CONGRUENCES ON REGULAR SEMIGROUPS

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by

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S U M M A R Y

It is well known that the set of congruences on a semigroup (or indeed on any Algebra) forms a lattice if ordered by set-theoretic inclusion. In this thesis certain results are presented about congruences on regular semigroups with particular emphasis on lattice-theoretic properties of the lattice of congruences.

Chapter I is of an introductory nature, in which we summarise some basic results in the theory of semigroups.

In Chapter II, certain special congruences on a regular semigroup are considered, such as the minimum group, band and semilattice congruences and the maximum idempotent-separating congruences. Interrelations among these congruences and their mutual intersections and joins are also examined. The special case of inverse semigroup is considered at the end of the chapter. Most of the material presented in this chapter is due to HOWIE [11] and HOWIE and LALLEMENT [12].

Chapter III is devoted to an account of the recent work of MUNN [22], REILLY [15] and MUNN and REILLY [21] on congruences on bisimple ω-semigroups.

In Chapter IV an account is given of TAMURA's [18] characterization of congruences on completely O-simple semigroups. This characterization is used to give a new proof of the semi-modularity of the lattice of congruences in a completely O-simple semigroup, a result due in the first instance to LALLEMENT [6].

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C O N T E N T S

INTRODUCTION

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CHAPTER I

§1 Let S be a non-empty set. A <u>relation</u> ρ on a set S is defined to be a set of ordered pairs (a, b); $a, b \in S$ where $(a, b) \in \rho$. The relation ρ can also be considered as a subset of the product set $S \times S$. We shall denote the identity relation $\{(a, a) \in S \times S\}$ by ι_S. The <u>inverse relation</u> ρ^{-1} of the relation ρ on S is defined by the set $\{(b, a) \in S \times S : (a, b) \in \rho\}$. The <u>composition</u> of two relations ρ and σ on S is defined as follows : $\rho \circ \sigma = \{(a, c) \in S \times S$, there exist $b \in S$ such that $(a, b) \in \rho$ and $(b, c) \in \sigma\}$.

<u>DEFINITION 1.1</u> The relation ρ on the set S is said to be an <u>equivalence</u> on S if and only if

I $(a, a) \in \rho$ for every a in S. (Reflexivity)

II $(a, b) \in \rho \Rightarrow (b, a) \in \rho$ (where $a, b \in S$) (Symmetry)

III $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$ (where $a, b, c \in S$) (Transitivity)

We note that if ϑ is a mapping from the set S into the set T, then the relation $\rho = \vartheta \circ \vartheta^{-1}$ on S defined by the rule $(a, b) \in \rho$ if and only if $a\vartheta = b\vartheta$, is an equivalence on the set S.

It is easily seen that the intersection of an arbitrary collection of equivalences on S is an equivalence on S. We define $\rho \vee \sigma$ to be the intersection of all equivalences on S which contain ρ and σ. Then $\rho \vee \sigma$ is called the <u>join</u> of the equivalences ρ and σ on S. It is easily seen that if $\rho \circ \sigma = \sigma \circ \rho$ then $\rho \vee \sigma = \rho \circ \sigma$.

DEFINITION 1.2 A semigroup is a set S that is closed under an associative binary operation.

Let S be a semigroup and ρ be an equivalence on S. We shall denote by $x\rho$ $(x \in S)$, the equivalence class under ρ containing the element x and by S/ρ, the set of all ρ-classes.

DEFINITION 1.3 An equivalence ρ on a semigroup S is said to be congruence if and only if it is left compatible and right compatible i.e. if and only if

$$(a, b) \in \rho \Rightarrow \begin{cases} (xa, xb) \in \rho & \text{(left compatibility)} \\ & \text{for every } x \in S \\ (ax, bx) \in \rho & \text{(right compatibility)} \end{cases}$$

Let ρ be a congruence on the semigroup S. Let $(a, b) \in \rho$ and $(c, d) \in \rho$. By left and right compatibility of ρ it follows that $(ac, bc) \in \rho$ and $(bc, bd) \in \rho$. The transitivity of ρ implies that $(ac, bd) \in \rho$. Thus $(a\rho)\circ(c\rho) \subseteq (ac)\rho$. Hence we can define an operation $(\circ)$ in S/ρ by the rule :

$$(a\rho) \circ (b\rho) = (ab)\rho \quad \text{for every } a, b \in S$$

Let us define a map $\rho^{\natural}$ from S onto S/ρ by the rule $a\rho^{\natural} = a\rho$ for every a in S. Clearly $\rho^{\natural}$ is a homomorphism of S onto S/ρ. Since S is a semigroup, it follows that S/ρ is a semigroup. Conversely it can be shown that if ϑ is a homomorphism from the semigroup S onto the semigroup T, then $\rho = \vartheta \circ \vartheta^{-1}$ is a congruence on S and that $a\rho \to a\vartheta$ $(a \in S)$ defines an isomorphism from S/ρ onto T.

It can be easily deduced that the intersection of an arbitrary collection of congruences on a semigroup S is itself a congruence on S. The join of two congruences ρ and σ on the semigroup S is defined as the intersection of all congruences on S containing ρ and σ.

Let S be a semigroup. An element $1 \in S$ is called an identity of S if and only if $x.1 = x = 1.x.$ for every $x \in S$. An element $0 \in S$ is called a zero of S if and only if $x.0 = 0 = 0.x$ for every $x \in S$. In a semigroup S, either or both of them may be absent. But it is easy to show that if either exists then it is unique. We define

$$S^1 = \begin{cases} S & \text{if S has the identity 1} \\ S \cup \{1\} & \text{if S does not have an identity} \end{cases}$$

and

$$S^0 = \begin{cases} S & \text{if S has a zero} \\ S \cup \{0\} & \text{if S does not have a zero.} \end{cases}$$

PROPOSITION 1.4 Let α be an equivalence on the semigroup S. We define $\alpha_b \subseteq S \times S$ by the rule that

$(x,y) \in \alpha_b \Leftrightarrow (s\,x\,t,\ s\,y\,t) \in \alpha \quad \forall\ s, t \in S^1$. Then α_b is the largest congruence on S contained in the equivalence α.

Proof It is easy to check that α_b is an equivalence on S. Let $z \in S$ and $(x, y) \in \alpha_b$. It follows that $(s\,x\,t,\ s\,y\,t) \in \alpha$ for every $s, t \in S^1$. In particular $((sz)\,xt,\ (sz)\,yt) \in \alpha$ for every $s, t \in S^1$ and hence $(zx, zy) \in \alpha_b$. Similarly $(xz, yz) \in \alpha_b$. Thus α_b is a congruence. Further if $(x, y) \in \alpha_b$ then $(1.\,x.\,1,\ 1.\,y.\,1) \in \alpha$ i.e. $(x, y) \in \alpha$ and hence $\alpha_b \subseteq \alpha$. On the other hand let ρ be a congruence on S such that $\rho \subseteq \alpha$. Let $(x, y) \in \rho$ then we can deduce by the compatibility of ρ that $(s\ x\ t,\ s\ y\ t)\ \varepsilon\ \rho \subseteq \alpha$ for every $s,\ t\ \varepsilon\ S^1$ and hence $(x,\ y)\ \varepsilon\ \alpha_p$. Thus $\rho \subseteq \alpha_p$.

Proved

If $\rho \circ \sigma \neq \sigma \circ \rho$, where ρ and σ are equivalences then $\rho \circ \sigma$ may fail to be an equivalence; specifically it is reflexive [redacted] but may fail to be symmetric and transitive. For an arbitrary reflexive relation ξ we define ξ^n inductively by $\xi^1 = \xi$, $\xi^n = \xi^{n-1} \circ \xi$. Then $\xi \subseteq \xi^2 \subseteq \xi^3$ ---, since ξ is reflexive.

THEOREM 1.5 If ρ and σ are equivalences on a set S, then $\rho \vee \sigma = \bigcup_{n=1}^{\infty} (\rho \circ \sigma)^n$

Proof Let us put $\varkappa = \bigcup_{n=1}^{\infty} (\rho \circ \sigma)^n$. We first note that $\varkappa$ is an equivalence. It is reflexive, since $\iota_S \subseteq \rho \circ \sigma \subseteq \varkappa$. [redacted]

First we show that $(x,y)\ \varepsilon\ \rho o \sigma$ implies that $(y,x)\ \varepsilon\ (\rho o \sigma)^2$. Clearly $(x,y)\ \varepsilon\ \rho o \sigma$ means that there exists z in S such that $(x,z)\ \varepsilon\ \rho$ and $(z,y)\ \varepsilon\ \sigma$. Since ρ and σ are equivalences it follows that $(z,x)\ \varepsilon\ \rho \subseteq \rho o \sigma$ and that $(y,z)\ \varepsilon\ \sigma \subseteq \rho o \sigma$. It follows that $(y,x)\ \varepsilon\ (\rho o \sigma)\ o\ (\rho o \sigma) = (\rho o \sigma)^2$. Now since $(x,y)\ \varepsilon\ \varkappa$ implies that $(x,y)\ \varepsilon\ (\rho o \sigma)^n$ for some $n \geq 1$. It follows that $(y,x)\ \varepsilon\ (\rho o \sigma)^{2n} \subseteq \bigcup_{n=1}^{\infty} (\rho o \sigma)^n = \varkappa$. Thus $\varkappa$ is symmetric.

[redacted] Suppose now that $(x,\ y)\ \varepsilon\ \varkappa$, $(y,\ z)\ \varepsilon\ \varkappa$. Then $(x,\ y)\ \varepsilon\ (\rho \circ \sigma)^m$ and $(y,\ z)\ \varepsilon\ (\rho \circ \sigma)^n$ for some $m,\ n \geq 1$. Thus there exist z_1, --, z_{m-1}, u_1, - -, u_{n-1} such that $(x_1,\ z_1)\ \varepsilon\ \rho \circ \sigma$, - -, $(z_{m-1},\ y)\ \varepsilon\ \rho \circ \sigma$, $(y,\ u_1)\ \varepsilon\ \rho \circ \sigma$, -- $(u_{n-1}, y)\ \varepsilon\ \rho \circ \sigma$. Hence $(x,\ y)\ \varepsilon\ (\rho \circ \sigma)^{n+m} \subseteq \varkappa$.

Also by reflexivity of ρ and σ, we have that $\rho \subseteq \rho \circ \sigma \subseteq \varkappa$, $\sigma \subseteq \rho \circ \sigma \subseteq \varkappa$. If ξ is an equivalence on S such that $\rho \subseteq \xi$, $\sigma \subseteq \xi$, then $\rho \circ \sigma \subseteq \xi \circ \xi = \xi$. Hence we see that $(\rho \circ \sigma)^2 = (\rho \circ \sigma) \circ (\rho \circ \sigma) \subseteq \xi \circ \xi = \xi$; and in general $(\rho \circ \sigma)^n \subseteq \xi$ $(n = 1,\ 2,\ --)$. Thus $\varkappa \subseteq \xi$ and so $\varkappa = \rho \vee \sigma$.

Proved

PROPOSITION 1.6 If S is a semigroup and ρ, σ are congruences on S, then $\kappa = \bigcup_{n=1}^{\infty} (\rho \circ \sigma)^n$ is a congruence on S and so is the smallest congruence containing ρ and σ.

Proof We know that κ is an equivalence. Suppose that $(x, y) \in \kappa$ and that z is an arbitrary element of S. Then $(x, y) \in (\rho \circ \sigma)^n$ for some n. and so there exist $u_1, --, u_{n-1} \in S$ such that

$$(x, u_1), (u_1, u_2), --, (u_{n-1}, y) \in \rho \circ \sigma.$$

Now $(a, b) \in \rho \circ \sigma$ implies that $(ca, cb) \in \rho \circ \sigma$ for every c in S; for there exists d $\in$ S such that $(a, d) \in \rho$ and $(d, b) \in \rho$ and so, by left compatibility of ρ and σ, $(ca, cd) \in \rho$, $(cd, cb) \in \sigma$ from which it follows that $(ca, cb) \in \rho \circ \sigma$. Thus $(zx, zu_1) \in \rho \circ \sigma$, $(zu_1, zu_2) \in \rho \circ \sigma$, - - $(zu_{n-1}, zy) \in \rho \circ \sigma$ and so $(zx, zy) \in (\rho \circ \sigma)^n \subseteq \kappa$. Similarly right compatibility of κ can be established and so κ is a congruence. Obviously it is the smallest congruence containing ρ and σ.

Proved

PROPOSITION 1.7 If ρ and ρ' are congruences on the semigroup S such that $\rho \subseteq \rho'$, then the relation $\rho'/_\rho$ on $S/_\rho$ defined by $\rho'/_\rho = \{(x\rho, y\rho) : (x, y) \in \rho'\}$ is a congruence on $S/_\rho$. Moreover $S/_{\rho'} \cong (S/_\rho) / (\rho'/_\rho)$. Conversely if δ is a congruence on $S/_\rho$, then there exists a congruence ρ' on S such that $\rho \subseteq \rho'$ and $\delta = \rho'/_\rho$.

Proof Clearly $\rho'/_\rho$ is an equivalence since ρ' is a congruence. Next let $(x\rho, y\rho) \in \rho'/_\rho$. It follows that $(x, y) \in \rho'$. Let $z\rho \in S/_\rho$ $(z \in S)$. Now we observe that $(x\rho)(z\rho) = (xz)\rho$ and $(y\rho)(z\rho) = (yz)\rho$. Compatibility of ρ' implies that $(xz, yz) \in \rho'$. It follows that

$((x\rho)(z\rho), (y\rho)(z\rho)) \in \rho'/_{\rho}$. Similarly the left compatibility of $\rho'/_{\rho}$ can be established. Thus $\rho'/_{\rho}$ is a congruence on $S/_{\rho}$. Clearly the mapping $\vartheta : S/_{\rho'} \longrightarrow (S/_{\rho}) / (\rho'/_{\rho})$ defined by the rule, $(x\rho')\vartheta = (x\rho)(\rho'/_{\rho})$ is an isomorphism.

Conversely suppose δ is a congruence on $S/_{\rho}$. Let us define ρ' on S by $\rho' = \{(x, y) : (x\rho, y\rho) \in \delta\}$. Then ρ' is easily verified to be a congruence on the semigroup S. Further let $(x, y) \in \rho$. It follows that $x\rho = y\rho$ and hence $(x\rho, y\rho) \in \delta$. Hence we have that $(x, y) \in \rho'$. Thus $\rho \subseteq \rho'$. Obviously $\delta = \rho'/_{\rho}$.

<u>Proved</u>

<u>NOTE 1.8</u>

It is easily seen that if γ and δ are congruences on the semigroup S, containing ρ, then $\gamma \cap \delta$ is a congruence containing ρ, and

$$(\gamma \cap \delta)/_{\rho} = (\gamma/_{\rho}) \cap (\delta/_{\rho})$$

Let P be a non-empty set. A <u>partial order relation</u> in P is a relation $\leqslant$ in P which is assumed to have the following properties

(a) $x \leqslant x, \ \forall \ x \in P$ (Reflexivity)

(b) $x \leqslant y$ and $y \leqslant x \Rightarrow x = y, \quad x, y \in P$ (Anti-symmetry)

(c) $x \leqslant y$ and $y \leqslant z \Rightarrow x \leqslant z, \quad x, y, z \in P$ (Transitivity)

A non-empty set P, in which there is defined a partial order relation is called a <u>partially ordered set</u>.

Let P be a partially ordered set. An element x in P is said to be maximal if $y \geq x \Rightarrow y = x$ i.e. if no element other than x itself is greater or equal to x. Similarly an element x in P is called minimal if $y \leq x \Rightarrow y = x$. Let A be a non-empty subset of P. An element x in P is called a lower bound of A if $x \leq a$ for each $a \in A$; and a lower bound of A is called a greatest lower bound of A, if it is greater than or equal to every lower bound of A. Similarly an element y in P is said to be an upper bound of A if $a \leq y$ for every $a \in A$, and a least upper bound of A is an upper bound of A which is less than or equal to every upper bound of A.

DEFINITION 1.9 A lattice is a partially ordered set L in which each pair of elements has a greatest lower bound and a least upper bound.

If x and y are any two elements in a lattice L we denote their least upper bound and greatest lower bound by $x \vee y$ and $x \wedge y$ respectively. We shall call $x \vee y$ and $x \wedge y$, the join and meet of x, y respectively. A partially ordered set P is called an upper [lower] semilattice if every pair x, y in P has a join [meet] in P. We shall use the term semilattice to mean lower semilattice unless the contrary is specified.

A lattice L is called modular if and only if its elements satisfy the following condition

(1) $$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \quad (x, y, z \in L)$$

There is an alternative characterization for the modularity which we shall use frequently.

A lattice L is modular if and only if

(2) $$[x \leq y,\ x \vee z = y \vee z,\ x \wedge z = y \wedge z] \Rightarrow x = y \quad (x, y, z \in L)$$

Let x, y be two arbitrary elements of a lattice L. We say that x <u>covers</u> y (and write $x \succ y$) if and only if (I) x is strictly greater than y i.e. $x > y$ and (II) there does not exist $z \in L$ such that $x > z > y$.

Let L be a lattice. We say that lattice L is <u>semimodular</u> if and only if

(3) $[x \succ x \wedge y$ and $y \succ x \wedge y] \Rightarrow [x \vee y \succ x$ and $x \vee y \succ y]$

where $x, y \in L$.

<u>PROPOSITION 1.10.</u> <u>Every modular lattice is semimodular.</u>

<u>Proof</u> Suppose that $x \succ x \wedge y$ and $y \succ x \wedge y$, and suppose by way of contradiction, that there exists $z \in L$ such that $x < z < x \vee y$. Then we have that $x \wedge y \leq z \wedge y \leq (x \vee y) \wedge y = y$

Now we observe that

$$z \wedge y = x \wedge y \Rightarrow x = x \vee (y \wedge x) = x \vee (y \wedge z) = (x \vee y) \wedge z \text{ (by modularity)}$$
$$= z \quad (\text{since } z < x \vee y)$$

which is a contradiction.

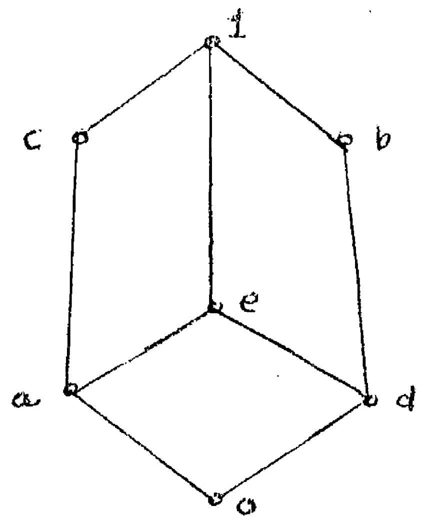
Further

$$z \wedge y = y \Rightarrow x \vee y = x \vee (y \wedge z) = (x \vee y) \wedge z \quad (\text{by modularity}) = z$$

a contradiction again. Hence $x \wedge y < z \wedge y < y$, a contradiction to the assumption that $y \succ x \wedge y$. We conclude that $x \vee y \succ x$; similarly we can show that $x \vee y \succ y$.

<u>Proved</u>

On the other hand NOT every semimodular lattice need be modular. To show this we construct the following example. Let us consider the lattice L = $\{0, 1, a\ b, c, d, e\}$ represented by the lattice-diagram



The lattice L is clearly semimodular, but it is not modular, since

$a < c$, but

$$a \vee (b \wedge c) = a \vee 0 = a$$

and

$$(a \vee b) \wedge c = 1 \wedge c = c$$

G. BIRKHOFF [5] has shown that the equivalences on a semigroup S form a semimodular lattice.

§2 Let S be a semigroup and let A and B be non-empty subsets of the semigroup S. Then we define

$$AB = \{x \in S : x = ab \text{ for some } a \in A \text{ and } b \in B\}$$

A subset A of the semigroup S is said to be <u>subsemigroup</u> of S if $A.A \subseteq A$. A subset A of the semigroup S is said to be <u>left ideal</u>

[right ideal] of S if $SA \subseteq A$ [$AS \subseteq A$]. A subset A of the semigroup S is said to be an ideal of S if $SAS \subseteq A$. A Principal left ideal is a left ideal which is generated by a single element. Principal right ideal and Principal ideal are defined similarly. Let a be an arbitrary element of the semigroup S. Then the following expressions are easily obtained.

The Principal left ideal of S generated by a is $S^1 a$.
" " right " " " " " " " $a S^1$.
" " ideal " " " " " " " $S^1 a S^1$.

Now we describe some relations on S which arise from the notion of [left, right] ideals. These relations were introduced by J. A. GREEN [9].

Let S by a semigroup and let $a, b \in S$. Then we define

$(a, b) \in \mathcal{L}$ iff $S^1 a = S^1 b$

$(a, b) \in \mathcal{R}$ iff $a S^1 = b S^1$

$(a, b) \in \mathcal{J}$ iff $S^1 a S^1 = S^1 b S^1$

$(a, b) \in \mathcal{H}$ iff $S^1 a = S^1 b$ and $a S^1 = b S^1$

Evidently $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. Also if S is commutative then $\mathcal{J} = \mathcal{H}$. All these relations are easily verified to be equivalences on the semigroup S.

We call an equivalence ρ on a semigroup S, a left congruence if and only if it is left compatible and ρ will be called right congruence if it is right compatible.

Clearly $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on S.

PROPOSITION 2.1 Let S be a semigroup. Let λ be a left congruence on S such that $\lambda \subseteq \mathcal{R}$ and ρ be a right congruence on S such that $\rho \subseteq \mathcal{L}$. Then $\lambda \circ \rho = \rho \circ \lambda$; in particular $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof In view of the symmetry, it is sufficient to show that $\rho \circ \lambda \subseteq \lambda \circ \rho$. Let $(a, c) \in \rho \circ \lambda$. It follows that there exists $b \in S$ such that $(a, b) \in \rho$ and $(b, c) \in \lambda$. Since $\rho \subseteq \mathcal{L}$, it follows that there exists $u \in S^1$ such that $a = ub$. Further $\lambda \subseteq \mathcal{R}$ implies that there exists $v \in S^1$ such that $c = bv$. Now we observe that

$av = (ub)v = u(bv) = uc$. Let $d = av = uc$.

Since λ is a left congruence, we deduce that $(ub, uc) \in \lambda$, i.e. that $(a, d) \in \lambda$. Further since ρ is a right congruence, it follows that $(av, bv) \in \rho$ i.e. that $(d, c) \in \rho$. It follows that $(a, c) \in \lambda \circ \rho$. Thus $\rho \circ \lambda \subseteq \lambda \circ \rho$. Conversely $\lambda \circ \rho \subseteq \rho \circ \lambda$ by symmetry and the required result follows.

Proved

We define $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Then it is clear that $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R}$. Since $\mathcal{J} \supseteq \mathcal{L}$ and $\mathcal{J} \supseteq \mathcal{R}$, we have that $\mathcal{D} \subseteq \mathcal{J}$.

Each $\mathcal{D}$-class of the semigroup S can be expressed as the union of $\mathcal{R}$-classes of S and $\mathcal{L}$-classes of the semigroup S. Further if L and R are $\mathcal{L}$- and $\mathcal{R}$-classes of the semigroup S, then $L \cap R \neq \varphi$ if and only if L and R are contained in the same $\mathcal{D}$-class of S. We shall denote the $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{J}$-classes of S containing a element $a \in S$ by H_a, L_a, R_a, D_a, and J_a respectively. Obviously $H_a \subseteq L_a \subseteq D_a \subseteq J_a$ and $H_a \subseteq R_a \subseteq D_a \subseteq J_a$.

DEFINITION 2.2 A semigroup S is said to be bisimple if it consists of a single $\mathcal{D}$ -class.

LEMMA 2.3 (J.A. GREEN) Let a and b be $\mathcal{R}$ -equivalent elements of the semigroup S and let $s, t \in S^1$ be such that $as = b$ and $bt = a$. Then the maps $\sigma : L_a \rightarrow L_b$ and $\tau : L_b \rightarrow L_a$ defined by the rules $x\sigma = xs$ $(x \in L_a)$ and $y\tau = yt$ $(y \in L_b)$ are mutually inverse $\mathcal{R}$ -class preserving bijections.

Proof Let $x \in L_a$. Then $(x, a) \in \mathcal{L}$ and since $\mathcal{L}$ is a right congruence we have that $(xs, as) \in \mathcal{L}$; i.e. $xs \in L_b$. Similarly $yt \in L_a$ for every $y \in L_b$. Now $(x, a) \in \mathcal{L}$ implies that there exist an element $u \in S^1$ such that $x = ua$. Now we can deduce that

$$x(\sigma\tau) = xst = uast = ubt = ua = x$$

Hence $\sigma\tau$ is the identity mapping of the set L_a. Similarly $\tau\sigma$ is the identity mapping of the set L_b. Thus σ and τ are mutually inverse bijections. Further since $xst = x$ and $xs = xs$, it follows that $(x, xs) \in \mathcal{R}$ for any $x \in L_a$. Thus σ is $\mathcal{R}$ -class preserving. Similarly τ is $\mathcal{R}$ -class preserving.

Proved

The dual of the above lemma can be proved similarly. The following two corollaries are immediate consequences of the lemma.

COROLLARY 2.4 *Let a, c, s be elements of S such that* $(a, as) \in \mathcal{R}$ *and* $(a, c) \in \mathcal{L}$. *Then the mapping* $\sigma' : H_c \rightarrow H_{cs}$ *defined by the rule* $x\sigma' = x s \ (x \in H_c)$ *is a bijection and so* $H_c S = H_{cs}$; *in particular* $H_a S = H_{as}$.

We omit the proof.

COROLLARY 2.5 *If a, c are $\mathcal{D}$-equivalent elements of S then H_a and H_c have the same cardinal.*

We omit the proof.

THEOREM 2.6 *Let H be an $\mathcal{H}$-class of the semigroup S, such that* $r s \in H$ *for some* $r, s \in H$. *Then H is a group.*

Proof It is sufficient to show that $aH = H = Ha$ for every $a \in H$. Let $r, s \in H$ such that $rs \in H$. Hence $(r, rs) \in \mathcal{H} \subseteq \mathcal{R}$. It follows by the Corollary 2.4 that $H_r S = H_{rs}$ that is $Hs = H$. Now let $a \in H$. Then $a s \in H$ and so $(a, as) \in \mathcal{H} \subseteq \mathcal{L}$. Hence $aH_s = H_{as}$ by the dual of the Corollary 2.4. Thus $aH = H$. Similarly $H = Ha$.

Proved

The following corollaries are immediate.

COROLLARY 2.7 *If e be an idempotent of the semigroup S, then H_e is the unique maximal subgroup of S with the identity e.*

COROLLARY 2.8 *An $\mathcal{H}$-class contains atmost one idempotent.*

Finally we prove a useful lemma.

<u>LEMMA 2.9</u> <u>Let ρ be a congruence on the semigroup S and let</u> $x, y \in S$. <u>Then</u> $(x, y) \in \mathcal{L}$ <u>implies that</u> $(x\rho, y\rho) \in \mathcal{L}$ <u>in</u> $S/_\rho$. <u>Dually</u> $(x, y) \in \mathcal{R}$ <u>implies that</u> $(x\rho, y\rho) \in \mathcal{R}$ <u>in</u> $S/_\rho$.

<u>Proof</u> Let $(x, y) \in \mathcal{L}$. It follows that there exist $s, t \in S^1$ such that $x = sy$ and $y = tx$. Hence we deduce that $(x\rho) = (s\rho)(y\rho)$ and $(y\rho) = (t\rho)(x\rho)$. It follows that $(x\rho, y\rho) \in \mathcal{L}$ in $S/_\rho$. Dual statement is similarly established.

<u>Proved</u>

The following Corollary is easily deduced.

<u>COROLLARY 2.10</u> <u>Let</u> ρ <u>be a congruence on the semigroup S.</u> <u>Then</u>

(I) $(x, y) \in \mathcal{H} \Rightarrow (x\rho, y\rho) \in \mathcal{H}$ in $S/_\rho$ $(x, y \in S)$

(II) $(x, y) \in \mathcal{J} \Rightarrow (x\rho, y\rho) \in \mathcal{J}$ in $S/_\rho$ $(x, y \in S)$

We omit the proof.

§3 Now we introduce the concept of <u>regularity</u> in semigroups.

<u>DEFINITION 3.1</u> An element a in a semigroup S is called <u>regular</u> if there exists an element b in S such that $aba = a$. We say that semigroup S is <u>regular</u> if all its elements are regular.

An element e in a semigroup S is called an <u>idempotent</u> iff $e^2 = e.e = e$.

Clearly if $aba = a$, then ab and ba are idempotents in S. It is easily shown that an element a in the semigroup S is regular if and only if there exists an idempotent e in S such that $(a, e) \in \mathcal{R}$; dually a is a regular element if and only if there exists an idempotent f in S such that $(a, f) \in \mathcal{L}$.

