THE ELLIPTICAL RESTRICTED THREE-BODY PROBLEM AS APPLIED TO THE MOTIONS OF OUTER SATELLITES OF JUPITER

by

R. B. Hunter

Thesis

submitted to the

University of Glasgow

for the degree of

Ph.D

Department of Astronomy
University of Glasgow

May 1966

Preface

The thesis deals with the application of the elliptical restricted three-dimensional three-body problem of Celestial Mechanics to the motions of asteroids and satellites of Jupiter, moving under the gravitational attraction of the Sun and Jupiter. From the integration of a large number of orbits an attempt is made to map out the regions around Jupiter of stable and unstable satellite and asteroid orbits.

In chapter I the orbits of the seven outermost satellites of Jupiter which are appreciably perturbed by the Sun are discussed and a general review is given of early numerical work by Darwin and others on the restricted three-body problem as well as of more recent work by Chevotarov and colleagues on the numerial computation of the orbits of distant satellites of Jupiter.

In chapter II the various computer programs which were developed in the course of the work are discussed.

In chapter III the manner of computing the starting values for some of the integrations is discussed.

In chapter IV the stability of the orbits of distant satellites of Jupiter is investigated according to their initial semi-major axes, eccentricities and inclinations to the plane of Jupiter's orbit about the Sun, as well as to whether the satellites' motions are direct or retrograde.

In chapter V the escape of satellites from Jupiter and the capture by Jupiter of asteroids is investigated. From the results of the integrations of the orbits of hypothetical asteroids just inside the orbit of Jupiter the existence of a second asteroid belt between the orbits of Jupiter and Saturn is postulated.

In chapter VI an attempt is made to explain the tendency for outer satellites of Jupiter to be in orbits which have mean angular motions which are commensurable with Jupiter's mean motion about the Sun. It is suggested that these orbits lie close to periodic orbits and an attempt is made to find one such periodic orbit by considering the relationship between the mean elements of a satellite's orbit and the rates of movement of the apse and the node of the orbit.

Chapter VII is an assessment of the results of the previous chapters and concludes with suggestions for further work along the lines of this thesis

The figures and tables referred to are at the end of the thesis and bear numbers of the form "5.4", the first integer being the number of the chapter to which the figure or table refers. References are at the end of each chapter and are numbered in a similar way, the first integer being the number of the chapter in which the reference is *first* quoted.

The work described in the thesis was carried out while the author was on the staff of the Department of Astronomy in the University of Glasgow. It is a pleasure to acknowledge here his indebtedness to Professor P.A. Sweet for the opportunity to do this work and for his encouragement throughout, to Dr. A.E. Roy for his help and advice at all stages of the work and to Dr. M.W. Ovenden for many useful discussions on points which arose during the work. The author would also like to thank the staff of the Department of Computing in the University of Glasgow for their help in writing and in running the computer programs.

CONTENTS.

Page.

CHAPTER I.

The Outer Satellites of Jupiter and the Restricted Three-Body Problem.

CHAPTER II.

The Computer Programs

Page

CHAPTER VII

Assessment of Results

CHAPTER I

The Outer Satellites of Jupiter and the Restricted Three-Body Problem

1.1. The Outer Satellites of Jupiter.

The seven outermost satellites of Jupiter have all been discovered photographically since 1904. These satellites (referred to as numbers VI to XII in order of their discovery) are very faint objects having apparent magnitudes between 14.7 and 19.0.

Satellites VI and VII were discovered by C.D. Perrine using the 36-inch Crossley reflector at Lick in 1904/5. Satellite VII was discovered by P.J. Melotte using the 36-inch Greenwich reflector in 1908, number IX by S.B. Nicholson with the Crossley reflector in 1914 and numbers X and XI also by S.B. Nicholson in 1938 using the 100 inch reflector at Mount Wilson. In 1951 S.B. Nicholson discovered Jupiter XII, again with the 100-inch at Mount Wilson. No more satellites have been discovered since that date.

The mean distances of these satellites from Jupiter are so large compared with their distances from the Sun (at least one in eighty) and the mass of the Sun is so much greater than that of Jupiter (1047:1) that these satellites do not move in fixed ellipses about Jupiter as if moving about a single mass point, but move in jovicentric orbits whose shape and orientation are continually changing due to the gravitational perturbations of the Sun on the satellites. Fig. 1.1 (after S.B. Nicholson) shows marked changes in some of the orbits during a single revolution. The orbit of Jupiter VIII due to its large eccentricity and semi-major axis is subject to particularly large perturbations. The orbit of this satellite was first well established by Grosch (ref. 1.1) who used two-dimensional diagrams to illustrate its complicated behaviour.

Mean elements of all seven satellites have been computed. References to recent determinations are given in sections 3.1 and 3.2. Table 1.1 shows some of the mean elements of the satellites in order of increasing semi-major axis. The orbits appear to fall into three groups -

Group (i) Satellites VI, X and VII which are in direct orbits with similar semi-major axes, eccentricities and inclinations to the plane of Jupiter's orbit.

Group (ii) Satellite XII which is in a retrograde orbit slightly inside the orbits of the group (iii) satellites.

Group (iii) Satellites XI, VIII and IX which are in retrograde orbits with similar elements.

Other authors including Kuiper(ref.1.2) prefer to think of the satellites as belonging to two groups, the direct satellites and the retrograde ones. Kuiper considers that the similarity in the elements of the satellites in each of the groups suggests that the seven satellites are the remains of two larger satellites which were captured by Jupiter while still a proto-planet and have since broken up. It seems likely, therefore, according to Kuiper, that other parts of the two satellites, presumably smaller and fainter,may also be in orbit about Jupiter in this area. Kuiper estimates that any so far undiscovered satellite of Jupiter could not have a diameter greater than 18 kilometres.

Kuiper considers that the retrograde satellites are so far from Jupiter as to be on the verge of instability and concludes that they could not have been captured by Jupiter at a time when Jupiter's mass had its present value,but must have been captured at a time when Jupiter was losing mass.

1.2 Commensurabilities.

Roy and Ovenden (ref.1.3) have discussed the occurrence of commensurabilities between the mean angular motions of these satellites and the mean angular motion of Jupiter about the Sun(see table 1.2). All the satellites of group (i) are seen to lie close to the commensurability 1/17 while the average of the three mean motions lies even closer. The satellite in group (ii) is seen to lie near the commensurability 1/7 and those of group (iii) near 1/6. Again the average of the three mean motions of the group (iii) satellites lies even nearer 1/6.

In general Roy and Ovenden have found that commensurabilities are much more common in the solar system than could be merely attributed to chance. In the case of satellites the commensurability is found between the satellite and the chief perturbing body(s). For one of the Galilean satellites of Jupiter these are the three other Galilean satellites and commensurabilities exist between the orbits of the four Galilean satellites. For an outer satellite of Jupiter however the chief perturbing body is the Sun and the commensurability is between the Sun (thought of as a satellite of Jupiter) and the satellite.

Roy and Ovenden suggest that the reason for this preference for near-commensurable orbits is that such orbits are relatively stable. They go on to prove the "Mirror Theorem" viz.

> "If n point masses are acted upon by their mutual gravitational forces only and at a certain epoch each radius vector from the (assumed stationary) centre of mass of the system is perpendicular to each velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch."

From this they deduce that orbits that satisfy mirror configurations (as defined above) at two separate epochs are periodic. They suggest that the smaller the interval between such mirror configurations the more quickly the cancellation of perturbations would take place so that the orbits would not wander too far from mean orbits and would therefore be more stable. They go on to show that near-commensurable orbits provide opportunities for such frequent mirror configurations.

1.3 The Restricted Three Body Problem.

The restricted three body problem is a special case of the classical three body problem which studies the motions of three massive bodies moving under their mutual gravitational attractions. The restricted problem stipulates that one of the bodies should be of infinitesimal mass so as not to affect the motion of the other two bodies, and that the two massive bodies should move in circular orbits about their common centre of gravity.

The problem was first formulated by Euler and other contributors have included Jacobi (first integral), Lagrange (analytic solutions), Poincaré (proof of non existence of other integrals) and Darwin (periodic orbits by numerical integration).

The equations of motion of the infinitesimal third body are given by Moulton (ref.1.4).

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - (1-\mu)\frac{(x-x_1)}{r_1^3} - \mu\frac{(x-x_2)}{r_2^3} \quad (1.3.1)$$

$$\frac{d^2y}{dt^2} - 2\frac{dx}{dt} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \quad (1.3.2)$$

$$\frac{d^2z}{dt^2} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \quad (1.3.3)$$

where the origin is taken to be the centre of gravity of the two massive bodies and the x-axis is taken so as always to pass through the two massive bodies whose masses are taken to be μ and $1-\mu$. The unit of time is chosen to make the constant of gravitation unity. x_1 and x_2 are the x-coordinates of the massive bodies and r_1 and r_2 the distances of the third body from the two massive bodies. The z-axis is perpendicular to the plane of motion of the massive bodies.

If the third body is taken to move in the x, y plane then equation (1.3.3) does not apply. From now on the phrase "restricted three-body problem" will be taken to imply that the third body moves in this plane. When this is not the case the problem will be referred to as the "three-dimensional restricted three-body problem". When the two massive bodies are taken to move in elliptical orbits about their common centre of gravity it will be referred to as "the elliptical restricted three-body problem" or "the three-dimensional elliptical restricted three-body problem" as the case may be.

The above equations give rise to an integral first given by Jacobi (ref 1.5) and also discussed by Hill (ref.1.6) in connection with his lunar theory.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + C \quad (1.3.4)$$

where C is a constant of integration.

This integral has been used to define surfaces of zero velocity over which the third body cannot cross and also by Tisserand for the re-identification of comets (ref.1.4).

The restricted three-body problem provides a reasonable model for many situations in the solar system and its relative simplicity has led to its extensive use in the numerical calculation of orbits in the solar system.

1.4 Use of the Jacobi Integral

Hagihara(ref.1.7) has investigated the motion of the Moon and other satellites in the solar system by means of the Jacobi integral. The equations of motion of the Moon as given by Hill's intermediate orbit are used viz.

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x = 0 \qquad (1.4.1)$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \mu\frac{y}{r^3} = 0 \qquad (1.4.2)$$

from which Jacobi's integral is obtained in the form.

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = \frac{\mu}{r} + \frac{3}{2}n'^2x^2 - C \qquad (1.4.3)$$

C being the constant of integration.

These equations refer to the Earth as origin and the x-axis is taken so that it always points to the Sun which is taken to move about the Earth with uniform angular velocity, n', at infinity. μ is the sum of the masses of the Earth and the Moon multiplied by the constant of gravitation.

From equation (1.4.3) it is deduced that the following inequality must hold:-

$$\frac{\mu}{(x^2+y^2)^{\frac{1}{2}}} + \frac{3}{2}n'^2x^2 - C \geqslant 0 \quad (1.4.4)$$

This is the equation of a curve of zero velocity for the Moon which it cannot cross. In fig.1.2 the curve is drawn for the following values of C.

$$(2C)^{3/2} > 9\mu n', \quad (2C)^{3/2} = 9\mu n',$$

$$(2C)^{3/2} < 9\mu n'$$

The curves appear to be rather distorted versions of the curves of zero velocity for the more general restricted three-body problem (see e.g. ref.1.8); the distortion being due to the assumptions involved in deriving (1.4.1) and (1.4.2).

For the case $(2C)^{3/2} > 9\mu n'$ part of the curve consists of a closed oval larger than the Moon's orbit. The value of C for the Moon is in this range and it is therefore deduced that the Moon cannot escape from the Earth since it is not able to cross this zero velocity curve.

For other satellite systems the same diagrams may be used with a suitable change of units. $9\mu n'$ is calculated for each planet and $(2C)^{3/2}$ for each satellite. For each satellite, except the four retrograde ones of Jupiter, it is found that

$$(2C)^{3/2} > 9\mu n'$$

and the satellite is contained within the oval. It is therefore deduced that none of these satellites can escape from the planet concerned.

In the case of the four retrograde satellites of Jupiter however

$$(2C)^{3/2} < 9\mu n'$$

and the zero velocity curve is of a different form and no such conclusions may be drawn regarding the stability of the orbits of these satellites.

The assumption is made throughout that the orbits of the satellites do not deviate sufficiently from Hill's intermediate orbit for any of the satellites with $(2C)^{3/2} > 9\mu n'$ to escape from the oval of zero velocity.

While this may be a reasonable assumption in the case of the Moon it is not obviously so in the case of the outer satellites of Jupiter given a sufficiently long time scale.

Other authors, e.g. Kuiper (ref.1.2(a) and 1.9) and Charlier (ref.1.10) have used the Jacobi integral of the restricted three-body problem as applied to the motion of satellites and asteroids moving in the gravitational field of Jupiter and the Sun. Kuiper, for instance, has used the Jacobi integral to show that asteroids with semi-major axes less than 4.2 astronomical units which are in almost circular orbits with small inclinations can never have semi-major axes greater than this value whereas for satellites with more eccentric orbits this critical value for the semi-major axis is rather less than 4.2 astronomical units. It is therefore deduced that the Trojan asteroids (a group forming an equilateral triangle with the Sun and Jupiter) could never have been part of the asteroid belt. Similarly it is shown that they could not have been satellites of Jupiter, unless in the meantime Jupiter had lost mass.

Ovenden and Roy (ref.1.11) have pointed out that the use of the Jacobi integral of the restricted three-body problem in the case where the two massive bodies move in elliptical orbits about their common centre of gravity, as in the Sun-Jupiter system, is not justified as the integral no longer holds. It is concluded that there is no reason to believe that conclusions drawn for long periods of time from the Jacobi integral are even approximately true in the elliptical case.

Kopal and Lyttleton (ref.1.12) express the same view and state that the argument based on the Jacobi integral that the Moon has always been a satellite of the Earth would appear to be invalid due to the non-zero eccentricity of the Earth's orbit about the Sun.

In this thesis the orbits of satellites and asteroids moving under the gravitational attraction of the Sun and Jupiter will be computed with the assumption that Jupiter moves in an elliptical orbit about the Sun. It will not in general be assumed that the Jacobi integral holds and in fact a watch will be kept for any evidence which contradicts the predictions of the circular Jacobi.

1.5 Darwin's Work

A great deal of work was done by Darwin (ref.1.13) on the numerical computation of orbits in the restricted three-body problem. Darwin considered two bodies which he called the Sun and Jove and the motion of an infinitesimal body which moved under the attraction of the other two bodies. Jove was taken to move in a circular orbit about the Sun and extensive use was made of the Jacobi integral to simplify the computations. The mass of the Sun was taken to be ten times that of Jove (cf.ratio of mass of Sun to mass of Jupiter is about 1,000). As the calculations were performed by means of four and five figure logarithms this relatively small ratio helped to ensure that significant results would be obtained within a reasonable period of time.

The integration was always begun with the satellite on the Sun-Jove line having a velocity perpendicular to this line. An orbit was defined to be simple periodic if the next time the satellite crossed this line its path was again at right angles to it i.e. the system satisfied mirror configurations on two successive revolutions of the satellite. Many of these simple periodic orbits were found, largely by a process of trial and error, and their stability was investigated.

This was done by varying a periodic orbit and noting whether the resultant orbit departed appreciably from the original one, in which case the periodic orbit was said to be unstable, or whether the resultant orbit oscillated about the periodic one with small oscillations, in which case the periodic orbit was said to be stable.

Periodic orbits were divided into three categories; those which were stable within the accuracy of the calculations although it was suspected that all orbits were ultimately unstable; those which were unstable with even instability, i.e. those in which the planet or satellite in the varied orbit crossed and recrossed the periodic orbit an even number of times in each revolution making an increasing number of excursions on each side; and those which were unstable with uneven instability (defined in an analogous manner to even instability). In the case of unstable orbits Hill's infinite determinant (ref.1.14) was used to assign a modulus of stability to each orbit.

A large number of orbits were computed for various values of the constant in the Jacobi integral. Attention however was in the main confined to direct orbits with values of the Jacobi constant such that the small body could move between being a satellite and being a planet.

1.6 The Copenhagen Problem

E Strömgren and others (see e.g. ref.1.15) have made exhaustive investigations of what is often referred to as the "Copenhagen Problem". This is the special case of the restricted three-body problem where the masses of the two large bodies are equal. From the equations of motion of the third body a large number of possible orbits were computed. Periodic orbits were obtained about the Langrangian libration points, about each of the masses separately and about both the masses together. In the last two cases both direct and retrograde orbits were considered. The work was done with the aid of more than thirty human computers in Denmark, Germany, Norway and Sweden.

Bartlett (ref.1.16) using an electronic computer and a modified Runge-Kutta method in the early 1960's recalculated most of Strömgren's orbits, found many new orbits in each class and discovered a number of new classes of orbits.

1.7 Three Dimensional Orbits.

Goudas (ref.1.17) has investigated periodic orbits in the three-dimensional restricted three-body problem using the Mercury electronic computer at Manchester; the integrations being performed again by a modified Runge-Kutta method. Three classes of symmetric orbits are defined and it is shown how such orbits may be found.

The stability of an orbit is related to the eigenvalues of a certain matrix - stability being defined in a similar way to that used by Darwin. If all the eigenvalues of the matrix have modulus less than unity or some are unity (necessarily an even number) and the rest are less than unity then the orbit is stable. If however any of the eigenvalues has modulus greater than unity then the orbit is unstable. The third possibility is that all the eigenvalues have modulus unity in which case the stability of the orbit is not determined without higher terms and the orbit is defined to be quasi-stable.

No stable orbits were discovered from the integrations but some orbits which were quasi-stable were discovered and it is shown that when such an orbit is perturbed by $O(\epsilon)$ a particle in the disturbed orbit will not depart from the original orbit by more than $O(\epsilon)$ for at least $A\epsilon^{-2}$ periods, where A is a suitable positive constant.

All the periodic orbits which were highly inclined to the plane of motion of the two massive bodies were found to be highly unstable. As the inclination was decreased the orbits became less and less unstable until for some inclination i_0 they became quasi-stable. It is deduced from this that after a certain time, not necessarily long, no orbits of high inclination will exist in such a system. Goudas concludes that such a system will eventually become almost flat and asserts that this is why the solar system is almost coplanar and therefore no conclusions can be drawn about the origin of the solar system merely from its flatness.

1.8 Direct and Retrograde Orbits.

Moulton (ref.1.18) has investigated whether a direct satellite of Jupiter with the period of Jupiter VIII (which is retrograde) would be more or less stable than Jupiter VIII. Stability being defined rigorously only for a periodic orbit and Jupiter VIII not being exactly periodic, the criterion for stability had to be applied to a periodic orbit close to that of Jupiter VIII and also to a periodic orbit with the same period but direct.

Jupiter was assumed to move in a circular orbit about the Sun and the satellite to move in the plane of Jupiter's orbit about the Sun. The equations of motion of the satellite (assumed to have infinitesimal mass) thus gave rise to the Jacobi integral. Unlike Darwin, Moulton used rectangular coordinates in his work and did not use the Jacobi integral to simplify the calculations. This made it possible to use the integral as a check on the numerical work. In fact it was found that the value of the constant in the integral never differed by more than one figure in the fifth decimal place throughout the computations.

Darwin had found in some cases, especially when the period of the periodic orbit was large, that it was not possible to calculate the value of Hill's infinite determinant and in these cases had either calculated the stability of the orbit only approximately or had left it underdetermined altogether. In this paper Moulton derives a new method of determining stability which avoids this difficulty.

The periodic orbit close to that of Jupiter VIII was found to be very stable indeed and it was therefore deduced that the orbit of Jupiter VIII was also very stable. After some eleven attempts a direct periodic with nearly the same period as Jupiter VIII was found and its stability was investigated. It was found to be unstable with even instability thus suggesting that retrograde orbits at this distance from Jupiter are more stable than direct ones.

1.9 Cometary Orbits in the Outer Solar System.

Chebotarev(ref.1.19) has applied the restricted three-body problem to the case of a comet moving in the other regions of the solar system under the attraction of the Sun and the Galaxy. The Galaxy was assumed to be a point mass moving in a clockwise direction in a circular orbit about the Sun. The comet was defined to be in a retrograde orbit if it also moved in a clockwise direction about the Sun. Otherwise the orbit was said to be direct. This convention was consistent with the major planets moving in direct orbits about the Sun, but if this application of the restricted three-body problem is compared with the case of the motion of a satellite of Jupiter, with the Sun moving round Jupiter in an anticlockwise direction, it is found that the terms direct and retrograde in the two cases must be interchanged. Chebotarev's conclusion therefore that direct orbits are stable at much greater distances from the Sun than retrograde ones is consistent with Moulton's result of Section 1.8 regarding the greater stability of "retrograde" orbits in the case of Jupiter's satellites.

All the orbits considered by Chebotarev were outside the sphere of influence of the Sun (radius 60,000 astronomical units). This is the sphere inside which, in calculating the orbit of the comet, it is convenient to take the Sun as the central body and the Galaxy as the perturbing body. Orbits were considered both inside and outside Hill's sphere (radius 230,000 a.u.): the radius of this sphere being the distance of the Sun from the nearest Langrangian point. According to Hill the orbit of a comet lying outside this sphere will be unstable. This does not necessarily mean that such a comet will leave the solar system, merely that it may do so. In fact the retrograde satellites of Jupiter are in unstable orbits according to Hill's criterion.

Cowell's method was used to integrate the equations of motion of the comet. In each case the integration was begun on the line Sun-centre of the Galaxy and the comet was given circular velocity. Six initial semi-major axes were chosen and each orbit was computed in the direct and in the retrograde manner. In the remainder of this section the terms "direct" and "retrograde" will be used in the sense defined by Chebotarev. The six semi-major axes were -

orbit 1	250,000 a.u.
orbit 2	230,000 a.u.
orbit 3	200,000 a.u.
orbit 4	150,000 a.u.
orbit 5	100,000 a.u.

A comet was defined to be in a stable orbit if it was able to make several revolutions about the Sun without escaping from the solar system. This would not appear to be a very exacting criterion for stability as there is no reason to believe that such a comet would not escape from the solar system after a few more revolutions about the Sun. In particular the conclusion that direct stable orbits are possible at a distance from the Sun of 230,000 a.u.(i.e. on Hill's sphere) based on the completion by a comet of just over two revolutions about the Sun(orbit 2) at the end of which time the comet was approaching the Sun with an orbital eccentricity of 0.99993 and a semi-major axis of 429×10^3 a.u. would not seem to be justified. It would seem likely that after a close approach to the Sun the comet, having such a large eccentricity, would then escape from the solar system. Admittedly after such a close approach to the Sun the comet could be perturbed into a stable orbit by one of the planets (e.g. Neptune), but this possibility would seem to be beyond the scope of the present problem.

Chebotarev's results are summarised below.

Orbit 1 direct and retrograde motion unstable.

Orbit 2 direct motion stable (after two revolutions), retrograde motion unstable.

<u>orbit 3</u> direct motion stable(after two revolutions),
retrograde motion unstable.

<u>Orbit 4</u> direct motion stable (after one revolution),
retrograde motion unstable.

<u>orbit 5</u> direct motion stable (after one revolution),
retrograde motion stable(after one revolution).

It is concluded that the maximum radius of a stable direct orbit about the Sun is 230,000 a.u. and the maximum radius of a stable retrograde orbit 100,000 a.u.

1.10 <u>Application to Outer Satellites of Jupiter.</u>

Chebotarev and others (ref.1.20) have published a number of papers on the stability of satellite orbits about Jupiter. The motions of the satellites are treated as examples of the elliptical restricted three-body problem and the Sun's coordinates, with respect to a stationary Jupiter, are obtained from tables. Integration of the equations of motion of the satellite is performed by Cowell's method with a variable integration step which always took one of the values 10.8, 21.6 or 43.2 days.

The first paper by Chebotarev and Bozhkova deals with direct initially-circular orbits within the sphere of influence of Jupiter which is calculated to have a radius of about 0.32 a.u., depending on the position of Jupiter in its orbit. It is shown that orbits within the sphere with radius

$$R^* = r_1 m_1^{\frac{1}{3}} \qquad (1.10.1)$$

r_1 being the distance of Jupiter from the Sun and m_1 being the ratio of the mass of Jupiter to the mass of the Sun, are stable. This sphere had radius about 0.16 a.u. Stability is again defined to be the ability of a satellite to perform several revolutions about Jupiter without escaping from the planet's influence.

Six satellites are started in initially-circular orbits on the line Sun-Jupiter with the following semi-major axes.

$a_0 = 0.30$ (satellite 1) $\quad a_0 = 0.28$(satellite 2)

$a_0 = 0.25$ (satellite 3) $\quad a_0 = 0.20$(satellite 4)

$a_0 = 0.15$ (satellite 5) $\quad a_0 = 0.10$(satellite 6)

Satellites 1 and 2 leave the vicinity of Jupiter almost immediately and the eccentricities of satellites 3 and 4 after half and orbit and one and a half orbits respectively grow so large (0.89 and 0.93) that it would seem that they must surely escape from Jupiter after another revolution or so; though in fact the integration is not continued any further. Satellites 5 and 6 complete one and two revolutions about Jupiter respectively and are taken to be in stable orbits. Again it would not seem possible to draw any long term conclusions regarding the stability of these two satellites from the length of the integration performed.

In the second paper, by Chebotarev and Volkov, the stability of initially circular retrograde orbits, both inside and outside the sphere of influence of Jupiter is considered and the conclusion is reached that motion inside the sphere of influence is stable. The method is similar to that of the previous paper and the orbits of five satellites are computed with the following initial semi-major axes.

a_0 = 0.20 (satellite 1) a_0 = 0.25 (satellite 2)

a_0 = 0.30 (satellite 3) a_0 = 0.35 (satellite 4)

a_0 = 0.40 (satellite 5)

Apparently the orbits of satellites 1, 2 and 3 are taken to be stable. The integration of the orbit of satellite 3 is terminated after it has completed a few revolutions and its orbital eccentricity has reached 0.847. From similar integrations performed by the author of this thesis (see Chapter 4) it would be expected that a satellite with such a high eccentricity would soon escape from the planet. The orbits of satellites 1 and 2 however appear to be perfectly stable from the integrations performed. Satellites 4 and 5 are clearly unstable and do not complete even one revolution about Jupiter.

The conclusion which might be drawn from papers one and two is that retrograde orbits are in general more stable than direct ones. Satellite 3 of the first paper, which is direct, has the same semi-major axis as satellite 2 of the second paper, which is retrograde. The results show quite clearly that for this value of semi-major axis, the retrograde orbit is more stable than the direct one.

In the third paper, by Chebotarev and Boshkova, the orbits of six direct satellites with initial eccentricities of 0.5 are considered. The same six values for the semi-major axis are used as in the first paper and the numbers 1-6 will be used to refer to the satellites in this paper which correspond to numbers 1-6 in the first paper.

Satellites 1-4 are highly unstable while satellites 5 and 6 complete three revolutions about Jupiter and are therefore taken to be stable. The effect of the non-zero eccentricity is best seen in the case of satellite 4. In paper one this satellite was able to perform more than one and a half revolutions about Jupiter whereas in paper 3 it is only able to complete three-quarters of a revolution. The conclusion from papers 1 and 3 is that for two direct orbits with the same initial semi-major axis the one with the larger eccentricity is likely to be less stable.

Two more papers by Chebotarev and Boskova on the motions of hypothetical satellites of Mars, Venus and Mercury in the ecliptic and perpendicular to it have been published, the treatment being very similar to that of the other three papers. Detailed consideration of these papers is however beyond the scope of this thesis.

<u>References in Chapter 1</u>

1.1 Grosch, H. R. J. Astronomical Journal, Vol.53, p.180, 1948.

1.2 (a) Kuiper, G. P. Vistas in Astronomy, Vol.II, p.1661, 1956.

(b) Kuiper, G. P. Planets and Satellites, Vol.III, p.585, 1960.

1.3 (a) Roy, A.E. and Ovenden, M. W. Monthly Notices of the Royal Astronomical Society, Vol.114, No.2, 1954.

(b) Roy, A. E.and Ovenden, M. W. Monthly Notices of the Royal Astronomical Society, Vol.115, No.3, 1955.

1.4 Moulton, F. R. Celestial Mechanics, Chapter VIII, 1958.

1.5 Jacobi, C. G. J. Comptes Rendues de l'Academie des Sciences de Paris, Vol.III, P.59, 1836.

1.6 Hill, G. W. American Journal of Mathematics Vol.1, p.18, 1878.

1.7 Hagihara, Y. Proceedings of the Japanese Academy, Vol.28, No.4, 1952.

1.8 Finlay-Freundlich, E. Celestial Mechanics, p.46, Pergamon, 1958.

1.9 Kuiper, G. P. Proceedings of the National Academy of Science, Vol.37, p.720, 1951.

1.10 Charlier, C. L. Die Mechanik des Himmels, Vol.2 p.293, 1927.

1.11 Ovenden, M. W. and Roy A. E. Monthly Notices of the Royal Astronomical Society, Vol.123, No.1, 1961.

1.12 Kopal, Z and Lyttleton R. A., Icarus Vol.1,1963.

1.13 Darwin, G. H. Scientific Papers, Vol.4, 1911.

1.14 Darwin, G. H. Scientific Papers, Vol.4, p.35,1911.

1.15 Strömgren, E. Publications of the Copenhagen Observatory, No.39, 1922.

1.16 Bartlett, J. H. Publications of the Copenhagen Observatory, No.179, 1964.

1.17 (a) Goudas, C. L. Bull. de la Soc. Mathematique de Grece, Nouvelle serie, Vol.2, 1961.

(b) Goudas, C. L. Astronomical Contributions from the University of Manchester, series III, No.103, 1963.

1.18 Moulton, F.R. Monthly Notices of the Royal Astronomical Society, vol. 75, 1915.

1.19 Chebotarev, G.A. Soviet Astronomy, vol. 8, no. 5., 1965

1.20 (a) Chebotarev, G.A. and Bozhkova, A. Bull. Inst. Theor. Astronomy Leningrad, Vol. VII, No. 8, 1961

(b) Chebotarev, G.A. and Volkov, M. Bull. Inst. Theor. Astronomy Leningrad, Vol. VIII, No. 2, 1961.

(c) Chebotarev, G. and Bozhkova, A. Bull. Inst. Theor. Astronomy Leningrad, Vol. VIII, No. 9, 1962.

(d) Chebotarev, G. and Bozhkova, A. Bull. Inst. Theor. Astronomy, Leningrad, Vol. IX, No. 6, 1964.

(e) Chebotarev, G. and Bozhkova, A. Bull. Inst. Theor. Astronomy Leningrad, Vol. IX, No. [illegible], 1965.

CHAPTER II

The Computer Programs

2.1 The Computers Used.

It will be necessary later to integrate numerically the equations of motion of the massless body in the three-dimensional elliptical restricted three-body problem and in this chapter the method of integration and the computer program used to perform the integration will be described in some detail.

The bulk of the work was done on the English Electric K.D.F.9 computer in the Department of Computing in the University of Glasgow, although a little preliminary work was done on the English Electric Deuce computer, the predecessor to K.D.F.9 in the University. The programs for Deuce were written in Deuce Alphacode and those for K.D.F.9 in K.D.F.9 Algol. With the most recent Algol compiler used (Intermediate Kidsgrove) it was possible to compute one satellite revolution about Jupiter per minute of computer time. About one quarter of the work was done before Kidsgrove Algol was available using the Whetstone compiler which was about fifteen times

slower than the intermediate Kidsgrove one. The speed of Deuce Alphacode was about thirty times slower than Whetstone Algol. In all about 2,000 satellite revolutions about Jupiter were computed.
revolutions about Jupiter were computed.

2. 2 The Equations of Motion

According to Kovalevsky (ref.2.1) the differential equations of motion of the massless body in the three-dimensional elliptical restricted three-body problem which will be called the satellite (the two massive bodies being the Sun and Jupiter) are of the form

$$x'' = -\frac{Gmx}{r^3} + GM\left(\frac{X-x}{\Delta^3} - \frac{X}{R^3}\right) \quad (2.2.1)$$

x, X being the x-coordinates of the satellite and the Sun with respect to axes fixed in space with the centre of Jupiter as origin, taking the Sun to move in an elliptic orbit about Jupiter. The positive direction of the x-axis is taken to be the direction of the Sun when it is at perijove and the z-axis is taken to be perpendicular to the plane of the Sun's orbit about Jupiter, to form a right-handed system. G is the constant of gravitation and r, Δ, R are the distances Jupiter-satellite, Sun-satellite and Jupiter-Sun respectively. The masses of Jupiter and the Sun are m and M respectively.

The equation involving y is exactly similar but since $\dot{z} = 0$ the equation in z becomes

$$z'' = -\frac{Gmz}{r^3} - \frac{GMz}{\Delta^3} \qquad (2.2.2)$$

For simplicity the following units were used -

M = 1

giving m = 1/1047.355 using the value of the mass of Jupiter adopted by Kovalevsky which includes the masses of the Galilean satellites, assumed to be located at the centre of mass of the system. The unit of distance was taken to be the Jupiter unit, defined to be the mean distance of Jupiter from the Sun.

From ref.2.1

1 Jupiter Unit = 5.2028 143 astronomical units

$\doteq$ 483,000,000 miles

The unit of time used was the Jupiter year i.e. the sidereal period of Jupiter.

