A DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF BOUNDARY-VALUE PROBLEMS

being a THESIS presented by

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PERSONAL FOREWORD

The work for this thesis which was started in September, 1957, and completed in June, 1961, was carried out in the Mathematics Department, Royal College of Science and Technology, Glasgow, under the supervision of Professor D.C.Pack and Mr.B.Noble.

Chapters 1 and 3 consist of original work carried out in collaboration with Mr.B.Noble and it is intended to publish these as joint papers.

The substance of Chapters 2, 4, and 5 consists in the main of original work which has not yet been published.

I wish to thank Mr.Noble for introducing me to the subject and both Professor Pack and Mr.Noble for their constant encouragement and advice.

I am also indebted to the staff of the Computing Laboratory, University of Glasgow, under Dr.D.C.Gilles, for all the facilities they so willingly made available, to Miss G.D.Williamson for her very responsible typing, and to Mr.G.Swan who drew all the graphs.

CONTENTS

INTRODUCTION

In the method of finite differences for the numerical solution of boundary-value problems, an elliptic partial differential equation is replaced by a set of simultaneous linear equations or "difference equations", usually simple in form but large in number. Two of the papers which pioneered this method are those of Richardson [1] and Liebmann [2].

One of the main practical difficulties of the finite-difference method lies in the solution of the large set of simultaneous equations. Many iterative methods have been proposed, ranging from procedures suitable for hand computation, such as relaxation (Southwell [3]), to systematic iterative procedures suitable for automatic computers, such as those described in Forsythe and Wasow [4].

The desire to avoid the iterative solution of large sets of simultaneous linear equations has led to the development of various direct methods of attack in which the concepts of matrix algebra usually play the major role (Bickley and McNamee [5]). In some of these methods (e.g. Cornock [6]) the technique is to replace the solution of the large number of simultaneous equations by the solution of a much smaller number of more complicated equations. This is usually carried out by manipulating the matrix of coefficients of the original system.

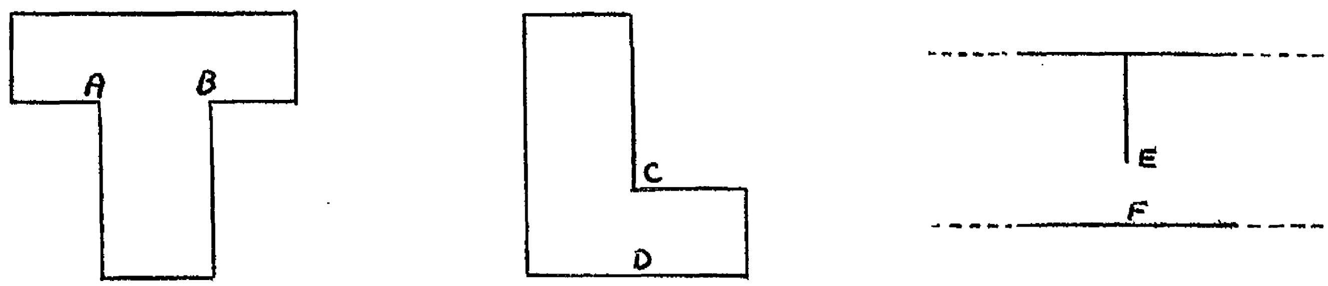
In this thesis a method of direct type is described and applied to the solution of boundary-value problems defined in rectangular regions and regions which may be divided into a number of rectangular sub-regions. Only three types of linear elliptic partial differential equations are considered, namely

Laplace's equation, $\nabla^2\phi = 0$, Poisson's equation, $\nabla^2\phi = f(x,y)$, and the steady-state wave equation, $\nabla^2\phi + k^2\phi = 0$.

The method is related to some of the other direct methods (e.g. Cornock [6], Karlqvist [7], Burgerhout [8]) but the distinctive feature is that the problem is reduced to the solution of a relatively small number of simultaneous equations by using methods analogous to those available for partial differential equations, in particular separation of variables. Separation of variables has long been used in solving difference equations, e.g. Ellis [9], 1844. More recently Hyman [10], 1952, has used the technique in solving boundary-value problems by difference methods. No reference has been found, however, in which the difference analogue of the related, but much more recent, transform technique has been applied. The foundation of the method described here lies in the use of such a difference analogue and the method is henceforth referred to as "the discrete transform method".

The plan of the thesis will now be given. In Chapter 1 the discrete transform method is developed by considering two simple boundary-value problems in rectangles, one involving Laplace's equation, the other Poisson's equation. It is shown that this method of solving sets of difference equations is exactly parallel to the separation of variables and transform method of solving the corresponding partial differential equations. In particular, the concept of a "discrete transform" is introduced and it is shown that, in exact analogy with the use of an integral transform in continuous analysis, the effect of a discrete transform is to reduce the dimension of the problem by one. By this is meant that the solution of a set of partial difference equations for function values at points in a plane is reduced to the solution of a set of second-order ordinary difference equations for unknown quantities situated on a line. It is also shown, in an appendix, that when these second-order difference equations are non-homogeneous, they may be solved by a method analogous to the Green's function method of solving non-homogeneous second-order differential equations.

Chapters 2 and 3 are concerned with "adjacent region" problems. These involve regions which can be sub-divided into rectangles such as the regions shown in the accompanying figures. As indicated in Fig. (c) these rectangles



(a) (b) (c)

may be infinite in one direction. In Chapter 2, two problems involving Laplace's equation are considered while a steady-state wave problem in an infinite strip containing an obstruction (Fig. (c)) is illustrated in Chapter 3. Using the discrete transform method as developed in Chapter 1, the solution of such problems is reduced to the solution of a set of linear simultaneous equations in unknown quantities introduced at mesh points on the common boundary of the two adjacent regions (e.g. AB, CD, EF in Figs. (a), (b), (c) respectively). In continuous analysis the solution of such problems leads to an integral equation for an unknown function introduced on the common boundary. Difficulties are encountered in solving such integral equations and it was in an attempt to avoid these that the discrete transform method was initially devised.

In the numerical examples considered in Chapters 2 and 3 it is shown that the accuracy of the discrete approximations is poor and that their rate of convergence to exact values as the net spacing is reduced is slow. Attention was therefore diverted from further consideration of steady-state wave problems to devise means of improving the accuracy of the approximate solutions obtained using differences. This has led to investigations into the numerical treatment of singular points in boundary-value problems.

Although many authors mention the fact that singular points need careful treatment, very few specific methods for doing this appear to have been published. In fact the only useful references which have been found are Southwell [3], who uses an "advance to a finer net" technique, and Motz [11], Jeffreys and Jeffreys [12], and Woods [13], who each describe methods for generating special difference equations at mesh points near the singularity. In section 2 of Chapter 5 some variants of these methods for treating singular points are described, including a new method of generating special equations which combines ideas of Woods and Jeffreys and Jeffreys.

To provide a flexible tool for using special difference equations in conjunction with the discrete transform method, an iterative method is developed in Chapter 4 which may be employed when special difference equations are used at any mesh points. It would also be possible to use this iterative method in conjunction with Fox's "difference correction process" [14] to increase the accuracy of discrete approximations obtained using straightforward techniques.

In section 3 of Chapter 5 numerical results, obtained using the method developed in Chapter 4, are given for a particular problem in which the special difference equations derived in the previous section are used. These results enable a comparison to be made, on an experimental basis, of the effectiveness of some of the methods proposed for dealing with singularities.

This completes a description of the contents of the thesis. To summarise, the object of the thesis is to investigate various aspects of the application of what is called "the discrete transform method" to "adjacent region problems". The type of problem for which the method is claimed to be advantageous is illustrated in Chapter 3 which deals with the solution of the steady-state wave equation, $\nabla^2\phi + k^2\phi = 0$, in a strip of infinite extent.

CHAPTER 1

FUNDAMENTALS OF THE DISCRETE TRANSFORM METHOD

1.1 Choice of Difference Net Patterns

The discrete transform method is developed here for boundary value problems defined on rectangular regions. For simplicity, only square lattices are used to cover the region and only three types of net patterns are dealt with:

(a) the five point square net pattern,

(b) the five point diagonal net pattern,

(c) the nine point net pattern.

For Laplace's equation, $\nabla^2\phi = 0$, and with reference to the regular array of points in fig. 1, these net patterns are:

(a) $4\phi_0 - \phi_1 - \phi_2 - \phi_3 - \phi_4 = 0,$

(b) $4\phi_0 - \phi_5 - \phi_6 - \phi_7 - \phi_8 = 0,$

(c) $20\phi_0 - 4(\phi_1 + \phi_2 + \phi_3 + \phi_4) - (\phi_5 + \phi_6 + \phi_7 + \phi_8) = 0,$

where ϕ_k is the value of ϕ at the point numbered 'k'.

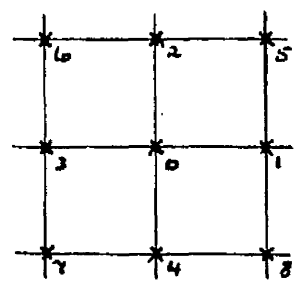


Fig. 1

We shall frequently refer to patterns (a), (b), and (c) simply as the square, diagonal, and nine point patterns respectively.

1.2 The Analogy between the Separation of Variables and Transform Techniques of Continuous Analysis and the Discrete Transform Method

Separation of variables is one of the oldest and most powerful methods of solving certain types of boundary value problems in continuous analysis. It has also been used to solve boundary-value problems numerically by difference methods, e.g. Hyman [10] and Berger & Lasher [16]. Integral transforms, a convenient means of application of the separation technique in continuous analysis, are of more recent development, the papers of Heaviside at the beginning of the present century on the mathematics of electrical transmission providing the real initial impetus. Many workers have been engaged in this field, particularly in the last twenty years, and many types of integral transforms are now in use. These methods are well described in numerous texts, e.g. Sneddon [17] and Irving & Mullineux [18]. The effect of employing an integral transform to solve a partial differential equation is (ultimately) to reduce the problem to that of solving an ordinary differential equation. It seems that no corresponding technique has been used for difference equations.

In the following sections it is shown that a matrix analogue of the integral transform technique does exist for difference equations and that its effect is to reduce the solution of the set of simultaneous equations to the solution of a single (second-order) difference equation. To illustrate the analogy the closed solutions of a simple Dirichlet problem in two dimensions, problem 'A', are obtained using both techniques. On the left hand side of the page the solution is worked out by the standard separation of variables and transform technique. On the right hand side it is shown that the corresponding difference equations may be solved by exactly the same technique. Both these solutions are illustrated in much greater detail than is necessary in practice purely to clarify the parallelism of the two techniques. Further the more important corresponding equations are similarly numbered except that those for the discrete case have a dash superscript on the equation number e.g. (1.3.5') in the discrete case corresponds to (1.3.5) in the continuous.

A second simple boundary-value problem, problem 'B', is then introduced to show that when the second-order difference equation resulting from the application of the discrete transform is non-homogeneous a discrete Green's function technique may be used to obtain its solution. This is in analogy with the Green's function method of solving the second-order, non-homogeneous differential equation arising from an integral transform attack on the original partial differential equation. Problem B has also been chosen in order to illustrate the most convenient way of positioning the square lattice when the conditions on two parallel boundaries are of Neumann type.

In this chapter, and elsewhere in the thesis, the common practice of writing discrete variables as subscripts is adopted as far as is convenient.

1.3 Problem A. Dirichlet Problem for Laplace's Equation $\nabla^2\phi = 0$ in a Rectangle

We consider the rectangle to be $0 \le x \le a$, $0 \le y \le b$ and take $\phi = 0$ on $x = a$, $y = 0$, $y = b$ while $\phi = g(y)$ on $x = 0$ [fig. 2]. The solution of of the general Dirichlet problem in a rectangle may be obtained by superposition of the solutions of four problems of this simple form.

For the discrete case (fig. 2') we cover the rectangle with a square lattice co-incident with the boundaries so that $a = (m+1)h$, $b = (n+1)h$, where h is the mesh length and m, n are positive integers. Step measures r and s are taken in the x and y directions respectively and the value of ϕ at the mesh point (rh, sh) is denoted by $\phi_{r,s}$. The boundary conditions then become

$$\phi_{r,0} = 0, \ r = 1,2,\cdots,m \ ; \quad \phi_{0,s} = g_s, \ s = 1,2,\cdots,n,$$

$$\phi_{r,n+1} = 0, \ r = 1,2,\cdots,m \ ; \quad \phi_{m+1,s} = 0, \ s = 1,2,\cdots,n.$$

We use the 5 point square net pattern so that, at any interior point (rh, sh), (§ 1.1)

$$4\phi_{r,s} - \phi_{r+1,s} - \phi_{r,s+1} - \phi_{r-1,s} - \phi_{r,s-1} = 0, \quad 1 \le r \le m, \ 1 \le s \le n.$$

(i) SEPARATION OF VARIABLES AND TRANSFORM METHOD

(ii) THE DISCRETE TRANSFORM METHOD

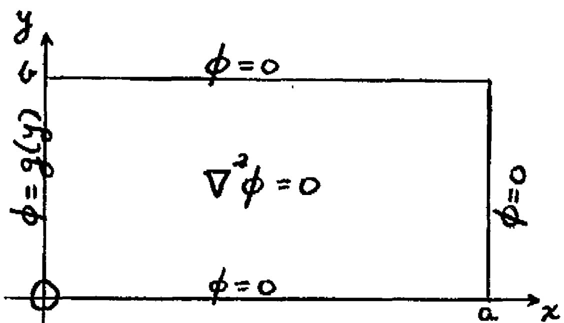


FIG. 2

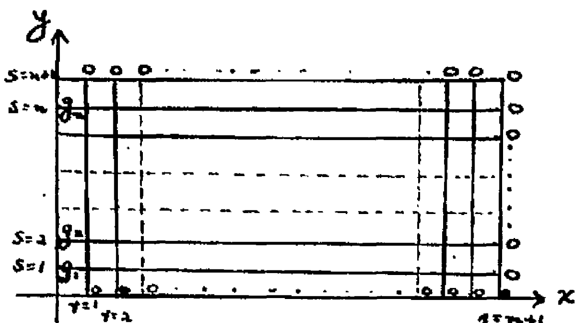


FIG 2'

Equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad (1.3.1)$$

Separate the variables

i.e. let $\phi(x,y)=X(x)Y(y)$

Substitute in (1.3.1) to get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

Let $\frac{1}{Y}\frac{d^2Y}{dy^2}=-\lambda^2$, (1.3.2)

where λ is any real number

i.e. $$\frac{d^2Y}{dy^2}+\lambda^2Y=0 \qquad (1.3.3)$$

$$\therefore Y(y)=Ae^{i\lambda y}+Be^{-i\lambda y}$$

[Exponential functions are used in preference to trigonometric functions merely for the purpose of the analogy.]

Apply boundary conditions on $y=0$ and $y=b$.

$Y(0)=0 \Rightarrow A=-B$

$Y(b)=0 \Rightarrow A(e^{i\lambda b}-e^{-i\lambda b})=0.$

Equations are

$$4\phi_{r,s}-\phi_{r+1,s}-\phi_{r,s+1}-\phi_{r-1,s}-\phi_{r,s-1}=0,$$

where $1\le r\le m$, $1\le s\le n$. (1.3.1')

Separate the variables

i.e. let $\phi_{r,s}=R(r)S(s)$

Substitute in (1.3.1') to get

$$4-\frac{1}{R(r)}\big(R(r+1)+R(r-1)\big)-\frac{1}{S(s)}\big(S(s+1)+S(s-1)\big)=0$$

Let $\frac{1}{S(s)}\big(S(s+1)+S(s-1)\big)=2\lambda$, (1.3.2')

where λ is any real number

i.e. $$S(s+1)-2\lambda S(s)+S(s-1)=0 \qquad (1.3.3')$$

$$\therefore S(s)=Ap^s+Bq^s,$$

where p, q are solutions of

$$\alpha^2-2\lambda\alpha+1=0.$$

Apply boundary conditions on $s=0$ and $s=n+1$

$S(0)=0 \Rightarrow A=-B$

$S(n+1)=0 \Rightarrow A(p^{n+1}-q^{n+1})=0.$

i.e. $$\left(\lambda+\sqrt{\lambda^2-1}\right)^{n+1}-\left(\lambda-\sqrt{\lambda^2-1}\right)^{n+1}=0$$

Let $\lambda=\cos\theta$

$$\therefore \cos(n+1)\theta+i\sin(n+1)\theta-\cos(n+1)\theta+i\sin(n+1)\theta=0.$$

i.e. $\sin \lambda b = 0$

$$\therefore \lambda = \frac{l\pi}{b}, \quad l = 0,1,2,\ldots$$

$l=0$ is not permissible since this implies $\phi \equiv 0$.

We thus obtain an infinite set of eigenvalues with corresponding eigenfunctions

$$Y(y) = A(l) \sin \frac{l\pi y}{b}$$

i.e. $\sin (n+1)\theta = 0$

i.e. $$\theta_l = \frac{l\pi}{n+1}, \quad l = 0,1,2,\ldots,n.$$

$l=0$ is not permissible since this implies $\lambda = 1$ and hence $\phi \equiv 0$.

We thus obtain n distinct eigenvalues, say λ_l, $l = 1,2,\ldots,n$, with corresponding discrete eigenfunctions

$$S_l(s) = A_l(p_l^s - q_l^s).$$

Let $\underset{\sim}{S}_l = \{S_l(1), S_l(2), \ldots\ldots, S_l(n)\}$,

where $\{ \ \}$ denotes a column vector, and define the $n \times n$ matrix $\underset{\sim}{A}$ by

$$\underset{\sim}{A} = \begin{bmatrix} 4 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & 4 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 4 & -1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & -1 & 4 & -1 \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & 4 \end{bmatrix}$$

$\underset{\sim}{A}$ will be called the <u>Basic Matrix of the problem for the square net pattern.</u>

For $s = 1,2,\ldots\ldots,n$, equations (1.3.5') may be written

$$\underset{\sim}{A}\,\underset{\sim}{S}_l = (4 - 2\lambda_l)\underset{\sim}{S}_l \qquad (\alpha)$$

Since $\underset{\sim}{A}$ is real and symmetric its latent roots are real (equation (1.3.2')).

Since these latent roots are also distinct we may normalise $\underset{\sim}{S}_{\ell}$ so that

$$\underset{\sim}{S}_k^T \underset{\sim}{S}_\ell = \delta_k^\ell$$

where $\underset{\sim}{S}^T$ is the transpose of $\underset{\sim}{S}$ and δ_k^ℓ is the Kronecker delta function.

Define $\underset{\sim}{\Phi}_r = \{\phi_{r,1}, \phi_{r,2}, \cdots\cdots, \phi_{r,n}\}$

Equations (1.3.1') may then be written in the matrix form

$$\underset{\sim}{A}\,\underset{\sim}{\Phi}_r - \underset{\sim}{\Phi}_{r+1} - \underset{\sim}{\Phi}_{r-1} = \underset{\sim}{0} \quad , 1 \le r \le m \quad (\beta)$$

where $\underset{\sim}{0}$ is the $n \times 1$ vector with all elements zero.

Equation (β) will be referred to as the (Matrix) Governing Equation of the problem for the square net pattern.

The transform for this problem is, therefore,

$$F(\ell,x) = \int_0^b \phi(x,y) \sin\frac{\ell\pi y}{b}\, dy \quad (1.3.4)$$

and the inversion theorem gives

$$\phi(x,y) = \frac{2}{b}\sum_{\ell=1}^{\infty} F(\ell,x) \sin\frac{\ell\pi y}{b} \quad (1.3.5)$$

Apply this transform to Laplace's equation (1.3.1) by multiplying through by $\sin\frac{\ell\pi y}{b}$ and integrating with respect to y between limits

Define the discrete transform by

$$F_{\ell,r} = \underset{\sim}{S}_\ell^T \underset{\sim}{\Phi}_r \quad (1.3.4')$$

then, by the orthogonality of vectors $\underset{\sim}{S}_\ell$,

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} F_{\ell,r}\, \underset{\sim}{S}_\ell \quad (1.3.5')$$

Apply the discrete transform to the set of difference equations (1.3.1')(i.e. equation (β)) by pre-multiplying by $\underset{\sim}{S}_\ell^T$.

0 and b. Thus

$$\int_0^b \frac{\partial^2\phi}{\partial x^2}\sin\frac{l\pi y}{b}\,dy + \int_0^b \frac{\partial^2\phi}{\partial y^2}\sin\frac{l\pi y}{b}\,dy = 0 \quad (1.3.6)$$

Integrating by parts,

$$\int_0^b \frac{\partial^2\phi}{\partial y^2}\sin\frac{l\pi y}{b}\,dy = -\left(\frac{l\pi}{b}\right)^2\int_0^b \phi\sin\frac{l\pi y}{b}\,dy$$

$$= -\left(\frac{l\pi}{b}\right)^2 F(l,x) \quad (1.3.7)$$

(1.3.6) becomes

$$\frac{d^2}{dx^2}\int_0^b \phi\sin\frac{l\pi y}{b}\,dy - \left(\frac{l\pi}{b}\right)^2 F(l,x) = 0$$

i.e. $$\frac{d^2}{dx^2}F(l,x) - \left(\frac{l\pi}{b}\right)^2 F(l,x) = 0 \quad (1.3.8)$$

$$\therefore F(l,x) = C(l)e^{\frac{l\pi x}{b}} + D(l)e^{-\frac{l\pi x}{b}}$$

To evaluate $C(l)$ and $D(l)$ we have the boundary conditions on $x=a$ and $x=0$.

On $x=a$, $\phi=0 \Rightarrow F(l,a)=0$.

i.e. $$C(l)e^{\frac{l\pi a}{b}} = -D(l)e^{-\frac{l\pi a}{b}}$$

$$= E(l), \quad \text{say}$$

Thus

$$\underset{\sim}{S}_l^T \underset{\sim}{A}\,\underset{\sim}{\Phi}_r - \underset{\sim}{S}_l^T \underset{\sim}{\Phi}_{r+1} - \underset{\sim}{S}_l^T \underset{\sim}{\Phi}_{r-1} = 0 \quad (1.3.6')$$

Equation (α) shows that

$$\underset{\sim}{S}_l^T \underset{\sim}{A} = (4-2\lambda_l)\underset{\sim}{S}_l^T \qquad (\gamma)$$

hence $$\underset{\sim}{S}_l^T \underset{\sim}{A}\,\underset{\sim}{\Phi}_r = (4-2\lambda_l)F_{l,r} \quad (1.3.7')$$

(1.3.6') becomes

$$(4-2\lambda_l)F_{l,r} - F_{l,r+1} - F_{l,r-1} = 0$$

i.e. $$F_{l,r+1} - (4-2\lambda_l)F_{l,r} + F_{l,r-1} = 0 \quad (1.3.8')$$

$$\therefore F_{l,r} = C_l P_l^{\,r} + D_l Q_l^{\,r},$$

where P_l and Q_l are solutions of

$$\alpha^2 - (4-2\lambda_l)\alpha + 1 = 0 \qquad (\delta)$$

To evaluate C_l and D_l we have the boundary conditions on $r=m+1$ and $r=0$.

On $r=m+1$, $\underset{\sim}{\Phi}_{m+1} = \underset{\sim}{0} \Rightarrow F_{l,m+1}=0$.

i.e. $$C_l P_l^{\,m+1} = -D_l Q_l^{\,m+1}$$

$$= E_l, \quad \text{say}$$

$$\therefore F(\ell,x) = E(\ell)\left(e^{\frac{\ell\pi}{b}(a-x)} - e^{-\frac{\ell\pi}{b}(a-x)}\right) \quad (1.3.9)$$

On $x=0$, $\phi = g(y)$

$$\therefore F(\ell,0) = \int_0^b g(y)\sin\frac{\ell\pi y}{b}\,dy$$

$$= E(\ell)\left(e^{-\frac{\ell\pi a}{b}} - e^{\frac{\ell\pi a}{b}}\right)$$

$$\therefore F(\ell,x) = \frac{\sinh\frac{\ell\pi}{b}(a-x)}{\sinh\frac{\ell\pi a}{b}}\int_0^b g(y)\sin\frac{\ell\pi y}{b}\,dy$$

From (1.3.5)

$$\phi(x,y) = \frac{2}{b}\sum_{\ell=1}^{\infty}\left[\left\{\left(\frac{\sinh\frac{\ell\pi}{b}(a-x)}{\sinh\frac{\ell\pi a}{b}}\right)\int_0^b g(\eta)\sin\frac{\ell\pi\eta}{b}\,d\eta\right\} \times \sin\frac{\ell\pi y}{b}\right] \quad (1.3.10)$$

$$\therefore F_{\ell,r} = E_\ell\left(P_\ell^{r-m-1} - \varphi_\ell^{r-m-1}\right) \quad (1.3.9')$$

On $r=0$, $\underset{\sim}{\Phi}_0 = \{g_1, g_2, \cdots\cdots, g_n\} = \underset{\sim}{G}$, say

$$\therefore F_{\ell,0} = \underset{\sim}{S}_\ell^T\underset{\sim}{G}$$

$$= E_\ell\left(P_\ell^{-m-1} - \varphi_\ell^{-m-1}\right)$$

From (S), $P_\ell\varphi_\ell = 1$

$$\therefore F_{\ell,r} = \frac{\left(P_\ell^{m+1-r} - \varphi_\ell^{m+1-r}\right)}{\left(P_\ell^{m+1} - \varphi_\ell^{m+1}\right)}\underset{\sim}{S}_\ell^T\underset{\sim}{G}$$

From (1.3.5')

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n}\left(\frac{P_\ell^{m+1-r} - \varphi_\ell^{m+1-r}}{P_\ell^{m+1} - \varphi_\ell^{m+1}}\right)\left[\underset{\sim}{S}_\ell^T\underset{\sim}{G}\right]\underset{\sim}{S}_\ell \quad (1.3.10')$$

Both solutions (1.3.10) and (1.3.10') were obtained by transforming between the lower and upper boundaries. In the continuous case this is usually referred to as a transform "in the y direction". We shall adopt a similar terminology for the discrete case and say that the discrete transform has been applied 'in the s direction". Problem A could, however, also be solved in a very similar manner by transforming in the x (or r) direction. We now illustrate this

<u>(i) The Continuous Case</u>

Instead of (1.3.2) we take

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2, \quad (1.3.11)$$

<u>(ii) The Discrete Case</u>

Instead of (1.3.2') we take

$$\frac{1}{R(r)}\left(R(r+1) + R(r-1)\right) = 2\lambda, \quad (1.3.11')$$

where λ is any real number.

$$\therefore X(x) = Ae^{i\lambda x} + Be^{-i\lambda x}.$$

In the manner of the previous method we obtain the eigenfunctions

$$X(x) = A(k)\sin\frac{k\pi x}{a}, \quad k = 1, 2, \cdots$$

by assuming $X(0) = X(a) = 0$.

[Note that, in this case, the eigenvalues λ are given by

$$\lambda = \frac{k\pi}{a}, \quad k = 1, 2, \ldots\ldots \quad]$$

where λ is any real number.

$$\therefore R(r) = Ap^r + Bq^r.$$

In the manner of the previous method we obtain the m discrete eigenfunctions

$$R_k(r) = A_k(p_k^r - q_k^r), \quad k = 1, 2, \cdots, m,$$

by assuming $R(0) = R(m+1) = 0$.

[Note that, in this case, we have m distinct eigenvalues, λ_k say, where

$$\lambda_k = \cos\frac{k\pi}{m+1}, \quad k = 1, 2, \cdots\cdots, m.]$$

Define $\underset{\sim}{R}_k = \{R_k(1), R_k(2), \cdots\cdots, R_k(m)\}$

then, for $r = 1, 2, \cdots\cdots m$, equations (1.3.11') may be written

$$\underset{\sim}{A}\,\underset{\sim}{R}_k = (4 - 2\lambda_k)\underset{\sim}{R}_k.$$

The basic matrix is of the same form as previously but is now of type $(m \times m)$

$$\therefore \underset{\sim}{R}_k^T\underset{\sim}{A} = (4 - 2\lambda_k)\underset{\sim}{R}_k^T. \qquad (*)$$

and we may normalise $\underset{\sim}{R}_k$ such that

$$\underset{\sim}{R}_k^T\underset{\sim}{R}_\ell = \delta_k^\ell.$$

Let $\underset{\sim}{\Phi}_s = \{\phi_{1,s}, \phi_{2,s}, \cdots\cdots, \phi_{m,s}\}$

to give the governing equation

$$\underset{\sim}{A}\,\underset{\sim}{\Phi}_s - \underset{\sim}{\Phi}_{s+1} - \underset{\sim}{\Phi}_{s-1} = \underset{\sim}{\Gamma}_s, \quad 1 \le s \le n,$$

Define the integral transform

$$G(k,y) = \int_0^a \phi(x,y) \sin\frac{k\pi x}{a}\, dx .$$

giving

$$\phi(x,y) = \frac{2}{a}\sum_{k=1}^{\infty} G(k,y) \sin\frac{k\pi x}{a} \qquad (1.3.12)$$

Applying the transform we have

$$\int_0^a \frac{\partial^2\phi}{\partial x^2} \sin\frac{k\pi x}{a}\, dx + \int_0^a \frac{\partial^2\phi}{\partial y^2} \sin\frac{k\pi x}{a}\, dx = 0 \qquad (1.3.13)$$

Now $\displaystyle\int_0^a \frac{\partial^2\phi}{\partial x^2} \sin\frac{k\pi x}{a}\, dx$

$$= \left[\frac{\partial\phi}{\partial x}\sin\frac{k\pi x}{a}\right]_0^a - \left(\frac{k\pi}{a}\right)\int_0^a \frac{\partial\phi}{\partial x}\cos\frac{k\pi x}{a}\, dx$$

$$= -\left(\frac{k\pi}{a}\right)\left\{\left[\phi\cos\frac{k\pi x}{a}\right]_0^a + \left(\frac{k\pi}{a}\right)\int_0^a \phi \sin\frac{k\pi x}{a}\, dx\right\}$$

$$= \frac{k\pi}{a}\, g(y) - \left(\frac{k\pi}{a}\right)^2 \int_0^a \phi \sin\frac{k\pi x}{a}\, dx .$$

since $\phi(0,y) = g(y)$.

$\therefore$ (1.3.13) becomes

$$\frac{d^2}{dy^2} G(k,y) - \left(\frac{k\pi}{a}\right)^2 G(k,y) = -\left(\frac{k\pi}{a}\right) g(y). \qquad (1.3.14)$$

where $\underset{\sim}{\Gamma}_s = \{g_s, 0, \cdots\cdots 0\}$ of type $(m\times 1)$

i.e. the governing equation has now a <u>non-zero right hand side</u>.

Define the discrete transform by

$$G_{k,s} = \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_s$$

giving

$$\underset{\sim}{\Phi}_s = \sum_{k=1}^{m} G_{k,s} \underset{\sim}{R}_k \qquad (1.3.12')$$

Applying the transform we have

$$\underset{\sim}{R}_k^T \underset{\sim}{A}\, \underset{\sim}{\Phi}_s - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s+1} - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s-1} = \underset{\sim}{R}_k^T \underset{\sim}{\Gamma}_s . \qquad (1.3.13')$$

By (*)

$$(4-2\lambda_k)\underset{\sim}{R}_k^T \underset{\sim}{\Phi}_s - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s+1} - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s-1} = \underset{\sim}{R}_k^T \underset{\sim}{\Gamma}_s$$

i.e. $(4-2\lambda_k) G_{k,s} - G_{k,s+1} - G_{k,s-1} = \underset{\sim}{R}_k^T \underset{\sim}{\Gamma}_s$.

(1.3.13') becomes

$$G_{k,s+1} - (4-2\lambda_k) G_{k,s} + G_{k,s-1} = -\underset{\sim}{R}_k^T \underset{\sim}{\Gamma}_s . \qquad (1.3.14')$$

i.e. a <u>non-homogeneous</u> second order differential equation

This equation may be solved by a Green's function technique

The solution is

$$G(k,y) = \int_0^b g(k,\eta)\, K(k;y|\eta)\, d\eta$$

where

$$K(k;y|\eta) = \begin{cases} \dfrac{\left(e^{\frac{k\pi}{a}(b-\eta)} - e^{-\frac{k\pi}{a}(b-\eta)}\right)\left(e^{\frac{k\pi y}{a}} - e^{-\frac{k\pi y}{a}}\right)}{2\left(e^{\frac{k\pi b}{a}} - e^{-\frac{k\pi b}{a}}\right)} & y \leq \eta, \\[2ex] \dfrac{\left(e^{\frac{k\pi}{a}(b-y)} - e^{-\frac{k\pi}{a}(b-y)}\right)\left(e^{\frac{k\pi\eta}{a}} - e^{-\frac{k\pi\eta}{a}}\right)}{2\left(e^{\frac{k\pi b}{a}} - e^{-\frac{k\pi b}{a}}\right)} & y \geq \eta. \end{cases}$$

i.e. a <u>non-homogeneous</u> second order difference equation.

A non-homogeneous difference equation will always result when the matrix governing equation has a non-zero right hand side. This will always be the case when dealing, for example, with Poisson's equation, $\nabla^2\phi = f(x,y)$. The solution of non-homogeneous difference equations is a fairly elaborate process and is illustrated in APPENDIX A. The method is, however, analogous to the Green's function method of solving second-order differential equations. It will suffice, at the moment, to quote the solution of equation (1.3.14').

The solution is

$$G_{k,s} = \sum_{j=1}^{n} K_{k,s}^{(j)} \left[\underset{\sim}{R}_k^T \underset{\sim}{G}_j \right]$$

where

$$K_{k,s}^{(j)} = \begin{cases} \dfrac{\left(p_k^{n+1-j} - q_k^{n+1-j}\right)\left(p_k^{s} - q_k^{s}\right)}{\left(p_k - q_k\right)\left(p_k^{n+1} - q_k^{n+1}\right)}, & s \leq j \\[2ex] \dfrac{\left(p_k^{n+1-s} - q_k^{n+1-s}\right)\left(p_k^{j} - q_k^{j}\right)}{\left(p_k - q_k\right)\left(p_k^{n+1} - q_k^{n+1}\right)}, & s \geq j \end{cases}$$

P and Q are, as before, solutions of

$$\alpha^2-(4-2\lambda)\alpha+1=0.$$

From (1.3.12)

$$\phi(x,y)=\frac{2}{a}\sum_{k=1}^{\infty}\left[\left\{\int_0^b g(k,\eta)K(k;y|\eta)\,d\eta\right\}\times\sin\frac{k\pi x}{a}\right] \quad (1.3.15)$$

From (1.3.12')

$$\underset{\sim}{\Phi}_s=\sum_{k=1}^{m}\left\{\sum_{j=1}^{n}\left\{K_{k,s}^{(j)}\left(\underset{\sim}{R}_k^T\underset{\sim}{G}_j\right)\right\}\right\}\underset{\sim}{R}_k. \quad (1.3.15')$$

It is evident in both the continuous and discrete cases that closed solutions of Dirichlet problems in rectangles are more easily obtained by applying the transform between parallel boundaries on which the function values are zero. From a computational point of view, the closed solutions obtained for problem A by tranforming between zero boundaries are more easily evaluated than those deduced by transforming in the other direction (compare equation (1.3.10') with (1.3.15')). In fact, it is a general rule that both integral and discrete transforms are more simply applied if the transform is taken in the direction of that variable whose axis is perpendicular to those boundaries on which the boundary conditions are most simple.

COROLLARY

It is not necessary that $\phi=0$ on one of the boundaries parallel to the direction of transform and this was introduced only to simplify the illustration. For example, we could have postulated that $\phi=k(y)$ on $x=a$ [Fig. 2 , p 4] The appropriate boundary conditions for the discrete case [Fig. 2'] would then be

$$\phi_{m+1,s}=k_s \quad , \; s=1,2,\dots,n.$$

This would give the solutions

(a) in the continuous case

$$\phi(x,y) = \frac{2}{b}\sum_{l=1}^{n}\left\{\sinh\frac{l\pi x}{b}\int_0^b h(\eta)\sin\frac{l\pi\eta}{b}\,d\eta + \sinh\frac{l\pi}{b}(a-x)\int_0^b g(\eta)\sin\frac{l\pi\eta}{b}\,d\eta\right\}\frac{\sin\frac{l\pi y}{b}}{\sin\frac{l\pi a}{b}},$$

(b) in the discrete case

$$\underset{\sim}{\Phi}_s = \sum_{l=1}^{n}\left\{\frac{(P_l^{r}-\varphi_l^{r})\underset{\sim}{S}_l^T\underset{\sim}{H} + (P_l^{m+1-r}-\varphi_l^{m+1-r})\underset{\sim}{S}_l^T\underset{\sim}{G}}{P_l^{m+1}-\varphi_l^{m+1}}\right\}\underset{\sim}{S}_l \, .$$

where $\underset{\sim}{H} = \{h_1, h_2, \cdots, h_n\}$ of type $(n \times 1)$.

The solution of the general Dirichlet problem in a rectangle could be obtained by superposing these latter solutions with the corresponding solutions of the problem in which ϕ is specified to be non-zero on the upper and lower boundaries. In this case our transform would be in the x (or r) direction.

1.4 Problem A using Diagonal and Nine Point Net Patterns

The method of solution of boundary value problems using diagonal and nine point net patterns is analogous in technique to the use of the square net pattern. Thus, the solution of problem A using these nets will only be sketched briefly. Further, we shall show that the discrete solutions for the diagonal and nine point patterns can be expressed in forms identical with those already obtained (§ 1.3) using a square net. Before proceeding with this, however, the effect of non-zero function values at the corners of the rectangle should be considered.

