Game-theoretical Approaches to Social Interaction

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ABSTRACT

This thesis consists of three chapters of economic analyses on social interaction, which mainly cover the topics of users' online interaction and cross-cultural interaction in multicultural societies.

Chapter 1 studies users' online interaction. Friend-based social media users create their own content and browse their friends' creating content with advertisements sent by the social media owner. Only users admitting friendships can interact; their interaction is associated with intrapersonal and asymmetrical interpersonal externalities cross the dual activities. This chapter considers the social media owner as a monopolist discriminating users by sending different quantities of advertisements to the users while they are browsing. More generally, the monopolist can subsidize users' creating and prices their browsing. The pricing browsing plan depends on users' degree centralities; the pricing and subsidizing plan depends on users' Katz-Bonacich centralities of degree two. The owner benefits from denser networks and intrapersonal externalities increasing.

Chapter 2 studies cross-cultural interaction. In this chapter, the acculturation game is proposed to study the cultural learning behavior of individuals from minority groups in the acculturation process, by which the individuals integrate themselves into a multicultural society. In the game, agents are endowed with certain levels of cultural knowledge of the society; the agents acquire cultural knowledge to improve their ability to coordinate with each other, but experience difficulties or/and acculturative stress in order to learn. With difficulties in learning only, the agents endowed with low levels of cultural knowledge learn while those endowed with high levels do not; and the opposite is true for agents who experience only acculturative stress. The presence of both difficulties in learning and stress leads to a 'double-threshold strategy', where the agents endowed with low and high levels of cultural knowledge do not learn, while the agents endowed with middle levels learn. Many countries impose the tests of language, culture and history for candidates in acquiring citizenship. We show that this policy improves social welfare in most cases; if the policy does not improve social welfare, agents experience high levels of stress in learning.

Chapter 3 extends the acculturation game with considering heterogeneous marginal costs. The heterogeneous marginal cost function increases with the distances from the cultural knowledge level agents trying to acquire to their endowed cultural knowledge levels. The results show that the shape of equilibrium depends on the population density distribution over agents' endowed cultural knowledge levels: if the distribution is "flat", the equilibrium strategy increases with agents' endowed cultural knowledge levels; if the distribution peaks at some intervals, then agents, whose endowed cultural knowledge levels are at the intervals and the right neighbourhood of the intervals, conform to expand to the same level of cultural knowledge. The social welfare increasing from adaptation is independent of the density distribution function if the equilibrium strategy is strictly increasing. An equilibrium strategy with agents conforming to expand to the same level does not necessarily cause gains or loss of the social welfare from adaptation; we show with an example that the social welfare from adaptation decreases with the population density distribution over the agents with high endowed cultural knowledge levels.

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Place: Glasgow

Date: 14/02/2019 **Li Nie**

DECLARATION

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Print Name:

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Date:

Chapter 1

Pricing Asymmetrical Network Effects on Social Media

1.1 Introduction

Network externalities are the key determinants to users' online behavior. Friend-based social media, such as Facebook and WeChat, are internet platforms where friends and acquaintances share their personal life events with each other. The authentic identity, innovative ways of social interaction exhibited on friend-based social media platforms, and associated social network externalities led to their rapid growth and commercial success.[1] The goal of this paper is to study how the platform owners price users by sending different quantities of advertisements to the users. Users' dual activities-content creating and content borrowing display the asymmetric interpersonal effects and intrapersonal effects across the two activities on friend-based social media. Therefore, how will these network effects determine the platform owners' pricing strategy or pricing and subsidizing strategy?

Content creating and browsing are users' basic online behaviors. In friend-based social media context, a user's creating behavior includes creating her/his profile, updating her/his statuses, posting photos/videos, sharing articles/news and reacting to her/his friends' posting with a "like", "comment", or "share". A user's creating content can be seen by her/his friends. However, a user's browsing behavior is private

[1] For the innovative ways, Joinson (2008) identifies the 7 unique usages and gratification of users on Facebook, which includes social connection, shared identities, content, social investigation, social network surfing, and status updating.

For the growth and success of Facebook, Facebook was founded in 2004 and had its IPO in May 2012. Facebook was ranked as 5th largest internet company in the world by market value with 2.3 billion monthly active users in 2018 (Wikipedia (2019), Seth (2018) and Facebook (2019)). For the case of WeChat, it was launched in 2011, which is owned by one of the largest internet companies Tencent in China.WeChat had 1.1 billion monthly active users all over the world in 2018 (Tencent Holdings Limited (2019)).

information to herself/himself, which means it can not be seen by her/his friends or any other user on social media; and a user's browsing behavior includes browsing her/his friends' posts, and browsing the reactions to her/his own posts and those to her/his friends'.

The privacy of browsing behavior on friend-based social media is a unique design among online social media.[2] The reason for this particular design of platforms could be that: users use their authentic identities on friend-based social media, and the majority of one user's friends know or at least meet the user in real life.[3] Revealing a user's friends' and her/his acquaintances' browsing behavior might have some negative repercussions. For example, a new friend of a user might not want to know more about him/her by searching his/her posting, if the friend's browsing information is revealed to the user. This is due to the fact that the friend might concern that her/his particular attention shown on the user causes over-interpretation or misunderstanding, which affects their real-life interaction. Thus, friend-based social media owners do not reveal the browsing behavior information on the platforms to make users unaware of being searched or feel free to search other users on the platforms.

Two effects appear in users' dual-activity interaction on friend-based social media by their unique design. One of these two effects is *an asymmetric interpersonal cross-activity externality*. It arises since users spend more time on browsing, if their friends post more content; and the interpersonal externality is asymmetric since users' private browsing behavior has no direct impact on any user's creating.[4] The evidence of interpersonal cross-activity externality is from an empirical study by Joinson (2008), who states that users spend time on social media because of the content on it.

The other effect is *intrapersonal cross-activity externalities*. Each user's dual activities could be substitutes due to time constraints; they could also be complements since: (i) users react to their friends' posts after browsing, and (ii) users check their

[2] On other widely used social media without asking for authentic identities except for celebrities, such as Quora, Twitter, and Weibo, users can observe the number of times their created content being watched.

[3] Facebook asks its users to register with their authentic names; for WeChat, not only WeChat verify the ID numbers of users to use their associated payment system, but also users can add each other as friends conditioning on that they share their account numbers with each other (Graziani (2017) and WeChat Help Center (2019)). It is more demanding for two users to have real-life interaction to become friends on WeChat than on Facebook.

[4] A user's browsing behavior has an indirect impact on her/his friends' creating behavior due to the intrapersonal externality. For more details, please see the remaining of this paragraph and the model in Section 1.2.

social media homepages more often after posting. Aggregately, the two types of intrapersonal effects, which are the substitute and complementary effects of each user's dual activities, might neutralize each other, or one type of effects dominates the other. The empirical evidence of different types of intrapersonal cross-activity externalities is from Viswanath et al. (2009), who find that users' frequencies of interaction with their friends are different. Note that we may use "intrapersonal externalities" or "intrapersonal effects" for short interchangeably with "intrapersonal cross-activity externalities" in the following discussion.

Social media owners provide their users with the information service, with which the users can interact with their friends online; meanwhile, the owners "price" their users by posting advertisements to the users while they are browsing. The friend-based social media owners act as monopolists in their markets due to the legislative barriers and huge social network externalities of existing users;[5] they also have *all the social network information of users* in such a way that they can deliver the users created content to their friends only. footnoteSeveral empirical studies, e.g., Viswanath et al. (2009), Schneider et al. (2009), collected their network information data on social media platform by themselves by using computer programs such as "crawler". Therefore, we can also infer that the social media owner has all the social network information.

There are some maturing business tactics for subsidizing content generation on friend-based social media. For example, users receive discounts or electronic money by sharing their consumption information of certain products on friend-based social media. Some of the subsidized contents are not pure advertisements, as friends can also receive discounts or electronic money by clicking the subsidized content. For example, there are popular online games developed and published by social media based firms (Web Desk (2018)). footnoteFor example, Facebook has popular game "Candy Crush" (Kafka (2014)); and WeChat's owner Tencent has PvP game "Honor of Kings" (Webster (2017)). Players, who are also friend-based social media users, share their gaming activities on friend-based social media to gain electronic money of the game, and their friends could also receive electronic money by clicking the content. However,

[5]Chinese government banned Facebook's access to Chinese market in 2009 (Pham (2016)). Tencent operates WeChat on international markets, however, there is no updating information of the total number of monthly active users on international markets since 2013, when WeChat had 1 million monthly active users on the international market (Iqbal (2017)).

there might be more advertisements on the web pages linked to the subsidized content. In such ways, the friends derive utilities from watching the subsidized content, and the monopolist could send more advertisements to her/his users. In this paper, we assume that users would like to gain the electronic money in the games and focus on users' behavior on friend-based social media only. The research question is: will the social media owners discriminate users by their network information and how?

We analyze a social media owner's pricing discriminating plan across connected users in our stylized model of social media through a game-theoretical approach. In the model, users choose time spending on two interdependent activities-creating and browsing activities. The interaction of users' dual activities is associated with an asymmetrical interpersonal cross-activity externality and intrapersonal cross-activity externalities on friend-based social media. The goal is to understand if network information is valuable to the social media owner as a monopolist. In particular, we want to understand the role of the three types of intrapersonal externalities. Further, we consider a more general setting, where the monopolist can subsidize users' creating behavior and price their browsing behavior.

In comparative static studies, we analyze the effects of denser networks and increase of intrapersonal cross-activity externalities on the monopolist's profit, and we analyze how users' utilities depend on the intrapersonal externalities. The two studies correspond to two operational ways of social media owners: (i) building denser networks corresponds to the owner's service of recommending new friends to users; and (ii) raising intrapersonal externalities corresponds to that both social media owners and users are trying to improve the ways of online interaction by developing stickers and GIF online, such that users have higher willingness to express their feelings after browsing on social media.

We show the existence of a unique subgame-perfect equilibrium. If the monopolist prices only users' browsing behavior, the pricing plan for each user is linear in the user's weighted degree centrality and the intrapersonal externalities. In the cases where the monopolist both prices users' browsing and subsidizes their creating without the intrapersonal effects, the pricing and subsidizing plan for each user depends on a new network measure: Katz-Bonacich centrality of degree 2.[6] We are able to

[6] If there exist intrapersonal externalities, then the monopolist's plan and users' behavior are related to the users' quasi-Katz-Bonacich centralities of degree 2, shown at Section 1.7.3. Please see the Sec-

use a simply way to show how Katz-Bonacich centrality and Katz-Bonacich centrality of degree 2 are related. Moreover, we show that nodes' ranking of Katz-Bonacich centrality in a given undirected network are not necessarily preserved in that of Katz-Bonacich centrality of degree 2. In comparative static studies, the results show that the monopolist's profit increases with new links added to the existing network in most cases, and that both users' utilities and the monopolist's profit increase with intrapersonal cross-activity externalities. Furthermore, the monopolist prefers networks with more diverse degree distributions.

Some empirical studies investigate users' behaviors on social media. Benevenuto et al. (2009) and Schneider et al. (2009) show that users spend most of their time on browsing, not on creating on social media. Lewis et al. (2008) show that a user with a higher degree or centrality would be more active on social media. Joinson (2008) identifies content, status updating, social network investigation, surfing and other three uses as the unique uses and gratifications for users on Facebook.

A strand of literature concerns players' local network interaction with a setting where players' best replies are linear in the sum of their neighbors' actions. The largest eigenvalues of adjacency matrices of networks determine users' equilibrium strategy in the game of complements (e.g., Ballester et al. (2006) and Corbo et al. (2007)); the smallest eigenvalues of those do in the game of substitutes (e.g., Bramoullé et al. (2014)). The largest and the smallest eigenvalues are still central in our analysis.

The relevant works on the monopolist's pricing on social networks are Candogan et al. (2012) and Bloch and Quérou (2013). In their models, players have only one action with a symmetric local network externality. Their studies show that the price is uniform in undirected networks.[7] Candogan et al. (2012) also study the case in which the monopolist discriminate consumers with a full price and a discount price, and they find it is NP-hard. Moreover, Zhou and Chen (2018) study monopolist's dynamic pricing of social goods on network, and they show that the monopolist can gain more profit if she/he offers sequentially to the customers of certain groups. For a summary and further discussion of this strand literature, please see Bloch (2016).

For the analysis of multiple activities on networks, Chen et al. (2018b) study the

tion 1.7.1 for more information about Katz-Bonacich centrality of degree 2 and quasi-Katz-Bonacich centrality of degree 2.

[7] In Candogan et al. (2012)'s model, the price is uniform when valuation is the same for all consumers.

case where connected players experience the interpersonal externality of each of their multiple activities. Gagnon and Goyal (2017) are interested in explaining the interaction between the market and social ties. In their case, players choose a market action and a network action.

The most relevant studies on pricing multiple activities consider cases with different numbers of pricing firms and different types of interpersonal and intrapersonal externalities. We clarify some terms of interpersonal effects to facilitate following discussion on relevant literature. The interpersonal effects are of two types. We name the first one as the interpersonal externality *within each activity*, which is studied by Chen et al. (2015), Chen et al. (2018a), and Chen et al. (2018b). The externality is interpersonal but *within each activity*. The other one is interpersonal effects *across two activities*, which is what we study in this paper. In Chen et al. (2015), Chen et al. (2018a) and Chen et al. (2018b)'s papers, the terms which captures the interpersonal effect in user's utility function is $t_i \sum_{j \in N(i)} t_j$ and $v_i \sum_{j \in N(i)} v_j$ in which t_i and v_i denote player i's two activities respectively and $N(i)$ denotes the set of players who have a link to user i. For a real life example, consider teenagers' allocation of time between studying and watching TV. Teenagers spend more time on studying if their friends do, and they also spend more time on watching TV if their friends watch more; it is not cross activity, since the teenagers do not spend more time on studying if their friends spend more time on watching TV. For the interpersonal cross-activity externality, the interpersonal effect is denoted by $v_i \sum_{j \in N(i)} t_j$. The example is the case studied in this chapter: users spend more time on browsing if their friends' total time spending on creating increases. The interpersonal cross-activity externality is symmetrical if each behavior has the same local network effects on the other, for example social media, such as Weibo, Twitter and Quora etc, reveal the users' browsing information.

Chen et al. (2018a) consider the users' dual activities on two platforms with interpersonal externalities *within each activity* and intrapersonal substitutes cross two activities. Their study show that the pricing plan is network-dependent and the more central users are priced less than other users. However, their study shows that the duopoly competition is essential to derive the network-dependent pricing plan; in the monopolist's cases, the pricing plan collapses to uniform pricing as in Bloch and Quérou (2013) and Candogan et al. (2012)'s cases. We are able to show that the interpersonal asymmetric across dual activities preserves the network-dependent pricing plan in monopolist's cases. We will make a more comprehensive comparison between our results and Chen et al. (2018b) in Section 1.4.

Fainmesser and Galeotti (2016), Leduc et al. (2017), Ushchev and Zenou (2018), Shin (2017), Sääskilahti (2007), and Ghiglino and Goyal (2010) study price problems with local network externalities in different specific settings. Also, this chapter enriches platform competition literature (Katz and Shapiro (1985), Farrell and Saloner (1985), Caillaud and Jullien (2003), Rochet and Tirole (2003), and Armstrong (2006))by considering the local network externalities between the participants from one side (user side)of the platforms.

This chapter unfolds as follows. Section 1.2 introduces the model. Section 1.3 solves the equilibria of monopolist's pricing plan on browsing. Section 1.4 studies the monopolist's dual activity pricing and subsidizing plan. The comparative static studies are in Section 1.5. In detail, Section 1.5.1 analyzes the effect of network structure on the monopolist's profit and on users' behavior; Section 1.5.2 examines the intrapersonal effect on users' utilities and on the monopolist's profit. Section 1.6 concludes.

1.2 The Model

Consider a set of users N connected in a network $\mathbf{G}$, such that $\mathbf{G}$ is an undirected and unweighted $|N| \times |N|$ adjacency matrix which states the connectedness of the network. For all $i, j \in N$, g_{ij} denotes the entry in row i and column j of matrix $\mathbf{G}$, such that: (i) for all $i \neq j \in N$, $g_{ij} = g_{ji} = 1$ if user i and user j are connected as friends, otherwise, $g_{ij} = g_{ji} = 0$; (ii) for all $i \in N$, $g_{ii} = 0$. We abuse notation by using $N(i) = \{j \in N \mid g_{ij} = 1\}$ to denote i's friends on the network, and we assume that every user i has at least one friend in network $\mathbf{G}$ such that for all i, $|N(i)| \geq 1$.

User i chooses both the time spending on creating content t_i and that on browsing/viewing content v_i such that $(t_i, v_i) \in \mathbb{R}^2_+$. The monopolist chooses an advertisement plan p_i for user i; p_i is the quantity of advertisements sent to user i per unit time. Let $\mathbf{p}^T = \left(p_1, p_2, ..., p_{|N|}\right) \in \mathbb{R}^{|N|}_+$ denote the monopolist's pricing plan for all users,

The game is a two-stage extensive game with perfect information and simultaneous moves. The **timing of events** is: at the first stage, the monopolist chooses the pricing plan $\mathbf{p}^T$; for all $i \in N$, user i chooses the time to create t_i and the time to browse v_i

simultaneously at the second stage.

Let vector $\mathbf{t}_{-i} \in \mathbb{R}_+^{|N|-1}$ and $\mathbf{v}_{-i} \in \mathbb{R}_+^{|N|-1}$ denote profiles of time spending on creating and browsing of all users except for user i. The utility function of user i is defined as follows:[8]

$$U\left((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i\right) =$$
$$\underbrace{\alpha_i t_i - \frac{1}{2}\beta t_i^2}_{\text{Creating Utilities}} + \underbrace{\overbrace{v_i \sum_{j \in N(i)} t_j}^{\text{Interpersonal Externality}} - \eta p_i v_i - \frac{1}{2}\beta v_i^2}_{\text{Browsing Utilities}} + \overbrace{\gamma t_i v_i}^{\text{Intrapersonal Externality}}, \tag{1.1}$$

in which $\alpha_i, \beta > 0$, and $\beta > |\gamma|$. Eq. (1.1) captures the utilities user i deriving from spending time on creating and browsing with the intrapersonal cross-activity externality. Before we explain each term in details in the following paragraphs, please note that what factor we include in user i's utility function should be information that i can have from the platforms. In other words, readers may infer some other externalities embedded in users' interaction, however, they should not be included in the utility functions if the information of one party incurring externalities are not revealed to users on these platforms.

□ **Create original content.** Linear-quadratic term $\alpha_i t_i - \beta t_i^2/2$ denotes the utilities user i derives by spending time t_i on creating irrespective of the intrapersonal externalities. Parameter α_i is user i marginal benefits of spending time on creating, which captures all the exogenous factors boosting i's original creating willingness. The quadratic term $-\beta t_i^2/2$ captures i's diminishing return in creating activity on social media. We name β as the index of rate of diminishing return.

□ **Browsing friends' created content.** Users browse the content available on social media, however, their browsing behavior is discouraged by the advertisements on the web pages of social media website. For the first term $v_i \sum_{j \in N(i)} t_j$ in browsing utilities, the sum of her/his friends' time spent on creating content $\sum_{j \in N(i)} t_j$ can be interpreted as the content available to user i, which is the marginal benefits of user i spending time on browsing. The second term $-\eta p_i v_i$ captures user i's disutilities of watching advertisements on social media, in which p_i represents the quantity of advertisements

[8] The linear-quadratic utility function of each user is essential to derive the results of this paper. It is not robust to other forms of users' utility functions.

sent to user i per unit time, such that $p_i v_i$ is the total advertisements user i watches while she/he is browsing, and $\eta > 0$ captures the users' intolerance to advertisements. The third term $-\beta v_i^2/2$ captures i's diminishing return in browsing activity on social media.

□ **Intrapersonal effects: the aggregate of intrapersonal substitutes and complementarities.**

- The intrapersonal effects $\gamma t_i v_i$ can be rewritten as $-\beta t_i v_i + (\gamma + \beta) t_i v_i$.

- Adding the first term $-\beta t_i v_i$ to the sum of diminishing return terms $-\beta t_i^2/2$ and $-\beta v_i^2/2$, we have term $-\beta (t_i + v_i)^2 /2$, which denotes the diminishing return of user i by spending time on the platforms. Therefore, $-\beta t_i v_i$ captures the intrapersonal substituting effects cross user i's dual activities due to time constraints.

- Term $(\beta + \gamma) t_i v_i$ such that $\beta + \gamma > 0$ captures the complementary intrapersonal effect between i's browsing behavior and her/his reaction to the post. The *complementary intrapersonal effect* $(\beta + \gamma) t_i v_i$ combined with *the asymmetric interpersonal cross-activity externalities* $v_i \sum_{j \in N(i)} t_j$ constitutes *the pattern of users' interaction online*. Consider two users i, j on the network who are connected as friends such that $j \in N(i)$. Users i and j's interaction is as follows: user i posts content such that t_i increases; friend j browse i's post such that v_j increases (by $v_j \sum_{k \in N(j)} t_k$ and $i \in N(j)$); friend j comment on i's post after browsing such that t_j increases (by $(\beta + \gamma) t_j v_j$); and i browses j's comment and replies to it such that v_i increases and t_i increases (by $v_i \sum_{k \in N(i)} t_k$ with $j \in N(i)$ and $(\beta + \gamma) t_i v_i$),....and so on so forth. Note that the browsing behavior of user j has *indirectly effects* on her/his friend i's creating behavior though the process of commenting and replying interaction on the platform.

- Aggregately, the substitute effect due to time constraints $-\beta t_i v_i$ in $\beta (t_i + v_i)^2 /2$ cancels out the term $\beta t_i v_i$ in the complementary effect $(\beta + \gamma) t_i v_i$, and we have the intrapersonal effects/externalities denoted as $\gamma t_i v_i$. If $\gamma = 0$, the substitute and complementary intrapersonal effects cancel each other out; if $\gamma < 0$, the substitute effect dominates the complementary effect; and the complementary effect dominates the substitute effect, if $\gamma > 0$. Further, we assume that $|\gamma| < \beta$, such that utility function U is strictly concave and has a unique maximum point with a negative definite Hessian matrix.

□ **Monopolist's pricing plan.** The monopolist gains profit from advertisers by the times users watching the advertisers' advertisements. Assume that the advertisers' demand to reach every user on social media is high enough. Then, all advertisement spaces on social media can be sold. Without loss of generality, each advertisement space is assumed to be sold at \$1, which is the opportunity cost of outside advertisement sponsoring options. Let $\mathbf{v}^T = \left(v_1, v_2, ..., v_{|N|}\right) \in \mathbb{R}_+^{|N|}$ denote the profile of all users' time spending on browsing. Then, the monopolist's profit function is $\pi = \mathbf{p}^T\mathbf{v}$.

Several remarks of the model are made below:

(i) As we mentioned in introduction, Facebook and WeChat do not reveal users' browsing time to their friends to make users unaware of being searched and to make them feel free to search other users. Therefore, the other users' browsing time spending $\mathbf{v}_{-i}$ is not included in the expression of eq. (1.1). Moreover, user i can infer that her/his friend j browsed i's content if j commented on i's posting, and it is captured by the intrapersonal effects. See the third paragraph of **intrapersonal effect** above for detailed explanations.

(ii) The social media owner does not reveal users' browsing information ($\mathbf{v}$) to their friends. However, users know how many friends they have ($\mathbf{G}$) on friend-based social media. The more friends a user has, the more the user will create. The users' degrees str network-dependent and exogenous; therefore, the degree effect of users on their creating is captured by α_i. The other network-independent factors captured by α_i contain users' personalities, chances of events in daily life etc.

(iii) if $\eta = 1$ and $\gamma = 0$, the terms depending on v_i in the utility function are $\left(\sum_{j\in N(i)} t_j - p_i\right) v_i - \beta v_i^2/2$. As $\sum_{j\in N(i)} t_j$ is the total content available for i, readers may think that p_i should be interpreted as the total advertisements seen by user i; user i spends no time on browsing, if the total content available is equal to the total advertisements $\sum_{j\in N(i)} t_j = p_i$. This statement makes an assumption that user i's utility function corresponds to browsing advertisements and browsing content in the same way, which is not the case here.[9]

1.3 Subgame-perfect Equilibria

We choose subgame-perfect equilibrium (SPE) as the solution concept for the game, which is an extensive game with perfect information and simultaneous moves (which is also used by Bloch and Quérou (2013) and Chen et al. (2018a)). The game can be solved by backward induction. Given the monopolist's pricing plan, we solve the strategy of users' dual activities at the second stage. Then, the monopolist's optimal pricing strategy will be determined.

We start with the simplest case $\gamma = 0$. In equilibrium, a user' creating time depends on the her/his own willingness to create, and the user's browsing time depends on her/his friends' creating. The results show the monopolist's pricing plan for each user is proportional to his/her degree. Then, we study general cases where $\gamma \neq 0$. The intrapersonal externality has no network-dependent effect on the monopolist's pricing plan, but induces each user's activities having global effects on the social network and both of the user's dual activities depending her/his Katz-Bonacich centrality.

1.3.1 Asymmetric Interpersonal Effects Only

If $\gamma = 0$, the utility function is given by $U\left((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i\right) = \alpha_i t_i + v_i \sum_{j \in N(i)} t_j - \beta\left(t_i^2 + v_i^2\right)/2 - p_i v_i$. We solve each user's utility maximization problem, then we solve the equilibrium given each user's best respond in the game.

Maximize U w.r.t t_i and v_i by deriving U's F.O.C.s:

$$t_i = \frac{\alpha_i}{\beta}$$

and

$$v_i = \begin{cases} \sum_{j \in N(i)} t_j/\beta - \eta p_i/\beta & \sum_{j \in N(i)} t_j > \eta p_i \\ 0 & \sum_{j \in N(i)} t_j \leq \eta p_i \end{cases}. \tag{1.2}$$

[9] If this is the case, the utility function is $v_i \sum_{j \in N(i)} t_j - \beta v_i^2/2 - \eta\left(p_i v_i\right) v_i$ in which p_i is still the density of advertisements for user i. The best response would be $v_i^* = \sum_{j \in N(i)} t_j / \left(\beta + 2\eta p_i\right)$. The monopolist's profit from i $p_i v_i^*$ strictly increases with $p_i \in [0, +\infty)$, which is not sound. Moreover, if we use p_i to denote the total advertisements sent to i, then the monopolist profit is $\mathbf{p}^T \mathbf{1}$. The monopolist can choose a p_i slightly smaller than $\sum_{j \in N(i)} t_j$, and she/he gains all the profit from advertisements on the website, which is not sound neither.

User i's optimal creating time t_i and browsing time v_i are independent.

Consider eq. (1.2). If p_i is less than $\sum_{j\in N(i)} t_j/\eta$, v_i is positive and is proportional to the difference between her/his friends' sum of creating time and the density of advertisements sent to her/him; otherwise, v_i is at its lower bound 0. The second case where $v_i = 0$ driven by a high p_i will not appear, since the monopolist has no incentive to choose a $p_i \geq \sum_{j\in N(i)} t_j$.[10]

Let $\boldsymbol{\alpha} = \left(\alpha_1, \alpha_2, ..., \alpha_{|N|}\right)^T$ be the vector of users' original willingness to create. Replace $\sum_{j\in N(i)} t_j$ in eq. (1.2) with $\sum_{j\in N(i)} \alpha_i/\beta$, and we have the users' best responses of browsing time $v_i = \sum_{j\in N(i)} \alpha_j/\beta^2 - \eta p_i/\beta$. In matrix form, $\mathbf{v} = \left(\mathbf{G}\boldsymbol{\alpha}/\beta - \eta\mathbf{p}\right)/\beta$. Combining that the monopolist's profit function is $\pi = \mathbf{p}^T\mathbf{v}$, the monopolist's profit maximization problem is: $\max_{\mathbf{p}\in\mathbb{R}_+^{|N|}} \mathbf{p}^T\left(\mathbf{G}\boldsymbol{\alpha}/\beta - \eta\mathbf{p}\right)/\beta$. Solving the profit maximization problem, we have:

Proposition 1.1. *Let $\gamma = 0$. The monopolist's optimal price strategy is given by $\mathbf{p}^* = \mathbf{G}\boldsymbol{\alpha}/\left(2\beta\eta\right)$, such that users' dual activities are given by*

$$\mathbf{t}^* = \frac{1}{\beta}\boldsymbol{\alpha} \textit{ and } \mathbf{v}^* = \frac{1}{2\beta^2}\mathbf{G}\boldsymbol{\alpha}$$

in SPE.

Proof. See Section 1.7.2.

This proposition is not hard to derive. However, it reveals some fundamental intuition for the monopolist's pricing plan and users' dual activities with asymmetric interpersonal effects.

The monopolist's discriminating pricing plan on each user is proportional to her/his weighted degree, which is the sum of her/his friends' exogenous willingness to create. It differs from the cases of Bloch and Quérou (2013) and Candogan et al. (2012), which consider one action with a symmetrical interpersonal network externality and derive a uniform pricing plan. In their cases, the symmetry of local network externality leads to the uniform pricing plan, since the monopolist's incentives to price a user more induced by the local network externality cancel out her/his incentives to subsidize

[10] If $v_i = 0$, the monopolist gains no profit from user i, and the monopolist's profit from any other user on the network is independent from her/his choice of p_i. See also the proof of Proposition 1.1 in the Section 1.7.2.

the user induced by that. However, for the dual activities of users in our cases, the monopolist prices the behavior which is locally and unilaterally affected by the other behavior. Thus, the monopolist has the incentive to price the browsing behavior but no incentive to subsidize it due to the asymmetry of the browsing and creating behavior. Therefore, pricing plans in our cases depend on each user's degree centrality.

1.3.2 Generalization

In this section, we study the cases where the interpersonal network externality of creating behavior appears ($\gamma \neq 0$). We focus on the unique and interior solutions such that we can compare the effects of γ on users' behavior and the monopolist pricing plan in equilibrium.

The second stage of the game is solved by envelope theorem and potential game. The reason for using potential game is due to the existence of corner solutions and the multiplicities of those in the games with local network substitutes. If the potential function of the game is strictly concave and some more conditions are satisfied, we are able to focus on the unique and interior solutions for further comparative static studies. It is firstly introduced by Bramoullé et al. (2014) to analyze the games with local network substitutes by potential game.

We firstly show the condition for the uniqueness of solutions in equilibria, and the full characterization of unique and interior solutions for both $\gamma > 0$ and $\gamma < 0$ follows.

By the envelope theorem, a user's one action corresponds to her/his other action accordingly in her/his utility maximization problem. Here, we choose each user's creating behavior to solve the equilibrium at the second stage. The details are as follows.

The partial derivative of U with respect to v_i is:

$$\frac{\partial U}{\partial v_i} = \sum_{j \in N(i)} t_j + \gamma t_i - \eta p_i - \beta v_i. \tag{1.3}$$

Solve $\partial U / \partial v_i = 0$, and we have user i's browsing time as a function of creating time t_i presenting as :

$$v(t_i) = \frac{\sum_{j \in N(i)} t_j + \gamma t_i - \eta p_i}{\beta}. \tag{1.4}$$

Assume that $v(t_i) > 0$. We replace v_i in eq. (1.1) with $v(t_i)$ denoted by eq. (1.4) resulting in:

$$U\left(\left(t_i, v\left(t_i\right)\right); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i\right) = \alpha_i t_i - \frac{1}{2}\beta t_i^2 + \frac{1}{2\beta}\left(\sum_{j\in N(i)} t_j + \gamma t_i - \eta p_i\right)^2. \quad (1.5)$$

The F.O.C. of eq. (1.5) w.r.t t_i is as follows:

$$\frac{dU\left(\left(t_i, v\left(t_i\right)\right); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i\right)}{dt_i} = \alpha_i - \beta t_i + \frac{\gamma}{\beta}\left(\sum_{j\in N(i)} t_j + \gamma t_i - \eta p_i\right) = 0.$$

Let $\delta = \left(\beta^2 - \gamma^2\right)^{-1}$. Therefore, if $t_i, v\left(t_i\right) > 0$, we can write user i's best response of creating time as $t_i = \beta\delta\alpha_i + \gamma\delta\sum_{j\in N(i)} t_j - \gamma\delta\eta p_i$. Note that: (i) if $\gamma > 0$, $v\left(t_i\right) > 0$ implies $t_i > \alpha_i/\beta$; (ii) if $\gamma < 0$, $v\left(t_i\right) > 0$ implies $t_i < \alpha_i/\beta$.

Assume that for all $k \in N$, $t_k, v\left(t_k\right) > 0$. In matrix form, we have $\mathbf{t} = \beta\delta\boldsymbol{\alpha} + \gamma\delta\mathbf{G}\mathbf{t} - \gamma\delta\eta\mathbf{p}$. In the best responses of creating time, the local network externality factor within creating time is $\gamma\delta$, which can be regarded as a index of relative magnitude of intrapersonal effect with respect to the rate of diminishing return. The intrapersonal cross-activity externality γ increases the interpersonal effects within users' creating activity $\gamma\delta$; the index of rate of diminishing return β reduces each activity, by which it also reduces the interpersonal effects within users' creating activity $\gamma\delta$.

Monderer and Shapley (1996) state that for payoff functions $\widetilde{U}$ which are continuous and twice-differentiable, there exists a potential function, if and only if for all $i \neq j$, $\partial^2\widetilde{U}_i/\partial t_i\partial t_j = \partial^2\widetilde{U}_j/\partial t_j\partial t_i$.

Let U_i and U_j denote $U\left(\left(t_i, v\left(t_i\right)\right); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i\right)$ and $U\left(\left(t_j, v\left(t_j\right)\right); \mathbf{t}_{-j}, \mathbf{v}_{-j}, p_j\right)$ as shown in eq. (1.5) respectively. We take the mixed partial derivatives of U_i and U_j w.r.t. t_i and t_j. Thus, we have $\partial^2 U_i/\partial t_i\partial t_j = \partial^2 U_j/\partial t_j\partial t_i = \gamma g_{ij}/\beta$, which satisfies the condition of Monderer and Shapley (1996). Therefore, given a pricing plan $\mathbf{p}$, we have potential function:

$$\varphi\left(\mathbf{t}; \boldsymbol{\alpha}, \mathbf{p}, \mathbf{G}\right) = \delta\mathbf{t}^T\left(\beta\boldsymbol{\alpha} - \eta\gamma\mathbf{p}\right) - \frac{1}{2}\mathbf{t}^T\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{t}.$$

The maxima and saddle points of φ correspond to the set of equilibria of the game (Bramoullé et al. (2014) Lemma 1.). In our case, $\mathbf{t}$ satisfies Kuhn-Tucker conditions:

(i) Assume $\gamma > 0$, then $t_i \geq \alpha_i/\beta$. If φ is strictly concave, then $t_i = \alpha_i/\beta \Rightarrow$

$\partial \varphi / \partial t_i \leq 0$, and $t_i > \alpha_i/\beta \Rightarrow \partial \varphi / \partial t_i = 0$; if φ is convex, for all i $t_i \to +\infty$.

(ii) Assume $\gamma < 0$, then $0 \leq t_i \leq \alpha_i/\beta$. Therefore, $t_i = 0 \Rightarrow \partial \varphi / \partial t_i \leq 0$, $0 < t_i < \alpha_i/\beta \Rightarrow \partial \varphi / \partial t_i = 0$, and $t_i = \alpha_i/\beta \Rightarrow \partial \varphi / \partial t_i \geq 0$.[11]

Potential function φ has a unique maximum on $\mathbb{R}_+^{|N|}$ if and only if $\mathbf{H}(\varphi) = \nabla^2 \varphi = -(\mathbf{I} - \gamma\delta\mathbf{G})$, the Hessian matrix of φ, is negative definite. As $-(\mathbf{I} - \gamma\delta\mathbf{G})$ is a Hermitian matrix, it is negative definite if and only if all of its eigenvalues are negative. Let $\mathbf{A}$ denote an $|N| \times |N|$ squared matrix, and let $\lambda(\mathbf{A})$ denote an eigenvalue of matrix $\mathbf{A}$. Therefore, if the maximal eigenvalue of $\mathbf{H}(\varphi)$ $\lambda_{\max}(\mathbf{H}(\varphi))$is smaller than 0, we have a unique solution for the second stage of the game. Then, $\lambda_{\max}(\mathbf{H}(\varphi)) = \lambda_{\max}(-\mathbf{I} + \gamma\delta\mathbf{G}) < 0$, which is equivalent to $-1 + \delta\lambda_{\max}(\gamma\mathbf{G}) < 0$. Therefore:

Lemma 1.1. *If $\lambda_{\max}(\gamma\mathbf{G}) < \delta^{-1}$, for any price vector $\mathbf{p}$, there exists a unique equilibrium in the second stage of game.*

If $\gamma > 0$, $\lambda_{\max}(\gamma\mathbf{G}) = \gamma\lambda_{\max}(\mathbf{G})$. Condition $\gamma\lambda_{\max}(\mathbf{G}) < \delta^{-1}$ is the sufficient and necessary condition for the existence and uniqueness of the equilibrium. The largest eigenvalue of $\mathbf{G}$ determines the equilibrium. If $\lambda_{\max}(\mathbf{G}) > (\gamma\delta)^{-1}$, φ is not strictly concave in $\mathbb{R}_+^{|N|}$ such that the cumulative network externalities are large enough to drive every user's time spending to infinite. The condition and results are derived by Ballester et al. (2006), Corbo et al. (2007), Ballester and Calvó-Armengol (2010) et al.

