A thesis presented by

GEORGE W. SWAN

in fulfilment of the requirements

of the degree of Doctor of Philosophy

of the University of Glasgow

Department of Mathematics
University of Strathclyde*
Glasgow

March 1965

* Formerly the Royal College of Science and Technology, Glasgow.

SUMMARY OF THESIS

The Method of Characteristics is applied to the study of the non-linear partial differential equations which govern the two-dimensional steady motion of a fully ionised gas, which is idealised as a perfect fluid of infinite electrical conductivity, in the presence of a magnetic field. Certain results are derived for the flow over slender wedges. For the steady flow over a convex corner of infinitesimal angle (of Prandtl - Meyer type in ordinary gas dynamics) it is shown that no steady state solution exists. The reasons for the breakdown of this type of flow are investigated via the propagation of magnetic disturbances in the non-conducting solid wall and the disturbances in the gas. Reflected and transmitted waves are given as solutions of a singular integral equation.

A solution is now presented to the non-linear problem of the attached shock-wave configuration which appears when a non-conducting symmetric wedge of finite angle travels through the gas. The applied magnetic field is oblique to and is in the same plane as the incident stream. The presence of the magnetic field non-aligned with the stream renders the shock-wave pattern on the upper half-plane different from that in the lower half-plane. There is no symmetry in the flow. From the jump conditions appropriate to these plane magneto-gasdynamic shock waves expressions for the unknown parameters downstream of the shocks are given in terms of the known (in general) quantities upstream. Direct analytic solution is not feasible. Perturbations are made from the known solution for the case when the magnetic field is aligned with the stream and depends on the solution of 24 equations in 26 unknowns. To obtain sufficient equations it is required to match the solution for the flow with that found in the wedge. The effect of the coupling of the flows above and below the wedge via the boundary conditions on the magnetic field is demonstrated. These equations are solved numerically and solutions are presented for angles of inclination of the magnetic field to the stream up to 12°. The computations are executed for the case of a weak magnetic field and some tabulated results are given.

The physical stability of the shock-wave solutions found is then investigated. This final analysis includes the dismissal of the upstream facing thermodynamically stable shock waves which are predicted by the aligned fields theory.

ACKNOWLEDGMENTS

The author desires to acknowledge his gratitude for the assistance and guidance given by Professor D. C. Pack of the Department of Mathematics in the University of Strathclyde.

The author is indebted to the Trustees of the Sir James Caird's Scholarships Trust for a Junior Scholarship in Aeronautics held from October 1961 to December 1962.

Since January 1963 to September 1964 the research has been supported by the United States Air Force under Grants Nos. 66-22 and 64-6.

But for these grants the author would not have contemplated doing research work. The author is also grateful to Professor D. C. Pack for the provision of excellent working facilities in the Department.

GENERAL INTRODUCTION

We consider the equations of two-dimensional steady flow of a fully ionised gas. In particular we obtain the characteristics of these equations together with their conditions of compatibility. Certain results dependent on linearised theory are derived. There is an application of the theory to the flow over a thin non-conducting wedge for certain configurations of the applied magnetic field. Also the flow over a sharp convex corner is considered. In ordinary gas dynamics this kind of flow is classified as being of the Prandtl-Meyer type. It is demonstrated that, in general, we cannot have flows of this type in magneto-gasdynamics. The reasons for this are indicated, and in the second part of the thesis a more detailed analysis is given and the physical mechanism involved is explained.

The third part is devoted to the method of construction of a solution to the non-linear shock wave problem which arises when a non-conducting wedge travels through a fully ionised gas. Since characteristics are weak shocks of vanishing strength we show first that for a thin wedge certain of the results derived in Part I can be deduced from the equations holding across the shock waves. Also the analysis of Part I aids in the formulation of a perturbation technique which is applied directly to the full system of shock wave equations.

The solution (with numerical results given in graphic form) to the attached shock wave problem when the angle between the magnetic field and the incident stream is zero was given by Cabannes [6]. We allow this angle to increase from its zero value and arrange our expansions for p_2/p_1, v_2/v_1, etc., in terms of it. Thus the zero-order terms are those which come direct from the Cabannes theory. The expansions to first order terms are given. We can show analytically that only in the case when the gas speed exceeds the Alfvén speed do we obtain a physically meaningful solution. The perturbation method breaks down when the gas speed is equal to or is less than the Alfvén speed. Extensive computations have been executed for the case of a weak magnetic field, a wedge of semi-vertex angle 20° and for angles of inclinations of magnetic field to incident stream at 2° intervals from 0° to 12°.

In the final part of the thesis the stability of these attached shock waves is investigated. The numerical results obtained in the perturbation analysis aid in resolving some of the difficulties which are posed when we try to satisfy the conditions for stability.

The substance of part II of the thesis is to appear in a forthcoming publication of the Q.J.M.A.M. (See Vol XVIII, Part 2, (May 1965) pp 243-255.)

INDEX

NOTATION AND SYMBOLS

Rationalised MKS units are used throughout.

$\underline{H}$	magnetic intensity or field strength
H_x, H_y	x,y - components of magnetic field
$\underline{E}$	electric intensity or field strength
$\underline{j}$	current density vector
μ	magnetic permeability
σ	electrical conductivity, assumed independent of time and position
$\underline{V}$	fluid velocity
u,v	x,y - components of velocity
ρ	fluid density
p	fluid pressure
a	local sound speed
M	Mach number ($= V/a$)
χ	angle between $\underline{H}$ and $\underline{V}$
θ	flow direction

PART I

LINEARISED THEORY OF STEADY TWO-DIMENSIONAL MAGNETO-GASDYNAMIC FLOW

INTRODUCTION

Many qualitative results can be obtained from linear theories. In magneto-gasdynamics the basic partial differential equations are more complicated than those of ordinary gas dynamics because of the presence of those terms which arise from the consideration of the magnetic field. One method of analysing these equations is via the theory of characteristi We look at this theory afresh with a view to decreasing the actual physica effort which is involved in the evaluation of the determinants of large order. A method is evolved which is particularly suited to problems involving two independent variables. Since disturbances created in the flow propagate along the characteristics we can examine the effect of introducing a slender body into the flow. The flow over a wall of some arbitrary shape which varies slowly may also be examined. When the body is no longer thin the non-linear effects begin to dominate and the created disturbance is of finite size. Accordingly in the following sections we obtain first of all an understanding of the processes involved in a linear theory. Knowledge of these results will help in the construction of the solution to the non-linear problem.

1.1 BASIC EQUATIONS

We shall concern ourselves with the macroscopic motion of a fully ionised gas, which we shall assume to be adequately represented as a perfectly conducting inviscid non-heat-conducting perfect gas. The temperature of the gas is assumed to be sufficiently low and density high enough in order that effects due to Hall current may be neglected. All gas velocities are assumed to be small in comparison with the speed of light. A consequence of this assumption is that we can neglect the charge density and the displacement current terms from the partial differential equations. For simplification we also assume that the motion of the gas is not influenced by any external forces. The equation of motion for the gas are then those of magneto-gasdynamics. With the notation and under the assumptions outlined above we write these equation in the following form:

$$\rho \frac{D\underline{V}}{Dt} = -\nabla p + \mu \, \underline{j} \wedge \underline{H} \quad , \qquad \text{equation of motion,} \tag{1.1}$$

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \underline{V} = 0 , \qquad \text{equation of continuity ,} \tag{1.2}$$

$$\frac{Dp}{Dt} = a^2 \frac{D\rho}{Dt} \quad , \qquad \text{adiabatic condition .} \tag{1.3}$$

We also have the Maxwell equations:

$$\operatorname{div} \underline{H} = 0 , \tag{1.4}$$

$$\operatorname{curl} \underline{E} = - \frac{\partial \underline{B}}{\partial t} , \tag{1.5}$$

$$\operatorname{curl} \underline{H} = \underline{j} , \tag{1.6}$$

$$\underline{E} + \mu \, \underline{V} \wedge \underline{H} = \underline{0} \quad , \quad \sigma = \infty \qquad \text{assumed,}$$

$$\operatorname{div} \underline{E} = 0 . \tag{1.7}$$

These equations are true for an isotropic medium with $\underline{B} = \mu\underline{H}$. The operator D/Dt is defined as $\partial/\partial t + (\underline{V}.\nabla)$.

1.2 THEORY OF CHARACTERISTICS

Many sets of conservation laws for physical systems involving two independent variables x, y can be expressed in the form:

$$P \frac{\partial A}{\partial x} + Q \frac{\partial A}{\partial y} = 0, \tag{1.8}$$

where P, Q are matrices of some particular order and A is a column vector. The quantities $\partial/\partial x$, $\partial/\partial y$ operate on each element of A. If we specify the members of A to be on a curve in the x-y plane determined by the parametric equations $x = x(\tau)$, $y = y(\tau)$, then

$$\frac{dA}{d\tau} = \frac{\partial A}{\partial x}\frac{dx}{d\tau} + \frac{\partial A}{\partial y}\frac{dy}{d\tau}. \tag{1.9}$$

In this equation the differentiations are taken with respect to τ along the curve. On premultiplying both sides of equation (1.9) by P and using (1.8) we obtain

$$\left(P\frac{dy}{d\tau} - Q\frac{dx}{d\tau}\right)\frac{\partial A}{\partial y} = P\frac{dA}{d\tau}. \tag{1.10}$$

We have in this equation a means of finding the first order partial derivatives with respect to y of the elements of A. This solution is unique unles

$$\det\left|Py' - Q\right| = 0. \tag{1.11}$$

From this equation the values of y' ($= dy/dx$) give the characteristic curves of (1.8). When these values are real the system of equations (1.8) is of hyperbolic type.

If we replace any column of the determinant on the left side of (1.11) by the quantity $P\ dA/dx$, found from (1.10), and equate the modified determinant to zero we have the condition for compatibility. Note that the result of the analysis is the same if we premultiply (1.9) by Q instead of P.

The important feature of this method is that the order of the determinant in (1.11) is equal to the number of elements of A. Comparison with the usual methods of obtaining characteristics shows that in this case the order of the determinant involved has been halved, and this consequently results in a large saving in the labour of evaluation. With this method the characteristics for ordinary two-dimensional isentropic steady gas flows can be written down immediately from (1.11). The method is extremely powerful in the investigation of systems containing a large number of partial differential equations. This is demonstrated in the next section.

1.3 CHARACTERISTICS OF 2-DIMENSIONAL STEADY FLOW

For steady two-dimensional motion of the gas with $\underline{H}$ in the plane of the flow the equations (1.1) - (1.7) in component form when $\sigma = \infty$ become

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu H_y \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right),$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - \mu H_x \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right),$$

$$\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 , \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 ,$$

$$u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} - a^2 \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) = 0 , \qquad u H_y - v H_x = K . \tag{1.12 a,b}$$

The last equation in this group is an integral of (1.5) and K is a constant which is determined by the initial conditions or by the condition at upstream infinity. To shorten the labour we can eliminate the derivatives $\partial H_y/\partial x$, $\partial H_y/\partial y$ from this system by using this last equation. We find that

$$(\rho u^2 - \mu H_y^2) \frac{\partial u}{\partial x} + \rho v u \frac{\partial u}{\partial y} + u \frac{\partial p}{\partial x} - \mu u H_y \frac{\partial H_x}{\partial y} + \mu v H_y \frac{\partial H_x}{\partial x} + \mu H_x H_y \frac{\partial v}{\partial x} = 0 ,$$

$$(\rho u^2 - \mu H_x^2) \frac{\partial v}{\partial x} + \rho v u \frac{\partial v}{\partial y} + u \frac{\partial p}{\partial y} + \mu u H_x \frac{\partial H_x}{\partial y} - \mu v H_x \frac{\partial H_x}{\partial x} + \mu H_x H_y \frac{\partial u}{\partial x} = 0 , \tag{1.13}$$

$$\rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 , \qquad u \frac{\partial H_x}{\partial x} + v \frac{\partial H_x}{\partial y} + H_x \frac{\partial v}{\partial y} - H_y \frac{\partial u}{\partial y} = 0$$

$$u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} - a^2 u \frac{\partial \rho}{\partial x} - a^2 v \frac{\partial \rho}{\partial y} = 0 .$$

It is readily seen that this system of equations is of the same general form as the set given by equation (1.8). With A chosen as the column vector $\{u, v, \rho, p, H_x\}$, on using (1.11) we see that the characteristics of the set (1.13) are given by

$$\begin{vmatrix} (\rho u^2-\mu H_y^2)y'-\rho v u & \mu H_x H_y y' & 0 & u y' & \mu v H_y y'+\mu u H_y \\ \mu H_x H_y y' & (\rho u^2-\mu H_x^2)y'-\rho v u & 0 & -u & -\mu v H_x y'-\mu u H_x \\ \rho y' & -\rho & uy'-v & 0 & 0 \\ H_y & -H_x & 0 & 0 & uy'-v \\ 0 & 0 & -a^2(uy'-v) & uy'-v & 0 \end{vmatrix} = 0. \qquad (1.14)$$

A straightforward expansion of this determinant gives for $\lambda = uy' -$ the results:

$$uy'-v = 0\;, \qquad (1.15)$$

$$\lambda^2\rho\left\{a^2(1+y'^2)-\lambda^2\right\} = \mu(1+y'^2)\left\{a^2(y'H_x-H_y)^2-\lambda^2 H^2\right\}. \qquad (1.16)$$

The first of these, equation (1.15), is the equation for the streamline. The second, which is of degree four in y', is of a complicated nature and investigation of the roots is difficult. Before analysing (1.16) in detail we derive certain results which follow immediately from it.

On setting $\underline{H} = \underline{0}$ in equation (1.16) we recover immediately the equation for the characteristics for ordinary steady two-dimensional gas dynamic flows. Note that two of the characteristics of (1.16) now become streamlines.

Consider the special case when $\underline{H}$ is parallel to $\underline{V}$ (the aligned fields case). The condition for parallelism of the vectors at infinity gives $K = 0$; and

$$u/H_x = v/H_y = V/H, \quad V^2 = u^2 + v^2, \quad H^2 = H_x^2 + H_y^2 .$$

If we write

$$\epsilon^2 = (\text{Alfvén speed/sound speed})^2 = \mu H^2 \rho^{-1}/a^2 \tag{1.17}$$

then equation (1.16) reduces, for $M = V/a$, the Mach number, to

$$\lambda^2 \left[\lambda^2/a^2 - (1+y'^2)\left\{ M^2 - \epsilon^2(1-M^2) \right\}/M^2 \right] = 0 . \tag{1.18}$$

We see from equation (1.18) that two of the characteristics have now collapsed onto the streamline. The converse of this result is also true. That is, if we start from an aligned fields flow, and allow the magnetic field to deviate by a small amount from its parallel direction with $\underline{V}$, then two of the characteristics in the now non-aligned fields set-up have sprung from the streamline. This latter result forms an important part in the perturbation analysis of Part III. If we define σ as the angle between a characteristic and a streamline then we have

$$dy/dx = \tan(\theta + \sigma) , \qquad \text{where} \qquad \tan\theta = v/u ,$$

and equation (1.18) becomes with this notation

$$\tan^2\sigma = \left\{ M^2 + \epsilon^2(M^2-1) \right\} / (M^2-\epsilon^2)(M^2-1) . \tag{1.19}$$

From equation (1.19) it is possible to determine regions, specified by values of ϵ^2 and M^2, for which the angle σ is real or imaginary. This has been done by various authors.

For the case when $\underline{H}$ and $\underline{V}$ have a random orientation it is possible to choose a moving system of co-ordinates in which $\underline{H}$ and $\underline{V}$ are now aligned. Thus instead of using (1.16) directly we could make interpretations on the behaviour of the characteristics via equation (1.19). Precisely this technique was used by Kogan [1]; however he commenced his analysis with linearised forms of the equations (1.1) - (1.7) and not from the non-linear equations. If we replace

$$\underline{H} \text{ by } (H_{x_0}+H_x, H_{y_0}+H_y), \ \underline{V} \text{ by } (V_0+u, v), \ p \text{ by } p_0+p, \ \rho \text{ by } \rho_0+\rho \tag{1.20}$$

in equation (1.16) we find that the linearised form is

$$\begin{aligned} &\left\{(M_0^2-\varepsilon_{x_0}^2)(1-M_0^2)+M_0^2\varepsilon_{y_0}^2\right\}y'^4+2\varepsilon_{x_0}\varepsilon_{y_0}y'^3+ \\ &+\left\{M_0^2-\varepsilon_0^2(1-M_0^2)\right\}y'^2+2\varepsilon_{x_0}\varepsilon_{y_0}y'-\varepsilon_{y_0}^2 = 0 . \end{aligned} \tag{1.21}$$

In this equation y' denotes the inclination of a characteristic to the velocity vector and $\epsilon_{xo}^2 = H_{xo}^2/\rho_o a_o^2$.

