THE SOLUTION OF LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

A Thesis presented on application for the
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by

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CONTENTS.

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CHAPTER I: Introduction

The purpose of this Dissertation is to give a detailed account of Martin's method for the solution of linear second order partial differential equations of hyperbolic type. For a thorough understanding of this method, the use of characteristics as co-ordinate axes, as well as Riemann's method of which Martin's method is an extension, are fully investigated.

In Chapter II, all the relevant facts about characteristics as co-ordinate axes in spaces of two and three dimensions are discussed. In the case of two dimensions, the method of classifying any linear partial differential equation by reducing it to canonical form is explained and illustrated by examples. Also, basic principles relating to the Cauchy problem are developed for hyperbolic partial differential equations in normal form, for systems of simultaneous partial differential equations and for hyperbolic partial differential equations with weak discontinuities. Once the details and results of characteristics in the space of three dimensions are well understood, the application of characteristics in spaces of higher dimension is easily understood and appreciated. In particular, the use of characteristic "surfaces" in the solution of the Cauchy problem in the space of n dimensions becomes a simple exercise.

In Chapter III, Riemann's method of solution is presented in a form in which it is easy to extend the results to Martin's method and vice-versa. The supporting examples illustrate in some detail how the Riemann function is obtained. Most of these examples are also discussed in Chapter IV and their results suggest a connection between the Riemann and Martin functions.

Chapter IV is a detailed account of Martin's method, with applications to the solution of the Cauchy problem for the wave equation in spaces of two, three and n dimensions. Two other examples are also discussed: a particular example whose Martin function is obtained by the method of solution in series, and the Euler-Darboux equation which Martin originally considered.

Chapter V contains an important general formula which connects the Riemann and Martin functions of any hyperbolic partial differential equation. The formula obtained by Professor A.G. Mackie in the case of the Euler-Poisson equation [14]* is shown to be a particular case of this general formula. The supporting examples illustrate how each function can be derived from the other.

Professor E.T. Copson [13] has made a survey of all the known methods of deriving Riemann functions of various equations of hyperbolic type, and so we can assume that Martin functions of these equations are also known.

Apart from a particular generalisation of Martin's method [12], which enables us to solve parabolic and elliptic equations as well as hyperbolic equations not necessarily in normal form, this Dissertation is an account, supported by various examples, of all that is known about Martin's method.

This generalisation of Martin's method has no special merit for hyperbolic equations and I have therefore omitted its discussion.

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*Numbers in rectangular brackets refer to References at the end of the Dissertation.

CHAPTER II: The Characteristics of Linear Partial Differential Equations

2.1. Second Order Partial Differential Equations With Variable Coefficients.

In this Section we investigate the properties of the second order partial differential equation of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \qquad (2.1.1)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$. We shall restrict ourselves to the case in which the variable coefficients R, S and T are functions of x and y only. We shall also assume that these coefficients are continuous functions of x and y and possess continuous partial derivatives of the orders we want in the region under consideration. With these restrictions equation (2.1.1) is said to be linear.

Let us change the independent variables from x, y to ξ, η. By elementary partial differentiation we find that

$$r = \xi_x^2 \frac{\partial^2 z}{\partial \xi^2} + 2\xi_x \eta_x \frac{\partial^2 z}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2 z}{\partial \eta^2},$$

$$s = \xi_x \xi_y \frac{\partial^2 z}{\partial \xi^2} + (\xi_x \eta_y + \xi_y \eta_x) \frac{\partial^2 z}{\partial \xi \partial \eta} + \eta_x \eta_y \frac{\partial^2 z}{\partial \eta^2}$$

$$t = \xi_y^2 \frac{\partial^2 z}{\partial \xi^2} + 2\xi_y \eta_y \frac{\partial^2 z}{\partial \xi \partial \eta} + \eta_y^2 \frac{\partial^2 z}{\partial \eta^2}$$

Equation (2.1.1) is then transformed into the equation

$$A(\xi_x, \xi_y) \frac{\partial^2 z}{\partial \xi^2} + 2\, B(\xi_x, \eta_x; \xi_y, \eta_y) \frac{\partial^2 z}{\partial \xi \partial \eta} + C(\eta_x, \eta_y) \frac{\partial^2 z}{\partial \eta^2} = F(\xi, \eta, z, z_\xi, z_\eta), \qquad (2.1.2)$$

where $A(u, v) = Ru^2 + Suv + Tv^2$

$$B(u_1, u_2; v_1, v_2) = Ru_1u_2 + \tfrac{1}{2}S(u_1v_2 + u_2v_1) + Tv_1v_2$$

$$C(u, v) = A(u, v)$$

and $F(\xi, \eta, z, z_\xi, z_\eta)$ is the function which $f(x, y, z, p, q)$ takes in the new variables.

The problem now is to choose the new variables ξ and η so that equation 2.1.2) reduces to the simples form. This problem is solved by considering the sign of the discriminant $\Delta(x, y)$ defined by $\Delta(x, y) = S^2 - 4RT$. This is because the sign of this function remains invariant under the above transformation, for it can be shown that

$$4B^2 - 4AC = (S^2 - 4RT)(\xi_x\eta_y - \xi_y\eta_x)^2.$$

We shall, however, use other elementary considerations instead of this identity in obtaining our main results.

Case (i). Suppose that $\Delta(x, y)$ is positive for all points (x, y) in region $\mathscr{D}$. Then the equation $R\alpha^2 + S\alpha + T = 0$ has two distinct real roots $\alpha_1(x, y)$ and $\alpha_2(x, y)$. If we choose $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that $\xi_x = \alpha_1\xi_y$, $\eta_x = \alpha_2\eta_y$ then

$$A(\xi_x, \xi_y) = \xi_y^2(R\alpha_1^2 + S\alpha_1 + T) = 0$$

$$C(\eta_x, \eta_y) = \eta_y^2(R\alpha_2^2 + S\alpha_2 + T) = 0$$

Since the equation $R\alpha^2 + S\alpha + T = 0$ has two unequal roots α_1, α_2 we must have $R \neq 0$, $\alpha_1 + \alpha_2 = -S/_R$ and $\alpha_1\alpha_2 = T/_R$. Hence

$$B(\xi_x, \eta_x; \xi_y, \eta_y) = \xi_y\eta_y\{R\alpha_1\alpha_2 + \tfrac{1}{2}S(\alpha_1+\alpha_2) + T\}$$

$$= \frac{\xi_y\eta_y}{2R}(4RT - S^2)$$

$\neq 0$, since ξ, η are functions of both x and y.

Hence equation (2.1.2) reduces to the simplest form

$$\frac{\partial^2 z}{\partial\xi\partial\eta} = G(\xi, \eta, z, z_\xi, z_\eta). \tag{2.1.3}$$

<u>Case (ii)</u>. Suppose $\Delta(x, y)$ is negative for all points (x, y) in a region $\mathcal{D}$. Then the equation $R\alpha^2 + S\alpha + T = 0$ has two complex roots α_1, α_2. As in Case (i), equation (2.1.2) reduces to

$\dfrac{\partial^2 z}{\partial\xi\partial\eta} = G(\xi, \eta, z, z_\xi, z_\eta)$ where now ξ, η are complex conjugate variables. We therefore introduce new real variables α, β given by $\alpha = \frac{1}{2}(\xi + \eta)$, $\beta = \frac{1}{2}i(\eta - \xi)$. It then follows that

$\dfrac{\partial^2 z}{\partial\xi\partial\eta} = \frac{1}{4}\left(\dfrac{\partial^2 z}{\partial\alpha^2} + \dfrac{\partial^2 z}{\partial\beta^2} \right)$. Hence in this case, equation (2.1.2) reduces to the simplest form

$$\frac{\partial^2 z}{\partial\alpha^2} + \frac{\partial^2 z}{\partial\beta^2} = G(\alpha, \beta, z, z_\alpha, z_\beta) \tag{2.1.4}$$

Case (iii). Suppose $\Delta(x, y) = 0$ for all points (x, y) in a region $\mathscr{D}$. Then the equation $R\alpha^2 + S\alpha + T = 0$ has a repeated root $\alpha = \alpha_1$. We can therefore choose $\xi = \xi(x, y)$ so that $\xi_x = \alpha_1\xi_y$. Further, we let $\eta(x, y)$ be any other function of x and y which is independent of ξ. The equation (2.1.2) then reduces to

$$2\, B(\xi_x, \eta_x; \xi_y, \eta_y)\frac{\partial^2 z}{\partial\xi\partial\eta} + C(\eta_x, \eta_y)\frac{\partial^2 z}{\partial\eta^2} = F(\xi, \eta, z, z_\xi, z_\eta), \qquad (2.1.5)$$

Since equation $R\alpha^2 + S\alpha + T = 0$ has two equal roots, we must have $R \neq 0$, $2\alpha_1 = -S/_R$ and so

$$\begin{aligned} B(\xi_x, \eta_x; \xi_y, \eta_y) &= R\alpha_1\xi_y\eta_x + \tfrac{1}{2}S(\alpha_1\xi_y\eta_y + \xi_y\eta_x) + T\xi_y\eta_y \\ &= \tfrac{1}{4R}(4RT - S^2)\xi_y\eta_y \\ &= 0\ . \end{aligned}$$

Since $\eta(x, y)$ is independent of $\xi(x, y)$ we can choose $\eta = \eta(x, y)$ such that $C(\eta_x, \eta_y) \neq 0$ in the region $\mathscr{D}$. Hence (2.1.5) reduces to the simplest form

$$\frac{\partial^2 z}{\partial\eta^2} = G(\xi, \eta, z, z_\xi, z_\eta) \qquad (2.1.6)$$

If the equation (2.1.1) is such that $\Delta(x, y) > 0$ at all points (x, y) in a region $\mathscr{D}$, it is said to be hyperbolic in that region, and the equation (2.1.3) is called its canonical form. If $\Delta(x, y) < 0$, it is said to be elliptic and equation (2.1.4) is its canonical form. If, however, $\Delta(x, y) = 0$, it is said to be parabolic and equation (2.1.6) is its canonical form.

The curves $\xi(x, y) = a$, $\eta(x, y) = b$ where a, b are constants are called characteristics of the equation (2.1.1). Hence through every point of a region in which equation (2.1.1) is hyperbolic there pass two distinct characteristics, in which it is parabolic only one characteristic, and in which it is elliptic there are no real characteristics.

The variables ξ and η are called canonical, or characteristic, variables. From the equation $\xi(x, y) = a$ we find that $\xi_x d_x + \xi_y d_y = 0$ i.e. $\frac{dy}{dx} = \frac{\xi x}{\xi y} = -\alpha_1$. Similarly from $\eta(x, y) = b$ we have $\frac{dy}{dx} = -\alpha_2$. The differential equation of the characteristics is therefore given by

$$R \left(\frac{dy}{dx} \right)^2 - S \frac{dy}{dx} + T = 0$$

$$\text{i.e.} \qquad R \dot{y}^2 - S \dot{y} \dot{x} + T \dot{x}^2 = 0 \tag{2.1.7}$$

where $\dot{y} = \frac{dy(\tau)}{d\tau}$ and τ is a parameter of a characteristic.

An equation of type (2.1.1) can be hyperbolic in one part of the xy-plane, elliptic in a second and parabolic in a third. When this situation arises, the equation is said to be of mixed type.

We now apply the foregoing theory to two equations of mixed type, namely, $x^2 r = y^2 t$ and the Tricomi's equation $yr + t = 0$.

Since $\Delta(x, y) = 4x^2y^2$, the 1st equation is hyperbolic in the whole xy- plane excepting the axes $x = 0$, $y = 0$ which we call lines of parabolic degeneracy. Equations of the characteristics are given by the differential equations $x^2\left(\frac{dy}{dx}\right)^2 = y^2$ i.e. $\frac{dy}{dx} \pm \frac{y}{x} = 0$ whose solutions are $xy = a$, $\frac{y}{x} = b$, where a, b are constants. Some of the characteristics of this two parameter family are shown in Fig. 2a on page 9a.

Using characteristic variables as coordinates we let $\xi = xy$, $\eta = \frac{y}{x}$ when $x \neq 0$, $y \neq 0$. Then

$$x^2 \frac{\partial^2 z}{\partial x^2} = x^2y^2 \frac{\partial^2 z}{\partial \xi^2} - 2y^2 \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{y^2}{x^2} \frac{\partial^2 z}{\partial \eta^2} + 2 \frac{y}{x} \frac{\partial z}{\partial y}$$

$$y^2 \frac{\partial^2 z}{\partial y^2} = 2y^2 \frac{\partial^2 z}{\partial \xi \partial \eta} + x^2y^2 \frac{\partial^2 z}{\partial \xi^2} + \frac{y^2}{x^2} \frac{\partial^2 z}{\partial \eta^2} .$$

Hence the equation reduces to $\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{2\xi} \frac{\partial z}{\partial \eta}$.

Integrating along the characteristic ξ = constant partially with respect to η we get

$\frac{\partial z}{\partial \xi} = \frac{1}{2\xi} z + \sqrt{\xi}\, A(\xi)$ where $A(\xi)$ is an arbitrary function of ξ.

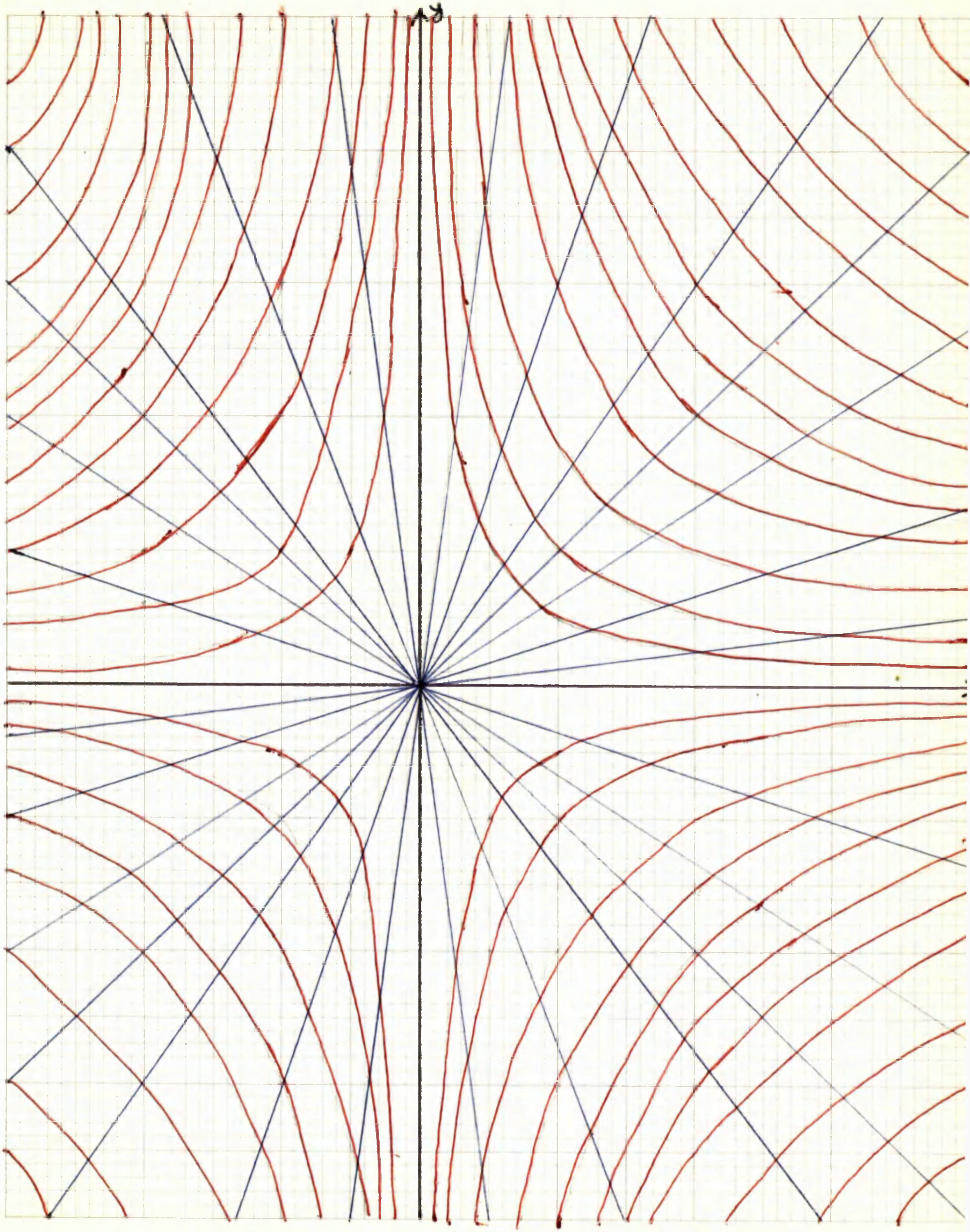
Integrating along the characteristic η = constant partially with respect to ξ we get

$z(\xi, \eta) = \sqrt{\xi}\, F(\xi) + \sqrt{\xi}\, G(\eta) + C$ where F, G are arbitrary functions of ξ and η and c is a constant.

Hence, in the xy-plane the solution is

$z(x, y) = \sqrt{xy}\, F(xy) + \sqrt{xy}\, G(y/x) + C$

where F, G are arbitrary functions of xy and y/x respectively.


y

In the case of Tricomi's equation $y \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, the discriminant $\Delta(x, y)$ is given by $\Delta(x, y) = -4y$. The equation is therefore hyperbolic in the lower half-plane $y < 0$, elliptic in the upper half-plane $y > 0$, and the x-axis is a curve of parabolic degeneracy. These facts are displayed in Fig. 2b on page 10a.

The differential equations of characteristics are $(-y)^{\frac{1}{2}} \frac{dy}{dx} \pm 1 = 0$ with solutions $x + \frac{2}{3}(-y)^{\frac{3}{2}} = a$, $x - \frac{2}{3}(-y)^{\frac{3}{2}} = b$. In the region in which the equation is hyperbolic, we let $\xi = x + \frac{2}{3}(-y)^{\frac{3}{2}}$, $\eta = x - \frac{2}{3}(-y)^{\frac{3}{2}}$. The equation is then reduced to

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{4(6\xi - 6\eta)^{\frac{1}{3}}} \left(\frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \right) .$$

2.2. The Relation of Characteristics to the Cauchy Problem.

The Cauchy problem for the equation (2.1.1) asks us to find a solution $z(x, y)$ of this equation which satisfies the condition that it and its normal derivative $\frac{\partial z}{\partial n}$ take on prescribed values

$$z = z_1(\tau), \quad \frac{\partial z}{\partial n} = g_1(\tau)$$

on a curve Γ in the xy-plane specified by the parametric equations $x = x(\tau)$, $y = y(\tau)$. This condition is equivalent to the condition that z and its first partial derivatives take on prescribed values

$$z = z_1(\tau), \; p = p_1(\tau), \quad q = q_1(\tau) \tag{2.2.1}$$

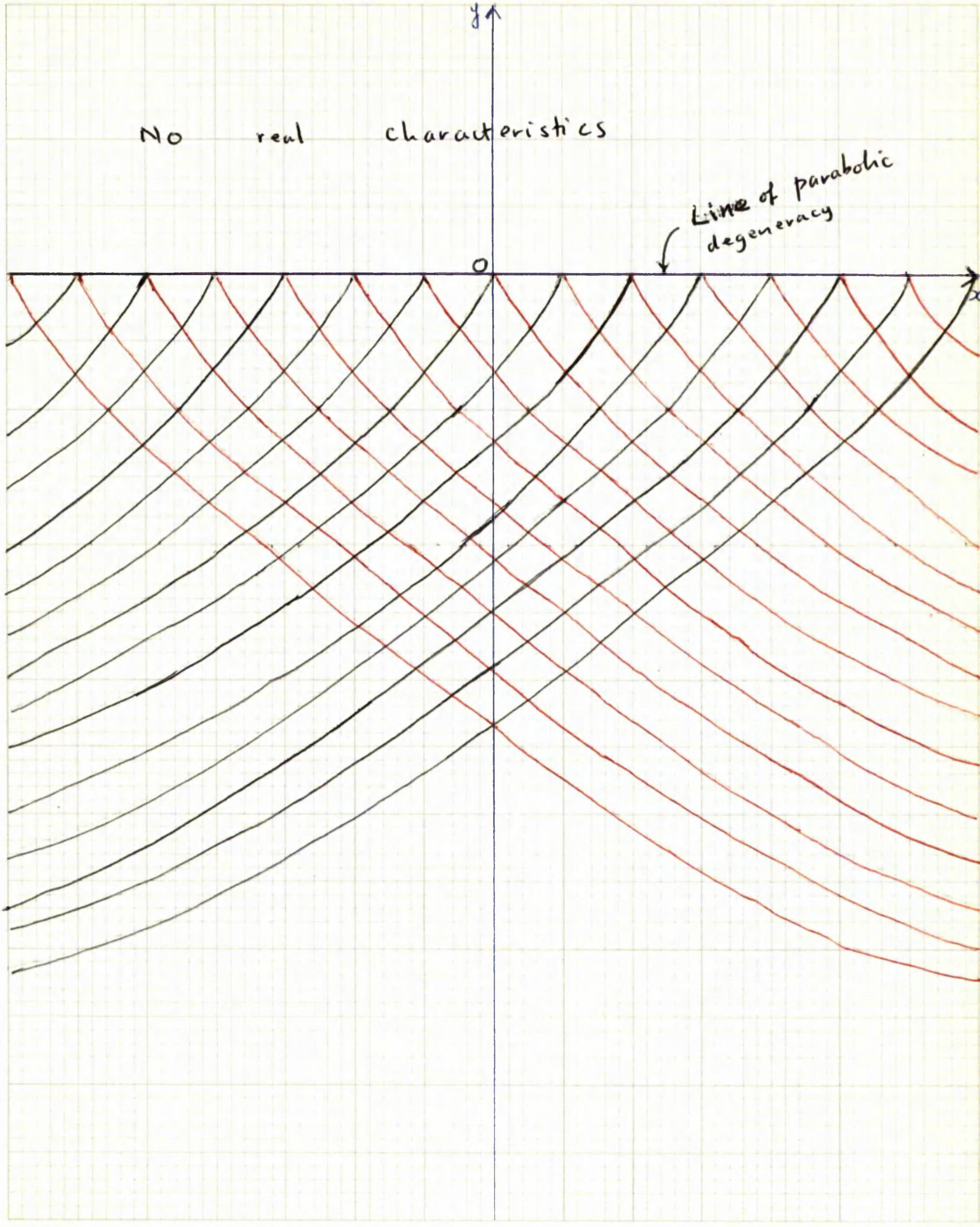


Fig. 2b

on the curve Γ. The functions $z_1(\tau)$, $p_1(\tau)$ and $q_1(\tau)$ are not independent, they are related by the compatibility condition

$$\dot{z}_1(\tau) = p_1(\tau)\,\dot{x}(\tau) + q_1(\tau)\,\dot{y}(\tau) \tag{2.2.2}$$

where $\quad \dot{z}_1(\tau) = \frac{dz_1}{d\tau}$, etc.

On the curve Γ, $p_1(\tau) = p\{x(\tau), y(\tau)\}$ and $q_1(\tau) = q\{x(\tau), y(\tau)\}$. Hence

$$\dot{p}_1(\tau) = r_1(\tau)\,\dot{x}(\tau) + s_1(\tau)\,\dot{y}(\tau) \tag{2.2.3}$$

$$\dot{q}_1(\tau) = s_1(\tau)\,\dot{x}(\tau) + t_1(\tau)\,\dot{y}(\tau) \tag{2.2.4}$$

From equation (2.1.1) we have a third relation

$$-f_1(\tau) = R_1(\tau)\,r_1(\tau) + S_1(\tau)\,s_1(\tau) + T_1(\tau)\,t_1(\tau) \tag{2.2.5}$$

where $\quad R_1(\tau) = R(x(\tau), y(\tau))$, etc.

From equations (2.2.3), (2.2.4) and (2.2.5) we conclude that $r_1(\tau)$, $s_1(\tau)$ and $t_1(\tau)$ are uniquely determined if and only if the determinant

$$\Delta = \begin{vmatrix} \dot{x}(\tau) & \dot{y}(\tau) & 0 \\ 0 & \dot{x}(\tau) & \dot{y}(\tau) \\ R_1(\tau) & S_1(\tau) & T_1(\tau) \end{vmatrix} \quad \text{does not vanish.}$$

From $r_1(\tau) = r(x(\tau), y(\tau)$ and $s_1(\tau) = s(x(\tau), y(\tau)$ we get

$$\dot{r}_1(\tau) = r_x\,\dot{x}(\tau) + s_x\,\dot{y}(\tau), \tag{2.2.6}$$

$$\dot{s}_1(\tau) = s_x\,\dot{x}(\tau) + t_x\,\dot{y}(\tau). \tag{2.2.7}$$

Also differentiating equation (2.2.5) with respect to x we get

$$-R_x r_1(\tau) - S_x s_1(\tau) - T_x t_1(\tau) - f_x = R_1(\tau) r_x + S_1(\tau) s_x + T_1(\tau) t_x. \tag{2.2.8}$$

The quantities on the left of these equations are assumed known and so the functions r_x, s_x, t_x are uniquely determined on the curve Γ provided that

$$\Delta = \begin{vmatrix} \dot{x}(\tau) & \dot{y}(\tau) & 0 \\ 0 & \dot{x}(\tau) & \dot{y}(\tau) \\ R_1(\tau) & S_1(\tau) & T_1(\tau) \end{vmatrix} \quad \text{does not vanish.}$$

By repeated application of the above procedure, all the higher derivatives of z can be uniquely obtained at each point of the curve Γ provided that the determinant Δ is not zero. We therefore conclude that if $\Delta \neq 0$ at each point of the curve Γ and the functions $z_1(\tau)$, $p_1(\tau)$, $q_1(\tau)$, $R_1(\tau)$, etc. are analytic at each point of this curve, then the Cauchy problem has a unique solution in the neighbourhood of the curve Γ obtained by expanding $z(x, y)$ in a Taylor's series.

If, however, $\Delta = 0$ then the values of $r_1(\tau)$, $s_1(\tau)$ and $t_1(\tau)$ as well as higher order derivatives of z on the curve Γ may exist but cannot be uniquely determined. The vanishing of the determinant on the curve Γ implies that

$$R_1\dot{y}^2 - S_1\dot{x}\dot{y} + T_1\dot{x}^2 = 0 \qquad (2.2.\ )$$

Comparing equations (2.1.7) and (2.2.) we conclude that if $\Delta = 0$

on the curve Γ then Γ is a characteristic of the equation (2.1.1). It therefore follows that the Cauchy problem does not have a unique solution if the curve along which the data are prescribed is a characteristic of the partial differential equation.