With the square net pattern no use is made of the function values at the corners of the rectangle since none of these appear in the difference equation for any interior point, e.g. at point $(1, 1)$ [Fig. 4] the difference equation is

$$4\phi_{1,1} - \phi_{2,1} - \phi_{1,2} - \phi_{0,1} - \phi_{1,0} = 0,$$

and does not involve the value $\phi_{0,0}$. We may, therefore, speak of zero boundaries for a square net even when the function values at the extremeties are non-zero.

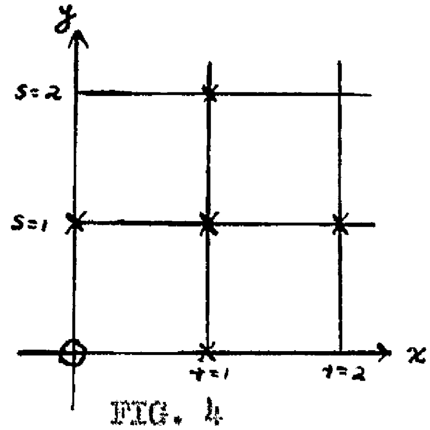


FIG. 4

Diagonal and nine point net patterns do, however, make use of the corner values, e.g. at point $(1, 1)$ [Fig. 4] the diagonal net pattern is

$$4\phi_{1,1} - \phi_{2,2} - \phi_{0,2} - \phi_{0,0} - \phi_{2,0} = 0.$$

Since none of the column vectors $\underset{\sim}{\Phi}$ (§ 1.3) involve corner values, in formulating the set of diagonal or nine point difference equations in matrix form corner values must be placed on the right hand side of the equation. This gives a matrix governing equation with a non-zero right hand side and the resulting second-order difference equation is non-homogeneous. For simplicity, in this section we shall consider the corner values to be zero and shall transform in the "s" direction. To illustrate the analogy in technique between the use of these patterns and the square net pattern equations are numbered in correspondence with those of the previous section.

(a) <u>The Diagonal Net</u>

The difference equations are (§ 1.1)

$$4\phi_{r,s} - \phi_{r+1,s+1} - \phi_{r-1,s+1} - \phi_{r-1,s-1} - \phi_{r+1,s-1} = 0, \quad (1.4.1'a)$$

where $1 \leq r \leq m$, $1 \leq s \leq n$. [Fig. 5]

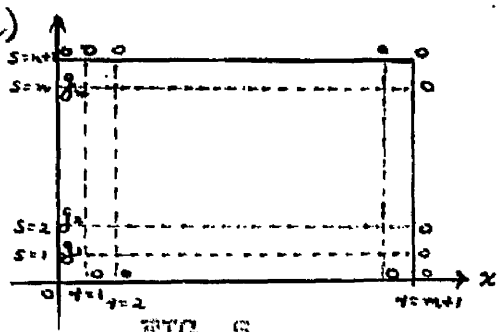


FIG. 5

Let $\phi_{r,s} = R(r)S(s)$.

(1.4.1'a) becomes

$$4 - \left(\frac{R(r+1) + R(r-1)}{R(r)}\right)\left(\frac{S(s+1) + S(s-1)}{S(s)}\right) = 0.$$

Let $$\frac{1}{S(s)}\left(S(s+1) + S(s-1)\right) = 2\lambda, \quad (1.4.2'a)$$

where λ is any real number.

We obtain $S_\ell(s) = A_\ell(h_\ell^s - q_\ell^s)$ where λ_ℓ, h_ℓ and q_ℓ are exactly the same as in § 1.3.

Define $\underset{\sim}{S}_\ell = \{S_\ell(1), S_\ell(2), \ldots\ldots, S_\ell(n)\}$ normalised as in § 1.3, and

$$\underset{\sim}{\Phi}_r = \{\phi_{r,1}, \phi_{r,2}, \ldots\ldots, \phi_{r,n}\}.$$

The <u>matrix governing equation of the problem for the diagonal net</u> is then

$$4\underset{\sim}{\Phi}_r - \underset{\sim}{B}\left(\underset{\sim}{\Phi}_{r+1} + \underset{\sim}{\Phi}_{r-1}\right) = \underset{\sim}{O}_{n\times 1}, \quad 1 \leq r \leq m, \quad (1.4.\beta a)$$

where the <u>basic matrix</u> $\underset{\sim}{B}$ is of type $(n \times n)$ and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & \ldots & 0 & 1 & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & 1 & 0 \end{bmatrix}$$

$$\therefore\ \underset{\sim}{B}\,\underset{\sim}{S}_\ell = 2\lambda_\ell\,\underset{\sim}{S}_\ell \qquad (1.4.\alpha a)$$

and, since $\underset{\sim}{B}$ is symmetric, $\underset{\sim}{S}_\ell^T\underset{\sim}{B} = 2\lambda_\ell\,\underset{\sim}{S}_\ell^T$. (1.4.γa)

Applying the discrete transform, $F_{\ell,r} = \underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_r$, to equation (1.4.βa), we obtain

$$4\,\underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_r - \underset{\sim}{S}_\ell^T\underset{\sim}{B}\left(\underset{\sim}{\Phi}_{r+1} + \underset{\sim}{\Phi}_{r-1}\right) = 0, \qquad (1.4.6'a)$$

i.e. $$4\,F_{\ell,r} - 2\lambda_\ell\left(F_{\ell,r+1} + F_{\ell,r-1}\right) = 0,$$

i.e. $$\lambda_\ell F_{\ell,r+1} - 2F_{\ell,r} + \lambda_\ell F_{\ell,r-1} = 0, \qquad (1.4.8'a)$$

$$\therefore\ F_{\ell,r} = C_\ell u_\ell^r + D_\ell V_\ell^r ,$$

where u_ℓ , V_ℓ are solutions of $\lambda_\ell\alpha^2 - 2\alpha + \lambda_\ell = 0$. (1.4.8a)

The boundary conditions on $r=0$ and $r=m+1$ then give

$$F_{\ell,r} = \frac{\left(u_\ell^{m+1-r} - V_\ell^{m+1-r}\right)}{\left(u_\ell^{m+1} - V_\ell^{m+1}\right)}\,\underset{\sim}{S}_\ell^T\underset{\sim}{G} ,$$

where $\underset{\sim}{G} = \{g_1, g_2, \cdots\cdots, g_n\}$.

By the inverse formula, $\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} F_{\ell,r}\,\underset{\sim}{S}_\ell$, we obtain the solution

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n}\left(\frac{u_\ell^{m+1-r} - V_\ell^{m+1-r}}{u_\ell^{m+1} - V_\ell^{m+1}}\right)\left[\underset{\sim}{S}_\ell^T\underset{\sim}{G}\right]\underset{\sim}{S}_\ell . \qquad (1.4.10'a)$$

For the square net pattern we recall that the solution is (equation (1.3.10'))

$$\underset{\sim r}{\Phi} = \sum_{\ell=1}^{n} \left(\frac{P_\ell^{m+1-r}(\lambda) - \varphi_\ell^{m+1-r}(\lambda)}{P_\ell^{m+1}(\lambda) - \varphi_\ell^{m+1}(\lambda)} \right) \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G} \right] \underset{\sim}{S}_\ell \qquad (*)$$

where, equation (δ) (§ 1.3), $P_\ell(\lambda) = 2 - \lambda_\ell + \sqrt{3 - 4\lambda_\ell + \lambda_\ell^2}$,

$$\varphi_\ell(\lambda) = 2 - \lambda_\ell - \sqrt{3 - 4\lambda_\ell + \lambda_\ell^2} .$$

Thus the solutions for the diagonal and square net patterns are similar. We can, however, make these solutions identical in form as follows.

From (1.4.8a), $u_\ell = \dfrac{1 + \sqrt{1 - \lambda_\ell^2}}{\lambda_\ell}$, $v_\ell = \dfrac{1 - \sqrt{1 - \lambda_\ell^2}}{\lambda_\ell}$.

Let $\mu_\ell = 2 - \dfrac{1}{\lambda_\ell}$ i.e. $\lambda_\ell = \dfrac{1}{2 - \mu_\ell}$ then

$$u_\ell(\mu) = 2 - \mu_\ell + \sqrt{3 - 4\mu_\ell + \mu_\ell^2} \qquad (\text{c.f. } P_\ell)$$

$$v_\ell(\mu) = 2 - \mu_\ell - \sqrt{3 - 4\mu_\ell + \mu_\ell^2} \qquad (\text{c.f. } \varphi_\ell)$$

We may therefore write the diagonal net solution as:

$$\underset{\sim r}{\Phi} = \sum_{\ell=1}^{n} \left(\frac{P_\ell^{m+1-r}(\mu) - \varphi_\ell^{m+1-r}(\mu)}{P_\ell^{m+1}(\mu) - \varphi_\ell^{m+1}(\mu)} \right) \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G} \right] \underset{\sim}{S}_\ell \qquad (1.4.10'a)$$

Solution (1.4.10'a) is identical with the square net solution (*) except that, in the former the evaluation of P_ℓ and φ_ℓ is carried out using μ_ℓ , in the latter λ_ℓ is used.

(b) The Nine Point Net

The difference equations are (§ 1.1)

$$20\phi_{r,s} - 4(\phi_{r+1,s} + \phi_{r,s+1} + \phi_{r-1,s} + \phi_{r,s-1}) - (\phi_{r+1,s+1} + \phi_{r-1,s+1} + \phi_{r-1,s-1} + \phi_{r+1,s-1}) = 0, \quad (1.4.1'b)$$

where $1 \le r \le m$, $1 \le s \le n$.

Let $\phi_{r,s} = R(r)\,S(s)$.

(1.4.1'b) becomes

$$20 - 4\left\{\frac{(R(r+1)+R(r-1))}{R(r)} + \frac{(S(s+1)+S(s-1))}{S(s)}\right\} - \left(\frac{R(r+1)+R(r-1)}{R(r)}\right)\left(\frac{S(s+1)+S(s-1)}{S(s)}\right) = 0.$$

Let $\frac{1}{S(s)}\left(S(s+1)+S(s-1)\right) = 2\lambda,$ (1.4.2'b)

where λ is any real number, to give $S_\ell(s) = A_\ell(h_\ell^{\,s} - \rho_\ell^{\,s})$ and λ_ℓ, h_ℓ and ρ_ℓ are as in § 1.3.

In the usual notation the matrix governing equation of the problem for the nine point net is

$$4\,\underset{\sim}{C}\,\underset{\sim}{\Phi}_r - \underset{\sim}{D}\left(\underset{\sim}{\Phi}_{r+1} + \underset{\sim}{\Phi}_{r-1}\right) = \underset{\sim}{0}_{n\times 1}, \quad 1 \le r \le m, \qquad (1.4.\beta b)$$

where basic matrices $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are of type ($n \times n$) and

$$\underset{\sim}{C} = \begin{bmatrix} 5 & -1 & 0 & \cdots & & & 0 \\ -1 & 5 & -1 & 0 & \cdots & & 0 \\ 0 & -1 & 5 & -1 & 0 & \cdots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdots & & 0 & -1 & 5 & -1 \\ 0 & \cdots & & & 0 & -1 & 5 \end{bmatrix}; \quad \underset{\sim}{D} = \begin{bmatrix} 4 & 1 & 0 & \cdots & & & 0 \\ 1 & 4 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdots & & 0 & 1 & 4 & 1 \\ 0 & \cdots & & & 0 & 1 & 4 \end{bmatrix}$$

$$\therefore\ \underset{\sim}{C}\,\underset{\sim}{S}_\ell = (5-2\lambda_\ell)\underset{\sim}{S}_\ell;\quad \underset{\sim}{D}\,\underset{\sim}{S}_\ell = (4+2\lambda_\ell)\underset{\sim}{S}_\ell \qquad (1.4.\text{c}b)$$

$$\therefore\ \underset{\sim}{S}_\ell^T\underset{\sim}{C} = (5-2\lambda_\ell)\underset{\sim}{S}_\ell^T;\quad \underset{\sim}{S}_\ell^T\underset{\sim}{D} = (4+2\lambda_\ell)\underset{\sim}{S}_\ell^T \qquad (1.4.\gamma b)$$

Applying the discrete transform, $F_{\ell,r} = \underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_r$, to equation (1.4.βb) we obtain

$$4\,\underset{\sim}{S}_\ell^T\underset{\sim}{C}\,\underset{\sim}{\Phi}_r - \underset{\sim}{S}_\ell^T\underset{\sim}{D}\left(\underset{\sim}{\Phi}_{r+1} + \underset{\sim}{\Phi}_{r-1}\right) = 0. \qquad (1.4.6'b)$$

i.e. $$4(5-2\lambda_\ell)F_{\ell,r} - (4+2\lambda_\ell)(F_{\ell,r+1} + F_{\ell,r-1}) = 0$$

i.e. $$(2+\lambda_\ell)F_{\ell,r+1} - 2(5-2\lambda_\ell)F_{\ell,r} + (2+\lambda_\ell)F_{\ell,r-1} = 0 \qquad (1.4.8'b)$$

$$\therefore \quad F_{\ell,r} = C_\ell M_\ell^{\,r} + D_\ell N_\ell^{\,r}$$

where M_ℓ, N_ℓ are solutions of $(2+\lambda_\ell)\alpha^2 - 2(5-2\lambda_\ell)\alpha + (2+\lambda_\ell) = 0.$ (1.4.8b)

The boundary conditions on $r=0$ and $r=m+1$ then give

$$F_{\ell,r} = \frac{(M_\ell^{\,m+1-r} - N_\ell^{\,m+1-r})}{(M_\ell^{\,m+1} - N_\ell^{\,m+1})}\, \underset{\sim}{S}_\ell^T \underset{\sim}{G}$$

and the solution is

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} \left(\frac{M_\ell^{\,m+1-r} - N_\ell^{\,m+1-r}}{M_\ell^{\,m+1} - N_\ell^{\,m+1}} \right) \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G}\right] \underset{\sim}{S}_\ell \qquad (1.4.10b)$$

This solution is again similar to the square net pattern (*) and may be made <u>identical</u> in form by the substitution

$$\nu_\ell = \frac{4\lambda_\ell - 1}{2+\lambda_\ell} \qquad \text{i.e.} \quad \lambda_\ell = \frac{2\nu_\ell + 1}{4 - \nu_\ell}\,.$$

We may then write the nine point net solution as

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} \left(\frac{P_\ell^{\,m+1-r}(\nu) - Q_\ell^{\,m+1-r}(\nu)}{P_\ell^{\,m+1}(\nu) - Q_\ell^{\,m+1}(\nu)} \right) \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G}\right] \underset{\sim}{S}_\ell \qquad (1.4.10'b)$$

Thus, for the nine point pattern solution P_ℓ and Q_ℓ are evaluated using ν_ℓ instead of λ_ℓ.

1.5 The Main Points of the Continuous-Discrete Analogy

The parallelism between the continuous and discrete techniques is evident and holds for any of the three net patterns. In particular, the example (§ 1.3) shows that

(i) pre-multiplying the matrix governing equation of the discrete problem by a row eigenvector of the basic matrix is equivalent to applying the integral transform to the partial differential equation (equations (1.3.4' - 1.3.8')),

(ii) the discrete inverse formula (equation (1.3.5')) exists and consists of only a finite linear combination of eigenvectors, c.f. equation (1.3.5),

(iii) the effect of pre-multiplying the basic matrix by one of its row eigenvectors (equation (1.3.7')) is analogous to integration by parts (equation (1.3.7)),

(iv) application of the discrete transform technique reduces the problem to the solution of a second-order difference equation (equation (1.3.8')), c.f. a second-order differential equation (equation (1.3.8)),

(v) (from APPENDIX A) the technique of solving a non-homogeneous second-order difference equation (equation (1.3.14')) is analogous to the Green's function method of solving a non-homogeneous second-order differential equation (equation (1.3.14)).

All the basic matrices arising in problem A were symmetric. Though convenient, this is not essential for the application of the discrete transform. If the basic matrices in a particular problem are not symmetric the idea of biorthogonality may be introduced as discussed in the next section. The direct occurrence of symmetric basic matrices <u>in the problems of this thesis</u> depends on two factors:

(a) the nature of the conditions on the parallel boundaries between which the transform is applied, and

(b) the positioning of the covering lattice in relation to these boundaries.

In problem A the basic matrices were symmetric because we chose the lattice to coincide with the boundaries at $s=0$ and $s=(n+1)h$ on which ϕ was specified. If the conditions on these boundaries had been that $\frac{\partial\phi}{\partial n}=0$, where $\frac{\partial\phi}{\partial n}$ is the derivative normal to the boundary, a non-symmetric matrix would have resulted. This is due to the fact that the reflection condition on such a boundary affects one of the off-diagonal elements of the matrix. For example, in problem A if the condition on $y=0$ had been that $\frac{\partial\phi}{\partial y}=0$, i.e. $\phi_{r,1}=\phi_{r,-1}$ (Fig. 2', p. 4), then the basic matrix $\underset{\sim}{A}$ would have been of type $(n+1)\times(n+1)$ (since we now have to consider difference equations on $s=0$) and

$$\underset{\sim}{A}=\begin{bmatrix} 4 & -2 & 0 & . & . & . & . & . & 0 \\ -1 & 4 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 4 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & -1 & 4 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 4 \end{bmatrix} \qquad (1.5.1)$$

Non-symmetric basic matrices can sometimes be converted into symmetric form, however, by a slight alteration in the definitions of the vectors $\underset{\sim}{S}$ and $\underset{\sim}{\Phi}$ (§ 1.3). This we now illustrate for the matrix $\underset{\sim}{A}$ (equation (1.5.1)). Proceeding as in § 1.3 we have, for the above amended problem A,

$$\underset{\sim}{A}\underset{\sim}{S}_{\ell}=(4-2\lambda_{\ell})\underset{\sim}{S}_{\ell}\ , \qquad (1.5.2)$$

$$\underset{\sim}{A}\underset{\sim}{\Phi}_{r}-\underset{\sim}{\Phi}_{r+1}-\underset{\sim}{\Phi}_{r-1}=\underset{\sim}{O}_{(n+1)\times 1}\ ,\quad 1\leq r\leq m, \qquad (1.5.3)$$

where $\underset{\sim}{S}_\ell = \{ S_\ell(0), S_\ell(1), \ldots\ldots\ldots, S_\ell(n) \}$, $\underset{\sim}{\Phi}_r = \{\phi_{r,0}, \phi_{r,1}, \ldots\ldots, \phi_{r,n}\}$, (1.5.4)

$\lambda_\ell = \cos\frac{(2\ell+1)\pi}{2(n+1)}$, $\ell = 0,1,2,\ldots, n$, and $\underset{\sim}{A}$ is given by (1.5.1).

If $\underset{\sim}{M}$ is any non-singular matrix of the same order as $\underset{\sim}{A}$, pre-multiplying equation (1.5.3) by $\underset{\sim}{M}$ gives

$$\underset{\sim}{M}\,\underset{\sim}{A}\,\underset{\sim}{\Phi}_r - \underset{\sim}{M}\,\underset{\sim}{\Phi}_{r+1} - \underset{\sim}{M}\,\underset{\sim}{\Phi}_{r-1} = \underset{\sim}{0}. \qquad (1.5.5)$$

Let $\underset{\sim}{M}\,\underset{\sim}{\Phi}_r = \underset{\sim}{\Phi}'_r$ to give $\underset{\sim}{\Phi}_r = \underset{\sim}{M}^{-1}\underset{\sim}{\Phi}'_r$. (1.5.6)

(1.5.5) becomes $\underset{\sim}{M}\,\underset{\sim}{A}\,\underset{\sim}{M}^{-1}\underset{\sim}{\Phi}'_r - \underset{\sim}{\Phi}'_{r+1} - \underset{\sim}{\Phi}'_{r-1} = \underset{\sim}{0}$,

i.e. $$\underset{\sim}{A}'\underset{\sim}{\Phi}'_r - \underset{\sim}{\Phi}'_{r+1} - \underset{\sim}{\Phi}'_{r-1} = \underset{\sim}{0}, \qquad (1.5.7)$$

where $\underset{\sim}{A}' = \underset{\sim}{M}\,\underset{\sim}{A}\,\underset{\sim}{M}^{-1}$.

We now require to determine $\underset{\sim}{M}$ so that $\underset{\sim}{A}'$ is symmetric. It is sufficient to choose $\underset{\sim}{M}$ as a diagonal matrix and, in fact,

$$\underset{\sim}{M} = \text{diag.}\left\{\frac{1}{\sqrt{2}}, 1, 1, \ldots, 1\right\} \text{ of order } (n+1)\times(n+1). \qquad (1.5.8)$$

$$\therefore\ \underset{\sim}{A}' = \begin{bmatrix} 4 & -\sqrt{2} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -\sqrt{2} & 4 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & -1 & 4 & -1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 4 \end{bmatrix}$$

From (1.5.4), (1.5.6), and (1.5.8), $\underset{\sim}{\Phi}'_r = \left\{\frac{\phi_{r,0}}{\sqrt{2}}, \phi_{r,1}, \ldots\ldots\ldots, \phi_{r,n}\right\}$.
Further, if we define $\underset{\sim}{S}'_\ell$ as $\underset{\sim}{S}'_\ell = \left\{\frac{S_\ell(0)}{\sqrt{2}}, S_\ell(1), \ldots, S_\ell(n)\right\}$ we can show that equation (1.5.2) may be written

$$\underset{\sim}{A}'\underset{\sim}{S}'_\ell = (4-2\lambda_\ell)\underset{\sim}{S}'_\ell,$$

and, therefore, $\underset{\sim}{S}_\ell'^T \underset{\sim}{A}' = (4-2\lambda_\ell)\underset{\sim}{S}_\ell'^T$.

The problem may then be solved in the usual way by applying the discrete transform, $F_{\ell,r}' = \underset{\sim}{S}_\ell'^T \underset{\sim}{\Phi}_r'$, to the modified governing equation (1.5.7), finding the particular solution of the resulting difference equation satisfying the boundary conditions at $r=0$ and $r=(m+1)h$, and substituting in the inverse transform formula, $\underset{\sim}{\Phi}_r' = \sum_{\ell=0}^{\tilde{m}} F_{\ell,r}' \underset{\sim}{S}_\ell'$.

It is possible, however, to position the lattice to give a symmetric basic matrix <u>directly</u> when a Neumann boundary condition exists perpendicular to the direction of transform. We shall discuss this in section 1.7.

<u>1.6 The Discrete Transform Method from a Matrix Stand-Point</u>

In the previous sections we introduced the discrete transform method as the analogue of the separation of variables and transform technique of continuous analysis. We might, however, have introduced the method purely from a matrix stand-point. The basic equation to be solved in problem A (§ 1.3) was equation (β), p. 7, namely

$$\underset{\sim}{A}\underset{\sim}{\Phi}_r - \underset{\sim}{\Phi}_{r+1} - \underset{\sim}{\Phi}_{r-1} = \underset{\sim}{0} \tag{1.6.1}$$

This was solved essentially by finding eigenvalues, $\mu_\ell = (4-2\lambda_\ell)$, and eigenvectors $\underset{\sim}{S}_\ell$ of $\underset{\sim}{A}$:

$$\underset{\sim}{A}\underset{\sim}{S}_\ell = (4-2\lambda_\ell)\underset{\sim}{S}_\ell.$$

Then on pre-multiplying equation (1.6.1) by $\underset{\sim}{S}_\ell^T$ we found

$$(4-2\lambda_\ell)\underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_r - \underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_{r+1} - \underset{\sim}{S}_\ell^T\underset{\sim}{\Phi}_{r-1} = 0. \tag{1.6.2}$$

When $\underset{\sim}{A}$ is non-symmetrical we need to introduce biorthogonal eigenvectors, $\underset{\sim}{S}_\ell$ and $\underset{\sim}{S}_\ell^*$, such that

$$\underset{\sim}{A}\,\underset{\sim}{S}_\ell = (4-2\lambda_\ell)\underset{\sim}{S}_\ell ,$$

$$\underset{\sim}{A}^T \underset{\sim}{S}_\ell^* = (4-2\lambda_\ell)\underset{\sim}{S}_\ell^* \text{ or } \underset{\sim}{S}_\ell^{*T}\underset{\sim}{A} = (4-2\lambda_\ell)\underset{\sim}{S}_\ell^{*T}.$$

On pre-multiplying equation (1.6.1) by $\underset{\sim}{S}_\ell^{*T}$ we find, instead of (1.6.2),

$$(4-2\lambda_\ell)\underset{\sim}{S}_\ell^{*T}\underset{\sim}{\Phi}_+ - \underset{\sim}{S}_\ell^{*T}\underset{\sim}{\Phi}_{++1} - \underset{\sim}{S}_\ell^{*T}\underset{\sim}{\Phi}_{+-1} = 0. \qquad (1.6.3)$$

The discrete transform is then $F_{\ell,+} = \underset{\sim}{S}_\ell^{*T}\underset{\sim}{\Phi}_+$ with the expansion formula $\underset{\sim}{\Phi}_+ = \sum F_{\ell,+}\underset{\sim}{S}_\ell$ and the extension of the analysis is obvious.

In order to give an example where a non-symmetrical matrix will always arise independent of the positioning of the net and the conditions on the boundaries, consider Laplace's equation in two dimensions in cylindrical polar co-ordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{\partial^2\phi}{\partial z^2} = 0. \qquad (1.6.4)$$

If we consider a square net of mesh length h and take step measures i and j in z and ρ directions respectively, then, in difference form (see e.g. Hartree [15]), equation (1.6.1) may be written

$$4\phi_{i,j} - \phi_{i+1,j} - \phi_{i-1,j} - \left(1+\frac{1}{2j}\right)\phi_{i,j+1} - \left(1-\frac{1}{2j}\right)\phi_{i,j-1} = 0 \qquad (1.6.5)$$

where $\phi_{i,j} = \phi(ih, jh)$.

Separate the variables by setting $\phi_{i,j} = Z(i)W(j)$ and suppose we wish to transform in the ρ (or j) direction. The difference equation from which the eigenvalues and eigenvectors are determined is then

$$\left(1+\frac{1}{2j}\right)W(j+1) - 2\lambda W(j) + \left(1-\frac{1}{2j}\right)W(j-1) = 0. \qquad (1.6.6)$$

This will give rise to a non-symmetrical matrix $\underset{\sim}{A}$. We cannot derive explicit expressions for the eigenvalues, λ, or the eigenvectors, $\underset{\sim}{W}$, from equation

(1.6.6) as we could in previous sections and, for any particular case, it may well be convenient to use standard machine programmes to obtain these.

We shall not approach any of the problems of this thesis from the purely matrix stand-point and shall not require to obtain eigenvalues or eigenvectors of basic matrices by any means other than the evaluation of explicit expressions derived by the use of separation of variables.

1.7 PROBLEM B A Boundary-Value Problem for Poisson's Equation in a Rectangle

This problem is introduced to illustrate two points:

(1) the positioning of the lattice to give directly a symmetric basic matrix when the transform is applied between boundaries where the conditions are of Neumann type,

(2) the analogy between the method of solution of a non-homogeneous second-order difference equation and the Green's function method of solving a non-homogeneous second-order differential equation.

Consider the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ [Fig. 6], with $\frac{\partial \phi}{\partial x} = 0$ on $x = 0$, $x = a$, and $\phi = 0$ on $y = 0$, $y = b$. To illustrate point (1) we will transform in the x direction.

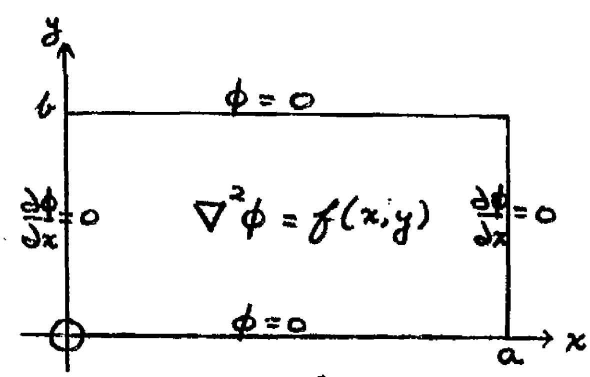


FIG. 6

The Discrete Problem B

It was pointed out (§ 1.5) that, with the net co-incident with a Neumann type boundary, the reflection condition affects one of the off-diagonal elements of the basic matrix resulting in non-symmetry. Thus, to obtain a symmetric basic matrix directly, the lattice must be positioned so that the reflection condition affects only a diagonal element. This may be accomplished by choosing the lattice to straddle such boundaries. Besides being more convenient the use of a straddling net permits a more accurate difference representation of the derivative boundary condition. Since the function values are specified on the upper and lower boundaries we shall choose our net to coincide with these.

Thus, we cover the rectangle with a square lattice such that the boundaries of the region are given by $r = -\frac{1}{2}$, $r = m + \frac{1}{2}$, $s = 0$ and $s = n+1$ [Fig. 7]. The boundary conditions are then

$$\phi_{-1,s} = \phi_{0,s} \; ; \; \phi_{m,s} = \phi_{m+1,s} \quad , s = 1, 2, \ldots\ldots, n, \tag{1.7.1}$$

$$\phi_{r,0} = \phi_{r,n+1} = 0 \quad , \quad r = 0, 1, 2, \ldots\ldots, m. \tag{1.7.2}$$

As was pointed out in § 1.4 the solution using square, diagonal and nine point net patterns may be obtained by the same techniques and we shall illustrate this problem using the square net pattern.

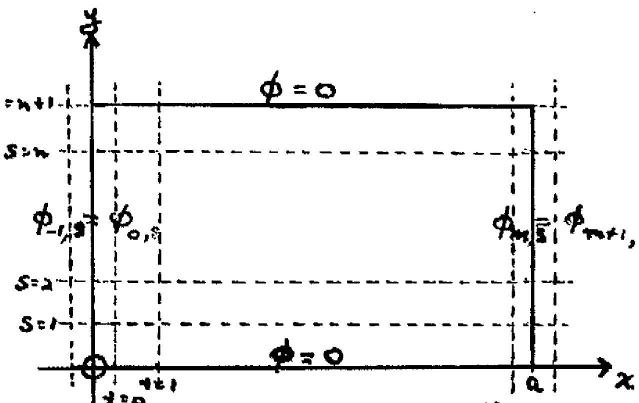


FIG. 7

[Since corner points of the rectangle are not mesh points there is no difficulty over corner values in the use of diagonal or nine point patterns in this problem.]

For Poisson's equation, $\nabla^2\phi = f(x,y)$, the corresponding set of difference equations will, therefore, be taken as

$$4\phi_{r,s} - \phi_{r+1,s} - \phi_{r,s+1} - \phi_{r-1,s} - \phi_{r,s-1} = -h^2 f_{r,s} \, ,$$

where $0 \leq r \leq m$, $1 \leq s \leq n$, and h is the mesh length.

The Solution of Problem B by the Discrete Transform Method

The method of solution is again analogous to the separation of variables and transform technique of continuous analysis.

The difference equations are, as above,

$$4\phi_{r,s} - \phi_{r+1,s} - \phi_{r,s+1} - \phi_{r-1,s} - \phi_{r,s-1} = -h^2 f_{r,s}, \qquad (1.7.3)$$

where $0 \leq r \leq m$, $1 \leq s \leq n$.

Let $\phi_{r,s} = R(r)S(s)$ and, since we are going to transform in the r direction, take

$$\frac{1}{R(r)}\left(R(r+1) + R(r-1)\right) = 2\lambda, \qquad (1.7.4)$$

where λ is any real number. (In the usual way, this is obtained by separation of variables in the homogeneous equation corresponding to (1.7.3).)

$$\therefore \quad R(r) = Ap^r + Bq^r,$$

where p, q are solutions of $\alpha^2 - 2\lambda\alpha + 1 = 0$.

On applying the boundary conditions (1.7.1) it can be shown that

$$R_k(r) = C_k\left(p_k^{r+1/2} + q_k^{r+1/2}\right), \; k = 0,1,2,\ldots,m, \qquad (1.7.5)$$

where $\lambda_k = \cos\frac{k\pi}{m+1}$.

Let $\underset{\sim}{R}_k = \{R_k(0), R_k(1), \ldots\ldots\ldots, R_k(m)\}$, and

$\underset{\sim}{\Phi}_s = \{\phi_{0,s}, \phi_{1,s}, \ldots\ldots\ldots, \phi_{m,s}\}$.

On account of straddling the boundaries at $r = -\frac{1}{2}$ and $r = m + \frac{1}{2}$ the basic matrix $\underset{\sim}{A}$ is

$$\underset{\sim}{A} = \begin{bmatrix} 3 & -1 & 0 & . & . & . & . & . & 0 \\ -1 & 4 & -1 & 0 & . & . & . & . & 0 \\ 0 & -1 & 4 & -1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & -1 & 4 & -1 \\ 0 & . & . & . & . & . & 0 & -1 & 3 \end{bmatrix}$$

of type $(m+1) \times (m+1)$ (c.f. equation (1.5.1)).

The <u>matrix governing equation of the problem for the square net</u> is then

$$\underset{\sim}{A}\underset{\sim}{\Phi}_s - \underset{\sim}{\Phi}_{s+1} - \underset{\sim}{\Phi}_{s-1} = -\underset{\sim}{F}_s \,, \quad 1 \leq s \leq n, \qquad (1.7.6)$$

where $\underset{\sim}{F}_s = h^2 \{ \zeta_{0,s}, \zeta_{1,s}, \ldots, \zeta_{m,s} \}$.

From equation (1.7.4) with $r = 0, 1, 2, \ldots, m$, we have

$$\underset{\sim}{A}\underset{\sim}{R}_k = (4 - 2\lambda_k)\underset{\sim}{R}_k \,,$$

$$\text{i.e. } \underset{\sim}{R}_k^T \underset{\sim}{A} = (4 - 2\lambda_k)\underset{\sim}{R}_k^T \,,$$

and we will regard the $\underset{\sim}{R}_k$ normalised to unit length. The discrete transform is then

$$G_{k,s} = \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_s \,,$$

and the inverse formula is $$\underset{\sim}{\Phi}_s = \sum_{k=0}^{m} G_{k,s}\underset{\sim}{R}_k \,. \qquad (1.7.7)$$

Applying this transform to equation (1.7.6) we have

$$\underset{\sim}{R}_k^T \underset{\sim}{A}\underset{\sim}{\Phi}_s - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s+1} - \underset{\sim}{R}_k^T \underset{\sim}{\Phi}_{s-1} = -\underset{\sim}{R}_k^T \underset{\sim}{F}_s \,.$$

$$\therefore \quad G_{k,s+1} - (4 - 2\lambda_k)G_{k,s} + G_{k,s-1} = \beta_{k,s} \,, \qquad (1.7.8)$$

where $\beta_{k,s} = \underset{\sim}{R}_k^T \underset{\sim}{F}_s$ i.e. a non-homogeneous second-order difference equation. From the boundary conditions (1.7.2) we require the solution of equation (1.7.8) subject to the conditions that $G_{k,0} = G_{k,n+1} = 0$. This may be obtained, as illustrated in Appendix A, by a "discrete Green's function method". [In fact, equation (1.7.8) is the example chosen for illustration in Appendix A (ii)]. We obtain

$$G_{k,s} = \sum_{j=1}^{n} K_{k,s}^{(j)} \beta_{k,j} \,, \qquad (1.7.9)$$

$$\text{where} \quad K_{k,s}^{(j)} = \begin{cases} \dfrac{(p_k^{\,j-n-1} - \varphi_k^{\,j-n-1})(p_k^{\,s} - \varphi_k^{\,s})}{(p_k - \varphi_k)(p_k^{\,n+1} - \varphi_k^{\,n+1})} \,, & s \leq j \,, \\[2ex] \dfrac{(p_k^{\,j} - \varphi_k^{\,j})(p_k^{\,s-n-1} - \varphi_k^{\,s-n-1})}{(p_k - \varphi_k)(p_k^{\,n+1} - \varphi_k^{\,n+1})} \,, & s \geq j \,, \end{cases} \qquad (1.7.10)$$

and the boundary conditions are satisfied.

The solution of problem B is then obtained from the inversion formula (1.7.7) as

$$\underset{\sim}{\Phi}_s = \sum_{k=0}^{m}\left(\sum_{j=1}^{n} K_{k,s}^{(j)} \beta_{k,j}\right) \underset{\sim}{R}_k . \tag{1.7.11}$$

<u>Note</u> As for problem A, the solutions of problem B for diagonal and nine point net patterns can be expressed in forms identical with equation (1.7.11). With these patterns, however, multiplying factors of 2 and 6 respectively occur in the definition of $\underset{\sim}{F}_s$ (equation (1.7.6)) from which $\beta_{k,s}$ (equation (1.7.8)) is derived. These factors arise from the sets of difference equations corresponding to Poisson's equation. Instead of equation (1.7.3) we have

(a) for the diagonal net pattern

$$4\phi_{r,s} - \phi_{r+1,s+1} - \phi_{r-1,s+1} - \phi_{r-1,s-1} - \phi_{r+1,s-1} = -2h^2\zeta_{r,s}, \tag{1.7.3a}$$

and (b) for the nine point net pattern

$$20\phi_{r,s} - 4(\phi_{r+1,s} + \phi_{r,s+1} + \phi_{r-1,s} + \phi_{r,s-1}) - (\phi_{r+1,s+1} + \phi_{r-1,s+1} + \phi_{r-1,s-1} + \phi_{r+1,s-1}) = -6h^2\zeta_{r,s} \tag{1.7.3b}$$

<u>1.8 Solving Boundary-Value Problems in Rectangles by the Discrete Transform Method</u>

In this section we list the basic steps involved in the use of the discrete transform method to solve the type of boundary value problems in rectangles which will occur later in this thesis. These steps have been developed in sections 1.3, 1.5 and 1.7 and are as follows:

(1) If possible, choose the network

(a) to coincide with those boundaries on which Dirichlet conditions are specified, and (b) to straddle those boundaries where conditions involve a derivative.

