THE RESPONSE OF LINEAR SYSTEMS TO NON-GAUSSIAN EXCITATION

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SUBMITTED AS A THESIS FOR THE DEGREE OF

MASTER OF SCIENCE

UNIVERSITY OF GLASGOW

1974

ProQuest Number: 10647896

ProQuest 10647896

SUMMARY

A theoretical method has been devised to predict the probability
characteristics of the response of linear systems to non-Gaussian stationary
excitation. Use is made of the higher product moments of the excitation process.
The information provided by higher order correlation functions of both Gaussian
and non-Gaussian processes is discussed.

An alternative more practical method of response prediction is established
using Numerical Methods. This technique is shown to have many applications in
practice and can be extended to handle non-linear systems and non-stationary
excitation.

CONTENTS

NOMENCLATURE

$\{X(t)\}$ — A Random Process.

$X^i(t)$ — A member function of $\{X(t)\}$.

$X^i(t_j)$ — Instantaneous value at time t_j of the member function $X^i(t)$.

X_j — The set of all $X^i(t_j)$.

$F_{\{X(t)\}}(\ )$ — The Probability Distribution of $\{X(t)\}$.

$\Phi_{\{X(t)\}}(\ )$ — The Characteristic Function of $\{X(t)\}$.

$E[\ ]$ — Expected value of.

$\langle\ \rangle$ — Mean value with respect to time.

$R(\ )$ — Moment function (of an ergodic process).

$h(\tau)$ — Impulsive Receptance of a linear system.

$a(if)$ — Complex frequency response of a linear system.

$a^*(if)$ — Complex conjugate of $a(if)$.

$S(\ )$ — Spectral Density Function (of an ergodic process).

μ — Mean value.

σ^2 — Variance.

y' — Derivative of y.

$f(x,y)$ — A function of x and y.

m — Mass.

k — Spring stiffness.

c — Damping coefficient.

c_c — Critical value of damping.

ζ	Damping Ratio c/c_c.
$F(t)$	Time-Amplitude representation of Force Excitation.
$N(\mu,\sigma^2)$	Normal (Gaussian) distribution, mean value μ, variance σ^2.
Δt	Sampling interval.
n	Number of Sample values.
T	Sample length $= n \times \Delta t$.
B_E	Bandwidth of Analysis.
ϵ_r	Random Error in Spectral Density Estimates.
f_m	Maximum Frequency of interest.
f_o	Resonant Frequency.
f_n	Natural Frequency (alternative to above).

SECTION 1

INTRODUCTION

1. INTRODUCTION

In many random vibration problems which occur in practice where the response of a given system to a single random excitation is required, it is reasonable to assume that the excitation process is Gaussian. This assumption considerably simplifies any analysis. If the system can be considered to be linear then the response to the Gaussian excitation will itself be Gaussian [1]. Since any Gaussian random process is completely characterised by definition of its first and second order moments, the excitation will thus be defined by its mean value and auto-correlation function (or spectral density). Knowledge of the complex frequency response $\alpha(if)$ of the system will then yield the spectral density of the response $S^O(f)$ using the relationship:-

$$S^O(f) = |\alpha(if)|^2 S^I(f) \qquad\qquad 1.1$$

The mean value of the response and excitation are also simply related (although for convenience are usually adjusted to zero):-

$$\frac{E[\text{response}]}{E[\text{excitation}]} = \alpha(o) \qquad\qquad 1.2$$

Clearly the response of linear systems to Gaussian excitation can be easily obtained once the system frequency response is known.

There are however, many instances where the assumption of Normality is not justified. Even where specific distributions approach the Gaussian - eg road surfaces - the approximation is often to clipped Gaussian rather than the true Gaussian distribution. It is not generally known to what extent various degrees of clipping of the Gaussian distribution affects the above arguments.

Where the excitation to a linear system is known to be non-Gaussian, it cannot be uniquely defined by its first two moments. Also, while equation 1.1 is still applicable, the response spectrum is

insufficient to define the response completely. Therefore, in such cases, both excitation and response are only partially defined and hence alternative methods are required to complete the description.

Various techniques are presently available for the prediction of linear system response to specific non-Gaussian excitations. See for example, references [2,3] which deal with cases where the excitation can be represented by a series of uncorrelated random pulses which occur at Poisson distributed time intervals - these excitations are generally known as shot noise. Use is made of the moments of the excitation to predict the corresponding response moments. These moments are combined to form a series which gives the Fourier Transform of the first-order probability density of the response. By employing Edgeworth's expansion an estimate of this probability is obtained. The success of this method is due to the fact that the required values of the excitation moment functions are defined (analytically) and the impulsive receptance of the linear system can be employed to give the corresponding values of the response moment functions. This method can be considered to be the one-dimensional case of a more general method whereby not just the first-order probability distribution can be estimated but also all the higher orders.

It will be shown that by considering single non-Gaussian excitations to be realisations of ergodic random processes, the probabilistic character of the responses of linear systems to these excitations can be estimated from certain statistical parameters. These parameters can be computed from the excitation time history.

In view of the practical difficulties in using these methods - in particular, performing the Fourier Inversions - only a need for detailed probabilistic information will justify their use. In certain fatigue studies this will be the case.

For the cases where less detailed probabilistic information is

required - eg in cases where only the mean value, auto-correlation function, spectral density and first order probability are needed - it is clear that an alternative method is required. With a view to the provision of such a method, Numerical Integration techniques were considered. These techniques permit a 'real-time' simulation of the response-prediction problem. The excitation is defined, the system is represented by sets of equations and these are then solved at various time intervals to give the response. Standard random analysis can be applied to this response to provide the information required.

Numerical Integration techniques have not been widely employed in the simulation of vibration problems where both excitation and response are of random nature. Consequently the testing of these techniques requires considerable care and errors which might arise in practical random simulation must be investigated.

One of the many integration 'packages' currently available was employed in the test. The accuracy of the simulation was determined and recommendations made concerning the use of such techniques in random response prediction.

SECTION 2

THEORETICAL RESPONSE PREDICTION

2. THEORETICAL RESPONSE PREDICTION

2.1 DESCRIPTION OF A RANDOM PROCESS

Consider a random process $\{X(t)\}$ whose sample functions are denoted:-

$$X^i(t), \; i = 1,2,\ldots$$

A schematic representation of such a process is given in Figure 1. For any time t_1 the instantaneous value of $X^i(t)$ is $X^1(t_1)$. The set of all $X^i(t_1)$ will be denoted $X(t_1)$ or simply X_1, the set of all $X^i(t_2)$, X_2, etc. The random process $\{X(t)\}$ can be considered to be composed of the set of random variables:-

$$X_i, \; i = 1,2,3,\ldots n \qquad \text{where } n \text{ tends to infinity.}$$

Complete probabilistic description of $\{X(t)\}$ may be achieved in two ways:-

(1) by specifying the n-dimensional distribution function for the random variables X_i, i.e.

$$F_{\{X(t)\}}(x_1,x_2,x_3,\ldots x_n) = \text{Prob}\left[X(t_1) \le x_1, X(t_2) \le x_2, \ldots X(t_n) \le x_n\right]$$

(2) or by definition of the n-dimensional characteristic function

$$\Phi_{\{X(t)\}}(u_1,u_2,\ldots u_n) = E\left[\exp(i(u_1 X(t_1) + u_2 X(t_2)\ldots u_n X(t_n)))\right]$$

where $E[\;\;]$ denotes the expected value.

$$= \int_{-\infty}^{\infty} \ldots \ldots \int_{-\infty}^{\infty} \exp(i(u_1 x_1 + u_2 x_2 \ldots u_n x_n) dF_{\{X(t)\}}(x_1 \ldots x_n) \qquad 2.1.1$$

$$\underbrace{\phantom{\int_{-\infty}^{\infty} \ldots \ldots \int_{-\infty}^{\infty}}}_{n \text{ fold}}$$

Definition of the n-dimensional distribution function or

characteristic function implies all corresponding functions of orders

less than n since,

$$F_{\{X(t)\}}(x_1, x_2, \ldots x_{n-1}) = \int_{-\infty}^{\infty} F_{\{X(t)\}}(x_1, x_2, \ldots x_n) dx_n$$

and

$$\Phi_{\{X(t)\}}(u_1, u_2, \ldots u_{n-1}) = \Phi_{\{X(t)\}}(u_1, u_2, u_3, \ldots u_{n-1}, 0) \qquad 2.1.2$$

The characteristic function defined by 2.1.1 may be expanded

$$\Phi_{\{X(t)\}}(u_1, \ldots u_n) = 1 + iu_j E[X_j] + \frac{i^2}{2!} u_j u_k E[X_j X_k] \ldots$$

$$+ \frac{i^n}{n!} u_j u_k u_l \ldots E[X_j X_k X_l \ldots]$$

$$\text{for } j, k, l, \ldots = 1, 2, 3, \ldots n$$

i.e. as a series of product moments of the random variables X_i from the

process $\{X(t)\}$. As an illustration assume $n = 2$. It follows that:-

$$\Phi_{\{X(t)\}}(u_1, u_2) = 1 + i(u_1 E[X_1] + u_2 E[X_2])$$

$$+ \frac{i^2}{2!} (u_1^2 E[X_1^2] + u_2^2 E[X_2^2] + u_1 u_2 E[X_1 X_2] + u_2 u_1 E[X_2 X_1])$$

$$+ \frac{i^3}{3!} (u_1^3 E[X_1^3] + u_2^3 E[X_2^3] + u_1 u_1 u_2 E[X_1 X_1 X_2]$$

$$+ u_1 u_2 u_1 E[X_1 X_2 X_1] + u_1 u_2 u_2 E[X_1 X_2 X_2] + u_2 u_1 u_1 E[X_2 X_1 X_1]$$

$$+ u_2 u_1 u_2 E[X_2 X_1 X_2] + u_2 u_2 u_1 E[X_2 X_2 X_1])$$

$$+ \frac{i^4}{4!} (u_1^4 E[X_1^4] \ldots \text{ etc.}$$

$$= 1 + i(u_1 E[X_1] + u_2 E[X_2]) + \frac{i^2}{2!} (u_1^2 E[X_1^2] + u_2^2 E[X_2^2] + 2u_1 u_2 E[X_1 X_2])$$

$$+ \frac{i^3}{3!} (u_1^3 E[X_1^3] + u_2^3 E[X_2^3] + 3u_1^2 u_2 E[X_1^2 X_2] + 3u_1 u_2^2 E[X_1 X_2^2])$$

$$+ \ldots \ldots \qquad 2.1.3$$

Fourier inversion of this function gives the 2-dimensional

probability distribution $F_{\{X(t)\}}(x_1, x_2)$ i.e. the probability that both

$X(t_1)$ is less than x_1 and $X(t_2)$ is less than x_2. As n increases, the

characteristic function tends to the continuous function, which implies

the continuous probability distribution i.e.

$$\Phi_{\{X(t)\}}(u(t)) \longrightarrow F_{\{X(t)\}}(x(t)) \qquad\qquad 2.1.4$$

clearly then the set of all product moments $E\left[X_1'\right]$, $E\left[X_1 X_2\right]$, $E\left[X_1 X_2 X_3\right]$

..... etc is sufficient to uniquely define the probabilistic nature

of a random process.

2.2 ERGODIC RANDOM PROCESSES

Assume now that the process $\{X(t)\}$ is stationary. It follows

that the probability characteristics of the set of variables $X(t_i)$,

$X(t_j)$, $X(t_k)$,.....$X(t_n)$ are identical to those of the set $X(t_a)$,

$X(t_b)$, $X(t_c)$,..... provided that $(t_j - t_i) = (t_b - t_a)$, $(t_k - t_j) = (t_c - t_b)$,

..... etc. i.e. the probability characteristics of the process are

invariant with a shift in the time reference and depend only on the

time increments between the random variables $X(t_i)$. Assume further

that $\{X(t)\}$ is ergodic. This demands that any probability parameter

computed across the ensemble of $\{X(t)\}$ is identical with the same

parameter computed along any member function e.g.

$$E\left[X(t_i)\right] = \langle X^j(t) \rangle \text{ for any i,j}, \qquad\qquad 2.2.1$$

where $\langle \ \rangle$ denotes averaging with time t.

and that:-

$$E\left[X(t_i)X(t_j)\right] = \langle X^k(t)X^k(t+\delta) \rangle \text{ for any i,j,k where } \delta = (t_j - t_i)$$

$$E\left[X(t_i)X(t_j)X(t_k)\right] = \langle X^k(t)X^k(t+\delta_1)X^k(t+\delta_1+\delta_2)\rangle$$

where $\delta_1 = t_j - t_i$, $\delta_2 = t_k - t_j$

$\vdots$

etc.

