Summary of thesis "Statistical Analysis of Congestion in Road Traffic at an Intersection", by D. H. Reid

The problem considered is that of delays caused to vehicles on a major two-lane road at its intersection with a minor road. There are no traffic lights or policemen at the intersection and delays are assumed tc occur purely as a result of right-turning traffic (where left-hand-drive is the convention observed). A survey of literature relevant to this problem is made in Chapter 1.

In Chapters 2, 3, 4 mathematical models of increasing complexity are proposed as idealisations of the actual situation. These models are constructed with the aims of simplicity and realism in view, so that a valid mathematical analysis of the steady-state behaviour of each may be made. The models are based on such simplifications as the replacement of vehicles by geometrical points, the generation of vehicle arrivals by a Poisson process, and rules for the interaction of vehicles at the intersection which, although probably unrealistic in model I, may provide a reasonable picture of actual behaviour in model III. The distributions o delay to vehicles is determined under the assumptions of each of models I - III, and programmes were constructed to calculate numerical values for certain aspects of these distributions.

Chapter 5 is concerned with the assessment of these models, particula model III, against observations. The observations are described and thei limitations discussed. The parameters defining model III are estimated, and the goodness-of-fit of model III to the data is considered. A comparison of model III with model I is attempted. The tentative conclus

is reached that on the basis of the data available model III may provide an adequately realistic picture of vehicle behaviour.

In a final chapter some practical applications of the work are briefly considered, and further related problems are described.

STATISTICAL ANALYSIS OF CONGESTION IN ROAD TRAFFIC AT AN INTERSECTION.

BY

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Contents.

Chapter 1.

Introduction and Description of Problem.

1.1. Congestion and mathematical models.

Situations are often encountered in which some type of agency provides service for individuals or groups of a population, and in which the demand for service at times exceeds that which the agency is able to supply. Congestion, in a broad sense, refers to the non-availability of service to the customer at the instant at which it is first required. Examples are the demands by drivers of cars for entry to a parking space, or by telephone subscribers for the routeing of calls through an exchange. In each of these cases it is possible that if the service is not available as soon as it is required, the immediate demand disappears.

The kind of situation of interest here, by contrast, has the property that a customer whose demand for service is not immediately met will wait until the service becomes available. In many cases the customers wait in an orderly manner, forming a queue or queues, and receiving service in the order of their positions in the queue. For customers delayed in this way, the interval of waiting for service often represents an economic loss, and so the minimisation of the total cost of the system may be a problem of some importance. In service- or congestion- systems of this kind, it would appear that efficiency might be improved by the careful weighing of the cost of providing better service facilities against the savings accruing to customers, as a result of reduced delays. In order to assess various possible designs for a typical congestion-system a detailed analysis is required, which might well initially be concerned with an existing system.

The first steps in the analysis of the system, as of any other physical system, should be to observe and to describe it as precisely as possible. The extent to which this can be achieved will vary with different types of system. In some cases, for example, it may be possible to describe the service-mechanism in detail. In others it may be necessary to use inferential methods to discover the nature of the service from observations on the input and output processes. For many systems a purely deterministic description will not be satisfactory and the language and ideas of the theory of stochastic processes will be appropriate.

The description of the system is then used to construct a conceptual, usually mathematical, model of the real situation. In the first stages of the analysis the model is chosen to be as simple as possible while remaining consistent with the salient features of the description. At a later stage it may be possible to modify the original model, in the light of the analysis, to correspond more closely with the real situation. The description of many models is in terms of certain parameters, to which values must be allocated at the discretion of the analyst, and in this way a considerable degree of flexibility may be built into a model of a particular type.

The inclusion of parameters in a model gives rise to problems of statistical inference. There are naturally problems of estimation, and an assessment of "goodness-of-fit" is desirable in order to decide whether the model really reflects the features of the situation which are of interest, or whether a further model should be considered.

Ideally a model may give guidance on aspects of a congestion-system other than the purely economic questions. A satisfactory model may provide information on such problems as the general behaviour of the system, its efficiency and capacity, and the effects of minor modifications on the system.

1.2. Situations in road traffic which generate congestion.

It is apparent that there are many situations in road traffic which result in congestion. Vehicles may be delayed by other moving or stationary vehicles. Delays to pedestrians are usually caused by moving vehicles.

1.2.1. Delays to vehicles on the open road.

Travel on the open road is often impeded by the presence of slower-moving vehicles in the same carriageway. Mathematical models for this situation have been proposed by Miller (1962), Tanner (1961), and Newell (1955), among others, and have been examined to determine the distributions of delays caused by differences in speeds of vehicles. These authors give more attention to the development of the models than to the systematic assessment of their adequacy and realism.

1.2.2. Delays to vehicles at intersections.

Vehicles may be delayed at intersections by other vehicles, or by pedestrians, policemen, or traffic lights. Several authors have constructed mathematical models to describe the behaviour of vehicles at T-junctions and other types of intersection. Usually the effects of pedestrians on vehicle flow are ignored, although consideration has been given to the delay caused to pedestrians wishing to cross a road. (Tanner, 1953 and Mayne, 1954). Here pedestrians will be assumed to have no appreciable effect on the flow of traffic, and we shall be concerned with intersections which are not controlled by policemen or traffic-lights.

Such uncontrolled intersections are often of what we may call the 'priority' type, in which there are 'halt' or 'stop' signs requiring vehicles in certain lanes at the intersection to give way to those in other lanes.

It is convenient to distinguish between the terms 'T-junction' and 'intersection'. The former will be used to refer to the junction of a minor road with one side of a major road, the latter to the junction of two minor roads with the opposite sides of a major road at a given position along its length.

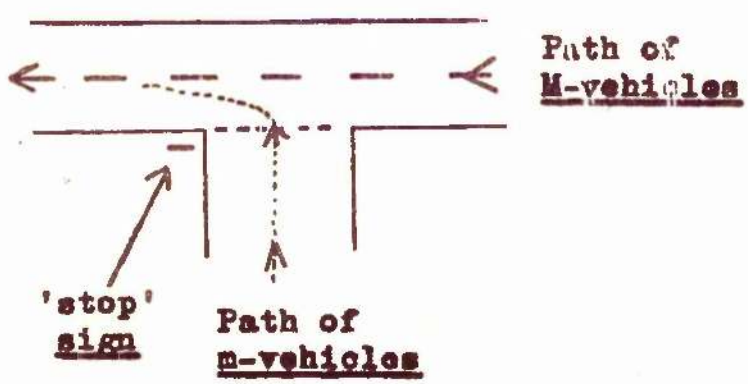
For brevity we shall sometimes refer to

'vehicles on the major road' as 'M-vehicles',

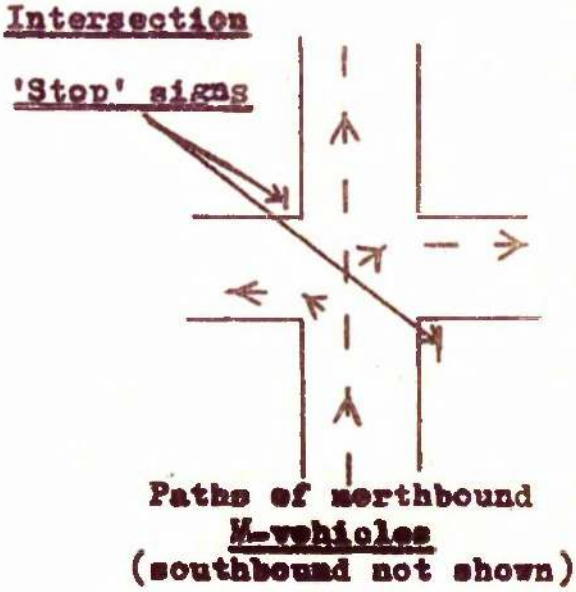
'vehicles on the minor road' as 'm-vehicles',

and for instance, 'the driver of a vehicle on the major road' as an 'M-driver'.

T-junction



Intersection



1.2.3. The main problem of the thesis.

The central problem of this thesis is to investigate the delay caused to M-vehicles at a priority intersection by right-turning vehicles* in the absence of controlling factors, such as traffic lights or policemen, and interfering factors, such as pedestrians. It is assumed that the major road has effectively a single lane in each direction, so that it is not possible for vehicles going straight on or turning left at the intersection to pass to the left of a vehicle awaiting an opportunity to complete its right turn.

The form of the thesis is as follows. In chapters 2, 3 and 4, mathematical models of increasing complexity are proposed for the problem. In chapter 5 the attempt is made to relate these models to data from an actual intersection, and to make an assessment of their realism.

1.3. Previous relevant work.

Very little appears to have been published on this problem, the relevant papers being summarised in section 1.3.2. There has been a fair amount of work on delays in simpler but similar situations, and many of the ideas and methods developed in these discussions are useful. A brief account of this work is given below.

*Left-hand drive is the convention observed.

1.3.1. Previous discussions of delays to m-vehicles at a priority intersection.

Many authors have considered the problem of the delay caused to vehicles entering a major road from a particular minor road at a priority intersection. The major road is almost always regarded as containing a single traffic stream, and this assumption is made throughout this section. Further assumptions often made are that the traffic in the major road is unaffected by the presence of vehicles in the minor road, so that for instance an M-driver would not give way to a waiting vehicle, and that the flow of traffic on the major road is not disrupted by the untimely emergence of vehicles from the minor road. Stochastic equilibrium of the system is assumed in most discussions.

Tanner (1962) proposes a model for this situation for which he derives a formula for the mean delay to m-vehicles. The time-intervals between successive arrivals of vehicles in both M- and m-streams follow an exponential distribution, and subject to the following restrictions, vehicles pass through the intersection instantaneously. An M- (m-) vehicle may not pass through the intersection within a fixed time β_1 (β_2) of the previous M- (m-) vehicle, but must, where appropriate, delay its passage to conform with this condition. An m-vehicle may not enter the intersection within a fixed time α ($> \beta_1$) of the previous M-vehicle. There is no restriction on how closely an M-vehicle may follow an m-vehicle.

The purpose of the parameter β_1 in this model is to allow traffic on the major road, as viewed by m-drivers, to be nonrandom in that it would conform to the output of a single-server queue with Poisson arrivals, in which the queue discipline is first-come-first-served, and the service time is constant. In this way the model simulates groups of vehicles at minimum separation followed by larger gaps. The result is that the exit from the minor road is 'blocked' for intervals distributed with the 'Borel-Tanner' distribution, and is free for intervals having an exponential distribution. Tanner's analysis is, however, valid for a general distribution of duration of block.

In order to extend Tanner's model and analysis to the situation in which m-vehicles seek to cross or merge with either of two streams of traffic on the major road, the distribution of blocks in the double M-stream must be specified. This problem does not appear to have been solved. For a two-stream extension of the Borel-Tanner distribution one requires the busy-period distribution of two queues in parallel, each of the above type, where the system is understood to be busy if at least one server is occupied.

Weiss and Maradudin (1962) propose a model which is based on the idea of a 'gap acceptance function', previously proposed by Cohen, Dearnaley and Hansel (1955). This function is defined as follows.

We suppose that at the instant when an m-vehicle arrives at the intersection, there is no queue in the minor road, and that the nearest approaching vehicle on the major road requires an interval of length t to reach the intersection. The gap acceptance function $\alpha(t)$ gives the probability that the driver of the m-vehicle accepts the gap and joins the M-stream.

We observe that Tanner's model uses a gap acceptance function which has the particular form of a step-function, because the model is essentially unaltered if a block is presumed to start an interval $\alpha'(< \alpha)$ ahead of the arrival of the first of a group of M-vehicles, and to finish an interval $\alpha - \alpha'$ after the last vehicle of the group. Weiss and Maradudin claim greater realism in their model on the grounds that an m-driver cannot infallibly distinguish between gaps greater than and less than α'.

Weiss and Maradudin exploit the gap acceptance function and the methods of Renewal Theory to derive solutions of a number of problems connected with the delay to m-vehicles at a priority intersection. They assume that the gaps between successive vehicles in the (single) M-stream are independently and identically distributed, and that M-vehicles cross the intersection instantaneously, so that the set of arrival instants forms a renewal process. As they are principally concerned with a single m-vehicle the distribution of arrival instants in the m-stream is not specified.

They determine the distribution of delay to a single m-vehicle which arrives at a random instant in time to find no other waiting m-vehicles. The probability that the m-vehicle moves off after the n^{th} subsequent M-vehicle is evaluated.

Several modifications are now made to the original model. By varying the form of the gap acceptance function used by an m-vehicle for successive delaying M-vehicles, they construct a model which includes an 'impatient driver' characteristic. The distribution of delay to a randomly arriving, non-queueing m-vehicle is found for this model. In another version of the model the distribution of successive time gaps between M-vehicles (in the original model) is made dependent on real time. From this the equations governing the distribution of delay to an m-vehicle, which arrived at real time T_o, say, to find no other queueing vehicle, are formulated in the nonstationary case. These equations are not solved.

Two alternative specifications of M-traffic are also considered. In the first of these, an attempt is made to simulate an n-lane major road by finding the distribution of time to the first arrival in an n-stream major road, when the distribution of arrivals in each stream is independent of arrivals in other streams, and forms a renewal process in time. In the other, the distribution of a gap in a single M-stream depends on the types of the vehicles which define the gap, each vehicle now being considered as belonging to one of R types.

Weiss and Maradudin then apply the simplest of their models to the case of a stream of m-vehicles delayed by the M-stream. They are unable to analyse the resulting model. They do succeed in generalising their simple model in a more limited way by deriving the distribution of delay experienced by each of a pair of m-vehicles which arrive simultaneously at the intersection.

Contemporary with the paper of Weiss and Maradudin is one by Cohen and Stam (1963). These authors analyse a model which makes some use of the gap acceptance function and also of what may be called a 'follow-on probability'. The latter is defined as follows. Suppose that an m-vehicle arrives at the intersection after waiting in a queue for a non-zero interval. Then, without reference to vehicles approaching on the major road, but on a purely random basis, the driver of this vehicle decides instantaneously whether or not to follow the preceding vehicle from the minor road. That is, with probability k, the 'follow-on probability', he follows the preceding vehicle, and with probability 1 - k he does not follow, but awaits the arrival of the next vehicle on the major road. Once he has made a decision to move, a driver completes his movement through the intersection instantaneously, and so the next driver in the queue (if any) is immediately in a position to make an independent decision in accordance with the same follow-on probability. It is possible in this way for a queue instantaneously to decrease in size by several vehicles.

Vehicles on the minor road which arrive at the intersection without queueing, or which have delayed to await the passage of one or more M-vehicles, use a gap acceptance function : all other m-vehicles use a follow-on probability. It will be observed that no driver uses the follow-on probability more than once. It is assumed that the arrival instants of M-vehicles form a Renewal process, and those of m-vehicles form a Poisson process.

The follow-on probability is intended to reflect to some extent the behaviour of drivers in the real situation, insofar as they find it easier to merge with the M-traffic when they are moving than when they are starting from rest. However, this rule has a further advantage. Cohen and Stam regard the total delay to a particular m-vehicle, which joins the end of a queue of n m-vehicles, as the sum of the 'completion times' of these n + 1 vehicles (partial completion time of the first), where 'completion time' refers to the interval elapsing from the arrival of a vehicle at the head of the queue until its departure. They shew that, for the model with follow-on probabilities, the completion times of the n + 1 queueing vehicles are independent. In this way they reduce the problem of delay to one involving the sum of independently distributed random variables. Although they evaluate only the mean delay, it would presumably be possible by this approach to determine higher moments of the delay distribution.

In a later section of the same paper the model proposed by Weiss and Maradudin is considered, in which the essential change is that the follow-on probabilities are replaced by gap acceptance functions; in this case the probability that of m (> n) queueing vehicles the first n only accept a gap of t is $\alpha^n(t)(1 - \alpha(t))$ instead of $\alpha(t)(1 - k)k^{n-1}$. It seems that the property of independence of the completion times of the vehicles in a queue is not possessed by this model. The determination of the delay distribution by the previous method would not be possible, although results concerning the mean delay would be unaffected.

Numerical results for both of the models proposed in the paper by Cohen and Stam are provided in a subsequent paper by Cohen and de Lange (1965).

Hawkes (1966) considers two congestion situations, the first of which concerns the delay to vehicles turning left or right from a minor road onto a major road which has two-way traffic, consisting of a single stream in each direction. Traffic on the major road is Poisson in each lane, and the arrivals of m-vehicles form a Poisson process. Vehicles on the minor road turn either left or right, and do so independently of previous vehicles. The completion of a left turn requires an interval of at least T_1 (a constant) between successive nearside vehicles on the major road, and drivers of right turning vehicles must await a gap of at least T_2 (a constant $\geq T_1$) in the combined flow of nearside and farside streams. It is assumed that vehicles depart instantaneously, and that there is a constant 'move-up time' a, which is the minimum separation with which successive m-vehicles may depart. Left- and right-turning vehicles form a single queue in the minor road.

Hawkes uses an embedded Markov Chain to find the distribution of delay to m-vehicles under the assumption that $a \leq T_1 \leq T_2 \leq 2a$. These restrictions are necessary to ensure the validity of the Markov formulation employed. If, for instance, $T_2 > 2a$, then the gap with which an m-vehicle arriving at the head of a queue is confronted may depend, not only on the previous m-vehicle, but on several previous to it.

Like Tanner, Hawkes makes use of a step-function as a gap acceptance function. The cost of generalising the analysis of Tanner to the two-lane major road is, in this case, that the analysis is valid only for random arrivals in the M-streams, in contrast to the analysis of Tanner. Hawkes states that the basic restrictions on the parameters of his model may not be satisfied in some actual situations, where it appears that T_2 has a value of about 2a. Although it would appear that this model would be most applicable at intersections where the behaviour of traffic, at least insofar as gap acceptance is concerned, is fairly uniform, as with the other models considered here any final assessment of the validity of the model would require detailed comparison of its calculated behaviour with actual observations.

The second model proposed in this paper by Hawkes is for delays to M-vehicles at a priority intersection and is more appropriately considered in the next section, 1.3.2.

In an earlier paper (1965) Hawkes discusses a model for the queueing by m-vehicles caused by a series of 'blocks' and 'gaps' in the single stream major road. Blocks have arbitrary known distribution and gaps are distributed exponentially. Vehicles in the minor road arrive in bunches at random instants of time. The number of m-vehicles in a bunch is distributed independently of the numbers in previous bunches with known distribution. m-vehicles may depart, singly, only during a gap, and after a departure a fixed time a, the 'move-up time' must elapse before another queueing vehicle may depart, should the exit be unblocked at this time. Movements of m-vehicles through the intersection are completed instantaneously. The distribution of delay to m-vehicles is determined.

Oliver and Bisbee (1962) assume that the distribution of gaps is general, that an m-vehicle requires a time interval T to move out, and that no two m-vehicles may depart in the same gap. For this model they determine the distribution of queue-size at the instants when an m-vehicle departs.

The model proposed by Weiss (1965) for delays to m-vehicles at an intersection makes use of a modification of the gap acceptance function described above. If is supposed that a driver at the head of a queue in the minor road requires a time interval of length T to turn into the major road. T is sampled from a distribution with density f(T). If the gap G presented to the driver is greater than T, the driver completes his turn, blocking the intersection for exactly time T. At the end of this time, the next driver in the queue (if any) bases his decision in a similar way on the gap G - T with which he is presented.

In this model both m- and M-vehicles arrive at random in time, and Weiss evaluates in the case of stationary behaviour the probability generating function for queue length at the instants at which M-vehicles cross the intersection.

1.3.2. Previous discussions of delays to M-vehicles at a priority intersection.

In this section we are concerned with the delay caused to vehicles on the major road by vehicles on this road which turn into the minor road. As the delays to other M-vehicles caused by those M-vehicles which turn left or go straight on are relatively insignificant, we classify M-vehicles into two groups: type I vehicles turn left or go straight through the intersection; type II vehicles turn right. We are now interested in the delays to M-vehicles arising from the movements of type II vehicles.

The second problem discussed by Hawkes (1966) is concerned with the delay to M-vehicles caused by type II vehicles at a T-junction, which we may regard as an intersection at which only one of the two M-streams contains type II vehicles. Queues may form behind a type II vehicle which does not cross immediately: it is assumed to block its lane completely. The model used by Hawkes is similar to that of the earlier section of the paper, which is discussed above in section 1.3.1.

Another model, originally discussed by Newell (1959) and subsequently modified by Haight (1963, §6.3), is as follows. The model is for a priority intersection, but time is regarded as divided into small intervals ('slots'), in each of which the system remains in a certain state, and transitions from one state to another are made discontinuously between one slot and the next. In each lane of the major road at a particular slot one of the following describes the situation at the intersection.

(a) A vehicle is ready to go straight through the intersection or to turn left.

(b) A vehicle is ready to turn right.

(c) No vehicle is present.

Natural rules are given which between each pair of slots allow the departure of one vehicle or both if at least one vehicle is present at the intersection. These rules are such that the departure of a straight or left-bound vehicle from one lane will prevent the departure of an opposing right-bound vehicle between any pair of slots.

Haight and Newell analyse this system by defining the 'states' at each slot as follows.

State 1 : no vehicle is delayed.

State 2 : a northbound vehicle is delayed.

State 3 : a southbound vehicle is delayed.

In order to describe the arrivals of vehicles in the system, it is convenient to make use of the positions occupied by vehicles immediately prior to their entry to the intersection. These positions are either empty or occupied. If empty, such a position may be occupied at each successive slot with constant probability, entirely independently of the other aspects of the system. Vehicles in these positions are of types I or II independently of the remainder of the system with probabilities constant for each lane.

A matrix of transition probabilities is defined on the states and state probabilities are obtained for the case of stochastic equilibrium. These may be used to find expressions for mean delays and to deduce conditions for stability of flow.

This model seems to be only an approximation to the actual situation, even if time is supposed to proceed by discrete steps, for the following reason. In an actual situation, the probability that the position adjacent to the intersection is filled appears to depend on the extent to which a previous right-turning vehicle in this position has been delayed. It is possible that a queue would form during any prolonged arrest of this vehicle. Thus the model applies exactly to the situation where a vehicle, which arrives in a given lane only to find that lane blocked by a delayed right-turning vehicle, merely disappears, and thus no queues form and the only delays in the system are caused by vehicles actually at the intersection, rather than waiting in a queue. While it is appreciated that all mathematical models are idealisations to varying degrees, it seems that an essential feature of the actual situation, namely the possibility of the formation of queues, has been neglected in this model.

1.3.3. The relevance of these models to the present problem.

(a) Input mechanism.

(i) Discrete time process. Although some authors have treated the arrival process as operating in discrete time, it is apparent that the characteristics of such a process must be based on an examination of the actual continuous process, with the result that most authors prefer to try to construct a continuous time model.

(ii) <u>Poisson Arrival Process in time.</u> As is well known, this process has several properties leading to mathematical simplification, and it has often been used in this context. The fact that the probability density for the inter-arrival distribution, $\lambda e^{-\lambda t}$, has its greatest value at the origin would lead to difficulties if vehicles of finite size were considered. Thus for this arrival distribution it is customary to replace vehicles by geometrical points in the model, and to attempt to compensate this further idealisation by the construction of rules for the behaviour of vehicles which simulate finiteness of size.

(iii) <u>Poisson arrival process with bunched arrivals.</u> In that it is more flexible this process would presumably be an improvement on the Poisson process. However the difficulties arising from the latter are considerable and there is not much prospect of progress with bunched arrivals at present.

(iv) <u>Renewal Process.</u> Similar remarks to those of (iii) are applicable.

<u>(b) Mechanism of Congestion.</u> In this lies the characteristic feature of the problem, and in the models which follow the attempt is made to achieve greater realism by modification of this mechanism. Some of the devices used by previous authors are applicable, notably those of Weiss (1965).

(c) Methods of Analysis. Except for a small section of the paper by Weiss and Maradudin (1962), all of the above work concerns the case of stochastic equilibrium, and the present analysis is confined to this case. The analysis uses the Markov Chain associated with an embedded sequence of regeneration points. (D. G. Kendall (1953)).

Chapter 2

Construction and Analysis of Model I.

2.1 Derivation of the model.

In attempting to construct a model for the type of intersection under investigation, we may take advantage of two characteristics of actual intersections which may lead to simplifications in any models to be considered.

The first such characteristic is that it is usually the case that little change in the delay to M-vehicles would result from the diversion of each left turning vehicle to a straight course. As in section 1.3.2, vehicles are therefore classified into only two types, and we recall that

> type I vehicles go straight on or turn left, type II vehicles turn right.

We would expect that in some sense type I vehicles would have priority over type II vehicles in the model.

The other simplification is a consequence of the fact that queues are not simultaneously present in both lanes at the intersection. At any instant, either vehicles are passing freely through the intersection without delay, or there is a queue present in one lane. The simplest method of incorporating this feature in a model is to stipulate that a vehicle at the intersection can be delayed by an approaching vehicle in the opposing lane only when the types of the two confronting vehicles are different, and that then at most one of the vehicles is delayed. Thus, in the model, a vehicle ready to cross the intersection (i.e. unimpeded by vehicles in the same lane) should be delayed only in the following circumstances, and not necessarily even then. These circumstances are that the vehicle about to cross should be of type II, and that the next vehicle to arrive in the opposite lane should be of type I.

In such a situation it might be conjectured that the driver of the stationary type II vehicle would base his turning decision on several factors, as well as the type of nearest vehicle in the opposite lane. The most obvious factor would be some measure of distance to the opposing type I vehicle, but distances to subsequent vehicles in the opposite lane might also be considered. It is a matter of observation that it is a comparatively infrequent occurrence for any consideration at all to be given by such a driver to vehicles other than the nearest in the opposing lane, and it is reasonable to suppose that his measure of the distance of this vehicle is equivalent to an assessment of the time due to elapse until its arrival at the intersection. In this first exploratory model the situation is deliberately simplified by the omission from the model of this feature of dependence of the decision on the size of the available gap.

2.2. Description of model I.

(i) Layout.

The model consists of the intersection of a two-lane major road with a minor road. Vehicles on the minor road have no effect on traffic on the major road, as there are halt signs on the minor road and the intersection is not controlled by traffic lights or by a policeman. Thus the minor road may be regarded as serving only as a means of departure of M-vehicles from the system.

The lanes of the major road are labelled 1 and 2 respectively.

(ii) Vehicles.

For the purposes of the following analysis m-vehicles are ignored.

Vehicles are represented by geometrical points. The following restrictions are imposed on the movements of these points.

a) They may not pass through each other.

b) They may not traverse the geometrical bounds of the model.