[The words "if possible" have been inserted here since, as we shall see, one is not always able to adhere strictly to rules (a) and (b).]

(2) Separate the variables in accordance with the direction of transform.

(3) Apply the conditions on the boundaries perpendicular to the direction of transform to obtain explicit expressions for the eigenvalues and corresponding eigenvectors of the basic matrix.

(4) Write down the matrix governing equation and apply the discrete transform i.e. premultiply throughout by a row eigenvector of the basic matrix. If the basic matrix is not symmetric it must be made so by amending the definitions of the column vectors $\underset{\sim}{\Phi}$ of function values and the expressions for the eigenvectors obtained in (3). (We have illustrated how this may be performed in § 1.5.)

(5) For the resulting second-order difference equation obtain the solution satisfying the conditions on the boundaries parallel to the direction of transform.

(6) Substitute the solution (5) in the inverse transform formula.

Note: In transforming between parallel boundaries on which Dirichlet conditions are specified it has already been pointed out (p. 13) that the discrete transform method is most simply applied when the boundary values ϕ are zero. If these function values are non-zero but constant we may always express ϕ as a linear function of ψ, say, where ψ has zero values on these boundaries, e.g. in problem A (§ 1.3, p. 4), if $\phi(x,y) = c, d$ on $y = 0, b$ respectively then, if we write $\phi(x,y) = \psi(x,y) + c + \left(\frac{d-c}{b}\right)y$, the problem is reduced to solving Laplace's equation for ψ in the rectangle with zero boundary conditions on

$y = 0, b$ and slightly modified Dirichlet conditions on $x = 0, a$. The true solution ϕ may then be obtained from the solution ψ by using the above relation.

1.9 Some General Comments

Problems A and B have served to illustrate that there is a step-by-step analogy between the discrete transform method of solving a boundary-value problem expressed in finite difference form and the separation of variables and transform method of solving the corresponding continuous problem. The method can obviously be used to solve the finite difference equations corresponding to any partial differential equation problem which can be solved by separation of variables, e.g. in cylindrical or other separable co-ordinate systems. In chapters 2 and 3 we will show how the method may be extended to problems in regions which can be divided into sub-regions in each of which the separation of variables technique is applicable.

It will be noticed that the solutions of problems A and B (equations (1.3.10') and (1.7.11)) involve only finite summations and lend themselves easily to numerical computation. The forms of these discrete solutions indicate that, if desired, we can obtain the solution over only a part of the total region of the problem and, in particular, we can obtain the solution at a specified net point. (There is no additional difficulty in dealing with problems involving infinite or semi-infinite strips - see the problem of chapter 3.) In addition, a definite high degree of accuracy in the solutions of the original set of difference equations may be obtained after a specified number of arithmetic operations. The accuracy of a discrete approximation to the true analytic solution may be improved by increasing the number of mesh points in the region. Thus, it is possible to obtain a good estimate of the true solution by extrapolating a curve obtained by plotting a series of solutions against the corresponding net spacings. An improvement on this latter technique is provided by the fact that different net patterns may be used to obtain different approximations. By obtaining series of solutions for different net patterns and different net spacings, a curve may be drawn for each pattern and the true solution estimated. Further, in certain circumstances the use of a diagonal net gives an upper limit to the true solution while a lower limit is obtained using a square net e.g. problems A and B. (It should be noted that the discrete transform method may be applied with rectangular net patterns in exactly the same way. These are, however, not used in the problems of this thesis.)

In the remainder of this section we discuss the relation between the discrete transform method and other methods which have appeared in the literature.

The discrete transform method of solving difference equations for boundary-value problems is claimed to be a new method only in the same sense that, though Fourier transforms are usually thought to be a relatively modern technique, they are really only a different way of applying the Fourier integral. It is really only a new method of application of a very old technique, namely separation of variables. The separation of variables technique itself was used to solve difference equations at least as early as 1844 (Ellis [9]). Of the more recent references we mention Hyman [10] and Berger and Lasher [16]. Hyman has used the technique to obtain starting values for his 'step-ahead' method of solving boundary-value problems while Berger and Lasher have employed it to derive expressions for discrete Green's functions for the difference equations corresponding to Poisson's equation. [The analytical summation required in Berger and Lasher's paper (Appendix 1) is avoided if the discrete transform method is used.]

The discrete transform method is closely linked to certain matrix methods described by Karlqvist [4], Cornock [16], Burgerhout [8], and Bickley and McNamee [5]. Each of these authors have illustrated ways of inverting what Bickley and McNamee call the 'big' matrix. The method illustrated in section 7 of Karlqvist's paper is most closely related to the discrete transform method but seems to be more complicated in derivation. The solution of non-homogeneous second-order difference equations has been illustrated by Moskovitz [19] by a method which is related to that illustrated in Appendix A (ii). Moskovitz does not, however, indicate that the method is the discrete analogue of the Green's function technique of solving non-homogeneous second-order differential equations.

1.10 Points of Computational Interest

We have already indicated that the discrete eigenvalues λ may be computed easily as cosines of integral multiples of a certain basic angle e.g. in problem A (§1.3) the λ_ℓ were shown to be cosines of integral multiples of the angle $\frac{\pi}{n+1}$. The corresponding discrete eigenfunctions can also be expressed in terms of trigonometric functions. To illustrate this consider again problem A. We saw that the discrete eigenfunctions were given by (p. 6)

$$S_\ell(s) = A_\ell(h_\ell^s - q_\ell^s), \quad \ell = 1, 2, \ldots, n, \tag{1.10.1}$$

where $h_\ell = \lambda_\ell + \sqrt{\lambda_\ell^2 - 1}$, $q_\ell = \lambda_\ell - \sqrt{\lambda_\ell^2 - 1}$, $\lambda_\ell = \cos\theta_\ell$, and $\theta_\ell = \frac{\ell\pi}{n+1}$. Thus

$$\begin{aligned} S_\ell(s) &= A_\ell(\cos s\theta_\ell + i\sin s\theta_\ell - \cos s\theta_\ell + i\sin s\theta_\ell) \\ &= B_\ell \sin s\theta_\ell . \end{aligned} \tag{1.10.2}$$

In practice the orthogonal eigenvectors $\underset{\sim}{S}_\ell$ are not normalised to be of unit length but such that $S_\ell(1) = 1$. On this basis we may write (1.10.2) as

$$S_\ell(s) = \frac{\sin s\theta_\ell}{\sin\theta_\ell}. \tag{1.10.3}$$

[The effect of this normalisation on the previous theory is merely to introduce a 'scalar product' factor, $\Delta_\ell^2 = \underset{\sim}{S}_\ell^T \underset{\sim}{S}_\ell$, into the denominator of each term in the finite series. This arises because we require to define an inverse formula as $\underset{\sim}{\Phi}_+ = \sum_{\ell=1}^{n} F_{\ell,+} \frac{\underset{\sim}{S}_\ell}{\Delta_\ell^2}$.] For hand computation and when using an autocode scheme on a digital computer it is easier to compute the eigenvectors $\underset{\sim}{S}_\ell$ using the expression (1.10.3). If the discrete transform solution is programmed for evaluation using the basic language of a particular computer, however, it may be more efficient to use the equivalent expression derived from equation (1.10.1), namely

$$S_\ell(s) = \frac{p_\ell^s - q_\ell^s}{p_\ell - q_\ell}. \qquad (1.10.4)$$

Thus, all quantities occurring in the solutions (1.3.10') and (1.7.11) are capable of simple arithmetical evaluation. Further, the quantities which we have denoted by $P(\lambda)$, $\varphi(\lambda)$ and which are solutions of the quadratic equation $\alpha^2-(4-2\lambda)\alpha+1=0$ are common to solutions of all problems by the discrete transform method (see e.g. (1.3.10'), (1.3.15'), (1.7.11)). We could, therefore, design a computer programme to compute tables of these quantities for various values of λ once and for all. It has already been pointed out (§ 1.4) that P and φ, regarded as functions of parameters related to λ, occur in the solutions of problems when diagonal and nine point net patterns are used. Consequently, the same computer programme may be used to compute tables of P and φ for use with these patterns.

The direct computation of solutions given by the discrete transform method does not require a machine with as large a storage capacity as may be thought at first sight. This is due to the fact that the only quantities which need be permanently stored are the eigenvectors of the basic matrix or quantities related to them. For example, consider the solution of problem A (§ 1.3). By equation (1.3.10') we have

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} \left(\frac{P_\ell^{m+1-r} - Q_\ell^{m+1-r}}{P_\ell^{m+1} - Q_\ell^{m+1}} \right) \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G} \right] \underset{\sim}{S}_\ell \qquad (1.10.5)$$

and by normalising the eigenvectors $\underset{\sim}{S}_\ell$ according to (1.10.3) or (1.10.4) this may be written

$$\underset{\sim}{\Phi}_r = \sum_{\ell=1}^{n} Z_{\ell,r} \underset{\sim}{T}_\ell \, . \qquad (1.10.6)$$

In this equation (1.10.6), $Z_{\ell,r} = \left(\frac{P_\ell^{m+1-r} - Q_\ell^{m+1-r}}{P_\ell^{m+1} - Q_\ell^{m+1}} \right)$, $\underset{\sim}{T}_\ell = \left[\underset{\sim}{S}_\ell^T \underset{\sim}{G} \right] \frac{\underset{\sim}{S}_\ell}{\Delta_\ell^2}$ and $\Delta_\ell^2 = \underset{\sim}{S}_\ell^T \underset{\sim}{S}_\ell$. Thus if we permanently store the column vectors $\underset{\sim}{T}_\ell$ (i.e. n^2 numbers) we can compute the solution for each column vector $\underset{\sim}{\Phi}_r$, $r = 1, 2, \ldots\ldots\ldots, m$, individually by using for column vector $\underset{\sim}{\Phi}_k$ only values of $Z_{\ell,k}$, $(\ell = 1, 2, \ldots\ldots, n)$. The $Z_{\ell,r}$ may themselves be computed individually for each r inside the machine or calculated from the tables mentioned in the previous paragraph. Similar remarks may be applied to the evaluation of the solution of problem B (equation (1.7.11)).

CHAPTER 2

THE SOLUTION OF BOUNDARY-VALUE PROBLEMS DEFINED ON ADJACENT RECTANGULAR REGIONS

2.1 Basis of the Method

In this chapter the discrete transform method is extended to deal with boundary-value problems defined on regions composed of adjacent rectangles. A simple example is given by the T-shaped region of Fig. 8 which can be split into the subregions ABCD and HEFG.

The discrete transform method of solving such problems is as follows. For simplicity, we will consider a region composed of only two adjacent rectangles (e.g. Fig. 8) so that we have only one common boundary (*HE*). The extension to more complicated regions will be obvious. We first cover the whole region with a suitably positioned lattice and introduce unknown quantities at the points of intersection of the mesh lines with the common boundary. Normally the mesh is chosen to coincide with the common boundary so that the points of intersection are mesh points and the unknown quantities are function values. This will always be the case in this chapter.* Suppose there are '*n*' mesh points on this boundary and let us denote

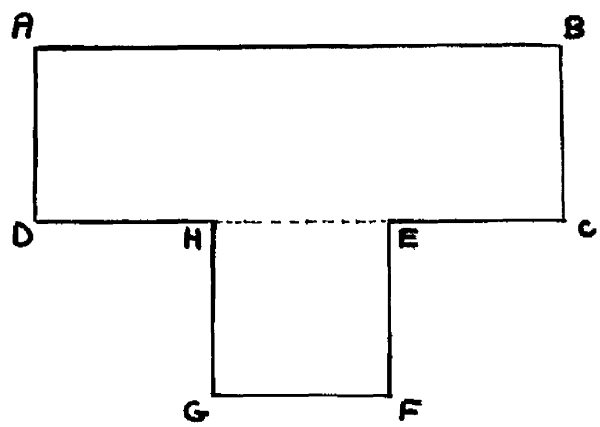


FIG. 8

* Occasionally, however, problems arise in which the normal derivative is specified over part of the common boundary between the two adjacent rectangular regions. In such cases greater accuracy is provided by positioning the lattice to straddle this boundary. Unknown derivatives are then introduced at the points of intersection of the mesh lines with that part of the common boundary on which no conditions are specified. The steady wave problem considered in chapter 3 is a case in point.

the 'n' unknown function values by f_i, $i = 1, 2, \ldots\ldots, n$. We now apply the discrete transform method in the manner illustrated in Chapter 1 to each of the sub-regions individually. This enables us to obtain solutions for both sub-regions in terms of the 'n' unknown function values f_i. The original difference equation of the problem must, however, be satisfied at each of the 'n' mesh points on the common boundary. (This is known as the "fitting" or "matching" process of the problem.) The problem is therefore reduced to the solution of a set of 'n' simultaneous linear equations in the 'n' unknowns f_i. We have, in fact, reduced the dimension of the problem by one in that the set of difference equations at points in a plane is replaced by a set of simultaneous equations at points on a line.

If the region is such that it can be sub-divided in more than one way (e.g. Fig. 9), the sub-division which will give rise to the smallest number of simultaneous equations should be used. In Fig. 9, for example, since DC is less than CG we would sub-divide the L-shaped region into the two rectangular regions ABCD and DEFH and use a matching process at mesh points on DC.

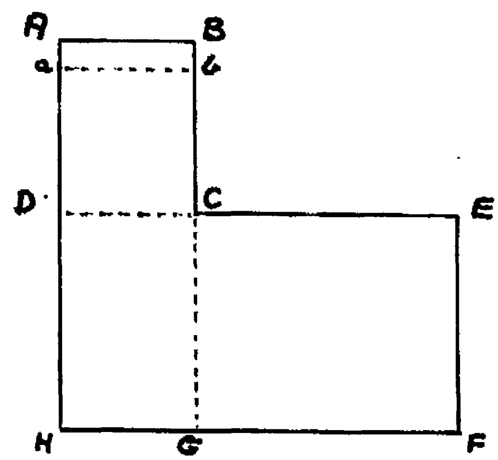


FIG. 9

The Discrete Transform Method can clearly be extended to deal with other types of region provided they may be divided into sub-regions in each of which separation of variables is applicable. Problems of this sort may also be solved by the continuous technique of separation of variables and transforms but, in this case, we must introduce an unknown <u>function</u> on the common boundary. The fitting conditions on this boundary then give rise to an integral equation. We may, therefore, extend our points of analogy (§ 1.5) between the discrete transform technique and the continuous separation of variables and transform technique to say that a set of simultaneous linear equations is the analogue of an integral equation

Comock [6] describes a method of solving these problems which is closely related to the discrete transform technique. He expresses the complete set of difference equations as a 'big' matrix equation. By pre-multiplying the 'big' matrix by another matrix to convert it into a certain lower triangular form, he is led to solve a set of si multaneous linear equations at mesh points on a mesh arm parallel to the common boundary e.g. in Fig. 9, if Dirichlet conditions are specified on AB, he would solve equations at mesh points on ab. The Discrete Transform Method seems to be simpler in derivation and application and more suitable for automatic machine computation.

The basic steps involved in solving boundary-value problems in rectangular regions have been listed in § 1.8 (p.30) and the reader would be well advised to refresh his memory on these at this point. In adjacent region problems it is sometimes impossible, on account of the boundary dimensions, to position a square lattice to coincide with Dirichlet boundaries while at the same time straddling boundaries where derivative conditions are specified (§ 1.8, step (1), (a) and (b)). It is more convenient, in such circumstances, to position the lattice to coincide with boundaries of the latter type. An alternative may be provided, however, by the use of a rectangular lattice but we shall not do this.

In this chapter, as in chapter 1, we will use the symbols P and Q to represent the solutions of the quadratic equations $\alpha^2-(4-2\lambda)\alpha+1=0$. Thus

$$P=2-\lambda+\sqrt{3-4\lambda+\lambda^2}\ ,\quad Q=2-\lambda-\sqrt{3-4\lambda+\lambda^2}. \qquad (2.1)$$

2.2 A Two-Dimensional Rectangular Box Condenser Problem

For simplicity we shall consider the case in which the rectangles are concentric and have corresponding sides parallel (Fig. 10). Further we shall assume the inner rectangle to have potential unity and the outer to be earthed. The problem is to determine the distribution of potential in the region bounded by the two rectangles.

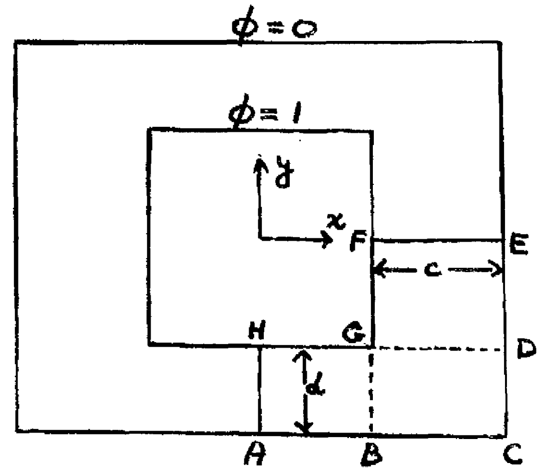


Fig. 10

By symmetry, we need only consider one quarter of the total region which we may take as the bottom right hand corner. The problem therefore reduces to the solution of Laplace's equation in the L-shaped region ABCDEFGH (Fig. 10) with the boundary conditions $\phi = 0$ on ABC, CDE, $\phi = 1$ on HG, GF, $\frac{\partial \phi}{\partial x} = 0$ on HA and $\frac{\partial \phi}{\partial y} = 0$ on FE, i.e. an adjacent region problem.

For the purposes of illustration suppose the dimensions of the problem are such that a square lattice may be chosen to coincide with AC, CE, FB, HD, and FE and to straddle HA (Fig. 11). Assume (Fig. 11) that $c > d$ so that we will get fewer simultaneous equations to solve if we divide the region into the two rectangles $ABGH$ and $BCDEFG$ and then match across BG. Let there be 'n' interior mesh points on

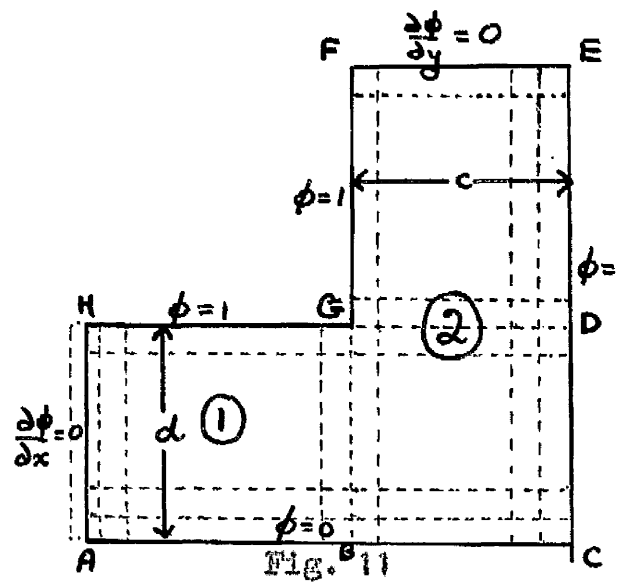


Fig. 11

BG, 'm' on GD and denote the 'n' unknown function values on BG by $\gamma_1, \gamma_2, \ldots \ldots \ldots, \gamma_n$. As explained in § 2.1 the technique is to obtain the solutions, in terms of these γ_i, of the boundary-value problem for Laplace's equation in each rectangle individually and to evaluate the γ_i by a matching process on BG.

The rectangle $ABGH$ will be known as region (1), rectangle $BCDEFG$ as region (2) and we shall use superscripts $^{(1)}$ and $^{(2)}$ to distinguish quantities derived for the two regions. We shall illustrate the solution using a square net pattern but solutions for diagonal and nine point patterns may be obtained in exactly the same way.

(1) The rectangle ABGH

Consider that the boundaries of this rectangle (Fig. 12) are given by $r = -\frac{1}{2}, r = k+1, s = 0, s = n+1.$ The set of difference equations to be solved is then

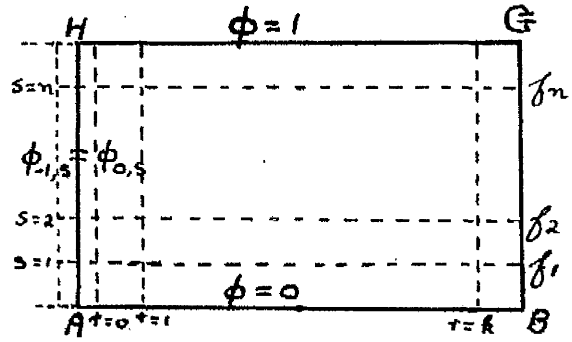


Fig. 12

$$4\phi^{(1)}_{r,s} - \phi^{(1)}_{r+1,s} - \phi^{(1)}_{r,s+1} - \phi^{(1)}_{r-1,s} - \phi^{(1)}_{r,s-1} = 0, \qquad (2.2.1a)$$

$0 \leq r \leq k$, $1 \leq s \leq n$, with the boundary conditions

$$\phi^{(1)}_{-1,s} = \phi^{(1)}_{0,s} \;;\; \phi^{(1)}_{k+1,s} = \gamma_s \quad , s = 1, 2, \ldots\ldots, n, \qquad (2.2.2a)$$

$$\phi^{(1)}_{r,0} = 0 \quad ; \quad \phi^{(1)}_{r,n+1} = 1 \quad , r = 0, 1, 2, \ldots\ldots, k. \qquad (2.2.3a)$$

We shall transform in the 's' direction and in order to introduce a function which is zero on the boundaries $s = 0$ and $s = (n+1)$, we set

$$\psi(x,y) = \phi^{(1)}(x,y) - \frac{y}{\alpha}\,, \quad \text{i.e.} \quad \psi_{r,s} = \phi^{(1)}_{r,s} - \frac{s}{n+1}\,. \tag{2.2.4}$$

In terms of $\psi_{r,s}$, therefore, the set of difference equations to be solved is

$$4\psi_{r,s} - \psi_{r+1,s} - \psi_{r,s+1} - \psi_{r-1,s} - \psi_{r,s-1} = 0, \tag{2.2.1b}$$

$0 \le r \le k$, $1 \le s \le n$, with the boundary conditions

$$\psi_{-1,s} = \psi_{0,s}\ ;\ \psi_{m+1,s} = \gamma_s - \frac{s}{n+1}\ , \quad s = 1,2,\ldots\ldots,n, \tag{2.2.2b}$$

$$\psi_{r,0} = \psi_{r,n+1} = 0\,, \quad r = 0,1,2,\ldots\ldots,k. \tag{2.2.3b}$$

We introduce the vector

$$\underset{\sim}{\Psi}_r = \{\psi_{r,1}, \psi_{r,2}, \ldots\ldots\ldots, \psi_{r,n}\}.$$

By the same method used to derive equation (1.3.10') it can be shown that

$$\underset{\sim}{\Psi}_r = \sum_{\ell=1}^{n} \alpha_{r,\ell} \left[\underset{\sim}{S}_\ell^{(1)T} \underset{\sim}{G}^{(1)}\right] \frac{\underset{\sim}{S}_\ell^{(1)}}{\Delta_\ell^{(1)2}}\,, \tag{2.2.5}$$

where $\alpha_{r,\ell} = \dfrac{P_\ell^{(1)r} + Q_\ell^{(1)r}}{P_\ell^{(1)k+1} + Q_\ell^{(1)k+1}}$, $\underset{\sim}{G}^{(1)} = \left\{\gamma_1 - \frac{1}{n+1}, \gamma_2 - \frac{2}{n+1}, \ldots\ldots, \gamma_n - \frac{n}{n+1}\right\}$, and

$\underset{\sim}{S}_\ell^{(1)} = \{S_\ell^{(1)}(1), S_\ell^{(1)}(2), \ldots\ldots\ldots\ldots, S_\ell^{(1)}(n)\}$. In these formulae $P_\ell^{(1)}$ and $Q_\ell^{(1)}$ are defined in equation (2.1.1) with $\lambda = \lambda_\ell^{(1)}$ where $\lambda_\ell^{(1)} = \cos\theta_\ell^{(1)}$ and $S_\ell^{(1)}(s) = \dfrac{\sin s\,\theta_\ell^{(1)}}{\sin\theta_\ell^{(1)}}$ (§ 1 10, p. 34), $\theta_\ell^{(1)} = \dfrac{\ell\pi}{n+1}$, $\Delta_\ell^{(1)2} = \underset{\sim}{S}_\ell^{(1)T}\underset{\sim}{S}_\ell^{(1)}$.

Define $\underset{\sim}{\Phi}_r^{(1)} = \{\phi_{r,1}^{(1)}, \phi_{r,2}^{(1)}, \ldots\ldots\ldots\ldots, \phi_{r,n}^{(1)}\}$ then from (2.2 4) and (2.2.5)

$$\underset{\sim}{\Phi}_r^{(1)} = \sum_{\ell=1}^{n} \alpha_{r,\ell} \left[\underset{\sim}{S}_\ell^{(1)T} \underset{\sim}{G}^{(1)}\right] \frac{\underset{\sim}{S}_\ell^{(1)}}{\Delta_\ell^{(1)2}} + \frac{1}{n+1}\underset{\sim}{N}, \tag{2.2.6}$$

where $\underset{\sim}{N} = \{1, 2, \ldots\ldots, n\}$. Thus the solution, $\underset{\sim r}{\Phi}^{(1)}$, of Laplace's equation in the region **ABGH** is expressed in terms of 'n' unknown function values $\gamma_1, \gamma_2, \ldots, \gamma_n$ contained in the matrix $\underset{\sim}{G}^{(1)}$ (2.2.5).

(2) The rectangle BCDEFG

Let the boundaries of this rectangle be (Fig. 13) $r = 0$, $r = (m+1)h$, $s = 0$, $s = (n+j+1)h$. The set of difference equations to be solved is then

$$4\phi^{(2)}_{r,s} - \phi^{(2)}_{r+1,s} - \phi^{(2)}_{r,s+1} - \phi^{(2)}_{r-1,s} - \phi^{(2)}_{r,s-1} = 0, \quad (2.2.7)$$

$1 \leq r \leq m$, $1 \leq s \leq n+j+1$, with the boundary conditions

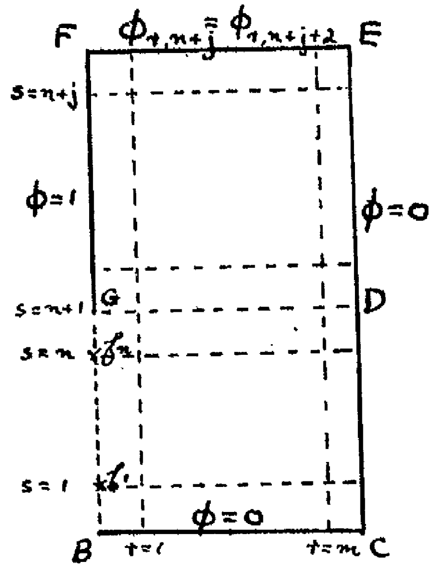


Fig 13

$$\phi^{(2)}_{0,s} = \begin{cases} \gamma_s, & s = 1, 2, \ldots\ldots, n, \\ 1, & s = n+1, n+2, \ldots, n+j+1, \end{cases} \quad ; \quad \phi^{(2)}_{m+1,s} = 0, \quad s = 1, 2, \ldots\ldots, (n+j+1), \quad (2.2.8)$$

$$\phi^{(2)}_{r,0} = 0 \quad ; \quad \phi^{(2)}_{r,n+j} = \phi^{(2)}_{r,n+j+2}, \quad r = 1, 2, \ldots\ldots\ldots, m. \quad (2.2.9)$$

Since the lattice is coincident with the Neumann boundary **FE** the basic matrix of the problem is not symmetrical. To make the matrix symmetric we require to define (c.f. § 1.5)

$$\underset{\sim}{S}^{(2)}_{\ell} = \left\{ S^{(2)}_{\ell}(1), S^{(2)}_{\ell}(2), \ldots\ldots, S^{(2)}_{\ell}(n+j), \frac{S^{(2)}_{\ell}(n+j+1)}{\sqrt{2}} \right\} \text{ and}$$

$$\underset{\sim r}{\Phi}^{(2)} = \left\{ \phi^{(2)}_{r,1}, \phi^{(2)}_{r,2}, \ldots\ldots\ldots, \phi^{(2)}_{r,n+j}, \frac{\phi^{(2)}_{r,n+j+1}}{\sqrt{2}} \right\}.$$

Proceeding in the manner indicated in § 1.5 it can be shown that

$$\underset{\sim r}{\Phi}^{(2)} = \sum_{\ell=1}^{n+j+1} \beta_{r,\ell} \left[\underset{\sim}{S}^{(2)T}_{\ell} \underset{\sim}{G}^{(2)} \right] \frac{\underset{\sim}{S}^{(2)}_{\ell}}{\Delta^{(2)^2}_{\ell}}, \quad (2.2.10)$$

where $\beta_{r,\ell} = \dfrac{P_\ell^{(2)^{m+1-r}} - Q_\ell^{(2)^{m+1-r}}}{P_\ell^{(2)^{m+1}} - Q_\ell^{(2)^{m+1}}}$ and $\underset{\sim}{G}^{(2)} = \{ g_1, g_2, \ldots, g_n, 1, 1, \ldots, 1 \}$

of type $(n+j+1) \times 1$. In these formulae, $P_\ell^{(2)}, Q_\ell^{(2)}$ are defined by (2.1.1) with $\lambda = \lambda_\ell^{(2)}$ where $\lambda_\ell^{(2)} = \cos\theta_\ell^{(2)}$ and $\theta_\ell^{(2)} = \dfrac{(2\ell+1)\pi}{2(n+j+1)}$. The elements $S_\ell^{(2)}(s)$ of $\underset{\sim}{S}_\ell^{(2)}$ are given by $S_\ell^{(2)}(s) = \dfrac{\sin s\theta_\ell^{(2)}}{\sin\theta_\ell^{(2)}}$ and $\Delta_\ell^{(2)^2} = \underset{\sim}{S}_\ell^{(2)^T} \underset{\sim}{S}_\ell^{(2)}$.

Thus the solution, $\underset{\sim}{\Phi}_+^{(2)}$ of Laplace's equation in the region BCDEFG is expressed in terms of the 'n' unknown function values g_i contained in the matrix $\underset{\sim}{G}^{(2)}$ (2.2.10).

The "Matching" Process

On BG (Fig 11) we have to satisfy the difference form of Laplace's equation at each of the mesh points Thus

$$4g_s - \phi_{1,s}^{(2)} - g_{s+1} - \phi_{k,s}^{(1)} - g_{s-1} = 0 , \quad s = 1, 2, \ldots, n \qquad (2.2.11)$$

where $g_0 = 0$ and $g_{n+1} = 1$.

From equation (2.2.10) $\quad \phi_{1,s}^{(2)} = \sum_{\ell=1}^{n+j+1} \beta_{1,\ell} \left[\underset{\sim}{S}_\ell^{(2)^T} \underset{\sim}{G}^{(2)} \right] \dfrac{S_\ell^{(2)}(s)}{\Delta_\ell^{(2)^2}}$.

$$\therefore \left\{ \phi_{1,s}^{(2)} = \sum_{i=1}^{n} a_{s,i} g_i + c_s \right\}, \qquad (2.2.12)$$

where $a_{s,i} = \sum_{\ell=1}^{n+j+1} \beta_{1,\ell} S_\ell^{(2)}(i) \dfrac{S_\ell^{(2)}(s)}{\Delta_\ell^{(2)^2}}$ and $c_s = \sum_{\ell=1}^{n+j+1} \beta_{1,\ell} \left(\sum_{i=n+1}^{n+j} S_\ell^{(2)}(i) + \dfrac{S_\ell^{(2)}(n+j+1)}{\sqrt{2}} \right) \dfrac{S_\ell^{(2)}(s)}{\Delta_\ell^{(2)^2}}$

From equation (2.2.6) $\quad \phi_{k,s}^{(1)} = \sum_{\ell=1}^{n} \alpha_{k,\ell} \left[\underset{\sim}{S}_\ell^{(1)^T} \underset{\sim}{G}^{(1)} \right] \dfrac{S_\ell^{(1)}(s)}{\Delta_\ell^{(1)^2}} + \dfrac{s}{n+1}$.

$$\therefore \left\{ \phi_{k,s}^{(1)} = \sum_{i=1}^{n} b_{s,i} g_i + \frac{s}{n+1} \right\}, \qquad (2.2.13)$$

where $b_{s,i} = \sum_{\ell=1}^{n} \alpha_{k,\ell} S_\ell^{(1)}(i) \dfrac{S_\ell^{(1)}(s)}{\Delta_\ell^{(1)^2}}$.

Substitute from (2.2.12) and (2.2.13) into (2.2.11) to obtain

$$4\gamma_s - \sum_{i=1}^{n} a_{s,i}\gamma_i - c_s - \gamma_{s+1} - \sum_{i=1}^{n} b_{s,i}\gamma_i - \frac{s}{n+1} - \gamma_{s-1} = 0, \quad s=1,2,\dots,n.$$

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$$\sum_{i=1}^{n} u_{s,i}\gamma_i = Z_s , \quad s=1,2,\dots,n, \qquad (2.2.14)$$

where $u_{s,i} = 4\delta_i^s - \delta_i^{s+1} - \delta_i^{s-1} - a_{s,i} - b_{s,i}$, $Z_s = c_s + \frac{s}{n+1} + \delta_s^n$.

We thus obtain a set of 'n' simultaneous linear equations (2.2.14) in the 'n' unknowns γ_i . The solution of the problem is then obtained by evaluating the vectors $\underset{\sim}{G}^{(1)}$ (equation (2.2.5)) and $\underset{\sim}{G}^{(2)}$ (equation (2.2.10)) and substituting these into solutions (2.2.6) and (2.2.10) respectively

In Fig. 14 (p. 46) results, obtained on a desk calculator, are recorded for the case $n=2$, $m=3$, $j=4$.

RESULTS FOR TWO-DIMENSIONAL RECTANGULAR BOX CONDENSER PROBLEM

CASE: n = 2, m = 3, j = 4

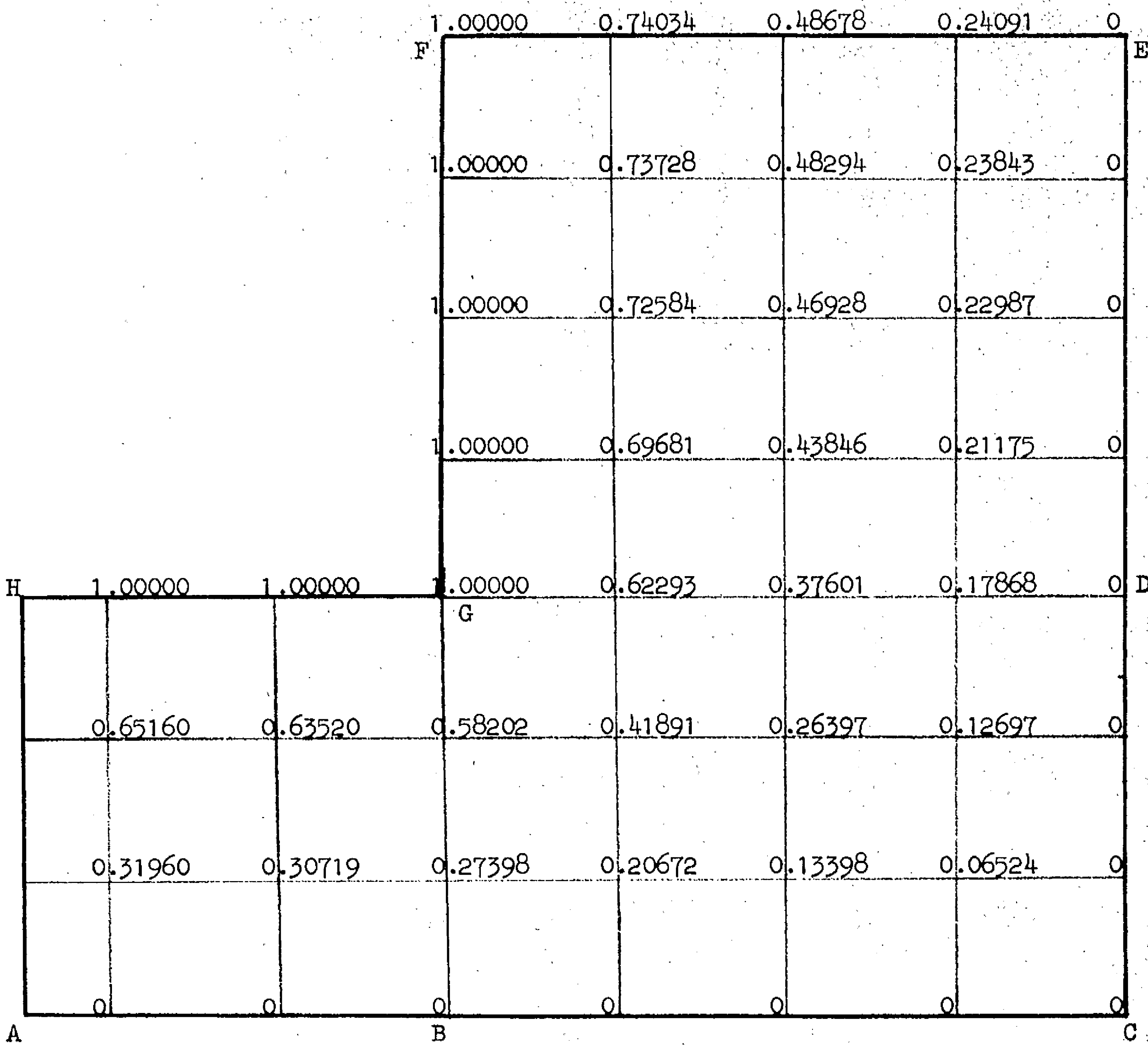

FIG. 14

2.3 A Two-Dimensional Square Box Condenser Problem

A particular case of the previous rectangular box condenser problem occurs when both rectangles are square. Though the method of § 2.2 may be used, this problem may be solved in another way since, by symmetry, we need now consider only one-eighth of the inter-box region and the problem may be depicted pictorially as in Fig. 15. ($\frac{\partial \phi}{\partial n}$ is used to denote the derivative of ϕ normal to the boundary.)