It follows from the above that any member function $X^i(t)$ of an ergodic process is sufficient to uniquely describe the process. Hence the characteristic function may be computed from such a member function. The set of product moments of the process $\{X(t)\}$ can therefore be defined from the sample function $X^i(t)$ as follows:-

$$E\left[X_1\right] = \langle X^i(t)\rangle$$

$$E\left[X_1 X_2\right] = \langle X^i(t)X^i(t+\tau_1)\rangle \quad \text{where } \tau_1 = t_2 - t_1 \qquad 2.2.2$$

$$E\left[X_1 X_2 X_3\right] = \langle X^i(t)X^i(t+\tau_1)X^i(t+\tau_1+\tau_2)\rangle \quad \text{where } \tau_2 = t_3 - t_2$$

$\vdots$

etc.

Of the above set of moments only the first and second are in general use. $E\left[X_1\right]$ is the mean value and $E\left[X_1 X_2\right]$ is the autocorrelation function $R(\tau)$.

The question now arises as to the values of the higher order moments of the class of ergodic processes whose distributions are Gaussian.

2.3   GAUSSIAN RANDOM PROCESSES

It is known [4] that definition of the first two moments of any Gaussian process implies all the higher orders. For a general

Gaussian process $\{X(t)\}$ with random variables $X(t_i) = X_i$ as before,

it has been shown [1] :-

$$E\left[X_i X_j X_k \dots X_n\right] = 0 \quad \text{if the number of terms in the bracket is odd}$$

<div align="right">2.3.1</div>

$$= \sum_{j,k,\dots=1}^{n} E\left[X_j X_k\right] E\left[X_r X_s\right] \dots \quad \text{if number is even}$$

Note that the number of terms is $\dfrac{n!}{\dfrac{n}{2}! \, (2)^{\frac{n}{2}}}$

i.e.

$$E\left[X_1 X_2 X_3\right] = 0$$

$$E\left[X_1 X_2 X_3 X_4\right] = E\left[X_1 X_2\right] E\left[X_3 X_4\right] + E\left[X_1 X_3\right] E\left[X_2 X_4\right] + E\left[X_1 X_4\right] E\left[X_2 X_3\right] \quad 2.3.2$$

$$E\left[X_1 X_2 X_3 X_4 X_5\right] = 0$$

$$\vdots$$

etc.

With the assumption of ergodicity these moments may be obtained from

any sample function $X(t)$ say, as follows:-

$$E\left[X_1 X_2\right] = \langle X(t)X(t+\tau)\rangle = R(\tau) \quad \text{where } \tau = t_2 - t_1$$

$$E\left[X_1 X_2 X_3\right] = \langle X(t)X(t+\tau_1)X(t+\tau_1+\tau_2)\rangle = R(\tau_1 \tau_2)$$

<div align="right">2.3.3</div>

$$\text{where } \tau_1 = t_2 - t_1, \; \tau_2 = t_3 - t_2$$

$$E\left[X_1 X_2 X_3 X_4\right] = \langle X(t)X(t+\tau_1)X(t+\tau_1+\tau_2)X(t+\tau_1+\tau_2+\tau_3)\rangle = R(\tau_1 \tau_2 \tau_3)$$

$$\text{where } \tau_3 = t_4 - t_3$$

These equations will serve to define the higher moments. It now

follows from equations 2.3.2 that for any member function of an ergodic

Gaussian process:-

$$R(\tau_1 \tau_2) = 0$$

$$R(\tau_1 \tau_2 \tau_3) = R(\tau_1)R(\tau_3) + R(\tau_1+\tau_2)R(\tau_2+\tau_3) + R(\tau_1+\tau_2+\tau_3)R(\tau_2)$$

<div align="right">2.3.4</div>

$$R(\tau_1 \tau_2 \tau_3 \tau_4) = 0$$

$$R(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5) = \sum E[X_j X_k] E[X_l X_m] E[X_n X_p]$$

$$= R(\tau_1)R(\tau_3)R(\tau_5) + R(\tau_1)R(\tau_3 + \tau_4)R(\tau_4 + \tau_5) + \ldots(15 \text{ terms})$$

etc.

Equations 2.3.4 apply to any member function of a Gaussian random process or in effect, to any Gaussian random variable. The investigations into the higher correlation functions of Gaussian random variables which prompted the above theoretical approach are described in Appendix A and will serve as practical verification of equations 2.3.4. Reference should now be made to Appendix A.

For any process which is other than Gaussian, equations 2.3.4 will not apply. The higher moments of some typically occurring signals were derived (analytically or practically) and are also given in Appendix A. A discussion of the individual usefulness of the higher moments in describing non-Gaussian signals is given in Section 2.6.

To recap, the probabilistic nature of a random process is defined by its characteristic function. This function may be expanded as a series of product moments of the process. These moments may be computed from a single sample function where the process is ergodic. It would clearly be simpler to monitor the probability distributions where a sample time history is available. However, it will now be shown that if there exists a relationship between the moments of the input and output of linear systems, the probability distributions of the output may be computed solely from a knowledge of the input by using the characteristic function.

2.4 SOME GENERAL PROPERTIES OF THE HIGHER MOMENT FUNCTIONS

1. $R(0,0) = \langle X^3(t) \rangle$ i.e. the mean cube value

   $R(0,0,0) = \langle X^4(t) \rangle$ i.e. the mean fourth power value

   etc.

2. $$R(-\tau_1-\tau_2) = \langle X(t)X(t-\tau_1)X(t-\tau_1-\tau_2)\rangle$$

Let t be replaced by $t+\tau_1+\tau_2$. This is valid if the process is stationary. It follows that:-

$$R(-\tau_1-\tau_2) = \langle X(t+\tau_1+\tau_2)X(t+\tau_2)X(t)\rangle = R(\tau_2\tau_1)$$

Similarly,

$$R(-\tau_1,-\tau_2,-\tau_3,\ldots.-\tau_n) = R(\tau_n,\tau_{n-1},\ldots.\tau_1) \qquad 2.4.1$$

Refer back to equations 2.3.4 and recall that $R(-\tau) = R(\tau)$. In the equation (for Gaussian processes):-

$$R(\tau_1\tau_2\tau_3) = R(\tau_1)R(\tau_3) + R(\tau_1+\tau_2)R(\tau_2+\tau_3) + R(\tau_1+\tau_2+\tau_3)R(\tau_2) \quad 2.4.2$$

Replacing τ_1,τ_2,τ_3 by $-\tau_1,-\tau_2,-\tau_3$ does not change the RHS. i.e.

$$R(-\tau_1-\tau_2-\tau_3) = R(\tau_1\tau_2\tau_3)$$

But from above,

$$R(-\tau_1-\tau_2-\tau_3) = R(\tau_3\tau_2\tau_1)$$

Therefore if equations 2.3.4 are valid it must follow that:-

$$R(\tau_3\tau_2\tau_1) = R(\tau_1\tau_2\tau_3)$$

but

$$R(\tau_3\tau_2\tau_1) = R(\tau_3)R(\tau_1) + R(\tau_3+\tau_2)R(\tau_2+\tau_1) + R(\tau_3+\tau_2+\tau_1)R(\tau_2)$$

which is identical to the RHS of 2.4.2 above. This confirms the validity of equations 2.3.4

2.5 PREDICTION OF THE RESPONSE OF LINEAR SYSTEMS

Consider a general ergodic process $\{X(t)\}$ whose distributions are non-Gaussian, and assume that the mean values of $X_i(t)$ are zero (this reduces the complication of the equations). The characteristic function uniquely defines the process. For convenience the 2-

dimensional characteristic function will be dealt with but the conclusions apply to all higher orders. Recall that (equation 2.1.3):-

$$\Phi_{\{X(t)\}}(u_1 u_2) = 1 + i(u_1 E[X_1] + u_2 E[X_2]) + \frac{i^2}{2!}(u_1^2 E[X_1^2] + u_2^2 E[X_2^2]$$

$$+ 2u_1 u_2 E[X_1 X_2]) + \frac{i^3}{3!}(u_1^3 E[X_1^3] + u_2^3 E[X_2^3]$$

$$+ 3u_1^2 u_2 E[X_1^2 X_2] + 3u_1 u_2^2 E[X_1 X_2^2]) + \frac{i^4}{4!}(\ldots\ldots \text{ etc.}$$

From the definition of the product moments by equation 2.3.3 in the previous section it is clear that:-

$$\Phi_{\{X(t)\}}(u_1 u_2) = 1 + i(0+0) + \frac{i^2}{2!}(R(0)(u_1^2 + u_2^2) + 2R(\delta)u_1 u_2)$$

$$+ \frac{i^3}{3!}(R(0,0)(u_1^3 + u_2^3) + 3R(\delta,0)u_1 u_2^2 + 3R(0,\delta)u_1^2 u_2)$$

$$+ \frac{i^4}{4!}(R(0,0,0)(u_1^4 + u_2^4) + 6R(0,\delta,0)u_1^2 u_2^2$$

$$+ 4R(\delta,0,0)u_1 u_2^3 + 4R(0,0,\delta)u_1^3 u_2) + \ldots \qquad 2.5.1$$

$$\text{where } \delta = t_2 - t_1$$

By way of illustration assume for the moment that $\{X(t)\}$ is Gaussian. By equation 2.3.4 it follows that:-

$$\Phi_{\{X(t)\}}(u_1 u_2) = 1 - \frac{1}{2}(R(0)(u_1^2 + u_2^2) + 2R(\delta)u_1 u_2)$$

$$+ \frac{1}{24}\Big[3R^2(0)(u_1^4 + u_2^4) + 6(R^2(0) + R^2(\delta) + R^2(\delta))u_1^2 u_2^2$$

$$+ 4((3R(0)R(\delta))u_1^3 u_2 + 3(R(\delta)R(0))u_1 u_2^3)\Big] + \ldots \qquad 2.5.2$$

but it is known [1] that the 2-dimensional function of a Gaussian process is given by:-

$$\Phi(u_1 u_2) = \exp(-\frac{1}{2}\sigma^2 u_1^2 - \frac{1}{2}\sigma^2 u_2^2 - R(\delta)u_1 u_2) \quad \text{where } \sigma^2 = R(0)$$

$$= \Big[1 - \frac{1}{2}\sigma^2 u_1^2 + \frac{1}{8}\sigma^4 u_1^4 \ldots\Big]\Big[1 - \frac{1}{2}\sigma^2 u_2^2 + \frac{1}{8}\sigma^4 u_2^4 \ldots\Big]$$

$$\cdot \left[1 - R(\delta)u_1 u_2 + \frac{1}{2} R^2(\delta)u_1{}^2 u_2{}^2 \ldots \right]$$

$$= \left[1 - \frac{1}{2} \left(R(0)u_1{}^2 + R(0)u_2{}^2 \right) - R(\delta)u_1 u_2 + \frac{1}{8} \left(\sigma^4 u_1{}^4 + \sigma^4 u_2{}^4 \right) \right.$$

$$+ \frac{1}{2} R^2(\delta)u_1{}^2 u_2{}^2 + \frac{1}{4} \sigma^4 u_1{}^2 u_2{}^2 + \frac{1}{2} \sigma^2 R(\delta)u_1{}^3 u_2$$

$$\left. + \frac{1}{2} \sigma^2 R(\delta)u_1 u_2{}^3 \ldots \text{etc.} \right] \qquad 2.5.3$$

Equations 2.5.2 and 2.5.3 are identical. Note that for Gaussian processes the characteristic functions are real valued since all the odd order product moments are zero.

Return now to the case of general non-Gaussian processes. The characteristic function may be obtained from a single record of an ergodic process by computation of the set of moment functions $R(\tau_1 \tau_2 \ldots \tau_n)$.

It may seem fruitless to compute the characteristic function when a sample function is available - it would be simpler to monitor the probability distributions. However, this approach provides the link whereby the probabilistic response of linear systems may be predicted solely from a knowledge of the input. Consider the following: if a relationship exists between the moments of the input and output of any given system it follows that the output moments may be obtained from a record of the input. Just as the input was characterised by definition of its moments, so too will the output. Clearly then for complete probabilistic description of the response to a given system, a record of the input and the relationship between moments of the input and output is sufficient.

The relationships required are those between the mean value of the output and input, between the auto-correlation function $R(\tau)$ of output and input, between the third product moment function $R(\tau_1 \tau_2)$ of output and input, etc.