From Allen (ref.2.2.)

1 Jupiter year = 11.86223 tropical years.

According to Smart(ref 2.3), Newton's equation for two-body motion is

$$\frac{4\pi^2 a^3}{T^2} = G(M+m) \qquad (2.2.3)$$

where G is the constant of gravitation, m, M the two masses, a the relative semi-major axis of their orbit and T its period.

Applying this equation to the motion of the Sun about Jupiter and using the units defined above (2.2.3) becomes

$$G = \frac{4\pi^2}{1+m} \qquad (2.2.4)$$

These units will be used from now on and the equations of motion therefore become

$$x'' = \frac{4\pi^2}{1+m}\left(-\frac{mx}{r^3} + \frac{X-x}{\Delta^3} - \frac{X}{R^3}\right) \qquad (2.2.5)$$

$$y'' = \frac{4\pi^2}{1+m}\left(-\frac{my}{r^3} + \frac{Y-y}{\Delta^3} - \frac{Y}{R^3}\right) \qquad (2.2.6)$$

$$z'' = \frac{4\pi^2}{1+m}\left(-\frac{mz}{r^3} - \frac{z}{\Delta^3}\right) \qquad (2.2.7)$$

2.3 The Method of Integration

It was necessary to choose a method of integrating the three non-linear simultaneous second-order differential equations. The fact that the first derivatives do not appear in the equations suggested De Vogelaere's method (ref.2.4). Others e.g. Kovalevsky(ref.2.1) and Chebotarev (ref.1.20) have used Cowell's method to integrate the equations but it was felt that De Vogelaere's method was more elegant, would require less programming and had a simple starting sequence. About the same average step-length was needed as was used by Kovalevsky (about 5 days) who was working to about the same accuracy (11 significant figures). Chebotarev used a larger step-length but was presumably not working to so many significant figures. Certainly as Chebotarev never continued the integration for more than a few satellite revolutions he would not require to work to so many significant figures since the growth of rounding off errors in such a short time would be small.

If the differential equation to be integrated is of the form

$$x'' = f(x, t) \qquad (2.3.1)$$

then De Vogelaere's method consists of the cyclic use of the following equations

$$x_{r+\frac{1}{2}} = x_r + \frac{1}{2} h x'_r + \frac{h^2}{24}\left(4 f_r - f_{r-\frac{1}{2}}\right) \qquad (2.3.2)$$

$$x_{r+1} = x_r + h x'_r + \frac{h^2}{6}\left(f_r + 2 f_{r+\frac{1}{2}}\right) \qquad (2.3.3)$$

$$x'_{r+1} = x'_r + \frac{h}{6}\left(f_r + 4 f_{r+\frac{1}{2}} + f_{r+1}\right) \qquad (2.3.4)$$

where f_r denotes $f(x_r, t_r)$ and h is the length of the integration step. The neglected terms are of order h^4, h^5, h^5 respectively and the method is comparable in accuracy with the fourth-order Runge-Kutta process. The function f however is only calculated twice per step.

To begin the sequence of operations the value of $f_{-\frac{1}{2}}$ is required and is readily obtained from $x_{-\frac{1}{2}}$ which is given with sufficient accuracy by

$$x_{-\frac{1}{2}} = x_0 - \frac{1}{2} h x'_0 + \frac{1}{8} h^2 f_0 \qquad (2.3.5)$$

The method can be extended to solve simultaneous second-order differential equations with the first derivatives missing like equations (2.2.5-7). These equations may be written in the form

$$x'' = f(x, y, z, t) \qquad (2.3.6)$$

$$y'' = g(x, y, z, t) \qquad (2.3.7)$$

$$z'' = k(x, y, z, t) \qquad (2.3.8)$$

assuming that X, Y can be expressed as simple functions of t. The solution of these equations involves applying each of equations (2.3.2-4) in turn to the equations(2.3.6-8).

The data required to start the program are the coordinates and velocity components of the satellite with respect to Jupiter and the initial size of the integration step. The position of the Sun was calculated at each step and not obtained from tables as Chebotarev did. The values of various constants required by the program (e.g. the mass of Jupiter) were incorporated in the program.

2.4 Calculation of the Sun's position

Each time the functions f and g were evaluated (i.e. twice per step) the coordinates of the Sun had to be calculated. This was done by the solution of Kepler's Equation using Newton's method of successive approximations.

The Sun's mean anomaly, M, is related to its eccentric anomaly, E, via Kepler's Equation(ref.2.5 p.114)

$$E - e \sin E = M \qquad (2.4.1)$$

where e is the eccentricity of the Sun's orbit about Jupiter. M increases uniformly with time and the initial value of M is read into the computer at the start of the program and after each half step, ΔM, the increment in M corresponding to half the step length, is added to the value of M and the corresponding value of E is obtained by the solution of (2.4.1). If at any time the value of M exceeds 2π then 2π is subtracted from it until the value of M is less than 2π.

The first approximation to E is given by

$$E = e \sin M + M \qquad (2.4.2)$$

According to Newton's method of successive approximations if x_n is an approximate solution of the equation $F(x) = 0$ then x_{n+1} is a better approximation where

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \qquad (2.4.3)$$

Let $$G(E) = E - e \sin E - M \qquad (2.4.4)$$

then according to Newton's method if E_n is an approximate solution of the equation $G(E) = 0$ then

$$E_{n+1} = E_n + \Delta E_n \qquad (2.4.5)$$

is a better solution where

$$\Delta E_n = -\frac{G(E_n)}{G'(E_n)} \qquad (2.4.6)$$

$$= \frac{M - E_n + e \sin E_n}{1 - e \cos E_n} \qquad (2.4.7)$$

$$\text{i.e. } E_{n+1} = \frac{M + e \sin E_n - E_n e \cos E_n}{1 - e \cos E_n} \qquad (2.4.8)$$

The second and succeeding approximations were calculated by means of equation (2.4.8) and the process was continued until two consecutive values of E differed by less than 5.10^{-11}. K.D.F.9 Algol uses about twelve decimal figures and agreement to this accuracy was usually achieved after three or four iterations. If the values of E did not converge after twenty iterations the computer was made to terminate the program and to print out the comment "Kepler". On the three or four occasions on which this occurred it was found to be due to a machine fault and on a subsequent run with the same initial data the E's converged as usual.

From the value of E the coordinates of the Sun with respect to Jupiter were calculated by means of the following equations (ref.2.5 p.112)

$$X = \cos E - e \qquad (2.4.9)$$

$$Y = b \sin E \qquad (2.4.10)$$

where b is the semi-minor axis of the Sun's orbit about Jupiter.

2.5 The Integration Step

The value of the integration step used in the program varied according to the distance of the satellite from Jupiter. The smallest value normally used was 1.6025641×10^{-4} Jupiter years, i.e. about 0.7 day. Every thirty steps the program checks the step-length by halving it and then calculating two steps with this new interval and comparing the coordinates and velocities so calculated with those obtained after one step with the previous interval. If any of the six coordinates and velocities differ by more than 4.10^{-12} times their value then the program will decide that the interval of integration is too large and will half it. However, assuming that at the previous check the interval has been correctly adjusted, it may be deduced that somewhere between the two checks the interval became too large. When this is discovered it is therefore necessary to go back to the previous check and to continue from there with half the interval.

If the interval does not need to be halved it is then checked to see if it can be doubled. Two steps are taken with the interval and compared with the result of one step with double the interval. If all the coordinates and velocities agree to within the same accuracy as before then it is deduced that the interval may be doubled. In this case the integration is continued with double the interval of integration used previously. Otherwise it is continued with the same interval as before.

For nearly circular orbits once every thirty steps is probably too often to check the interval for time will be wasted checking the interval when it does not need adjusting. On the other hand for highly elliptical orbits when the step length has to be changed frequently time will be wasted every time it is necessary to go back to the previous check and half the interval in which case it is advisable to check the step length fairly often. It is not usually possible to know in advance the shape of an orbit corresponding to particular starting values and it was felt that the figure thirty, representing about a thirtieth of a satellite revolution was a reasonable compromise between the two extreme cases mentioned above. No doubt by experimenting with different values of this constant a slightly more satisfactory value could be found but it is not thought that a great deal of computer time could be saved in this way. Actually the value of this constant was not built into the program but was one of the pieces of data fed into the computer at the beginning of each program run and so it would have been quite simple to have changed its value if a better one could have been found.

On a computer run the program always keeps track of whether the radius of the satellite's orbit is increasing or decreasing. When it is stationary the satellite is defined to be at perijove if it is at a minimum and apojove if it is at a maximum. The interval is always checked at perijove and apojove and the counter which counts the number of steps between interval checks is set back to zero. It is necessary to check the interval at perijove since otherwise the step length could be small enough at the check points on either side of perijove but not sufficiently small at perijove itself. Also as the coordinates and velocities of the satellite and its orbital elements can only be printed out immediately after an interval check, due to the manner in which the program was written, it was of interest to check the interval at perijove and apojove and to print out results at these points.

It would have been useful to have inserted in the program the condition that the interval could not be doubled between apojove and perijove and could not be halved between perijove and apojove. Occasionally this did take place and in each case on the subsequent check the computer reversed its previous decision thus wasting machine time as in both cases this involved the recalculation of thirty integration steps.

Each time the satellite crosses the x-axis from negative to positive the computer adds one to the revolution counter and prints out the number of revolutions which the satellite has completed about Jupiter.

Usually every fourth time but sometimes every eighth time, according to the initial data, that the interval is checked, the computer punches out fifteen results. These are -

the time in Jupiter years since the integration was begun,

the Sun's mean anomaly in radians,

the radius vector of the satellite's orbit,

the six jovicentric coordinates and velocity components of the satellite,

the osculating jovicentric elements of the satellite orbit.

These results are always punched out at apojove and perijove as well as elsewhere.

This gives about six to eight sets of results per satellite revolution; each satellite revolution being about 1,000 integration steps. The results are normally printed out to eight significant figures except at the end of a machine run when they are printed out, along with a few more results, to twelve significant figures to be used as data on a subsequent machine run.

After each interval check the computer prints out whether it has halved or doubled the interval or kept it the same. It also prints out "perijove" or "apojove" as appropriate when the radius of the satellite's orbit has a stationary point.

2.6 Jovicentric Elements

The following elements are used to define at a given instant the osculating elliptical orbit of a satellite about Jupiter (see fig. 2.1)

a - the semi-major axis of the orbit in Jupiter units

e - the eccentricity of the orbit

i - the inclination of the orbital plane to the plane of the Sun's orbit about Jupiter.

Ω - the longitude of the ascending node of the orbit on the plane of the Sun's orbit about Jupiter, measured from the positive direction of the x-axis

ω - the argument of perijove.

τ - the time of perijove passage

These elements are calculated from the coordinates and velocity components of the satellite via the following equations (see e.g. ref. 2.1. p.23)

$$c_x = y z' - z y' \tag{2.6.1}$$

$$c_y = z x' - x z' \tag{2.6.2}$$

$$c_z = x y' - y x' \tag{2.6.3}$$

$$h^2 = c_x^2 + c_y^2 + c_z^2 \tag{2.6.4}$$

$$p = \frac{h^2}{Gm} \tag{2.6.5}$$

$$r^2 = x^2 + y^2 + z^2 \tag{2.6.6}$$

$$\Omega = \tan^{-1}\left(\frac{-c_x}{c_y}\right) \tag{2.6.7}$$

where $\cos\Omega$ has the sign of c_y

$$i = \tan^{-1}\left\{\left(\frac{h^2 - c_z^2}{c_z^2}\right)^{\frac{1}{2}}\right\} \qquad (2.6.8)$$

$$e^2 = 1 + \left(\frac{x'^2 + y'^2 + z'^2}{Gm} - \frac{2}{r}\right)p \qquad (2.6.9)$$

$$a = \frac{p}{1-e^2} \qquad (2.6.10)$$

$$x_N = x\cos\Omega + y\sin\Omega \qquad (2.6.11)$$

$$y_N = -x\sin\Omega\cos i + y\cos\Omega\cos i + z\sin i \qquad (2.6.12)$$

$$\omega + v = \tan^{-1}\left(\frac{y_N}{x_N}\right) \qquad (2.6.13)$$

where $\cos(\omega + v)$ has the sign of x_N

$$\tan v = \sqrt{\frac{p}{Gm}}\left(\frac{xx' + yy' + zz'}{p - r}\right) \qquad (2.6.14)$$

where cos v has the sign of (p - r)

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{v}{2} \qquad (2.6.15)$$

$$M = E - e \sin E \qquad (2.6.16)$$

$$n^2 = \frac{Gm}{a^3} \qquad (2.6.17)$$

$$t - \tau = \frac{M}{n} \qquad (2.6.18)$$

where v is the satellite's true anomaly, n its mean angular motion and t is the time since the integration began.

These formulae take i to be measured from 0° to 180°, inclinations less than 90° referring to direct satellites and inclinations greater than 90° referring to retrograde satellites. It was thought to be simpler to consider only inclinations less than 90°, direct and retrograde, in which case the relation below equation (2.6.7) must be reversed for retrograde satellites. The program takes this into account.

If e becomes greater than unity so that the program would fail at equation (2.6.15) the computer misses out equations (2.6.15-18) and prints out the comment "eccentricity".

2.7 Checks on the Program

Before production runs were done with the program two temporary modifications were made to it to check the accuracy of the method and to find any outstanding programming errors which might be present.

Firstly the program was modified to make the Sun move in a circular orbit about Jupiter so that the Jacobi integral could be used as a check on the accuracy of the calculations. The orbit of a satellite in the vicinity of the group (i) satellites was integrated using this modified program for sixty integration steps with a step length of 8.0128 x 10^{-5}. The initial value of the constant in the integral was

4.08728109

and its values after 30 and 60 of the above steps were

4.08728111 and

4.08728112

respectively, while after 30 double length steps the value was

4.08728110

The same integration was performed using an ordinary Taylor's series expansion using the same step length and including terms involving the fourth derivative of the coordinates. The interval was chosen in the first place so that subsequent terms in the expansion were negligible. The equations involved are given in Appendix I.

The value of the constant after 30 Taylor series steps was

4.08728114

The Taylor's series method was very cumbersome but did serve as a check on the other program. The coordinates and velocities calculated by the two methods after thirty steps agreed to within a few units in the eighth significant figure.

This work was done on the Deuce computer which worked at about one five-hundredth of the rate of Kidsgrove Algol. It was therefore not possible to perform the checks after a much longer period of integration than described above. It was not thought worth continuing this work on K.D.F.9 as it was felt that check No.2 was a more satisfactory check of the program. One of the disadvantages of the Jacobi integral check is that the change in the value of the Jacobi constant after a given number of steps is no indication of the size of the corresponding errors in the coordinates, velocities and osculating elements after the same number of steps. In fact it was noticed in developing the program that programming errors which gave rise to relatively large errors in the calculated coordinates affected the value of the Jacobi constant to a much lesser extent.

The second check which was performed on K.D.F.9 using Whetstone Algol consisted of ignoring the Sun altogether so that what the program did was to calculate the successive positions etc. of a satellite moving under the influence of Jupiter alone. It would then be expected that the joviocentric elements of the satellite would remain constant throughout the integration.

As a further check on the program a satellite orbit with a relatively high eccentricity (0.6) was chosen so that the program would require to alter the size of the integration step frequently.

The initial elements chosen were

$$\Omega = 0$$

$$i = 5.2220\ 2086\ 322 \times 10^{-1} \text{ radians}$$

$$e = 5.9999\ 9919\ 561 \times 10^{-1}$$

$$a = 7.8741\ 2539\ 583 \times 10^{-3}$$

$$\omega = -3.1415\ 9265\ 360 \quad \text{radians}$$

$$\tau = 1.1306\ 2985\ 728 \times 10^{-2}$$

After 26 revolutions of the satellite about Jupiter (i.e. about 26,000 steps of integration) the values of the elements were

$$\Omega = -6.5600\ 6831\ 833 \times 10^{-11}$$

$$i = 5.2220\ 2086\ 440 \times 10^{-1}$$

$$e = 5.9999\ 9919\ 301 \times 10^{-1}$$

$$a = 7.8741\ 2543\ 852 \times 10^{-3}$$

$$\omega = -3.1415\ 9255\ 935$$

$$\tau = 1.1306\ 2968\ 873 \times 10^{-2}$$

In no case did the change in the elements, due presumably to rounding off and truncation errors reach the seventh significant figure. None of the satellite orbits in the real case will be integrated for more than a hundred revolutions and it will be assumed throughout that the jovicentric elements are accurate to five significant figures. In view of the above check this assumption would appear to be justified.

2.8 Heliocentric Elements

It will also be necessary to compute the heliocentric coordinates and elements of a satellite which has escaped from Jupiter from its jovicentric coordinates. Another program was written in Whetstone Algol to do this and is described in section 5.1.

References in Chapter II

2.1 Kovalevsky, J. Thèses présentées a la Faculté des Sciences de l'Université de Paris, 1959.

2.2 Allen, C. W. Astrophysical Quantities 2nd. edition p.142, 1963.

2.3 Smart, W. M. Foundations of Astronomy p.89, 1941

2.4 (a) Modern Computing Methods, National Physical Laboratory p.88, 1961

(b) De Vogelaere, R. A. method for the numerical integration of differential equations of second order without explicit first derivatives. Journal Res. Nat. Bur. Stand. Vol.54, 1955.

2.5 Smart, W. M. Spherical Astronomy, 1956.

CHAPTER III

The Starting Values and Mirror Configurations

3.1 The Real Satellites

It will be necessary to compute the orbits of hypothetical jovian satellites in orbits with the mean elements of the satellite groups (i) to (iii) of section 1.1. In this section it will be explained how the mean elements of the group (i) satellites were obtained and in section 3.2 it will be explained how those for the group (ii) and group (iii) satellites were obtained.

The elements for satellites VI and VII were taken from the "Explanatory Supplement to the Astronomical Ephemeris" (ref. 3.1) and are due to Bobone (ref. 3.2). The elements for satellite X are due to Lemechova (ref.3.3).

The elements i, e, n were as follows; n replacing a as an element through equation (2.6.17)

	i	e	n
VI	28°.436	0.15798	1.43675
VII	27°.750	0.20719	1.38647
X	28°.805	0.10739	1.38879

n being measured in degrees per day.

The inclinations in the cases of satellites VI and VII are with respect to the plane of Jupiter's orbit about the Sun but in the case of X the inclination is with respect to the ecliptic. Fig.3.1 shows the great circles representing the planes of the orbits of the satellite and the Sun on a jovicentric celestial sphere.

N' and N are the ascending nodes of the two planes on the ecliptic and Υ is the direction of the First Point of Aries. To find the inclination of the orbit of Jupiter X to the plane of Jupiter's orbit about the Sun the spherical triangle $NN'M$ must be solved for the angle M. In the triangle $NN'M$

angle N' is known $(=180^\circ - 28\overset{\circ}{.}805)$

angle N is given by Lemechova to be equal to $1\overset{\circ}{.}30614$

arc NN' is given by the difference in the longitudes of the two ascending nodes i.e.

$$NN' = 99\overset{\circ}{.}92939 - 80\overset{\circ}{.}73580 = 19\overset{\circ}{.}1936$$ using Lemechova's figures.

Thus in the spherical triangle two angles and the included side are known. Thus by the dual of the cosine formula of spherical trigonometry the angle opposite the known side may be obtained. This formula is obtained from the cosine formula given e.g. by Smart(ref.3.4) by substituting for each side in the formula the supplement of the opposite angle and for each angle the supplement of the opposite side. See e.g. ref. 3.5.

Applying the cosine formula to triangle $NN'M$

$$\cos m = \cos n \cos n' + \sin n \sin n' \cos M \quad (3.1.1.)$$

in the usual notation. Hence the dual of (3.1.1) is given by

$$-\cos M = \cos N' \cos N - \sin N' \sin N \cos N'N \quad (3.1.2.)$$

giving $M = 27\overset{\circ}{.}5748$

The average values of $i, e, n,$ for the three satellites were

$$i = 27\overset{\circ}{.}92$$

$$e = 0.1572$$

$$n = 1{\cdot}404 \text{ degrees/day}$$

$$= 106{\cdot}168 \text{ radians/Jupiter year}$$

From the mean value of n the corresponding semi-major axis was calculated from (2.6.17) giving

$$a = 1.4954 \times 10^{-2} \text{ Jupiter units.}$$

These three values of a, e, i were used to define the initial osculating elements of an orbit in the mean position of the group (i) satellites. The corresponding coordinates and velocity components were calculated from the following equations, taking the satellite to be initially at perijove and on the line Sun-Jupiter

$$x_0 = a(1-e) \tag{3.1.3}$$

$$y_0 = 0 \tag{3.1.4}$$

$$z_0 = 0 \tag{3.1.5}$$

$$x_0' = 0 \tag{3.1.6}$$

$$y_0' = \sqrt{\frac{\mu(1+e)}{a(1-e)}}\cos i \tag{3.1.7}$$

$$z_0' = \sqrt{\frac{\mu(1+e)}{a(1-e)}}\sin i \tag{3.1.8}$$

In this case $x_0 = 1.25986 \times 10^{-2}$

$y_0' = 1.644344$

$z_0' = 0.87137$

3.2 The Retrograde Satellites

The orbits of the three satellites in group (iii) of section 1.1 were treated in a similar manner. The elements of Jupiter VIII were calculated from the satellite's coordinates and velocities as given by Kovalevsky (ref 2.1 p17) using the equations of section 2.6. The elements of Jupiter IX were due to Nicholson (ref.3.6) and those of Jupiter XI were due to Herget (ref 3.7). In the cases of Jupiter IX and Jupiter XI the inclinations were given with respect to the ecliptic and the inclinations with respect to the plane of Jupiter's orbit were calculated as in section 3.1.

The mean elements for the three satellites were

$$i = 22^\circ\!.28$$

$$e = 0.302$$

$$n = 38.14 \text{ radians/Jupiter year}$$

$$a = 2.96081 \times 10^{-2}$$

Initial coordinates and velocities were calculated as in the previous section. These were -

$$x_0 = 2.06549 \times 10^{-2}$$

$$y_0' = -1.42636$$

$$z_0' = -0.58441$$

Initial coordinates and velocities and velocities were also calculated for Jupiter XII from its elements as given by Herrick (ref.3.8). These were

$$x_0 = 2.26534 \times 10^{-2}$$
$$y_0' = -1.17627$$
$$z_0' = -0.74902$$

3.3 Mirror Configurations

If a satellite integration is begun when the satellite is at a "mirror configuration" (see section 1.2) then the path of the satellite in the future will be the mirror image about some line or some plane of its path in the past. One advantage of starting off a satellite in such a configuration is that by computing its orbit for so many revolutions in the sense of increasing time information may also be obtained about the orbit of the satellite for as many revolutions in the past. For example if the satellite should at some time in the future escape from Jupiter then it may be deduced that at a corresponding time in the past the satellite must have been captured by Jupiter. In this way information may be obtained about the circumstances of capture of asteroids by Jupiter from the numerical integration of jovian satellites which eventually escape from Jupiter.

The particular mirror configuration which will be used a great deal will be to have the Sun at perijove and to begin the integration with the satellite on the line Jupiter-Sun i.e. on the positive half of the x-axis, with its velocity vector perpendicular to this line. It is easily seen that this configuration of the three bodies satisfies the conditions for a mirror configuration as given in section 1.2 and that the past behaviour of a satellite whose integration is begun at this particular mirror configuration is the reflection about the x-axis of its future behaviour.

3.4 Semi-Mirror Configurations

Useful though the idea of a mirror configuration may be it would appear to have some limitations. It is arguable, for instance, whether any new information is gained by deducing the past path of a satellite from its future path and, in considering the capture of asteroids by Jupiter it is a considerable restriction to be able to consider only those asteroids which after capture will sooner or later escape from Jupiter again, to become asteroids in exactly similar orbits to the ones in which they were in previously. For these reasons the idea is now introduced of a "semi-mirror configuration". No attempt is made to generalise the notion as Roy and Ovenden have done (section 1.2) for mirror configurations but one example of such a configuration which will be used later is given.

Suppose that the Sun is at perijove and the satellite is situated on the line Jupiter-Sun but now suppose that the velocity vector of the satellite makes an angle <u>not</u> equal to 90° with this line. The three bodies are then said to form a "semi-mirror configuration".

To integrate the past path of the satellite it would be necessary to reverse the velocity vector of the satellite and to make the Sun move in the opposite direction about Jupiter. However an exactly similar orbit to this (though the values of y and y' would have the opposite sign) may be obtained by reflecting the velocity vectors of the satellite and the Sun (for the "past" orbit) about the Sun-Jupiter line. The Sun would now move in the usual direction about Jupiter and the initial coordinates of the satellite would be the same as for the forward integration except that the satellite's velocity vector would be reflected about the line in the plane of the Sun's orbit about Jupiter passing through the satellite and perpendicular to the Sun-Jupiter line. Thus by making the transformation

$$x_1 = x_0$$
$$y_1 = y_0 = 0$$
$$z_1 = z_0 = 0$$
$$x_1' = -x_0'$$
$$y_1' = y_0'$$
$$z_1' = z_0'$$

to the original starting values x_0, y_0, z_0, x_0', y_0', z_0' the orbit corresponding to the past behaviour of the satellite may be obtained. By integrating the two "semi-mirror" orbits it is possible to obtain information about asteroids before and after capture by Jupiter.

References in Chapter III

3.1 Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, p.358 Her Majesty's Stationery Office 1961.

3.2 (a) Bobone, J. Astronomical Journal, vol.44, p.165, 1935.

(b) Bobone, J. Astronomical Journal, vol.45, p.189, 1936.

3.3 Lomochova, E. N. Bull. Inst. Theor. Astronomy Leningrad, vol.VIII No.2, 1961

3.4 Smart, W. M. Spherical Astronomy, C.U.P., p7,1956

3.5 Todhunter, I. and Leathem, J. G., Spherical Trigonometry p.14, Macmillan, 1960.

3.6 Nicholson, S. B. Astrophysical Journal, vol.100 p.57, 1944.

3.7 Herget, P. Publications of the Astronomical Society of the Pacific, vol.50, p.347, 1938.

3.8 Herrick, S. Publications of the Astronomical Society of the Pacific, vol.64, p.238, 1952.

CHAPTER IV

STABILITY OF ORBITS

4.1 Nomenclature

In this chapter the stability of orbits with various initial osculating elements will be considered. All the integrations will be started at the mirror configuration described in section 3.3 with the elements Ω, ω, τ initially zero. The object therefore will be to consider the effect of varying the elements a, e, i on the stability of an orbit.

It will be convenient to refer to an orbit by its initial values of a, e, i and the expression x/y/z will refer to the orbit with initial semi-major axis, $x.10^{-2}$ Jupiter units, initial eccentricity, y and initial inclination to the plane of the Sun's orbit about Jupiter z degrees. In fact the inclination used will often be $22^{\circ}.28$, the mean inclination for the group (iii) satellites, in which case the orbit of the satellite will be referred to by the expression x/y where x is the semi-major axis as before and y is the eccentricity.

In addition an orbit may be referred to by number corresponding to the starting values of its elements e.g. SV 22. This number may be used to find the initial elements and lifetime of the satellite in terms of revolutions about Jupiter from Appendix II. Generally speaking the SV number is an indication of the order in which the integrations were performed on the computer.

A satellite orbit is said to be stable with respect to escape from Jupiter if the satellite is able to complete fifty revolutions about Jupiter without escaping from the planet. A satellite is said to escape from Jupiter the moment its osculating jovicentric eccentricity becomes greater than unity. It is realised that a satellite could momentarily have an orbital eccentricity greater than unity and still not actually escape from the vicinity of Jupiter but this was never seen to happen in any of the integrations performed and it is felt that the above definition of escape from Jupiter is a reasonably satisfactory one. Each integration of an escaping satellite was continued until the satellite was at least one Jupiter unit from Jupiter so that there was no doubt that it was no longer moving predominately under the influence of the planet.

4.2 Comparison of Direct and Retrograde Orbits.

Table 4.1 compares the relative stability of the orbits of direct and retrograde satellites. Eight semi-major axes were chosen and the orbits of satellites with these initial semi-major axes and with the usual inclination were computed both in the direct and in the retrograde sense and with eccentricities 0 and 0.3.

As would presumably be expected the stability of a satellite orbit decreases with increasing semi-major axis. It is also to be noticed that retrograde satellites always remained in orbit about Jupiter for at least as great a number of revolutions and usually a greater number than the corresponding direct ones. This is broadly in agreement with Chebotarev's results summarised in section 1.10. It is of particular interest to note that when the orbit of the satellite corresponding to the mean of the group (iii) satellites but direct instead of retrograde is computed, the satellite escapes from Jupiter after 29 revolutions about the planet. This might have been expected from Moulton's paper summarised in section 1.8 and would certainly be consistent with the view that retrograde orbits at this distance from Jupiter are more stable than direct ones.

Not many orbits were found to be such that the satellite escaped from Jupiter after between three and fifty revolutions but the fact that a few such orbits were discovered suggests that it was rather premature of Chebotarev and others(section 1.10) to conclude that certain satellite orbits were stable after the integration of only a few satellite revolutions about Jupiter.

From table 4.1 it might be concluded that approximately circular orbits about Jupiter in the direct sense are stable out to a distance of 3.37×10^{-2} Jupiter units from Jupiter while retrograde orbits are stable out to a distance of 5.0×10^{-2} Jupiter units. This is perhaps not strictly the case as can be seen from table 4.2 where it is shown that the mean distance of some distant retrograde satellites from the planet seems to decrease after a number of revolutions. It would seem that the solar perturbations can be such as to bring a distant satellite closer in to the planet, a mechanism which may be extremely useful in explaining the present positions of the outer satellites of Jupiter.

The value of 3.368 for one of the semi-major axes in table 4.1 corresponds to the commensurability of 1/5 and the value 3.774 to the commensurability 1/4. The value 3.908 does not correspond to any commensurability but is sufficiently close to 3.774 to provide a comparison between adjacent commensurable and non-commensurable orbits. The larger values of the semi-major axis have no particular significance. It would appear that no conclusion can be drawn from table 4.1 to suggest that commensurable orbits are more or less stable than adjacent non-commensurable ones.

4.3 Effect of Eccentricity

It may also be seen from table 4.1 that a satellite with initial eccentricity zero always has at least as long a lifetime (in terms of numbers of revolutions about Jupiter) as the corresponding satellite with eccentricity 0.3. Chebotarev (see section 1.10) who used eccentricities 0 and 0.5 came to the same conclusion that, for a series of orbits all with the same initial semi-major axis and inclination, the effect of increasing the initial eccentricity of an orbit is to decrease the stability of the orbit.

Table 4.3 shows this more clearly. The first four orbits are <u>direct</u> and their semi-major axes correspond to the mean values for the group (ii) and group (iii) satellites of section 1.1 which are of course <u>retrograde</u> satellites. The appropriate eccentricities for the two groups are 0.15 and 0.30 respectively. It is to be noted that the only satellite which escapes from Jupiter after a relatively short time is the one with both the larger semi-major axis and the greater eccentricity.

The othersixteenorbits are retrograde and the pattern is similar. For both the values of semi-major axis used there appears to be a cut off in stability at an eccentricity between 0.25 and 0.30.

Since the lifetime of a satellite apparently decreases with increasing initial semi-major axis <u>and</u> with increasing initial eccentricity of its orbit it might be thought that the stability of a satellite orbit might merely be a decreasing function of its apojove distance in the initial osculating orbit, given by

$$a_o(1 + e_o)$$

However by calculating values for this expression from the data in tables 4.1 and 4.3 it is easily seen that this is not the case (see table 4.4).

4.4 Effect of Inclination

To investigate the effect of the initial orbital inclination on the lifetime of a satellite it was decided to take one direct and one retrograde satellite, each of which had a lifetime of about ten revolutions about Jupiter, and to investigate the effect of changing the initial inclinations of these satellites on their lifetimes. It was felt that these satellites might be fairly sensitive to changes in inclination.

The retrograde satellite chosen was SV 43 i.e. 5.0/0.25/22 which escaped from Jupiter after 12 revolutions about the planet. Orbits of retrograde satellites with the same semi-major axis and eccentricity and with the following inclinations were computed 5°, 15°, 45°, 60°, 75°; Their lifetimes in terms of revolutions about Jupiter and that of SV 43 were as follows

Inclination($^{\circ}$)	Lifetime
5	>50
15	>50
22	12
45	3
60	3
75	3

It would appear from these results that the smaller the inclination the greater the stability of the orbit and there would appear to be a sharp cut off in stability somewhere between 15 and 45 degrees of inclination. These results would support the conclusions of Goudas (section 1.7) who found in the three-dimensional three-body problem (not elliptical) that orbits with large inclinations were highly unstable.

The direct satellite chosen to investigate the effect of inclination was SV 5 i.e. 3.37/0.3/22 and direct satellite orbits were computed with inclinations 3°, 15°, 28°, 45°, 60°, 75° and with the same semi-major axis and eccentricity as SV 5. Their lifetimes and that of SV 5 were as follows -

inclination(°)	Lifetime
3	3
15	3
22	9
28	9
45	20
60	>50
75	>50

Here the pattern is exactly the opposite to the one for the retrograde satellites. It would appear that for direct satellite orbits those with high inclinations are *more* stable than those with low inclinations which would apparently contradict Goudas.