For $\epsilon_o^2 < 1$ in Figure 1.1a we see the regimes for which (1.21) has four real characteristics (fully hyperbolic flow), two real characteristics (quasi-hyperbolic flow) and no real characteristics. This figure was first given by Kogan [1]. For $\epsilon_o^2 > 1$ we have a similar figure, Figure 1.1b.

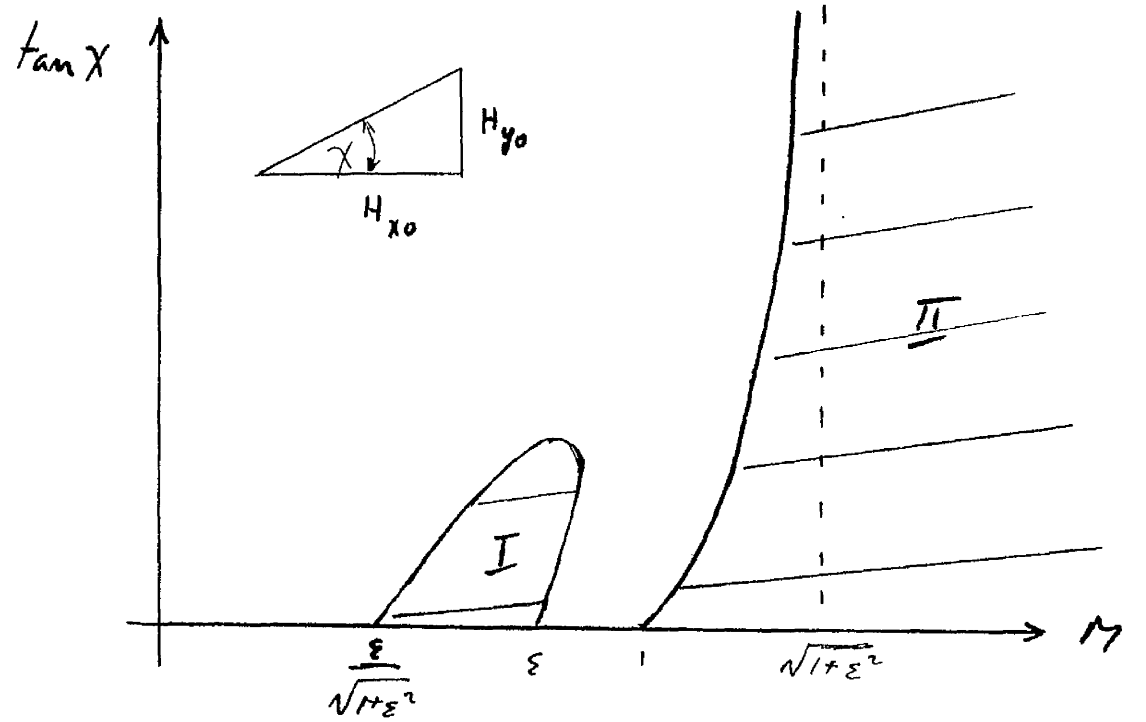

FIGURE 1.1a: REGIMES OF FLOW: I QUASI-HYPERBOLIC

II FULLY HYPERBOLIC $(\varepsilon < 1)$

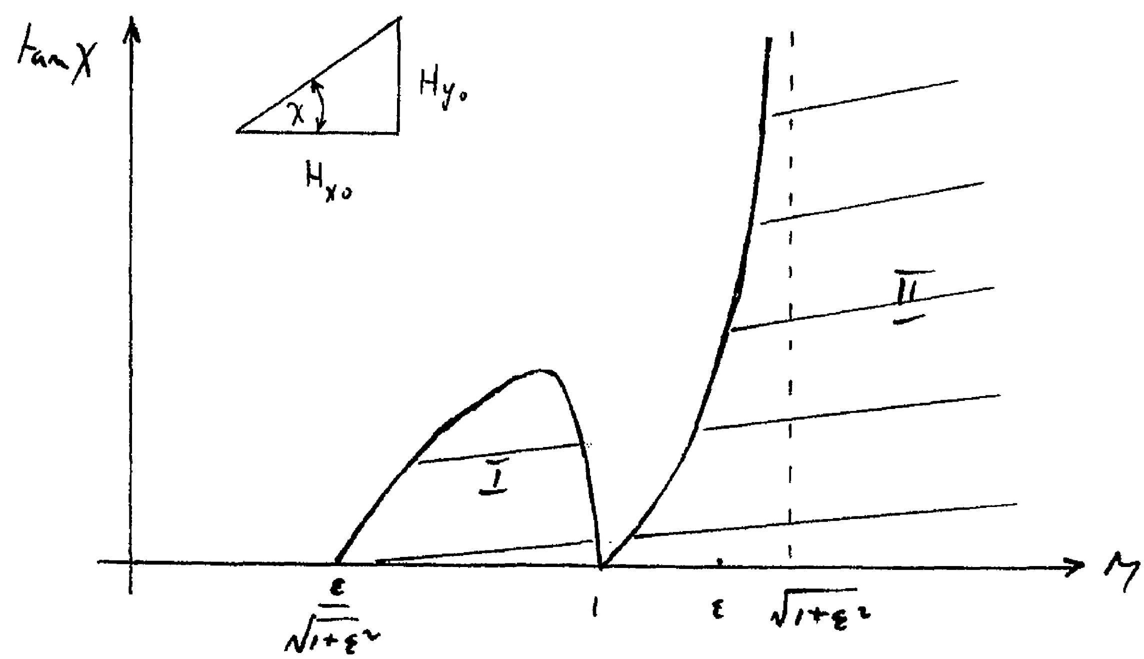

FIGURE 1.1b: REGIMES OF FLOW: I QUASI-HYPERBOLIC

II FULLY HYPERBOLIC $(\varepsilon > 1)$

When $\underline{H}_o$ is perpendicular to $\underline{V}_o$, so that H_{xo} is zero we have the crossed fields configuration, and for this case equation (1.21) becomes

$$(y'_{1,2})^2 = \frac{-\{M_o^2-\varepsilon_o^2(1-M_o^2)\}\pm\sqrt{\left[\{M_o^2-\varepsilon_o^2(1-M_o^2)\}^2+4\varepsilon_o^2M_o^2\{(1-M_o^2)+\varepsilon_o^2\}\right]}}{2M_o^2\{(1-M_o^2)+\varepsilon_o^2\}}. \quad (1.22)$$

Kogan shows that for this situation ($\underline{H}_o \perp \underline{V}_o$), hyperbolic flow exists for $M_o^2 \geq 1 + \epsilon_o^2$ (see Figure 1.1) and from equation (1.22) we see that four distinct values of y' exist; the suffix 1 corresponds to the smaller root and the suffix 2 to the larger one.

1.4 CONDITIONS OF COMPATIBILITY

By means of the notation introduced in section 1.2 we have

$$\rho\frac{dA}{dx} = \begin{bmatrix} (\rho u^2-\mu H_y^2)\frac{du}{dx}+\mu H_xH_y\frac{dv}{dx}+u\frac{dp}{dx}+\mu v H_y\frac{dH_x}{dx} \\ \mu H_xH_y\frac{du}{dx}+(\rho a^2-\mu H_x^2)\frac{dv}{dx}-\mu v H_x\frac{dH_x}{dx} \\ \rho\frac{du}{dx}+u\frac{d\rho}{dx} \\ u\frac{dH_x}{dx} \\ -a^2u\frac{d\rho}{dx}+u\frac{dp}{dx} \end{bmatrix} \quad (1.23)$$

If we replace the last column of the determinant (1.14) by the column vecto (1.23) and expand the new determinant we obtain the results

$$uy'-v=0, \quad (1.24)$$

$$K_1\,dp+K_2\,du+K_3\,dv+K_4\,dH_x=0, \quad (1.25)$$

where we have introduced

$$K_1 = u\{\lambda(uH_x + vH_y) - a^2(y'H_x - H_y)\}, \tag{1.26}$$

$$K_2 = \rho\lambda H_y\left[\frac{-\mu}{\rho\lambda}\{a^2(y'H_x - H_y)^2 - \lambda^2 H^2\} - u(u\lambda - a^2 y')\right] + a^2\rho u K, \tag{1.27}$$

$$K_3 = -\rho\lambda H_x\left[\frac{-\mu}{\rho\lambda}\{a^2(y'H_x - H_y)^2 - \lambda^2 H^2\} - u(u\lambda - a^2 y')\right] + a^2\rho u K y', \tag{1.28}$$

$$K_4 = -\frac{\mu V^2}{\lambda}\{a^2(y'H_x - H_y)^2 - \lambda^2 H^2\}, \tag{1.29}$$

$$uH_y - vH_x = K, \quad uy' - v = \lambda, \quad V^2 = u^2 + v^2, \quad H^2 = H_x^2 + H_y^2.$$

The differential relation (1.25) has been obtained from the non-linear equations, and the values of y' are those obtained from equation (1.16). If we set $\underline{H} = \underline{0}$ in (1.25) we do not recover at once the compatibility condition of ordinary gas dynamics. One way of obtaining this limit is the following. Let the fields become aligned so that $\underline{H} /\!/ \underline{V}$. When we now let $\underline{H}$ be zero the appropriate condition is found.

If we now linearise the differential expression (1.25) via the scheme (1.20) we obtain

$$(1+y'^2)\{(M_0^2 y' - 1)\varepsilon_{x_0} + \varepsilon_{y_0}\}\frac{dp}{\gamma p_0} + \varepsilon_{y_0} M_0^2\{(1+y'^2) - M_0^2 y'^2\}\frac{du}{V_0} +$$

$$+ y' M_0^2(y'\varepsilon_{x_0} M_0^2 + \varepsilon_{y_0})\frac{dv}{V_0} - \varepsilon_{x_0} M_0^2 y'\{(1+y'^2) - M_0^2 y'^2\}\frac{dH_x}{H_{x_0}} = 0. \tag{1.30}$$

This equation does not agree with the compatibility condition given by Kogan [1]. He states that along the characteristics the following equation holds:

$$-(y'^2 + y'^3 \varepsilon_{x_0}\varepsilon_{y_0} + y'\varepsilon_{x_0}\varepsilon_{y_0})\frac{dp}{\gamma p_0} + (y'\varepsilon_{x_0}\varepsilon_{y_0} + M_0^2\varepsilon_{y_0}^2 y'^2 - \varepsilon_{y_0}^2)\frac{du}{V_0} -$$

$$-\{y'^3(M_0^2 - \varepsilon_{x_0}^2) + y'^2\varepsilon_{x_0}\varepsilon_{y_0}(1+M_0^2)\}\frac{dv}{V_0} -$$

$$-(y'^2\varepsilon_{x_0}^2 + M_0^2\varepsilon_{x_0}\varepsilon_{y_0}y'^3 - y'\varepsilon_{x_0}\varepsilon_{y_0})\frac{dH_x}{H_{x_0}} = 0. \qquad (1.31)$$

We know from the analysis of the theory of equations that no new independent result can come from equation (1.11) by replacing a different column by PdA/dx as given by (1.23). However let us examine the result of placing the column vector PdA/dx into the first column of the determinant (1.11) and expanding the result. This time we find that

$$u[\mu(1+y'^2)H_x(vH_x - uH_y) + \rho\lambda\{(\lambda^2-a^2) - u\lambda y'\}]dp +$$

$$+[\mu\rho\{-u^2\lambda H_x^2(1+y'^2) - H_y^2\lambda(\lambda^2-a^2) - \lambda a^2 H_x H_y y' + a^2 u H_x(1+y'^2)(y'H_x - \dots$$

$$+\rho\lambda u\{u(\lambda^2-a^2) - a^2\lambda y'\}]du + \rho\lambda[\mu H_x\{(\lambda^2-a^2)H_y - u^2(1+y'^2)H_y$$

$$+a^2 y'H_x\} - \rho a^2 u y']dv - \mu\rho V^2\{\lambda^2 H_y + a^2(y'H_x - H_y)\}dH_x = 0. \qquad (1.32)$$

If we linearise this expression we obtain (1.31). Direct transformation of (1.30) into (1.31) can be achieved by a manipulation of the differential expression for the equations of the characteristics and the integral of the equations of motion (1.12b).

Kogan [1] uses the linearised results (1.22), (1.31) to investigate the steady flow over a slender non-conducting wedge. He demonstrates that for flows of this type ($H_o \neq V_o$), since characteristics are shock waves of vanishing strength, the pressure increase occurs through two successive shocks. It does not seem possible to generalise his analysis to the case where H_o is arbitrarily inclined to V_o. The reason for this is that the roots of the quartic (1.21) do not separate out and accordingly when we integrate (1.31) along each characteristic and solve for each of the perturbation quantities there is no simplification of the resulting ratios The regime in which there exists four real characteristics is marked on Figure 1.1 (area of fully hyperbolic flow).

In part III we undertake an investigation of the shock wave configuration when there is steady flow over a wedge which is not restricted to be slender. These shock waves are no longer weak and consequently we have a complicated non-linear problem to solve.

Before proceeding to this topic we first of all derive certain results from the linear theory.

Consider the two-dimensional steady flow over a non-conducting solid. Assume that the interface of the solid with the gas consists of two straig edged walls at an infinitesimal angle to each other (see Figure 1.2). We use equation (1.12b), which is

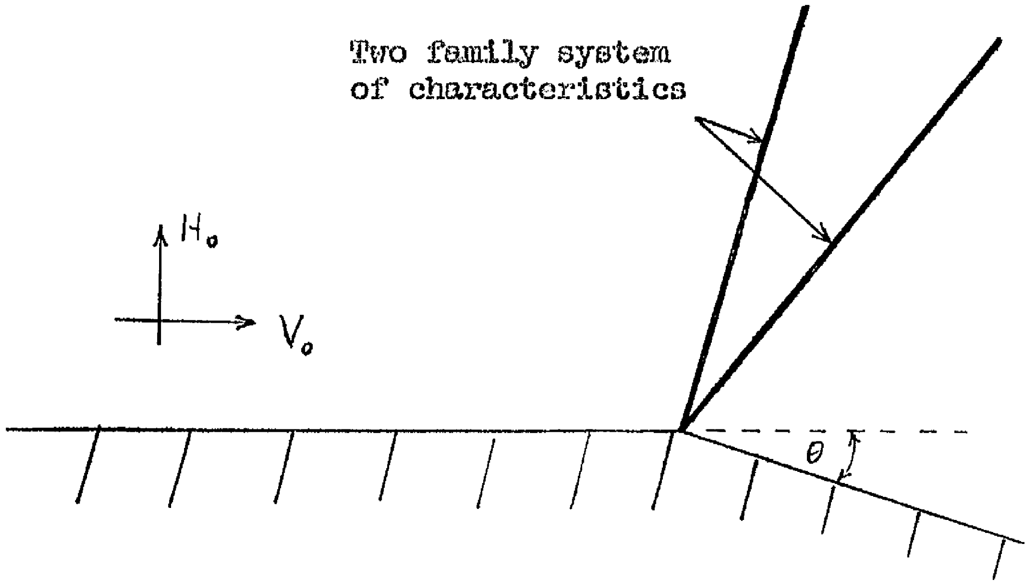

FIGURE 1.2: STEADY FLOW OVER A CONVEX CORNER

an integral of the equations of motion:

$$u H_y - v H_x = K = \text{constant} .$$

If we replace u by $V_o(1 + u)$, H_y by H_{yo}, H_x by $H_{xo} + H_x$ in this equation we obtain, for $v = - V_o\theta$ (since θ is small), the result

$$u H_{y_o} + \theta H_{x_o} = 0 .$$

Thus if H_{xo} is zero we require u to be zero. The substitution of $H_{xo} = 0$ into (1.31) gives

$$-\frac{dp}{\gamma p_o} - y' M_o^2 \frac{dv}{V_o} = 0 .$$

If we integrate this expression along each family of characteristics we obtain

$$p - p_o = \gamma p_o M_o^2 y_1' \theta \quad \text{and} \quad p - p_o = \gamma p_o M_o^2 y_2' \theta .$$

We see immediately that these equations are incompatible with each other. The solution for the magnetic field inside the non-conducting solid does not affect this conclusion.