Consider any linear system with impulsive receptance $h(\tau)$ and

complex frequency response $\alpha(if)$. It can be shown [5] that:-

$$R_y(\tau) = \iint_0^\infty h(\epsilon)h(\eta)R_x(\tau-\epsilon-\eta)d\epsilon d\eta$$

where $x(t)$ and $y(t)$ are the system input and output.

By extending the argument it can be simply shown that:-

$$R_y(\tau_1 \tau_2) = \iiint_0^\infty h(\epsilon)h(\eta)h(\delta)R_x(\epsilon-\eta+\tau_1, \eta-\delta+\tau_2)d\epsilon d\eta d\delta \qquad 2.5.4$$

and in general

$$R_y(\tau_1, \ldots \tau_n) = \int_0^\infty \underset{(n+1)\text{fold}}{\cdots} \int_0^\infty h(\epsilon)h(\eta) \ldots h(\alpha)h(\delta)$$

$$\cdot R_x(\epsilon - \eta + \tau_1, \ldots \alpha - \delta + \tau_n)d\epsilon\, d\eta\, d\alpha\, d\delta \qquad 2.5.5$$

Consequently, knowledge of $h(\tau)$ of the system together with the input moment functions implies all corresponding moment functions of the output.

Since the frequency composition of a process is often of very great importance the spectral density functions corresponding to the higher moment functions were defined. Input/output relationships similar to equation 2.5.5 but in the frequency domain were derived for linear systems in Appendix B, where it is shown:-

$$S_y(f_1 f_2 \ldots f_n) = \alpha(if_1)\alpha(i(f_2-f_1))\ldots\alpha(i(f_n-f_{n-1}))$$

$$\cdot \alpha^*(if_n)S_x(f_1 \ldots f_n)$$

Figure 19 gives a schematic representation of the steps required to derive the distributions of the output of linear systems from input information alone.

2.6   DISCUSSION OF HIGHER MOMENT FUNCTIONS

1   INDIVIDUAL MOMENT FUNCTIONS

At the outset of this research project it was decided to
determine what additional information on random signals could be
obtained from the higher moment functions with particular interest
in the third moment function $R(\tau_1 \tau_2)$.   There is a natural reluctance
to consider moment functions of order greater than three for general
use in view of the difficulties associated with their presentation.
It has been shown that the higher moment functions can be defined
for Gaussian processes (signals).   The third and all subsequent odd
order moment functions were shown to be zero regardless of the
frequency content of such processes.   The fourth and higher even
functions can also be expanded in terms of the second moment (or auto-
correlation) function - equations 2.3.4.   It was further shown that for
any finite digital sample of a Gaussian process the computed values
of the third moment function were not zero but were found to be of
the same order as the mean cube value of the data.   Clearly as the
sample size is increased the mean cube value will tend to zero as
will the third moment function.   Consequently unless the sample size
is very large, it would be difficult to distinguish between
deviations from zero in the third moment function caused by departures
from the Gaussian and those resulting from the finite sample.   Figure 20
gives an illustration of this.   The plot of $R(1,\tau_2)$ for the Gaussian
sample is superimposed on that for the sample clipped at 1.5 standard
deviations - i.e. all values in the sample greater than 1.5 are set
to 1.5.   It should be noted that allowance has been made for the
reduction in RMS value of the data due to the clipping by plotting
the Normalised third moment function which is defined:-

$$R^N(\tau_1 \tau_2) = R(\tau_1 \tau_2)/(RMS)^3$$

Little indication is given in the plots to suggest the sample has been quite severely clipped. It is not surprising that this is so. The normalised auto-correlation gives little indication either, and as shown in Figure 42 the effect of the clipping reduces the overall level of the spectrum within the specified frequency ranges, but has little effect on its shape: it should be noted that this refers only to Gaussian signals whose spectra are white in this range, and may not apply to narrow band processes. It can therefore be concluded that the higher moment functions, taken individually, can give little information on deviations from the Gaussian but taken collectively will define the distribution uniquely via the characteristic function.

It is perhaps worth noting at this stage, that one significant difference occurs between the second and third moment functions. Recall that for wide band random processes, where no single sine wave or set of sine waves predominates, the auto-correlation function falls off rapidly to zero. If, however, the process is narrow band the periodicity in the signal betrays itself by causing the auto-correlation function to fall to zero much less rapidly. The third moment function for a sine wave is zero - Appendix A. It follows that a Gaussian signal super-imposed upon a large amplitude sine wave (or close set of sine waves) can be expected to have a third moment function similar to that of the Gaussian signal alone.

In view of the fact that the third order correlation function is zero for a Gaussian signal with any frequency content and that finite samples of Gaussian variables produce third correlations with small non-zero values which are similar to those of other practically occurring signals, the third correlation function $R(\tau_1 \tau_2)$ is likely to be of little use on its own. The major disadvantage in this function for approximately

Gaussian signals is that it gives no indication of the frequency content as does the auto-correlation function. The fourth and higher correlations must be rejected for general use, in view of the computations required in their calculation.

The Fourier Transforms of the higher moment functions, $S(f_1 \ldots f_n)$, as defined in Appendix B, progressively build up a picture of the frequency composition of a random process. Just as the spectral density $S(f)$ of a random signal defines the relative proportions of each frequency present, the 2-dimensional spectrum $S(f_1 f_2)$ gives information on the phase existing between frequencies f_1 and f_2 for each f_1, f_2. It should be remembered that the higher moment functions are not symmetrical, i.e.

$$R(\tau_1, \tau_2, \ldots \tau_n) \neq R(-\tau_1, -\tau_2, \ldots -\tau_n)$$

Care must therefore be exercised in any definition of one-sided spectra.

2 MOMENT FUNCTIONS TAKEN COLLECTIVELY

It has been shown that prediction of the response of linear systems to general non-Gaussian excitation can, in theory, be achieved via the higher order moment functions of the input process. The following steps are required:-

a) Computation of the higher moment functions $R(\tau_1 \tau_2)$, $R(\tau_1 \tau_2 \tau_3) \ldots$ etc from a statistically adequate sample function of the input process the values of τ_1, τ_2 being determined by the order of the characteristic function required.

b) Derivation of the corresponding moments of the output process by either of the two methods outlined above, using either the system transfer function or impulsive receptance.

c) Estimation of the output characteristic function using (b).

d) Fourier inversion of the characteristic function to give the corresponding probability distribution function.

  e) Integration of the distribution function to give lower order

      distributions.

The above steps will now be considered in turn, outlining any practical

difficulties or simplifications where they exist in each case.

(a)   Computation of the moment functions can be simply achieved with

high speed digital computers. However, as the order of moment increases

so too does the computation required. If for example, in the computation

of $R(\tau_1 \tau_2 \tau_3)$ all values of τ_1, τ_2, τ_3, from 1 to 20 are required from a

sample of 3000 values say, then the number of quadruple multiplications

required is approximately 20^3 x 3000! Fortunately, this is not

necessary. Consider, for example, equation 2.5.1 where the 2-dimensional

characteristic function is expanded. It can be seen that only four

values of the fourth moment function are required, i.e. $R(0,0,0)$,

$R(\delta,0,0)$, $R(0,0,\delta)$.

(b)   In order that the above simplifications are applicable to the

response, the method used to compute the output moments must be that

given by equation 2.5.5, where a one-to-one relationship exists between

specific values of the moment functions of the input and output. This

one-to-one relationship does not exist in the spectral density approach

where a whole set of values of, say, $R(\tau_1 \tau_2)$ is necessary to compute

one value of $S(f_1 f_2)$. However, computation of higher spectral density

functions may be required for information on the frequency composition

of processes.

(c)   This is simply achieved once the moments are defined.

(d)   At the present time, there are methods of computing higher order

Fourier Transforms. The first order transformation is widely practised

in spectral density estimates and the necessary smoothing functions,

etc. are well documented. It is necessary however, that at least second

and third order Fourier Transformation techniques are developed
together with the corresponding spectral estimates and implemented on
a digital computer.

(e). This is again simply achieved.

The foregoing method can predict the probabilistic response of
linear systems to non-Gaussian excitation. It has been shown to
require considerable computation in its execution. However, with some
restriction on the order of distribution function required, it will be
of use in fatigue analysis where amplitude probability distributions
are of paramount importance and frequency content of secondary
importance. For the cases where the above method is unnecessary or
unjustified, another method must be found. Consequently, it was
decided to investigate Numerical Integration techniques as a method of
response prediction. Such methods are considered more amenable to
periodic functions, but it was thought necessary to determine to what
degree these methods could be applied to random vibration problems.

SECTION 3

NUMERICAL INTEGRATION TECHNIQUES

3. NUMERICAL INTEGRATION TECHNIQUES

3.1 A NEED FOR REAL-TIME RESPONSE PREDICTION

In Section 2, it was shown to be possible to predict probabilistic information on the response of given linear systems to general stationary random excitation. It was demonstrated that information on the input process was itself sufficient to achieve this. By utilising the higher product moments of the excitation process the corresponding product moments of the response process could be evaluated via the dynamic or impulsive receptance of the system. These response moments permitted estimation of the characteristic function of the response which, on Fourier Inversion, provided the required probability distributions.

In order that the product moments of the input process be calculated, a sample time history (or a digital representation of it) is necessary. From this time history of the excitation process the product moments of the response process may be derived. The above theoretical method, in providing the probability distributions from these response moments, at no time gives sufficient information to construct a time history of the response - only the probability that particular sequences of values will occur in such a record. While such information is sufficient in many instances there are many benefits to be gained from obtaining a real-time record of the response. Some of these advantages are as follows:-

i) those experienced in random analysis can draw much information on the characteristics of a random signal from a visual representation.

ii) the real-time record (in digital form) can, with a digital spectral analysis package, provide information on the frequency content of the process - the most important factor in many applications.

iii) in fatigue analysis real-time records permit computation of

parameters such as peak distributions, level crossings, time

between levels, and so on.

The foregoing theoretical method deprives the user of such

advantages. As will be shown later the technique of Numerical

Integration can provide real-time records when used as a method of

response prediction.

Before proceeding to outline the techniques of numerical

integration it is appropriate to describe a problem of response

prediction of current interest to the author and how it is hoped to

employ integration methods in the provision of a solution. It will

also provide a blueprint of the manner in which similar problems may

be handled.

The problem is that of estimating the response of a motor vehicle

as it moves at constant speed over a road surface. The vehicle may be

a car or truck and the road surface may be of any type. The problem

can be visualised in three separate stages:-

(a) Description of the input

Clearly the initial step in any response prediction must be

adequate definition of the input. A wide variety of road surfaces

have hitherto been surveyed both in Britain and on the Continent. The

surveying techniques may be simple, employing a civil engineering staff

and theodolite or may be more sophisticated, the surveying being

performed by a small wheel attached to a moving vehicle [6]. In either

case, a digital representation of the road surface heights, as a

function of distance measured along the stretch of road, is obtained.

Once the vehicle speed is defined it is a simple matter to obtain the

surface heights as a function of time. This is the amplitude - time

record of the input to the vehicle. It should be stressed that for a

realistic input four such records are necessary, one for each wheel on the vehicle. In order to define these records accurately, the cross-correlation between the two tracks is required. At present, information on this correlation is not available, except for a few isolated road surfaces, but it has been shown by the author [7] that assumptions of zero cross-correlation between the tracks does not introduce serious errors over such a wide frequency range in the response prediction as is the case when the tracks are taken to be identical* i.e. with maximum cross-correlation. Considerable phase information will, however, be available shortly as a result of work being currently undertaken by M.I.R.A. (Motor Industry Research Association). It will then be possible to describe road surface excitations to vehicles to a high degree of accuracy.

(b)   the second stage in response prediction is the defintion of the system - in this case a motor vehicle. In view of the great complexity of a motor vehicle certain assumptions must be made in its representation as a dynamic system. It is usual to represent a vehicle by a mathematical model whose components are linear approximations to those of the vehicle. Such models are currently being developed in the Department by the author. The mathematical models provide sets of differential equations (of the second order) which define the behaviour of the system. The road surface excitations are incorporated in these equations.

(c)   the final step is the solving of the set of differential equations representing the system to obtain the response of the system. It is at this stage that Numerical Integration techniques are hoped to be successfully employed in the solution of these equations. The means by which the numerical methods provide the solution will be outlined in the following section.

