dot{p}$ are given by

$$\begin{aligned} \underline{v} &= \underline{\psi}(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)}, \\ \dot{p} &= \varphi(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)}, \end{aligned} \tag{4.7.2}$$

that is the components of $\underline{v}$ are

$$v_i = \psi_i(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)}, \quad i \in \{1,2,3\}. \tag{4.7.3}$$

From equation $(4.7.1)_1$, we have

$$A_{0j_1l_1}\, v_{1,jl} + A_{0j_1l_2}\, v_{2,jl} + A_{0j_1l_3}\, v_{3,jl} - \dot{p}_{,1} = \rho\ddot{v}_1,$$

$$A_{0j_2l_1}\, v_{1,jl} + A_{0j_2l_2}\, v_{2,jl} + A_{0j_2l_3}\, v_{3,jl} - \dot{p}_{,2} = \rho\ddot{v}_2, \quad (4.7.4)$$

$$A_{0j_3l_1}\, v_{1,jl} + A_{0j_3l_2}\, v_{2,jl} + A_{0j_3l_3}\, v_{3,jl} - \dot{p}_{,3} = \rho\ddot{v}_3,$$

Also, from $(4.7.1)_2$, we get

$$-ik\cos\theta\ \psi_1(x_2) + \psi_1'(x_2) - ik\sin\theta\ \psi_2'(x_3) = 0. \qquad (4.7.5)$$

By differentiating (4.7.2) and (4.7.3) and substituting in (4.7.4), we obtain

$$-k^2\cos^2\theta\ A_{01111}\ \psi_1 + A_{02121}\ \psi_1'' - k^2\sin^2\theta\ A_{03131}\ \psi_1$$
$$- ik\cos\theta\ (A_{01122} + A_{02112})\psi_2' - k^2\cos\theta\ \sin\theta\ (A_{01133} + A_{03113})\psi_3$$
$$+ ik\cos\theta\ \varphi = -\rho\omega^2\ \psi_1,$$

$$- ik\cos\theta\ (A_{01221} + A_{02211})\psi_1' - k^2\cos\theta\ A_{01212}\ \psi_2$$
$$- k^2\sin^2\theta\ A_{03232}\ \psi_2 + A_{02222}\ \psi_2''$$
$$- ik\sin\theta\ (A_{02233} + A_{03223})\psi_3' + \varphi' = -\rho\omega^2\ \psi_2'', \quad (4.7.6)$$

$$- k^2\cos\theta\ \sin\theta\ (A_{01331} + A_{03311})\psi_1 - ik\sin\theta\ (A_{02332} + A_{03322})\psi_2'$$
$$- k^2\cos^2\theta\ A_{01313}\ \psi_3 + A_{02323}\ \psi_3'' - k^2\sin^2\theta\ A_{03333}\ \psi_3$$
$$+ ik\sin\theta\ \varphi = -\rho\omega^2\ \psi_3.$$

Suppose that

$$\psi_1 = A\, e^{-skx_2}, \qquad \psi_2 = B\, e^{-skx_2},$$
$$\qquad (4.7.7)$$
$$\psi_3 = C\, e^{-skx_2} \quad \text{and} \quad \varphi = D\, e^{-skx_2}.$$

So, equations (4.7.5) and (4.7.6) becomes

$$is\, B = \cos\theta\, A + \sin\theta\, C,$$

$$(\rho c^2 + s^2 A_{02121} - \cos^2\theta\, A_{01111} - \sin^2\theta\, A_{03131})\, A$$

$$+ is \cos\theta\, (A_{01122} + A_{02112})\, B - \cos\theta \sin\theta\, (A_{01133} + A_{02112}) C$$

$$+ i\, {}^{c}/_{\omega} \cos\theta\, D = 0,$$

$$- s \cos\theta\, (A_{01221} + A_{02211})\, A \tag{4.7.8}$$

$$+ (\rho c^2 + s^2 A_{02222} - \cos^2\theta\, A_{01212} - \sin^2\theta\, A_{03232})\, B$$

$$- s \sin\theta\, (A_{02233} + A_{03223})\, C - is\, {}^{c}/_{\omega}\, D = 0$$

$$- \cos\theta \sin\theta\, (A_{01331} + A_{03311})\, A + s \sin\theta\, (A_{02332} + A_{03322})\, B$$

$$+ (\rho c^2 + s^2 A_{02323} - \cos^2\theta\, A_{01313} - \sin^2\theta\, A_{03333})\, C$$

$$+ i\, {}^{c}/_{\omega} \sin\theta\, D = 0.$$

On elimination of B, equations (4.7.8) become

$$[\rho c^2 + s^2 A_{02121} + \cos^2\theta \, (A_{01122} + A_{02112} - A_{01111}) - \sin^2\theta \, A_{03131}]A$$

$$+ [\cos\theta \sin\theta (A_{01122} + A_{02112} - A_{01133} - A_{03113})] \, C$$

$$+ i \, {}^{c}/_{\omega} \cos\theta \, D = 0,$$

$$[\rho c^2 + s^2 (A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta \, A_{01212}$$

$$- \sin^2\theta \, A_{03232}] \, k\cos\theta \, A + [\rho c^2 + s^2 \, (A_{02222} - A_{01221} - A_{02211})$$

$$- \cos^2\theta \, A_{01212} - \sin^2\theta \, A_{03232}] \, k \sin\theta C - i \, {}^{c}/_{\omega} \, s^2 D = 0 \qquad (4.7.9)$$

$$\sin\theta \cos\theta (A_{02332} + A_{03322} - A_{01331} - A_{03311}) \, A$$

$$+ [\rho c^2 + s^2 A_{02323} - \cos^2\theta \, A_{01313} + \sin^2\theta (A_{02332} + A_{03322} - A_{03333})]C$$

$$+ i \, c/\omega \sin\theta \, D = 0.$$

For these equations (4.7.9) to have non-trivial solution for A, C and D we must have

$$\begin{vmatrix} \begin{matrix} \rho c^2 + s^2 A_{02121} \\ + \cos^2\theta(A_{01122} + A_{02112} \\ - A_{01111}) - \sin^2\theta\, A_{03131} \end{matrix} & \begin{matrix} \cos\theta \sin\theta(A_{01122} + A_{01133} \\ - A_{03113}) \end{matrix} & i^c/_\omega \cos\theta \\ \\ \begin{matrix} [\rho c^2 + s^2(A_{02222} \\ - A_{01221} - A_{02211}) \\ - \cos^2\theta\, A_{01212} \\ - \sin^2\theta\, A_{03232}]\, k\cos\theta \end{matrix} & \begin{matrix} [\rho c^2 + s^2(A_{02222} \\ - A_{01221} - A_{02211}) \\ - \cos^2\theta\, A_{01212} \\ - \sin^2\theta\, A_{03232}]\, k\sin\theta \end{matrix} & -i^c/_\omega\, s^2 \\ \\ \begin{matrix} \sin\theta \cos\theta(A_{02332} \\ + A_{03322} + A_{01331} \\ - A_{03311}) \end{matrix} & \begin{matrix} \rho c^2 + s^2 A_{02323} \\ - \cos^2\theta\, A_{01313} \\ + \sin^2\theta(A_{02332} \\ + A_{03322} - A_{03333}) \end{matrix} & -i^c/_\omega \sin\theta \end{vmatrix} = 0$$

i.e

$$[\rho c^2 + s^2 A_{02121} - \cos^2\theta(A_{01122} + A_{02112} - A_{01111}) - \sin^2\theta\, A_{03131}]$$

$$[s^2\{\rho c^2 + s^2 A_{02323} - \cos^2\theta\, A_{01313} + \sin^2\theta(A_{03322} - A_{03333})\}$$

$$-\sin^2\theta\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\, A_{01212}$$

$$- \sin^2\theta\, A_{03232}\}]$$

$$- \sin\theta\cos\theta\,(A_{01122} + A_{01133} - A_{03113})[\sin\theta\cos\theta\,\{\rho c^2$$

$$+ s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\, A_{01212} - \sin^2\theta\, A_{03232}\}$$

$$- s^2\sin\theta\cos\theta\,(A_{02332} + A_{03322} - A_{01331} - A_{03311})]$$

$$+ \cos\theta\,[\cos\theta\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\, A_{01212}$$

$$- \sin^2\theta\, A_{03232}\}\{\rho c^2 + s^2 A_{02323} - \cos^2\theta\, A_{01313} + \sin^2\theta(A_{02332}$$

$$+ A_{03322} - A_{03333})\} - \sin\theta k\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211})$$

$$- \cos^2\theta\, A_{01212} - \sin^2\theta\, A_{03232}\}\{\sin\theta\cos\theta\,(A_{02332} + A_{03322}$$

$$- A_{01331} - A_{03311})\}] = 0. \qquad (4.7.10)$$

This equation is a cubic equation for s^2. Let s_1, s_2 and s_3 be the three values of s with positive real part. Then we may write the solution in the form

$$\psi_1 = A_1\, e^{-s_1 kx_1} + A_2\, e^{-s_2 kx_2} + A_3\, e^{-s_3 kx_3},$$

$$\psi_2 = B_1\, e^{-s_1 kx_1} + B_2\, e^{-s_2 kx_2} + B_3\, e^{-s_3 kx_3},$$

$$\psi_3 = C_1\, e^{-s_1 kx_1} + C_2\, e^{-s_2 kx_2} + C_3\, e^{-s_3 kx_3}, \qquad (4.7.11)$$

$$\varphi = D_1\, e^{-s_1 kx_1} + D_2\, e^{-s_2 kx_2} + D_3\, e^{-s_3 kx_3}.$$

To obtain the speed of surface wave propagation in any direction for the general case we must deduce that from (4.7.8) the ratio A_i: B_i: C_i: D_i, $i \in \{1,2,3\}$ for A_1: A_2: A_3 to be non-zero solution, the boundary conditions yield the secular equation. Because of the cumbersome algebra involved we omit details of the general case here, but concentrate on the application to the neo-Hookean strain-energy function.

4.7.1 Propagation in general direction for a neo-Hookean material

For a neo-Hookean material the strain-energy function is given by

$$W = \tfrac{1}{2}\,\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$

and hence, from (2.6.13), we obtain

$$\begin{aligned}
&A_{01111} = A_{01212} = A_{01313} = \mu\,\lambda_1^2,\\
&A_{02222} = A_{02121} = A_{02323} = \mu\,\lambda_2^2,\\
&A_{03333} = A_{03131} = A_{03232} = \mu\,\lambda_3^2,\\
&A_{01122} = A_{01221} = A_{01331} = A_{02233} = A_{02332} = A_{03223} = 0
\end{aligned} \qquad (4.7.12)$$

Substituting (4.7.12) in (4.7.10) we obtain

$$(\rho c^2 + \mu\,\lambda_2^2\,s^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)^2(s^2 - 1) = 0. \qquad (4.7.13)$$

This equation is a cubic equation for s^2, which yields two distinct values of s^2 with positive real part, s_1^2 and s_2^2 say, where

$$s_1^2 = 1 \qquad \text{and} \qquad s_2^2 = (\mu\, \lambda_1^2 \cos^2\theta + \mu\, \lambda_3^2 \sin^2\theta - \rho c^2)/\mu\, \lambda_2^2. \tag{4.7.14}$$

We require

$$0 < \rho c^2/\mu < \lambda_1^2 \cos^2\theta + \lambda_3^2 \sin^2\theta. \tag{4.7.15}$$

From (4.7.13), we see that s_2 is repeated root, that is $s_2 = s_3$, so equations (4.7.11) become

$$\begin{aligned}
\psi_1 &= A_1\, e^{-s_1 k x_2} + (A_2\, x_2 + A_3)\, e^{-s_2 k x_2},\\
\psi_2 &= B_1\, e^{-s_1 k x_2} + (B_2\, x_2 + B_3)\, e^{-s_2 k x_2},\\
\psi_3 &= C_1\, e^{-s_1 k x_2} + (C_2\, x_2 + C_3)\, e^{-s_2 k x_2},\\
\varphi &= D_1\, e^{-s_1 k x_2} + (D_2\, x_2 + D_3)\, e^{-s_2 k x_2}.
\end{aligned} \tag{4.7.16}$$

Next, the incremental boundary conditions for propagation in any direction are

$$\dot{s}_{02i} = 0 \qquad \text{on } x_2 = 0.$$

On use of equations (4.1.4), (4.7.2) and (4.7.3) with the above boundary conditions we get

$$\begin{aligned}
&(A_{02112} + p)\, \frac{\partial v_2}{\partial x_1} + A_{02121}\, \frac{\partial v_1}{\partial x_2} = 0,\\
&A_{02211}\, \frac{\partial v_1}{\partial x_1} + (A_{02222} + p)\, \frac{\partial v_2}{\partial x_2} + A_{02233}\, \frac{\partial v_3}{\partial x_3} - \dot{p} = 0 \qquad \text{on } x_2 = 0,\\
&A_{02323}\, \frac{\partial v_3}{\partial x_2} + (A_{02332} + p)\, \frac{\partial v_2}{\partial x_3} = 0.
\end{aligned} \tag{4.7.17}$$

Since, from (2.6.13), $A_{02112} = A_{02121} - \dfrac{\partial W}{\partial \lambda_3}$, $A_{02112} + p = A_{02121} - \sigma_2$ and similarly $A_{02332} + p = A_{02323} - \sigma_2$. Also for the case $\sigma_2 = 0$ equations (4.7.17) reduce to

$$\begin{aligned}
&\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = 0 \\
&A_{02211}\frac{\partial v_1}{\partial x_1} + (A_{02222} + p)\frac{\partial v_2}{\partial x_2} + A_{02233}\frac{\partial v_3}{\partial x_3} - \dot{p} = 0, \qquad \text{on } x_2 = 0 \\
&\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} = 0.
\end{aligned} \tag{4.7.18}$$

For a neo-Hookean material these reduce to

$$\begin{aligned}
&\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = 0, \\
&2\mu\,\lambda_2^2\frac{\partial v_2}{\partial x_2} - \dot{p} = 0, \qquad \text{on } x_2 = 0 \\
&\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} = 0,
\end{aligned} \tag{4.7.19}$$

since $\sigma_2 = 0$ implies $p = \mu\,\lambda_2^2$.

From (4.7.2) and (4.7.3), the boundary conditions (4.7.19) can be written as

$$\begin{aligned}
&\psi_1' - ik\cos\theta\,\psi_2 = 0, \\
&2\mu\,\lambda_2^2\,\psi_2' - \varphi = 0, \qquad \text{on } x_2 = 0 \\
&\psi_3' - ik\sin\theta\,\psi_2 = 0.
\end{aligned} \tag{4.7.20}$$

On use of equations (4.7.16) in (4.7.20), we obtain

$$s_1k\, A_1 - A_2 + s_2k\, A_3 + ik\cos\theta\, B_1 + ik\cos\theta\, B_3 = 0,$$

$$2\mu\,\lambda_2^2\, s_1k\, B_1 - B_2 + 2\mu\,\lambda_2^2\, s_2k\, B_3 + D_1 + D_2 = 0, \text{on } x_2 = 0 \quad (4.7.21)$$

$$s_1k\, C_1 - C_2 + s_2k\, C_3 + ik\sin\theta\, B_1 + ik\sin\theta\, B_3 = 0.$$

Now, we wish to determine the ratio A_i: B_i: C_i: D_i from (4.7.8). For a neo-Hookean material the second and fourth equations of (4.7.8) reduce to

$$(\rho c^2 + \mu\,\lambda_2^2\, s^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)\, A + ic/\omega\cos\theta\, D = 0,$$

$$(4.7.22)$$

$$(\rho c^2 + \mu\,\lambda_2^2\, s^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)\, C + ic/\omega\sin\theta\, D = 0.$$

For $s = s_1$, (4.7.22) give

$$\frac{D_1}{A_1} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\, s_1^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\cos\theta}$$

$$\frac{D_1}{C_1} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\, s_1^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\sin\theta}$$

so that

$$\frac{C_1}{A_1} = \tan\theta.$$

For $s = s_2$ we have to consider

$$\psi_1 = (A_2\, x_2 + A_3)\, e^{-s_2kx_2},$$

$$\psi_2 = (B_2\, x_2 + B_3)\, e^{-s_2kx_2},$$

$$\psi_3 = (C_2\, x_2 + C_3)\, e^{-s_2kx_2}, \qquad (4.7.23)$$

$$\varphi = (D_2\, x_2 + D_3)\, e^{-s_2kx_2}.$$

Subsitution (4.7.23) into (4.7.6) shows that $A_2 = B_2 = C_2 = D_2 = 0$ and hence (4.7.22) applies with $s = s_2$, giving

$$\frac{D_3}{A_3} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\,s_2^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\,\cos\theta}$$

$$\frac{D_3}{C_3} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\,s_2^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\,\sin\theta}$$

and

$$\frac{C_3}{A_3} = \tan\theta .$$

From $(4.7.8)_1$, also we have

$$i\,s_1\,B_1 = \cos\theta\,A_1 + \sin\theta\,C_1 \qquad \text{for } s = s_1,$$

$$i\,s_2\,B_3 = \cos\theta\,A_3 + \sin\theta\,C_3 \qquad \text{for } s = s_2.$$

Thus, the boundary conditions (4.7.21) become

$$\begin{aligned} s_1\,A_1 + s_2\,A_3 + i\cos\theta\,B_1 + i\cos\theta\,B_3 &= 0, \\ 2\mu\,\lambda_2^2\,s_1 k\,B_1 + 2\mu\,\lambda_2^2\,s_2\,k\,B_3 + D_1 + D_2 &= 0, \qquad (4.7.24) \\ ik\,\sin\theta\,B_1 + ik\,\sin\theta\,B_3 + s_1\,C_1 + s_2\,C_3 &= 0. \end{aligned}$$

Substitution for B_1, B_3, C_1, C_3, D_1, D_3 in terms of A_1 and A_3 gives

$$is_1\,B_1 = \frac{A_1}{\cos\theta}, \qquad is_2\,B_2 = \frac{A_3}{\cos\theta}.$$

Equation (4.7.24) can then be written as

$$(s_1 + 1/s_1)\, A_1 + (s_2 + 1/s_2)\, A_3 = 0,$$

$$(4.7.25)$$

$$(\rho c^2 + \mu\, \lambda_2^2\, s_1^2 - 2\mu\, \lambda_2^2 - \mu\, \lambda_1^2 \cos^2\theta - \mu\, \lambda_3^2\ \sin^2\theta)\, A_1$$

$$+ (\rho c^2 + \mu\, \lambda_2^2\, s_2^2 - 2\mu\, \lambda_2^2 - \mu\, \lambda_1^2 \cos^2\theta - \mu\, \lambda_3^2 \sin^2\theta)\, A_3 = 0.$$

For A_1 and A_3 in (4.7.25) to be non-trivial solution we must have

$$\begin{vmatrix} s_2(s_1^2 + 1) & s_1(s_2^2 + 1) \\ \rho c^2 + \mu\, \lambda_2^2\, s_1^2 - 2\mu\, \lambda_2^2 & \rho c^2 + \mu\, \lambda_2^2\, s_2^2 - 2\mu\, \lambda_2^2 \\ - \mu\, \lambda_1^2 \cos^2\theta & - \mu\, \lambda_1^2 \cos^2\theta \\ - \mu\, \lambda_3^2 \sin^2\theta & - \mu\, \lambda_3^2 \sin^2\theta \end{vmatrix} = 0,$$

which gives

$$(s_1 - s_2)[\{\rho c^2 - \mu\, \lambda_1 \cos^2\theta - \mu\, \lambda_2 \sin^2\theta - 2\mu\, \lambda_2\}\ (s_1 s_2 - 1)$$

$$- \mu\, \lambda_2\, s_1 s_2 - \mu\, \lambda_2(s_1 + s_2 + s_1 s_2)\,] = 0.$$

Assuming $s_1 \neq s_2$, the above equation becomes

$$(\rho c^2 - \mu\, \lambda_1^2 \cos^2\theta - \mu\, \lambda_2^2 \sin^2\theta - 2\mu\, \lambda_2^2)(s_1 s_2 - 1) - \mu\, \lambda_2^2\, s_1\, s_1$$

$$- \mu\, \lambda_2^2(s_1^2 + s_2^2 + s_1 s_2) = 0. \quad (4.7.26)$$

This is the secular equation for the propagation of Rayleigh waves in any direction for a neo-Hookean material. (As in the case of propagation along a principal axis, $s_1 = s_2$ gives only the trivial result $A_1 = A_2 = 0$ etc.)

Next from (4.7.14), we have

$$s_1 = 1, \qquad s_2 = \{(\mu\ \lambda_1^2 \cos^2\theta + \mu\ \lambda_3^2 \sin^2\theta - \rho c^2)/\mu\lambda_2^2\}^{\frac{1}{2}}$$

so,

$$s_1^2 + s_2^2 = 1 + \frac{\mu\ \lambda_1^2\cos^2\theta + \mu\ \lambda_3^2 \sin^2\theta - \rho c^2}{\mu\ \lambda_2^2}$$

Hence,

$$\mu\ \lambda_1^2 \cos^2\theta + \mu\ \lambda_3^2 \sin^2\theta - \mu\ \lambda_2^2 - \rho c^2 =$$

$$\left[\frac{\mu\ \lambda_1^2 \cos^2\theta + \ \mu\ \lambda_3^2 \sin^2\theta - \rho c^2}{\mu\ \lambda_2^2}\right]^{\frac{1}{2}}$$

$$(\rho c^2 - \mu\ \lambda_1^2 \cos^2\theta - \mu\ \lambda_2^2 \sin^2\theta - 3\mu\ \lambda_2^2). \qquad (4.7.27)$$

On setting $\xi = \rho c^2/\mu\ \lambda_2^2$, $\eta = \dfrac{\mu\ \lambda_1^2 \cos^2\theta + \mu\ \lambda_3^2 \sin^2\theta}{\mu\ \lambda_2^2}$, the secular equation (4.7.27) is written as

$$(\eta - \xi - 1) = (\eta - \xi)^{\frac{1}{2}}(\xi - \eta - 3),$$

On squaring this becomes

$$\omega^3 + 5\ \omega^2 + 11\ \omega - 1 = 0, \qquad (4.7.28)$$

where $\omega = \eta - \xi$.

This secular equation gives only one positive, solution for ω, ω_0 say, so

$$\mu(\lambda_1^2 \cos^2\theta + \lambda_3^2 \sin^2\theta) - \rho c^2 = \omega_0\ \mu\ \lambda_2^2.$$

Hence,

$$\frac{\rho c^2}{\mu} = \lambda_1^2 \cos^2\theta + \lambda_3^2 \sin^2\theta - \omega_0\, \mu\, \lambda_2^2,$$

where $\omega_0 = 0.08738$. This result is equivalent to an equation given by Flavin (1963).

Equation (4.7.27), for $\theta = 0$, becomes

$$(\mu(\lambda_1^2 - \lambda_2^2) - \rho c^2) = \left[\frac{\mu\, \lambda_1^2 - \rho c^2}{\mu\, \lambda_2^2}\right]^{\frac{1}{2}} (\rho c^2 - \mu\, \lambda_1^2 - 3\mu\, \lambda_2^2).$$

This is the secular equation for Rayleigh surface waves for a neo-Hookean material propagating along a principal axis which is given by (4.4.3).

4.8 Analysis for a compressible materials

For compressible material the components of $\underline{\dot{s}}_0$ are given by (2.6.8) In general the components of $\underline{\dot{s}}_0$ are

$$\begin{aligned}
\dot{s}_{011} &= A_{01111}\, v_{1,1} + A_{01122}\, v_{2,2} + A_{01133}\, v_{3,3}, \\
\dot{s}_{022} &= A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2} + A_{02233}\, v_{3,3}, \\
\dot{s}_{033} &= A_{03311}\, v_{1,1} + A_{03322}\, v_{2,2} + A_{03333}\, v_{3,3}, \\
\dot{s}_{012} &= A_{01212}\, v_{2,1} + A_{01221}\, v_{1,2}, \\
\dot{s}_{021} &= A_{02121}\, v_{1,2} + A_{02112}\, v_{2,1}, \qquad (4.8.1) \\
\dot{s}_{013} &= A_{01313}\, v_{3,1} + A_{01331}\, v_{1,3}, \\
\dot{s}_{031} &= A_{03131}\, v_{1,3} + A_{03113}\, v_{3,1}, \\
\dot{s}_{023} &= A_{02323}\, v_{3,2} + A_{02332}\, v_{2,3}, \\
\dot{s}_{032} &= A_{03232}\, v_{2,3} + A_{03223}\, v_{3,2}.
\end{aligned}$$

4.8.1 Plane incremental motion

We take $v_3 \equiv 0$ and assume that v_1 and v_2 are independent of x_3. Equations (4.8.1) become

$$\begin{aligned}
\dot{s}_{011} &= A_{01111}\, v_{1,1} + A_{01122}\, v_{2,2},\\
\dot{s}_{022} &= A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2},\\
\dot{s}_{033} &= A_{03311}\, v_{1,1} + A_{03322}\, v_{2,2},\\
\dot{s}_{012} &= A_{01212}\, v_{2,1} + A_{01221}\, v_{1,2},\\
\dot{s}_{021} &= A_{01212}\, v_{1,2} + A_{02112}\, v_{2,1}.
\end{aligned} \tag{4.8.2}$$

By using the incremental equation of motion for compressible material (2.6.19) with equation (4.1.8) we get

$$\dot{s}_{0ji,j} = A_{0jilk}\, v_{k,lj} = \rho \ddot{v}_i. \tag{4.8.3}$$

From this equation, we obtain

$$\begin{aligned}
\dot{s}_{011,1} + \dot{s}_{021,2} &= \rho \ddot{v}_1,\\
\dot{s}_{012,1} + \dot{s}_{022,2} &= \rho \ddot{v}_2.
\end{aligned} \tag{4.8.4}$$

From (4.8.2), we get

$$\begin{aligned}
\dot{s}_{011,1} &= A_{01111}\, v_{1,11} + A_{01122}\, v_{2,21},\\
\dot{s}_{012,1} &= A_{01212}\, v_{2,11} + A_{01221}\, v_{1,21},\\
\dot{s}_{021,2} &= A_{02121}\, v_{1,22} + A_{02112}\, v_{2,12},\\
\dot{s}_{022,2} &= A_{02211}\, v_{1,12} + A_{02222}\, v_{2,22}.
\end{aligned} \tag{4.8.5}$$

Substituting (4.8.5) into (4.8.4), we have the required equations of plane incremental motion, namely

$$\rho\dot{v}_1 = A_{01111} v_{1,11} + A_{01122} v_{2,21} + A_{02121} v_{1,22} + A_{02112} v_{2,12},$$

$$\rho\dot{v}_2 = A_{01212} v_{2,11} + A_{01221} v_{1,21} + A_{02211} v_{1,12} + A_{02222} v_{2,22}. \tag{4.8.6}$$

4.8.2 Propagation along a principal axis

We now assume that v_1 and v_2 are given by

$$v_1 = A_1 e^{skx_2 + ikx_1 - i\omega t},$$

$$v_2 = A_2 e^{skx_2 + ikx_1 - i\omega t}. \tag{4.8.7}$$

Equations (4.8.6) yield

$$-\rho c^2 A_1 = (A_{02121} s^2 - A_{01111}) A_1 + is (A_{01122} + A_{02112}) A_2,$$

$$-\rho c^2 A_2 = is (A_{01221} + A_{02211}) A_1 + (A_{02222} s^2 - A_{01212}) A_2. \tag{4.8.8}$$

For these equations to have non-trivial solution for A_1 and A_2 we must have

$$\begin{vmatrix} A_{02121} s^2 - A_{01111} + \rho c^2 & is(A_{01122} + A_{02112}) \\ is(A_{01221} + A_{02211}) & A_{02222} s^2 - A_{01212} + \rho c^2 \end{vmatrix} = 0,$$

which gives

$$A_{02222}\,A_{02121}\,s^4 - [A_{01111}\,A_{02222} + A_{02121}\,A_{01212}$$

$$- (A_{01122} + A_{02112})^2 - \rho c^2\,(A_{02121} + A_{02222})]\,s^2$$

$$+ A_{01111}\,A_{01212} - \rho c^2\,(A_{01111} + A_{01212} - \rho c^2) = 0, \tag{4.8.9}$$

which is a quadratic equation for s^2. Suppose it has roots s_1^2 and s_2^2. Then

$$s_1^2 + s_2^2 = \frac{A_{01111}\,A_{02222} + A_{02121}\,A_{01212} - (A_{01122} + A_{02112})^2 - (A_{02121} + A_{02222})\rho c^2}{A_{02222}\,A_{02121}}$$

$$s_1^2 s_2^2 = \frac{A_{01111}\,A_{01212} - \rho c^2(A_{01111} + A_{01212} - \rho c^2)}{A_{02222}\,A_{02121}} \tag{4.8.10}$$

Now suppose that the underlying state of deformation corresponds to a pure homogeneous strain of a half-space which, in the deformed configuration, occupies the region $x_2 < 0$ with the boundary $x_2 = 0$. We take the incremental surface traction to vanish on the boundary, so that

$$\dot{s}_{021} = 0, \qquad \dot{s}_{022} = 0 \qquad \text{on } x_2 = 0.$$

From (4.8.2), we get

$$\left.\begin{aligned} A_{02121}\,v_{1,2} + A_{02112}\,v_{2,1} &= 0, \\ A_{02211}\,v_{1,1} + A_{02222}\,v_{2,2} &= 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \tag{4.8.11}$$

For Rayleigh surface waves we seek a solution for v_1 and v_2 in equation (4.8.7) which vanishes when $x_2 \longrightarrow -\infty$ and which also satisfies the above boundary conditions.

The general solutions for v_1 and v_2 are given by

$$v_1 = (A_1 e^{s_1 kx_2} + B_1 e^{s_2 kx_2})e^{ikx_1 - i\omega t},$$

$$v_2 = (A_2 e^{s_1 kx_2} + B_2 e^{s_2 kx_2})e^{ikx_1 - i\omega t}, \tag{4.8.12}$$

where s_1 and s_2 should have positive real part.

Next, substitute equations (4.8.12) into the boundary conditions (4.8.11) to give

$$A_{02121}\, s_1\, A_1 + iA_{02112}\, A_2 + A_{02121}\, s_2\, B_1 + iA_{02112}\, B_2 = 0,$$

$$iA_{02211}\, A_1 + A_{02222}\, s_1\, A_2 + iA_{02211}\, B_1 + A_{02222}\, s_2\, B_2 = 0. \tag{4.8.13}$$

From (4.8.8), we have

$$\frac{iA_2}{A_1} = \frac{A_{01111} - A_{02121}\, s_1^2 - \rho c^2}{s_1\,(A_{01122} + A_{02112})}$$

and similarly (4.8.14)

$$\frac{iB_2}{B_1} = \frac{A_{01111} - A_{02121}\, s_2^2 - \rho c^2}{s_2\,(A_{01122} + A_{02112})}$$

Next, on use of (4.8.14) in (4.8.13) we obtain

$$\{A_{02121}\, s_1 + \frac{A_{02112}(A_{01111} - A_{02121}\, s_1^2 - \rho c^2)}{s_1\,(A_{01122} + A_{02112})}\}\, A_1$$

$$+ \{A_{02121} s_2 + \frac{A_{02112}\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)}{s_2\,(A_{01122} + A_{02112})}\}\, B_1 = 0,$$

(4.8.15)

$$\{iA_{02211} + \frac{A_{02222}\,(A_{01111} - A_{02121}\, s_1^2 - \rho c^2)}{i\,(A_{01122} + A_{02112})}\}\, A_1$$

$$+ \{iA_{02211} + \frac{A_{02222}\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)}{i\,(A_{01122} + A_{02112})}\}\, B_1 = 0.$$

This can be written as

$$s_2\{A_{02121}(A_{01122} + A_{02112})s_1^2 + A_{02112}(A_{01111} - A_{02121}s_1^2 - \rho c^2)\}\, A_1$$

$$+ s_1\{A_{02121}\,(A_{01122} + A_{02112})\, s_2^2$$

$$+ A_{02112}\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)\}\, B_1 = 0,$$

(4.8.16)

$$\{A_{02211}\,(A_{01122} + A_{02112}) - A_{02222}\,(A_{01111} - A_{02121}\, s_1^2 - \rho c^2)\}\, A_1$$

$$+ \{A_{02211}\,(A_{01122} + A_{02112})$$

$$- A_{02222}\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)\}\, B_1 = 0.$$

For these to have a non-trivial solution for A_1 and B_1, we must have

$$
\begin{vmatrix}
\begin{array}{l} s_1^2 s_2\, A_{02121}\,(A_{01122} + A_{02112}) \\ + s_2\, A_{02112}\,(A_{01111} \\ \qquad - A_{02121}\, s_1^2 - \rho c^2) \end{array} & \begin{array}{l} s_2^2 s_1\, A_{02121}\,(A_{01122} + A_{02112}) \\ + s_1\, A_{02112}\,(A_{01111} \\ \qquad - A_{02121}\, s_2^2 - \rho c^2) \end{array} \\
 & \\
\begin{array}{l} A_{02211}(A_{01122} + A_{02112}) \\ - A_{02222}\,(A_{01111} \\ \qquad - A_{02121}\, s_1^2 - \rho c^2) \end{array} & \begin{array}{l} A_{02211}(A_{01122} + A_{02112}) \\ - A_{02222}\,(A_{01111} \\ \qquad - A_{02121}\, s_2^2 - \rho c^2) \end{array}
\end{vmatrix} = 0
$$

and hence

$$
\begin{aligned}
& s_2[\, s_1^2\,(A_{01122} + A_{02112})^2\, A_{02121}\, A_{02211} \\
& \quad + (A_{01122} + A_{02112})(A_{01111} + A_{02121}\, s_1^2 - \rho c^2)\, A_{02211}\, A_{02112} \\
& - s_1^2\,(A_{01122} + A_{02112})\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)\, A_{02222}\, A_{02121} \\
& \quad - (A_{01111} - A_{02121}\, s_1^2 - \rho c^2)(A_{01111} - A_{02121}\, s_2^2 - \rho c^2) \\
& \qquad\qquad A_{02222}\, A_{02121})\,] \\
& - s_1[\, s_2^2\,(A_{01122} + A_{02112})^2\, A_{02121}\, A_{02211} \\
& \quad + (A_{01122} + A_{02112})\,(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)\, A_{02211}\, A_{02112} \\
& - s_2^2\,(A_{01122} + A_{02112})\,(A_{01111} + A_{02121}\, s_1^2 - \rho c^2)\, A_{02222}\, A_{02121} \\
& \quad - (A_{01111} - A_{02121}\, s_1^2 - \rho c^2)\,(A_{01111} + A_{02121}\, s_2^2 - \rho c^2) \\
& \qquad\qquad A_{02222}\, A_{02121})\,] = 0.
\end{aligned}
$$

Gathering together like terms, we obtain

$$(A_{01122} + A_{02112})^2 A_{02121} A_{02211} s_1 s_2 (s_1 - s_2)$$

$$(A_{01122} + A^{02112}) A_{02211} A_{02112} [s_2(A_{01111} - A_{02121}s_1^2 - \rho c^2) - s_1(A_{01111} - A_{02121} s_2^2 - \rho c^2)]$$

$$+ (A_{01122} + A_{02112}) A_{02222} A_{02121} [s_1 s_2^2 (A_{01111} - A_{02121} s_1^2 - \rho c^2) - s_1^2 s_2 (A_{01111} - A_{02121} s_2^2 - \rho c^2)]$$

$$+ (A_{01111} + A_{02121} s_1^2 - \rho c^2)(A_{01111} + A_{02121} s_2^2 - \rho c^2) A_{02112} A_{02222} (s_1 - s_2) = 0.$$

i.e

$$(s_1 - s_2)[(A_{01122} + A_{02112})^2 A_{02121} A_{02211} s_1 s_2$$

$$- (A_{01122} + A_{02112}) A_{02112} A_{02211} (A_{01111} + A_{02121} s_1 s_2 - \rho c^2)$$

$$+ A_{02222} A_{02121} (A_{01122} + A_{02112})$$

$$(\rho c^2 s_1 s_2 - A_{02121} s_1^2 s_2^2 - A_{01111} s_1 s_2)$$

$$+ (A_{01111} - A_{02121} s_1^2 - \rho c^2) (A_{01111} - A_{02121} s_2^2 - \rho c^2)$$

$$A_{02112} A_{02222}] = 0. \qquad (4.8.17)$$

As for an incompressible material the case $s_1 = s_2$ does not lead to the existence of Rayleigh waves. We therefore assume $s_1 \neq s_2$, and hence

$$A_{02121}\, A_{02211}\, (A_{01122} + A_{02112})^2\; s_1 s_2$$

$$-\; A_{02112}\, A_{02211}\, (A_{01122} + A_{02112})\; (A_{01111} - \rho c^2)$$

$$-\; A_{02121}\, A_{02112}\, A_{02211}\, (A_{01122} + A_{02112})\; s_1 s_2$$

$$+\; A_{02222}\, A_{02121}\, (A_{01122} + A_{02112})\; (\rho c^2 - A_{01111})\; s_1 s_2$$

$$-\; A_{02222}\, A^2_{02121}\, (A_{01122} + A_{02112})^2\; s_1^2 s_2^2$$

$$+\; A_{02112}\, A_{02222}\, (A_{01111} - \rho c^2)^2$$

$$-\; A_{02121}\, A_{02222}\, A_{02112}\, (A_{01111} - \rho c^2)\; (s_2^2 + s_2^2\;)$$

$$+\; A^2_{02121}\, A_{02222}\, A_{02112}\; s_1^2 s_2^2 = 0. \tag{4.8.18}$$

i.e

$$s_1^2 s_2^2\, A^2_{02121}\{A_{02222}\, A_{02112} - A_{02222}\, (A_{01122} + A_{02112})\}$$

$$-\; (s_1^2 + s_2^2)\; A_{02121}\, A_{02222}\, A_{02112}\, (A_{01111} - \rho c^2)$$

$$+\; s_1 s_2\, A_{02121}\, (A_{01122} + A_{02112})\; \{A_{01122}\, A_{02211}$$

$$+\; A_{02222}\, (\rho c^2 - A_{01111})\}$$

$$+\; A_{02112}\, (A_{01111} - \rho c^2)\; \{A_{02222}\, (A_{01111} - \rho c^2)$$

$$-\; A_{02211}\, (A_{01122} + A_{02112})\} = 0. \tag{4.8.19}$$

This equation can be written as

$$s_1^2 s_2^2\, A_{02121}^2\, A_{02222}\, A_{01122}$$

$$+\ (s_1^2 + s_2^2)\, A_{02121}\, A_{02222}\, A_{02112}\, (A_{01111} - \rho c^2)$$

$$-\ A_{02112}\, (A_{01111} - \rho c^2)\{A_{02222}\, (A_{01111} - \rho c^2)$$

$$-\ A_{02211}\, (A_{01122} + A_{02112})\} =$$

$$s_1 s_2\, A_{02121}\, (A_{01122} + A_{02112})\{A_{01122}\, A_{02211}$$

$$+\ A_{02222}\, (\rho c^2 - A_{01111})\}. \tag{4.8.20}$$

From (4.8.10), we have

$$s_1^2 + s_2^2 =$$

$$\frac{A_{01111}\, A_{02222} + A_{02121}\, A_{01212} - (A_{01122} + A_{02112})^2 - \rho c^2 (A_{02121} + A_{02222})}{A_{02121}\, A_{02222}}$$

$$s_1^2 s_2^2 = \frac{(A_{01111} - \rho c^2)(A_{01212} - \rho c^2)}{A_{02121}\, A_{02222}} \tag{4.8.21}$$

From (2.7.18), for the classical linear theory, we have $A_{01111} = A_{02222} = \lambda + 2\mu$, $A_{0212} = A_{02121} = \mu$ and we get that

$$s_1^2 s_2^2 = \frac{(\lambda + 2\mu - \rho c^2)(\mu - \rho c^2)}{\mu(\lambda + 2\mu)},$$

since $s_1^2 s_2^2 > 0$, we obtain

$$\rho c^2 < \mu, \qquad \rho c^2 < \lambda + 2\mu.$$

We also require for the non-linear theory that $s_1^2 s_2^2 > 0$, as in incompressible theory in Section 4.2.1, so

$$\rho c^2 < A_{01212}, \qquad \rho c^2 < A_{01111}.$$

i.e.

$$\rho c^2 < \min\{A_{01212}, A_{01111}\} \tag{4.8.22}$$

The alternative to (4.8.22), that $\rho c^2 > \max(A_{01111}, A_{01212})$ is ruled out for reasons that will become apparent in the next section.

First of all, to simplify the notation we set

$$\begin{aligned} &\alpha_{11} = JA_{01111}, && \alpha_{22} = JA_{02222}, \\ &\alpha_{12} = JA_{01122}, && \gamma_1 = JA_{01212}, \\ &\gamma_2 = JA_{02121}, && \delta = JA_{01122} + JA_{02112}, \end{aligned} \tag{4.8.23}$$

and

$$2\beta = \alpha_{11}\,\alpha_{22} + \gamma_1\,\gamma_2 - \delta^2.$$

Recall from (2.6.6) that

$$\alpha_{12} = \gamma_2 - \tau_2, \tag{4.8.24}$$

where

$$\tau_2 = J\sigma_2.$$

Equations (4.8.20) and (4.8.21) now can be written as

$$\begin{aligned} &s_1^2\,s_2^2\,\gamma_2^2\,\alpha_{12}\,\alpha_{22} + (s_1^2 + s_2^2)\,(\gamma_2 - \tau_2)\,(\alpha_{11} - \rho c^2)\gamma_2\,\alpha_{22} \\ &- (\gamma_2 - \tau_2)\,(\alpha_{11} - \rho c^2)\,\{\alpha_{22}\,(\alpha_{11} - \rho c^2) - \alpha_{12}\,\delta\} \\ &= s_1\,s_2\,\gamma_2\,\delta\,\{\alpha_{12}^2 + \alpha_{22}\,(\rho c^2 - \alpha_{11})\}, \end{aligned} \tag{4.8.25}$$

and

$$\begin{aligned} s_1^2 + s_2^2 &= \frac{2\beta - (\gamma_2 + \alpha_{22})\,\rho c^2}{\gamma_2\,\alpha_{22}}, \\ s_1^2 s_2^2 &= \frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2\,\alpha_{22}}, \end{aligned} \tag{4.8.26}$$

respectively

Also, the necessary and sufficient conditions (2.8.12) become

$$\alpha_{11} > 0, \qquad \alpha_{22} > 0, \qquad \gamma_1 > 0, \qquad \gamma_2 > 0 \tag{4.8.27}$$

and

$$\beta + (\alpha_{11}\, \alpha_{22}\, \gamma_1\, \gamma_2)^{\frac{1}{2}} > 0. \tag{4.8.28}$$

We note that in (2.8.12) we assumed $\gamma_1 = \gamma_2$, but in general we note from (2.6.6) that

$$\gamma_1\, \lambda_2^2 = \gamma_2\, \lambda_1^2, \tag{4.8.29}$$

On use of the notations (4.8.23) it is easy to see that (4.8.28) can be written as

$$\left\{\sqrt{\alpha_{11}\, \alpha_{22}} + \sqrt{\gamma_1\, \gamma_2} + \delta\right\}\left\{\sqrt{\alpha_{11}\, \alpha_{22}} + \sqrt{\gamma_1\, \gamma_2} - \delta\right\} > 0, \tag{4.8.30}$$

In the linear theory it follows from (2.7.18) and (4.8.23) that

$$\begin{gathered} \alpha_{11} = \alpha_{22} = \lambda + 2\mu, \qquad \alpha_{12} = \lambda, \qquad \gamma_1 = \gamma_2 = \mu, \\ \delta = \lambda + \mu, \qquad \beta = \mu\, (\lambda + 2\mu). \end{gathered} \tag{4.8.31}$$

Then, the two factors on the left-hand side of (4.8.30) are $2(\lambda + 2\mu)$ and 2μ respectively, both of which are taken to be positive. Hence, by continuity, each factor must remain positive on a path of deformation from the natural configuration (where $\lambda_i = 1$ and $\sigma_i = 0$ for $i \in \{1,2,3\}$) if (4.8.30) is to be maintained. Thus (4.8.28) can be replaced by

$$\sqrt{\alpha_{11}\, \alpha_{22}} + \sqrt{\gamma_1\, \gamma_2} \pm \delta > 0. \tag{4.8.32}$$

In terms of the strain-energy function W, the necessary and sufficient condtions (4.8.27) and (4.8.32) can be written as

$$W_{11} > 0, \qquad W_{22} > 0, \qquad \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1^2 - \lambda_2^2} > 0, \qquad (4.8.33)$$

and

$$\sqrt{W_{11} W_{22}} - W_{12} + \frac{W_1 + W_2}{\lambda_1 + \lambda_2} > 0,$$

$$(4.8.34)$$

$$\sqrt{W_{11} W_{22}} + W_{12} - \frac{W_1 - W_2}{\lambda_1 - \lambda_2} > 0,$$

respectively, where $W_i = \partial W / \partial \lambda_i$, $W_{ij} = \partial^2 W / \partial \lambda_i \partial \lambda_j$, we note here (4.8.33) are given in $(2.8.11)_{1-3}$.

For the special case in which $\lambda_1 = \lambda_2$, the inequalities (4.8.33) and (4.8.34) reduce to the pair of inequalities

$$W_{11} > 0, \qquad W_{11} - W_{12} + \lambda_1^{-1} W_1 > 0, \qquad (4.8.35)$$

or, equivalently,

$$\alpha > 0, \qquad \gamma > 0, \qquad (4.8.36)$$

where

$$\alpha = \alpha_{11} = \alpha_{22}, \qquad \gamma = \gamma_1 = \gamma_2. \qquad (4.8.37)$$

On substituting (4.8.26) in (4.8.25), we have

$$
\begin{aligned}
&(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)\ \alpha_{12}\ \gamma_2 \\
&+ \{2\beta - (\gamma_2 + \alpha_{22})\rho c^2\}\ (\gamma_2 - \tau_2)(\alpha_{11} - \rho c^2) \\
&- \{\alpha_{22}(\alpha_{11} - \rho c^2) - \alpha_{12}\ \delta\}\ (\gamma_2 - \tau_2)(\alpha_{11} - \rho c^2) \\
&= \left\{\frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2\ \alpha_{22}}\right\}^{\frac{1}{2}} \left\{\alpha_{12}^2 + \alpha_{22}\ (\rho c^2 - \alpha_{11})\right\}\ \gamma_2\ \delta .
\end{aligned}
$$

The above equation, after substituting the expression for 2β, can be written as

$$
\begin{aligned}
&(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)\ \alpha_{12}\ \gamma_2 \\
&+ \{\alpha_{11}\ \alpha_{22} + \gamma_1\ \gamma_2 - \delta^2 - (\gamma_2 + \alpha_{22})\rho c^2\}(\gamma_2 - \tau_2)\ (\alpha_{11} - \rho c^2) \\
&- \{\alpha_{22}(\alpha_{11} - \rho c^2) - \alpha_{12}\ \delta\}\ (\gamma_2 - \tau_2)(\alpha_{11} - \rho c^2) \\
&= \left\{\frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2\ \alpha_{22}}\right\}^{\frac{1}{2}} \left\{\alpha_{12}^2 + \alpha_{22}\ (\rho c^2 - \alpha_{11})\right\}\ \gamma_2\ \delta ,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&(\alpha_{11} - \rho c^2)\Big[(\gamma_1 - \rho c^2)\ \gamma_2\ \alpha_{12} + (\gamma_2 - \tau_2)\Big\{\gamma_2\ (\gamma_1 - \rho c^2) \\
&\qquad\qquad + \delta\ (\alpha_{12} - \delta)\Big\}\Big] \\
&= \left\{\frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2\ \alpha_{22}}\right\}^{\frac{1}{2}} \left\{\alpha_{12}^2 + \alpha_{22}\ (\rho c^2 - \alpha_{11})\right\}\ \gamma_2\ \delta ,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&(\alpha_{11} - \rho c^2)\left[\gamma_2\ (\gamma_1 - \rho c^2) - (\gamma_2 - \tau_2)^2\right] \\
&= \left\{\frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2\ \alpha_{22}}\right\}^{\frac{1}{2}} \left\{\alpha_{12}^2 + \alpha_{22}\ (\rho c^2 - \alpha_{11})\right\}\ \gamma_2 ,
\end{aligned}
$$

i.e.

$$\alpha_{22}\,(\alpha_{11} - \rho c^2)[\gamma_2\,(\gamma_1 - \rho c^2) - (\gamma_2 - \tau_2)^2]$$

$$= [\alpha_{12}^2 + \alpha_{22}\,(\rho c^2 - \alpha_{11})][\alpha_{22}\,\gamma_2\,(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)]^{\frac{1}{2}}, \tag{4.8.38}$$

this secular eqution will be investigated in the next section.

