SOME PROBLEMS IN THE

MATHEMATICAL THEORY OF THERMOELASTICITY.

by

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SUMMARY.

The mathematical theory of thermoelasticity entered a new phase in 1956 when M. A. Biot rediscovered the form of the heat conduction equation for an elastic solid — an equation containing a term in the displacement vector, thereby linking this equation to the well known thermoelastic equations of motion. This thesis is concerned with obtaining formal solutions to some problems in thermoelasticity, using the full linked equations and using integral transforms as the means of solution. In addition two approximate solutions are considered: the classical solution (obtained by omitting the linking term from the heat conduction equation) and the quasi-static solution (obtained by omitting the inertia terms from the equations of motion). These solutions and the 'complete linked' solution coincide in steady-state problem

The first chapter, which contains an introduction to the basic equations and a summary of published work in this field, is followed by the solution of the steady-state problem for the half-space, the thick plate and the elastic layer on a rigid foundation. In each case formal solutions are obtained for arbitrary temperature distributions on the traction free boundaries, and in the special cases considered the thermoelastic analogues of the isochromatic lines of photoelasticity are constructed.

The next chapter deals with simple dynamical problems — the sphere, spherical shell and infinite medium with a spherical cavity, subjected to radially symmetrical thermal and elastic disturbances. For the case of the sphere under an exponentially time-dependent surface temperature, it was possible to compare the three types of solution mentioned in the first paragraph above. In the special case considered it was found that the error introduced by neglecting the inertia terms was of the same order as that introduced by neglecting the linking term. The linked quasi-static solution for the cavity problem was then compared with the corresponding classical solution for the case where the boundary is subjected to a sudden rise in temperature.

Chapter IV contains the description of a method of obtaining solutions to a class of boundary value problems, by considering modified problems in an infinite space. This method is used in the following chapter to study the thermoelastic effects of longitudinal wave propagation in infinitely long circular cylinders and tubes. The thermoelastic equivalents of Rayleigh surface waves are also considered in this chapter and a brief summary is given of another author's work on plane-wave propagation.

The foregoing problems were all simplified by some special feature — steady-state, radial symmetry, wave propagation of a prescribed form. In the last two chapters of the thesis, a beginning is made on the difficult task of solving the linked equations in their most general form. The infinite medium under the action of time dependent heat sources and body forces is treated first, and then a formal solution is obtained for the semi-infinite medium subjected to arbitrary heat sources, body forces, thermal and elastic boundary conditions. In all of these cases the solutions appear in the form of multiple integrals, and in any particular application it would seem to be necessary to evaluate these integrals numerically on a digital computer. In the thesis a very special example has been considered as an application of the solutions for the semi-infinite medium, and here the integrals reduce to a simple form. Within the quasi-static theory, it was found that the linked problem for the infinite medium is identical with a classical problem, in which an extra 'equivalent heat source' is introduced and one of the elastic constants is suitably changed.

PREFACE.

The mathematical theory of thermoelasticity entered a new phase in 1956 when M. A. Biot rediscovered the form of the heat conduction equation for an elastic solid - an equation which includes a term linking it to the well known thermoelastic equations of motion. This thesis is concerned with the formal solution of some problems in thermoelasticity, and throughout use is made of integral transforms.

Much of the work described here has been published already or has been accepted for publication in the near future. Reference has been made to those publications at the relevant points in the text. The contents of Chapters II and VI are due to the joint efforts of Professor I. N. Sneddon and the present author. The remainder of the thesis is the work of the candidate, unless specific reference has been made to the contrary.

Vectors are denoted by underlining the particular symbol with a 'wavy' line, thus: $\underset{\sim}{u}$. It is rarely necessary to use cross references between the equations of different chapters, so that equation numbers mentioned in the text in general refer to equations in the same chapter. On the rare occasions when a cross reference is made, the equation number is preceded by a Roman numeral denoting the chapter. Thus (II-12) refers to equation (12) of Chapter II.

I am greatly indebted to the Department of Scientific and Industrial Research for the award of a Research Studentship during the period in which the work was done. I would also like to offer my sincere thanks to Professor I. N. Sneddon who suggested the problems described here, and who supervised my work.

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CHAPTER ONE.

THE EQUATIONS OF THERMOELASTICITY.

1.1 Introduction.

When an elastic solid is subjected to a nonuniform distribution of temperature, neighbouring elements of the solid will tend to expand (or contract) by differing amounts. Due to the elastic properties of the material, the thermal expansion will not be able to take place freely, and the resulting deformation must therefore be considered as the combination of a thermal and an elastic effect.

Thus the presence of heat sources in an elastic body will in general imply the existence of a stress field as well as a distribution of temperature; and should the body be loaded by applied forces, the resulting state of stress will be modified by the presence of the heat sources. Not only does a temperature distribution produce a stress field, but under certain circumstances applied forces produce temperature variations. We shall see that this effect occurs when the body is subjected to time dependent forces, in which case the work done on the body is partly converted into heat energy.

It is with this 'linking' between temperature and stress that the theory of thermoelasticity is concerned. In this chapter we shall give an outline of the main contributions which have been made to this theory to date.

It should be mentioned that, parallel to the classical theory of elasticity, we shall be concerned with external causes which produce effects consistent with a linear theory, and in addition it is assumed that the elastic 'constants' of the material are not sensibly changed by the physical and thermal changes produced in it.

1.2 Derivation of the Steady-State Equations.

1.2 We consider a homogeneous isotropic body which is subjected to

We consider a homogeneous isotropic body which is subjected to
a system of body forces $\underset{\sim}{F} = (F_1, F_2, F_3)$ per unit volume, a distributed heat source q and certain known boundary conditions. The components of the body force have been taken parallel to the coordinate directions x_1, x_2, x_3. In a thermoelastic deformation it is assumed that the total strain is made up of two components: that due to a purely thermal expansion and that due to the elastic deformation. Thus we may write a typical component of the total strain as

$$\gamma_{ij} = \gamma_{ij}^{t} + \gamma_{ij}^{e} \qquad (i, j = 1, 2, 3) \qquad (1)$$

where the superscripts $\underline{t}$ and $\underline{e}$ refer to the thermal and elastic components respectively.

Now under free thermal expansion an element of length l_{io} parallel to the x_i-axis will deform into an element of length l_i (also parallel to the x_i-axis) given by

$$l_i = l_{io}(1 + \alpha\theta)$$

in which $\underline{\alpha}$ is the coefficient of linear expansion for the elastic material and $\underline{\theta}$ is the (small) change of temperature from that of the unstrained state. Thus, for example

$$\gamma_{11}^{t} = (l_1 - l_{10})/l_{10} = \alpha\theta$$

and in general

$$\gamma_{ij}^{t} = \alpha\theta\delta_{ij} \qquad (2)$$

where δ_{ij} is the Kronecker delta. The non-diagonal elements of the thermal strain tensor are zero because the angle between any two lines is preserved during free thermal expansion i.e. no shearing occurs.

Thus, using (1) and (2), we can write the components of the elastic strain tensor in the form

$$\gamma_{ij}^{e} = \gamma_{ij} - \alpha\theta\delta_{ij} \qquad (3)$$

so that the elastic dilatation Δ^e is given by

$$\Delta^e = \Delta - 3\alpha\theta \tag{4}$$

where the dilatation $\Delta = \gamma_{ii} = u_{i,i}$ and $\underset{\sim}{u} = (u_1, u_2, u_3)$ is the displacement vector. Here the comma in the suffix denotes differentiation with respect to the variable whose suffix follows the comma, i.e. $u_{i,j} = \partial u_i/\partial x_j$, and summation is implied by the repeated suffix.

We can now substitute the expressions obtained for γ^e_{ij} and Δ^e into the elastic stress/strain relationship of Hooke's Law, which can be written in the form

$$\tau_{ij} = \lambda\Delta^e\delta_{ij} + 2\mu\gamma^e_{ij} \tag{5}$$

where τ_{ij} denotes a typical component of stress and λ and μ are Lame's elastic constants for the material. On making this substitutuon we get

$$\tau_{ij} = (\lambda\Delta - \gamma\theta)\,\delta_{ij} + 2\mu\gamma_{ij} \tag{6}$$

with the notation $\gamma = \alpha(3\lambda + 2\mu)$.

Equation (6) is known as the Duhamel-Neumann law. It was discovered independently by these two authors, NEUMANN (1885) using the method described above, whilst DUHAMEL (1838) developed the theory by regarding an elastic solid as a system of material points.

Making use of the definition $\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ in equation (6) and then substituting the resulting expression for τ_{ij} into the equations of equilibrium

$$\tau_{ij,j} + F_i = 0$$

we finally arrive at the set of equations

$$\mu\nabla^2 u_i + (\lambda + \mu)\,\Delta_{,i} - \gamma\theta_{,i} + F_i = 0 \tag{7}$$

that is, a set of three partial differential equations from which the three components of displacement produced by the distribution of temperature θ and the body forces $\underset{\sim}{F}$ are to be determined.

In many problems the distribution θ is not given, but is known to be due to certain heat sources. In such problems the equation giving the temperature distributuon within the material is, in this steady-state theory, a simple form of the heat conduction equation, namely

$$Q + \kappa\nabla^2\theta = 0 \qquad (8)$$

where κ is the diffusivity of the material, and $Q(x_1,x_2,x_3)$ is related to the quantity $q(x_1,x_2,x_3)$ of heat generated per unit volume by the relation $Q = q/\rho c$. Here ρ and c are respectively the density and specific heat per unit mass for the material.

Thus, having found the distribution of temperature using equation (8) and the thermal boundary condition, the steady-state thermoelastic problem is reduced to the solution of the set of equations (7) subject to the given elastic boundary conditions.

1.3 Dynamical Problems: Classical Solutions.

In the previous section we derived the equations by which the steady-state thermoelastic problem is governed. We seek now the equations which govern the transient or time-dependent problem. That is, we wish, for example, to find the stress and displacement fields produced by a time-dependent heat source, or to find the temperature distribution which results from variations with time of the applied forces.

The analysis leading to the Duhamel-Neumann law (6) is still valid for this more general set of problems, and the natural way to form a set of governing equations would be to substitute this expression for τ_{ij} into the equations of motion

$$\tau_{ij,j} + F_i = \rho\ddot{u}_i \qquad (9)$$

(the double dot denoting the second partial derivative with respect to time) and to use the more general form of the Fourier heat conduction equation

$$Q + \kappa\nabla^2\theta = \dot{\theta} \tag{10}$$

Thus the set of equations governing the transient problem would be

$$\begin{aligned} \mu\nabla^2\underset{\sim}{u} + (\lambda + \mu)\ \mathrm{grad}\,\Delta - \gamma\ \mathrm{grad}\,\theta + \underset{\sim}{F} &= \rho\ddot{\underset{\sim}{u}} \\ Q + \kappa\nabla^2\theta &= \dot{\theta} \end{aligned} \tag{11}$$

However it can be seen that the use of these equations to solve the problem of the application solely of variable body forces would produce no temperature field at all, since the temperature field can be obtained from the last of equations (11) without any consideration of the nature of the applied forces. Now this is contrary to physical intuition, since we would expect a body subjected to fluctuations in its loading to become heated. By thermodynamical reasoning it has been shown by BIOT (1956) that the last of equations (11) is incomplete and should in fact contain an additional term in the dilatation Δ, thus linking this equation with the others of (11). Strangely enough, this result was obtained by VOIGT (1910) and JEFFREYS (1929), but these modified equations do not seem to have received much attention until interest in them was revived by BIOT. We shall give BIOT's derivation of the heat conduction equation for an elastic medium in the next section.

The solutions obtained from equations (11) should therefore be regarded only as approximations, and we shall call these solutions the 'classical' solutions of the thermoelastic problem. (In general they are close approximations, but it should nevertheless be realised that they are in fact approximations). It should, perhaps, also be mentioned here that the term missing from the last of equations (11) is $\partial\Delta/\partial t$ so that this modification has no effect on the steady-state problem discussed earlier.

The classical solutions to several problems have been given in the literature. In some cases the authors do not seem to have been aware of the linking term, whilst others omitted it because of its small effect. These contributions are discussed in section 1.11.

1.4. Thermodynamical Reasoning and the Linked Dynamical Equations.

If we consider an element of the body having unit size, and denote by U and h respectively its internal energy and the heat absorbed by it, then the first law of thermodynamics (conservation of energy) requires that

$$dh = dU - \sum_{ij} \tau_{ij} d\gamma_{ij} \tag{12}$$

The summation here is taken over all distinct pairs i,j. Thus if the absolute temperature is T_1 we can write the change in entropy ds as

$$ds = \frac{dh}{T_1} = \frac{dU}{T_1} - \frac{1}{T_1}\sum_{ij} \tau_{ij} d\gamma_{ij}$$

$$= \frac{1}{T_1}\frac{\partial U}{\partial T_1} dT_1 + \frac{1}{T_1}\sum_{ij}\left\{ \frac{\partial U}{\partial \gamma_{ij}} - \tau_{ij} \right\} d\gamma_{ij} \tag{13}$$

Now the second law of thermodynamics requires that ds should be an exact differential in T_1 and in γ_{ij}. Hence

$$\frac{\partial s}{\partial T_1} = \frac{1}{T_1}\frac{\partial U}{\partial T_1} \qquad \text{and} \qquad \frac{\partial s}{\partial \gamma_{ij}} = \frac{1}{T_1}\left\{ \frac{\partial U}{\partial \gamma_{ij}} - \tau_{ij} \right\}$$

Therefore eliminating s

$$\frac{1}{T_1}\frac{\partial^2 U}{\partial T_1 \partial \gamma_{ij}} = \frac{1}{T_1}\left\{ \frac{\partial^2 U}{\partial \gamma_{ij}\partial T_1} - \frac{\partial \tau_{ij}}{\partial T_1} \right\} - \frac{1}{T_1^2}\left\{ \frac{\partial U}{\partial \gamma_{ij}} - \tau_{ij} \right\}$$

that is

$$\frac{\partial U}{\partial \gamma_{ij}} = \tau_{ij} - T_1 \frac{\partial \tau_{ij}}{\partial T_1} \tag{14}$$

Also, if c is the specific heat per unit mass in the absence of deformation, then

$$\rho c = \frac{dh}{dT_1} = \frac{\partial U}{\partial T_1} \tag{15}$$

where we have used equation (12). With the aid of equations (13),(14) and (15) we can now write

$$ds = \rho c \frac{dT_1}{T_1} - \sum_{ij} \frac{\partial \tau_{ij}}{\partial T_1} dy_{ij}$$

i.e. $$ds = \rho c \frac{dT_1}{T_1} + \gamma d\Delta \tag{16}$$

since, from equation (6),

$$\frac{\partial \tau_{ij}}{\partial T_1} = \frac{\partial \tau_{ij}}{\partial \theta} = -\gamma \delta_{ij}$$

Upon integration (16) becomes

$$s = \rho c \log\left(1 + \frac{\theta}{T}\right) + \gamma\Delta \tag{17}$$

where we have written $T_1 = T + \theta$, so that $\underline{T}$ is the temperature for zero stress and strain and $\underline{\theta}$, as defined before, is the deviation of the temperature from this value. The constant of integration has been chosen to make $s = 0$ when $\theta = 0$. For small variations of the temperature from the unstressed state, i.e. for small values of $\underline{\theta}$, we may replace the logarithm by the first term in its expansion and then write the heat absorbed $\underline{h}$ in the form

$$h = Ts = \rho c\theta + T\gamma\Delta \tag{18}$$

The law of heat conduction

$$q + k\nabla^2\theta = \frac{\partial h}{\partial t}$$

(where $\underline{k}$ is the thermal conductivity of the material) then becomes

$$q + k\nabla^2\theta = \rho c \frac{\partial \theta}{\partial t} + T\gamma \frac{\partial \Delta}{\partial t}$$

i.e. $$Q + \kappa\nabla^2\theta = \frac{\partial \theta}{\partial t} + \gamma' \frac{\partial \Delta}{\partial t} \tag{19}$$

since $\kappa = k/\rho c$, and where we have defined $\gamma' = T\gamma/\rho c$.

Thus the basic equations of thermoelasticity are

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu)\ \mathrm{grad}\,\Delta - \gamma\,\mathrm{grad}\,\theta + \underset{\sim}{F} = \rho\ddot{\underset{\sim}{u}}$$

$$Q + \kappa\nabla^2\theta = \dot{\theta} + \gamma'\,\dot{\Delta} \qquad (20)$$

and the components of stress can be obtained from

$$\tau_{ij} = (\lambda\Delta - \gamma\theta)\,\delta_{ij} + 2\mu\gamma_{ij} \qquad (6)$$

$$= (\lambda\Delta - \gamma\theta)\,\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

1.5 Biot's Variational Formulation.

BIOT (1956) has shown that the equations of thermoelasticity can be derived from a general variational principle in thermodynamics (BIOT 1955). We shall only give this derivation in broad outline.

In the quasi-static case (i.e. when the inertia of the moving material is neglected) and in the absence of body forces BIOT's variational principle reduces to the case of minimising the integral

$$I = V + D - \iint_{\sigma} (\underset{\sim}{T}.\underset{\sim}{u} + \theta\underset{\sim}{n}.\underset{\sim}{S})\ d\sigma \qquad (21)$$

In this expression $\underset{\sim}{u}$ and θ are as defined previously, $\underset{\sim}{T}$ is the boundary force per unit area, $\underset{\sim}{n}$ is the unit normal to the boundary directed towards the interior, $\underset{\sim}{S}$ is a vector called the entropy flow or entropy displacement and is given by $s = -\,\mathrm{div}\,\underset{\sim}{S}$. The integration is taken over the entire boundary σ. The functions V and D are invariants defined by

$$V = \iiint_{\tau} \left\{ W + \frac{1}{2}\frac{\rho c}{T}\theta^2 \right\} d\tau \qquad (22)$$

$$D = \frac{1}{2}\iiint_{\tau} \frac{T}{k}\left\{\frac{\partial S}{\partial t}\right\}^2 d\tau \qquad (23)$$

in which $W = \frac{1}{2}\tau_{ij}\gamma_{ij}$ is the isothermal mechanical energy and the integrations are taken over the volume τ.

The quantities V and D may be interpretted in terms of irreversible thermodynamics. It may be shown that

$$V = \iiint_{\tau} (U - sT)\, d\tau$$

where, for uniform temperatures, $U - sT$ is the classical thermodynamic free energy. Thus V is a generalised free energy. Also

$$D = \frac{1}{2} \iiint_{\tau} TR\, d\tau$$

where R is the rate of entropy production. Therefore D, a generalised dissipation function, measures the irreversible properties of the medium.

In applying the variational principle $\delta I = 0$ to thermoelasticity BIOT (1956) replaced expression (23) for D by

$$D' = \frac{1}{2}p \iiint_{\tau} \frac{T}{k} (\underset{\sim}{s})^2 d\tau \qquad (24)$$

where $p = d/dt$. The operator p is treated as a constant and it is only in the final differential equations that it is replaced by an actual differential. The only justification given for this procedure is the fact that it leads to the correct differential equations.

BIOT then claims that the modified variational principle leads to the appropriate form of equations (20). The interested reader can find this in the reference given. We shall not discuss it further here, since we prefer the derivation of section 1.4. or that of CHADWICK (1959) which is mentioned in the next section.

It is, however, worth mentioning that, in the presence of body forces $\underset{\sim}{F}$, and when taking into account the inertia of the moving material, the variational principle from which BIOT derives the dynamical equatioss takes the form $\delta I = 0$ where

$$I = V + D - \iint_{\sigma} (\underset{\sim}{T}.\underset{\sim}{u} + \theta \underset{\sim}{n}.\underset{\sim}{S})\, d\sigma - \iiint_{\tau} \rho \underset{\sim}{F}.\underset{\sim}{u}\, d\tau + \iiint_{\tau} \rho \frac{\partial^2 \underset{\sim}{u}}{\partial t^2}.\underset{\sim}{u}\, d\tau \qquad (25)$$

1.6 CHADWICK's Formulation of the Thermoelastic Equations.

It is not the purpose of this chapter to give in detail alternative derivations of the same set of equations. However for the sake of completeness, we refer the reader to a derivation of the thermoelastic equations due to CHADWICK (1959).

This derivation rests upon four fundamental equations based upon the thermodynamic theory of irreversible processes. The equations express the conservation of mass, linear momentum and energy, and the second law of thermodynamics. Making the assumption that the amplitudes of all disturbances are small, CHADWICK shows that these equations lead to a form of the heat conduction equation, linking the temperature, entropy and heat source distribution. He then obtains the thermoelastic Hooke's law by expanding the Helmholtz free energy f in a Taylor series about the reference state, and noting that

$$\sigma_{ij} = \rho \left\{\frac{\partial f}{\partial \gamma_{ij}}\right\}_T$$

Finally he uses the second law of thermodynamics, and manipulates with differential relations to obtain an equation of the form (17). From the equations obtained it is possible to write the thermoelastic equations in the form derived previously.

1.7 Dimensionless Form of the Equations.

In working with the equations (20) it is often more convenient to write them in dimensionless form. Following SNEDDON and BERRY (1958) (p.123) we take a typical length l and a typical time τ as our units of length and time, and we take T and μ respectively as the units of temperature and stress. Equations (20) can then be written in the dimensionless forms

$$\begin{aligned} \nabla^2 \underset{\sim}{u} + (\beta^2 - 1)\,\mathrm{grad}\,\Delta - b\,\mathrm{grad}\,\theta + \underset{\sim}{X} &= a\ddot{\underset{\sim}{u}} \\ \Theta + \nabla^2\theta &= f\dot{\theta} + g\dot{\Delta} \end{aligned} \qquad (26)$$

and the components of stress are

$$\tau_{ij} = \left\{(\beta^2 - 2)\Delta - b\theta\right\}\delta_{ij} + 2\gamma_{ij} \qquad (27)$$

where β is the ratio of the velocity of P-waves to that of S-waves i.e.

$$\beta = v_p/v_s = \left\{(\lambda + 2\mu)/\mu\right\}^{1/2}$$

and

$$\underset{\sim}{X} = \frac{l}{\mu}\underset{\sim}{F} \qquad \Theta = \frac{Ql^2}{kT}$$

$$a = \left\{\frac{l}{v_s\tau}\right\}^2, \quad b = \frac{\gamma T}{\mu}, \quad f = \frac{l^2}{k\tau}, \quad g = \frac{\gamma l^2}{k\tau} \qquad (28)$$

An idea of the relative magnitudes of a,b,f,g can be obtained from the following table which is taken from EASON and SNEDDON (1959) and is calculated for l = 1 cm., τ = 1 sec., T = 293°K.

	Aluminium	Copper	Iron	Lead
a	1.034×10^{-11}	2.166×10^{-11}	1.532×10^{-11}	2.034×10^{-10}
b	0.0639	0.0417	0.00890	0.2320
f	1.168	0.899	5.208	4.152
g	2.687	1.497	8.035	12.25
$\epsilon = bg/f\beta^2$	3.56×10^{-2}	1.68×10^{-2}	2.97×10^{-4}	7.33×10^{-2}

Some problems are better solved using other choices for the units. In particular we mention here the system of units introduced by CHADWICK and SNEDDON (1958). We choose as the units of time and length the quantities $1/\omega^*$ and v_p/ω^* respectively, where ω^* is the particular frequency

$$\omega^* = \rho c v_p^2/k$$

We again choose $\underset{\sim}{T}$ and $\underset{\sim}{\mu}$ as the units of temperature and stress.

The thermoelastic equations (20) can then be written in the dimensionless forms

$$\begin{aligned} &\nabla^2 \underset{\sim}{u} + (\beta^2 - 1) \text{ grad } \Delta - b \text{ grad } \theta + \underset{\sim}{X} = \beta^2 \ddot{\underset{\sim}{u}} \\ &\Theta + \nabla^2 \theta = \dot{\theta} + g \Delta \end{aligned} \tag{29}$$

and the components of stress are

$$\tau_{ij} = \left\{ (\beta^2 - 2)\Delta - b\theta \right\} \delta_{ij} + 2\gamma_{ij} \tag{30}$$

where

$$\underset{\sim}{X} = \frac{v_p}{\omega^* \mu} \underset{\sim}{F} \quad , \qquad \Theta = \frac{Q}{Tb^*}$$

and

$$b = \frac{\gamma \underset{\sim}{T}}{\mu} \quad , \qquad g = \frac{\gamma}{\rho c} \tag{31}$$

All the frequencies $\underset{\sim}{\omega}$ which are obtainable in practice are much smaller than ω^*, so that in this system of units $\omega \ll 1$, a fact which is extremely useful in obtaining approximate solutions to problems whose exact solutions we are unable to find.

It should be noted that the dimensionless forms (26) can be transformed into the forms (29) by the set of transformations

$$\beta \to \beta \quad , \quad b \to b \quad , \quad a \to \beta^2 \quad , \quad f \to 1 \quad , \quad g \to g$$

Thus it is easy to obtain the solutions of a particular problem in one set of units from the solutions corresponding to the other set of units.

1.8 Methods of Solution for the Steady-State Equations.

The problems considered in the following chapters of this thesis will be tackled by the method of integral transforms which is a very powerful tool when applied to suitable problems. In this section and section 1.9 we shall give a brief review of some of the other methods used in the literature for solving thermoelastic problems. The present section is devoted to the steady-state equations (i.e. no time-dependence) and the following section touches on the full dynamical equations. Since it is to be the main tool employed in this thesis, the method of integral transforms is discussed separately in section 1.10.

Solution by the Determination of a Newtonian Potential.

By comparing the equations (6) and (7) with the corresponding equations for the non-thermal problem, and remembering that the boundary forces T_i can be expressed as

$$T_i = \tau_{ij}\nu_j \quad , \qquad \text{on the boundary}$$

(the ν_j are direction-cosines) it is easily seen that the steady-state problem is equivalent to the elastostatic problem with body forces P_i and surface tractions S_i given by

$$P_i = F_i - \gamma\theta_{,i} \qquad\qquad S_i = T_i + \gamma\theta$$

Thus if we can find a particular solution of the equation

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu)\underset{\sim}{\nabla}\Delta \;+ \underset{\sim}{P} = 0 \tag{33}$$

the problem is reduced to that of solving the homogeneous form of Navier's equation (equation (33) with $\underset{\sim}{P} = 0$), the solution for which can be obtained by standard methods.

It is particularly easy to find a particular integral of (33) when the body forces F_i can be written in terms of a scalar potential, i.e. $F_i = -\Phi_{,i}$. If we then look for solutions of the form $u = \phi_{,i}$

equation (33) can be written in the form

$$(\lambda + 2\mu)\nabla^2\phi_{,i} = (\Phi + \gamma\theta)_{,i}$$

Integration of these equations with respect to x_i gives

$$\nabla^2\phi = \frac{1}{(\lambda + 2\mu)}(\Phi + \gamma\theta) \tag{34}$$

where we have omitted the constant of integration since we are interested only in finding a particular integral. Now it will be seen that (34) is merely Poisson's Equation $\nabla^2\phi = -4\pi\rho$ with

$$\rho = -\frac{\Phi + \gamma\theta}{4\pi(\lambda + 2\mu)}$$

and it therefore has a solution of the form

$$\phi = \int_{\tau'} \frac{\rho(\underset{\sim}{x}')\, d\tau(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

where $\underset{\sim}{x}$ is the radius vector of the point (x_1, x_2, x_3) and the integration is taken over all the parts of the body which are subjected to body forces and/or temperature variations.

Thus the determination of the particular integral is equivalent to the determination of the Newtonian potential for a mass distribution of known density.

In the case where $\underset{\sim}{F}$ cannot be written as the gradient of a scalar potential we introduce the scalar and vector potentials ϕ, $\underset{\sim}{\psi}$, Φ, $\underset{\sim}{\Psi}$ and write

$$\begin{aligned} \underset{\sim}{u} &= \underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla}\wedge\underset{\sim}{\psi} \\ \underset{\sim}{F} &= \underset{\sim}{\nabla}\Phi + \underset{\sim}{\nabla}\wedge\underset{\sim}{\Psi} \end{aligned} \tag{35}$$

$\underset{\sim}{F}$ can be expressed in the above form by putting

$$\Phi = -\frac{1}{4\pi}\int_{\tau'} \underset{\sim}{P'}\cdot\underset{\sim}{\nabla}\left\{\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r'}|}\right\} d\tau'$$

$$\underset{\sim}{\Psi} = -\frac{1}{4\pi}\int_{\tau'} \underset{\sim}{P'}\wedge\underset{\sim}{\nabla}\left\{\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r'}|}\right\} d\tau'$$

Now with the definitions (35) equations (33) become

$$(\lambda + 2\mu)\frac{\partial}{\partial x}(\nabla^2\phi) + \mu\left(\frac{\partial}{\partial y}\nabla^2\psi_z - \frac{\partial}{\partial z}\nabla^2\psi_y\right) + \left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Psi_z}{\partial y} - \frac{\partial\Psi_y}{\partial z}\right) = 0 \qquad \text{etc.}$$

and particular solutions of these equations can be found from particular solutions of the set of equations

$$\begin{aligned}(\lambda + 2\mu)\nabla^2\phi + \Phi &= 0\\ \mu\nabla^2\psi_x + \Psi_x &= 0\\ \mu\nabla^2\psi_y + \Psi_y &= 0\\ \mu\nabla^2\psi_z + \Psi_z &= 0\end{aligned}$$

which are all examples of Poisson's equation.

Thus the determination of a particular integral is again reduced to a problem mathematically identical to the determination of a Newtonian potential. It will have been noted that the simple case considered first is but a special case of the more general problem, the solution for which was first given by LORD KELVIN (1848).

In any problem the state represented by the particular integral will depend for its maintenance on a system of surface forces as well as the given temperature distribution and body forces. These forces are easily calculated. Thus in using Navier's equations to find the complimentary solutions we must introduce extra boundary forces equal and opposite to those required by the particular integral. This however is just a problem in the ordinary theory of elasticity and can be treated by standard methods. One of the first clear expositions on this method of solution applied to the thermoelastic problem was given by GOODIER (1937).

The Boussinesq-Papkovich solution.

In this method of solution it is assumed that the equation (33) has a solution of the form

$$\underset{\sim}{u} = A \text{ grad } (\phi + \underset{\sim}{r}.\underset{\sim}{\psi}) + B\underset{\sim}{\psi}$$

where ϕ and $\underset{\sim}{\psi}$ are the Boussinesq-Papkovich potentials. It can then be shown (See e.g. SNEDDON and BERRY (1958) p.90) that the displacement can be written as

$$\underset{\sim}{u} = \text{grad } (\phi + \underset{\sim}{r}.\underset{\sim}{\psi}) - 4(1 - \nu)\underset{\sim}{\psi}$$

as long as ϕ and $\underset{\sim}{\psi}$ satisfy the equations

$$4(1 - \nu)\mu\nabla^2\phi + (\underset{\sim}{r}.\underset{\sim}{P}) = 0$$

$$4(1 - \nu)\mu\nabla^2\underset{\sim}{\psi} - \underset{\sim}{P} = 0 \qquad (36)$$

where ν is the Poisson ratio $\nu = \lambda/2(\lambda + \mu)$.

The problem is thus reduced to the solution of equations (36), several simple cases of which are of particular interest (See SNEDDON and BERRY (1958) p.91).

Green's method and the Boussinesq logarithmic potential.

It is convenient in some problems to introduce yet another potential function: the Boussinesq logarithmic potential χ of the form, for example

$$\chi(\underset{\sim}{r}) = \int_{\tau'} P(\underset{\sim}{r}') \log (|\underset{\sim}{r} - \underset{\sim}{r}'| + z) \, d\tau' \qquad (37)$$

and it is well known that Green's functions can often be conveniently used in the solution of partial differential equations. Rather than describe the use of these devices here, we refer to a paper by STERNBERG and McDOWELL (1957) in which the problem of the half-space subjected to a known temperature distribution on a part of its boundary is treated by a combination of these methods. In essence the treatment consists of the following operations.

(a) The determination, using the Green's function, of the particular solution of the thermoelastic equations due to a unit point source of temperature at a point on the boundary. In this case the Boussinesq -Papkovich potentials ϕ and $\underset{\sim}{\psi}$ are easily found, so that the expressions for the displacement and stress fields can be written down.

(b) The determination of the solution corresponding to an arbitrary surface temperature distribution by an integration over the boundary.

It is found that all the integral expressions in this solution can be expressed in terms of a function χ (and its derivatives) where

$$\chi(\underset{\sim}{x}) = \int_{\sigma'} f(\underset{\sim}{x}') \log (|\underset{\sim}{x} - \underset{\sim}{x}'| + z) \, d\sigma'$$

Here $f(\underset{\sim}{x}')$ is the form of the known surface temperature distribution and the integration is taken over the boundary.

Thus the problem is reduced to the determination of the Boussinesq logarithmic potential for a disc whose mass density corresponds to the given temperature distribution. Exactly the same potential arrises in Boussinesq's problem of the half-space subjected to surface tractions which have the same form as the surface temperature distribution in the present problem. Thus the solutions to several important thermoelastic problems can be obtained using the logarithmic potentials derived by LOVE (1929) in solving the corresponding Boussinesq problems.

1.9 BIOT's Solutions of the Equations of Thermoelasticity.

From (20) the quasi-static thermoelastic equations, in the absence of body forces and heat sources, are

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu)\ \text{grad}\,\Delta - \gamma\ \text{grad}\,\theta = 0$$

$$\kappa\nabla^2\theta = \dot{\theta} + \gamma'\dot{\Delta} \qquad (38)$$

Also, upon dividing throughout (18) by ρc, we get

$$\theta = \frac{\gamma'}{\gamma} s - \gamma'\Delta \qquad (39)$$

and substitution of this expression into (38) gives

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu + \gamma\gamma')\ \text{grad}\,\Delta - \gamma'\ \text{grad}\,s = 0$$

$$\kappa\nabla^2 s - \kappa\gamma\nabla^2\Delta = \dot{s} \qquad (40)$$

By applying the operator grad throughout the first of (40), we find that

$$(\lambda + 2\mu + \gamma\gamma')\nabla^2\Delta - \gamma'\nabla^2 s = 0 \qquad (41)$$

Finally we take the first of equations (40) together with the equation obtained by eliminating Δ between (41) and the second of equations (40). In this way we get the equations

$$\mu\nabla^2 \underset{\sim}{u} + (\lambda + \mu + \gamma\gamma')\ \text{grad}\,\Delta - \gamma'\ \text{grad}\,s = 0$$

$$\frac{\kappa(\lambda + 2\mu)}{\lambda + 2\mu + \gamma\gamma'}\nabla^2 s = \dot{s} \qquad (42)$$

Thus $\underset{\sim}{s}$ satisfies the diffusion equation, which may be interpreted by saying that disorder is propagated by a process of diffusion.