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* This assumption arises when a 2-dimensional vehicle model is used

Numerical Integration techniques have hitherto been successfully applied to dynamic response problems similar to the one above. However, in these cases the inputs were either single, transient inducing, impulses or were of a periodic nature, and hence the responses were periodic. It would indeed be a step forward if such methods could be applied to random response prediction in general.

Consequently, it was necessary to test a typical numerical integration package with a problem in random response prediction of known solution. In the subsequent sections, a brief description of the theory of numerical integration will be given together with the particular test problem performed on the selected package.

It is perhaps worth stressing at this point that while the scope of this report is the testing of random response problems with linear systems and stationary excitations, there is no reason why the techniques should not be applicable to non-linear systems with non-stationary excitations. It is the purpose of the following sections to establish a method of evaluating the techniques, using the particular (but perhaps most common) problem, that of linear systems subjected to stationary excitation.

3.2   THEORY OF NUMERICAL INTEGRATION METHODS

It must firstly be stated that the theory of Numerical Integration methods is very complex. While the fundamental ideas behind the integration techniques are simple, their practical application is indeed very complicated. This is especially true of such factors as error estimation, predictor correctors, and similar parameters. Considerable research has been carried out to develop these techniques and several computer 'packages' are currently available. The specific details of the theory on which these packages are based need not concern the vibration analyst. It will be sufficient to have a general grasp of the principles involved so that prudent choice of the user-specified variables in such packages may be made.

To this end a brief description of the theory will be given here and will be of a sufficiently general nature to be applicable to a wide range of integration packages.

As stated above, it is required to solve a set of differential equations. These equations are generally of second order, but to avoid needless complication a single equation of the first order will serve to illustrate the techniques for the time being.

Consider the initial value problem.

$$y' = f(x,y) \qquad y(x_0) = y_0 \qquad \qquad 3.2.1$$

The first step is to find the value y_1 which is the solution of the differential equation for $x = x_0 + h$, where h is some fixed value known as the step length (size). Recall Taylor's series to estimate y_1.

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \ldots\ldots \qquad \qquad 3.2.2$$

Writing $y' = f(x,y) = f:-$

$$y(x+h) = y(x) + hf + \frac{h^2}{2!} f' + \frac{h^3}{3!} f'' + \ldots\ldots \qquad 3.2.3$$

where f, f', f'', are evaluated at $(x, y(x))$.

If h is small, terms in $h^2, h^3, \ldots$etc may be neglected giving the approximation:-

$$y(x+h) = y(x) + hf \qquad 3.2.4$$

The computation may now proceed:-

$$y(x_0+h) = y_1 = y_0 + hf(x_0, y_0)$$

$$y(x_0+2h) = y_2 = y_1 + hf(x_0+h, y_1) \qquad 3.2.5$$

$$\vdots$$

$$y(x_0+nh) = y_n = y_{n-1} + hf(x_0+(n-1)h, y_{n-1})$$

Figure 21 illustrates this technique graphically. In an attempt to follow the curve from the point (x_0, y_0) a distance h, in the x - direction, is moved at gradient $f(x_0, y_0)$. This defines the new point (x_0+h, y_1). A further distance h is moved with gradient $f(x_0+h, y_1)$ to the point (x_0+2h, y_2). This process is repeated throughout the range of integration. It can be seen that as the integration proceeds the estimated points gradually move away from the curve. For convenience, the step length h was kept constant but this need not be the case. This very simple process is the Euler method. It is a first-order method since only terms up to the first power of h are considered in the Taylor's series. The omission of orders of h, greater than or equal to 2, causes a truncation error of order h^2. Further error is introduced since the estimation of $y(x_n)$ is based upon the value $y(x_{n-1})$ which is itself an estimate. As illustrated graphically, these errors can mount up as the integration proceeds.

However, a much improved solution can be obtained by a simple

modification. This is the adaption of a correction procedure. Refer to Figure 22. As in the previous method, the point y_1 is calculated. The derivative at this point is determined i.e. $f(x_0+h,y_1)$. The average of the derivatives at y_0 and y_1 is found and a new approximation $y_1^{(2)}$ is obtained, i.e.

$$y_1^{(2)} = y_0 + \frac{h}{2} \left(f(x_0,y_0) + f(x_0+h,y_1) \right) \qquad 3.2.6$$

Further approximations y_1 can be made by using the current value y_1 i.e.

$$y_1 = y_0 + \frac{h}{2} \left(f(x_0,y_0) + f(x_0+h,y_1) \right) \qquad 3.2.7$$

After sufficient corrections have been made, the final value of y_1 can be used as a starting point to obtain the point y_2. Corrective procedures can now be similarly applied to y_2. This process is repeated over the entire interval of integration. This is known as the Euler-trapezoidal method. It provides a simple example of a predictor-corrector formula. The predictor-corrector methods employed in the integration packages are based upon the same principle but information on several previous values are used to obtain an increase in accuracy.

The Euler method would not be used in practice, in view of its simplicity and resultant inaccuracy. More refined versions based on the same principle are however, currently in use.

The other class of methods in widespread use is the Runge-Kutta type. Many versions of this type exist but are all based on estimates of the derivative, not only at the ends of the interval h, but also at intermediate points. A further feature of these methods is that the calculation uses only the values at the initial point and previous estimates are neither used nor stored. A typical example of a Runge-Kutta method will be outlined to indicate the method of computation.

The example is a fourth-order version. The step size will be

assumed constant but, again need not be so - in a variable step version the length of the step would be determined in accordance with some error criterion. Use of this method requires four quantities to be calculated at each integration interval, namely:-

$$a_n = hf(x_n, y_n)$$

$$b_n = hf(x_n + \frac{h}{2}, y_n + \frac{a_n}{2})$$

$$c_n = hf(x_n + \frac{h}{2}, y_n + \frac{b_n}{2})$$

$$d_n = hf(x_n + h, y_n + c_n)$$

3.2.8

The value y_{n+1} is given by:-

$$y_{n+1} = y_n + \frac{1}{6}(a_n + 2b_n + 2c_n + d_n)$$

Since $y' = f(x,y) = \Delta y/h$ it can be seen that a_n, b_n, c_n, d_n, are estimates of the increments in y at the left hand, twice at the centre, and at the right-hand of the integration interval. In calculating y_{n+1}, the previous estimates a_n, b_n, c_n, d_n, are available and a weighted-average value of these increments in y provide the estimate of the value y_{n+1}.

Most dynamic systems can be represented by sets of second order differential equations of the form,

$$y'' = f(x,y,y')$$

3.2.9

The derivative of Taylor's series is required:-

$$y'(x+h) = y'(x) + hy''(x) + \frac{h^2}{2!}y'''(x) \ldots\ldots$$

The Runge-Kutta method when applied to second order equations requires computation of the following variables at each step:-

$$a_n = \frac{h}{2}f(x_n, y_n, y'_n)$$ noting that y_n, y'_n are evaluated up to the

term in h^4 in the relevant Taylor expansion

$$b_n = \frac{h}{2} f(x_n + \frac{h}{2}, y_n + a_n, y'_n + a_n) \quad \text{where } a_n = \frac{h}{2}(y_n + \frac{a_n}{2})$$

$$c_n = \frac{h}{2} f(x_n + \frac{h}{2}, y_n + a_n, y'_n + b_n)$$

$$d_n = \frac{h}{2} f(x_n + h, y_n + \beta_n, y'_n + 2c_n) \quad \text{where } \beta_n = h(y'_n + c_n)$$

The new value y_{n+1} is now given by,

$$y_{n+1} = y_n + h(y'_n + R_n) \quad \text{where } R_n = \frac{1}{3}(a_n + b_n + c_n)$$

and

$$y'_{n+1} = y'_n + R^*_n \quad \text{where } R^*_n = \frac{1}{3}(a_n + 2b_n + 2c_n + d_n)$$

The next step in the routine will require y'_{n+1}. Repetition of the

above sequence is performed throughout the interval of integration.

In the methods described above, the step length h has been

assumed constant. The facility of a variable step length is offered

by many routines and will be mentioned here. The operation of these

methods is as follows. A value of h is chosen within the program -

usually as some fraction of a user-specified parameter - and the first

step executed. The value of the step length is reduced and the step

re-executed. If the difference between the two computed values

satisfies the error criterion, the solution is accepted. If it is

outwith the specified error, h is further reduced until the error

criterion is satisfied. The specified error may be a relative or an

absolute error and in many packages is specified either by the user or

by default. Integration routines vary considerably in the type of

error criterion adapted.

It must be emphasised that while the simplest integration method

will certainly give only approximate solutions, very sophisticated

routines need not give highly accurate estimates. It is a fact that

in striving for a high degree of accuracy, problems of _stability_ can

arise: i.e. the errors become greatly magnified as the computational

sequence progresses until the routine breaks down. It is practically impossible to state with any certainty, that a particular routine will give good results with a particular type of problem, without prior testing. Clearly great care must be taken when using numerical integration methods - this cannot be stressed to highly - since the margin between a very good estimation and a very poor one can be slight.

The above outline of the theory of integration methods is, of necessity, severely curtailed. However, it should be sufficient for most applications when considered in conjunction with the information provided in package user-manual.

3.3   THE C.S.M.P. PACKAGE

In order to ascertain whether numerical integration methods are applicable to random response prediction, a typical commercial package was chosen. The package was the IBM 'Continuous Systems Modelling Program' known as C.S.M.P. This package is implemented on most IBM computers and as such has wide application. The facilities provided by C.S.M.P. are well documented in the user's manual [8], but, for completeness, a brief description is appropriate.

C.S.M.P. is a general purpose package with a wide range of applications. It offers a choice of seven integration routines varying in complexity from a simple rectangular method to a Milne method, written in double precision with predictor-corrector and variable step length to satisfy the user-specified error criterion. The language of C.S.M.P. allows the user to derive C.S.M.P. statements which represent the differential equations of the system of interest. These statements are very simple to construct as will be illustrated later. C.S.M.P. also provides a translation phase which avoids the user having to specify the statements in correct computational sequence i.e. to avoid any phase lag occurring in the execution.

A standard library of interval function generators is provided for specific purposes, e.g. delay time, random number generators, dead space limiter etc.

One particular facility is worthy of note. It is possible with C.S.M.P. to specify external functions: these take the form of sets of co-ordinates which are representative of the functions. This facility provides the link between traditional integration methods and their application to random analysis. If an external function is specified as a large number of amplitude - time co-ordinate pairs, that function

can be representative of a random signal. Consider the function so specified to be the input to a dynamic system translated into C.S.M.P. language. The simulation is executed as follows. It is clear that the statements will involve the input function. As the integration is performed and the time parameter progresses, the function is accessed at each step to obtain the value of the input to system at that particular time. If it should happen that the program step size is smaller than the user-specified time intervals in the function data, then the program estimates the required value by linear interpolation. This raises an important point. If the step length should be set to a value smaller than the data spacing Δt, or if the integration method should be variable step, then values of the excitation between the specified values will be taken to lie on the line joining the surrounding specified points. There is no justification for this. Ideally, the integrations should be performed only at times where the excitation is known i.e. at multiples of the data spacing. If this is adhered to, then only fixed step routines can be used. Unfortunately, variable step routines are desirable for reasons other than their variable step facility being generally of a higher order than the fixed step methods available in the packages. During the preliminary trials two facts emerged:-

1)   when using the fixed step method (4th order) setting the step size much less than the data spacing produced <u>very little</u> difference in the simulation accuracy, when compared with setting the step length equal to the data spacing (although wasteful of computing time). This implies that the system response to the waveform obtained by joining the digitisations by straight lines is little different from the response obtained without specifying the shape of the waveform between the digitised values. Presumably, the choice of the sampling interval for the excitation - avoiding aliasing problems etc. - is sufficient to

adequately define the excitation within the frequency range associated with the sampling interval. Particular methods of joining the digitisations will however produce different response outwith this frequency range. Note that this argument applies to <u>linear</u> systems - it need not apply to non-linear systems.

2)   the 5th order variable step method gave responses similar (more accurate) to the above and consideration of the computing time required to execute the same problem as above suggested that the step length stabilised at a value equal to the sampling interval. This is reasonable. If the step length is initially set much smaller than the sampling interval, the routine will then find the excitation 'very predictable' - in fact, a straight line. The routine will therefore lengthen the step until some error criterion is about to be violated. Clearly, this value will be equal to the sampling interval (or very close to it).

It therefore appears that variable step methods can be used since the problem chosen here was perhaps the most difficult to simulate. If the excitation is narrow band, the variable step methods will reduce computing costs by allowing the step size to be <u>greater</u> than the sampling interval if this is perhaps chosen needlessly small.