As a check on this conclusion about direct orbits the orbit of SV 2 i.e. 2.96/0.3/22 was recomputed with an initial inclination of 3° instead of 22°. This integration was given the number SV 54. Whereas SV 2 escaped from Jupiter after 29 revolutions about the planet, SV 54 escaped after only 23 revolutions. This would appear to support the conclusion that for direct orbits with the same semi-major axis and eccentricity the higher the inclination the greater the stability.

4.5 Direct Orbits.

In this section and the next the orbits of some of the satellites, particularly those which remained for a relatively short time in the vicinity of Jupiter, will be discussed in more detail.

Of the initially circular direct orbits SV 8, 3.77/0 was the one with the smallest semi-major axis which escaped from Jupiter after a few revolutions. Fig.4.1 is the projection of this orbit on the x-y plane.

The Sun is on the positive half of the x-axis at the beginning of the integration and its relative position after each revolution of the satellite is given in the diagram in the bottom right hand corner of the figure. In all the diagrams the continuous lines represent the parts of the orbit above the x-y plane and the dotted lines the parts below the plane.

It can be seen from the figure that the orbit is initially almost circular but soon becomes more elliptical and after a fairly close approach to Jupiter the satellite leaves the vicinity of the planet altogether.

SV 24, 4.5/0(see fig.4.2) has a rather similar orbit but in this case the satellite makes a close approach to Jupiter after less than one revolution and then leaves the vicinity if the planet altogether.

SV 34, 6.0/0 (see fig.4.3) never really revolves round Jupiter at all. Clearly there is no point in considering direct initially circular orbits with semi-major axes greater than this.

It seems to be a feature of the manner of escape of direct satellites from Jupiter, that between the beginning of the integration and the moment of escape, the oscillations in the orbital eccentricity should build up and that the satellite should escape from Jupiter after a fairly close approach to the planet. The satellite will usually attain a fairly large apojove distance along with a large orbital eccentricity (about 0.8) about one revolution before escape. It will then come in towards perijove still with this large eccentricity which may even continue to rise after leaving apojove. Once the satellite is within 2.5×10^{-2} Jupiter units from Jupiter its orbital elements are seen to be more or less constant due to the preponderance of the gravitational attraction of Jupiter over that of the Sun at this distance from the planet, so that the satellite's motion is essentially a fixed ellipse about Jupiter. If the satellite reaches this distance from Jupiter with a large orbital eccentricity on its way in from apojove it will therefore be more or less committed to a very close approach to Jupiter at perijove. After this close approach the radius vector of the satellite's orbit will increase again and, since the satellite still has a large orbital eccentricity, it will have a large potential apojove distance. However it would appear that the satellite rarely manages to reach this apojove before the solar perturbations become sufficiently large for it to escape from Jupiter altogether. The same pattern is seen in Chebotarev's work (section 1.10). Fig. 4.4 is one of his integrations which according to the notation here would be 5.8/0.5/0.

Although on these close approaches to Jupiter the satellite would come very close to Jupiter compared for instance with its initial joviocentric distance on no occasion were near collision orbits discovered. On the closest approach of a satellite to Jupiter that was recorded the satellite was at about the distance of the outermost galilean satellite from Jupiter i.e. about 0.0126 astronomical units or about 2.4×10^{-3} Jupiter units. Fig. 4.5 shows typical fluctuations of a and e for a short lived direct initially circular orbit, in this case SV 8.

The inclinations of these short lived direct satellites tended to remain fairly constant. For example in the case of SV 24, 4.5/0, with initial inclination 0.4873 radians, the inclination until just before the moment of escape(after one revolution about Jupiter) always lay between 0.44 and 0.49.

In the case of SV 8, 3.77/0 the inclination always lay between 0.3 and 0.4 until just before escape after two and a half revolutions.

The pattern for short lived direct satellites with initial eccentricity 0.3 was rather similar to that described above except that satellites with this eccentricity always escaped from Jupiter after fewer revolutions than the corresponding ones with eccentricity zero. On a few occasions the integration was begun with the satellite at opposition instead of conjunction with respect to the Sun; for example SV 26, 4.5/0.30 and SV 28, 4.5/0.30(figs. 4.6 and 4.7). However, this distinction apparently made little difference to the lifetime of a satellite which e.g. was three revolutions for both SV 26 and for SV 28.

SV 20, 5.0/0.3 (fig.4.8) provides an interesting case of a closed orbit. It might be tempting to conclude from the integration of a single revolution that the orbit is very stable since it is closed in three-dimensional space. However, since the satellite's velocity will not necessarily be the same after one revolution as it was initially and since the position of the Sun will be different after one revolution of the satellite and consequently its perturbing effect also there is no reason to believe that the satellite will continue in this closed curve in the future. In fact as can be seen from the figure the satellite escapes from Jupiter after one and a half revolutions.

4.6 Retrograde Orbits

Whereas for short lived direct satellites the usual lifetime for an escaping satellite was about three revolutions about Jupiter, for short lived retrograde satellites lifetimes of four, five or six revolutions were much more common. Again the pattern was usually that of a relatively circular orbit becoming more elliptical until after a relatively close approach to Jupiter the satellite would escape from the planet altogether.

Figs. 4.9 to 4.11 show the projections of the orbits of some of these retrograde satellites on the x-y plane. It can be seen from some of the figures that the orbital inclinations of these satellites suffer much greater variations than in the case of the direct ones. For example in the case of SV 21, 5.0/0.3(fig 4.9) with initial inclination 0.487 radians, the inclination varies between 0.386 radians and 1.0 radians in the first three revolutions of the satellite about Jupiter.

Another way in which the retrograde orbits differ from the direct ones is the rate at which the line of nodes revolves. The join of a dotted line and a continuous line on any of the figures indicates the direction of one end of the line of nodes. In the case of retrograde orbits the line of nodes can revolve through 180° in about three revolutions of the satellite about Jupiter (see e.g. SV 21, fig.4.9) whereas for direct orbits with a similar lifetime the rate is usually about half of this (see e.g. SV 8, fig.4.1).

These two differences between direct and retrograde orbits in the rate of movement of the nodes and the changes in the inclinations should not have been unexpected. What have been compared are direct and retrograde satellites with similar lifetimes about Jupiter, namely 3 or 4 revolutions. As has already been seen the retrograde satellites concerned are in orbits which are much further from Jupiter than the direct ones and would therefore be expected to be much more perturbed by the attraction of the Sun than the direct ones. As the changes in the inclinations and the movement of the nodes are caused entirely by the Sun's perturbations on an orbit it is not surprising that these effects are greater for the retrograde satellites than for the direct ones. If now the change in the inclination of the direct satellite SV 8, 3.774/0/0.392 rads. with a range in inclination of 0.35 to 0.49 radians is compared with that of the retrograde satellite SV 9, 3.774/0/0.392 radians with a range in inclination of 0.39 to 0.48 radians, both taken over the first two and a half revolutions of the integrations, it is seen that direct and retrograde satellites at the same distance from Jupiter suffer similar changes in their inclinations due to the perturbations of the Sun.

The orbit which suffered the most dramatic changes in its inclination was SV 53, 5.0/0.25/7[illegible] which departed from Jupiter after three revolutions about the planet. Fig.4.12 shows how the inclination of this orbit varied.

4.7 Circular Orbits

In the course of the integrations two almost circular retrograde orbits were discovered. These were SV 37, 4.5/0.15 and SV 39, 5.0/0.15. The projection of SV 39 on the x-y plane is shown in fig.4.13. The integration was continued for fifty revolutions of the satellite about Jupiter but gradually the orbit became less circular. For the first three revolutions of the satellite the radius vector always lay between 4.21 and 4.79 x 10^{-2} Jupiter units which was much more constant than in any of the other integrations. Instead of the radius vector having about one maximum and one minimum per satellite revolution there were ten to twelve apojoves and perijoves in each revolution. After a few revolutions the orbit became less circular and after ten revolutions there were about five apojoves and perijoves per revolution and the radius vector ranged between 3.9 and 5.3 x 10^{-2} Jupiter units. After forty-five revolutions the orbit was much more typical with one apojove and one perijove per revolution and a radius vector ranging from 3.6 to 5.7 x 10^{-2} Jupiter units. The point of interest is the near circularity of the initial part of the orbit and it is conjectured that there may be an orbit in this region which would remain almost circular for an even longer period of time.

The orbit of SV 37 was similar though not so markedly circular.

4.8 Conclusions

The object of this chapter has been to consider which initial osculating elements will give rise to stable satellite orbits about Jupiter under the attraction of Jupiter and the Sun. Various values of semi-major axis, eccentricity and inclination have been considered and a fairly clear picture has emerged.

For direct satellites with initial orbital eccentricity zero and inclination about 30° stability would appear to be possible for semi-major axes up to about 3×10^{-2} Jupiter units while the corresponding figure for retrograde satellites is about 5×10^{-2} Jupiter units. These distances correspond to distances of 1.5×10^{-2} and 3.0×10^{-2} of the most distant direct and retrograde satellites of Jupiter known at present. It seems possible therefore that there may be more distant retrograde satellites not yet discovered but unlikely that there could be any more direct ones beyond the distance of the outermost retrograde satellites from Jupiter.

It is now possible to consider Chebotarev's conclusions(section 1.10) that direct initially circular orbits are stable within the sphere given by

$$R^* = r_1 m_1^{\frac{1}{2}} \qquad (4.8.1)$$

but not outside this sphere and that retrograde initially circular orbits are stable inside Jupiter's sphere of influence (radius R).

In terms of Jupiter units the radii of these two spheres are respectively

$$R^* = 3.1 \times 10^{-2}, \quad R = 6.0 \times 10^{-2}$$

Thus the conclusion regarding the stability of direct orbits seems to be in agreement with the work described here while the one concerning retrograde orbits does not. The reason for the difference would appear to lie in the less stringent conditions which Chebotarev requires for the stability of an orbit, namely the ability of the satellite to perform about three revolutions about Jupiter without escaping from the planet compared, with the fifty revolutions required here and it is felt that the more stringent requirements of this chapter provide a more realistic assessment of the stability of a satellite orbit. Of course the figure of fifty for the number of revolutions which a satellite must complete for its orbit to be stable is still somewhat arbitrary but to have continued the integrations of all the orbits which lasted for fifty revolutions for, say, another fifty revolutions, would have been quite prohibitive of computer time and, as can be seen from Appendix II, the probability of a satellite escaping from Jupiter seems to decrease with the number of revolutions it has performed so that the probability of any of the "stable" satellites escaping from Jupiter after another fifty or a hundred revolutions would seem to be small. An example of this is the fact that in all the integrations performed only one satellite escaped from Jupiter between its 25th and its 50th revolution although more than twenty completed fifty revolutions (see also table 4.5).

The results may also be compared with work done by Kuiper (ref.4.1) who states that the average stability limit for a satellite is $0.5R_A$, where R_A is given by

$$\log \frac{R_A}{a} = 0.318 \log \mu - 0.327 \qquad (4.8.2)$$

where a is the semi-major axis of the planet's orbit and μ is given by

$$\mu = \frac{M_p}{M_\odot + M_p} \qquad (4.8.3)$$

For Jupiter

$$0.5\, R_A = 2.57 \times 10^{-2} \text{ Jupiter units.}$$

Kuiper adds however that since the orbits of retrograde satellites are more stable than those of direct ones and since satellites in eccentric orbits can wander further from their parent planet than the value of their semi-major axis, it is worth looking for satellites out to a distance of 0.75 R_A from the planet. For Jupiter this is a distance of 3.85×10^{-2} Jupiter units. From the results of this chapter this would seem to be a fairly conservative estimate.

Kuiper also points out that a satellite could not remain in orbit at a distance R_A from a planet since such a satellite would not be at rest in a rotating frame. For Jupiter $R_A = 5.14 \times 10^{-2}$ Jupiter units and the results described in this chapter are certainly consistent with the above statement.

In section 4.3 the effect of varying the initial orbital eccentricity of a satellite with given initial semi-major axis and inclination was considered. For both direct and retrograde satellites it was found that the lifetime of the satellite decreased as the eccentricity was increased(at least for eccentricities in the range 0 to 0.3). This is not perhaps surprising when it is realised that a satellite with a larger orbital eccentricity is able to wander further from the planet and will therefore suffer greater perturbations due to the Sun.

Chebotarov came to the same conclusion regarding the greater stability of circular orbits but it is difficult to compare the two sets of integrations since Chebotarov considered only orbits in the plane of Jupiter's orbit about the Sun and the orbits considered here are for the main part inclined to this plane. It is tempting however to deduce from Chebotarov's work that this result holds for eccentricities up to 0.5 (the value Chebotarov used for his elliptical orbits) and possibly for higher values of the eccentricity as well.

The results of section 4.4 on varying the initial inclination are interesting but perhaps not too much inference should be drawn from them. It should be remembered that satellite orbits were chosen which it was felt might be fairly sensitive to changes in their inclinations. Thus the direct satellite chosen for this investigation in no way corresponded to the retrograde one chosen (the semi-major axes were 3.37 x 10^{-2} and 5.0 x 10^{-2} Jupiter units respectively) and it might be dangerous to draw conclusions about the effect of varying the inclinations of satellites with different semi-major axes and eccentricities from the two cases considered. The one piece of corroborative evidence was given by the rerun of the orbit of the direct satellite SV 2, 2.96/0.3 with a smaller inclination and its subsequent earlier escape from Jupiter. To sum up it might be tempting to conclude from the results of section 4.4 that direct satellite orbits are more stable the greater the inclination and that retrograde ones are more stable the smaller the inclination but clearly the subject needs further investigation.

Since the planets in the solar system all revolve round the Sun in the same direction, the orbit of a planet about the Sun being perturbed in its motion by the other planets is more nearly comparable to the case of a direct(rather than a retrograde) satellite of Jupiter being perturbed in its motion by the Sun.

It might therefore be deduced from the results of section 4.4 that planetary orbits with large inclinations to, say, the plane of Jupiter's orbit (the most massive planet), would be more stable and therefore more probably than orbits in or near the plane of Jupiter's orbit. This would appear to contradict Goudas' conclusion of section 1.7 that the solar system would be expected to become "flat" due to the mutual perturbations of the planets and that therefore no conclusions could be drawn about the origin of the system from its flatness.

References in Chapter IV

4.1 Kuiper, G. P., Planets and Satellites p.577, University of Chicago Press, 1961

CHAPTER V

ASTEROID ORBITS

5.1 Calculation of Heliocentric Elements.

In this chapter the nature of the heliocentric orbits of the satellites which escaped from Jupiter will be investigated and the possibility of capture by Jupiter of asteroids will be discussed. It was necessary to write a short program for K.D.F.9 which would calculate the heliocentric elements of an asteroid (or a satellite) from its jovicentric space and velocity coordinates and the corresponding mean anomaly of Jupiter in its elliptical orbit about the Sun. The program was written in Whetstone Algol and the calculation of each set of heliocentric elements required less than one second of computer time.

The jovicentric coordinates of the Sun were calculated by the method of section 2.4 and the Sun's velocity components from

$$X' = \frac{n \sin E}{e \cos E - 1} \qquad (5.1.1)$$

$$Y' = \frac{nb\cos E}{1 - e\cos E} \qquad (5.1.2)$$

obtained by differentiating equations (2.4.1), (2.4.9) and (2.4.10) and using equation (2.6.13). From the jovicentric coordinates and velocity components of the Sun and the asteroid (or the satellite) the heliocentric coordinates and velocity components of the asteroid with respect to parallel and similarly directed axes through the Sun were easily obtained. From these the heliocentric orbital elements of the asteroid(or satellite) were determined by means of equations(2.6.1) to (2.6.13).

In all cases the calculation of the heliocentric elements of an escaped satellite was delayed until the satellite (or asteroid) was at least one Jupiter unit from the planet so that for all practical purposes the body was moving solely under the influence of the Sun. In most cases no attempt was made to continue the orbital computation beyond this point and it is realised that these initial heliocentric elements could be subject to considerable perturbations due to later close approaches of the asteroid to Jupiter. In the next two sections the initial asteroid elements of escaped direct and retrograde satellites will be considered and the problem of further encounters between the asteroid and Jupiter will be discussed later in the chapter.

5.2 Asteroid Elements of Direct Satellites.

In table 5.1 the initial joviocentric elements and the heliocentric elements, calculated soon after escape, of the direct satellites which escaped from Jupiter are given along with the number of revolutions which each satellite completed before escaping from Jupiter. The difference between SV's 26 and 28 is that the integration of one was started with the satellite at conjunction(C) and the other with the satellite at opposition(O).

It is to be noted that the inclinations of the resultant asteroids to the plane of Jupiter's orbit about the Sun are all very small (< 0.05 radians, i.e. $< 3^{\circ}$) despite the fact that the inclinations of the original satellites were sometimes fairly large. SV 46, 3.37/0.3/45, the satellite with the largest inclination which escaped from Jupiter, turns out to have a relatively large asteroid inclination of 0.05 radians or

3°, while SV's 40 and 54, 3.37/0.3/5 and 2.96/0.3/3 which had small satellite inclinations have also small asteroid ones ($<$ 0.005 radians). From the table there would seem to be some correlation between the magnitude of a satellite's inclination and its inclination as an asteroid after escape from Jupiter. In general the asteroid inclinations found would appear to be typical of the inclinations of the real asteroids.

The heliocentric eccentricities of the asteroids are also fairly small ($<$ 0.3) and are again typical of those of the real asteroids.

The heliocentric semi-major axes of the asteroids appear to fall into two groups. From table 5.1 it can be seen that those asteroids with semi-major axes greater than unity appear to have values around 1.42 Jupiter units and those with semi-major axes less than unity appear to have values around 0.73 Jupiter units.

5.3 Asteroid Elements of Retrograde Satellites.

Table 5.2 gives the same information for escaped retrograde satellites as table 5.1 gave for direct ones. The inclinations and eccentricities of these asteroids tend to be rather larger than for the direct ones though they are still relatively small and not untypical of the real asteroids. For the retrograde satellites there seems to be no correlation between a satellite's jovicentric inclination and its heliocentric inclination after escape, as there was for the direct satellites.

The values of the semi-major axes this time tend to cluster around three values, 0.8, 1.18 and 1.79 Jupiter units. Two of the asteroids, SV 29 and SV 35, have elements such that at aphelion they would be outside the orbit of Saturn, semi-major axis 1.833 Jupiter units.

9.4 <u>Asteroid Capture.</u>

It should be noted from tables 9.1 and 9.2 that the asteroids may be divided into four mutually exclusive sets according to the values of their semi-major axes. These are -

(1) Those asteroids with $a < 0.76$ which were all direct satellites

(2) Those asteroids with $0.76 < a < 1.3$ which were all retrograde satellites.

(3) Those asteroids with $1.3 < a < 1.5$ which were all direct satellites

(4) Those asteroids with $1.5 < a$ which were all retrograde satellites.

It is felt that this clustering of "direct" and "retrograde" semi-major axes is not likely to be merely due to chance and that some significance must be attached to it. It should be remembered that all the integrations were begun at the same mirror configuration (see section 3.3) and because of this and the fact that the satellites all escaped from Jupiter after a comparatively small number of revolutions about the planet, it may be inferred that each satellite was captured by Jupiter the same number of revolutions before the start of the integration from a heliocentric orbit with the same a, e, i as the orbit into which it later escaped.

Thus there exist heliocentric orbits with the values of a, e, i given in tables 5.1 and 5.2 from which asteroids may be captured by Jupiter. However since an orbit is not defined uniquely without specifying the values of Ω, ω, τ it does not follow that any asteroid in an orbit with these values of a, e, i will be captured by Jupiter. The asteroid must in addition have appropriate values of Ω, ω, τ in which case it will inevitably be captured by Jupiter for a few revolutions before escaping again into a heliocentric orbit. An asteroid with suitable a, e, i but unsuitable Ω, ω, τ to be captured and pass through a mirror configuration may never attain a suitable configuration with the Sun and Jupiter to be captured or it may be captured and not pass through a mirror configuration, in which case it will not inevitably escape from Jupiter after a few revolutions but could possibly become a permanent satellite of the planet. This latter possibility would appear to be more likely for the case when Ω, ω, τ are close to values which would lead to the asteroid being captured and passing through a mirror configuration.

It would seem reasonable to conclude from the clustering of the values of the heliocentric semi-major axes of the escaped direct and retrograde satellites and the equivalence of pre-capture and post-escape heliocentric orbits for these satellites that there is a correlation between the semi-major axis of an asteroid and its "decision" whether to become a direct or a retrograde satellite if captured by Jupiter. To be more precise it appears that asteroids with semi-major axes in the range $0 < a < 0.76$ or $1.3 < a < 1.5$ are likely to become direct satellites, if captured by Jupiter, while asteroids with semi-major axes in the range $0.76 < a < 1.3$ or $1.5 < a$ are likely to become retrograde satellites if so captured.

The actual capture orbits corresponding to the escape orbits of tables 5.1 and 5.2 will all lead to only temporary capture by Jupiter but it seems likely that small changes in the elements of those orbits could give rise in some cases to permanent captures.

5.5 <u>Manner of Escape of Satellites</u>

For each satellite that escaped from Jupiter the position of the Sun was noted at the moment of escape (i.e. when the satellite's osculating jovicentric eccentricity was unity). From the rectangular coordinates of the satellite with respect of Jupiter the mean anomaly of the satellite was obtained and the difference between this and the corresponding mean anomaly of the Sun was computed to give the angular distance between the Sun and the satellite at the moment of escape. Fig. 5.1 shows this quantity for the direct satellites which escaped. The crosses correspond to satellites which became asteroids with semi-major axes less than that of Jupiter (i.e. those that escaped towards the Sun) and the circles correspond to satellites which became asteroids with semi-major axes greater than that of Jupiter (i.e. those that escaped away from the Sun).

Fig.5.2 gives the same information for the retrograde satellites which escaped from Jupiter. The shaded circles represent satellites which became asteroids only just outside the orbit of Jupiter.

There would certainly seem to be a correlation between the positions of the Sun at the moment of escape and the semi-major axes of the resultant asteroid orbits. The correspondence in the case of the direct satellites is more or less what might have been expected while that for the retrograde satellites would seem to be slightly "out of phase" with what might have been expected. It should be pointed out that unfortunately the computer program did not print out the requisite data for this investigation at the moment of escape, so that the mean anomalies of the Sun and the satellite at the moment of escape had to interpolated from their values printed out a little before and a little after the moment of escape, thus introducing some uncertainty as to their exact values at the moment of escape.

It was of interest to plot the zero-velocity curve from the Jacobi integral of the restricted three-body problem (see sections 1.3 and 1.4) for a typical escaped satellite.

The constant in the integral was calculated for several satellites from the equation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - 2n\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 2G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - C \qquad (5.5.1)$$

which is the Jacobi integral expressed in non-rotating coordinates as given in ref.5.1. m_1 and m_2 are the masses of the Sun and Jupiter and r_1 and r_2 the distances of the satellite (or asteroid) from the Sun and Jupiter respectively, n being the mean motion of Jupiter about the Sun.

The values of C for a number of escaped satellites was as follows

		C
SV 21	5.0/0.3 R	118.77
SV 24	4.5/0 D	119.33

SV 26	4.5/0.3 D	119.28
SV 35	6.0/0 R	118.93
SV 43	5.0/0.25R	118.95
SV 54	2.96/0.3/3 D	121.78

The zero-velocity curves were drawn for C = 119 by the method described by Moulton (ref.5.2). Using bipolar coordinates and changing the units so that n = 1 and the masses become μ and $1 - \mu$ the equation of the zero-velocity curves is given by Moulton to be

$$(1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \mu\left(r_2^2 + \frac{2}{r_2}\right) = C_1 + \mu(1-\mu) \quad (5.5.2)$$

where C_1 is C expressed in the units defined above. The curves were drawn by choosing values for r_1 and solving equation (5.5.2) graphically for r_2 or vice versa. The zero velocity curves for C = 119 are shown in fig. 5.3. It can be seen that, according to the Jacobi integral, a satellite with this value of C is free to escape from Jupiter either towards or away from the Sun but that it cannot move from an orbit inside Jupiter's orbit to an orbit outside that of Jupiter without passing close to Jupiter and therefore possibly being captured, if only temporarily, by Jupiter.

It should be pointed out again (see section 1.4) that, due to the elliptical nature of Jupiter's orbit about the Sun, no long term conclusions should be drawn regarding the possible future path of a satellite or an asteroid based on the Jacobi integral of the restricted three-body problem.

Roth (ref.5.3) has shown that the perihelia of asteroid orbits tend to lie in the direction of the perihelion of Jupiter's orbit. Figure 5.4 (after Roth) shows the distribution of the perihelion longitudes of 1,626 minor planets compared with the perihelion direction of Jupiter's orbit. It can be seen that three quarters of the asteroids have their direction of perihelion within 90° of that of Jupiter.

When the same data is plotted for the thirty satellites which escaped from Jupiter it is found that twenty one of them have perihelia within 90° of Jupiter's perihelion (see fig.5.5).

A probability argument will indicate how likely it is that these results are merely due to chance. Assume that the probability of the perihelion of an asteroid orbit lying within 90° of the perihelion of Jupiter's orbit is 1/2. Then out of n asteroid orbits the probability of r orbits ($r < n$) having their perihelia in this range is given by ${}^nC_r\ (1/2)^n$. Hence it may be calculated that the probability of 21 out of 30 of the escaped satellites having perihelia in this range is about 1.3×10^{-2} and of three-quarters of the 1,626 real asteroids having this property is about $1/10^{90}$. It is therefore assumed that the above results are significant and cannot merely be attributed to chance.

It would be wrong however to assume that the reasons for the two sets of asteroids having their perihelia predominately in this range were necessarily the same. In the case of the real asteroids it would appear that in the course of many revolutions of Jupiter's perihelion direction about the Sun(due to the perturbations of other planets) the perturbations of Jupiter on the asteroid orbits have tended to bring their perihelia into line with that of Jupiter. In the case of the escaped satellites, however, there has been no time for such a mechanism to have had effect since the heliocentric elements of the satellites have been calculated immediately after their escape from Jupiter. The explanation in this case may lie in the particular configuration of the three bodies which existed at the start of each integration(the usual mirror configuration) which was symmetrical about the perihelion direction of Jupiter's orbit.

To investigate this matter further it would be necessary to compute the orbits of satellites of Jupiter with initial perijoves making various angles with that of the Sun and to analyse the distribution of the directions of the perihelia of the satellites which escaped from the planet. If the perihelia were again found to lie predominately in the direction of Jupiter's perihelion then some alternative explanation to the one given above would require to be found.

5.6 Distribution of Asteroids

Fig. 5.6 (after Brouwer ref.5.4) shows the distribution of asteroids with respect to their mean motions about the Sun in seconds of arc per day. The distribution shows the familiar Kirkwood gaps corresponding to mean motions which would be commensurable with that of Jupiter. In particular there is a marked lack of asteroids with mean motions between 550 and 300" per day and Brouwer explains this as being due to the clustering of commensurabilities in this region. The small concentration of asteroids with mean motions about 450" per day (i.e. at the 3/2 commensurability) is explained as being due to interference from nearby commensurabilities . In fact a mean motion of 450" per day corresponds to a heliocentric semi-major axis of 0.762 Jupiter units, so it can be seen that this concentration of asteroids lies between the group (1) and the group (2) asteroids of section 5.4.

This fact would appear to suggest that while asteroids with semi-major axes which would include them in group (1) tend to be captured by Jupiter to become direct satellites and asteroids with semi-major axes which would include them in group (2) tend to be captured by Jupiter to become retrograde satellites, asteroids with semi-major axis about 0.76 Jupiter units which would place them between the two groups would not appear likely to be captured by Jupiter at all. In fact if there is anything significant about the conclusions of section 5.4 regarding the distribution of potential direct and retrograde satellites with regard to their semi-major axes as asteroids, it is difficult to imagine what would become of an asteroid on the boundary of two of the groups except that it would collide with Jupiter, which is not thought to be very probably, or that it would not be captured by Jupiter at all but remain in its heliocentric orbit.

While not disputing the accepted explanation for the Kirkwood gaps in the asteroid distribution as being due to commensurabilities, Brouwer's explanation of the lack of asteroids with mean motions between 550 and 300" per day as being due to the clustering of commensurabilities in this region is not felt to be very convincing. It would seem that any asteroids which were initially in this region would suffer such large perturbations by Jupiter as to be sent into completely different orbits, either by being temporarily or permanently captured by Jupiter or by colliding with Jupiter or by being perturbed by Jupiter into "smaller" heliocentric orbits further from the influence of the planet. There is evidence for this not only from the asteroid capture orbits already discussed but also from the asteroid orbits close to Jupiter which are described later in this chapter. There would seem to be no need therefore to explain the lack of asteroids in this region in the way Brouwer has done.

Another possible way of explaining the concentration of asteroids at the 3/2 commensurability is as follows. Suppose that at some time in the past there was an appreciable density of asteroids in the 600 to 300" per day region, say comparable to that in the 850 to 750" per day region, thus giving a roughly gaussian distribution of asteroids over the whole range. In particular the density of asteroids at the 3/2 commensurability would be much greater than at present. According to the previous arguments, the asteroids with mean motions less than 550" per day which attain (after a few revolutions) suitable configurations with the Sun and Jupiter for capture by Jupiter(i.e. had suitable values of λ, ω, τ) would then be captured by the planet. This would be the case for both commensurable and non-commensurable orbits. In the same way some asteroids would be perturbed by Jupiter into "smaller" orbits.

Of the asteroids which were not removed from this region after a few revolutions those in non-commensurable orbits would probably sooner or later achieve suitable configurations with the Sun and Jupiter for temporary or permanent capture by Jupiter or to be sent into "smaller" heliocentric orbits. Those in commensurable orbits, however, would not achieve any new configurations with the Sun and Jupiter after the first few revolutions so unless captured early on could not later be captured by Jupiter or sent into "smaller" orbits ,except in so far as allowance is made for more gradual jovian perturbations changing the elements of the asteroid orbits.

Thus it would be expected that this region of the asteroid belt would in a matter of some hundred Jupiter years, say, be cleared of asteroids except perhaps for a few remaining particularly at the strong commensurabilities. It can be seen from fig. 5.6 that as well as the concentration of asteroids at the 3/2 commensurability there is evidence of a smaller concentration at the 4/3 commensurability.

It should however be pointed out that the argument that non-commensurable orbits satisfy more new configurations with the Sun and Jupiter after one synodic period of the asteroid with respect to Jupiter only holds if the effect of the asteroid's eccentricity and inclination and the eccentricity of Jupiter's orbit are taken into account so that e.g. all oppositions of an asteroid with respect to Jupiter are not identical but depend on the positions of the asteroid and Jupiter in their respective orbits at the time concerned.

5.7 Further Integrations

It would be of interest to examine more closely the effect of jovian perturbations on asteroids with semi-major axes close to unity and also to consider the possibility of an asteroid being temporarily captured by Jupiter to be later sent into a heliocentric orbit between Jupiter and Saturn and, if possible, to find such an orbit.

Search was made for such an orbit by starting a number of satellites at "semi-mirror" configurations (see section 3.4) and performing the integrations with two sets of initial conditions, the one being derived from the other via the transformations of section 3.4, so that effectively the orbit was computed in the sense of increasing and decreasing time without having to alter the program in any way or having to make the step length negative. Values of the initial conditions were chosen so that the satellite would be expected to escape from the planet after a few revolutions in each direction.

For example two satellites, which will be referred to as A and B, were started on the Sun-Jupiter line with the following initial coordinates

		x	y	z
coordinates :	A	3.15×10^{-2}	0	0
	B	3.15×10^{-2}	0	0
velocities :	A	$+9.603 \times 10^{-2}$	-1.098	-0.584
	B	-9.603×10^{-2}	-1.098	-0.584

These starting values are similar to those of the retrograde satellite SV 27, 4.5/0.3 which escaped from Jupiter after four revolutions. Satellite A escaped after ten revolutions and satellite B after seven revolutions. Both become asteroids inside the orbit of Jupiter. Their orbital elements were as follows

	a	e	i(radians)
A	0.795	0.125	0.03
B	0.709	0.323	0.03

The two parts of the integration if taken together provide an example of an asteroid being captured by Jupiter performing seventeen revolutions about Jupiter and then escaping from Jupiter to become an asteroid with rather different heliocentric elements.

The behaviour of satellites C and D which were treated in a similar manner and whose initial coordinates and velocities are given below was rather more complex though this may be due merely to the fact that the integrations were continued for some time after the initial escape of the two satellites.

		x	y	z
coordinates :	C	6×10^{-2}	0	0
	D	6×10^{-2}	0	0
velocities :	C	+0.3502	−0.6065	−0.3711
	D	−0.3502	−0.6065	−0.3711

Satellite D escaped after two revolutions to become an asteroid with the following elements:

a	e	i
1.892	0.453	0.06

and its integration was not continued any further. With such a large heliocentric semi-major axis it would be unlikely to suffer further perturbations by Jupiter.