A solution for the displacement field can be found by introducing the Boussinesq-Papkovich potentials (see section 1.8). We write the displacement field in the form

$$\underset{\sim}{u} = -\text{grad}\,(\phi + \underset{\sim}{x}.\underset{\sim}{\psi}) + B\underset{\sim}{\psi} \qquad (43)$$

where

$$\nabla^2 \underset{\sim}{\psi} = 0$$

and

$$B = \frac{2(\lambda + 2\mu + \gamma\gamma')}{(\lambda + \mu + \gamma\gamma')} \tag{44}$$

Substituting expression (43) into the first of (42) we obtain the equation

$$\text{grad}\left\{(\lambda + 2\mu + \gamma\gamma')\nabla^2\phi + \gamma' s\right\} = 0 \tag{45}$$

The grad operator may be dropped since this amounts to adding a constant to the right hand side of the equation, hence also to adding to ϕ a quadratic function of the coordinates. However this is also equivalent to adding a linear function of the coordinates to $\underset{\sim}{\psi}$. Thus

$$(\lambda + 2\mu + \gamma\gamma')\nabla^2\phi + \gamma' s = 0 \tag{46}$$

and hence, because of the last of (42)

$$\nabla^2(K_0\nabla^2 - \frac{\partial}{\partial t})\phi = 0$$

where

$$K_0 = \frac{\mu(\lambda + 2\mu)}{\lambda + 2\mu + \gamma\gamma'}$$

Finally we may write ϕ in the form $\phi = \phi_1 + \phi_2$ where ϕ_1 and ϕ_2 satisfy

$$\nabla^2\phi_1 = 0$$

$$(K_0\nabla^2 - \frac{\partial}{\partial t})\phi_2 = 0 \tag{47}$$

Wave propagation.

CHADWICK (1959) noted that if we express the displacement vector $\underset{\sim}{u}$ as the sum of irrotational and solenoidal components

$$\underset{\sim}{u} = \underset{\sim}{\nabla\phi} + \text{curl } \underset{\sim}{A} \tag{48}$$

then in the absence of body forces and heat sources, equations (20) are equivalent to

$$\rho \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = \mu \nabla^2 \underset{\sim}{A} \tag{49}$$

$$\rho \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \phi - \gamma \text{ grad } \theta \tag{50}$$

$$\kappa \nabla^2 \theta = \dot{\theta} + \gamma' \frac{\partial}{\partial t} \nabla^2 \phi \tag{51}$$

Equation (48) is used in the theory of elastic waves to separate the longitudinal and transverse modes. The scalar potential function ϕ describes compressional waves in which volume changes occur and particle motions are parallel to the direction of propagation. The vector potential $\underset{\sim}{A}$ generates shear waves which produce no volume changes but are subject to polarization. We see from (49) that shear waves are not affected by the ability of the medium to conduct heat. Equations (50) and (51) show that purely elastic compressional waves are modified by thermal straining and that, conversely, some of the mechanical energy expended in volume changes is converted into heat.

We shall look at the effect of thermal properties on the propagation of certain types of wave in Chapter Five.

1.10 The Method of Integral Transforms.

If the function $f(x)$ is defined in the range $[a,b]$ then we can define a function $\bar{f}(\xi)$ where

$$\bar{f}(\xi) = \int_a^b f(x)\, K(x,\xi)\, dx \tag{52}$$

Here the function $K(x,\xi)$ is an arbitrary kernel and $\bar{f}(\xi)$ is called the integral transform of $f(x)$ with respect to the kernel $K(x,\xi)$. If the integral (52) is divergent then we say that the transform of $f(x)$ with respect to $K(x,\xi)$ does not exist. This idea is important if there exists some inverse relationship expressing $f(x)$ as an integral transform of $\bar{f}(\xi)$. That is, if we can write

$$f(x) = \int_{\alpha}^{\beta} \bar{f}(\xi)\, K_1(x,\xi)\, d\xi \tag{53}$$

Transforms of the type (52) are given different names depending on the nature of the kernel $K(x,\xi)$. In this thesis we shall use five types of transform, and these are listed below along with their inverse transforms.

1) Fourier sine transform.

$$\begin{aligned} \bar{f}(\xi) &= (2/\pi)^{1/2} \int_0^{\infty} f(x) \sin(\xi x)\, dx \\ f(x) &= (2/\pi)^{1/2} \int_0^{\infty} \bar{f}(\xi) \sin(\xi x)\, d\xi \end{aligned} \tag{54}$$

2) Fourier cosine transform.

$$\begin{aligned} \bar{f}(\xi) &= (2/\pi)^{1/2} \int_0^{\infty} f(x) \cos(\xi x)\, dx \\ f(x) &= (2/\pi)^{1/2} \int_0^{\infty} \bar{f}(\xi) \cos(\xi x)\, d\xi \end{aligned} \tag{55}$$

3) Complex Fourier transform.

$$\begin{aligned} \bar{f}(\xi) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx \\ f(x) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \bar{f}(\xi)\, e^{-i\xi x}\, d\xi \end{aligned} \tag{56}$$

4) Hankel transform.

$$\bar{f}(\xi) = \int_0^{\infty} x f(x)\; J_\nu(\xi x)\; dx$$
$$f(x) = \int_0^{\infty} \xi \bar{f}(\xi)\; J_\nu(\xi x)\; d\xi \qquad (57)$$

5) Laplace transform.

$$\bar{f}(\xi) = \int_0^{\infty} f(x)\; e^{-\xi x}\; dx$$
$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(\xi)\; e^{\xi x}\; d\xi \qquad (58)$$

where $\underline{c}$ is greater than the real parts of all singularities of $\bar{f}(\xi)$.

The transforms described above are one-dimensional transforms. If we have a function $f(x_1,x_2,\ldots x_n)$ then we can define a multiple transform

$$\bar{f}(\xi_\alpha,\xi_\beta,\ldots\xi_\gamma) = \int_{a_\alpha}^{b_\alpha}\int_{a_\beta}^{b_\beta}\cdots\int_{a_\gamma}^{b_\gamma} K_\alpha(x_\alpha,\xi_\alpha)K_\beta(x_\beta,\xi_\beta)\ldots K_\gamma(x_\gamma,\xi_\gamma)f(x_1,x_2,\ldots x_n)\; dx_\alpha dx_\beta \ldots dx_\gamma \qquad (59)$$

where we have applied a transform of the type (52) to some or all of the variables $x_1,x_2,\ldots x_n$. The individual transforms are chosen from (54)-(58) to suit the particular problem, and the transform inverse to (59) is built up in a similar way from the transforms inverse to those making up (59).

As a simple illustration of the use of integral transforms in solving partial differential equations we shall consider the simple heat conduction equation

$$\kappa \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} \qquad (60)$$

and we shall suppose the boundary conditions to be

$$\theta(0,t) = \theta_0(t)$$
$$\theta \to 0,\ \partial\theta/\partial x \to 0 \quad \text{as } x \to \infty \tag{61}$$
$$\theta(x,0) = 0$$

Then by multiplying throughout (60) by $(2/\pi)^{1/2}\sin(\xi x)\,e^{-pt}$ and applying the integrations $\int_0^\infty \int_0^\infty dx\,dt$ we get

$$\kappa(2/\pi)^{1/2}\int_0^\infty e^{-pt}dt\int_0^\infty \frac{\partial^2\theta}{\partial x^2}\sin(\xi x)dx = (2/\pi)^{1/2}\int_0^\infty e^{-pt}\frac{\partial\theta}{\partial t}dt\int_0^\infty \sin(\xi x)dx \tag{62}$$

which can be reduced using integration by parts to read

$$\Phi - \xi^2\kappa\bar{\bar{\theta}} = p\bar{\bar{\theta}} \tag{63}$$

where $\bar{\bar{\theta}}$ is the multiple transform

$$\bar{\bar{\theta}}(\xi,p) = (2/\pi)^{1/2}\int_0^\infty \theta(x,t)\sin(\xi x)\,dx\int_0^\infty e^{-pt}\,dt \tag{64}$$

and

$$\Phi = \kappa(2/\pi)^{1/2}\xi\int_0^\infty \theta_0\,e^{-pt}\,dt$$

Thus

$$\bar{\bar{\theta}} = \Phi/(\xi^2\kappa + p) \tag{65}$$

and using the transform inverse to (64) we finally have the solution

$$\theta(x,t) = (2/\pi)^{1/2}\frac{1}{2\pi i}\int_0^\infty \sin(\xi x)\,d\xi\int_{c-i\infty}^{c+i\infty}\frac{\Phi}{\xi^2\kappa + p}e^{pt}\,dp \tag{66}$$

in terms of the function Φ which can be evaluated from the known function θ_0.

It will have been noticed that the application of the integral transform reduced the differential equation (60) in θ to an algebraic equation (63) in $\bar{\theta}$. In many applications it is possible to reduce a set of differential equations to a set of algebraic equations which are easier to solve. The required quantities are then obtained from their transforms by applying the inverse transforms.

Of course, it is not always possible to find a transform suitable for application to any given problem, but we shall see how some problems in thermoelasticity can be tackled directly by this method. The medium considered in the example above occupied the region $0 \leq x \leq \infty$ and the problem was therefore suitable for a transform over that range. Some problems concerning finite bodies can be tackled using finite transforms i.e. transforms of the type (52) where the integration is taken over a finite range. This will be mentioned again in Chapter Four.

For more exhaustive treatments of the use of integral transforms the reader is referred to TRANTER (1951), SNEDDON (1957), etc.

1.11 The Literature.

The consideration of thermal stresses in practical applications has become of much greater importance within the last few years, and this is reflected in the recent increase in the number of publications on this subject. In this section we shall endeavour to give an impression of the work which has been done already in this field. It cannot, however, be thought of as a complete coverage, and should only be looked upon as a guide to the more important contributions made up to the time of writing.

In the steady-state theory McDOWELL and STERNBERG (1957) have considered the thermal stresses and displacements in a spherical shell due to an arbitrary axisymmetric distribution of surface temperature, obtaining their solutions in the form of a series. Apart from this the only other steady-state problems which have attracted much attention are those of the semi-infinite medium, and the thick plate, in which the boundaries are free from stress but are subjected to a known steady temperature distribution. Solutions for the problem of the semi-infinite medium were obtained by STERNBERG and McDOWELL (1957) and for the plate problem by McDOWELL (1957), the method of solution in each case being a combination of the use of the Green's function and the Boussinesq logarithmic potential. As a particular example they supposed the boundary to be subjected to a certain circular region of exposure. Their results, which appeared in the form of elliptic integrals, were not particularly suited to numerical interpretation.

These two problems were also considered by MUKI (1956) who obtained solutions for the unsymmetrical problems, and by KNOPS (1959) who was able to obtain solutions to some particular problems by taking the difference between two isothermal elastic solutions for problems which differed only in the value of Poisson's ratio for the material. The plate problem has also been tackled by SHARMA (1956) using a direct integration of the governing equations, and NOWACKI (1957a and 1957b) used a thermoelastic displacement potential to solve the problems of the infinite and semi-infinite spaces subjected to a given temperature field. Recently SNEDDON and LOCKETT (1959a) were able to obtain solutions to these problems by using two-dimensional integral transforms. Their solutions, which were in the form of two-dimensional inverse transforms, were found to reduce to a form suitable for numerical work in the special cases considered, and the three dimensional analogues of the isochromatic lines used in photoelasticity were derived from them. These authors also considered (1959b) the elastic layer resting on a rigid foundation and then subjected to thermal conditions.

We have defined the 'classical' solution of the dynamical thermo-elastic problem to be the solution obtained when the linking term $\partial\Delta/\partial t$ is omitted from the heat conduction equation. The work of GOODIER

(1937) is applicable to these equations and has been used by MINDLIN and CHENG (1950) to examine the half-space problem. Thermal shock in the half-space has also been treated by DANILOVSKAYA (1950 and 1952) and by IGNACZAK (1957), the latter using the Fourier transform as the mathematical tool. NOWACKI (1957c) considered the action of time dependent heat sources in the infinite medium, and obtained solutions to the cylindrically symmetrical problem by using a thermoelastic potential and integral transforms. In his treatment of the plate problem SHARMA (1956) also considered the classical solution of the dynamical problem. Finally STERNBERG (1957) and STERNBERG and CHAKRAVORTY (1958) considered the thermal shock in an infinite elastic body when the surface of a spherical cavity in the medium was subjected to a sudden rise in temperature. In the former paper the inertia terms were neglected from the equations of motion, these effects being accounted for in the latter publication.

Finally we come to the class of solutions obtained by using the complete linked equations which were formulated by BIOT (1956), and which have been derived in different forms or by different approaches by CHADWICK (1959), LESSEN (1956) and others. Simple solutions of these equations can be found by considering wave propagation of a prescribed form. Plane thermoelastic waves have been treated by CHADWICK and SNEDDON (1958), DERESIEWICZ (1957) and LESSEN (1959), the last named author showing how other solutions can be built up from these simple solutions and also giving a brief discussion of thermal shock in the half-space and in the infinite cylinder. LOCKETT (1958) examined the thermoelastic equivalent of the classical Rayleigh surface waves and CHADWICK (1959) considered the propagation of longitudinal waves in solid circular cylinders. This latter problem was also considered by LOCKETT (1959a) who also considered propagation in hollow cylinders and in the infinite medium with a cylindrical cavity. SNEDDON (1959a) treated the problem of the propagation of thermal stresses in thin metallic rods, whilst ZORSKI (1958) was concerned with stress propogation in an infinite space due to the action of a thermal impulse.

The solution of the linked problem for the infinite space has been treated by several authors. NOWACKI (1959) obtained solutions to the special class of problems having exponential time dependence and cylindrical symmetry. General solutions were obtained using integral transforms by EASON and SNEDDON (1959) and LOCKETT and SNEDDON (1959/60), the former considering the action of arbitrary heat sources and the latter considering arbitrary body forces. EASON and SNEDDON (1959) and NOWACKI (1959a) were able to obtain solutions to some special problems of the half-space. PARIA (1959) considered the cylindrically symmetrical problem of the half-space free from surface traction and LOCKETT (1959b) has obtained a formal solution to the problem of the half-space submitted to arbitrary heat sources, body forces, surface tractions and thermal boundary conditions.

These are the main contributions to the solution of the thermo-elastic problem. Both CHADWICK (1959) and SNEDDON (1959b) have reviewed some of this work, and for the sake of completeness it should be mentioned that CHU (1957) and NOWACKI (1959b) have extended the basic theory to visco-elasticity, the former also considering the extension to finite deformation in the theory of thermoelasticity.

CHAPTER TWO.

STEADY-STATE PROBLEMS.

2.1 Introduction.

In Chapter One it was seen that the term linking the heat conduction equation with the thermoelastic equations of motion does not appear in the steady-state equations. The following chapters of this thesis will be concerned with dynamical problems in which this term does occur. However partly for the sake of completeness, we shall discuss here a steady-state problem - the effect of imposing a steady temperature distribution on the surface(s) of a semi-infinite medium and a thick plate. These problems have been considered before - the case of the semi-infinite medium by STERNBERG and McDOWELL (1957) and the thick plate by McDOWELL (1957). In each case the problem was tackled by the use of the Green's function.

The analysis presented in this chapter is due to SNEDDON and LOCKETT (1959). The results of STERNBERG and McDOWELL are rediscovered by a simpler analysis, which is suited better to numerical work. A special case discussed by these authors and other cases are considered, and in each case the three-dimensional analogue of the isochromatic lines is constructed.

2.2 General Solutions.

For the steady-state type of problem considered in this chapter the thermoelastic field equations (I-26) reduce to the form

$$\nabla^2 \underset{\sim}{u} + (\beta^2 - 1) \operatorname{grad} \Delta = b \operatorname{grad} \theta \tag{1}$$

$$\nabla^2 \theta = 0 \tag{2}$$

where $\underset{\sim}{u} = (u, v, w)$ denotes the displacement vector with components in the (x, y, z) directions. and $\underset{\sim}{\theta}$ and $\underset{\sim}{\Delta}$ have the meanings assigned to them previously.

If we multiply throughout equation (2) by the expression

$(2\pi)^{-1}\exp\left\{i(\xi x + \eta y)\right\}$, integrate over x and y from $-\infty$ to $+\infty$, and introduce the notation $\bar{\bar{f}}$ for the two-dimensional Fourier transform

$$\bar{\bar{f}}(\xi, \eta, z) = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} f(x, y, z)\, e^{i(\xi x + \eta y)}\, dx\, dy \tag{3}$$

equation (2) transforms to

$$(D^2 - \zeta^2)\,\bar{\bar{\theta}} = 0 \tag{4}$$

where $D = d/dz$, $D^2 = d^2/dz^2$, and $\zeta^2 = \xi^2 + \eta^2$.

Equation (4) shows that the transform of the temperature can be expressed in the form

$$\bar{\bar{\theta}} = Ee^{-\zeta z} + E'e^{\zeta z} \tag{5}$$

Similarly we can transform equations (1) to obtain the set

$$\begin{aligned}
&(D^2 - \beta^2\xi^2 - \eta^2)\bar{\bar{u}} - (\beta^2 - 1)\xi\eta\bar{\bar{v}} - (\beta^2 - 1)i\xi D\bar{\bar{w}} = -bi\xi\,\bar{\bar{\theta}} \\
&-(\beta^2 - 1)\xi\eta\bar{\bar{u}} + (D^2 - \xi^2 - \beta^2\eta^2)\bar{\bar{v}} - (\beta^2 - 1)i\eta D\bar{\bar{w}} = -bi\eta\,\bar{\bar{\theta}} \\
&-(\beta^2 - 1)i\xi D\bar{\bar{u}} - (\beta^2 - 1)i\eta D\bar{\bar{v}} + (\beta^2 D^2 - \xi^2 - \eta^2)\bar{\bar{w}} = bD\bar{\bar{\theta}}
\end{aligned} \tag{6}$$

and by eliminating the unknowns we find that $\bar{\bar{u}}$, $\bar{\bar{v}}$, $\bar{\bar{w}}$ satisfy the equations

$$\begin{aligned}
\beta^2(D^2 - \zeta^2)^3\bar{\bar{u}} &= -ib\xi(D^2 - \zeta^2)^2\bar{\bar{\theta}} \\
\beta^2(D^2 - \zeta^2)^3\bar{\bar{v}} &= -ib\eta(D^2 - \zeta^2)^2\bar{\bar{\theta}} \\
\beta^2(D^2 - \zeta^2)^3\bar{\bar{w}} &= bD(D^2 - \zeta^2)^2\bar{\bar{\theta}}
\end{aligned} \tag{7}$$

Thus because of (4), $\bar{u}$, $\bar{v}$, $\bar{w}$ all satisfy the equation

$$(D^2 - \zeta^2)^3 f = 0 \tag{8}$$

so they can be written in the form

$$\begin{aligned}
\bar{u} &= (A_1 + B_1 z + C_1 z^2)\, e^{-\zeta z} + (A_1' + B_1' z + C_1' z^2)\, e^{\zeta z} \\
\bar{v} &= (A_2 + B_2 z + C_2 z^2)\, e^{-\zeta z} + (A_2' + B_2' z + C_2' z^2)\, e^{\zeta z} \\
\bar{w} &= (A_3 + B_3 z + C_3 z^2)\, e^{-\zeta z} + (A_3' + B_3' z + C_3' z^2)\, e^{\zeta z}
\end{aligned} \tag{9}$$

It is obvious that all of these eighteen coefficients cannot be described arbitrarily. If we substitute the expressions (9) back into (6) and equate to zero all the coefficients of $e^{-\zeta z}$, $e^{\zeta z}$, $ze^{-\zeta z}$, $ze^{\zeta z}$ $z^2 e^{-\zeta z}$, $z^2 e^{\zeta z}$ (since the equations must be satisfied for all values of z) then we get the set of equations

$$-\frac{2(C_1 - \zeta B_1)}{(\beta^2 - 1)\xi} + \xi A_1 + \eta A_2 + i(-\zeta A_3 + B_3) = \frac{ibE}{\beta^2 - 1}$$

$$\frac{4C_1\zeta}{(\beta^2 - 1)\xi} + \xi B_1 + \eta B_2 + i(-\zeta B_3 + 2C_3) = 0$$

$$\xi C_1 + \eta C_2 - i\zeta C_3 = 0 \; ^*$$

$$-\frac{2(C_2 - \zeta B_2)}{(\beta^2 - 1)\eta} + \xi A_1 + \eta A_2 + i(-\zeta A_3 + B_3) = \frac{ibE}{\beta^2 - 1}$$

$$\frac{4C_2\zeta}{(\beta^2 - 1)\eta} + \xi B_1 + \eta B_2 + i(-\zeta B_3 + 2C_3) = 0 \tag{10}$$

an equation identical with *

$$-\frac{2i\beta^2(C_3 - \zeta B_3)}{(\beta^2 - 1)\zeta} + \xi A_1 + \eta A_2 - i\zeta A_3 - \frac{\xi B_1 + \eta B_2}{\zeta} = \frac{ibE}{\beta^2 - 1}$$

$$\frac{4i\beta^2 C_3}{(\beta^2 - 1)} + \xi B_1 + \eta B_2 - i\zeta B_3 - \frac{2(\xi C_1 + \eta C_2)}{\zeta} = 0$$

an equation identical with *

plus a further nine equations which can be obtained from (10) by replacing each coefficient by the corresponding dashed quantity and ζ by $-\zeta$.

From these equations it can be shown that

$B_1 = \xi P$, $B_2 = \eta P$, $B_3 = -i\zeta P$, $C_1 = C_2 = C_3 = 0$

where

$$P = -\frac{(\beta^2 - 1)}{(\beta^2 + 1)\zeta}\left\{ \xi A_1 + \eta A_2 - i\zeta A_3 - \frac{ibE}{\beta^2 - 1} \right\} \qquad (11)$$

and there is a similar set of equations for the dashed quantities.

Hence the solutions (9) can be finally written in the form

$$\begin{aligned} \bar{u} &= (A_1 + \xi Pz)\, e^{-\zeta z} + (A_1' + \xi P' z)\, e^{\zeta z} \\ \bar{v} &= (A_2 + \eta Pz)\, e^{-\zeta z} + (A_2' + \eta P' z)\, e^{\zeta z} \\ \bar{w} &= (A_3 - i\zeta Pz)\, e^{-\zeta z} + (A_3' + i\zeta P' z)\, e^{\zeta z} \end{aligned} \qquad (12)$$

where P is given by (11) and

$$P' = \frac{(\beta^2 - 1)}{(\beta^2 + 1)\zeta}\left\{ \xi A_1' + \eta A_2' + i\zeta A_3' - \frac{ibE'}{\beta^2 - 1} \right\} \qquad (13)$$

The equations (12) are, of course, only suitable for problems in which the solid region under consideration is bounded by planes normal to the z-axis. In such problems it is possible to find the coefficients A_j, A_j' by applying the transformed boundary conditions to (12). The physical quantity can then be obtained from these equations by applying the transform inverse to (3). We shall obtain solutions to the half-space problem and the thick plate problem in the following sections.

2.3 Solution for the Half-Space.

If we assume that the components of the displacement vector and the temperature distribution each tend to zero as $z \to \infty$, the solution corresponding to the half-space $z \geq 0$ is

$$\bar{\theta} = E\, e^{-\zeta z}$$

$$\bar{u} = (A_1 + \xi P z)\, e^{-\zeta z}$$

$$\bar{v} = (A_2 + \eta P z)\, e^{-\zeta z} \qquad (14)$$

$$\bar{w} = (A_3 - i\zeta P z)\, e^{-\zeta z}$$

where A_1, A_2, A_3 are to be determined from the boundary conditions and P is given by equation (11).

Using equations (I-27) and (14) the transforms of the stress components τ_{xz}, τ_{yz}, σ_z can be written

$$\bar{\tau}_{xz} = (-\zeta A_1 - \xi\zeta P z + \xi P - i\xi A_3 - \xi\zeta P z)\, e^{-\zeta z}$$

$$\bar{\tau}_{yz} = (-\zeta A_2 - \eta\zeta P z + \eta P - i\eta A_3 - \eta\zeta P z)\, e^{-\zeta z} \qquad (15)$$

$$\bar{\sigma}_z = \left\{ -i(\beta^2 - 2)(\xi A_1 + \xi^2 P z + \eta A_2 + \eta^2 P z) + \beta^2(-\zeta A_3 + i\zeta^2 P z - i\zeta P) - bE \right\} e^{-\zeta z}$$

Thus, if the boundary $z = 0$ is free from stress we have the set of equations

$$\zeta A_1 - \xi P + i\xi A_3 = 0$$

$$\zeta A_2 - \eta P + i\eta A_3 = 0 \qquad (16)$$

$$i(\beta^2 - 2)(\xi A_1 + \eta A_2) + \beta^2(\zeta A_3 + i\zeta P) + bE = 0$$

for the determination of the A_i. The solutions of this set of equations are found to be

$$A_1 = \frac{i b E \xi}{2(\beta^2 - 1)\zeta^2} \qquad A_2 = \frac{i b E \eta}{2(\beta^2 - 1)\zeta^2} \qquad A_3 = \frac{-bE}{2(\beta^2 - 1)\zeta} \qquad (17)$$

from which it follows that $P = 0$.

If the thermal boundary condition is the imposition of a surface temperature distribution $\phi(x,y)$ then the first of (14) shows that

$$E = \bar{\phi}(\xi,\eta) \tag{18}$$

Therefore, using (14), (17) and (18), we see that

$$\bar{u} = \frac{ib\bar{\phi}\xi}{2(\beta^2 - 1)\zeta^2}\, e^{-\zeta z}, \quad \bar{v} = \frac{ib\bar{\phi}\eta}{2(\beta^2 - 1)\zeta^2}\, e^{-\zeta z}, \quad \bar{w} = \frac{-b\bar{\phi}}{2(\beta^2 - 1)\zeta}\, e^{-\zeta z} \tag{19}$$

and hence the displacement vector is given by

$$\underset{\sim}{u} = \frac{b}{4\pi(\beta^2 - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (i\xi,\ i\eta,\ -\zeta)\, \frac{\bar{\phi}}{\xi^2 + \eta^2}\, e^{-i(\xi x + \eta y) - \zeta z}\, d\xi\, d\eta \tag{20}$$

and the temperature distribution is

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\phi}(\xi,\eta)\, e^{-i(\xi x + \eta y) - \zeta z}\, d\xi\, d\eta \tag{21}$$

From the expressions (15) and (17) it is obvious that

$$\bar{\tau}_{xz} = \bar{\tau}_{yz} = \bar{\sigma}_z = 0\ , \qquad z \geq 0$$

showing that the stress field is plane and parallel to the boundary, in agreement with the result of STERNBERG and McDOWELL.

Axially symmetrical solution.

The solutions to the problem in which the prescribed surface temperature $\phi(x,y)$ is axially symmetrical could be obtained from first principles by writing the equations (1) and (2) in cylindrical polar coordinates. However it is an easy matter to obtain them directly from the solutions derived above.

Because of axial symmetry we may write $\phi(x,y) = \phi(\rho)$ where $\rho = \sqrt{(x^2 + y^2)}$. Then

$$\bar{\phi}(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^{\infty} \rho\phi(\rho)\, e^{i\zeta\rho\cos(\psi-\chi)}\, d\rho$$

where ψ is the angular coordinate and we have written $\xi = \zeta\cos\chi$, $\eta = \zeta\sin\chi$. Therefore, since

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\zeta\rho\cos(\psi-\chi)}\, d\psi = J_0(\zeta\rho)$$

we see that $\bar{\phi}(\xi,\eta) = \bar{\phi}(\zeta)$, where $\bar{\phi}(\zeta)$ denotes the zero order Hankel transform of the function $\phi(\rho)$. Denoting the components of $\underset{\sim}{u}$ in the ρ-direction by u_ρ we have

$$\begin{aligned} u_\rho &= u\cos\psi + v\sin\psi \\ &= \frac{b}{4\pi(\beta^2-1)}\int_0^{\infty}\bar{\phi}(\zeta)\, e^{-\zeta z} d\zeta \int_0^{2\pi} i\cos(\psi-\chi)\, e^{-i\zeta\rho\cos(\psi-\chi)}\, d\psi \\ &= \frac{b}{2(\beta^2-1)}\int_0^{\infty}\bar{\phi}(\zeta)\, J_1(\zeta\rho)\, e^{-\zeta z} d\zeta \end{aligned} \tag{22}$$

Also

$$w = -\frac{b}{2(\beta^2-1)}\int_0^{\infty}\bar{\phi}(\zeta)\, J_0(\zeta\rho)\, e^{-\zeta z} d\zeta \tag{23}$$

As a special example of the use of these formulae, we consider the case which is considered in some detail by STERNBERG and McDOWELL, namely that in which

$$\phi(\rho) = \begin{cases} \phi_0, & 0 \leq \rho \leq 1 \\ 0, & \rho > 1, \end{cases}$$

in which case $\bar{\phi}(\zeta) = \phi_0 J_1(\zeta)/\zeta$ and, in the notation of EASON, NOBLE and SNEDDON (1955), we have

$$u_\rho = \frac{b\phi_0}{2(\beta^2-1)} J(1,1;-1), \qquad w = -\frac{b\phi_0}{2(\beta^2-1)} J(1,0;-1) \tag{24}$$

where

$$\mathcal{J}(\mu,\nu;\lambda) = \int_0^\infty \bar{\phi}(\zeta)\, J_\nu(\zeta\rho)\, e^{-\zeta z}\, \zeta^\lambda\, d\zeta \tag{25}$$

The integrals $\mathcal{J}(1,1;-1)$ and $\mathcal{J}(1,0;-1)$ have been tabulated for ranges of values of ρ and z in Tables 9, 10, 11 on pp. 545-546 of EASON, NOBLE and SNEDDON (1955), so it is an easy matter to calculate the components of the displacement vector at any point. Figs. 1, 2 & 3 show the variations of w, θ and u_ρ in planes parallel to the boundary.

Using the formulae developed in the reference mentioned above, it is a simple matter to show that the equations (24) are in agreement with the expressions given by STERNBERG and McDOWELL.

It is of more direct interest to calculate the difference of the principal stresses. In our system of units, we have

$$\sigma_\rho - \sigma_\psi = 2\left\{\frac{\partial u_\rho}{\partial \rho} - \frac{u_\rho}{\rho}\right\}$$

so that, in the general case,

$$\sigma_\rho - \sigma_\psi = \frac{b}{\beta^2 - 1}\int_0^\infty \zeta\bar{\phi}(\zeta)\left\{J_0(\rho\zeta) - \frac{2}{\zeta\rho}J_1(\rho\zeta)\right\} e^{-\zeta z}\, d\zeta \tag{26}$$

and in the case in which $\zeta\bar{\phi}(\zeta) = \phi_0 J_1(\zeta)$

$$\sigma_\rho - \sigma_\psi = \frac{b\phi_0}{\beta^2 - 1}\left\{\mathcal{J}(1,0;0) - \frac{2}{\rho}\mathcal{J}(1,1;-1)\right\} \tag{27}$$

The values of this stress difference at a grid of points in the $z\rho$-plane can be calculated readily from the Tables 6 and 11 of EASON et al., and it is then a simple matter to plot the lines joining points with the same value of this stress difference. The resulting set of curves will be the curves obtained by cutting planes of equal maximum shearing stress by a plane ψ = constant, and will correspond to the isochromatic lines of two dimensional photoelasticity. The contours are shown in Fig. 4 for the simple problem we have considered here.

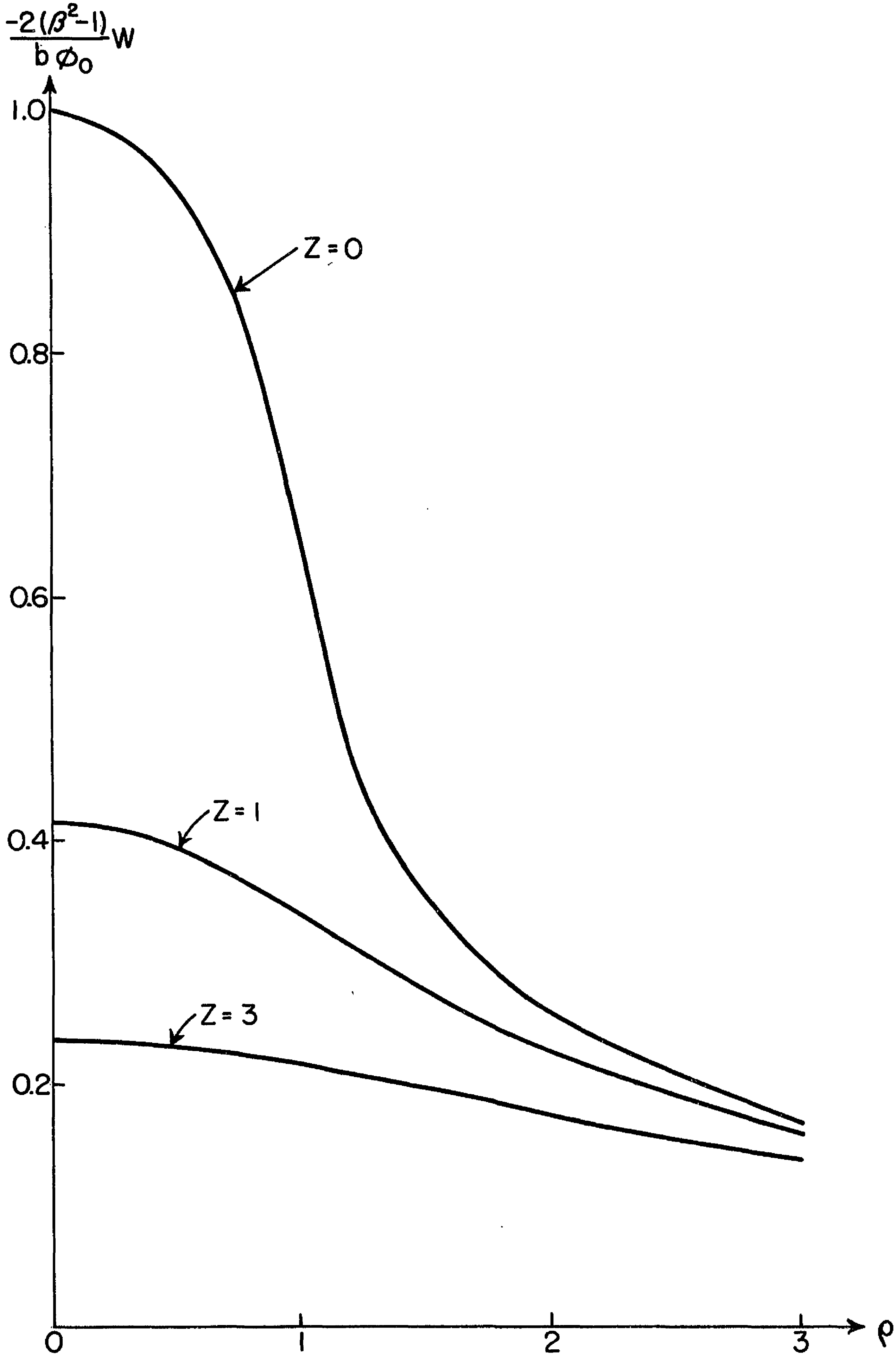


Fig. I. Variation of the normal component of the displacement in planes parallel to the boundary.

