Quantum Group Actions on Singular Plane Curves

by

Angela Ankomaah Tabiri

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School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

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Abstract

In [28, 38], the coordinate ring of the cusp $y^2 = x^3$ is seen to be a quantum homogeneous space. Using this as a starting example, the coordinate ring of the nodal cubic $y^2 = x^2 + x^3$ was shown to be a quantum homogeneous space in [40]. This thesis focuses on finding singular plane curves which are quantum homogeneous spaces.

We begin by discussing the background theory of Hopf algebras, algebraic groups and the set up for Bergman's Diamond Lemma [9]. Next, we recall the theory of quantum homogeneous spaces in the commutative (classical) and noncommutative (nonclassical) settings. Examples and theorems on these spaces are stated.

Then main theorem in this thesis is that decomposable plane curves (curves of the form $f(y) = g(x)$) of degree less than or equal to five are quantum homogeneous spaces. In order to prove this, we construct two new families of Hopf algebras, $A(x, a, g)$ and $A(g, f)$. Then we use Bergman's Diamond lemma to prove that $A(g, f)$ is faithfully flat over the coordinate ring of $f(y) = g(x)$.

These new Hopf algebras that we have discovered have nice properties when $\deg(g)$, $\deg(f) \leq 3$. The properties include being noetherian domains, finite Gelfand-Kirillov dimensions, AS-regular and finite modules over their centres. We derive these properties from the isomorphism between $A(x, a, g)$ and well studied algebras, the localised quantum plane and down-up algebras [7] when $\deg(g) = 2, 3$.

Declaration

Name: Angela Ankomaah Tabiri
Registration Number: 2184490

I certify that the thesis presented here for examination for a PhD degree of the University of Glasgow is solely my own work other than where I have clearly indicated that it is the work of others and that the thesis has not been edited by a third party beyond what is permitted by the University's PGR Code of Practice.

I declare that the thesis does not include work forming part of a thesis presented successfully for another degree.

I declare that this thesis has been produced in accordance with the University of Glasgow's Code of Good Practice in Research.

I acknowledge that if any issues are raised regarding good research practice based on review of the thesis, the examination may be postponed pending the outcome of any investigation of the issues.

Signature: ....
Date: ....

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Contents

Chapter 1

Introduction

Let k be an algebraically closed field of characteristic 0. An affine homogeneous space over k is an affine variety with a group acting transitively on it. Quantum homogeneous spaces are noncommutative "analogues" of homogeneous spaces. There is a classical correspondence between the category of affine commutative Hopf algebras over k and the category of affine algebraic groups [81]. Under this correspondence, given an affine algebraic group G, the coordinate ring $\mathcal{O}(G)$ is a commutative Hopf algebra.

The surjectivity of the map $\pi : G \to G/G'$ corresponds to the faithful flatness of the Hopf algebra $H = \mathcal{O}(G)$ over the coideal subalgebra $\mathcal{O}(G/G')$, where G' is a closed normal subgroup of G. One of the reasons coideal subalgebras are so important in the study of noncommutative Hopf algebras is that noncommutative Hopf algebras do not have "enough" Hopf subalgebras [45]. This shortage of Hopf subalgebras is evident for quantised enveloping algebras.

In [45], Letzter studies coideal subalgebras of quantised enveloping algebras and their connections with quantum homogeneous spaces. These coideal subalgebras are related with quantum symmetric pairs. For instance in [36], some quantum symmetric pairs are found which correspond with coideal subalgebras of quantised enveloping algebras.

Takeuchi showed that commutative Hopf algebras are faithfully flat over their Hopf subalgebras [78, Theorem 3.1]. Hopf subalgebras are in particular right and left coideal subalgebras. Masuoka and Wigner showed in [53, Theorem 3.4] that a commutative Hopf algebra is flat over its right coideal subalgebras but in general it is not faithfully flat over all its right or left coideal subalgebras. Thus, right or left coideal subalgebras of a Hopf algebra H over which H is faithfully flat form a special class and we will see in subsequent paragraphs that this special class has nice geometric properties.

The result by Takeuchi in the paragraph above was first proved by Demazure and Gabriel in the setting of affine algebraic groups. In [21], Demazure and Gabriel showed that for an affine algebraic group G, the quotient G/G' by an affine normal subgroup G' of G is an affine algebraic group and $\mathcal{O}(G)$ is faithfully flat over $\mathcal{O}(G/G')$.

Hopf subalgebras in commutative Hopf algebras, correspond to quotient groups when we consider commutative Hopf algebras as affine algebraic groups. When we keep the faithful flatness property, many desirable properties still survive when we move from the commutative to the noncommutative setting.

Turning now to the noncommutative setting, we use the definition of a quantum group as being a noncommutative noncocommutative Hopf algebra. A Hopf algebra H' is called a *quantum subgroup* of a Hopf algebra H if there is a Hopf algebra epimorphism $\pi: H \to H'$. Then H' is a quotient Hopf algebra. By a quantum quotient space, we mean a subalgebra of H of all elements which are fixed under the coaction of H' on H induced by π.

We could go with the obvious definition of a quantum homogeneous space as the quotient space derived from the quotient of a quantum group by some quantum subgroup. It turns out that this definition would be too restrictive. This is because we can construct some quantum spaces which are "homogeneous" in the noncommutative setting but cannot be defined as the quotient of a quantum group by a quantum subgroup. In [16], Brzeziński gives examples of the quantum two sphere and the quantum plane which are homogeneous in the noncommutative setting but are not realised as quotients of a quantum group by a quantum subgroup. Podleś shows in [63] the construction of spaces which are homogeneous under the action of the quantum $SU(2)$ group. This example is discussed in Example 3.4.2.

In the analytic setting, an ergodic action of a compact quantum group $\mathbb{G}$ on an operator algebra A is called a quantum homogeneous space. Varilly showed in [80] that the noncommutative spheres of Connes and Landi are quantum homogeneous spaces for certain compact quantum groups. For more on quantum homogeneous spaces in the analytic setting, see [20, 34, 72].

In this thesis, we take the view that the "right" definition of a quantum homogeneous space is the following. A *quantum homogeneous space* B is a right or left coideal subalgebra of a Hopf algebra H such that H is faithfully flat as a left and right B-module. More general definitions of a quantum homogeneous space are given in [63] and [50].

Suppose H is a connected graded Hopf algebra over an algebraically closed field k of characteristic 0 and suppose H has finite Gelfand-Kirillov dimension. Then in [12], right or left coideal subalgebras of H are studied by Brown and Gilmartin. Homological properties of these quantum homogeneous spaces of connected Hopf algebras are also discussed. We do not assume that the Hopf algebras that we work with in this thesis are connected and indeed, our main examples are not connected.

Given a space X, classical symmetries on X are realised when we find a group G which acts on X. For instance, $GL(2,k)$, the group of invertible two by two matrices acts on the plane k^2. When we consider the algebraic variety $y^2 = x^3$ referred to as the cusp, the only symmetry it has in the plane is reflection long the x-axis while the group of matrices of the form

$$\begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^3 \end{bmatrix}, \qquad \lambda \in k^\times$$

acts on the cusp.

When we move from groups acting on spaces to the algebraic setting, the group G corresponds to the coordinate ring $\mathcal{O}(G)$ of the group and an algebraic variety X corresponds with the coordinate ring $\mathcal{O}(X)$ of the variety. The coordinate ring $\mathcal{O}(G)$ is a commutative Hopf algebra and $\mathcal{O}(X)$ is a commutative algebra. The action of G on X corresponds with the coaction of $\mathcal{O}(G)$ on $\mathcal{O}(X)$. Due to the extra structure of the counit, comultiplication and antipode that $\mathcal{O}(G)$ possesses, classical symmetries of X are better understood by studying the coaction of $\mathcal{O}(G)$ on $\mathcal{O}(X)$.

We now move to the noncommutative setting and consider noncommutative analogues of an action of a group on a space, we get the notion of quantum symmetry. Quantum symmetry studies the coaction by a noncommutative noncocommutative Hopf algebra on an algebra. We say an object has genuine quantum symmetry if the coaction of the Hopf algebra does not factor through a cocommutative Hopf algebra. In the case of a finite dimensional semisimple Hopf algebra coacting on a commutative domain, Etingof and Walton in [23, Theorem 1.3] showed that there is no genuine quantum symmetry in this instance. On the other hand, when we consider finite dimensional pointed (not necessarily semisimple) Hopf algebras coacting on a commutative domain, there can exist genuine quantum symmetries [24, 25]. For more on quantum symmetry, see the following papers [19, 22–24]. The result in [23, Theorem 1.3] is in contrast with the examples of quantum homogeneous spaces that we discover in this thesis.

Contrary to results of no genuine quantum symmetries discussed in paragraphs above, this thesis studies a completely new class of quantum homogeneous spaces. Our interest is in singular plane curves which are quantum homogeneous spaces. Examples include the coordinate rings of the cusp $y^2 = x^3$ [28] and the nodal cubic $y^2 = x^2 + x^3$ [40]. These two examples were the starting point of this thesis. Initially, the intuition was that we can only find quantum homogeneous spaces arising from irreducible plane curves. Then, we found a reducible quantum homogeneous space, the coordinate ring of the coordinate crossing $xy = 0$.

In order to define a noncommutative Hopf algebra which contained the coordinate ring of $xy = 0$ as a right coideal subalgebra, we transformed $xy = 0$ to $y^2 = x^2$. A common feature of all these examples of quantum homogeneous spaces are that they are given by polynomials of the form $f(y) = g(x)$. This led us to discover two new families of Hopf algebras which we call $A(x, a, g)$ and $A(g, f)$.

Given a decomposable plane curve (a curve given by $f(y) = g(x)$) with $\deg(f) = m$, $\deg(g) = n$, we construct two auxiliary Hopf algebras; $A(x, a, g)$ and $A(y, b, f)$. These auxiliary Hopf algebras are defined as quotients of the algebra $k[x] * k[a^{\pm 1}]$ by defining relations which make $g(x)$ a a^n-skew primitive element and central in $A(x, a, g)$ and $f(y)$ b^m-skew primitive in $A(y, b, f)$. We then derive $A(g, f)$ as a quotient of the tensor product of $A(x, a, g)$ and $A(y, b, f)$ by the relations $f(y) = g(x)$ and $b^m = a^n$.

We conjecture that the coordinate ring B of a decomposable plane curve $f(y) = g(x)$ becomes a right coideal subalgebra of $A(g, f)$. We prove that $A(g, f)$ is faithfully flat as a left and right $B-$module when $\deg f, \deg g \leq 5$ using Bergman's Diamond Lemma to show that there is a basis for $A(g, f)$ over B.

The structure of the thesis is as follows. In Chapter 2, we discuss background topics needed. These include the definitions of an algebra, coalgebra, module, comodule and a Hopf algebra. Examples of these algebraic structures are given and some theorems on them stated.

As discussed in paragraphs above, a quantum homogeneous space is a right coideal subalgebra B of a Hopf algebra H such that H is faithfully flat over B. Chapter 3 develops the theory by first discussing the faithful flatness property, then we consider homogeneous spaces in the commutative setting. Afterwards, we move to the noncommutative setting to discuss quantum homogeneous spaces in detail. Classical results such as the faithful flatness of a commutative Hopf algebra over its Hopf subalgebras and the faithful flatness of a pointed Hopf algebra over a right coideal subalgebra which is stable under the antipode are discussed.

The definition and properties of the Hopf algebra $A(x, a, g)$ are discussed in Chapter 4. As we explain below, when the degree of g is less than or equal to three, $A(x, a, g)$ turns out to be the localised quantum plane or deformations of the localised down-up algebra. This identification of $A(x, a, g)$ with these well studied algebras helps us to determine properties of $A(x, a, g)$ when $\deg(g) \leq 3$.

When $\deg(g) = 2$, $A(x, a, g)$ is isomorphic to a localised quantum plane $k\langle a^{\pm 1}, x\rangle$ at the parameter -1. Thus, $A(x, a, g)$ is a noetherian AS-regular domain of Gelfand-Kirillov and global dimension 2. Also, $A(x, a, g)$ is a finite module over its central Hopf subalgebra $k[a^{\pm 2}][g]$.

Similar results hold when $\deg(g) = 3$. That is, $A(x, a, g)$ is a PBW deformation of a down-up algebra. Down-up algebras were first defined by Benkart and Roby in [7], and have been the subject of much research since then. Consequently from results on down-up algebras, $A(x, a, g)$ is a noetherian AS-regular domain of Gelfand-Kirillov and global dimensions 3.

We also prove using Bergman's Diamond Lemma that there is a PBW basis for $A(x, a, g)$ over $k[x]$ when $\deg(g) \leq 5$. Conjecturally, it remains true that $A(x, a, g)$ has a PBW basis over $k[x]$ when $\deg(g) \geq 5$, but currently we are unable to show this.

In Chapter 5, given a decomposable plane curve $f(y) = g(x)$, we define the Hopf algebra

$A(g,f)$ as a quotient of the tensor product of $A(x,a,g)$ and $A(y,b,f)$. Nice properties for $A(g,f)$ such as finite Gelfand-Kirillov dimension are obtained when $\deg(g)$, $\deg(f) \leq 3$. Examples of $A(g,f)$ are given for a general degree three decomposable plane curve, cusps, the nodal cubic and the lemniscate.

Goodearl and Zhang in [28, Construction 1.2] defined new families of Hopf algebras $B(n,p_0,p_1,\cdots,p_s,q)$. We prove that for the case of cusps $y^m = x^n$, the Hopf algebra $B(1,1,n,m,q)$ is a factor Hopf algebra of $A(x^n,y^m)$.

This new family of Hopf algebras $A(g,f)$ that we have discovered has a PBW basis when $\deg(g),\deg(f) \leq 5$. From this PBW theorem, we deduce that the coordinate ring B of a decomposable plane curve $f(y) = g(x)$ embeds in $A(g,f)$. Thus, decomposable plane curves which have degree at most five are quantum homogeneous spaces.

It is conjectured in [40] that all plane curves are quantum homogeneous spaces. In [51, Theorem 1.3(a)], Masuoka showed that for a pointed Hopf algebra, if the coradical of a right coideal subalgebra is stable under the antipode, then the pointed Hopf algebra is faithfully flat as a left and right module over the right coideal subalgebra. The Hopf algebras $A(g,f)$ which we construct are pointed. The only missing piece to enable us to show that decomposable plane curves are quantum homogeneous spaces is our inability to prove that the coordinate ring of any decomposable plane curve $f(y) = g(x)$ embeds in $A(g,f)$.

In Chapter 6, we list open questions which arise from this thesis. We state in Chapter 4 without proof that $A(x,a,g)$ has a PBW basis when $\deg(g) \leq 5$. The appendix chapter fills in the remaining details of this proof by showing that ambiguities are resolvable. The details of this proof are relegated to the appendix because the argument requires long calculations to confirm that the ambiguities (in Bergman's sense) arising from the application of the relations are indeed resolvable.

Chapter 2

Preliminaries

In this chapter, we define the key words and notation used throughout the thesis.

2.1 Notation

We work over a field of characteristic 0 and represent this by k. Unadorned tensor products are over k. We use the notation $k\langle x, y\rangle$ for the free algebra on two generators and $k[x, y]$ for polynomials in two variables. The ideal generated by f is denoted by $\langle f\rangle$.

2.2 Algebras and Coalgebras

As we will see later on, Hopf algebras generalise groups in the classical correspondence between affine commutative Hopf algebras and affine algebraic groups over k. We can therefore think of some Hopf algebras as deformations of the coordinate rings of algebraic groups. A Hopf algebra is an algebra and a coalgebra, together with an antihomomorphism called an antipode such that some compatibility conditions are satisfied.

Considering some Hopf algebras now as deformations of algebraic groups into the coordinate ring of the group, the multiplication structure of the group corresponds with the comultiplication map of the coordinate ring of the group. The inverse map of the group is identified with the antipode map while the identity element of the group is identified with the counit map of the Hopf algebra structure of the coordinate ring of the group. Thus, the associative property of the group is translated into the coassociativity of the comultiplication in the coalgebra while the existence of left and right inverse of elements in the group is associated with the compatibility condition of the antipode. The identity element which is identified with the counit map gives the compatibility condition of the counit map.

Reversing the multiplication and unit maps of an algebra leads to the dual notion, called a coalgebra. The associativity and unit properties of an algebra correspond with dual notions of coassociativity and counit properties. The coalgebra structure of a Hopf algebra is important because it contributes to the “locally finite” property of Hopf algebras which algebras generally do not possess. This “locally finite” property of coalgebras is also referred to as the fundamental theorem of coalgebras which states that every coalgebra is a union of its finite dimensional subcoalgebras [65, Theorem 2.2.3].

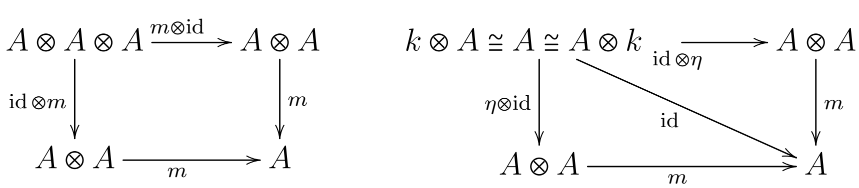
One motivation for studying Hopf algebras is that, generally, the tensor product of representations of an algebra is not always a representation of that algebra, but in the case of Hopf algebras, the coalgebra structure provides an action on the tensor product of representations of the underlying algebra of the Hopf algebra.

Our motivation for studying Hopf algebras starts from the classical correspondence between affine commutative Hopf algebras and affine algebraic groups over k. Under this correspondence, the space of cosets of the quotient of an affine algebraic group by a closed subgroup in the category of affine algebraic groups corresponds with homogeneous spaces in the category of affine commutative Hopf algebras. Homogeneous spaces are spaces on which a group acts transitively. Moving now to the noncommutative setting, a quantum homogeneous space is a right coideal subalgebra B of a Hopf algebra H such that H is faithfully flat as a left and right B-module. More details on this are discussed in Chapter 3.

2.3 Definitions and Examples

The definitions in this section are adapted from [57, 65].

Definition 2.3.1. An *algebra A over a field k* is a tuple (A, m, η), where A is a vector space over k and $m : A \otimes A \to A$, $\eta : k \to A$ are linear maps such that the diagrams below commute



The commutativity of the first and second diagrams are referred to as the *associativity* and *unit* properties of an algebra. We write $m(a \otimes b) = ab$ for all $a, b \in A$ and $\eta(1_k) = 1$.

Example 2.3.2. The group algebra kG of a group G over a field k is a k-vector space with basis G written as

$$kG = \{\sum_{g \in G} \lambda_g \, g \mid \lambda_g \in k \text{ with } \lambda_g = 0 \text{ except for finitely many } g \in G\}$$

and multiplication $*$ given by the multiplication in the group, that is,

$$\left(\sum_{g\in G}\lambda_g\, g\right) * \left(\sum_{h\in G}\mu_h\, h\right) = \sum_{g'=gh}\lambda_g\mu_h\, g'$$

The unit map on kG is given by $\eta(1) = e$ where e is the identity element in the group G.

Definition 2.3.3. The *flip map* on an algebra A is defined as $\tau_{A,A} : A \otimes A \to A \otimes A$ with $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$. From the multiplication map of an algebra (A, m, η), we define a multiplication $m^{op} : A \otimes A \to A$ with $m^{op} := m \circ \tau_{A,A}$. The tuple (A, m^{op}, η) is an algebra over k called the *opposite algebra*.

Definition 2.3.4. A *commutative algebra over* k is an algebra (A, m, η) over k such that $m^{op} = m$.

For example, the group algebra kG of an abelian group G over a field k is an example of a commutative algebra.

Definition 2.3.5. Let (A, m, η) be an algebra over k. The tuple (M, μ) is called a *right* A*-module* if M is a k-vector space and μ is an *action* $\mu : M \otimes A \to M$ that is, the associativity axiom

$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_M \otimes m)$$

and unitary axiom

$$\mu \circ (\mathrm{id}_M \otimes \eta) = \mathrm{id}_M$$

hold. We write the associativity and unitary axioms respectively as

$$(m \cdot a) \cdot b = m \cdot (ab), \quad \text{and} \quad m \cdot 1 = m$$

for all $m \in M$, $a, b \in A$.

A left A-module is defined similarly as the definition of a right A-module as a tuple (M, μ) where $\mu : A \otimes M \to M$ is an action such that the associativity and unitary axioms are satisfied. We denote left and right A-modules by ${}_AM$ and M_A respectively.

A left A-module ${}_AM$ which also has the structure of a right B-module M_B is referred to as an $A - B$-bimodule if the left and right actions are compatible that is

$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$

for all $a \in A$, $b \in B$ and $m \in M$. We denote an $A - B$-bimodule M by ${}_AM_B$.

Suppose M is a right (resp. left) A-module with structure map $\mu : M \otimes A \to M$ (resp. $\mu : A \otimes M \to M$). A subspace $N \subset M$ is called a *submodule* if the restriction of μ to N is contained in N, that is $\mu(A \otimes N) \subseteq N$.

Example 2.3.6. An algebra (A, m, η) over k is a right and left A-module with $M = A$, $\mu = m$ and an $A - A$-bimodule denoted by ${}_AA_A$. A module A is a left or right module over any subalgebra.

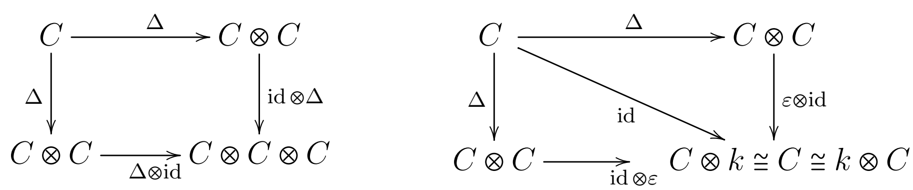
Definition 2.3.7. Let (A, m_A, η_A) and (B, m_B, η_B) be algebras over k. A k-linear map $f : A \to B$ is called a *morphism of algebras* if the following conditions hold:

$$f \circ m_A = m_B \circ (f \otimes f), \quad f(1_A) = 1_B$$

for all $a, b \in A$.

Dual to the notion of an algebra is that of a coalgebra which we get by reversing the arrows in the commuting diagrams in the definition of an algebra.

Definition 2.3.8. A *coalgebra C over a field k* is a tuple (C, Δ, ε) where C is a vector space over k and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are linear maps such that the diagrams below commute



The facts that the first and second diagrams commute are known respectively as the *coassociativity* and *counit* properties of a coalgebra.

Example 2.3.9. 1. Let k be a field. Then k is a coalgebra with

$$\Delta(1) = 1 \otimes 1, \qquad \varepsilon(1) = 1.$$

2. The group algebra kG of a group G over a field k is a coalgebra with the coproduct and counit on G given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad \text{for all } g \in G.$$

We obtain the coproduct of kG by extending the coproduct and counit on G linearly.

We usually denote the coproduct $\Delta(c) \in C \otimes C$ of $c \in C$ by the *Sweedler notation* $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. Note that $c_{(1)} \otimes c_{(2)}$ is not an elementary tensor. Sometimes, the summation sign is dropped to get $\Delta(c) = c_{(1)} \otimes c_{(2)}$. The coassociativity and counit properties are expressed in Sweedler's notation as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$$

and

$$\varepsilon\left(c_{(1)}\right)c_{(2)} = c = c_{(1)}\varepsilon\left(c_{(2)}\right)$$

respectively.

Definition 2.3.10. A nonzero subspace D of a coalgebra C over k is called a *subcoalgebra* if $\Delta(D) \subseteq D \otimes D$. If the only subcoalgebras of C are the trivial ones 0 and C, then C is called a *simple coalgebra* of C.

For instance if H is a subgroup of a group G, then the group algebra kH is a subcoalgebra of the coalgebra kG.

The next definition is inspired by the form the comultiplication takes for the special case of the group algebra.

Definition 2.3.11. A *grouplike element* of a coalgebra C over k is an element $c \in C$ such that

$$\Delta(c) = c \otimes c \qquad \text{and} \qquad \varepsilon(c) = 1.$$

The set of grouplike elements of C is denoted by $G(C)$. For instance for the group algebra kG, $G(kG) = G$. We will see later on that the set of grouplike elements $G(C)$ is actually a group when C is Hopf algebra, thus the name.

Definition 2.3.12. A *skew primitive element* of a coalgebra C over k is an element $s \in C$ such that

$$\Delta(s) = g \otimes s + s \otimes h$$

for some grouplike elements $g, h \in C$. If in particular both g and h in the coproduct of s are equal to 1, then we call s a *primitive element* of C. The set of primitive elements of a coalgebra C is denoted by $P(C)$.

Definition 2.3.13. Let C and D be coalgebras over k with coproducts and counits Δ_C, ε_C and Δ_D, ε_D respectively.

1. The coalgebra C is called *cocommutative* if $\tau_{C,C} \circ \Delta_C = \Delta_C$
2. A *coalgebra homomorphism* is a map $f : C \to D$ such that

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f, \qquad \varepsilon_C = \varepsilon_D \circ f$$

Definition 2.3.14. A *coideal* (respectively *right coideal*, *left coideal*) B of a coalgebra A is a subspace $B \subset A$ with $\Delta(B) \subset B \otimes A + A \otimes B$ and $\varepsilon(B) = 0$ (respectively $\Delta(B) \subset B \otimes A$, $\Delta(B) \subset A \otimes B$).

For instance, the set of primitive elements $P(A)$ of a coalgebra A over k is a coideal of A.

Definition 2.3.15. The *coradical* C_0 of a coalgebra C over k is the sum of simple subcoalgebras of C.

We will see later on that these two sets $G(C)$ and $P(C)$ are important in determining the structure of a cocommutative Hopf algebra. For instance if H is a cocommutative pointed Hopf algebra over a field k of characteristic 0, then H can be decomposed into a smash product of the universal enveloping algebra of $P(H)$ and the group algebra $kG(H)$. This theorem is attributed to Cartier, Gabriel and Kostant.

Definition 2.3.16. A *pointed coalgebra* over k is a coalgebra C over k whose simple subcoalgebras are one-dimensional.

Example 2.3.17. The group algebra kG of a group G is a pointed coalgebra.

Next, we define the dual formulation of Definition 2.3.5.

Definition 2.3.18. Let (C, Δ, ε) be a coalgebra over k. A tuple (M, ρ) is called a *right C-comodule* if M is a k-vector space and $\rho : M \to M \otimes C$ is a coaction such that the coassociativity axiom

$$(\rho \otimes \mathrm{id}_C) \circ \rho = (\mathrm{id}_M \otimes \Delta) \circ \rho$$

and counit axiom

$$(\mathrm{id}_M \otimes \varepsilon) \circ \rho = \mathrm{id}_M$$

hold. We write the coassociativity and counit axioms in Sweedler notation as

$$m_{(0)(0)} \otimes m_{(0)(1)} \otimes m_{(1)} = m_{(0)} \otimes m_{(1)(1)} \otimes m_{(1)(2)} \quad \text{and} \quad m_{(0)}\varepsilon(m_{(1)}) = m$$

for all $m \in M$ with $\rho(m) = m_{(0)} \otimes m_{(1)}$.

A left C-comodule is defined similarly as the definition of a right C-comodule as a tuple (M, ρ) where $\rho : M \to M \otimes C$ is a coaction, that is, the coassociativity and counitary axioms are satisfied. We denote left and right C-comodules by ${}^AM$ and M^A respectively.

A left C-comodule ${}^CM$ and a right D-comodule M^D is referred to as a *C-D-bicomodule* if the left and right coactions are compatible. We denote a $C-D$-bicomodule M by ${}^CM^D$.

A subspace N of a right (resp. left) C-comodule M with structure map $\rho : M \to M \otimes C$ (resp. $\rho : M \to C \otimes M$) is called a *subcomodule* if $\rho(N) \subseteq N \otimes C$ (resp. $\rho(N) \subseteq C \otimes N$).

Example 2.3.19. A coalgebra (C, Δ, ε) over k is a right and left C-comodule and a $C-C$-bicomodule with $M = C$ and $\rho = \Delta$. A subcoalgebra of a coalgebra C is an example of a left or right C-comodule.

2.4 Bialgebras

When we combine both the algebra and coalgebra structures on the vector space A over k and impose some compatibility conditions, we get a bialgebra structure on A. Most of this section is adapted from [57].

Definition 2.4.1. A *bialgebra* A over the field k is a tuple $(A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is an algebra and (A, Δ, ε) is a coalgebra such that any of the following equivalent conditions hold:

1. The algebra maps m and η are coalgebra homomorphisms.
2. The coalgebra maps Δ and ε are algebra homomorphisms.

Definition 2.4.2. A *biideal* of a bialgebra A over a field k is a subspace of A which is both an ideal and a coideal of A.

Suppose I is a biideal of a bialgebra A, then the quotient algebra and coalgebra structures endow A/I with a unique bialgebra structure such that the projection $\pi : A \to A/I$ is an algebra and coalgebra morphism (bialgebra morphism).

Most of the examples of bialgebras that we consider throughout the thesis will be defined as a quotient of a bialgebra by a biideal as seen below.

Example 2.4.3. (i) [57, Example 1.3.2] We already saw above that the group algebra kG of a group G over a field k is both an algebra and a coalgebra. The algebra and coalgebra structures on kG are compatible making kG a bialgebra.

(ii) [Example 1.5.8]montgomery Let $0 \neq q \in k$ and let $B = \mathcal{O}_q(k^2) = k\langle x, y\rangle/\langle xy - qyx\rangle$. Then B is a bialgebra with

$$\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.$$

We call $\mathcal{O}_q(k^2)$ the *quantum plane.*

(iii) [57, Example 1.3.3] Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ be a Lie algebra and $T(\mathfrak{g})$ the free algebra $k\langle \mathfrak{g}\rangle$ generated by $\mathfrak{g}$. The *universal enveloping algebra* $U(\mathfrak{g})$ is defined as

$$U(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g})$$

where $I(\mathfrak{g})$ is the two sided ideal of $F(\mathfrak{g})$ generated by $xy - yx - [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$. Given any associative algebra A and a Lie algebra homomorphism $f : \mathfrak{g} \to A$, by the universal property, there exists a unique map $\hat{f} : U(\mathfrak{g}) \to A$ such that $\hat{f} \circ \iota = f$ where ι is the inclusion map of $\mathfrak{g}$ in $U(\mathfrak{g})$. For each $x \in \mathfrak{g}$, we define

$$\Delta(x) = 1 \otimes x + x \otimes 1, \qquad \varepsilon(x) = 0.$$

This extends linearly to $U(\mathfrak{g})$ via

$$\Delta(x_1 x_2 \cdots x_n) := \Delta(x_1)\Delta(x_2)\cdots\Delta(x_n)$$

for $x_1 x_2 \cdots x_n \in F(\mathfrak{g})$. This is a canonical cocommutative bialgebra algebra structure on $U(\mathfrak{g})$.

Definition 2.4.4. Let A be a bialgebra over the field k. A sub-bialgebra of A is a subspace B of A which is simultaneously a subalgebra and a subcoalgebra of A.

Remark 2.4.5. A bialgebra A is *commutative* if the underlying algebra structure of A is commutative and *cocommutative* if A is cocommutative as a coalgebra.

Definition 2.4.6. A *pointed bialgebra* over k is a bialgebra over k whose underlying coalgebra structure is pointed.

The following theorem has useful applications in finding the set of all grouplike elements of a pointed coalgebra and in proving that certain bialgebras are pointed.

Theorem 2.4.7. [65, Corollary 5.1.14] *Suppose A is a bialgebra over k generated by $\mathcal{S} \cup \mathcal{P}$, where $\mathcal{S} \subseteq G(A)$ and $\mathcal{P}$ consists of skew-primitives x which satisfy $\Delta(x) = s \otimes x + x \otimes s'$ for some $s, s' \in \mathcal{S}$. Then:*

a) A is pointed.

b) $G(A)$ is the multiplicative submonoid of A generated by $\mathcal{S}$.

Let A be a k-algebra and let H be a k-bialgebra.

Definition 2.4.8. Suppose A is a left H-module. If the action map $H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$ satisfies the properties

$$\begin{aligned} h \cdot (ab) &= (h_{(1)} \cdot a)(h_{(2)} \cdot b) \\ h \cdot 1 &= \varepsilon(h)1 \end{aligned}$$

for all $h \in H$, $a, b \in A$, then A is called a *left H-module algebra* or we say *H acts on A*.

Let A be a left H-module algebra. Then $A^H := \{a \in A \mid \forall h \in H : h \cdot a = \varepsilon(h)a\}$ is the subalgebra of *H-invariant elements*. Dually, we define a comodule algebra as follows.

Definition 2.4.9. Let A be a right H-comodule with coaction $\rho : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$. Then A is a *right H-comodule algebra* and H coacts on A if ρ is an algebra map.

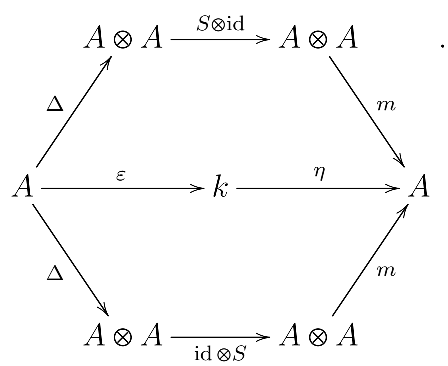
If A is a right H-comodule algebra, then $A^{\mathrm{co}\, H} := \{a \in A \mid \rho(a) = a \otimes 1\}$ is the algebra of *H-coinvariant elements* of A.

Example 2.4.10. Let $\pi : A \to A/I = H$, $a \mapsto \overline{a}$, be a surjective map of bialgebras. Then A is a right Hcomodule algebra with coaction

$$\rho : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}\otimes\pi} A \otimes H.$$

2.5 Definition and Examples of Hopf Algebras

Definition 2.5.1. A Hopf algebra is a bialgebra $A = (A, m, \eta, \Delta, \varepsilon)$ over a field k together with a linear map $S : A \to A$ called the *antipode* such that the following diagram commutes



The commutativity of the diagram above is expressed as

$$m \circ (S \otimes \mathrm{id}_A) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id}_A \otimes S) \circ \Delta$$

or

$$S(a_{(1)})a_{(2)} = \varepsilon(a)1 = a_{(1)}S(a_{(2)})$$

in Sweedler notation.

Definition 2.5.2. A Hopf algebra H is called *commutative* if the underlying algebra structure is commutative and *cocommutative* if the underlying coalgebra structure is cocommutative.

The antipode S of a Hopf algebra H is unique [65, Definition 7.1.1], it is always an antihomomorphism by [57, Proposition 1.5.10]. If H is a commutative or cocommutative Hopf algebra, then the antipode of H is of order 2, that is $S^2 = id_H$ [57, Corollary 1.5.12]. If H is a finite dimensional Hopf algebra, then the antipode S of H gives a bijection of H [65, Theorem 7.1.14]. However, the antipode in general is not bijective [77, Theorem 11].

Definition 2.5.3. A *Hopf subalgebra* of a Hopf algebra H over k with antipode S is a sub-bialgebra B of H such that $S(B) \subseteq B$.

Next, we define the left (resp. right) adjoint action and coaction of a Hopf algebra on itself.

Definition 2.5.4. Let H be a Hopf algebra. Then the *left adjoint action* of H on itself is defined as

$$\begin{aligned} \mu_\ell : H \otimes H &\to H \\ x \otimes y &\mapsto x_{(1)} y S(x_{(2)}), \end{aligned}$$

resp. the *right adjoint action*

$$\begin{aligned} \mu_r : H \otimes H &\to H \\ x \otimes y &\mapsto S(x_{(1)}) y x_{(2)}. \end{aligned}$$

Definition 2.5.5. Let H be a Hopf algebra. The *left adjoint coaction* and the *right adjoint coaction* of H on itself are defined as

$$\begin{aligned} \rho_\ell : H &\to H \otimes H \\ x &\mapsto x_{(1)} S(x_{(3)}) \otimes x_{(2)}, \end{aligned}$$

and

$$\begin{aligned} \rho_r : H &\to H \otimes H \\ x &\mapsto x_{(2)} \otimes S(x_{(1)}) x_{(3)}. \end{aligned}$$

respectively.

Definition 2.5.6. Let H be a Hopf algebra and $K \subseteq H$, $I \subseteq H$ be sub vector spaces. Then K is said to be *left normal* (resp. *right normal*) if it is stable under the left (resp. right) adjoint action. Dually, I if *left* (resp. *right*) *conormal* if it is stable under the left (resp. right) adjoint coaction.

Definition 2.5.7. Let B be a Hopf subalgebra of a Hopf algebra H. Then B is called a *normal Hopf subalgebra* if B is stable under the right and left adjoint actions, that is $S(h_{(1)}) B h_{(2)} \subseteq B$ and $h_{(1)} B S(h_{(2)}) \subseteq B$ respectively for all $h \in H$.

Let A be a k-algebra. If H is a finite-dimensional Hopf algebra with dual H^*, there is a bijective correspondence between coactions $A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$ and actions $H^* \otimes A \to A$, $p \otimes a \mapsto p \cdot a$, given by $p \cdot a = a_{(0)} p(a_{(1)})$. Under this correspondence, $A^{\text{co } H} = A^{H^*}$[70, §1].

Most of the Hopf algebras that we discuss have their algebra structures defined as the quotient of a free algebra H by an ideal I generated by a set of relations. In order for there to be a unique Hopf algebra structure on the quotient algebra and coalgebra structure on

H/I such that the projection $\pi : H \to H/I$ is a Hopf algebra map, we need the following definition.

Definition 2.5.8. A *Hopf ideal* of a Hopf algebra H is a biideal I of H which is stable under the antipode; that is $S(I) \subseteq I$. If I is a Hopf ideal, then there is a unique Hopf algebra structure on the quotient H/I such that the projection $\pi : H \to H/I$ is a Hopf algebra map.

Definition 2.5.9. A Hopf ideal I of a Hopf algebra H is called *normal* if both

$$\rho_\ell(I) \subseteq H \otimes I \qquad \text{and} \qquad \rho_r(I) \subseteq I \otimes H$$

with ρ_ℓ and ρ_r defined in Definition 2.5.5.

Classical examples of Hopf algebras are either commutative or cocommutative. The first nonclassical example is Sweedler's Hopf algebra.

Example 2.5.10. We saw in Example 2.4.3 that the group algebra kG of a group G is a bialgebra. The antipode S exists for G and it is defined by

$$S(g) = g^{-1}$$

for all $g \in G$. This extends linearly to kG to give the antipode on kG. The maps Δ and ε are algebra homomorphisms and S is an antihomomorphism. Thus, kG is a Hopf algebra. This Hopf algebra is always cocommutative, and is commutative if and only if G is abelian.

For any subgroup G' of G, the group algebra kG' is a Hopf subalgebra of kG.

Example 2.5.11. When we localise the quantum plane $\mathcal{O}_q(k^2)$ defined in Example 2.4.3(ii) above, we get $H = \mathcal{O}_q(k^2)[x^{-1}]$. This H is a Hopf algebra with antipode

$$S(x) = x^{-1}, \qquad S(y) = -yx^{-1}.$$

Example 2.5.12. [57, Example 1.5.4] Recall from Example 2.4.3 that the *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a bialgebra with the coproduct and counit on generators defined by

$$\Delta(x) = 1 \otimes x + x \otimes 1, \qquad \varepsilon(x) = 0, \qquad \text{for all } x \in \mathfrak{g}.$$

There is an antipode on $U(\mathfrak{g})$ defined by

$$S(x) = -x, \qquad \text{for all } x \in \mathfrak{g}.$$

This extends linearly to $U(\mathfrak{g})$, and defines a cocommutative Hopf algebra structure on $U(\mathfrak{g})$. In particular, this is the unique cocommutative Hopf algebra structure on $U(\mathfrak{g})$ as we will see in the following theorem.

Before we state this theorem, we define the smash product algebra as follows.

Definition 2.5.13. Let A be a left H-module algebra. Then the *smash product algebra* $A\#H$ is defined as follows, for all $a, b \in A$, $h, k \in H$:

1. as k-vector spaces, $A\#H = A \otimes H$. We write $a\#h$ for the element $a \otimes h$
2. multiplication given by

$$(a\#h)(b\#k) = \sum a(h_{(1)} \cdot b)\#h_{(2)}k. \tag{2.1}$$

Consequently, we have $A \cong A\#1$ and $H \cong 1\#H$; for this reason we abbreviate the element $a\#h$ by ah. In this notation, we sometimes write $ha = \sum (h_{(1)} \cdot a)h_{(2)}$ using (2.1).

The following theorem on the decomposition of a cocommutative pointed Hopf algebra is attributed to Cartier, Gabriel and Kostant.

Theorem 2.5.14. [65, Theorem 15.3.2] *Let H be a cocommutative pointed Hopf algebra over the field k with coradical H_0, and let $G = G(H)$ and $K = \Delta^{-1}(H \otimes H_0 + H_0 \otimes H)$. Then the smash product $K\#k[G]$ is a Hopf algebra with the tensor product coalgebra structure and there is an isomorphism $F : K\#k[G] \to H$ of Hopf algebras determined by $F(a \otimes g) = ag$ for all $a \in K$ and $g \in k[G]$.*

Next, we consider a family of finite dimensional Hopf algebras which are neither commutative nor cocommutative.

Example 2.5.15. [65, §7.3] Let $n \geq 1$ and $q \in k^*$ be a primitive nth root of unity. The Taft algebra $H_{n,q}$ is defined as the quotient of the free algebra $k\langle g, x\rangle$ by the ideal generated by the relations

$$x^n = 0, \qquad g^n = 1, \qquad gx = qxg.$$

The Hopf algebra structure of $H_{n,q}$ is determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = 1 \otimes x + x \otimes g,$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g^{-1}, \quad S(x) = -xg^{-1}.$$

When $n = 2$, we call $H_{2,-1}$ Sweedler's 4-dimensional Hopf algebra. This is the first example of a Hopf algebra which was neither commutative nor cocommutative.

Example 2.5.16. [84, Example 3.1] Let $t \in \mathbb{Z}$ and $F(t) = k\langle x, a^{\pm 1}\rangle$ be the algebra with defining relations $aa^{-1} = a^{-1}a = 1$. It is straightforward to check that the algebra $k\langle x, a^{\pm 1}\rangle$ is a Hopf algebra with the coproduct, counit and antipode defined on the generators by

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = 1 \otimes x + x \otimes a^t$$
$$\varepsilon(x) = 0, \quad \varepsilon(a) = 1, \quad S(a) = a^{-1}, \quad S(x) = -xa^{-t}.$$

The ideal generated by the relations $aa^{-1} = a^{-1}a = 1$ is a Hopf ideal making $F(t)$ a Hopf algebra. We refer to $F(t)$ as *free pointed Hopf algebra.* It has a k-linear basis

$$\{a^{i_1}xa^{i_2}x\cdots xa^{i_{n+1}} \mid (i_1, i_2, \cdots, i_{n+1}) \in \mathbb{Z}^{n+1},\ n \in \mathbb{N}\}.$$

When $t = 0$, $F(t)$ becomes cocommutative, otherwise, $F(t)$ is neither commutative nor cocommutative.

Definition 2.5.17. A Hopf algebra H is called a *pointed Hopf algebra* if the underlying coalgebra is pointed.

In the setting of finite dimensional Hopf algebras, it is conjectured in [2, Conjecture 5.7] that a Hopf algebra is pointed if and only if it is generated by grouplike and skew primitive elements. However, in the general setting, we get one side of the equivalence. That is if a Hopf algebra H is generated by grouplike and skew primitive elements, then H is pointed [65, Corollary 5.1.14].

Example 2.5.18. [57, Lemma 5.5.5] The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a pointed Hopf algebra.

2.6 Tensor products of H-modules and H-comodules

Let H be an algebra. Then the tensor product of H-modules is generally not a H-module. But if H is a Hopf algebra, then the coalgebra structure on H enables the tensor product of H-modules to become a H-module. This is defined as follows.

Definition 2.6.1. Let H be a Hopf algebra, and V and W left H-modules. Then $V \otimes W$ is also a left H-module, via

$$h \cdot (v \otimes w) = \sum (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w)$$

for all $h \in H$, $v \in V$, $w \in W$.

Analogously, the tensor product of right H-modules is again a right H-module.

Definition 2.6.2. Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras over k. There is a natural coalgebra structure on $C \otimes D$ defined as follows. The flip map $\tau : C \otimes D \to D \otimes C$ with $\tau(c \otimes d) = d \otimes c$ for all $c \in C, d \in D$ induces a coproduct on $C \otimes D$ defined as by

$$\Delta_\tau : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{id_C \otimes \tau \otimes id_D} C \otimes D \otimes C \otimes D$$

with

$$\begin{aligned} \Delta_\tau(c \otimes d) &= (id_C \otimes \tau \otimes id_D) \circ (\Delta_C \otimes \Delta(D))(c \otimes d) \\ &= (id_C \otimes \tau \otimes id_D)(c_{(1)} \otimes c_{(2)} \otimes d_{(1)} \otimes d_{(2)}) \\ &= c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}. \end{aligned}$$

Coassociativity of Δ_C and Δ_D ensures that Δ_τ is coassociative. Next, we define the map $\varepsilon_\tau := \varepsilon_C \otimes \varepsilon_D : C \otimes D \to k \otimes k \cong k$ with $\varepsilon_\tau(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d)$. Then from the counit properties of ε_C and ε_D, ε_τ satisfies the counit property for $C \otimes D$. We call $(C \otimes D, \Delta_\tau, \varepsilon_\tau)$ the natural coalgebra structure on the tensor product of the coalgebras C and D.

Next, we define a Hopf module.

Definition 2.6.3. For a Hopf algebra H over k, a *right H-Hopf module* is a k-space M such that

1. M is a right H-module
2. M is a right H-comodule via $\rho : M \to M \otimes H$
3. ρ is a right H-module map, where $M \otimes H$ is a right H-module as Definition 2.6.1, and where H acts on itself by right multiplication.

A left H-Hopf module is defined analogously.

Remark 2.6.4. [57, Definition 1.9.1] More generally, if we replace H in module part of Definition 2.6.3 by any Hopf subalgebra K of H; M then becomes a *right* (H,K)*–Hopf module*. The category of all right (resp. left) (H,K)-Hopf modules is denoted $\mathcal{M}_K^H$ (resp. ${}_K^H\mathcal{M}$).

2.7 Algebraic Groups and Commutative Hopf Algebras

An algebraic group is a group defined by a collection of polynomials. In characteristic 0, affine commutative Hopf algebras correspond to affine algebraic groups. This translates to the fact that Hopf algebras generalise groups since the coordinate rings of affine algebraic groups are Hopf algebras in a canonical way. In this section, we discuss this classical correspondence and its consequences.

Most of the content of this section is adapted from [29] and we assume that all fields k in this section are algebraically closed.

2.7.1 Algebraic Groups

Definition 2.7.1. Let $n \in \mathbb{Z}_{\geq 1}$. Then the *affine n-space* is the set

$$\mathbb{A}^n = \{(a_1, a_2, \cdots, a_n) \mid a_i \in k, 1 \leq i \leq n\}.$$

When $n = 1$, $\mathbb{A}^1$ is called the *affine line* and $\mathbb{A}^2$ is called the *affine plane*.

Definition 2.7.2. Let $f \in k[x_1, x_2, \cdots, x_n]$ be a polynomial. The *zeros* of f is defined as

$$Z(f) = \{a \in \mathbb{A}^n \mid f(a) = 0\}.$$

More generally, if P be a set of polynomials in $k[x_1, x_2, \cdots, x_n]$. We define the *zero set of* P to be the common zeros of all the elements of P, namely

$$Z(P) = \{a \in \mathbb{A}^n \mid f(a) = 0 \ \forall \ f \in P\}.$$

If $\mathfrak{a}$ is an ideal of $k[x_1, x_2, \cdots, x_n]$ generated by P, then $Z(P) = Z(\mathfrak{a})$. Since the ring $k[x_1, x_2, \cdots, x_n]$ is a noetherian ring, any ideal $\mathfrak{a}$ has a finite set of generators $f_1, f_2, \cdots, f_r$. Thus, $Z(P)$ can be expressed as the zero set of the finite set of polynomials $f_1, f_2, \cdots, f_r$.

Definition 2.7.3. An *affine algebraic set* of $\mathbb{A}^n$ is a subset $X \subseteq \mathbb{A}^n$ for which there exists $P \subseteq k[x_1, x_2, \cdots, x_n]$ such that X = Z(P).

Example 2.7.4. The affine $n-$space $\mathbb{A}^n$ is an affine algebraic set if we consider the set $P = \{0\} \subseteq k[x_1, x_2, \cdots, x_n]$.

Example 2.7.5. Let $c \in k^\times$. Then the empty set $\varnothing$ is an affine algebraic set for $P = \{c\} \subseteq k[x_1, x_2, \cdots, x_n]$.

Since the affine $n-$space $\mathbb{A}^n$ and the empty set $\varnothing$ are algebraic sets, this suggests we might have a topology on $\mathbb{A}^n$. In fact, $\mathbb{A}^n$ is a topological space with closed sets given by algebraic sets [29, Proposition 1.1]. This topology on $\mathbb{A}^n$ is called *Zariski topology.* We get this from the following

$$Z(S \cup T) = Z(S) \cap Z(T), \qquad Z(ST) = Z(S) \cup Z(T)$$

for subsets $S, T \subseteq k[x_1, x_2, \cdots, x_n]$. There are fewer open set in the Zariski topology than in the usual metric topology. Closed sets in the Zariski topology are closed in the usual metric topology since closed sets are given by the zeros of polynomials which are continuous in the usual metric topology. For instance, the nontrivial closed sets in $\mathbb{A}^1$ are finite subsets of k.

Definition 2.7.6. An algebraic set X is said to be *irreducible* if it cannot be written as a union of proper closed (in the Zariski topology) subsets. The empty set is not considered to be irreducible.

Example 2.7.7. The affine line $\mathbb{A}^1$ is irreducible since its only closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

However, the algebraic set $Z(xy) \subset \mathbb{A}^2$ is not irreducible since $Z(xy) = Z(x) \cup Z(y)$.

Definition 2.7.8. An *affine algebraic variety* (or simply *affine variety*) is an irreducible closed subset of $\mathbb{A}^n$.

Definition 2.7.9. For any subset $X \subseteq \mathbb{A}^n$, the *ideal* of X in $k[x_1, x_2, \cdots, x_n]$ is defined by

$$I(X) = \{f \in k[x_1, x_2, \cdots, x_n] \mid f(a) = 0 \text{ for all } a \in X\}$$

Thus, finding the zeros of a set of polynomials gives a map Z from subsets of the polynomial ring $k[x_1, x_2, \cdots, x_n]$ to algebraic sets, and finding the ideal of a subset of $\mathbb{A}^n$ gives a map I which maps subsets of $\mathbb{A}^n$ to ideals.

Example 2.7.10. The ideal of the affine n-space $\mathbb{A}^n$, $I(\mathbb{A}^n) = 0$ while the ideal of the empty set $\varnothing$, $I(\varnothing) = k[x_1, \cdots, x_n]$.

Definition 2.7.11. Let $\mathfrak{a} \subseteq k[x_1, x_2, \cdots, x_n]$ be an ideal. The *radical* of $\mathfrak{a}$ is defined as

$$\sqrt{\mathfrak{a}} = \{f \in k[x_1, x_2, \cdots, x_n] \mid f^r \in \mathfrak{a} \text{ for some } r > 0\}.$$

Definition 2.7.12. An ideal I of a commutative ring R is

(i) *radical* if it is equal to its radical

$$\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some } n > 0\}.$$

(ii) *prime* if for any $f, g \in R$, $fg \in I$ implies either $f \in I$ or $g \in I$.

(iii) *maximal* if J is an ideal containing I implies $J = R$ or $J = I$.

The following theorem gives a relationship between the ideal of the zeros of an ideal and the radical of the ideal. It is referred to as "Nullstellensatz", the German word for "zeros of points theorem".

Theorem 2.7.13 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and $I \subseteq k[x_1, x_2, \cdots, x_n]$ an ideal. Then $I(Z(I)) = \sqrt{I}$.*

We refer to Theorem 2.7.13 as the "strong Nullstellensatz". If I is a maximal ideal in $k[x_1, x_2, \cdots, x_n]$, then I is radical by Theorem 2.7.13. The weak Hilbert's Nullstellensatz states that maximal ideals of $k[x_1, x_2, \cdots, x_n]$ are of the form $I = \langle x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n\rangle$ which correspond with points in $\mathbb{A}^n$. This equivalence is stated as follows:

Theorem 2.7.14. [29, Corollary 1.4] *There is a one-to-one inclusion-reversing similarity between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1, \cdots, x_n]$, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Definition 2.7.15. The *coordinate ring* $\mathcal{O}(X)$ of an algebraic set $X \subseteq \mathbb{A}^n$ is defined as

$$\mathcal{O}(X) = \{f|_X : X \to k \mid f \in k[x_1, x_2, \cdots, x_n]\} = k[x_1, x_2, \cdots, x_n]/I$$

where $f|_X$ is the evaluation of the polynomial f on the algebraic set X and I is the ideal $I(X)$.