Because the uniform flow cannot be deviated through an infinitesimal amount round a sharp convex corner one consequence is that we cannot use this linear theory to build up flows similar to those of the Prandtl-Meyer type which exist in ordinary gas dynamics. Here, the corner engenders disturbances in the gas and also in the solid. For the latter since we do not at this stage have sufficient knowledge of their behaviour, we suspect that they contribute to the breakdown of the steady flow pattern. It is felt that these magnetic disturbances in the solid propagate everywhere and because of the boundary conditions applicable across the interface they make their presence known in the gas. A more thorough examination of these effects is set down in the next part of the thesis. Although we commence this investigation from a different standpoint the gradual unfolding of formulae similar to the above serves to unify the whole theory of such flows.

PART II

UPSTREAM INFLUENCE EFFECTS IN THE FLOW OF A CONDUCTING FLUID OVER AN INSULATING WALL

SUMMARY

In conventional gas dynamics there is no great difficulty encountered when studying the propagation of a disturbance through the two-dimensional steady supersonic flow of a perfect gas past a straight-edged wall. A two-dimensional disturbance introduced into the flow travels along the appropriate Mach line to the wall from which it is then reflected downstream. The flow of a uniform stream past a wall convex to the stream is achieved by means of the well-known Prandtl-Meyer expansion. Flows in channels and similar problems can be analysed by the techniques of the method of characteristics.

We have seen at the end of part I, that in magneto-gasdynamics, however, the corresponding problems for the flow of an infinitely conducting gas indicate that situations arise in which disturbances not only propagate upstream in the gas but can also propagate in all directions in the solid wall. One immediate consequence is that the conditions upstream can be continually modified. To obtain an understanding of these processes a mathematical analysis is presented below.

After this work was written up there appeared a paper by Chu [4] who considered a similar sort of problem. In his work the flow everywhere is uniform and the disturbances are created only at the boundary between the gas and the non-conducting solid. The present analysis includes the work of Chu [4] as a particular case.

INTRODUCTION

We assume that an infinitely conducting, non-viscous perfect gas is moving over a stationary non-conducting wall of infinite extent with equation $y = F(x)$. At infinity upstream the gas has a uniform velocity U parallel to the x-axis. A uniform magnetic field of magnitude H_o is assumed to exist, and for the sake of simplicity we suppose that it is orientated at right angles to the flow. This disturbance-free configuration is perturbed only slightly so that the linearised magneto-gasdynamic equations may be used. Let the velocity vector be $\underline{V} = \underline{U} + \underline{v}$ in the gas and the magnetic field intensity vectors be $\underline{H} = \underline{H}_o + \underline{h}$ in the gas and $\underline{H} = \underline{H}_o + \underline{h}_s$ in the non-conducting solid. Under the conditions $|\underline{v}| \ll |\underline{U}|$; $|\underline{h}|, |\underline{h}_s| \ll |\underline{H}_o|$ the current is also a small quantity.

We introduce $\underline{\xi} = \text{curl}\ \underline{H}$. On combining (1.2) with (1.3) we obtain

$$\text{div}\ \underline{v} = \frac{-1}{\rho_\infty a_\infty^2} \cdot \frac{Dp}{Dt}. \qquad (2.1$$

The equation of motion (1.1) is

$$\frac{D\underline{v}}{Dt} + \frac{1}{\rho_\infty} \nabla p = \frac{\mu}{\rho_\infty} \underline{\xi} \wedge \underline{H}. \qquad (2.2$$

Take the divergence of this equation and use (2.1) together with the fact that the flow is isentropic ($dp/d\rho = a_\infty^2$), then

$$\nabla^2 p - \frac{1}{a_\infty^2} \left(\frac{D}{Dt}\right)^2 p = \mu\, \underline{H} \cdot \text{curl}\ \underline{\xi}. \qquad (2.3$$

Take the curl of (2.2). We find that

$$\frac{D}{Dt}(\operatorname{curl} \underline{v}) = \frac{\mu}{\rho_\infty}(\underline{H}_0 . \nabla)\underline{\xi} . \tag{2.4}$$

The elimination of $\underline{E}$ between (1.5) and (1.6) gives, since $\underline{j} = \sigma(\underline{E} + \mu \underline{V}_\wedge \underline{H})$,

$$\frac{\partial \underline{H}}{\partial t} = (\underline{H}_0 . \nabla)\underline{v} - \underline{H}_0 \operatorname{div} \underline{v} + \frac{1}{\mu\sigma} \nabla^2 \underline{H} . \tag{2.5}$$

If we take the curl of (2.5) we obtain

$$\frac{D\underline{\xi}}{Dt} = (\underline{H}_0 . \nabla) \operatorname{curl} \underline{v} - \frac{1}{\rho_\infty a_\infty^2}(\underline{H}_{0\wedge}\nabla)\frac{Dp}{Dt} + \frac{1}{\mu\sigma}\nabla^2 \underline{\xi} . \tag{2.6}$$

We can now eliminate curl $\underline{v}$ between (2.4) and (2.6). If we operate on the result with $\nabla^2 - (1/a_\infty^2)(D/Dt)^2$ and use (2.3) we have an equation for $\underline{\xi}$ alone:

$$\left\{ \nabla^2 - \frac{1}{a_\infty^2}\left(\frac{D}{Dt}\right)^2 \right\} \left\{ \frac{1}{\mu\sigma}\nabla^2 \frac{D\underline{\xi}}{Dt} - \frac{D^2\underline{\xi}}{Dt^2} + \frac{\mu}{\rho_\infty}(\underline{H}_0 . \nabla)^2 \underline{\xi} \right\} =$$

$$= \frac{\mu}{\rho_\infty a_\infty^2}(\underline{H}_{0\wedge}\nabla)\left[\frac{D^2}{Dt^2}\left\{ \underline{H}_0 \cdot \operatorname{curl} \underline{\xi} \right\} \right] . \tag{2.7}$$

For steady two-dimensional flows with $\underline{H}_0$ in the plane of the flow and $\sigma = \infty$ we have curl $\underline{h} = \underline{\xi} = (0,0,\xi)$ and hence from (2.7) we have the following partial differential equation for the current ξ:

$$M^2(1+\varepsilon^2-M^2)\frac{\partial^4 \xi}{\partial x^4} + \left\{ M^2(1+\varepsilon^2) - \varepsilon^2 \right\} \frac{\partial^4 \xi}{\partial x^2 \partial y^2} - \varepsilon^2 \frac{\partial^4 \xi}{\partial y^4} = 0 , \tag{2.8}$$

where we have introduced the notation

$$M^2 = U^2/a_\infty^2 \quad , \qquad \varepsilon^2 = \mu H_0^2/\rho_\infty a_\infty^2 . \tag{2.9a,b}$$

Equation (2.8) was given by McCune and Resler [2], but a different notation was used for the coefficients. We now write the left-hand side of equation (2.8) as the product of two operators:

$$\left(\frac{\partial^2}{\partial x^2} - R^2 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} - S^2 \frac{\partial^2}{\partial y^2}\right)\xi = 0 , \tag{2.1}$$

where,

$$R^2 = \frac{-\{M^2(1+\varepsilon^2)-\varepsilon^2\} - \sqrt{[\{M^2(1+\varepsilon^2)-\varepsilon^2\}^2 + 4\varepsilon^2 M^2(1+\varepsilon^2-M^2)]}}{2M^2(1+\varepsilon^2-M^2)} . \tag{2.1}$$

and S^2 has the same form but with a plus sign in front of the radical.

For gas flows with infinite electrical conductivity, when the applied magnetic field is perpendicular to the main stream, there exist four real characteristics for $M^2 \geq 1 + \epsilon^2$ and in particular the tangents of the angles of inclination of one family of characteristics are given from (2.11) above by $\pm$ R. The inclinations of the other family are given by $\pm$ S. These results were derived in part I, section 3, c.f. (1.22) with (2.11), and it was stated there that Kogan gave (1.22) in [1].

In what follows we shall consider the case when both operators of equation (2.10) are hyperbolic. There are then four characteristics through any point (x,y).

Equation (2.10) can be solved in a quite general manner and we readily find that

$$\frac{\xi}{H_0} = \frac{(1+R^2)}{R^2}\left\{ F_1'(x-R^{-1}y) + F_2'(x+R^{-1}y)\right\} + $$

$$+ \frac{(1+S^2)}{S^2}\left\{ G_1'(x-S^{-1}y) + G_2'(x+S^{-1}y)\right\}, \qquad (2.1$$

where we have chosen the constants and the functions in a manner which will simplify the later working.

FULLY HYPERBOLIC FLOW

2.1 DISTURBANCES IN THE GAS

A small steady disturbance is assumed to originate at some point in the stream and to propagate along each respective characteristic or Mach line to the wall. At the boundary there is a reflection of the disturbance and the effect is carried away on the two individual characteristics which are inclined downstream. The transmitted part of the disturbance passes into the non-conducting solid, where it immediately propagates in all directions and consequently gives rise to further disturbances in the magneto-supersonic region by transmission back across the wall. We will assume that the disturbance has been maintained for some time and will attempt to find a solution with steady flow. (See Figure 2.1).

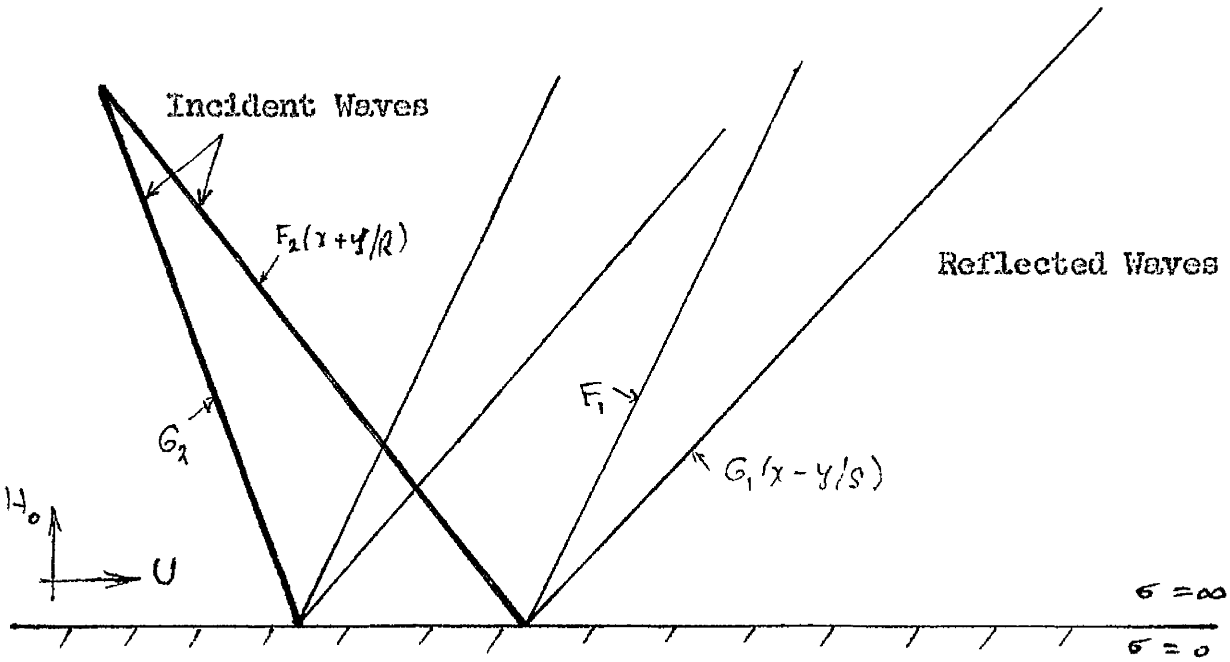

FIGURE 2.1: PROPAGATION OF DISTURBANCE THROUGH THE GAS

Since $\xi = \partial h_y/\partial x - \partial h_x/\partial y$ we can find h_x and h_y from (2.12). From (1.6), when $\sigma = \infty$, $\underline{E} + \underline{V} \wedge \underline{B} = \underline{O}$ and this gives $u H_0 + h_y U = 0$. Thus we can find u and if we substitute into the momentum equation (2.2) we can find p. Finally v is obtained from (2.1). In particular, these perturbation quantities may be written in the form:

$$\frac{h_x}{H_0} = \frac{1}{R}\left\{F_1(x-R^{-1}y) - F_2(x+R^{-1}y)\right\} + \frac{1}{S}\left\{G_1(x-S^{-1}y) - G_2(x+S^{-1}y)\right\}, \quad (2.1$$

$$h_y = H_0\left\{F_1(x-R^{-1}y) + F_2(x+R^{-1}y) + G_1(x-S^{-1}y) + G_2(x+S^{-1}y)\right\}, \quad (2.1$$

$$u = -U\left\{F_1(x-R^{-1}y) + F_2(x+R^{-1}y) + G_1(x-S^{-1}y) + G_2(x+S^{-1}y)\right\}, \quad (2.1$$

$$v = -U\Big[R\left\{1-M^2+\varepsilon^2(1+R^{-2})\right\}\left\{F_1(x-R^{-1}y) - F_2(x+R^{-1}y)\right\} +$$

$$+ S\left\{1-M^2+\varepsilon^2(1+S^{-2})\right\}\left\{G_1(x-S^{-1}y) - G_2(x+S^{-1}y)\right\}\Big], \quad (2.1$$

$$p - p_\infty = -\rho_\infty U^2 \left[(R^2\rho^2+\varepsilon^2)\{F_1+F_2\} + (S^2\rho^2+\varepsilon^2)\{G_1+G_2\} \right]$$

where $\rho^2 = M^2(1+\varepsilon^2-M^2)$. (2.1

The equations (2.12) - (2.17) were given by McCune and Resler [2] in their own notation.

2.2 DISTURBANCES IN THE SOLID

For the non-conducting solid we have the Maxwell equations

$$\text{curl } \underline{h}_s = \underline{0}, \quad \text{div } \underline{h}_s = 0, \tag{2.1}$$

and the nature of these equations makes possible a perturbation potential ϕ such that

$$\underline{h}_s = \text{grad } \phi, \ \nabla^2\phi = 0. \tag{2.19a,b}$$

We specify that on the interface ($y = 0$), $\partial\phi/\partial y = f(x)$. The form of $f(x)$ will be examined later, but for the moment we assume that it is known. The transmitted disturbance propagates instantaneously through the solid as an electromagnetic wave and to ensure the correct behaviour at infinit we need to impose some form of radiation condition. We have an elliptic problem and the appropriate form of the radiation condition in this case is that $\partial\phi/\partial y \to 0$ as $y \to -\infty$. With this requirement there is no build up of disturbances in the solid.

Apply the Fourier transform

$$\Phi(\alpha, y) = \int_{-\infty}^{\infty} \phi(x,y)\, e^{i\alpha x}\, dx$$

to (2.19b). Application of the boundary conditions gives

$$\Phi(\alpha, y) = \frac{e^{|\alpha|y}}{|\alpha|} \int_{-\infty}^{\infty} f(\xi)\, e^{i\alpha\xi}\, d\xi$$

and inversion yields

$$\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} \frac{e^{|\alpha|y}}{|\alpha|} e^{i\alpha(\xi - x)}\, d\alpha\, d\xi . \qquad (2.2$$

Since we work in terms of magnetic field we therefore deduce that

$$(h_x)_{solid} = \frac{-i}{2\pi} \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} \frac{\alpha}{|\alpha|} e^{|\alpha|y} e^{i\alpha(\xi - x)}\, d\alpha\, d\xi$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\xi - x) f(\xi)}{y^2 + (\xi - x)^2}\, d\xi , \qquad (2.2$$

$$(h_y)_{solid} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} e^{|\alpha|y} e^{i\alpha(\xi - x)}\, d\alpha\, d\xi$$

$$= -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\, d\xi . \qquad (2.2$$

These integrals may be interpreted by means of the theory of generalised functions (Lighthill [3]). The main result needed is that

$$\lim_{y \to 0} \frac{y}{y^2 + (\xi - x)^2} = \pi \operatorname{sgn} y \; \delta(\xi - x) , \qquad (2.2$$

where δ denotes the Dirac delta-function. It follows at once that on $y = 0$

$$(h_x)_{solid} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi , \qquad (2.2$$

$$(h_y)_{solid} = f(x) , \qquad (2.2$$

where P signifies the principal value, and f(x) must satisfy a Lipschitz condition.