<u>PROPOSITION 3.2</u> <u>Let D be a $\mathcal{D}$ -class of the semigroup S and let D contain a regular element of S.</u> <u>Then every element of D is regular.</u> <u>Moreover every $\mathcal{R}$ -class and every $\mathcal{L}$-class in D contain idempotents.</u>

<u>Proof</u> Let a be a regular element of D, and let $(a, b) \in \mathcal{D}$. It follows that there exists $c \in S$ such that $(a, c) \in \mathcal{R}$ and $(c, b) \in \mathcal{L}$. Further since a is regular, there exists an idempotent e in S such that $(a, e) \in \mathcal{R}$. It follows that $(e, c) \in \mathcal{R}$ and hence c is a regular element. Hence there exists an idempotent f in S such that $(c, f) \in \mathcal{L}$. Thus $(b, f) \in \mathcal{L}$ and hence b is regular. Finally it follows that there exists an idempotent g in S such that $(b, g) \in \mathcal{R}$.

<u>Proved</u>

<u>DEFINITION 3.3</u> Two elements a and b of a semigroup S are said to be <u>inverses</u> of each other if

$$aba = a \quad \text{and} \quad bab = b$$

PROPOSITION 3.4 An element a in the semigroup S has an inverse if and only if a is regular.

Proof Suppose a has an inverse a' in S. Then clearly $aa'a = a$ and $a'aa' = a'$ and hence a is regular. Conversely suppose a is regular. It follows that there exists an element b such that $aba = a$. Let us write $a' = bab$. Then we observe that

$$a a' a = a b \underline{a b a} = a b a = a \quad \text{and}$$

$$a' a a' = b a b \underline{a b a} b = b \underline{a b a} b = b a b = a'$$

Hence a has an inverse a' in S.

Proved

PROPOSITION 3.5 If e and f are idempotents in a regular semigroup S, then ef has an idempotent inverse g in S such that $ge = fg = g$.

Proof Since S is regular, ef has an inverse x (say) in S, that is $ef\,x\,ef = ef$ and $x\,e\,f\,x = x$. We further note that $(f\,x\,e)^2 = f\,(xe.\,fx)\,e = f\,x\,e$ and hence $f\,x\,e$ is an idempotent in S. Further we observe that $ef.\,fxe.\,ef = ef.\,x.\,ef = ef$ and

$$fxe.\,ef.\,fxe = f\,(xefx)\,e = f\,x\,e$$

Hence $f\,x\,e$ is an inverse of ef with the required properties, that is $g = f\,x\,e$.

Proved

The next theorem helps us in locating the position of an inverse.

<u>THEOREM 3.6</u> <u>Let a be a regular element of S.</u>

(I) If a' is an inverse of a, then $(a, a') \in \mathcal{D}$. Moreover the $\mathcal{H}$-classes $R_a \cap L_{a'}$ and $R_{a'} \cap L_a$ contain idempotents.

(II) Let e, f be idempotents such that $(a, e) \in \mathcal{R}$ and $(a, f) \in \mathcal{L}$. Then $R_f \cap L_e$ contain exactly one inverse a' of a. Also $aa' = e$ and $a'a = f$.

We omit the proof.

§4 Now we introduce the concept of an inverse semigroup.

<u>DEFINITION 4.1</u> A semigroup S is called an <u>inverse semigroup</u> provided each element in S has a <u>unique</u> inverse in S.

The following theorem due to VAGNER [1952b] and MUNN and PENROSE [1955] characterizes an inverse semigroup.

<u>THEOREM 4.2</u> <u>The following conditions on a semigroup S are equivalent</u>

(I) <u>S is regular and its idempotents commute</u>

(II) <u>each $\mathcal{R}$-class and each $\mathcal{L}$-class in S contains exactly one idempotent</u>

(III) <u>S is an inverse semigroup</u>.

We omit the proof.

It is clear that the product ef of two idempotents e and f in an inverse semigroup S is itself an idempotent.

PROPOSITION 4.3 Let S be an inverse semigroup and let e, f be idempotents in S. Then $Se \cap Sf = Sef = Sfe$

Proof Obviously $Sef = Sfe \subseteq Se \cap Sf$. Conversely let $a \in Se \cap Sf$. Hence there exist $x, y \in S$ such that $a = xe$ and $a = yf$. Next we observe that

$$ae = x\,e\,e = xe = a \quad \text{and} \quad af = y\,f\,f = y\,f = a.$$

Now we can deduce that $aef = af = a$. Hence $a \in Sef$. Thus $Se \cap Sf = Sef$.

Proved

PROPOSITION 4.4 Suppose e is an idempotent in an inverse semigroup S. Then $x^{-1}\,e\,x$ and $x\,e\,x^{-1}$ are idempotents in S for every element x in S.

Proof Making use of commutativity of idempotents in the inverse semigroup S we easily deduce that

$$\begin{aligned}(x^{-1}ex)^2 &= x^{-1}e\ (xx^{-1})\ ex \\ &= x^{-1}\,(xx^{-1})\ e\ e\ x \qquad \because xx^{-1} \text{ is idempotent} \\ &= x^{-1}\ ex\end{aligned}$$

Similarly we observe that

$$\begin{aligned}(x\ e\ x^{-1})^2 &= x\ e\ (x^{-1}.\ x)\ ex^{-1} \\ &= xx^{-1}\ x\ e\ x^{-1} \qquad \because x\,x^{-1} \text{ is idempotent.} \\ &= x\ e\ x^{-1}\end{aligned}$$

Next we construct an example of an inverse semigroup.

EXAMPLE 4.5 Let N denote the set of non-negative integers and let $B = N \times N$. We define multiplication in B as follows :

$$(m, n)(p, q) = (m + p - r,\ n + q - r)$$

where $r = \min\{n, p\}$. Associativity is easily verified and we call the semigroup B, the Bicyclic semigroup. The elements of B can be expressed in an array of rows and columns as follows :

(0, 0)	(0, 1)	(0, 2)	(0, 3)	. . . .
(1, 0)	(1, 1)	(1, 2)	(1, 3)	. . . .
(2, 0)	(2, 1)	(2, 2)	(2, 3)	. . . .
(3, 0)	(3, 1)	(3, 2)	(3, 3)	. . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . .
. . . . . .	. . . . . .	. . . . . .	. . . . . .	. . . .

All elements in the same row are $\mathcal{R}$-equivalent and all elements in the same column are $\mathcal{L}$-equivalent. Moreover since $R \cap L \neq \varphi$ for each $\mathcal{R}$-class R and $\mathcal{L}$-class L, it follows that B consists of a single $\mathcal{D}$-class, i.e. B is bisimple. Element (0, 0) is the identity of the semigroup B. Let $(m, n) \in B$. Then (n, m) is the unique inverse of (m, n) in B. Thus B is an inverse semigroup with identity. The elements (m, m) for every $m \in N$ are idempotents of the semigroup B.

Let S be an inverse semigroup. A subsemigroup T of S is said to be an inverse subsemigroup provided each $a \in T$ has its unique inverse a^{-1} in T.

Let a, b be two arbitrary elements of an inverse semigroup S.

We define a partial ordering $\leqslant$ by the rule

(4) ——— $a \leqslant b$ iff there exists an idempotent e such that $a = e\,b$

It can be shown that the above condition is equivalent to any one of the following conditions

$a^{-1}a = a^{-1}b$ or $a^{-1}a = b^{-1}a$ or $a\,a^{-1} = b\,a^{-1}$ or $a\,a^{-1} = a\,b^{-1}$

This ordering was introduced by VAGNER [19] and we shall call it the natural partial ordering in the elements of an inverse semigroup. If we restrict this ordering to the set E of idempotents of an inverse semigroup S, then an alternative characterization is

(5) ——— $e \leqslant f$ iff $e = e\,f = f\,e$ $(e, f \in E)$

Further if $a \leq b$ then $a^{-1} \leq b^{-1}$. Also if e is an idempotent in S then $ea \leqslant a$, $a\,e \leqslant a$, $a\,e\,b \leqslant a\,b$ for arbitrary elements $a, b \in S$.

We call a semigroup S a band if all its elements are idempotents. Since idempotents in an inverse semigroup commute, the set E of idempotents of an inverse semigroup S is a commutative band. It can be easily proved that a commutative band is a semilattice with respect to the natural partial ordering $\leqslant$ of S. The $e \wedge f$ of two elements in E is just their product $e.\,f$. Conversely a semilattice is a commutative band with respect to the meet operation. Let H be an arbitrary subset of S. Then we define $H\omega = \{ a \in S\ ;\ a \geqslant h$ for some $h \in H \}$. Then $H\omega$ is called the closure of H under the order relation $\geqslant$. A set H will be said to be closed if $H = H\omega$. An inverse subsemigroup H of an inverse semigroup S will be called self-conjugate if $x \in H$ implies that $z x z^{-1} \in H$ for every element $z \in S$.

PROPOSITION 4. 6 Let K be a closed, self-conjugate, inverse subsemigroup of an inverse semigroup S and let $K \supseteq E$ where E is the set of idempotents of S. Then the relation ρ_k defined by the rule that

$$(x, y) \in \rho_k \quad \text{iff} \quad x y^{-1} \in K \quad \text{for every} \quad x, y \in S$$

is a congruence on S.

Proof Since $x x^{-1} \in E \subseteq K$, reflexivity is obvious. Further, since K is an inverse subsemigroup, $x y^{-1} \in K$ implies $y x^{-1} \in K$. Thus ρ_k is symmetric. Next let $(x, y) \in \rho_k$ and $(y, z) \in \rho_k$. It follows that $x y^{-1} \in K$ and $y z^{-1} \in K$. It follows that $x y^{-1} y z^{-1} \in K$. But $x y^{-1} y z^{-1} \leq x z^{-1}$; hence $x z^{-1} \in K$, since K is closed. Hence $(x, z) \in \rho_k$. Thus ρ_k is an equivalence. Next let $(x, y) \in \rho_k$ and hence $x y^{-1} \in K$. Let $z \in S$. Then we observe that $z (x y^{-1}) z^{-1} = z x (z y)^{-1} \in K$, since K is self-conjugate. Thus $(z x, z y) \in \rho_k$. On the other hand we observe that

$$(x z) (y z)^{-1} = x z z^{-1} y^{-1} = x z z^{-1} y^{-1} y y^{-1}$$
$$= x y^{-1} y z z^{-1} y^{-1} = (x y^{-1}) (y z z^{-1} y^{-1}) \in K. \; E \subseteq K ;$$

and hence $(x z, y z) \in \rho_k$. Thus ρ_k is congruence on S.

Proved

By a decomposition of a semigroup S, we mean a partition of S into the union of disjoint subsemigroups S_α $(\alpha \in \Omega)$. Suppose that $S = \bigcup \{S_\alpha : \alpha \in \Omega\}$ is a decomposition of S such that, for every pair of elements α, β of the index set Ω, there is an element γ of Ω such that $S_\alpha \cdot S_\beta \subseteq S_\gamma$. If we define a product in Ω by $\alpha\beta = \gamma$ if $S_\alpha \cdot S_\beta \subseteq S_\gamma$, then Ω becomes thereby a band. We say that S is the <u>union of the band Ω of semigroups S_α</u> $(\alpha \in \Omega)$. The mapping φ defined by $a\varphi = \alpha$ if $a \in S_\alpha$ is a homomorphism of S upon Ω and the S_α are the congruence-classes of the congruence $\varphi \circ \varphi^{-1}$. Conversely, if φ is a homomorphism of a semigroup S upon a band Ω, then the inverse image $S_\alpha = \alpha\varphi^{-1}$ of each element α of Ω is a subsemigroup of S and S is the union of the band Ω of semigroups S_α $(\alpha \in \Omega)$. If Ω is commutative we say that S is the <u>union of the semilattice Ω of semigroups</u> S_α $(\alpha \in \Omega)$. Generally we shall use the abbreviated expression, S is band [semilattice] of semigroups of the type $\mathcal{C}$, to mean that S is the union of a band [semilattice] Ω of semigroups S_α $(\alpha \in \Omega)$ where S_α is a semigroup of type $\mathcal{C}$. We shall very often make use of the following theorem (§4.2 [1].)

<u>THEOREM 4.7</u> <u>The following assertions concerning a semigroup S are mutually equivalent</u> :

(I) <u>S is a union of groups.</u>

(II) <u>Every $\mathcal{H}$-class of S is a group.</u>

(III) <u>S is a semilattice Y of completely simple semigroups S_α $(\alpha \in Y)$, where Y is the semilattice of principal ideals of S, and each S_α is a $\mathcal{J}$-class of S.</u>

We omit the proof.

If S an inverse semigroup is a union of groups, then $x x^{-1} = x^{-1} x$

for every $x \varepsilon S$. If e is an idempotent and a an arbitrary element of S then we see that $(e\, a)(e\, a)^{-1} = (ea)^{-1}(e\, a)$; i.e. $e\, a\, a^{-1}\, e = a^{-1}\, e.\, e\, a$; i.e. $a\, a^{-1}\, e = a^{-1}\, e\, a$.

Thus $a\, e = a\,(a^{-1}\, a\, e) = a\,(a\, a^{-1}\, e) = a\, a^{-1}\, e\, a = e\, a\, a^{-1}\, a$

$= e\, a$.

Hence each idempotent e is in the centre of S. Conversely if the idempotents are in the centre, then for any $a \varepsilon S$, we have that $a\, a^{-1} = a\, a^{-1}\, a\, a^{-1} = a^{-1}\, a\, a\, a^{-1} = a^{-1}\, a\, a^{-1}\, a = a^{-1}\, a$

Thus S is a union of groups. Thus we have proved the following

<u>PROPOSITION 4.8</u> <u>Let S be an inverse semigroup.</u> <u>Then S is a union of groups if and only if every idempotent of S is in the centre of S</u>.

CHAPTER II

§1 Let ρ be a congruence defined on a regular semigroup S. Then we know that the set of all ρ-classes in S, with the operation defined by the rule

$$(x\rho) \circ (y\rho) = (xy)\rho \quad \text{for every } x, y \text{ in } S$$

is a regular semigroup and we denote it by $S/_\rho$. If e is an idempotent element in S, then we observe that

$$(e\rho)^2 = (e\rho) \circ (e\rho) = (e^2)\rho = e\rho$$

Hence $e\rho$, the ρ-class containing the element e, is an idempotent element in $S/_\rho$.

Conversely, for an idempotent element $a\rho$ in $S/_\rho$, $(a \varepsilon S)$ the existence of an idempotent element of S in $a\rho$ is guaranteed by the following

<u>LEMMA 1.1</u> (G. LALLEMENT) <u>Let</u> ρ <u>be a congruence from a regular semigroup S and let</u> $a\rho$ <u>be an idempotent element of</u> $S/_\rho$ $(a \varepsilon S)$. <u>Then</u> $a\rho$ <u>contains an idempotent of S</u>.

<u>Proof</u> Since S is regular and $a \varepsilon S$, it follows that a has an inverse a' (say) in S.

Let us write $f = aa'$ and $g = a'a$ $(f, g \varepsilon S)$.

Then we note that $ag = aa'a = a$ and

that $fa = aa'a = a$. Further, since $gf \varepsilon S$, it has an inverse z (say) in S.

Let us write $e = fzg$ $(e \varepsilon S)$.

Then we note that

$$e^2 = ee = (fzg)(fzg) = f(zgfz)g = fzg = e$$

Thus e is an idempotent element in S. It remains to show that $(a, e) \in \rho$.

Now we deduce that

$$aea = (ag)(fzg)(fa) = a(gfzgf)a$$
$$= a(gf)a = (ag)(fa) = aa = a^2$$

We further observe that

$$e = fzg = aa'za'a$$
$$= au \quad \text{where } u = a'za'a \in S$$
$$= va \quad \text{where } v = aa'za' \in S$$

Hence we have that

$$e = e^2 = ee = (au)(va) = auva$$

and it follows that

$$a^2 = aea = a(auva)a = a^2uva^2.$$

Now, by hypothesis, we have that $a\rho = (a\rho)^2 = a^2\rho$, i.e. $(a, a^2) \in \rho$. It follows by the compatibility of ρ that $(auv, a^2uv) \in \rho$. Since ρ is a congruence it follows that $(auva, a^2uva^2) \in \rho$ i.e. $(e, a^2) \in \rho$. Finally, the transitivity of ρ implies that $(e, a) \in \rho$.

<u>Proved</u>

<u>NOTE 1.2</u>

In the proof of the foregoing lemma, we see that the idempotent element e also has the property : $e \in aS \cap Sa$.

Now we proceed to introduce the concept of an idempotent-separating congruence.

DEFINITION 1.3 A congruence ρ on a semigroup S is said to be idempotent-separating if each ρ-class contains at most one idempotent.

Next we proceed to obtain necessary and sufficient condition for a congruence on a regular semigroup to be idempotent-separating.

PROPOSITION 1.4 If ρ is a congruence on an arbitrary semigroup S, such that $\rho \subseteq \mathcal{H}$, then ρ is an idempotent-separating congruence on S.

Proof Let e, f be idempotent elements in S and let $(e, f) \in \rho$.

Then it follows that $(e, f) \in \mathcal{H}$.

But an $\mathcal{H}$-class contains at most one idempotent, hence $e = f$.

Thus ρ is idempotent-separating.

Proved

For a regular semigroup, the converse situation is specified by the following

PROPOSITION 1.5 Let ρ be an idempotent-separating congruence on a regular semigroup S. Then $\rho \subseteq \mathcal{H}$.

Proof Let $(x, y) \in \rho$. Since S is regular, x has an inverse x' (say). It follows by the compatibility of ρ that $(xx', yx') \in \rho$, i.e. $(xx')\rho = (yx')\rho$. But xx' is an idempotent; hence the

ρ -class $(yx')\rho$ is an idempotent in $S/_{\rho}$. Then by the lemma 1.1 and the note 1.2, it follows that there exists an idempotent e in S such that $(yx', e) \varepsilon \rho$ and that $e \varepsilon yx' S \cap S yx'$. Transitivity of ρ implies that $(xx', e) \varepsilon \rho$. But ρ is an idempotent-separating and hence $e = xx'$. It follows that $xx' \varepsilon yx' S \subseteq y S$. Thus we have that $x = xx'x \varepsilon y S$. Similarly we can show that $y \varepsilon x S$. It follows that $(x, y) \varepsilon \mathcal{R}$. In the same way we can show that $(x, y) \varepsilon \mathcal{L}$. Thus $(x, y) \varepsilon \mathcal{R} \cap \mathcal{L} = \mathcal{H}$. It follows that $\rho \subseteq \mathcal{H}$.

Proved

The foregoing proposition helps us to guarantee the existence of a maximum-idempotent-separating congruence, i.e. an idempotent-separating congruence which contains every other idempotent-separating congruence on a regular semigroup.

COROLLARY 1.6 A regular semigroup S admits a maximum idempotent-separating congruence μ.

Proof We define the relation μ on S by the rule

$$(x, y) \varepsilon \mu \Leftrightarrow (sxt, syt) \varepsilon \mathcal{H} \quad \text{for every } s, t \varepsilon S^1.$$

Then by the proposition I.1.4 it follows that μ is the largest congruence contained in $\mathcal{H}$.

Then the proposition 1.4 and proposition 1.5 imply that μ is the maximum idempotent-separating congruence on S.

Proved

Next we proceed to establish a fundamental theorem which leads to the result that idempotent-separating congruences form a modular sublattice of the lattice of all congruences.

THEOREM 1.7 (G. LALLEMENT). The set $\Sigma_{\mathcal{H}}$ of all congruences ρ on an arbitrary semigroup S, such that $\rho \subseteq \mathcal{H}$, forms a modular sublattice of Λ, the lattice of all congruences on S.

Proof By the proposition I.1.4 we know that $\Sigma_{\mathcal{H}}$ has a maximal element defined by the rule

$$(x, y) \in \mathcal{H}_b \Leftrightarrow (sxt, syt) \in \mathcal{H} \quad \forall\ s, t \in S^1$$

which is the maximum congruence contained in $\mathcal{H}$. Further, if ρ and λ are congruences on S such that $\rho \subseteq \mathcal{H}$ and $\lambda \subseteq \mathcal{H}$, then it is obvious that $\rho \cap \lambda \subseteq \mathcal{H}$ and $\rho \vee \lambda \subseteq \mathcal{H}$. Thus $\Sigma_{\mathcal{H}}$ is a sublattice of Λ.

To show that $\Sigma_{\mathcal{H}}$ is modular, it is sufficient to prove that

$\rho \subseteq \tau \quad \Rightarrow \quad \rho \vee (\lambda \cap \tau) \supseteq (\rho \vee \lambda) \cap \tau \qquad \rho, \lambda, \tau \in \Sigma_{\mathcal{H}}$

Now let $\rho, \lambda, \tau \in \Sigma_{\mathcal{H}}$ be such that $\rho \subseteq \tau$.

Then since ρ is right congruence contained in $\mathcal{L}$ and λ is left congruence contained in $\mathcal{R}$ it follows by the proposition I.2.1 that

$\rho \vee \lambda = \rho \circ \lambda$.

It follows that $\rho \vee (\lambda \cap \tau) = \rho \circ (\lambda \cap \tau)$.

Let $(x, y) \in (\rho \vee \lambda) \cap \tau$, i.e. $(x, y) \in \rho \circ \lambda$ and $(x, y) \in \tau$.

It follows that there exists $z \in S$ such that

$(x, z) \in \rho$ and $(z, y) \in \lambda$. Since $\rho \subseteq \tau$, we have that $(x, z) \in \tau$.

It follows that $(z, y) \in \tau$. Hence we have that $(x, y) \in \rho \circ (\lambda \cap \tau)$, i.e. $(x, y) \in \rho \vee (\lambda \cap \tau)$. Thus we have that

$$\rho \vee (\lambda \cap \tau) \supseteq (\rho \vee \lambda) \cap \tau.$$

Proved

<u>COROLLARY 1.8</u> <u>The idempotent-separating congruences on a regular semigroup S form a modular sublattice of Λ.</u>

<u>Proof</u> By propositions 1.4 and 1.5, $\Sigma_{\mathcal{H}}$ is precisely the set of all idempotent-separating congruences on S.

<u>Proved</u>

<u>COROLLARY 1.9</u> <u>The lattice of congruences, Λ on a group S is modular.</u>

<u>Proof</u> If S is a group, then $\mathcal{H} = S \times S$, the universal congruence on S and so $\Sigma_{\mathcal{H}} = \Lambda$.

<u>Proved</u>

It is interesting to note that the lattice of congruences need not be modular on an inverse semigroup. To show this we construct the following example.

<u>EXAMPLE 1.10</u> Let $S = \{e, f, a, b\}$ be the semigroup whose multiplication table is

	e	f	a	b
e	e	f	a	b
f	f	f	b	b
a	a	b	e	f
b	b	b	f	f

Then it is easily verified that S is an inverse semigroup and is a union of the two groups $\{e, a\}$ and $\{f, b\}$. The congruences on S are as follows:

$\sigma = \{(e, e), (a, a), (f, f), (b, b), (e, f), (f, e), (a, b), (b, a)\}$

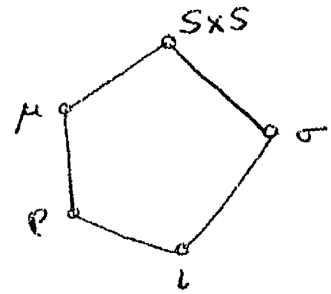
$\mu = \{(e, e), (a, a), (f, f), (b, b), (e, a), (a, e), (f, b), (b, f)\}$

$\rho = \{(e, e), (a, a), (f, f), (b, b), (f, b), (b, f)\}$

$\iota = \{(e, e), (a, a), (f, f), (b, b)\}$

$\omega = \{ S \times S \}$

The lattice diagram of Λ is



and clearly $\rho \subseteq \mu$, but

$(\rho \vee \sigma) \cap \mu = \mu$ and $\rho \vee (\sigma \cap \mu) = \rho$

Thus $(\rho \vee \sigma) \cap \mu \neq \rho \vee (\sigma \cap \mu)$

and so Λ is not modular.

Finally, we prove a useful lemma.

<u>LEMMA 1.11</u> Let S be an arbitrary regular semigroup. If $(a, b) \in \mathcal{H}$, then for every inverse a' of a there exists an inverse b' of b such that $a'a = b'b$.

<u>Proof</u> Let $(a, b) \in \mathcal{H}$ and let a' be an inverse of a . By the dual of GREEN'S lemma, $x \longrightarrow a'x$ and $y \longrightarrow ay$ are mutually inverse one-to-one mappings of H_a onto $H_{a'a}$ and $H_{a'a}$ onto H_a respectively. In particular $ba' \in H_{a'a}$ and so $(a'a, a'b) \in \mathcal{H}$. By a theorem of MILLER and CLIFFORD (Theorem 2.18 (II), [1]) there exists an inverse b' of b such that $(a', b') \in \mathcal{H}$. Again from GREEN's lemma we deduce that $(a'b, b'b) \in \mathcal{H}$. It follows that $(a'a, b'b) \in \mathcal{H}$. Since $\mathcal{H}$ is idempotent-separating, we obtain that $a'a = b'b$ as required.

<u>Proved</u>

§2 Now we proceed to introduce some special classes of congruences on a semigroup S, whose minimal elements carry the most importance. Following HOWIE and LALLEMENT (1966), we find it convenient to use the abbreviated expressions for regular semigroups with particular conditions imposed. Throughout this section we shall denote by E the set of idempotents of the semigroup S. We shall call a semigroup S an <u>RIS-semigroup</u> if S is regular and its idempotents form a subsemigroup of S. Equivalently, we can say that a regular semigroup S is an <u>RIS-semigroup</u> if and only if

$$[x^2 = x, \ y^2 = y] \Rightarrow (xy)^2 = (xy) \qquad x, y \in S.$$

<u>DEFINITION 2.1</u> A congruence ρ on a semigroup S is said to be an <u>RIS-congruence</u> if $S/_\rho$ is an RIS-semigroup.

Since every homomorphic image of a regular semigroup is a regular semigroup, it is clear that a congruence ρ on a regular semigroup S is a RIS-congruence if and only if

(1) $[(x, x^2) \in \rho, \ (y, y^2) \in \rho] \Rightarrow ((xy)^2, (xy)) \in \rho \qquad x, y \in S$

<u>PROPOSITION 2.2</u> <u>A regular semigroup S admits a minimum RIS-congruence λ</u>.

<u>Proof</u> Let $\{\lambda_i : i \in I\}$ denote the family of all RIS-congruences on S where I is an index set. It is non-empty since $S \times S$ obviously is an RIS-congruence. Let $\lambda = \bigcap_{i \in I} \lambda_i$. Since the intersection of congruences is a congruence, it is clear that λ is a congruence on S, and obviously $S/_\lambda$ is regular. Since λ_i is an RIS-congruence for every $i \in I$, it follows that condition (1) holds for every

λ_i $(i \in I)$. Hence conditions (1) certainly holds for $\lambda = \bigcap_{i \in I} \lambda_i$. Thus λ is an RIS-congruence, and obviously λ is contained in every RIS-congruence on S.

Proved

PROPOSITION 2.3 A homomorphic image of an RIS-semigroup is an RIS-semigroup.

Proof Let S be an RIS-semigroup and $T = S\vartheta$, where ϑ is a homomorphism of S onto T. Obviously T is regular.

Now let $x = x^2$ and $y = y^2$ $(x, y \in T)$.

Then it follows by the lemma 1.1 that there exist idempotents $a, b \in S$ such that $x = a\vartheta$ and $y = b\vartheta$.

Now we deduce that

$$(xy)^2 = [\,(a\vartheta)(b\vartheta)\,]^2 = [\,(ab)\vartheta\,]^2 = (ab)^2\vartheta$$
$$= (ab)\vartheta = [\,(a)\vartheta\,(b)\vartheta\,] = xy$$

Hence T is an RIS-semigroup.

Proved

The following definition is due to DUBREIL (1941).

DEFINITION 2.4 A subset U of semigroup S is called left [right] unitary if and only if $us \in U$ $[su \in U]$ and $u \in U$ imply that $s \in U$. U is called unitary if it is both left and right unitary.

We shall call a semigroup S an RU-semigroup if S is regular and its idempotents form a unitary subsemigroup of S. We stop to prove two useful lemmas on RU-semigroups.

LEMMA 2.5 *If a, b are elements of an RU-semigroup S, then* $ab \in E$ *if and only if* $ba \in E$.

Proof Let $ab \in E$ and let b' be an inverse of b.

Now we deduce that

$$(babb')^2 = ba\,\underline{bb'\,ba}\,bb' = \underline{ba}\,babb' = babb'.$$

It follows that $ba\,bb' \in E$. But $bb' \in E$ and since E is unitary we have that $ba \in E$. The converse follows by symmetry.

Proved

LEMMA 2.6 *A regular semigroup S is an RU-semigroup if and only if E is a left unitary set.*

Proof If S is an RU-semigroup, then it is trivial that E is left unitary.

Conversely we assume that E is left unitary.

Let $e \in E$ and let $xe = f \in E$.

Then we note that $(efx)^2 = ef\underline{x.e}fx = efffx = ef^3x = ef^2x = efx.$

Thus $efx \in E$.