If X is an affine variety, then $\mathcal{O}(X)$ is an integral domain. Furthermore, $\mathcal{O}(X)$ is a finitely generated k-algebra.

Definition 2.7.16. Given an algebraic set X, the *dimension of* X (dim(X)) is the maximal length d of chains $X_0 \subset X_1 \subset \cdots \subset X_d$ of distinct nonempty irreducible algebraic sets contained in X.

For instance the algebraic set $Z(xy) \subset \mathbb{A}^2$ which is decomposed as $Z(xy) = Z(x) \cup Z(y)$ has dimension one.

Definition 2.7.17. Given algebraic sets $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, a function $\varphi : X \to Y$ is called a *morphism of algebraic sets* if there are polynomials $\psi_1, \psi_2, \cdots, \psi_m \in k[x_1, x_2, \cdots, x_n]$ such that $\varphi(x) = (\psi_1(x), \psi_2(x), \cdots, \psi_m(x))$ for every $x \in X$. Examples of morphisms between an algebraic set $X \subseteq \mathbb{A}^n$ and the affine line $\mathbb{A}^1$ are polynomials evaluated on X.

Definition 2.7.18. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic sets with

$$V = Z(S) = \{a \in \mathbb{A}^n \mid f(a) = 0 \ \forall \ f \in S\}, \quad W = Z(T) = \{b \in \mathbb{A}^m \mid g(a) = 0 \ \forall \ g \in T\}.$$

Then the *product of V and W* is defined as

$$V \times W = \{(a_1, \cdots, a_n, b_1, \cdots, b_m) \in \mathbb{A}^{m+n} \mid f(a_1, \cdots, a_n) = 0, g(b_1, \cdots, b_m) = 0 \text{ for all } f \in S,\ g \in T\}.$$

The product $V \times W$ is an algebraic set of $\mathbb{A}^{m+n}$.

Definition 2.7.19. An algebraic set G is called an *algebraic group* if G is a group such that the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (g,h) &\mapsto gh, \end{aligned}$$

and the inverse map

$$\begin{aligned} S : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are morphisms of algebraic sets.

Example 2.7.20. An example of an algebraic group is the group of invertible $n \times n$ matrices over a field k denoted by $GL_n(k)$ for $n \geq 1$. The coordinate ring $\mathcal{O}(GL_n(k))$ of $GL_n(k)$ is

$$\mathcal{O}(GL_n(k)) = \frac{k[x_{11}, x_{12}, \cdots, x_{nn}, t]}{\langle \det(x_{ij})t - 1 \rangle}.$$

In particular, when $n = 1$, $GL_1(k) = k^{\times}$, the multiplicative field and $\mathcal{O}(GL_1(k)) = k[x, x^{-1}]$.

Remark 2.7.21. The coordinate ring $\mathcal{O}(GL_n(k))$ of $GL_n(k)$ is a Hopf algebra. We derive the coproduct by dualising the usual matrix multiplication in $GL_n(k)$, yielding

$$\Delta(x_{ij}) = \sum_{1 \leq \ell \leq n} x_{i\ell} \otimes x_{\ell j}.$$

Similarly, dualising the identity of matrix multiplication gives the counit.

$$\varepsilon(x_{ij}) = \delta_{ij}.$$

The antipode is defined as $S(X) = X^{-1}$, that is $S(x_{ij})$ is the ij^{th} entry of X^{-1} for $X = (x_{ij}) \in GL_n(k)$.

Definition 2.7.22. [32, §15.1, 17.5] An invertible endomorphism is called *unipotent* if it is the sum of the identity and a nilpotent endomorphism, or equivalently, if its sole eigenvalue is 1. A subgroup of an algebraic group is called *unipotent* if all its elements are unipotent.

We will see in the next subsection the classical correspondence between algebraic groups and commutative Hopf algebras over an algebraically closed field of characteristic 0, that the dual of the multiplication map m of an algebraic group gives a coproduct on the coordinate ring $\mathcal{O}(G)$ of G , turning $\mathcal{O}(G)$ into a Hopf algebra.

2.7.2 Commutative Hopf Algebras

A commutative ring R is *semiprime* if and only if the zero ideal is radical. Hence, there is a $1:1$ correspondence between algebraic sets and commutative semiprime rings. Cartier proved that affine commutative Hopf algebras over fields of characteristic 0 are always semiprime [81, Theorem 11.4]. In the following results about affine commutative Hopf algebras, we use the property of them being semiprime.

Let H be an affine commutative Hopf algebra over an algebraically closed field k of characteristic 0. Then $H \cong k[x_1, x_2, \cdots, x_n]/I$ for some semiprime ideal $I \subset k[x_1, x_2, \cdots, x_n]$. We define the algebraic set $Z(I)$ as

$$Z(I) := \{\underline{a} \in \mathbb{A}^n \mid f(\underline{a}) = 0 \ \forall \ f \in I\}.$$

The maximum spectrum of H which is denoted by $\mathrm{Maxspec}(H)$ is defined as the set of maximal ideals in H which by the weak Nullstellensatz is equivalent to

$$\mathrm{Maxspec}(H) = \{f : H \to k \mid f \text{ is an algebra homomorphism}\}.$$

The weak Nullstellensatz yields the identification $Z(I)$ =$\mathrm{Maxspec}(H)$. There is a one to one correspondence between an element $\underline{m} \in \mathrm{Maxspec}(H)$ and an algebra homomorphism $H \to H/\underline{m}$. Given two algebra homomorphisms f, g from H to k, we define their product to be

$$(f \cdot g)(h) := f(h_{(1)})g(h_{(2)})$$

and the inverse of f to be

$$f^{-1}(h) := f \circ S(h)$$

for all $h \in H$ with $\Delta(h) = h_{(1)} \otimes h_{(2)}$ and S the antipode of the Hopf algebra H. This gives a group structure on $\mathrm{Maxspec}(H)$ and hence a group structure on $Z(I)$. Thus, yielding a functor from the category of affine commutative Hopf algebras to the category of affine algebraic groups over an algebraically closed field k of characteristic 0.

Turning now to the category of affine algebraic groups over an algebraically closed field k of characteristic 0, for an algebraic group G, by [56, §2.15], we have an isomorphism between the coordinate ring of $G \times G$ and the tensor product of the coordinate ring of G given by

$$\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G).$$

Thus, when we dualise the multiplication map $m : G \times G \to G$ on an algebraic group G, we get a coalgebra structure on $\mathcal{O}(G)$ with

$$\Delta : G \to \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$$

and $\Delta(f)(x,y) := f(m(x,y))$ for all $x, y \in G$, $f \in \mathcal{O}(G)$. The counit and antipode on $\mathcal{O}(G)$ are defined by $\varepsilon(f) = f(1_G)$, $S(f)(x) = f(x^{-1})$ for $f \in \mathcal{O}(G), x \in G$. This makes $\mathcal{O}(G)$ a Hopf algebra and since $\mathcal{O}(G)$ is a commutative algebra, we get a commutative Hopf algebra. This yields a functor from the category of affine algebraic groups over k to the category of affine commutative Hopf algebras over k.

Affine algebraic groups are smooth as varieties [17]. When we translate this algebraic property to commutative Hopf algebras, we deduce that affine commutative Hopf algebras have finite global dimension [29, Theorem 5.1].

We summarise this section as follows. Given a commutative affine Hopf algebra H, we can express H as a quotient $k[x_1, x_2, \cdots, x_n]/I$. The zero set $Z(I)$ is an algebraic group. Thus, we get a functor from the category of affine commutative Hopf algebras over k to the category of affine algebraic groups over k.

In terms of functors between the two categories, the classical correspondence is given by

$$\begin{array}{rcl} \left\{\begin{matrix}\text{commutative affine} \\ \text{Hopf algebras over } k\end{matrix}\right\} & \longleftrightarrow & \left\{\begin{matrix}\text{affine algebraic} \\ \text{groups over } k\end{matrix}\right\} \\ H \cong k[x_1, x_2, \cdots, x_n]/I & \mapsto & Z(I) \\ \mathcal{O}(G) & \mapsfrom & G \end{array}$$

In summary, if H is an affine commutative Hopf algebra, then H is the coordinate ring $\mathcal{O}(G)$ of an algebraic group G over k and conversely [81].

2.8 Manin's Approach

Given a k-algebra A defined as a quotient of a free algebra $k\langle x_1, \cdots, x_N\rangle$ by an ideal of relations I, is there a Hopf algebra H which coacts on A, taking the vector space spanned by the x_is to a one-sided coideal? That is, is there a map

$$\begin{array}{rcl} \rho : A & \to & A \otimes H \\ x_i & \mapsto & \sum_j x_j \otimes h_{ji} \end{array}$$

which turns A into an H-comodule algebra? Since we want ρ to be a coaction, we need ρ to satisfy

$$\sum_j \rho(x_j) \otimes h_{ji} = \sum_j x_j \otimes \Delta(h_{ji}).$$

The coassociativity constraint gives

$$x_\ell \otimes \sum_j h_{\ell j} \otimes h_{ji} = \sum_\ell x_\ell \otimes \Delta(h_{\ell i}).$$

This gives the definition of the coproduct of the generators of $k\langle h_{ji}\rangle$ given by matrix multiplication. Thus, $k\langle h_{ji}\rangle$ is a bialgebra with

$$\Delta(h_{ji}) = \sum_\ell h_{j\ell} \otimes h_{\ell i}, \quad \varepsilon(h_{ij}) = \delta_{ij}.$$

Define H as the k-algebra $k\langle h_{ji}\rangle/J$. Given the ideal of relations I of A, we want to find J such that ρ is an algebra map and J is a biideal. This will enable us to conclude that H is a bialgebra. In addition, we want the relations of J to be such that the quotient algebra is a Hopf algebra, that is, admits an antipode. The ideal J is generated by the relations defined by I and additional relations which make the coproduct, counit and antipode on H respect the relations defined by I.

Our motivation for studying Hopf algebras is to find universal Hopf algebras which contain the coordinate ring of a plane curve as a right coideal subalgebra. This enables us to prove that some class of plane curves are quantum homogeneous spaces. The following example is that of the cusp, $y^2 = x^3$, which follows the universal construction by Manin described in the example above.

Example 2.8.1. Let A be the coordinate ring of the cusp $y^2 = x^3$. Suppose there is a coaction $\rho : A \to A \otimes H$ of a Hopf algebra H on A with

$$\rho(x) = 1 \otimes a_1 + x \otimes a_2, \qquad \rho(y) = 1 \otimes b_1 + y \otimes b_2.$$

Then the coassociativity constraint

$$(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho$$

and the fact that we want A to be a right coideal subalgebra ensures that

$$a_1 = x, \quad b_1 = y, \quad \Delta(a_2) = a_2 \otimes a_2, \quad \Delta(b_2) = b_2 \otimes b_2$$

so that ρ is the inclusion. Also, since we want the antipode to exist on the generators of H, we need the grouplike elements a_2 and b_2 of H to be invertible. Thus, H is a quotient of the free algebra $k\langle x, y, a_2, b_2, a_2^{-1}, b_2^{-1}\rangle$ by some ideal J. We also need the coalgebra to respect the relation $y^2 = x^3$ of I and the commutation relation $xy = yx$. In order for the coproduct of xy given by

$$\Delta(xy) = 1 \otimes xy + x \otimes a_2 y + y \otimes x b_2 + xy \otimes a_2 b_2$$

to be equal to the coproduct of yx given by

$$\Delta(yx) = 1 \otimes yx + x \otimes y a_2 + y \otimes b_2 x + yx \otimes b_2 a_2,$$

we need to add the relations

$$a_2 y = y a_2, \quad x b_2 = b_2 x, \quad a_2 b_2 = b_2 a_2.$$

In addition, we want the coproduct of y^2 given by

$$\Delta(y^2) = 1 \otimes y^2 + y \otimes (b_2 y + y b_2) + y^2 \otimes b_2^2$$

to be equal to the coproduct of x^3 given by

$$\Delta(x^3) = 1 \otimes x^3 + x \otimes (a_2 x^2 + x a_2 x + x^2 a_2) + x^2 \otimes (a_2^2 x + a_2 x a_2 + x a_2^2) + x^3 \otimes a_2^3,$$

we need to add the relations

$$b_2 y + y b_2 = 0, \quad a_2 x^2 + x a_2 x + x^2 a_2 = 0, \quad a_2^2 x + a_2 x a_2 + x a_2^2 = 0, \quad b_2^2 = a_2^3.$$

Thus, the ideal J is defined by the relations

$$y^2 = x^3, \quad xy = yx, \quad a_2 y = y a_2, \quad x b_2 = b_2 x, \quad a_2 b_2 = b_2 a_2,$$
$$b_2 y + y b_2 = 0, \quad a_2 x^2 + x a_2 x + x^2 a_2 = 0, \quad a_2^2 x + a_2 x a_2 + x a_2^2 = 0, \quad b_2^2 = a_2^3.$$

2.9 Gelfand-Kirillov Dimension

In this section, we will define and state results about the Gelfand-Kirillov dimension (GKdim) of an algebra. This measures the "rate of growth" of an algebra with respect to any finite generating set. The content and results we state are from [41].

Definition 2.9.1. Let Φ denote the set of all functions $f : \mathbb{N} \to \mathbb{R}$ which are eventually monotone increasing and positive valued, that is, for which there exists $n_0 = n_0(f) \in \mathbb{N}$, such that

$$f(n) \in \mathbb{R}^+ \quad \text{and} \quad f(n+1) \geq f(n) \quad \text{for all} \quad n \geq n_0.$$

For $f, g \in \Phi$ set $f \leq^* g$ if and only if there exist $c, m \in \mathbb{N}$ such that

$$f(n) \leq cg(mn) \quad \text{for almost all} \quad n \in \mathbb{N},$$

and $f \sim g$ if and only if $f \leq^* g$ and $g \leq^* f$. For $f \in \Phi$ the equivalence class $\mathcal{G}(F) \in \Phi/\sim$ is called the *growth* of f. The partial ordering on the set $\Phi/\sim$ induced by $\leq^*$ is denoted by $\leq$.

The "growth" of a finitely generated k-algebra is independent irrespective of the choice of a finite dimensional generating subspace. This is seen in the Lemma below. The following is Lemma 1.1 in [41]

Lemma 2.9.2. *Let A be a finitely generated algebra with finite dimensional generating subspaces V and W. If $d_V(n)$ and $d_W(n)$ denote the dimensions of $\sum_{i=0}^{n} V^i$ and $\sum_{i=0}^{n} W^i$, respectively, then $\mathcal{G}(d_V(n)) = \mathcal{G}(d_W(n))$.*

Remark 2.9.3.

1. If f and g are polynomial functions, then f and g have the same growth if and only if $\deg(f) = \deg(g)$. For a real number $\gamma \geq 0$ the growth of the function $p_\gamma : n \to n^\gamma$ is denoted by $\mathcal{P}_\gamma$.
2. For $\varepsilon \in \mathbb{R}^+$ the growth of $q_\varepsilon : n \to e^{n^\varepsilon}$ is denoted by $\mathcal{E}_\varepsilon$.

Definition 2.9.4. Let A be a finitely generated k-algebra with a finite dimensional generating subspace V. Then $\mathcal{G}(A) := \mathcal{G}(d_V)$ is called the *growth* of A, and A is said to have

polynomial growth if $\mathcal{G}(A) = \mathcal{P}_m$, for some $m \in \mathbb{N}$,

exponential growth if $\mathcal{G}(A) = \mathcal{E}_1$,

subexponential growth if $\mathcal{G}(A) < \mathcal{E}_1$, yet $\mathcal{G}(A)\mathcal{P}_m$ for all $m \in \mathbb{N}$.

Example 2.9.5. Let $A = k\langle x, y\rangle$ be the free algebra on two generators. Then $V = kx + ky$ is a generating subspace for A and

$$d_V(n) = \dim_k \left(\sum_{i=0}^{n} V^i \right) = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Thus, $\mathcal{G}(A) = \mathcal{E}_1$.

Example 2.9.6. Consider $A' = k[x_1, x_2, \cdots, x_d]$, the commutative polynomial algebra. The vector space $V' = kx_1 + kx_2 + \cdots kx_d$ is a generating subspace for A'. We easily verify that

$$\dim(V'^{(n+1)}) = \binom{n+1+d-1}{d-1} = \binom{n+d}{d-1}$$

is a polynomial of degree $d-1$. Since $\dim(V'^{(n+1)}) = d_{V'}(n+1) - d_{V'}(n)$. It follows from [41, Lemma 1.5(b)] that $d_{V'}(n)$ is a polynomial of degree d so that $\mathcal{G}(A') = \mathcal{P}_d$.

Next, we define the growth of an algebra.

Definition 2.9.7. The *Gelfand-Kirillov dimension* (GKdim) of a k-algebra A is

$$\mathrm{GKdim}(A) = \sup_V \overline{\lim} \log_n d_V(n),$$

where the supremum is taken over all the finite dimensional subspaces V of A.

The Gelfand-Kirillov dimension of the algebras in Examples 2.9.5 and 2.9.6 are ∞ and d respectively.

One natural question we can ask is: Does the growth of an algebra depend on the generating subspace that we choose? This question is answered in the remark below.

Remark 2.9.8. It is shown in [41, Lemma 1.1] that for a finitely generated generated algebra B with finite dimensional generating subspace V, the growth of B is independent of the particular choice of V.

All the algebras that we consider in this thesis are finitely generated so we can choose any finite dimensional generating subspace to compute the Gelfand-Kirillov dimension of the algebra under consideration.

It turns out that the Gelfand-Kirillov dimension of an algebra A cannot be strictly between 1 and 2. Again, for any real number $r \geq 2$, we can find an algebra with Gelfand-Kirillov dimension r. These two are due to Warfield and Bergman. We write GKdim instead of Gelfand-Kirillov dimension in the rest of the text.

Theorem 2.9.9. [10, Bergman] *No algebra has Gelfand-Kirillov dimension strictly between* 1 *and* 2*.*

Theorem 2.9.10. [66, Theorem 2] *For any real number* $r \geq 2$*, there exists a two generator algebra* $A := k\langle x, y\rangle/(Z)$ *with GKdim*$(A) = r$ *where* (Z) *is the ideal generated by a set of monomials* Z*.*

Returning to the classical correspondence that an affine commutative Hopf algebra H is the coordinate ring $\mathcal{O}(G)$ of an algebraic group G over k, from [41], we get the following relationship between the Gelfand-Kirillov dimension of H and the dimension of G,

$$\mathrm{GKdim}(H) = \dim(G).$$

2.10 The Diamond Lemma and the PBW Theorem

In this section, we recall the notations and definitions needed for Bergman's Diamond Lemma. Most of the content of this section are adapted from [9] and [13].

We recall the set-up needed to apply Bergman's diamond lemma [9]. For more details, see for example [13, pp. 97-101]. Let $\langle X \rangle$ denote the free semigroup on a set X and $k\langle X \rangle$ the free k-algebra with generators X. Let R be the quotient of $k\langle X \rangle$ by a set of relations Σ. Suppose that every relation $\sigma \in \Sigma$ can be written in the form $W_\sigma = f_\sigma$ with $W_\sigma \in \langle X \rangle$ and $f_\sigma \in k\langle X \rangle$. We write Σ as the set of pairs of the form $\sigma = (W_\sigma, f_\sigma)$.

For each $\sigma \in \Sigma$ and $A, B \in \langle X \rangle$, let $r_{A\sigma B}$ denote the linear endomorphism of $k\langle X \rangle$ which fixes all the elements of $\langle X \rangle$ other than $AW_\sigma B$, and which sends this basis element of $k\langle X \rangle$ to $Af_\sigma B$. We call Σ a *reduction system*, with the maps $r_{A\sigma B} : k\langle X \rangle \to k\langle X \rangle$ called *elementary reductions*, and a composition of elementary reductions called a *reduction*. An elementary reduction $r_{A\sigma B}$ acts *trivially* on an element $\alpha \in k\langle X \rangle$ if the coefficient of $AW_\sigma B$ in α is zero, and we call α *irreducible* (under Σ) if every elementary reduction is trivial on α. The k-vector space of irreducible elements of $k\langle X \rangle$ is denoted by $k\langle X \rangle_{irr}$.

A *semigroup ordering* on $\langle X \rangle$ is a partial order $\leq$ such that if $a, b, c, d \in \langle X \rangle$ and $a < b$, then $cad < cbd$. A semigroup ordering on $\langle X \rangle$ is *compatible with* Σ if, for all $\sigma \in \Sigma$, f_σ is a linear combination of words W with $W < W_\sigma$.

Define an *overlap ambiguity* of Σ to be a 5-tuple (σ, τ, A, B, C) with $\sigma, \tau \in \Sigma$ and $A, B, C \in \langle X \rangle$, such that $W_\sigma = AB$, $W_\tau = BC$. An overlap ambiguity (σ, τ, A, B, C) is *resolvable* if there exist compositions r and r' of reductions such that $r(f_\sigma C) = r'(Af_\tau)$. An *inclusion ambiguity* is a 5-tuple (σ, τ, A, B, C) with $\sigma, \tau \in \Sigma$ and $A, B, C \in \langle X \rangle$, such that $ABC = W_\sigma$ and $B = W_\tau$. An inclusion ambiguity (σ, τ, A, B, C) is *resolvable* if there are reductions r, r' such that $r \circ r_{1\sigma 1}(ABC) = r' \circ r_{A\tau C}(ABC)$. Observe that if the W_σ for $\sigma \in \Sigma$ are distinct words of the same length, then there are no non-trivial inclusion ambiguities.

Bergman's theorem [9, Theorem 1.2] can now be stated, as follows.

Theorem 2.10.1. *With the above notation and terminology, suppose that $\leq$ is a semigroup ordering on $\langle X \rangle$ which is compatible with Σ and satisfies the descending chain condition. Suppose that all overlap and inclusion ambiguities are resolvable. Let I be the ideal $\langle W_\sigma - f_\sigma : \sigma \in \Sigma \rangle$ of the free k-algebra $k\langle X \rangle$. Then the map $\omega \mapsto \omega + I$ gives a vector space isomorphism from $k\langle X \rangle_{irr}$ to $k\langle X \rangle / I$; that is, the irreducible words in $\langle X \rangle$ map bijectively to a k-basis of $k\langle X \rangle / I$.*

Example 2.10.2. Let $H := k\langle x, y, a, b, a^{-1} \rangle / I$ where I is the ideal generated by the following relations:

$$aa^{-1} = a^{-1}a = 1, \qquad ba = ab, \qquad b^2 = a^3, \qquad y^2 = x^2 + x^3, \qquad ay = ya, \qquad bx = xb,$$
$$by = -yb, \qquad yx = xy, \qquad a^2x = -(xa^2 + axa + a^2) + a^3, \qquad ax^2 = -(x^2a + xax + ax + xa).$$

There are two overlap ambiguities resulting from the two routes needed to find the word $a^2x^2 \in k\langle x, y, a, b, a^{-1}\rangle$. These routes are pre multiplying ax^2 with a or post multiplying a^2x with x. After computing these, we get the same linear combination of words for a^2x^2 whichever route we take. Hence, by the Bergman's Diamond Lemma, the irreducible words form a basis for H which was stated in Example 2.10.2 above.

Next, we state the Poincaré-Birkhoff-Witt(PBW) Theorem for the universal enveloping algebra of a Lie algebra. This helps to find the basis of an algebra given by generators and relations in terms of irreducible monomials as defined in Bergman's Diamond Lemma above.

Theorem 2.10.3 (Poincaré-Birkhoff-Witt(PBW))**.** [11] *For any basis $\{x_i : i \in I\}$ of a Lie algebra $\mathfrak{g}$ with ordered index set I, the monomials*

$$x_{i_1}^{e_1} \cdots x_{i_n}^{e_n}$$

where $i_1 < \cdots < i_n$ and $e_i > 0$ form a basis for the universal enveloping algebra $U(\mathfrak{g})$.

Example 2.10.4. [40] *The algebra in Example 2.10.2 has PBW basis*

$$\{x^i y^j (ax)^l a^{i'} b^{j'} \mid i, l \in \mathbb{Z}_{\geq 0},\ i' \in \mathbb{Z},\ j, j' \in \{0, 1\}\}.$$

Chapter 3

Quantum Homogeneous Spaces

3.1 Introduction

Let H be a Hopf algebra and let $B \subset H$ be a right coideal subalgebra of H. We are interested in cases where the extension $B \subset H$ satisfies the faithful flatness property. The inclusion $B \subset H$ defines a quotient map $G \to X$ where G is a quantum group and X is a quantum space with right G-action or a right G-space [59, §0]. The quantum space X is not usually the quotient of B by some quantum subgroup but if H is faithfully flat over B, we get B back from the quotient map $H \to H/HB^+$ as the H/HB^+-coinvariant elements of H where $B^+ = B \cap \ker \varepsilon$. We call such right coideal subalgebras satisfying the faithful flatness property *quantum homogeneous spaces.*

In the classical setting of commutative Hopf algebras, there is a bijection between right coideal subalgebras B over which H is right faithfully flat and ideals and left coideals $I \subset H$ such that H is faithfully coflat over H/I [70, Theorem 3.1.6]. Such right coideal subalgebras satisfying this property are the coinvariants of the quotient H/I. We call them *homogeneous spaces.*

In particular, under the classical correspondence between affine commutative Hopf algebras and affine algebraic groups over a field k of characteristic zero, a Hopf algebra H corresponds to the coordinate ring $\mathcal{O}(G)$ of an affine algebraic group. If G' is a closed subgroup of G, then the space of left and right cosets may or may not be an affine variety. But if G' is a closed normal subgroup of G, then the space of cosets is an affine algebraic variety so that $\mathcal{O}(G/G')$ is a Hopf subalgebra of $H = \mathcal{O}(G)$. Takeuchi proved in [78] that this is a one to one correspondence between normal closed subgroups G' of G and Hopf quotients H/I of H with I a normal ideal so that G' corresponds to $H/I = \mathcal{O}(G')$. Also, Takeuchi proves in [78] that in this case, H is always faithfully flat over the ring of coinvariants $H^{co\, H/I}$. We call this ring of coinvariants $H^{co\, H/I}$ or $\mathcal{O}(G/G')$ a *homogeneous space.* Correspondingly, a left or right coideal subalgebra B of H has H as a flat B-module [53] but not always faithfully flat. The right or left coideal subalgebras B over which H

is faithfully flat are called *homogeneous spaces*.

When we tensor modules, some properties arise on the tensor product. One such property is the faithful flatness property which we discuss in detail in §3.2. The definition of a homogeneous space in the commutative setting is discussed in §3.3. Then in §3.3.1, we discuss homogeneous spaces under the classical correspondence between affine commutative Hopf algebras and affine algebraic groups over a field k of characteristic 0.

The final section of this chapter is §3.4 where we discuss the theory of homogeneous spaces in the noncommutative setting. Here, homogeneous spaces are referred to as quantum homogeneous spaces. Classical examples of quantum homogeneous spaces are stated.

Our motivation for studying quantum homogeneous spaces is to find out which singular plane curves are quantum homogeneous spaces. The nodal cubic was the starting example. We describe in detail a Hopf algebra which is faithfully flat over the coordinate ring of the nodal cubic. Then in Chapter 5, we prove that the family of decomposable plane curves of degree at most 5 are quantum homogeneous spaces.

3.2 Flatness and Faithful Flatness

In this section, we discuss some modules which arise from tensor products. These modules are flat and faithfully flat modules. Faithful flatness is a key property we need in order to define homogeneous and quantum homogeneous spaces later in this chapter. An extension $B \subset H$ of a Hopf algebra H such that H is faithfully flat as a left and right B-module has nice properties that we are interested in.

Most of the content of this section is adapted from [68].

Definition 3.2.1. If R is a ring, then a right R-module A is *flat* if $A \otimes_R -$ is an exact functor; that is, whenever

$$0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$$

is an exact sequence of left R-modules, then

$$0 \to A \otimes_R B' \xrightarrow{\mathrm{id}_A \otimes i} A \otimes_R B \xrightarrow{\mathrm{id}_A \otimes p} A \otimes_R B'' \to 0$$

is an exact sequence of abelian groups. Flatness of left R-modules is defined analogously.

Definition 3.2.2. A right R-module A is called *faithfully flat* if

(i) A is a flat module; and

(ii) the converse of Definition 3.2.1 is true or equivalently, for all left R-modules X, if $A \otimes_R X = \{0\}$, then $X = \{0\}$.

The following proposition gives some examples of flat modules.

Proposition 3.2.3. [68, Proposition 3.46] *Let R be an arbitrary ring.*

(i) The right R-module R is a flat right R-module.

(ii) The direct sum $\bigoplus_j M_j$ of right R-modules is flat if and only if each M_j is flat.

(iii) Every projective right R-module P is flat.

(iv) If every finitely generated submodule of a right R-module M is flat, then M is flat

The proof of Proposition 3.2.3(ii) uses the fact that direct sums and tensor products commute. However, tensor products and direct products do not commute as we will see in the next example.

Example 3.2.4. [68, Example 3.52] The $\mathbb{Z}$-module $\mathbb{Q}$ is a flat module since it is a torsion free $\mathbb{Z}$-module over the principal ideal domain $\mathbb{Z}$ [68, Corollary 3.50]. Indeed, the flatness of $\mathbb{Q}$ as a $\mathbb{Z}$-module follows at once from Proposition 3.2.3(i) and (ii), since every finitely generated $\mathbb{Z}$-submodule of $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$. Let $\mathbb{I}_n$ denote the integers modulo n. The following holds

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n\geq 2} \mathbb{I}_n \ncong \prod_{n\geq 2} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{I}_n).$$

The right hand side is $\{0\}$ because $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{I}_n = \{0\}$ for all n, by [68, Proposition 2.7.3]. That is, if we consider the $\mathbb{Z}$-module $\mathbb{I}_n$ for $n \in \mathbb{Z}$, $n \neq 0, \pm 1$, then for all $q \in \mathbb{Q}$, $a \in \mathbb{I}_n$,

$$q \otimes a = \frac{q}{n} \otimes na = 0.$$

Thus, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{I}_n = 0$ though $\mathbb{I}_n \neq 0$. On the other hand, $\prod_{n\geq 2} \mathbb{I}_n$ contains an element of infinite order: if $\mathbb{I}_n = \langle a_n \rangle$, (where $a_n = 1 + n\mathbb{Z}$ is a generator of $\mathbb{I}_n$ for all n) then there is no positive integer m with $0 = m(a_n) = (ma_n)$; hence, there is an exact sequence

$$0 \to \mathbb{Z} \to \prod_{n\geq 2} \mathbb{I}_n.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, by [68, Corollary 3.50], there is exactness of

$$0 \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n\geq 2} \mathbb{I}_n.$$

But $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$, and so $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n\geq 2} \mathbb{I}_n \neq \{0\}$.

Example 3.2.5. The flat $\mathbb{Z}$-module $\mathbb{Q}$ is not faithfully flat since as shown above, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{I}_n = \{0\}$ for all $n \in \mathbb{Z} - \{0, \pm 1\}$ though $\mathbb{I}_n \neq \{0\}$.

Example 3.2.6. Let R be a nonzero ring.

(i) Then R is a flat module since $R \otimes_R V = V$ for all R-modules V. In particular, R is a faithfully flat R-module.

(ii) Since the direct sum of faithfully flat R-modules is again faithfully flat, it follows from (i) that free R-modules are faithfully flat.

Remark 3.2.7. It is not true that projective modules are faithfully flat. For example, let k be a field and

$$R = k \oplus k.$$

Let $e = (1, 0) \in R$; so $R = eR \oplus (1 - e)R$. Then eR and $(1 - e)R$ are projective R-modules but

$$eR \otimes_R R(1 - e) = 0$$

since if $r, s \in R$, then

$$\begin{aligned} er \otimes_R s(1-e) &= ere \otimes_R s(1-e) \\ &= er \otimes_R se(1-e) \\ &= 0. \end{aligned}$$

3.3 Homogeneous Spaces in the Commutative Setting

Most of the content in this section is adapted from [32].

For a curve in affine space, the space of tangents at a point has vector space dimension 1 unless the point is "singular", and then the dimension goes up. The dimension of the tangent of X at x, $\dim \mathcal{T}(X)_x \geq \dim X$ for a variety X and all points $x \in X$ [32, Theorem 5.2]. If equality holds, x is called a *simple point* of X. If all points of X are simple, X is called *smooth* or *nonsingular*.

Recall the definition of an affine algebraic group as a group G which also has the structure of an affine variety over k such that the maps $m : G \times G \to G$ where $m(x, y) = xy$ and $i : G \to G$, where $i(x) = x^{-1}$, are morphisms of varieties. The translation by an element $y \in G$ $(x \mapsto xy)$ yields an isomorphism of varieties $G \to G$. Thus, geometric properties at one point of G can be transferred to any other point, by suitable choice of y. In particular, since G has simple points [32, §5.2], all points must be simple, that is G is smooth [32, §7.1].

If G is a group and X a set, we say G *acts* on X if there is a map $\varphi : G \times X \to X$ denoted by $\varphi(x, y) = x \cdot y$, such that

$$x_1 \cdot (x_2 \cdot y) = (x_1 x_2) \cdot y, \quad \mathrm{id}_G \cdot y = y$$

for all $x_1, x_2 \in G$, $y \in X$. For $y \in X$, we define the *isotropy group* (or *stabilizer*) of y to be the subgroup $G_y = \{x \in G \mid x \cdot y = y\}$. The *orbit map* $G \to G \cdot y$ defined by $x \mapsto x \cdot y$ induces a bijection between G/G_y and $G \cdot y$. Let H be a closed subgroup of an algebraic group G. Then there is a transitive action of G on the space of left cosets G/H ($yH \mapsto xyH$), H being the isotropy group of the coset H. In view of this, every transitive action of G is essentially of this form. We call the space of left cosets G/H with this natural transitive action of G a *homogeneous space*.

Given an algebraic group G and a closed subgroup H, how can the homogeneous space G/H be endowed with a "reasonable" structure of a variety? See [32, §12.1] for details on the fact that the variety so constructed has the properties demanded of a "quotient".

It is sometimes helpful to embed an affine algebraic group into the general linear group as a closed subgroup and then exploit the special properties of matrices to prove theorems about affine algebraic groups. This linearisation of affine groups is possible due to the following theorem.

Theorem 3.3.1. [32, Theorem 8.6] *Let G be an affine algebraic group. Then G is isomorphic to a closed subgroup of some $GL(n,k)$.*

Turning our attention back to the task to put a reasonable structure on G/H to turn the quotient space into a variety, the following theorem enables us to do this when H is a normal subgroup of G.

A morphism $\varphi : G \to GL(n,k)$ of algebraic groups is called a *rational representation*.

Theorem 3.3.2. [32, Theorem 11.5] *Let G be an algebraic group, N a closed normal subgroup of G. Then there is a rational representation $\psi : G \to GL(W)$ such that $N = \ker \psi$.*

With the aid of this theorem, we can give the abstract group G/N the structure of an affine algebraic group by identifying it with $\psi(G)$. However, some further work has to be done to guarantee that this process is independent of the choices made and leads to good universal properties. For details on this, see [32, Chapter IV]

3.3.1 Homogeneous spaces under the classical correspondence

We saw in §2.7 the classical correspondence between affine commutative Hopf algebras and affine algebraic groups over an algebraically closed field k of characteristic zero. In [32, Chapter IV], the universal construction of the quotient space G/H for an algebraic group G and its closed and normal subgroup H is discussed. The quotient space (space of cosets) G/H is what we call a *homogeneous space*.

In this subsection, we are going to discuss what a homogeneous space in the category of affine algebraic groups corresponds to in the category of affine commutative Hopf algebras.

Let H be a commutative Hopf algebra. When does

$$\left\{\begin{matrix} K\colon\ K \text{ is a normal} \\ \text{Hopf subalgebra of } H \end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix} I\colon\ I \text{ is a normal} \\ \text{Hopf ideal of } H. \end{matrix}\right\}$$

where $\phi(K) = HK^+$ and $\psi(I) = {}^{coH/I}H$ give inverse bijections? Takeuchi in [78] proves this bijective correspondence. This gave an algebraic proof of the same theorem by Demazure and Gabriel in [21]. This correspondence holds for Hopf algebras which are either commutative or have cocommutative coradicals.

Theorem 3.3.3. [78, Theorem 4.3][57, Theorem 3.4.6] *Let H be any Hopf algebra. Then ϕ and ψ are inverse bijections if either H is commutative or if the coradical H_0 of H is cocommutative.*

Most of the content of the rest of this section is adapted from [70].

Definition 3.3.4. Let H be a Hopf algebra, $I \subseteq H$ a coideal and a right ideal, and $\pi : H \to Q := H/I$ the quotient map. Thus, Q is a quotient of H as a coalgebra and as a right H-module. Let $\delta : A \to A \otimes H$ be a right H-comodule algebra. Then A is a right Q-comodule with structure map $\delta_Q := (\mathrm{id} \otimes \pi) : A \xrightarrow{\delta} A \otimes H \xrightarrow{\mathrm{id}\otimes\pi} A \otimes Q$. We define the algebra of *Q-coinvariant elements* by $B := A^{coQ} := \{a \in A \mid \delta_Q(a) = a \otimes \pi(1)\}$.

We sometimes write the Q-coinvariant elements of A, A^{coQ} as $A^{co\,\pi}$. Similarly, if B is a right coideal subalgebra of H, then the coinvariant elements of H under the projection $H \to H/HB^+$ (where $B^+ = B \cap \ker(\varepsilon)$) is a right coideal subalgebra of H as stated in the proposition below.

Proposition 3.3.5. [79, Proposition 1] *If $B \subset H$ is a right coideal subalgebra of H, H/HB^+ is a quotient left H-module coalgebra of H. Let $\pi_B : H \to H/HB^+$ be the projection. If $\pi : H \to \pi(H)$ is a quotient left H-module coalgebra, $B_\pi = \{h \in H \mid \pi(h_{(1)}) \otimes h_{(2)} = \pi(1) \otimes h\}$ is a left coideal subalgebra of H.*

Proof. Let $a, b \in B_\pi$. Then

$$\begin{aligned} (\pi \otimes \mathrm{id}) \circ \Delta(ab) &= \big(\pi(a_{(1)}) \otimes a_{(2)}\big)\big(\pi(b_{(1)}) \otimes b_{(2)}\big) \\ &= (\pi(1) \otimes a)(\pi(1) \otimes b) \\ &= \pi(1) \otimes ab. \end{aligned}$$

Thus, $ab \in B_\pi$ and we conclude that B_π is a subalgebra of H. Since $(\pi \otimes \mathrm{id}) \circ \Delta(B_\pi) \subseteq H \otimes B_\pi$, we conclude that B_π is a left coideal subalgebra of H. □

Next, we define the cotensor product of comodules and use it to define the dual of flatness and faithful flatness property of a module.

Definition 3.3.6. Let W and V be right and left H-comodules with comodule structure maps $\rho_W : W \to W \otimes H$ and $\rho_V : V \to H \otimes V$ respectively. Then the *cotensor product of W and V* is

$$W \,\square_H\, V := \{\sum w_i \otimes v_i \in W \otimes V \mid \sum \rho_W(w_i) \otimes v_i = \sum w_i \otimes \rho_V(v_i)\}.$$

Equivalently, the cotensor product of W and V, $W \,\square_H\, V$ is defined as the kernel of

$$\rho_W \otimes \mathrm{id} - \mathrm{id} \otimes \rho_V : W \otimes V \to W \otimes H \otimes V.$$

Definition 3.3.7. Let H be a coalgebra, A a right H-comodule algebra, and Q a coalgebra and right H-module quotient of H. Then H is called *left Q-coflat* (resp. *left faithfully Q-coflat*) for $\mathcal{M}_A^Q$ (Remark 2.6.4), if the functor $\square_Q H$ preserves (resp. preserves and reflects) exact sequences in $\mathcal{M}_A^Q$.

Theorem 3.3.8. [70, Theorem 3.1.3] *Let H be a Hopf algebra, $K \subseteq H$ a left coideal subalgebra, and $Q := H/K^+H$. Then the following are equivalent:*

1. *The coinduction functor ${}^Q\mathcal{M} \to {}^H_K\mathcal{M}$, $V \mapsto H \,\square_Q\, V$, is an equivalence.*
2. *H is right faithfully flat over K for ${}^H_K\mathcal{M}$ (Remark 2.6.4).*
3. *H is right faithfully coflat over Q and $K = H^{coQ}$.*

In this case, H is projective as a right K-module, and K is a right K-direct summand in H.

The dual version of this theorem is [70, Theorem 3.1.2]. In §3.1 of [70], a correspondence is described between left coideal subalgebras $K \subseteq H$ and quotients of H of the form H/I where $I \subseteq H$ is a coideal and right ideal. We state this correspondence in Theorem 3.3.10.

In general, the antipode of a Hopf algebra H is not bijective. For instance in [77], Takeuchi constructed a free Hopf algebra generated by a coalgebra whose antipode is injective but not surjective. Schauenburg gave a counterexample in [69] of a Hopf algebra with an antipode which is surjective but not injective. Skyrabin has conjectured that, every noetherian Hopf algebra has bijective antipode [74]. In [74], sufficient conditions for the bijectivity of an antipode is given using ring theoretic conditions on the Hopf algebra.

In cases when the antipode is not bijective, this helps to prove results about the Hopf algebra. For example, Schauenburg showed in [69] that if H is a Hopf algebra with an injective but nonsurjective antipode, then there exists a Hopf subalgebra B such that H is not faithfully flat as a B-module.

We state the following lemma on a correspondence between left and right coideal subalgebras of a Hopf algebra with bijective antipode. Here, we discover that left and right coideal subalgebras coincide when the Hopf algebra has a bijective antipode. In the following Lemma, part 1.(a) is attributed to Koppinen [37, Lemma 3.1] for right coideals.

Lemma 3.3.9. [70, Lemma 3.1.4] *Let H be a Hopf algebra with antipode S.*

1. *Let $K \subseteq H$ be a left coideal. Then*
   (a) *$MK^+ = MS(K)^+$ for every right H-module M.*
   (b) *If S is bijective, then $S(K^+H) = HK^+$.*
   (c) *If $HK^+ \subseteq K^+H$, then K^+H is a Hopf ideal of H.*
2. *Let $I \subseteq H$ be a left ideal. Then*
   (a) *$M^{co\,H/I} = \{m \in M \mid m_{(0)} \otimes \overline{S(m_{(1)})} = m \otimes \overline{1} \in M \otimes H/I\}$ for every right H-comodule M.*
   (b) *If S is bijective, then $S(H^{co\,H/I}) = {}^{co\,H/I}H$.*
   (c) *If ${}^{co\,H/I}H \subseteq H^{co\,H/I}$, then ${}^{co\,H/I}H$ is a Hopf subalgebra of H.*

We introduce the following notation before stating the next theorem. Let

$$\begin{aligned}
\mathcal{K}(H) &:= \{K \mid K \subseteq H \text{ is a left coideal subalgebra}\}\\
\mathcal{Q}(H) &:= \{I \mid I \subseteq H \text{ is a coideal and a right } H\text{-submodule}\}\\
\mathcal{K}(H)_{right} &:= \{K \in \mathcal{K}(H) \mid H \text{ is right faithfully flat over } K\}\\
\mathcal{Q}(H)_{right} &:= \{I \in \mathcal{Q}(H) \mid H \text{ is right faithfully coflat over } H/I\}
\end{aligned}$$

We define $\mathcal{K}(H)_{left}$ and $\mathcal{Q}(H)_{left}$ analogously. The following theorem was first formulated in this generality by Masuoka in [52] but not proved explicitly.

Theorem 3.3.10. [70, Theorem 3.1.6] *Let H be a Hopf algebra with bijective antipode. Then*

1. *The maps*
$$\mathcal{K}(H)_{right} \leftrightarrow \mathcal{Q}(H)_{right},\ K \mapsto K^+H,\ and\ I \mapsto H^{co\,H/I},$$
   are inverse bijections.
2. *If $K \in \mathcal{K}(H)_{right}$, $I = K^+H$ and $Q = H/I$, the coinduction functor*
$${}^{Q}\mathcal{M} \to {}^{H}_{K}\mathcal{M},\ V \mapsto H \,\Box_Q\, V$$
   is an equivalence, H is projective as a right K-module, and K is a right K-direct summand in H.

Applying Theorem 3.3.10 to the category of affine commutative Hopf algebras, we deduce that a right (resp. left) coideal subalgebra B of a Hopf algebra H over which H is right (resp. left) faithfully flat corresponds to quotient Hopf algebras H/I of H over which H is right (resp. left) faithfully coflat. In the category of affine algebraic groups over an algebraically closed field, this correspondence is interpreted as $H = \mathcal{O}(G)$ and $I = \mathcal{O}(G/T)$ such that H is faithfully flat over I where T is a closed subgroup of an algebraic group G. In particular, right coideal subalgebras B over which H is faithfully flat are of the form $H^{co\, H/B^+H}$.

Similarly, we saw in §2.7 that the coordinate ring $H = k[G]$ of an affine variety G corresponds with an algebraic group structure on G. As described in [38, §1.3], a faithfully flat embedding $B = k[X] \subset H$ corresponds with a surjection $G \twoheadrightarrow X$ [54, Theorem 7.3]. Since $\Delta(B) \subset B \otimes H \cong k[X \times G]$, Δ defines an algebraic action $X \times G \to X$ of G on X for which the quotient map $G \twoheadrightarrow X$ is equivariant. Thus, the action is transitive and $X \cong G/H$ for a closed subgroup $H \subset G$ so that X is actually a homogeneous space of G. The geometric property of transitive group actions corresponds with the algebraic property of faithful flatness described in the preceding paragraph. Thus, we get a motivation for the definition of a quantum homogeneous space as a right coideal subalgebra B of a Hopf algebra H over which H is left and right faithfully flat.

Next, we discuss what normal Hopf subalgebras (Definition 2.5.7) correspond to in the theory of homogeneous spaces.

Another version of Lemma 3.3.9(2)(b) is as follows. If a Hopf algebra H has a bijective antipode S, then there is bijection between left and right coideal subalgebras of H. That is, given a left coideal subalgebra B of H, then $S(B)$ is a right coideal subalgebra of H since for any $b \in B$,

$$\Delta(S(b)) = \tau \circ (S \otimes S) \circ \Delta = S(b_{(2)}) \otimes S(b_{(1)}) \in S(B) \otimes H.$$

Thus, $S(B)$ is a right coideal subalgebra of H. The inverse map of the bijection is given by the inverse S^{-1} of the antipode which turns a right coideal subalgebra into a left coideal subalgebra.

In particular, as noted in [27], every coideal subalgebra of $H = \mathcal{O}(G)$ for an affine unipotent algebraic group G arises as above. That is if K is a left coideal subalgebra of H, then K^+H is a Hopf ideal of H so $H/K^+H = \mathcal{O}(D)$ for some closed subgroup D of G and $K = H^{co\,\pi}$ where $\pi : H \to H/K^+H$. Moreover, as algebras $K \cong \mathcal{O}(G/D)$, the coordinate ring of a homogeneous space of G [27, Theorem 5.3.2].

The following theorem is a special case of Theorem 3.3.10(1), since if H is a commutative Hopf algebra and K is a Hopf subalgebra H, then H is a faithfully flat K-module by Theorem 3.3.13. Thus, the collection $\{K\}$ in the subsequent theorem is a subset of $\mathcal{K}(H)_{right}$ in Theorem 3.3.10(1). Hence, the following theorem is a restatement of the commutative part of Theorem 3.3.10.

Theorem 3.3.11. [78, Theorem 4.3] *The correspondence $K \mapsto K^+H$ is a bijection from the set of all Hopf subalgebras of a commutative Hopf algebra H onto the set of all normal Hopf ideals of H where K^+ is the kernel of $\varepsilon : K \to k$.*

In the classical setting of commutative or cocommutative Hopf algebras, Takeuchi proved that there is a family of homogeneous spaces which is stated as follows.

Theorem 3.3.12. [78, Theorem 3.1] *Let H be a Hopf algebra over a field k and K be a Hopf subalgebra of H. If H is commutative or cocommutative, then H is a faithfully flat right (or left) K-module.*

Any Hopf subalgebra of a Hopf algebra H is in particular a left and right coideal subalgebra of H. However, we only get the flatness property and not the faithful flatness property when we consider right coideal subalgebras over a commutative Hopf algebra as stated as follows. This result is due to Masuoka and Wigner.

Theorem 3.3.13. [53, Theorem 3.4] *A commutative Hopf algebra is a flat module every right coideal subalgebra.*

Though from Theorem 3.3.13 we see that a commutative Hopf algebra is flat over its right coideal subalgebras, it is not true in general that a commutative Hopf algebra is faithfully flat over its right coideal ideal subalgebras. We see an example of a right coideal subalgebra which does not satisfy the faithful flatness property in Example 3.4.8. The special class of right coideal subalgebras which satisfy the faithful flatness property is what we call a quantum homogeneous space.

We saw in §3.2.3 above that free modules are faithfully flat modules. Though we saw in Theorem 3.3.12 that commutative or cocommutative Hopf algebras are faithfully flat over its Hopf subalgebras, Oberst and Schneider gave an example of a commutative Hopf algebra which is not free over a Hopf subalgebra. This is stated as follows:

Example 3.3.14. [61][57, Example 3.5.2] Let $F \subset E$ be a Galois field extension of degree 2, with Galois group $G = \{1, \sigma\}$. Let σ act on $\mathbb{Z}$ by $z \mapsto -z$. Then G acts on the group algebra $E\mathbb{Z}$ by acting on both E and $\mathbb{Z}$. Let $H = (E\mathbb{Z})^G$ and $K = (E(n\mathbb{Z}))^G \subset H$. The algebra $H = (E\mathbb{Z})^G$ is a Hopf algebra by [57, Lemma 3.5.1]. If n is even, then H is not free over K.

Thus, freeness of a Hopf algebra over its Hopf subalgebras seems to be a strong and restrictive property than the faithful flatness property.

We conclude this section with a family of quantum homogeneous spaces when H is a pointed commutative Hopf algebra. This result is due to Takeuchi.

Theorem 3.3.15. [79, Corollary 4] *If H is a commutative pointed Hopf algebra, there is a one to one correspondence between right coideal subalgebras over which H is free and Hopf ideals of H.*

The results from this section are summarised in the diagrams below: Let H be a commutative Hopf algebra, then we have the following diagram.

$$\left\{\begin{array}{l} K\text{, a Hopf} \\ \text{subalgebra of } H \end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{l} H/I\text{, a Hopf quotient} \\ \text{of } H\text{, with } I \text{ normal.} \end{array}\right\} \tag{3.1}$$

$$\begin{array}{ccc} K & \mapsto & H/K^{+}H \\ H^{co\,H/I} & \leftarrow\!\shortmid & H/I \end{array}$$

$$\bigcap\!| \qquad\qquad\qquad\qquad\qquad\qquad \bigcap\!|$$

$$\left\{\begin{array}{l} K\text{, a faithfully} \\ \text{flat left coideal} \\ \text{subalgebra of } H \end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{l} H/I\text{, a Hopf quotient} \\ \text{of } H \text{ with } H \text{ faithfully} \\ \text{coflat over } H/I. \end{array}\right\} \tag{3.2}$$

$$\begin{array}{ccc} K & \mapsto & H/K^{+}H \\ H^{co\,H/I} & \leftarrow\!\shortmid & H/I \end{array}$$

$$\updownarrow \quad \downarrow^{K}_{S(K)} \ \uparrow^{S(K)}_{K} \qquad\qquad\qquad\qquad\qquad \|$$

$$\left\{\begin{array}{l} K\text{, a faithfully} \\ \text{flat right coideal} \\ \text{subalgebra of } H \end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{l} H/I\text{, a Hopf quotient} \\ \text{of } H \text{ with } H \text{ faithfully} \\ \text{coflat over } H/I. \end{array}\right\} \tag{3.3}$$

$$\begin{array}{ccc} K & \mapsto & H/K^{+}H \\ {}^{co\,H/I}H & \leftarrow\!\shortmid & H/I \end{array}$$

The one-to-one correspondence in (3.1) is due to Theorem 3.3.11 which is equivalent to part of Theorem 3.3.3. That of (3.2) and (3.3) are due to Theorem 3.3.10(1). The inclusion of the left hand side of (3.1) into the left hand side of (3.2) is due to Theorem 3.3.12. However, the inclusion of the right hand side of (3.1) into the right hand side of (3.2) follows by following round three sides of the top square, that is using the arrow $\leftarrow$ in (3.1), followed by the inclusion and then finishing with the arrow $\rightarrow$ as in the diagram

$$\begin{array}{cc} & \longleftarrow \\ \bigcap\!| & \\ & \longrightarrow. \end{array}$$

The one-to-one correspondence of the left hand side of (3.2) with the left hand side of (3.3) is due to Lemma 3.3.9(2)(b). In fact, the inclusion of the left hand side of (3.1) in the left hand side of (3.2) and (3.3) is because a Hopf subalgebra K is a left and right coideal subalgebra such that $S(K) = K$.