If $\gamma < 0$, $\lambda_{\max}(\gamma\mathbf{G}) = \gamma\lambda_{\min}(\mathbf{G})$ in which $\lambda_{\min}(\mathbf{G})$ is the smallest eigenvalue of $\mathbf{G}$. Condition $|\lambda_{\min}(\mathbf{G})| < |\gamma\delta|^{-1}$ is a sufficient condition only. The condition and results are derived by Bramoullé et al. (2014). If $|\lambda_{\min}(\mathbf{G})| > |\gamma\delta|^{-1}$, there exists at least one equilibrium with some agent who is either inactive in creating or inactive in browsing.

The condition can be rewritten as $|\lambda_{\max}(\mathbf{G})| < |\gamma\delta|^{-1}$ if $\gamma > 0$ or $|\lambda_{\min}(\mathbf{G})| < |\gamma\delta|^{-1}$if $\gamma < 0$. In both cases, given a network $\mathbf{G}$, the magnitude of relative intrapersonal effects $|\gamma\delta|$ should be smaller enough to ensure the existence and uniqueness of equilibria.

[11] If $\gamma < 0$ and for some user i $t_i = \alpha_i/\beta$, then user i's browsing time binds such that $v_i = v(t_i) = 0$.

We focus on the unique and interior solutions. If the solutions are unique and interior. Then, by solving $\arg\max_{\mathbf{t}\in\mathbb{R}_+^{|N|}} \varphi(\mathbf{t};\boldsymbol{\alpha},\mathbf{p},\mathbf{G})$, creating time is:

$$\mathbf{t} = \delta(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}(\beta\boldsymbol{\alpha}-\eta\gamma\mathbf{p}),$$

and the browsing time is

$$\mathbf{v} = \delta(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}(\mathbf{G}\boldsymbol{\alpha}+\gamma\boldsymbol{\alpha}-\beta\eta\mathbf{p}).$$

The monopolist's profit function is $\pi = \mathbf{p}^T\mathbf{v}$. Combining with the two previous sentences, monopolist's profit maximization problem is:

$$\arg_{\mathbf{p}\in\mathbb{R}_+^N} \mathbf{p}^T\delta(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}(\mathbf{G}\boldsymbol{\alpha}+\gamma\boldsymbol{\alpha}-\beta\eta\mathbf{p}).$$

The Hessian matrix of π is $\mathbf{H}(\pi) = \nabla^2\pi = -\beta\eta\delta(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}$, which is negative definite. Therefore, it has a unique profit maximization point. We solve the profit maximization problem, and have the necessary conditions for unique and interior solutions. Combining with the sufficiency of these conditions, we have the following proposition:[12]

Proposition 1.2. *Let $\lambda_{\max}(\gamma\mathbf{G}) < \delta^{-1}$. If and only if:*

(i) $\gamma > 0$, *or*

(ii) $\gamma < 0$ *and for all* i $((\mathbf{G}+\gamma\mathbf{I})\boldsymbol{\alpha})_i > 0$, *and* $-\alpha_i < \beta^2\delta\left((\mathbf{I}-\gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha}\right)_i < \alpha_i$;

the SPE has a unique and interior solution, in which the monopolist's pricing strategy is $\mathbf{p}^* = (\mathbf{G}+\gamma\mathbf{I})\boldsymbol{\alpha}/(2\eta\beta)$, *users' strategy of their dual activities is given by*

$$\mathbf{t}^* = \frac{1}{2\beta}\left(\mathbf{I}+\beta^2\delta(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}\right)\boldsymbol{\alpha} \text{ and } \mathbf{v}^* = \frac{\delta}{2}(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}(\mathbf{G}+\gamma\mathbf{I})\boldsymbol{\alpha}.$$

Proof. See Section 1.7.2.

The intuition for the condition of the proposition is as follows. The solution of a strictly concave function is unique and interior if and only if the solution satisfies the

[12] Please see the proof of Proposition 1.2 at Section 1.7.2 for details.

F.O.C.s and is interior. In our cases, the solution satisfies that for all i $p_i, t_i, v_i > 0$. Assuming $\gamma > 0$. Then, every entry of pricing plan $\mathbf{p}^* = (\mathbf{G} + \gamma\mathbf{I})\,\boldsymbol{\alpha}/\,(2\eta\beta)$ is positive; users' dual activity, every entry of creating time $\mathbf{t}^* = \left(\mathbf{I} + \beta^2\delta\,(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\right)\boldsymbol{\alpha}/\,(2\beta)$ is positive and so is every entry of $\mathbf{v}^* = \delta\,(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\,(\mathbf{G}\boldsymbol{\alpha} + \gamma\boldsymbol{\alpha})\,/2$, since every entry of $(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}$ is positive if $\gamma > 0$. Therefore, the solution of equilibria with $\gamma > 0$ is unique and interior.

If $\gamma < 0$, at the maximum of profit function, we have $\mathbf{p}^* = (\mathbf{G} + \gamma\mathbf{I})\,\boldsymbol{\alpha}/\,(2\eta\beta)$, $\mathbf{t}^* = \left(\mathbf{I} + \beta^2\delta\,(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\right)\boldsymbol{\alpha}/\,(2\beta)$ and $\mathbf{v}^* = \delta\,(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\,(\mathbf{G}\boldsymbol{\alpha} + \gamma\boldsymbol{\alpha})\,/2$. The necessity and sufficiency of the condition that for all i $((\mathbf{G} + \gamma\mathbf{I})\,\boldsymbol{\alpha})_i > 0$, and $-\alpha_i < \beta^2\delta\left((\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha}\right)_i < \alpha_i$ are straightforward. In details, the condition that for all i, $((\mathbf{G} + \gamma\mathbf{I})\,\boldsymbol{\alpha})_i > 0$ is for $\mathbf{p}^*$ to be interior; if condition that for all i $-\alpha_i < \beta^2\delta\left((\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha}\right)_i < \alpha_i$ is satisfied, then for all i it is for $0 < t_i^* < \alpha_i/\beta$ such that for all i $v_i^* > 0$.

To compare with the case $\gamma = 0$, we subtract the $\mathbf{p}^*$ with $\mathbf{G}\boldsymbol{\alpha}/\,(2\beta\eta)$, which is the monopolist's pricing plan in cases where $\gamma = 0$ (See alsoProposition 1.1), which results in $\Delta\mathbf{p} = \mathbf{p}^* - \mathbf{G}\boldsymbol{\alpha}/\,(2\beta\eta) = \gamma\boldsymbol{\alpha}/\,(2\eta\beta)$. It means that the intrapersonal cross-activity externality does not bring network-dependent factors into the monopolist's pricing plan. The result coincides with the cases studied by Candogan et al. (2012) and Bloch and Quérou (2013). The intuition of the results is: the local externality of users' creating behavior induced by intrapersonal externalities is symmetrical in monopolist's profit function's F.O.C.s, such that the profit increasing from pricing one user more is canceled out by the profit increasing from pricing the user less for the purposes of pricing the user's friends more.

Both of users' two activities are linear in Katz-Bonacich centrality in equilibrium. Compared with the case without intrapersonal externality where $\mathbf{t}^* = \boldsymbol{\alpha}/\beta$, the creating time with intrapersonal effect is $\mathbf{t}^* = \boldsymbol{\alpha}/\,(2\beta) + \beta^2\delta\,(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\,\boldsymbol{\alpha}/\,(2\beta)$. User i's creating time can be jointly described by i's willingness to create α_i and i's Katz-Bonacich centrality–the cumulative network effects on user i. It is supported by the following intuition: user i's creating time is induced by her/his willingness to create (the first term), and it is boosted by i's interaction with her/his friends on the social media (the second term). With neutralized intrapersonal effects ($\gamma = 0$), the boosted creating time (the second term) equals to the induced creating time (the first term), which means that even though users spend time on creating, their behavior depends on their own willingness to create only. If user i's dual activities are complements ($\gamma > 0$), the boosted creating time (the second term) is larger than the induced creating time (the first term); if user i's dual activities are substitutes ($\gamma < 0$), the boosted time (the second term) depends on the overall substitute and complementary network effects on the users.[13]

The browsing time of users is $\mathbf{v}^* = \delta\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\mathbf{G}\boldsymbol{\alpha} + \gamma\boldsymbol{\alpha}\right)/2$. User i's browsing time depends on the sum of cumulative network effects on her/his friends, and it increases (decreases) with that effect on herself/himself if $\gamma > 0$ ($\gamma < 0$). The results are supported by that any user i's browsing behavior is boosted by her/his friends' creating. But why does the first term of a user's creating time has dependency on original creating willingness but it is not the case that the user browsing time is boosted by her/his friends' original willingness to create? The reason is that: with pricing plan $\mathbf{p}^* = \left(\mathbf{G} + \gamma\mathbf{I}\right)\boldsymbol{\alpha}/\left(2\eta\beta\right)$, the monopolist extracts the benefit of users' boosted by original creating of her/his friends; and she/he prices the users own original willingness to create if $\gamma > 0$ but the monopolist subsidizes users' browsing behavior by pricing less if $\gamma < 0$. Therefore, user i's browsing time depends on the cumulative network effects on his/her friends, but it does not depend on their original willingness to create.

In empirical studies, we observe that most users on friend-based social media spend 95% of their time on browsing (Benevenuto et al. (2009) and Schneider et al. (2009)). It means that the case we study where the intrapersonal cross-activity substitute appears $\gamma < 0$ fits better to real-life cases. An average user has 338 friends on Facebook (Mazie (2014)). The huge amount of available content on social media consumes their time on browsing, which deduce users' content creating activity.

1.4 Dual-activity Pricing and Subsidizing Plan

Consider that the monopolist not only prices the users' browsing activity, but also subsidizes their creating activity. In this section, we show that, in SPE, both the monopo-

[13] As *Katz-Bonacich centrality counts all the paths with both oddly and evenly numbered steps of one node to reach all nodes in the networks.* If $\gamma < 0$, the paths with oddly numbered steps to i contribute negative effects to the cumulative network effects on i, and the paths with evenly numbered steps contribute positively to that.

list' plan and users' dual activities depend on a new network measure– Katz-Bonacich centrality of degree 2. Then, we will explain its implication in our model. In Section 1.4.2, we show how the Katz-Bonacich centrality of degree 2 differs from other centrality measures.

We use $\mathbf{q}^T = \left(q_1, q_2, q_3, ..., q_{|N|}\right) \in \mathbb{R}_+^{|N|}$ to denote the monopolist's subsidizing plan, such that $q_i \in \mathbb{R}_+$ represents how much the monopolist subsidizes user i for her/his one unit time spending on creating. For every user i, her/his creating utilities increase by $\kappa q_i t_i$ due to the monopolist's subsidizing, in which $\kappa \in (0, \eta)$. Parameter κ is smaller than η, which means that users' utilities' increase from one unit of subsidies is smaller than her/his disutilities' from watching advertisements once.[14]

The utility function of user i with subsidizing denoted by $\hat{U}$ is:

$$\hat{U}\left((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i, q_i\right) = \alpha_i t_i - \frac{1}{2}\beta t_i^2 + \kappa q_i t_i + v_i \sum_{j \in N(i)} t_j - \frac{1}{2}\beta v_i^2 - \eta p_i v_i + \gamma t_i v_i. \tag{1.6}$$

Then, the monopolist's profit maximization problem is:

$$\max_{\mathbf{p}, \mathbf{q}} \mathbf{p}^T \mathbf{v} - \mathbf{q}^T \mathbf{t}. \tag{1.7}$$

We study the case with $\gamma = 0$. For the cases where $\gamma \neq 0$, please see Section 1.7.3.

1.4.1 Equilibrium Analysis

We focus on the unique and interior solutions. If $\gamma = 0$, user i's utility function with q_i $\hat{U}$ is:

$$\hat{U}\left((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i, q_i\right) = \alpha_i t_i - \frac{1}{2}\beta t_i^2 + \kappa q_i t_i + v_i \sum_{j \in N(i)} t_j - \frac{1}{2}\beta v_i^2 - \eta p_i v_i.$$

The F. O. C.s of $\hat{U}$ are:

$$\frac{\partial \hat{U}}{\partial t_i} = \alpha_i + \kappa q_i - \beta t_i = 0;$$

[14] If $\kappa > \eta$, the monopolist can gain infinity profit by choosing a friend of i $j \in N(i)$ and making a plan as $p_j > q_i$ and $q_i \to +\infty$.

$$\frac{\partial \hat{U}}{\partial v_i} = \sum_{j \in N(i)} t_j - \eta p_i - \beta v_i = 0.$$

Solve $\partial \hat{U}/\partial t_i = 0$ and $\partial \hat{U}/\partial v_i = 0$ simultaneously, and write them in matrix form. We have:

$$\mathbf{t} = \frac{1}{\beta}\boldsymbol{\alpha} + \frac{\kappa}{\beta}\mathbf{q} \tag{1.8}$$

and

$$\mathbf{v} = \frac{1}{\beta^2}\mathbf{G}\boldsymbol{\alpha} + \frac{\kappa}{\beta^2}\mathbf{G}\mathbf{q} - \frac{\eta}{\beta}\mathbf{p}. \tag{1.9}$$

Replace $\mathbf{t}$ and $\mathbf{v}$ from eq. (1.8) and eq. (1.9) in the profit function eq. (1.7):

$$\pi = -\mathbf{q}^T\left(\frac{\boldsymbol{\alpha}}{\beta} + \frac{\kappa\mathbf{q}}{\beta}\right) + \mathbf{p}^T\left(\frac{\mathbf{G}\boldsymbol{\alpha}}{\beta^2} + \frac{\kappa\mathbf{G}\mathbf{q}}{\beta^2} - \frac{\eta\mathbf{p}}{\beta}\right). \tag{1.10}$$

Let $\mathbf{x}^T = \left(\mathbf{p}^T, \mathbf{q}^T\right)$ denote the monopolist's pricing and subsidizing plans on users. Also, we define vectors or matrices as follows: , $\mathbf{I}_{u,l} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\mathbf{I}_{b,r} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$, $\mathbf{I}_{u,r} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix}$. Then, the profit function eq. (1.10) can be rewritten as:

$$\pi\left(\mathbf{p}, \mathbf{q}\right) =$$
$$-\frac{1}{\beta}\mathbf{x}^T\mathbf{I}_{b,r}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\mathbf{x}^T\mathbf{I}_{u,l}\mathbf{B}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} - \frac{\kappa}{\beta}\mathbf{x}^T\mathbf{I}_{b,r}\mathbf{x} + \frac{\kappa}{\beta^2}\mathbf{x}^T\mathbf{I}_{u,r}\mathbf{B}\mathbf{x} - \frac{\eta}{\beta}\mathbf{x}^T\mathbf{I}_{u,l}\mathbf{x}. \tag{1.11}$$

The Hessian matrix of the monopolist's profit function eq. (1.11) is:

$$\mathbf{H}\left(\pi\right) = \frac{1}{\beta^2}\begin{bmatrix} -2\eta\beta\mathbf{I} & \kappa\mathbf{G} \\ \kappa\mathbf{G} & -2\kappa\beta\mathbf{I} \end{bmatrix}.$$

The unique maximum of π exists if and only if its Hessian matrix is negative definite. Since $\mathbf{H}\left(\pi\right)$ a Hermitian matrix, it is negative definite if and only if the largest eigenvalue of $\mathbf{H}\left(\pi\right)$ is negative, which means $\lambda_{\max}\left(\mathbf{G}\right) < 2\beta\sqrt{\eta/\kappa}$.[15]

The F. O. C.s of eq. (1.11) are as follows:

[15] For the details of calculation, see the proof of Proposition 1.3 in Section 1.7.2.

$$\nabla\pi = -\frac{1}{\beta}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\begin{pmatrix} \mathbf{G}\boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\begin{bmatrix} -2\eta\beta\mathbf{I} & \kappa\mathbf{G} \\ \kappa\mathbf{G} & -2\kappa\beta\mathbf{I} \end{bmatrix}\mathbf{x}.$$

Solve $\nabla\pi = \mathbf{0}$. We have the following proposition:

Proposition 1.3. *Let $\gamma = 0$. If and only if $\lambda_{\max}(\mathbf{G}) < 2\beta\sqrt{\eta/\kappa}$, the monopolist's profit function is strictly concave such that it has a unique maximum. Further, if and only if $\lambda_{\max}(\mathbf{G}) < 2\beta\sqrt{\eta/\kappa}$ and for all i, for all i $\left(\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2 - \mathbf{I}\right)^{-1}\boldsymbol{\alpha}\right)_i >$ 0, the SPE has an interior solution, in which the monopolist's optimal pricing and subsidizing plan is given by:*

$$\mathbf{p}^* = \frac{1}{4\eta\beta}\mathbf{G}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha} \text{ and } \mathbf{q}^* = \frac{1}{2\kappa}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha} - \frac{1}{\kappa}\boldsymbol{\alpha};$$

the users' creating time and browsing time are given by:

$$\mathbf{t}^* = \frac{1}{2\beta}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha} \text{ and } \mathbf{v}^* = \frac{1}{4\beta^2}\mathbf{G}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha}.$$

Proof. See Section 1.7.2.

Condition that for all i $\left(\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2 - \mathbf{I}\right)^{-1}\boldsymbol{\alpha}\right)_i > 0$ is sufficient and necessary for the results. The SPE solution is interior if and only if the solution satisfies the F.O.C.s and for all i $p_i, q_i, t_i, v_i > 0$. The expressions for $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{t}^*$ ensure that all of their entries are positive, since every entry of matrix $\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2 - \mathbf{I}\right)^{-1}$ is positive. For every entry of subsidizing plan $\mathbf{q}^*$ to be positive, the sufficiency and necessity of condition that for all i $\left(\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2 - \mathbf{I}\right)^{-1}\boldsymbol{\alpha}\right)_i > 0$ are straightforward.

We name κ/η as users' cross-activity responding rate to the monopolist's plan or responding rate for short. By the condition ensuring the positiveness of $\mathbf{q}^*$, i. e., for all i such that $\left(\mathbf{G}^2\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha}\right)_i \kappa/\eta > 4\beta^2\alpha_i$, we know that the monopolist has the incentive to subsidize all users, if users' responding rate to her/his dual activity pricing and subsidizing plan is high enough.

Consider the condition for the strict concavity of profit function, i. e., $\lambda_{\max}(\mathbf{G}) < 2\beta\sqrt{\eta/\kappa}$. The existence of equilibria depends on the network structure $\mathbf{G}$, the rate of diminishing return β and users' responding rate κ/η. Given a network $\mathbf{G}$, if the index of rate of diminishing return is low or users' responding rate κ/η is high enough such that $\lambda_{\max}(\mathbf{G}) \geq 2\beta\sqrt{\eta/\kappa}$, the monopolist can use pricing and subsidizing plan $(\mathbf{p}, \mathbf{q})$ to drive users' time spending and the profit to infinity. By the observation of real life, we could infer that users' responding rate is low in real life.

The Katz-Bonacich centralities $\mathbf{b}(\mathbf{G}, \theta) = \mathbf{M}(\mathbf{G}, \theta)\mathbf{1}$ are with matrix $\mathbf{M}(\mathbf{G}, \theta) = (\mathbf{I} - \theta\mathbf{G})^{-1}$ such that $\theta \in \mathbb{R}$ and $(\mathbf{I} - \theta\mathbf{G})^{-1}$ is contracting. In the SPE with dual-activity pricing and subsidizing plans, the centralities are with matrix $\mathbf{M}(\mathbf{G}^2, \theta) = (\mathbf{I} - \theta\mathbf{G}^2)^{-1}$ in which $\theta \in \mathbb{R}$ and $(\mathbf{I} - \theta\mathbf{G}^2)^{-1}$ is contracting. We name $\mathbf{b}(\mathbf{G}^2, \theta) = (\mathbf{I} - \theta\mathbf{G}^2)^{-1}\mathbf{1}$ as the vector of Katz-Bonacich centrality with degree 2 of network $\mathbf{G}$. See also Section 1.7.1. We will firstly explain the intuition of the results and then compare our cases with Chen et al. (2018a). For the studies of Katz-Bonacich Centrality of degree 2 on typical network structures, please see Section 1.4.2.

For the subsidizing plan $\mathbf{q}^*$, term $-\boldsymbol{\alpha}/\kappa$ in $\mathbf{q}^*$ implies that the monopolist neutralizes agents' original willingness to create and boosts users' willingness to create by their Katz-Bonacich of centrality of degree 2 $(\mathbf{I} - \kappa/(4\eta\beta^2)\mathbf{G}^2)^{-1}\boldsymbol{\alpha}/(2\kappa)$.

The results are supported by the following intuition. *Katz-Bonacich centrality counts all the paths with both odd and even numbered steps of one node to reach all nodes in the networks. However, Katz-Bonacich centrality of degree 2 counts the paths with even numbered steps of that only.* To understand why the monopolist chooses a pricing and subsidizing plan depending on Katz-Bonacich centrality of degree 2, let us consider what comprises the number of paths with 2 steps on a network for a user. For user i, the first composition is the number of i's friends, since a user can always go to his/her neighborhood and go back to herself/himself with two steps; the second composition is the number of user i' friends' friends. That is to say, the monopolist chooses to subsidy the users who have high degrees and also have high correctness to their friends' friends. The monopolist's incentive to subsidy the users with high degrees is easy to understand. For subsidizing a user's friends' friends, the monopolist can price a user's friends more by the combining effects of boosting the user's post and also her/his friends' friends' post. The monopolist has no incentive to subsidize the users who are with 3 steps from user i due to the lack of the combining effects. However, by subsidizing the users who are with 4 steps away from user i, the subsidies on users who are 2 steps away from user i become more valuable to the monopolist, which also make the subsidies on user i become more valuable. It is the same idea for the users who reach user i with 6 steps, 8 steps, 10 steps...... Therefore, the monopolist uses subsidizing plans which are dependent on Katz-Bonacich centrality of degree 2.

If the monopolist can price one activity only, her/his pricing plan extracts the value of users' browsing activity boosted by their friends' willingness to create ($\mathbf{p}^* = (\mathbf{G} + \gamma \mathbf{I})\, \boldsymbol{\alpha} / (2\eta\beta)$) at most. If the monopolist can price and subsidize at the same time, she/he can stimulate users' creating activity with global network effects with neutralizing their creating activity boosted by their own willingness to create (the monopolist's subsidizing plan $\mathbf{q}^* = \left(\mathbf{I} - \kappa / (4\eta\beta^2)\, \mathbf{G}^2\right)^{-1} \boldsymbol{\alpha} / (2\kappa) - \boldsymbol{\alpha}/\kappa$ and users' creating time $\mathbf{t}^* = \left(\mathbf{I} - \kappa / (4\eta\beta^2)\, \mathbf{G}^2\right)^{-1} \boldsymbol{\alpha} / (2\beta)$). By using the subsidizing plan with global network effects, the monopolist can gain more profit with pricing users' browsing activity, which is boosted by their friends' creating activity with global network effects (users' browsing time $\mathbf{v}^* = \mathbf{G} \left(\mathbf{I} - \kappa / (4\eta\beta^2)\, \mathbf{G}^2\right)^{-1} \boldsymbol{\alpha} / (4\eta\beta)$ and the monopolist's pricing plan $\mathbf{p}^* = \mathbf{G} \left(\mathbf{I} - \kappa / (4\eta\beta^2)\, \mathbf{G}^2\right)^{-1} \boldsymbol{\alpha} / (4\beta^2)$).

The asymmetric network effects in each profit function are essential to derive network-dependent pricing plan both in Chen et al. (2015)'s case and in our cases. In Chen et al. (2018a)'s case, the asymmetry is captured by two profit maximization goals due to duopoly competition; in our case, the asymmetry is captured by the asymmetric interpersonal across dual activities. In Chen et al. (2018a)'s duopoly setting with symmetrical interpersonal effects within each activity($t_i \sum_{j\in N(i)} t_j$ and $v_i \sum_{j\in N(i)} v_j$), firm A gains profit from pricing activity A only. Thus, the network effects of activity B on activity A in the F.O.C.s of firm A's profit function is asymmetric, and the same for firm B. Therefore, the network effect can not be cancelled out and results in that the pricing plan is proportional to Katz-Bonacich centrality. However, these network effects vanish in pricing in monopolist setting. The two activities and the two pricing plans are symmetric in the F.O.C.s of monopolist's profit function; therefore, the network effects are all cancelled out by monopolist's balancing the profits of two activities. In our cases, the asymmetry of network effects preserves in the monopolist's profit function, which enable us to derive the network-dependent subsidizing and pricing plan in monopolist setting.

1.4.2 Katz-Bonacich Centrality of degree 2 on simple networks

For the simplicity of discussion, we use KB to denote Katz-Bonacich centrality and KB-2 to denote Katz-Bonacich centrality of degree 2 in this section. We focus on the network structure's effects on KB and KB-2, such that we assume that $\boldsymbol{\alpha} = \mathbf{1}^T$ and $\theta > 0$ is small enough, such that for a graph $\mathbf{G}$, the KB of $\mathbf{G}$ is denoted as $\mathbf{b}(\mathbf{G}, \theta) = (\mathbf{I} - \theta\mathbf{G})^{-1}\mathbf{1}$ and KB-2 of $\mathbf{G}$ is denoted as $\mathbf{b}(\mathbf{G}^2, \theta) = (\mathbf{I} - \theta\mathbf{G}^2)^{-1}\mathbf{1}$.

The KB ranking of nodes in a network is not necessarily preserved in their KB-2 ranking. The rankings of KB and KB-2 could be different even in simple networks, such as stars. Figure 1.1 shows the process we analyze KB-2 of a 4-node star: we derive a graph, whose KB ranking is equivalent to KB-2's ranking of the 4-node star. Therefore, for a graph $\mathbf{G}$, we derive a graph $\mathbf{G}'$ such that $\mathbf{b}(\mathbf{G}^2, \theta)\mathbf{1} = \mathbf{b}(\mathbf{G}', \theta)\mathbf{1}$. The process of deriving the equivalent graph and explanation of that are as follows.

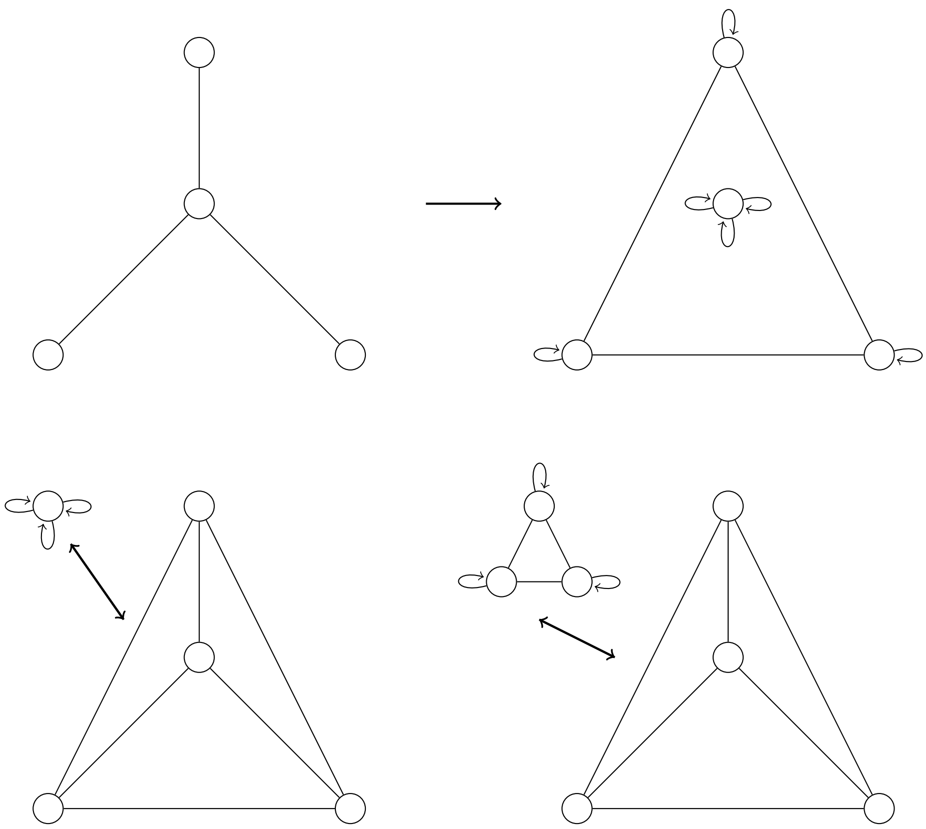

Figure 1.1 The KB Equivalent Form of 4-node star's KB-2.

The network on the top left is a star network with 4 nodes in Figure 1.1. *We link two nodes if they can reach each other in a path with two steps*. The periphery nodes in the star can reach other periphery nodes with two steps, and they can also reach themselves with two steps by traveling to the center node and then come back; for the node in the center, it can reach itself though three different paths with two steps by traveling to the three periphery nodes and come back. Therefore, we have the network on the top right, whose KB is equivalent to the KB-2 of the top left 4-node star network. footnoteThis process of simplification works well for simple graphs. For the analysis of more complex graphs, the process does not necessarily help to reduce the calculation of finding the KB-2, e.g., the nested networks in König et al. (2014).

The two graphs at the bottom of Figure 1.1 show the equivalent forms of each disconnected component at the top right graph. For the graph at the bottom left, a node with three loops' KB equals to the KB of a node in 4-node complete graph: for each node in the complete network, there are three edges to reach the next node; and it is the same for a single node with 3 loops, since it has three paths to go back to itself. For the graph at the bottom right, any node's KB in a complete 3-node network with each node having a loop equals to KB of any node in a 4-node complete network. This is due to that: for a complete 3-node network with each node having a loop, consider replacing all the loops with a hypothetical node which links to every original node. Therefore, for each node, they can still reach themselves (hypothetical one) and go to any node including themselves.

Therefore, every node shares the same KB-2 in the 4-star node network, due to the symmetry of two complete 4-node networks.

Figure 1.2 shows an example where the ranking of KB preserves in KB-2. In Figure 1.2, the network on the left is a 4-node line. The two nodes in the middle have higher KB than the other two periphery nodes have. For the KB-2, please see the following analysis. The network in the middle of the figure results from we linking the nodes which reach each other by two steps. The network on the right is the equivalent network sharing the same KB with the network in the middle. The nodes in the middle of line correspond to the nodes at the bottom left of each component in the right graph; and the periphery nodes in the line correspond to the nodes at top right of each component in the right graph. The top left node in a component is the hypothetical node which connects to both nodes. The node in the middle of the 4-node line has two loops, therefore, it has an additional hypothetical node which is directed to the periphery node. Therefore, we have that the nodes in the middle of the 4-node line

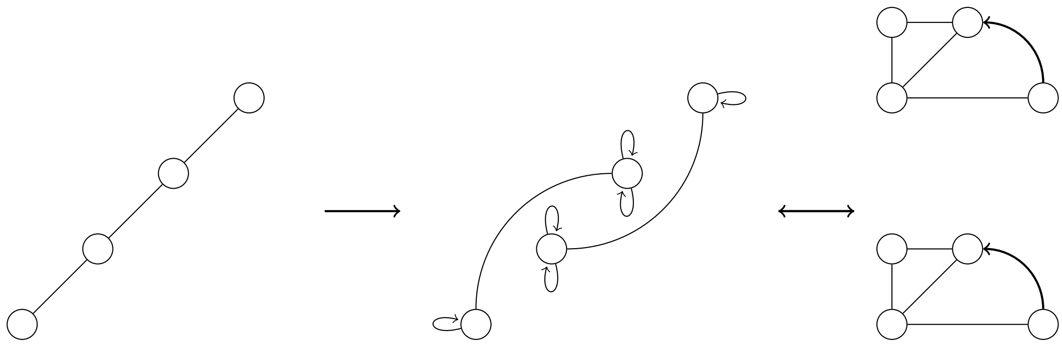

Figure 1.2 The KB Equivalent Form of 4-node line's KB-2.

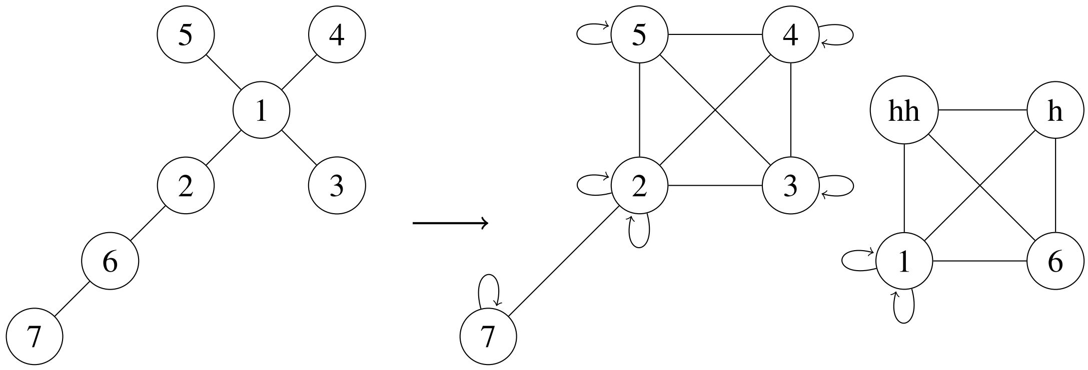

Figure 1.3 The KB Equivalent Form of Example 1.1's KB-2.

have higher KB-2 than the nodes at the periphery, such that the KB ranking preserves for KB-2 in the 4-node line.

Now, we use the following Example 1.1 to show that for some nodes in a network, their KB ranking and KB-2 ranking could be reversed.

Example 1.1 Consider the network in Figure 1.3, where user 1 and user 2 are the most central two users in the network, and user 1's KB is larger than user 2's. For KB-2, please see the graph on the right whose KB is equivalent to KB-2 of the network on the left. In the graph on the right, node 2, which has two loops to itself, is connected to 4 nodes with one step, all the nodes node 2 connected to have one loop and node 3, 4, and 5 have a link to each other; however, node 1, which also has two loops to itself, are connected to 3 nodes, which have links to each other.

It is strict to have the following statements for the rankings of KB-2 of Example 1.1: (i) node 2 is with the highest ranking; (ii) node 6 is ranked below node 3, 4 and 5.

Table 1.1 shows how the rankings reversed with numbers with $\theta = 1/6$.

The first column of Table 1.1 shows each user's index corresponding to her/his

i	$p'_i = \frac{1}{3}\lvert N_i\rvert$	$p_i = \frac{1}{6}\left(\mathbf{Gb}\left(\mathbf{G}^2, \frac{1}{6}\right)\right)_i$	$q_i = \frac{1}{2}\left(\mathbf{b}\left(\mathbf{G}^2, \frac{1}{6}\right)\right)_i$	$\left(\mathbf{b}\left(\mathbf{G}, \frac{1}{6}\right)\right)_i$
1	1.33	2.57	1.14	1.92
2	0.67	1.14	1.36	1.56
3,4,5	0.33	0.71	0.79	1.32
6	0.67	1.14	0.29	1.47
7	0.33	0.43	0.07	1.24

Table 1.1 Pricing only plan $\mathbf{p}'$ and dual-activity pricing and subsidizing plan $\mathbf{p}, \mathbf{q}$

representing node's index in Figure 1.3. Given $\beta = 1$, $\gamma = 0$, $\eta = 1.5$ and $\boldsymbol{\alpha} = \mathbf{1}^T$, Table 1.1's second column shows the monopolist's pricing only strategy $\mathbf{p}'$, which is proportional to users' degree; the third and fourth columns of Table 1.1 show her/his dual-activity pricing and subsidizing strategy $\mathbf{p}$ and $\mathbf{q}$, in which $\mathbf{p}$ is proportional to the sum of users' friends' Katz-Bonacich centralities of degree 2, and $\mathbf{q}$ is linear in users' Katz-Bonacich centralities of degree 2; the last column shows the KB. See also Section 1.7.1.

For $\mathbf{p}$ and $\mathbf{q}$, user 2 has the highest subsidizing plan and the second highest pricing plan among all users. However, user 1 has the second highest subsidizing plan, but the highest pricing plan. Moreover, user $3, 4, 5$ have higher subsidizing plan, but lower pricing plan than user 6 does.

The monopolist has higher willingness to subsidize the user i who reaches more users with 2 steps, such that with boosting their creating (user i and the users with 2 steps from her/him), the monopolist can price more on user i's friends.

Compare pricing only plan $\mathbf{p}'$ and $\mathbf{p}$, which is the pricing plan in dual activity pricing and subsidizing plan. For every user i in the network, $p_i > p'_i$. For other centralities measures, the rankings of users' degrees, KB and the sum of users' neighbors' KB-2 are the same.