2.3 BOUNDARY CONDITIONS

When the magnetic and velocity vectors are non-aligned an appropriat assumption for an infinitely conducting non-viscous gas in motion over a non-conducting body is that the gas cannot tolerate the Lorentz force on its boundary that would be produced by surface currents on the wall-gas interface. A consequence of this is that we require continuity of the tangential (t) and normal (n) components of magnetic field across the interface. (This gives the jump conditions

$$[H_t] = 0 , \qquad [H_n] = 0 .$$

A more detailed discussion of these boundary conditions is presented in part III of the thesis; see section 3.5.)

Hence equations (2.13) and (2.24), (2.14) and (2.25) give, respectively

$$H_0 \left[\frac{1}{R}\{F_1(x) - F_2(x)\} + \frac{1}{S}\{G_1(x) - G_2(x)\} \right] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi , \qquad (2.2$$

$$H_0 \{ F_1(x) + F_2(x) + G_1(x) + G_2(x) \} = f(x) . \qquad (2.2$$

If we assume that the equation of the interface between the gas and the solid is of the form $y = F(x)$, where $|F'(x)| \ll 1$ and $F'(x) \to 0$ as $x \to \pm\infty$ then we require $v = F'(x)$ on $y = 0$. Hence from equation (2.16) we have

$$X\{F_1(x) - F_2(x)\} + Y\{G_1(x) - G_2(x)\} = -SR\,F'(x) , \qquad (2.$$

where we have introduced the notation

$$X = S\{R^2(1+\varepsilon^2 - M^2) + \varepsilon^2\} , \quad Y = R\{S^2(1+\varepsilon^2-M^2)+\varepsilon^2\} .$$

If we define

$$\omega = Y/X , \qquad \Omega = -1/X \qquad (2.29a$$

then equation (2.28) gives

$$F_1(x) = F_2(x) + \omega G_2(x) - \omega G_1(x) + SR\,\Omega\,F'(x) . \qquad (2.$$

Substitution for $F_1(x)$ in (2.26) gives

$$H_0\left[K\{G_1(x) - G_2(x)\} + S\,\Omega\,F'(x)\right] = \frac{1}{\pi}\,P\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\,d\xi , \qquad (2.$$

where we have written

$$K = 1/S - \omega/R . \qquad (2.$$

Now substitute (2.30) into (2.27) and write

$$f(x) = H_0 \left\{ 2F_2(x) + (\omega+1)G_2(x) - (\omega-1)G_1(x) + SR\Omega F'(x) \right\}. \quad (2.$$

Eliminating f(x) between (2.31) and (2.33) and setting

$$L = \omega - 1, \quad (2.$$

we readily find that, for $-\infty < x < \infty$

$$K G_1(x) + \frac{L}{\pi} \mathcal{P}\int_{-\infty}^{\infty} \frac{G_1(\xi)}{\xi - x} d\xi = k(x), \quad (2.$$

where we have written

$$k(x) = K G_2(x) - S\Omega F'(x) + \frac{2}{\pi} \mathcal{P}\int_{-\infty}^{\infty} \frac{F_2(\xi)}{\xi - x} d\xi +$$

$$+ \frac{(\omega+1)}{\pi} \mathcal{P}\int_{-\infty}^{\infty} \frac{G_2(\xi)}{\xi - x} d\xi + \frac{SR\Omega}{\pi} \mathcal{P}\int_{-\infty}^{\infty} \frac{F'(\xi)}{\xi - x} d\xi. \quad (2.$$

Since F_2 and G_2 are incident waves, and can have a prescribed form, equation (2.35) is a singular integral equation, with a Cauchy-type kernel, for the reflected wave G_1. Once G_1 is determined we can use (2.30) to find the other reflected wave F_1. Consequently the appropriat form of f(x) can be found from (2.33) and the expressions for the transmitted disturbances in the solid are then calculated from (2.21) and (2.22).

Back substitution enables us to arrive at representations for all the perturbation quantities.

Perhaps the most direct method for deriving the solution of the singular integral equation (2.35) is by making diligent use of the Hilbert transform pair:

$$\Phi(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\phi(\xi)}{\xi - x} d\xi \;,\quad \phi(x) = \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\Phi(\xi)}{\xi - x} \;, \qquad (2.37a,$$

where the relationship is skew-symmetrical, i.e. $-\phi(x)$ is conjugate to $\Phi(x)$.

Define

$$g(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{G_1(\xi)}{\xi - x} d\xi \qquad (2.3$$

so that equation (2.35) becomes

$$K G_1(x) + L g(x) = k(x) \,. \qquad (2.3$$

Replace x by ξ in equation (2.39), multiply through by $1/\pi(\xi - x)$ and integrate the result with respect to ξ from $-\infty$ to ∞. This yields the result

$$K g(x) - L G_1(x) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{k(\xi)}{\xi - x} d\xi \,. \qquad (2.4$$

Elimination of g(x) between (2.39) and (2.40) immediately gives

$$G_1(x) = \left(\frac{K}{K^2 + L^2}\right) k(x) - \left(\frac{L}{K_1^2 + L^2}\right) \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{k(\xi)}{\xi - x} d\xi \,. \qquad (2.4$$

That this is the required solution of (2.35) can easily be verified by direct substitution, and making use of the Poincaré-Bertrand formula (derived from (2.37)) for the compounding of singular integrals:

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-x}\int_{-\infty}^{\infty}\frac{H(\zeta)}{\zeta-\xi}\,d\zeta = -H(x) \quad (2.42)$$

We can now use (2.30) to discover the appropriate representation forwaves are then found by replacing x by x - $\mathcal{S}^{-1}$y for G_1 and x by x - R^{-} waves are then found by replacing x by x - $\mathcal{S}^{-1}$y for G_1 and x by x - R^{-1}y for F_1.

To facilitate evaluation of the reflected wave $G_1(x)$ it is more convenient to express (2.41) in the form

$$(K_1^2+L^2)\,G_1(x) =$$

$$= (K_1^2+\omega^2-1)\,G_2(x) + 2L\,F_2(x) + S\Omega\,(LR-K)\,F'(x) +$$

$$+\frac{2K_1}{\pi}\,P\int_{-\infty}^{\infty}\frac{F_2(\xi)}{\xi-x}\,d\xi + \frac{2K_1}{\pi}\,P\int_{-\infty}^{\infty}\frac{G_2(\xi)}{\xi-x}\,d\xi + \frac{S\Omega}{\pi}(L+KR)\,P\int_{-\infty}^{\infty}\frac{F'(\xi)}{\xi-x}$$

where we have substituted (2.36) into (2.41) and used the relation (2.42)

From this expression we can see directly the contributions to the reflected wave $G_1(x)$ which not only come from the disturbances created in the gas but also come from the disturbances created at the gas-solid interface. If we assume that all the disturbances are created at this interface then there are no incident waves and accordingly $F_2 = G_2 = 0$. Hence from above we have

$$(K^2+L^2)\,G_1(x) = S\Omega\,(LR-K)\,F'(x) + \frac{S\Omega}{\pi}(L+KR)\,P\int_{-\infty}^{\infty}\frac{F'(\xi)}{\xi-x}\,d\xi\,.$$

Thus if we prescribe $F'(x)$ we can determine $G_1(x)$ and $F_1(x)$. Consequently $f(x)$ can be found and the magnetic disturbances in the non-conducting solid are given by (2.21) and (2.22).

A similar problem was considered by Chu [4] in detail.

In the remaining sections we shall be concerned with the situations in which disturbances created in the gas propagate to the wall. For this case we can assume $F(x) = 0$ so that the expression for $G_1(x)$ may now be written as

$$(K^2+L^2)G_1(x) = (K^2+\omega^2-1)G_2'(x) + 2LF_2(x) +$$

$$+\frac{2K}{\pi}P\int_{-\infty}^{\infty}\frac{F_2(\xi)}{\xi-x}d\xi + \frac{2K}{\pi}P\int_{-\infty}^{\infty}\frac{G_2(\xi)}{\xi-x}d\xi . \qquad (2.$$

The other expressions which contained $F'(x)$ are modified accordingly.

2.4 LOCAL DISTURBANCE EFFECTS

We now consider the important case when the incident disturbance is confined to the triangle QAB (see Figure 2.2) and for such a disturbance we assume that the incident waves $F_2(x)$ and $G_2(x)$ vanish for $|x| > \alpha$, $|x| > \beta$, respectively, say.

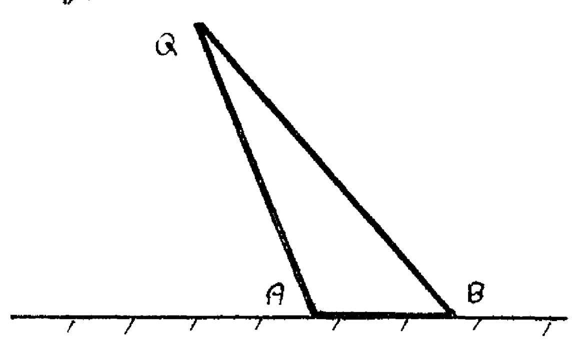

FIGURE 2: REGION OF PRESCRIBED INCIDENT DISTURBANCE QAB

Thus for large x, we have from equation (2.43)

$$G_1(x) \sim \frac{-2K}{\pi x(K^2+L^2)} \left\{ \int_{-\alpha}^{\alpha} F_2(\xi)\,d\xi + \int_{-\beta}^{\beta} G_2(\xi)\,d\xi \right\}. \qquad (2.$$

Also, the other reflected wave, $F_1(x)$ is found from equation (2.30) to have the asymptotic form

$$F_1(x) \sim -\omega\, G_1(x). \qquad (2.$$

We can now determine, for x sufficiently large, on using (2.44) and (2.45) the asymptotic representation for the perturbation quantities given in equations (2.13) - (2.17). In particular we examine the pressures in the reflected waves. Thus the pressure on the wall in the reflected wave $G_1(x)$ is given by equations (2.17) and (2.44) as

$$p-p_\infty \sim \frac{2K\rho_\infty U^2(S^2P^2+\varepsilon^2)}{\pi x(K^2+L^2)} \left\{ \int_{-\alpha}^{\alpha} F_2(\xi)\,d\xi + \int_{-\beta}^{\beta} G_2(\xi)\,d\xi \right\}, \qquad (2.$$

whereas the pressure on the wall in the other reflected wave $F_1(x)$ is given by

$$p-p_\infty \sim \frac{-2K\rho_\infty U^2\omega(R^2P^2+\varepsilon^2)}{\pi x(K^2+L^2)} \left\{ \int_{-\alpha}^{\alpha} F_2(\xi)\,d\xi + \int_{-\beta}^{\beta} G_2(\xi)\,d\xi \right\}. \qquad (2.$$

It can be readily seen that in each respective representation for the disturbance in the gas, for large x, they behave as $O(1/x)$.

Thus the effect of each reflected wave is felt upstream as well as downstream of the two original waves and the importance of "upstream" propagation of disturbances in both the gas and the solid is verified. Physically, the disturbance created in the gas flow propagates along the appropriate Mach line or characteristic to the interface. In the non-conducting solid the transmitted disturbance propagates everywhere with the speed of light (assumed infinite in the present approximations) and accordingly its presence has consequences everywhere upstream and downstream of the specified disturbance of the gas.

The expressions for the magnetic disturbance in the solid are found from equations (2.21) and (2.22). For example we have, from (2.21) on using (2.33) and simplifying the result:

$$\frac{(h_x)_s}{H_0} = \frac{2K_1^2}{\pi(K_1^2+L^2)}\int_{-\alpha}^{\alpha}\frac{(\xi-x)F_2(\xi)}{y^2+(\xi-x)^2}d\xi + \frac{2K_1^2}{\pi(K_1^2+L^2)}\int_{-\beta}^{\beta}\frac{(\xi-x)G_2(\xi)}{y^2+(\xi-x)^2}d\xi$$

$$-\frac{2KL}{\pi^2(K^2+L^2)}\int_{-\infty}^{\infty}\frac{(\xi-x)}{y^2+(\xi-x)^2}\left\{P\int_{-\alpha}^{\alpha}\frac{F_2(\zeta)}{\zeta-\xi}d\zeta + P\int_{-\beta}^{\beta}\frac{G_2(\zeta)}{\zeta-\xi}d\zeta\right\}d\xi$$

If we examine the expression in the braces for large ξ, and evaluate the resulting improper integral we can show that for x sufficiently large

$$(h_x) \sim \frac{-2K^2H_0x}{\pi(K^2+L^2)(x^2+y^2)}\left\{\int_{-\alpha}^{\alpha}F_2(\xi)\,d\xi + \int_{-\beta}^{\beta}G_2(\xi)\,d\xi\right\}. \quad (2.$$

Similarly, we have

$$(h_y)_s \sim \frac{-2K^2H_0y}{\pi(K^2+L^2)(x^2+y^2)}\left\{\int_{-\alpha}^{\alpha}F_2(\xi)d\xi + \int_{-\beta}^{\beta}G_2(\xi)\,d\xi\right\}. \quad (2.$$

From (2.49) we see that $(h_y)_s \to 0$ as $y \to -\infty$. Hence the requirement $\partial\phi/\partial y \to 0$ as $y \to -\infty$ which we imposed on the disturbance in the solid is verified.

Suppose now that $F_2(x)$ and $G_2(x)$ are each small and positive for $|x| < \alpha$, $|x| < \beta$, respectively. We shall confine our attention to the pressure disturbance. From the definitions of R^2, S^2 (see equation (2.11)) we have after some algebra

$$R^2(M^2-1-\varepsilon^2)-\varepsilon^2 > 0, \quad S^2(M^2-1-\varepsilon^2)-\varepsilon^2 < 0 .$$

It follows, on using these results, that ω defined by equation (2.29a) is always negative for the cases under consideration. Also, from (2.32 we may write

$$K = (R^2-S^2)(M^2-1-\varepsilon^2)/S\{R^2(M^2-1-\varepsilon^2)-\varepsilon^2\}$$

and on an incoming characteristic this quantity is negative. We also have the results:

$$R^2\rho^2+\varepsilon^2 < 0, \quad S^2\rho^2+\varepsilon^2 > 0 .$$

By way of illustration we now give several examples which serve to show the importance of the results of this section. For example, if we let $F_2(x)$ and $G_2(x)$ have the forms

$$F_2(x) = \lambda(\alpha-x), \quad 0 \le x \le \alpha; \quad F_2(x) = F_2(-x),$$

$$G_2(x) = \mu(\beta-x), \quad 0 \le x \le \beta; \quad G_2(x) = G_2(-x),$$

where λ, μ, α and β are constants, it can be readily verified that

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{F_2(\xi)}{\xi - x} d\xi =$$

$$= \left(\frac{\lambda}{\pi}\right) \left\{ 2x \ln |x| - (\alpha + x) \ln |\alpha + x| + (\alpha - x) \ln |\alpha - x| \right\} .$$

If F_2 is replaced by G_2 we obtain a corresponding expression. From equation (2.43), on using the above results, we see that for large x

$$G_1(x) \sim - \frac{2K}{\pi} (\lambda \alpha^2 + \mu \beta^2) / x(K^2 + L^2)$$

and this result is the same as that obtained on using (2.44).

For convenience let $\lambda = \alpha = 1$ in the expression for $F_2(x)$ and let $G_2(x) = 0$. For a value of Mach number, $M = 2$, and a weak magnetic fiel $\epsilon^2 = 0.1$ we can calculate the total incident pressure and the total reflected pressure (for $R, S > 0$). The results are represented graphically in Figure 2.3.