3.4 Noncommutative Setting

Now, suppose H be an arbitrary Hopf algebra and $B \subset H$ a right coideal subalgebra of H. From the algebraic point of view, the inclusion $B \subset H$ only has good properties if H is faithfully flat as a left or right B-module [52, 72, 79]. Suppose H has a bijective antipode, then by [52, Theorem 1.11], the mapping

$$\begin{Bmatrix} B\colon\ B \subset H \text{ is a right coideal} \\ \text{subalgebra such that } H \text{ is} \\ \text{left faithfully flat over } B. \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} H/I\colon\ H/I \text{ is a quotient left} \\ H\text{-module coalgebra over} \\ \text{which } H \text{ is left faithfully coflat.} \end{Bmatrix} \tag{3.4}$$

$$\begin{array}{ccc} B & \mapsto & H/HB^{+} \\ H^{co\, H/I} & \leftarrow & H/I \end{array}$$

is a bijection between right coideal subalgebras and quotient objects H/I of H which satisfy the faithful flatness and coflatness properties. Let H be a commutative or cocommutative Hopf algebra and B a Hopf subalgebra of H. We saw in Theorem 3.3.12 that Hopf subalgebras of commutative or cocommutative Hopf algebras are examples of subobjects which satisfy the faithful flatness property.

In the setting of an arbitrary Hopf algebra (or quantum group) H, there is some inconsistency in the definition of a quantum homogeneous space. For instance, Krähmer in [38] defines a quantum homogeneous space as a right coideal subalgebra of a Hopf algebra over which the Hopf algebra is right faithfully flat. We use the definition in [47] of a quantum homogeneous space being a right coideal subalgebra B of H such that H is faithfully flat as a left and right B-module.

Definition 3.4.1. A *right quantum homogeneous space* is a right coideal subalgebra B of a Hopf algebra A such that A is faithfully flat as a right and left B-module. A *left quantum homogeneous space* is a left coideal subalgebra B of a Hopf algebra A such that A is faithfully flat as a right and left B-module

Usually, we write quantum homogeneous spaces in short for right or left quantum homogeneous spaces.

Due to the one-to-one correspondence (3.4) between right coideal subalgebras over which a Hopf algebra H is left faithfully flat and coideals I which are left ideals over which H is left faithfully coflat over H/I, this yields a definition of a quantum homogeneous spaces as coinvariants of H/I over which H is faithfully coflat. That is a quantum homogeneous space B of a Hopf algebra H is written as

$$B = \{h \in H \mid (id_H \otimes \pi) \circ \Delta(h) = h \otimes \pi(1)\}$$

where π is the projection from H to H/HB^+ with $B^+ := B \cap \ker(\varepsilon)$.

Our first example of a quantum homogeneous space is Podleś Standard Quantum 2-Sphere which is an $\mathrm{SU}_q(2)$-space which are analogues of the classical 2-sphere denoted by $\mathrm{SU}(2)/\mathrm{SO}(2)$.

Example 3.4.2 (Podleś Standard Quantum 2-Sphere)**.** [63] Fix $q \in \mathbb{C}^\times$ not a root of unity and consider the quantised coordinate ring $H := \mathbb{C}_q[\mathrm{SL}(2)]$ defined by generators a, b, c, d satisfying the relations;

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc,$$

$$ad - qbc = 1, \quad da - q^{-1}bc = 1.$$

Define the coproduct, counit and antipode of the generators by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes a + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d$$

$$\varepsilon(a) = \varepsilon(b) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0$$

$$S(a) = d, \quad S(b) = q^{-1}b, \quad S(c) = -qc, \quad S(d) = a.$$

The algebra B generated by

$$y_{-1} := ca, \quad y_0 := bc, \quad y_1 := bd$$

and defining relations

$$y_0 y_{\pm 1} = q^{\pm 2} y_{\pm 1} y_0, \qquad y_{\pm 1} y_{\mp 1} = q^{\mp 2} y_0^2 + q^{\mp 1} y_0$$

is a quantum homogeneous space known as *Podleś Standard Quantum 2-Sphere* [63]. Podleś proved this using analytical methods. From the algebraic point of view, by [59, Theorem 2.2], Müller and Schneider proved that the extension $B \subset H$ has the crucial property of faithful flatness.

In the next part of this section, we discuss examples of general families of quantum homogeneous spaces which have been exhibited for Hopf algebras satisfying some properties.

The first of Kaplansky's "Ten conjectures on Hopf algebras" [33] was that a Hopf algebra H over a field k was free as a module over any Hopf subalgebra. We saw in

Example 3.3.14 above that this fails to be true in general. However, in the case when H is a finite dimensional Hopf algebra, the conjecture holds. This result is due to Nichols and Zoeller and it is stated as follows.

Theorem 3.4.3. [60, Theorem 7] *Let H be a finite dimensional Hopf algebra over a field k, and let B be a Hopf subalgebra. Then H is free as a left B-module.*

In particular, Hopf subalgebras of a finite dimensional Hopf algebra are quantum homogeneous spaces.

The next theorem is about Hopf algebras with cocommutative coradicals. This result is due to Masuoka.

Theorem 3.4.4. [51, Theorem 1.3(a)] *Let H be a Hopf algebra with the antipode S. Suppose the coradical H_0 of H is cocommutative. If B is a right coideal subalgebra of H such that $S(B_0) = B_0$, then H is faithfully flat as a left and right B-module.*

Next, we state the following Lemma by Masuoka.

Lemma 3.4.5. [51, Lemma 2.6] *Let H be as in Theorem 3.4.4 and $\pi : H \to \pi(H)$ a quotient H-module coalgebra. Put $B = H^{co\pi}$.*

1. $S(B_0) = B_0$
2. $H/HB^+ = \pi(H)$
3. *H is faithfully coflat as a right or left $\pi(H)$-comodule.*

Corollary 3.4.6. *Let H be as in Theorem 3.4.4 and $\pi : H \to \pi(H)$ a quotient H-module coalgebra. Put $B = H^{co\pi}$. Then H is faithfully flat as a left or right B-module. In particular, coinvariants $H^{co\pi}$ of a pointed Hopf algebra are quantum homogeneous spaces.*

A similar result has been shown by Radford for pointed Hopf algebras as follows:

Theorem 3.4.7. [64, Theorem 4] *Let A be a pointed Hopf algebra and $B \subseteq A$ any Hopf subalgebra. Then A is a free left (and right) B-module. In particular, A is faithfully flat as a left (and right) B-module.*

However, it is not true that an arbitrary Hopf algebra is faithfully flat as a left or right module over its right or left coideal subalgebras. We see this in the following example.

Example 3.4.8. Let $H = k\langle x, x^{-1}\rangle$ be the group algebra of the infinite cyclic group. Then H is a Hopf algebra with

$$\Delta(x) = x \otimes x, \quad \Delta(x^{-1}) = x^{-1} \otimes x^{-1}, \quad \varepsilon(x) = \varepsilon(x^{-1}) = 1, \quad S(x) = x^{-1}, \quad S(x^{-1}) = x^{-1}.$$

Then the subalgebra $B = k[x]$ is a right coideal subalgebra of H but not a Hopf subalgebra of H since $S(B) \notin B$. In addition, H is not faithfully flat as a left or right B-module since for $N = k[x]/\langle x^2\rangle$, $H \otimes_B N = 0$ even though $N \neq 0$.

Our focus in this thesis is to investigate which plane curves are quantum homogeneous spaces. We have a favourable result, that is, a special class of plane curves are quantum homogeneous spaces. Details of this are given in the next two chapters.

The starting point of our research was the following example which says the coordinate ring of the cusp $y^2 = x^3$ is a quantum homogeneous space.

Example 3.4.9. [28, Construction 1.2] Let $H := k\langle x, y, a, a^{-1}\rangle/I$ where I is the ideal generated by the following relations:

$$aa^{-1} = a^{-1}a = 1, \qquad ay = -ya, \qquad ax = \lambda xa,$$
$$yx = xy, \qquad y^2 = x^3$$

where $\lambda \in k$ is a primitive third root of unity. Define the coproduct, counit and antipode on the generators by:

$$\Delta(a) = a \otimes a, \quad \Delta(a^{-1}) = a^{-1} \otimes a^{-1}, \quad \Delta(x) = 1 \otimes x + x \otimes a^2, \quad \Delta(y) = 1 \otimes y + y \otimes a^3$$
$$\varepsilon(a) = \varepsilon(a^{-1}) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad S(a) = a^{-1}, \quad S(x) = -xa^{-2}, \quad S(y) = -ya^{-3}.$$

It is straightforward to check that I is a Hopf ideal. Thus H is a Hopf algebra. It can be shown that the coordinate ring B of the cusp $y^2 = x^3$ is a right coideal subalgebra of H and H is faithfully flat as a left and right B-module. We get the faithful flatness property of H over B when we use the Diamond lemma to show that a basis of H over B is

$$\{x^i y^j a^{i'} \mid i \in \mathbb{Z}_{\geq 0},\ i' \in \mathbb{Z},\ j \in \{0, 1\}\}.$$

Thus, B is a quantum homogeneous space. The Hopf algebra H appeared in a paper by Goodearl and Zhang in [28, Construction 1.2] as the Hopf algebra $B(1, 1, 2, 3, q)$ and in [38] by Krähmer . Some nice properties of H are that H is a noetherian domain of GK-dimension two.

Using the cusp $(y^2 = x^3)$ as a starting point, we wanted to investigate in [40] whether the nodal cubic $(y^2 = x^2 + x^3)$ could have its coordinate ring being a quantum homogeneous space. We started by using the Hopf algebra structure of the cusp defined in Example 3.4.9. This failed to work. We had to introduce a new generator b with $b^2 = a^3$. Again, we had to change the commutation relation between a and x from $ax = \lambda xa$ to $a^2x = -(axa + xa^2 + a^2) + a^3$ and $ax^2 = -(xax + x^2a + ax + xa)$. This new Hopf algebra has GK-dimension three as compared with the Hopf algebra in Example 3.4.9 which has GK-dimension two. Specific details of the nodal cubic being a quantum homogeneous space are in the paper [40] coauthored with Ulrich Krähmer. This result for the nodal cubic is a special case of general results on plane curves which are quantum homogeneous spaces that we discuss in Chapter 5.

The next example gives details of the Hopf algebra used to show that the nodal cubic is a quantum homogeneous space.

Example 3.4.10. Recall from Example 2.10.2 the algebra $H := k\langle x, y, a, b, a^{-1}\rangle/I$ where I is the ideal generated by the following relations:

$$aa^{-1} = a^{-1}a = 1, \qquad ba = ab, \qquad b^2 = a^3, \qquad y^2 = x^2 + x^3, \qquad ay = ya, \qquad bx = xb,$$
$$by = -yb, \qquad yx = xy, \qquad a^2x = -(xa^2 + axa + a^2) + a^3, \qquad ax^2 = -(x^2a + xax + ax + xa).$$

Then H is a Hopf algebra with the coproduct, counit and antipode defined on the generators by:

$$\Delta(a) = a \otimes a, \qquad \Delta(a^{-1}) = a^{-1} \otimes a^{-1}, \qquad \Delta(b) = b \otimes b,$$
$$\Delta(x) = 1 \otimes x + x \otimes a, \qquad \Delta(y) = 1 \otimes y + y \otimes a,$$
$$\varepsilon(a) = \varepsilon(a^{-1}) = \varepsilon(b) = 1, \qquad \varepsilon(x) = \varepsilon(y) = 0,$$
$$S(a) = a^{-1}, \qquad S(a^{-1}) = a, \qquad S(b) = b^{-1}, \qquad S(x) = -xa^{-1}, \qquad S(y) = -yb^{-1}.$$

We shall see later on in §5.3.3 that H has GK-dimension 3.

The coordinate ring B of the nodal cubic is a right coideal subalgebra of H. Moreover, H is faithfully flat as a left and right B module since we have a basis of H over B given by

$$\{x^i y^j (ax)^l a^{i'} b^{j'} \mid i, l \in \mathbb{Z}_{\geq 0},\ i' \in \mathbb{Z},\ j, j' \in \{0, 1\}\}.$$

This is shown using Bergman's Diamond Lemma. Consequently, H is free over B. Thus, the nodal cubic B is a quantum homogeneous space.

All the facts are special cases of more general results to be proved in Chapter 5.

Chapter 4

The Hopf Algebra $A(x, a, g)$

4.1 Introduction

As stated in Chapter 3, the motivation of this thesis is to investigate which plane curves are quantum homogeneous spaces. Our main result is that decomposable plane curves with degree less than or equal to 5 are quantum homogeneous spaces. In this chapter, we construct an auxiliary Hopf algebra $A(x, a, g)$ which will be used in Chapter 5 to prove the main theorem.

Recall that a *decomposable* plane curve $\mathcal{C}$ is a curve $\mathcal{C}$ of the form

$$\mathcal{C} = \{(x_0, y_0) \in k^2 \mid f(y_0) = g(x_0)\}.$$

Without loss of generality, we may assume both f and g have no constant term. We construct auxiliary Hopf algebras $A(x, a, g)$ and $A(y, b, f)$ using the left and right hand sides of the polynomial equation defining the decomposable curve $\mathcal{C}$. Define $A(x, a, g)$ as a quotient of the free product $k[x] * k[a^{\pm 1}]$ of the polynomial ring in the variable x and the Laurent polynomial in a, where in the free product, a is grouplike and x is $(1, a)$-primitive. Let $g(x)$ have degree $n \geq 2$. Then the quotient is by an ideal generated by $(n-1)$ relations which make the coset of $g(x)$ in the factor to be $(1, a^n)$-skew primitive.

The Hopf algebra $A(x, a, g)$ has many interesting properties, some of which we summarise here. For a discussion on what we mean by PBW basis, see §2.10.1. Similarly, for Gelfand-Kirillov dimension, down-up algebra, maximal order, AS-regular, GK-Cohen Macaulay, see §2.9, §4.6.1 and §4.6.12.

Theorem 4.1.1. *Let $g = g(x)$ be a monic polynomial of degree n, with $n \geq 2$ and $g(0) = 0$.*

(i) The Hopf algebra $A(x, a, g)$ is pointed, generated by the group-like a and the $(1, a)$ -primitive element x.

(ii) If $n \leq 5$ then $A(x, a, g)$ has an explicit PBW basis.

(iii) Suppose $n \le 3$. Then $A(x,a,g)$ is noetherian, a finite module over its affine centre, and has Gel'fand-Kirillov dimension n.

(iv) When $n = 2$, $A(x,a,g)$ is isomorphic as a Hopf algebra to the Borel in $U_{-1}(\mathfrak{sl}(2))$, or equivalently to the quantum plane at parameter $q = -1$, localised at powers of one of the two generators.

(v) Suppose $g(x) = x^3$. Then $A(x,a,g)$ is isomorphic as an algebra to the localisation of the down-up algebra $A(-1,-1,0)$ at powers of a generator. It is a noetherian domain of Gel'fand-Kirillov dimension 3, finite over its centre, a maximal order, AS-regular and GK-Cohen Macaulay.

(vi) Suppose $n = 3$. Then $A(x,a,g)$ is a PBW deformation of $A(x,a,x^3)$, (and so of the localised down-up algebra), having the same properties as listed above for $A(x,a,x^3)$.

Part (i) of Theorem 4.1.1 is Proposition 4.3.1(ii), part (ii) is Corollary 4.4.8, part (iii) comes from Propositions 4.6.11 and 4.6.23, part (iv) from Proposition 4.6.11, and parts (v) and (vi) from Proposition 4.6.14.

4.2 Definition of the Hopf algebra $A(x,a,g)$

4.2.1 Generators and relations

Let $F = k[x] * k[a^{\pm 1}]$ be the free product of the polynomial algebra $k[x]$ with the Laurent polynomial algebra $k[a^{\pm 1}]$. It is easy to check that F is a Hopf algebra with x defined to be $(1,a)$-primitive and a group-like. More precisely, we have the following lemma.

Lemma 4.2.1. *The algebra F admits a unique Hopf algebra structure whose coproduct Δ, counit ε and antipode S satisfy:*

$$\Delta(x) = 1 \otimes x + x \otimes a, \qquad \Delta(a) = a \otimes a,$$
$$\varepsilon(x) = 0, \quad \varepsilon(a) = 1, \qquad S(x) = -xa^{-1}, \quad S(a) = a^{-1}.$$

The proof of Lemma 4.2.1 is a routine check which is left to the reader.
Define a $(\mathbb{Z}, \mathbb{Z}_{\ge 0})$-grading on F by giving a and x the degrees $(1,0)$ and $(0,1)$ respectively. For $i,j \in \mathbb{Z}_{\ge 0}$, let $P(j,i)_{(a,x)}$ denote the sum of all monomials in F of degree (j,i); in particular we set $P(0,0)_{(a,x)} = 1$. It is convenient in the proof of Lemma 4.2.2 to extend the argument of $P(j,i)_{(a,x)}$ to $\mathbb{Z} \times \mathbb{Z}$, by setting $P(j,i)_{(a,x)} = 0$ if i or j is less than 0. We will omit the suffix (a,x) whenever the variables involved are clear from the context. For instance,

$$P(2,1) = a^2x + axa + xa^2 \quad \text{and} \quad P(1,2) = ax^2 + xax + x^2a.$$

Let $n \geq 2$ and $g := g(x) = \sum_{i=1}^{n} r_i x^i \in k[x]$ with $r_n \neq 0$. Define $A(x,a,g)$ to be the quotient of F by the ideal I generated by the elements

$$\sigma_j = \sigma_j(x,a,g) := \sum_{i=j}^{n} r_i P(j, i-j)_{(a,x)} - r_j a^n, \qquad j = 1, \ldots, n-1. \tag{4.1}$$

As indicated, we will omit the variables and simply write σ_j whenever possible. It will be useful also for us sometimes to view the elements σ_j of F as members of the free subalgebra $E := k\langle x, a\rangle$ of F. We define $I_g := \langle \sigma_j : 1 \leq j \leq n-1\rangle$, an ideal of E, and

$$A_0(x,a,g) := E/I_g. \tag{4.2}$$

It is worth noting that the ideal of relations I can be expressed in terms of I_g as $I = FI_gF$ and $I_gF \subseteq I$. Throughout the paper, we shall slightly abuse notation by using the same symbols for the elements x and a of E and F and for their cosets in $A_0(x,a,g)$ and $A(x,a,g)$. We shall on some occasions as, for example, in the statement of Theorem 4.1.1, find it convenient to assume that $g(x)$ is normalised so that its highest coefficient r_n equals 1. The form of the relations (4.1) ensures that this does not affect the definition of $A_0(x,a,g)$ or $A(x,a,g)$.

4.2.2 Examples of the algebra $A(x,a,g)$

We give here two very simple examples, for g with $\deg(g) = 2, 3$, which we shall return to later in Section 4.6.2.

(i) Let $g_2 := x^2$, then $A(x,a,g_2)$ is the quotient of F by the relation

$$ax = -xa.$$

Thus, the algebra $A(x,a,g_2)$ is a quantum Borel in $U_{-1}(\mathfrak{sl}(2))$ (see [13, Chapter I.3] for more on $U_{-1}(\mathfrak{sl}(2))$ and the quantum Borel subalgebra).

(ii) Let $g_3 := x^2 + x^3$. Then $A(x,a,g_3)$ is the quotient of F by the relations

$$a^2x = -(a^2 + xa^2 + axa) + a^3,$$
$$ax^2 = -(ax + xa + x^2a + xax).$$

4.2.3 $A(x,a,g)$ is a Hopf algebra

To prove that the Hopf algebra structure on F descends to $A(x,a,g)$ we need the following lemma.

Lemma 4.2.2. *Let $g(x) \in k[x]$ with $g(x) = \sum_{i=1}^{n} r_i x^i$, with $r_n \neq 0$ and $n \geq 2$. Retain the notation of §4.2.1.*

(i) In F, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$\Delta(x^\ell) = \sum_{s=0}^{\ell} x^s \otimes P(s, \ell - s).$$

(ii) In F, for all $j, t \in \mathbb{Z}_{\geq 0}$

$$\Delta(P(j,t)) = \sum_{\ell=0}^{t} P(j,\ell) \otimes P(j+\ell, t-\ell).$$

(iii) Modulo $I \otimes F + F \otimes I$,

$$\Delta(g) \equiv 1 \otimes g + g \otimes a^n.$$

(iv) Let $j \in \mathbb{Z}$ with $1 \leq j \leq n-1$. Modulo $I \otimes F + F \otimes I$,

$$\Delta(\sum_{\ell=j}^{n} r_\ell P(j, \ell - j)) \equiv r_j a^n \otimes a^n.$$

Proof. (i) Let $\ell \in \mathbb{Z}_{\geq 0}$ Then

$$\Delta(x^\ell) = \Delta(x)^\ell = (1 \otimes x + x \otimes a)^\ell = \sum_{s=0}^{\ell} x^s \otimes P(s, \ell - s).$$

(ii) By coassociativity, the following holds:

$$(\mathrm{id} \otimes \Delta) \circ \Delta(x^{j+t}) = (\Delta \otimes \mathrm{id}) \circ \Delta(x^{j+t}) \tag{4.3}$$

From (i) above,

$$\Delta(x^{j+t}) = \sum_{s=0}^{j+t} x^s \otimes P(s, (j+t) - s).$$

Thus, the left hand side of (4.3) becomes

$$(\mathrm{id} \otimes \Delta) \circ \Delta(x^{j+t}) = \sum_{s=0}^{j+t} x^s \otimes \Delta(P(s, (j+t) - s)). \tag{4.4}$$

Similarly, the right hand side of (4.3) becomes

$$(\Delta \otimes \mathrm{id}) \circ \Delta(x^{j+t}) = \sum_{s=0}^{j+t} \Delta(x^s) \otimes P(s, (j+t) - s)$$

which expands by (i) to

$$(\Delta \otimes \mathrm{id}) \circ \Delta(x^{j+t}) = \sum_{s=0}^{j+t} \left(\sum_{\ell=0}^{s} x^{\ell} \otimes P(\ell, s-\ell) \right) \otimes P(s, (j+t)-s). \tag{4.5}$$

The component of (4.4) with left hand side tensor x^j is

$$x^j \otimes \Delta(P(j,t)).$$

Turning now to (4.5), the component with left hand entry x^j here is obtained from the terms in the sum on the right where $s = j, j+1, \cdots, j+t$. Thus, the component of (4.5) with left hand tensor x^j is

$$x^j \otimes P(j,0) \otimes P(j,t) + x^j \otimes P(j,1) \otimes P(j+1,t-1) + \cdots + x^j \otimes P(j,t) \otimes P(j+t,0) + \cdots$$

$$= x^j \otimes \sum_{\ell=0}^{t} P(j,\ell) \otimes P(j+\ell, t-\ell).$$

Hence, by the equality in (4.3), we deduce that

$$\Delta(P(j,t)) = \sum_{\ell=0}^{t} P(j,\ell) \otimes P(j+\ell, t-\ell).$$

(iii) It is convenient to define $r_0 = 0$. By part (i), in F,

$$\begin{aligned} \Delta(g) &= \sum_{\ell=0}^{n} r_\ell \Delta(x^\ell) \\ &= \sum_{\ell=0}^{n} r_\ell \left(\sum_{s=0}^{\ell} (x^s \otimes P(s, \ell - s) \right) \\ &= \sum_{s=0}^{n} x^s \otimes \left(\sum_{\ell=s}^{n} r_\ell P(s, \ell - s) \right). \end{aligned}$$

Thus, recalling the generators (4.1) of I and noting that $r_0 = 0$, the above identity in F implies that, *mod* $(I \otimes F + F \otimes I)$,

$$\begin{aligned} \Delta(g) &\equiv 1 \otimes \left(\sum_{\ell=1}^{n} r_\ell P(0,\ell) \right) + \sum_{s=1}^{n} x^s \otimes r_s a^n \\ &\equiv 1 \otimes \left(\sum_{\ell=1}^{n} r_\ell x^\ell \right) + \left(\sum_{s=1}^{n} r_s x^s \right) \otimes a^n, \end{aligned}$$

proving (iii).

(iv) Let $j \in \mathbb{Z}_{\geq 0}$, with $1 \leq j \leq n-1$. We calculate, using (ii) for the second equality, regrouping terms for the third, and then by two applications of the relations (4.1), that

$$\begin{aligned}
\Delta\left(\sum_{i=j}^{n} r_i P(j,i-j)\right) &= \sum_{i=j}^{n} r_i \Delta(P(j,i-j)) \\
&= r_j P(j,0)\otimes P(j,0) + r_{j+1}\left(\sum_{i=0}^{1} P(j,i)\otimes P(j+i,1-i)\right) + \\
&\qquad \cdots + r_n \sum_{i=0}^{n-j} (P(j,i)\otimes P(j+i,n-j-i)) \\
&= P(j,0)\otimes \sum_{s=j}^{n} r_s P(j,s-j) + P(j,1)\otimes \sum_{s=j+1}^{n} r_s P(j+1,s-j-1) \\
+\cdots+ &\quad P(j,n-1-j)\otimes\left(\sum_{s=n-1}^{n} r_\ell P(n-1,s-n+1)\right) \\
+ &\quad P(j,n-j)\otimes r_n P(n,0) \\
&\equiv (P(j,0)\otimes r_j P(n,0)) + (P(j,1)\otimes r_{j+1}P(n,0)) \\
&\qquad +\cdots+ (P(j,n-j)\otimes r_n P(n,0)) \qquad (mod\, I).
\end{aligned}$$

We may thus conclude that, for $j = 1, \ldots, n-1$,

$$\begin{aligned}
\Delta(\sum_{i=j}^{n} r_i P(j,i-j)) &\equiv \left(\sum_{i=j}^{n} r_i P(j,i-j)\right)\otimes P(n,0) \\
&\equiv r_j a^n \otimes a^n,
\end{aligned}$$

as required.

□

It is now a simple matter to deduce the following theorem. Notice in particular that the existence of the counit includes the implication that $A(x,a,g)$ is not $\{0\}$, a conclusion to be strengthened in Proposition 4.3.1(iv).

Theorem 4.2.3. *Assume the notation and hypotheses from §4.2.1. Then the k-algebra $A := A(x,a,g)$ is a Hopf algebra with the coproduct, counit and antipode defined in Lemma 4.2.1.*

Proof. By Lemma 4.2.1, it suffices to show that, for each generator σ_j of I_g,

$$\Delta(\sigma_j) \in I_g F\otimes F + F\otimes I_g F; \quad S(\sigma_j)\in I_g F; \quad \textit{and} \quad \varepsilon(\sigma_j) = 0.$$

First, for $j = 1, \ldots n-1$, $\Delta(\sigma_j) \in I_g F\otimes F + F\otimes I_g F$ by Lemma 4.2.2(iv), and it is easy to check that $\varepsilon(\sigma_j) = 0$. Thus $A(x,a,g)$ is a bialgebra, which by §4.2.1 is generated by the invertible grouplike element a, and the $\{1,a\}$-primitive element x. Hence, by [65, Proposition 7.6.3], it is a Hopf algebra, with antipode given by the formulae in Lemma 4.2.1. □

Example 4.2.4. The algebra $A(x, a, x^2 + x^3)$ defined in Section 4.2.2 above is a Hopf algebra. This Hopf algebra first appeared in the paper [40] by Krähmer and Tabiri where it was used to show that the coordinate ring of the nodal cubic $y^2 = x^2 + x^3$ is a quantum homogeneous space.

4.2.4 Scaling isomorphisms

In the hypothesis of Theorem 4.1.1, we assume that $g(x) = \sum_{i=1}^{n} r_i x^i$ is a monic polynomial. The purpose of Lemma 4.2.5 is to show that for any $g(x)$ not necessarily monic, there is an isomorphism between $A(x, a, g)$ and $A(x, a, g')$ where $g' = g(\lambda x)$, with $\lambda = r_n^{-1/n}$, is the monic polynomial we derive from g.

Lemma 4.2.5. *Assume the notation and hypotheses from §4.2.1, and let $\lambda \in k \setminus \{0\}$. Define a k-algebra automorphism θ_λ of $E = k\langle x, a\rangle$ by $\theta_\lambda(a) = a$, $\theta_\lambda(x) = \lambda x$. Set*

$$g^\lambda := g(\lambda x) = \sum_{i=1}^{n} r_i (\lambda x)^i.$$

(i) $\theta_\lambda(I_g) = I_{g^\lambda}$.

(ii) θ_λ induces an algebra isomorphism from $A_0(x, a, g)$ to $A_0(x, a, g^\lambda)$, and a Hopf algebra isomorphism from $A(x, a, g)$ to $A(x, a, g^\lambda)$.

Proof. (i) One checks easily that, for $j = 1, \ldots, n-1$,

$$\theta_\lambda(\sigma_j(x, a, g)) = \lambda^{-j} \sigma_j(x, a, g^\lambda).$$

(ii) It is immediate from (i) that θ_λ induces an algebra isomorphism from $A_0(x, a, g)$ to $A_0(x, a, g^\lambda)$, which clearly extends to $A(x, a, g)$ since θ_λ extends to an automorphism of F. It is routine to check that the map respects the Hopf operations. □

Remark 4.2.6. Lemma 4.2.5 permits us to adjust the defining polynomial g, for example by ensuring that the polynomial is monic, as we shall frequently do in the sequel, to ease calculations.

4.3 First properties of $A(x, a, g)$ and $A_0(x, a, g)$

We begin this section by discussing the universality of the Hopf algebra $A(x, a, g)$ following Manin's approach. For more details on this, see [49]. Then we talk about other properties of $A(x, a, g)$. All the properties that we discuss here hold in general for any $A(x, a, g)$ irrespective of the degree of g. In subsequent sections, we discuss properties of $A(x, a, g)$ for specific degrees of g such as $\deg(g) = 2, 3$. These include the existence of a PBW basis and the fact that $A(x,a,g)$ is a noetherian domain of finite global and Gelfand-Kirillov dimension when $\deg(g) = 2,3$.

4.3.1 The origin of the defining relations for $A(x,a,g)$

Given a decomposable plane curve defined by the polynomial $f(y) = g(x)$, our aim was to find a Hopf algebra U which contained the coordinate ring B of the curve as a right coideal subalgebra such that U is faithfully flat as a left and right B-module. The process of finding this universal bialgebra which respects the relation $f(y) = g(x)$ is attributed to Manin [49] and referred to as Manin's approach.

In §2.8, we saw an example of Manin's approach for the cusp $y^2 = x^3$. After using the same approach for the nodal cubic $y^2 = x^2 + x^3$ in [40], we generalised this to any decomposable plane curve $f(y) = g(x)$. This gives a new family of Hopf algebras, $A(x,a,g)$ which was defined at the beginning of this chapter and $A(g,f)$ which we will see in Chapter 5.

Generalising the approach in Example 2.8.1 to a general decomposable plane curve $f(y) = g(x)$, we derive a Hopf algebra which conjecturally contains the coordinate ring of $f(y) = g(x)$ as a right coideal subalgebra. This Hopf algebra with coproduct, counit and antipode the same as Example 2.8.1 is a quotient of $k\langle x, y, a^{\pm 1}, b^{\pm 1}\rangle$ by the ideal generated by the following relations:

$$f(y) = g(x), \qquad yx = xy, \qquad ay = ya, \qquad bx = xb, \qquad ba = ab$$
$$\sum_{i=j}^{m} s_i P_{(b,y)}(j, i-j) = s_j b^m, \qquad \sum_{i=j'}^{n} r_i P_{(a,x)}(j', i-j) = r_{j'} a^n, \qquad b^m = a^n$$

where $f(y) = \sum_{i=1}^{m} s_i y^i$, $s_m \neq 0$, $g(x) = \sum_{i=1}^{n} r_i x^i$, $r_n \neq 0$, $1 \leq j \leq m-1$ and $1 \leq j' \leq n-1$.

Similar to Example 2.8.1, the relations

$$ay = ya, \qquad bx = xb, \qquad ba = ab$$

are derived from the fact that we want the coproduct to be the same on xy and yx. Also, in order for the coproduct on $f(y)$ and $g(x)$ to be equal, we derive the remaining relations

$$\sum_{i=j}^{m} s_i P_{(b,y)}(j, i-j) = s_j b^m, \qquad \sum_{i=j'}^{n} r_i P_{(a,x)}(j', i-j') = r_{j'} a^n, \qquad b^m = a^n.$$

We can see from the defining relations that there are "nice" relations between all the generators except the relations between b and y and that between x and a. The rest of the relations look like the coordinate rings of the decomposable plane curves $f(y) = g(x)$ and $b^m = a^n$. This guided our choice to consider an algebra with generators x, a and another algebra with generators y, b separately before bringing them together to find this algebra

U which gives us the universal property we are looking for.

4.3.2 Other properties

Here are some elementary consequences of the defining relations for the Hopf algebras $A(x,a,g)$ and $A_0(x,a,g)$, valid for all choices of $g = g(x)$.

Proposition 4.3.1. *Let $A := A(x,a,g)$ be as defined in §4.2.1, so g has degree $n \geq 2$. Assume that g is monic. Then the following hold.*

(i) The element g of A is $(1,a^n)$-primitive.

(ii) A is a pointed Hopf algebra.

(iii) $k\langle g, a^{\pm n}\rangle$ of A is a central Hopf subalgebra of A.

(iv) The polynomial algebra $k[x]$ embeds in $A_0(x,a,g)$.

Proof. (i) This is immediate from Lemma 4.2.2(iii).
(ii) The Hopf algebra A is pointed by [65, Corollary 5.1.14(a)], since it is generated by the grouplike element a and the skew-primitive element x.
(iii) In view of (i) and the fact that a^n is grouplike, it suffices to show that a^n commutes with x and that g commutes with a.

From the defining relations (4.1) in §4.2.1, the elements a, x of A satisfy σ_{n-1}, namely

$$r_{n-1}a^{n-1} + P(n-1,1) = r_{n-1}a^n.$$

Thus

$$a^{n-1}x = -(xa^{n-1} + axa^{n-2} + \cdots + a^{n-2}xa) - r_{n-1}a^{n-1} + r_{n-1}a^n, \tag{4.6}$$

and so, pre-multiplying (4.6) by a,

$$a^n x = -(axa^{n-1} + a^2xa^{n-2} + \cdots + a^{n-1}xa) - r_{n-1}a^n + r_{n-1}a^{n+1}.$$

Using (4.6) to replace the term $a^{n-1}xa = (a^{n-1}x)a$ in this identity yields

$$\begin{aligned} a^n x \quad = \quad & -(axa^{n-1} + a^2xa^{n-2} + \cdots + a^{n-2}xa^2) - r_{n-1}a^n + r_{n-1}a^{n+1} \\ & +xa^n + axa^{n-1} + a^2xa^{n-2} + \cdots + a^{n-2}xa^2 + r_{n-1}a^n - r_{n-1}a^{n+1}. \end{aligned}$$

That is, $a^n x = xa^n$ as required.

To show that g commutes with a, begin with relation σ_1 from (4.1), namely

$$\sum_{i=1}^{n} r_i P(1, i-1) = r_1 a^n.$$

That is, in A,

$$ax^{n-1} = -(x^{n-1}a + x^{n-2}ax + \cdots + xax^{n-2}) - \sum_{i=1}^{n-1} r_i P(1, i-1) + r_1 a^n. \tag{4.7}$$

Post-multiplying (4.7) by x and then using (4.7) to replace the term xax^{n-1} in the resulting identity, we obtain

$$\begin{aligned} ax^n &= -(x^{n-1}ax + x^{n-2}ax^2 + \cdots + x^2ax^{n-2}) - \sum_{i=1}^{n-1} r_i P(1, i-1)x + r_1 a^n x \\ &\quad +(x^n a + x^{n-1}ax + \cdots + x^2 ax^{n-2}) + \sum_{i=1}^{n-1} r_i x P(1, i-1) - r_1 x a^n. \end{aligned}$$

The monomials in the above identity which begin and end with x cancel, so that, in A,

$$ax^n = x^n a - a\left(\sum_{i=1}^{n-1} r_i x^i\right) + \left(\sum_{i=1}^{n-1} r_i x^i\right) a + r_1(a^n x - xa^n).$$

Since a^n is central as proved above, it follows that $ag = ga$, so $g \in Z(A)$. Therefore, by (i), the proof of (iii) is complete.
(iv) Recall that E denotes the free algebra $k\langle x, a\rangle$. Since $\sigma_j \in EaE$ for all $j = 1, \ldots, n-1$, it follows that

$$A_0(x, a, g)/A_0(x, a, g)aA_0(x, a, g) \cong E/EaE \cong k[x].$$

Hence, $k[x]$ is a subalgebra of $A_0(x, a, g)$.

□

Remark 4.3.2. The restriction of (iv) of the above proposition to $A_0(x, a, g)$ rather than $A(x, a, g)$ constitutes a gap in our analysis: we would like to be able to say that no equation of the form

$$a^m h(x) = 0, \tag{4.8}$$

for $m > 0$ and $h(x) \in k[x] \setminus \{0\}$, is valid in $A_0(x, a, g)$. This would then imply that $k[x]$ embeds as a right coideal subalgebra of the localisation $A(x, a, g)$ of $A_0(x, a, g)$, and hence, by [51, Theorem 1.3], that $A(x, a, g)$ is a faithfully flat $k[x]$-module. While we shall prove below that (4.8) cannot occur in many important cases, the general statement remains open.

4.4 The PBW theorem for $A(x, a, g)$, $n \leq 5$

In this section we first obtain a PBW theorem, when $g(x)$ has degree at most 5, for the bialgebras $A_0(x, a, g)$, as defined in §4.2.1, and then use localisation to obtain a similar result for the corresponding Hopf algebras $A(x,a,g)$.

Fix an integer $n > 1$, and consider the setup of §4.2.1, with $g(x)$ normalised so that its top coefficient $r_n = 1$, as permitted by Lemma 4.2.5. Thus $X = \{x, a\}$ and $E = k\langle x, a\rangle = k\langle X\rangle$ is the free algebra, with $E \subset F = k[x] * k[a^{\pm 1}]$. For non-negative integers m and q, define

$$Q(m,q) \quad := \quad P(m,q) - a^m x^q \in k\langle X\rangle; \tag{4.9}$$

in particular, $Q(0,q) = Q(m,0) = 0$ for all $q, m \geq 0$. The generators (4.1) in F of the defining ideal I of $A(x,a,g)$ can thus be regarded as a set $\Sigma_g = \{\sigma_1, \dots, \sigma_{n-1}\}$ of relations for the free algebra $k\langle X\rangle$; namely, we view them as

$$\sigma_j : \qquad a^j x^{n-j} = -Q(j, n-j) - \sum_{i=j}^{n-1} r_i P(j, i-j) + r_j a^n, \qquad 1 \leq j \leq n-1. \tag{4.10}$$

For $j = 1, \cdots, n-1$, denote the left side of the relation σ_j by ω_j. We shall view Σ_g as a reduction system on $k\langle X\rangle$, as in §2.10.

A convenient semigroup ordering on the free monoid $\langle X\rangle$ to use with Σ_g is a *weighted graded lexicographic order*, which we denote by $>_{grlex+}$, and define as follows.

Definition 4.4.1. Let $w = x_1 \cdots x_t \in \langle X\rangle$, where $x_j \in \{a, x\}$ for $j = 1, \dots, t$.

(i) Define

- the *length* $|w|$ to be t;
- the *x-weight* $wt_x(w) := |\{i : x_i = x\}|$;
- the *lexicographic order* on $\langle X\rangle$ to be given by declaring $a >_{lex} x$.

(ii) For $u, v \in \langle X\rangle$, set

$$u >_{grlex+} v \iff (|u| > |v|) \vee (|u| = |v| \wedge wt_x(u) > wt_x(v)) \vee (|u| = |v| \wedge$$

$$wt_x(u) = wt_x(v) \wedge u >_{lex} v).$$

Lemma 4.4.2. *Retain the notation and hypotheses as above.*

(i) $>_{grlex+}$ *is a semigroup ordering on* $\langle X\rangle$.

(ii) $>_{grlex+}$ *satisfies the descending chain condition.*

(iii) $>_{grlex+}$ *is compatible with* Σ_g.

(iv) Σ_g *has no nontrivial inclusion ambiguities.*

(v) The only overlap ambiguities in Σ_g *are* $(\sigma_j, \sigma_{j+t}, a^t, a^j x^{n-j-t}, x^t)$*, for* $1 \leq j < j+t \leq n-1$.

Proof. (i)-(iv) are easy to check. For (v), the listed cases are indeed clearly overlap ambiguities. For the converse, note that ω_j has total degree n, for $j = 1, \dots, n-1$. Now every overlap ambiguity has the form

$$ABC = \overbrace{a^{r'}(a^r x^s}^{\omega_{j'}}) \underbrace{x^{s'}}_{} \quad \text{with } \omega_j = (a^r x^s) x^{s'}$$

for some $j' > j$, since $\{\sigma_1, \dots, \sigma_{n-1}\}$ are the only relations. Comparing degrees,

$$n = r' + r + s = r + s + s'.$$

Therefore, $r' = s' := t$, say, and then $j = r = j' - t$. Thus $j' = j + t$ as claimed. □

Retain the integer n, $n > 1$, and continue with $X = \{x, a\}$. Define a subset of $\langle X \rangle$,

$$\mathcal{L}_n := \{a^i x^j \mid 0 < i, j < n,\ i + j < n\}. \tag{4.11}$$

Lemma 4.4.3. *Keep the above notation.*

(i) $|\mathcal{L}_n| = \frac{1}{2}(n-1)(n-2)$.

(ii) The subsemigroup $\langle \mathcal{L}_n \rangle$ *of* $\langle X \rangle$ *generated by* $\mathcal{L}_n$ *is free of rank* $\frac{1}{2}(n-1)(n-2)$.

(iii) The set of irreducible words in $\langle X \rangle$ *with respect to the reduction system* Σ_g *is*

$$\langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle := \{x^i \omega a^\ell : i, \ell \in \mathbb{Z}_{\geq 0},\ \omega \in \langle \mathcal{L}_n \rangle\}.$$

Proof. (i) This is an easy induction.
(ii) Write u_{ij} for the element $a^i x^j$ of $\mathcal{L}_n$, and consider $w = u_{i_1 j_1} \cdots u_{i_m j_m} \in \langle \mathcal{L}_n \rangle$. Since each element of $\mathcal{L}_n$ begins with a and ends with x, the given expression for w as an element of $\langle \mathcal{L}_n \rangle$ is unique. Thus $\langle \mathcal{L}_n \rangle$ is free with basis $\mathcal{L}_n$.
(iii) A word in $\langle X \rangle$ is Σ_g-reducible if and only if it contains ω_j as a subword for some $j = 1, \dots, n-1$. Thus $\langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle$ consists of irreducible words. For, ω_j starts with a and ends with x, so if ω_j is a subword of $u = x^\ell u_0 a^m \in \langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle$, where $u_0 \in \langle \mathcal{L}_n \rangle$, then ω_j is a subword of u_0. But this is impossible since every word in $\mathcal{L}_n$ has length less than n, starts with a and ends with x.

Conversely, if $\omega \in \langle X \rangle$ and $\omega \notin \langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle$, then it must contain a subword of the form $a^i x^j$ for $i, j > 0$ and $i + j > n$. Thus ω is reducible. □

4.4.1 Resolving ambiguities

The aim in this subsection is to prove the following result.

Theorem 4.4.4. *Retain the notation and definitions of §4.2.1, §2.10 and §4.4. Let $n \in \mathbb{Z}$, $2 \le n \le 5$. Then $A_0(x,a,g)$ has PBW basis the set of monomials $\langle x\rangle\langle \mathcal{L}_n\rangle\langle a\rangle$ defined in Lemma 4.4.3(iii).*

Note that it follows from the discussion in §2.10 and §4.4, in particular from Lemma 4.4.3(iii), that, for *every* $n \ge 2$, $\langle x\rangle\langle \mathcal{L}_n\rangle\langle a\rangle$ is a spanning set for the vector space $A(x,a,g)$. By Bergman's Theorem 2.10.1 and by Lemma 4.4.2(iv), to prove that this set is linearly independent and hence a k-basis it remains only to show that the overlap ambiguities in Σ_g listed in Lemma 4.4.2(v) are resolvable. We shall achieve this for $n \le 5$ in Proposition A.0.2, for which a couple of preliminary lemmas are needed.

Lemma 4.4.5. *Let $n \ge 4$ and let $r,t \in \{1,2,\cdots,n-3\}$ with $r+t < n$. Let w and v be words of length t in a and x. Then $wP(r,n-t-r)v$ is reducible $\Leftrightarrow$ there exist $i,j \in \mathbb{Z}_{\ge 0}$ with $i+j=t$ such that w ends with a^i and v starts with x^j.*

Proof. Let $i,j \in \mathbb{Z}_{\ge 0}$, $i+j=t$, $w = w_1a^i$ and $v = x^jv_2$. Then $wP(r,n-t-r)v$ contains the reducible subword $a^i(a^rx^{n-t-r})x^j$ as claimed.
Conversely, suppose $wP(r,n-t-r)v$ contains a reducible word. Then, since the defining relations (4.10) are homogeneous of total degree n, this reducible word must occur as one of the following subwords of total degree n:

$$wP(r,n-t-r), \qquad P(r,n-t-r)v \qquad \text{and} \qquad w_2P(r,n-t-r)v_1.$$

The left hand sides of the defining relations begin with a^j and end with x^{n-j} for $1 \le j \le n-1$. This forces w in $wP(r,n-t-r)$ to be a^t since $r \ge 1$ and w has length t. Similarly, v in $P(r,n-t-r)v$ must be x^t since $t+r < n$. Finally, a reducible subword of the form $w_2P(r,n-t-r)v_1$, with w_2 and v_1 both non-identity words, must have $w_2 = a^i$ and $v_1 = x^j$ for $i+j=t$, since $r \ge 1$ and $t+r<n$. □

Lemma 4.4.6. *For $n \ge 4$, $r,t \in \{1,2,\cdots,n-3\}$ with $r+t<n$, w,v words of length t in a and x, all the words in $wQ(r,n-t-r)v$ are irreducible.*

Proof. This is a corollary of Lemma 4.4.5 since reducible words occurred there only from the word $a^rx^{n-t-r} \in P(r,n-t-r)$, which no longer appears in $Q(r,n-t-r)$. □

In the following proof we will use the symbol " $\to$ " whenever we replace a monomial ω_j (that is a^jx^{n-j}) with the right hand side $\left(-\left(Q(j,n-j) + \sum_{i=j}^{n-1} r_iP(j,i-j)\right) + r_ja^n\right)$ of the jth relation. Whenever a linear combination of irreducible words appears during the reduction process, we will underline it. For instance, $wQ(r,n-t-r)v$ from Lemma 4.4.6 above is irreducible so is written as $\underline{wQ(r,n-t-r)v}$. We use the symbol $\in$ to indicate that a word ω appears in an element of $k\langle X\rangle$: for example, $a^jx^{n-j} \in P(j,n-j)$.

The arguments used to prove the following proposition are elementary, but long and involved beyond $t = 1$. So we include only the proof for the case $t = 1$ here, relegating the proofs of the cases $t = 2$, $t = 3$ to the Appendix.

Proposition 4.4.7. *Retain the notation of §2.10 and §4.4. Then the overlap ambiguities*

$$(\sigma_j, \sigma_{j+t}, a^t, a^j x^{n-j-t}, x^t)$$

are resolvable when

(i) $t = 1$ *and* $n \geq 3$*;*

(ii) $t = 2$ *and* $n \geq 4$*;*

(iii) $t = 3$ *and* $n \geq 5$*.*

Proof. (i) Let $1 \leq j \leq n-1$, and consider the overlap ambiguity $\{\omega_j, \omega_{j+1}\}$ for $A(x,a,g)$. That is we consider the word in the free algebra $k\langle a, x\rangle$ given by

$$a\omega_j = a(a^j x^{n-j}) = (a^{j+1}x^{n-j-1})x = \omega_{j+1}x.$$

We show that applying either the relation σ_j or the relation σ_{j+1} to this word leads to the same linear combination of irreducible words in $A_0(x,a,g)$.

In what follows we shall frequently use the trivial identities

$$P(r,s) \quad = \quad P(r,s-1)x + P(r-1,s)a, \tag{4.12}$$

which hold for all $r, s \geq 0$ and

$$Q(r,s) \quad = \quad Q(r,s-1)x + P(r-1,s)a \tag{4.13}$$

which holds for $r \geq 0, s > 0$.