Satellite C escaped from Jupiter after one revolution to become an asteroid with the following orbital elements

a	e	i
1.314	0.195	0.02

As this asteroid was still in the vicinity of Jupiter's orbit it seemed to be of interest to continue its integration about the Sun to investigate the effect of possible further encounters with Jupiter. The asteroid made a number of further encounters with Jupiter and on the occasions on which it found itself with a jovicentric eccentricity less than unity it will be said to have been temporarily captured by Jupiter. This is consistent with the definitions given to the terms "capture" and "escape" in section 4.1.

After escaping from Jupiter to become an asteroid with the above elements asteroid G made a close approach to Jupiter two and a half Jupiter years later and was captured by the planet. It attained the following values for its perijove distance, jovian eccentricity and jovian semi-major axis

p.d.	e'	a'
7.9×10^{-3}	0.945	1.45×10^{-1}

It then escaped from Jupiter without performing a complete revolution about the planet to become an asteroid with the elements.

a	e	i
0.874	0.01	0.01

Thus an example has been found of an asteroid after one encounter with Jupiter having its semi-major axis changed from 1.31 to 0.874 and clearly the reverse process would be equally possible. In fact it would be possible by combining the orbits of satellites C and D to have an asteroid with semi-major axis 0.874 becoming an asteroid with semi-major axis 1.89 after two encounters with Jupiter.

At asteroid C's next close approach to Jupiter it came to within 0.29 Jupiter units from the planet but was not captured by it. When the asteroid reached a distance of 1.84 Jupiter units from Jupiter immediately after this close approach its heliocentric elements were calculated to be

a	e	i
0.803	0.13	0.01

thus showing that an asteroid's orbital elements can be considerably altered in a single revolution about the Sun due to the perturbations of Jupiter without the asteroid actually being captured by Jupiter.

About nine Jupiter years later the asteroid was again captured by Jupiter to have the following jovian elements.

p.d.	e'	a'
2.36×10^{-2}	0.816	0.128

The asteroid immediately escaped from Jupiter and its new heliocentric elements were calculated to be

a	e	i
1.184	0.035	0.02

On its next close approach to Jupiter the asteroid was again captured and came to within 1.42×10^{-1} Jupiter units of the planet. On immediate escape its asteroid elements were

a	e	i
1.307	0.187	0.02

On its next close approach to Jupiter the asteroid was once more captured and at perijove had the following elements

pd.	e'	a'
1.93×10^{-3}	0.965	5.5×10^{-2}

the perijove distance being small enough to bring the satellite within the orbit of the outermost Galilean satellite. On this occasion the satellite did not immediately escape from Jupiter but went through two more perijoves before escaping. The elements and distances at the perijoves and apojoves were as follows

p.d.	a.d.	e'	a'
1.93×10^{-3}		0.965	5.5×10^{-2}
	7.87×10^{-2}	0.999	3.9×10^{-2}
8.85×10^{-3}		0.740	3.4×10^{-2}
	8.04×10^{-2}	0.920	4.2×10^{-2}
7.55×10^{-3}		0.871	5.9×10^{-2}

On subsequent escape the heliocentric elements of the asteroid were

a	e	i
1.34	0.190	0.03

The integration was continued for another thirteen revolutions of the asteroid about the Sun and its heliocentric elements were calculated at each apojove. The asteroid was not captured by Jupiter during this period. Table 5.5 gives the values of the elements at each apojove and the value of the perijove distances between each apojove. It is to be noted that the largest changes in the heliocentric elements occur after the closest approaches to Jupiter as would be expected.

By the end of the integration the asteroid appeared to be in a fairly stable orbit and did not seem likely to make any more close approaches to Jupiter due to its relatively large semi-major axis. It would however be possible for successive perturbations by Jupiter to change the orbit sufficiently to allow another capture by Jupiter some time in the future.

As another example of the effect of Jupiter on the orbits of asteroids with semi-major axis close to unity the orbit of SV 46, 3.37/0.3/45, which escaped after twenty revolutions about Jupiter was continued for forty-five Jupiter years after escape. The heliocentric elements, always calculated at apojove, after various periods of time, are given in table 5.4. On five occasions the asteroid passed close enough to Jupiter for its joviocentric eccentricity to be less than unity. The sets of elements calculated immediately after each of these close approaches are marked with an asterisk.

5.8 Conclusions.

Clearly it would be desirable to have much more data on the effect of Jupiter on the orbits of asteroids with semi-major axes close to one Jupiter unit. From the results described in this chapter it would appear that asteroids with semi-major axes lying between say 0.75 and 1.4 Jupiter units would be likely not only to have their orbits greatly perturbed by Jupiter but even to be captured temporarily or permanently by or (probably less likely) collide with Jupiter. The results of the integration of the orbits of satellites C and D of the last section would suggest that asteroids close to Jupiter's orbit would be likely to suffer multiple captures by Jupiter until they are removed from the vicinity of the planet altogether. In this way asteroids in this critical region could be dispersed throughout the solar system either into heliocentric orbits well inside that of Jupiter(although no example has been found of this occurring) or into heliocentric orbits well outside that of Jupiter, possibly even as far out as the orbit of Saturn. Occasionally a capture by Jupiter might become permanent due to successive solar perturbations decreasing the initial jovicentric semi-major axis of a captured asteroid until the asteroid found itself in a stable jovicentric orbit. It seems likely that in this way the four outermost satellites of Jupiter were captured from the asteroid belt.

It would seem reasonable to suppose that asteroids in orbits near Saturn could suffer similar encounters with Saturn to those which have been described between asteroids and Jupiter. It has already been assumed that some of the original asteroid belt may now be in orbit near Saturn and presumably such asteroids could now be captured temporarily or permanently by Saturn or, through successive encounters with Saturn, be sent into even further reaches of the solar system. For instance it seems possible that Phoebe, the outermost retrograde satellite of Saturn may have originated in the asteroid belt.

References in Chapter V

5.1 Brouwer, D and Clemence, G. M. Methods of Celestial Mechanics p.255, Academic Press, 1961.

5.2 Moulton, F. R. Celestial Mechanics p.288, Macmillan 1914.

5.3 Roth, G. D. The System of Minor Planets p.90, Faber, 1962.

5.4 Brouwer, D. Astronomical Journal Vol.68, p.152, 1963.

CHAPTER VI

A PERIODIC ORBIT

6.1 Delaunay Series.

In this chapter an attempt will be made to find a periodic orbit in the vicinity of the group (i) satellites of Jupiter. The method used will involve discussion of methods of determining the rates of movements of the apses and nodes of jovian satellites. In this section one such method will be considered.

Delaunay (ref.6.1) has derived series for the calculation of the mean rate of movement of the Moon's node and its apse line from the mean values of its mean motion, its eccentricity and its inclination. One series gives the mean rate of change of the longitude of the ascending node and the other the mean rate of change of the longitude of perigee. The series are in powers of the ratio of the mean motion of the Sun to the mean motion of the Moon (m), the eccentricity of the Sun's orbit about the Earth (e'), the eccentricity of the Moon's orbit about the Earth (e) and the sine of half the inclination of the Moon's orbit to the plane of the Sun's orbit about the Earth(γ). Up to seventh order of these small quantities are included.

Bobone (ref.6.2) has derived values for these rates for the orbits of Jupiter VI and Jupiter VII from his theory of the motions of these satellites by treating the rates as two of the unknowns to be determined from the observations. Proskurin(ref.6.3) has used the Delaunay series to recalculate the rates for Jupiter VI (see table 6.1). Proskurin gives the coefficient of the term in $\gamma^2 m^3$ in the series for the rate of movement of perijove as 189/9 compared with Delaunay's value of 189/8(ref.6.1). This would appear to be a misprint, as Proskurin's value for this rate is more nearly in agreement with his having used Delaunay's value of the coefficient rather than his having used the value given in his paper (see table 6.1.).

Lemechova (ref.6.4) has determined new values for the elements of Jupiter X and from them has derived the rates of movement of its apse and node using the Delaunay series, reference being made to both Proskurin and Delaunay. However when the rates are recalculated using both the Proskurin and the Delaunay versions of the series for the rate of movement of perijove the value obtained using the series given by Proskurin (including the misprint) is more nearly in agreement with Lemechova's result than the value obtained using the original Delaunay series, thus suggesting that Lemechova used the incorrect series given by Proskurin instead of the original Delaunay series (see table 6.2).

Brown (ref.6.5) pointed out that the coefficients of e^2m^6 and e^2m^7 in the Delaunay series for the rate of movement of perijove were in error. These terms, however, make only a small contribution to the rate for the group (i) satellites of Jupiter. Brown has also discussed (ref.6.6) the application of the Delaunay series to the orbit of Jupiter's eighth satellite (a group (iii) satellite) where the various small quantities are considerably larger than in the case of the Moon or the Group (i) satellites of Jupiter.

6.2 Determination of Rates of Movement of Apses and Nodes from the Numerical Integrations.

In this section a method of determining the rates of movement of the apses and nodes of jovian satellites in the region of the group (i) satellites from the numerical integration of the orbits of such satellites will be described, and the effect of varying the initial elements of such an orbit on the resultant values of its rates will be considered.

λ will be defined to be the mean rate of change of the longitude of a satellite's ascending node on the plane of the Sun's apparent orbit about Jupiter and μ will be defined to be the mean rate of change of the satellite's argument of perijove (the argument of perijove being the angular distance between the ascending node and the direction of perijove on a jovicentric celestial sphere). The Delaunay series therefore give λ and $\lambda + \mu$ from which λ and μ can easily be obtained.

For Jupiter X Lemechova (ref.6.4) gives the following values for λ and μ, viz.

$$\lambda = -0.25341$$

$$\mu = 0.57327 \text{ or } 0.56983 \text{(corrected value)}$$

in the usual units. It should be noted that for all satellites in the region of the group (i) satellites μ is about double λ and is positive whereas λ is negative. Further reference will be made to these points in section 6.4.

All the orbits discussed in this chapter were begun at the usual mirror configuration i.e. with Λ, ω, τ all initially zero. It should be possible therefore to calculate approximate values for λ and μ at any stage of an integration merely by dividing the current values of $\mathcal{L}$ and ω by the time, t, which has elapsed since the beginning of the integration. Suppose that at any stage of the integration the value of $\mathcal{L}$ is given by

$$\mathcal{L} = \lambda t + \text{periodic terms} \qquad (6.2.1)$$

then presumably the greater the value of t the greater the accuracy to which λ may be determined by this method. In fact none of the integrations were continued much beyond one hundred revolutions of the satellite about Jupiter and it can be seen from fig.6.1 that around the hundredth revolution of the satellite the value of λ calculated by the above method varied considerably from one revolution to the next due to the effect of the periodic terms. The problem therefore was to discover some method of using the values of $\mathcal{L}$ and ω over the first hundred revolutions to determine as accurately as possible the values of λ and μ.

As has been already seen the values of λ and μ calculated towards the end of an integration would be expected to be more accurate than those calculated near the beginning. It was therefore decided to use the values of Ω and ω computed at perijove over about the last thirty revolutions of each integration to determine λ and μ, which would be given by the weighted mean of the values of Ω/M and ω/M over this period (M being the mean anomaly of the Sun at the appropriate instant plus a suitable multiple of 2π making it a measure of the time since the start of the integration).

Before deciding exactly which part of each integration would be used for this calculation graphs were drawn of the variations of Ω/M and ω/M over the last thirty or forty satellite revolutions computed (see fig.6.1). From each graph a rough mean line could be drawn i.e. a line parallel to the time axis with roughly equal areas of the curve above and below it. A length of the curve was then chosen which was roughly symmetrical about this line and the values of Ω/M and ω/M corresponding to this length of the curve were used to calculate λ and μ. The reason for this was to try and remove any "local property" of the range of the integration chosen for this purpose.

The weighted mean of the Ω/M's was obtained by multiplying each value of Ω/M by the appropriate value of M and by dividing the sum of the quantities so obtained by the sum of the M's i.e. each value of Ω/M was weighted in proportion to the time since the start of the integration. In this way λ was given by

$$\lambda = \frac{\sum\left(\frac{\Omega}{M}\right).M}{\sum M} = \frac{\sum \Omega}{\sum M} \qquad (6.2.2)$$

and similarly μ was obtained from

$$\mu = \frac{\sum \omega}{\sum M} \qquad (6.2.3)$$

From the Delaunay series it can be seen that, for a series of satellites of a given planet, λ and μ will be functions of $\bar{n}$, $\bar{e}$, $\bar{i}$, the mean values of the mean motion, the eccentricity and the inclination of the various satellite orbits. In all the orbits considered in this chapter the integration was begun at the usual mirror configuration and the only available parameters were the values of n, e, i, at the beginning of the integration, referred to as n_o, e_o, i_o. $\bar{n}$, $\bar{e}$, $\bar{i}$, will therefore be functions of n_o, e_o, i_o and so λ and μ may be expressed as functions of n_o, e_o, i_o, say in the form

$$\lambda = f(n_0, e_0, i_0) \quad (6.2.4)$$

$$\mu = F(n_0, e_0, i_0) \quad (6.2.5)$$

Since initial elements are used on the right hand sides of equations (6.2.4) and (6.2.5) instead of the mean elements f and F will not be given exactly by the Delaunay series, though for the orbits considered n_0, e_0, and i_0 will not differ by much from $\bar{n}$, $\bar{e}$ and $\bar{i}$ so that f and F will be given approximately by the Delaunay series.

It is now possible to consider the effect of varying the values of n_0, e_0 and i_0 on the values of λ and μ as determined from the numerical integrations. The orbit of SV 4, 1.495/0.157/0.487r, the orbit corresponding to the mean of the group (i) satellites, was integrated for a hundred revolutions about Jupiter. Three other similar orbits(SV's 1,16,17) were integrated over a similar period, each orbit differing from SV 4 by a small amount in one of the parameters. The effect of varying the initial elements on the values of λ and μ is shown in table 6.3.

6.3 Applications of the Method

As has already been mentioned Lemechova (ref.6.4), has determined the values of λ and μ for Jupiter X using the Delaunay series with new values for the mean elements of the satellite. As a check on the method of section 6.2 for determing the values of λ and μ from the numerical integrations it was decided to compute the orbit of Jupiter X using as initial elements the mean elements given by Lemechova and to compare the values of λ and μ obtained from the numerical integration with those given by Lemechova. The results were as follows -

from numerical integration $\lambda = -2.5004 \times 10^{-1}$

$\mu = 5.5086 \times 10^{-1}$

Lemechova (after correction) $\lambda = -2.5341 \times 10^{-1}$

$\mu = 5.6983 \times 10^{-1}$

There are three ways in which the discrepancy between the two sets of figures may be explained.

(i) Possible errors caused by not considering further terms in the Delaunay series used in obtaining Lemechova's results. This is not thought to be a major factor as the smallest terms used in the series hardly contribated to the result.

(ii) Errors in obtaining the values of λ and μ from the numerical integration. Although every attempt was made to eliminate the periodic terms contributing to λ and μ it is not to be expected that the effect of some long period terms was completely removed. In particular, since the period of the apse line is seen from the numerical integrations to be equivalent to about 400 satellite revolutions, it would be expected that λ would have a periodic term about this length.

(iii) The fact that in the numerical integration of the satellite's orbit the mean elements of Jupiter X were used as starting elements so that the computed orbit would have slightly different mean elements from those used by Lemechova in the Delaunay series.

To investigate further the effect of factor (iii) it was desirable in the case of one of the integrations to compare the values of λ and μ computed by the method of section 6.2 with their values obtained from the Delaunay series using the attained mean values of the elements from the numerical integration. This was done for SV 56, 1.49/0.159/0.53r. The initial and mean elements of this orbit and the values of λ and μ calculated by the two methods were as follows

	n	e	i
initial	106.814	0.1590	0.5310
mean	107.079	0.1621	0.5119

	λ	μ
computed by method of section 6.2	−0.2590	0.5234
Delaunay series using mean elements	−0.2570	0.5354

In these results the effect of factor (iii) has now been more or less eliminated and the remaining discrepancy between the two sets of values for λ and μ is thought to be mainly due to the effect of periodic terms of long period. It is likely too that the mean values for n, e, i are still in error by a small amount due to the effect of long period terms. The main reason for the discrepancy therefore is taken to be the relatively short period of the integration. Unfortunately to have continued the integration for say a thousand revolutions of the satellite would have been quite prohibitive of computer time.

As the difference between the two sets of values for λ and μ is rather smaller than the two sets given above for Jupiter X, which included the effect of factor (iii), it is felt that in the results for Jupiter X the effect of factor (iii) was appreciable.

6.4 A Periodic Orbit

It has already been suggested (see section 1.2) that the preference for near-commensurable satellite orbits might be due to the fact that these orbits are close to periodic orbits and will therefore be relatively stable orbits. If a periodic orbit could be found in the region, say, of the group (i) satellites of Jupiter and it could be shown that periodic orbits were more likely to occur at commensurabilities than elsewhere then this preference for near-commensurable orbits might be explained, it being assumed that any satellite in an orbit near the group (i) satellites would tend to be perturbed into an orbit of greater stability in the course of time. In the following argument the various conditions required for a satellite to satisfy two successive mirror configurations and hence be in a periodic orbit will be discussed.

Suppose first of all that the Sun moves in a circular orbit about Jupiter. Then a satellite moving in a circular orbit about Jupiter in the plane of the Sun's apparent orbit about Jupiter will satisfy a mirror configuration with the Sun and Jupiter twice in each synodic period of the satellite with respect to the Sun i.e. each time it crosses the Sun-Jupiter line. If however the eccentricity of Jupiter's orbit is taken into account then a satellite started on the Sun-Jupiter line in a circular orbit in the plane of the Sun's apparent orbit about Jupiter when the Sun is at perijove or apojove will not satisfy another mirror configuration with the Sun and Jupiter until it crosses the Sun-Jupiter line when the Sun is again at perijove or apojove. For a satellite whose mean motion is commensurable with that of the Sun this will certainly happen after a small number of revolutions of the satellite about Jupiter and hence such an orbit will be periodic.

Suppose now that the orbit of a satellite moving in the plane of the Sun's apparent orbit about Jupiter, with non-zero orbital eccentricity, is computed with the satellite initially on the Sun-Jupiter line and at perijove to form a mirror configuration with Jupiter and the Sun, which is also at perijove. Suppose too that the ratio of the mean motions of the Sun and the satellite is exactly $1/n$ where n is a small (< 30 say) integer. Then, but for the movement of the apse line of the satellite's orbit, which must of course be taken into account, another mirror configuration involving the three bodies would occur when the satellite had completed n revolutions about Jupiter. Taking into account the movement of the apse line another mirror configuration will occur when the apse line is again along the major axis of the Sun's apparent orbit about Jupiter and the satellite and the Sun are both on this line. If the period of the apse line is an even multiple of the period of the Sun about Jupiter, then for a commensurable orbit these conditions will be satisfied after half of the apse period since this will correspond to an integral number of Jupiter years after which time the Sun and the satellite will certainly be on the apse line of Jupiter's orbit and have velocity vectors perpendicular to it. This implies that the period between the two mirror configurations will be m/2 Jupiter years where m is an even integer and the period of the apse of the satellite's orbit is m Jupiter years.

The case of a satellite in an orbit inclined to the plane of the Sun's apparent orbit about Jupiter is slightly more complicated. Consider the orbit of a satellite, again initially at perijove and on the Sun-Jupiter line with the Sun at perijove to form the usual mirror configuration with the Sun and Jupiter, but this time with a velocity vector not in the plane of the Sun's apparent orbit about Jupiter. Suppose again that the satellite is at the commensurability 1/n. In this case allowance must be made for the movement of the node as well as the movement of the apse of the satellite's orbit between the two mirror configurations. If, for example, the period of the apse was an even number of Jupiter years and the period of the node was a simple simple fraction(say 1/2)of the period of the apse then for a commensurable orbit the second mirror configuration would occur $q/2$ Jupiter years after the first, where q was the period of the apse of the satellite's orbit. After this period the satellite would again lie on the line of nodes which would have rotated through 360° and would again lie on its apse line which would by then have rotated through 180° with respect to the line of nodes. In addition the satellite would once more lie on the Sun-Jupiter line since it was in a commensurable orbit.

As has been seen, the rates of movement of the lines of apses and nodes of the group (i) satellites do in fact almost bear a simple ratio to one another, μ being just about double λ though of the opposite sign. It now remains to see if a periodic orbit exists with suitable values of λ, μ and n (the mean motion of the satellite in its orbit), the three parameters defining the periodic orbit, in the region of the group (i) satellites. Suppose n is taken to be seventeen times the mean motion of

Jupiter (see section 1.2). Then

$$n = 17 \times 2\pi = 106.8141 \text{ radians / Jupiter year.}$$

The values of λ and μ which satisfy the above conditions and are nearest to those computed for the group (i) satellites are

$$\lambda = -0.2617994$$

$$\mu = 0.5235988$$

giving $12\lambda = -\pi$ and $12\mu = 2\pi$ therefore after twelve Jupiter years a satellite with these values of the parameters should satisfy another mirror configuration.

The problem now is to find values of n_o, e_o and i_o which give rise to these values of n, λ and μ. It has already been seen that

$$\lambda = f(n_o, e_o, i_o) \qquad (6.4.1)$$

$$\mu = F(n_o, e_o, i_o) \qquad (6.4.2)$$

Then if Δn_o, Δe_o, and Δi_o represent small changes in the values of n_o, e_o and i_o respectively, then the corresponding changes in λ and μ are given, to the first order by

$$\Delta\lambda = \frac{\partial f}{\partial n_o}\Delta n_o + \frac{\partial f}{\partial e_o}\Delta e_o + \frac{\partial f}{\partial i_o}\Delta i_o \qquad (6.4.3)$$

$$\Delta\mu = \frac{\partial F}{\partial n_0}\Delta n_0 + \frac{\partial F}{\partial e_0}\Delta e_0 + \frac{\partial F}{\partial i_0}\Delta i_0 \qquad (6.4.4)$$

The values of n_0, e_0 and i_0 for SV 4 were taken as standard, it being assumed that the orbit of the mean of the group (i) satellites would lie fairly close to the periodic orbit if it exists. The orbits of SV 1, SV 16 and SV 17 (see table 6.3) were considered as small variations of the orbit of SV 4 and the differences in the values of λ, μ, n, e, i, between each of the orbits and those of SV 4 were used to form three pairs of equations like (6.4.3.) and (6.4.4.) in which the only unknowns were the six partial derivatives. For example SV 1 which only differed from SV 4 in its initial value of i_0 gave rise to equations of the form

$$(\Delta\lambda)_1 = \frac{\partial f}{\partial i_0}\Delta i_0 \qquad (6.4.5)$$

$$(\Delta\mu)_1 = \frac{\partial F}{\partial i_0}\Delta i_0 \qquad (6.4.6)$$

and the other four equations took a similar simple form. The values of the partial derivatives were calculated from these equations and are given in table 6.4.

It was now possible to determine the values of n_0, e_0 and i_0, if any existed, which would give rise to the desired values of n, λ and μ from equations (6.4.3) and (6.4.4). The partial derivatives were now known, Δn_0 was given (approximately) by the difference between the required n and the value of n_0 for SV 4 and $\Delta\lambda$ and $\Delta\mu$ were given by the differences between the required values of λ and μ and those obtained from SV 4. Equations (6.4.3) and (6.4.4) therefore gave Δe_0 and Δi_0, the differences between the initial values of e and i in SV 4 and those in the periodic orbit.

e_0 and i_0 for the periodic orbit were thus found to be

$$e_0 = 0.15905$$

$$i_0 = 0.53115$$

and the value of a_0 corresponding to the sought for value of n_0 was

$$a_0 = 1.4893 \times 10^{-2}$$

The orbit with these initial elements, SV 56, 1.49/0.159/0.531r, was integrated for 110 revolutions about Jupiter and the resultant values of λ and μ were calculated in the usual way. These were found to be

$$\lambda = -0.2590209$$

$$\mu = 0.5233895$$

The situation is summarised in table 6.3. It is to be noted that the values of λ and μ computed for SV 56 were relatively close to the predicted values compared with, say, the differences between the values for SV 4 and those for SV's 1, 16 and 17. This would suggest that the result is significant and that the differences in the λ's and the μ's for the comparison orbits are really due to differences in their initial conditions and are not merely due to the uncertainties in the method by which they were determined. The effect of the long period terms referred to in the last section should not affect this "differential" method of obtaining the periodic orbit too greatly as all the "close" orbits will tend to be "in phase" as far as those are concerned.

It is therefore seen that to an accuracy of 1% or better values of n_0, e_0, and i_0 can be predicted which will give rise to particular values of λ and μ. By choosing the increments in n_0, e_0 and i_0 more carefully in the comparison orbits and by using more comparison orbits and solving the resultant equations by a least squares method it is felt that the partial derivatives could be determined more accurately.

However, in view of the uncertainties of the method due to the relatively short length of the integrations, it is not thought that, without using considerably more computer time, the required values of λ and μ could be attained with much greater accuracy.

What would be required would be to continue the integration of the comparison orbits for at least twenty-four Jupiter years or four hundred revolutions of the satellites, which is the period of the sought for periodic orbit, over which time it ought to be possible to eliminate the long period terms. In addition many more comparison orbits would require to be used. It is felt however that there is some evidence from the results of this section that such an orbit exists and that it could be discovered by a method similar to that described here.

6.5 Differentiation of the Delaunay Series.

As a check on the significance of the results of the last section it was decided to compare the value of $\partial l/\partial n$ as obtained by the differential method described in the last section with the value obtained by partially differentiating the Delaunay series for the rate of movement of the node. The mean values of n, e, i for SV 56 were substituted in the Delaunay expression for $\partial f/\partial n$ which was evaluated to be

$$-2.349 \times 10^{-3}$$

compared with the value according to the differential method of

$$-2.003 \times 10^{-3}$$

This would appear to indicate that the partial derivatives calculated in section 6.4 are reasonably accurate, though it may be that the ones which were determined from comparison orbits with relatively close values of λ and μ were less well determined. $\partial f/\partial i$, for example, may not be so well determined.

It should be noted that exact agreement between the values of $\partial f/\partial n$ given above would not have been expected as the function "f" used in the differential method was a function of the initial elements and therefore did not exactly correspond to the Delaunay series which, as already pointed out, involved the mean elements.

References in Chapter VI

References in Chapter VI

6.1 Delaunay, M. Comptes Rendus de l'Academie des Sciences, vol.74, 1872.

6.2 Bohone, J. Astronomical Journal vol.44 p165, 1934-5
Astronomical Journal vol.45 p189, 1935-6

6.3 Proskurin, B.F. Bull. Inst. Theor. Astronomy Leningrad vol.VI, 1955.

6.4 Lemechova, E.N. Bull. Inst. Theor. Astronomy Leningrad vol.VIII, 1961.

6.5 Brown, E.W. Monthly Notices of the Royal Astronomical Society, vol.57 p 336, 1897.

6.6 Brown, E.W. Astronomical Journal vol.35, 1924.

CHAPTER VII

Assessment of Results.

7.1. Background.

The aim of this thesis has been to use the results obtained from the numerical integration of the orbits of hypothetical satellites of Jupiter to try and map out areas of stable satellite and asteroid orbits, and to explain the observed distribution of outer satellites of Jupiter and of asteroids in orbits close to Jupiter. The object of this chapter is to consider to what extent this aim has been achieved.

A number of authors including Darwin (ref. 1.13), Strömgren (ref. 1.15), Goudas (ref. 1.17) and Chebotarev (ref.1.20) have computed large numbers of orbits of the massless body in the restricted three body problem, in some cases with reference to the outer satellites of Jupiter. Other authors, such as Bobone (ref. 6.2), Lemechova (ref. 6.4) and Kovalevsky (ref. 2.1), have computed the orbits of particular outer satellites of Jupiter with considerable accuracy.

The approach here has lain somewhere between those of the two sets of authors. On the one hand the orbits computed have been more like the orbits of the real Jovian satellites than those computed by the first set of authors, e.g. only Goudas considered orbits not in the plane of motion of the two massive bodies and only Chebotarev considered one of the massive bodies to move in an elliptical orbit about the other. On the other hand it was felt that more information might be obtained regarding the stability of orbits if a large number of similar orbits were computed, so that the effect of varying the various parameters on the stability of an orbit might be investigated. The second set of authors were interested in the orbit of one particular jovian satellite. Here the idea was to investigate the areas in which stable orbits might exist with particular reference to the positions of the outer satellites of Jupiter.

The method of integration used (De Vogelaere's method) was found to be very convenient and has not, as far as is known, been used for this purpose before. A sufficiently small step length was used and a sufficiently large number of significant figures were retained that the method could be used to calculate the orbit of a satellite for a few hundred revolutions about Jupiter. It would appear to be a weakness of Chebotarev's work that such a large step length was used that after a few revolutions the results became of little significance due to the growth of rounding off errors and also that the large size of the step length made it impossible to deal with close approaches of a satellite to Jupiter.

7.2 The Group (i) Satellites

From the integrations of the orbits of satellites in the region of the group (i) satellites it has been seen that such orbits are extremely stable. For example the range of values of a, e and i during a hundred revolutions of SV 4 were as follows -

a	:	1.479×10^{-2}	-	1.503×10^{-2}
e	:	0.111	-	0.200
i	:	$25^\circ.8$	-	$27^\circ.9$

It has been suggested that these orbits lie very close to a periodic orbit, so that orbits in this region may be thought of as small variations of the periodic orbit and are hence very stable. In chapter VI it was shown that a periodic orbit probably exists in this region and could possibly be discovered by a method similar to the one described in that chapter. It was also pointed out that periodic orbits tend to be associated with commensurabilities which would help to explain why the seven outermost satellites of Jupiter all lie close to commensurabilities.

The foregoing argument should not be taken to suggest that orbits just outside the position of the group (i) satellites would be particularly unstable, for it was seen in chapter IV that direct orbits out to about double the distance of the group (i) satellites from Jupiter would be stable in the sense that a satellite in such an orbit would be able to complete more than fifty revolutions about Jupiter without escaping from the planet, though of course it would suffer much greater oscillations in its elements during this period than a typical group (i) satellite. In fact there is no great weight of evidence from the orbits computed in this vicinity to suggest that commensurable orbits are more stable than neighbouring non-commensurable ones (see section 4.2). It is more from the presence of satellites at these commensurabilities and the apparent likelihood of periodic orbits existing at these commensurabilities that it is felt that such orbits are more stable than neighbouring ones.

It is thought that further work along the lines of that described in chapter VI might result in the discovery of periodic orbits in the regions of the group (i), group (ii) and group (iii) satellites and in this way the preference for commensurabilities amongst the orbits of the seven outermost satellites of Jupiter might be explained.

7.3 Stable and Unstable Orbits

In chapter IV the effect of varying the initial conditions of a satellite orbit on the stability of the orbit was investigated in some detail. The result was a fairly comprehensive picture of the values of a_o, e_o and i_o which would give rise to stable and unstable orbits. From these results it would almost be possible to deduce for any given values of a_o, e_o and i_o whether the corresponding direct or retrograde satellite orbit would be stable or not.

Groves and Shaikh(ref.7.1) have used the Jacobi integral to investigate the stability of lunar satellites perturbed by the Earth and Earth satellites perturbed by the Moon, the criterion for stability being that the zero-velocity curve for the satellite should be a closed curve about the parent body and that the satellite should be inside this curve. The authors admit that no long term predictions may be made due to the assumptions involved in the use of the Jacobi integral.

It might have been expected however that although the "scale" of the system is different from the Sun-Jupiter system, the effect of changes in the initial conditions on the stability of a lunar satellite perturbed by the Earth would be similar to the results obtained for a Jovian satellite perturbed by the Sun. The results of Groves and Shaikh are however considerably different from those obtained in chapter IV. For instance Groves and Shaikh found that direct orbits are stable out to considerably greater distances from the Moon than retrograde ones and that the effect of increasing the inclination of an orbit is <u>always</u> to decrease its stability. These conclusions differ so much from the numerical results of chapter IV and from Chebotarev's work that it is felt that, while the analysis in the paper is no doubt quite valid, the criterion adopted for the stability of an orbit is not very useful.

It would appear that there is no simple method of determining the location of stable (with the meaning of chapter IV) orbits other than by direct numerical experiment along the lines of chapter IV. The results of chapter IV were found to be very self consistent and it should be possible, from the integration of a few more carefully chosen orbits, to map out the space completely between stable and unstable orbits.

7.4. <u>Capture of Asteroids.</u>

The results of chapter V would tend to suggest that the outer four satellites of Jupiter are captured asteroids. It was not possible to re-enact such a capture but there was some evidence from chapter IV that asteroids captured into more distant satellite orbits, if they did not later escape from Jupiter, might be successively perturbed into orbits of this type which would be relatively stable.

The results of chapter V would also seem to explain the lack of asteroids in the vicinity of Jupiter's orbit as being due to the dissipation of asteroids in this region by Jupiter and it would not seem to be necessary to rely on the clustering of commensurabilities to provide an explanation. If this explanation is accepted, however, the existence of asteroids in orbits between those of Jupiter and Saturn must now be postulated and an explanation why no such asteroids have been discovered must be offered.