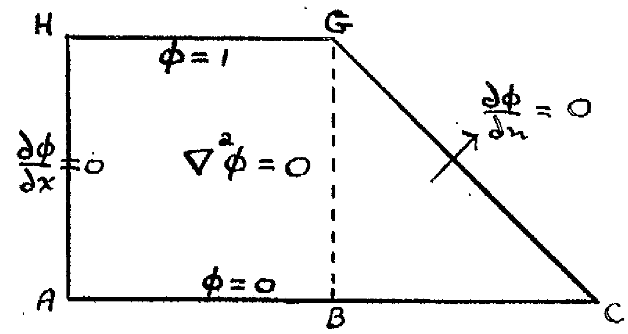

Fig. 15

In the discrete transform method of solving this problem we choose a square lattice to coincide with AC and HG and such that GC coincides with a node on each mesh row. Thus if there are 'n' mesh lines between AC and HG there will be 'n' mesh points on GC . The net may either coincide with AH (Fig. 16) or straddle AH depending on the dimensions of the problem. We first ignore the boundary GC and use the discrete transform method (transforming in the "s" direction) to obtain a solution involving 'n' arbitrary constants in the semi-infinite rectangle $IAHJ$ (Fig. 16). (We assume $\phi = 0, 1$ on AI, HJ respectively).

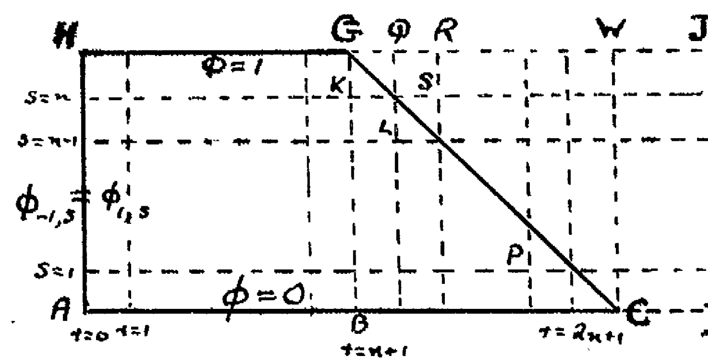

Fig. 16

These arbitrary constants are then evaluated by using the difference equations at mesh points on GC which are specified by the conditions of symmetry of the solution about this line. As in the previous adjacent region example this problem is therefore reduced to the solution of a set of simultaneous linear equations at points on a line.

To illustrate the method we shall consider the case in which the sides of the outer square are twice the length of those of the inner so that the lattice must coincide with AH (Fig 16).

Consider the semi-infinite rectangle $IAHJ$ and suppose that its boundaries are given by $r=0$, $s=0$, $s=(n+1)$.

We set $$\psi_{r,s} = \phi_{r,s} - \frac{s}{n+1}, \quad s=0,1,2,\ldots,n+1, \; r \geq 0. \tag{2.3.1}$$

The difference equations for ψ are

$$4\psi_{r,s} - \psi_{r+1,s} - \psi_{r,s+1} - \psi_{r-1,s} - \psi_{r,s-1} = 0, \tag{2.3.2}$$

$r \geq 0$, $1 \leq s \leq n$, with the boundary conditions

$$\psi_{r,0} = \psi_{r,n+1} = 0, \quad r = 0,1,2,\ldots\ldots, \tag{2.3.3}$$

$$\psi_{-1,s} = \psi_{1,s}, \quad s = 1,2,\ldots\ldots,n. \tag{2.3.4}$$

If we define $\underset{\sim}{\Psi}_r = \{\psi_{r,1}, \psi_{r,2}, \ldots\ldots, \psi_{r,n}\}$ then on applying a discrete transform in the "s" direction to equations (2.3.2) and using the above boundary conditions it can be shown that

$$\underset{\sim}{\Psi}_r = \sum_{l=1}^{n} c_l V_l^{(r)} \frac{\underset{\sim}{S}_l}{\Delta_l^2}. \tag{2.3.5}$$

In (2.3.5) the c_l are arbitrary constants while $V_l^{(r)} = P_l^r + Q_l^r$, ($P_l$, Q_l being defined by equation (2.1.1) with $\lambda = \lambda_l$ and $\lambda_l = \cos\theta_l$, $\theta_l = \frac{l\pi}{n+1}$). Further $\Delta_l^2 = \underset{\sim}{S}_l^T \underset{\sim}{S}_l$ and $\underset{\sim}{S}_l = \{S_l(1), S_l(2), \ldots, S_l(n)\}$ where $S_l(s) = \frac{\sin s\theta_l}{\sin\theta_l}$.

Let $\underset{\sim}{\Phi}_r = \{\phi_{r,1}, \phi_{r,2}, \ldots\ldots, \phi_{r,n}\}$, then by (2.3.5) and (2.3.1) we have

$$\underset{\sim}{\Phi}_r = \sum_{l=1}^{n} c_l V_l^{(r)} \frac{\underset{\sim}{S}_l}{\Delta_l^2} + \frac{1}{n+1}\underset{\sim}{N}, \tag{2.3.6}$$

$\underset{\sim}{N} = \{1, 2, \ldots\ldots, n\}$. Hence

$$\phi_{r,s} = \sum_{\ell=1}^{n} C_\ell V_\ell^{(r)} \frac{S_\ell(s)}{\Delta_\ell^2} + \frac{s}{n+1} . \qquad (2.3.7)$$

To evaluate the arbitrary constants C_ℓ we now use the difference equations at mesh points on GC (Fig. 16). Any mesh point on GC has co-ordinates of the form $((n+1+k)h, (n+1-k)h)$, where $k = 1, 2, \cdots, n$. Thus, by the reflection condition on this boundary, the difference equations on GC are

$$4\phi_{n+1+k, n+1-k} - 2\phi_{n+k, n+1-k} - 2\phi_{n+1+k, n-k} = 0, \; k = 1, 2, \ldots, n . \qquad (2.3.8)$$

Substitution of (2.3.7) in (2.3.8) gives

$$2\sum_{\ell=1}^{n} C_\ell V_\ell^{(n+1+k)} \frac{S_\ell(n+1-k)}{\Delta_\ell^2} - \sum_{\ell=1}^{n} C_\ell V_\ell^{(n+k)} \frac{S_\ell(n+1-k)}{\Delta_\ell^2} - \sum_{\ell=1}^{n} C_\ell V_\ell^{(n+1+k)} \frac{S_\ell(n-k)}{\Delta_\ell^2} = -\frac{1}{n+1}$$

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$$\sum_{\ell=1}^{n} u_{k\ell} C_\ell = -\frac{1}{n+1} , \; k = 1, 2, \cdots, n, \qquad (2.3.9)$$

where $u_{k\ell} = \left(2V_\ell^{(n+1+k)} - V_\ell^{(n+k)}\right) \dfrac{S_\ell(n+1-k)}{\Delta_\ell^2} - V_\ell^{(n+1+k)} \dfrac{S_\ell(n-k)}{\Delta_\ell^2}$.

(2.3.9) constitutes a set of 'n' simultaneous linear equations in the 'n' arbitrary constants C_ℓ . Evaluation of these constants and substitution in equation (2.3.6) then gives the required solution .

It should be pointed out that, though the solutions $\underset{\sim}{\Phi}_r$ (equation (2.3.6)), $r = 0, 1, 2, \ldots, 2n+1$, (Fig. 16), may be evaluated to give function values at all points in the inter-square region $CAHW$, only those function values for points lying in the domain of the original problem are correct. (Note that CW coincides with a side of the outer square). That function values at mesh points within the inter-square region GCW are incorrect is due to the method adopted to solve this problem. In order to obtain the solution in the semi-infinite rectangle $IAHJ$ we required to set $\phi = 1$ on GJ. Thus, irrespective of the values of the arbitrary constants in this latter solution the difference equation at the point S, for example, will only be satisfied if $\phi_R = 1$ which is not the case The function values at mesh points on the boundary GC are only correct because, by our manner of evaluating the arbitrary constants C_ℓ, we have imposed the correct difference equations at these points

Certain difficulties arise in the above method when diagonal or nine point net patterns are used. These difficulties are due to the fact that the difference equations at the points K, L, ..., P [Fig. 16] now involve function values at points on the opposite side of GC. In particular, the difference equation at the point K involves the function value at the point Q and to apply the method we require to set $\phi_Q = 1$. Thus the solution obtained would give function values such that the difference equation at K will only be satisfied if Q is at unit potential. The solution would, therefore, be incorrect. It is possible to overcome these difficulties by using special net patterns at the points K, L, ..., P, when solving in the semi-infinite rectangle $IAHJ$, but we will not discuss this here. Solutions for the diagonal and nine point net patterns may be obtained by using the adjacent region method (§ 2.2) where the above difficulties do not arise.

2.4 Computation of Solutations for the Square Box Condenser Problem

In this section we discuss the automatic computation of solutions for the square box condenser problem illustrated in the previous section.

Since in problems of this sort we are usually interested in the potential values at all mesh points of the region we first group the solutions (2.3.6) for

$r = 0, 1, 2, \ldots, 2n+1$, in the form of a single matrix equation. We define

$$\underset{\sim}{\Phi} = \left\{ \underset{\sim}{\Phi}_0^T, \underset{\sim}{\Phi}_1^T, \ldots\ldots, \underset{\sim}{\Phi}_{2n+1}^T \right\} \text{ of type } (2n+2)\times n,$$

$$\underset{\sim}{V} = \left[V_\ell^{(r)} \right] \text{ where r is the row number, } \ell \text{ the column number,}$$

$$0 \leq r \leq 2n+1, \quad 1 \leq \ell \leq n,$$

$$\underset{\sim}{S} = \left\{ \frac{\underset{\sim}{s}_1^T}{\Delta_1^2}, \frac{\underset{\sim}{s}_2^T}{\Delta_2^2}, \ldots\ldots, \frac{\underset{\sim}{s}_n^T}{\Delta_n^2} \right\} \text{ of type } n\times n,$$

$$\underset{\sim}{C}^D = \text{diag.} \left\{ c_1, c_2, \ldots, c_n \right\},$$

$$\underset{\sim}{M} = \frac{1}{n+1} \left\{ \underset{\sim}{N}^T, \underset{\sim}{N}^T, \ldots\ldots, \underset{\sim}{N}^T \right\} \text{ of type } (2n+2)\times n.$$

It can then be shown that
$$\underset{\sim}{\Phi} = \underset{\sim}{V}\,\underset{\sim}{C}^D \underset{\sim}{S} + \underset{\sim}{M}. \qquad (2.4.1)$$

The matrix array $\underset{\sim}{\Phi}$ gives function values at all points in the rectangular region CAHW (Fig. 16) but, as was pointed out in the previous section, values at points on the right of GC are neither correct nor required The first row of $\underset{\sim}{\Phi}$ gives values of $\phi_{r,s}$ at points on $r=0$, the second on $r=1$, and so on.

The set of simultaneous equations (2.3.9) for the evaluation of the 'n' arbitrary constants c_ℓ may be written in the matrix form

$$\underset{\sim}{U}\,\underset{\sim}{C} = \underset{\sim}{Z}, \qquad (2.4.2)$$

where $\underset{\sim}{U} = [u_{k\ell}]$ of type $n \times n$, $\underset{\sim}{C} = \{ c_1, c_2, \ldots\ldots, c_n \}$, and $\underset{\sim}{Z}$ is the $(n\times 1)$ column vector each of whose elements has the value $\left(-\frac{1}{n+1} \right)$. Hence

$$\underset{\sim}{C} = \underset{\sim}{U}^{-1} \underset{\sim}{Z}. \qquad (2.4.3)$$

This problem has been programmed for the Deuce electronic computer (Mark II version). Two of the simplified programming schemes available were used, namely

Alphacode and the General Interpretive Programme (G.I.P.). Alphacode is one of the normal types of auto-code scheme and works entirely in floating point binary arithmetic. G.I.P. is a powerful auto-code scheme devised for matrix calculations. It also works in floating point binary arithmetic but two forms may be used: fully-floating and block-floating. A matrix is said to be fully floated when each element is individually expressed as a floating point number; it is said to be block floated when all elements have the same exponent, the exponent for a particular matrix being determined by its largest element. Though slower, it is more accurate to work with matrices in fully-floating form and this was the form used for matrix calculations in our problem except in finding the inverse of the matrix of coefficients $\underset{\sim}{U}$ of the fitting equations (2.4.2). (It was necessary to block float in finding this inverse since no G.I.P. subroutine (or "brick") exists for inverting a fully-floating matrix.) Another advantage of using fully-floating matrices is that data and results in this form are common to both auto-code schemes.

Since two auto-code schemes were used, the programme for the square condenser problem was divided into two parts. In part (1) the matrices $\underset{\sim}{V}$, $\underset{\sim}{S}$, $\underset{\sim}{U}$, $\underset{\sim}{Z}$, and $\underset{\sim}{M}$ were computed using Alphacode and punched on cards in fully-floating form so that they were immediately available as data for the G.I.P. part of the programme. In part (2), G.I.P. was used (a) to invert $\underset{\sim}{U}$, (b) to calculate $\underset{\sim}{C}$ and hence $\underset{\sim}{C}^{D}$ and, finally, (c) to compute the solution $\underset{\sim}{\Phi}$ given by equation (2.4.1). Flow diagrams for these two auto-code programmes are given on p. 56.

It will be noticed from these flow diagrams that the only data which need be manually punched are the numbers 'n' and 'π' and it is consequently a very easy matter to compute solutions for various net spacings i.e. various values of 'n'. Results have been obtained on the Deuce Computer at Glasgow University for n = 3, 5, 7, 9 and the solution for n = 3 is shown in Fig. 17, p. 55. The manner of computation of the matrices $\underset{\sim}{V}$ and $\underset{\sim}{S}$ has already been indicated in § 1.10 It is clearly essential that the matrix of coefficients $\underset{\sim}{U}$ of the fitting equations (2.4.2) should be computed accurately. No difficulty was experienced in this, however, due to the Alphacode programme working in fully-floating arithmetic and no scaling was necessary. The resulting range of the elements of this matrix and the necessity to block-float, however, led to the use of an iterative formula in finding the inverse of $\underset{\sim}{U}$ in the cases n = 7 and n = 9. The iterative formula used was

$$\underset{\sim}{W}_{n+1} = (2\underset{\sim}{I} - \underset{\sim}{W}_n . \underset{\sim}{U})\underset{\sim}{W}_n$$

where $\underset{\sim}{W}_0$ is a first approximation to the true inverse of $\underset{\sim}{U}$ and $\underset{\sim}{I}$ is the unit matrix. It should be pointed out that the equations themselves were not badly ill-conditioned. Results were obtained to an accuracy of 7 of the 9 decimal places carried by the machine.

A feature of these results is the near-linear convergence of the approximate ϕ values at a particular point as the net spacing is reduced. This is shown by the graph on p. 55 which has been obtained by plotting the ϕ values at the mid-point of the boundary GC (Fig. 16) against $\frac{1}{(n+1)}$ for values of n = 3, 5, 7, and 9. [Note that $\frac{1}{(n+1)}$ is proportional to the net spacing 'h', since (n + 1) h = d (Fig. 16)]. The figures from which this graph is drawn are contained in the tables in Appendix C.

No theory seems to exist to explain this rate of convergence. As far as the author is aware the published papers have dealt only with either Dirichlet or Neumann problems but not with problems in which Dirichlet conditions are specified on part of the boundary while Neumann conditions are specified on the remainder. We recall that the latter was the case when we solved the square box condenser problem by considering only one-eighth of the inter-square region (§ 2.3). The real problem, however, i.e. one square inside another, is a Dirichlet problem in a doubly-connected region in which four of the interior angles are greater than π. (These angles are the angles at the corners of the inner square.) It seems that only Laasonen [20] has considered the rate of convergence of discrete approximations to the exact value in Dirichlet problems where interior angles $\alpha_+\pi$, $\alpha_+ \geqslant 1$, occur. Further Laasonen has considered only problems defined in simply-connected regions. His theory shows that the truncation error, i.e. the difference between the exact value and an approximate value obtained using finite differences, should vary as $h^{1/\alpha}$ where $\alpha\pi$, $\alpha > 1$, is the greatest interior angle. In our case $\alpha = \frac{3}{2}$ so that, for Laasonen's theory to be valid in the doubly-connected region of our problem we should expect the error to vary as $h^{2/3}$. Instead, using the fact that the true potential value at the mid-point of GC is $0 \cdot 1945$ (Chapter 5), we have found that the error varies as $h^{1 \cdot 14}$.

It would be worth-while trying to justify the value of this exponent theoretically but the accuracy of the discrete approximations is so poor that it was thought preferable to investigate methods for improving this accuracy. This is done in Chapter 5.

POTENTIAL DISTRIBUTION IN THE TWO-DIMENSIONAL SQUARE BOX CONDENSER PROBLEM WITH n = 3.

(Results quoted to only 5 decimal places)

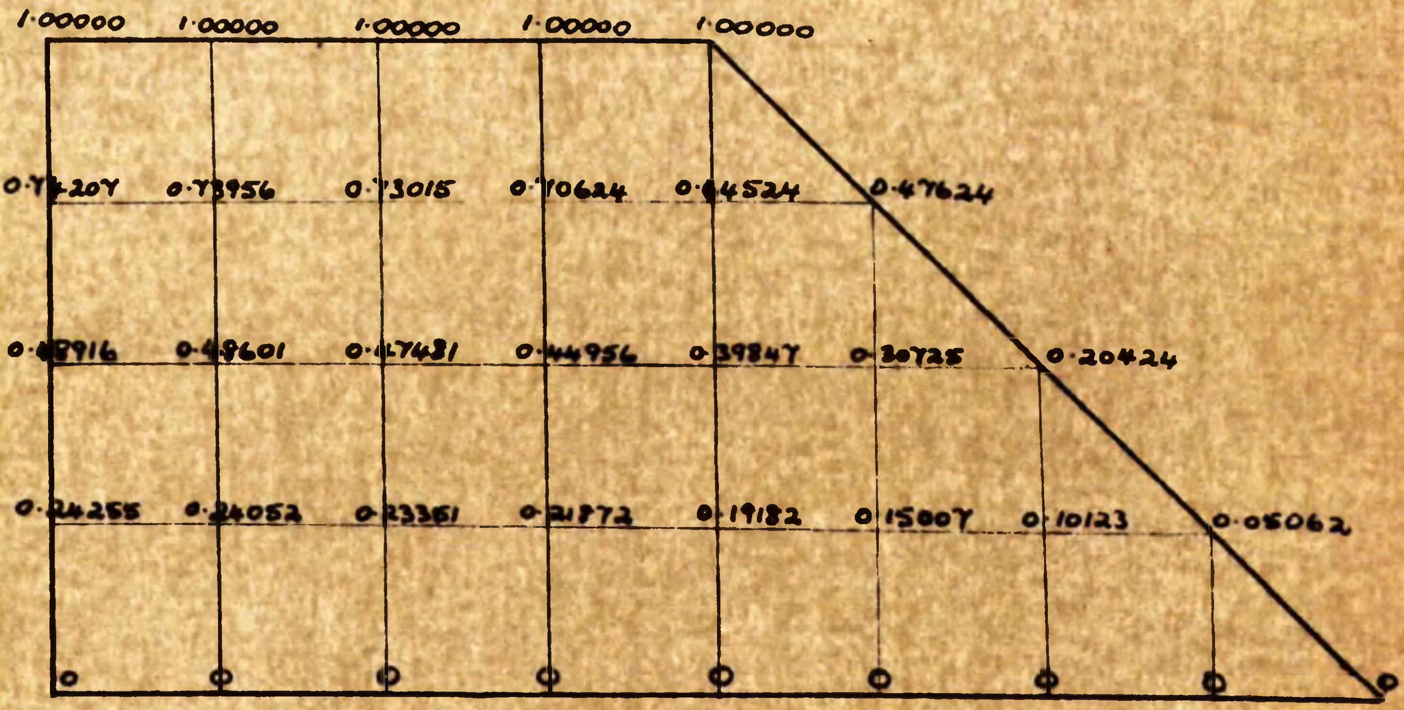
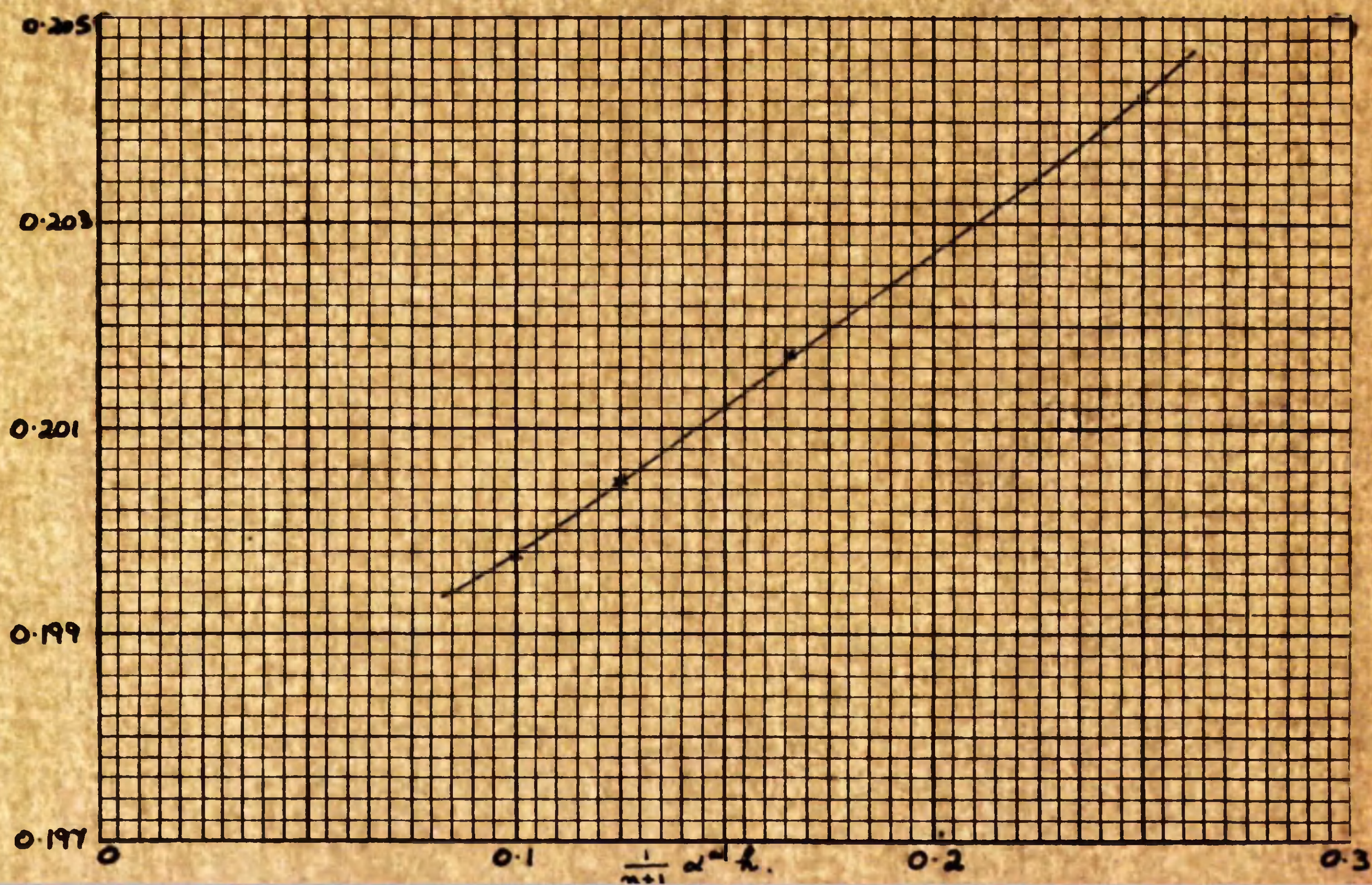


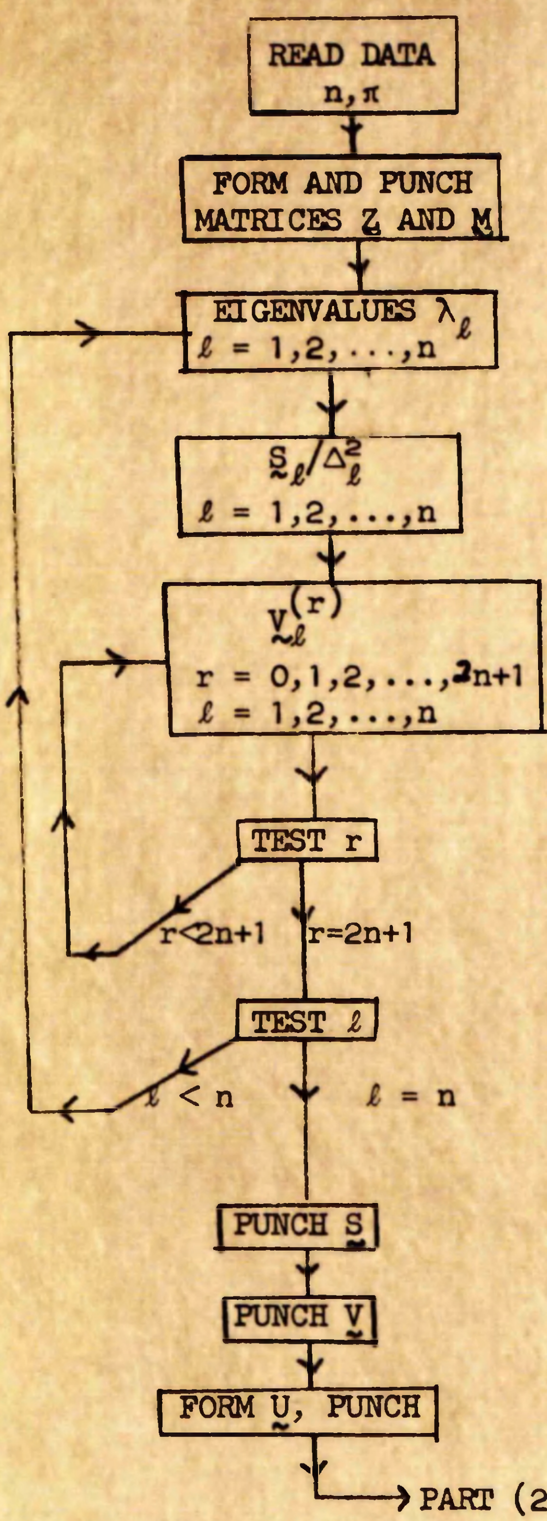
FIG. 17

CONVERGENCE GRAPH FOR SQUARE BOX CONDENSER PROBLEM

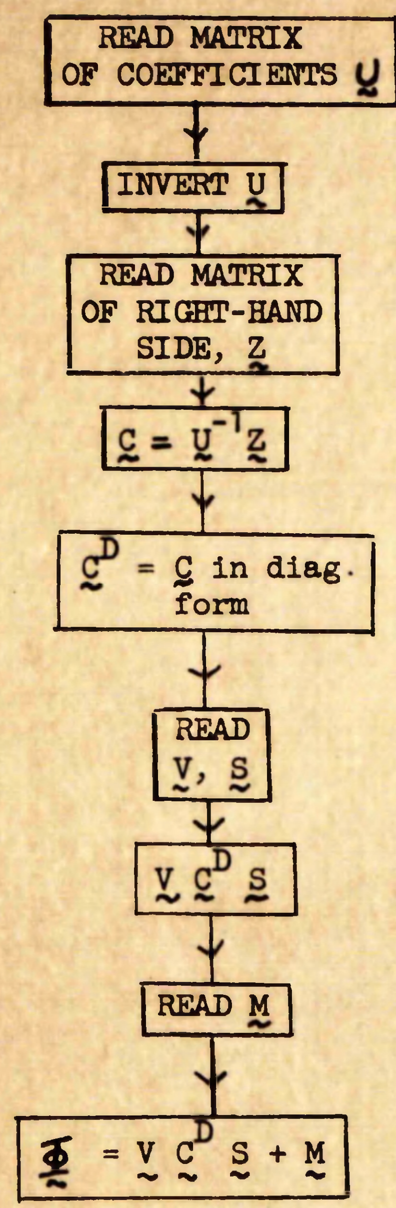


FLOW DIAGRAMS FOR THE COMPUTATION ON THE DEUCE COMPUTER OF SOLUTIONS OF THE SQUARE BOX CONDENSER PROBLEM USING A SQUARE NET PATTERN

PART (1) ALPHACODE



PART (2) G.I.P.



CHAPTER 3

A STEADY-STATE WAVE PROBLEM: THE HELMHOLTZ EQUATION, $\nabla^2\phi + k^2\phi = 0$.

3.1 Introduction

In chapters 1 and 2 we have illustrated the discrete transform method of solving boundary-value problems in rectangular regions and regions composed of adjacent rectangles. We show in this chapter how the same techniques may be applied to obtain solutions of steady-state wave problems in similar types of region.

Before giving details (in § 3.2) of the particular problem used for illustration, in this section we shall examine some of the ground-work involved in its solution. Consider the propagation of waves in the two-dimensional duct, $0 \le y \le 2b$, $-\infty < x < \infty$, [Fig. 18]. With wave function $\psi(x,y,t)$ the equation of the wave motion is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$

where c is the velocity of the waves, and we have $\frac{\partial\psi}{\partial y} = 0$ on $y = 0, 2b$. Taking a time factor $e^{\pm i\omega t}$ and letting $\psi(x,y,t) = \phi(x,y)e^{\pm i\omega t}$, the steady-state wave equation is

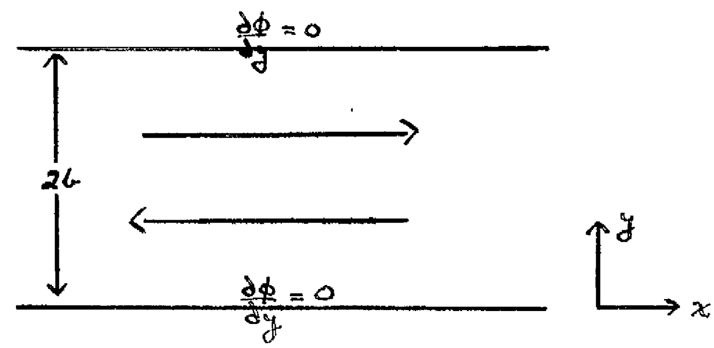


Fig. 18

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \tag{3.1.1}$$

where $k^2 = \frac{\omega^2}{c^2}$, and $\frac{\partial\phi}{\partial y} = 0$ on $y = 0, 2b$. Separation of variables gives the solution

$$\phi = \cos\frac{l\pi y}{2b}\, e^{\pm\gamma_l x}, \quad l = 0, 1, 2, \ldots. \tag{3.1.2}$$

where $\gamma_\ell = \left[\left(\frac{\ell\pi}{2b}\right)^2 - k^2\right]^{1/2}$. We shall confine ourselves to the case in which there is only one propagating node, all others being exponentially attenuated. This means that, for $\ell > 0$, γ_ℓ must be real while $\gamma_0 = ik$ and therefore k must be such that

$$0 < 2bk \leq \pi. \tag{3.1.3}$$

Using the time factor $e^{-i\omega t}$, the steady-wave function $\phi = e^{ikx}$ corresponds to the progressive wave $\psi = e^{i(kx - \omega t)}$ which is a wave travelling from left to right while $\phi = e^{-ikx}$ corresponds to a wave travelling in the opposite direction. We shall now obtain results corresponding to equations (3.1.2) and (3.1.3) using a discrete transform attack on the set of difference equations corresponding to the Helmholtz equation (3.1.1).

We first cover the region with a square lattice positioned so that it straddles the upper and lower edges of the duct (Fig. 19). Take step measures r and s in the x and y directions respectively and suppose the edges of the duct correspond to $s = -\frac{1}{2}$ and $s = n + \frac{1}{2}$. Using a square net pattern, the set of difference equations corresponding to equation (3.1.1) is

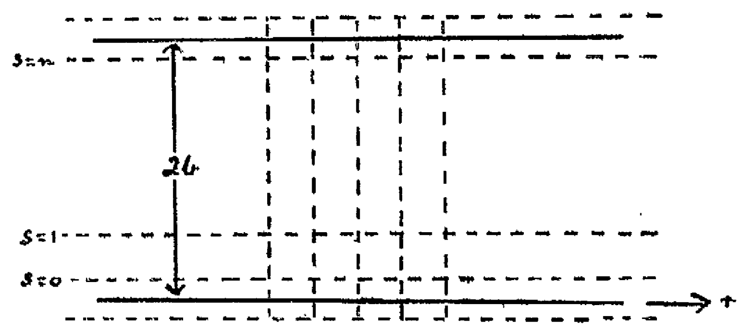


Fig. 19

$$(4 - k^2h^2)\phi_{r,s} - \phi_{r+1,s} - \phi_{r,s+1} - \phi_{r-1,s} - \phi_{r,s-1} = 0, \tag{3.1.4}$$

where h is the net spacing, $0 \leq s \leq n$, and r can have any integral value. In the "s" direction we have the boundary conditions

$$\phi_{r,-1} = \phi_{r,0} \; ; \; \phi_{r,n} = \phi_{r,n+1} \tag{3.1.5}$$

Separating the variables and transforming in the "s" direction it can be shown, as in problem B, § 1.7, p. 28, that the eigenvalues of the separation constant 2λ are given by

$$\lambda_\ell = \cos\Theta_\ell \quad ; \quad \Theta_\ell = \frac{\ell\pi}{n+1}, \quad \ell = 0, 1, 2, \ldots, n, \tag{3.1.6}$$

with corresponding discrete eigenfunctions $S_\ell(s)$ given by (§ 1.10, p. 34),

$$S_\ell(s) = \frac{\cos(2s+1)\frac{\Theta_\ell}{2}}{\cos\frac{\Theta_\ell}{2}}. \tag{3.1.7}$$

Further, if we define the vector $\underset{\sim}{S}_\ell$ by

$$\underset{\sim}{S}_\ell = \{S_\ell(0), S_\ell(1), \ldots\ldots, S_\ell(n)\}. \tag{3.1.8}$$

$$\text{then} \quad \underset{\sim}{A}\,\underset{\sim}{S}_\ell = (4-2\lambda_\ell)\underset{\sim}{S}_\ell \quad \text{or} \quad \underset{\sim}{S}_\ell^T\underset{\sim}{A} = (4-2\lambda_\ell)\underset{\sim}{S}_\ell^T \tag{3.1.9}$$

where the basic matrix $\underset{\sim}{A}$ is of type $(n+1) \times (n+1)$ and

$$\underset{\sim}{A} = \begin{bmatrix} 3 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 4 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdots & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & \cdots & \cdots & 0 & -1 & 4 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 3 \end{bmatrix} \tag{3.1.10}$$

The matrix governing equation of this problem for a square net is

$$\underset{\sim}{A}\,\underset{\sim}{\Phi}_r - k^2\ell^2\underset{\sim}{\Phi}_r - \underset{\sim}{\Phi}_{r+1} - \underset{\sim}{\Phi}_{r-1} = \underset{\sim}{O}_{(n+1)\times 1}. \tag{3.1.11}$$

where the column vector $\underset{\sim}{\Phi}_r$ is

$$\underset{\sim}{\Phi}_r = \{\phi_{r,0}, \phi_{r,1}, \ldots\ldots, \phi_{r,n}\}. \tag{3.1.12}$$

By applying the discrete transform, solving the resulting second-order difference equation, and substituting in the inverse formula, we can show that

$$\underset{\sim}{\Phi}_r = \sum_{\ell=0}^{n}\left(C_\ell U_\ell^r + D_\ell V_\ell^r\right)\underset{\sim}{S}_\ell \tag{3.1.13}$$

where C_ℓ, D_ℓ are arbitrary constants. In this equation the quantities U_ℓ, V_ℓ are given by

$$U_\ell = \frac{1}{2}\left\{(4-k^2h^2-2\lambda_\ell)+\sqrt{(4-k^2h^2-2\lambda_\ell)^2-4}\right\},$$
$$V_\ell = \frac{1}{2}\left\{(4-k^2h^2-2\lambda_\ell)-\sqrt{(4-k^2h^2-2\lambda_\ell)^2-4}\right\}, \tag{3.1.14}$$

and $U_\ell V_\ell = 1$, $\ell = 0, 1, \ldots, n$. Equation (3.1.13) is a general solution involving ($n+1$) discrete modes and corresponds to the continuous solution (3.1.2). If we assume that we have only one propagating mode, only one U_ℓ and, therefore, one V_ℓ can be complex. From equation (3.1.6), max $\lambda_\ell = \lambda_0 = 1$ so that U_0 , V_0 are complex since $kh \neq 0$. In order that all other U_ℓ , V_ℓ should be real and correspond to attenuated modes, k must be such that

$$(4-k^2h^2-2\lambda_\ell)^2 \geqslant 4, \quad \ell = 1, 2, \ldots, n.$$

$$\text{i.e. } (4-k^2h^2-2\lambda_\ell) \geqslant 2.$$

(The negative root is impossible since $2\lambda_\ell < 2$ and the above relation must hold for all h .)

$$\text{i.e. } k^2h^2 \leq 2-2\lambda_\ell = 2(1-\cos\Theta_\ell) = 4\sin^2\frac{\Theta_\ell}{2}.$$

$$\text{i.e. } 0 < kh \leq 2\sin\frac{\Theta_\ell}{2} \quad \text{since } kh \text{ is positive.}$$

Hence, from (3.1.6) we have

$$0 < kh \leq 2\sin\left(\frac{\pi}{2(n+1)}\right) \text{since } \ell = 1 \text{ gives smallest value of } \Theta_\ell .$$

$$\therefore \quad 0 < 2bk \leq 2(n+1)\sin\left(\frac{\pi}{2(n+1)}\right). \tag{3.1.15}$$

since $2b = (n+1)h$ [Fig. 19]. (3.1.15) corresponds to (3.1.3) since, as $n \to \infty$, $2(n+1)\sin\left(\frac{\pi}{2(n+1)}\right) \to \pi$.