In the case when the determinant vanishes, equations (2.2.3), (2.2.4) and (2.2.5) will have a solution, in fact, a multiplicity of solutions, provided that these equations are consistent. From the theory of linear algebraic equations, this condition is satisfied when the rank of the augmented matrix

$$\begin{bmatrix} \mathring{x}(\tau) & \mathring{y}(\tau) & 0 & \mathring{p}_1(\tau) \\ 0 & \mathring{x}(\tau) & \mathring{y}(\tau) & \mathring{q}_1(\tau) \\ R_1(\tau) & S_1(\tau) & T_1(\tau) & -f_1(\tau) \end{bmatrix}$$

is two. This condition will be satisfied if all the 3×3 determinants vanish and each of at least two of them contains one non-vanishing 2×2 determinant. But the determinant Δ contains $\begin{vmatrix} \mathring{x}(\tau) & \mathring{y}(\tau) \\ 0 & \mathring{x}(\tau) \end{vmatrix}$ and $\begin{vmatrix} \mathring{y}(\tau) & 0 \\ 0 & \mathring{x}(\tau) \end{vmatrix}$ and both cannot vanish and so in addition to $\Delta = 0$ we must have, if the Cauchy problem for the data given on the characteristic Γ is to have a solution, the additional condition

$$\begin{vmatrix} \mathring{p}_1(\tau) & \mathring{y}(\tau) & 0 \\ \mathring{q}_1(\tau) & \mathring{x}(\tau) & \mathring{y}(\tau) \\ -f_1(\tau) & S_1(\tau) & T_1(\tau) \end{vmatrix} = 0 \qquad (2.2.9)$$

or the additional condition

$$\begin{vmatrix} \dot{x}(\tau) & \dot{p}_1(\tau) & 0 \\ 0 & \dot{q}_1(\tau) & \dot{y}(\tau) \\ R_1(\tau) & -f_1(\tau) & T_1(\tau) \end{vmatrix} = 0 , \qquad (2.2.10)$$

or the additional condition

$$\begin{vmatrix} \dot{x}(\tau) & \dot{y}(\tau) & \dot{p}_1(\tau) \\ 0 & \dot{x}(\tau) & \dot{q}_1(\tau) \\ R_1(\tau) & S_1(\tau) & -f_1(\tau) \end{vmatrix} = 0, \qquad (2.2.11)$$

taking that 3×3 determinant which contains a non-zero 2×2 determinant.

We have therefore proved that when the initial data are given along a characteristic, a solution of the Cauchy problem exists only if a further relation of the type (2.2.9), (2.2.10) or (2.2.11) holds along the characteristic. We may therefore define a characteristic of the equation (2.1.1) as a curve in the xy-plane along which the specification of consistent data is not sufficient to ensure a unique solution to the Cauchy problem.

2.3 Propagation of Weak Discontinuities.

We consider again the equation (2.1.1) in which R, S and T are continuous functions of x and y only and f is a continuous function of x, y, z, p and q.

Definition. A solution of the equation (2.1.1) whose first order partial derivatives are continuous but not all of whose second order derivatives are continuous is said to have a weak discontinuity.

Suppose a function $g(x, y)$ is uniquely defined at all points (x, y) of a region $\mathcal{D}$ except on a curve Γ in $\mathcal{D}$ with parametric equations $x = x(\tau)$, $y = y(\tau)$, see Fig. 2C.

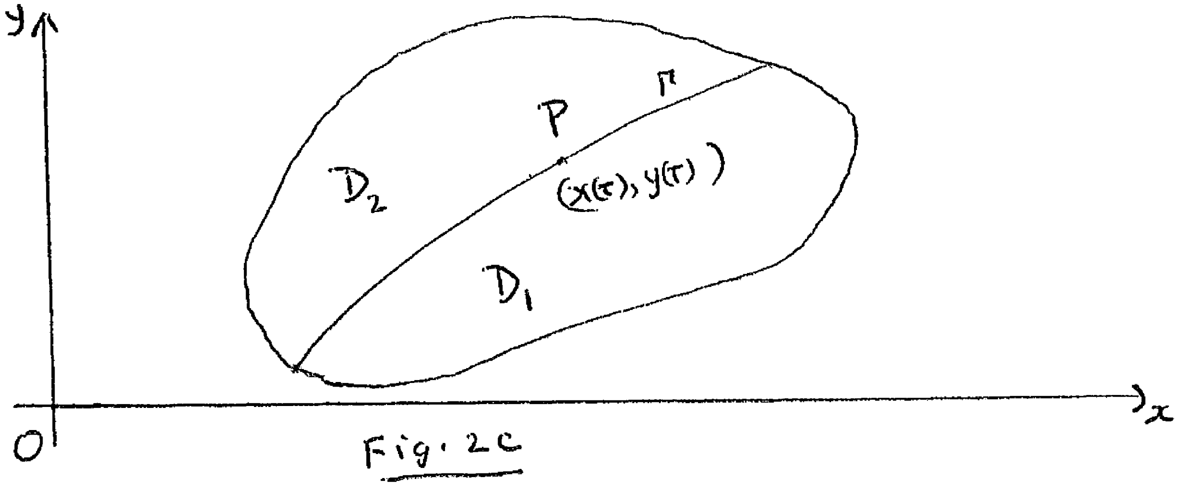


Fig. 2C

At a point $P(x(\tau), y(\tau))$ on Γ, define

$$g_1(x(\tau), y(\tau)) = \lim_{\substack{x_1 \to x(\tau) \\ y_1 \to y(\tau) \\ (x_1, y_1) \in \mathcal{D}_1}} g(x, y), \qquad g_2(x(\tau), y(\tau)) = \lim_{\substack{x_2 \to x(\tau) \\ y_2 \to y(\tau) \\ (x_2, y_2) \in \mathcal{D}_2}}$$

and $[g(\tau)] = g_2(x(\tau), y(\tau)) - g_1(x(\tau), y(\tau))$ where we suppose that both the limits g_1 and g_2 exist. $[g(\tau)]$ is called the saltus of the function g and is the jump in the function g as we cross the curve Γ from the side 1 to the side 2. If $[g(\tau)] = 0$ then g is continuous on Γ.

Suppose a solution $z(x, y)$ of the equation (2.1.1) has a weak discontinuity on the curve Γ, then $[z] = 0$, $[p] = 0$, $[q] = 0$ (2.3.1)

These equations can be written in the alternative forms

$$z_1(x(\tau),y(\tau)) = z_2(x(\tau),\ y(\tau)), \qquad (2.3.2)$$

$$p_1(x(\tau),y(\tau)) = p_2(x(\tau),\ y(\tau)), \qquad (2.3.3)$$

$$q_1(x(\tau),y(\tau)) = q_2(x(\tau),\ y(\tau)). \qquad (2.3.4)$$

Differentiating both sides of equation (2.3.3) with respect to τ we get

$$r_1(x(\tau),\ y(\tau))\dot{x}(\tau) + s_1(x(\tau),\ y(\tau))\dot{y}(\tau) = r_2(x(\tau),y(\tau))\dot{x}(\tau) + s_2(x(\tau),y(\tau))\dot{y}$$

which may be written in the form

$$[r]\ \dot{x}(\tau) + [s]\ \dot{y}(\tau) = 0 \qquad (2.3.5)$$

Similarly from the equation (2.3.4) we find that

$$[s]\ \dot{x}(\tau) + [t]\ \dot{y}(\tau) = 0 \qquad (2.3.6)$$

Since the functions R, S, T and f as well as z, p and q are continuous, equation (2.1.1) gives rise to the relation

$$R[r] + S[s] + T[t] = 0 \qquad (2.3.7)$$

The solution of the equations (2.3.5) and (2.3.6) is

$$[r] = \lambda\dot{y}^2,\quad [s] = -\lambda\dot{x}\dot{y},\quad [t] = \lambda\dot{x}^2, \qquad (2.3.8)$$

where λ is a non-zero constant. For this solution to satisfy equation (2.3.7) we must have equation

$$R\dot{y}^2 - S\dot{x}\dot{y} + T\dot{x}^2 = 0 \qquad (2.3.9)$$

satisfied on the curve Γ. Comparing equations (2.39) and (2.1.7) we conclude that, as far as equations of type (2.1.1) are concerned if weak discontinuities occur at all they must occur along the characteristics. If weak discontinuitieslutions of elliptic equations of type (2.1.1) cannot have weak discontinuities.

If we now assume that equation (2.1.1) is hyperbolic we can reduce it to the canonical form

$$\frac{\partial^2 z}{\partial x \partial y} = g(x, y, z, p, q). \tag{2.3.10}$$

Let us take Γ to be the characteristic $x(\tau) = \tau$, $y(\tau) = a$ where a is a constant. Then from equations (2.3.8) we have $[r] = 0$, $[s] = 0$, $[t] = \lambda$. Differentiating both sides of equation (2.3.10) with respect to y we get

$$\frac{\partial t}{\partial x} = \frac{\partial g}{\partial y} + q\frac{\partial g}{\partial z} + s\frac{\partial g}{\partial p} + t\frac{\partial g}{\partial q} .$$

Hence $\quad \frac{\partial}{\partial x}[t]_{y=c} = k(x, c)\,[t]_{y=c}$ where $k(x,c) = \left(\frac{\partial g}{\partial q}\right)_{y=c}$

and so $\quad [t]_{y=c} = A \exp\left(\int_0^x k(u, c)\, du\right)$.

From this result we deduce the following results:

(i) if $[t]$ is non-zero at one point of the characteristic Γ it is non-zero at every point of Γ

(ii) if $[t]$ is zero at one point of the characteristic Γ it is zero at all points of Γ.

2.4. Systems of Linear First Order simultaneous Equations.

2.4. Systems of Linear First Order simultaneous Equations.

$$\sum_{j=1}^{n} (P_{ij}p_j + Q_{ij}q_j) = f_i(x, y, z_1, z_2, \ldots, z_n), \quad (i = 1, 2, \ldots, n) \qquad (2.4.1)$$

where $p_j = \frac{\partial z_j}{\partial x}$, $q_j = \frac{\partial z_j}{\partial y}$ and P_{ij}, Q_{ij} are functions of x and y only.

The Cauchy problem for this system of equations is to determine $\underset{\sim}{z} = \{z_1(x, y), z_2(x, y), \ldots, z_n(x, y)\}$ given the value of $\underset{\sim}{z}$ on a curve Γ with parametric equations $x = x(\tau)$, $y = y(\tau)$.

Suppose on Γ, $z_j(x(\tau), y(\tau)) = \zeta_j(\tau)$. Then differentiating both sides of this equation with respect to τ we obtain

$$p_j + y' q_j = \zeta_j' \qquad (2.4.2)$$

where $y' = \frac{\dot{y}}{\dot{x}}$ and $\zeta_j' = \frac{\dot{\zeta}_j}{\dot{x}}$. Substituting from (2.4.2) into (2.4.1) we find that on the curve Γ,

$$\sum_{j=1}^{n} (Q_{ij} - y'P_{ij})q_j = f_i - \sum_{j=1}^{n} P_{ij}\zeta_j' \qquad (2.4.3)$$

Hence the Cauchy problem will have a unique solution if det $(Q_{ij} - y'P_{ij})$ does not vanish at any point of the curve Γ.

If $\det(Q_{ij} - y'P_{ij})$ does vanish, then the Cauchy problem will not have a unique solution. It follows from this that the Cauchy problem will not have a unique solution if the initial data are given on a curve Γ such that $y' = \lambda$ where λ is a root of the equation $\det(Q_{ij} - \lambda P_{ij}) = 0$. The curves Γ satisfying these conditions are called characteristics of the system (2.4.1).

For the set of equations (2.4.3) to be consistent on a characteristic Γ the augmented matrix must be of rank $n-1$. Hence the matrix

$$\begin{bmatrix} Q_{11} - \lambda P_{11} & Q_{12} - \lambda P_{12} & \cdots & Q_{1n} - \lambda P_{1n} & f_1 dx - \sum_{j=1}^{n} P_{1j}\, dz_j \\ Q_{21} - \lambda P_{21} & Q_{22} - \lambda P_{22} & \cdots & Q_{2n} - \lambda P_{2n} & f_2 dx - \sum_{j=1}^{n} P_{2j}\, dz_j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Q_{n1} - \lambda P_{n1} & Q_{n2} - \lambda P_{n2} & \cdots & Q_{nn} - \lambda P_{nn} & f_n dx - \sum_{j=1}^{n} P_{nj}\, dz_j \end{bmatrix}$$

must be of rank $n-1$.

Let us denote $\det(Q_{ij} - \lambda P_{ij})$ by Δ and the determinant obtained from Δ by replacing the i th column by the last column of the above matrix by Δ_i. Then along a characteristic the functions $z_1(x, y)$, $z_2(x, y), \ldots, z_n(x, y)$ are connected by the $(n+1)$ equations:

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$$\Delta = 0, \quad \Delta_1 = 0, \quad \Delta_2 = 0, \ldots, \Delta_n = 0 \qquad (2.4.4)$$

These relations are not independent since, from the theory of linear algebraic equations, they can be replaced by the two equations

$$\Delta = 0, \quad \Delta_k = 0 \qquad (2.4.5)$$

where the determinant Δ_k contains a non-zero $(n-1) \times (n-1)$ determinant and k is one of the numbers $1, 2, \ldots, n$.

The equation $\Delta = 0$ is called the equation of the direction of the characteristics and $\Delta_k = 0$ the differential expression of the characteristics

For reasons which are explained in [12], let us, for example, consider the simultaneous equations

$$\begin{aligned} \rho_1 \frac{\partial u}{\partial x_1} + \sigma_1 \frac{\partial u}{\partial x_2} &= f \\ \rho \frac{\partial v}{\partial x_1} + \sigma \frac{\partial v}{\partial x_2} &= g \end{aligned} \qquad (2.4.6)$$

where $\rho, \sigma, \rho_1, \sigma_1, u, v$ are functions of x_1 and x_2 only and ρ, ρ_1 do not vanish in the region under consideration.

Then $P_{11} = \rho_1$, $P_{12} = 0$ and $Q_{11} = \sigma_1$, $Q_{12} = 0$

$P_{21} = 0$, $P_{22} = \rho$ $\qquad$ $Q_{21} = 0$, $Q_{22} = \sigma$

Therefore the characteristics are given by

$$\begin{vmatrix} \sigma_1 - \lambda \cdot \rho_1 & 0 - \lambda \cdot 0 \\ 0 - \lambda \cdot 0 & \sigma - \lambda \cdot \rho \end{vmatrix} = 0$$

i.e. $\lambda = \frac{\sigma_1}{\rho_1}$ or $\lambda = \frac{\sigma}{\rho}$

Hence the differential equations of the characteristics are $\frac{dx_2}{dx_1} = \frac{\sigma}{\rho}$ and $\frac{dx_2}{dx_1} = \frac{\sigma_1}{\rho_1}$. Hence characteristics are given by the equations $\rho x_2 - \sigma x_1 = \text{constant}$ and $\rho_1 x_2 - \sigma_1 x_1 = \text{constant}$.

Let $\xi = \rho x_2 - \sigma x_1$, $\eta = \rho_1 x_2 - \sigma_1 x_1$. Then

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial \xi} \cdot (-\sigma) + \frac{\partial u}{\partial \eta} \cdot (-\sigma_1)$$

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial \xi} \cdot \rho + \frac{\partial u}{\partial \eta} \cdot \rho_1$$

$$\therefore \quad \rho_1 \frac{\partial u}{\partial x_1} + \sigma_1 \frac{\partial u}{\partial x_2} = (\rho\sigma_1 - \rho_1\sigma) \frac{\partial u}{\partial \xi} \qquad (2.4.7)$$

similarly $\rho \frac{\partial v}{\partial x_1} + \sigma \frac{\partial v}{\partial x_2} = (\sigma\rho_1 - \rho\sigma_1) \frac{\partial v}{\partial \eta}$ (2.4.8)

If the rank of the matrix $\begin{bmatrix} \rho & \sigma \\ \rho_1 & \sigma_1 \end{bmatrix}$ is 2, then $\rho_1\sigma - \rho\sigma_1 \neq 0$ and equations (2.4.6) reduce to

$$\frac{\partial u}{\partial \xi} = \frac{f}{\rho\sigma_1 \;\; \rho_1\sigma} \qquad (2.4.9)$$

$$\frac{\partial v}{\partial \eta} = - \frac{g}{\rho\sigma_1 \;\; \rho_1\sigma}$$

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It is shown in [12], that if the rank of the matrix

$$\begin{bmatrix} a^{11} & 2a^{12} & a^{22} \\ b^{11} & 2b^{12} & b^{22} \end{bmatrix}$$ is 2, then two simultaneous hyperbolic equations

$$L(u) = a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a^i \frac{\partial u}{\partial x_i} + cu = 0 \qquad (i, j = 1, 2)$$

$$M(v) = b^{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + b^i \frac{\partial v}{\partial x_i} + cv = 0 \qquad (i, j = 1, 2)$$

where $a^{ij} = a^{ji}$, $b^{ij} = b^{ji}$, a^i, b^i are functions of x_1, x_2 can be reduced to a pair of first order simultaneous equation of the type (2.4.9) which can be solved by integrating along the characteristics.

2.5. Characteristic Surfaces of Equation In Three Independent Variables

In this section we shall consider the linear equation

$$\sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad (2.5.1)$$

where a_{ij}, b_i and c are functions of x_1, x_2, x_3.

The Cauchy problem for this equation would ask us to find a solution $u = u(x_1, x_2, x_3)$ such that it and its normal derivative $\frac{\partial u}{\partial n}$ are prescribed on a surface S whose equation is given by

$$f(x_1, x_2, x_3) = 0. \qquad (2.5.2)$$

Suppose the freedom equations of S are $x_i = \bar{x}_i(\tau_1, \tau_2)$, $i = 1, 2, 3$. Then the boundary conditions on S may be written as

$$\bar{u} = F(\tau_1, \tau_2), \quad \frac{\partial \bar{u}}{\partial n} = G(\tau_1, \tau_2) \qquad (2.5.3)$$

On the surface S, equation (2.5.2) gives rise to

$$\sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \tau_1} d\tau_1 + \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \tau_2} d\tau_2 \right) = 0,$$ and since τ_1, τ_2 are arbitrary

we must have

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial \tau_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial \tau_1} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial \tau_1} = 0 \qquad (2.5.4)$$

$$\frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial \tau_2} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial \tau_2} + \frac{\partial f}{\partial x_3} \frac{\partial x_3}{\partial \tau_2} = 0 \qquad (2.5.5)$$

Solving for $\frac{\partial f}{\partial x_i}$, $i = 1, 2, 3$ we get

$$\frac{\frac{\partial f}{\partial x_1}}{\Delta_1} = \frac{\frac{\partial f}{\partial x_2}}{\Delta_2} = \frac{\frac{\partial f}{\partial x_3}}{\Delta_3} = \rho \qquad (2.5.6)$$

where $\rho \neq 0$, since otherwise $\frac{\partial f}{\partial x_i} = 0$ $(i = 1, 2, 3)$

and $\Delta_1 = \frac{\partial(x_2, x_3)}{\partial(\tau_1, \tau_2)}$, $\Delta_2 = \frac{\partial(x_3, x_1)}{\partial(\tau_1, \tau_2)}$, $\Delta_3 = \frac{\partial(x_1, x_2)}{\partial(\tau_1, \tau_2)}$

cannot all be zero.

Also on S,

$$d\bar{u} = \sum_{i=1}^{3} \left(\frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_1} \partial \tau_1 + \frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_2} d\tau_2 \right)$$

Hence $\displaystyle\sum_{i=1}^{3} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_1} = \frac{\partial F}{\partial \tau_1}, \quad \sum_{i=1}^{3} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_2} = \frac{\partial F}{\partial \tau_2}$.

From equation (2.5.2) we have

$\displaystyle\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} = 0$. But $\displaystyle\frac{\partial x_1}{\partial n}, \frac{\partial x_2}{\partial n}$ and $\displaystyle\frac{\partial x_3}{\partial n}$ are direction-cosines of the normal to the surface S at any of its points (x_1, x_2, x_3) and so $\displaystyle\frac{\partial x_1}{\partial n} = \lambda_1 \frac{\partial f}{\partial x_1}, \quad \frac{\partial x_2}{\partial n} = \lambda_1 \frac{\partial f}{\partial x_2}$ and $\displaystyle\frac{\partial x_3}{\partial n} = \lambda_1 \frac{\partial f}{\partial x_3}$.

Therefore

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial n} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial n} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial n}$$

$$= \lambda_1 \frac{\partial f}{\partial x_1} \frac{\partial u}{\partial x_1} + \lambda_1 \frac{\partial f}{\partial x_2} \frac{\partial u}{\partial x_2} + \lambda_1 \frac{\partial f}{\partial x_3} \frac{\partial u}{\partial x_3} .$$

$$\therefore \quad \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial u}{\partial x_i} = \frac{G}{\lambda_1}$$

But $\displaystyle\left(\frac{\partial x_1}{\partial n} \right)^2 + \left(\frac{\partial x_2}{\partial n} \right)^2 + \left(\frac{\partial x_3}{\partial n} \right)^2 = 1$

$$\therefore \quad \lambda_1 = \left\{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2\right\}^{-\frac{1}{2}}$$

It follows that the given conditions on the surface S are equivalent to the following three conditions

$$\sum_{i=1}^{3} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_1} = \frac{\partial F}{\partial \tau_1} \tag{2.5.7}$$

$$\sum_{i=1}^{3} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial x_i}{\partial \tau_2} = \frac{\partial F}{\partial \tau_2} \tag{2.5.8}$$

$$\sum_{i=1}^{3} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial f}{\partial x_i} = G \left\{\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\right)^2\right\}^{\frac{1}{2}} \tag{2.5.9}$$

Now

$$\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial x_1}{\partial \tau_1} & \frac{\partial x_2}{\partial \tau_1} & \frac{\partial x_3}{\partial \tau_1} \\ \frac{\partial x_1}{\partial \tau_2} & \frac{\partial x_2}{\partial \tau_2} & \frac{\partial x_3}{\partial \tau_2} \end{vmatrix} = \frac{\partial f}{\partial x_1} \cdot \Delta_1 + \frac{\partial f}{\partial x_2} \cdot \Delta_2 + \frac{\partial f}{\partial x_3} \cdot \Delta_3$$

$$= \rho(\Delta_1{}^2 + \Delta_2{}^2 + \Delta_3{}^2)$$

$$\neq 0, \text{ since } \rho \neq 0$$

and $\Delta_1, \Delta_2, \Delta_3$ are not all zero. It therefore follows from equations (2.5.7), (2.5.8) and (2.5.9) that $\frac{\partial \bar{u}}{\partial x_i}$, $i = 1, 2, 3$, are uniquely determined by the Cauchy data (2.5.3) when prescribed on the surface S.

We now attempt to determine second partial derivatives of u at all points of S. To do this we apply the above procedure to $\frac{\partial \bar{u}}{\partial x_1}$, $\frac{\partial \bar{u}}{\partial x_2}$ and $\frac{\partial \bar{u}}{\partial x_3}$ in succession as we have just applied to $\bar{u}$. Applying the method specifically to $\frac{\partial \bar{u}}{\partial x_1}$ we get

$$\sum_{r=1}^{3} \frac{\partial}{\partial x_r}\left(\frac{\partial \bar{u}}{\partial x_1}\right)\frac{\partial x_r}{\partial \tau_1} = \frac{\partial}{\partial \tau_1}\left(\frac{\partial \bar{u}}{\partial x_1}\right)$$

$$\sum_{r=1}^{3} \frac{\partial}{\partial x_r}\left(\frac{\partial \bar{u}}{\partial x_1}\right)\frac{\partial x_r}{\partial \tau_2} = \frac{\partial}{\partial \tau_2}\left(\frac{\partial \bar{u}}{\partial x_1}\right)$$

i.e. $$\frac{\partial^2 \bar{u}}{\partial x_1^2}\frac{\partial x_1}{\partial \tau_1} + \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2}\frac{\partial x_2}{\partial \tau_1} + \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_3}\frac{\partial x_3}{\partial \tau_1} = \frac{\partial}{\partial \tau_1}\left(\frac{\partial \bar{u}}{\partial x_1}\right) \qquad (2.5.10)$$

$$\frac{\partial^2 \bar{u}}{\partial x_1^2}\frac{\partial x_1}{\partial \tau_1} + \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2}\frac{\partial x_2}{\partial \tau_2} + \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_3}\frac{\partial x_3}{\partial \tau_2} = \frac{\partial}{\partial \tau_2}\left(\frac{\partial \bar{u}}{\partial x_1}\right) \qquad (2.5.11)$$

These two equations are not sufficient for the determination of $\frac{\partial^2 \bar{u}}{\partial x_1^2}$, $\frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 \bar{u}}{\partial x_1 \partial x_3}$. We therefore introduce *numerical* constants $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \frac{\partial^2 \bar{u}}{\partial x_1^2} + \alpha_2 \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2} + \alpha_3 \frac{\partial^2 \bar{u}}{\partial x_1 \partial x_3} = \gamma \qquad (2.5.12)$$

where γ is a parameter depending on α_1, α_2 and α_3 and in such a way that the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \frac{\partial x_1}{\partial \tau_1} & \frac{\partial x_2}{\partial \tau_1} & \frac{\partial x_3}{\partial \tau_1} \\ \frac{\partial x_1}{\partial \tau_2} & \frac{\partial x_2}{\partial \tau_2} & \frac{\partial x_3}{\partial \tau_2} \end{vmatrix} \text{ does not vanish}$$

on the surface S. Solving equations (2.5.10), (2.5.11) and (2.5.12) we get

$$\frac{\frac{\partial^2 \bar{u}}{\partial x_1^2}}{\Delta_1} = \frac{\frac{\partial^2 \bar{u}}{\partial x_1 \partial x_3}}{\Delta_2} = \frac{\frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2}}{\Delta_3} = \rho_1 \qquad (2.5.11)$$

Similarly applying the procedure to $\frac{\partial \bar{u}}{\partial x_2}$ and $\frac{\partial \bar{u}}{\partial x_3}$ separately we get

$$\frac{\frac{\partial^2 \bar{u}}{\partial x_2^2}}{\Delta_2} = \frac{\frac{\partial^2 \bar{u}}{\partial x_2 \partial x_3}}{\Delta_3} = \frac{\frac{\partial^2 \bar{u}}{\partial x_2 \partial x_1}}{\Delta_1} = \rho_2 \qquad (2.5.12)$$

$$\frac{\frac{\partial^2 \bar{u}}{\partial x_3^2}}{\Delta_3} = \frac{\frac{\partial^2 \bar{u}}{\partial x_3 \partial x_1}}{\Delta_1} = \frac{\frac{\partial^2 \bar{u}}{\partial x_3 \partial x_2}}{\Delta_2} = \rho_3 \qquad (2.5.13)$$

Hence $\frac{\rho_1}{\rho_2} = \frac{\Delta_1}{\Delta_2} = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$ by (2.5.6)

$\therefore \quad \rho_1 = \mu \frac{\partial f}{\partial x_1}, \quad \rho_2 = \mu \frac{\partial f}{\partial x_2}$ where μ is now a constant of proportionality. Similarly $\rho_3 = \frac{\partial f}{\partial x_3}$. It immediately follows from these equations and (2.5.6) that

$$\frac{\partial^2 \bar{u}}{\partial x_1{}^2} = \rho_1 \Delta_1 = \mu \Delta_1 \frac{\partial f}{\partial x_1} = \frac{\mu}{\rho} \left(\frac{\partial f}{\partial x_1} \right)^2 = \lambda \left(\frac{\partial f}{\partial x_1} \right)^2$$

where $\lambda = \frac{\mu}{\rho}$. Similarly

$$\frac{\partial^2 \bar{u}}{\partial x_1 \partial x_2} = \lambda \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2}, \quad \text{etc.}$$

Substituting from these equations into the equation (2.5.1) we get

$$\lambda \sum_{i,j=1}^{3} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum_{i=1}^{3} b_i p_i + cu = 0. \qquad (2.5.14)$$

We can therefore determine the value of λ from equation (2.5.14) provided that the function

$$\Phi(f) = \sum_{i,j=1}^{3} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \qquad (2.5.15)$$

does not vanish on the surface S. If this condition is satisfied then the second partial derivatives of $\bar{u}$ on S can be uniquely determined. By a repeated application of this procedure it can be shown that all the higher partial derivatives of $\bar{u}$ can be found provided the function $\Phi(f)$ does not vanish. Then by a Taylor's series expansion of $u = u(x_1, x_2, x_3)$ there exists in a neighbourhood of the surface S a unique solution of the equation (2.5.1) subject to the Cauchy data (2.5.3).