Hence, either

$$\delta = 0, \tag{4.8.39}$$

or

$$\rho c^2 = \alpha_{11}, \tag{4.8.40}$$

or

$$\{\alpha_{22}\,(\alpha_{11} - \rho c^2)\}^{\frac{1}{2}}\,\{\gamma_2\,(\gamma_1 - \rho c^2) - (\gamma_2 - \tau_2)^2\}$$

$$- \{\gamma_2\,(\gamma_1 - \rho c^2)\}^{\frac{1}{2}} + \{\alpha_{22}(\alpha_{11} - \rho c^2) - \alpha_{12}^2\} = 0. \tag{4.8.41}$$

On squaring and rearranging (4.8.41)

$$\begin{aligned}
&\gamma_2\,\alpha_{22}\,(\alpha_{22} - \gamma_2)(\rho c^2)^3 \\
&+ \gamma_2\,\alpha_{22}[(\alpha_{11}\gamma_2 - \alpha_{22}\,\gamma_1) + 2\{\gamma_1\gamma_2 + (\gamma_2 - \tau_2)\,\} \\
&\qquad\qquad + 2(\alpha_{12}^2 - \alpha_{11}\alpha_{22})](\rho c^2)^2 \\
&+ [\gamma_2\,\alpha_{22}(\alpha_{11}^2\,\alpha_{22} - \gamma_1^2\,\gamma_2) + 2\,\gamma_1\,\gamma_2\,\alpha_{22}(\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{22}) \\
&\quad + 2\,\gamma_2\,\alpha_{22}\,\{\gamma_1\,(\gamma_2 - \tau_2)^2 - \alpha_{11}\,\alpha_{12}^2\} - 2\,\gamma_1\,\alpha_{22}\,\alpha_{11}\{(\gamma_2 - \tau_2)^2 \\
&\qquad\qquad + \gamma_1\gamma_2\} + (\gamma_2 - \tau_2)^4(\alpha_{22} + \gamma_2)]\rho c^2 \\
&+ \gamma_1\,\gamma_2(\alpha_{11}\,\alpha_{22}\,\gamma_1\gamma_2 - \alpha_{12}^4) + 2\,\alpha_{11}\,\alpha_{22}\,\gamma_1\,\gamma_2\,\{\alpha_{12}^2 - (\gamma_2 - \tau_2)^2\} \\
&\qquad\qquad - \alpha_{11}\,\alpha_{22}\{(\gamma_2 - \tau_2)^4 + \alpha_{11}^2\,\alpha_{22}^2\} = 0.
\end{aligned} \tag{4.8.42}$$

By dividing this equation by $(\rho c^2)^3$, we have

$$[\ \gamma_1\ \gamma_2(\alpha_{11}\ \alpha_{22}\ \gamma_1\gamma_2 - \alpha_{12}^4) + 2\ \alpha_{11}\ \alpha_{22}\ \gamma_1\ \gamma_2\ \{\alpha_{12}^2 - (\gamma_2 - \sigma_2)^2\}$$

$$- \alpha_{11}\ \alpha_{22}\{(\gamma_2 - \sigma_2)^4 + \alpha_{11}^2\ \alpha_{22}^2\}\ ]\ \zeta^3$$

$$+ [\gamma_2\ \alpha_{22}(\alpha_{11}^2\ \alpha_{22} - \gamma_1^2\ \gamma_2) + 2\ \gamma_1\ \gamma_2\ \alpha_{22}(\alpha_{11}\ \alpha_{22} - \alpha_{12}\ \alpha_{22})$$

$$+ 2\ \gamma_2\ \alpha_{22}\ \{\gamma_1\ (\gamma_2 - \sigma_2)^2 - \alpha_{11}\ \alpha_{12}^2\} - 2\ \gamma_1\ \alpha_{22}\ \alpha_{11}\{(\gamma_2 - \sigma_2)^2$$

$$+ \gamma_1\gamma_2\} + (\gamma_2 - \sigma_2)\ (\alpha_{22} + \gamma_2)\ ]\ \zeta^2$$

$$+ \gamma_2\ \alpha_{22}\ [\ (\alpha_{11}\ \gamma_2 - \alpha_{22}\ \gamma_1) + 2\{\gamma_1\gamma_2 + (\gamma_2 - \sigma_2)\ \}$$

$$+ 2(\alpha_{12}^2 - \alpha_{11}\alpha_{22})\ ]\ \zeta$$

$$+ \gamma_2\ \alpha_{22}\ (\alpha_{22} - \gamma_2) = 0, \tag{4.8.43}$$

where $\zeta = 1/\rho c^2$.

Equation (4.8.43) is the secular equation for Rayleigh surface waves propagating along a principal axis in a pre-stressed compressible elastic medium. An equivalent result, but in terms of the strain-energy function W and the invariants I_i $i \in \{1,2,3\}$, was given by Hayes and Rivlin (1961b). Note, however, that the squaring process can introduce spurious solutions, as in the incompressible case and we do not therefore use (4.8.43).

4.9 Analysis of the secular equation

In this section we shall concentrate on the secular equation of the form (4.8.38)

4.9.1 The general case

Here, we shall study the general case of the secular equation (4.8.38) for the cases $\delta \neq 0$ and $\delta = 0$ separately.

First of all we assume that $\delta \neq 0$, and the secular equation (4.8.38) can be written as

$$\gamma_2 (\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2) - (\alpha_{11} - \rho c^2)(\gamma_2 - \tau_2)^2$$

$$+ \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \left[\alpha_{11} - \rho c^2\right]^{\frac{1}{2}} \left[\gamma_1 - \rho c^2\right]^{\frac{1}{2}} \left\{\alpha_{22} (\alpha_{11} - \rho c^2) - \alpha_{12}^2\right\} = 0. \tag{4.9.1}$$

Now, for the above secular equation we shall consider the following special cases

a) $0 \leqslant \rho c^2 \leqslant \gamma_1 < \alpha_{11}$, b) $0 \leqslant \rho c^2 \leqslant \alpha_{11} = \gamma_1$,

c) $0 \leqslant \rho c^2 \leqslant \alpha_{11} < \gamma_1$,

for both cases when $\gamma_2 \neq \tau_2$ and $\gamma_2 = \tau_2$ separately.

a) The case $0 \leqslant \rho c^2 \leqslant \gamma_1 < \alpha_{11}$

For this case and $\gamma_2 \neq \tau_2$, the secular equation (4.9.1) becomes

$$\gamma_2(\gamma_1 - \rho c^2) - (\gamma_2 - \tau_2)^2 + \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \left[\frac{\gamma_1 - \rho c^2}{\alpha_{11} - \rho c^2}\right]^{\frac{1}{2}}$$

$$\left\{\alpha_{22} (\alpha_{11} - \rho c^2) - \alpha_{12}^2\right\} = 0. \tag{4.9.2}$$

We set

$$\eta = \left\{\frac{\gamma_1 - \rho c^2}{\alpha_{11} - \rho c^2}\right\}^{\frac{1}{2}}, \tag{4.9.3}$$

so that

$$0 < \eta < \sqrt{\frac{\gamma_1}{\alpha_{11}}}\ , \tag{4.9.4}$$

the left-hand limit in the above corresponds to $\rho c^2 = \gamma_1$ and the righ-hand limit to $c = 0$.

We also have

$$\rho c^2 = \frac{\alpha_{11}\ \eta^2 - \gamma_1}{\eta^2 - 1}. \tag{4.9.5}$$

Equation (4.9.2) now can be rearranged as

$$f(\eta) \equiv \left[\frac{\gamma_2}{\alpha_{11}}\right]^{\frac{1}{2}} \alpha_{12}^2\ \eta^3 + \left\{\left[\gamma_2 - \tau_2\right]^2 + \gamma_2\ \left[\alpha_{11} - \gamma_1\right]\right\} \eta^2$$
$$+ \left[\frac{\gamma_2}{\alpha_{11}}\right]^{\frac{1}{2}} \left\{\alpha_{22}\ \left[\alpha_{11} - \gamma_1\right] - \alpha_{12}^2\right\} \eta - (\gamma_2 - \tau_2)^2 = 0. \tag{4.9.6}$$

We seek solutions of (4.9.6) subject to (4.9.4). We have

$$f(0) = -\ (\gamma_2 - \tau_2)^2 < 0, \tag{4.9.7}$$

and

$$f\left(\sqrt{\frac{\gamma_1}{\alpha_{11}}}\right) \equiv (\alpha_{11} - \gamma_1)\left\{\left[\frac{\gamma_1\ \gamma_2}{\alpha_{11}\ \alpha_{22}}\right]^{\frac{1}{2}}\left[\alpha_{11}\ \alpha_{22} - \alpha_{12}^2\right] + \gamma_1\gamma_2 - (\gamma_2 - \tau_2)^2\right\}. \tag{4.9.8}$$

Since, $\alpha_{11} - \gamma_1 > 0$, we deduce that $f(\eta) = 0$ has a solution in the interval (4.9.4) if $f\left(\sqrt{\frac{\gamma_1}{\alpha_{11}}}\right) > 0$, i.e. if

$$\left[\frac{\gamma_1\ \gamma_2}{\alpha_{11}\ \alpha_{22}}\right]^{\frac{1}{2}}\left[\alpha_{11}\ \alpha_{22} - \alpha_{12}^2\right] + \gamma_1\gamma_2 - (\gamma_2 - \tau_2)^2 \geqslant 0. \qquad (4.9.9)$$

If equality (4.9.9) holds then this corresponds to zero wave speed (c = 0).

The following argument shows that the solution guaranteed by (4.9.9) is unique. First we note that

$$f'(\eta) = 3\sqrt{\frac{\gamma_2}{\alpha_{22}}}\ \alpha_{12}^2\ \eta^2 + 2\left\{\gamma_2(\alpha_{11} - \gamma_1) + (\gamma_2 - \tau_2)^2\right\}\eta$$

$$+ \sqrt{\frac{\gamma_2}{\alpha_{22}}}\left\{\alpha_{22}(\alpha_{11} - \gamma_1) - \alpha_{12}^2\right\}, \qquad (4.9.10)$$

and

$$f'(0) = \sqrt{\frac{\gamma_2}{\alpha_{22}}}\left\{\alpha_{22}(\alpha_{11} - \gamma_1) - \alpha_{12}^2\right\}, \qquad (4.9.11)$$

If $f'(0) \geqslant 0$ then $f'(\eta) > 0$ for $\eta > 0$, if $f'(0) < 0$ then minimum of $f(\eta)$ occurs for $\eta < 0$ (and the maximum for $\eta < 0$). In either cases the solution is unique.

Thus, the secular equation (4.9.6) has a unique solution in the interval (4.9.4) if and only if (4.9.9) holds.

In the natural configuration the left-hand limit of (4.9.9) is $4\mu^2\ (\lambda + \mu)/(\lambda + 2\mu)$. This is positive if the shear modulus μ and bulk modulus $\lambda + (^2/_3)\mu$ are positive, which we assume to be the case. By continuity, the strict inequality

$$\sqrt{\frac{\gamma_1\ \gamma_2}{\alpha_{11}\ \alpha_{22}}}\ (\alpha_{11}\ \alpha_{22} - \alpha_{12}^2) + \gamma_1\ \gamma_2 - (\gamma_2 - \tau_2)^2 > 0, \qquad (4.9.12)$$

therefore holds on a path of quasi-static deformation from the natural configuration. The connected region, which includes the natural configuration, in $(\lambda_1, \lambda_2, \lambda_3)$-space defined by this inequality is bounded by the surface defined by

$$\sqrt{\frac{\gamma_1 \gamma_2}{\alpha_{11} \alpha_{22}}} (\alpha_{11} \alpha_{22} - \alpha_{12}^2) + \gamma_1 \gamma_2 - (\gamma_2 - \tau_2)^2 = 0. \tag{4.9.13}$$

When equation (4.9.13) holds the unique solution of $f(\eta) = 0$ is $\eta = \sqrt{\frac{\gamma_1}{\alpha_{11}}}$, which corresponds to $c = 0$. The solution (4.9.12), with $\omega = 0$, is then interpreted as a quasi-static surface deformation. The inequality (4.9.12) is an *exlusion condition*, which excludes the existence of non-trivial quasi-static surface deformations of the considered type. In the other words, bifurcation from the underlying homogeneous deformation into a mode of deformation of the form (4.8.7), with $\omega = 0$, is prevented by the inequality (4.9.12), but becomes possible when (4.9.13) is met. In a limited sense (4.9.12) also guarantees infinitesimal stability of the underlying configuration, with (4.9.13) corresponding to configurations exhibiting neutral stability.

In terms of the strain-energy function W, (4.9.9) can be written as

$$W_{11} W_{22} - W_{12}^2 + \sqrt{W_1 W_2} \left\{\frac{W_1^2 - W_2^2}{\lambda_1 W_1 - \lambda_2 W_2}\right\} > 0. \tag{4.9.14}$$

It is interesting to note that, given the inequalities (4.8.27), the inequality (4.8.28) is a consequence of (4.9.12) as we shall see shortly. This echoes a results in the incompressible theory as given in Section 4.4, but is less immediate in the incompressible case.

Here, we shall prove that (4.9.12) implies (4.8.28).

We set

$$a = W_{11}, \qquad c = W_{22}, \qquad b = W_{12}, \qquad d = \frac{W_1 + W_2}{\lambda_1 + \lambda_2}, \qquad e = \frac{W_1 - W_2}{\lambda_1 - \lambda_2},$$

the inequality (4.9.12) can written

$$ac - b^2 - 2\sqrt{ac}\,\frac{d\,e}{d + e} > 0,$$

where, from (4.8.27) $a > 0$, $c > 0$ and $d + e > 0$. This can be rearranged as

$$\left\{\sqrt{ac} + \frac{d\,e}{d + e}\right\}^2 > b^2 + \frac{d^2\,e^2}{(d + e)^2} \geqslant b^2.$$

and hence

$$\left\{\sqrt{ac} + b + \frac{d\,e}{d + e}\right\}\left\{\sqrt{ac} - b + \frac{d\,e}{d + e}\right\} > 0.$$

since each factor is positive in the natural configrution we deduce that

$$\sqrt{ac} \pm b + \frac{de}{d + e} > 0.$$

hence

$$\sqrt{ac} + b + e > \frac{e^2}{d + e} \geqslant 0,$$

and

$$\sqrt{ac} - b + d > \frac{d^2}{d + e} \geqslant 0,$$

i.e. the inequalities (4.8.34) hold, and these are equivalent to (4.8.28).

b) <u>The case $0 < \rho c^2 < \alpha_{11} = \gamma_1$</u>

For the case

$$0 < \rho c^2 < \alpha_{11} = \gamma_1, \tag{4.9.15}$$

the secular equation (4.9.1) reduces to

$$\gamma_2 \; (\gamma_1 - \rho c^2) - (\gamma_2 - \tau_2)^2 + \left\{\frac{\gamma_2}{\alpha_{22}}\right\}^{\frac{1}{2}} \left[\alpha_{22} \; (\gamma_1 - \rho c^2) - \alpha_{12}^2\right] = 0,$$

which yields either

$$\rho c^2 = \gamma_1, \tag{4.9.16}$$

or

$$(\gamma_1 - \rho c^2)(\gamma_2 + \sqrt{\alpha_{22}\,\gamma_2}) = (\gamma_2 - \tau_2)^2 + \sqrt{\frac{\gamma_2}{\alpha_{22}}}\;\alpha_{12}^2. \tag{4.9.17}$$

If (4.9.16) holds then, by (4.8.26), we may take $s_1 = 0$ and $s_2 = -\,\delta^2 / \alpha_{22}\,\gamma_2 < 0$, which does not give a surface wave. The limiting speed $c = c_L$ satisfying (4.9.16) is associated with a plane body wave.

Hence, when $\alpha_{11} = \gamma_1$, ρc^2 is given uniquely by (4.9.17). When $c = 0$, (4.9.17) yields

$$\gamma_1(\gamma_2 + \sqrt{\alpha_{22}\,\gamma_2}) = (\gamma_2 - \tau_2)^2 + \sqrt{\frac{\gamma_2}{\alpha_{22}}}\;\alpha_{12}^2, \tag{4.9.18}$$

which is also given by (4.9.13) in this case.

c) The case $0 < \rho c^2 < \alpha_{11} < \gamma_1$

Finally in this section we consider the inequalities

$$0 < \rho c^2 < \alpha_{11} < \gamma_1, \tag{4.9.19}$$

the secular equation (4.9.1) may written as

$$\gamma_2(\alpha_{11} - \rho c^2)^{\frac{1}{2}} (\gamma_1 - \rho c^2) - (\alpha_{11} - \rho c^2)^{\frac{1}{2}} (\gamma_2 - \tau_2)^2$$

$$+ \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \left[\gamma_1 - \rho c^2\right]^{\frac{1}{2}} \left\{\alpha_{22}(\alpha_{11} - \rho c^2) - \alpha_{12}^2\right\} = 0.$$

Since $\alpha_{11} < \gamma_1$, the above secular equation may written as

$$\gamma_2 (\gamma_1 - \rho c^2) \left[\frac{\alpha_{11} - \rho c^2}{\gamma_1 - \rho c^2}\right]^{\frac{1}{2}} - (\gamma_2 - \tau_2)^{\frac{1}{2}} \left[\frac{\alpha_{11} - \rho c^2}{\gamma_1 - \rho c^2}\right]^{\frac{1}{2}}$$

$$+ \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \left\{\alpha_{22} (\alpha_{11} - \rho c^2) - \alpha_{12}^2\right\} = 0. \tag{4.9.20}$$

Here, we set

$$\eta = \left\{\frac{\alpha_{11} - \rho c^2}{\gamma_1 - \rho c^2}\right\}^{\frac{1}{2}}, \tag{4.9.21}$$

so that

$$0 < \eta < \sqrt{\frac{\alpha_{11}}{\gamma_1}}, \tag{4.9.22}$$

the left-hand limit in the above corresponds to $\rho c^2 = \alpha_{11}$ and the right-hand limit to $c = 0$, also we have

$$\rho c^2 = \frac{\alpha_{11} - \gamma_1 \eta^2}{1 - \eta}, \tag{4.9.23}$$

so, equation (4.9.20) can be written as

$$h(\eta) \equiv (\gamma_2 - \tau_2)^2 \eta^3 + \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \left\{\alpha_{22} (\gamma_1 - \alpha_{11}) + \alpha_{12}^2\right\} \eta^2$$

$$+ \left\{\gamma_2 (\gamma_1 - \alpha_{11}) - (\gamma_2 - \tau_2)^2\right\} \eta - \left[\frac{\gamma_2}{\alpha_{22}}\right]^{\frac{1}{2}} \alpha_{12}^2 = 0. \tag{4.9.24}$$

Hence,

$$h(0) = -\sqrt{\frac{\gamma_2}{\alpha_{22}}}\,\alpha_{12}^2,$$

and

$$h\left(\sqrt{\frac{\alpha_{11}}{\gamma_1}}\right) \equiv (\gamma_1 - \alpha_{11})\left[\sqrt{\frac{\alpha_{11}}{\gamma_1}}\left\{\gamma_1\,\gamma_2 - (\gamma_2 - \tau_2)^2\right\}\right.$$

$$\left. + \sqrt{\frac{\gamma_2}{\alpha_{22}}}\left\{\alpha_{11}\,\alpha_{22} - \alpha_{12}^2\right\}\right].$$

Since we have $\gamma_1 - \alpha_{11} > 0$, we obtain that $h(\eta) = 0$ has a solution in the interval (4.9.22) if $h\left(\sqrt{\frac{\alpha_{11}}{\gamma_1}}\right) \geqslant 0$, i.e. If

$$\sqrt{\frac{\alpha_{11}}{\gamma_1}}\left\{\gamma_1\,\gamma_2 - (\gamma_2 - \tau_2)^2\right\} + \sqrt{\frac{\gamma_2}{\alpha_{22}}}\left[\alpha_{11}\,\alpha_{22} - \alpha_{12}^2\right] \geqslant 0. \quad (4.9.25)$$

If $\rho c^2 \neq \alpha_{11}$ then (4.9.25) shows that (4.9.9) is again necessary and sufficient for the existence of a unique surface wave. Under the inequalities (4.8.27) no non-trivial surface waves with $\rho c^2 = \alpha_{11}$ is possible.

4.9.2 The case $\gamma_2 = \tau_2$

When $\gamma_2 = \tau_2$ and $0 \leqslant \rho c^2 \leqslant \gamma_1 < \alpha_{11}$ holds, the secular equation (4.9.6) can be written as

$$f(\eta) \equiv \eta\, g(\eta) = 0, \qquad (4.9.26)$$

where

$$g(\eta)=\left[\frac{\gamma_2}{\alpha_{11}}\right]^{\frac{1}{2}}\alpha_{12}^2\,\eta^2 + \gamma_2(\alpha_{11} - \gamma_1)\,\eta + \left[\frac{\gamma_2}{\alpha_{11}}\right]^{\frac{1}{2}}\left\{\alpha_{22}(\alpha_{11} - \gamma_1) - \alpha_{12}^2\right\} = 0.$$

$$(4.9.27)$$

If $g(0) \equiv f'(0) > 0$ then $g'(0) > 0$ for $\eta > 0$ so that $g(\eta) = 0$ has no solution for η in the interval (4.9.4). Thus, if

$$\alpha_{22}\,(\alpha_{11} - \gamma_1) - \alpha_{12}^2 > 0, \tag{4.9.28}$$

then $\eta = 0$ is the only solution of (4.9.27). When (4.9.28) holds we may therefore take $s_1 = 0$ and $s_2^2 = \dfrac{\alpha_{22}\,(\alpha_{11} - \gamma_1) - \alpha_{12}^2}{\gamma_2\,\alpha_{22}} > 0$, and the wave speed is given by $\rho c^2 = \gamma_1$. However, application of the boundary conditions shows that $v_1 = 0$ and $v_2 = B_1 e^{i(\omega t - k\,x_1)}$, which describes a plane shear waves, not a surface wave.

Next, we note that

$$g\left(\sqrt{\frac{\gamma_1}{\alpha_{11}}}\right) = \frac{\alpha_{11} - \gamma_1}{\alpha_{11}}\sqrt{\frac{\gamma_2}{\alpha_{22}}}\left[\alpha_{11}\,\alpha_{22} - \alpha_{12}^2 + \sqrt{\alpha_{11}\,\alpha_{22}\,\gamma_1\,\gamma_2}\right]. \tag{4.9.29}$$

Since $g(\eta)$ is increasing for $\eta > 0$ we deduce that $g(0) > 0$ implies that $g\left(\sqrt{\frac{\gamma_1}{\alpha_{11}}}\right) > 0$. Thus, necessary and sufficient conditions for $g(\eta) = 0$ to have a unique solution for η in the interval (4.9.4) are

$$\alpha_{22}(\alpha_{11} - \gamma_1) - \alpha_{12}^2 < 0, \tag{4.9.30}$$

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}^2 + \sqrt{\alpha_{11}\,\alpha_{22}\,\gamma_1\,\gamma_2} > 0. \tag{4.9.31}$$

jointly. The solution corresponding to $\eta = 0$ is then excluded. This result corresponds to results given for incompressible materials.

If $\alpha_{11} = \gamma_1$ the solution $\rho c^2 = \gamma_1$ of (4.8.38) does not correspond to a surface wave, as in Section 4.9.1; the surface wave speed is then given uniquely by (4.9.17) with $\gamma_2 = \tau_2$.

If $0 < \rho c^2 < \alpha_{11} < \gamma_1$ then $\rho c^2 = \alpha_{11}$ is a solution of (4.8.38) with $s_1 = 0$ and $s_2^2 = \dfrac{\gamma_2(\gamma_1 - \alpha_{11}) - \alpha_{12}^2}{\gamma_2 \alpha_{22}}$. However, as for the case discussed in the paragraph following (4.9.28), application of the boundary conditions demonstrates that the resulting wave is not a surface wave, it corresponds to a plane longitudinal wave with $v_1 = A_1 e^{i(\omega t - kx_1)}$, $v_2 = 0$. A unique surface wave with $\rho c^2 < \alpha_1$ exists if and only if the exclusion condition

$$\alpha_{11} \alpha_{22} - \alpha_{12}^2 + \sqrt{\alpha_{11} \alpha_{22} \gamma_1 \gamma_2} > 0. \qquad (4.9.32)$$

holds. Quasi-static surface deformations become possible when (4.9.32) first fails on a path of deformation from the natural configuration, i.e. when

$$\alpha_{11} \alpha_{22} - \alpha_{12}^2 + \sqrt{\alpha_{11} \alpha_{22} \gamma_1 \gamma_2} \geqslant 0. \qquad (4.9.33)$$

4.9.3 The case $\delta = 0$

In the above discussion we considered $\delta \neq 0$. In this section we shall consider the special case in which $\delta = 0$, so equations (4.8.6) reduce to

$$\rho \ddot{v}_1 = \alpha_{11} v_{1,11} + \gamma_2 v_{1,22},$$
$$\rho \ddot{v}_2 = \gamma_1 v_{1,11} + \alpha_{22} v_{2,11}, \qquad (4.9.34)$$

and on use of (4.8.7), we get

$$\rho c^2 = \alpha_{11} - \gamma_2 s^2, \qquad \rho c^2 = \gamma_1 - \alpha_{22} s^2, \qquad (4.9.35)$$

equation $(4.9.35)_1$, for $s = s_1$, gives

$$s_1^2 = \frac{\alpha_{11} - \rho c^2}{\gamma_2}, \qquad (4.9.36)$$

equation $(4.9.35)_2$, for $s = s_2$, gives

$$s_2^2 = \frac{\gamma_1 - \rho c^2}{\alpha_{22}}, \tag{4.9.37}$$

Equations (4.8.14) can be written, on use of the notations (4.8.23) as

$$\begin{aligned} i\, s\, \delta\, A_2 &= (\alpha_{11} - \gamma_2\, s_1^2 - \rho c^2) A_1, \\ i\, s\, \delta\, B_2 &= (\alpha_{11} - \gamma_2\, s_2^2 - \rho c^2) B_1, \end{aligned} \tag{4.9.38}$$

so, equations (4.8.12) are replaced by

$$\begin{aligned} v_1 &= A_1\, e^{s_1 k x_2 - i(kx_1 - \omega t)}, \\ v_2 &= A_2\, e^{s_2 k x_2 - i(kx_1 - \omega t)}. \end{aligned} \tag{4.9.39}$$

On use of the notations (4.8.23), the boundary conditions (4.8.11) become

$$\left.\begin{aligned} \gamma_2\, v_{1,2} + (\gamma_2 - \tau_2)\, v_{2,1} &= 0, \\ \alpha_{12}\, v_{1,1} + \alpha_{22}\, v_{2,2} &= 0. \end{aligned}\right\} \quad \text{on } x_2 = 0 \tag{4.9.40}$$

Substituting (4.9.39) in (4.9.40), we have

$$\begin{aligned} \gamma_2\, s_1\, A_1 + i\, (\gamma_2 - \tau_2) A_2 &= 0, \\ i\, \alpha_{12}\, A_1 + \alpha_{22}\, s_2\, A_2 &= 0. \end{aligned} \tag{4.9.41}$$

For these equations to have non-trivial solutions for A_1, A_2 we must have

$$\begin{vmatrix} \gamma_2 \, s_1 & i\,(\gamma_2 - \tau_2) \\ i\,\alpha_{12} & \alpha_{22}\, s_2 \end{vmatrix} = 0,$$

which gives

$$\gamma_2\, \alpha_{22}\, s_1 s_2 + \alpha_{12}(\gamma_2 - \tau_2) = 0, \tag{4.9.42}$$

On use of (4.9.36) and (4.9.37) in (4.9.42), the secular equation becomes

$$\{\gamma_2\, \alpha_{22}\, (\alpha_{11} - \rho c^2)\, (\gamma_1 - \rho c^2)\}^{\frac{1}{2}} + \alpha_{12}(\gamma_2 - \tau_2) = 0, \tag{4.9.43}$$

since we consider $\delta = 0$, we then have

$$\gamma_2 - \tau_2 = -\alpha_{12}, \tag{4.9.44}$$

equation (4.9.43) becomes

$$\{\gamma_2\, \alpha_{22}\, (\alpha_{11} - \rho c^2)\, (\gamma_1 - \rho c^2)\}^{\frac{1}{2}} - \alpha_{12}^2 = 0. \tag{4.9.45}$$

This equation is the secular equation for compressible Rayleigh surface waves for the special case $\delta = 0$.

Next, equation (4.8.41), for $\delta = 0$, becomes

$$\{\alpha_{22}\, (\alpha_{11} - \rho c^2)\}^{\frac{1}{2}}\, \{\gamma_2\, (\gamma_1 - \rho c^2) - \alpha_{12}^2\}$$

$$- \{\gamma_2\, (\gamma_1 - \rho c^2)\}^{\frac{1}{2}}\, \{\alpha_{22}(\alpha_{11} - \rho c^2) - \alpha_{12}^2\} = 0. \tag{4.9.46}$$

i.e.

$$\{\alpha_{22}(\alpha_{11}-\rho c^2)(\gamma_1-\rho c^2)\}^{\frac{1}{2}}[\{\alpha_{22}(\alpha_{11}-\rho c^2)\}^{\frac{1}{2}}+\{\gamma_2(\gamma_1-\rho c^2)\}]$$

$$-\alpha_{12}^2[\{\alpha_{22}(\alpha_{11}-\rho c^2)\}^{\frac{1}{2}}+\{\gamma_2(\gamma_1-\rho c^2)\}^{\frac{1}{2}}]=0,$$

from this equation we can deduce equation (4.9.45).

Equation (4.9.42) embraces the situations in which $s_1 = 0$, $\alpha_{12} = \gamma_2 - \tau_2 = 0$, $A_2 = 0$ (corresponding to a plane longitudinal wave) and $s_2 = 0$, $\alpha_{12} = \gamma_2 - \tau_2 = 0$, $A_1 = 0$ (corresponding to a plane shear wave).

A solution with $c = 0$ exists in configurations for which

$$\sqrt{\alpha_{11}\,\alpha_{22}\,\gamma_1\,\gamma_2} - (\gamma_2 - \tau_2)^2 = 0. \qquad (4.9.47)$$

while the exclusion condition takes the form

$$\sqrt{\alpha_{11}\,\alpha_{22}\,\gamma_1\,\gamma_2} - (\gamma_2 - \tau_2)^2 > 0. \qquad (4.9.48)$$

Rearrangement of the (4.9.45) yields the unique solution

$$\rho c^2 = \tfrac{1}{2}\left[\alpha_{11} + \gamma_1 - \left\{(\alpha_{11} - \gamma_1) + 4\,\frac{\alpha_{12}^2}{\alpha_{11}\,\alpha_{22}}\right\}^{\frac{1}{2}}\right], \qquad (4.9.49)$$

satisfying (4.8.22)

4.10 Results for some special deformations

Here we shall summarize the results for some special deformations, for which the secular equation simplifies.

4.10.1 Equibiaxial deformation

We now specialize the underlying deformation so that $\lambda_1 = \lambda_2$, and introduce the notation

$$\alpha = \alpha_{11} = \alpha_{22}, \qquad \gamma = \gamma_1 = \gamma_2, \qquad \tau = J\sigma_1 = J\sigma_2. \quad (4.10.1)$$

Then from (2.6.6) and (4.8.23), we obtain

$$\alpha_{12} = \alpha - 2\gamma + \tau, \qquad \delta = \alpha - \gamma, \qquad \beta = \alpha\gamma, \quad (4.10.2)$$

and equation (4.8.9) becomes

$$\alpha\,\gamma\,s^4 - \{2\alpha\,\gamma - \rho c^2(\gamma + \alpha)\}\,s^2 + \alpha\,\gamma - \rho c^2(\alpha + \gamma - \rho c^2) = 0. \quad (4.10.3)$$

which gives

$$s_1^2 = 1 - \frac{\rho c^2}{\gamma}, \qquad s_2^2 = 1 - \frac{\rho c^2}{\alpha}. \quad (4.10.4)$$

For $\gamma \neq \tau$ and $\alpha \neq \gamma$ the inequality (4.9.12) yields

$$\frac{-\,2\gamma\,(\alpha - \gamma)}{\alpha + \gamma} < \tau < 2\gamma. \quad (4.10.5)$$

For $\alpha = \gamma$ (and hence $\delta = 0$), and $\gamma \neq \tau$, the secular equation (4.8.38) reduces to

$$(\alpha - \rho c^2)[\alpha(\alpha - \rho c^2) - (\alpha - \tau)^2] = 0, \quad (4.10.6)$$

so that either $\rho c^2 = \alpha$ or $\rho c^2 = \frac{\tau(2\alpha - \tau)}{\alpha}$. The latter is also obtained directly from (4.9.17), and requires $0 < \tau < 2\alpha$ for the existence of surface wave. If $\rho c^2 = \alpha$ the boundary conditions ensure that $v_1 = v_2 = 0$, so that no wave corresponds to this case.

When $\alpha > \gamma = \tau$ then $\rho c^2 = \gamma$ and $\rho c^2 = \alpha$ are the only (positive, real) solutions of the secular equation. The first of these gives a plane shear wave, not a surface wave, the second again gives the trivial solution $v_1 = v_2 = 0$. The same is true if $\tau = \gamma > \alpha$. However, in this case a unique surface wave also, exists provided $3\alpha > \gamma$, and its speed is given by

$$\rho c^2 = \frac{\gamma \left[3\sqrt{\alpha} - \sqrt{4\gamma - 3\alpha}\right]}{2\sqrt{\alpha}}. \tag{4.10.7}$$

Quasi-static surface deformations are possible if $\gamma = 3\alpha$.

If $\gamma = \alpha = \tau$ then no surface wave exists. The only solution of the secular equation is $\rho c^2 = \alpha$ and the boundary conditions admit plane waves with either $v_1 = 0$, $v_2 = B\, e^{i(\omega t - kx_1)}$ or $v_1 = A\, e^{i(\omega t - kx_1)}$, $v_2 = 0$.

The results described in this section also apply when stress is purely hydrostatic and $\lambda_1 = \lambda_2 = \lambda_3$.

Finally, for the special case $\lambda_1 = \lambda_2$ and for the case $\tau_2 = 0$, equation (4.8.42) reduces to

$$\gamma\alpha\,(\rho c^2)^3 - 8\,\gamma^2\,\alpha\,(\rho c^2)^2 + 8\,\gamma^3\,(3\alpha - 2\gamma)\,(\rho c^2) + 16\,\gamma^4\,(\gamma - \alpha) = 0. \tag{4.10.8}$$

This equation is an equivalent to that arising in the linear theory as we shall see in the next case.

4.10.2 The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Equation (4.10.8) can be written as

$$\frac{\gamma}{\alpha}\left[\frac{\rho c^2}{\alpha}\right]^3 - 8\left[\frac{\gamma}{\alpha}\right]^2\left[\frac{\rho c^2}{\alpha}\right]^2 + 8\left[\frac{\gamma}{\alpha}\right]\left[3 - 2\,\frac{\gamma}{\alpha}\right]\left[\frac{\rho c^2}{\alpha}\right] + 16\left[\frac{\gamma}{\alpha}\right]^4\left[\frac{\gamma}{\alpha} - 1\right] = 0. \tag{4.10.9}$$

For this case $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have $\alpha_{11} = \lambda + 2\mu \equiv \rho c_L^2$, $\gamma_2 = \mu \equiv \rho c_T^2$ (as given in (2.7.18)), where μ and λ are the Lamé constants of the linear theory, so that equation (4.10.9) becomes

$$\left[\frac{c_T^2}{c_L^2}\right]\left[\frac{c^2}{c_L^2}\right]^3 - 8\left[\frac{c_T^2}{c_L^2}\right]^2\left[\frac{c^2}{c_L^2}\right]^2 + 8\left[\frac{c_T^2}{c_L^2}\right]^3\left[3 - \frac{c_T^2}{c_L^2}\right]\left[\frac{c^2}{c_L^2}\right] + 16\left[\frac{c_T^2}{c_L^2}\right]^4\left[\frac{c_T^2}{c_L^2} - 1\right] = 0.$$

On rearranging this equation becomes

$$\left[\frac{c}{c_T}\right]^6 - 8\left[\frac{c}{c_T}\right]^4 + \left[24 - 16\,\frac{c_T^2}{c_L^2}\right]\left[\frac{c}{c_T}\right]^2 + 16\left[1 - \frac{c_T^2}{c_L^2}\right] = 0, \tag{4.10.10}$$

which is the corresponding result of linear theory (see section 3.6). Also (4.8.46) can be written in terms of the Lamé constants as

$$\left[\frac{\rho c^2}{\mu}\right]^3 - 8\left[\frac{\rho c^2}{\mu}\right]^2 + \left[24 - 16\,\frac{\mu}{\lambda + \mu}\right]\left[\frac{\rho c^2}{\mu}\right] - 16\left[1 - \frac{\mu}{\lambda + \mu}\right] = 0. \tag{4.10.11}$$

4.11 Application to specific strain-energy functions

For the case $\tau_2 = 0$ surface waves in a restricted Hadamard material have been examined in detail by Chadwick and Jarvis (1979); and also by Willson (1973b), who confined attention to equibiaxial deformations corresponding, in our notation to $\lambda_3 = \lambda_1$. Their results for propagation along a principal axis can be recovered by appropiate specialization of formulae given in Section 4.9. When c = 0 the criterion (4.9.13) for the existence of surface deformations reduces (when $\tau_2 = 0$) to a result of Usmani and Beatty (1974) for the same strain-energy function. The restricted Hadamard material is characterized by the strain-energy function

$$W = \tfrac{1}{2}\,\mu\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \kappa\, f(J), \qquad (4.11.1)$$

where μ (> 0) and κ (> 0) are respectively the shear and bulk modulus of the material in the natural configuration, and the function f is such that

$$f(1) = 0, \qquad f'(1) = -\,{}^{\mu}/_{\kappa}, \qquad f''(1) = 1 + {}^{1}/_{3}\,{}^{\mu}/_{\kappa}. \qquad (4.11.2)$$

Possible inequalities on $f'(J)$ and $f''(J)$ have been considered by Willson and Chadwick and Jarvis; we do not impose such inequalities here.

We assume that $\tau_2 \neq 0$ and examine two special cases from Section 4.9.

a) If $\alpha_{11} = \gamma_1$ then $\delta = 0$ and $f''(J) = 0$. From (4.9.17) or (4.9.47) it follows that

$$\rho c^2 = \gamma_1 - \alpha_{12}^2 / \gamma_2, \qquad (4.11.3)$$

and the exclusion condition is

$$\gamma_1 \gamma_2 - \alpha_{12}^2 \equiv (\mu \lambda_1 \lambda_2 + \kappa J f') (\mu \lambda_1 \lambda_2 - \kappa J f') > 0. \tag{4.11.4}$$

Surface deformations are possible where either $\mu \lambda_1 \lambda_2 = \kappa J f'(J)$ or $\mu \lambda_1 \lambda_2 = - \kappa J f'(J)$, J being given by $f''(J) = 0$. Only one of these equations can hold for a given J since $\mu \lambda_1 \lambda_2$ must be positive.

b) If $\gamma_2 = \tau_2$ then $f'(J) = 0$ and we obtain

$$\alpha_{22} (\alpha_{11} - \gamma_1) - \alpha_{12}^2 = \gamma_2 \delta,$$

where $\delta = \kappa J^2 f''(J)$. From Section 4.9.2 we find that when $\delta > 0$ a necessary condition for the existence of a surface wave is $\delta < 0$ (recall (4.9.30)). Thus, no surface wave exists if $\delta > 0$. On the other hand, when $\delta < 0$ the existence of a unique surface wave is guaranteed if (4.9.45) holds. In particular, if $\lambda_1 = \lambda_2$ the equality (4.9.32) yields $\delta > - {}^2/_3\, \gamma$, which is stronger than the requirement $\alpha \equiv \gamma + \delta > 0$.

As second example we consider the strain-energy function

$$W = 2\mu (\lambda_1 + \lambda_2 + \lambda_3 + J^{-1} - 4), \tag{4.11.5}$$

with $\lambda_3 = 1$. The left-hand side of (4.9.12) now has the form

$$\frac{2\mu^2 \lambda_1^{-2} \lambda_2^{-2} [7 \lambda_1^2 \lambda_2^2 - 2(\lambda_1 + \lambda_2)]}{\lambda_1 + \lambda_2}.$$

so the exclusion condition is

$$7 \lambda_1^2 \lambda_2^2 > 2(\lambda_1 + \lambda_2). \tag{4.11.6}$$

If $\alpha_{11} = \gamma_1$ then $2(\lambda_1 + \lambda_2) = \lambda_1^3 \lambda_2$ and (4.11.6) reduces to $7\lambda_2 > 2\lambda_1$.

When $\delta = 0$ we obatin $2(\lambda_1 + \lambda_2) = \lambda_1^2 \lambda_2^2$ and (4.11.6) is automatically satisfied.

When $\gamma_2 = \tau_2$ then $\lambda_1 + \lambda_2 = \lambda_1^2 \lambda_2^2$ and the exclusion condition (4.9.32) holds. Also $\alpha_{11} - \gamma_1$ has the sign of $2\lambda_2 - \lambda_1$; consideration of (4.9.30) shows that a surface wave exists if $\lambda_2 < {}^2/_3 \lambda_1$.

To illustrate the results for the strain-energy function (4.11.5), the curves $\alpha_{11} = \gamma_1$, $\gamma_2 = \tau_2$ and $\delta = 0$ are shown in (λ_1, λ_2)-plane for the region where (4.11.6) holds in Fig. 4.15.

Finally, we note if that $\lambda_1 = \lambda_2$ then $\gamma = \mu \lambda$, $\alpha = 4\mu \lambda^{-2}$ and $\tau = 2\gamma - \frac{1}{2} \alpha$; if $\alpha = \gamma$ then $\delta = 0$ and $\lambda^3 = 4$, and the wave speed is given by $\rho c^2 = 3 (4)^{-1/3} \mu = \frac{3}{4} \gamma$. If $\gamma = \tau$ then $\lambda^3 = 2$, $\alpha > \gamma$ and no surface wave is possible.

Figure 4.15 illustrates the results for the strain-energy function (4.11.5). In particular, the curves (a) $\alpha_{11} = \gamma_1$, (b) $\gamma_2 = \tau_2$, (c) $\delta = 0$, (d) $\alpha_{22} (\alpha_{11} - \gamma_1) = \alpha_{12}^2$, (e) equation (4.9.13) in the (λ_1, λ_2)-plane in respect of the strain-energy function (4.11.5). The exclusion condition (4.9.12) holds above the curve (e). The broken part of (b) is where (4.9.30) fails.

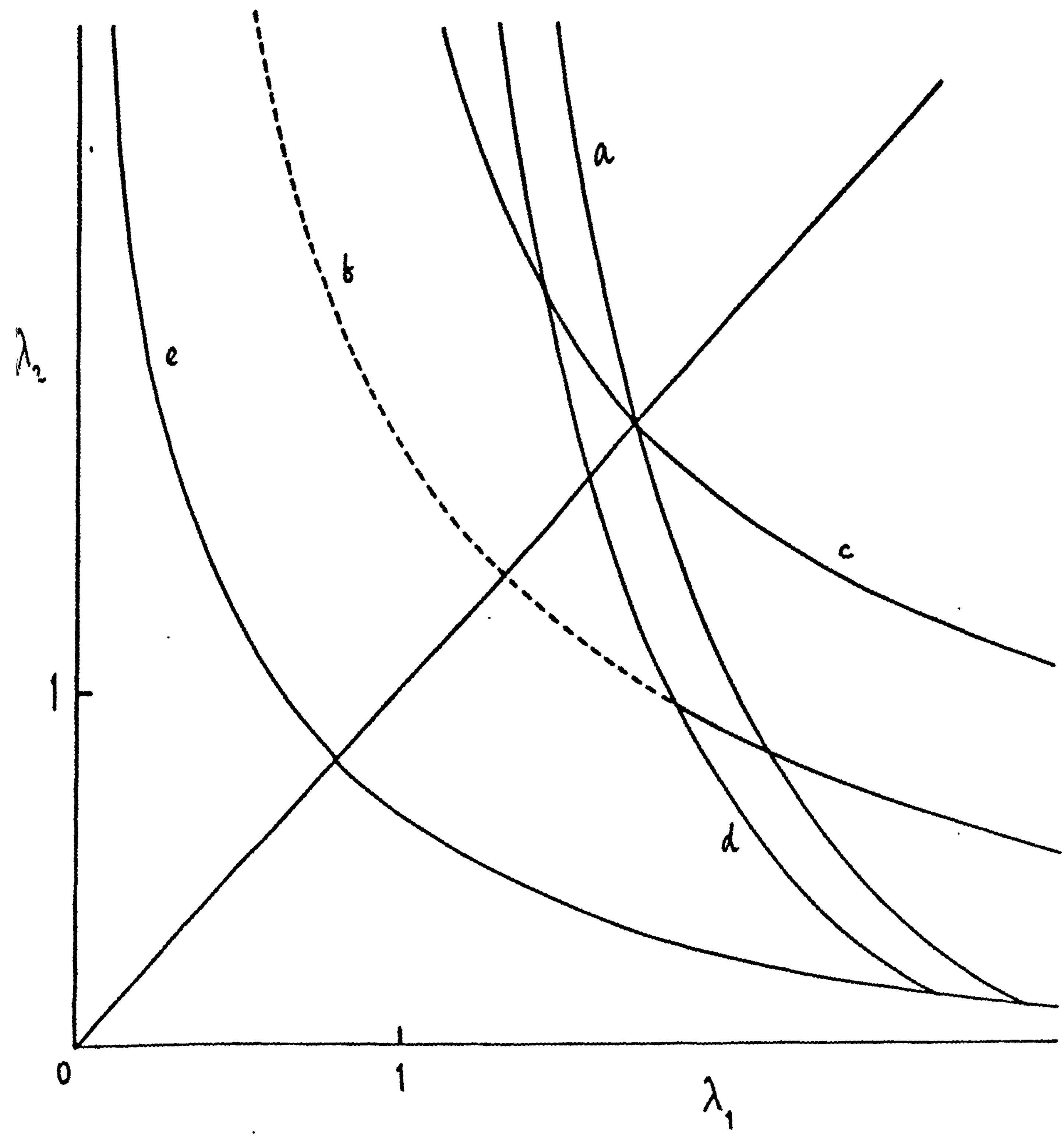


Figure 4.15

Chapter 5

Love waves and Rayleigh waves on a layered half-space

In this Chapter we shall consider Love waves in a pre-stressed layered half-space, where the layer and half-space consist of different materials. Also, we shall consider Rayleigh surface waves on a layered half-space.