1.5 Comparative Static Studies

Social media owners recommend new friends to build denser social networks, and both the owners and their users exert efforts to improve the ways of interacting on the social media, which increases intrapersonal cross-activity externalities. We examine the two effects in comparative static studies. In details, Section 1.5.1 studies the effects of network structures on the monopolist's profit and users' time spending. In Section 1.5.2, we study the effect of intrapersonal cross-activity externality on the monopolist's profit and users' dual activities.

The comparative static studies apply to the cases where the monopolist prices users' browsing behavior such that $\kappa = 0$.

1.5.1 Network Structures

Adding one link between user i and user j has two effects: (i) the change of adjacency matrix $\mathbf{G}$ has an impact on users' behavior and the monopolist's pricing plan; (ii) α_i and α_j might be affected, since α_i captures the exogenous network-dependent factors as mentioned in Remark (ii) at the end of Section 1.2. We study each effect separately, then we consider the aggregate effects. The results show the monopolist's profit increases if $\gamma \geq 0$, and for $\gamma < 0$, more information are needed to justify the effects of structural changes of networks on the monopolist's profit. Moreover, we study the profitability of networks with the same number of nodes and links but different degree distributions.

The comparative studies on network structures changes are discrete mathematical analysis. If $\gamma \geq 0$, the analysis is straightforward. However, if $\gamma < 0$, the analysis becomes complex. The mathematical technique we used in this section is introduced by Bramoullé et al. (2014) in which they study the effect of adding links on total activity level in games with local network substitutes. We extend their technique of potential functions to analyze the effect of network structure on the monopolist's profit.

Adding Links

Consider network $\mathbf{G}$ and its supergraph $\mathbf{G}'$, for which there exists exactly one pair of users i and j such that i and j are not connected in $\mathbf{G}$ $l_{ij} \notin \mathbf{G}$, but they are connected in $\mathbf{G}'$such that $\mathbf{G}' = \mathbf{G} + l_{ij}$. Consider that both equilibria on the two networks $\mathbf{G}$ and $\mathbf{G}'$ are unique and interior solutions.

The effects of adding a new link to user i and user j are different as γ varies. If $\gamma = 0$, a new link added between users i and j increases their browsing activities and the pricing plans on them. Thus, the monopolist's profit increases. If $\gamma > 0$, a new link added between users i and j boosts all users' time spending on their dual activities; also, the pricing plans on users i and j increase accordingly. Thus, the monopolist's profit increases. If $\gamma < 0$, user i and user j's browsing time increases; and their creating time decreases, which affects their friends' dual activities and so on so forth. Therefore, there is ambiguity of how adding a link will affect the time spending and the monopolist's profit on the network.

We use potential function to solve the question. Consider a $|N| \times |N|$ diagonal matrix $\mathbf{C}$ such that $C_{ii} \in \mathbf{C}$ and $C_{ii} = (\alpha_i)^{-1}$ and a new potential function $\psi(\mathbf{x}, \mathbf{G}) = \mathbf{x}^T\mathbf{1} - \mathbf{x}^T\mathbf{C}(\mathbf{I} - \gamma\delta\mathbf{G})\mathbf{x}/2$. The Hessian matrix of ψ denoted $\mathbf{H}(\psi)$ is:

$$\mathbf{H}(\psi) = \frac{1}{2}\mathbf{C}(\mathbf{I} - \gamma\delta\mathbf{G}) + \frac{1}{2}(\mathbf{I} - \gamma\delta\mathbf{G})\mathbf{C}.$$

If matrix $(\mathbf{I} - \gamma\delta\mathbf{G})$ is positive definite, then matrices $\mathbf{H}(\psi(\mathbf{x}, \mathbf{G}))$ and $\mathbf{H}(\psi(\mathbf{x}, \mathbf{G}'))$ are positive definite.[16] If the maximizers of $\psi(\mathbf{x}, \mathbf{G})$ and $\psi(\mathbf{x}, \mathbf{G}')$ are interior, then $\psi(\mathbf{x}^*, \mathbf{G}) = \boldsymbol{\alpha}^T(\mathbf{I} + \gamma\delta\mathbf{G})^{-1}\mathbf{1}/2$, which is the part of $\sum t_i^*$ varying from different $\mathbf{G}$, we need to analyze the maximum $\psi(\mathbf{x}, \mathbf{G})$ and $\psi(\mathbf{x}'^*, \mathbf{G}')$ only. That is to say, if $\psi(\mathbf{x}^*, \mathbf{G}) > \psi(\mathbf{x}'^*, \mathbf{G}')$, then $\sum t_i^* > \sum t_i'^*$. Graph $\mathbf{G}$ is a subgraph of $\mathbf{G}'$, such that $\psi(\mathbf{x}'^*, \mathbf{G}') - \psi(\mathbf{x}'^*, \mathbf{G}) = \gamma\delta x_i'^* x_j'^* < 0$. Thus, we have $\psi(\mathbf{x}'^*, \mathbf{G}') < \psi(\mathbf{x}'^*, \mathbf{G}) < \psi(\mathbf{x}^*, \mathbf{G})$. Therefore, $\sum x_i^* > \sum x_i'^*$ and $\sum t_i^* > \sum t_i'^*$. For the sum of browsing time, we have:

$$\begin{aligned}\mathbf{v}^T\mathbf{1} &= \frac{\delta}{2}\boldsymbol{\alpha}^T(\mathbf{G} + \gamma\mathbf{I})(\mathbf{I} - \gamma\delta\mathbf{G})^{-1}\mathbf{1} \\ &= \frac{1}{2\gamma}\boldsymbol{\alpha}^T\mathbf{1} - \frac{1}{\gamma}\beta^2\delta\mathbf{x}^{*T}\mathbf{1}.\end{aligned}$$

The value of second term correspond negatively to ψ. Thus, $\sum v_i'^* > \sum v_i^*$.

For the analysis on monopolist's profit in the cases where $\gamma < 0$, please see the proof of Proposition 1.4 in Section 1.7.2. The results show that, if $\gamma\delta > -2\beta$, the overall effects on the monopolist's profit is positive. Therefore:

Proposition 1.4. *Consider a network* $\mathbf{G}$ *and a* $\mathbf{G}'$ *such that* $\mathbf{G}' = \mathbf{G} + l_{ij}$ *and* $l_{ij} \notin \mathbf{G}$. *Given a* γ *and an vector* $\boldsymbol{\alpha}$, $\mathbf{t}^*$ *and* $\mathbf{v}^*$ *are assigned by the equilibrium strategy of*

[16] We prove that the product of two positive definite symmetric matrices is also positive definite. Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric and positive definite. Suppose λ is the eigenvalue of $\mathbf{AB}$ and its corresponding eigenvector is $\mathbf{x}$, then $\mathbf{ABx} = \lambda\mathbf{x}$. Thus, $\mathbf{BABx} = \lambda\mathbf{Bx} \Rightarrow \mathbf{x}^T\mathbf{BABx} = \lambda\mathbf{x}^T\mathbf{Bx}$. Consider $\mathbf{y} = \mathbf{Bx}$, then we have $\mathbf{y}^T\mathbf{Ay} = \lambda\mathbf{x}^T\mathbf{Bx}$. Since $\mathbf{A}$ and $\mathbf{B}$ are both positive definite matrices, then $\mathbf{y}^T\mathbf{Ay} > 0$ and $\mathbf{x}^T\mathbf{Bx} > 0$. Thus, we have for any eigenvalue $\lambda = (\mathbf{y}^T\mathbf{Ay})/(\mathbf{x}^T\mathbf{Bx}) > 0$ and $\mathbf{AB}$ is positive definite. Thus, $\mathbf{BA}$ is also positive definite. As $\mathbf{C}$ and $(\mathbf{I} - \gamma\delta\mathbf{G})$ are positive definite and the sum of two positive definite matrices are also positive definite, we have that matrices $\mathbf{H}(\psi(\mathbf{x}, \mathbf{G}))$ and $\mathbf{H}(\psi(\mathbf{x}, \mathbf{G}'))$ are positive definite.

users with network $\mathbf{G}$*, and the monopolist's profit with pricing plan* $\mathbf{p}^*$ *is* π^**; and* $\mathbf{t}'^*$ *and* $\mathbf{v}'^*$ *are assigned by the equilibrium strategy of users with network* $\mathbf{G}'$*, and the monopolist's profit with pricing plan* $\mathbf{p}'^*$ *is* π'^**. If matrices* $\mathbf{I} - \gamma\delta\mathbf{G}$ *and* $\mathbf{I} - \gamma\delta\mathbf{G}'$ *are positive definite and both* $(\mathbf{p}^*, \mathbf{t}^*, \mathbf{v}^*)$ *and* $(\mathbf{p}'^*, \mathbf{t}'^*, \mathbf{v}'^*)$ *are interior solutions, then:*

(i) Assume $\gamma = 0$*. For the users' dual activities:* $\mathbf{t}^* = \mathbf{t}'^*$*,* $v^*_{i,j} < v'^*_{i,j}$ *and for all* k *in* $N \setminus \{i, j\}$*,* $v^*_k = v'^*_k$*. The monopolist's profits satisfy* $\pi^* < \pi'^*$*.*

(ii) Assume $\gamma > 0$*. For all* i*,* $t^*_i < t'^*_i$ *and* $v^*_i < v'^*_i$*. The monopolist's profits satisfy* $\pi^* < \pi'^*$*.*

(iii) Assume $\gamma < 0$*. For the users' dual activities,* $\sum_{j\in N} t^*_j > \sum_{j\in N} t'^*_j$ *and* $\sum v^*_i < \sum v'^*_i$*. If* $\gamma\delta/(2\beta) > -1$*, then the monopolist's profits satisfy* $\pi^* < \pi'^*$*.*

Proof. See Section 1.7.2.

The condition $\gamma\delta/(2\beta) > -1$ ensures the increase of monopolist's profit with denser networks in the case where $\gamma < 0$.[17] The intuition is as follows. The increase of monopolist' profit is ensured if the increase of users i and j's browsing time does not discourage their creating time spending so much, by which it demands that the relative magnitude of intrapersonal effect to be small enough.[18]

In Bramoullé et al. (2014)'s analysis on local network substitutes, the overall activity level increases. In our model, if the dual activities of each user are substitutes, the effect of adding a new link to the existing network increases the overall activity level of browsing behavior, but it decreases that of creating behavior. The intuition of that is: with asymmetric cross-activity externality, browsing behavior which is directly affected by the network structure, increases with network density; however, the other behavior (creating behavior), which is indirectly affected by the network structure, decreases with network density due to the intrapersonal substitutes between the dual activities.

[17]Condition $\gamma\delta/(2\beta) > -1$ is satisfied if $\beta > 1/2$, since $\gamma\delta > -1$ as the smallest eigenvalue of any undirected graph is smaller than or equal to -1. If $\beta \leq 1/2$, we solve $\gamma\delta/(2\beta) > -1$ and have $(4\beta)^{-1} - \sqrt{(16\beta^2)^{-1} + \beta^2} < \gamma < 0$.

[18]If $\beta > 1/2$, the relative magnitude can be ensured by the condition for the uniqueness of equilibria, which is essentially a index of relative magnitude of intrapersonal effects with the index of rate of diminishing return.

We now consider the second effect of denser network. Assume that original willingness to create of users i and j increases with a new link added between them. We start with the case where $\exists \epsilon_i > 0$ such that $\alpha_i' = \alpha_i + \epsilon_i$ and $\alpha_j' = \alpha_j$, then it could be generalized to the case that $\exists \epsilon_i, \epsilon_j > 0$ such that $\alpha_i' = \alpha_i + \epsilon_i$ and $\alpha_j' = \alpha_j + \epsilon_j$. If the intrapersonal effects are neutralized $\gamma = 0$, the increasing of one user i's willingness to create leads to the monopolist raising the pricing plan on the user's friends; and it also lead s to the increase of the user's friends' browsing time spending. If the intrapersonal effects are complementary $\gamma > 0$, the positive intrapersonal cross-activity externality drives all users' both time spending to a higher level.

If the intrapersonal effects are substitutes $\gamma < 0$, the negative intrapersonal externality drives lower browsing time spending of user i; however, it drives browsing time of her/his friends to a higher level. Therefore, the overall effect remains unclear. We invoke $\psi(\mathbf{x}, \mathbf{G})$ to $\psi(\mathbf{x}, \mathbf{C})$. We use $|N| \times |N|$ diagonal matrix $\mathbf{C}'$ such that $C_{ii}' \in \mathbf{C}'$ and $C_{ii}' = \alpha_i'^{-1}$, such that the unique maximal interior solution of $\psi(\mathbf{x}, \mathbf{C}')$ correspond to the total creating time of users on the network. Using the same technique of proof in Proposition 1.4 to compare the maximums of $\psi(\mathbf{x}, \mathbf{C})$ and $\psi(\mathbf{x}', \mathbf{C}')$, we find that the overall effect on creating is positive. The increase of willingness to create boosts the overall creating activity level on the network, even though creating activity is locally substitute in SPE. However, the overall effects on users' total browsing time and the monopolist's profit remain unclear.

Proposition 1.5. *Given a γ such that $\lambda_{\max}(\gamma\mathbf{G}) < 1/\delta$, consider an $\boldsymbol{\alpha}$, with which $\mathbf{t}^*$ and $\mathbf{v}^*$ are assigned by the equilibrium strategy for users, and the monopolist's profit with pricing plan $\mathbf{p}^*$ is π^*; and consider an $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\epsilon}$ such that $\boldsymbol{\epsilon} = (0, .., 0, \epsilon_i, 0, .., 0)$ and $\epsilon_i > 0$, then $\mathbf{t}'^*$ and $\mathbf{v}'^*$ are assigned by users' equilibrium strategy with $\boldsymbol{\alpha}'$, and the monopolist's profit with pricing plan $\mathbf{p}'^*$ is π'^*. If matrices $(\mathbf{I} - \gamma\delta\mathbf{G})$ is positive definite and both $(\mathbf{p}^*, \mathbf{t}^*, \mathbf{v}^*)$ and $(\mathbf{p}'^*, \mathbf{t}'^*, \mathbf{v}'^*)$ are interior solutions, then:*

(i) Assume $\gamma = 0$. For the users' dual activities, $t_i^* < t_i'^*$, for all $j \neq i$ in N, $t_j^* = t_j'^*$, for all k in N but not in $N(i)$, $v_k^* = v_k'^*$, and for all l in $N(i)$, $v_l^* < v_l'^*$. The monopolist's profits satisfy $\pi^* < \pi'^*$.

(ii) Assume $\gamma > 0$. For the users' dual activities for all j in N, $t_j^* < t_j'^*$ and $v_j^* < v_j'^*$. The monopolist's profit satisfy $\pi^* < \pi'^*$.

(iii) Assume $\gamma < 0$*, then* $\sum_{j\in N} t_j^* < \sum_{j\in N} t_j'^*$.

Proof. See Section 1.7.2.

The intuition for the increase of users' total time spending on creating on the network is: the total creating time is directly driven by the willingness to create. Compared to the effect of network density on time spending, the ambiguity on how total time of users spending on browsing behavior change is due to that creating behavior changing has larger effects than that of network density in the first-degree influence of network (nodes that given a user can reach by one step),[19] which is the direct and major effects. A link added to a network affect the two newly connected users in the first-degree influence of the network; but the increase of a user's creating time affects the user and all of her/his friends in the first-degree influence of the network. Therefore, the browsing time could increase or decrease depending on how many friends the user has.

We further study the cases where $\gamma < 0$, and we find a condition under which the profit increases. However, it is not the full characterization of nonnegative profit changes.

Consider a new potential function:

$$\phi\left(\mathbf{x}, \mathbf{D}\right) = \mathbf{x}^T\mathbf{1} - \frac{1}{2}\mathbf{x}^T\mathbf{D}\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{x},$$

where $\mathbf{D}$ is a diagonal matrix and $D_{ii} = \left(\sum_{j\in N(i)} \alpha_j + \gamma\alpha_i\right)^{-1}$. Matrix $\mathbf{D}'$ is the diagonal matrix such that $D'_{ii} = \left(\sum_{j\in N(i)} \alpha'_j + \gamma\alpha'_i\right)^{-1}$. If matrix $(\mathbf{I} - \gamma\delta\mathbf{G})$ is positive definite, the Hessian matrix of ϕ of $\mathbf{D}$ $\mathbf{H}\left(\phi\left(\mathbf{x}, \mathbf{D}\right)\right) = \mathbf{D}\left(\mathbf{I} - \gamma\delta\mathbf{G}\right) + \left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{D}$ and that of $\mathbf{D}'$ $\mathbf{H}\left(\phi\left(\mathbf{x}, \mathbf{D}'\right)\right) = \mathbf{D}'\left(\mathbf{I} - \gamma\delta\mathbf{G}\right) + \left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{D}'$ are positive definite, as shown in Footnote 16.

If the maximizers of $\phi\left(\mathbf{x}^*, \mathbf{D}\right)$ and $\phi\left(\mathbf{x}'^*, \mathbf{D}'\right)$ are interior, then $\phi\left(\mathbf{x}^*, \mathbf{D}\right)$ and $\phi\left(\mathbf{x}'^*, \mathbf{D}'\right)$ correspond to $\sum v_i^*$ and $\sum v_i'^*$ respectively.

With condition that for all j and k such that $\sum_{j\in N(k)} \alpha_j + \gamma\alpha_k \leq \sum_{j\in N(k)} \alpha'_j + \gamma\alpha'_k$, there is no decrease of pricing plan. The results show that the overall browsing time increases, and the monopolist's profit increases. The following steps of proof are

[19] Note that we use first-degree influence of network to restrict the scope of analysis on a given user and her/his friends. The aim is to distinguish it with cumulative network effects on a user.

almost the same as those of Proposition 1.5. For the details, please see the appendix.

Corollary 1.5.1. *Given a $\gamma < 0$, consider an $\boldsymbol{\alpha}$, with which $\mathbf{t}^*$ and $\mathbf{v}^*$ are assigned by the equilibria strategy of users, and the monopolist's profit is $\pi^*$with pricing plan $\mathbf{p}^*$; and consider an $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\epsilon}$ such that $\boldsymbol{\epsilon} = (0, .., 0, \epsilon_i, 0, .., 0, \epsilon_j, 0, ...0)$ and $\epsilon_i, \epsilon_j > 0$, then $\mathbf{t}'^*$ and $\mathbf{v}'^*$ are assigned by users' equilibrium strategy with $\boldsymbol{\alpha}'$, and the monopolist's profit is π'^* with pricing plan $\mathbf{p}'^*$. If matrix $\mathbf{I}-\gamma\delta\mathbf{G}$ is positive definite, for both i and j such that $\sum_{k\in N(i)} \alpha_k+\gamma\alpha_i \leq \sum_{k\in N(i)} \alpha'_k+\gamma\alpha'_i$ and $\sum_{k\in N(j)} \alpha_k+\gamma\alpha_j \leq \sum_{k\in N(j)} \alpha'_k + \gamma\alpha'_j$, and $(\mathbf{p}^*, \mathbf{t}^*, \mathbf{v}^*)$ and $(\mathbf{p}'^*, \mathbf{t}'^*, \mathbf{v}'^*)$ are interior, then $\sum_{k\in N} v^*_k < \sum_{k\in N} v'^*_k$ and $\pi^* < \pi'^*$.*

Proof. See Section 1.7.2.

The results is supported by the following intuition. Even though users i's and j's browsing time v_i and v_j decrease due to the boosting of their creating time, the nondecreasing pricing plans for i and j and strictly increasing pricing plans on i's and j's friends ensure the overall profit to increase.

To conclude the effect of adding a link to the existing networks, the monopolist's profit increases if $\gamma \geq 0$; however, if $\gamma < 0$ and the condition in Corollary 1.5.1 holds, then the monopolist's profit increases. For other cases, more information are needed to justify the profit changes.

Degree distribution of Networks

In this section, we consider an abstract comparative static study. Users and links are the sources of the monopolist's profit. If we fix the users and the total number of links in the social networks, there are $\binom{|N|\left(|N|-1\right)/2}{m}$ possibilities of network structures in which the number of links is m. Even we just consider the connected networks, the number of connected networks is still huge. A question is raised: which network can the monopolist gain more profit from? For the networks with the same numbers of nodes and edges, we find that: (i) if the intrapersonal effects are not neutralized $\gamma \neq 0$, the monopolist gains more profit from a network with more diverse degree distribution; (ii) if the intrapersonal effects are neutralized $\gamma = 0$, the monopolist gains the same profit from any network.

Let P and P' denote the degree distributions of $\mathbf{G}$ and $\mathbf{G}'$. Let $|V(\mathbf{G})|$ and $|E(\mathbf{G})|$ denote the number of users and the number of links in the network $\mathbf{G}$ respectively. We have the following proposition:

Proposition 1.6. *Consider two graphs $\mathbf{G}$ and $\mathbf{G}'$ such that $|V(\mathbf{G})| = |V(\mathbf{G}')|$, $|E(\mathbf{G})| = |E(\mathbf{G}')|$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}' = \alpha(1, 1, ..., 1)^T$ such that $\alpha > 0$ and $|\gamma\delta|$ is small enough, and the SPEs in both graphs have unique and interior solutions. If P''s variance of degree distribution is higher than that of P, then in SPEs, the monopolist gains more profit from $\mathbf{G}'$ than from $\mathbf{G}$.*

Proof. See Section 1.7.2.

Given $\mathbf{G}$ and $\mathbf{G}'$ having the same number of nodes and edges and a small enough $|\gamma\delta|$, the profits gained from network $\mathbf{G}$ and $\mathbf{G}'$ is determined by term $\mathbf{1}^T\mathbf{G}^2\mathbf{1}$ and $\mathbf{1}^T\mathbf{G}'^2\mathbf{1}$ respectively, whatever γ is. The variance of the degree distribution is equivalent to the overall network connectivities with 2 steps, such that the profit of the network with higher variance $\mathbf{G}'$ is higher than that of network $\mathbf{G}$ by $\mathbf{1}^T\mathbf{G}'^2\mathbf{1} > \mathbf{1}^T\mathbf{G}^2\mathbf{1}$.

We give an example of why the variance of degree distribution affecting the profitability of a network. Consider a 5-node star which is the network with most diverse degree distribution among all networks with 5 nodes and 4 edges. Periphery nodes can reach each other by 2 steps and the center node can research itself 4 times by 2 steps. Any alternating edges (changing either end of a link) will decrease the 2-step-connectivity of the network. Therefore, the monopolist prefers networks with more diverse degree distributions to other networks sharing the same number of nodes and links.

We have one remark: the variance of degree distribution corresponds to the sum of Katz-Bonacich centrality of degree 2 of users in network $\mathbf{G}$ with a small enough $\gamma\delta$. We can see that not only the Katz-Bonacich centrality of degree 2 captures the combining effects of subsidizing plan on boosting creating activity, but also its sum of a network captures the profitability of the network generated by the second degree influence of the network (the variance of degree distribution).[20]

[20] If $|\gamma\delta|$ is small enough, the cumulative effect by Katz-Bonacich centrality is restrict to a node's second degree influence. The second-degree influence is restrict to a given user and her/his friends' friends.

1.5.2 Intrapersonal Cross-activity Externalities

Our motivation to analyze γ's effect on users and the monopolist is that both the social media owner and users try to improve the ways of interaction online. Intuitively, more ways to express users' feeling and thoughts, more comments users would like to make when they browse. The results show that both users and the monopolist benefit from the increase of intrapersonal externalities.

By analyzing the first derivative of each user's utility function and the monopolist's profit, we have the following proposition:

Proposition 1.7. *If $\delta\lambda_{\max}(\gamma\mathbf{G}) < 1$, the monopolist's profit increases with γ; if (i) $\gamma > 0$ or (ii) $\gamma < 0$ and $\gamma \to 0$, each user's utilities increase with γ.*

Proof. See Section 1.7.2.

1.6 Conclusion

We consider a certain type of interaction on social media. The design and the unique usage of friend-based social media are characterized by the mutual consent of any two users to establish a link and the asymmetrical interpersonal effects cross users' dual activities. Both the pricing browsing plans and the pricing and subsidizing plans are network-dependent. The pricing browsing plan on each user is linear in each user's weighted degree; and the pricing and subsidizing plan on each user is linear in each user's Katz-Bonacich of degree 2.

The games with an asymmetrical interpersonal externality we study here are two cases in between pricing one activity with the local network externality and pricing two activities with symmetric interpersonal effects. Comparing our pricing browsing plan with the network-independent uniform pricing plan for one activity with the local network externality (Bloch and Quérou (2013) and Candogan et al. (2012)), the pricing plan on each user is network-dependent in our cases. The intrapersonal externalities present in a linear form in the pricing browsing plan, but users' behavior are linear in their Katz-Bonacich centralities of a parameter depending on the intrapersonal externality. It coincides with the case of Bloch and Quérou (2013) and Candogan et al. (2012).

For pricing and subsidizing plan, the Katz-Bonacich centralities of degree 2 appear in the cases where intrapersonal effects are cancelled out.

For future studies, two extension cases would be studied. A user's friends on social media could not only contain her/his friends in real life but also contain her/his enemies, therefore, it is interesting to study how users distinguish different the content to friends in different groups. Besides, we could study the case where the subsidized content does not benefit users' friends.

Moreover, the full characterization of the dual activities should be expected. These include: (i) for how the exogenous factors affect the dual activities, it could be the cases that one of the dual activities is affected, both activities are affected, or neither of the activities are affected by exogenous factors; (ii) for the interpersonal externalities, we could consider the cases with symmetrical/asymmetrical interpersonal effects which are within each activity or cross-activity; (iii) there exist intrapersonal effects or not; (iv) for the game participants, how many firms price and subsidize on the network, and more real-life examples are needed for applications.

1.7 Appendix

1.7.1 Centrality

Consider a scalar θ and a $n \times n$ symmetric matrix $\mathbf{A}$ such that $n \in \mathbb{Z}_{++}$ and $\mathbf{I} - \theta\mathbf{A}$ is invertible, then a matrix $\mathbf{M}$ is contracting and defined as:

$$\mathbf{M}\left(\mathbf{A}, \theta\right) = \left(\mathbf{I} - \theta\mathbf{A}\right)^{-1} = \sum_{k=0}^{+\infty} \theta^k \mathbf{A}^k = \mathbf{I} + \theta\mathbf{A} + \theta^2\mathbf{A}^2 + ... + \theta^n\mathbf{A}^n + ...$$

If the symmetric matrix $\mathbf{A}$ is a network's adjacency matrix $\mathbf{G}$, a vector of *Katz-Bonacich centralities* of all nodes in network $\mathbf{G}$ with local externality parameter θ is defined as $\mathbf{b}\left(\mathbf{G}, \theta\right) = \left(\mathbf{I} - \theta\mathbf{G}\right)^{-1}\mathbf{1}$; and a vector of weighted Katz-Bonacich centralities of all nodes in network $\mathbf{G}$ is defined as $\mathbf{b}_{\boldsymbol{\alpha}}\left(\mathbf{G}, \theta\right) = \left(\mathbf{I} - \theta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}$, in which $\forall \alpha_i \in \boldsymbol{\alpha}$, $\alpha_i > 0$ and $\exists i \neq j \in N$, $\alpha_i \neq \alpha_j$.

If $\mathbf{A}$ is $\mathbf{G}^2$, a vector of *Katz-Bonacich centralities of degree* 2 for all nodes in network $\mathbf{G}$ with local externality parameter θ is defined as $\mathbf{b}\left(\mathbf{G}^2, \theta\right) = \left(\mathbf{I} - \theta\mathbf{G}^2\right)^{-1}\mathbf{1}$; and a vector of weighted Katz-Bonacich centralities of degree 2 in network $\mathbf{G}$ is defined as $\mathbf{b}_{\boldsymbol{\alpha}}\left(\mathbf{G}^2, \theta\right) = \left(\mathbf{I} - \theta\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha}$, in which $\forall \alpha_i \in \boldsymbol{\alpha}$, $\alpha_i > 0$ and $\exists i \neq j \in N$, $\alpha_i \neq \alpha_j$. For matrix $\mathbf{G}^2$, the entry in row i and column j is the number of paths of length 2 between user i and user j in network $\mathbf{G}$.

If $\mathbf{A}$ is the $(\upsilon\mathbf{I} + \mathbf{G})^2$ s.t. $\upsilon \in \mathbb{R}$, a vector of *quasi-Katz-Bonacich centralities of degree* 2 of all nodes in network $\mathbf{G}$ with local externality parameter θ is $\mathbf{b}\left((\upsilon\mathbf{I} + \mathbf{G})^2, \theta\right) = \left[\mathbf{I} - \theta\left(\upsilon\mathbf{I} + \mathbf{G}\right)^2\right]^{-1}\mathbf{1}$; and a vector of weighted Katz-Bonacich centralities of degree 2 of all nodes on network $\mathbf{G}$ is defined as $\mathbf{b}_{\boldsymbol{\alpha}}\left((\upsilon\mathbf{I} + \mathbf{G})^2, \theta\right) = \left[\mathbf{I} - \theta\left(\upsilon\mathbf{I} + \mathbf{G}\right)^2\right]^{-1}\boldsymbol{\alpha}$, in which $\forall \alpha_i \in \boldsymbol{\alpha}$, $\alpha_i > 0$ and $\exists i \neq j \in N$, $\alpha_i \neq \alpha_j$.

1.7.2 Proofs

PROOF OF PROPOSITION 1.1. We firstly check that the monopolist has no incentive to choose a pricing plan $\mathbf{p}$ such that there exists an i for which $p_i \geq \sum_{j\in N(i)} t_j/\eta$ and $v_i = 0$. Suppose the monopolist chooses $\mathbf{p}^*$ such that there exist an i for which $p_i^* = \sum_{j\in N(i)} t_j/\eta$ and $v_i^* = 0$; and $\mathbf{p}^*$ maximizes the monopolist's profit denoted as π^*. The profit gained from user i is $\pi_i^* = p_i^* v_i^* = 0$ due to that $v_i = 0$.

Consider a new price vector $\mathbf{p}' = \left(\mathbf{p}^*_{-i}, p'_i\right)$ in which $\exists \epsilon > 0$ such that $p'_i = p_i^* - \epsilon > 0$. In $\mathbf{p}'$, the pricing plan p'_i on i is slightly smaller than the original pricing plan p_i^*; for any other user, the pricing plan is the same as in $\mathbf{p}^*$. Therefore, user i's browsing time $v'_i = \epsilon/\beta > 0$. Then, the profit gained from i with pricing plan p'_i is $\pi'_i = p'_i v'_i = p'_i \epsilon/\beta > 0 = \pi_i^*$. For users except i, the pricing plans for them are the same in $\mathbf{p}'_{-i} = \mathbf{p}^*_{-i}$, and it results in their same time spending on browsing $\mathbf{v}'_{-i} = \mathbf{v}^*_{-i}$. Thus, the profit from users except user i are the same $\mathbf{p}'^T_{-i}\mathbf{v}'_{-i} = \mathbf{p}^{*T}_{-i}\mathbf{v}^*_{-i}$. Comparing the profit with $\mathbf{p}^*$ and that with $\mathbf{p}'$, we have $\pi' = \mathbf{p}'^T_{-i}\mathbf{v}'_{-i} + p'_i v'_i > \mathbf{p}^{*T}_{-i}\mathbf{v}^*_{-i} + p_i^* v_i^* = \pi^*$, which contradicts the assumption.

The F. O. Cs of the monopolist's profit function $\pi = \mathbf{p}^T\left(\mathbf{G}\boldsymbol{\alpha} - \eta\beta\mathbf{p}\right)/\beta^2$ w.r.t. $\mathbf{p}$ is:

$$\nabla\pi = \left(\mathbf{G}\boldsymbol{\alpha} - 2\eta\beta\mathbf{p}\right)/\beta^2 = \mathbf{0}$$

Solve $\nabla\pi = \mathbf{0}$. We have $\mathbf{p}^* = \mathbf{G}\boldsymbol{\alpha}/\left(2\eta\beta\right)$ and $\mathbf{v}^* = \mathbf{G}\boldsymbol{\alpha}/\left(2\beta^2\right)$. Q.E.D.

PROOF OF PROPOSITION 1.2. We check the sufficiency of the conditions.

Consider $\gamma > 0$. For any user i, $t_i \geq \alpha_i/\beta$. We firstly check that the monopolist has no incentive to choose a vector $\mathbf{p}$ such that $\exists i$ $p_i \geq \sum_{j\in N(i)} t_j/\eta + \gamma t_i/\eta$ and

$v_i = 0$. We fix $\mathbf{p}_{-i}$. There exists an $\epsilon > 0$ such that $p_i' = \sum_{j \in N(i)} t_j/\eta + \gamma t_i/\eta - \epsilon > 0$ and $v_i = \epsilon/\beta > 0$. Thus, the profit from agent i increases. User i's browsing time v_i increases, which increases her/his creating time t_i such that any $j \in N(i)$, v_j is nondecreasing (v_j could be zero with a large enough p_j). Thus, the profit from any of i's friends is nondecreasing. Therefore, the total profit with pricing plan p_i' is larger than that of p_i. Thus, for all i $v_i > 0$, then $t_i > \alpha_i/\beta$. Therefore, both $\mathbf{t}^*$ and $\mathbf{v}^*$ are unique and interior, and we can just use $\mathbf{v} = \delta\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\mathbf{G}\boldsymbol{\alpha} + \gamma\boldsymbol{\alpha} - \beta\eta\mathbf{p}\right)$ to solve the monopolist' profit maximization problem.

Consider $\gamma < 0$. We prove the sufficiency of the conditions by contradiction. This is to say, we prove that statement "if $\gamma < 0$, and for all i $\left(\left(\mathbf{G} + \gamma\mathbf{I}\right)\boldsymbol{\alpha}\right)_i > 0$ and $-\alpha < \beta^2\delta\left(\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)_i < \alpha_i$, the SPE does not have a unique and interior solution" produces contradiction. The uniqueness has been proved by the strict concavity of potential function φ with $\lambda_{\max}\left(\gamma\mathbf{G}\right) < \delta^{-1}$. We only need to check whether there exist an interior solution. If $\gamma < 0$, condition that for all i $\beta^2\delta\left(\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)_i > -\alpha_i$ could be rewritten as for all i $\left(\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\beta\boldsymbol{\alpha} - \gamma\left(\mathbf{G} + \gamma\mathbf{I}\right)\boldsymbol{\alpha}/2\beta\right)\right)_i > 0$. If $\mathbf{p} = \left(\mathbf{G} + \gamma\mathbf{I}\right)/\left(2\eta\beta\right)$, then $\mathbf{t}$ is interior. So is $\mathbf{p}$ by condition that for all i $\left(\left(\mathbf{G} + \gamma\mathbf{I}\right)\boldsymbol{\alpha}\right)_i > 0$. By the condition that for all i $\beta^2\delta\left(\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)_i < \alpha_i$, for all i, $t_i < \alpha_i/\beta$ and $v_i > 0$. Pricing plan $\mathbf{p} = \left(\mathbf{G} + \gamma\mathbf{I}\right)/\left(2\eta\beta\right)$ satisfies π's F.O.C.s conditioning on an interior solution of browsing time as $\mathbf{v} = \delta\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\mathbf{G}\boldsymbol{\alpha} + \gamma\boldsymbol{\alpha} - \eta\beta\mathbf{p}\right)$. Thus, we have interior solution under the given conditions, which contradicts the statement.

The more intuitive explanation is in the main body of the paper. Q.E.D.

PROOF OF PROPOSITION 1.3. We firstly prove the following lemma. Then, we show the steps of solving the equilibrium.

Lemma 1.2. *Matrix* $\mathbf{H}\left(\pi\right) = \beta^{-2}\begin{bmatrix} -2\eta\beta\mathbf{I} & \kappa\mathbf{G} \\ \kappa\mathbf{G} & -2\kappa\beta\mathbf{I} \end{bmatrix}$ *is negative definite, if and only if* $\lambda_{\max}\left(\mathbf{G}\right) < 2\beta\sqrt{\eta/\kappa}$.

Proof. Matrix $\mathbf{H}\left(\pi\right)$ is a Hemitian matrix, thus it is negative definite if and only if all of its eigenvalue is negative. Therefore, we need to show that the largest eigenvalue of $\mathbf{H}\left(\pi\right)$ is negative.

We need to find the expression of $\lambda_{\max}\left(\mathbf{H}\left(\pi\right)\right)$ in terms of $\lambda_{\max}\left(\mathbf{G}\right)$ to derive the condition.