If, however, the surface (2.5.2) is such that $\Phi(f) = 0$, then λ cannot be found and so second and higher partial derivatives of $\bar{u}$ on the surface cannot be determined. It follows from this that the Cauchy problem cannot be uniquely determined on the surface (2.5.2) if $\Phi(f)=0$. The equation $\Phi(f) = 0$ is called the equation of characteristic surfaces and any surface satisfying this equation is a characteristic surface of the linear partial differential equation (2.5.1).

Jacobi's auxilliary equations of the first order partial differential equation $\Phi(f) = 0$ are

$$\frac{dx_1}{\Phi(f)_{fx_1}} = \frac{dx_2}{\Phi(f)_{fx_2}} = \frac{dx_3}{\Phi(f)_{fx_3}} = \frac{dfx_1}{-\Phi(f)_{x_1}} = \frac{dfx_2}{-\Phi(f)_{x_2}} = \frac{dfx_3}{-\Phi(f)_{x_3}}$$

The integrals of these equations satisfying correct initial conditions at a given point define lines called the bicharacteristics of the equation (2.5.1). These lines generate a surface called a conicoid, which reduces to a characteristic cone when the a_{ij}'s are constants.

Let us, for an example, consider the wave equation

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0 \qquad (2.5.16)$$

The condition for characteristic surface $\Phi(x, y, z) = 0$ is

$$\Phi(f) \equiv \phi_t^2 - \phi_x^2 - \phi_y^2 - \phi_z^2 = 0 \qquad (2.5.17)$$

This is a first order partial differential equation. Jacobi's auxilliary equations are

$$\frac{dx}{-2\phi_x} = \frac{dy}{-2\phi_y} = \frac{dz}{-2\phi_z} = \frac{d\phi_x}{0} = \frac{d\phi_y}{0} = \frac{d\phi_z}{0} = \frac{d\phi_t}{0}$$

$\therefore \quad d\phi_x = 0, \quad d\phi_y = 0, \quad d\phi_z = 0, \quad d\phi_t = 0$

Hence $u_t = \alpha_0$, $u_x = \alpha_1$, $u_y = \alpha_2$, $u_z = \alpha_3$ where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are constants

$\therefore \quad d\phi = \alpha_0 d_t + \alpha_1 d_x + \alpha_2 d_y + \alpha_3 d_z$

and so $\phi = \alpha_0 t + \alpha_1 x + \alpha_2 y + \alpha_3 z + \beta \qquad (2.5.18)$

where the condition (2.5.17) shows that

$$\alpha_0^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 .$$

The planes represented by the complete integral (2.5.18) therefore depend only on four parameters $\alpha_1, \alpha_2, \alpha_3$ and β. These planes are the characteristic surfaces and so also must be the envelope of the family of these planes and since they pass through a fixed point, say the point (x_0, y_0, z_0, t_0) in the three-dimensional space-time, we have

$$\phi = \alpha_0 t + \alpha_1 x + \alpha_2 y + \alpha_3 z + \beta = \alpha_0 t_0 + \alpha_1 x_0 + \alpha_2 y_0 + \alpha_3 z_0 + \beta.$$

Hence $\alpha_0(t-t_0) + \alpha_1(x-x_0) + \alpha_2(y-y_0) + \alpha_3(z-z_0) = 0.$

i.e.

$$\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\ (t-t_0) + \alpha_1(x-x_0) + \alpha_2(y-y_0) + \alpha_3(z-z_0) = 0.$$

Differentiating with respect to $\alpha_1, \alpha_2, \alpha_3$ in succession we get

$$\frac{\alpha_1}{\alpha_0}(t-t_0) + x - x_0 = 0, \quad \frac{\alpha_2}{\alpha_0}(t-t_0) + y - y_0 = 0 \quad \text{and}$$

$$\frac{\alpha_3}{\alpha_0}(t-t_0) + z - z_0 = 0. \quad \text{Hence}$$

$$(t-t_0)^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \qquad (2.5.19)$$

This shows that (2.5.19) is the equation of the surface of the characteristic cone through the point (x_0, y_0, z_0) at time t_0 of the three-dimensional wave equation (2.5.16).

The results we have just obtained above for the wave equation in three dimensions can easily be extended for the wave equation in spaces of higher dimension.

III. Riemann's Method of Solution for Equation with Two Independent Variables.

1. A description of Riemann's Method

Riemann's method was first used by Bernhard Riemann to solve the Cauchy problem for a second order linear partial differential equation of the hyperbolic type which, as shown in Section II, can be transformed into the equation

$$\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = f \qquad (3.1.1)$$

where a, b, c and f are given functions of x and y only. The Cauchy problem is to find a solution $u = u(x, y)$, when the values of u and of its first derivatives are prescribed on a curve C which has the property that no characteristic cuts it in more than one point.

Define operators L and M by

$$L(u) = \frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu$$

$$M(v) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv$$

Then $M(v) = 0$ is called the adjoint equation of (3.1.1).

$$v\,L(u) - u\,M(v) = v\frac{\partial^2 u}{\partial x\partial y} - u\frac{\partial^2 v}{\partial x\partial y} + \left(av\,\frac{\partial u}{\partial x} + u\,\frac{\partial av}{\partial x}\right) + \left(bv\,\frac{\partial u}{\partial y} + u\,\frac{\partial bv}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(v\,\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\,\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}(auv) + \frac{\partial}{\partial y}(buv).$$

Also $v\,L(u) - u\,M(v) = \frac{\partial}{\partial y}\left(v\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}(auv) + \frac{\partial}{\partial y}(buv)$. Hence

$$v\,L(u) - u\,M(v) = \tfrac{1}{2}\frac{\partial}{\partial x}\left\{v\,\frac{\partial u}{\partial y} - u\,\frac{\partial v}{\partial y} + 2a\,uv\right\} - \tfrac{1}{2}\frac{\partial}{\partial y}\left\{u\,\frac{\partial u}{\partial x} - v\,\frac{\partial u}{\partial x} - 2\,buv\right\} \quad (3.1.$$

The characteristics of the equation (3.1.1) are given by $dx\,dy = 0$ i.e. by the straight lines x = constant, y = constant. Suppose we wish to find a solution u of the equation (3.1.1) at the point $P(\xi,\eta)$ in the xy-plane given a curve C satisfying the stated condition above on which u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are prescribed. (See Figure 3a.)

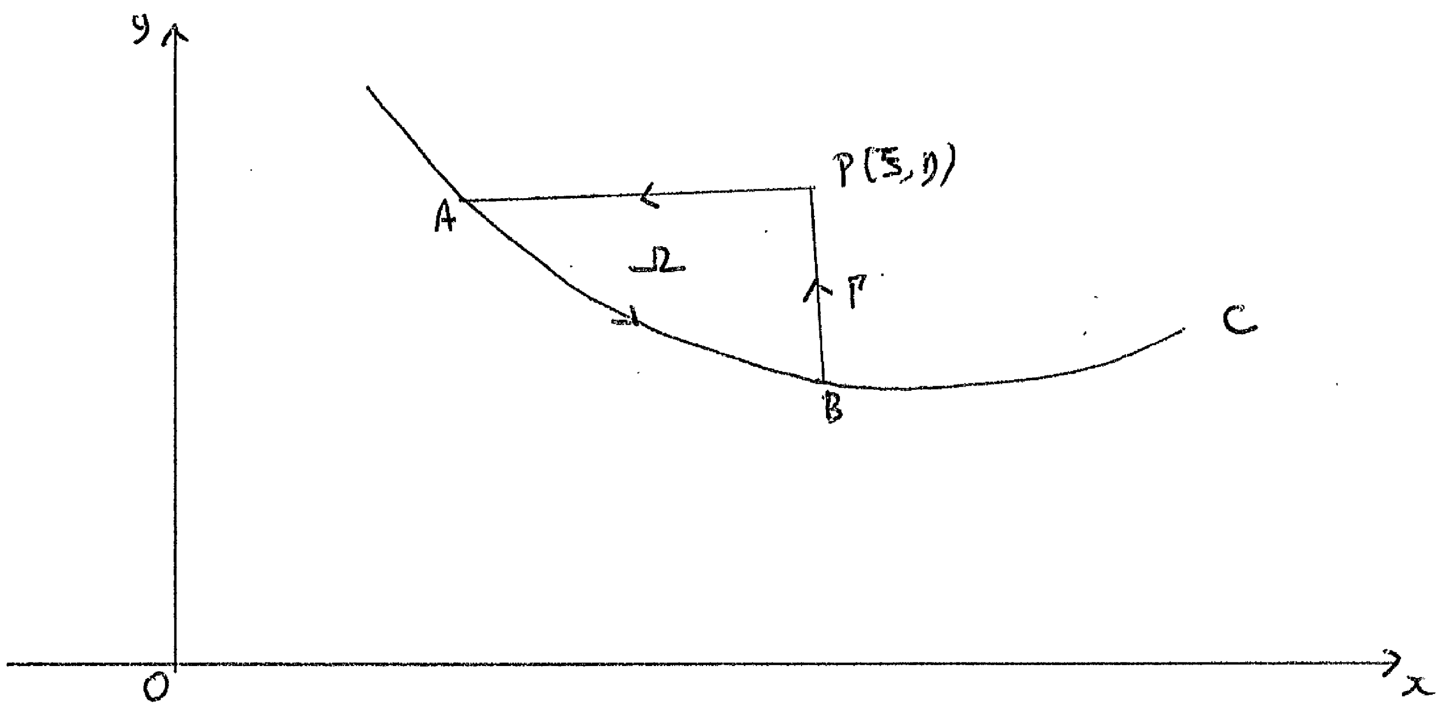

Fig. 3a.

Then PA, PB are portions of the characteristics through P cutting the curve C in the points A and B. Let Γ be the boundary of the region Ω enclosed by the lines PA, PB and the curve C. Let us assume that u satisfies the conditions of Green's theorem inside the region Ω. Then by (3.1.2) we have

$$\iint_{\Omega} \{v\ L(u) - u\ M(v)\}\,dx\,dy = \tfrac{1}{2}\int_{\Gamma} \left[\{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\}\,dy + \{u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} - 2buv\}\right.$$

where the integration around the contour Γ is taken in the anti-clockwise direction. If in the region Ω, $M(v) = 0$ then we must have

$$\tfrac{1}{2}\int \left\{ v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv \right\} dy + \left\{ u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} - 2buv \right\} dx = \iint fv\,dx\,dy. \quad (3.1.3)$$

Along PA, $dy = 0$ and $\displaystyle\int_{PA} \left\{ \tfrac{1}{2}u \frac{\partial v}{\partial x} - \tfrac{1}{2}v \frac{\partial u}{\partial x} - buv \right\} dx = - \frac{u(A)v(A)}{2} + \frac{u(P)v(P)}{2} +$

$$\int_{PA} u\left(\frac{\partial v}{\partial x} - bv\right)dx.$$

Along BP, $dx = 0$ and $\displaystyle\int_{BP} \left(\tfrac{1}{2}v \frac{\partial u}{\partial y} - \tfrac{1}{2}u \frac{\partial v}{\partial y} + auv\right) dy = \frac{u(P)v(P)}{2} - \frac{u(B)v(B)}{2} -$

$$\int_{BP} u\left(\frac{\partial v}{\partial y} - av\right)\,dy.$$

Now, following Riemann, we choose v to satisfy the equations:

$$M(v) = 0 \qquad (3.1.5)$$

$$\frac{\partial v}{\partial x} = bv \quad \text{on} \quad y = \eta, \qquad (3.1.6)$$

$$\frac{\partial v}{\partial y} = av \quad \text{on} \quad x = \xi, \qquad (3.1.7)$$

$$v = 1 \quad \text{at} \quad p. \qquad (3.1.8)$$

It follows from these equations that v is independent of the curve C carrying the Cauchy data and that

$$u(P) = \frac{u(A)\ v(A) + u(B)\ v(B)}{2} + \tfrac{1}{2}\int_C \left[(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv)dx - (v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv)dy \right] + \iint_\Omega fv\ dx\ dy. \qquad (3.1.9)$$

Equation (3.1.9) is therefore the integral representation of the solution of the equation (3.1.1) appropriate to the Cauchy data on the curve C provided that the function v, called the Riemann function, which satisfies conditions (3.1.5), (3.1.6), (3.1.7) and (3.1.8), exists and is unique. This is in fact the case.

2. Proof of the Existence of the Riemann's Solution.

Suppose that w is a solution of

$$L(w) = \frac{\partial^2 w}{\partial x \partial y} + a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} + cw = 0 \tag{3.2.1}$$

where w is defined in the rectangle PA DB whose sides DA, DB are the characteristics through $D(x_0, y_0)$ into which the curve AB of

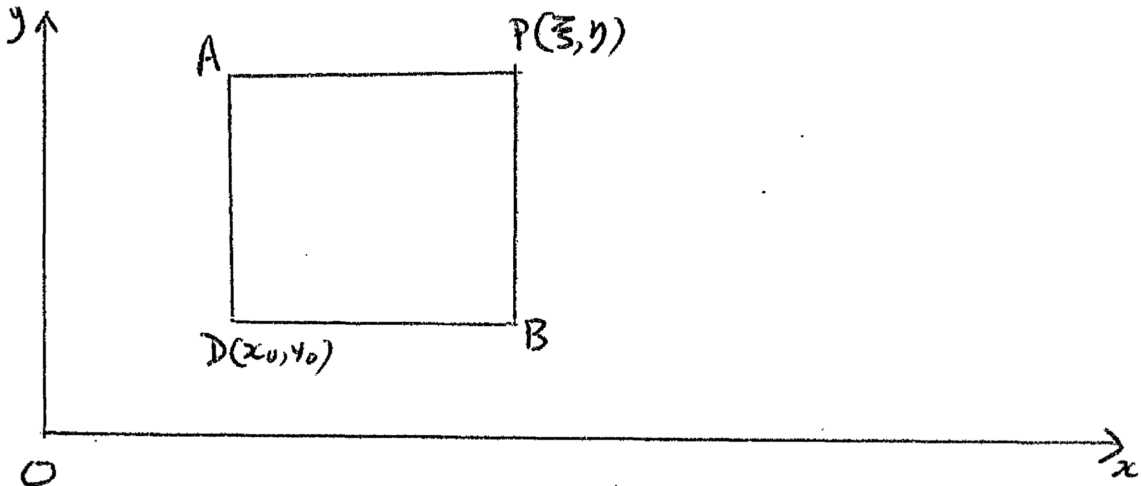

Fig. 3.5.

the previous section degenerates. Riemann's representation formula (3.1.9) then becomes

$$w(P) = \frac{w(A)\ v(A) + w(B)\ v(B)}{2} - \frac{1}{2}\int_{AD} \left(v\frac{\partial w}{\partial y} - w\frac{\partial v}{\partial y} + 2awv\right)dy$$

$$- \frac{1}{2}\int_{DB} \left(w\frac{\partial v}{\partial x} - v\frac{\partial w}{\partial x} - 2bwv\right)dx$$

$$= \frac{w(A)\ v(A) + w(B)\ v(B)}{2} + \tfrac{1}{2}[wv]_A^D - \tfrac{1}{2}[vw]_D^B - \int_{AD} v\ (\frac{\partial w}{\partial y} + aw)dy$$

$$+ \int_{DB} v(\frac{\partial w}{\partial x} + bw)dx$$

$$\text{i.e. } w(P) = w(D)\ v(D) - \int_{AD} v(\frac{\partial w}{\partial y} + aw)dy + \int_{DB} v(\frac{\partial w}{\partial x} + bw)dx \qquad (3.2.2)$$

Let w be the Riemann function of the adjoint equation $M(v) = 0$ with respect to the point $D(x_0, y_0)$ and so

$$w = w(x, y;\ \ x_0, y_0)$$

$$L(w) = 0$$

$$\frac{\partial w}{\partial y} + aw = 0$$

$$\frac{\partial w}{\partial x} + bw = 0$$

$$w(x_0, y_0; x_0, y_0) = 1$$

It follows from these equations and (3.2.2) that $w(\xi, \eta;\ \ x_0, y_0) = v(x_0, y_0; \xi, \eta)$ and so v considered as a function of ξ, η satisfies

$$\frac{\partial^2 v}{\partial\xi\partial\eta} + a(\xi, \eta)\ \frac{\partial v}{\partial\xi} + b(\xi, \eta)\ \frac{\partial v}{\partial\eta} + cv = 0 \qquad (3.2.3)$$

To prove the existence of a solution of the equation (3.1.1) subject to the Cauchy data it is sufficient to establish the existence of a solution of the equation (3.1.1) under the conditions that on the curve $y = \mu(x), |\mu'(x)| < 1$, there exists a function w such that w together with its first-order derivatives vanish on the curve C. This is attained by the relation

$$w = u - \varphi(x) - [y - \mu(x)]\,\psi(x) \text{ where } u = \varphi(x) \text{ and}$$

$\frac{\partial u}{\partial y} = \psi(x)$ are the given values on the curve C.

Hence $u(x, y) = w(x, y) + \varphi(x) + [y - \mu(x)]\,\psi(x)$ would be the solution of the equation (3.1.1) satisfying the above conditions. In the case when $w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}$ vanish on the curve C, Riemann's representation formula (3.1.9) reduces to

$$w(\xi, \eta) = \iint_{\Omega} vf(x, y)\,dx\,dy$$

$$= \int_{x=\mu^{-1}(y)}^{\xi} \left(\int_{BP} v(x, y; \xi, \eta)\, f(x, y) dy \right) dx$$

$$\frac{\partial w}{\partial \xi} = \int_{BP} v(\xi, y; \xi, \eta)\, f(\xi, y) dy + \iint_{\Omega} \frac{\partial v}{\partial \xi} f(x, y)\,dx\,dy \quad (3.2.4)$$

Similarly $$\frac{\partial w}{\partial \eta} = \int_{AP} v(x, \eta; \xi, \eta)\, f(x, \eta) dx + \iint_{\Omega} \frac{\partial v}{\partial \eta} f(x, y)\,dx\,dy \quad (3.2.5)$$

when (ξ, η) is a point on the curve C, $\Omega = 0$, B and P coincides and A and P coincide and so on the curve C,

$$w(\xi, \eta) = 0$$

$$\frac{\partial w}{\partial \xi} = 0$$

$$\frac{\partial w}{\partial \eta} = 0$$

Also by (3.2.4),

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = v(\xi, \eta; \xi, \eta)\ f(\xi, \eta) + \int_{BP} \frac{\partial v}{\partial \eta} f(\xi, y)dy + \int_{AP} \frac{\partial v}{\partial \xi} f(x, \eta)dx$$

$$+ \iint_{\Omega} \frac{\partial^2 v}{\partial \xi \partial \eta} f(x, y)dx\ dy.$$

Hence,

$$\frac{\partial^2 w}{\partial \xi \partial \eta} + a(\xi, \eta)\frac{\partial w}{\partial \xi} + b(\xi, \eta)\frac{\partial w}{\partial \eta} + c(\xi, \eta)w$$

$$= f(\xi, \eta) + \int_{BP}\left\{\frac{\partial v}{\partial \eta} + av(\xi, \eta)\right\} f(\xi, y)dy + \int_{AP}\left\{\frac{\partial v}{\partial \xi} + bv(x, \eta)\right\} f(x, y)dx$$

$$+ \iint_{\Omega}\left(\frac{\partial^2 v}{\partial \xi \partial \eta} + a(\xi, \eta)\frac{\partial v}{\partial \xi} + b(\xi, \eta)\frac{\partial v}{\partial \eta} + cv\right) f(x, y)dx\ dy$$

$$= f(\xi, \eta).$$

Hence a solution $w(\xi, \eta) = \iint_\Omega V(x, y; \xi, \eta) f(x, y)\, dx\, dy$ of equation (3.1.1) satisfying the Cauchy data

$$w(\xi, \eta) = 0, \quad \frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial \eta} = 0 \quad \text{on the curve} \quad y = \mu(x) \text{ exists.}$$

It therefore follows that

$$w(\xi, \eta) = \varphi(\xi) + [\eta - \mu(\xi)]\psi(\xi) + \iint_\Omega v(x, y; \xi, \eta) f(x, y)\, dx\, dy$$

is a solution of

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + a \frac{\partial u}{\partial \xi} + b \frac{\partial u}{\partial \eta} + cu = f(\xi, \eta)$$

satisfying on the given curve $y = \mu(x)$ the conditions

$$u = \phi(\xi), \quad \frac{\partial u}{\partial n} = \psi(\xi), \quad \frac{\partial u}{\partial \xi} = \phi'(\xi) - \mu'(\xi)\,\psi(\xi).$$

This proves the assertion that a solution of equation (3.1.1) satisfying the given Cauchy data exists and is unique.

3. Evaluation of the Riemann Function

As a first example, let us consider the equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \qquad (3.3.1)$$

Characteristics are given by $x = \text{const.}$, $y = \text{const.}$ The Riemann function v satisfies, see Figure 3C,

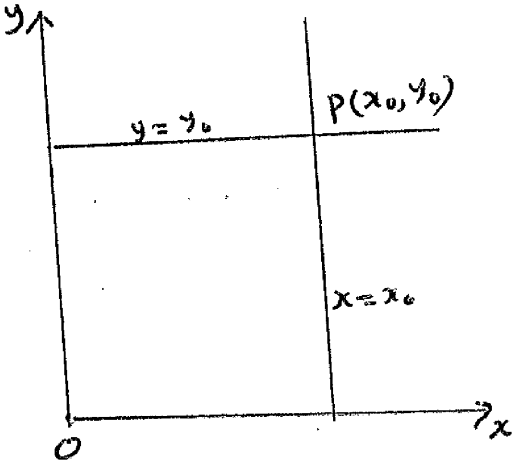

Fig. 3C

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial v}{\partial x} = 2v \quad \text{on} \quad y = y_0,$$

$$\frac{\partial v}{\partial y} = -v \quad \text{on } x = x_0,$$

$$v(x_0, y_0; x_0, y_0) = 1.$$

It therefore follows that $v = e^{2(x-x_0)}$ on $y = y_0$ and $v = e^{-(y-y_0)}$ on $x = x_0$. We may therefore try $v = e^{2(x-x_0) - (y-y_0)}F(s)$ where $s = (x-x_0)(y-y_0)$. Then from the above adjoint equation, $F(s)$ satisfies the differential equation

$$\frac{d^2F}{ds^2} + \frac{1}{s}\frac{dF}{ds} + \frac{2}{s}F(s) = 0.$$

Let $s = t^2$. Then $F(t)$ satisfies the differential equation

$$\frac{d^2F}{dt^2} + \frac{1}{t}\frac{dF}{dt} + 8F(t) = 0.$$

This is Bessel's differential equation of order 0 with solution $F(t) = J_0(2\sqrt{2}\ t)$. Hence the Riemann function is given by

$$v(x, y; x_0, y_0) = e^{2(x-x_0)-(y-y_0)}J_0(2\sqrt{2(x-x_0)(y-y_0)}). \tag{3.3.2}$$

As a second example, let us take the equation of damped waves

$$\frac{\partial^2 u}{\partial x \partial y} - u = 0 \tag{3.3.3}$$

The Riemann function v satisfies

$$\frac{\partial^2 u}{\partial x \partial y} - v = 0,$$

$$\frac{\partial v}{\partial x} = 0 \text{ on } y = y_0,$$

$$\frac{\partial v}{\partial y} = 0 \text{ on } x = x_0,$$

$$v(x_0,\ y_0;\ x_0,\ y_0) = 1.$$

It therefore follows that $v = 1$ on $x = x_0$ and on $y = y_0$. We may therefore try $v = F(s)$ where $s = (x-x_0)(y-y_0)$. Then $F(s)$ satisfies the differential equation

$$\frac{d^2F}{ds^2} + \frac{1}{s}\frac{dF}{ds} - \frac{1}{s}F = 0.$$

Let $s = t^2$. Then $F(t)$ satisfies the differential equation

$$\frac{d^2F}{dt^2} + \frac{1}{t}\frac{dF}{dt} - 4F = 0.$$

This is Bessel's differential equation of order 0 of imaginary argument with solution $F(t) = I_0(2t)$ where $I_0(t) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}t)^{2m}}{(m!)^2}$.

Hence the Riemann function of the equation (3.3.3) is

$$v(x,\ y;\ \ x_0,\ y_0) = I_0(2\sqrt{(x-x_0)(y-y_0)}) \qquad (3.3.4)$$

It is interesting to compare this elementary method with the transform method applied by Titchmarsh [13] to the same equation.

Let $X = x - x_0$, $Y = y - y_0$ then the equation (3.3.3) becomes

$$\frac{\partial^2 u}{\partial Y \partial Y} - u = 0,$$ and the Riemann function v now satisfies

$\frac{\partial^2 v}{\partial X \partial Y} - v = 0$ and $v = 1$ on $X = 0$ and on $Y = 0$.