For our future convenience we shall re-write solution (3.1.13) in the form

$$\underset{\sim}{\Phi}_+ = (C_0U_0^+ + D_0V_0^+)\underset{\sim}{S}_0 + \sum_{\ell=1}^{n}(C_\ell U_\ell^+ + D_\ell V_\ell^+)\underset{\sim}{S}_\ell , \tag{3.1.16}$$

where u_0, V_0 are complex and correspond to the propagating mode and u_ℓ, V_ℓ $(\ell = 1, 2, \cdots, n)$ are real and correspond to attenuated modes. By analogy with the continuous case, taking time factor $e^{-i\omega t}$, u_0 corresponds to a wave travelling from left to right and V_0 to a wave travelling in the opposite direction.

We will now turn to the solution of a particular steady-state wave problem.

3.2 A Steady-State Wave Problem

Consider the propagation of waves in a two-dimensional duct containing an obstruction perpendicular to the direction of propagation. In particular suppose the duct to be given by $0 \le y \le 2b$, $-\infty < x < \infty$ (i.e. as in § 3.1) and let the obstruction lie on the linear segment $x = 0$, $c \le y \le 2b$ where $0 < c < 2b$ (Fig. 20)

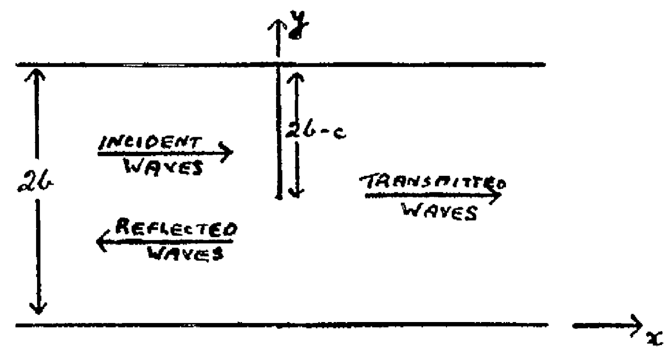


Fig. 20

Due to the presence of the obstruction the incident waves are split into reflected and transmitted portions so that we have the following "radiation conditions":

(a) on the left of the obstruction waves may travel in both the positive and negative "x" directions, and

(b) on the right of the obstruction waves may only travel in the positive "x" direction.

Taking a time factor $e^{-i\omega t}$ our problem is then to solve the Helmholtz equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0, \tag{3.2.1}$$

in the infinite strip $0 \le y \le 2b$, $-\infty < x < \infty$ such that

(i) $\frac{\partial \phi}{\partial y} = 0$ on $y = 0, 2b$,

(ii) $\frac{\partial \phi}{\partial x} = 0$ on $x = 0-$, $x = 0+$, $c \le y \le 2b$ (i.e. zero normal derivatives on both sides of the obstruction).

(iii) the radiation conditions (a) and (b) are satisfied, and

(iv) ϕ is finite at $x = \pm\infty$.

For simplicity we shall assume that $0 < 2bk \leq \pi$ so that, by equation (5.1.5), we have only one propagating mode.

This problem is an adjacent region problem with sub-regions consisting of the two semi-infinite rectangles on either side of the obstruction. It differs from the adjacent region problems discussed in chapter 2 since we are now concerned with the Helmholtz equation in infinite regions with the consequent introduction of radiation conditions and we now have a derivative condition specified over part of the common boundary.

5.3 The Discrete Transform Method of Solving the Steady-State Wave Problem

Our method of solving this adjacent region problem is similar to those discussed in chapter 2 in that we obtain solutions in terms of unknown quantities for each of the sub-regions and then use a matching process to evaluate those quantities. It differs, however, in that unknown quantities need only be introduced over part of the common boundary and the matching process need only be applied over this part.

Mindful of the derivative conditions specified we first cover the region with a square lattice to straddle the boundaries of the duct and the obstruction and such that the end-point of the obstruction lies at the centre of a mesh (Fig. 21). Let the boundaries of the duct be given by $s = -\frac{1}{2}$, $s = n + \frac{1}{2}$, and take the end-point of the obstruction to be at the centre of the square bounded by $r = 0-$, $r = 0+$, $s = jh$, $s = (j+1)h$, where h is the net spacing. Using a square

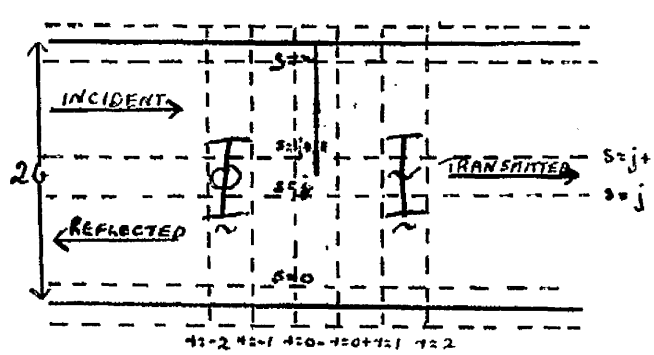


Fig. 21

net pattern the set of difference equations which must be solved is

$$(4-k^2h^2)\,\phi_{r,s}-\phi_{r+1,s}-\phi_{r,s+1}-\phi_{r-1,s}-\phi_{r,s-1}=0, \tag{3.3.1}$$

where $0 \leq s \leq n$ and r can have any positive or negative integral value or may be zero (Fig. 21). Since we assume only one propagating mode $2bk$ must satisfy the relation (3.1.15).

Using only the Neumann conditions on the edges of the duct, equation (3.1.16) gives a general solution in each semi-infinite rectangle involving $(2n+2)$ arbitrary constants. By the radiation conditions ((a) and (b) § 3.2) and the conditions of finiteness at infinity in the x-direction, for the semi-infinite rectangle on the left of the obstruction we may write

$$\underset{\sim}{\Phi}_r = \left(U_0^r + R_0 V_0^r\right)\underset{\sim}{S}_0 + \sum_{\ell=1}^{n} R_\ell U_\ell^r \underset{\sim}{S}_\ell\ , \tag{3.3.2a}$$

where $R_0, R_1, \ldots, R_n$ are arbitrary constants, $\underset{\sim}{\Phi}_r$ is given by (3.1.12), $\underset{\sim}{S}_\ell$ by (3.1.8) and U_ℓ, V_ℓ by (3.1.14). (In equation (3.3.2a) we have assumed the incident wave to have unit intensity.) The solution takes this form since, as was pointed out in § 3.1, U_0, V_0 are complex, corresponding to waves travelling to the right and left respectively, while U_ℓ is real and $U_\ell > 1$, $\ell = 1, 2, \ldots, n$. In the same way, for the semi-infinite rectangle on the right of the obstruction we may write

$$\underset{\sim}{\Psi}_r = T_0 U_0^r \underset{\sim}{S}_0 + \sum_{\ell=1}^{n} T_\ell V_\ell^r \underset{\sim}{S}_\ell\ , \tag{3.2.2b}$$

where $T_0, T_1, \ldots, T_n$ are arbitrary constants and $\underset{\sim}{\Psi}_r$ is used instead of $\underset{\sim}{\Phi}_r$ simply to avoid confusion. We have thus obtained solutions in both sub-regions, each solution involving $(n+1)$ arbitrary constants.

Our procedure is to express these solutions in terms of unknown quantities introduced on the common boundary and then to evaluate these using a matching process. Before we show in detail how this may be carried out, however, we shall first discuss the logic behind the matching process used and, to clarify this logic, shall consider the analogous matching procedures which would be adopted in the continuous case.

In the continuous case the boundary conditions on $x=0$ are:

(i) ϕ is continuous across $x=0,\ 0\le y\le c$,

(ii) $\frac{\partial\phi}{\partial x}$ is continuous across $x=0,\ 0\le y\le c$,

(iii) $\frac{\partial\phi}{\partial x}=0$ on $x=0-$ and $x=0+$, $c\le y\le 2b$.

There are two procedures which could be adopted:

(a) If we assume that $\left(\frac{\partial\phi}{\partial x}\right)_{x=0}$ equals the unknown function $f(y)$ for $0\le y\le c$ we can express the potentials in the two regions $x\geqslant 0$ and $x\le 0$ in terms of $f(y)$. An integral equation for $f(y)$ is then obtained using condition (i) that ϕ is continuous across $x=0,\ 0\le y\le c$.

(b) If we assume that the discontinuity in ϕ across $x=0$ is given by the unknown function $g(y)$ for $c\le y\le 2b$ we can express the potentials on either side of $x=0$ in terms of $g(y)$. By using condition (iii) that $\frac{\partial\phi}{\partial x}$ is continuous across the obstacle we can obtain an integral equation for $g(y)$.

Discrete analogues of both (a) and (b) exist as we now indicate.

(A) The discrete analogue of method (a) is as follows. We introduce $(j+1)$ unknown quantities, α_s, defined by

$$\alpha_s=\psi_{0,s}-\phi_{0,s}\ ,\quad s=0,1,2,\dots,j\ , \tag{3.3.3}$$

where $\phi_{0,s}$ and $\psi_{0,s}$ denote potentials at the points indicated in Fig. 22.

[To avoid confusion, in the remainder of this chapter the symbols ϕ and ψ will be used to denote potentials in the semi-infinite rectangles on the left and right of the obstruction respectively. There is, therefore, no ambiguity in allowing two mesh lines to have the step measure $r=0$.]

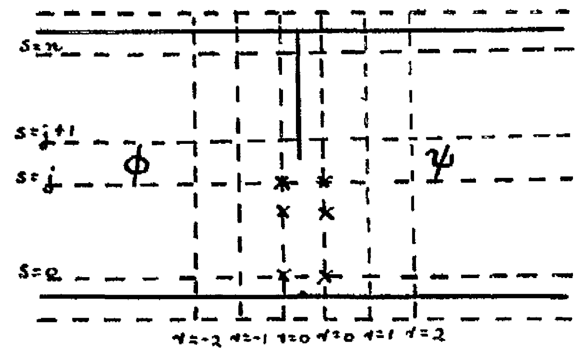


Fig. 22

Except for a factor h (i.e. the net spacing) the quantities α_s so defined are approximate normal derivatives on horizontal mesh lines between the end of the obstruction and the lower edge of the duct. As we shall show in detail later the potentials $\phi_{r,s}$ and $\psi_{r,s}$ can be expressed in terms of these quantities and, in particular,

$$\phi_{0,s} = \sum_{i=0}^{j} a_{s,i}\,\alpha_i + c_s\,, \tag{3.3.4}$$

and

$$\psi_{0,s} = \sum_{i=0}^{j} b_{s,i}\,\alpha_i\,. \tag{3.3.5}$$

The condition of continuity across the gap between the end of the obstruction and lower edge of the duct is now that these values must be related by equation (3.3.3). Substituting (3.3.4) and (3.3.5) into equation (3.3.3) we obtain the set of simultaneous linear equations

$$\alpha_s + \sum_{i=0}^{j} \left(a_{s,i} - b_{s,i}\right)\alpha_i = -c_s\,, \quad s = 0,1,2,\ldots,j. \tag{3.3.6}$$

Instead of an integral equation we thus require to solve a set of $(j+1)$ linear equations in the unknown "derivatives" α_s, where $(j+1)$ is the number of horizontal mesh lines between the end of the obstruction and the lower edge of the duct.

(B) A discrete analogue of the continuous method (b) can be obtained in a similar way. We shall not go into this in detail but merely state that in this case we require to solve a set of $(n-j)$ simultaneous linear equations where $(n-j)$ is the number of mesh lines cutting the obstruction. This method would therefore be preferable if the obstruction extends less than half-way across the duct.

We shall now return to the solution of our problem and will use the discrete matching procedure (A).

On account of the Neumann conditions on the obstruction and on the upper boundary of the duct, the difference equations at mesh points with $s > j$ on $r = 0-$ (Fig. 21) may be written in the form:

$$\left.\begin{aligned} -\phi_{0,s-1} + (4-k^2h^2)\phi_{0,s} - \phi_{0,s+1} - \phi_{0,s} - \phi_{-1,s} &= 0, \quad s = j+1, \ldots, n-1, \\ -\phi_{0,n-1} + (3-k^2h^2)\phi_{0,n} \qquad - \phi_{0,n} - \phi_{-1,n} &= 0 \end{aligned}\right\} \quad (3.3.7)$$

At mesh points with $s \leq j$ the corresponding difference equations are

$$\left.\begin{aligned} (3-k^2h^2)\phi_{0,0} - \phi_{0,1} - \psi_{0,0} - \phi_{-1,0} &= 0 \\ -\phi_{0,s-1} + (4-k^2h^2)\phi_{0,s} - \phi_{0,s+1} - \psi_{0,s} - \phi_{-1,s} &= 0, \quad s = 1, 2, \ldots, j \end{aligned}\right\} \quad (3.3.8)$$

If we introduce α_s, $(s = 0, 1, 2, \cdots, j)$, defined by equation (3.3.3), the set of difference equations (3.3.8) becomes

$$\left.\begin{aligned} &(3-k^2h^2)\phi_{0,0} - \phi_{0,1} - \phi_{0,0} - \phi_{-1,0} = \alpha_0 , \\ -\phi_{0,s-1} + &(4-k^2h^2)\phi_{0,s} - \phi_{0,s+1} - \phi_{0,s} - \phi_{-1,s} = \alpha_s , \quad s = 1, 2, \cdots, j \end{aligned}\right\} \quad (3.3.9)$$

Equations (3.3.7) and (3.3.9) may then be written as the matrix difference equation:

$$\underset{\sim}{A}\underset{\sim}{\Phi}_0 - (1+k^2h^2)\underset{\sim}{\Phi}_0 - \underset{\sim}{\Phi}_{-1} = \underset{\sim}{G} , \qquad (3.3.10a)$$

where $\underset{\sim}{G} = \{\alpha_0, \alpha_1, \cdots, \alpha_j, 0, 0, \cdots, 0\}$ of type $(n+1) \times 1$ and $\underset{\sim}{A}$ is given by equation (3.1.10). Proceeding in a similar manner we can show that the set of difference equations on $t = 0+$ (Fig. 21) gives the matrix difference equation

$$\underset{\sim}{A}\underset{\sim}{\Psi}_0 - (1+k^2h^2)\underset{\sim}{\Psi}_0 - \underset{\sim}{\Psi}_1 = -\underset{\sim}{G} \qquad (3.3.10b)$$

where $\underset{\sim}{\Psi}$ is used instead of $\underset{\sim}{\Phi}$ on the right of the obstruction to avoid confusion.

From equation (3.3.2a), we have

$$\underset{\sim}{\Phi}_0 = (1+R_0)\underset{\sim}{S}_0 + \sum_{\ell=1}^{n} R_\ell \underset{\sim}{S}_\ell , \qquad (3.3.11a)$$

$$\underset{\sim}{\Phi}_{-1} = (U_0^{-1} + R_0 V_0^{-1})\underset{\sim}{S}_0 + \sum_{\ell=1}^{n} R_\ell U_\ell^{-1} \underset{\sim}{S}_\ell . \qquad (3.3.12a)$$

Pre-multiplying equation (3.3.10a) by $\underset{\sim}{S}_\ell^T, (\ell = 0, 1, \cdots, n)$, and using the above equations and equation (3.1.9) we have, by the orthogonality of the vectors $\underset{\sim}{S}_\ell$

$$R_0 = \frac{\underset{\sim}{S}_0^T \underset{\sim}{G}}{\Delta_0^2 \beta_0} - \mu_0 \; ; \quad R_\ell = \frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell} \; , \; \ell = 1, 2, \cdots, n, \qquad (3.3.13)$$

where $\Delta_\ell^2 = \underset{\sim}{S}_\ell^T \underset{\sim}{S}_\ell$, $\beta_0 = (3 - k^2 h^2 - 2\lambda_0 - V_0^{-1})$, $\beta_\ell = (3 - k^2 h^2 - 2\lambda_\ell - U_\ell^{-1})$, and $\mu_0 = \frac{1}{\beta_0}(3 - k^2 h^2 - 2\lambda_0 - U_0^{-1})$. By similarly treating equation (3.3.10b) we can show that:

$$T_0 = -\frac{\underset{\sim}{S}_0^T \underset{\sim}{G}}{\Delta_0^2 \beta_0} \; ; \quad T_\ell = -\frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell} \; , \; \ell = 1, 2, \cdots, n. \qquad (3.3.14)$$

[The 'scalar product' factors $\Delta_\ell^2 = \underset{\sim}{S}_\ell^T \underset{\sim}{S}_\ell$ occurring in equations (3.3.13) and (3.3.14) are due to the fact that we have not normalised the vectors $\underset{\sim}{S}_\ell$ (equation (3.1.8)) to have unit length but such that $S_\ell(0) = 1$ (equation (3.1.7)).] Solutions (3.3.2a) and (3.3.2b) are then

$$\underset{\sim}{\Phi}_+ = \left(U_0^+ + \left(\frac{\underset{\sim}{S}_0^T \underset{\sim}{G}}{\Delta_0^2 \beta_0} - \mu_0\right) V_0^+\right) \underset{\sim}{S}_0 + \sum_{\ell=1}^{n} \left(\frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell}\right) U_\ell^+ \underset{\sim}{S}_\ell \, , \qquad (3.3.15a)$$

$$\underset{\sim}{\Psi}_+ = -\frac{\underset{\sim}{S}_0^T \underset{\sim}{G}}{\Delta_0^2 \beta_0} U_0^+ \underset{\sim}{S}_0 - \sum_{\ell=1}^{n} \left(\frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell}\right) V_\ell^+ \underset{\sim}{S}_\ell \, . \qquad (3.3.15b)$$

<u>The solutions in both semi-infinite rectangles are thus expressed in terms of the $(j+1)$ unknown "derivatives" α_s, $s = 0, 1, \cdots, j$</u>, contained in the vector $\underset{\sim}{G}$ (equation (3.3.10a)).

To evaluate the unknowns α_s , we use the matching process given by equation (3.3.3). From equation (3.3.15a),

$$\phi_{0,s} = (1 - \mu_0) S_0(s) + \sum_{\ell=0}^{n} \left(\frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell}\right) S_\ell(s) \qquad (3.3.16)$$

$$\text{i.e.} \quad \phi_{o,s} = \sum_{i=0}^{j} a_{s,i} \alpha_i + c_s, \tag{3.3.17}$$

where $a_{s,i} = \sum_{\ell=0}^{\tilde{n}} \frac{S_\ell(i) S_\ell(s)}{\Delta_\ell^2 \beta_\ell}$ and $c_s = (1-\mu_o) S_o(s)$ and these give explicit expressions for the quantities $a_{s,i}$, c_s occurring in equation (3.3.4). Similarly from equation (3.3.15b) we have

$$\psi_{o,s} = - \sum_{\ell=0}^{\tilde{n}} \left(\frac{\underset{\sim}{S}_\ell^T \underset{\sim}{G}}{\Delta_\ell^2 \beta_\ell} \right) S_\ell(s)$$

$$\text{i.e.} \quad \psi_{o,s} = \sum_{i=0}^{j} b_{s,i} \alpha_i , \tag{3.3.18}$$

where $b_{s,i} = -a_{s,i}$. We thus obtain explicit expressions for the quantities $b_{s,i}$ occurring in equation (3.3.5). The simultaneous equations (3.3.6) may then be written

$$\sum_{i=0}^{j} (2a_{s,i} + \delta_i^s) \alpha_i = -c_s, \quad s = 0,1,2,\dots,j \tag{3.3.19}$$

<u>Our problem is now reduced to the solution of a set of $(j+1)$ simultaneous linear equations in the $(j+1)$ unknown derivatives α_s, $s = 0,1,2,\dots,j$</u>

Solving these equations and substituting the quantities α_s, $s = 0,1,2,\dots,j$ into the vector $\underset{\sim}{G}$ occurring in equations (3.3.15a) and (3.3.15b) then gives the solution of the problem.

The solution of equations (3.3.19) is not as complicated as may be thought. Though the coefficients are complex, each has the same imaginary part. This follows from the fact that in the finite series determining $a_{s,i}$ (equation (3.3.17)) only β_o (equation (3.3.13)) is complex and by definition, (equation (3.1.7)), $S_o(s) \equiv 1$. Thus the coefficients in (3.3.19) may be written:

$$2a_{s,i} + \delta_i^s = \frac{2}{\Delta_0^2 \beta_0} + 2\sum_{\ell=1}^{n} \frac{S_\ell(i) S_\ell(s)}{\Delta_\ell^2 \beta_\ell} + \delta_i^s ,$$

where only the first term on the right hand side is complex and is independent of both i and s. Further, since $S_0(s) \equiv 1$, the right hand sides of equations (3.3.19) have the same complex value $(\mu_0 - 1)$ (equation (3.3.17)). We may, therefore, subtract one of these equations from all the others to give a set of $(j+1)$ equations which consists of one equation with complex coefficients and a complex right hand side while the remaining "j" equations have real coefficients and zero right hand sides.

Clearly the number of equations which we require to solve for any given net spacing depends on the number of mesh lines between the end of the obstruction and the lower boundary of the duct. The longer the obstruction, the fewer equations which must be solved. Results, for n = 3, 5, 7, and 9 [Fig. 21], have been obtained on a desk calculator for the case in which the obstruction extends half-way across the duct so that we required to solve sets of 2, 3, 4 and 5 equations respectively. (The value of k was assumed to be such that $2bk = 2$ (equation (3.1.15)). In Fig. 23, p.70a, we record the results for the case in which n = 3.

§ 3.4 Solution of the Steady-State Wave Problem using Diagonal and Nine Point Net Patterns

The solution of the problem of § 3.2 using diagonal and nine point net patterns may be obtained in the manner of the previous section except for minor modifications in the fitting process. Our method of solution using the square net pattern was dependent on the formation of the matrix difference equations (3.3.10a) and (3.3.10b). It was through these equations that we were able to

RESULTS FOR WAVE PROBLEM, WITH THE OBSTRUCTION STRADDLED,

USING A SQUARE NET PATTERN

1.18501 -0.58488i	1.75505 -0.59730i	2.03243 -0.56862i	-0.15743 +1.05275i	-0.34880 +0.96039i	-0.59916 +0.73617i
1.10059 -0.52232i	1.60886 -0.48905i	1.80171 -0.39779i	0.07329 +0.88192i	-0.20260 +0.85214i	-0.51466 +0.67361i
0.97765 -0.43129i	1.37586 -0.31654i	1.31341 -0.03626i	0.56159 +0.52038i	0.03040 +0.67963i	-0.39171 +0.58258i
0.90178 -0.37512i	1.25955 -0.23043i	1.18612 +0.05798i	0.68888 +0.42614i	0.14670 +0.59352i	-0.31584 +0.52640i

Fig. 23

express the solutions in each of the sub-regions in terms of the $(j+1)$ unknown quantities introduced on the lower part of the common boundary. In fact, the unknown quantities were defined in the manner of equation (3.3.3) to make the formation of these matrix difference equations possible. When diagonal or nine point net patters are used, however, the corresponding matrix difference equations can only be formed if we introduce $(j+2)$ unknown quantities, defined in a slightly different manner, on the lower part of the common boundary. For example, in the nine point pattern case, the $(j+2)$ unknown quantities which must be introduced on mesh lines $s=0,1,2,\dots,j+1$ [Fig. 22] are given by

$$\beta_s = 4(\psi_{0,s}-\phi_{0,s}) + (\psi_{0,s+1}-\phi_{0,s+1}) + (\psi_{0,s-1}-\phi_{0,s-1}), \quad s=0,1,2,\dots j \tag{3.4.1}$$

$$\beta_{j+1} = \psi_{0,j}-\phi_{0,j} \ .$$

To solve the problem using diagonal or nine point net patterns we therefore require to solve a set of $(j+2)$ simultaneous equations, these equations being obtained using a matching process determined by the definitions of the unknown quantities, e.g. equations (3.4.1).

We do not prove here why $(j+2)$ unknown quantities are required but merely state that the reason lies in the geometrical arrangement of the points involved in these patterns (§ 1.1). Results, for n = 3, 5, 7, and 9, corresponding to those for the square net pattern, have been obtained using a nine point net pattern.

§ 3.5 Reflection and Transmission Coefficients

From equation (3.1.2) and on account of the radiation and finiteness conditions given in § 3.2, p. 61, the continuous solutions in the semi-infinite rectangles on either side of the obstruction may be written as:

$$\begin{aligned} &\text{(a) on the left:} \quad \phi = e^{ikx} + R_0 e^{-ikx} + \sum_{\ell=1}^{\infty} R_\ell \cos\frac{\ell\pi y}{2b} e^{\gamma_\ell x}, \\ &\text{(b) on the right:} \quad \phi = T_0 e^{ikx} + \sum_{\ell=1}^{\infty} T_\ell \cos\frac{\ell\pi y}{2b} e^{-\gamma_\ell x}, \end{aligned} \qquad (3.5.1)$$

where R_ℓ, T_ℓ, $(\ell = 0, 1, 2, \ldots\ldots)$, are arbitrary constants. (Solutions (a) and (b) correspond to the discrete solutions (3.3.2a) and (3.3.2b)). Constants of considerable interest in this problem are the (energy) reflection and transmission coefficients, $|R_0|^2$ and $|T_0|^2$(equation (3.5.1)). Baldwin and Heins [21] have shown, using an integral equation method, that, when the obstruction extends half-way across the duct, the true values of these quantities are given by $|R_0|^2 = \sin^2\rho$, $|T_0|^2 = \cos^2\rho$ where $\rho = \sum_{m=1}^{\infty} (-1)^{m+1} \arcsin\frac{2bk}{m\pi}$. Summing this series for ρ by Euler's Transform method these formulae give, for the case in which $2bk = 2$

$$|R_0|^2 = 0.2212; \; |T_0|^2 = 0.7788.$$

Using the discrete transform method approximate values for these quantities have been obtained for various net spacings and for both square and nine point net patterns. In the square net case the formulae for R_0 and T_0 given in equations (3.3.13) and (3.3.14) were employed for this purpose while corresponding formulae were used in the nine point case. The net spacings employed corresponded to values of n equal to 3, 5, 7 and 9 [Fig. 21]. By plotting the approximate values of $|R_0|^2$ and $|T_0|^2$ against $\frac{1}{n+1}$, graphs may be drawn to illustrate the convergence of the discrete solutions to the true solution as the net spacing is reduced. On p.75a, a number of such graphs are reproduced, those numbered (1S) and (1N) corresponding to the method illustrated in the previous sections using square and nine point net patterns respectively. The graphs (IS) and (IN) show that by this method convergence is near linear but slow.

§ 3.6 Other methods

In § 3.3, on account of the Neumann conditions on the obstruction, we chose the lattice to straddle the common boundary. (For convenience we shall call this case (1)). Due to the slow rate of convergence mentioned in the last section, it seemed worth-while to consider the solution of the problem when the lattice is chosen to be coincident with the common boundary. Two cases have been considered which we shall refer to as cases (2) and (3) respectively.

In case (2) the lattice was chosen so that the end-point of the obstruction was coincident with a mesh point (Fig. 24). The manner of solution was the same as in § 3.3 except that the continuity condition giving the $(j+1)$ simultaneous equations in the $(j+1)$ unknown derivatives introduced on $t=0$ at $s=0, 1, 2, \ldots, j$, was expressed as

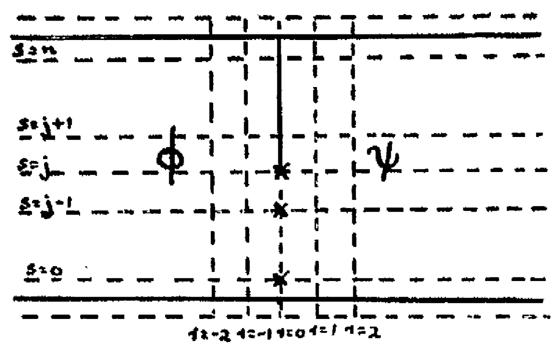


Fig. 24

$$\phi_{0,s} = \psi_{0,s}, \quad s=0,1,2,\ldots,j$$

(c.f. equation (3.3.3)).

In case (3) the lattice was chosen so that the end-point of the obstruction lay mid-way between two mesh points (Fig. 25). A slightly different matching process was used in this case which we shall briefly illustrate.

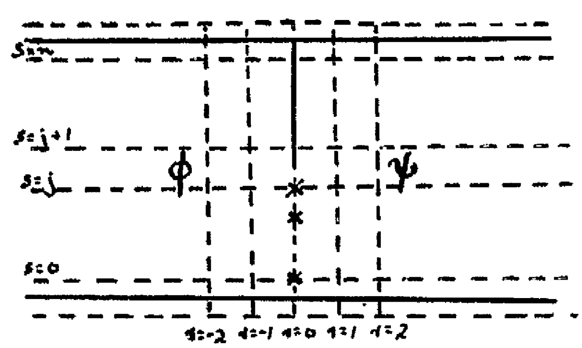


Fig. 25

Assume, on the left of the obstruction line $r = 0$, that

$$h\left(\frac{\partial\phi(r,s)}{\partial x}\right)_{r=0} = a\,\phi_{0,s} + b\,\phi_{-1,s} + c\,\phi_{-2,s}\ ,\quad s = 0,1,2,\cdots,n\ .$$

Then by Taylor series expansions about the points $(0, sh)$, $s = 0,1,\cdots,n$, it can be shown that

$$h\left(\frac{\partial\phi}{\partial x}(r,s)\right)_{r=0} = \frac{3}{2}\phi_{0,s} - 2\phi_{-1,s} + \frac{1}{2}\phi_{-2,s} = \gamma_s\ ,\quad s = 0,1,\cdots,n, \tag{3.6.1}$$

where $\gamma_s = 0$, $s > j$, and the quantities γ_s so defined are, except for a factor h, approximate normal derivatives on the left of $r = 0$. Writing equations (3.6.1) in matrix form we have

$$\frac{3}{2}\underset{\sim}{\Phi}_0 - 2\underset{\sim}{\Phi}_{-1} + \frac{1}{2}\underset{\sim}{\Phi}_{-2} = \underset{\sim}{H}, \tag{3.6.2}$$

where $\underset{\sim}{\Phi}_r = \{\phi_{r,0}, \phi_{r,1}, \cdots, \phi_{r,n}\}$ and $\underset{\sim}{H} = \{\gamma_0, \gamma_1, \cdots, \gamma_j, 0, 0, \cdots, 0\}$ of type $(n+1) \times 1$. Similarly by deriving expansions for the approximate normal derivatives on the right of $r = 0$ and using the continuity of derivatives across $r = 0$, we obtain

$$\frac{3}{2}\underset{\sim}{\Psi}_0 - 2\underset{\sim}{\Psi}_1 + \frac{1}{2}\underset{\sim}{\Psi}_2 = -\underset{\sim}{H} \tag{3.6.3}$$

where $\underset{\sim}{\Psi}_r = \{\psi_{r,0}, \psi_{r,1}, \cdots, \psi_{r,n}\}$.

Equations (3.6.2) and (3.6.3) correspond to the matrix difference equations (3.3.10a) and (3.3.10b) of § 3.3 and are treated in exactly the same way to express the solutions $\underset{\sim}{\Phi}_r$ and $\underset{\sim}{\Psi}_r$ in the semi-infinite rectangle on the left and on the right of the obstruction respectively in terms of the $(j+1)$ unknown derivatives γ_s, $(s = 0, 1, 2, \ldots j)$. As in § 3.3 these unknown derivatives are then evaluated using the set of $(j+1)$ simultaneous linear equations obtained using the condition of continuity of the solution across the gap between the end of the obstruction and the lower edge of the duct. In this case this condition is expressed in the form:

$$\phi_{0,s} = \psi_{0,s}\ ,\quad s = 0,1,2,\cdots,j\ .$$

The steady-state wave problem has also been solved by a semi-continuous method by leaving the "x" variable in the wave function $\phi(x,y)$ continuous but using difference approximations in the "y" direction. Only horizontal mesh lines were therefore used and, as in case (3), these were positioned so that the end-point of the obstruction lay mid-way between the two lines given by $s = j$ and $s = (j+1)$ (Fig. 25). The method of solution used was similar to case (3) but the matrix difference equations (3.6.2) and (3.6.3) were written simply as

$$\left(\frac{d\underset{\sim}{\Phi}(x)}{dx} \right)_{x=0-} = \left(\frac{d\underset{\sim}{\Psi}(x)}{dx} \right)_{x=0+} = \underset{\sim}{H} ,$$

where $\underset{\sim}{\Phi}(x) = \{ \phi(x,0), \phi(x,h), \ldots\ldots, \phi(x,nh) \}$, $\underset{\sim}{\Psi}(x) = \{ \psi(x,0), \psi(x,h), \ldots, \psi(x,nh) \}$

Graphs, corresponding to those of case (1), are also shown on p. 75a for cases (2) and (3) using various net patterns. To distinguish the graphs we have annotated them by means of a number and a letter The number corresponds to the case (i.e. 1, 2, or 3) and the letter to the type of net pattern (S ≡ square, D ≡ diagonal, N ≡ nine point). For example, (3S) is the graph obtained in case (3) using a square net pattern. (It should be noted that in obtaining these graphs we assumed the obstruction to extend half-way across the duct. In case (2), therefore, only even values of "n" are possible while in cases (1) and (3) "n" must be odd.) Graphs for the semi-continuous method are indistinguishable, on the scale used, from those obtained in case (3) using a diagonal net i.e. curves (3D).

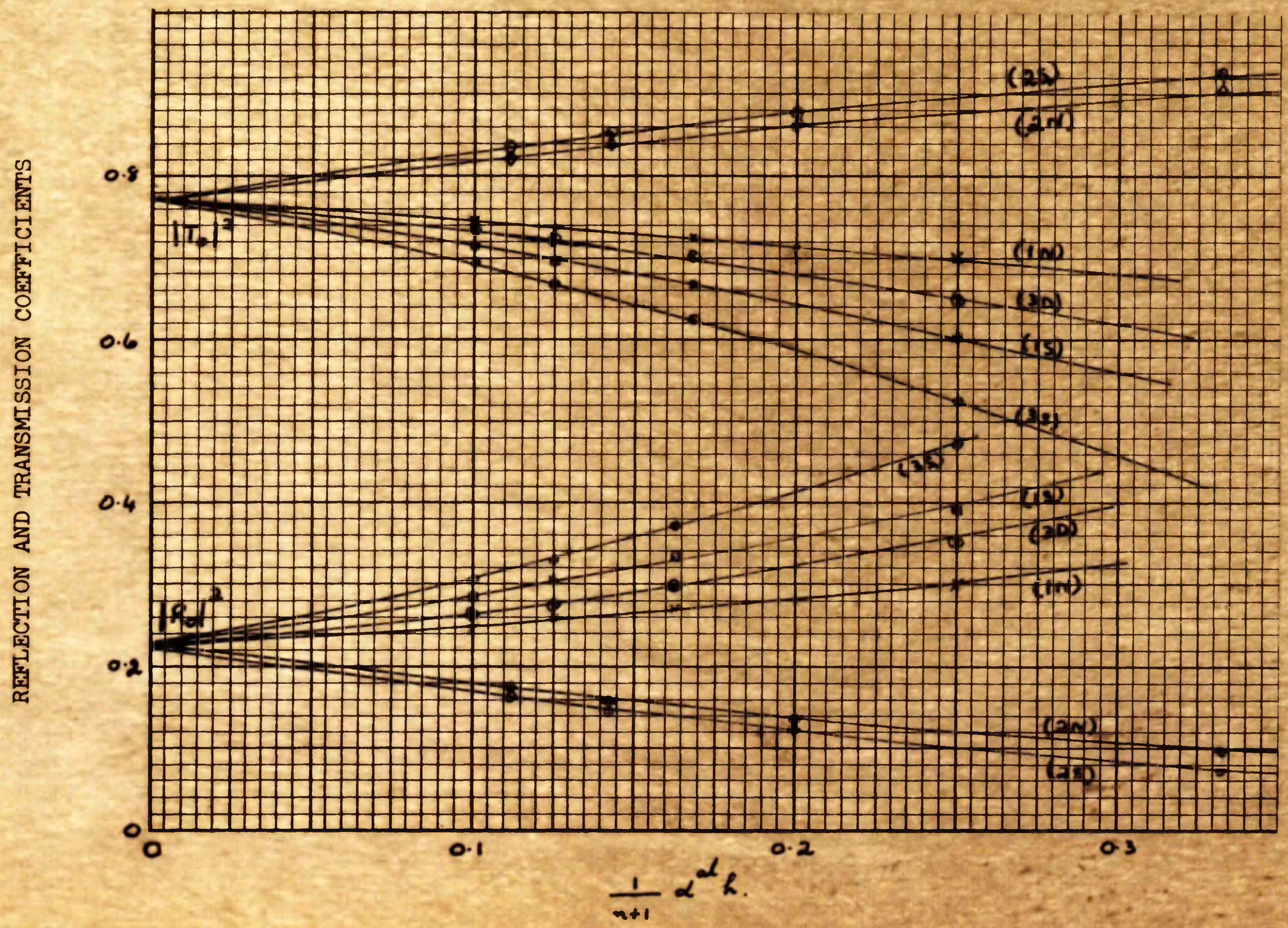
These graphs illustrate two main points:

(a) in each case convergence is near-linear, but the accuracy of the discrete approximations is poor.