5.1 Love waves

5.1.1 Results for an incompressible material

We consider a pre-stressed half-space defined by $x_2 < 0$ on which there is a layer of different pre-stressed material of uniform thickness h with boundaries $x_2 = 0$ and $x_2 = h$.

We assume that $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*)$ are the stretches of the deformation in the half-space and the layer respectively and let W and W^* be the corresponding strain-energy functions. Now consider propagation along a principal axis by solving equation (2.6.20) with the boundary conditions given by

$$\begin{aligned} &\dot{s}_{02i} = 0 && \text{on } x_2 = h, \\ &\underline{v} = \underline{v}^*, \quad \dot{s}_{02i} = \dot{s}^*_{02i} && \text{on } x_2 = 0, \end{aligned} \tag{5.1.1}$$

where $\underline{v}$ and $\underline{v}^*$ are given by

$$\begin{aligned} \underline{v} &= (v_1, v_2, v_3) = (0,\ 0,\ A\, e^{skx_2 - ikx_1 + i\omega t}), \\ \underline{v}^* &= (v_1^*, v_2^*, v_3^*) = (0,\ 0,\ (A' \cos s^*kx_2 + A''\sin s^*kx_2)). \end{aligned} \tag{5.1.2}$$

On use of $(5.1.2)_1$ in the equations of motion (2.6.20), we deduce that

$$\dot{p}_{,1} = 0,$$

$$\dot{p}_{,2} = 0, \tag{5.1.3}$$

$$A_{01313}\, v_{3,11} + A_{02323}\, v_{3,22} = \rho \ddot{v}_3.$$

Substituting $\underline{v}$ in $(5.1.3)_3$, we get

$$A_{02323}\, s^2 - A_{01313} = -\rho c^2, \tag{5.1.4}$$

which gives

$$s^2 = \frac{A_{01313} - \rho c^2}{A_{02323}}.$$

Similarly

$$s^{*2} = \frac{\rho^* c^2 - A^*_{01313}}{A^*_{02323}}.$$

Then on use of the notation given by (2.7.17), the above equations become

$$s^2 = \frac{c_{13}^2 - c^2}{c_{23}^2}, \qquad s^{*2} = \frac{c^2 - c^{*2}_{13}}{c^{*2}_{23}}. \tag{5.1.5}$$

Substituting (5.1.5) into the boundary conditions (5.1.1), we have

$$\dot{s}_{023} \equiv A_{02323}\, v_{3,2} + (A_{02332} + p)\, v_{2,3} = A_{02323}\, v_{3,2}. \tag{5.1.6}$$

The boundary conditions $(5.1.1)_1$ give $v^*_{3,2} = 0$ on $x_3 = h$, and from $(5.1.2)_2$ we obtain

$$A' \sin s^* kh - A'' \cos s^* kh = 0. \tag{5.1.7}$$

Also, from $(5.1.1)_2$, we get

$$A = A', \qquad s\, A_{02323}\, A = s^*\, A^*_{02323}\, A''. \tag{5.1.8}$$

Substituting (5.1.7) and (5.1.8), we deduce the secular equation

$$\cot s^* kh = \frac{s^*\, A^*_{02323}}{s\, A_{02323}} \equiv \frac{s^*\, \rho^* c^{*2}_{23}}{s\, \rho c^2_{23}}.$$

i.e.

$$\tan s^* kh = \frac{s\, \rho c^2_{23}}{s^* \rho^* c^{*2}_{23}}, \tag{5.1.9}$$

where

$$c^{*2}_{13} < c^2 < c^2_{13}. \tag{5.1.10}$$

On use of (5.1.5) in (5.1.9), the secular equation becomes

$$\tan\left\{ kh \sqrt{\frac{c^2 - c^{*2}_{13}}{c^{*2}_{23}}} \right\} = \frac{\rho c_{23} \sqrt{c^2_{13} - c^2}}{\rho^* c^*_{23} \sqrt{c^2 - c^{*2}_{13}}} \tag{5.1.11}$$

This is the secular equation for incompressible Love waves propagating along a principal axis in respect of the general strain-energy function.

5.1.2 Limiting cases

Here, we wish to investegate the following limiting cases

a) The case $c = c_{13}$

For this limiting case equation (5.1.11) reduces to

$$\tan \left\{ kh \sqrt{\frac{c_{13}^2 - c_{13}^{*2}}{c_{23}^{*2}}} \right\} = 0,$$

this equation leads to

$$kh \sqrt{\frac{c_{13}^2 - c_{13}^{*2}}{c_{23}^{*2}}} = n\,\pi, \qquad n \in \mathbb{N}.$$

For the general case equation (5.1.11) will have real root for c when $c < c_{13}$. The solution of (5.1.11) may have many real roots, every real root corresponds to a particular mode of oscillation of the layer, see for example Eringen and Suhubi (1975). From the above limiting case we establish that the different modes correspond to

0,1,2,3,.... as shown in Fig. 5.1.

b) The limit $c \longrightarrow c_{13}^*$

For this case equation (5.1.11) leads to

$$kh \longrightarrow \infty \quad \text{as} \quad c \longrightarrow c_{13}^* .$$

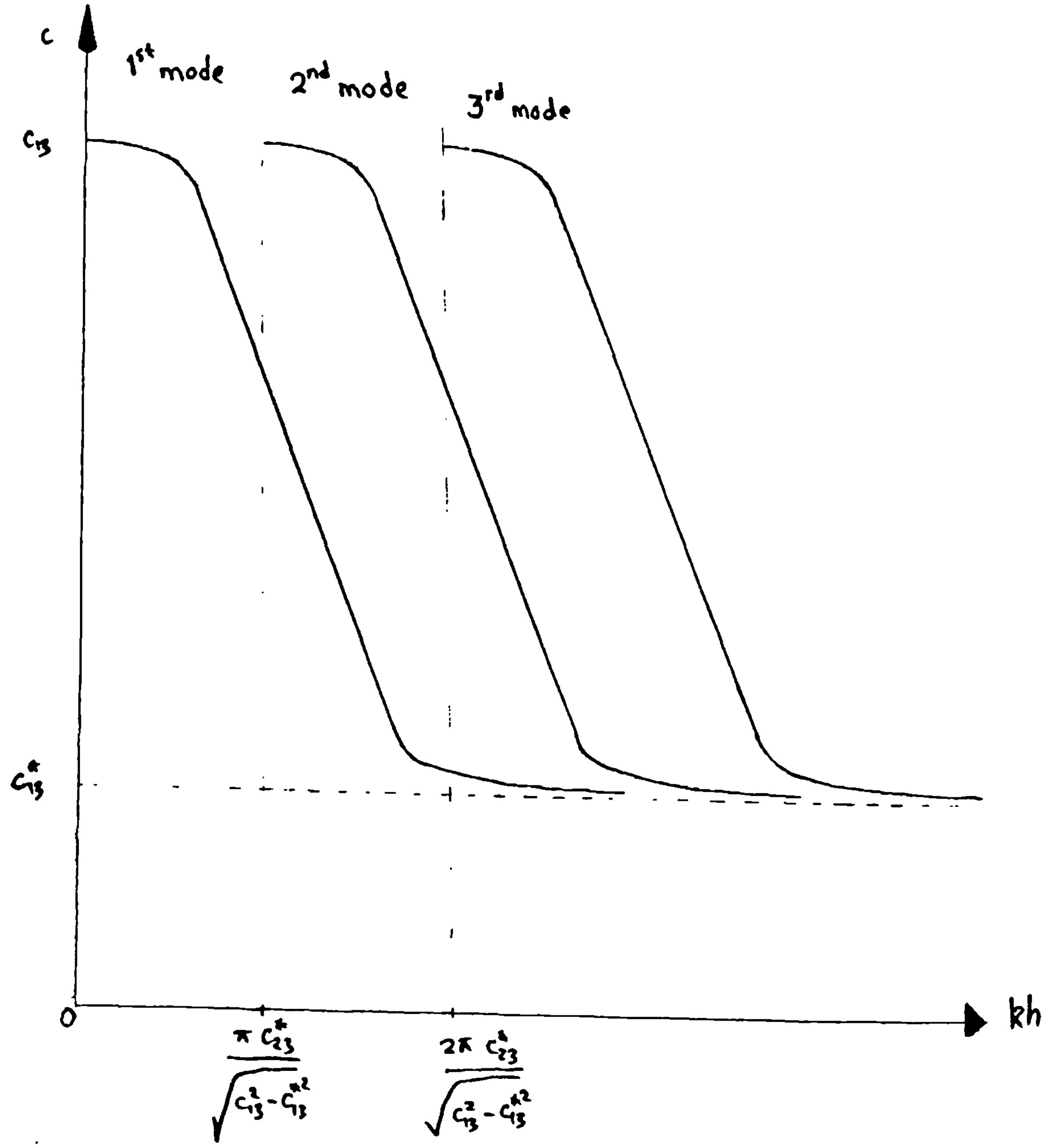

c
1st mode
2nd mode
3rd mode
c_{13}
c_{13}^*
0
kh
$\frac{\pi c_{23}^*}{\sqrt{c_{13}^2 - c_{13}^{*2}}}$
$\frac{2\pi c_{23}^*}{\sqrt{c_{13}^2 - c_{13}^{*2}}}$

Figure 5.1

5.1.3 Love waves for a neo-Hookean material

Next, to illustrate the general results above we consider a neo-Hookean material, so that

$$\rho c_{ij}^2 = \mu\, \lambda_i^2, \qquad \rho^* c_{ij}^{*2} = \mu^*\, \lambda_i^{*2},$$

and hence equations (5.1.11) and (5.1.10) become

$$\tan\left\{ kh \sqrt{\frac{(\rho^* c^2/\mu^*) - \lambda_1^{*2}}{\lambda_2^{*2}}} \right\} = \frac{\mu\, \lambda_2}{\mu^* \lambda_2^*} \sqrt{\frac{\lambda_1^2 - (\rho c^2/\mu)}{(\rho^* c^2/\mu^*) - \lambda_1^{*2}}} \qquad (5.1.12)$$

Subject to

$$\frac{\rho\mu^*}{\mu\rho^*}\, \lambda_1^{*2} < \frac{\rho c^2}{\mu} < \lambda_1^2. \qquad (5.1.13)$$

On setting $\alpha = \dfrac{\rho^*}{\rho}$, $\beta = \dfrac{\mu}{\mu^*}$ and $\xi = \dfrac{\rho c^2}{\mu}$ equations (5.1.12) and (5.1.13) become

$$\tan\left\{ kh \sqrt{\frac{\alpha\, \beta\, \xi - \lambda_1^{*2}}{\lambda_2^{*2}}} \right\} = \beta\, \frac{\lambda_2}{\lambda_2^*} \sqrt{\frac{\lambda_1^2 - \xi}{\alpha\, \beta\, \xi - \lambda_1^{*2}}},$$

subject to (5.1.14)

$$\frac{\lambda_1^{*2}}{\alpha\, \beta} < \xi < \lambda_1^2.$$

Finally in this section we present some numerical results based on the secular equation (5.1.14). First of all we specify values of λ_1, λ_2 and λ_1^* , λ_2^* subject to

$$\frac{\lambda_1^{*2}}{\alpha\,\beta} < \lambda_1^2. \tag{5.1.15}$$

When we take $\lambda_2^* = 1$, $\lambda_2 = 0.75$ then $(5.1.14)_1$ can be written as

$$\tan\left\{ kh\,\sqrt{\alpha\,\beta\,\xi - 1}\right\} = \frac{3\beta}{4}\sqrt{\frac{\lambda_1^2 - \xi}{\alpha\,\beta\,\xi - 1}}, \tag{5.1.16}$$

subject to $(5.1.14)_2$.

5.1.4 Equibiaxial deformation $\lambda_1 = \lambda_3$ and $\lambda_1^* = \lambda_3^*$

By choosing $\lambda_1 = \lambda_3$, $\lambda_1^* = \lambda_3^*$ and using the incompressibility constraint (2.2.4) we deduce $\lambda_1^{*2} = 1$ and $\lambda_2^2 = 4/3 \simeq 1.333$, so the secular equation (5.1.16) and $(5.1.14)_2$ can be written as

$$\tan\left\{ kh\,\sqrt{\alpha\,\beta\,\xi - 1}\right\} = \frac{3\beta}{4}\sqrt{\frac{1.333 - \xi}{\alpha\,\beta\,\xi - 1}},$$

subject to (5.1.17)

$$\frac{1}{\alpha\,\beta} < \xi < 1.333.$$

Next, we solved this equation (5.1.17) to plot ξ as a function of kh for the following values of α and β. For the case $\alpha = \beta = 1$ the results are given in Fig. 5.2; for $\alpha = 1$ and $\beta = 2$ the results are shown in Fig. 5.3; for $\alpha = 1$ and $\beta = 3$ see Fig. 5.4; for $\alpha = 2$ and $\beta = 1$ results are in Fig. 5.5; for $\alpha = \beta = 2$ the results are in Fig. 5.6; for $\alpha = 2$ and $\beta = 3$ the results are in Fig. 5.7; for $\alpha = 3$ and $\beta = 1$ the results are in Fig. 5.8; for $\alpha = 3$ and $\beta = 2$ the results in fig. 5.9; and for $\alpha = \beta = 3$ the results are in Fig. 5.10. Also equation (5.1.17) was solved for $\alpha = 1$ with $\beta = 5$, 10, 50 and the results are given in Figs. 5.11 - 5.13.

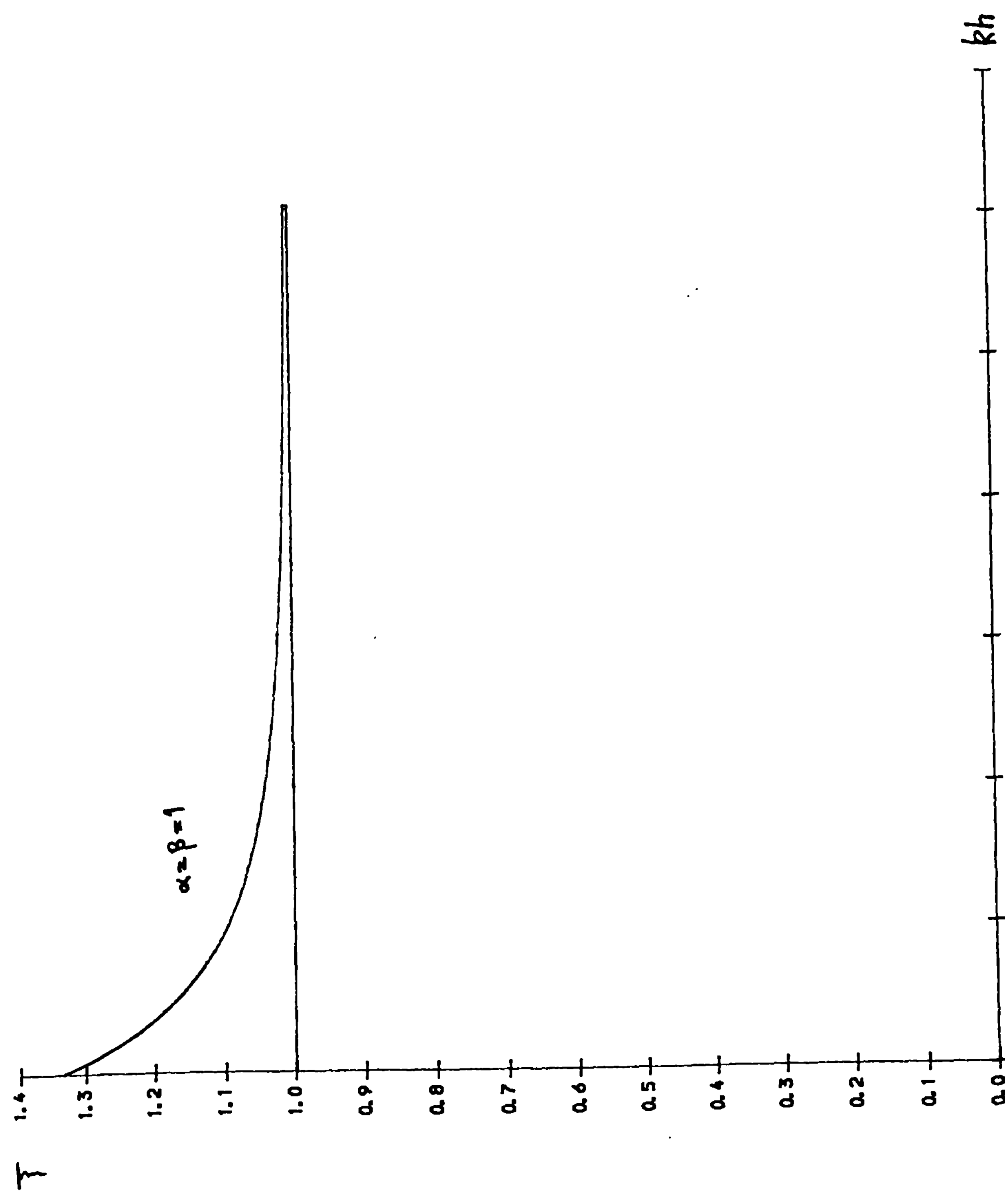


Figure 5.2

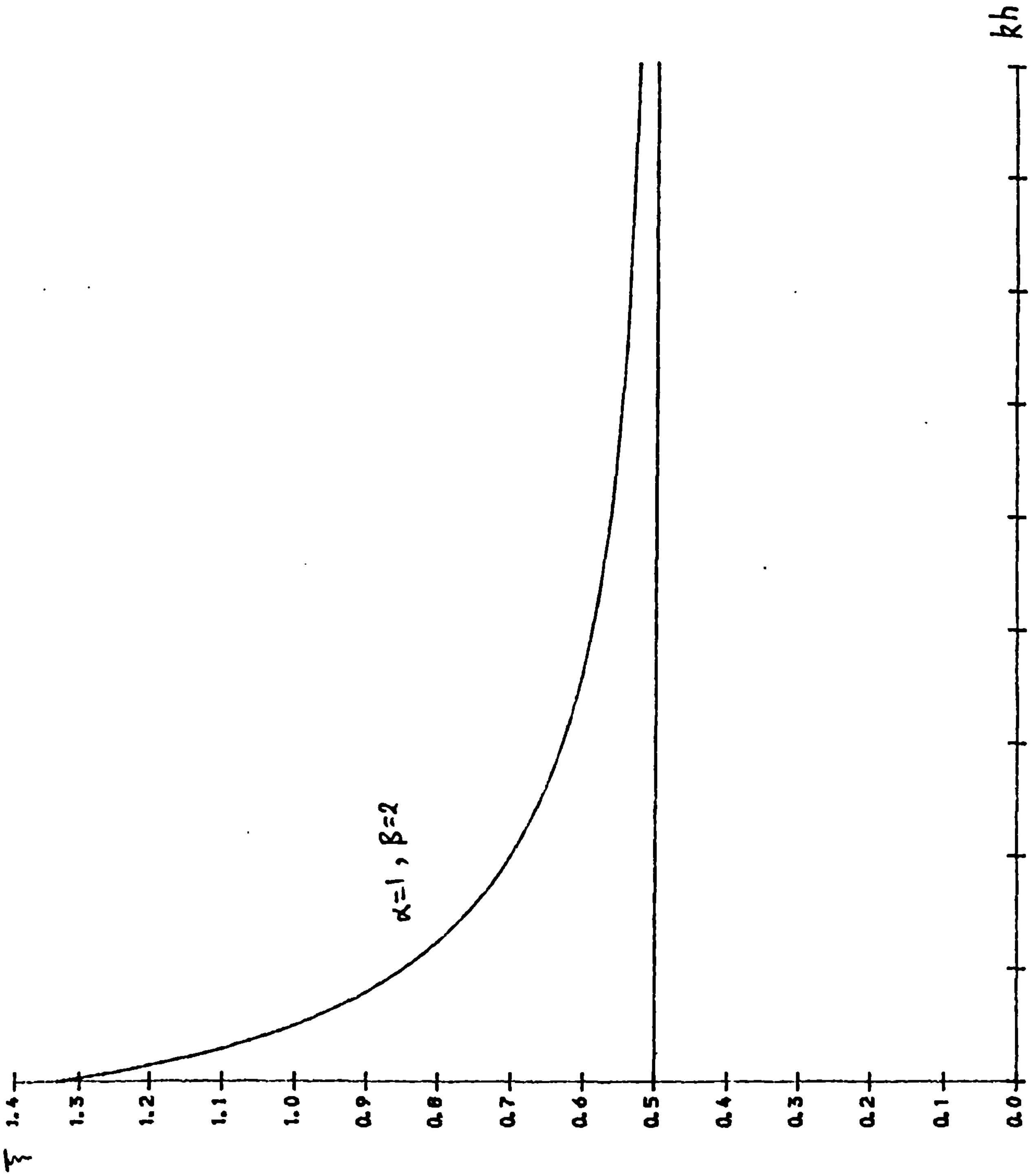

α=1 , β=2
kh
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4

Figure 5.3

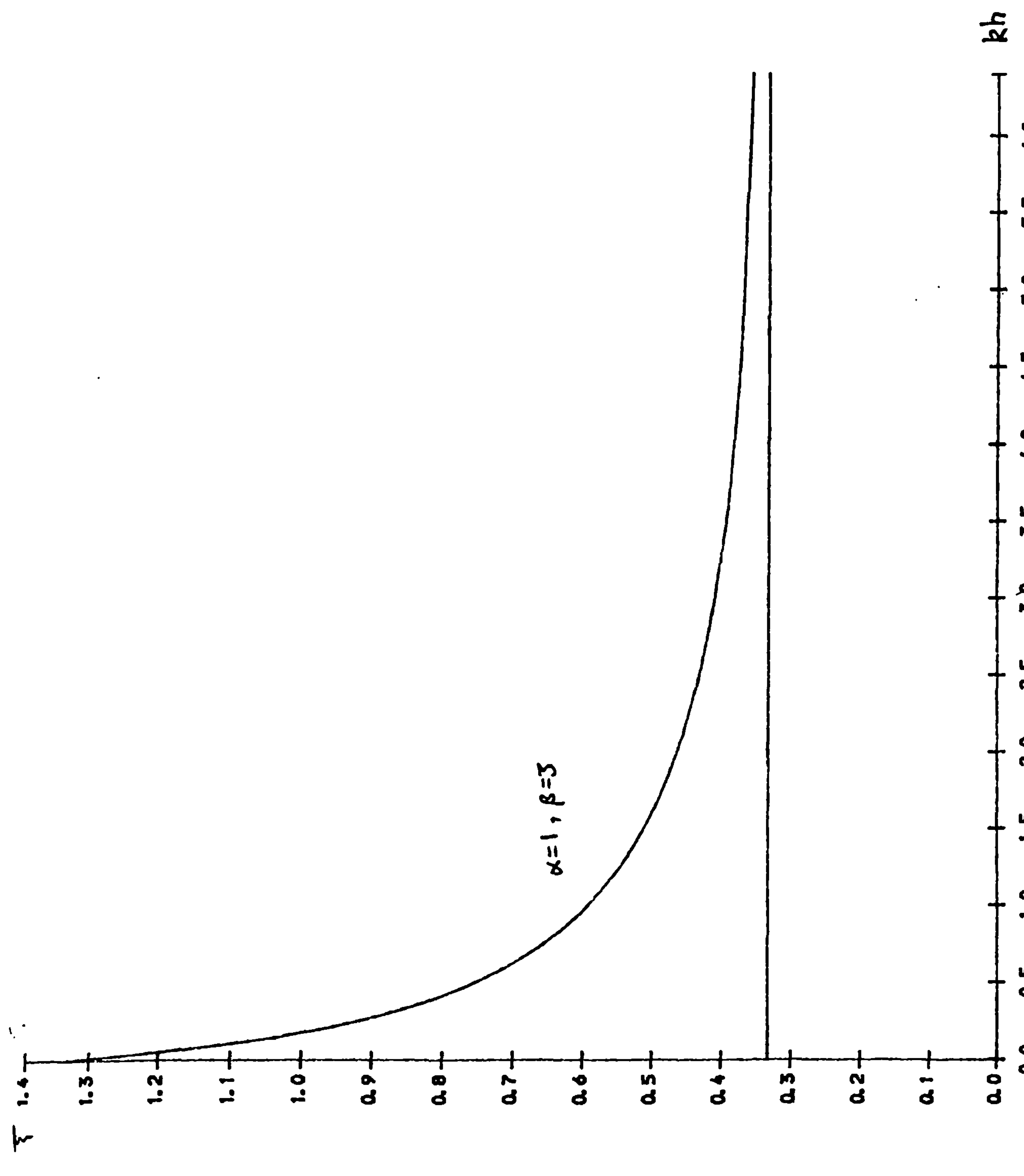

kh
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
α=1 , β=3

Figure 5.4

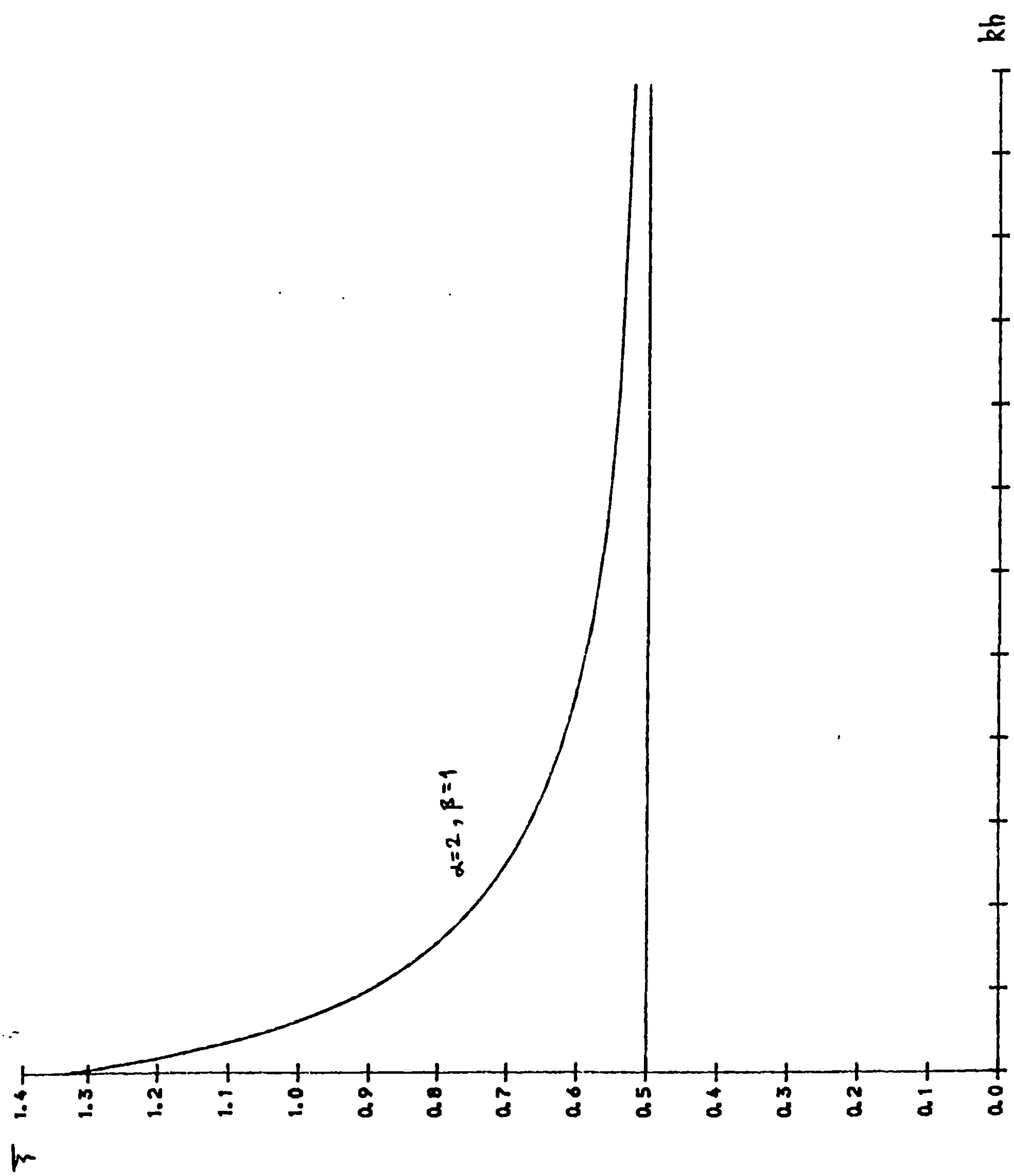
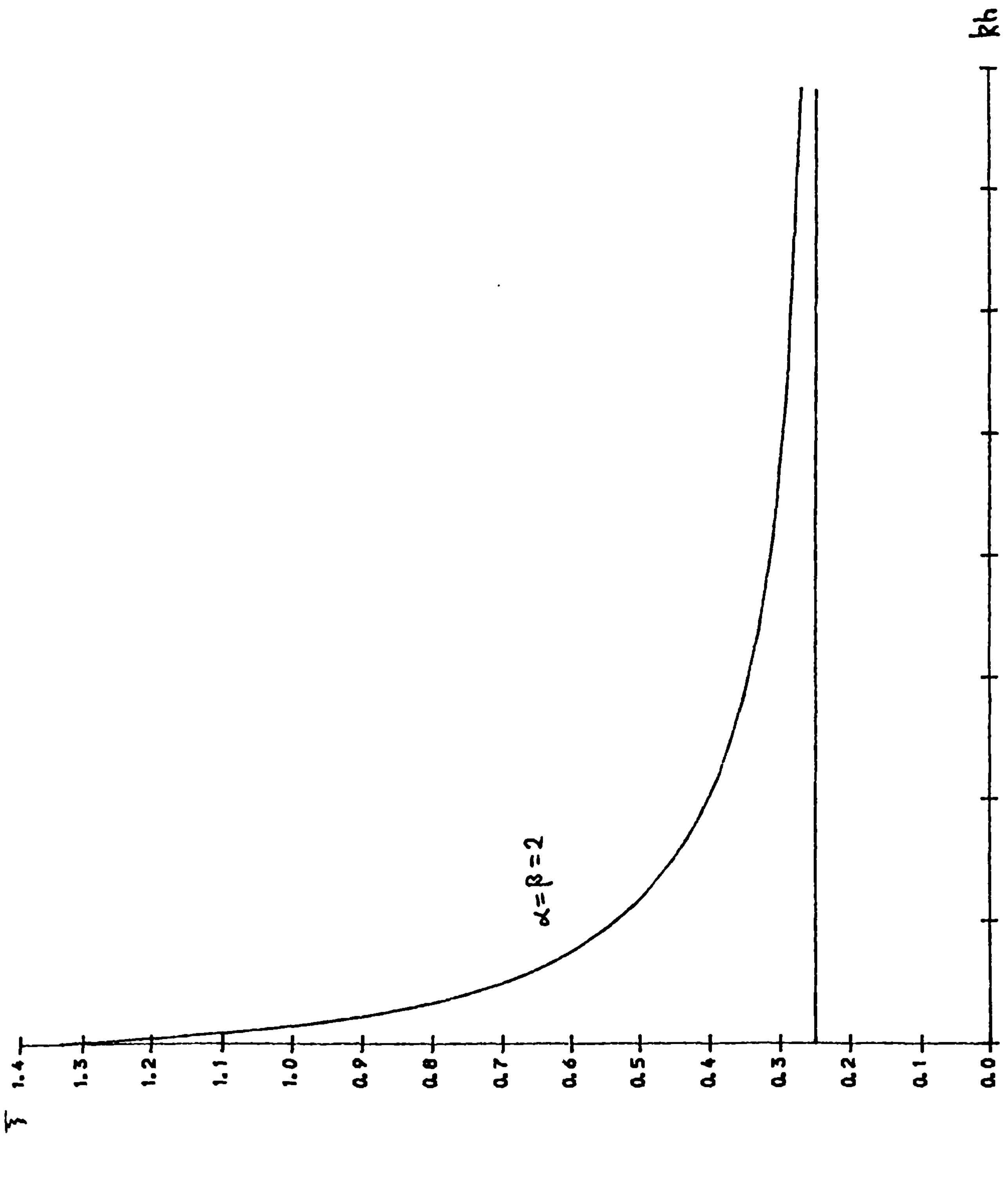

α=2, β=1
kh
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4

Figure 5.5



α=β=2
kh
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4

Figure 5.6

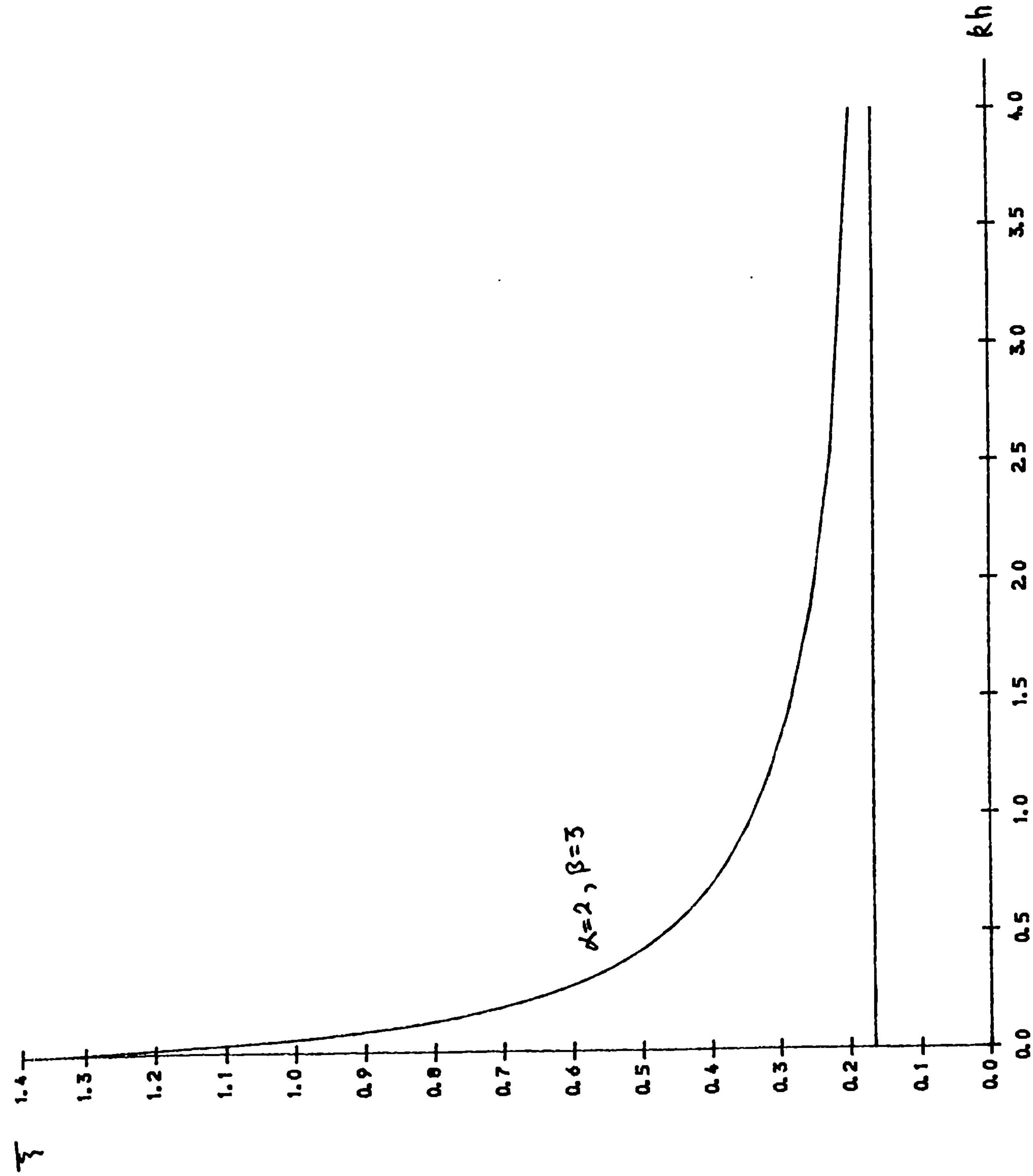
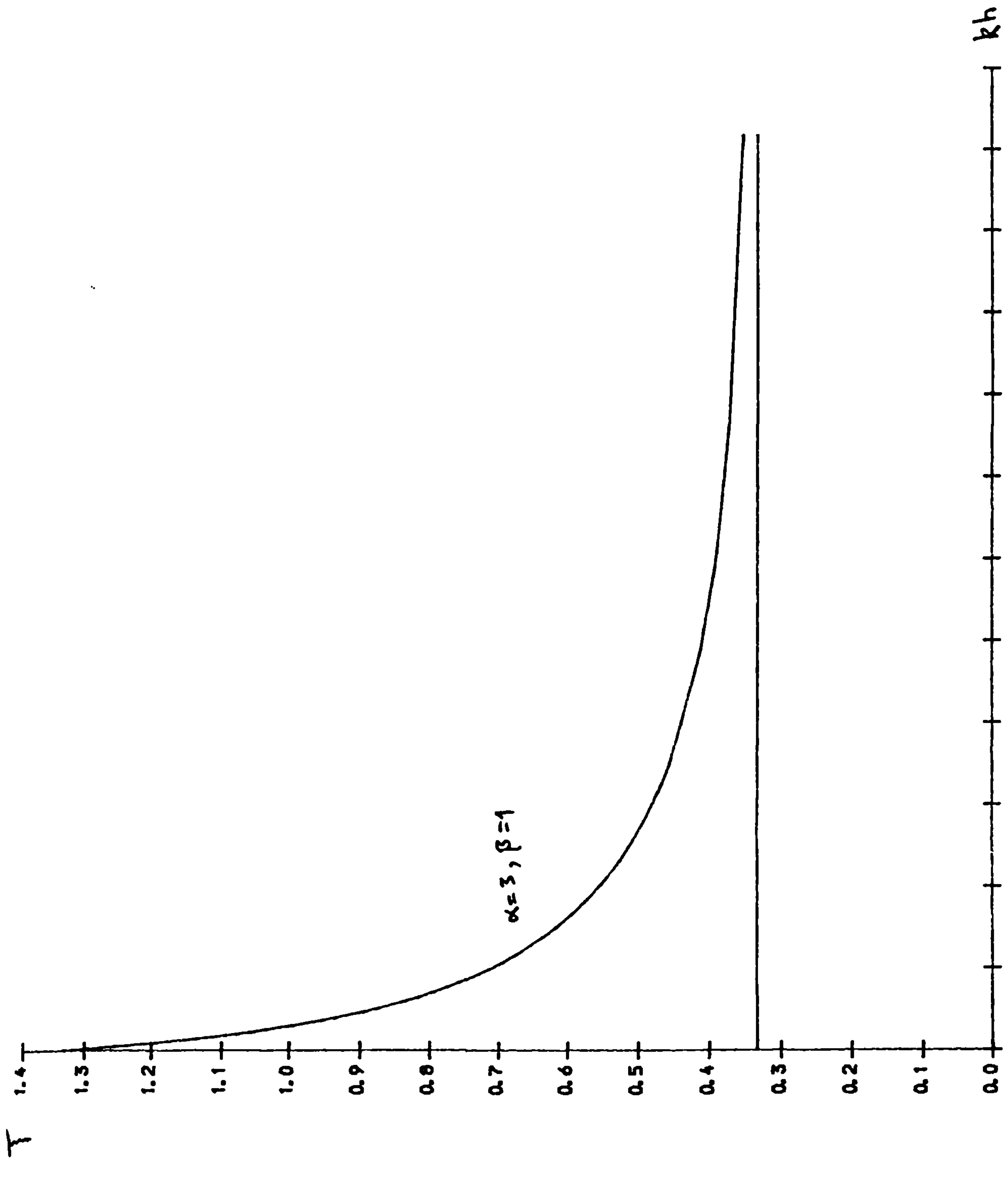
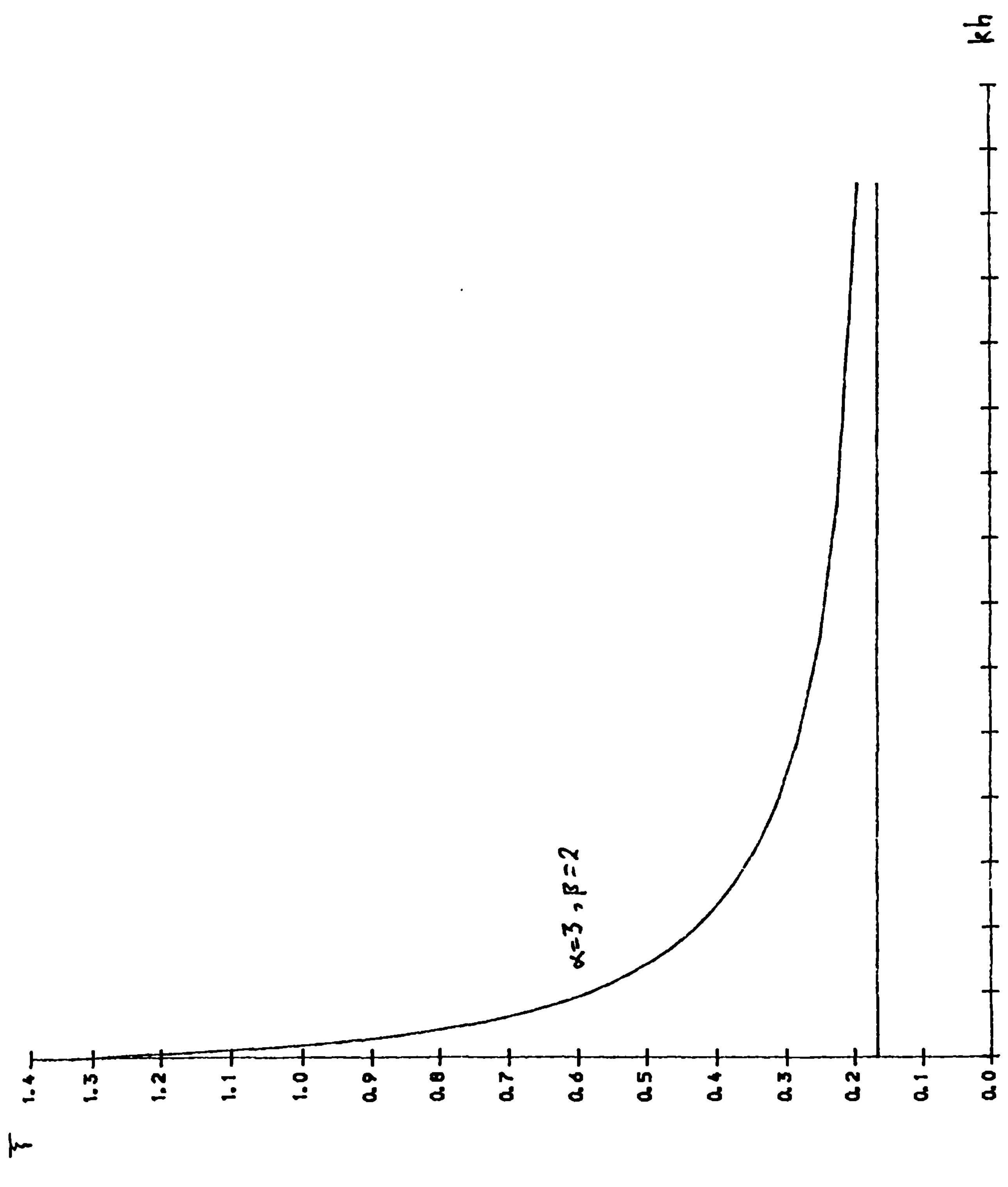

kh
$\alpha=2$, $\beta=3$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4

Figure 5.7



T
kh
$\alpha=3, \beta=1$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4

Figure 5.8



kh
α=3 , β=2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0

Figure 5.9

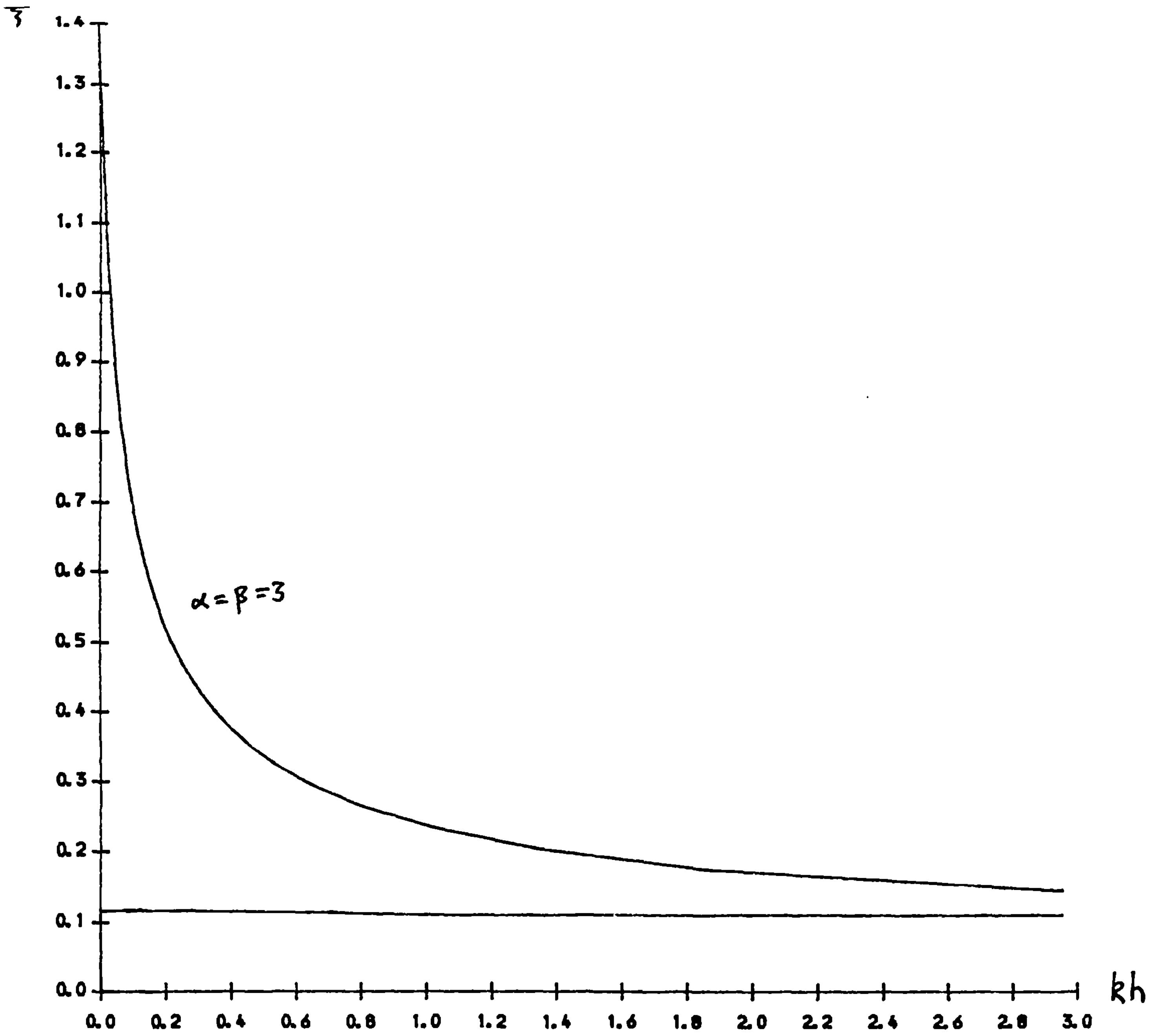
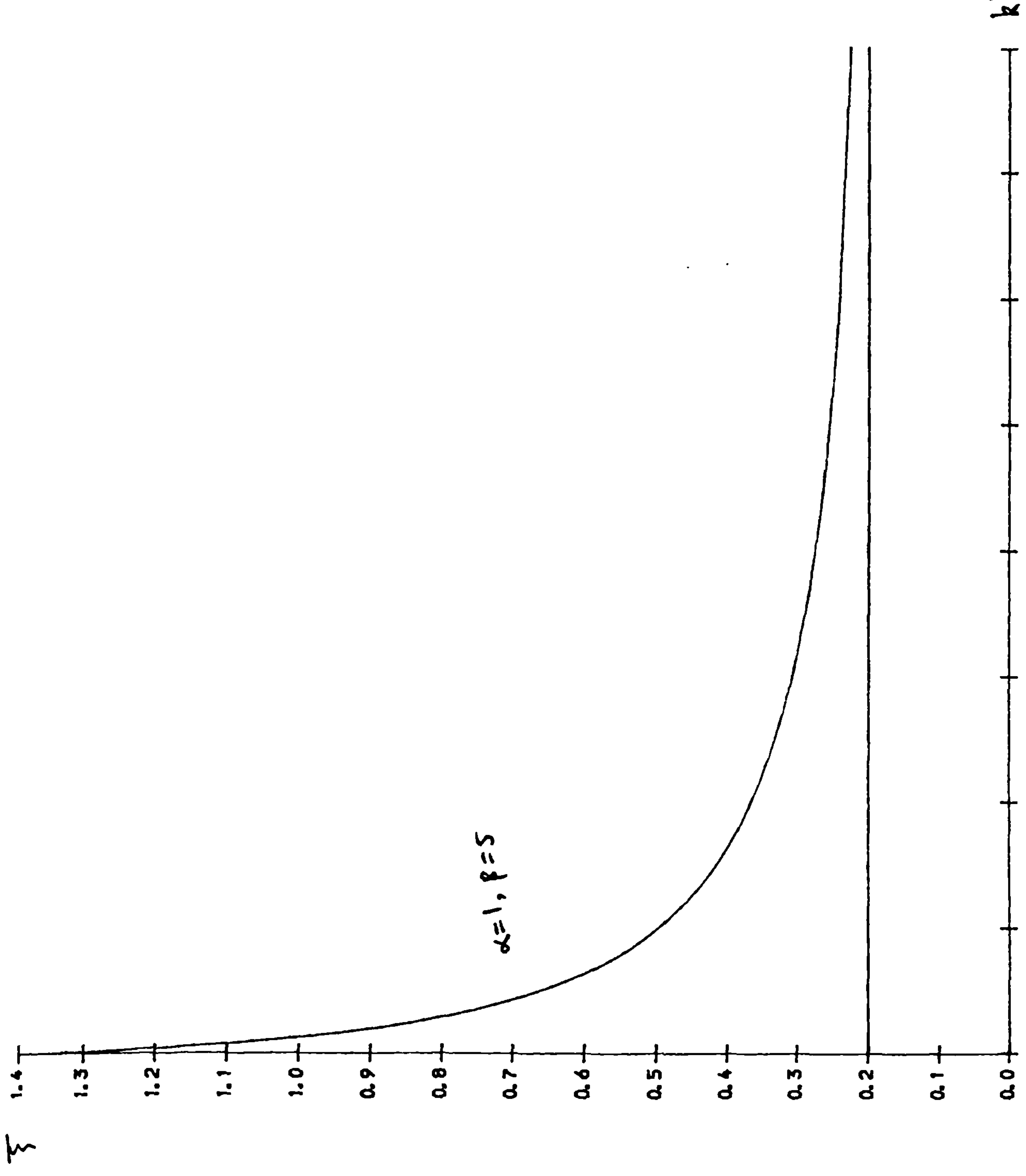