Although outwith the scope of this report, the external function facility can be used in non-stationary random analysis. If, for example, the response of a vehicle accelerating over a road surface is required, it would be a simple task to adjust the time co-ordinates in the function representative of the input at constant speed, to conform to that of the input to the accelerating vehicle. The response predicted in this manner would be no less accurate than that for the constant velocity case.

The C.S.M.P. package offers fixed format printing and print-plotting of the solutions. A sample output is shown in Figure 23. The

time (or equivalent) increments for plotting are, again, user-specified, although it is usual to set them equal to those specified in the external function data. The print-plot facility is especially useful for visual interpretation of the output and can be usefully employed to estimate transient - decay time which will be mentioned later.

A simple problem was devised to test the package when applied to random simulation.

3.4   THE TEST PROBLEM

The next task was to devise a realistic problem of random response prediction with which to evaluate the C.S.M.P. package in particular and numerical integration methods in general. Consider the necessary attributes of any such test problem: the major ones must be:-

a)   it must be a typical problem. This satisfies the criterion of general application.

b)   the exact solution to the problem must be available in order that the accuracy of the technique under test might be assessed.

c)   the problem should be designed to accentuate any suspected weaknesses in the proposed technique.

If a), b), and c) are strictly adhered to the accuracy of the technique as indicated by the test problem will be a lower bound. In view of the fact that testing such a technique as numerical integration, with all its attendant variables, is likely to involve considerable computer time, it was desirable to keep the test problem as simple as possible without violating the conditions above.

As a result, the test problem was chosen to be that of predicting the response of a single degree of freedom linear system to Gaussian white noise excitation. The reasons for adopting this particular problem were:-

i)   it is a typical problem and it is a simple one - this satisfies condition a).

ii)   it is known [1] that the response of any linear system to Gaussian excitation is itself Gaussian. The response is therefore completely characterised by its spectrum. The dynamic receptance $\alpha(if)$ of the system is easily defined and with use of the equation,

$$S_R(f) = | \alpha(if) |^2 S_E(f)$$

where R and E denote response
and excitation respectively

together with the excitation spectrum, the theoretical response
spectrum can be evaluated, satisfying condition b).

iii) it is stated in the C.S.M.P. user's manual that integration
techniques 'cannot be expected to handle wildly oscillating
functions with extreme precision'. It is clear that such functions
might introduce serious errors in response estimation.
Consequently, the most serious errors are likely to be caused by
functions which exhibit no dominant periodicity: the excitation
forcing function was therefore chosen with a white spectrum.

Figure 24 gives a representation of the test system – a simple
mass/spring/damper arrangement subject to forced excitation. The
differential equation defining the behaviour of the system is:-

$$m\ddot{x} + c\dot{x} + kx = F(t)$$ with the usual notation.

The forced excitation $F(t)$ was represented by a digital record
consisting of 3000 samples whose distribution was Gaussian and whose
frequency spectrum was white within the 0-25Hz range. The sample size
of 3000 was selected since this number is the lowest which can be
considered statistically representative of a random signal for the
present purpose. However, for most applications, this sample size
should be considerably higher. This is in accordance with condition
iii) above. An amplitude-time plot (with the assumed sampling interval)
of the forcing function is given in Figure 2.

The system parameters m and k were set to give the system a
resonant frequency of just over 1Hz which, together with the assumed
sampling frequency of the excitation (50Hz) gave 1 second of output per
page on the print-plots. This facilited recognition of the resonant

frequency in the response, at one cycle per page of print-plot output.

The system transfer function $\alpha(if)$ is given by:-

$$\alpha(if) = \frac{\frac{1}{k}}{1 - \left(\frac{f}{f_n}\right)^2 + i2\zeta\frac{f}{f_n}} \qquad\qquad 3.4.1$$

where ζ is the damping ratio, f_n the resonant frequency.

The modulus of this function is sometimes known as the _magnification_ _factor_ of the system. The differential equation of the system was translated into C.S.M.P. statements. The basis of these statements is the integral function, INTGRL. The appropriate equations are:-

X = INTGRL(0,XDOT)   where the first argument is the initial value

XDOT = INTGRL(0,X2DOT)

$$\qquad\qquad\qquad\qquad 3.4.2$$

F = AFGEN(FORCE,TIME)

X2DOT = (F-c*XDOT-k*X)/m

where $X = x$, $XDOT = \frac{dx}{dt}$, $X2DOT = \frac{d^2x}{dt^2}$, and F is the external function

(arbitary function generator - AFGEN) named FORCE with independent variable TIME. FORCE is specified as the set of input time-amplitude co-ordinates in the following manner:-

FORCE = 0.0,0.0,0.02,0.594,0.04, 0.372,.....60.0,0.921

The simplicity of the above statements is commendable. Complicated dynamic systems can therefore be represented on C.S.M.P. with little effort.

The choice of the damping factor, c, required more consideration. Assume the damping is set very high, say several times the critical value. The response $x(t)$ will then be very similar in frequency content to that of the excitation since the system is effectively 'solid'. As the damping is gradually reduced to say, $\frac{1}{2}\%$ critical, the shape of the response spectrum will become progressively more narrow-band. It was

important therefore to choose a range of values of damping and to assess
the accuracy of the simulation in each case. The values of the damping
ratio chosen were:-

$\zeta = 0.5, 0.25, 0.1, 0.05, 0.02, 0.005$

This range should be sufficient to cover the majority of response
prediction problems.

.Once the program was set up, a preliminary check on its operation
was made. Since it is known that numerical integration works well with
'predictable' inputs, a sine wave input of 1Hz frequency was selected as
the force input. The value of ζ was set to 0.25 for this test. Use was
made of the variable step Runge-Kutta method offered in the package.
Any linear system excited by a pure sine wave will produce a response
which is a sine wave of the same frequency but with different amplitude
and phase (with respect to a time reference).

The response produced by C.S.M.P. for this preliminary test was
found to be very accurate, being a pure sine wave with amplitude and
phase almost exactly as predicted by equation 3.4.1.

C.S.M.P. was now tested with the random input problem.

3.5   EXPERIMENTAL PROCEDURES AND RESULTS

1. EXPERIMENTAL PROCEDURES

Since the random problem is likely to be the most difficult to solve with numerical integration methods, it was appropriate to select the most sophisticated routine offered in the C.S.M.P. package. This is the Milne method. It is a fifth-order method with variable step length incorporating predictor-corrector. The version used, as implemented on the IBM 370/158, has been rewritten in double-precision.

Using this method the random test problem was executed with the various values of damping ratio and the response amplitude print-plots obtained. On scrutinising the plots it was apparent that a certain degree of non-stationarity existed in the responses. In each case the peak amplitudes at the beginning of the plots were greater than those for the remainder. It was also apparent that the time taken for the peak amplitudes to settle down was greater for the lower values of the damping ratio. In the case where the damping ratio was 0.005, the peak amplitudes remained more or less constant throughout the integration interval. It was clear that the cause was transient response, the decay time increasing with reduced damping. For the case where the damping was 0.005 critical it was presumed that the transients were still dominant at the end of the simulation, i.e. after 60 secs (60 cycles at 1Hz).

It is a simple matter to determine the number of cycles taken for these transients to reduce to some percentage of their initial value. Where the damping ratio is 0.005, the number of cycles for the transient to reduce to say, 10%, of their initial value, is given by:-

$$n = \ln \frac{X_o}{X_n}$$ where n = number of cycles, X_o, X_n are the initial and nth amplitude values.

i.e.

n x 0.005 = ln(10) ≐ 2.3

n = 460 cycles, which is equivalent to 460 secs in this case.
However, since the integration method has a variable step length, it
is not possible to obtain, with any accuracy, the initial value of the
transient. It is therefore difficult to assess at what fraction of
their initial value can the transients be neglected. Another method
of transient estimation was devised.

A separate run is required with a new forcing function as follows.
Instead of 3000 amplitude-time co-ordinates, only the first input force
amplitude is used. The remainder are set to zero. This is simply
achieved by the following statement:-

FORCE = 0.0,0.0,0.02,0.594,0.04,0.0,60.0,0.0.

In effect the first (transient inducing) force amplitude is allowed to
act on the system and is then quickly reduced to zero and remains at
this value throughout the integration interval. The print-plot is now
representative of the transient decay (Fig 23). Now, by comparing the
relative magnitudes of the response in the two cases the point at
which the transient can be neglected is determined. The response on
the original plot can now be taken to be steady-state. For the present
tests the transients were neglected when the relative magnitude was
about 2%. A disadvantage of this method is that the response sample
size is effectively reduced. It was found that for $\zeta = 0.5$, 2750 values
of the response were reliable, whereas the number was just less than
2000 for $\zeta = 0.02$. This disadvantage can be overcome as will be
described in the section on FURTHER DEVELOPMENTS.

The reliable response values so obtained were analysed with a
separate statistical and spectral analysis package. The accuracy of
the simulation was assessed by considering two factors:-

a) PROBABILITY DISTRIBUTION

As stated previously, the response distribution should be Gaussian, i.e. similar to the excitation. It is not possible, however, to exactly represent a Gaussian distribution with a finite sample. The 3000 valued input sample is therefore only an approximation to the Gaussian distribution, although a good one for the size of the sample. This fact makes the test more realistic: not only is the integration technique under test, the adequacy of representing a continuous distribution with a finite sample is also under evaluation.

Since, at the outset, these tests were designed to illustrate any weaknesses in the simulation technique a thorough assessment of probability distribution was essential. In plotting probability histograms, the choice of the number of class intervals is of great importance. If the intervals are chosen too wide, derivations from the time distribution can be masked; it often happens that the values occurring within a wide class interval can be badly skewed and yet the centre of the interval coincides with the true Gaussian curve. Consequently, the number of class intervals for a sample size N was selected by the formula,

$$NCI = 1.87N^{.4}$$

which is the number required by the Chi-square test for a 95% confidence level [9]. This gave 45 intervals for a 3000 valued sample. The histograms were preferred to the Chi-square test of Hypothesis Acceptance. This test either passes or fails the sample: it does not give any indication of which amplitudes are well represented or those that are not. If, for example, some data fails the test for Normality, no indication is given of which amplitudes caused the failure. This is of general interest since in some disciplines, high amplitude derivations are unacceptable, e.g. certain problems in fatigue, whereas

in others such deviations are of less importance.

It must therefore be stressed that this distribution test is a severe one. The probability histogram of the forced excitation is given in Figure 25 with the true Gaussian distribution superimposed (full line).

b) SPECTRA

The response spectra were computed for comparison with the theoretical spectra. The data was analysed using the standard Blackman-Tukey method up to 25Hz - this limit is imposed by the sampling interval to avoid aliasing problems [5]. The random error ϵ_r of the computed spectra is given by:-

$$\epsilon_r = \frac{1}{\sqrt{B_e T}}$$

where B_e is the analysis bandwidth and T the record length in seconds.

i.e.

$$\epsilon_r = \frac{1}{\sqrt{15 \cdot 60}} = \frac{1}{\sqrt{30}}$$

18% for the 3000 valued sample

This should be remembered in comparing the measured and theoretical spectral densities. There is an unfortunate characteristic produced by the Hanning smoothing function which is associated with the Blackman-Tukey spectral analysis procedure. If the spectrum of the signal under analysis falls off rapidly with increasing frequency the estimates produced can eventually become unstable. The instability is manifest by negative-valued estimates of spectral density.

This phenomenon occurred in certain cases in the present analysis but fortunately did not do so until the values of the spectral density had fallen several orders of magnitude from the resonant values.

The spectrum of the forced excitation is shown in Figure [3].

2. RESULTS

For each value of damping ratio, three response plots were produced:-

a) an amplitude-time plot (where time zero is the point at which the transients were neglected).

b) the probability histogram with the true (continuous) Gaussian line superimposed.

c) the response spectral density (full line) with the theoretical* spectrum superimposed (dotted line).

The above set of plots are given consecutively for each value of damping in Figures 26 - 40 . (Note the value $\zeta = 0.005$ is absent since the response was entirely contaminated by the transient)

---

* as given by the equation $S_R(f) = |\, a(if)\,|^2 \, S_E(f)$

3.6   DISCUSSION OF THE RESULTS

It can be seen from the amplitude-time plots of the responses that their form is what would be expected. Compare, for example, the force excitation and the response of the system with damping ratio 0.02. The wide band characteristic in the excitation has been filtered out by the system to produce a response that is typical of narrow-band random with its characteristic fluctuating amplitude sine wave. It is clear also that the dominant frequency in all the responses is 1Hz which is the system resonant frequency. Perhaps the only criticism which can be levelled at the responses is that in some of the plots - especially where the damping is low - there appears to be a little non-stationarity in the initial stages. This is almost certainly due to residual effects of the transient which can easily be removed with larger samples. This may be achieved by allowing some time to elapse in the responses, after the transients appear to have died away, before assuming the response to be steady state. This will counteract the residual effects of the transients which may remain for some time within the integration routine.