It follows that $fx \in E$, since E is left unitary, and hence $x \in E$.

Thus we have that E is right unitary. Thus E is unitary.

Further, let $x, y \in E$ and let f be an inverse of xy.

Then it follows that $xyf \in E$ and hence $yf \in E$ since $x \in E$.

Further, since $y \in E$, it follows that $f \in E$. Finally $(xy) f \in E$ and $f \in E$ imply that $xy \in E$; i.e. E is a subsemigroup of S. Thus S is an RU-semigroup.

Proved

Next we proceed to introduce RU-congruences.

DEFINITION 2.7 A congruence ρ on a semigroup S is called an RU-congruence if S/ρ is an RU-semigroup.

From the foregoing lemma it is clear that a congruence ρ on a regular semigroup S is an RU-congruence if and only if

$$(2) \qquad [\,(x^2, x) \in \rho\,,\ (\,(xy)^2, xy) \in \rho\,] \Rightarrow (y^2, y) \in \rho \qquad (x, y \in S)$$

A parallel argument to the one used in the proof of the proposition 2.2 establishes the following

PROPOSITION 2.8 A regular semigroup S admits a minimum RU-congruence $\varkappa$.

We omit the proof.

It is interesting to note that a homomorphism does not necessarily preserve the property of being an RU-semigroup. To show this we consider the following example.

EXAMPLE 2.9 Let S be the semigroup of the type described in the theorem which is the disjoint union of two non-trivial isomorphic groups G_1 and G_2. Then G_2 is a 2-sided ideal of S and S/G_2 is the Rees factor semigroup of S modulo G_2. Obviously S/G_2 is not an RU-semigroup since it contains zero, although S is an RU-semigroup.

Next we present the concept of group congruences. These were first studied by VAGNER [19] and MUNN [20].

DEFINITION 2.10 A congruence ρ on a semigroup S is called a group congruence if $S/_\rho$ is a group.

THEOREM 2.11 A regular semigroup S admits a minimum group congruence γ.

Proof Let $\{\gamma_i : i \in I\}$ be the family of all group congruences on S, where I is an index set. Obviously I is not empty since S x S is a group congruence. Let $\gamma = \bigcap_{i \in I} \gamma_i$. Clearly γ is a congruence on S.

Next let e, f be idempotents in the semigroup S. Then clearly $e\gamma_i$ and $f\gamma_i$ are idempotents in S/γ_i for every $i \in I$. But S/γ_i is a group for every $i \in I$ and since a group contains only one idempotent, namely its identity, it follows that $e\gamma_i = f\gamma_i$ for every $i \in I$, i.e. $(e, f) \in \gamma_i$ for every $i \in I$. Thus $(e, f) \in \bigcap_{i \in I} \gamma_i = \gamma$.

Thus all the idempotents of S belong to the same γ-class in $S/_\gamma$. Let us denote it by $\mathbf{1}$.

Now for any $x \in S$, we have that

$$\mathbf{1} \circ (x\gamma) = (xx')\gamma \circ (x\gamma) = (xx'x)\gamma = x\gamma$$

$$(x\gamma) \circ \mathbf{1} = (x)\gamma \circ (x'x)\gamma = (xx'x)\gamma = x\gamma$$

Thus $\mathbf{1}$ is the (unique) 2-sided identity of $S/_\gamma$. Further if for every $x \in S$, x' denotes an inverse of x, then we have that

$$(x\gamma) \circ (x'\gamma) = (xx')\gamma = \mathbf{1} = (x'x)\gamma = (x'x) \circ (x\gamma),$$

i.e. $x'\gamma$ is the (unique) 2-sided group inverse for every $x\gamma$ in $S/_\gamma$.

Thus $S/_\gamma$ is a group and γ is a group congruence. Obviously it is contained in every other group congruence on S.

Proved

Finally we discuss the concept of band congruence and semilattice congruence on an arbitrary semigroup.

DEFINITION 2.12 A congruence ρ on a semigroup S is called a band congruence if $S/_\rho$ is a band.

It is clear that a congruence ρ is a band congruence if and only if

(3) ____ $(x, x^2) \in \rho$ for every $x \in S$.

PROPOSITION 2.13 Every semigroup S has a minimum band congruence β.

We omit the proof.

DEFINITION 2.14 A congruence ρ on a semigroup S is said to be a semilattice congruence if $S/_\rho$ is a semilattice.

It follows that a congruence ρ on a semigroup S is a semilattice congruence if and only if

(4) ____ $\begin{cases} (x, x^2) \in \rho & \text{for every } x \in S; \text{ and} \\ (xy, yx) \in \rho & \text{for every } x, y \in S. \end{cases}$

PROPOSITION 2.15 Every semigroup admits a minimum semilattice congruence η.

We omit the proof.

§3 Before proceeding to introduce some more new congruences on a regular semigroup S, we stop by to prove two important results which establish a useful link between the minimum band congruence β and the minimum semilattice congruence η on the one hand, and Green's relations on the other. These results were jointly discovered by HOWIE and LALLEMENT (1966). Following their notation, we shall denote the congruence generated by an equivalence ρ on a semigroup S, by ρ^*.

<u>THEOREM 3.1</u> <u>If</u> β <u>is the minimum band congruence on a regular semigroup S, then</u>

$$\mathcal{H}^* \subseteq \beta \subseteq \mathcal{R}^* \cap \mathcal{L}^*.$$

<u>Proof</u> Let $(x, y) \in \mathcal{H}$ for some $x, y \in S$.

Since β is a congruence on S, it follows by the corollary I.2.10 that $(x\beta, y\beta) \in \mathcal{H}$ in $S/_\beta$. But an $\mathcal{H}$-class can contain at most one idempotent; hence $x\beta = y\beta$, i.e. $(x, y) \in \beta$. Thus $\mathcal{H} \subseteq \beta$. By definition $\mathcal{H}^*$ is the smallest congruence on S containing $\mathcal{H}$; hence $\mathcal{H}^* \subseteq \beta$.

Next let x be an arbitrary element of the semigroup S and let x^{-1} denote an inverse of x in S. Then we know that

$$(x, xx^{-1}) \in \mathcal{R} \subseteq \mathcal{R}^*.$$

By compatibility of $\mathcal{R}^*$, it follows that $(xx, xx^{-1}x) \in \mathcal{R}^*$; i.e. $(x^2, x) \in \mathcal{R}^*$ for every $x \in S$. It follows that $S/_{\mathcal{R}^*}$ is a band. Since β is the minimum band congruence on S, it follows that $\beta \subseteq \mathcal{R}^*$. Similarly we can show that $\beta \subseteq \mathcal{L}^*$. It follows that $\beta \subseteq \mathcal{R}^* \cap \mathcal{L}^*$. Thus $\mathcal{H}^* \subseteq \beta \subseteq \mathcal{R}^* \cap \mathcal{L}^*$.

<u>Proved</u>

COROLLARY 3.2 In a regular semigroup S we have that $\mu \subseteq \beta$.

Proof It follows immediately from the ~~lemma~~ Corollary 1.6 that

$$\mu \subseteq \mathcal{H} \subseteq \mathcal{H}^* \subseteq \beta$$

Proved

THEOREM 3.3 If η is the minimum semilattice congruence on a regular semigroup S, then

$$\eta = \mathcal{D}^* = \mathcal{J}^*$$

Proof It is sufficient to show that

$$\eta \subseteq \mathcal{D}^* \subseteq \mathcal{J}^* \subseteq \eta$$

By the foregoing theorem 3.1, we know that $S/\mathcal{R}^*$ is a band since $\beta \subseteq \mathcal{R}^*$; hence $S/\mathcal{D}^*$ is a band since $\mathcal{R}^* \subseteq \mathcal{D}^*$. To prove commutativity in $S/\mathcal{D}^*$, we need the following

LEMMA 3.4 If e and f are idempotents of a regular semigroup S, then $(ef, fe) \in \mathcal{D}^*$.

Proof By proposition I.3.5, we know that ef has an inverse g in S such that $ge = fg = g$. Then we have that $(ef, ef.g) \in \mathcal{R}$, and similarly $(ef.g, g) \in \mathcal{L}$. It follows that $(ef, g) \in \mathcal{R} \circ \mathcal{L} = \mathcal{D} \subseteq \mathcal{D}^*$. By compatibility of $\mathcal{D}^*$, it follows that

$$(f.ef.e, f.g.e) \in \mathcal{D}^* \quad \text{i.e.} \quad ((fe)^2, g) \in \mathcal{D}^*.$$

Since $S/\mathcal{D}^*$ is a band, we have that $((fe)^2, fe) \in \mathcal{D}^*$. Hence the transitivity of $\mathcal{D}^*$ implies that $(fe, g) \in \mathcal{D}^*$. It follows that $(ef, fe) \in \mathcal{D}^*$.

Proved

Continuing with the proof of the theorem, we consider two arbitrary elements a, b of S and let a' and b' be arbitrarily chosen inverses of a, b respectively. Then we know that aa' and bb' are idempotents and it follows by the lemma 3.4 that

$$(aa'bb', bb'aa') \in \mathcal{D}^*.$$

Also, clearly we have that

$$(a, aa') \in \mathcal{R} \subseteq \mathcal{D}^*, \text{ and } (b, bb') \in \mathcal{R} \subseteq \mathcal{D}^*.$$

Hence it follows that $(ab, aa'bb') \in \mathcal{D}^*$ and $(ba, bb'aa') \in \mathcal{D}^*$. It follows that $(ab, ba) \in \mathcal{D}^*$ for every $a, b \in S$, and hence $S/_{\mathcal{D}^*}$ is a commutative band, i.e. a semilattice. But η is the minimum semilattice congruence on S, and hence it follows that $\eta \subseteq \mathcal{D}^*$.

Next, we know that $\mathcal{D} \subseteq \mathcal{J}$ for an arbitrary semigroup S. Hence obviously $\mathcal{D}^* \subseteq \mathcal{J}^*$. Next let $(x, y) \in \mathcal{J}$ where $x, y \in S$. Since η is a congruence on S, it follows by the corollary I.2.10 that $(x\eta, y\eta) \in \mathcal{J}$ in $S/_\eta$. By the commutativity of $S/_\eta$, it is clear that $\mathcal{H} = \mathcal{J} = i_{S/_\eta}$; hence $x\eta = y\eta$ i.e. $(x, y) \in \eta$. Thus we have that $\mathcal{J} \subseteq \eta$. Since η is a congruence on S containing it follows that $\mathcal{J}^* \subseteq \eta$. Thus we finally get that

$$\eta \subseteq \mathcal{D}^* \subseteq \mathcal{J}^* \subseteq \eta.$$

<u>Proved</u>

§4 So far we have introduced some special congruences in the lattice of congruences on a regular semigroup S. Operations of join and meet applied to these congruences give rise to some further congruences. This section is devoted to determining the nature of such congruences. We recall that the join of two arbitrary congruences ρ and τ in Λ, the lattice of congruences on a semigroup S, is the smallest congruence on S containing ρ and τ, and will be denoted by $\rho \vee \tau$. The meet of ρ and τ is just their intersection $\rho \cap \tau$.

It is convenient to refer to a band of groups (in the sense of Chapter I, §4) as a <u>BG-semigroup</u> and to call a congruence ρ on an arbitrary semigroup S a <u>BG-congruence</u> if $S/_\rho$ is a BG-semigroup.

<u>THEOREM 4.1</u> <u>A regular semigroup S admits a minimum BG-congruence.</u>

<u>Proof</u> Let β denote the minimum band congruence on the regular semigroup S and let E be the set of idempotents of S. We define $\alpha = \beta \cap (E \times E)$ and let $\pi = \alpha^*$; i.e. π is the smallest congruence on S containing α. We intend to show that π is the required minimum BG-congruence on S. Obviously $\beta \subseteq \pi$.
Let $\beta/_\pi = \{(x\pi, y\pi) : (x, y) \in \beta\}$; then it follows by the proposition I.1.7 that $S/_\beta \cong (S/_\pi)/(\beta/_\pi)$ and hence $\beta/_\pi$ is a band congruence on $S/_\pi$. Further, if ρ is any other band congruence on $S/_\pi$, then by proposition I.1.7 there exists a congruence σ on S such that $\rho = \sigma/_\pi$. Clearly σ is a band congruence on S and hence $\sigma \supseteq \beta$. It is immediate then that $\rho \supseteq \beta/_\pi$ and hence $\beta/_\pi$ is the minimum band congruence on the semigroup $S/_\pi$. Now suppose $e\pi$ and $f\pi$ are any two idempotents in $S/_\pi$. Then without any loss of generality

we can assume that $e, f \in E$ (by lemma 1.1).

Let $(e\pi, f\pi) \in \beta/_{\pi}$. Then $(e, f) \in \beta$ and also $(e, f) \in E \times E$ hence $(e, f) \in \beta \cap (E \times E) = \alpha \subseteq \pi$. Thus $e\pi = f\pi$, and hence $\beta/_{\pi}$ is idempotent separating in $S/_{\pi}$. Hence $\beta/_{\pi} \subseteq \mathcal{H}$ in $S/_{\pi}$. But by the theorem 3.1, $\mathcal{H} \subseteq \mathcal{H}^* \subseteq \beta/_{\pi}$, it follows that $\beta/_{\pi} = \mathcal{H}$ in $S/_{\pi}$. It follows that $\mathcal{H}$ is a congruence in $S/_{\pi}$ and each $\mathcal{H}$-class in $S/_{\pi}$ is an idempotent and hence (by the lemma 1.1) contains an idempotent element of $S/_{\pi}$. Consequently each $\mathcal{H}$-class is a group.

Thus $S/_{\pi}$ is a band of groups. To show the minimality, let us assume that π' is a BG-congruence on S. Thus $S/_{\pi'}$ is a band of groups. Let β' be the minimum band congruence on $S/_{\pi'}$; then it follows that $\beta' = \gamma/_{\pi'}$ where γ is a band congruence on S such that $\gamma \supseteq \pi'$. By definition of β, we have that $\beta \subseteq \gamma$. Also since $S/_{\pi'}$ is a band of groups, we have that $\beta' = \mathcal{H}$ in $S/_{\pi'}$. Now it is sufficient to show that $\alpha \subseteq \pi'$ (since π' is a congruence and $\pi = \alpha^*$). Let $(x, y) \in \alpha$. Then $(x, y) \in \beta$ and $x, y \in E$. Now $(x, y) \in \beta \subseteq \gamma$ and hence $(x\pi', y\pi') \in \gamma/_{\pi'} = \beta' = \mathcal{H}$. But $x\pi'$ and $y\pi'$ are idempotents since $x, y \in E$; hence $x\pi' = y\pi'$, i.e. $(x, y) \in \pi'$. Thus $\alpha \subseteq \pi'$. Thus π is the minimum BG-congruence on the regular semigroup S.

<u>Proved</u>

<u>THEOREM 4.2</u> <u>Every homomorphic image of a BG-semigroup is a BG-semigroup.</u>

<u>Proof</u> First of all we need to establish the following lemma due to CLIFFORD [3].

LEMMA 4.3 A subgroup S is a BG-semigroup if and only if

(α) $a \in Sa^2 \cap a^2S$ for every $a \in S$

(β) $Sba = Sba^2$, $abS = a^2bS$ for every $a, b \in S$

Proof We assume that S is a BG-semigroup. Then a and a^2 both belong to the same subgroup G of S, so that $a \in Ga^2 \cap a^2G \subseteq Sa^2 \cap a^2S$. Next let $a, b \in S$. From the fact that a and a^2 belong to the same subgroup of S, and the assumption that S is a band of groups, we conclude that ba and ba^2 belong to the same subgroup of S and hence $Sba = Sba^2$. Similarly $abS = a^2bS$.

Suppose conversely that conditions (α) and (β) hold in S. Condition (α) clearly implies that $(a, a^2) \in \mathcal{L}$ and $(a, a^2) \in \mathcal{R}$, that is $(a, a^2) \in \mathcal{H}$. By Green's theorem 2.16 ([1]) it follows that the $\mathcal{H}$-class Ha containing a is a group. Since $\mathcal{H}$-classes are disjoint it follows that S is a disjoint union of groups. To show that S is a band of groups, it is sufficient to show that $\mathcal{H}$ is a congruence. Let $(a, a') \in \mathcal{H}$ and let $b \in S$. It follows that a and a' belong to the same subgroup G_α (say) of S. Let e_α be the identity of G_α and a^{-1} be the group inverse of a in G_α. Replacing b by $a^{-1}b$ in the second part of (β), we get that $e_\alpha bS = abS$. Similarly $e_\alpha bS = a'bS$. Thus $abS = a'bS$ and hence $(ab, a'b) \in \mathcal{R}$. Similarly $(ab, a'b) \in \mathcal{L}$. Thus $(ab, a'b) \in \mathcal{H}$. Similarly left compatibility can be verified and hence $\mathcal{H}$ is a congruence.

Proved

Now it is trivial that any homomorphic image of a BG-semigroup S also inherits the properties (α) and (β) and hence is a BG-semigroup.

Proved

We shall call a semigroup S an SG-semigroup if it is a semilattice of groups in the sense of Chapter I, §4. Further a congruence ρ on a semigroup S is called an SG-congruence if $S/_\rho$ is an SG-semigroup. We recall from proposition I.4.8 that a regular semigroup S is a semilattice of groups if and only if its idempotents are in the centre of S. It follows that a congruence ρ on a semigroup S is an SG-congruence if and only if

(5) $\quad (x, x^2) \in \rho \implies (xy, yx) \in \rho \qquad \forall\ x, y \in S.$

An argument similar to the one used in the proof of the proposition 2.2 now establishes the following

PROPOSITION 4.4 A regular semigroup S admits a minimum SG-congruence ξ.

We omit the proof.

By virtue of LALLEMENT's lemma 1.1, it is easily seen that the property that idempotents are central is inherited by the homomorphic images. Hence we have that every homomorphic image of an SG-semigroup is an SG-semigroup.

Next we introduce the concept of an ISBG-congruence on a semigroup. A BG-semigroup S in which idempotents form a subsemigroup will be called an ISBG-semigroup. Further a congruence ρ on a semigroup S will be called an ISBG-congruence if $S/_\rho$ is an ISBG-semigroup.

<u>THEOREM 4.5</u> <u>A regular semigroup S admits a minimum ISBG-congruence</u> ζ.

<u>Proof</u> By the proposition 2.2 and the theorem 4.1, we know that a regular semigroup S has a minimum RIS-congruence λ and a minimum BG-congruence π. Let us write $\zeta = \lambda \vee \pi$, which is the smallest congruence on S containing λ and π. Since S is an RIS-semigroup, it follows by the proposition 2.3 that $S/_{(\lambda \vee \pi)}$ is an RIS-semigroup. Further, since a homomorphic image of a BG-semigroup is a BG-semigroup (Theorem 4.2), we have that $S/_{(\lambda \vee \pi)}$ is a BG-semigroup. Thus $S/_{(\lambda \vee \pi)}$ is an ISBG-semigroup. Further, if ρ is an ISBG-congruence on the semigroup S, then obviously ρ is also an RIS-congruence and a BG-congruence on S. It follows that $\lambda \subseteq \rho$ and $\pi \subseteq \rho$; and hence $(\lambda \vee \pi) \subseteq \rho$. Thus $\zeta = \lambda \vee \pi$ is the minimum ISBG-congruence on the semigroup S.

<u>Proved</u>

<u>NOTE 4.6</u>

(a) From the foregoing proof it is clear that every homomorphic image of an ISBG-semigroup is an ISBG-semigroup.

(b) For any congruence ρ on a regular semigroup S, we have that $\rho \supseteq \zeta \iff \rho$ is an ISBG-congruence on S.

By a special case of Fantham's Theorem as described by PETRICH [14] an ISBG-semigroup S can be described in terms of a band B with maximum semilattice homomorphic image $B/_{\eta} = Y$ and η-classes E_{α} $(\alpha \in Y)$, a collection of groups G_α indexed by Y, and a system of homomorphisms $\varphi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ for all α, β in Y such that $\varphi_{\alpha,\beta}\,\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha \geqslant \beta \geqslant \gamma$. (The homomorphism $\varphi_{\alpha,\alpha}$ is the identity automorphism of the group G_α for every α in Y.) The semigroup S is the disjoint union of the "rectangular groups" $E_\alpha \times G_\alpha$, the product in the semigroup S of (e_α, a_α) and (f_β, b_β) being $(e_\alpha f_\beta, (a_\alpha \varphi_{\alpha,\gamma})(b_\beta \varphi_{\beta,\gamma}))$ where $\gamma = \alpha\beta$ in Y. The product $e_\alpha f_\beta$ is evaluated in the band B, while the product $(a_\alpha \varphi_{\alpha,\gamma})(b_\beta \varphi_{\beta,\gamma})$ is evaluated in the group G_γ.

We shall call a BG-semigroup S a UBG-semigroup if and only if its idempotents form a unitary subsemigroup of S.

<u>THEOREM 4.7</u> <u>An ISBG-semigroup is a UBG-semigroup if and only if all its structure homomorphisms are one-to-one</u>.

<u>Proof</u> Suppose $S = \bigcup \{E_\alpha \times G_\alpha ;\ \alpha \in Y\}$ is the given ISBG-semigroup and $\varphi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ are its structure homomorphisms. We assume that S is a UBG-semigroup. Let I_α denote the identity of the group G_α for each α in Y. Then obviously the set of idempotents of S is $E = \{(e_\alpha, I_\alpha) : e_\alpha \quad E_\alpha,\ \alpha \in Y\}$.

Now if $a_\alpha \varphi_{\alpha,\beta} = I_\beta \quad (\alpha \geqslant \beta)$. Then

$$(e_\beta, I_\beta)(f_\alpha, a_\alpha) = (e_\beta f_\alpha, I_\beta (a_\alpha \varphi_{\alpha,\beta})) = (e_\beta f_\alpha, I_\beta) \in E.$$

Since E is unitary it follows that $(f_\alpha, a_\alpha) \in E$. Thus $a_\alpha = I_\alpha$ and hence $\varphi_{\alpha,\beta}$ is one-to-one.

Conversely we assume that the structure homomorphisms $\varphi_{\alpha,\beta}$ are one-to-

$(e_\beta, I_\beta)(f_\alpha, a_\alpha) \in E$ i.e. $(e_\beta f_\alpha, (I_\beta\, \varphi_{\beta,\gamma})(a_\alpha \varphi_{\alpha,\gamma})) \in E$

where $\gamma = \alpha\beta$ i.e. that $(e_\beta f_\alpha, I_\gamma\, (a_\alpha, \varphi_{\alpha,\gamma})) \in E$; hence $\in$ E

where $\gamma = \alpha\beta$ i.e. that $(e_\beta f_\alpha, I_\gamma\, (a_\alpha, \varphi_{\alpha,\gamma})) \in E$; hence

$a_\alpha \varphi_{\alpha,\gamma} = I_\gamma$. But since $\varphi_{\alpha,\gamma}$ is one-to-one, we have that $a_\alpha = I_\alpha$, and thus $(f_\alpha, a_\alpha) \in E$. Thus E is left unitary. By Lemma 2.6 it is immediate that S is a U.B.G.-semigroup.

Proved

Suppose S is a regular semigroup. Since a group has only one idempotent, namely its identity, it is clear that $\gamma \vee \beta = S \times S$ and $\gamma \vee \eta = S \times S$ where the symbols have their usual meaning. Next we proceed to consider the intersection of the minimum group congruence γ and the minimum band congruence β on a regular semigroup S. First we prove the following useful lemma.

<u>LEMMA 4.8</u> <u>If S is an RU-semigroup, then E, the set of idempotents of S, is a γ-class.</u>

<u>Proof</u> Let x, y be two arbitrary elements of S. Suppose $xey \in E$ for some idempotent e in S. Then by the lemma 2.5 and the fact that E is unitary, we deduce that $eyx \in E$ and $yx \in E$. Since E is a subsemigroup, $yxf \in E$ for every idempotent $f \in S$ and hence $xfy \in E$. Thus we have seen that the condition

$$xEy \cap E \neq \varphi \Rightarrow xEy \subseteq E$$

holds for any x, y in S. Then by the theorem [13] it follows that there exists a congruence ρ on S such that E is a ρ-class. By lemma 1.1, every idempotent of S/ρ contains an idempotent of S and it follows that S/ρ has exactly one idempotent, namely E. Since S/ρ is regular it follows that S/ρ is a group. Thus $\gamma \subseteq \rho$ where γ denotes the minimum group congruence on S. But obviously the identity of S/γ contains E and hence it follows that E is a γ-class.

Proved

THEOREM 4.9 Suppose γ and β denote the minimum group congruence and minimum band congruence on a regular semigroup S respectively, then $\gamma \cap \beta = \iota_S$ if and only if S is a UBG-semigroup.

Proof We assume that S is a UBG-semigroup and let $S = \bigcup \{E_\alpha \times G_\alpha : \alpha \in Y\}$. Let $((e_\alpha, a_\alpha), (e_\beta, b_\beta)) \in \gamma \cap \beta$.

Now it is easily seen that in a UBG-semigroup, $\mathcal{H}$ is a congruence and $\mathcal{H} = \beta$, and hence $((e_\alpha, a_\alpha), (e_\beta, b_\beta)) \in \mathcal{H}$. Further since each $\mathcal{H}$-class is the maximal subgroup containing the idempotent (e_α, I_α) for each α in Y it follows that $\alpha = \beta$ and hence $e_\alpha = e_\beta$. Further since γ is compatible, we have that

$$((e_\alpha, a_\alpha)(e_\alpha, a_\alpha^{-1}), (e_\alpha, b_\alpha)(e_\alpha, a_\alpha^{-1})) \in \gamma,$$

where a_α^{-1} denotes the group inverse of a_α in the group G_α. Thus we have that $((e_\alpha, I_\alpha), (e_\alpha, b_\alpha a_\alpha^{-1})) \in \gamma$. But by the foregoing lemma 4.8, the idempotents in S form a γ-class; hence we have $b_\alpha a_\alpha^{-1} = I_\alpha$, which implies that $a_\alpha = b_\alpha$. Thus we have shown that $\gamma \cap \beta = \iota_S$.

Conversely we assume that $\gamma \cap \beta = \iota_S$. Suppose that e, f are idempotents in S such that $(e, f) \in \beta$. Also $(e, f) \in \gamma$ always holds and hence $(e, f) \in \gamma \cap \beta = \iota_S$. Thus $e = f$. Thus β is idempotent-separating in S. It follows that $\beta \subseteq \mathcal{H}$ which together with theorem 3.1 implies that $\beta = \mathcal{H}$. Now it follows that each $\mathcal{H}$-class is an idempotent and hence by the lemma 1.1 contains an idempotent of S and so is a group. Thus S is a union of groups. Also since $\mathcal{H}$ is a congruence, it follows that S is a band of groups. To show that the idempotents of S form a unitary subsemigroup of S, in view of the lemma 4.8 it is sufficient to show that E, the set of idempotents of S, is a γ-class. Clearly $E \subseteq E'$, where E' is the identity of $S/_{\gamma}$. Conversely if $x \in E'$, ~~then $x \in E'$~~, then $x \in He$ for some e in E; hence $(x, e) \in \mathcal{H} \cap \gamma = \beta \cap \gamma = \iota_S$ which means $x = e \in E$. Thus we have $E = E'$.

<u>Proved</u>

The next proposition guarantees the existence of a minimum UBG-congruence on a regular semigroup S.

<u>PROPOSITION 4.10</u> <u>If γ and β are the minimum group congruence and the minimum band congruence respectively on a regular semigroup S, then $\gamma \cap \beta$ is the minimum UBG-congruence on the semigroup S.</u>

<u>Proof</u> It is easily checked that the minimum group congruence and the minimum band congruence on $S/_{(\gamma \cap \beta)}$ are $\gamma/_{(\gamma \cap \beta)}$ and $\beta/_{(\gamma \cap \beta)}$ respectively. Moreover we observe that

$$(\gamma/_{(\gamma \cap \beta)}) \cap (\beta/_{(\gamma \cap \beta)}) = (\gamma \cap \beta)/_{(\gamma \cap \beta)} = \iota_{S/(\gamma \cap \beta)}$$

Hence by the theorem 4.9, it follows that $S/_{(\gamma \cap \beta)}$ is a UBG-semigroup; i.e. $\gamma \cap \beta$ is a UBG-congruence on S.

If ρ is any UBG-congruence on S, then $S/_{\rho}$ is a UBG-semigroup. Now if τ' and σ' are the minimum group congruence and the minimum band congruence on $S/_{\rho}$, then by the proposition I.1.7, $\tau' = \tau/_{\rho}$ and $\sigma' = \sigma/_{\rho}$ where τ is a group congruence on S containing ρ and σ is a band congruence on S containing ρ. Also clearly $\tau \supseteq \gamma$ and $\sigma \supseteq \beta$. It follows that $\gamma \cap \beta \subseteq \tau \cap \sigma$. Also, since $S/_{\rho}$ is a UBG-semigroup, wehave that $\iota_{S/\rho} = \tau' \cap \sigma' = (\tau/_\rho) \cap (\sigma/_\rho) = (\tau \cap \sigma)/_\rho$. Hence we have $\tau \cap \sigma = \rho$. Thus $\rho \supseteq \gamma \cap \beta$ and we conclude that $\gamma \cap \beta$ is the minimum UBG-congruence on S.