(b) in some cases convergence is "from above" while in others it is "from below". In particular, convergence from above occurs when the net is chosen such that the end-point of the obstruction coincides with a mesh point. Graphs corresponding to diagonal and square net patterns do not converge from opposite directions although one might expect this from the error terms in the respective difference approximations to the partial differential equation.

It seems that no theory exists at the present moment to explain these results but, in case such a theory is produced by someone other than the author of this thesis, the figures from which these graphs are drawn are given in Appendix B.

GRAPHS OF REFLECTION AND TRANSMISSION COEFFICIENTS AGAINST NET SPACING



CHAPTER 4

AN ITERATIVE PROCEDURE USING THE DISCRETE TRANSFORM METHOD

4.1 Introduction

In the square box condenser problem of Chapter 2 and the steady-state wave problem of Chapter 3 it has already been pointed out that the accuracy of the results obtained by using standard finite difference equations is poor. Moreover the rate of convergence of discrete approximations to the exact values as the net spacing is reduced is slow. The reason for these facts is the occurrence of singularities due to the presence of sharp corners. It will be shown in Chapter 5 that, at least in the condenser problem, both the accuracy and the rate of convergence can be improved by using special difference equations at mesh points near the singularity. In order to provide a flexible tool for using these special equations in conjunction with the discrete transform method, we illustrate, in this chapter, an iterative procedure for solving the condenser problem when special difference equations are used at any mesh points.

The idea on which this iterative procedure is based is of general application. It may be used to solve the finite difference equations corresponding to any elliptic partial differential equation problem which can be solved by separation of variables when some or all of these difference equations are of special form. The method may also be used in conjunction with Fox's 'difference correction process' [14] to increase the accuracy of discrete approximations obtained using straightforward techniques.

4.2 The Iterative Procedure

Before considering the specific problem in § 4.3 we state the general method. Suppose that instead of the standard square net pattern (§ 1.1) for

Laplace's equation at the point (rh, sh) we have a more general equation:

$$4\phi_{r,s} - (\alpha_1\phi_{r+1,s} + \alpha_2\phi_{r,s+1} + \alpha_3\phi_{r-1,s} + \alpha_4\phi_{r,s-1}) - (\beta_1\phi_{r+1,s+1} + \beta_2\phi_{r-1,s+1} + \beta_3\phi_{r-1,s-1} + \beta_4\phi_{r+1,s-1}) - \ldots\ldots\ldots\ldots = 0, \quad (4.2.1)$$

where the α_i, β_i, ... are constants but their values may depend on the particular point under consideration. We shall write this general equation as

$$\mathcal{L}^*\phi_{r,s} = 0 \quad (4.2.2)$$

where $\mathcal{L}^*$ is a linear difference operator. Equation (4.2.1) may be written in the form:

$$4\phi_{r,s} - \phi_{r+1,s} - \phi_{r,s+1} - \phi_{r-1,s} - \phi_{r,s-1} = (\alpha_1 - 1)\phi_{r+1,s} + (\alpha_2 - 1)\phi_{r,s+1} + (\alpha_3 - 1)\phi_{r-1,s} + (\alpha_4 - 1)\phi_{r,s-1} + \beta_1\phi_{r+1,s+1} + \beta_2\phi_{r-1,s+1} + \beta_3\phi_{r-1,s-1} + \beta_4\phi_{r+1,s-1} + \ldots\ldots \quad (4.2.3)$$

$$\text{i.e.} \quad \mathcal{L}\phi_{r,s} = \mathcal{M}\phi_{r,s} \quad (4.2.4)$$

where $\mathcal{L}$, $\mathcal{M}$ are linear difference operators such that

$$\mathcal{L} - \mathcal{M} = \mathcal{L}^* \quad (4.2.5)$$

$\mathcal{L}$ is, in fact, the standard square net pattern linear operator for Laplace's equation.

Choosing an initial guess, $\phi_{r,s}^{(0)}$, for the function values at all mesh-points of the region we define an iterative procedure by

$$\mathcal{L}\phi_{r,s}^{(p+1)} = \mathcal{M}\phi_{r,s}^{(p)}, \quad p = 0,1,2,\ldots\ldots \quad (4.2.6)$$

where a superscript $^{(p)}$ denotes the pth iterate. Further, let

$$\delta_{r,s}^{(h+1)} = \phi_{r,s}^{(h+1)} - \phi_{r,s}^{(h)} . \tag{4.2.7}$$

On subtracting $\mathcal{L}\phi_{r,s}^{(h)}$ from each side of equation (4.2.6), we obtain

$$\mathcal{L}\,\delta_{r,s}^{(h+1)} = -(\mathcal{L}-\mathcal{M})\phi_{r,s}^{(h)} ,$$

$$= -\mathcal{L}^{*}\phi_{r,s}^{(h)} ,$$

i.e. $$\mathcal{L}\,\delta_{r,s}^{(h+1)} = -R_{r,s}^{(h)} \quad \text{say,} \tag{4.2.8}$$

where $R_{r,s}^{(h)}$ is the "residual" at the pth iteration, i.e. the non-zero quantity obtained when the approximate values of $\phi_{r,s}$ found at the pth iteration are substituted in the lefthand side of equation (4.2.2).

The final formulae for the iterative procedure are therefore:

$$\left.\begin{aligned} \phi_{r,s}^{(h+1)} &= \phi_{r,s}^{(h)} + \delta_{r,s}^{(h+1)} , && \text{(a)} \\ \text{where}\quad \mathcal{L}\,\delta_{r,s}^{(h+1)} &= -R_{r,s}^{(h)} . && \text{(b)} \end{aligned}\right\} \tag{4.2.9}$$

The basic principle involved in the iterative procedure (4.2.9) is that we compute accurately residuals at each mesh point of the region using the difference equation appropriate to that point but reduce or "relax" these residuals using the simple equation for the square net pattern. (This principle has previously been adopted by various users of relaxation methods of solving difference equations, e.g. Woods [13].) As in ordinary relaxation, the iterative process is continued until the residuals, $R_{r,s}^{(h)}$, at all mesh points of the region are less than ε, say, where ε is any small number previously specified.

In this procedure, residuals, $R_{r,s}^{(k)}$, at the pth stage of the iteration, are computed for all points in the region and then the corrections $\delta_{r,s}^{(k+1)}$ are computed for all points using equation (b)(4.2.9). The procedure is, therefore, related to the method of simultaneous displacements for solving boundary-value problems in finite differences (see, e.g. Forsythe and Wasow [4], p. 220). We shall now illustrate how this "simultaneous relaxation of residuals" may be carried out using the discrete transform method.

4.3 The Square Box Condenser Problem with Special Difference Equations at Any Mesh Points

The problem considered in § 2.3 consisted of the solution of Laplace's equation in the region between two concentric squares with corresponding sides parallel and such that the sides of the inner square were half the length of those of the outer. The inner square was assumed to have potential unity while the outer square was assumed to be earthed. It was shown that, on account of the symmetry of the problem, we needed to consider only one-eighth of the inter-square region (see Fig. 15, p. 47). We consider the same problem in this section except that, instead of using the ordinary square net pattern at each mesh point of a square lattice covering the region of the problem, we allow the use of <u>any</u> appropriate difference equation at <u>any</u> of the mesh points.*

* Simple examples of appropriate difference equations are the diagonal and nine point net pattern equations (§ 1.1). It was pointed out, in § 2.3, p. 50, that difficulties are encountered when we try to solve this problem as a single region problem with these difference patterns. We suggested that solutions could be mor[e] easily obtained by considering one-quarter of the inter-square region and regarding the problem as an adjacent region problem. However, solutions using diagonal or nine point net patterns can easily be obtained by the method of this section where the problem is regarded as a single-region problem. This has been carried out for the nine-point pattern. See § 4.4.

Since the dimensions of the region and the conditions on the boundaries are the same as in § 2.3, we position the covering lattice in the same manner. That is, the lattice is positioned (Fig. 26) to coincide with AC, HG, and AH and such that GC coincides with a node on each of the 'n' mesh rows.

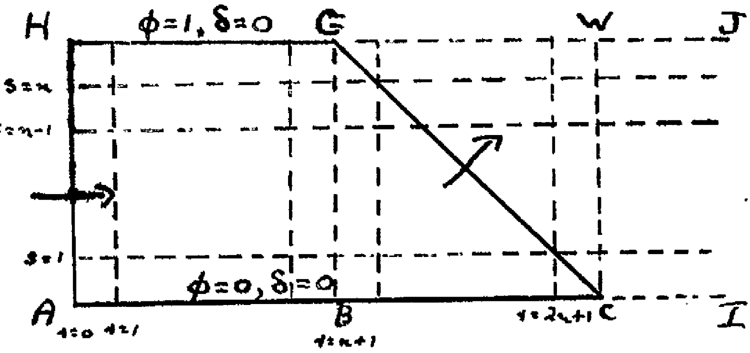


Fig. 26

The iterative procedure (4.2.9) is used in the following way. At any stage in the iteration values $\phi_{t,s}^{(h)}$ are known for the potentials at all mesh points. From these, residuals $R_{t,s}^{(h)}$ are computed at each point using the difference equation appropriate to that point. Then the corrections $\delta_{t,s}^{(h+1)}$ are computed from:

$$4\,\delta_{t,s}^{(h+1)} - \delta_{t+1,s}^{(h+1)} - \delta_{t,s+1}^{(h+1)} - \delta_{t-1,s}^{(h+1)} - \delta_{t,s-1}^{(h+1)} = -R_{t,s}^{(h)}\,, \qquad (4.3.1)$$

subject to the conditions $\delta = 0$ on AC and HG and Neumann conditions on AH and GC as indicated by the arrows in Fig. 26. The new potential values are then obtained from

$$\phi_{t,s}^{(h+1)} = \phi_{t,s}^{(h)} + \delta_{t,s}^{(h)} \qquad (4.3.2)$$

To start the process we guess or compute suitable initial values $\phi_{t,s}^{(0)}$.

Equations (4.3.1) are solved using the discrete transform method as follows. We first obtain values of the residuals at mesh points in the interior of the region GCW using the reflection condition on GC (Fig. 26). The boundary GC is then ignored for the present and, by transforming in the "s" direction and using the Neumann condition on AH, we obtain a solution in the semi-infinite rectangle $IAHJ$ involving 'n' arbitrary constants (we assume $\delta = 0$ on AI and HJ and zero residuals at mesh points with $r > 2n+1$). An '$n \times n$' set of simultaneous equations is then obtained for these arbitrary constants by using the condition of symmetry about GC. The remainder of this section is devoted to the technical details of this procedure.

The solution in the semi-infinite rectangle $IAHJ$ can be obtained by the method already discussed in § 1.3. In matrix form, equations (4.3.1) are:

$$\underset{\sim}{A}\, \underset{\sim}{\bar{\delta}}_{r}^{(k+1)} - \underset{\sim}{\bar{\delta}}_{r+1}^{(k+1)} - \underset{\sim}{\bar{\delta}}_{r-1}^{(k+1)} = -\underset{\sim}{R}_{r}^{(k)}, \quad r \geq 0, \tag{4.3.3}$$

where $\underset{\sim}{A}$ has been given on p. 6 and

$$\underset{\sim}{\bar{\delta}}_{r}^{(k+1)} = \left\{ \delta_{r,1}^{(k+1)}, \delta_{r,2}^{(k+1)}, \ldots\ldots, \delta_{r,n}^{(k+1)} \right\},$$

$$\underset{\sim}{R}_{r}^{(k)} = \left\{ R_{r,1}^{(k)}, R_{r,2}^{(k)}, \ldots\ldots, R_{r,n}^{(k)} \right\}.$$

Since the values of $\delta_{r,s}^{(k+1)}$ on $s = 0$ and $s = (n+1)$ are zero, as in § 1.3 we use eigenvalues λ_{ℓ} and eigenvectors $\underset{\sim}{S}_{\ell}$ such that

$$\underset{\sim}{S}_{\ell}^{T} \underset{\sim}{A} = (4 - 2\lambda_{\ell}) \underset{\sim}{S}_{\ell}^{T}.$$

On pre-multiplying equation (4.3.3) by $\underset{\sim}{S}_{\ell}^{T}$ we obtain:

$$F_{\ell,r+1}^{(k+1)} - (4 - 2\lambda_{\ell}) F_{\ell,r}^{(k+1)} + F_{\ell,r-1}^{(k+1)} = \beta_{\ell,r}^{(k)}, \tag{4.3.4}$$

where $F_{\ell,r}^{(k+1)} = \underset{\sim}{S}_{\ell}^{T} \underset{\sim}{\bar{\delta}}_{r}^{(k+1)}$ and $\beta_{\ell,r}^{(k)} = \underset{\sim}{S}_{\ell}^{T} \underset{\sim}{R}_{r}^{(k)}$. When the $F_{\ell,r}^{(k+1)}$ are determined, the $\underset{\sim}{\bar{\delta}}_{r}^{(k+1)}$ can be found from the inverse formula

$$\tilde{\bar{S}}_{+}^{(h+1)} = \sum_{\ell=1}^{n} F_{\ell,+}^{(h+1)} \frac{\tilde{S}_{\ell}}{\Delta_{\ell}^{2}} , \tag{4.3.5}$$

where $\Delta_\ell^2 = \tilde{S}_\ell^T \tilde{S}_\ell$ since, in practice (§ 1.10), $\tilde{S}_\ell$ is normalised such that $\tilde{S}_\ell(1) = 1$.

We next solve the equation (4.3.4) subject to the boundary condition

$$F_{\ell,-1}^{(h+1)} = F_{\ell,1}^{(h+1)} , \quad \ell = 1, 2, \cdots, n. \tag{4.3.6}$$

By using methods similar to those developed in Appendix A it can be shown that a particular solution of equation (4.3.4) satisfying the condition (4.3.6) is

$$\Gamma_{\ell,+}^{(h)} = \sum_{i=0}^{2n+1} K_{\ell,+}^{(i)} \beta_{\ell,i}^{(h)} , \tag{4.3.7}$$

where
$$K_{\ell,+}^{(i)} = \begin{cases} 0 , & |+| \leq i , \\ \dfrac{P_\ell^{|+|-i} - \varphi_\ell^{|+|-i}}{P_\ell - \varphi_\ell} , & |+| \geq i , \ i \neq 0 , \\ \dfrac{1}{2} \dfrac{P_\ell^{|+|-i} - \varphi_\ell^{|+|-i}}{P_\ell - \varphi_\ell} , & |+| \geq i , \ i = 0 , \end{cases} \tag{4.3.8}$$

P_ℓ , φ_ℓ being solutions of $\alpha^2 - (4 - 2\lambda_\ell)\alpha + 1 = 0$. On adding a complementary function satisfying the condition (4.36), the complete solution of (4.3.4) is found to be:

$$F_{\ell,+}^{(h+1)} = C_\ell^{(h)} V_\ell^{(+)} + \Gamma_{\ell,+}^{(h)} , \tag{4.3.9}$$

where $V_\ell^{(+)} = P_\ell^{+} + \varphi_\ell^{+}$ and $C_\ell^{(h)}$ is an arbitrary constant.

From equation (4.3.5) we have

$$\tilde{\bar{S}}_{+}^{(h+1)} = \sum_{\ell=1}^{n} \left\{ C_\ell^{(h)} V_\ell^{(+)} + \Gamma_{\ell,+}^{(h)} \right\} \frac{\tilde{S}_\ell}{\Delta_\ell^2} , \tag{4.3.10}$$

i.e. the correction for the function value at the point (rh, sh) at the $(h+1)$th iteration is

$$\delta_{r,s}^{(h+1)} = \sum_{\ell=1}^{n} C_\ell^{(h)} V_\ell^{(r)} \frac{S_\ell(s)}{\Delta_\ell^2} + D_{r,s}^{(h)}, \qquad (4.2.11)$$

where $$D_{r,s}^{(h)} = \sum_{\ell=1}^{n} \Gamma_{\ell,r}^{(h)} \frac{S_\ell(s)}{\Delta_\ell^2}. \qquad (4.3.12)$$

To evaluate the arbitrary constants $C_\ell^{(h)}$ we now use the difference equations at mesh points on GC (Fig. 26). These are:

$$4\,\delta_{n+1+k,n+1-k}^{(h+1)} - 2\,\delta_{n+k,n+1-k}^{(h+1)} - 2\,\delta_{n+1+k,n-k}^{(h+1)} = -R_{n+1+k,n+1-k}^{(h)}, \; k=1,2,\dots,n \qquad (4.3.13)$$

Equations (4.3.11) and (4.3.13) are analogous to equations (2.3.7) and (2.3.8) and following the procedure illustrated for these latter equations to obtain equations (2.3.9) in § 2.3, we obtain the analogous equations:

$$\sum_{\ell=1}^{n} u_{k\ell}\, C_\ell^{(h)} = \gamma_k^{(h)}, \quad k=1,2,\dots,n, \qquad (4.3.14)$$

where $u_{k\ell}$ are defined in the line immediately following equation (2.3.9) and

$$\gamma_k^{(h)} = -\frac{1}{2} R_{n+1+k,n+1-k}^{(h)} - \sum_{\ell=1}^{n} \left\{ 2\Gamma_{\ell,n+1+k}^{(h)} - \Gamma_{\ell,n+k}^{(h)} \right\} \frac{S_\ell(n+1-k)}{\Delta_\ell^2} + \sum_{\ell=1}^{n} \Gamma_{\ell,n+1+k}^{(h)} \frac{S_\ell(n-k)}{\Delta_\ell^2}. \qquad (4.3.15)$$

An important feature of the fitting equations (4.3.14) is that the coefficients $u_{k\ell}$ are independent of h, so that the matrix of coefficients need be inverted once and for all.

The correction at the point (rh, sh) is then obtained by substituting these solutions into equation (4.3.11) or, alternatively, into (4.3.10).

§ 4.4 Computation of Solutions for Square Box Condenser Problem using the Iterative Procedure

In order to describe a convenient method for carrying out the computations involved in the procedure of § 4.3 on an automatic computer, we express the formulae of § 4.3 in matrix form.

We first write the iterative formula (4.3.2) in matrix form by introducing the matrices $\underset{\sim}{\bar{\Phi}}$ and $\underset{\sim}{\bar{\delta}}$ given by

$$\underset{\sim}{\bar{\Phi}} = \left\{ \underset{\sim}{\bar{\Phi}}_0^T, \underset{\sim}{\bar{\Phi}}_1^T, \ldots\ldots, \underset{\sim}{\bar{\Phi}}_{2n+1}^T \right\},$$

$$\underset{\sim}{\bar{\delta}} = \left\{ \underset{\sim}{\bar{\delta}}_0^T, \underset{\sim}{\bar{\delta}}_1^T, \ldots\ldots, \underset{\sim}{\bar{\delta}}_{2n+1}^T \right\},$$

where $\underset{\sim}{\bar{\Phi}}_+ = \{ \phi_{+,1}, \phi_{+,2}, \ldots, \phi_{+,n} \}$, $\underset{\sim}{\bar{\delta}}_+ = \{ \delta_{+,1}, \delta_{+,2}, \ldots, \delta_{+,n} \}$. $\underset{\sim}{\bar{\Phi}}$ and $\underset{\sim}{\bar{\delta}}$ are, therefore, both of type $(2n+2) \times n$. The iterative formula (4.3.2) may then be written as:

$$\underset{\sim}{\bar{\Phi}}^{(h+1)} = \underset{\sim}{\bar{\Phi}}^{(h)} + \underset{\sim}{\bar{\delta}}^{(h+1)}. \qquad (4.4.1)$$

The matrix array $\underset{\sim}{\bar{\delta}}^{(h+1)}$ gives corrections, at the $(h+1)$th iteration, for all points of the rectangular region CAHW (Fig. 26) but, since we have assumed $\delta = 0$ on GJ, corrections at mesh points on the right of GC are neither correct nor required. (This may be shown by an argument similar to that given in connection with the direct method of solving the square box condenser problem using the ordinary five point square net pattern (pp. 49, 50).) Consequently function values given by (4.4.1) for the $(h+1)$th iterate on the right of GC are incorrect. No use is made of these however in computing residuals in this region since, as mentioned previously, these residuals are obtained by reflection across GC.

We shall now derive a matrix expression for $\bar{\underset{\sim}{\delta}}^{(h+1)}$. Define

$\underset{\sim}{Z}^{(h)} = \left[C_\ell^{(h)} V_\ell^{(r)} + \Gamma_{\ell,r}^{(h)} \right]$, where r is the row number and ℓ the column number, $0 \le r \le 2n+1$, $1 \le \ell \le n$,

and $\underset{\sim}{S} = \left\{ \frac{\underset{\sim}{S}_1^T}{\Delta_1^2}, \frac{\underset{\sim}{S}_2^T}{\Delta_2^2}, \ldots, \frac{\underset{\sim}{S}_n^T}{\Delta_n^2} \right\}$ of type $(n \times n)$,

then, by equation (4.3.10),

$$\bar{\underset{\sim}{\delta}}^{(h+1)} = \underset{\sim}{Z}^{(h)} \underset{\sim}{S} . \tag{4.4.2}$$

Further define $\underset{\sim}{V} = \left[V_\ell^{(r)} \right]$, where r is the row number, ℓ the column number,

$\underset{\sim}{C}^{(h)^D} = \text{diag} \left\{ C_1^{(h)}, C_2^{(h)}, \ldots, C_n^{(h)} \right\}$,

$\underset{\sim}{\Gamma}^{(h)} = \left[\Gamma_{\ell,r}^{(h)} \right]$, of type $(2n+2) \times n$, i.e. $0 \le r \le 2n+1$, $1 \le \ell \le n$,

then

$$\underset{\sim}{Z}^{(h)} = \underset{\sim}{V} \underset{\sim}{C}^{(h)^D} + \underset{\sim}{\Gamma}^{(h)} . \tag{4.4.3}$$

From equations (4.4.2) and (4.4.3),

$$\bar{\underset{\sim}{\delta}}^{(h+1)} = \left(\underset{\sim}{V} \underset{\sim}{C}^{(h)^D} + \underset{\sim}{\Gamma}^{(h)} \right) \underset{\sim}{S} . \tag{4.4.4}$$

Now if we define $\underset{\sim}{R}^{(h)} = \left[R_{r,s}^{(h)} \right]$, r the row number, s the column number i.e. of type $(2n+2) \times n$,

$\underset{\sim}{\Delta} = \text{diag} \left\{ \Delta_1^2, \Delta_2^2, \ldots, \Delta_n^2 \right\}$,

$\underset{\sim}{\beta}^{(h)} = \underset{\sim}{R}^{(h)} (\underset{\sim}{\Delta} \underset{\sim}{S})^T$,

$\underset{\sim}{K}_\ell = \left[K_{\ell,r}^{(i)} \right]$ where r is the row number, i the column number so that matrix $\underset{\sim}{K}_\ell$ is of type $(2n+2) \times (2n+2)$ (equation (4.3.8)), and since $1 \le \ell \le n$, there are 'n' matrices $\underset{\sim}{K}_\ell$.

and $\underset{\sim}{g}_{\ell}$ = an $(n\times n)$ matrix whose only non-zero element is (ℓ,ℓ) with value unity,

it can be shown that

$$\underset{\sim}{\Gamma}^{(h)} = \sum_{\ell=1}^{n} \underset{\sim}{K}_{\ell}\, \underset{\sim}{\beta}^{(h)} \underset{\sim}{g}_{\ell} \cdot \tag{4.4.5}$$

From equations (4.4.4) and (4.4.5),

$$\overline{\underset{\sim}{S}}^{(h+1)} = \Big(\underset{\sim}{V}\, \underset{\sim}{c}^{(h)^{D}} + \sum_{\ell=1}^{n} \underset{\sim}{K}_{\ell}\, \underset{\sim}{\beta}^{(h)} \underset{\sim}{g}_{\ell} \Big) \underset{\sim}{S}\, , \tag{4.4.6}$$

and this is the form used for computation.

The set of simultaneous equations (4.3.14) may be written in the form

$$\underset{\sim}{U}\, \underset{\sim}{c}^{(h)} = \underset{\sim}{\gamma}^{(h)}, \tag{4.4.7}$$

where $\underset{\sim}{U} = [u_{k\ell}]$ of type $(n\times n)$, $\underset{\sim}{c}^{(h)} = \{c_1^{(h)}, c_2^{(h)}, \ldots, c_n^{(h)}\}$ and $\underset{\sim}{\gamma}^{(h)} = \{\gamma_1^{(h)}, \gamma_2^{(h)}, \ldots \ldots \gamma_n^{(h)}\}$. Hence

$$\underset{\sim}{c}^{(h)} = \underset{\sim}{U}^{-1} \underset{\sim}{\gamma}^{(h)}. \tag{4.4.8}$$

This completes the matrix formulation of the method of § 4.3. We next discuss the computational procedure on the Deuce electronic computer (mark II version) using the two auto-code schemes, Alphacode and G.I.P., referred to in § 2.4.

The matrices $\underset{\sim}{S}'$, $\underset{\sim}{V}$, $\underset{\sim}{\Delta}$, $\underset{\sim}{K}_{\ell}$, $\ell = 1, 2, \ldots, n$, and $\underset{\sim}{U}^{-1}$ do not alter during the iterative procedure and are, therefore, computed once and for all prior to the start of the main iterative programme. The computation of the matrices $\underset{\sim}{S}$, $\underset{\sim}{V}$, and $\underset{\sim}{U}^{-1}$ has already been indicated in § 2.4 and in the flow diagram on p. 56. By minor modifications of the Alphacode part of the programme in § 2.4 we can also compute the matrices $\underset{\sim}{\Delta}$ and $\underset{\sim}{K}_{\ell}$, $\ell = 1, 2, \ldots\ldots, n$. The elements $\Delta_{\ell}^{2} = \underset{\sim}{s}_{\ell}^{T} \underset{\sim}{s}_{\ell}$ of $\underset{\sim}{\Delta}$ are required in the computation of the matrix $\underset{\sim}{S}$

so that we need only arrange to store these appropriately to obtain $\underset{\sim}{\Delta}$. Each matrix $\underset{\sim}{K}_\ell$ may be formed while computing the matrix $\underset{\sim}{V}$. This follows from the fact that elements of both $\underset{\sim}{K}_\ell$ and $\underset{\sim}{V}$, i.e. $K_{\ell,r}^{(i)}$ and $V_\ell^{(r)}$ respectively, are expressed in terms of the quantities P_ℓ and φ_ℓ (see equations (4.3.8) and (4.3.9)). In practice it is easier to compute the transpose of $\underset{\sim}{K}_\ell$ since we then require to compute only the first and second rows, all other rows being deduced from the second row. The reasons for this are: (a) in Alphacode, matrices are stored by row, (b) a facility exists for moving blocks of numbers held in consecutive stores, and (c), from equation (4.3.8), $\underset{\sim}{K}_\ell^T$ is of the form:

$$\underset{\sim}{K}_\ell^T = \begin{bmatrix} 0, & \frac{1}{2}\frac{P_\ell - \varphi_\ell}{P_\ell - \varphi_\ell}, & \frac{1}{2}\frac{P_\ell^2 - \varphi_\ell^2}{P_\ell - \varphi_\ell}, & \ldots, & \cdots\cdots & , \frac{1}{2}\frac{P_\ell^{2n+1} - \varphi_\ell^{2n+1}}{P_\ell - \varphi_\ell} \\ 0, & 0, & \frac{P_\ell - \varphi_\ell}{P_\ell - \varphi_\ell}, & \frac{P_\ell^2 - \varphi_\ell^2}{P_\ell - \varphi_\ell}, & \cdots\cdots & , \frac{P_\ell^{2n} - \varphi_\ell^{2n}}{P_\ell - \varphi_\ell} \\ 0, & 0, & 0, & \frac{P_\ell - \varphi_\ell}{P_\ell - \varphi_\ell}, & \cdots\cdots & , \frac{P_\ell^{2n-1} - \varphi_\ell^{2n-1}}{P_\ell - \varphi_\ell} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0, & \cdots\cdots & , 0 \end{bmatrix}$$

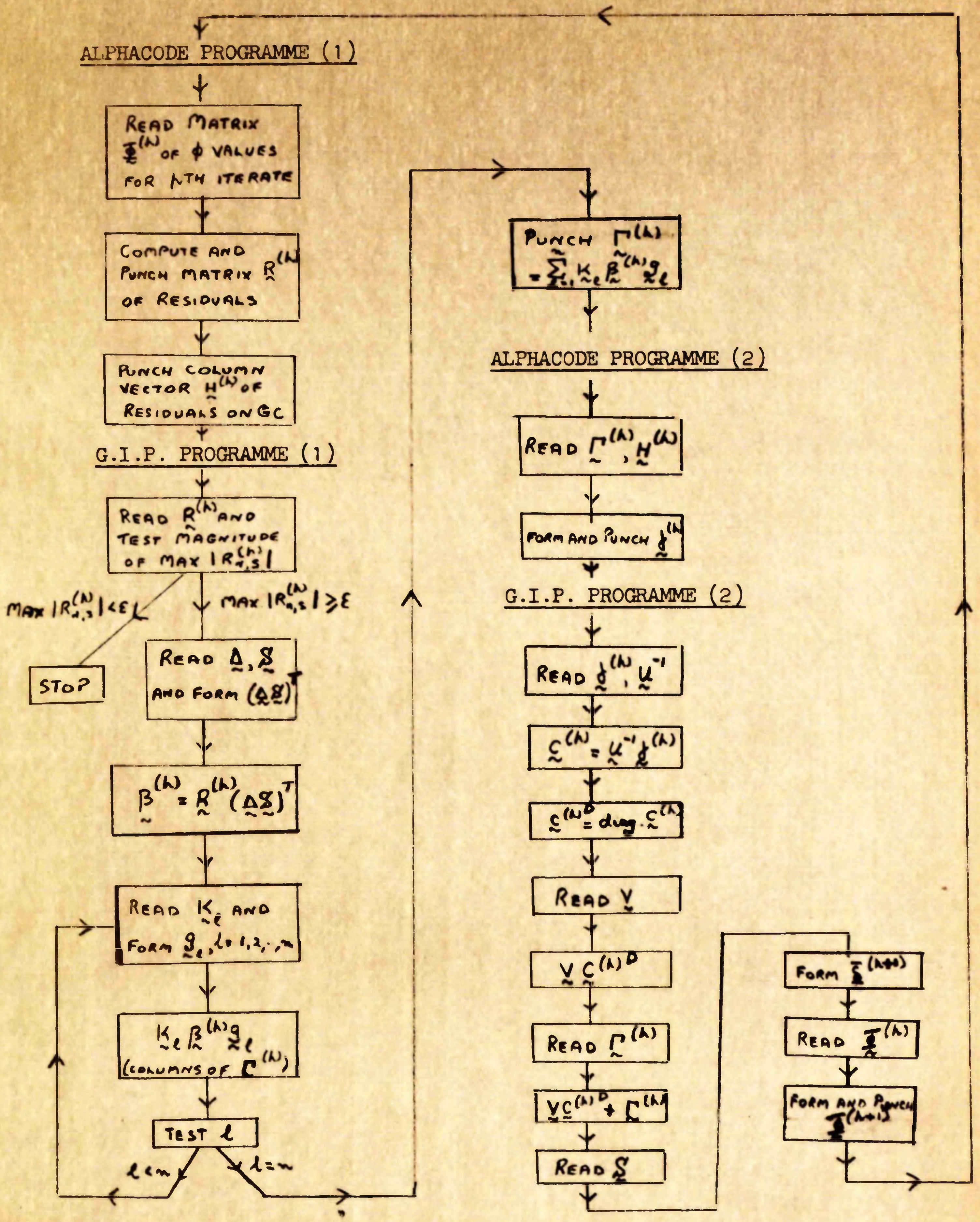
The iterative procedure involves the use of four auto-code programmes, two in Alpha-code and two in G.I.P. In Alphacode programme (1) the matrix of residuals $\underset{\sim}{R}^{(h)}$ is computed and punched out together with a column vector $\underset{\sim}{H}^{(h)}$ of residuals at mesh points on the boundary GC (Fig. 26), the latter being required for later computation of the matrix $\underset{\sim}{\gamma}^{(h)}$ used in the fitting equations. G.I.P. programme (1) is then used to test the numerical magnitude of the largest residual, to compute the matrix $\underset{\sim}{\beta}^{(h)}$, to form the matrices $\underset{\sim}{g}_\ell$, $\ell = 1, 2, \ldots, n$, and then to compute the matrix $\underset{\sim}{\Gamma}^{(h)}$ (equation (4.4.5)). Using the matrices $\underset{\sim}{H}^{(h)}$ and $\underset{\sim}{\Gamma}^{(h)}$ the matrix $\underset{\sim}{\gamma}^{(h)}$ is computed in Alphacode programme (2). Finally G.I.P. programme (2) is used (a) to compute the column vector $\underset{\sim}{C}^{(h)}$, (equation (4.4.8)), (b) to diagonalise $\underset{\sim}{C}^{(h)}$, (c) to compute the matrix $\underset{\sim}{\bar{S}}^{(h+1)}$ (equation (4.4.4)), and finally (d) to obtain the next iterate $\underset{\sim}{\bar{\Phi}}^{(h+1)}$ (equation (4.4.1)). Flow diagrams for these programmes, showing only the main steps are given on p. 89 .

Each of these programmes is arranged to work in fully-floating arithmetic, it being necessary to block-float only when performing matrix additions in G.I.P. (1) and G.I.P. (2). No loss in accuracy is caused by this, however, since elements of any one of the matrices involved are of the same order of magnitude.

A more efficient standard Deuce programme [22], written in basic machine language and working in fixed-point arithmetic, exists for solving problems of this nature. This standard programme is, however, able to deal with difference equations involving only potentials at the five points of the ordinary square net pattern (§ 1.1) and can only give results accurate to five decimal digits. The auto-code programmes discussed in this section can deal with <u>any</u> difference approximations to Laplace's equation involving potentials at any of the points of the nine point net pattern and has been used to obtain results accurate to seven of the nine decimal digits carried by the machine.

As an example of the use of the above iterative procedure results have been obtained for the condenser problem when the special difference equations were taken as the difference equations of the ordinary nine point net pattern. Two cases have been dealt with corresponding to values of n = 3, 5 where 'n' is the number of mesh lines between the boundaries AC and HG (Fig. 26) and results for the case n = 3 are shown in Fig. 27. In this problem, and also in the problems solved in Chapter 5, it has been found that the maximum residual is reduced by a factor of approximately 6 after each iteration.