The theoretical and measured spectra show good agreement. Where deviations do occur from the theoretical spectrum, at the low damping values, it is in the magnitude rather than in the shape of the spectrum. The reduced sample sizes for these cases will account for the discrepancies.

The probability histograms show that the response distributions are approximately Gaussian. The plot for $\zeta = 0.02$ can be seen to exhibit a tendency to a sine-wave distribution. This indicates that the transient is still present in the response to some extent.

It is clear that increasing the sample size of the inputs to say,

20,000 values would produce extremely accurate responses, even for the present problem of narrow-band filtering. Where the problem does not demand such a sharp change in spectral shape the simulation will be even better.

In conclusion, the technique of random response simulation by numerical integration can only be considered to have great potential. It will be possible with this method to predict linear system response to non-Gaussian excitation, non-linear system response to both Gaussian and non-Gaussian excitation and even non-stationary problems could be handled.

3.7 RECOMMENDED PROCEDURES FOR RANDOM SIMULATION BY NUMERICAL METHODS

The following procedures are strongly recommended for any random response problem which is to be solved with Numerical Integration.

1. REPRESENTATION OF THE INPUT

Spectral analysis procedures impose constraints upon the sampling interval and record length of the input [5]. The sampling frequency must be at least twice that of the highest frequency of interest in the analysis. The record length T which defines the sample size $(n=T/\Delta t$, Δt = sampling interval) must be at least that determined by the desired accuracy ϵ_r of the spectral estimates and by the analysis bandwidth B_E, i.e.

$$\epsilon_r = \frac{1}{\sqrt{B_E T}}$$

For narrow-band response simulation, the minimum recommended sample size n, is derived in Appendix C. It is given by:-

$$n \geq \frac{25 f_m}{f_o \zeta}$$

where f_m is the maximum frequency of interest, f_o and ζ is the resonant frequency and associated damping whose product is minimum.

This sample size does not make allowance for transients.

2. CHOICE OF INTEGRATION METHOD

It would be dangerous to recommend particular types of integration routine for general application - the choice is largely dependent upon the specific type of problem. However, the following procedure may be helpful in such a choice:-

1) Select the most sophisticated routine offered in the package - usually a variable step method.

2) Perform a trial run over part of the integration interval using

this method.

3) Repeat with the second most sophisticated method. If it happens to be a fixed step method, set the step size to the data spacing.

4) Compare the responses produced. They should be similar. If so, accept the original method. If not, and further reduction in the step size of the second method gives no improvement, discard the original method and repeat the steps with the second and third methods, and so on with all pairs of methods until some agreement is obtained.

It is essential that a 'second opinion' be obtained, since it is dangerous to accept unqualified responses where the general form of the response is unknown.

3. TRANSIENT REMOVAL

Adopt the procedure outlined in Section 3.5 (1).

4. COST

The cost of the simulation is largely dependent upon the complexity of the system, the step size, the interval of integration - in effect the excitation record length - and the method of integration. For the one degree of freedom system used, with the 3000 valued excitation sample, the cost was approximately £3 (£6 to commercial users). It is likely to be less than this in general for such a problem, since the step size was of necessity very low in view of the white spectrum of the excitation.

3.8　RESPONSE OF A SINGLE DEGREE OF FREEDOM LINEAR SYSTEM TO CLIPPED

GAUSSIAN EXCITATION

It was possible within the time available to assess the response

of the test system to clipped Gaussian excitation. The values of

damping chosen were $\zeta = 0.25$ and 0.1 to keep the effective response

sample as large as possible - about 2750 in this case. Two degrees of

clipping were chosen - 1.5 sigma and 1.0 sigma, where sigma is the

standard deviation, unity in this case. A plot of the 1.5 sigma

clipped sample is given in Figure 41 and the corresponding spectral

density plot in Figure 42.

It is interesting to note that the effect of this clipping is to

reduce the overall level of the spectrum, but it retains its shape, i.e.

it remains white. While this is the case up to 25Hz, it may be that

the higher frequencies are attenuated by the clipping.

The responses were predicted for the two damping values and the

plots were produced as before and are given Figures 43 - 54.

It is immediately apparent from the probability distributions

that the responses are approximately Gaussian. The spectral density

plots show good agreement with the theoretical spectrum.

These results suggest that a linear system tends to restore the

Normality to a clipped Gaussian excitation.

3.9   FURTHER DEVELOPMENT OF THE C.S.M.P. PACKAGE

The encouraging nature of the test results of C.S.M.P. has justified further development of the numerical simulation technique. Since the completion of the work reported upon here, the following refinements have been made:-

1) it is now possible to handle samples of up to about 100,000 values.

2) the output from C.S.M.P. (which may be more than one variable) is read automatically written onto a temporary file without sacrificing the print-plot(s). This file is accessed during the same run by a compatible statistical and spectral analysis package.

3) computer plots of the following parameters are available in the same run:-

        Auto-correlation function
        Spectral density
        Probability histogram with Gaussian superimposed (option)
        Amplitude-time signal
        Peak distribution
        Level crossings
        Coherency and Phase (if two outputs produced by C.S.M.P.)

The flexibility of the entire package will permit much basic research to be done in the near future.

SECTION 4

CONCLUSIONS

4. CONCLUSIONS

The conclusions may be summarised as follows:-

1)   Complete probabilistic information on an ergodic process can be obtained from the set of product moments of the process. These product moments can be obtained from a single realisation of the process. When arranged in a certain manner, the product moments define the characteristic function of the process. Fourier Inversion of this function yields the probability distributions of the process.

2)   For Gaussian processes, the higher product moments are simply defined. For ergodic Gaussian processes, the higher order correlation functions $R(\tau_1, \tau_2, \ldots \tau_n)$ of any realisation of the process can be expressed as simple functions of the auto-correlation function $R(\tau)$ of that realisation (or of any other).

3)   By using higher order input/output relations for linear systems, either in the time or frequency domain, probabilistic information on the response can be obtained from a knowledge of the excitation, where the excitation is assumed to be a member function of an ergodic process.

4)   The computation required to obtain the distributions, especially for the higher orders, is considerable, but need not be prohibitive with the use of a modern digital computer. The technique is likely to be of use in fatigue analysis where amplitude distributions are most important.

5)   The third order product moment functions of random processes, or equivalently, the third order correlation functions $R(\tau_1 \tau_2)$ of random variables are theoretically zero for zero mean Gaussian variables and for single sine waves. In practice, when computed from finite digital

samples, the third order correlations are not zero but fluctuate in a random manner with amplitudes of the order of the mean cube value of the sample. Random variables which can be considered as a Gaussian variable superimposed upon a sine wave function have third order correlations also approximately zero. Many other random variables have third order correlations which are very small. Deviations from zero in the third order correlations of finite samples do not therefore necessarily imply deviations from the Gaussian. For random variables which are approximately Gaussian, the third order correlation function is of little use.

6) A more practical method of random response prediction is Numerical Integration. Even when using limited digitisations of the excitations, the response of linear systems can be predicted with acceptable accuracy. An increase in the excitation sample size will yield a corresponding increase in simulation accuracy.

7) Numerical Integration Techniques can be applied to non-linear systems and non-stationary excitations.

8) The response of a single degree of freedom system to excitations, whose distributions are sharply clipped Gaussian and whose spectra are white over the frequency range where the system response is high, tends to the Gaussian for certain values of the damping ratio.

ACKNOWLEDGEMENTS

REFERENCES

1. LIN, Y.K.   'PROBABILISTIC THEORY OF STRUCTURAL DYNAMICS'
                     McGraw Hill New York 1967.

2. ROBERTS, J.B.   'ON THE RESPONSE OF A SIMPLE OSCILLATOR TO RANDOM
                     IMPULSES'
                     Journal of Sound & Vibration(1966) 4(1), 51-61.

3. JANSSEN &   'NUMERICAL CALCULATION OF SOME RESPONSE STATISTICS FOR A
    LAMBERT   LINEAR OSCILLATOR - UNDER IMPULSIVE-NOISE EXCITATION'
                     JASA Vol. 41 No. 4 1967 (827-835).

4. PARZEN, E.   'STOCHASTIC PROCESSES'
                     Holden Day 1962.

5. BENDAT, J. &   'RANDOM DATA: MEASUREMENT AND ANALYSIS PROCEDURES'
    PIERSOL, A.G.   Wiley/Interscience 1971.

6.   'MEASUREMENT & ANALYSIS OF ROAD SURFACE ROUGHNESS'
                     MIRA Report No. 1970/5.

7. RENUCCI, M.P.   'DEDUCTIONS FROM A 13 DEGREE OF FREEDOM VEHICLE MODEL'
                     SRC Contract B/RG/2684 Report No. 1974/3.

8. MCARTHUR, C.D.   'A USER'S GUIDE TO CSMP'
                     University of Edinburgh SSP Library Cat. No. 19.410.200
                     March 1972 (First Edition).

9. LUCKE, W.J. &   'FATIGUE DATA PROCESSING WITH A SMALL DIGITAL COMPUTER'
    BROWN, G.W.   ISA Transactions Vol. 11 2(128-134).

BIBLIOGRAPHY

1. RICE, S.O.   'MATHEMATICAL ANALYSIS OF RANDOM NOISE'
   Bell System Technical Journal 23(1944), 282-332;
   24(1945), 46-156.

2. GRENANDER, U. &   'STATISTICAL ANALYSIS OF STATIONARY TIME SERIES'
   ROSENBLATT, M.   Wiley & Sons, New York 1957.

3. INYENGAR, R.N.   'FIRST PASSAGE PROBABILITY DURING RANDOM VIBRATION'
   Journal of Sound & Vibration 1973 31(2) 185-193.

4. LANING &   'RANDOM PROCESSES IN AUTOMATIC CONTROL'
   BATTIN   McGraw Hill 1956.

5. ROBSON, J.D.   'AN INTRODUCTION TO RANDOM VIBRATION'
   Edinburgh University Press 1963.

6. MCFADDEN, J.A.   'PROBABILITY DENSITY OF THE OUTPUT OF A FILTER WHEN
   THE INPUT IS A RANDOM TELEGRAPHIC SIGNAL'
   IRE Trans. Information Theory 1959.

7. CRAMER &   'STATIONARY AND RELATED STOCHASTIC PROCESSES'
   LEADBETTER   Wiley Series in Probability and Mathematical Statistics
   1967.

8. CRANDALL &   'RANDOM VIBRATION IN MECHANICAL SYSTEMS'
   MARK   Academic Press 1963.

9. DOOB, J.L.   'STOCHASTIC PROCESSES'
   Wiley 1953.

10. PAPOULIS, A.   'PROBABILITY, RANDOM VARIABLES AND STOCHASTIC PROCESSES'
    McGraw Hill 1965.

11. KREYSIG, E.   'ADVANCED ENGINEERING MATHEMATICS'
    Wiley International 1967.

12. WILLIAMS   'NUMERICAL COMPUTATION'
                     University of Manchester, Institute of Science &
                     Technology 1972.

13. HAMMING, W.   'NUMERICAL METHODS FOR SCIENTISTS AND ENGINEERS'
                     McGraw Hill New York 1962.

14. VERNON, J.B.   'LINEAR VIBRATION THEORY - GENERALISED PROPERTIES
                     AND NUMERICAL METHODS'
                     Wiley & Sons 1967.

15. VOLTERRA &   'DYNAMICS OF VIBRATION'
     ZACHMANOGLOU   Merrill Books Incorporated, Ohio 1965.

APPENDIX A

A.1 GENERATION OF THE GAUSSIAN SAMPLE

In order to compute higher correlation functions of a Gaussian
random variable, it is first necessary to obtain a digital sample of a
time history of the random variable. It would have been possible to
obtain a tape recording of some practically occurring signal such as
the response at some point on a motor vehicle which could then have
been digitised. While it is usual to assume that such responses are
Gaussian there is insufficient justification for this assumption to
base research upon it. In the absence of practically occurring Gaussian
signals, it was convenient to generate a set of numbers (amplitudes)
whose distribution exhibited a close approximation to the Gaussian. An
IBM random number generator was used and 10,000 numbers generated with
an $N(0,1)$ distribution.