Proved

Analogous results hold when we consider SG-semigroups. We shall call an SG-semigroup S a USG-semigroup if the set of idempotents of S is a unitary subsemigroup of S.

THEOREM 4.11 An SG-semigroup S is a USG-semigroup if and only if its structure homomorphisms are one-to-one.

This follows trivially from the theorem 4.7.

THEOREM 4.12 If γ and η denote the minimum group congruence and the minimum semilattice congruence respectively on a regular semigroup S, then $\gamma \cap \eta = \iota_S$ if and only if S is a USG-semigroup.

Proof If S is a USG-semigroup, then obviously $\beta = \eta$ and hence by the theorem 4.9, we have that $\gamma \cap \eta = \gamma \cap \beta = \iota_S$
Conversely assume $\gamma \cap \eta = \iota_S$ holds in S. Since $\beta \subseteq \eta$ clearly $\gamma \cap \beta = \iota_S$ and hence S is a UBG-semigroup by the theorem 4.9. If e, f are arbitrary idempotents of S, then certainly ef and fe are idempotents and thus $(ef, fe) \in \gamma$. Also since $S/_\eta$ is commutative we have $(ef, fe) \in \eta$. Thus we have $(ef, fe) \in \gamma \cap \eta = \iota_S$ and hence $ef = fe$. Thus S is a USG-semigroup.

Proved

A congruence ρ on a semigroup S will be called a USG-congruence if $S/_\rho$ is an USG-semigroup.

An argument exactly parallel to the one used in the proof of the proposition 4.10, establishes the following

PROPOSITION 4.13 *If γ and η are the minimum group congruence and the minimum semilattice congruence respectively on a regular semigroup S, then $\gamma \cap \eta$ is the minimum USG-congruence on S.*

We omit the proof.

We recall that by the example 2.9, it was shown that the homomorphic image of an RU-semigroup need not be an RU-semigroup. However,

THEOREM 4.14 *If S is an RU-semigroup and ξ is the minimum SG-congruence on S, then $S/_\xi$ is a USG-semigroup.*

Proof Let us define $\mathfrak{X} = \{ (ea, ae);\ e \in E, a \in S \}$ where E is the set of idempotents of S. Let $\mathfrak{X}^*$ denote the congruence generated by $\mathfrak{X}$ on S. First we show that $\mathfrak{X}^* = \xi$ on S. Let $(ea, ae) \in \mathfrak{X}$. Since $S/_\xi$ is an SG-semigroup, its idempotents are central by the proposition I.4.8. Hence $(e\xi)(a\xi) = (a\xi)(e\xi)$ for arbitrary e in E and a in S. Thus $(ea, ae) \in \xi$ i.e. $\mathfrak{X} \subseteq \xi$. It is immediate that $\mathfrak{X}^* \subseteq \xi$. On the other hand by the lemma 1.1, every idempotent of $S/_{\mathfrak{X}^*}$ is of the form $e\mathfrak{X}^*$ for some e in E. Obviously $(e\mathfrak{X}^*)(a\mathfrak{X}^*) = (a\mathfrak{X}^*)(e\mathfrak{X}^*)$ for every e in E and $a \in S$. Thus idempotents of $S/_{\mathfrak{X}^*}$ are central in $S/_{\mathfrak{X}^*}$ and by the proposition I.4.8,

$S/_{\mathfrak{X}^*}$ is an SG-semigroup i.e. $\xi \subseteq \mathfrak{X}^*$. Thus $\mathfrak{X}^* = \xi$. Now we know by the theorem 1.8 [1] that $(a, b) \in \mathfrak{X}^*$ $(a, b \in S)$ if and only if b can be obtained from a by a finite sequence of elementary $\mathfrak{X}$-transitions. Suppose f, an idempotent of S is transformed by an elementary $\mathfrak{X}$-transition : i.e. $f = peaq \longrightarrow paeq$ where $p, q \in S$. (Other possibilities while either or both p and q may be absent can be dealt with similarly.) Making use of the lemma 2.5 and the fact that E is unitary, from $peaq \in E$ we easily deduce that $eaqp \in E$, $aqp \in E$ and $qpa \in E$. Also E is a subsemigroup, hence $qpae \in E$ and $paeq \in E$. Thus we have established that an element obtained by means of an elementary $\mathfrak{X}$-transition from an idempotent of S is itself an idempotent of S. Repetition of this argument a finite number of times finally gives the result that any element $a \in S$ such that $(a, f) \in \mathfrak{X}^*$ for some idempotent f is itself an idempotent.

Now since $S/_\xi$ obviously is an SG-semigroup, by lemma 2.6 it is sufficient to show that the set of idempotents of the semigroup $S/_\xi$ is left unitary. Suppose that $(e\xi)(a\xi) = (f\xi)$ for some $a \in S$ and where $e\xi$ and $f\xi$ denote the arbitrary idempotents of $S/_\xi$ for some e, f in E. Since $\mathfrak{X}^* = \xi$, it is clear that $(ea, f) \in \mathfrak{X}^*$ and since f is an idempotent in S it follows that ea is an idempotent in S. But S is an RU-semigroup and hence <u>a</u> is idempotent in S i.e. that $(a\xi)$ is idempotent in $S/_\xi$. Thus $S/_\xi$ is a USG-semigroup.

<u>Proved</u>

A parallel result holds for UBG-semigroups.

THEOREM 4.15 If ζ denotes the minimum ISBG-congruence on a regular semigroup S, then S/ζ is a UBG-semigroup.

Proof Without any loss of generality we can take two arbitrary idempotents of S/ζ as $(e\zeta)$ and $(f\zeta)$ for some e, f in E. Also let $(e\zeta)(a\zeta) = (f\zeta)$ for some $a \in S$. Obviously for the RU-semigroup S, $\zeta \subseteq \xi$ and hence $(ea, f) \in \xi$. As in the proof of the theorem 4.14, this implies that ea is idempotent and consequently a is an idempotent since E is unitary. Thus $(a\zeta)$ is idempotent in S/ζ. Thus the set of idempotents in S/ζ is left unitary and so by the lemma 2.6, it follows that S/ζ is a UBG-semigroup.

Proved

The last two theorems help us to determine the nature of $\zeta \vee \varkappa$ and $\xi \vee \varkappa$.

THEOREM 4.16 Let S be a regular semigroup and ζ and $\varkappa$ denote the minimum ISBG-congruence and the minimum RU-congruence respectively on S. Then $\zeta \vee \varkappa$ is the minimum UBG-congruence on S.

Proof From Note 4.6, any congruence containing ζ is an ISBG-congruence on S, hence $\zeta \vee \varkappa$ is an ISBG-congruence on S, i.e. $S/(\zeta \vee \varkappa)$ is an ISBG-semigroup. Now by the proposition I.1.7, we have that

$$S/(\zeta \vee \varkappa) \cong (S/\varkappa)/((\zeta \vee \varkappa)/\varkappa)$$

it follows that $(\zeta \vee \varkappa)/\varkappa$ is an ISBG-congruence on $S/\varkappa$, since any homomorphic image of an ISBG-semigroup is an ISBG-semigroup. Suppose

ρ' is some ISBG-congruence on $S/_{\varkappa}$. Then by the proposition I.1.7, it follows that $\rho' = \rho/_{\varkappa}$ where ρ is some ISBG-congruence on S containing $\varkappa$. Also $\rho \supseteq \zeta$ and hence $\rho \geq \zeta \vee \varkappa$. Thus $(\rho \vee \varkappa)/_{\varkappa} \leq \rho'$ i.e. $(\rho \vee \varkappa)/_{\varkappa}$ is the minimum ISBG-congruence on $S/_{\varkappa}$. By the theorem 4.15, it follows that $(S/_{\varkappa})/((\zeta \vee \varkappa)/\varkappa)$ is a UBG-semigroup, and obviously $\zeta \vee \varkappa$ is the minimum UBG-congruence on S.

<u>Proved</u>

An exactly parallel argument based on the theorem 4.14, proves the analogous result for USG-semigroups.

<u>THEOREM 4.17</u> <u>Let S be a regular semigroup and let ξ and $\varkappa$ denote the minimum SG-congruence and the minimum RU-congruence respectively on S. Then $\xi \vee \varkappa$ is the minimum USG-congruence on S.</u>

We omit the proof.

<u>PROPOSITION 4.18</u> <u>On a regular semigroup S</u>

$$\gamma \cap \beta = \zeta \vee \varkappa \quad \underline{\text{and}} \quad \gamma \cap \eta = \xi \vee \varkappa$$

<u>where the symbols have their usual meaning.</u>

<u>Proof</u> Comparing the results of the theorem 4.10 and the theorem 4.16, we immediately obtain that $\gamma \cap \beta = \zeta \vee \varkappa$. Similarly the theorem 4.17 and the proposition 4.13 together give $\gamma \cap \eta = \xi \vee \varkappa$.

<u>Proved</u>

By the formula (2) and the Note 4.6, it is easily verified that an intersection of arbitrary UBG-congruences on a semigroup S is a

UBG-congruence. Also clearly an arbitrary group congruence ρ and an arbitrary band congruence τ are UBG-congruences. Hence $\rho \cap \tau$ is a UBG-congruence on S. The converse situation is specified by the following.

<u>THEOREM 4.19</u> <u>Any UBG-congruence ρ on a regular semigroup S can be uniquely expressed as the intersection</u> $\tau \cap \sigma$ <u>of a group congruence</u> τ <u>and a band congruence</u> σ <u>on S.</u>

<u>Proof</u> Suppose τ' and σ' are the minimum group congruence and the minimum band congruence on the regular semigroup $S/_\rho$. Then by the proposition I.1.7, it follows that $\tau' = \tau/_\rho$ and $\sigma' = \sigma/_\rho$ where τ is a group congruence on S, containing ρ and σ is a band congruence on S containing ρ. Since $S/_\rho$ is a UBG-semigroup, it follows by the theorem 4.9 that $\iota_{S/_\rho} = \tau' \cap \sigma' = (\tau/_\rho) \cap (\sigma/_\rho) = (\tau \cap \sigma)/_\rho$. Hence $\tau \cap \sigma = \rho$. To see the uniqueness, let $\tau_2 \cap \sigma_2 = \tau_1 \cap \sigma_1$ where τ_1 and τ_2 are group congruences and σ_1 and σ_2 are band congruences on S respectively. Obviously $\sigma_1 \cap \sigma_2$ is a band congruence on S.

Let $(a, b) \in \sigma_1$. Now by the lemma 1.1, it follows that there exist idempotents e, f in S such that $(a, e) \in \sigma_1 \cap \sigma_2$ and $(b, f) \in \sigma_1 \cap \sigma_2$. By the transitivity of σ_1 we have that $(e, f) \in \sigma_1$. Also $(e, f) \in \tau_1$ and so it follows that

$$(e, f) \in \tau_1 \cap \sigma_1 = \tau_2 \cap \sigma_2 \subseteq \sigma_2.$$

Transitivity of σ_2 implies that $(a, b) \in \sigma_2$. Thus $\sigma_1 \subseteq \sigma_2$. By symmetry it is immediate that $\sigma_2 \subseteq \sigma_1$. Thus $\sigma_1 = \sigma_2 = \sigma$ (say). Next let $(a, b) \in \tau_1$. By compatibility of τ_1, we have

$$(a.a\,ba,\ a.b.ba) \in \tau_1 .$$

Also since $S/_\sigma$ is a band, it follows that

$(a\sigma)(a\sigma)(b\sigma)(a\sigma) = (a\sigma)(b\sigma)(a\sigma) = (a\sigma)(b\sigma)(b\sigma)(a\sigma)$

Hence we have $(aaba, abba) \in \tau_1 \cap \sigma = \tau_2 \cap \sigma \subseteq \tau_2$. Suppose $\underline{a}'$ and $(\underline{ba})'$ denote arbitrarily chosen inverses of $\underline{a}$ and $\underline{ba}$ respectively in the regular semigroup S. Then by the compatibility of τ_2, we have that $(a'.aaba\,(ba)', a'.abba.(ba)') \in \tau_2$. But since τ_2 is a group congruence, we have that

$(a'a)\tau_2 = ((ba)(ba)')\tau_2$ = identity of S/τ_2. Therefore we have $a\tau_2 = ((a'a)\tau_2)(a\tau_2)((ba)(ba)'\tau_2) = (a'aaba\,(ba)')\tau_2$

$= (a'.abba\,(ba)')\tau_2 = ((a'a)\tau_2)(b\tau_2)((ba\,(ba)'\tau_2) = (b\tau_2)$

Thus we have that $(a, b) \in \tau_2$; i.e. $\tau_1 \subseteq \tau_2$. By symmetry it is immediate that $\tau_2 \subseteq \tau_1$. Thus $\tau_1 = \tau_2 = \tau$ (say)

<u>Proved</u>

A similar line of argument establishes the following.

<u>THEOREM 4.20</u> <u>Any USG-congruence ρ on a regular semigroup S can be uniquely expressed as the intersection $\tau \cap \delta$ of a group congruence τ and a semilattice congruence δ on S.</u>

We omit the proof.

Finally we restrict our attention to a regular semigroup S, in which the set of idempotents of S forms a unitary subsemigroup of S, (what we are calling an RU-semigroup. Many results of the preceding sections are strikingly simplified in this case.

PROPOSITION 4.21 Let S be an RU-semigroup. Then the following holds :

I) The minimum RU-congruence $\varkappa$ on S is the identity congruence ι_S.

II) The minimum SG-congruence ξ on S is equal to $\gamma \cap \eta$.

III) The minimum ISBG-congruence ζ on S is equal to $\gamma \cap \beta$.

IV) $\xi \cap \mathcal{H} = \zeta \cap \mathcal{H} = \gamma \cap \mu = \xi \cap \mu = \zeta \cap \mu = \iota_S$.

Proof Part (I) is trivial. Part (II) and Part (III) are immediately obtained by substituting $\varkappa = \iota_S$ in the proposition 4.18.

Since it is clear that $\zeta \subseteq \xi \subseteq \gamma$ and $\mu \subseteq \mathcal{H}$, to prove Part (IV), it is sufficient to show that $\gamma \cap \mathcal{H} = \iota_S$

First we observe that if $(x, e) \in \gamma \cap \mathcal{H}$ for some $x \in S$ and $e \in E$ where E is the set of idempotents of S, then $x \in E$, since E is γ-class in RU-semigroup S. Further since $\mathcal{H}$-class cannot contain more than one idempotent, it follows that $x = e$. Now let $(a, b) \in \gamma \cap \mathcal{H}$ $(a, b \in S)$. Suppose that a' denotes an inverse of a in S. Then we know that $(a, aa') \in \mathcal{R}$. It follows by Green's lemma I.2.3, that $x \to xa'$ and $y \to ya$ are mutually inverse one-to-one maps of $H_a \to H_{aa'}$ and $H_{aa'} \to H_a$ respectively. In particular $(aa', ba') \in \mathcal{H}$. Also by the compatibility of γ it follows $(aa', ba') \in \gamma$. Thus we have $(aa', ba') \in \gamma \cap \mathcal{H}$. But aa' is an idempotent; hence we have $aa' = ba'$. Also by Lemma 1.11 there exists an inverse b' of b such that $a'a = b'b$. Hence we have $a = \underline{aa'}a = b\underline{a'a} = bb'b = b$. Thus we conclude that

$$\gamma \cap \mathcal{H} = \iota_S$$

Proved.

§5 The nature of many of the congruences introduced in the preceding sections is more specifically determined in the case of an inverse semigroup. By E we shall mean the semilattice of idempotents of the inverse semigroup S. The unique inverse of an element a in the inverse semigroup S will be denoted by a^{-1}. Clearly if e is an idempotent then $e^{-1} = e$; also $(ab)^{-1} = b^{-1}a^{-1}$.

PROPOSITION 5.1 A homomorphic image of an inverse semigroup is itself an inverse semigroup.

Proof Suppose that ϑ is a homomorphism of an inverse semigroup S onto a semigroup T i.e. $T = S\vartheta$. Obviously T is regular. Suppose that $x^2 = x$ and $y^2 = y$ where $x, y \in T$. Then by the lemma 1.1 there exist $e, f \in S$ such that $e^2 = e$, $f^2 = f$ and $e\vartheta = x$, $f\vartheta = y$. Now we have $xy = (e\vartheta)(f\vartheta) = (ef)\vartheta = (fe)\vartheta = yx$. Thus idempotents commute in T i.e. T is an inverse semigroup.

Proved

PROPOSITION 5.2 Let ρ be a congruence on an inverse semigroup S. Then $(x, y) \in \rho$ if and only if $(x^{-1}, y^{-1}) \in \rho$, where $x, y \in S$.

Proof Let $(x, y) \in \rho$ and hence $x\rho = y\rho$. Also we observe that

$$(x\rho)(x^{-1}\rho)(x\rho) = (xx^{-1}x)\rho = x\rho$$

and $$(x^{-1}\rho)(x\rho)(x^{-1}\rho) = (x^{-1}xx^{-1})\rho = x^{-1}\rho.$$

It follows that $(x\rho)^{-1} = (x^{-1}\rho)$ since inverse is unique in S. Hence we deduce that $(x^{-1}\rho) = (x\rho)^{-1} = (y\rho)^{-1} = (y^{-1}\rho)$, which implies that $(x^{-1}, y^{-1}) \in \rho$. Converse follows by the symmetry.

Proved

HOWIE (1964) has provided following two characterizations for the maximum idempotent-separating congruence μ on an inverse semigroup S.

THEOREM 5.3 Let S be an inverse semigroup. The relation μ defined on S by the rule

(6) $$(x, y) \in \mu \Longleftrightarrow x^{-1}ex = y^{-1}ey \quad \text{for every } e \text{ in } E$$

is the maximum idempotent-separating congruence on S.

Proof Obviously μ is an equivalence on S. Let $(x, y) \in \mu$; equivalently $x^{-1}ex = y^{-1}ey$ for every idempotent e in S. Let $z \in S$. Premultiplying by z^{-1} and postmultiplying by z on both sides, we have that $z^{-1}x^{-1}exz = z^{-1}y^{-1}eyz$ for every e in E, and for every $z \in S$. Thus $(xz)^{-1} e (xz) = (yz)^{-1} eyz$ for every $e \in E$, and hence $(xz, yz) \in \mu$ for every z in S. On the other hand we know by Proposition I.4.4 that $z^{-1}ez$ is an idempotent for every $e \in E$ and $z \in S$. Hence $x^{-1}z^{-1}ezx = y^{-1}z^{-1}ezy$ holds for every z in S, i.e. $(zx)^{-1} e (zx) = (zy)^{-1} e (zy)$ and hence $(zx, zy) \in \mu$. Thus we have established that μ is a congruence. Suppose e, f are idempotents in S such that $(e, f) \in \mu$. Then we observe that $e^{-1}ee = f^{-1}ef$ and $e^{-1}fe = f^{-1}ff$ i.e. that $e = ef$, $ef = f$ and hence $e = f$. Hence μ is an idempotent-separating congruence on S. Next let ρ be an idempotent-separating congruence on S, and let $(x, y) \in \rho$ where $x, y \in S$. By the proposition 5.2, it follows that $(x^{-1}, y^{-1}) \in \rho$ and hence $(x^{-1}e, y^{-1}e) \in \rho$ for every idempotent e in S, since ρ is compatible. It follows that $(x^{-1}ex, y^{-1}ey) \in \rho$. But since ρ is idempotent-separating we have that $x^{-1}ex = y^{-1}ey$, where e is an arbitrary idempotent in S. Thus $(x, y) \in \mu$ and we have $\rho \subseteq \mu$.

Proved

THEOREM 5.4 Let μ be the maximum idempotent-separating congruence on an inverse semigroup S. Then

$$(x, y) \in \mu \Longleftrightarrow x^{-1}x = y^{-1}y \text{ and } xy^{-1} \in E\zeta$$

where $E\zeta$ is the centralizer of E in S. Dually

(7) ——— $$(x, y) \in \mu \Longleftrightarrow xx^{-1} = yy^{-1} \text{ and } x^{-1}y \in E\zeta.$$

Proof Suppose $(x, y) \in \mu$. Then $(x^{-1}, y^{-1}) \in \mu$ by the proposition 5.2 and hence by (6) it follows that $xex^{-1} = yey^{-1}$ for every e in E, where E is the semilattice of idempotents in S. Now since the idempotents commute we easily deduce that

$$x^{-1}x = x^{-1}x.\ x^{-1}x.\ x^{-1}x = x^{-1}yx^{-1}xy^{-1}x = y^{-1}yx^{-1}xy^{-1}y = x^{-1}x.y^{-1}y$$

Similarly we can deduce that $y^{-1}y = x^{-1}x.y^{-1}y$. Thus we have that $x^{-1}x = y^{-1}y$. On the other hand from $x^{-1}ex = y^{-1}ey$, we easily deduce that $xx^{-1}exy^{-1} = xy^{-1}eyy^{-1}$ and hence since idempotents commute $exx^{-1}xy^{-1} = xy^{-1}yy^{-1}e$ and hence $exy^{-1} = xy^{-1}e$. This holds for every $e \in E$. Hence $xy^{-1} \in E\zeta$.

Conversely we assume $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\zeta$ for some x, y in S. Then for an arbitrary idempotent e in E we have $xy^{-1}e = exy^{-1}$ and it follows that $x^{-1}(xy^{-1}e)y = x^{-1}(exy^{-1})y$; hence $y^{-1}yy^{-1}ey = x^{-1}exx^{-1}x$, which gives $y^{-1}ey = x^{-1}ex$. This holds for any arbitrary idempotent e. Hence by (6) we have that $(x, y) \in \mu$. The dual statement is immediate by 5.2 and the fact that $(x^{-1})^{-1} = x$ and $(y^{-1})^{-1} = y$.

Proved

THEOREM 5.5 Let μ be the maximum idempotent-separating congruence on an inverse semigroup S. Then $S/_{\mu} \cong E$ if and only if E is central in S.

Proof We assume that E is central in S. Let us consider the map $\vartheta : e \rightarrow e\mu$ from E into $S/_{\mu}$. Since μ is idempotent-separating, it follows that ϑ is one-one. We further observe that for any x in S, $x^{-1}x = (x^{-1}x)^{-1} x^{-1}x$ and $x (x^{-1}x)^{-1} = xx^{-1}x = x \in E\zeta$, since $E\zeta = S$. Hence by (7) it follows that $(x, x^{-1}x) \in \mu$. Thus each μ-class $x\mu$ $(x \in S)$ contains an idempotent- $x^{-1}x$. Thus ϑ is ONTO. Finally ϑ is a homomorphism since for any e, f in E, we observe that $(ef)\vartheta = (ef)\mu = (e\mu)(f\mu) = (e\vartheta)(f\vartheta)$. Thus we conclude that $S/_{\mu} \cong E$.

Conversely we assume that ϑ is an isomorphism of E onto $S/_{\mu}$. Let x be an arbitrary element of S. Since ϑ is ONTO, there exists an idempotent e in E such that $(x, e) \in \mu$. It follows by (7) that $x^{-1}x = e^{-1}e = e$ and $xe^{-1} = xe \in E\zeta$. Then we can deduce that $x = xx^{-1}x = xe \in E\zeta$. Thus $S \subseteq E\zeta$. Hence we have that $E\zeta = S$ i.e. E is central in S.

Proved

The next theorem provides a necessary and sufficient condition for μ to be an identity relation.

THEOREM 5.6 Let μ be the maximum idempotent-separating congruence on an inverse semigroup S. Then $\mu = \iota_S$ if and only if $E\zeta = E$, where $E\zeta$ is the centralizer of E in S.

Proof First we assume that $\mu = \iota_S$ in S. Obviously in the inverse semigroup S, $E \subseteq E\zeta$. On the other hand let x be an arbitrary element of $E\zeta$. Clearly $x^{-1}x = (x^{-1}x)^{-1} x^{-1}x$ and $x(x^{-1}x)^{-1} = xx^{-1}x = x \in E\zeta$; hence by (7) it follows that $(x, x^{-1}x) \in \mu$ and we have $x = x^{-1}x \in E$, since $\mu = \iota_S$. Thus $E\zeta \subseteq E$. Finally we conclude that $E = E\zeta$.

Conversely let us assume that $E\zeta = E$ and let $(x, y) \in \mu$, where x, y are in S. By (7) it follows that $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\zeta = E$. Since xy^{-1} is idempotent we have $(x^{-1}y) = (xy^{-1})^{-1} = yx^{-1}$; also from the proposition 5.2 and (6), it is immediate that $xex^{-1} = yey^{-1}$ holds for every idempotent e in S. Now we easily deduce that

$$xx^{-1} = xx^{-1}xx^{-1} = yx^{-1}xy^{-1} = (xy^{-1})^{-1}xy^{-1} = xy^{-1}.$$

Hence we have that $x = xx^{-1}x = xy^{-1}x = yx^{-1}x = yy^{-1}y = y$. Thus $\mu = \iota_S$.

Proved

NOTE 5.7

Any idempotent-separating congruence ν on S/μ is equal to the identity relation.

For if $\nu \neq \iota_{S/\mu}$, then the congruence ν' on S defined by the rule $(x, y) \in \nu' \Leftrightarrow (x\mu, y\mu) \in \nu$ is clearly idempotent-separating such that $\nu' \supset \mu$ which is a contradiction to the maximality of μ.

The next theorem provides a useful characterization of the minimum group congruence on an inverse semigroup.

THEOREM 5.8 (MUNN) *Let S be an inverse semigroup. The relation on S defined by the rule*

(8) $(x, y) \in \gamma \Longleftrightarrow ex = ey$ *for some idempotent e in S*

is the minimum group congruence on S.

Dually γ *can be defined as follows :*

$(x, y) \in \gamma \Longleftrightarrow xf = yf$ *for some idempotent f in S.*

Proof It is sufficient to show the first of the two dual statements. Obviously γ defined by (8) is reflexive and symmetric. Next let $(x, y) \in \gamma$ and $(y, z) \in \gamma$. Then there exist idempotents e, f in S such that $ex = ey$ and $fy = fz$. Since S is an inverse semigroup, it follows that $ef \in E$, $fe \in E$ and $ef = fe$.

Now we can deduce that

$efx = fex = fey = efy = efz$, and hence we have that $(x, z) \in \gamma$. Thus γ is transitive. Further if $(x, y) \in \gamma$, then it is immediate from (8) that $exz = eyz$ for any z in S and thus $(xz, yz) \in \gamma$ for every z. On the other hand, for any z, we also have that $zex = zey$ or $zz^{-1}zex = zz^{-1}zey$ which gives $(zez^{-1})zx = (zez^{-1})zy$ since idempotents commute. But we know that zez^{-1} is idempotent in S and hence it follows that $(zx, zy) \in \gamma$. Thus we have established that γ is a congruence on S. Also for any two idempotents e, f in S, it is clear that $efe = eff$; that is $(e, f) \in \gamma$. Thus all the idempotents in S belong to the same γ-class. Then as in the proof of the theorem 2.11, it can be easily shown that γ is a group congruence on S. Further let ρ be a group congruence on S and let $(x, y) \in \gamma$.

It follows that $ex = ey$ for some $e \in E$. Hence we deduce that $(e\rho)(x\rho) = (e\rho)(y\rho)$, but since $S/_{\rho}$ is a group, $e\rho$ is the identity of $S/_{\rho}$ and hence we have that $x\rho = y\rho$, which means $(x, y) \in \rho$. Thus $\gamma \subseteq \rho$.

Proved

HOWIE (1964) has given another very useful characterization of the minimum group congruence on an inverse semigroup S. In the notation of Chapter I, §4, we shall denote by $E\omega$, the closure of E, under the natural order relation defined by I.(4) where E is the semilattice of idempotents in the inverse semigroup S.

THEOREM 5.9 Let γ denote the minimum group congruence on an inverse semigroup S. Then

(9) $$(x, y) \in \gamma \iff xy^{-1} \in E\omega$$

Proof Let $(x, y) \in \gamma$ where x, y are in S. By (8), it follows that $ex = ey$ for some idempotent e in S. It is immediate that $exy^{-1} = eyy^{-1} \in E$. But we have that $exy^{-1} \leqslant xy^{-1}$. Hence we have that $xy^{-1} \in E\omega$. Conversely let $xy^{-1} \in E\omega$. Then there exists an idempotent f such that $xy^{-1} \geqslant f$, i.e. such that $fxy^{-1} = ff = f$. Let us put $e = fxy^{-1}yx^{-1}$. Then clearly

$$e^2 = (fxy^{-1}yx^{-1})(fxy^{-1}yx^{-1}) = ffxy^{-1}yx^{-1}xy^{-1}yx^{-1}$$

$$= fxx^{-1}xy^{-1}yy^{-1}yx^{-1} = fxy^{-1}yx^{-1} = e$$

and so $e \in E$. Also $ef = fxy^{-1}yx^{-1}f = fxy^{-1}yx^{-1} = e$, since idempotents commute in S.