$\bar{\xi}$
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$\alpha=\beta=3$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
kh

Figure 5.10



ξ
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
8
kh
$\alpha=1,\ \beta=5$

Figure 5.11

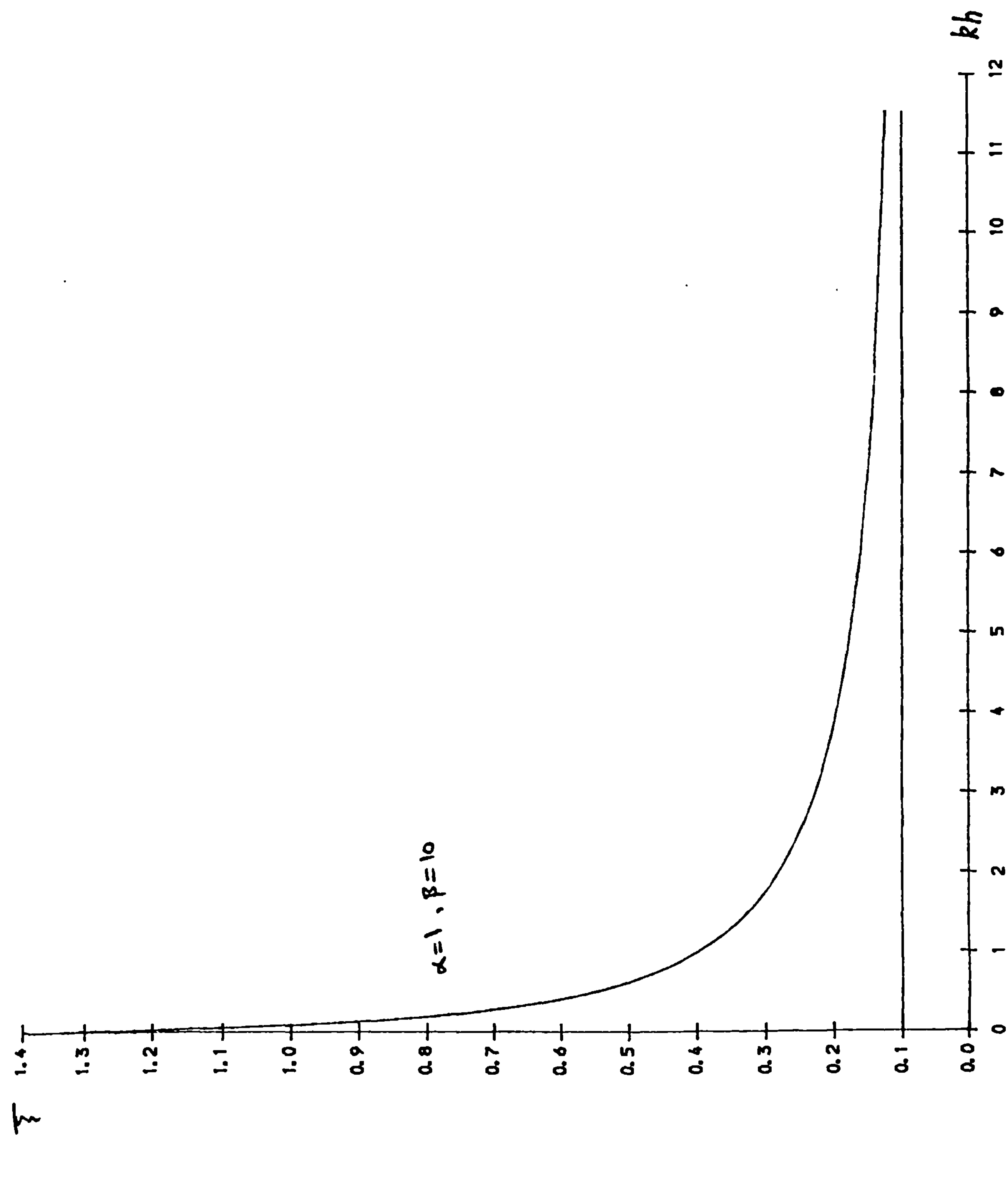
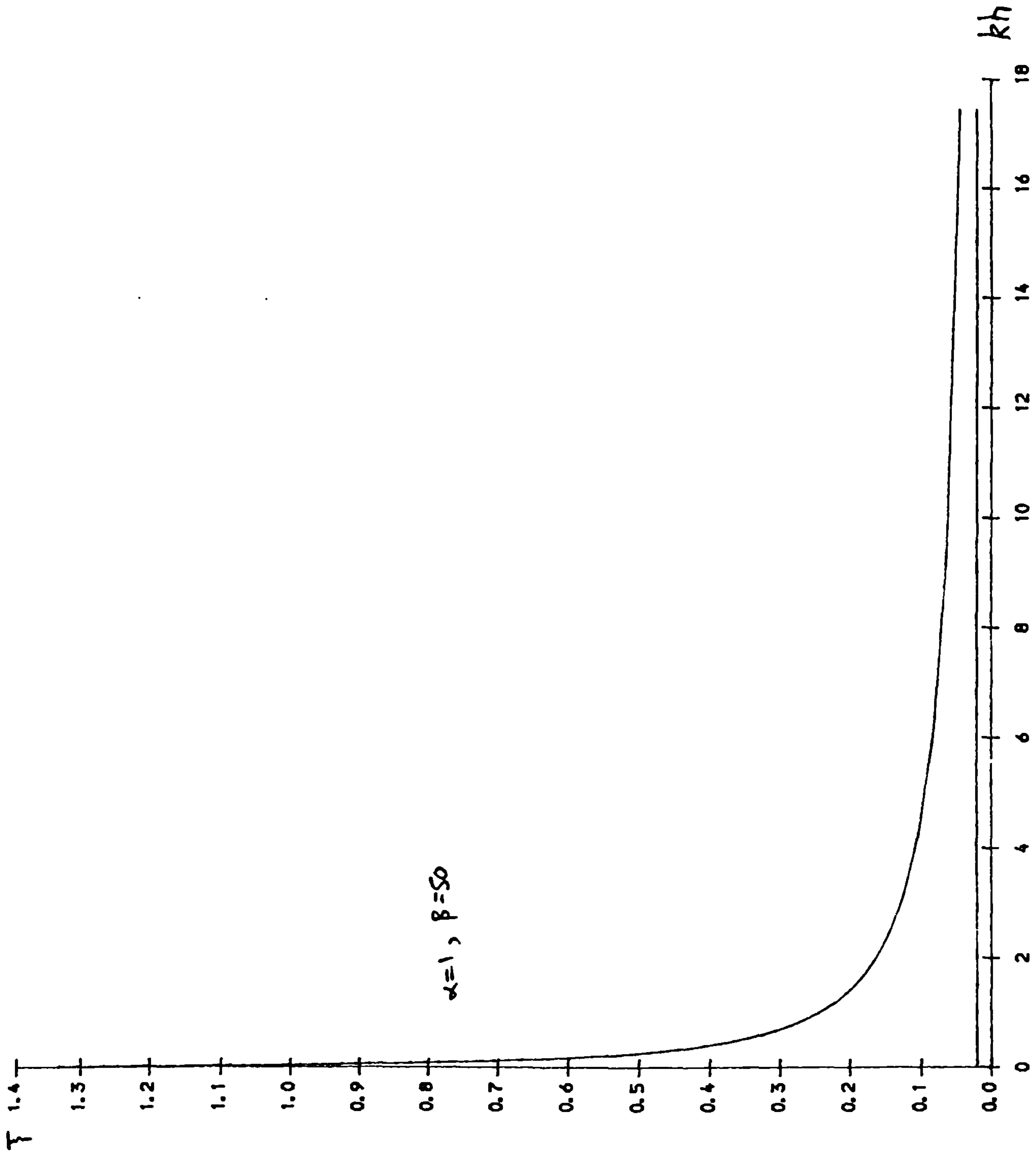

$\alpha=1$, $\beta=10$
kh
0 1 2 3 4 5 6 7 8 9 10 11 12
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4

Figure 5.12



T
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
2
4
6
8
10
12
14
16
18
kh
α=1, β=50

Figure 5.13

5.1.5 Results for the linear theory

Here we note that for the special deformation in which $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$, the secular equation (5.2.12) reduces to

$$\tan\left\{ kh \left[\frac{\rho^* c^2}{\mu^*} - 1 \right]^{\frac{1}{2}} \right\} - \frac{\mu}{\mu^*} \left[\frac{1 - (\rho c^2/\mu)}{1 - (\rho^* c^2/\mu^*)} \right] = 0, \qquad (5.1.18)$$

which is the result for the classical linear theory of Love waves which has been discussed in sections 3.4

5.1.6 Results for a compressible material

In this section we shall consider the corresponding results for compressible material by solving the equations of motion (2.6.19) with the boundary coditions

$$\begin{aligned} &\dot{s}_{02i} = 0 && \text{on } x_2 = h, \\ &\underline{v} = \underline{v}^*, \quad \dot{s}_{02i} = \dot{s}^*_{02i} && \text{on } x_2 = 0, \end{aligned} \qquad (5.1.19)$$

where $\underline{v}$ and $\underline{v}^*$ given by (5.1.2)

From (2.6.19), the secular equations of motion in the half-space and the layer are

$$\begin{aligned} A_{01313}\, v_{3,11} + A_{02323}\, v_{3,22} &= \rho \ddot{v}_3, \\ A^*_{01313}\, v^*_{3,11} + A^*_{02323}\, v^*_{3,22} &= \rho \ddot{v}^*_3, \end{aligned} \qquad (5.1.20)$$

respectively.

On use of $(5.1.2)_1$ into $(5.1.20)_1$, we have

$$A_{02323}\, s^2 - A_{01313} + \rho c^2 = 0,$$

i.e.

$$s^2 = \frac{A_{01313} - \rho c^2}{A_{02323}}$$

Similarly

$$s^{*2} = \frac{A^*_{01313} - \rho^* c^2}{A^*_{02323}}$$

Also, by using the notations (2.7.17), these become the same as the incompressible case (5.1.5) namely

$$s^2 = \frac{c^2_{13} - c^2}{c^2_{23}}, \qquad s^{*2} = \frac{c^2 - c^{*2}_{13}}{c^{*2}_{23}}. \qquad (5.1.21)$$

The boundary conditions $(5.1.1)_1$, leads

$$\dot{s}_{023} = A_{02323}\, v_{3,2} + A_{02323}\, v_{2,3} = A_{02323}\, v_{3,2}.$$

Also, the boundary conditions $(5.1.1)_1$ gives $v_{3,2} = 0$ on $x_2 = h$, from $(5.1.2)_2$, we get the same results as in an incompressible case, namely

$$\tan\left\{ kh \sqrt{\frac{c^2 - c^{*2}_{13}}{c^{*2}_{23}}} \right\} = \frac{\rho c^2_{23} \sqrt{c^2_{13} - c^2}}{\rho^* c^{*2}_{23} \sqrt{c^2 - c^{*2}_{13}}}.$$

This results but not the corresponding results for an incompressible materaial, was given by Hayes and Rivlin (1961b), but in different notation.

5.2 Rayleigh surface waves on a layered half-space for an incompressible material

In this section, we shall consider Rayleigh waves on a layered half-space, i.e. we are seeking waves with $\underline{v} = (v_1, v_2, 0)$ for both incompressible and compressible materials.

Let us assume that $(\lambda_1, \lambda_2, \lambda_3)$ and $(\overset{*}{\lambda}_1, \overset{*}{\lambda}_2, \overset{*}{\lambda}_3)$ are the stretches of the deformation in the half-space and the layer respectively, and W, W^* are the corresponding strain-energy functions. Also, we assume that the elastic modulus tensor in the layer is $\overset{*}{\underline{A}}$ and in the half-space $\underline{A}$.

We also assume that

$$\underline{v} = (v_1, v_2, 0), \qquad \underline{v}^* = (v_1^*, v_2^*, 0). \tag{5.2.1}$$

We wish to solve the incremental equations of motion for an incompressible material, namely

$$A_{0jilk} \frac{\partial^2 v_k}{\partial x_l \partial x_j} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{v}_i, \tag{5.2.2}$$

with the boundary conditions

$$\begin{aligned} &\dot{s}^*_{021} = 0, \quad \dot{s}^*_{022} = 0 && \text{on } x_2 = h, \\ &\underline{v} = \underline{v}^*, \quad \dot{s}_{02i} = \dot{s}^*_{02i} && \text{on } x_2 = 0, \end{aligned} \tag{5.2.3}$$

where $\dot{s}_{0ij}$ are similar to equations given in Section 4.1, in all the following cases the superscript * indicating quantities for the region $x_2 \geqslant 0$.

5.2.1 Plane incremental motion

We take $v_3 = 0$ and $v_3^* = 0$ and assume v_1, v_2 and v_1^*, v_2^* are independent of x_3. Thus, in the layer there is a function $\psi^*(x_1, x_2, t)$ such that

$$v_1^* = \psi^*_{,2}, \qquad v_2^* = -\psi^*_{,1}. \tag{5.2.4}$$

As in Section 4.1.1, the equations of motion (5.2.2) leads to

$$\alpha\, \psi_{,1111} + 2\beta\, \psi_{,1122} + \gamma\, \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}),$$

and similarly, (5.2.5)

$$\alpha^*\, \psi^*_{,1111} + 2\beta^*\, \psi^*_{,1122} + \gamma^*\, \psi^*_{,2222} = \rho^*(\ddot{\psi}^*_{,11} + \ddot{\psi}^*_{,22}).$$

Recall the notations (4.1.12)

$$\alpha = A_{01212}, \qquad \gamma = A_{02121},$$

$$2\beta = A_{01111} + A_{02222} - 2A_{01221} - 2A_{01122},$$

and similarly, (5.2.6)

$$\alpha^* = A_{01212}, \qquad \gamma^* = A_{02121},$$

$$2\beta^* = A^*_{01111} + A^*_{02222} - 2A^*_{01221} - 2A^*_{01122}.$$

The boundary conditions $(5.2.3)_1$ become

$$A^*_{02121}\, \psi^*_{,22} - (A^*_{02112} + p^*)\, \psi^*_{,11} = 0,$$

$$\text{on } x_2 = h \tag{5.2.7}$$

$$(A^*_{02211} - A^*_{02222} - p^*)\, \psi^*_{,12} - \dot{p}^* = 0.$$

Similarly, as in Section 4.1.1, to eliminate $\dot{p}^*$ we must differentiate equations (5.2.7) with respect to x_1 and on use of the notations (5.2.6) and equations (2.6.13), the boundary conditions (5.2.7) and $(5.2.3)_2$ become

$$\gamma^* \psi_{,22} + (\gamma^* - \sigma_2^*)\, \psi^*_{,22} = 0,$$

on $x_2 = h$

$$\gamma^* \psi^*_{,22} + (2\beta^* + \gamma^* - \sigma_2^* - \rho^* c^2)\, k^2\, \psi^*_{,2} = 0,$$

$$\gamma\, \psi_{,22} + (\gamma - \sigma_2)\, k^2\, \psi - \gamma^* \psi^*_{,22} - (\gamma^* - \sigma_2^*)\, k^2\, \psi^* = 0,$$

$$\gamma\, \psi_{,222} - (2\beta + \gamma - \sigma_2 - \rho c^2)\, k^2\, \psi - \gamma^* \psi^*_{,222} \qquad (5.2.8)$$

$$+ (2\beta^* + \gamma^* - \sigma_2^* - \rho^* c^2) k^2\, \psi^*_{,2} = 0,$$

on $x_2 = 0$

$$\psi_{,2} - \psi^*_{,2} = 0,$$

$$\psi_{,1} - \psi^*_{,1} = 0.$$

<u>5.2.2</u> <u>Propagation along a principal axis</u>

We now assume that $\psi(x_1, x_2, t)$ in the half-space has the form

$$\psi = A\, e^{skx_2 - ikx_1 + i\omega t}, \qquad (5.2.9)$$

and $\psi^*(x_1, x_2, t)$ in the layer has the form

$$\psi^* = A^*\, e^{-s^* kx_2 - ikx_2 + i\omega t}, \qquad (5.2.10)$$

these lead to

$$s_1^2 + s_2^2 = \frac{2\beta - \rho c^2}{\gamma}, \qquad s_1^2 s_2^2 = \frac{\alpha - \rho c^2}{\gamma}, \qquad (5.2.11)$$

as given in (4.1.18).

Similarly,

$$s_1^{*2} + s_2^{*2} = \frac{2\beta^* - \rho^* c^2}{\gamma^*}, \qquad s_1^{*2} s_2^{*2} = \frac{\alpha^* - \rho^* c^2}{\gamma^*}. \qquad (5.2.12)$$

For Rayleigh surface waves on a layered half-space we must have a solution for ψ in (5.2.5), which decays when $x_2 \longrightarrow -\infty$ and which satisfies the boundary conditions (5.2.8).

Suppose now that, the general solution for the half- space is

$$\psi = (A_1\ e^{s_1 k x_2} + A_2\ e^{s_2 k x_2})\ e^{i\omega t - ikx_1}, \quad (5.2.13)$$

and for the layer

$$\psi^* = (A_1^*\ e^{s_1^* k x_2} + B_1^*\ e^{-s_1^* k x_2} + A_2^*\ e^{s_2^* k x_2} + B_2^*\ e^{-s_2^* k x_2})\ e^{i\omega t - ikx_1} \quad (5.2.14)$$

Next, on use of equations (5.2.13) and (5.2.14) into the boundary conditions (5.2.8) we get

$$\{\gamma^* s_1^{*2} + (\gamma^* - \sigma_2^*)\} e^{s_1^* kh} A_1^* + \{\gamma^* s_2^{*2} + (\gamma^* - \sigma_2^*) e^{s_2^* kh} A_2^*$$

$$+ \{\gamma^* s_1^{*2} + (\gamma^* - \sigma_2^*)\} e^{-s_1^* kh} B_1^* + \{\gamma^* s_2^{*2} + (\gamma^* - \sigma_2^*)\} e^{s_2^* kh} B_2^* = 0,$$

$$\text{on } x_2 = h$$

$$\{\gamma^* s_1^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) + \rho^* c^2\} s_1^* e^{s_1^* kh} A_1^*$$

$$+ \{\gamma^* s_2^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) + \rho^* c^2\} s_2^* e^{s_2^* kh} A_2^*$$

$$- \{\gamma^* s_1^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) + \rho^* c^2\} s_1^* e^{-s_1^* kh} B_1^*$$

$$- \{\gamma^* s_2^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) + \rho^* c^2\} s_2^* e^{-s_2^* kh} B_2^* = 0,$$

$$\{\gamma s_1^2 + \gamma - \sigma_2\} A_1 + \{\gamma s_2^2 + \gamma - \sigma_2\} A_2 - \{\gamma^* s_1^{*2} + \gamma^* - \sigma_2^*\} A_1^*$$

$$- \{\gamma^* s_1^{*2} + \gamma^* - \sigma_2^*\} A_2^* - \{\gamma^* s_2^{*2} + \gamma^* - \sigma_2^*\} B_1^*$$

$$- \{\gamma^* s_2^{*2} + \gamma^* - \sigma_2^*\} B_2^* = 0,$$

$$\{\gamma s_1^2 - (2\beta + \gamma - \sigma_2) + \rho c^2\} A_1 + \{\gamma s_2^2 - (2\beta + \gamma - \sigma_2) + \rho c^2\} s_2 A_2$$

$$+ \{2\beta^* + \gamma^* - \sigma_2^* - \rho^* c^2 - \gamma^* s_1^{*2}\} s_1^* A_1^*$$

$$+ \{\gamma^* s_1^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) - \rho^* c^2\} s_1^* A_2^*$$

$$+ \{2\beta^* + \gamma^* - \sigma_2^* - \rho^* c^2 - \gamma^* s_2^{*2}\} s_2^* B_1^*$$

$$+ \{ \gamma^* s_2^{*2} - (2\beta^* + \gamma^* - \sigma_2^*) + \rho^* c^2\} s_2^* B_2^* = 0, \qquad \text{on } x_2 = 0$$

$$s_1 A_1 + s_2 A_2 - s_1^* A_1^* + s_1^* B_1^* - s_2^* A_2^* + s_2^* B_2^* = 0,$$

$$A_1 + A_2 - A_1^* - B_1^* - A_2^* - B_2^* = 0.$$

On use of equations (5.2.11) and (5.2.12), the above equations become

$$\{\gamma^* (s_1^{*2} + 1) - \sigma_2^*\} e^{s_1^* kh} A_1^* + \{\gamma^* (s_2^{*2} + 1) - \sigma_2^*\} e^{s_2^* kh} A_2^*$$

$$+ \{\gamma^* (s_1^{*2} + 1) - \sigma_2^*\} e^{-s_1^* kh} B_1^* + \{\gamma^* (s_2^{*2} + 1) - \sigma_2^*\} e^{-s_2^* kh} B_2^* = 0,$$

$$\{\gamma^* s_1^* (s_2^{*2} + 1) - \sigma_2^* s_1^*\} e^{s_1^* kh} A_1^* \qquad \text{on } x_2 = h$$

$$+ \{\gamma^* s_2^* (s_1^{*2} + 1) - \sigma_2^* s_2^*\} e^{s_2^* kh} A_2^*$$

$$- \{\gamma^* s_1^* (s_2^{*2} + 1) - \sigma_2^* s_1^*\} e^{-s_1^* kh} B_1^*$$

$$- \{\gamma^* s_2^* (s_1^{*2} + 1) - \sigma_2^* s_2^*\} e^{-s_2^* kh} B_2^* = 0,$$

$$\{\gamma (s_1^2 + 1) - \sigma_2\} A_1 + \{\gamma (s_2^2 + 1) - \sigma_2\} A_2 - \{\gamma^* (s_1^{*2} + 1) - \sigma_2^*\} A_1^*$$

$$- \{\gamma^* (s_1^{*2} + 1) - \sigma_2^*\} A_2^* - \{\gamma^* (s_2^{*2} + 1) - \sigma_2^*\} B_1^*$$

$$- \{\gamma^* (s_2^{*2} + 1) - \sigma_2^*\} B_2^* = 0,$$

$$\{\gamma s_1 (s_2^2 + 1) - \sigma_2 s_1\} A_1 + \{\gamma s_2 (s_1^2 + 1) - \sigma_2 s_2\} A_2$$

$$- \{\gamma^* s_1^* (s_2^{*2} + 1) - \sigma_2^* s_1^*\} A_1^* + \{\gamma^* s_1^* (s_2^{*2} + 1) - \sigma_2^* s_1^*\} A_2^*$$

$$- \{\gamma^* s_2^* (s_1^{*2} + 1) - \sigma_2^* s_2^*\} B_1^* + \{\gamma^* s_2^* (s_1^{*2} + 1) - \sigma_2^* s_2^*\} B_2^* = 0,$$

$$s_1 A_1 + s_2 A_2 - s_1^* A_1^* + s_1^* A_2^* - s_2^* B_1^* + s_2^* B_2^* = 0,$$

$$A_1 + A_2 - A_1^* - A_2^* - B_1^* - B_2^* = 0.$$

For the special case $\sigma_2 = 0$ and $\sigma_2^* = 0$ these reduce to

$$\gamma^* \, (s_1^{*2} + 1) \, e^{s_1^* kh} \, A_1^* + \gamma^* \, (s_2^{*2} + 1) \, e^{s_2^* kh} \, A_2^*$$

$$+ \, \gamma^* \, (s_1^{*2} + 1) \, e^{-s_1^* kh} \, B_1^* + \gamma^* \, (s_2^{*2} + 1) \, e^{-s_2^* kh} \, B_2^* = 0,$$

$$\gamma^* \, s_1^* (s_2^{*2} + 1) \, e^{s_1^* kh} \, A_1^* + \gamma^* \, s_2^* (s_1^{*2} + 1) \, e^{s_2^* kh} \, A_2^*$$

$$- \, \gamma^* \, s_1^* (s_2^{*2} + 1) \, e^{-s_1^* kh} \, B_1^* - \gamma^* \, s_1^* (s_2^{*2} + 1) \, e^{-s_2^* kh} \, B_2^* = 0,$$

$$\gamma \, (s_1^2 + 1) \, A_1 + \gamma \, (s_2^2 + 1) \, A_2 - \gamma^* \, (s_1^{*2} + 1) \, A_1^* - \gamma^* \, (s_1^{*2} + 1) \, A_2^*$$

$$- \, \gamma^* \, (s_2^{*2} + 1) B_1^* - \gamma^* \, (s_2^{*2} + 1) B_2^* = 0,$$

$$\gamma \, s_1 (s_2^2 + 1) \, A_1 + \gamma \, s_2 (s_1^2 + 1) \, A_2 - \gamma^* \, s_1^* (s_2^{*2} + 1) \, A_1^*$$

$$+ \, \gamma^* \, s_1^* (s_2^{*2} + 1) \, A_2^* - \gamma^* \, s_2^* (s_1^{*2} + 1) \, B_1^* - \gamma^* \, s_2^* (s_1^{*2} + 1) \, B_2^* = 0,$$

$$s_1 \, A_1 + s_2 \, A_2 - s_1^* \, A_1^* + s_1^* \, A_2^* - s_2^* \, B_1^* + s_2^* \, B_2^* = 0,$$

$$A_1 + A_2 - A_1^* - A_2^* - B_1^* - B_2^* = 0.$$

For these equations to have non-trivial solutions for A_i, A_i^*, B_i^*, $i \in \{1, 2\}$, we must have

$$\begin{vmatrix} 0 & 0 & \gamma^*(s_1^{*2}+1)e^{s_1^*kh} & \gamma^*(s_2^{*2}+1)e^{s_2^*kh} & \gamma^*(s_1^{*2}+1)e^{-s_1^*kh} & \gamma^*(s_2^{*2}+1)e^{-s_2^*kh} \\ 0 & 0 & \gamma^*s_1^*(s_2^{*2}+1)e^{s_1^*kh} & \gamma^*s_2^*(s_1^{*2}+1)e^{s_2^*kh} & -\gamma^*s_1^*(s_2^{*2}+1)e^{-s_1^*kh} & -\gamma^*s_2^*(s_1^{*2}+1)e^{-s_2^*kh} \\ \gamma(s_1^2+1) & \gamma(s_2^2+1) & -\gamma^*(s_1^{*2}+1) & -\gamma^*(s_1^{*2}+1) & -\gamma^*(s_2^{*2}+1) & -\gamma^*(s_2^{*2}+1) \\ \gamma s_1(s_2^2+1) & \gamma s_2(s_1^2+1) & -\gamma^*s_1^*(s_2^{*2}+1) & \gamma^*s_1^*(s_2^{*2}+1) & -\gamma^*s_2^*(s_1^{*2}+1) & \gamma^*s_2^*(s_1^{*2}+1) \\ s_1 & s_2 & -s_1^* & s_1^* & -s_2^* & s_2^* \\ 1 & 1 & -1 & -1 & -1 & -1 \end{vmatrix} \quad (5.2.15)$$

This equation (5.2.15) is the secular equation of Rayleigh surface waves on a layered half-space propagating along a principal axis Because of complicated algebra involved we omit details here.

5.2.3 Specialization to the linear theory

In this section, we shall specialize the theory using equation (2.7.19), so equations (5.2.11) and (5.2.12) reduce to

$$s_1^2 + s_2^2 = 2 - \frac{\rho c^2}{\mu}, \qquad s_1^2 s_2^2 = 1 - \frac{\rho c^2}{\mu},$$

$$s_1^{*2} + s_2^{*2} = 2 - \frac{\rho^* c^2}{\mu^*}, \qquad s_1^{*2} s_2^{*2} = 1 - \frac{\rho^* c^2}{\mu^*}, \tag{5.2.16}$$

i.e.

$s_1^2 = 1$, $s_2^2 = 1 - (\rho c^2/\mu)$, $s_1^{*2} = 1$ and $s_2^{*2} = 1 - (\rho^* c^2/\mu^*)$, which was given in section 3.2. Also, the secular equation (5.2.16) becomes

$$\begin{vmatrix} 0 & 0 & \mu^*(s_1^{*2}+1)e^{s_1^* kh} & \mu^*(s_2^{*2}+1)e^{s_2^* kh} & \mu^*(s_1^{*2}+1)e^{-s_1^* kh} & \mu^*(s_2^{*2}+1)e^{-s_2^* kh} \\ 0 & 0 & \mu^* s_1^*(s_2^{*2}+1)e^{s_1^* kh} & \mu^* s_2^*(s_1^{*2}+1)e^{s_2^* kh} & -\mu^* s_1^*(s_2^{*2}+1)e^{-s_1^* kh} & -\mu^* s_2^*(s_1^{*2}+1)e^{-s_2^* kh} \\ \mu(s_1^2+1) & \mu(s_2^2+1) & -\mu^*(s_1^{*2}+1) & -\mu^*(s_1^{*2}+1) & -\mu^*(s_2^{*2}+1) & -\mu^*(s_2^{*2}+1) \\ \mu s_1(s_2^2+1) & \mu s_2(s_1^2+1) & -\mu^* s_1^*(s_2^{*2}+1) & \mu^* s_1^*(s_2^{*2}+1) & -\mu^* s_2^*(s_1^{*2}+1) & \mu^* s_2^*(s_1^{*2}+1) \\ s_1 & s_2 & -s_1^* & s_1^* & -s_2^* & s_2^* \\ 1 & 1 & -1 & -1 & -1 & -1 \end{vmatrix}$$

This is the secular eqution of Rayleigh waves on a layered haf-space for linear theory. Also because of complicated algebra involved we omit details here.

5.3 Rayleigh surface waves on a layered half-space for a compressible material

As in Section 4.8, the equations of incremental motion lead to

$$\rho\ddot{v}_1 = A_{01111}\, v_{1,11} + A_{01122}\, v_{2,21} + A_{02121}\, v_{1,22} + A_{02112}\, v_{2,12},$$
$$(5.3.1)$$
$$\rho\ddot{v}_2 = A_{01212}\, v_{2,11} + A_{01221}\, v_{1,12} + A_{02211}\, v_{1,12} + A_{02222}\, v_{2,22},$$

as given (4.8.6), for the half-space.

Similarly,

$$\rho^*\ddot{v}_1^* = A^*_{01111}\, v^*_{1,11} + A^*_{01122}\, v^*_{2,21} + A^*_{02121}\, v^*_{1,22} + A^*_{02112}\, v^*_{2,12},$$
$$(5.3.2)$$
$$\rho^*\ddot{v}_2^* = A^*_{01212}\, v^*_{2,11} + A^*_{01221}\, v^*_{1,21} + A^*_{02211}\, v^*_{1,12} + A^*_{02222}\, v^*_{2,22},$$

for the layer.

Assume now v_1, v_2 and v_1^*, v_2^* are given by

$$v_1 = A_1\, e^{skx_2 + i\omega t - ikx_1},$$

$$v_2 = A_2\, e^{skx_2 + i\omega t - ikx_1},$$

$$v_1^* = A_1^*\, e^{-skx_2 + i\omega t - ikx_1}, \qquad (5.3.3)$$

$$v_2^* = A_2^*\, e^{-skx_2 + i\omega t - ikx_1}.$$

Also, as in Section 4.8 these lead to

$$s_1^2 + s_2^2 = \frac{A_{01111}\,A_{02222} + A_{02121}\,A_{01212} - (A_{01122} + A_{02112})^2 - (A_{02121} + A_{02222})\rho c^2}{A_{02222}\,A_{02121}},$$

$$s_1^2 s_2^2 = \frac{(A_{01111} - \rho c^2)(A_{01212} - \rho c^2)}{A_{02222}\,A_{02121}}, \tag{5.3.4}$$

and similarly

$$s_1^{*2} + s_2^{*2} = \frac{A^*_{01111}\,A^*_{02222} + A^*_{02121}\,A^*_{01212} - (A^*_{01122} + A^*_{02112})^2 - (A^*_{02121} + A^*_{02222})\rho^* c^2}{A^*_{02222}\,A^*_{02121}},$$

$$s_1^{*2} s_2^{*2} = \frac{(A^*_{01111} - \rho^* c^2)(A^*_{01212} - \rho^* c^2)}{A^*_{02222}\,A^*_{02121}}. \tag{5.3.5}$$

Next, we wish to solve the incremental equations of motion (2.6.19) with the boundary conditions given by

$$\begin{aligned} &\dot{s}^*_{021} = 0, \qquad \dot{s}^*_{022} = 0 \qquad && \text{on } x_2 = h, \\ &\dot{s}_{02i} = \dot{s}^*_{02i}, \qquad \underline{v} = \underline{v}^* \qquad && \text{on } x_2 = 0, \end{aligned} \tag{5.3.6}$$

also, s_{02i} are similar to equations given in Section 4.8.

i.e.

$$A^*_{02112}\, v^*_{2,1} + A^*_{02121}\, v^*_{1,2} = 0,$$

on $x_2 = h$

$$A^*_{02211}\, v^*_{1,1} + A^*_{02222}\, v^*_{2,2} = 0,$$

$$A_{02112}\, v_{2,1} + A_{02121}\, v_{1,2} - (A^*_{02112}\, v^*_{2,1} + A^*_{02121}\, v^*_{1,2}) = 0, \tag{5.3.6}$$

$$A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2} - (A^*_{02211}\, v^*_{1,1} + A^*_{02222}\, v^*_{2,2}) = 0,$$

on $x_2 = 0$

$$v^*_1 - v^*_1 = 0,$$

$$v^*_2 - v^*_2 = 0.$$

For Rayleigh surface waves on a layered half-space we must have a solution for v_1, v_2 in equations (5.3.3) which vanishes when $x_2 \longrightarrow -\infty$ and which satisfies the boundary conditions (5.3.6).

Suppose now that

$$v_1 = (A_1 e^{s_1 k x_2} + B_1 e^{s_2 k x_2})e^{ikx_1 - i\omega t},$$

$$v_2 = (A_2 e^{s_1 k x_2} + B_2 e^{s_2 k x_2})e^{ikx_1 - i\omega t}, \tag{5.3.7}$$

for the half-space, and

$$v^*_1 = (A^*_1\, e^{s_1 k x_2} + B^*_1\, e^{-s_1 k x_2} + A^*_1\, e^{s_2 k x_2} + B^*_1\, e^{-s_2 k x_2})\, e^{i\omega t - ikx_1},$$

$$v^*_2 = (A^*_2\, e^{s_1 k x_2} + B^*_2\, e^{-s_1 k x_2} + A^*_2\, e^{s_2 k x_2} + B^*_2\, e^{-s_2 k x_2})\, e^{i\omega t - ikx_1}, \tag{5.3.8}$$

for the layer.

To get the Rayleigh waves on a layered half-space for a compressible material, we must diffrentiate equations (5.3.7) and (5.3.8) and substitute them into the boundary conditions (5.3.6) using equations given by (4.8.14) namely

$$\frac{iA_2}{A_1} = \frac{A_{01111} - A_{02121}\, s_1^2 - \rho c^2}{s_1\,(A_{01122} + A_{02112})},$$

$$\frac{iB_2}{B_1} = \frac{A_{01111} - A_{02121}\, s_2^2 - \rho c^2}{s_2\,(A_{01122} + A_{02112})},$$

and similarly

$$\frac{iA_2^*}{A_1^*} = \frac{-(A_{01111}^* - A_{02121}^*\, s_1^{*2} - \rho^* c^2)}{s_1^*\,(A_{01122}^* + A_{02112}^*)},$$

$$\frac{iB_2^*}{B_1^*} = \frac{-(A_{01111}^* - A_{02121}^*\, s_2^{*2} - \rho^* c^2)}{s_2^*\,(A_{01122}^* + A_{02112}^*)}.$$

But also, because of the algebra getting more complicated than for the incompressible case we not give details here.

Chapter 6

Interfacial waves at the boundary between two pre-stressed incompressible elastic half-spaces

6.1 Stoneley waves propagating along a principal axis

Here, we shall discuss Stoneley waves on an interface between two pre-stressed elastic half-spaces of incompressible elastic material bonded together.

6.1.1 Analysis for interfacial waves

We consider two half-spaces of incompressible isotropic elastic material. The half-spaces are subjected to pure homogeneous strains and then bonded along their common (plane) boundary in such a way that the principal directions of strain are aligned, one direction being normal to the interface. In rectangular Cartesian coordinates we take the interface to be $x_2 = 0$. Let B denote the deformed half-space in the region $x_2 < 0$ and B^* that in $x_2 > 0$. The x_1 and x_3 coordinate axes are taken to coincide with the principal directions of the pure strain that are parallel to the interface.

Let λ_1, λ_2, λ_3 and λ_1^*, λ_2^*, λ_3^* denote the principal stretches of the deformations in B and B^* respectively. Then, by incompressibility, we have

$$\lambda_1 \lambda_2 \lambda_3 = 1, \qquad \lambda_1^* \lambda_2^* \lambda_3^* = 1. \tag{6.1.1}$$

We suppose that $W(\lambda_1,\lambda_2,\lambda_3)$ and $W^*(\lambda_1^*,\lambda_2^*,\lambda_3^*)$ are the strain energies of the material per unit volume in B and B^* respectively. Let σ_1, σ_2, σ_3 and σ_1^*, σ_2^*, σ_3^* be the corresponding principal Cauchy stresses; then

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p,$$

$$\sigma_i^* = \lambda_i \frac{\partial W^*}{\partial \lambda_i^*} - p^*, \qquad i \in \{1, 2, 3\}, \tag{6.1.2}$$

where p and p^* are arbitrary hydrostatic pressures in B and B^* respectively.

Continuity of traction across the interface requires that

$$\sigma_2^* = \sigma_2. \tag{6.1.3}$$

Let $\underline{v}$ be the velocity in B associated with an incremental (infinitesimal) motion superimposed on the underlying deformation.

The incremental motion is governed by

$$\dot{s}_{0ji,j} = \rho \ddot{v}_i. \tag{6.1.4}$$

Using the incremental constitutive equation (2.6.11) we get

$$A_{0jilk} \frac{\partial^2 v_k}{\partial x_l \partial x_j} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{v}_i. \tag{6.1.5}$$

Since the motion is isochoric the displacement is governed by

$$v_{1,1} + v_{2,2} + v_{3,3} = 0. \tag{6.1.6}$$

6.1.2 Plane incremental motion

We now take $v_3 = 0$ and assume v_1, v_2 are functions of x_1, x_2 and time t only. Equation (6.1.6) reduces to

$$v_{1,1} + v_{2,2} = 0, \tag{6.1.7}$$

and hence there exists a function $\psi(x_1, x_2, t)$ such that

$$v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1},$$

and similarly (6.1.8)

$$v_1^* = \psi^*_{,2}, \qquad v_2^* = -\psi^*_{,1},$$

for B^*, where $\underline{v}^*$ is the velocity in B^*.

As in Section 4.1, the incremental equation of motion (6.1.4) leads to

$$(A_{01111} - A_{01122} - A_{02112})\ \psi_{,112} + A_{02121}\ \psi_{,222} - \dot{p}_{,1} = \rho\ddot{\psi}_{,2},$$

(6.1.9)

$$(A_{01221} + A_{02211} - A_{02222})\ \psi_{,122} - A_{01212}\ \psi_{,111} - \dot{p}_{,2} = \rho\ddot{\psi}_{,1}.$$

Recall the notations given by (4.1.2)

$$\alpha = A_{01212}, \qquad \gamma = A_{02121},$$

$$2\beta = A_{01111} + A_{02222} - 2A_{01221} - 2A_{01122}.$$

Similarly, we introduce corresponding notation for B^* (6.1.10)

$$\alpha^* = A_{01212}, \qquad \gamma^* = A_{02121},$$

$$2\beta^* = A^*_{01111} + A^*_{02222} - 2A^*_{01221} - 2A^*_{01122}.$$

As in Section 4.1, the incremental equations (6.1.5) lead to

$$\alpha\ \psi_{,1111} + 2\beta\ \psi_{,1122} + \gamma\ \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}), \qquad (6.1.11)$$

where ρ is the mass density of the material in B and a superposed dot denotes differentiation with respect to t. The coefficients α,β,γ on the left-hand side of (6.1.11) are (constant) material parameters defined, in terms of the strain-energy function W and the principal stretches, by

$$\alpha\,\lambda_2^2 = \gamma\,\lambda_1^2 = \frac{(\lambda_1 W_1 - \lambda_2 W_2)\,\lambda_1^2\,\lambda_2^2}{\lambda_1^2 - \lambda_2^2} \tag{6.1.12}$$

$$2\beta + 2\gamma = \lambda_1^2 W_{11} + \lambda_2^2 W_{22} - 2\,\lambda_1\,\lambda_2\,W_{12} + 2\,\lambda_2\,W_2, \tag{6.1.13}$$

where $W_i = \dfrac{\partial W}{\partial \lambda_i}$, $W_{ij} = \dfrac{\partial^2 W}{\partial \lambda_i\,\partial \lambda_j}$. When $\lambda_1 = \lambda_2$ these reduce to

$$\alpha = \beta = \gamma = \tfrac{1}{2}\,(\lambda_1^2 W_{11} - \lambda_1^2 W_{12} + \lambda_1 W_1), \tag{6.1.14}$$

where the right-hand side is evaluated for $\lambda_1 = \lambda_2$. Moreover, in the undeformed configuration, where $\lambda_1 = \lambda_2 = \lambda_3 = 1$, equation (6.1.14) reduces to

$$\alpha = \beta = \gamma = \mu, \tag{6.1.15}$$

where μ (> 0) is the ground-state shear modulus of the material.