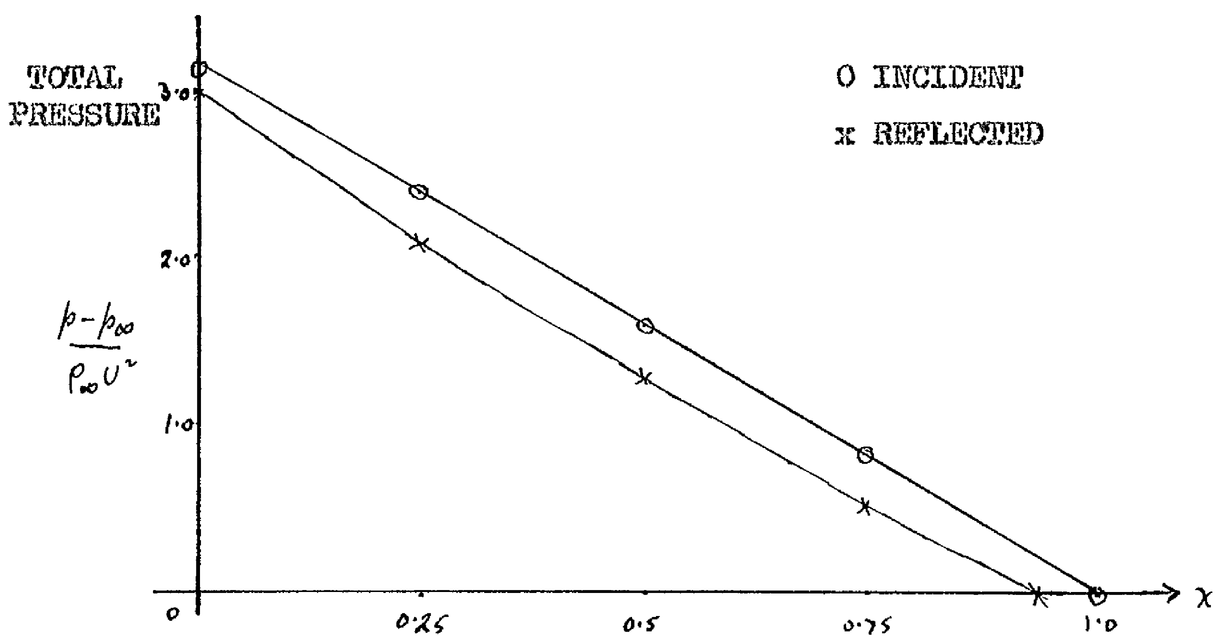

FIGURE 2.3: TOTAL INCIDENT AND TOTAL REFLECTED PRESSURES FOR $M = 2$ AND $\epsilon^2 = 0.1$

As a second example we examine the effect of a compression wave, created at some point of the ionised gas, which is incident on the interface of the semi-infinite solid.

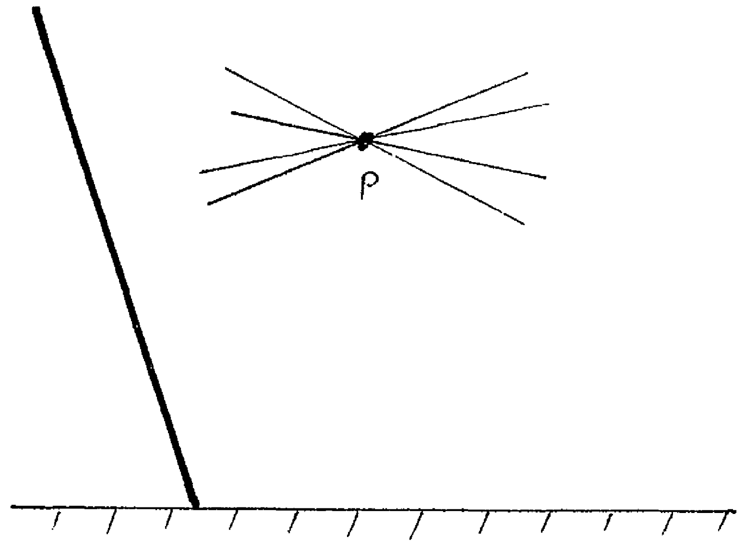

FIGURE 2.4: COMPRESSION WAVE INCIDENT ON SOLID
(TWO FAMILY SYSTEM OF CHARACTERISTICS THROUGH A POINT P)

To simplify the investigation we consider the case when $G_2(x) = 0$ and only $F_2(x)$ is present. We prescribe $F_2(x)$ to have the form

$$F_2(x+R^{-1}y) = \begin{cases} 0, & x+R^{-1}y < 0, \\ \left(\frac{m}{\pi}\right)^{\frac{1}{2}} \exp\left\{-m(x+R^{-1}y)^2\right\}, & x+R^{-1}y > 0 . \end{cases}$$

Since F_2 is piecewise continuous we must appeal to the theory of generalised functions, e.g. Lighthill [3] in order to interpret the meaning of the integrals which arise.

Thus we have from equation (2.43) the result that $G_1(x) = 0$ for $x < 0$, and

$$(K^2+L^2)\,G_1(x) = \left(\frac{n}{\pi}\right)^{\frac{1}{2}} e^{-nx^2} + \frac{2K}{\pi}\,P\int_{-\infty}^{\infty} \frac{\left(\frac{n}{\pi}\right)^{\frac{1}{2}} e^{-n\xi^2}}{\xi - x}\,d\xi \quad , x > 0$$

Since (see, for example, Lighthill [3])

$$\lim_{n\to\infty} \left(\frac{n}{\pi}\right)^{\frac{1}{2}} e^{-nx^2} = \delta(x)$$

we obtain, if we let $n \to \infty$ in the above expression:

$$(K^2+L^2)\,G_1(x) = \delta(x) - \frac{2K}{\pi x} .$$

This result is to be interpreted as follows. When there is an incident compression wave, then the expression for the reflected wave involves a term which gives rise to a shock-wave (we obtain a delta-function since we are dealing with a linearised theory) and also to a term which indicates that there is a finite, eventually vanishing upstream influence effect.

By similar methods the results from two compression waves, created by a slender wedge fixed in the flow, could be worked out.

Chu and Lynn [5] consider the steady flow past a non-conducting slender wedge. They also examine the possibility of steady flow over a sharp convex corner. For the latter case they indicate by means of a counting procedure that the number of equations to be satisfied exceeds the number of unknowns by one. The implication of this result is as shown above in section 1.4, that flows of the Prandtl-Meyer type cannot exist. They attribute this situation to the feed-back of magnetic disturbances in the non-conducting solid. The situation is very similar to the one considered in this section and Chu and Lynn's result is an immediate consequence of the presence of non-uniform disturbances induced by the propagation of electromagnetic waves in the solid.

OTHER FLOW REGIMES

When the applied magnetic field is orientated at an arbitrary angle to the stream the operator in the partial differential equation for the current ξ can still be written as the product of two quadratic operators [2] by a process similar to that which gave rise to equation (2.10). Values of M and ϵ occur which allow these operators to be both hyperbolic and if we follow through the analysis of the above sections no new feature arises.

When the values of M and ϵ lead to the product of an elliptic and a hyperbolic operator then the solution for ξ depends on the sum of the two parts and we write

$$\xi = (\quad)_e + (\quad)_h .$$

Detailed expressions for ξ are given in [2]. We assume that the boundary condition (as $y \rightarrow -\infty$) on the magnetic field in the solid is the same as that used previously (see section 2). If we also assume that the jump relations of section 3 are also valid then it is possible to obtain a singular integral equation for the single reflected wave. This equation is identical in form to that given by equation (2.35) but the quantities K, L, and k(x) are now different. The complete solution is readily found and we can show that when the disturbance is localised the upstream effect is again of order $1/x$ for large x.

PART III

MAGNETO-GASDYNAMIC FLOW OVER A WEDGE

3.1 INTRODUCTION

The basic problems in fluid mechanics are the behaviour of a fluid in motion either with free boundaries or over a solid body. For flow over a solid body a fundamental problem, useful for a general understanding of the properties of the fluid under consideration, is that of a uniform infinite stream impinging upon a semi-infinite wedge. The study of this flow led to important results in conventional gas dynamics, and the same may be expected from the corresponding study in magneto-gasdynamics. Some work has already appeared on the subject. First, Cabannes [6] presented the solution to the problem of the steady flow of a perfectly conducting fluid over a symmetrical wedge at zero angle of attack when there is an applied magnetic field aligned with the oncoming stream. By a well-known theorem, a magnetic field aligned with the stream everywhere upstream at infinity remains in this configuration in an inviscid, perfectly conducting fluid.* This problem is the simplest possible extension of gas dynamics. The attached plane stationary magneto-gasdynamic shocks are two in number and symmetrically

---

* The theorem is easy to prove from the basic equations of continuous flow. For example, from (1.12b), which is an integral of the equations o motion, we can see that $u/H_x = v/H_y = V/H$. It is not, however, immediately obvious that it remains true across a shock-wave. An examination of the jump relations across a shock-wave (see paragraph 3.4 below) shows quite simply that it is, in fact, valid.

placed as in gas dynamics. The flow field and the magnetic field inside the wedge remain uncoupled for a magnetic field aligned with the stream that it is not necessary to specify the conductivity of the wedge. In t absence of a component of magnetic force normal to the surface of the wedge there is no tangential Lorentz force acting on the inviscid fluid particles in contact with the wedge; hence the presence of a current sheet is permissible and such a surface, in fact, separates the body of moving fluid from the solid boundary. Expressions were derived by Cabannes for the velocity, density and pressure jumps in terms of the shock angle β and the semi-vertex angle of the wedge, θ. The trigonometric equation for the shock angle was found to be of fifth order in $\tan \beta$ and required to be solved numerically.

The corresponding problem for an applied magnetic field oblique to the stream has received considerable attention from Kogan [1] who restricted attention to thin wedges and thin aerofoils, for which linearisation of the equations is possible and the exercise becomes one involving the theory of characteristics. This paper requires careful reading because of the numerous errors and mis-prints it contains. Recently, Chu and Lynn [5] considered the problem of the two-dimensional steady flow of an infinitely conducting fluid past a non-conducting wedge with a magnetic field non-aligned with the oncoming stream. By means of a counting procedure they indicated that to obtain sufficient equations to solve for the number of unknown parameters it was required to match the solution for the flow with that found in the wedge. They considered the jump conditions which hold across weak shocks (characteristics) and restricted their analysis to thin wedges. Their prime object in this linear theory was to demonstrate the effect of the coupling of the flows above and below the wedge via the boundary conditions on the magnetic field. In a more recent paper, Mimura [8] presented a solution to the non-linear problem of the shock-wave configuration on a non-conducting wedge of finite angle in the presence of an incident perfectly conducting steady stream. In this case, however, the magnetic field was applied perpendicular to the uniform flow and was assumed to be weak. He indicated that the flow required to pass through four shock waves, two for the upper surface and two for the lower.

In the following sections, the equations for the problem of the flow of a fully ionised inviscid gas past an infinite non-conducting wedge are developed in full generality. They are then used to show how flow with four attached shock waves develops from the solutions of Cabannes when the magnetic field ahead of the wedge becomes oblique to the stream. A method of perturbation is found for small obliquity χ_1 which illustrates how the current sheets, lying along the surfaces of the wedge, move out into the stream to give the additional shock waves. In the wedge a magnetic field, inclined at a finite angle to the wedge axis is set up. When the parameter $k_1 (= \epsilon_1/M_1) < 1$ expressions can be obtained for the perturbation quantities in the regions between the second shocks and the wedge surfaces. These have been calculated up to the second significant power in χ_1. Perturbation solutions of this kind could not be found for $k_1 \geq 1$; and it is argued in part IV of the thesis that for these values of k_1 the shocks found by Cabannes are either unstable or physically unrealisable, and the results obtained in this part lend support to these views.

3.2 NOTATION AND SYMBOLS

We now introduce some further notation and symbols:

ν coefficient of fluid viscosity.

S specific entropy.

χ angle between $\underline{H}$ and $\underline{V}$.

θ flow direction.

β inclination of first shock to wedge axis.

δ inclination of second shock to wedge axis.

suffix 1 refers to conditions upstream.

suffix 2 refers to conditions between the first and second shocks.

suffix 3 refers to conditions between the second shock and the wedge quantities which are dashed, e.g. H', refer to regions II and III below the wedge (see Figure 3.1).

b Alfvén speed, $= (\mu H^2/\rho)^{\frac{1}{2}}$.

ϵ non-dimensional parameter ($= b/a$).

k non-dimensional parameter ($= \epsilon/M$).

suffix n refers to the normal to the wave front.

suffix t refers to the tangential direction along the wave front.

3.3 STATEMENT OF THE PROBLEM

Consider the two-dimensional steady flow of a fully ionised gas here idealised as a perfect, inviscid fluid of infinite electrical conductivity in irrotational motion over a stationary, semi-infinite, straight-walled, non-conducting symmetric wedge at zero angle of attack to the oncoming stream. Without loss of generality the permeability of the body may be assumed to be the same as that of the incident stream. Diamagnetic effects are ignored and Maxwell's equations are used in their usual form in conjunction with the basic approximations and equations of magneto-gasdynamics. The applied magnetic field, of magnitude H_1, is orientated at an angle χ_1 to the incident uniform flow, which has a uniform speed V_1 at infinity upstream and is directed along the axis of the wedge (Figure 3.1). The non-conducting wedge is assumed to be symmetrical with semi-vertex angle θ_3. The restriction to a symmetrical wedge is not necessary (the field is, in any case, unsymmetrical) but leads to some simplification of very complicated equations and makes it easier to draw comparison with the results of conventional gas dynamics. The two-dimensional flow is assumed to be of restricted type, i.e. the magnetic field is assumed to lie entirely in the plane of the flow (the x,y-plane), which is supposed to be normal to the leading edge of the wedge. The addition of a third component independent of z, while making the equations more complicated, is straightforward from a theoretical point of view and will not be considered here. As pointed out by Chu and Lynn [5] this removes from the flow field a pair of Alfvén waves, one above and one below the body.

On the basis of a linearised theory, Kogan [1] has shown that when the equations of motion are "fully hyperbolic" there are four real characteristics (other than streamlines) through every point. A discussion of those parts of Kogan's work which are relevant in this investigation was presented in part I section 1.3; see Figure 1.1. In a full non-linear theory the characteristics through the apex of the wedge, representing weak discontinuities for the thin wedge, may be expected to be replaced by shock waves, two above and two below the wedge. The fluid flow has to be such that the magnetic fields on the surfaces of the wedge are compatible with the field inside the insulating wedge, which is governed by an elliptic differential equation. The lack of alignment in the magnetic field induces different shock and flow patterns on the upper and lower surfaces of the wedge. The solution will be sought, as indicated in Figure 3.1, by the juxtaposition of uniform regions of perfectly conducting fluid separated by shock waves.