Let $\lambda(\mathbf{H}(\pi))$ denote an eigenvalue $\mathbf{H}(\pi)$ and $\mathbf{y}^T = (\mathbf{y}_1^T, \mathbf{y}_2^T)$denote $\lambda(\mathbf{H}(\pi))$'s corresponding eigenvector, for which $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^{|N|}$. Thus,

$$\mathbf{H}(\pi)\mathbf{y} = \lambda(\mathbf{H}(\pi))\mathbf{y} \iff \begin{array}{c} -2\eta\beta\mathbf{y}_1 + \kappa\mathbf{G}\mathbf{y}_2 = \beta^2\lambda(\mathbf{H}(\pi))\mathbf{y}_1 \\ \kappa\mathbf{G}\mathbf{y}_1 - 2\kappa\beta\mathbf{y}_2 = \beta^2\lambda(\mathbf{H}(\pi))\mathbf{y}_2 \end{array}.$$

Solve the two systems of linear equations above simultaneously, resulting in:

$$\kappa^2\mathbf{G}^2\mathbf{y}_1 = \left(\beta^2\lambda(\mathbf{H}(\pi)) + 2\eta\beta\right)\left(\beta^2\lambda(\mathbf{H}(\pi)) + 2\kappa\beta\right)\mathbf{y}_1 \tag{1.12}$$

Therefore, vector $\mathbf{y}_1$ is an eigenvector $\mathbf{G}^2$. Vector $\mathbf{y}_1$ is also an eigenvector of $\mathbf{G}$; and eigenvalue of $\mathbf{G}^2$ $\lambda(\mathbf{G}^2)$ is equal to $\lambda^2(\mathbf{G})$. To prove that, suppose $\mathbf{x}$ is an eigenvector of $\mathbf{G}$, then $\mathbf{G}\mathbf{x} = \lambda(\mathbf{G})\mathbf{x}$. Thus, $\mathbf{G}^2\mathbf{x} = \lambda(\mathbf{G})\mathbf{G}\mathbf{x} = \lambda^2(\mathbf{G})\mathbf{x}$. Then, we have $\kappa^2\mathbf{G}^2\mathbf{y}_1 = \kappa^2\lambda^2(\mathbf{G})\mathbf{y}_1$. Combining eq. (1.12) with the previous sentence, we have $(\beta^2\lambda(\mathbf{H}(\pi)) + 2\eta\beta)(\beta^2\lambda(\mathbf{H}(\pi)) + 2\kappa\beta)\mathbf{y}_1 = \kappa^2\lambda^2(\mathbf{G})\mathbf{y}_1$.

Then, we have a quadratic function of $\lambda(\mathbf{H}(\pi))$ as:

$$\left(\beta^2\lambda(\mathbf{H}(\pi)) + 2\eta\beta\right)\left(\beta^2\lambda(\mathbf{H}(\pi)) + 2\kappa\beta\right) = \kappa^2\lambda^2(\mathbf{G}).$$

Solve the quadratic equation of $\lambda(\mathbf{H}(\pi))$ in terms of β, κ, η and $\lambda(\mathbf{G})$, and we have:

$$\lambda(\mathbf{H}(\pi)) = \pm\frac{1}{\beta^2}\sqrt{\beta^2(\eta-\kappa)^2 + \kappa^2\lambda^2(\mathbf{G})} - \frac{\eta+\kappa}{\beta}.$$

We have the largest eigenvalue of $\mathbf{H}(\pi)$ $\lambda_{\max}(\mathbf{H}(\pi))$ with positive sign of square root in the expression above and $\lambda(\mathbf{G}) = \lambda_{\max}(\mathbf{G})$. The expression of $\lambda_{\max}(\mathbf{H}(\pi))$ is as follows:

$$\lambda_{\max}(\mathbf{H}(\pi)) = \frac{1}{\beta^2}\sqrt{\beta^2(\eta-\kappa)^2 + \kappa^2\lambda_{\max}^2(\mathbf{G})} - \frac{\eta+\kappa}{\beta}.$$

Solving $\lambda_{\max}(\mathbf{H}(\pi)) < 0$, we have $\lambda_{\max}(\mathbf{G}) < 2\beta\sqrt{\eta/\kappa}$. Q.E.D.

We solve the equilibrium now. If $\gamma = 0$, user i's utility function with q_i $\hat{U}$ is:

$$\hat{U}((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i, q_i) = \alpha_i t_i - \frac{1}{2}\beta t_i^2 + \kappa q_i t_i + v_i \sum_{j\in N(i)} t_j - \frac{1}{2}\beta v_i^2 - \eta p_i v_i.$$

The F. O. C.s of $\hat{U}$ are:

$$\frac{\partial \hat{U}}{\partial t_i} = \alpha_i + \kappa q_i - \beta t_i = 0;$$

$$\frac{\partial \hat{U}}{\partial v_i} = \sum_{j \in N(i)} t_j - \eta p_i - \beta v_i = 0.$$

Solve $\partial \hat{U}/\partial t_i = 0$ and $\partial \hat{U}/\partial v_i = 0$ simultaneously, and write them in matrix form. We have:

$$\mathbf{t} = \frac{1}{\beta}\boldsymbol{\alpha} + \frac{\kappa}{\beta}\mathbf{q} \tag{1.13}$$

and

$$\mathbf{v} = \frac{1}{\beta^2}\mathbf{G}\boldsymbol{\alpha} + \frac{\kappa}{\beta^2}\mathbf{G}\mathbf{q} - \frac{\eta}{\beta}\mathbf{p}. \tag{1.14}$$

Combining the monopolist's profit function $\pi = -\mathbf{q}^T\mathbf{t} + \mathbf{p}^T\mathbf{v}$, we have the profit function as:

$$\pi = -\mathbf{q}^T\left(\frac{\boldsymbol{\alpha}}{\beta} + \frac{\kappa\mathbf{q}}{\beta}\right) + \mathbf{p}^T\left(\frac{\mathbf{G}\boldsymbol{\alpha}}{\beta^2} + \frac{\kappa\mathbf{G}\mathbf{q}}{\beta^2} - \frac{\eta\mathbf{p}}{\beta}\right).$$

We use a vector $\mathbf{x}^T = (\mathbf{p}^T, \mathbf{q}^T)$, $\mathbf{I}_{u,l} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\mathbf{I}_{b,r} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$, $\mathbf{I}_{u,r} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix}$, the profit function is now:

$$\pi = -\frac{1}{\beta}\mathbf{x}^T\mathbf{I}_{b,r}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\mathbf{x}^T\mathbf{I}_{u,l}\mathbf{B}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} - \frac{\kappa}{\beta}\mathbf{x}^T\mathbf{I}_{b,r}\mathbf{x} + \frac{\kappa}{\beta^2}\mathbf{x}^T\mathbf{I}_{u,r}\mathbf{B}\mathbf{x} - \frac{\eta}{\beta}\mathbf{x}^T\mathbf{I}_{u,l}\mathbf{x}.$$

The Hessian matrix of π is:

$$\mathbf{H}(\pi) = -2\frac{\kappa}{\beta}\mathbf{I}_{b,r} - 2\frac{\eta}{\beta}\mathbf{I}_{u,l} + \frac{\kappa}{\beta^2}\mathbf{I}_{u,r}\mathbf{B} + \frac{\kappa}{\beta^2}\mathbf{I}_{u,r}^T\mathbf{B}.$$

Equivalently,

$$\mathbf{H}(\pi) = \frac{1}{\beta^2}\begin{bmatrix} -2\eta\beta\mathbf{I} & \kappa\mathbf{G} \\ \kappa\mathbf{G} & -2\kappa\beta\mathbf{I} \end{bmatrix}.$$

By Lemma 1.2, if and only if $\lambda_{\max}(\mathbf{G}) < 2\beta\sqrt{\eta/\kappa}$, $\lambda_{\max}(\mathbf{H}(\pi)) < 0$. Thus, the

$\mathbf{H}(\pi)$ is negative definite. We have a unique maximum of profit function. The first derivatives of π with respect to $\mathbf{x}$ is:

$$\frac{\partial \pi}{\partial \mathbf{x}} = -\frac{1}{\beta}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\begin{pmatrix} \mathbf{G}\boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} + \frac{1}{\beta^2}\begin{bmatrix} -2\eta\beta\mathbf{I} & \kappa\mathbf{G} \\ \kappa\mathbf{G} & -2\kappa\beta\mathbf{I} \end{bmatrix}\mathbf{x}.$$

Solve $\partial\pi/\partial\mathbf{x} = \mathbf{0}$, and we have

$$\mathbf{p}^* = \frac{1}{4\eta\beta}\mathbf{G}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha}$$

and

$$\mathbf{q}^* = \frac{1}{2\kappa}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha} - \frac{1}{\kappa}\boldsymbol{\alpha}.$$

Therefore, by the best responses of creating time and browsing time, i. e., eq. (1.13) and eq. (1.14) and, $\mathbf{p}^*$ and $\mathbf{q}^*$, users' creating time are given by:

$$\mathbf{t}^* = \frac{1}{2\beta}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha},$$

and their browsing time are given by:

$$\mathbf{v}^* = \frac{1}{4\beta^2}\mathbf{G}\left(\mathbf{I} - \frac{\kappa}{4\eta\beta^2}\mathbf{G}^2\right)^{-1}\boldsymbol{\alpha}.$$

The interior solutions must satisfy the F.O.C.s and be positive. Then, the sufficient and necessary condition is for all i, for all i $\left(\left(\mathbf{I} - \kappa\left(4\eta\beta^2\right)^{-1}\mathbf{G}^2 - \mathbf{I}\right)^{-1}\boldsymbol{\alpha}\right)_i > 0$. Q.E.D.

Proof of Proposition 1.4. For the monopolist's profit, consider a new potential function Φ:

$$\Phi(\mathbf{x}, \mathbf{G}) = \mathbf{x}^T(\mathbf{G} + \gamma\mathbf{I})\boldsymbol{\alpha} - \frac{1}{2}\mathbf{x}^T(\mathbf{I} - \gamma\delta\mathbf{G})\mathbf{x}.$$

By the positive definiteness of $\mathbf{I} - \gamma\delta\mathbf{G}$ and $\mathbf{I} - \gamma\delta\mathbf{G}'$, $\Phi(\mathbf{x}, \mathbf{G})$ and $\Phi(\mathbf{x}', \mathbf{G}')$ are strictly concave functions.[21] The maximizers of $\Phi(\mathbf{x}^*, \mathbf{G})$ and $\Phi(\mathbf{x}'^*, \mathbf{G}')$ are interior, then the values of potential functions $\Phi(\mathbf{x}^*, \mathbf{G})$ and $\Phi(\mathbf{x}'^*, \mathbf{G}')$ correspond to the monopolist's profits π^* on network $\mathbf{G}$ and π'^* on network $\mathbf{G}'$ respectively. As

[21] See Footnote 16 for the proof of that the product of two positive definite matrices is positive definite.

$l_{ij} \notin \mathbf{G}$ and $\mathbf{G}' = \mathbf{G} + l_{ij}$, we have

$$\begin{aligned}\Phi\left(\mathbf{x}^*, \mathbf{G}'\right) - \Phi\left(\mathbf{x}^*, \mathbf{G}\right) &= x_i^* \alpha_j + x_j^* \alpha_i + \gamma\delta x_j^* x_i^* \\ &= x_i^* \left(\alpha_j + \gamma\delta x_j^*/2\right) + x_j^* \left(\alpha_i + \gamma\delta x_i^*/2\right) \\ &> x_i^* \left(\alpha_j + \gamma\delta\alpha_j / \left(2\beta\right)\right) + x_j^* \left(\alpha_i + \gamma\delta\alpha_i / \left(2\beta\right)\right).\end{aligned}$$

If $\gamma\delta / \left(2\beta\right) > -1$, we have

$$\Phi\left(\mathbf{x}^*, \mathbf{G}'\right) - \Phi\left(\mathbf{x}^*, \mathbf{G}\right) > x_i^* \left(\alpha_j + \gamma\delta\alpha_j / \left(2\beta\right)\right) + x_j^* \left(\alpha_i + \gamma\delta\alpha_i / \left(2\beta\right)\right) > 0.$$

Then, by $\Phi\left(\mathbf{x}'^*, \mathbf{G}'\right) \geq \Phi\left(\mathbf{x}^*, \mathbf{G}'\right) > \Phi\left(\mathbf{x}^*, \mathbf{G}\right)$. Therefore, $\pi'^* > \pi^*$.

We now discuss the condition $\gamma\delta / \left(2\beta\right) > -1$. Consider the condition for positive definiteness of matrix $\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)$. Since the smallest eigenvalue of a undirected graph is smaller than or equal to -1, we have $\gamma\delta > -1$. Therefore, if $\beta > 1/2$, $\gamma\delta / \left(2\beta\right) > -1$. If $\beta < 1/2$, then we solve the $\gamma\delta / 2\beta > -1$, which results in $\left(4\beta\right)^{-1} - \sqrt{\left(16\beta^2\right)^{-1} + \beta^2} < \gamma < 0$. Q.E.D.

Proof of Proposition 1.5. By condition that $\psi\left(\mathbf{x}, \mathbf{C}\right)$ and $\psi\left(\mathbf{x}', \mathbf{C}'\right)$ are strictly concave, and the maximizers of $\psi\left(\mathbf{x}, \mathbf{C}\right)$ and $\psi\left(\mathbf{x}', \mathbf{C}'\right)$ are interior, then $\psi\left(\mathbf{x}^*, \mathbf{C}\right)$ and $\psi\left(\mathbf{x}'^*, \mathbf{C}'\right)$ correspond to $\sum t_i^*$ and $\sum t_i'^*$ respectively. The partial derivative of ψ w.r.t. C_{ii} is negative: $\partial\psi / \partial C_{ii} = -x_i^2/2 + \gamma\delta \sum_{j \in N(i)} x_i x_j / 2 < 0$. Therefore, $\psi\left(\mathbf{x}^*, \mathbf{C}\right) < \psi\left(\mathbf{x}^*, \mathbf{C}'\right)$, since $C_{ii}' = \left(\alpha_i'\right)^{-1} \leq \left(\alpha_i\right)^{-1} = C_{ii}$. Thus, $\psi\left(\mathbf{x}'^*, \mathbf{C}'\right) \geq \psi\left(\mathbf{x}^*, \mathbf{C}'\right) > \psi\left(\mathbf{x}^*, \mathbf{C}\right)$. Therefore, $\sum t_i'^* > \sum t_i^*$. Q.E.D.

Proof of Corollary 1.5.1. Consider a new potential function ϕ:

$$\phi\left(\mathbf{x}, \mathbf{D}\right) = \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T \mathbf{D} \left(\mathbf{I} - \gamma\delta\mathbf{G}\right) \mathbf{x},$$

where $\mathbf{D}$ is a diagonal matrix and $D_{ii} = \left(\sum_{j \in N(i)} \alpha_j + \gamma\alpha_i\right)^{-1}$. We have matrix $\mathbf{D}'$ with $D_{ii}' = \left(\sum_{j \in N(i)} \alpha_j' + \gamma\alpha_i'\right)^{-1}$. If matrix $\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)$ is positive definite, Hessian matrices of $\phi\left(\mathbf{x}, \mathbf{D}\right)$ and $\phi\left(\mathbf{x}', \mathbf{D}'\right)$ –$\mathbf{H}\left(\phi\left(\mathbf{x}, \mathbf{D}\right)\right) = \mathbf{D}\left(\mathbf{I} - \gamma\delta\mathbf{G}\right) + \left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{D}$ and $\mathbf{H}\left(\phi\left(\mathbf{x}, \mathbf{D}'\right)\right) = \mathbf{D}'\left(\mathbf{I} - \gamma\delta\mathbf{G}\right) + \left(\mathbf{I} - \gamma\delta\mathbf{G}\right)\mathbf{D}'$ are positive definite.[22]

[22] See Footnote 16 for the proof of that the product of two positive definite matrices is positive definite.

Potential functions $\phi(\mathbf{x}, \mathbf{D})$ and $\phi(\mathbf{x}', \mathbf{D}')$ are strictly concave and the maximizers of $\phi(\mathbf{x}, \mathbf{D})$ and $\phi(\mathbf{x}', \mathbf{D}')$ are interior. Thus, $\phi(\mathbf{x}^*, \mathbf{D})$ and $\phi(\mathbf{x}'^*, \mathbf{D}')$ correspond to $\sum v_i^*$ and $\sum v_i'^*$ respectively. The partial derivative of ϕ w.r.t. D_{ii} is negative: $\partial\phi/\partial D_{ii} = -x_i^2/2 + \gamma\delta\sum_{j\in N(i)} x_i x_j/2 < 0$. Since $D'_{ii} > \phi(\mathbf{x}^*, \mathbf{D})$, it follows that $\phi(\mathbf{x}^*, \mathbf{D}) < \phi(\mathbf{x}^*, \mathbf{D}')$. Thus, we have $\phi(\mathbf{x}^*, \mathbf{D}) < \phi(\mathbf{x}^*, \mathbf{D}') \leq \phi(\mathbf{x}'^*, \mathbf{D}')$. Therefore, $\sum v_i^* < \sum v_i'^*$.

For the monopolist's profit, consider the potential function $\Phi(\mathbf{x}, \boldsymbol{\alpha})$. Potential functions $\Phi(\mathbf{x}, \boldsymbol{\alpha})$ and $\Phi(\mathbf{x}', \boldsymbol{\alpha}')$ are strictly concave functions and the maximizers of $\Phi(\mathbf{x}, \boldsymbol{\alpha})$ and $\Phi(\mathbf{x}', \boldsymbol{\alpha}')$ are interior. The values of $\Phi(\mathbf{x}^*, \boldsymbol{\alpha})$ and $\Phi(\mathbf{x}'^*, \boldsymbol{\alpha}')$ correspond to the monopolist's profit π^* with $\boldsymbol{\alpha}$ and π'^* with $\boldsymbol{\alpha}'$ respectively. We show that $\Phi(\mathbf{x}^*, \boldsymbol{\alpha}) < \Phi(\mathbf{x}^*, \boldsymbol{\alpha}')$; then, by $\Phi(\mathbf{x}^*, \boldsymbol{\alpha}') \leq \Phi(\mathbf{x}'^*, \boldsymbol{\alpha}')$, $\Phi(\mathbf{x}^*, \boldsymbol{\alpha}) < \Phi(\mathbf{x}'^*, \boldsymbol{\alpha}')$ is straightforward.

Consider

$$\Phi(\mathbf{x}^*, \boldsymbol{\alpha}') - \Phi(\mathbf{x}^*, \boldsymbol{\alpha}) = (\boldsymbol{\alpha}' - \boldsymbol{\alpha})^T (\mathbf{G} + \gamma\mathbf{I})^2 (\mathbf{I} - \gamma\delta\mathbf{G})^{-1} \boldsymbol{\alpha}.$$

For both i and j such that $\sum_{k\in N(i)} \alpha_k + \gamma\alpha_i \leq \sum_{k\in N(i)} \alpha'_k + \gamma\alpha'_i$ and $\sum_{k\in N(j)} \alpha_k + \gamma\alpha_j \leq \sum_{k\in N(j)} \alpha'_k + \gamma\alpha'_j$, then $\left((\boldsymbol{\alpha}' - \boldsymbol{\alpha})^T (\mathbf{G} + \gamma\mathbf{I})\right)_{i,j} \geq 0$. For any user k who is either i's or j's friend, the pricing plan p'_k is strictly increasing. Equivalently, for $k \in \{N(i), N(j)\}$, $\left((\boldsymbol{\alpha}' - \boldsymbol{\alpha})^T (\mathbf{G} + \gamma\mathbf{I})\right)_k > 0$. Combined with the condition that any user's browsing time is positive, we have:

$$(\boldsymbol{\alpha}' - \boldsymbol{\alpha})^T (\mathbf{G} + \gamma\mathbf{I}) (\mathbf{G} + \gamma\mathbf{I}) (\mathbf{I} - \gamma\delta\mathbf{G})^{-1} \boldsymbol{\alpha} > 0.$$

Thus, $\Phi(\mathbf{x}^*, \boldsymbol{\alpha}) < \Phi(\mathbf{x}^*, \boldsymbol{\alpha}') \leq \Phi(\mathbf{x}'^*, \boldsymbol{\alpha}')$, which implies $\pi'^* > \pi^*$.

Q.E.D.

PROOF OF PROPOSITION 1.6. Let π^* and π'^* denote the monopolist's profit from network $\mathbf{G}$ and $\mathbf{G}'$ respectively. Without loss of generality, we assume $\alpha = 1$.

If $\gamma = 0$, by Proposition 1.1 and the monopolist's profit function $\pi = \mathbf{p}^T\mathbf{v}$, the monopolist's profit from network $\mathbf{G}$ π^* is equal to $\mathbf{1}^T\mathbf{G}^2\mathbf{1}/(4\beta^3\eta)$; and her/his profit from network $\mathbf{G}'$ π'^* is equal to $\mathbf{1}^T\mathbf{G}'^2\mathbf{1}/(4\beta^3\eta)$. For the term $\mathbf{1}^T\mathbf{G}^2\mathbf{1}$ in π^*, it can be reorganized as $|E(\mathbf{G})| + \sum_i d_i(d_i - 1)$. Since $\mathbf{G}'$ is with higher variance of degree than $\mathbf{G}$ but the same mean, we have $\sum_i d'_i(d'_i - 1) > \sum d_i(d_i - 1)$ such that

$\mathbf{1}^T\mathbf{G}^2\mathbf{1} < \mathbf{1}^T\mathbf{G}'^2\mathbf{1}$. Therefore, $\pi^* < \pi'^*$.

If $\gamma = 0$, the monopolist's profit from network $\mathbf{G}$ by discrimination pricing across each node is given by $\pi^* = \delta\boldsymbol{\alpha}^T(\mathbf{G}+\gamma\mathbf{I})^2(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha}/(4\beta)$. Reorganizing the equation, it yields

$$\pi^* = \frac{\beta^3\delta}{(4\gamma^2)}\boldsymbol{\alpha}^T(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha} - \frac{\beta}{(4\gamma^2)}\boldsymbol{\alpha}^T\boldsymbol{\alpha} - \frac{1}{(4\beta\gamma)}\boldsymbol{\alpha}^T(\mathbf{G}+\gamma\mathbf{I})\boldsymbol{\alpha}.$$

Then, the profit from network $\mathbf{G}'$ is :

$$\pi'^* = \frac{\beta^3\delta}{(4\gamma^2)}\boldsymbol{\alpha}^T(\mathbf{I}-\gamma\delta\mathbf{G}')^{-1}\boldsymbol{\alpha} - \frac{\beta}{(4\gamma^2)}\boldsymbol{\alpha}^T\boldsymbol{\alpha} - \frac{1}{(4\beta\gamma)}\boldsymbol{\alpha}^T(\mathbf{G}'+\gamma\mathbf{I})\boldsymbol{\alpha}.$$

Recall that $|V(\mathbf{G})| = |V(\mathbf{G}')|$ and $|E(\mathbf{G})| = |E(\mathbf{G}')|$. Compare the second and third terms of π^* and π'^*, it yields

$$\frac{-\beta}{(4\gamma^2)}\boldsymbol{\alpha}^T\boldsymbol{\alpha} - \frac{1}{(4\beta\gamma)}\boldsymbol{\alpha}^T(\mathbf{G}+\gamma\mathbf{I})\boldsymbol{\alpha} = \frac{-\beta}{(4\gamma^2)}\boldsymbol{\alpha}^T\boldsymbol{\alpha} - \frac{1}{(4\beta\gamma)}\boldsymbol{\alpha}^T(\mathbf{G}'+\gamma\mathbf{I})\boldsymbol{\alpha}.$$

Hence, the comparison of π^* and π'^* depends on the sum of users' Katz-Bonacich centralities.

Since $\alpha = 1$, the first term of π^* $\boldsymbol{\alpha}^T(\mathbf{I}-\gamma\delta\mathbf{G})^{-1}\boldsymbol{\alpha}$ is equal to $\mathbf{1}^T\mathbf{1}+\gamma\delta\mathbf{1}^T\mathbf{G}\mathbf{1}+\gamma^2\delta^2\mathbf{1}^T\mathbf{G}^2\mathbf{1}+\mathcal{O}(\gamma^3\delta^3)$; and the first term of π'^* $\boldsymbol{\alpha}^T(\mathbf{I}-\gamma\delta\mathbf{G}')^{-1}\boldsymbol{\alpha}$ is equal to $\mathbf{1}^T\mathbf{1}+\gamma\delta\mathbf{1}^T\mathbf{G}'\mathbf{1}+\gamma^2\delta^2\mathbf{1}^T\mathbf{G}'^2\mathbf{1}+\mathcal{O}(\gamma^3\delta^3)$. We compare the two expression above. The first terms of them are the same, since $\mathbf{1}^T\mathbf{1} = |V(\mathbf{G})| = |V(\mathbf{G}')|$; the second terms $\gamma\delta\mathbf{1}^T\mathbf{G}\mathbf{1}$ and $\gamma\delta\mathbf{1}^T\mathbf{G}'\mathbf{1}$ are also equal to each other, since $\mathbf{1}^T\mathbf{G}\mathbf{1} = |E(\mathbf{G})| = |E(\mathbf{G}')| = \mathbf{1}^T\mathbf{G}'\mathbf{1}$. We consider the third term $\gamma^2\delta^2\mathbf{1}^T\mathbf{G}^2\mathbf{1}$ and reorganize it as $\gamma^2\delta^2(|E(\mathbf{G})|+\sum_i d_i(d_i-1))$. Since $\mathbf{G}'$ is with higher variance of degree than $\mathbf{G}$ but the same mean, we have $\sum_i d_i'(d_i'-1) > \sum d_i(d_i-1)$ such that $\gamma^2\delta^2\mathbf{1}^T\mathbf{G}^2\mathbf{1} < \gamma^2\delta^2\mathbf{1}^T\mathbf{G}'^2\mathbf{1}$. Therefore, $\pi^* < \pi'^*$. Q.E.D.

PROOF OF PROPOSITION 1.7. For the monopolist's profit, we have the first derivative of Φ w.r.t γ:

$$\frac{\partial\Phi}{\partial\gamma} = \sum_{i\in N} x_i\alpha_i + \delta\sum_{i\in N}\sum_{j\in N(i)} x_ix_j/2 > 0$$

If $\gamma < \gamma'$, $\Phi(\mathbf{x}^*,\gamma) < \Phi(\mathbf{x}^*,\gamma')$. Thus, $\Phi(\mathbf{x}'^*,\gamma') \geq \Phi(\mathbf{x}^*,\gamma') > \Phi(\mathbf{x}^*,\gamma)$. Therefore, $\pi'^* > \pi^*$.

User's utility is given by:

$$\mathbf{U}^{*} = \frac{3}{8\beta}\boldsymbol{\alpha}^{\circ 2} + \frac{\beta\delta}{8}\left(\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)^{\circ 2} + \frac{\beta\delta^{2}}{8}\left(\mathbf{G}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)^{\circ 2}.$$

Take the first derivative with respect to γ on $\mathbf{U}^{*}$, then it follows that

$$\begin{aligned}\frac{\partial \mathbf{U}}{\partial \gamma} =& \frac{\gamma\beta\delta^{3}}{2}\left(\mathbf{G}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)^{\circ 2} + \frac{\beta\delta^{2}}{8}\left(\beta^{2}+\gamma^{2}\right)\delta^{2}\left(\mathbf{G}^{2}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-2}\boldsymbol{\alpha}\right)^{\circ 2} \\ &+ \frac{\gamma\beta\delta^{2}}{4}\left(\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\boldsymbol{\alpha}\right)^{\circ 2} + \frac{\beta\delta}{8}\left(\beta^{2}+\gamma^{2}\right)\delta^{2}\left(\mathbf{G}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-2}\boldsymbol{\alpha}\right)^{\circ 2}.\end{aligned}$$

By checking the the derivatives, we have $(\partial \mathbf{U}/\partial \gamma)_i > 0$, provided $\gamma > 0$ or $\gamma \to 0^{-}$. Also, it holds that $\partial \pi/\partial \gamma > 0$ when $\gamma > 0$.

Solve $\left[\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\right]'$, then

$$\begin{aligned}&\left[\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\right]' \\ &= \left(\beta^{2}+\gamma^{2}\right)\delta^{2}\left(\mathbf{G} + 2\gamma\delta\mathbf{G}^{2} + ... + (\gamma\delta)^{|N|-1}\mathbf{G}^{|N|} - ...\right) \\ &= \left(\beta^{2}+\gamma^{2}\right)\delta^{2}\mathbf{G}\left(\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1} + \gamma\delta\mathbf{G}\left(\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1} + \gamma\delta\mathbf{G}\left(..\right)..\right)\right) \\ &= \left(\beta^{2}+\gamma^{2}\right)\delta^{2}\mathbf{G}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-1}\left(\mathbf{I} + \gamma\delta\mathbf{G} + ... + (\gamma\delta)^{|N|-1}\mathbf{G}^{|N|-1}..\right) \\ &= \left(\beta^{2}+\gamma^{2}\right)\delta^{2}\mathbf{G}\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)^{-2}.\end{aligned}$$

Q.E.D.

1.7.3 Supplements for Section 1.4 – Generalization

We have the utility function $\hat{U}$ denoted as follows:

$$\hat{U}\left((t_i, v_i); \mathbf{t}_{-i}, \mathbf{v}_{-i}, p_i, q_i\right) = \alpha_i t_i - \frac{1}{2}\beta t_i^{2} + \kappa q_i t_i + v_i \sum_{j \in N(i)} t_j - \frac{1}{2}\beta v_i^{2} - \eta p_i v_i + \gamma t_i v_i. \tag{1.15}$$

We take the first partial derivative of $\hat{U}$ w.r.t v_i such that

$$\frac{\partial \hat{U}_i}{\partial v_i} = \sum_{j \in N(i)} t_j - \beta v_i - \eta p_i + \gamma t_i = 0$$

Solve the equation above, and we have user i's browsing time as a function of her/his creating time t_i:

$$v\left(t_i\right)=\frac{\sum_{j\in N(i)}t_j-\eta p_i+\gamma t_i}{\beta}. \tag{1.16}$$

Assume that $v\left(t_i\right)>0$. We replace v_i in eq. (1.15) with $v\left(t_i\right)$ denoted as eq. (1.16):

$$\hat{U}\left(\left(t_i,v\left(t_i\right)\right);\mathbf{t}_{-i},\mathbf{v}_{-i},p_i,q_i\right)=\alpha_i t_i-\frac{1}{2}\beta t_i^2+\kappa q_i t_i+\frac{1}{2\beta}\left(\sum_{j\in N(i)}t_j-\eta p_i+\gamma t_i\right)^2. \tag{1.17}$$

The F. O. C of eq. (1.17) w.r.t t_i is as follows:

$$\frac{d\hat{U}\left(\left(t_i,v\left(t_i\right)\right);\mathbf{t}_{-i},\mathbf{v}_{-i},p_i,q_i\right)}{dt_i}=\alpha_i-\beta t_i+\kappa q_i+\frac{\gamma}{\beta}\left(\sum_{j\in N(i)}t_j-\eta p_i+\gamma t_i\right)=0$$

Therefore, we have user i's best response of creating time with q_i and p_i as $t_i=\delta\beta\alpha_i+\delta\kappa\beta q_i+\gamma\delta\sum_{j\in N(i)}t_j-\gamma\delta\eta p_i$, if $t_i,v\left(t_i\right)>0$. Note that: if $\gamma>0$, $v\left(t_i\right)>0$ implies that $t_i>\alpha_i/\beta+\kappa q_i/\beta$; if $\gamma<0$, $v\left(t_i\right)>0$ implies that $t_i<\alpha_i/\beta+\kappa q_i/\beta$.

Assume that for all $k\in N$, $t_k,v\left(t_k\right)>0$. In matrix form, we have $\mathbf{t}=\gamma\delta\mathbf{G}\mathbf{t}+\delta\left(\beta\boldsymbol{\alpha}-\eta\gamma\mathbf{p}+\kappa\beta\mathbf{q}\right)$. We take the mixed partial derivatives of $\hat{U}_i$ and $\hat{U}_j$ respectively. Thus, $\partial^2\hat{U}_i/\partial t_i\partial t_j=\partial^2\hat{U}_j/\partial t_i\partial t_j=\gamma g_{ij}/\beta$, which satisfies the condition of Monderer and Shapley (1996). Therefore, we have the potential function φ is given by:

$$\varphi\left(\mathbf{t};\boldsymbol{\alpha},\mathbf{p},\mathbf{q},\mathbf{G}\right)=\delta\mathbf{t}^T\left(\beta\boldsymbol{\alpha}-\eta\gamma\mathbf{p}+\kappa\beta\mathbf{q}\right)-\frac{1}{2}\mathbf{t}^T\left(\mathbf{I}-\gamma\delta\mathbf{G}\right)\mathbf{t},$$

in which $\mathbf{t}$ satisfies Kuhn-Tucker condition:

(i) Assume $\gamma>0$, then $t_i\geq\alpha_i/\beta+\kappa q_i/\beta$. If φ is strictly concave, then $t_i=\alpha_i/\beta+\kappa q_i/\beta\Rightarrow\partial\varphi/\partial t_i\leq 0$, and $t_i>\alpha_i/\beta+\kappa q_i/\beta\Rightarrow\partial\varphi/\partial t_i=0$; if φ is convex, for all i $t_i\to+\infty$.

(ii) Assume $\gamma<0$, then $0\leq t_i\leq\alpha_i/\beta$. $t_i=0\Rightarrow\partial\varphi/\partial t_i\leq 0$, $0<t_i<\alpha_i/\beta+\kappa q_i/\beta\Rightarrow\partial\varphi/\partial t_i=0$, and $t_i=\alpha_i/\beta+\kappa q_i/\beta\Rightarrow\partial\varphi/\partial t_i\geq 0$.[23]

If $\lambda_{\max}\left(\gamma\mathbf{G}\right)<\delta^{-1}$, $\mathbf{H}\left(\varphi\right)$ is negative definite. Then φ is strictly concave and

[23] If $\gamma<0$ and for some user i $t_i=\alpha_i/\beta+\kappa q_i/\beta$, then user i's browsing time binds such that $v_i=v\left(t_i\right)=0$.

there exists a unique maximum of φ. Assume that $\mathbf{t}$ and $\mathbf{v}$ are interior. We have

$$\mathbf{t} = \delta\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\beta\boldsymbol{\alpha} - \eta\gamma\mathbf{p} + \kappa\beta\mathbf{q}\right)$$

and

$$\mathbf{v} = \frac{\beta\delta}{\gamma}\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\beta\boldsymbol{\alpha} - \eta\gamma\mathbf{p} + \kappa\beta\mathbf{q}\right) - \frac{1}{\gamma}\boldsymbol{\alpha} - \frac{\kappa}{\gamma}\mathbf{q}.$$

Replace $\mathbf{t}$ and $\mathbf{v}$ in the profit function:

$$\begin{aligned}\pi =& -\mathbf{q}^T\delta\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\beta\boldsymbol{\alpha} - \eta\gamma\mathbf{p} + \kappa\beta\mathbf{q}\right)\\ &+ \frac{\delta\beta}{\gamma}\mathbf{p}^T\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\left(\beta\boldsymbol{\alpha} - \eta\gamma\mathbf{p} + \kappa\beta\mathbf{q}\right) - \frac{1}{\gamma}\mathbf{p}^T\boldsymbol{\alpha} - \frac{\kappa}{\gamma}\mathbf{p}^T\mathbf{q}.\end{aligned}$$

We use a vector $\mathbf{x}^T = \left(\mathbf{p}^T, \mathbf{q}^T\right)$, $\mathbf{I}_{u,l} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$, $\mathbf{I}_{b,r} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}$, $\mathbf{I}_{u,r} = \begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}\left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1} & \mathbf{0}\\ \mathbf{0} & \left(\mathbf{I} - \gamma\delta\mathbf{G}\right)^{-1}\end{pmatrix}$, the profit function is now:

$$\begin{aligned}\pi =&\mathbf{x}^T\mathbf{B}\left[-\delta\beta\mathbf{I}_{b,r}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}\end{pmatrix} + \delta\eta\gamma\mathbf{I}_{u,r}\mathbf{x} - \delta\kappa\beta\mathbf{I}_{b,r}\mathbf{x}\right]\\ &+ \mathbf{x}^T\mathbf{B}\left[\frac{\delta\beta^2}{\gamma}\mathbf{I}_{u,l}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}\end{pmatrix} - \delta\beta\eta\mathbf{I}_{u,l}\mathbf{x} + \frac{\delta\kappa\beta^2}{\gamma}\mathbf{I}_{u,r}\mathbf{x}\right] - \frac{1}{\gamma}\mathbf{x}^T\mathbf{I}_{u,l}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}\end{pmatrix} - \frac{\kappa}{\gamma}\mathbf{x}^T\mathbf{I}_{u,r}\mathbf{x}.\end{aligned}$$

The Hessian matrix is:

$$\mathbf{H}\left(\pi\right) = \delta\left(\mathbf{I} - \delta\gamma\mathbf{G}\right)^{-1}\begin{bmatrix}-2\eta\beta\mathbf{I} & \gamma\left(\eta + \kappa\right)\mathbf{I} + \kappa\mathbf{G}\\ \gamma\left(\eta + \kappa\right)\mathbf{I} + \kappa\mathbf{G} & -2\kappa\beta\mathbf{I}\end{bmatrix}.$$

Assume that $-\gamma^2\left(1 + \eta/\kappa\right) + 2\beta|\gamma|\sqrt{\eta/\kappa} > 0$.[24] The largest eigenvalue of $\mathbf{H}\left(\pi\right)$ is negative, if and only if $\lambda_{\max}\left(\gamma\mathbf{G}\right) < -\gamma^2\left(1 + \eta/\kappa\right) + 2\beta|\gamma|\sqrt{\eta/\kappa}$.[25]

[24] If $-\gamma^2\left(1 + \eta/\kappa\right) + 2\beta|\gamma|\sqrt{\eta/\kappa} < 0$, then $\mathbf{H}\left(\pi\right)$is not negative definite. The profit function does not have a unique maximum point.