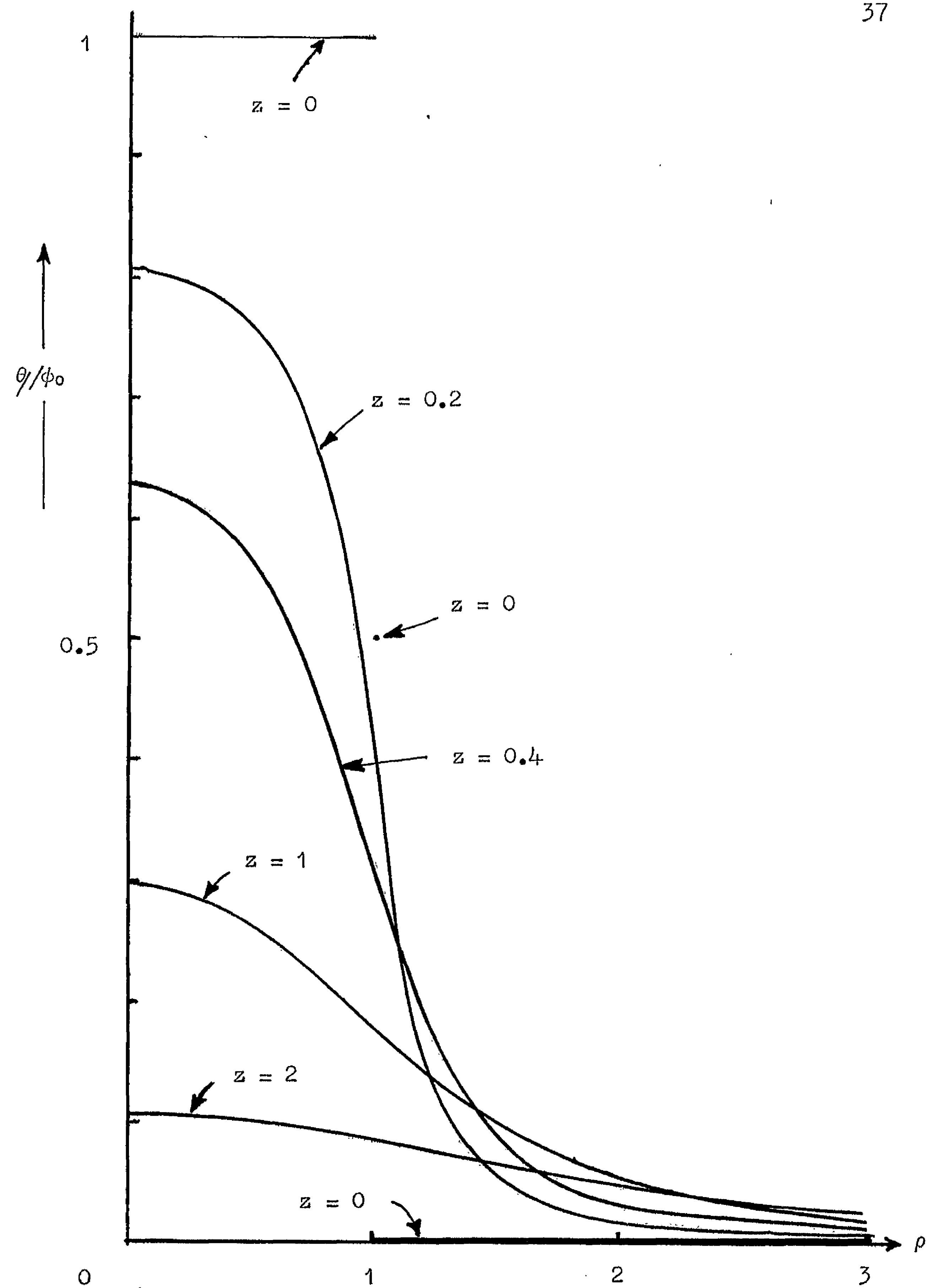


Fig.2. Temperature distribution on planes parallel to the boundary.

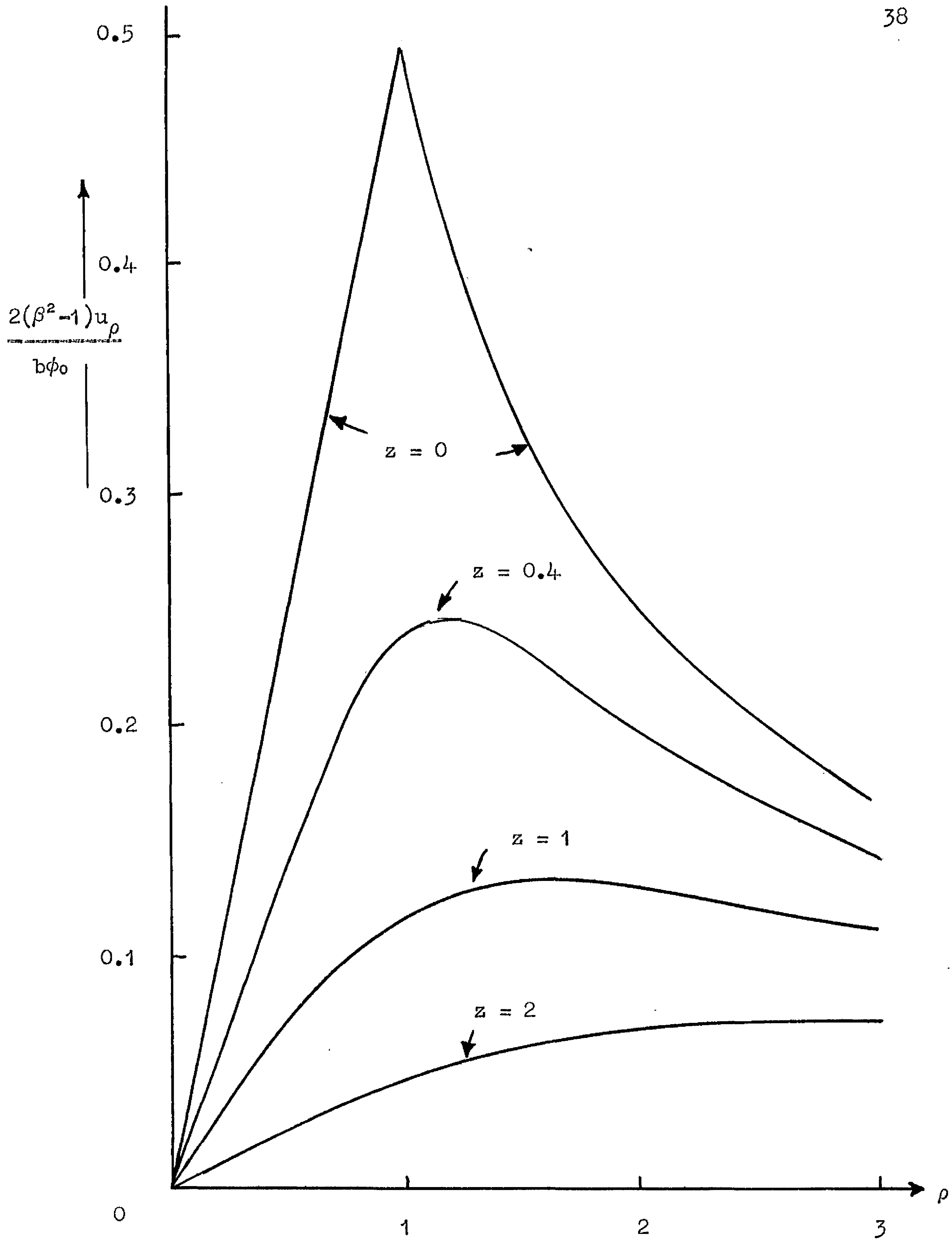


Fig. 3. Radial displacement on planes parallel to the boundary.

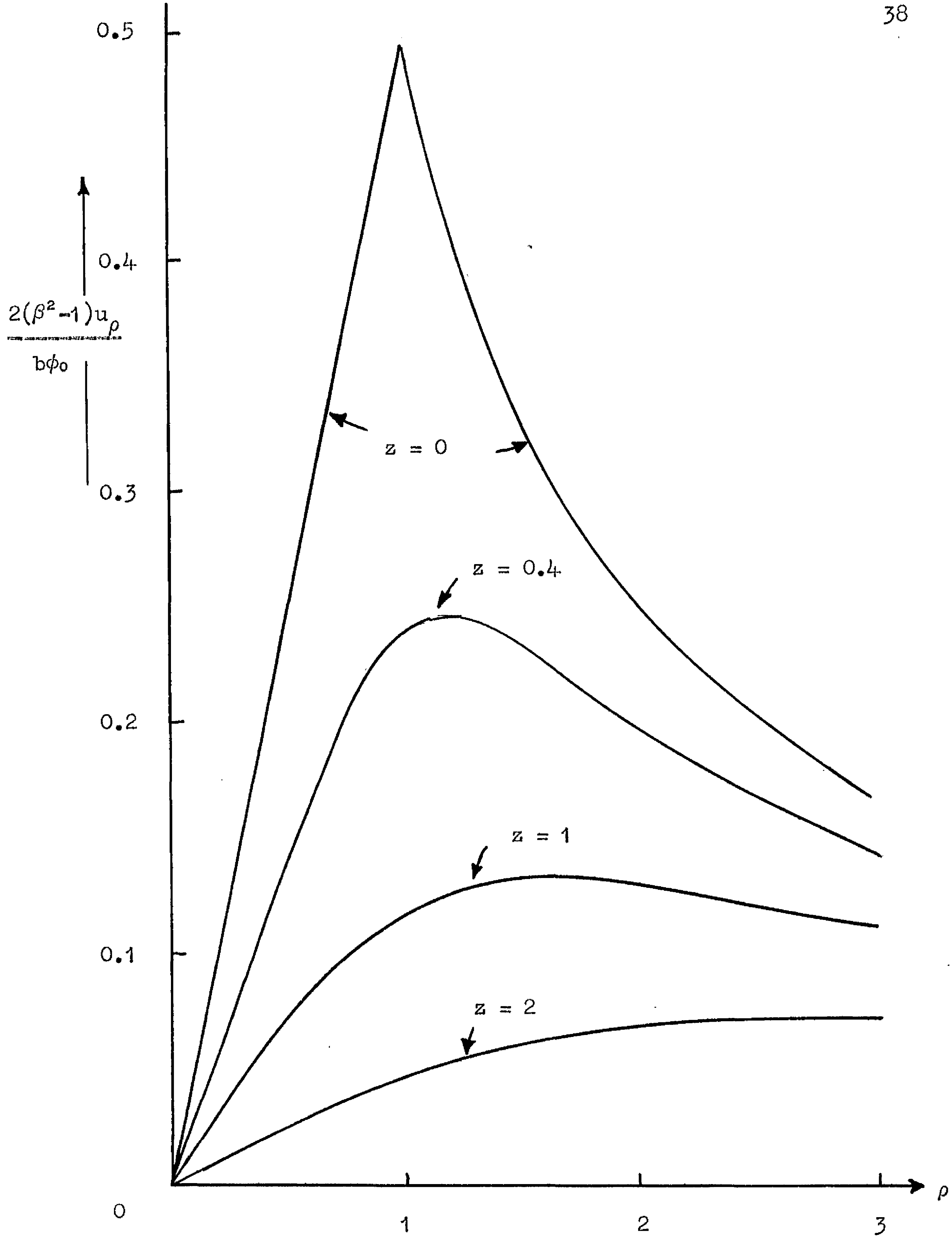


Fig. 3. Radial displacement on planes parallel to the boundary.

TABLE I. SEMI-INFINITE MEDIUM. VALUES OF $|\sigma_\rho - \sigma_\phi|$.

ρ \ z	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	3.0
0	0	0	0	0	0	0	0	0	0	0	0	0
0.2	0	0.0027	0.0039	0.0039	0.0037	0.0024	0.0021	0.0015	0.0010	0.0006	0.0005	0.0000
0.4	0	0.0131	0.0186	0.0177	0.0142	0.0105	0.0073	0.0056	0.0038	0.0028	0.0021	0.0003
0.6	0	0.0384	0.0483	0.0414	0.0313	0.0227	0.0161	0.0115	0.0082	0.0062	0.0045	0.0012
0.8	0	0.1057	0.1017	0.0760	0.0540	0.0378	0.0268	0.0193	0.0139	0.0101	0.0074	0.0020
1.0	0.5000	0.2736	0.1737	0.1147	0.0776	0.0535	0.0376	0.0268	0.0196	0.0145	0.0109	0.0031
1.2	0.6945	0.3778	0.2236	0.1437	0.0966	0.0669	0.0475	0.0340	0.0252	0.0187	0.0142	0.0043
1.4	0.5101	0.3470	0.2298	0.1551	0.1073	0.0761	0.0547	0.0404	0.0300	0.0227	0.0174	0.0053
1.6	0.3906	0.2931	0.2123	0.1525	0.1103	0.0806	0.0596	0.0446	0.0333	0.0260	0.0202	0.0066
1.8	0.3087	0.2449	0.1883	0.1426	0.1074	0.0812	0.0618	0.0473	0.0365	0.0285	0.0225	0.0066
2.0	0.2500	0.2056	0.1648	0.1300	0.1016	0.0791	0.0618	0.0485	0.0382	0.0303	0.0244	0.0087
3.0	0.1111	0.0992	0.0876	0.0766	0.0665	0.0573	0.0492	0.0420	0.0355	0.0298	0.0260	0.0119

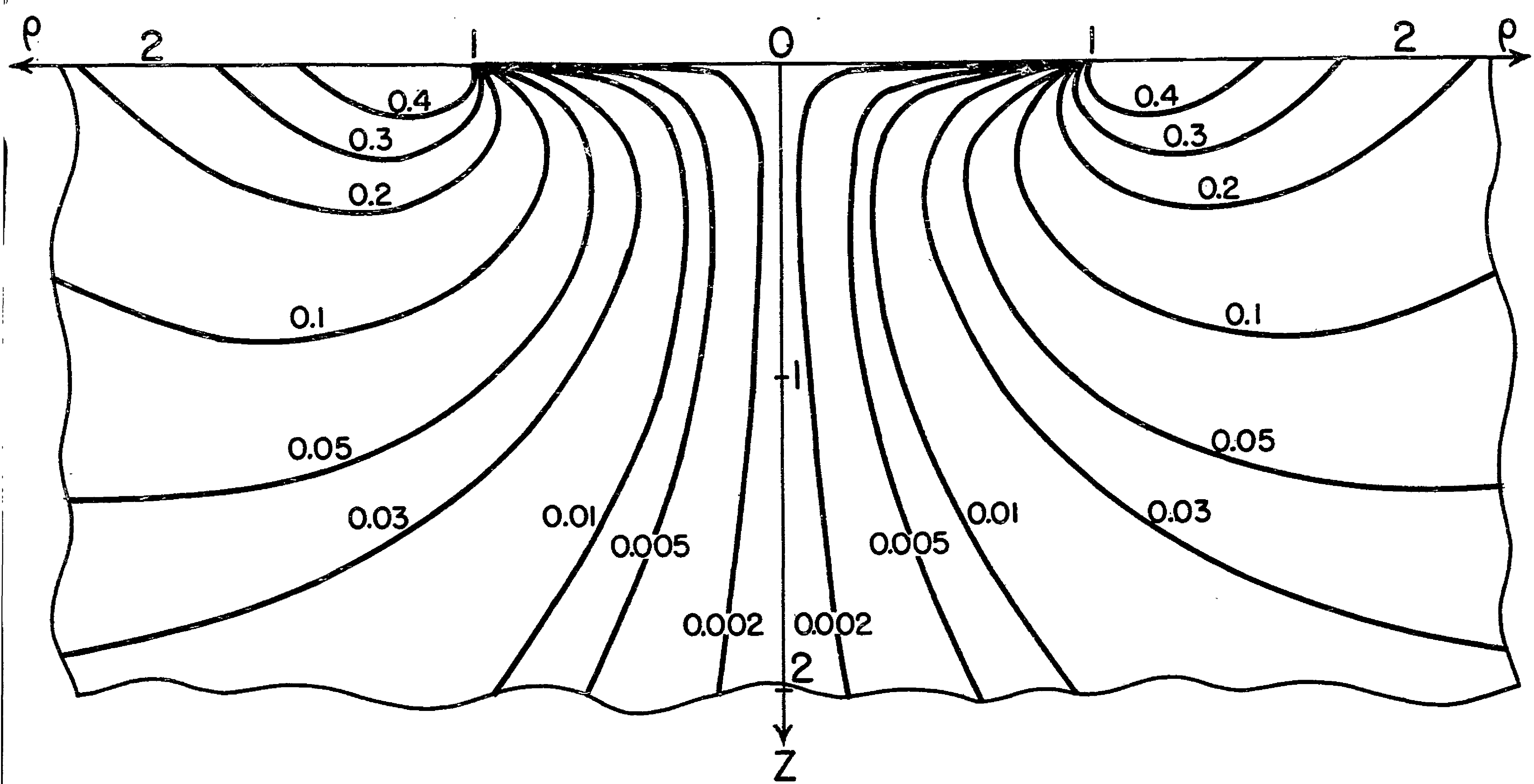


Fig. 4. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant in a semi-infinite solid. The numbers refer to the values of $(\sigma_\psi - \sigma_\rho)(\beta^2 - 1)/b\Phi_0$

2.4 Solution for the Thick Plate.

The general solutions (12) can be written in a form which is more suitable to plate problems. The transforms of the components of the displacement vector due to the temperature distribution given by

$$\bar{U} = E \cosh(\zeta z) + E' \sinh(\zeta z) \tag{28}$$

can be written in the form

$$\begin{aligned}
\bar{u} &= A_1 \cosh(\zeta z) + A_1' \sinh(\zeta z) + P' \xi z \cosh(\zeta z) + P \xi z \sinh(\zeta z) \\
\bar{v} &= A_2 \cosh(\zeta z) + A_2' \sinh(\zeta z) + P' \eta z \cosh(\zeta z) + P \eta z \sinh(\zeta z) \\
\bar{w} &= A_3 \sinh(\zeta z) + A_3' \cosh(\zeta z) + i\zeta z P' \sinh(\zeta z) + i\zeta z P \cosh(\zeta z)
\end{aligned} \tag{29}$$

where

$$P' = \frac{\beta^2 - 1}{(\beta^2 + 1)\zeta}\left\{\xi A_1' + \eta A_2' + i\zeta A_3' - \frac{ibE'}{\beta^2 - 1}\right\}, \quad P = \frac{\beta^2 - 1}{(\beta^2 + 1)\zeta}\left\{\xi A_1 + \eta A_2 + i\zeta A_3 - \frac{ibE}{\beta^2 - 1}\right\} \tag{30}$$

The components of stress τ_{xz}, τ_{yz}, σ_z due to this displacement field are given by the equations

$$\begin{aligned}
\bar{\tau}_{xz} &= (\zeta A_1 - i\xi A_3 + P\xi)\sinh(\zeta z) + 2P\xi\zeta z \cosh(\zeta z) + (\zeta A_1' - i\xi A_3' + P'\xi)\cosh(\zeta z) \\
&\quad + 2P'\xi\zeta z \sinh(\zeta z) \\
\bar{\tau}_{yz} &= (\zeta A_2 - i\eta A_3 + P\eta)\sinh(\zeta z) + 2P\eta\zeta z \cosh(\zeta z) + (\zeta A_2' - i\eta A_3' + P'\eta)\cosh(\zeta z) \\
&\quad + 2P'\eta\zeta z \sinh(\zeta z) \\
\bar{\sigma}_z &= -i(\beta^2 - 2)\left\{(A_1\xi + A_2\eta)\cosh(\zeta z) + P\zeta^2 z \sinh(\zeta z)\right\} - bE\cosh(\zeta z) + \beta^2 A_3\zeta\cosh(\zeta z) \\
&\quad + \beta^2 i\zeta P\cosh(\zeta z) + \beta^2 i\zeta^2 z P\sinh(\zeta z) - i(\beta^2 - 2)\left\{(A_1'\xi + A_2'\eta)\sinh(\zeta z) \right. \\
&\quad \left. + P'\zeta^2 z \cosh(\zeta z)\right\} - bE'\sinh(\zeta z) + \beta^2 A_3'\zeta\sinh(\zeta z) + \beta^2 i\zeta P'\sinh(\zeta z) \\
&\quad + \beta^2 i\zeta^2 z P'\cosh(\zeta z).
\end{aligned} \tag{31}$$

We now consider a plate bounded by the planes $z = \pm d$, and require that these boundaries should be free from traction. That is, we have the conditions

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0, \quad \text{on } z = \pm d.$$

Since the expressions on the right hand sides of (31) consist of the sum of an odd and an even part, these conditions lead to three pairs of equations of the type a + b = 0, a − b = 0, which imply that a = 0 and that b = 0. In this way we get the six equations

$$
\begin{aligned}
&(\zeta A_1 - i\xi A_3 + P\xi)\sinh(\zeta d) + 2P\xi\zeta d\cosh(\zeta d) = 0 \quad * \\
&(\zeta A_1' - i\xi A_3' + P'\xi)\cosh(\zeta d) + 2P'\xi\zeta d\sinh(\zeta d) = 0 \\
&(\zeta A_2 - i\eta A_3 + P\eta)\sinh(\zeta d) + 2P\eta\zeta d\cosh(\zeta d) = 0 \quad * \\
&(\zeta A_2' - i\eta A_3' + P'\eta)\cosh(\zeta d) + 2P'\eta\zeta d\sinh(\zeta d) = 0 \\
&-i(\beta^2-2)\left\{(A_1\xi + A_2\eta)\cosh(\zeta d) + P\zeta^2 d\sinh(\zeta d)\right\} - bE\cosh(\zeta d) \\
&\qquad + \beta^2 A_3\zeta\cosh(\zeta d) + \beta^2 i\zeta P\cosh(\zeta d) + \beta^2 i\zeta^2 dP\sinh(\zeta d) = 0 \quad * \\
&-i(\beta^2-2)\left\{(A_1'\xi + A_2'\eta)\sinh(\zeta d) + P'\zeta^2 d\cosh(\zeta d)\right\} - bE'\sinh(\zeta d) \\
&\qquad + \beta^2 A_3'\zeta\sinh(\zeta d) + \beta^2 i\zeta P'\sinh(\zeta d) + \beta^2 i\zeta^2 dP'\cosh(\zeta d) = 0
\end{aligned}
\tag{32}
$$

The equations marked * together with the second of (30) are sufficient for the determination of A_1, A_2, A_3. It is found that

$$A_1 = \frac{ibE\xi}{2\zeta^2(\beta^2-1)}\,, \quad A_2 = \frac{ibE\eta}{2\zeta^2(\beta^2-1)}\,, \quad A_3 = \frac{bE}{2\zeta(\beta^2-1)}\,, \quad P = 0\,. \tag{33}$$

From the remainder of (32) we find a set of expressions for the A_i' which are identical with those for the A_i if E is replaced by E′. It is then easily seen from (29) and (33) that the components of the displacement vector can be obtained by applying the inverse transformations to the expressions

$$
\begin{aligned}
\bar{\bar{u}} &= \frac{ib\xi}{2\zeta^2(\beta^2-1)}\left\{E\cosh(\zeta z) + E'\sinh(\zeta z)\right\} \\
\bar{\bar{v}} &= \frac{ib\eta}{2\zeta^2(\beta^2-1)}\left\{E\cosh(\zeta z) + E'\sinh(\zeta z)\right\} \\
\bar{\bar{w}} &= \frac{b}{2\zeta(\beta^2-1)}\left\{E\sinh(\zeta z) + E'\cosh(\zeta z)\right\}
\end{aligned}
\tag{34}
$$

Substituting the values for the constants into the equations (31), we find that

$$\bar{\tau}_{xz} = \bar{\tau}_{yz} = \bar{\sigma}_z = 0\ , \qquad -d \leq z \leq d,$$

so that the STERNBERG-McDOWELL result that the stress field induced by an arbitrary distribution of surface temperature is plane and parallel to the boundary holds for a thick plate as well as for a semi-infinite solid.

Axially symmetrical solution.

As in the case of the half-space we can easily derive the solution in the case when the temperature field is axially symmetrical. To illustrate the procedure, we shall consider the simple situation in which

$$\theta = \phi(\rho), \quad \text{on} \quad z = d, \qquad \theta = 0 \quad \text{on} \quad z = -d. \tag{35}$$

For this distribution of surface temperature we find that

$$\begin{aligned} \theta &= \int_0^\infty \xi\bar{\phi}(\xi)\, \frac{\sinh\xi(z+d)}{\sinh 2\xi d}\, J_0(\xi\rho)\, d\xi \\ u_\rho &= \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}(\xi)\, \frac{\sinh\xi(z+d)}{\sinh 2\xi d}\, J_1(\xi\rho)\, d\xi \\ w &= \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}(\xi)\, \frac{\cosh\xi(z+d)}{\sinh 2\xi d}\, J_0(\xi\rho)\, d\xi \end{aligned} \tag{36}$$

Our unit of length is unspecified as yet - we merely assumed that it was a "typical" length l. If we now take $l = d$, so that all lengths are measured as ratios of half the thickness of the plate, we find that equations (36) assume the simpler form

$$\begin{aligned} \theta &= \int_0^\infty \xi\bar{\phi}(\xi)\, \frac{\sinh\xi(z+1)}{\sinh 2\xi}\, J_0(\xi\rho)\, d\xi \\ u_\rho &= \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}(\xi)\, \frac{\sinh\xi(z+1)}{\sinh 2\xi}\, J_1(\xi\rho)\, d\xi \\ w &= \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}(\xi)\, \frac{\cosh\xi(z+1)}{\sinh 2\xi}\, J_0(\xi\rho)\, d\xi \end{aligned} \tag{37}$$

For any given distribution of temperature on the upper surface of the plate, we can calculate the zero-order Hankel transform $\bar{\phi}(\zeta)$ of the function $\phi(\rho)$, and, inserting it in equations (37), calculate the temperature field and the displacement vector within the plate. In the general case the evaluation of these integrals would be pretty complicated because of the occurrence of sinh 2ζ in the denominator of the integrand. By suitably choosing the function $\phi(\rho)$ we can, however, obtain integrals which can be easily evaluated and obtain the solution of a representative problem.

For example, if we take the surface distribution of temperature to be

$$\phi(\rho) = \frac{(k^2-4)^2\theta_0}{8k}\left\{\frac{k-2}{[\rho^2+(k-2)^2]^{3/2}} - \frac{k+2}{[\rho^2+(k+2)^2]^{3/2}}\right\} \qquad k > 2 \qquad (38)$$

then the temperature at the point $z = d$, $\rho = 0$, or in our system of units $z = 1$, $\rho = 0$, is θ_0, and the zero-order Hankel transform is

$$\bar{\phi}(\zeta) = \frac{(k^2-4)^2\theta_0}{4k}\, e^{-k\zeta}\sinh 2\zeta \qquad (39)$$

If, now, we substitute the expression (39) into (37) and evaluate the simple integrals so obtained, we find that

$$\theta = \frac{(k^2-4)^2\theta_0}{8k}\left\{\frac{k-1-z}{[\rho^2+(k-1-z)^2]^{3/2}} - \frac{k+1+z}{[\rho^2+(k+1+z)^2]^{3/2}}\right\} \qquad (40)$$

$$u_\rho = \frac{(k^2-4)^2 b\theta_0}{16k(\beta^2-1)}\left\{\frac{k+1+z}{\rho\sqrt{[\rho^2+(k+1+z)^2]}} - \frac{k-1-z}{\rho\sqrt{[\rho^2+(k-1-z)^2]}}\right\} \qquad (41)$$

From the latter equation we in turn deduce that the difference in the principal stresses is

$$\sigma_\rho - \sigma_\psi = \frac{(k^2-4)^2 b\theta_0}{8k(\beta^2-1)}\left\{\frac{k-1-z}{[\rho^2+(k-1-z)^2]^{3/2}} - \frac{k+1+z}{[\rho^2+(k+1+z)^2]^{3/2}}\right.$$

$$\left. + \frac{2(k-1-z)}{\rho^2[\rho^2+(k-1-z)^2]^{1/2}} - \frac{2(k+1+z)}{\rho^2[\rho^2+(k+1+z)^2]^{1/2}}\right\} \qquad (42)$$

In a similar way, we find that

$$w = \frac{(k^2-4)^2 b\theta_0}{16k(\beta^2-1)}\left\{\frac{1}{\sqrt{[\rho^2+(k-1-z)^2]}}+\frac{1}{\sqrt{[\rho^2+(k+1+z)^2]}}\right\} \qquad (43)$$

By suitably choosing the constant k (it must be greater than 2 to ensure the convergence of the integrals) we can obtain several interesting problems. For example a value of k slightly greater than 2 gives a temperature distribution which is concentrated in the neighbourhood of the point $\rho = 0$. On the other hand a large value of k produces a temperature distribution which is almost constant over the central portion of the plate. Numerical calculations have been carried out on the cases $k = 2.1$, $k = 3$, and k very large. The temperature distribution and the isochromatic surfaces corresponding to these distributions are shown in Figs. 5 - 7. The values of $\sigma_\rho - \sigma_\phi$ from which these surfaces were plotted are given in tables II and III (cases $k = 2.1$ and $k = 3$). When k is very large $\sigma_\rho - \sigma_\phi$ is proportional to $\rho^2(1+z)$ - the values of this simple function have not been tabulated here.

2.5 The Elastic Layer on a Rigid Foundation.

As a further application of the general solutions developed above we shall consider a problem not treated by previous authors. We suppose a layer of elastic material to be resting on a rigid frictionless foundation. The upper surface ($z = 0$) is free from traction and is subjected to a known temperatute distribution ϕ_1, the flux of heat $\partial\theta/\partial z = \phi_2$ being known across the lower surface ($z = d$). Thus the boundary conditions are

$$\sigma_3 = \tau_{3x} = \tau_{3y} = 0\ ,\ \theta = \phi_1 \qquad \text{on } z = 0$$

$$w = \tau_{3x} = \tau_{3y} = 0\ ,\ \partial\theta/\partial z = \phi_2 \qquad \text{on } z = d$$

TABLE II. THICK PLATE. CASE k = 2.1 VALUES OF $|\sigma_p - \sigma_\phi|$.