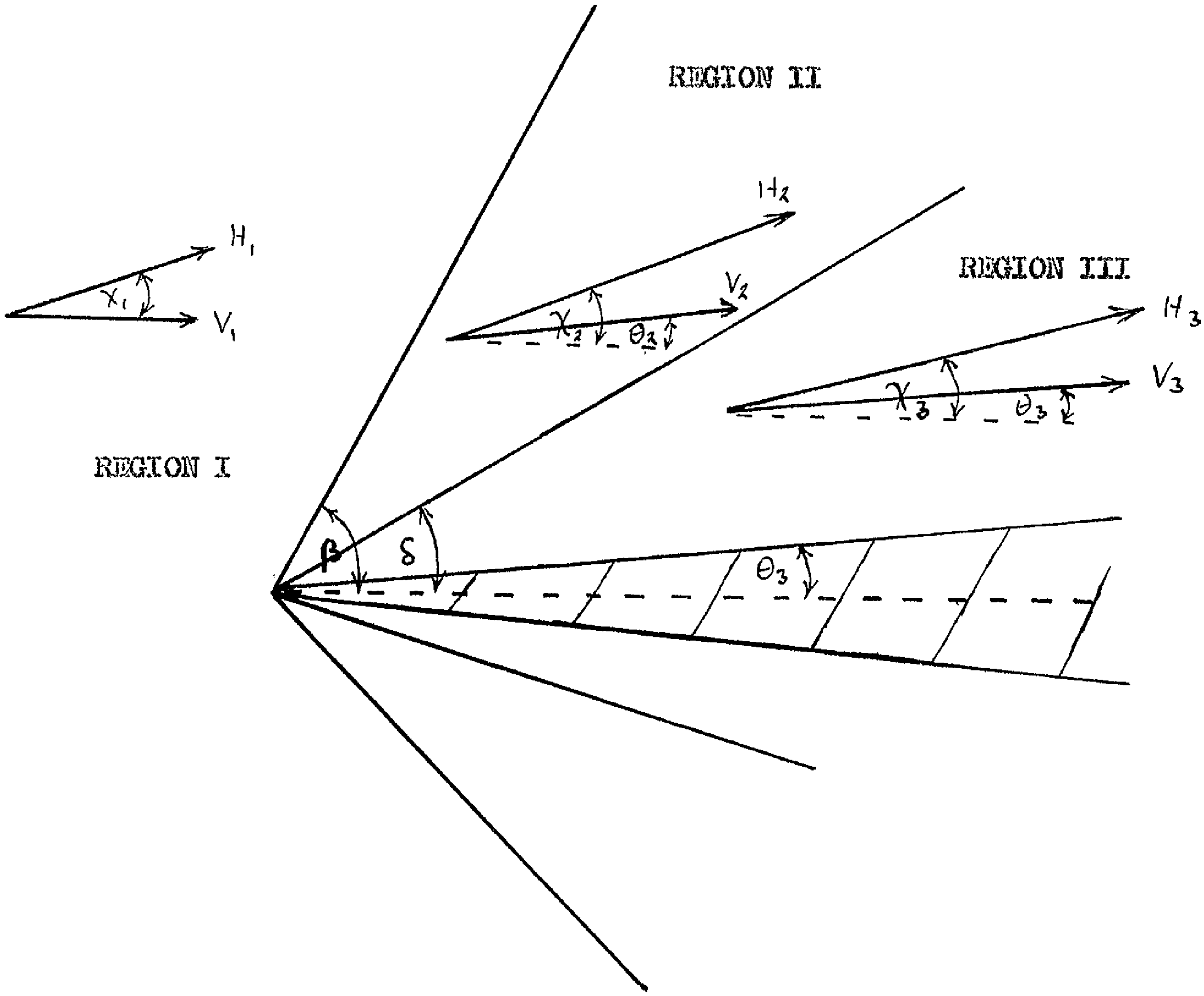

FIGURE 3.1: FLOW OVER A WEDGE - SHOCK-WAVE CONFIGURATION

3.4 BASIC EQUATIONS

Under the assumptions made Maxwell's equations take the form:

$$\operatorname{div} \underline{H} = 0, \tag{3.1}$$

$$\operatorname{curl} \underline{H} = \underline{j}, \tag{3.2}$$

$$\underline{j} = \sigma(\underline{E} + \mu \underline{V}_\wedge \underline{H}). \tag{3.3}$$

The equations holding across a plane stationary shock-wave in an infinitely conducting gas are (see, for example, Bazer and Ericson [9])

$$[H_n] = 0, \tag{3.4}$$

$$[\rho V_n] = 0, \tag{3.5}$$

$$[\rho V_n \underline{V} + (p + \tfrac{1}{2}\mu H^2)\underline{n} - \mu H_n \underline{H}] = 0, \tag{3.6}$$

$$[\rho V_n (\tfrac{1}{2}V^2 + p/\rho + \mu H^2/\rho) - \mu H_n (\underline{H}.\underline{V})] = 0, \tag{3.7}$$

$$[V_n \underline{H} - H_n \underline{V}] = 0, \tag{3.8}$$

$$\rho V_n [S] \geqslant 0. \tag{3.9}$$

Here the suffix n indicates a component normal to the shock-wave. The square brackets are used to indicate the change, across the shock-wave, in the enclosed quantity.

A count of unknown quantities shows that in region II downstream of the first shock there are the seven unknowns H_2, V_2, ρ_2, p_2, χ_2, θ_2 and β, and in region III downstream of the second shock the six unknowns

H_3, V_3, ρ_3, p_3, X_3 and δ. Thus there are thirteen unknowns for the solution on the upper surface; there are also thirteen from the two regions below, giving a combined total of twenty-six unknowns. An examination of the jump relations (3.4) - (3.8) shows that across any single shock there are but six independent basic scalar equations; there are therefore twenty-four equations altogether and twenty-six unknowns. Accordingly we are led to the conclusion that for non-aligned fields the solution in the fluid depends on the solution of the boundary-value problem within the non-conducting wedge. It is easy to show (Swan [10], Chu and Lynn [5]) that the magnetic field inside the semi-infinite non-conducting wedge must be constant. In consequence, the magnitude and direction of the magnetic field in the wedge are the same on the upper and lower surfaces of the wedge. This result supplies two further relations once the connection between the values of the magnetic fields in the fluid and the wedge at the interface have been established. This matter is investigated in the next section.

3.5 CONDITIONS AT THE FLUID-WEDGE INTERFACE

At the interface between the two media the normal component of the magnetic induction is required to be continuous. Because there is no change in the permeability this implies continuity of H_n (the suffix n always indicates the normal component across the interface between two adjoining regions). The tangential component of $\underline{H}$ may or may not be continuous. If it is not, then a current sheet lies on the interface. It can be verified that in the upper layer the flow of current is equal and opposite to that in the lower layer so that the net contribution at the apex is zero. In the event that $H_n \neq 0$, the current sheet and magnetic field together produce a Lorentz force acting on the layer of particles in contact with and moving along the wedge. It has been customary to rule this possibility out on the grounds that an inviscid fluid cannot support a surface traction. If this is accepted, then for $H_n \neq 0$, no current sheet is permissible; in consequence, the tangential component of $\underline{H}$, and hence $\underline{H}$ itself, must be continuous across the interface. For the problem under consideration this implies that the vector $\underline{H}$ has the same value in the fluid on both the upper and lower surfaces of the wedge. This result supplies the two additional conditions required to bring the number of equations up to the number of unknown quantities and thus to make the problem theoretically soluble.

Before proceeding it is useful to point out that the correct tangential boundary condition to be satisfied at the interface is not quite so straightforward as has sometimes been supposed. Stewartson [11] has discussed at some length the nature of the limiting condition at an interface between solid and fluid as the viscosity in the fluid tends to zero and the electrical conductivity tends to infinity. The jump condition to be satisfied by the magnetic and velocity vectors across the interface is

$$[H_t] = (\rho \nu \sigma)^{\frac{1}{2}} \; [V_t] \; .$$

In general the values of σ and ν are such, that the limit of $\nu\sigma$ may be taken to be zero. Thus $[H_t] = 0$ and consequently $[\underline{H}] = \underline{0}$ as assumed above. However, in some astrophysical applications the limit may well be finite.* Then a discontinuity in $\underline{H}$ at the interface is necessary and the surface force inevitable. The usual concept of an inviscid fluid must therefore be modified in this case in order to allow a correct representation of the boundary conditions to be made. This state of affairs does not materially affect the solution of the problem under discussion, the result being merely a modification of the two additional conditions on $\underline{H}_3$, $\underline{H}_3^1$ (involving also $\underline{V}_3$ and $\underline{V}_3^1$).

When $H_n = 0$ no difficulties of the above kind arise, the fields in the fluid and wedge being completely uncoupled. Cabannes' problem was therefore capable of solution without any reference to the nature of the wedge or the field inside it.

In what follows it will be assumed, for the sake of definiteness and simplicity that $\lim_{\substack{\sigma \to \infty \\ \nu \to 0}} \nu\sigma = 0$ and that the continuity of $\underline{H}$ across the interface between the fluid and the wedge has to be assured.

3.6 EQUATIONS HOLDING ACROSS SHOCKS

The first shock on the upper surface

From (3.4), $H_{1n} = H_{2n}$ or

$$H_2/H_1 = \sin(\beta-\chi_1)/\sin(\beta-\chi_2) \quad . \qquad (3.$$

---

* The author is indebted to Professor K. Stewartson for this observation

From (3.5), $\rho_1 V_{1n} = \rho_2 V_{2n}$, a statement of the continuity of the mass flux across the shock, and this may be written as

$$\rho_2/\rho_1 = V_1 \sin\beta / V_2 \sin(\beta-\theta_2) . \qquad (3.$$

By use of the tangential component of (3.6) and the equation (3.10) it follows that

$$\frac{V_2}{V_1} = \frac{\cos\beta}{\cos(\beta-\theta_2)} - \frac{\varepsilon_1^2}{M_1^2}\frac{\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)}{\sin\beta\cos(\beta-\theta_2)\sin(\beta-\chi_2)} . \qquad (3.$$

The quantity ϵ/M is seen to be the ratio of the Alfvén to the flow speed. Equations (3.11) and (3.12) may be combined to give

$$\frac{\rho_1}{\rho_2} = \frac{\tan(\beta-\theta_2)}{\tan\beta} - \frac{\varepsilon_1^2}{M_1^2}\frac{\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)\sin(\beta-\theta_2)}{\sin^2\beta\cos(\beta-\theta_2)\sin(\beta-\chi_2)} . \qquad (3.1$$

The normal component of (3.6) combined with (3.12) leads after some little algebra to the result

$$p_2/p_1 + \tfrac{1}{2}\gamma\varepsilon_1^2(H_2/H_1)^2 - 1 =$$

$$= \gamma M_1^2\frac{\sin\beta\sin\theta_2}{\cos(\beta-\theta_2)} + \tfrac{1}{2}\gamma\varepsilon_1^2\left\{1 + \frac{2\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)\sin(\beta-\theta_2)}{\sin(\beta-\chi_2)\cos(\beta-\theta_2)}\right\} \qquad (3.1$$

Equation (3.7) gives

$$\tfrac{1}{2}(1-V_2^2/V_1^2) + \frac{\varepsilon_1^2}{M_1^2}(1-\rho_1 H_2^2/\rho_2 H_1^2) + \frac{1}{M_1^2(\gamma-1)}(1-a_2^2/a_1^2) -$$

$$-\frac{\varepsilon_1^2}{M_1^2}\left\{\cos\chi_1 - (V_2H_2/V_1H_1)\cos(\chi_2-\theta_2)\right\}\frac{\sin(\beta-\chi_1)}{\sin\beta} = 0 . \qquad (3.1$$

When the flow direction θ_2 and the inclination χ_2 of the magnetic field are known this equation gives an expression for the shock angle β. When the fields are aligned it reduces to a quintic equation in $\tan\beta$, the equation found by Cabannes and solved by him via numerical methods to give the complete solution.

The component of (3.8) tangential to the shock gives

$$\frac{V_2}{V_1}\sin(\chi_2-\theta_2) = \frac{H_1}{H_2}\sin\chi_1 \quad . \tag{3.1}$$

Substitution of (3.12) and (3.10) in (3.16) leads after some little effort to an equation of the fourth degree in $\tan\beta$ with coefficients involving given quantities and $\tan\chi_2$, $\tan\theta_2$. It may be arranged in the form

$$Q\tan\theta_2 = P, \tag{3.1}$$

where

$$P = (\tan\chi_2-\tan\chi_1)\Big\{\tan^2\beta\sec^2\chi_1(\tan\beta-\tan\chi_2) + \\ + \frac{\xi_1^{\,2}}{M_1^{\,2}}\tan\chi_2\sec^2\beta(\tan\beta-\tan\chi_1)^2\Big\}, \tag{3.17}$$

and

$$Q = \Big\{(1-\tan\chi_1\tan\chi_2)\tan^2\beta + (\tan^2\beta-1)\tan\chi_1\tan\beta\Big\}\sec^2\chi_1(\tan\beta-\tan\chi_2) + \\ + \frac{\xi_1^{\,2}}{M_1^{\,2}}(\tan\beta-\tan\chi_1)^2(\tan\chi_2-\tan\chi_1)(1+\tan^2\beta) . \tag{3.17}$$

The six equations (3.10), (3.12), (3.13), (3.14), (3.15) and (3.16) will form the basis of the analysis in subsequent sections. The remaining

18 shock equations may be derived very simply from these by the transformations indicated below.

The second shock on the upper surface

The angle between the magnetic field and the direction of flow upstream of the second shock on the upper surface is $\chi_2 - \theta_2$. By referring all angles to the flow direction in this region, the equations appropriate to this shock follow from those obtained above by means of the substitutions:

$$\delta - \theta_2 \text{ for } \beta,\ \theta_3 - \theta_2 \text{ for } \theta_2,\ \chi_3 - \theta_2 \text{ for } \chi_2 \text{ and } \chi_2 - \theta_2 \text{ for } \chi_1.$$

The shocks on the lower surface

The equations for the shocks on the lower surface may be obtained directly from those established for the upper surface. The simplest form results if we measure directions downwards from the wedge axis in Figure 3.1 and add a dash to the variables β, δ, χ_2, ... to mark quantities in the lower half-plane. The equality of the magnetic field vector for both half-planes ahead of the leading shocks, and also behind the second shocks, is then provided for by writing $-\chi_1$ for χ_1 and $-\chi_3$ for χ_3.

3.7 CERTAIN LIMITS FOR $\chi_1 = 0$ AND $\chi_1 = \frac{1}{2}\pi$

Before proceeding to the investigation of the perturbation analysis we first of all utilise the equations (3.10) - (3.17) to obtain certain limiting forms. These limits provide useful checks.

(a) Reduction to Cabannes

When we put $\chi_1 = 0$ in (3.17) we have the result that $\chi_2 = \theta_2 = \theta$, say, the semi-vertical angle of the wedge. However, see the later analysis which surrounds (3.26). Note that $\theta = \theta_3$ but for convenience in this section we drop the subscript. Equation (3.10) now becomes

$$H_2/H_1 = \sin\beta/\sin(\beta - \theta). \qquad (3.$$

Equations (3.12), (3.13) and (3.14) become respectively:

$$\frac{V_2}{V_1} = \frac{\cos\beta}{\cos(\beta-\theta)} + \frac{\xi_1^2}{M_1^2}\cdot\frac{\sin\theta}{\cos(\beta-\theta)\sin(\beta-\theta)}, \qquad (3.$$

$$\frac{\rho_1}{\rho_2} = \frac{\tan(\beta-\theta)}{\tan\beta} + \frac{\xi_1^2}{M_1^2}\left\{1 - \frac{\tan(\beta-\theta)}{\tan\beta}\right\}, \qquad (3.$$

$$p_2 + \tfrac{1}{2}\mu H_2^2 = p_1 + \rho_1 V_1^2\frac{\sin\beta\sin\theta}{\cos(\beta-\theta)} + \tfrac{1}{2}\mu H_1^2\frac{\cos(\beta+\theta)}{\cos(\beta-\theta)}. \qquad (3.$$

Substitution of these ratios into the energy equation (3.15) and simplifying the result gives

$$AM_1^4 + (B\xi_1^2 - 1)M_1^2 + \xi_1^2(C\xi_1^2 + 1) = 0, \qquad (3.$$

where

$$A = \sin\beta\cos\beta\left[\tan\beta - \left\{\tfrac{1}{2}(\gamma-1)\tan^2\beta + \tfrac{1}{2}(\gamma+1)\right\}\tan\theta\right]\Big/\left(1+\tan\beta/\tan\theta\right),$$

$$B = -\frac{2\tan\beta - \gamma(1+2\tan^2\beta)\tan\theta + (\gamma+2)\tan\beta\tan^2\theta}{2(\tan\beta-\tan\theta)(1+\tan\beta\tan\theta)},$$

$$C = \frac{-(1+\tan^2\beta)\tan\theta\{(\gamma+1)\tan\beta - \gamma\tan\theta + \tan\beta\tan^2\theta\}}{2(1+\tan\beta\tan\theta)(\tan\beta-\tan\theta)^2} .$$

The results (3.19) - (3.22) were given in this form by Cabannes [6].