Similarly for the material in B^*, α^*, β^* and γ^* are defined analogously to (6.1.12) and (6.1.13), and the velocity components v_1^* and v_2^* are given by $(6.1.8)_{3,4}$ where ψ^* satisfies the equation

$$\alpha^*\,\psi^*_{,1111} + 2\beta^*\,\psi^*_{,1122} + \gamma^*\,\psi^*_{,2222} = \rho^*(\ddot{\psi}^*_{,11} + \ddot{\psi}^*_{,22}) \quad \text{for } x_2 > 0. \tag{6.1.16}$$

6.1.3 Propagation along a principal axis

An interfacial wave is characterized by the fact that its amplitude decays rapidly away from the interface; we therefore seek solutions for which $\psi \to 0$ as $x_2 \to -\infty$ and $\psi^* \to 0$ as $x_2 \to \infty$. For simplicity we consider time-harmonic waves propagating in the x_1-direction and, in B, we take ψ to have the form

$$\psi = A\,e^{skx_2 - ikx_1 + i\omega t}.$$

Then equation (6.1.11) yields

$$\gamma\, s^4 - (2\beta - \rho c^2)\, s^2 + \alpha - \rho c^2 = 0, \tag{6.1.17}$$

and if s_1^2 and s_2^2 are the roots of the above quadratic equation then

$$s_1^2 + s_2^2 = \frac{2\beta - \rho c^2}{\gamma}, \qquad s_1^2 s_2^2 = \frac{\alpha - \rho c^2}{\gamma}. \tag{6.1.18}$$

Either s_1^2 and s_2^2 are complex conjugates or, for consistency with the requirements on ψ, s_1^2 and s_2^2 are positive. Allowing, exceptionally, for the possibility of one of s_1 or s_2 vanishing, we deduce from (6.1.17) that

$$0 \leqslant \rho c^2 \leqslant \alpha, \tag{6.1.19}$$

it being assumed that the inequalities of the strong ellipticity conditions (2.5.18) hold.

The solution for ψ in B may now be written

$$\psi = \varphi(x_2)\, e^{i\omega t - ikx_1} \qquad x_2 \leqslant 0, \tag{6.1.20}$$

where

$$\varphi(x_2) = A_1\, e^{s_1 k x_2} + A_2\, e^{s_2 k x_2}, \tag{6.1.21}$$

A_1 and A_2 are constants, and s_1 and s_2 are the solutions of (6.1.17) with positive real part.

Similarly, in B^*,

$$\psi^* = \varphi^*(x_2)\, e^{i\omega t - ikx_1} \qquad x_2 \geqslant 0, \tag{6.1.22}$$

with

$$\varphi^*(x_2) = A_1^*\, e^{-s_1^* k x_2} + A_2^*\, e^{-s_2^* k x_2}, \tag{6.1.23}$$

where A_1^* and A_2^* are constants, and s_1^* and s_2^* are the solutions with positive real part of the analogue of (6.1.17) for B^*. Thus,

$$s_1^{*2} + s_2^{*2} = \frac{2\beta^* - \rho^* c^2}{\gamma^*}, \qquad s_1^{*2} s_2^{*2} = \frac{\alpha^* - \rho^* c^2}{\gamma^*}. \tag{6.1.24}$$

and

$$0 < \rho^* c^2 < \alpha^*, \tag{6.1.25}$$

in parallel with (6.1.18) and (6.1.19)

Next, for the two-dimensional motion considered here the active components of the nominal traction rate on $x_2 = 0$, as measured in B, are $\dot{s}_{021}$ and $\dot{s}_{022}$. For B^* the corresponding components are $\dot{s}^*_{021}$ and $\dot{s}^*_{022}$. We assume that on $x_2 = 0$ the velocity and traction rate are continuous, so that the boundary coditions are

$$\underline{v} = \underline{v}^*, \qquad \dot{s}_{02i} = \dot{s}^*_{02i} \qquad \text{on } x_2 = 0, \quad i = 1,2$$

that is

$$v_1 - v_1^* = 0,$$

$$v_2 - v_2^* = 0,$$

$$A_{02121}\, v_{1,2} + (A_{01221} + p)\, v_{2,1} - A^*_{02121}\, v^*_{1,2} - (A^*_{01221} + p^*)\, v^*_{2,1} = 0, \tag{6.1.26}$$

$$A_{01122}\, v_{1,1} + (A_{02222} - p)\, v_{2,2} - \dot{p} - A^*_{01122}\, v^*_{1,1} - (A^*_{02222} - p^*)\, v^*_{2,2} - \dot{p}^* = 0.$$

On use of the notations in (6.1.10), (2.6.18) and (6.1.8) the above equations may be rewritten as

$$\psi_{,2} - \psi^*_{,2} = 0,$$

$$\psi_{,1} - \psi^*_{,1} = 0, \tag{6.1.27}$$

$$\gamma\, \psi_{,22} - (\gamma - \sigma_2)\, \psi_{,11} - \gamma^*\, \psi^*_{,22} + (\gamma^* - \sigma^*_2)\, \psi^*_{,11} = 0,$$

$$(A_{01122} - A_{02222} - p)\, \psi_{,12} - \dot{p} + (A^*_{02222} - A^*_{01122} + p^*)\, \psi^*_{,12} + \dot{p}^* = 0.$$

From (6.1.9) we have

$$\dot{p}_{,1} = [A_{01111} - A_{01122} - A_{02112}]\, \psi_{,112} + A_{02121}\, \psi_{,222} - \rho\ddot{\psi}_{,2},$$

and similarly (6.1.28)

$$\dot{p}^*_{,1} = [A^*_{01111} - A^*_{01122} - A^*_{02112}]\, \psi^*_{,112} + A^*_{02121}\, \psi^*_{,222} - \rho^*\, \ddot{\psi}^*_{,2}.$$

To eliminate $\dot{p}$ and $\dot{p}^*$ we must differentiate equations $(6.1.27)_4$ with respect to x_1 and using equations (6.1.28) and the notations in (6.1.10), the fourth equation of (6.1.27) becomes

$$\gamma\, \psi_{,222} - (2\beta + \gamma - \sigma_2)\psi_{,112} - \rho\ddot{\psi}_{,2} - \gamma^*\, \psi^*_{,222}$$

$$- (2\beta^* + \gamma^* - \sigma^*_2)\psi^*_{,112} + \rho^*\, \ddot{\psi}^*_{,2} = 0. \tag{6.1.29}$$

On use of equations (6.1.20) and (6.1.22) in the boundary conditions $(6.1.27)_{1-3}$ and (6.1.29) and also use (6.1.18) and (6.1.24) in (6.1.29) , we have

$$s_1 A + s_2 B + s_1^* A^* + s_2^* B^* = 0,$$

$$A + B - A^* - B^* = 0,$$

$$[\gamma (s_1^2 + 1) - \sigma_2] A + [\gamma (s_2^2 + 1) - \sigma_2] B - [\gamma^* (s_1^{*2} + 1) - \sigma_2^*] A^* - [\gamma^* (s_2^{*2} + 1) - \sigma_2^*] B^* = 0, \tag{6.1.30}$$

$$s_1 [\gamma (s_2^2 + 1) - \sigma_2] A + s_2 [\gamma (s_1^2 + 1) - \sigma_2] B - s_1^* [\gamma^* (s_2^{*2} + 1) - \sigma_2^*] A^* - s_2^* [\gamma^* (s_1^{*2} + 1) - \sigma_2^*] B^* = 0.$$

For these equations (6.1.30), to have non-trivial solutions we must have

$$\begin{vmatrix} s_1 & s_2 & s_1^* & s_2^* \\ 1 & 1 & -1 & -1 \\ \gamma(s_1^2 + 1) - \sigma_2 & \gamma(s_2^2 + 1) - \sigma_2 & -\gamma^*(s_1^{*2} + 1) - \sigma_2^* & -\gamma^*(s_2^{*2} + 1) - \sigma_2^* \\ s_1\{\gamma(s_1 + 1) - \sigma_2\} & s_2\{\gamma(s_2 + 1) - \sigma_2\} & s_1\{\gamma^*(s_1 + 1) - \sigma_2\} & s_2\{\gamma^*(s_2 + 1) - \sigma_2\} \end{vmatrix} = 0,$$

which gives

$$\gamma^2\ (s_1 - s_2)(s_1^* - s_2^*)\Big[1 - 2\ s_1 s_2 - s_1 s_2\ \big[s_1^2 + s_2^2 + s_1 s_1\big]\Big]$$

$$+ \gamma\ \gamma^*\ (s_1 - s_2)(s_1^* - s_2^*)\Big[\big[s_1 + s_2\big]\big[s_2^* + s_2^*\big]\big[s_1 s_2 + s_1^* s_2^*\big]$$

$$- 2\big[\ s_1 s_2 + s_1^* s_2^* - s_1 s_2 s_1^* s_2^* - 1\big]\Big]$$

$$+ 2\gamma\ (s_1 - s_2)(s_1^* - s_2^*)\Big[s_1 s_2\big[\sigma_2 - \sigma_2^*\big] - \big[\sigma_2 - \sigma_2^*\big]\Big]$$

$$+ \gamma^{*2}(s_1 - s_2)(s_1^* - s_2^*)\Big[1 - 2\ s_1^* s_2^* - s_1^* s_2^*\ \big[s_1^{*2} + s_2^{*2} + s_1^* s_1^*\big]\Big]$$

$$+ 2\gamma^*\ (s_1 - s_2)(s_1^* - s_2^*)\Big[\big[\sigma_2 - \sigma_2^*\big] - s_1^* s_2^*\big[\sigma_2 - \sigma_2^*\big]\Big]$$

$$+ (s_1 - s_2)(s_1^* - s_2^*)\Big[\sigma_2^2 - 2\sigma_2\sigma_2^* + \sigma_2^{*2}\Big] = 0,$$

As in Section 4.1, we assume that $s_1 \neq s_2$ and $s_1^* \neq s_2^*$. On removal of the factors $(s_1 - s_2)$ and $(s_1^* - s_2^*)$ the above equation reduces to

$$\gamma^2\ \Big[1 - 2\ s_1 s_2 - s_1 s_2\ \big[s_1^2 + s_2^2 + s_1 s_2\big]\Big]$$

$$+\gamma\gamma^*\Big[\big[s_1 + s_2\big]\big[s_1^* + s_2^*\big]\big[s_1 s_2 + s_1^* s_2^*\big] - 2\big[\ s_1 s_2 + s_1^* s_2^* - s_1 s_2 s_1^* s_2^* - 1\big]\Big]$$

$$+ 2\gamma\ \Big[s_1 s_2\big[\sigma_2 - \sigma_2^*\big] - \big[\sigma_2 - \sigma_2^*\big]\Big]$$

$$+ \gamma^{*2}\ \Big[1 - 2\ s_1^* s_2^* - s_1^* s_2^*\ \big[s_1^{*2} + s_2^{*2} + s_1^* s_1^*\big]\Big]$$

$$+ 2\gamma^*\ \Big[\big[\sigma_2 - \sigma_2^*\big] - s_1^* s_2^*\big[\sigma_2 - \sigma_2^*\big]\Big] + \Big[\sigma_2^2 - 2\sigma_2\sigma_2^* + \sigma_2^{*2}\Big] = 0.$$

Also, this equation for the case $\sigma_2^* = \sigma_2$ reduces to

$$\gamma^2 \left[s_1^2 s_2^2 - 1 + s_1 s_2 \left[s_1^2 + s_2^2 + 2\right]\right]$$

$$- \gamma\gamma^* \left[2\left[s_1 s_2 + s_1 s_2 - s_1 s_2 s_1 s_2 - 1\right] - \left[s_1 + s_2\right]\left[s_2^* + s_2^*\right]\left[s_1 s_2 + s_1^* s_2^*\right]\right]$$

$$+ \gamma^{*2} \left[s_1^{*2} s_2^{*2} - 1 + s_1^* s_2^* \left[s_1^{*2} + s_2^{*2} + 2\right]\right] = 0. \tag{6.1.31}$$

On use of (6.1.18) and (6.1.24), the secular equation (6.1.31) becomes

$$\gamma^2 \left[\frac{\alpha - \gamma - \rho c^2}{\gamma} + \frac{2(\beta + \alpha) - \rho c^2}{\gamma}\left[\frac{\alpha - \rho c^2}{\gamma}\right]^{\frac{1}{2}}\right]$$

$$- \gamma\, \gamma^* \left[2\left\{\left[\frac{\alpha - \rho c^2}{\gamma}\right]^{\frac{1}{2}} + \left[\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}} - \left[\frac{\alpha - \rho c^2}{\gamma}\right]^{\frac{1}{2}} - \left[\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}} - 1\right\} \right.$$

$$- \left\{\frac{2\beta - \rho c^2}{\gamma} + 2\left[\frac{\alpha - \rho c^2}{\gamma}\right]^{\frac{1}{2}}\right\}\left\{\frac{2\beta^* - \rho^* c^2}{\gamma^*} + 2\left[\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right\}$$

$$\left.\left\{\left[\frac{\alpha - \rho c^2}{\gamma}\right]^{\frac{1}{2}} \left[\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right\}\right]$$

$$+ \gamma^{*2}\left[\frac{\alpha^* - \gamma^* - \rho^* c^2}{\gamma^*} + \frac{2(\beta^* + \alpha^*) - \rho^* c^2}{\gamma^*} \left[\frac{\alpha^* - \rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right] = 0. \tag{6.1.32}$$

i.e.

$$f(c) \equiv (2\beta + 2\gamma - \rho c^2)\{\gamma\ (\alpha - \rho c^2)\}^{\frac{1}{2}} + \gamma\ (\alpha - \rho c^2) - \gamma^2$$

$$+ (2\beta^* + 2\gamma^* - \rho^* c^2)\{\gamma^*\ (\alpha^* - \rho^* c^2)\}^{\frac{1}{2}} + \gamma^*\ (\alpha^* - \rho^* c^2) - \gamma^{*2}$$

$$+ 2[\gamma - \{\gamma\ (\alpha - \rho c^2)\}^{\frac{1}{2}}]\ [\gamma^* - \{\gamma^*\ (\alpha^* - \rho^* c^2)\}^{\frac{1}{2}}]$$

$$+ [\{\gamma^*\ (\alpha - \rho c^2)\}^{\frac{1}{2}} + \{\gamma\ (\alpha^* - \rho^* c^2)\}^{\frac{1}{2}}\ ]$$

$$[2\beta - \rho c^2 + 2\{\gamma\ (\alpha - \rho c^2)\}^{\frac{1}{2}}\ ]^{\frac{1}{2}}$$

$$[2\beta^* - \rho^* c^2 + 2\{\gamma^*\ (\alpha^* - \rho^* c^2)\}^{\frac{1}{2}}\ ]^{\frac{1}{2}} = 0, \qquad (6.1.33)$$

where the left-hand identity defines the function f.

Equation (6.1.33) is the *secular equation* governing the speed of interfacial waves. Whether or not such waves exist depends crucially on the values of α, β, γ, α^*, β^*, γ^*, ρ and ρ^*. Equation (6.1.33) is new although special cases of it have been considered previously by Chadwick and Jarvis (1979a,b). In the linear theory the specialization of (6.1.33) coincides with Stoneley's (1924) original result for incompressible materials but is expressed in different notation.

From (6.1.19) and (6.1.25) we have

$$0 < c^2 < \min \{\frac{\alpha}{\rho}, \frac{\alpha^*}{\rho^*}\}. \qquad (6.1.34)$$

By introducing the notation

$$c_L^2 = \frac{\alpha}{\rho}, \qquad c_L^{*2} = \frac{\alpha^*}{\rho^*} \qquad (6.1.35)$$

and, without loss of generality, taking c, c_L and c_L^* to be non-negative, we obtain

$$0 < c < \min\{c_L,\ c_L^*\}. \qquad (6.1.36)$$

The <u>limiting speeds</u> c_L and c_L^* for the separate half-spaces are the speeds of plane shear waves in B and B^* respectively.

We note that the inequality

$$\gamma\ (s_1 + s_2)^2 \equiv 2\beta - \rho c^2 + 2\{\gamma(\alpha - \rho c^2)\}^{\frac{1}{2}} > 0$$

is a consequence of the strong ellipticity condition $(2.8.15)_2$ and our assumptions concerning s_1 and s_2, and when $c = 0$ it reduces to $(2.8.15)_3$. We also note that (6.1.33) does not depend explicitly on the interfacial traction σ_2, a fact also pointed out by Chadwick and Jarvis (1979a) in the case they considered.

6.2 Analysis of the secular equation

Here we shall examine the secular equation (6.1.33) in detail, but we shall consider first some specializations of (6.1.33).

6.2.1 Surface waves on a half-space

We specialize to the case in which $\alpha^* = \beta^* = \gamma^* = 0$ corresponding to the situation in which there is no material in $x_2 < 0$. The secular equation (6.1.33) then reduces to

$$\gamma^2\ g(c) \equiv (2\beta + 2\gamma - \rho c^2)\{\gamma(\alpha - \rho c^2)\} + \gamma\ (\alpha - \rho c^2) - \gamma^2 = 0, \qquad (6.2.1)$$

where the function g defined as $g(c^2)$ in Chapter 4.

This is the equation derived in Chapter 4 for the speed of propagation of Rayleigh surface waves on a half-space with $\sigma_2 = 0$. We now note some features of (6.2.1) which were discussed in detail in Section 4.2.

First, necessary and sufficient conditions for (6.2.1) to have a unique solution $c \in [0, c_L)$ are

$$\gamma > 0, \qquad 2(\beta + \gamma)\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}} + \alpha - \gamma \geq 0, \tag{6.2.2}$$

with equality corresponding to $c = 0$. Together, the inequalities (6.2.2) imply that the strong ellipticity inequalities (2.8.15) hold.

In the notation

$$\eta = \{(\alpha - \rho c^2)/\gamma\}^{\frac{1}{2}}, \tag{6.2.3}$$

equation (6.2.1) can be expressed

$$g(c) \equiv \eta^3 + \eta^2 + \frac{2\beta + 2\gamma - \alpha}{\gamma}\,\eta - 1 = 0, \tag{6.2.4}$$

and η must lie in the interval

$$0 \leq \eta \leq \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}} \equiv \frac{\lambda_1}{\lambda_2}. \tag{6.2.5}$$

From (6.2.3) we also have

$$\rho c^2 = \alpha - \gamma\,\eta^2. \tag{6.2.6}$$

Given $(6.2.1)_1$, the inequality $(6.2.2)_2$ can be expressed as

$$g(0) \geq 0. \tag{6.2.7}$$

In the special case $\lambda_1 = \lambda_2$, use of (6.1.14) shows that

$$g(c) = \eta^3 + \eta^2 + 3\eta - 1, \tag{6.2.8}$$

and the unique positive solution, η_0 say, of $g(c) = 0$ in this case is approximately

$$\eta_0 \simeq 0.2956, \tag{6.2.9}$$

and

$$\frac{\rho c^2}{\alpha} = 1 - \eta_0^2 \simeq 0.9126. \tag{6.2.10}$$

6.2.2 Interfacial deformations

The possibility of interfacial deformations arises if (6.1.33)is satisfied for $c = 0$. With $c = 0$, in the notation g and its analogue g^* for B^*, defined by

$$g^*(c) \equiv \eta^{*3} + \eta^{*2} + \frac{2\beta^* + 2\gamma^* - \alpha^*}{\gamma^*}\eta^* - 1, \tag{6.2.11}$$

where

$$\eta^* = \{(\alpha^* - \rho^* c^2)/\gamma^*\}^{\frac{1}{2}}, \tag{6.2.12}$$

equation (6.1.33) can be written as

$$\begin{aligned} f(0) \equiv \gamma^2\, g(0) + \gamma^{*2}\, g^*(0) + 2(\gamma\,\gamma^*)^{\frac{1}{2}}\,(\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}}) \\ + \left[\frac{\gamma\,\gamma^*}{\alpha\,\alpha^*}\right]^{\frac{1}{4}}\left\{(\gamma^*\,\alpha)^{\frac{1}{2}} + (\gamma\,\alpha^*)^{\frac{1}{2}}\right\}\left\{\gamma\, g(0) + (\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^2\right\}^{\frac{1}{2}} \\ \left\{\gamma^*\, g^*(0) + (\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}})^2\right\}^{\frac{1}{2}} = 0. \end{aligned} \tag{6.2.13}$$

Interfacial deformations described by (6.1.20)-(6.1.23) with $\omega = 0$ and k arbitrary are possible in configurations for which $f(0) = 0$.

When $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$ the left-hand identity in (6.2.13) becomes

$$f(0) \equiv 4(\gamma + \gamma^*)^2, \tag{6.2.14}$$

which reduces to $4(\mu + \mu^*)^2$ in the reference configuration $\lambda_i = \lambda_i^* = 1$, $i \in \{1,2,3\}$, where μ^* is the ground-state shear modulus for B^*. Since $f(0) > 0$ in this reference configuration it follows by continuity that interfacial deformations are excluded on any path of deformation in $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*)$-space, from the reference configuration along which $f(0) > 0$, with interfacial deformations emerging if $f(0)$ vanishes. We refer to the inequality $f(0) > 0$, with $f(0)$ given by (6.2.13), as the *exclusion condition*. For *surface* deformations in the separate half-spaces, B and B^*, the exclusion conditions are $g(0) > 0$ and $g^*(0) > 0$ respectively. In the context of surface deformations the exclusion condition has been examined in detail in Chapter 4.

From (6.2.14) we conclude that interfacial deformations cannot exist in configurations for which $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$ (and hence $\alpha = \gamma$, $\alpha^* = \gamma^*$). We are now in a position to establish the following result.

THEOREM 1. If $g(0) \geqslant 0$ and $g^*(0) \geqslant 0$ then $f(0) > 0$.

Proof If $\alpha = \gamma$ then $g(0) = 4$ and

$$f(0) = 4\,\gamma^2 + \gamma^{*2}\, g^*(0) + 2\,\gamma\,(\gamma^{*\frac{1}{2}} + \alpha^{*\frac{1}{2}})(\gamma^*/\alpha^*)^{\frac{1}{4}} \{\gamma^*\, g^*(0) + (\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}})^2\}^{\frac{1}{2}}.$$

It follows that $f(0) > 0$ if $g^*(0) \geqslant 0$ (this includes the case in which $\alpha^* = \gamma^*$).

If $\alpha \neq \gamma$, $\alpha^* \neq \gamma^*$, $g(0) > 0$ and $g^*(0) \geqslant 0$ then two possibilities arise: either (a) $(\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}}) > 0$, in which case $f(0) > 0$ follows immediately from (6.2.13), or (b) $(\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}}) < 0$, and it follows that

$$f(0) > \left[\frac{\gamma\ \gamma^*}{\alpha\ \alpha^*}\right]^{\frac{1}{4}} \left|(\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}})\right| \left\{\left[\gamma^*\ \alpha\right]^{\frac{1}{4}} - \left[\gamma\ \alpha^*\right]^{\frac{1}{4}}\right\}^2, \tag{6.2.15}$$

and hence the required result.

If $g(0) = g^*(0)$ then $f(0)$ can vanish when $\gamma^*\ \alpha = \gamma\ \alpha^*$, but this is inconsistent with the inequality $(\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}}) < 0$ in (b). Hence

$$f(0) = \left[\frac{\gamma\ \gamma^*}{\alpha\ \alpha^*}\right]^{\frac{1}{4}} (\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\gamma^{*\frac{1}{2}} - \alpha^{*\frac{1}{2}})\left\{\left[\gamma^*\ \alpha\right]^{\frac{1}{4}} + \left[\gamma\ \alpha^*\right]^{\frac{1}{4}}\right\},$$

which is positive.

It follows from this theorem that *a necessary condition for the existence of interfacial deformations is that either* $g(0) < 0$ *or* $g^*(0) < 0$.

6.2.3 Limiting cases

With reference to (6.1.33) we assume, without loss of generality, that

$$0 \leqslant c \leqslant c_L \leqslant c_L^*. \tag{6.2.16}$$

We consider two limiting cases:

(a) $c = c_L < c_L^*$

From (6.1.18) we have $s_1^2\ s_2^2 = 0$, so we take $s_2 = 0$ and $s_1^2 = (2\beta - \alpha)/\gamma$. It follows from (6.1.20)-(6.1.23) that

$$\psi = (A_1\ e^{s_1 k x_2} + A_2\ e^{s_2 k x_2})\ e^{i(\omega t - k x_1)}, \tag{6.2.17}$$

$$\psi^* = (A_1^*\ e^{s_1^* k x_2} + A_2^*\ e^{s_2^* k x_2})\ e^{i(\omega t - k x_1)}, \tag{6.2.18}$$

and the boundary conditions are adjusted accordingly.

For the first term in (6.2.17) to represent an interfacial wave we require $2\beta > \alpha$. The second term in (6.2.17) corresponds to a plane shear wave in B. In general these two waves cannot exist independently: if $A_1 = 0$, it is easily shown that $A_2^* = A_1^* = A_2^* = 0$ follows from the boundary conditions; similary, if either $A_2 = 0$ or $s_1 = 0$ $(2\beta = \alpha)$ then the trivial solution again follows unless either $s_1^* = 0$ or $s_2^* = 0$, as exemplified in case (b) below.

Thus, in configurations in which $f(c_L) = 0$, a wave of the form (6.2.17) in B and (6.2.18) in B^* can propagate with limiting speed c_L, but (6.2.17) does not represent a true interfacial wave.

(b) $c = c_L = c_L^*$

In this case (6.1.33) gives

$$f(c_L) = -(\gamma - \gamma^*)^2. \tag{6.2.19}$$

We take $s_2 = s_2^* = 0$, $s_1^2 = \dfrac{2\beta - \alpha}{\gamma}$, $s_1^{*2} = \dfrac{2\beta^* - \alpha^*}{\gamma^*}$, and the boundary conditions yield the trivial solution unless $\gamma = \gamma^*$. If $\gamma = \gamma^*$ it follows that $A_1 = A_1^* = 0$ and $A_2^* = A_2$. The result is a plane shear wave given by

$$\psi = \psi^* = A_2 \, e^{i(\omega t - kx_1)}. \tag{6.2.20}$$

Such a wave is only possible under the rather restrictive conditions $\gamma = \gamma^*$ and $\alpha/\rho = \alpha^*/\rho^*$.

In the linear specialization the result in (a) above can be compared with that in Case 3 of Barnett at al. (1985, p161). On the other hand, the restriction in (b) reduces to $\mu^* = \mu$, $\rho^* = \rho$ in the linear theory, i.e. the two materials are indistinguishable (see Barnett et al., 1985, Case 4). Note, however that the work of Barnett et al. was set in the context of *compressible* linear elasticity.

6.2.4 The general case

The secular equation (6.1.33) can be expressed more compactly in terms of the notation (6.2.3) and its counterpart for B^*. Thus, with

$$\eta = \{(\alpha - \rho c^2)/\gamma\}^{\frac{1}{2}}, \qquad \eta^* = \{(\alpha^* - \rho^* c^2)/\gamma^*\}^{\frac{1}{2}}, \tag{6.2.21}$$

equation (6.1.33) becomes

$$f(c) \equiv \gamma^2\, g(c) + \gamma^{*2}\, g^*(c) + 2\,\gamma\,\gamma^*\,(1 - \eta)(1 - \eta^*)$$

$$+ \gamma\,\gamma^*\,(\eta + \eta^*)\,\eta^{-\frac{1}{2}}\,\eta^{*-\frac{1}{2}}\,[g(c) + (\eta - 1)^2]^{\frac{1}{2}}\,[g^*(c) + (\eta^* - 1)^2]^{\frac{1}{2}} = 0. \tag{6.2.22}$$

We assume that the exclusion condition $f(0) > 0$, discussed in Section 6.2.2, holds. In a limited sense this means that the underlying homogeneous configuration is stable. If we also assume that the limiting speeds are ordered as in (6.2.16) it follows that a *sufficient* condition for the existence of an interfacial wave with $0 < c < c_L$ is therefore

$$f(c_L) < 0, \tag{6.2.23}$$

where

$$f(c_L) = -\gamma^2 + \gamma^{*2}\, g^*(c_L) + 2\,\gamma\,\gamma^*(1 - \eta_L^*)$$
$$+ \gamma\,\gamma^*\,\eta_L^{*\frac{1}{2}}\,[(2\beta - \alpha)/\gamma]^{\frac{1}{2}}\,[g^*(c_L) + (\eta_L^* - 1)^2]^{\frac{1}{2}} \tag{6.2.24}$$

and

$$\eta_L^* = \left\{\frac{\rho\,\alpha^* - \rho^*\,\alpha}{\rho\,\gamma^*}\right\}^{\frac{1}{2}}. \tag{6.2.25}$$

The limiting case $c = c_L$ was discussed in Section 6.2.3 and is not considered further here.

Clearly, circumstances in which (6.2.23) can be satisfied depend very much on the values of α, β, γ, α^*, β^*, γ^* and ρ^*/ρ. In particular, (6.2.23) can be satisfied if γ^*/γ is sufficiently small. Whether or not (6.2.23) is also necessary for the existence of an interfacial wave depends on the properties of $f(c)$ for $0 < c < c_L$. For a single half-space, it has been established in Chapter 4 that if a surface wave exists it is a unique. The question of uniqueness in the present situation is not clear because $f(c)$ is more complicated than $g(c)$; at this point it has not proved possible to settle the question in general, although, as we see below, uniqueness can be established in some special cases. However, some general conclusions can nevertheless be drawn.

Firstly, if $c_L = c_L^*$ then

$$f(c_L) = -(\gamma - \gamma^*)^2.$$

If $\gamma \neq \gamma^*$ then the existence of an interfacial wave is guaranteed. The case $\gamma = \gamma^*$ was discussed in Section 6.2.3

Let c_R and c_R^* be the (unique) positive surface (Rayleigh) wave speeds for B and B^* respectively, so that

$$g(c_R) = g^*(c_R^*) = 0 \qquad (6.2.26)$$

and

$$\begin{aligned} g(0) &> 0 \quad \text{for} \quad 0 \leqslant c \leqslant c_R, \\ g^*(0) &> 0 \quad \text{for} \quad 0 \leqslant c \leqslant c_R^*. \end{aligned} \qquad (6.2.27)$$

THEOREM 2. (a) If $c_R \neq c_R^*$ then

$$f(c) > 0 \text{ for } 0 \leqslant c \leqslant \min(c_R, c_R^*). \tag{6.2.28}$$

(b) If $c_R = c_R^*$ then

$$f(c) \geqslant 0 \text{ for } 0 \leqslant c \leqslant c_R = c_R^* \tag{6.2.29}$$

with equality holding if and only if $c = c_R$ and *either*

$$\rho c_R^2 = 2\beta + 2\gamma = \alpha - \gamma, \; g^*(c_R) = 0 \tag{6.2.30}$$

or

$$\rho^* c_R^2 = 2\beta^* + 2\gamma^* = \alpha^* - \gamma^*, \; g(c_R) = 0. \tag{6.2.31}$$

Exceptionally, (6.2.30) and (6.2.31) can hold simultanceously.

Proof (a) This result follows immediately by using an argument parallel to that required for the proof of Theorem 1.

(b) If $c_R = c_R^*$ then, by the same argument as in (a), we have $f(c) > 0$ for $0 \leqslant c < c_R$; we also obtain

$$f(c_R) = \gamma\,\gamma^*(\eta\,\eta^*)^{-\frac{1}{2}} \left|(\eta - 1)(\eta^* - 1)\right| (\eta^{\frac{1}{2}} + \sigma\,\eta^{*\frac{1}{2}})^2,$$

where $\sigma = \text{sign}\,\{(\eta - 1)(\eta^* - 1)\}$, and η and η^* are evaluted for $c = c_R$. The solution $\eta = \eta^*$ (for $\sigma = -1$) of $f(c_R) = 0$ is inconsistent with $\sigma = -1$. Hence $f(c_R) \geqslant 0$ with equality if and only if either $\eta = 1$ or $\eta^* = 1$ (or both).

We deduce from (6.2.3), (6.2.4) and (6.2.26) that (6.2.30) holds. Similarly, for $\eta^* = 1$, equations (6.2.31) hold. Exceptionally, $\eta = 1$ and $\eta^* = 1$ can hold simultaneously if

$$c_R^2 = (2\beta + 2\gamma)/\rho = (\alpha - \gamma)/\rho = (2\beta^* + 2\gamma^*)/\rho^* = (\alpha^* - \gamma^*)/\rho^* \,. \tag{6.2.32}$$

This completes the proof.

For $c_R \neq c_R^*$ it follows that if an interfacial (Stoneley) wave exists, with speed c_s say, then

$$c_s > \min\,(c_R,\ c_R^*), \tag{6.2.33}$$

i.e. *the Stoneley wave speed is greater than the speed of the slowest of the Rayleigh waves for the separate half-spaces* (for the same underlying state of deformation). This extends to the nonlinear theory a result which is well known in the linear theory (see Barnett et al., 1985).

The inequality $c_L \leqslant c_L^*$, adopted in (6.2.16), does not in general imply that $c_R \leqslant c_R^*$. If $c_R \neq c_R^*$, then the following wavespeed orderings are possible, depending on the material properties and the underlying deformations of the two half-spaces:

$$\begin{aligned}
&\text{(i)} && c_R < c_s < c_R^* < c_L \leqslant c_L^*,\\
&\text{(ii)} && c_R < c_R^* < c_s < c_L \leqslant c_L^*,\\
&\text{(iii)} && c_R < c_s < c_L < c_R^* < c_L^*,\\
&\text{(iv)} && c_R^* < c_s < c_R < c_L \leqslant c_L^*,\\
&\text{(v)} && c_R^* < c_R < c_s < c_L \leqslant c_L^*.
\end{aligned} \tag{6.2.34}$$

Case (iii) is excluded if $c_L = c_L^*$.

If $c_R = c_R^*$ then any possible Stoneley wave has speed c_s such that

$$c_R = c_R^* \leqslant c_s < c_L \leqslant c_L^*. \tag{6.2.35}$$

In the case in which $c_s = c_R$ it is worth noting, with reference to (6.2.3), that the strict form of the inequality $(6.2.2)_2$ is equivalent to $\alpha - \gamma > 0$.

The solution $c_S = c_R = c_R^*$ corresponding to (6.2.30) yields $s_1 = i$, $s_2 = -i$, and hence, from (6.1.20) and (6.1.21),

$$\psi = (A_1 \, e^{ikx_2} + A_2 \, e^{-ikx_2}) \, e^{i(\omega t - kx_1)}, \tag{6.2.36}$$

while, for $\eta^* \neq 1$, ψ^* is given by (6.1.22) and (6.1.23), with s_1^* and s_2^* evaluated for $c = c_R$, provided $g^*(c_R) = 0$ holds. The boundary conditions (6.1.30) show that A_1^*, A_2^* and A_2 can be expressed in terms of A_1. Although (6.2.36) does not describe an interfacial wave, it can be interpreted in the following way: the terms in A_1 and A_2 correspond to plane shear waves propagating in B in directions bisecting the coordinate axes, are a wave incident on the plane boundary $x_2 = 0$ and the other reflected from it with changed amplitude. The incident wave also generates a transmitted wave which, in B^*, corresponds to an interfacial wave propagating in the x_1-direction.

If $\eta = \eta^* = 1$ then the above solution degenerates to

$$\psi = \psi^* = (A_1 \, e^{ikx_2} + A_2 \, e^{-ikx_2}) \, e^{i(\omega t - kx_1)}, \tag{6.2.37}$$

with A_1 and A_2 independent. This represents two independent plane shear waves which cross the boundary $x_2 = 0$ without change in direction or amplitude.

At this point it is worth noting that the condition $2\beta + 2\gamma = \alpha - \gamma$ arising in (6.2.30) can be expressed compactly in terms of the strain-energy function. If we write

$$\hat{W}(\lambda_1, \lambda_3) = W(\lambda_1, \lambda_1^{-1} \lambda_3^{-1}, \lambda_3) \tag{6.2.38}$$

then, from (6.1.12) and (6.1.13), it follows that

$$2\beta + 2\gamma = \lambda_1^2 \hat{W}_{11}, \qquad \alpha - \gamma = \lambda_1 \hat{W}_1. \tag{6.2.39}$$

Hence, (6.2.30) requires that

$$\lambda_1 \hat{W}_{11} = \hat{W}_1. \tag{6.2.40}$$

6.2.5 Results for biaxial deformations

If the underlying deformations in the two half-spaces are biaxial with $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$, then, from (6.1.14), we have $\alpha = \beta = \gamma$, and, similarly, $\alpha^* = \beta^* = \gamma^*$. Equation (6.2.22) then reduces to

$$f(c) \equiv \gamma^2 (\eta^3 + \eta^2 + 3\eta - 1) + \gamma^* (\eta^{*3} + \eta^{*2} + 3\eta^* - 1)$$

$$+ 2 \gamma \gamma^* (1 - \eta)(1 - \eta^*) + \gamma \gamma^* (\eta + \eta^*)(\eta + 1)(\eta^* + 1) = 0, \tag{6.2.41}$$

where

$$\eta = \left[1 - \frac{\rho c^2}{\alpha}\right]^{\frac{1}{2}}, \qquad \eta^* = \left[1 - \frac{\rho^* c^2}{\alpha^*}\right]^{\frac{1}{2}}. \tag{6.2.42}$$

Also, recalling (6.1.14) and (6.1.15) and their counterparts for B^*, we obtain

$$c_R = c_L(1 - \eta_0^2)^{\frac{1}{2}}, \quad c_R^* = c_L^*(1 - \eta_0^2)^{\frac{1}{2}}. \tag{6.2.43}$$

It follows that $c_R < c_R^*$ if and only if $c_L < c_L^*$ and that

$$c_R\, c_L^* = c_R^*\, c_L. \tag{6.2.44}$$

Of the five possibilities in (6.2.34), numbers (iv) and (v) are now excluded. Moreover, if $c_L = c_L^*$ then $c_R = c_R^*$ and, if a Stoneley wave exists, its speed c_S must satisfy $c_R = c_R^* < c_S < c_L = c_L^*$ since (6.2.30) cannot hold in this case.

From (6.2.14) we have $f(0) > 0$, while for $c = c_L \leq c_L^*$ $(\eta = 0)$ we have

$$f(c_L) \equiv -\gamma^2 + \gamma\gamma^* (\eta_L^{*2} - \eta_L^* + 2) + \gamma^{*2} (\eta_L^{*3} + \eta_L^{*2} + 3\eta_L^* - 1),$$

where, specializing (6.2.25), we have $\eta_L^* = (1 - \rho^*\alpha/\rho\alpha^*)^{\frac{1}{2}}$.

If $c_L = c_L^*$ this reduces to $-(\gamma - \gamma^*)^2$, as in the more general case discussed in Section 6.4. But then, for the biaxial deformations considered here, $\eta = \eta^*$ and (6.2.41) reduces to

$$f(c) \equiv (\gamma^2 + \gamma^{*2})(\eta^3 + \eta^2 + 3\eta - 1) + 2\gamma\gamma^*(\eta^3 + 3\eta^2 - \eta + 1),$$

which is easily shown to be monotonic for $0 \leq c \leq c_L$.

Thus, *if* $c_L = c_L^*$ *and the underlying deformation is biaxial, a unique Stoneley wave exists.*

The following theorem generalizes this result.

THEOREM 3. If $c_L \leq c_L^*$ then, for biaxial deformations $(\lambda_1 = \lambda_2, \lambda_1^* = \lambda_2^*)$, the inequality $f(c_L) < 0$ is necessary and sufficient for the existence of a unique interfacial wave.

Proof Since $f(c_L) < 0$ is sufficient for existence we require to establish uniqueness. First, we note that (6.2.22) can be factorized in the form

$$f(c) = [\{p(c)\,q(c)\}^{\frac{1}{2}} - r(c)][\{p(c)\,q(c)\}^{\frac{1}{2}} + r(c)], \tag{6.2.45}$$

where

$$p(c) = \gamma\{\eta^2 + 2\eta + (2\beta - \alpha)/\gamma\}^{\frac{1}{2}} + \gamma^*\{\eta^{*2} + 2\eta^* + (2\beta^* - \alpha^*)/\gamma^*\}^{\frac{1}{2}} \tag{6.2.46}$$

$$q(c) = \gamma\eta\{\eta^2 + 2\eta + (2\beta - \alpha)/\gamma\}^{\frac{1}{2}} + \gamma^*\eta^*\{\eta^{*2} + 2\eta^* + (2\beta^* - \alpha^*)/\gamma^*\}^{\frac{1}{2}} \tag{6.2.47}$$

$$r(c) = \gamma(1 - \eta) - \gamma^*(1 - \eta^*). \tag{6.2.48}$$

For biaxial deformations (6.2.46) and (6.2.47) reduce to

$$p(c) = \gamma\ (\eta + 1) + \gamma^*\ (\eta^* + 1), \tag{6.2.49}$$

$$q(c) = \gamma\ \eta\ (\eta + 1) + \gamma^*\ \eta^*\ (\eta^* + 1), \tag{6.2.50}$$

while r(c) is unchanged.

Clearly, $p(c) > 0$ and $q(c) > 0$ for $0 \leq c < c_L$, but $r(c)$ may be positive or negative, with $r(0) = 0$. Also, p and q are decreasing functions of c, and hence $(pq)^{\frac{1}{2}}$ is a decreasing function of c. In order to consider the signs of the two factors in (6.2.45) we investigate the properties of r(c). From (6.2.42) and (6.2.48) we obtain

$$\frac{dr(c)}{dc} = c\left[\frac{\rho}{\eta} - \frac{\rho^*}{\eta^*}\right]. \tag{6.2.51}$$

Since (for $c_L < c_L^*$) $\eta^* > \eta$, it follows that r is an increasing function of c if $\rho > \rho^*$. In this case the second factor in (6.2.45) is positive and hence $f(c_L) < 0$ if and only if

$$\{p(c_L)\ q(c_L)\}^{\frac{1}{2}} - r(c_L) < 0. \tag{6.2.52}$$

Thus, when $\rho > \rho^*$, p, q and r are increasing functions so that (6.2.52) is necessary and sufficient for the existence of a unique interfacial wave.

It remains to consider $\rho < \rho^*$. In this case (6.2.51) vanishes where

$$\rho\ \eta^* = \rho^*\ \eta. \tag{6.2.53}$$

This corresponds to a value of c ($= c_m$, say) given by

$$c_m^2 = \frac{\alpha\,\alpha^* \, (\rho^{*2} - \rho^2)}{\rho\,\rho^* \, (\rho^* \alpha^* - \rho\,\alpha)}, \tag{6.2.54}$$

with $c_m < c_L$ for $c_L < c_L^*$. Thus, r has a minimum at $c = c_m$, $r(c) \leqslant 0$ for $0 \leqslant c \leqslant c_m$ and $dr(c)/dc > 0$ for $c_m < c \leqslant c_L$.

If $r(c_L) \leqslant 0$ the first factor in (6.2.45) is positive for $0 \leqslant c \leqslant c_L$. From (6.2.48) it follows that $\gamma^* \geqslant \gamma/(1 - \eta_L^*)$ and, after some manipulation, that

$$p(c_L)\; q(c_L) - [r(c_m)]^2 > 0. \tag{6.2.55}$$

Similarly, if $r(c_L) > 0$, we have $\gamma^* > \gamma > \gamma^* \, (1 - \eta_L^*)$, and again it can be shown that (6.2.55) holds. In this case we also have

$$\{p(c_L)\; q(c_L)\}^{\frac{1}{2}} - r(c_L) > 0. \tag{6.2.56}$$

Thus, if $\rho < \rho^*$, $f(c) > 0$ for $0 \leqslant c \leqslant c_L$, and the theorem is established.

We remark that it has not as yet been possible to draw similar conclusions for more general deformations with p and q as defined in (6.2.46) and (6.2.47).

In the reference configuration, $\alpha = \mu$ and $\alpha^* = \mu^*$ and the results of this section therefore carry over to the linear theory. We note that, for the *compressible* linear theory, Barnett et al. (1985) have established that if a (subsonic) Stoneley wave exists then it is unique.

It is worth noting here that the boundary conditions (6.1.30) can be expressed in terms of the Stroh formulation (Stroh, 1962; Barnett at al., 1985). The secular equation is then equivalent to the vanishing of a 2x2 determinant whose eigenvalues are the two factors on the right-hand side of (6.2.45). For biaxial deformation it is easily shown that each eigenvalue is a decreasing function of c.

6.2.6 Application to the neo-Hookean material

For the neo-Hookean strain-energy function, we get

$$\alpha = \mu\,\lambda_1^2, \qquad \gamma = \mu\,\lambda_2^2 \quad \text{and} \quad 2\beta = \mu(\lambda_1^2 + \lambda_2^2), \tag{6.2.57}$$

and similarly for α^*, β^*, γ^*, the secular equation (6.1.32) become

$$\begin{aligned}
&(\mu\,\lambda_2^2)\left[\frac{\mu(\lambda_1^2-\lambda_2^2)-\rho c^2}{\mu\,\lambda_2^2}+\frac{\mu(\lambda_1^2+3\lambda_2^2-\rho c^2)}{\mu\,\lambda_2^2}\left[\frac{\mu\,\lambda_1^2-\rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}\right]\\
&-\mu\mu^*\left[2\left\{\left[\frac{\mu\,\lambda_1-\rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}+\left[\frac{\mu^*\,\lambda_1^{*2}-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}\right]^{\frac{1}{2}}-\left[\frac{\mu\,\lambda_1-\rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}\left[\frac{\mu^*\lambda_1-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}\right]^{\frac{1}{2}}-1\right\}\right.\\
&\qquad-\left\{\frac{\mu(\lambda_1^2+\lambda_2^2)-\rho c^2}{\mu\,\lambda_2^2}+2\left[\frac{\mu\,\lambda_1^2-\rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\\
&\qquad\left\{\frac{\mu^*(\lambda_1^{*2}+\lambda_2^{*2})-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}+2\left[\frac{\mu^*\,\lambda_1^{*2}-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\\
&\qquad\left.-\left\{\left[\frac{\mu\,\lambda_1-\rho c^2}{\mu\,\lambda_2^2}\right]^{\frac{1}{2}}+\left[\frac{\mu^*\,\lambda_1^{*2}-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}\right]^{\frac{1}{2}}\right\}\right]\\
&+\mu^*\,\lambda_2^{*2}\left[\frac{\mu^*(\lambda_1^{*2}-\lambda_2^{*2})-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}+\frac{\mu^*(\lambda_1^{*2}+3\lambda_2^{*2}-\rho^*c^2)}{\mu^*\,\lambda_2^{*2}}\left[\frac{\mu^*\,\lambda_1^{*2}-\rho^*c^2}{\mu^*\,\lambda_2^{*2}}\right]^{\frac{1}{2}}\right]=0,
\end{aligned} \tag{6.2.58}$$

or, the secular equation of the form (6.2.22) reduces to

$$\begin{aligned}
f(c) \equiv{}& \gamma^2\,(\eta^3+\eta^2+3\eta-1)+\gamma^*\,(\eta^{*3}+\eta^{*2}+3\eta^*-1)\\
&+2\,\gamma\,\gamma^*\,(1-\eta)(1-\eta^*)+\gamma\,\gamma^*\,(\eta+\eta^*)(\eta+1)(\eta^*+1)=0,
\end{aligned} \tag{6.2.59}$$

This has the same form as for the biaxial case discussed in Section 6.2.5, but here η and η^* are given by

$$\eta = \left[\frac{\mu\ \lambda_1^2 - \rho c^2}{\mu\ \lambda_2^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \eta^* = \left[\frac{\mu^*\ \lambda_1^{*2} - \rho^* c^2}{\mu^*\ \lambda_2^{*2}}\right]^{\frac{1}{2}}, \qquad (6.2.60)$$

Equations (6.2.43) are replaced by

$$c_R = c_L\ (1 - \gamma\ \eta_0^2/\ \alpha)^{\frac{1}{2}}, \quad c_R^* = c_L^*\ (1 - \gamma^*\ \eta_0^{*2}/\alpha^*)^{\frac{1}{2}}, \qquad (6.2.61)$$

and (6.2.44) holds if and only if $\gamma\ \alpha^* = \gamma^*\ \alpha$, i.e. $\lambda_2\ \lambda_1^* = \lambda_2^*\ \lambda_1$.