Beginning with the resolution of $a\omega_j$, write ω_j with the aid of (4.12) and (4.13) as

$$\begin{aligned}
a^j x^{n-j} &\rightarrow -\left(\sum_{i=j}^{n-1} r_i P(j, i-j) + Q(j, n-j)\right) + r_j a^n \\
&= -\sum_{i=j}^{n-1} r_i P(j-1, i-j)a - \sum_{i=j+1}^{n-1} r_i P(j, i-j-1)x \\
&\qquad -Q(j, n-j-1)x + P(j-1, n-j)a + r_j a^n \\
&= -\left(\sum_{i=j}^{n} r_i P(j-1, i-j)a + \sum_{i=j+1}^{n-1} r_i P(j, i-j-1)x + Q(j, n-j-1)x\right) + r_j a^n.
\end{aligned}$$

Now premultiply the above by a, and use Lemmas 4.4.5 and 4.4.6 to separate reducible and irreducible words, yielding

$$(\gamma)$$

$$\begin{aligned} a(a^j x^{n-j}) &\to -\left(\sum_{i=j}^{n} r_i aP(j-1,i-j)a + \sum_{i=j+1}^{n-1} r_i aP(j,i-j-1)x + \underline{aQ(j,n-j-1)x}\right) \\ &\qquad +r_j \underline{a^{n+1}} \\ &= -\sum_{i=j}^{n-1} r_i \underline{aP(j-1,i-j)a} - a^j x^{n-j} a - \underline{aQ(j-1,n-j)a} \\ &\qquad - \sum_{i=j+1}^{n-1} r_i aP(j,i-j-1)x - \underline{aQ(j,n-j-1)x} + r_j \underline{a^{n+1}}. \end{aligned}$$

As indicated by underlining above, the only irreducible words in (γ) are $a^j x^{n-j} a \in aP(j-1,n-j)a$ and $a^{j+1}x^{n-j-1} \in aP(j,n-j-2)x$. We reduce the first of these, first applying the relation σ_j to give

$$\begin{aligned} -a^j x^{n-j} &\to \sum_{i=j}^{n-1} r_i P(j,i-j) + Q(j,n-j) - r_j a^n \\ &= \sum_{i=j}^{n-1} r_i aP(j-1,i-j) + \sum_{i=j+1}^{n-1} r_i xP(j,i-j-1) + aQ(j-1,n-j) + \end{aligned}$$

$$\begin{aligned} & xP(j,n-j-1) - r_j a^n \\ = & \sum_{i=j}^{n-1} r_i \underline{aP(j-1,i-j)} + \sum_{i=j+1}^{n} r_i \underline{xP(j,i-j-1)} + \underline{aQ(j-1,n-j)} - r_j \underline{a^n}. \end{aligned}$$

Postmultiplying this by a yields

$$-a^j x^{n-j} a \to \sum_{i=j}^{n-1} r_i \underline{aP(j-1,i-j)a} + \underline{aQ(j-1,n-j)a} + \sum_{i=j+1}^{n} r_i \underline{xP(j,i-j-1)a} - r_j \underline{a^{n+1}},$$

where irreducibility is assured by Lemmas 4.4.5 and 4.4.6. Substitute this reduction in the reduction (γ) for $a(a^j x^{n-j})$, to obtain

$$(\alpha)$$

$$\begin{aligned} a(a^j x^{n-j}) &\to -\sum_{i=j}^{n-1} r_i \underline{aP(j-1,i-j)a} - \underline{aQ(j-1,n-j)a} - \sum_{i=j+1}^{n-1} r_i \underline{aP(j,i-j-1)x} \\ &\qquad -\underline{aQ(j,n-j-1)x} + r_j \underline{a^{n+1}} + \sum_{i=j}^{n-1} r_i \underline{aP(j-1,i-j)a} \\ &\qquad +\underline{aQ(j-1,n-j)a} + \sum_{i=j+1}^{n} r_i \underline{xP(j,i-j-1)a} - r_j \underline{a^{n+1}} \\ &= -\sum_{i=j+1}^{n-1} r_i aP(j,i-j-1)x + \sum_{i=j+1}^{n} r_i \underline{xP(j,i-j-1)a} - \underline{aQ(j,n-j-1)x}. \end{aligned}$$

Consider now the second side of the ambiguity, namely $\omega_{j+1}x$. The relation for ω_{j+1} is

$$\begin{aligned}
a^{j+1}x^{n-j-1} &\rightarrow -\left(\sum_{i=j+1}^{n-1} r_i P(j+1, i-j-1) + Q(j+1, n-j-1)\right) + r_{j+1}\underline{a^n} \\
&= -\sum_{i=j+1}^{n-1} r_i aP(j, i-j-1) - aQ(j, n-j-1) - xP(j+1, n-j-2) \\
&\qquad - \sum_{i=j+2}^{n-1} r_i xP(j+1, i-j-2) + r_{j+1}a^n \\
&= -\left(\sum_{i=j+1}^{n-1} r_i \underline{aP(j, i-j-1)} + \underline{aQ(j, n-j-1)} + \sum_{i=j+2}^{n} r_i \underline{xP(j+1, i-j-2)}\right) \\
&\quad + r_{j+1}\underline{a^n}.
\end{aligned}$$

Postmultiplying this reduction by x and using Lemmas 4.4.5 and 4.4.6 to separate reducible and irreducible words yields

$$\begin{aligned}
(\beta) \qquad (a^{j+1}x^{n-j-1})x &\rightarrow -(\sum_{i=j+1}^{n-1} r_i aP(j, i-j-1)x + \underline{aQ(j, n-j-1)x} + \\
&\qquad \sum_{i=j+2}^{n} r_i xP(j+1, i-j-2)x) + r_{j+1}\underline{a^n x}.
\end{aligned}$$

Now $xa^{j+1}x^{n-j-1} \in xP(j+1, n-j-2)x$ and $a^{j+1}x^{n-j-1} \in aP(j, n-j-2)x$ are the only reducible words remaining the above reduction. To reduce $xa^{j+1}x^{n-j-1}$, we first write ω_{j+1} as

$$\begin{aligned}
-a^{j+1}x^{n-j-1} &\rightarrow \sum_{i=j+1}^{n-1} r_i \underline{P(j, i-j-1)a} + \sum_{i=j+2}^{n-1} r_i \underline{P(j+1, i-j-2)x} + \underline{Q(j+1, n-j-2)x} \\
&\qquad + \underline{P(j, n-j-1)a} - r_{j+1}\underline{a^n} \\
&= \sum_{i=j+1}^{n} r_i \underline{P(j, i-j-1)a} + \sum_{i=j+2}^{n-1} r_i \underline{P(j+1, i-j-2)x} + \underline{Q(j+1, n-j-2)x} \\
&\qquad - r_{j+1}\underline{a^n}.
\end{aligned}$$

Premultiplying this reduction by x and using the centrality of a^n, Proposition 4.3.1(iii), we get

$$\begin{aligned}
-xa^{j+1}x^{n-j-1} &\rightarrow (\sum_{i=j+1}^{n} r_i \underline{xP(j, i-j-1)a} + \sum_{i=j+2}^{n-1} r_i \underline{xP(j+1, i-j-2)x} + \\
&\qquad \underline{xQ(j+1, n-j-2)x}) - r_{j+1}\underline{a^n x}.
\end{aligned}$$

Substituting this reduction in (β) yields

$$(\tau) \qquad (a^{j+1}x^{n-j-1})x \to \sum_{i=j+1}^{n} r_i \underline{xP(j,i-j-1)a} - \sum_{i=j+1}^{n-1} r_i aP(j,i-j-1)x - \underline{aQ(j,n-j-1)x}. \tag{4.14}$$

Comparing the processes (α) and (τ), we conclude that the overlap ambiguity $\{\omega_j, \omega_{j+1}\}$ is resolvable for all $j = 1, \cdots, n-1$. This proves (i). □

Proof of Theorem 4.4.4. This is immediate from Theorem 2.10.1 and Proposition A.0.2. □

It is now a simple matter to deduce the PBW theorem for the corresponding algebras $A(x,a,g)$, as follows.

Corollary 4.4.8. *Let $n \in \mathbb{Z}$ with $2 \le n \le 5$, and let $A := A(x,a,g)$ be defined as in §4.2.1. Then A has k-basis*

$$\{x^\ell \langle \mathcal{L}_n \rangle a^j : \ell \in \mathbb{Z}_{\ge 0}, j \in \mathbb{Z}\}.$$

Proof. Consider the algebra $A_0(x,a,g)$, defined in §4.4. By Theorem 4.4.4, $A_0(x,a,g)$ has vector space basis $\langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle$, from which it follows that $A_0(x,a,g)$ is a free right $k[a]$-module on the basis $\langle x \rangle \langle \mathcal{L}_n \rangle$. Since a^n is central in $A_0(x,a,g)$ by the proof of Proposition 4.3.1(iii), $A_0(x,a,g)$ is thus a free left and right $k[a^n]$-module. It follows in particular that a^n is not a zero divisor in $A_0(x,a,g)$, so that we can form the partial quotient algebra Q of $A_0(x,a,g)$ got by inverting the central regular elements $\{a^{nt} : t \ge 0\}$. It is clear that (a) Q has vector space basis $\{x^\ell \langle \mathcal{L}_n \rangle a^j : \ell \in \mathbb{Z}_{\ge 0}, j \in \mathbb{Z}\}$, and ($b$) Q has the same generators and relations as $A(x,a,g)$. This proves the corollary. □

4.5 The Hopf algebras $A(x,a,x^n)$

In this section we examine the Hopf algebras $A(x,a,g)$ when $g(x)$ is a power of x. Although the PBW theorem, Corollary 4.4.8, is only proved for $A(x,a,x^n)$ when $n \le 5$, we can use this to obtain structural information for all values of n, starting from the simple observation below. More precise information for $n \le 3$ is then obtained in §4.6.2.

Lemma 4.5.1. *Retain the notation of §4.2.1 and 4.4. Let $n \in \mathbb{Z}$ with $n \ge 2$.*

(i) $A(x,a,x^n)$ is spanned as a vector space by $\langle x \rangle \langle \mathcal{L}_n \rangle \langle a \rangle$.

Proof. (i) This is simply a restatement of Lemma 4.4.3(iii), which does not require the degree of $g(x)$ to be at most 5. □

Recall from §3.4 that a *quantum homogeneous space* B of a Hopf algebra H is a right coideal subalgebra of H such that H is a faithfully flat right and left B-module.

Proposition 4.5.2. *Retain the notation of §4.2.1 and §4.4, so $F = k[x] * k[a^{\pm 1}]$. Let $n \in \mathbb{Z}$ with $n \geq 2$.*

(i) Construct $A(x,a,x^n)$ as the factor $F/I(n)F$. Choose a primitive nth root of unity q in k. Then $A(x,a,x^n)$ has as a quotient Hopf algebra the localised quantum plane

$$k_q\langle x, a^{\pm 1}\rangle := F/\langle xa - qax\rangle.$$

(ii) The commutative subalgebra $k[x, a^{\pm n}]$ of the Hopf algebra $A(x,a,x^n)$ is a quantum homogeneous subspace of $A(x,a,x^n)$.

(iii) The polynomial subalgebra $k[x]$ is a quantum homogeneous subspace of $A(x,a,x^n)$, and the Laurent polynomial algebra $k[a^{\pm 1}]$ is a Hopf subalgebra over which $A(x,a,x^n)$ is faithfully flat.

Proof. (i) We need to show that in F, for $j = 1, \dots, n-1$,

$$P(j, n-j) \subseteq \langle xa - qax\rangle. \tag{4.15}$$

In $k_q\langle x, a^{\pm 1}\rangle = F/\langle xa - qax\rangle$, the quantum binomial theorem for q−commuting variables which is attributed to M.P. Schützenberger [73] in [71, §1] is given by

$$(x+a)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i a^{n-i},$$

where
$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1)\cdots(q - 1)}.$$

Thus, since q is a primitive nth root of unity,

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = 0$$

for $1 \leq i \leq n-1$ so that $(x+a)^n = x^n + a^n$. But also, again in $k_q\langle x, a^{\pm 1}\rangle$,

$$(x+a)^n = x^n + a^n + \sum_{j=1}^{n-1} P(j, n-j).$$

Thus $P(j, n-j) = 0$ in $k_q\langle x, a^{\pm 1}\rangle$ for $1 \leq j \leq n-1$, since no cancellation can occur between these terms when straightening out the monomials on the right side of the above identity, as the number of a's and x's in a monomial does not change when applying the q-commutation identity. Thus (4.15) is proved.

(ii) Let q be a primitive nth root of unity in k. From (i), the subalgebra $C := k\langle x, a^{\pm n}\rangle$ of $A(x, a, x^n)$ maps, under the factor by $\langle xa - qax\rangle$, onto the subalgebra D of the localised quantum plane generated by x and $a^{\pm n}$. But the latter algebra is precisely $k[x, a^{\pm n}]$. However C is commutative because a^n is central in $A(x, a, x^n)$ by Proposition 4.3.1(iii), so the map from C onto D must be an isomorphism, since every proper factor of the domain $k[x, a^{\pm n}]$ has GK-dimension strictly less than 2. Since x is $(1, a)$–primitive and a^n is grouplike, C is a right coideal subalgebra of $A(x, a, x^n)$. Since $S(C_0) = S(k[a^{\pm n}]) \subseteq C$, it follows from [51, Theorem 1.3] that $A(x, a, x^n)$ is a faithfully flat right and left C-module.

(iii) The arguments are similar to those for (ii) and are left to the reader. □

4.6 The Hopf algebras $A(x, a, g)$, for $g(x)$ of degree at most 3

In this section, we discuss properties of the Hopf algebra $A(x, a, g)$ first when $\deg(g) = 2$ and then $\deg(g) = 3$. We discover that $A(x, a, g)$ is related with the quantum plane when $\deg(g) = 2$ and a PBW-deformation of the down-up algebra when $\deg(g) = 3$. This relationship between $A(x, a, g)$ and the two well studied algebras enable us to derive properties of $A(x, a, g)$ when $\deg(g) = 2, 3$.

We recall definitions of the properties we want to discuss before we consider the properties of $A(x, a, g)$ when $\deg(g) = 2$ and $\deg(g) = 3$.

4.6.1 Homological algebra

In this subsection, we discuss some nice properties which are noncommutative analogues of well studied commutative properties. These include Auslander Gorenstein, Auslander regular, Artin-Schelter regular, and GK-Cohen Macaulay algebras.

Auslander Gorenstein and Auslander-regular rings

A ring R is *quasi-Frobenius* if it is left and right artinian, and left and right self-injective and we call R a *Gorenstein ring* if and only if R has finite right and left injective dimension. An Auslander Gorenstein ring can be viewed as a noncommutative analogue of a commutative Gorenstein ring and as a generalisation of a quasi-Frobenius ring. Several recent results in noncommutative ring theory suggest that the Auslander Gorenstein property is a fundamental homological property that relates to other properties such as being a domain, localisable, etc. [1, §0]. We state and give some examples of Auslander Gorenstein algebras in this subsection. For more on the background of this, see [68, Chapter 8].

We start by defining the grade of a module.

Definition 4.6.1. Let R be a noetherian ring. The *grade (or j-number)* of a finitely generated R-module M is defined as

$$j(M) := \inf\{j \geq 0 \mid \operatorname{Ext}^j_R(M,R) \neq 0\}$$

and

$$j(M) := \infty \quad \text{if} \quad \operatorname{Ext}^j_R(M,R) = 0 \ \forall\, j \geq 0.$$

Example 4.6.2. If a ring R has finite global or injective dimension, then every nonzero finitely generated R-module has finite grade [46, Remark 2.2(1)].

Definition 4.6.3. [1, Definition 0.1] A ring A

- satisfies the *Auslander condition* if for every noetherian A-module M and for all $i \geq 0$, we have $j(N) \geq i$ for all submodules $N \subset \operatorname{Ext}^i(M,A)$;
- is *Auslander Gorenstein* if A is two-sided noetherian, satisfies the Auslander condition and has finite left and right injective dimension;
- is *Auslander regular* if it is Auslander Gorenstein, and has finite global dimension.

Example 4.6.4. [46, §6.7] Let S be the Sklyanin algebra as in [75, Theorem 5.4]. The algebra S is Auslander regular of dimension 4.

Artin-Schelter regular algebras

Artin-Schelter regular algebras were originally defined in [3] for k-algebras which are connected graded, i.e., algebras of the form $A = k \bigoplus A_1 \bigoplus A_2 \cdots$ where A_i, $i \in \mathbb{N}$ are finite dimensional k-vector spaces with $A_i A_j \subseteq A_{i+j}$ $\forall i,j$. We use the more general definition of Artin-Schelter regular regular algebra for augmented algebras which are defined as follows.

Let A be a ring and M be a left A-module. In the following definitions, we use the notation $\operatorname{inj.dim}({}_AM)$ to denote the injective dimension of M as a left A-module and $\operatorname{gl.dim}(A)$ to denote the global dimension of the ring A.

Definition 4.6.5. [44, Chapter IV, §1] Let A be a k-algebra which is *augmented* i.e. there is a distinguished algebra homomorphism $\pi : A \to k$. Then A becomes a left A-module using the multiplication in A and k becomes a left A-module using the homomorphism $\pi : A \to k$. Let ${}_AA$ and ${}_Ak$ denote these modules respectively. Then A is called *Artin-Schelter Gorenstein of dimension n* if

1. $\operatorname{inj.dim}({}_AA) = n < \infty$
2. $\operatorname{Ext}^i_A({}_Ak, {}_AA) \cong \delta_{in} k$ as vector spaces

and the analogous conditions hold for right modules instead of left modules.

In the case of a connected graded algebra $A = \bigoplus_{i\geq 0} A_i$, the distinguished algebra homomorphism is the projection

$$\begin{aligned} \pi : A &\to A_0 \\ a_0 + a_1 + \cdots + a_n &\mapsto a_0 \end{aligned}$$

where $A_0 = k$.

Artin-Schelter regular algebras are a class of noncommutative algebras introduced by Artin and Schelter in [3]. These algebras are noncommutative analogues of a commutative polynomial ring which has finite global and Gelfand-Kirillov dimensions.

Definition 4.6.6. Let A be an augmented algebra over a field k. The algebra A is called *Artin-Schelter regular of dimension* d if it satisfies the following properties:

(i) Artin-Schelter Gorenstein

(ii) A has finite global dimension d: every A-module has projective dimension $\leq d$.

We write AS-regular in place of Artin-Schelter regular throughout the thesis. The original definition of AS-regular algebras in [3] is for connected graded algebras A, that is $A = k \oplus A_1 \oplus A_2 \cdots$. This original definition is a special case of Definition 4.6.6. Here are some examples of AS-regular algebras.

Example 4.6.7. [3, §0] Connected graded AS-regular algebras of dimension 2 are of the form $A = k\langle x, y\rangle/(f)$ where f is one of the following polynomials:

$$yx - cxy, \quad c \neq 0, \quad yx - xy - x^2$$

where $c \in k^{\times}$.

A complete list of connected graded AS-regular algebras of dimension three, generated in degree one has been assembled by Artin and Schelter in [3] and Artin, Tate and Van den Bergh in [4].

Example 4.6.8. [3, §0] The algebra $A := k\langle x, y\rangle/(f_1, f_2)$ where

$$f_1 = y^2x - 2yxy + xy^2, \qquad f_2 = x^2y - 2xyx + yx^2$$

is AS-regular of global dimension three.

GK-Cohen Macaulay rings

Definition 4.6.9. A k-algebra A is called *GK-Cohen Macaulay* if for all finitely generated nonzero A-modules M,

$$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A).$$

Example 4.6.10. [46, §6.7] Let A be an AS-regular algebra of dimension 3 and type A as in [5]. Then A is Auslander regular and GK-Cohen Macaulay of dimension 3.

4.6.2 $g(x)$ of degree 2: the quantum Borel.

In this subsection, we discuss the properties of $A(x,a,g)$ when $\deg(g) = 2$. It turns out that in this case, independently of the precise form of $g(x)$, $A(x,a,g)$ is isomorphic to the Borel in $U_{-1}(\mathfrak{sl}(2))$, or equivalently to a localised quantum plane $k\langle a^{\pm 1}, x'\rangle$ at the parameter -1 as is stated in the proposition below.

Proposition 4.6.11. *Keep the notation of §4.2.1, and let $n = 2$, so that $g(x) = x^2 + r_1 x$, for $r_1 \in k$.*

(i) The Hopf algebra structure of $A(x,a,g)$ is independent of g, that is of the parameter r_1. Namely, given $r_1 \in k$, set $x' := x + \frac{r_1}{2}(1-a)$. Then $A(x,a,g) = k\langle a^{\pm 1}, x'\rangle$ is isomorphic as a Hopf algebra to the Borel in $U_{-1}(\mathfrak{sl}(2))$, or equivalently to a localised quantum plane $k\langle a^{\pm 1}, x'\rangle$ at the parameter -1. Moreover, $A_0(x,a,g)$ is isomorphic as an algebra to the quantum plane with parameter -1.

(ii) $A(x,a,g)$ is a noetherian AS-regular domain of Gelfand-Kirillov and global dimension 2.

(iii) $A(x,a,g)$ is a finite module over its central Hopf subalgebra $k[a^{\pm 2}][g]$, where g is the square of the uninverted $(1,a)$-primitive generator of the quantum Borel.

Proof. (i) Recall the relation $\sigma_1(x,a,g)$ from (4.1),

$$r_1 P(1,0) + P(1,1) - r_1 a^2 = 0.$$

Rewrite this as

$$a\left(x + \frac{r_1}{2}(1-a)\right) + \left(x + \frac{r_1}{2}(1-a)\right)a = 0, \tag{4.16}$$

and define $x' := x + \frac{r_1}{2}(1-a) \in A(x,a,g)$. Thus $A(x,a,g)$ is the quantum plane $k_{-1}\langle x', a^{\pm 1}\rangle$ with a grouplike and x' which is $(1,a)$-primitive, as required. The corresponding statement regarding $A_0(x,a,g)$ is also immediate from (4.16).

(ii) Being an iterated skew polynomial algebra by (i), $A(x,a,g)$ has finite global dimension [58, Theorems 7.5.3 and 7.5.5]. The algebra $A(x,a,g)$ is a finite module over $k[a^{\pm 2}, x'^2]$ so by [58, Corollary 13.1.13], $A(x,a,g)$ is a polynomial identity ring. Also, by [58, Theorem 1.2.9(iv)], $A(x,a,g)$ is noetherian. Since it is an affine noetherian Hopf algebra satisfying a polynomial identity, it is AS-Gorenstein and GK-Cohen Macaulay by [14, §6.2]. Thus, $A(x,a,g)$ is AS-regular. Since it has GK-dimension 2 by virtue of being a finite module over $k[a^{\pm 2}, x'^2]$, its global dimension is also 2 since it is GK-Cohen Macaulay.

(iii) It is easy to check that $k\langle a^{\pm 2}, x'^2\rangle = k\langle a^{\pm 2}, g\rangle$, and that $A(x,a,g)$ is a finite module over this subalgebra. Its structure as a Hopf subalgebra is well-known and easy to check; or one can use Proposition 4.3.1(i). □

4.6.3 $g(x)$ of degree 3: localised down-up algebras and their deformations.

In this subsection, we discuss properties of the algebra $A(x,a,g)$ when $\deg(g) = 3$. We derive these properties of $A(x,a,g)$ by relating it with the well studied down-up algebra which has nice properties.

Let $g(x) = r_1 x + r_2 x^2 + x^3$, with $r_1, r_2 \in k$. Recall from (4.1) in §4.2.1 that the defining relations in the free algebra $k\langle x, a\rangle$ of the subalgebra $A_0(x,a,g)$ of $A(x,a,g)$ are

$$\begin{aligned} \sigma_1 : \qquad ax^2 &= -xax - x^2 a - r_2(ax + xa) - r_1 a + r_1 a^3 , \\ \sigma_2 : \qquad a^2 x &= -axa - xa^2 - r_2 a^2 + r_2 a^3 . \end{aligned}$$

Compare the above relations with the following relations of a down-up algebra:

Definition 4.6.12 (Benkart, Roby [7]). Let $\alpha, \beta, \gamma \in \mathbb{C}$. The *down-up algebra* $A = A(\alpha, \beta, \gamma)$ is the $\mathbb{C}$-algebra with generators d, u and relations

$$\begin{aligned} du^2 &= \alpha udu + \beta u^2 d + \gamma u , \\ d^2 u &= \alpha dud + \beta ud^2 + \gamma d. \end{aligned}$$

The relation between the two presentations is encompassed in the following concept, introduced in [8, Section 3]. Here, we slightly weaken the usual requirement that the generators of the free algebra are assigned degree 1, in order to allow for the terms in a^3 in the relations for $A_0(x,a,g)$.

The following set up for the definition of PBW-deformations is adapted from [26, §1].

Definition 4.6.13. Let V be a finite dimensional vector space over k. Let R be a subspace of $V^{\otimes N}$ where N is some integer ≥ 2, and $\langle R\rangle$ the two-sided ideal generated by R in the tensor algebra $A = T(V)/\langle R\rangle$.

Instead of a homogeneous space of relation R, we may consider a non-homogeneous space of relations P in $\bigoplus_{i \leq N} V^{\otimes i}$, and get a non-homogeneous algebra $U = T(V)/\langle P\rangle$.

We are interested in the case when this algebra is a deformation of A of a particular kind, a PBW-deformation. We then assume P intersects $\bigoplus_{i\le N-1} V^{\otimes i}$ trivially and that R is the image of P by the natural projection of $\bigoplus_{i\le N-1} V^{\otimes i}$ to $V^{\otimes N}$. Then there are linear maps $R \xrightarrow{\alpha_i} V^{\otimes N-i}$ such that we may write

$$P = \{r + \alpha_1(r) + \dots + \alpha_N(r) \mid r \in R\}.$$

There is a natural filtration on the tensor algebra by letting $F^\ell T(V)$ be $\bigoplus_{i\le \ell} V^{\otimes i}$. This induces a natural filtration $F^\ell U$ on the quotient algebra U, and in the situation described there is a surjection

$$A \twoheadrightarrow \mathrm{gr} U$$

We say that U is a *PBW-deformation* of A if this map is an isomorphism.

Proposition 4.6.14. *Retain the notation of §4.2.1 and of (4.11) in §4.4 and let* $k = \mathbb{C}$. *Let* $g(x) = r_1 x + r_2 x^2 + x^3 \in k[x]$.

(i) $A_0(x, a, x^3)$ *is isomorphic to the down-up algebra* $A(-1, -1, 0)$.

(ii) $A_0(x, a, x^3)$ *is a noetherian domain.*

(iii) $A_0(x, a, x^3)$ *is Auslander regular of global dimension 3.*

(iv) $A_0(x, a, x^3)$ *has PBW basis* $\langle x\rangle\langle \mathcal{L}_3\rangle\langle a\rangle$; *that is, it has basis*

$$\{x^\ell (ax)^i a^j : \ell, i, j \in \mathbb{Z}_{\ge 0}\}.$$

(v) $\mathrm{GKdim}(A_0(x, a, x^3)) = 3$, *and* $A_0(x, a, x^3)$ *is GK-Cohen Macaulay.*

(vi) $A_0(x, a, g)$ *is a PBW deformation of* $A_0(x, a, x^3)$. *That is,* $A_0(x, a, g)$ *is a PBW deformation of the down-up algebra* $A(-1, -1, 0)$.

(vii) Statements (ii)-(v) apply verbatim to $A_0(x, a, g)$ *and to* $A(x, a, g)$. *Moreover, the algebra* $A(x, a, g)$ *is a PBW deformation of* $A(x, a, x^3)$ *with PBW basis* $\{x^\ell (ax)^i a^j : \ell, i \in \mathbb{Z}_{\ge 0}, j \in \mathbb{Z}\}$..

(viii) $A(x, a, g)$ *is AS-regular of dimension 3.*

Proof. (i) is clear from the definitions.

(ii) This is immediate from (i) and [35, Main Theorem], where it is proved that $A(\alpha, \beta, \gamma)$ is a noetherian domain provided $\beta \ne 0$.

(iii) Immediate from (i) and [35, Theorem 4.1 and Lemma 4.2(i)].

(iv) This is a special case of Theorem 4.4.4. In the light of (i), it is also a special case of [7, Theorem 3.1].

(v) For any down-up algebra, its GK-dimension is 3 by [7, Corollary 3.2]. Let $\beta \neq 0$, then the down-up algebra $A = A(\alpha, \beta, \gamma)$ is Cohen Macaulay by [35, Lemma 4.2(ii)]. Applying these two results to the down-up algebra $A(-1, -1, 0)$, we deduce the result that $A_0(x, a, x^3)$ is GK-Cohen Macaulay.

(vi) For all g of degree 3, $U := A_0(x, a, g)$ has the same PBW basis as described in (iv), by Theorem 4.4.4. Let $V = ka \oplus kx$ and filter $T(V)$ by setting x to have degree 2 and a to have degree 1. Then the surjection referred to before the proposition, in this case from $A := A_0(x, a, x^3)$ to $\mathrm{gr}U$, is an isomorphism, as claimed.

(vii) We prove these statements for $A_0(x, a, g)$. Then we extend the conclusions to the localisation $A(x, a, g)$ of $A_0(x, a, g)$ at the central regular Ore set $\{a^{3\ell} : \ell \in \mathbb{Z}_{\geq 0}\}$. By [58, Proposition 2.13], Gelfand-Kirillov dimension is preserved after localising $A_0(x, a, g)$ at the central regular Ore set $\{a^{3\ell} : \ell \in \mathbb{Z}_{\geq 0}\}$ to get $A(x, a, g)$.

That $A_0(x, a, g)$ is a noetherian domain follows from (ii), (vi) and [58, Proposition 1.6.6 and Theorem 1.6.9]. Moreover, the fact (vi) that $A_0(x, a, g)$ has the same PBW basis as $A_0(x, a, x^3)$ ensures that $\mathrm{GKdim} A_0(x, a, g) = 3$, from (v); indeed, this also follows from the following paragraph.

Denoting the filtration of $A_0(x, a, g)$ defined in the proof of (vi) by $\mathcal{F}$, we know from (vi) that this is a finite dimensional filtration whose associated graded algebra is affine and noetherian by (vi) and (ii). A filtration of an $A_0(x, a, g)$-module M is called a *good filtration* if $\mathrm{gr}M$ is a finitely generated $\mathrm{gr}A_0(x, a, g)$-module. Thus, for any finitely generated $A_0(x, a, g)$-module M given a good $\mathcal{F}$-filtration,

$$\mathrm{GKdim}_{A_0(x,a,g)}(M) = \mathrm{GKdim}_{\mathrm{gr}_{\mathcal{F}} A_0(x,a,g)}(\mathrm{gr}_{\mathcal{F}}(M))$$

by [58, Proposition 8.6.5].

For a module M over the ring R, recall from Definition 4.6.1 that the homological grade of M, $j_R(M) = \inf\{i : \mathrm{Ext}^i_R(M, R) \neq 0\}$. For every finitely generated $A_0(x, a, g)$-module M,

$$j_{A_0(x,a,g)}(M) = j_{\mathrm{gr}_{\mathcal{F}} A_0(x,a,g)}(\mathrm{gr}_{\mathcal{F}}(M))$$

by [31, Chapter III, §2.5, Theorem 2].

Let M be a finitely generated $A_0(x,a,g)$-module. Then, by (v) and the above equalities,

$$\begin{aligned} j_{A_0(x,a,g)}(M) + \mathrm{GKdim}_{A_0(x,a,g)}(M) &= j_{\mathrm{gr}_{\mathcal{F}}A_0(x,a,g)}(\mathrm{gr}_{\mathcal{F}}(M)) + \mathrm{GKdim}_{\mathrm{gr}_{\mathcal{F}}A_0(x,a,g)}(\mathrm{gr}_{\mathcal{F}}(M)) \\ &= j_{A_0(x,a,x^3)}(\mathrm{gr}_{\mathcal{F}}(M)) + \mathrm{GKdim}_{A_0(x,a,x^3)}(\mathrm{gr}_{\mathcal{F}}(M)) \\ &= \mathrm{GKdim}(A_0(x,a,x^3)) \\ &= \mathrm{GKdim}(A_0(x,a,g)) = 3, \end{aligned}$$

so that $A_0(x,a,g)$ is GK-Cohen Macaulay.

By (vi), (iii) and [58, Corollary 7.6.18], writing gl.dim for the global dimension of an algebra,

$$\mathrm{gl.dim} A_0(x,a,g) \le \mathrm{gl.dim} A_0(x,a,x^3) = 3. \tag{4.17}$$

Finally, to see that $\mathrm{gl.dim} A_0(x,a,g) \ge 3$, apply the Cohen Macaulay property with M equal to the trivial module k.

(viii) Note that $A(x,a,g)$ is a noetherian Hopf algebra by Theorem 4.2.3. Thus (viii) follows from (iii), (v), (vii) and [15, Lemma 6.1], which states that an Auslander regular noetherian Hopf algebra is AS-regular. □

In this final part of the chapter, our aim is to prove Proposition 4.6.23. The fact that the down-up algebra $A(-1,-1,0)$ is finite over its centre was proved by Kulkarni in [42]. Extending this result to $A_0(x,a,g)$ and $A(x,a,g)$ when g is an arbitrary degree 3 polynomial requires some work.

We start by defining terms needed to define a maximal order, one of the properties of $A(x,a,g)$ stated in Proposition 4.6.23. These definitions are from [58].

Definition 4.6.15. [58, §§2.1.2] An element x of a ring R is *right regular* if $xr = 0$ implies $r = 0$ for $r \in R$. A *left regular* element of R is defined analogously and $x \in R$ is *regular* if it is both left and right regular.

Definition 4.6.16. [58, §§3.1.1] A ring Q is called a *quotient ring* if every regular element of Q is a unit.

An example of a quotient ring is a right Artinian ring Q [58, Proposition 3.1.1].

Definition 4.6.17. Given a quotient ring Q, a subring R (not necessarily containing 1) is called a *right order* in Q if each $q \in Q$ is of the form rs^{-1} for some $r, s \in R$. A *left order* is defined similarly, and a left and right order is called an *order*.

We now consider a lemma which describes relations between right orders in the same quotient ring.

Lemma 4.6.18. [58, Lemma 3.1.6] *Let R be a right order in a quotient ring Q and let S be a subring of Q (not necessarily with 1). Suppose further that there are units a, b of Q such that $aRb \subseteq S$. Then S is also a right order in Q.*

The relationship described in Lemma 4.6.18 leads to an equivalence relation on right orders R_1, R_2 in a fixed quotient ring Q with relation defined by $R_1 \sim R_2$ if there are units $a_1, a_2, b_1, b_2 \in Q$ such that $a_1R_1b_1 \subseteq R_2$ and $a_2R_2b_2 \subseteq R_1$. Then R_1 and R_2 are called *equivalent* right orders.

Definition 4.6.19. [58, §5.1.1] Let Q be a quotient ring and R a right order in Q. Then R is called a *maximal* right order if it is maximal within the equivalence class described after Lemma 4.6.18.

We now turn our attention to the down-up algebra $A(-1, -1, 0)$, which is a finite module over its centre and affine by [42, Corollary 2.0.2 and Lemma 2.0.1], [83, Theorem 1.3(f)]. The next result generalises these facts to the algebras $A_0(x, a, g)$ and $A(x, a, g)$, for an arbitrary degree 3 polynomial $g(x)$. The proof proceeds via a result of independent interest, also obtained by Kulkarni [43] in the down-up case: namely we show that the algebras $A_0(x, a, g)$ and $A(x, a, g)$ are maximal orders when $g(x)$ has degree at most 3[1]. The definition of the term *maximal order* as applied to a prime noetherian ring R is given for example in Definition 4.6.19. In [58, Theorem 5.3.13] it is shown that, when such a ring R is a finite module over its centre Z, this definition of maximal order coincides with the concept of a *maximal classical Z-order*. The proof that these algebras are finite over their centres is derived from well-known results, but we state them explicitly. The key point, that the maximal order property lifts to a filtered deformation, is due to Chamarie, [18].

Next, we define terms needed in the definition of a normal domain, a property discussed in subsequent theorems.

Definition 4.6.20. [58, §5.3.12 and 5.3.3] Let A and B be rings with A contained in the centre of B and let $b \in B$. If there is a monic polynomial $f(x) \in A[x]$ with $f(b) = 0$, then b is said to be *integral* over A, and B is *integral* over A if this is true for all $b \in B$. Similarly, if $A[b]$ is contained in some finitely generated A-module, then b is said to be *c-integral* over A, and if this is true for all $b \in B$, then B is *c-integral* over A. In the particular case when B is the field of fractions of an integral domain, A is *integrally closed* if the elements of B which are integral over A all belong to A. A *normal domain* is an integrally closed noetherian domain.

Theorem 4.6.21. *Let k be a field and let R be an affine k-algebra satisfying a polynomial identity. Suppose that R has a $\mathbb{Z}_{\geq 0}$-filtration $\mathcal{F}$ such that $\mathrm{gr}_{\mathcal{F}}(R)$ is a domain and a*

[1]When $g(x)$ has degree 2, so that $A_0(x, a, g)$ and $A(x, a, g)$ are (localised) quantum planes by Proposition 4.6.11(i), this is immediate from [55, Corollaire V.2.6]

noetherian maximal order. Then R is a domain and a noetherian maximal order, and is a finite module over its centre $Z(R)$, which is a normal affine domain.

Proof. That the stated properties of $\mathrm{gr}_{\mathcal{F}}(R)$ all lift to R is guaranteed, respectively, by [58, Proposition 1.6.6, Theorem 1.6.9 and Theorem 5.1.6]. But now, since R is a maximal order satisfying a polynomial identity, it is its own trace ring by [58, Proposition 13.9.8]. Thus [58, Proposition 13.9.11(ii)] implies that R is a finite module over its centre $Z(R)$, with $Z(R)$ an affine k-algebra because R is affine, thanks to the Artin-Tate lemma [58, Lemma 13.9.10(ii)]. Finally, the fact that R is a maximal order implies that $Z(R)$ is normal, [58, Proposition 5.1.10(b)]. □

The pay-off in part (iv) below now follows easily by combining the above with Kulkarni's work on the down-up algebra but before we discuss this, we now consider hyperbolic rings. Hyperbolic rings appeared as a result of an attempt to single out the biggest natural commutative subring in $U_q(\mathfrak{sl}(2))$, $U(\mathfrak{sl}(2))$ and some other algebras [67, Chapter IV §1.2].

Definition 4.6.22. [67, Chapter II] Let R be a commutative ring and let θ be a ring automorphism of R. Let $\xi \in R$. A *hyperbolic ring* $R\{\theta, \xi\}$ is an associative ring generated by indeterminates x, y satisfying the relations

$$xr = \theta(r)x, \qquad ry = y\theta(r) \qquad \forall\, r \in R,$$
$$xy = \xi, \qquad yx = \theta^{-1}(\xi).$$

The subring generated by R and x, $R[x, \theta]$ is a twisted polynomial ring with its commutation relation $xr = \theta(r)x$. By [67, Lemma 3.1.6,Chapter II], $R\{\theta, \xi\}$ is a free R-bimodule which gives a basis for the left and right module structures. If R is a domain, then $R\{\theta, \xi\}$ is a domain by [42, Corollary 2.0.1].

Proposition 4.6.23. *Keep the hypotheses and notation of Proposition 4.6.14, so we have in particular $g(x) = r_1x + r_2x^2 + x^3$.*

(i) Let λ be a primitive cube root of unity in k. The centre of $A_0(x, a, x^3)$ is the subalgebra

$$\begin{aligned} Z(A_0(x,a,x^3)) \;=\; & k\langle a^3, x^3, (xa)^3 - 3\lambda^2(xa)^2(ax) + 3\lambda(xa)(ax)^2 - (ax)^3, \\ & (xa)^3 - 3\lambda(xa)^2(ax) + 3\lambda^2(xa)(ax)^2 - (ax)^3, (ax)^2 - x^2a^2\rangle; \end{aligned}$$

and the centre of $A(x, a, x^3)$ is $Z(A_0(x, a, x^3))[a^{-3}]$.

(ii) The centres of $A_0(x,a,g)$ *and of* $A(x,a,g)$ *contain the subalgebra*

$$k\langle g, a^3, (ax)^2 - x^2a^2 - r_2xa^2 - r_1a^2\rangle.$$

(iii) Both $A_0(x,a,g)$ *and* $A(x,a,g)$ *are domains and maximal orders.*

(iv) Both $A_0(x,a,g)$ *and* $A(x,a,g)$ *are finite modules over their centres, which are affine normal domains.*

Proof. (i) By Proposition 4.6.14(i), $A_0(x,a,x^3)$ is isomorphic to the down-up algebra $A(-1,-1,0)$. By [42, Proposition 4.0.3], if $\alpha^2 + 4\beta \neq 0$, the centre of the down-up algebra $A(\alpha,\beta,\gamma)$ is the subalgebra generated by $\{u^3, d^3\}$ and $\{w_1^i w_2^j \mid \lambda_1^i \lambda_2^j = 1\}$ where λ_i are roots of the quadratic equation $t^2 + \frac{\alpha}{\beta}t - \frac{1}{\beta} = 0$ and

$$w_i = \lambda_i ud + \frac{1}{\beta}du - \frac{\lambda_i\gamma}{(\lambda_i - 1)\beta}$$

for $i = 1,2$. When we consider the down-up algebra $A(-1,-1,0)$, we deduce $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^2$ where λ is a primitive third root of unity. The possibilities of i and j satisfying $\lambda_1^i\lambda_2^j = 1$ are

$$i = j = 1, \qquad i = 3, j = 0 \qquad \text{and} \qquad i = 0, j = 3.$$

The set $\{w_1^i w_2^j \mid \lambda_1^i \lambda_2^j = 1\}$ becomes

$$\{(du)^2 - u^2d^2, (ud)^3 - 3\lambda(ud)^2(du) + 3\lambda^2(ud)(du)^2 - (du)^3,$$

$$(ud)^3 - 3\lambda^2(ud)^2(du) + 3\lambda(ud)(du)^2 - (du)^3\}.$$

Under the isomorphism between $A_0(x,a,x^3)$ and the down-up algebra $A(-1,-1,0)$, $u \mapsto x$ and $d \mapsto u$. Thus, the set $\{w_1^i w_2^j \mid \lambda_1^i \lambda_2^j = 1\}$ becomes

$$\{(ax)^2 - x^2a^2, (xa)^3 - 3\lambda(xa)^2(ax) + 3\lambda^2(xa)(ax)^2 - (ax)^3,$$

$$(xa)^3 - 3\lambda^2(xa)^2(ax) + 3\lambda(xa)(ax)^2 - (ax)^3\}.$$

Hence, the centre of $A_0(x,a,x^3)$ is

$$\begin{aligned} Z(A_0(x,a,x^3)) \quad = \quad & k\langle a^3, x^3, (xa)^3 - 3\lambda^2(xa)^2(ax) + 3\lambda(xa)(ax)^2 - (ax)^3, \\ & (xa)^3 - 3\lambda(xa)^2(ax) + 3\lambda^2(xa)(ax)^2 - (ax)^3, (ax)^2 - x^2a^2\rangle \end{aligned}$$

as claimed.

The second part of (i) follows easily from the first, since $A(x,a,x^3) = A_0(x,a,x^3)\langle a^{-3}\rangle$, with a^3 central.

(ii) By Proposition 4.3.1(iii), $g, a^3 \in Z(A_0(x,a,g))$. It is straightforward to check that $axax - x^2a^2 - r_2xa^2 - r_1a^2$ commutes with a and x. Since $A(x,a,g)$ is the localisation of $A_0(x,a,g)$ at the central regular Ore set $\{a^{3\ell} : \ell \geq 0\}$, the listed elements are central in $A(x,a,g)$.

(iii) Consider the down-up algebra $A(\alpha,\beta,\gamma)$ and let θ be the automorphism of $\mathbb{C}[\xi,\eta]$ given by

$$\theta(\xi) = \frac{\eta - \alpha\xi - \gamma}{\beta}, \qquad \theta(\eta) = \xi$$

where $\xi = ud$ and $\eta = du$. Then $ur = \theta(r)u$, $dr = \theta^{-1}(r)d$ for any $r \in \mathbb{C}[\xi,\eta]$. Thus, the down-up algebra $A(\alpha,\beta,\gamma)$ is isomorphic to the hyperbolic ring $R\{\xi,\theta\}$ [42, Proposition 3.0.1]. The centre of $A(\alpha,\beta,\gamma)$ is generated by $\{u^m, d^m\}$ over $(\mathbb{C}[\xi,\eta])^\theta$ where $w_i \in \mathbb{C}[\xi,\eta]$,

$$\theta(w_1) = \lambda_1 w_1, \qquad \theta(w_2) = \lambda_2 w_2,$$

m is the smallest positive integer such that $\theta^m = \mathrm{id}$ with λ_i and w_i defined in (i) above. We saw in (i) above that for the down-up algebra $A(-1,-1,0)$, λ_1 and λ_2 are primitive third roots of unity so the smallest such m is $m = 3$. By [43, Theorem 2.6], if the automorphism θ is of finite order, then $A(\alpha,\beta,\gamma)$ is a maximal order. Thus, since θ has order 3, we conclude that $A(-1,-1,0)$ and hence $A_0(x,a,x^3)$ are maximal orders. The maximal order property lifts from $A_0(x,a,x^3)$ to its filtered deformation $A_0(x,a,g)$ by [58, Theorem 5.1.6].

From the definition of a maximal order, we deduce that the localisation of a maximal order at a central regular Ore set is again a maximal order, so the desired conclusion for $A(x,a,g)$ also follows.

(iv) Given (iii) and Theorem 4.6.21, the desired results will follow if $A_0(x,a,g)$ (and hence also its localisation $A(x,a,g)$) satisfy a polynomial identity which is explained as follows. By (ii), the subalgebra

$$C := k\langle g, a^3, (ax)\rangle$$

of $A_0(x,a,g)$ is commutative, and from the PBW theorem for $A_0(x,a,g)$, Theorem 4.4.4, $A_0(x,a,g)$ is a finitely generated right or left C-module. Therefore $A_0(x,a,g)$ satisfies a polynomial identity by [58, Corollary 13.1.13(iii)], as required. □

Remark 4.6.24. Consider the k-algebra involution of $A_0(x,a,x^3)$ which interchanges a and x. The generators listed in (i) above for $Z(A_0(x,a,x^3))$ are permuted by this involution and the third listed element of $Z(A_0(x,a,g))$ in (ii) is a lift of the involution invariant generator $(ax)^2 - x^2a^2$ of $Z(A_0(x,a,x^3))$. It would be interesting to determine $Z(A_0(x,a,g))$ for a general polynomial $g(x)$ of degree 3, and in particular to discover whether it contains elements which are lifts of the other listed generators of $Z(A_0(x,a,x^3))$. We leave this as an open question from this thesis.

Chapter 5

The Hopf Algebra $A(g, f)$

5.1 Introduction

In this chapter, we discuss the construction of the Hopf algebra $A(g, f)$ from the tensor product of $A(x, a, g)$ and $A(y, b, f)$ which we derive from a plane curve of the form $\mathcal{C} := f(y) = g(x)$ with $f(y) = \sum_{i=1}^{m} s_i y^i \in k[y]$, $g(x) = \sum_{i=1}^{n} r_i x^i \in k[x]$, where $r_i, s_i \in k$, and $r_n, s_m \in k \setminus \{0\}$. This gives us a family of Hopf algebras $A(g, f)$ and when $\deg(f), \deg(g) \leq 3$, the algebra $A(g, f)$ has nice properties such as being a noetherian polynomial identity algebra, and having finite Gelfand-Kirillov and global dimensions. Using Bergman's Diamond lemma, we prove that $A(g, f)$ has a basis over the coordinate ring $\mathcal{O}(\mathcal{C}) = k[x, y]/\langle f(y) - g(x)\rangle$ of the plane curve $\mathcal{C}$ when $\deg(f), \deg(g) \leq 5$. This is how we prove our main theorem that the coordinate rings of plane curves of the form $f(y) = g(x)$ are quantum homogeneous spaces when $\deg(f), \deg(g) \leq 5$.

We conjecture that $A(g, f)$ is free over $\mathcal{O}(\mathcal{C})$ irrespective of the degree of the plane curve $\mathcal{C}$. That is, we show that the coordinate ring of the plane decomposable curve $\mathcal{C}$, defined by $f(y) = g(x)$ is a quantum homogeneous space in a Hopf algebra. When n or m is 1 this is of course trivial, since then $\mathcal{O}(\mathcal{C}) = k[t]$ is itself a Hopf algebra. Our current proof of the main theorem uses Lemma A.0.2 which says that ambiguities are resolvable when $\deg(g), \deg(f) \leq 5$. Details of this proof is provided in the appendix. Though the proof is very long, we anticipate that there is a shorter proof in the general case. In order to prove the main result, we shall assume in §5.2.1 that

(H) *each polynomial $g(x)$ and $f(y)$ either has degree at most 5 or is a power of x resp. y.*

This ensures that $k[x]$ and $k[y]$ are quantum homogeneous subspaces of $A(x, a, g)$ and $A(y, b, f)$ respectively.

5.2 The construction of $A(g,f)$

We begin, though, without assuming **(H)**. To construct $A(g,f)$, given polynomials $f(y)$ and $g(x)$ as in the first sentence of §5.1 above, first form the Hopf algebras $A(x,a,g)$ and $A(y,b,f)$ as in §4.2.1. Then consider their tensor product

$$T \quad := \quad A(x,a,g) \otimes_k A(y,b,f).$$

Thus T is a Hopf k-algebra, inheriting the relevant structures from its component Hopf subalgebras $A(x,a,g)$ and $A(y,b,f)$ in view of Theorem 4.2.3. Moreover, T is affine, $T = k\langle x, y, a^{\pm 1}, b^{\pm 1}\rangle$, where, here and henceforth, we simplify notation by writing x for $x \otimes 1$, y for $1 \otimes y$, etc. Since a and b are grouplike and x and y are skew primitive, by Lemmas 4.2.1 and 4.2.2, T is generated by grouplike and skew primitive elements, and is therefore pointed, by [65, Corollary 5.1.14(a)]. The elements f, g, a^n, b^m are in the centre of T by Proposition 4.3.1(iii), so that the right ideal

$$I \quad := \quad T(g-f) + T(a^n - b^m)$$

of T is actually two-sided. We can therefore define the k-algebra

$$A(g,f) \quad := \quad T/I. \tag{5.1}$$

In the theorem below and later, we will continue with the abuse of notation used earlier, writing x, a and so on for the images of these elements of T in various factor algebras, in situations where we believe confusion is unlikely. For the reader's convenience the relations for $A(g,f)$ are listed in Theorem 5.2.1(i), even though they are easily read off from (4.1) and (5.1).

Theorem 5.2.1. *Keep the notation introduced in the above paragraphs, but don't assume* **(H)**.

(i) $A(g,f)$ is the factor k-algebra of the free product $k\langle x,y\rangle * k\langle a^{\pm 1}, b^{\pm 1}\rangle$ by the ideal generated by the relations

$$[x,y] = [x,b] = [a,b] = [a,y] = 0; \qquad f(y) = g(x), \quad a^n = b^m,$$
$$\sum_{i=j}^{n} r_i P(j, i-j)_{(a,x)} - r_j a^n, \qquad (j = 1, \dots, n-1),$$
$$\sum_{\ell=p}^{m} r_\ell P(p, \ell-p)_{(b,y)} - r_p b^m, \qquad (p = 1, \dots, m-1).$$

(ii) The k-algebra $A(g,f)$ inherits a Hopf algebra structure from T. Thus its coproduct

Δ, *counit* ε *and antipode* S *satisfy:*

$$\Delta(x) = 1 \otimes x + x \otimes a, \qquad \Delta(y) = 1 \otimes y + y \otimes b,$$
$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 0, \quad \varepsilon(a) = \varepsilon(b) = 1,$$
$$S(x) = -xa^{-1}, \quad S(y) = -yb^{-1}, \quad S(a) = a^{-1} \qquad S(b) = b^{-1}.$$

Proof. Given the above discussion and the results of §4.2, it is enough to show that I is a Hopf ideal of T. This is an easy consequence of the facts that a^n and b^m are grouplike, and g and f are respectively $(1, a^n)$-skew primitive and $(1, b^m)$-skew primitive, by Proposition 4.3.1(i). To see that $S(I) \subseteq I$ one can either calculate directly or appeal to [65, Proposition 7.6.3]. □

5.2.1 Properties of $A(g, f)$ under hypothesis (H)

To describe the PBW basis for $A(g, f)$ it is necessary to decorate the notation for the PBW generators of $A(x, a, g)$ introduced at (4.11) in §4.4. Namely, for $g(x)$ of degree n and $f(y)$ of degree m, define subsets of (respectively) the free semigroups on generators $\{x, a\}$ and $\{y, b\}$,

$$\mathcal{L}_n(a, x) \coloneqq \{a^i x^j : i, j > 0,\ i + j < n\},$$

and

$$\mathcal{L}_m(b, y) \coloneqq \{b^i y^j : i, j > 0,\ i + j < m\}.$$

As before, let $\langle \mathcal{L}_n(a, x) \rangle$ and $\langle \mathcal{L}_m(b, y) \rangle$ denote the free subsemigroups of $\langle a, x \rangle$ and $\langle b, y \rangle$ generated by these sets.

In the proof of the next theorem we use some elementary properties of faithful flatness whose proofs we have not been able to locate in the literature, although closely related statements in a commutative setting can be found for example in [30]. Namely, let R and S be rings, I an ideal of R and M a left R-module. The following proof is from [30].

Proposition 5.2.2. *(i) If M is a faithfully flat R-module, then M/IM is a faithfully flat R/I-module for a proper ideal I of R.*

(ii) If $\theta : R \longrightarrow S$ is a ring homomorphism and S is a faithfully flat (left) R-module, then θ is injective.

Proof. (i) Let M be a faithfully flat R-module. Then $M/IM = R/I \otimes_R M \neq 0$ since M is a faithfully flat R-module. If X is a non-zero right R/I-module, then

$$\begin{aligned} X \otimes_{R/I} (R/I \otimes_R M) &= \big(X \otimes_{R/I} R/I\big) \otimes_R M \\ &= X \otimes_R M \\ &\neq 0. \end{aligned}$$

A similar argument shows that $- \otimes_{R/I} (R/I \otimes_R M)$ preserves exactness of exact sequences of right R/I modules.

(ii) Let $\theta : R \longrightarrow S$ be a ring homomorphism and let S be a faithfully flat (left) R-module. For any ideal J of R, we have an injection $J \hookrightarrow R$ which yields an injection $J \otimes_R S \hookrightarrow S$ when we apply $- \otimes_R S$. The image of the injection is JS, so that $J \otimes_R S \cong JS$. If J is the kernel of θ, we then have $J \otimes_R S \cong JS = 0$. Since S is faithfully flat, this implies $J = 0$. Hence, θ is injective.

□

Theorem 5.2.3. *Retain the notations introduced in §5.2, and assume hypothesis* **(H)**.

(i) The coordinate ring $\mathcal{O}(\mathcal{C})$ of the plane curve $\mathcal{C}$ is a quantum homogeneous space of the Hopf algebra $A(g,f)$.

(ii) $A(x,a,g)$ and $A(y,b,f)$ are Hopf subalgebras of $A(g,f)$, and $A(g,f)$ is faithfully flat over these subalgebras.

(iii) Assume that f and g have degrees m and n respectively, with $2 \leq m,n \leq 5$. Let $\{c_\ell : \ell \in \mathbb{Z}_{\geq 0}\}$ be a k-basis for $\mathcal{O}(\mathcal{C})$.