After some further effort (3.22) can be written in the form

$$a\tan^5\beta + b\tan^4\beta + c\tan^3\beta + d\tan^2\beta + e\tan\beta + f = 0 , \qquad (3.23)$$

where (we drop the subscript 1 on M and ϵ)

$$a = (M^2-\varepsilon^2)\{-\tfrac{1}{2}(\gamma-1)M^2 + \tfrac{1}{2}(\gamma+1)\varepsilon^2 - 1\}\tan\theta - \tfrac{1}{2}\varepsilon^4\tan^3\theta ,$$

$$b = (M^2-\varepsilon^2)(M^2-1) + [(\gamma-1)M^4 - \{\tfrac{1}{2}(3\gamma+2)\varepsilon^2 - 2\}M^2 + (\tfrac{1}{2}\gamma\varepsilon^2 - 2)\varepsilon^2]\tan^2\theta ,$$

$$c = [-\tfrac{1}{2}(\gamma+5)M^4 + \{\tfrac{1}{2}(3\gamma+2)\varepsilon^2 + 1\}M^2]\tan\theta - \varepsilon^2(\gamma\varepsilon^2 + \varepsilon^2 + 1)\tan\theta +$$

$$+ [-\tfrac{1}{2}(\gamma-1)M^4 + \{\tfrac{1}{2}(\gamma+2)\varepsilon^2 - 1\}M^2 - \varepsilon^2(\varepsilon^2-1)]\tan^3\theta ,$$

$$d = -M^2(1+\varepsilon^2) + \varepsilon^2 + \{(\gamma+2)M^4 - (2\gamma\varepsilon^2 + \varepsilon^2 - 1)M^2 + \varepsilon^2(\gamma\varepsilon^2 - 1)\}\tan^2\theta ,$$

$$e = [\{\tfrac{1}{2}(\gamma+2)\varepsilon^2 + 2\}M^2 - \varepsilon^2\{\tfrac{1}{2}(\gamma+1)\varepsilon^2 + 2\}]\tan\theta +$$

$$+ (M^2-\varepsilon^2)\{-\tfrac{1}{2}(\gamma+1)M^2 + \tfrac{1}{2}\varepsilon^2 - 1\}\tan^3\theta ,$$

$$f = -(M^2-\varepsilon^2)(\tfrac{1}{2}\gamma\varepsilon^2 + 1)\tan^2\theta .$$

A comparison of the above coefficients with those given by Cabannes in a later paper [7] indicates that there are two errors in the value of c given by him. A quick check to verify that this claim is correct is to proceed thus: since the coefficients of ϵ_1^2, ϵ_1^4 agree in all the comparisons we can examine the effect as $\epsilon_1 \to 0$. Thus from (3.23) we have, when $\epsilon_1 = 0$,

$$(\tan\beta - \tan\theta)^2 \left[\{(\gamma-1)M_1^2 + 2\}\tan\theta\tan^3\beta + 2(1-M_1^2)\tan^2\beta + \right.$$

$$\left. + \{(\gamma+1)M_1^2 + 2\}\tan\theta\tan\beta + 2 \right] = 0$$

Since $\beta \neq \theta$ we can dismiss the repeated first factor. The remaining cubic equation is the familiar equation which one has to solve for the shock angle β in conventional gas dynamics. The solution to the Cabannes problem is straightforward. We specify the semi-vertex angle of the wedge θ, the incident Mach number M_1 and the strength of the magnetic field, ϵ_1. The quintic for β, (3.23) can then be solved numerically. Since Cabannes did not give tabulated values of β it was necessary to calculate some of them for comparison with his graphs, in order to establish the validity of these. His analysis is the starting point of the perturbation method which will be presented in detail in later sections.

(b) Kogan limits for weak shocks, $\chi_1 = \frac{1}{2}\pi$

From (3.17) when $\chi_1 = \frac{1}{2}\pi$ we have the result:

$$\tan\theta_2 \tan^2\beta \tan^2\chi_2 + \{k_1^2 - (1-k_1^2)\tan^2\beta + \tan\theta_2 \tan\beta(1-2\tan^2\beta)\}\tan\chi_2 +$$

$$+ \tan^3\beta + \tan\theta_2\{(\tan^2\beta - 1)\tan^2\beta - k_1^2(1+\tan^2\beta)\} = 0 . \qquad (3.2$$

When the semi-vertex angle of the wedge, $\theta_3 \ll 1$ the flow deviation in region II, $\theta_2 \ll 1$. Since characteristics are shock waves of vanishing strength we may write $\beta = \sigma + \lambda\theta_2$, where σ is the angle between a characteristic and a streamline and λ is some constant. We can write (3.24) to terms of order θ_2, thus

$$\theta_2 \tan^2\sigma \tan^2\chi_2 + \left[\left\{ k_1^2 - (1-k_1^2)\tan^2\sigma \right\} + \theta_2 \left\{ \tan\sigma - 2\tan^3\sigma - \right.\right.$$

$$\left.\left. - 2\lambda(1-k_1^2)\tan\sigma \sec^2\sigma \right\} \right] \tan\chi_2 + \tan^3\sigma +$$

$$+ \theta_2 \left\{ 3\lambda \tan^2\sigma \sec^2\sigma + (\tan^4\sigma - \tan^2\sigma) - k_1^2 \sec^2\sigma \right\} = 0 .$$

The root of this quadratic in $\tan\chi_2$ which $\rightarrow \infty$ as $\theta_2 \rightarrow 0$ is

$$\tan\chi_2 = \left\{ A + B\theta_2 + O(\theta_2^2) \right\} / C\theta_2 , \qquad (3.25)$$

where,

$$A = (1-k_1^2)\tan^2\sigma - k_1^2 , \qquad C = \tan^2\sigma ,$$

$$B = -\tan\sigma + 2\tan^3\sigma + 2\lambda(1-k_1^2)\tan\sigma \sec^2\sigma - \frac{1}{A}\tan^5\sigma .$$

Substitution of (3.25) into (3.10) - (3.14) gives, respectively:

$$H_2/H_1 = (\tan^2\sigma/A)\theta_2 + O(\theta_2^2), \quad V_2/V_1 = 1 - (\tan^3\sigma/A)\theta_2 + O(\theta_2^2),$$

$$\frac{\rho_1}{\rho_2} = 1 - \frac{\sec^2\sigma(\tan^2\sigma - k_1^2)}{A\tan\sigma}\theta_2 + O(\theta_2^2), \quad \frac{p_2}{p_1} = 1 + \gamma M_1^2 \tan\sigma . \theta_2 + O(\theta_2^2).$$

These values substituted into (3.15) give eventually,

$$\{M_1^2(1-M_1^2)+\xi_1^2 M_1^2\}\tan^4\sigma + \{M_1^2(1+\xi_1^2)-\xi_1^2\}\tan^2\sigma - \xi_1^2 + O(\theta_2) = 0 .$$

This quartic equation gives the characteristic angle σ when $\chi_1 = \frac{1}{2}\pi$, and was discussed in detail in part I (c.f. (1.22)).

As already explained there are 26 dependent variables for which there are 24 equations derived above and 2 boundary conditions. The transcendental nature of the equations involved renders a direct analytic approach virtually impossible. The equations could be tackled on a fairly large electronic computer, but again the number of parameters suggests that a considerable amount of complicated interpolation would be necessary in order to obtain results. All solutions must of course be subjected finally to the thermodynamic test of non-diminishing entropy laid down by (3.9), and the choice of branches where two possible shock directions exist has also to be made. In view of this it seemed worthwhile to try to narrow the problem to that of finding how the general configuration begins to develop from a known solution by making a small alteration in some parameter and attempting an analytic approach.

The starting point chosen was Cabannes' solution for a magnetic field aligned with the stream. The variation introduced was in the direction of the magnetic field upstream of the wedge. The non-alignment of the field provides interesting insight into the adjustment of the fields in the wedge and in the fluid. As the inclination χ_1 of the magnetic field to the stream tends to zero the configuration has to pass from one in which there is continuity of magnetic field at the interface to one in which a current sheet lies on the surface.

3.8 PERTURBATION OF CABANNES' SOLUTION

Since the field inside the non-conducting wedge is constant, the condition of zero normal component on both upper and lower surfaces of the wedge for aligned fields in the fluid requires that there shall be no magnetic field inside the wedge. Corresponding to the collapse of the second family of characteristics it is to be expected that the second shock wave will fall on to the wedge surface and that this will provide the source of the current sheet appearing in Cabannes' solution. (See part I, section 3 for a similar approach). Another way of looking at this is to consider that the magnetic field in region II will orient itself so that $\underline{H}_2$ is parallel to the second shock in the limit as $\chi_1 \to 0$ while there will be no magnetic field in region III in this limit. Under these circumstances on putting $\delta = \chi_2$ in the equation corresponding to (3.17) for the second shock on the upper surface the condition

$$\tan^2(\chi_2-\theta_2)\left\{1+\tan(\chi_2-\theta_2)\tan(\theta_3-\theta_2)\right\}\left\{\tan(\chi_3-\theta_2)-\tan(\chi_2-\theta_2)\right\}^2=0$$

is obtained, and if shock angles greater than $\frac{1}{2}\pi$ are ignored this has the roots

$$\chi_2=\theta_2, \quad \chi_2=\chi_3. \qquad (3.2$$

The first of these is consistent with the Cabannes limit; when this is preserved the analysis outlined below shows that $\chi_2 \neq \chi_3$ in the limit as $\chi_1 \to 0$ and the angle χ_3 is therefore not restricted to approach θ_3 as $\chi_1 \to 0$.

As will be seen below the values of quantities in regions III and III' may not all be found to order χ_1 unless the perturbations from the Cabannes limits are calculated to order χ_1^2. The following perturbations are therefore introduced, the subscript c representing the (known) Cabannes values:

$$
\begin{aligned}
H_2/H_1 &= (H_2/H_1)_c + b_2\chi_1 + B_2\chi_1^2, \\
V_2/V_1 &= (V_2/V_1)_c + c_2\chi_1 + C_2\chi_1^2, \\
\rho_1/\rho_2 &= (\rho_1/\rho_2)_c + d_2\chi_1 + D_2\chi_1^2, \\
p_2/p_1 &= (p_2/p_1)_c + e_2\chi_1 + E_2\chi_1^2, \\
\beta &= \beta_c + f_2\chi_1 + F_2\chi_1^2, \\
\theta_2 &= \theta_3 + l_2\chi_1 + L_2\chi_1^2, \\
\chi_2 &= \theta_3 + g_2\chi_1 + G_2\chi_1^2.
\end{aligned}
\qquad (3.2
$$

Here b_2, B_2, c_2, C_2, ..., l_2, L_2 are constants to be determined. A set of six linear equations is obtained by equating terms of first order in χ_1 after substitution of (3.27) in (3.10), (3.12), (3.11), (3.14), (3.15) and (3.16):

$$\sin^2\omega\, b_2 + \sin\theta f_2 - \sin\beta\cos\omega\, g_2 = -\cos\beta\sin\omega, \qquad (3.2$$

$$\cos^2\omega\sin^2\omega\, c_2 + (1+k_1^2\cos 2\omega)\sin\theta\, f_2 - k_1^2\sin\beta\cos\omega\, g_2 +$$
$$+ (\cos\beta\sin\omega + k_1^2\sin\theta)\sin^2\omega\, h_2 = -k_1^2\operatorname{cosec}\beta\sin(\beta+\theta)\sin\omega\cos\omega, \quad (3.29)$$

$$\sin\beta\sin\omega\, c_2 - \sin^2\beta\, d_2 + (V_2/V_1)_c\sin\theta\, f_2 - (V_2/V_1)_c\cos\omega\sin\beta\, h_2 = 0, \quad (3.30)$$

$$-\gamma\xi_1^2(H_2/H_1)_c\cos^2\omega\sin\omega\, b_2 + \gamma(M_1^2-\xi_1^2)\sin\theta\cos\theta\sin\omega\, f_2 -$$
$$-\gamma\xi_1^2\sin^2\beta\cos\omega\, g_2 + \gamma(M_1^2\sin\beta\cos\beta\sin\omega + \xi_1^2\sin\beta\sin\theta)\, h_2 -$$
$$-\cos^2\omega\sin\omega\, e_2 = -\gamma\xi_1^2\sin(\beta+\theta)\cos\omega\sin\omega, \quad (3.31)$$

$$k_1^2(V_2/V_1)_c\, b_2 + \{(V_2/V_1)_c - k_1^2(H_2/H_1)_c\}\, c_2 +$$
$$+ \{k_1^2(H_2/H_1)_c^2 + (p_2/p_1)_c/M_1^2(\gamma-1)\}\, d_2 + \{(\rho_1/\rho_2)_c/M_1^2(\gamma-1)\}\, e_2 =$$
$$= k_1^2\{1-(V_2/V_1)_c(H_2/H_1)_c\}\cot\beta, \quad (3.32)$$

$$g_2 - h_2 = (V_1H_1/V_2H_2)_c > 0, \quad (3.33)$$

(necessarily positive since V and H are scalar resultants). For convenienc we have set $\omega = \beta - \theta$ and have omitted the subscript c on β and 3 on θ.

The same procedure in region II' with (3.27) modified to read

$$H_2'/H_1 = (H_2/H_1)_c - b_2'\chi_1 + B_2'\chi_1^2, \quad \text{etc.},$$

leads to the result

$$g_2' - h_2' = g_2 - h_2 > 0. \quad (3.34)$$

The flow near the wedge

For the transition to region III the assumptions consistent with the previous analysis are

$$\left.\begin{aligned} \delta &= \chi_2 + f_3\chi_1 + F_3\chi_1^2, \\ \chi_3 &= \alpha + g_3\chi_1 + G_3\chi_1^2, \end{aligned}\right\} \quad (3.35)$$

where α is the, as yet unknown, orientation of the magnetic field in region III and is O(1). (The particular form chosen for δ, with the perturbation measured from χ_2 instead of θ_3, aids in simplifying the algebra.) We have

$$H_3/H_2 = \sin(\delta-\chi_2)/\sin(\delta-\chi_3)$$

$$= f_3\chi_1/\sin(\theta_3-\alpha), \quad (3.36)$$

$$V_3/V_2 = \cos(\delta-\theta_2)/\cos(\delta-\theta_3) - k_2^2 H_3\sin(\chi_2-\chi_3)/H_2\cos(\delta-\theta_3)\sin(\delta-\theta$$

$$= 1 - f_3 k_{2c}^2 / (f_3+g_2-h_2), \quad (3.37)$$

$$\rho_2/\rho_3 = V_3\sin(\delta-\theta_3)/V_2\sin(\delta-\theta_2)$$

$$= \frac{f_3+g_2}{f_3+g_2-h_2}\left\{1 - \frac{f_3 k_{2c}^2}{(f_3+g_2-h_2)}\right\}, \quad (3.38)$$

$$p_3/p_2 = 1-\tfrac{1}{2}\gamma\varepsilon_2^2(H_3/H_2)^2 + \gamma M_2^2\sin(\delta-\theta_2)\sin(\theta_3-\theta_2)/\cos(\delta-\theta_3) +$$

$$+\tfrac{1}{2}\gamma\varepsilon_2^2\left\{1+2H_3\sin(\chi_2-\chi_3)\sin(\delta-\theta_3)/H_2\cos(\delta-\theta_3)\right\}$$

$$= 1 + \tfrac{1}{2}\gamma\varepsilon_{2c}^2, \quad (3.39)$$

$$(V_3/V_2)\sin(\chi_3-\theta_3) = (H_2/H_3)\sin(\chi_2-\theta_2)$$

or

$$f_3 + g_2 - l_2 = \pm f_3 k_{2c}, \tag{3.40}$$

where $k_{2c} = (\epsilon_2/M_2)_c$.

Different cases now arise according as $k_{2c} <, =,$ or > 1.

(i) $k_{2c} < 1$

The two values of f_3 given by (3.40) are negative. A necessary and sufficient condition for the flow in region II to intersect the second shock wave, as it must do in a physically real situation, is that $\delta > \theta_2$ or $f_3 + g_2 - l_2 > 0$. Accordingly, since $f_3 < 0$, the upper sign in (3.40) has to be dismissed and thus

$$f_3 = -(g_2 - l_2)/(1 + k_{2c}). \tag{3.41}$$

On substitution of the ratios (3.36) - (3.39) into the energy equation one finds that

$$(f_3+g_2)(1+k_{2c})/(-k_{2c}f_3) = \{1+\tfrac{1}{2}(\gamma-1)\epsilon_{2c}^2\}/(1+\tfrac{1}{2}\gamma\epsilon_{2c}^2) < 1. \tag{3.42}$$

This equation yields the value of g_2, and when it is used in conjunction with equations (3.28) - (3.33) we can solve for the seven parameters $b_2, c_2, \ldots, l_2$.