As in section 2.1, vehicles are classified into types I and II.

(iii) Vehicle-arrivals.

The arrivals of individual vehicles at the intersection are specified by the instants at which they arrive either at the intersection, or at the tail of a queue of vehicles. Since in the model vehicles possess zero length, the latter is geometrically equivalent to arrival at the intersection. The idea of a queue of vehicles, as a time-ordered group of one or more stationary vehicles in a particular lane, is however preserved.

In a given lane the set of arrival instants forms a stochastic process in time, and the intervals between arrivals follow independent exponential distributions, with parameter λ_i for lane i, $i = 1,2$.

With probability p_i an arriving vehicle in lane i is of type I and with probability q_i it is of type II, where $p_i + q_i = 1$, $i = 1,2$. The type of this vehicle is independent of the type of previous arrivals.

(iv) Vehicle-interactions.

To simulate the movement of individual vehicles through an actual intersection, it is assumed in the model that as soon as a vehicle is 'ready' to move it does so and clears the intersection instantaneously. It is 'ready' to move only when it is the leading vehicle of a queue (which could consist solely of this vehicle), and the driver of the vehicle has decided to move. His decision is made instantaneously upon his arrival in the leading position in accordance with the following rules.

a) A type I vehicle at the head of a queue in either lane moves as soon as it attains that position.

b) The driver of a type II vehicle at the head of a queue in lane i ($i = 1,2$) takes into account in making his decision only the type of the nearest opposing vehicle. If this is of type II he moves immediately. If it is of type I, with probability r_i he decides to move immediately, and with probability $1 - r_i$ he decides to await the passage through the intersection of the opposing type I vehicle. The driver of a vehicle which delays makes a fresh decision, following this same rule, as soon as the opposing type I vehicle has completed its crossing. Each decision made by the driver of a type II vehicle confronted by several type I vehicles in succession is independent of any previous decisions made by him.

These rules suffice to determine completely the behaviour of vehicles in the model. The only delays occurring are those due to the failure of a type II vehicle to move when faced with an approaching type I, i.e. those due to right-turning vehicles.

2.3. Steady-state behaviour of Model I.

2.3.1. Selection of regeneration points.

Before attempting to select suitable instants of time for consideration as regeneration points for the model, we observe certain general aspects of its behaviour which follow immediately from the description above. It is apparent that, at any instant of time, either there is a queue in one lane of the model, or there is no queue in either lane. A queue is necessarily headed by a type II vehicle, and the nearest opposing vehicle is of type I. Queues may decrease in size only at the instants immediately following the departure of an opposing type I vehicle. At such an instant one or several vehicles may depart instantaneously, or the driver of the leading vehicle may decide not to move. Queues may increase in size at any instant.

It is convenient to choose as regeneration points the instants at which a vehicle crosses the intersection in either lane. Each such instant is described by (i) the lane from which the vehicle departs, (ii) the number of vehicles (if any) queueing at that instant, and (iii) the type of the nearest vehicle in the other lane, if there is no queue in that lane. To avoid ambiguity in the cases where several vehicles depart simultaneously, item (i) will be taken to refer to the lane which contains the single type I vehicle whose passage has released the queue, and it will be the initial or largest size of the queue which is quoted in item (ii). Item (iii) is made necessary by the requirement that the description of the system at a regeneration point should be complete in the sense that no further information about its previous history is required to specify probabilities relating to its future behaviour. The omission of item (iii) from the description of the system would mean that this requirement was not always met, for example in the case where a II in lane 1 decided to move when opposed by a I in lane 2, and before the arrival of this I a further II arrived in lane 1.

It is in fact exactly those situations which give rise to the necessity for item (iii) above which cause difficulty in the attempt to analyse, by a similar system of regeneration points, a more complicated model based on the gap acceptance methods of Cohen and Stam. For certain of these points to carry an adequate description of the system, it would be necessary for this description to include the length of the time interval due to elapse before the arrival of the nearest opposing vehicle. As the use of a continuous variable for descriptive purposes at a regeneration point would imply that the number of such points is not denumerable, the method of the Embedded Markov Chain cannot be used.

2.3.2. Notation for steady state analysis.

The following notation is used for reference to the set of regeneration points of §2.3.1.

$R_i(n)$ is the label given to an instant at which a vehicle departs from lane i, and at which there are $n(\geq 1)$ vehicles queueing in the opposing lane. It is understood that the remarks of §2.3.1. concerning i and n will apply.

$R_i(-1)$, $R_i(-2)$ refer to instants at which a vehicle departs from lane i with no queue in the opposing lane, in which the next arrival is of type I, II, respectively.

The interval between successive regeneration points will be called a transition interval, and it may be partly specified by reference to the types of regeneration points at its extremities. Thus an interval commencing with an $R_i(m)$ and ending with an $R_j(n)$, $(i, j = 1, 2; \ m, n = -2, -1, 1, 2, \ldots)$ will be referred to as of type $\{i, j; \ m, n\}$. A transition interval is defined to include the initial but not the final regeneration point, so that vehicles may leave the intersection only at the initial instant of a transition interval.

It is assumed that the behaviour of the system is stochastically stationary, and that parameter values are consistent with this assumption. We define 'state probabilities' as follows:

$\rho_i(n)$ denotes the probability that a regeneration point selected at random is of type $R_i(n)$, $i = 1, 2; \ n = -2, -1, 1, 2, \ldots$

It is convenient also to define transition probabilities for the probability that a transition interval is of a particular type, conditional on the type of the initial regeneration point of the interval.

Additional variables may be used to specify a transition interval with greater precision, thus we refer to an interval of type $\{i, j; \ m, n\}$, of duration τ, and having a queue of length k in lane i immediately following the initial instant, as being of type $\{i, j; \ m, n; \ \tau, k\}$. The probability density that an interval is of type $\{i, j; \ m, n; \ \tau, k\}$, conditional on the initial regeneration point being of type $R_i(m)$, will be denoted by $\pi(i,j;\ m,n;\ \tau,k)$; and $\pi(i,j;\ m,n)$ will denote the probability that an interval is of type $\{i,j;\ m,n\}$, conditional on the initial point being of type $R_i(m)$.

2.3.3. Transition probabilities.

The transition probabilities are determined by arguments similar to the following for $\pi(1,1;\ m,n)$, $m,n > 0$.

The procedure is to determine separately the quantities

$$\pi(1,1;\ m,n;\ \tau,\ m-r), \quad 1 \leq r \leq m-1;$$

$$\pi(1,1;\ m,n;\ \tau,\ m);$$

$$\pi(1,1;\ m,n;\ \tau,\ 0);$$

and to make use of the fact that

$$\pi(1,1;\ m,n) = \int_0^{\infty} \sum_{r=0}^{m} \pi(1,1;\ m,n;\ \tau,\ m-r)\, d\tau .$$

The quantity $\pi(1,1;\ m,n;\ \tau,\ m-r)$, $1 \leq r \leq m-1$, refers to transitions from an $R_1(m)$ to an $R_1(n)$. It is the probability density for the joint occurrence of the events listed below, conditional on the presence of a queue of m vehicles in lane 2 and the departure of a type I vehicle from lane 1 at the initial instant. These events are

(i) the next vehicle in lane 1 is of type I and it arrives at a time τ after the initial instant;

(ii) the leading vehicle (of type II) in lane 2 moves prior to the arrival of this type I vehicle, as do the following r-1 vehicles (of assorted type) in the queue - i.e. they all move at the initial instant;

(iii) the next vehicle in the queue is of type II and the driver decides to await the passage through the intersection of the opposing type I vehicle;

(iv) the number of arrivals in lane 2 during the subsequent time interval τ is n-m+r.

It follows that

$$\pi(1,1;\ m,n;\ \tau,\ m-r)$$
$$= \lambda_1 p_1 e^{-\lambda_1\tau}.r_2(p_2+q_2r_2)^{r-1}.q_2(1-r_2).\ \frac{e^{-\lambda_2\tau}(\lambda_2\tau)^{n-m+r}}{(n-m+r)!}\ .\varepsilon(n-m+r),$$
$$1 \leqslant r \leqslant m-1,$$

where $\varepsilon(n) = 1,\quad n \geqslant 0,$
$$= 0,\quad n < 0.$$

Similarly $\pi(1,1;\ m,n;\ \tau,\ m)$ and $\pi(1,1;\ m,n;\ \tau,\ 0)$ are evaluated.

Writing $f_i = p_i + q_ir_i$, $\lambda = \lambda_1 + \lambda_2$, we have

$$\pi(1,1;\ m,n;\ \tau,\ m\} = p_1\lambda_1(1-r_2)e^{-\lambda\tau}\ \frac{(\lambda_2\tau)^{n-m}}{(n-m)!}\ \varepsilon(n-m)\ ,$$

$$\pi(1,1;\ m,n;\ \tau,\ 0\} = \int_0^\infty \lambda_1 p_1 e^{-\lambda_1(t+\tau)}.\lambda_2 q_2 e^{-\lambda_2 t}(1-r_2)e^{-\lambda_2\tau}\frac{(\lambda_2\tau)^{n-1}}{(n-1)!}$$
$$.\ f_2^{\,m-1}r_2\ dt,$$

and so

$$\pi(1,1;\ m,n) = p_1\lambda_1(1-r_2)\left\{\frac{1}{\lambda}\left(\frac{\lambda_2}{\lambda}\right)^{n-m} \varepsilon(n-m)\right.$$

$$+ \frac{r_2q_2f_2^{\,m-n-1}}{\lambda_1+\lambda_2q_2(1-r_2)}\left[1-\left(\frac{\lambda_2f_2}{\lambda}\right)^{n}\right]\varepsilon(m-n-1)$$

$$+ \left(\frac{\lambda_2}{\lambda}\right)^{n-m+1}\frac{r_2q_2}{\lambda_1+\lambda_2q_2(1-r_2)}\left[1-\left(\frac{\lambda_2f_2}{\lambda}\right)^{m-1}\right]\varepsilon(n-m)$$

$$\left. + \frac{r_2q_2}{\lambda}f_2^{\,m-1}\left(\frac{\lambda_2}{\lambda}\right)^{n}\right\};\quad m,\ n > 0.$$

The complete set of 36 transition probabilities for model I is given in Appendix I. A check on the transition probabilities is provided by the set of consistency relations

$$\sum_{\substack{n=-2\\ n\neq 0}}^{\infty}\sum_{j=1}^{2} \pi(i,j;\ m,n) = 1,\quad i = 1,2;\quad m = -2,-1,1,2,\ldots$$

2.3.4. Evaluation of state probabilities.

The state probabilities satisfy the stationarity equations

$$\rho_j(n) = \sum_{\substack{m=-2\\ m\neq 0}}^{\infty}\sum_{i=1}^{2} \pi(i,j;\ m,n)\,\rho_i(m),\quad j = 1,2;\quad n = -2,-1,1,2,\ \ldots, \tag{1}$$

together with the normalising relation

$$\sum_{i=1}^{2}\sum_{\substack{n=-2\\ n\neq 0}}^{\infty}\rho_i(n) = 1. \tag{2}$$

These equations are now solved for the state probabilities.

If we define $\widetilde{R}_i(x) = \sum_{n=1}^{\infty} \rho_i(n)\, x^n$, $i = 1,2$; $|x| \leq 1$, multiplication of each of the equations of (1) by an appropriate power of x followed by summation reduces the set (1) to the equivalent set

$$\lambda_1 p_2 r_1 \rho_2(-2) - \left[\lambda_2 + \lambda_1 q_1(1-r_1)\right]\rho_1(-1) + \lambda_1 p_2 \rho_2(-1)$$

$$+ \lambda_1 p_2 q_1 r_1 \widetilde{R}_1(1) + (\lambda_1 p_1 p_2 r_2/f_2)\widetilde{R}_1(f_2) + \lambda_1 p_2 r_1 \widetilde{R}_2(f_1) = 0, \qquad (3)$$

$$\widetilde{R}_1(x)\left[\frac{p_1\lambda_1(1-r_2)(p_2-x)}{(\lambda-\lambda_2 x)(f_2-x)} - 1\right] + \frac{\lambda_1\lambda_2(1-r_2)}{\lambda} \cdot \frac{x}{(\lambda-\lambda_2 x)} \cdot \left[p_1\rho_1(-2)+q_2\rho_2(-1)\right]$$

$$+ \widetilde{R}_1(f_2)\,\frac{p_1\lambda_1(1-r_2)r_2q_2}{f_2} \cdot \frac{x}{(\lambda-\lambda_2 x)}\left(\frac{1}{f_2-x} + \frac{\lambda_2}{\lambda}\right)$$

$$+ \widetilde{R}_1\left(\frac{\lambda_2 f_2}{\lambda}\, x\right) \cdot \frac{p_1(1-r_2)r_2q_2\lambda\lambda_2}{f_2\left[\lambda_1+\lambda_2 q_2(1-r_2)\right]} \cdot \frac{(x-1)}{(\lambda-\lambda_2 x)}$$

$$+ \widetilde{R}_2(1) \cdot \frac{p_1 q_2(1-r_2)\lambda_1\lambda_2}{\lambda} \cdot \frac{x}{(\lambda-\lambda_2 x)} = 0, \qquad (4)$$

and

$$\lambda_2\rho_1(-2)-\lambda_1 q_2\rho_2(-2)-\lambda_1 q_2\rho_2(-1)-\lambda_1 q_1 q_2\widetilde{R}_1(1)-\lambda_1 q_2\widetilde{R}_2(1)-\frac{\lambda_1 q_2 p_1 r_2}{f_2}\,\widetilde{R}_1(f_2) = 0. \qquad (5)$$

Equation (2) is

$$\rho_1(-2)+\rho_2(-2)+\rho_1(-1)+\rho_2(-1)+\widetilde{R}_1(1)+\widetilde{R}_2(1) = 1. \qquad (6)$$

Corresponding to each of equations (3), (4), (5) there is an equation obtained by interchanging lane suffices. These equations are labelled (3^*), (4^*), (5^*) respectively.

The construction of equations (3), (4), (5) has introduced certain constants which are themselves combinations of the unknown state probabilities. These are $\tilde{R}_1(1)$, $\tilde{R}_2(1)$, $\tilde{R}_1(f_2)$ and $\tilde{R}_2(f_1)$. Before proceeding further we evaluate these constants, together with the quantities $\rho_1(-2)$, $\rho_2(-2)$, $\rho_1(-1)$, $\rho_2(-1)$. For this purpose, we have equations (3), (3^*), (5), (5^*), together with an additional pair, found by setting $x = 1$ in (4), (4^*),

$$\lambda_2(1-r_2)\left[p_1\rho_1(-2) + q_2\rho_2(-1)\right] - \frac{p_1 r_2}{f_2}(\lambda_1+\lambda_2 f_2)\tilde{R}_1(f_2)$$
$$+ p_1 q_2(1-r_2)\lambda_2\tilde{R}_2(1) - q_1\lambda\tilde{R}_1(1) = 0, \tag{7}$$

with a corresponding equation (7^*).

When the coefficients of these six equations are inspected, it is found that only five are linearly independent, so that, on taking into account equation (6), we have in all six independent relations between the eight unknown quantities. A further independent pair of equations may be derived as follows by an argument based on the fact that the functions $\tilde{R}_i(z)$ are analytic within the unit circle.

Writing k for $\lambda_2 f_2/\lambda$, so that $k < 1$, equations (4) and (7) reduce to

$$\tilde{R}_1(x) = \frac{x\ell(x)}{(x-\alpha)(x-\beta)} + \frac{c(f_2-x)(x-1)}{(x-\alpha)(x-\beta)}\tilde{R}_1(kx), \tag{8}$$

where $\ell(x) = \frac{\lambda_1}{\lambda_2}\frac{p_1 r_2}{f_2}(1-x)\tilde{R}_1(f_2) + \frac{\lambda_1}{\lambda_2}q_1(f_2-x)\tilde{R}_1(1)$,

$$c = p_1 q_2 r_2(1-r_2)\lambda/f_2(\lambda_1+\lambda_2 q_2(1-r_2)),$$

and $\lambda_2(x-\alpha)(x-\beta) = \lambda_2 x^2 - x(\lambda_2 f_2+\lambda-\lambda_1 p_1(1-r_2))+\lambda f_2 - p_1 p_2\lambda_1(1-r_2)$,

$$\equiv S(x) \text{ say.}$$

Since $S(f_2) > 0$, $S(1) < 0$ and the coefficient of x^2 in $S(x)$ is positive, it follows that one root of $S(x) = 0$ lies between f_2 and 1 and the other is greater than 1, i.e. $\alpha < 1$, $\beta > 1$.

Equation (8) may be further rewritten

$$\tilde{R}_1(x) = h(x) + g(x)\,\tilde{R}_1(kx),$$

to which a formal solution, obtained by repeated substitution, is

$$\begin{aligned}\tilde{R}_1(x) &= h(x) + g(x)h(kx) + g(x)g(kx)h(k^2x) + \dots\\ &= \frac{x}{(x-\alpha)(x-\beta)}\left\{\ell(x) + \sum_{n=1}^{\infty} \frac{k^n c^n \ell(k^n x) \prod_{i=0}^{n-1} (f_2-k^i x)(k^i x-1)}{\prod_{i=1}^{n}(k^i x-\alpha)(k^i x-\beta)}\right\}.\end{aligned}$$

By definition, $\tilde{R}_1(z)$ is an analytic function of z at $z = \alpha$, so that its numerator must vanish at $z = \alpha$, i.e.

$$\ell(\alpha) + \sum_{n=1}^{\infty} k^n c^n \ell(k^n\alpha) \prod_{i=0}^{n-1} \left\{(f_2-k^i\alpha)(k^i\alpha-1)/(k^{i+1}\alpha-\alpha)(k^{i+1}\alpha-\beta)\right\} = 0,$$

which is equivalent to

$$\begin{aligned}&\frac{p_1 r_2}{f_2}\tilde{R}_1(f_2)\left\{1-\alpha+\sum_{n=1}^{\infty} k^n c^n (1-k^n\alpha) \prod_{i=0}^{n-1} \left[(f_2-k^i\alpha)(k^i\alpha-1)/(k^{i+1}\alpha-\alpha)(k^{i+1}\alpha-\beta)\right]\right\}\\ &+q_1\tilde{R}_1(1)\left\{f_2-\alpha+\sum_{n=1}^{\infty} k^n c^n (f_2-k^n\alpha) \prod_{i=0}^{n-1} \left[(f_2-k^i\alpha)(k^i\alpha-1)/(k^{i+1}\alpha-\alpha)(k^{i+1}\alpha-\beta)\right]\right\}\\ &= 0. \qquad (9)\end{aligned}$$

Equation (9) and the corresponding equation (9^*) constitute the additional pair of linear equations required to evaluate the eight constants, $\tilde{R}_1(1)$, ..., $\rho_2(-1)$.

By means of equation (8) we may now calculate the state probabilities $\rho_1(n)$, $n \geq 1$. We define

$$y_1 = \frac{p_1\lambda_1 r_2}{f_2}\tilde{R}_1(f_2) + q_1\lambda_1 f_2\tilde{R}_1(1),$$

$$y_2 = \frac{p_1\lambda_1 r_2}{f_2}\tilde{R}_1(f_2) - q_1\lambda_1\tilde{R}_1(1),$$

$$a_0 = p_1\lambda_1(1-r_2) - \lambda_2 f_2 - \lambda,$$

$$a_1 = \lambda f_2 - p_1 p_2\lambda_1(1-r_2),$$

$$b_0 = p_1(1-r_2)r_2 q_2\lambda\lambda_2/f_2(\lambda_1+\lambda_2 q_2(1-r_2)),$$

$$b_1 = -(1+f_2)b_0,$$

$$b_2 = f_2 b_0.$$

Writing for convenience $\rho_1(n) = u_n$, so that $\tilde{R}_1(x) = \sum_{i=1}^{\infty} u_i x^i$, (8) is equivalent to

$$u_1(a_1 + kb_2) = y_1,$$

$$u_2(a_1 + k^2 b_2) + u_1(a_0 + kb_1) = y_2, \qquad (10)$$

$$u_n(a_1 + k^n b_2) + u_{n-1}(a_0 + k^{n-1}b_1) + u_{n-2}(\lambda_2 + k^{n-2}b_0) = 0, \quad n \geq 3.$$

If we regard the first n equations of the set (10) as non-homogeneous linear equations in the unknowns u_1, u_2, ..., u_n, we may write the solution for u_n, $n \geq 3$, in the determinantal form

$$u_n = (-1)^{n+1} \Delta_n / \Delta'_n ,$$

where $\Delta'_n = \prod_{i=1}^{n} (a_1 + k^i b_2)$,

and
$$\Delta_n = \begin{vmatrix} a_1 + kb_2 & 0 & 0 & 0 & 0 & y_1 \\ a_0 + kb_1 & a_1 + k^2 b_2 & 0 & --- \; 0 & 0 & y_2 \\ \lambda_2 + kb_0 & a_0 + k^2 b_1 & a_1 + k^3 b_2 & 0 & 0 & 0 \\ 0 & \lambda_2 + k^2 b_0 & a_0 + k^3 b_1 & 0 & 0 & 0 \\ & - & - & - & & \\ 0 & 0 & 0 \; --- & \lambda_2 + k^{n-2} b_0 & a_0 + k^{n-1} b_1 & 0 \end{vmatrix}$$

see e.g. Jordan (1950), p.587. In practice, however, the numerical evaluation of Δ_n for large n seems to be most easily achieved by regarding Δ_n as the solution of a system of recurrence relations, and so it is just as satisfactory to attempt the solution of (10) directly.

It is not possible to solve (10) by calculation of $u_1, u_2, u_3, \ldots$ from successive equations of the set, as it is found that an ultimately divergent sequence is obtained. This may be understood upon consideration of the asymptotic form of (10). The recurrence relation becomes almost exactly

$$a_1 u_n + a_0 u_{n-1} + \lambda_2 u_{n-2} = 0 \qquad (11)$$

for moderately large n, and (11) has an auxiliary quadratic with roots α^{-1}, β^{-1}. Thus $u_n \sim A\beta^{-n} + B\alpha^{-n}$ and since $\alpha^{-1} > 1$, $\beta^{-1} < 1$, it is the solution with B exactly zero which is required. This solution may not be reached by the forward solution of (10) from u_1, u_2, owing to the accumulation of round-off error.

The procedure adopted here involves the backward solution of the system (10). We assume that, for some number N, the values of u_n $(n > N)$ are sufficiently small to make a negligible contribution to the value of the expression for whose calculation the $\{u_n\}$ is required. In the determination of the mean delay it is found that the quantity most likely to be affected by inaccuracies in the $\{u_n\}$ is the second moment of the distribution $\{u_n\}$. This second moment is calculated repeatedly, for a selection of suitably spaced and increasing values of N, using the values of $\{u_n\}$ approximated by the method to be described. This process is continued until substantial increases in the value of N result in comparatively small changes in the second moment. At this point we consider N adequately determined.

To complete the approximate determination of $\{u_i\}$, we note that for large n, the ratio of u_{n-1} to u_n is effectively β. A set of quantities u_n', $n = 1, \ldots N$, satisfying (10) and with $u_N' = 1$, $u_{N-1}' = \beta$, are found by backward solution of (10), the first pair of equations of the set being ignored. The first equation of (10) gives a value for u_1, and the approximation to u_n is then

$$\begin{aligned} u_n &= u_n' \times u_1/u_1', \quad && n \leqslant N, \\ &= 0, && n > N. \end{aligned}$$

Checks on these approximations are provided by

(i) comparison of the value for u_2 with the value given by the the second equation of (10);

(ii) comparison of $\sum_{n=1}^{N} u_n$ with the known value of $\tilde{R}_1(1)$.

2.3.5. The mean delay.

By 'delay' to a vehicle we understand the interval elapsing from its arrival at the tail of a queue, or at the intersection if no queue is present, until the instant it crosses the intersection. In this section the mean delay to vehicles in lane 2 is determined. In view of the Poisson arrival process in lane 2, the formula

mean delay = (mean queue length) / (arrival rate),

(Kendall 1951)

may be used for this purpose, where 'mean queue length' refers to the mean queue length left by a departing vehicle.

The method used here to determine this mean queue length is to average its value for a particular type of transition interval over all transition intervals. The average formed is weighted for each interval by the number of vehicles leaving lane 2 during the interval, since by 'mean' we imply selection of a vehicle rather than a point in time. Thus the number of vehicles leaving lane 2 during an interval must be included in the description of the interval.

In any interval, the only instant at which departures in lane 2 occur is the initial instant. If there are n vehicles queueing in lane 2 just prior to the initial instant, and if in the initial instant the number of queueing vehicles is reduced to k, the mean queue length as seen by a vehicle departing in this interval is $\frac{1}{2}(n+k-1)$.

We denote by $\pi(1, j;\ m, n;\ k)$ the probability that a regeneration point of type $R_1(m)$ is followed by a point of type $R_j(n)$, and that there are k vehicles in lane 2 immediately following the initial instant. If we further define by ω_2 the proportion of transition intervals having as initial point $R_2(n)$, $n = -2, -1, 1, 2, \ldots$, then

Prob. [a vehicle selected at random in lane 2 departs during an interval of type $\{1, j;\ m, n;\ k\}$]

$$= \frac{(m-k)\,\pi_1(m)\,\pi(1, j;\ m, n;\ k)}{\omega_2 + \sum\limits_{j'=1}^{2}\sum\limits_{m'=1}^{\infty}\sum\limits_{\substack{n'=-2\\ n'\neq 0}}^{\infty}\sum\limits_{k'=0}^{m'-1}(m'-k')\pi_1(m')\pi(1, j';\ m', n';\ k')},$$

$$m \geqslant 1;\quad k = 0, 1, \ldots m-1.$$

Thus the mean queue length left by a vehicle departing from lane 2 is

$$\left\{\sum_{j,m,n}\sum_{k=0}^{m-1}(m+k-1)(m-k)\pi_1(m)\pi(1, j;\ m, n;\ k)\right\} \Big/ 2\left\{\omega_2 + \sum_{j',m',n'}\sum_{k'=0}^{m'-1}(m'-k')\pi_1(m')\pi(1, j';\ m', n';\ k')\right\},$$

as no contribution is made to the numerator by those intervals which commence with the passage of zero or one vehicles in lane 2. From this expression numerical values for the mean queue length and hence mean delay may be derived.