ρ \ z	−1.0	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1.0
0	0	0	0	0	0	0	0	0	0	0	0
0.2	0	.0012	.0027	.0051	.0097	.0192	.0419	.1094	.3744	1.9969	18.6949
0.4	0	.0045	.0102	.0189	.0345	.0653	.1343	.3104	.8340	2.5995	8.0412
0.6	0	.0090	.0203	.0369	.0649	.1164	.2197	.4425	.9481	2.0761	4.1970
0.8	0	.0139	.0307	.0545	.0920	.1559	.2702	.4815	.8709	1.5448	2.5450
1.0	0	.0181	.0396	.0685	.1113	.1784	.2870	.4638	.7442	1.1588	1.7001
1.2	0	.0214	.0460	.0777	.1218	.1861	.2815	.4220	.6222	.8895	1.2129
1.4	0	.0234	.0497	.0822	.1250	.1834	.2640	.3739	.5187	.6992	.9072
1.6	0	.0243	.0511	.0829	.1228	.1744	.2414	.3274	.4344	.5613	.7029
1.8	0	.0243	.0506	.0808	.1172	.1621	.2176	.2855	.3663	.4588	.5596
2.0	0	.0236	.0479	.0770	.1097	.1484	.1945	.2488	.3112	.3808	.4552

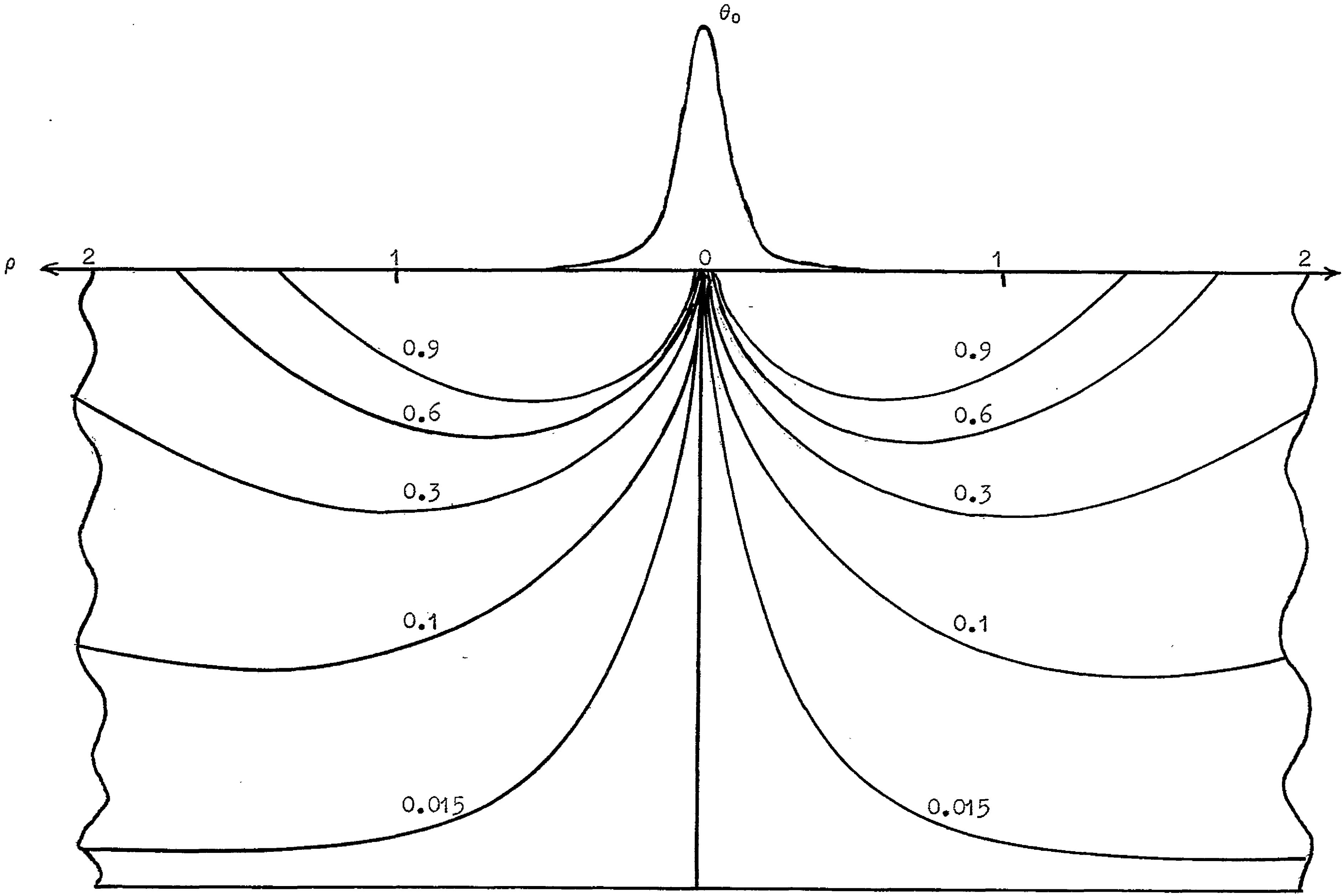


Fig. 5. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant in a thick plate. The numbers refer to the values of $-8k(\sigma_\rho - \sigma_\psi)(\beta^2 - 1)/b\theta_0(k^2 - 4)^2$, (for $k = 2.1$).

TABLE III. THICK PLATE. CASE k = 3. VALUES OF $|\sigma_\rho - \sigma_\phi|$.

ρ \ z	-1.0	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1.0
0	0	0	0	0	0	0	0	0	0	0	0
0.2	0	.0003	.0004	.0008	.0012	.0017	.0027	.0043	.0075	.0138	.0280
0.4	0	.0008	.0016	.0028	.0043	.0066	.0102	.0162	.0272	.0483	.0934
0.6	0	.0016	.0035	.0058	.0090	.0136	.0207	.0324	.0524	.0891	.1608
0.8	0	.0027	.0057	.0095	.0145	.0217	.0324	.0492	.0768	.1239	.2079
1.0	0	.0039	.0081	.0134	.0202	.0296	.0433	.0640	.0961	.1474	.2311
1.2	0	.0050	.0105	.0170	.0254	.0366	.0524	.0752	.1089	.1594	.2358
1.4	0	.0060	.0125	.0202	.0298	.0423	.0592	.0826	.1157	.1625	.2289
1.6	0	.0068	.0142	.0227	.0331	.0463	.0636	.0866	.1176	.1595	.2157
1.8	0	.0075	.0155	.0246	.0354	.0488	.0658	.0876	.1160	.1527	.1998
2.0	0	.0080	.0164	.0258	.0368	.0500	.0563	.0866	.1120	.1439	.1832

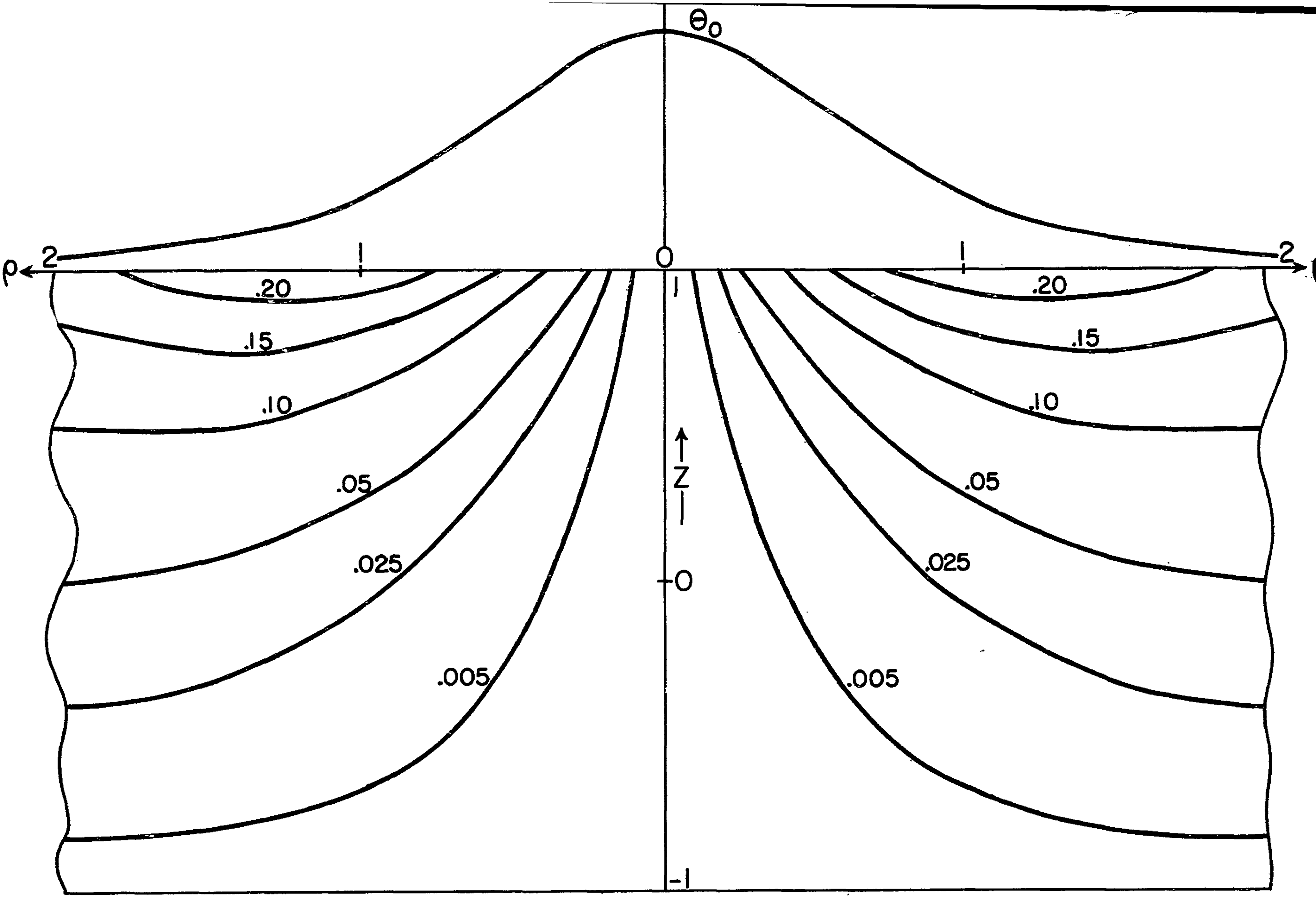


Fig. 6. Section by a plane ψ=constant of the surfaces $|\sigma_\rho - \sigma_\psi|$=constant in a thick plate. The numbers refer to the values of $8k(\sigma_\psi - \sigma_\rho)(\beta^2-1)/b\Theta_0(k^2-4)^2$, (for k=3)

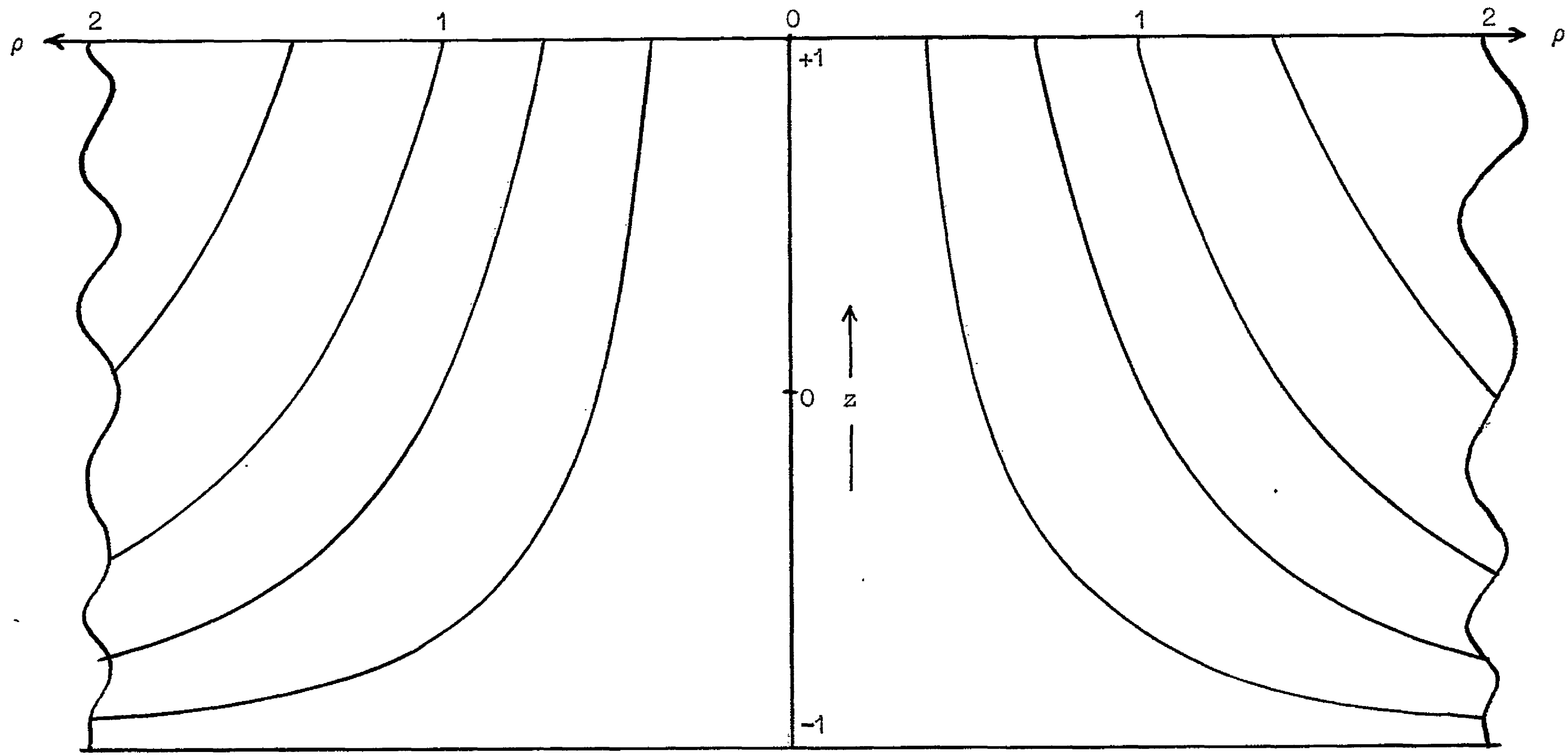


Fig. 7. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant in a thick plate. The surface temperature distribution corresponds to k large i.e. an almost uniform distribution.

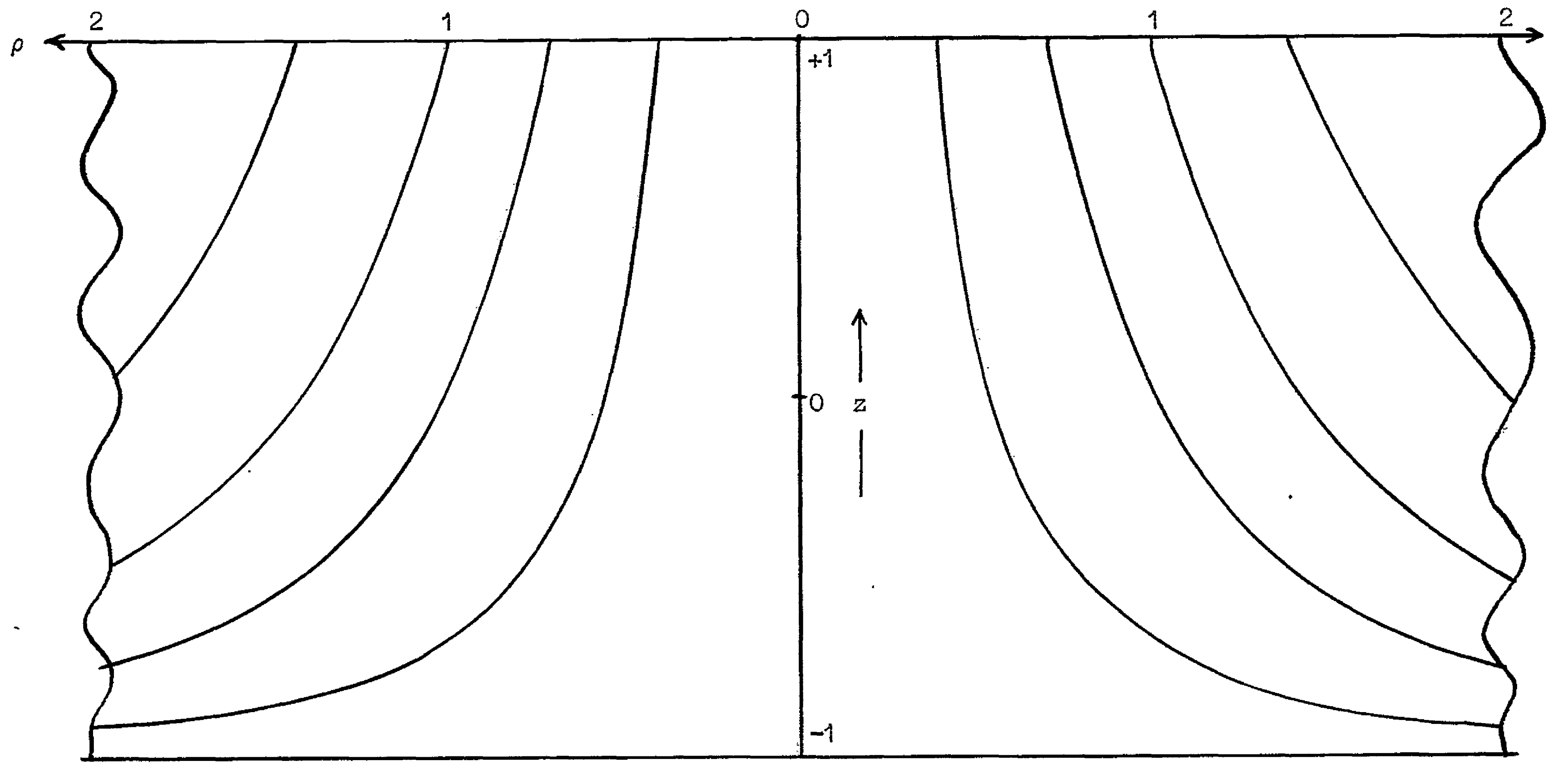


Fig. 7. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant in a thick plate. The surface temperature distribution corresponds to $\underline{k}$ large i.e. an almost uniform distribution.

Using the general solutions in the form (29) the elastic conditions imply that

$$
\begin{aligned}
&-i(\beta^2-2)(A_1\xi+A_2\eta)-bE+\beta^2A_3\zeta+\beta^2 i\zeta P=0\\
&\zeta A_1'-i\xi A_3'+P'\xi=0\\
&\zeta A_2'-i\eta A_3'+P'\eta=0\\
&(\zeta A_1-i\xi A_3+P\xi)S+2P\xi\zeta dC+2P'\xi\zeta dS=0\\
&(\zeta A_2-i\eta A_3+P\eta)S+2P\eta\zeta dC+2P'\eta\zeta dS=0\\
&(A_3+iP'\zeta d)S+(A_3'+iP\zeta d)C=0
\end{aligned}
\tag{44}
$$

where we have used the notation $S = \sinh(\zeta d)$, $C = \cosh(\zeta d)$ and where P and P' are given by (30).

The solutions of these equations can be written in terms of a quantity A defined by

$$
A=-\frac{bE(S+dC\zeta)+dS\zeta bE'}{2i\zeta^2(\beta^2-1)(S-\zeta S^2C^{-1}d+dC\zeta)}=-\frac{bE(S+dC\zeta)+dS\zeta bE'}{2i\zeta^2(\beta^2-1)(S+\zeta C^{-1}d)}
\tag{45}
$$

In terms of $\underline{A}$

$$
A_1=\xi A \qquad A_2=\eta A \qquad A_3=\frac{bE-2i\zeta^2A}{2\beta^2\zeta}
$$

$$
A_1'=\left\{\frac{ibE'-2\zeta^2ASC^{-1}}{2\beta^2\zeta^2}\right\}\xi \qquad A_2'=\left\{\frac{ibE'-2\zeta^2ASC^{-1}}{2\beta^2\zeta^2}\right\}\eta \qquad A_3'=i\zeta ASC^{-1}
\tag{46}
$$

$$
P=\frac{2\zeta^2A(\beta^2-1)-ibE}{2\zeta\beta^2} \qquad P'=-\frac{2\zeta^2ASC^{-1}(\beta^2-1)+ibE'}{2\zeta\beta^2}
$$

<u>General thermal boundary conditions.</u>

Since the temperature distribution $\underline{\theta}$ is given by

$$
\bar{\theta}=E\cosh(\zeta z)+E'\sinh(\zeta z)
$$

the thermal boundary conditions, $\theta=\phi_1$ on $z=0$, $\partial\theta/\partial z=\phi_2$ On $z=d$, require that

$$
\bar{\phi}_1=E, \qquad \bar{\phi}_2=\zeta(ES+E'C)
$$

That is

$$E = \bar{\phi}_1 \qquad\qquad E' = \frac{\bar{\phi}_2}{\zeta C} - \frac{\bar{\phi}_1 S}{C} \tag{47}$$

The expression (45) for $\underline{A}$ then takes the form

$$A = \frac{-b\bar{\phi}_1}{2i\zeta^2(\beta^2-1)} - \frac{bdSC^{-1}\bar{\phi}_2}{2i\zeta^2(\beta^2-1)(S+\zeta dC^{-1})} \tag{48}$$

Thermally insulated boundary.

The condition $\phi_2 = 0$ corresponds to the problem where the foundation is made of some non-conducting material, so that the elastic layer is thermally insulated at its lower surface.

In this case

$$A = \frac{ib\bar{\phi}_1}{2\zeta^2(\beta^2-1)}$$

and it follows that $P = P' = 0$ and

$$A' = \frac{-ib\bar{\phi}_1 SC^{-1}}{2\zeta^2(\beta^2-1)} \qquad A_3' = \frac{-b\bar{\phi}_1 SC^{-1}}{2\zeta(\beta^2-1)} \qquad A_3 = \frac{b\bar{\phi}_1}{2\zeta(\beta^2-1)}$$

Then we can finally write

$$\begin{aligned}
\bar{u} &= \frac{ib\bar{\phi}_1\xi}{2\zeta^2(\beta^2-1)}\left\{\cosh(\zeta z) - SC^{-1}\sinh(\zeta z)\right\}\\
\bar{v} &= \frac{ib\bar{\phi}_1\eta}{2\zeta^2(\beta^2-1)}\left\{\cosh(\zeta z) - SC^{-1}\sinh(\zeta z)\right\}\\
\bar{w} &= \frac{b\bar{\phi}_1}{2\zeta(\beta^2-1)}\left\{\sinh(\zeta z) - SC^{-1}\cosh(\zeta z)\right\}\\
\bar{\theta} &= \bar{\phi}_1\left\{\cosh(\zeta z) - SC^{-1}\sinh(\zeta z)\right\}
\end{aligned} \tag{49}$$

Cylindrical symmetry.

By a similar procedure to that adopted before we can easily obtain the symmetrical solutions directly from equations (49). It is found that

$$u_\rho = \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}_1(\zeta) \left\{ \cosh(\zeta z) - SC^{-1} \sinh(\zeta z) \right\} J_1(\zeta\rho)\, d\zeta$$

$$w = \frac{b}{2(\beta^2-1)} \int_0^\infty \bar{\phi}_1(\zeta) \left\{ \sinh(\zeta z) - SC^{-1} \cosh(\zeta z) \right\} J_0(\zeta\rho)\, d\zeta \qquad (50)$$

$$\theta = \int_0^\infty \zeta\, \bar{\phi}_1(\zeta) \left\{ \cosh(\zeta z) - SC^{-1} \sinh(\zeta z) \right\} J_0(\zeta\rho)\, d\zeta$$

where $\bar{\phi}_1(\zeta)$ is the zero-order Hankel transform of $\phi_1(\rho)$.

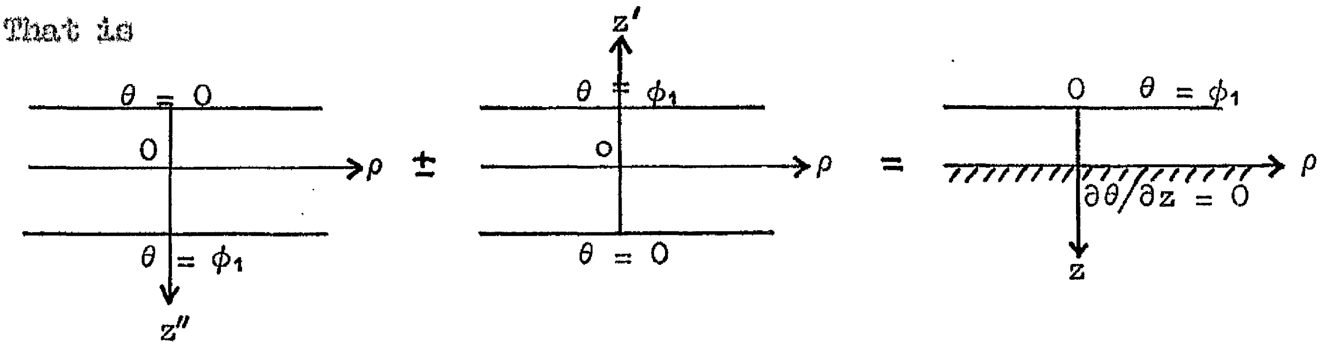
Now it can easily be shown that

$$\cosh(\zeta z) - SC^{-1}\sinh(\zeta z) = \frac{\sinh\zeta(z''+d)}{\sinh(2\zeta d)} + \frac{\sinh\zeta(z'+d)}{\sinh(2\zeta d)}$$

$$\sinh(\zeta z) - SC^{-1}\cosh(\zeta z) = \frac{\cosh\zeta(z''+d)}{\sinh(2\zeta d)} - \frac{\cosh\zeta(z'+d)}{\sinh(2\zeta d)} \qquad (51)$$

where $z'' = z - d$, $z' = -z + d$.

If we substitute from (51) into equations (50), and compare the resulting expressions with equations (36) of section 2.4, then we see that the results for the present problem can be obtained by superposition of two solutions of the kind dealt with in section 2.4. That is



The minus sign is to be used in evaluating w and the plus sign in evaluating u_ρ and θ (and also $\sigma_\rho - \sigma_\phi = 2(\partial u_\rho/\partial\rho - u_\rho/\rho)$).

In section 2.4 we considered the case

$$\bar{\phi}_1 = \frac{(k^2-4)^2\theta_0}{4k}\, e^{-k\zeta}\sinh(2\zeta)$$

By superposition we can draw the isochromatic lines for the layer problem with this surface temperature distribution on the upper surface. The values of $\sigma_{\beta} = \sigma_{\psi}$ for $k = 2.1$ and $k = 3$ are tabulated in Tables IV and V, and the corresponding curves are shown in Figs. 8-9.

The problem considered above was greatly simplified by the fact that we took $\phi_2 = 0$. In general, when this function is not set equal to zero, the solutions (49), (50) will contain a term in $\bar{\bar{\phi}}_2$. Since this will also contain the expression $(S + \xi d\sigma^{-1})$ in the denominator of the integrand, the numerical work will not be as simple as that in the case considered above. In such cases it would probably be best to perform the integrations numerically on a computer.

TABLE IV. LAYER ON A RIGID FOUNDATION. CASE k = 2.1 VALUES OF $|\sigma_\rho - \sigma_\psi|$.

ρ \ z	0	0.2	0.4	0.6	0.8	1.0
0	0	0	0	0	0	0
0.2	18.6949	1.9981	.3771	.1145	.0516	.0384
0.4	8.0412	2.6040	.8442	.3293	.1688	.1306
0.6	4.1970	2.0851	.9684	.4794	.2846	.2328
0.8	2.5450	1.5587	.9016	.5360	.3622	.3118
1.0	1.7001	1.1769	.7838	.5323	.3983	.3568
1.2	1.2129	.9109	.6682	.4997	.4033	.3722
1.4	.9072	.7226	.5684	.4561	.3890	.3668
1.6	.7029	.5856	.4855	.4103	.3642	.3488
1.8	.5596	.4831	.4169	.3663	.3348	.3242
2.0	.4552	.4044	.3591	.3258	.3042	.2968

TABLE V. LAYER ON A RIGID FOUNDATION. CASE k = 3. VALUES OF $|\sigma_\rho - \sigma_\psi|$.

ρ \ z	0	0.2	0.4	0.6	0.8	1.0
0	0	0	0	0	0	0
0.2	.0280	.0144	.0079	.0051	.0039	.0034
0.4	.0934	.0491	.0288	.0190	.0145	.0132
0.6	.1608	.0907	.0559	.0382	.0297	.0272
0.8	.2079	.1266	.0825	.0587	.0469	.0434
1.0	.2311	.1513	.1042	.0774	.0635	.0592
1.2	.2358	.1644	.1194	.0922	.0778	.0732
1.4	.2289	.1685	.1282	.1028	.0890	.0846
1.6	.2157	.1663	.1318	.1093	.0967	.0926
1.8	.1998	.1602	.1315	.1122	.1012	.0976
2.0	.1832	.1519	.1284	.1124	.1031	.1000

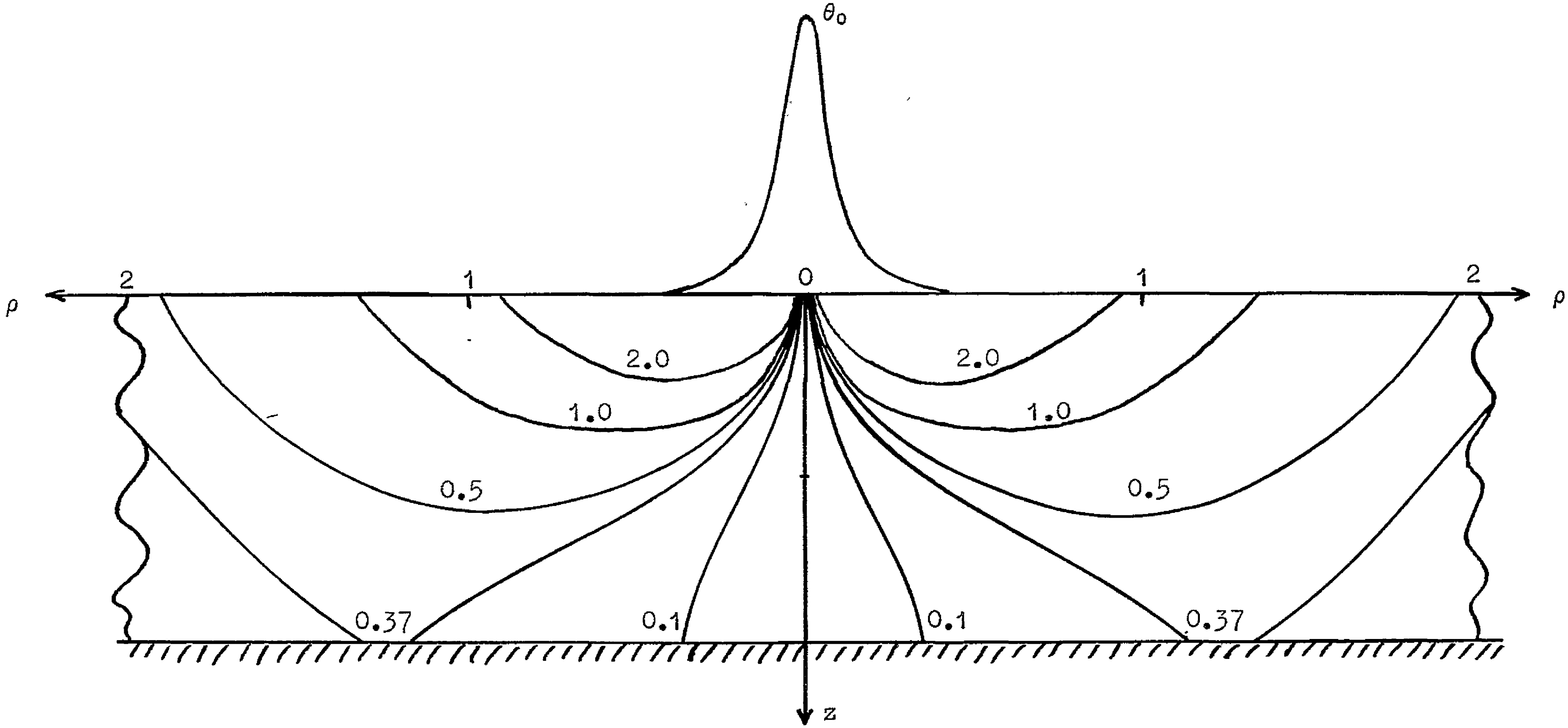


Fig. 8. Layer on rigid foundation. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant. The numbers refer to the values of $8k(\sigma_\psi - \sigma_\rho)(\beta^2 - 1)/b\theta_0(k^2 - 4)^2$, for k = 2.1

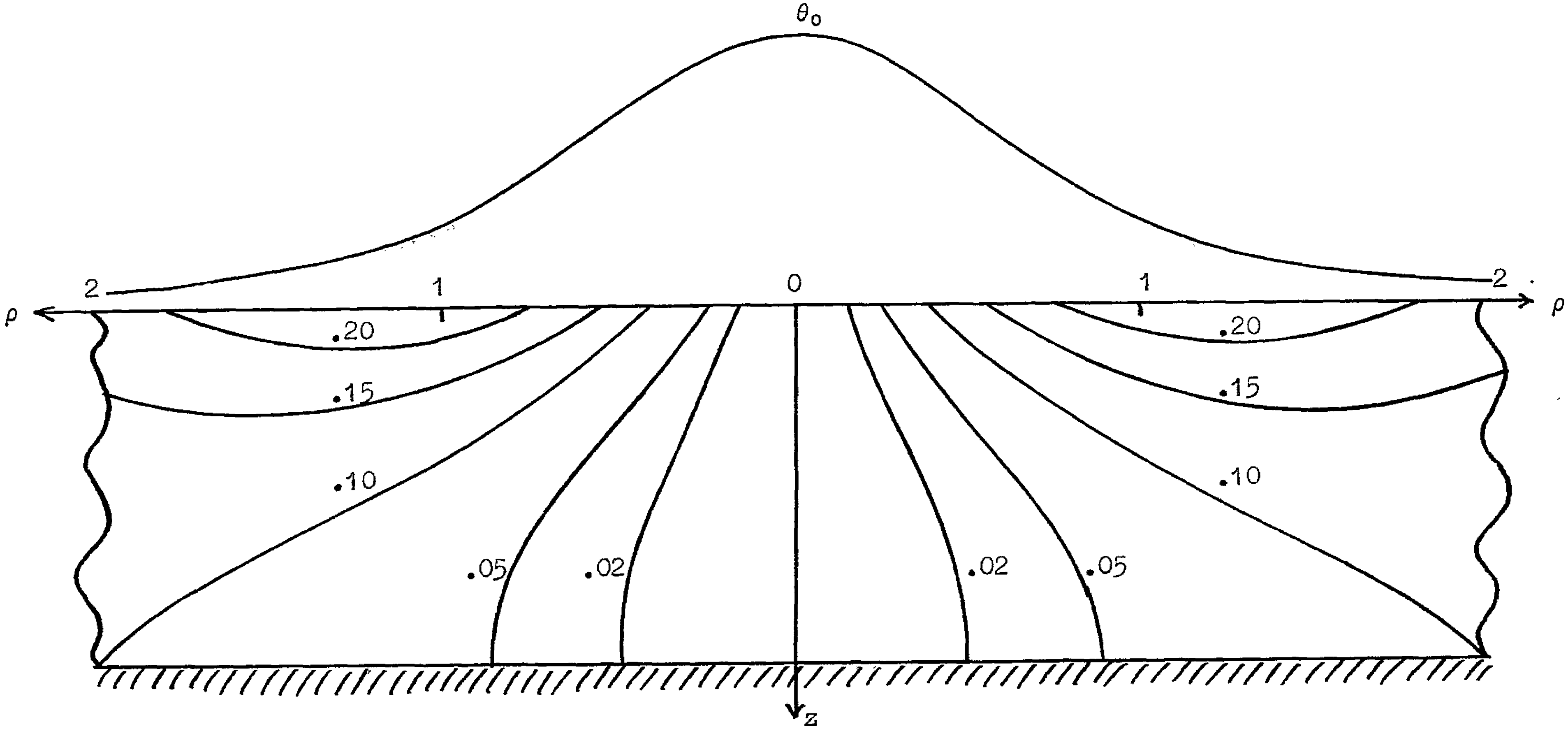


Fig. 9. Layer on rigid foundation. Section by a plane ψ = constant of the surfaces $|\sigma_\rho - \sigma_\psi|$ = constant. The numbers refer to the values of $8k(\sigma_\psi - \sigma_\rho)(\beta^2 - 1)/b\theta_0(k^2 - 4)^2$, for $k = 3$.