Let $$\bar{v}\,(\zeta, Y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty v(X, Y) e^{i\zeta X} dX,$$

where $\zeta = \xi + i\eta$ and $\eta > 0$. Then

$$\frac{\partial \bar{v}}{\partial Y} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\partial v}{\partial Y}\, e^{i\zeta X} dX = \frac{1}{\sqrt{2\pi}} \left[\frac{\partial v}{\partial Y} \frac{e^{i\zeta X}}{i\zeta} \right]_0^\infty - \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\partial^2 v}{\partial X \partial Y} \frac{e^{i\zeta X}}{i\zeta}\, dX$$

But $\frac{\partial v}{\partial Y} = 0$ on $X = 0$ and so

$$\frac{\partial \bar{v}}{\partial Y} = - \frac{1}{i\zeta\sqrt{2\pi}} \int_0^\infty v\, e^{i\zeta X} dX = - \frac{1}{i\zeta}\, \bar{v}\,.$$

Hence $\bar{v}(\zeta, Y) = A(\zeta) e^{\frac{iY}{\zeta}}$. It follows from this that

$$A(\zeta) = \bar{v}(\zeta, 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty v(0, Y) e^{i\zeta X} dX$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\zeta X} dX = - \frac{1}{i\zeta\sqrt{2\pi}}$$

Hence $\bar{v}(\zeta, Y) = -\frac{1}{i\zeta\sqrt{2\pi}} e^{\frac{iY}{\zeta}}.$

Therefore $v(X, Y) = -\frac{1}{2\pi i}\int_{i\gamma-\infty}^{i\gamma+\infty} e^{\frac{iY}{\zeta}} e^{-i\zeta X} \frac{d\zeta}{\zeta} = I_0(2\sqrt{XY})$

i.e. $v(x, y; x_0, y_0) = I_0(2\sqrt{(x-x_0)(y-y_0)}).$

We shall in the remainder of this section evaluate by three different methods the Riemann function of the Euler-Poisson equation

$$\frac{\partial^2 u}{\partial x\partial y} + \frac{n}{x+y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \qquad (3.3.5)$$

where n is a constant.

Method 1. The characteristics of the equation (3.3.5) through the point $P(\xi, \eta)$ are those shown in Fig. 3d.

The Riemann function $v(x, y; \xi, \eta)$ satisfies

$$\frac{\partial^2 v}{\partial x\partial y} - \frac{\partial}{\partial x}\left(\frac{nv}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{nv}{x+y}\right) = 0 \quad \text{i.e.}$$

$$\frac{\partial^2 v}{\partial x\partial y} - \frac{n}{x+y}\frac{\partial v}{\partial x} - \frac{n}{x+y}\frac{\partial v}{\partial y} + \frac{2nv}{(x+y)^2} = 0 \qquad (3.3.6a)$$

$$\frac{\partial v}{\partial x} = \frac{nv}{x+y} \text{ on } y = \eta \qquad (3.3.6b)$$

$$\frac{\partial v}{\partial y} = \frac{nv}{x+y} \text{ on } x = \xi \qquad (3.3.6c)$$

$$v(\xi, \eta; \xi, \eta) = 1 \qquad (3.3.6d)$$

By (3.3.6b) integrating along the characteristic $y = \eta$ we get $\frac{dv}{v} = \frac{n\,dx}{x+\eta}$ from which it follows that

$$v = A(\xi, \eta)(x + \eta)^n.$$

Similarly $v = A(\xi, \eta)(\xi + y)^n$. Then by (3.3.6d) we get

$$A(\xi,\eta) = \frac{1}{(\xi+y)^n}.$$ We therefore write

$$v(x, y); \ \xi, \eta) = \frac{(x+y)^n}{(\xi+\eta)^n} F(x, y; \xi, \eta) \tag{3.3.7}$$

where F is to be determined. Substituting from (3.3.7) into (3.3.6a) we find, after simplification, that $(x+y)^2 \frac{\partial^2 F}{\partial x \partial y} + n(1-n) F = 0$ (3.3

Let $F(x, y; \xi, \eta) = F(\mu)$ where $\mu = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)}$. Then (3.3.8) becomes

$$(x+y)^2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{d^2 F}{d\mu^2} + (x+y)^2 \frac{\partial^2 \mu}{\partial x \partial y} \frac{dF}{d\mu} + n(1-n)F = 0.$$

Now $\frac{\partial \mu}{\partial x} = -\frac{y-\eta}{(x+y)(\xi+\eta)} + \frac{(x-\xi)(y-\eta)}{(x+y)^2(\xi+\eta)}$

$$\frac{\partial \mu}{\partial y} = -\frac{x-\xi}{(x+y)(\xi+\eta)} + \frac{(x-\xi)(y-\eta)}{(x+y)^2(\xi+\eta)}$$

$$\therefore \quad (x+y) \frac{\partial \mu}{\partial x} = -\frac{y-\eta}{\xi+\eta} - \mu$$

$$(x+y) \frac{\partial \mu}{\partial y} = -\frac{x-\xi}{\xi+\eta} - \mu$$

It follows from this that

$$(x+y)^2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} = \mu^2 - \mu$$

Similarly

$$(x+y)^2 \frac{\partial^2 \mu}{\partial x \partial y} = 2\mu - 1$$

The equation then reduces to

$$\mu(\mu-1) \frac{d^2 F}{d\mu^2} + (2\mu-1) \frac{dF}{d\mu} + n(1-n)F = 0. \qquad (3.3.9)$$

This is the hypergeometric equation whose solution is given by

$F(\mu) = {}_2F_1(1-n,\ n;\ 1;\ \mu)$. Hence the Riemann function of the equation is given by

$$v(x,\ y;\ \xi,\ \eta) = \left(\frac{x+y}{\xi+\eta} \right) {}_2F_1(1-n,\ n;\ 1;\ \mu), \qquad (3.3.10)$$

where $\mu = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)}$.

Further, if n is a positive integer and we let $x = 1-2\mu$ then it follows from (3.3.9) that

$$(1-x^2) \frac{d^2 F}{dx^2} - 2x \frac{dF}{dx} + (n-1)nF = 0$$

This is the Legendre differential equation of order $(n-1)$ with solution

$$F(x) = P_{n-1}(x) = P_{n-1}(1-2\mu).$$

The Riemann function, for integral values of n, then becomes

$$v(x,\ y;\ \xi,\ \eta) = \left(\frac{x+y}{\xi+\eta} \right)^n P_{n-1}(1-2\mu) \qquad (3.3.11)$$

where P_{n-1} is the Legendre polynomial of degrees $n-1$.

The second method we shall study for the construction of the Riemann function applies only to those equations which can be solved by the method of separation of variables. We shall describe the method and apply it again to the equation (3.3.5).

Suppose that we are given the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2a\,\frac{\partial u}{\partial x} - 2b\,\frac{\partial u}{\partial y} + (p-q)u = 0$$

The method of separation of variables then gives

$$\frac{d^2\psi}{dx^2} + 2a\frac{d\psi}{dx} + (p+\lambda^2)\psi = 0 \qquad (3.3.12)$$

and

$$\frac{d^2\phi}{dy^2} + 2b\frac{d\phi}{dy} + (q+\lambda^2)\phi = 0, \qquad (3.3.13)$$

where λ^2 is the separation constant. Let $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ be linearly independent solutions of (3.3.12) and let

$$W' = \begin{vmatrix} \psi_1(x) & \psi_2(x) \\ \dot{\psi}_1(x) & \dot{\psi}_2(x) \end{vmatrix},$$ the Wronskian of the two solutions, and

where dots denote differentiation with respect to x.

Let $\phi_1(y, \lambda)$ and $\phi_2(y, \lambda)$ be linearly independent solutions of (3.3.13) and let

$$W'' = \begin{vmatrix} \phi_1(y) & \phi_2(y) \\ \dot{\phi}_1(y) & \dot{\phi}_2(y) \end{vmatrix}$$ be their Wronskian.

Let us consider the integral equation

$$u(x, y) = \int_L \{f_1(\lambda)\psi_1(x, \lambda) + f_2(\lambda)\psi_2(x, \lambda)\}\phi_1(y, \lambda)\, d\lambda \qquad (3.3.14)$$

where the range of integration L will be determined by the nature of the problem. The problem of Cauchy which we shall solve is to determine (3.3.14) such that

$u = 0,\ \frac{\partial u}{\partial x} = F(y)$ when $x = x_0$. Hence

$$\int_L \{f_1(\lambda)\psi_1(x_0, \lambda) + f_2(\lambda)\psi_2(x_0, \lambda)\}\phi_1(y, \lambda)d\lambda = 0$$

$$\int_L \{f_1(\lambda)\dot{\psi}_1(x_0, \lambda) + f_2(\lambda)\dot{\psi}_2(x_0, \lambda)\}\phi_1(y, \lambda)d\lambda = F(y)$$

Let us, apart from an arbitrary constant, suppose that

$$F(y) = \int_{L'} f(\lambda)\, \phi_1(y, \lambda)d\lambda \qquad (3.3.15)$$

Hence we may assume that

$$f_1(\lambda)\psi_1(x_0, \lambda) + f_2(\lambda)\psi_2(x_0, \lambda) = 0 \qquad (3.3.16)$$

$$f_1(\lambda)\dot{\psi}_1(x_0, \lambda) + f_2(\lambda)\dot{\psi}_2(x_0, \lambda) = f(\lambda) \qquad (3.3.17)$$

where $f(\lambda)$ is the integral solution of the equation (3.3.15) which we may write apart from an arbitrary constant as

$$f(\lambda) = \int_{L'} F(y)\, \bar{\phi}_1(y, \lambda)dy \qquad (3.3.18)$$

Solving equations (3.3.16) and (3.3.17) we get

$$\frac{f_1(\lambda)}{\begin{vmatrix} 0 & \psi_2(x_0, \lambda) \\ f(\lambda) & \dot{\psi}_2(x_0, \lambda) \end{vmatrix}} = \frac{f_2(\lambda)}{\begin{vmatrix} \psi_1(x_0 \ \ \lambda) & 0 \\ \dot{\psi}_1(x_0 \ \ \lambda) & f(\lambda) \end{vmatrix}} = \frac{1}{w'(x_0, \lambda)}$$

i.e. $f_1(\lambda) = -\dfrac{f(\lambda)\psi_2(x_0, \lambda)}{w'(x_0, \lambda)}$, $f_2(\lambda) = \dfrac{f(\lambda)\psi_1(x_0, \lambda)}{w'(x_0, \lambda)}$.

It follows from (3.3.14) and (3.3.18) that

$$U(X, Y) = \int_L \frac{f(\lambda)}{w'(x_0, \lambda)} \left\{ -\psi_2(x_0, \lambda)\, \psi(X, \lambda) + \psi_1(x_0, \lambda)\, \psi_2(X, \lambda) \right\} \phi_1(Y, \lambda) d\lambda$$

$$= \int_L \left(\int_{L'} f(y)\, \bar{\phi}_1(y, \lambda) dy \right) \left\{ \frac{\psi_1(x_0, \lambda)\, \psi_2(X, \lambda) - \psi_2(x_0, \lambda)\, \psi(X, \lambda)}{w'(x_0, \lambda)} \right\} \phi_1(Y$$

$$= \int_L \int_{L'} \frac{f(y)\phi_1(Y, \lambda)\phi_1(y, \lambda)}{w'(x_0, \lambda)} \left\{ \psi_1(x_0, \lambda)\, \psi_2(X, \lambda) - \psi_2(x_0, \lambda)\, \psi_1(X, \lambda) \right\} dy$$

(3.3.19)

But from a result similar to (3.1.9) we find that

$$U(X, Y) = \tfrac{1}{2} \int_{Y-X+x_0}^{Y+X-x_0} F(y) v(x_0, y; X, Y) dy.$$

Comparing these two results we get

$$v(x, y; X, Y) = \pm 2 \int_L \frac{\phi_1(Y, \lambda)\, \bar{\phi}_1(y, \lambda)}{w'(x, \lambda)} \left\{ \psi_1(x, \lambda)\, \psi_2(X, \lambda) - \psi_2(x, \lambda)\psi_1(X, \lambda) \right\} d$$

(3.3.20)

This result holds when $Y - X + x_0 \leq y \leq Y + X - x_0$. The positive sign is taken if $X > x$ and the negative sign if $X < x$. If y is outside this interval, the integral vanishes.

Let us consider again the equation (3.3.5) whose Riemann function has been derived in the form (3.3.11). Letting $x = \xi + \eta$, $y = \xi - \eta$ this equation reduces to

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + \frac{\eta}{\xi}\frac{\partial u}{\partial \xi} = 0.$$

Instead of (3.3.5) we shall consider the transformed equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} = 0 \quad \text{where} \quad n = 2\alpha. \qquad (3.3.21)$$

Let $u = \psi(x)\ (y)$. Then associated equations of (3.3.21) are

$$\ddot{\psi} + \frac{2\alpha}{x}\dot{\psi} + \lambda^2\psi = 0, \quad \ddot{\phi} + \lambda^2\phi = 0$$

where λ^2 is the separation constant.

Let $\psi = x^{\frac{1}{2}-\alpha}R(x)$. Then R satisfies $R'' + \frac{1}{x}R' + \left(\lambda^2 - \frac{(\frac{1}{2}-\alpha)^2}{x^2}\right)R = 0$

This shows that we can take as a first pair of fundamental solutions

$$\psi_1(x, \lambda) = x^{\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}}(\lambda x), \quad \psi_2(x, \lambda) = x^{\frac{1}{2}-\alpha}J_{\frac{1}{2}-\alpha}(\lambda x). \quad \text{Hence}$$

$$w'(x, \lambda) = \begin{vmatrix} x^{\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}} & x^{\frac{1}{2}-\alpha}J_{\frac{1}{2}-\alpha} \\ (\frac{1}{2}-\alpha)x^{-\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}} + \lambda x^{\frac{1}{2}-\alpha}J'_{\alpha-\frac{1}{2}} & (\frac{1}{2}-\alpha)x^{-\frac{1}{2}-\alpha}J_{\frac{1}{2}-\alpha} + \lambda x^{\frac{1}{2}-\alpha}J'_{\frac{1}{2}-\alpha} \end{vmatrix}$$

$$= \lambda x^{1-2\alpha}\left(J_{\alpha-\frac{1}{2}}\, J'_{\frac{1}{2}-\alpha} - J_{\frac{1}{2}-\alpha} J'_{\alpha-\frac{1}{2}}\right)$$

$$= \lambda x^{1-2\alpha}\ \frac{-2\sin(\alpha-\frac{1}{2})\pi}{\pi\ \lambda\ x} = \frac{2\cos\alpha\pi}{x^{2\alpha}}$$

Let us assume that $\alpha - \frac{1}{2}$ is not an integer or zero. Then

$$w'(x, \lambda) = \frac{2\cos\alpha\pi}{x^{2\alpha}} \text{ does not vanish.}$$

From the second equation we have

$$\phi_1(y, \lambda) = e^{i\lambda y}, \quad \phi_2(y, \lambda) = e^{-i\lambda y}, \quad w'(y, \lambda) = \begin{vmatrix} e^{i\lambda y} & e^{-i\lambda y} \\ i\lambda e^{i\lambda y} & -i\lambda e^{-i\lambda y} \end{vmatrix} = -2$$

If in (3.3.15) we take

$$F(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)\,\phi_1(y, \lambda)\, d\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda) e^{i\lambda y} d\lambda,$$

then by Fourier inversion theorem we have that

$$f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(y) e^{-i\lambda y} dy.$$

It follows from (3.3.20) that

$$v(x; y; X, Y) = \pm\frac{x^{\alpha-\frac{1}{2}} X^{\frac{1}{2}-\alpha}}{\sqrt{2\pi}\cos\alpha\pi}\int_{-\infty}^{\infty} e^{i\lambda(Y-y)}\left\{J_{\alpha-\frac{1}{2}}(\lambda x)\, J_{\frac{1}{2}-\alpha}(\lambda X) - J_{\alpha-\frac{1}{2}}(\lambda X)\, J_{\frac{1}{2}-\alpha}(\lambda x)\right\} d$$

(3.3.22)

We shall content ourselves with this particular solution; for other solutions and the evaluation of the integral occurring in the solution one may consult [13].

Let us consider again the equation (3.3.5). The Riemann function $v(x, y; x_0, y_0)$ satisfies, according to (3.2.3), the equation

$$\frac{\partial^2 u}{\partial x_0 \, \partial y_0} + \frac{n}{x_0+y_0}\left(\frac{\partial u}{\partial x_0} + \frac{\partial u}{\partial y_0}\right) = 0. \qquad (3.3.23)$$

We notice that $w_1 = (z - x_0)^{-n}(z + y_0)^{-n}$, where z is a complex number, is a particular solution of (3.2.23). We further notice that $w_2 = (x + y)(z - x)^{n-1}(z + y)^{n-1}$ is a particular solution of the adjoint equation (3.3.6a). We may therefore take as the Riemann function, the function

$$v(x, y; x_0, y_0) = \frac{x + y}{2 \pi i} \int_C \frac{(z-x)^{n-1} (z+y)^{n-1}}{(z-x_0)^n \, (z+y_0)^n} f(z) dz \qquad (3.3.24)$$

where $f(z)$ has to be determined by the boundary conditions (3.3.6b), (3.3.6d) i.e. as we have already shown by

$$v(x_0, y; x_0, y_0) = \frac{(x_0+y)^n}{(x_0+y_0)^n} \qquad (3.3.25a)$$

$$v(x, y_0; x_0, y_0) = \frac{(x + y_0)^n}{(x_0 + y_0)^n} \qquad (3.3.25b)$$

For non-integral values of n, the integrand in (3.2.24) has branch points at x, x_0, $-y$, $-y_0$. Let us assume that x, x_0, y, y_0 are all positive. Join the points $-y$ and $-y_0$ by a cut and x and x_0 by another cut. Then if $f(z)$ is analytic in the whole z-plane, the integrand is analytic in the z-plane cut in this way. Let the contour C_1 surround the cut on the right and C_2 the cut on the left of the Figure 3d. Let us also assume that C_1 and C_2 are in opposite directions.

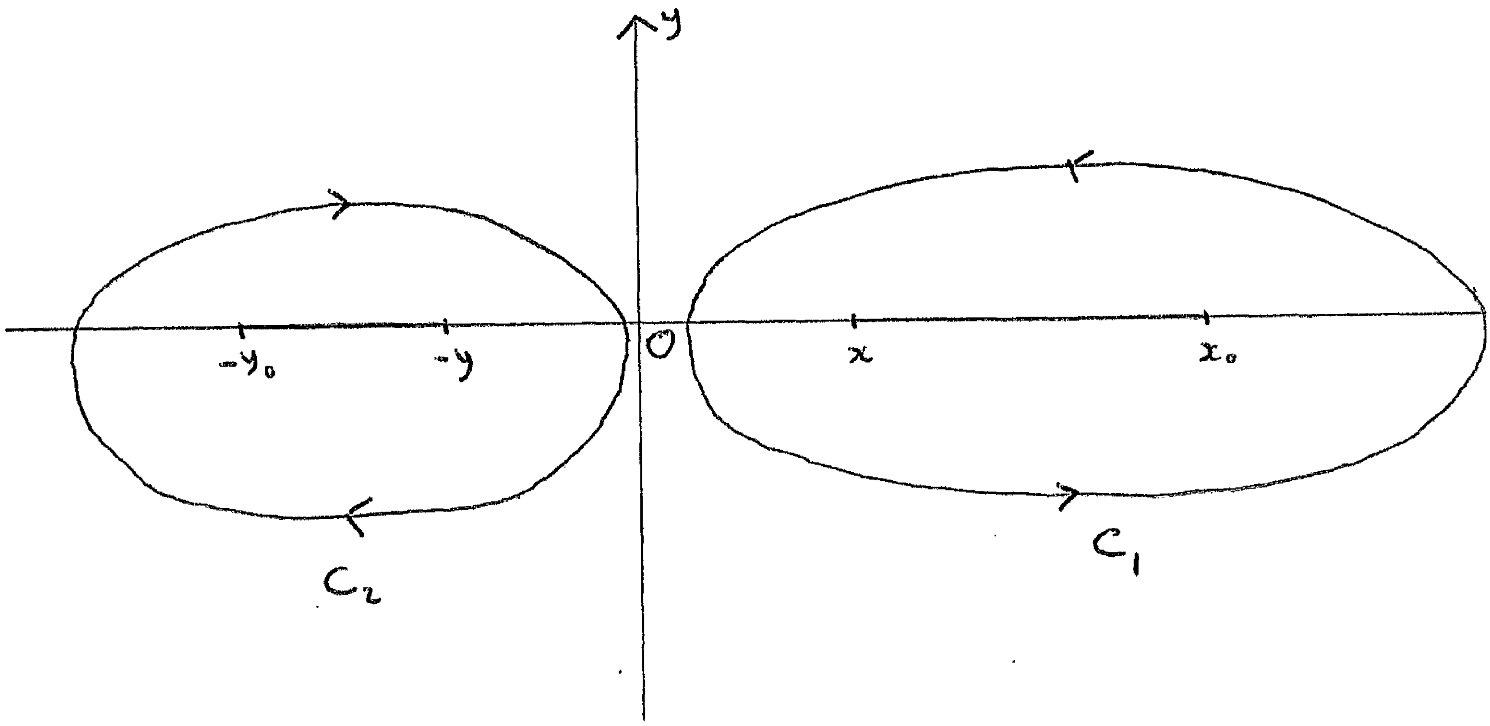

Figure 3d.

Taking the contour C_1, we have from (3.3.24) that

$$v(x_0, y; x_0, y_0) = \frac{x_0+y}{2\pi i}\int_{C_1} \frac{(z+y)^{n-1}}{(z+y_0)^n} \cdot \frac{1}{z-x_0} \cdot f(z)dz$$

$$= \frac{(x_0+y)^n}{(x_0+y_0)^n} f(x_0)$$

To satisfy (3.3.25a), $f(x_0) = 1$ and so we must take $f(z) = 1$ and

$$v(x, y; x_0, y_0) = \frac{x+y}{2\pi i}\int_{C_1} \frac{(z-x)^{n-1}}{(z-x_0)^n} \frac{(z+y)^{n-1}}{(z+y_0)^n}\, dz \qquad (3.3.26)$$

To satisfy (3.3.25b) we replace C_1 by C_2. This is justified because the integrand in (3.3.26) is of order $\frac{1}{z^2}$ for large $|z|$ and as a result each curve can be deformed into the other.

Calculation shows that if we put $z = x + \frac{(x_0-x)(x+y_0)t}{x_0+y_0-(x_0-x)t}$ in (3.3.26) we get

$$v(x, y; x_0, y_0) = \frac{(x+y)^n}{(x_0+y_0)^n} \cdot \frac{1}{2\pi i}\int_{C_3} t^{n-1}(t-1)^{-n}(1-\mu t)^{n-1}\, dt$$

where $\mu = -\frac{(x-x_0)(y-y_0)}{(x_0+y_0)(x+y)}$ and C_3 is a closed contour enclosing the points 0 and 1 but not the point $\frac{1}{\mu}$. It therefore follows that

$v(x, y; x_0, y_0) = \frac{(x+y)^n}{(x_0+y_0)^n}\, {}_2F_1(1-n, n; 1; \mu)$. This is the result (3.3.10) we derived by elementary considerations.

As a last example, let us consider the Euler-Darboux equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{n}{x-y}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \qquad (3.3.27)$$

where m is a constant. The adjoint equation is

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{m}{x-y}\frac{\partial v}{\partial x} - \frac{m}{x-y}\frac{\partial v}{\partial y} - \frac{2mv}{(x-y)^2} = 0. \tag{3.3.28}$$

The boundary conditions are

$$\frac{\partial v}{\partial x} = \frac{mv}{x-y} \quad \text{on} \quad y = y_0,$$

$$\frac{\partial v}{\partial y} = -\frac{mv}{x-y} \quad \text{on } x = x_0,$$

$$v(x_0, y_0; x_0, y_0) = 1.$$

It follows from these conditions that

$$v(x_0, y) = \frac{(x_0-y)^m}{(x_0-y_0)^m}, \quad v(x, y_0) = \frac{(x-y_0)^m}{(x_0-y_0)^m} .$$

Let us write $v(x, y; x_0, y_0) = \frac{(x-y)^m}{(x_0-y_0)^m} F(\mu)$ where in this case we take $\mu = \frac{(x-x_0)(y-y_0)}{(x-y)(x_0-y_0)}$. Then from (3.3.28), $F(\mu)$ satisfies the differential equation

$$\mu(1-\mu)\frac{d^2F}{d\mu^2} + (1-2\mu)\frac{dF}{d\mu} + (m-1)m\,F(\mu) = 0.$$

This is the hypergeometric equation with solution

$$F(\mu) = {}_2F_1(m-1, m; 1; \mu).$$

It follows from this that the Riemann function of the equation (3.3.27) is

$$v(x, y; x_0, y_0) = \frac{(x-y)^m}{(x_0-y_0)^m}\, {}_2F_1\left(m-1, m; 1; \frac{(x-x_0)(y-y_0)}{(x-y)(x_0-y_0)}\right). \tag{3.3.29}$$

If m is a positive integer and if we let $t = 1-2\mu$, then $F(t)$ satisfies the differential equation

$$(1-t^2)\frac{d^2F}{dt^2} - 2t\frac{dF}{dt} + (m-1)mF = 0$$

This is Legendre's differential equation with solution $F(t) = P_{m-1}(t)$. The Riemann function of the given equation is then

$$v(x, y; x_0, y_0) = \frac{(x-y)^m}{(x_0-y_0)^m} P_{m-1}(1-2\mu),$$

where $\mu = \frac{(x-x_0)(y-y_0)}{(x-y)(x_0-y_0)}$.

Apart from minor variations, we have therefore shown that the Euler-Poisson equation (3.3.5) and the Euler-Darboux equation (3.3.27) may be solved by the same method.

1. Motivation For Martin's Method

The main features of Riemann's method which form the basis of the

The main features of Riemann's method which form the basis of the Martin's solution of the Cauchy problem for a hyperbolic second order linear partial differential equation of the type

$$L(u) = u_{xy} + au_x + bu_y + cu = 0, \qquad (4.1.1)$$

where a, b and c are functions of x and y only, are as follows:-

(i) The introduction of characteristics as coordinate lines.

(ii) A properly chosen solution $v = v(x, y)$ of the adjoint equation

$$M(v) = v_{xy} - (av)_x - (bv)_y + cv = 0$$

(iii) The identity $vL(u) - uM(v) = \frac{\partial}{\partial x}A + \frac{\partial}{\partial y}B$ which ensures that the line integral

$$I = \int_{\Gamma} (B\,dx - A\,dy)$$

vanishes around any closed path Γ in the interior of a domain within which u and v are regular solutions of $L(u) = 0$, $M(v) = 0$ respectively.

(iv) The functions $A(x, y)$ and $B(x, y)$ are bilinear forms in u, u_x, u_y and v, v_x, v_y.

M.H. Martin took different bilinear forms for A and B, and the adjoint equation of the Riemann's method is replaced by a different but similar equation, called the <u>associate equation</u>. A solution of the associate equation, now called Martin's function, is the analogue of Riemann's function. One of the merits of Martin's method is that each solution ϕ of the original equation $L(u) = 0$ gives rise to an associate equation. The other is that, unlike Riemann's method, it can be used to solve the Cauchy problem in spaces of higher dimension.

2. Martin's Solution of The Hyperbolic Equation in Two Independent Variables

For simplicity we shall take the equation (4.1.1) in the form

$$L(u) \equiv u_{xy} - au_x - bu_y = 0 \qquad (4.2.1)$$

where a and b are functions of x and y. We shall, using Martin's method, find the solution of this equation which satisfies the Cauchy data that u, u_x and u_y are given functions of x and y on a non-characteristic curve C.

The associate equation to (4.1.1) is taken in the form

$$M(v) \equiv v_{xy} - \alpha v_x - \beta v_y = 0$$

where α, β are functions of x and y which we shall determine.