FLOW DIAGRAMS FOR SQUARE BOX CONDENSER PROBLEM USING ITERATIVE METHOD (ITERATIVE PART ONLY)



SQUARE CONDENSER PROBLEM WITH ORDINARY NINE POINT NET PATTERN

CASE, n = 3. (RESULTS QUOTED TO FIVE DECIMAL PLACES)

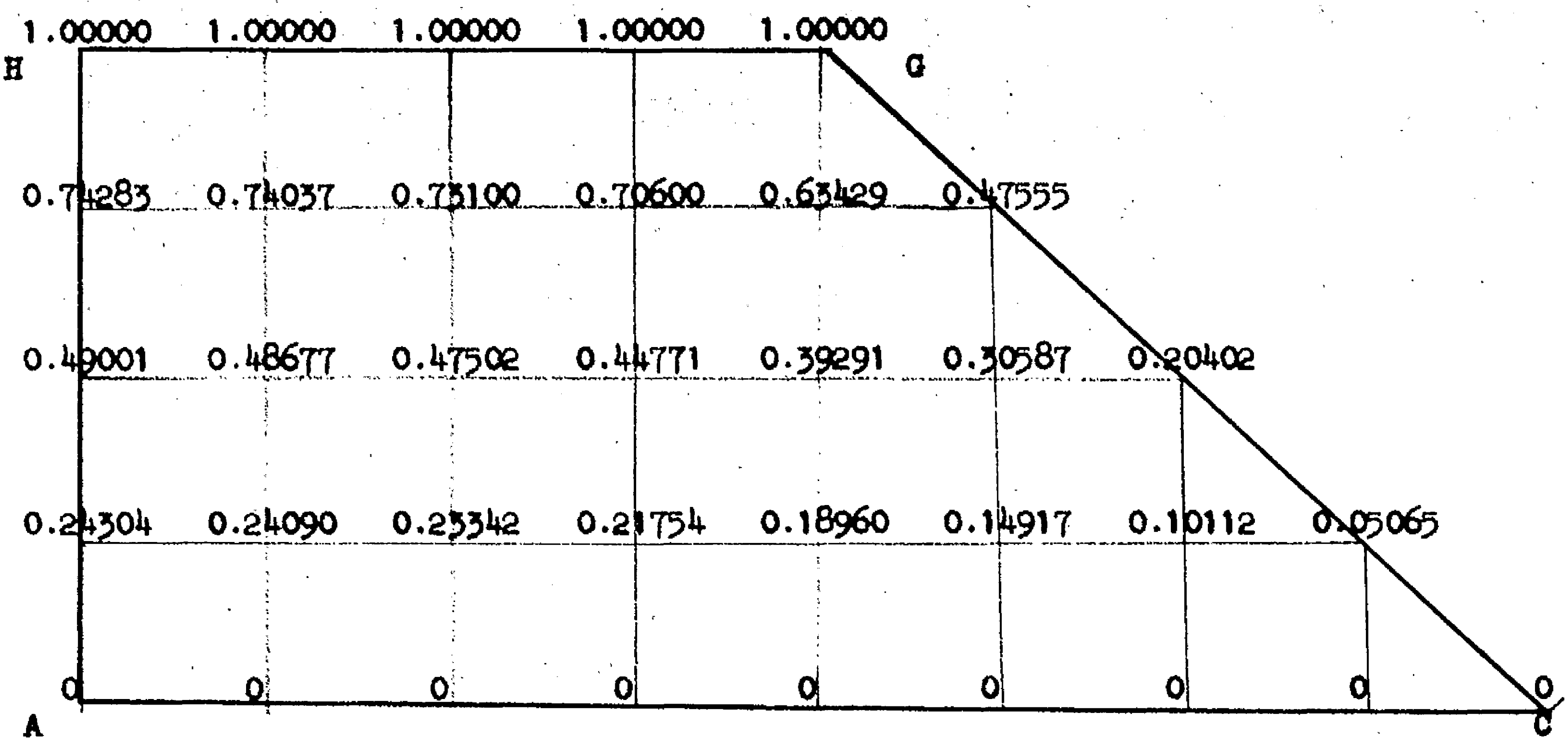


Fig. 27

CHAPTER 5

THE TREATMENT OF THE SINGULARITY IN THE SQUARE BOX CONDENSER PROBLEM

5.1 Introduction

Although many authors mention the fact that singular points in boundary-value problems require careful treatment, very few specific methods for doing this appear to have been published. In fact the only useful references which have been found are Southwell [3] who uses an 'advance to a finer net' technique, and Motz [11], Jeffreys and Jeffreys [12] and Woods [13] who each describe methods for generating special difference equations at mesh points in the vicinity of the singularity.

The objects of this chapter are two-fold:

(a) to describe methods for generating special difference equations which are variants of those of the above authors and to describe a new method in which ideas of Woods and Jeffreys and Jeffreys are combined,

(b) to present numerical results for the square box condenser problem when the singular point is treated by all the methods in (a); this enables a comparison to be made on an experimental basis, of the effectiveness of these methods in increasing the accuracy of discrete approximations to the true potential values. (The published numerical results are few in number and there seems to be practically no discussion of the improvement in accuracy obtainable by using these special methods.)

We confine ourselves to singularities of the kind which Woods has classified as Type I, i.e. function values near the singularity are finite but derivatives tend to infinity on approaching the singular point.

5.2 The Generation of Special Difference Equations

In this section we first describe the methods of Motz and Jeffreys and Jeffreys and indicate how these have been modified for our purposes. Woods's method is then introduced and this leads us into a description of the new method of generating special difference equations. We finally show how the finer net idea of Southwell may also be used to generate special difference equations at mesh points of the original, regular lattice. A singularity of Type I exists in the square box condenser problem of Chapter 2 and this is used for the purposes of illustration.

The reason for the inaccuracy and the slow convergence of the discrete approximations in the condenser problem is the rapid variation of the potential field near the sharp corner, i.e. the point G in Fig. 28. This means that the ordinary equations for a square lattice (§ 1.1) are not adequate difference representations of the partial differential equation in this region. In order to develop more accurate representations, it is convenient to take explicit account of the special behaviour of the field near the singularity. To do this we use the fact that it is possible to represent the field in the neighbourhood of G by a series about G of the form:

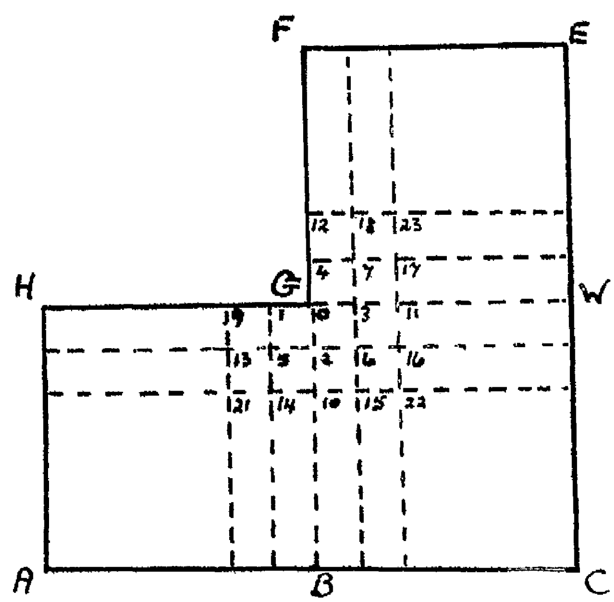


Fig. 28

$$\phi = A_0 + A_1 r^{2/3} \sin \tfrac{2}{3}\theta + A_2 r^2 \sin 2\theta + A_3 r^{10/3} \sin \tfrac{10}{3}\theta + \cdots \qquad (5.2.1)$$

where $\theta = 0$ corresponds to GH and the A_i are constants. This series is obtained by separating the variables, ignoring the boundary $ABCW$.

In the method of Motz [11] it is assumed that the field near the singularity is adequately represented by a series of the above type with a finite number of terms. If the finite series has 'n' terms then 'n' points some distance from the singularity are chosen and the 'n' arbitrary constants are expressed in terms of the potentials at these points. The finite series is then used to express the potentials at points in the neighbourhood of the singularity in terms of the potentials at the more distant points. For example, if $n = 3$, we may express A_0, A_1, and A_2 in terms of ϕ_{21}, ϕ_{10}, and ϕ_{22} (Fig. 28) and the finite series may then be written:

$$\phi = \gamma_1(r,\theta)\,\phi_{21} + \gamma_2(r,\theta)\,\phi_{10} + \gamma_3(r,\theta)\,\phi_{22}. \qquad (5.2.2)$$

By substituting appropriate values of 'r' and 'θ' for the point '5', say, in equation (5.2.2) we obtain the special difference equation:

$$4\phi_5 = \alpha_1\phi_{21} + \alpha_2\phi_{10} + \alpha_3\phi_{22}, \qquad (5.2.3)$$

where the α_i are numerical constants. Special difference equations for the points '2' and '6' may be obtained in similar fashion.

We have modified Motz's method slightly by expressing the potential at any point near the singularity in terms of potentials at immediately neighbouring points. For example, we write:

$$4\phi_5 = \beta_1\phi_{13} + \beta_2\phi_{14} + \beta_3\phi_2 + \beta_4\phi_1, \qquad (5.2.4)$$

or, to obtain a more accurate formula,

$$4\phi_5 = \gamma_1\phi_{13} + \gamma_2\phi_{21} + \gamma_3\phi_{14} + \gamma_4\phi_{10} + \gamma_5\phi_2 + \gamma_6\phi_0 + \gamma_7\phi_1 + \gamma_8\phi_9 . \qquad (5.2.5)$$

The coefficients β_i and γ_i are determined by a method precisely similar to that described above for the original Motz method. Numerical values of these coefficients are given in Appendix D (i) for the special equations at the points '5', '2' and '6'. (In this appendix, equations like (5.2.4) are referred to as five point special equations and those like (5.2.5) as nine point special equations.)

In both the original Motz method and its adaptation, the number of undetermined coefficients A_i in the finite series:

$$\phi = A_0 + A_1 r^{2/3}\sin\tfrac{2}{3}\theta + A_2 r^2\sin 2\theta + \cdots\cdots + A_n r^{\frac{4n-2}{3}}\sin\left(\tfrac{4n-2}{3}\right)\theta$$

is exactly equal to the number of potentials in terms of which the A_i are expressed. In the method of Jeffreys and Jeffreys [12], however, the number of A_i is less than the number of potentials and the method of least squares is used to determine the optimum values of the A_i. This is carried out as follows. The first few terms of the series are taken, say

$$\phi = A_0 + A_1 r^{2/3}\sin\tfrac{2}{3}\theta + A_2 r^2\sin 2\theta , \qquad (5.2.6)$$

and the residual, R_j, at the point 'j' is defined by:

$$R_j = \phi_j - A_0 - A_1 r_j^{2/3}\sin\tfrac{2}{3}\theta_j - A_2 r_j^2\sin 2\theta_j .$$

The sum of the squares of these residuals at the points with $j =$ 13, 14, 10, 15, 16, 11, 17, and 18 (Fig. 28) are then minimized in the usual way and the optimum values of A_0, A_1, and A_2 expressed in terms of the potentials at these points. The special difference equations at the points '5', '2' and '6' are obtained by substituting these optimum expressions into equation (5.2.6) applied to ϕ_5, ϕ_2 and ϕ_6 in turn.

We have also modified Jeffreys and Jeffreys method in order to obtain special difference equations expressing the potential at each point in the vicinity of the singularity in terms of the potentials at points in its immediate neighbourhood. For example, to obtain the five point special difference equation at the point '5' (Fig. 28) we minimize the sum $S_5^2 = \sum_j R_j^2$ where $j =$ 5, 13, 14, 2, and 1 or, for the more accurate nine point equation, $j =$ 5, 13, 21, 14, 10, 2, 0, 1, and 9. Substitution of the optimum expressions obtained for A_0, A_1 and A_2 into equation (5.2.6) applied to the point '5' then gives the special difference equation:

$$4\phi_5 = \beta_1\phi_{13} + \beta_2\phi_{14} + \beta_3\phi_2 + \beta_4\phi_1$$

in the first case, and in the more accurate case, the equation

$$4\phi_5 = \gamma_1\phi_{13} + \gamma_2\phi_{21} + \gamma_3\phi_{14} + \gamma_4\phi_{10} + \gamma_5\phi_2 + \gamma_6\phi_0 + \gamma_7\phi_1 + \gamma_8\phi_9 .$$

Special difference equations at the points '2' and '6' are derived in similar fashion. Numerical values of the coefficients occurring in both the five point and nine point equations are given for each of the points '5', '2' and '6' in Appendix D(i).

Before proceeding further we point out that in the Motz and Jeffreys and Jeffreys methods, and in their adaptations, special difference equations are obtained in which all coefficients have definite fixed values. These special equations may therefore be used in both iterative and direct methods of solving the problem. The method of Woods [13], as he describes it, may only be used iteratively since the special difference equations change at each stage of the iteration. We shall now illustrate one of Woods's methods for generating special difference equations in the neighbourhood of the singularity and shall describe it, as Woods has done, for use in conjunction with the relaxation method of solving difference equations.

Woods uses only the term in $r^{2/3} \sin 2/3\,\theta$ of the series expansion (5.2.1) and writes the potential function ϕ as

$$\phi = \alpha r^{2/3} \sin 2/3\,\theta + \eta \; , \tag{5.2.7}$$

where α is a constant and η is a harmonic function which is assumed to be well behaved in that it can be represented by one of the standard square net formulae (§ 1.1) even at mesh points in the vicinity of the singularity. The solution ignoring the singularity, i.e. η, is assumed known and, at the pth stage of the relaxation process, residuals, $R_j^{(p)}$, at each point ' j ' of the region are computed from the special difference equations

$$\mathcal{L}\phi_j^{(p)} - \alpha^{(p)} \mathcal{L} r_j^{2/3} \sin \tfrac{2}{3}\theta_j = R_j^{(p)} \; , \tag{5.2.8}$$

where $\mathcal{L}$ is the Laplace difference operator for the ordinary five point square net pattern. Equation (5.2.8) follows from equation (5.2.7) since, by our initial assumption, $\mathcal{L}\eta_j = 0$. It can be shown that $\mathcal{L} r_j^{2/3} \sin 2/3\,\theta_j$ is negligible for points not in the immediate neighbourhood of the singularity so that, as in the Motz and Jeffreys and Jeffreys methods, special difference equations are only used at the points '5', '2', and '6' in Fig. 28. These equations change, however, since, as indicated in equation (5.2.8), the value of the "constant" α depends on the stage of the relaxation process. ($\alpha^{(p)}$ is evaluated using a difference equation centred on the singular point G (Fig. 28) and the reader is referred to Woods's original paper for a description of how this is done.)

We now describe a new method of generating special difference equations at mesh points in the vicinity of the singularity. This method has been developed for various reasons including the fact that, in three-dimensional problems involving sharp corners, it is possible to establish the nature of the singularity at a corner but it is difficult to obtain more than the first term of the systematic series expansion about the singular point. Woods's idea of using only the dominant term in the series expansion has therefore been incorporated in this new method but the iteration inherent in his method has been avoided by using Jeffreys and Jeffreys least squares technique. The method is consequently referred to as the Least-Squares-Dominant-Term method or the L.S.D.T. method for brevity.

We write

$$\phi = Ar^{2/3}\sin\tfrac{2}{3}\theta + a + bx + cy + d(x^2-y^2) + exy, \qquad (5.2.9)$$

where A, a, b, c, d, and e are arbitrary constants. In this equation the first term is the dominant term of the series expansion about the corner point while the remainder is a Taylor series expansion about the point at which the special equation is to be computed. (If the special equation is to be computed at the point '5' (Fig. 28), then for the point '2', for example, the expression on the right of equation (5.2.9) is $Ar_2^{2/3}\sin\tfrac{2}{3}\theta_2 + a + bh + dh^2$ where h is the net spacing.) We now define the residual R_j at the point 'j' by:

$$R_j = \phi_j - Ar_j^{2/3}\sin\tfrac{2}{3}\theta_j - a - bx_j - cy_j - d(x_j^2 - y_j^2) - ex_jy_j . \qquad (5.2.10)$$

Suppose we wish to compute the special difference equation at the point '5', (Fig. 28). We write

$$S_5^2 = \sum_j \omega_j R_j^2 \tag{5.2.11}$$

where ω_j is a weighting factor to be discussed later and the summation is over the points with j = 5, 13, 21, 14, 10, 2, 0, 1, and 9. A set of six simultaneous linear equations in the six arbitrary constants A, a, b, c, d, and e is then obtained by minimizing S_5^2 with respect to each of these constants in turn. Solution of these equations enables us to express the arbitrary constants in terms of the function values ϕ_5, ϕ_{13}, ϕ_{21}, ϕ_{14}, ϕ_{10}, ϕ_2, ϕ_0, ϕ_1, and ϕ_9. (The solution of the equations is easy in practice since only one equation involves all six unknowns. Each of the others involves only A and one other unknown.) Substitution of these optimum expressions in equation (5.2.9) applied to the point '5' then gives the required special equation which is of the form:

$$4\phi_5 = \gamma_1\phi_{13} + \gamma_2\phi_{21} + \gamma_3\phi_{14} + \gamma_4\phi_{10} + \gamma_5\phi_2 + \gamma_6\phi_0 + \gamma_7\phi_1 + \gamma_8\phi_9 . \tag{5.2.12}$$

Only two sets of weighting factors ω_j have been used. For the point '5', for example, there are:

(a) $\omega_j \equiv 1$, and

(b) $\omega_5 = 20$, $\omega_{13} = \omega_{14} = \omega_2 = \omega_1 = 4$, $\omega_{21} = \omega_{10} = \omega_0 = \omega_9 = 1$,

i.e. the weighting factors of the ordinary nine point net pattern.

Special difference equations for the points numbered '2' and '6' (Fig. 28) can be obtained in exactly the same manner as that illustrated above for the point '5'. Numerical values for the coefficients in the special difference equations at the points '5', '2', and '6', using both sets of weighting factors, are given in Appendix D(i).

It should be noted that special difference equations involving only potential values at five mesh points cannot be obtained by this method. This is due to the fact that in the least squares process we require to minimize with respect to six unknowns so that, on substitution in equation (5.2.9), trivial expressions (e.g. $\phi_5 = \phi_5$) will be obtained if potential values at less than seven mesh points are involved.

So far we have described methods of generating special difference equations in which explicit account of the behaviour of the field near the singularity has been taken. Southwell's method of 'advance to a finer net' takes no such explicit account of this behaviour but introduces a net of smaller mesh size in the region where the field is changing rapidly (Fig. 29). By so doing the error in using the ordinary square net formulae as approximations to the partial differential equation is reduced. The idea of a refined mesh over part of the region of the problem cannot be used directly in conjunction with the discrete transform method for which a regular net is essential. Nevertheless it is possible to develop suitable

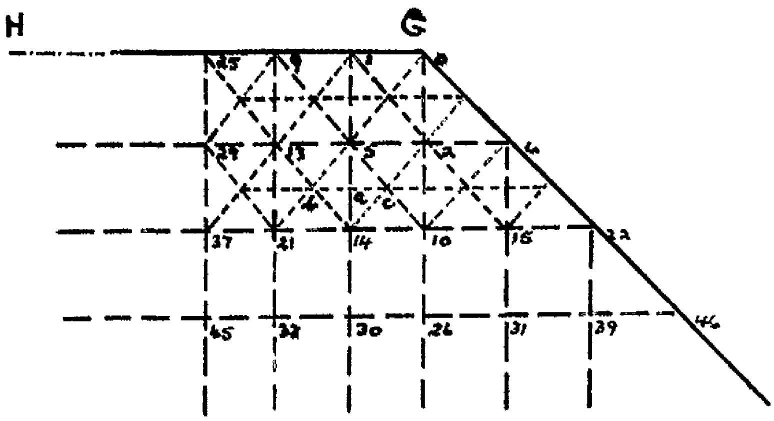


Fig. 29

special difference equations at mesh points of the original, regular net by combining difference equations for the refined net.

For example, consider the mesh point '14' in Fig. 29. The difference equations at the points 'a', 'b', 'c' and '14' are:

$$\phi_a = \frac{1}{4}(\phi_5 + \phi_6 + \phi_{14} + \phi_c) \;, \quad \text{(ordinary square net pattern)},$$

$$\phi_b = \frac{1}{4}(\phi_{13} + \phi_{21} + \phi_{14} + \phi_5) \;, \quad \text{(ordinary diagonal net pattern)},$$

$$\phi_c = \frac{1}{4}(\phi_5 + \phi_{14} + \phi_{10} + \phi_2) \;, \quad \text{(ordinary diagonal net pattern)},$$

and
$$\phi_{14} = \frac{1}{6}\phi_{21} + \frac{2}{9}\phi_{30} + \frac{1}{6}\phi_{10} + \frac{4}{9}\phi_a .$$

This last equation is a "non-uniform mesh" equation obtained in the usual way by Taylor series expansions about the point '14'. The special difference equation at the point '14' involving only potential values at points of the original, regular net is then obtained by eliminating the three values ϕ_a, ϕ_b, and ϕ_c in the above four equations. It is found that this procedure gives

$$4\phi_{14} = 0.93333\phi_{21} + 1.06667\phi_{30} + 0.93333\phi_{10} + 0.13333\phi_2 + 0.80000\phi_5 + 0.13333\phi_{13}.$$

Special difference equations have been obtained in a similar manner for the points 21, 10, 15, 22, 6, 2, 5, and 13 (Fig. 29) and these are given in Appendix D(ii).

[Special difference equations were computed for nine mesh points since we have only used a slight refinement of the net in the vicinity of the singularity. To be exact, we have merely halved the net spacing. As a result of this refinement the equations at the points '6', '2', '5' and '13' turn out to be those given by the ordinary nine point net pattern. It was nevertheless thought worth-while to use these equations for the purposes of experiment.]

5.3 Numerical Procedure and Results

Various results have been obtained for the square box condenser problem using the special difference equations at mesh points in the neighbourhood of the singularity developed in the last section. The computations have been performed on the Deuce computer at Glasgow University using the iterative procedure described in Chapter 4.

As we recall (p. 78), in this procedure we compute accurately the residuals at each mesh point using the difference equation appropriate to that point, and then relax these residuals using the simple square net pattern equations. It is not a source of difficulty if the difference equations change from one iterate to the next since these are only used in the computation of the residuals. Woods's method, in which varying special difference equations arise, therefore required only a slightly different Alphacode programme to compute the matrix of residuals (see § 4.4 and the flow diagram on p. 89), the constant $\alpha^{(h)}$ being computed using the method described in Woods's original paper.

For the adapted Motz method, the adapted Southwell method, and the Woods method, results have been obtained only for cases in which the ordinary five point square net pattern equations (§ 1.1) were used at mesh points other than those at which special patterns were used. (Such points will be referred to as "ordinary" points while points at which special difference equations are used will be referred to as "special" points.) Using the adapted Jeffreys and Jeffreys method and the Least-Squares-Dominant-Term method, results have been obtained for cases in which the ordinary five point net pattern equations were used at ordinary points and also for cases in which the ordinary nine point net pattern equations were used at these points. Furthermore, in most cases results have been obtained for values of n = 3,5,7, and 9, where 'n' is the number of mesh lines between the side of the outer square (AC (Fig.28)) and the corresponding side of the inner (EG) and parallel to them.

To illustrate the effectiveness of these methods in increasing the accuracy and the rate of convergence of the discrete approximations, two sets of graphs have been drawn.

The first set (set I, p.104) shows graphs of the potential values ϕ at the mid-point of GC (Fig.28, p.92) plotted against $\frac{1}{(n+1)}$, which is proportional to the net spacing. These graphs are numbered from (1) to (5) and correspond to the use of the following difference equations:

(1) normal five point square net pattern at <u>all</u> mesh points, i.e. no provision made for the special behaviour of the field near the singularity,

(2) normal nine point net pattern equations at <u>all</u> mesh points,

(3) normal five point square net pattern equations at ordinary points, special equations obtained by the adapted Southwell method at points numbered 21, 14, 10, 15, 22, 6, 2, 5, and 13 in Fig.29, p.99,

(4) normal <u>nine</u> point pattern equations at ordinary points, special equations obtained by the L.S.D.T. method with weighting factors of normal nine point pattern, (i.e.(b), p.98), at points numbered 5, 2, and 6 in Fig.29.

The curve numbered (5) is a representative curve corresponding to the use of the normal five point square net pattern equations at ordinary points and special equations obtained by any of the series expansion methods at points numbered 5, 2, and 6. (It is practically impossible to distinguish the curves for the various series expansion methods on the scale used for this set of graphs).

The main features illustrated by the graphs of Set I are as follows.

(i) If no account is taken of the special behaviour of the field near the singularity no significant improvement in either the accuracy or the rate of convergence of the discrete approximations to the exact value is obtained if the nine point net pattern equation is used at each point instead of the five point square net pattern equation, (graphs (1) and (2)).

[Attempts have been made to calculate the exact value by transforming the trapezoidal region ABCGH (Fig.28) into a rectangular region as outlined in Chapter X of Bowman's "Elliptic Functions" [25]. Unfortunately success has not yet been achieved. Graphical estimates, however, give a value of 0.1975±0.0001.]

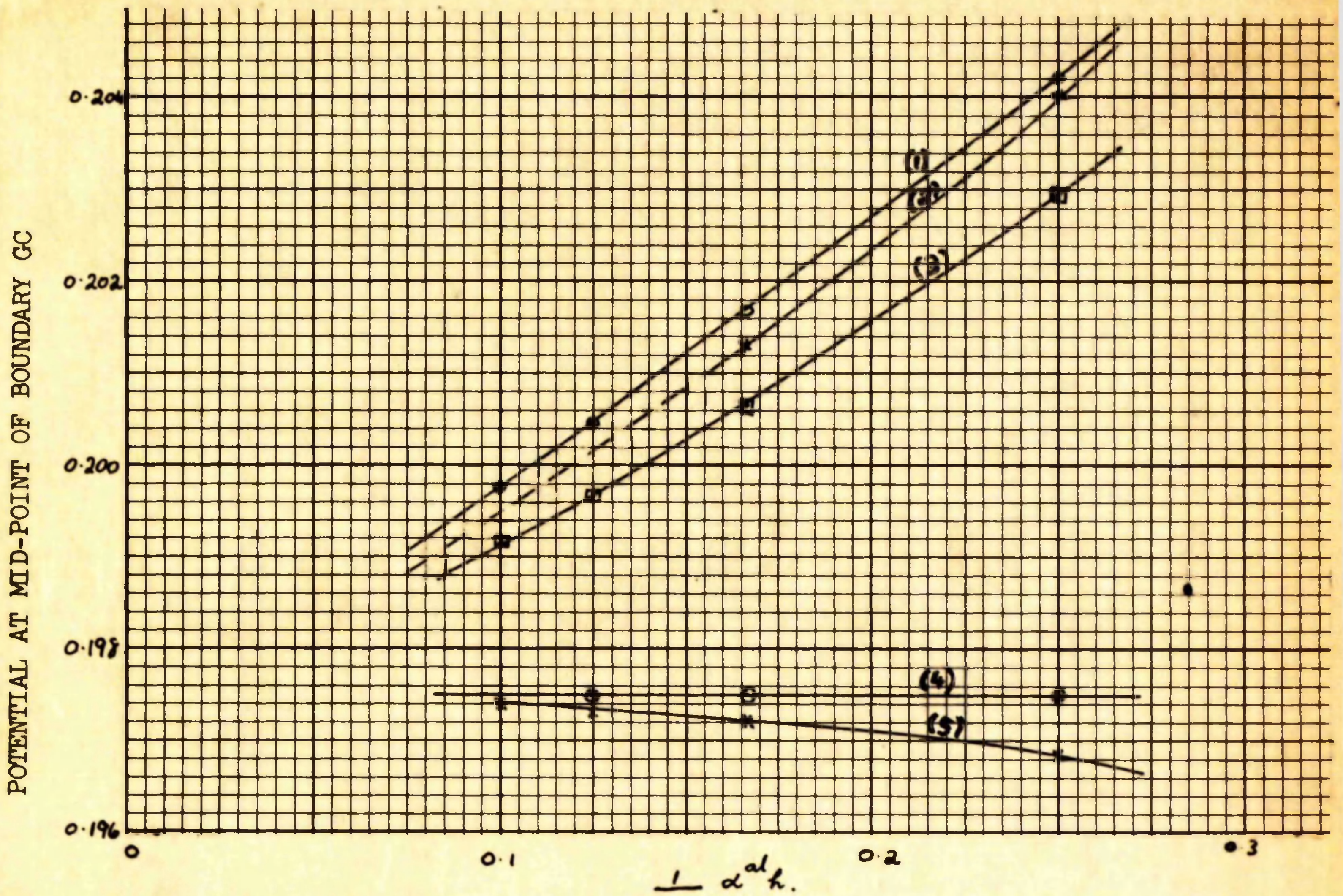
(ii) Graph (3) shows that the adapted Southwell method is not very effective if the net in the vicinity of the singularity is refined by merely halving the mesh size. To obtain the accuracy of the series expansion methods (graphs (4) and (5)) we would require to use a much greater refinement. This has not been attempted since the amount of labour involved would be greater than that required for the series expansion methods.

(iii) If the singularity is treated by <u>any</u> of the series expansion methods, the absolute accuracy is much improved, (graphs (4) and (5)).

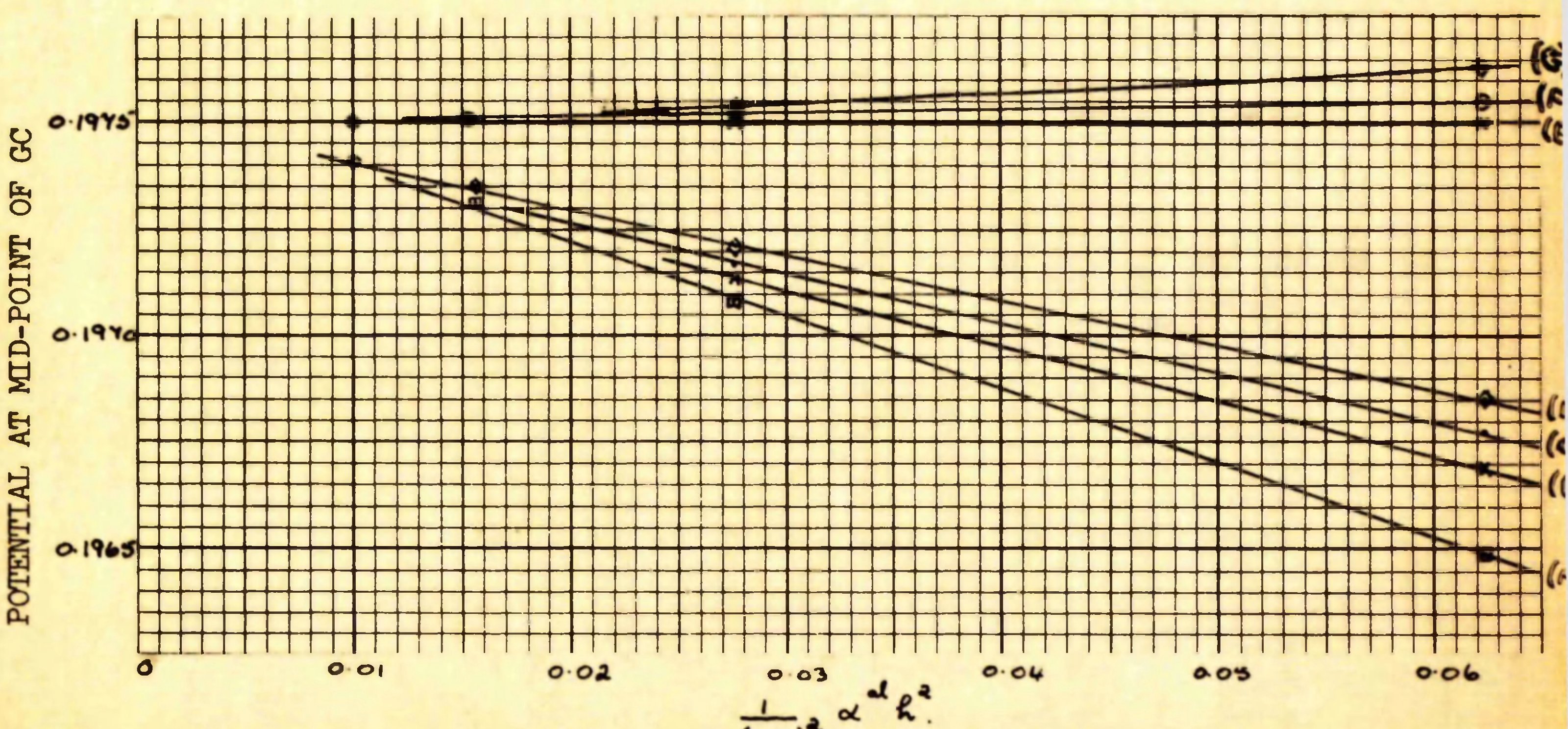
The second set of graphs (Set II, p.104) corresponds to the use of the various series expansion methods of treating the singularity so that in all cases <u>only three special points</u> are involved, namely the points numbered 5, 2, and 6 in Fig.28. These graphs show plots of the potential values ϕ at the mid-point of GC against $\frac{1}{(n+1)^2}$ which is proportional to ℓ^2. The scale on the ϕ- axis is four

GRAPHS FOR CONDENSER PROBLEM WITH SPECIAL EQUATIONS

SET I



SET II



times as fine as that on the ϕ axis in set I and, for reasons of clarity, not all plotted points are shown.

The lettering of the graphs corresponds to the use of the difference equations given in the following table where:

'S' indicates that the normal five point square net pattern equations were used at ordinary points,

'N' indicates that the normal nine point net pattern equations were used at ordinary points,

'm' is the number of function values in the computation of the special difference equations, and

'(W)' indicates that the weighting factors of the normal nine point net pattern were used with the L.S.D.T. method.

Graph	Net Pattern at Ord. Points	Method of Computing Special Equations	m
A	S	Adapted Motz.	5
B	S	{Woods / Adapted Jeffreys and Jeffreys.	5
C	S	{Adapted Motz. / L.S.D.T.(W).	9
D	S	Adapted Jeffreys and Jeffreys.	9
E	N	L.S.D.T.(W).	9
F	N	Adapted Jeffreys and Jeffreys.	9
G	N	L.S.D.T.	9

The main features illustrated by the graphs of Set II are as follows.

(i) The rate of convergence of the discrete approximations to the exact value as the net spacing is reduced is of the order of h^2 (c.f. the order of h if no special equations are used).

(ii) If five point square net pattern equations are used at ordinary points, there is only a small improvement in the accuracy of the discrete approximations when nine point special difference equations are used instead of five point equations (e.g. graphs B and D). The accuracy is much improved, however, if nine point net pattern equations are used at ordinary points, (e.g. graphs C and E). We re-call from Set I that this is not the case when no special equations are used.

(iii) If the same type of net patterns are used (see (ii)), no one series expansion method of treating the singularity is significantly more effective than the others in increasing the accuracy. In particular, the L.S.D.T. method is as effective as any of the methods proposed by previous authors.

No theory seems to exist which will explain these results but, in case such a theory should be produced, the figures from which the graphs of both Sets I and II are drawn are given in Appendix C.

In Fig. 30,p.107, results are recorded for the case n = 3 when nine point net pattern equations are used at ordinary points and the special equations are computed using the L.S.D.T.(W) method. These results are in almost exact agreement with those given on p. 175 of Woods's paper [13]. (Woods, however, only gives results to four decimal places).

SQUARE CONDENSER PROBLEM WITH (1) SPECIAL EQUATIONS COMPUTED BY L.S.D.T.(W) METHOD, (2) NINE POINT NET PATTERN EQUATIONS AT ORDINARY POINTS

CASE n = 3 (RESULTS QUOTED TO FIVE DECIMAL PLACES)

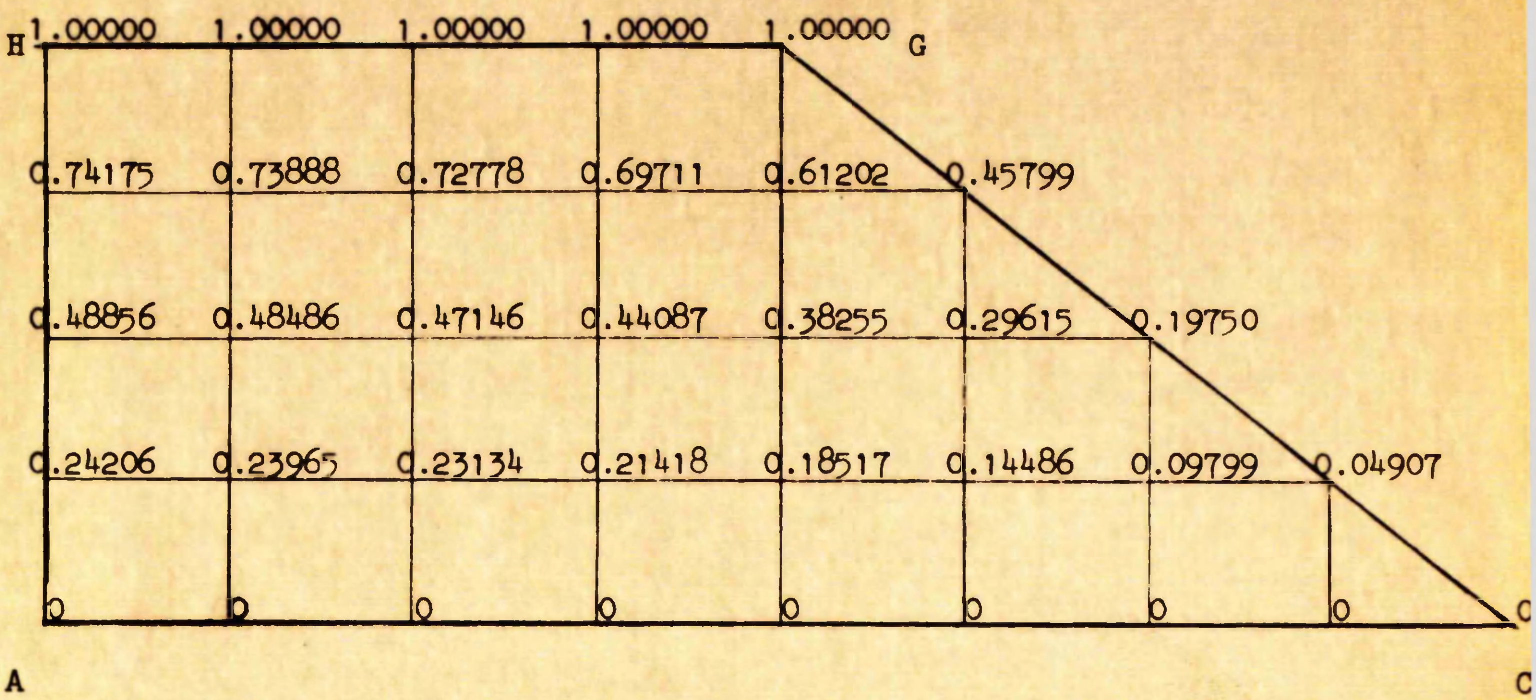


FIG. 30

5.4 A Direct Method of Solving the Square Box Condenser Problem when Special Difference Equations are used at a few mesh points.