Now we can deduce that

$$ex = eex = efxy^{-1}yx^{-1}x = efxx^{-1}xy^{-1}y = efxy^{-1}y = efy = ey$$

Then it is immediate by (8) that $(x, y) \in \gamma$.

Proved

Now we turn our attention to the intersection and the join of these two congruences, namely $\gamma \cap \mu$ and $\gamma \vee \mu$. The next theorem provides a necessary and sufficient condition for $\gamma \cap \mu$ to be equal to the identity relation.

THEOREM 5.11 On an inverse semigroup S

$$\gamma \cap \mu = \iota_S \quad \text{iff} \quad E\zeta \cap E\omega = E$$

where the symbols have their usual meaning.

Proof We assume that $E\zeta \cap E\omega = E$ and let $(x, y) \in \gamma \cap \mu$ for some x, y in S. By (7) and (9), it follows that $x^{-1}x = y^{-1}y$; also $xy^{-1} \in E\zeta$ and $xy^{-1} \in E\omega$ i.e. $xy^{-1} \in E\zeta \cap E\omega = E$. Hence $(xy^{-1})^{-1} = yx^{-1} = xy^{-1}$ since xy^{-1} is an idempotent. Now it is easily deduced that $xx^{-1} = xx^{-1}.xx^{-1} = yx^{-1}xy^{-1} = (xy^{-1})^{-1}xy^{-1} = xy^{-1}$. Hence we have that $x = xx^{-1}x = xy^{-1}x = yx^{-1}x = yy^{-1}y = y$. Thus $\gamma \cap \mu = \iota_S$. Conversely let us assume that $\gamma \cap \mu = \iota_S$. Obviously $E \subseteq E\zeta \cap E\omega$. On the other hand, let $x \in E\zeta \cap E\omega$. Now since $x^{-1}x = (x^{-1}x)^{-1}x^{-1}x$ and $x(x^{-1}x)^{-1} = xx^{-1}x = x \in E\zeta$ it follows by (7) that $(x, x^{-1}x) \in \mu$. Also since $x(x^{-1}x)^{-1} = xx^{-1}x = x \in E\omega$, hence we have that $(x, x^{-1}x) \in \gamma$. Thus $(x, x^{-1}x) \in \gamma \cap \mu = \iota_S$ and it follows that $x = x^{-1}x \in E$.

Thus we conclude that $E = E\zeta \cap E\omega$

Proved

Next we proceed to investigate $\gamma \vee \mu$. First we prove the following very useful proposition.

PROPOSITION 5.12 Let γ denote the minimum group-congruence and let ξ be an arbitrary congruence on an inverse semigroup S. Then

$$\xi \vee \gamma = \gamma \circ \xi \circ \gamma$$

Proof Obviously $\gamma \circ \xi \circ \gamma \subseteq \xi \vee \gamma$. On the other hand $\gamma \circ \xi \circ \gamma \supseteq \gamma$ and $\gamma \circ \xi \circ \gamma \supseteq \xi$. Thus all we need to show is that $\gamma \circ \xi \circ \gamma$ is a congruence on S. Clearly it is reflexive and symmetric. Next let $(x, y) \in \gamma \circ \xi \circ \gamma$ and $(y, z) \in \gamma \circ \xi \circ \gamma$. It follows that there exist elements a, b, c, d in S such that $(x, a) \in \gamma$, $(a, b) \in \xi$, $(b, y) \in \gamma$ and $(y, c) \in \gamma$, $(c, d) \in \xi$, and $(d, z) \in \gamma$. By the transitivity of γ we have that $(b, c) \in \gamma$. Hence by (8), there exists an idempotent e such that $eb = ec$. Also by the compatibility of ξ , we have $(ea, ed) \in \xi$. Again by (8) and since $ea = eea$ we have that $(ea, a) \in \gamma$. Hence by transitivity of γ , we have $(x, ea) \in \gamma$. Similarly $(ed, z) \in \gamma$. Hence we conclude that $(x, z) \in \gamma \circ \xi \circ \gamma$. Thus $\gamma \circ \xi \circ \gamma$ is transitive. The compatibility of $\gamma \circ \xi \circ \gamma$ is immediate from the compatibility of γ and ξ .

Proved

THEOREM 5.13 Let a relation ρ on an inverse semigroup S be defined by the rule

(10) $$(x, y) \in \rho \iff xy^{-1} \in (E\zeta)\,\omega$$

where the symbols have their usual meaning. Then ρ is equal to $\gamma \vee \mu$ on S.

Proof Suppose $x, y \in E\zeta$; then it follows that for any e in E, $xye = xey = exy$ and hence $xy \in E\zeta$. Also if $xe = ex$, then taking inverses we have $ex^{-1} = x^{-1}e$. Thus $x^{-1} \in E\zeta$. Further let $x \in E\zeta$ and z be an arbitrary element of S, then by the commutativity of the idempotents on S, we deduce that

$$zxz^{-1}e = zxz^{-1}zz^{-1}e = (zxz^{-1})(e)zz^{-1} = ezxz^{-1}zz^{-1} = ezxz^{-1}$$

Hence we have that $zxz^{-1} \in E\zeta$. Thus we conclude that $E\zeta$ is a self-conjugate inverse subsemigroup of S. Then ~~by Proposition~~ it readily follows that $(E\zeta)\,\omega$ is a self-conjugate inverse subsemigroup of S. Further clearly $E \subseteq (E\zeta)\,\omega$. Now by Proposition I.4.6 we conclude that the relation ρ defined by (10) is a congruence on inverse semigroup S. Now let $(x, y) \in \gamma$. Then by (9), $xy^{-1} \in E\omega \subseteq (E\zeta)\,\omega$ and hence $(x, y) \in \rho$. Also if $(x, y) \in \mu$ then by (7) we have $xy^{-1} \in E\zeta \subseteq (E\zeta)\omega$ and hence $(x, y) \in \rho$. Thus ρ is a congruence containing γ and μ ; it follows that $\gamma \vee \mu \subseteq \rho$. In view of the proposition 5.12, it is sufficient to show that $\rho \subseteq \gamma \circ \mu \circ \gamma$.

Let $(x, y) \in \rho$. Then $xy^{-1} \in (E\zeta)\omega$ and hence there exists an element $z \in E\zeta$ such that $xy^{-1} \geq z$. Let us write $u = zy$ and $v = z^{-1}zy$. Clearly $xu^{-1} = xy^{-1}z^{-1}$. But $xy^{-1}z^{-1} \geq zz^{-1}$. Hence $xu^{-1} \in E\,\omega$. It follows that $(x, u) \in \gamma$. Now it is easily deduced that $v^{-1}v = y^{-1}z^{-1}zz^{-1}zy = y^{-1}z^{-1}zy = u^{-1}u$. Also, since $z \in E\zeta$, we have that for any idempotent e in S that

$$uv^{-1}e = zyy^{-1}z^{-1}ze = zeyy^{-1}z^{-1}z = ezyy^{-1}z^{-1}z = euv^{-1}.$$

It follows that $uv^{-1} \in E\zeta$ and hence by (7) we conclude that $(u, v) \in \mu$. Also it is clear that $vy^{-1} = z^{-1}zyy^{-1} \in E \subseteq E\omega$ and hence $(u, v) \in \gamma$. Thus we conclude that $(x, y) \in \gamma \circ \mu \circ \gamma$.

Proved

PROPOSITION 5.14 Let S be an inverse semigroup and suppose its idempotents form a unitary subsemigroup of S. Then

$$\gamma \cap \mathcal{R} = \gamma \cap \mathcal{L} = \gamma \cap \mathcal{H} = \iota_S$$

Proof It is clear by the ~~theorem~~ PROPOSITION 4.21, that $\gamma \cap \mathcal{H} = \iota_S$. In view of the dual argument it is sufficient to show that $\gamma \cap \mathcal{R} = \iota_S$. Let $(x, y) \in \gamma \cap \mathcal{R}$, where $x, y \in S$. Then we know that $(x, xx^{-1}) \in \mathcal{R}$ and $(y, yy^{-1}) \in \mathcal{R}$. By the transitivity of $\mathcal{R}$ it follows that $(xx^{-1}, yy^{-1}) \in \mathcal{R}$. But by THEOREM I.4.7, each $\mathcal{R}$-class in S contains a unique idempotent and hence we have that $xx^{-1} = yy^{-1}$. Further, by compatibility of γ, it is immediate that $(x^{-1}x, x^{-1}y) \in \gamma$. Since E is a γ-class in S (~~Proposition~~ LEMMA 4.8) we have that $x^{-1}y \in E$. It follows that

$$x^{-1}y = (x^{-1}y)^{-1} = y^{-1}x$$

Now we can easily deduce that

$$x^{-1}x = x^{-1}x.x^{-1}x = x^{-1}y.y^{-1}x = (x^{-1}y)(x^{-1}y)^{-1} = x^{-1}y.$$

It follows that $x = xx^{-1}x = xx^{-1}y = yy^{-1}y = y$. Thus $\gamma \cap \mathcal{R} = \iota_S$

Proved.

However it should be noted that equality of the theorem 4.14 need not necessarily hold in an arbitrary RU-semigroup. For example, we consider a rectangular band $B = X \times Y$, where X and Y are non-empty sets having more than one element. The operation in B is defined by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1, y_2)$$

It is easily seen that $((x_1, y_1), (x_2, y_2)) \varepsilon \mathcal{R}$ if and only if $x_1 = x_2$. Thus if y_1, y_2 are two distinct elements in Y, we have $((x, y_1), (x, y_2)) \varepsilon \mathcal{R} \cap \gamma$, since $\gamma = B \times B$. Similarly $\gamma \cap \mathcal{L} \neq \iota_S$ in B.

CHAPTER III

§1 This chapter will be devoted to the study of congruences on a particular class of regular semigroups, namely bisimple ω-semigroups.

We recall that the set E of idempotents of a semigroup S is partially ordered under the natural ordering $\leq$ defined by the rule :

$$e \leq f \qquad \text{iff} \qquad ef = fe = e;$$

where e, f are arbitrary idempotents of the semigroup S. In an inverse semigroup, the set of idempotents forms a commutative semilattice under this ordering.

<u>DEFINITION 1.1</u> A semigroup S is said to be an ω-semigroup if its idempotents form a simple descending chain

$$e_0 > e_1 > e_2 > ---$$

<u>PROPOSITION 1.2</u> <u>A regular ω-semigroup S is an inverse semigroup with an identity.</u>

<u>Proof</u> Suppose $E = \{ e_i : i = 0, 1, 2 -- \}$, is the set of idempotents in the regular ω-semigroup S, where $e_0 > e_1 > e_2 > ---$.
linearly ordered, it follows that either $e_i \leq e_j$ or $e_j \leq e_i$;
linearly ordered, it follows that either $e_i \leq e_j$ or $e_j \leq e_i$;
that is, either $e_i e_j = e_j e_i = e_i$ or $e_i e_j = e_j e_i = e_j$. Thus idempotents in S commute, and hence by the theorem I.4.2 it follows that S is an inverse semigroup. Next we consider an arbitrary element <u>a</u> in S and let a^{-1} be its inverse in S. Then we have that $aa^{-1} = e_i$ for some i in N, where N is the set of non-negative integers. Now we can deduce that $e_0 a = e_0 aa^{-1}a = e_0 e_i a = e_i a = a$. Similarly

it can be verified that $ae_0 = a$. Thus e_0 is the identity element in the semigroup S.

Proved

We recall that a semigroup S is called bisimple if it has only one $\mathcal{D}$-class. Since a bisimple semigroup having an idempotent is regular (Theorem 2.11 [1]) it follows that a bisimple ω-semigroup is an inverse semigroup with an identity. The bicyclic semigroup B described in Example I.4.5 is clearly a bisimple ω- semigroup.

A more general example of a bisimple ω-semigroup can be constructed as follows.

EXAMPLE 1.3 Let G be a group and α an endomorphism of G. Let

$$S = \{(m, g, n) : m, n \in N, g \in G\}$$

where N is the set of non-negative integers. Multiplication in S is defined by the rule:

(1) $$(m, g, n)(p, h, q) = (m + p - r, g\alpha^{p-r} h\alpha^{n-r}, n + q - r)$$

where $m, n, p, q \in N$, $g, h \in G$; $r = \min(n, p)$ and by α^0 we mean the identity automorphism of the group G. The semigroup S is said to be generated by the group G and endomorphism α and will be denoted by $S(G, \alpha)$. REILLY [15] has shown that $S(G, \alpha)$ is a bisimple ω-semigroup. Further, two elements (m, g, n) and (p, h, q) in $S(G, \alpha)$ are $\mathcal{R}$-equivalent if and only if $m = p$. Similarly they are $\mathcal{L}$-equivalent if and only if $n = q$. Thus it follows that

(2) $$((m, g, n), (p, h, q)) \in \mathcal{H} \iff m = p \text{ and } n = q$$

The set of idempotent elements of S is given by $E = \{(m, 1, m) : m \in N\}$

P.T.O

where I is the identity of the group G. Obviously the $\mathcal{H}$-class containing e_0 is given by : $U = \{(0, g, 0) : g \in G\}$. The unique inverse of an element (m, g, n) in S is given by (n, g^{-1}, m) where g^{-1} denotes the group-inverse of g in G. REILLY [15] has further shown that if S is an arbitrary bisimple ω-semigroup and G is the group of units of S, then there exists an endomorphism α of G such that S is isomorphic to $S(G, \alpha)$. For an example in the case of the bicyclic semigroup B, we have that $G = \{I\}$ and α is the identity automorphism.

Suppose that $S_1 = S(G_1, \alpha)$ and $S_2 = S(G_2, \beta)$ are two bisimple ω-semigroups where α, β are endomorphisms of the groups G_1 and G_2 respectively. Then REILLY [15] has shown that there exists an isomorphism φ of S_1 onto S_2 if and only if there exists an isomorphism ϑ of G_1 onto G_2 such that $\alpha\vartheta = \vartheta\beta\lambda_z$, where λ_z is an inner automorphism of G_2 for some element z in G_2.

In the rest of this chapter, all the symbols attached to semigroup $S = S(G, \alpha)$ will continue to carry the meaning prescribed to them in this section.

§2 Now we proceed to consider Λ, the lattice of congruences on a bisimple ω-semigroup $S = S(G, \alpha)$. Let us denote by Λ_{IS} and Λ_G, the set of all idempotent-separating congruences on S and the set of all group-congruences on S respectively. Clearly $\Lambda_{IS} \cap \Lambda_G = \varphi$ MUNN and REILLY [21] have shown that Λ is the disjoint union of Λ_{IS} and Λ_G.

<u>THEOREM 2.1</u> <u>A congruence ρ on the semigroup $S = S(G, \alpha)$ is either a group-congruence or an idempotent-separating congruence.</u>

<u>Proof</u> Let us assume that ρ is not idempotent-separating. In order to show that ρ is a group-congruence it is sufficient to show that all the idempotents of S lie in the same ρ-class of S (Theorem II.2.11). Since ρ is not idempotent-separating, it follows that $(e_m, e_{m+k}) \in \rho$ for some $m \in N$ and $k > 0$. Let us put $x = (0, 1, m)$. Then we have that

$$xe_m x^{-1} = (0, 1, m)(m, 1, m)(m, 1, 0) = (0, 1, 0) = e_0$$

and

$$xe_{m+k} x^{-1} = (0, 1, m)(m + k, 1, m + k)(m, 1, 0) = (k, 1, k) = e_k.$$

Also clearly $e_0 e_1 = e_1$ and $e_k e_1 = e_k$. Now from the compatibility of ρ it follows that $(e_0, e_k) \in \rho$ and also that $(e_1, e_k) \in \rho$. Hence it follows that $(e_0, e_1) \in \rho$. Now we make use of the law of induction and let us assume that $(e_0, e_n) \in \rho$ for some $n \in N$. Let $y = (n, 1, 0)$. Then we have that $ye_1y^{-1} = e_{n+1}$ and $ye_0y^{-1} = e_n$. By compatibility of ρ, it follows that $(e_n, e_{n+1}) \in \rho$ and transitivity of ρ implies that $(e_0, e_{n+1}) \in \rho$. Thus by induction $(e_0, e_m) \in \rho$ for all m in N, and hence all idempotents of S lie in the same ρ-class.

<u>Proved</u>

Many properties of a congruence λ on the semigroup $S = S(G,\alpha)$ can be conveniently expressed in terms of the subset A_λ of the group G, defined as follows :

$$A_\lambda = \{g \in G : ((0, g, 0), e_0) \in \lambda\}$$

PROPOSITION 2.2 For any congruence λ on a semigroup $S = S(G, \alpha)$, the subset A_λ is an α-admissible normal subgroup of the group G.

Proof Let us define $\lambda_0 = \lambda \cap (U \times U)$ where U is the $\mathcal{H}$-class of S, containing e_0. Clearly λ_0 is a congruence on U. Also e_0 is the identity element of the subgroup U and hence $e_0\lambda_0$, the λ_0-class containing e_0 is a normal subgroup of U. The mapping φ defined by the rule $g\varphi = (0, g, 0)$ is clearly an isomorphism from G onto U and obviously $A_\lambda\varphi = e_0\lambda_0$. It follows that A_λ is a normal subgroup of G.

Next suppose that $g \in A_\lambda$: that is $(x, e_0) \in \lambda$ where $x = (0, g, 0)$. Let $z = (0, 1, 1)$. Then we deduce that

$$zxz^{-1} = (0, 1, 1)(0, g, 0)(1, 1, 0) = (0, g\alpha, 0).$$

Also we have that $ze_0z^{-1} = e_0$. By compatibility of λ it follows that $((0, g\alpha, 0), e_0) \in \lambda$ and hence $g\alpha \in A_\lambda$. Thus A_λ is α-admissible.

Proved

We shall denote by A the set of all α-admissible normal subgroups of the group G.

PROPOSITION 2.3 Let λ, λ' be two arbitrary congruences on the semigroup $S = S(G, \alpha)$ such that $\lambda \subseteq \lambda'$. Then $A_\lambda \subseteq A_{\lambda'}$.

Proof Let $g \in A_{\lambda}$. It follows that $((0, g, 0), e_0) \in \lambda$, and hence that $((0, g, 0), e_0) \in \lambda'$, since $\lambda \subseteq \lambda'$. Then it is immediate that $g \in A_{\lambda'}$. Thus $A_{\lambda} \subseteq A_{\lambda'}$.

Proved

§3 Firstly, we restrict our attention to the consideration of idempotent-separating congruences on the semigroup $S = S(G, \alpha)$.

PROPOSITION 3.1 Let λ be an idempotent-separating congruence on the semigroup $S = S(G, \alpha)$. Then

$((m, g, n), (p, h, q)) \in \lambda$ iff $m = p$, $n = q$ and $gh^{-1} \in A_{\lambda}$

Proof Let us put $x = (m, g, n)$ and $y = (p, h, q)$. Since λ is idempotent-separating, we have by the proposition II.1.5 that $\lambda \subseteq \mathcal{H}$. Now $(x, y) \in \mathcal{H}$ and by (2) it follows that $m = p$ and $n = q$. Let $a = (0, 1, m)$ and $b = (n, 1, 0)$. Then we have that $axb = (0, 1, m)(m, g, n)(n, 1, 0) = (0, g, 0)$ and that $ayb = (0, 1, m)(p, h, q)(n, 1, 0) = (0, h, 0)$. Hence by the compatibility of λ we have that $((0, g, 0), (0, h, 0)) \in \lambda$. Let h^{-1} denote the group-inverse of h in G. Then by the compatibility of λ it follows that

$$((0, g, 0)(0, h^{-1}, 0), (0, h, 0)(0, h^{-1}, 0)) \in \lambda$$

and hence $((0, gh^{-1}, 0), e_0) \in \lambda$. Hence we have that $gh^{-1} \in A_{\lambda}$.

Conversely we assume that $m = p$, $n = q$ and $gh^{-1} \in A_{\lambda}$. Now $gh^{-1} \in A_{\lambda}$ implies that $((0, gh^{-1}, 0)\, e_0) \in \lambda$. Hence by the compatibility of λ, it follows that

$$((m, 1, 0)(0, gh^{-1}; 0)(0, h, 0)(0, 1, n), (p, 1, 0)(0, 1, 0)(0, h, 0)(0, 1, q)) \in \lambda$$

that is $((m, 1, 0)(0, g, 0)(0, 1, n), (p, 1, 0)(0, h, 0), (0, 1, q)) \varepsilon \lambda$ and hence $((m, g, n), (p, h, q)) \varepsilon \lambda$.

Proved

COROLLARY 3.2 Let λ, λ' be idempotent-separating congruences on the semigroup $S = S(G, \alpha)$, such that $A_\lambda \subseteq A_{\lambda'}$. Then $\lambda \subseteq \lambda'$.

Proof Let us assume that $((m, g, n), (p, h, q)) \varepsilon \lambda$, where $m, n, p, q \varepsilon N$ and $g, h \varepsilon G$. Since λ is idempotent- separating, it follows by the proposition 3.1 that $m = p$, $n = q$ and $gh^{-1} \varepsilon A_\lambda$. It follows that $gh^{-1} \varepsilon A_{\lambda'}$ since $A_\lambda \subseteq A_{\lambda'}$. Again by the proposition 3.1, it follows that $((m, g, n), (p, h, q)) \varepsilon \lambda'$, since λ' is idempotent-separating. Thus $\lambda \subseteq \lambda'$.

Proved

PROPOSITION 3.3 Let A be an α-admissible normal subgroup of the group G. Then there exists an idempotent-separating congruence λ on $S = S(G, \alpha)$, such that $A = A_\lambda$.

Proof Let us define a relation λ on the semigroup $S = S(G, \alpha)$ by the rule

$$((m, g, n), (p, h, q)) \varepsilon \lambda \Longleftrightarrow m = p, n = q \text{ and } gh^{-1} \varepsilon A$$

where $m, h, p, q \varepsilon N$ and $g, h \varepsilon G$. Obviously λ is reflexive and symmetric. Let $((m, g, n), (m, h, n)) \varepsilon \lambda$ where $gh^{-1} \varepsilon A$. Also let $((m, h, n), (m, k, n)) \varepsilon \lambda$ where $k \varepsilon G$. Clearly $hk^{-1} \varepsilon A$. Since A is a subgroup, it follows that

$$gk^{-1} = g.1.k^{-1} = gh^{-1}. hk^{-1} \varepsilon A$$

and hence $((m, g, n), (m, k, n)) \in \lambda$. Thus λ is transitive. Next we assume that $((m, g, n), (m, h, n)) \in \lambda$, where $gh^{-1} \in A$. Let us choose an arbitrary element (p, k, q) in the semigroup S. Now we have that

$$(m, g, n)(p, k, q) = (m + p - r, g\alpha^{r-p} k\alpha^{r-q}, h + q - r)$$

where $r = \min(n, p)$. Similarly we have that

$$(m, h, n)(p, k, q) = (m + p - r, h\alpha^{r-p} k\alpha^{r-q}, n + q - r).$$

We further deduce that

$$(g\alpha^{r-p} k\alpha^{r-n})(h\alpha^{r-p} k\alpha^{r-n})^{-1}$$

$$= (g\alpha^{r-p})(k\alpha^{r-n})(k\alpha^{r-n})^{-1}(h\alpha^{r-p})^{-1}$$

$$= (g\alpha^{r-p})(h\alpha^{r-p})^{-1} = (gh^{-1})\alpha^{r-p}$$

But since $gh^{-1} \in A$ and A is α-admissible, it follows that $(gh^{-1})\alpha^{r-p} \in A$. Hence it follows that

$$((m, g, n)(p, k, q), (m, h, n)(p, k, q)) \in \lambda.$$

Similarly the left compatibility of λ can be verified. Thus λ is a congruence on $S = S(G, \alpha)$. Obviously $\lambda \subseteq \mathcal{H}$ and hence it is idempotent-separating. Obviously for any $g \in A$, we have $((0, g, 0), (0, 1, 0)) \in \lambda$ and hence $g \in A_\lambda$. On the other hand if $g \in A_\lambda$, then $((0, g, 0), e_0) \in \lambda$ and hence $g \in A$. Thus $A = A_\lambda$.

<u>Proved</u>

<u>PROPOSITION 3.4</u> <u>Let</u> μ <u>be the maximum idempotent-separating congruence on the semigroup</u> $S = S(G, \alpha)$. <u>Then</u> $\mu = \mathcal{H}$ <u>and</u> $S/\mathcal{H} \cong B$, <u>where B is the bicyclic semigroup described in the example</u>

Proof Let us consider the mapping ϑ from S onto B defined by the rule : $(m, g, n)\vartheta = (m, n)$, where $m, n \in N$ and $g \in G$. Obviously ϑ is a homomorphism. Also we know that $((m, g, n), (p, h, q)) \in \mathcal{H}$ if and only if $m = p$ and $n = q$, that is $(m, g, n)\vartheta = (p, h, q)\vartheta$. It follows that $\vartheta \circ \vartheta^{-1} = \mathcal{H}$ i.e. that $\mathcal{H}$ is a congruence. But μ is the largest congruence contained in $\mathcal{H}$, hence we have that $\mu = \mathcal{H}$. Further, by the fundamental theorem of homomorphisms, it is immediate that $S/\mathcal{H} \cong B$.

Proved

Let A be an α-admissible normal subgroup of the group G. We define the mapping $\alpha/A : G/A \to G/A$ by the rule : $(Ag)\alpha = A(g\alpha)$ for every $g \in G$. Clearly α/A is an endomorphism of the quotient group G/A. Similarly we can define α^k/A and it is easily checked that $\alpha^k/A = (\alpha/A)^k$.

THEOREM 3.5 Let λ be an idempotent-separating congruence on the semigroup $S = S(G,\alpha)$. Then $S/\lambda \cong S(G/A_\lambda, \alpha/A_\lambda)$.

Proof We define a mapping ϑ from the semigroup S into the semigroup $S(G/A_\lambda, \alpha/A_\lambda)$ by the rule : $(m, g, n)\vartheta = (m, A_\lambda g, n)$. Clearly ϑ is an ONTO mapping. Further, since A_λ is an α-admissible normal subgroup and $(A_\lambda g)\,\alpha^r/A_\lambda = A_\lambda(g\alpha^r)$, we easily deduce that

$$[\,(m,\ g,\ n)\,(p,\ h,\ q)\,]\,\vartheta$$
$$= [\,(m + p - r,\ g\alpha^{p-r}\ h\alpha^{n-r},\ n + q - r)]\,\vartheta \qquad r = \min\,(n,\ p)$$
$$= (m + p - r,\ A_\lambda\ (g\alpha^{p-r}\ h\,\alpha^{n-r}),\ n + q - r)$$
$$= (m + p - r,\ \ [A_\lambda\ (g\alpha^{p-r})\ ]\,[A_\lambda\ (h\alpha^{n-r})\ ],\qquad n + q - r)$$
$$= (m + p - r,\ \ [\ (A_\lambda g)\,\alpha^{p-r}/A_\lambda]\ [\ (A_\lambda h)\,\alpha^{n-r}/A_\lambda],\qquad n + q - r)$$
$$= (m + p - r,\ (A_\lambda g)\ (\alpha/A_\lambda\,)^{p-r}\ (A_\lambda h)\ (\alpha/A_\lambda\,)^{n-r},\qquad n + q - r)$$
$$= (m,\ A_\lambda g,\ n)\ (p,\ A_\lambda h,\ q)$$
$$= (m,\ g,\ n)\,\vartheta \quad (p,\ h,\ q)\,\vartheta$$

Hence it follows that ϑ is a homomorphism.

Also we have that $(m,\ g,\ n)\ \vartheta\ =\ (p,\ h,\ q)\ \vartheta$, that is, $(m,\ A_\lambda g,\ n)\ =\ (p,\ A_\lambda h,\ q)$ if and only if $m = p,\ n = q$ and $A_\lambda g = A_\lambda h$ i.e. if and only if $gh^{-1}\ \varepsilon\ A_\lambda$ since A_λ is a normal subgroup. Making use of the proposition 3.1, we conclude that $(m,\ g,\ n)\,\vartheta\ = (p,\ h,\ q)\,\vartheta$ if and only if $(\ (m,\ q,\ n),\ (p,\ h,\ q)\ )\ \varepsilon\ \lambda$ and hence $\vartheta\circ\vartheta^{-1}\ =\ \lambda$

By the fundamental theorem of homomorphism, it is immediate that

$$S/_\lambda\ \cong\ S\ (G/A_\lambda\,,\ \alpha/A_\lambda\,)$$

<u>Proved</u>

§4 Now we consider the group-congruences on the semigroup $S\ =\ S\ (G,\alpha\ )$.

Let A be an α-admissible normal subgroup of the group G.