Consider the implications of attempting to relate these numbers
to a digitised set of amplitudes derived from some time history of a
Gaussian process. Broadly speaking, if we have a large, or even infinite,
set of numbers which are assumed to be digitised from some time history,
the individual values determine the probability distribution of the
record and the order in which the numbers appear defines the 'frequency
composition' of the sample - this is then related to the time history.
If, for example, the numbers in the sample increase and then decrease
periodically, the time history can be considered to be narrow band. If,
however, the numbers exhibit no periodic pattern, they will be
representative of wide-band random. There must exist only one infinite
set of discrete values which have a specific Gaussian distribution.
The set may however may be arranged in an infinite number of ways giving
different frequency contents when each number is associated with a time
co-ordinate. In this instance, consider a 10,000 valued sample of the
infinite set of numbers which would be generated from the subroutine if

allowed to run for all time. Consider the first 3000 values of this sample. Since no time co-ordinates are implied with the sample, they must be arbitrary. We can consider the 3000 values to be a representation of the first micro-second of the infinite record or the first 60 seconds. To associate the sample with a particular record length merely defines the shape of the spectrum of the infinite signal within a <u>specific</u> <u>frequency range</u>. If the record length is taken as 1 micro-second then we have a very short record of the signal defining very high frequencies. Taking the length to be 60 secs gives a larger record length defining much lower frequencies.

The sample (3000 values) was assumed to be a 60 second record with sampling interval 1/50 sec. The spectrum was calculated with these assumptions and was found to be white within the frequency range 0 - 25Hz as defined by the sampling interval. It should be noted that the random error in the spectral estimates is about 20% and is given by:-

$$\epsilon_r = \frac{1}{\sqrt{B_E T}}$$

where B_E is the bandwidth of analysis and T the record length.

It is apparent that for any sampling interval the shape of the spectrum within the associated frequency range will also be white. The subroutine can therefore generate sets of numbers which, by correct choice of associated time co-ordinates, can represent a signal which is white within the specified range.

The higher correlation functions for the sample were computed as outlined below. <u>One point must be stressed</u>: it will be seen from the graphs representing these higher correlations that the x-axis is labelled 'data spacing units'. The reason for this is since the time co-ordinates are arbitrary so too are the lag values. The values of the correlation functions are computed only at multiples of the data spacing (for obvious reasons). However, for visual interpretation,

these values are joined by straight lines. <u>This does not imply that the continuous correlation function has this form.</u> In fact, it is apparent that the continuous graph could not possibly have such a form since there is no reason why these values should not occur within, say the first lag length by suitable choice of sampling interval.

A.2   ESTIMATION OF THE HIGHER CORRELATION FUNCTIONS

As mentioned above for frequency analysis, the 3000 valued sample was considered to be a 60 second record of Gaussian noise in the range 0 - 25Hz. This range is determined by the sampling interval, since higher frequencies cannot be represented unless the data spacing is reduced. The amplitude-time and spectral density plots are given in Figures 2,3.

The third correlation function $R(\tau_1 \tau_2)$ for values of $\tau_1, \tau_2 = n\delta$ where δ is the data spacing and $n = 0,1,2,.....20$ was computed using the following equation:-

$$R(\tau_1 \tau_2) = \sum_{i=1}^{N-(\tau_1+\tau_2)} \frac{x_i \; x_{i+\tau_1} \; x_{i+\tau_1+\tau_2}}{N - (\tau_1+\tau_2)}$$

where N is the sample size and x_i the individual samples. Initially, the computation was performed with the first 1000 values of the sample.

A selection of the plots obtained for this sample are given in Figures 4-7. At this stage, a check was made on the effect of altering the order of the sample - in effect, altering the frequency content of the signal. The numbers were arranged in certain sequences which had the effect of reducing the predominant bandwidth of the signal and the third correlation function recomputed. Again a selection of the plots for this set (2) are given in Figures 8-11. It is apparent from the plots of Set 2 that, while the individual values of $R(\tau_1 \tau_2)$ differ from those of Set 1, the plots have a similar nature. Other arrangements of the numbers were tried and considerable time was spent in attempting to relate the values of $R(\tau_1 \tau_2)$ to those of the auto-correlation function $R(\tau)$. No correlation could be found. It was then suspected that the

third correlation function was <u>independent</u> of the auto-correlation function. Bearing in mind that it has been long known that the mean value and auto-correlation function define all higher orders, this could only mean that $R(\tau_1 \tau_2) = 0$ for all τ_1, τ_2 for a Gaussian variable. The values obtained were clearly being caused by the non-continuity of the distribution - ie since the sample is finite.

The sample was now increased to 3000 and then to 10,000, the third correlation function being computed each time. See, for example, Figures 12-14 which show selected plots for the 10,000 valued sample. It is immediately apparent that for each τ_1, τ_2 the value of $R(\tau_1 \tau_2)$ is much less than for the 1000 sample. It was noticed too, that the individual values of $R(\tau_1 \tau_2)$ were typically of the same order as the mean cube value of the data. Clearly then, as the distribution approaches the true Gaussian - as the sample gets very large - then the mean cube value will tend to zero as will the third correlation function $R(\tau_1 \tau_2)$.

It is worth noting at this stage that where the distribution is $N(\mu, \sigma^2)$, $R(\tau_1 \tau_2)$ is given by:-

$$R_{(\mu,\sigma^2)}(\tau_1 \tau_2) = \mu^3 + \mu\sigma^2 (R(\tau_1) + R(\tau_1 + \tau_2) + R(\tau_2))$$

where $R(\ )$ refers to the $N(0,1)$ distribution. Computation of the fourth correlation function $R(\tau_1 \tau_2 \tau_3)$ was conducted in a similar manner:-

$$R(\tau_1 \tau_2 \tau_3) = \sum_{i=1}^{N-(\tau_1 + \tau_2 + \tau_3)} \frac{x_i \, x_{i+\tau_1} \, x_{i+\tau_1+\tau_2} \, x_{i+\tau_1+\tau_2+\tau_3}}{N - (\tau_1 + \tau_2 + \tau_3)}$$

On computing this function for the 1000 valued sample, it was immediately apparent that some of the values were approximately zero (10^{-2}) while other values were either 1 or 3. Since the auto-correlation of the

sample is given by:-

$$R(\tau) = 1 \qquad \tau = 0$$

$$R(\tau) \doteq 0 \qquad \tau > 0$$

the following relationship was found by a process of elimination:-

$$R(\tau_1 \tau_2 \tau_3) = R(\tau_1)R(\tau_2) + R(\tau_1 + \tau_2)R(\tau_2 + \tau_3) + R(\tau_1 + \tau_2 + \tau_3)R(\tau_2)$$

It was thought that this relation, if correct, could have been derived analytically and the investigations into the theory of random processes showed these relations to be correct. A comparison of selected values of the theoretical and experimental computations of $R(\tau_1 \tau_2 \tau_3)$ are given in Table 1.

Again, where the distribution is $N(\mu, \sigma^2)$, the fourth correlation function is given by:-

$$R(\tau_1 \tau_2 \tau_3) = \sigma^4 R(\tau_1 \tau_2 \tau_3) + \mu^2 \sigma^2 \Big[R(\tau_1) + R(\tau_2) + R(\tau_3)$$

$$+ R(\tau_1 + \tau_2) + R(\tau_2 + \tau_3) + R(\tau_1 + \tau_2 + \tau_3) \Big] + \mu^4$$

where $R(\ )$ relates to the $N(0,1)$ distribution.

A.3   ANALYTICAL DERIVATION OF HIGHER PRODUCT MOMENTS OF SINUSOIDAL FUNCTIONS

A.3.1 SINGLE SINUSOID

Let $X(t) = a\cos(t)$. Recall that $R(\tau) = \dfrac{a^2}{2}\cos(\tau)$

Now,

$$R(\tau_1 \tau_2) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)X(t+\tau_1)X(t+\tau_1+\tau_2)\,dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} a^3 \cos(t)\cos(t+\tau_1)\cos(t+\tau_1+\tau_2)\,dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} I\,dt$$

It can be shown that:-

$$I = \frac{a^3}{4}\left[\cos(t+2\tau_1+\tau_2) - \cos(t+\tau_2) + \cos(3t+\tau_2) + \cos(t-2\tau_1-\tau_2)\right].$$

and hence it follows that:-

$$R(\tau_1 \tau_2) = 0$$

Similarly it can be shown that:-

$$R(\tau_1 \tau_2 \tau_3) = \frac{a^4}{8}\left[\cos(\tau_1-\tau_2-\tau_3) + \cos(\tau_2-\tau_3-\tau_1) + \cos(\tau_3-\tau_1-\tau_2)\right].$$

and in general for $X(t) = a\cos(t)$:-

$$R(\tau_1 \tau_2 \tau_3 \ldots \tau_n) = \frac{a^{n+1}}{2^n}\sum_{r=1}^{n}\cos\left(2\tau_r - \sum_{j=1}^{n}\tau_j\right) \quad \text{for n odd}$$

A.3.1.1

$$= 0 \qquad\qquad\qquad \text{for n even}$$

A.3.2 GENERAL PERIODIC FUNCTION

Let $X(t) = \cos t + \cos(2t + \psi)$. Recall that $R(\tau) = \frac{1}{2}(\cos\tau + \cos 2\tau)$.

After considerable algebra it can be shown that:-

$$R(\tau_1 \tau_2) = \frac{1}{4}\left(\cos(\tau_1 + 2\tau_2 + \psi) + \cos(2\tau_1 + \tau_2 - \psi) + \cos(\tau_1 - \tau_2 + \psi)\right)$$

Note that the above function is dependent upon the phase angle ψ in contrast to the auto-correlation function. It can be further shown that all the odd moments are dependent on the phase angle whereas the even moments are not.

In general then, for $X(t) = \cos t + \cos(2t + \psi_1) + \cos(3t + \psi_2) \ldots$:-

$$R(\tau_1 \ldots \tau_{2n}) = F(\tau_1, \tau_2 \ldots \tau_{2n}, \psi_1, \psi_2 \ldots \psi_n) \qquad \text{A.3.2.1}$$

and,

$$R(\tau_1 \ldots \tau_{2n+1}) = F(\tau_1 \tau_2 \ldots \tau_{2n+1}) \quad \text{for all } n = 1, 2, \ldots.$$

A.3.3 PSEUDO RANDOM BINARY FUNCTION

A digital sample of a P.R.B signal was obtained. The third order product moment function $R(\tau_1 \tau_2)$ was computed as before and typical plots are given in Figures 15-17.

APPENDIX   B

B.1   HIGHER ORDER INPUT-OUTPUT SPECTRAL RELATIONS

Consider the linear system in Figure 18 with complex frequency response $a(if)$, impulsive receptance $W(\tau)$, and input-output $P(t)$, $x(t)$. It follows that:-

$$x(t) = \int_0^\infty W(\tau)P(t-\tau)d\tau \qquad W(\tau)=0, \ \tau<0 \qquad (1)$$

Now $R_x(\tau_1 \tau_2) = <x(t)x(t+\tau_1)x(t+\tau_1+\tau_2)>$

but $x(t) = \int_0^\infty W(\tau_\alpha)P(t-\tau_\alpha)d\tau_\alpha$

$x(t+\tau_1) = \int_0^\infty W(\tau_\beta)P(t-\tau_\beta+\tau_1)d\tau_\beta$

$x(t+\tau_1+\tau_2) = \int_0^\infty W(\tau_\gamma)P(t-\tau_\gamma+\tau_1+\tau_2)d\tau_\gamma$

$$\Longrightarrow R_x(\tau_1 \tau_2) = \int_0^\infty W(\tau_\alpha) \int_0^\infty W(\tau_\beta) \int_0^\infty W(\tau_\gamma)$$

$$\cdot <P(t-\tau_\alpha)P(t-\tau_\beta+\tau_1)P(t-\tau_\gamma+\tau_1+\tau_2)>d\tau_\alpha d\tau_\beta d\tau_\gamma$$