The bilinear forms A and B in the line integral $\int (B\,dx - A\,dy)$ are taken as

$$A = -\lambda^{-1} u_y v_y, \quad B = -\mu^{-1} u_x v_x \qquad (4.2.3)$$

where λ and μ are non-zero differentiable functions of x and y which are also to be determined. Motivated by the identity $vL(u) - uM(v) =$

$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}$ of the Riemann's method, we simplify $\frac{\partial}{\partial y}(\lambda^{-1}u_x v_x) - \frac{\partial}{\partial x}(\mu^{-1}u_y v_y)$.

We then find that

$$\frac{\partial}{\partial y}(\lambda^{-1}u_x v_x) - \frac{\partial}{\partial x}(\mu^{-1}u_y v_y) = (\lambda^{-1}v_x - \mu^{-1}v_y)L(u) + (\lambda^{-1}u_x - \mu^{-1}u_y)M(v)$$

$$+ \lambda^{-1}u_x v_x(a + \alpha - \frac{1}{\lambda}\lambda_y) + u_x v_y\left(-\frac{a}{\lambda} + \frac{b}{\lambda}\right)$$

$$+ u_y v_x\left(\frac{b}{\lambda} \quad \frac{\alpha}{\mu}\right) + \mu^{-1}u_y v_y(-b-\beta + \frac{1}{\mu}\mu_x).$$

We are therefore led to require that the boundary functions α, β, λ, μ, should satisfy the four equations

$$\begin{aligned} &a\lambda - \beta\mu = 0 \qquad && \lambda_y = (a + \alpha)\lambda, \\ &\alpha\lambda - b\mu = 0, \qquad && \mu_x = (b + \beta)\mu. \end{aligned} \qquad (4.2.4)$$

Eliminating α and β we get $\lambda_y = a\lambda + b\mu = \mu_x$. This shows that the expression $\lambda dx + \mu dy$ is an exact differential i.e. there exists a function $\phi(x, y)$ such that $\lambda dx + \mu dy = \phi_x dx + \phi_y dy$ i.e.

$\lambda = \phi_x$, $\mu = \phi_y$. We also notice that

$L(\phi) = \phi_{xy} - a\phi_x - b\phi_y = \lambda_y - a\lambda - b\mu = 0$ by (4.1.4)

Thus is a solution of (4.1.1). We take only that solution of (4.1.1) which satisfies $\phi_x \neq 0$, $\phi_y \neq 0$ in the region Ω under considerati

From (4.1.4) we find that $\alpha = \phi_x^{-1}\phi_y b$, $\beta = \phi_x \phi_y^{-1} a$.

With this choice of α, β, λ, μ we find that

$$\frac{\partial}{\partial y}(\phi_x^{-1}u_x v_x) - \frac{\partial}{\partial x}(\phi_y^{-1}u_y v_y) = (\phi_x^{-1}v_x - \phi_y^{-1}v_y)L(u) + (\phi_x^{-1}u_x - \phi_y^{-1}u_y)M(v) \qquad (4.2.5)$$

where $\quad M(v) \equiv v_{xy} - b\phi_x^{-1}\phi_y v_x - a\phi_x\phi_y^{-1}v_y = 0 \qquad (4.2.6)$

is the associate equation which arises from a non-zero solution $u = \phi(x, y)$ of $L(u) = 0$.

Just as in the Riemann method we apply the method of characteristics to the equation (4.1.1). The characteristics are given by x = constant y = constant. For a point $P(\bar{x}, \bar{y})$ in the xy-plane and a non-characteristic curve C carrying the Cauchy data we construct Fig. 4a as shown

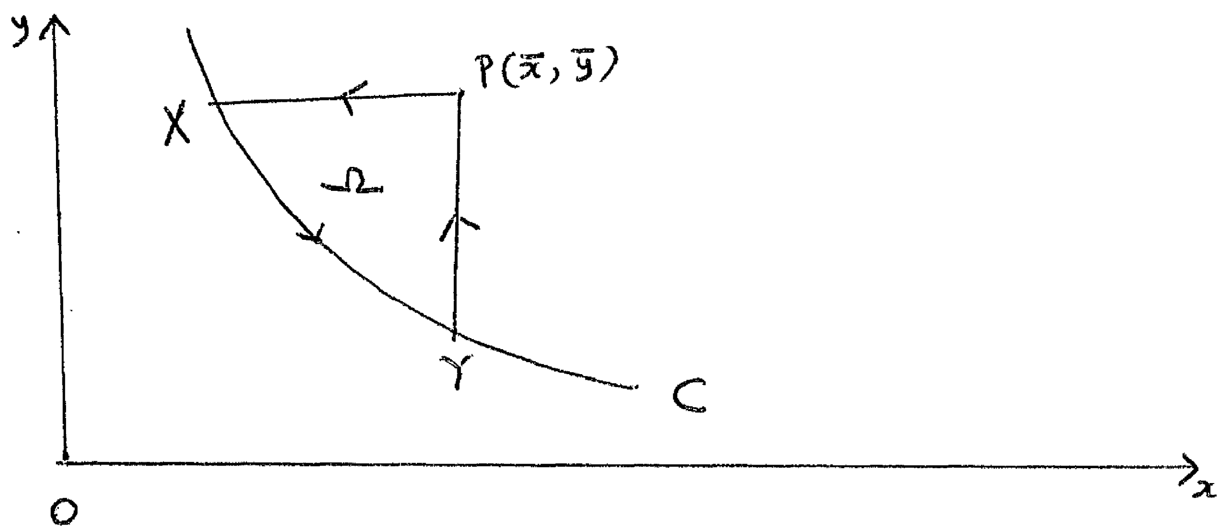

Fig. 4a.

Then if $L(u) = 0$, $M(v) = 0$ inside and on the boundary Γ of a region Ω, equation (4.1.5) yields

$$\int_P^X \phi_x^{-1} u_x v_x dx + \int_Y^P \phi_y^{-1} u_y v_y dy + \int_X^Y \{\phi_x^{-1} u_x v_x dx + \phi_y^{-1} u_y v_y dy\} = 0.$$

The boundary conditions for v on the characteristics through P are different from those of the Riemann's method. Here we require v to satisfy:

$$v_x = \phi_x \quad \text{on} \quad y = \bar{y}, \quad v_y = -\phi_y \quad \text{on} \quad x = \bar{x} . \tag{4.2.7}$$

From this it follows that

$$\int_P^X u_x dx - \int_Y^P + u_y dy + \int_X^Y (\phi_x^{-1} u_x v_x dx + \phi_y^{-1} u_y v_y dy) = 0 \quad \text{and so}$$

$$u(\bar{x}, \bar{y}) = \frac{u(X) + u(Y)}{2} + \frac{1}{2} \int_X^Y (\phi_x^{-1} u_x v_x dx + \phi_y^{-1} u_y v_y dy). \tag{4.2.8}$$

Apart from an arbitrary constant which does not affect the solution (4.2.8), v is uniquely determined by the equations (4.2.6) and (4.2.7) and the solution of the Cauchy problem is given in the integral form (4.2.8).

3. Martin's Solution of the Wave Equation in two dimensions.

We shall use Martin's method to solve the wave equation

$$u_{xx} + u_{yy} - u_{tt} = 0 \tag{4.3.1}$$

subject to the Cauchy data $u(x, y, 0) = f(x, y)$, $u_t(x, y, 0) = g(x, y)$

where f and g are functions of x and y only.

By (2.5.19), a point $\bar{p}(\bar{x}, \bar{y}, \bar{t})$ of the (x, y, t)-space is vertex of the characteristic cone

$$(x - \bar{x})^2 + (y - \bar{y})^2 = (t - \bar{t})^2.$$

The region PXY of the Riemann's method in Chapter III is now replaced by the points of the lower sheet of the characteristic cone with vertex at $\bar{p}(\bar{x}, \bar{y}, \bar{t})$ defined by

$$C: \quad 0 \leqslant t \leqslant \bar{t}, \ (x - \bar{x})^2 + (y - \bar{y})^2 \leqslant (t - \bar{t})^2 \qquad (4.3.2)$$

as shown in Figure 4b.

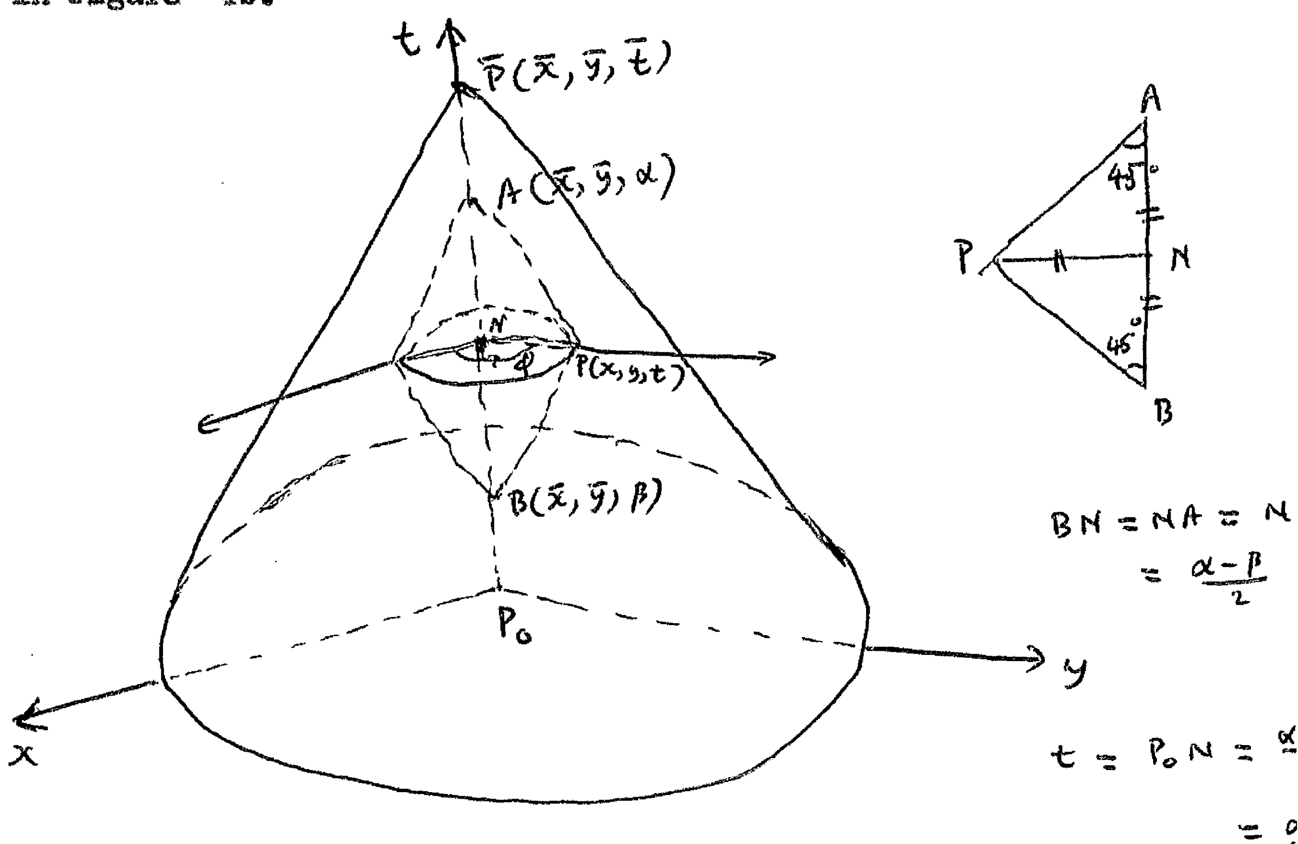

Fig. 4b.

Curvilinear coordinates α, β, φ are now introduced in order to fix the position of a point $p(x, y, t)$ in C. Here α and β determine two points A and B on the axis $\bar{p}\, p_0$ given by $A = A(\bar{x}, \bar{y}, \alpha)$, $B = B(\bar{x}, \bar{y}, \beta)$. The lower sheet of the characteristic cone with vertex at A intersects the upper sheet of the characteristic cone with vertex at B in a circle through P. The position of P on this circle is then fixed by the angle φ as shown in Fig. 4b.

It then follows that

$$x = \bar{x} + \frac{\alpha - \beta}{2} \cos \varphi \;, \quad y = \bar{y} + \frac{\alpha - \beta}{2} \sin \varphi, \quad t = \frac{\alpha + \beta}{2} \, . \qquad (4.5.3)$$

The coordinates α, β, φ of the point P are therefore characteristic coordinates. The equation (4.5.1) in the (x, y, t)-space is then transformed into an equivalent one in the (α, β, φ)-space. We then find that

$$u_{xx} = u_{\alpha\alpha} \cdot \alpha_x^2 + 2u_{\alpha\beta} \cdot \alpha_x \beta_x + u_{\alpha\varphi} \cdot \alpha_x \varphi_x + u_{\beta\beta} \cdot \beta_x^2 + u_{\beta\varphi} \cdot \beta_x \varphi_x$$
$$+ u_{\alpha\varphi} \cdot \alpha_x \varphi_x + u_{\beta\varphi} \cdot \beta_x \varphi_x + u_{\varphi\varphi} \cdot \varphi_x^2 \, .$$

But $\alpha + \beta = 2t$ and so $\alpha_x + \beta_x = 0$ and $\alpha_{xx} + \beta_{xx} = 0$.

$$\therefore \quad u_{xx} = (u_{\alpha\alpha} - 2u_{\alpha\beta} + u_{\beta\beta})\alpha_x^2 + 2(u_{\alpha\varphi} - u_{\beta\varphi})\alpha_x \varphi_x + u_{\varphi\varphi} \cdot \varphi_x^2 + (u_\alpha - u_\beta)\alpha_{xx}$$
$$+ u_\varphi \varphi_{xx} \, .$$

Similarly

$$u_{yy} = (u_{\alpha\alpha} - 2u_{\alpha\beta} + u_{\beta\beta})\alpha_y^2 + 2(u_{\alpha\varphi} - u_{\beta\varphi})\alpha_y \varphi_y + u_{\varphi\varphi}\, \varphi_y^2 + (u_\alpha - u_\beta)\alpha_{yy} + u_\varphi\, \varphi_{yy} \, .$$

$$\therefore \quad u_{xx} + u_{yy} = (u_{\alpha\alpha} - 2u_{\alpha\beta} + u_{\beta\beta})(\alpha_x^2 + \alpha_y^2) + 2(u_{\alpha\varphi} - u_{\beta\varphi})(\alpha_x\varphi_x + \alpha_y\,\varphi_y)$$
$$+ u_{\varphi\varphi}(\varphi_x^2 + \varphi_y^2) + (u_\alpha - u_\beta)(\alpha_{xx} + \alpha_{yy}) + u_\varphi(\varphi_{xx} + \varphi_{yy}).$$

Let $\rho = \frac{\alpha - \beta}{2}$. Then $\alpha_x - \beta_x = 2\frac{\partial\rho}{\partial x} = 2\,\frac{x - \bar{x}}{\rho}$, $\quad \alpha_x + \beta_x = 0$ and so

$\alpha_x = \frac{x - \bar{x}}{\rho}$, $\alpha_y = \frac{y - \bar{y}}{\rho}$.

$$\therefore \quad \alpha_x^2 + \alpha_y^2 = \frac{(x-\bar{x})^2 + (y-\bar{y})^2}{\rho^2} = 1 \quad \text{and} \quad \alpha_{xx} + \alpha_{yy} = \frac{1}{\rho}.$$

Also from $\tan\varphi = \frac{y - \bar{y}}{x - \bar{x}}$ we get

$$\varphi_x = -\frac{y - \bar{y}}{\rho^2}, \quad \varphi_y = \frac{x - \bar{x}}{\rho^2}. \quad \text{Hence}$$

$$\varphi_x^2 + \varphi_y^2 = \frac{1}{\rho^2}, \quad \text{and} \quad \alpha_x\varphi_x + \alpha_y\varphi_y = 0.$$

It then follows that

$$u_{xx} + u_{yy} = u_{\alpha\alpha} - 2u_{\alpha\beta} + u_{\beta\beta} + \frac{u_\alpha - u_\beta}{\rho} + \frac{u_{\varphi\varphi}}{\rho^2}.$$

Also $\quad u_t = u_\alpha + u_\beta, \quad u_{tt} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}.$

The equation (4.3.1) then becomes

$$L(u) = u_{\alpha\beta} - \frac{1}{2(\alpha-\beta)}(u_\alpha - u_\beta) - \frac{1}{(\alpha-\beta)^2}u_{\varphi\varphi} = 0 \qquad (4.3.4).$$

The above transformation carries the conical region (4.3.2) of the (x, y, t)-space into the wedge defined by

$$W: \; 0 \leq \alpha \leq \bar{t}, \; -\alpha \leq \beta \leq \alpha, \; 0 \leq \varphi \leq 2\pi \qquad (4.3.5)$$

of the (α, β, φ)-space as shown in the Figure 4c below.

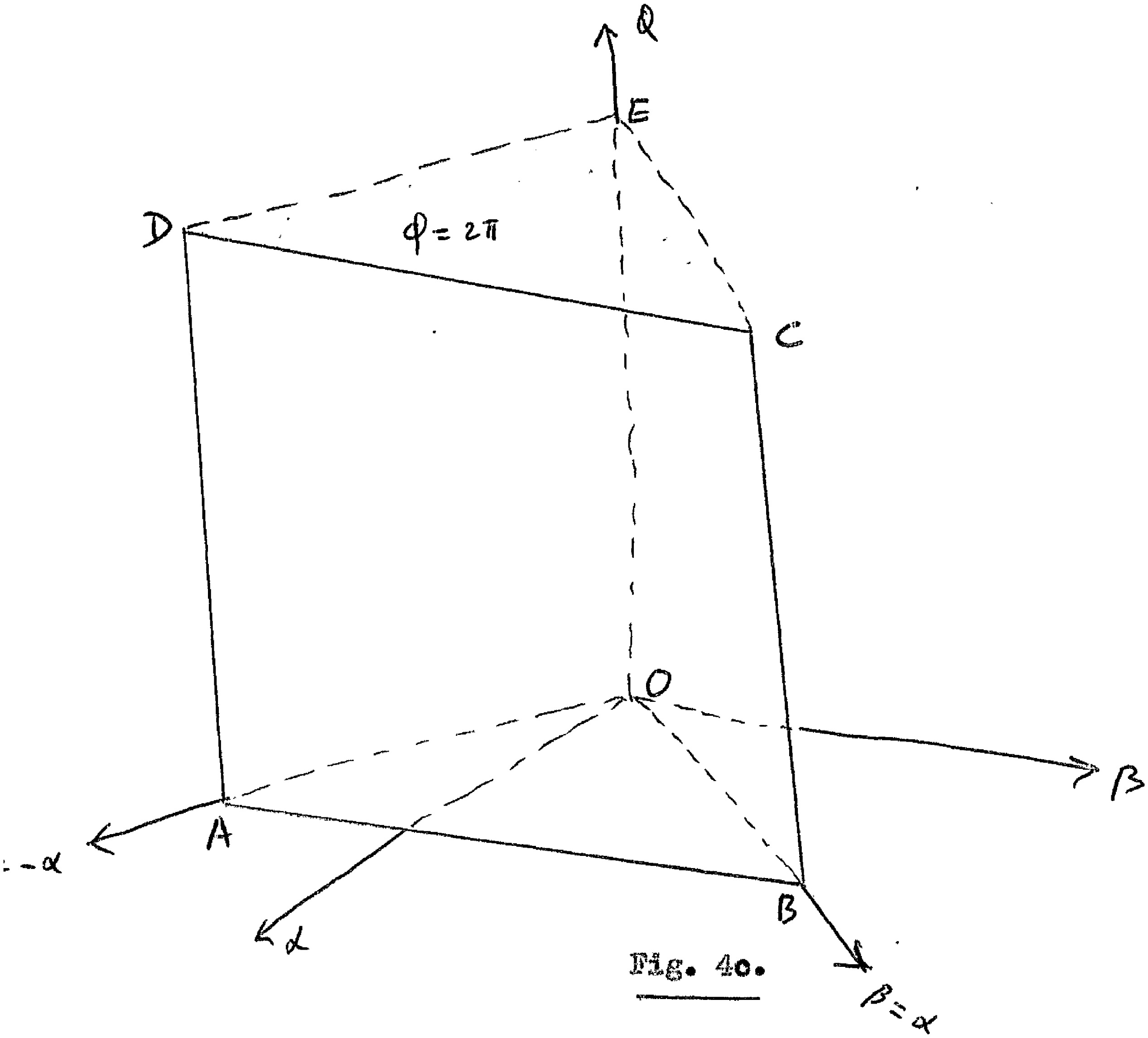

Fig. 4c.

From $t = \frac{\alpha + \beta}{2}$ we conclude that the base of the cone (4.3.2), the carrier of the Cauchy data, is transformed into the vertical face $\beta = -\alpha$ (AOED) which now becomes the carrier of the Cauchy data in (α, β, φ)-space. The axis $\bar{P}P_0$ is transformed into the vertical plane

$\beta = \alpha$(OBCE). The conical sheet is transformed into the vertical plane $\alpha = \bar{t}$ (ABCD). The boundary of the base of the cone is transformed into the edge $\alpha = \bar{t}$, $\beta = -\bar{t}$ (AD). The horizontal faces $\varphi = 0$, $\varphi = 2\pi$ are the transform of the plane through $\bar{p}(\bar{x}, \bar{y}, \bar{t})$ parallel to the base of the cone.

The solution of the equation (4.3.1) subject to the Cauchy data $u(x, y, 0) = f(x, y)$, $u_t(x, y, 0) = g(x, y)$ now becomes the solution of the equation (4.3.4) subject to the Cauchy data (which we proceed to obtain) on the carrier $\alpha + \beta = 0$. From (4.3.3) we have

$$x = \bar{x} + \rho \cos\varphi, \quad y = \bar{y} + \rho \sin\varphi, \quad t = \frac{\alpha + \beta}{2}, \quad \rho = \frac{\alpha - \beta}{2},$$

$$u_\alpha = u_\rho \cdot \tfrac{1}{2} + u_t \cdot \tfrac{1}{2} + 0, \quad u_\beta = u_\rho \cdot (-\tfrac{1}{2}) + u_t \cdot \tfrac{1}{2} + 0.$$

Therefore on the carrier $\beta = -\alpha$ we have

$$u_\alpha = \frac{f_\rho + g}{2}, \quad u_\beta = -\frac{f_\rho - g}{2}, \quad u_\varphi = [u_\varphi]_{\beta = -\alpha} \qquad (4.3.6)$$

The problem of computing the value of $u(\bar{x}, \bar{y}, \bar{t})$ at the vertex $\bar{P}$ of the characteristic cone (4.3.2) from a knowledge of the Cauchy data given over the circular region intercepted by the characteristic cone on the plane $t = 0$ (see Figure 4b) now becomes the problem of computing the value of a solution $u = u(\alpha, \beta, \varphi)$ of the equation (4.3.4) along the edge BC, the transform of $\bar{P}$, of Figure 4c, when the Cauchy data (4.3.6) are prescribed on the face $\beta = -\alpha$.

In an attempt to find a surface integral which vanishes when taken over closed surfaces we proceed as follows: Let

$K = u_\beta v_\beta$, $L = -u_\alpha v_\alpha$ where $u = u(\alpha, \beta\ \varphi)$ is a solution of (4.3.4) and $v = v(\alpha, \beta)$ is a solution of the associate equation

$$M(v) \equiv v_{\alpha\beta} + \frac{1}{2(\alpha-\beta)}(v_\alpha - v_\beta) = 0. \qquad (4.3.7)$$

Then $K_\alpha + L_\beta = -u_{\alpha\beta}(v_\alpha - v_\beta) - v_{\alpha\beta}(u_\alpha - u_\beta)$

$$= (v_\alpha - v_\beta)\left[-\frac{1}{2(\alpha-\beta)}(u_\alpha - u_\beta)\right] + (u_\alpha - u_\beta)\left[\frac{1}{2(\alpha-\beta)}(v_\alpha - v_\beta)\right] + (v_\alpha - v_\beta)\left[-\frac{1}{(\alpha-\beta)^2}u_{\varphi\varphi}\right]$$

$$= -\frac{v_\alpha - v_\beta}{(\alpha-\beta)^2}u_{\varphi\varphi}.$$

It therefore follows that if we let $M = \dfrac{v_\alpha - v_\beta}{(\alpha-\beta)^2}u_\varphi$ then

$K_\alpha + L_\beta + M_\varphi = 0$. Hence by Green's Theorem,

$$0 = \iiint_W (K_\alpha + L_\beta + M_\varphi)\,dw$$

$$= \iint_S \{K \cos(\alpha, \underset{\sim}{n}) + L \cos(\beta, \underset{\sim}{n}) + M \cos(\varphi, \underset{\sim}{n})\}\ d\sigma$$

where $d\sigma$ is the element of area on the surface S of the wedge W and where $d\sigma$ is always taken positive and $\underset{\sim}{n}$ is the outward drawn normal to the surface.

We therefore integrate the surface integral

$$I = \iint_S \{K \cos(\alpha, \underset{\sim}{n}) + L \cos(\beta, \underset{\sim}{n}) + M \cos(\varphi, \underset{\sim}{n})\}\, d\sigma \qquad (4.3.8)$$

over the surface of the wedge W. We then obtain

$$I_{\alpha=\bar{t}} + I_{\beta=-\alpha} + I_{\beta=\alpha} + I_{\varphi=0} + I_{\varphi=2\pi} = 0 \qquad (4.3.9)$$

Now $\mathbf{u}\varphi$ has period 2π in φ and $\mathbf{v}_\alpha$, $\mathbf{v}_\beta$ are independent of φ and so $I_{\varphi=0} + I_{\varphi=2\pi} = 0$, since these integrals are equal and in opposite directions.

On $\alpha = \bar{t}$, $d\alpha = 0$, the sign of cos (α, n) is positive and that of $\cos(\beta, \underset{\sim}{n})$ is also positive while $\cos(\varphi, \underset{\sim}{n}) = 0$. It follows from these and (4.3.8) that

$$I_{\alpha=\bar{t}} = \iint_{ABCD} (u_\beta v_\beta d_\beta d\varphi - u_\alpha v_\alpha d_\alpha d_\varphi) = \iint_{ABCD} u_\beta v_\beta d_\beta d\varphi \;.$$

Choose v such that $v_\beta = 1$ on $\alpha = \bar{t}$. Then

$$I_{\alpha=\bar{t}} = \int_0^{2\pi} d\varphi \int_{-\bar{t}}^{\bar{t}} u_\beta d_\beta = \int_0^{2\pi} u(\bar{t}, \bar{t}, \varphi) d\varphi - \int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi) d\varphi.$$

Now $(\bar{t}, \bar{t}, \varphi)$ is the edge BC of Fig. 4c, the transform of $\bar{P}(\bar{x}, \bar{y}, \bar{t})$, and so $u(\bar{t}, \bar{t}, \varphi) = u(\bar{x}, \bar{y}, \bar{t})$. Hence

$$I_{\alpha=\bar{t}} = 2\pi u(\bar{x}, \bar{y}, \bar{t}) - \int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi)\, d\varphi$$

On $\beta = \alpha$, $\cos(\alpha, n)$ is negative, $\cos(\beta, n)$ is positive and $\cos(\varphi, n) =$
It follows from these and (4.3.8) that

$$I_{\beta=\alpha} = \iint_{OBCE} (-u_\beta v_\beta\, d\beta\, d\varphi - u_\alpha v_\alpha\, d\alpha\, d\varphi + 0) = 0 \quad \text{if we let}$$

$v_\alpha = v_\beta = 0$ on $\beta = \alpha$.