CHAPTER THREE.

RADIALLY SYMMETRIC SOLUTIONS OF THE LINKED THERMOELASTIC EQUATIONS.

We consider media which have boundaries of the form r = constant, where r is the radial coordinate in a spherical polar system. That is, we consider the solid sphere, the infinite medium with a spherical cavity, and the spherical shell of arbitrary thickness. Further we shall only work with problems in which the applied system of stress, displacement and/or temperature is radially symmetric.

The dynamic field equations of thermoelasticity are

$$\nabla^2 \underset{\sim}{u} + (\beta^2 - 1) \text{ grad } \Delta - b \text{ grad } \theta = a \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} \tag{1}$$

$$\nabla^2 \theta = f \frac{\partial \theta}{\partial t} + g \frac{\partial \Delta}{\partial t} \tag{2}$$

in the absence of heat sources and body forces. If we apply the operator div to (1) then these equations can be written as

$$\begin{aligned} \beta^2 \nabla^2 \Delta - b \nabla^2 \theta &= a \ddot{\Delta} \\ \nabla^2 \theta &= f \dot{\theta} + g \dot{\Delta} \end{aligned} \tag{3}$$

involving only θ and Δ .

3.1 Exponential Time Dependence.

Let us first suppose that all the physical quantities have an exponential time dependence of the form $e^{\alpha t}$, so that

$$\theta = \theta' \, e^{\alpha t}, \qquad \Delta = \Delta' \, e^{\alpha t} \tag{4}$$

where, in the radially symmetric problems under consideration here, θ' and Δ' are functions of $\underline{r}$ only.

It should be noted that since the derivations of equations (1) and (2) depended upon the assumption of small effects, solutions of this form will only be valid for sufficiently small values of the time $\underline{t}$. For example, the expression $\log(1 + \theta/T)$ was approximated by the first term in its expansion (θ/T), which gives an error of the order $\theta^2/2T^2$. Expressed as a percentage error this is approximately

$$\left\{ \frac{\theta^2}{2T^2} \Big/ \frac{\theta}{T} \right\} \times 100 = \frac{50\,\theta}{T} = \frac{50\,\theta' \, e^{\alpha t}}{T}$$

Thus if the maximum permissible error is E% then t_{max} is given by

$$\frac{50\,\theta' \, e^{\alpha t_{max}}}{T} = E \qquad \text{i.e.} \quad t_{max} = \frac{1}{\alpha} \log_e \frac{TE}{50\,\theta'}$$

which depends on both α and θ'. The solutions obtained will only be valid for values of $\underline{t}$ less than this maximum value.

Substituting the expressions (4) into (3) we see that θ' and Δ' satisfy

$$\begin{aligned} (\beta^2\nabla^2 - a\alpha^2)\Delta' &= b\nabla^2\theta' \\ (\nabla^2 - \alpha f)\theta' &= \alpha g \Delta' \end{aligned} \tag{5}$$

from which it follows that θ' and Δ' both satisfy

$$\left\{ (\beta^2\nabla^2 - a\alpha^2)(\nabla^2 - \alpha f) - \alpha b g \nabla^2 \right\} F = 0 \tag{6}$$

We shall write this equation in the form

$$\beta^2(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\, F = 0 \tag{7}$$

where k_1^2 and k_2^2 are the roots of the quadratic

$$\beta^2 x^2 - (a\alpha^2 + \alpha f\beta^2 + \alpha bg)x + a\alpha^3 f = 0 \tag{8}$$

and for k_1 and k_2 we take the positive roots of k_1^2 and k_2^2 .

Now the equation

$$(\nabla^2 - k^2)\, F = \frac{\partial^2 F}{\partial r^2} + \frac{2}{r}\frac{\partial F}{\partial r} - k^2 F = 0$$

has the solutions

$$F = \frac{1}{r}\, e^{\pm kr}$$

Thus θ' and Δ' are of the form

$$\begin{aligned} r\theta' &= A_1 e^{k_1 r} + B_1 e^{-k_1 r} + A_2 e^{k_2 r} + B_2 e^{-k_2 r} \\ r\Delta' &= C_1 e^{k_1 r} + D_1 e^{-k_1 r} + C_2 e^{k_2 r} + D_2 e^{-k_2 r} \end{aligned} \tag{9}$$

Not all of these coefficients are independent. If we substitute back from (9) into (5) then we find relations between them, and deduce that (9) can be written

$$\begin{aligned} 2r\theta' &= (\beta^2 k_1^2 - a\alpha^2)Pe^{k_1 r} + (\beta^2 k_1^2 - a\alpha^2)Qe^{-k_1 r} + (\beta^2 k_2^2 - a\alpha^2)Re^{k_2 r} + (\beta^2 k_2^2 - a\alpha^2)Se^{-k_2 r} \\ 2r\Delta' &= bk_1^2 Pe^{k_1 r} + bk_1^2 Qe^{-k_1 r} + bk_2^2 Re^{k_2 r} + bk_2^2 Se^{-k_2 r} \end{aligned} \tag{10}$$

Equations (10) represent the general solution to the class of problems considered. The coefficients P, Q, R, S have to be determined from the boundary conditions of the particular problem under consideration.

3.2 The Solid Sphere.

To obtain the solution corresponding to the solid sphere we must take $Q = -P$ and $S = -R$, to avoid infinite values of θ' and Δ' at the origin ($r = 0$). The expressions for θ' and Δ' are then

$$r\theta' = (\beta^2 k_1^2 - a\alpha^2)P \sinh(k_1 r) + (\beta^2 k_2^2 - a\alpha^2)R \sinh(k_2 r)$$
$$r\Delta' = bk_1^2 P \sinh(k_1 r) + bk_2^2 R \sinh(k_2 r) \qquad (11)$$

To obtain the radial displacement u' we note that

$$\frac{\partial}{\partial r}(r^2 u') = r^2\Delta' = bk_1^2 Pr \sinh(k_1 r) + bk_2^2 Rr \sinh(k_2 r)$$

so that

$$r^2 u' = bP\left\{k_1 r \cosh(k_1 r) - \sinh(k_1 r)\right\} + bR\left\{k_2 r \cosh(k_2 r) - \sinh(k_2 r)\right\} \qquad (12)$$

The constant term which appears due to the integration must be set equal to zero to avoid an infinite displacement at the origin.

Finally we can write the radial component of the stress tensor σ_r' as

$$\sigma_r' = (\beta^2 - 2)\Delta' - b\theta' + 2\frac{\partial u'}{\partial r}$$
$$= bP\left\{ar^{-1}\alpha^2 \sinh(k_1 r) + 4r^{-3}\sinh(k_1 r) - 4k_1 r^{-2}\cosh(k_1 r)\right\}$$
$$+ bR\left\{ar^{-1}\alpha^2 \sinh(k_2 r) + 4r^{-3}\sinh(k_2 r) - 4k_2 r^{-2}\cosh(k_2 r)\right\} \qquad (13)$$

As an example of the use of these equations we consider a sphere of unit radius which is subjected to a temperature distribution $\theta_0 e^{\alpha t}$ on its traction-free surface. Thus, from equations (11) and (13), the equations giving P and R are

$$\theta_0 = (\beta^2 k_1^2 - a\alpha^2)P\,\sinh(k_1) + (\beta^2 k_2^2 - a\alpha^2)R\,\sinh(k_2)$$

$$0 = P\left\{a\alpha^2\sinh(k_1) + 4\,\sinh(k_1) - 4k_1\cosh(k_1)\right\} + R\left\{a\alpha^2\sinh(k_2) + 4\,\sinh(k_2) - 4k_2\cosh(k_2)\right\} \qquad (14)$$

For the specific case of lead, the constants in the equations of thermoelasticity have the values

$$\beta^2 = 3, \quad b = 0.232, \quad f = 4.152, \quad g = 12.25, \quad a = 2.034\times10^{-10}$$

We shall consider two separate values for α:

Case (i): For $\alpha = 0.1$ equation (8) gives the values

$$k_1 = 0.714 \qquad k_2 = 4.616\times10^{-7}$$

Equations (14) can now be used to determine P and R, and when these values are substituted into (11) it is found that the space-dependent part of the temperature distribution is given by

$$\theta' = \frac{1.827\ \theta_0\sinh(0.714\ r)}{r} - 0.418\ \theta_0 \qquad (15)$$

Case (ii): For $\alpha = 0.6$ equation (8) gives the values

$$k_1 = 1.746 \qquad k_2 = 2.770\times10^{-6}$$

and the temperature distribution is now given by

$$\theta' = \frac{0.484\, \theta_0 \sinh(1.746\, r)}{r} - 0.344\, \theta_0 \tag{16}$$

Values of the functions (15) and (16) are given in the table on page 65 and their graphs are shown in Fig. 10 on page 66.(they are shown as the 'complete linked solutions'.)

3.3 Quasi-Static Solutions for Exponential Time Dependence.

The solution in which inertia terms are neglected from the equations of motion is known as the quasi-static solution and is characterised by setting a = 0 in equation (1). The quasi-static equivalent of equation (6) is

$$\beta^2 \nabla^2 (\nabla^2 - \alpha f - \alpha b g/\beta^2)\, F = 0 \tag{17}$$

so that the solutions for θ' and Δ' are

$$r\theta' = A_1 e^{kr} + B_1 e^{-kr} + A_2 + B_2 r$$

$$r\Delta' = C_1 e^{kr} + D_1 e^{-kr} + C_2 + D_2 r$$

where

$$k^2 = \alpha f + \alpha b g/\beta^2 \tag{18}$$

Again, all these coefficients are not independent, and we find that

$$r\theta' = \beta^2 P\, e^{kr} + \beta^2 Q\, e^{-kr} + gR + gSr$$
$$r\Delta' = bP\, e^{kr} + bQ\, e^{-kr} - fR - fSr \tag{19}$$

Solid sphere. Returning to the sphere problem, we must take

$$r\theta' = 2\beta^2 P \sinh(kr) + gSr$$

$$r\Delta' = 2bP \sinh(kr) - fSr \qquad (20)$$

to avoid infinite values of θ' and Δ' at the origin.

The displacement u' is then given by

$$\frac{\partial}{\partial r}(r^2 u') = r^2\Delta' = 2bPr \sinh(kr) - fSr^2$$

so that

$$r^2 u' = 2bPk^{-2}\left\{kr \cosh(kr) - \sinh(kr)\right\} - \tfrac{1}{3}fSr^3 \qquad (21)$$

and we have again set the constant of integration equal to zero to avoid a singularity in the displacement at $r = 0$.

Using the expressions (20) and (21) it is found that the stress σ'_r is given by

$$r\sigma'_r = Sr(2f - \beta^2 f - bg - \tfrac{2}{3}f) + 8bPk^{-2}\left\{r^{-2}\sinh(kr) - kr^{-1}\cosh(kr)\right\}$$

so that the conditions $\theta' = \theta_0$, $\sigma'_r = 0$ on $r = 1$ give the equations

$$\theta_0 = 2\beta^2 P \sinh(k) + gS$$

$$0 = S(2f - \beta^2 f - bg - \tfrac{2}{3}f) + 8bPk^{-2}\left\{\sinh(k) - k\cosh(k)\right\} \qquad (22)$$

Using thespecial values given previously for lead, we find that

<u>Case (i)</u>: $\alpha = 0.1$ gives $k = 0.714$ and

$$\theta' = \frac{1.473\ \theta_0 \sinh(0.714\ r)}{r} - 0.143\ \theta_0 \qquad (23)$$

Case (ii): $\alpha = 0.6$ gives $k = 1.746$ and

$$\theta' = \frac{0.404\ \theta_0 \sinh(1.746\ x)}{x} - 0.122\ \theta_0 \qquad (24)$$

Classical Quasi-static Solution.

The classical solution is obtained by neglecting in equations (2) the linking term $g\partial A/\partial t$, i.e. by setting $g = 0$. From equations (18) and (22) we can obtain the classical quasi-static solutions ($a = g = 0$):

Case (i): $\alpha = 0.1$ gives $k = 0.644$ and

$$\theta' = \frac{1.450\ \theta_0 \sinh(0.644\ x)}{x} \qquad (25)$$

Case (ii): $\alpha = 0.6$ gives $k = 1.578$ and

$$\theta' = \frac{0.431\ \theta_0 \sinh(1.578\ x)}{x} \qquad (26)$$

Values of the functions (23) - (26) are given in the following table and their graphs are shown in Fig. 10 on the following page - the linked quasi-static solution being shown as a full line, the classical quasi-static solution being shown as a broken line.

α	Solution \ x	0	0.2	0.4	0.6	0.8	1.0
0.1	complete linked	.886	.891	.904	.927	.958	1.000
	quasi-static	.909	.912	.923	.941	.967	1.000
	classical quasi-static	.934	.937	.944	.958	.976	1.000
0.6	complete linked	.501	.518	.571	.664	.803	1.000
	quasi-static	.583	.597	.642	.719	.836	1.000
	classical quasi-static	.680	.692	.727	.787	.876	1.000

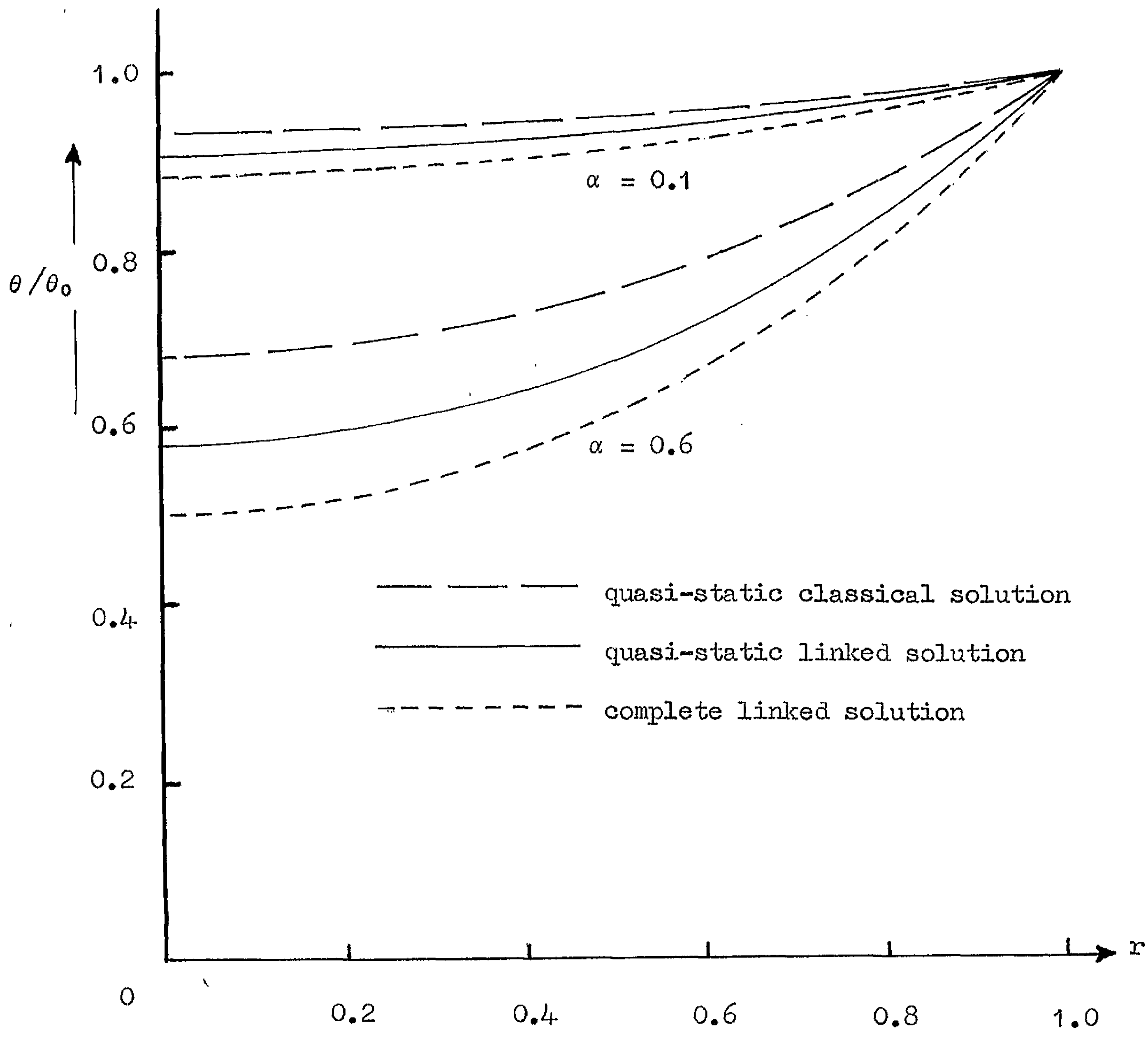


Fig. 10. Comparison of the three solutions (for $\alpha = 0.1$ and $\alpha = 0.6$)

It will have been noted that the agreement of the three solutions is not very good. However we have chosen values for g which give a rapidly varying time dependence – so we would expect sizable errors due to neglecting time dependent terms. For more practicable (i.e. smaller) values of g the solutions show a better agreement and in the limiting case of $c = 0$, (i.e. no time dependence) it is obvious from equations (1) and (2) that the three solutions coincide. A point worth noting from the graphs plotted overleaf is that the error introduced by neglecting the linking term in (2) is of approximately the same size as the error introduced by neglecting the inertia term in (1).

3.4 Solutions for General Time Dependence.

To deal with the case of general time dependence we must use the theory of the integral transform to reduce equations (3) to a set of differential equations in the single variable z. If we define the Laplace transform $\bar{f}$ of a function f by

$$\bar{f} = \int_0^\infty f \, e^{-ct} \, dt \tag{27}$$

then equations (3) can be transformed to the form

$$\begin{aligned} \beta^2 \nabla^2 \bar{\Delta} - b \, \nabla^2 \bar{\theta} &= a \, c^2 \bar{\Delta} \\ \nabla^2 \bar{\theta} &= c f \bar{\theta} + c g \bar{\Delta} \end{aligned} \tag{28}$$

which is identical with the set (5) with $\bar{\Delta}$, $\bar{\theta}$ replacing Δ', θ'.
By comparison with the previous case we know that

$$2r\bar{\theta} = (\beta^2 k_1^2 - a\alpha^2)Pe^{k_1 r} + (\beta^2 k_1^2 - a\alpha^2)Qe^{-k_1 r} + (\beta^2 k_2^2 - a\alpha^2)Re^{k_2 r} + (\beta^2 k_2^2 - a\alpha^2)Se^{-k_2 r} \tag{29}$$

$$2r\bar{\Delta} = bk_1^2 Pe^{k_1 r} + bk_1^2 Qe^{-k_1 r} + bk_2^2 Re^{k_2 r} + bk_2^2 Se^{-k_2 r}$$

In this case we have to use the <u>transformed</u> boundary conditions to solve for P, Q, R, S.

<u>3.5 Quasi-Static Solution for General Time Dependence.</u>

If we replace θ', Δ' in section 3.4 by $\bar{\theta}$ and $\bar{\Delta}$ we get

$$r\bar{\theta} = \beta^2 P\, e^{kr} + \beta^2 Q\, e^{-kr} + gR + gSr$$

$$r\bar{\Delta} = bP\, e^{kr} + bQ\, e^{-kr} - fR - fSr \tag{30}$$

where

$$k^2 = \alpha(f + bg/\beta^2) = \alpha f(1 + \epsilon) = \alpha f_1$$

and $\underline{k}$ is the positive root of k^2. With the appropriate expressions for P, Q, R, S, the expressions (30) lead, upon inversion, to the general quasi-static solutions.

<u>3.6 Infinite Medium with Spherical Cavity: Quasi-Static Solutions.</u>

To ensure that $\bar{\theta}$ and $\bar{\Delta}$ tend to zero as $r \to \infty$ we must take

$$r\bar{\theta} = \beta^2 Q\, e^{-kr} + gR$$

$$r\bar{\Delta} = bQ\, e^{-kr} - fR \tag{31}$$

and the equation $\frac{\partial}{\partial r}(r^2\bar{u}) = r^2\bar{\Delta}$ gives

$$\bar{u} = -bQk^{-2}(kr^{-1} + r^{-2})\, e^{-kr} - \tfrac{1}{2}fR + \phi r^{-2}$$

where ϕ is a constant of integration. If we require that $\bar{u}$ should also tend to zero as $r \to \infty$ then we must take $R = 0$. Hence we have

$$r\bar{\theta} = \beta^2 Q \, e^{-kr}$$

$$\bar{u} = -bQk^{-2}(kr^{-1} + r^{-2})\, e^{-kr} + \phi r^{-2} \qquad (32)$$

It is obvious from the first of these equations that, in this quasi-static approximation, the temperature distribution can be calculated from the thermal boundary condition alone, irrespective of the surface tractions acting.

As a particular example we shall consider the infinite medium with a spherical cavity of unit radius, which is loaded thermally on its traction free surface, i.e. $\theta = \theta_0$, $\sigma_r = 0$ on $r = 1$. These conditions lead to the expressions

$$Q = \beta^{-2} e^{k} \bar{\theta}_0 \qquad \phi = bk^{-2}(k+1)\beta^{-2}\bar{\theta}_0 \qquad (33)$$

STERNBERG (1957) obtained the classical quasi-static solution to the problem where

$$\theta_0 = \begin{cases} 0, & t = 0. \\ 1, & t > 0. \end{cases}$$

for which $\bar{\theta}_0 = \alpha^{-1}$ (34)

so that

$$r\bar{\theta} = \alpha^{-1} e^{-(r-1)(\alpha f_1)^{1/2}} \qquad (35)$$

since $k^2 = \alpha f_1$. We can now apply the inverse transform to obtain the expression (ERDELYI (1954) Vol. I.p245(3))

$$\theta = \frac{1}{r}\,\mathrm{Erfc}\left\{ \frac{(r-1)f_1^{1/2}}{2t^{1/2}} \right\} \qquad (36)$$

which is the linked quasi-static solution. When $f = 1$ and $g = 0$ this expression agrees with the classical result obtained by STERNBERG.

Also we can use equations (32), (33) and (34) to obtain expressions for $\bar{u}$, $\bar{\sigma}_r$, $\bar{\sigma}_\psi$, $\bar{\sigma}_\phi$. We find that they can all be written in terms of a function $\bar{F}(r,\alpha)$

$$\bar{F}(r,\alpha) = -k^{-2}\left\{ (kr+1)\, e^{-k(r-1)} - (k+1)\right\} \alpha^{-1} \tag{37}$$

so that upon inversion

$$\begin{aligned} u &= b\beta^{-2} r^{-2} F(r,t) \\ \sigma_r &= -4b\beta^{-2} r^{-3} F(r,t) \\ \sigma_\psi = \sigma_\phi &= 2b\beta^{-2} r^{-3} F(r,t) - 2b\beta^{-2}\theta \end{aligned} \tag{38}$$

Finally we use the fact that $k^2 = \alpha f_1$ and invert the expression (37) for $\bar{F}(r,\alpha)$ to obtain (ERDELYI, pp245-6)

$$F(r,t) = \tfrac{1}{2}(r^2 - 2f_1^{-1}t - 1)\,\mathrm{Erfc}(\xi) - f_1^{-1/2}(t/\pi)^{1/2}(r+1)e^{-\xi^2} + f_1^{-1}t + 2(t/f_1\pi)^{1/2} \tag{39}$$

where

$$\xi = \tfrac{1}{2} f_1^{1/2}(r-1)\,t^{-1/2} \tag{40}$$

Equations (36), (38), (39), (40) constitute the linked quasi-static solutions to the problem. In all cases they reduce to STERNBERG's results if $f = 1$, $g = 0$ (except for multiplicative constants resulting from a different choice for the units of length and stress)

We now note, from (36)-(40), that the quantities f_1 and t always appear in the form f_1/t. This leads us to the following reasoning. Suppose that the classical quasi-static value of F (or θ) takes on a certain value at a time t_0. Then the linked quasi-static value of F (or θ)

takes on the same value at a time t given by

$$\frac{r}{t_0} = \frac{r_1}{t} = \frac{r(1+\epsilon)}{t}$$

That is

$$t = (1+\epsilon)t_0$$

Thus we have proved that a configuration of temperature, displacement and stress, which exists within the classical quasi-static theory at a time t_0, will be reached in the linked quasi-static theory at a time ϵt_0 later.

Obviously the set of curves depicting the variations of u, σ_r, σ_ϕ, σ_ϕ with r and t will be of the same form for the linked solution as for the classical solution. To obtain the curves for the former from those for the latter we need only multiply the t-graduations by the factor $(1+\epsilon)$. The curves for the classical solution have already been given by STERNBERG.

3.7 Quasi-Static Solutions of the Linked Equations.

All solutions referred to in this section will be quasi-static solutions i.e. $a = 0$. We saw in the example of the last section that the linked solution was the same as the classical solution with the time scale magnified by the factor $(1+\epsilon)$. We look now to see if this is true in general, i.e. for any kind of body, and under arbitrary elastic and thermal conditions.

The thermoelastic equations are

$$\begin{aligned} &\nabla^2 \underline{u} + (\beta^2 - 1)\,\text{grad}\,\Delta - b\,\text{grad}\,\theta + \text{grad}\,\phi = 0 \\ &\theta + \nabla^2\theta = f\dot{\theta} + g\dot{\Delta} \end{aligned} \qquad (44)$$

where we have assumed that the body force $\underline{X}$ can be expressed in terms of a potential function, $\underset{\sim}{X} = \text{grad}\,\phi$.

Applying the operator div to the first of (41) gives

$$\beta^2\nabla^2\Delta - b\nabla^2\theta + \nabla^2\phi = 0$$

so that

$$\beta^2\Delta - b\theta + \phi = \text{function of the coordinates}$$

Thus

$$\beta^2\mathring{\Delta} - b\mathring{\theta} + \mathring{\phi} = 0 \tag{42}$$

We may now eliminate $\mathring{\Delta}$ between (42) and the second of (41) giving

$$\Theta + g\mathring{\phi}/\beta^2 + \nabla^2\theta = f(1 + \epsilon)\mathring{\theta}$$

Thus the set (41) is equivalent to the set

$$\nabla^2\underset{\sim}{u} + (\beta^2 - 1)\ \text{grad}\ \Delta - b\ \text{grad}\ \theta + \text{grad}\ \phi = 0$$

$$\Theta + g\mathring{\phi}/\beta^2 + \nabla^2\theta = f(1 + \epsilon)\mathring{\theta} \tag{43}$$

We can see that, in general, the substitution $t = (1 + \epsilon)\tau$ does not reduce these equations to the classical equations, since the quantities Θ and ϕ (and also the boundary conditions) depend on $\underset{\sim}{t}$. The cavity problem considered in the last section was a special case, since Θ and ϕ were both identically zero, as was the surface stress. Also the boundary temperature

$$\theta_0(t) = \begin{cases} 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

is invariant under the transformation $t = (1 + \epsilon)\tau$. Thus it was accidental that we should have chosen a particular problem for which the equations (43) and the boundary conditions both reduce to the classical problem under the transformation $t = (1 + \epsilon)\tau$.

A better interpretation of equations (43) is to say that the linked problem can be reduced to a classical problem if we consider an extra heat source equal to $g\dot{\lambda}/\beta^2$ and replace f by $f_1 = f(1 + \epsilon)$. This fact was noted by LOCKETT and SNEDDON (1959/60) in their analysis of the linked thermoelastic problem for the infinite medium (see Chap.VI.).

CHAPTER FOUR.

A METHOD OF DERIVING SOLUTIONS OF A CLASS OF BOUNDARY VALUE PROBLEMS.

4.1 Introduction.

In many branches of applied mathematics there exists a class of problems which depend for their solution upon the integration of a set of simultaneous linear partial differential equations subject to certain boundary conditions. In all but the simplest cases it is not practicable to deal with these equations by standard methods. For problems involving infinite regions, solutions can often be found by the use of integral transforms. However, in many problems we are concerned with media of finite extent, so that if we are to make a direct application of this method, we shall have to use finite transforms, and under certain conditions these are much more difficult to apply than transforms over an infinite range.

This situation exists in thermoelasticity. The full equations are too numerous and too complicated to tackle by direct elimination of the unknowns. Even if, as in the solution of the steady-state plate problem of Chapter Two, we were to transform the equations with respect to all but one of the variables, we would be left with a system of four second order linear ordinary differential equations - still a formidable problem to solve. The problem for the infinite medium can be solved by transforming the equations with respect to all of the variables (see Chapter Six). However, for bodies of finite extent (and to some extent even for the semi-infinite body), difficulties arise if we try to transform with respect to all of the variables. Firstly, we would need to use finite transforms which, even if suitable ones are available for the particular problem, are often more difficult to invert than the transforms over an infinite range. Secondly, the use of these transforms would demand a knowledge of the values of some of the unknowns at the limit points of the integrals. In most problems these values are not known. Thus, the 'infinite problem' has some advantages over the other problems.

In this chapter we shall show how, by considering modified problems in the entire (infinite) space, we can use transforms over the infinite range to obtain the solutions to some problems for finite media. The method, which is due to LOCKETT (1959c) is given in general form, for application to any suitable branch of applied mathematics. It is used to solve specific problems in section 5.3 and Chapter Seven.

4.2 Illustrative Example.

Before attempting to state the principles of the method in general terms we shall use the method to solve a relatively simple problem. It should then be easier for the reader to understand the general formulation, by comparing it with this solution.

Temperature distribution in an infinite cylinder due to a surface temperature $\theta_0(z,t)$.

As our example we shall find the temperature distribution in an infinitely long cylinder due to the application of a boundary temperature $\theta_0(z,t)$ on the surface $r = a$. Thus the problem is mathematically equivalent to finding a solution, in the region $r \leq a$, of the heat conduction equation

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \kappa\frac{\partial\theta}{\partial t} \qquad (1)$$

subject to the boundary condition that $\theta = \theta_0(z,t)$ on $r = a$.

Instead of solving this problem directly, we shall consider a modified problem in an infinite medium. We shall find the effect of a (for the moment unknown) heat source $\Theta(z,t)$ which is concentrated on the radius $r = a$. We shall then choose $\Theta(z,t)$ in such a way that $\theta = \theta_0(z,t)$ on $r = a$. Our solution will therefore satisfy

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \kappa\frac{\partial\theta}{\partial t} - \Theta(z,t)\,\delta(r - a) \tag{2}$$

subject to the condition that $\theta = \theta_0(z,t)$ on $r = a$.

Now within the region $r < a$, equations (1) and (2) are identical, so that, within this region, our solutions for the modified problem are also the solutions required for the cylinder problem.

If we define the transforms

$$\bar{\theta}(\xi,\zeta,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\zeta z+\omega t)}dz\,dt\int_0^{\infty} r\theta(r,z,t)\,J_0(\xi r)\,dr \tag{3}$$

$$f^\circ(\zeta,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,t)\,e^{i(\zeta z+\omega t)}\,dz\,dt \tag{4}$$

equation (2) transforms to

$$(\xi^2 + \zeta^2 - i\omega\kappa)\bar{\theta} = aJ_0(\xi a)\Theta^\circ \tag{5}$$

so that

$$\begin{aligned} \theta^\circ(r,\zeta,\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta(r,z,t)\,e^{i(\zeta z+\omega t)}\,dz\,dt \qquad (6)\\ &= \int_0^{\infty}\xi J_0(\xi r)\,\bar{\theta}(\xi,\zeta,\omega)\,d\xi \\ &= a\Theta^\circ\int_0^{\infty}\xi(\xi^2+\zeta^2-i\omega\kappa)^{-1}J_0(\xi a)\,J_0(\xi r)\,d\xi \\ &= a\Theta^\circ\, I_0(kr)\,K_0(ka) \qquad \text{for } r \leq a \qquad (7)\end{aligned}$$

where $k^2 = \zeta^2 - i\omega\kappa$ and we choose for k the branch with the positive real part.

We can now apply the transformed boundary condition

$$\theta^*(a,\zeta,\omega) = \theta_0^*(\zeta,\omega)$$

to get

$$c^{D} = \theta_0^*/I_0(ka)\,K_0(ka)$$

and substitution of this expression into (7) gives

$$\theta^*(r,\zeta,\omega) = \theta_0^* I_0(kr)\,/\,I_0(ka) \qquad (8)$$

Finally we use (8) and the transforms inverse to (6) to give us the temperature distribution

$$\theta(r,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_0^*(\zeta,\omega)\,\frac{I_0\{r(\zeta^2 - i\omega\kappa)^{1/2}\}}{I_0\{a(\zeta^2 - i\omega\kappa)^{1/2}\}}\; e^{-i(\zeta z + \omega t)}\,d\zeta\,d\omega \qquad (9)$$

where we take the square root with the positive real part.

It is easily verified that this expression satisfies equation (1) and the boundary condition $\theta = \theta_0(z,t)$ on $r = a$.

4.3 General Physical Description of Method.

Suppose that the medium in our problem occupies the region D and is acted upon by various 'causes' which we shall denote by C. These causes may be body forces, heat sources, electric or magnetic fields, sources, sinks, etc. On the boundaries S_1, S_2,... of D there are a set of boundary conditions which we shall denote by T. The set of equations T may express the fact that stress components, temperature, temperature gradient, potential, etc., are known on these boundaries. Suppose that our physical knowledge of the problem allows us to formulate it as a simultaneous set of linear partial differential equations E, which we have to solve within the region D in accordance with the boundary conditions T.