An asteroid in such an orbit would of course be considerably fainter than one in the asteroid belt. Consider, for instance, the change in an asteroid's apparent magnitude, as seen from the Earth at opposition, on being moved from a heliocentric orbit of radius 2.5 astronomical units to one of radius 8 astronomical units (radius of Saturn's orbit = 9.54 astronomical units). Since the asteroid shines by reflected sunlight its absolute brightness will be decreased in the ratio

$$\frac{B_1}{B_2} = \left(\frac{2.5}{8}\right)^2 \qquad (7.4.1)$$

where B_1 is the absolute brightness of the asteroid in the larger orbit and B_2 the absolute brightness in the smaller one. But the distance of the asteroid from the Earth at opposition is greater for the larger orbit than for the smaller one so that the ratio of the apparent brightnesses of the asteroid, as seen from the Earth at opposition, in the two orbits is given by

$$\frac{b_1}{b_2} = \left(\frac{2.5}{8}\right)^2 \left(\frac{1.5}{7}\right)^2 \qquad (7.4.2)$$

where b_1 is the apparent brightness of the asteroid in the larger orbit and b_2 its apparent brightness in the smaller orbit. If m_1 and m_2 are the apparent magnitudes corresponding to b_1 and b_2 then it follows that

$$m_1 - m_2 = 2.5 \log_{10}(0.00449) = 5.86 \qquad (7.4.3)$$

Thus it appears that an asteroid which would have had apparent magnitude 9.5, when viewed from the Earth at opposition, (only ten asteroids have magnitudes less than this - ref. 7.2) in a heliocentric orbit of radius 2.5 astronomical units would have apparent magnitude 15.36 in a heliocentric orbit of radius 8 astronomical units.

It would seem possible that objects with such large apparent magnitudes might not have been discovered. Of the thousand minor planets listed in reference 7.2 none had as large an apprent magnitude. On the other hand it does seem possible that such asteroids could be found with modern equipment and there is no doubt that if even a few such asteroids could be discovered considerably weight would be lent to the theories of chapter V. It is hoped that within the next few years some attempt will be made to search for such asteroids.

7.5 Further Work

Much work remains to be done in mapping the region around Jupiter for stable and unstable orbits, in looking for a periodic orbit in the region of the group (i) satellites and in trying to re-enact the capture of one of the outer satellites of Jupiter. It is suggested that these problems might be solved by work along the lines of that done in this thesis. The computer programs developed in the course of this work could be used more generally to investigate the orbits of asteroids away from the vicinity of Jupiter and to compute the orbits of satellites of other planets which are subject to perturbations by the Sun.

References in Chapter VII

7.1 Groves, G.V. and Shaikh, N.A. Journal of the British Interplanetary Society, vol.20, 1965.

7.2 Publications of the Lick Observatory vol. XIX, 1935

APPENDIX I.

In section 2.7 it was explained how the orbit of a satellite was computed from a Taylor's series expansion involving up to fourth derivatives of the satellite's joviocentric co-ordinates. The third and fourth drivetives were obtained by differentiating the basic equations of motion as given by Kovalevsky (equation (2.2.1)) and were of the form -

$$x''' = Gm\left(-\frac{x'}{r^3} + \frac{3xr'}{r^4}\right)$$

$$+GM\left(-\frac{3\Delta'(X-x)}{\Delta^4} + \frac{X'-x'}{\Delta^3}\right.$$

$$\left.-\frac{X'}{R^3} + \frac{3XR'}{R^4}\right) \qquad (A.1.1)$$

$$x^{iv} = -Gm\left(\frac{x''}{r^3} + \frac{6x'r'}{r^4} + \frac{3xr''}{r^4} - \frac{12x(r')^2}{r^5}\right)$$

$$+GM\left(-\frac{3(X-x)\Delta''}{\Delta^4} - \frac{6(X'-x')\Delta'}{\Delta^4}\right.$$

$$+ \frac{12(\Delta')^2(X-x)}{\Delta^5} + \frac{X''-x''}{\Delta^3}$$

$$- \frac{X''}{R^3} + \frac{6X'R'}{R^4}$$

$$+ \frac{3XR''}{R^4} - \frac{12X(R')^2}{R^5}\Bigg) \qquad (A.1.2)$$

An expression for the fifth derivative was also obtained viz.

$$x^{v} = Gm\Bigg(-\frac{x'''}{r^3} + \frac{9x''r'}{r^4} - \frac{36x'(r')^2}{r^5}$$

$$+ \frac{9x'r''}{r^4} - \frac{36xr'r''}{r^5} + \frac{60x(r')^3}{r^6} + \frac{3xr'''}{r^4}\Bigg)$$

$$+GM\Bigg(\frac{X'''-x'''}{\Delta^3} - \frac{9(X''-x'')\Delta'}{\Delta^4}$$

$$+ \frac{36(X'-x')(\Delta')^2}{\Delta^5} - \frac{9(X'-x')\Delta''}{\Delta^4}$$

$$+ \frac{36(X-x)\Delta''\Delta'}{\Delta^5} - \frac{60(X-x)(\Delta')^3}{\Delta^6}$$

$$- \frac{3(X-x)\Delta'''}{\Delta^4} - \frac{X'''}{R^3} + \frac{9X''R'}{R^4}$$

$$- \frac{36X'(R')^2}{R^5} + \frac{9X'R''}{R^4} - \frac{36XR'R''}{R^5}$$

$$\left. + \frac{60X(R')^3}{R^6} + \frac{3XR'''}{R^4} \right) \qquad (A.1.3)$$

APPENDIX II

Starting Values and Lifetimes of Satellites

S.V.	a	e	i	D or R	O or C	No.of revs
1	1.49	0.158	29.9	D	C	>100
2	2.96	0.30	22.3	D	C	29
3	2.73	0.17	32.6	D	C	> 50
4	1.49	0.158	27.9	D	C	>100
5	3.37	0.30	22.3	D	C	9
6	3.37	0.30	22.3	R	C	>50
7	3.37	0.00	22.3	D	C	>50
8	3.77	0.00	22.3	D	C	3
9	3.77	0.00	22.3	R	C	>50
10	3.91	0.00	22.3	D	C	3
11	3.91	0.00	22.3	R	C	>50
12	3.91	0.30	22.3	D	C	2
13	3.91	0.30	22.3	R	C	>50
14	3.77	0.30	22.3	D	C	2
15	3.77	0.30	22.3	R	C	>50
16	1.42	0.158	27.9	D	C	>100
17	1.49	0.168	27.9	D	C	>100
18	5.00	0.00	22.3	D	C	5
19	5.00	0.00	22.3	R	C	>50
20	5.00	0.30	22.3	D	C	1
21	5.00	0.30	22.3	R	C	3

S.V.	a	e	l	D or R	O or C	No.of revs
22	5.00	0.30	22.3	D	O	1
23	5.00	0.30	22.3	R	O	3
24	4.50	0.00	22.3	D	C	1
25	4.50	0.00	22.3	R	C	>50
26	4.50	0.30	22.3	D	C	3
27	4.50	0.30	22.3	R	C	4
28	4.50	0.30	22.3	D	O	3
29	4.50	0.30	22.3	R	O	5
30	2.96	0.158	27.9	D	C	>100
31	2.96	0.30	22.3	R	C	>100
32	2.72	0.3	22.3	D	C	>100
33	2.72	0.168	32.6	R	C	>100
34	6.00	0.00	22.3	D	C	0
35	6.00	0.00	22.3	R	C	4
36	1.51	0.107	28.8	D	O	>100
37	4.50	0.15	22.3	R	C	>50
39	5.00	0.15	22.3	R	C	>50
40	3.37	0.30	3.0	D	C	3
41	4.50	0.25	22.3	R	C	>50
42	3.37	0.30	15.0	D	C	3
43	5.00	0.25	22.3	R	C	12
44	3.37	0.30	28.0	D	C	9
45	5.00	0.25	5.0	R	C	>50

S.V.	a	e	i	D or R	O or C	No.of revs.
46	3.37	0.30	45.0	D	C	20
47	5.00	0.25	60.0	R	C	3
49	5.00	0.25	15.0	R	C	> 50
50	3.37	0.30	60.0	D	C	> 50
51	5.00	0.25	45.0	R	C	3
52	3.37	0.30	75.0	D	C	> 50
53	5.00	0.25	75.0	R	C	3
54	2.96	0.30	3.0	D	C	23
55	Satellite A of section 5.7					10
56	1.49	0.159	30.4	D	C	> 100
57	Satellite B of section 5.7					7
63	Satellite C of section 5.7					1
65	Satellite D of section 5.7					2

D - Direct, R - Retrograde, O - satellite started at opposition, C - satellite started at conjunction.

a is given in terms of 10^{-2} Jupiter units

i is in degrees

KIDSGROVE ALGOL VERSION OF MAIN PROGRAM

```
→ ESTABLISH DA0700100KP4;
P33DK;O/P 8;→
begin
library A0,A6;
real h,e,b,M,DM,E1,E2,pt,R,r,D,G,m,cx,cy,cz,ha,bom,xn,yn,
    P,tt,pp,ecc,inc,pi,n,ma,to,wpv,a,t,v,u,som,rp,eps,dM;
integer i,q,p,jk,x,j,c,kj,F,J,K,per,rep,jp,jd,SV,s,ss,
     d,w,l,kd;
real array y,yh,yd,f,fh,fm,g,k,X,st,std,hy,hyd,prev,
           prevd[1:3];
    open(20); open(30);
    ss:=read(20);   d:=read(20);
    jd:=read(20); kd:=read(20);
    h:=read(20); DM:=read(20);t:=read(20);
    M:=read(20); per:=read(20); SV:=read(20);
    y[1]:=read(20); y[2]:=read(20);y[3]:=read(20);
    yd[1]:=read(20); yd[2]:=read(20); yd[3]:=read(20);
    G:= read (20);
    m:= 9.5478 6102 ₁₀-4;
    pi:=3.1415 9265 36;
    b:= 9.9882 798 ₁₀-1;
    e:=  4.8401 1 ₁₀-2;
    pt:= 5.0 ₁₀-11;
    eps:= 4.0 ₁₀-12;
    x:=1;
    rep:=0;
    l:=0;
    s:=ss-1;
    j:=0;
    M:= M+DM;
    for i:=1 step 1 until 3 do
    begin   prev[i]:=y[i];
            prevd[i]:=yd[i]; end;
    goto ELS;
BB: l:=l+1;
    if l=kd+1 then goto DD;
    kj:=0;
    jk:=4;
    rep:=0;
    j:=0;
B:  fori:=1step1until3do
    k[i]:=y[i];
    p:=1;
    goto S;
```

```
A:  fori:=1step1until3do
    begin f[i]:=g[i];
          yh[i]:=y[i]-h×yd[i]/2+h↑2×g[i]/8;
          k[i]:=yh[i]; end;
    M:=M-2×DM;
    p:=0;
    goto S;
C:  fori:=1step1until3do
    fh[i]:= g[i];
    M:=M+DM;
V:  if y[1]<0 then w:=0 else w:=1;
    fori:=1step1until3do
    begin yh[i]:=y[i]+h×yd[i]/2+(4×f[i]-fh[i])×h↑2/24;
          k[i]:=yh[i];end ;
    q:=1;
    p:=2;
    goto S;
W:  fori:=1step1until3do
    begin fh[i]:=g[i];
          y[i]:=y[i]+h×yd[i]+(f[i]+2 ×fh[i])×h↑2/6;
          k[i]:=y[i];end;
    q:=0;
    gotoS;
U:  for i:=1step 1 until 3 do
    begin fm[i]:=f[i];
          f[i]:=g[i];
          yd[i]:=yd[i]+(fm[i]+f[i]+4×fh[i])×h/6;end;
    goto Z;
S:  M:=M-entier(M/2/pi)×2×pi;
    E1:=M+e×sin(M);
    c:=0;
T:  E2:=(M+e×sin(E1)-E1×e×cos(E1))/(1-e×cos(E1));
    if abs(E2-E1)>pt then
    begin E1:=E2; c:=c+1;
          if c=20 then
          begin write text(30,[Kepler]);
                goto DD; end;gotoT; end;
    X[1]:=cos(E2)-e;
    X[2]:=b×sin(E2);
    X[3]:=0;
    R:=sqrt(X[1]↑2+X[2]↑2);
    rp:=r;
    r:=sqrt(k[1]↑2+k[2]↑2+k[3]↑2);
    D:=sqrt((X[1]-k[1] )↑2+(X[2]-k[2])↑2+k[3]↑2);
    fori:=1step1until3do
    g[i]:=G×((X[i]-k[i])/D↑3-X[i]/R↑3)-G×m×k[i]/r↑3;
    M:=M+DM;
    ifp=1then gotoA;
    if p=0 thengotoC;
    if q=1then goto W else goto U;
Z:  if y[1]>0 and w=0  and kj=0 then
    begin d:=d+1;
```

```
            x:=0;
            write text(30,[orbit]);
            write(30,format([nndc]),d); end;
      jk:=jk+1;
      j:=j+1;
      if rep=2 then goto E;
      if j=jd then goto ELS;
E:    if jk=2 then goto STEP;
      ifjk=3 then goto COMPARE;
      if j=1 and per=2 and rp>r then
      begin per:=1;
            write text(30,[OKa[c]]);end;
      if per=1 and rp<r and kj=0 then
      begin write text(30,[perijove[c]]);
            jp:=j;
            rep:=2;
            s:=ss-1;
            goto ELS; end;
      if per=0 and rp>r and kj=0 then
      begin  per:=2;
             rep:=2;
             write text(30,[apojove[c]]);
             s:=ss-1;
             goto ELS;end;
      goto V;
CI:   M:=M-DM;
      kj:=1;
      jk:=0;
      DM:=DM/2;
      h:=h/2;
      for i:=1step 1 until 3do
      begin st[i]:=y[i];
            std[i]:=yd[i]; end;
      goto B;
STEP: M:=M-5×DM;
      DM:=DM×2;
      h:=h×2;
      for i:=1step 1until3 do
      begin hy[i]:=y[i];
            hyd[i]:=yd[i];
            y[i]:=st[i];
            yd[i]:=std[i]; end;
      goto B;
COMPARE: for i:=1 step 1 until 3 do
      if abs((hy[i]-y[i])/r)>eps then goto HI;
      if kj= 2 then goto DI;
      for i:=1 step 1 until 3 do
      begin y[i]:=st[i];
            yd[i]:=std[i]; end;
```

```
    jk:=0;
    kj:=2;
    M:=M-3×DM;
    goto B;
DI: write text(30,[double[c]]);
    if rep=2 then goto I;
    M:=M-3×DM;
    for i:=1step1until 3 do
    begin y[i]:=st[i];
          yd[i]:=std[i];
          prev[i]:=y[i];
          prevd[i]:=yd[i];     end;
    x:=1;
    goto BB;
HI: h:=h/2;
    DM:=DM/2;
    M:=M-6×DM;
    if kj=2 then goto I;
    write text (30,[half[c]]);
    dM:=(j-3)×DM×4;
    M:=M-dM;
    for i:=1 step 1 until 3 do
    begin y[i]:=prev[i];
          yd[i]:=prevd[i];end;
    t:=t-2×(j-3)×h;
    rep:=0;
    if x=0 then d:=d-1;
    goto BB;
I:  write text(30,[keep[c]]);
    for i:=1 step 1 until 3 do
    begin y[i]:=st[i];
          yd[i]:=std[i];
          prev[i]:=y[i];
          prevd[i]:=yd[i];end;
    if rep =2 and per = 1 then
    begin per:=0;
          write text(30,[OKp[c]]); end;
    x:=1;
    goto BB;
ELS:t:=t+j×h;
    s:=s+1;
    if l=kd then s:=ss;
    if s≠ss then goto CI;
    cx:=y[2]×yd[3]-y[3]×yd[2];
    cy:=y[3]×yd[1]-y[1]×yd[3];
    cz:=y[1]×yd[2]-y[2]×yd[1];
    ha:=sqrt(cx↑2+cy↑2+cz↑2);
    pp:=ha↑2/(G×m);
    r:=sqrt(y[1]↑2+y[2]↑2+y[3]↑2);
```

```
    bom:=arctan(0-cx/cy);
    if cy>0 then bom:=bom+pi;
    inc:=arctan(sqrt((ha/cz)↑2-1));
    ecc:=sqrt(1+((yd[1]↑2+yd[2]↑2+yd[3]↑2)/(G×m)-2/r)×pp);
    a:=pp/(1-ecc↑2);
    xn:=y[1]×cos(bom)+y[2]×sin(bom);
    yn:=y[2]×cos(bom)×cos(inc)-y[1]×sin(bom)×cos(inc)+
         y[3]×sin(inc);
    wpv:=arctan(yn/xn);
    if xn<0 then wpv:=wpv+pi;
    v:=arctan(sqrt(pp/(G×m))×(y[1]×yd[1]+y[2]×yd[2]
        +y[3]×yd[3])/(pp-r));
    if pp-r<0 then v:=v+pi;
    som:=wpv-v;
    if 1-ecc<0then
    begin write text(30,[eccentricity[c]]);
         goto L ;end;
    u:=2×arctan(sqrt((1-ecc)/(1+ecc))×sin(v/2)/cos(v/2));
    ma:=u-ecc×sin(u);
    n:=sqrt(G×m/a↑3);
    P:=2×pi/n;
    tt:=t-(entier(t/P))×P;
    to:=tt-ma/n;
L:  if l=0 then
    begin output(30,SV); output(30,m); output(30,pi);
          output(30,b); output(30,e);  output(30,G);
          output(30,pt); output(30,eps); output(30,jd);end;
    if l≠0 and l<(kd-ss) then goto H;
    output(30,ss);output(30,d);output(30,jd);output(30,kd);
    output(30,h); output(30,DM);  output(30,t);
    output(30,M-DM); output(30,per); output(30,SV);
    for i:=1 step 1 until 3 do
    output(30,y[i]);
    for i:=1 step 1 until 3 do
    output(30,yd[i]);
    output(30,bom); output(30,inc); output(30,ecc);
    output(30,a);  output(30,som);  output(30,to);
    s:=0; goto CI;
H:  F := format([+d.ddddddd₁₀+nd;]);
    J := format([8s+d.ddddddd₁₀+nd;]);
    K := format([8s+d.ddddddd₁₀+nd;c]);
    write(30,F,t); write(30,J,M-DM); write(30,K,r);
    write(30,F,y[1]); write(30,J,y[2]); write(30,K,y[3]);
    write(30,F,yd[1]); write(30,J,yd[2]); write(30,K,yd[3]);
    write(30,F,bom); write(30,J,inc); write(30,K,ecc);
    write(30,F,a); write(30,J,som); write(30,K,to);
    s:=0; goto CI;
DD: close(20); close(30);
end→
```

Table 1.1

Outer satellites of Jupiter.

Satellite	Semi-major axis (a.u.)	Sidereal period (yrs.)	Inclination	Eccentricity
VI	0.0767	0.686	28°	0.158
X	0.0783	0.710	29°	0.107
VII	0.0785	0.711	28°	0.207
XII	0.142	1.728	33°R	0.169
XI	0.151	1.895	17°R	0.207
VIII	0.157	2.037	32°R	0.410
IX	0.158	2.075	23°R	0.275

Table 1.2.

Satellites	$\frac{n_2}{n_1}$	A_2	A_1	$\frac{n_2}{n_1} - \frac{A_2}{A_1}$
Sun J VIII	0.17055	1	6	+0.00388
Sun J IX	0.17196	1	6	+0.00529
Sun J XI	0.15982	1	6	-0.00684
Sun and mean of J VIII, J IX, J XI	0.16725	1	6	+0.00059
Sun J VI	0.05784	1	17	-0.00098
Sun J VII	0.06002	1	17	+0.00120
Sun J X	0.05867	1	17	-0.00015
Sun and mean of J VI, J VII, J X	0.05883	1	17	-0.00001
Sun J XII	0.14564	1	7	+0.00278

Note: $\frac{1}{16} - \frac{1}{17} = 0.00368$

$\frac{1}{6} - \frac{1}{7} = 0.02381$

Table 4.1

Direct and Retrograde.

Semi-major axis 10^{-2} J.U.	No. of revs. $e = 0$ D	R	No. of revs. $e = 0.3$ D	R
2.725 (XII)	-	-	>50	-
2.959 (VIII, IX and XI)	-	-	29	>50
3.368	>50	-	9	>50
3.774	3	>50	2	>50
3.908	3	>50	2	>50
4.50	1	>50	3	4
5.00	5	>50	-	3
6.0	0	4	-	-

D - direct.

R - retrograde.

Table 4.2

D or R	initial elements a	e	Mean distance after 40 revolutions.
R	5.0	0	3.64
R	4.5	0	3.76

Distances in terms of 10^{-2} Jupiter units

Table 4.3

Effect of Eccentricity

D or R	semi-major axis	Number of revolutions			
		e = 0	e = 0.15	e = 0.25	e = 0.3
D	2.725	-	> 50	-	> 50
D	2.959	-	> 50	-	29
R	4.5	> 50	> 50	> 50	4
R	5.0	> 50	> 50	12	3

Distances in terms of 10^{-2} Jupiter units.

Table 4.4

Stability of Orbits as a Function of their Initial Osculating Apojove Distances.

Direct Orbits.

S.V.	a(1 + e)	No. of revolutions.
2	3.85	29
5	4.38	9
18	5.00	5
8	3.77	3
10	3.92	3

Retrograde Orbits.

S.V.	a(1 + e)	No. of revolutions.
43	6.25	12
35	6.00	4
27	5.85	4
21	6.50	3

Distances are in terms of 10^{-2} Jupiter units

Table 4.5

Numbers of Satellites which completed various Numbers of Revolutions about Jupiter

No. of revs.	No. of Satellites
0 - 3	17
4 - 6	4
7 - 10	3
11 - 20	1
21 - 30	2
31 - 40	0
41 - 50	0
> 50	27

Table 5.1

Direct Escapes

S.V.	Jovicentric Elements a	e	i	No. of revolutions	Heliocentric Elements a	e	i
5	3.37	0.3	22	9	0.71	0.17	2
44	3.37	0.3	28	9	0.71	0.21	2
2	2.96	0.3	22	29	0.73	0.15	1
10	3.91	0.0	22	3	0.73	0.23	2
54	2.96	0.3	3	23	0.73	0.16	0
20	5.00	0.3	22	1	0.74	0.26	2
34	6.00	0.0	22	0	0.74	0.15	2
8	3.77	0.0	22	3	0.75	0.16	2
24	4.50	0.0	22	1	0.75	0.23	2
14	3.77	0.3	22	2	1.40	0.16	1
46	3.37	0.3	45	20	1.40	0.20	3
12	3.91	0.3	22	2	1.41	0.16	1
28	4.50(0)	0.3	22	3	1.42	0.21	2
18	5.00	0.0	22	5	1.42	0.17	2
26	4.50	0.3	22	3	1.47	0.26	2
42	3.37	0.3	15	3	1.48	0.27	1
40	3.37	0.3	5	3	1.50	0.27	0

Inclinations are in degrees

The jovicentric semi-major axes are in terms of 10^{-2} Jupiter units.

Table 5.2

Retrograde Escapes.

S.V.	Jovicentric Elements a	e	i	No. of revs.	Heliocentric Elements a	e	i	a(1 + e)
23	5.0(e)	0.30	22	3	0.78	0.23	3	0.96
27	4.5	0.30	22	4	0.83	0.14	3	0.95
21	5.0	0.30	22	3	0.86	0.10	3	0.95
51	5.0	0.25	45	3	0.86	0.08	4	0.93
43	5.0	0.25	22	12	1.15	0.03	3	1.18
47	5.0	0.25	60	3	1.17	0.04	3	1.22
53	5.0	0.25	75	3	1.23	0.12	5	1.38
35	6.0	0.00	22	4	1.74	0.38	4	2.42
29	4.5(e)	0.30	22	5	1.77	0.40	4	2.48

Inclinations are in degrees

The jovicentric semi-major axes are in terms of 10^{-2} Jupiter units.

Table 5.3

SV 63

Distance from Jupiter at previous perijove.	Heliocentric elements at apojove a	e	i
	1.37	0.209	0.03
6.5×10^{-1}	1.37	0.208	0.03
5.2×10^{-1}	1.41	0.220	0.03
6.9×10^{-1}	1.41	0.227	0.03
4.1×10^{-1}	1.44	0.238	0.03
7.9×10^{-1}	1.44	0.237	0.03
5.6×10^{-1}	1.47	0.248	0.03

Distances are in Jupiter units, inclinations are in radians.

Table 5.4

SV 46

Time (J.yrs.)	a	e	i
5.25	1.398	0.199	0.05
8.15	1.400	0.202	0.05
11.07	1.342*	0.157*	0.05*
13.24	1.447*	0.225*	0.05*
17.36	1.445	0.227	0.05
23.22	1.390*	0.189*	0.05*
30.03	1.487*	0.245*	0.05*
37.22	1.518	0.263	0.05
44.36	1.372*	0.162*	0.05*
49.37	1.403	0.187	0.05

Table 6.1

Rate of change of longitude of the perijove of Jupiter VI

Bobone	$1\overset{\circ}{.}354$/year
Proskurin	$1\overset{\circ}{.}370$/year
Delaunay series (using Proskurin's elements)	$1\overset{\circ}{.}370$/year
Incorrect Delaunay series (using Proskurin's elements)	$1\overset{\circ}{.}386$/year

Table 6.2

Rate of change of longitude of the perijove (Jupiter X)

Lemechova	$1^{\circ}\!.545$/year
Delaunay series (using Lemechova's elements).	$1^{\circ}\!.529$/year
Incorrect Delaunay series (using Lemechova's elements)	$1^{\circ}\!.545$/year

Table 6.3

Values of λ and μ for various n_o, e_o, i_o

SV.	n_o	e_o	i_o(rads.)	λ	μ	
4	106.8	0.1575	0.4873	−0.2633	0.5826	(i)
1	106.8	0.1575	0.5222	−0.2627	0.5397	(i)
16	114.8	0.1575	0.4873	−0.2474	0.5071	(i)
17	106.8	0.1675	0.4873	−0.2669	0.5892	(i)
36 (J X)	104.7	0.1070	0.5028	−0.2500	0.5509	(i)
				−0.2470	0.5410	(ii)
56	106.8	0.1590	0.5310	−0.2590	0.5254	(i)
				−0.2618	0.5256	(ii)
				−0.2570	0.5355	(iii)

(i) Values of λ and μ obtained from the numerical integrations by the method of section 6.2

(ii) Values of λ and μ obtained by the differential method from SV's 1, 4, 16 and 17.

(iii) Values of λ and μ obtained from the Delaunay series using the attained mean values of the elements.

Table 6.4

Values of the Partial Derivatives

	∂f	∂F
∂n	2.003×10^{-3}	-9.537×10^{-3}
∂c	-3.566×10^{-1}	6.573×10^{-1}
∂i	1.742×10^{-2}	-1.228

FIG. 1.1

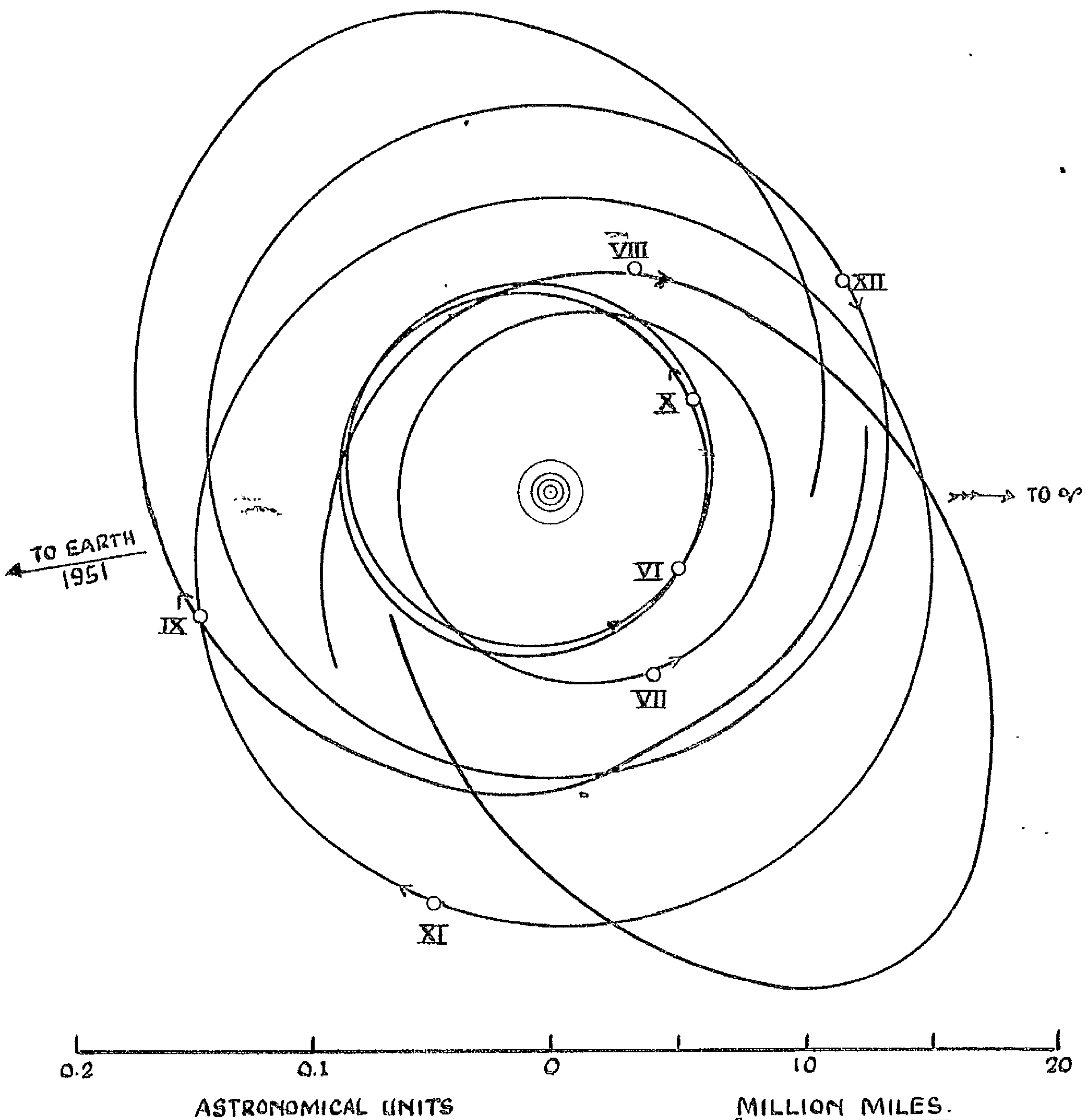
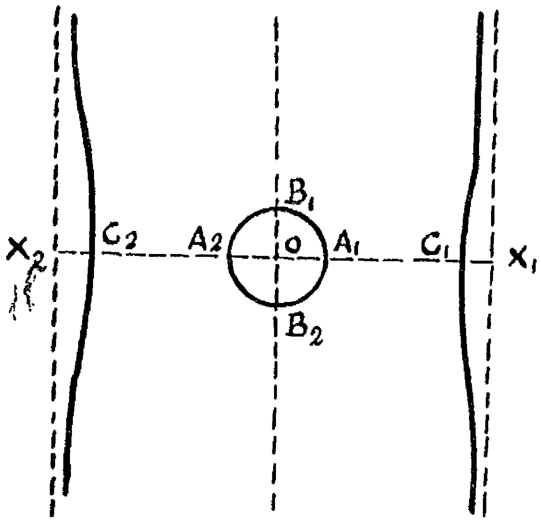
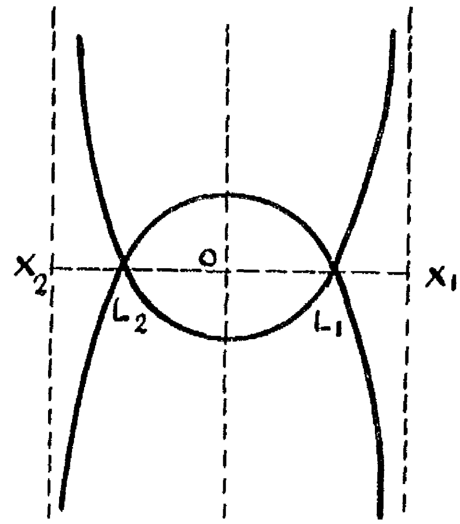

VIII
XII
X
VI
VII
IX
XI
TO EARTH 1951
TO ♈
0.2
0.1
0
10
20
ASTRONOMICAL UNITS
MILLION MILES.