(a) $A(g,f)$ has PBW basis

$$\{c_\ell \langle \mathcal{L}_n(a,x)\rangle \langle \mathcal{L}_m(b,y)\rangle a^i b^j : \ell \in \mathbb{Z}_{\geq 0}, i \in \mathbb{Z}, 0 \leq j < m\}.$$

(b) $A(g,f)$ is a free left $\mathcal{O}(\mathcal{C})$−module with basis

$$\{\langle \mathcal{L}_n(a,x)\rangle \langle \mathcal{L}_m(b,y)\rangle a^i b^j : i \in \mathbb{Z}, 0 \leq j < m\}.$$

Proof. (i) Under hypothesis **(H)**, the subalgebra $k\langle x, a^{\pm n}\rangle$ of $A(x,a,g)$ is isomorphic to $k[x, a^{\pm n}]$; this follows from the centrality of $a^{\pm n}$, Proposition 4.3.1(iii), together with the PBW theorem Corollary 4.4.8 when $g(x)$ has degree at most 5, and by Proposition 4.5.2(ii) when $g(x) = x^n$. The same applies to the subalgebra $k\langle y, b^{\pm m}\rangle = k[y, b^{\pm m}]$ of $A(y,b,f)$. Thus $R := k[x,y,a^{\pm n}, b^{\pm m}]$ is a subalgebra of T. Note that R is a quantum homogeneous space of T, just as was the case in Proposition 4.5.2(ii) for its two components in their respective Hopf algebras - that is, it is a right coideal subalgebra of T which contains the inverses of all its grouplike elements, so Masuoka's theorem [51, Theorem 1.3] applies. In particular, T is a faithfully flat left and right R-module.

It follows from Proposition 5.2.2(i) above that the algebra $T/(a^n - b^m)T$ is a faithfully flat $R/(a^n - b^m)R$-module. Observe that $R/(a^n - b^m)R$ is the group ring $k[x,y]G$, where G is the infinite cyclic group generated by a^n, that is $G = \langle a^{\pm n}\rangle$. In particular, $R/(a^n - b^m)R$ is a free $k[x,y]$-module, so that $T/(a^n - b^m)T$ is a faithfully flat $k[x,y]$-module. Hence, by Proposition 5.2.2(ii) above, $k[x,y]$ embeds in $T/(a^n - b^m)T$. A second application of Proposition 5.2.2(i) and (ii), this time to the ideal $(g-f)k[x,y]$ of $k[x,y]$, now implies that $T/(a^n - b^m)T + (g-f)T$ is a faithfully flat $\mathcal{O}(\mathcal{C})$-module. That is, again appealing to Proposition 5.2.2(ii), $\mathcal{O}(\mathcal{C})$ embeds in $A(g,f)$ and is a quantum homogeneous space of $A(g,f)$.

(ii) In a similar way to (i), $k[f,b^{\pm m}]$ is a right coideal subalgebra of $A(y,b,f)$ and hence, again using [51, Theorem 1.3], $A(y,b,f)$ is a faithfully flat left and right $k[f,b^{\pm m}]$-module. Define

$$D := A(x,a,g) \otimes k[f,b^{\pm m}] \subseteq T,$$

so that $D \cong A(x,a,g)[f,b^{\pm m}]$ and T is a faithfully flat left and right D-module. Let $J = (f-g)D + (b^m - a^n)D$, an ideal of D with $D/J \cong A(x,a,g)$ and $T/JT = A(g,f)$. By Proposition 5.2.2(i) above, T/JT is a faithfully flat left and right $A(x,a,g)$-module. In particular, by Proposition 5.2.2(ii) above, $A(x,a,g)$ embeds in $A(g,f)$.

The argument for $A(y,b,f)$ is exactly similar.

(iii) Both parts are similar to (i), but easier: thanks to Corollary 4.4.8 there is an explicit PBW basis for T under the given hypotheses. Thus one can simply retrace the proof of (i), replacing "faithful flatness" by "free over an explicitly stated basis" at each occurrence of the former term. □

Remark 5.2.4. In fact there is less of a gap between parts (i) and (iii) of the above result than at first appears. For, notice that the proof of (i) starts with the fact that $R = k[x,y,a^{\pm n},b^{\pm m}]$ is a quantum homogeneous space of T. In particular T is a faithfully flat left R-module, and hence, by [53, Theorem 2.1], T is a projective generator for R. Therefore, by the Quillen-Suslin theorem on projective modules over polynomial algebras, as strengthened by Swan [76, Corollary 1.4] to allow Laurent polynomial generators, T is a free left R-module. Finally, freeness is preserved by factoring by $(a^n - b^m)R + (g-f)R$. So the only extra feature in (iii)(b) as compared with (i) is the explicit description of a free basis.

Fundamental properties of the Hopf algebras $A(g,f)$ can be read off from our knowledge gained about the algebras $A(x,a,g)$ in §§4.2-4.6, provided hypothesis **(H)** is in play. The following result summarises the basic facts.

Theorem 5.2.5. *Retain the notation of §5.2, so $g(x)$ and $f(y)$ are polynomials of degree n and m respectively. Assume hypothesis* **(H)** *for parts (ii) - (v). Then $A(g,f)$ satisfies the following properties.*

(i) $A(g,f)$ is a pointed Hopf algebra, with its grouplikes being the finitely generated abelian group $\langle a^{\pm 1}, b^{\pm 1} : [a,b] = 1, a^n = b^m \rangle$.

(ii) Consider the statements:

(a) $\max\{n, m\} \leq 3$;

(b) $A(g, f)$ is a finite module over its centre.

(c) $A(g, f)$ satisfies a polynomial identity;

(d) $\mathrm{GKdim}(A(g, f)) < \infty$;

(e) $A(g, f)$ does not contain a noncommutative free subalgebra;

Then

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e)$$

and

$$(e) \Longrightarrow (a)$$

if $\max\{n, m\} \leq 5$.

(iii) If the equivalent conditions in (ii) hold, then $A(g, f)$ is noetherian.

(iv) Suppose $\max\{n, m\} \leq 3$. *Then the algebra $A(g, f)$ is AS-Gorenstein and GK-Cohen Macaulay, with*

$$\mathrm{inj.dim}(A(g, f)) = \mathrm{GKdim}(A(g, f)) = n + m - 2.$$

(v) If $\mathcal{C}$ has a singular point at the origin then $\mathrm{gl.dim}(A(g, f)) = \infty$

Proof. (i) It has already been shown in Theorem 5.2.1 that $A(g, f)$ is a Hopf algebra. It is generated by the grouplike elements a, b and the skew primitives x and y. Since x is $(1, a)$ skew primitive and y is $(1, b)$ skew primitive, by [65, Corollary 5.1.14(a)] $A(g, f)$ is pointed and the grouplike elements of $A(g, f)$, $G(A(g, f))$ is the multiplicative submonoid of $A(g, f)$ generated by a and b. Hence $G(A(g, f))$ is as stated.

(ii) $(a) \Longrightarrow (b)$: Suppose $\max\{n, m\} \leq 3$. Then both $A(x, a, g)$ and $A(y, b, f)$ are finite modules over their centres, by Propositions 4.6.11(ii) and 4.6.23(iv). The same is thus clearly true of the tensor product T of these algebras, and so of its factor algebra $A(g, f)$.
$(b) \Longrightarrow (c)$: [58, Corollary 13.1.13(i)].
$(c) \Longrightarrow (d)$: [41, Corollary 10.7].
$(d) \Longrightarrow (e)$: Suppose that $\mathrm{GKdim}(A(g, f))$ is finite. Then $A(g, f)$ cannot contain a noncommutative free subalgebra, since such an algebra has infinite GK-dimension, [41, Example 1.2].
$(e) \Longrightarrow (a)$: Suppose that $A(g, f)$ does not contain a noncommutative free subalgebra. By Theorem 5.2.3(ii), the same is true for the subalgebras $A(x, a, g)$ and $A(y, b, f)$. By Corollary 4.4.8, $\max\{n, m\} \leq 3$.

(iii) Suppose that (ii)(b) holds. Since $A(g,f)$ is an affine k-algebra, its centre Z is also affine, by (ii)(b) and the Artin-Tate lemma [58, Lemma 13.9.10(ii)]. Thus Z is noetherian by the Hilbert basis theorem, and so $A(g,f)$ is a noetherian algebra since it is a finite Z-module.

(iv) Since $\max\{n,m\} \leq 3$, the Hopf algebra $A(g,f)$ is a finite module over its affine centre by (ii). Indeed, the same is also true for $T = A(x,a,g) \otimes_k A(y,b,f)$. Therefore T and $A(g,f)$ are both AS-Gorenstein and GK-Cohen-Macaulay, by [82, Theorem 0.2]. Now

$$\mathrm{GKdim}(T) = \mathrm{GKdim}(A(x,a,g)) + \mathrm{GKdim}(A(y,b,f)) = n + m, \tag{5.2}$$

by [41, Corollary 10.17] for the first equality and the second by Propositions 4.6.11(ii) and 4.6.14(v),(vii).

Next, the central elements $f - g$ and $a^n - b^m$ of the GK-Cohen-Macaulay algebra T form a regular central sequence in T, using the PBW theorem, Theorem 5.2.3(ii)(a). So, by (5.2) and [14, Proposition 2.11 and Theorem 4.8$(i) \Longleftrightarrow (iii)$],

$$\mathrm{GKdim}(A(g,f)) = \mathrm{GKdim}(T) - 2 = n + m - 2, \tag{5.3}$$

as required. Finally, since $A(g,f)$ is an AS-Gorenstein GK-Cohen-Macaulay algebra, its injective and Gel'fand-Kirillov dimensions are equal, [82, Theorem 0.2].

(v) Suppose that $\mathcal{C}$ has a singularity at the origin. Thus, letting $\mathfrak{m} = \langle x,y \rangle \triangleleft \mathcal{O}(\mathcal{C}) \subset A(g,f)$,

$$\mathrm{pr.dim}_{\mathcal{O}(\mathcal{C})}(\mathcal{O}(\mathcal{C})/\mathfrak{m}) = \infty. \tag{5.4}$$

Suppose that $\mathrm{gl.dim}(A(g,f)) < \infty$. Then, in particular, $\mathrm{pr.dim}_{A(g,f)}(k_{\mathrm{tr}}) < \infty$, where k_{tr} denotes the trivial module. By Theorem 5.2.3(i) and (**H**), $A(g,f)$ is a flat $\mathcal{O}(\mathcal{C})$-module, so the restriction to $\mathcal{O}(\mathcal{C})$ of a projective $A(g,f)$-resolution of k_{tr} yields a finite flat $\mathcal{O}(\mathcal{C})$-resolution of k_{tr}.

But $\varepsilon(x) = \varepsilon(y) = 0$, so that $\mathfrak{m}k_{\mathrm{tr}} = 0$. Hence, bearing in mind that flat dimension and projective dimension are the same for modules over a noetherian ring [68, Corollary 8.28],

$$\mathrm{pr.dim}_{\mathcal{O}(\mathcal{C})}(\mathcal{O}(\mathcal{C})/\mathfrak{m}) < \infty.$$

This contradicts (5.4), and so $\mathrm{gl.dim}(A(g,f)) = \infty$, as required. □

Presumably Theorem 5.2.5(v) remains true for all singular plane decomposable curves $\mathcal{C}$, but we have been unable so far to prove this.

5.3 Examples

In this section we gather together and discuss some special cases of the construction described in §5.2 when $\deg(g), \deg(f) \leq 5$. All the examples we consider are plane curves with singularities though the construction works also for smooth plane curves.

We begin by noting that, given a decomposable plane curve $\mathcal{C}$ with equation $g(x) = f(y)$, we may always use a linear change of coordinates to assume that the curve passes through the origin, so that f and g have constant term 0. We can further assume that both f and g are monic. For, recall the scaling maps $\{\theta_\lambda : x \mapsto \lambda x : \lambda \in k \setminus \{0\}\}$ of $k\langle x, a^{\pm 1}\rangle$ introduced in §4.2.4, where we wrote g^λ for $\theta(g)$. The following lemma extends this scaling procedure to the Hopf algebras $A(g, f)$.

Lemma 5.3.1. *Let $\mathcal{C}$ be the decomposable plane curve with equation $f(y) = g(x)$, where f and g are polynomials with constant term 0, respectively of degree m, n with $m, n \geq 2$. Let $\lambda, \mu \in k \setminus \{0\}$.*

(i) $\theta_\lambda \otimes \mathrm{Id}_{A(y,b,f)}$ and $\mathrm{Id}_{A(x,a,g)} \otimes \theta_\mu$ are commuting automorphisms of $k\langle x, a^{\pm 1}\rangle \otimes k\langle y, b^{\pm 1}\rangle$, whose composition induces an isomorphism of Hopf algebras

$$\theta_{\lambda,\mu} : A(x, a, g) \otimes A(y, b, f) \longrightarrow A(x, a, g^\lambda) \otimes A(y, b, f^\mu).$$

(ii) The map $\theta_{\lambda,\mu}$ induces an isomorphism of Hopf algebras

$$\overline{\theta_{\lambda,\mu}} : A(g, f) \longrightarrow A(g^\lambda, f^\mu),$$

under which the quantum homogeneous space $k[x,y]/\langle g - f\rangle$ of $A(g, f)$ is mapped to the quantum homogeneous space $k[x,y]/\langle g^\lambda - f^\mu\rangle$ of $A(g^\lambda, f^\mu)$.

Proof. (i) This follows from Lemma 4.2.5(ii) and from the fact that a composition of isomorphisms is an isomorphism.

(ii) The following holds:

$$\theta_{\lambda,\mu}(g \otimes 1 - 1 \otimes f) = \left(\mathrm{Id}_{A(x,a,g)} \otimes \theta_\mu\right)\left(g^\lambda \otimes 1 - 1 \otimes f\right) = g^\lambda \otimes 1 - 1 \otimes f^\mu,$$

and

$$\theta_{\lambda,\mu}(a^n \otimes 1 - 1 \otimes b^m) = \left(\mathrm{Id}_{A(x,a,g)} \otimes \theta_\mu\right)(a^n \otimes 1 - 1 \otimes b^m) = a^n \otimes 1 - 1 \otimes b^m.$$

Thus, $A(g, f)$ and $A(g^\lambda, f^\mu)$ are isomorphic as algebras. It can be shown that the composition of $(\theta_{\lambda,\mu} \otimes \theta_{\lambda,\mu})$ with the coproduct of $A(g, f)$ is equal to the composition of the coproduct of $A(g^\lambda, f^\mu)$ with $\theta_{\lambda,\mu}$, that is

$$(\theta_{\lambda,\mu} \otimes \theta_{\lambda,\mu}) \circ \Delta_{A(g,f)} = \Delta_{A(g^\lambda, f^\mu)} \circ \theta_{\lambda,\mu}.$$

Similar arguments hold for the counit and antipode. This yields the conclusion that $\overline{\theta_{\lambda,\mu}}$ induces an isomorphism of Hopf algebras whose restriction gives an isomorphism between the quantum homogeneous spaces $k[x,y]/\langle g - f\rangle$ of $A(g,f)$ and $k[x,y]/\langle g^\lambda - f^\mu\rangle$ of $A(g^\lambda, f^\mu)$.

□

5.3.1 $A(g,f)$ for degree 2 polynomials

Let $\mathcal{C}$ be an arbitrary decomposable plane curve of degree 2 - that is, $\mathcal{C}$ has defining equation $g(x) = f(y)$ with $\deg f = \deg g = 2$. After an application of Lemma 5.3.1 we can assume without loss of generality that g and f are monic. Thus the equation of $\mathcal{C}$ has the form

$$rx + x^2 \;=\; sy + y^2, \tag{5.5}$$

where $(r,s) \in k^2$. The possibilities for $\mathcal{C}$ are described as follows. The Jacobian matrix is given by

$$\begin{bmatrix} 2x + r & -2y - s \end{bmatrix}$$

The Jacobian criterion states that a point (x,y) on $\mathcal{C}$ is singular if it satisfies the equations

$$\begin{aligned} 2x + r &= 0 \\ -2y - s &= 0 \end{aligned}$$

Thus, a point (x,y) on $\mathcal{C}$ is smooth if and only if $r \neq \pm s$; and if $r = \pm s$, $(x,y) = (-r/2, -s/2)$ is a unique singular point. Then, by the linear change of variables

$$u = x - y + \frac{1}{2}(r - s),\; v = x + y + \frac{1}{2}(r + s),$$

one sees that the coordinate ring $\mathcal{O}(\mathcal{C})$ is isomorphic to $k[u,v]/\langle uv + C\rangle$, where C is a constant which is non-zero in the smooth case and 0 in the singular case, the latter being the coordinate crossing.

Consider now the corresponding Hopf algebra $A(g,f)$. As in Proposition 4.6.11(i), proceed by changing the variables x and y in $A(x,a,g)$ and $A(y,b,f)$ to

$$x' := x + \frac{r}{2}(1 - a), \qquad y' := y + \frac{s}{2}(1 - b). \tag{5.6}$$

These elements are respectively $(1,a)$– and $(1,b)$–primitive, and by Proposition 4.6.11(i) we have

$$\begin{aligned} T &= A(x,a,g)\otimes A(x,b,f) \\ &= k\langle a^{\pm 1}, b^{\pm 1}, x', y' : x'a + ax' = 0,\ y'b + by' = 0, \\ &\qquad [a,b] = [x',y'] = [a,y'] = [x',b] = 0\rangle. \end{aligned}$$

One calculates that, in T,

$$x'^2 - y'^2 = f - g + \frac{1}{4}(r^2 - s^2) - \frac{1}{4}(r^2a^2 - s^2b^2).$$

Therefore, in $A(g,f)$, that is *modulo*$\langle f - g, a^2 - b^2\rangle$,

$$x'^2 - y'^2 \equiv \frac{1}{4}(r^2 - s^2)(1 - a^2). \tag{5.7}$$

The outcome is summarised in the next result.

Proposition 5.3.2. *Let $\mathcal{C}$ be the decomposable degree 2 plane curve embedded in the plane by the equation (5.5). Then $\mathcal{O}(\mathcal{C})$ is a quantum homogeneous space of the Hopf algebra $A(g,f)$, where $A(g,f)$ has the following properties.*

(i) Defining x' and y' as in (5.6), $A(g,f)$ has presentation

$$\begin{aligned} A(g,f) &= k\langle a^{\pm 1}, b^{\pm 1}, x', y' : x'a + ax' = 0,\ y'b + by' = 0, \\ &\qquad [a,b] = [x',y'] = [a,y'] = [x',b] = 0, \\ &\qquad a^2 = b^2,\ \ x'^2 - y'^2 = \frac{1}{4}(r^2 - s^2)(1 - a^2)\rangle. \end{aligned}$$

Here, a and b are grouplike, x' is $(1,a)$-primitive and y' is $(1,b)$-primitive.

(ii) $A(g,f)$ is an affine noetherian pointed Hopf algebra, is a finite module over its centre, and is AS-Gorenstein and GK-Cohen-Macaulay, with injective and Gel'fand-Kirillov dimensions equal to 2.

(iii) $\mathrm{gl.dim}(A(g,f)) < \infty \iff r \neq \pm s$, *that is, if and only if $\mathcal{C}$ is smooth. In this case,* $\mathrm{gl.dim}(A(g,f)) = 2$.

(iv) Up to an isomorphism of Hopf algebras there are only two possible algebras $A(g,f)$ - the smooth case and the singular case.

(v) $A(g,f)$ is prime but is not a domain.

Proof. (i) This is sketched in the discussion before the proposition.

(ii) These are all special cases of Theorem 5.2.5(i),(ii), (iv), (v).

(iii) Suppose first that $r^2 = s^2$. Then the relation (5.7) of $A(g,f)$ becomes $(x'-y')(x'+y') = 0$. By Theorem 5.2.3(ii), $A(g,f)$ is left free over $\mathcal{O}(\mathcal{C})$. Note that, by Masuoka's theorem [51, Theorem 1.3], $A(g,f)$ is faithfully flat over its right coideal subalgebra $k\langle x', y'\rangle$. As in the proof of Theorem 5.2.5(v), if the trivial $A(g,f)$-module had a finite projective resolution then the same would be true for the $k\langle x', y'\rangle$-module $k\langle x', y'\rangle/\langle x', y'\rangle$ since the latter is the restriction of the former. But this is manifestly false, since $\langle x', y'\rangle$ defines the singular point of this curve. So $\mathrm{gl.dim}(A(g,f)) = \infty$.

Conversely, suppose that $r^2 \neq s^2$. Define $R := k\langle x', y', a^{\pm 2}\rangle$ to be the commutative subalgebra of $A(g,f)$. Observe that, setting

$$X := x' - y',\ Y := x' + y',\ Z := \frac{1}{4}(r^2 - s^s)(1 - a^2),$$

R is isomorphic to the localisation of

$$k[X, Y, Z]/\langle XY - Z\rangle$$

at the powers of $Z - \frac{1}{4}(r^2 - s^2)$. In particular, (for example by the Jacobian criterion), R is smooth, $\mathrm{gl.dim}(R) = 2$. Consider the augmentation ideal $R^+ := \langle X, Y, Z\rangle$ of R. By the defining relations of $A(g,f)$, the right ideal $R^+A(g,f)$ of $A(g,f)$ is a two sided ideal. Then one easily checks that

$$A(g,f)/R^+A(g,f) \ \cong\ k(\mathbb{Z}_2 \times \mathbb{Z}_2), \tag{5.8}$$

the group algebra of the Klein 4-group K, with generators the images of a and ab^{-1}. In fact, $A(g,f)$ is a crossed product $R * K$, though we don't need this. Rather, it is enough to note that $A(g,f)$ is faithfully flat as a right and left R-module, being a quantum homogeneous space using as usual [51, Theorem 1.3].

The simple R-module R/R^+ has a finite R-projective resolution $\mathcal{P}$, by smoothness of R. Faithful flatness ensures exactness of $- \otimes_R A(g,f)$, yielding a finite projective resolution $\mathcal{P} \otimes_R A(g,f)$ of the $A(g,f)$-module $A(g,f)/R^+A(g,f)$. However, $A(g,f)/R^+A(g,f)$ is semisimple Artinian, by (5.8) and Maschke's theorem, and hence contains the trivial module k of the Hopf algebra $A(g,f)$ as a direct summand. Therefore $\mathrm{pr.dim}_{A(g,f)}(k) < \infty$. By [48, §2.4], $\mathrm{gl.dim}(A(g,f)) < \infty$, as required. That the global dimension is equal to 2 follows either from the fact that the injective dimension is 2, as shown in (ii), or from the fact that $\mathrm{pr.dim}_R(R/R^+) = 2$ since R is the coordinate ring of a smooth surface.

(iv) It is clear from the presentation in (i) how to define a Hopf algebra isomorphism between any two members of the smooth family simply by scaling the generators x' and y'. On the other hand, the coefficients disappear from the relations when $r = \pm s$, so the result is clear in this case also.

(v) It is clear from the defining relations and the PBW theorem that $A(g,f)$ is not a domain, since $a \neq \pm b$, but $(a-b)(a+b) = 0$. One way to see that $A(g,f)$ is prime is to use the crossed product description of $A(g,f)$ found in the proof of (iii). Namely, it was shown there that $A(g,f)$ contains a commutative subalgebra $R := k[X,Y,Z]/\langle XY - tZ\rangle$, where $t \in k$ is 0 in the singular case and non-zero in the smooth case. Then $A(g,f)$ is a crossed product $R * \Gamma$ where $\Gamma = \langle \overline{a}, \overline{b}\rangle$ is a Klein 4-group such that $a : X \leftrightarrow -Y$ and $b : X \leftrightarrow Y$. When $t \neq 0$, R is a domain and Γ acts faithfully on its quotient field Q. Hence $Q * \Gamma$ is a simple ring by [6], see [62, Exercise 6, p. 48]. When $t = 0$, R is Γ-prime and primeness of $R * \Gamma$ follows by passing to the quotient ring $Q(R) * \Gamma$, where $Q(R) = k(X,Z) \oplus k(Y,Z)$ is Γ-simple and Γ acts faithfully on $Q(R)$. Thus $Q(R) * \Gamma$ can easily be checked to be a simple ring by direct calculation. So in all cases $R * \Gamma$ has a simple quotient ring and hence is a prime ring. □

5.3.2 The cusps $y^m = x^n$

Let n and m be coprime integers, with $m > n \geq 2$. The cusp $y^m = x^n$ was shown to be a quantum homogeneous space in a pointed affine noetherian Hopf k-algebra domain, in [28, Construction 1.2]. In the notation introduced by Goodearl and Zhang in [28], the Hopf algebra constructed is labelled $B(1,1,n,m,q)$, where q is a primitive nmth root of unity in k. The algebra $B(1,1,n,m,q)$ is constructed as the skew group algebra of the infinite cyclic group whose coefficient ring is the coordinate ring of the cusp, with the twisting automorphism acting on the generators of the coordinate ring by multiplication by appropriate powers of q. In particular, this means that $B(1,1,n,m,q)$ has GK-dimension two [28, Proposition 0.2] and is a finite module over its centre. It is also straightforward to see from its construction that $B(1,1,n,m,q)$ is a factor Hopf algebra of a suitable localised quantum 4-space $Q := k_{\mathbf{q}}[x_1^{\pm 1}, x_2^{\pm 1}, x_3, x_4]$, where $[x_1, x_2] = [x_3, x_4] = 0$, the other pairs of generators q-commute, and the factoring relations are

$$x_1^n - x_2^m, \qquad \text{and} \qquad x_3^n - x_4^m. \tag{5.9}$$

Now it is also not hard to deduce from Proposition 4.5.2(i) that the *same* localised quantum 4-space Q is a factor Hopf algebra of $T := A(x, a, x^n) \otimes A(y, b, y^m)$. Moreover, the relations used to define $A(x^n, y^m)$ as a factor of T have images in Q which are exactly the elements listed in (5.9).

We give a detailed exposition of the above remarks.

Definition 5.3.3. [28, Construction 1.2] Let $n, p_0, p_1, \cdots, p_s$ be positive integers and $q \in k^{\times}$ with the following properties:

(a) $s \geq 2$ and $1 < p_1 < p_2 < \cdots < p_s$;

(b) $p_0|n$ and $p_0, p_1, \cdots, p_s$ are pairwise relatively prime;

(c) q is a primitive ℓ-th root of unity where $\ell = (n/p_0)p_1p_2\cdots p_s$.

Set $m = p_1p_2\cdots p_s$ and $m_i = m/p_i$ for $i = 1, 2, \cdots, s$. Choose an indeterminate y and consider the subalgebra $A = k[y_1, y_2, \cdots, y_s]$ of $k[y]$, where $y_i := y^{m_i}$ for $i = 1, 2, \cdots, s$. The k-algebra automorphism of $k[y]$ sending $y \mapsto qy$ restricts to an automorphism σ of A. There is a unique Hopf algebra structure on the skew Laurent polynomial ring $B = A[x^{\pm 1}; \sigma]$ such that x is grouplike and the y_i are skew primitive, with

$$\Delta(y_i) = y_i \otimes 1 + x^{m_i n} \otimes y_i$$

for $i = 1, 2, \cdots, s$. We denote this Hopf algebra by $B(n, p_0, p_1, \cdots, p_s, q)$.

Remark 5.3.4. The Hopf algebra $B(n, p_0, p_1, \cdots, p_s, q)$ can be presented as the quotient of the k-algebra $k\langle y_1, y_2, \cdots, y_s\rangle * k[x^{\pm 1}]$ by the following relations:

$$\begin{aligned} xx^{-1} &= x^{-1}x = 1 \\ xy_i &= q^{m_i} y_i x \qquad (1 \le i \le s) \\ y_i y_j &= y_j y_i \qquad (1 \le i < j \le s) \\ y_i^{p_i} &= y_j^{p_j} \qquad (1 \le i < j \le s) \end{aligned}$$

The following example is the simplest case of Definition 5.3.3.

Example 5.3.5. Let $n = p_0 = 1$, $p_1 = 2$ and $p_2 = 3$. Then the resulting Hopf algebra $B(1, 1, 2, 3, q)$ is the quotient of the algebra $k\langle y_1, y_2, x^{\pm 1}\rangle$ by the relations:

$$xy_1 = q^3 yx, \qquad xy_2 = q^2 y_2 x, \qquad y_1y_2 = y_2y_1, \qquad y_1^2 = y_2^3$$

where q is a primitive 6-th root of unity.

Lemma 5.3.6. *With the above notation, the Hopf algebra $B(1, 1, n, m, q)$ is a factor Hopf algebra of $A(x^n, y^m)$.*

Proof. Recall from the proof of Proposition 4.5.2(i) that the following hold

$$P_{(a,x)}(j, n-j) = 0 \quad \text{and} \quad P_{(b,y)}(j, m-j) = 0$$

for $1 \le j \le n-1$ in $k_n\langle x, a^{\pm 1}\rangle$ and for $1 \le j \le m-1$ in $k_m\langle y, b^{\pm 1}\rangle$ respectively. Now, if r and s are primitive n-th and m-th roots of unity respectively, then

$$r^{mn} = (r^n)^m = (1)^m = 1, \quad s^{mn} = (s^m)^n = (1)^n = 1.$$

Thus, primitive nth and mth roots of unity are also mnth roots of unity. Hence

$$P_{(a,x)}(j, n-j) = 0 \quad \text{and} \quad P_{(b,y)}(j, m-j) = 0$$

in $B(1,1,n,m,q)$ by Proposition 4.5.2(i). □

Note that even in the "smallest" case, $(n,m) = (2,3)$, $B(1,1,2,3,q)$ is a *proper* factor of $A(x^2,y^3)$ - here, both algebras are finite over their centres, but $A(x^2,y^3)$ has GK-dimension 3 by Theorem 5.2.5 and $B(1,1,2,3)$ is a domain of GK-dimension 2.

5.3.3 The nodal cubic

As stated in § 3.4 above, the starting point of this thesis was to check whether the coordinate ring of the nodal cubic, $y^2 = x^2 + x^3$ is a quantum homogeneous space, following the example of the cusp $y^2 = x^3$. Since $(n,m) = (2,3)$, the results of §§5.2,5.2.1 apply. The presentation of $A(x^2 + x^3, y^2)$ is the quotient of $k\langle x, y, a^{\pm 1}, b^{\pm 1}\rangle$ by the ideal generated by the following relations:

$$\begin{gathered} y^2 = x^2 + x^3,\ a^3 = b^2, \\ [x,y] = [x,b] = [a,y] = [a,b] = 0, \\ ax + xa + ax^2 + xax + x^2a = 0, \\ a^2x + axa + xa^2 = a^3 - a^2, \\ yb + by = 0. \end{gathered}$$

Recall from definition of the Hopf algebra A in [40]. Fix $(q,p) \in k^2$ satisfying $p^2 = q^2 + q^3$. Then the unital associative k-algebra A with generators x, y, a, a^{-1}, b satisfying the relations

$$\begin{gathered} aa^{-1} = a^{-1}a = 1, \qquad y^2 = x^2 + x^3, \qquad b^2 = a^3 \\ ba = ab, \qquad ya = ay, \qquad bx = xb, \qquad yx = xy, \qquad by = -yb + 2pb^2 \\ a^2x = -(xa^2 + axa + a^2) + (1+3q)a^3 \\ ax^2 = -(ax + xa + x^2a + xax) + (2+3q)qa^3 \end{gathered}$$

admits a Hopf algebra structure with a and b grouplike, and $(x - qa)$ and $(y - pb)$ $(1,a)$ and $(1,b)$ skew-primitive respectively. Then the Hopf algebra $A(x^2 + x^3, y^2)$ is precisely the Hopf algebra A presented in [40], for the parameters $p = q = 0$. As noted in [39, §2.3], the other values of p and q yield isomorphic Hopf algebras. From the results of §5.2.1 we see that $A(x^2 + x^3, y^2)$ is an affine noetherian Hopf algebra of GK-dimension 3, which is a finite module over its centre. It is AS-Gorenstein and GK-Cohen Macaulay of injective dimension 3, but has infinite global dimension. The Hopf algebra constructed in [39] is a quotient Hopf algebra of $A(x^2+x^3,y^2)$ of GK-dimension 1 which still admits the nodal cubic as a quantum homogeneous space, demonstrating that $A(x^2+x^3,y^2)$ is *not* minimal with this property.

5.3.4 The lemniscate

The Lemniscate of Gerono is usually presented by the equation $y^2 = x^2 - x^4$. Applying Lemma 5.3.1 with $\mu = 1$ and λ a primitive 8^{th} root of 1 in k, we can work with the presentation

$$y^2 = x^4 + \lambda^2 x^2 \tag{5.10}$$

of the lemniscate. The outcome is as follows, recalling that in §5.2.1 we defined $\langle \mathcal{L}_4(a,x)\rangle$ to denote the free subsemigroup of the free semigroup $\langle a,x\rangle$ generated by

$$\{(ax),(ax^2),(a^2x)\}.$$

Lemma 5.3.7. *Let the lemniscate $\mathcal{L}$ be presented by the equation (5.10), so $f(y) = y^2$ and $g(x) = x^4 + \lambda^2 x^2$.*

(i) The coordinate ring $\mathcal{O}(\mathcal{L})$ is a quantum homogeneous space in $A(x^4+\lambda^2x^2,y^2)$, a Hopf algebra with generators $x, a^{\pm 1}, y, b^{\pm 1}$ and relations

$$[a,b] = [x,y] = [a,y] = [x,b] = 0,$$
$$by + yb = 0, \qquad ax^3 + xax^2 + x^2ax + x^3a + \lambda^2(xa+ax) = 0,$$
$$\lambda^2a^2 + a^2x^2 + x^2a^2 + xaxa + ax^2a + xa^2x + axax - \lambda^2a^4 = 0,$$
$$a^3x + a^2xa + axa^2 + xa^3 = 0.$$

(ii) The algebra $A(x^4+\lambda^2x^2,y^2)$ has PBW basis

$$\{x^r y^{\epsilon_1}\langle \mathcal{L}_4(a,x)\rangle a^s b^{\epsilon_2} : r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}, \epsilon_j \in \{0,1\}, j = 1,2\}.$$

(iii) The algebra $A(x^4+\lambda^2x^2,y^2)$ is not a domain and has infinite Gelfand-Kirillov and global dimensions.

Proof. (i) Theorem 5.2.1.

(ii) The PBW basis is given by Theorem 5.2.3(iii)(a), noting that $\mathcal{O}(\mathcal{L}) = k[x,y]/\langle y^2 - x^4 - \lambda^2x^2\rangle$ is a free $k[x]$-module on the basis $\{1,y\}$.

(iii) The group-like element $a^{-2}b$ of $A(x^4+\lambda^2x^2,y^2)$ has order 2, so $(a^{-2}b-1)(a^{-2}b+1) = 0$ and $A(x^4+\lambda^2x^2,y^2)$ is not a domain. By (ii), $A(x^4+\lambda^2x^2,y^2)$ contains the noncommutative free algebra $k\langle\mathcal{L}_4(a,x)\rangle$, and hence has infinite Gel'fand-Kirillov dimension by [41, Example 1.2]. Since the lemniscate has a singularity at the origin, $\mathrm{gl.dim}(A(x^4+\lambda^2x^2,y^2)) = \infty$ by Theorem 5.2.5(v). $\square$

Chapter 6

Open Questions

In this chapter, we discuss open questions which arise from this thesis.

6.1 Questions about the Hopf algebra $A(x,a,g)$

Question 6.1.1. *What is the centre of $A(x,a,g)$ when $\deg(g) = 3$? Can we use the techniques used in* [42] *and* [83] *to find the centres of a down-up algebra to compute this ?*

Question 6.1.2. *When is the algebra $A(x,a,g)$ a domain?*

Question 6.1.3. *The Hopf algebra $A(x,a,g)$ is a quotient of the Hopf algebra $F(t)$ defined in* [84]. *In* [84], *Zhuang proved that over a field of characteristic zero, a Hopf algebra H which is a domain with $2 \leq GKdim H \leq \infty$ has a Hopf subalgebra of GKdim two. If we are able to show that the central Hopf subalgebra $k\langle g, a^{\pm 1}\rangle$ of $A(x,a,g)$ has GKdim two, does this mean we can prove in general that $A(x,a,g)$ has a Hopf algebra of GKdim two?*

Question 6.1.4. *Can we add additional relations to those used to define $A(x,a,g)$ in order to get rid of the free subalgebra we get when $\deg(g) \geq 4$?*

Question 6.1.5. *In defining the algebras $A(x,a,g)$ and $A(g,f)$, we used Manin's approach to get the universal Hopf algebra which contains the coordinate ring of a decomposable plane curve defined by $f(y) = g(x)$ as a right coideal subalgebra with the coproduct of x and y given by*

$$\Delta(x) = 1 \otimes a_1 + x \otimes a_2, \quad \Delta(y) = 1 \otimes b_1 + y \otimes b_2.$$

The coassociativity constraint enabled us to find a_1, a_2, b_1, b_2. It would be great to use a similar approach to find a similar universal Hopf algebra which contains the coordinate ring of a general plane curve defined by $f(y) = g(x)$ as a right coideal subalgebra with the coproduct of x and y given by

$$\Delta(x) = \sum_{i=0}^{r} x^i \otimes a_i, \quad \Delta(y) = \sum_{i=0}^{s} y^i \otimes b_i$$

with $r, s \geq 2$. This will provide a proof to the conjecture by Kraehmer and Tabiri in [40] *that all plane curves are quantum homogeneous spaces.*

Question 6.1.6. *What is the space of skew primitive elements of the Hopf algebras $A(x, a, g)$ and $A(g, f)$?*

Question 6.1.7. *When are the conditions for $A(x, a, g)$ and $A(x, a, g')$ to be isomorphic as algebras and Hopf algebras for arbitrary g and g'?*

Question 6.1.8. *Can we decompose $A(x, a, g)$ as a smash or crossed products of well studied algebras such as down-up algebras, quantum planes or the universal enveloping algebra of a Lie algebra?*

6.2 Questions about the Hopf algebra $A(g, f)$

Question 6.2.1. *Is the Hopf algebra $A(g, f)$ for the nodal cubic with $g(x) = x^2 + x^3$ and $f(y) = y^2$ a domain? Can we prove in general that if both $g(x)$ and $f(y)$ are irreducible and the $\gcd(\deg(g), \deg(f)) = 1$, then $A(g, f)$ is a domain?*

Appendix A

Computations

The PBW computations are as follows: In the following proofs, we will use the symbol " $\to$ " whenever we replace the monomial $a^j x^{n-j}$ with the right hand side of the defining relations. Whenever we have a linear combination of irreducible words during the reduction process, we will underline it. For instance, $wQ(r, n-t-r)v$ from Lemma 4.4.6 above is irreducible so we write it as $\underline{wQ(r, n-t-r)v}$. Recall Proposition 4.4.7 as follows.

Proposition A.0.2. *Retain the notation of §2.10 and §4.4. Then the overlap ambiguities*

$$(\sigma_j, \sigma_{j+t}, a^t, a^j x^{n-j-t}, x^t)$$

are resolvable when

(i) $t = 1$ *and* $n \geq 3$*;*

(ii) $t = 2$ *and* $n \geq 4$*;*

(iii) $t = 3$ *and* $n \geq 5$.

We present here details of the proofs of the cases $t = 2$ and $t = 3$ of Proposition 2.8.
(ii) Let $t = 2$ and $n \geq 4$. We use the following identities throughout the proofs:

$$P(r,s) = P(r-1,s-1)xa + P(r-1,s-1)ax + P(r-2,s)a^2 + P(r,s-2)x^2, \quad \text{(A.1)}$$
$$P(r,s) = xaP(r-1,s-1) + axP(r-1,s-1) + a^2P(r-2,s) + x^2P(r,s-2), \quad \text{(A.2)}$$
$$P(r,s) = xP(r-1,s-1)a + aP(r-1,s-1)x + aP(r-2,s)a + xP(r,s-2)x. \quad \text{(A.3)}$$

We resolve the ambiguities in three cases.
(a) The ambiguity arises from the two routes to resolve the word

$$a^2\omega_1 = a^2(ax^{n-1}) = (a^3x^{n-3})x^2 = \omega_3 x^2$$

in the free algebra $k\langle a,x\rangle$ using the relations σ_1 and σ_3. Considering first σ_1, use (A.1) to write it as

$$\omega_1 = ax^{n-1} \to -(\sum_{i=2}^{n-1} r_i(\underline{P(0,i-2)ax} + \underline{P(0,i-2)xa}) + \sum_{i=3}^{n-1} r_i\underline{P(1,i-3)x^2} +$$

$$\underline{Q(1,n-3)x^2} + \underline{P(0,n-2)ax} + \underline{P(0,n-2)xa} + r_1\underline{a}) + r_1\underline{a^n}$$

$$= -(\sum_{i=2}^{n} r_i(\underline{P(0,i-2)ax} + \underline{P(0,i-2)xa}) + \sum_{i=3}^{n-1} r_i\underline{P(1,i-3)x^2} + \underline{Q(1,n-3)x^2} + r_1\underline{a}) + r_1\underline{a^n}.$$

Premultiply this by a^2, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yielding

$$a^2\omega_1 \to -((\sum_{i=2}^{n} r_i(a^2P(0,i-2)ax + a^2P(0,i-2)xa) + \sum_{i=3}^{n-1} r_i a^2P(1,i-3)x^2 + \qquad (\alpha)$$

$$\underline{a^2Q(1,n-3)ax} + r_1\underline{a^3}) + r_1\underline{a^{n+2}}.$$

The following words in (α) of length $n+2$ are reducible:

$$-a^2x^{n-2}ax \qquad \text{and} \qquad -a^2x^{n-2}xa.$$

Using (A.2), we write σ_2 as

$$-a^2x^{n-2} \to \sum_{i=3}^{n-1} r_i(\underline{xaP(1,i-3)} + \underline{axP(1,i-3)}) + \sum_{i=4}^{n-1} r_i\underline{x^2P(2,i-4)} + \sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)}$$

$$+\underline{xaP(1,n-3)} + \underline{axP(1,n-3)} + \underline{x^2P(2,n-4)} - r_2\underline{a^n}$$

$$= \sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)} + \underline{axP(1,i-3)}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)} + \sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)} - r_2\underline{a^n}.$$

Post multiplying this by ax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^2x^{n-2}ax \to \sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)ax} + \qquad (\beta)$$

$$\sum_{i=2}^{n-1} r_i\underline{a^2P(0,i-2)ax} - r_2\underline{axa^n}$$

Similarly, post multiplying the relation for $-a^2x^{n-2}$ above with xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^2x^{n-2}xa \to \sum_{i=3}^{n} r_i(xaP(1,i-3)xa + \underline{axP(1,i-3)xa}) + \sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)xa} + \qquad (\tau)$$

$$\sum_{i=2}^{n-1} r_i a^2 P(0,i-2)xa - r_2\underline{xa^{n+1}}.$$

To reduce $xa^2x^{n-2}a$, note first that using (A.3), the left hand side of σ_2 is written as

$$a^2x^{n-2} \to -(\sum_{i=2}^{n-1} \underline{aP(0,i-2)a} + \sum_{i=3}^{n-1} r_i(\underline{xP(1,i-3)a} + \underline{aP(1,i-3)x}) + \sum_{i=4}^{n-1} \underline{xP(2,i-4)x}$$

$$+\underline{xP(1,n-3)a} + \underline{aQ(1,n-3)x} + \underline{xP(2,n-4)x} + \underline{aP(0,n-2)a}) + r_2\underline{a^n}.$$

$$= -(\sum_{i=2}^{n} \underline{aP(0,i-2)a} + \sum_{i=3}^{n-1} r_i(\underline{xP(1,i-3)a} + \underline{aP(1,i-3)x}) + \sum_{i=4}^{n} \underline{xP(2,i-4)x}$$

$$+\underline{xP(1,n-3)a} + \underline{aQ(1,n-3)x}) + r_2\underline{a^n}.$$

Pre and post multiplying this by x and a respectively and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$xa^2x^{n-2}a \to -(\sum_{i=2}^{n} \underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i(\underline{x^2P(1,i-3)a^2} + \underline{xaP(1,i-3)xa})+ \qquad (\gamma)$$

$$\sum_{i=4}^{n} \underline{x^2P(2,i-4)xa} + \underline{x^2P(1,n-3)a^2} + \underline{xaQ(1,n-3)xa}) + r_2\underline{xa^{n+1}}.$$

Thus, the reduction process ends here. Substituting (β), (τ) and (γ) into (α) and simplifying yields

$$a^2\omega_1 \to (\sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax} + \underline{axP(1,n-3)xa})+ \qquad (\chi)$$

$$\sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)ax} + r_1\underline{a^{n+2}}) - (\sum_{i=2}^{n} r_i\underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i a^2P(i,i-3)x^2$$

$$+\sum_{i=3}^{n} r_i\underline{x^2P(1,i-3)a^2} + \underline{a^2Q(1,n-3)x^2} + r_1\underline{a^3} + r_2\underline{axa^n}).$$

Turning now to ω_3x^2, using (A.2) to write the right hand side of σ_3 yields

$$\omega_3 \to -(\sum_{i=3}^{n-1} r_i\underline{a^2P(1,i-3)} + \underline{a^2Q(1,n-3)} + \sum_{i=4}^{n-1} r_i(\underline{xaP(2,i-4)} + \underline{axP(2,i-4)})+$$

$$\underline{xaP(2,n-4)} + \underline{axP(2,n-4)} + \sum_{i=5}^{n-1} r_i\underline{x^2P(3,i-5)} + \underline{x^2P(3,n-5)}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i\underline{a^2P(1,i-3)} + \underline{a^2Q(1,n-3)} + \sum_{i=4}^{n} r_i(\underline{xaP(2,i-4)} + \underline{axP(2,i-4)})+$$

$$\sum_{i=5}^{n} r_i\underline{x^2P(3,i-5)}) + r_3\underline{a^n}.$$

When we post multiply this by x^2 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, this yields

$$\omega_3 x^2 \to -(\sum_{i=3}^{n-1} r_i a^2 P(1,i-3)x^2 + \underline{a^2 Q(1,n-3)x^2} + \sum_{i=4}^{n} r_i (xaP(2,i-4)x^2 \qquad (\alpha')$$

$$+axP(2,i-4)x^2) + \sum_{i=5}^{n} r_i x^2 P(3,i-5)x^2) + r_3 \underline{x^2 a^n}.$$

We get the following reducible words of length $n+2$ from (α'):

$$xaa^2x^{n-2}, \quad axa^2x^{n-2}, \quad x^2a^3x^{n-3}$$

from $xaP(2,n-4)x^2$, $axP(2,n-4)x^2$ and $x^2P(3,n-5)x^2$ respectively. Using (A.1), the right hand side of σ_2 is written as

$$a^2x^{n-2} \to -(\sum_{i=2}^{n-1} r_i \underline{P(0,i-2)a^2} + \sum_{i=3}^{n-1} r_i (\underline{P(1,i-3)ax} + \underline{P(1,i-3)xa}) + \sum_{i=4}^{n-1} r_i \underline{P(2,i-4)x^2}$$

$$+\underline{Q(2,n-4)x^2} + \underline{P(0,n-2)a^2} + \underline{P(1,n-3)ax} + \underline{P(1,n-3)xa}) + r_2 \underline{a^n}$$

$$= -(\sum_{i=2}^{n} r_i \underline{P(0,i-2)a^2} + \sum_{i=3}^{n} r_i (\underline{P(1,i-3)ax} + \underline{P(1,i-3)xa}) + \sum_{i=4}^{n-1} r_i \underline{P(2,i-4)x^2}$$

$$+\underline{Q(2,n-4)x^2}) + r_2 \underline{a^n}.$$

Pre multiplying this by xa and ax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-xaa^2x^{n-2} \to (\sum_{i=2}^{n} r_i \underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n} r_i (\underline{xaP(1,i-3)ax} + xaP(1,i-3)xa) + \qquad (\beta')$$

$$\sum_{i=4}^{n-1} r_i xaP(2,i-4)x^2 + \underline{xaQ(2,n-4)x^2}) - r_2 \underline{xa^{n+1}}$$

and

$$-axa^2x^{n-2} \to (\sum_{i=2}^{n} r_i axP(0,i-2)a^2 + \sum_{i=3}^{n} r_i (\underline{axP(1,i-3)ax} + \underline{axP(1,i-3)xa}) + \qquad (\gamma')$$

$$\sum_{i=4}^{n-1} r_i \underline{axP(2,i-4)x^2} + \underline{axQ(2,n-4)x^2}) - r_2 \underline{axa^n}$$

respectively. Recall from (γ) above that the reducible word $xa^2x^{n-2}a$ is given by,

$$xa^2x^{n-2}a \to -(\sum_{i=3}^{n-1} r_i (\underline{x^2P(1,i-3)a^2} + \underline{xaP(1,i-3)xa}) + \sum_{i=4}^{n} \underline{x^2P(2,i-4)xa} + \qquad (\eta')$$

$$\sum_{i=2}^{n} \underline{xaP(0,i-2)a^2} + \underline{x^2P(1,n-3)a^2} + \underline{xaQ(1,n-3)xa}) + r_2\underline{xa^{n+1}}.$$

Turning now to the reducible word $ax^{n-1}a^2$ in (γ'), using (A.1), the right hand side of σ_1 is written as

$$ax^{n-1} \to -(\sum_{i=2}^{n-1} r_i(\underline{axP(0,i-2)} + \underline{xaP(0,i-2)}) + \underline{xaP(0,n-2)} + \sum_{i=3}^{n-1} r_i\underline{x^2P(1,i-3)} +$$

$$\underline{x^2P(1,n-3)} + r_1\underline{a}) + r_1\underline{a^n}$$

so that

$$ax^{n-1}a^2 \to -(\sum_{i=2}^{n-1} r_i(\underline{axP(0,i-2)a^2} + \underline{xaP(0,i-2)a^2}) + \underline{xaP(0,n-2)a^2} + \qquad (\tau')$$

$$\sum_{i=3}^{n} r_i\underline{x^2P(1,i-3)a^2} + r_1\underline{a^3}) + r_1\underline{a^{n+2}}.$$

Using (A.1), the right hand side of σ_3 becomes,

$$a^3x^{n-3} \to -(\sum_{i=4}^{n-1} r_i\underline{P(1,i-3)a^2} + \sum_{i=4}^{n-1} r_i(\underline{P(2,i-4)ax} + \underline{P(2,i-4)xa}) + \sum_{i=5}^{n-1} r_i\underline{P(3,i-5)x^2}$$

$$+\underline{Q(3,n-5)x^2} + \underline{P(1,n-3)a^2} + \underline{P(2,n-4)ax} + \underline{P(2,n-4)xa} + r_3\underline{a^3}) + r_3\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{P(1,i-3)a^2} + \sum_{i=4}^{n} r_i(\underline{P(2,i-4)ax} + \underline{P(2,i-4)xa}) + \sum_{i=5}^{n-1} r_i\underline{P(3,i-5)x^2}$$

$$+\underline{Q(3,n-5)x^2} + r_3\underline{a^3}) + r_3\underline{a^n}.$$

Premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-x^2a^3x^{n-3} \to (\sum_{i=4}^{n} r_i\underline{x^2P(1,i-3)a^2} + \sum_{i=4}^{n} r_i(\underline{x^2P(2,i-4)ax} + \underline{x^2P(2,i-4)xa}) + \qquad (\chi')$$

$$\sum_{i=5}^{n-1} r_i\underline{x^2P(3,i-5)x^2} + \underline{x^2Q(3,n-5)x^2} + r_3\underline{x^2a^3}) + r_3\underline{x^2a^n}.$$

Thus, the reduction process ends here. Substituting (β'), (γ'), $(\eta'$, (τ') and (χ') into (α') and simplifying yields

$$\omega_3x^2 \to (\sum_{i=3}^{n} r_i(\underline{xaP(1,i-3)ax} + \underline{axP(1,i-3)ax} + \underline{axP(1,n-3)xa}) + \qquad (\alpha')$$

$$\sum_{i=4}^{n} r_i\underline{x^2P(2,i-4)ax} + r_1\underline{a^{n+2}}) - (\sum_{i=2}^{n} r_i\underline{xaP(0,i-2)a^2} + \sum_{i=3}^{n-1} r_ia^2P(i,i-3)x^2$$

$$+\sum_{i=3}^{n} r_i \underline{x^2P(1,i-3)a^2} + \underline{a^2Q(1,n-3)x^2} + r_1\underline{a^3} + r_2\underline{axa^n}).$$

Comparing (α) and (α'), we conclude that the overlap ambiguity $\{\omega_1, \omega_3\}$ is resolvable.