Since $\gamma > 0$ it follows from (6.2.58) that (6.2.30) cannot hold for the neo-Hookean material.

From (6.2.47) and (6.2.48) we obtain

$$\begin{aligned} f(0) = {} & \mu^2\ (\lambda_1^3\ \lambda_2 + \lambda_1^2\ \lambda_2^2 + 3\ \lambda_1\ \lambda_3^3 - \lambda_2^4) \\ & + \mu^{*2}\ (\lambda_1^{*3}\ \lambda_2^* + \lambda_1^{*2}\ \lambda_2^{*2} + 3\ \lambda_1^*\ \lambda_3^{*3} - \lambda_2^{*4}) \\ & + 2\ \mu\ \mu^*\ \lambda_2\ \lambda_2^*\ (\ \lambda_2 - \lambda_1)(\lambda_2^* - \lambda_1^*) \\ & + \mu\ \mu^*\ (\lambda_1 + \lambda_2)(\lambda_1^* + \lambda_2^*)(\lambda_1\ \lambda_2^* + \lambda_1^*\ \lambda_2). \end{aligned} \qquad (6.2.62)$$

The exclusion condition requires that $f(0) > 0$. For a fixed value of the ratio μ^*/μ, the neutral surface, N say, given by $f(0) = 0$, is a surface in $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*)$-space. Note that N is independent of ρ^*/ρ.

For fixed values of μ^*/μ and ρ^*/ρ the limiting surface, L say, corresponding to $c_L < c_L^*$, can also regarded as a surface in $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*)$-space, and its equation is obtained by setting $\eta = 0$ and $\eta^* = \eta_L^*$, where

$$\eta_L^* = \frac{(\lambda_1^{*2} - \kappa^2\ \lambda_1^2)^{\frac{1}{2}}}{\lambda_2^*} \tag{6.2.63}$$

and

$$\kappa^2 = \frac{\rho^*\ \mu}{\rho\ \mu^*} \tag{6.2.64}$$

$(\kappa > 0)$ in (6.2.59). Thus

$$f(c_L) \equiv -\gamma^2 + \gamma^* \ \ (\eta_L^{*3} + \eta_L^{*2} + 3\eta_L^* - 1) + \gamma\gamma^* \ (\eta_L^{*2} - \eta_L^* + 2) = 0.$$

Writing

$$\epsilon = \frac{\mu^*}{\mu},$$

we can express this in the form

$$F\ (\lambda_1,\ \lambda_2,\ \lambda_1^*,\ \lambda_2^*,\ \epsilon,\ \kappa) = 0, \qquad \lambda_1^* \geqslant \kappa\ \lambda_1. \tag{6.2.65}$$

The equation for the limiting surface, L^* say, corresponding to $c_L^* < c_L$ is obtained in a similar way and can also be expressed in terms of the function F by means of a suitable transformation of variables. Thus, L^* is described by

$$F\ (\lambda_1^*,\ \lambda_2^*,\ \lambda_1,\ \lambda_2,\ \epsilon^{-1},\ \kappa^{-1}) = 0, \qquad \lambda_1^* \leqslant \kappa\ \lambda_1. \tag{6.2.66}$$

For $\mu = \mu^*$ and $\rho = \rho^*$, Chadwick and Jarvis have examined the structures of N and L (and L^*) in detail (for the neo-Hookean material) as curves in $(\lambda_1,\ \lambda_1^*)$-space for biaxial deformations with $\lambda_3 = \lambda_1$, $\lambda_2 = \lambda_1^{-2}$, $\lambda_3^* = \lambda_1^*$, $\lambda_2^* = \lambda_1^{-2}$ (1979a) and for fixed values of λ_2, λ_2^* (1979b).

In Section 6.2.7 we provide numerical results illustrating the dependence of N, L and L^* on μ^*/μ and ρ^*/ρ to complement those provided by Chadwick and Jarvis for $\mu^*/\mu = \rho^*/\rho = 1$. In particular, we confine attention to biaxial deformations of the form $\lambda_1 = \lambda_3 = \lambda$, $\lambda_2 = \lambda^{-2}$, $\lambda_1^* = \lambda_3^* = \lambda^*$, $\lambda_2^* = \lambda^{*-2}$. We use the notation

$$\hat{F}(\lambda, \lambda^*, \epsilon, \kappa) \equiv F(\lambda, \lambda^{-2}, \lambda^*, \lambda^{*-2}, \epsilon, \kappa), \tag{6.2.67}$$

so that L, as a curve in (λ, λ^*)-space for fixed ϵ and κ, is described by the equation

$$\hat{F}(\lambda, \lambda^*, \epsilon, \kappa) = 0, \qquad \lambda^* \geqslant \kappa\,\lambda. \tag{6.2.68}$$

Explicitly, (6.2.68) gives

$$\epsilon\,\lambda^4\,\lambda^{*4}\,(\lambda^{*2} - \kappa^2\,\lambda^2)(\epsilon\lambda^4 + \lambda^{*4}) - (\epsilon\,\lambda^4 + \lambda^{*4})^2$$

$$+ \epsilon\,\lambda^4\,\lambda^{*2}\,(\lambda^{*2} - \kappa^2\,\lambda^2)^{\frac{1}{2}}\,\{3\,\epsilon\,\lambda^4 + \epsilon\,\lambda^4\,\lambda^{*4}(\lambda^{*2} + \kappa^2\,\lambda^2) - \lambda^{*4}\} = 0, \tag{6.2.69}$$

The curve L^* is defined by

$$\hat{F}(\lambda^*, \lambda, \epsilon^{-1}, \kappa^{-1}) = 0, \qquad \lambda^* \leqslant \kappa\,\lambda. \tag{6.2.70}$$

6.2.7 Numerical results

In this section we solve the equation for N, namely

$$\lambda^{*8}\,(1 - 3\,\lambda^3 - \lambda^6 - \lambda^9)$$

$$- \epsilon\,\lambda^4\,\lambda^{*4}\,(2 - \lambda^3 - \lambda^{*3} + \lambda^6 + 4\,\lambda^3\,\lambda^{*3} + \lambda^{*6} + \lambda^6\,\lambda^{*3} + \lambda^3\,\lambda^{*6}$$

$$+ \epsilon^2\,\lambda^8\,(1 - 3\,\lambda^{*3} - \lambda^{*6} - \lambda^{*9}) = 0. \tag{6.2.71}$$

for different values of ϵ (ϵ = 1,0.8, 0.5, 0.4, 0.2) for λ as a function of λ^* and λ^* as a function of λ.

Also, we solve the equation of L, namely

$$\lambda^4 \lambda^{*2} (\lambda^{*2} - \kappa^2 \lambda^2)^{\frac{1}{2}} [\epsilon^{-1} \lambda^{*4} - 3 \lambda^4 + \kappa^2 \lambda^6 \lambda^{*4} - \lambda^4 \lambda^{*6}]$$

$$+ (\epsilon^{-1} \lambda^{*4} - \lambda^4)^2 - (\lambda^{*2} - \kappa^2 \lambda^2)[\epsilon^{-1} \lambda^4 \lambda^{*4} + \lambda^8 \lambda^{*4}] = 0, \tag{6.2.72}$$

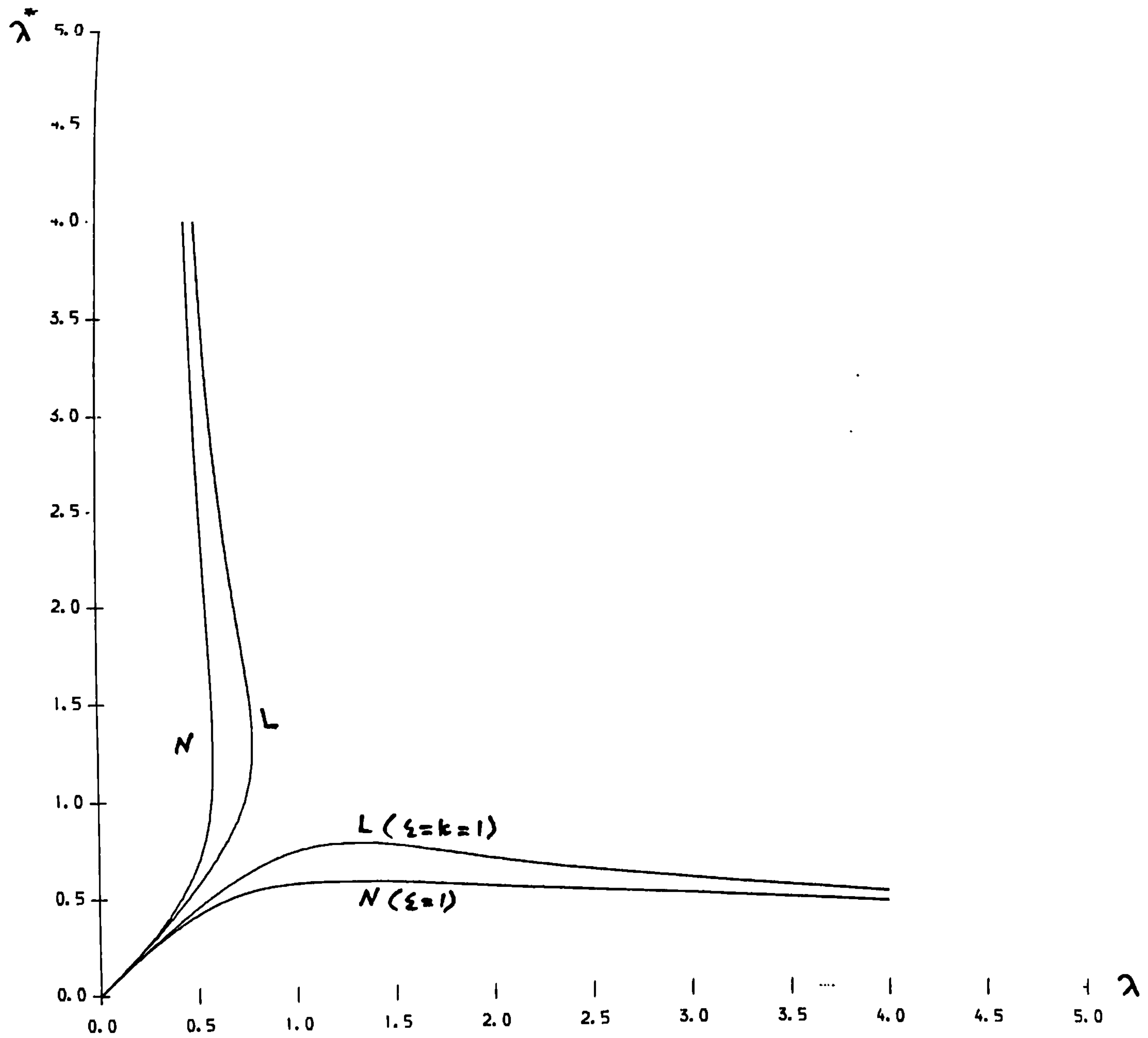
for different values of ϵ and κ to get the first curve of L and

finally in this section we solve

$$\epsilon \lambda^4 \lambda^{*2} (\lambda^{*2} - \kappa^2 \lambda^2)^{\frac{1}{2}} (\lambda^{*4} + \epsilon \kappa^4 \lambda^6 \lambda^{*4} - \epsilon \lambda^4 \lambda^{*6} - 3 \epsilon \lambda^4)$$

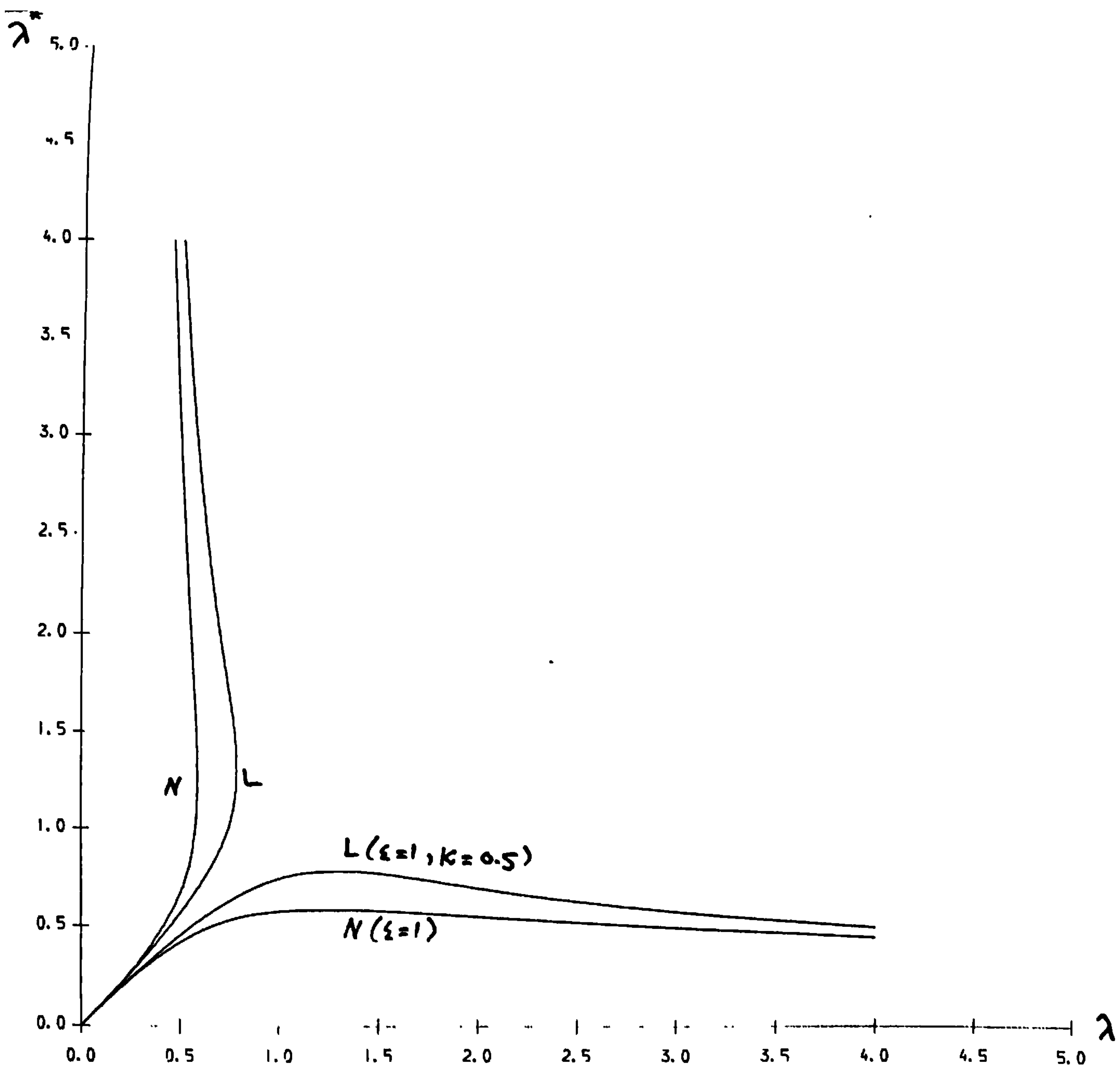
$$+ (\epsilon \lambda^4 - \lambda^{*4})^2 - \epsilon \lambda^4 \lambda^{*4} (\lambda^{*2} - \kappa^2 \lambda^2)(\epsilon \lambda^4 + \lambda^{*4}) = 0, \tag{6.2.73}$$

for the same values of ϵ and κ to get the second curve of L and the results are illustrate in Figures 6.1-6.15. We note that results do not show any significant dependence on κ.



λ*
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
N
L
L (ξ=k=1)
N (ξ=1)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
λ

Figure 6.1



$\bar{\lambda}^*$
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
N
L
L(ξ=1, k=0.5)
N(ξ=1)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
λ

Figure 6.2

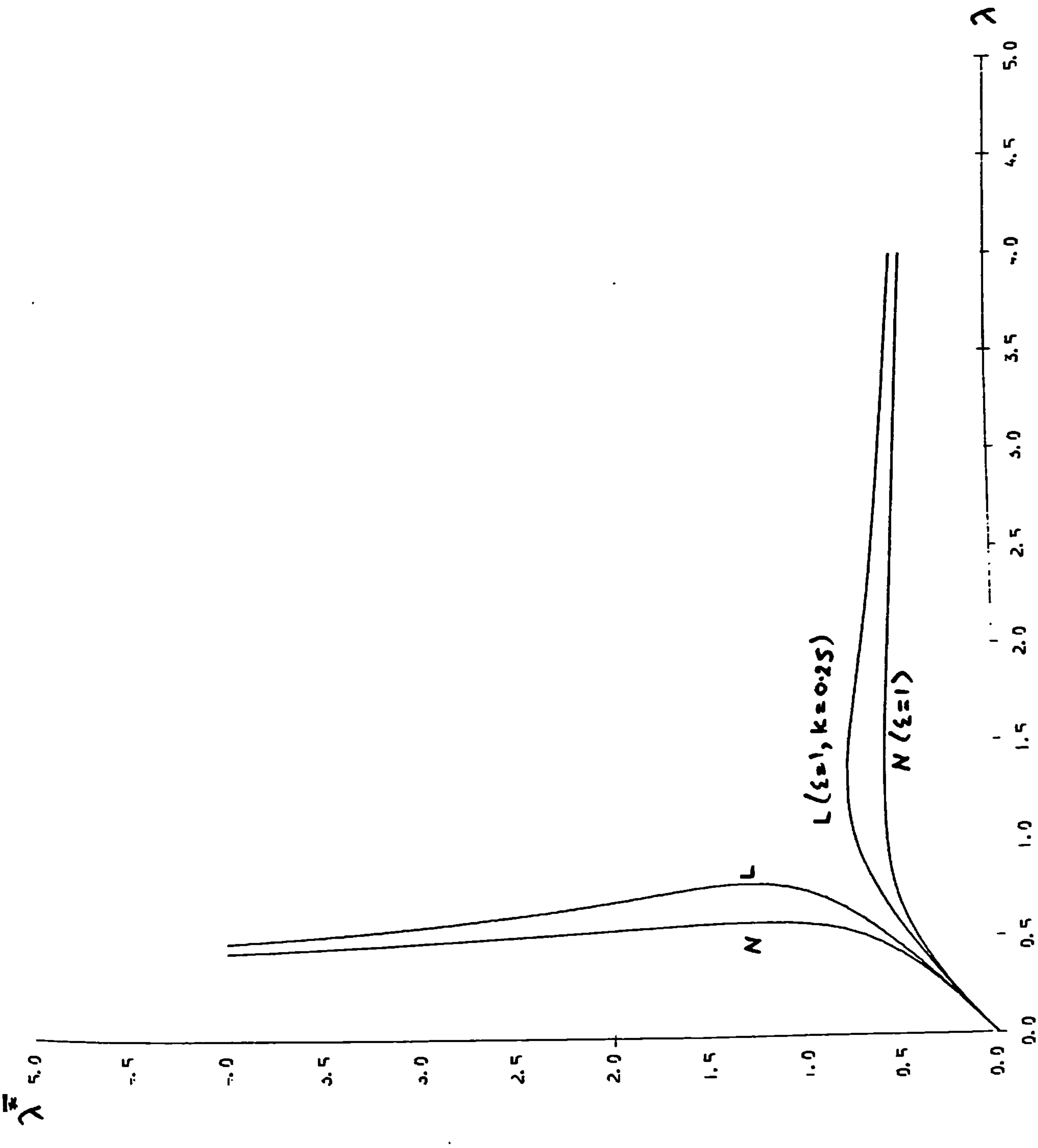


Figure 6.3

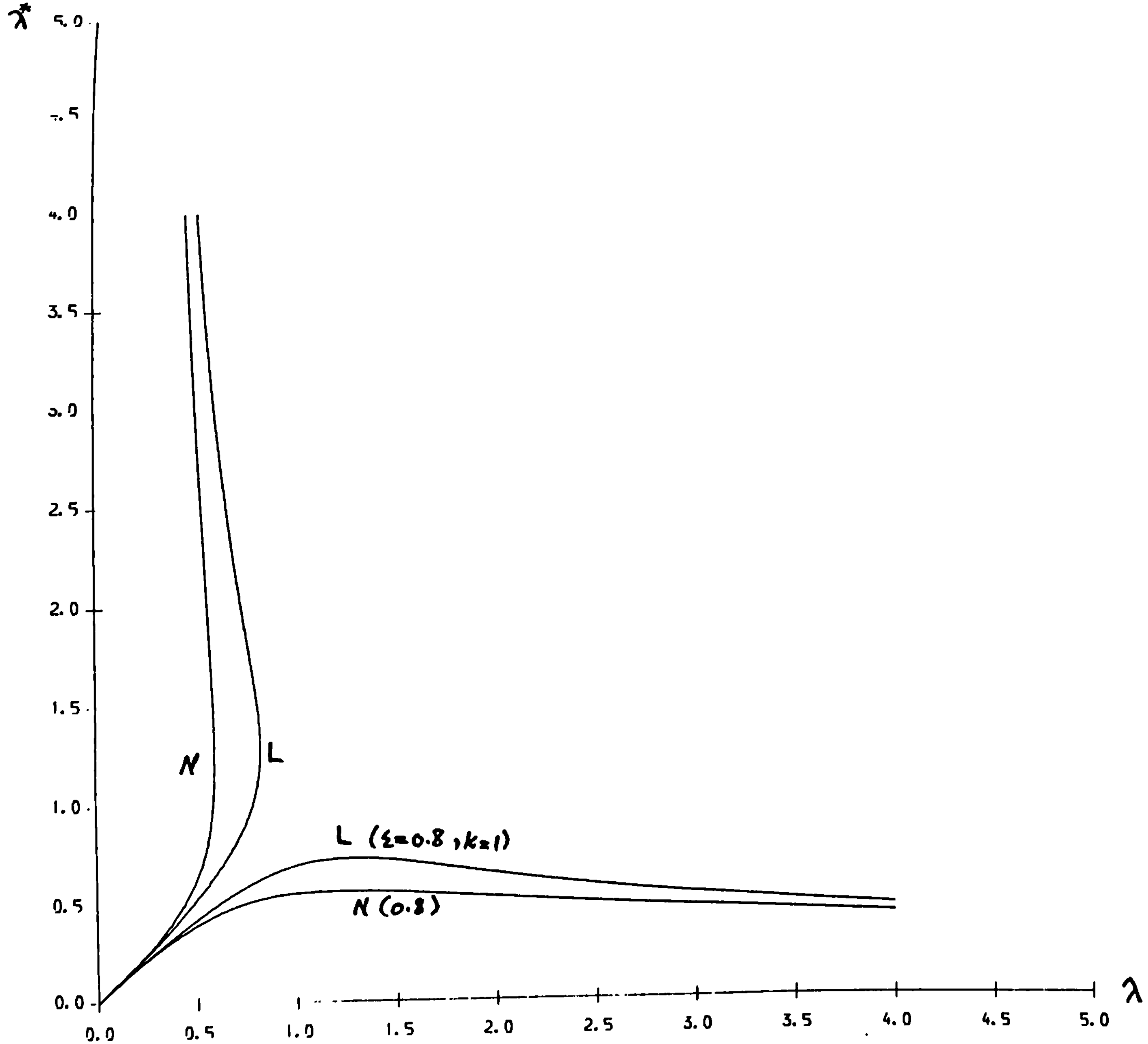


Figure 6.4

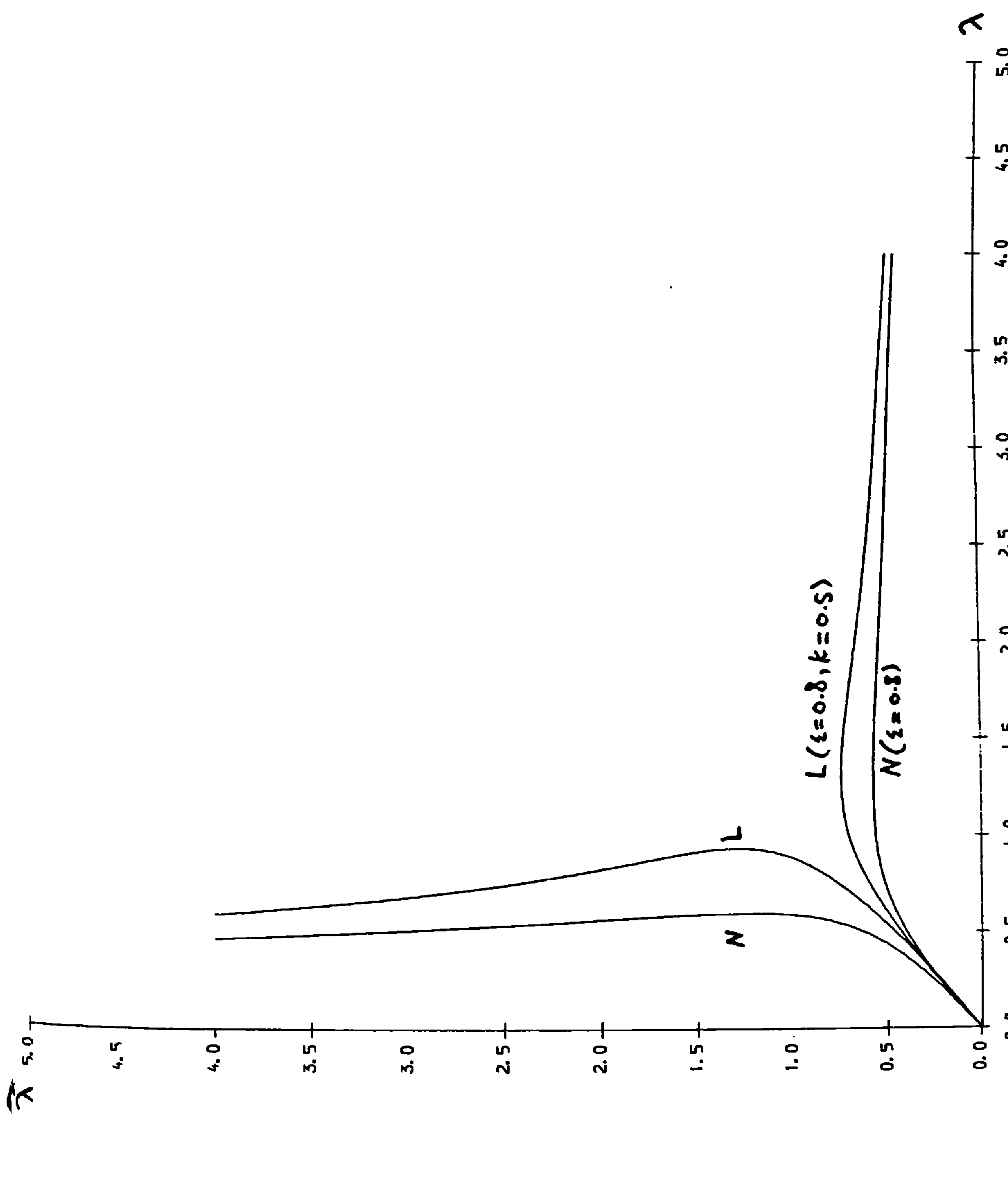


Figure 6.5

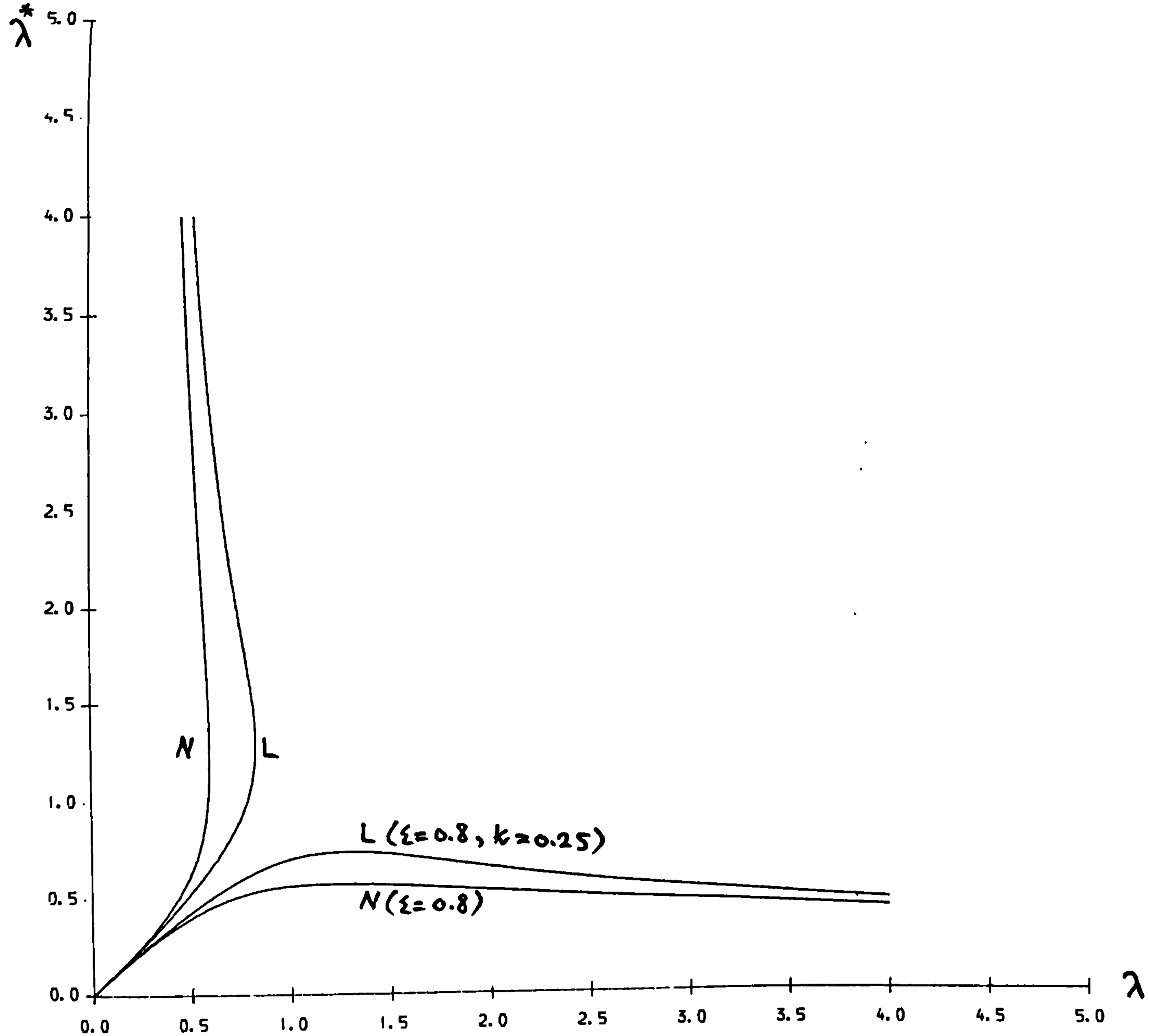

λ*
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
N
L
L (ξ=0.8, k=0.25)
N(ξ=0.8)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
λ

Figure 6.6

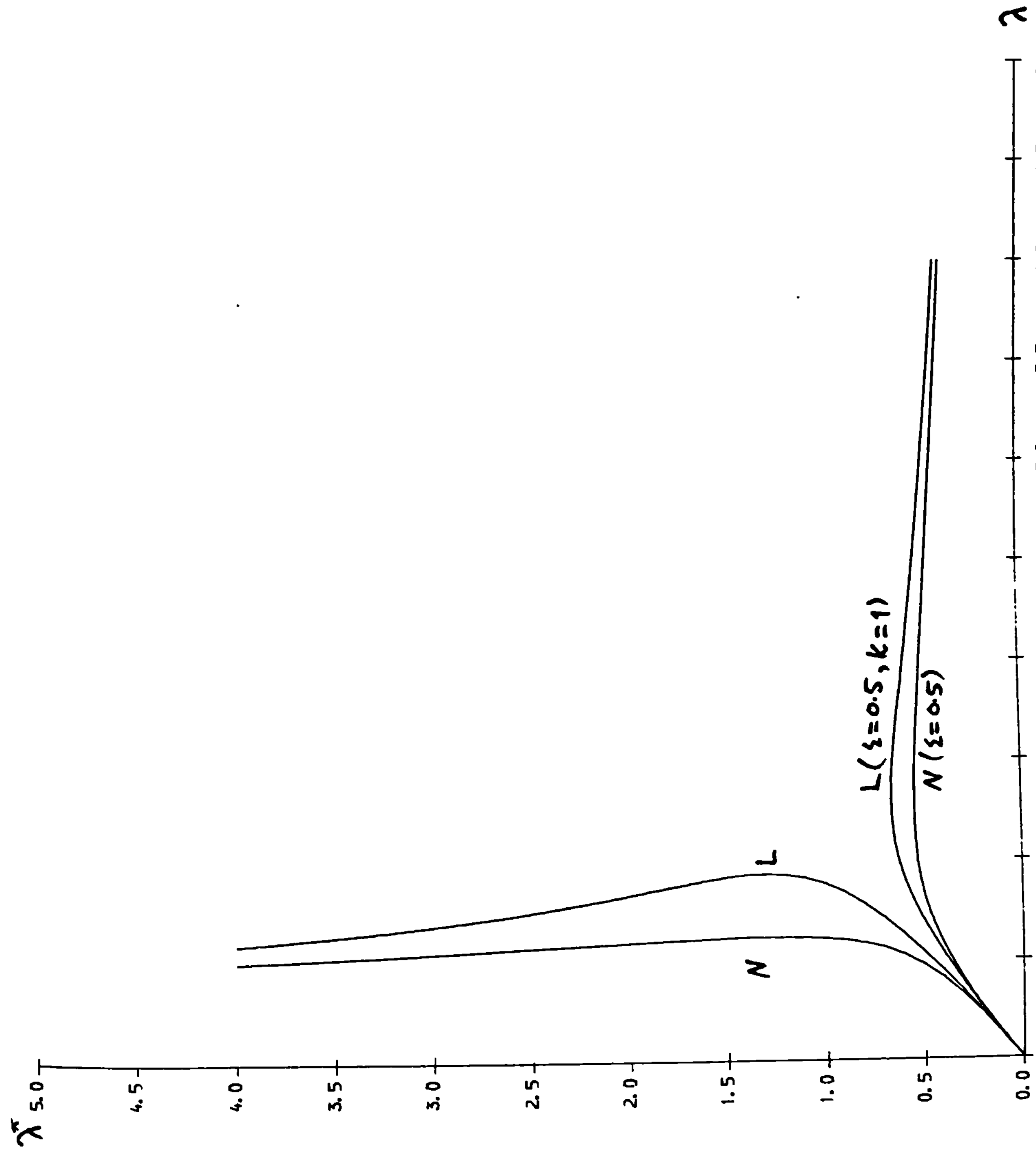


Figure 6.7

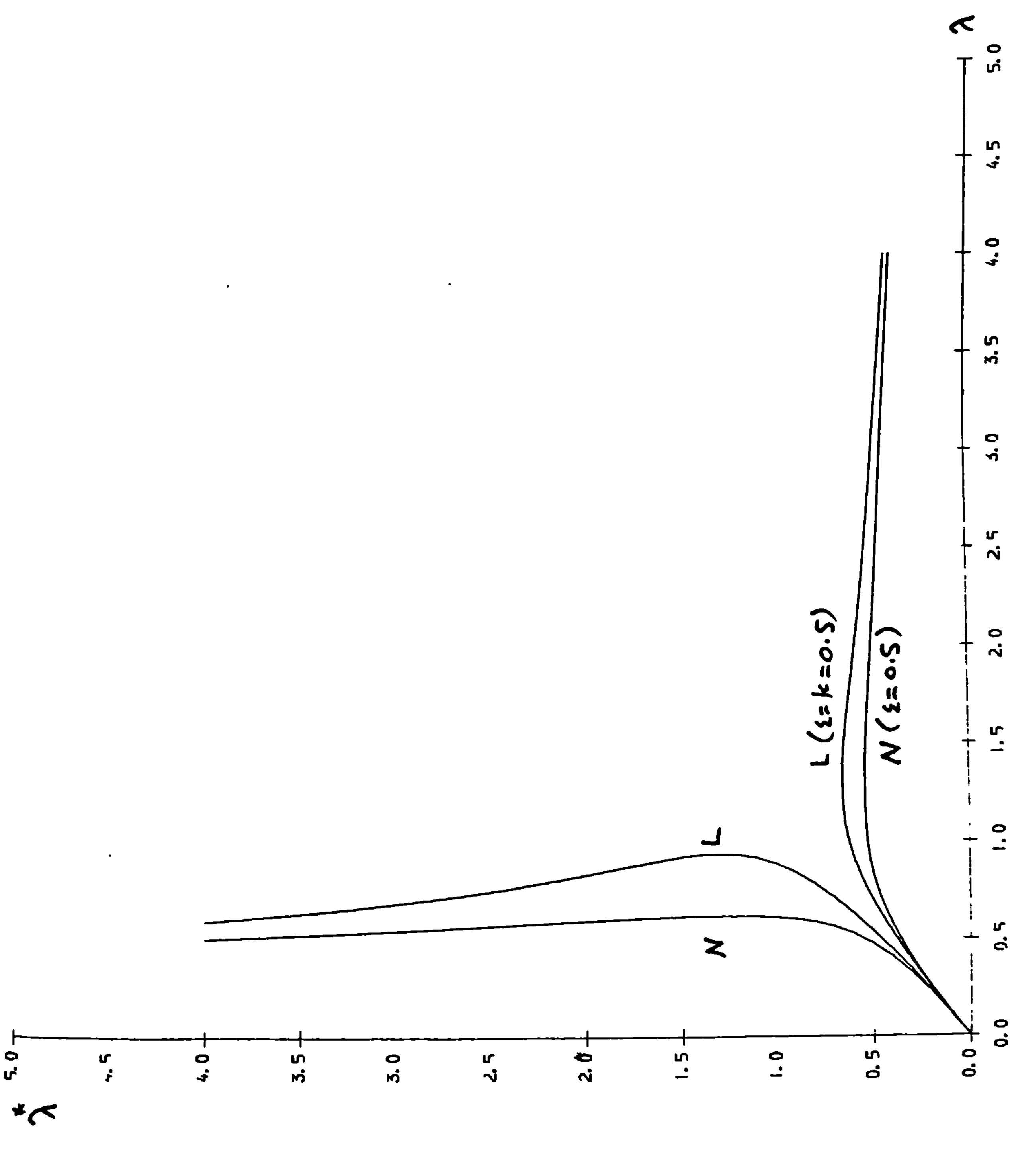


Figure 6.8

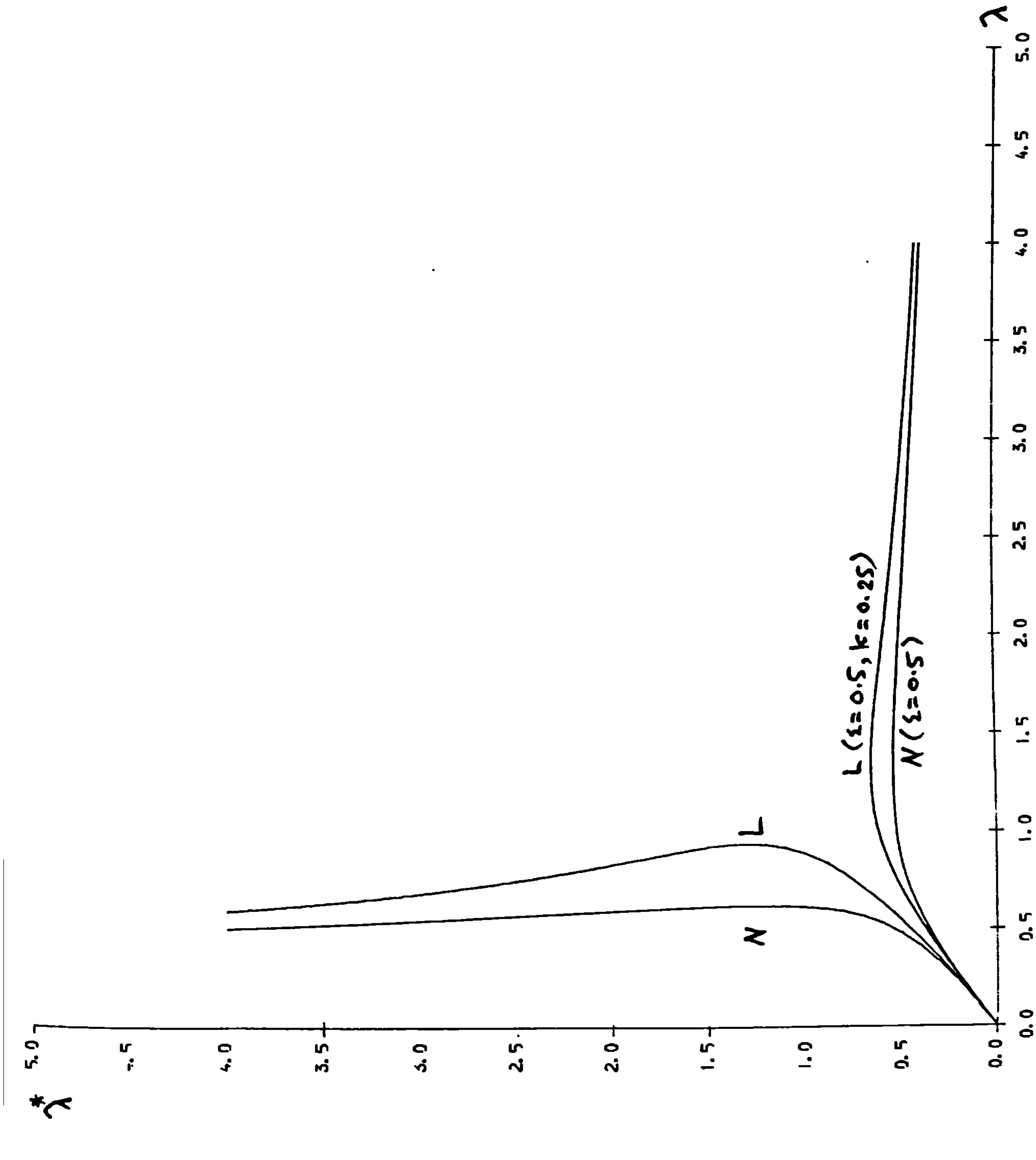
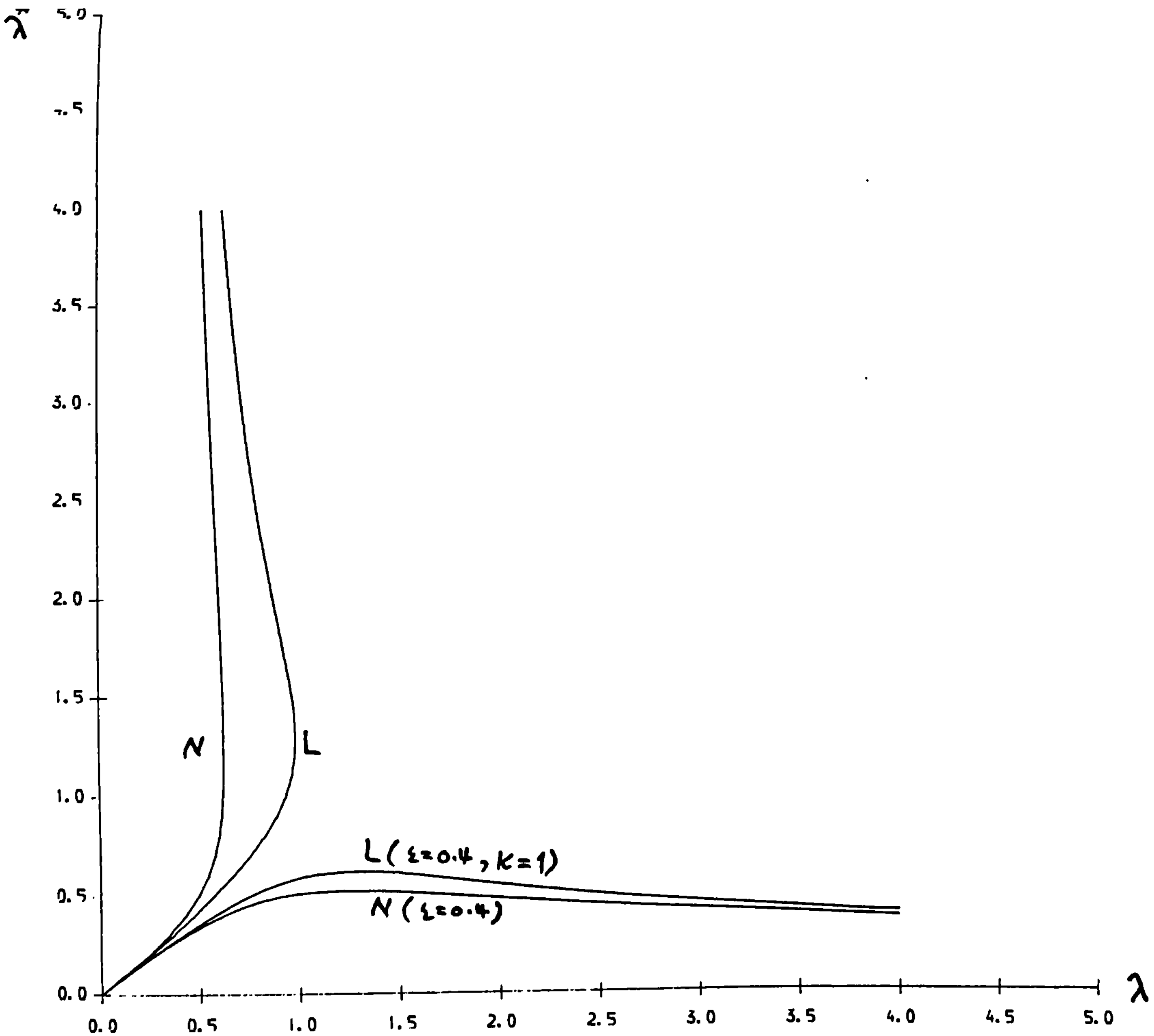