<u>DEFINITION 4.1</u> The <u>radical of A</u> is defined to be the set of those elements of the group G for which $g\alpha^n$ belongs to A for some integer n; that is

$$\text{rad } A = \{\ g\ \varepsilon\ G\ :\ \ g\alpha^n\ \varepsilon\ A\ \ \text{for some}\ \ n\ \varepsilon\ N\ \}$$

The concept of radical of A plays an important part in the investigation of group congruences on $S = S(G, \alpha)$. Some basic facts about the radical of an α-admissible normal subgroup A ($A \in \mathcal{A}$) are expressed in the following

PROPOSITION 4.2 Let $A, A' \in \mathcal{A}$. Then the following hold:

(I) $\text{rad } A \in \mathcal{A}$

(II) $A \subseteq \text{rad } A$

(III) $A \subseteq A' \Rightarrow \text{rad } A \subseteq \text{rad } A'$

(IV) $\text{rad rad } A = \text{rad } A$

(V) $A \text{ rad } I \subseteq \text{rad } A$

(VI) $\text{rad } A = \text{rad } (A \text{ rad } I)$

Proof (I) Let $g \in \text{rad } A$ and hence $g\alpha^n \in A$ for some integer n, equivalently $(g\alpha)\alpha^{n-1} \in A$ and hence we have that $g\alpha \in \text{rad } A$. Thus rad A is α-admissible. Further let $g, h \in \text{rad } A$. It follows that $g\alpha^n \in A$ and $h\alpha^m \in A$ for some integers n, m and
(*) let $n > m$ such that $n = m + k$. Since A is a subgroup and α is an endomorphism, we have that $(h\alpha^m)^{-1} = h^{-1}\alpha^m \in A$ and further $h^{-1}\alpha^m\alpha^k = h^{-1}\alpha^n \in A$ since A is α-admissible. It follows that $(g\alpha^n)(h^{-1}\alpha^n) = (gh^{-1})\alpha^n \in A$. Thus $gh^{-1} \in \text{rad } A$ and hence rad A is a subgroup of the group G.

Finally let $g \in \text{rad } A$ and h be an arbitrary element of the group G. Let h^{-1} denote the inverse of h on G. Now we observe that

$(h^{-1}gh)\alpha^n = (h^{-1}\alpha^n)(g\alpha^n)(h\alpha^n) \in A$; since $g\alpha^n \in A$ and A is normal. It follows that $h^{-1}gh \in \text{rad } A$ and hence we conclude that

(*) This proof also holds if $k = 0$ i.e. when $m = n$, since α^0 is the identity automorphism of the group G.

rad A is normal in G. Thus rad A ε $\mathcal{A}$

(II) Since A is α-admissible, it is obvious that $A \subseteq$ rad A.

(III) Let $A \subseteq A'$ and suppose that $g \,\varepsilon$ rad A. It follows that $g\alpha^n \,\varepsilon\, A$ for some integer n and hence $g\alpha^n \,\varepsilon\, A'$, which immediately gives that $g \,\varepsilon$ rad A′. Thus rad A $\subseteq$ rad A′.

(IV) By the part (I) and part (II), it is clear that rad A $\subseteq$ rad (rad A). On the other hand let $g \,\varepsilon$ rad (rad A). It follows that $g\alpha^n \,\varepsilon$ rad A for some $n \,\varepsilon\, N$, and hence $(g\alpha^n)\alpha^m = g\alpha^{n+m} \,\varepsilon\, A$ for some integers n, m. Hence we have that $g \,\varepsilon$ rad A. Thus rad (rad A) $\subseteq$ rad A and hence rad A = rad rad A.

(V) Clearly rad I $= \bigcup_{k=1}^{\infty} \ker \alpha^k$ where I is the identity of the group G. Let $g \,\varepsilon$ A rad I, that is $g = hj$ for some $h \,\varepsilon\, A$ and $j \,\varepsilon$ rad I. It follows that $j\alpha^n = I$ for some integer n. Now we observe that

$g\alpha^n = (hj)\,\alpha^n = (h\alpha^n)(j\alpha^n) = h\alpha^n.I = h\alpha^n \,\varepsilon\, A$;

since A is α-admissible. Thus $g \,\varepsilon$ rad A and we have that A rad I $\subseteq$ rad A.

(VI) From parts (III) (IV) and (V), it is clear that

$$\text{rad (A rad I)} \subseteq \text{rad rad A} = \text{rad A}$$

On the other hand since A and rad I are normal subgroups, we have that $A \subseteq$ A rad I and hence rad A $\subseteq$ rad (A rad I). Thus we conclude that rad A = rad (A rad I).

<u>Proved</u>

We shall denote by $\mathcal{A}^*$ the set of all α-admissible normal subgroups A ($A \,\varepsilon\, \mathcal{A}$) for which rad A = A holds.

PROPOSITION 4.3 Let τ be a group-congruence on the semigroup $S = S(G, \alpha)$. Then

(I) $(x, y) \in \tau$ iff $xy^{-1} \in e_0 \tau$ $(x, y \in S)$

(II) $A_\tau \in \mathcal{A}^*$

Proof (I) We assume that $(x, y) \in \tau$. Let y^{-1} be an inverse of y in the semigroup $S = S(G, \alpha)$. By compatibility of τ, we have that $(xy^{-1}, yy^{-1}) \in \tau$. Also since τ is a group congruence, we have that $(yy^{-1}, e_0) \in \tau$. By transitivity of τ, it is immediate that $(xy^{-1}, e_0) \in \tau$. Conversely let us assume that $xy^{-1} \in e_0\tau$. Then by compatibility of τ, we have that $(xy^{-1}y, e_0y) \in \tau$. Also since $(e_0, y^{-1}y) \in \tau$, it follows that $(x, xy^{-1}y) \in \tau$. Then the transitivity of τ implies that $(x, y) \in \tau$.

(II) We know by the proposition 2.2, that A_τ is α-admissible normal subgroup of G. Clearly $A_\tau \subseteq \text{rad } A_\tau$. Thus all we need to show is that $\text{rad } A_\tau \subseteq A_\tau$. Let $g \in \text{rad } A_\tau$. It follows that $g\alpha^n \in A_\tau$ for some integer n, that is $((0, g\alpha^n, 0)\, e_0) \in \tau$. Now we observe that $(n, 1, 0)(0, g\alpha^n, 0)(0, 1, n) = (n, g\alpha^n, n)$ and $(n, 1, 0)(0, 1, 0)(0, 1, n) = (n, 1, n) = e_n$. It follows by the compatibility of τ, that $((n, g^n, n), e_n) \in \tau$. Also since τ is a group congruence, we have that $(e_n, e_0) \in \tau$. Again by compatibility of τ, we have that

$((n, 1, n)(0, g, 0), e_0(0, g, 0)) \in \tau$, i.e. $((n, g^n, n),(0,g,0)) \in$

By transitivity of τ, we immediately get that $((0, g, 0), e_0) \in \tau$ and hence $g \in A_\tau$. Thus $\text{rad } A_\tau \subseteq A_\tau$.

Proved

Our next task is to determine the structure of the maximum group homomorphic image of the semigroup $S = S(G,\alpha)$.

Let us consider the Cartesian product $G \times N$, where N is the set of non-negative integers. We define a relation ρ on $G \times N$ by the rule that

$$((a, i), (b, j)) \in \rho \quad \text{iff} \quad a\alpha^{r-i} = b\alpha^{r-j}$$

for some $r \geqslant i, j$ (and hence for all sufficiently large r). It is clear that

(3) $$(a, i)\rho = (a\alpha^{n}, i + n)\rho .$$

for every integer n. Obviously the relation ρ is reflexive and symmetric. Next let $((a, i), (b, j)) \in \rho$ and $((b, j), (c, k)) \in \rho$. Thus we have that $a\alpha^{r-i} = b\alpha^{r-j}$ for $r \geqslant i, j$ and that $b\alpha^{s-j} = c\alpha^{s-k}$ for some $s \geqslant j, k$. Let $t = \max\{r, s\}$. Then clearly $a\alpha^{t-i} = b\alpha^{t-j} = c\alpha^{t-k}$ and hence it follows that $((a, i), (c, k)) \in \rho$. Thus ρ is an equivalence on the set $G \times N$. Let us consider the Quotient set $(G \times N)/\rho$, and we define the multiplication in $(G \times N)/\rho$ by the rule :

$$(a, i)\rho\ (b, j)\rho = (a\alpha^{m-i} \circ b\alpha^{m-j}, m)\rho$$

where $m = \max(i, j)$. Associativity of the operation is easily checked. Also $(a, i)\rho \circ (I, 0)\rho = (a, i)\rho$ and $(I, 0)\rho\,(a, i)\rho = (a, i)\rho$ and hence $(I, 0)\rho$ is the two-sided identity of $(G \times N)/\rho$. Let $(a, i)\rho \in (G \times N)/_{\rho}$ and a^{-1} denote the group-inverse of a in G. Then we notice that

$$(a, i)\rho\,(a^{-1}, i)\rho = (I, i)\rho = (I, \theta)\rho$$

by virtue of (3). Similarly $(a^{-1}, i)\rho\,(a, i)\rho = (I, \theta)\rho$. Thus

$(G \times N)/\rho$ is a group; we call it the <u>direct α-limit</u> of the group G and denote it by G_α. Next we define a mapping $\tilde{\alpha} : G_\alpha \longrightarrow G_\alpha$ by the rule that $[(a, i)\rho]\tilde{\alpha} = (a\alpha, i)\rho$.

It should be noted here that for any $p \in N$, we have that

$$[(a, i)\rho]\tilde{\alpha}^p = (a\alpha^p, i)\rho$$

It is easily seen that the mapping $\varphi : G_\alpha \longrightarrow G_\alpha$ defined by the rule:

(4) ____ $$[(a, i)\rho]\varphi = (a, i+1)\rho$$

is a two-sided inverse of the map $\tilde{\alpha}$ that is $\varphi = \tilde{\alpha}^{-1}$ and hence $\tilde{\alpha}$ is a one-one onto map.

Next we observe that

$$[(a, i)\rho]\tilde{\alpha}\,[(b, j)\rho]\tilde{\alpha} = [(a\alpha, i)\rho][(b\alpha, j)\rho]$$

$$= ((a\alpha)\alpha^{m-1}\,(b\alpha)\alpha^{m-j}, m)\rho$$

$$= [((a\alpha^{m-1})(b\alpha^{m-j}), m)\rho]\tilde{\alpha}$$

$$= ([(a, i)\rho][(b, j)\rho])\tilde{\alpha}$$

Thus $\tilde{\alpha}$ is an automorphism.

Also by (4) it is clear that for any integer q

$[(a, i)\rho]\tilde{\alpha}^{-q} = (a, i+q)$ and we can deduce that for any integers p, q

(5) ____ $$[(a, i)\rho]\tilde{\alpha}^{p-q} = (a\alpha^p, i+q)$$

We shall denote the set of integers by Z. Let H be an arbitrary group and β an automorphism of the group H. We define a multiplication in the set $Z \times H$ by the rule $(i, a)(j, b) = (i+j, a\beta^j b)$ where $i, j \in Z$ and $a, b \in H$. Associativity of this multiplication is immediate from the associativity of addition of integers, and the fact that β is an automorphism of the group H. Let $\hat{1}$ denote the identity of the group H. Then for any element $(i, a) \in Z \times H$, we have that

$$(i, a)(0, 1) = (i, a) \quad \text{and} \quad (0, 1)(i, a) = (i, a)$$

Thus $(0, 1)$ is the identity of $Z \times H$. Also let a^{-1} denote the inverse of the element <u>a</u> in G, and let β^{-1} denote the inverse map of automorphism β. Then we deduce that

$$(i, a)(-i, a^{-1}\beta^{-1}) = (0, a\beta^{-1}a^{-1}\beta^{-1}) = (0, 1)$$

Similarly $(-i, a^{-1}\beta^{-1})(i, a) = (o, 1)$. It follows that $(-i, a^{-1}\beta^{-1})$ is a two-sided inverse of the element (i, a) in $Z \times H$, and thus $Z \times H$ is a group. We denote it by $H \uparrow \beta$.

<u>THEOREM 4.4</u> <u>Let</u> γ <u>be the minimum group congruence on the semigroup</u> $S = S(G,\alpha)$. <u>Then</u> $S/\gamma \cong G_\alpha \uparrow \tilde{\alpha}$ <u>where the symbols have their usual meanings.</u>

<u>Proof</u> We define a map $\vartheta : S \longrightarrow G_\alpha \uparrow \tilde{\alpha}$ by the rule

$$(m, g, n)\vartheta = (m-n, (g, n)\rho).$$

Suppose $(i, (g, j)\rho)$ is an arbitrary element of the group $G_\alpha \uparrow \tilde{\alpha}$. If $i \geqslant o$, then we note that $(i, (g, j)\rho) = (i + j, g, j)\vartheta$, and on the other hand if $i \leqslant o$, then we see that

$$(i, (g, j)\rho) = (i, (g\alpha^{-i}, j-i)\rho) \qquad \text{by (3)}$$

$= (j, g\alpha^{-i}, j-i)\vartheta$. It follows that ϑ is an onto mapping from S onto the group $G_\alpha \uparrow \tilde{\alpha}$.

Next let (m, g, n) and (p, h, q) be two arbitrary elements of the semigroup $S = S(G, \alpha)$. Then we can deduce that

$(m, g, n)\vartheta \; (p, h, q)\vartheta$

$= (m-n, (g, n)\rho) \; (p-q, (h, q)\rho)$

$= (m-n + p-q, (g, n)\rho\,\tilde{\alpha}^{p-q}(h, q)\rho)$

$= (m-n + p-q, (g\alpha^p, n + q)\rho\,(h, q)\rho)$ by virtue of (5)

$= (m - n + p - q, (g\alpha^{p} h\alpha^{n}, n + q)\rho)$

$= (m - n + p - q, (g\alpha^{p-r} h\alpha^{n-r}, n + q - r)\rho)$ by (3)

$r = \min\{n, p\}$

$= (m + p - r, g\alpha^{p-r} h\alpha^{n-r}, n + q - r)\vartheta$

$= [(m, g, n)(p, h, q)]\vartheta$

Thus ϑ is a homomorphism.

Clearly $\vartheta \circ \vartheta^{-1}$ is a group congruence on S and hence $\gamma \subseteq \vartheta \circ \vartheta^{-1}$. On the other hand let $((m, g, n), (p, h, q)) \varepsilon \vartheta \circ \vartheta^{-1}$. This means that $(m-n, (g, n)\rho) = (p-q, (h, q)\rho)$ and hence we have that $m - n = p - q$ and $(g, n)\rho = (h, q)\rho$. It follows that $g\alpha^{k-n} = h\alpha^{k-q}$ for some integer $k \geqslant n, q$. Now we can deduce that

$(m, g, n) e_k = (m + k - n, g\alpha^{k-n}, k)$

$= (p + k - q, h\alpha^{k-q}, k) = (p, h, q) e_k$

Then by the theorem II.5.8 it follows that $((m, g, n)(p, h, q)) \varepsilon \gamma$

Thus $\vartheta \circ \vartheta^{-1} \subseteq \gamma$. Now by the fundamental theorem of the homomorphisms, it is immediate that $S/\gamma \cong G_\alpha \uparrow \tilde{\alpha}$.

Proved

DEFINITION 4.5 The mapping α of the group G into itself is called stable provided the following hold

(I) $G\alpha^{k} = G\alpha^{k+1}$ for some integer k.

II) $\alpha/G\alpha^{k}$ is an automorphism of the group $G\alpha^{k}$.

The smallest integer k for which these conditions hold is called the index of stability of the map α.

NOTE 4.6

(I) α is stable if it is an automorphism of the group G.

II) α is stable if it is nilpotent, i.e. $\alpha^n = \zeta$ for some n, where ζ is the zero map.

III) α' is stable if the group G is finite.

The structure of the group $S/_\gamma$ takes a simpler form if α is stable.

<u>THEOREM 4.7</u> <u>Let the endomorphism</u> α <u>of the group G be stable with the index of stability k, and let</u> $\beta = \alpha /_{G\alpha^k}$. <u>Then</u> $G_\alpha \uparrow \tilde{\alpha} \cong G\alpha^k \uparrow \beta$.

WE OMIT THE PROOF.

§5 Now we proceed to consider the congruences arising by the operations of join and intersection of the group-congruences and idempotent-separating congruences on the bisimple ω-semigroup $S = S(G, \alpha)$ generated by the group G and an endomorphism α of the group G.

<u>PROPOSITION 5.1</u> <u>Let</u> γ <u>be the minimum group congruence on the semigroup</u> $S = S(G,\alpha)$ <u>and I denote the identity of the group G.</u> <u>Then</u>

$$A_{\gamma \cap \mathcal{H}} = A_\gamma = \bigcup_{k=1}^{\infty} \ker \alpha^k = \text{rad } I$$

<u>Proof</u> We know that $A_\gamma = \{ g \in G : ((o, g, o), e_o) \in \gamma \}$. Now since the $\mathcal{H}$-class of S containing e_o is given by

$$U = \{ (o, g, o) \in S : g \in G \}$$

it is clear that $A_{\gamma \cap \mathcal{H}} = A_\gamma$. Also it is clear by the definition 4.1 that rad I $= \bigcup_{k=1}^{\infty} \ker \alpha^b$. Thus all we need to show is that $A_\gamma = \bigcup_{k=1}^{\infty} \ker \alpha^k$. Suppose g is an element of the group G. Now we deduce that $g \in A_\gamma \iff ((0, g, 0)\ e_0) \in \gamma \iff$

P.T.O

i.e. $\Longleftrightarrow$ $e_m(0, g, 0) = e_m e_0$ for some m, by Theorem II.5.8

i.e. $\Longleftrightarrow$ $(m, g\alpha^m, m) = e_m$ for some m

i.e. $\Longleftrightarrow$ $g\alpha^m = I$ for some m

i.e. $\Longleftrightarrow$ $g \in \bigcup_{k=1}^{\infty} \ker \alpha^k$

Hence we conclude that $A_\gamma = \bigcup_{k=1}^{\infty} \ker \alpha^k$

Proved

COROLLARY 5.2 Let $S = S(G, \alpha)$ and let $K = \bigcup_{k=1}^{\infty} \ker \alpha^k$

Then $S/_{(\gamma \cap \mathcal{H})} \cong S(G/_K, \alpha/_K)$

Proof Since $\gamma \cap \mathcal{H} \subseteq \mathcal{H}$, it follows that $\gamma \cap \mathcal{H}$ is an idempotent-separating congruence on S. Also by the proposition 5.1, we have that $A_{\gamma \cap \mathcal{H}} = K$. Now making use of the theorem 3.5, we immediately get that

$$S/_{\gamma \cap \mathcal{H}} \cong S(G/_K, \alpha/_K)$$

Proved

PROPOSITION 5.3 Let γ be the minimum group congruence on the semigroup $S = S(G, \alpha)$. Then

$$((m, g, n)(p, h, q)) \in \gamma \vee \mathcal{H} \quad \text{iff} \quad m - n = p - q$$

Proof Let $x = (m, g, n)$ and $y = (p, h, q)$. We assume that $(x, y) \in \gamma \vee \mathcal{H}$. By Proposition II.5.12, we know that $\gamma \vee \mathcal{H} = \gamma \circ \mathcal{H} \circ \gamma$. It follows that there exist two elements $a = (m', g', n')$ and $b = (p', h', q')$ in the semigroup S, such that $(x, a) \in \gamma$, $(a, b) \in \mathcal{H}$ and $(b, y) \in \gamma$. Now since $(a, b) \in \mathcal{H}$, it follows by (2) that $m' = p'$ and $n' = q'$. Next since $(x, a) \in \gamma$, it follows by the theorem II.5.8 that $e_k x = e_k a$ for some integer k, and hence for any integer greater than k, since S is a ω-semigroup. Thus we can assume that $k > m, m'$. Now we see that

$$e_k x = (k + m - r, g\alpha^{k-r}, k + n - r) = (k + m' - r', g'\alpha^{k-r'}, k + n' - r') = e_k a$$

where $r = \min\{k, m\}$ and $r' = \min\{k, m'\}$. Then we have that $(k, g\alpha^{k-m}, k + n - m) = (k, g'\alpha^{k-m'}, k + n' - m')$. This implies that $k + n - m = k + n' - m'$ and hence $m - n = m' - n'$. Also since $(b, y) \in \gamma$, we can likewise deduce that $p' - q' = p - q$. Now it is immediate that $m - n = m' - n' = p' - q' = p - q$. Conversely we assume that $m - n = p - q$. Let us suppose that $m \leq p$. Now we note that $e_p x = (p, g\alpha^{p-m}, p + n - m) = (p, g\alpha^{p-m}, q)$ Hence it follows that $(e_p x, y) \in \mathcal{H}$. Also since $e_p x = e_p e_p x$, it follows that $(x, e_p x) \in \gamma$. Thus we conclude that

$$(x, y) \in \gamma \circ \mathcal{H} \subseteq \gamma \vee \mathcal{H}.$$

A similar argument holds for the case when $m \geq p$.

Proved

PROPOSITION 5.4 *Let* $S = S(G,\alpha)$. *Then* $S/_{\gamma \vee \mathcal{H}} \cong \mathcal{C}$ *where* $\mathcal{C}$ *is the infinite cyclic semigroup.*

Proof We take $\mathcal{C}$ to be the group $\mathbb{Z}$ of integers with addition. Let us define a mapping ϑ from the semigroup S onto the group $\mathbb{Z}$ by the rule $(m, g, n)\vartheta = m - n$. Now we note that

$[(m, g, n)\vartheta][(p, h, q)\vartheta] = (m - n) + (p - q) = (m + p) - (n + q)$
$= (m + p - r) - (n + q - r)$, where $r = \min\{n, p\}$

On the other hand we see that $[(m, g, n).(p, h, q)]\vartheta$
$= [(m + p - r, g\alpha^{p-r} h\alpha^{n-r}, n + q - r)]\vartheta = (m + p - r) - (n + q - r)$.

It follows that $[(m, g, n)\vartheta][(p, h, q)\vartheta] = [(m, g, n)(p, h, q)]\vartheta$

Hence ϑ is a homomorphism. Further, by the proposition 5.3, it is clear that $\vartheta \circ \vartheta^{-1} = \gamma \vee \mathcal{H}$. Now by the fundamental theorem of homomorphisms, it is immediate that $S/_{\gamma \vee \mathcal{H}} \cong \mathcal{C}$.

<u>Proved</u>

<u>PROPOSITION 5.5</u> <u>Let</u> $\tau, \tau' \in [\gamma, \mathcal{H} \vee \gamma]$. <u>Then the following hold.</u>

(I) $e_0\tau = \{ (m, g, m) \in S : g \in A_\tau, m \in N \}$

(II) <u>If</u> $A_\tau \subseteq A_{\tau'}$ <u>then we have that</u> $\tau \subseteq \tau'$.

<u>Proof</u> (I) Let us put $x = (m, g, n)$ and suppose that $x \in e_0\tau$. By hypothesis $\tau \subseteq \mathcal{H} \vee \gamma$; hence $(x, e_0) \in \mathcal{H} \vee \gamma$. By Proposition 5.3 it follows that $m = n$. Further, by the compatibility of τ, we deduce that $((0, 1, m)\, e_0\, (m, 1, 0), (0, 1, m)\, x\, (m, 1, 0)) \in \tau$; equivalently $(e_0, (0, g, 0)) \in \tau$ and hence $g \in A_\tau$. Conversely let $x = (m, g, m)$, where $g \in A_\tau$. It follows that $((0, g, 0)\, e_0) \in \tau$. Again using the compatibility of τ, we deduce that

$$((m, 1, 0)(0, g, 0)(0, 1, m), (m, 1, 0)\, e_0\, (0, 1, m)) \in \tau;$$

equivalently $((m, g, m), e_m) \in \tau$. Since $(e_m, e_0) \in \tau$, it is immediate that $(x, e_0) \in \gamma$.

(II) Suppose $A_\tau \subseteq A_{\tau'}$. By part (I), it is obvious that $e_0\tau \subseteq e_0\tau'$. Let $(x, y) \in \tau$, where $x, y \in S$. Suppose that y^{-1} is the inverse of the element y in the semigroup $S = \mathcal{S}(G, \alpha)$. Then by the compatibility of τ , we have that $(xy^{-1}, yy^{-1}) \varepsilon \tau$. Also since τ is a group congruence we have that $(yy^{-1}, e_0) \varepsilon \tau$. It follows that $xy^{-1} \varepsilon e_0 \tau \subseteq e_0 \tau'$. Now from the proposition 4.3, it follows that $(x, y) \varepsilon \tau'$. Thus $\tau \subseteq \tau'$.

Proved

COROLLARY 5.6 Let $\tau \varepsilon [\gamma, \mathcal{H} \vee \gamma]$ and let $x = (m, g, n)$ and $y = (p, h, q)$ be elements in the semigroup $S = S(G, \alpha)$. Then $(x, y) \varepsilon \tau$ iff $m - n = p - q$ and $(g\alpha^{q-r})(h^{-1}\alpha^{n-r}) \varepsilon A_\tau$ where $r = \min\{n, q\}$.

Proof Let $(x, y) \varepsilon \tau$. Since $\tau \subseteq \mathcal{H} \vee \gamma$, we have that $(x, y) \varepsilon \mathcal{H} \vee \gamma$. It is immediate by the proposition 5.3, that $m - n = p - q$. We further observe that $xy^{-1} = (m, g, n)(q, h^{-1}, p)$ $= (m + q - r, g\alpha^{q-r} . h^{-1}\alpha^{n-r}, n + p - r)$. Now since τ is a group congruence, it follows by the proposition 4.3, that $xy^{-1} \varepsilon e_0\tau$. Making use of the part (I) of the proposition 5.5, we obtain that $g\alpha^{q-r}. h^{-1}\alpha^{n-r} \varepsilon A_\tau$.

Conversely suppose $m - n = p - q$ and $(g\alpha^{q-r})(h^{-1}\alpha^{n-r}) \varepsilon A_\tau$. It follows that $m + q - r = n + p - r$ and hence by part (I) of the proposition 5.5, we get that $xy^{-1} \varepsilon e_0 \tau$. Again by the proposition 4.3, we finally get that $(x, y) \varepsilon \tau$.

Proved

PROPOSITION 5.7 Let A be an α-admissible normal subgroup of the group G, such that $\operatorname{rad} A = A$. Then there exists a congruence $\tau \varepsilon [\gamma, \mathcal{H} \vee \gamma]$ such that $A = A_\tau$.

<u>Proof</u> Let us define $K = \{(m, g, m) \in S;\ g \in A, m \in N\}$ and let $X = (m, g, m)$, $y = (n, h, n)$ be two arbitrary elements of K, where $g, h \in A$ and $m, n \in N$. We observe that

$$xy = (m + n - r, g\alpha^{n-r} . h\alpha^{m-r}, m + n - r) \text{ where } r = \min\{m, n\}.$$

Now since $g, h \in A$ and A is α-admissible subgroup, it follows that $(g\alpha^{n-r})(h\alpha^{m-r}) \in A$ and hence $xy \in K$. We know that the inverse of the element x in the semigroup $S = S(G, \alpha)$ is given by $x^{-1} = (m, g^{-1}, m)$ and it clearly belongs to K since $g^{-1} \in A$. Thus K is an inverse subsemigroup of the semigroup S.

Next let $x = (m, g, m) \in K$ and $y = (p, h, q)$ be an arbitrary element of the semigroup S. Now if $q \geqslant m$, then we observe that

$$yxy^{-1} = (p, h, q)(m, g, m)(q, h^{-1}, p)$$
$$= (p, h(g\alpha^{q-m}) q)(q, h^{-1}, p)$$
$$= (p, h(g\alpha^{q-m}) h^{-1}, p)$$

and if $q \leqslant m$, then we have that

$$yxy^{-1} = (p, h, q)(m, g, m)(g, h^{-1}, p)$$
$$= (p + m - q, (h\alpha^{m-q}) g, m)(q, h^{-1}, p)$$
$$= (p + m - q, (h\alpha^{m-q}) g (h^{-1}\alpha^{m-q}), p + m - q)$$

Since $g \in A$ and A is α-admissible, we have that $g\alpha^{q-m} \in A$. Further since A is normal in G, it follows that $h(g\alpha^{q-m}) h^{-1} \in A$. On the other hand we see that

$$(h\alpha)^{m-q} g (h^{-1}\alpha^{m-q}) = (h\alpha^{m-q}) g (h\alpha^{m-q})^{-1} \in A$$

since A is normal in G. Thus we conclude that $yxy^{-1} \in K$ for every $x \in K$ and $y \in S$, i.e. K is a self-conjugate subsemigroup of S. Next suppose $e_p x \in K$ for some idempotent e_p in S. We know that

$e_p x = (p + m - r, g\alpha^{p-r}, p + n - r)$ where $r = \min \{m, p\}$. It follows that $p + m - r = p + n - r$ i.e. $m = n$ and $g\alpha^{p-r} \varepsilon A$, which means that $g \varepsilon$ rad A. But we have that rad A = A; hence $g \varepsilon A$. Thus $x \varepsilon A$. Hence we conclude that K is a closed subsemigroup in the sense of Chapter I, §4. Clearly K contains the set E of idempotents of the semigroup S.