The Martin function $v(\alpha, \beta)$ must therefore satisfy equation (4.3.7) subject to the boundary conditions

$$v_\beta = 1 \text{ on } \alpha = \bar{t}, \quad v_\alpha = v_\beta = 0 \quad \text{on} \quad \beta = \alpha, \qquad (4.3.10)$$

where $\alpha = \bar{t}$, $\beta = \alpha$ are the characteristic planes intersecting in BC, the transform of $\bar{P}(\bar{x}, \bar{y}, \bar{t})$.

On $\beta = -\alpha$, $\cos(\alpha, n)$ is negative, $\cos(\beta, n)$ is negative while $\cos(\varphi, n) = 0$. It therefore follows

$$I_{\beta=-\alpha} = \iint_{AOED} (-u_\beta v_\beta + u_\alpha v_\alpha)\, d\alpha\, d\varphi \quad \text{since} \quad L = -u_\alpha v_\alpha \quad \text{and} \quad |d\beta\, d\varphi| = d\alpha\, d\varphi$$

$$= \int_0^{2\pi} d\varphi \int_0^{\bar{t}} (u_\alpha v_\alpha - u_\beta v_\beta) d\alpha \,.$$

Equation (4.3.9) therefore reduces to

$$2\pi u(\bar{x}, \bar{y}, \bar{t}) - \int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi) d\varphi + \int_0^{2\pi} (u_\alpha v_\alpha - u_\beta v_\beta) d\alpha = 0 \quad \text{i.e.}$$

$$u(\bar{x}, \bar{y}, \bar{t}) = \frac{1}{2\pi} \int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi) d\varphi - \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\bar{t}} (u_\alpha v_\alpha - u_\beta v_\beta) d\alpha. \quad (4.3.1$$

We now proceed to find the solution of the associate equation (4.3.7) subject to the boundary conditions (4.3.10). We notice that $v = \alpha + \beta$ is a solution of (4.3.7). The boundary conditions suggest that we try

$v = \alpha + \beta + A(\bar{t} - \alpha)^m (\bar{t} - \beta)^n$ where $m > 0$ and A is a constant.

Then

$$v_\beta = 1 - Am(\bar{t} - \alpha)^m \quad (\bar{t} - \beta)^{m-1}$$

when $\beta = \alpha$, $v_\beta = 1 - Am(t - \alpha)^{2m-1} = 0$ for all α and $A \neq 0$

$\therefore \quad 2m-1 = 0, \quad Am = 1 \,.$

Hence $m = \frac{1}{2}$, $A = 2$.

$\therefore \quad v = \alpha + \beta + 2(\bar{t} - \alpha)^{\frac{1}{2}} (\bar{t} - \beta)^{\frac{1}{2}}$ is the appropriate solution of the associate equation (4.3.7). This is justified by a direct verification. Hence when $\beta = -\alpha$, we have $v_\alpha = 1 - \left(\frac{\bar{t} - \beta}{\bar{t} - \alpha}\right)^{\frac{1}{2}}$, $v_\beta = 1 - \left(\frac{\bar{t} - \alpha}{\bar{t} + \alpha}\right)^{\frac{1}{2}}$. H

$$u(\bar{x},\bar{y},\bar{t}) = \frac{1}{2\pi}\int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi)d\varphi - \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{2\pi} \Big\{ u_\alpha\Big[1 - \Big(\frac{\bar{t}+\alpha}{\bar{t}-\alpha}\Big)^{\frac{1}{2}}\Big]$$

$$- u_\beta\Big[1 - \Big(\frac{\bar{t}-\alpha}{\bar{t}+\alpha}\Big)^{\frac{1}{2}}\Big]\Big\} d\alpha \qquad (4.3.12)$$

This formula expresses the value of $u(x, y, t)$ of the solution of the equation (4.3.4) in terms of the Cauchy data prescribed on the carrier plane $\beta = -\alpha$. We have therefore solved the Cauchy problem in (α, β, φ)-space.

We note that $(\bar{t}, -\bar{t}, \varphi)$ is the transform of the boundary of the base of the cone of Fig. 4b and so by the mean value theorem

$$\frac{1}{2\pi}\int_0^{2\pi} u(\bar{t}, -\bar{t}, \varphi)d\varphi = \frac{1}{2\pi}\int_\Gamma u(x, y, 0)ds$$

$$= u(\bar{x}, \bar{y}, 0)$$

$$= f(\bar{x}, \bar{y}).$$

Also by (4.3.6), $u_\alpha = \frac{f_\rho + g}{2}$, $u_\beta = -\frac{f_\rho - g}{2}$, $\rho = \alpha$ on $\beta = -\alpha$ and $d\rho = d\alpha$. Hence

$$u\alpha - u\beta - u\alpha\left(\frac{\bar{t}+\alpha}{\bar{t}-\alpha}\right)^{\frac{1}{2}} + u_\beta\left(\frac{\bar{t}-\alpha}{\bar{t}+\alpha}\right)^{\frac{1}{2}}$$

$$= f\rho - \frac{f_\rho + g}{2}\left(\frac{\bar{t}+\rho}{\bar{t}-\rho}\right)^{\frac{1}{2}} - \frac{f_\rho - g}{2}\left(\frac{\bar{t}-\rho}{t+\rho}\right)^{\frac{1}{2}}$$

$$= f_\rho - \frac{\bar{t}f_\rho + \rho g}{(\bar{t}-\rho)^{\frac{1}{2}}} \quad .$$

Also $\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\bar{t}} f_\rho d\rho d\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(\bar{t}, \varphi)d\varphi - \frac{1}{2\pi}\int_0^{2\pi} f(0, \varphi)d\varphi$

$$= f(\bar{x}, \bar{y}) - \lim_{\rho\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(\rho, \varphi)d\varphi$$

$$= f(\bar{x}, \bar{y}) - f(\bar{x}, \bar{y})$$

$$= 0.$$

In the (x, y, t)-space, it follows from (4.3.12) that

$$u(\bar{x}, \bar{y}, \bar{t}) = f(\bar{x}, \bar{y}) + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\bar{t}} \frac{\bar{t}\, f\rho + \rho g}{(\bar{t}^2 - \rho^2)^{\frac{1}{2}}} d\rho d\varphi \,, \qquad (4.3.13)$$

where $f\rho = \frac{\partial}{\partial\rho} f(\bar{x} + \rho \cos\varphi, \ \bar{y} + \rho \sin\varphi)$.

We now show that the solution (4.3.13) is equivalent to the classical solution given by Posson's formula

$$u(\bar{x}, \bar{y}, \bar{t}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\bar{t}} \frac{g(\bar{x}+\rho \cos\varphi, \bar{y}+\rho \sin\varphi)}{(\bar{t}^2 - \rho^2)^{\frac{1}{2}}} \rho d\rho d\varphi + \frac{1}{2\pi}\frac{\partial}{\partial \bar{t}}$$

$$\int_0^{2\pi}\int_0^{\bar{t}} \frac{f}{(\bar{t} - \rho^2)^{\frac{1}{2}}} \rho d\rho d\varphi \qquad (4.3.14)$$

Let $F_r(f) = \int_0^{2\pi}\int_0^{r} \frac{f(\bar{x}+\rho \cos\varphi, \bar{y}+\rho \sin\varphi)}{(r^2 - \rho^2)^{\frac{1}{2}}} \rho d\rho d\varphi$. We cannot differentiate this with respect to r as it stands because the integrand has a singularity at $\rho = r$. Let $\rho = r\tau$. Then

$$F_r(f) = r\int_0^{2\pi}\int_0^1 \frac{f(\bar{x} + r\tau\cos\varphi,\ \bar{y} + r\tau\sin\varphi)}{(1-\tau)^{\frac{1}{2}}}\,\tau d\tau d\varphi$$

$$\frac{\partial F_r(f)}{\partial r} = \int_0^{2\pi}\int_0^1 \frac{f}{(1-\tau^2}\,\tau d\tau d\varphi + r\int_0^{2\pi}\int_0^1 \frac{f\rho\cdot\frac{\partial\rho}{\partial r}}{(1-\tau^2)^{\frac{1}{2}}}\,\tau d\tau d\varphi$$

$$= \frac{1}{r}\int_0^{2\pi}\int_0^r \frac{\rho f}{(r^2-\rho^2)^{\frac{1}{2}}}\,d\rho d\varphi + \frac{1}{r}\int_0^{2\pi}\int_0^r \frac{\rho^2\ f\rho}{(r^2-\rho^2)^{\frac{1}{2}}}\,d\rho d\varphi$$

$$= I_1 + I_2 \quad \text{where}$$

$$I_1 = \frac{1}{r}\int_0^{2\pi}\left\{\left[-f(r^2-\rho^2)^{\frac{1}{2}}\right]_{\rho=0}^{r} + \int_0^r (r^2-\rho^2)^{\frac{1}{2}}\ f\rho d\rho\right\} d\varphi$$

$$= \frac{1}{r}\int_0^{2\pi} r\,[f]_{\rho=0} d\varphi + \frac{1}{r}\int_0^{2\pi}\int_0^r \frac{r^2-\rho^2}{(r^2-\rho^2)^{\frac{1}{2}}}\ f\rho d\rho d\varphi$$

$$= 2\pi\ f(\bar{x},\ \bar{y}) + \int_0^{2\pi} \frac{rf\rho}{(r^2-\rho^2)^{\frac{1}{2}}}\,d\rho d\varphi - I_2\ .$$

Equation (4.3.14) therefore reduces to

$$u(\bar{x},\ \bar{y},\ \bar{t}) = f(\bar{x},\ \bar{y}) + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\bar{t}} \frac{\bar{t}f\rho + \rho g}{(\bar{t}-\rho^2)^{\frac{1}{2}}}\,d\rho d\varphi.$$

This shows that the solution (4.3.13) of the Cauchy problem obtained by Martin's method is equivalent to the classical Poisson solution (4.3.14).

4. Martin's Solution of the Wave Equation in Three dimensions

In this section we shall extend Martin's method to the solution of the wave equation

$$L(u) \equiv u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0, \quad t > 0 \qquad (4.4.1)$$

subject to the Cauchy data $u(x, y, z, 0) = f(x, y, z)$, $u_t(x, y, z, 0) = g(x, y, z)$ where f and g are given functions of x, y, z.

The problem consists in the determination of the value of u at any point $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$, $\bar{t} > 0$, in terms of the assigned Cauchy data. In (x, y, z, t)-space we shall employ the cylindrical coordinates (r, θ, φ, t) where θ is the latitude and θ is the longitude. The transformation equations are therefore

$$x = \bar{x} + r \sin\theta \cos\varphi, \quad y = \bar{y} + r \sin\theta \sin\varphi, \quad z = \bar{z} + r \cos\theta.$$

Equation (4.4.1) then becomes

$$L(u) \equiv u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 \sin\theta}\left(\sin\theta \; u_\theta\right)_\theta + \left(\frac{1}{r^2 \sin^2\theta} u_\varphi\right)_\varphi - u_{tt} = 0 \quad (4.4.$$

As in the case of the wave equation in two dimensions we let

$$M(v) \equiv v_{rr} - \frac{2}{r} v_r - v_{tt} = 0 \qquad (4.4.3)$$

be the associate equation. Then

$$v_r L(u) = v_r u_{rr} + \frac{2}{r} u_r v_r + \frac{v_r}{r^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta + v_r\left(\frac{1}{r^2 \sin^2\theta} u_\varphi\right)_\varphi - v_r u_{tt}$$

$$u_r M(v) = u_r v_{rr} - \frac{2}{r} u_r v_r - u_r v_{tt} \ .$$

This suggests that we add rather than subtract. Then

$$v_r L(u) + u_r M(v) = (v_r u_{rr} + u_r v_{rr}) - v_r u_{tt} - u_r v_{tt} + \frac{v_r}{r^2 \sin\theta}\left(\sin\theta u_\theta\right)_\theta + v_r\left(\frac{1}{r^2 \sin^2\theta} u_\varphi\right)_\varphi$$

Now $v_r u_{rr} + u_r v_{rr} - v_r u_{tt} - u_r v_{tt} = (u_r v_r)_r - (v_r u_t + u_r v_t)_t + u_t v_{rt} + u_{rt} v_t$

$$= (u_r v_r + u_t v_t)_r - (v_r u_t + u_r v_t)_t \ .$$

We therefore take as a generalisation of the Lagrange-identity, the identity

$$v_r L(u) + u_r M(v) = (u_r v_r + u_t v_t)_r - (u_t v_r + u_r v_t)_t + \frac{v_r}{r^2 \sin\theta}\left(\sin\theta u_\theta\right)_\theta + v_r\left(\frac{1}{r^2 \sin^2\theta} u_\varphi\right)_\varphi \qquad (4.4.4)$$

As in the case of the two-dimensional wave equation, we denote the four-dimensional cone with vertex at $\bar{P}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ by

$$C: \quad 0 \leq t \leq \bar{t}, \quad (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 \leq (\bar{t} - t)^2 \qquad (4.4.5)$$

and we transform this into a four-dimensional wedge

$$W: 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq r \leq \bar{t} - t, \quad 0 \leq t \leq \bar{t} . \qquad (4.4.6)$$

The base of the cone C is then

$$D: (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 \leq \bar{t}^2. \qquad (4.4.7)$$

It then follows from the identity (4.4.4) that

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \int_{t=0}^{\bar{t}} \int_{r=0}^{\bar{t}-t} \left[\frac{v_r L(u) + u_r M(v)}{r^2} \right] r^2 \sin\theta \, d\theta \, d\varphi \, dt \, dr$$

$$= \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \int_{t=0}^{\bar{t}} \Big[u_r v_r + u_t v_t \Big]_{r=0}^{\bar{t}-t} \sin\theta \, d\theta \, d\varphi \, dt - \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \int_{t=0}^{\bar{t}} \int_{r=0}^{\bar{t}-t}$$

$$(u_t v_r + u_r v_t)_t \, dt \sin\theta \, d\theta \, d\varphi \, dr + 0. \qquad (4.4.8)$$

But $\displaystyle\int_0^{\bar{t}} \Big[u_r v_r + u_t v_t \Big]_{r=0}^{\bar{t}-t} dt = \int_0^{\bar{t}} \Big[u_r v_r + u_t v_t \Big]_{r=\bar{t}-t} dt - \int_0^{\bar{t}} \Big[u_r v_r + u_t v_t \Big]_{r=0} dt$

$$= I_1 - I_2, \quad \text{say.}$$

Also by changing the order of integration,

$$\int_{t=0}^{\bar{t}} dt \int_{r=0}^{\bar{t}-t} (u_t v_r + u_r v_t)_t \, dr = \int_{r=0}^{\bar{t}} dr \int_{t=0}^{\bar{t}-r} (u_t v_r + u_r v_t)_t \, dt$$

$$= \int_{r=0}^{\bar{t}} \Big[u_t v_r + u_r v_t \Big]_{t=\bar{t}-r} - \int_{r=0}^{\bar{t}} \Big[u_t v_r + u_r v_t \Big]_{t=0} dr$$

$$= I_3 - I_4 .$$

If $L(u) = 0$ and $M(v) = 0$ then it readily follows from (4.4.8) that

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{t=0}^{\bar{t}}\Big[u_t v_t + u_r v_r\Big]_{r=0} dt = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{r=0}^{\bar{t}}\Big[u_t v_r + u_r v_t\Big]_{t=0} dr \qquad (4.4.9)$$

We now solve the associate equation (4.4.3) subject to the boundary conditions determined by $[u_t v_t + u_r v_r]_{r=0}$ and $[u_t v_r + u_r v_t]_{t=0}$. Since we have to use the terms in the first brackets to find u we let $v_r = 0$ when $r = 0$ and v_t be a function of t alone. Let $v_t = \bar{t} - t$ when $r = 0$. We could let $v_t = 1$ but the solution we then obtain will not be the same as the one we require involving $\bar{t}$. From the terms in the second brackets we let $v_r = r$, $v_t = \bar{t}$ when $t = 0$. These latter boundary conditions are motivated by the solution (4.3.13) of t last Section.

Try $v = Ar^2 + Br + Ct^2 + Dt + E$

$v_r = 2Ar + B$, $\quad v_t = 2Ct + D$

when $r = 0$, $v_r = B = 0$

when $t = 0$, $v_t = D = \bar{t}$

when $r = 0$, $v_t = \bar{t} - t = 2Ct + D$

$\therefore \quad D = \bar{t}$, $C = -\frac{1}{2}$, $A = \frac{1}{2}$ $\quad B = 0$.

Hence $v = \frac{1}{2}r^2 - \frac{1}{2}t^2 + \bar{t}t + E$ and this satisfies equation (4.4.3) for any constant E. Taking $E = -\frac{\bar{t}^2}{2}$ we have $v = \frac{1}{2}r^2 - \frac{1}{2}(\bar{t} - t)^2$ as the appropriate solution of the associate equation (4.4.3). Hence

$$[u_t v_r + u_r v_t]_{t=0} = [r u_t + \bar{t} u_r]_{t=0}$$

$$[u_t v_t + u_r v_r]_{r=0} = [(\bar{t}-t)u_t + r u_r]_{r=0} = (\bar{t}-t)[u_t]_{t=0} \ .$$

It then follows from equation (4.4.9) that

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{r=0}^{\bar{t}}\left[\frac{r u_t + \bar{t} u_r}{r^2}\right]_{t=0} r^2 \sin\theta \, d\theta \, dr = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\int\left[(\bar{t}-t)u_t\right]_{t=0} \sin\theta \, d\theta d$$

$$= 4\pi \int_0^{\bar{t}} (\bar{t}-t)u_t(\bar{x}, \bar{y}, \bar{z}, t) \, dt$$

Differentiating each side with respect to $\bar{t}$ we have

$$4\pi \int_0^{\bar{t}} u_t(\bar{x}, \bar{y}, \bar{z}, t)dt = \frac{\partial}{\partial \bar{t}} \iiint_B \left(\frac{g(x,y,z)}{r} + \frac{\bar{t} f_r(x,y,z)}{r^2} \right) dx \, dy \, dz$$

i.e.

$$u(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = f(\bar{x}, \bar{y}, \bar{z}) + \frac{1}{4\pi}\frac{\partial}{\partial \bar{t}} \iiint_B \left(\frac{g(x,y,z)}{r} + \frac{\bar{t} f_r(x,y,z)}{r^2}\right) dx \, dy \, dz. \quad (4.4.$$

This is the solution to our Cauchy problem. It can also be shown, as in Section 3 of this Chapter, that this solution is equivalent to the classical solution as given by Poisson formula.

5. Martin's Solution of the Wave Equation in Spaces of Higher Dimension

In this Section we study the extension of Martin's Method to the solution of the wave equation in n dimensions. We shall solve the wave equation

$$u_{x_1 x_1} + \dots + u_{x_n x_n} - u_{tt} = 0 \qquad (4.5.1)$$

subject to the Cauchy data

$$\begin{aligned} u(x_1, x_2, \dots, x_n, 0) &= f(x_1, x_2, \dots, x_n), \\ u_t(x_1, x_2, \dots, x_n, 0) &= g(x_1, x_2, \dots, x_n). \end{aligned} \qquad (4.5.2)$$

This generalised method holds for $n \geq 3$ and the method of Section 4 is merely a simplified version of it. In three dimensions we transformed variables from Cartesian rectangular coordinates to polar coordinates by the equations $x = r \cos\varphi \sin\varphi$, $y = r \sin\varphi \sin\theta$, $z = r\cos\theta$ where $0 \leq \varphi \leq 2\pi$ $0 \leq \theta \leq \pi$, $r \geq 0$. In space of n dimensions the generalisation of this is the transformation

$$\begin{aligned}
x_1 &= r \cos\varphi \sin\theta_1 \sin\theta_2 \sin\theta_3 \dots \sin\theta_{n-2} \\
x_2 &= r \sin\varphi \sin\theta_1 \sin\theta_2 \sin\theta_3 \dots \sin\theta_{n-2} \\
x_3 &= r \cos\theta_1 \sin\theta_2 \sin\theta_3 \dots \sin\theta_{n-2} \\
x_4 &= r \cos\theta_2 \sin\theta_3 \dots \sin\theta_{n-2} \\
&\vdots \\
x_{n-1} &= r \cos\theta_{n-3} \sin\theta_{n-2} \\
x_n &= r \cos\theta_{n-2}
\end{aligned}$$

where $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta_j \leqslant \pi$ $(j = 1, 2, \ldots, n-2)$, $r \geqslant 0$.

Let us write $y_1 = \varphi$, $y_2 = \theta_1, \ldots, y_{n-1} = \theta_{n-2}$, $y_n = r$. Then, using a result of elementary vector analysis,

$$h_1{}^2 = \left(\frac{\partial x_1}{\partial \varphi}\right)^2 + \left(\frac{\partial x_2}{\partial \varphi}\right)^2 + \ldots + \left(\frac{\partial x_n}{\partial \varphi}\right)^2$$

$$= r^2 \sin^2\varphi \sin^2\varphi_1 \ldots\ldots \sin^2\theta_{n-2} + r^2\cos^2\varphi \sin^2\theta_1 \ldots\ldots \sin^2\theta_{n-2} + 0$$

$$\therefore \quad h_1 = r \sin\theta_1 \sin\theta_2 \ldots\ldots \sin\theta_{n-2}.$$

Similarly
$$h_2 = r \sin\theta_2 \sin\theta_3 \ldots\ldots \sin\theta_{n-2}$$
$$\vdots$$
$$h_{n-2} = r \sin\theta_{n-2}$$
$$h_{n-1} = r$$
$$h_n = 1$$

$$\therefore h_1 h_2 \ldots\ldots h_n = r^{n-1} \sin\theta_1 \sin^2\theta_2 \sin^3\theta_3 \ldots\ldots \sin^{n-2}\theta_{n-2} = g, \text{ say.}$$

Then by the transformation rule,

$$u_{x_1x_1} + \ldots\ldots + u_{x_nx_n} = \nabla^2 u = \frac{1}{h_1 h_2 \ldots\ldots h_n} \sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left[\frac{h_1 h_2 \ldots\ldots h_n}{h_i{}^2} \frac{\partial u}{\partial y_i}\right]$$

$$= \frac{1}{g} \sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left(\frac{g}{h_i{}^2} \frac{\partial u}{\partial y_i}\right)$$

$$= \frac{1}{g}\frac{\partial}{\partial \varphi}\left(\frac{g}{h_1{}^2} u_\varphi\right) + \frac{1}{g}\left[\sum_{j=1}^{n-2} \frac{\partial}{\partial \theta_j}\left(\frac{g}{h_{j+1}{}^2} u_{\theta_j}\right)\right] + \frac{1}{g}\frac{\partial}{\partial r}\left(\frac{g}{h_n{}^2} u_r\right).$$

But $\frac{1}{g}\frac{\partial}{\partial r}(gu_r) = u_{rr} + \frac{1}{g}u_r\left[(n-1)r^{n-2}\sin\theta_1 \sin^2\theta_2 \ldots \sin^{n-2}\theta_{n-2}\right]$

$$= u_{rr} + \frac{n-1}{r}u_r .$$

Let us set $\frac{g}{h_i^2} = r^{n-3}f_{i-1}$. Then

$$\nabla_2 u = \frac{r^{n-1}}{g}\frac{\partial}{\partial\varphi}\left(\frac{f_0}{r^2}u_\varphi\right) + \frac{r^{n-1}}{g}\sum_{j=1}^{n-2}\frac{\partial}{\partial\theta_j}\left(\frac{f_j}{r^2}u_{\theta_j}\right) + u_{rr} + \frac{n-1}{r}u_r$$

Now $\frac{g}{r^{n-1}} = \sin\theta_1 \sin^2\theta_2 \ldots\ldots \sin^{n-2}\theta_{n-2} = f_{n-2} = f$, say

Therefore

$$\nabla_2 u = \frac{1}{f}\left[\frac{\partial}{\partial\varphi}\left(\frac{f_0}{r^2}u_\varphi\right) + \sum_{j=1}^{n-2}\frac{\partial}{\partial\theta_j}\left(\frac{f_j}{r^2}u_{\theta_j}\right)\right] + u_{rr} + \frac{n-1}{r}u_r$$

Also by elementary vector analysis the $(n-1)$-dimensional element of surface area on the unit sphere $r=1$ is given by

$$dw_n = h_1 h_2 \ldots h_n\, d\varphi\, d\theta_1\, d\theta_2 \ldots d\theta_{n-2}$$

$$= f\, d\varphi\, d\theta_1\, d\theta_2 \ldots d\theta_{n-2}, \quad \text{since } g = f \text{ when } r = 1.$$

Hence the $(n-1)$-dimensional area of the unit sphere is given by

$$w_n = \int_0^\pi\int_0^\pi \ldots\ldots \int_0^\pi\int_0^{2\pi} f\, d\varphi\, d\theta_1 \ldots d\theta_{n-2} .$$

Since f is independent of φ,

$$w_n = 2\pi\int_0^\pi\int_0^\pi \ldots \int_0^\pi f\, d\theta_1\, d\theta_2 \ldots d\theta_{n-2}$$

$$n = 3, \quad w_3 = 2\pi \int_0^{\pi} \sin\theta_1 \, d\theta_1 = 4\pi = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}$$

$$n = 4, \quad w_4 = 2\pi \cdot 4 \cdot \int_0^{\pi/2} \sin\theta_1 \, d\theta_1 \cdot \int_0^{\pi/2} \sin^2\theta/_2 \; d\theta_2$$

$$= 2\pi^2 = 2 \; \frac{\pi^{\frac{4}{2}}}{\Gamma(\frac{4}{2})}$$

Suppose $\quad w_m = \dfrac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$. Then

$$w_{m+1} = \frac{2\pi^{\frac{m}{2}}}{(\frac{m}{2})} \cdot 2 \int_0^{\pi/2} \sin^{m-1}\theta_{m-1} \, d\theta_{m-1}$$

Let $\; m = 2r, \;$ then

$$w_{m+1} = \frac{2^2 \cdot \pi^r}{(r)} \cdot \frac{(2r-2)(2r-4)\ldots\ldots 2}{(2r-1)(2r-3)\ldots\ldots 3} = 2^{2r}\pi^r \; \frac{(r)}{(2r)}$$

$$= 2^{2r} \; \pi^r \; \frac{(\frac{1}{2})}{2^{2r-1} \; (r+\frac{1}{2})}$$

$$= 2 \; \frac{\pi^{\frac{m+1}{2}}}{(\frac{m+1}{2})}$$

Hence by induction, $\quad w_n = \dfrac{2\pi^{\frac{n}{2}}}{(\frac{n}{2})}$.