We consider now a different problem. Let the medium occupy the entire space X, which may be of one, two or three dimensions according to the problem in question. In addition to the causes C acting within the region D, we now consider a further system of causes C^* acting on the surfaces S_1, S_2,... For each boundary C^* contains a complete set of causes concentrated on this boundary. By a 'complete' set we mean that it should contain all types of cause relevant to the particular branch of the subject. Thus, in an elastic problem, it should contain body forces in all posible directions, whilst in a thermoelastic problem it should also contain a concentrated heat source. This ensures that there are as many members in the C^* as there are boundary conditions.

Suppose that the physics of the modified problem leads to a set of governing equations E^*. Within the interior of the region D the sets of equations E^* and E are identical since the C^* are zero within this region. We now use the transform method to obtain solutions of the E^* within the entire space X.These solutions will depend on the C^*, which up to now are unknown functions. We now choose expressions for them so that the conditions T are satisfied.

Our solutions therefore satisfy the boundary conditions, and within D they are solutions of E. Thus we have found solutions to our original problem. As far as the finite medium problem is concerned the solutions have no meaning whatsoever outside the region D.

4.4 Restrictions on the Type of Problem Considered.

The method is only applicable to a certain class of problems. The restrictions which must be placed on the type of problem considered are mainly concerned with the type of boundary and the type of boundary condition which are permitted. In fact, we only require that the governing equations should be a set of linear partial differential equations which can be reduced to a set of algebraic equations by the application of a suitable multiple integral transform over the entire space. We place no restriction either on the number of independent variables or on the number of unknowns (as long as this is equal to the number of equations!)

The boundary conditions may be given by specifying the boundary values of functions of the unknowns and/or their derivatives, but each such specification must be made over the whole of that boundary. Thus it is not permissible to have mixed boundary conditions on any boundary, although the conditions on one boundary may be entirely different from those on another boundary.

The other restriction is the form which the boundary itself can take. Each boundary must be of the form : some coordinate x_1 = constant, and in any one problem all the boundaries must be given by putting the same coordinate equal to different constant values. Thus it is not permissible to consider the quarter plane bounded by $x = 0$ $(y > 0)$ and $y = 0$ $(x > 0)$, though it is possible to consider this domain if we use polar coordinates, since the boundaries can then be written $\theta = 0$ and $\theta = \pi/2$. This last condition does of course restrict the number of problems which can be tackled by the method. However we can see that by using suitable coordinates we can tackle problems involving the semi-infinite plane, infinite strip, semi-infinite space, infinite plate, interior and exterior of circle, circular ring, infinite sector, infinite circular cylinder and tube, infinite space with cylindrical cavity, infinite wedge, sphere, shell, infinite cone, etc. Thus an interesting class of problems still remains.

4.5 General Mathematical Formulation.

In section 4.3 we described the method in physical terms. It is, of course, possible to give a purely mathematical formulation, and this is the purpose of the present section.

Suppose that we have a set of linear partial differential equations

$$\underset{\sim}{L}\,\underset{\sim}{y} - \underset{\sim}{f} = \underset{\sim}{0} \tag{10}$$

where $\underset{\sim}{L}$ is an $n \times n$ matrix of linear partial differential operators in the independent variables $x_1, x_2, \dots x_r$. $\underset{\sim}{y}$ is the column matrix $\{y_1\ y_2 \dots y_n\}$ of unknown functions $y_i = y_i(x_1, x_2, \dots x_r)$ and $\underset{\sim}{f}$ is the column matrix $\{f_1\ f_2 \dots f_n\}$ of the known functions $f_i = f_i(x_1, x_2, \dots x_r)$. We wish to find solutions of these equations within the domain $a_1 \leq x_1 \leq a_2$, all $x_2, \dots x_r$ (time may be included in the independent variables) subject to certain boundary conditions on $x_1 = a_1$ and $x_1 = a_2$. These conditions may include functions differentiated with respect to any of the variables.

Consider now the set of equations

$$\underset{\sim}{L}\,\underset{\sim}{y} - \underset{\sim}{f} = \underset{\sim}{\phi} \tag{11}$$

where $\underset{\sim}{\phi}$ is a column matrix whose elements are of the form

$$\phi_i = g_{1i}(x_2, \dots x_r)\ \delta(x_1 - a_1) + g_{2i}(x_2, \dots x_r)\delta(x_1 - a_2) \tag{12}$$

so that $\phi_i = 0$ for $a_1 < x_1 < a_2$.

Equation (12) can be written in matrix form as

$$\underset{\sim}{\phi} = \underset{\sim}{g_1}\ \delta(x_1 - a_1) + \underset{\sim}{g_2}\ \delta(x_1 - a_2) \tag{13}$$

The solution of the problem can now be completed in four steps:

(i) Transform equations (11) over the entire range of $x_1, x_2, \dots x_r$ so that

$$\underset{\sim}{\bar{L}}\,\underset{\sim}{\bar{y}} - \underset{\sim}{\bar{F}} = \underset{\sim}{\bar{\Phi}} \tag{14}$$

Suppose that

$$\bar{\underset{\sim}{L}}\,\bar{\underset{\sim}{y}} = \underset{\sim}{m}(\xi_1,\ldots,\xi_r)\bar{\underset{\sim}{y}}$$

$$\bar{\underset{\sim}{\phi}} = \underset{\sim}{g}_1^o(\xi_2,\ldots,\xi_r)h_1(\xi_1,a_1) + \underset{\sim}{g}_2^o(\xi_2,\ldots,\xi_r)h_2(\xi_1,a_2)$$

where the superscript ^{o} denotes the transform with respect to all coordinates except x_1, and h_1 and h_2 are the transforms of the delta functions with respect to x_1. Equation (14) then gives

$$\bar{\underset{\sim}{y}} = \underset{\sim}{m}^{-1}\left\{\bar{\underset{\sim}{f}} + \underset{\sim}{g}_1^o h_1 + \underset{\sim}{g}_2^o h_2\right\} \tag{15}$$

(ii) Denote by $\mathcal{J}^{-1}$ the inverse transform with respect to ξ_1. Then (15) gives

$$\mathcal{J}^{-1}\bar{\underset{\sim}{y}} = \underset{\sim}{y}^o = \mathcal{J}^{-1}\left\{\underset{\sim}{m}^{-1}\bar{\underset{\sim}{f}}\right\} + \mathcal{J}^{-1}\left\{\underset{\sim}{m}^{-1}h_1\right\}\underset{\sim}{g}_1^o + \mathcal{J}^{-1}\left\{\underset{\sim}{m}^{-1}h_2\right\}\underset{\sim}{g}_2^o \tag{16}$$

since $\underset{\sim}{g}_1^o$ and $\underset{\sim}{g}_2^o$ are not functions of ξ_1. When these transforms are evaluated (16) can be written in the form

$$\underset{\sim}{y}^o(x_1,\xi_2,\ldots,\xi_r) = \underset{\sim}{u}(x_1,\xi_2,\ldots,\xi_r) + \underset{\sim}{v}(x_1,\xi_2,\ldots,\xi_r)\,\underset{\sim}{g}_1^o(\xi_2,\ldots,\xi_r)$$

$$+ \underset{\sim}{w}(x_1,\xi_2,\ldots,\xi_r)\,\underset{\sim}{g}_2^o(\xi_2,\ldots,\xi_r) \tag{17}$$

(iii) Apply the transformed boundary conditions to (17) and solve the resulting equations for $\underset{\sim}{g}_1^o$ and $\underset{\sim}{g}_2^o$. Substitute these values into (17) to obtain an expression for $\underset{\sim}{y}^o$.

(iv) Apply the inverse transform to $\underset{\sim}{y}^o(x_1,\xi_2,\ldots,\xi_r)$ and obtain an expression for $\underset{\sim}{y}(x_1,\ldots,x_r)$.

Within the region $a_1 < x_1 < a_2$ these solutions satisfy (10) since $\underset{\sim}{\phi} = \underset{\sim}{0}$ there, and they also satisfy the boundary conditions on $x_1 = a_1$ and $x_1 = a_2$. They are therefore the required solutions.

4.6 Discussion.

The simple example given in section 4.2 is easily solved by the standard method. In fact it is only necessary to transform (1) with respect to x and t to obtain a differential equation whose solution is easily seen to be of the form (8). Thus we should give some justification for producing a method which at first sight seems to be more complicated than the standard ones.

Theoretically all problems of the type envisaged here can be solved by standard methods, and it is best that simple problems should be solved in this way. However, for more complicated problems involving several unknown functions the standard procedure becomes very difficult to apply. As an example, let us consider the amount of work necessitated by the standard method and by the present method for the following problem.

We require the solutions to a system of n simultaneous m'th order linear partial differential equations in n unknowns and r independent variables.

(i) Standard method: Transform each equation by an $(r-1)$ dimensional transform, giving n simultaneous mth order differential equations. Eliminate the unknowns to give n (mn)th order differential equations. Solve these equations and apply the $(r-1)$ dimensional inverse transforms.

(ii) Present method: Transform each equation by an r dimensional transform, giving n simultaneous algebraic equations. Solve this set of equations and apply the r dimensional inverse transforms.

Thus, at the expense of a one dimensional transform on each equation and a one dimensional inverse transform in each solution, we gain the advantage of having to solve a set of algebraic equations rather than a set of differential equations which are not easy to obtain. Even for fairly small values of m, n and r this advantage, and the fact that the present method is more systematic, greatly outweigh the disadvantages.

As an example the reader is referred to an application of this method in section 5.3. In particular we refer to equations (26) of that section, which, apart from the delta functions, are the equations whose solutions are required.

Nothing has been said so far about the values which should be taken for the unknown functions at the limits of integration, when transforming the differential equations. In many problems the values at one limit will be given by the corresponding finite medium problem, whilst the values at the other limit will be arbitrary. For example, in the example of section 4.2 the values at $r = 0$ are given, but it can easily be verified that the result (9) is independent of the conditions introduced at $r = \infty$. This is to be expected from the physical reasoning.

The method described in sections 4.3 and 4.5 also suggests the possibility of extension to the case of concentrating the causes C^* outside the boundary on the surface $x_1 = d$. However, we must then be more careful in order to retain mathematical rigour of the analysis. For instance, in the example of 4.2 we could have chosen a heat source on the radius $r = d$, where $d > a$. The analysis then continues as before and leads to the result (9). However, it can be seen that the expression for Θ is now

$$d\Theta^{\circ} = \frac{\theta_0^{\circ}}{I_0(ka)\,K_0(kd)} + \frac{A^{\circ}(a,\xi)}{K_0(kd)} \tag{18}$$

where A° is the transform of a function of the conditions introduced at $r = \infty$. If $A = 0$, this will lead to an integral expression for Θ which, for many choices of θ_0 will be divergent. When $d = a$ this mathematical difficulty disappears. Otherwise it may be possible to choose conditions at $r = \infty$ in such a way that the expression (18) converges.

In the application given in section 5.3, the causes C^* have been placed on the radius $r = d$. Mathematical rigour can be attained either by introducing suitable conditions at the limits of integration, or by taking $d = a$, though it will be seen that this does not make any difference to the final results.

CHAPTER FIVE.

FREE WAVES IN AN ELASTIC SOLID CONDUCTING HEAT.

The propagation of elastic waves is a subject which can be studied within the linked thermoelastic theory. Not all of the energy of deformation is used in producing a further disturbance in the material,- the remainder is used in producing a temperature distribution, and this in turn modifies the displacement field.

In this chapter we shall discuss three types of wave propagation. The analysis of plane waves given in section 5.1 is due to CHADWICK and SNEDDON (1958). The other two sections contain the thermal treatment of the type of surface wave named after Rayleigh, and a study of the propagation of longitudinal waves in cylinders and tubes. These discussions are due to LOCKETT (1958 and 1959a).

5.1 Plane Waves in an Elastic Solid Conducting Heat.

Using the dimensionless forms introduced by CHADWICK and SNEDDON (1958), the thermoelastic field equations are (see (I-29))

$$(\beta^2 - 1)\ \text{grad}\ \Delta + \nabla^2 \underset{\sim}{u} - b\ \text{grad}\ \theta = \beta^2 \ddot{\underset{\sim}{u}} \qquad (1)$$

$$\nabla^2 \theta = \dot{\theta} + g\dot{\Delta} \qquad (2)$$

in the absence of body forces and heat sources.

If we define the rotation vector $\underset{\sim}{\omega}$ = curl $\underset{\sim}{u}$ and remember that curl grad Δ = 0, then we find, on applying the operator curl throughout equation (1), that

$$\nabla^2 \underset{\sim}{\omega} = \beta^2 \frac{\partial^2 \underset{\sim}{\omega}}{\partial t^2} \tag{3}$$

which shows that $\underset{\sim}{\omega}$ is propagated with velocity β^{-1} in the dimensionless units, i.e. v_S in conventional units. This implies that transverse waves undergo no thermal modification. This result is to be expected since transverse waves cause no change in volume, and it is the dilatation which is the linking term in equation (2).

If we apply the operator div throughout equation (1) we get the equation

$$\beta^2 \nabla^2 \Delta - b\nabla^2 \theta = \beta^2 \frac{\partial^2 \Delta}{\partial t^2} \tag{4}$$

which, with (2), forms a pair of coupled equations for the determination of θ and Δ.

The analysis is now restricted to harmonic plane waves of the form

$$\begin{aligned} \Delta &= \Delta^\circ \exp\left\{ i\omega t + p(\underset{\sim}{a}.\underset{\sim}{r}) \right\} \\ \theta &= \theta^\circ \exp\left\{ i\omega t + p(\underset{\sim}{a}.\underset{\sim}{r}) \right\} \end{aligned} \tag{5}$$

where $\underset{\sim}{r}$ = (x,y,z), $\underset{\sim}{a}$ is a constant unit vector and ω is a real constant.

Since $v = \omega/\mathcal{I}(p)$ and $q = \mathcal{R}(p)$ are respectively the phase velocity and attenuation coefficient of the waves, we must choose values of p which have negative real parts.

If the expressions (5) are now substituted into (2) and (4) and the ratio $\theta^\circ/\Delta^\circ$ eliminated between the resulting equations, then we obtain the relation

$$p^4 + p^2 \left\{ \omega^2 - i\omega(1 + \epsilon) \right\} - i\omega^3 = 0 \tag{6}$$

for the determination of the possible values of p.

Chadwick and Sneddon considered approximate results corresponding to values of ω which are either much larger or much smaller than unity. They showed that one of the roots of the quadratic (6) corresponds to a modification of the purely thermal wave, whilst the other root corresponds to a modification of the purely elastic P-wave. When $\omega \gg 1$ the modified P-wave is transmitted with velocity (i.e. phase velocity) v_p but has an attenuation coefficient which, in conventional units, is

$$q_\infty = \tfrac{1}{2}\epsilon(\omega^*/v_p)$$

When $\omega \ll 1$ the phase velocity and attenuation coefficient are given, in conventional units, by

$$v = (1 + \tfrac{1}{2}\epsilon)v_p, \qquad q = \tfrac{1}{2}(2 - 5\epsilon)q_\infty(\omega/\omega^*)^2$$

where the fact that ϵ is small has also been used.

For the intermediate case of $\omega = 1$ (i.e. $\omega = \omega^*$ in conventional units) it is found that

$$v = (1 + \tfrac{1}{4}\epsilon)v_p, \qquad q = \tfrac{1}{2}q_\infty$$

These observations indicate that the attenuation coefficient of thermally modified P-waves is an increasing function of the frequency ω, varying like ω^2 at low frequencies and approaching the value q_∞ asymptotically as $\omega \to \infty$. This, and other observations made from the approximate results, have been demonstrated to be valid by a series of accurate numerical calculations carried out by the authors of the paper to which reference has been made.

The range of attainable frequencies is actually limited above by the cut-off frequency of the Debye spectrum. Also the characteristic frequency ω^* is very much greater than the frequencies attainable in practice even in experiments employing ultrasonic pulses. Thus, at the moment at least, we may assume that in practice $\omega \ll \omega^*$. Therefore, for instance, we could expect the measured velocity of longitudinal elastic waves to be $(1 + \frac{1}{2}\epsilon)v_p$. This has an interesting physical consequence.

The elastic constants λ and μ can be (and are) found experimentally in an experiment based on the accurate measurement of the phase velocity. Thus from the result above, the dynamical value of $(\lambda + 2\mu)$ should be calculated from the equation

$$(\lambda + 2\mu)_d = \rho v^2 = (1 + \epsilon)\rho v_p^2$$

Therefore, since the statical value is given by

$$(\lambda + 2\mu)_s = \rho v_p^2$$

we see that

$$\frac{(\lambda + 2\mu)_d}{(\lambda + 2\mu)_s} = 1 + \epsilon$$

In the case of lead it is found that the dynamical value will be nearly $7\frac{1}{2}\%$ higher than the statical value, and this is certainly greater than the experimental error of measurement made by the ultrasonic pulse technique.

5.2 Effect of Thermal Properties of a Solid on the Velocity of Rayleigh Waves.

We consider a semi-infinite solid $z \geq 0$ whose boundary $z = 0$ is free from stress. The classical theory of Rayleigh waves producing a displacement field of the type

$$\underset{\sim}{u} = \underset{\sim}{a}\, e^{-\alpha z + iq(x - ct)} \tag{1}$$

can be found in an article by SNEDDON and BERRY (1958)(p.109). In this section we shall consider how the form of these waves is affected when we take into account the thermal properties of the solid.

It is again convenient to use the dimensionless forms of the field equations due to Chadwick and Sneddon:

$$\begin{aligned} \nabla^2\underset{\sim}{u} + (\beta^2 - 1)\ \text{grad div}\ \underset{\sim}{u} - b\ \text{grad}\ \theta &= \beta^2\ddot{\underset{\sim}{u}} \\ \nabla^2\theta = \dot{\theta} + g\frac{\partial}{\partial t}\ \text{div}\ \underset{\sim}{u} \end{aligned} \tag{2}$$

We now assume a solution of these equations of the form

$$\underset{\sim}{u} = (a_1, 0, a_3)\, e^{i\phi}, \qquad \theta = \theta_1\, e^{i\phi} \tag{1a}$$

where $\phi = q(x - ct) + i\alpha z$.

We could have taken the more general displacements (1) but we would soon have discovered that $a_2 = 0$. If we substitute from equations (1a) into equations (2) we find that

$$\begin{aligned} &\left\{\alpha^2 - q^2\beta^2(1 - c^2)\right\}a_1 - iq\alpha(\beta^2 - 1)a_3 - ibq\theta_1 = 0 \\ &- iq\alpha(\beta^2 - 1)a_1 + \left\{\beta^2\alpha^2 - (1 - \beta^2c^2)q^2\right\}a_3 + b\alpha\theta_1 = 0 \\ &gcq^2a_1 + igcqa_3\alpha - (\alpha^2 - q^2 + iqc)\theta_1 = 0 \end{aligned} \tag{3}$$

Eliminating the ratios a_1/θ_1, a_3/θ_1 from these equations gives

$$\left\{\alpha^2 - (1 - \beta^2c^2)q^2\right\}\left[\beta^2\alpha^4 - (2\beta^2q^2 - iqc\beta^2 - \beta^2c^2q^2 - ibqgc)\alpha^2 + \left\{(1 - c^2)\beta^2q^2(q^2 - iqc) - ibq^3gc\right\}\right] = 0$$

The roots α_1^2, α_2^2, α_3^2 of this equation are given by the formulae

$$\alpha_1^2 = (1 - \beta^2 c^2)q^2,$$

$$\alpha_2^2 + \alpha_3^2 = (2q^2 - iqc - c^2q^2 - iqc\epsilon), \quad \alpha_2^2\alpha_3^2 = q^2(1 - c^2)(q^2 - iqc) - iq^3c\epsilon. \tag{4}$$

where we have written $\epsilon = bg/\beta^2$. The required square roots α_1, α_2, α_3 are those with positive real parts. To each of these roots α_i there corresponds a set of values of a_1, a_3, θ_1; we shall denote those values by a_{1i}, a_{3i}, θ_{1i}.

Case (i): $\alpha_i \neq \alpha_1$.

From the first two equations of the set (3) we find that, if $\alpha_i \neq \alpha_1$,

$$\alpha_i a_{1i} + iqa_{3i} = 0 \tag{5}$$

and

$$ibq\theta_{1i} = \beta^2\left\{ \alpha_i^2 - (1 - c^2)q^2 \right\}a_{1i} \tag{6}$$

Case (ii): $\alpha_i = \alpha_1$.

From the second and third equations of the set (3) we find that $\theta_{11} = 0$ and that

$$a_{31}q(1 - \beta^2c^2) = i\alpha_1 a_{11}. \tag{7}$$

The general solution of the field equations is then of the form $\underset{\sim}{u} = (u, 0, w)$ and θ where

$$u = (a_{11}e^{-\alpha_1 z} + a_{12}e^{-\alpha_2 z} + a_{13}e^{-\alpha_3 z})\, e^{iq(x - ct)}$$

$$w = (a_{31}e^{-\alpha_1 z} + a_{32}e^{-\alpha_2 z} + a_{33}e^{-\alpha_3 z})\, e^{iq(x - ct)} \tag{8}$$

$$\theta = (\theta_{11}e^{-\alpha_1 z} + \theta_{12}e^{-\alpha_2 z} + \theta_{13}e^{-\alpha_3 z})\, e^{iq(x - ct)}$$

Radiation Condition on the Boundary.

The boundary conditions are that the surface $z = 0$ is free from stress and that the radiation condition

$$\frac{\partial\theta}{\partial z} + h\theta = 0$$

holds over the boundary. With the forms (8) the shearing stress τ_{yz} is identically zero, and the three conditions $\tau_{xz} = \sigma_z = 0$, $\partial\theta/\partial z + h\theta = 0$ on $z = 0$ are satisfied if

$$(\alpha_1 a_{11} + \alpha_2 a_{12} + \alpha_3 a_{13}) = iq(a_{31} + a_{32} + a_{33}),$$

$$(\beta^2 - 2)iq(a_{11} + a_{12} + a_{13}) + \beta^2(\alpha_1 a_{31} + \alpha_2 a_{32} + \alpha_3 a_{33}) - b(\theta_{11} + \theta_{12} + \theta_{13}) = 0,$$

$$\alpha_1\theta_{11} + \alpha_2\theta_{12} + \alpha_3\theta_{13} = h(\theta_{11} + \theta_{12} + \theta_{13}).$$

With the aid of (4), (5), (6) and (7) we can write these equations in the form

$$-(\beta^2c^2 - 2)\alpha_1 a_{11} + 2(1 - \beta^2c^2)\alpha_2 a_{12} + 2(1 - \beta^2c^2)\alpha_3 a_{13} = 0,$$

$$-2a_{11} + (\beta^2c^2 - 2)a_{12} + (\beta^2c^2 - 2)a_{13} = 0,$$

$$(h-\alpha_2)\left\{\alpha_2^2 - (1-c^2)q^2\right\}a_{12} + (h-\alpha_3)\left\{\alpha_3^2 - (1-c^2)q^2\right\}a_{13} = 0$$

If we eliminate the ratios a_{12}/a_{11}, a_{13}/a_{11} and assume that $\alpha_2 \neq \alpha_3$

we find that

$$\left\{(\beta^2c^2-2)^2\alpha_1 h + 4(1-\beta^2c^2)\alpha_2\alpha_3\right\}\left\{\alpha_2^2+\alpha_3^2+2\alpha_2\alpha_3\right\}^{\frac{1}{2}}$$

$$= 4(1-\beta^2c^2)h\left\{\alpha_2\alpha_3+(1-c^2)q^2\right\} + (\beta^2c^2-2)^2\alpha_1\left\{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3-(1-c^2)q^2\right\} \qquad (9)$$

where α_1, α_2, α_3 are given by (4). Equation (9), with the substitutions made from (4), is an equation giving c in terms of q. We therefore find that dispersion occurs in the complete thermoelastic theory.

<u>Insulated Boundary.</u>

The results corresponding to zero flux of heat across the boundary may be obtained by putting $h = 0$ in the equations of the last section. If we put $h = 0$ in (9), square both sides of the equation and substitute the appropriate expressions for α_1^2 and $\alpha_2^2+\alpha_3^2$ from (4), we obtain

$$16(1-\beta^2c^2)\left\{(1-c^2)(\omega^2/c^2-i\omega)-i\omega\epsilon\right\}(2\omega^2/c^2-i\omega-\omega^2-i\omega\epsilon+2\alpha_2\alpha_3)$$

$$= (\beta^2c^2-2)^4(\omega^2/c^2-i\omega-i\omega\epsilon+\alpha_2\alpha_3)^2 \qquad (10)$$

where $\omega = qc$, and

$$\alpha_2\alpha_3 = \left\{\frac{\omega^4}{c^4}-\frac{\omega^4}{c^2}-\frac{i\omega^3}{c^2}+i\omega^3-\frac{i\omega^3}{c^2}\epsilon\right\}^{\frac{1}{2}} \qquad (11)$$

Now ω is 2π times the frequency of the wave and in the present system of units $\omega \ll 1$, so that a reasonable approximation can be expected by retaining only the lowest power of ω occurring in (10). In this way we obtain

$$16(1-\beta^2c^2)(1-c^2+\epsilon) = (\beta^2c^2-2)^4(1+\epsilon) \qquad (12)$$

This approximate equation holds not only for the special case but also for the general case considered in the last section. The
This approximate equation holds not only for the special case but also for the general case considered in the last section. The
proof is straightforward and is omitted here.

Comparison with the Classical Theory.

To compare (12) with the classical theory we first write it in the form

$$(1+\epsilon)\beta^2(\beta c)^6 - 8(1+\epsilon)\beta^2(\beta c)^4 + 8\left\{3(1+\epsilon)\beta^2 - 2\right\}(\beta c)^2 - 16\left\{(1+\epsilon)\beta^2 - 1\right\} = 0 \tag{13}$$

and then return to conventional units. If the velocity of S-waves in the solid is denoted by v_S this becomes

$$(1+\epsilon)\beta^2\left\{\frac{c}{v_S}\right\}^6 - 8(1+\epsilon)\beta^2\left\{\frac{c}{v_S}\right\}^4 + 8\left\{3(1+\epsilon)\beta^2 - 2\right\}\left\{\frac{c}{v_S}\right\}^2 - 16\left\{(1+\epsilon)\beta^2 - 1\right\} = 0 \tag{14}$$

On comparing this with the equation for the determination of the velocity of the Rayleigh waves in the classical theory (SNEDDON and BERRY (1958) equation (71.16)) we see that (14) is merely the classical equation with β^2 replaced by $(1+\epsilon)\beta^2$.

We now consider a typical case. For the typical values $\beta^2 = 3$, $\epsilon = 0.05$, equation (14) becomes

$$3.15\ x^6 - 25.20\ x^4 + 59.60\ x^2 - 34.40 = 0 \tag{15}$$

whereas the corresponding classical equation is

$$3\ x^6 - 24\ x^4 + 56\ x^2 - 32 = 0 \tag{16}$$

where $x = c/v_S$. The root of (16) lying between 0 and 1 is 0.9194 and this gives $\alpha_1 = 0.3933q$, $\alpha_2 = 0.8475q$. The corresponding expressions for the components of the displacement vector are

$$u = A(0.5774\, e^{-0.3933qz} - e^{-0.8475qz})\, e^{iq(x-ct)}$$

$$w = iA(1.4679\, e^{-0.3933qz} - 0.8475\, e^{-0.8475qz})\, e^{iq(x-ct)} \qquad (17)$$

The other roots make α_1 and α_2 purely imaginary and do not lead to Rayleigh waves. The corresponding root of (15) is 0.9224 so that the percentage difference in the value of the velocity of propagation of Rayleigh waves is less than one per cent (it is in fact about $\frac{1}{3}$%). This root leads to the values $\alpha_1 = 0.3863q$, $\alpha_2 = 0.8422q$ and, if q is small, to the value $\alpha_3 = 0.5288(1-i)q^{\frac{1}{2}}$, where we have neglected terms of order $q^{3/2}$ and above.

In the limiting case of q extremely small, these roots lead to the expressions

$$u = A(0.5745\, e^{-0.3863qz} - e^{-0.8422qz})\, e^{iq(x-ct)}$$

$$w = iA(1.4875\, e^{-0.3863qz} - 0.8422\, e^{-0.8422qz})\, e^{iq(x-ct)} \qquad (18)$$

$$\theta = -0.0213\, iqb^{-1}A\, e^{-0.8422qz + iq(x-ct)}$$

A comparison of the sets of equations (17) and (18) shows that the coefficients calculated on the thermoelastic theory differ from those calculated on the classical theory by amounts of the order of one per cent.

5.3 Longitudinal Elastic Waves in Cylinders and Tubes.

Introduction.

Longitudinal elastic waves in solid circular cylinders were first investigated by POCHHAMMER (1876) and CHREE (1889) but they did not take into account the effects that thermal properties have on the propagation of these waves. In this section we shall consider waves in solid and hollow cylinders as well as in the infinite medium with a cylindrical cavity, and in each case we shall take account of the thermoelastic effects.

The type of wave considered can be characterised in cylindrical polar coordinates by the components of displacement

$$u_r = R(r)\, e^{i(qz + pt)}, \qquad u_z = Z(r)\, e^{i(qz + pt)}, \qquad u_\psi = 0 \tag{1}$$

Here the z-axis is the axis of the cylinder and we consider wave propagation with cylindrical symmetry. R and Z are functions of the radial coordinate r only, and the problem is to find expressions for them such that the field equations and the boundary conditions are satisfied. In all the problems considered here the boundary conditions are such that the components of stress σ_r and τ_{rz} vanish on the cylindrical boundaries. There is also a thermal radiation condition.

We shall again use the dimensional forms (I-29) of the field equations and we shall write them in the form

$$\begin{aligned} \beta^2 \text{grad div } \underset{\sim}{u} - 2 \text{ curl } \underset{\sim}{\omega} - b \text{ grad } \theta &= \beta^2 \ddot{\underset{\sim}{u}} - \underset{\sim}{F} \\ \nabla^2 \theta &= \dot{\theta} + g \frac{\partial}{\partial t} (\text{div } \underset{\sim}{u}) - \Psi \end{aligned} \tag{2}$$

where $\underset{\sim}{\omega} = \frac{1}{2}$ curl $\underset{\sim}{u}$, and we have used $\underset{\sim}{F}$ and Ψ to denote the dimensionless forms of the body forces and heat sources respectively.

In an analysis where thermal effects are ignored it is obvious that equations (2) are replaced by the simpler vector equation

$$\beta^2 \text{grad div } \underset{\sim}{u} - 2 \text{ curl } \underset{\sim}{\omega} = \beta^2 \ddot{\underset{\sim}{u}} - \underset{\sim}{F} \tag{3}$$

In the actual problems considered there are no body forces or heat sources present. However we need to retain them in equations (2) and (3) to enable us to apply the method of solution given by LOCKETT (1959c) which has been described in Chapter IV of this thesis.

NON-THERMAL ANALYSES.

(i) Solid Circular Cylinder.

The results of POCHHAMMER (1876) and CHREE (1889) for this problem are well known. However, we shall re-derive their results here, since it illustrates the method on a relatively simple example whose result is already known, and because the results of the other two problems are quickly obtainable from this analysis.

Following the method of Chapter IV, we consider the infinite medium subjected to wave propagation of the type (1) and to the action of body forces concentrated on the radius $r = d$, where $d > a$. We write these body forces in the form

$$F_r = Ad^{-1}\delta(r - d)\, e^{i(qz-pt)}, \quad F_z = Bd^{-1}\delta(r - d)\, e^{i(qz-pt)} \tag{4}$$

where $\delta(x)$ is the Dirac Delta function and A and B are to be chosen so that the components of stress σ_r and τ_{rz} should vanish on $r = a$. The exponential dependence is necessitated by the choice of the expressions (1) and the term d^{-1} is put in for convenience at a later stage.

Substitution from (1) and (4) into (3) gives us the set of differential equations

$$\beta^2\left(R'' + \frac{1}{r}R' - \frac{1}{r^2}R + iqZ'\right) + iq(iqR - Z') = -\beta^2 p^2 R - Ad^{-1}\delta(r - d)$$

$$\beta^2 iq\left(R' + \frac{1}{r}R + iqZ\right) - \frac{1}{r}(iqR + iqR' - Z' - rZ'') = -\beta^2 p^2 Z - Bd^{-1}\delta(r - d) \qquad (5)$$

which can be solved by the transform method.

Defining the Hankel transforms

$$\bar{R}(\xi) = \int_0^\infty rJ_1(\xi r)R(r)\,dr, \qquad \bar{Z}(\xi) = \int_0^\infty rJ_0(\xi r)Z(r)\,dr \qquad (6)$$

equations (7) may be transformed and simplified to read

$$(\beta^2 p^2 - \beta^2\xi^2 - q^2)\bar{R} - (\beta^2 - 1)iq\xi\bar{Z} = -AJ_1(\xi d)$$

$$(\beta^2 - 1)iq\xi\bar{R} + (\beta^2 p^2 - \beta^2 q^2 - \xi^2)\bar{Z} = -BJ_0(\xi d) \qquad (7)$$

The solutions of these algebraic equations can be expressed in the form

$$\bar{R} = -\beta^{-2}p^{-2}iqB\xi J_0(\xi d)\left\{(\xi^2 + k_1^2)^{-1} - (\xi^2 + k_2^2)^{-1}\right\}$$

$$- \beta^{-2}AJ_1(\xi d)\left\{-p^{-2}q^2(\xi^2 + k_1^2)^{-1} + p^{-2}k_2^2(\xi^2 + k_2^2)^{-1}\right\}$$

$$\bar{Z} = -\beta^{-2}BJ_0(\xi d)\left\{p^{-2}k_1^2(\xi^2 + k_1^2)^{-1} - p^{-2}q^2(\xi^2 + k_2^2)^{-1}\right\} \qquad (8)$$

$$+ \beta^{-2}p^{-2}iqA\xi J_1(\xi d)\left\{(\xi^2 + k_1^2)^{-1} - (\xi^2 + k_2^2)^{-1}\right\}$$

where

$$k_1^2 = q^2 - \beta^2 p^2, \qquad k_2^2 = q^2 - p^2 \qquad (9)$$

and these expressions are in a convenient form for applying the inverse transforms

$$R = \int_0^\infty \xi \bar{R} J_1(\xi r)\, d\xi, \qquad Z = \int_0^\infty \xi \bar{Z} J_0(\xi r)\, d\xi \tag{10}$$

If we use the notation

$$I_{mn} = I_m(k_n r), \qquad K_{mn} = K_m(k_n d) \tag{11}$$

and regroup the terms in the resulting expressions, these integrations give

$$\begin{aligned} R &= -\beta^{-2}p^{-2}\left\{(k_2^2 A K_{12} + iqB k_2 K_{02}) I_{12} - (Aq^2 K_{11} + iqB k_1 K_{01}) I_{11}\right\} \\ Z &= -\beta^{-2}p^{-2}\left\{(iqk_2 A K_{12} - q^2 B K_{02}) I_{02} - (iqA k_1 K_{11} - B k_1^2 K_{01}) I_{01}\right\} \end{aligned} \tag{12}$$

The required integrals can be found in ERDELYI (1954) (Vol. II p.49) and have been evaluated for $0 < r < d$. Thus it is at this stage that the physical requirement (that the body forces should be outside or on the radius $r = a$) enters the mathematics.