It follows easily that, to a first approximation (independent of χ_1),

$$V_3/V_2 = 1 + k_{2c},$$

$$\rho_3/\rho_2 = (1+\tfrac{1}{2}\gamma\epsilon_{2c}^2)/\{1+\tfrac{1}{2}(\gamma-1)\epsilon_{2c}^2\}$$

and

$$p_3/p_2 = 1+\tfrac{1}{2}\gamma\epsilon_{2c}^2.$$

Thus V_3/V_2, ρ_3/ρ_2, p_3/p_2 are all greater than unity. The last two of these results are required for a shock wave. The first shows that the flow is actually accelerated through the second shock and it is interestin to recall that Kogan [1] showed that an acceleration could occur across a magneto-gasdynamic shock when he applied linearised theory to the flow past a thin wedge with $\chi_1 = \frac{1}{2}\pi$ (he found such accelerations through the second shocks both above and below the wedge). The requirement of increas of entropy across the shock means that the inequality

$$\ln\left\{\left(\frac{p_3}{p_2}\right)\left(\frac{\rho_2}{\rho_3}\right)^{\gamma}\right\} > 0 \tag{3.43}$$

must be satisfied for the shock to be thermodynamically stable. Now

$$\left(1-\frac{\frac{1}{2}\varepsilon_{2c}^2}{1+\frac{1}{2}\gamma\varepsilon_{2c}^2}\right)^{\gamma} > 1-\frac{\frac{1}{2}\gamma\varepsilon_{2c}^2}{1+\frac{1}{2}\gamma\varepsilon_{2c}^2}, \quad \text{if} \quad \gamma > 1$$

$$= 1/\left(1+\tfrac{1}{2}\gamma\varepsilon_{2c}^2\right),$$

i.e. $$\left(1+\tfrac{1}{2}\gamma\varepsilon_{2c}^2\right)\left[\left\{1+\tfrac{1}{2}(\gamma-1)\varepsilon_{2c}^2\right\}/\left(1+\tfrac{1}{2}\gamma\varepsilon_{2c}^2\right)\right]^{\gamma} > 1,$$

and hence the above inequality (3.43) is satisfied.

Before discussing the cases $k_{2c} \gtrless 1$ it is convenient at this stage to give the analysis for the flow near the lower surface of the wedge.

In region III' (in the lower half-plane) the equations corresponding to (3.35) are

$$\left.\begin{aligned} \delta' &= \chi_2' - f_3'\chi_1 + F_3'\chi_1^2, \\ \chi_3' &= -(\alpha + g_3\chi_1 + G_3\chi_1^2), \end{aligned}\right\} \tag{3.41}$$

where we have used the result that $\underline{H}$ is constant throughout the non-conducting wedge.

Analysis carried out just as for the upper half-plane gives $f_3' + g_2' - l_2' = \pm k_{2c} f_3'$ and again we need to investigate whether $k_{2c} <, =,$ or > 1. For the moment we restrict k_{2c} to be less than unity. The choice of sign is found from the requirement of the flow in region II' to intersect the second shock. This yields

$$f_3' = -(g_2 - l_2)/(1 - k_{2c}) < 0. \qquad (3.45)$$

The substitution of the ratios H_3'/H_2' etc. into the energy equation gives

$$(f_3' + g_2')(1 - k_{2c})/k_{2c} f_3' = \{1 + \tfrac{1}{2}(\gamma - 1)\varepsilon_{2c}^2\}/(1 + \tfrac{1}{2}\gamma\varepsilon_{2c}^2) < 1, \qquad (3.46)$$

and from the equality we can find g_2'. As before the quantities $b_2', c_2', \ldots, l_2'$ can be found. Also

$$V_3'/V_2' = 1 - k_{2c}, \qquad (3.47)$$

while ρ_3'/ρ_2' and p_3'/p_2' have the same values as the corresponding quantities in region III. Equation (3.47) shows that the flow is decelerated through the second shock in contrast with what occurs on the upper side of the wedge. The shock is thermodynamically stable.

The value of α, to order χ_1^2 follows from the equality $H_3 = H_3'$ by writing

$$(H_3/H_2)(H_2/H_1) = (H_3'/H_2')(H_2'/H_1)$$

which gives

$$f_3/\sin(\theta_3 - \alpha) + f_3'/\sin(\theta_3 + \alpha) = 0 . \qquad (3.4$$

When f_3 and f_3' are replaced by their respective values from (3.41) and (3.45) it follows at once that

$$\tan \alpha = (1/k_{2c}) \tan \theta_3. \qquad (3.49$$

This implies that $\alpha > \theta_3$ (and incidentally verifies that $H_3/H_2 > 0$ as required).

We now continue with the investigation of the remaining values of k_{2c}.

<u>(ii) $k_{2c} = 1$</u>

When $k_{2c} = 1$ we have $f_3 + g_2 - l_2 = \pm f_3$. We dismiss the upper sign because of the result (3.33). By inspection the lower sign is found to be admissible. However we also require $f_3' + g_2' - l_2' = \pm f_3'$ and we can dismis the upper sign for the same reason as before (3.34). The lower sign gives $f_3' < 0$. But for the flow in region II' to intersect the shock we require $f_3' + g_2' - l_2'$ to be negative; there is thus a contradiction. We cannot therefore with this value of k_{2c} find a shock-wave solution of the type sought. It will be demonstrated in part IV that this value of k_{2c} is associated with shock waves which are physically unstable in the Cabannes problem.

<u>(iii) $k_{2c} > 1$</u>

Mathematical consistency now demands that $f_3 = (g_2 - l_2)/(k_{2c} - 1)$ and $f_3' = -(g_2 - l_2)/(k_{2c} + 1)$. This implies that $f_3' + g_2' - l_2' > 0$, with the consequence that the flow behind the first (lower) shock wave cannot meet the postulated second shock. There cannot therefore be two shock waves on the lower surface of the wedge. We shall show also in part IV that this value of k_{2c} gives rise to shock waves which, if they exist, are not physically stable.

3.9 EXTENSION TO $O(\chi_1)$ TERMS

In the last section we expressed V_3/V_2, ρ_3/ρ_2 and p_3/p_2 to first order terms only. To ensure consistency in our analysis we must develop the perturbation of these quantities to $O(\chi_1)$ terms. In general we have

$$\varepsilon_2^2 = \varepsilon_1^2 (p_1/p_2)(H_2/H_1)^2 , \quad M_2^2 = M_1^2 (p_1/p_2)(V_2/V_1)^2 (\rho_2/\rho_1) ,$$

and hence

$$\varepsilon_2^2 = \varepsilon_{2c}^2 (1+\mu_2 \chi_1) , \quad M_2^2 = M_{2c}^2 (1-\tau_2 \chi_1) , \quad k_2^2 = k_{2c}^2 (1+\lambda_2 \chi_1) ,$$

where for conciseness we have written

$$\mu_2 = 2 b_2 (H_1/H_2)_c - e_2 (p_1/p_2) , \tag{3.50}$$

$$\tau_2 = e_2 (p_1/p_2)_c + d_2 (\rho_1/\rho_2)_c - 2 c_2 (V_1/V_2)_c , \tag{3.50}$$

$$\lambda_2 = \mu_2 + \tau_2 . \tag{3.50}$$

Equation (3.16) may be used again (c.f. derivation of (3.33)) and this time we equate coefficients of χ_1:

$$G_2 - L_2 = -(g_2 - h_2)^2 \{ b_2 (V_2/V_1)_c + c_2 (H_2/H_1)_c \} . \tag{3.51}$$

To obtain all expansions to $O(\chi_1)$ terms we have to express H_3/H_2 to $O(\chi_1^2)$ and

$$H_3/H_2 = f_3 \chi_1 (1 + \lambda_3 \chi_1) / \sin(\theta_3 - \alpha),$$

where we have introduced λ_3 so that

$$f_3 \lambda_3 = F_3 - f_3 (f_3 + g_2 - g_3) \cot(\theta_3 - \alpha). \qquad (3.5$$

The result (3.40) was obtained from the constant terms in the expansion. If we now equate the coefficients of χ_1 we obtain

$$2(1 + k_{2c}) \lambda_3 f_3 +$$
$$+ k_{2c} f_3 \lambda_2 + 2(G_2 - L_2) + \{ f_3^2 + (2g_3 - g_2)(g_2 - h_2) \} \cot(\theta_3 - \alpha) = 0. \qquad (3.5$$

We can use (3.40) to simplify the ratio V_3/V_2 and we write

$$V_3/V_2 = 1 + k_{2c} + c_3 \chi_1,$$

where

$$f_3 c_3 = f_3 k_{2c} \{ \lambda_2 + \lambda_3 + (g_2 - g_3) \cot(\theta_3 - \alpha) \} + F_3 + G_2 - L_2. \qquad (3.5$$

Also,

$$\rho_2/\rho_3 = \{ 1 + \tfrac{1}{2}(\gamma - 1) \varepsilon_{2c}^2 \} / (1 + \tfrac{1}{2} \gamma \varepsilon_{2c}^2) + d_3 \chi_1,$$

where

$$k_{2c}^2 f_3^2 d_3 + c_3 (f_3 + g_2) k_{2c} f_3 +$$
$$+ (1 + k_{2c}) \{ (F_3 + G_2) k_{2c} f_3 + (f_3 + g_2)(F_3 + G_2 - L_2) \} = 0, \qquad (3.5$$

and

$$p_3/p_2 = 1 + \tfrac{1}{2} \gamma \varepsilon_{2c}^2 (1 + \mu_2 \chi_1).$$

For the lower surface we have

$$H_3'/H_2' = -f_3'\chi_1\left(1-\lambda_3'\chi_1\right)/\sin(\theta_3+\alpha),$$

where

$$f_3'\lambda_3' = F_3' - f_3'\left(f_3'+g_2'-g_3\right)\cot(\theta_3+\alpha). \qquad (3.5$$

The result, analogous to (3.53) is

$$2(1-k_{1c})\lambda_3'f_3' - k_{2c}f_3'\lambda_2' + 2(G_2'-L_2') +$$

$$+\left\{f_3'^2+(2g_3-g_2')(g_2'-h_2')\right\}\cot(\theta_3+\alpha) = 0. \qquad (3.5$$

We now have

$$V_3'/V_2' = 1 - k_{2c} + c_3'\chi_1,$$

where

$$f_3'c_3' = f_3'k_{2c}\left\{\lambda_2'+\lambda_3'+(g_2'-g_3)\cot(\theta_3+\alpha)\right\} - (F_3'+G_2'-L_2'). \qquad (3.5$$

Also

$$p_3'/p_2' = 1+\tfrac{1}{2}\gamma q_{2c}^2\left(1-\mu_2'\chi_1\right).$$

If we now apply $H_3 = H_3'$, the constant terms give (3.49) but the coefficients of χ_1 yield

$$(\lambda_3+\lambda_3')(H_2/H_1)_c + b_2 + b_2' = 0. \qquad (3.5$$

The values of f_3 and f_3' are known from (3.41) and (3.45) and from these we can determine g_2 and g'_2 from (3.42) and (3.46), respectively. It follows that the quantities b_2, c_2, ..., l_2, b_2', c_2', ...,l_2' can now be found from the two sets of linear equations. The value of α is found from (3.49) and the value of $G_2 - L_2$ may be obtained from (3.51). (The value of $G_2^{\prime} - L_2^{\prime}$ is obtained from the corresponding equation for the lower surface.) Also from (3.50) we know λ_2 and $\lambda_2^{\prime}$. Thus if we eliminat λ_3 and $\lambda_3^{\prime}$ from (3.59) by means of (3.53) and (3.57) we obtain an expressi for g_3 and its value is known:

$$g_3 = \frac{-\left\{N_2 + 2(H_1/H_2)_c (g_2 - h_2)(b_2 + b_2')\right\}}{2(g_2 - h_2)\left\{\cot(\theta_3 - \alpha) + \cot(\theta_3 + \alpha)\right\}}, \qquad (3.6$$

where

$$N_2 = k_{2c}(f_3\lambda_2 - f_3'\lambda_2') + 2(G_2 - L_2 + G_2' - L_2') + $$
$$+ \left\{f_3^2 - g_2(g_2 - h_2)\right\}\cot(\theta_3 - \alpha) + \left\{f_3'^2 - g_2'(g_2' - h_2')\right\}\cot(\theta_3 + \alpha)$$

Back substitution enables us to find λ_3 and $\lambda_3^{\prime}$ in terms of known quantities. Hence from (3.52) and (3.56) we can find F_3 and $F_3^{\prime}$, respectively, and when this is done we utilise the results to obtain expressions for c_3 and $c_3^{\prime}$ in terms of known quantities. By inspection it is clear that we have now obtained expansions to $O(\chi_1)$ for all the ratios, etc., save for ρ_3/ρ_2 and $\rho_3^{\prime}/\rho_2^{\prime}$. From (3.55) we see that on the right-hand side the quantity G_2 appears. To find G_2 (and also $G_2^{\prime}$) it is necessary to substitute for all the ratios in the energy equation and equate coefficients of χ_1. It is felt that there is nothing really to be gained in reproducing this analysis here. The algebra was in fact executed in the hope that some simplification would occur (c.f. the results (3.42) and (3.46)) but no simple grouping of parameters appeared.

At the beginning of section 3.8 it was stated that we required to take the perturbations laid down by (3.27). The results (3.53) and (3.59), for example, serve to illustrate that terms to $O(\chi_1^2)$ are required.

3.10 METHOD OF COMPUTATION

For any computation we require to specify M_1, ϵ_1, χ_1 and θ_3. Across each of the leading shock waves we have six equations in seven unknowns. Accordingly at this stage it would appear that if we specified one of these unknowns for each of the regions II and II' we could theoretically then determine the solution. Some kind of iteration procedure would then need to be introduced to obtain re-estimates of the two postulated quantities. However a close inspection of the equations which hold across the leading shocks reveals that, because of the transcendental nature of the equations and the way in which the groupings of the unknown parameters occur, we need to specify two unknowns for each of regions II and II'. For example, if we give values for β and χ_2 we can obtain θ_2 from (3.17) and hence we can find H_2/H_1; θ_2' is obtained the same way. Because of the number of parameters it is not feasible to pursue this line of approach.

By means of the results of the perturbation of Cabannes' solution we now present certain numerical results which reveal the consistency of the analytic approach.

Cabannes [6] gave numerical results of the solution of (3.23) for $\theta_3 = 20°$ when $\epsilon_1^2 = 0.1$ (weak magnetic field), $\epsilon_1^2 = 1$, and $\epsilon_1^2 = 10$ (strong magnetic field). He presented curves of β versus M_1 for each of the values of ϵ_1 in turn. We chose the case $\epsilon_1^2 = 0.1$, $\theta_3 = 20°$, a value of $\gamma = 1.4$ was selected and the coefficients in (3.23) were computed. The values of M_1 chosen were 1.85, 1.9, 2.0, 2.25, 2.5, 2.75 and 3.0; see Figure 3.2. Following the procedure in ordinary gas dynamics where one looks for the shock angle which is appropriate to the weak attached shock wave the values of β in (3.23) which gave rise to the weak shock branch were computed. It is possible to obtain an approximate value of β from [6]; iteration via the Newton method secured convergence to the root. With the values of β_c now found we obtained numerical values for $(H_2/H_1)_c$, $(V_2/V_1)_c$ etc. These values were checked against values which were represented graphically by Cabannes.

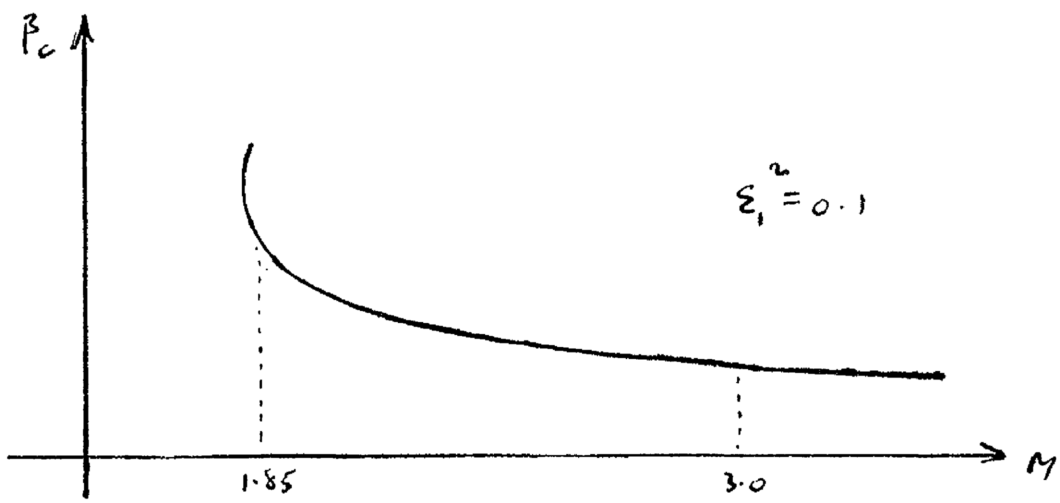

FIGURE 3.2: ANGLE OF SHOCK (WEAK SHOCK BRANCH)

The coefficients b_2, c_2, ..., l_2 were computed now from the set of linear equations. The solution of these equations was effected by the method of successive elimination; checks were executed via hand computation on a desk machine. From (3.27) all the ratios and angles were found