It has been pointed out that the mean queue length as seen by a departing vehicle is in fact equal to the mean queue length in time. If we denote the respective means by $E(q_d)$ and $E(q_t)$, then in the equilibrium case, we have for the time interval $(t, t+\delta t)$,

expected total delay experienced = expected total delay generated,

$$\text{i.e.} \quad E\left\{q_t \delta t\right\} = \lambda_2 \delta t E\left\{v\right\},$$

where v is the delay of a given vehicle,

$$\text{so that} \quad E\left\{q_t\right\} = \lambda_2 E(v) = E\left\{q_d\right\}.$$

2.3.6. Numerical results.

Using a Ferranti Sirius computer equations (10) were solved, and the mean delays computed for several sets of parameter values. In table 2.1, λ_1 refers to number of vehicles per unit time, and the mean delay is measured in these time units.

Table 2.1.

Results of calculations for mean delay.

p_1	r_1	λ_1	p_2	r_2	λ_2	Mean delay in lane 2.
.5	.4	10	.5	.4	18	.201
.5	.4	26	.5	.4	10	.030
.5	.4	10	.5	.1	10	.207
.5	.4	10	.3	.2	10	.203 .. (a)
.5	.4	10	.3	.2	5	.184
.5	.4	20	.3	.2	10	.092 .. (b)

2.4. Remarks on model I.

As explained in section 2.1, model I was constructed with mathematical tractability in mind, and in order to see whether an analysis of its steady-state behaviour might be possible. Although such an analysis has been achieved, it is of interest to assess the practical value of the model before it is abandoned as merely a step towards more realistic ones. We consider this from two points of view, namely the plausibility of the initial assumptions and of the results of calculations of the mean delay.

2.4.1. Validity of the initial assumptions.

In the model, the replacement of vehicles by geometrical points is intended to simplify the mathematical description of both the arrival process and the mechanism of congestion. In an actual intersection, congestion largely results from the fact that the dimensions of vehicles and the widths of lanes are of the same order of magnitude. There is no provision in the model, other than the rule for delaying type II vehicles, for simulating this aspect of the actual situation. In fact, any unrealistic behaviour of the model resulting from the assumption of point-vehicles would tend to be emphasised by the additional assumptions of zero crossing-times and decision-times for vehicles and drivers.

It is just possible, of course, that these assumptions may be reasonable approximations in some situations where, on the average, delay-times considerably exceed crossing-times. Again, the rule which allocates to right-turning vehicles the same probability of moving, no matter how near an opposing I might be, might be regarded as reasonable in a situation where the visibility of drivers is restricted, or where vehicles are moving very slowly.

The assumption of random arrivals may not be completely justified by the actual traffic flow at a particular intersection, but it is nevertheless the case that a fundamental simplification in the analysis of the model is dependent upon this assumption. It should therefore be retained in some form if at all possible. There is a considerable mass of experimental evidence of varying quality suggesting that the Poisson model is adequate for vehicle arrivals at points along roads some distance from any congestion mechanism. This form of vehicle distribution may be demonstrated to be the asymptotic form assumed by traffic possessing certain fairly plausible initial velocity and position characteristics. (See e.g. Haight (1963) Chapter 4).

The assumed constancy of arrival rates should reasonably approximate the actual situation, at least over limited periods of observation.

2.4.2. Predicted behaviour of model I.

It is characteristic of actual intersections that the total number of vehicles per unit time which may cross the intersection is limited. This leads to the idea of the capacity of an intersection, and traffic engineers are much concerned with an appropriate definition of this quantity. If the vehicular flow to the intersection exceeds its capacity, the behaviour of the system ceases to be stationary.

It is interesting to determine what restrictions may be necessary on the values of the parameters in order that the behaviour of the model should be stationary. We see that, with probability zero, queues increase indefinitely in this model (for non-zero p_i, q_i), since any queue may be completely released, instantaneously, by the arrival of an opposing II. Stationary behaviour of the system is therefore possible for any set of parameter values. This characteristic of the model may be regarded as a serious deficiency, and it would tend to restrict the successful application of the model to an actual intersection to those situations where traffic is relatively light.

Inspection of table 2.1 suggests that it may be possible, for at least some values of p_i, r_i, to decrease delay in lane 2 by increasing the arrival rate of vehicles in lane 1, other parameters remaining fixed. (c.f. rows (a), (b)). This apparently unrealistic behaviour may be understood by observing that by increasing λ_1 we merely offer more frequent crossing opportunities to a delayed II in lane 2, with a consequent reduction in mean waiting-time in lane 2. In practice an increase in λ_1 would imply a decrease in r_2 - the existence of a relationship between r_1 and λ_2 and between r_2 and λ_1 might be considered.

The behaviour of model I does not seem to correspond more closely with actual traffic than one would expect from the considerably idealised initial assumptions. The simplicity of the model would allow of its analysis by methods other than that of the Embedded Markov Chain, but it appears that the value of such an exercise would be mainly academic.

A description of model I is given in Reid (1967).

Chapter III

Construction and Analysis of Model II.

3.1. Derivation of the model.

A serious defect of model I is the assumption that a driver's decision to turn right is not affected by the proximity of an opposing vehicle. We have seen in section 2.3.1. that modifications to model I based on the gap acceptance function as used by, e.g. Weiss and Maradudin (1962), are unlikely to be tractable, partly because of the manner in which such a function requires drivers to make use of the entire available gap in reaching their decision.

It seems reasonable to suppose that the interest of a turning driver in a gap in the opposing lane is concerned solely with its adequacy for his intended crossing. We may imagine that he has in mind an interval α, which will be called his 'gap-requirement time', and which is the minimum time he requires for starting up and moving clear of the opposing stream of vehicles. His decision whether to cross is made by comparing the available gap with α: if it exceeds α, he crosses without hesitation, otherwise postponing his decision until the opposing vehicle has cleared the intersection. If we regard the intersection as blocked in both lanes for the gap-requirement time α, and make the assumption of Poisson traffic, the interval from the end of this blocked period until the passage of the original opposing vehicle will have an exponential distribution. Thus any vehicles arriving at the intersection during this interval, possibly from a queue, will be confronted with random opposing arrivals, simplifying considerably the analysis of a model based on such a scheme.

It is a matter of observation, however, that drivers tend to overestimate the time they require to complete a crossing. Thus in an actual situation there is a certain interval T following the block caused by a turning vehicle which is the remainder of the interval estimated by the driver to be necessary for his movement. Although, in a model, T might be regarded as part of the gap requirement time α, this would imply that the driver of the vehicle queueing behind the vehicle which has crossed and given rise to T would not start to make his decision until the end of the interval T. This might be reasonable if he was required to make a distinct movement to the head of the queue before making his decision, and if the time he required for this movement was on the average about equal to T.

Although it would be possible to discuss methods of interpreting α in greater detail, possibly by investigating the relative sizes of separable components of α, and by considering the accuracy of a driver's assessment of the distance of an opposing vehicle, it is felt that little would be gained because of the impracticability of measurements of these quantities. We leave further discussion of the suitability of this method of describing the gap-acceptance mechanism until chapter 5. Similar types of gap-acceptance mechanism have been proposed by Gaver (1965) and Weiss (1965).

In model II the driver of a turning vehicle will be taken to make use of a gap requirement time in the manner described above, so long as his position at the head of a queue is a consequence either of his arrival at the intersection with no queue in either lane, or of a previous decision which he has made not to move in the face of an opposing vehicle. Drivers in a given lane are assumed to have access to a population of gap-requirement times, from which a particular driver selects one for a given situation uninfluenced by any previous choices of himself or other drivers.

Those drivers in model II who arrive at the head of a queue when one or more preceding members of the queue move off, make an immediate decision in the same way as drivers in model I, and if they decide to cross, do so instantaneously. This assumption is intended to reflect the tendency of actual drivers to 'follow-on', i.e. to follow instantaneously almost without regard to oncoming traffic a preceding vehicle which is making the crossing. One would expect a model incorporating such an assumption to be easier to analyse than one in which all turning drivers made use of a gap requirement criterion.

3.2. Description of model II.

(i) Layout, vehicles, vehicle-arrivals.

Model II differs from model I only in the manner in which vehicles interact at the intersection. The description of the layout, vehicles, and arrival process is as given in section 2.2, (i), (ii), (iii).

(ii) Vehicle-interactions.

We distinguish the vehicles which arrive at the intersection to find one or more preceding vehicles delayed in the same lane, from those whose arrival at the intersection is uninterrupted by preceding vehicles in their lane : we refer to arrivals 'from a queue' in the former case and 'in free flow' in the latter. 'Crossing' and 'turning' refer to movements complete to the extent that the intersection is left clear.

The rules for the interaction of vehicles at the intersection are :

(a) A type I vehicle, which arrives at the intersection either in free flow or from a queue, crosses immediately.

(b) A type II vehicle which arrives at the intersection in free flow or from a queue, or which has already delayed its departure from the head of a queue by either of rules (c), (d) below, turns immediately when the nearest vehicle in the opposite lane is also of type II.

(c) The driver of a type II vehicle in lane i which arrives at the intersection in free flow, or which has already delayed its crossing by either of the rules to be described here and in rule (d), acts as follows when the nearest vehicle in the opposite lane is of type I.

He measures the time interval due to elapse, of length τ, from the instant at which his vehicle is in a position to move (i.e. either its instant of arrival or the instant at which an opposing type I vehicle departs), until the arrival of the opposing vehicle. He selects a 'gap-requirement time' α from a distribution with density $f_i(\alpha)$. If $\alpha < \tau$, he delays for an interval α and then instantaneously completes his turn, thus blocking lane i for an interval of length α. If $\alpha \geqslant \tau$ he awaits the arrival and crossing of the opposing type I vehicle, before again assessing the situation by means of rules (b), (c). Each decision made by a driver in this way is independent of any previous decisions made by him or other drivers.

(d) A type II vehicle in lane i, which arrives at the intersection from a queue and which finds the nearest opposing vehicle to be of type I, turns immediately with probability k_i, or awaits the crossing of the type I vehicle with probability $1 - k_i$, whereupon it uses whichever of rules (b), (c) is appropriate.

The quantity k_i is the 'follow-on probability'. Each decision made by a driver in this way is independent of any previous decisions made by him or others.

Rules (a) - (d) suffice to determine completely the interaction of vehicles in the model. As in model I, the only delays to M-vehicles are those due to the presence of right-turning vehicles.

3.3. Steady-state behaviour of Model II.

3.3.1. Selection of regeneration points : notation.

It is obvious that, at any instant of time, a queue may exist in at most one lane of the model. Queues may be augmented at any instant of time. The instants at which queues may decrease in size or vanish are of two types. A queue in lane i may disappear immediately after the crossing of an opposing type I vehicle, if the next opposing vehicle is of type II. Otherwise, a delayed type II vehicle may turn after pausing for its gap-requirement time, to be followed immediately by several other vehicles of assorted type. Thus the complete or partial release of a queue in model II is not always immediately preceded by the crossing of a I in the other lane. Recognition of this aspect of the behaviour of the model is of assistance in the construction of an Embedded Markov Chain.

It is assumed that the system has attained a state of stochastic stationarity. As in model I we identify as regeneration points those instants at which a vehicle crosses the intersection in either lane. In the case where several vehicles depart simultaneously, we regard the 'first' vehicle to cross the intersection, in the sense of section 2.2 (iii), as defining the regeneration point. The description of the system at a regeneration point must include the following information :

(i) the lane from which the first vehicle departs,

(ii) if a queue exists, its lane and size,

(iii) if there is no queue, the type of the nearest vehicle in the lane other than that from which the vehicle departs.

The notation used to represent regeneration points is as follows :

$R_i(n)$ refers to an instant at which a vehicle necessarily of type I, crosses in lane i, in the presence of a queue of n vehicles in the <u>other</u> lane $(n = 1, 2, \ldots)$;

$S_i(n)$ refers to an instant at which a vehicle necessarily of type II, crosses in lane i, in the presence of an <u>additional</u> n queueing vehicles in the same lane $(n = 1, 2, \ldots)$.

$\left.\begin{matrix} R_i(-1) \\ R_i(-2) \end{matrix}\right\}$ refer to instants at which a vehicle of either type crosses in lane i, there being no queue present, and the next opposing vehicle being of type $\left\{\begin{matrix} I \\ II \end{matrix}\right.$.

Thus for instance $S_1(n)$ refers to an instant at which a vehicle of type II turns from lane 1, after delaying in this lane for its gap-requirement time α, in the presence of a further queue of n vehicles in lane 1. The next arrival in lane 2 is a type I vehicle.

Because of the properties of the exponential distribution, the description of the system at each regeneration point is sufficiently complete to enable all subsequent probabilistic behaviour of the model to be inferred without knowledge of its previous history. Vehicles cross the intersection only at the regeneration points : they may arrive at any instant of time.

We define the state probabilities for the Embedded Markov Chain as follows :

$$\left.\begin{array}{l}\rho_i(n)\\ \sigma_i(n)\\ \rho_i(-1)\\ \rho_i(-2)\end{array}\right\}\quad \begin{array}{l}\text{is the probability that a randomly}\\ \text{selected regeneration point is of type}\end{array}\quad \left\{\begin{array}{l}R_i(n)\\ S_i(n)\\ R_i(-1)\\ R_i(-2)\end{array}\right.$$

$$(n = 1, 2, \ldots).$$

3.3.2. Transition Probabilities.

We now consider in detail the chain of events which determines the type of regeneration point which follows a regeneration point of given type, and evaluate corresponding transition probabilities. There are 64 types of transition such as $R_1(n) \rightarrow R_1'(m)$, but by symmetry we may reduce the number for study to 32. The example below illustrates the method :

$S_1(m) \rightarrow R_2(n)$

At the initial instant a type II vehicle is instantaneously completing its turn from lane 1, with a further queue of m vehicles in this lane; the driver of this type II vehicle has already used the information that the next arrival in lane 2 will be of type I.

We write for brevity "at τ" for "in $(\tau, \tau+d\tau)$". For the next regeneration point to be of type $R_2(n)$, one of the following sequences of events must occur.

Either $r(0 \leq r \leq m-1)$ vehicles depart from lane 1 immediately following the initial vehicle, after which the driver of the $(r+1)^{th}$ vehicle decides to await the arrival of the opposing type I vehicle, distant time τ, and during this time a further n-m+r vehicles arrive in lane 1.

Or, all the vehicles in lane 1 cross or turn immediately and the next vehicle in lane 1 (due an interval τ_1 later) is of type II, and arrives prior to the arrival of the type I vehicle in lane 2 (which occurs after an interval $\tau+\tau_1$, say). The driver of this type II vehicle decides to await the arrival of the opposing type I, and in the remaining interval, of length τ, a further n-1 vehicles accumulate in lane 1.

With p_i, q_i, λ_i, λ, s, defined as for model I (section 2.2 (iii), 2.3.3), if we write $F_i(\tau) = \int_0^{\tau} f_i(t)dt$, $s_i = p_i + q_i k_i$, the probability of this transition is

$$\int_0^\infty \sum_{r=0}^{m-1} \lambda_2 e^{-\lambda_2\tau}(p_1+q_1k_1)^r q_1(1-k_1)e^{-\lambda_1\tau} \frac{(\lambda_1\tau)^{n-m+r}}{(n-m+r)!} d\tau\, \varepsilon(n-m+r)$$

$$+ \int_0^\infty \int_0^\infty \lambda_2 e^{-\lambda_2(\tau+\tau_1)}(p_1+q_1k_1)^m e^{-\lambda_1\tau_1}\lambda_1 q_1 \left[1-F_1(\tau)\right] e^{-\lambda_1\tau}\frac{(\lambda_1\tau)^{n-1}}{(n-1)!} d\tau d\tau_1$$

$$= \frac{\lambda_2 q_1(1-k_1)}{\lambda} \left(\frac{\lambda_1}{\lambda}\right)^{n-m} \sum_{r=0}^{m-1} \left(\frac{s_1\lambda_1}{\lambda}\right)^r \varepsilon(n-m+r)$$

$$+ \frac{\lambda_2 q_1}{\lambda} \frac{\lambda_1{}^n s_1{}^m}{(n-1)!} \int_0^\infty \left[1 - F_1(\tau)\right] e^{-\lambda\tau}\tau^{n-1} d\tau .$$

It is found that of the 32 basic types of transition, 9 have probability 0. The transition $R_1(m) \rightarrow R_2(-1)$ has positive probability only in the case $m = 1$. The complete set of transition probabilities for model II is given in Appendix I. The transition probabilities may be verified by use of the consistency relations of section 2.3.3.

3.3.3. Determination of State Probabilities.

The state probabilities satisfy the usual set of stationarity equations together with a normalising condition.

We define

$$\tilde{R}_i(x) = \sum_{n=1}^{\infty} x^n \rho_i(n), \qquad |x| \leq 1;$$

$$\tilde{S}_i(x) = \sum_{n=1}^{\infty} x^n \sigma_i(n), \qquad |x| \leq 1;$$

and $$f_i^*(s) = \int_0^\infty e^{-st} f_i(t)dt, \qquad \mathcal{R}(s) > 0.$$

The stationarity equations then reduce to equations (1) to (4) below, together with equations (1^*) to (4^*) obtained from these by an interchange of lane suffices. The equations are

$$(\lambda-\lambda_2x)\tilde{R}_1(x) = \lambda_1\left[1-f_2^*(\lambda-\lambda_2x)\right]\left\{\frac{\lambda_2}{\lambda}x(p_1\rho_1(-2)+q_2\rho_2(-1))+p_1\tilde{R}_1(x)\right.$$

$$\left.+\frac{\lambda_2}{\lambda}p_1q_2x\tilde{R}_2(1)+\frac{\lambda_2}{\lambda}q_2x\tilde{S}_2(s_2)\right\}$$

$$+\frac{\lambda_1q_2(1-k_2)x}{(s_2-x)}\left\{\tilde{S}_2(s_2)-\tilde{S}_2(x)\right\}, \qquad (1)$$

(by considering transitions into $R_1(n)$);

$$\tilde{S}_1(x) = \frac{\lambda_1}{\lambda}\left(f_1^*(\lambda-\lambda_1x)-f_1^*(\lambda)\right)\left\{q_1\rho_1(-1)+p_2\rho_2(-2)+q_1p_2\tilde{R}_1(1)\right.$$

$$\left.+q_1\tilde{S}_1(s_1)\right\}$$

$$+\frac{p_2}{x}\tilde{R}_2(x)f_1^*(\lambda-\lambda_1x)-p_2f_1^*(\lambda)\rho_2(1),$$

(transitions into $S_1(n)$); (2)

$$\lambda\rho_1(-1) = \lambda_1(p_1+q_1f_1^*(\lambda))(\rho_1(-1)+\tilde{S}_1(s_1))+\lambda_1p_2(\rho_2(-1)+f_1^*(\lambda)\rho_2(-2)$$

$$+\tilde{S}_2(s_2)+q_1f_1^*(\lambda)\tilde{R}_1(1))+\lambda p_2f_1^*(\lambda)\rho_2(1), \qquad (3)$$

(transitions into $R_1(-1)$); and

$$\lambda_2\rho_1(-2) = \lambda_1q_2(\rho_2(-1)+\rho_2(-2)+q_1\tilde{R}_1(1)+\tilde{R}_2(1)+\tilde{S}_2(s_2)),$$

(transitions into $R_1(-2)$). (4)

Equations (1) and (2^*) and (1^*) and (2) may be solved simultaneously for $\widetilde{R}_i(x)$, $\widetilde{S}_i(x)$, and these functions are then determined as rational functions of x and known transforms, but containing as unknown quantities the set of constants $\rho_1(-2)$, $\rho_2(-2)$, $\rho_1(-1)$, $\rho_2(-1)$, $R_1(1)$, $\widetilde{R}_2(1)$, $\widetilde{S}_1(s_1)$, $\widetilde{S}_2(s_2)$, $\rho_1(1)$ and $\rho_2(1)$.

It is convenient to use equations (3), (3^*) to substitute for the last pair of this set in terms of the others, leaving eight unknowns. We proceed to find a nonsingular set of eight equations linear in these unknowns, which we shall denote by $u_1, u_2, \ldots u_8$, respectively.

Setting x = 1 in the derived expression for $\widetilde{R}_1(x)$ gives

$$p_1\lambda_2u_1 + p_1\lambda_2u_3 - \lambda_1u_4 - \lambda q_1u_5 + p_1q_2\lambda_2u_6 + p_1\lambda_2u_7 - \lambda_1u_8 = 0, \quad (5)$$

together with a corresponding equation (5^*) derived from $\widetilde{R}_2(x)$. The set (4), (4^*), (5), (5^*) are linearly dependent, since the coefficients of u_i sum to zero (i = 1, 2, ... 8).

If in the normalising condition

$$\rho_1(-2) + \rho_2(-2) + \rho_1(-1) + \rho_2(-1) + \widetilde{R}_1(1) + \widetilde{R}_2(1) + \widetilde{S}_1(1) + \widetilde{S}_2(1) = 1,$$

the last two terms of the L.H.S. are replaced by formulae derived from the expressions for $\widetilde{S}_i(x)$, we have a further equation in the $\{u_i\}$.

Writing

$$\xi_1 = \lambda_1(s_2 - 1)(q_1 + p_1 f_2^*(\lambda_1)),$$
$$\phi_1 = \lambda_1(s_2 - 1)(f_2^*(\lambda_1) - q_1 f_2^*(\lambda) - p_1 f_2^*(\lambda_1) f_2^*(\lambda)),$$

with similar definition of ξ_2, ϕ_2 for interchanged lane suffices, we have

$$\sum_{i=1}^{8} c_i u_i = 1, \tag{6}$$

where
$$c_1 = 1 + \frac{\lambda_2 p_1}{\lambda\lambda_1 q_1(s_2-1)} \left\{\phi_1 + f_2^*(\lambda)\xi_1\right\},$$

$$c_3 = 1 + \frac{1}{\lambda\lambda_1 q_1(s_2-1)} \left\{\lambda_2 p_1 \xi_1 + \lambda_1 q_1 \phi_2 - \lambda_1 \xi_2 \left[\frac{\lambda}{\lambda_1} - (p_1 + q_1 f_1^*(\lambda))\right]\right\},$$

$$c_5 = 1 + \frac{\lambda_1 q_1 p_2}{\lambda\lambda_2 q_2(s_1-1)} \left\{\phi_2 + f_1^*(\lambda)\xi_2\right\},$$

$$c_7 = \frac{\lambda_2\, p_1\, \xi_1}{\lambda\lambda_1 q_1(s_2-1)} + \frac{\lambda_1}{\lambda\lambda_2 q_2(s_1-1)} \left\{q_1\phi_2 + \frac{\lambda\lambda_2 p_2(1-s_1) f_1^*(\lambda_2)}{\lambda_1} + \left[p_1 + q_1 f_1^*(\lambda)\right]\xi_2\right\},$$

and $c_{2r} = c_{2r-1}$ with lane subscripts interchanged, $r = 1, \ldots 4$.

The statement $\tilde{S}_2(0) = 0$ is equivalent to

$$p_1 s_2 \lambda f_2^*(\lambda) u_1 + p_1 s_2(\lambda - \lambda_1 p_1 + \lambda_1 p_1 f_2^*(\lambda)) u_3 + s_2\left[\lambda_1 p_1(1 - f_2^*(\lambda))\left(\frac{\lambda}{\lambda_2} - p_2\right) - \lambda\left(\frac{\lambda}{\lambda_2} - p_2 - q_2 f_2^*(\lambda)\right)\right] u_4 + p_1 q_2 s_2 \lambda f_2^*(\lambda) u_6 + p_1 s_2(\lambda - \lambda_1 p_1 + \lambda_1 p_1 f_2^*(\lambda)) u_7$$
$$+ \left[\frac{\lambda\lambda_1}{\lambda_2} p_1(1-s_2) f_2^*(\lambda) + \lambda s_2 q_2 f_2^*(\lambda) + s_2 p_2(\lambda - \lambda_1 p_1 + \lambda_1 p_1 f_2^*(\lambda))\right] u_8 = 0. \tag{7}$$

There is a corresponding equation (7^*).

The expressions for $\tilde{R}_1(x)$, $\tilde{S}_2(x)$ have a common denominator $D_1(x)$, where $D_1(x) = (s_2-x)(\lambda-\lambda_1 p_1-\lambda_2 x) + \lambda_1 p_1(1-x)f_2^{*}(\lambda-\lambda_2 x)$, after removal of irrelevant factors. Since $D_1(x)$ is continuous and $D_1(0) > 0$, $D_1(1) < 0$, there is at least one zero of $D_1(x)$ in $(0, 1)$. It is shown in section 3.3.6 that $D_1(x)$ has exactly one zero in this interval, at say $x = \alpha_1$. It follows by the definition of the functions $\tilde{R}_1(x)$, $\tilde{S}_2(x)$ that the numerators of the expressions for these functions must also vanish at $x = \alpha_1$. If we write

$$\psi_1 = \alpha_1 - s_2 + (1-\alpha_1)f_2^{*}(\lambda-\lambda_2\alpha_1)$$

the condition for $\tilde{R}_1(x)$ is

$$p_1\psi_1 u_1 + p_1(1-s_2)u_3 + (q_2\psi_1 - (1-s_2)(\frac{\lambda}{\lambda_2} - p_2))u_4$$
$$+ p_1 q_2\psi_1 u_6 + p_1(1-s_2)u_7 + (q_2\psi_1 - (1-s_2)(\frac{\lambda}{\lambda_2} - p_2))u_8 = 0. \qquad (8)$$

A similar equation (8^{*}) may be derived by consideration of $\tilde{R}_2(x)$ at $x = \alpha_2$; the equations based on the numerators of $\tilde{S}_2(\alpha_1)$, $\tilde{S}_1(\alpha_2)$ give no new information. (This statement and the next are based on numerical work: complexity has so far defeated analytic investigation.)

Equations (4), (4^{*}), (5), (6), (7), (7^{*}), (8), (8^{*}) constitute a linearly independent set from which $\{u_i\}$ may be determined. We have thus determined $\rho_i(-2)$, $\rho_i(-1)$, $\tilde{R}_i(x)$ and $\tilde{S}_i(x)$, which are required for the determination of the distribution of delay.

3.3.4. The distribution of delay.

The distribution function for the delay experienced by vehicles in lane 1 is denoted by $B_1(x)$, where it is understood as in §2.3.5 that the delay to a vehicle is the total time during which the vehicle is at rest at the intersection. The delay to vehicles in lane 2 is considered in this section.