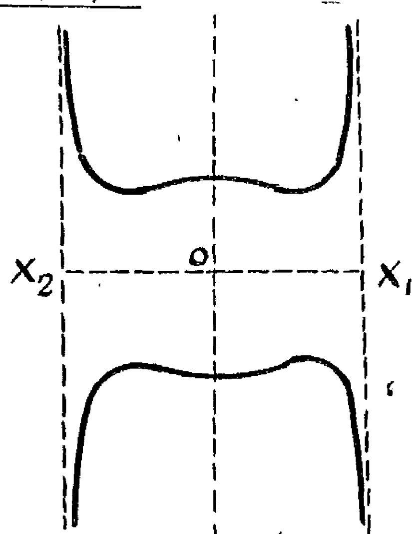
FIG. 1.2



$(2c)^{3/2} > 9\mu n'$

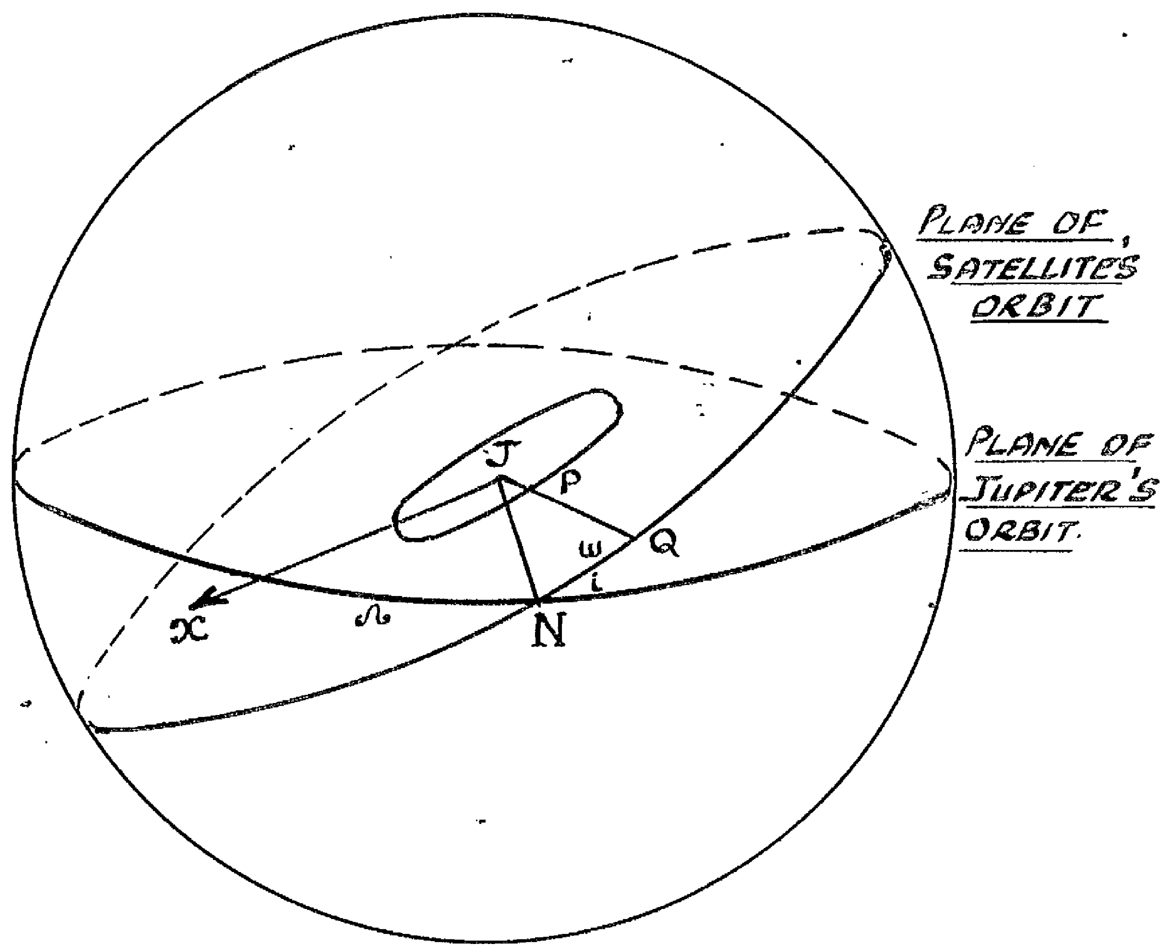


$(2c)^{3/2} = 9\mu n'$



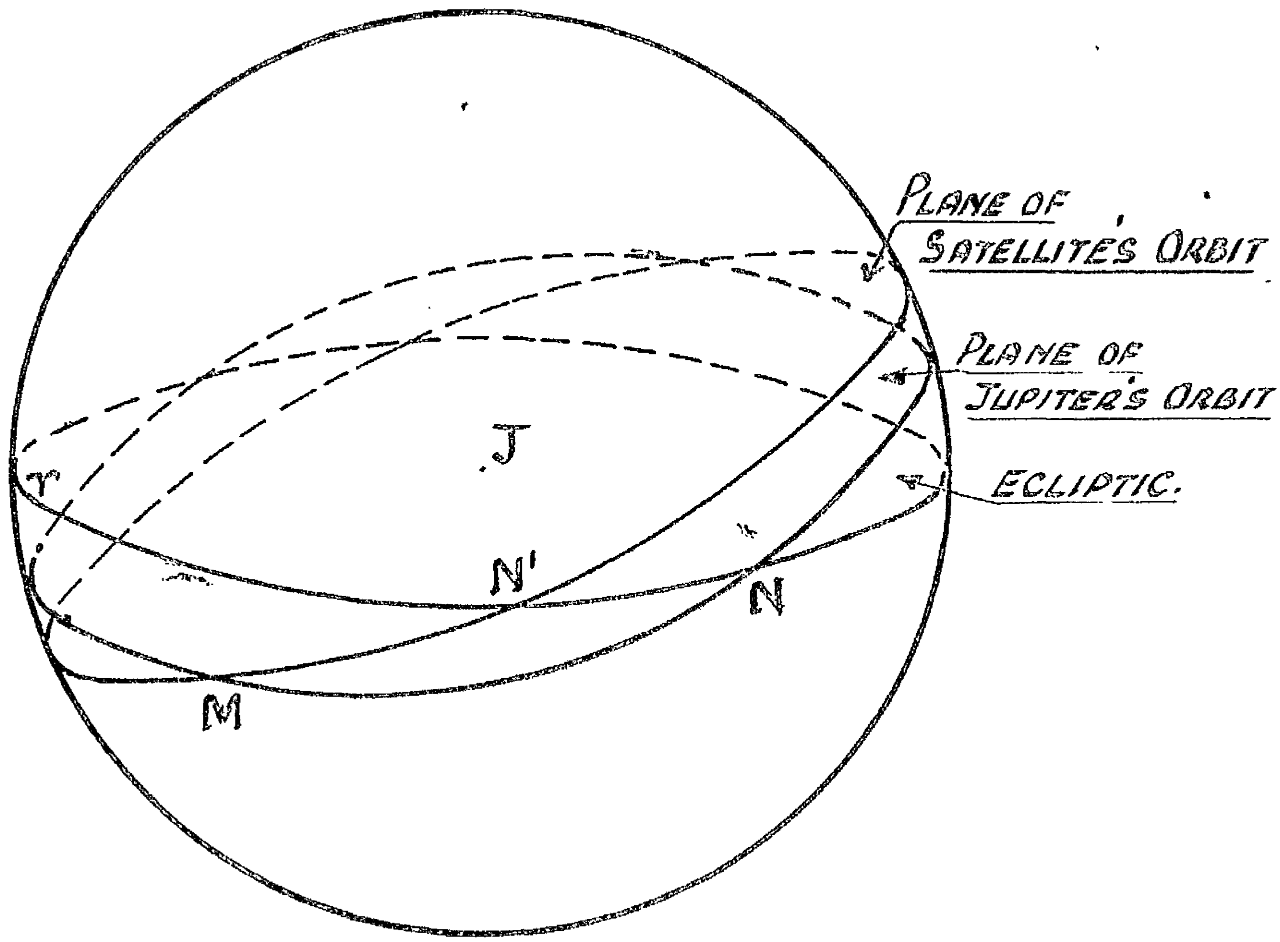
$(2c)^{3/2} < 9\mu n'$

FIG. 2.1



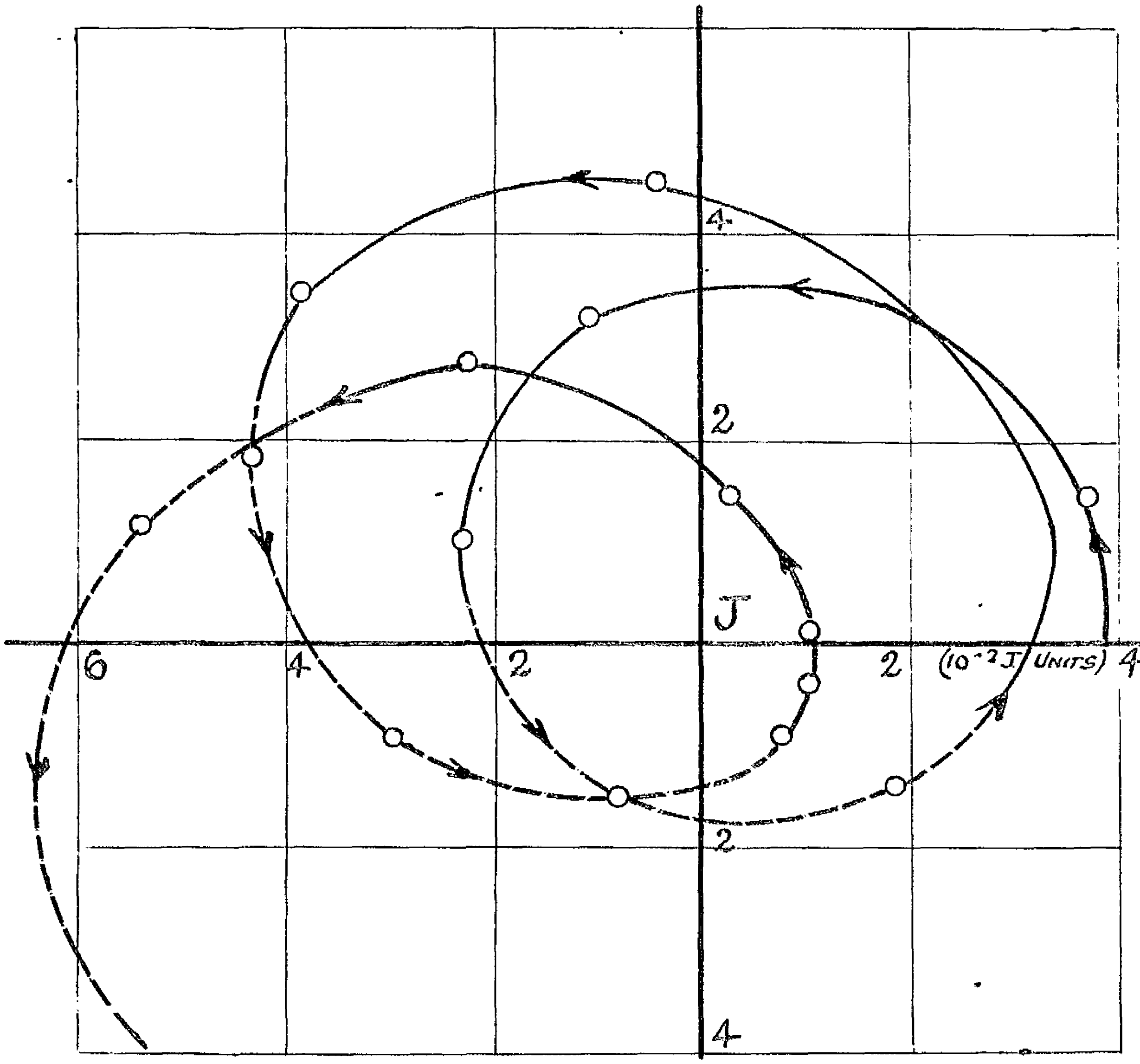
J. P. DIRECTION OF SATELLITE'S PERIJOVE.

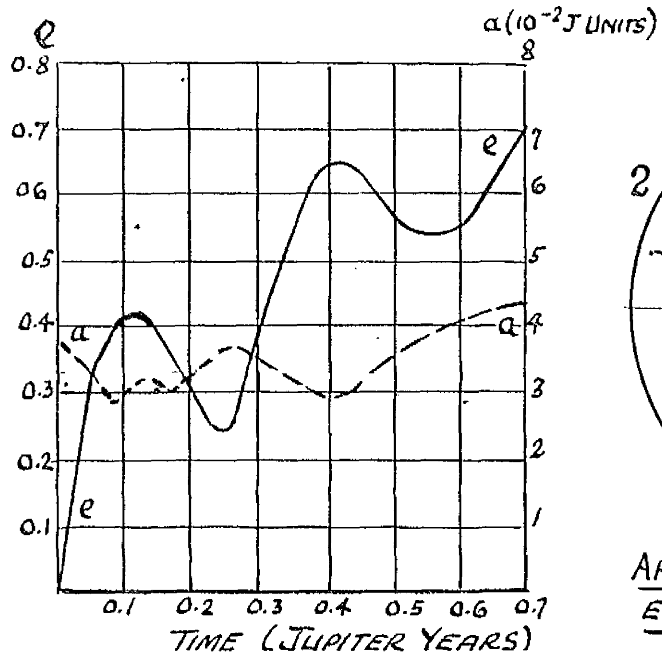
FIG. 3.1

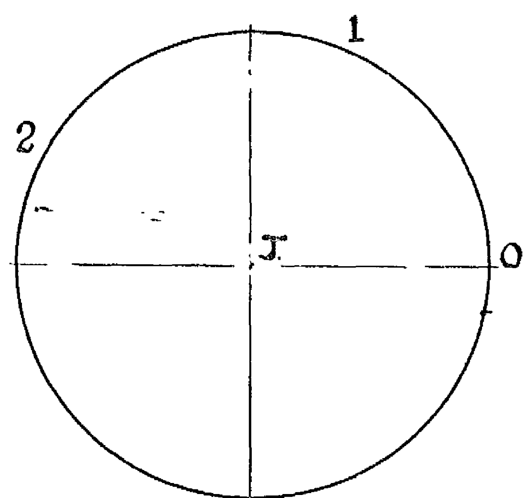

PLANE OF
SATELLITE'S ORBIT
PLANE OF
JUPITER'S ORBIT
ECLIPTIC.
J
♈
N'
N
M

S V. 8.

FIG. 4.1



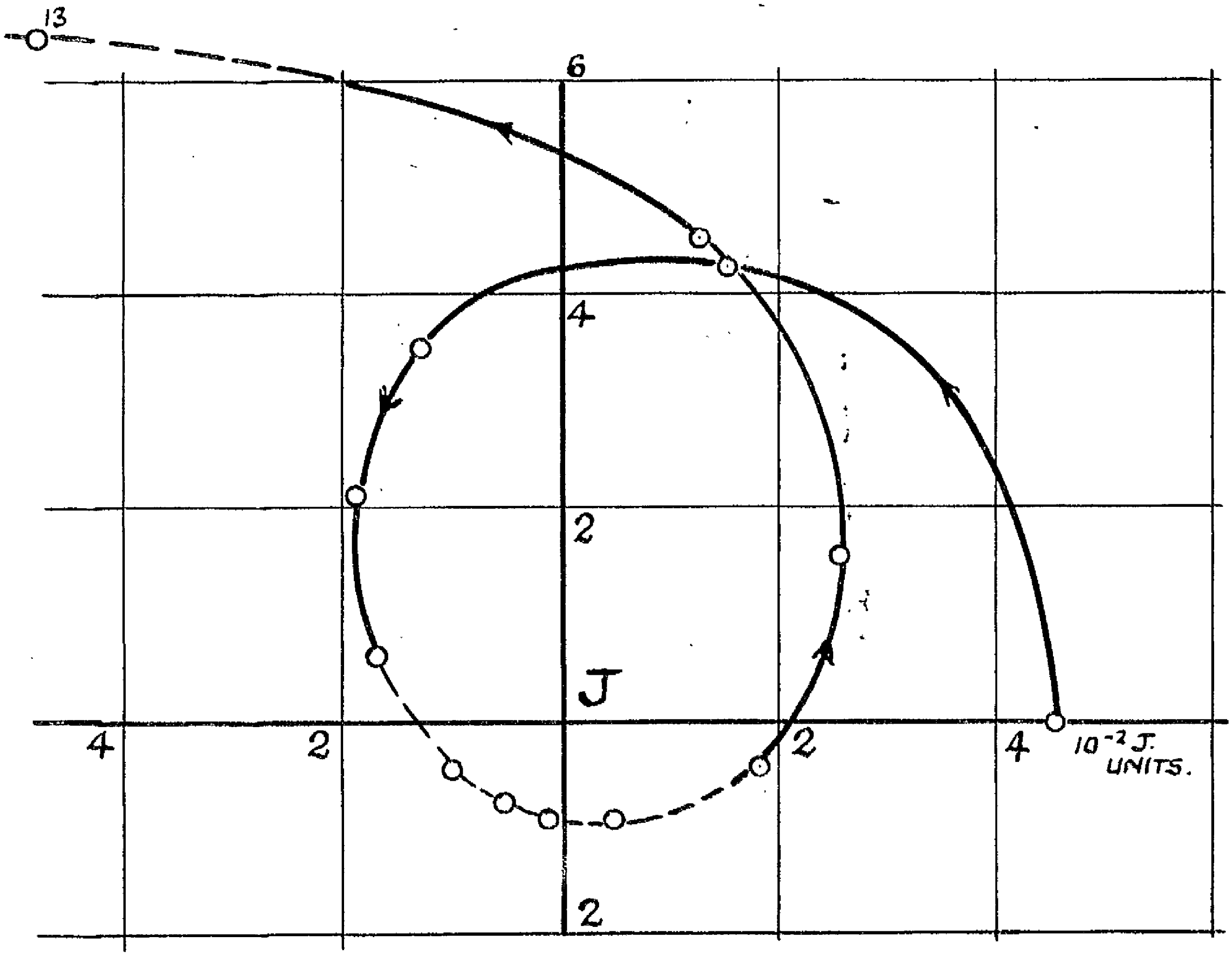


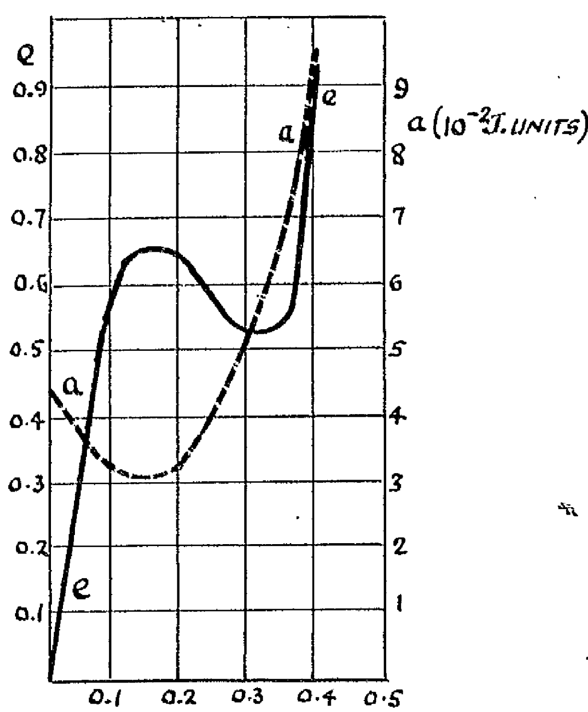


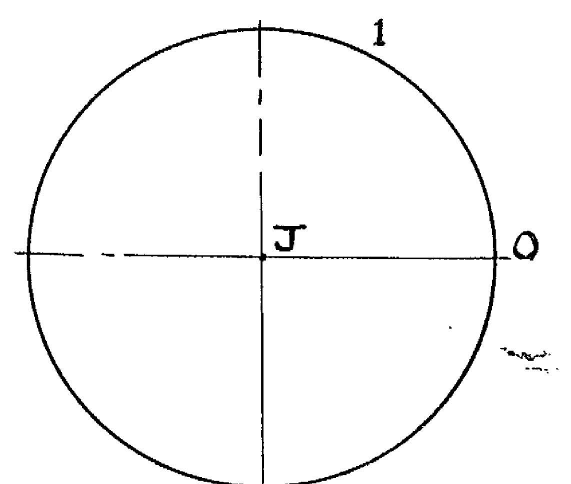
APPROX. POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION.

FIG. 4.2

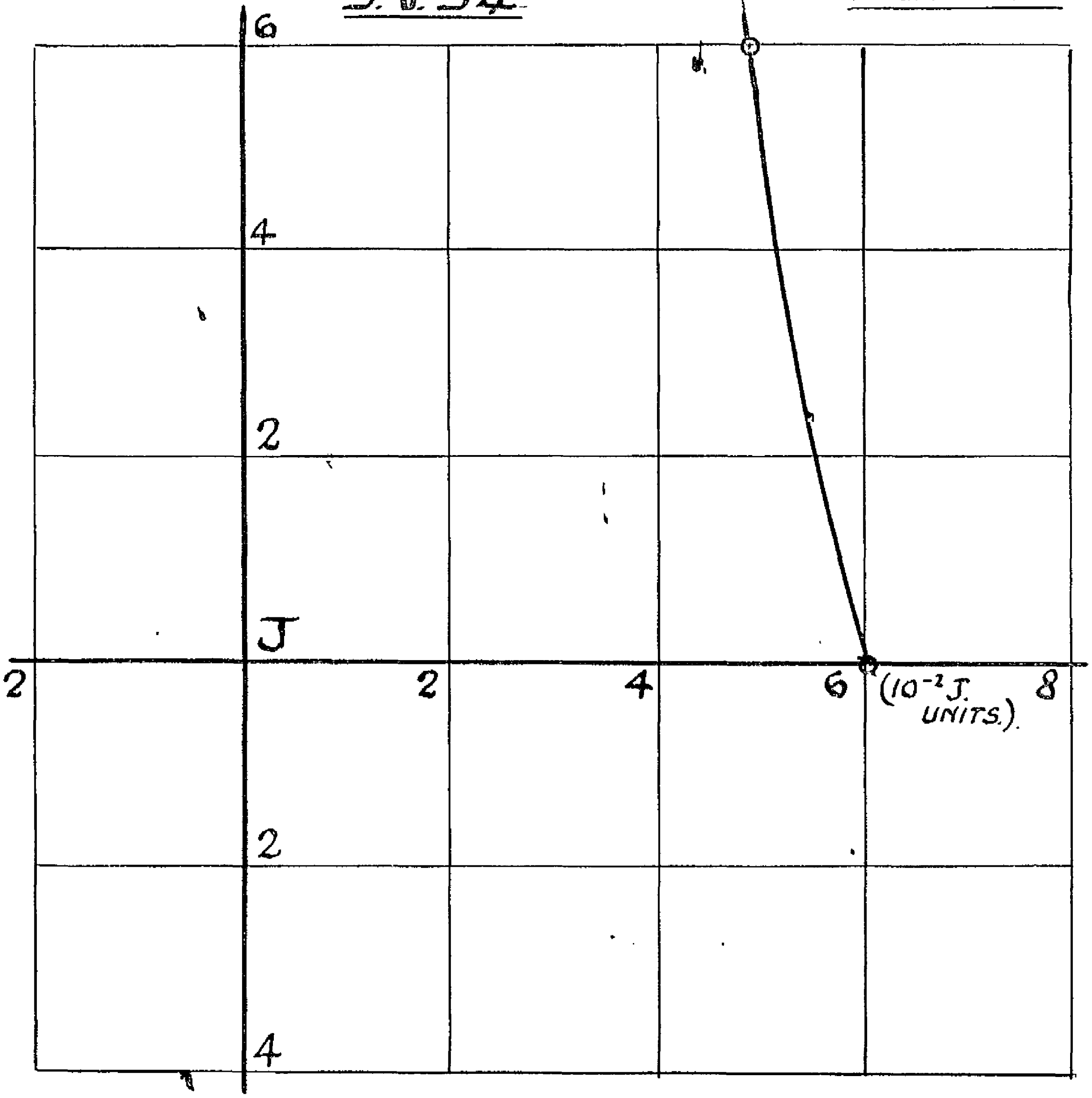
S.V. 24

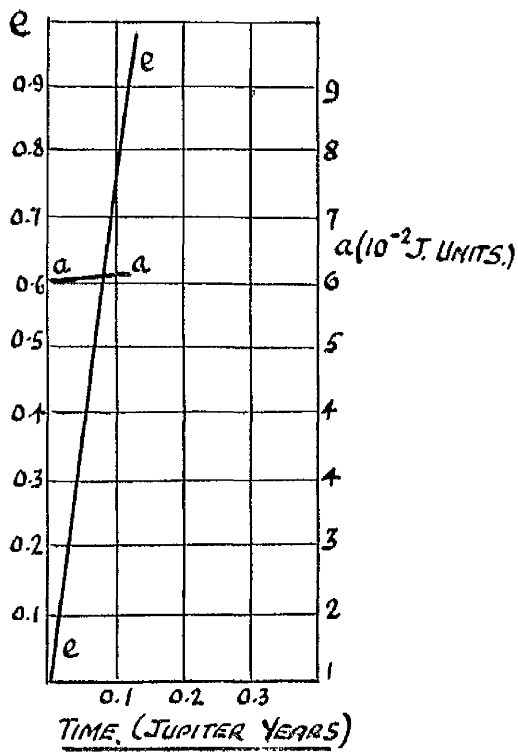


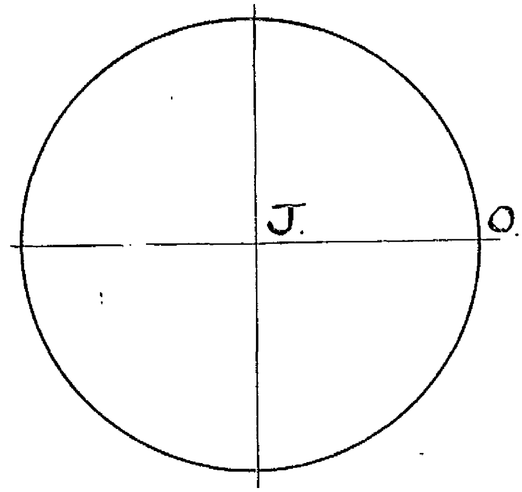




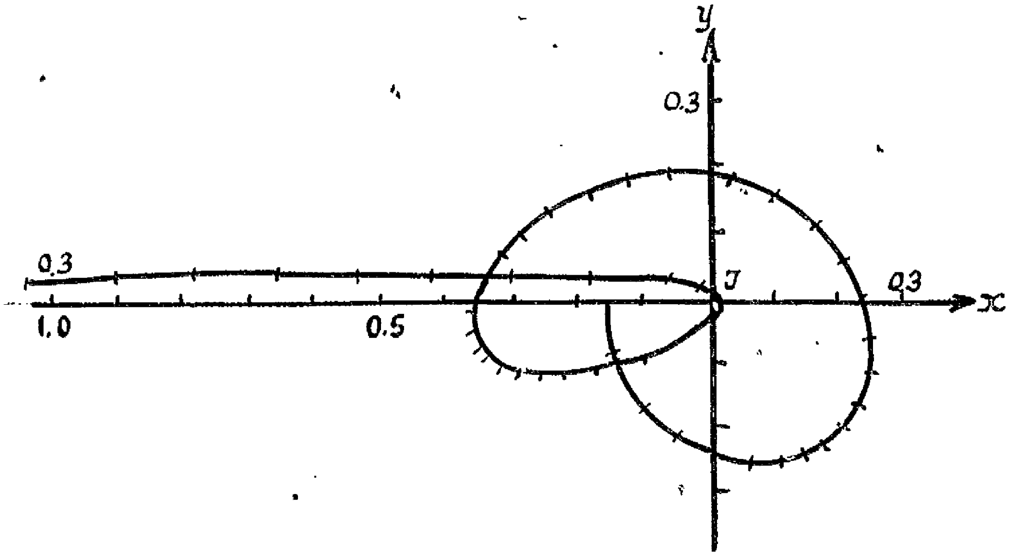
APPROX. POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION.







APPROX. POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION.



ONE OF CHEBOTAREV'S ORBITS

FIG. 4.5

S.V 8

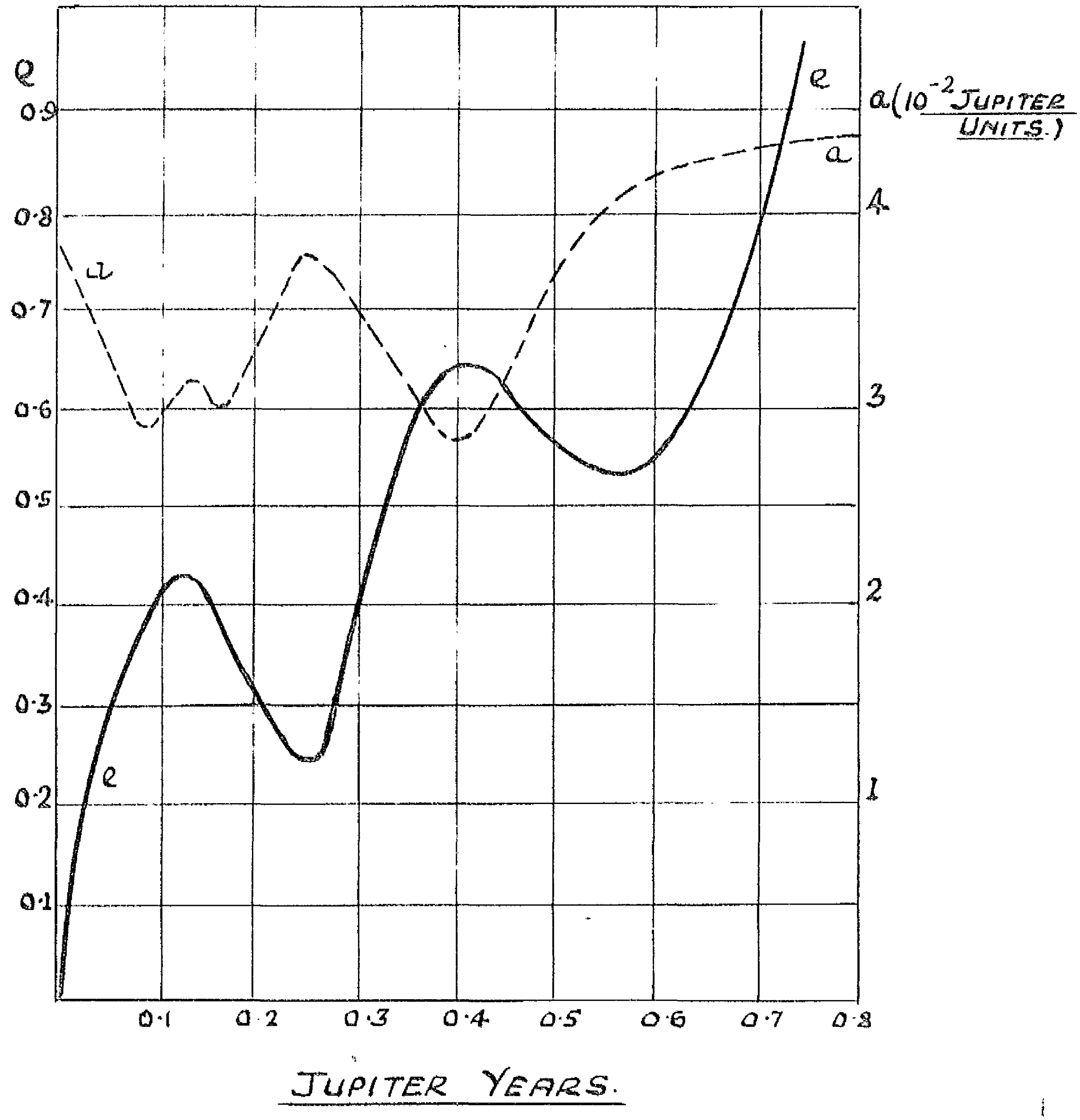
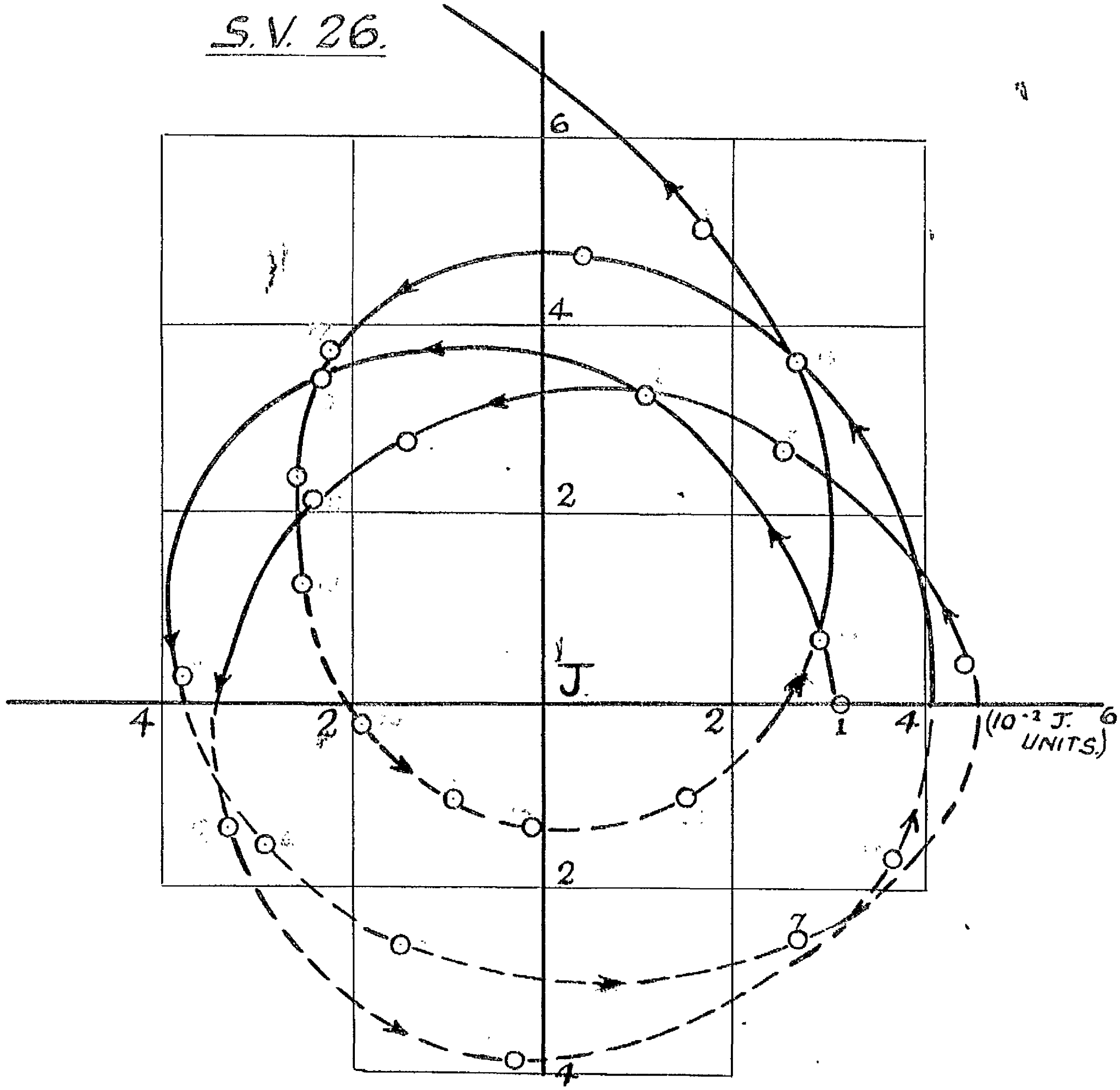
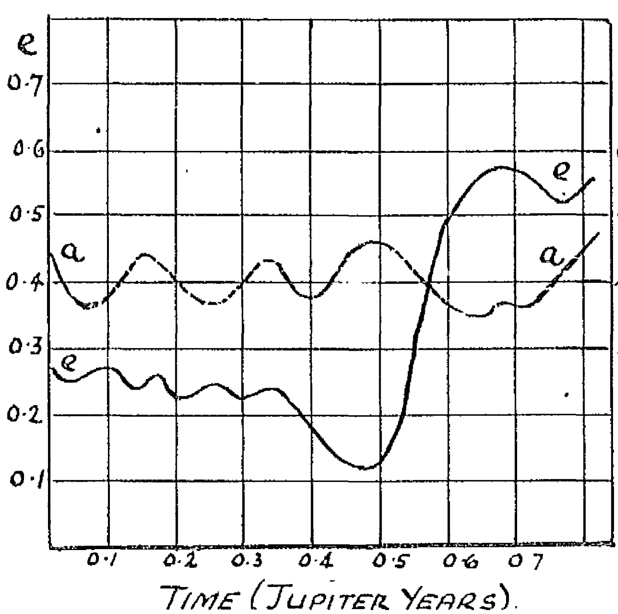
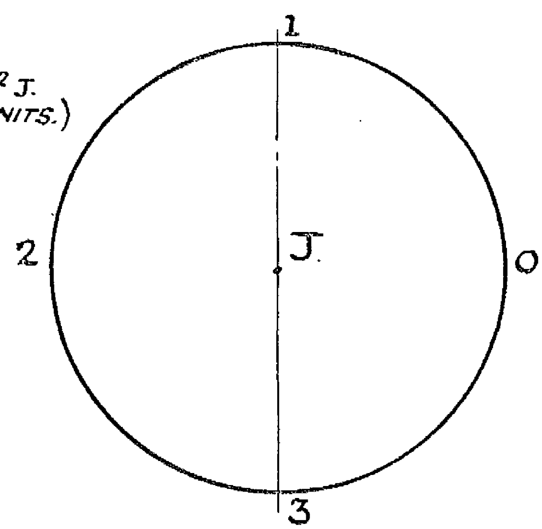

e
a(10^{-2} Jupiter Units.)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
4
3
2
1
e
a
a
e
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Jupiter Years.