Let us make a further transformation $\alpha = t + r$, $\beta = t - r$ so that we have a transformation from $(\varphi, \theta_1, \theta_2, \ldots, \theta_{n-2})$-space to $(\alpha, \beta, \varphi, \theta_1, \theta_2, \ldots, \theta_{n-2})$-space.

Then $u_t = u_\alpha + u_\beta$, $u_r = u_\alpha - u_\beta$,

$$u_{tt} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}, \quad u_{rr} = u_{\alpha\alpha} - 2u_{\alpha\beta} + u_{\beta\beta}.$$

$$\therefore \quad u_{tt} - \nabla_2 u = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta} - u_{\alpha\alpha} + 2u_{\alpha\beta} - u_{\beta\beta} + \frac{(-u_\alpha + u_\beta)2(n-1)}{\alpha - \beta}$$

$$- \frac{1}{f}\left[\frac{\partial}{\partial\varphi}\left(\frac{f_0}{r^2} u_\varphi\right) + \sum_{j=1}^{n-2} \frac{\partial}{\partial\theta_j}\left(\frac{f_j}{r^2} u_{\theta_j}\right)\right]$$

$$= 4u_{\alpha\beta} - \frac{2(u_\alpha - u_\beta)(n-1)}{\alpha - \beta} - \frac{4}{f(\alpha-\beta)^2}\left[\frac{\partial}{\partial\varphi}(f_0 u_\varphi) + \sum_{j=1}^{n-2}\frac{\partial}{\partial\theta_j}\left(f_j u_{\theta_j}\right)\right]$$

i.e.

$$\frac{f}{4}(u_{tt} - \nabla_2 u) = \left[u_{\alpha\beta} - \frac{(u_\alpha - u_\beta)(n-1)}{2(\alpha-\beta)}\right] f - \frac{1}{(\alpha-\beta)^2}\left[\frac{\partial}{\partial\varphi}(f_0 u_\varphi) + \sum_{j=1}^{n-2} \frac{\partial}{\partial\theta_j}(f_j u_{\theta_j})\right]$$

and so

$$L(u) = \left[u_{\alpha\beta} - \frac{n-1}{2(\alpha-\beta)}(u_\alpha - u_\beta)\right] f - \frac{1}{(\alpha-\beta)^2}\left[\frac{\partial}{\partial\varphi}(f_0 u_\varphi) + \sum_{j=1}^{n-2}\frac{\partial}{\partial\theta_j}(f_j u_{\theta_j})\right] \tag{4.5.3}$$

The associate equation of (4.5.3) is therefore

$$M(v) = v_{\alpha\beta} + \frac{n-1}{2(\alpha-\beta)} (v_\alpha - v_\beta). \tag{4.5.6}$$

We now look for a surface integral which vanishes over closed surfaces. Following the procedure for $n = 2$, we let

$$A = f\, u_\beta v_\beta, \quad B = -f\, u_\alpha v_\alpha\,, \quad \Phi = f_0 \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_\varphi\,.$$

Then $A_\alpha + B_\beta + \Phi_\varphi = f u_{\alpha\beta} v_\beta + f u_\beta v_{\alpha\beta} - f u_{\alpha\beta} v_\alpha - f u_\alpha v_{\alpha\beta} + f_0 \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_{\varphi\varphi}$

$$= (v_\beta - v_\alpha)\left[L(u) + \frac{n-1}{2(\alpha-\beta)}(u_\alpha - u_\beta) f + \frac{1}{(\alpha-\beta)^2}\left[\frac{\partial}{\partial \varphi}(f_0 u_\varphi) + \sum_{j=1}^{n-2} \frac{\partial}{\partial \theta_j}(f_j u_{\theta_j})\right]\right.$$

$$- f(u_\alpha - u_\beta)\left[M(v) - \frac{n-1}{(\alpha-\beta)}(v_\alpha - v_\beta)\right] + f_0 \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_{\varphi\varphi}$$

$$= (v_\beta - v_\alpha) L(u) + f(u_\beta - u_\alpha) M(v) - \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} \sum_{j=1}^{n-2} \frac{\partial}{\partial \theta_j}(f_j u_{\theta_j}).$$

It follows that if we let $\Theta_j = \sum_{j=1}^{n-2} \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} f_j u_{\theta_j}$ we get

$$(v_\beta - v_\alpha) L(u) + f(u_\beta - u_\alpha) M(v) = A_\alpha + B_\beta + \Phi_\varphi + \sum_{j=1}^{n-2} \frac{\partial}{\partial \theta_j} \Theta_j. \tag{4.5.7}$$

This now plays the role of the Lagrange identity. Hence if $L(u) = 0$, $M(v) = 0$ in an $(n+1)$-dimensional domain of the $(\alpha, \beta, \varphi, \theta_1, \ldots \theta_{n-2})$-

space, we have by the generalised Green's theorem,

$$0 = \int_{V_{n+1}} \left(A_\alpha + B_\beta + \Phi_\varphi + \sum_{j=1}^{n-2} \frac{\partial}{\partial \theta_j} \Theta_j \right) d\alpha d\beta d\varphi d\theta_1 \ldots\ldots d\theta_{n-2}$$

$$= \int_{S_n} (A d\beta d\varphi d\theta_1 \ldots\ldots d\theta_{n-2} + B d\alpha d\varphi d\theta_1 \ldots d\theta_{n-2} + \Phi d\alpha d\beta d\theta_1 \ldots d\theta_{n-2}$$

$$+ \Theta_1 d\alpha d\beta d\varphi d\theta_2 \ldots\ldots d\theta_{n-2} + \ldots\ldots + \Theta_{n-2} d\alpha d\beta \ldots\ldots d\theta_{n-3}).$$

Hence

$$I_n \equiv \int_{S_n} (A\nu_\alpha + B\nu_\beta + \Phi\nu_\varphi + \Theta_1\nu_{\theta_1} + \ldots\ldots + \Theta_{n-2}\nu_{\theta_{n-2}})\, dS_n = 0 \qquad (4.5.8)$$

where $\nu_\alpha, \nu_\beta \ldots\ldots \nu_{\theta_{n-2}}$ are the components of the unit outward normal to S_n and dS_n is a positive element of area on the surface S_n. We note that each of A, $B, , \ldots\ldots, \Theta_{n-2}$ is a bilinear form in the partial derivatives u and v with respect to $\alpha, \beta, \varphi, \theta_1, \ldots\ldots, n-2$.

We are now in a position to solve the wave equation (4.5.1) subject to the Cauchy data (4.5.2). Let $\bar{P}(\bar{x}_1, \bar{x}_2, \ldots\ldots, \bar{x}_n, \bar{t})$ denote the point $(\bar{x}_1, \ldots\ldots, \bar{t})$. We now try to solve the Cauchy problem by expressing $u(\bar{x}_1, \bar{x}_2, \ldots\ldots, \bar{t})$ as an integral in terms of the initial data f and g on the carrier hyperplane $t = 0$ contained within the lower mantle of the characteristic half-cone with vertex at $\bar{P}$ i.e. in terms of the initial data prescribed on

$$(x_1 - \bar{x}_1)^2 + \dots\dots + (x_n - \bar{x}_n)^2 \leq \bar{t}^2, \ \bar{t} > 0, \ t = 0. \qquad (4.5.9)$$

Consider the (n+1)-dimensional conical volume C bounded in space-time by the characteristic hypercone with vertex at $\bar{P}$ and $t = 0$. The axis of C is the straight line $\bar{P}P_0$ traced out by $P(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, t)$ as t ranges from 0 to $\bar{t}$. Let us introduce at each point $P(x_1, x_2, \dots, x_n, t)$, as in the case for $n = 2$, the polar coordinates $\varphi, \theta_1, \theta_2, \dots\dots, \theta_{n-2}r$ with pole at P. Then the conical volume C is described by

$$C: 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta_j \leq \pi \ (j = 1, 2, \dots, n-2), \ 0 \leq r \leq \bar{t}-t, \ 0 \leq t \leq \bar{t} \qquad (4.5.10)$$

Now take $\alpha, \beta, \varphi, \theta_1, \dots\dots, \theta_{n-2}$, where $\alpha = t + r$, $\beta = t - r$, as rectangular coordinates in an $(n + 1)$-dimensional space. Then C is transformed into the wedge

$$W: 0 \leq \alpha \leq \bar{t}, \ -\alpha \leq \beta \leq \alpha, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta_j \leq \pi \ (j=1,2,\dots,n-2)$$

As in the case for $n = 2$, the following also hold:

The conical mantle of C is transformed into $\alpha = \bar{t}$ of W

The base $t = 0$ of C is transformed into the face $\beta = -\alpha$ of W

The axis $\bar{P}P_0$ of C is transformed into the face $\beta = \alpha$ of W

The vertex $\bar{P}$ of C is transformed into the edge $\alpha = \beta = \bar{t}$ of W

The perphery of the base of C is transformed into the edge $\alpha = -\beta = \bar{t}$ of W.

The centre P_0 of the base of C is transformed into the edge $\alpha = \beta = 0$ of W

The carrier $t = 0$ of the Cauchy data in C is transformed into the face $\beta = -\alpha$, the carrier of Cauchy data in W. Also

$$u_\alpha = u_t \cdot \tfrac{1}{2} + u_r \cdot \tfrac{1}{2} = \frac{u_r + u_t}{2}$$

$$u_\beta = u_r(-\tfrac{1}{2}) + u_t \cdot \tfrac{1}{2} = -\tfrac{1}{2}(u_r - u_t).$$

Therefore from the given Cauchy data on $t = 0$ in C we get

$$u_\alpha = \frac{f_r + g}{2}, \; u_\beta = -\frac{f_r - g}{2} \qquad (4.5.12)$$

on the carrier $\beta + \alpha = 0$ of W.

Since $\alpha = \beta = \bar{t}$ is the transform of $\bar{P}$, we now try to find a solution u of (4.5.3) along the edge $\alpha = \beta = \bar{t}$ in terms of the initial data (4.5.12). Equation (4.5.8) now becomes

$$\underset{\beta=\alpha}{I_n} + \underset{\beta=-\alpha}{I_n} + \underset{\alpha=\bar{t}}{I_n} + \underset{\varphi=0}{I_n} + \underset{\varphi=2\pi}{I_n} + \sum_{j=1}^{n-2}\left(\underset{\theta_j=0}{I_n} + \underset{\theta_j=\pi}{I_n} \right) = 0 \qquad (4.5.13)$$

As in the case for $n = 2$ $\underset{\varphi=0}{I_n} + \underset{\varphi=2\pi}{I_n} = 0.$

Also each Θ_j, $j = 1, 2, \ldots, n-2$, contains $\sin\theta_j$ as a factor and so

$$\underset{\theta_j=0}{I_n} = 0 = \underset{\theta_j=\pi}{I_n}$$

Hence (4.5.13) reduces to

$$\underset{\beta=\alpha}{I_n} + \underset{\beta=-\alpha}{I_n} + \underset{\alpha=\bar{t}}{I_n} = 0 \qquad (4.5.14)$$

where $I_n = \int_{S_n} (A\nu_\alpha + B\nu_\beta + \Phi\nu_\varphi + \Theta_1\nu_{\theta_1} + \dots + \Theta_{n-2}\nu_{\theta_{n-2}})dS_n.$

Using the same reasoning as in the case for $n = 2$, we find that

$\underset{\beta=\alpha}{I_n} = \int_0^{\bar{t}}\int_{W_n} (-A + B)f^{-1}d\alpha dw_n + 0$, since the normal to $\beta = \alpha$ is perpendicular to the remaining $(n-1)$ directions;

$\underset{\alpha=\bar{t}}{I_n} = \int_{S_n} Af^{-1}d\beta dw_n + o = \int_{-\bar{t}}^{\bar{t}}\int_{W_n} f^{-1}d\beta dw_n$, since $d\alpha = 0$;

and $\underset{\beta=-\alpha}{I_n} = \int_{S_n} (-Af^{-1}d\beta dw_n - Bf^{-1}d\alpha dw_n) + o$

$$= \int_0^{\bar{t}}\int_{W_n} (A + B)f^{-1}d\alpha dw_n$$

Hence we have, since $A = fu_\beta v_\beta$ and $B = -fu_\alpha v_\alpha$,

$$-\int_0^{\bar{t}}\int_{W_n} [u_\alpha v_\alpha + u_\beta v_\beta]_{\beta=\alpha} dw_n d\alpha + \int_{-\bar{t}}^{\bar{t}}\int_{W_n} [u_\beta v_\beta]_{\alpha=\bar{t}}\, d\beta dw_n +$$

$$\int_0^{\bar{t}}\int_{W_n} [u_\alpha v_\alpha - u_\beta v_\beta]_{\beta=-\alpha} dw_n d\alpha = 0 \qquad (4.5.15)$$

As in the case for $n = 2$, the characteristic hyperplanes through $\alpha = \beta = \bar{t}$ over which the boundary conditions which a solution of (4.5.6) must satisfy are $\alpha = \bar{t}$ and $\beta = \alpha$. From the second integral on the left of (4.5.15) we are motivated to let $v_\beta = 0$ on $\alpha = \bar{t}$. Using the procedure we adopted in the special simpler case for $n = 3$, we let

$v_\alpha = v_\beta = (\bar{t} - \alpha)^{n-2}$ when $\beta = \alpha$. This is because for $n = 3$, the appropriate v satisfied $v_\alpha = (\bar{t} - t)$ when $\beta = \alpha$. But when $\beta = \alpha$, $\alpha = t$, $r = 0$. We are therefore led to solve equation (4.5.6) subject to the boundary conditions

$$v_\beta = 0 \quad \text{on} \quad \alpha = \bar{t}, \quad v_\alpha = v_\beta = (\bar{t} - t)^{n-2} \quad \text{on} \quad \beta = \alpha \qquad (4.5.16)$$

These conditions suggest that we try, following the case for $n = 2$,

$$v = A(\bar{t} - \alpha)^m(\bar{t} - \beta)^m$$

Then

$$v_\alpha = -mA(\bar{t} - \alpha)^m(\bar{t} - \beta)^{m-1}, \quad v_\beta = -mA(\bar{t} - \alpha)^m(\bar{t} - \beta)^{m-1}.$$

when $\beta = \alpha$, $v_\alpha = -mA(\bar{t} - \alpha)^{2m-1} = (\bar{t} - \alpha)^{n-2}$ for fixed n.

Hence $m = \frac{n-1}{2}$, $A = -\frac{2}{n-1}$. It follows that

$$v = -\frac{2}{n-1}(\bar{t} - \alpha)^{\frac{n-1}{2}}(\bar{t} - \beta)^{\frac{n-1}{2}} \qquad (4.5.17)$$

It is easily verified that (4.5.17) satisfies the associate equation (4.5.6).

$$v_\alpha = (\bar{t} - \alpha)^{\frac{n-3}{2}} (\bar{t} - \beta)^{\frac{n-1}{2}} , \quad v_\beta = (\bar{t} - \alpha)^{\frac{n-1}{2}} (\bar{t} - \beta)^{\frac{n-3}{2}} .$$

Therefore when $\beta = \alpha$, then $\alpha = t$ and

$$u_\alpha v_\alpha + u_\beta v_\beta = (\bar{t} - t)^{n-2}[u_\alpha + u_\beta]_{\beta=\alpha}$$

$$= (\bar{t} - t)^{n-2}[u_t]_{r=0}$$

$$= (\bar{t} - t)^{n-2} u_t(\bar{x}_1, \ldots, \bar{x}_n, t).$$

when $\beta = -\alpha$, $t = 0$, $\alpha = r$ and so

$$u_\alpha v_\alpha - u_\beta v_\beta = (\bar{t}-r)^{\frac{n-3}{2}} (\bar{t}+r)^{\frac{n-3}{2}} [\bar{t}(u_\alpha - u_\beta) + r(u_\alpha + u_\beta)]_{\beta=-\alpha}$$

$$= (\bar{t}^2 - r^2)^{\frac{n-3}{2}} [\bar{t} f_r + rg]$$

Equation (4.5.15) therefore reduces to

$$\int_0^{\bar{t}} \int_{W_n} (\bar{t} - t)^{n-2} u_t(\bar{x}_1, \ldots, \bar{x}_n, t) dw_n dt = \int_0^{\bar{t}} \int_{W_n} (\bar{t} - r^2)^{\frac{n-3}{2}} (\bar{t} f_r + rg) \, dw_n dt.$$

Differentiating this w.r.t. $\bar{t}$ $(n-2)$ times we get

$$(n-2)! \; w_n \int_0^{\bar{t}} u_t(\bar{x}_1, \ldots, \bar{x}_n, t) dt = \frac{\partial^{n-2}}{\partial \bar{t}^{n-2}} \int_0^{\bar{t}} \int_{W_n} (\bar{t}^2 - r^2)^{\frac{n-3}{2}} (\bar{t} f_r + rg) \, dw_n dt.$$

i.e.

$$u(\bar{P}) = u(P_0) + \frac{1}{(n-2)!w_n} \frac{\partial^{n-2}}{\partial \bar{t}^{n-2}} \int_0^{\bar{t}} \int_{w_n} \left(\bar{t}-r^2\right)^{\frac{n-3}{2}} \left(\bar{t}f_r + rg\right) dw_n dt \qquad (4.5.18)$$

where $w_n = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

The classical solution for this problem is

$$u(\bar{P}) = \frac{1}{(n-2)!w_n} \frac{\partial^{n-1}}{\partial \bar{t}^{n-1}} \int_0^{\bar{t}} \int_{w_n} \left(\bar{t}^2-r^2\right)^{\frac{n-3}{2}} rf \, dw_n \, dr$$

$$+ \frac{1}{(n-2)!w_n} \frac{\partial^{n-2}}{\partial \bar{t}^{n-2}} \int_0^{\bar{t}} \int_{w_n} \left(\bar{t}^2-r^2 \right)^{\frac{n-3}{2}} rg \, dw_n \, dr. \qquad (4.5.19)$$

It can also be proved, as in the case for $n = 2$, that (4.5.18) and (4.5.19) represent the same solution of the given problem. This also shows that the solution is unique.

6. Some particular examples

As a first example, let $L(u) \equiv u_{xy} - u_x + 2u_y = 0$ (4.6.1)

$\varphi = 2x + y$ is an elementary solution of the equation. Also $a = 1$, $b = 2$, $\varphi_x = 2$, $\varphi_y = 1$. Hence Martin function v satisfies

$$M(v) \equiv v_{xy} + v_x - 2v_y = 0, \tag{4.6.2a}$$

$$v_x = \varphi_x = 2 \quad \text{on} \quad y = \bar{y}, \tag{4.6.2b}$$

$$v_y = -\varphi_y = -1 \quad \text{on} \quad x = \bar{x}. \tag{4.6.2c}$$

For simplicity, let us first derive v at the point $\bar{x} = 0$, $\bar{y} = 0$.

Let $v = a_0(y) + a_1(y)x + a_2(y)\frac{x^2}{2!} + a_3(y)\frac{x^3}{3!} + \dots + a_n(y)\frac{x^n}{n!} + \dots$

Then $v_x = a_1(y) + 2a_2(y)\frac{x}{2!} + \dots\dots$

$v_x(x, 0) = a_1(0) + a_2(0)\frac{x}{2!} + \dots\dots$

$= 2$, by (4.6.2b).

$\therefore$ $a_1(0) = 2$, $a_n(0) = 0$, $n = 2, 3, \dots\dots$

Also $v_y = \mathring{a}_0(y) + \mathring{a}_1(y)x + \mathring{a}_2(y)\frac{x^2}{2!} + \dots\dots$

$v_y(0, y) = \mathring{a}_0(y) = -1$, by (4.6.2c)

Hence $a_0(y) = -y + c_0$ where c_0 is an arbitrary constant. Since in the Martin solution (4.2.8), c_0 does not affect the solution of the Cauchy problem we may take $c_0 = 0$.

Since $v = \sum_{r=0}^{\infty} a_r \frac{x^r}{r!}$ has to satisfy the associate equation (4.6.2a)

we must have

$$\sum_{r=1}^{\infty} \mathring{a}_r \frac{x^{r-1}}{(r-1)!} + \sum_{r=1}^{\infty} \ddot{a}_r \frac{x^{r-1}}{(r-1)!} - \sum_{r=0}^{\infty} 2\mathring{a}_r \frac{x^r}{r!} = 0$$

Hence $\mathring{a}_r + \ddot{a}_r = 2\mathring{a}_{r-1} \qquad r = 1, 2, 3, \ldots\ldots$

When $r = 1 \qquad \mathring{a}_1 + a_1 = 2\mathring{a}_0 = -2$

$$\mathring{a}_1 e^y + a_1 e^y = -2e^y$$

$$\therefore \quad a_1 e^y = -2e^y + c_1$$

$$a_1 = -2 + c_1 e^{-y}$$

$$a_1(o) = -2 + c_1 = 2, \quad c_1 = 4$$

$$\therefore a_1(y) = -2 + 4e^{-y}$$

Similarly $a_2(y) = -8ye^{-y}, \quad a_3(y) = (-16y + 8y^2)e^{-y}$

$$a_4(y) = (-32y + 32y^2 - \frac{16}{3}y^3)e^{-y}$$

$$a_5(y) = (-64y + 96y^2 - 32y^3 + \frac{8}{3}y^4)e^{-y}, \quad \text{etc.}$$

Hence $v = -y - 2x - 4e^{-y}\Big\{-x + 2y\frac{x^2}{2!} + (4y - 2y^2)\frac{x^3}{3!} + (8y - 8y^2 + \frac{4}{3}y^3)\frac{x^4}{4!}$

$$+ (16y - 24y^2 + 8y^3 - \frac{2}{3}y^4)\frac{x^5}{5!} + \ldots\ldots \qquad (4.6.3)$$

Martin function is then obtained from (4.6.3) by replacing x and y by $x - \bar{x}$ and $y - \bar{y}$ respectively.

It is possible in this case, and in some similar examples, to obtain an integral representation of the Martin function.

Let $f = -x + 2y\frac{x^2}{2!} + \left(4y - 4 \cdot \frac{y^2}{2!}\right)\frac{x^3}{3!} + \left(8y - 16\frac{y^2}{2!} + 8\frac{y^3}{3!}\right)\frac{x^4}{4!}$

$$+ \left(16y - 48\frac{y^2}{2!} + 48\frac{y^3}{3!} - 16\frac{y^4}{4!}\right)\frac{x^5}{5!} + \cdots\cdots$$

For fixed x, let us define the Laplace transform of f by

$$\mathcal{L}(f) = \int_0^\infty e^{-sy} f(y)\,dy.$$

It then follows that

$$\mathcal{L}(f) = -\frac{x}{s} + \frac{2x^2}{2!}\frac{1}{s^2} + \frac{x^3}{3!}\left(\frac{4}{s^2} - \frac{4}{s^3}\right) + \frac{x^4}{4!}\left(\frac{8}{s^2} - \frac{16}{s^3} + \frac{8}{s^4}\right)$$

$$+ \frac{x^5}{5!}\left(\frac{16}{s^2} - \frac{48}{s^3} + \frac{48}{s^4} - \frac{16}{s^5}\right) + \cdots\cdots$$

$$= -\frac{x}{s} + \frac{x^2}{2!}\cdot\frac{2}{s^2} + \frac{1}{2s^2\left(1-\frac{1}{s}\right)^2}\left[\frac{(2x)^3}{3!}\left(1-\frac{1}{s}\right)^3 + \frac{(2x)^4}{4!}\left(1-\frac{1}{s}\right)^4 + \frac{(2x)^5}{5!}\left(1-\frac{1}{s}\right)^5 + \right.$$

$$= -\frac{x}{s} + \frac{2x^2}{s^2 2!} + \frac{1}{2(s-1)^2}\left[1 + \frac{2x}{1}\left(1-\frac{1}{s}\right) + \frac{(2x)^2}{2!}\left(1-\frac{1}{s}\right)^2 + \frac{(2x)^3}{3!}\left(1-\frac{1}{s}\right)^3 + \cdot\right.$$

$$- \frac{1}{2(s-1)^2} - \frac{1}{2(s-1)^2}\cdot\frac{2x}{1}\cdot\left(1-\frac{1}{s}\right) - \frac{1}{2(s-1)^2}\frac{(2x)^2}{2!}\left(1-\frac{1}{s}\right)^2.$$

$$= -\frac{x}{s-1} - \frac{1}{2(s-1)^2} + \frac{1}{2(s-1)^2}\,e^{2x\left(1-\frac{1}{s}\right)}$$

$$= -\frac{x}{s-1} - \frac{1}{2(s-1)^2} + \frac{e^{2x}}{2}\cdot\frac{s}{(s-1)^2}\cdot\frac{e^{-2\frac{x}{s}}}{s}$$

$$\mathcal{L}^{-1}\frac{x}{s-1} = xe^{y},\ \mathcal{L}^{-1}\frac{1}{(s-1)^2} = ye^{y}\ ,$$

$$\mathcal{L}^{-1}\frac{s}{(s-1)^2} = (1+y)e^{y} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{e^{\frac{2x}{s}}}{s}\right) = J_0(2\sqrt{2xy})$$

It follows from the convolution theorem that

$$\mathcal{L}^{-1}\left(\frac{s}{(s-1)^2}\cdot\frac{e^{\frac{2x}{s}}}{s}\right) = \int_0^y e^{y-t}(1+y-t)\,J_0(\sqrt{8xt})dt\ .$$

Hence $\quad f = \mathcal{L}^{-1}\,\mathcal{L}(f)$

$$= -xe^{y} - \tfrac{1}{2}ye^{y} + \frac{e^{2x}}{2}\int_0^y e^{y-t}(1+y-t)\,J_0(\sqrt{8xt})\,dt$$

Equation (4.6.3) therefore becomes

$$v = -y - 2x + 4x + 2y - 2e^{2x-y}\int_0^y e^{y-t}(1+y-t)\,J_0(\sqrt{8xt})dt$$

i.e.

$$v = y + 2x - 2\int_0^y e^{2x-t}(1+y-t)\,J_0(\sqrt{8xt})dt\ .$$

Replacing x by $x-\bar{x}$, y by $y-\bar{y}$, the Martin function for the equation (4.6.1) is

$$v(x,\ y;\ \bar{x},\ \bar{y}) = y - \bar{y} + 2(x-\bar{x}) - 2\int_0^{y-\bar{y}} e^{2(x-\bar{x})-t}(1+y-\bar{y}-t)\,J_0(\sqrt{8(x-\bar{x})t})dt$$

(4.6.4)

As a second example, let us consider the equation

$$L(u) \equiv u_{xx} - 4u_{yy} + u_x + u_y = 0 \qquad (4.6.5)$$

Characteristics are given by the equations $y \pm 2x = \text{const.}$ Let $\xi = y + 2x$, $\eta = y - 2x$. Then equation (4.6.5) is transformed into

$$16\, u_{\xi\eta} - 3u_\xi + u_\eta = 0 \qquad (4.6.6).$$

We can now apply the method of the first example to (4.6.6).