Let us define

v_r = prob. [a vehicle departing in lane 2 leaves a queue of length r in lane 2],

$$\Lambda_2(x) = \sum_{r=0}^{\infty} v_r x^r, \quad |x| \leq 1,$$

and $B_2^*(s) = \int_0^\infty e^{-st} dB_2(t)$.

Since the queue in lane 2 left by a driver departing from this lane consists of vehicles which have arrived during his delay time, of duration v, say, the distribution of the number of queueing vehicles, n, conditional on v is

$$\frac{(\lambda_2 v)^n}{n!} e^{-\lambda_2 v}, \quad n = 0, 1, 2, \ldots,$$

and the unconditional distribution of n has probability generating function

$$\sum_{n=0}^{\infty} x^n \int_0^\infty \frac{(\lambda_2 v)^n}{n!} e^{-\lambda_2 v} dB_2(v)$$

$$= B_2^*(\lambda_2(1-x)) = \Lambda_2(x).$$

Thus

$$B_2^*(x) = \Lambda_2\left(\frac{\lambda_2 - x}{\lambda_2}\right), \quad |x| \leq 1, \text{ a standard result.}$$

We now construct an expression for $\Lambda_2(x)$.

The only regeneration points at which a vehicle departing from lane 2 may leave a non-empty queue in this lane are of type $R_1(n)$ or $S_2(n)$. Let k denote the total number of departures from lane 2 associated with such a point. For $R_1(n)$, k assumes one of the values zero and n, since at such a regeneration point the only way in which a non-zero number of vehicles may depart in lane 2 is by the next arrival in lane 1 being of type II. For departures in lane 2, if H_2 is a normalising constant,

$$\text{prob.}\left[\begin{array}{l}\text{remaining queue is of length r and regeneration point is of}\\ \text{type } R_1(n)\end{array}\right]$$

$$= \frac{1}{n} \times \frac{n\rho_1(n)q_1}{H_2} \; \varepsilon(n-1-r) \quad .$$

Similarly,

$$\text{prob.}\left[\begin{array}{l}\text{remaining queue is of length r, regeneration point is of}\\ \text{type } S_2(n)\text{, with a total of k departures, } 1 \leq k \leq n\end{array}\right]$$

$$= \frac{1}{k} \times \frac{k\sigma_2(n)s_2^{\,k-1}(1-s_2)}{H_2} \; \varepsilon(n-r)\, \varepsilon(r-n+k-1)$$

and

$$\text{prob.}\left[\begin{array}{l}\text{remaining queue is of length r, regeneration point is of}\\ \text{type } S_2(n)\text{, total of n+1 departures}\end{array}\right]$$

$$= \frac{1}{(n+1)} \times \frac{(n+1)\,\sigma_2(n)s_2^{\,n}}{H_2} \; \varepsilon(n-r) \quad .$$

Here

$$H_2 = \rho_2(-2)+\rho_2(-1)+\sum_{n=1}^{\infty}\rho_2(n)+\sum_{n=1}^{\infty}\sum_{k=1}^{n}k\sigma_2(n)s_2^{k-1}(1-s_2)$$

$$+\sum_{n=1}^{\infty}(n+1)\sigma_2(n)s_2^{n} + q_1\sum_{n=1}^{\infty}n\rho_1(n)$$

$$= \rho_2(-2)+\rho_2(-1)+\tilde{R}_2(1)+\frac{1}{(1-s_2)}\left\{\tilde{S}_2(1)-s_2\tilde{S}_2(s_2)\right\}+q_1\tilde{R}_1{}'(1),$$

where the prime denotes differentiation, (c.f. 'length-biased sampling' in Cox and Lewis (1966), p. 61). H_2 is the mean number of vehicles departing from lane 2 at a randomly chosen regeneration point. Thus we have after a few lines of algebra

$$H_2A_2(x) = \frac{q_1}{(1-x)}\left\{\tilde{R}_1(1)-\tilde{R}_1(x)\right\} + \frac{1}{(s_2-x)}\left\{s_2\tilde{S}_2(s_2)-x\tilde{S}_2(x)\right\}$$

$$+\rho_2(-2)+\rho_2(-1)+\tilde{R}_2(1),$$

and $B_2^{*}(s)$ is therefore determined.

It may be deduced from this that the mean delay to arrivals in lane 2 is

$$\frac{1}{2H_2\lambda_2}\left\{q_1\tilde{R}_1{}''(1) + \frac{2s_2}{(1-s_2)^2}\left[\tilde{S}_2(s_2)-\tilde{S}_2(1)\right] + \frac{2}{(1-s_2)}\tilde{S}_2{}'(1)\right\} .$$

and that the second moment about the origin of the delay distribution is, writing $\delta_2 = \lambda_2(1-s_2)$,

$$\frac{1}{H_2}\left\{\frac{q_1}{3\lambda_2^2}\tilde{R}_1{}'''(1) - \frac{2\lambda_2 s_2}{\delta_2^3}\left[\tilde{S}_2(s_2)-\tilde{S}_2(1)\right] - \frac{2s_2}{\delta_2^2}\tilde{S}_2{}'(1)\right.$$

$$\left.+\frac{1}{\lambda_2\delta_2}\tilde{S}_2{}''(1)\right\} .$$

3.3.5. Numerical Results.

A Ferranti Sirius computer was used to calculate the mean and variance of the distribution of delay to vehicles in lane 2, for a selection of parameter values, under the assumption that the distribution of gap-requirement times had a translated negative exponential form. i.e.

$$f_i(\alpha) = 0, \qquad \alpha \leq T_i,$$

$$= \nu_i \exp(-\nu_i(\alpha - T_i)), \qquad \alpha > T_i.$$

This choice of distribution was suggested by Herman and Weiss (1961), and is discussed in chapter 5.

For each of the sets of parameter values the value of k_i is chosen to be equal to the probability that a II at the intersection, faced with an opposing I sampled from the opposing stream, would accept the gap with which it is presented if it were to make use of the gap-requirement distribution for its lane. Thus k_i is fixed in terms of λ_i, ν_i, T_i, and, e.g. we set $k_1 = f_1^*(\lambda_2)$.

Results are quoted only for the case of lane-symmetric traffic, and are shown in table 3.1.

Comments on table 3.1.

An example of the effects of changes in the value of λ_i on the delay distribution is given in rows 2, 5, 19, 13, 20, 21. It appears that, as λ_i increases from the value 0.1, the mean delay

Table 3.1.

Results of calculations.

row	parameter values †					delay distn. in lane i	
	p_i	λ_i	ν_i	T_i	k_i	mean	variance
1	.2	.1	1.0	2.5	.708	.596	10.62
2	.5					.864	91.37
3	.8					.005	1214.85
4	.2	.3			.363	.504	2.50
5	.5					1.236	7.82
6	.8					1.372	22.93
7	.9					.864	70.72
8	.93					.610	142.33
9	.97					.21	¤
10	.10	.9			.056	.112	.24
11	.20					.240	.55
12	.35					.467	1.22
13	.50					.772	2.44
14	.65					1.239	5.23
15	.80					2.142	14.56
16	.95					4.505	84.25
17	.98					3.853	99.90
18	.99					2.494	¤
19	.5	.6			.139	1.014	4.15
20		1.5			.009	.494	1.04
21		2.5			.001	.300	.39
22		.3	.3		.236	1.697	12.56
23			.05		.067	2.287	22.39
24		.9	1.0	5	.006	.833	2.94

† Certain values are omitted for clarity - in such cases the relevant value is the nearest above in the same column.

¤ For this set of parameter values it was not possible to obtain a reliable value for the variance because of round-off error.

increases, passes through a maximum and then decreases. To explain this, we observe that for those sets of parameter values which correspond to mean delay decreasing with λ_i, the values of λ_i are such that the mean inter-arrival interval is smaller than T_i, so that it is unlikely that type II vehicles are released by either the gap-acceptance or follow-on mechanism. In this situation larger values of λ_i correspond merely to the more frequent arrivals of type II vehicles, with consequent opportunities for the complete dispersion of any opposing queues. It is therefore plausible that mean delay should be a decreasing function of λ_i. For comparatively small values of λ_i, by contrast (for the same values of p_i, ν_i, T_i), the gap-acceptance and follow-on probabilities play an appreciable part in dispersing queues, and an increase in the value of λ_i, while implying the more frequent arrival of type II vehicles, may increase congestion, on balance, by sufficiently decreasing the probability that a type II vehicle at the head of a queue accepts the gap to an opposing I. We would expect this effect to be reduced for smaller values of p_i, and the calculations of rows 1, 4, 11; 3, 6, 15, support this conjecture.

With regard to the effect of the parameter p_i on the flow of vehicle through the intersection, we intuitively expect the mean delay to tend to zero as p_i tends to zero or unity, since in these limiting cases there should be no interference between vehicles each of the same type. This behaviour is reflected in rows 1, 2, 3; 4 - 9; 10 - 18. In each case, so far as it has been calculated, the variance is an increasing function of p_i. Computational difficulties prevent the determination of the variance for values of p_i close to unity.

Inspection of rows 5, 22, 23 confirms that, as expected, decreasing ν_i results in an increase in the mean of the delay distribution. A similar result follows from an increase in T_i (rows 13, 24).

3.3.6. Proof of result used in section 3.3.3.

To show that $D_1(x)$ has only one zero in $(0, 1)$.

If not, since $D_1(0) > 0$, $D_1(1) < 0$ and $D_1(x)$ is continuous, there are at least three zeros in this interval, so that $D_1''(x)$ must vanish at an interior point.

Now

$$D_1''(x) = 2\lambda_2 + \lambda_1 p_1 \left(-2 \frac{d}{dx} f_2^*(\lambda-\lambda_2 x) + (1-x) \frac{d^2}{dx^2} f_2^*(\lambda-\lambda_2 x)\right),$$

and
$$\frac{d^2}{dx^2} f_2^*(\lambda-\lambda_2 x) = \lambda_2^2 \int_0^\infty e^{-(\lambda-\lambda_2 x)u} u^2 f_2(u)du,$$
$$> 0.$$

Also since $ue^{-ku} \leq k^{-1}e^{-1}$ for $k > 0$, $u > 0$, and $\int_0^\infty f_2(u)du = 1$,

$$\frac{d}{dx} f_2^*(\lambda-\lambda_2 x) = \lambda_2 \int_0^\infty e^{-(\lambda-\lambda_2 x)u} u f_2(u)du,$$
$$\leq \lambda_2(\lambda-\lambda_2 x)^{-1} e^{-1}.$$

It follows that, for $0 < x < 1$,

$$D_1''(x) > 2\lambda_2 - 2\lambda_1 p_1 \lambda_2 (\lambda - \lambda_2 x)^{-1} e^{-1} ,$$

$$= \frac{2}{e(\lambda-\lambda_2 x)} \left(\lambda_2(\lambda-\lambda_2 x)e - \lambda_1\lambda_2 p_1\right),$$

$$= \frac{2}{e(\lambda-\lambda_2 x)} \left(\lambda_1\lambda_2 q_1 + \lambda_1\lambda_2(e-1) + \lambda_2{}^2 e(1-x)\right),$$

$$> \; 0.$$

Hence $D_1(x)$ has only one zero in $0 < x < 1$.

3.4. Remarks on Model II.

The use of the gap-acceptance mechanism in this model should result in a model which is both more flexible, in that greater allowance may be made for variations among individual drivers, and more realistic, than model I. In model II, the driver of a stationary right-turning vehicle may take a non-zero interval of time both to assess the situation with which he is confronted, and to complete his turn if he so decides.

On the other hand it is apparent that there are several aspects of the behaviour of vehicles at an actual intersection which are not represented in the model. For instance, in rule (ii) of section 3.2., when a type II vehicle in lane 1 is about to execute its turn, the nearest approaching vehicle in lane 2 being also of type II, no account is taken of the proximity of the vehicle immediately following the type II vehicle in lane 2. Also in rule (iv) of the same section, one would prefer the number of vehicles simultaneously 'following-on' to be limited in some way by the size of the available gap in the opposite stream. although the rule as it stands might be credible in those actual situations where drivers of type I vehicles occasionally give way to drivers of opposing type II vehicles. The seriousness of omissions such as these will vary with actual intersections under study.

The fact that the behaviour of model II is ergodic for all reasonable sets of parameter values, which is an immediate consequence of its property that queues of any length may be dissipated instantaneously, is a more general difficulty. As in model I this characteristic would make the model unsuitable for the simulation of high-flow situations.

It is interesting that model II, which appears to be more realistic than model I, leads to an analysis which is little more difficult than that of the earlier model.

Chapter IV

Construction and Analysis of Model III.

4.1. Derivation of the model.

Perhaps the most important defect of models I and II is their inability to reflect the non-stationary behaviour of an actual intersection under certain traffic conditions, and model III is particularly concerned with an attempt to remedy this deficiency.

We recall that the stationary character of the previous models is a consequence of the immediate dissipation of any queues present by the certain arrival of a type II vehicle in the opposing lane. A method is required which would, at least in some cases, replace this instantaneous process by a more gradual dissipation of the queue over a non-zero interval of time. One way of arranging this is to require at least some of the vehicles in a queue, for preference the type II vehicles, to delay and to use gap-requirement times when faced with opposing II's. The stipulation that all type II vehicles should make their decision in this way would lead to unrealistic behaviour. An example would be the situation where the driver of a II in lane 1, having arrived at the intersection without waiting in a queue, selects a gap-requirement time and on the basis of this awaits the passage of an approaching II in lane 2 before re-assessing the situation.

A type II vehicle will, therefore, be taken to make use of a gap-requirement criterion only when it is faced with an opposing I, or when the type II vehicle arrives at the intersection from a queue, and it is faced with an opposing II. There will be no 'follow-on' behaviour of the kind occurring in model II, where it is possible for a turning vehicle to follow immediately the preceding member of a queue through the intersection. The selection of gap-requirement times by type II vehicles faced with an opposing I will also be influenced by whether these vehicles arrive at the intersection from free flow or from a queue.

4.2. Description of model III.

(i) The layout is as in models I, II.

(ii) Vehicles. As in the previous models, these will be regarded as geometrical points of types I, II. Vehicles are further classified as to whether their arrival at the intersection is or is not consequent upon delay in a queue - the former are of type S (stationary) and the latter of type M (moving).

(iii) Vehicle-arrivals occur as in models I, II.

(iv) Vehicle-interactions. The behaviour of vehicles at the intersection is described below. We denote by "g. r. t.: f" the result that the vehicle at the intersection uses a gap-requirement time (as defined in section 3.2.(ii)(c)), chosen from a population with density $f(t)$.

Type of vehicle at intersection in lane i	Type of nearest vehicle in opposing lane	Result
I, M or S	I or II	No delay to veh. at intersection.
II, M	I	g.r.t. : f_i
II, M	II	No delay to veh. at intersection.
II, S	I	g.r.t. : g_i
II, S	II	g.r.t. : h_i

This system of rules describes the behaviour of traffic in any situation which can arise. It is not possible for queues to exist simultaneously in both lanes, since any queue requires as its leading vehicle either a II (M or S) faced with an opposing I or a II (S) faced with an opposing II. The only delays occurring in the model are those caused by right-turning vehicles.

4.3. Steady-state behaviour of model III.

4.3.1. Selection of regeneration points : notation.

As in model II, it is possible for a queue in either lane in model III to be augmented at any instant. The instants at which queues decrease in size may be classified once again into those at which the initial departure (in the sense of section 2.3.1.) occurs in the lane in which the queue exists and those at which this departure occurs in the lane opposite the queue. It is of importance in assessing the future behaviour of the system that the type of the opposing vehicle should be known in the former case, since this vehicle may be of type I or II in model III, and this information would already have been used by the driver of the leading vehicle of the queue in making a decision to cross.

We shall consider the behaviour of the system under the assumption of stochastic stationarity, and we identify as regeneration points those instants at which a vehicle crosses the intersection in either direction, by noting at each such instant the following information :

(i) the lane from which the 'initial' vehicle departs,

(ii) if a queue exists, its lane and size,

(iii) if the queue is in the same lane as (i), the type of the nearest opposing vehicle,

(iv) if there is no queue, the type of the nearest opposing vehicle.

The notation used is as follows :

$R_i(n)$ refers to an instant at which a vehicle which may be of either type crosses in lane i, in the presence of a queue of n vehicles in the other lane, $(n = 1, 2, \ldots)$;

$\left.\begin{matrix} S_i(n) \\ T_i(n) \end{matrix}\right\}$ refers to an instant at which a vehicle necessarily of type II crosses in lane i, in the presence of an additional n queueing vehicles in lane, i, the nearest vehicle in the other lane being of type $\begin{cases} \text{I} \\ \text{II} \end{cases}$, $(n = 1, 2, \ldots)$;

$\left.\begin{matrix} R_i(-1) \\ R_i(-2) \end{matrix}\right\}$ refers to instants at which a vehicle crosses in lane i, there being no queue present, and the nearest opposing vehicle being of type $\begin{cases} \text{I} \\ \text{II} \end{cases}$.

The remarks of section 3.3.1 concerning the validity of regeneration points are relevant here. State probabilities for this embedded Markov Chain are defined as follows :

$$\left.\begin{array}{l}\rho_i(n)\\ \sigma_i(n)\\ \tau_i(n)\\ \rho_i(-1)\\ \rho_i(-2)\end{array}\right\} \quad \text{is the probability that a randomly selected regeneration point is of type} \quad \left\{\begin{array}{l}R_i(n)\\ S_i(n)\\ T_i(n)\\ R_i(-1)\\ R_i(-2)\end{array}\right. ,$$

$$n = 1, 2, \ldots$$

4.3.2. Transition Probabilities.

There are 100 types of transition such as $R_2(m) \rightarrow T_1(n)$, $R_1(-1) \rightarrow S_2(n)$. Considerations of symmetry reduce the number for study to 50.

As an example we evaluate the conditional probability that a point of type $S_1(m)$ is followed by a point of type $R_1(-1)$. The sequences of events which would give rise to this situation are as follows (writing 'at x' for 'in (x, x+dx)').

(a) Of the m vehicles queueing in lane 1 immediately following the departure of the initial II from this lane, the first m-1 are of type I and therefore cross immediately. The remaining vehicle is of type II, its driver measures the interval until the arrival of the nearest opposing vehicle (of necessity of type I) as $\tau+\alpha$, and selects a gap-requirement time of length α, using the density $g_1(\alpha)$. It thereupon crosses after an interval α, during which there is no arrival in lane 1, causing an $R_1(-1)$.

(b) All m+1 vehicles at the intersection in lane 1 cross immediately, i.e. they are all of type I except the first. The next arrival at the intersection is in lane 1 and is of type I.

(c) All m+1 vehicles at the intersection in lane 1 cross immediately. The next arrival at the intersection is in lane 1, is of type II, and is at time τ(measured from the initial $S_1(m)$). This II selects a gap-requirement time of length α, using the density $f_1(\alpha)$. The next arrival in lane 2, of type I, is at time $\tau+\alpha+\tau_1$. No vehicles arrive in lane 1 during the interval α for which the II is stationary, and it is the departure of the II which causes the $R_1(-1)$.

The transition probability associated with $S_1(m) \to R_1(-1)$ is therefore

$$\frac{\lambda_1}{\lambda} p_1^{m+1} + q_1 p_1^{m-1} \int_0^\infty g_1(\alpha) e^{-\lambda\alpha} d\alpha + p_1^m q_1 \frac{\lambda_1}{\lambda} \int_0^\infty f_1(\alpha) e^{-\lambda\alpha} d\alpha.$$

The complete set of transition probabilities is listed in Appendix I. These may be checked by the consistency relations of section 2.3.3. Of the 50 types of transition, it is found that 23 have zero probability. The transitions $R_1(m) \to R_2(-2)$, $R_2(m) \to R_1(-2)$, $R_1(m) \to R_2(-1)$, $R_2(m) \to R_1(-1)$ are possible only when $m = 1$.

<u>4.3.3. Determination of state probabilities.</u>

We define

$$\tilde{R}_i(x) = \sum_{n=1}^{\infty} x^n \rho_i(n), \quad \tilde{S}_i(x) = \sum_{n=1}^{\infty} x^n \sigma_i(n), \quad \tilde{T}_i(x) = \sum_{n=1}^{\infty} x^n \tau_i(n),$$

$$|x| \leq 1;$$

and
$$\left.\begin{aligned} f_i^*(s) &= \int_0^\infty e^{-st} f_i(t)\,dt, \\ g_i^*(s) &= \int_0^\infty e^{-st} g_i(t)\,dt, \\ h_i^*(s) &= \int_0^\infty e^{-st} h_i(t)\,dt, \end{aligned}\right\} \quad \Re(s) > 0.$$

Then after some rearrangement the stationarity equations reduce to equations (1) to (5) below, together with a corresponding set (1^*) to (5^*) with interchanged lane suffices.

If we write
$$\xi_2(x) = 1 - g_2^*(\lambda - \lambda_2 x),$$
$$\eta_2(x) = 1 - h_2^*(\lambda - \lambda_2 x),$$
$$\phi_2(x) = 1 - f_2^*(\lambda - \lambda_2 x),$$

the equations are

$$\tilde{R}_1(x)(\lambda-\lambda_2 x-\lambda_1 p_1 \xi_2(x)-\lambda_1 q_1 \eta_2(x)) - \tilde{S}_2(x)\,\frac{\lambda_1 q_2 x}{x-p_2}\,\xi_2(x)$$

$$- \tilde{T}_2(x)\,\frac{\lambda_1 q_2 x}{x-p_2}\,\eta_2(x) = A_1(x), \qquad (1)$$

$$-\tilde{R}_1(x)\frac{p_1}{x} g_2^*(\lambda-\lambda_2 x) + \tilde{S}_2(x)\left(1-\frac{q_2}{x-p_2}\, g_2^*(\lambda-\lambda_2 x)\right) = B_1(x), \qquad (2)$$

$$-\tilde{B}_1(x)\frac{q_1}{x}h_2^*(\lambda-\lambda_2 x) + \tilde{T}_2(x)\left(1-\frac{q_2}{x-p_2}\,h_2^*(\lambda-\lambda_2 x)\right) = C_1(x), \qquad 5)$$

$$p_2 g_1^*(\lambda)\rho_2(1) + \left[\frac{\lambda_1 p_1}{\lambda} + \frac{q_1}{p_1}\,g_1^*(\lambda) + \frac{q_1\lambda_1}{\lambda}\,f_1^*(\lambda)\right]\tilde{S}_1(p_1)$$

$$+ \frac{\lambda_1}{\lambda}\,p_2\tilde{S}_2(p_2) + \frac{\lambda_1 p_2}{\lambda}\,f_1^*(\lambda)\tilde{T}_2(p_2) + \rho_1(-1)\left[\frac{\lambda_1}{\lambda}\left(p_1+q_1 f_1^*(\lambda)\right)\right.$$

$$\left. + \frac{\lambda_1}{\lambda}\,p_2\rho_2(-1) + \frac{\lambda_1 p_2}{\lambda}\,f_1^*(\lambda)\rho_2(-2)\right] = 0, \qquad 4)$$

$$q_2 h_1^*(\lambda)\rho_2(1) + \frac{\lambda_1 q_2}{\lambda}\,\tilde{S}_2(p_2) + \left[\frac{\lambda_1}{\lambda} + \frac{q_1}{p_1}\,h_1^*(\lambda)\right]\tilde{T}_1(p_1)$$

$$+ \frac{\lambda_1 q_2}{\lambda}\,\tilde{T}_2(p_2) + \frac{\lambda_1 q_2}{\lambda}\rho_2(-1) - \frac{\lambda_2}{\lambda}\rho_1(-2) + \frac{\lambda_1 q_2}{\lambda}\rho_2(-2) = 0, \qquad 6)$$

where

$$A_1(x) = \lambda_1 x\left\{q_2\left(\frac{\lambda_2}{\lambda}\phi_2(x) - \frac{\xi_2(x)}{x-p_2}\right)\tilde{S}_2(p_2) + \frac{\lambda_2 p_1}{\lambda}\,\phi_2(x)\tilde{T}_1(p_1)\right.$$

$$\left. - \frac{q_2}{x-p_2}\,\eta_2(x)\tilde{T}_2(p_2) + \frac{\lambda_2}{\lambda}\phi_2(x)\left(q_2\rho_2(-1)+p_1\rho_1(-2)\right)\right\},$$

$$B_1(x) = -p_1\rho_1(1)g_2^*(\lambda) + \tilde{S}_2(p_2)\left[\frac{\lambda_2 q_2}{\lambda}\left(f_2^*(\lambda-\lambda_2 x)-f_2^*(\lambda)\right) - \frac{q_2}{x-p_2}\,g_2^*(\lambda-\lambda_2 x)\right.$$

$$\left. - \frac{q_2}{p_2}\,g_2^*(\lambda)\right] + \frac{p_1\lambda_2}{\lambda}\left(f_2^*(\lambda-\lambda_2 x)-f_2^*(\lambda)\right)\tilde{T}_1(p_1)$$

$$+ \frac{\lambda_2}{\lambda}\left(f_2^*(\lambda-\lambda_2 x)-f_2^*(\lambda)\right)\left(\rho_2(-1)q_2+\rho_1(-2)p_1\right),$$

$$C_1(x) = -q_1\rho_1(1)h_2^*(\lambda) - \left[\frac{q_2}{x-p_2}\,h_2^*(\lambda-\lambda_2 x)+\frac{q_2}{p_2}\,h_2^*(\lambda)\right]\tilde{T}_2(p_2).$$

Equations (1), (2), (3) may be solved simultaneously to give $\tilde{R}_1(x)$, $\tilde{S}_2(x)$, $\tilde{T}_2(x)$ in terms of the constants $\tilde{S}_1(p_1)$, $\tilde{S}_2(p_2)$, $\tilde{T}_1(p_1)$, $\tilde{T}_2(p_2)$, $\rho_1(1)$, $\rho_2(1)$, $\rho_1(-1)$, $\rho_2(-1)$, $\rho_1(-2)$, $\rho_2(-2)$, which we denote by $u_1, \ldots u_{10}$ respectively. In similar manner $\tilde{R}_2(x)$, $\tilde{S}_1(x)$, $\tilde{T}_1(x)$ are determined. We proceed to evaluate these constants.