[25] Using the same technique in the proof of Proposition 1.3:

$$\lambda\left(\mathbf{H}\left(\pi\right)\right) = \pm\frac{\sqrt{\beta^2\left(\eta - \kappa\right)^2 + \left[\gamma\left(\eta + \kappa\right) + \kappa\lambda\left(\mathbf{G}\right)\right]^2}}{1 - \delta\lambda\left(\gamma\mathbf{G}\right)} - \frac{\beta\left(\eta + \kappa\right)}{1 - \delta\lambda\left(\gamma\mathbf{G}\right)},$$

such that:

If $\lambda_{\max}(\gamma\mathbf{G}) < \min\left\{\delta^{-1}, -\gamma^2(1+\eta/\kappa)+2\beta|\gamma|\sqrt{\eta/\kappa}\right\}$, we have unique and interior solution of SPE. Let $\mathbf{W} = \gamma(1+\eta/\kappa)\mathbf{I}+\mathbf{G}$. Solve $d\pi/d\mathbf{x} = \mathbf{0}$ resulting in:

$$\mathbf{p}^* = \left(-\frac{\gamma}{2\beta\kappa}\mathbf{I}+\frac{1}{4\beta\eta}\mathbf{W}\right)\left(\mathbf{I}-\frac{\kappa}{4\beta^2\eta}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}$$

and

$$\mathbf{q}^* = \left(\frac{1}{2\kappa}\mathbf{I}-\frac{\gamma}{4\beta^2\kappa}\mathbf{W}\right)\left(\mathbf{I}-\frac{\kappa}{4\beta^2\eta}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}-\frac{1}{\kappa}\boldsymbol{\alpha};$$

and the users' creating time and browsing time are:

$$\mathbf{t}^* = \frac{1}{2\beta}\left(\mathbf{I}-\frac{\kappa}{4\beta^2\eta}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}$$

and

$$\mathbf{v}^* = \frac{1}{4\beta^2}\mathbf{W}\left(\mathbf{I}-\frac{\kappa}{4\beta^2\eta}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}.$$

If $\gamma > 0$, the following assumption must hold to ensure that $\forall i, t_i, v_i, p_i, q_i > 0$:

Assumption 1.1. *Assume $\gamma > 0$. If and only if for all i,*

(i) $\left(\left(-\gamma\mathbf{I}+\kappa(2\eta)^{-1}\mathbf{W}\right)\left(\mathbf{I}-\kappa(4\beta^2\eta)^{-1}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}\right)_i > 0$;

(ii) $\left(\left(\mathbf{I}/2-\gamma(4\beta^2)^{-1}\mathbf{W}\right)\left(\mathbf{I}-\kappa(4\beta^2\eta)^{-1}\mathbf{W}^2\right)^{-1}\boldsymbol{\alpha}\right)_i > \alpha_i$.

The first condition of Assumption 1.1 ensures that every entry $\mathbf{p}^*$ is positive, and the second condition of Assumption 1.1 ensures that every entry of $\mathbf{q}^*$ is positive.

If the intrapersonal substitutes appear $\gamma < 0$, then the following assumption must hold to ensure that for all $i, t_i, v_i, p_i, q_i > 0$:

Assumption 1.2. *If $\gamma < 0$, if and only if for all i,*

(i) assume $\gamma > 0$,

$$\lambda_{\max}(\mathbf{H}(\pi)) = \frac{\sqrt{\beta^2(\eta-\kappa)^2+[\gamma(\eta+\kappa)+\kappa\lambda_{\max}(\mathbf{G})]^2}}{1-\gamma\delta\lambda_{\max}(\mathbf{G})} - \frac{\beta(\eta+\kappa)}{1-\gamma\delta\lambda_{\max}(\mathbf{G})},$$

in which $\lambda_{\max}(\mathbf{G}) < -\gamma(1+\eta/\kappa)+2\beta\sqrt{\eta/\kappa}$ implies $\lambda_{\max}(\mathbf{H}(\pi)) < 0$.
(ii) assume $\gamma < 0$,

$$\lambda_{\max}(\mathbf{H}(\pi)) = \frac{\sqrt{\beta^2(\eta-\kappa)^2+[\gamma(\eta+\kappa)+\kappa\lambda_{\min}(\mathbf{G})]^2}}{1-\gamma\delta\lambda_{\min}(\mathbf{G})} - \frac{\beta(\eta+\kappa)}{1-\gamma\delta\lambda_{\min}(\mathbf{G})},$$

where $\lambda_{\min}(\mathbf{G}) > -\gamma(1+\eta/\kappa)-2\beta\sqrt{\eta/\kappa}$ implies $\lambda_{\max}(\mathbf{H}(\pi)) < 0$.
Therefore, if $\lambda_{\max}(\gamma\mathbf{G}) < -\gamma^2(1+\eta/\kappa)+2\beta|\gamma|\sqrt{\eta/\kappa}$, we have $\lambda_{\max}(\mathbf{H}(\pi)) < 0$.

(i) $\left(\left(-\gamma\mathbf{I}+\kappa\left(2\eta\right)^{-1}\mathbf{W}\right)\left(\mathbf{I}-\kappa\left(4\beta^{2}\eta\right)^{-1}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}\right)_{i}>0$*;*

(ii) $\left(\left(\mathbf{I}/2-\gamma\left(4\beta^{2}\right)^{-1}\mathbf{W}\right)\left(\mathbf{I}-\kappa\left(4\beta^{2}\eta\right)^{-1}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}\right)_{i}>\alpha_{i}$*.*

(iii) $\left(\left(\mathbf{I}-\kappa\left(4\beta^{2}\eta\right)^{-1}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}\right)_{i}>0$

(iv) $\left(\mathbf{W}\left(\mathbf{I}-\kappa\left(4\beta^{2}\eta\right)^{-1}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}\right)_{i}>0.$

In details, the first condition ensures all the entries in $\mathbf{p}^*$ are positive; the second one ensures evey entry in $\mathbf{q}^*$ is positive; the third and fourth conditions ensure $\mathbf{t}^*$, and $\mathbf{v}^*$'s entries to be positive respectively.

Now we have the proposition:

Proposition 1.8. *Assume that* $\zeta=-\gamma^{2}\left(1+\eta/\kappa\right)+2\beta|\gamma|\sqrt{\eta/\kappa}>0$ *and* $\lambda_{\max}\left(\gamma\mathbf{G}\right)<\min\left\{\delta^{-1},\zeta\right\}$*. If and only if either (i) Assumption 1.1 or (ii) Assumption 1.2 holds, in the unique SPE, the monopolist's pricing strategy is:*

$$\mathbf{p}^{*}=\left(-\frac{\gamma}{2\beta\kappa}\mathbf{I}+\frac{1}{4\beta\eta}\mathbf{W}\right)\left(\mathbf{I}-\frac{\kappa}{4\beta^{2}\eta}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}$$

and

$$\mathbf{q}^{*}=\left(\frac{1}{2\kappa}\mathbf{I}-\frac{\gamma}{4\beta^{2}\kappa}\mathbf{W}\right)\left(\mathbf{I}-\frac{\kappa}{4\beta^{2}\eta}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}-\frac{1}{\kappa}\boldsymbol{\alpha};$$

and the users' creating time and browsing time are:

$$\mathbf{t}^{*}=\frac{1}{2\beta}\left(\mathbf{I}-\frac{\kappa}{4\beta^{2}\eta}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}\ \textit{and}\ \mathbf{v}^{*}=\frac{1}{4\beta^{2}}\mathbf{W}\left(\mathbf{I}-\frac{\kappa}{4\beta^{2}\eta}\mathbf{W}^{2}\right)^{-1}\boldsymbol{\alpha}.$$

Chapter 2

Social Integration and the Acculturation Game

2.1 Introduction

Social integration is of great concern in multicultural societies. The fundamental issue of marginalization and separation of societies is the miscoordination in the dimension of culture between the members of the societies. The miscoordination results from the issues of group identity and members' lack of ability to coordinate with each other.[1] An example of the second issue is that immigrants or familial immigrants, who look for a position in mainstream societies, agree on the professional, economic, and academic ways of the mainstream life, but they need to improves their ability to coordinate by learning the cultural knowledge of mainstream to fit in.[2] This paper considers a framework where an individual's payoffs from coordination depends on her/his ability to understand others individuals have the willingness to improve their own ability to interact and coordinate with the others, but experiences different types of costs in order to do so.

Interaction and coordination are the essential ways for individuals to participate in social and economic life.[3] However, interaction and coordination are not culturally free. Cultural knowledge helps individuals to build a better understanding and be more predictive of their coordinators' or colleagues' behaviors, which enable individuals to

[1] The majority of theoretical economic studies on culture issue with aspect to economic interaction in public life of multicultural societies are based on the identity theory(Akerlof and Kranton (2000)). This paper studies the effects of improvement of members' ability to coordinate on social integration. Please see Section 2.2 for detailed discussion about these two issues.

[2] Seminal political philosophic study Kymlicka (1995) points out that individuals, such as immigrant or familial immigrants, seek for greater recognition of ethnic identities, but they want to participate in the modern industrialized form of social life.

[3] Rawls (2009) states that: "a society is a cooperative venture for mutual advantage".

make suitable responses in social coordination. We think of *cultural knowledge* as *the knowledge about local social context*, which facilitates individuals' social interaction and coordination. It includes: *the proficiency in local languages, the common beliefs, the organization of public life, historical events, current affairs etc.*

Individuals who live in a cultural social context different from their origins, need to cope with acculturative stress and exert efforts to overcome the difficulties in learning the mainstream culture (Sam and Berry (2006)). Acculturative stress is the negative emotion individuals feel when they experience failures of social interaction due to cultural differences. Individuals experience less stress when they become adaptive. However, the more cultural knowledge an individual has, the more difficult it is for her/him to gain new.

Psychologists use "acculturation" to capture individuals' psychological, cognitive and behavioral changes in the process of cultural adaptation (Sam and Berry (2006)). We use "acculturation game" to propose a stylized model through a game-theoretical approach to describe the behavior of individuals, who acquire knowledge about the social context to improve their ability to coordinate with each other. In the model, we use the interval from zero to one to represent the cultural knowledge spectrum of the society. Each agent is endowed with cultural knowledge from zero to a certain level and derives utilities from interacting with others. The utilities derived by each of the two interacting agents are proportional to the length of the cultural knowledge spectrum they have in common; that is to say, the minimum of the two agents' cultural knowledge spectrum dominates the utilities of interacting for both of them. Agents can expand their cultural knowledge with cost. We are interested in what level of cultural knowledge each agent chooses to gain. Further, a cultural test, as a type of integration policies, sets a minimum cultural knowledge level for all agents; we are interested in when this policy improves social welfare.

We analyze payoff dominant equilibria of three cases of acculturation game with different types of marginal cost functions. Each of the three cases corresponds to the situation where agents experience: (i) difficulties, (ii) stress, or (iii) both difficulties and stress in learning.

In Section 2.4.1, we study the case where agents experience difficulties only. This case corresponds to the game with strictly increasing marginal cost functions, which capture increasing difficulties for agents to gain new knowledge. In equilibria, the agents endowed with low levels of cultural knowledge learn, and those endowed with high levels do not. This case applies to the socialization of children. Children interact with their parents and playmates in the process of socialization, by which the children gain cultural identities for coordination. Children acquire their first languages at their earliest stage of socialization. The seminal study by Brown (1973) states that a child's prelinguistic grasp of concepts and meanings through interacting with her/his parents aids her/his language acquisition. He also mentions that there was no evidence that children's speech into lines is impelled by any kind of pressure.[4]

In Section 2.4.2, we study the case where agents experience stress only. This case corresponds to the game with strictly decreasing marginal cost functions, which captures the decreasing of stress experienced by agents, as they become more adaptive to the dominant culture. The second case applies to the acculturation of second-generation immigrant adolescents. The second-generations have easy access to cultural knowledge of the mainstreams from their education systems; however, the culture parts of the second-generations' upbringing are closer to their parents' origins, which are different from the dominant mainstream culture. Therefore, the second-generation immigrant adolescents experience high levels of acculturative stress.[5] In equilibria, the agents endowed with high levels of cultural knowledge acquire all cultural knowledge and the others do not acquire any. The results of equilibria can be interpreted as the marginalization or separation of agents endowed with low levels of cultural knowledge.

In Section 2.4.3, we study the game with U-shaped marginal cost functions. A U-shaped function is the sum of two functions such that one of the two is strictly increasing and the other is strictly decreasing. Agents experience both difficulties and acculturative stress in order to learn. In equilibria, only the agents endowed with middle levels of cultural knowledge learn, while both the agents endowed with low and high levels of cultural knowledge do not. The third case applies to the acculturation of first-generation immigrants: they experience acculturative stress and difficulties in learning in social and economic life.[6]

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[4] To the best of our knowledge, this statement still holds.

[5] Hovey and King (1996) found that one fourth of second-generation immigrants adolescents of Latino, who answers their questionnaire, reported critical levels of depression due to acculturative stress.

[6] Empirical studies of sociology and economics show the acculturation outcomes of first- and second-

For policy implication in Section 2.5, we consider well-defined marginal cost functions. In the case where agents experience difficulties only, the optimal policy level decreases with the difficulties in learning, and increases with the proportions of agents endowed with higher levels of cultural knowledge. In the case where agents experience only stress, a low level policy can induce all agents to acquire all cultural knowledge, if there exist some agents acquiring all cultural knowledge without policy. In the cases where agents experience both difficulties and stress, we use a diagram to show the policy level depends on difficulties in learning, levels of acculturative stress and the population distributions of agents' endowed knowledge levels. In the cases where the policy does not improve social welfare, agents experience high levels of acculturative stress.

2.2 Literature Review

Integration policy refers to the policy which concerns immigrants after their arrival at the host countries. For the policy specifically concerning citizenship awarding, it contains birthright citizenship, descendent citizenship and naturalization citizenship. Host countries award or abandon certain rights of citizenship over time depending on their histories and current affairs. Algan et al. (2010) review the integration policies of the UK, France and Germany, and find that the UK takes a positive multicultural approach with a high tolerance of the cultural differences of immigrants' original ways of life.[7] The UK began to impose tests on candidates who seek for citizenship in 2005, due to the fact that the they did not join the wider society beyond their communities. France and Germany, although starting with different approaches, have a convergence to adopt the same test policy for citizenship. The test is also adopted for naturalization citizenship by other migration counties such as Australia, Canada, the USA etc. The tests are criticized widely for their content. For example, the citizenship test "Life in the UK test" is criticized for its inconsistency, obsolescence, and gender imbalance

generation immigrants are different due to their main acculturative ages and education backgrounds with cross-country evidence (Zhou (1997), Portes and Zhou (2012) and Algan et al. (2010)).

[7] For the political framework of multicultural societies, Kymlicka (1995) thinks that: immigrants voluntarily choose their movements to host countries; they seek the right for ethnic identities but agree on economic, professional, academic interactive ways in the mainstream. Indigenous people seek the right of self-governance. Refugees' willingness is hard to identify, as the time of returning to their home countries is indecisive. We focus on the individuals who choose to integrate whichever group they come from, since they could make different choices from the majorities of their groups.

(Brooks (2013)).

Culture is widely studied across the whole spectrum of social sciences. The word 'culture' has its own definitional issues in academic research. As Throsby (2001) stated, " 'culture' is a word which is employed in a variety of senses in everyday use but without a tangible and generally agreed core meaning". Even we narrow down the meaning of 'culture' to the aspect of the variety of human behavior in social life across different groups, economic researches on culture-dependent outcomes diverse with analyses across psychology, individual, group levels with different assumptions and approaches.

A strand of theoretical literature considers the effect of cultural/ethnic identities and peer effects of identities on the resilience of conforming to social norms/social context of the mainstream. The literature covers the studies on cultural identities on students' performance across different ethnicity in school(Akerlof and Kranton (2002)); and the peer effects on job seekers' job finding outcome in labor market(Battu et al. (2007)). However, in the process of cultural adaption, individuals face other identity-free difficulties for individuals in order to improve their status in social context. For example, it is hard for a good number of nonnative English speaking scholars to write language satisfying papers, even after they spend many years living in English speaking countries.

Another strand of theoretical literature with premises that family is the main reason of resilience of mainstream culture, as parents choose their children's culture traits with imperfect empathy (Bisin and Verdier (2001) and Bisin and Verdier (2011)). Some cultural related preference or social norms fit into their context are related to highly family-dependent life practices, such as fertility practices. Moreover, the setting that parents' choice of time spending with children is key to determine children's cultural traits does not taking into account the mutual interest of both parents and children to spend time together. In other words, children might not want to spend time with their parents. This chapter is more focused on the public life of individuals, and is to uncover how social coordination as the nature of society will affect individuals' choice of the degree of conforming to mainstream public life.

The existence of culture dependent but not cultural identity dependent factors on coordination outcomes are observed in experimental and labor economic studies on intercultural interaction; and they are not integrated into theoretical analysis of microeconomics. Experimental study conducted by Jackson and Xing (2014) observes payoff differences in cross-cultural coordination without revealing the cultural identities of the players: participants with American and Indian cultural backgrounds are asked to play a revised coordination game, and they do not know the cultural background of the one they play against with. Their results show that the expected payoff of individuals is higher if the one they play against with is from the same culture.

For empirical studies, seminal study by Chiswick (1978) shows that there are earning gaps between immigrants and native born individuals; however, the gaps are closed, and immigrants' earnings surpass the natives' after immigrants' several years of living experience in the host countries. Further, a number of labor economic studies emphasize the positive effects of language proficiency on immigrants' earnings assimilation.[8] There are ambiguities of the effects on income is culture-dependent or culture identity dependent. In other words, it is so hard to say whether the first generation immigrants at least partially change their cultural identities or not in their cultural adaption. However, we can conclude that the ability of the first-generation immigrants to fit in the society have improved.

The theoretical economic literature on behavior assimilation from a coordination perspective, which is identity-free, is of two strands. One strand of literature studies the evolution of social convention, and the other strand of literature studies network effects on the convergence of consensus. These studies do not apply to the culture-related issues of social integration. Seminal theoretical study by Young (1993) analyzes the evolution of social conventions by studying a coordination game played by many matched pairs of agents randomly drawn from a population. Their behavior is self-enforcing due to utility maximization.[9] He shows that the social convention which is one equilibrium of many can be achieved dynamically. However, modern social interaction and coordination are vastly connected with belief, attitude and behavior in a cultural aspect, such that individuals or players' cost is not only the loss of no cooperation, but also the cost of learning. Our study takes the outcomes of evolution of social conventions as given and focuses on individuals' cost of learning associated with stress

[8] Weiermair (1976), Kossoudji (1988), Dustmann (1994), Dustmann (1997), Chiswick and Miller (1995), Beenstock (1996), Chiswick (1998), Gonzalez (2000), Hayfron (2001), Berman et al. (2003) etc.

[9] See also Young (1996), Crawford and Haller (1990), and Crawford (1995).

and difficulties. Golub and Jackson (2012) show the network effects of homophily on the convergence of consensus: it takes more time to convergent to a consensus in a network with higher densities of links within groups than those among groups. We show that the assimilation of behaviors within the whole society in a cultural aspect is hard to achieve due to the cost in learning even without network effects.

The remainder of this chapter is organized as follows. Section 2.3 introduces the model. Section 2.4 shows payoff dominant equilibria with different marginal cost functions. Section 2.5 analyzes the policy and social welfare. Section 2.6 concludes.

2.3 The Model

Consider a set of agents N in a society. For agent $i \in N$, she/he is born with a type $\theta_i \in \Theta$ where $\Theta = \{\theta \mid 0 \leq \theta \leq 1\}$ is the cultural knowledge spectrum of the society; and $[0, \theta_i]$ is the endowed cultural knowledge spectrum of agent i. An agent with born type 1 is fully adaptive to the cultural aspect of society; and an agent with type 0 is born as "blank slates", or she/he is totally foreign to the society.[10] θ_i is independently and identically distributed on Θ. The cumulative population distribution function w.r.t agents' born types is continuous and defined as $F(\theta) := \int_0^\theta f(x)dx$ in which $f : \Theta \to \mathbb{R}_+$ is the population density distribution function. Cumulative distribution function $F(\theta)$ is common knowledge in the game.

Agent i's action set is $A(\theta_i) = \{a_i \mid \theta_i \leq a_i \leq 1\}$. Agent i expands her/his cultural knowledge spectrum from $[0, \theta_i]$ to $[0, a_i]$, if $a_i > \theta_i$; otherwise, agent i stays with her/his born cultural spectrum. In the game, agent i meets a fixed number of other agents who are independently drawn from F. The utilities agent i derives, when she/he meets agent j, are the length of the cultural knowledge spectrum they have in common, which is $\min(a_i, a_j)$. As shown in Figure 2.1, even though agent i and agent j have willingness to improve their interaction and coordination, their utilities are bounded by agent i's ability to make agent j understand or to understand agent j (Dustmann (1994)).

The cost to choose a_i is

$$C(a_i, \theta_i) = c(a_i) - c(\theta_i),$$

[10] Locke (1690) describes the mind at birth as blank slates.

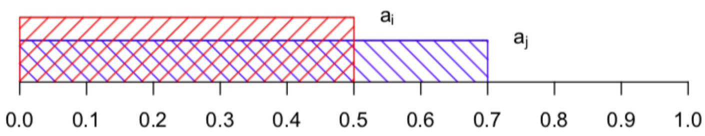

Figure 2.1 Cultural Knowledge Spectrum

in which $c : [0, 1] \to \mathbb{R}_+$ and $c(\cdot)$ is strictly increasing and differentiable. Consider the marginal cost function such that $\partial C(a_i, \theta_i)/\partial a_i = dc(a_i)/da_i$. Thus, the marginal cost for each agent is independent from her/his born type. We rewrite it as $mc(a) = dc(a)/da$. The independence of marginal cost from agents' born types implies that the learning opportunities are equal for all agents in acquiring cultural knowledge on a certain interval of the spectrum.[11]

As mentioned in the introduction, we study three types of marginal cost function; then, we use $mc^i(\cdot)$, $mc^d(\cdot)$ and $mc^u(\cdot)$ to denote strictly increasing, strictly decreasing and U-shaped marginal cost functions respectively. For $mc^u(\cdot)$ formally, there exists a $\theta^* \in (0, 1)$ such that $mc^u(\cdot)$ strictly decreases at interval $[0, \theta^*)$ and strictly increases at interval $(\theta^*, 1]$.

We normalize the number of agents with whom agent i meets. Given a profile of actions of agents excluding i denoted as $a_{-i} \in \times_{j \in N \setminus \{i\}} A(\theta_j)$, the utilities agent i deriving by choosing a_i are the difference between expected payoff of interacting with other agents $E_{a_{-i}}(\min\{a_i, a_{-i}\})$ and cost $C(a_i, \theta_i)$. Agent i's utility function is:

$$u(a_i, \theta_i, a_{-i}) := E_{a_{-i}}(\min\{a_i, a_{-i}\}) - C(a_i, \theta_i). \tag{2.1}$$

We focus on pure strategy symmetric Bayesian equilibrium. Then, the strategy could be present as $s : \Theta \to A(\Theta)$ such that s is a mapping from agent i's born type θ_i to i's action set $A(\theta_i) = \{s_i \mid \theta_i \leq s_i \leq 1\}$. Therefore, we can rewrite eq. (2.1) as:

$$u(s_i, \theta_i, s, f) = \int_0^1 \min(s_i, s(\theta)) f(\theta)\, d\theta - C(s_i, \theta_i). \tag{2.2}$$

We use u_i to denote user i's utility function interchangeably with $u(\cdot, \theta_i, s, f)$ in the following discussion.

[11] In Chapter 3, we consider another type of marginal cost function, for which agents' marginal cost to learn knowledge on a certain interval increases with the distances between the interval and the agents' born types.

The symmetric Bayesian equilibrium results in multiple solutions in many cases of the acculturation game. We focus on *payoff dominant equilibrium*, which has a *unique* solution for the game. The general statement and full characterization of the *payoff dominant equilibrium* is given by Proposition 2.5 at Section 2.4.4. We will walk readers through: from the examples and characterizations of three cases with $mc^i(\cdot)$, $mc^d(\cdot)$ and $mc^u(\cdot)$ to the characterization of general cases.

The definition of *payoff dominate equilibrium* is as follows:

Definition 2.1. A strategy s^* comprises a *payoff dominant equilibrium*, if s^* is Pareto superior to all other symmetric equilibria. That is, s^* is a *payoff dominant equilibrium*, if the following conditions hold:

(i) $s^* \in S$ in which $S = \left\{ s \in A(\Theta)^{\Theta} \mid \forall i,\ s(\theta_i) = \arg\max_{s_i \in A(\theta_i)} u(s_i, \theta_i, s, f) \right\}$;

(ii) $\forall s \in S \backslash \{s^*\}$ and $\forall \theta_i$, $u(\theta_i, s^*, f) \geq u(\theta_i, s, f)$, and $\exists \theta_i$ such that $u(\theta_i, s^*, f) > u(\theta_i, s, f)$.

Set S contains all symmetric equilibria. Every agent's expected utilities in equilibrium s^* are at least as much as hers/his in the other symmetric equilibria, and at least one agent is strictly better off in equilibrium s^*.

The reasons why we should expect the payoff dominant equilibrium in the acculturation game is as follows. As we mentioned in the literature review, immigrants, as the agents with low levels of cultural knowledge, are more risk-loving than the native born residents and have high willingness to improve their living condition. Therefore, they would prefer to enforce the payoff dominant equilibrium.

The following Example 2.1 shows the multiplicities of symmetric equilibria and the uniqueness of payoff dominant equilibrium of the acculturation game with a strictly increasing marginal cost function.

Example 2.1 Consider the game with uniform distribution $f(\theta) = 1$ and strictly increasing marginal cost function $mc^i(\theta) = 2\theta$ such that $C(s_i, \theta_i) = s_i^2 - \theta_i^2$. In symmetric equilibria:

$$s(\theta) = \begin{cases} \overline{\theta} & 0 \leq \theta < \overline{\theta} \\ \theta & \overline{\theta} \leq \theta \leq 1 \end{cases}, \tag{2.3}$$

in which $\overline{\theta}$ can be any real number in $[1/3, 1/2]$. The left graph of Figure 2.2 shows the equilibrium strategy with $\overline{\theta} = 1/3,\ 2/5$, and $1/2$ respectively, such that agents born in $\left[0, \overline{\theta}\right]$ expand to $\overline{\theta}$, and agents born in $\left[\overline{\theta}, 1\right]$ stay with their born types.

The symmetric equilibrium s in eq. (2.3) with $\overline{\theta} = 1/3$ is derived from the assuming that no agents will expand in equilibrium $s(\theta) = \theta$. The agent with born type 0 has incentives to expand to $1/3$ with utility function $s_i - 3s_i^2/2$ given $s(\theta) = \theta$. Agent 0's expansion to level $1/3$ increases the marginal utilities of agents with born types in $[0, 1/3]$ to expand to level $1/3$. Therefore, we have the equilibrium s with $\overline{\theta} = 1/3$.

The symmetric equilibrium s in eq. (2.3) with $\overline{\theta} = 1/2$ is derived from the assumption that every agent chooses as least as high as any other agent in the society. Agent with born type 0 expands to $1/2$ with utility function $s_i - s_i^2$ under the assumption. So does any other agent in the society. As $1/2$ is not in the action sets of agents with born types higher than $1/2$, we have the symmetric equilibrium s with $\overline{\theta} = 1/2$.

Therefore, it is not hard to get that s with any $\overline{\theta} \in [1/3, 1/2]$ is a symmetric equilibrium: any equilibrium with a $\overline{\theta} \in (1/3, 1/2)$ leads to that for all i, her/his marginal utilities are strictly larger than 0 on $\left[\theta_i, \overline{\theta}\right]$, and are strictly less than 0 on $\left[\overline{\theta}, 1\right]$.

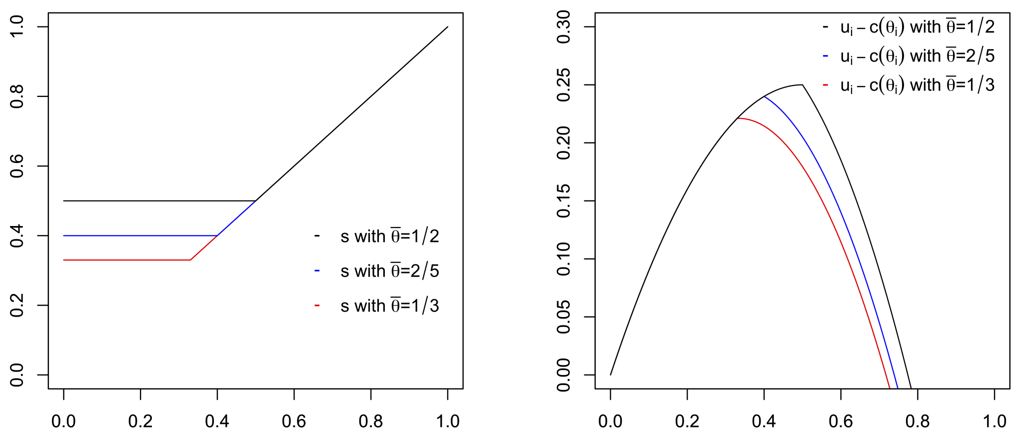

Figure 2.2 Equilibrium strategy eq. (2.3) and auxiliary utility functions eq. (2.4).

The payoff dominant equilibrium of Example 2.1 is s^* with $\overline{\theta}^* = 1/2$. Given a equilibrium strategy $s(\cdot)$ in eq. (2.3), the shape of utility function, which shows the relationship between each agents' utilities and the strategy she/he chooses, is the same for any two agents on the action set they have in common. Therefore, for any agent

i, we only need to analyze the universal auxiliary utility function $u(\cdot, \theta_i, s, f) - \theta_i^2$ but on i's action set $[\theta_i, 1]$ to find the utility maximization point. The auxiliary utility function is present as:

$$u(s_i, \theta_i, s, f) - \theta_i^2 = \begin{cases} s_i - s_i^2 & \theta_i \le s_i < \overline{\theta} \\ \overline{\theta}^2/2 + s_i - 3s_i^2/2 & \max\left\{\theta_i, \overline{\theta}\right\} \le s_i \le 1 \end{cases} . \qquad (2.4)$$

The graph on the right of Figure 2.2 shows $u - \theta_i^2$ with $\overline{\theta} = 1/3$, $2/5$, and $1/2$.

Every agent's utilities increase with $\overline{\theta}$. Therefore, we have the payoff dominant equilibrium s^* with $\overline{\theta} = 1/2$.

2.4 Shapes of Equilibria

Before we analyze the equilibrium strategy for marginal cost function $mc^i(\cdot)$, $mc^d(\cdot)$ and $mc^u(\cdot)$ in Section 2.4.1, Section 2.4.2 and Section 2.4.3, and show the full characterization of payoff dominant equilibrium with general costs in Section 2.4.4, we show some general properties of the payoff dominant equilibrium strategy.

The payoff dominant equilibrium strategy in Example 2.1 is non-decreasing and displays the conformity of expansion level for agents at certain intervals. Proposition 2.1 shows that the two features hold in general.

Proposition 2.1. *In the acculturation game, payoff dominant equilibrium $s^*(\cdot)$ is a nondecreasing function; and if there exist θ_i, a_i such that $0 \le \theta_i < a_i \le 1$ and $s^*(\theta_i) = a_i$, then for all $\theta \in [\theta_i, a_i]$, $s^*(\theta) = a_i$.*

If an agent θ_i expands to a_i in payoff dominant equilibrium, then the agents with born types in $[\theta_i, a_i]$ expand to a_i. The proof is as follows.

Proof. We prove the conformity of users' behavior firstly, which is the second part of proposition. Given a strategy $s^*(\cdot)$ played by other agents, the marginal utilities from the term $\int_0^1 \min(s_i, s^*(\theta))\, dF(\theta)$ is the same for agent i and agent j. Combined with that the marginal cost is also the same for them, the marginal utilities are the same for any two agents $\theta_i < \theta_j$ of expanding to any level $a \in (\theta_j, 1]$. Then, agent θ_i and agent θ_j reach their local utility maximization points at interval $(\theta_j, 1]$ at the same time. Therefore, if agent θ_i expands to a level $a_i > \theta_j$ in equilibrium, then agent θ_j chooses the same expansion level by the definition of payoff dominant equilibrium. Suppose agent θ_i gains the same amount of utilities from two different utility maximization points a and a' at $[\theta_j, 1]$ such that $a < a'$. Then, both agent θ_i and θ_j choose a' in payoff dominant equilibria. This is due to that u_i is a continuous and nondecreasing function of $s_{-i} \in \times_{j \in N \setminus \{i\}} A(\theta_j)$ and the utility function of agent with born type 1 is strictly increasing with s_{-i}; and that s_i is also a non-decreasing function of s_{-i}. Considering that θ_j could be any real number at interval $[\theta_i, a_i]$.

For the nondecreasing of equilibrium strategy: it is possible that agent θ_i expands to a level lower than or equal to θ_j, or stay with born type θ_i, if the total cost of expanding to $a > \theta_j$ is too high to be covered by the increment of interacting utilities. Therefore, we have Proposition 2.1. Q.E.D.

2.4.1 Strictly Increasing Marginal Cost Functions

Consider the game with a strictly increasing marginal cost function $mc^i(\cdot)$. In payoff dominant equilibrium, the agents born with low levels of cultural knowledge expand while the agents with high levels do not.

As shown in Example 2.1, the payoff dominant equilibrium with $mc^i(\cdot)$ is derived with the assumption that every agent at least chooses an action as high as any user. Therefore, if there exists an $a \in [0, 1]$ such that $mc^i(a) = 1$ and we have $mc^i(\cdot)$'s inverse function $mc^{i^{-1}} : [mc^i(0), 1] \to [0, 1]$, then agent $\theta_i \in \left[0, mc^{i^{-1}}(1)\right]$ expands to $mc^{i^{-1}}(1)$, while agent $\theta_i \in \left(mc^{i^{-1}}(1), 1\right]$ stays with her/his born type. Combine with two other cases :(i) if $mc^i(1) \leq 1$, all agents expand to 1; or (ii) if $mc^i(0) > 1$, all agents stays with their born types. We have:

Proposition 2.2. *If the marginal cost function is strictly increasing denoted as $mc^i(\cdot)$, then in payoff dominant equilibrium:*

(i) if $\exists a \in (0, 1]$ such that $mc^i(a) = 1$, then

$$s^*(\theta) = \begin{cases} mc^{i^{-1}}(1) & 0 \leq \theta < mc^{i^{-1}}(1) \\ \theta & mc^{i^{-1}}(1) \leq \theta \leq 1 \end{cases};$$

(ii) if $mc^i(1) \leq 1$, for all θ, $s^*(\theta) = 1$;

(iii) if $mc^i(0) > 1$, for all θ, $s^*(\theta) = \theta$.

For the details of proof, please see Section 2.7.1.

Agents acquire more but not all cultural knowledge, if the marginal cost of gaining new cultural knowledge at some level is higher than their marginal utilities of interacting. The expansion level $mc^{i^{-1}}(1)$ is independent of the distribution function $F(\cdot)$.

The first case with strictly increasing marginal cost functions applies to the socialization of children. They do not have any primitive cultural knowledge or any established cultural identity. Children interact with people to acquire their first languages and form social manners.

2.4.2 Strictly Decreasing Marginal Cost Functions

This section starts with Example 2.2, which shows the shape of equilibrium strategy with a strictly decreasing marginal cost function. Then, we analyze the equilibrium strategy with general $mc^d(\cdot)$.

Example 2.2 If the game is with density distribution function $f(\theta) = 2\theta$ and marginal cost function $mc^d(\theta) = 3 - 3\theta$, then in payoff dominant equilibrium:

$$s^*(\theta_i) = \begin{cases} \theta & 0 \le \theta < 1/2 \\ 1 & 1/2 \le \theta \le 1 \end{cases}. \tag{2.5}$$

The equilibrium strategy is present at the left graph of Figure 2.3. The agents with low levels of cultural knowledge stay with their born types in equilibrium. The intuition is that the cost is too high to be covered by the increment of interacting utilities from expansion.