FIG. 4.6

S.V. 26.



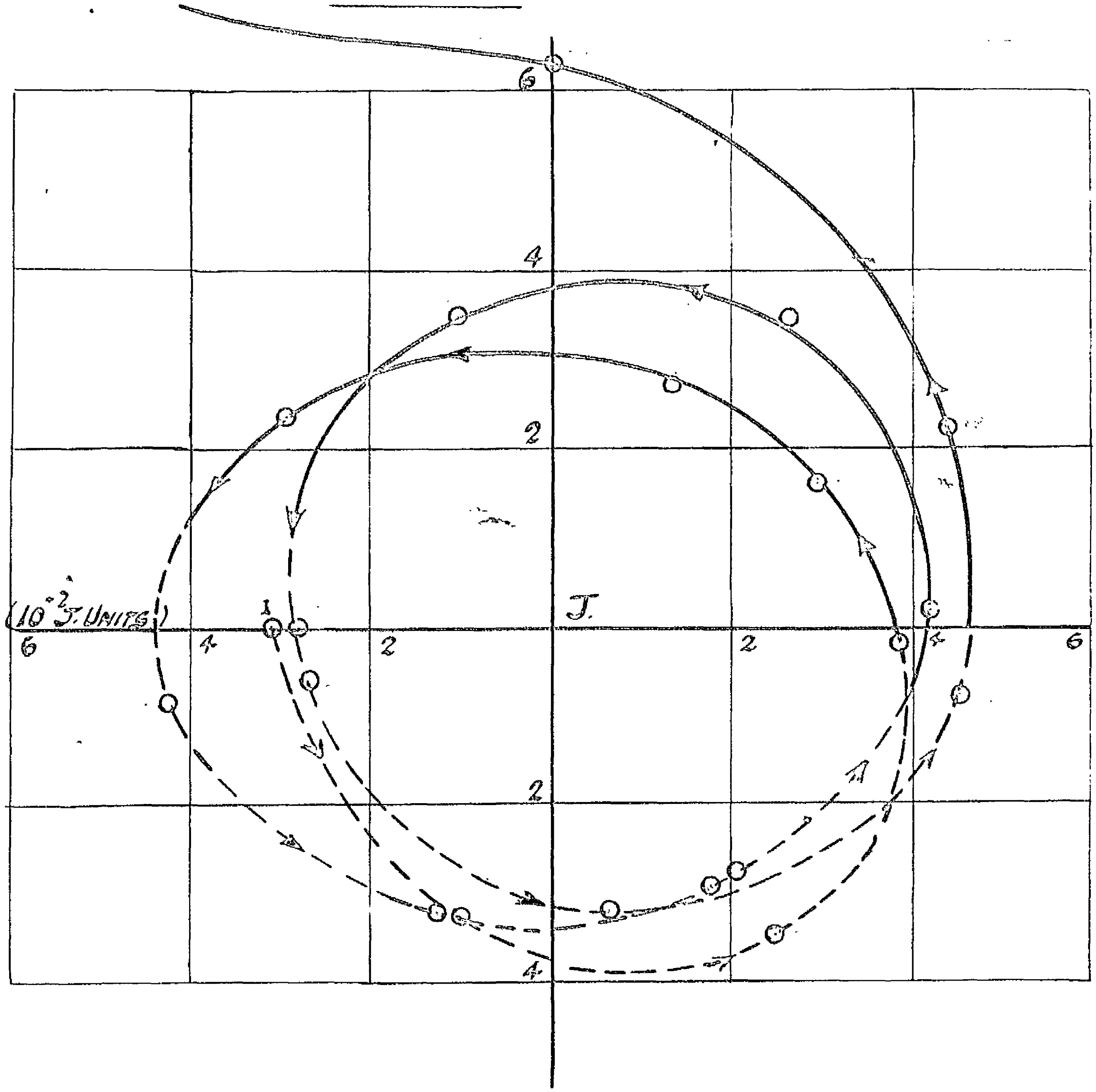


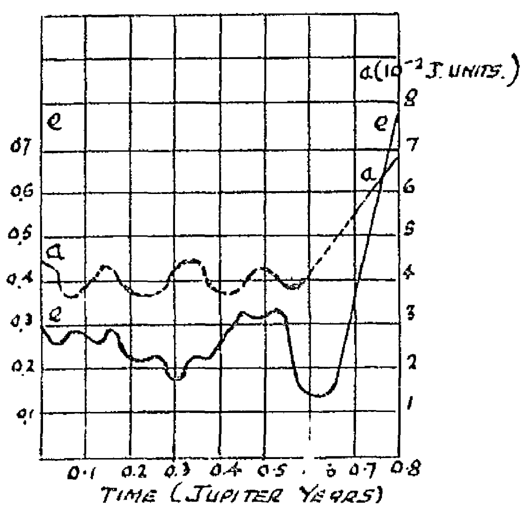


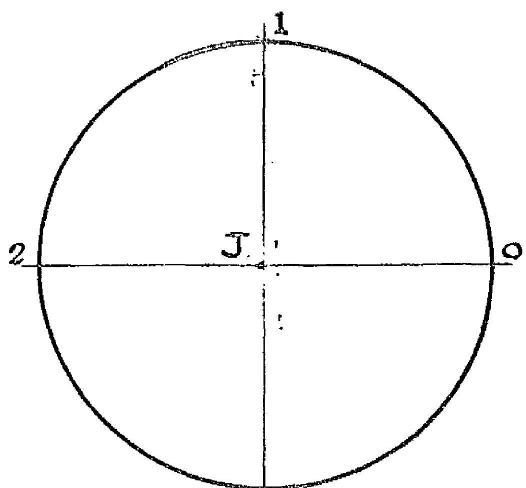
APPROX. POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION.

S V. 28

FIG. 4.7



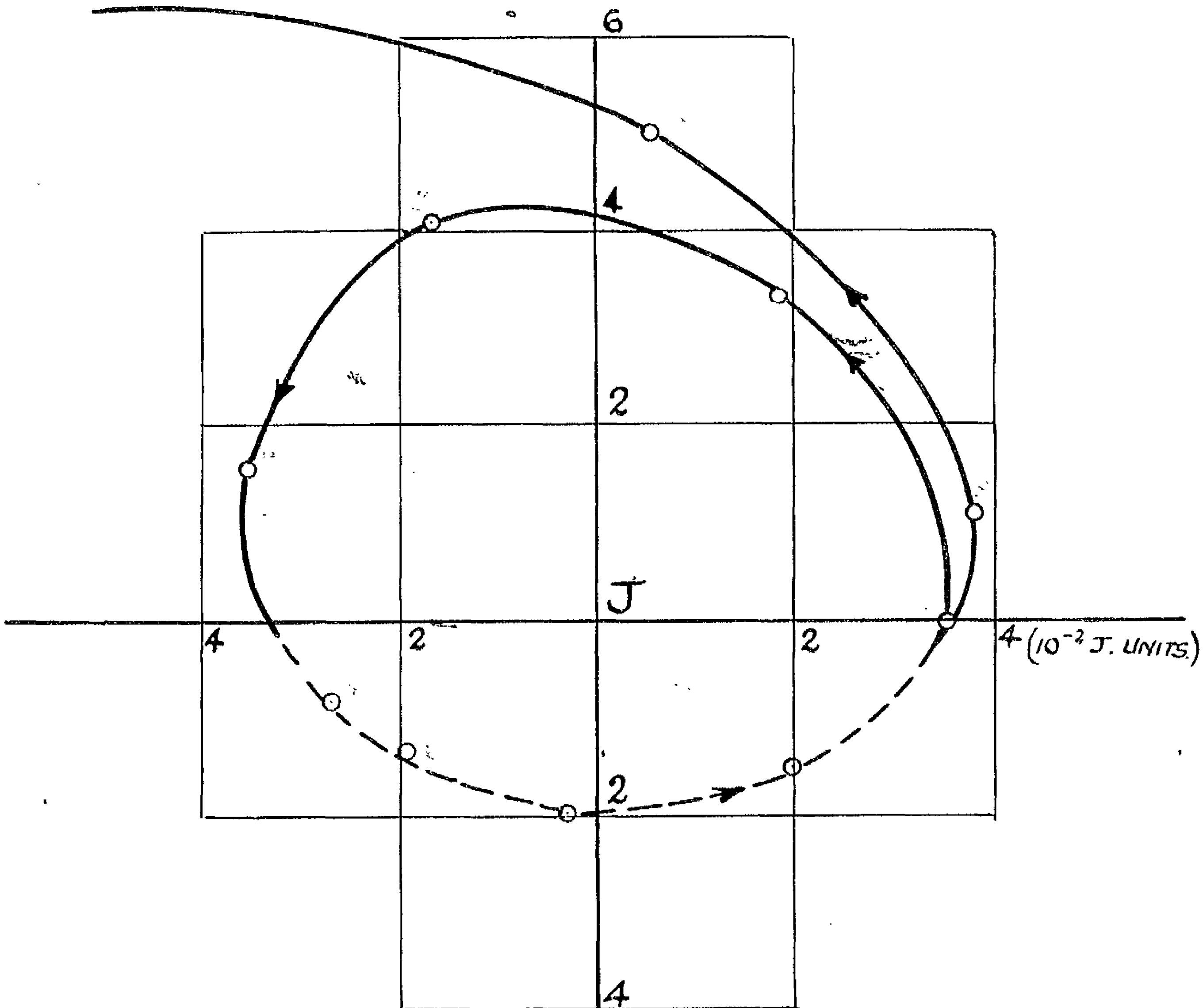


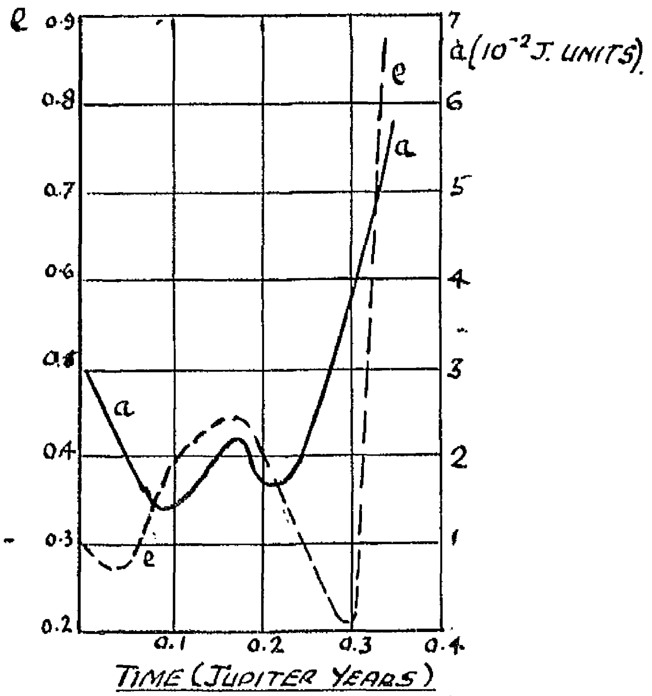


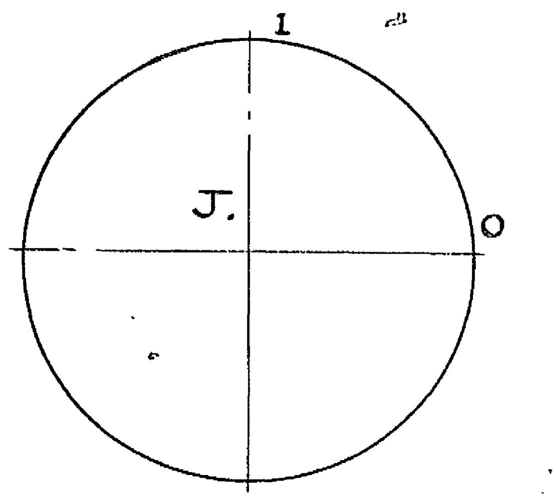
APPROX POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION

FIG. 4.8

S.V. 20



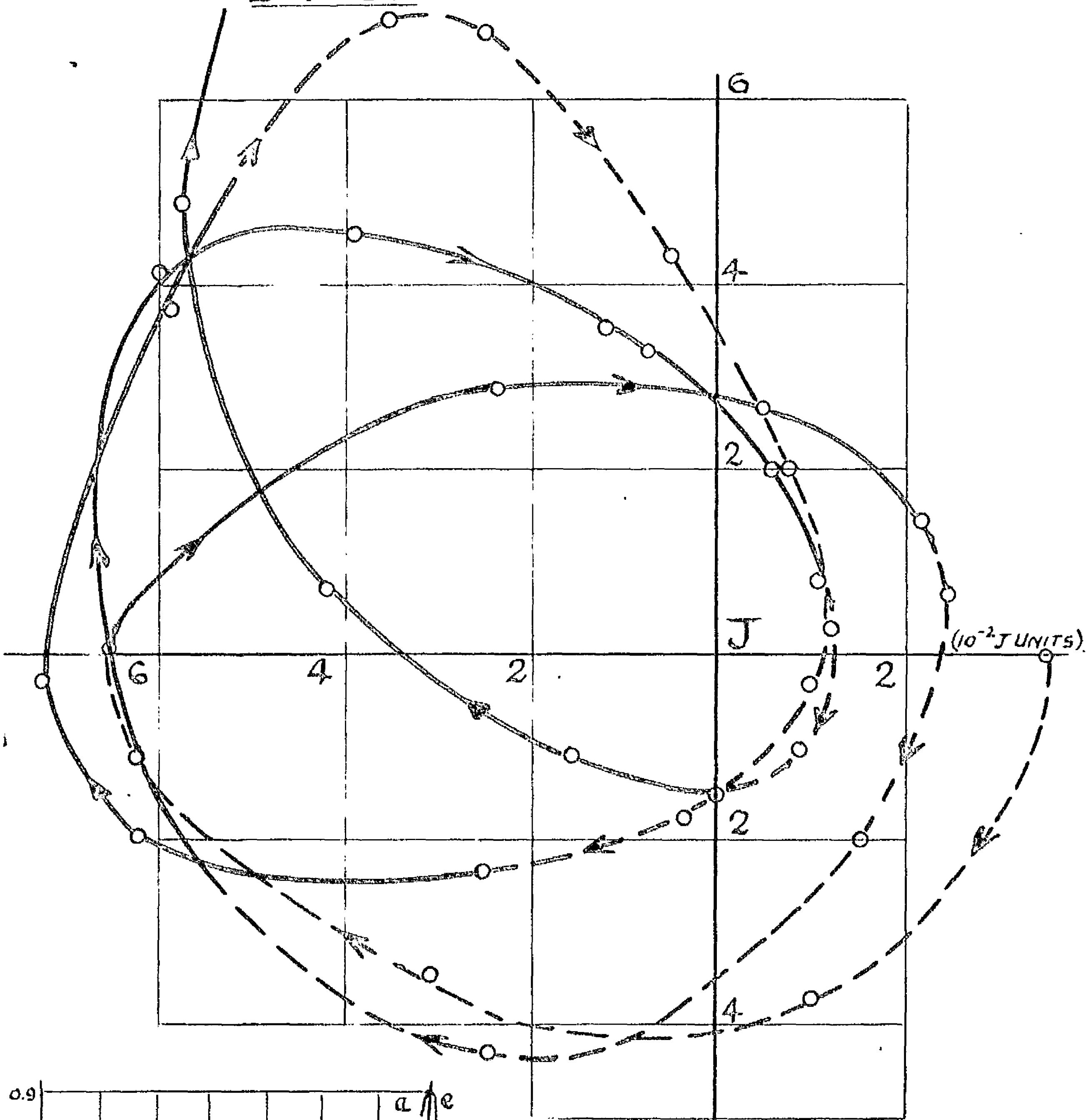


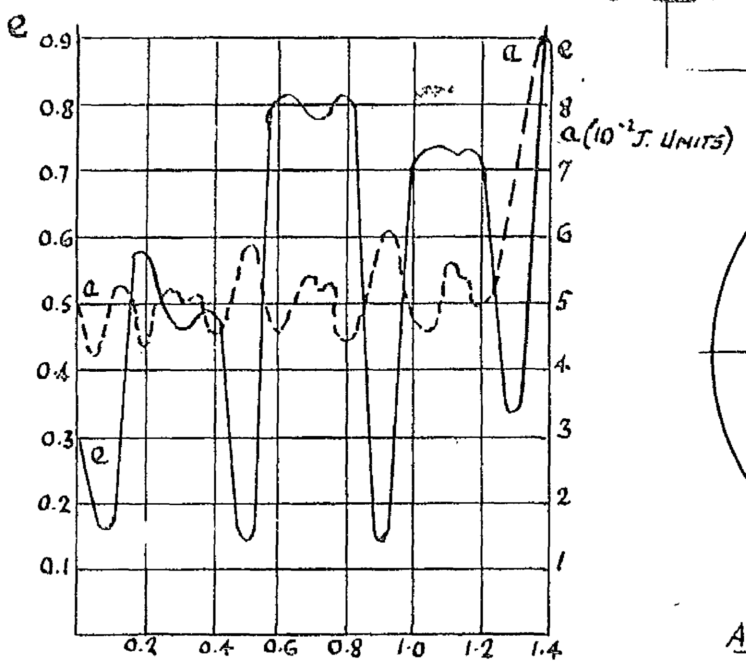


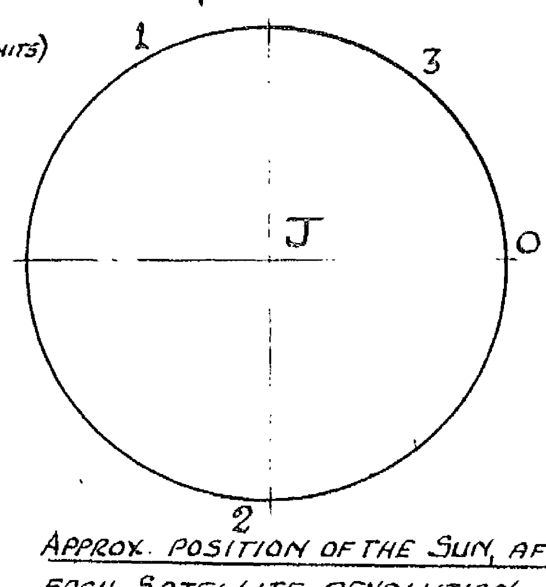
APPROX. POSITION OF SUN, AFTER EACH SATELLITE REVOLUTION

SV. 21

FIG. 4.9.

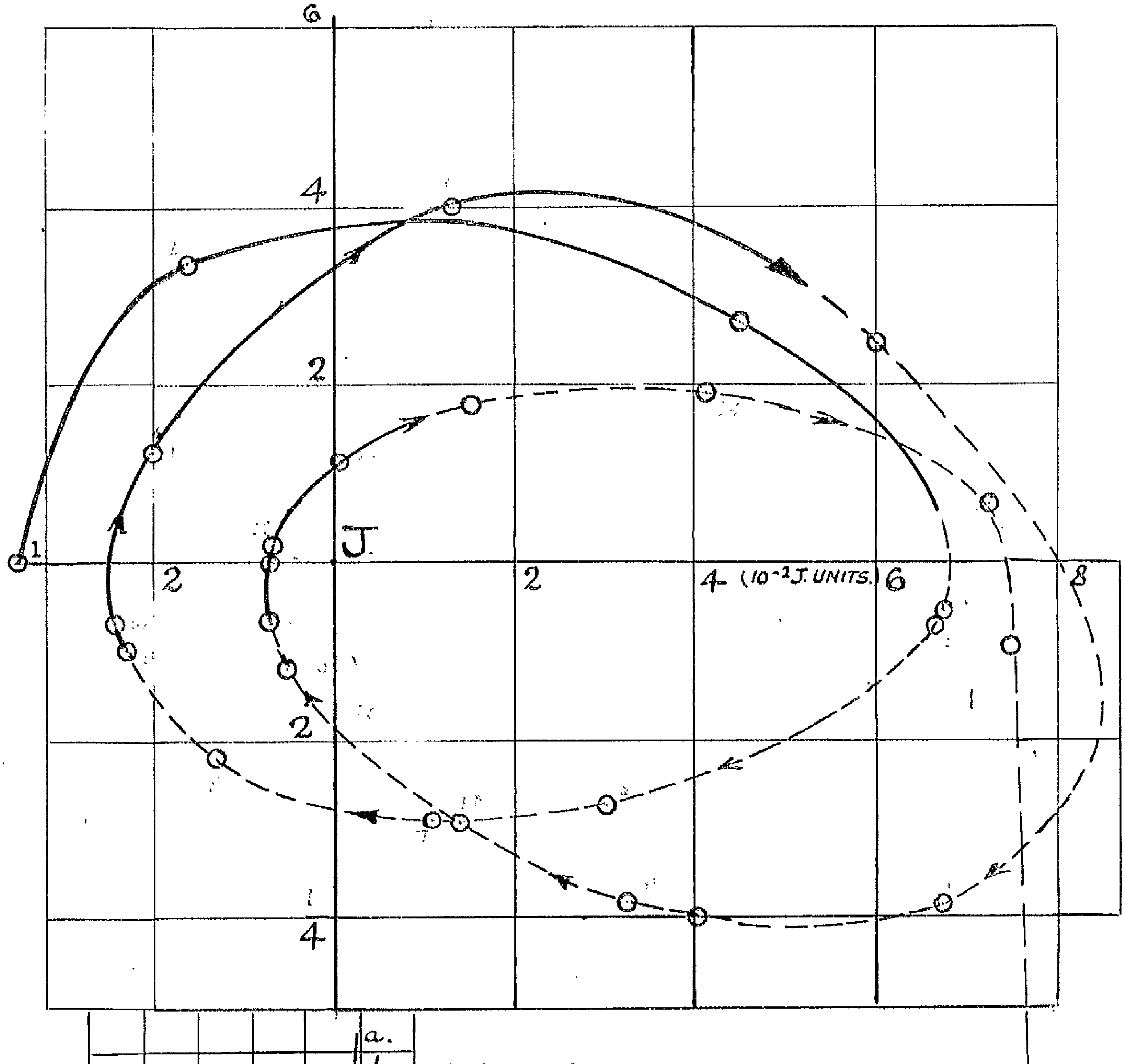


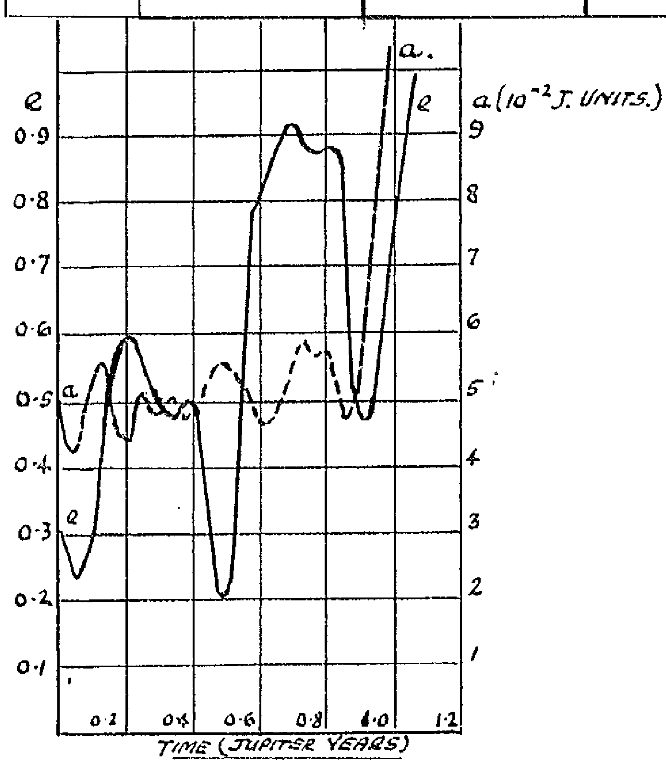


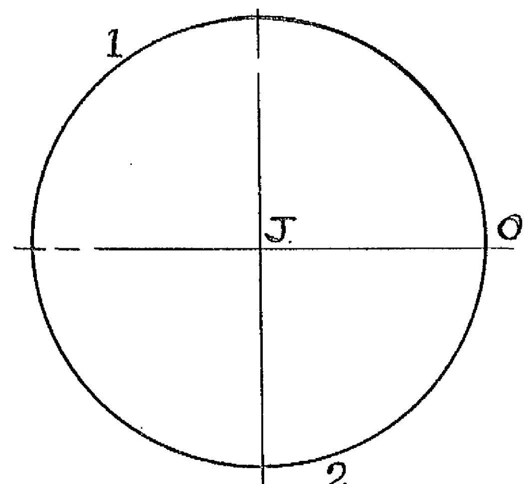


APPROX. POSITION OF THE SUN, AFTER EACH SATELLITE REVOLUTION.

S.V. 23.



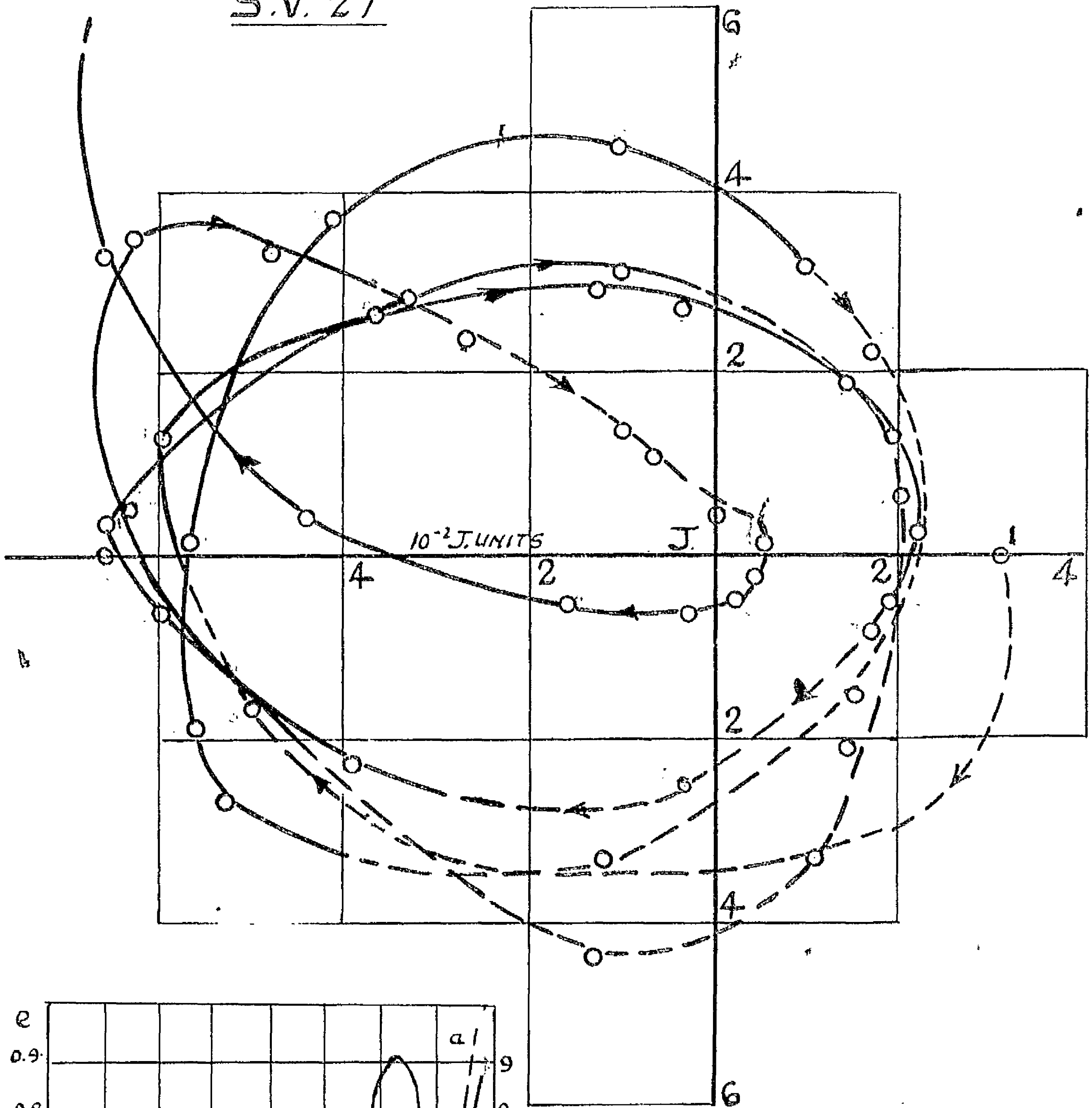


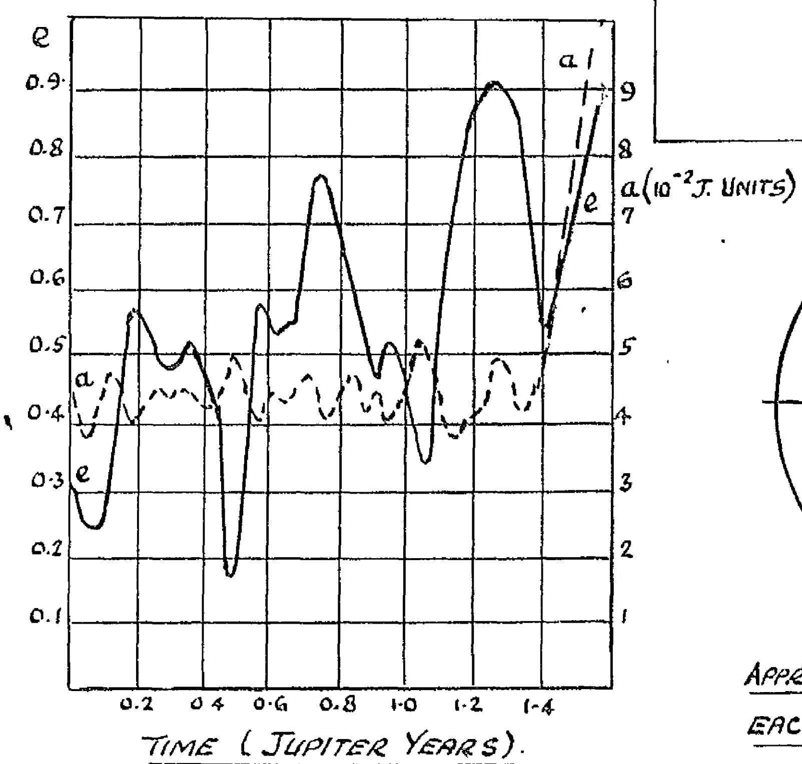


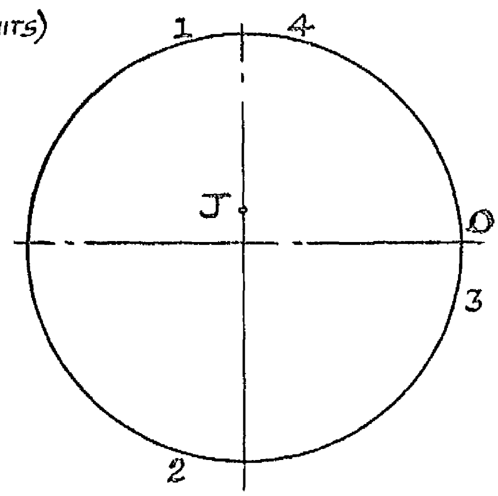
APPROX. POSITION OF THE SUN, AFTER EACH SATELLITE REVOLUTION.