If we write

$$J_1(x) = \lambda-\lambda_2 x-\lambda_1 p_1 \xi_2(x)-\lambda_1 q_1 \eta_2(x),$$

$$D_1(x) = x-p_2-q_2 g_2^{*}(\lambda-\lambda_2 x),$$

$$E_1(x) = x-p_2-q_2 h_2^{*}(\lambda-\lambda_2 x),$$

equations (1) to (3) may be written

$$\begin{bmatrix} (x-p_2)J_1(x) & -\lambda_1 q_2 x \xi_2(x) & -\lambda_1 q_2 x \eta_2(x) \\ \frac{-p_1}{x}(x-p_2) g_2^{*}(\lambda-\lambda_2 x) & D_1(x) & 0 \\ \frac{-q_1}{x}(x-p_2) h_2^{*}(\lambda-\lambda_2 x) & 0 & E_1(x) \end{bmatrix} \begin{bmatrix} \tilde{R}_1(x) \\ \tilde{S}_2(x) \\ \tilde{T}_2(x) \end{bmatrix} = (x-p_2) \begin{bmatrix} A_1(x) \\ B_1(x) \\ C_1(x) \end{bmatrix} \tag{6}$$

and if we denote by $\left[\Gamma_{ij}(x)\right]$ the matrix of minors of the left hand matrix,

$$\tilde{R}_1(x) = \frac{A_1(x)\Gamma_{11}(x) - B_1(x)\Gamma_{21}(x) + C_1(x)\Gamma_{31}(x)}{\Delta_1(x)} \tag{7}$$

where $\Delta_1(x) = (1-x)\Big\{(\lambda_1+\lambda_2 q_2) q_2 g_2^{*}(\lambda-\lambda_2 x) h_2^{*}(\lambda-\lambda_2 x)$

$-(x-p_2)\Big[(q_1\lambda_1+q_2\lambda_2) h_2^{*}(\lambda-\lambda_2 x)+(p_1\lambda_1+q_2\lambda_2) g_2^{*}(\lambda-\lambda_2 x)\Big]$

$+\lambda_2(x-p_2)^2\Big\}$.

In order that $\lim_{x\to 1} \tilde{R}_1(x) < \infty$, the numerator of the expression (7) must vanish at $x = 1$. This leads to

$$A_1(1) + \lambda_1(B_1(1) + C_1(1)) = 0,$$

or equivalently

$$q_2\left(\frac{\lambda_2}{\lambda}(1-f_2^*(\lambda)) - \frac{1}{q_2} - \frac{g_2^*(\lambda)}{p_2}\right)u_2 + p_1\frac{\lambda_2}{\lambda}(1-f_2^*(\lambda))u_3$$

$$-q_2\left(\frac{1}{q_2} + \frac{h_2(\lambda)}{p_2}\right)u_4 - (p_1g_2^*(\lambda) + q_1h_2^*(\lambda))u_5$$

$$+\frac{\lambda_2}{\lambda}(1-f_2^*(\lambda))(q_2u_8 + p_1u_9) = 0, \tag{8}$$

with a corresponding equation (8^*). The same equation (8) arises from similar consideration of $\tilde{S}_2(1)$, $\tilde{T}_2(1)$. Equations (4), (4^*), (5), (5^*), (8), (8^*) are linearly dependent.

The conditions $\tilde{R}_1(0) = \tilde{S}_2(0) = \tilde{T}_2(0) = 0$ are all equivalent to

$$\lambda_1q_2\left(\frac{\lambda_2}{\lambda}\phi_2(0) + \frac{\xi_2(0)}{p_2}\right)u_2 + \frac{\lambda_1\lambda_2}{\lambda}p_1\phi_2(0)u_3 + \frac{\lambda_1q_2}{p_2}\eta_2(0)u_4$$

$$-J_1(0)u_5 + \frac{\lambda_1\lambda_2}{\lambda}q_2\phi_2(0)u_8 + \frac{\lambda_1\lambda_2}{\lambda}p_1\phi_2(0)u_9 = 0, \tag{9}$$

with a corresponding equation.

By definition, the functions $\tilde{R}_1(x)$, $\tilde{S}_2(x)$, $\tilde{T}_2(x)$ are bounded for $0 \leq x \leq 1$. Since $\Delta_1(x)$ may have one or more zeros in this interval, further conditions may have to be met by the numerators of these functions.

Let us suppose for the present that $\Delta_1(x)$ (and $\Delta_2(x)$) vanish exactly once for $0 < x < 1$ at $x = a_1, (a_2)$. We require the numerators of each of $\tilde{R}_1(a_1)$, $\tilde{S}_2(a_1)$, $\tilde{T}_2(a_1)$, expressed in the form of (7), to vanish. This results in a further three linear conditions on $u_1, \ldots, u_{10}$. It has been found numerically that the first of these conditions yields no new information about $u_1, \ldots u_{10}$. The second does yield a further linearly independent relationship, , and the third is linearly dependent on the second and the previous conditions on $u_1, \ldots u_{10}$. Although an algebraic demonstration of these remarks is desirable, the complexity of the resulting equations has been such as to resist all attempts to construct one.

Thus we have, writing $\theta_i(x) = f_i^{*}(\lambda-\lambda_i x) - f_i^{*}(\lambda)$,

$$
\begin{aligned}
& u_2 \cdot \lambda_1 q_1 q_2 h_2^{*}(\lambda-\lambda_2 a_1) \Bigg[D_1(a_1) \left\{ \frac{\lambda_2}{\lambda} \phi_2(a_1) - \frac{\zeta_2(a_1)}{a_1 - p_2} \right\} + \zeta_2(a_1) \left\{ \frac{\lambda_2 q_2}{\lambda} \theta_2(a_1) \right. \\
& \qquad \left. - \frac{q_2}{a_1 - p_2} g_2^{*}(\lambda-\lambda_2 a_1) - \frac{q_2}{p_2} g_2^{*}(\lambda) \right\} \Bigg] + \Big[p_1 u_3 + q_2 u_8 + p_1 u_9 \Big] \frac{\lambda_1 \lambda_2}{\lambda} q_1 h_2^{*}(\lambda-\lambda_2 a_1) \\
& \qquad \Big\{ D_1(a_1) \phi_2(a_1) + q_2 \zeta_2(a_1) \theta_2(a_1) \Big\} - u_4 \Bigg[\frac{q_1}{a_1 - p_2} h_2^{*}(\lambda-\lambda_2 a_1) D_1(a_1) \lambda_1 q_2 \zeta_2(a_1) \\
& \qquad + \Big\{ J_1(a_1) D_1(a_1) - \lambda_1 p_1 q_2 g_2^{*}(\lambda-\lambda_2 a_1) \zeta_2(a_1) \Big\} \left\{ \frac{q_2 h_2^{*}(\lambda-\lambda_2 a_1)}{a_1 - p_2} + \frac{q_2 h_2^{*}(\lambda)}{p_2} \right\} \Bigg] \\
& - q_1 u_5 \Bigg[\lambda_1 q_2 \zeta_2(a_1) h_2^{*}(\lambda-\lambda_2 a_1) p_1 g_2^{*}(\lambda) + h_2^{*}(\lambda) \Big\{ J_1(a_1) D_1(a_1) - \lambda_1 p_1 q_2 g_2^{*}(\lambda-\lambda_2 a_1) \\
& \qquad \zeta_2(a_1) \Big\} \Bigg] = 0, \qquad (10)
\end{aligned}
$$

with a corresponding equation (10^{*}), obtained from $\tilde{T}_1(a_2)$.

The normalising condition is

$$\tilde{R}_1(1)+\tilde{S}_2(1)+\tilde{T}_2(1)+\tilde{R}_2(1)+\tilde{S}_1(1)+\tilde{T}_1(1)+u_7+u_8+u_9+u_{10} = 1. \qquad (11)$$

By means of L'Hopital's rule we find that

$$\tilde{R}_1(1)+\tilde{S}_2(1)+\tilde{T}_2(1) = \frac{1}{\Delta_1'(1)} \sum_{i=1}^{3} (A_1'\Gamma_{1i}-B_1'\Gamma_{2i}+C_1'\Gamma_{3i}+A_1\Gamma_{1i}'$$

$$-B_1\Gamma_{2i}'+C_1\Gamma_{3i}'),$$

where we write $A_1' = A_1'(1)$, $\Gamma_{2i} = \Gamma_{2i}(1)$ etc., for brevity, i.e.

$$\begin{aligned}
\tilde{R}_1(1)+\tilde{S}_2(1)+\tilde{T}_2(1) = \frac{1}{\Delta_1'(1)} \Bigg\{ & u_2 \Bigg[\left\{ \frac{\lambda_1\lambda_2}{\lambda} q_2\phi_2'(1)+\frac{\lambda_1\xi_2(1)}{q_2} - \lambda_1\xi_2'(1) \right\} \Sigma\Gamma_{1i}(-1)^{i+1} \\
& + \left\{ \frac{\lambda_2^2}{\lambda} q_2 f_2^{*\prime}(\lambda_1) - \frac{g_2^{*}(\lambda_1)}{q_2} - \lambda_2 g_2^{*\prime}(\lambda_1) \right\} \Sigma\Gamma_{2i}(-1)^{i+1} \\
& + \left\{ \frac{\lambda_1\lambda_2}{\lambda} q_2\phi_2(1) - \lambda_1\xi_2(1) \right\} \Sigma(\Gamma_{1i}+\Gamma_{1i}')(-1)^{i+1} \\
& - \left\{ \frac{\lambda_2 q_2}{\lambda} \theta_2(1) - g_2^{*}(\lambda_1) - \frac{q_2}{p_2} g_2^{*}(\lambda) \right\} \Sigma\Gamma_{2i}'(-1)^{i+1} \Bigg] \\
& + \left[p_1u_3+q_2u_8+p_1u_9 \right] \Bigg[\frac{\lambda_1\lambda_2}{\lambda} p_1\phi_2'(1)\Sigma\Gamma_{1i}(-1)^{i+1} + \frac{\lambda_2^2}{\lambda} p_1 f_2^{*\prime}(\lambda_1)\Sigma\Gamma_{2i}(-1)^{i+1} \\
& + \frac{\lambda_1\lambda_2}{\lambda} p_1\phi_2(1)\Sigma(\Gamma_{1i}+\Gamma_{1i}')(-1)^{i+1} - \frac{p_1\lambda_2}{\lambda} \theta_2(1)\Sigma\Gamma_{2i}'(-1)^{i+1} \Bigg] \\
& + u_4 \Bigg[\lambda_1 \left\{ \frac{\eta_2(1)}{q_2} - \eta_2'(1) \right\} \Sigma\Gamma_{1i}(-1)^{i+1} + \left\{ \frac{h_2^{*}(\lambda_1)}{q_2} + \lambda_2 h_2^{*\prime}(\lambda_1) \right\} \Sigma\Gamma_{3i}(-1)^{i+1} \\
& - \lambda_1\eta_2(1)\Sigma(\Gamma_{1i}+\Gamma_{1i}')(-1)^{i+1} - \left\{ h_2^{*}(\lambda_1) + \frac{q_2}{p_2} h_2^{*}(\lambda) \right\} \Sigma\Gamma_{3i}'(-1)^{i+1} \Bigg] \\
& + u_5 \left[p_1 g_2^{*}(\lambda)\Sigma\Gamma_{2i}'(-1)^{i+1} - q_1 h_2^{*}(\lambda)\Sigma\Gamma_{3i}'(-1)^{i+1} \right] \Bigg\} .
\end{aligned}$$

We may now solve equations (4), (4^{x}), (5), (5^{x}), (9), (9^{x}), (10), (10^{x}), (8), (11) for $\{u_{i_j}\}$, in the case when $\Delta_1(x), \Delta_2(x)$ each have a single zero in (0,1). From the resulting values, by means of equations (1), (2), (3), (1^{x}), (2^{x}), (3^{x}), the functions $\tilde{R}_i(x)$, $\tilde{S}_i(x)$, $\tilde{T}_i(x)$ may be constructed, and it is found that the solutions in this case satisfy the requirements pertaining to a probability distribution with regard to positivity and boundedness by unity.

For some parameter values, however, it is found that $\Delta_i(x)$ has two zeros in (0,1), for at least one value of i. This may lead to the addition of further equations to (10), (10^{x}), and it is found that by taking different pairs of these equations, together with the other eight linearly independent equations, differing solutions for $u_1, \ldots u_{10}$ are obtained which do not satisfy the basic requirements on these constants, such as positivity. We conclude that it is not possible to find a bounded solution to the stationarity equations in such a case. A similar conclusion would be expected in the case where more than two zeros of either of $\Delta_1(x), \Delta_2(x)$ lie in (0,1). There is also the possibility that either of $\Delta_i(x)$, $i = 1,2$, does not have a zero in (0,1). In this case it appears impossible to derive a unique stationary distribution.

From this discussion it seems that a necessary and sufficient condition for ergodic behaviour of the system is that each of $\Delta_1(x)$, $\Delta_2(x)$ should have a single zero in (0,1).

4.3.4. The distribution of delay.

As in section 3.3.4 we consider the distribution of delay to vehicles in lane 2, which may be deduced from the distribution of lengths of queues in lane 2 left by vehicles departing from that lane. Vehicles depart only at regeneration points, and vehicles in lane 2 only at points of the types $R_2(n)$, $S_2(n)$, $T_2(n)$, $R_2(-2)$, $R_2(-1)$. Points at which vehicles depart in lane 2, and at which a queue remains in this lane, are of the types $S_2(n)$, $T_2(n)$. Thus if

v_r = prob $\left\{\begin{matrix}\text{a vehicle departing in lane 2 leaves a queue of length} \\ r \text{ in lane 2}\end{matrix}\right\}$,

$$\Lambda_2(x) = \sum_{r=0}^{\infty} v_r x^r, \quad |x| \leq 1,$$

and $\tilde{V}_i(x) = \tilde{S}_i(x) + \tilde{T}_i(x)$,

then

$$v_r = \frac{1}{H_2}\left\{q_2 \sum_{n=1}^{\infty} \sum_{k=1}^{n} (\sigma_2(n)+\tau_2(n))p_2^{k-1}\varepsilon(n-r)\varepsilon(r-n+k-1) + \sum_{n=1}^{\infty}(\sigma_2(n)+\tau_2(n))p_2^{n}\varepsilon(n-r) + (\rho_2(-2)+\rho_2(-1)+\tilde{R}_2(1))\varepsilon(-r)\right\},$$

so that

$$\Lambda_2(x) = \frac{1}{H_2}\left\{\frac{p_2\tilde{V}_2(p_2)-x\tilde{V}_2(x)}{p_2-x} + \rho_2(-2)+\rho_2(-1)+\tilde{R}_2(1)\right\},$$

where $H_2 = \frac{1}{q_2}(\tilde{V}_2(1)-\tilde{V}_2(p_2)) + \tilde{V}_2(p_2)+\tilde{R}_2(1)+\rho_2(-2)+\rho_2(-1)$.

The Laplace transform of the delay distribution in lane 2 is $B_2^*(s) = \Lambda_2\left(\frac{\lambda_2 - s}{\lambda_2}\right)$ from which the first and second moments of the delay distribution may be derived as

$$\mu_1' = \frac{1}{H_2\lambda_2}\left\{\frac{p_2}{q_2^2}\left[\tilde{V}_2(p_2)-\tilde{V}_2(1)\right] + \frac{1}{q_2}\tilde{V}_2'(1)\right\},$$

and $$\mu_2' = \frac{1}{\lambda_2^2 q_2 H_2}\left\{\frac{2p_2}{q_2^2}\left[\tilde{V}_2(1)-\tilde{V}_2(p_2)\right] - \frac{2p_2}{q_2}\tilde{V}_2'(1)+\tilde{V}_2''(1)\right\}.$$

4.3.5. The probability that a vehicle in lane 1 is not delayed.

This probability is of some interest in matching the model to data, chiefly because of its immediate interpretation in the practical situation. It is considered further in section 5.4.2.

In order that a vehicle in lane 1 should not be delayed it must cross the intersection at a regeneration point of type $R_1(-1)$, $R_1(-2)$ or $R_1(n)$. If it crosses at a point of type $R_1(n)$, then it has not been delayed. However it is possible that a vehicle of type II crossing at an $R_1(-1)$ or $R_1(-2)$ could be either the last vehicle in a queue in lane 1 or could (at an $R_1(-1)$) accept a gap but take a finite time to do so. (All type II vehicles faced with an opposing I are delayed).

The situations in which a vehicle experiences no delay are as follows. A vehicle which arrives in lane 1 at a situation other than these is necessarily delayed.

Regeneration point at which vehicle crosses.		Previous regeneration point.
Type of point.	Type of vehicle.	
$R_1(n)$	any	any
$R_1(-1)$	I	any
$R_1(-2)$	I	any
$R_1(-2)$	II	$R_1(-2)$ $R_2(-2)$ $T_1(k)$ with k queueing type I's $T_2(k)$ with k queueing type I's

These considerations lead to the following expression for the probability that a vehicle in lane 1 suffers no delay at the intersection.

$$\text{Prob.}\left\{\text{no delay in lane } 1\right\}$$

$$= \frac{1}{H_1}\left\{\tilde{R}_1(1)+\rho_1(-2)\left(p_1+\frac{\lambda_1}{\lambda}\,q_1\right)+p_1\rho_1(-1)+\frac{\lambda_1}{\lambda}q_1q_2\rho_2(-2)\right.$$

$$\left.+\frac{\lambda_1}{\lambda}q_1\left(\tilde{T}_1(p_1)+q_2T_2(p_2)\right)\right\},$$

where H_1 is defined in section 4.3.4.

4.3.6. Numerical results.

An ICT 1905 computer was programmed in Algol to calculate (a) any zeros of the denominators of $\tilde{R}_1(x)$; (b) the set of subsidiary constants $u_1, \ldots, u_{10}$; (c) the mean and variance of the delay distribution and (d) the probability of no delay for each lane.

These calculations were performed for the particular case in which each of the gap requirement time distributions possessed a density function of translated negative exponential form.

In the evaluation of (b), the quantities Γ_{ij}, Γ_{ij}' required for equation (11) of section 4.3.3 were found by writing the left hand matrix of equation (6) of that section in the form $\left[\Lambda_{ij}\right]$ where $\Lambda_{ij} = \Lambda_{ij}(x)$. $\Lambda_{ij}(1)$, $\Lambda_{ij}'(1)$, $\Gamma_{ij}(1)$, $\Gamma_{ij}'(1)$ may then be determined.

The calculation of (c) is simplified by writing

$$\Delta_1(x).\tilde{V}_2(x) = \Big[A_1(x)(\Gamma_{13}(x)-\Gamma_{12}(x))-B_1(x)(\Gamma_{23}(x)-\Gamma_{22}(x))+C_1(x)(\Gamma_{33}(x) - \Gamma_{32}(x))\Big]$$

in the form

$$\begin{aligned}\tilde{V}_2(x) = \Big[\,&\tilde{S}_2(p_2)\Big\{\lambda_1\left[q_2h_2^*(\lambda-\lambda_2x)g_2^*(\lambda-\lambda_2x)-(x-p_2)(p_1g_2^*(\lambda-\lambda_2x)+q_1h_2^*(\lambda-\lambda_2x))\right]\\ &+\lambda_2(x-p_2-q_2h_2^*(\lambda-\lambda_2x))(x-p_2-q_2g_2^*(\lambda-\lambda_2x)-\frac{\lambda_2q_2}{\lambda}(x-p_2)(1-f_2^*(\lambda-\lambda_2x))\Big\}\\ &-\frac{\lambda_2^2}{\lambda}(x-p_2)(x-p_2-q_2h_2^*(\lambda-\lambda_2x))(1-f_2^*(\lambda-\lambda_2x))(p_1\tilde{T}_1(p_1)+q_2\rho_2(-1)+\\ &\qquad p_1\rho_1(-2))\\ &+\tilde{T}_2(p_2)\Big\{-\lambda_1\left[(x-p_2)(q_1h_2^*(\lambda-\lambda_2x)+p_1g_2^*(\lambda-\lambda_2x))-q_2h_2^*(\lambda-\lambda_2x)g_2^*(\lambda-\lambda_2x)\right]\\ &+(x-p_2)\lambda_2(x-p_2-q_2h_2^*(\lambda-\lambda_2x))-q_2\lambda_2h_2^*(\lambda-\lambda_2x)(x-p_2-q_2g_2^*(\lambda-\lambda_2x))\\ &-\frac{\lambda_2q_2^2}{p_2}h_2^*(\lambda)(x-p_2)(h_2^*(\lambda-\lambda_2x)-g_2^*(\lambda-\lambda_2x))\Big\}\\ &+\rho_1(1)q_1q_2\lambda_2h_2^*(\lambda)(x-p_2)(g_2^*(\lambda-\lambda_2x)-h_2^*(\lambda-\lambda_2x))\Big](1-x)\Big/\Delta_1(x),\end{aligned}$$

and cancelling (1-x) in numerator and denominator.

The results are listed in table 4.1.

Table 4.1.

Results of calculations for model III.

The distributions of gap-acceptance times are assumed to have translated exponential form. All parameters are assumed to be lane-symmetric.

row	Parameter values †								Distn. of delay in lane i		
	p_i	λ_i	a_i	b_i	c_i	d_i	e_i	k_i	Mean	Variance	Pr. $\{$no delay$\}$
1	.5	.2	.1	2	1	5	.1	10	.058	1.17	.684
2		.3							.071	.65	.681
3		.5							.112	.44	.674
4		.9							.338	1.15	.647
5		1.5							3.283	42.15	.551
6		3.0							*	*	*
7		5.0							*	*	*
8		.2					1		.291	5.54	.673
9							3		.686	12.10	.661
10							6		1.094	18.37	.656
11							10		1.434	23.89	.654
12							1	20	.279	5.33	.673
13	.2	.5					.1	10	.083	.14	.782
14	.8								.057	1.26	.792
15	.1	.3	3.5	.7	3.5	.7	3.5	.7	*	*	*
16	.3								*	*	*
17	.5								*	*	*
18	.7								49.843	7604.73	.507
19	.9								3.317	60.40	.679
20	.95								1.633	25.96	.802
21	.5	.1	3.5	.6	3.5	.6	3.5	.6	2.641	24.17	.633
22				.3		.3		.3	4.724	80.82	.610
23				.1		.1		.1	91.225	26021.44	.515
24				.05		.05		.05	*	*	*
25	.5	.3	3.5	.3	3.5	.3	3.5	.3	*	*	*
26				.1		.1		.1	*	*	*
27				.05		.05		.05	*	*	*

† These parameters are defined in chapter 4. Certain values are omitted for clarity - in such cases the relevant value is the nearest above in the same column.

x For these sets of parameter values there is not a unique zero of $\Delta_1(x)$ in (0,1).

Comments on table 4.1.

(i) It appears that mean delay is critically related to λ_1, and is an increasing function of λ_1. For large λ_1, a unique zero of $\Delta_1(x)/(1-x)$ in $0 < x < 1$ does not exist, and it may be inferred that the behaviour of the system is not ergodic. (Rows (1) - (7)).

(ii) Rows (13), (3), (14); and (15) - (20) illustrate possible effects of variations of p_1 on the delay distribution. For the situations of rows (13), (3), (14), the gap-acceptance distributions are such that the probability of a type II vehicle accepting a gap with which it is presented is quite high. Thus we expect the mean delay to be small for p_1 near zero. On the other hand, in the situations of rows (15) - (20), the arrival rates and gap-requirement distributions are such that it is not probable that a turning vehicle accepts a random gap with which it is presented. Since all queueing type II vehicles make use of a gap-requirement distribution, we might expect congestion to increase in this case as p_i tends to zero. This supposition is confirmed by the results of rows (15) - (20): it appears that the increase in congestion is so great that the behaviour of the model is not ergodic for the smaller values of p_i.

(iii) The effect of changes in the gap-acceptance parameters b_i, d_i, k_i is shown in rows (21) - (24). Decreasing these parameters has the effect of decreasing the probability that a type II vehicle accepts a gap with which it is presented. As expected, the mean delay is found to increase substantially as b_i, d_i, k_i are decreased, to the extent that for small values of these parameters, ergodic behaviour of the system is not possible.

(iv) The behaviour of the prob. $\{$no delay$\}$ for variations in the parameters is consistent with that of the mean delay.

(v) It would be interesting to investigate the circumstances in which the change from ergodic to non-ergodic behaviour of the system occurs. Although an analytic investigation of this does not seem feasible, it is suggested by the data in table 4.1 that broadly speaking the system is not ergodic when it is overloaded. It seems that, since ergodicity is dependent on the parameters through the functions $\Delta_i(x)$, it is not affected by the form of the functions $f_i(x)$. This situation might have been expected from intuitive considerations.

4.4. Remarks on Model III.

Model III should deal with some of the criticisms of model II made in section 3.4. In particular the 'follow-on' behaviour of vehicles has been made sufficiently realistic to introduce non-ergodic behaviour for certain parameter values.

It is possible to select various aspects of the model which may not at times be in agreement with observation. For instance, the release of a queue consisting entirely of type I vehicles does not occur instantaneously in practice and some delay may be caused by these vehicles to vehicles in the other lane. Some drivers may base their decisions not merely on the nearest opposing vehicle but also on the vehicle following it. In some situations courteous type I drivers may give way to opposing type II's. In general, however, the occurrence of such events is comparatively infrequent, and model III may well give a reasonable overall picture of behaviour at the intersection.

Other aspects of model III which could represent a more serious lack of conformity with the actual situation are that the postulated gap acceptance behaviour may be unrealistic, as discussed in section 3.1, that the arrival process may not be Poisson, and that the classification of vehicles into types I and II may not be valid. Some of these points are further discussed in section 5.4 in the light of the data.

4.5. 'Impatient' queueing type I vehicles : model III(a).

Observation on actual intersections suggests that model III could be made more generally applicable by allowing for the tendency of some type I vehicles, which in model III would join the end of a queue, to pass alongside the queue and make a crossing without delay. This would lead effectively to consideration of the intersection of a four lane major road with a minor road.

In an attempt to fit model III to an intersection at which the proportion of these 'impatient' type I's is small, observed queues could be slightly modified by the inclusion of hypothetical vehicles corresponding to these type I's, provided they did not interfere with turning traffic.

Another approach to this problem would be to modify model III to simulate this type of vehicle behaviour, again under the assumption that the 'impatient' type I vehicles do not interfere with turning traffic. (If this assumption is not applicable there is a possibility of two simultaneous queues.) The modification to model III depends on its characteristic that, as soon as a type I vehicle has joined a queue, it ceases to exist, at least so far as any further influence it has on the behaviour of the system is concerned. Its presence is of course noted by the previous analysis of model III.