Consider the game with a strictly decreasing marginal cost function $mc^d(\cdot)$. We use $\Theta(s^*(\cdot)) = \{\theta_i \in [0,1] \mid s^*(\theta_i) > \theta_i \text{ and } s^*(\theta_i) > \max_{\theta<\theta_i} s^*(\theta)\}$ to denote the set of agents, each of whom is the one with the smallest born type expands to a certain level. If $\Theta = \emptyset$, no agent expands in equilibrium; and if $\Theta \neq \emptyset$, there are some agents expanding in equilibrium.

In the game with a $mc^d(\cdot)$, agents either stay with their born types or expand to level 1 in equilibria, such that $|\Theta| \le 1$. If $\Theta \neq \emptyset$, we use $\underline{\theta}$ to denote the unique element in Θ, in which $\underline{\theta} = \min\left\{x \in [0,1) \mid (1-x)[1-F(x)] = \int_x^1 mc^d(\theta)\, d\theta\right\}$.

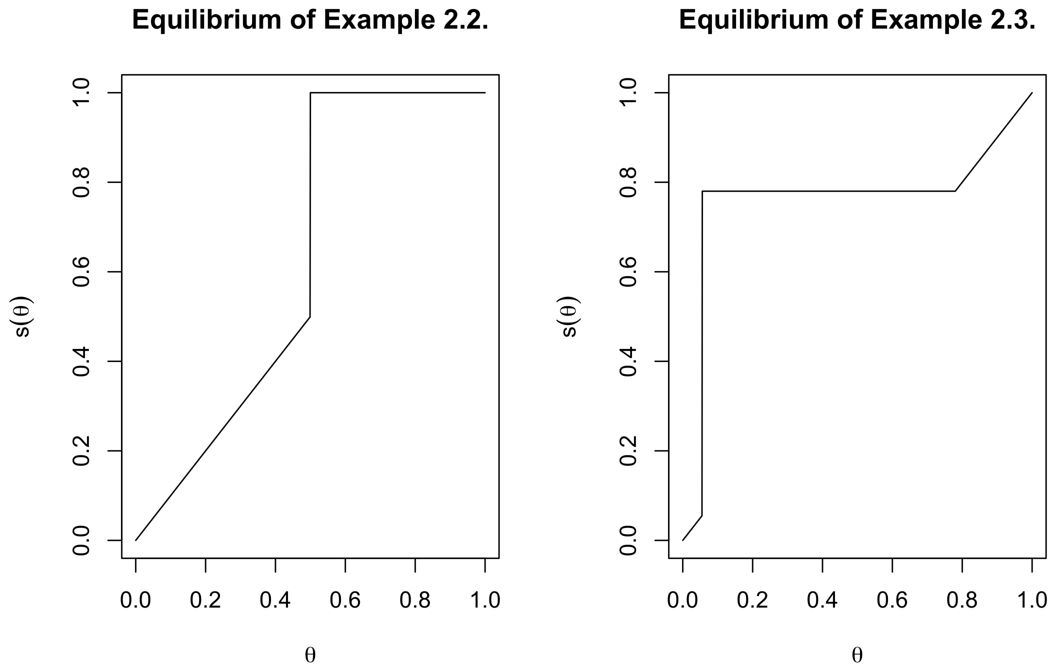

Figure 2.3 Equilibria of Example 2.2 and Example 2.3.

The smallest real number $\underline{\theta}$ solves equation $(1-\underline{\theta})\left[1-F\left(\underline{\theta}\right)\right]=\int_{\underline{\theta}}^{1} mc^{d}\left(\theta\right)d\theta$.[12] It means that: different from the agents with born types in $[0,\underline{\theta})$ who choose to stay with their born types, agent $\underline{\theta}$ switches to expand to level 1, since her/his total interacting utility increment of expanding to level 1 is equal to the total cost of that. Therefore:

Proposition 2.3. *If the marginal cost function is strictly decreasing denoted as $mc^{d}\left(\cdot\right)$, in payoff dominant equilibria:*

(i) if $\Theta=\emptyset$, then for all θ, $s^{*}\left(\theta\right)=\theta$;

(ii) if $\Theta\neq\emptyset$, then

$$s^{*}\left(\theta\right)=\begin{cases}\theta & 0\leq\theta<\underline{\theta}\\ 1 & \underline{\theta}\leq\theta\leq1\end{cases}.$$

Please see Section 2.7.1 for the details of the proof.

Agents with low levels of cultural knowledge stay with their born types, as the cost of expanding to level 1 is too high to be covered by the interacting utilities' increments from expansion.

[12]The equation could have other solutions, which are not the thresholds of payoff dominant equilibrium, and some of which are not the thresholds of symmetric equilibrium.

The acculturation game with strictly decreasing marginal cost functions applies to the acculturation of second-generation immigrant adolescents. Our results in equilibria explain the marginalization and separation of individuals from some minority groups in their acculturation process.

2.4.3 U-shaped Marginal Cost Functions

We start with Example 2.3, which shows the shape of equilibrium strategy with a U-shaped marginal cost function. General cases with U-shaped function are solved below the example.

Example 2.3 Consider $mc^u(\theta) = 3.25\theta^2 - 3.5\theta + 1.75$ and $f(\rho) = 2\theta$. In payoff dominant equilibrium, there are two thresholds $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} = 7/29 + 2\sqrt{1222}/377 \approx 0.056$ and $\overline{\theta} = 7/13 + 2\sqrt{2.5 - 13\underline{\theta}^2}/13 \approx 0.78$, then:

$$s^*(\theta) = \begin{cases} \theta & \theta < \underline{\theta} \\ \overline{\theta} & \underline{\theta} < \theta < \overline{\theta} \\ \theta & \theta \geq \overline{\theta} \end{cases}.$$

The equilibrium strategy is present at the right graph of Figure 2.3. The effects from U-shaped marginal cost result in that both agents with low and high born types do not expand, while the agents with middle born types expand.

In the equilibrium of the game with a U-shaped marginal cost function, there are two "thresholds" $\underline{\theta}, \overline{\theta} \in [0,1]$ such that $\underline{\theta} \leq \overline{\theta}$; the agents with born types lower than $\underline{\theta}$ or higher than $\overline{\theta}$ stay with their born types, while the agents with born types in between $\underline{\theta}$ and $\overline{\theta}$ expand to $\overline{\theta}$. In the following paragraphs, we further characterize the equilibrium strategy with U-shaped marginal cost functions.

Consider the game with a U-shaped marginal cost function.

We reorganize the utility eq. (2.2) as follows:

$$u(s_i^*, s, \theta_i) = \underbrace{\overbrace{s_i^*[1 - F(\theta_i)]}^{\text{the least motivation to expand}} + \int_0^{\theta_i} s^*(\theta)\, dF(\theta)}_{\text{utilities of interacting}} - C(s_i^*, \theta_i). \tag{2.6}$$

Non-decreasing of equilibrium strategy in Proposition 2.1 implies that no matter what agent θ_i chooses in equilibrium, all agents with born types higher than θ_i choose the same or higher levels. Therefore, $1 - F(\theta_i)$, the proportion of agents with born types higher than θ_i, is the least marginal interacting utilities of expanding for agent θ_i in payoff dominant equilibrium.[13] The second term $\int_0^{\theta_i} s^*(\theta)\, dF(\theta)$ has impacts on θ_i's expansion level in equilibrium, if there is an agent with born type less than θ_i expanding to a level higher than θ_i.

We define its inverse function as such: if $mc^u(\theta^*) \leq 1$, the inverse function of $mc^u(\cdot)$ is defined as $mc^{u^{-1}}(\cdot) : [mc^u(\theta^*), 1] \to [\theta^*, 1]$, in which if $mc^u(1) < 1$, for all $\theta \in [mc^u(1), 1]$, $mc^{u^{-1}}(\theta) = 1$. If $\theta_i \in \Theta$, then $s^*(\theta_i) = mc^{u^{-1}}(1 - F(\theta_i))$, where agent θ_i 's marginal utilities equal to 0.

If the marginal cost function is $mc^u(\cdot)$, there exists at most one element in Θ. If $\Theta \neq \emptyset$, let $\overline{x} = mc^{u-1}(1 - F(\underline{x}))$, then we have

$$\underline{\theta} = \min\left\{\underline{x} \in [0,1) \mid (\overline{x} - \underline{x})[1 - F(\underline{x})] = \int_{\underline{x}}^{\overline{x}} mc^u(\theta)\, d\theta\right\},$$

such that $\underline{\theta}$ is the unique element of Θ and $\overline{\theta} = mc^{u^{-1}}(1 - F(\underline{\theta}))$. The smallest real number $\underline{\theta}$ in $[0,1]$ solves equation $\left(\overline{\theta} - \underline{\theta}\right)(1 - F(\underline{\theta})) = \int_{\underline{\theta}}^{\overline{\theta}} mc^u(\theta)\, d\theta$, which means that agent $\underline{\theta}$ chooses to expand to $\underline{\theta}$ since her/his utility increments of expanding to $\overline{\theta}$ equal to her/his total cost of that.[14] The agents with born types lower than $\underline{\theta}$ and those with born types higher than $\overline{\theta}$ do not expand, due to no positive gains from expanding. Therefore:

Proposition 2.4. *If the marginal cost function is U-shaped denoted as $mc^u(\cdot)$, in payoff dominant equilibrium:*

(i) if $\Theta \neq \emptyset$, then

$$s^*(\theta) = \begin{cases} \theta & \theta < \underline{\theta} \\ \overline{\theta} & \underline{\theta} < \theta < \overline{\theta} \\ \theta & \theta \geq \overline{\theta} \end{cases};$$

(ii) if $\Theta = \emptyset$, for all θ, $s^*(\theta) = \theta$.

[13] Note that $1 - F(\theta_i)$ is not the least marginal interacting utilities of expanding for agent θ_i in symmetric equilibria.

[14] The equation could have other solutions, which are not the thresholds of payoff dominant equilibrium, and some of which are not the thresholds of symmetric equilibrium.

Please see Section 2.7.1 for details of the proof.

The game with U-shaped marginal cost functions applies to the acculturation of first-generation immigrants. They interact with individuals who have different established cultural identities at working environment. First-generation immigrants experience both acculturative stress and difficulties in order to acquire cultural knowledge of the mainstream.

2.4.4 General Costs

For any $mc(\cdot)$, agent i compares her/his utilities at local utility maximization levels and the boundary points $\{\theta_i, 1\}$. We use set $A^*(\theta_i)$ to collect all the possible choices of i, such that:

$$A^*(\theta_i) := \{a \in [\theta_i, 1] \mid mc(a) = 1 - F(\theta_i) \text{ or } a \in \{\theta_i, 1\}\}.$$

We use function $\Delta u(\theta_i)$ to represent the largest non-negative utility difference of i between choosing from $A^*(\theta_i)$ and staying with her/his born type. Then:

$$\Delta u(\theta_i) := \max_{a \in A^*(\theta_i)} u(a, s, \theta_i) - u(\theta_i, s, \theta_i) = \max_{a \in A^*(\theta_i)} [1 - F(\theta_i)](a - \theta_i) - C(a, \theta_i).$$

Furthermore, we define a function $s^{\max}(\theta_i) = \max \arg \Delta u(\theta_i)$ to pick the maximal level among all utility maximization levels of agent θ_i.

Combined with the cases where there exists an agent with born type smaller than θ_i choose a level higher than θ_i, we have the following proposition:

Proposition 2.5. *There exists a unique payoff dominant equilibrium and the equilibrium strategy is* $s^*(\theta_i) = \max_{\theta \in [0, \theta_i]} s^{\max}(\theta)$ *for all* θ_i *in* $[0, 1]$.

For the proof of the uniqueness, please see Section 2.7.1.

An equilibrium strategy is a piecewise function. The function has several jumping points. Between the jumping points, the function firstly stays constant and then increases; the increasing parts of the function overlap with the $45°$ line through the origin point 0. With Θ denoted as $\{\underline{\theta}_1, ..., \underline{\theta}_{|\Theta|}\}$ such that $\underline{\theta}_1 < ... < \underline{\theta}_{|\Theta|}$, the equilibrium strategy is represented as

$$s^{*}(\theta)=\begin{cases}\theta & 0\leq\theta<\underline{\theta}_{1}\\ s\left(\underline{\theta}_{1}\right) & \underline{\theta}_{1}\leq\theta<s\left(\underline{\theta}_{1}\right)\\ \ldots & \ldots\\ s\left(\underline{\theta}_{|\Theta|}\right) & \underline{\theta}_{|\Theta|}\leq\theta<s\left(\underline{\theta}_{|\Theta|}\right)\\ \theta & s\left(\underline{\theta}_{|\Theta|}\right)\leq\theta\leq 1\end{cases}.$$

2.5 Policy and Social Welfare

Consider an integration policy which sets a minimum level on the cultural knowledge spectrum for all agents to achieve. The action sets of agents whose born types are smaller than the policy level are constrained. In this section, we define payoff dominant equilibrium with policy constraints; then we analyze the equilibrium strategy and the optimal policy for the three cases. The characterizations of equilibrium strategy are present in lemmas, and the results of optimal policy are present in propositions and/or with diagrams.

2.5.1 Policy setting and General solutions

We use $\tau\in[0,1]$ to denote the minimum level knowledge for all agents to achieve. Agents' possible choice sets are constrained to be equal to or more than τ such that $A(\theta,\tau):=\{a_i\mid\max\{\tau,\theta\}\leq a_i\leq 1\}$.

Definition 2.2. A strategy $s^{*}(.,\tau)$ comprises a *τ-payoff dominant equilibrium*, if for a policy level τ, strategy $s^{*}(\theta_i,\tau)$ is Pareto superior to all other symmetric equilibrium with $A(,\tau)$. That is, $s^{*}(.,\tau)$ is a *τ-payoff dominant equilibrium* for a policy level τ, if for every θ_i displayed by any typical agent i, the following conditions hold:

(i) $s^{*}(,\tau)\in S(,\tau)$ such that:

$$S(.,\tau)=\left\{s(.,\tau)\in A(\Theta,\tau)^{\Theta}\mid\forall\theta_i,\ s(\theta_i,\tau)=\arg\max_{s_i\in A(\theta_i,\tau)}u\left(s_i,\theta_i,s(.,\tau),f\right)\right\};$$

(ii) $\forall s(.,\tau)\in S(.,\tau)\setminus\{s^{*}(,\tau)\}$ and $\forall\theta_i$, $u(\theta_i,s^{*}(.,\tau),f)\geq u(\theta_i,s(.,\tau),f)$, and $\exists\theta$ such that $u(\theta,s^{*}(.,\tau),f)>u(\theta,s(.,\tau),f)$.

Set $S(., \tau)$ contains all symmetric equilibria constrained by τ. Every agent's expected utilities in equilibrium $s^*(., \tau)$ are at least as much as hers/his in the other symmetric equilibria with constraint τ; and at least one agent is strictly better off in equilibrium $s^*(., \tau)$.

The agents endowed with smaller than τ knowledge levels compare choices from the constrained possible choice sets and the minimum knowledge level τ, such that:

$$A^*(\theta, \tau) := \{a \in [\max\{\tau, \theta\}, 1] \mid mc(a) = 1 - F(\theta) \ \ or \ a \in \{\tau, \theta, 1\}\}.$$

only the agents whose born types are larger than τ compare the choices from possible choice sets. Therefore, function $\Delta u(\theta, \tau)$ is defined as follows:

$$\Delta u(\theta, \tau) = \begin{cases} \max_{a \in A^*(\theta,\tau)} [1 - F(\theta)](a - \tau) - C(a, \tau) & \theta < \tau \\ \Delta u(\theta) & \theta \geq \tau \end{cases}.$$

Thus, we define a function $s^{\max}(\theta, \tau) = \max \arg \Delta u(\theta, \tau)$ to pick the largest choice for each agent from her/his utility maximization points.

Proposition 2.6. *Given a $\tau \in (0, 1]$, there exists a unique political payoff dominant equilibrium; and the equilibrium strategy is $s^*(\theta_i, \tau) = \max_{\theta \in [0, \theta_i]} s^{max}(\theta, \tau)$.*

The proof follows that of Proposition 2.5.

Consider there is a social planner of the society, who impose a policy level on all individuals before they choose their cultural knowledge expansion levels. The social welfare is $W(\tau, f) = \int_0^1 u(s^*(\theta, \tau), \theta, f) f(\theta) d\theta$. The social planner aims to choose an optimal policy level τ^* to maximize the social welfare such that:

$$\tau^* = \arg \max_{\tau \in [0,1]} W(\tau, f).$$

We consider population density distribution $f(\theta) = 2\rho\theta + 1 - \rho$ such that $\forall \rho \in [0, 1]$, $F(1) = \int_0^1 (2\rho\theta + 1 - \rho) d\theta \equiv 1$. If $\rho = 0$, $f(\theta)$ follows a uniform distribution; if $\rho = 1$, $f(\theta) = 2\theta$. The higher ρ is, the steeper the density distribution is.

In the following three sections, we show political payoff dominant equilibria and optimal policy levels of three cases of the game with strictly increasing, strict decreasing and U-shaped marginal cost functions.

2.5.2 Strictly Increasing Marginal Cost Functions

In this section, we show agents' equilibrium strategy in Lemma 2.1 with any $mc^i(\cdot)$. For the optimal policy, we study the game with a well-defined linearly increasing marginal cost function. The results in Proposition 2.7 show that the policy improves social welfare in all cases with imposing a higher expansion level than $\overline{\theta}$ in equilibrium without policy.

If $mc^i(\cdot)$ and $\tau \leq \overline{\theta}$, the strategy in political payoff dominant equilibria is the same as that in payoff dominant equilibria; if $\tau > \overline{\theta}$, the agents with born types lower than τ choose the policy level τ, while other agents stay with their born types. Therefore,

Lemma 2.1. *If the marginal cost function is $mc^i(\cdot)$, in political payoff dominant equilibrium:*

(i) if $\tau \leq \overline{\theta}$, $s^*(\theta, \tau) = s^*(\theta)$ for all $\theta \in [0,1]$;

(ii) if $\tau > \overline{\theta}$,

$$s^*(\theta, \tau) = \begin{cases} \tau & \theta \leq \tau \\ \theta & \theta > \tau \end{cases}.$$

We now further specify $mc^i(\theta) = 2\lambda_1\theta$ such that $\lambda_1 \geq 1/2$.[15] The higher λ_1 is, the higher the difficulties in learning is.

Proposition 2.7. *For all $\rho \in [0,1]$, $\lambda_1 \in (1/2, +\infty)$, there exists a unique optimal policy variable $\tau^* \in \left(\overline{\theta}, 1\right]$ such that*

$$\tau^* = \begin{cases} \frac{2}{1+2\lambda_1} & \rho = 0 \\ \frac{1}{2\rho}\left(\sqrt{(\rho - 2\lambda_1 - 1)^2 + 8\rho} - 2\lambda_1 + \rho - 1\right) & 0 < \rho < 1 \\ \sqrt{\lambda_1^2 + 2} - \lambda_1 & \rho = 1 \end{cases}$$

in which $\partial\tau^*/\partial\lambda_1 < 0$ and $\partial\tau^*/\partial\rho > 0$.

[15] If $\lambda_1 \leq 1/2$, $s^*(\theta) = 1$ for all θ, then no matter what τ is, it has no effect on the equilibria and social welfare.

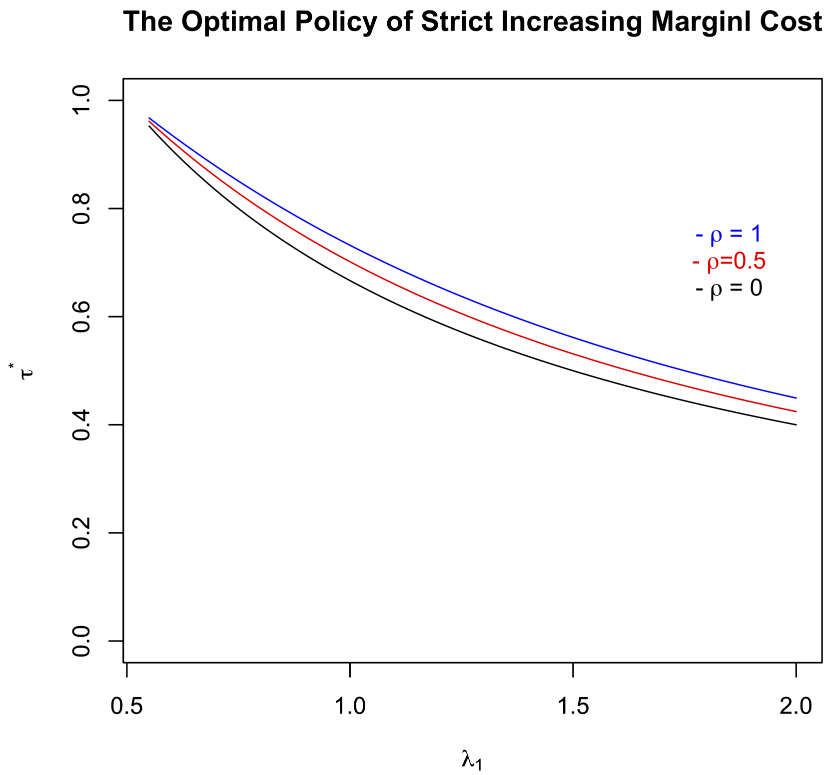

Figure 2.4 τ^* for $\rho = 0, 0.5, 1$ with $mc^i(\theta) = 2\lambda_1\theta$.

Figure 2.4 shows optimal level τ^* with $\rho = 0,\ 0.5$ and 1 respectively. As the difficulty level in learning λ_1 increases, τ^* decreases in all cases; τ^* increases with ρ, given a λ_1 holds.

The optimal levels are higher than the values of $\overline{\theta} = mc^{i^{-1}}(1)$ in all cases.

2.5.3 Decreasing Marginal Cost Functions

We study a well-defined linearly decreasing marginal cost function. Agents' equilibrium strategy are summarized at Lemma 2.3. For the optimal policy in Proposition 2.8 and Section 2.5.3, fully adapting policy improves social welfare in most cases; however, if the level of acculturative stress is too high, no policy should be imposed.

In political payoff dominant equilibrium with strictly decreasing marginal cost functions, either (i) all agents choose to learn all cultural knowledge; or (ii) agents whose born types are at $[0, \tau]$ choose τ and the other agents choose the same as they do in payoff dominant equilibrium.

We further assume the marginal cost function is: $mc^d(\theta) = 2\lambda_2 - 2\lambda_2\theta$. The higher λ_2 is, the higher the level of acculturative stress is.

We need to study the equilibria without policy for further understanding of equilibria with policy. When $\lambda_2 - \rho < 1$, some agents expand; when $\lambda_2 - \rho \geq 1$, no agents expand.

Lemma 2.2. *If marginal cost function $mc^d(\theta) = 2\lambda_2 - 2\lambda_2\theta$, in payoff dominant equilibrium :*

(i) if $\lambda_2 - \rho < 1$, $s^*(\theta) = \begin{cases} \theta & 0 \leq \theta < \underline{\theta} \\ 1 & \underline{\theta} \leq \theta \leq 1 \end{cases}$ in which

$$\underline{\theta} = -\frac{1}{2\rho}\sqrt{(\rho + \lambda_2 - 1)^2 + 4\rho(\rho + \lambda_2 - 1)} + \frac{\lambda_2}{2\rho} + \frac{1}{2} - \frac{1}{2\rho};$$

(ii) if $\lambda_2 - \rho \geq 1$, for all ρ, there is no adaptation in payoff dominant equilibrium.

Consider the equilibria with policy. If $\tau < 1 - 1/\lambda_2$, agents with born types lower than τ expand to τ, while agents with other born types expand to the same level as they do in equilibria without policy; if $\tau \geq 1 - 1/\lambda_2$, then all agents expand to 1. Combined with the two different cases shown at Lemma 2.2, we have political payoff dominant equilibria as follows:

Lemma 2.3. *If the marginal cost function is $mc^d(\theta) = 2\lambda_2 - 2\lambda_2\theta$, then:*

(i) if $\lambda_2 - \rho < 1$, in political payoff dominant equilibrium:

(a) if $\tau < 1 - 1/\lambda_2$, $s^*(\theta, \tau) = \begin{cases} \tau & \theta \leq \tau \\ \theta & \tau < \theta \leq \underline{\theta} \\ 1 & \underline{\theta} < \theta \leq 1 \end{cases}$;

(b) if $1 - 1/\lambda_2 \leq \tau \leq 1$, $s^*(\theta, \tau) = 1$.

(ii) if $\lambda_2 - \rho > 1$, $\Theta = \emptyset$; and in political payoff dominant equilibrium:

(a) if $\tau < 1 - 1/\lambda_2$, $s^*(\theta, \tau) = \begin{cases} \tau & \theta \leq \tau \\ \theta & \tau < \theta \leq 1 \end{cases}$;

(b) if $1 - 1/\lambda_2 \leq \tau \leq 1$, $s^*(\theta, \tau) = 1$;

That $\underline{\theta} > 1 - 1/\lambda_2$ is checked in the proof of Lemma 2.3 at Section 2.7.1.

The results imply that with policy level $1 - 1/\lambda_2$ which is less than 1, all the agents fully adapt to the society.

We have the following proposition characterize τ^*:

Proposition 2.8. *For all $\rho \in [0, 1]$, if $\lambda_2 - \rho \leq 1$, τ^* is any level at interval $[1 - 1/\lambda_2, 1]$; if $\lambda_2 - \rho > +1$:*

(i) if $\lambda_2 \leq (20 + 5\rho + \rho^2) / (10 - 5\rho)$, τ^* is any level in $[1 - 1/\lambda_2, 1]$;

(ii) if $\lambda_2 > (20 + 5\rho + \rho^2) / (10 - 5\rho)$, $\tau^* = 0$.

For the details of calculation, please see Section 2.7.1.

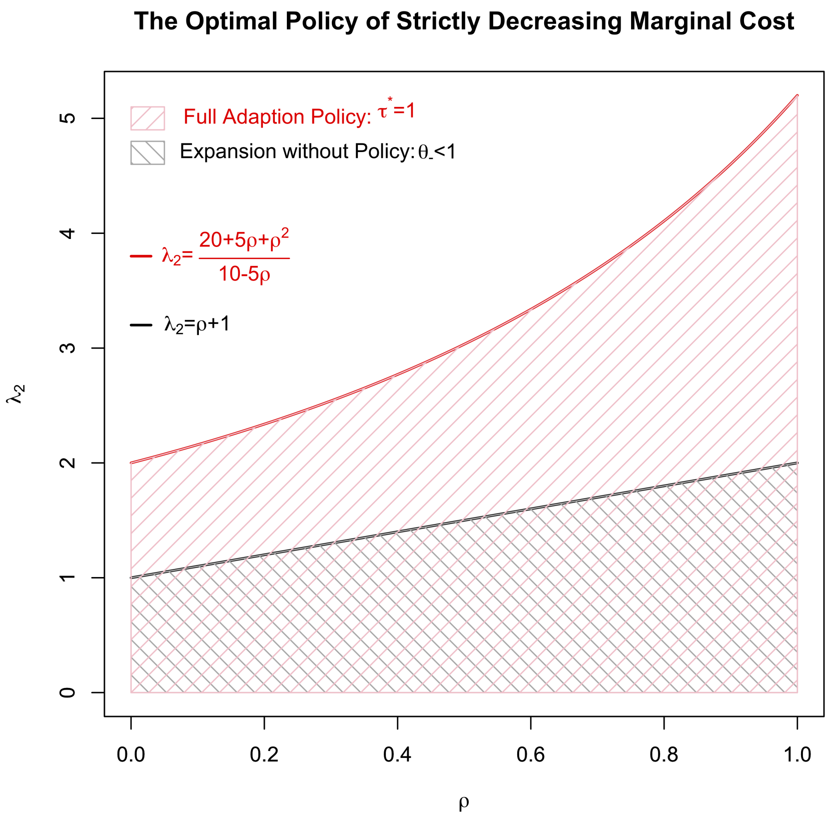

Figure 2.5 Phase Diagram with $mc^d(\theta) = 2\lambda_2 - 2\lambda_2\theta$.

Section 2.5.3 summarizes Lemma 2.3 and Proposition 2.8 by showing equilibrium without policy and optimal policies as λ_2 and ρ range. In the grey area, some agents acquire all cultural knowledge in equilibrium without policy. In the red area, the optimal policy level τ^* is at level 1. The grey area is a sub area of the red area. Therefore, if there exist some agents expanding their knowledge spectrum without policy, the optimal policy $\tau^* = 1$ should be used.

2.5.4 U-shaped Marginal Cost Functions

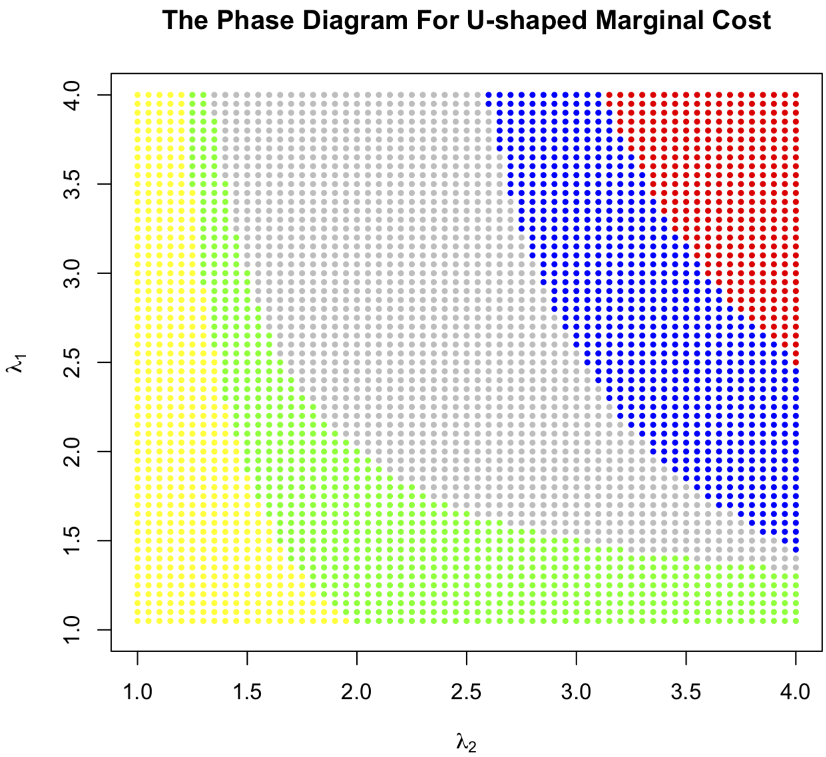

Figure 2.6 Phase Diagram with U-shape Marginal Cost eq. (2.7).

Consider a U-shaped marginal cost function defined as

$$mc^u(\theta) = \lambda_1 \theta^2 + \lambda_2 (1-\theta)^2, \tag{2.7}$$

such that $1 < \lambda_1, \lambda_2 \leq 4$; λ_1 denotes the difficulty levels, and λ_2 denotes the stress levels in learning. The results of optimal policy levels are present in Figure 2.6 and Figure 2.7; and please see the explanation of the two Figures in the next two paragraph.[16]

The social welfare maximization problem can not be solved analytically. We show the solutions of optimal policies in Figure 2.6 and Figure 2.7. Figure 2.6 presents

[16]For the political payoff dominant equilibrium, it has four types of strategies depending the values of λ_1 and λ_2, the existence of Θ, and the value of $\underline{\theta}$ if Θ is not empty. Please see Section 2.7.2 for further analysis of equilibrium strategy.

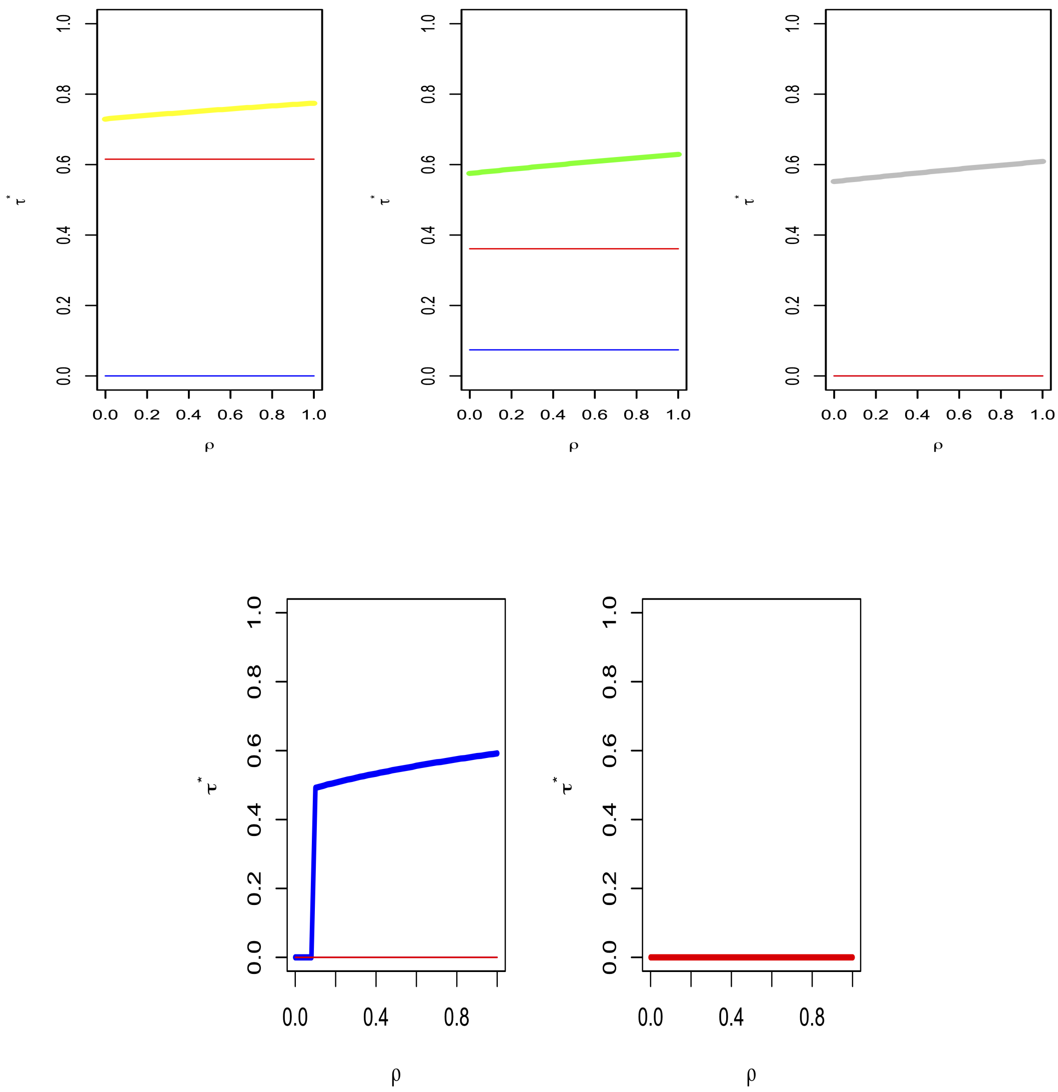

Figure 2.7 Phase Diagram with U-shape Marginal Cost eq. (2.7).

five areas and each area presents a type of policy changing with ρ. Figure 2.7 shows a typical optimal policy for each area as ρ increases. Colors of the thick lines in Figure 2.7 correspond to the colors of areas in Figure 2.6. For the grey, green and yellow areas, a policy level should be set whatever ρ is. In the yellow area, $\underline{\theta} = 0$ such that agents with born type 0 expand without policy; in the green area, $\underline{\theta} > 0$ or $\Theta = \emptyset$; in the grey area, $\min mc^u(\theta) > 1$ such that $\Theta = \emptyset$ and no agent expands without policy. In the blue area, the policy level should be set if ρ is above some threshold. In the red area, no policy should be set whatever ρ is. Both blue and red area have no expanding agents without policy. The blue thin lines in Figure 2.7 are θ_t and the red line is $mc^{u^{-1}}(1)$.[17]

We can see from Figure 2.6 that at a given λ_1 there exist at least two phases as λ_2 increases and at most five phases as λ_2 increases; at a given λ_2 there exists at least one phase as λ_2 increases and at most four phases as λ_1 increases. In general, the effectiveness of the policy is impeded more by λ_2, the indicator of levels of acculturative stress, than λ_1, the indicator of difficulties in learning.

2.6 Conclusion

We use a game-theoretical approach to describe agents' cultural knowledge learning behavior in acculturation and regard cultural tests as a type of policy improving social welfare by setting minimum levels of cultural knowledge of the mainstream for all agents. We model the culture knowledge on the interval $[0, 1]$. Agents are endowed with certain levels of cultural knowledge, and they acquire the cultural knowledge to coordinate with other agents. We show the uniqueness and full characterization of payoff dominant equilibria and those of political payoff dominant equilibria. In particular, we study equilibria and optimal policies of three cases of the game with different types of marginal cost functions.