S.V. 27

FIG. 4.11

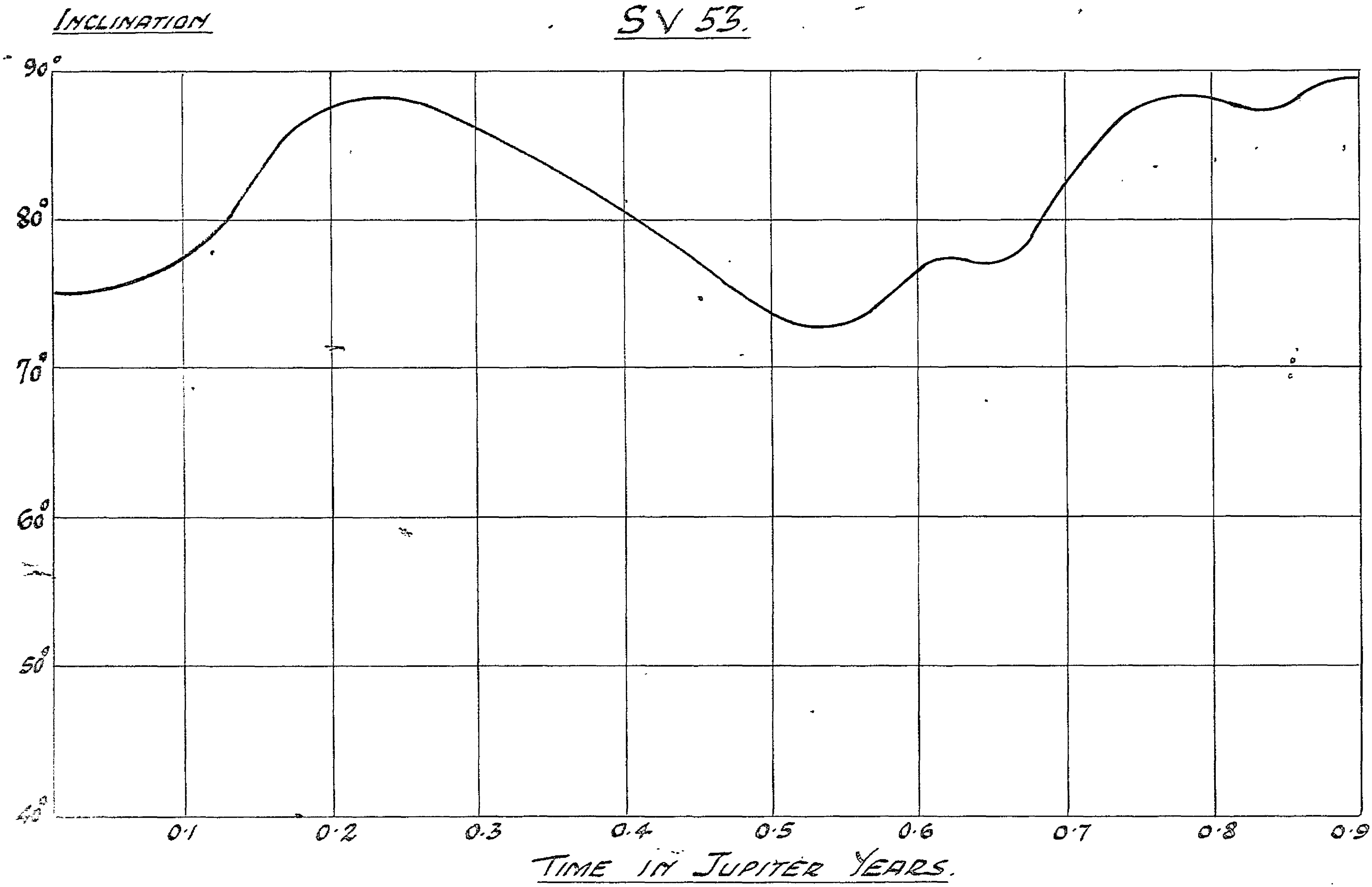






APPROX. POSITION OF THE SUN, AFTER EACH SATELLITE REVOLUTION.

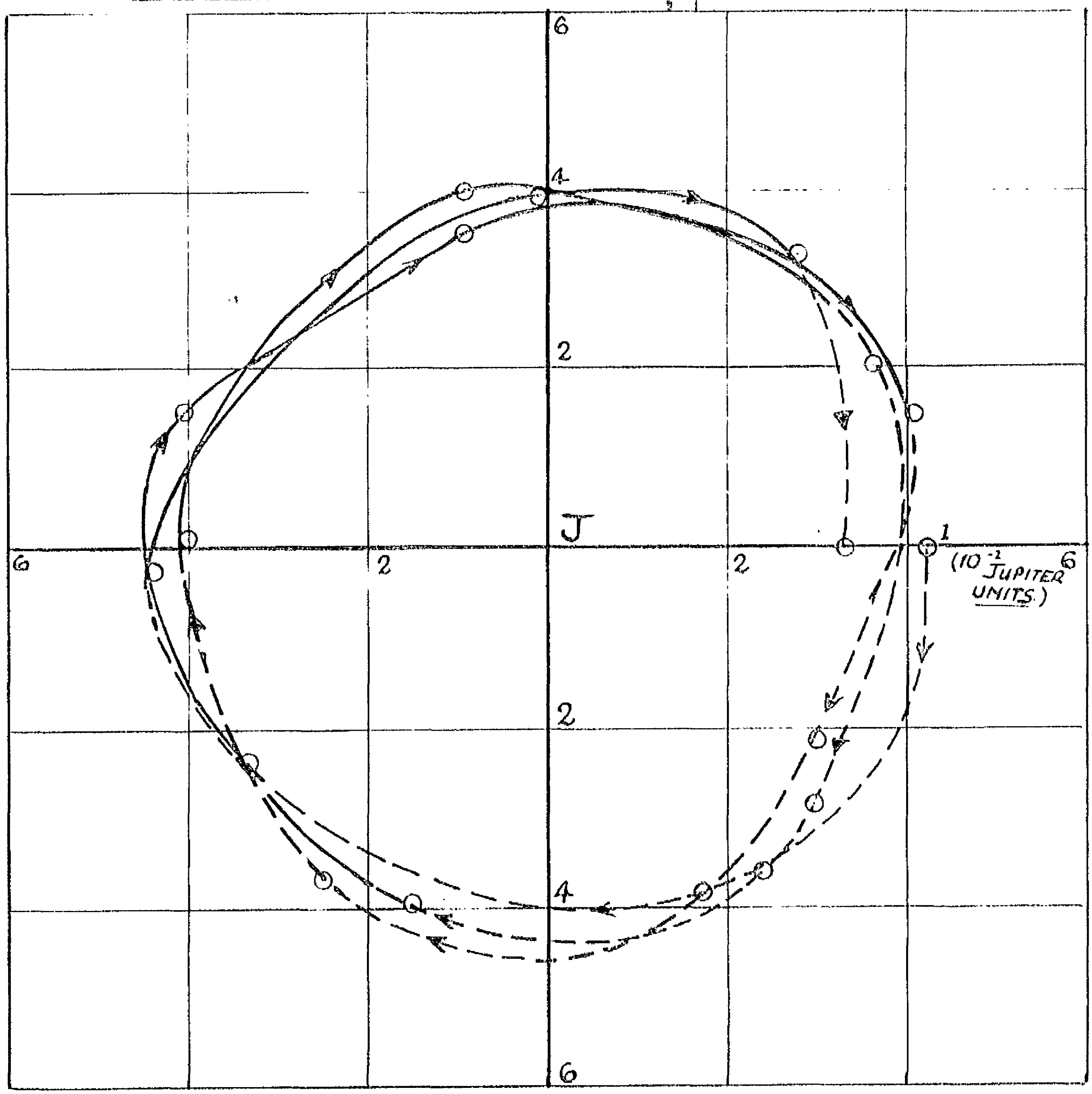
SV 53.

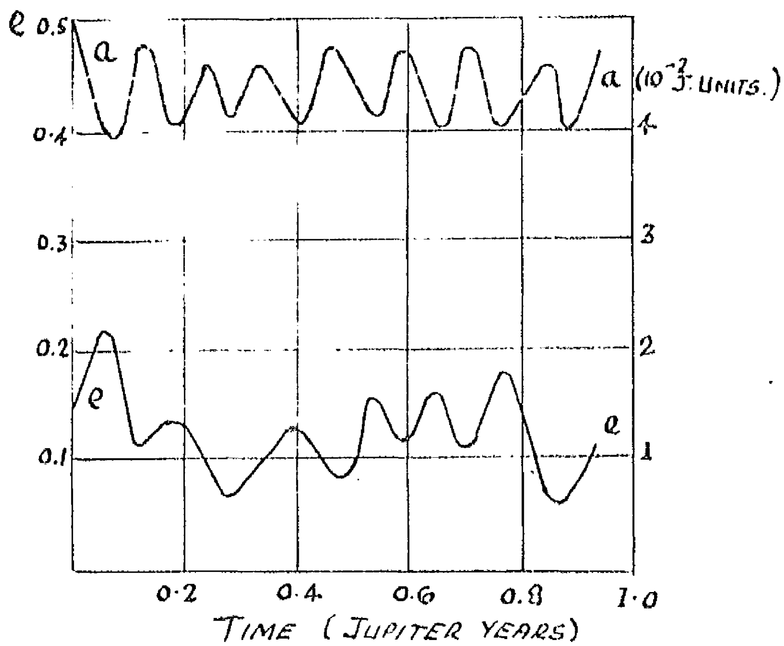


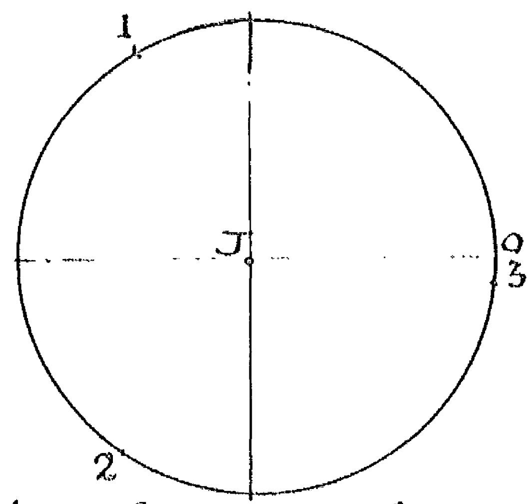
INCLINATION
90°
80°
70°
60°
50°
40°
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
TIME IN JUPITER YEARS.

S.V.39 (FIRST FEW REVOLUTIONS).

FIG. 4.13.



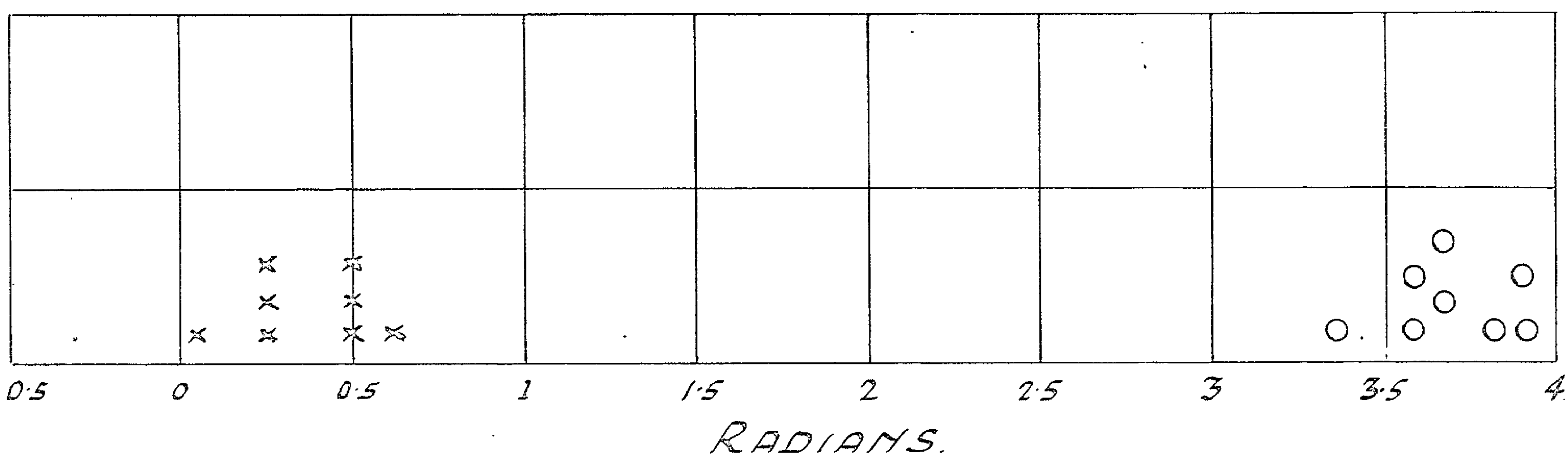




APPROX. POSITION OF THE SUN AFTER EACH SATELLITE REVOLUTION

JOVICENTRIC ANGULAR DISTANCE BETWEEN SUN AND SATELLITE AT MOMENT OF ESCAPE

DIRECT SATELLITES



JOVICENTRIC ANGULAR DISTANCE BETWEEN SUN

AND SATELLITE AT MOMENT OF ESCAPE

RETROGRADE SATELLITES

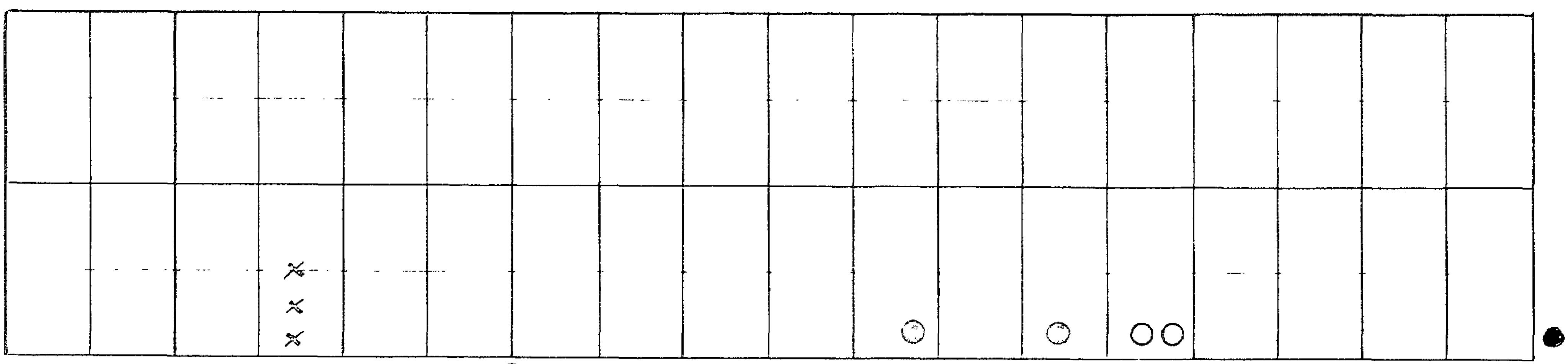
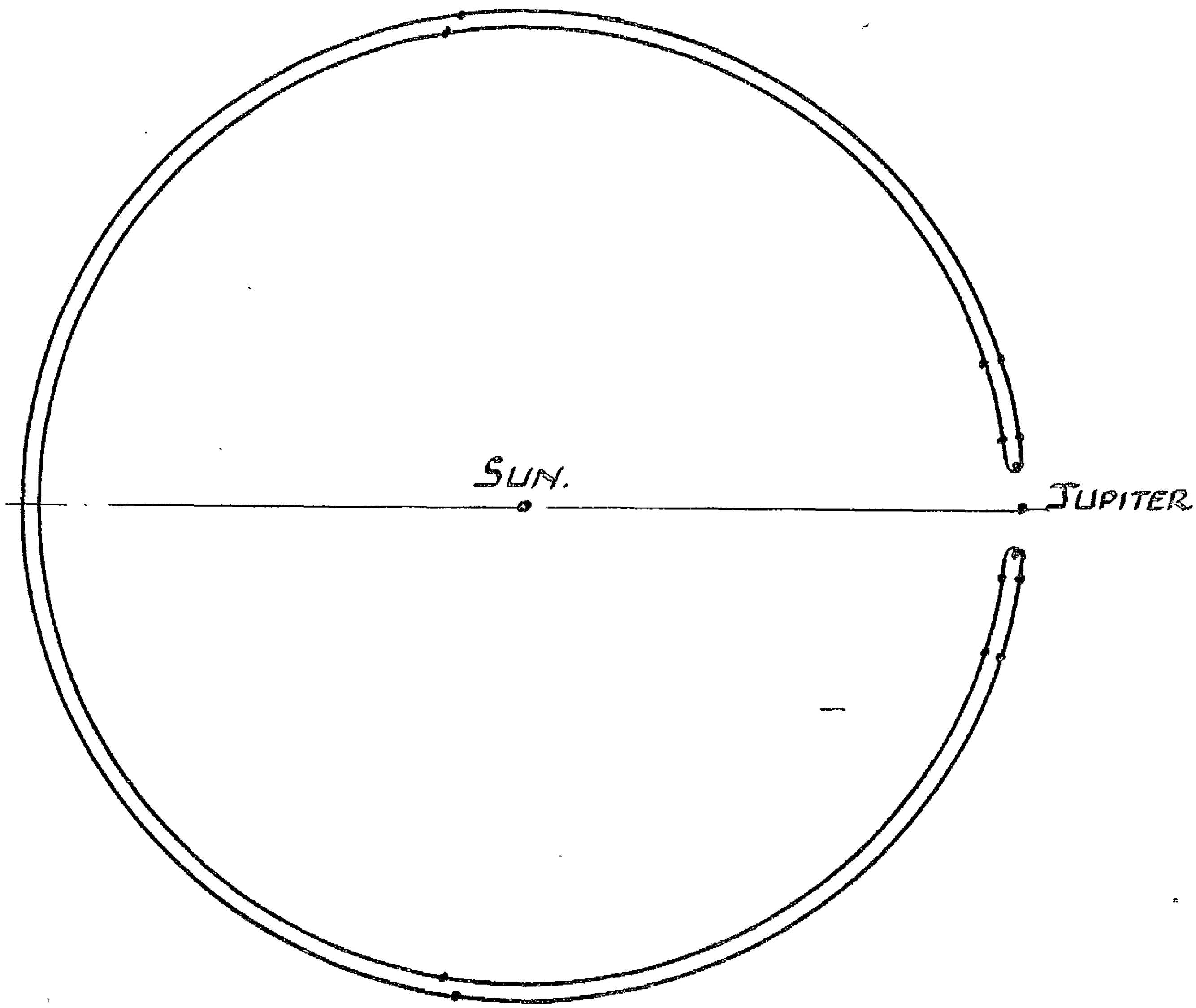


FIG. 5.2

FIG. 5.3

ZERO VELOCITY CURVES OF JACOBI INTEGRAL FOR C = 119



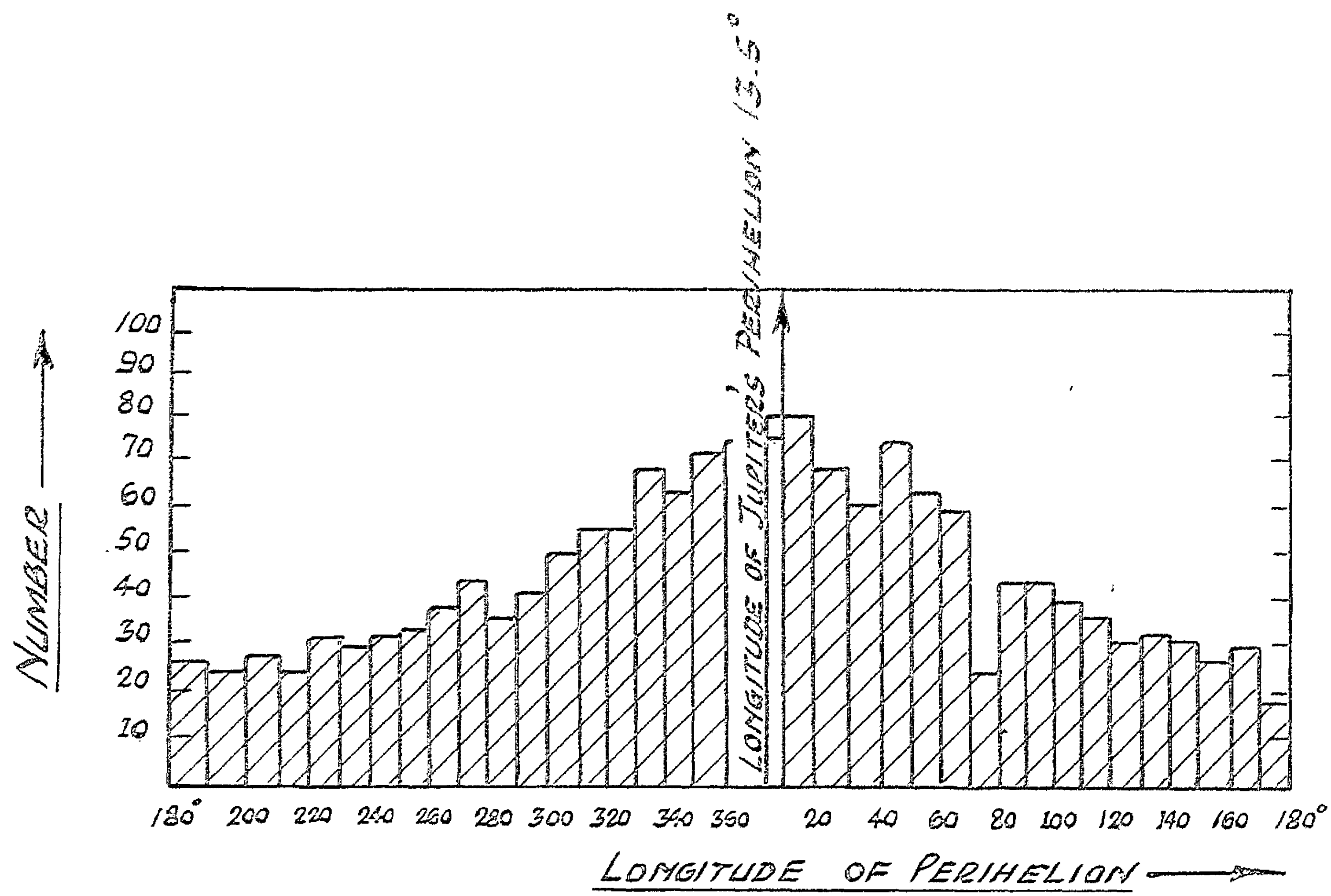
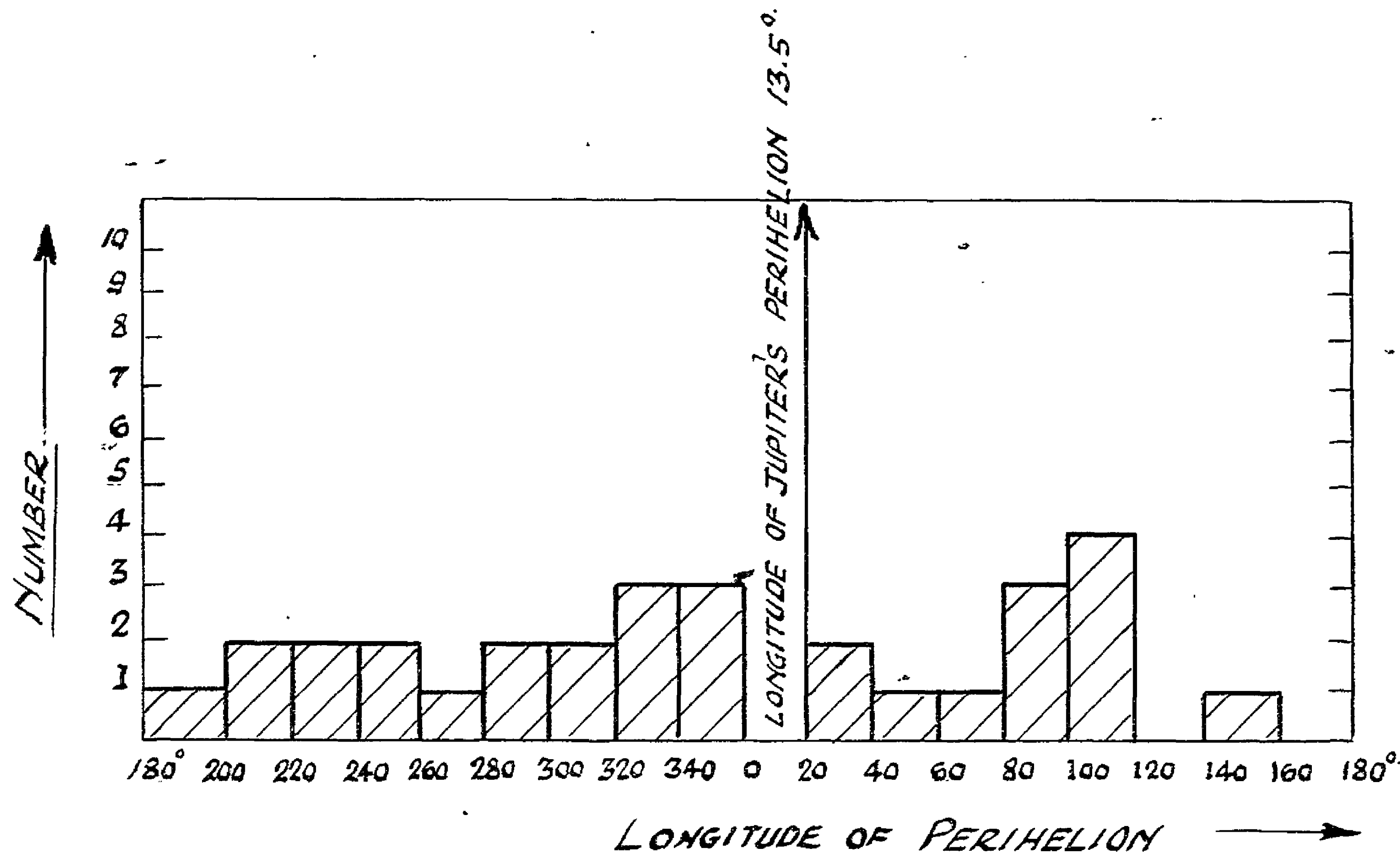
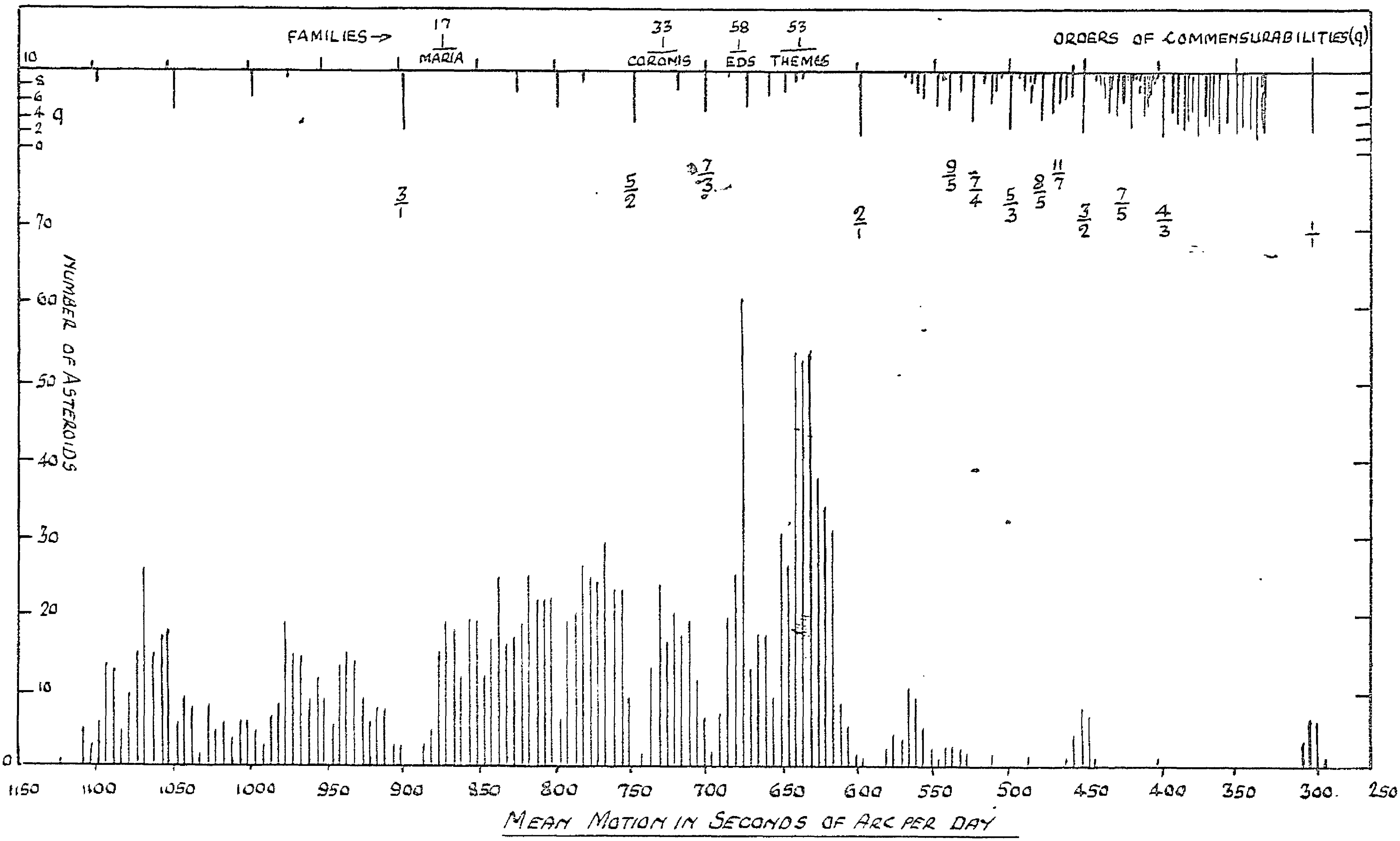

NUMBER
100
90
80
70
60
50
40
30
20
10
LONGITUDE OF JUPITER'S PERIHELION 13.5°
180° 200 220 240 260 280 300 320 340 360 20 40 60 80 100 120 140 160 180°
LONGITUDE OF PERIHELION

FIG. 5.5

DISTRIBUTION OF PERIHELIA OF ESCAPED SATELLITES





FAMILIES →
17 MARIA
33 CORONIS
58 EOS
53 THEMIS
ORDERS OF COMMENSURABILITIES (q)
q
3/1
5/2
7/3
2/1
9/5
7/4
5/3
8/5
11/7
3/2
7/5
4/3
1/1
NUMBER OF ASTEROIDS
MEAN MOTION IN SECONDS OF ARC PER DAY

FIG. 6.1

S.V. 16

VARIATIONS OF Ω/M. AND ω/M OVER LAST 30 SATELLITE REVOLUTIONS