In Chapter 4 we have shown how the discrete transform method can be used in an iterative procedure to solve the square box condenser problem when special difference equations are used at any or all of the mesh points. In this section we show how the method may also be used to obtain the solution directly when special equations are used at only a few mesh points. (We re-call (§§ 5.2, 5.3) that in dealing with the singularity only a few special equations were used).

The lattice is positioned in the same way as before (Fig.31) so that there are 'n' mesh points on the boundary GC. For simplicity we shall assume that a special difference equation is used at only one mesh point, say (hk, qk), which does not lie on GC. (Special equations at mesh points on GC are considered later). At all other mesh points the normal five point square net pattern equations are used.

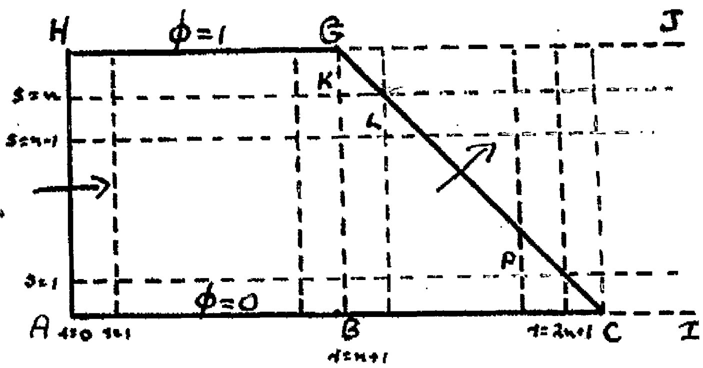


Fig. 31

We first arrange the special equation in a form convenient for use with the discrete transform method. As in § 4.2 we represent this equation in operator form by

$$\mathcal{L}^*\phi_{h,q} = 0 \tag{5.4.1}$$

where $\mathcal{L}^*$ is the appropriate linear difference operator. Equation (5.4.1) may be written as

$$\mathcal{L}\phi_{h,q} = m\phi_{h,q} \tag{5.4.2}$$

where $\mathcal{L}$ is the standard five point square net pattern linear difference operator for Laplace's equation and $\mathcal{M}$ is a linear difference operator such that $\mathcal{L}-\mathcal{M}=\mathcal{L}^*$. In order to obtain a discrete function which is zero on the upper and lower boundaries of the region, we set

$$\psi_{r,s} = \phi_{r,s} - \frac{s}{n+1} \; . \qquad (5.4.3)$$

In terms of ψ, equations (5.4.1) and (5.4.2) are then respectively

$$\mathcal{L}^*\psi_{h,q} = k_{h,q} \, , \qquad (5.4.4)$$

$$\text{and} \quad \mathcal{L}\,\psi_{h,q} = \mathcal{M}\psi_{h,q} + k_{h,q} \, , \qquad (5.4.5)$$

where $k_{h,q}$ is a <u>known</u> constant whose value may depend on the particular point (hl, qh) under consideration. We now set

$$\mathcal{L}\psi_{h,q} = D_{h,q} \, , \qquad (5.4.6)$$

$$\text{where} \quad D_{h,q} = \mathcal{M}\psi_{h,q} + k_{h,q} \, , \qquad (5.4.7)$$

and $D_{h,q}$ is regarded as a constant to be determined.

Our problem is therefore to solve the set of difference equations :

$$4\psi_{r,s} - \psi_{r+1,s} - \psi_{r,s+1} - \psi_{r-1,s} - \psi_{r,s-1} = \delta_r^h \, \delta_s^q \, D_{h,q} \; , \qquad (5.4.8)$$

defined at mesh points of the region ABCGH subject to the boundary conditions $\psi_{r,0} = \psi_{r,n+1} = 0$, $\psi_{-1,s} = \psi_{1,s}$ with Neumann conditions on ϕ on GC. The solution is then obtained using equation (5.4.3).

In the usual way we first ignore the boundary GC and obtain a solution, involving arbitrary constants, in the semi-infinite rectangle IABJ. (We assume $\psi=0$ on BI and GJ). The matrix governing equation is

$$\underset{\sim}{A}\,\underset{\sim}{\Psi}_r - \underset{\sim}{\Psi}_{r+1} - \underset{\sim}{\Psi}_{r-1} = \delta_r^h \, \underset{\sim}{D}_h \, , \quad r \geqslant 0 \, , \qquad (5.4.9)$$

where $\underset{\sim}{A}$ is given on p.6, $\underset{\sim}{\Psi}_r = \{\psi_{r,1}, \psi_{r,2}, \dots, \psi_{r,n}\}$ and $\underset{\sim}{D}_h$ is an $(n \times 1)$ column vector whose only non-zero element is the qth one with (unknown) value $D_{h,q}$. Equation (5.4.9) is of the same form and subject to the same boundary conditions as equation (4.3.3), p. 81, so that its solution may be found in exactly the same manner.

Thus, as in § 1.3, we determine eigenvalues λ_ℓ, $\ell = 1, 2, \dots, n$, and eigenvectors $\underset{\sim}{S}_\ell = \{S_\ell(1), S_\ell(2), \dots\dots, S_\ell(n)\}$ such that

$$\underset{\sim}{S}_\ell^T \underset{\sim}{A} = (4 - 2\lambda_\ell)\, \underset{\sim}{S}_\ell^T .$$

Pre-multiplying equation (5.4.9) by $\underset{\sim}{S}_\ell^T$ we obtain

$$\begin{aligned} F_{\ell,r+1} - (4 - 2\lambda_\ell) F_{\ell,r} + F_{\ell,r-1} &= - \underset{\sim}{S}_\ell^T \delta_r^k \underset{\sim}{D}_k \\ &= - \delta_r^k \beta_{\ell,k} , \end{aligned} \tag{5.4.10}$$

where $F_{\ell,r} = \underset{\sim}{S}_\ell^T \underset{\sim}{\Psi}_r$ and $\beta_{\ell,k} = \underset{\sim}{S}_\ell^T \underset{\sim}{D}_k = S_\ell(q) D_{k,q}$. Using a method similar to that developed in Appendix A, we find that a particular solution of equation (5.4.10) satisfying the condition $F_{\ell,-1} = F_{\ell,1}$ is given by

$$\Gamma_{\ell,r} = - \sum_{i=0}^{2n+1} K_{\ell,r}^{(i)} \delta_i^k \beta_{\ell,k} ,$$

where $K_{\ell,r}^{(i)}$ is defined in equation (4.3.8). Hence

$$\begin{aligned} \Gamma_{\ell,r} &= - K_{\ell,r}^{(k)} \beta_{\ell,k} \\ &= - K_{\ell,r}^{(k)} S_\ell(q)\, D_{k,q} . \end{aligned} \tag{5.4.11}$$

On adding a complementary function satisfying the condition that $F_{\ell,-1} = F_{\ell,1}$, the complete solution of equation (5.4.10) is

$$F_{\ell,r} = C_\ell V_\ell^{(r)} - K_{\ell,r}^{(k)} S_\ell(q)\, D_{k,q} , \tag{5.4.12}$$

where C_ℓ is an arbitrary constant and $V_\ell^{(r)} = P_\ell^r + Q_\ell^r$, ($P_\ell$ and Q_ℓ being solutions of $\alpha^2 - (4 - 2\lambda_\ell)\alpha + 1 = 0$). Using the inverse formula, $\underset{\sim}{\Psi}_r = \sum_{\ell=1}^{n} F_{\ell,r} \frac{\underset{\sim}{S}_\ell}{\Delta_\ell^2}$, where $\Delta_\ell^2 = \underset{\sim}{S}_\ell^T \underset{\sim}{S}_\ell$, we then have

$$\underset{\sim}{\Psi}_r = \sum_{\ell=1}^{n} \left(C_\ell V_\ell^{(r)} - K_{\ell,r}^{(k)} S_\ell(q) D_{k,q} \right) \frac{\underset{\sim}{S}_\ell}{\Delta_\ell^2} . \tag{5.4.13}$$

Hence

$$\psi_{r,s} = \sum_{\ell=1}^{n} \left(C_\ell V_\ell^{(r)} - K_{\ell,r}^{(k)} S_\ell(q) D_{k,q} \right) \frac{S_\ell(s)}{\Delta_\ell^2} , \tag{5.4.14}$$

and, by equations (5.4.3) and (5.4.14),

$$\phi_{r,s} = \sum_{\ell=1}^{n} \left(C_\ell V_\ell^{(r)} - K_{\ell,r}^{(k)} S_\ell(q) D_{k,q} \right) \frac{S_\ell(s)}{\Delta_\ell^2} + \frac{s}{n+1} . \tag{5.4.15}$$

The solution $\phi_{r,s}$ in the semi-infinite rectangle IAHJ is thus expressed in terms of the $(n+1)$ arbitrary constants $C_1, C_2, \cdots, C_n$ and $D_{h,q}$.

As in § 2.3 a set of 'n' simultaneous linear equations with those $(n+1)$ arbitrary constants as unknowns is obtained by using the difference equations at mesh points on GC, these "fitting" equations being specified by the condition of symmetry of the solution about this boundary. The $(n+1)$ th equation is obtained by superposing the condition that the special equation (5.4.1), i.e. $\mathcal{L}^{*}\phi_{h,q} = 0$, must be satisfied. (Alternatively equation (5.4.7) may be used). Solution of these equations and substitution in equation (5.4.15) then gives the solution of the problem.

With only one special equation therefore, the problem reduces to the solution of a set of $(n+1)$ simultaneous linear equations. Similarly, if two special points not on GC are involved we may solve the problem by introducing two arbitrary constants, say D_1 and D_2, in the manner illustrated in equations (5.4.1) - (5.4.7). We are then led to solve a set of $(n+2)$ simultaneous linear equations, 'n' of these equations being obtained from the fitting equations at mesh points on GC and the remaining two from the special equations themselves. The technique could clearly be extended to deal with cases where special difference equations are used at any number of mesh points, but, as each special point requires the introduction of a new arbitrary constant and consequently an additional simultaneous linear equation, the method is practical only when a few special points are involved.

A special difference equation at any mesh point on the boundary GC however does not require the introduction of a new arbitrary constant and therefore does not increase the number of equations which must eventually be solved. This is due to the fact that, in order to obtain a solution involving arbitrary constants in the semi-infinite rectangle IAHJ, we ignore the presence of the boundary GC and assume that the normal five point square net pattern equations are used at mesh points co-incident with it. The difference equations actually used at these mesh points are only imposed on this solution as fitting equations. Thus any special equation at a mesh point on GC may be used in its original form, i.e. $\mathcal{L}^*\phi_{r,s}=0$, as the fitting equation at that point.

In § 5.2 we showed how the singularity in the condenser problem may be taken account of by using special difference equations at three mesh points in its immediate vicinity, one of which lies on the boundary GC. Thus in solving the condenser problem with these special equations and with the normal five point square net pattern equations at ordinary points, we require to solve a set of $(n+2)$ simultaneous linear equations if the direct method is used. ('n' is the number of fitting equations on the boundary GC, (Fig. 31)). Results have been obtained on a desk calculator for the simple case n = 3 when the singularity is treated by the adapted Motz and by the adapted Jeffreys and Jeffreys methods.

This direct method may be used in a similar way when diagonal or nine point net pattern equations are used at ordinary points if, instead of equation (5.4.2), we take

$$\mathcal{L}^D\phi_{h,q}=\mathcal{M}^D\phi_{h,q} \quad \text{or} \quad \mathcal{L}^N\phi_{h,q}=\mathcal{M}^N\phi_{h,q},$$

where $\mathcal{L}^D$ and $\mathcal{L}^N$ are respectively the standard diagonal and nine point net pattern linear difference operators for Laplace's equation and $\mathcal{M}^D$ and $\mathcal{M}^N$ are such that $\mathcal{L}^*=\mathcal{L}^D-\mathcal{M}^D=\mathcal{L}^N-\mathcal{M}^N$. AS was pointed out at the end of § 2.3, (p.50), however, special difference equations must be used at the points K, L, ..., P in Fig. 31. Thus, to obtain the solution for the ordinary diagonal or nine point net pattern, we are led to solve a set of 2n simultaneous linear equations.

5.5 Concluding Remarks

The purpose of this thesis has been to introduce and to illustrate the "discrete transform method" of obtaining the numerical solution of boundary-value problems expressed in terms of finite differences. The method has been shown to be the difference analogue of the separation of variables and transform technique of continuous analysis. It may be applied to solve the finite difference equations corresponding to <u>any</u> partial differential equation problem defined in a region which can be divided into a number of sub-regions in each of which separation of variables is applicable.

To give the reader some perspective however, it should be emphasised that the simple problems considered in Chapters 1 and 2 were introduced only for the purpose of developing the method and to bring out the points of analogy between the discrete and the continuous techniques. The discrete transform method is <u>not</u> advocated as the best method of solving such problems. One of the advantages of the method is that it can be used to solve problems defined in regions which are infinite in one direction. This was illustrated in Chapter 3 where we considered a steady-state wave problem in an infinite strip. The author hopes that, having shown how the method may be applied, it may prove a useful tool for solving problems of practical importance in wave-guide theory for which no convenient numerical methods exist at present.

It seems difficult to obtain any theoretical explanation of the results obtained in the square box condenser problem and the steady-state wave problem, and it was thought better to investigate methods of improving the accuracy of the difference approximations. This decision was also taken in view of the scarcity of published papers on methods of treating singularities in boundary-value problems and, in particular, since there has been practically no discussion on the effectiveness of these methods in improving accuracy. The latter part of the thesis has therefore been concerned, in the main, with "numerical experimentation", and with the introduction of the L.S.D.T. method of treating singular points. Results have been given only for the simple condenser problem but investigations have also been carried out into the application of the L.S.D.T. method of treating the singularity in the steady-state wave problem. It was hoped to include results for this latter problem but difficulties have been encountered with the Bessel functions of various orders involved and no satisfactory special equations have as yet been obtained. The solution of the steady-state wave problem itself with special equations at mesh points near the end of the obstruction presents no difficulty, a direct method similar to that illustrated in § 5.4 for the condenser problem being used.

In summary, the practical criteria satisfied by the discrete transform method may be listed as follows. The method is

(a) able to deal with equations of more complicated form than Laplace's or Poisson's equation,

(b) able to deal with problems where boundaries may be at infinite distance,

(c) capable of allowing the treatment of singularities by means of special difference equations,

(d) suitable for automatic machine computation.

The author hopes that enough has been done to reveal the potentialities of this new direct method of solving sets of difference equations corresponding to boundary-value problems and also to illustrate the effectiveness of the L.S.D.T.

method of treating singularities. In particular, since the effect of the discrete transform is to reduce the dimension of the problem by one, the solution of a set of partial difference equations at mesh points in space will be reduced to the solution of a set of ordinary difference equations for unknown quantities at mesh points in a plane. The discrete transform method may therefore prove practical for solving boundary-value problems in three dimensions.

APPENDIX A

The Particular Solutions of Certain Non-Homogeneous Second-Order Difference Equations

Particular solutions of the non-homogeneous second-order difference equations occurring in this thesis may be obtained by a method analogous to the Green's function method of obtaining particular solutions of non-homogeneous second-order differential equations. We will, therefore, refer to the method as the discrete Green's function method. To illustrate this analogy we now obtain

(i) a particular solution of a simple second-order differential equation using the continuous Green's function method, and

(ii) a particular solution of a simple second-order difference equation using the discrete technique.

Corresponding equations are similarly numbered.

(i) The Continuous Green's Function Method

Consider the differential equation

$$\frac{d^2G(z)}{dz^2} - \alpha^2 G(z) = F(z), \qquad (A(i)1)$$

where $G(z)$, $F(z)$ are defined in the interval $0 \leq z \leq c$ and α is a constant. Suppose we have the boundary conditions that $G(0) = G(c) = 0$.

We seek a Green's function, $K(z,\xi)$ say, which is such that (a) $K(z,\xi)$, regarded as a function of z, satisfies the homogeneous form of (A(i)1) except at $z = \xi$.

$$\text{i.e.} \quad \frac{\partial^2 K(z,\xi)}{\partial z^2} - \alpha^2 K(z,\xi) = 0, \qquad (A(i)2)$$

except at $z = \xi$,

(b) $K(z,\xi)$ satisfies the boundary conditions imposed on $G(z)$.

i.e. $$K(0,\xi) = K(c,\xi) = 0 , \qquad (A(i)3)$$

(c) $K(z,\xi)$ is a continuous function of z at $z = \xi$,

(d) the discontinuity in derivative of K with respect to z at $z = \xi$ is a constant which we may take as unity

i.e. $$\left[\frac{\partial K}{\partial z}\right]_{z=\xi+} - \left[\frac{\partial K}{\partial z}\right]_{z=\xi-} = 1 . \qquad (A(i)4)$$

Under these conditions it can be shown that (see e.g. Bateman [23], Morse & Feshbach [24])

$$K(z,\xi) = K(\xi,z) , \qquad (A(i)5)$$

and $$G(z) = \int_0^c K(z,\xi)\,F(\xi)\,d\xi . \qquad (A(i)6)$$

Since we chose K to satisfy the boundary conditions (A(i)3) the solution $G(z)$ automatically satisfies the same conditions.

From (A(i)2) and (A(i)3) we deduce

$$K(z,\xi) = \begin{cases} A(\xi)\left(e^{\alpha z} - e^{-\alpha z}\right) , & z < \xi , \\ B(\xi)\left(e^{\alpha(c-z)} - e^{-\alpha(c-z)}\right) , & z > \xi , \end{cases} \qquad (A(i)7)$$

where $A(\xi)$, $B(\xi)$ are arbitrary constants with respect to z .

Conditions (c) and (d) are then used to evaluate $A(\xi)$, $B(\xi)$ and we obtain

$$K(z,\xi) = \begin{cases} \dfrac{\left(e^{\alpha(\xi-c)} - e^{-\alpha(\xi-c)}\right)\left(e^{\alpha z} - e^{-\alpha z}\right)}{2\alpha\left(e^{\alpha c} - e^{-\alpha c}\right)} , & z \leq \xi , \\ \dfrac{\left(e^{\alpha\xi} - e^{-\alpha\xi}\right)\left(e^{\alpha(z-c)} - e^{-\alpha(z-c)}\right)}{2\alpha\left(e^{\alpha c} - e^{-\alpha c}\right)} , & z \geq \xi . \end{cases} \qquad (A(i)8)$$

The particular solution, $G(z)$, of (A(i)1) is then given by substituting (A(i)8) into (A(i)6).

(ii) The Discrete Green's Function Method

Consider the difference equation

$$G_{t+1} - (4-2\lambda)G_t + G_{t-1} = F_t, \qquad \text{(A(ii)1)}$$

where $1 \leq t \leq h$ and λ is a constant. Let the boundary conditions be that $G_0 = G_{h+1} = 0$.

We will denote the discrete Green's function by $K_t^{(v)}$. (A superscript is used for the second discrete variable since subscripts in addition to "t" are usually involved in practice, e.g. the subscript normally associated with the eigenvalue λ in equation (A(ii)1))

$K_t^{(v)}$ is chosen such that

(a') $K_t^{(v)}$ satisfies the homogeneous form of equation (A(ii)1) except at $t=v$,

i.e. $$K_{t+1}^{(v)} - (4-2\lambda)K_t^{(v)} + K_{t-1}^{(v)} = 0, \qquad \text{(A(ii)2)}$$

except at $t=v$.

(b') $K_t^{(v)}$ satisfies the boundary conditions imposed on G_t,

i.e. $$K_0^{(v)} = K_{h+1}^{(v)} = 0, \qquad \text{(A(ii)3)}$$

(c') the expressions for $K_t^{(v)}$ derived from (a') and (b') on the left and on the right of $t=v$ are equal at $t=v$, i.e. $K_t^{(v)}$ is "continuous" at $t=v$.

(d') at $t=\nu$, $K_t^{(\nu)}$ satisfies equation (A(ii)1) with unity on the right hand side,

i.e. $$K_{\nu+1}^{(\nu)} - (4-2\lambda)K_{\nu}^{(\nu)} + K_{\nu-1}^{(\nu)} = 1, \qquad (A(ii)4)^{*}$$

Under these conditions we will now show that

(A) $K_t^{(\nu)}$ is symmetric in t and ν, i.e. $K_t^{(\nu)} = K_\nu^{(t)}$ (A(ii)5)

(B) the particular solution of (A(ii)1) is given by

$$G_t = \sum_{\nu=1}^{h} K_t^{(\nu)} F_\nu \ . \qquad (A(ii)6)$$

Note that since $K_t^{(\nu)}$ is chosen to satisfy the boundary conditions (A(ii)3) G_t automatically satisfies the same conditions.

(A) To show that $K_t^{(\nu)} = K_\nu^{(t)}$.

Let $K_j^{(t)}$ and $K_j^{(\nu)}$ be any two discrete functions satisfying conditions (a') - (d'). Thus, from (A(ii)2) and (A(ii)4),

$$K_{j+1}^{(t)} - (4-2\lambda)K_j^{(t)} + K_{j-1}^{(t)} = \delta_j^t , \qquad (1)$$

and $$K_{j+1}^{(\nu)} - (4-2\lambda)K_j^{(\nu)} + K_{j-1}^{(\nu)} = \delta_j^\nu , \qquad (2)$$

---

* (A(ii)4) may be written

$$\frac{K_{\nu+1}^{(\nu)} - K_\nu^{(\nu)}}{h} - \frac{K_\nu^{(\nu)} - K_{\nu-1}^{(\nu)}}{h} - \frac{2(1-\lambda)}{h} K_\nu^{(\nu)} = \frac{1}{h} .$$

This is the analogue of the derivative discontinuity condition (d) of the continuous case since $(1-\lambda)$ is of the order of h^2 (e.g. in problem B (§ 1.7, p. 28) we showed that

$$\lambda_k = \cos\theta_k = \cos\left(\frac{k\pi}{m+1}\right) = \cos\left(\frac{k\pi h}{(m+1)h}\right) = \cos\left(\frac{k\pi}{a}\right)h = \cos(\alpha h).$$

Series expansion of this cosine shows that $(1-\lambda)$ is of the order of h^2).

where δ_j^i is the Kronecker delta function.

Multiply (1) by $K_j^{(v)}$, (2) by $K_j^{(t)}$ and subtract. We obtain

$$K_j^{(v)}K_{j+1}^{(t)} - K_{j-1}^{(v)}K_j^{(t)} + K_j^{(v)}K_{j-1}^{(t)} - K_{j+1}^{(v)}K_j^{(t)} = K_j^{(v)}\delta_j^t - K_j^{(t)}\delta_j^{(v)}. \quad (3)$$

Sum (3) from $j=1$ to $j=h$ and use the "continuity" condition (c'), then

$$\sum_{j=1}^{h}\left\{K_j^{(v)}K_{j+1}^{(t)} - K_{j-1}^{(v)}K_j^{(t)} + K_j^{(v)}K_{j-1}^{(t)} - K_{j+1}^{(v)}K_j^{(t)}\right\} = \sum_{j=1}^{h}\left\{K_j^{(v)}\delta_j^t - K_j^{(t)}\delta_j^{(v)}\right\}.$$

The middle terms cancel in this summation and we obtain

$$K_1^{(v)}K_0^{(t)} - K_0^{(v)}K_1^{(t)} + K_h^{(v)}K_{h+1}^{(t)} - K_{h+1}^{(v)}K_h^{(t)} = K_t^{(v)} - K_v^{(t)}. \quad (4)$$

But, by the boundary conditions (A(ii)3), the left hand side of (4) is zero

$$\therefore \quad K_t^{(v)} = K_v^{(t)}. \quad (A(ii)5)$$

Thus property (A) holds.

(B) To show that $G_t = \sum_{v=1}^{h} K_t^{(v)}F_v$.

From (A(ii)1) we have $$G_{v+1} - (4-2\lambda)G_v + G_{v-1} = F_v . \quad (5)$$

From (A(ii)2), (A(ii)4), and the symmetry of $K_t^{(v)}$ we have

$$K_{v+1}^{(t)} - (4-2\lambda)K_v^{(t)} + K_{v-1}^{(t)} = \delta_v^t \quad (6)$$

$(5)\times K_v^{(t)} - (6)\times G_v$

$$K_v^{(t)}G_{v+1} - K_{v-1}^{(t)}G_v + K_v^{(t)}G_{v-1} - K_{v+1}^{(t)}G_v = K_v^{(t)}F_v - G_v\delta_v^t \quad (7)$$

Summing (7) from $v=1$ to $v=h$ and using boundary conditions on K and G we obtain

$$G_t = \sum_{\nu=1}^{h} K_\nu^{(t)} F_\nu$$

$$= \sum_{\nu=1}^{h} K_t^{(\nu)} F_\nu . \qquad (A(ii)6)$$

Thus property (B) is proved.

Having obtained the preliminary results (A) and (B) we now obtain the particular solution of equation (A(ii)1).

From (A(ii)2) and (A(ii)3)

$$K_t^{(\nu)} = \begin{cases} A_\nu (P^t - Q^t) , & t < \nu, \\ B_\nu (P^{h+1-t} - Q^{h+1-t}) , & t > \nu, \end{cases} \qquad (A(ii)7)$$

where A_ν , B_ν are constants independent of t and P , Q are solutions of

$$\beta^2 - (4-2\lambda)\beta + 1 = 0 . \qquad (x)$$

Conditions (c') and (d') are then used to evaluate A_ν , B_ν , noting from (x) that $4-2\lambda = P+Q$, and we obtain

$$K_t^{(\nu)} = \begin{cases} \dfrac{(P^{\nu-h-1} - Q^{\nu-h-1})(P^t - Q^t)}{(P-Q)(P^{h+1} - Q^{h+1})} , & t \leq \nu, \\ \dfrac{(P^\nu - Q^\nu)(P^{t-h-1} - Q^{t-h-1})}{(P-Q)(P^{h+1} - Q^{h+1})} , & t \geq \nu. \end{cases} \qquad (A(ii)8)$$

The particular solution, G_t , of (A(ii)1) is then given by substituting (A(ii)8) into (A(ii)6).

Thus, the example illustrates that the discrete Green's function method of obtaining particular solutions of non-homogeneous, second-order difference equations is exactly parallel to the continuous Green's function method used for non-homogeneous, second-order differential equations. It should be noted that conditions (a), (c), and (d) in the continuous case, or (a'), (c'), and (d') in the discrete, are common for the solution of any non-homogeneous, second-order differential, or difference, equation. The equation (A(i)3), or (A(ii)3), will be different for differential, or difference, equations subject to boundary conditions other than those present in the example illustrated but the techniques used in solution are exactly the same.

APPENDIX B

STEADY-STATE WAVE PROBLEM: APPROXIMATIONS TO THE (ENERGY) REFLECTION AND TRANSMISSION COEFFICIENTS

CASE 1

(a) Square Net Pattern

n	R_o^2	T_o^2
3	0.3962	0.6038
5	0.3318	0.6682
7	0.3020	0.6980
9	0.2849	0.7151

(b) Nine Point Net Pattern

n	R_o^2	T_o^2
3	0.2977	0.7023
5	0.2696	0.7304
7	0.2614	0.7386
9	0.2552	0.7448

CASE 2

(a) Square Net Pattern

n	R_o^2	T_o^2
2	0.0744	0.9256
4	0.1217	0.8783
6	0.1467	0.8533
8	0.1618	0.8382

(b) Nine Point Net Pattern

n	R_o^2	T_o^2
2	0.0938	0.9062
4	0.1376	0.8624
6	0.1593	0.8407
8	0.1721	0.8279

CASE 3

(a) Square Net Pattern

n	R_o^2	T_o^2
3	0.4720	0.5280
5	0.3731	0.6269
7	0.3294	0.6706
9	0.3052	0.6948

(b) Diagonal Net Pattern

n	R_o^2	T_o^2
3	0.3485	0.6515
5	0.2962	0.7038
7	0.2742	0.7258
9	0.2637	0.7363

(c) Semi-Continuous Net

n	R_o^2	T_o^2
3	0.3419	0.6581
5	0.2981	0.7019
7	0.2776	0.7224
9	0.2651	0.7349

APPENDIX C

SQUARE BOX CONDENSER PROBLEM: POTENTIAL VALUES AT MID-POINT OF GC

(a) Ignoring Singularity

n	5 Point Net Pattern	9 Point Net Pattern
3	0.20424	0.20402
5	0.20172	0.20131
7	0.20048	---
9	0.19977	---

(b) Singularity Treated in Manner of (a) Southwell, (b) Motz, (c) Woods

(All with five point square net pattern at ordinary points).

n	(a)	(b) with 5 Pt. Spec. Eqns.	(b) with 9 Pt. Spec. Eqns.	(c)
3	0.20298	0.19648	0.19677	0.19669
5	0.20062	0.19707	0.19717	0.19714
7	0.19967	0.19732	0.19734	0.19732
9	0.19914	0.19740	0.19740	0.19739

(c) Singularity Treated in Manner of Jeffreys and Jeffreys

n	5 Pt. Eqns. at Ord. Pts. 5 Pt. Eqns. at Spec. Pts.	5 Pt. Ordinary 9 Pt. Special	9 Pt. Ordinary 9 Pt. Special
3	0.19669	0.19685	0.19755
5	0.19714	0.19721	0.19752
7	0.19732	0.19735	0.19751
9	0.19739	0.19742	0.19751

(d) Singularity Treated by Least-Squares-Dominant-Term Method

n	9 Pt. Ordinary 9 Pt. Special	5 Pt. Ord., Weighted 9 Pt. Special	9 Pt. Ord., Weighted 9 Pt. Special
3	0.19764	0.19678	0.19750
5	0.19752	0.19716	0.19751
7	0.19751	0.19735	0.19751

(Weighted = Weighting Factors of Ordinary Nine Point Net Pattern)

APPENDIX D(i)

SPECIAL DIFFERENCE EQUATIONS FOR SQUARE BOX CONDENSER PROBLEM

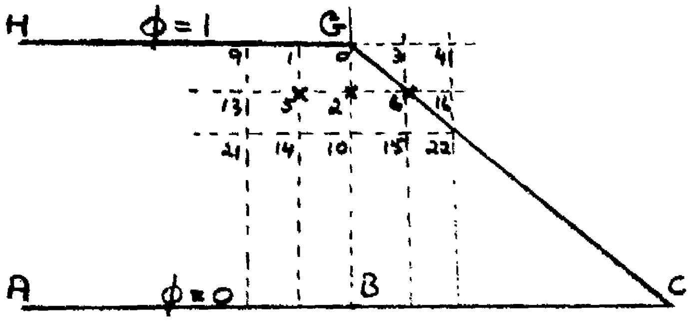
Abbreviations

A.M. = Adapted Motz Method.

A.J.J. = Adapted Jeffreys and Jeffreys Method.

L.S.D.T. = Least-Squares-Dominant-Term Method, $w_j \equiv 1$.

L.S.D.T.(W) = Least-Squares-Dominant-Term Method with weighting factors of ordinary nine point net pattern.



Special equations were used at the points numbered 5, 2, and 6 in the above figure.

(1)(a) Five point special difference equations at the point '5'

Since $\phi = 1$ on HG these special equations are of the form

$$4\phi_5 = \beta_1 \phi_{13} + \beta_2 \phi_{14} + \beta_3 \phi_2 + \beta_4$$

METHOD	β_1	β_2	β_3	β_4
A.M.	1.00978	0.99022	0.98752	1.01248
A.J.J.	1.02387	0.97613	0.99769	1.00231

(1)(b) Nine point special difference equations at the point '5'

Since $\phi = 1$ on HG these special equations are of the form

$$4\phi_5 = \gamma_1 \phi_{13} + \gamma_2 \phi_{21} + \gamma_3 \phi_{14} + \gamma_4 \phi_{10} + \gamma_5 \phi_2 + \delta$$

METHOD	γ_1	γ_2	γ_3	γ_4	γ_5	δ
A.M.	0.80016	0.20000	0.79984	0.19856	0.82024	1.18120
A.J.J.	0.46501	0.50725	0.52049	0.54023	0.49544	1.47158
L.S.D.T.	0.32764	0.56499	0.54239	0.28540	0.88681	1.39277
L.S.D.T(W)	0.75059	0.22203	0.80536	0.15077	0.89313	1.17812

(2)(a) Five point special difference equations at the point '2'

Since $\phi = 1$ on HG, these special equations are of the form

$$4\phi_2 = \beta_1\phi_5 + \beta_2\phi_{10} + \beta_3\phi_6 + \beta_4$$

METHOD	β_1	β_2	β_3	β_4
A.M.	1.10556	1·00000	1.10556	0.78888
A.J.J.	1.06111	1.06111	1.06111	0.81667

(2)(b) Nine point special difference equations at the point '2'

Since $\phi = 1$ on HG and $\phi_2 = \phi_3$ these equations are of the form

$$4\phi_2 = \gamma_1\phi_5 + \gamma_2\phi_{14} + \gamma_3\phi_{10} + \gamma_4\phi_{15} + \gamma_5\phi_6 + \delta$$

METHOD	γ_1	γ_2	γ_3	γ_4	γ_5	δ
A.M.	0.88512	0.21390	0.86404	0.21390	0.88512	0.93792
A.J.J.	0.56919	0.56919	0.56919	0.56919	0.56919	1.15405
L.S.D.T.	0.57379	0 57238	0.55401	0.56864	0.58127	1.14991
L.S.D.T(W)	0.84887	0.16056	1.04641	0.19791	0.77416	0.97209

(3)(a) Five point special difference equations at the point '6'

Since $\phi_2 = \phi_3$, $\phi_{15} = \phi_{16}$, these special equations are of the form

$$4\phi_6 = \beta_1\phi_2 + \beta_2\phi_{15} + \mu$$

METHOD	β_1	β_2	μ
A.M.	2.04217	1.95783	0
A.J.J.	1.96088	2.00000	0.03912

(3)(b) Nine point special difference equations at the point '6'

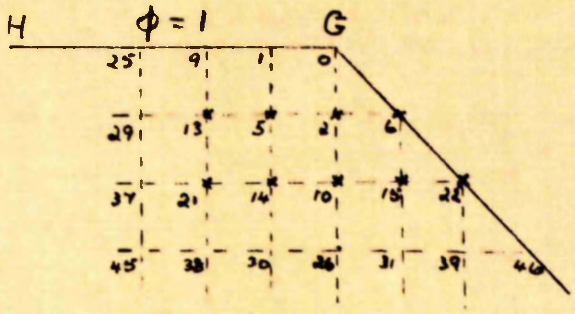
Since $\phi = 1$ on HG, $\phi_2 = \phi_3$, $\phi_{10} = \phi_{11}$, $\phi_{15} = \phi_{16}$, these special equations are of the form

$$4\phi_6 = \gamma_1\phi_2 + \gamma_2\phi_{10} + \gamma_3\phi_{15} + \gamma_4\phi_{22} + \delta$$

METHOD	γ_1	γ_2	γ_3	γ_4	δ
A.M.	1.64272	0.39503	1.60107	0.19946	0.16172
A.J.J.	1.07848	1.03727	1.01331	0.49336	0.37758
L.S.D.T	1.70211	0.73059	0.83671	0.58164	0.14895
L.S.D.T.(W)	1.76760	0.34427	1.54385	0.22807	0.11621

APPENDIX D(ii)

Special Difference Equations for Square Box Condenser Problem using the Adapted Southwell Method



Special difference equations were used at all points marked with crosses in the above figure and were as follows:

At point '21'

$$4\phi_{21} - 0.93333\phi_{37} - 1.06667\phi_{38} - 0.93333\phi_{10} - 0.13333\phi_{2} - 0.80000\phi_{5} - 0.13333\phi_{13} = 0$$

while equations at points '14' and '10' had the same coefficients.

At point '15'

$$4\phi_{15} - 0.96552\phi_{10} - 1.10344\phi_{31} - 0\ 96552\phi_{22} - 0.82759\phi_{6} - 0.13793\phi_{2} = 0.$$

At point '22'

$$4\phi_{22} - 1.73333\phi_{15} - 0.13333\phi_{31} - 2.00000\phi_{39} - 0.13333\phi_{6} = 0.$$

At point '13'

$$4\phi_{13} - 0.80000\phi_{29} - 0.20000\phi_{37} - 0.80000\phi_{21} - 0.20000\phi_{14} - 0\ 80000\phi_{5} - 1.20000 = 0$$

while the equation at the point '5' had the same coefficients.

At point '2'

$$4\phi_{2} - 0.84211\phi_{5} - 0.21053\phi_{14} - 0.84211\phi_{10} - 0.21053\phi_{15} - 0.84211\phi_{6} - 1.05262 = 0.$$

At point '6'

$$4\phi_{6} - 1.60000\phi_{2} - 0.40000\phi_{10} - 1.60000\phi_{15} - 0.20000\phi_{22} - 0.20000 = 0.$$

The constant terms occurring in the special equations at points numbered 13, 5, 2 and 6 are due to the fact that $\phi = 1$ on HG in the above figure.

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