(b) Let $j = n-3$, and consider the overlap ambiguity $\{\omega_{n-3}, \omega_{n-1}\}$. We may assume without loss of generality that $n \geq 5$ since we have dealt with $1 = 4-3$ in Proposition 2.8(i). Using (A.1), we write the right hand side of σ_{n-3} as

$$\omega_{n-3} \to -(\sum_{i=n-3}^{n-1} r_i \underline{P(n-5,i-(n-3))a^2} + \sum_{i=n-2}^{n-1} r_i(\underline{P(n-4,i-(n-2))xa}+$$

$$\underline{P(n-4,i-(n-2))ax}) + r_{n-1}\underline{P(n-3,0)x^2} + \underline{Q(n-3,1)x^2} + \underline{P(n-5,3)a^2}+$$

$$\underline{P(n-4,2)xa} + \underline{P(n-4,2)ax}) + r_{n-3}\underline{a^n}$$

$$= -(\sum_{i=n-3}^{n} r_i \underline{P(n-5,i-(n-3))a^2} + \sum_{i=n-2}^{n} r_i(\underline{P(n-4,i-(n-2))xa}+$$

$$\underline{P(n-4,i-(n-2))ax}) + r_{n-1}\underline{P(n-3,0)x^2} + \underline{Q(n-3,1)x^2}) + r_{n-3}\underline{a^n}$$

Thus, premultiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$a^2\omega_{n-3} \to -(\sum_{i=n-3}^{n} r_i a^2P(n-5,i-(n-3))a^2 + \sum_{i=n-2}^{n} r_i(a^2P(n-4,i-(n-2))xa+ \qquad (I)$$

$$a^2P(n-4,i-(n-2))ax) + r_{n-1}a^2P(n-3,0)x^2 + \underline{a^2Q(n-3,1)x^2}) + r_{n-3}\underline{a^{n+2}},$$

The reducible words of length $n+2$ above are :

$$a^{n-3}x^3a^2, \qquad a^{n-2}x^2ax, \qquad a^{n-2}x^2xa,$$

from $a^2P(n-5,3)a^2$, $a^2P(n-4,2)ax$ and $a^2P(n-4,2)xa$ respectively.

Using (A.2), we write the right hand side of σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-5,i-(n-3))} + \sum_{i=n-2}^{n-1} r_i(\underline{axP(n-4,i-(n-2))}+$$

$$\underline{xaP(n-4,i-(n-2))}) + r_{n-1}\underline{x^2P(n-3,0)} + \underline{a^2Q(n-5,3)} + \underline{axP(n-4,2)}+$$

$$\underline{xaP(n-4,2)} + \underline{x^2P(n-3,1)}) + r_{n-3}\underline{a^n}$$

$$= -(\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-5,i-(n-3))} + \sum_{i=n-2}^{n} r_i(\underline{axP(n-4,i-(n-2))}+$$

$$\underline{xaP(n-4,i-(n-2))}) + \sum_{i=n-1}^{n} r_i \underline{x^2P(n-3,i-(n-1))} + \underline{a^2Q(n-5,3)}) + r_{n-3}\underline{a^n}.$$

Post multiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^{n-3}x^3a^2 \to (\sum_{i=n-3}^{n-1} r_i\underline{a^2P(n-5,i-(n-3))a^2} + \sum_{i=n-2}^{n} r_i(\underline{axP(n-4,i-(n-2))a^2}+ \quad (II)$$

$$\underline{xaP(n-4,i-(n-2))a^2}) + \sum_{i=n-1}^{n} r_i\underline{x^2P(n-3,i-(n-1))a^2} + \underline{a^2Q(n-5,3)a^2}) - r_{n-3}\underline{a^{n+2}}.$$

Similarly, using (A.2), the right hand side of σ_{n-2} becomes

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-4,i-(n-2))} + r_{n-1}(\underline{xaP(n-3,i-(n-1))}+$$

$$\underline{axP(n-3,i-(n-1))}) + \underline{xaP(n-3,1)} + \underline{axP(n-3,1)} + \underline{x^2P(n-2,0)}+$$

$$\underline{a^2Q(n-4,2)}) + r_{n-2}\underline{a^n}.$$

Thus, post multiplying this by ax and xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-a^{n-2}x^2ax \to (\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-4,i-(n-2))ax} + \sum_{i=n-1}^{n} r_i(\underline{xaP(n-3,i-(n-1))ax} \quad (III)$$

$$+\underline{axP(n-3,i-(n-1))ax}) + x^2P(n-2,0)ax + \underline{a^2Q(n-4,2ax)}) - r_{n-2}\underline{axa^n}.$$

and

$$-a^{n-2}x^2xa \to (\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-4,i-(n-2))xa} + \sum_{i=n-1}^{n} r_i(xaP(n-3,i-(n-1))xa \quad (IV)$$

$$+\underline{axP(n-3,i-(n-1))xa}) + \underline{x^2P(n-2,0)xa} + \underline{a^2Q(n-4,2)xa}) - r_{n-2}\underline{xa^{n+1}}$$

respectively. Turning now to the reducible word $x^2a^{n-1}x$, using (A.1), the right hand side of σ_{n-1} becomes

$$a^{n-1}x \to -\left(\underline{P(n-2,0)xa} + \underline{P(n-3,1)a^2} + r_{n-1}\underline{a^{n-1}}\right) + r_{n-1}\underline{a^n}.$$

Premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$x^2a^{n-1}x \to -\left(\underline{x^2P(n-2,0)xa} + \underline{x^2P(n-3,1)a^2} + r_{n-1}\underline{x^2a^{n-1}}\right) + r_{n-1}\underline{x^2a^n}. \quad (V)$$

Returning to (IV), the reducible word $xa^{n-2}x^2a \in xaP(n-3,1)xa$ is reduced as follows.

We use (A.3) to write the right hand side of σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i \underline{aP(n-4,i-(n-2))a} + r_{n-1}(\underline{aP(n-3,0)x} + \underline{xP(n-3,0)a})$$

$$+\underline{xP(n-2,0)x} + \underline{aQ(n-3,1)x} + \underline{aP(n-4,2)a} + \underline{xP(n-3,1)a}) + r_{n-2}\underline{a^n}.$$

$$= -(\sum_{i=n-2}^{n} r_i \underline{aP(n-4,i-(n-2))a} + r_{n-1}\underline{aP(n-3,0)x} + \sum_{i=n-1}^{n} r_i \underline{xP(n-3,i-(n-1))a}$$

$$+\underline{xP(n-2,0)x} + \underline{aQ(n-3,1)x}) + r_{n-2}\underline{a^n}.$$

Therefore, pre and post multiplying this by x and a respectively and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$xa^{n-2}x^2a \to -(\sum_{i=n-2}^{n} r_i \underline{xaP(n-4,i-(n-2))a^2} + r_{n-1}\underline{xaP(n-3,0)xa} + \qquad (VI)$$

$$\sum_{i=n-1}^{n} r_i \underline{x^2P(n-3,i-(n-1))a^2} + \underline{x^2P(n-2,0)xa} + \underline{xaQ(n-3,1)xa}) + r_{n-2}\underline{xa^{n+1}}.$$

Thus, the reduction process stops here and we get the following after assembling (I), (II), (III), (IV), (V), (VI) and simplifying yields:

$$a^2\omega_{n-3} \to (\sum_{i=n-1}^{n} r_i(\underline{xaP(n-3,i-(n-1))ax} + \underline{axP(n-3,i-(n-1))ax} + \qquad (\Gamma)$$

$$\underline{axP(n-3,i-(n-1))xa}) + \sum_{i=n-2}^{n} r_i \underline{axP(n-4,i-(n-2))a^2} + r_{n-1}\underline{x^2a^n}) - (r_{n-2}\underline{axa^n} +$$

$$r_{n-1}a^2P(n-3,0)x^2 + \underline{a^2Q(n-3,1)x^2} + \underline{x^2P(n-2,0)xa} + \sum_{i=n-1}^{n} r_i \underline{x^2P(n-3,i-(n-1))a^2}).$$

Now, consider the alternate grouping of the overlap ambiguity, given by $\omega_{n-1}x^2 = (a^{n-1}x)x^2$. Using (A.2), the words on the right hand side of σ_{n-1} are written as

$$\omega_{n-1} \to -\left(\underline{a^2Q(n-3,1)} + \underline{axP(n-2,0)} + \underline{xaP(n-2,0)} + r_{n-1}\underline{a^{n-1}}\right) + r_{n-1}\underline{a^n}.$$

Postmultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\omega_{n-1}x^2 \to -\left(\underline{a^2Q(n-3,1)x^2} + axP(n-2,0)x^2 + xaP(n-2,0)x^2 + r_{n-1}a^{n-1}x^2\right) \qquad (a)$$

$$+r_{n-1}\underline{x^2a^n}.$$

The reducible words of length $n+2$ above are

$$axa^{n-2}x^2 \qquad \text{and} \qquad xaa^{n-2}x^2.$$

Using (A.1), the right hand side of σ_{n-2} becomes

$$a^{n-2}x^2 \to -(\sum_{i=n-1}^{n} r_i(\underline{P(n-4,i-(n-2))a^2} + \underline{P(n-3,i-(n-1))xa}$$

$$+\underline{P(n-3,i-(n-1))ax} + r_{n-2}\underline{a^{n-2}}) + r_{n-2}\underline{a^n}.$$

Premultiplying this by ax and xa gives

$$-axa^{n-2}x^2 \to (\sum_{i=n-1}^{n} r_i(\underline{axP(n-4,i-(n-2))a^2} + \underline{axP(n-3,i-(n-1))xa}+ \qquad (b)$$

$$\underline{axP(n-3,i-(n-1))ax}) + r_{n-2}\underline{axa^{n-2}}) - r_{n-2}\underline{axa^n}$$

and

$$-xaa^{n-2}x^2 \to (\sum_{i=n-1}^{n} r_i(\underline{xaP(n-4,i-(n-2))a^2} + xaP(n-3,i-(n-1))xa+ \qquad (c)$$

$$\underline{xaP(n-3,i-(n-1))ax}) + r_{n-2}\underline{xa^{n-1}}) + r_{n-2}\underline{xa^{n+1}}$$

respectively. The reducible word $xa^{n-2}x^2a$ is given by (VI). Thus, the reduction process ends here and assembling (a), (b), (c) and (VI), we obtain

$$\omega_{n-1}x^2 \to (\sum_{i=n-1}^{n} r_i(\underline{xaP(n-3,i-(n-1))ax} + \underline{axP(n-3,i-(n-1))ax}+ \qquad (\Gamma')$$

$$\underline{axP(n-3,i-(n-1))xa}) + \sum_{i=n-2}^{n} r_i\underline{axP(n-4,i-(n-2))a^2} + r_{n-1}\underline{x^2a^n}) - (r_{n-2}\underline{axa^n}+$$

$$r_{n-1}a^2P(n-3,0)x^2 + \underline{a^2Q(n-3,1)x^2} + \underline{x^2P(n-2,0)xa} + \sum_{i=n-1}^{n} r_i\underline{x^2P(n-3,i-(n-1))a^2}).$$

Therefore, comparing (Γ) and (Γ'), we conclude that the overlap ambiguity $\{\omega_{n-3},\omega_{n-1}\}$ is resolvable.

(c) Suppose now that $1<j<n-3$, so that $n\geq 6$. Consider the overlap ambiguity $\{\omega_j,\omega_{j+2}\}$. We use (A.1) to write the right hand side of σ_j as

$$\omega_j \to -(\sum_{i=j}^{n-1} r_i\underline{P(j-2,i-j)a^2} + \sum_{i=j+1}^{n-1} r_i\left(\underline{P(j-1,i-j-1)ax} + \underline{P(j-1,i-j-1)xa}\right)+$$

$$\sum_{i=j+2}^{n-1}\underline{P(j,i-j-2)x^2Q(j,n-j-2)x^2}+\underline{P(j-2,n-j)a^2}+\underline{P(j-1,n-j-1)ax}$$

$$+\underline{P(j-1,n-j-1)xa})+r_j\underline{a^n}$$

$$=-(\sum_{i=j}^{n}r_i\underline{P(j-2,i-j)a^2}+\sum_{i=j+1}^{n}r_i\left(\underline{P(j-1,i-j-1)ax}+\underline{P(j-1,i-j-1)xa}\right)+$$

$$\sum_{i=j+2}^{n-1}\underline{P(j,i-j-2)x^2}+\underline{Q(j,n-j-2)x^2})+r_j\underline{a^n}$$

Premultiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$a^2\omega_j\to-(\sum_{i=j}^{n}r_i\underline{a^2P(j-2,i-j)a^2}+\sum_{i=j+1}^{n}r_i(a^2P(j-1,i-j-1)ax+\qquad(\mu)$$

$$a^2P(j-1,i-j-1)xa)+\sum_{i=j+2}^{n-1}a^2P(j,i-j-2)x^2+\underline{a^2Q(j,n-j-2)x^2})+r_j\underline{a^{n+2}}$$

We have the following reducible words of length $n+2$ from (μ):

$$a^{j+1}x^{n-j-1}ax,\ a^{j+1}x^{n-j-1}xa,\ a^jx^{n-j}a^2\qquad(M)$$

from $a^2P(j-1,n-j-1)ax$, $a^2P(j-1,n-j-1)xa$ and $a^2P(j-2,n-j)a^2$ respectively. The word $a^{j+1}x^{n-j-1}xa$ is $(a^{j+1}x^{n-j-1})xa$ and $a(a^jx^{n-j})a$. So it involves an overlap ambiguity, but for $t=1$ and this overlap ambiguity has been resolved in (3) and (4). Thus, we using the route $(a^{j+1}x^{n-j-1})xa$ will lead to the same result.

We treat the reducible words in (M) in turn, starting with the following: $a^{j+1}x^{n-j-1}ax$ and $a^{j+1}x^{n-j-1}xa$. Expand σ_{j+1} using $(A.2)$ as

$$a^{j+1}x^{n-j-1}\to-(\sum_{i=j+1}^{n-1}\underline{a^2P(j-1,i-j-1)}+\underline{a^2Q(j-1,n-j-1)}+$$

$$\sum_{i=j+2}^{n-1}r_i\left(\underline{axP(j,i-j-2)}+\underline{xaP(j,i-j-2)}\right)+\underline{axP(j,n-j-2)}+\underline{xaP(j,n-j-2)}+$$

$$\sum_{i=j+3}^{n-1}\underline{x^2P(j+1,i-j-3)}+\underline{x^2P(j+1,n-j-3)})+r_{j+1}\underline{a^n}.$$

$$=-(\sum_{i=j+1}^{n-1}\underline{a^2P(j-1,i-j-1)}+\underline{a^2Q(j-1,n-j-1)}+$$

$$\sum_{i=j+2}^{n}r_i\left(\underline{axP(j,i-j-2)}+\underline{xaP(j,i-j-2)}\right)+\sum_{i=j+3}^{n}\underline{x^2P(j+1,i-j-3)})+r_{j+1}\underline{a^n}.$$

Thus, post multiplying this by ax and xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$-a^{j+1}x^{n-j-1}ax \to (\sum_{i=j+1}^{n-1} \underline{a^2P(j-1,i-j-1)ax} + \underline{a^2Q(j-1,n-j-1)ax} + \tag{ω}$$

$$\sum_{i=j+2}^{n} r_i \Big(\underline{axP(j,i-j-2)ax} + \underline{xaP(j,i-j-2)ax}\Big) + \sum_{i=j+3}^{n} \underline{x^2P(j+1,i-j-3)ax}) - r_{j+1}\underline{axa^n}$$

and

$$-a^{j+1}x^{n-j-1}xa \to (\sum_{i=j+1}^{n-1} \underline{a^2P(j-1,i-j-1)xa} + \underline{a^2Q(j-1,n-j-1)xa} + \tag{ζ}$$

$$\sum_{i=j+2}^{n} r_i \Big(\underline{axP(j,i-j-2)xa} + \underline{xaP(j,i-j-2)xa}\Big) + \sum_{i=j+3}^{n} \underline{x^2P(j+1,i-j-3)xa}) - r_{j+1}\underline{xa^{n+1}}$$

respectively. Turning now to the reduction of $xa^{j+1}x^{n-j-1}a$ in (ζ), use (A.3) to expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i\underline{aP(j-1,i-j-1)a} + \underline{aP(j-1,n-j-1)a} + \sum_{i=j+2}^{n-1} r_i\underline{aP(j,i-j-2)x} +$$

$$\underline{aQ(j,n-j-2)x} + \sum_{i=j+2}^{n-1} r_i\underline{xP(j,i-j-2)a} + \underline{xP(j,n-j-2)a} + \sum_{i=j+3}^{n} r_i\underline{xP(j+1,i-j-3)x}$$

$$+\underline{xP(j+1,n-j-3)x}) + r_{j+1}\underline{a^n}$$

$$= -(\sum_{i=j+1}^{n} r_i\underline{aP(j-1,i-j-1)a} + \sum_{i=j+2}^{n-1} r_i\underline{aP(j,i-j-2)x} + \underline{aQ(j,n-j-2)x}$$

$$+ \sum_{i=j+2}^{n} r_i\underline{xP(j,i-j-2)a} + \sum_{i=j+3}^{n} r_i\underline{xP(j+1,i-j-3)x}) + r_{j+1}\underline{a^n}.$$

Pre and post multiplying this by x and a respectively and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$xa^{j+1}x^{n-j-1}a \to -(\sum_{i=j+1}^{n} r_i\underline{xaP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n-1} r_i\underline{xaP(j,i-j-2)xa} + \tag{N}$$

$$\underline{xaQ(j,n-j-2)xa} + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i\underline{x^2P(j+1,i-j-3)xa}) + r_{j+1}\underline{xa^{n+1}}.$$

Turning now to the reducible word $a^jx^{n-j}a^2$, the third reducible term in (M), use (A.2) to expand σ_j as

$$a^jx^{n-j} \to -(\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)} + \underline{a^2Q(j-2,n-j)} + \sum_{i=j+1}^{n-1} r_i(\underline{axP(j-1,i-j-1)} +$$

$$\underline{xaP(j-1,i-j-1)}) + \underline{axP(j-1,n-j-1)} + \underline{xaP(j-1,n-j-1)} +$$

$$\sum_{i=j+2}^{n-1} r_i\underline{x^2P(j,i-j-2)} + \underline{x^2P(j,n-j-2)}) + r_j\underline{a^n}$$

$$= -(\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)} + \underline{a^2Q(j-2,n-j)} + \sum_{i=j+1}^{n} r_i(\underline{axP(j-1,i-j-1)}$$

$$+\underline{xaP(j-1,i-j-1)}) + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)}) + r_j\underline{a^n}.$$

Thus, post multiplying this by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$-a^jx^{n-j}a^2 \to (\sum_{i=j}^{n-1} r_i\underline{a^2P(j-2,i-j)a^2} + \underline{a^2Q(j-2,n-j)a^2} + \sum_{i=j+1}^{n} r_i(\underline{axP(j-1,i-j-1)a^2}$$

$$+\underline{xaP(j-1,i-j-1)a^2}) + \sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)a^2}) - r_j\underline{a^{n+2}}. \qquad (P)$$

Hence, the reduction process ends here and assembling (μ), (M), (ω), (ζ), (N), and (P), we obtain

$$a^2\omega_j \to (\sum_{i=j+1}^{n} r_i\underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i(\underline{xaP(j,i-j-2)ax} + \underline{axP(j,i-j-2)ax} \qquad (Q)$$

$$+\underline{axP(j,i-j-2)xa}) + \sum_{i=j+3}^{n} r_i\underline{x^2P(j+1,i-j-3)ax}) - (\sum_{i=j+2}^{n-1} r_ia^2P(j,i-j-3)x^2 +$$

$$\underline{a^2Q(j,n-j-2)x^2} + r_{j+1}\underline{axa^n})$$

Turning now to the other half of the overlap ambiguity, use (A.2) to expand σ_{j+2} as

$$\omega_{j+2} \to -(\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j,i-j-2)} + \underline{a^2Q(j,n-j-2)} + \sum_{i=j+3}^{n-1} r_i(\underline{axP(j+1,i-j-3)} +$$

$$\underline{xaP(j+1,i-j-3)}) + \underline{axP(j+1,n-j-3)} + \underline{xaP(j+1,n-j-3)} +$$

$$\sum_{i=j+4}^{n-1} r_i\underline{x^2P(j+2,i-j-4)} + \underline{x^2P(j+2,n-j-4)}) + \underline{r_{j+2}a^n}$$

$$= -(\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j,i-j-2)} + \underline{a^2Q(j,n-j-2)} + \sum_{i=j+3}^{n} r_i(\underline{axP(j+1,i-j-3)} +$$

$$\underline{xaP(j+1,i-j-3)}) + \sum_{i=j+4}^{n} r_i\underline{x^2P(j+2,i-j-4)}) + \underline{r_{j+2}a^n}.$$

Thus, post multiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$\omega_{j+2}x^2 \to -(\sum_{i=j+2}^{n-1} r_i a^2 P(j,i-j-2)x^2 + \underline{a^2 Q(j,n-j-2)x^2} + \sum_{i=j+3}^{n} r_i (axP(j+1,i-j-3)x^2 \quad (R)$$

$$+xaP(j+1,i-j-3)x^2) + \sum_{i=j+4}^{n} r_i x^2 P(j+2,i-j-4)x^2) + r_{j+2}\underline{a^n x^2}.$$

The reducible words in (R) of length $n+2$ are

$$xaa^{j+1}x^{n-j-1},\ axa^{j+1}x^{n-j-1},\ x^2a^{j+2}x^{n-j-2}$$

which belong to $xaP(j+1,n-j-3)x^2$, $axP(j+1,n-j-3)x^2$ and $x^2P(j+2,n-j-4)x^2$ respectively. The word $xaa^{j+1}x^{n-j-1}$ is $xa(a^{j+1}x^{n-j-1})$ and $x(a^{j+2}x^{n-j-2})x$. So it involves an overlap ambiguity, but for $t=1$ and this overlap ambiguity has been resolved in (3) and (4).

To deal with the first two reducible words, first, use (A.1) to expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i \underline{P(j-1,i-j-1)a^2} + \underline{P(j-1,n-j-1)a^2} + \sum_{i=j+2}^{n-1} r_i (\underline{P(j,i-j-2)ax} +$$

$$\underline{P(j,i-j-2)xa}) + \underline{P(j,n-j-2)ax} + \underline{P(j,n-j-2)xa} + \sum_{i=j+3}^{n-1} \underline{P(j+1,i-j-3)x^2}$$

$$\underline{Q(j+1,n-j-3)x^2}) + r_{j+1}\underline{a^n}$$

$$= -(\sum_{i=j+1}^{n} r_i \underline{P(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i \left(\underline{P(j,n-j-2)ax} + \underline{P(j,n-j-2)xa}\right) +$$

$$\sum_{i=j+3}^{n-1} \underline{P(j+1,i-j-3)x^2} + \underline{Q(j+1,n-j-3)x^2}) + r_{j+1}\underline{a^n}$$

Thus, premultiplying this xa and ax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yield,

$$-xaa^{j+1}x^{n-j-1} \to (\sum_{i=j+1}^{n} r_i \underline{xaP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i (\underline{xaP(j,n-j-2)ax} + \qquad (S)$$

$$xaP(j,n-j-2)xa) + \sum_{i=j+3}^{n-1} xaP(j+1,i-j-3)x^2 + \underline{xaQ(j+1,n-j-3)x^2}) - r_{j+1}\underline{xa^{n+1}},$$

and

$$-axa^{j+1}x^{n-j-1} \to (\sum_{i=j+1}^{n} r_i \underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i (\underline{axP(j,n-j-2)ax} + \qquad (T)$$

$$\underline{axP(j,n-j-2)xa}) + \sum_{i=j+3}^{n-1} \underline{axP(j+1,i-j-3)x^2} + \underline{axQ(j+1,n-j-3)x^2}) - r_{j+1}\underline{axa^n}$$

respectively. The reduction of $xa^{j+1}x^{n-j-1}a$ to irreducible words is given in (N). To reduce $x^2a^{j+2}x^{n-j-2}$, first, use (A.1) to expand σ_{j+2} as

$$a^{j+2}x^{n-j-2} \to -(\sum_{i=j+2}^{n-1} r_i\underline{P(j,i-j-2)a^2} + \underline{P(j,n-j-2)a^2} + \sum_{i=j+3}^{n} r_i(\underline{P(j+1,i-j-3)ax}+$$

$$\underline{P(j+1,i-j-3)xa}) + \underline{P(j+1,n-j-3)ax} + \underline{P(j+1,n-j-3)xa}+$$

$$\sum_{i=j+4}^{n-1} r_i\underline{P(j+2,i-j-4)x^2} + \underline{Q(j,n-j-4)x^2}) + \underline{r_{j+2}a^n}$$

$$= -(\sum_{i=j+2}^{n} r_i\underline{P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i\left(\underline{P(j+1,i-j-3)ax} + \underline{P(j+1,i-j-3)xa}\right) +$$

$$\sum_{i=j+4}^{n-1} r_i\underline{P(j+2,i-j-4)x^2} + \underline{Q(j,n-j-4)x^2}) + r_{j+2}\underline{a^n}.$$

Thus, premultiplying this by x^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words yields,

$$-x^2a^{j+2}x^{n-j-2} \to (\sum_{i=j+2}^{n} r_i\underline{x^2P(j,i-j-2)a^2} + \sum_{i=j+3}^{n} r_i(\underline{x^2P(j+1,i-j-3)ax}+ \qquad (U)$$

$$\underline{x^2P(j+1,i-j-3)xa}) + \sum_{i=j+4}^{n-1} r_i\underline{x^2P(j+2,i-j-4)x^2} + \underline{x^2Q(j,n-j-4)x^2}) + r_{j+2}\underline{x^2a^n}.$$

Thus, the reduction process ends here and when we assemble (R), (S), (T), (N) and (U), we obtain

$$\omega_{j+2}x^2 \to (\sum_{i=j+1}^{n} r_i\underline{axP(j-1,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i(\underline{xaP(j,i-j-2)ax}+ \qquad (V)$$

$$\underline{axP(j,i-j-2)ax} + \underline{axP(j,i-j-2)xa}) + \sum_{i=j+3}^{n} r_i\underline{x^2P(j+1,i-j-3)ax})$$

$$-(\sum_{i=j+2}^{n-1} r_ia^2P(j,i-j-2)x^2 + \underline{a^2Q(j,n-j-2)x^2} + r_{j+1}\underline{axa^n}).$$

Comparing (Q) and (V), we conclude that the overlap ambiguity $\{\omega_j, \omega_{j+2}\}$ is resolvable for all j with $1 < j < n-3$. Thus, Proposition 2.8(ii) follows from cases (a), (b) and (c).

(iii) We get an ambiguity from the two routes to resolve the word

$$a^3\omega_j = a^3(a^jx^{n-j}) = (a^{j+3}x^{n-j-3})x^3 = \omega_{j+3}x^3$$

in the free algebra $k\langle a,x\rangle$ using the relations σ_j and σ_{j+3} for $1 \le j \le n-4$. Throughout the proofs, we use the following identities:

$$P(r,s) = P(r-3,s)a^3 + P(r-2,s-1)(a^2x+axa+xa^2) + P(r-1,s-2)(ax^2+xax+x^2a) \quad \text{(A.4)}$$
$$+P(r,s-3)x^3,$$

$$P(r,s) = a^3P(r-3,s) + (a^2x+axa+xa^2)P(r-2,s-1) + (ax^2+xax+x^2a)P(r-1,s-2) \quad \text{(A.5)}$$
$$+x^3P(r,s-3),$$

$$P(r,s) = \left(x^2P(r-1,s-2) + axP(r-2,s-1) + xaP(r-2,s-1) + a^2P(r-3,s)\right)a +$$
$$\left(x^2P(r,s-3) + axP(r-1,s-2) + xaP(r-1,s-2) + a^2P(r-2,s-1)\right)x \quad \text{(A.6)}$$

$$P(r,s) = a\left(P(r-1,s-2)x^2 + P(r-2,s-1)ax + P(r-2,s-1)xa + P(r-3,s)a^2\right) +$$
$$x\left(P(r,s-3)x^2 + P(r-1,s-2)ax + P(r-1,s-2)xa + P(r-2,s-1)a^2\right). \quad \text{(A.7)}$$

We resolve the ambiguities in four cases.

(a) Let $j=1$ and consider the overlap ambiguity $\{\omega_1,\omega_4\}$. Use (A.4) to expand σ_1 as

$$\omega_1 \to -(\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)(x^2a+xax+ax^2)} + \underline{P(0,n-3)(x^2a+xax+ax^2)}$$
$$+\underline{Q(1,n-4)x^3} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)x^3} + r_2\underline{P(1,1)} + r_1\underline{a}) + r_1\underline{a^n}$$
$$= -(\sum_{i=3}^{n} r_i\underline{P(0,i-3)(x^2a+xax+ax^2)} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)x^3} + \underline{Q(1,n-4)x^3} + r_2\underline{P(1,1)}$$
$$+r_1\underline{a}) + r_1\underline{a^n}.$$

Thus, premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$a^3\omega_1 \to -(\sum_{i=3}^{n} r_ia^3P(0,i-3)(x^2a+xax+ax^2) + \sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)x^3} + \quad \text{(A.8)}$$
$$\underline{a^3Q(1,n-4)x^3} + r_2\underline{a^3P(1,1)} + r_1\underline{a^4}) + r_1\underline{a^{n+3}}.$$

The following words in $(A.8)$ of length $n+3$ are reducible:

$$a^3x^{n-1}a,\ a^3x^{n-3}ax^2,\ a^3x^{n-2}ax.$$

We first reduce $a^3x^{n-1}a$ as follows. Use (A.5) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)} + \sum_{i=4}^{n-1} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)}+$$

$$\underline{(xa^2+axa+a^2x)P(1,n-4)} + \sum_{i=5}^{n-1} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)}+$$

$$\underline{(x^2a+xax+ax^2)P(2,n-5)} + \sum_{i=6}^{n-1} r_i\underline{x^3P(3,i-6)}) + \underline{x^3P(3,n-6)}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)} + \sum_{i=4}^{n} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)}+$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)}) + r_3\underline{a^n}$$

Post multiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^3x^{n-1}a \to (\sum_{i=3}^{n-1} r_ia^3P(0,i-3)x^2a + \sum_{i=4}^{n} r_i(xa^2+axa+a^2x)P(1,i-4)x^2a+ \tag{A.9}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)x^2a} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)x^2a}) - r_3\underline{x^2a^{n+1}},$$

$$-a^3x^{n-2}ax \to (\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)xax} + \sum_{i=4}^{n} r_i(xa^2+axa+a^2x)P(1,i-4)xax+ \tag{A.10}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)xax} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xax}) - r_3\underline{xaxa^n}$$

and

$$-a^3x^{n-3}ax^2 \to (\sum_{i=3}^{n-1} r_i\underline{a^3P(0,i-3)ax^2} + \sum_{i=4}^{n} r_i\underline{(xa^2+axa+a^2x)P(1,i-4)ax^2}+ \tag{A.11}$$

$$\sum_{i=5}^{n} r_i\underline{(x^2a+xax+ax^2)P(2,i-5)ax^2} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)ax^2}) - r_3\underline{ax^2a^n}$$

respectively. The reducible words in (A.9)-(A.11) of length $n+3$ are as follows:

1. $axa^2x^{n-2}a \in axaP(1,n-4)x^2a$,

2. $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$,

3. $xa^3x^{n-3}xa \in xa^2P(1,n-4)x^2a$,

4. $xa^3x^{n-3}ax \in xa^2P(1,n-4)xax$.

Turning now to reduce $axa^2x^{n-2}a$, use (A.7) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i(\underline{aP(0,i-3)ax} + \underline{aP(0,i-3)xa} + \underline{xP(0,i-3)a^2}) + \underline{aP(0,n-3)ax} +$$

$$\underline{aP(0,n-3)xa} + \underline{xP(0,n-3)a^2} + \sum_{i=4}^{n-1} r_i\underline{aP(1,i-4)x^2} + \underline{aQ(1,n-4)x^2} +$$

$$\sum_{i=4}^{n-1} r_i(\underline{xP(1,i-4)ax} + \underline{xP(1,i-4)xa}) + \underline{xP(1,n-4)ax} + \underline{xP(1,n-4)xa} +$$

$$\sum_{i=5}^{n-1} r_i\underline{xP(2,i-5)x^2} + \underline{xP(2,n-5)x^2} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i(\underline{aP(0,i-3)ax} + \underline{aP(0,i-3)xa} + \underline{xP(0,i-3)a^2}) + \sum_{i=4}^{n-1} r_i\underline{aP(1,i-4)x^2} +$$

$$\underline{aQ(1,n-4)x^2} + \sum_{i=4}^{n} r_i(\underline{xP(1,i-4)ax} + \underline{xP(1,i-4)xa}) + \sum_{i=5}^{n} r_i\underline{xP(2,i-5)x^2} +$$

$$r_2\underline{a^2}) + r_2\underline{a^n}.$$

Pre and postmultiply this by ax and a respectively and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$axa^2x^{n-2}a \to -(\sum_{i=3}^{n} r_i(\underline{axaP(0,i-3)axa} + \underline{axaP(0,i-3)xa^2} + ax^2P(0,i-3)a^3) + \tag{A.12}$$

$$\sum_{i=4}^{n-1} r_i\underline{axaP(1,i-4)x^2a} + \underline{axaQ(1,n-4)x^2a} + \sum_{i=4}^{n} r_i(\underline{ax^2P(1,i-4)axa} +$$

$$\underline{ax^2P(1,i-4)xa^2}) + \sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)x^2a} + r_2\underline{axa^3}) + r_2\underline{axa^{n+1}}.$$

The only reducible word above is $ax^{n-1}a^3 \in ax^2P(0,n-3)a^3$. To reduce $ax^{n-1}a^3$, use (A.5) to expand σ_1 as

$$ax^{n-1} \to -(\sum_{i=4}^{n-1} r_i\underline{x^3P(1,i-4)} + \underline{x^3P(1,n-4)} + \sum_{i=3}^{n-1} r_i\underline{(x^2a + xax + ax^2)P(0,i-3)}$$

$$+r_2\underline{P(1,1)} + r_1\underline{a}) + r_1\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{x^3P(1,i-4)} + \sum_{i=3}^{n-1} r_i\underline{(x^2a + xax + ax^2)P(0,i-3)} + r_2\underline{P(1,1)} + r_1\underline{a}) + r_1\underline{a^n}.$$

Post multiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-ax^{n-1}a^3 \to (\sum_{i=4}^{n} r_i \underline{x^3P(1,i-4)a^3} + \sum_{i=3}^{n-1} r_i \underline{(x^2a+xax+ax^2)P(0,i-3)a^3} \tag{A.13}$$

$$+r_2\underline{P(1,1)a^3} + r_1\underline{a^4}) - r_1\underline{a^{n+3}}.$$

Turning now to the reduction of $xa^3x^{n-3}ax$ and $xa^3x^{n-3}xa$, use (A.6) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{a^2P(0,i-3)a} + \underline{a^2P(0,n-3)a} + \sum_{i=4}^{n-1} r_i\underline{a^2P(1,i-4)x} + \underline{a^2Q(1,n-4)x}$$

$$+\sum_{i=4}^{n-1} r_i(\underline{axP(1,i-4)a} + \underline{xaP(1,i-4)a}) + \underline{axP(1,n-4)a} + \underline{xaP(1,n-4)a}+$$

$$\sum_{i=5}^{n-1} r_i(\underline{x^2P(2,i-5)a} + \underline{xaP(2,i-5)x} + \underline{axP(2,i-5)x}) + \underline{x^2P(2,n-5)a}+$$

$$\underline{xaP(2,n-5)x} + \underline{axP(2,n-5)x} + \sum_{i=6}^{n-1} r_i\underline{x^2P(3,i-6)x} + \underline{x^2P(3,n-6)x}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i\underline{a^2P(0,i-3)a} + \sum_{i=4}^{n-1} r_i\underline{a^2P(1,i-4)x} + \underline{a^2Q(1,n-4)x} + \sum_{i=4}^{n} r_i(\underline{axP(1,i-4)a}$$

$$+\underline{xaP(1,i-4)a}) + \sum_{i=5}^{n} r_i(\underline{x^2P(2,i-5)a} + \underline{xaP(2,i-5)x} + \underline{axP(2,i-5)x})+$$

$$\sum_{i=6}^{n} r_i\underline{x^2P(3,i-6)x}) + r_3\underline{a^n}.$$

Thus, premultiplying this by x and post multiplying by ax and xa and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$xa^3x^{n-3}ax \to -(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^2x} + \sum_{i=4}^{n-1} r_i\underline{xa^2P(1,i-4)xax} + \underline{xa^2Q(1,n-4)xax}+ \tag{A.14}$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^2x} + \underline{x^2aP(1,i-4)a^2x}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^2x} + \underline{x^2aP(2,i-5)xax}$$

$$+\underline{xaxP(2,i-5)xax}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xax}) + r_3\underline{xaxa^n}.$$

and

$$xa^3x^{n-3}xa \to -(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)axa} + \sum_{i=4}^{n-1} r_i\underline{xa^2P(1,i-4)x^2a} + \underline{xa^2Q(1,n-4)x^2a}+ \tag{A.15}$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)axa} + \underline{x^2aP(1,i-4)axa}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)axa} + x^2aP(2,i-5)x^2a$$

$$+\underline{xaxP(2,i-5)x^2a}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)x^2a}) + r_3\underline{x^2a^{n+1}}$$

respectively. All the words in (A.14) and (A.15) are irreducible except $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$. But $x^2a^3x^{n-3}a$ appears with opposite sign to the same reducible word in (A.9) so they cancel out. Thus, the reduction process ends here and substituting (A.9), (A.10), (A.11), (A.12), (A.13), (A.14), (A.15) into (A.8) and simplifying gives

$$a^3\omega_1 \to (r_2\underline{xa^4} + r_2\underline{axa^{n+1}} + \sum_{i=3}^{n-1} r_i\underline{(x^2a+xax)P(0,i-3)a^3} + \sum_{i=4}^{n} r_i(\underline{x^3P(1,i-4)a^3}+ \tag{A.16}$$

$$\underline{a^2xP(1,i-4)(ax^2+x^2a+xax)} + \underline{axaP(1,i-4)(ax^2+xax)} + \underline{xa^2P(1,i-4)ax^2})+$$

$$\sum_{i=5}^{n} r_i(\underline{(ax^2+x^2a+xax)P(2,i-5)ax^2} + \underline{ax^2P(2,i-5)xax}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)ax^2}$$

$$-(r_2\underline{a^3P(1,1)} + r_3\underline{ax^2a^n} + \sum_{i=3}^{n} r_i(\underline{xa^2P(0,i-3)(a^2x+axa)} + \underline{axaP(0,i-3)(axa+xa^2)})+$$

$$\sum_{i=4}^{n} r_i(\underline{(xax+ax^2+x^2a)P(1,i-4)axa} + \underline{(xax+x^2a)P(1,i-4)a^2x} + \underline{ax^2P(1,i-4)xa^2})$$

$$\sum_{i=4}^{n-1} r_ia^3P(1,i-4)x^3 + \underline{a^3Q(1,n-4)x^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)(a^2x+axa)}.$$

Moving now to ω_4x^3, use (A.5) to write σ_4 as

$$\omega_4 \to -(\sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(2,i-5)}+$$

$$\underline{(a^2x+axa+xa^2)P(2,n-5)} + \sum_{i=6}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)}+$$

$$\underline{(ax^2+xax+x^2a)P(3,n-6)} + \sum_{i=7}^{n-1} r_i\underline{x^3P(4,i-7)} + \underline{x^3P(4,n-7)}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n-1} r_i\underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n} r_i\underline{(a^2x+axa+xa^2)P(2,i-5)}$$

$$+\sum_{i=6}^{n} r_i\underline{(ax^2+xax+x^2a)P(3,i-6)} + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)}) + r_4\underline{a^n}.$$

Post multiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\omega_4x^3 \to -(\sum_{i=4}^{n-1} r_ia^3P(1,i-4)x^3 + \underline{a^3Q(1,n-4)x^3} + \sum_{i=5}^{n} r_i(a^2x+axa+xa^2)P(2,i-5)x^3 \tag{A.17}$$

$$+\sum_{i=6}^{n} r_i(ax^2+xax+x^2a)P(3,i-6)x^3 + \sum_{i=7}^{n} r_ix^3P(4,i-7)x^3) + r_4\underline{x^3a^n}.$$

We get the following reducible words of length $n+3$ from (A.17):

1. $xa^2(a^2x^{n-2}) \in xa^2P(2,n-5)x^3$,
2. $axa(a^2x^{n-2}) \in axaP(2,n-5)x^3$,
3. $a^2x(a^2x^{n-2}) \in a^2xP(2,n-5)x^3$,
4. $x^2a(a^3x^{n-3}) \in x^2aP(3,n-6)x^3$,
5. $ax^2(a^3x^{n-3}) \in ax^2P(3,n-6)x^3$,
6. $xax(a^3x^{n-3}) \in xaxP(3,n-6)x^3$,
7. $x^3(a^4x^{n-4}) \in x^3P(4,n-7)x^3$.

Use (A.4) to expand σ_2 as

$$\begin{aligned} a^2x^{n-2} \to -(&\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)(xa^2+axa+a^2x)} + \underline{P(0,n-3)(xa^2+axa+a^2x)}+ \\ &\sum_{i=4}^{n-1} r_i\underline{P(1,i-4)(ax^2+xax+x^2a)} + \underline{P(1,n-4)(ax^2+xax+x^2a)}+ \\ &\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}. \\ = -(&\sum_{i=3}^{n} r_i\underline{P(0,i-3)(xa^2+axa+a^2x)} + \sum_{i=4}^{n} r_i\underline{P(1,i-4)(ax^2+xax+x^2a)}+ \\ &\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}. \end{aligned}$$

Thus, premultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$\begin{aligned} -xa^2(a^2x^{n-2}) \to (&\sum_{i=3}^{n} r_ixa^2P(0,i-3)(xa^2+axa+a^2x) + \sum_{i=4}^{n} r_ixa^2P(1,i-4)(ax^2+xax+x^2a) \\ &+ \sum_{i=5}^{n-1} r_ixa^2P(2,i-5)x^3 + \underline{xa^2Q(2,n-5)x^3} + r_2\underline{xa^4}) - r_2\underline{xa^{n+2}}, \end{aligned} \tag{A.18}$$

$$\begin{aligned} -axa(a^2x^{n-2}) \to (&\sum_{i=3}^{n} r_i\underline{axaP(0,i-3)(xa^2+axa+a^2x)} + \sum_{i=4}^{n} r_iaxaP(1,i-4)(ax^2+xax+x^2a) \\ &+ \sum_{i=5}^{n-1} r_i\underline{axaP(2,i-5)x^3} + \underline{axaQ(2,n-5)x^3} + r_2\underline{axa^3}) - r_2\underline{axa^{n+1}}, \end{aligned} \tag{A.19}$$

and

$$-a^2x(a^2x^{n-2}) \to (\sum_{i=3}^{n} r_ia^2xP(0,i-3)(xa^2+axa+a^2x)+ \tag{A.20}$$

$$\sum_{i=4}^{n} r_i \underline{a^2xP(1,i-4)(ax^2+xax+x^2a)} + \sum_{i=5}^{n-1} r_i \underline{a^2xP(2,i-5)x^3} + \underline{a^2xQ(2,n-5)x^3}$$

$$+r_2\underline{a^2xa^2}) - r_2\underline{a^2xa^n}$$

respectively. Also, use (A.4) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i \underline{P(0,i-3)a^3} + \underline{P(0,n-3)a^3} + \sum_{i=4}^{n-1} r_i \underline{P(1,i-4)(a^2x+axa+xa^2)}+$$

$$\underline{P(1,n-4)(a^2x+axa+xa^2)} + \sum_{i=5}^{n-1} r_i \underline{P(2,i-5)(ax^2+xax+x^2a)}+$$

$$\underline{P(2,n-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i \underline{P(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{P(1,i-4)(a^2x+axa+xa^2)}+$$

$$\sum_{i=5}^{n} r_i \underline{P(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}) + r_3\underline{a^n}.$$

Premultiplying this by x^2a, ax^2 and xax and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-x^2a(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i \underline{x^2aP(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{x^2aP(1,i-4)(a^2x+axa+xa^2)}+ \tag{A.21}$$

$$\sum_{i=5}^{n} r_i x^2aP(2,i-5)(ax^2+xax+x^2a) + \sum_{i=6}^{n-1} r_i \underline{x^2aP(3,i-6)x^3} + \underline{x^2aQ(3,n-6)x^3})$$

$$-r_3\underline{x^2a^{n+1}},$$

$$-ax^2(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i ax^2P(0,i-3)a^3 + \sum_{i=4}^{n} r_i \underline{ax^2P(1,i-4)(a^2x+axa+xa^2)}+ \tag{A.22}$$

$$+\sum_{i=5}^{n} r_i \underline{ax^2P(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{ax^2P(3,i-6)x^3} + \underline{ax^2Q(3,n-6)x^3})$$

$$-r_3\underline{ax^2a^n},$$

and

$$-xax(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i \underline{xaxP(0,i-3)a^3} + \sum_{i=4}^{n} r_i \underline{xaxP(1,i-4)(a^2x+axa+xa^2)}+ \tag{A.23}$$

$$+\sum_{i=5}^{n} r_i \underline{xaxP(2,i-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i \underline{xaxP(3,i-6)x^3} + \underline{xaxQ(3,n-6)x^3})$$

$$-r_3\underline{xaxa^n}$$

respectively. Moreover, use (A.4) to expand σ_4 as

$$a^4x^{n-4} \to -(\sum_{i=4}^{n-1} r_i \underline{P(1,i-4)a^3} + \underline{P(1,n-4)a^3} + \sum_{i=5}^{n-1} r_i \underline{P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\underline{P(2,n-5)(a^2x+axa+xa^2)} + \sum_{i=6}^{n-1} r_i \underline{P(3,i-6)(ax^2+xax+x^2a)}$$

$$\underline{P(3,n-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i \underline{P(1,i-4)a^3} + \sum_{i=5}^{n} r_i \underline{P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\sum_{i=6}^{n} r_i \underline{P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}.$$

Premultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-x^3(a^4x^{n-4}) \to (\sum_{i=4}^{n} r_i \underline{x^3P(1,i-4)a^3} + \sum_{i=5}^{n} r_i \underline{x^3P(2,i-5)(a^2x+axa+xa^2)}+ \tag{A.24}$$

$$\sum_{i=6}^{n} r_i \underline{x^3P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i \underline{x^3P(4,i-7)x^3} + \underline{x^3Q(4,n-7)x^3}) - r_4\underline{x^3a^n}.$$

We get the following reducible words of length $n+3$ from (A.18)-(A.24):

1. $xa^3x^{n-2}a \in xa^2P(1,n-4)x^2a$,
2. $xa^3x^{n-3}ax \in xa^2P(1,n-4)xax$,
3. $axa^2x^{n-2}a \in axaP(1,n-4)x^2a$,
4. $a^2x^{n-1}a^2 \in a^2xa^2x^{n-2}$,
5. $a^2x^{n-2}a^2x \in a^2xa^2x^{n-2}$,
6. $a^2x^{n-2}axa \in a^2xa^2x^{n-2}$,
7. $x^2a^3x^{n-3}a \in x^2aP(2,n-5)x^2a$,
8. $ax^{n-1}a^3 \in ax^2a^3x^{n-3}$,
9. $xa^2x^{n-2}a^2 \in xa^2a^2x^{n-2}$.

Recall from (A.15) that

$$xa^3x^{n-3}xa \to -(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)axa} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)x^2a} + \underline{xa^2Q(1,n-4)x^2a}+$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)axa} + \underline{x^2aP(1,i-4)axa}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)axa} +$$

$$x^2aP(2,i-5)x^2a + \underline{xaxP(2,i-5)x^2a}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)x^2a}) + r_3\underline{x^2a^{n+1}}.$$

All the words in (A.15) except $-x^2a^3x^{n-3}a \in -x^2aP(2,n-5)x^2a$. Thus this term cancels out with the (vii) term in the list above because they appear with opposite signs.

Similarly, recall from (A.14) that

$$xa^3x^{n-3}ax \to -(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^2x} + \sum_{i=4}^{n-1} r_i\underline{xa^2P(1,i-4)xax} + \underline{xa^2Q(1,n-4)xax} +$$

$$\sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^2x} + \underline{x^2aP(1,i-4)a^2x}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^2x} +$$

$$\underline{x^2aP(2,i-5)xax} + \underline{xaxP(2,i-5)xax}) + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)xax}) + r_3\underline{xaxa^n}.$$

Furthermore, use (A.5) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} + \sum_{i=3}^{n-1} r_i\underline{(xa^2+axa)P(0,i-3)} + \underline{(xa^2+axa)P(0,n-3)}$$

$$+\sum_{i=4}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \underline{(ax^2+xax+x^2a)P(1,n-4)} +$$

$$\sum_{i=5}^{n-1} r_i\underline{x^3P(2,i-5)} + \underline{x^3P(2,n-5)} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)} + \sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)} + \sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)}$$

$$+\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

Post multiply this by a^2x, xa^2 and axa and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$a^2x^{n-2}a^2x \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)a^2x} + \sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)a^2x} + \tag{A.25}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^2x} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^2x} + r_2\underline{a^4x}) + r_2\underline{a^2xa^n},$$

$$a^2x^{n-2}xa^2 \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)xa^2} + \sum_{i=3}^{n} r_i(xa^2+axa)P(0,i-3)xa^2 + \tag{A.26}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)xa^2}+\underline{x^3P(2,n-5)xa^2}+xa^2x^{n-2}a^2+\underline{axax^{n-2}a^2})$$

and

$$a^2x^{n-2}axa \to -(\sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)axa}+\sum_{i=3}^{n} r_i\underline{(xa^2+axa)P(0,i-3)axa}+ \tag{A.27}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)axa}+\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)axa}+r_2\underline{a^3xa})+r_2\underline{axa^{n+1}}$$

respectively. The reducible word $(-xa^2x^{n-2}a^2)$ in (A.26) cancels out with $xa^2x^{n-2}a^2$ which is the (ix) term in the list of reducible words above.

Recall from (A.12) that

$$axa^2x^{n-2}a \to -(\sum_{i=3}^{n} r_i(\underline{axaP(0,i-3)axa}+\underline{axaP(0,i-3)xa^2}+ax^2P(0,i-3)a^3)+$$

$$\sum_{i=4}^{n-1} r_i\underline{axaP(1,i-4)x^2a}+\underline{axaQ(1,n-4)x^2a}+\sum_{i=4}^{n} r_i(\underline{ax^2P(1,i-4)axa}+$$

$$\underline{ax^2P(1,i-4)xa^2})+\sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)x^2a}+r_2\underline{axa^3})+r_2\underline{axa^{n+1}}.$$

All the words in (A.12) are irreducible except $-ax^2P(0,i-3)a^3$ which can be written as $-ax^{n-1}a^3$. But this cancels out with $ax^{n-1}a^3$ which is the term (h) in the list of reducible words above.

Thus, the reduction process ends here and when we substitute (A.12), (A.14), (A.15), (A.18), (A.19), (A.20), (A.21), (A.22), (A.23), (A.24), (A.25), (A.26) and (A.27) into (A.17), we get

$$\omega_4x^3 \to (r_2\underline{xa^4}+r_2\underline{axa^{n+1}}+\sum_{i=3}^{n-1} r_i\underline{(x^2a+xax)P(0,i-3)a^3}+\sum_{i=4}^{n} r_i(\underline{x^3P(1,i-4)a^3}+ \tag{A.28}$$

$$\underline{a^2xP(1,i-4)(ax^2+x^2a+xax)}+\underline{axaP(1,i-4)(ax^2+xax)}+\underline{xa^2P(1,i-4)ax^2})+$$

$$\sum_{i=5}^{n} r_i(\underline{(ax^2+x^2a+xax)P(2,i-5)ax^2}+\underline{ax^2P(2,i-5)xax})+\sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)ax^2}$$

$$-(r_2\underline{a^3P(1,1)}+r_3\underline{ax^2a^n}+\sum_{i=3}^{n} r_i(\underline{xa^2P(0,i-3)(a^2x+axa)}+\underline{axaP(0,i-3)(axa+xa^2)})+$$

$$\sum_{i=4}^{n} r_i(\underline{(xax+ax^2+x^2a)P(1,i-4)axa}+\underline{(xax+x^2a)P(1,i-4)a^2x}+\underline{ax^2P(1,i-4)xa^2})$$

$$\sum_{i=4}^{n-1} r_ia^3P(1,i-4)x^3+\underline{a^3Q(1,n-4)x^3}+\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)(a^2x+axa)}.$$

Comparing (A.16) and (A.28), we conclude that the overlap ambiguity $\{\omega_1,\omega_4\}$ is resolvable.