Two ways in which model III could be modified are the following:

(i) The analysis of the model is as above up to section 4.3.4. At this point we suppose that a type I vehicle may leave a queue immediately upon its arrival with probability s_i for lane i. The analysis of section 4.3.4 is modified to find the distribution of delay to vehicles which do not leave a queue upon arrival : from this the distribution of delay to all vehicles in a given lane is constructed.

(ii) Alternatively, an earlier modification to the model would arrange the input of type I vehicles to queues at a different rate to that obtaining in the free flow situation. This would lead to a modification of the transition probabilities.

Chapter 5.

Statistical Analysis of Observations.

5.1. Introduction.

In any situation in which an attempt is made to analyse mathematically a practical problem, the success of the analysis depends on the extent to which its results agree with actual observations. In the present case we have three models for an intersection, each of which is a considerable idealisation of the actual situation. It is considered unlikely that close agreement between observed and predicted values for models I and II would be obtained, but that model III may perhaps be flexible and realistic enough to justify the effort involved in assessing its results against observation. It seems highly desirable to assess the progress of the work so far, before launching on the construction of further models.

This chapter briefly describes the observations which were made, the statistical processes by which the characteristics of model III were compared with them, and any conclusions regarding the value of the model which seemed to be justified. There is a brief reference to model I in section 5.4.3, for purposes of comparison.

5.2. The observations.

In Appendix II an account is given of the method used to obtain the observations, and of certain difficulties which were encountered in their interpretation. Here it is enough to note that the observations, which refer to only one intersection, consist of a series of pictures of the situation at the intersection, taken at a constant interval of approximately one second. From the filmed record it is possible to observe queue-sizes at any instant and to measure delays experienced by vehicles. These measurements are subject to the various kinds of experimental error discussed in Appendix II.

5.3. Estimation problems.

5.3.1. Estimation of p_i, λ_i.

The estimation of these parameters is comparatively simple. Each refers to a characteristic of the model which has a well defined counterpart in the actual situation. Under the assumptions of a Poisson input of vehicles of randomly assorted type in each lane, appropriate estimates consist, respectively, of the sample proportion of type I vehicles, and the observed rate of vehicle arrivals, in each lane.

These estimates could be modified, by the method to be described in section 5.3.3, in order to secure a better fit of the model to the data. Unless the modified estimates were found not to differ significantly from the original estimates, their adoption would be inconsistent with the natural interpretation of the parameters.

Consequently, subject to the adherence of the observed input processes to the assumptions of the model (section 5.4.1), the intuitive estimates were accepted as final. Their values are as follows.

	p_i	λ_i (vehicles/second)
Lane 1	.63855	.19280
Lane 2	.96988	.20441 .

5.3.2. Estimation of other parameters in model III.

For the description of model III, details of the functions h_i, g_i, f_i are necessary. In the model these functions describe both the driver's estimate of his gap-requirement time and the time required by him to complete his crossing. As it is very likely that these intervals will differ in the actual situation, the practical interpretation of the functions h_i, etc., is not obvious. It is apparent also that direct information concerning the estimates formed by drivers in these circumstances will not be easily available, and it is therefore likely that the interpretation given to the gap-requirement distributions will be influenced by the types of relevant observations possible in practice.

The following are quantities which are relevant to gap-acceptance behaviour and upon which measurements may be made.

(a) The sizes of gaps which are accepted or rejected by type II vehicles.

(b) The intervals actually used by turning vehicles to complete their manoeuvres.

From the film it is found that measurements of the intervals (b) usually have values of less than three secs. In view of the practical difficulties in making these measurements, and of the frame separation of about one second, such data are not considered to contain useful information.

Measurements of (a) are available only for the situations which in the model would require the use of g_1 or f_1. They are listed in table 5.1. We observe that gap sizes of interest are in the range 4-7 secs. for g_1, 3-5 secs. for f_1, and that the frame separation of approximately one second is not small enough to permit any but the crudest inferences about g_1 and f_1 to be made, even ignoring the considerable subjective judgement involved in constructing these data.

Table 5.1.

Gaps accepted and rejected by type II vehicles.

(i) Stationary type II in lane 1 with approaching type I in lane 2(g_1).

Gap size (secs.)	2	3	4	5	6	7	8	9
Proportion of gaps accepted.	0	0	0	.6	.75	1	1	1
Number of gaps observed.	24	8	3	5	4	1	1	1

(ii) Moving type II in lane 1 with approaching type I in lane 2 (f_1).

Gap size.	1	2	3	4	5	6	7
Proportion of gaps accepted.	0	0	0	.33	1	1	1
Number of gaps observed.	3	9	4	3	1	5	2

The initial approximations to g_1, f_1, are therefore based on table 5.1., and on the work of others in this field, who have suggested several possible forms for these functions. (See e.g. Cohen, Dearnaley and Hansel (1955) who proposed the function erf (log t-A)/B for the probability that a driver accepts a gap of size t.) Herman and Weiss (1961) suggest that, from data obtained in a controlled experiment, the form of the functions h_1, g_1, f_1 may be approximated by a translated exponential distribution, and in view of the computational advantages of this distribution, it was decided to adopt it here. Specifically, it is assumed that

$$
\begin{aligned}
h_1(t) &= 0, \; t \leq a_1; \\
&= b_1 \exp(-b_1(t-a_1)), \; t > a_1; \\
g_1(t) &= 0, \; t \leq c_1; \\
&= d_1 \exp(-d_1(t-c_1)), \; t > c_1; \\
f_1(t) &= 0, \; t \leq e_1; \\
&= k_1 \exp(-k_1(t-e_1)), \; t > e_1.
\end{aligned}
$$

The problem is now reduced to one of estimating the 12 parameters $a_1, b_1, c_1, d_1, e_1, k_1$.By comparing a plot of the logarithm of the proportion of gaps rejected against size of gap for the data of table 5.1 with the logarithm of the survivor function of the appropriate gap-requirement distribution we obtain the following estimates.

Parameter :	c_1	d_1	e_1	k_1
Estimate :	3.5	.45	3.0	.50

Although it would be possible to apply more sophisticated methods to the data of table 5.1, such as the maximum likelihood method, it is considered that, because of the unreliability of the data, little would be gained.

5.3.3. Improvements on the crude estimates of the gap-acceptance parameters.

Since, in the model, the parameters considered in the previous section, together with p_1, λ_1, determine the entire probabilistic character of model III, it is likely that inferences about these parameters could be based on aspects of the behaviour of an actual intersection which are less immediately concerned with gap-acceptance. One could, for instance, consider the construction of maximum likelihood estimates of the gap-acceptance parameters, but it is found that the entire history of the system is too complex for the construction of a likelihood function to be practicable.

In the lack of a likelihood function for the complete set of observations, we might consider basing estimates on a likelihood function for a limited set of observations, which might nevertheless contain sufficient information about the parameters to be useful. Here we make use of observations on the embedded Markov Chain, and observation of the system is confined to the sequence of regeneration points defined in §4.3.1. The resulting data consist only of the lengths and types of queues at these regeneration points and are available with reasonable accuracy. A further advantage of this procedure is that observations on variables in the model, whose interpretation in the actual situation is uncertain, are not required.

The likelihood function for such observations may be constructed as follows. Transition intervals are first classified according to the types of the regeneration points defining the interval. Thus we have, e.g.,

<u>Transition</u>	<u>classification number (θ)</u> ($\theta = 1, \ldots 54$).
$R_1(-2) \to R_1(n)$	1
$R_2(-2) \to R_2(n)$	2
.	.
.	.
.	.
.	.
$R_1(m) \to R_1(n)$	9
.	.
.	.
.	.

We denote by $n_\theta(i, j)$ the number of observed transitions of type θ, where i and j may be used to denote relevant queue lengths: for some transitions one or both are not required. $p_\theta(i, j)$ denotes the corresponding transition probability, and the logarithm of the likelihood (conditional on the types of the regeneration points beginning the sequences of observations - see Appendix II.) is taken as

$$\sum_{\theta=1}^{54} \underbrace{\sum_{i} \sum_{j}}_{\text{where relevant}} n_{\theta}(i,j) \log p_{\theta}(i,j).$$

$p_{\theta}(i,j)$ must be completely evaluated for the particular functions h_i, g_i, f_i assumed: for the assumptions of section 5.3.2, we have, for example,

$$p_9(m,n) = \lambda_1 \lambda_2^{n-m} \left[\frac{p_1 e^{d_2 c_2}}{(d_2+\lambda)^{n-m+1}} \Gamma((d_2+\lambda)c_2, n-m+1) + \frac{q_1 e^{a_2 b_2}}{(b_2+\lambda)^{n-m+1}} \Gamma((b_2+\lambda)a_2, n-m+1) \right] \varepsilon(n-m),$$

where $\Gamma(x,n) = \int_x^{\infty} e^{-t} t^{n-1} dt/(n-1)!$.

A procedure was developed to compute the value of the log-likelihood for a given set of parameter values and a given vector of observed transitions. This was incorporated into a standard program for maximising a function by the steepest-descent method (E.J.Wasscher, 1963).

The results of the calculations carried out are noted in table 5.2, and detailed comments are made in the accompanying notes. The following considerations are relevant here.

(i) It was considered desirable to make some attempt to investigate the multiplicity of maxima of the likelihood function, and to run the program from a variety of starting points in the parameter space. Since very long runs were required to reach a maximum with respect to the whole set of gap-acceptance parameters, the maximisation was usually carried out with respect to a smaller number of variable parameters.

(ii) It seemed a possibility that the likelihood function would contain comparatively little information about some of the parameters under investigation. It was to be expected, therefore, that its use might lead to spurious estimates of these parameters. Certain methods were used to try to prevent this.

More specifically, it was observed that there were relatively few type II vehicles which arrived in lane 2 during the period of observation; in fact there were 5 from a total of 176 vehicles. It would be reasonable to expect the information available about a parameter such as e_2, specifying the probability that a II in lane 2 accepts a gap, to be slight, and that this might be reflected in the likelihood function by the absence of factors which are functions of e_2. However if we consider the total of 255 transitions extracted from film, it is found that 42 are of types whose transition probabilities are functions of e_2. A study of the 42 transitions reveals that 37 are of the type $R_2(-1) \Rightarrow R_2(-1)$ with associated probability $\lambda_2\left\{p_2+(q_2k_2e^{-\lambda e_2}/(\lambda+k_2))\right\}/\lambda$, a decreasing function of e_2, and we accordingly expect this function to have a considerable influence on the value of the maximum likelihood estimate of e_2.

It may seem surprising that e_2 should occur in all 42 times in the interpretation of data for which it is known that the total number of II's observed in lane 2 is 5. The explanation is that the transition probabilities must allow for the fact that the type of the vehicle in lane 2, which defines the end of the transition interval, is not specified by the description of the interval used, and may be type II. In fact for the data such a vehicle very rarely is of type XI. Thus it would appear very likely that estimates obtained from the maximisation of this likelihood function are fundamentally affected by the inadequacy of the way in which the system is described at some regeneration points. This disadvantage of the estimation process is of course magnified by the particular set of data which are available, and in which so large a proportion of transitions are among regeneration points where no queue is present.

(iii) In view of these doubts concerning the validity of the estimation process based on this particular likelihood function, it was considered prudent to make some assessment of the realism of each of the sets of estimated parameter values. This took two forms.

(a) The mean and variance of the distribution of delay in each lane of model III, together with the probability of no delay, were computed using the programs of §4.4. The resulting values were compared with the corresponding sample statistics. (At this stage a visual comparison was used - a more detailed comparison is discussed in §5.4.2.)

(b) Although the data of table 5.1 do not provide sufficient information to estimate the parameters with acceptable precision, they do suggest approximate bounds for functions which might be considered as gap-acceptance distribution functions. From inspection of the data we might reasonably require such a function to satisfy the following conditions.

(1) Prob.$\{$a driver accepts a gap of 2 seconds$\} \sim 0$

(2) Prob.$\{$a driver accepts a gap of 9 seconds$\} \sim 1$

(3) $\int_0^x h_1(t)dt < \int_0^x f_1(t)dt, \quad 2 < x < 9.$

Table 5.2.

Maximum Likelihood estimates of parameters

Restrictions on maximisation (see notes)	Parameter values at maximum		Moments of delay distributions of fitted Model III					
	a_1 b_1 c_1 d_1 e_1 k_1	a_2 b_2 c_2 d_2 e_2 k_2	Lane 1			Lane 2		
			mean	var	pr {no delay}	mean	var	pr {no delay
(a)	3.2 4.5 1.0* 0.1* 1.0* 0.1*	4.0 5.0 4.0 5.0 3.0 10.0						
(b)	3.5 .178 3.5 .179 3.0 .054	3.5 .45 3.5 .45 3.0 .50						
(c)	1.152 0.185 .001* .140 .001* .071	.062 .252 .001* .987 1.341 .001*	3.09	20.9	.38	.41	15.8	.94
(d)	.038 .191 .001* .155 .001* .058	.038 .191 .001* .155 .001* .058	9.16	139.4	.27	.25	10.5	.96
(e)	.001* .156 .001* .156 .001* .058	.001* .156 .001* .156 .001* .058	9.10	137.7	.27	.27	4.01	.96
†			5.05	44.9	.42	.432		.94

*Value at boundary of parameter range.

**Moments calculated only for those estimates of parameters which seemed reasonably lane-symmetrical

† Corresponding sample statistics.

Notes on table 5.2.

Maximisation (a). The parameters varied for these first runs were a_1, b_1, c_1, d_1, e_1, k_1, as it was expected that there would be little information relevant to the others in the likelihood function for the data. Several sets of starting values were chosen, a typical set being, respectively, 4, 5, 4, 5, 3, 10, lane symmetry being assumed. The results for these values are shown. It is necessary to impose arbitrary positive lower limits on the range of variation of the parameters (to avoid log 0), and it will be observed that several of the estimates listed have values at boundary points of the range of parameter variation. As it was expected that any realistic estimates of these parameters would possess a degree of lane-symmetry, determination of moments of the delay distributions for these values was not considered worthwhile.

Maximisation (b). The lower bounds for the parameters used in (a) were decreased. In view of consideration (ii) above, values of the parameters a_1, c_1, e_1 (the translation parameters), were fixed, and the maximisation was carried out with respect to b_1, d_1, k_1, from starting values given by the graphical method of section 5.3.2, it being assumed initially that gap-acceptance parameter values are independent of lane, and that $a_1 = c_1$, $b_1 = d_1$.

Maximisation (c). In this all twelve parameters a_i, ... k_i were varied. Starting values were chosen as for (b).

<u>Maximisation (d).</u> Consideration (ii) above suggests it is unlikely that useful estimates of the parameters for gap-acceptance by lane 2 vehicles would be obtained from the likelihood function. The physical characteristics of the intersection were such that one would expect the gap-acceptance parameters to be nearly lane-symmetric. The program was arranged to maximise with respect to the variables a, b, c, d, e, k, where $a = a_1 = a_2$ etc.

<u>Maximisation (e).</u> The scarcity of information in the likelihood function about a_i, b_i, suggests that as a first approximation we consider the behaviour of a II, stationary at the intersection, to be independent of the type of the approaching vehicle. Thus we maximise as in (d), subject to the further restriction that $a_i = c_i$, $b_i = d_i$, $i = 1, 2$.

If a provisional assessment of the results quoted in table 5.2 is made along the lines indicated in (iii) (b) above, it is found that none of the estimates of rows (a) to (e) may be regarded as acceptable. The question of the acceptability of these estimates according to other criteria is discussed in section 5.4.

In view of the known deficiencies of the likelihood function, it was decided to make ad hoc modifications to the maximum likelihood estimates, in order to bring them into conformity with requirements (iii) (b). Several sets of parameter values were produced in this way, and the corresponding moments of the delay distribution were calculated : the results are shown in table 5.3.

Table 5.3.

Modified M.L. estimates of parameters

(Values of p_i, λ_i as before)

Set	Estimate of parameter**						Lane 1			Lane 2		
	a_i	b_i	c_i	d_i	e_i	k_i	mean	var	pr{no delay}	mean	var	pr{no delay}
1	1.0	.20	0.0	.20	.5	.05	6.52	70.9	.30	.24	19.9	.96
2	2.5	.30	2.5	.30	1.5	.40	13.10	278.9	.22	.187	2.4	.97
3	1.5	.30	1.5	.30	1.5	.40	5.55	64.89	.38	.15	1.44	.96
4	0	.30	0	.30	0	.40	1.80	11.5	.50	.07	.48	.97
5	0	.20	0	.20	0	.30	3.60	40.2	.41	.12	1.16	.96
*	*						5.05	44.9	.42	.432		.94

* Corresponding sample statistics.

** Lane symmetry assumed.

5.4. The assessment of goodness-of-fit.

In any attempt to assess the adequacy of model III as a model for the behaviour of actual traffic, certain basic questions must be considered. The model is based on a number of assumptions concerning the behaviour of individual vehicles - how realistic are these assumptions and do they reflect sufficiently those aspects of this behaviour which are most relevant to a study of congestion? From these basic assumptions certain deductions are made about the behaviour of traffic. To what extent are these deductions consistent with the behaviour of actual traffic?

Perfect agreement between model and reality is not expected. One of the purposes of building models, such as those of this thesis, is to provide a simple explanation of complex situations which is nevertheless sufficiently realistic to provide approximate agreement with observations on the actual system. We are now trying to assess whether model III gives a sufficiently realistic picture of an actual intersection to make further development of the model unnecessary.

The plausibility of some aspects of model III has been discussed in §4.5 and we now reconsider the predictions of the model in the light of the data.

5.4.1. The distribution of arrivals.

With the purpose of comparing observed arrivals of vehicles in

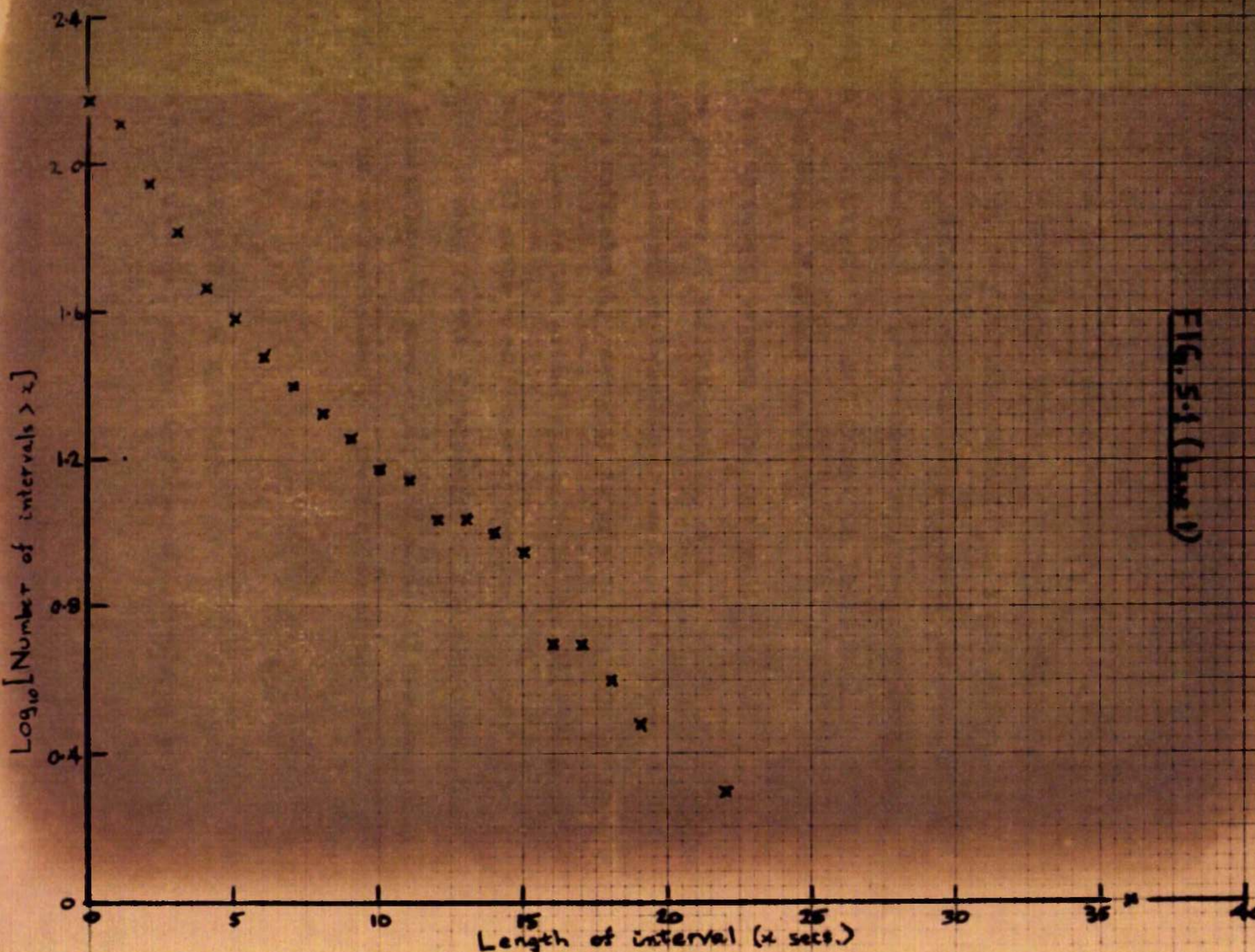
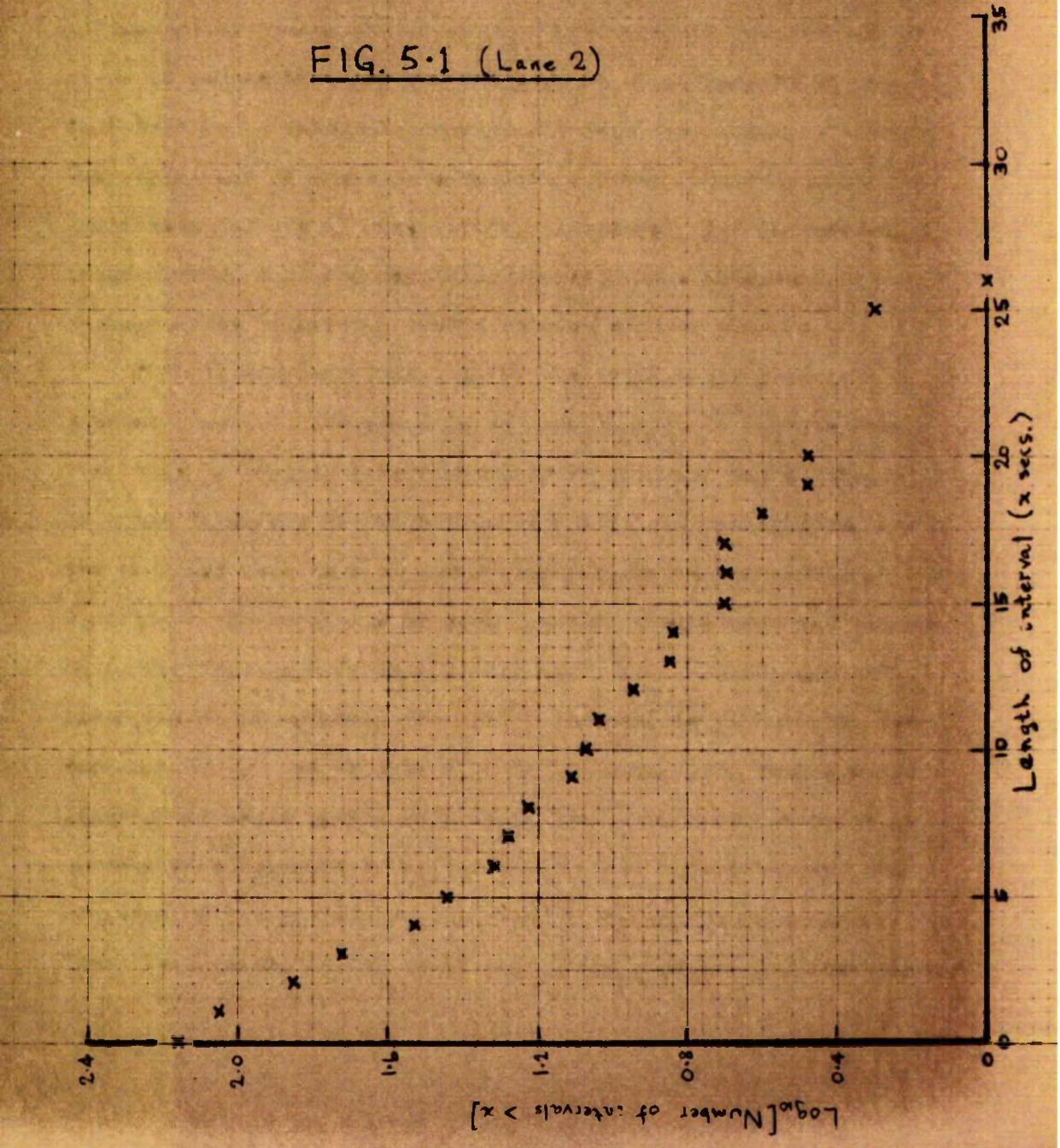

Log_{10}[Number of intervals > x]
Length of interval (x secs.)
2·4
2·0
1·6
1·2
0·8
0·4
0
5
10
15
20
25
30
35
40

FIG. 5·4 (Lane 1)

FIG. 5·1 (Lane 2)



Length of interval (x secs.)
0
5
10
15
20
25
30
35
Log10[Number of intervals > x]
2.4
2.0
1.6
1.2
0.8
0.4
0

each lane with the postulated Poisson process, frequency distributions for the inter-arrival intervals (shown in Appendix III) were constructed. In this context, an arrival is defined as the passage of a vehicle past a fixed point chosen as far distant from the intersection as possible, so as to reduce to a minimum the effects of congestion on the arrival distribution. Intervals between arrivals are listed in Appendix III. The separation of these intervals into groups, corresponding to continuous periods of observation, is ignored, and the sequence of inter-arrival instants resulting for each lane is examined in order to assess its conformity with a Poisson arrival process.

The log survivor function for the inter-arrival intervals is plotted (fig.5.1) for arrivals in each lane. It will be observed that this is roughly linear as would be expected for a Poisson input. In order to assess the significance of the deviations from linearity the modified mean test of Lewis (1965) based on the statistic S' is applied to the intervals of each lane. In each case the value of S' is far from significant. (In lane 1, S' = 83.30, and the distribution of S' under the null hypothesis is normal with mean 82.5, variance 13.75: in lane 2, S' = 73.70 (mean 72.5, variance 12)). Although as Lewis points out the S' test should detect serially correlated alternatives, it was thought useful to determine the correlation coefficient of lag one for the intervals of lanes 1 and 2. These were respectively .00645 and .00436, and are far from significant if assessed by the criterion of Cox and Lewis (1966), p.165.