These forms for the integrals, as stated in the published tables, also require that $\mathcal{R}\, k_1 > 0$ and $\mathcal{R}\, k_2 > 0$. However, it can be verified that the results (14) satisfy the given conditions even when these conditions are not satisfied.

We can now introduce the parameters

$$\begin{aligned} L &= -\beta^{-2}p^{-2}(k_2 A K_{12} + iqB K_{02}) \\ M &= -\beta^{-2}p^{-2}(iqA K_{11} - B k_1 K_{01}) \end{aligned} \tag{13}$$

and write

$$
\begin{aligned}
R &= k_2 L I_{12} + iqM I_{11} \\
Z &= iqL I_{02} - k_1 M I_{01}
\end{aligned} \tag{14}
$$

Since

$$
\begin{aligned}
\sigma_r &= (\beta^2 - 2)\left\{\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right\} + 2\frac{\partial u_r}{\partial r} \\
\tau_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}
\end{aligned} \tag{15}
$$

the expressions (14) and the boundary conditions, $\sigma_r = \tau_{rz} = 0$ on $r = a$, imply that

$$
\begin{aligned}
&2iqk_1 I'_{11} L - \left\{(\beta^2 - 2)p^2 I_{02} - 2k_2^2 I'_{12}\right\} M = 0 \\
&(\beta^2 p^2 - 2q^2) I_{11} L + 2iqk_2 I_{12} M = 0
\end{aligned} \tag{16}
$$

where the modified Bessel functions are now evaluated for $r = a$. These are the equations determining the ratio L/M. (The initial amplitude of the wave is arbitrary).

The compatibility of equations (16) leads to the frequency equation

$$
F = \begin{vmatrix} 2iqk_1 I'_1(k_1 a) & -(\beta^2 - 2)p^2 I_0(k_2 a) + 2k_2^2 I'_1(k_2 a) \\ (\beta^2 p^2 - 2q^2) I_1(k_1 a) & 2iqk_2 I_1(k_2 a) \end{vmatrix} = 0 \tag{17}
$$

i.e.

$$
4q^2 k_1 k_2 I_{12} I'_{11} = (\beta^2 p^2 - 2q^2) I_{11} \left\{(\beta^2 - 2)p^2 I_{02} - 2k_2^2 I'_{12}\right\} \tag{18}
$$

which apart from slight differences of notation is the result obtained by previous authors (see e.g. SNEDDON and BERRY (1958) Eq.(73.24)).

(ii) Infinite Medium with Cylindrical Cavity.

The only difference between this and the previous analysis is that $d \leq a \leq r$ and we are therefore interested in evaluating the integrals (10) for $d < r$. It can be seen from the published tables that the result of this inequality is to reverse the roles of the modified Bessel functions I and K in the previous analysis. Thus R and Z are given by expressions of the form

$$R = k_2 LK_{12} + iqMK_{11}\,,$$
$$Z = iqLK_{02} - k_1 MK_{01}\,, \qquad (19)$$

and the frequency equation becomes

$$Q = \begin{vmatrix} 2iqk_1 K_1'(k_1 a) & -(\beta^2 - 2)p^2 K_0(k_2 a) + 2ik_2^2 K_1'(k_2 a) \\ (\beta^2 p^2 - 2q^2)K_1(k_1 a) & 2iqk_2 K_1(k_2 a) \end{vmatrix} = 0 \qquad (20)$$

(iii) Hollow circular cylinder.

We now have to consider the extra body forces

$$F_r = \left\{ A_1 d_1^{-1} \delta(r - d_1) + A_2 d_2^{-1} \delta(r - d_2) \right\} e^{i(qz + pt)}$$
$$F_z = \left\{ B_1 d_1^{-1} \delta(r - d_1) + B_2 d_2^{-1} \delta(r - d_2) \right\} e^{i(qz + pt)} \qquad (21)$$

where

$$d_1 \geq a_1 > a_2 \geq d_2$$

a_1 and a_2 being respectively the external and internal radii of the tube.

The components with suffix "1" behave like the forces in the solid cylinder analysis, and those with suffix "2" behave like the forces in the cavity analysis. It is easily seen that their combined effect leads to the expressions

$$R = k_2 L_1 I_{12} + iqM_1 I_{11} + k_2 L_2 K_{12} + iqM_2 K_{11}$$

$$Z = iqL_1 I_{02} - k_1 M_1 I_{01} + iqL_2 K_{02} - k_1 M_2 K_{01} \qquad (22)$$

We now have to apply the conditions $\sigma_r = \tau_{rz} = 0$ on both $r = a_1$ and $r = a_2$, and this leads to a frequency equation

$$\begin{vmatrix} P(a_1) & Q(a_1) \\ P(a_2) & Q(a_2) \end{vmatrix} = 0 \qquad (23)$$

where the minors P and Q are defined in (17) and (20), and the arguments a_1 and a_2 show that they have been evaluated for $r = a_1$ and $r = a_2$ respectively.

THERMOELASTIC EFFECTS.

(i) Solid Circular Cylinder.

To the forms (1) we now have to add the temperature distribution θ in the form

$$\theta = \Theta(r)\, e^{i(qz + pt)} \qquad (24)$$

Since there is an extra (thermal) condition to apply we have also to consider a concentrated heat source

$$\Psi = Qd^{-1}\delta(r - d)\, e^{i(qz + pt)} \qquad (25)$$

in addition to the body forces (4).

Substitution of (1), (4), (24) and (25) into (2) gives the set of differential equations

$$\beta^2\left(R'' + \frac{1}{r}R' - \frac{1}{r^2}R + iqZ'\right) + iq(iqR - Z') - b\Theta' = -\beta^2 p^2 R - Ad^{-1}\delta(r-d)$$

$$\beta^2 iq\left(R' + \frac{1}{r}R + iqZ\right) - \frac{1}{r}(iqR + iqrR' - Z' - rZ'') - biq\Theta = -\beta^2 p^2 Z - Bd^{-1}\delta(r-d) \qquad (26)$$

$$\Theta'' + \frac{1}{r}\Theta' - q^2\Theta = ip\Theta + gip\left(R' + \frac{1}{r}R + iqZ\right) - Cd^{-1}\delta(r-d)$$

which are the thermal counterparts of equations (5).

We can now transform these equations to the form

$$(\beta^2p^2 - \beta^2\xi^2 - q^2)\bar{R} - (\beta^2 - 1)iq\xi\bar{Z} + b\xi\bar{\Theta} = -AJ_1(\xi d)$$

$$(\beta^2 - 1)iq\xi\bar{R} + (\beta^2p^2 - \beta^2q^2 - \xi^2)\bar{Z} - biq\bar{\Theta} = -BJ_0(\xi d) \qquad (27)$$

$$gip\xi\bar{R} - gpq\bar{Z} + (\xi^2 + q^2 + ip)\bar{\Theta} = -CJ_0(\xi d)$$

where $\bar{R}$ and $\bar{Z}$ are defined by (6) and

$$\bar{\Theta}(\xi) = \int_0^\infty r\, J_0(\xi r)\, \Theta(r)\, dr \qquad (28)$$

The solutions of equations (27) are found to be

$$\bar{R} = \Big[D\left\{(\xi^2+q^2+ip)(\xi^2+\beta^2q^2-\beta^2p^2)+ipbgq^2\right\} - iC\left\{(\beta^2-1)(\xi^2+q^2+ip)+ipbg\right\}\xi q$$
$$- K(\xi^2+q^2-\beta^2p^2)b\xi\Big] \Big/ (\xi^2+q^2-\beta^2p^2)\mathcal{D}$$

$$\bar{Z} = \Big[iD\left\{(\beta^2-1)(\xi^2+q^2+ip)+ipbg\right\}\xi q + C\left\{(\xi^2+q^2+ip)(\beta^2\xi^2+q^2-\beta^2p^2)+ipbg\xi^2\right\}$$
$$+ iK(\xi^2+q^2-\beta^2p^2)bq\Big] \Big/ (\xi^2+q^2-\beta^2p^2)\,\mathcal{D} \qquad (29)$$

$$\bar{S} = \left\{ -Dipg\xi + Gpgq - K(\beta^2\xi^2+\beta^2q^2-\beta^2p^2) \right\} / \mathcal{D} \qquad (29)$$

where

$$D = A\, J_1(\xi a), \qquad G = B\, J_0(\xi a), \qquad K = C\, J_0(\xi a)$$

and $\qquad (30)$

$$\mathcal{D} = \beta^2(\xi^2+q^2-p^2)(\xi^2+q^2+4p) + 4pbg(\xi^2+q^2)$$

We now write $\mathcal{D}$ in the form

$$\mathcal{D} = \beta^2(\xi^2+k_3^2)(\xi^2+k_4^2) \qquad (31)$$

where k_3^2 and k_4^2 are the roots of the equation

$$\beta^2x^2 - (2\beta^2q^2+4pg^2-\beta^2p^2+4pbg)x + \beta^2(q^2+4p)(q^2-p^2) + 4pbgq^2 = 0 \qquad (32)$$

It is then easily seen that

$$\beta^2/\mathcal{D} = (k_4^2-k_3^2)^{-1}\left\{(\xi^2+k_3^2)^{-1}-(\xi^2+k_4^2)^{-1}\right\}$$

$$\beta^2\xi^2/\mathcal{D} = (k_4^2-k_3^2)^{-1}\left\{-k_3^2(\xi^2+k_3^2)^{-1}+k_4^2(\xi^2+k_4^2)^{-1}\right\}$$

$$\beta^2/\mathcal{D}\,(\xi^2+k_1^2) = X(\xi^2+k_1^2)^{-1} + Y(\xi^2+k_3^2)^{-1} + Z(\xi^2+k_4^2)^{-1}$$

$$\beta^2\xi^2/\mathcal{D}\,(\xi^2+k_1^2) = -k_1^2X(\xi^2+k_1^2)^{-1} - k_3^2Y(\xi^2+k_3^2)^{-1} - k_4^2Z(\xi^2+k_4^2)^{-1} \qquad (33)$$

$$\beta^2\xi^4/\mathcal{D}\,(\xi^2+k_1^2) = k_1^4X(\xi^2+k_1^2)^{-1} + k_3^4Y(\xi^2+k_3^2)^{-1} + k_4^4Z(\xi^2+k_4^2)^{-1}$$

where

$$X = \left\{(k_3^2-k_1^2)(k_4^2-k_1^2)\right\}^{-1}, \quad Y = -\left\{(k_3^2-k_1^2)(k_4^2-k_3^2)\right\}^{-1}, \quad Z = \left\{(k_4^2-k_1^2)(k_4^2-k_3^2)\right\}^{-1}$$

Thus the expressions (29) can all be written as the sums of terms like

$$Q\xi^{p}(\xi^{2}+k_{i}^{2})^{-1}J_{\mu}(\xi d)$$

where Q is a coefficient not involving ξ, and where p and μ can take the values 0 or 1, and i can take the value 1, 3 or 4.

If we now apply to the new forms of (29) the transforms inverse to (6) and (28) we find that R, Z and Θ are the sums of terms of the type

$$Q\int_{0}^{\infty}\xi^{p+1}(\xi^{2}+k_{i}^{2})^{-1}J_{\mu}(\xi d)\ J_{\nu}(\xi r)\ d\xi$$

which, when evaluated for $r < d$, is of the form

$$(\text{coefficient}) \times K_{\mu}(k_{i}d)\ I_{\nu}(k_{i}r)$$

In each of the expressions for R, Z and Θ, we then collect together the terms involving the same Bessel function $I_{n}(k_{n}r)$, so that these expressions can be written in the form

$$(\text{coefficient } P_{mn}) \times I_{n}(k_{n}r)$$

Although we do not give the details of the algebra here, it is found that the coefficients P_{mn}, which depend on d, can all be written down in terms of three new parameters L, M and N. In so doing it is necessary to remember that k_3^2 and k_4^2 are roots of the equation (32). We then get

$$R = iqLI_{11} + bk_{3}MI_{13} + bk_{4}NI_{14}$$

$$Z = -k_{1}LI_{01} + ibqMI_{03} + ibqNI_{04} \qquad (34)$$

$$\Theta = \beta^{2}(k_{3}^{2} + p^{2} - q^{2})MI_{03} + \beta^{2}(k_{4}^{2} + p^{2} - q^{2})NI_{04}$$

and it can be verified that these expressions satisfy (26) for $r \leq a$.

If we now apply the boundary conditions $\sigma_r = 0$, $\tau_{rz} = 0$, $\partial\theta/\partial r + h\theta = 0$ on $r = a$, we obtain three equations in L, M and N, the compatibility of which demands that

$$U = \begin{vmatrix} 2iqk_1 I'_{11} & b\left\{2k_3^2 I'_{13} + (2q^2 - 2k_3^2 - \beta^2 p^2) I_{03}\right\} & b\left\{2k_4^2 I'_{14} + (2q^2 - 2k_4^2 - \beta^2 p^2) I_{04}\right\} \\ (\beta^2 p - 2q^2) I_{11} & 2ibqk_3 I_{13} & 2ibqk_4 I_{14} \\ 0 & (q^2 - p^2 - k_3^2)(k_3 I_{13} + hI_{03}) & (q^2 - p^2 - k_4^2)(k_4 I_{14} + hI_{04}) \end{vmatrix} = 0 \qquad (35)$$

where the modified Bessel functions are evaluated for $r = a$, that is $I_{mn} = I_m(k_n a)$. Equation (35) is the thermoelastic frequency equation.

(ii) <u>Infinite Medium with Cylindrical Cavity.</u>

Having already derived the solutions to problems (ii) and (iii) from the solution of problem (i) for the non-thermal case, it is not necessary to describe how this is done for the thermal case. It is easily seen that the thermoelastic frequency equation is

$$V = \begin{vmatrix} 2iqk_1 K'_{11} & b\left\{2k_3^2 K'_{13} + (2q^2 - 2k_3^2 - \beta^2 p^2) K_{03}\right\} & b\left\{2k_4^2 K'_{14} + (2q^2 - 2k_4^2 - \beta^2 p^2) K_{04}\right\} \\ (\beta^2 p^2 - 2q^2) K_{11} & 2ibqk_3 K_{13} & 2ibqk_4 K_{14} \\ 0 & (q^2 - p^2 - k_3^2)(k_3 K_{13} + hK_{03}) & (q^2 - p^2 - k_4^2)(k_4 K_{14} + hK_{04}) \end{vmatrix} = 0 \qquad (36)$$

(iii) Hollow Circular Cylinder.

From the results of problems (i) and (ii) and the reasoning given in the non-thermal theory we see that the new frequency equation is of the form

$$\begin{vmatrix} U(a_1) & V(a_1) \\ U(a_2) & V(a_2) \end{vmatrix} = 0 \qquad (37)$$

where the minors U and V are defined in (35) and (36) and the arguments a_1 and a_2 indicate that the Bessel functions are evaluated for $r = a_1$ and $r = a_2$ respectively.

CHAPTER SIX.

PROPAGATION OF THERMAL STRESSES IN AN INFINITE SOLID.

The problems considered in the previous chapters were all simplified by some special feature -- steady-state, radial symmetry, wave propagation of a prescribed form. In the last two chapters of this thesis a beginning will be made on the difficult task of solving the linked equations in their most general form.

In this chapter we shall consider the effect of arbitrary heat sources and body forces applied to a medium of infinite extent. The classical solutions to this problem have been given by GOODIER (1937), NOWACKI (1957c) and others, whilst the problem of heat sources in the infinite medium has been treated, using the linked equations, by EASON and SNEDDON (1959). In this discussion we shall use the full dynamical equations of BIOT (1956), and we shall consider the effect of both heat sources and body forces. Thus the results of EASON and SNEDDON for heat sources only will be rederived, together with the results for variable body forces which are due to LOCKETT and SNEDDON (1959/60).

6.1 General Theory.

Rectangular Cartesian Coordinates.

Using rectangular cartesian coordinates and the dimensionless forms due to SNEDDON and BERRY the thermoelastic equations can be written as

$$\tau_{ij,j} + X_i = a\ddot{u}_i \qquad i,j = 1,2,3. \qquad (1)$$

$$\tau_{ij} = \left\{(\beta^2 - 2)\Delta - b\theta\right\}\delta_{ij} + 2\gamma_{ij} \qquad i,j = 1,2,3. \tag{2}$$

$$\Theta + \nabla^2\theta = f\dot{\theta} + g\dot{\Delta} \tag{3}$$

where the rectangular cartesian coordinates are denoted by x_1, x_2, x_3 and the other quantities have a similar convention.

If we eliminate the stress between (1) and (2) we get an equation

$$(\beta^2 - 2)\Delta_{,i} - b\theta_{,i} + 2\gamma_{ij,j} + X_i = a\ddot{u}_i \tag{4}$$

which can be differentiated with respect to x_i to give

$$\beta^2\Delta_{,ii} - b\theta_{,ii} + X_{i,i} = a\ddot{\Delta} \tag{5}$$

We now define the multiple integral transform

$$\bar{f}(\xi_1,\xi_2,\xi_3,\omega) = \frac{1}{4\pi^2}\int_{V_4} f(x_1,x_2,x_3,t)\,\exp\left\{i(\xi_j x_j + \omega t)\right\}\,dV \tag{6}$$

where $dV = dx_1 dx_2 dx_3 dt$ and where the integral is taken over the entire $x_1 x_2 x_3 t$-space.

Then by multiplying throughout equations (3),(4),(5) by $(4\pi^2)^{-1}\exp\left\{i(\xi_j x_j + \omega t)\right\}$ and integrating over V_4, we get the transformed equations

$$\bar{\Theta} - \underset{\sim}{\xi}^2\bar{\theta} = -i\omega f\bar{\theta} - i\omega g\bar{\Delta} \tag{7}$$

$$\bar{X}_i - i\xi_i(\beta^2 - 2)\bar{\Delta} + ib\xi_i\bar{\theta} - \underset{\sim}{\xi}^2\bar{u}_i - i\xi_i\bar{\Delta} = -a\omega^2\bar{u}_i \tag{8}$$

$$-i\xi_i\bar{X}_i - \beta^2\underset{\sim}{\xi}^2\bar{\Delta} + b\underset{\sim}{\xi}^2\bar{\theta} = -a\omega^2\bar{\Delta} \tag{9}$$

where $\underset{\sim}{\xi}^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

The solutions of equations (7) and (9) are

$$\bar{\theta} = \frac{\omega g \xi_n \bar{X}_n + (\beta^2 \underset{\sim}{\xi}^2 - a\omega^2)\bar{\Theta}}{\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)} \tag{10}$$

$$\bar{\Delta} = \frac{-\xi_n \bar{X}_n(\underset{\sim}{\xi}^2 - i\omega f) + b\underset{\sim}{\xi}^2\bar{\Theta}}{\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)} \tag{11}$$

and we can now use (8) to give

$$\bar{u}_i = \frac{\bar{X}_i}{\underset{\sim}{\xi}^2 - a\omega^2} - \frac{\left\{(\beta^2 - 1)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega b g\right\}\xi_i \xi_n \bar{X}_n}{(\underset{\sim}{\xi}^2 - a\omega^2)\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)} + \frac{i b \xi_i \bar{\Theta}}{\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)} \tag{12}$$

where we have defined

$$\mathfrak{D}(\omega, \underset{\sim}{\xi}^2) = (\beta^2 \underset{\sim}{\xi}^2 - a\omega^2)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega b g \underset{\sim}{\xi}^2 \tag{13}$$

Finally application of the inverse transforms to expressions (10) and (12) gives us the displacement and temperature fields

$$u_i = \frac{1}{4\pi^4}\int_{W_4} \left\{\frac{\bar{X}_i}{\underset{\sim}{\xi}^2 - a\omega^2} - \frac{\left\{(\beta^2 - 1)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega b g\right\}\xi_i \xi_n \bar{X}_n}{(\underset{\sim}{\xi}^2 - a\omega^2)\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)} + \frac{i b \xi_i \bar{\Theta}}{\mathfrak{D}}\right\} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{14}$$

$$\theta = \frac{1}{4\pi^4}\int_{W_4} \left\{\frac{\omega g \xi_n \bar{X}_n + (\beta^2 \underset{\sim}{\xi}^2 - a\omega^2)\bar{\Theta}}{\mathfrak{D}(\omega, \underset{\sim}{\xi}^2)}\right\} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{15}$$

where $dW = d\xi_1 d\xi_2 d\xi_3 d\omega$ and the integration is taken over the entire $\xi_1\xi_2\xi_3\omega$-space.

Problems with Axial Symmetry.

It is often interesting to consider problems in which, if we use cylindrical polar coordinates, there is symmetry about the z-axis. We shall therefore consider this type of problem in its most general form.

Assuming symmetry about the z-axis, the thermoelastic field equations can be written in the form

$$\beta^2\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} + \frac{\partial^2 u}{\partial z^2} + (\beta^2 - 1)\frac{\partial^2 w}{\partial r \partial z} + X_r - b\frac{\partial \theta}{\partial r} = a\frac{\partial^2 u}{\partial t^2} \tag{16}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \beta^2\frac{\partial^2 w}{\partial z^2} + (\beta^2 - 1)\frac{\partial}{\partial z}\left\{\frac{\partial u}{\partial r} + \frac{u}{r}\right\} + X_z - b\frac{\partial \theta}{\partial z} = a\frac{\partial^2 w}{\partial t^2} \tag{17}$$

$$\Theta + \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} = f\frac{\partial \theta}{\partial t} + g\frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right\} \tag{18}$$

Thus if we make use of the transforms

$$(\bar{\bar{u}}, \bar{\bar{X}}_r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\zeta z + \omega t)}\,dz\,dt \int_0^{\infty} r\,J_1(\xi r)(u, X_r)\,dr \tag{19}$$

$$(\bar{\bar{w}}, \bar{\bar{\theta}}, \bar{\bar{\Theta}}, \bar{\bar{X}}_z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\zeta z + \omega t)}\,dz\,dt \int_0^{\infty} r\,J_0(\xi r)(w, \theta, \Theta, X_z)\,dr \tag{20}$$

then equations (16),(17) and (18) transform to

$$(\beta^2\xi^2 + \zeta^2 - a\omega^2)\bar{\bar{u}} - i(\beta^2 - 1)\xi\zeta\bar{\bar{w}} - \bar{\bar{X}}_r = b\xi\bar{\bar{\theta}} \tag{21}$$

$$i(\beta^2 - 1)\xi\zeta\bar{\bar{u}} + (\xi^2 + \beta^2\zeta^2 - a\omega^2)\bar{\bar{w}} - \bar{\bar{X}}_z = ib\zeta\bar{\bar{\theta}} \tag{22}$$

$$-\bar{\bar{\Theta}} + (\xi^2 + \zeta^2 - i\omega f)\bar{\bar{\theta}} = i\omega g(\xi\bar{\bar{u}} - i\zeta\bar{\bar{w}}) \tag{23}$$

The solutions of these equations are

$$\bar{\bar{u}} = \frac{\left\{(\xi^2+\beta^2\zeta^2-a\omega^2)(\xi^2+\zeta^2-i\omega f)-i\omega bg\zeta^2\right\}\bar{\bar{X}}_r}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}$$

$$+\frac{i\left\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\right\}\xi\zeta\bar{\bar{X}}_z}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}+\frac{ib\xi\bar{\bar{\theta}}}{\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)} \qquad (24)$$

$$\bar{\bar{w}} = \frac{-i\left\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\right\}\xi\zeta\bar{\bar{X}}_r}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}$$

$$+\frac{\left\{(\beta^2\xi^2+\zeta^2-a\omega^2)(\xi^2+\zeta^2-i\omega f)-i\omega bg\xi^2\right\}\bar{\bar{X}}_z}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}+\frac{ib\zeta\bar{\bar{\theta}}}{\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)} \qquad (25)$$

$$\bar{\theta} = \frac{g\omega(i\xi\bar{\bar{X}}_r+\zeta\bar{\bar{X}}_z)+(\beta^2\underset{\sim}{\xi}^2-a\omega^2)\bar{\bar{\theta}}}{\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)} \qquad (26)$$

where $\underset{\sim}{\xi}^2 = \xi^2+\zeta^2$ and $\mathfrak{D}$ is defined by (13).

The expressions for u, w and θ can now be obtained from (24), (25) and (26) by means of the transforms inverse to (19) and (20). In some of the most interesting applications the radial component of the body force X_r is zero. The expressions for the components of displacement and for the temperature distribution then become

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta\, d\omega\int_0^{\infty}\frac{i\left\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\right\}\xi\zeta\bar{\bar{X}}_z+b\xi\bar{\bar{\theta}}(\underset{\sim}{\xi}^2-a\omega^2)}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}\xi J_1(\xi r)\,d\xi \qquad (27)$$

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta\, d\omega\int_0^{\infty}\frac{\left\{(\beta^2\xi^2+\zeta^2-a\omega^2)(\underset{\sim}{\xi}^2-i\omega f)-i\omega bg\xi^2\right\}\bar{\bar{X}}_z+ib\zeta\bar{\bar{\theta}}(\underset{\sim}{\xi}^2-a\omega^2)}{(\underset{\sim}{\xi}^2-a\omega^2)\mathfrak{D}(\omega,\underset{\sim}{\xi}^2)}$$

$$\times\ \xi J_0(\xi r)\,d\xi \qquad (28)$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z - \omega t)}\, d\zeta\, d\omega \int_0^{\infty} \frac{a\omega\zeta \bar{X}_z + (\beta^2 \xi^2 - a\omega^2)\bar{S}}{\mathcal{D}(\omega, \xi^2)}\, \xi J_0(\xi r)\, d\xi \tag{29}$$

6.2 Effects Due to Uneven Heating.

Rectangular Cartesian Coordinates.

It is easily seen from the foregoing work, that the components of displacement and the temperature distribution produced by a heat source $\Theta(x_1, x_2, x_3, t)$ in the absence of body forces are given by

$$u_i = \frac{ib}{4\pi^2}\int_{W_4} \frac{\xi_i \bar{\Theta}}{\mathcal{D}} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{30}$$

$$\theta = \frac{1}{4\pi^2}\int_{W_4} \frac{(\beta^2 \xi^2 - a\omega^2)\bar{\Theta}}{\mathcal{D}} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{31}$$

which are the expressions derived by EASON and SNEDDON. There are two special cases which are of particular interest. For completeness we shall quote these results here — the derivations can be found in the reference mentioned.

The Steady-State Solution.

Using (30) and (31) it is found that the components of displacement and the temperature distribution due to a steady heat source $\Theta(x_1, x_2, x_3)$ are

$$u_i = \frac{ib}{(2\pi)^{3/2}}\int_{W_3} \frac{\xi_i \bar{\Theta}}{\beta^2 \xi^2} \exp(-i\xi_n x_n)\, d\xi \tag{32}$$

$$\theta = \frac{1}{(2\pi)^{3/2}}\int_{W_3} \frac{\bar{\Theta}}{\xi^2} \exp(-i\xi_n x_n)\, d\xi \tag{33}$$

where $d\underset{\sim}{\xi} = d\xi_1 d\xi_2 d\xi_3$ and

$$\bar{S} = \frac{1}{(2\pi)^{3/2}} \int_{V_3} \vartheta \exp(i\xi_n x_n)\, dx_1 dx_2 dx_3 \tag{34}$$

The two dimensional Problem.

The solutions are easily shown to be

$$u_1 = \frac{ib}{(2\pi)^{3/2}} \int_{T_3} \frac{\xi_1 \bar{S} \exp\left\{-i(\xi_1 x_1 + \xi_2 x_2 + \omega t)\right\}}{(\beta^2\xi_1^2 + \beta^2\xi_2^2 - a\omega^2)(\xi_1^2 + \xi_2^2 - i\omega f) - i\omega b g(\xi_1^2 + \xi_2^2)}\, dT \tag{35}$$

$$\theta = \frac{1}{(2\pi)^{3/2}} \int_{T_3} \frac{(\beta^2\xi_1^2 + \beta^2\xi_2^2 - a\omega^2)\bar{S} \exp\left\{-i(\xi_1 x_1 + \xi_2 x_2 + \omega t)\right\}}{(\beta^2\xi_1^2 + \beta^2\xi_2^2 - a\omega^2)(\xi_1^2 + \xi_2^2 - i\omega f) - i\omega b g(\xi_1^2 + \xi_2^2)}\, dT \tag{36}$$

where $dT = d\xi_1 d\xi_2 d\omega$, $\vartheta(x_1, x_2, t)$ is the heat source, the integration is taken over the entire $\xi_1\xi_2\omega$-space, and

$$\bar{S} = \frac{1}{(2\pi)^{3/2}} \int_{S_3} \vartheta \exp\left\{i(\xi_1 x_1 + \xi_2 x_2 + \omega t)\right\} dS \tag{37}$$

Problems with Axial Symmetry.

From equations (27)-(29) it is immediately seen that the components of displacement and the temperature distribution due solely to the action of a heat source $\bar{\vartheta}$ which is symmetrical about the z-axis are

$$u = \frac{b}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z + \omega t)} d\zeta\, d\omega \int_0^{\infty} \xi^2 D^{-1} \bar{\vartheta}\, J_1(\xi r)\, d\xi \tag{38}$$

$$w = \frac{ib}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta e^{-i(\zeta z + \omega t)} d\zeta\, d\omega \int_0^{\infty} \xi \bar{\vartheta} D^{-1} J_0(\xi r)\, d\xi \tag{39}$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_{0}^{\infty} (\beta^2 \underset{\sim}{\xi}^2 - \omega^2)\mathcal{D}^{-1}\,\bar{G}\,\xi J_0(\xi r)\;d\xi \tag{40}$$

where $\bar{G}$ is defined by (20).

6.3 Effects Produced by Time Dependent Body Forces.

Rectangular Cartesian Coordinates.

It is readily seen that the components of displacement and the temperature distribution due to body forces $\underset{\sim}{X}$ are

$$u_i = \frac{1}{4\pi^2}\int_{W_4}\left\{\frac{\bar{X}_i}{\underset{\sim}{\xi}^2-\omega^2} - \frac{\left\{(\beta^2-1)(\underset{\sim}{\xi}^2-i\omega f)-i\omega bg\right\}\xi_i\xi_n\bar{X}_n}{(\underset{\sim}{\xi}^2-\omega^2)\mathcal{D}}\right\}\exp\left\{-i(\xi_n x_n+\omega t)\right\}\,dW \tag{41}$$

$$\theta = \frac{g}{4\pi^2}\int_{W_4}\frac{\omega\xi_n\bar{X}_n}{\mathcal{D}}\exp\left\{-i(\xi_n x_n+\omega t)\right\}\,dW \tag{42}$$

where $\bar{X}_n$ is defined by (6). It may be noted that, since the classical equations may be obtained from the linked equations by putting $b = g = 0$, the classical solution for the temperature distribution will be $\theta = 0$. This is, of course, obviously true, since in the classical solution the temperature is given by the heat conduction equation quite independently of the other equations, which contain the mechanical effects.

Steady-State Problem.

If the body forces do not depend on the variable t, so that

$$X_i = F_i(x_1, x_2, x_3)$$

then we find that

$$\bar{X}_i = (2\pi)^{1/2}\bar{F}_i\,\delta(\omega)$$

where $\bar{F}_i$ is the three dimensional Fourier transform of F_i.

On substituting this value into (41) and (42) we see that $\theta = 0$, showing that a steady body force does not produce a thermal effect, and

$$u_i = \frac{1}{(2\pi)^{3/2}a^2\beta^2} \int_{W_3} \frac{\beta^2 \xi^2 \bar{F}_i - (\beta^2-1)\xi_i \xi_n \bar{F}_n}{(\xi^2)^2} \exp\left\{-i\xi_n x_n\right\} d\xi \tag{43}$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$. This latter expression can be converted to the standard expression for the statical solution (see e.g. MASON et al.(1956) p 581).

The Two Dimensional Problem.

The solutions for the two dimensional problem can be obtained by putting $\bar{X}_3 = 0$ and

$$\bar{X}_i = (2\pi)^{1/2}\bar{F}_i(\xi_1, \xi_2, \omega)\,\delta(\xi_3) \qquad (i = 1,2)$$

where $\bar{F}_i$ is the three dimensional Fourier transform of the components of the body force $F_i(x_1, x_2, t)$.

These solutions are

$$u_i = \frac{1}{(2\pi)^{3/2}} \int_{W_3} \left\{ \frac{\bar{F}_i}{\chi^2 - \omega^2} - \frac{\left\{(\beta^2-1)(\chi^2 - i\omega f) - i\omega b g\right\}\xi_i \xi_n \bar{F}_n}{(\chi^2-\omega^2)\mathfrak{D}(\omega,\chi^2)} \right\} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{44}$$

$$\theta = \frac{g}{(2\pi)^{3/2}} \int_{W_3} \frac{\omega \xi_n \bar{F}_n}{\mathfrak{D}(\omega,\chi^2)} \exp\left\{-i(\xi_n x_n + \omega t)\right\} dW \tag{45}$$

where $dW = d\xi_1 d\xi_2 d\omega$ and $\chi^2 = \xi_1^2 + \xi_2^2$.

Problems with Axial Symmetry.