(b) Let $j = 2$ and consider the overlap ambiguity $\{\omega_2, \omega_5\}$. Use (A.4) to expand σ_2 as

$$\omega_2 \to -(\sum_{i=3}^{n-1} r_i \underline{P(0,i-3)(a^2x + axa + xa^2)} + \underline{P(0,n-3)(a^2x + axa + xa^2)}$$

$$\sum_{i=4}^{n-1} r_i \underline{P(1,i-4)(ax^2 + xax + x^2a)} + \underline{P(1,n-4)(ax^2 + xax + x^2a)} +$$

$$\sum_{i=5}^{n-1} r_i \underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i \underline{P(0,i-3)(a^2x + axa + xa^2)} + \sum_{i=4}^{n} r_i \underline{P(1,i-4)(ax^2 + xax + x^2a)} +$$

$$\sum_{i=5}^{n-1} r_i \underline{P(2,i-5)x^3} + \underline{Q(2,n-5)x^3} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

Premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$a^3\omega_2 \to -(\sum_{i=3}^{n} r_i a^3 P(0,i-3)(a^2x + axa + xa^2) + \sum_{i=4}^{n} r_i a^3 P(1,i-4)(ax^2 + xax + x^2a) + \quad \text{(A.29)}$$

$$\sum_{i=5}^{n-1} r_i a^3 P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3} + r_2\underline{a^5}) + r_2\underline{a^{n+3}}.$$

The following words of length $n+3$ in (A.29) are reducible:

1. $(a^4x^{n-4})x^2a \in a^3P(1,n-4)x^2a$
2. $(a^4x^{n-4})xax \in a^3P(1,n-4)xax$
3. $(a^4x^{n-4})ax^2 \in a^3P(1,n-4)ax^2$
4. $a^3x^{n-3}axa$
5. $a^3x^{n-3}a^2x$
6. $a^3x^{n-2}a^2$.

Using (A.5), we expand σ_4 as

$$a^4x^{n-4} \to -(\sum_{i=4}^{n-1} r_i \underline{a^3P(1,i-4)} + \underline{a^3Q(1,n-4)} + \sum_{i=5}^{n-1} r_i \underline{(a^2x + axa + xa^2)P(2,i-5)} +$$

$$\underline{(a^2x + axa + xa^2)P(2,n-5)} + \sum_{i=6}^{n-1} r_i \underline{(ax^2 + xax + x^2a)P(3,i-6)} +$$

$$\underline{(ax^2+xax+x^2a)P(3,n-6)}+\sum_{i=7}^{n-1}r_i\underline{x^3P(4,i-7)}+\underline{x^3P(4,n-7)})+r_4\underline{a^n}$$

$$=-(\sum_{i=4}^{n-1}r_i\underline{a^3P(1,i-4)}+\underline{a^3Q(1,n-4)}+\sum_{i=5}^{n}r_i\underline{(a^2x+axa+xa^2)P(2,i-5)}+$$

$$\sum_{i=6}^{n}r_i\underline{(ax^2+xax+x^2a)P(3,i-6)}+\sum_{i=7}^{n-1}r_i\underline{x^3P(4,i-7)})+r_4\underline{a^n}.$$

Thus, post multiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-(a^4x^{n-4})x^2a\to(\sum_{i=4}^{n-1}r_ia^3P(1,i-4)x^2a+\sum_{i=5}^{n}r_i(a^2x+axa+xa^2)P(2,i-5)x^2a+ \tag{A.30}$$

$$\sum_{i=6}^{n}r_i\underline{(ax^2+xax+x^2a)P(3,i-6)x^2a}+\sum_{i=7}^{n}r_i\underline{x^3P(4,i-7)x^2a}+\underline{a^3Q(1,n-4)x^2a})$$

$$-r_4\underline{x^2a^{n+1}},$$

$$-(a^4x^{n-4})xax\to(\sum_{i=4}^{n-1}r_i\underline{a^3P(1,i-4)xax}+\sum_{i=5}^{n}r_i(a^2x+axa+xa^2)P(2,i-5)xax+ \tag{A.31}$$

$$\sum_{i=6}^{n}r_i\underline{(ax^2+xax+x^2a)P(3,i-6)xax}+\sum_{i=7}^{n}r_i\underline{x^3P(4,i-7)xax}+\underline{a^3Q(1,n-4)xax})$$

$$-r_4\underline{xaxa^n},$$

and

$$-(a^4x^{n-4})ax^2\to(\sum_{i=4}^{n-1}r_i\underline{a^3P(1,i-4)ax^2}+\sum_{i=5}^{n}r_i\underline{(a^2x+axa+xa^2)P(2,i-5)ax^2}+ \tag{A.32}$$

$$\sum_{i=6}^{n}r_i\underline{(ax^2+xax+x^2a)P(3,i-6)ax^2}+\sum_{i=7}^{n}r_i\underline{x^3P(4,i-7)ax^2}+\underline{a^3Q(1,n-4)ax^2})$$

$$-r_4\underline{ax^2a^n}$$

respectively. Similarly, use (A.5) to expand σ_3 as

$$a^3x^{n-3}\to-(\sum_{i=3}^{n-1}r_i\underline{a^3P(0,i-3)}+\sum_{i=4}^{n-1}r_i\underline{(a^2x+axa+xa^2)P(1,i-4)}+$$

$$\underline{(a^2x+axa+xa^2)P(1,n-4)}+\sum_{i=5}^{n-1}r_i(\underline{(ax^2+xax+x^2a)P(2,i-5)})+$$

$$\underline{(ax^2+xax+x^2a)P(2,n-5)}+\sum_{i=6}^{n}r_i\underline{x^3P(3,i-6)})+r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n-1} r_i \underline{a^3 P(0,i-3)} + \sum_{i=4}^{n} r_i \underline{(a^2x + axa + xa^2)P(1,i-4)}$$

$$+ \sum_{i=5}^{n} r_i (\underline{(ax^2 + xax + x^2a)P(2,i-5)}) + \sum_{i=6}^{n} r_i \underline{x^3 P(3,i-6)}) + r_3 \underline{a^n}.$$

Post multiply this by axa, a^2x and xa^2, and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$-a^3x^{n-3}axa \to (\sum_{i=3}^{n-1} r_i \underline{a^3 P(0,i-3)axa} + \sum_{i=4}^{n} r_i \underline{(a^2x + axa + xa^2)P(1,i-4)axa} \tag{A.33}$$

$$+ \sum_{i=5}^{n} r_i (\underline{(ax^2 + xax + x^2a)P(2,i-5)axa}) + \sum_{i=6}^{n} r_i \underline{x^3 P(3,i-6)axa}) - r_3 \underline{axa^{n+1}},$$

$$-a^3x^{n-3}a^2x \to (\sum_{i=3}^{n-1} r_i \underline{a^3 P(0,i-3)a^2x} + \sum_{i=4}^{n} r_i \underline{(a^2x + axa + xa^2)P(1,i-4)a^2x} \tag{A.34}$$

$$+ \sum_{i=5}^{n} r_i (\underline{(ax^2 + xax + x^2a)P(2,i-5)a^2x}) + \sum_{i=6}^{n} r_i \underline{x^3 P(3,i-6)a^2x}) - r_3 \underline{a^2xa^n}$$

and

$$-a^3x^{n-3}xa^2 \to (\sum_{i=3}^{n-1} r_i \underline{a^3 P(0,i-3)xa^2} + \sum_{i=4}^{n} r_i (a^2x + axa + xa^2)P(1,i-4)xa^2 \tag{A.35}$$

$$+ \sum_{i=5}^{n} r_i (\underline{(ax^2 + xax + x^2a)P(2,i-5)xa^2}) + \sum_{i=6}^{n} r_i \underline{x^3 P(3,i-6)xa^2}) - r_3 \underline{xa^{n+2}}.$$

The following words of length $n+3$ from (A.30), (A.31), (A.32), (A.33), (A.34) and (A.35) are reducible:

1. $x^2a^4x^{n-4}a \in x^2aP(3,n-6)x^2a$
2. $x(a^4x^{n-4})xa \in xa^2P(2,n-5)x^2a$
3. $axa^3x^{n-3}a \in axaP(2,n-5)x^2a$
4. $xa^4x^{n-4}ax \in xa^2P(2,n-5)xax$
5. $xa^3x^{n-3}a^2 \in xa^2P(1,n-4)xa^2$.

Use (A.6) to expand σ_4 as

$$a^4x^{n-4} \to -(\sum_{i=4}^{n-1} r_i \underline{a^2P(1,i-4)a} + \underline{a^2P(1,n-4)a} + \sum_{i=5}^{n-1} r_i \underline{a^2P(2,i-5)x} + \underline{a^2Q(2,n-5)x} +$$

$$\sum_{i=5}^{n-1} r_i (\underline{axP(2,i-5)a} + \underline{xaP(2,i-5)a}) + \underline{axP(2,n-5)a} + \underline{xaP(2,n-5)a} +$$

$$\sum_{i=6}^{n-1} r_i(\underline{x^2P(3,i-6)a} + \underline{axP(3,i-6)x} + \underline{xaP(3,i-6)x}) + \underline{x^2P(3,n-6)a} +$$

$$\underline{axP(3,n-6)x} + \underline{xaP(3,n-6)x} + \sum_{i=7}^{n-1} r_i\underline{x^2P(4,i-7)x} + \underline{x^2P(4,n-7)x}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{a^2P(1,i-4)a} + \sum_{i=5}^{n-1} r_i\underline{a^2P(2,i-5)x} + \sum_{i=5}^{n} r_i(\underline{axP(2,i-5)a} + \underline{xaP(2,i-5)a})$$

$$+\sum_{i=6}^{n} r_i(\underline{x^2P(3,i-6)a} + \underline{axP(3,i-6)x} + \underline{xaP(3,i-6)x}) + \sum_{i=7}^{n} r_i\underline{x^2P(4,i-7)x}$$

$$+\underline{a^2Q(2,n-5)x}) + r_4\underline{a^n}.$$

Thus, premultiplying this by x and post multiplying this by xa and ax, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$xa^4x^{n-4}xa \to -(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)axa} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)x^2a} + \underline{xa^2Q(2,n-5)x^2a} \quad \text{(A.36)}$$

$$+\sum_{i=5}^{n} r_i(\underline{xaxP(2,i-5)axa} + \underline{x^2aP(2,i-5)axa}) + \sum_{i=6}^{n} r_i(\underline{x^3P(3,i-6)axa} +$$

$$\underline{xaxP(3,i-6)x^2a} + x^2aP(3,i-6)x^2a) + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)x^2a}) + r_4\underline{x^2a^{n+1}}$$

and

$$xa^4x^{n-4}ax \to -(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)a^2x} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)xax} + \underline{xa^2Q(2,n-5)xax} \quad \text{(A.37)}$$

$$+\sum_{i=5}^{n} r_i(\underline{xaxP(2,i-5)a^2x} + \underline{x^2aP(2,i-5)a^2x}) + \sum_{i=6}^{n} r_i(\underline{x^3P(3,i-6)a^2x} +$$

$$\underline{xaxP(3,i-6)xax} + \underline{x^2aP(3,i-6)xax}) + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)xax}) + r_4\underline{xaxa^n}$$

respectively. The reducible word $-x^2a^4x^{n-4}a$ in (A.36) has an opposite sign with $x^2a^4x^{n-4}a$, the term (i) in the list above so they cancel out.

Again, use (A.7) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{aP(0,i-3)a^2} + \underline{aP(0,n-3)a^2} + \sum_{i=4}^{n-1} r_i(\underline{aP(1,i-4)ax} + \underline{aP(1,i-4)xa}$$

$$+\underline{xP(1,i-4)a^2}) + \underline{aP(1,n-4)ax} + \underline{aP(1,n-4)xa} + \underline{xP(1,n-4)a^2} +$$

$$\sum_{i=5}^{n-1} r_i\underline{aP(2,i-5)x^2} + \underline{aQ(2,n-5)x^2} + \sum_{i=5}^{n-1} r_i(\underline{xP(2,i-5)ax} + \underline{xP(2,i-5)xa}) +$$

$$\underline{xP(2,n-5)ax} + \underline{xP(2,n-5)xa} + \sum_{i=6}^{n-1} r_i \underline{xP(3,i-6)x^2} + \underline{xP(3,n-6)x^2}) + r_3 \underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i \underline{aP(0,i-3)a^2} + \sum_{i=4}^{n} r_i(\underline{aP(1,i-4)ax} + \underline{aP(1,i-4)xa} + \underline{xP(1,i-4)a^2})+$$

$$\sum_{i=5}^{n-1} r_i \underline{aP(2,i-5)x^2} + \sum_{i=5}^{n} r_i(\underline{xP(2,i-5)ax} + \underline{xP(2,i-5)xa}) + \sum_{i=6}^{n} r_i \underline{xP(3,i-6)x^2}$$

$$+\underline{aQ(2,n-5)x^2}) + r_3 \underline{a^n}.$$

Pre and post multiplying this by ax and a respectively, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$axa^3x^{n-3}a \to -(\sum_{i=3}^{n} r_i \underline{axaP(0,i-3)a^3} + \sum_{i=4}^{n} r_i(\underline{axaP(1,i-4)axa} + \underline{axaP(1,i-4)xa^2} \quad \text{(A.38)}$$

$$+\underline{ax^2P(1,i-4)a^3}) + \sum_{i=5}^{n-1} r_i \underline{axaP(2,i-5)x^2a} + \sum_{i=5}^{n} r_i(\underline{ax^2P(2,i-5)axa} +$$

$$\underline{ax^2P(2,i-5)xa^2}) + \sum_{i=6}^{n} r_i \underline{ax^2P(3,i-6)x^2a} + \underline{axaQ(2,n-5)x^2a}) + r_3 \underline{axa^{n+1}}.$$

Turning now to the reduction of $xa^3x^{n-3}a^2$, use (A.6) to expand σ_3 as

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i \underline{a^2P(0,i-3)a} + \underline{a^2P(0,n-3)a} + \sum_{i=4}^{n-1} r_i \underline{a^2P(1,i-4)x} + \underline{a^2Q(1,n-4)x} +$$

$$\sum_{i=4}^{n-1} r_i(\underline{axP(1,i-4)a} + \underline{xaP(1,i-4)a}) + \underline{axP(1,n-4)a} + \underline{xaP(1,n-4)a}$$

$$+ \sum_{i=5}^{n-1} r_i(\underline{x^2P(2,i-5)a} + \underline{axP(2,i-5)x} + \underline{xaP(2,i-5)x})+$$

$$\underline{x^2P(2,n-5)a} + \underline{axP(2,n-5)x} + \underline{xaP(2,n-5)x}) + r_3 \underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i \underline{a^2P(0,i-3)a} + \sum_{i=4}^{n-1} r_i \underline{a^2P(1,i-4)x} + \sum_{i=4}^{n} r_i(\underline{axP(1,i-4)a} + \underline{xaP(1,i-4)a})$$

$$+ \sum_{i=5}^{n} r_i(\underline{x^2P(2,i-5)a} + \underline{axP(2,i-5)x} + \underline{xaP(2,i-5)x}) + \underline{a^2Q(1,n-4)x}) + r_3 \underline{a^n}$$

Pre and post multiplying this by x and a^2 respectively, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$xa^3x^{n-3}a^2 \to -(\sum_{i=3}^{n} r_i \underline{xa^2P(0,i-3)a^3} + \sum_{i=4}^{n-1} r_i \underline{xa^2P(1,i-4)xa^2} + \sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^3} \quad \text{(A.39)}$$

$$+\underline{x^2aP(1,i-4)a^3}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^3} + \underline{xaxP(2,i-5)xa^2} + \underline{x^2aP(2,i-5)xa^2}$$

$$+\underline{xa^2Q(1,n-4)xa^2}) + r_3\underline{xa^{n+2}}.$$

Thus, the reduction process ends here and when we substitute (A.30), (A.31), (A.32), (A.33), (A.34), (A.35), (A.36), (A.37), (A.38) and (A.39) into (A.29), we get

$$a^3\omega_2 \to (r_2\underline{a^{n+3}} + \sum_{i=4}^{n} r_i(\underline{a^2xP(1,i-4)(a^2x+axa+xa^2)} + \underline{axaP(1,i-4)a^2x}) + \tag{A.40}$$

$$\sum_{i=5}^{n} r_i(\underline{a^2xP(2,i-5)(ax^2+xax+x^2a)} + \underline{axaP(2,i-5)(ax^2+xax)} + \underline{xa^2P(2,i-5)ax^2}$$

$$+\underline{ax^2P(2,i-5)a^2x}) + \sum_{i=6}^{n} r_i(\underline{ax^2P(3,i-6)xax} + \underline{(ax^2+xax+x^2a)P(3,i-6)ax^2}) +$$

$$\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)ax^2} - (r_2\underline{a^5} + r_3\underline{a^2xa^n} + r_4\underline{ax^2a^n} + \sum_{i=3}^{n} r_i(\underline{axa+xa^2)P(0,i-3)a^3} +$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3}$$

$$+\sum_{i=5}^{n-1} r_ia^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3}).$$

Turning now to the reduction of ω_5x^3, use (A.5) to expand σ_5 as

$$\omega_5 \to -(\sum_{i=5}^{n-1} r_i\underline{a^3P(2,i-5)} + \underline{a^3Q(2,n-5)} + \sum_{i=6}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(3,i-6)} +$$

$$\underline{(a^2x+axa+xa^2)P(3,n-6)} + \sum_{i=7}^{n-1} r_i\underline{(ax^2+xax+xa^2)P(4,i-7)} +$$

$$\underline{(ax^2+xax+xa^2)P(4,n-7)} + \sum_{i=8}^{n-1} r_i\underline{x^3P(5,i-8)} + \underline{x^3P(5,n-8)}) + r_5\underline{a^n}$$

$$= -(\sum_{i=5}^{n-1} r_i\underline{a^3P(2,i-5)} + \underline{a^3Q(2,n-5)} + \sum_{i=6}^{n} r_i\underline{(a^2x+axa+xa^2)P(3,i-6)} +$$

$$\sum_{i=7}^{n} r_i\underline{(ax^2+xax+xa^2)P(4,i-7)} + \sum_{i=8}^{n} r_i\underline{x^3P(5,i-8)}) + r_5\underline{a^n}.$$

Post multiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\omega_5x^3 \to -(\sum_{i=5}^{n-1} r_ia^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3} + \sum_{i=6}^{n} r_i(a^2x+axa+xa^2)P(3,i-6)x^3 + \tag{A.41}$$

$$\sum_{i=7}^{n} r_i(ax^2+xax+xa^2)P(4,i-7)x^3 + \sum_{i=8}^{n} r_ix^3P(5,i-8)x^3) + r_5\underline{a^{n+3}}.$$

The following words in (A.41) of length $n+3$ are reducible:

1. $x^3a^5x^{n-5} \in x^3P(5,n-8)x^3$

2. $x^2a(a^4x^{n-4}) \in x^2aP(4,n-7)x^3$

3. $xaxa^4x^{n-4} \in xaxP(4,n-7)x^3$

4. $ax^2a^4x^{n-4} \in ax^2P(4,n-7)x^3$

5. $xa^2(a^3x^{n-3}) \in xa^2P(3,n-6)x^3$

6. $a^2xa^3x^{n-3} \in a^2xP(3,n-6)x^3$

7. $axa(a^3x^{n-3}) \in axaP(3,n-6)x^3$.

In order to reduce (b), (c) and (d) in the list above, use (A.4) to expand σ_4 as

$$a^4x^{n-4} \to -(\sum_{i=4}^{n-1} r_i\underline{P(1,i-4)a^3} + \underline{P(1,n-4)a^3} + \sum_{i=5}^{n-1} r_i\underline{P(2,i-5)(a^2x+axa+xa^2)}$$

$$+\underline{P(2,n-5)(a^2x+axa+xa^2)} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)(ax^2+xax+x^2a)}+$$

$$\underline{P(3,n-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}$$

$$= -(\sum_{i=4}^{n} r_i\underline{P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{P(2,i-5)(a^2x+axa+xa^2)}+$$

$$\sum_{i=6}^{n} r_i\underline{P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)x^3} + \underline{Q(4,n-7)x^3}) + r_4\underline{a^n}.$$

Premultiply this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-x^2a(a^4x^{n-4}) \to (\sum_{i=4}^{n} r_i\underline{x^2aP(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^2aP(2,i-5)(a^2x+axa+xa^2)} \tag{A.42}$$

$$+\sum_{i=6}^{n} r_ix^2aP(3,i-6)(ax^2+xax+x^2a) + \sum_{i=7}^{n-1} r_i\underline{x^2aP(4,i-7)x^3}+$$

$$\underline{x^2aQ(4,n-7)x^3}) - r_4\underline{x^2a^{n+1}},$$

$$-xax(a^4x^{n-4}) \to (\sum_{i=4}^{n} r_i\underline{xaxP(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{xaxP(2,i-5)(a^2x+axa+xa^2)} \tag{A.43}$$

$$+\sum_{i=6}^{n} r_i\underline{xaxP(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{xaxP(4,i-7)x^3}$$

$$+\underline{xaxQ(4,n-7)x^3}) - r_4\underline{xaxa^n},$$

and

$$-ax^2(a^4x^{n-4}) \to (\sum_{i=4}^{n} r_i\underline{ax^2P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{ax^2P(2,i-5)(a^2x+axa+xa^2)} \qquad \text{(A.44)}$$

$$+\sum_{i=6}^{n} r_i\underline{ax^2P(3,i-6)(ax^2+xax+x^2a)} + \sum_{i=7}^{n-1} r_i\underline{ax^2P(4,i-7)x^3}$$

$$+\underline{ax^2Q(4,n-7)x^3}) - r_4\underline{ax^2a^n}$$

respectively. Similarly, use (A.4) to expand σ_5 as

$$a^5x^{n-5} \to -(\sum_{i=5}^{n-1} r_i\underline{P(2,i-5)a^3} + \underline{P(2,n-5)a^3} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)(a^2x+axa+xa^2)}$$

$$+\underline{P(3,n-6)(a^2x+axa+xa^2)} + \sum_{i=7}^{n-1} r_i\underline{P(4,i-7)(ax^2+xax+ax^2)}+$$

$$\underline{P(4,n-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{P(5,i-8)x^3} + \underline{Q(5,n-8)x^3}) + r_5\underline{a^n}$$

$$= -(\sum_{i=5}^{n} r_i\underline{P(2,i-5)a^3} + \sum_{i=6}^{n} r_i\underline{P(3,i-6)(a^2x+axa+xa^2)}+$$

$$\sum_{i=7}^{n} r_i\underline{P(4,i-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{P(5,i-8)x^3} + \underline{Q(5,n-8)x^3}) + r_5\underline{a^n}.$$

Premultiply this by x^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$-x^3a^5x^{n-5} \to (\sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3} + \sum_{i=6}^{n} r_i\underline{x^3P(3,i-6)(a^2x+axa+xa^2)}+ \qquad \text{(A.45)}$$

$$\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)(ax^2+xax+ax^2)} + \sum_{i=8}^{n-1} r_i\underline{x^3P(5,i-8)x^3}$$

$$+\underline{x^3Q(5,n-8)x^3}) - r_5\underline{x^3a^n}.$$

Similarly, expand σ_3 using (A.4) to get

$$a^3x^{n-3} \to -(\sum_{i=3}^{n-1} r_i\underline{P(0,i-3)a^3} + \underline{P(0,n-3)a^3} + \sum_{i=4}^{n-1} r_i\underline{P(1,i-4)(a^2x+axa+xa^2)}+$$

$$\underline{P(1,n-4)(a^2x+axa+xa^2)} + \sum_{i=5}^{n-1} r_i\underline{P(2,i-5)(ax^2+xax+x^2a)}+$$

$$\underline{P(2,n-5)(ax^2+xax+x^2a)} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)x^3}$$

$$+\underline{Q(3,n-6)x^3}) + r_3\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i\underline{P(0,i-3)a^3} + \sum_{i=4}^{n} r_i\underline{P(1,i-4)(a^2x + axa + xa^2)}+$$

$$\sum_{i=5}^{n} r_i\underline{P(2,i-5)(ax^2 + xax + x^2a)} + \sum_{i=6}^{n-1} r_i\underline{P(3,i-6)x^3} + \underline{Q(3,n-6)x^3}) + r_3\underline{a^n}.$$

Thus, premultiplying this by xa^2, a^2x and axa, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-xa^2(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^3} + \sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)(a^2x + axa + xa^2)}+ \tag{A.46}$$

$$\sum_{i=5}^{n} r_i xa^2P(2,i-5)(ax^2 + xax + x^2a) + \sum_{i=6}^{n-1} r_i\underline{xa^2P(3,i-6)x^3}$$

$$+\underline{xa^2Q(3,n-6)x^3}) - r_3\underline{xa^{n+2}},$$

$$-a^2x(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i a^2xP(0,i-3)a^3 + \sum_{i=4}^{n} r_i\underline{a^2xP(1,i-4)(a^2x + axa + xa^2)}+ \tag{A.47}$$

$$\sum_{i=5}^{n} r_i\underline{a^2xP(2,i-5)(ax^2 + xax + x^2a)} + \sum_{i=6}^{n-1} r_i\underline{a^2xP(3,i-6)x^3}$$

$$+\underline{a^2xQ(3,n-6)x^3}) - r_3\underline{a^2xa^n}$$

and

$$-axa(a^3x^{n-3}) \to (\sum_{i=3}^{n} r_i\underline{axaP(0,i-3)a^3} + \sum_{i=4}^{n} r_i\underline{axaP(1,i-4)(a^2x + axa + xa^2)}+ \tag{A.48}$$

$$\sum_{i=5}^{n} r_i axaP(2,i-5)(ax^2 + xax + x^2a) + \sum_{i=6}^{n-1} r_i\underline{axaP(3,i-6)x^3}$$

$$+\underline{axaQ(3,n-6)x^3}) - r_3\underline{axa^{n+1}}$$

respectively. The following words in (A.42)-(A.48) of length $n+3$ are irreducible:

1. $x^2a^4x^{n-4}a \in x^2aP(3,n-6)x^2a$

2. $x(a^4x^{n-4})xa \in xa^2P(2,n-5)x^2a$

3. $xa^4x^{n-4}ax \in xa^2P(2,n-5)xax$

4. $xa^3x^{n-3}a^2 \in xa^2P(1,n-4)xa^2$

5. $a^2x^{n-2}a^3$

6. $axa^3x^{n-3}a \in axaP(2,n-5)x^2a$

Recall (A.36), (A.37), (A.38) and (A.39) as follows:

$$xa^4x^{n-4}xa \to -(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)axa} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)x^2a} + \underline{xa^2Q(2,n-5)x^2a}$$
$$+\sum_{i=5}^{n} r_i(\underline{xaxP(2,i-5)axa} + \underline{x^2aP(2,i-5)axa}) + \sum_{i=6}^{n} r_i(\underline{x^3P(3,i-6)axa}+$$
$$\underline{xaxP(3,i-6)x^2a} + x^2aP(3,i-6)x^2a) + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)x^2a}) + r_4\underline{x^2a^{n+1}},$$

$$xa^4x^{n-4}ax \to -(\sum_{i=4}^{n} r_i\underline{xa^2P(1,i-4)a^2x} + \sum_{i=5}^{n-1} r_i\underline{xa^2P(2,i-5)xax} + \underline{xa^2Q(2,n-5)xax}$$
$$+\sum_{i=5}^{n} r_i(\underline{xaxP(2,i-5)a^2x} + \underline{x^2aP(2,i-5)a^2x}) + \sum_{i=6}^{n} r_i(\underline{x^3P(3,i-6)a^2x}+$$
$$\underline{xaxP(3,i-6)xax} + \underline{x^2aP(3,i-6)xax}) + \sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)xax}) + r_4\underline{xaxa^n},$$

$$axa^3x^{n-3}a \to -(\sum_{i=3}^{n} r_i\underline{axaP(0,i-3)a^3} + \sum_{i=4}^{n} r_i(\underline{axaP(1,i-4)axa} + \underline{axaP(1,i-4)xa^2}$$
$$+\underline{ax^2P(1,i-4)a^3}) + \sum_{i=5}^{n-1} r_i\underline{axaP(2,i-5)x^2a} + \sum_{i=5}^{n} r_i(\underline{ax^2P(2,i-5)axa}+$$
$$\underline{ax^2P(2,i-5)xa^2}) + \sum_{i=6}^{n} r_i\underline{ax^2P(3,i-6)x^2a} + \underline{axaQ(2,n-5)x^2a}) + r_3\underline{axa^{n+1}},$$

and

$$xa^3x^{n-3}a^2 \to -(\sum_{i=3}^{n} r_i\underline{xa^2P(0,i-3)a^3} + \sum_{i=4}^{n-1} r_i\underline{xa^2P(1,i-4)xa^2} + \sum_{i=4}^{n} r_i(\underline{xaxP(1,i-4)a^3}$$
$$+\underline{x^2aP(1,i-4)a^3}) + \sum_{i=5}^{n} r_i(\underline{x^3P(2,i-5)a^3} + \underline{xaxP(2,i-5)xa^2} + \underline{x^2aP(2,i-5)xa^2}$$
$$+\underline{xa^2Q(1,n-4)xa^2}) + r_3\underline{xa^{n+2}}.$$

In order to reduce $a^2x^{n-2}a^3$, use (A.5) to expand σ_2 as

$$a^2x^{n-2} \to -(\sum_{i=3}^{n-1} r_i\underline{(axa+xa^2)P(0,i-3)} + \underline{(axa+xa^2)P(0,n-3)} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)}$$

$$+\sum_{i=4}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \underline{(ax^2+xax+x^2a)P(1,n-4)}+$$

$$\sum_{i=5}^{n-1} r_i\underline{x^3P(2,i-5)} + \underline{x^3P(2,n-5)} + r_2\underline{a^2}) + r_2\underline{a^n}$$

$$= -(\sum_{i=3}^{n} r_i\underline{(axa+xa^2)P(0,i-3)} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)}+$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)} + r_2\underline{a^2}) + r_2\underline{a^n}.$$

Post multiply this by a^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words to get

$$a^2x^{n-2}a^3 \to -(\sum_{i=3}^{n} r_i\underline{(axa+xa^2)P(0,i-3)a^3} + \sum_{i=3}^{n-1} r_i\underline{a^2xP(0,i-3)a^3}+ \tag{A.49}$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3} + r_2\underline{a^5}) + r_2\underline{a^{n+3}}.$$

The reducible word $-x^2a^4x^{n-4}a$ in (A.36) has an opposite sign with $x^2a^4x^{n-4}a$, the term (i) in the list above so they cancel out. Hence, the reduction process ends here and when we substitute (A.42)-(A.49) and (A.36)-(A.39) into (A.31), yielding

$$\omega_5x^3 \to (r_2\underline{a^{n+3}} + \sum_{i=4}^{n} r_i(\underline{a^2xP(1,i-4)(a^2x+axa+xa^2)} + \underline{axaP(1,i-4)a^2x})+ \tag{A.50}$$

$$\sum_{i=5}^{n} r_i(\underline{a^2xP(2,i-5)(ax^2+xax+x^2a)} + \underline{axaP(2,i-5)(ax^2+xax)} + \underline{xa^2P(2,i-5)ax^2}$$

$$+\underline{ax^2P(2,i-5)a^2x}) + \sum_{i=6}^{n} r_i(\underline{ax^2P(3,i-6)xax} + \underline{(ax^2+xax+x^2a)P(3,i-6)ax^2})+$$

$$\sum_{i=7}^{n} r_i\underline{x^3P(4,i-7)ax^2} - (r_2\underline{a^5} + r_3\underline{a^2xa^n} + r_4\underline{ax^2a^n} + \sum_{i=3}^{n} r_i(\underline{axa+xa^2)P(0,i-3)a^3}+$$

$$\sum_{i=4}^{n} r_i\underline{(ax^2+xax+x^2a)P(1,i-4)a^3} + \sum_{i=5}^{n} r_i\underline{x^3P(2,i-5)a^3}$$

$$+\sum_{i=5}^{n-1} r_ia^3P(2,i-5)x^3 + \underline{a^3Q(2,n-5)x^3}).$$

Comparing (A.40) and (A.50), we conclude that the overlap ambiguity $\{\omega_2, \omega_5\}$ is resolvable.

(c) Let $3 \le j < n-4$ and consider the overlap ambiguity $\{\omega_j, \omega_{j+3}\}$. Using (A.4), expand σ_j to get

$$\omega_j \to -(\sum_{i=j}^{n-1} r_i\underline{P(j-3,i-j)a^3} + \underline{P(j-3,n-j)a^3} + \sum_{i=j+1}^{n-1} r_i\underline{P(j-2,i-j-1)(a^2x+axa+xa^2)}$$

$$+\underline{P(j-2,n-j-1)(a^2x+axa+xa^2)}+\sum_{i=j+2}^{n-1} r_i\underline{P(j-1,i-j-2)(ax^2+xax+x^2a)}+$$

$$\underline{P(j-1,n-j-2)(ax^2+xax+x^2a)}+\sum_{i=j+3}^{n-1} r_i\underline{P(j,i-j-3)x^3}+\underline{Q(j,n-j-3)x^3})+r_j\underline{a^n}$$

$$=-(\sum_{i=j}^{n} r_i\underline{P(j-3,i-j)a^3}+\sum_{i=j+1}^{n} r_i\underline{P(j-2,i-j-1)(a^2x+axa+xa^2)}+$$

$$\sum_{i=j+2}^{n} r_i\underline{P(j-1,i-j-2)(ax^2+xax+x^2a)}+\sum_{i=j+3}^{n-1} r_i\underline{P(j,i-j-3)x^3}+\underline{Q(j,n-j-3)x^3})+r_j\underline{a^n}.$$

Premultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$a^3\omega_j \to -(\sum_{i=j}^{n} r_i a^3P(j-3,i-j)a^3+\sum_{i=j+1}^{n} r_i a^3P(j-2,i-j-1)(a^2x+axa+xa^2) \qquad \text{(A.51)}$$

$$+\sum_{i=j+2}^{n} r_i a^3P(j-1,i-j-2)(ax^2+xax+x^2a)+\sum_{i=j+3}^{n-1} r_i a^3P(j,i-j-3)x^3+$$

$$\underline{a^3Q(j,n-j-3)x^3})+r_j\underline{a^{n+3}}.$$

The following words in (A.51) of length $n+3$ are reducible:

1. $a^jx^{n-j}a^3 \in a^3P(j-3,n-j)a^3$
2. $(a^{j+2}x^{n-j-2})x^2a \in a^3P(j-1,n-j-2)x^2a$
3. $(a^{j+2}x^{n-j-2})xax \in a^3P(j-1,n-j-2)xax$
4. $(a^{j+2}x^{n-j-2})ax^2 \in a^3P(j-1,n-j-2)ax^2$
5. $(a^{j+1}x^{n-j-1})xa^2 \in a^3P(j-2,n-j-1)xa^2$
6. $(a^{j+1}x^{n-j-1})axa \in a^3P(j-2,n-j-1)axa$
7. $(a^{j+1}x^{n-j-1})a^2x \in a^3P(j-2,n-j-1)a^2x.$

Using (A.5), expand σ_j as follows:

$$a^jx^{n-j} \to -(\sum_{i=j}^{n-1} r_i\underline{a^3P(j-3,i-j)}+\underline{a^3Q(j-3,n-j)}+$$

$$\sum_{i=j+1}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(j-2,i-j-1)}+\underline{(a^2x+axa+xa^2)P(j-2,n-j-1)}+$$

$$\sum_{i=j+2}^{n-1} r_i\underline{(ax^2+xax+x^2a)P(j-1,i-j-2)}+\underline{(ax^2+xax+x^2a)P(j-1,n-j-2)}+$$

$$\sum_{i=j+3}^{n-1} r_i \underline{x^3P(j,i-j-3)} + \underline{x^3P(j,n-j-3)}) + r_j\underline{a^n}$$

$$= -(\sum_{i=j}^{n-1} r_i \underline{a^3P(j-3,i-j)} + \underline{a^3Q(j-3,n-j)} + \sum_{i=j+1}^{n} r_i \underline{(a^2x+axa+xa^2)P(j-2,i-j-1)}$$

$$+ \sum_{i=j+2}^{n} r_i \underline{(ax^2+xax+x^2a)P(j-1,i-j-2)} + \sum_{i=j+3}^{n} r_i \underline{x^3P(j,i-j-3)}) + r_j\underline{a^n}.$$

Postmultiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^jx^{n-j}a^3 \to (\sum_{i=j}^{n-1} r_i \underline{a^3P(j-3,i-j)a^3} + \underline{a^3Q(j-3,n-j)a^3} \tag{A.52}$$

$$+ \sum_{i=j+1}^{n} r_i \underline{(a^2x+axa+xa^2)P(j-2,i-j-1)a^3} +$$

$$\sum_{i=j+2}^{n} r_i \underline{(ax^2+xax+x^2a)P(j-1,i-j-2)a^3} + \sum_{i=j+3}^{n} r_i \underline{x^3P(j,i-j-3)a^3}) - r_j\underline{a^{n+3}}.$$

Using (A.5), expand σ_{j+2} as

$$a^{j+2}x^{n-j-2} \to -(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)} + \underline{a^3Q(j-1,n-j-2)} +$$

$$\sum_{i=j+3}^{n-1} r_i \underline{(a^2x+axa+xa^2)P(j,i-j-3)} + \underline{(a^2x+axa+xa^2)P(j,n-j-3)} +$$

$$\sum_{i=j+4}^{n-1} r_i \underline{(ax^2+xax+x^2a)P(j+1,i-j-4)} + \underline{(ax^2+xax+x^2a)P(j+1,n-j-4)} +$$

$$+ \sum_{i=j+5}^{n-1} r_i \underline{x^3P(j+2,i-j-5)} + \underline{x^3P(j+2,n-j-5)}) + r_{j+2}\underline{a^n}$$

$$= -(\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1,i-j-2)} + \underline{a^3Q(j-1,n-j-2)} +$$

$$\sum_{i=j+3}^{n} r_i \underline{(a^2x+axa+xa^2)P(j,i-j-3)} + \sum_{i=j+4}^{n} r_i \underline{(ax^2+xax+x^2a)P(j+1,i-j-4)}$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)}) + r_{j+2}\underline{a^n}$$

Thus, post multiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-(a^{j+2}x^{n-j-2})x^2a \to (\sum_{i=j+2}^{n-1} r_i a^3P(j-1,i-j-2)x^2a + \underline{a^3Q(j-1,n-j-2)x^2a} + \tag{A.53}$$

$$\sum_{i=j+3}^{n} r_i(a^2x + axa + xa^2)P(j, i-j-3)x^2a + \sum_{i=j+4}^{n} r_i(ax^2 + xax + x^2a)P(j+1, i-j-4)x^2a$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2, i-j-5)x^2a}) - r_{j+2}\underline{x^2a^{n+1}},$$

$$-(a^{j+2}x^{n-j-2})xax \to (\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1, i-j-2)xax} + \underline{a^3Q(j-1, n-j-2)xax} + \quad \text{(A.54)}$$

$$\sum_{i=j+3}^{n} r_i(a^2x + axa + xa^2)P(j, i-j-3)xax + \sum_{i=j+4}^{n} r_i \underline{(ax^2 + xax + x^2a)P(j+1, i-j-4)xax}$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2, i-j-5)xax}) - r_{j+2}\underline{xaxa^n}$$

and

$$-(a^{j+2}x^{n-j-2})ax^2 \to (\sum_{i=j+2}^{n-1} r_i \underline{a^3P(j-1, i-j-2)ax^2} + \underline{a^3Q(j-1, n-j-2)ax^2} + \quad \text{(A.55)}$$

$$\sum_{i=j+3}^{n} r_i \underline{(a^2x + axa + xa^2)P(j, i-j-3)ax^2} + \sum_{i=j+4}^{n} r_i \underline{(ax^2 + xax + x^2a)P(j+1, i-j-4)ax^2}$$

$$+ \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2, i-j-5)ax^2}) - r_{j+2}\underline{ax^2a^n}$$

respectively. Again, using (A.5), expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i \underline{a^3P(j-2, i-j-1)} + \underline{a^3Q(j-2, n-j-1)} +$$

$$\sum_{i=j+2}^{n-1} r_i \underline{(a^2x + axa + xa^2)P(j, i-j-3)} + \underline{(a^2x + axa + xa^2)P(j, n-j-3)} +$$

$$\sum_{i=j+3}^{n-1} r_i \underline{(ax^2 + xax + x^2a)P(j, i-j-3)} + \underline{(ax^2 + xax + x^2a)P(j, n-j-3)} +$$

$$\sum_{i=j+4}^{n-1} r_i \underline{x^3P(j+1, i-j-4)} + \underline{x^3P(j+1, n-j-4)}) + r_{j+1}\underline{a^n}.$$

$$= -(\sum_{i=j+1}^{n-1} r_i \underline{a^3P(j-2, i-j-1)} + \underline{a^3Q(j-2, n-j-1)} +$$

$$\sum_{i=j+2}^{n} r_i \underline{(a^2x + axa + xa^2)P(j, i-j-3)} + \sum_{i=j+3}^{n} r_i \underline{(ax^2 + xax + x^2a)P(j, i-j-3)} +$$

$$\sum_{i=j+4}^{n} r_i \underline{x^3P(j+1, i-j-4)}) + r_{j+1}\underline{a^n}.$$

Hence post multiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-(a^{j+1}x^{n-j-1})xa^2 \to (\sum_{i=j+1}^{n-1} r_i \underline{a^3P(j-2,i-j-1)xa^2} + \underline{a^3Q(j-2,n-j-1)xa^2} + \tag{A.56}$$

$$\sum_{i=j+2}^{n} r_i(a^2x+axa+xa^2)P(j,i-j-2)xa^2 + \sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)xa^2}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)xa^2}) - r_{j+1}\underline{xa^{n+2}},$$

$$-(a^{j+1}x^{n-j-1})axa \to (\sum_{i=j+1}^{n-1} r_i \underline{a^3P(j-2,i-j-1)axa} + \underline{a^3Q(j-2,n-j-1)axa} + \tag{A.57}$$

$$\sum_{i=j+2}^{n} r_i\underline{(a^2x+axa+xa^2)P(j,i-j-2)axa} + \sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)axa}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)axa}) - r_{j+1}\underline{axa^{n+1}}$$

and

$$-(a^{j+1}x^{n-j-1})a^2x \to (\sum_{i=j+1}^{n-1} r_i \underline{a^3P(j-2,i-j-1)a^2x} + \underline{a^3Q(j-2,n-j-1)a^2x} + \tag{A.58}$$

$$\sum_{i=j+2}^{n} r_i\underline{(a^2x+axa+xa^2)P(j,i-j-2)a^2x} + \sum_{i=j+3}^{n} r_i\underline{(ax^2+xax+x^2a)P(j,i-j-3)a^2x}$$

$$+\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)a^2x}) - r_{j+1}\underline{a^2xa^n}$$

respectively. The words in (A.52)-(A.58) of length $n+3$ which are reducible are as follows:

1. $x^2a^{j+2}x^{n-j-2}a \in x^2aP(j+1,n-j-4)x^2a$
2. $x(a^{j+2}x^{n-j-2})xa \in xa^2P(j,n-j-3)x^2a$
3. $x(a^{j+2}x^{n-j-2})ax \in xa^2P(j,n-j-3)xax$
4. $xa^{j+1}x^{n-j-1}a^2 \in xa^2P(j,n-j-3)xa^2$
5. $axa^{j+1}x^{n-j-1}a \in axaP(j,n-j-3)x^2a.$

Using (A.6), expand σ_{j+2} as

$$a^{j+2}x^{n-j-2} \to -(\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)a} + \underline{a^2P(j-1,n-j-2)a}$$

$$+\sum_{i=j+3}^{n-1} r_i\underline{a^2P(j,i-j-3)x} + \underline{a^2Q(j,n-j-3)x} + \sum_{i=j+3}^{n-1} r_i(\underline{axP(j,i-j-3)a}+$$

$$\underline{xaP(j,i-j-3)a}) + \underline{axP(j,n-j-3)a} + \underline{xaP(j,n-j-3)a}+$$

$$\sum_{i=j+4}^{n-1} r_i(\underline{axP(j+1,i-j-4)x} + \underline{xaP(j+1,i-j-4)x} + \underline{x^2P(j+1,i-j-4)a})$$

$$+\underline{axP(j+1,n-j-4)x} + \underline{xaP(j+1,n-j-4)x} + \underline{x^2P(j+1,n-j-4)a}+$$

$$\sum_{i=j+5}^{n-1} r_i\underline{x^2P(j+2,i-j-5)x} + \underline{x^2P(j+2,n-j-5)x}) + r_{j+2}\underline{a^n}$$

$$= -(\sum_{i=j+2}^{n} r_i\underline{a^2P(j-1,i-j-2)a} + \sum_{i=j+3}^{n-1} r_i\underline{a^2P(j,i-j-3)x} + \underline{a^2Q(j,n-j-3)x}+$$

$$\sum_{i=j+3}^{n} r_i(\underline{axP(j,i-j-3)a} + \underline{xaP(j,i-j-3)a}) + \sum_{i=j+4}^{n} r_i(\underline{axP(j+1,i-j-4)x}+$$

$$\underline{xaP(j+1,i-j-4)x} + \underline{x^2P(j+1,i-j-4)a}) + \sum_{i=j+5}^{n} r_i\underline{x^2P(j+2,i-j-5)x}) + r_{j+2}\underline{a^n}.$$

Thus, premultiplying this by x and postmultiplying by xa and ax, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$x(a^{j+2}x^{n-j-2})xa \to -(\sum_{i=j+2}^{n} r_i\underline{xa^2P(j-1,i-j-2)axa} + \sum_{i=j+3}^{n-1} r_i\underline{xa^2P(j,i-j-3)x^2a} \quad \text{(A.59)}$$

$$+\underline{xa^2Q(j,n-j-3)x^2a} + \sum_{i=j+3}^{n} r_i(\underline{xaxP(j,i-j-3)axa} + \underline{x^2aP(j,i-j-3)axa})$$

$$+\sum_{i=j+4}^{n} r_i(\underline{xaxP(j+1,i-j-4)x^2a} + x^2aP(j+1,i-j-4)x^2a+$$

$$\underline{x^3P(j+1,i-j-4)axa}) + \sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)x^2a}) + r_{j+2}\underline{x^2a^{n+1}}$$

and

$$x(a^{j+2}x^{n-j-2})ax \to -(\sum_{i=j+2}^{n} r_i\underline{xa^2P(j-1,i-j-2)a^2x} + \sum_{i=j+3}^{n-1} r_i\underline{xa^2P(j,i-j-3)xax} \quad \text{(A.60)}$$

$$+\underline{xa^2Q(j,n-j-3)xax} + \sum_{i=j+3}^{n} r_i(\underline{xaxP(j,i-j-3)a^2x} + \underline{x^2aP(j,i-j-3)a^2x})$$

$$+\sum_{i=j+4}^{n} r_i(\underline{xaxP(j+1,i-j-4)xax} + \underline{x^2aP(j+1,i-j-4)xax}+$$

$$\underline{x^3P(j+1,i-j-4)a^2x}) + \sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)xax}) + r_{j+2}\underline{xaxa^n}$$

respectively. Similarly, expand σ_{j+1} using (A.6) to get

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i\underline{a^2P(j-2,i-j-1)a} + \underline{a^2P(j-2,n-j-1)a} +$$

$$\sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)x} + \underline{a^2Q(j-1,n-j-2)x} +$$

$$\sum_{i=j+2}^{n-1} r_i(\underline{axP(j-1,i-j-2)a} + \underline{xaP(j-1,i-j-2)a}) +$$

$$\underline{axP(j-1,n-j-2)a} + \underline{xaP(j-1,n-j-2)a} +$$

$$\sum_{i=j+3}^{n-1} r_i(\underline{axP(j,i-j-3)x} + \underline{xaP(j,i-j-3)x} + \underline{x^2P(j,i-j-3)a}) +$$

$$\underline{axP(j,n-j-3)x} + \underline{xaP(j,n-j-3)x} + \underline{x^2P(j,n-j-3)a} +$$

$$\sum_{i=j+4}^{n-1} r_i\underline{x^2P(j+1,i-j-4)x} + \underline{x^2P(j+1,n-j-4)x}) + r_{j+1}\underline{a^n}.$$

$$= -(\sum_{i=j+1}^{n} r_i\underline{a^2P(j-2,i-j-1)a} + \sum_{i=j+2}^{n-1} r_i\underline{a^2P(j-1,i-j-2)x} +$$

$$\underline{a^2Q(j-1,n-j-2)x} + \sum_{i=j+2}^{n} r_i(\underline{axP(j-1,i-j-2)a} + \underline{xaP(j-1,i-j-2)a}) +$$

$$\sum_{i=j+3}^{n} r_i(\underline{axP(j,i-j-3)x} + \underline{xaP(j,i-j-3)x} + \underline{x^2P(j,i-j-3)a}) +$$

$$\sum_{i=j+4}^{n} r_i\underline{x^2P(j+1,i-j-4)x}) + r_{j+1}\underline{a^n}.$$

Thus, premultiplying this by x and postmultiplying by a^2 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$xa^{j+1}x^{n-j-1}a^2 \to -(\sum_{i=j+1}^{n} r_i\underline{xa^2P(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n-1} r_i\underline{xa^2P(j-1,i-j-2)xa^2} + \quad \text{(A.61)}$$

$$\underline{xa^2Q(j-1,n-j-2)xa^2} + \sum_{i=j+2}^{n} r_i(\underline{xaxP(j-1,i-j-2)a^3} + \underline{x^2aP(j-1,i-j-2)a^3})$$

$$+ \sum_{i=j+3}^{n} r_i(\underline{xaxP(j,i-j-3)xa^2} + \underline{x^2aP(j,i-j-3)xa^2} + \underline{x^3P(j,i-j-3)a^3}) +$$

$$\sum_{i=j+4}^{n} r_i\underline{x^3P(j+1,i-j-4)xa^2}) + r_{j+1}\underline{xa^{n+2}}.$$

Similarly, expand σ_{j+1} using (A.7) to get

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i\underline{aP(j-2,i-j-1)a^2} + \underline{aP(j-2,n-j-1)a^2} +$$

$$\sum_{i=j+2}^{n-1} r_i(\underline{xP(j-1,i-j-2)a^2} + \underline{aP(j-1,i-j-2)ax} + \underline{aP(j-1,i-j-2)xa}) +$$

$$\underline{xP(j-1,n-j-2)a^2} + \underline{aP(j-1,n-j-2)ax} + \underline{aP(j-1,n-j-2)xa} +$$

$$\sum_{i=j+3}^{n-1} r_i(\underline{xP(j,i-j-3)ax} + \underline{xP(j,i-j-3)xa}) + \underline{xP(j,n-j-3)ax} +$$

$$+\underline{xP(j,n-j-3)xa} + \sum_{i=j+3}^{n-1} r_i\underline{aP(j,i-j-3)x^2} + \underline{aQ(j,n-j-3)x^2} +$$

$$\sum_{i=j+4}^{n-1} r_i\underline{xP(j+1,i-j-4)x^2} + \underline{xP(j+1,n-j-4)x^2}) + r_{j+1}\underline{a^n}$$

$$= -(\sum_{i=j+1}^{n} r_i\underline{aP(j-2,i-j-1)a^2} + \sum_{i=j+2}^{n} r_i(\underline{xP(j-1,i-j-2)a^2} +$$

$$\underline{aP(j-1,i-j-2)ax} + \underline{aP(j-1,i-j-2)xa}) + \sum_{i=j+3}^{n} r_i(\underline{xP(j,i-j-3)ax} +$$

$$\underline{xP(j,i-j-3)xa}) + \sum_{i=j+3}^{n-1} r_i\underline{aP(j,i-j-3)x^2} + \underline{aQ(j,n-j-3)x^2} +$$

$$\sum_{i=j+4}^{n} r_i\underline{xP(j+1,i-j-4)x^2}) + r_{j+1}\underline{a^n}.$$

Premultiplying this by ax and postmultiplying by a and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$axa^{j+1}x^{n-j-1}a \to -(\sum_{i=j+1}^{n} r_i\underline{axaP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i(\underline{ax^2P(j-1,i-j-2)a^3} \tag{A.62}$$

$$+\underline{axaP(j-1,i-j-2)axa} + \underline{axaP(j-1,i-j-2)xa^2}) + \sum_{i=j+3}^{n} r_i(\underline{ax^2P(j,i-j-3)axa}$$

$$+\underline{ax^2P(j,i-j-3)xa^2}) + \sum_{i=j+3}^{n-1} r_i\underline{axaP(j,i-j-3)x^2a} + \underline{axaQ(j,n-j-3)x^2a}$$

$$+ \sum_{i=j+4}^{n} r_i\underline{ax^2P(j+1,i-j-4)x^2a}) + r_{j+1}\underline{axa^{n+1}}$$

The reducible word $-x^2a^{j+2}x^{n-j-2}a$ in (A.59) has an opposite sign with $x^2a^{j+2}x^{n-j-2}a$, the term (i) in the list above so they cancel out. Thus, the reduction process ends here and when we substitute (A.52)-(A.62) into (A.51), we get

$$a^3\omega_j \to (\sum_{i=j+1}^{n} r_i\underline{a^2xP(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i(\underline{a^2xP(j,i-j-2)(a^2x+axa+xa^2)}+ \tag{A.63}$$

$$\underline{axaP(j,i-j-2)a^2x})+\sum_{i=j+3}^{n} r_i(\underline{a^2xP(j,i-j-3)(ax^2+xax+x^2a)}+$$

$$\underline{axaP(j,i-j-3)(ax^2+xax)}+\underline{xa^2P(j,i-j-3)ax^2}+\underline{ax^2P(j,i-j-3)a^2x})+$$

$$\sum_{i=j+4}^{n} r_i(\underline{ax^2P(j+1,i-j-4)(ax^2+xax)}+\underline{(xax+x^2a)P(j+1,i-j-4)ax^2})+$$

$$\sum_{i=j+5}^{n} r_i\underline{x^3P(j+2,i-j-5)ax^2})-(r_{j+1}\underline{a^2xa^n}+r_{j+2}\underline{ax^2a^n}+$$

$$\sum_{i=j+3}^{n-1} r_ia^3P(j,i-j)x^3+\underline{a^3Q(j,n-j)x^3}).$$

Similarly, use (A.5) to expand σ_{j+3} as

$$\omega_{j+3} \to -(\sum_{i=j+3}^{n-1} r_i\underline{a^3P(j,i-j-3)}+\underline{a^3Q(j,n-j-3)}+\sum_{i=j+4}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(j+1,i-j-4)}$$

$$+\underline{(a^2x+axa+xa^2)P(j+1,n-j-4)}+\sum_{i=j+5}^{n} r_i\underline{(ax^2+xax+x^2a)P(j+2,i-j-5)}$$

$$+\sum_{i=j+6}^{n-1} r_i\underline{x^3P(j+3,i-j-6))}+\underline{x^3P(j+3,n-j-6)})) + r_{j+3}\underline{a^n}$$

$$= -(\sum_{i=j+3}^{n-1} r_i\underline{a^3P(j,i-j-3)}+\underline{a^3Q(j,n-j-3)}+\sum_{i=j+4}^{n} r_i\underline{(a^2x+axa+xa^2)P(j+1,i-j-4)}$$

$$+\sum_{i=j+5}^{n} r_i\underline{(ax^2+xax+x^2a)P(j+2,i-j-5)}+\sum_{i=j+6}^{n} r_i\underline{x^3P(j+3,i-j-6)})) + r_{j+3}\underline{a^n}.$$

Post multiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\omega_{j+3}x^3 \to -(\sum_{i=j+3}^{n-1} r_ia^3P(j,i-j-3)x^3+\underline{a^3Q(j,n-j-3)x^3}+ \tag{A.64}$$

$$\sum_{i=j+4}^{n} r_i(a^2x+axa+xa^2)P(j+1,i-j-4)x^3+$$

$$\sum_{i=j+5}^{n} r_i(ax^2+xax+x^2a)P(j+2,i-j-5)x^3+$$

$$\sum_{i=j+6}^{n} r_i x^3 P(j+3, i-j-6))x^3) + r_{j+3}\underline{a^{n+3}}.$$

The reducible words in (A.64) of length $n+3$ are:

1. $x^2a(a^{j+2}x^{n-j-2}) \in x^2aP(j+2, n-j-5)x^3$
2. $xax(a^{j+2}x^{n-j-2}) \in xaxP(j+2, n-j-5)x^3$
3. $ax^2(a^{j+2}x^{n-j-2}) \in ax^2P(j+2, n-j-5)x^3$
4. $xa^2(a^{j+1}x^{n-j-1}) \in xa^2P(j+1, n-j-4)x^3$
5. $axa(a^{j+1}x^{n-j-1}) \in axaP(j+1, n-j-4)x^3$
6. $a^2x(a^{j+1}x^{n-j-1}) \in a^2xP(j+1, n-j-4)x^3$
7. $x^3a^{j+3}x^{n-j-3} \in x^3P(j+3, n-j-6)x^3$.