The first case of acculturation game considers strictly increasing marginal cost functions. It applies to the socialization of children. Children acquire their first languages through interaction without pressure but with difficulties in learning. In equilibria, agents acquire more but not all cultural knowledge if the marginal cost of gaining new cultural knowledge is higher than the marginal utilities of interacting.

The second case of the game considers strictly decreasing marginal cost functions. It applies to the acculturation of second-generation immigrant adolescents. Their cultural parts of upbringing are different from the dominant culture. They experience acculturative stress from interacting with their classmates and playmates with the dominant cultural background. In equilibria, the agents endowed with low levels of knowledge are marginalized, in the sense that they do not gain any more cultural knowledge to improve their abilities to coordinate. In empirical studies, segmented assimilations are observed. However, our model can not capture factors, such as the sizes of minority

[17] θ_t is the transition point of τ for all agents expanding to $mc^{u^{-1}}(1)$. For more detail, please see Section 2.7.2.

groups which have an impact on the outcomes of acculturation at this stage.

The third case of acculturation game with U-shape marginal cost functions applies to the acculturation of first-generation immigrants. They experience acculturative stress and difficulties in learning. In equilibria, the agents endowed with middle levels of cultural knowledge choose to acquire cultural knowledge but not all, and the agents endowed with low and high levels of cultural knowledge do not acquire any cultural knowledge. The studies of labor economics on immigrants' earnings assimilation observe that immigrants' earnings surpass the earnings of native-born individuals. They suggest that there are self-selective effects of immigrants: immigrants have higher willingness to improve their living conditions and are more risk-loving than native-born individuals. Our model cannot capture the heterogeneity of immigrants at this stage.

We have closed-form solutions of optimal policies for the first and second cases with well-defined linear functions. Optimal policies of the first case set minimum policy levels higher than the adaptation levels without policy; however, the optimal policy level decreases with the difficulties in learning. In the second case where some agents acquire all cultural knowledge without policy, optimal policies are less than one but induce all agents to acquire all cultural knowledge; however, if the stress level is too high, no policy should be implemented. For U-shaped marginal cost functions, we study quadratic marginal cost functions. In general, the effectiveness of minimum level policy is impeded more by high levels of acculturative stress than by high difficulties in learning.

The game-theoretical analysis of inter-cultural social coordination is still at an early stage. Our model can be extended in many ways including taking into account of the heterogeneity of agents, analyzing the dynamics of adapting etc. We also expect to test our model with data and estimate cost functions.

2.7 Appendix

2.7.1 Proofs

PROOF OF PROPOSITION 2.2. Consider the case where there exists an $a \in [0,1]$ such that $mc^i(a) = 1$. For agent with born type 0, $u(s_i, 0, s, f) = s_i - c(s_i)$ by the nondecreasing of strategy in payoff dominant equilibrium. The maximum of $u(s_i, 0, s, f)$ is at $s_i = mc^{i^{-1}}(1)$ by the first order condition. By Proposition 2.1, we have that agent $\theta_i \in \left[0, mc^{i^{-1}}(1)\right]$ reaches her/his utility maximization point at $mc^{i^{-1}}(1)$, while the utility maximization point of agent $\theta_i \in \left(mc^{i^{-1}}(1), 1\right]$ is her/his born type. It is the payoff dominant equilibrium, since $s_i - c(s_i)$ is also the upper bound of utility function given any strategy.

Consider the case where $mc^i(1) \leq 1$. For the agent with born type 0, her/his utility function $u(s_i, 0, s, f) = s_i - c(s_i)$ is strictly increasing on interval $[0, 1]$. Therefore, agent 0 expands to 1, and so do other agents.

Consider the case where $mc^i(0) > 1$. For the agent with born type 0, her/his utility function $u(s_i, 0, s, f) = s_i - c(s_i)$ is strictly decreasing on interval $[0, 1]$. Therefore, agent 0 do not expand, and other agents do not, neither. Q.E.D.

Proof of Proposition 2.3. In a game with $mc^d(\cdot)$, agent θ_i either (i) expands to 1 if $u(1, s^*, \theta_i) \geq u(\theta_i, s^*, \theta_i)$, or (ii) stays with their born types if $u(1, s^*, \theta_i) < u(\theta_i, s^*, \theta_i)$. To prove the statement above, consider agent $\theta_i = 0$, the marginal utility increases since the marginal utilities of interacting from expansion is 1 in all payoff dominant equilibria. Therefore, the utility maximization point is either 0 or 1. Now, suppose there exists an agent $\theta_j \in [0, 1)$ such that $s^*(\theta_j) = a_j$ and $a_j \in (\theta_j, 1)$. Then, the marginal utilities at a_j for agent θ_j is 0 and marginal utilities is negative at $[a_j, 1]$. So are for agents with born types smaller than θ_j, which includes agent $\theta_i = 0$. It contradicts the strictly increasing of agent θ_i's marginal utilities.

Then, $|\Theta| \leq 1$ in the game with strictly decreasing functions, since agents' choices are limited to 1 or their born types; if an agent expands to 1, the agents with born types higher than her/him expand to 1 by Proposition 2.1.

For $\underline{\theta}$, it solves equation $(1 - \underline{\theta})[1 - F(\underline{\theta})] = \int_{\underline{\theta}}^{1} mc^d(\theta)\, d\theta$ in the payoff dominant equilibrium, since it considers the largest set of agents expand. Q.E.D.

Proof of Proposition 2.4. We prove the uniqueness of Θ by contradiction. Suppose that there are two agents θ_i and θ_j such that $\theta_{i,j} \in \Theta$ and $\theta_i < \theta_j$; θ_i expands to $s^*(\theta_i) > \theta_i$ and θ_j expands to $s^*(\theta_j) > \theta_j$. By the least motivation to expand, we have $1 - F(\theta_{i,j}) - mc^u(s^*(\theta_{i,j})) = 0$ at $[\theta_i, 1]$. As $mc^{u^{-1}}(1 - F(\theta))$ is non-decreasing with respect to θ, we have:

$$s^*(\theta_i) = mc^{u^{-1}}(1 - F(\theta_i)) > mc^{u^{-1}}(1 - F(\theta_j)) = s^*(\theta_j).$$

It contradicts Proposition 2.1.

For $\underline{\theta}$ and $\overline{\theta}$, if $\Theta \neq \emptyset$, let $\overline{x} = mc^{u-1}(1 - F(\underline{x}))$, then:

$$\underline{\theta} = \min\left\{\underline{x} \in [0,1) \mid (\overline{x} - \underline{x})[1 - F(\underline{x})] = \int_{\underline{x}}^{\overline{x}} mc^u(\theta)\, d\theta\right\}.$$

The equation above considers the largest set of agents who expand.

Q.E.D.

PROOF OF PROPOSITION 2.5. Each agent's minimum choice is assigned by $s^{\max}(\cdot)$ in payoff dominant equilibrium. Therefore, $\max_{\theta\in[0,\theta_i]} s^{\max}(\theta)$ assigns the minimum choice to each agent in payoff dominant equilibrium.

Function $\max_{\theta\in[0,\theta_i]} s^{\max}(\theta)$ also assigns the maximal choices to each agent, since $s^{\max}(0)$ is the also maximal choice agent 0 plays in payoff dominant equilibrium and $\max_{\theta\in[0,\theta_i]} s^{\max}(\theta)$ preserves all the maximal choices of all agents with born types larger than 0. Since $s^{\max}(0)$ is maximal choice agent 0 plays in payoff dominant equilibrium, then for agent $\theta < s^{\max}(0)$, $s^*(\theta) = s^*(0)$ which assigns maximal choices to the agents; For agent $\theta > s^{\max}(0)$, check if $s^{\max}(\theta) > \theta$: (i) if it is the case, then agent $\theta' \in [\theta, s^{\max}(\theta)]$ such that $s^*\left(\theta'\right) = s^{\max}(\theta)$ which assigns maximal choices to the agents; then the same for $\theta'' > s^{\max}(\theta)$...; (ii) if it is not, then $s^*(\theta) = \theta$ which still assign maximal choices to the agents. Q.E.D.

PROOF OF PROPOSITION 2.7. Given $mc^i(\theta) = 2\lambda_1\theta$ and $\lambda_1 \geq 1/2$, the social welfare function for $\tau > \overline{\theta} = 1/(2\lambda_1)$:

$$\begin{aligned} W(\tau;\rho) = &\left[\frac{1}{3} + \frac{1}{6}\rho + \frac{1}{30}\rho^2 + \tau^2(1-\rho) + \tau^3\left(-\frac{1}{3} + \frac{4}{3}\rho - \frac{1}{3}\rho^2\right)\right] \\ &+ \left[\tau^4\left(-\frac{1}{2}\rho + \frac{1}{2}\rho^2\right) - \frac{1}{5}\rho^2\tau^5\right] - \lambda_1\left[\frac{1}{2}\rho\tau^4 + \frac{2}{3}(1-\rho)\tau^3\right]. \end{aligned}$$

The first order condition with respect to τ is:

$$2\tau(1-\rho) - \tau^2\left(1+2\lambda_1-4\rho-2\lambda_1\rho+\rho^2\right) + \tau^3\left(-2\lambda_1\rho-2\rho+2\rho^2\right) - \tau^4\rho^2 = 0. \tag{2.8}$$

When $\rho = 0$, eq. (2.8) results in:

$$\tau_{\rho=0} = \begin{cases} 0 \\ 2/(1+2\lambda_1) \end{cases}.$$

At $[0, 2/(1+2\lambda_1))$, $\partial W/\partial\tau > 0$, and at $(2/(1+2\lambda_1), 1]$, $\partial W/\partial\tau < 0$. Therefore, the optimal policy level is $\tau^*_{\rho=0} = 2/(1+2\lambda_1)$.

When $\rho = 1$, eq. (2.8) results in:

$$\tau^1_{\rho=1} = \begin{cases} 0 \\ \tau^2_{\rho=1} = \sqrt{\lambda_1^2+2} - \lambda_1 \\ \tau^3_{\rho=1} = -\sqrt{\lambda_1^2+2} - \lambda_1 \end{cases}.$$

We have $\partial W/\partial\tau > 0$ at interval $\left[0, \tau^2_{\rho=1}\right)$; and $\partial W/\partial\tau < 0$ at interval $\left(\tau^2_{\rho=1}, 1\right]$. Therefore, the optimal policy level is $\tau^*_{\rho=1} = \sqrt{\lambda_1^2+2} - \lambda_1$.

When $\rho \in (0,1)$, eq. (2.8) results in:

$$\tau_\rho = \begin{cases} (\rho-1)/\rho \\ -\sqrt{(-\rho+2\lambda_1+1)^2+8\rho}/(2\rho) + 1/2 - \lambda_1/\rho - 1/(2\rho) \\ \left(\sqrt{(-\rho+2\lambda_1+1)^2+8\rho} + \rho - 2\lambda_1 - 1\right)/(2\rho) \\ 0 \end{cases}.$$

We have $\partial W/\partial\tau > 0$ at interval $\left[0, \tau^3_\rho\right)$; and $\partial W/\partial\tau < 0$ at interval $\left(\tau^3_\rho, 1\right]$. The optimal policy is:

$$\tau^*_\rho = \frac{1}{2\rho}\left(\sqrt{(\rho-2\lambda_1-1)^2+8\rho} - 2\lambda_1 + \rho - 1\right).$$

By the continuity of function W, for all τ^*,

$$\frac{\partial \tau^*}{\partial \rho} = 2\lambda_1 + 1 - \frac{4\lambda_1^2 + 4\lambda_1 + 1 + 3\rho - 2\rho\lambda_1}{\sqrt{\left(-\rho + 2\lambda_1 + 1\right)^2 + 8\rho}}.$$

We prove the ratio of the third term to the sum of the first and second term is smaller than 1:

$$\frac{2\lambda_1 + 1 - \rho + \frac{2\rho\lambda_1}{2\lambda_1+1}}{\sqrt{\left(-\rho + 2\lambda_1 + 1\right)^2 + 8\rho}} < 1.$$

By $2\lambda_1 / \left(2\lambda_1 + 1\right) < 1 < 8$. Therefore, $\partial \tau^* / \partial \rho > 0$.

Now consider the derivatives of λ_1:

$$\frac{\partial \tau^*}{\partial \lambda_1} = -\frac{1}{\rho}\left(1 - \frac{2\lambda_1 + 1 - \rho}{\sqrt{\left(-\rho + 2\lambda_1 + 1\right)^2 + 8\rho}}\right);$$

we have $\partial \tau^* / \partial \lambda_1 < 0$ by $\left(2\lambda_1 + 1 - \rho\right) / (\sqrt{\left(-\rho + 2\lambda_1 + 1\right)^2 + 8\rho} < 1$. Q.E.D.

PROOF OF LEMMA 2.2. If $\max_\theta \Delta u\left(\rho, \lambda_2, \theta\right) > 0$, there exists a $\underline{\theta}$; Otherwise, there is no expansion behavior:

$$\Delta u\left(\rho, \lambda_2, \theta\right) = \left(\theta - 1\right)\left(\rho\theta^2 + \left(1 - \rho - \lambda_2\right)\theta + \lambda_2 - 1\right).$$

F.O.C. of $\Delta u\left(\rho, \lambda_2, \theta\right)$, we have:

$$3\rho\theta^2 + 2\left(1 - 2\rho - \lambda_2\right)\theta + 2\left(\lambda_2 - 1\right) + \rho = 0.$$

Solve the equation. The local extreme point are $\theta' = 1$ or $\theta'' = 2\left(\lambda_2 - 1\right) / \left(3\rho\right) + 1/3$.

If $\theta'' > 1$ such that $\lambda_2 - 1 > \rho$, $\Delta u\left(\rho, \lambda_2, \theta''\right) = 4\left(1 + \rho - \lambda_2\right)^3 / \left(27\rho^2\right) < 0$. No agent expand without policy.

If $\theta'' \leq 1$ such that $\lambda_2 - 1 \leq \rho$, $\Delta u\left(\rho, \lambda_2, \theta''\right) = 4\left(1 + \rho - \lambda_2\right)^3 / \left(27\rho^2\right) > 0$. There exist agents who expand without policy.

Now we find $\underline{\theta}$ with condition $\lambda_2 - 1 \leq \rho$.

When $\rho = 0$, the solution for $\Delta u\left(\rho, \lambda_2, \theta''\right) = 0$ is $\theta = 1$. There is no adoption when $\rho = 0$.

When $0 < \rho < 1$, the transition point satisfies $\Delta u = 0$ which results in:

$$\theta = \begin{cases} 1 \\ \frac{1}{2\rho}\sqrt{(\rho+\lambda_2-1)^2 - 4\rho(\lambda_2-1)} + \frac{\lambda_2}{2\rho} + \frac{1}{2} - \frac{1}{2\rho} \\ -\frac{1}{2\rho}\sqrt{(\rho+\lambda_2-1)^2 - 4\rho(\lambda_2-1)} + \frac{\lambda_2}{2\rho} + \frac{1}{2} - \frac{1}{2\rho} \end{cases}.$$

Therefore, we have $\underline{\theta}$ equals to the third expression above:

$$\underline{\theta} = -\frac{1}{2\rho}\sqrt{(\rho+\lambda_2-1)^2 - 4\rho(\lambda_2-1)} + \frac{\lambda_2}{2\rho} + \frac{1}{2} - \frac{1}{2\rho},$$

since the third term is smaller than the second term and it is larger than 0.

Q.E.D.

PROOF OF LEMMA 2.3. For the political payoff dominant equilibria, the agent chooses to expand the cultural knowledge spectrum to 1 if $\Delta u(s_i = \tau)$ is equal to $\Delta u(s_i = 1)$. Therefore,

$$\left[1 - \rho\theta^2 - (1-\rho)\theta\right](1-\tau) - \lambda_2 + \left(2\lambda_2\tau - \lambda_2\tau^2\right) = 0.$$

As the difference of cost between τ and 1 and the difference between marginal utilities of interacting from expanding between τ and 1 is the same for every agent, the agent with the lowest type 0 have the highest utilities of interacting from expanding for every τ. Therefore, we set $\theta = 0$. We have $\tau = 1 - 1/\lambda_2$. Q.E.D.

PROOF OF PROPOSITION 2.8. When $\lambda_2 - \rho > 1$ and $\tau < (\lambda_2 - 1)/\lambda_2$, the social welfare function is present as follows:

$$W(\rho,\tau) = \left[\frac{1}{3} + \frac{1}{6}\rho + \frac{1}{30}\rho^2 + \tau^2(1-\rho) + \tau^3\left(-\frac{1}{3} + \frac{4}{3}\rho - \frac{1}{3}\rho^2\right)\right] + \left[\tau^4\left(-\frac{1}{2}\rho + \frac{1}{2}\rho^2\right) - \frac{1}{5}\rho^2\tau^5\right] - \lambda_2\left[-\frac{1}{2}\rho\tau^4 - \frac{2}{3}(1-2\rho)\tau^3 + (1-\rho)\tau^2\right].$$

F.O.C of W w.r.t. τ is:

$$-\rho^2\tau\left(\tau - 1 + \frac{1}{\rho}\right)\left[\rho\tau^2 + (1-\rho-2\lambda_2)\tau + 2\lambda_2 - 2\right] = 0.$$

If $\rho = 0$,

$$\tau_{\rho=0} = \begin{cases} 0 \\ 2(\lambda_2 - 1)/(2\lambda_2 - 1) \end{cases}.$$

Then, social welfare W decreases in $[0, 2(\lambda_2 - 1)/(2\lambda_2 - 1)]$, and it increases in $[2(\lambda_2 - 1)/(2\lambda_2 - 1), 1]$. Then, the optimum level is either at $\tau = 0$ or at $\tau = 1$.

We have:

$$W(\rho, 1) - W(\rho, 0) = 1 - \frac{\lambda_2}{6}(2 - \rho) - \frac{1}{3} + \frac{\rho}{6} + \frac{\rho^2}{30}$$

We have:

(i) if $\lambda_2 \leq (20 + 5\rho + \rho^2)/(10 - 5\rho)$, $\tau^* = 1$;

(ii) if $\lambda_2 > (20 + 5\rho + \rho^2)/(10 - 5\rho)$, $\tau^* = 0$.

When $\rho \in (0, 1]$, we have

$$\tau_\rho = \begin{cases} 1 - 1/\rho \\ -\sqrt{(\rho - 2\lambda_2 + 1)^2 + 4\rho}/(2\rho) + 1/2 + \lambda_2/\rho - 1/(2\rho) \\ \sqrt{(\rho - 2\lambda_2 + 1)^2 + 4\rho}/(2\rho) + 1/2 + \lambda_2/\rho - 1/(2\rho). \end{cases}.$$

In $\left[0, \tau_\rho^2\right]$, W is decreasing; in $\left[\tau_\rho^2, 1\right]$, W is increasing. Then, the optimum level is either at $\tau = 0$ or at $\tau = 1$. The threshold to choose to level 1 or 0 is the same as the case where $\rho = 0$.

When $\lambda_2 - \rho < 1$, we assume the social welfare is W'. When $\tau < 1 - 1/\lambda_2$, W' is fixed. Therefore, the derivative of social welfare is the same as the case of $\lambda_2 - \rho > 0$. Therefore, the policy depends on $W(\rho, 1) - W(\rho, 0)$. As $\lambda_2 < \rho + 1 < 2 + \rho < (20 + 5\rho + \rho^2)/(10 - 5\rho)$. For all $\lambda_2 - \rho < 1$, $\tau^* > 1 - 1/\lambda_2$. Q.E.D.

2.7.2 Supplements for Section 2.5.4

There are four types of political payoff dominant equilibria with U-shaped marginal costs.

(i) if the cost is high enough, the policy drives only the agents whose born types are below the policy level to expand;

(ii) if agent 0 expands without policy, the policy plays the same role in determining the shape of equilibria for the case with strictly increasing marginal costs;

(iii) if there is no agent expanding her/his cultural knowledge spectrum and the cost is not high, then if the policy variable τ is either high ($\tau > mc^{u-1}(1)$) or low, the policy drives the agents whose born types are below the policy level to expand; if the policy level is at middle levels, the policy drives the agents to expand to $mc^{u-1}(1)$;

(iv) if the born types of the expanding agents with lowest born types are not 0, the policy variable plays the same role as it is in the third case. However, due to the expanding without policy, the presentation is different.

We start from the simplest case: if $\lambda_1\lambda_2 > \lambda_1 + \lambda_2$ such that $mc^u(\theta^*) > 1$, no agent expands in equilibrium without policy, and no agent chooses a level higher than τ unless their born types are higher than τ. Therefore, if the marginal cost function is $mc^u(\theta) = (\lambda_1 + \lambda_2)\theta^2 - 2\lambda_2\theta + \lambda_2$ and $\lambda_1\lambda_2 > \lambda_1 + \lambda_2$, in political payoff dominant equilibrium:

$$s^*(\theta,\tau) = \begin{cases} \tau & 0 \le \theta < \tau \\ \theta & \tau \le \theta \le 1 \end{cases}.$$

For the second case, if $\lambda_1\lambda_2 \le \lambda_1+\lambda_2$ and $\underline{\theta} = 0$, τ's impact on strategy of political payoff dominant equilibrium is the same as that of the game with strictly increasing marginal cost functions such that:

(i) if $\tau < {mc^u}^{-1}(1)$, then:

$$s^*(\theta,\tau) = \begin{cases} {mc^u}^{-1}(1) & 0 \le \theta < {mc^u}^{-1}(1) \\ \theta & {mc^u}^{-1}(1) \le \theta \le 1 \end{cases}.$$

(ii) if $\tau \ge {mc^u}^{-1}(1)$, then:

$$s^*(\theta,\tau) = \begin{cases} \tau & 0 \le \theta < \tau \\ \theta & \tau \le \theta \le 1 \end{cases}.$$

If $mc^u(\theta^*) \leq 1$, $\underline{\theta} > 0$ and $\tau \in (0, \lambda_2/(\lambda_1+\lambda_2)]$, agents' utilities derived from expanding to τ may equal to the utilities from choosing their local extremes at $[\theta^*, 1]$. For all $\tau \in (0, \lambda_2/(\lambda_1+\lambda_2)]$, $\Delta u(\theta, \tau)$ decreases with θ in $[0, \underline{\theta}]$.[18] It means that only $\Delta u(0, \tau)$ needs to be checked. If there exists a $\tau \in (0, \lambda_2/(\lambda_1+\lambda_2)]$ such that $\Delta u(0, \tau) \geq 0$, the agents whose born types are below or equal to ${mc^u}^{-1}(1 - F(0))$ choose ${mc^u}^{-1}(1 - F(0))$; if for all $\tau \in (0, \lambda_2/(\lambda_1+\lambda_2)]$ such that $\Delta u(0, \tau) < 0$, the agents whose born types are at $[0, \tau]$ choose τ and the other agents choose the same as they do in payoff dominant equilibrium.

For further analysis of the third and the fourth cases, we firstly show the conformity of expansion level with policy:

Corollary 2.8.1. *If there exists a $\theta_i < a_i < 1$ such that $s^*(\theta_i, \tau) = a_i$ in the unique political payoff dominant equilibrium, then for all $\theta \in [\theta_i, a_i]$, $s^*(\theta, \tau) = a$.*

We use θ_t to denote the transition point for agent 0 to choose ${mc^u}^{-1}(1 - F(0))$ such that ${mc^u}^{-1}(1) - \theta_t = \int_{\theta_t}^{{mc^u}^{-1}(1)} mc^u(\theta)\, d\theta$ such that $\theta_t \in (0, \theta^*)$, since ${mc^u}^{-1}(1) - 0 < \int_0^{{mc^u}^{-1}(1)} mc^u(\theta)\, d\theta$, if $\underline{\theta} > 0$; ${mc^u}^{-1}(1) - \theta^* > \int_{\theta^*}^{{mc^u}^{-1}(1)} mc^u(\theta)\, d\theta$ holds for all $mc^u(\theta^*) \leq 1$. Then, given any $\tau \in \left(\theta_t, {mc^u}^{-1}(1)\right)$, the equilibrium strategy is the same as that with $\tau^* = \theta_t$ does. When $\tau \in \left[{mc^u}^{-1}(1), 1\right]$, the agents with born types smaller than τ choose τ and the other agents stay with their born types. Note that if $\Theta \neq \emptyset$ and $\underline{\theta} < \theta^*$, then $\min(\Theta) = \underline{\theta} > \theta_t$. Otherwise, consider a $\tau \in (\underline{\theta}, \theta_t)$; $s^*(\theta, \tau) = \tau$ and for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, $s^*(\theta, \tau) = \overline{\theta}$, which contradicts Corollary 2.8.1.

If the marginal cost function is $mc^u(\theta) = (\lambda_1 + \lambda_2)\theta^2 - 2\lambda_2\theta + \lambda_2$, $\lambda_1\lambda_2 \leq \lambda_1 + \lambda_2$ and $\Theta = \emptyset$, in political payoff dominant equilibrium:

(i) if $\tau < \theta_t$ and $\tau > {mc^u}^{-1}(1)$, then $s^*(\theta, \tau) = \begin{cases} \tau & 0 \leq \theta < \tau \\ \theta & \tau \leq \theta \leq 1 \end{cases}$;

(ii) if $\theta_t < \tau < {mc^u}^{-1}(1)$, then $s^*(\theta, \tau) = \begin{cases} {mc^u}^{-1}(1) & 0 \leq \theta < {mc^u}^{-1}(1) \\ \theta & {mc^u}^{-1}(1) \leq \theta \leq 1 \end{cases}$.

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[18] We prove $\partial \Delta u(0, \tau)/\partial\theta < 0$. We have $\Delta u(\theta, \tau)$ expressed in with U-shaped homogeneous marginal cost function $mc^u(\theta)$ as: $\Delta u(\theta, \tau) = [1 - F(\theta)]\left[{mc^u}^{-1}(1 - F(\theta)) - \tau\right] - \int_\tau^{{mc^u}^{-1}(1-F(\theta))} mc^u(\theta)\, d\theta$. The partial derivative of $\Delta u(0, \tau)$ w.r.t θ is: $\partial \Delta u(0, \tau)/\partial\theta = -F'(\theta)\left[{mc^u}^{-1}(1 - F(\theta)) - \tau\right]$. As $\tau \in [0, \lambda_2/(\lambda_1+\lambda_2)]$ and ${mc^u}^{-1}(1 - F(\theta)) \in \left[\lambda_2/(\lambda_1+\lambda_2), {mc^u}^{-1}(1)\right]$, $\partial \Delta u(0, \tau)/\partial\theta < 0$.

Lastly, if the marginal cost function is $mc^u(\theta) = (\lambda_1 + \lambda_2)\theta^2 - 2\lambda_2\theta + \lambda_2$, $\lambda_1\lambda_2 \leq \lambda_1 + \lambda_2$ and $\Theta \neq \emptyset$, in political payoff dominant equilibrium:

(i) if $\tau < \theta_t$, then $s^*(\theta,\tau) = \begin{cases} \tau & 0 \leq \theta < \tau \\ \theta & \tau \leq \theta \leq \underline{\theta} \\ \overline{\theta} & \underline{\theta} \leq \theta < \overline{\theta} \\ \theta & \overline{\theta} < \theta \leq 1 \end{cases}$;

(ii) if $\theta_t < \tau < mc^{u^{-1}}(1)$, then $s^*(\theta,\tau) = \begin{cases} mc^{u^{-1}}(1) & 0 \leq \theta < mc^{u^{-1}}(1) \\ \theta & mc^{u^{-1}}(1) \leq \theta \leq 1 \end{cases}$;

(iii) if $\tau > mc^{u^{-1}}(1)$, then $s^*(\theta,\tau) = \begin{cases} \tau & 0 \leq \theta < \tau \\ \theta & \tau \leq \theta \leq 1 \end{cases}$.

Chapter 3

The Acculturation Game: Density Effects under Heterogeneous Marginal Learning Costs

3.1 Introduction

In the process of acculturation, individuals from minority groups constantly exert efforts to explore host countries and to learn the host countries' cultural knowledge. Both ethnic densities at individuals' living areas and the similarities between individuals' culture of origins and that of host country have impacts on their learning process and the acculturation outcomes.

For an individual who is from a cultural background which is more similar to the host country's culture, she/he can gain the resource of learning the cultural knowledge more easily. The advantage of gaining the resource of learning comes from the endowed social network, which can provide more relevant information of cultural learning. Studies suggest that the difficulty to learn the host country's culture is less for an individual, if her/his original cultural background is more similar to the host country's (Furnham and Bochner (1982) and Ward et al. (2001)).

The frequency of intercultural contacts with people from the host country's cultural background improves the acculturation outcome (Clément and Kruidenier (1985)). Therefore, the ethnic densities at individuals' living areas could be a key factor determining the acculturation outcomes. If the ethnic densities of the individuals' origins are high at their living areas, the individuals have higher chances to meet the individuals who are from the same ethnicity, which could substitutes the time and opportunities to explore the new host environment and interact with the natives. From another perspective, the low density of native individuals results in the low incentives of individuals from minority groups to acquire cultural knowledge. This is due to that the chances of coordinating with natives are low such that the benefits of cultural learning are less than the costs of that.

In Chapter 2, we study the acculturation game and shows that the shape of acculturative outcomes in equilibrium depends on the shape of marginal cost function (strictly increasing, strictly decreasing or U-shaped). The limitation of the model is that it does not capture the heterogeneity of the marginal costs with respect to agents' born types.

We extend acculturation game model in Chapter 2 by considering the heterogeneous marginal cost. The marginal cost to gain new cultural knowledge increases with the distance between the cultural knowledge at certain levels and agents' born cultural knowledge levels. We are interested in how the density distribution and heterogeneous marginal costs will affect the shape of equilibrium. We are also interested in the impact of the shapes of equilibrium strategy on the social welfare increasing from adaptation, which is the difference between the social welfare in equilibrium and the endowed social welfare.

The result shows that if the population density distribution is "flat" such that it is below some thresholds, the equilibrium strategy of every agent is strictly increasing with her/his born cultural knowledge level; if the population density function peaks at some intervals such that it is above the thresholds, there exists the conformity of expansion level of the agents whose born cultural knowledge levels are on the intervals and the right neighbourhood of these intervals. The thresholds are determined by the marginal cost function and the population cumulative distribution function over agents' cultural knowledge levels.

For the social welfare from adaptation, if the equilibrium strategy is strictly increasing, the social welfare from adaptation is independent of the density distributions over the cultural knowledge levels. The conformity of equilibrium strategy does not necessarily cause gains or loss of social welfare from adaptation. We show with an example that the social welfare from adaptation decreases with the density distribution over the agents endowed with high levels of cultural knowledge.

There is little empirical literature we could find studying the ethnic density's effects on the individuals' acculturation outcome.[1]

[1] Social psychology literature focuses on the ethnic density effect on the mental health of immigrants (Murphy (1977), Faris and Dunham (1939) and Mintz and Schwartz (1964)). Sociological studies by

A strand of empirical economic literature relevant to ethnic density focusing on the ethnic densities' effect on school segregation (Orfield (1983) and Echenique et al. (2006)). Echenique et al. (2006) shows that school segregation is not linear in the ratio of students from minority groups, which contradicts the prediction of Orfield (1983). Echenique et al. (2006) also shows that school segregation (low interracial friendship rate) has negative effects on Hispanics and African Americans, however, it has no effects on Asians.

This chapter contributes to the existing literature of behavior assimilation by show that the level of behavior assimilation depending on the population density distribution. Theoretical economic studies on evolution of convention specify the process of matching and take an interest in its effects on the economic and evolutionary outcome. For the uniform matching process, Young (1993) and Kandori et al. (1993) shows that the evolutionary outcome is history dependent if agents' action is with noise or mutation. For the matching process in which agents have higher probabilities to meet their neighbours, Ellison (1993) show the agents converge to the risk dominant equilibrium. In network economic literature, Golub and Jackson (2012) shows that it takes more time to reach opinion consensus if the density of links within groups are higher than that between groups. We shows that with heterogeneous marginal cost of learning, the high density of population of types prevents a high level of behavioral assimilation.

The remainder of this chapter is unfolded as follows. Section 3.2 introduces the model. Section 3.3 shows the shape of payoff dominant equilibria under density distribution functions over the cultural knowledge spectrum. Section 3.4 analyzes the social welfare from adaptation. Section 3.5 concludes.

3.2 The Model with Heterogeneous Marginal Costs

Consider a set of agents N in a society. For agent $i \in N$, she/he is born with a type $\theta_i \in \Theta$ where $\Theta = \{\theta \mid 0 \leq \theta \leq 1\}$ is the cultural knowledge spectrum of the society; and $[0, \theta_i]$ is the endowed cultural knowledge spectrum of agent i. Agent i's born type θ_i is independently and identically distributed on Θ. The cumulative population distribution function w.r.t agents' born types is continuous and defined as

Zhou (1997) and Portes and Zhou (2012) investigate the segmented assimilations of different ethnic groups in the United States. However, they do not provide any information about how the sizes of ethnic groups affect the acculturation outcome.

$F(\theta) := \int_0^\theta f(x)dx$ in which $f : \Theta \to \mathbb{R}_+$ is population density distribution function. Density distribution function f is right continuous, such that for all $\theta \in [0,1]$ such that $\lim_{x\to\theta^+} f(x) = f(\theta)$. The cumulative distribution function $F(\theta)$ is common knowledge in the game.

Agent i's action set is $A(\theta_i) = \{a_i \mid \theta_i \leq a_i \leq 1\}$. Agent i expands her/his cultural knowledge spectrum from $[0,\theta_i]$ to $[0,a_i]$, if $a_i > \theta_i$; otherwise, agent i stays with her/his born cultural spectrum. In the game, agent i meets a fixed number of other agents who are independently drawn from F. The utilities agent i derives, when she/he meets agent j, are the length of the cultural knowledge spectrum they have in common, which is $\min(a_i, a_j)$.

The cost to choose a_i is

$$C(a_i, \theta_i) = c(a_i - \theta_i)$$

in which $c : [0,1] \to \mathbb{R}_+$ and $c(\cdot)$ is strictly increasing and twice-differentiable. Consider the marginal cost function such that $\partial C(a_i, \theta_i)/\partial a_i = \partial c(a_i - \theta_i)/\partial a_i$. Thus, the marginal cost for agent i to acquire knowledge level a_i is the distance from θ_i to a_i. The marginal cost is a function of the difference between the action agents choose and their born types, which implies that the learning opportunities and resources are easier to get for the agents with higher born types in acquiring cultural knowledge on a certain interval of the spectrum.

We normalize the number of agents with whom agent i meets. Given a profile of actions of agents excluding i denoted as $a_{-i} \in \times_{j\in N\setminus\{i\}} A(\theta_j)$, the utilities agent i deriving by choosing a_i are the difference between expected payoff of interacting with other agents $E_{a_{-i}}(\min\{a_i, a_{-i}\})$ and cost $C(a_i, \theta_i)$. Agent i's utility function is:

$$u(a_i, \theta_i, a_{-i}) := E_{a_{-i}}(\min\{a_i, a_{-i}\}) - C(a_i, \theta_i). \tag{3.1}$$

We focus on pure strategy symmetric Bayesian equilibrium. Then, the strategy could be present as $s : \Theta \to A(\Theta)$ such that s is a mapping from agent i's born type θ_i to i's action set $A(\theta_i) = \{s_i \mid \theta_i \leq s_i \leq 1\}$. Therefore, we can rewrite eq. (3.1) as:

$$u(s_i, \theta_i, s, f) = \int_0^1 \min(s_i, s(\theta)) f(\theta)\, d\theta - C(s_i, \theta_i). \tag{3.2}$$

We use u_i to denote user i's utility function interchangeably with $u(\cdot, \theta_i, s, f)$ in the following discussion.

The symmetric Bayesian equilibrium results in multiple solutions in many cases of the acculturation game.[2] We focus on *payoff dominant equilibrium*, which has a *unique* solution for the game.

The definition of *payoff dominate equilibrium* is as follows:

Definition 3.1. A strategy s^* comprises a *payoff dominant equilibrium*, if s^* is Pareto superior to all other symmetric equilibria. That is, s^* is a *payoff dominant equilibrium*, if the following conditions hold:

(i) $s^* \in S$ in which $S = \left\{ s \in A(\Theta)^{\Theta} \mid \forall i,\ s(\theta_i) = \arg\max_{s_i \in A(\theta_i)} u(s_i, \theta_i, s, f) \right\}$;

(ii) $\forall s \in S \backslash \{s^*\}$ and $\forall \theta_i$, $u(\theta_i, s^*, f) \geq u(\theta_i, s, f)$, and $\exists \theta_i$ such that $u(\theta_i, s^*, f) > u(\theta_i, s, f)$.

Set S contains all symmetric equilibria. Every agent's expected utilities in equilibrium s^* are at least as much as hers/his in the other symmetric equilibria, and at least one agent is strictly better off in equilibrium s^*.

Before we discuss how the density distribution function $f(\cdot)$ affects the shape of equilibrium. We first show that the nondecreasing of equilibrium strategy of the acculturation game with heterogeneous marginal cost.