From equations (27)-(29) we see that the solutions corresponding to the application of a body force X_z are

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_0^{\infty} \frac{i\left\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\right\}\xi\zeta\bar{X}_z\,\xi J_1(\xi r)\,d\xi}{(\xi^2+\zeta^2-\omega^2)\mathfrak{D}} \qquad (46)$$

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_0^{\infty} \frac{\left\{(\beta^2\xi^2+\zeta^2-\omega^2)(\xi^2-i\omega f)-i\omega bg\xi^2\right\}\bar{X}_z\,\xi J_0(\xi r)\,d\xi}{(\xi^2+\zeta^2-\omega^2)\mathfrak{D}} \qquad (47)$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_0^{\infty} \frac{g\omega\zeta\bar{X}_z}{\mathfrak{D}}\;\xi J_0(\xi r)\;d\xi \qquad (48)$$

whilst the complete solution, corresponding to the application of both components X_r and X_z, can be easily obtained by inverting the relevant terms of expressions (24)-(26).

The Quasi-Static Solutions.

It will be seen from the table on page 11 that the constant g is usually very small for problems in which the c.g.s. system of units is the natural system to use. We would therefore expect to get a good approximation to the exact solution by neglecting the terms in which g occurs. It can be seen that, in the quasi-static approximation,

$$\mathfrak{D} = \beta^2\xi^2(\xi^2 - i\omega f_1)$$

where $f_1 = f(1+\epsilon)$ and $\epsilon = bg/\beta^2 f$.

Using (48) the quasi-static approximation to the temperature field produced by a body force X_z is

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_{0}^{\infty} \frac{\omega g\zeta \bar{X}_z}{\beta^2 \underset{\sim}{\xi}^2 (\underset{\sim}{\xi}^2 - i\omega f_1)}\,\xi J_0(\xi r)\,d\xi \qquad (49)$$

Equivalent Heat Source.

Let us consider the classical solution of the problem of the temperature distribution produced by a distributed heat source. The solution satisfies the heat conduction equation

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \Theta = f\,\frac{\partial\theta}{\partial t}$$

which can be transformed, using (20), to the form

$$-\underset{\sim}{\xi}^2\bar{\theta} + \bar{\Theta} = -i\omega f\bar{\theta}$$

so that

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_{0}^{\infty} \frac{\bar{\Theta}}{\underset{\sim}{\xi}^2 - i\omega f}\,\xi J_0(\xi r)\,d\xi \qquad (50)$$

Thus, by comparing equations (48) and (50), we can see that the same temperature distribution would be given by the body force X_z as would be given, in the classical theory, by a heat source Θ obeying the relation

$$\bar{\Theta} = \frac{\omega g\zeta(\underset{\sim}{\xi}^2 - i\omega f)}{\mathfrak{D}}\,\bar{X}_z \qquad (51)$$

We call the heat source Θ given by (51), the equivalent heat source for the body force X_z. Having found the equivalent heat source, the problem is identical with a problem in the classical heat conduction equation.

Similarly we can see from (49) and (50) that the quasi-static solution can be obtained from a solution of the classical heat conduction equation. All we need to do, is to replace f in that equation by f_1, and consider a heat source Θ given by

$$\bar{\Theta} = \frac{\omega g\bar{X}_z\zeta}{\beta^2(\xi^2 + \zeta^2)} \qquad (52)$$

<u>Equivalent Heat Source for a Point Force.</u>

As an illustration of the above, we shall calculate the heat source equivalent to the quasi-static treatment of a point force at the origin. Thus

$$X_z = \frac{1}{2\pi r}\,\delta(r)\,\delta(z)\,f(t)$$

and

$$\bar{X}_z = \frac{1}{(2\pi)^{3/2}}\,G(\omega) \qquad (53)$$

where

$$G(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,dt \qquad (54)$$

Inversion of (54) and differentiation with respect to t gives the relation

$$f'(t) = \frac{-i}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \omega G(\omega)\,e^{-i\omega t}\,d\omega \qquad (55)$$

Now, using (52) and (53), the required heat source is given by

$$\bar{\theta} = \frac{\varepsilon\omega\zeta}{\beta^2(\xi^2+\zeta^2)}\,\frac{G(\omega)}{(2\pi)^{3/2}}$$

so that

$$\theta = \frac{\varepsilon}{\beta^2(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega\int_0^{\infty}\xi J_0(\xi r)\,\frac{\omega\zeta G(\omega)}{\xi^2+\zeta^2}\,d\xi$$

$$= \frac{i\varepsilon f'(t)}{\beta^2(2\pi)^2}\int_{-\infty}^{\infty}\zeta e^{-i\zeta z}\,d\zeta\int_0^{\infty}\xi(\xi^2+\zeta^2)^{-1}J_0(\xi r)\,d\xi$$

using (55). Evaluating the integrals we finally get

$$\Theta = \frac{gf'(t)}{4\pi\beta^2} z(r^2 + z^2)^{-3/2} \tag{56}$$

This result can be immediately generalised to give, in the quasi-static approximation, the heat source equivalent to the distributed body force

$$X_z = F(x,y,z)\ f(t)$$

This heat source is given by

$$\Theta = \frac{gf'(t)}{4\pi\beta^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(z - z')\ F(x',y',z')\ dx'\,dy'\,dz'}{\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{3/2}} \tag{57}$$

CHAPTER SEVEN.

PROPAGATION OF THERMAL STRESSES IN A SEMI-INFINITE MEDIUM.

7.1 Introduction.

In this chapter we shall use the equations in the dimensionless forms:

$$\nabla^2 \underset{\sim}{u} + (\beta^2 - 1) \text{ grad } \Delta - b \text{ grad } \theta + \underset{\sim}{X} = a\ddot{\underset{\sim}{u}} \tag{1}$$

$$\Theta + \nabla^2 \theta = f\dot{\theta} + g\dot{\Delta} \tag{2}$$

and we shall make use of the method of solution described in chapter IV. To apply this method we need the results for the thermoelastic problem for the infinite space. These were derived in Chapter VI and can be summarized by saying that the displacement field $\underset{\sim}{u} = (u_1, u_2, u_3)$ and temperature distribution θ due to a combination of body forces $\underset{\sim}{X} = (X_1, X_2, X_3)$ and heat sources Θ are given by

$$\bar{u}_i = \frac{\bar{X}_i}{\underset{\sim}{\xi}^2 - a\omega^2} - \frac{\left\{(\beta^2-1)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega bg\right\}\xi_i \xi_r \bar{X}_r}{(\underset{\sim}{\xi}^2 - a\omega^2)\mathcal{D}} + \frac{ib\xi_i\bar{\Theta}}{\mathcal{D}} \tag{3}$$

$$\bar{\theta} = \frac{\omega g\xi_r \bar{X}_r + (\beta^2\underset{\sim}{\xi}^2 - a\omega^2)\bar{\Theta}}{\mathcal{D}} \tag{4}$$

where

$$\underset{\sim}{\xi}^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \tag{5}$$

$$\mathcal{D} = (\beta^2\underset{\sim}{\xi}^2 - a\omega^2)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega bg\underset{\sim}{\xi}^2 \tag{6}$$

and the bar denotes the four-dimensional complex Fourier transform

$$\bar{\bar{F}}(\xi_1, \xi_2, \xi_3, \omega) = \frac{1}{(4\pi^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x_1, x_2, x_3, t) \exp\left\{ i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \omega t)\right\} dx_1\, dx_2\, dx_3\, dt \tag{7}$$

These results assume that all the unknowns and their derivatives vanish at infinity.

Using the system of units adopted in equations (1) and (2), the normal components of stress are given by

$$\sigma_i = (\beta^2 - 2)\Delta - b\theta + 2\frac{\partial u_i}{\partial x_i} \tag{8a}$$

(no summation implied in the last term.), and the shear components are

$$\tau_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{8b}$$

7.2 Statement of the Problem.

We shall consider the semi-infinite medium $x_3 \geq 0$ subjected to body forces $\underset{\sim}{F} = (F_1, F_2, F_3)$ and a heat source distribution Φ. Further there may be any set of physically possible boundary conditions on the boundary $x_3 = 0$. We shall therefore look for solutions to the linked equations (1) and (2) subject to the specification on $x_3 = 0$ of three of the quantities u_1, u_2, u_3, σ_3, τ_{31}, τ_{32} plus one of the quantities θ, $\partial\theta/\partial x_3$, $(\partial\theta/\partial x_3 + h\theta)$.

The standard method of using a four-dimensional integral transform on the given equations (a sine or cosine transform in the x_3-direction) will only work in certain special cases, because of our lack of knowledge of the behaviour of many of the unknowns and their derivatives at $x_3 = 0$.(For an example where this does work see EASON and SNEDDON

(1959)). Further the method of transforming the equations with respect to x_1, x_2 and t, and then solving the resulting four differential equations in the single variable x_3, is too cumbersome in this case. Instead we shall use the method of Chapter IV, which is described in the next section in the form relevant to this problem. This solution is due to LOCKETT (1959b).

7.3 Method of Solution.

We wish to find expressions for $\underset{\sim}{u}$ and θ which, in the region $x_3 > 0$, satisfy the equations (1) and (2) with $\underset{\sim}{X} = \underset{\sim}{F}$ and $\Theta = \Phi$ and which satisfy the given boundary conditions on $x_3 = 0$. Instead we look for solutions which satisfy (1) and (2) in the whole space $-\infty \leq x_3 \leq \infty$ when

$$X_i = F_i + (2\pi)^{1/2} P_i(x_1, x_2, t)\, \delta(x_3)$$
$$\Theta = \Phi + (2\pi)^{1/2} \psi(x_1, x_2, t)\, \delta(x_3) \qquad (9)$$

where $\delta(x)$ is the Dirac Delta function. Thus we consider the infinite medium subjected to $\underset{\sim}{F}$ and Φ in the region $x_3 > 0$ and to additional concentrated body forces $(2\pi)^{1/2}\underset{\sim}{P}$ and heat sources $(2\pi)^{1/2}\psi$ on the plane $x_3 = 0$. We shall then choose $\underset{\sim}{P}$ and ψ in such a way that the boundary conditions of the original problem are satisfied on $x_3 = 0$.

Within the region $x_3 > 0$ the solutions to the modified problem satisfy the equations (1) and (2) with $\underset{\sim}{X} = \underset{\sim}{F}$, $\Theta = \Phi$ (since $\delta(x_3) = 0$ for $x_3 \neq 0$) and on the plane $x_3 = 0$ the given boundary conditions are satisfied. Thus for $x_3 > 0$ the solutions to the modified problem are also the solutions to the original problem. The concentrated forces and sources $\underset{\sim}{P}$ and ψ could have been taken off the boundary $x_3 = 0$ by considering the Dirac Delta function in the form $\delta(x_3 + d)$ where $d > 0$, so that they lie on the plane $x_3 = -d$. However, when the analysis is carried through we find that the solutions are independent of d, which we would expect on physical grounds, and so we content ourselves with the case $d = 0$.

7.4 Solution in Rectangular Cartesian Coordinates.

For the expressions (9) we find that

$$\bar{\bar{X}}_1 = \bar{\bar{F}}_1 + P_1^o \qquad \bar{\bar{\theta}} = \bar{\bar{G}} + \psi^o \tag{10}$$

where $\bar{\bar{f}}$ is defined by (7) and f^o is the transform with respect to x_1, x_2, t only. That is

$$f^o = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \exp i(\xi_1 x_1 + \xi_2 x_2 + \omega t)\, dx_1\, dx_2\, dt \tag{11}$$

It is obvious from (7) and (11) that

$$\bar{\bar{f}} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f^o \exp(i\xi_3 x_3)\, dx_3 \text{ so that } f^o = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{\bar{f}} \exp(-i\xi_3 x_3)\, d\xi_3 \tag{12}$$

Using the expressions (3), (4), (10) we can write the solutions to the modified problem in the form

$$u_i = u_{i1} + u_{i2}, \qquad \theta = \theta_1 + \theta_2 \tag{13}$$

where

$$\bar{\bar{u}}_{i1} = \frac{\bar{\bar{F}}_i}{\underset{\sim}{\xi}^2 - a\omega^2} + \frac{\left\{(\beta^2-1)(\underset{\sim}{\xi}^2 - i\omega f) - i\omega bg\right\}\xi_i \xi_q \bar{\bar{F}}_q}{(\underset{\sim}{\xi}^2 - a\omega^2)\,\mathcal{D}} + \frac{ib\xi_i \bar{\bar{G}}}{\mathcal{D}}$$

$$\bar{\bar{\theta}}_1 = \frac{\omega g \xi_q \bar{\bar{F}}_q + (\beta^2 \underset{\sim}{\xi}^2 - a\omega^2)\bar{\bar{G}}}{\mathcal{D}} \tag{14}$$

and

$$\bar{u}_{12} = \frac{F_1^o}{\xi^2 - a\omega^2} - \frac{\left\{(\beta^2 - 1)(\xi^2 - i\omega f) - i\omega b g\right\}\xi_1 \xi_q F_q^o}{(\xi^2 - a\omega^2)\mathcal{D}} + \frac{ib\xi_1 \psi^o}{\mathcal{D}}$$

$$\bar{\theta}_2 = \frac{\omega g \xi_q F_q^o + (\beta^2 \xi^2 - a\omega^2)\psi^o}{\mathcal{D}} \tag{15}$$

Since in any particular problem F_i and ψ are known, $\bar{u}_{11}$ and $\bar{\theta}_1$, can be evaluated and can therefore be treated as known functions. From (12) and (15)

$$(2\pi)^{\frac{1}{2}} u_{12}^o = F_1^o \int_{-\infty}^{\infty} \frac{e^{-i\xi_3 x_3}}{\xi^2 - a\omega^2}\, d\xi_3 - F_q^o \int_{-\infty}^{\infty} \frac{\left\{(\beta^2 - 1)(\xi^2 - i\omega f) - i\omega b g\right\}\xi_1 \xi_q}{(\xi^2 - a\omega^2)\mathcal{D}}\, e^{-i\xi_3 x_3}\, d\xi_3 + ib\psi^o \int_{-\infty}^{\infty} \frac{\xi_1 e^{-i\xi_3 x_3}}{\mathcal{D}}\, d\xi_3$$

$$(2\pi)^{\frac{1}{2}} \quad = \omega g F_q^o \int_{-\infty}^{\infty} \frac{\xi_q e^{-i\xi_3 x_3}}{\mathcal{D}}\, d\xi_3 + \psi^o \int_{-\infty}^{\infty} \frac{(\beta^2 \xi^2 - a\omega^2)}{\mathcal{D}}\, e^{-i\xi_3 x_3}\, d\xi_3 \tag{16}$$

since F_1^o and ψ^o are not functions of ξ_3. If we now write

$$\xi^2 - a\omega^2 = \xi_3^2 + k_1^2 \tag{17}$$

so that $k_1^2 = \eta^2 - a\omega^2$ where $\eta^2 = \xi_1^2 + \xi_2^2$ (18)

and put

$$\mathcal{D} = \beta^2(\xi_3^2 + k_2^2)(\xi_3^2 + k_3^2) \tag{19}$$

the integrals in (16) can be solved by reducing the integrands by partial fractions. If we evaluate these integrals for $x_3 > 0$ (the only range in which we are interested) and, for convenience, introduce new unknowns A, B, C, D which depend on F_1^o and ψ^o, then we find that

$$
\begin{aligned}
u^0_{12} &= ik_1Ae^{-k_1x_3} + ib\xi_1Ce^{-k_2x_3} + ib\xi_1De^{-k_3x_3}\\
u^0_{22} &= ik_1Be^{-k_1x_3} + ib\xi_2Ce^{-k_2x_3} + ib\xi_2De^{-k_3x_3}\\
u^0_{32} &= (\xi_1A + \xi_2B)e^{-k_1x_3} + bk_2Ce^{-k_2x_3} + bk_3De^{-k_3x_3}\\
\theta^0_2 &= (\beta^2\eta^2 - a\omega^2 - \beta^2k_2^2)Ce^{-k_2x_3} + (\beta^2\eta^2 - a\omega^2 - \beta^2k_3^2)De^{-k_3x_3}
\end{aligned}
\tag{20}
$$

Thus using (13) the three-dimensional transforms of the displacement field and the temperature distribution are given by

$$
\begin{aligned}
u^0_1 &= u^0_{11} + ik_1Ae^{-k_1x_3} + ib\xi_1Ce^{-k_2x_3} + ib\xi_1De^{-k_3x_3}\\
u^0_2 &= u^0_{21} + ik_1Be^{-k_1x_3} + ib\xi_2Ce^{-k_2x_3} + ib\xi_2De^{-k_3x_3}\\
u^0_3 &= u^0_{31} + (\xi_1A + \xi_2B)e^{-k_1x_3} + bk_2Ce^{-k_2x_3} + bk_3De^{-k_3x_3}\\
\theta^0 &= \theta^0_1 + (\beta^2\eta^2 - a\omega^2 - \beta^2k_2^2)Ce^{-k_2x_3} + (\beta^2\eta^2 - a\omega^2 - \beta^2k_3^2)De^{-k_3x_3}
\end{aligned}
\tag{21}
$$

where u^0_{i1} and θ^0_1 can be found using (12) and (14).

In any particular problem, we have now only to apply (21) to the transformed boundary conditions to obtain four equations giving the values of A, B, C and D. Substituting these values back into (21) and applying the transform inverse to (11) we obtain expressions for the required quantities u and θ.

<u>Transforms of stress components.</u>

Since the boundary conditions to many problems specify one or more of the components of stress σ_3, τ_{31}, τ_{32} we give here their three dimensional transforms:

$$
\begin{aligned}
\sigma^0_3 &= -i(\beta^2-2)(\xi_1u^0_1 + \xi_2u^0_2) - b\theta^0 + \beta^2\,\partial u^0_3/\partial x_3\\
\tau^0_{31} &= \partial u^0_1/\partial x_3 - i\xi_1u^0_3\\
\tau^0_{32} &= \partial u^0_2/\partial x_3 - i\xi_2u^0_3
\end{aligned}
\tag{22}
$$

7.5 Example.

As an example of the use of the equations derived above we shall consider the problem considered by EASON and SNEDDON (1959). For this $\underset{\sim}{F} = 0$ and the boundary conditions are $u_1 = u_2 = \sigma_3 = 0$, $\theta = \theta_0(x_1, x_2, t)$ on $x_3 = 0$. A heat source $\Phi(x_1, x_2, x_3, t)$ is assumed to be acting in the medium $x_3 > 0$.

Thus from (14)

$$\bar{\bar{u}}_{i1} = ib\xi_i\bar{\bar{\Phi}}/\mathcal{D} \qquad \bar{\bar{\theta}}_1 = (\beta^2\underset{\sim}{\xi}^2 - a\omega^2)\bar{\bar{\Phi}}/\mathcal{D} \tag{23}$$

and the boundary conditions can be written

$$\begin{aligned}
&\frac{ib\xi_1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3 + ik_1 A + ib\xi_1 C + ib\xi_1 D = 0\\
&\frac{ib\xi_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3 + ik_1 B + ib\xi_2 C + ib\xi_2 D = 0\\
&\frac{-b(2\eta^2 - a\omega^2)}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3 - 2k_1\xi_1 A - 2k_1\xi_2 B - b(2\eta^2 - a\omega^2)(C + D) = 0\\
&\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{(\beta^2\underset{\sim}{\xi}^2 - a\omega^2)\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3 + (\beta^2\eta^2 - a\omega^2 - \beta^2k_3^2)C + (\beta^2\eta^2 - a\omega^2 - \beta^2k_3^2)D = \bar{\theta}_0
\end{aligned} \tag{24}$$

from which we find that $A = B = 0$ and

$$\begin{aligned}
C &= (k_1^2 - k_3^2)^{-1}\left\{-\beta^{-2}\bar{\theta}_0 + (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{(\xi_3^2 + k_3^2)\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3\right\}\\
D &= (k_1^2 - k_3^2)^{-1}\left\{\beta^{-2}\bar{\theta}_0 - (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{(\xi_3^2 + k_2^2)\bar{\bar{\Phi}}}{\mathcal{D}}\,d\xi_3\right\}
\end{aligned} \tag{25}$$

We may now use (21) and the inverse tranforms to show that

$$\underset{\sim}{u} = -\operatorname{grad}\psi$$

where

$$\psi = \frac{b}{4\pi^2}\iiint_{-\infty}^{\infty} e^{-i(\xi_1 x_1+\xi_2 x_2+\omega t)}\,d\xi_1\,d\xi_2\,d\omega\left\{\int_{-\infty}^{\infty} e^{-i\xi_3 x_3}\frac{\bar{\bar{Q}}}{D}\,d\xi_3+(2\pi)^{\frac{1}{2}}\left(Ce^{-k_2 x_3}+De^{-k_3 x_3}\right)\right\} \tag{26}$$

which is a formal solution to the problem, and it can be shown that this is equivalent to the result obtained by Eason and Sneddon. However it should be noted that the present method will produce formal solutions to any of the problems mentioned earlier.

7.6 Solutions for the Axially Symmetrical Problem.

The solutions to this class of problems could be found by running through a similar procedure to the one outlined above, using the forms of the equations when expressed in cylindrical polar coordinates. However it is not a difficult matter to obtain them directly from the results expressed in rectangular coordinates.

We denote by u_r and w the components of displacement in the r and z directions, and use ϕ to denote the polar coordinate. The components of the body force are denoted by F_r and F_z and the temperature distribution and heat sources are represented by θ and Q as before.

We define the transforms

$$(u_r^o,\ F_r^o) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\omega t}dt\int_0^{\infty} r(u_r,\ F_r)\ J_1(\xi r)\ dr \tag{27}$$

$$(w^o,\ \theta^o,\ F_z^o,\ Q^o) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{i\omega t}dt\int_0^{\infty} r(w,\ \theta,\ F_z,\ Q)\ J_0(\xi r)\ dr \tag{28}$$

and

$$\bar{f} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f e^{i\xi z}\ dz \tag{29}$$

If we make the following substitutions from the rectangular system to the polar system

$$x_1 = r\cos\psi \qquad x_2 = r\sin\psi \qquad x_3 = z$$
$$u_1 = u_r\cos\psi \qquad u_2 = u_r\sin\psi \qquad u_3 = w \qquad (30)$$
$$F_1 = F_r\cos\psi \qquad F_2 = F_r\sin\psi \qquad F_3 = F_z$$

and put

$$\xi_1 = \xi\cos\phi \qquad \xi_2 = \xi\sin\phi \qquad \xi_3 = \zeta$$

then we find that, for instance

$$\xi_1\bar{F}_1+\xi_2\bar{F}_2=\frac{1}{(2\pi)^2}\iiiint_{-\infty}^{\infty}(\xi_1F_1+\xi_2F_2)\exp\left\{i(\xi_1x_1+\xi_2x_2+\xi_3x_3+\omega t)\right\}dx_1\,dx_2\,dx_3\,dt$$

$$=\frac{1}{(2\pi)^2}\iiiint_{-\infty}^{\infty}\xi F_r\cos(\phi-\psi)\exp\left\{i(\xi_1x_1+\xi_2x_2+\xi_3x_3+\omega t)\right\}dx_1\,dx_2\,dx_3\,dt$$

$$=\frac{1}{2\pi}\iint_{-\infty}^{\infty}e^{i(\zeta z+\omega t)}dz\;dt\int_0^{\infty}\xi F_r\,r\,dr\frac{1}{2\pi}\int_0^{2\pi}\cos(\phi-\psi)e^{i\xi r\cos(\phi-\psi)}d\psi$$

$$=\frac{i\xi}{2\pi}\iint_{-\infty}^{\infty}e^{i(\zeta z+\omega t)}dz\;dt\int_0^{\infty}rF_rJ_1(\xi r)\;dr$$

Therefore

$$\bar{F}_r = (\xi_1\bar{F}_1+\xi_2\bar{F}_2)/i\xi$$

Similarly (31)

$$\bar{u}_r = (\xi_1\bar{u}_1+\xi_2\bar{u}_2)/i\xi, \quad \bar{w} = \bar{u}_3, \quad \bar{F}_z = \bar{F}_3$$

and $\bar{\sigma}$ and $\bar{\tau}$ defined by (28) and (29) are identical to the definitions given by (7). A similar set of relations holds where $\bar{F}$ is replaced by f^o. From these expressions and equations (14) and (21) it is easily seen that u_r, w and θ are given by

$$u_r^o = u_{r1}^o + k_1 L e^{-k_1 z} + i\xi M e^{-k_2 z} + b\xi N e^{-k_3 z}$$

$$w^o = w_1^o + \xi L e^{-k_1 z} + b k_2 M e^{-k_2 z} + b k_3 N e^{-k_3 z} \qquad (32)$$

$$\theta^o = \theta_1^o + (\beta^2\xi^2 - a\omega^2 - \beta^2 k_2^2) M e^{-k_2 z} + (\beta^2\xi^2 - a\omega^2 - \beta^2 k_3^2) N e^{-k_3 z}$$

where

$$k_1^2 = \xi^2 - a\omega^2$$

$$\beta^2(\xi^2 + k_2^2)(\xi^2 + k_3^2) = (\beta^2\xi^2 + \beta\zeta^2 + a\omega^2)(\xi^2 + \zeta^2 - i\omega f) - i\omega b g(\xi^2 + \zeta^2) \qquad (32a)$$

and u_{r1}^o, w_1^o, θ_1^o are obtained from (12) and the expressions

$$\bar{\bar{u}}_{r1} = \frac{\bar{\bar{F}}_r}{(\xi^2 + \zeta^2 - a\omega^2)} + \frac{i\xi\left\{(\beta^2 - 1)(\xi^2 + \zeta^2 - i\omega f) - i\omega b g\right\}(i\xi\bar{\bar{F}}_r + \zeta\bar{\bar{F}}_z)}{(\xi^2 + \zeta^2 - a\omega^2)\mathcal{D}} + \frac{b\xi\bar{\bar{\Theta}}}{\mathcal{D}}$$

$$\bar{\bar{w}}_1 = \frac{\bar{\bar{F}}_z}{(\xi^2 + \zeta^2 - a\omega^2)} - \frac{\left\{(\beta^2 - 1)(\xi^2 + \zeta^2 - i\omega f) - i\omega b g\right\}\zeta(i\xi\bar{\bar{F}}_r + \zeta\bar{\bar{F}}_z)}{(\xi^2 + \zeta^2 - a\omega^2)\mathcal{D}} + \frac{i b\zeta\bar{\bar{\Theta}}}{\mathcal{D}} \qquad (33)$$

$$\bar{\bar{\theta}}_1 = \frac{i\omega g\xi\bar{\bar{F}}_r + \omega g\zeta\bar{\bar{F}}_z + (\beta^2\xi^2 + \beta^2\zeta^2 - a\omega^2)\bar{\bar{\Theta}}}{\mathcal{D}}$$

The final results to any particular problem can now be found by
(i) applying the boundary conditions to equations (32) to obtain L, M, N.
(ii) calculating u_{r1}^o, w_1^o, θ_1^o from (12) and (33)
(iii) substituting these values into (32)
(iv) inverting (32) using the transforms inverse to (27) and (28).

Transforms of stress components.

With the notation used above the two dimensional transforms of the components of stress σ_z and τ_{rz} are

$$\sigma_z^\circ = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t}dt \int_0^{\infty} r\sigma_z J_0(\xi r)dr = (\beta^2-2)\xi u_r^\circ - b\theta^\circ + \beta^2 \frac{\partial w^\circ}{\partial z}$$

$$\tau_{rz}^\circ = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t}dt \int_0^{\infty} r\tau_{rz} J_1(\xi r)dr = \frac{\partial u_r^\circ}{\partial z} - \xi w^\circ \qquad (34)$$

7.7 Simple Numerical Example.

With the equations derived above it is possible to write down a formal solution to any of the problems mentioned in section 7.2. However, in most problems, it will be necessary to use numerical methods to put the results in a meaningful form. We shall be satisfied here to look at the result of taking into account thermoelastic effects for a problem which, though not of obvious practical interest, reduces the numerical work to a minimum.

Consider a semi-infinite elastic medium resting upon a rigid frictionless foundation, so that $w = \tau_{rz} = 0$ on $z = 0$. We suppose too that the rate of change of the temperature across the boundary is known, i.e. $\partial\theta/\partial z = \phi(r)$ on $z = 0$. We wish to find the pressure exerted by the solid onto the foundation (It is assumed that the medium is prevented from separating from the foundation). The equations for the evaluation of L, M, N take the form

$$\xi L + bk_2M + bk_3N = 0$$

$$(\xi^2 + k_1^2)L + 2b\xi k_2M + 2b\xi k_3N = 0$$

$$-(\beta^2\xi^2 - a\omega^2 - \beta^2k_2^2)k_2M - (\beta^2\xi^2 - a\omega^2 - \beta^2k_3^2)k_3N = \phi^\circ$$

so that

$$L = 0, \qquad k_2M = -k_3N = \phi^\circ/\beta^2(k_2^2 - k_3^2) \qquad (35)$$

We now choose the problem in which ϕ is equal to

$$PBe^{At}J_0(Br)$$

(where P, A, B are constants), which is obtained by placing

$$\phi^{0} = (2\pi)^{\frac{1}{2}} P \,\delta(\omega - iA)\, \delta(\xi - B) \tag{36}$$

in the formula for ϕ in terms of its transform. Using (32), (35) and (36) we find the solutions

$$u_r = PBbg\left\{\beta^2(k_2^2 - k_3^2)k_2k_3\right\}^{-1}(k_3e^{-k_2z} - k_2e^{-k_3z})\; e^{At}J_1(Br)$$

$$w = PBb\left\{\beta^2(k_2^2 - k_3^2)\right\}^{-1}(e^{-k_2z} - e^{-k_3z})\; e^{At}J_0(Br) \tag{37}$$

$$\theta = PB\left\{\beta^2(k_2^2 - k_3^2)\right\}^{-1}\left\{k_2^{1}(\beta^2B^2 - aA^2 - \beta^2k_2^2)e^{-k_2z} - k_3^{1}(\beta^2B^2 - aA^2 - \beta^2k_3^2)e^{-k_3z}\right\} \times e^{At}J_0(Br)$$

where k_2 and k_3, are to be evaluated for $\omega = iA$, $\xi = B$. This is not a rigorous derivation of these results, but it can be verified that they satisfy all the required equations and conditions (remembering that k_2, k_3 satisfy (32a)).

Using (34) it is now easily seen that on the boundary $z = 0$

$$\sigma_z = \frac{PBb(2B^2 - aA^2)}{\beta^2(k_2 + k_3)k_2k_3}\, e^{At}\, J_0(Br) \tag{38}$$

Thus, in this problem, the discrepancy between the results obtained using the linked equations and that obtained using the classical equations ($g = 0$) occurs only in the constant multiplying factor, and the ratio of these constants is

$$R = \left\{ \frac{\left\{(k_2 + k_3)k_2k_3\right\}_{g\,=\,0}}{(k_2 + k_3)k_2k_3} \right\}_{\substack{\omega = iA \\ \xi = B}} \tag{39}$$

It is more convenient to use the system of units due to Chadwick and Sneddon, which corresponds to placing $a = \beta^2$, $f = 1$, $\epsilon = bg/\beta^2$.

In this system of units, the units of length and time are very small ($\sim 10^{-6}$). Thus in formulating a problem which may be encountered in practice, we need to take $A, B \ll 1$. Otherwise the variations of the physical quantities with time would be too rapid, and the r-dependence would be too localized.

The ratio (39) can now be written as

$$R = \frac{\left\{(k_2 + k_3)k_2 k_3\right\}_{\varepsilon = 0}}{(k_2 + k_3)k_2 k_3} \qquad (40)$$

where k_2 and k_3 are given by

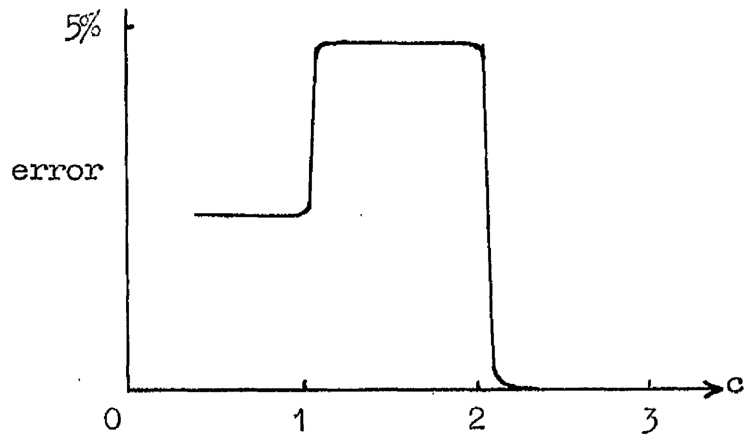
$$(\xi^2 + k_2^2)(\xi^2 + k_3^2) = (\xi^2 + B^2 + A^2)(\xi^2 + B^2 + A) + A\varepsilon(\xi^2 + B^2) \qquad (41)$$

The following table shows, for a range of values of A and B, and with $\varepsilon = 0.05$ (the value for copper is 0.017 and the value for lead is 0.0733), the percentage error of the classical solution with respect to the solution obtained from the linked equations (error = $100(1 - R)$).

Table. Percentage errors in the surface value of σ_z.

B \ A	10^{-5}	10^{-10}	10^{-15}	10^{-20}	10^{-25}	10^{-30}
10^{-5}	3.6	1.9	0	0	0	0
10^{-10}	2.4	3.6	4.8	1.9	0	0
10^{-15}	2.4	2.4	3.6	4.8	4.8	1.9
10^{-20}	2.4	2.4	2.4	3.6	4.8	4.8
10^{-25}	2.4	2.4	2.4	2.4	3.6	4.8
10^{-30}	2.4	2.4	2.4	2.4	2.4	3.6

From this table it is obvious that, as long as A, B << 1, the error depends only on a quantity *c* defined by the relation $A = B^c$. The variation of the error with *c* takes the form



showing that the maximum error is approximately 5% and occurs through the range $1 < c < 2$. Further, for $c > 2$ the error is negligible and it is approximately 2½% for $c < 1$ (but *c* must not be so small as to violate the condition A << 1).

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