Let us define a relation τ on the semigroup S, by the rule

$$(x, y) \varepsilon \tau \quad \text{iff} \quad xy^{-1} \varepsilon K.$$

Since K is a closed, self-conjugate, inverse subsemigroup of the inverse semigroup $S = S(G, \alpha)$ and $K \supseteq E$ it follows by the proposition I.4.6 that τ is a congruence on S. Obviously all the idempotents $\{e_m : m \varepsilon N\}$ belong to the same τ- class of S and hence by the same argument employed in the proof of the theorem II.2.11, we can deduce that τ is a group congruence on S. Thus $\tau \supseteq \gamma$. Let $x = (m, g, n)$ and $y = (p, h, q)$ and suppose that $(x, y) \varepsilon \tau$. It follows that

$xy^{-1} = (m + q - r, (g\alpha^{q-r})(h^{-1}\alpha^{n-r}), n + p - r) \varepsilon K$

where $r = \min \{n, q\}$. Hence $m + q - r = n + p - r$ which gives $m - n = p - q$. By the proposition 5.3, it follows that $(x, y) \varepsilon \mathcal{H} \vee \gamma$ and hence $\tau \subseteq \mathcal{H} \vee \gamma$. Finally we observe that $(x, e_0) \varepsilon \tau$ iff $x \varepsilon K$. Hence $K = e_0\tau$. Since τ is a group congruence, it follows from the part (I) of the proposition 5.5 and from the definition of the subset K that $A = A_\tau$

Proved

PROPOSITION 5.8 Let λ and τ be congruences on the semigroup $S = S(G,\alpha)$, such that $\lambda \varepsilon [\iota, \mathcal{H}]$ and $\tau \varepsilon [\gamma, \mathcal{H} \vee \gamma]$. Then the following hold

(I) $\lambda \vee \tau \in [\gamma, \mathcal{H} \vee \gamma]$ and $A_{\lambda \vee \tau} = \text{rad}\,(A_\lambda A_\tau)$

(II) $\lambda \cap \tau \in [\iota, \mathcal{H}]$ and $A_{\lambda \cap \tau} = A_\lambda \cap A_\tau$

Proof It is clear that

$$\gamma \subseteq \tau \subseteq \lambda \vee \tau \subseteq \lambda \vee (\mathcal{H} \vee \gamma) \subseteq \mathcal{H} \vee \gamma$$

and hence we have that $\lambda \vee \tau \in [\gamma, \mathcal{H} \vee \gamma]$. We know by the proposition 4.2 that rad $(A_\lambda A_\tau)$ is an α-admissible normal subgroup of the group G. Also since rad rad $(A_\lambda A_\tau)$ = rad $(A_\lambda A_\tau)$ (by proposition 4.2) it follows that rad $(A_\lambda A_\tau) \in \mathcal{A}^*$. Now by the proposition 5.7 it follows that there exists a congruence ξ on the semigroup S such that $\xi \in [\gamma, \gamma \vee \mathcal{H}]$ and that A_ξ = rad $(A_\lambda A_\tau)$. Now let us put $x = (m, g, n)$ and $y = (p, h, q)$ and suppose that $(x, y) \in \lambda$. It follows by the proposition 3.1, that $m = p$, $n = q$ and $gh^{-1} \in A_\lambda$. Obviously $gh^{-1} \in A_\lambda \subseteq A_\lambda A_\tau \subseteq \text{rad}\,(A_\lambda A_\tau) = A_\xi$, since A_λ and A_τ are normal subgroups in G. Further we observe that $xy^{-1} = (m, gh^{-1}, m)$. It follows by the part (I) of the proposition 5.3 that $xy^{-1} \in e_0\,\xi$ and hence by the proposition 4.3, we have that $(x, y) \in \xi$. Thus $\lambda \subseteq \xi$. Also it is clear that $A_\tau \subseteq A_\lambda A_\tau \subseteq A_\xi$ and hence by the proposition 5.5 (II) that we have that $\tau \subseteq \xi$. Combining these we get that $\lambda \vee \tau \subseteq \xi$ and hence by proposition 2.3, that $A_{\lambda \vee \tau} \subseteq A_\xi$. Obviously $\lambda \subseteq \lambda \vee \tau$ and $\tau \subseteq \lambda \vee \tau$. It follows by proposition 2.3 that $A_\lambda \subseteq A_{\lambda \vee \tau}$ and $A_\tau \subseteq A_{\lambda \vee \tau}$ and hence we have that $A_\lambda A_\tau \subseteq A_{\lambda \vee \tau}$, since A_λ and A_τ are normal subgroups. It follows by the proposition 4.2 that rad $(A_\lambda A_\tau) \subseteq \text{rad}\,(A_{\lambda \vee \tau}) = A_{\lambda \vee \tau}$. Thus we have established that $A_\xi = \text{rad}\,(A_\lambda A_\tau) \subseteq A_{\lambda \vee \tau} \subseteq A_\xi$ and hence $A_{\lambda \vee \tau} = \text{rad}\,(A_\lambda A_\tau)$.

(II) Obviously $\lambda \cap \tau \leq \lambda \quad \varepsilon \quad [\iota, \mathcal{H}]$. Next let $x = (0, g, 0) \in S$. Then we observe that $g \in A_{\lambda \cap \tau}$ if and only if $(x, e_0) \in \lambda$ and $(x, e_0) \in \tau$ that is, if and only if $g \in A_\lambda \cap A_\tau$. Hence we conclude that $A_{\lambda \cap \tau} = A_\lambda \cap A_\tau$

<u>Proved</u>

<u>COROLLARY 5.9</u>

$$A_{\lambda \vee \gamma} = \text{rad } A_\lambda \quad \underline{\text{for any}} \ \lambda \in [\iota, \mathcal{H}]$$

<u>Proof</u> It is shown in the foregoing proposition that $A_{\lambda \vee \gamma} = \text{rad } (A_\lambda A_\gamma)$. But $A_\gamma = \text{rad } I$ (Proposition 5.1) and further by the proposition 4.2, $\text{rad } (A_\lambda \text{ rad } I) = \text{rad } A_\lambda$. Hence we conclude that $A_{\lambda \vee \gamma} = \text{rad } A_\lambda$.

<u>Proved</u>

§6 We recall from the proposition 2.1 that Λ, the lattice of congruences on the bisimple ω-semigroup $S = S(G, \alpha)$ can be expressed as the disjoint union of Λ_{IS}, the sublattice of idempotent-separating congruences on S and Λ_G, the sublattice of group-congruences on S.

It is clear from the corollary that Λ_{IS} is modular. Next let Δ denote the lattice of congruences on the group $S/_\gamma$, where γ is the minimum group congruence on the semigroup S. Suppose ρ is a group congruence on S i.e. $\rho \geq \gamma$. Then we know from the proposition I.1.7 that there exists a congruence $\rho/_{\gamma}$ on the group $S/_{\gamma}$ defined by the rule : $(x, y) \in \rho$ iff $(x\gamma, y\gamma) \in \rho/_{\gamma}$. Let us define a mapping $\phi : \rho \longrightarrow \rho/_{\gamma}$ from the lattice Λ_G to the lattice Δ. We also know that for any congruence σ on the group $S/_{\gamma}$, the relation ρ defined by the rule : $(x, y) \in \rho$ iff $(x\gamma, y\gamma) \in \sigma$ for every $x, y \in S$ is a congruence on S such that $\rho \supseteq \gamma$, i.e. $\rho \in \Lambda_G$. It follows that ϕ is a one-one onto mapping. Further let $\rho, \rho' \in \Lambda_G$ such that $\rho \subseteq \rho'$. Then we see that $(x\gamma, y\gamma) \in \rho/_{\gamma}$ implies $(x, y) \in \rho$ and hence $(x, y) \in \rho'$ which gives $(x\gamma, y\gamma) \in \rho'/_{\gamma}$ and so $\rho\phi \subseteq \rho'\phi$. Similarly it can be verified that ϕ^{-1} is order-preserving. Then it follows that ϕ is a lattice-isomorphism. Now since by the corollary II.1.9, Δ is modular, it follows that Λ_G is a modular sublattice of Λ.

<u>LEMMA 6.1</u> <u>On the semigroup</u> $S = S(G,\alpha)$, <u>the following hold</u>.

$$\mathcal{H} \subseteq \gamma \iff G = \bigcup_{k=1}^{\infty} \ker \alpha^k$$

<u>where the symbols have their usual meanings</u>.

<u>Proof</u> First we assume that $\mathcal{H} \subseteq \gamma$, and let us put $K = \bigcup_{k=1}^{\infty} \ker \alpha^k$. Obviously $K \subseteq G$. Now let $g \in G$. Then we have that

$$((0, g, 0)\ e_0) \in \mathcal{H} \subseteq \gamma$$

and hence $g \in A_{\gamma}$. It follows by the proposition 5.1, that $g \in A_{\gamma} = K$. Thus we have that $G = K$.

Conversely suppose that $G = K$ holds and let $(x, y) \in \mathcal{H}$, where $x = (m, g, n)$ and $y = (m, h, n)$. Obviously $gh^{-1} \in G = K$ and hence there exists an integer k such that $(gh^{-1})\alpha^k = I$. It follows that $(g\alpha^k)(h^{-1}\alpha^k) = (g\alpha^k)(h\alpha^k)^{-1} = I$ and hence we have that $g\alpha^k = h\alpha^k$. Now we observe that

$e_{m+k}x = (m + k, g\alpha^k, n + k) = (m + k, h\alpha^k, n + k) = e_{m+k}y$

It follows by the proposition II.5.8, that $(x, y) \varepsilon \gamma$. Hence $\mathfrak{H} \subseteq \gamma$.

Proved

The next theorem provides necessary and sufficient conditions for the lattice of congruences on the semigroup $S = S(G,\alpha)$ to be modular.

THEOREM 6.2 (MUNN) On the semigroup $S = S(G,\alpha)$, the following conditions are equivalent.

(I) rad A = A rad I for every $A \varepsilon \mathfrak{A}$

(II) $[\iota, \mathfrak{H} \vee \gamma]$ forms a modular sublattice of Λ, the lattice of congruences on S.

(III) Λ, the lattice of congruences on the semigroup S, is modular.

Proof [a] First we show that condition (I) implies the condition (II). We assume that rad A = A for every α-admissible normal subgroup A of the group G, and we shall show that

(6) $[\lambda, \subseteq \lambda', \lambda \vee \tau = \lambda' \vee \tau, \lambda \cap \tau = \lambda' \cap \tau] \Rightarrow \lambda = \lambda'$

where λ, λ', τ are arbitrary congruences in the sublattice $[\iota, \mathfrak{H} \vee \gamma]$.

From the proposition 2.1, it is clear that

$$[\iota, \mathfrak{H} \vee \gamma] = [\iota, \mathfrak{H}] \cup [\gamma, \mathfrak{H} \vee \gamma]$$

where $[\iota, \mathfrak{H}] \cap [\gamma, \mathfrak{H} \vee \gamma] = \phi$. Since the sublattice of idempotent-separating congruences on S, and the sublattice of the group congruences on S are shown to be modular, it follows that when $\lambda, \lambda', \tau \in [\iota, \mathfrak{H}]$ or when $\lambda, \lambda', \tau \in [\gamma, \mathfrak{H} \vee \gamma]$, then (6) certainly holds. Next we consider the case when $\lambda, \tau \in [\iota, \mathfrak{H}]$ and $\lambda' \in [\gamma, \mathfrak{H} \vee \gamma]$. Then we note that $\mathfrak{H} \supseteq \lambda \vee \tau = \lambda' \vee \tau \supseteq \gamma$ i.e. $\mathfrak{H} \supseteq \gamma$, which is impossible.

Similarly in the case when $\lambda \in [\iota, \mathfrak{H}]$ and $\lambda', \tau \in [\gamma, \mathfrak{H} \vee \gamma]$ we deduce that $\mathfrak{H} \supseteq \lambda \cap \tau = \lambda' \cap \tau \supseteq \gamma$ i.e. $\mathfrak{H} \supseteq \gamma$. Now we are left with only the two following possibilities.

<u>CASE I</u>. Let $\lambda, \lambda' \in [\iota, \mathfrak{H}]$ and $\tau \in [\gamma, \mathfrak{H} \vee \gamma]$.

It is clear by the hypothesis that $\text{rad}(A_\tau A_\lambda) = A_\tau A_\lambda \text{ rad } I$. Since τ is a group congruence, it follows by the proposition 4.3 (II) that $A_\tau = \text{rad } A_\tau \supseteq \text{rad } I$ (by Proposition 4.2 (III)). It follows that $A_\lambda A_\tau \text{ rad } I \subseteq A_\lambda A_\tau$. Hence we have that

$$A_\lambda A_\tau \subseteq \text{rad}(A_\lambda A_\tau) = A_\lambda A_\tau \text{ rad } I \subseteq A_\lambda A_\tau$$

Thus we have that $A_\lambda A_\tau = \text{rad}(A_\lambda A_\tau)$. It follows by the proposition 5.8 that $A_\lambda A_\tau = A_{\lambda \vee \tau}$. Similarly we can obtain that $A_{\lambda'} A_\tau = A_{\lambda' \vee \tau}$ Since $\lambda \vee \tau = \lambda' \vee \tau$, it follows by Proposition 2.3, that $A_{\lambda \vee \tau} = A_{\lambda' \vee \tau}$ and hence we have that $A_\lambda A_\tau = A_{\lambda'} A_\tau$. Further since $\lambda \cap \tau = \lambda' \cap \tau$, it follows that $A_{\lambda \cap \tau} = A_{\lambda' \cap \tau}$. Now it follows by the proposition 5.8, that $A_\lambda \cap A_\tau = A_{\lambda'} \cap A_\tau$. Also since $\lambda \subseteq \lambda'$, it is clear that $A_\lambda \subseteq A_{\lambda'}$. Now from the modularity of the lattice of normal subgroups on the group G, it follows that $A_\lambda = A_{\lambda'}$. It is immediate by the corollary 3.2 that $\lambda = \lambda'$.

<u>CASE II</u>. Let $\lambda, \lambda' \in [\gamma, \mathfrak{H} \vee \gamma]$ and $\tau \in [\iota, \mathfrak{H}]$. Since λ is a group congruence, we have by Proposition 4.3 that

$A_\lambda = \text{rad } A_\lambda \supseteq \text{rad } I$. Hence we have that

$$A_\tau A_\lambda \subseteq \text{rad } (A_\tau A_\lambda) = A_\tau A_\lambda \text{ rad } I \subseteq A_\tau A_\lambda ,$$

which gives us $A_\tau A_\lambda = \text{rad } (A_\tau A_\lambda)$. Now by the proposition 5.8, we have that $A_{\tau \vee \lambda} = \text{rad } (A_\tau A_\lambda)$; hence we have that $A_\tau A_\lambda = A_{\tau \vee \lambda}$. Similarly we can deduce that $A_\tau A_{\lambda'} = A_{\tau \vee \lambda'}$. But $\lambda \vee \tau = \lambda' \vee \tau$, it follows by the proposition 2.3 that $A_\lambda A_\tau = A_{\lambda'} A_\tau$. By the same argument employed in Case I, we can deduce that $A_\lambda \cap A_\tau = A_{\lambda'} \cap A_\tau$ and $A_\lambda \subseteq A_{\lambda'}$. Now by the modularity of the lattice of normal subgroups in the group G, it follows that $A_\lambda = A_{\lambda'}$. Now it is immediate from the proposition 5.5 that $\lambda = \lambda'$. Thus we conclude that (6) holds and hence the sublattice $[\iota, \mathfrak{N} \vee \lambda]$ is modular.

[b] Now we proceed to show that condition (II) implies condition (III). We assume that the sublattice $[\iota, \mathfrak{N} \vee \gamma]$ is modular and we will show that for arbitrary congruences λ, λ', τ in Λ, the lattice of congruences on S

$$[\lambda \subseteq \lambda', \ \lambda \vee \tau = \lambda' \vee \tau, \ \lambda \cap \tau = \lambda' \cap \tau] \qquad \lambda = \lambda'.$$

In view of the argument employed in part [a], it is clearly sufficient to consider only the following two cases.

<u>CASE I</u>. Let $\lambda \subseteq \lambda' \subseteq \mathfrak{N}$ and $\gamma \subseteq \tau$

Now we deduce that

$$(\lambda \vee \tau) \cap (\mathfrak{N} \vee \gamma) = (\lambda \vee (\gamma \vee \tau)) \cap (\mathfrak{N} \vee \gamma) \quad \text{since } \gamma \subseteq \tau$$

$$= ((\lambda \vee \gamma) \vee \tau) \cap (\mathfrak{N} \vee \gamma)$$

$$= (\lambda \vee \gamma) \vee (\tau \cap (\mathfrak{N} \vee \gamma)); \quad \text{since } \Lambda_G \text{ is modular}$$

$$= \lambda \vee (\tau \cap (\mathfrak{N} \vee \gamma)) \qquad \gamma \subseteq \tau \cap (\mathfrak{N} \vee \gamma)$$

Similarly we can deduce that

$$(\lambda' \vee \tau) \cap (\mathcal{H} \vee \gamma) = \lambda' \vee (\tau \cap (\mathcal{H} \vee \gamma))$$

Now since $\lambda \vee \tau = \lambda' \vee \tau$, it follows that

$$(\lambda \vee \tau) \cap (\mathcal{H} \vee \gamma) = (\lambda' \vee \tau) \cap (\mathcal{H} \vee \gamma).$$

Hence we have that

$$\lambda \vee (\tau \cap (\mathcal{H} \vee \gamma)) = \lambda' \vee (\tau \cap (\mathcal{H} \vee \gamma))$$

Further since $\lambda \cap \tau = \lambda' \cap \tau$, it follows that

$$\lambda \cap (\tau \cap (\mathcal{H} \vee \gamma)) = \lambda' \cap (\tau \cap (\mathcal{H} \vee \gamma)).$$

Clearly λ, λ' and $(\tau \cap (\mathcal{H} \vee \gamma))$ belong to $[\iota, \mathcal{H} \vee \gamma]$, hence by the modularity of $[\iota, \mathcal{H} \vee \gamma]$, it follows that $\lambda = \lambda'$

<u>CASE II</u>. Let $\tau \subseteq \mathcal{H}$ and $\gamma \subseteq \lambda \subseteq \lambda'$. Now we deduce that

$(\lambda \cap \tau) \vee \gamma = (\lambda \cap ((\mathcal{H} \vee \gamma) \cap \tau)) \vee \gamma$ since $\tau \subseteq \mathcal{H} \subseteq \mathcal{H} \vee \gamma$

$= ((\lambda \cap (\mathcal{H} \vee \gamma)) \cap \tau) \vee \gamma$

$= (\lambda \cap (\mathcal{H} \vee \gamma)) \cap (\tau \vee \gamma)$ since $[\iota, \mathcal{H} \vee \gamma]$ is modular

$= \lambda \cap ((\mathcal{H} \vee \gamma) \cap (\tau \vee \gamma))$

$= \lambda \cap (\tau \vee \gamma)$ since $\tau \subseteq \mathcal{H}$

Similarly we can show that $(\lambda' \cap \tau) \vee \gamma = \lambda' \cap (\tau \vee \gamma)$. But $\lambda \cap \tau = \lambda' \cap \tau$ and hence $(\lambda' \cap \tau) \vee \gamma = (\lambda \cap \tau) \vee \gamma$

It follows that $\lambda \cap (\tau \vee \gamma) = \lambda' \cap (\tau \vee \gamma)$. Also since $\lambda \vee \tau = \lambda' \vee \tau$, it is clear that $\lambda \vee (\tau \vee \gamma) = \lambda' \vee (\tau \vee \gamma)$.

Clearly λ, λ', and $\tau \vee \gamma$ are group congruences and since Λ_G is modular, it follows that $\lambda = \lambda'$. Thus we have established that (6) holds for arbitrary congruences in the lattice of congruences on S, i.e. Λ is modular.

[c] Finally we proceed to show that condition (III) implies condition (I). We assume that Λ is modular and suppose that A is an α-admissible normal subgroup in the group G. Let us put B = rad A and C = A rad I. It is clear from the proposition 4.2 that B and C are α-admissible normal subgroups of G such that rad B = B and rad C = C. Now it follows by the proposition 3.3 that there exist two congruences λ, λ' in $[\iota, \mathcal{H}]$ such that $B = A_{\lambda'}$ and $C = A_{\lambda}$. It is clear from the proposition 4.2 that $C \subseteq B$ and hence $A_{\lambda'} \supseteq A_{\lambda}$. Now from the corollary 3.2, we infer that $\lambda \leq \lambda'$. Now we deduce that

$A_{\lambda \vee \gamma}$

$=$ rad A_{λ} by Corollary 5.9

$=$ rad (A rad I)

$=$ rad A by Proposition 4.2

$=$ rad (rad A)

$=$ rad B $=$ rad $A_{\lambda'}$

$= A_{\lambda' \vee \gamma}$ by Corollary 5.9

It follows by the proposition 5.5, that $\lambda \vee \gamma = \lambda' \vee \gamma$.

We further deduce that

$A_{\lambda \cap \gamma} = A_{\lambda} \cap A_{\gamma}$ by Proposition 5.8

$=$ (A rad I) $\cap$ (rad I)

$=$ rad I

$=$ (rad A) $\cap$ (rad I)

$= A_{\lambda'} \cap A_{\gamma} = A_{\lambda' \cap \gamma}$ by Proposition 5.8

It follows by the corollary 3.2 that $\lambda \cap \gamma = \lambda' \cap \gamma$. Now since Λ_G is modular, we conclude that $\lambda = \lambda'$ and hence by the proposition 2.3, we have that $A_{\lambda} = A_{\lambda'}$. Thus rad A $=$ A rad I.

Proved

COROLLARY 6.3 *If rad I = G, then Λ is modular.*

Proof Clearly then

$$\text{rad } A = \text{rad } (A \text{ rad } I)$$
$$= \text{rad } (A.\ G) = \text{rad } (G) = G$$
$$= A.\ G = A \text{ rad } I$$

for every $A \in \mathcal{A}$, and hence Λ is modular by the foregoing theorem.

Proved

COROLLARY 6.4 *If G has no proper α-admissible normal subgroup, then Λ is modular; in particular this holds if G is simple.*

Proof Obviously rad $A = A$ rad I holds for $A = G$ and $A = \{I\}$

Proved

COROLLARY 6.5 *If α is an inner automorphism of the group G, then Λ is modular.*

Proof Since α is one-one onto, it is clear that rad $I = \bigcup_{k=1}^{\infty} \ker \alpha^k = I$

Further since A is a normal subgroup, it follows immediately from $(h^{-1})^n\, g\, (h)^n \in A$ for some $n \in N$ and some $h \in G$, that $g \in A$, and hence rad $A = A$. Thus we have rad $A = A$ rad I and hence Λ is modular.

Proved

COROLLARY 6.6 *If* $\alpha^k = \alpha^{k+l}$ *for some* $k \geq 0$ *and* $l > 0$ *then* Λ *is modular.*

Proof Let A be an α-admissible normal subgroup of the group G and let $g \in$ rad A. It follows that $g\alpha^n \in A$. Let us choose an integer m such that $lm \geq n$. Now we have that $h = g\alpha^{lm} \in A$ since A is

α -admissible. But we observe that

$(h^{-1}g)\alpha^{k} = (h^{-1}\alpha^{k})(g\alpha^{k})$

$= (g^{-1}\alpha^{lm+k})(g\alpha^{k}) = I$

since $g\alpha^{lm+k} = g\alpha^{k}$.

Hence we have that $h^{-1}g \in$ rad I. It follows that $g = h(h^{-1}g) \in$ A rad I.

Thus rad A   A rad I. But by the proposition 4.2, we have that

rad A $\supseteq$ A rad I. Thus rad A = A rad I and hence Λ is modular.

<u>Proved</u>

In particular we note that if G is finite then $\alpha^{k} = \alpha^{k+l}$ for some $k \geq 0$ and $l > 0$ and hence Λ is modular.

CHAPTER IV

§1 In this chapter we propose to study the form of the congruences on completely 0-simple semigroups.

<u>DEFINITION 1.1</u> A semigroup S is said to be <u>0-simple</u> if

(I) S has a zero 0

(II) $S^2 \neq \{0\}$

(III) the only ideals of S are S and $\{0\}$

<u>DEFINITION 1.2</u> Let S be a semigroup with zero 0. A [left, right] ideal M of S is said to be <u>0-minimal</u> in S if and only if

(I) $M \neq \{0\}$

(II) the only [left, right] ideals of S contained in M are M and $\{0\}$.

<u>DEFINITION 1.3</u> A completely 0-simple semigroup is a 0-simple semigroup which contains a 0-minimal left ideal and a 0-minimal right ideal.

In order to describe the structure of a completely 0-simple semigroup, we need the concept of a "Rees matrix semigroup over a group with zero". Let G be a group and let $G^0 = G \cup \{0\}$, the semigroup obtained from G by adjoining a zero 0. Let I and Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a fixed $\Lambda \times I$ matrix over G^0. Let us consider the set $G^0 \times I \times \Lambda$ of triplets (a, i, λ) where $a \in G^0$, $i \in I$, $\lambda \in \Lambda$ and define the multiplication by the rule

$$(a, i, \lambda) \quad (b, j, \mu) = (ap_{\lambda j} b, i, \mu)$$

The associativity can be easily checked. Now the set $0 \times I \times \Lambda$ is an ideal and we take the Rees factor semigroup of $G^0 \times I \times \Lambda$ modulo $0 \times I \times \Lambda$. We call the factor semigroup the Rees $I \times \Lambda$ matrix semigroup over G^0 and denote it by $\mathcal{M}^0(G, I, \Lambda, P)$. The semigroup $\mathcal{M}^0(G, I, \Lambda, P)$ is regular if and only if there exists a non-zero element of G^0 in each row and column of P. For this reason we shall say that the matrix P is regular if and only if each row and column of P contains a non-zero element of G^0. Rees has proved the following structure theorem for completely 0-simple semigroups (Theorem 3.5 [I]).

THEOREM 1.4 A semigroup S is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero.

We omit the proof.

§2 Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be an arbitrary completely 0-simple semigroup.

We define a relation $\mathcal{E}_I$ on the set I by the rule

$$(i, j) \in \mathcal{E}_I \quad \text{iff} \quad [p_{\lambda i} = 0 \quad \text{iff} \quad p_{\lambda j} = 0] \quad \text{where } \lambda \in \Lambda.$$

Obviously $\mathcal{E}_I$ is an equivalence on the set I. Similarly the relation $\mathcal{E}_\Lambda$ on the set Λ defined by the rule

$$(\lambda, \mu) \in \mathcal{E}_\Lambda \quad \text{iff} \quad [p_{\lambda i} = 0 \quad \text{iff} \quad p_{\mu i} = 0] \quad \text{where } i \in I$$

is an equivalence on the set Λ. We shall denote the quotient set $I/\mathcal{E}_I$ by I^* and the quotient set $\Lambda/\mathcal{E}_\Lambda$ by Λ^*. Clearly $I^* \times \Lambda^*$

is a partition of the set $I \times \Lambda$. Let $(i^*, \lambda^*) \in I^* \times \Lambda^*$. Thus it easily follows that either $p_{\lambda i} = 0$ for every $(i, \lambda) \in (i^*, \lambda^*)$ or that $p_{\lambda i} \neq 0$ for every $(i, \lambda) \in (i^*, \lambda^*)$. In the latter case we shall say that the equivalence class (i^*, λ^*) is <u>non-zero</u>. From the regularity of the matrix P it is easily deduced that for each $\lambda^* \in \Lambda^*$, there exists an $i^* \in I^*$ such that (i^*, λ^*) is non-zero, and vice-versa.

<u>DEFINITION 2.1</u> The sandwich matrix P is said to be <u>normal</u> if

(I) For every $\lambda^* \in \Lambda^*$, there exists an $i \in I$, such that $p_{\lambda i} = e$ for every $\lambda \in \lambda^*$.

(II) For every $i^* \in I^*$, there exists a $\lambda \in \Lambda$, such that $p_{\lambda i} = e$ for every $i \in i^*$.

<u>THEOREM 2.2</u> <u>Every completely 0-simple semigroup S is isomorphic to a regular Rees matrix semigroup whose sandwich matrix is normal.</u>

<u>Proof</u> Let S be a given completely 0-simple semigroup. It follows by Rees theorem 1.4 that S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^{\circ}(G, I, \Lambda, P)$ (say). Suppose that i^* and λ^* are equivalence classes of I and Λ respectively as described in the previous paragraph. By the axiom of choice we can choose a set of representatives one from each equivalence class i^* in I^* and one from each equivalence class λ^* in Λ^*. We shall denote the chosen representative element of i^* by i^0 and the representative element of λ^* by λ^0.

For each $\lambda^* \in \Lambda^*$, let us define $\alpha(\lambda^*) = i^0$, where i^* is such that $(i^*, \lambda^*) \neq 0$. For each $i^* \varepsilon I^*$, we define $\beta(i^*) = \lambda^0$, where λ^* is such that $\alpha(\lambda^*) = i^0$ if such a λ^* exists; otherwise we take $\beta(i^*) = \mu^0$ where μ^* is such that $(i^*, \mu^*) \neq 0$.