As shown in the following proposition, the nondecreasing of equilibrium strategy in the acculturation game with homogeneous marginal cost in Chapter 2 still holds for the acculturation game with cost function $c(a_i - \theta_i)$.

Proposition 3.1. *In the acculturation game with cost function $c(a_i - \theta_i)$, payoff dominant equilibrium strategy $s^*(\theta_i)$ is a nondecreasing function of θ_i.*

PROOF OF PROPOSITION 3.1. Given a strategy $s^*(\cdot)$ played by other agents, for any two agents i and j, the marginal utilities from $\int_0^1 \min(s_i, s^*(\theta))\, dF(\theta)$ is the same. Combined with that the marginal cost decreases with agents' born types, the marginal utilities to expand to a certain level for agents are increasing with their born types.

For any two agents $\theta_i < \theta_j$, (i) if agent θ_i expands to a level $a_i \in (\theta_j, 1]$, agent θ_j expands at least at a_i by the increasing of marginal utilities with agents' born types;

[2] For examples to illustrate the multiplicity of symmetric equilibria, please see Example 3.1 in Section 3.3 and notes for Footnote 3 in Section 3.6.

(ii) it is possible that agent θ_i expands to a level lower than or equal to θ_j, or stay with born type θ_i, if the total cost of expanding to $a > \theta_j$ is too high to be covered by the increment of interacting utilities. Therefore, we have Proposition 3.1. Q.E.D.

3.3 Shapes of Equilibria

In the acculturation game of Chapter 2, if agent θ_i expand to $a_i > \theta_i$, then all the agents with born types in $[\theta_i, a_i]$ conform to expand to a_i. Although the nondecreasing of equilibrium strategy still holds in the acculturation game with heterogeneous cost (Proposition 3.1), the condition for agents to expand and that for the conformity of expansion level is different from Chapter 2. We use the following example to show.

Example 3.1 Consider the cost function is in a quadratic form, such that $c(a_i - \theta_i) = 3(a_i - \theta_i)^2/5$. We have well-defined utility function presented as follows:

$$U(s_i, \theta_i) = \int_0^1 f(\theta) \min(s_i, s(\theta))\, d\theta - \frac{3}{5}(s_i - \theta_i)^2 .$$

Consider two different density distribution functions $f^1(\cdot)$ and $f^2(\cdot)$:

$$f^1(\theta) = \begin{cases} 0.90 & 0 \le \theta < 1/3 \\ 1.05 & 1/3 \le \theta \le 1 \end{cases}$$

and

$$f^2(\theta) = \begin{cases} 0.90 & 0 \le \theta < 1/3 \\ 1.50 & 1/3 \le \theta < 2/3 \\ 0.60 & 2/3 \le \theta \le 1 \end{cases}.$$

f^1 and f^2 have the same density distribution on interval $[0, 1/3]$. For interval $[1/3, 1]$, f^1 is constant; however, f^2 is peak at interval $[1/3, 2/3]$ and then drops to 0.6 at interval $[2/3, 1]$, which is shown at the left diagram of Figure 3.1.

The right diagram of Figure 3.1 shows the equilibrium strategy for f^1 and f^2 respectively. To be more specific:

In the payoff dominant equilibrium with f^1 denoted by black line in Figure 3.1, the equilibrium strategy $s^{*1}(\cdot)$is:

$$s^{*1}(\theta_i) = \begin{cases} \frac{5}{6} + \frac{1}{4}\theta_i & 0 \leq \theta < 1/3 \\ \frac{7}{8} + \frac{1}{8}\theta_i & 1/3 \leq \theta \leq 1 \end{cases}.$$

The equilibrium strategy with f^2 denoted by red line in Figure 3.1 is present as follows:

$$s^{*2}(\theta_i) = \begin{cases} \frac{5}{6} + \frac{1}{4}\theta_i & 0 \leq \theta < 1/3 \\ \frac{11}{12} & 1/3 \leq \theta < 5/6 \\ \frac{1}{2}\theta_i + \frac{1}{2} & 5/6 \leq \theta \leq 1 \end{cases}.$$

There is a kink at $1/3$ in both cases. However, for s^* with f^1 , it increases at interval $[1/3, 1]$; for s^* with f^2, it keeps constant at interval $[1/3, 5/6]$ and then increases to 1. Note that that the set of agents' born type who choose the constant expansion level $11/12$ not just includes the agents with born types at the peak interval $[1/3, 2/3]$ but also includes its right neighbourhood $[2/3, 5/6]$.

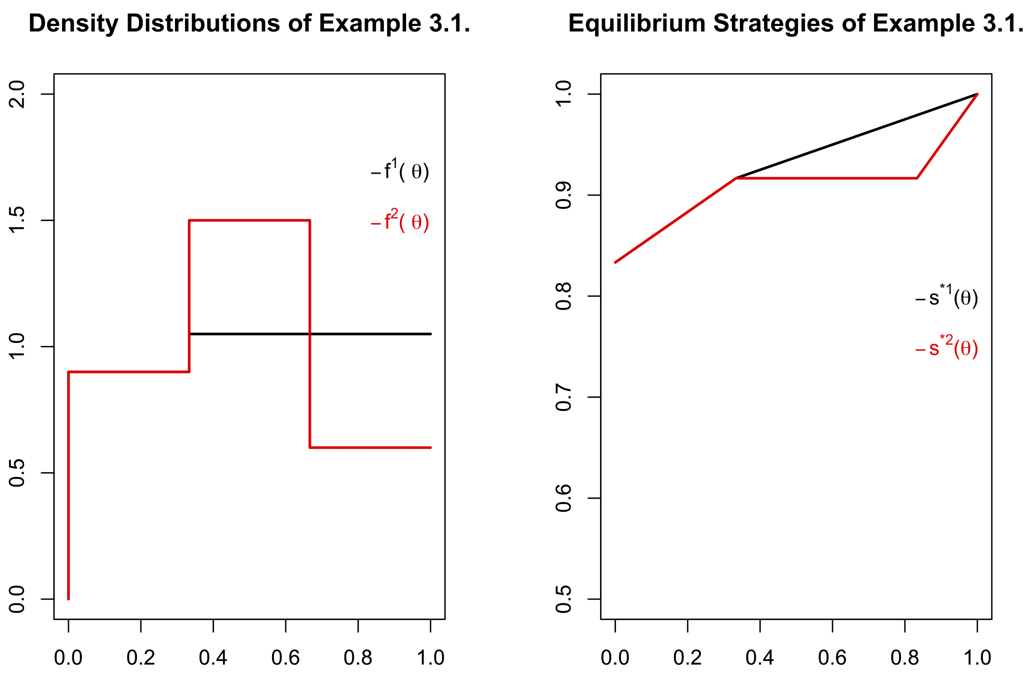

Figure 3.1 Density distributions and Equilibrium strategies of Example 3.1.

The intuition is as follows: we reorganize the utility eq. (3.2) as follows by the nondecreasing of equilibrium strategy:

$$u\left(s_i^*, s, \theta_i\right) = \underbrace{\overbrace{s_i^*\left[1 - F\left(\theta_i\right)\right]}^{\text{the least motivation to expand}} + \int_0^{\theta_i} s^*\left(\theta\right) dF\left(\theta\right)}_{\text{utilities of interacting}} - \frac{3}{5}\left(s_i^* - \theta_i\right). \tag{3.3}$$

The least marginal motivation to expand results in F.O.C:

$$1 - F\left(\theta_i\right) = \frac{6}{5}\left(s_i - \theta_i\right).$$

Solve the F.O.C. of the utility function, we have $s\left(\theta\right) = \theta_i + 5\left[F\left(1\right) - F\left(\theta_i\right)\right]/6$. If for all θ_i, $f\left(\theta_i\right) \leq 6/5$ and $\forall \theta_i^{'} \in (0,1)$ such that $f\left(\theta_i'\right) = 6/5$, $\left(\theta_i', s\left(\theta_i'\right)\right)$ is a saddle point, then $s(\theta_i)$ is a strictly increasing function, which is the case for f^1; however, if there exists a pair $\theta', \theta'' < 1$ such that for all $\theta \in (\theta', \theta'')$, $f\left(\theta\right) > 6/5$, then $s^*\left(\cdot\right)$ is a decreasing function. We combine with the nondecreasing of the equilibrium strategy, we have the strategy denoted as:

$$s^*(\theta_i) = \min\left\{1, \max_{\theta \in [0,\theta_i]}\left(\theta + \frac{5}{6}\left(F\left(1\right) - F\left(\theta\right)\right)\right)\right\}. \tag{3.4}$$

which is the case for f^2 such that agents with born types in $[1/3, 5/6]$ expand to $11/12$.

The intuition for the differences in equilibria of the two cases is that the density distribution peak at interval $[1/3, 2/3]$ of f^2 results in no incentives of agents with born types in $[1/3, 5/6]$ to expand from the low population distribution of agents with high born types; therefore, they conform with the cultural knowledge expansion level of agent with born type $1/3$. [3]

Now we derive the condition for different shapes of equilibria under convex cost function $c\left(\cdot\right)$.

We use $mc\left(\cdot\right)$ to denote the first derivative of the cost function $c\left(\cdot\right)$. If $c\left(\cdot\right)$ is convex and $mc\left(\cdot\right)$ is strictly increasing, then we use $mc^{-1} : [0,1] \rightarrow [0,1]$to denote the inverse function of $mc\left(\cdot\right)$, and $mc'\left(\cdot\right)$ to denote the second derivative of $c\left(\cdot\right)$.

The following proposition shows how population density function affects the shape of equilibrium strategy.

[3]For readers who have interests about the multiplicities of symmetric equilibrium in this case, please see Section 3.6.

Proposition 3.2. *In the acculturation game with cost function $c(a_i - \theta_i)$ such that $c(\cdot)$ is convex,*

(i) if for all θ in $[0,1]$, $f(\theta) \le mc'(mc^{-1}(1 - F(\theta)))$, and for all θ_i' in $(0,1)$ such that $f(\theta_i') = mc'(mc^{-1}(1 - F(\theta)))$, $(\theta_i', s(\theta_i'))$ is a saddle point, then $s^*(\theta_i)$ is a strictly increasing function of θ_i and is given by

$$s^*(\theta) = \theta + mc^{-1}(1 - F(\theta));$$

(ii) if there exists $\theta' < \theta''$ in $(0,1)$ such that for all θ in $[\theta', \theta'']$, the density distribution function satisfies $f(\theta) \ge mc'(mc^{-1}(1 - F(\theta)))$, then there exist $\theta''' > \theta''$ such that for all $\theta_i \in [\theta', \theta''']$, $s^*(\theta)$ is constant. The equilibrium strategy is :

$$s^*(\theta) = \min\left\{1, \max_{\theta' \in [0,\theta]} \theta' + mc^{-1}(1 - F(\theta'))\right\}.$$

For the details of proof, please see Section 3.6.

The conformity of expansion level happens where density distribution at a interval is higher than the threshold $mc'(mc^{-1}(1 - F(\theta)))$. The threshold is a nondecreasing function of θ: the high population density at higher level is with more or with equal probability to cause the conformity of expansion level.

Although the shape of equilibrium is different, the characterization of equilibrium strategy for general costs will be not very different from that in Chapter 2.

We use set $A^*(\theta_i)$ to collect all the possible choices of i, such that:

$$A^*(\theta_i) := \{a \in [\theta_i, 1] \mid mc(a - \theta_i) = 1 - F(\theta_i) \text{ or } a \in \{\theta_i, 1\}\}.$$

We use function $\Delta u(\theta_i)$ to represent the largest non-negative utility difference of i between choosing from $A^*(\theta_i)$ and staying with her/his born type. Then:

$$\Delta u(\theta_i) := \max_{a \in A^*(\theta_i)} u(a, s, \theta_i) - u(\theta_i, s, \theta_i) = \max_{a \in A^*(\theta_i)} [1 - F(\theta_i)](a - \theta_i) - c(a - \theta_i).$$

Furthermore, we define a function $s^{\max}(\theta_i) = \max \arg \Delta u(\theta_i)$ to pick the maximal level among all utility maximization levels of agent θ_i.

Combined with the cases where there exists an agent with born type smaller than

θ_i choose a level higher than θ_i, we have the following proposition:

Proposition 3.3. *There exists a unique payoff dominant equilibrium and the equilibrium strategy is* $s^*(\theta_i) = \max_{\theta \in [0,\theta_i]} s^{\max}(\theta)$ *for all* θ_i *in* $[0, 1]$.

For the uniqueness of payoff dominant equilibrium, please see Section 2.7.1.

3.4 Social Welfare From Adaptation

As readers might notice, agents, except for the agents with type 0, can gains utilities without expanding their knowledge spectrum. The utilities come from their endowed knowledge levels. A question raises: does ex-post social welfare mainly come from the endowed social welfare or the social welfare from adaption gained by expanding the cultural knowledge spectrum? In this section, we formally define the social welfare from adaptation; then, we show that the social welfare is constant for all density distribution function if the equilibrium strategy is strictly increasing. Further, we use example to show the conformity of expanding does not necessarily cause gains or loss compared to the constant social welfare from adaption; and we also show with an example that the social welfare from adaptation decreases with the density distribution of agents born in the high knowledge levels.

The *adaptive social welfare* is defined as the integral of utility differences of all agents between the social welfare after adaption and the social welfare with assumption that no agent expanding their cultural knowledge spectrum:

$$\Delta SW(f) = \int_0^1 u(s^*, \theta) - u(\theta, \theta)\, dF(\theta)\,.$$

The following proposition states the constant social welfare from adaptation for all increasing expanding strategies in equilibrium.

Proposition 3.4. *Given a cost function* $c(\cdot)$*, the adaptive social welfare* ΔSW *is constant for all equilibria with strictly increasing equilibrium strategies.*

PROOF OF PROPOSITION 3.4. We just need to show for any increasing function $g : [0, 1] \to [0, 1]$ such that $s^*(\theta) = \theta + g(1 - F(\theta))$, ΔSW is constant.

For every agent θ', the welfare is:

$$u\left(s^*\left(\theta'\right),\theta\right)-u\left(\theta',\theta\right)=\int_0^1\left[\min\left(s^*\left(\theta'\right),s^*\left(\theta\right)\right)-\min\left(\theta',\theta\right)\right]dF\left(\theta\right)-c\left(s^*\left(\theta\right)-\theta\right)$$

Replace $s^*\left(\theta\right)$ with $\theta+g\left(1-F\left(\theta\right)\right)$, we have:

$$u\left(s^*\left(\theta'\right),\theta\right)-u\left(\theta',\theta\right)=$$
$$\int_0^{\theta'}g\left(1-F\left(\theta\right)\right)dF\left(\theta\right)+g\left(1-F\left(\theta'\right)\right)\left(1-F\left(\theta'\right)\right)-c\left(g\left(1-F\left(\theta'\right)\right)\right).$$

Therefore,

$$\Delta SW=$$
$$\int_0^1\int_0^{\theta'}g\left(1-F\left(\theta\right)\right)dF\left(\theta\right)+g\left(1-F\left(\theta'\right)\right)\left(1-F\left(\theta'\right)\right)-c\left(g\left(1-F\left(\theta'\right)\right)\right)dF\left(\theta'\right).$$

Therefore, ΔSW is constant given c, as every item in the integral is a function of $F\left(\theta'\right)$ and the integral is on a fixed interval. Q.E.D.

The intuition is that the utilities from adaptation for all agents is a function of their accumulative distribution function $F\left(\cdot\right)$ if the strategy is strictly increasing. Therefore, as the integral is $dF\left(\theta\right)$ and on fixed interval $[0,1]$, the density effects on social welfare is canceled out.

For the cases where there exists the conformity of expansion level, the adaptive social welfare could be more than the constant ΔSW with strictly increasing equilibrium strategy; however, it could be also lower than the constant ΔSW depending on the distribution function. Consider distribution function f^3 as follows in the same setting of Example 3.1:

$$f^3\left(\theta\right)=\begin{cases}0.6 & 0\leq\theta<2/3\\ 1.8 & 2/3\leq\theta\leq 1\end{cases}.$$

At interval $[0,2/3]$ the density is low and constant at 0.6, and then the density peaks at interval $[2/3,1]$ and is equal to 1.8.

The approximate numbers of social welfare from adaptation for the three cases are 0.56, 0.57, and 0.49 for f^1, f^2 and f^3 respectively. Both equilibrium strategies with

f^2 and f^3 have the conformity of behavior; however, $\Delta SW\left(f^2\right) > \Delta SW\left(f^1\right)$, but $\Delta SW\left(f^3\right) < \Delta SW\left(f^1\right)$.

We give a simple example to show how the social welfare from adaptation depends on the density distribution function:

Example 3.2 We consider population density distribution $f(\theta) = 2\rho\theta + 1 - \rho$ such that $\forall \rho \in [0,1]$, $F(1) = \int_0^1 (2\rho\theta + 1 - \rho)\, d\theta \equiv 1$. If $\rho = 0$, $f(\theta)$ follows a uniform distribution; if $\rho = 1$, $f(\theta) = 2\theta$. The higher ρ is, the steeper the density distribution is. The cost function is quadratic with $\beta = 1$, then the utility function is as follows:

$$u(s_i, \theta_i) = \int_0^1 f(\theta) \min(s_i, s(\theta))\, d\theta - \frac{1}{2}(s_i - \theta_i)^2 . \tag{3.5}$$

Therefore, we have: given the density distribution function as $f(\theta) = 2\rho\theta + 1 - \rho$ and utility function as eq. (3.5), the adaptive social welfare $\Delta SW(\rho)$ is strictly decreasing with ρ. The calculation is shown below.

The threshold for the conformity of expansion level with eq. (3.5) is 1. By the increasing of $f(\cdot)$, if $f(\theta') = 1$, then for all $\theta > \theta'$, $s^*(\theta) = 1$. With $2\rho\theta' + 1 - \rho = 1$, we have $\theta' = 1/2$ whatever ρ is in this case. Therefore, the social welfare function is denoted as follows:

$$\begin{aligned}
&\Delta SW(\rho) = \\
&\quad \int_0^{\frac{1}{2}} \left(\int_0^{\theta_i} 1 - F(\theta)\, dF(\theta) + (1 - F(\theta_i))^2 - \frac{1}{2}(1 - F(\theta_i))^2 \right) dF(\theta_i) \\
&\quad + \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} 1 - F(\theta)\, dF(\theta) + (1 - \theta_i)(1 - F(\theta_i)) - \frac{1}{2}(1 - \theta_i)^2 \right) dF(\theta_i) \\
&\quad = \frac{11}{24} - \frac{1}{16}\rho - \frac{19}{480}\rho^2 - \frac{1}{128}\rho^3.
\end{aligned}$$

From the equation showed above, $\Delta SW(\theta)$ is a strictly decreasing function of ρ.

3.5 Conclusion

We study the acculturation game with heterogeneous marginal cost, which is increasing with the distances between the level of cultural knowledge agents acquire and their born cultural knowledge levels.

We show that the density distribution function over the cultural knowledge levels play a key role in determining the shape of equilibrium strategy. If the density distribution is below some thresholds, the equilibrium strategy is strictly increasing; otherwise, the equilibrium strategy for agents at the peak interval and its right neighbourhood conforms to the same level. The thresholds are determined by the marginal cost function and the population cumulative distribution function over agents' cultural knowledge levels.

For social welfare analysis, we show for all equilibrium which is strictly increasing, the social welfare from adaptation is constant under any distribution function. However, the conformity of equilibrium strategy does not necessarily cause gains and loss of the social welfare from adaptation. We further show that the social welfare is decreasing with density distribution at high levels of cultural knowledge.

At this stage, the acculturation game model oversimplifies the learning in the process of intercultural contact and the matching process of interacting agents. Future researches could be thought of to studies the effects of them. Moreover, empirical studies could be considered to exam the validity of the mechanism proposed by the acculturation game.

3.6 Appendix

NOTES FOR FOOTNOTE 3. There exists multiple symmetric equilibria for the cases with f^2. The symmetric equilibrium is denoted by:

$$s(\theta_i) = \begin{cases} \frac{5}{6} + \frac{1}{4}\theta_i & 0 \leq \theta < \overline{\theta} \\ \frac{5}{6} + \frac{1}{4}\overline{\theta} & \overline{\theta} \leq \theta < \frac{2}{3} + \frac{1}{2}\overline{\theta} \\ \frac{1}{2}\theta_i + \frac{1}{2} & \frac{2}{3} + \frac{1}{2}\overline{\theta} \leq \theta \leq 1 \end{cases}$$

in which $\overline{\theta} \in [0, 1/3]$.

In the symmetric equilibrium, the expansion strategy of agents endowed with low levels of cultural knowledge is strictly increasing with their born types; the agents endowed with middle levels of cultural knowledge expand to the same level, and the agent endowed with high cultural knowledge expand to a higher level and the expansion strategy strictly increases with agents' born types.

Every agent's utility increases with $\overline{\theta}$. Therefore, the payoff dominant equilibrium s^* is with $\overline{\theta} = 1/3$.

In symmetric equilibria, the set of agents choosing the conforming expansion level contains either agents in the interval $[0, 1/3]$ or agents born in interval $[2/3, 1]$, or both. The population of conformity reaches its infimum in payoff dominant equilibrium by $2/3 + \overline{\theta}/2 - \overline{\theta} = 2/3 - \overline{\theta}/2$, and the population density function is lower in high cultural knowledge levels than in the low cultural knowledge level. Q.E.D.

PROOF OF PROPOSITION 3.2. The least marginal utility of agent i to expand is $1 - F(\theta_i) - mc(s_i - \theta_i)$. If $c(\cdot)$ is convex, then the least expansion level for agent i is $s_i = \theta_i + mc^{-1}(1 - F(\theta_i))$.

Consider the first statement. If for all $\theta \in [0, 1]$ $f(\theta) \leq mc'(mc^{-1}(1 - F(\theta)))$, and for all $\theta_i' \in (0, 1)$ $f(\theta_i') = mc'(mc^{-1}(1 - F(\theta)))$ $(\theta_i', s(\theta_i'))$ is a saddle point, then $s(\theta)$ is strictly increasing. This is due to that the first derivatives of $s(\theta)$, $ds/d\theta = 1 - f(\theta)/mc'(mc^{-1}(1 - F(\theta))) \geq 0$ and any θ such that $ds/d\theta = 0$ is a saddle point. Therefore, the least marginal utility for every agent is also the the most marginal utility for her/her. Therefore, we have $s^*(\theta) = \theta + mc^{-1}(1 - F(\theta))$.

Consider the second statement. If there exists $\theta_i < \theta_j \in (0, 1)$ such that for all $\theta \in [\theta', \theta'']$, $f(\theta) \geq mc'(mc^{-1}(1 - F(\theta)))$, then $s(\theta) = \theta_i + mc^{-1}(1 - F(\theta_i))$ is strictly decreasing at interval $[\theta', \theta'']$. Then, $s(\theta)$ begins to increase. Therefore, there exists a $\theta''' > \theta''$ such that $s(\theta') = s(\theta''')$. For agent not in $[\theta', \theta''']$, the expansion strategy is given by $s(\theta)$; for agent in $[\theta', \theta''']$, their expansion strategy is the same and equal to $s(\theta')$ due to the nondecreasing of strategy (their motivation to expand is higher than $1 - F(\theta)$, since agents has low born types expand higher than $s(\theta)$). Combined with the expansion level bounded by 1, we have:

$$s^*(\theta) = \min\left\{1, \max_{\theta' \in [0,\theta]} \theta' + mc^{-1}(1 - F(\theta'))\right\};$$

Q.E.D.

REFERENCES

Akerlof, G. A. and R. E. Kranton (2000, August). Economics and Identity. *Quarterly Journal of Economics 115*(3), 715–753.

Akerlof, G. A. and R. E. Kranton (2002, December). Identity and Schooling: Some Lessons for the Economics of Education. *Journal of Economic Literature 40*(4), 1167–1201.

Algan, Y., C. Dustmann, A. Glitz, and A. Manning (2010). The Economic Situation of First and Second-Generation Immigrants in France, Germany and the United Kingdom*. *The Economic Journal 120*(542), F4–F30.

Armstrong, M. (2006). Competition in Two-Sided Markets. *The RAND Journal of Economics 37*(3), 668–691.

Ballester, C. and A. Calvó-Armengol (2010). Interactions with hidden complementarities. *Regional Science and Urban Economics 40*(6), 397–406.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006). Who's Who in Networks. Wanted: The Key Player. *Econometrica 74*(5), 1403–1417.

Battu, H., M. Mwale, and Y. Zenou (2007, July). Oppositional identities and the labor market. *Journal of Population Economics 20*(3), 643–667.

Beenstock, M. (1996). The Acquisition of Language Skills by Immigrants: The Case of Hebrew in Israel. *International Migration 34*(1), 3–30.

Benevenuto, F., T. Rodrigues, M. Cha, and V. Almeida (2009). Characterizing User Behavior in Online Social Networks. In *Proceedings of the 9th ACM SIGCOMM Conference on Internet Measurement Conference*, IMC '09, New York, NY, USA, pp. 49–62. ACM.

Berman, E., K. Lang, and E. Siniver (2003). Language-skill complementarity: returns to immigrant language acquisition. *Labour Economics 3*(10), 265–290.

Bisin, A. and T. Verdier (2001, April). The Economics of Cultural Transmission and the Dynamics of Preferences. *Journal of Economic Theory 97*(2), 298–319.

Bisin, A. and T. Verdier (2011, January). Chapter 9 - The Economics of Cultural Transmission and Socialization. In J. Benhabib, A. Bisin, and M. O. Jackson (Eds.), *Handbook of Social Economics*, Volume 1, pp. 339–416. North-Holland.

Bloch, F. (2016). *Targeting and Pricing in Social Networks*. Oxford University Press.

Bloch, F. and N. Quérou (2013). Pricing in social networks. *Games and Economic Behavior 80*, 243–261.

Bramoullé, Y., R. Kranton, and M. D'Amours (2014). Strategic Interaction and Networks. *The American Economic Review 104*(3), 898–930.

Brooks, T. (2013). *The 'life in the United Kingdom' citizenship test: is it unfit for purpose?* Durham: Durham University.

Brown, R. (1973). *A first language: The early stages*. Oxford, England: Harvard University Press.

Caillaud, B. and B. Jullien (2003). Chicken & Egg: Competition among Intermediation Service Providers. *The RAND Journal of Economics 34*(2), 309–328.

Candogan, O., K. Bimpikis, and A. Ozdaglar (2012). Optimal Pricing in Networks with Externalities. *Operations Research 60*(4), 883–905.

Chen, Y.-J., Y. Zenou, and J. Zhou (2015). Multiple Activities for Socially-Connected Criminals. SSRN Scholarly Paper ID 2630157, Social Science Research Network, Rochester, NY.

Chen, Y.-J., Y. Zenou, and J. Zhou (2018a). Competitive pricing strategies in social networks. *The RAND Journal of Economics 49*(3), 672–705.

Chen, Y.-J., Y. Zenou, and J. Zhou (2018b). Multiple Activities in Networks. *American Economic Journal: Microeconomics 10*(3), 34–85.

Chiswick, B. R. (1978). The Effect of Americanization on the Earnings of Foreign-born Men. *Journal of Political Economy 86*(5), 897–921.

Chiswick, B. R. (1998). Hebrew language usage: Determinants and effects on earnings among immigrants in Israel. *Journal of Population Economics 11*(2), 253–271.

Chiswick, B. R. and P. W. Miller (1995). The Endogeneity between Language and Earnings: International Analyses. *Journal of Labor Economics 13*(2), 246–288.

Clément, R. and B. G. Kruidenier (1985). Aptitude, Attitude and Motivation in Second Language Proficiency: A Test Of Clément's Model. *Journal of Language and Social Psychology 4*(1), 21–37.

Corbo, J., A. Calvó-Armengol, and D. C. Parkes (2007). The Importance of Network Topology in Local Contribution Games. In *SpringerLink*, pp. 388–395. Springer Berlin Heidelberg.

Crawford, V. P. (1995). Adaptive Dynamics in Coordination Games. *Econometrica 63*(1), 103–143.

Crawford, V. P. and H. Haller (1990). Learning How to Cooperate: Optimal Play in Repeated Coordination Games. *Econometrica 58*(3), 571–595.

Dustmann, C. (1994). Speaking fluency, writing fluency and earnings of migrants. *Journal of Population Economics 7*(2), 133–156.

Dustmann, C. (1997). The effects of education, parental background and ethnic concentration on language. *Quarterly Review of Economics and Finance 37*, 245–262.

Echenique, F., R. G. Fryer, and A. Kaufman (2006). Is School Segregation Good or Bad? *The American Economic Review 96*(2), 265–269.

Ellison, G. (1993). Learning, Local Interaction, and Coordination. *Econometrica 61*(5), 1047–1071.

Facebook (2019, January). 2018 Annual Report. Technical report.

Fainmesser, I. P. and A. Galeotti (2016). Pricing Network Effects. *Review of Economic Studies 83*(1), 165–198.

Faris, R. E. L. and H. W. Dunham (1939). *Mental disorders in urban areas: an ecological study of schizophrenia and other psychoses*. Oxford, England: University of Chicago Press.

Farrell, J. and G. Saloner (1985). Standardization, Compatibility, and Innovation. *The RAND Journal of Economics 16*(1), 70–83.

Furnham, A. and S. Bochner (1982). Social difficulty in a foreign culture: An empirical analysis of culture shock. In S. Bochner (Ed.), *Culture in Contact: Studies in cross cultural interaction.*, pp. 161–198. Oxford.: Pergamon.

Gagnon, J. and S. Goyal (2017). Networks, Markets, and Inequality. *American Economic Review 107*(1), 1–30.

Ghiglino, C. and S. Goyal (2010). Keeping up with the Neighbors: Social Interaction in a Market Economy. *Journal of the European Economic Association 8*(1), 90–119.

Golub, B. and M. O. Jackson (2012). How Homophily Affects the Speed of Learning and Best-Response Dynamics. *Quarterly Journal of Economics 127*(3), 1287–1338.

Gonzalez, A. (2000). The acquisition and labor market value of four English skills: new evidence from NALS. *Contemporary Economic Policy 18*(3), 259–269.

Graziani, T. (2017, January). How to set up WeChat Payment? A Simple Guide - WalktheChat. https://walkthechat.com/wechat-payment-5-reasons-tencent-might-kill-alipay/.

Hayfron, J. E. (2001). Language training, language proficiency and earnings of immigrants in Norway. *Applied Economics 33*(15), 1971–1979.

Hovey, J. D. and C. A. King (1996). Acculturative Stress, Depression, and Suicidal Ideation among Immigrant and Second-Generation Latino Adolescents. *Journal of the American Academy of Child & Adolescent Psychiatry 35*(9), 1183–1192.

Iqbal, M. (2017, August). WeChat Revenue and Usage Statistics (2019). http://www.businessofapps.com/data/wechat-statistics/.

Jackson, M. O. and Y. Xing (2014). Culture-dependent strategies in coordination games. *PNAS 111*(Supplement 3), 10889–10896.

Joinson, A. N. (2008). Looking at, Looking Up or Keeping Up with People?: Motives and Use of Facebook. In *Proceedings of the SIGCHI Conference on Human Factors in Computing Systems*, CHI '08, New York, NY, USA, pp. 1027–1036. ACM.

Kafka, P. (2014). The Candy Crush Effect: Facebook Revs Up Its Long-Awaited Ad Network, Fueled by Ads for Apps. https://www.vox.com/2014/1/22/11622586/the-candy-crush-effect-facebook-revs-up-its-long-awaited-ad-network.

Kandori, M., G. J. Mailath, and R. Rob (1993). Learning, Mutation, and Long Run Equilibria in Games. *Econometrica 61*(1), 29–56.

Katz, M. L. and C. Shapiro (1985). Network Externalities, Competition, and Compatibility. *The American Economic Review 75*(3), 424–440.

König, M. D., C. J. Tessone, and Y. Zenou (2014). Nestedness in networks: A theoretical model and some applications. *Theoretical Economics 9*(3), 695–752.

Kossoudji, S. A. (1988). English Language Ability and the Labor Market Opportunities of Hispanic and East Asian Immigrant Men. *Journal of Labor Economics 6*(2), 205–228.

Kymlicka, W. (1995). *Multicultural Citizenship: A Liberal Theory of Minority Rights*. Clarendon Press.

Leduc, M. V., M. O. Jackson, and R. Johari (2017). Pricing and referrals in diffusion on networks. *Games and Economic Behavior 104*, 568–594.

Lewis, K., J. Kaufman, M. Gonzalez, A. Wimmer, and N. Christakis (2008). Tastes, ties, and time: A new social network dataset using Facebook.com. *Social Networks 30*(4), 330–342.

Locke, J. (1996[1690]). *An Essay Concerning Human Understanding* (Abridged. K. P. Winkler ed.). Indianapolis: Hackett Publishing.

Mazie, S. (2014, October). Do You Have Too Many Facebook Friends? - Big Think. https://bigthink.com/praxis/do-you-have-too-many-facebook-friends.

Mintz, N. L. and D. T. Schwartz (1964). Urban Ecology and Psychosis: Community Factors in the Incidence of Schizophrenia and Manic-Depression Among Italians in Greater Boston. *International Journal of Social Psychiatry 10*(2), 101–118.

Monderer, D. and L. S. Shapley (1996). Potential Games. *Games and Economic Behavior 14*(1), 124–143.

Murphy, H. B. M. (1977). Migration, culture and mental health1. *Psychological Medicine 7*(4), 677–684.

Orfield, G. (1983). Public School Desegregation in the United States, 1968 - 1980. *UCLA: The Civil Rights Project / Proyecto Derechos Civiles*.

Pham, S. (2016, May). Banned! 11 things you won't find in China. https://money.cnn.com/gallery/technology/2016/05/23/banned-china-10/.

Portes, A. and M. Zhou (2012). The New Second Generation: Segmented Assimilation and Its Variants. *Annals of the American Academy of Political and Social Science 530*, 74–96.

Rawls, J. (2009, July). *A Theory of Justice*. Harvard University Press. Google-Books-ID: kvpby7HtAe0C.

Rochet, J.-C. and J. Tirole (2003). Platform Competition in Two-Sided Markets. *Journal of the European Economic Association 1*(4), 990–1029.

Sääskilahti, P. (2007). Monopoly pricing of social goods. Technical Report 3526, University Library of Munich, Germany.

Sam, D. L. and J. W. Berry (2006). *The Cambridge Handbook of Acculturation Psychology*. Cambridge University Press.

Schneider, F., A. Feldmann, B. Krishnamurthy, and W. Willinger (2009). Understanding Online Social Network Usage from a Network Perspective. In *Proceedings of the 9th ACM SIGCOMM Conference on Internet Measurement Conference*, IMC '09, New York, NY, USA, pp. 35–48. ACM.

Seth, S. (2018, October). World's Top 10 Internet Companies.

Shin, E. (2017). Monopoly pricing and diffusion of social network goods. *Games and Economic Behavior 102*, 162–178.

Tencent Holdings Limited (2019, January). 2018 Annual Report. Technical report.

Throsby, D. (2001). *Economics and Culture*. Cambridge University Press. Google-Books-ID: msDdVdklvDAC.

Ushchev, P. and Y. Zenou (2018). Price competition in product variety networks. *Games and Economic Behavior 110*, 226–247.

Viswanath, B., A. Mislove, M. Cha, and K. P. Gummadi (2009). On the Evolution of User Interaction in Facebook. In *Proceedings of the 2Nd ACM Workshop on Online Social Networks*, WOSN '09, New York, NY, USA, pp. 37–42. ACM.

Ward, C. A., S. Bochner, and A. Furnham (2001). *The Psychology of Culture Shock*. Sussex: Routledge.

Web Desk (2018). The Merging of Social Media and Gaming. https://www.digitalinformationworld.com/2018/10/the-merging-of-social-media-and-gaming.html.

Webster, A. (2017). Tencent is bringing China's biggest game to the rest of the world - The Verge. https://www.theverge.com/2017/12/18/16469736/tencent-arena-of-valor-honor-of-kings-north-america-launch.

WeChat Help Center (2019). How do I add WeChat friends? https://help.wechat.com/cgi-bin/micromsg-bin/oshelpcenter?t=help_center/topic_detail&opcode=2&plat=3&lang=en&id=1208117b2mai141024BRbUFJ&Channel=helpcenter.

Weiermair, K. (1976). The Economic Effects of Language Training to Immigrants: A Case Study. *The International Migration Review 10*(2), 205–219.

Wikipedia (2019, April). Facebook. https://en.wikipedia.org/wiki/Facebook.

Young, H. P. (1993). The Evolution of Conventions. *Econometrica 61*(1), 57–84.

Young, H. P. (1996). The Economics of Convention. *Journal of Economic Perspectives 10*(2), 105–122.

Zhou, J. and Y.-J. Chen (2018, July). Technical Note– Optimal Pricing with Sequential Consumption in Networks. *Operations Research 66*(5), 1218–1226.

Zhou, M. (1997). Segmented Assimilation: Issues, Controversies, and Recent Research on the New Second Generation. *International Migration Review 31*(4), 975–1008.