Using (A.4), expand σ_{j+2} as

$$a^{j+2}x^{n-j-2} \to -(\sum_{i=j+2}^{n-1} r_i \underline{P(j-1, i-j-2)a^3} + \underline{P(j-1, n-j-2)a^3} +$$

$$\sum_{i=j+3}^{n-1} r_i \underline{P(j, i-j-3)(a^2x + axa + xa^2)} + \underline{P(j, n-j-3)(a^2x + axa + xa^2)} +$$

$$\sum_{i=j+4}^{n-1} r_i \underline{P(j+1, i-j-4)(ax^2 + xax + x^2a)} + \underline{P(j+1, n-j-4)(ax^2 + xax + x^2a)} +$$

$$\sum_{i=j+5}^{n-1} r_i \underline{P(j+2, i-j-5)x^3} + \underline{Q(j+2, n-j-5)x^3}) + r_{j+2}\underline{a^n}$$

$$= -(\sum_{i=j+2}^{n} r_i \underline{P(j-1, i-j-2)a^3} + \sum_{i=j+3}^{n} r_i \underline{P(j, i-j-3)(a^2x + axa + xa^2)} +$$

$$\sum_{i=j+4}^{n} r_i \underline{P(j+1, i-j-4)(ax^2 + xax + x^2a)} + \sum_{i=j+5}^{n-1} r_i \underline{P(j+2, i-j-5)x^3}$$

$$+\underline{Q(j+2, n-j-5)x^3}) + r_{j+2}\underline{a^n}.$$

Thus, premultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-x^2a(a^{j+2}x^{n-j-2}) \to (\sum_{i=j+2}^{n} r_i \underline{x^2aP(j-1, i-j-2)a^3} + \tag{A.65}$$

$$\sum_{i=j+3}^{n} r_i \underline{x^2aP(j, i-j-3)(a^2x + axa + xa^2)} + \sum_{i=j+4}^{n} r_i x^2aP(j+1, i-j-4)(ax^2 + xax + x^2a)$$

$$+\sum_{i=j+5}^{n-1} r_i \underline{x^2aP(j+2,i-j-5)x^3} + \underline{x^2aQ(j+2,n-j-5)x^3}) - r_{j+2}\underline{x^2a^{n+1}},$$

$$-xax(a^{j+2}x^{n-j-2}) \to (\sum_{i=j+2}^{n} r_i \underline{xaxP(j-1,i-j-2)a^3}+ \tag{A.66}$$

$$\sum_{i=j+3}^{n} r_i \underline{xaxP(j,i-j-3)(a^2x+axa+xa^2)} + \sum_{i=j+4}^{n} r_i \underline{xaxP(j+1,i-j-4)(ax^2+xax+x^2a)}$$

$$+\sum_{i=j+5}^{n-1} r_i \underline{xaxP(j+2,i-j-5)x^3} + \underline{xaxQ(j+2,n-j-5)x^3}) - r_{j+2}\underline{xaxa^n}$$

and

$$-ax^2(a^{j+2}x^{n-j-2}) \to (\sum_{i=j+2}^{n} r_i \underline{ax^2P(j-1,i-j-2)a^3}+ \tag{A.67}$$

$$\sum_{i=j+3}^{n} r_i \underline{ax^2P(j,i-j-3)(a^2x+axa+xa^2)} + \sum_{i=j+4}^{n} r_i \underline{ax^2P(j+1,i-j-4)(ax^2+xax+x^2a)}$$

$$+\sum_{i=j+5}^{n-1} r_i \underline{ax^2P(j+2,i-j-5)x^3} + \underline{ax^2Q(j+2,n-j-5)x^3}) - r_{j+2}\underline{ax^2a^n}$$

respectively. Again, using (A.4), expand σ_{j+1} as

$$a^{j+1}x^{n-j-1} \to -(\sum_{i=j+1}^{n-1} r_i \underline{P(j-2,i-j-1)a^3} + \underline{P(j-2,n-j-1)a^3}+$$

$$\sum_{i=j+2}^{n-1} r_i \underline{P(j-1,i-j-2)(a^2x+axa+xa^2)} + \underline{P(j-1,n-j-2)(a^2x+axa+xa^2)}+$$

$$\sum_{i=j+3}^{n-1} r_i \underline{P(j,i-j-3)(ax^2+xax+x^2a)} + \underline{P(j,n-j-3)(ax^2+xax+x^2a)}+$$

$$\sum_{i=j+4}^{n-1} r_i \underline{P(j+1,i-j-4)x^3} + \underline{Q(j+1,n-j-4)x^3}) + r_{j+1}\underline{a^n}.$$

$$= -(\sum_{i=j+1}^{n} r_i \underline{P(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i \underline{P(j-1,i-j-2)(a^2x+axa+xa^2)}+$$

$$\sum_{i=j+3}^{n} r_i \underline{P(j,i-j-3)(ax^2+xax+x^2a)} + \sum_{i=j+4}^{n-1} r_i \underline{P(j+1,i-j-4)x^3}$$

$$+\underline{Q(j+1,n-j-4)x^3}) + r_{j+1}\underline{a^n}.$$

Hence, premultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-xa^2(a^{j+1}x^{n-j-1}) \to (\sum_{i=j+1}^{n} r_i \underline{xa^2P(j-2,i-j-1)a^3}+\sum_{i=j+2}^{n} r_i xa^2P(j-1,i-j-2)(a^2x+axa+xa^2) \tag{A.68}$$

$$+\sum_{i=j+3}^{n} r_i xa^2 P(j,i-j-3)(ax^2+xax+x^2a) + \sum_{i=j+4}^{n-1} r_i \underline{xa^2 P(j+1,i-j-4)x^3}$$

$$+\underline{xa^2 Q(j+1,n-j-4)x^3}) - r_{j+1}\underline{xa^{n+2}},$$

$$-axa(a^{j+1}x^{n-j-1}) \to (\sum_{i=j+1}^{n} r_i \underline{axaP(j-2,i-j-1)a^3}+ \tag{A.69}$$

$$\sum_{i=j+2}^{n} r_i \underline{axaP(j-1,i-j-2)(a^2x+axa+xa^2)} + \sum_{i=j+3}^{n} r_i axaP(j,i-j-3)(ax^2+xax+x^2a)$$

$$+\sum_{i=j+4}^{n-1} r_i \underline{axaP(j+1,i-j-4)x^3} + \underline{axaQ(j+1,n-j-4)x^3}) - r_{j+1}\underline{axa^{n+1}}$$

and

$$-a^2x(a^{j+1}x^{n-j-1}) \to (\sum_{i=j+1}^{n} r_i \underline{a^2xP(j-2,i-j-1)a^3}+ \tag{A.70}$$

$$\sum_{i=j+2}^{n} r_i \underline{a^2xP(j-1,i-j-2)(a^2x+axa+xa^2)} + \sum_{i=j+3}^{n} r_i \underline{a^2xP(j,i-j-3)(ax^2+xax+x^2a)}$$

$$+\sum_{i=j+4}^{n-1} r_i \underline{a^2xP(j+1,i-j-4)x^3} + \underline{a^2xQ(j+1,n-j-4)x^3}) - r_{j+1}\underline{a^2xa^n}$$

respectively. Also, use (A.4) to expand σ_{j+3} as

$$a^{j+3}x^{n-j-3} \to -(\sum_{i=j+3}^{n-1} r_i \underline{P(j,i-j-3)a^3} + \underline{P(j,n-j-3)a^3}+$$

$$\sum_{i=j+4}^{n-1} r_i \underline{P(j+1,i-j-4)(a^2x+axa+xa^2)} + \underline{P(j+1,n-j-4)(a^2x+axa+xa^2)}+$$

$$\sum_{i=j+5}^{n-1} r_i \underline{P(j+2,i-j-5)(ax^2+xax+x^2a)} + \underline{P(j+2,n-j-5)(ax^2+xax+x^2a)}+$$

$$\sum_{i=j+6}^{n-1} r_i \underline{P(j+3,i-j-6)x^3} + \underline{Q(j+3,n-j-6)x^3}) + r_{j+3}\underline{a^n}$$

$$= -(\sum_{i=j+3}^{n} r_i \underline{P(j,i-j-3)a^3} + \sum_{i=j+4}^{n} r_i \underline{P(j+1,i-j-4)(a^2x+axa+xa^2)}+$$

$$\sum_{i=j+5}^{n} r_i \underline{P(j+2,i-j-5)(ax^2+xax+x^2a)} + \sum_{i=j+6}^{n-1} r_i \underline{P(j+3,i-j-6)x^3}$$

$$+\underline{Q(j+3,n-j-6)x^3}) + r_{j+3}\underline{a^n}$$

Premultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-x^3a^{j+3}x^{n-j-3} \to (\sum_{i=j+3}^{n} r_i \underline{x^3P(j,i-j-3)a^3} + \sum_{i=j+4}^{n} r_i \underline{x^3P(j+1,i-j-4)(a^2x+axa+xa^2)} \tag{A.71}$$

$$+\sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)(ax^2+xax+x^2a)} + \sum_{i=j+6}^{n-1} r_i \underline{x^3P(j+3,i-j-6)x^3}$$

$$+\underline{x^3Q(j+3,n-j-6)x^3}) - r_{j+3}\underline{x^3a^n}.$$

The words in (A.65)-(A.71) of length $n+3$ which are reducible are as follows:

1. $x^2a^{j+2}x^{n-j-2}a \in x^2aP(j+1,n-j-4)x^2a$
2. $x(a^{j+2}x^{n-j-2})xa \in xa^2P(j,n-j-3)x^2a$
3. $x(a^{j+2}x^{n-j-2})ax \in xa^2P(j,n-j-3)xax$
4. $xa^{j+1}x^{n-j-1}a^2 \in xa^2P(j,n-j-3)xa^2$
5. $axa^{j+1}x^{n-j-1}a \in axaP(j,n-j-3)x^2a.$

Recall from (A.59), (A.60), (A.61) and (A.62) that

$$x(a^{j+2}x^{n-j-2})xa \to -(\sum_{i=j+2}^{n} r_i \underline{xa^2P(j-1,i-j-2)axa} + \sum_{i=j+3}^{n-1} r_i \underline{xa^2P(j,i-j-3)x^2a} +$$

$$\underline{xa^2Q(j,n-j-3)x^2a} + \sum_{i=j+3}^{n} r_i (\underline{xaxP(j,i-j-3)axa} + \underline{x^2aP(j,i-j-3)axa}) +$$

$$\sum_{i=j+4}^{n} r_i (\underline{xaxP(j+1,i-j-4)x^2a} + x^2aP(j+1,i-j-4)x^2a + \underline{x^3P(j+1,i-j-4)axa})$$

$$+\sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)x^2a}) + r_{j+2}\underline{x^2a^{n+1}},$$

$$x(a^{j+2}x^{n-j-2})ax \to -(\sum_{i=j+2}^{n} r_i \underline{xa^2P(j-1,i-j-2)a^2x} + \sum_{i=j+3}^{n-1} r_i \underline{xa^2P(j,i-j-3)xax} +$$

$$\underline{xa^2Q(j,n-j-3)xax} + \sum_{i=j+3}^{n} r_i (\underline{xaxP(j,i-j-3)a^2x} + \underline{x^2aP(j,i-j-3)a^2x}) +$$

$$\sum_{i=j+4}^{n} r_i (\underline{xaxP(j+1,i-j-4)xax} + \underline{x^2aP(j+1,i-j-4)xax} + \underline{x^3P(j+1,i-j-4)a^2x}) +$$

$$\sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)xax}) + r_{j+2}\underline{xaxa^n},$$

$$
\begin{aligned}
xa^{j+1}x^{n-j-1}a^2 \to -(& \sum_{i=j+1}^{n} r_i \underline{xa^2P(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n-1} r_i \underline{xa^2P(j-1,i-j-2)xa^2} + \\
& \underline{xa^2Q(j-1,n-j-2)xa^2} + \sum_{i=j+2}^{n} r_i(\underline{xaxP(j-1,i-j-2)a^3} + \underline{x^2aP(j-1,i-j-2)a^3}) + \\
& \sum_{i=j+3}^{n} r_i(\underline{xaxP(j,i-j-3)xa^2} + \underline{x^2aP(j,i-j-3)xa^2} + \underline{x^3P(j,i-j-3)a^3}) + \\
& \sum_{i=j+4}^{n} r_i \underline{x^3P(j+1,i-j-4)xa^2}) + r_{j+1}\underline{xa^{n+2}}
\end{aligned}
$$

and

$$
\begin{aligned}
axa^{j+1}x^{n-j-1}a \to -(& \sum_{i=j+1}^{n} r_i \underline{axaP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i(\underline{ax^2P(j-1,i-j-2)a^3} + \\
& \underline{axaP(j-1,i-j-2)axa} + \underline{axaP(j-1,i-j-2)xa^2}) + \sum_{i=j+3}^{n} r_i(\underline{ax^2P(j,i-j-3)axa} + \\
& \underline{ax^2P(j,i-j-3)xa^2}) + \sum_{i=j+3}^{n-1} r_i \underline{axaP(j,i-j-3)x^2a} + \underline{axaQ(j,n-j-3)x^2a} + \\
& \sum_{i=j+4}^{n} r_i \underline{ax^2P(j+1,i-j-4)x^2a}) + r_{j+1}\underline{axa^{n+1}}
\end{aligned}
$$

The reducible word $-x^2a^{j+2}x^{n-j-2}a$ in (A.59) appears with a sign opposite to that of $x^2a^{j+2}x^{n-j-2}a$, the term (i) in the list above which makes them cancel out. Hence, the reduction process ends here and when we substitute (A.59)-(A.61) and (A.65)-(A.71) into (A.64), we get

$$
\begin{aligned}
\omega_{j+3}x^3 \to (& \sum_{i=j+1}^{n} r_i \underline{a^2xP(j-2,i-j-1)a^3} + \sum_{i=j+2}^{n} r_i(\underline{a^2xP(j,i-j-3)(a^2x + axa + xa^2)} + \\
& \underline{axaP(j,i-j-3)a^2x}) + \sum_{i=j+3}^{n} r_i(\underline{a^2xP(j,i-j-3)(ax^2 + xax + x^2a)} + \\
& \underline{axaP(j,i-j-3)(ax^2 + xax)} + \underline{xa^2P(j,i-j-3)ax^2} + \underline{ax^2P(j,i-j-3)a^2x}) + \\
& \sum_{i=j+4}^{n} r_i(\underline{ax^2P(j+1,i-j-4)(ax^2 + xax)} + \underline{(xax + x^2a)P(j+1,i-j-4)ax^2}) \\
& + \sum_{i=j+5}^{n} r_i \underline{x^3P(j+2,i-j-5)ax^2}) - (r_{j+1}\underline{a^2xa^n} + r_{j+2}\underline{ax^2a^n} + \\
& \sum_{i=j+3}^{n-1} r_i a^3P(j,i-j)x^3 + \underline{a^3Q(j,n-j)x^3}).
\end{aligned}
\tag{A.72}
$$

Comparing (A.64) and (A.72), we conclude that the overlap ambiguity $\{\omega_j, \omega_{j+3}\}$ is resolvable for $3 \le j < n-4$

(d) Let $j = n-4$ and consider the overlap ambiguity $\{\omega_{n-4}, \omega_{n-1}\}$. Using (A.4), expand σ_{n-4} as

$$\omega_{n-4} \to -(\sum_{i=n-4}^{n-1} r_i \underline{P(n-7, i-n+4)a^3} + \underline{P(n-7,4)a^3} +$$

$$\sum_{i=n-3}^{n-1} r_i \underline{P(n-6, i-n+3)(a^2x + axa + xa^2)} + \underline{P(n-6,3)(a^2x + axa + xa^2)} +$$

$$\sum_{i=n-2}^{n-1} r_i \underline{P(n-5, i-n+2)(ax^2 + xax + x^2a)} + \underline{P(n-5,2)(ax^2 + xax + x^2a)} +$$

$$r_{n-1}\underline{P(n-4,0)x^3} + \underline{Q(n-4,1)x^3}) + r_{n-4}\underline{a^n}$$

$$= -(\sum_{i=n-4}^{n} r_i \underline{P(n-7, i-n+4)a^3} + \sum_{i=n-3}^{n} r_i \underline{P(n-6, i-n+3)(a^2x + axa + xa^2)} +$$

$$\sum_{i=n-2}^{n} r_i \underline{P(n-5, i-n+2)(ax^2 + xax + x^2a)} + r_{n-1}\underline{P(n-4,0)x^3} + \underline{Q(n-4,1)x^3}) + r_{n-4}\underline{a^n}$$

Premultiply this by a^3 and use Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$a^3\omega_{n-4} \to -(\sum_{i=n-4}^{n} r_i a^3 P(n-7, i-n+4)a^3 + \sum_{i=n-3}^{n} r_i a^3 P(n-6, i-n+3)(a^2x + axa + xa^2) \quad \text{(A.73)}$$

$$+ \sum_{i=n-2}^{n} r_i a^3 P(n-5, i-n+2)(ax^2 + xax + x^2a) + r_{n-1}a^3 P(n-4,0)x^3$$

$$+ \underline{a^3 Q(n-4,1)x^3}) + r_{n-4}\underline{a^{n+3}}$$

The reducible words above of length $n+3$ are as follows:

1. $a^{n-4}x^4a^3 \in a^3P(n-7,4)a^3$

2. $(a^{n-2}x^2)x^2a \in a^3P(n-5,2)x^2a$

3. $(a^{n-2}x^2)xax \in a^3P(n-5,2)xax$

4. $(a^{n-2}x^2)ax^2 \in a^3P(n-5,2)ax^2$

5. $(a^{n-3}x^3)xa^2 \in a^3P(n-6,3)xa^2$

6. $(a^{n-3}x^3)axa \in a^3P(n-6,3)axa$

7. $(a^{n-3}x^3)a^2x \in a^3P(n-6,3)a^2x$.

Using (A.5) expand σ_{n-4} as

$$a^{n-4}x^4 \to -(\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)} + \underline{a^3 Q(n-7,4)} +$$

$$\sum_{i=n-3}^{n-1} r_i \underline{(a^2x + axa + x^2a)P(n-6, i-n+3)} + \underline{(a^2x + axa + x^2a)P(n-6,3)} +$$

$$+ \sum_{i=n-2}^{n-1} r_i \underline{(ax^2 + xax + x^2a)P(n-5, i-n+2)} + \underline{(ax^2 + xax + x^2a)P(n-5,2)} +$$

$$\sum_{i=n-1}^{n} r_i \underline{x^3 P(n-4, i-n+1)}) + r_{n-4}\underline{a^n}$$

$$= -(\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)} + \underline{a^3 Q(n-7,4)} + \sum_{i=n-3}^{n} r_i \underline{(a^2x + axa + x^2a)P(n-6, i-n+3)}$$

$$+ \sum_{i=n-2}^{n} r_i \underline{(ax^2 + xax + x^2a)P(n-5, i-n+2)} + \sum_{i=n-1}^{n} r_i \underline{x^3 P(n-4, i-n+1)}) + r_{n-4}\underline{a^n}.$$

Post multiplying this by a^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-a^{n-4}x^4a^3 \to (\sum_{i=n-4}^{n-1} r_i \underline{a^3 P(n-7, i-n+4)a^3} + \underline{a^3 Q(n-7,4)a^3} + \tag{A.74}$$

$$\sum_{i=n-3}^{n} r_i \underline{(a^2x + axa + x^2a)P(n-6, i-n+3)a^3} +$$

$$\sum_{i=n-2}^{n} r_i \underline{(ax^2 + xax + x^2a)P(n-5, i-n+2)a^3} +$$

$$\sum_{i=n-1}^{n} r_i \underline{x^3 P(n-4, i-n+1)a^3}) + r_{n-4}\underline{a^{n+3}}.$$

Use (A.5) to expand σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i \underline{a^3 P(n-5, i-n+2)} + \underline{a^3 Q(n-5,2)} +$$

$$\sum_{i=n-1}^{n} r_i \underline{(a^2x + axa + xa^2)P(n-4, i-n+1)} +$$

$$\underline{(ax^2 + xax + x^2a)P(n-3,0)}) + r_{n-2}\underline{a^n}.$$

Thus, post multiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-(a^{n-2}x^2)x^2a \to (\sum_{i=n-2}^{n-1} r_i a^3 P(n-5, i-n+2)x^2a + \underline{a^3 Q(n-5,2)x^2a} + \tag{A.75}$$

$$\sum_{i=n-1}^{n} r_i(a^2x+axa+xa^2)P(n-4,i-n+1)x^2a+\underline{(ax^2+xax+x^2a)P(n-3,0)x^2a})$$

$$-r_{n-2}\underline{x^2a^{n+1}},$$

$$-(a^{n-2}x^2)xax \to (\sum_{i=n-2}^{n-1} r_i\underline{a^3P(n-5,i-n+2)xax}+\underline{a^3Q(n-5,2)xax}+ \tag{A.76}$$

$$\sum_{i=n-1}^{n} r_i(a^2x+axa+xa^2)P(n-4,i-n+1)xax+\underline{(ax^2+xax+x^2a)P(n-3,0)xax})$$

$$-r_{n-2}\underline{xaxa^n}$$

and

$$-(a^{n-2}x^2)ax^2 \to (\sum_{i=n-2}^{n-1} r_i\underline{a^3P(n-5,i-n+2)ax^2}+\underline{a^3Q(n-5,2)ax^2}+ \tag{A.77}$$

$$\sum_{i=n-1}^{n} r_i\underline{(a^2x+axa+xa^2)P(n-4,i-n+1)ax^2}+(ax^2+xax+x^2a)P(n-3,0)ax^2)$$

$$-r_{n-2}\underline{ax^2a^n}$$

respectively. Also, using (A.5), expand σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3)}+\underline{a^3Q(n-6,3)}+\underline{x^3P(n-3,0)}+$$

$$\sum_{i=n-2}^{n-1} r_i\underline{(a^2x+axa+xa^2)P(n-5,i-n+2)}+\underline{(a^2x+axa+xa^2)P(n-5,2)}+$$

$$\sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1)})+r_{n-3}\underline{a^n}.$$

$$=-(\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3)}+\underline{a^3Q(n-6,3)}+\underline{x^3P(n-3,0)}+$$

$$\sum_{i=n-2}^{n} r_i\underline{(a^2x+axa+xa^2)P(n-5,i-n+2)}+\sum_{i=n-1}^{n} r_i\underline{(ax^2+xax+x^2a)P(n-4,i-n+1)})$$

$$+r_{n-3}\underline{a^n}.$$

Hence, post multiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-(a^{n-3}x^3)xa^2 \to (\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6,i-n+3)xa^2}+\underline{a^3Q(n-6,3)xa^2}+ \tag{A.78}$$

$$\underline{x^3P(n-3,0)xa^2} + \sum_{i=n-2}^{n} r_i(a^2x + axa + xa^2)P(n-5, i-n+2)xa^2 +$$

$$\sum_{i=n-1}^{n} r_i\underline{(ax^2 + xax + x^2a)P(n-4, i-n+1)xa^2}) - r_{n-3}\underline{xa^{n+2}},$$

$$-(a^{n-3}x^3)axa \to (\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6, i-n+3))axa} + \underline{a^3Q(n-6,3))axa} + \tag{A.79}$$

$$\underline{x^3P(n-3,0))axa} + \sum_{i=n-2}^{n} r_i\underline{(a^2x + axa + xa^2)P(n-5, i-n+2))axa} +$$

$$\sum_{i=n-1}^{n} r_i\underline{(ax^2 + xax + x^2a)P(n-4, i-n+1))axa}) - r_{n-3}\underline{axa^{n+1}}$$

and

$$-(a^{n-3}x^3)a^2x \to (\sum_{i=n-3}^{n-1} r_i\underline{a^3P(n-6, i-n+3)a^2x} + \underline{a^3Q(n-6,3)a^2x} + \tag{A.80}$$

$$x^3P(n-3,0)a^2x + \sum_{i=n-2}^{n} r_i\underline{(a^2x + axa + xa^2)P(n-5, i-n+2)a^2x} +$$

$$\sum_{i=n-1}^{n} r_i\underline{(ax^2 + xax + x^2a)P(n-4, i-n+1)a^2x}) - r_{n-3}\underline{a^2xa^n}$$

respectively. The following words of length $n+3$ from (A.74)-(A.80) are reducible:

1. $x^2a^{n-2}x^2a$ on right hand side of the relation for $-(a^{n-2}x^2)x^2a$.
2. $x(a^{n-2}x^2)xa \in xa^2P(n-4,1)x^2a$
3. $axa^{n-3}x^3a \in axaP(n-4,1)x^2a$
4. $xa^{n-2}x^2ax \in xa^2P(n-4,1)xax$
5. $xa^{n-3}x^3a^2 \in xa^2P(n-5,2)xa^2$
6. $x^3a^{n-1}x$ on right hand side of the relation for $-(a^{n-3}x^3)a^2x$
7. $x^2a(a^{n-2}x^2)$ on right hand side of the relation for $-(a^{n-2}x^2)ax^2$
8. $xax(a^{n-2}x^2)$ on right hand side of the relation for $-(a^{n-2}x^2)ax^2$
9. $ax^2(a^{n-2}x^2)$ on right hand side of the relation for $-(a^{n-2}x^2)ax^2$.

Using (A.6), expand σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i\underline{a^2P(n-5, i-n+2)a} + \underline{a^2P(n-5,2)a} +$$

$$
\begin{aligned}
&\underline{x^2P(n-3,0)a} + \underline{axP(n-3,0)x} + \underline{xaP(n-3,0)x} + \sum_{i=n-1}^{n} r_i(\underline{axP(n-4,i-n+1)a} \\
&+\underline{xaP(n-4,i-n+1)a}) + r_{n-1}\underline{a^2P(n-4,0)x} + \underline{a^2Q(n-4,1)x}) + r_{n-2}\underline{a^n} \\
&= -(\sum_{i=n-2}^{n} r_i\underline{a^2P(n-5,i-n+2)a} + \underline{x^2P(n-3,0)a} + \underline{axP(n-3,0)x} + \underline{xaP(n-3,0)x}+ \\
&\sum_{i=n-1}^{n} r_i(\underline{axP(n-4,i-n+1)a} + \underline{xaP(n-4,i-n+1)a})+ \\
&r_{n-1}\underline{a^2P(n-4,0)x} + \underline{a^2Q(n-4,1)x}) + r_{n-2}\underline{a^n}.
\end{aligned}
$$

Thus, premultiplying this by x and postmultiplying this by xa and ax, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$
\begin{aligned}
x(a^{n-2}x^2)xa \to -(&\sum_{i=n-2}^{n} r_i\underline{xa^2P(n-5,i-n+2)axa} + \underline{x^3P(n-3,0)axa}+ \\
&\underline{xaxP(n-3,0)x^2a} + x^2aP(n-3,0)x^2a + \sum_{i=n-1}^{n} r_i(\underline{xaxP(n-4,i-n+1)axa}+ \\
&\underline{x^2aP(n-4,i-n+1)axa}) + r_{n-1}\underline{xa^2P(n-4,0)x^2a} + \underline{xa^2Q(n-4,1)x^2a}) + r_{n-2}\underline{x^2a^{n+1}}
\end{aligned}
\tag{A.81}
$$

and

$$
\begin{aligned}
x(a^{n-2}x^2)ax \to -(&\sum_{i=n-2}^{n} r_i\underline{xa^2P(n-5,i-n+2)a^2x} + x^3P(n-3,0)a^2x + \underline{x^2aP(n-3,0)xax}+ \\
&\underline{xaxP(n-3,0)xax} + \sum_{i=n-1}^{n} r_i(\underline{xaxP(n-4,i-n+1)a^2x} + \underline{x^2aP(n-4,i-n+1)a^2x}) \\
&+r_{n-1}\underline{xa^2P(n-4,0)xax} + \underline{xa^2Q(n-4,1)xax}) + r_{n-2}\underline{xaxa^n}
\end{aligned}
\tag{A.82}
$$

respectively. Similarly, using (A.7), expand σ_{n-3} as

$$
\begin{aligned}
a^{n-3}x^3 \to -(&\sum_{i=n-3}^{n-1} r_i\underline{aP(n-6,i-n+3)a^2} + \underline{aP(n-6,3)a^2}+ \\
&\sum_{i=n-2}^{n-1} r_i(\underline{aP(n-5,i-n+2)ax} + \underline{aP(n-5,i-n+2)xa} + \underline{xP(n-5,i-n+2)a^2}) \\
&+\underline{aP(n-5,2)ax} + \underline{aP(n-5,2)xa} + \underline{xP(n-5,2)a^2} + \sum_{i=n-1}^{n} r_i(\underline{xP(n-4,i-n+1)ax} \\
&+\underline{xP(n-4,i-n+1)xa}) + r_{n-1}\underline{aP(n-4,0)x^2} + \underline{aQ(n-4,1)x^2} + \underline{xP(n-3,0)x^2}) + r_{n-3}\underline{a^n} \\
= -(&\sum_{i=n-3}^{n} r_i\underline{aP(n-6,i-n+3)a^2} + \sum_{i=n-2}^{n} r_i(\underline{aP(n-5,i-n+2)ax} + \underline{aP(n-5,i-n+2)xa} \\
&+\underline{xP(n-5,i-n+2)a^2}) + \sum_{i=n-1}^{n} r_i(\underline{xP(n-4,i-n+1)ax} + \underline{xP(n-4,i-n+1)xa})
\end{aligned}
$$

$$+r_{n-1}\underline{aP(n-4,0)x^2} + \underline{aQ(n-4,1)x^2} + \underline{xP(n-3,0)x^2}) + r_{n-3}\underline{a^n}.$$

Pre and post multiplying this by ax and a respectively, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$ax a^{n-3}x^3 a \to -(\sum_{i=n-3}^{n} r_i \underline{axaP(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i(\underline{axaP(n-5,i-n+2)axa} \tag{A.83}$$

$$+\underline{axaP(n-5,i-n+2)xa^2} + \underline{ax^2P(n-5,i-n+2)a^3}) + \sum_{i=n-1}^{n} r_i(\underline{ax^2P(n-4,i-n+1)axa}$$

$$+\underline{ax^2P(n-4,i-n+1)xa^2}) + r_{n-1}\underline{axaP(n-4,0)x^2a} + \underline{axaQ(n-4,1)x^2a}$$

$$+\underline{ax^2P(n-3,0)x^2a}) + r_{n-3}\underline{axa^{n+1}}.$$

Again, using (A.6), expand σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i \underline{a^2P(n-6,i-n+3)a} + \underline{a^2P(n-6,3)a}+$$

$$\sum_{i=n-2}^{n-1} r_i(\underline{axP(n-5,i-n+2)a} + \underline{xaP(n-5,i-n+2)a}) + \underline{axP(n-5,2)a} + \underline{xaP(n-5,2)a}$$

$$+\sum_{i=n-2}^{n-1} r_i \underline{a^2P(n-5,i-n+2)x} + \underline{a^2Q(n-5,2)x} + \sum_{i=n-1}^{n} r_i(\underline{x^2P(n-4,i-n+1)a}+$$

$$\underline{axP(n-4,i-n+1)x} + \underline{xaP(n-4,i-n+1)x}) + \underline{x^2P(n-3,0)x}) + r_{n-3}\underline{a^n}$$

$$= -(\sum_{i=n-3}^{n} r_i \underline{a^2P(n-6,i-n+3)a} + \sum_{i=n-2}^{n} r_i(\underline{axP(n-5,i-n+2)a} + \underline{xaP(n-5,i-n+2)a})$$

$$+\sum_{i=n-1}^{n} r_i(\underline{x^2P(n-4,i-n+1)a} + \underline{axP(n-4,i-n+1)x} + \underline{xaP(n-4,i-n+1)x})+$$

$$\sum_{i=n-2}^{n-1} r_i \underline{a^2P(n-5,i-n+2)x} + \underline{a^2Q(n-5,2)x} + \underline{x^2P(n-3,0)x}) + r_{n-3}\underline{a^n}.$$

Pre and post multiplying this by x and a^2 respectively and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$xa^{n-3}x^3a^2 \to -(\sum_{i=n-3}^{n} r_i \underline{xa^2P(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i(\underline{xaxP(n-5,i-n+2)a^3}+ \tag{A.84}$$

$$\underline{x^2aP(n-5,i-n+2)a^3}) + \sum_{i=n-2}^{n-1} r_i \underline{xa^2P(n-5,i-n+2)xa^2} + \underline{xa^2Q(n-5,2)xa^2}+$$

$$\sum_{i=n-1}^{n} r_i(\underline{x^3P(n-4,i-n+1)a^3} + \underline{xaxP(n-4,i-n+1)xa^2} + \underline{x^2aP(n-4,i-n+1)xa^2})$$

$$+\underline{x^3P(n-3,0)xa^2}) + r_{n-3}\underline{xa^{n+2}}$$

Also, using (A.4), expand σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i\underline{P(n-5,i-n+2)a^3} + \underline{P(n-5,2)a^3}+$$

$$\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(a^2x+axa+xa^2)}$$

$$+\underline{P(n-3,0)(xax+x^2a)}) + r_{n-3}\underline{a^n}.$$

$$= -(\sum_{i=n-2}^{n} r_i\underline{P(n-5,i-n+2)a^3} + \sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(a^2x+axa+xa^2)}$$

$$+\underline{P(n-3,0)(xax+x^2a)}) + r_{n-3}\underline{a^n}.$$

Thus, premultiplying this by x^2a, xax and ax^2, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$x^2a(a^{n-2}x^2) \to -(\sum_{i=n-2}^{n} r_i\underline{x^2aP(n-5,i-n+2)a^3}+ \tag{A.85}$$

$$\sum_{i=n-1}^{n} r_i\underline{x^2aP(n-4,i-n+1)(a^2x+axa+xa^2)}+$$

$$x^2aP(n-3,0)(xax+x^2a)) + r_{n-3}\underline{x^2a^{n+1}},$$

$$xax(a^{n-2}x^2) \to -(\sum_{i=n-2}^{n} r_i\underline{xaxP(n-5,i-n+2)a^3}+ \tag{A.86}$$

$$\sum_{i=n-1}^{n} r_i\underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)}+$$

$$\underline{xaxP(n-3,0)(xax+x^2a)}) + r_{n-3}\underline{xaxa^n}$$

and

$$ax^2(a^{n-2}x^2) \to -(\sum_{i=n-2}^{n} r_i\underline{ax^2P(n-5,i-n+2)a^3}+ \tag{A.87}$$

$$\sum_{i=n-1}^{n} r_i\underline{ax^2P(n-4,i-n+1)(a^2x+axa+xa^2)}+$$

$$\underline{ax^2P(n-3,0)(xax+x^2a)}) + r_{n-3}\underline{ax^2a^n}$$

respectively. The word in (A.81)-(A.87) of length $n+3$ which are reducible are as follows:

1. $-x^2a^{n-2}x^2a \in x(a^{n-2}x^2)xa$

2. $-x^3a^{n-1}x \in x(a^{n-2}x^2)ax$

3. $-x^2a^{n-2}x^2a \in x^2a(a^{n-2}x^2)$.

But terms (i) and (ii) in the preceding list have opposite signs with the terms (i) and (iv) in the list of nine reducible words above so they cancel out. Thus, we only have to reduce the term (iii). Using (A.7), expand σ_{n-2} as

$$a^{n-2}x^2 \to -(\sum_{i=n-2}^{n-1} r_i \underline{aP(n-5,i-n+2)a^2} + \underline{aP(n-5,2)a^2}+$$

$$\sum_{i=n-1}^{n} r_i(\underline{aP(n-4,i-n+1)ax} + \underline{aP(n-4,i-n+1)xa} + \underline{xP(n-4,i-n+1)a^2})$$

$$+\underline{xP(n-3,0)ax} + \underline{xP(n-3,0)xa}) + r_{n-2}\underline{a^n}$$

$$= -(\sum_{i=n-2}^{n} r_i \underline{aP(n-5,i-n+2)a^2} + \sum_{i=n-1}^{n} r_i(\underline{aP(n-4,i-n+1)ax}+$$

$$\underline{aP(n-4,i-n+1)xa} + \underline{xP(n-4,i-n+1)a^2}) + \underline{xP(n-3,0)ax} + \underline{xP(n-3,0)xa})$$

$$+r_{n-2}\underline{a^n}.$$

Pre and post multiplying this by x^2 and a respectively, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-x^2a^{n-2}x^2a \to (\sum_{i=n-2}^{n} r_i \underline{x^2aP(n-5,i-n+2)a^3} + \sum_{i=n-1}^{n} r_i(\underline{x^2aP(n-4,i-n+1)axa}+ \tag{A.88}$$

$$\underline{x^2aP(n-4,i-n+1)xa^2} + \underline{x^3P(n-4,i-n+1)a^3}) + \underline{x^3P(n-3,0)axa}$$

$$+\underline{x^3P(n-3,0)xa^2}) - r_{n-2}\underline{x^2a^{n+1}}.$$

Hence, the reduction process ends here and when we substitute (A.74)-(A.88) into (A.73), we get

$$a^3\omega_{n-4} \to (\sum_{i=n-3}^{n} r_i \underline{a^2xP(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i(\underline{a^2xP(n-5,i-n+2)(a^2x+axa+xa^2)} \tag{A.89}$$

$$+\underline{axaP(n-5,i-n+2)a^2x}) + \sum_{i=n-1}^{n} r_i(\underline{a^2xP(n-4,i-n+1)(ax^2+xax+x^2a)}+$$

$$\underline{axaP(n-4,i-n+1)(ax^2+xax)} + \underline{xa^2P(n-4,i-n+1)ax^2} + \underline{x^3P(n-4,i-n+1)a^3})+$$

$$\underline{x^3P(n-3,0)(axa+xa^2} + r_{n-2}\underline{xaxa^n}) - (\sum_{i=n-2}^{n} r_i \underline{xax(P(n-5,i-n+2)a^3}+$$

$$\sum_{i=n-1}^{n} r_i(\underline{ax^2P(n-4,i-n+1)(axa+xa^2)} + \underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)}+$$

$$\underline{x^2aP(n-4,i-n+1)a^2x}) + r_{n-1}a^3P(n-4,0)x^3 + \underline{a^3Q(n-4,1)x^3} + \underline{xaxP(n-3,0)(xax+x^2a)}$$

$$+\underline{x^2aP(n-3,0)xax} + \underline{ax^2(P(n-3,0)x^2a+P(n-5,2)a^3)} + r_{n-3}\underline{a^2xa^n}).$$

Similarly, using (A.4), expand σ_{n-1} as

$$\omega_{n-1} \to -(r_{n-1}\underline{a^3P(n-4,0)} + \underline{a^3Q(n-4,1)} + \underline{(a^2x+axa+xa^2)P(n-3,0)}) + r_{n-1}\underline{a^n}.$$

Post multiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$\omega_{n-1}x^3 \to -(r_{n-1}a^3P(n-4,0)x^3 + \underline{a^3Q(n-4,1)x^3} + (a^2x+axa+xa^2)P(n-3,0)x^3) \quad \text{(A.90)}$$
$$+r_{n-1}\underline{a^nx^3}.$$

The following words in (A.90) of length $n+3$ are reducible:

1. $xa^{n-1}x^3$
2. $axa^{n-2}x^3$
3. $a^2xa^{n-3}x^3$.

Using (A.4), expand σ_{n-3} as

$$a^{n-3}x^3 \to -(\sum_{i=n-3}^{n-1} r_i\underline{P(n-6,i-n+3)a^3} + \underline{P(n-6,3)a^3}+$$
$$\sum_{i=n-2}^{n-1} r_i\underline{P(n-5,i-n+2)(a^2x+axa+xa^2)} + \underline{P(n-5,2)(a^2x+axa+xa^2)}+$$
$$\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(ax^2+xax+x^2a)}) + r_{n-3}\underline{a^n}.$$
$$= -(\sum_{i=n-3}^{n} r_i\underline{P(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i\underline{P(n-5,i-n+2)(a^2x+axa+xa^2)}+$$
$$\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)(ax^2+xax+x^2a)}) + r_{n-3}\underline{a^n}.$$

Thus, premultiplying this by xa^2, axa and a^2x, and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yield

$$-xa^2(a^{n-3}x^3) \to (\sum_{i=n-3}^{n} r_i\underline{xa^2P(n-6,i-n+3)a^3}+ \quad \text{(A.91)}$$
$$\sum_{i=n-2}^{n} r_i\underline{xa^2P(n-5,i-n+2)(a^2x+axa+xa^2)}+$$
$$\sum_{i=n-1}^{n} r_ixa^2P(n-4,i-n+1)(ax^2+xax+x^2a)) - r_{n-3}\underline{xa^{n+2}},$$

$$-axa(a^{n-3}x^3) \to (\sum_{i=n-3}^{n} r_i \underline{axaP(n-6,i-n+3)a^3}+ \tag{A.92}$$
$$\sum_{i=n-2}^{n} r_i \underline{axaP(n-5,i-n+2)(a^2x+axa+xa^2)}+$$
$$\sum_{i=n-1}^{n} r_i \underline{axaP(n-4,i-n+1)(ax^2+xax+x^2a)}) - r_{n-3}\underline{axa^{n+1}}$$

and

$$-a^2x(a^{n-3}x^3) \to (\sum_{i=n-3}^{n} r_i \underline{a^2xP(n-6,i-n+3)a^3}+ \tag{A.93}$$
$$\sum_{i=n-2}^{n} r_i \underline{a^2xP(n-5,i-n+2)(a^2x+axa+xa^2)}+$$
$$\sum_{i=n-1}^{n} r_i \underline{a^2xP(n-4,i-n+1)(ax^2+xax+x^2a)}) - r_{n-3}\underline{a^2xa^n}$$

respectively. The reducible words in (A.91)-(A.93) of length $n+3$ are as follows:

1. $xa^{n-2}x^2xa \in xa^2P(n-4,1)x^2a$
2. $xa^{n-2}x^2ax \in xa^2P(n-4,1)xax$
3. $xa^{n-3}x^3a^2 \in xa^2P(n-4,1)ax^2$
4. $axa^{n-3}x^3a \in axaP(n-4,1)x^2a$.

Recall from (A.81)-(A.84) that

$$x(a^{n-2}x^2)xa \to -(\sum_{i=n-2}^{n} r_i \underline{xa^2P(n-5,i-n+2)axa} + \underline{x^3P(n-3,0)axa}+$$
$$\underline{xaxP(n-3,0)x^2a} + x^2aP(n-3,0)x^2a + \sum_{i=n-1}^{n} r_i(\underline{xaxP(n-4,i-n+1)axa}+$$
$$\underline{x^2aP(n-4,i-n+1)axa}) + r_{n-1}\underline{xa^2P(n-4,0)x^2a} + \underline{xa^2Q(n-4,1)x^2a})$$
$$+r_{n-2}\underline{x^2a^{n+1}}.$$

$$x(a^{n-2}x^2)ax \to -(\sum_{i=n-2}^{n} r_i \underline{xa^2P(n-5,i-n+2)a^2x} + x^3P(n-3,0)a^2x+$$
$$\underline{x^2aP(n-3,0)xax} + \underline{xaxP(n-3,0)xax} + \sum_{i=n-1}^{n} r_i(\underline{xaxP(n-4,i-n+1)a^2x}+$$
$$\underline{x^2aP(n-4,i-n+1)a^2x}) + r_{n-1}\underline{xa^2P(n-4,0)xax} + \underline{xa^2Q(n-4,1)xax})$$
$$+r_{n-2}\underline{xaxa^n}.$$

$$axa^{n-3}x^3a \to -(\sum_{i=n-3}^{n} r_i\underline{axaP(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i(\underline{axaP(n-5,i-n+2)axa}+$$

$$\underline{axaP(n-5,i-n+2)xa^2} + \underline{ax^2P(n-5,i-n+2)a^3}) + \sum_{i=n-1}^{n} r_i(\underline{ax^2P(n-4,i-n+1)axa}$$

$$+\underline{ax^2P(n-4,i-n+1)xa^2}) + r_{n-1}\underline{axaP(n-4,0)x^2a} + \underline{axaQ(n-4,1)x^2a}+$$

$$\underline{ax^2P(n-3,0)x^2a}) + r_{n-3}\underline{axa^{n+1}}$$

and

$$xa^{n-3}x^3a^2 \to -(\sum_{i=n-3}^{n} r_i\underline{xa^2P(n-6,i-n+3)a^3} + \sum_{i=n-2}^{n} r_i(\underline{xaxP(n-5,i-n+2)a^3}+$$

$$\underline{x^2aP(n-5,i-n+2)a^3}) + \sum_{i=n-2}^{n-1} r_i\underline{xa^2P(n-5,i-n+2)xa^2} + \underline{xa^2Q(n-5,2)xa^2}+$$

$$\sum_{i=n-1}^{n} r_i(\underline{x^3P(n-4,i-n+1)a^3} + \underline{xaxP(n-4,i-n+1)xa^2} + \underline{x^2aP(n-4,i-n+1)xa^2})$$

$$+\underline{x^3P(n-3,0)xa^2}) + r_{n-3}\underline{xa^{n+2}}.$$

The reducible words in (A.81)-(A.84) of length $n+3$ are:

1. $-x^2a^{n-2}x^2a \in x(a^{n-2}x^2)xa$
2. $-x^3a^{n-1}x \in x(a^{n-2}x^2)ax.$

Recall from (A.88) that

$$-x^2a^{n-2}x^2a \to (\sum_{i=n-2}^{n} r_i\underline{x^2aP(n-5,i-n+2)a^3} + \sum_{i=n-1}^{n} r_i(\underline{x^2aP(n-4,i-n+1)axa}+$$

$$\underline{x^2aP(n-4,i-n+1)xa^2} + \underline{x^3P(n-4,i-n+1)a^3}) + \underline{x^3P(n-3,0)axa}$$

$$+\underline{x^3P(n-3,0)xa^2}) - r_{n-2}\underline{x^2a^{n+1}}.$$

Also, using (A.4), expand σ_{n-1} as

$$a^{n-1}x \to -(\sum_{i=n-1}^{n} r_i\underline{P(n-4,i-n+1)a^3} + \underline{P(n-3,0)(axa+xa^2)}) + r_{n-1}\underline{a^n}.$$

Premultiplying this by x^3 and using Lemmas 2.6 and 2.7 to separate reducible and irreducible words, yields

$$-x^3a^{n-1}x \to (\sum_{i=n-1}^{n} r_i\underline{x^3P(n-4,i-n+1)a^3} + \underline{x^3P(n-3,0)(axa+xa^2)}) - r_{n-1}\underline{x^3a^n}. \quad (A.94)$$

Thus, the reduction process ends here and when we substitute (A.81)-(A.84), (A.88), (A.91)-(A.94) into (A.90), we get

$$\omega_{n-1}x^3 \to (\sum_{i=n-3}^{n} r_i \underline{a^2xP(n-6,i-n+3)a^3}+ \tag{A.95}$$

$$\sum_{i=n-2}^{n} r_i(\underline{a^2xP(n-5,i-n+2)(a^2x+axa+xa^2)}+$$

$$\underline{axaP(n-5,i-n+2)a^2x}) + \sum_{i=n-1}^{n} r_i(\underline{a^2xP(n-4,i-n+1)(ax^2+xax+x^2a)}+$$

$$\underline{axaP(n-4,i-n+1)(ax^2+xax)} + \underline{xa^2P(n-4,i-n+1)ax^2} + \underline{x^3P(n-4,i-n+1)a^3})+$$

$$\underline{x^3P(n-3,0)(axa+xa^2} + r_{n-2}\underline{xaxa^n}) - (\sum_{i=n-2}^{n} r_i\underline{xax(P(n-5,i-n+2)a^3}$$

$$+\sum_{i=n-1}^{n} r_i(\underline{ax^2P(n-4,i-n+1)(axa+xa^2)} + \underline{xaxP(n-4,i-n+1)(a^2x+axa+xa^2)}+$$

$$\underline{x^2aP(n-4,i-n+1)a^2x}) + r_{n-1}a^3P(n-4,0)x^3 + \underline{a^3Q(n-4,1)x^3}+$$

$$\underline{xaxP(n-3,0)(xax+x^2a)} + \underline{x^2aP(n-3,0)xax} + \underline{ax^2(P(n-3,0)x^2a+P(n-5,2)a^3)}$$

$$+r_{n-3}\underline{a^2xa^n}).$$

Comparing (A.89) and (A.95), we conclude that the overlap ambiguity $\{\omega_{n-4}, \omega_{n-1}\}$ is resolvable.

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