Next we define

$$v_\lambda = p^{-1}_{\lambda, \alpha(\lambda^*)} \qquad \text{for every } \lambda \in \lambda^*$$

and

$$u_i = \begin{cases} e & \text{if } i^0 = \alpha(\lambda^*) \text{ for some } \lambda \\ p^{-1}_{\beta(i^*), i} \; p_{\beta(i^*), \alpha(\beta(i^*)^*)} & \text{if } i^0 \neq \alpha(\lambda^*) \text{ for any } \lambda \end{cases}$$

and $$p'_{\lambda i} = v_\lambda p_{\lambda i} u_i$$

Now we make use of the following lemma (Lemma 3.6 [I]) whose proof we omit.

<u>LEMMA 2.3</u> <u>Two Rees $I \times \Lambda$ matrix semigroups</u> $\mathcal{M}^0(G, I, \Lambda, P)$ <u>and</u> $\mathcal{M}^0(G, I, \Lambda, P')$ <u>over a group with zero G^0, are isomorphic if there exists a mapping</u> $i \longrightarrow u_i$ <u>of I into G and a mapping</u> $\lambda \longrightarrow v_\lambda$ <u>of Λ into G such that</u> $p'_{\lambda i} = v_\lambda p_{\lambda i} u_i$ <u>for all</u> $i \varepsilon I$ <u>and</u> $\lambda \varepsilon \Lambda$, <u>where</u> $P = (p_{\lambda i})$ <u>and</u> $P' = (p'_{\lambda i})$.

It follows from the lemma 2.3 that

$S \cong \mathcal{M}^0(G, I, \Lambda, P')$ where $P' = (p'_{\lambda i})$. Now it remains to show that the sandwich matrix P' is normal.

Let $\lambda^* \varepsilon \Lambda^*$ and let λ be an arbitrary element of λ^*. Then we observe that

$$p'_{\lambda, \alpha(\lambda^*)} = p^{-1}_{\lambda, \alpha(\lambda^*)} \; p_{\lambda, \alpha(\lambda^*)} \quad e = e$$

Next let $i^* \varepsilon I^*$ and let i be an arbitrary element of i^*. Now if $i^0 \neq \alpha(\lambda^*)$ for any λ, then we see that

$$p'_{\beta(i^*), i} = p_{\beta(i^*), \alpha(\beta(i^*)^*)} \; p_{\beta(i^*), i} \; p^{-1}_{\beta(i^*), i} \; p_{\beta(i^*), \alpha(\beta(i^*)}$$

$$= e$$

On the other hand if $i^0 = \alpha(\lambda^*)$ for some λ^*, then we note that $\beta(i^*) = \lambda^0$ and hence

$$p'_{\beta(i^*),\, i} = p'_{\lambda^0, i} = p'_{\lambda^0,\, \alpha(\lambda^*)} = e$$

It follows that the sandwich matrix $\mathcal{P}$ is normal.

Proved

NOTE 2.4

The foregoing theorem enables us to represent an arbitrary completely 0-simple semigroup as a Rees matrix semigroup over a group with zero whose sandwich matrix is normal.

We shall say a congruence ρ on a semigroup S is non-trivial provided $\rho \neq \iota_S$, the identity relation and $\rho \neq S \times S$, the universal relation.

PROPOSITION 2.5 Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be a completely 0-simple semigroup where P is normal and let ρ be a non-trivial congruence on S. If $((a, i, \lambda), (b, j, \mu)) \in \rho$ then $(i, j) \in \mathcal{E}_I$ and $(\lambda, \mu) \in \mathcal{E}_\Lambda$ and hence there exist $k \in I$ and $\nu \in \Lambda$ such that $p_{\nu i} = p_{\nu j} = e$ and $p_{\lambda k} = p_{\mu k} = e$.

Proof First we show that $(i, j) \in \mathcal{E}_I$. Let us assume on the contrary that there exists $\beta \in \Lambda$ such that $p_{\beta i} \neq 0$ and $p_{\beta j} = 0$. Let (c, l, η) be an arbitrary non-zero element of the semigroup S. Now there exists an $m \in I$ such that $p_{\lambda m} \neq 0$, since P is regular. Now we see that

$$((a^{-1} p_{\beta i}^{-1}, l, \beta)(a, i, \lambda)(p_{\lambda m}^{-1}, m, \eta) = (c, l, \eta) \text{ and}$$

that $((a^{-1}p_{\beta i}^{-1}, 1, \beta)(b, j, \mu)(p_{\lambda m}^{-1}, m, \eta) = 0$

It follows by the compatibility of ρ that $((i, 1, \eta), 0) \in \rho$. Hence transitivity of ρ implies that $\rho = S \times S$ which is a contradiction. Hence $(i, j) \in \mathcal{E}_I$. Similarly $(\lambda, \mu) \in \mathcal{E}_\Lambda$. The remaining part of the proposition is immediate since P is normal.

Proved

Next we define the following relations :

(I) ρ_G on group G by the rule that

$(x, y) \in \rho_G$ if and only if there exist $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $((x, i, \lambda), (y, j, \mu)) \in \rho$;

(II) ρ_I on the set I by the rule that

$(x, y) \in \rho_I$ if and only if there exist $x, y \in G$ and $\lambda, \mu \in \Lambda$ such that $((x, i, \lambda), (y, j, \mu)) \in \rho$;

(III) ρ_Λ on the set Λ by the rule that

$(\lambda, \mu) \in \rho_\Lambda$ if and only if there exist $x, y \in G$ and $i, j \in I$ such that $((x, i, \lambda), (y, j, \mu)) \in \rho$.

PROPOSITION 2.6 Let ρ be a non-trivial congruence on a completely 0-simple semigroup $S = \mathcal{M}^0(G, I, \Lambda, P)$ where the matrix P is normal. Then the relation ρ_G is a congruence on G and ρ_I and ρ_Λ are equivalences on the sets I and Λ respectively.

Proof Since ρ is a congruence on the semigroup S, the reflexivity and symmetry of all the relations is obvious by (I), (II) and (III). First we verify the transitivity of ρ_G. Let $(x, y) \in \rho_G$ and $(y, z) \in \rho_G$. It follows that there exist i, j, k, l in I and λ, μ, ν, π in Λ

such that

$((x, i, \lambda), (y, j, \mu)) \in \rho$ and $((y, k, \nu), (z, l, \pi)) \in \rho$.

It follows by the proposition 2.5 that there exist $m \in I$ and $\varphi \in \Lambda$ such that $p_{\lambda m} = p_{\mu m} = e = p_{\varphi i} = p_{\varphi j}$. Next we observe that

$$(e, k, \varphi)(x, l, \lambda)(e, m, \nu) = (x, k, \nu) \quad \text{and that}$$

$$(e, k, \varphi)(y, j, \mu)(e, m, \nu) = (y, k, \nu)$$

It follows by the compatibility of ρ that $((x, k, \nu), (y, k, \nu)) \in \rho$, and hence transitivity of ρ implies that $((x, k, \nu), (z, l, \pi)) \in \rho$. Thus $(x, z) \in \rho_G$. Hence ρ_G is an equivalence.

Next let $(x, y) \in \rho_G$. It follows that there exist $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $((x, i, \lambda), (y, j, \mu)) \in \rho$. Further by the proposition 2.5, there exist $\xi \in \Lambda$ such that $p_{\xi i} = p_{\xi j} = e$. Let $z \in G$ and $k \in I$. Then we note that

$$(z, k, \xi)(x, i, \lambda) = (zx, k, \lambda) \text{ and } (z, k, \xi)(y, j, \mu) = (zy, k, \mu)$$

Hence by the compatibility of ρ we have that $((zx, k, \lambda), (zy, k, \mu)) \in \rho$. Hence $(zx, zy) \in \rho_G$. Similarly we can show that $(xz, yz) \in \rho_G$. Thus ρ_G is a congruence on the group G.

Next we prove the transitivity of ρ_I. Let $(i, j) \in \rho_I$ and $(j, k) \in \rho_I$. It follows that there exist elements x, y, z, t in G and λ, μ, ν, π in Λ such that $((x, i, \lambda), (y, j, \mu)) \in \rho$ and $((z, j, \nu), (t, k, \pi)) \in \rho$. By proposition 2.5 there exists an $m \in I$ such that $p_{\lambda m} = p_{\mu m} = e$. Now since ρ is compatible, it follows that

$$((x, i, \lambda)(y^{-1}z, m, \nu), (y, j, \mu)(y^{-1}z, m, \nu)) \in \rho$$

i.e. $((xy^{-1}z, i, \nu), (z, j, \nu)) \in \rho$.

It follows that $((xy^{-1}z, i, \nu), (t, k, \pi)) \in \rho$, and hence $(i, k) \in \rho_I$.

Thus ρ_I is an equivalence. Similarly we can show that ρ_Λ is an equivalence.

Proved

NOTE 2.7

It is clear by the proposition 2.5, that $\rho_I \subseteq \mathcal{E}_I$ and $\rho_\Lambda \subseteq \mathcal{E}_\Lambda$ for any non-trivial congruence ρ on the semigroup S.

Let N, a normal subgroup of the group G, be the ρ_G-class containing the identity e of G. Thus $(x, y) \in \rho_G$ if and only if $xy^{-1} \in N$

DEFINITION 2.8 A triplet $\{N, \rho_I, \rho_\Lambda\}$ consisting of a normal subgroup N of the group G, an equivalence ρ_I on the set I such that $\rho_I \subseteq \mathcal{E}_I$ and an equivalence ρ_Λ on the set Λ such that $\rho_\Lambda \subseteq \mathcal{E}_\Lambda$ is called a linked triplet provided the following hold:

(i) $[(i, j) \in \rho_I$ and $p_{\lambda i} \neq 0] \Rightarrow p_{\lambda i}\, p_{\lambda j}^{-1} \in N$

(ii) $[(\lambda, \mu) \in \rho_\Lambda$ and $p_{\lambda i} \neq 0] \Rightarrow p_{\lambda i}\, p_{\mu i}^{-1} \in N$

THEOREM 2.9 Let $S = \mathcal{M}^0(G, I, \Lambda, P)$ be a completely 0-simple semigroup, where the sandwich matrix P is normal. Then every non-trivial congruence ρ on S determines a linked triplet $\{N, \rho_I, \rho_\Lambda\}$ in the manner described above. Conversely if $\{N, \rho_I, \rho_\Lambda\}$ is a linked triplet, then the relation ρ defined by the rule

$((x, i, \lambda), (y, j, \mu)) \in \rho$ iff $xy^{-1} \in N$, $(i, j) \in \rho_I$ and $(\lambda, \mu) \in \rho_\Lambda$ is a congruence on S.

<u>Proof</u> Suppose ρ is a non-trivial congruence on the semigroup S. We have already seen that N, being the ρ_G-class containing the identity e of G is a normal subgroup of G, and ρ_I and ρ_Λ are equivalences on the sets I and Λ respectively. Also $\rho_I \subseteq \mathcal{E}_I$ and $\rho_\Lambda \subseteq \mathcal{E}_\Lambda$ by the note 2.7.

Next suppose that $(i, j) \in \rho_I$ and $\lambda \in \Lambda$ be such that $p_{\lambda i} \neq 0$. Then $p_{\lambda j} \neq 0$ since $\rho_I \subseteq \mathcal{E}_I$. Also there exist $x, y \in G$ and $\mu, \nu \in \Lambda$ such that $((x, i, \mu), (y, j, \nu)) \in \rho$. It follows that $(x, y) \in \rho_G$ and so $yx^{-1} \in N$. Since $(\mu, \nu) \in \rho_\Lambda \subseteq \mathcal{E}_\Lambda$ and P is normal, there exists $k \in I$, such that $p_{\mu k} = p_{\nu k} = e$.

By compatibility of ρ, we have that

$((e, i, \lambda)(x, i, \mu)(x^{-1}, k, \mu), (e, i, \lambda)(y, j, \nu)(x^{-1}, k, \mu)) \in \rho$

that is $((p_{\lambda i}, i, \mu), (p_{\lambda j}\, yx^{-1}, i, \mu)) \in \rho$

Hence $p_{\lambda i}^{-1} (p_{\lambda j}\, yz^{-1}) \in N$. But $yx^{-1} \in N$ and so $p_{\lambda i}^{-1} p_{\lambda j} \in N$.

Thus $p_{\lambda i}\, p_{\lambda j}^{-1} = [p_{\lambda i} (p_{\lambda i}^{-1} p_{\lambda j})\, p_{\lambda j}^{-1}]^{-1} \in N$ as required.

Similarly we can show that

$[(\mu, \nu) \in \rho_\Lambda, \text{ and } p_{\mu i} \neq 0] \Rightarrow p_{\mu i}\, p_{\nu i}^{-1} \in N$

Thus $\{N, \rho_I, \rho_\Lambda\}$ is a linked triplet.

Conversely since N is a normal subgroup of G and ρ_I and ρ_Λ are equivalences on the sets I and Λ respectively, it is obvious that ρ is an equivalence on S. Let $((x, i, \lambda), (y, j, \mu)) \in \rho$ and let (z, k, ν) be an arbitrary element of the semigroup S. Now we observe that

$(z, k, \nu)(x, i, \lambda) = (zp_{\nu i}\, x, k, \lambda) = 0$ if and only if

$p_{\nu i} = 0$ if and only if $p_{\nu j} = 0$ (since $(i, j) \in \rho_I \subseteq \mathcal{E}_I$)

i.e. if and only if $(z, k, \nu)(y, j, \mu) = 0$

Hence if both products are zero, then the result is trivial. If $p_{\nu i} \neq 0$, then we have that $p_{\nu i} p_{\nu j}^{-1} \varepsilon N$ by the linking property. Also $xy^{-1} \varepsilon N$ by definition of ρ. Hence

$$(zp_{\nu i} x)(zp_{\nu j} y)^{-1} = zp_{\nu i} xy^{-1} p_{\nu j}^{-1} z^{-1}$$

$$= z [p_{\nu i} p_{\nu j}^{-1} p_{\nu j} (xy^{-1}) p_{\nu j}^{-1}] z^{-1} \varepsilon N$$ and so since $(\lambda, \mu) \varepsilon \rho_\Lambda$

we have that

$$((zp_{\nu i} x, k, \lambda), (zp_{\nu j} y, k, \mu)) \varepsilon \rho.$$

Hence ρ is left compatible. Similarly the right compatibility of ρ can be established. Thus ρ is a congruence on S.

<u>P roved</u>

§3 Now we consider the lattice of congruences on the completely 0-simple semigroup $S = \mathcal{M}^0 (G, I, \Lambda, P)$, where the sandwich matrix P is normal.

We shall represent a congruence ρ on the semigroup S by means of the linked triplet $[N, \rho_I, \rho_\Lambda]$ where N, ρ_I, ρ_Λ are determined in accordance with the theorem

The following proposition is immediate from the definition.

<u>PROPOSITION 3.1</u> (I) <u>Let</u> $\{N, \rho_I, \rho_\Lambda\}$ <u>be a linked triplet.</u> <u>Then the triplet</u> $\{M, \rho_I, \rho_\Lambda\}$ <u>is linked if</u> $N \subseteq M$

(II) Let $\{N, \rho_I, \rho_\Lambda\}$ be a linked triplet. Then the triplet $\{N, \sigma_I, \sigma_\Lambda\}$ is linked if $\sigma_I \subseteq \rho_I$ and $\sigma_\Lambda \subseteq \rho_\Lambda$.

Next we prove the following

PROPOSITION 3.2 Let $\rho = [M, \rho_I, \rho_\Lambda]$ and $\sigma = [N, \sigma_I, \sigma_\Lambda]$ be congruences on the semigroup $S = \mathcal{M}^0 (G, I, \Lambda, P)$ where P is normal. Then

(I) $\rho \subseteq \sigma \iff M \subseteq N, \rho_I \subseteq \sigma_I, \rho_\Lambda \subseteq \sigma_\Lambda$

(II) $\rho \cap \sigma = [M \cap N, \rho_I \cap \sigma_I, \rho_\Lambda \cap \sigma_\Lambda]$

(III) $\rho \vee \sigma = [MN, \rho_I \vee \sigma_I, \rho_\Lambda \vee \sigma_\Lambda]$

Proof (I) It can be easily verified.

(II) Let us consider the triplet $\{M \cap N, \rho_I \cap \sigma_I, \rho_\Lambda \cap \sigma_\Lambda\}$ Clearly $M \cap N$ is a normal subgroup of G and $\rho_I \cap \sigma_I$ and $\rho_\Lambda \cap \sigma_\Lambda$ are equivalences on the sets I and Λ respectively. Since $\rho_I \subseteq \mathcal{E}_I$ and $\sigma_I \subseteq \mathcal{E}_I$, it follows that $\rho_I \cap \sigma_I \subseteq \mathcal{E}_I$. Similarly $\rho_\Lambda \cap \sigma_\Lambda \subseteq \mathcal{E}_\Lambda$. Next let $(i, j) \in \rho_I \cap \sigma_I$ and let $p_{\lambda i} \neq 0$. It follows that $(i, j) \in \rho_I$, $(i, j) \in \sigma_I$ and $p_{\lambda i} \neq 0$. Now since ρ and σ are congruences on S, it follows that $p_{\lambda i} p_{\lambda j}^{-1} \in N$ and that $p_{\lambda i} p_{\lambda j}^{-1} \in M$. Thus $p_{\lambda i} p_{\lambda j}^{-1} \in N \cap M$. Similarly the second condition of the linking property can be established. Thus $\{M \cap N, \rho_I \cap \sigma_I, \rho_\Lambda \cap \sigma_\Lambda\}$ is a linked triplet and hence a congruence ξ (say) on S. It is clear by part (I) that $\xi \subseteq \rho$ and $\xi \subseteq \sigma$. Hence $\xi \subseteq \rho \cap \sigma$. On the other hand let $\eta = [Q, \eta_I, \eta_\Lambda]$ be a congruence on S such that $\eta \subseteq \rho$ and $\eta \subseteq \sigma$. Then it follows by part (I) that $\eta_I \subseteq \rho_I$, $\eta_I \subseteq \sigma_I$, $\eta_\Lambda \subseteq \rho_\Lambda$, $\eta_\Lambda \subseteq \sigma_\Lambda$ and $Q \subseteq M$, $Q \subseteq N$. It follows that $Q \subseteq M \cap N$, $\eta_I \subseteq \rho_I \cap \sigma_I$ and $\eta_\Lambda \subseteq \rho_\Lambda \cap \sigma_\Lambda$. Thus $\eta \subseteq \xi$.

Hence $\xi = \rho \cap \sigma$.

(III) Let us consider the triplet $\{MN, \rho_I \vee \sigma_I, \rho_\Lambda \vee \sigma_\Lambda\}$.

Clearly MN is a normal subgroup of G, and $\rho_I \vee \sigma_I$, $\rho_\Lambda \vee \sigma_\Lambda$ are equivalences on the sets I and Λ respectively. Since $\rho_I \subseteq \mathcal{E}_I$ and $\sigma_I \subseteq \mathcal{E}_I$, it follows that $\rho_I \vee \sigma_I \subseteq \mathcal{E}_I$. Similarly $\rho_\Lambda \vee \sigma_\Lambda \subseteq \mathcal{E}_\Lambda$. Next let $(i, j) \in \rho_I \vee \sigma_I$ and let $p_{\lambda i} \neq 0$. It follows from the ~~proposition~~ THEOREM I.1.5 that there exist $k_1, k_2, -- k_n \in I$ such that $(i, k_1) \in \rho_I$, $(k_1, k_2) \in \sigma_I$, $(k_2, k_3) \in \rho_I$, --- , $(k_n, j) \in \sigma_I$, and $p_{\lambda i} \neq 0$. It follows that $p_{\lambda i}\, p_{\lambda k_1}^{-1} \in M$, $p_{\lambda k_1}\, p_{\lambda k_2}^{-1} \in N$, --- $p_{\lambda k_n}\, p_{\lambda j}^{-1} \in N$. Now since N and M are normal subgroups, we have that $p_{\lambda i}\, p_{\lambda k_1}^{-1}\, p_{\lambda k_1}\, p_{\lambda k_2}^{-1}$ --- $p_{\lambda k_2}\, p_{\lambda j}^{-1} \in MN$ and hence $p_{\lambda i}\, p_{\lambda j}^{-1} \in MN$. Similarly the second condition of the linking property can be established. Thus $\{MN, \rho_I \vee \sigma_I, \rho_\Lambda \vee \sigma_\Lambda\}$ is a linked triplet and hence a congruence ξ (say) on the semigroup S. It is clear by part (I) that $\rho \subseteq \xi$ and $\sigma \subseteq \xi$ and hence $\rho \vee \sigma \subseteq \xi$. On the other hand, suppose that $\eta = [Q, \eta_I, \eta_\Lambda]$ is a congruence on S such that $\eta \supseteq \rho$ and $\eta \supseteq \sigma$. It follows by part (I) that $Q \supseteq M$, $Q \supseteq N$; $\eta_I \supseteq \rho_I$, $\eta_I \supseteq \sigma_I$, $\eta_\Lambda \supseteq \rho_\Lambda$ and $\eta_\Lambda \supseteq \sigma_\Lambda$. It follows that $Q \supseteq MN$, $\eta_I \supseteq \rho_I \vee \sigma_I$, $\eta_\Lambda \supseteq \rho_\Lambda \vee \sigma_\Lambda$ and hence by part (I) we infer that $\eta \supseteq \xi$. It follows that $\xi = \rho \vee \sigma$.

Proved

NOTE 3.3

It is clear from part (I) of the theorem 3.3 that if ρ strictly contains σ, then at least one of the inclusions $M \subseteq N$, $\rho_I \subseteq \sigma_I$, $\rho_\Lambda \subseteq \sigma_\Lambda$ should be strict.

Let Δ denote the lattice of all congruences on the completely 0-simple semigroup $S = \mathcal{M}^0(G, I, \Lambda, P)$ where P is normal. We proceed to show that Δ is semimodular. First we need to establish the following.

<u>LEMMA 3.4</u> <u>Let $\xi = [L, \xi_I, \xi_\Lambda]$ and $\eta = [M, \eta_I, \eta_\Lambda]$ be congruences on the completely 0-simple semigroup</u> $S = \mathcal{M}^0(G, I, \Lambda, P)$ <u>where P is normal. Then ξ covers η (i.e. $\xi \succ \eta$) if and only if any one of the following conditions hold</u> :

(I) $L \succ M$, $\xi_I = \eta_I$ $\xi_\Lambda = \eta_\Lambda$

(II) $L = M$, $\xi_I \succ \eta_I$ $\xi_\Lambda = \eta_\Lambda$

(III) $L = M$, $\xi_I = \eta_I$ $\xi_\Lambda \succ \eta_\Lambda$

<u>Proof</u> We assume that $\xi \succ \eta$. It follows from part (I) of the proposition 3.2, that

(1) ——— $L \supseteq M$, $\xi_I \supseteq \eta_I$, and $\xi_\Lambda \supseteq \eta_\Lambda$

Further by definition of <u>cover</u> ($\succ$) since $\xi \supset \eta$, it follows from the note 3.3 that at least one of the inequalities in the expression (1) should be strict.

First let us assume that $L \supset M$. It follows by the proposition 3.1, that $\{L, \eta_I, \eta_\Lambda\}$ is a linked triplet and hence a congruence ζ (say) on S. Further by the proposition 3.2, we infer that $\eta \subset \zeta \subseteq \xi$. Since ξ covers η, it follows that $\xi = \zeta$ and hence $\xi_I = \eta_I$ and $\xi_\Lambda = \eta_\Lambda$ Further if N is a normal subgroup of G such that $L \supset N \supset M$, then $\tau = [N, \eta_I, \eta_\Lambda]$ is a congruence by the proposition 3.1, such that $\xi \supset \tau \supset \eta$. This is a contradiction.

and hence $L \succ M$. Conversely it is clear that if condition (I) holds then ξ covers η.

Next we assume that $\xi_I \supset \eta_I$. It follows that $L = M$ (since if $L \supset M$, then $\xi_I = \eta_I$). Further $\{L, \xi_I\ \eta_\Lambda\}$ is clearly a linked triplet by the proposition 3.1 and further we infer from the proposition 3.2 that the congruence $[L, \xi_I, \eta_\Lambda] = \zeta$ (say) is such that $\xi \supseteq \zeta \supseteq \eta$. Since $\xi \succ \eta$, it follows that $\xi = \zeta$ and hence $\xi_\Lambda = \eta_\Lambda$. Further if there exist an equivalence τ_I on the set I such that $\xi_I \supset \tau_I \supset \eta_I$, then $\tau = [L, \tau_I, \eta_\Lambda]$ is a congruence on S, such that $\xi \supset \tau \supset \eta$, which is a contradiction. Hence $\xi_I \succ \eta_I$. Conversely if the condition (II) holds then obviously ξ covers η. A similar argument establishes condition (III).

Proved

THEOREM 3.5 The lattice of all congruences on a completely 0-simple semigroup S is semimodular.

Proof In accordance with the theorem 1.4, we can assume that $S \cong \mathfrak{M}^0(G, I, \Lambda, P)$ where P is normal. Now let $\xi = [L, \xi_I, \xi_\Lambda]$ and $\eta = [M, \eta_I, \eta_\Lambda]$ be two congruences on the semigroup S such that $\xi \succ \xi \cap \eta$ and $\eta \succ \xi \cap \eta$. By virtue of the proposition 3.2 we know that $\xi \cap \eta = [L \cap M, \xi_I \cap \eta_I, \xi_\Lambda \cap \eta_\Lambda]$ and that

$$\xi \vee \eta = [LM, \xi_I \vee \eta_I, \xi_\Lambda \vee \eta_\Lambda].$$

In view of the dual arguments, it will be sufficient to consider only the four following distinct possibilities arising in accordance with the lemma 3.4.

$$\text{(I)} \quad L \succ L \cap M, \quad \xi_I = \xi_I \cap \eta_I, \quad \xi_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$
$$\text{and} \quad M \succ L \cap M, \quad \eta_I = \xi_I \cap \eta_I, \quad \eta_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$

$$\text{(II)} \quad L \succ L \cap M, \quad \xi_I = \xi_I \cap \eta_I, \quad \xi_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$
$$\text{and} \quad M = L \cap M, \quad \eta_I \succ \xi_I \cap \eta_I, \quad \eta_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$

$$\text{(III)} \quad L = L \cap M, \quad \xi_I \succ \xi_I \cap \eta_I, \quad \xi_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$
$$\text{and} \quad M = L \cap M, \quad \eta_I \succ \xi_I \cap \eta_I, \quad \eta_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$

$$\text{(IV)} \quad L = L \cap M, \quad \xi_I \succ \xi_I \cap \eta_I, \quad \xi_\Lambda = \xi_\Lambda \cap \eta_\Lambda$$
$$\text{and} \quad M = L \cap M, \quad \eta_I = \xi_I \cap \eta_I, \quad \eta_\Lambda \succ \xi_\Lambda \cap \eta_\Lambda$$

Let us denote LM by N, $\xi_I \vee \eta_I$ by ζ_I and $\xi_\Lambda \vee \eta_\Lambda$ by ζ_Λ. Thus $\xi \vee \eta = [N, \zeta_I, \zeta_\Lambda]$

<u>CASE I</u>. It is obvious that $\zeta_I = \xi_I = \eta_I$ and $\zeta_\Lambda = \xi_\Lambda = \eta_\Lambda$

Further, since the lattice of normal subgroups is modular and hence semimodular (Theorem V.i [4]) it follows that $N = LM \succ L$ and $N = LM \succ M$. It follows by the lemma 5.4 that $\xi \vee \eta \succ \xi$ and $\xi \vee \eta \succ \eta$.

<u>CASE II</u>. Clearly in this case $N = LM = L$; $\zeta_I \succ \xi_I$ and $\zeta_\Lambda = \xi_\Lambda$ and hence we have that $\xi \vee \eta \succ \xi$. Further $N \succ M$, $\zeta_I = \eta_I$ and $\zeta_\Lambda = \eta_\Lambda$ and hence $\xi \vee \eta \succ \eta$.

CASE III. In this case we see that $N = L = M$ and $\zeta_\Lambda = \xi_\Lambda = \eta_\Lambda$.

Further since the lattice of equivalences is semimodular (§16 [5]) it follows that $\zeta_I \succ \xi_I$ and $\zeta_I \succ \eta_I$ and hence $\xi \vee \eta \succ \xi$ and $\xi \vee \eta \succ \eta$.

CASE IV. Clearly in this case $N = L$, $\zeta_I = \xi_I$ and $\zeta_\Lambda \succ \xi_\Lambda$. Hence we have that $\xi \vee \eta \succ \xi$. Further we observe that $N = L = M$, $\zeta_I \succ \eta_I$ and $\zeta_\Lambda = \eta_\Lambda$ and hence we have that $\xi \vee \eta \succ \eta$. Thus the lattice Δ is semimodular.

Proved

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