Figure 6.9



$\tilde{\lambda}$
λ
N
L
L(ε=0.4, k=1)
N(ε=0.4)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0

Figure 6.10

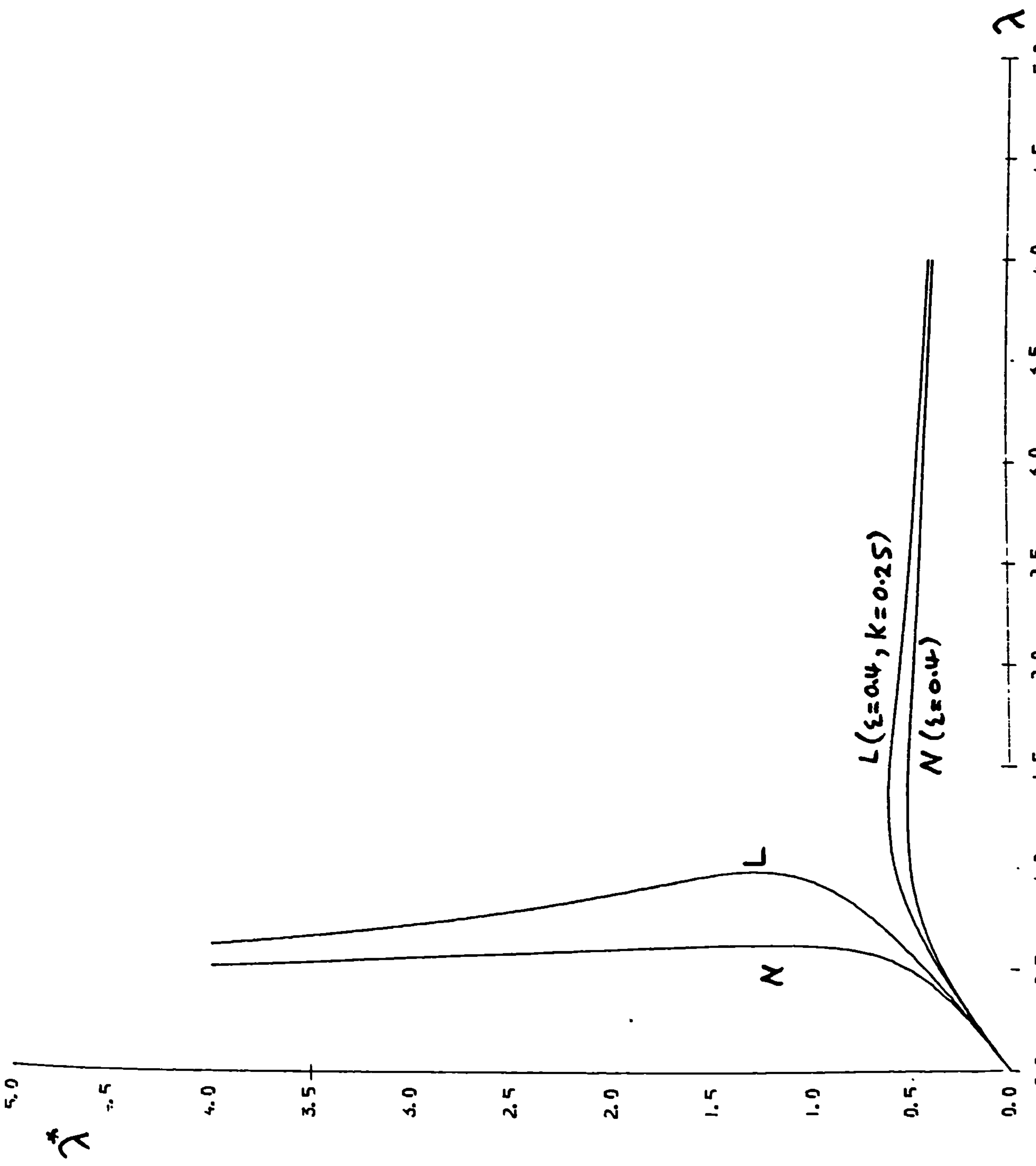


Figure 6.12

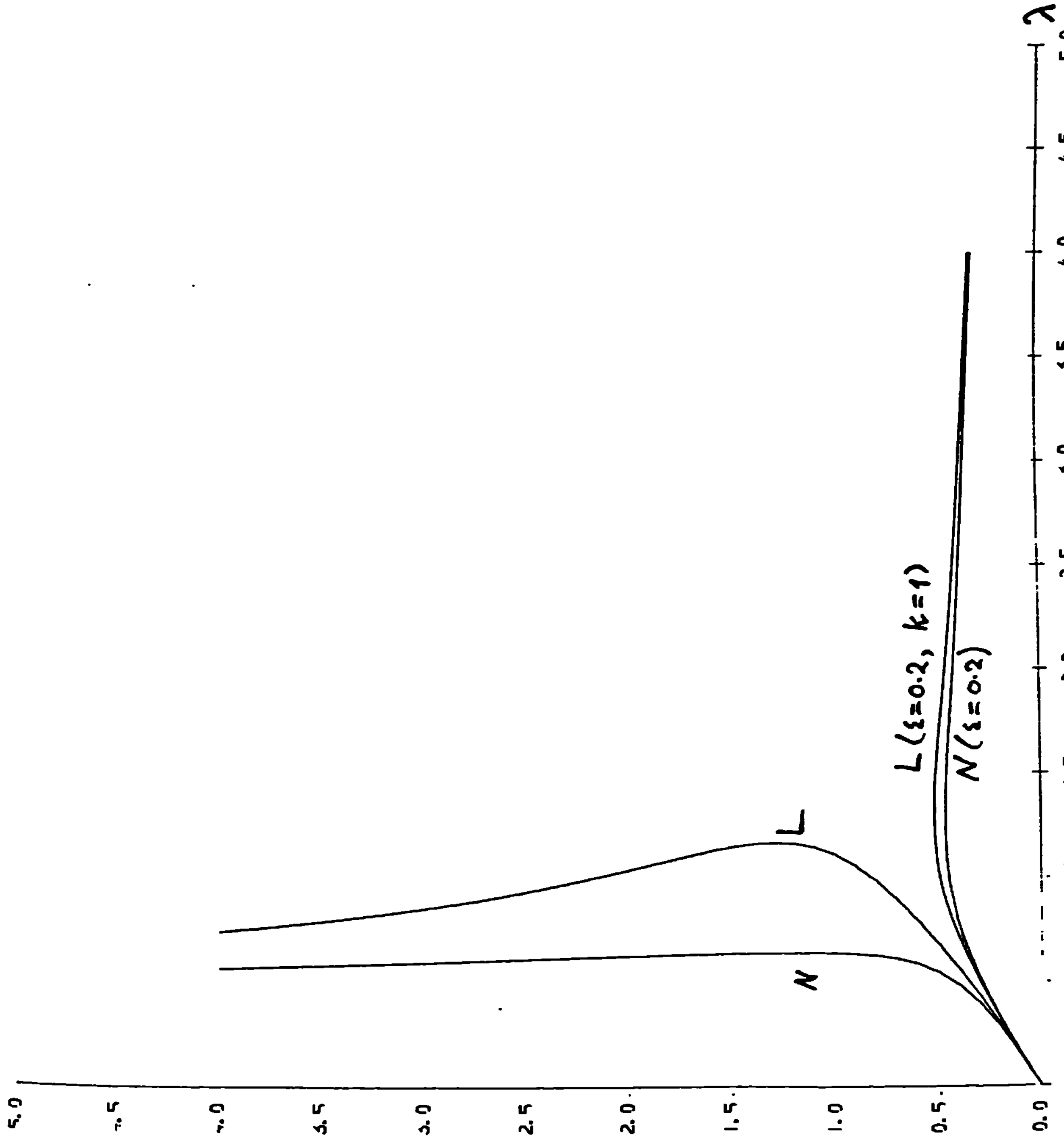


Figure 6.13

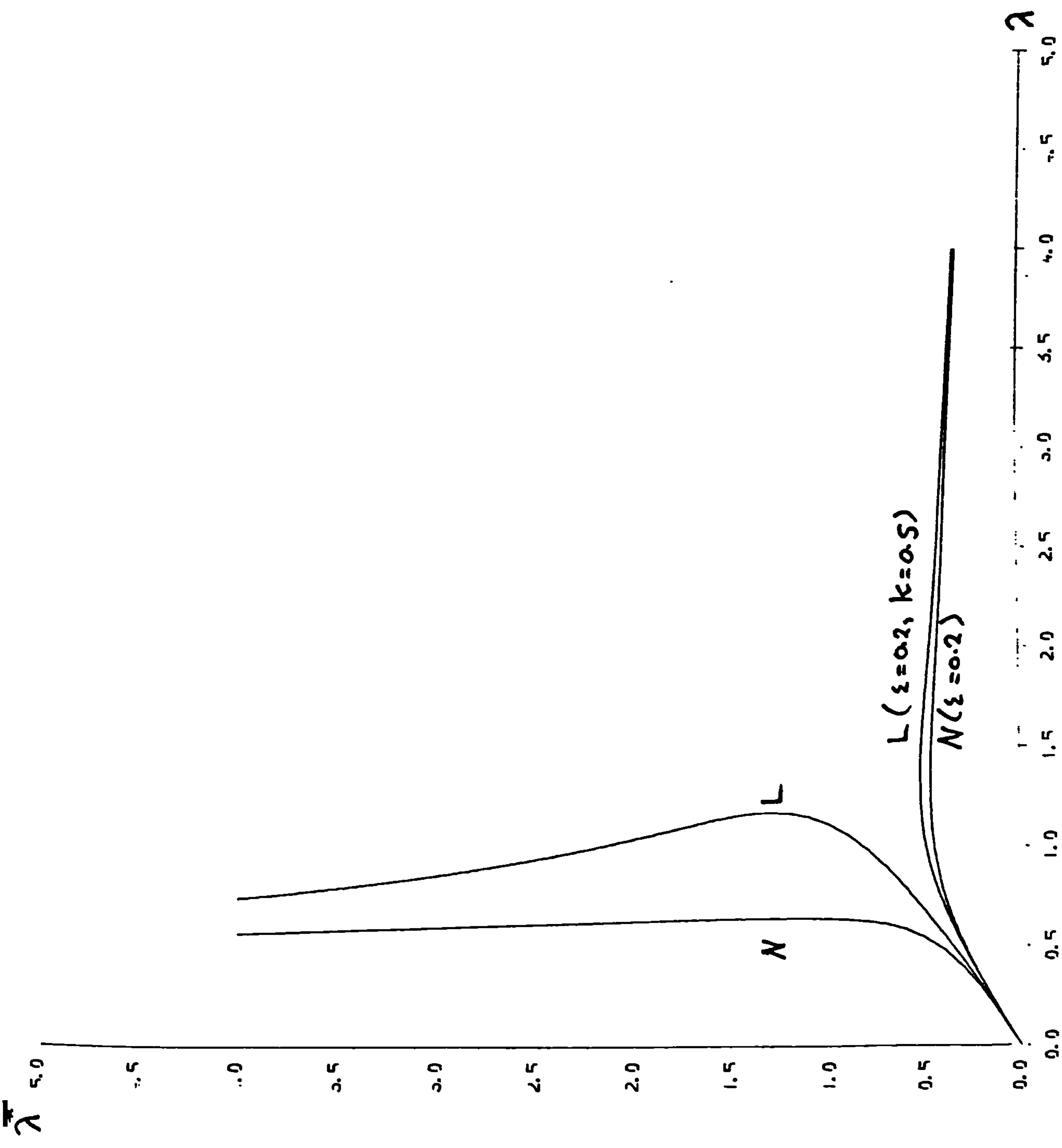


Figure 6.14

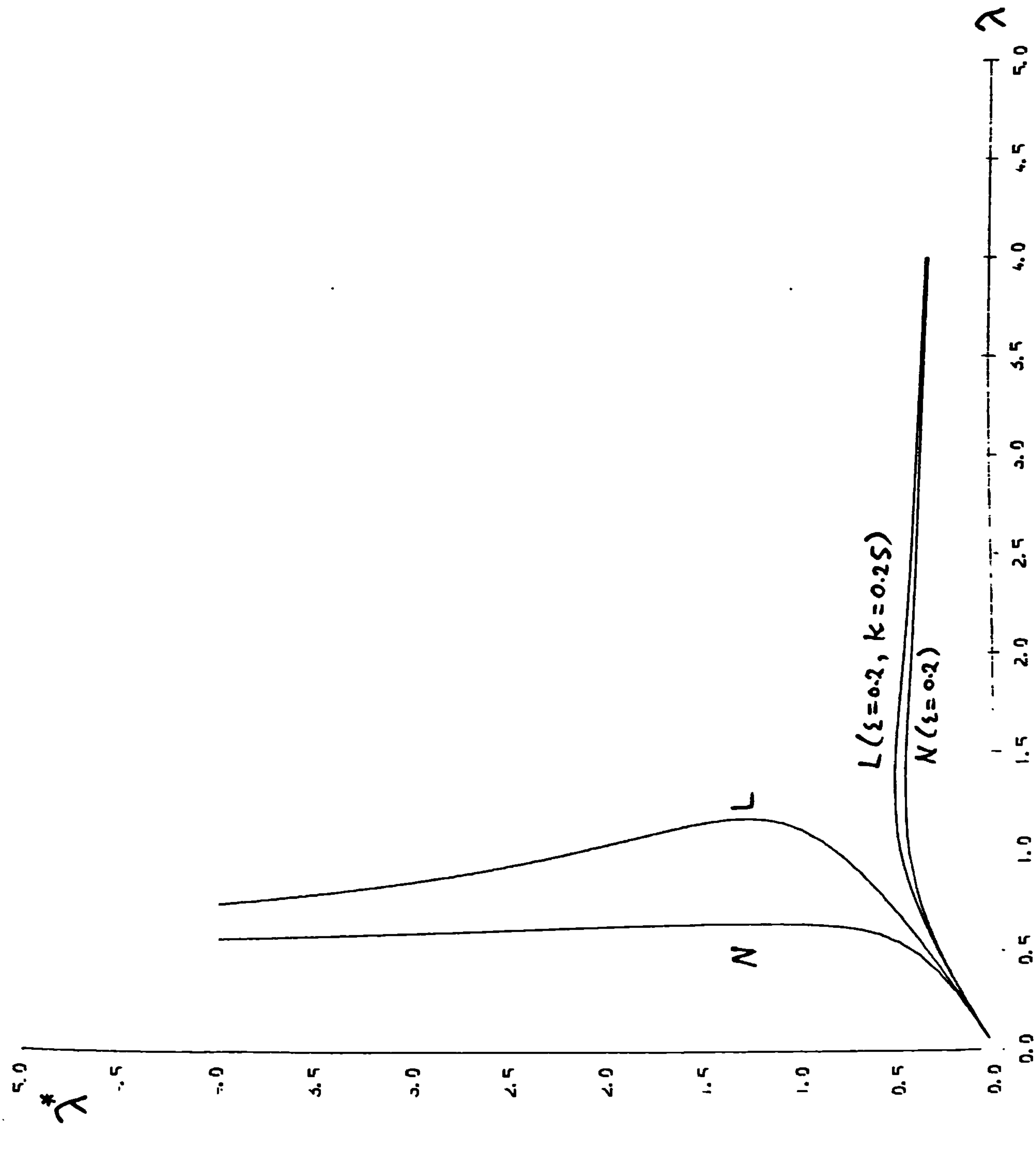


Figure 6.15

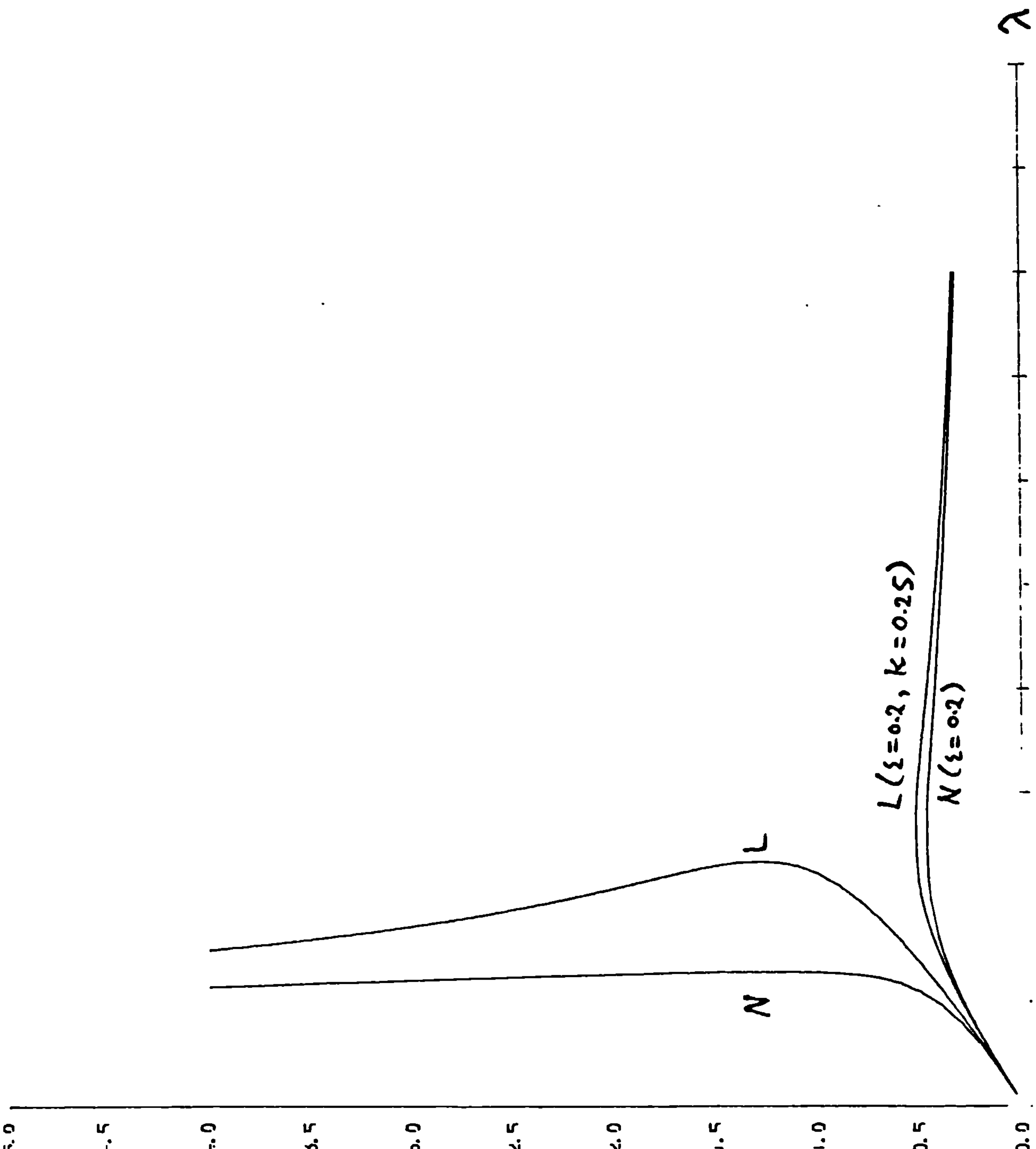


Figure 6.15

6.2.8 The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$

Before we consider the corresponding result of the classical linear theory, we note from Section 6.2.5 that for the deformation in which $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$,

we get

$$A_{01111} = A_{02222}, \qquad \alpha = \gamma,$$

$$\gamma = {}^1/_2\ (A_{01111} - A_{01122} + \lambda_2 \frac{\partial W}{\partial \lambda_2})$$

and hence

$$\beta = \alpha = \gamma.$$

Similarly, for $\lambda_1^* = \lambda_2^*$

$$\beta^* = \alpha^* = \gamma^*.$$

On use of the above notations in equation (6.1.32), the secular equation becomes

$$\gamma^2\left[\frac{\rho c^2}{\gamma} - \left[4 - \frac{\rho c^2}{\gamma}\right]\left[1 - \frac{\rho c^2}{\gamma}\right]^{\frac{1}{2}}\right]$$

$$- \gamma\,\gamma^* \left[2\left\{\left[1 - \frac{\rho c^2}{\gamma}\right]^{\frac{1}{2}} + \left[1 - \frac{\rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}} - \left[1 - \frac{\rho c^2}{\gamma}\right]^{\frac{1}{2}}\left[1 - \frac{\rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}} - 1\right\}\right.$$

$$- \left\{2 - \frac{\rho c^2}{\gamma} + 2\left[1 - \frac{\rho c^2}{\gamma}\right]^{\frac{1}{2}}\right\}\left\{1 - \frac{\rho^* c^2}{\gamma^*} + 2\left[1 - \frac{\rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right.$$

$$\left.\left\{\left[1 - \frac{\rho c^2}{\gamma}\right]^{\frac{1}{2}} + \left[1 - \frac{\rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right]$$

$$+ \gamma^{*2}\left[\frac{\rho^* c^2}{\gamma^*} - \left[4 - \frac{\rho^* c^2}{\gamma^*}\right]\left[1 - \frac{\rho^* c^2}{\gamma^*}\right]^{\frac{1}{2}}\right] = 0. \qquad (6.2.71)$$

On setting $\xi = \dfrac{\rho c^2}{\gamma}$, $\alpha = \dfrac{\gamma}{\gamma^*}$, $\beta = \dfrac{\rho}{\rho^*}$ and hence $\dfrac{\rho^* c^2}{\gamma^*} = \dfrac{\beta\xi}{\alpha}$, equation (6.2.71) is written as

$$(4 - \xi)(1 - \xi)^{\frac{1}{2}} - \xi$$

$$- \alpha \left[2\left\{(1 - \xi)^{\frac{1}{2}} + \left[1 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}} - (1- \xi)^{\frac{1}{2}}\left[1 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}} - 1\right\}\right.$$

$$\left. - 2\left\{2 - \xi + 2(1 - \xi)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{2 - \frac{\beta\xi}{\alpha} + 2\left[1 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{(1 - \xi)^{\frac{1}{2}} + \left[1 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}}\right\}\right]$$

$$+ \alpha^2 \left[\left[4 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}}\left[1 - \frac{\beta\xi}{\alpha}\right]^{\frac{1}{2}} - \frac{\beta\xi}{\alpha}\right] = 0. \qquad (6.2.72)$$

This equation is the secular equation for Stoneley waves for the special deformation $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$. It is worth noting that this is the same equation as given in Section 3.3 but for different values of α, β and ξ so the solution in Fig. 3.1 is the same as for this equation.

In this case, we get $\alpha = \gamma = \mu$ so equation (6.1.32) becomes

$$(4 - \frac{\rho c^2}{\mu})(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} - \frac{\rho c^2}{\mu}$$

$$- \frac{\mu^*}{\mu}\left[2\left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} + (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - (1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}}(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - 1\right\}\right.$$

$$- \left\{2 - \frac{\rho c^2}{\mu} + 2(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{2 - \frac{\rho^* c^2}{\mu^*} + 2(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\left.\left\{(1 - \frac{\rho c^2}{\mu})^{\frac{1}{2}} (1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}}\right\}\right]$$

$$+ (\frac{\mu^*}{\mu})^2\left[(4 - \frac{\rho^* c^2}{\mu^*})(1 - \frac{\rho^* c^2}{\mu^*})^{\frac{1}{2}} - \frac{\rho^* c^2}{\mu^*}\right] = 0. \qquad (6.2.73)$$

This equation corresponds to the that in linear theory, as discussed in Section 3.3.

6.3 Propagation in a general direction

In this section we wish to obtain the secular equation for Stoneley waves propagating in a general direction in the (x_1,x_3)-plane, in which the direction of the propagation has the direction $(\cos\theta, \sin\theta)$.

As in Section 4.7, for an incompressible material the incremental equations of motion are given by

$$A_{0jilk}\, v_{k,jl} - \dot{p}_{,i} = \rho \ddot{v}_i,$$

$$v_{i,i} = 0,$$

and similarly (6.3.1)

$$A^*_{0jilk}\, v^*_{k,jl} - \dot{p}^*_{,i} = \rho^* \ddot{v}^*_i,$$

$$v^*_{i,i} = 0.$$

Also, assuming v_i, $\dot{p}$, v^*_i and $\dot{p}^*$ are given by

$$\underline{v} = \underline{\psi}(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)},$$

$$\dot{p} = \varphi(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)},$$

$$i \in \{1,2,3\} \qquad (6.3.2)$$

$$\underline{v}^* = \underline{\psi}^*(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)},$$

$$\dot{p}^* = \varphi^*(x_2)\, e^{i(\omega t - k\cos\theta x_1 - k\sin\theta x_3)}.$$

Equations (6.3.1) lead to

$$A_{0j_1l_1}\, v_{1,jl} + A_{0j_1l_2}\, v_{2,jl} + A_{0j_1l_3}\, v_{3,jl} - \dot{p}_{,1} = \rho\ddot{v}_1,$$

$$A_{0j_2l_1}\, v_{1,jl} + A_{0j_2l_2}\, v_{2,jl} + A_{0j_2l_3}\, v_{3,jl} - \dot{p}_{,2} = \rho\ddot{v}_2, \quad (6.3.3)$$

$$A_{0j_3l_1}\, v_{1,jl} + A_{0j_3l_2}\, v_{2,jl} + A_{0j_3l_3}\, v_{3,jl} - \dot{p}_{,3} = \rho\ddot{v}_3,$$

Also, suppose that

$$\psi_1 = A\, e^{-skx_2},$$

$$\psi_2 = B\, e^{-skx_2},$$

$$\psi_3 = C\, e^{-skx_2}, \quad (6.3.4)$$

and

$$\varphi = D\, e^{-skx_2}.$$

By differentiating $(6.3.2)_{1,2}$ and subsituting in (6.3.3) and using equations (6.3.4), we shall get the same cubic equation for s^2 given by (4.7.10) namely

$$[\rho c^2 + s^2 A_{02121} - \cos^2\theta(A_{01122} + A_{02112} - A_{01111}) - \sin^2\theta\ A_{03131}]$$

$$[s^2\{\rho c^2 + s^2 A_{02323} - \cos^2\theta\ A_{01313} + \sin^2\theta(A_{03322} - A_{03333})\}$$

$$-\sin^2\theta\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\ A_{01212}$$

$$- \sin^2\theta\ A_{03232}\}]$$

$$- \sin\theta\cos\theta\ (A_{01122} + A_{01133} - A_{03113})[\sin\theta\cos\theta\ \{\rho c^2$$

$$+ s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\ A_{01212} - \sin^2\theta\ A_{03232}\}$$

$$- s^2\sin\theta\cos\theta\ (A_{02332} + A_{03322} - A_{01331} - A_{03311})]$$

$$+ \cos\theta\ [\cos\theta\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211}) - \cos^2\theta\ A_{01212}$$

$$- \sin^2\theta\ A_{03232}\}\{\rho c^2 + s^2 A_{02323} - \cos^2\theta\ A_{01313} + \sin^2\theta(A_{02332}$$

$$+ A_{03322} - A_{03333})\} - \sin\theta k\{\rho c^2 + s^2(A_{02222} - A_{01221} - A_{02211})$$

$$- \cos^2\theta\ A_{01212} - \sin^2\theta\ A_{03232}\}\{\sin\theta\cos\theta\ (A_{02332} + A_{03322}$$

$$- A_{01331} - A_{03311})\}] = 0. \qquad (6.3.5)$$

Let s_1, s_2, s_3 be the values of s with positive real part and write the solution as given in Section 4.7, namely

$$\begin{aligned}
\psi_1 &= A_1\ e^{s_1 kx_1} + A_2\ e^{s_2 kx_2} + A_3\ e^{s_3 kx_3}, \\
\psi_2 &= B_1\ e^{s_1 kx_1} + B_2\ e^{s_2 kx_2} + B_3\ e^{s_3 kx_3}, \\
\psi_3 &= C_1\ e^{s_1 kx_1} + C_2\ e^{s_2 kx_2} + C_3\ e^{s_3 kx_3}, \\
\varphi &= D_1\ e^{s_1 kx_1} + D_2\ e^{s_2 kx_2} + D_3\ e^{s_3 kx_3}.
\end{aligned} \qquad (6.3.6)$$

$$\psi_1^* = A_1^* e^{-s_1^* kx_1} + A_2^* e^{-s_2^* kx_2} + A_3^* e^{-s_3^* kx_3},$$

$$\psi_2^* = B_1^* e^{-s_1^* kx_1} + B_2^* e^{-s_2^* kx_2} + B_3^* e^{-s_3^* kx_3}, \tag{6.3.7}$$

$$\psi_3^* = C_1^* e^{-s_1^* kx_1} + C_2^* e^{-s_2^* kx_2} + C_3^* e^{-s_3^* kx_3},$$

$$\varphi^* = D_1^* e^{-s_1^* kx_1} + D_2^* e^{-s_2^* kx_2} + D_3^* e^{-s_3^* kx_3}.$$

To obtain the speed of Stoneley wave propagation in any direction for the general case we must deduce from (4.7.8) the ratio A_i: B_i: C_i: D_i and A_i^*: B_i^*: C_i^*: D_i^*, $i \in \{1,2,3\}$ and from the boundary conditions obtain the ratio A_1: A_2: A_3: A_1^*: A_2^*: A_3^* and the secular equation. Also, because of the cumbersome algebra involved we omit details of the general case here, as in Rayleigh waves we shall concentrate on the application to the neo-Hookean material.

6.3.1 Propagation in a general direction for a neo-Hookean material

For a neo-Hookean material, equation (6.3.5) becomes

$$(\rho c^2 + \mu \lambda_2^2 s^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta)^2 (s^2 - 1) = 0, \tag{6.3.8}$$

which is a cubic equation for s^2 and let s_1 and s_2 are the values of s^2 with positive real part, as in (4.7.14) s_1 and s_2 are given by

$$s_1^2 = 1 \quad \text{and} \quad s_2^2 = (\mu \lambda_1^2 \cos^2\theta + \mu \lambda_3^2 \sin^2\theta - \rho c^2)/\mu \lambda_2^2.$$

and similarly (6.3.9)

$$s_1^{*2} = 1 \quad \text{and} \quad s_2^{*2} = (\mu^* \lambda_1^{*2} \cos^2\theta + \mu^* \lambda_3^{*2} \sin^2\theta - \rho^* c^2)/\mu^* \lambda_2^{*2},$$

these require

$$0 < \rho c^2/\mu < \lambda_1^2 \cos^2\theta + \lambda_3^2 \sin^2\theta,$$

and similarly (6.3.10)

$$0 < \rho^* c^2/\mu^* < \lambda_1^{*2} \cos^2\theta + \lambda_3^{*2} \sin^2\theta.$$

Also, as in Section 4.7, s_2 and s_2^* are repeated roots, that is $s_2 = s_3$ and $s_2^* = s_3^*$, so equations (6.3.6) and (6.3.7)become

$$\psi_1 = A_1\, e^{s_1 k x_2} + (A_2\, x_2 + A_3)\, e^{s_2 k x_2}\ ,$$

$$\psi_2 = B_1\, e^{s_1 k x_2} + (B_2\, x_2 + B_3)\, e^{s_2 k x_2}\ ,$$

$$\psi_3 = C_1\, e^{s_1 k x_2} + (C_2\, x_2 + C_3)\, e^{s_2 k x_2}\ ,$$

$$\varphi = D_1\, e^{s_1 k x_2} + (D_2\, x_2 + D_3)\, e^{s_2 k x_2}\ ,$$

and (6.3.11)

$$\psi_1^* = A_1^*\, e^{-s_1^* k x_2} + (A_2^*\, x_2 + A_3^*)\, e^{-s_2^* k x_2},$$

$$\psi_2^* = B_1^*\, e^{-s_1^* k x_2} + (B_2^*\, x_2 + B_3^*)\, e^{-s_2^* k x_2},$$

$$\psi_3^* = C_1^*\, e^{-s_1^* k x_2} + (C_2^*\, x_2 + C_3^*)\, e^{-s_2^* k x_2},$$

$$\varphi^* = D_1^*\, e^{-s_1 k x_2} + (D_2^*\, x_2 + D_3^*)\, e^{-s_2 k x_2}.$$

Next, let the incremental boundary conditions for propagation in any direction be

$$\underline{v} = \underline{v}^*, \qquad \dot{s}_{02i} = \dot{s}_{02i}^* \qquad \text{on } x_2 = 0.$$

That is,

$$v_1 - v_1^* = 0,$$

$$v_2 - v_2^* = 0,$$

$$v_3 - v_3^* = 0,$$

$$\text{on } x_2 = 0$$

$$\left[A_{02112} + p\right] \frac{\partial v_2}{\partial x_1} + A_{02121} \frac{\partial v_1}{\partial x_2} - \left[A^*_{02112} + p^*\right] \frac{\partial v_2^*}{\partial x_1} - A^*_{02121} \frac{\partial v_1^*}{\partial x_2} = 0,$$

$$A_{02211} \frac{\partial v_1}{\partial x_1} + \left[A_{02222} + p\right] \frac{\partial v_2}{\partial x_2} + A_{02233} \frac{\partial v_3}{\partial x_3} - \dot{p}$$

$$- A^*_{02211} \frac{\partial v_1^*}{\partial x_1} - \left[A^*_{02222} + p^*\right] \frac{\partial v_2^*}{\partial x_2} - A^*_{02233} \frac{\partial v_3^*}{\partial x_3} + \dot{p}^* = 0,$$

$$A_{02323} \frac{\partial v_3}{\partial x_2} + \left[A_{02323} + p\right] \frac{\partial v_2}{\partial x_3} - A^*_{02323} \frac{\partial v_3^*}{\partial x_2} - \left[A^*_{02332} + p^*\right] \frac{\partial v_2^*}{\partial x_3} = 0,$$

since $A_{02112} + p = A_{02121} - \sigma_2$, $A_{02332} + p = A_{02323} - \sigma_2$, and similarly $A^*_{02112} + p^* = A^*_{02121} - \sigma_2^*$, $A^*_{02332} + p^* = A^*_{02323} - \sigma_2^*$, and for the special case $\sigma_2^* = \sigma_2 = 0$, the above boundary conditions become

$$v_1 - v_1^* = 0,$$

$$v_2 - v_2^* = 0,$$

$$v_3 - v_3^* = 0,$$

$$A_{02121}\left[\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right] - A^*_{02121}\left[\frac{\partial v_1^*}{\partial x_2} + \frac{\partial v_2^*}{\partial x_1}\right] = 0, \quad \text{on } x_2 = 0 \qquad (6.3.12)$$

$$A_{02211}\frac{\partial v_1}{\partial x_1} + \left[A_{02222} + p\right]\frac{\partial v_2}{\partial x_2} + A_{02233}\frac{\partial v_3}{\partial x_3} - \dot{p}$$

$$- A^*_{02211}\frac{\partial v_1^*}{\partial x_1} - \left[A^*_{02222} + p^*\right]\frac{\partial v_2^*}{\partial x_2} - A^*_{02233}\frac{\partial v_3^*}{\partial x_3} + \dot{p}^* = 0,$$

$$A_{02323}\left[\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right] - A^*_{02323}\left[\frac{\partial v_2^*}{\partial x_3} + \frac{\partial v_3^*}{\partial x_2}\right] = 0.$$

For a neo-Hookean material the boundary conditions (6.3.12) reduce to

$$v_1 - v_2^* = 0,$$

$$v_2 - v_3^* = 0,$$

$$v_3 - v_3^* = 0,$$

$$\mu\,\lambda_2^2\left[\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right] - \mu^*\lambda_2^{*2}\left[\frac{\partial v_1^*}{\partial x_2} + \frac{\partial v_2^*}{\partial x_1}\right] = 0, \quad \text{on } x_2 = 0 \qquad (6.3.13)$$

$$2\mu\,\lambda_2^2\,\frac{\partial v_2}{\partial x_2} - \dot{p} - 2\mu^*\,\lambda_2^{*2}\,\frac{\partial v_2^*}{\partial x} + \dot{p}^* = 0,$$

$$\mu\,\lambda_2^2\left[\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right] - \mu^*\lambda_2^{*2}\left[\frac{\partial v_2^*}{\partial x_3} + \frac{\partial v_3^*}{\partial x_2}\right] = 0,$$

since $\sigma_2 = 0 \Rightarrow p = \mu\lambda_2^2$ and $\sigma_2^* = 0 \Rightarrow p^* = \mu^*\,\lambda_2^{*2}$.

By differentiating (6.3.2) and substituting in (6.3.13), we have

$$\psi_1 - \psi_1^* = 0,$$

$$\psi_2 - \psi_2^* = 0,$$

$$\psi_3 - \psi_3^* = 0,$$

$$\text{on } x_2 = 0 \qquad (6.3.14)$$

$$\mu \lambda_2^2 \psi'_1 - ik \cos\theta \, \mu \lambda_2^2 \psi_2 - \mu^* \lambda_2^{*2} \psi_1^{*'} + ik \cos\theta \, \mu^* \lambda_2^{*2} \psi_2^* = 0,$$

$$\mu \lambda_2^2 \psi'_2 - \varphi - 2\mu^* \lambda_2^{*2} \psi_2^{*'} + \varphi^* = 0,$$

$$ik \sin\theta \, \mu \lambda_2^2 \psi_2 + \mu \lambda_2^2 \psi'_3 - ik \sin\theta \, \mu^* \lambda_2^{*2} \psi_2^* + \mu^* \lambda_2^{*2} \psi_3^{*'} = 0.$$

Next, on use of (6.3.11) in (6.4.14), the above boundary conditions become

$$A_1 + A_3 - A_1^* - A_3^* = 0,$$

$$B_1 + B_3 - B_1^* - B_3^* = 0,$$

$$C_1 + C_3 - C_1^* - C_3^* = 0,$$

$$\mu\,\lambda_2^2\, s_1\, k\, A_1 + \mu\,\lambda_2^2\, A_2 + \mu\,\lambda_2^2\, s_2\, k\, A_3 - ik\cos\theta\,\mu\,\lambda_2^2\, B_1$$

$$- ik\cos\theta\,\mu\,\lambda_2^2\, B_3 + \mu^*\,\lambda_2^{*2}\, s_1^*\, k\, A_1^* - \mu^*\,\lambda_2^{*2}\, A_2^* + \mu^*\,\lambda_2^{*2}\, s_2^*\, k\, A_3^*$$

$$+ ik\cos\theta\,\mu^*\,\lambda_2^{*2}\, B_1^* + ik\cos\theta\,\mu^*\,\lambda_2^{*2}\, B_3^* = 0, \tag{6.3.15}$$

$$2\mu\,\lambda_2^2\, s_1\, k\, B_1 + 2\mu\,\lambda_2^2\, B_2 + 2\mu\,\lambda_2^2\, s_2\, k\, B_3 - D_1 - D_3 + 2\mu^*\,\lambda_2^{*2}\, s_1^*\, k\, B_1^*$$

$$- 2\mu^*\,\lambda_2^{*2}\, B_2^* + 2\mu^*\,\lambda_2^{*2}\, s_2^*\, k\, B_3^* + D_1^* + D_3^* = 0,$$

$$ik\sin\theta\,\mu\,\lambda_2^2\, B_1 + ik\sin\theta\,\mu\,\lambda_2^2\, B_3 - \mu\,\lambda_2^2\, s_1\, k\, C_1 - \mu\,\lambda_2^2\, C_2$$

$$- \mu\,\lambda_2^2\, s_2\, k\, C_3 - ik\sin\theta\,\mu^*\,\lambda_2^{*2}\, B_1^* - ik\sin\theta\,\mu^*\,\lambda_2^{*2}\, B_3^*$$

$$- \mu^*\,\lambda_2^{*2}\, s_1^*\, k\, C_1^* + \mu^*\,\lambda_2^{*2}\, C_2^* - \mu^*\,\lambda_2^{*2}\, s_2^*\, k\, C_3^* = 0.$$

As for Rayleigh waves, the ratios A_i: B_i: C_i: D_i for A_1: A_2: A_3 are given in Section 4.7 namely,

for $s = s_1$,

$$\frac{D_1}{A_1} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\, s_1^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\cos\theta}$$

$$\frac{D_1}{C_1} = \frac{-(\rho c^2 + \mu\,\lambda_2^2\, s_1^2 - \mu\,\lambda_1^2\cos^2\theta - \mu\,\lambda_3^2\sin^2\theta)k}{i\sin\theta}$$

$$\frac{C_1}{A_1} = \tan\theta.$$

Similarly, for $s^* = s_1^*$,

$$\frac{D_1^*}{A_1^*} = \frac{-(\rho^* c^2 + \mu^* \lambda_2^{*2} s_1^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta)k}{i \cos\theta}$$

$$\frac{D_1^*}{C_1^*} = \frac{-(\rho^* c^2 + \mu^* \lambda_2^{*2} s_1^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta)k}{i \sin\theta}$$

$$\frac{C_1^*}{A_1^*} = \tan\theta .$$

Also, as in Section 4.7 for the case $s = s_2$, we have $A_2 = B_2 = C_2 = D_2 = 0$ and similarly for $s = s_2^*$ we have $A_2^* = B_2^* = C_2^* = D_2^* = 0$, and the ratios for this case are as given in Section 4.7:

$$\frac{D_3}{A_3} = \frac{-(\rho c^2 + \mu \lambda_2^2 s_2^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta)k}{i \cos\theta}$$

$$\frac{D_3}{C_3} = \frac{-(\rho c^2 + \mu \lambda_2^2 s_2^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta)k}{i \sin\theta}$$

and

$$\frac{C_3}{A_3} = \tan\theta .$$

Similarly, for $s^* = s_2^*$

$$\frac{D_3^*}{A_3^*} = \frac{-(\rho^* c^2 + \mu^* \lambda_2^{*2} s_2^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta)k}{i \cos\theta}$$

$$\frac{D_3^*}{C_3^*} = \frac{-(\rho^* c^2 + \mu^* \lambda_2^{*2} s_2^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta)k}{i \sin\theta}$$

and

$$\frac{C_3^*}{A_3^*} = \tan\theta .$$

Also, from Section 4.7, we have

$$is_1 B_1 = \cos\theta A_1 + \sin\theta C_1 \qquad \text{for } s = s_1,$$

$$is_2 B_3 = \cos\theta A_3 + \sin\theta C_3 \qquad \text{for } s = s_2 = s_3.$$

Similarly,

$$i s_1^* B_1^* = \cos\theta A_1^* + \sin\theta C_1^* \qquad \text{for } s^* = s_1^*,$$

$$i s_2^* B_3^* = \cos\theta A_3^* + \sin\theta C_3^* \qquad \text{for } s^* = s_2^* = s_3^*.$$

Now, the boundary conditions (6.3.15) can be written as

$$A_1 + A_3 - A_1^* - A_3^* = 0,$$

$$B_1 + B_3 - B_1^* - B_3^* = 0,$$

$$C_1 + C_3 - C_1^* - C_3^* = 0,$$

$$\mu \lambda_2^2 s_1 A_1 + \mu \lambda_2^2 s_2 A_3 - i\cos\theta\, \mu \lambda_2^2 B_1 - i\cos\theta\, \mu \lambda_2^2 B_3$$
$$+\mu^* \lambda_2^{*2} s_1^* A_1^* + \mu^* \lambda_2^{*2} s_2^* A_3^* + i\cos\theta\, \mu^* \lambda_2^{*2} B_1^* + i\cos\theta\, \mu^* \lambda_2^{*2} B_3^* = 0, \tag{6.3.16}$$

$$2\mu \lambda_2^2 s_1 k B_1 + 2\mu \lambda_2^2 s_2 k B_3 - D_1 - D_3 + 2\mu^* \lambda_2^{*2} s_1^* k B_1^*$$
$$+ 2\mu^* \lambda_2^{*2} s_2^* k B_3^* + D_1^* + D_3^* = 0,$$

$$i\sin\theta\, \mu \lambda_2^2 B_1 + i\sin\theta\, \mu \lambda_2^2 B_3 + \mu \lambda_2^2 s_1 C_1 + \mu \lambda_2^2 s_2 C_3$$
$$- i\sin\theta\, \mu^* \lambda_2^{*2} B_1^* - i\sin\theta\, \mu^* \lambda_2^{*2} B_3^* - \mu^* \lambda_2^{*2} s_1^* C_1^*$$
$$- \mu^* \lambda_2^{*2} s_2^* C_3^* = 0.$$

Also, as in Section 4.7, substitution for B_1, B_3, C_1, C_3, D_1, D_3 in terms of A_1 and A_3 we get

$$is_1\, B_1 = \frac{A_1}{\cos\theta}, \qquad is_2\, B_2 = \frac{A_3}{\cos\theta},$$

and similarly,

$$is_1^*\, B_1^* = \frac{A_1^*}{\cos\theta}, \qquad is_2^*\, B_2^* = \frac{A_3^*}{\cos\theta}.$$

Thus, equations (6.3.16) become

$$A_1 + A_3 - A_1^* - A_3^* = 0,$$

$$\frac{A_1}{s_1} + \frac{A_3}{s_2} - \frac{A_1^*}{s_1^*} - \frac{A_3^*}{s_2^*} = 0,$$

$$\mu\,\lambda_2^2\,(s_1 - \frac{1}{s_1})A_1 + \mu\,\lambda_2^2\,(s_2 - \frac{1}{s_2})\,A_2 + \mu^*\,\lambda_2^{*2}\,(s_1^* + \frac{1}{s_2^*})\,A_1^*$$
$$+ \mu^*\,\lambda_2^{*2}\,(s_2^* + \frac{1}{s_2^*})\,A_3 = 0,$$

$$(\rho c^2 + 2\mu\,\lambda_2^2 + \mu\,\lambda_2^2\,s_1^2 - \mu\,\lambda_1^2\,\cos^2\theta - \mu\,\lambda_3^2\,\sin^2\theta\,)A_1$$

$$+ (\rho c^2 + 2\mu\,\lambda_2^2 + \mu\,\lambda_2^2\,s_2^2 - \mu\,\lambda_1^2\,\cos^2\theta - \mu\,\lambda_3^2\,\sin^2\theta\,)A_3$$

$$- (\rho^* c^2 - 2\mu^*\,\lambda_2^{*2} + \mu^*\,\lambda_2^{*2}\,s_1^{*2} - \mu^*\,\lambda_1^{*2}\,\cos^2\theta - \mu^*\,\lambda_3^{*2}\,\sin^2\theta\,)A_1^*$$

$$- (\rho^* c^2 - 2\mu^*\,\lambda_2^{*2} + \mu^*\,\lambda_2^{*2}\,s_2^{*2} - \mu^*\,\lambda_1^{*2}\,\cos^2\theta - \mu^*\,\lambda_3^{*2}\,\sin^2\theta\,)A_3^* = 0,$$

these equations can be written as

$$A_1 + A_3 - A_1 - A_3 = 0,$$

$$s_2 s_1^* s_2^* A_1 + s_1 s_1^* s_2^* A_3 - s_1 s_2 s_2^* A_1^* - s_1 s_2 s_1^* A_3^* = 0,$$

$$\mu \lambda_2^2 (s_1^2 - 1) s_2 s_1^* s_2^* A_1 + \mu \lambda_2^2 (s_2^2 - 1) s_1 s_1^* s_2^* A_3 + \mu^* \lambda_2^{*2} (s_1^{*2} + 1) s_1 s_2 s_2^* A_3^*$$

$$+ \mu^* \lambda_2^{*2} (s_2^{*2} + 1) s_1 s_2 s_1^* A_3^* = 0,$$

$$(\rho c^2 + 2\mu \lambda_2^2 + \mu \lambda_2^2 s_1^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta) A_1 \qquad (6.3.17)$$

$$+ (\rho c^2 + 2\mu \lambda_2^2 + \mu \lambda_2^2 s_2^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta) A_3$$

$$- (\rho^* c^2 - 2\mu^* \lambda_2^{*2} + \mu^* \lambda_2^{*2} s_1^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta) A_1^*$$

$$- (\rho^* c^2 - 2\mu^* \lambda_2^{*2} + \mu^* \lambda_2^{*2} s_2^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta) A_3^* = 0.$$