It appears that on the data available the arrival process in each lane conforms with a Poisson process. A more detailed assessment of the adequacy of the Poisson model for arrivals seems inappropriate here partly because of the weakness of the techniques available for assessing goodness-of-fit of predictions resulting from the model, and the inaccuracies of the data, but also because it is to be hoped that the applicability of model III would not be too fundamentally affected by slight deviations of the input processes from complete randomness.

5.4.2. Realism of predictions from model III.

We consider next the comparison of derived aspects of model III with the data, in particular the comparison with sample values of certain predicted parameters of the delay distributions.

Means and variances of the delay distributions, both theoretical and empirical, are shown in tables 5.2, 5.3. The remarks of Appendix II concerning difficulties in measuring delay times are relevant here, and we note that an average bias of, say, 2 seconds in lane 1, arising from misinterpretation of 'delay', is not improbable. We would expect that biases for lanes 1 and 2 would differ because of greater difficulty in assessing the positions of vehicles in lane 2.

Differences in the values of the sample and predicted means might therefore be accounted for as due to a combination of four factors: lack of realism in the model, biased interpretation of 'delay', imprecision of parameter estimates in the fitted model, and sampling fluctuations of mean delay about its expected value. If the bias in observations of delay for a given lane is roughly constant, the variance of the delay distribution should be almost unaffected by it, and might provide a better statistic than the sample mean upon which to base a comparison of data with the model.

It is known that the estimate of the variance of the delay distribution calculated by regarding the observed sequence of delays as a random sample will be influenced to an unknown extent by the autocorrelations of the observations. Jowett (1955) has proposed a method of obtaining approximately unbiased estimates of the population variance and of the sampling variance of the mean from a section of a stationary time series, and it seems that this technique might be useful here. We denote the observed sequence of delays, in order, by x_i, $i = 1, \ldots n$. The s^{th} serial variation parameter, δ_s, may be defined by

$$\delta_s = E\left\{\tfrac{1}{2}(x_n - x_{n+s})^2\right\} = \sigma^2(1 - \rho_s),$$

where σ^2 is the variance of the marginal delay distribution and ρ_s the serial correlation coefficient of lag s of the joint delay distribution in lane 1. Jowett suggests that it should often be possible to find an s_o such that, for some suitable small ε, $|\delta_s - \sigma^2| < \varepsilon$, in which case we may write $E\left\{\frac{1}{2}(x_i - x_j)^2\right\} = \sigma^2 + \theta\varepsilon$ where $|\theta| \leqslant 1$, provided that $|i - j| \geqslant s_o$. His estimate of σ^2 is the average of all semi-squared differences which are provided by the sample, i.e.

$$\sigma_{s_o}^{2*} = \left\{\sum_{i,j:|i-j| \geqslant s_o} \tfrac{1}{2}(x_i - x_j)^2\right\} \bigg/ \left\{\sum_{i,j:|i-j| \geqslant s_o} 1\right\}$$

$$= \left\{n \sum_{i=1}^{n} (x_i - \bar{x})^2 - \sum_{s=1}^{s_o-1} D_s\right\} \bigg/ (n - s_o)(n - s_o + 1),$$

where $D_s = \sum_{i=1}^{n-s} (x_i - x_{i+s})^2$.

$\sigma_{s_o}^{2*}$ has a bias of at most ε.

He suggests that, in practice, as s_o is unknown, a value for s_o may be selected by inspection of sample estimates of δ_s given by $d_s = \frac{1}{2}D/(n-s)$.

For the data under discussion a delicate assessment of the most appropriate value for s_o is not necessary, since, for purposes of comparison with the theoretical model, $\sigma_{s_o}^{2*}$ is effectively constant, save for sampling fluctuations, for $s_o \geqslant 6$.

Table 5.4.

Analysis of Semi-squared differences of Delays.

s_o	d_{s_o}	$\sigma_{s_o}^{2*}$
0	0.0	43.70
1	15.79	43.97
2	27.65	44.32
3	33.79	44.53
4	36.99	44.67
5	37.24	44.76
6	40.70	44.86
7	45.12	44.91
8	47.03	44.91
9	45.53	44.88
10	42.45	44.88

From Table 5.4, the estimate of σ^2 for lane 1 has a value of about 44.9.

Observations of delay experienced by vehicles in lane 2 are too few to permit a sample estimate of the variance of the delays in that lane to be made.

A comparison of the variance for lane 1 derived in this way with the predicted variances of Tables 5.2 and 5.3 reveals that the entries of rows (c), 1, 3, 5 appear consistent with the sample value.

Jowett shows in addition that an approximately unbiased estimate of the sampling variance of the mean of a series of correlated observations is given by $\sigma_{s_0}^{2*} - v$, where $v = \frac{1}{2n^2} \sum_{i,j=1}^{n} (x_i - x_j)^2$. Thus for lane 1 the estimated standard deviation of the mean observed delay is $\sqrt{44.9 - 43.7} = 1.10$. This assists in the comparison of the sample mean with the predicted means: those of rows (1), (3), (5), (Table 5.3) are within about 1.5 estimated standard deviations of the sample mean, whereas the mean of row (c) (Table 5.2) is distant almost two standard deviations.

Inspection of the sequence of observed delays to vehicles in lane 1 suggests the likelihood of considerable correlation between delays of successive vehicles. It would be interesting, therefore, to match the serial correlation coefficients of the model against those of the data, possibly by means of a suitable kind of harmonic analysis. Unfortunately when the attempt is made to evaluate the serial correlation coefficient of lag one in the model, difficulties are encountered which are of such a substantial nature that it would appear that the evaluation of serial correlation coefficients of lags greater than one, by similar methods at least, is not feasible.

We are thus reduced to consideration of the single lag 1 correlation coefficient, and the derivation of this is discussed in some detail in section 5.4.4. This coefficient was not evaluated numerically for the following reasons.

(i) Quantities such as $\tilde{S}_2'(1)$, $\tilde{S}_2''(1)$ are required. In the evaluation of the mean and variance of the delay distribution only $\tilde{S}_2'(1) + \tilde{T}_2'(1)$ and $\tilde{S}_2''(1) + \tilde{T}_2''(1)$ were required and the algebra and computation were arranged in order to take advantage of any simplifications which might ensue from considering the sum $\tilde{V}_2(x) = \tilde{S}_2(x) + \tilde{T}_2(x)$. The evaluation of $\tilde{S}_2(x)$, though certainly possible, would necessitate repetition of much of this work.

(ii) The comparison of an observed with a predicted correlation coefficient would, it appears, be merely on an inspection basis in the lack of a suitable test of significance.

Despite these difficulties, it is the case that, as with the sampling variance of the delay distribution, the sample serial correlation coefficient of lag one should not be unduly influenced by the existence of an approximately constant bias in the observation on delay. It would appear to be possible to expand Jowett's work slightly to obtain an approximately unbiased estimate of the lag 1 serial correlation coefficient by a suitable averaging of the quantity

$$\sum_{\substack{i,j: \\ |i-j| \geq s_0}} \tfrac{1}{2}(x_i - x_j)(x_{i+1} - x_{j+1})$$

since $\gamma_s = E\left\{\tfrac{1}{2}(x_i - x_{i+s})(x_{i+1} - x_{i+s+1})\right\} = \sigma^2(\rho_1 - \frac{\rho_{s-1} + \rho_{s+1}}{2})$.

s_0 would be chosen so that $\left|\gamma_s - \sigma^2\rho_1\right| < \varepsilon$ for $s \geq s_0$.

The previous paragraphs have been concerned with attempts to assess how well the model predicts fairly general aspects of the distribution of delays in a given lane. However the question of proportionally how many vehicles pass through the intersection without delay is of considerable practical interest, and is another aspect of the data which might be compared with the model. Since only the first two moments of the marginal delay distributions have been considered, information derived from this comparison may be largely independent of previous results. The theoretical probabilities of no delay under the assumption of model III are determined in section 4.3.5, and calculated values listed in Tables 5.2 and 5.3.

In order to construct a sample value for the probability of no delay in lane 1, we associate with the n^{th} successive vehicle crossing the intersection after the commencement of the period of observation a dummy random variable z_i defined by

$$z_i = 1 \text{ if this vehicle is not delayed,}$$
$$= 0 \text{ if it is delayed.}$$

The probability of no delay is estimated by $\bar{z}$ for a large sample, and once again we may invoke the work of Jowett to provide an estimated standard error for this mean, since the z_i form a correlated sequence of observations. Table 5.5 gives for the estimated standard error of this probability a value of about .06 (with $s_o \doteq 10$).

From tables 5.2 and 5.3, we may therefore regard the entries for 'probability of no delay' in each of rows (c), (3) and (5), which are each within one estimated standard deviation of the sample value, as indicating consistency of the model with the data.

Table 5.5.

Analysis of Semi-squared differences of $\{z_i\}$.

s_o	d_{s_o}	$\sigma^{2*}_{s_o}$
0	0.0000	0.2410
1	0.1080	0.2424
2	0.1615	0.2441
3	0.2063	0.2451
4	0.2201	0.2456
5	0.2405	0.2460
6	0.2293	0.2460
7	0.2468	0.2462
8	0.2645	0.2462
9	0.2792	0.2460
10	0.2810	0.2456
11	0.2632	0.2451
12	0.2450	0.2449

In the preceding paragraphs we have attempted to compare certain significant aspects of the observations with the model. In spite of difficulties in finding suitable estimates of the parameters specifying the model, and although the interpretation given to certain aspects of the model in the actual situation is probably not the most suitable, it has been possible to demonstrate a considerable degree of conformity of predicted with observed behaviour. There is every likelihood that a more direct method of estimation of parameters in the model, possibly based on a minimisation of a measure of distance of observed delay frequency distributions from their theoretical counterparts, might lead to an even closer fit of the model to the data.

5.4.3. A comparison of model III with model I.

A likelihood function similar to that constructed for model III was programmed for model I, and a sequence of regeneration points of the type defined in model I was extracted from the film. In order to make this sequence of points compatible with the assumptions of model I a slight adjustment of the data was required. The likelihood function was maximised with respect to the parameters r_i, $i = 1, 2$. The maximised log-likelihood for model I was found to be -312.7. The comparable figure for model III is -174.8.

If these values were regarded as calculated for the entire record of the system which is available, then (Bartlett 1967) they might be considered to indicate the superiority of model III to model I. However the likelihoods refer to observed sets of regeneration points, which are not identical for both models, and thus the likelihoods are not directly comparable.

We may obviate this difficulty by observing that the set of regeneration points for model I (set I) is included in the set for model III (set 3), and thus if we consider set 1 under the assumptions of model III the value of the likelihood obtained in this way will be greater than or equal to that calculated for set 3 under model III.

It would appear that model III represents a substantial improvement on model I.

5.4.4. The correlation coefficient of successive delays to vehicles in lane 2 of model III.

Let A and B refer to consecutive vehicles in lane 2, selected at random. The vehicles are delayed for intervals of length t, t', and leave queues of lengths q, q', respectively. The joint distribution function of t, t' is $B_2(t, t')$; $g_2(q, q')$ is the probability of a particular pair of values q, q'.

Suppose first that q is non-zero, i.e. B arrives before A departs, say an interval τ after the arrival of A. Then in the interval $t-\tau$ immediately following the arrival of A, $q-1$ vehicles arrive in lane 2 and in the succeeding interval $t'-t+\tau$ a further $q'-q+1$ vehicles arrive in lane 2.

Thus

$$g(q,q') = \int_0^\infty \int_0^\infty \int_{\tau=0}^{t} \lambda_2 e^{-\lambda_2 \tau} \frac{\left[\lambda_2(t-\tau)\right]^{q-1}}{(q-1)!} e^{-\lambda_2(t-\tau)} \frac{\left[\lambda_2(t'-t+\tau)\right]^{q'-q+1}}{(q'-q+1)!}$$

$$e^{-\lambda_2(t'-t+\tau)} d\tau \, dB_2(t,t'), \quad q' \geq q-1, q \neq 0.$$

Similarly

$$g(0,q') = \int_0^\infty \int_0^\infty e^{-\lambda_2 t} e^{-\lambda_2 t'} \frac{(\lambda_2 t')^{q'}}{q'!} dB_2(t,t') \quad .$$

Let us define $\quad G_2(x,x') = \sum_{q=1}^{\infty} \sum_{q'=q-1}^{\infty} g_2(q,q') x^q x'^{q'} + \sum_{q'=0}^{\infty} g_2(0,q') x'^{q'},$

and $\quad B_2^*(s,s') = \int_0^\infty \int_0^\infty e^{-(st+s't')} dB_2(t,t').$

It follows that

$$G_2(x,x') = \lambda_2 x \int_0^\infty \int_0^\infty \int_0^t e^{-\lambda_2(t'+\tau)} e^{\lambda_2 x x'(t-\tau)} e^{\lambda_2 x'(t'-t+\tau)} d\tau \, dB_2(t,t')$$

$$+ \int_0^\infty \int_0^\infty e^{-\lambda_2(t'+t-t'x')} dB_2(t,t')$$

$$= \frac{x}{(1+xx'-x')} \left[B_2^*(\lambda_2 x'(1-x), \lambda_2(1-x')) - B_2^*(\lambda_2, \lambda_2(1-x')) \right] + B_2^*(\lambda_2, \lambda_2(1-x'))$$

This expression reduces immediately to the expression of §4.3.4 for the generating function of the marginal distributions of q, q'.

If we write $y = \lambda_2 x'(1-x)$, $y' = \lambda_2(1-x')$, and define further

$$K_2(y,y') = (\lambda_2-y)(\lambda_2-y')G_2\left(1 - \frac{y}{\lambda_2-y'},\ 1 - \frac{y'}{\lambda_2}\right),$$

we have, when $y = y' = 0$,

$$\frac{\partial^2}{\partial y \partial y'} K_2 = \lambda_2^2 \frac{\partial^2}{\partial y \partial y'} B_2^*(y,y') - \lambda_2 \frac{\partial}{\partial y} B_2^*(y,y') - \lambda_2 \frac{\partial}{\partial y'} B_2^*(y,y')$$

$$+ B_2^*(\lambda_2, y').$$

From this relation $E\{tt'\}$ may be recovered in terms of $\frac{\partial^2}{\partial y \partial y'} K_2(y,y')$ and known quantities. (Since e.g.

$$\left.\frac{\partial}{\partial y'} B_2^*(y,y')\right|_{y=y'=0} = E(t')\Big).$$

We proceed to describe how $G_2(x,x')$ may be determined. Vehicle A departs either at or immediately following a regeneration point which is called the 'preceding regeneration point'. Only the component of G_2 which arises from a preceding regeneration point of type $S_2(n)$ is evaluated: similar methods may be used for the other components of G_2. It is convenient to denote by k the <u>total</u> number of vehicles which leave at the $S_2(n)$ preceding the departure of A. H_2 is the normalising constant defined in §4.3.4.

Case (i): A and B both have as preceding regeneration point the same $S_2(n)$, with a total of k departures.

$\Pr\left\{\text{case (i) and } q, q' \text{ with } q' = q-1\right\}$

$$= \frac{k\sigma_2(n)}{H_2} p_2^{k-1} q_2 . \frac{1}{k}\, \varepsilon\,(n-q)\ \varepsilon\,(q-n+k-2),\quad 2 \leq k \leq n;$$

$$= \frac{\sigma_2(n)}{H_2} p_2^{n}\ \varepsilon\,(n-q)\ \varepsilon\,(q-1), \qquad k = n+1.$$

Case (ii): A is the last vehicle to leave at its preceding regeneration point, and the next vehicle to depart in either lane is B.

B may now have as preceding regeneration point, with non-zero probability, either $S_2(m)$ (for certain m), or $R_2(-1)$. We have, e.g.,

$$\Pr\left\{\text{case (ii), } 2^{nd} \text{ regeneration point is of type } S_2(m),\ q = n+1-k,\ q' = m\right\}$$

$$= \frac{\sigma_2(n)}{H_2}\, p_2^{k-1} q_2\, \frac{g(m-n+k)}{(m-n+k)!} \int_0^\infty e^{-\lambda a} (\lambda_2 a)^{m-n+k} g_2(a)\,da, \quad 1 \leqslant k \leqslant n;$$

$$= \frac{\sigma_2(n)}{H_2}\, p_2^{n} q_2\, \frac{\lambda_2^{m+1}}{\lambda m!} \int_0^\infty f_2(a) e^{-\lambda a} a^m\,da, \qquad k = n+1.$$

For the remaining cases((iii) - (v)) it is convenient to make use of probability generating functions from the outset. The following p.g.f's are useful and refer to the distribution of the number of arrivals in lane 2 in an interval defined by the initial and final conditions detailed. (I.C. and F.C.). In each situation I.C. refers to a regeneration point, with the exceptions of (δ), (ε), (ζ).

<u>Situation (α).</u>

I.C. : $S_2(n)$ at which at least one vehicle does not depart. Gap to I in 1 rejected.

F.C. : Departure of this I in 1.

$$\Pr\left\{r \text{ arrivals in lane 2 during interval of length } t\right\}$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t} (1 - G_2(t)) \frac{(\lambda_2 t)^r}{r!}\, e^{-\lambda_2 t}\,dt,$$

$$\therefore\ p_\alpha(z) = \lambda_1 (1 - g_2^*(\lambda - \lambda_2 z))/(\lambda - \lambda_2 z).$$

Situation (β).

I.C. : Queue exists in lane 2. Next vehicle in 1 (of unspecified type) has gap rejected.

F.C. : Departure of this vehicle in 1.

$$p_\beta(z) = \lambda_1(1 - p_1 g_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z) - q_1 h_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z))/(\lambda-\lambda_2 z).$$

Situation (γ).

I.C. : Queue in 2. Gap to next vehicle in lane 1 is accepted.

F.C. : Departure of leading vehicle in lane 2.

$$p_\gamma(z) = p_1 g_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z) + q_1 h_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z).$$

Situation (δ).

I.C. : No queues: arrival in lane 2. Nearest vehicle in 1 is a I, and gap to this vehicle is rejected.

F.C. : Departure of this I in 1.

(Omit possibility that first lane 2 arrival is a I: already covered in Case (ii) above.)

$$p_\delta(z) = \lambda_1\lambda_2 q_2 z(1 - f_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z))/\lambda(\lambda-\lambda_2 z).$$

Situation (ε).

I.C. : No queues: arrival in lane 2 of type II. Gap to next arrival in 1 (of type I) is rejected.

F.C. : Departure of this I in 1.

$$p_\varepsilon(z) = \lambda_1 z(1 - f_2^{\text{\texttt{x}}}(\lambda-\lambda_2 z))/(\lambda-\lambda_2 z).$$

Situation (ζ).

I.C. : No queues: arrival in lane 2 of type II. Gap to next arrival in 1 (of type I) is accepted.

F.C. : Departure of this II in 2.

$$p_\zeta(z) = zf_2^{H}(\lambda-\lambda_2 z).$$

Situation (η).

I.C. : No queues in lane 2: arrival in lane 2 of type I.

F.C. : Departure of this I in 2.

$$p_\eta(z) = z.$$

Situation (θ).

I.C. : No queues: arrival in lane 2 of type II. Next in 1 is of type II.

F.C. : Departure of this II in 2.

$$p_\theta(z) = z.$$

We now return to consideration of possible developments of the system from an $S_2(n)$.

Case (iii): A is last vehicle to leave at an $S_2(n)$, at which not all queueing vehicles depart. The next vehicle to cross the intersection is in lane 1.

This corresponds to situation (α) followed by possibly several situations (β), terminating with a situation (γ).

If we define

r_α = no. of arrivals in lane 2 during situation (α)

r_{β_i} = no. of arrivals in lane 2 during i^{th} situation (β)

r_γ = no. of arrivals in lane 2 during situation (γ),

we have

$$q' = n+1-k+r_\alpha+r_\gamma+\sum_{i=1}^{p} r_{\beta_i} -1$$

where $p = 0, 1, \ldots$ is the number of situations (β) which occur. Since r_{α}, r_{β_i}, r_γ are independent,

$$p.g.f.(q' \mid q, S_2)(x') = x'^{(q-1)} \frac{\lambda_1(1-g_2^*(\lambda-\lambda_2 x'))}{(\lambda-\lambda_2 x')} \cdot \frac{\left[p_1 g_2^*(\lambda-\lambda_2 x')+q_1 h_2^*(\lambda-\lambda_2 x')\right] (\lambda-\lambda_2}{\left[\lambda_2(1-x')+\lambda_1(p_1 g_2^*(\lambda-\lambda_2 x')+q_1 h_2^*(\lambda-\lambda_2 x')\right.}$$

and so the contribution to $G_2(x,x')$ is

$$\frac{\lambda_1 q_2 x}{H_2(p_2-xx')}\left\{\tilde{S}_2(p_2)-\tilde{S}_2(xx')\right\} \frac{(1-g_2^*(\lambda-\lambda_2 x')\left\{p_1 g_2^*(\lambda-\lambda_2 x')+q_1 h_2^*(\lambda-\lambda_2 x')\right\}}{\left\{\lambda_2(1-x')+\lambda_1(p_1 g_2^*(\lambda-\lambda_2 x')+q_1 h_2^*(\lambda-\lambda_2 x'))\right\}}$$

Case (iv): A is the last vehicle to leave at an $S_2(n)$ at which all queueing vehicles depart. The next vehicle to arrive in either lane is B, which is delayed for a non-zero interval of time.

This corresponds to situation (b) followed by possibly several situations (β) and situation (γ).

Case (v): A is the last vehicle to leave at an $S_2(n)$ at which all queueing vehicles depart. The next vehicle to arrive is in lane 1 (and of type I). There may be several further arrivals in lane 1 prior to the next arrival in lane 2 (the arrival of B). Immediately following the arrival B one of situations (ϵ), (ζ), (η) or (θ) obtains. Situation (ϵ) would be followed by possibly several situations (β) and a situation (γ).

The contributions to $G_2(x,x')$ arising from cases (iv) and (v) total to

$$\frac{\lambda_1}{\lambda}\ \frac{\tilde{S}_2(p_2)}{H_2}\left\{\frac{q_2(1-f_2^*(\lambda-\lambda_2x'))(\lambda_2+\lambda_1p_1)(p_1g_2^*(\lambda-\lambda_2x')+q_1h_2^*(\lambda-\lambda_2x'))}{\left[\lambda_2(1-x')+\lambda_1(p_1g_2^*(\lambda-\lambda_2x')+q_1h_2^*(\lambda-\lambda_2x'))\right]}\right.$$
$$\left.+p_1q_2f_2^*(\lambda-\lambda_2x')+p_2+q_2q_1\right\} .$$

The total contribution to $G_2(x,x')$ arising from a preceding regeneration point of type $S_2(n)$ consists of the sum of the contributions arising from cases (iii) to (v) (listed above) and the contributions from cases (i) and (ii), which are :

$$\frac{1}{H_2}\left[\frac{xp_2}{p_2-xx'}\ (\tilde{S}_2(p_2)-\tilde{S}_2(xx'))+\ \frac{xq_2}{p_2}\ g_2^*(\lambda-\lambda_2x')\left\{\frac{p_2\tilde{S}_2(xx')-xx'\tilde{S}_2(p_2)}{xx'-p_2}+\tilde{S}_2(p_2)\right\}\right.$$
$$\left.+\tilde{S}_2(p_2)\ \frac{\lambda_2}{\lambda}\ (q_2f_2^*(\lambda-\lambda_2x')+p_2)\right] .$$

The components of $G_2(x,x')$ arising from preceding regeneration points of types $T_2(n)$, $R_2(n)$, $R_2(-1)$, $R_2(-2)$ may be discussed in a similar way. For the reasons given in §5.4.2, this calculation is not completed here.

As a check on this complicated analysis, one might use the well-known properties of a bivariate probability generating function. On inspection of the component of G_2 above, after multiplication by H_2, it is found that

a) with $x = x' = 1$ this reduces exactly to those terms in H_2 which may be regarded as arising from $S_2(n)$.

b) with $x' = 1$, this quantity reduces to the corresponding part of the marginal p.g.f. of q, evaluated in §4.3.4.

Thus two partial checks on the calculations are available. When the attempt is made to obtain the relevant part of the marginal p.g.f. of x', the expression obtained is not immediately identifiable as in cases (a), (b). It seems likely, however, that, with appropriate use of the basic equations of the system, the entire function $G_2(x, x')$ would satisfy this further condition.

Chapter 6.

Conclusion

6.1. Discussion of chapters 2-5: problems outstanding and conclusions.

Chapters 2-5 appear to demonstrate that it is at least possible to construct plausible mathematical models for the problem of delay due to right-turning vehicles, and to make some progress with their analysis.

Further analysis of these models and in particular of model III might be directed towards answering the questions:

(i) what is the distribution of delay caused to vehicles arriving in the minor road?

(ii) is a time-dependent analysis of these models possible?

(iii) could model III be supplemented by a model for pedestrian behaviour at the intersection?

It might for instance be possible to utilise the formulae of Tanner (1962) for the mean delay to vehicles in a minor road caused by major road traffic, if it were possible to derive the first two moments of the distribution of the 'busy periods' of the intersection.

With regard to the results of fitting model III to an actual intersection, a claim could be made that the model reflects in a meaningful way those aspects of the real situation arising from right-turning vehicles. Given sufficient resources, one could envisage a much more comprehensive statistical analysis of the model. Data from several intersections might be available, possibly consisting of sequences of observations at different times of day reflecting a range of parameter values in the model, and in particular for situations in which the proportion of type II vehicles in each lane is substantial. These data would be obtained on film with a frame speed sufficiently high to give the detail required for an empirical study of the gap-acceptance mechanisms in use. It would be interesting to select firstly intersections to which the assumptions of the model (particularly concerning randomness of arrivals) might provide a good approximation, and secondly, a group of intersections to which these assumptions would apply to a progressively lesser extent. In this way information as to the range of validity of the model might be obtained.

At the same time a variety of distributional forms for gap-requirement times could be considered, and the most suitable determined. It seems that the likelihood function used in chapter 5 is not adequate and that modifications to this based on a more complete description of the system should be made. For instance, either the number of vehicles departing at a regeneration point, or the duration of the transition interval might be used to augment the description used above.

6.2. Practical applications of Model III.

It is possible to envisage certain applications of model III, although the construction and the analysis of the models of this paper is intended to be not so much a discussion of practical problems as a demonstration of the feasibility of this type of approach to problems arising from actual road layouts.