As a third and last example, let us again consider the Euler-Darboux equation (3.3.27), i.e. the equation

$\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{m}{x-y}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} \right) = 0$, where m is a constant. Since $\varphi = x + y$ satisfies (3.3.27), the resolvent equation is given by

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{m}{x-y}\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right) = \qquad (4.6.7)$$

The boundary conditions which (4.6.7) must satisfy are

$v_x = \varphi_x = 1$ on $y = y_0$

$v_y = -\varphi_y = -1$ on $x = x_0$

$v(x, y;\ x_0, y_0) = 0.$

It follows from these conditions that

$$v(x, y_0;\ x_0, y_0) = x - x_0, \qquad (4.6.8a)$$

$$v(x_0, y;\ x_0, y_0) = y_0 - y. \qquad (4.6.8b)$$

The procedure we adopted in the case of the Riemann function fails here, since the boundary conditions do not involve m. However, we find that $v = (x-a)^m(a-y)^m$ is a particular solution of the equation (4.6.7) where a is any fixed number.

Since $y = x$ is a line of singularity of the equation (4.6.7) we shall solve this equation in the half-plane $x > y$. In this region let us first take $x > x_0 > 0$, $y < y_0$ and $m = -\lambda$, $0 < \lambda < 1$.

Since a is a parameter,

$v = \int_y^x f(a)\ (x-a)^{-\lambda}(a-y)^{-\lambda}\ da$, where $f(a)$ is an arbitrary function of a, is a solution of (4.6.7).

It follows from the above restrictions that $y < y_0 < x_0 < x$, and so

$$v = \int_y^{y_0} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da + \int_{y_0}^{x_0} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da + \int_{x_0}^{x} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da.$$

Let us suppose that $f(a) = 0$ when $y_0 \leqslant a \leqslant x_0$. Then

$$v = \int_{x_0}^{x} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da + \int_{y}^{y_0} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da, \qquad (4.6.10)$$

where $f(a)$ is to be determined from the integral equations

$$v(x,\ y_0;\ x_0,y_0) = \int_{x_0}^{x} f(a)(x-a)^{-\lambda}(a-y_0)^{-\lambda}da = x-x_0, \qquad (4.6.11a)$$

$$= \int_{x_0}^{x}$$

$$v(x_0, y; x_0, y_0) = \int_y^{y_0} f(a)(x-a)^{-\lambda}(a-y)^{-\lambda}da = y_0 - y. \qquad (4.6.11b)$$

Let $\psi(t) = 0$, $\varphi(t) = 0$ when $0 \leqslant t \leqslant y_0$,

$\psi(t) = f(t)(t-y_0)^{-\lambda}$, $\varphi(t) = t-t_0$ when $t \geqslant x_0$.

It then follows from (4.6.11a) that

$$\int_0^x \frac{\psi(t)}{(x-t)^\lambda}dt = \varphi(x), \quad 0 < \lambda < 1.$$

This is Abel's integral equation with solution

$$\psi(t) = \frac{\sin \lambda \pi}{\pi} \frac{d}{dt} \int_0^t \frac{\varphi(x)\,dx}{(t-x)^{1-\lambda}} = \frac{\sin \lambda \pi}{\pi} \frac{d}{dt} \int_{x_0}^t \frac{(x-x_0)dx}{(t-x)^{1-\lambda}}$$

$$= \frac{\sin \pi \lambda}{\pi} \frac{d}{dt} \left(\frac{(t-x_0)^{\lambda+1}}{\lambda(\lambda + 1)} \right) = \frac{\sin \pi \lambda}{\pi\lambda} (t-x_0)^\lambda.$$

Hence $f(t) = \frac{\sin \pi \lambda}{\pi\lambda} (t-x_0)^\lambda (t-y_0)^\lambda$. (4.6.12)

The function $f(t)$ defined by (4.6.12) also satisfies (4.6.11b).

It follows from (4.6.10) that $v(x, y; x_0, y_0) = I_1 + I_2$ where

$$I_1 = \frac{\sin \pi \lambda}{\pi\lambda} \int_{x_0}^x (a-x_0)^\lambda (a-y_0)^\lambda (x-a)^{-\lambda} (a-y)^{-\lambda} da.$$ Let $a = x_0 + (x-x_0)t$,

then

$$I_1 = \frac{\sin \pi\lambda}{\pi\lambda} \int_0^1 (x-x_0)^{\lambda+1} t^{\lambda} [x_0 - y_0 + (x-x_0)t]^{\lambda} [x - x_0 - (x-x_0)t]^{-\lambda} [x_0 - y + (x-x_0)t]^{-\lambda}\, dt$$

$$= \frac{\sin \pi\lambda}{\pi\lambda} (x-x_0) \int_0^1 \frac{t^{\lambda}[x_0 - y_0 + (x-x_0)t]^{\lambda}}{(1-t)^{\lambda}[x_0 - y + (x-x_0)t]^{\lambda}}\, dt$$

$$= \frac{\Gamma(2)(x-x_0)}{\Gamma(1+\lambda)\Gamma(1-\lambda)} \left(\frac{x_0 - y_0}{x_0 - y}\right)^{\lambda} \int_0^1 \frac{t^{\lambda}\left(1 - \frac{x-x_0}{y_0 - x_0} t\right)^{\lambda}}{(1-t)^{\lambda}\left(1 - \frac{x-x_0}{y-x_0} t\right)^{\lambda}}\, dt, \quad \text{since}$$

$$\frac{\pi z}{\sin \pi z} = z\Gamma(z)\,\Gamma(1-z) = \frac{\Gamma(1+z)\Gamma(1-z)}{\Gamma(2)} .$$

Now if $\mathrm{Re}\,\alpha > 0$, $R_e(\gamma - \alpha) > 0$, it can be proved that

$$\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}\, dt = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha;\ \beta, \beta';\ \gamma;\ x, y),$$

where
$$F_1(\alpha;\ \beta, \beta';\ \gamma;\ x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}(1)_m(1)_n} x^m y^n \qquad (4.6.13)$$

is Appel's hypergeometric function of two variables, and $(\alpha)_r = \alpha(\alpha+1)\ldots(\alpha+r-1)$. It follows that if we let $\alpha - 1 = \lambda$, $\gamma - \alpha - 1 = -\lambda$, $\beta = -\lambda$, $\beta' = \lambda$, then $\gamma = 2$ and

$$I_1 = (x-x_0)\left(\frac{x_0 - y_0}{x_0 - y}\right)^{\lambda} F_1\left(1+\lambda;\ -\lambda, \lambda;\ 2;\ \frac{x-x_0}{y_0 - x_0}, \frac{x-x_0}{y-x_0}\right).$$

Similarly
$$I_2 = -(y-y_0)\left(\frac{x_0 - y_0}{x - y_0}\right)^{\lambda} F_1\left(1+\lambda, -\lambda, \lambda;\ 2;\ \frac{y-y_0}{x_0 - y_0}, \frac{y-y_0}{x-y_0}\right).$$

Hence the Martin function of the equation (3.3.27) subject to the above restrictions is given by

$$v(x,\ y;\ x_0,\ y_0) = (x-x_0)\left(\frac{x_0-y_0}{x_0-y}\right)^{\lambda} F_1\left(1+\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{x-x_0}{y_0-x_0},\ \frac{x-x_0}{y-x_0}\right)$$

$$-(y-y_0)\left(\frac{x_0-y_0}{x-y_0}\right)^{\lambda} F_1\left(1+\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{y-y_0}{x_0-y_0},\ \frac{y-y_0}{y-x_0}\right). \qquad (4.6.14)$$

It remains only to show that from (4.6.14) the Martin function of the equation (3.3.27) can be constructed throughout the region $x > y$.

It can be shown that

$F_1(\alpha;\ \beta,\ \beta';\ \gamma;\ x,\ y) = (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha;\ \beta,\ \beta';\ \gamma;\ \frac{x}{x-1},\ \frac{y}{y-1}\right)$. Hence

$$I_1 = (x-x_0)\left(\frac{x_0-y_0}{x_0-y}\right)^{\lambda}\left(1-\frac{x-x_0}{y_0-x_0}\right)^{\lambda}\left(1-\frac{x-x_0}{y-x_0}\right)^{-1} F_1\left(1-\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{x-x_0}{x-y_0},\ \frac{x-x_0}{x-y}\right)$$

$$= (x-x_0)\left(\frac{x-y_0}{x-y}\right)^{\lambda} F_1\left(1-\lambda;\ -\lambda;\ \lambda;\ 2;\ \frac{x-x_0}{x-y_0},\ \frac{x-x_0}{x-y}\right) \qquad (4.6.15)$$

Similarly

$$I_2 = -(y-y_0)\left(\frac{x_0-y}{x-y}\right)^{\lambda} F_1\left(1-\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{y-y_0}{y-x_0},\ \frac{y-y_0}{y-x}\right). \qquad (4.6.16)$$

It follows from (4.6.15) and (4.6.16) that in addition to (4.6.14) there are three other possible expressions for v i.e.

$$v = (x-x_0)\left(\frac{x-y_0}{x-y}\right)^{\lambda} F_1\left(1-\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{x-x_0}{x-y_0},\ \frac{x-x_0}{x-y}\right) - (y-y_0)\left(\frac{x_0-y_0}{x-y_0}\right)^{\lambda}$$

$$F_1\left(1+\lambda;\ -\lambda, \lambda;\ 2;\ \frac{y-y_0}{x_0-y_0},\ \frac{y-y_0}{x-y_0}\right), \qquad (4.6.17)$$

$$v = (x-x_0)\left(\frac{x_0-y_0}{x_0-y}\right)^{\lambda} F_1\left(1+\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{x-x_0}{y_0-x_0},\ \frac{x-x_0}{y-x_0}\right) - (y-y_0)\left(\frac{x_0-y}{x-y}\right)^{\lambda}$$

$$F_1\left(1-\lambda;\ -\lambda,\ \lambda;\ 2;\ \frac{y-y_0}{y-x_0},\ \frac{y-y_0}{y-x}\right), \qquad (4.6.18)$$

$$v = (x-x_0)\left(\frac{x-y_0}{x-y}\right)^{\lambda} F_1\left(1-\lambda;\ \lambda,\ \lambda;\ 2;\ \frac{x-x_0}{x-y_0},\ \frac{x-x_0}{x-y}\right) - (y-y_0)\left(\frac{x_0-y}{x-y}\right)^{\lambda}$$

$$F_1\left(1-\lambda;\ -\lambda\ \ \lambda;\ 2;\ \frac{y-y_0}{y-x_0},\ \frac{y-y_0}{y-x}\right). \qquad (4.6.19)$$

The formula (4.6.13) is convergent for $-1 < x < 1$, $-1 < y < 1$ provided that γ is not an integer and α, β, γ are any real complex numbers.

From these results, it can be shown by the principle of analytic continuation [6] that the solution (4.6.14) can be continued throughout the region $x > y$ for all real values of λ.

CHAPTER V: The Relation Between Riemann and Martin Functions.

1. Let us for the purpose of this Section consider the equation

$$L(u) \equiv u_{xy} + \alpha(x, y)(u_x + u_y) = f(x, y), \qquad (5.1.1)$$

where $\alpha(x, y)$, $f(x, y)$ are functions of x and y only. As $\varphi = x-y$ is a particular solution of $L(u) = 0$, the associate equation of (5.1.1) is

$$v_{xy} - a(x, y)(v_x + v_y) = 0. \qquad (5.1.2)$$

Taking the arbitrary constant zero, the Martin function v therefore satisfies the additional conditions

$$v_x = \varphi_x = 1 \quad \text{on} \quad y = y_0, \qquad (5.1.3a)$$

$$v_y = -\varphi_y = 1 \quad \text{on} \quad x = x_0, \qquad (5.1.3b)$$

$$v(x, y; \; x_0, y_0) = 0. \qquad (5.1.3.c)$$

Multiplying (5.1.1) by $(v_x + v_y)$ and (5.1.2) by $(u_x + u_y)$ and adding we get

$$\frac{\partial}{\partial y}(u_x \, v_x) + \frac{\partial}{\partial x}(u_y \, v_y) = (v_x + v_y) \, f(x, y).$$

It follows from Green's theorem and figure 5a that

$$\int_\Gamma (-u_x v_x dx + u_y v_y dy) = \iint_\Omega (v_x + v_y) f(x, y) dx \, dy. \qquad (5.1.4)$$

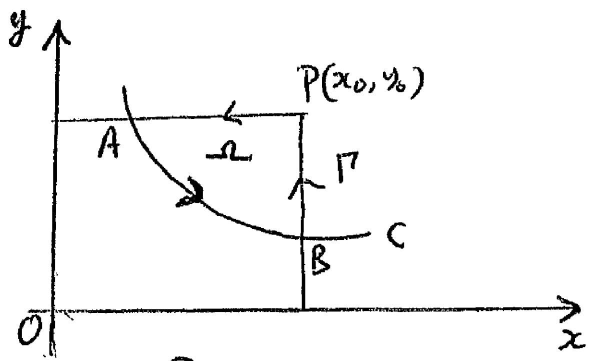

Fig. 5a

Along PA i.e. on $y = y_0$; $dy = 0$ and

$$\int_{PA} -u_x v_x dx = -\int_{PA} u_x dx = -u(A) + u(x_0, y_0).$$

Along BP i.e. on $x = x_0$; $dx = 0$ and

$$\int_{BP} u_y v_y dy = \int_{BP} u_y dy = u(B) + u(x_0, y_0).$$

It therefore follows from (5.1.4) that

$$u(x_0, y_0) = \frac{u(A) + u(B)}{2} + \frac{1}{2}\int_{AB} (u_x v_x dx - u_y v_y dy) + \frac{1}{2}\iint_\Omega (v_x + v_y) f(x, y) dx \, dy. \qquad (5.1.4)$$

This is an extension of the Martin's method which we have already established for $f(x,y) = 0$. We are now in a position to compare the solution (3.1.9) of the Riemann's method when $a = b = \alpha$ and (5.1.4) of the Martin's method. For the remainder of this chapter, we shall denote Riemann function by R and Martin function by M. Then (3.1.9) and (5.1.4) may be written as

$$u(x_0,y_0) = \frac{u(A)R(A) + u(B)R(B)}{2} + \frac{1}{2}\int_{AB} [Ru_x - uR_x + 2\alpha uR)dx - (Ru_y - uR_y + 2\alpha uR)dy]$$

$$+ \iint_{\Omega} rf \, dx\, dy, \qquad (5.1.5)$$

$$u(x_0,y_0) = \frac{u(A) + u(B)}{2} + \frac{1}{2}\int_{AB} (u_x M_x dx - u_y M_y d_y) + \frac{1}{2}\iint_{\Omega} (M_x + M_y) f \, dx\, dy \qquad (5.1.6)$$

Comparing the integrands in the double integrals in (5.1.5) and (5.1.6), let us tentatively assume that

$$R = \frac{1}{2}(M_x + M_y). \qquad (5.1.7)$$

Then $Ru_x - uR_x + 2\alpha uR = \frac{u_x}{2}(M_x + M_y) - \frac{u}{2}(M_{xx} + M_{xy}) + \alpha u(M_x + M_y)$

$$= \frac{u_x}{2}(M_x + M_y) - \frac{u}{2}(M_{xx} - M_{xy}), \quad \text{using (5.1.2)}$$

Similarly $Ru_y - uR_y + 2\alpha uR = \frac{u_x}{2}(M_x + M_y) - \frac{u}{2}(M_{yy} - M_{xy})$.

Now $\frac{1}{2}\int_{AB} \{-\frac{u}{2}(M_{xx}d_x + M_{xy}d_y) + \frac{u}{2}(M_{xy}d_x + M_{yy}d_y)\}$

$$= \frac{1}{2}\left[-\frac{u}{2}M_x + \frac{u}{2}M_y\right]_A^B + \frac{1}{2}\int_{AB}\left\{\frac{u_x d_x + u_y d_y}{2}M_x - \frac{u_x d_x + u_y d_y}{2}M_y\right\}$$

$$= \frac{-u(B)M_x(B)+u(B)M_y(B)+u(A)M_x(A)-u(A)M_y(A)}{4} + \frac{1}{4}\int_{AB}\left\{(u_xM_x-u_xM_x)dx+(u_yM_x-u_yM_y)d_y\right\}$$

Also $\frac{u(A)\ R(A)+u(B)\ R(B)}{2} = \frac{u(A)M_x(A)+u(A)M_y(A)+u(B)M_x(B)+u(B)M_y(B)}{4}$.

Now by (5.1.3a) and (5.1.3b), $M_x(A) = 1$, $M_y(B) = 1$ and therefore from (5.1.5) we have

$$u(x_0,y_0) = \frac{2u(A) + 2u(B)}{4} + \frac{1}{4}\int_{AB}(2u_x M_x d_x - 2u_y M_y d_y)$$

$$= \frac{u(A) + u(B)}{2} + \frac{1}{2}\int_{AB}(u_x M_x d_x - u_y M_y d_y).$$

We have thus used (5.1.7) to derive Martin's solution (5.1.6) from the Riemann's solution (5.1.5). The reverse process is also true. For, from (5.1.7) and (5.1.2), we have $M_{xy} = \alpha(M_x + M_y) = 2\alpha R$. Hence by (5.1.3a), (5.1.3b) and (5.1.3c) we have

$$M(x, y; x_0, y_0) = 2\int_{x_0}^{x}\int_{y_0}^{y}\alpha(u, v)R(u,v; x_0,y_0)dudv + x+y - x_0-y_0 \qquad (5.1.8)$$

2. As a first example, let us again consider the Eulen-Poisson Equation

$$U_{xy} + \frac{n}{x+y}(U_x + U_y) = 0 \qquad (5.2.1)$$

We have already shown that the Riemann function of (5.2.1) for positive integral values of n is

$$R(x, y; x_0, y_0) = \left(\frac{x+y}{x_0+y_0}\right) P_{n-1}(1-2\mu), \quad \mu = -\frac{(x-x_0)(y-y_0)}{(x+y)(x_0+y_0)}.$$

when $n = 1$ $R(x, y; x_0, y_0) = \dfrac{x+y}{x_0+y_0}$.

Hence the Martin function of (5.2.1) is given by

$$\frac{x+y}{x_0+y_0} = \tfrac{1}{2}(M_x+M_y).$$

But Martin function M satisfies the associate equation

$$M_{xy} - \frac{1}{x+y}(M_x + M_y) = 0.$$

Hence $M_{xy} = \dfrac{2}{x_0+y_0}$. It follows from this and the boundary conditions that

$$M(x, y; x_0, y_0) = \frac{(x-x_0)(x_0+y) + (y-y_0)(x+y_0)}{x_0 + y_0}, \quad \text{for } n = 1. \qquad (5.2.2)$$

When $n = 2$, $R = \left(\dfrac{x+y}{x_0+y_0}\right)^2 (1-\mu) = \left(\dfrac{x+y}{x_0+y_0}\right)^2 + \dfrac{2(x+y)(x-x_0)(y-y_0)}{(x_0+y_0)^3}$.

Hence $M_{xy} = \dfrac{4(x+y)}{(x_0+y_0)^2} + \dfrac{8(x-x_0)(y-y_0)}{(x_0+y_0)^3}$.

Integrating w. r. t. y,

$$M_x = \frac{2(x+y)^2}{(x_0+y_0)^2} + \frac{4(x-x_0)(y-y_0)^2}{(x_0+y_0)^3} + F(x)$$

when $y = y_0$, $M_x = 1 = \frac{2(x+y_0)^2}{(x_0+y_0)^2} + F(x)$

$$\therefore \quad M_x = \frac{2(x+y)^2}{(x_0+y_0)^2} + \frac{4(x-x_0)(y-y_0)}{(x_0+y_0)^3} + 1 - \frac{2(x+y_0)^2}{(x_0+y_0)^2} \ .$$

Integrating again partially w. r. t. x,

$$M = \frac{2(x+y)^3}{3(x_0+y_0)^2} + \frac{2(x-x_0)^2(y-y_0)^2}{(x_0+y_0)^3} + x - \frac{2(x+y_0)^3}{3(x_0+y_0)^2} + G(y)$$

From boundary conditions we get

$$G'(y) = 1 - \frac{2(x_0+y)^2}{(x_0+y_0)^2} \ , \quad G(y) = y - \frac{2(x_0+y)^3}{3(x_0+y_0)^2} - \frac{x_0+y_0}{3} \ . \quad \text{Hence}$$

$$M(x,y;x_0,y_0) = \frac{2(x+y)^3 - 2(x+y_0)^3 - 2(x_0+y)^3 - (x_0+y_0)^3}{3(x_0+y_0)^2} + \frac{2(x-x_0)^2(y-y_0)^2}{(x_0+y_0)^3} + x + y, \ldots \quad (5.2.3)$$

for $n = 2$.

If n is an integer greater than 2, or any other real number, the determination of the Martin function from the Riemann function (3.3.11) becomes very involved. In this case we can use the integral representation (3.3.26) of the Riemann function of the equation (5.2.1). From (3.3.26) and (5.1.8) it follows that

$$M = - \frac{1}{2\pi i n} \int_{c_1+c_2} \frac{(z-x)^n (z+y)^n}{(z-x_0)^n (z+y_0)^n} \, dz,$$

where c_1 and c_2 are the contours defined in Fig. 3d. When

$$n = 1, \quad M = -\frac{1}{2\pi i}\int_{C_1+C_2} \frac{(z-x)(z+y)}{(z-x_0)(z+y_0)}\, dz.$$

Let $f(z) = \frac{(z-x)(z+y)}{(z-x_0)(z+y_0)}$

Residue at $z = x_0$ of $f(z) = \lim_{z\to x_0} \frac{(z-x)(z+y)}{z+y_0} = \frac{(x_0-x)(x_0+y)}{x_0+y_0}$

Residue at $z = -y_0$ of $f(z) = \frac{(-y_0-x)(-y_0+y)}{(-y_0-x_0)} = \frac{(x+y_0)(y-y_0)}{x_0+y_0}$.

Hence, since C_1 and C_2 are in opposite directions,

$$M(x,y,x_0,y_0) = -\frac{1}{2\pi i}\;\frac{2\pi i[-(x-x_0)(x_0+y) - (x+y_0)(y-y_0)]}{x_0+y_0}$$

$$= \frac{(x-x_0)(x_0+y) + (x+y_0)(y-y_0)}{x_0+y_0}, \qquad (5.2.4)$$

for $n = 1$, which agrees with the result (5.2.2).

As a second example, let us consider the Martin function (4.6.4) of the equation (4.6.1) i.e.

$$M = 2(x-x_0) + y - y_0 - 2\int_0^{y-y_0} e^{2(x-x_0)-t}(1+y-y_0-t)\, J_0\left(\sqrt{8(x-x_0)t}\right) dt.$$

For simplicity, let us write $X = x-x_0$, $Y = y-y_0$ and $\lambda = \sqrt{8Xt}$.

Then $M = 2X + Y - 2\int_0^Y e^{2X-t}(1+Y-t)J_0(\lambda)\, dt$, and so

$$M_X = 2 - 4\int_0^Y e^{2X-t}(1+Y-t)J_0(\lambda)\, dt - 2\int_0^Y e^{2X-t}(1+Y-t)J_0'(\lambda)\cdot\sqrt{\frac{2t}{X}}\cdot dt,$$

since $\frac{\partial\lambda}{\partial X} = \sqrt{\frac{2t}{X}}$. It follows from $\frac{\partial\lambda}{\partial t} = \sqrt{\frac{2X}{t}}$ and

$$t \frac{\partial}{\partial t} J_0(\lambda) = \sqrt{2Xt}\; J_0{}'(\lambda) \quad \text{that}$$

$$\tfrac{1}{2}\left[2-4\int_0^Y e^{2X-t} J_0(\lambda)dt - M_x\right]$$

$$= 2\int_0^Y e^{2X-t}(Y-t)J_0(\lambda)dt + \int_0^Y e^{2X-t}(1+Y-t)J_0{}'(\lambda)\sqrt{\frac{2t}{X}}\,dt$$

$$= 2\int_0^Y e^{2X-t}(Y-t)J_0(\lambda)dt - \frac{1}{X}\left[t\,\frac{\partial J_0(\lambda)}{\partial t}\cdot(Y-t)e^{2X-t}\right]_0^Y$$

$$+ \frac{1}{X}\int_0^Y (Y-t)e^{2X-t}\frac{\partial}{\partial t}\left(t\,\frac{\partial J_0(\lambda)}{\partial t}\right)dt$$

$$= \int_0^Y e^{2X-t}(Y-t)\left\{\frac{1}{X}\frac{\partial}{\partial t}\left(t\,\frac{\partial J_0(\lambda)}{\partial t}\right) + 2J_0(\lambda)\right\} dt,$$

since $\lim\limits_{t\to 0} \; t\,\dfrac{\partial J_0(\lambda)}{\partial t} = -\lim\limits_{t\to 0}\sqrt{2Xt}\; J_1(\lambda) = 0.$

It follows from $t\,\dfrac{\partial J_0(\lambda)}{\partial t} = \sqrt{2Xt}\; J_0{}'(\lambda)$ that

$$\frac{1}{X}\frac{\partial}{\partial t}\left(t\,\frac{\partial J_0(\lambda)}{\partial t}\right) = \frac{1}{\sqrt{2Xt}}\,J_0{}'(\lambda) + 2\,J_0{}''(\lambda)$$

$$= 2\left(\frac{1}{\lambda}J_0{}'(\lambda) + J_0{}''(\lambda)\right)$$

$$= -2\,J_0(\lambda)\quad .$$

Hence $M_X = 2 - 4\int_0^Y e^{2X-t} J_0(\lambda)\,dt$, and so

$$M_x = 2 - 4\int_0^{y-y_0} e^{2(x-x_0)-t}\, J_0\left(\sqrt{8(x-x_0)t}\right) dt.$$ Also

$$M_y = 1-2\int_0^{y-y_0} e^{2(x-x_0)-t} J_0\left(\sqrt{8(x-x_0)t}\right) dt - 2e^{2(x-x_0)-(y-y_0)} J_0\left(\sqrt{8(x-x_0)(y-\right.$$

Hence $M_x - 2M_y = 4e^{2(x-x_0)-(y-y_0)}\, J_0\left(\sqrt{8(x-x_0)(y-y_0)}\right)$

$= 4\,R$, by (5.3.2).

Hence the relevant relation for the equation (4.6.1) is

$$R = \frac{1}{4}(M_x - 2\,M_y). \tag{5.2.5}$$

Similarly it can be shown that in the case of the equation

$$u_{xy} + \alpha u_x + \beta u_y = 0, \tag{5.2.6}$$

the relation between Riemann and Martin functions is given by

$$R = \frac{1}{2}\left(\frac{M_x}{\varphi_x} - \frac{M_y}{\varphi_y}\right), \tag{5.2.7}$$

where $u = \varphi(x, y)$ is a particular solution of (5.2.6). That (5.2.5) can be deduced from (5.2.7) follows from taking $\varphi = 2x + y$. It is also evident that (5.1.7) is a particular case of (5.2.7) by taking $\varphi = x-y$.

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