For A_1, A_3, A_1, A_3 in equations (6.3.17) to be non-trivial solution we must have

$$
\begin{vmatrix}
1 & 1 & -1 & -1 \\[2em]
s_2 s_1^* s_2^* & s_1 s_1^* s_2^* & -s_1 s_2 s_2^* & -s_1 s_2 s_1^* \\[2em]
\mu\ \lambda_2^2(s_1^2 - 1) & \mu\ \lambda_2^2(s_2^2 - 1) & \mu^*\ \lambda_2^{*2}(s_1^{*2} + 1) & \mu^* \lambda_2^{*2}(s_2^{*2} + 1) \\[2em]
s_2 s_1^* s_2^* & s_1 s_1^* s_2^* & s_1 s_2 s_2^* & s_1 s_2 s_1^* \\[2em]
\begin{array}{l} \rho c^2 + 2\mu\ \lambda_2^2 \\ \quad + \mu\ \lambda_2^2\ s_1^2 \\ \quad - \mu\ \lambda_1^2\ \cos^2\theta \\ \quad - \mu\ \lambda_3^2\ \sin^2\theta \end{array} &
\begin{array}{l} \rho c^2 + 2\mu\ \lambda_2^2 \\ \quad + \mu\ \lambda_2^2\ s_2^2 \\ \quad - \mu\ \lambda_1^2\ \cos^2\theta \\ \quad - \mu\ \lambda_3^2\ \sin^2\theta \end{array} &
\begin{array}{l} -(\rho^* c^2 - 2\mu^*\ \lambda_2^{*2} \\ \quad + \mu^*\ \lambda_2^{*2} s_1^* \\ \quad - \mu^*\ \lambda_1^{*2} \cos^2\theta \\ \quad - \mu^*\ \lambda_3^{*2} \sin^2\theta) \end{array} &
\begin{array}{l} -(\rho^* c^2 \\ \quad - 2\mu^*\ \lambda_2^{*2} \\ \quad + \mu^*\ \lambda_2^{*2}\ s_2^* \\ \quad - \mu^*\ \lambda_1^{*2} \cos^2\theta \\ \quad - \mu^* \lambda_3^{*2} \sin^2\theta) \end{array}
\end{vmatrix}
$$

On rearranging this becomes

$$s_1\, s_2\, s_1^*\, s_2^*\, (s_1 - s_2)(s_1^* - s_2^*)\Big[\mu^* \lambda_2^{*2}\, s_1^*\, s_2^*\, (2\mu^* \lambda_2^{*2} + \mu^* \lambda_1^{*2} \cos^2\theta$$

$$+ \mu^* \lambda_3^{*2} \sin^2\theta + \mu^* \lambda_3^{*2}\, s_1^*\, s_2^* - \rho^* c^2) + \mu^* \lambda_2^{*2} \{\rho^* c^2 + \mu^* \lambda_2^{*2} (s_1^{*2} + s_2^{*2}$$

$$+ s_1^*\, s_2^*) - 2\mu^* \lambda_2^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_2^{*2} \sin^2\theta\}$$

$$+ \mu\, \lambda_2^2\, s_1\, s_2\, (\rho c^2 - \mu\, \lambda_2^2\, s_1\, s_2 - 2\mu\, \lambda_2^2 - \mu\, \lambda_1^2 \cos^2\theta - \mu\, \lambda_3^2 \sin^2\theta)$$

$$- \mu\, \lambda_2^2\, \{\rho c^2 - \mu\, \lambda_2^2 (s_1^2 + s_2^2 + s_1\, s_2) - 2\mu\, \lambda_2^2 - \mu\, \lambda_1^2 \cos^2\theta - \mu\, \lambda_2^2 \sin^2\theta\}$$

$$+ \mu\, \lambda_2^2\, (s_1\, s_2 + 1)\{\rho^* c^2 - 2\mu^*\, \lambda_2^2 + \mu^* \lambda_2^{*2}\, (s_1^{*2} + s_2^{*2} + s_1^*\, s_1^*)$$

$$- \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta\}$$

$$- \mu^* \lambda_2^{*2}\, (s_1^*\, s_2^* + 1)\{\rho c^2 - 2\mu\, \lambda_2^2 + \mu\, \lambda_2^2\, (s_1^2 + s_2^2 + s_1\, s_1) - \mu\lambda_1^2 \cos^2\theta$$

$$- \mu\, \lambda_3^2\, \sin^2\theta\}$$

$$+ s_1\, s_2 (s_1 + s_2)(s_1^* + s_2^*)\, \mu\, \lambda_2^2\, \mu^*\, \lambda_2^{*2}$$

$$+ s_1^*\, s_2^* (s_1 + s_2)(s_1^* + s_2^*)\, \mu\, \lambda_2^2\, \mu^*\, \lambda_2^{*2}\Big] = 0,$$

Also, assuming that $s_1 \neq s_2$ and $s_1^* \neq s_2^*$, the above secular equation reduces to

$$\mu \lambda_2^2 (\rho c^2 - 2\mu \lambda_2^2 - \mu \lambda_1^2 \cos^2\theta - \mu \lambda_3^2 \sin^2\theta)(s_1 s_2 - 1)$$
$$- \mu^2 \lambda_2^4 (s_1^2 + s_2^2 + s_1^2 s_2^2 + s_1 s_2)$$
$$+ \mu^* \lambda_2^{*2}(\rho^* c^2 - 2\mu^* \lambda_2^{*2} - \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta)(s_1^* s_2^* - 1)$$
$$- \mu^{*2} \lambda_2^{*2} (s_1^{*2} + s_2^{*2} + s_1^{*2} s_2^{*2} + s_1^* s_2^*)$$
$$+ \mu \lambda_2^2 (s_1 s_2 + 1)\{\rho^* c^2 + \mu^* \lambda_2^{*2} (s_1^{*2} + s_2^{*2} + s_1^* s_2^* - 2)$$
$$- \mu^* \lambda_1^{*2} \cos^2\theta - \mu^* \lambda_3^{*2} \sin^2\theta\}$$
$$- \mu^* \lambda_2^{*2} (s_1^* s_2^* - 1)\{\rho c^2 + \mu \lambda_2^2 (s_1^2 + s_2^2 + s_1 s_2 + 2) - \mu \lambda_1^2 \cos^2\theta$$
$$- \mu \lambda_3^2 \sin^2\theta\}$$
$$+ (s_1 + s_2)(s_1^* + s_2^*) (s_1 s_1 - s_1^* s_2^*)\mu \lambda_2^2 \mu^* \lambda_2^{*2} = 0, \qquad (6.3.18)$$

From (6.3.9), we have

$$s_1^2 + s_2^2 = 1 + \frac{\mu \lambda_1^2 \cos^2\theta + \mu \lambda_3^2 \sin^2\theta - \rho c^2}{\mu \lambda_2^2},$$

$$s_1 s_2 = \left[\frac{\mu \lambda_1^2 \cos^2\theta + \mu \lambda_3^2 \sin^2\theta - \rho c^2}{\mu \lambda_2^2}\right]^{\frac{1}{2}}$$

$$s_1 + s_2 = \left[1 + \frac{\mu \lambda_1^2 \cos^2\theta + \mu \lambda_3^2 \sin^2\theta - \rho c^2}{\mu \lambda_2^2} + 2\left[\frac{\mu \lambda_1^2 \cos^2\theta + \mu \lambda_3^2 \sin^2\theta - \rho c^2}{\mu \lambda_2^2}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

(6.3.19)

$$s_1^{*2} + s_2^{*2} = 1 + \frac{\mu^* \lambda_1^{*2} \cos^2\theta + \mu^* \lambda_3^{*2} \sin^2\theta - \rho^* c^2}{\mu^* \lambda_2^{*2}},$$

$$s_1 s_2 = \left[\frac{\mu^* \lambda_1^{*2} \cos^2\theta + \mu^* \lambda_3^{*2} \sin^2\theta - \rho^* c^2}{\mu^* \lambda_2^{*2}}\right]^{\frac{1}{2}}$$

$$s_1^* + s_2^* = \left[1 + \frac{\mu^* \lambda_1^{*2} \cos^2\theta + \mu^* \lambda_3^{*2} \sin^2\theta - \rho^* c^2}{\mu^* \lambda_2^{*2}} + 2\left[\frac{\mu^* \lambda_1^{*2} \cos^2\theta + \mu^* \lambda_3^{*2} \sin^2\theta - \rho^* c^2}{\mu^* \lambda_2^{*2}}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

On setting

$$\eta = \frac{\mu\ \lambda_1^2 \cos^2\theta + \mu\ \lambda_3^2 \sin^2\theta}{\mu\ \lambda_2^2}, \qquad \xi = \frac{\rho c^2}{\mu\ \lambda_2^2}, \quad m = \frac{\mu\ \lambda_2^2}{\mu^*\ \lambda_2^{*2}},$$

$$\eta^* = \frac{\mu^*\ \lambda_1^{*2} \cos^2\theta + \mu^*\ \lambda_3^{*2} \sin^2\theta}{\mu^*\ \lambda_2^{*2}}, \qquad \xi^* = \frac{\rho^* c^2}{\mu^*\ \lambda_2^{*2}}, \quad m^* = \frac{1}{m}, \tag{6.3.20}$$

equation (6.3.18) reduces to

$$m\ \{(\xi - \eta - 2)(1 - s_1\ s_2) + s_1^2\ s_2^2 + s_1^2 + s_2^2 + s_1\ s_2\}$$

$$+ m^*\ \{(\xi^* - \eta^* - 2)(1 - s_1^*\ s_2^*) + s_1^{*2}\ s_2^{*2} + s_1^{*2} + s_2^{*2} + s_1^*\ s_2^*\}$$

$$- (s_1\ s_2 + 1)(s_1^*\ s_2^* - 1) - (s_1^*\ s_2^* - 1)(s_1\ s_2 - 3)$$

$$+ (s_1 + s_1)(s_1^* + s_2^*)(s_1\ s_1 + s_1^*\ s_1^*) = 0,$$

which can be written as

$$\xi - \eta - 2 + s_1^2\ s_2^2 + s_1^2 + s_2^2 - (\xi - \eta - 3)\ s_1\ s_2$$

$$+ m^{*2}\ \{\xi^* - \eta^* - 2 + s_1^{*2}\ s_2^{*2} + s_1^{*2} + s_2^{*2} - (\xi^* - \eta^* - 3)\ s_1^*\ s_2^*\}$$

$$- m^*\ \{2(s_1\ s_2 + s_1^*\ s_2^* - s_1\ s_2\ s_1^*\ s_2^* - 1) - (s_1 + s_2)(s_1^* + s_2^*)$$

$$(s_1\ s_2 + s_1^*\ s_2^*)\} = 0. \tag{6.3.21}$$

Also, equations (6.3.19) reduce to

$$s_1^2 + s_2^2 = 1 + \eta - \xi, \qquad s_1\ s_2 = (\eta - \xi)^{\frac{1}{2}}$$

$$s_1 + s_2 = \{1 + \eta - \xi + 2(\eta - \xi)^{\frac{1}{2}}\}^{\frac{1}{2}},$$

$$s_1^{*2} + s_2^{*2} = 1 + \eta^* - \xi^*, \qquad s_1^*\ s_2^* = (\eta^* - \xi^*)^{\frac{1}{2}} \tag{6.3.22}$$

$$s_1^* + s_2^* = \{1 + \eta^* - \xi^* + 2(\eta^* - \xi^*)^{\frac{1}{2}}\}^{\frac{1}{2}},$$

Next, substitute (6.3.22) in (6.3.21), the secular equation becomes

$$\eta - \xi - 1 - (\xi - \eta - 3)(\eta - \xi)^{\frac{1}{2}}$$

$$+ \left[\frac{\mu^* \lambda_2^{*2}}{\mu \lambda_2^2}\right]^2 \left[\eta^* - \xi^* - 1 - (\xi^* - \eta^* - 3)(\eta^* - \xi^*)^{\frac{1}{2}}\right]$$

$$- \left[\frac{\mu^* \lambda_2^{*2}}{\mu \lambda_2^2}\right] \left[2 \left\{\left[\eta - \xi\right]^{\frac{1}{2}} + \left[\eta^* - \xi^*\right]^{\frac{1}{2}} - \left[\eta - \xi\right]^{\frac{1}{2}}\left[\eta^* - \xi^*\right]^{\frac{1}{2}} - 1\right\}\right.$$

$$- \left\{1 + \eta - \xi + 2\left[\eta - \xi\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \left\{1 + \eta^* - \xi^* + 2\left[\eta^* - \xi^*\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\left.\left\{\left[\eta - \xi\right]^{\frac{1}{2}} + \left[\eta^* - \xi^*\right]^{\frac{1}{2}}\right\}\right] = 0. \qquad (6.3.23)$$

On putting $\alpha = (\eta - \xi)^{\frac{1}{2}}$ and $\alpha^* = (\eta^* - \xi^*)^{\frac{1}{2}}$, equation (6.3.23) reduces to

$$\mu^2 \lambda_2^4 (1 - 3\alpha - \alpha^2 - \alpha^3)$$

$$- \mu \mu^* \lambda_2^2 \lambda_2^{*2} (2 - \alpha - \alpha^* + \alpha^2 + 4 \alpha \alpha^* + \alpha^{*2} + \alpha^2 \alpha^* + \alpha \alpha^{*2})$$

$$+ \mu^{*2} \lambda_2^{*4} (1 - 3 \alpha^* - \alpha^{*2} - \alpha^{*3}) = 0. \qquad (6.3.24)$$

This equation is an equivalent to an equation given by Chadwick and Jarvis (1979), although they considered the same shear modulus for both regions $x_2 > 0$ and $x_2 < 0$ and they took the interface to be $x_3 = 0$. Also this equation is the same equation (6.2.59), which is given in Section 6.2, so all the numerical results given in Section 6.2.7 are also solution for this equation.

From equation (6.3.23), by taking the limit $m^* \longrightarrow 0$, we get the corresponding equation for Rayleigh waves propagting in a general direction namely

$$\omega^3 + 5\omega^2 + 11\omega - 1 = 0,$$

where $\omega = \eta - \xi$, (see Section 4.7).

6.4 Interfacial waves at the boundary between two pre-stressed compressible elastic half-spaces

Finally in this chapter we shall obtain the corresponding equations to these given in Section 6.1 for a compressible material.

As in Section 4.8, the incremental equations of motion for a compressible material lead to

$$\rho\ddot{v}_1 = A_{01111}\, v_{1,11} + A_{01122}\, v_{2,21} + A_{02121}\, v_{1,22} + A_{02112}\, v_{2,12},$$
$$(6.4.1)$$
$$\rho\ddot{v}_2 = A_{01212}\, v_{2,11} + A_{01221}\, v_{1,21} + A_{02211}\, v_{1,12} + A_{02222}\, v_{2,22},$$

for $x_2 < 0$.

Similarly

$$\rho^*\ddot{v}_1^* = A^*_{01111}\, v^*_{1,11} + A^*_{01122}\, v^*_{2,21} + A^*_{02121}\, v^*_{1,22} + A^*_{02112}\, v^*_{2,12},$$
$$(6.4.2)$$
$$\rho^*\ddot{v}_2^* = A^*_{01212}\, v^*_{2,11} + A^*_{01221}\, v^*_{1,21} + A^*_{02211}\, v^*_{1,12} + A^*_{02222}\, v^*_{2,22},$$

for $x_2 > 0$.

6.4.1 Propagation along a principal axis

Also, as in Section 4.8 we assume that v_1^*, v_2^* and v_1, v_2 are given by

$$v_1 = A_1 e^{skx_2 + ikx_1 - i\omega t},$$

$$v_2 = A_2 e^{skx_2 + ikx_1 - i\omega t},$$
$$(6.4.3)$$
$$v_1^* = A_1^* e^{s^*kx_2 + ikx_1 - i\omega t},$$

$$v_2^* = A_2^* e^{s^*kx_2 + ikx_1 - i\omega t}.$$

As in Section, 4.8 (6.4.1) and $(6.4.5)_{1,2}$ lead to

$$s_1^2 + s_2^2 =$$

$$\frac{A_{01111}\,A_{02222} + A_{02121}\,A_{01212} - (A_{01122} + A_{02112})^2 - \rho c^2 (A_{02121} + A_{02222})}{A_{02121}\,A_{02222}},$$

$$s_1^2 s_2^2 = \frac{(A_{01111} - \rho c^2)(A_{01212} - \rho c^2)}{A_{02121}\,A_{02222}}, \tag{6.4.4}$$

for $x_2 < 0$.

Similarly

$$s_1^{*2} + s_2^{*2} =$$

$$\frac{A^*_{01111}\,A^*_{02222} + A^*_{02121}\,A^*_{01212} - (A^*_{01122} + A^*_{02112})^2 - \rho^* c^2 (A^*_{02121} + A^*_{02222})}{A^*_{02121}\,A^*_{02222}},$$

$$s_1^{*2} s_2^{*2} = \frac{(A^*_{01111} - \rho^* c^2)(A^*_{01212} - \rho^* c^2)}{A^*_{02121}\,A^*_{02222}}, \tag{6.4.5}$$

for $x_2 > 0$.

Next, let us consider the incremental boundary conditions

$$v_i = v_i^*, \qquad \dot{s}_{02i} = \dot{s}^*_{02i} \qquad \text{on } x_2 = 0,$$

that is

$$v_1 - v_1^* = 0,$$

$$v_2 - v_2^* = 0,$$

$$\text{on } x_2 = 0 \quad (6.4.6)$$

$$A_{02121}\, v_{1,2} + A_{02112}\, v_{2,1} - (A^*_{02121}\, v^*_{1,2} + A^*_{02112}\, v^*_{2,1}) = 0,$$

$$A_{02211}\, v_{1,1} + A_{02222}\, v_{2,2} - (A^*_{02211}\, v^*_{1,1} + A^*_{02222}\, v^*_{2,2}) = 0.$$

For Stoneley waves we seek a solutions for v_1, v_2 and v_1^*, v_2^*, which decay when $x_2 \longrightarrow \pm\infty$ and satisfying the boundary conditions (6.4.8). Suppose that the general solution for v_1, v_2 and v_1^*, v_2^* are given by

$$v_1 = (A_1 e^{s_1 k x_2} + B_1 e^{s_2 k x_2}) e^{i\omega t - ikx_1},$$

$$v_2 = (A_2 e^{s_1 k x_2} + B_2 e^{s_2 k x_2}) e^{i\omega t - ikx_1},$$

$$v_1^* = (A_1^* e^{-s_1^* k x_2} + B_1^* e^{-s_2^* k x_2}) e^{i\omega t - ikx_1}, \quad (6.4.7)$$

$$v_2^* = (A_2^* e^{-s_1^* k x_2} + B_2^* e^{-s_2^* k x_2}) e^{i\omega t - ikx_1},$$

On use of (6.4.7) in (6.1.6), the boundary condtions become

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$A_2 + B_2 - A_2^* - B_2^* = 0,$$

$$A_{02121}\, s_1\, A_1 + A_{02121}\, s_2\, B_1 - i\, A_{02112}\, A_2 - i\, A_{02112}\, B_2$$

$$\text{on } x_2 = 0 \quad (6.4.8)$$

$$+ A^*_{02121}\, s_1^*\, A_1^* + A^*_{02121}\, s_2^*\, B_1^* + i\, A^*_{02112}\, A_2^* + i\, A^*_{02112}\, B_2^* = 0,$$

$$i\, A_{02211}\, A_1 + i\, A_{02211}\, B_1 - A_{02222}\, s_1\, A_1 - A_{02222}\, s_2\, B_2$$

$$- i\, A^*_{02211}\, A_1^* - i\, A^*_{02211}\, B_1^* - A^*_{02222}\, s_1^*\, A_1^* - A^*_{02222}\, s_2^*\, B_2^* = 0.$$

Recalling from Section 4.8 that

$$\frac{iA_2}{A_1} = \frac{A_{01111} - A_{02121}\, s_1^2 - \rho c^2}{s_1\,(A_{01122} + A_{02112})},$$

and similarly (6.4.9)

$$\frac{iB_2}{B_1} = \frac{A_{01111} - A_{02121}\, s_2^2 - \rho c^2}{s_2\,(A_{01122} + A_{02112})},$$

for $x_2 < 0$.

Similarly

$$\frac{iA_2^*}{A_1^*} = \frac{-(A_{01111}^* - A_{02121}^*\, s_1^{*2} - \rho^* c^2)}{s_1^*\,(A_{01122}^* + A_{02112}^*)},$$

(6.4.10)

$$\frac{iB_2^*}{B_1^*} = \frac{-(A_{01111}^* - A_{02121}^*\, s_2^{*2} - \rho^* c^2)}{s_2^*\,(A_{01122}^* + A_{02112}^*)},$$

for $x_2 > 0$.

Subsitituting (6.4.9) and (6.4.10) into the boundary conditions (6.4.8), we get

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$\left[\frac{A_{01111} - A_{02121}\, s_1^2 - \rho c^2}{s_1(A_{01122} + A_{02112})}\right] A_1 + \left[\frac{A_{01111} - A_{02121}\, s_2^2 - \rho c^2}{s_2(A_{01122} + A_{02112})}\right] B_1$$

$$+ \left[\frac{A^*_{01111} - A^*_{02121}\, s_1^{*2} - \rho^* c^2}{s_1^* \; (A^*_{01122} + A^*_{02112})}\right] A_1^* + \left[\frac{A^*_{01111} - A^*_{02121}\, s_2^{*2} - \rho^* c^2}{s_2^* \; (\overset{*}{A}_{01122} + {}^{*}A_{02112})}\right] B_1^* = 0,$$

$$\left[A_{02121}\; s_1 - \frac{A_{02112}(A_{01111} - A_{02121}\, s_1^2 - \rho c^2)}{s_1(A_{01122} - A_{02112})}\right] A_1$$

$$+ \left[A_{02121}\; s_2 - \frac{A_{02112}(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)}{s_2(A_{01122} - A_{02112})}\right] B_1 \quad \text{on } x_2 = 0 \quad (6.4.11)$$

$$+ \left[A^*_{02121}\; s_1^* - \frac{A^*_{02112}(A^*_{01111} - A^*_{02121}\, s_1^{*2} - \rho^* c^2)}{s_1^*(A^*_{01122} - A^*_{02112})}\right] A_1^*$$

$$+ \left[A^*_{02121}\; s_2^* - \frac{A^*_{02112}(A^*_{01111} - A^*_{02121}\, s_2^{*2} - \rho^* c^2)}{s_2^*(A^*_{01122} - A^*_{02112})}\right] B_1^* = 0,$$

$$\left[A_{02211} + \frac{A_{02222}(A_{01111} - A_{02121}\, s_1^2 - \rho c^2)}{A_{01122} + A_{02112}}\right] A_1$$

$$+ \left[A_{02211} + \frac{A_{02222}(A_{01111} - A_{02121}\, s_2^2 - \rho c^2)}{A_{01122} + A_{02112}}\right] B_1$$

$$- \left[A^*_{02211} + \frac{A^*_{02222}(A^*_{01111} - A^*_{02121}\, s_1^{*2} - \rho^* c^2)}{A^*_{01122} + A^*_{02112}}\right] A_1^*$$

$$- \left[A^*_{02211} + \frac{A^*_{02222}(A^*_{01111} - A^*_{02121}\, s_2^{*2} - \rho^* c^2)}{A^*_{01122} + A^*_{02112}}\right] B_1^* = 0.$$

We may rewrite these equations as

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$\begin{aligned}
&s_2\ s_1^*\ s_2^*\ (A^*_{01122} + A^*_{02112})(A_{01111} - A_{2121}\ s_1^2 - \rho c^2)\ A_1 \\
&+ s_1\ s_1^*\ s_2^*\ (A^*_{01122} + A^*_{02112})(A_{01111} - A_{2121}\ s_2^2 - \rho c^2)\ B_1 \\
&+ s_1\ s_2\ s_2^*\ (A_{01122} + A_{02112})(A^*_{01111} - A^*_{2121}\ s_1^{*2} - \rho^* c^2)\ A_2^* \\
&+ s_1\ s_2\ s_1^*\ (A_{01122} + A_{02112})(A^*_{01111} - A^*_{2121}\ s_2^{*2} - \rho^* c^2)\ B_2^* = 0,
\end{aligned}$$

$$\begin{aligned}
&\{s_1^2\ s_2\ s_1^*\ s_2^*\ A_{02121}\ (A_{01122} + A_{02112})(A^*_{01122} + A^*_{02112}) \\
&- s_2\ s_1^*\ s_2^*\ A_{02112}\ (A^*_{01122} + A^*_{02112})\ (A_{01111} - A_{02121}\ s_1^2 - \rho c^2)\}\ A_1 \\
&+ \{s_1\ s_2^2\ s_1^*\ s_2^*\ A_{02121}\ (A_{01122} + A_{02112})(A^*_{01122} + A^*_{02112}) \\
&- s_1\ s_1^*\ s_2^*\ A^*_{02112}\ (A^*_{01122} + A^*_{02112})\ (A_{01111} - A_{02121}\ s_2^2 - \rho c^2)\}\ B_1 \\
&+ \{s_1\ s_2\ s_1^{*2}\ s_2^*\ A_{02121}\ (A_{01122} + A_{02112})(A^*_{01122} + A^*_{02112}) \\
&- s_1\ s_2\ s_2^*\ A^*_{02112}\ (A_{01122} + A_{02112})\ (A^*_{01111} - A^*_{02121}\ s_1^{*2} - \rho^* c^2)\}\ A_1^* \\
&+ \{s_1\ s_2\ s_1^*\ s_2^{*2}\ A_{02121}\ (A_{01122} + A_{02112})\ (A^*_{01122} + A^*_{02112}) \\
&- s_1\ s_2\ s_1^*\ A^*_{02112}\ (A_{01122} + A_{02112})\ (A^*_{01111} - A^*_{02121}\ s_2^{*2} \\
&\qquad - \rho^* c^2)\}\ B_1 = 0,
\end{aligned}$$

$$\{A_{02211}\ (A_{01122} + A_{02112})\ (A^{*}_{01122} + A^{*}_{02112}) + A_{02222}(A_{01122} + A_{02112})$$

$$(A_{01111} - A_{02121}\ s_1^2 - \rho c^2)\}\ A_1$$

$$+\{A_{02211}\ (A_{01122} + A_{02112})(A^{*}_{01122} + A^{*}_{02112}) + A_{02222}(A_{01122} + A_{02112})$$

$$(A_{01111} - A_{02121}\ s_2^2 - \rho c^2)\}\ B_1$$

$$-\{A^{*}_{02211}\ (A_{01122} + A_{02112})(A^{*}_{01122} + A^{*}_{02112}) + A^{*}_{02222}(A_{01122} + A_{02112})$$

$$(A^{*}_{01111} - A^{*}_{02121}\ s_1^{*2} - \rho^{*} c^2)\}\ A^{*}_1$$

$$-\{A^{*}_{02211}\ (A_{01122} + A_{02112})(A^{*}_{01122} + A^{*}_{02112}) + A^{*}_{02222}(A_{01122} + A_{02112})$$

$$(A^{*}_{01111} - A^{*}_{02121}\ s_2^{*2} - \rho^{*} c^2)\}\ B^{*}_1 = 0.$$

Also, recall from Section 4.8 that

$$\alpha_{11} = A_{01111}, \qquad \alpha_{22} = A_{02222},$$

$$\alpha_{12} = A_{01122}, \qquad \alpha_{21} = A_{02211},$$

$$\gamma_1 = A_{01212}, \qquad \gamma_2 = A_{02121},$$

$$\delta = \alpha_{12} + \alpha_{21}, \qquad 2\beta = \alpha_{11}\ \alpha_{22} + \gamma_1\ \gamma_2 - \delta^2,$$

for $x_2 < 0$.

Similarly

$$\alpha_{11}^* = A_{01111}^*, \qquad \alpha_{22}^* = A_{02222}^*,$$

$$\alpha_{12}^* = A_{01122}^*, \qquad \alpha_{21}^* = A_{02211}^*,$$

$$\gamma_1^* = A_{01212}^*, \qquad \gamma_2^* = A_{02121}^*,$$

$$\delta^* = \alpha_{12}^* + \alpha_{12}^*, \qquad 2\beta^* = \alpha_{11}^* \, \alpha_{22}^* + \gamma_1^* \, \gamma_2^* - \delta^{*2},$$

for $x_2 > 0$.

Equations (6.4.4), (6.4.5) and (6.4.12) now can be written as

$$s_1^2 + s_2^2 = \frac{2\beta - (\gamma_2 + \alpha_{22}) \, \rho c^2}{\gamma_2 \, \alpha_{22}},$$

$$s_1^2 s_2^2 = \frac{(\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)}{\gamma_2 \, \alpha_{22}},$$

$$s_1^{*2} + s_2^{*2} = \frac{2\beta^* - (\gamma_2^* + \alpha_{22}^*) \, \rho^* c^2}{\gamma_2^* \, \alpha_{22}^*}, \tag{6.4.13}$$

$$s_1^{*2} s_2^{*2} = \frac{(\alpha_{11}^* - \rho^* c^2)(\gamma_1^* - \rho^* c^2)}{\gamma_2^* \, \alpha_{22}^*}.$$

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_2 s_1^* s_2^* \delta^* (\alpha_{11} - \gamma_2 s_1^2 - \rho c^2) A_1 + s_1 s_1^* s_2^* \delta^* (\alpha_{11} - \gamma_2 s_2^2 - \rho c^2) B_1$$

$$+ s_1 s_2 s_2^* \delta (\alpha_{11}^* - \gamma_2^* s_1^{*2} - \rho^* c^2) A_1^*$$

$$+ s_1 s_2 s_1^* \delta (\alpha_{11}^* - \gamma_2^* s_2^{*2} - \rho^* c^2) B_1^* = 0,$$

$$s_2 s_1^* s_2^* \delta^* \{s_1^2 \gamma_2 \delta - \alpha_{21} (\alpha_{11} - \gamma_2 s_1^2 - \rho c^2)\} A_1$$

$$+ s_1 s_1^* s_2^* \delta^* \{s_2^2 \gamma_2 \delta - \alpha_{21} (\alpha_{11} - \gamma_2 s_2^2 - \rho c^2)\} B_1$$

$$+ s_1 s_2 s_2^* \delta \{s_1^{*2} \gamma_2^* \delta^* - \alpha_{21}^* (\alpha_{11}^* - \gamma_2^* s_1^{*2} - \rho^* c^2)\} A_1^* \qquad (6.4.14)$$

$$+ s_1 s_2 s_1^* \delta \{s_2^{*2} \gamma_2^* \delta^* - \alpha_{21}^* (\alpha_{11}^* - \gamma_2^* s_2^{*2} - \rho^* c^2)\} B_1^* = 0.$$

$$\delta^* \{\alpha_{12} \delta + \alpha_{22} (\alpha_{11} - \gamma_2 s_1^2 - \rho c^2)\} A_1$$

$$+ \delta^* \{\alpha_{12} \delta + \alpha_{22} (\alpha_{11} - \gamma_2 s_2^2 - \rho c^2)\} B_1$$

$$- \delta \{\alpha_{12}^* \delta^* + \alpha_{22}^* (\alpha_{11}^* - \gamma_2^* s_1^{*2} - \rho^* c^2)\} A_1^*$$

$$- \delta \{\alpha_{12}^* \delta^* + \alpha_{22}^* (\alpha_{11}^* - \gamma_2^* s_2^{*2} - \rho^* c^2)\} B_1^* = 0.$$

For these equations to have non-trivial solutions for A_1, B_1, A_1^* and B_1^*, we must have

$$
\left|
\begin{array}{cccc}
1 & 1 & -1 & -1 \\
s_2 s_1^* s_2^* \, \delta^* \, (\alpha_{11} - \gamma_2 \, s_1^2 - \rho c^2) & s_1 s_1^* s_2^* \, \delta^* \, (\alpha_{11} - \gamma_2 \, s_2^2 - \rho c^2) & s_1 s_2 s_2^* \, \delta \, (\alpha_{11}^* - \gamma_2^* \, s_1^{*2} - \rho^* c^2) & s_1 s_2 s_1^* \delta \, (\alpha_{11}^* - \gamma_2^* \, s_2^{*2} - \rho c^2) \\
s_2 s_1^* s_2^* \, \delta^* \{ s_1^2 \, \gamma_2 \, \delta - \alpha_{21} (\alpha_{11} - \gamma_2 \, s_1^2 - \rho c^2) \} & s_1 s_1^* s_2^* \, \delta^* \{ s_2^2 \, \gamma_2 \, \delta - \alpha_{21} (\alpha_{11} - \gamma_2 \, s_2^2 - \rho c^2) \} & s_1 s_2 s_2^* \, \delta \{ s_1^{*2} \gamma_2^* \, \delta^* - \alpha_2^* (\alpha_{11}^* - \gamma_2^* \, s_1^{*2} - \rho^* c^2) \} & s_1 s_2 s_1^* \, \delta \{ s_2^{*2} \, \gamma_2^* \, \delta^* - {}^*\alpha_{21} (\alpha_{11}^* - \gamma_2^* \, s_2^{*2} - \rho^* c^2) \} \\
\delta^* \, \{ \alpha_{12} \, \delta + \alpha_{22} \, (\alpha_{11} - \gamma_2 \, s_1^2 - \rho c^2) \} & \delta^* \, \{ \alpha_{12} \, \delta + \alpha_{22} (\alpha_{11} - \gamma_2 \, s_2^2 - \rho c^2) \} & -\delta \, \{ \alpha_{12}^* \, \delta^* + \alpha_{22}^* (\alpha_{11}^* - \gamma_2^* \, s_1^{*2} - \rho^* c^2) \} & -\delta \, \{ \alpha_{12}^* \, \delta^* + \alpha_{22}^* (\alpha_{11}^* - \gamma_2^* \, s_2^{*2} - \rho^* c^2) \}
\end{array}
\right|
$$

6.4.2 The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$

In this section we wish to get the corresponding equation for compressible Stoneley waves in the linear theory by considering the special deformation in which $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$. So from equation (2.7.18), we have

$$A_{02121} = A_{01212} = \mu, \qquad A_{02222} = \lambda + 2\mu \qquad \text{and} \qquad A_{02211} = \lambda,$$

and similarly (6.4.13)

$$A^*_{02121} = A^*_{01212} = \mu^*, \qquad A^*_{02222} = \lambda^* + 2\mu^* \qquad \text{and} \qquad A^*_{02211} = \lambda^*,$$

On use of this equation in the boundary conditions (6.4.8), we get

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$A_2 + B_2 - A_2^* - B_2^* = 0,$$

on $x_2 = 0$ (6.4.14)

$$\mu\, s_1\, A_1 + \mu\, s_2\, B_1 - i\,\mu\, A_2 - i\,\mu\, B_2 + \mu^*\, s_1^*\, A_1^* + \mu^*\, s_2^*\, B_1^* - i\,\mu^*\, A_2^* - i\,\mu^*\, B_2^* = 0,$$

$$i\,\lambda\, A_1 + i\,\lambda\, B_1 - (\lambda + 2\mu)\, s_1\, A_1 - (\lambda + 2\mu)\, s_2\, B_2 - i\,\lambda^*\, A_1^* - i\,\lambda^*\, B_1^* - (\lambda^* + 2\mu^*)\, s_1^*\, A_1^* - (\lambda^* + 2\mu^*)\, s_2^*\, B_2^* = 0.$$

Also, recalling (2.7.20) and (2.7.21) that

$$\rho c_L^2 = \lambda + 2\mu, \qquad \rho c_T^2 = \mu,$$

and hence (6.4.15)

$$\lambda = \rho c_L^2 - 2\rho c_T^2.$$

Similarly

$$\rho^* c_L^{*2} = \lambda^* + 2\mu^*, \qquad \rho^* c_T^{*2} = \mu^*,$$

and hence (6.4.16)

$$\lambda^* = \rho^* c_L^{*2} - 2\rho^* c_T^{*2}.$$

Next, substituting (6.4.15) and (6.4.16) in (6.4.9) and (6.4.10), we have

$$\frac{iA_2}{A_1} = \frac{c_L^2 - c_T^2 s_1^2 - c^2}{s_1(c_L^2 - 2c_T^2)},$$

$$\frac{iB_2}{B_1} = \frac{c_L^2 - c_T^2 s_2^2 - c^2}{s_2(c_L^2 - 2c_T^2)}, \tag{6.4.17}$$

for $x_2 < 0$.

Similarly

$$\frac{iA_2^*}{A_1^*} = \frac{c_L^{*2} - c_T^{*2} s_1^{*2} - c^2}{s_1^*(c_L^{*2} - 2c_T^{*2})},$$

$$\frac{iB_2^*}{B_2^*} = \frac{c_L^{*2} - c_T^{*2} s_2^{*2} - c^2}{s_2^*(c_L^{*2} - 2c_T^{*2})}, \tag{6.4.18}$$

for $x_2 > 0$.

From the discussion in Sections 3.5 and 3.7, equations (6.4.17) and (6.4.18) can be written as

$$\frac{iA_2}{A_1} = -s_1, \qquad \frac{iB_2}{B_1} = -\frac{1}{s_2}, \qquad \text{for } x_2 < 0,$$

(6.4.19)

$$\frac{iA_2^*}{A_1^*} = s_1^*, \qquad \frac{iB_2^*}{B_1^*} = \frac{1}{s_2^*}, \qquad \text{for } x_2 < 0,$$

Equtions (6.4.14) now can be written as

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_1A_1 + \frac{B_1}{s_2} - s_1^*A_1^* - \frac{B_1^*}{s^*_2} = 0,$$

$$\text{on } x_2 = 0 \qquad (6.4.20)$$

$$2\mu s_1A_1 + \mu(s_2 + \frac{1}{s_2}) B_1 + 2\mu^* s_1^* A_1^* + \mu^*(s_2^* + \frac{1}{s_2^*})B_1^* = 0,$$

$$\{ \lambda - (\lambda + 2\mu)s_1^2\}A_1 - 2\mu B_1 + \{(\lambda^* + 2\mu^*)s_1^{*2} - \lambda^*\} A_1^* + 2\mu^* B_1^* = 0.$$

On use of equations (6.4.15) and (6.4.16) in (6.4.20), we have

$$A_1 + B_1 - A_1^* - B_1^* = 0,$$

$$s_1A_1 + \frac{B_1}{s_2} - s_1^*A_1^* - \frac{B_1^*}{s_2^*} = 0,$$

$$(6.4.21)$$

$$2s_1A_1 + (2 - c^2/c_T^2)\frac{B_1}{s_2} + 2\frac{\mu^*}{\mu}s_1^*A_1^* + \frac{\mu^*}{\mu}(2 - c^2/c_T^{*2})\frac{B_1^*}{s_2^*} = 0,$$

$$(2 - c^2/c_T^2)A_1 + 2B_1 - \frac{\mu^*}{\mu}(2 - c^2/c_T^{*2})A_1^* - 2\frac{\mu^*}{\mu}B_1^* = 0.$$

For these equations to have non-trivial solutions for A_i, B_i, A_i^*, B_i^*, $i \in \{1, 2\}$ we must have

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ s_1 & \frac{1}{s_2} & -s_1^* & -\frac{1}{s_2^*} \\ 2s_1 & \left(2 - \frac{c^2}{c_T^2}\right)\frac{1}{s_2} & 2\,\frac{\mu^*}{\mu}\, s_1 & \frac{\mu^*}{\mu}\left(2 - \frac{c^2}{c_T^{*2}}\right)\frac{1}{s_2^*} \\ 2 - c^2/c_T^2 & 2 & \frac{\mu^*}{\mu^{*2}}\left(2 - \frac{c^2}{c_T^{*2}}\right) & -2\,\frac{\mu^*}{\mu} \end{vmatrix} = 0$$

This secular equation in this form has been given in Section 3.7, which is the secular equation of compressible Stoneley waves in the classical linear theory.

REFERENCES

ACHENBACH, J.D. 1984 Wave propagation in elastic solids. North-Holland.

BARNETT, D.M and LOTHE, J. 1985 Free surface (Rayleigh) waves in anisotropic elastic half-space: the surface impedance method. Proc. R. Soc. A 402, 135-152

BARNETT, D.M, LOTHE, J., GAVAZZA, S. D. and MUSGRAVE, M. J. P. 1985 Consideration of the existence of interfacial (Stoneley) waves in bonded anisotropic elastic half-spaces, Proc. R. Soc. Lond. A 402, 153-166.

BIOT, M.A. 1965 Mechanics of Incremental Deformations. New York: Wiley.

CAGNIARD, L. 1962 Reflection and Refraction of Progressive Seismic waves. McGraw-Hill.

CHADWICK, P. 1985 A general analysis of transonic states in an anisotropic elastic body. Proc. R. Soc. A 401, 203-223

CHADWICK, P. & JARVIS, D.A. 1979 Surface waves in an a pre-stressed elastic body. Proc. R. Soc. A 366, 517-536.

CHADWICK, P. & JARVIS, D.A. 1979a Interfacial waves in a pre-stressed neo-Hookean body, I. Biaxial states of strain, Quart. J. Mech. Appl. Math. 32, 387-399.

CHADWICK, P. & JARVIS, D.A. 1979b Interfacial waves in a pre-stressed neo-Hookean body, II. Triaxial states of strain, Quart. J. Mech. Appl. Math. 32, 401-418.

CHADWICK, P. & SMITH, G.D. 1977 Foundations of the theory of surface waves in anisotropic elastic materials. Adv. Appl. Mech. 17, 303-376.

CHADWICK, P. & SMITH, G.D. 1982 Surface waves in cubic elastic materials, in Mechanics of Solids, The Rodney Hill 60th anniversary volume (eds. H.G. Hopkins and M.J. Sewell) 47-100. Oxford: Pergamon.

DUNWOODY, J., 1989 Elastic interfacial standing waves, in Elastic Wave Propagation (eds. M. F. McCarthy and M. A. Hayes), pp. 107-112. Elsevier (North-Holland), 1989.

DUNWOODY, J., and VILLAGIO, P., 1988 Self straining at the interface of two bonded hyperelastic half-spaces uniformly pre-stressed, Quart. J. Mech. Appl. Math. 41, 347-361.

EWING, W.M. , JARDETZKY, W.S. & PRESS, F. 1957 Elastic waves in Layered Media . New York: McGraw Hill.

FLAVIN, J.N. 1963 Surface waves in a pre-stressed Mooney material Quart. J. Mech. Appl. Math. 16, 441-449.

HAYES, M. and RIVLIN R.S. 1961a Propagation of a plane wave in an isotropic elastic material subject to pure homogeneous deformation. Arch. Ration. Mech. Analysis. 8, 15-22.

HAYES, M. and RIVLIN R.S. 1961b Surface waves in deformed elastic materials. Arch. Ration. Mech. Analysis.8, 358-380.

HILL, R. 1979 On the theory of plane finitely deformed compressible materials, Math. Proc.Camb. Phil. Soc. 86, 161 - 178.

HILL, R. and HUTCHINSON, J.W. 1975 Bifurcation phenomena in the plane tension test. J. Mech. Phys. Solids. 23, 239-264.

KNOWLES, J. K. and STERNBERG, E. 1977, On the failure of ellipticity of the equations of finite elastostatic plane strain. Arch. Ration. Mech. Anal. 63, 321-336.

NOWINSKI, J.L. 1969a On the surface instability of an isotropic highly elastic half-space. J. Math. Mech. 18, 1-10.

NOWINSKI, J.L. 1969b Surface instability of a half-space under high two-dimensional compression. J. Franklin Inst. 288, 367-376.

OGDEN, R.W. 1972 Large deformation isotropic elasticity: on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. A 326, 565-584.

OGDEN, R.W. 1984 Non-linear Elastic Deformations. Chichester: Ellis Horwood.

REDDY, B.D. 1982 Surface instabilities on an equibiaxially stretched elastic half-space. Math. Proc. Camb. Phil. Soc. 91, 491-501.

REDDY, B.D. 1983 The occurrence of surface instabilities and shear bands in plane-strain deformation of an elastic half-space. Quart. J. Mech. Appl. Math. 36, 337-350.

SCHOLTE, J. G., 1942 On the Stoneley wave equation, Proc. Kon. Acad.sci. Amsterdam 45, 159-164.

SCHOLTE, J. G., 1947 The range of existence of Rayleigh and Stoneley waves, Mon. Not. R. Astr. Soc. Geohys. Suppl. 5, 120-126.

SEZAWA, K. and KANAI, K. 1939 The range of possible existence of Stoneley waves, and some related problems, Bull. Earthq. Res. Inst. Tokyo Univ. 17, 1-8.

Stoneley, R., 1924 Elastic waves at the surface of separation of two solids, Proc. R. Soc. Lond. A 106, 416-428.

STROH, A. N., 1962 Steady state problems in anisotropic elasticicity, J. Math. Phys. 41, 77-103.

TRUESDELL, C.A. and NOLL, W. 1965 The non-linear field theories of mechanics. Handbuch der Physik, vol III/3, Springer.

USMANI, S.A. and BEATTY, M.F. 1974 On the surface instability of a highly elastic half-space. J. Elast. 4, 249-263.

WHITWORTH, A.M. and CHADWICK, P. 1984a The effect of inextensibility on elastic surface waves. Wave Motion 6, 289-302.

WHITWORTH, A.M. and CHADWICK, P. 1984b Surface waves in an elastic body with restricted shear. Arbian J. Sci. Engng. 9, 67-76.

WILLSON, A.J. 1972a Wave-propagation in uniaxially-stressed elastic media, Pure Appl. Geophys. 93, 5-8.

WILLSON, A.J. 1972b Wave-propagation in biaxially-stressed elastic media, Pure Appl. Geophys. 95, 48-58.

WILLSON, A.J. 1973a Surface and plate waves in biaxially-stressed elastic media, Pure Appl. Geophys. 102, 182-192.

WILLSON, A.J. 1973b Surface waves in restricted Hadamard materials Pure Appl. Geophys. 110, 1967-1976.

WILLSON, A.J. 1974a Surface waves in uniaxially-stressed Mooney material, Pure Appl. Geophys. 112, 352-364.

WILLSON, A.J. 1974b The anomalous surface waves in uniaxially -stressed elastic material, Pure Appl. Geophys. 112, 352-364.