An obvious application is to the design of a road system. The model could be used to give an indication of how much traffic a particular intersection could be expected to deal with, assumptions about gap-acceptance being based on experience with similar intersections elsewhere. The question of the optimal loading of an intersection would become important if some form of traffic routeing through a network were available, and the model might give information on this.

We note that it is frequently the case that the intersection of a four lane major road with a minor road may be regarded as the intersection of a two lane major road with a minor road, because of the reduction in useful width of the major road caused by parked vehicles. Model III or III(a) might be applicable to such situations.

At a particular intersection for which suitable data are to hand, model III could give information concerning (i) the choice of gap-requirement times by drivers, (ii) the capacity(corresponding to extreme values of parameters) of the intersection. If such data were available for a large number of intersections for which, in addition, accident statistics were available, it might be possible to relate either some aspect of the vector of parameters describing traffic at a particular intersection, or some measure of the loading of the intersection relative to its capacity, to aspects of the accident record of the intersection. Possibly a multivariate technique such as canonical correlations might be useful in this respect.

6.3. Further Problems.

A problem related to that considered in this thesis concerns the delays caused by right turning vehicles at the priority intersection of a four lane major road with a minor road.

It is possible to propose a set of rules of vehicular behaviour in a model of this situation which appear to make its analysis comparatively straightforward, by reducing the system to two independent subsystems. This may be achieved by the following assumptions (with the usual notation):

(i) Type I vehicles are confined to the outer lanes, and type II to the inner lanes, at the intersection.

(ii) Type II vehicles base their decision to cross on a gap-requirement time applied only to the stream of opposing type I's, i.e. to the opposing outer lane.

(iii) Vehicles arriving at the intersection are of randomly assorted type.

With this set of rules, each of the two subsystems consisting of type I vehicles together with the opposing type II vehicles, is independent of the other, at least so far as delays are concerned. The four lane intersection may be regarded as a combination of two independent 'T-junctions', models for which are discussed in Chapter 1. The model might be plausible for some situations, in particular when lane markings are present.

It is however a matter of common experience that turning traffic may be delayed by opposing queueing type I vehicles. It is not difficult to construct a model along similar lines which would incorporate this feature. It is not obvious in what way a system of regeneration points may be constructed to assist in its analysis. The difficulties are twofold: a) the possibility of queues simultaneously present in two lanes would presumably increase the number of types of regeneration point, with a coresponding increase in the number of transitions to be considered; and b) it would appear necessary to incorporate into the description of the system at any instant when two queues existed a measure of the time due to elapse before the arrival of the next vehicle in a particular lane. Thus it appears that the analysis of such a system might be more appropriately based on that of Cohen and Stam (1963).

Appendix I.

Transition probabilities for model I.

Transition	Probability
$R_1(-1),\ R_1(-1)$	$\lambda_1 f_1/\lambda$
$R_1(-1),\ R_2(-1)$	$\lambda_2 p_1/\lambda$
$R_1(-1),\ R_1(-2)$	0
$R_1(-1),\ R_2(-2)$	$\lambda_1 q_2/\lambda$
$R_1(-1),\ R_1(n)$	0
$R_1(-1),\ R_2(n)$	$\frac{\lambda_2 q_1}{\lambda}(1-r_1)\left(\frac{\lambda_1}{\lambda}\right)^n$
$R_1(-2),\ R_1(-1)$	0
$R_1(-2),\ R_2(-1)$	$\lambda_2 p_1 r_2/\lambda$
$R_1(-2),\ R_1(-2)$	λ_1/λ
$R_1(-2),\ R_2(-2)$	$\lambda_2 q_1/\lambda$
$R_1(-2),\ R_1(n)$	$\frac{\lambda_1 p_1}{\lambda}(1-r_2)\left(\frac{\lambda_2}{\lambda}\right)^n$
$R_1(-2),\ R_2(n)$	0

$R_1(n),\ R_1(-1)$ $\qquad \dfrac{\lambda_1 p_2}{\lambda}\left(p_1 r_2 f_2^{n-1} + q_1 r_1\right)$

$R_1(n),\ R_2(-1)$ $\qquad \lambda_2 p_1 r_2 f_2^{\,n}/\lambda$

$R_1(n),\ R_1(-2)$ $\qquad \dfrac{\lambda_1 q_2}{\lambda}\left(q_1 + p_1 r_2 f_2^{n-1}\right)$

$R_1(n),\ R_2(-2)$ $\qquad \lambda_2 q_1/\lambda$

$R_1(m),\ R_1(n)$

$$p_1\lambda_1(1-r_2)\Bigg\{\frac{1}{\lambda}\left(\frac{\lambda_2}{\lambda}\right)^{n-m}\epsilon(n-m)$$

$$+\ \frac{r_2 q_2 f_2^{\,m-n-1}}{\lambda_1+\lambda_2 q_2(1-r_2)}\left[1-\left(\frac{\lambda_2 f_2}{\lambda}\right)^{n}\right]\epsilon(m-n-1)$$

$$+\left(\frac{\lambda_2}{\lambda}\right)^{n-m+1}\cdot\frac{r_2 q_2}{\lambda_1+\lambda_2 q_2(1-r_2)}\left[1-\left(\frac{\lambda_2 f_2}{\lambda}\right)^{m-1}\right]\epsilon(n-m)$$

$$+\ \frac{\lambda_2^{\,n} q_2 r_2 f_2^{\,m-1}}{\lambda^{n+1}}\Bigg\}$$

$R_1(m),\ R_2(n)$ $\qquad \dfrac{\lambda_2 p_2 q_1}{\lambda}(1-r_1)\left(\dfrac{\lambda_1}{\lambda}\right)^{m}$

<u>Transition probabilities for model II.</u>

$R_1(-1),\ R_1(-1)$ $\qquad \dfrac{\lambda_1}{\lambda}\left(p_1 + q_1 f_1^{\,m}(\lambda)\right)$

$R_1(-1),\ R_2(-1)$ $\qquad \lambda_2 p_1/\lambda$

$R_1(-1),\ R_1(-2)$ $\qquad 0$

$R_1(-1),\ R_2(-2)$ $\qquad \lambda_2 q_1/\lambda$

$R_1(-1),\ R_1(n)$ $\qquad 0$

$R_1(-1)$, $R_2(n)$	$\frac{\lambda_2\lambda_1^n q_1}{\lambda(n-1)!}\int_0^\infty (1-F_1(\tau))e^{-\lambda\tau}\tau^{n-1}d\tau$
$R_1(-1)$, $S_1(n)$	$\frac{\lambda_1^{n+1} q_1}{\lambda . n!}\int_0^\infty f_1(\sigma)\, e^{-\lambda\sigma}\, \sigma^n\, d\sigma$
$R_1(-1)$, $S_2(n)$	0

$R_1(-2)$ $R_1(-1)$	0
$R_1(-2)$, $R_2(-1)$	$\lambda_2 p_1 f_2^*(\lambda)/\lambda$
$R_1(-2)$, $R_1(-2)$	λ_1/λ
$R_1(-2)$, $R_2(-2)$	$\lambda_2 q_1/\lambda$
$R_1(-2)$, $R_1(n)$	$\frac{\lambda_1\lambda_2^n p_1}{\lambda(n-1)!}\int_0^\infty (1-F_2(\tau))e^{-\lambda\tau}\tau^{n-1}d\tau$
$R_1(-2)$, $R_2(n)$	0
$R_1(-2)$, $S_1(n)$	0
$R_1(-2)$, $S_2(n)$	$\frac{\lambda_2^{n+1} p_1}{\lambda\, n!}\int_0^\infty f_2(\sigma)\, e^{-\lambda\sigma}\, \sigma^n\, d\sigma$

$R_1(n)$, $R_1(-1)$	$\lambda_1 q_1 p_2 f_1^*(\lambda)/\lambda$
$R_1(n)$, $R_2(-1)$	$p_1 f_2^*(\lambda)\ \varepsilon\ (1-m)$
$R_1(n)$, $R_1(-2)$	$\lambda_1 q_1 q_2/\lambda$
$R_1(n)$, $R_2(-2)$	$\lambda_2 q_1/\lambda$
$R_1(m)$, $R_1(n)$	$\frac{\lambda_1\lambda_2^{n-m} p_1}{(n-m)!}\int_0^\infty e^{-\lambda\tau}(1-F_2(\tau))\tau^{n-m}d\tau . \varepsilon(n-m)$

$R_1(m)$, $R_2(n)$	$\frac{\lambda_2 \lambda_1^{\ n} p_2 q_1}{\lambda(n-1)!} \int_0^\infty e^{-\lambda\tau}(1-F_1(\tau))\tau^{n-1} d\tau$
$R_1(m)$, $S_1(n)$	$\frac{\lambda_1^{\ n+1} q_1 p_2}{\lambda n!} \int_0^\infty f_1(\tau)\, \tau^n\, e^{-\lambda\tau}\, d\tau$
$R_1(m)$, $S_2(n)$	$\frac{\lambda_2^{\ n-m+1} p_1}{(n-m+1)!} \int_0^\infty f_2(\tau)\tau^{n-m+1}\, e^{-\lambda\tau} d\tau . \varepsilon(n-m+1)$

$S_1(n)$, $R_1(-1)$	$\frac{\lambda_1}{\lambda}\, s_1^{\ n}\, (p_1 + q_1 f_1^{\ *}(\lambda))$
$S_1(n)$, $R_2(-1)$	$\lambda_2 p_1 s_1^{\ n}/\lambda$
$S_1(n)$, $R_1(-2)$	0
$S_1(n)$, $R_2(-2)$	$\lambda_2 q_1 s_1^{\ n}/\lambda$
$S_1(m)$, $R_1(n)$	0
$S_1(m)$, $R_2(n)$	$\frac{\lambda_2 q_1}{\lambda} \left\{ (1-k_1)\left(\frac{\lambda_1}{\lambda}\right)^{n-m} \sum_{r=0}^{m-1} \left(\frac{s_1\lambda_1}{\lambda}\right)^r \varepsilon(n-m+r) + \frac{\lambda_1^{\ n} s_1^{\ m}}{(n-1)!} \int_0^\infty \left[1-F_1(\tau)\right] e^{-\lambda\tau}\tau^{n-1} d\tau \right\}$
$S_1(m)$, $S_1(n)$	$\frac{\lambda_1^{\ n+1} q_1 s_1^{\ m}}{\lambda n!} \int_0^\infty f_1(\tau) e^{-\lambda\tau}\tau^n d\tau$
$S_1(m)$, $S_2(n)$	0

Transition probabilities for model III.

$R_1(-1), \; R_1(-1)$	$\frac{\lambda_1}{\lambda} \; (p_1 + q_1 f_1^{*}(\lambda))$
$R_1(-1), \; R_2(-1)$	$\lambda_2 p_1/\lambda$
$R_1(-1), \; R_1(-2)$	0
$R_1(-1), \; R_2(-2)$	$\lambda_2 q_1/\lambda$
$R_1(-1), \; R_1(n)$	0
$R_1(-1), \; R_2(n)$	$\frac{\lambda_2 q_1 \lambda_1^{n}}{\lambda(n-1)!} \int_0^{\infty} (1-F_1(\tau))e^{-\lambda\tau}\tau^{n-1}d\tau$
$R_1(-1), \; S_1(n)$	$\frac{\lambda_1^{n+1} q_1}{\lambda n!} \int_0^{\infty} f_1(\tau)e^{-\lambda\tau}\tau^{n}d\tau$
$R_1(-1), \; S_2(n)$	0
$R_1(-1), \; T_1(n)$	0
$R_1(-1), \; T_2(n)$	0

$R_1(-2), \; R_1(-1)$	0
$R_1(-2), \; R_2(-1)$	$\lambda_2 p_1 f_2^{*}(\lambda)/\lambda$
$R_1(-2), \; R_1(-2)$	λ_1/λ
$R_1(-2), \; R_2(-2)$	$\lambda_2 q_1/\lambda$
$R_1(-2), \; R_1(n)$	$\frac{\lambda_1 p_1 \lambda_2^{n}}{\lambda(n-1)!} \int_0^{\infty} (1-F_2(\tau))e^{-\lambda\tau}\tau^{n-1}d\tau$

$R_1(-2)$, $R_2(n)$	0
$R_1(-2)$, $S_1(n)$	0
$R_1(-2)$, $S_2(n)$	$\frac{\lambda_2^{n+1} p_1}{\lambda n!} \int_0^\infty f_2(\tau) e^{-\lambda\tau} \tau^n d\tau$
$R_1(-2)$, $T_1(n)$	0
$R_1(-2)$, $T_2(n)$	0

$R_1(n)$, $R_1(-1)$	0
$R_1(n)$, $R_2(-1)$	$p_1 g_2^*(\lambda)\ \varepsilon(1-n)$
$R_1(n)$, $R_1(-2)$	0
$R_1(n)$, $R_2(-2)$	$q_1 h_2^*(\lambda)\ \varepsilon(1-n)$
$R_1(m)$, $R_1(n)$	$\frac{\lambda_1 \lambda_2^{n-m}}{(n-m)!} \int_0^\infty \tau^{n-m} e^{-\lambda\tau} \left\{1 - p_1 G_2(\tau) - q_1 H_2(\tau)\right\} d\tau\, \varepsilon(n-m)$
$R_1(m)$, $R_2(n)$	0
$R_1(m)$, $S_1(n)$	0
$R_1(m)$, $S_2(n)$	$\frac{p_1 \lambda_2^{n-m+1}}{(n-m+1)!} \int_0^\infty \tau^{n-m+1} e^{-\lambda\tau} g_2(\tau) d\tau\ \varepsilon(n-m+1)$
$R_1(m)$, $T_1(n)$	0
$R_1(m)$, $T_2(n)$	$\frac{q_1 \lambda_2^{n-m+1}}{(n-m+1)!} \int_0^\infty \tau^{n-m+1} e^{-\lambda\tau} h_2(\tau) d\tau\ \varepsilon(n-m+1)$

$S_1(n),\ R_1(-1)$	$\frac{\lambda_1}{\lambda} p_1^{n}(p_1+q_1f_1^{*}(\lambda)) + q_1p_1^{n-1}g_1^{*}(\lambda)$
$S_1(n),\ R_2(-1)$	$\lambda_2p_1^{n+1}/\lambda$
$S_1(n),\ R_1(-2)$	0
$S_1(n),\ R_2(-2)$	$\lambda_2p_1^{n}q_1/\lambda$
$S_1(m),\ R_1(n)$	0
$S_1(m),\ R_2(n)$	$\lambda_2q_1\left\{\sum_{r=0}^{m-1} p_1^{r} \int_0^{\infty}\left[1-G_1(\tau)\right] e^{-\lambda\tau}\frac{(\lambda_1\tau)^{n-m+r}}{(n-m+r)!}\, d\tau.\ \varepsilon(n-m+r)+\frac{\lambda_1^{n}p_1^{m}}{\lambda(n-1)!}\int_0^{\infty}\left[1-F_1(\tau)\right]e^{-\lambda\tau}\tau^{n-1}d\tau\right\}$
$S_1(m),\ S_1(n)$	$\frac{\lambda_1^{n+1}q_1p_1^{m}}{\lambda n!}\int_0^{\infty}f_1(\tau)e^{-\lambda\tau}\tau^{n}d\tau+q_1\sum_{r=0}^{m-1}\frac{\varepsilon(n-m+r+1)}{(n-m+r+1)!}\,p_1^{r} \int_0^{\infty}e^{-\lambda a}(\lambda_1\tau)^{n-m+r+1}\, g_1(\tau)d\tau$
$S_1(m),\ S_2(n)$	0
$S_1(m),\ T_1(n)$	0
$S_1(m),\ T_2(n)$	0

$T_1(n),\ R_1(-1)$	0
$T_1(n),\ R_2(-1)$	$\lambda_2 p_1^{n+1} f_2^{*}(\lambda)/\lambda$
$T_1(n),\ R_1(-2)$	$p_1^{n-1}\left(\frac{\lambda_1 p_1}{\lambda} + q_1 h_1^{*}(\lambda)\right)$
$T_1(n),\ R_2(-2)$	$\lambda_2 p_1^{n} q_1/\lambda$
$T_1(m),\ R_1(n)$	$\frac{\lambda_1 \lambda_2^{n} p_1^{m+1}}{\lambda (n-1)!} \int_0^{\infty} (1-F_2(\tau)) e^{-\lambda\tau} \tau^{n-1} d\tau$
$T_1(m),\ R_2(n)$	$\lambda_2 q_1 \sum_{r=0}^{m-1} p_1^{r} \int_0^{\infty} \left[1-H_1(\tau)\right] e^{-\lambda\tau} \frac{(\lambda_1\tau)^{n-m+r}}{(n-m+r)!} d\tau . \varepsilon(n-m+r)$
$T_1(m),\ S_1(n)$	0
$T_1(m),\ S_2(n)$	$\frac{\lambda_2^{n+1} p_1^{m+1}}{\lambda n!} \int_0^{\infty} e^{-\lambda\tau} f_2(\tau) \tau^{n} d\tau$
$T_1(m),\ T_1(n)$	$q_1 \sum_{r=0}^{m-1} \frac{\varepsilon(n-m+r+1)}{(n-m+r+1)!} p_1^{r} \int_0^{\infty} e^{-\lambda\tau} (\lambda_1\tau)^{n-m+r+1} h_1(\tau)\, d\tau$
$T_1(m),\ T_2(n)$	0

Appendix II. Practical Difficulties.

1. The Intersection.

Great difficulty was encountered in finding an intersection at which conditions were approximately those assumed in the model. The reasons for this were (i) there are not many intersections in the Glasgow area at which the major road contains only two lanes, with literally no opportunity for vehicles to pass to the left of a queue, (ii) almost all intersections of interest were controlled, or were subject to interfering factors such as pedestrian crossings, (iii) the traffic intensity or the proportion of turning vehicles in at least one lane was usually far too small. In the event, only one suitable intersection was found, and even it was not entirely satisfactory - some vehicles did by-pass the queue, and the proportion of type II vehicles in one lane was unexpectedly small.

2. The observations.

Pictures were taken with a time-lapse cine camera at a constant interval of about one second for about 75 minutes at a peak period, resulting in approximately 4500 frames.

Upon inspection, it was observed that for intervals of considerable length the intersection was quite free of congestion. This was found to be a result of the occasional blocking of each of lanes 1,2 some miles upstream of the intersection. It was decided, therefore, to ignore these parts of the record, and considerable care was exercised in selecting for consideration intervals (beginning and ending with free flow) during which the flow of traffic to the intersection in both lanes appeared to be unobstructed. It was hoped, by this means, to bring the arrival processes into reasonable conformity with the postulated Poisson process, and this was in fact achieved.

3. Interpretation of the record.

(i) Queues.

The interpretation of a 'queue' in the model is not quite so well defined in the actual situation as might be supposed. Drivers tend to avoid periods of stationary queueing by slowing their approach to an intersection if congestion is observed ahead. The film shows clearly the brake-lights of vehicles in lane 1, and this information may be used when necessary to estimate the queue length at a given instant. Moving vehicles may be considered to be queueing - the criterion is not one of speed, but whether a vehicle would or would not have joined the end of a stationary queue had it continued to move with the speed at which it initially approached the intersection.

Some difficulty was experienced in distinguishing individual vehicles at the intersection during periods of heavy congestion, but it was usually possible to deduce queue lengths to within reasonable accuracy.

(ii) Delays.

Several interpretations of delay in the actual situation are available. The one considered most meaningful and adopted here is as follows.

The delay experienced by a given vehicle is the difference between the times taken by this vehicle and by a vehicle of similar type, unimpeded by congestion and travelling initially at the same speed, to complete a journey through the intersection. This journey is from a point sufficiently far upstream to be clear of any congestion, to a point clear of the intersection in the exit lane.

The practical difficulties in estimating the delay of a vehicle from the film are considerable. The number of frames taken by the vehicle to complete its journey was counted, and the nearest unobstructed vehicle was used to provide a comparison. It was found that the speeds of unobstructed vehicles through the intersection varied considerably.

Thus observations on delay are subject to several sources of error. Likelihood of bias arising from an improper interpretation of delay in the actual situation, the limitations imposed by frame speed, and the considerable subjective element in measuring delays, are some of the components contributing to discrepancies of observed delays from their predicted values.

Appendix III. Observations.

(i) Inter-arrival intervals.

Measurements made to the nearest frame-separation (approximately 1 second). Commas define continuous periods of observation. (Data to be read horizontally.)

Lane 1.

9 3 4 5, 8 7 1 8 6 8, 2 6 39 1 2 6
22 9 2 2 14 2 6 4 6 10, 2 1, 36 4, 2 3
16 5 1 4 1 5 2, 2 2 1 2 4 2 16 2 2
3 1 5 4 1 1 2 1 3 1 1 3 1 2 2 3 3
3 2 1 3 1 2 1 9 6 10, 12 1 4 3 4 7
1 10 16 6, 15, 3 5 4 4, 2 2 1 1 3 3
7 3 2 2, 2 4 1 3 2 2 16, 4 2 4 12
7 6 2 2, 3 3 4 5 5, 2 1 3 1 2 4
2 1 3 4 1 3 2 19, 2 5 2 3 6 7 11 2
1 2 1, 2 2 3 18 4 4 2 3, 2 1 3 1
12 1 1 8 2 1 1 4

Lane 2.

1 11 1 1 10 15 2 1 2, 1 1 1 2 1 1
2 3 3 18 2, 1 2 3 6 1 1 2 2 2
6 25 1 1 1 4 1 19 2 4 3 2 3
4 7 1 4 2 2, 2 2 4 1 3 3 4 3

2 2 2 6 7, 5 2 2, 2 2 35 5
4 4 15 5 2 2 3 5, 1 12 1 8 2,
1 13, 4 4 1 8 6, 2 3 6 12, 2, 3
4 2 4 2 2 1, 1 1 2 2 1 3 9 4 2
6 9, 26 3, 1 2 1 3 1 13, 2 3 3 2 2
2 3 4 2 5 2 3 4 1 6 9, 4 1 1 1
2 4 4 2 1 9 5

(ii) Delays to vehicles.

Lane 1.

0 0 5 6 6 4 0 0, 19 12 5 4, 0 0
0 0 0 0 0 0 0 22 19 0 0 0 0, 0
1 0 0, 0 0 0, 0 0 0 5 6 0, 2 2
10 8 6 2 1 0 0 0 0 0 0 0 0
11 10 10 9 8 7 6 5 4 3 18 17 16
14 10 8 10 9 2 1 0, 0 0 0 0
1 4 1 5 4 2, 0 7, 0 11, 0 0 0
0 0 2 1 7 7 3 8, 2 2 0 0 0,
0 0 19 17 4 0, 0 15 11 12 7 2 7,
0 0 3 3 4 2 0 0 0 3 0 0 0
6 0, 4 2 11 5 2, 30 28 26 20 5
6 8 6 2, 0 0 0 18 17 15 15 20 19
18 13 12 11 11 9

Lane 2. (omitting zero delays).

5 2, 11 7 8 8 2, 11 9 8 5

Appendix IV. References.

BARTLETT, M.S. (1967), Inference and stochastic processes, J.R.Statist.Soc.A, 130, 457-474.

COHEN, J., DEARNALEY, E.J., HANSEL, C.E.M., (1955), The risk taken in crossing a road, Operational Research Quarterly, 6, 120-128.

COHEN, J.W., DE LANGE, S.J., (1965), Numerical results for "Queueing for gaps in a traffic stream", Mathematical Institute, Technological University, Delft.

COHEN, J.W., STAM, A.J., (1963), Queueing for gaps in a traffic stream, Mathematical Institute, Technological University, Delft.

COX, D.R., LEWIS, P.A.W., (1966), The Statistical Analysis of Series of Events, Methuen, London.

GAVER, D.P., Jr., (1965), Methods for describing time-dependent waits at traffic merges, Third International Symposium on the Theory of Traffic Flow, New York.

HAIGHT, F.A., (1963), Mathematical Theories of Traffic Flow, Academic Press, New York.

HAWKES, A.G., (1965), Queueing for gaps in traffic, Biometrika, 52, 79-85.

HAWKES, A.G., (1966), Delay at traffic intersections, J.R.Statist. Soc.B, 28, 202-212.

HERMAN, R., WEISS, G.H., (1961), Comments on the highway crossing problem, Operations Research, 9, 828-840.

JORDAN, C., (1950), Calculus of Finite Differences, Chelsea Publishing Co., New York.

JOWETT, G.H., (1955), The comparison of means of sets of observations from sections of independent stochastic series, J.R.Statist.Soc.B, 17, 208-227.

KENDALL, D.G., (1951), Some problems in the theory of queues, J.R. Statist.Soc.B, 13, 151-173.

KENDALL, D.G., (1953), Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov Chain, Ann.Math.Stat. 24, 338-354.

LEWIS, P.A.W., (1965), Some results on tests for Poisson processes, Biometrika, 52, 67-78.

MAYNE, A.J., (1954), Some further results in the theory of pedestrians and road traffic, Biometrika, 41, 375-389.

MILLER, A.J., (1962), Road traffic flow considered as a stochastic process, Proc.Camb.Phil.Soc., 58, 312-325.

NEWELL, G.F., (1955), Mathematical models for freely-flowing highway traffic, Operations Research, 3, 176-186.

NEWELL, G.F., (1959), The effect of left turns on the capacity of a traffic intersection, Quart.Appl.Math., 17, 67-76.

OLIVER, R.M., BISBEE, E.F., Queueing for gaps in high flow traffic streams, Operations Research, 10, 105-115.

REID, D.H., (1967), A mathematical model for delays caused by right-turning vehicles at an uncontrolled intersection, Journal of Applied Probability, 4, 180-191.

TANNER, J.C., (1953), A problem of interference between two queues, Biometrika, 40, 58-69.

TANNER, J.C., (1961), Delays on a two-lane road, J.R.Statist.Soc., B, 23, 38-63.

TANNER, J.C., (1962), A theoretical analysis of delays at an uncontrolled intersection, Biometrika, 49, 163-170.

WASSCHER, E.J., (1963), Algorithm 203: Steep 1, Communications of the Association for Computing Machinery, 6, 517-519.

WEISS, G.H., (1965), A survey of some recent research in road traffic, Proc.of Symp.on Congestion Theory, University of North Carolina Press, 253-280.

WEISS, G.H., MARADUDIN, A.A., (1962), Some problems in traffic delay, Operations Research, 10, 74-104.