Replacing $(t-\tau_\alpha)$ by t (since process is stationary)

$$R_x(\tau_1 \tau_2) = \int_0^\infty W(\tau_\alpha) \int_0^\infty W(\tau_\beta) \int_0^\infty W(\tau_\gamma)$$

$$\cdot <P(t)P(t+\tau_\alpha-\tau_\beta+\tau_1)P(t+\tau_\alpha+\tau_1+\tau_2-\tau_\gamma)>d\tau_\alpha d\tau_\beta d\tau_\gamma$$

$$= \int_0^\infty W(\tau_\alpha) \int_0^\infty W(\tau_\beta) \int_0^\infty W(\tau_\gamma)R_p(\tau_1+\tau_\alpha-\tau_\beta,\tau_2+\tau_\beta-\tau_\gamma)d\tau_\alpha d\tau_\beta d\tau_\gamma$$

Defining $S_x(f_1 f_2) = \iint_{-\infty}^\infty R_x(\tau_1 \tau_2)e^{i2\pi(f_1\tau_1+f_2\tau_2)}d\tau_1 d\tau_2 \qquad (2)$

i.e. the double Fourier Transform of $R_x(\tau_1 \tau_2)$

$$\Rightarrow \quad S_x(f_1 f_2) = \int_0^\infty W(\tau_\alpha) \int_0^\infty W(\tau_\beta) \int_0^\infty W(\tau_\gamma)$$

$$\cdot \int_{-\infty}^\infty R_p(\tau_1 + \tau_\alpha - \tau_\beta, \tau_2 + \tau_\beta - \tau_\gamma) e^{i2\pi(f_1 \tau_1 + f_2 \tau_2)} d\tau_\alpha d\tau_\beta d\tau_\gamma$$

$$= \int_0^\infty W(\tau_\alpha) e^{-i2\pi f_1 \tau_\alpha} \int_0^\infty W(\tau_\beta) e^{-i2\pi(f_2 - f_1)\tau_\beta} \int_0^\infty W(\tau_\gamma) e^{i2\pi f_2 \tau_\gamma}$$

$$\cdot \int_{-\infty}^\infty R_p(\tau_1 + \tau_\alpha - \tau_\beta, \tau_2 + \tau_\beta - \tau_\gamma)$$

$$\cdot e^{i2\pi(f_1(\tau_1 + \tau_\alpha - \tau_\beta) + f_2(\tau_2 + \tau_\beta - \tau_\gamma))} d\tau_\alpha d\tau_\beta d\tau_\gamma$$

$$= \int_0^\infty W(\tau_\alpha) e^{-i2\pi f_1 \tau_\alpha} \int_0^\infty W(\tau_\beta) e^{-i2\pi(f_2 - f_1)\tau_\beta} \int_0^\infty W(\tau_\gamma) e^{i2\pi f_2 \tau_\gamma}$$
$$\times S_p(f_1 f_2)$$

$$= a(if_1) a(i(f_2 - f_1)) a^*(if_2) S_p(f_1 f_2),$$

$$\therefore \quad S_x(f_1 f_2) = a(if_1) a(i(f_2 - f_1)) a^*(if_2) S_p(f_1 f_2) \tag{3}$$

Similarly it can be shown:-

$$S_x(f_1 f_2 \ldots f_n) = a(if_1) a(i(f_2 - f_1)) \ldots a(i(f_n - f_{n-1})) a^*(if_n) S_p(f_1 f_2) \tag{4}$$

Note that substitution of $f_n = 0$ and $a(if_n) = 0$ in (4) yields the

(n-1)th response relation.

For $n = 1$, equation (4) gives:-

$$S_x(f_1) = a(if_1) a^*(if_1) S_p(f_1) = |a(if)|^2 S_p(f)$$

APPENDIX   C

RECOMMENDED MINIMUM SAMPLE SIZE FOR NARROW-BAND RESPONSE SIMULATION

The minimum sample size recommendation is based upon the assumption that in using numerical integration in the simulation of response, the output sample size is the same as the specified excitation sample size - this is clearly desirable.

Consider a narrow-band response problem. It will be assumed that the spectrum has a similar shape to the magnification factor of the system. Spectral analysis of this response will impose conditions upon the sample size. In Figure 55 the response spectrum of a typical narrow-band signal is given. The width of the half-power points is $2f_o\zeta$ where f_o is the resonant frequency. It is desirable to have a bandwidth of analysis less than this value, i.e.

$$B_E < 2f_o\zeta$$

Now,

$$n = \frac{T}{\Delta t}$$ where T is the record length, Δt the sampling interval, n the sample size

but,

$$\Delta t = \frac{1}{2f_m}$$ where f_m is the maximum frequency of interest in the record

$$n = 2Tf_m$$

i.e.

$$T = \frac{n}{2f_m}$$

Now for a 20% error in spectral density estimates:-

$$\frac{1}{25} = \frac{1}{B_E T} = \frac{2f_m}{2f_o\zeta n}$$

$$\therefore n \geq \frac{25f_m}{f_o\zeta}$$

If the system response has more than one response peak, the value of n is chosen for the case where the product $f_o\zeta$ is a minimum.
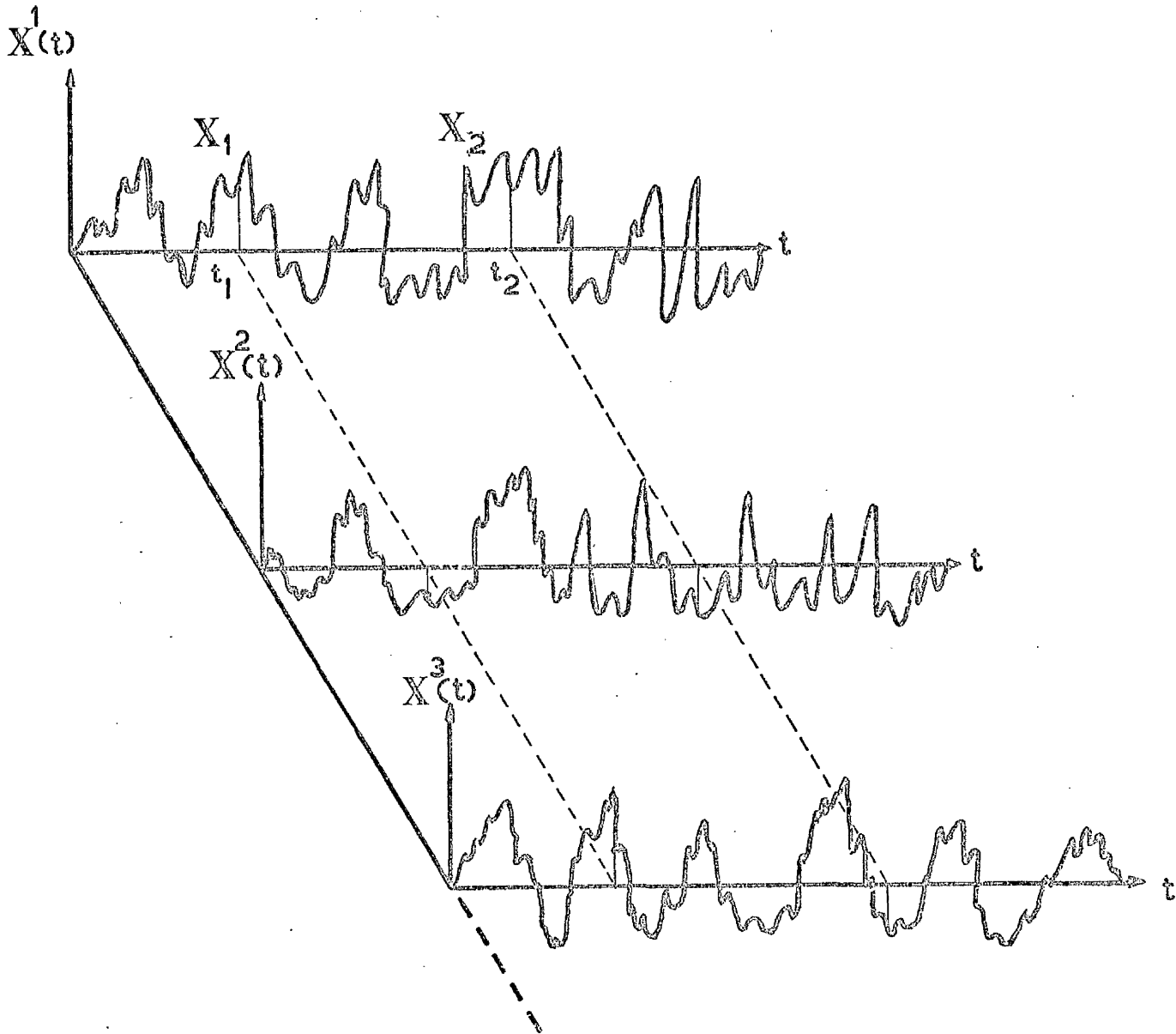
ENSEMBLE OF THE RANDOM PROCESS $\{X(t)\}$

FIGURE 1
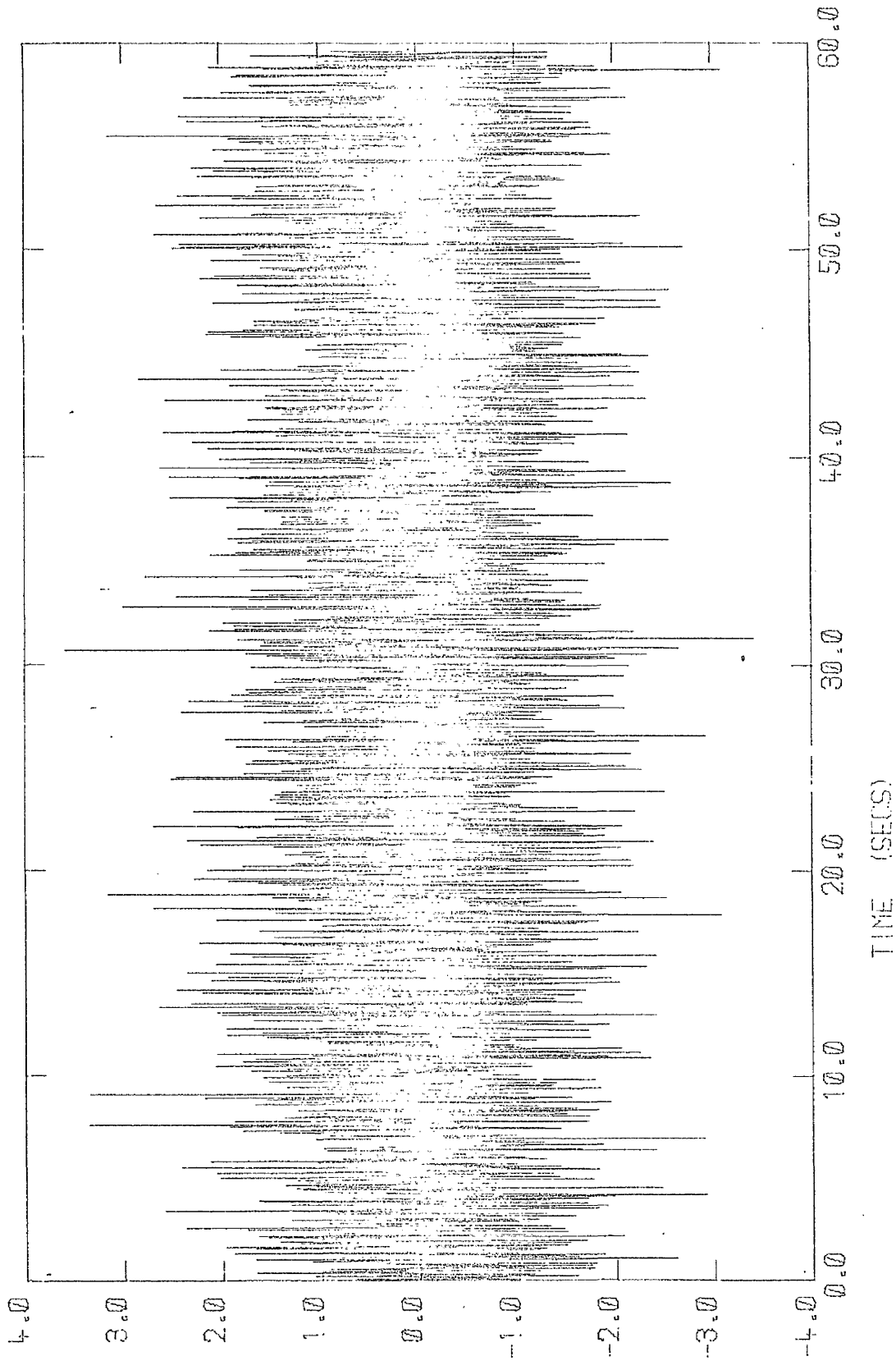
GAUSSIAN 3000 INPUT

TIME (SECS)

AMPLITUDE
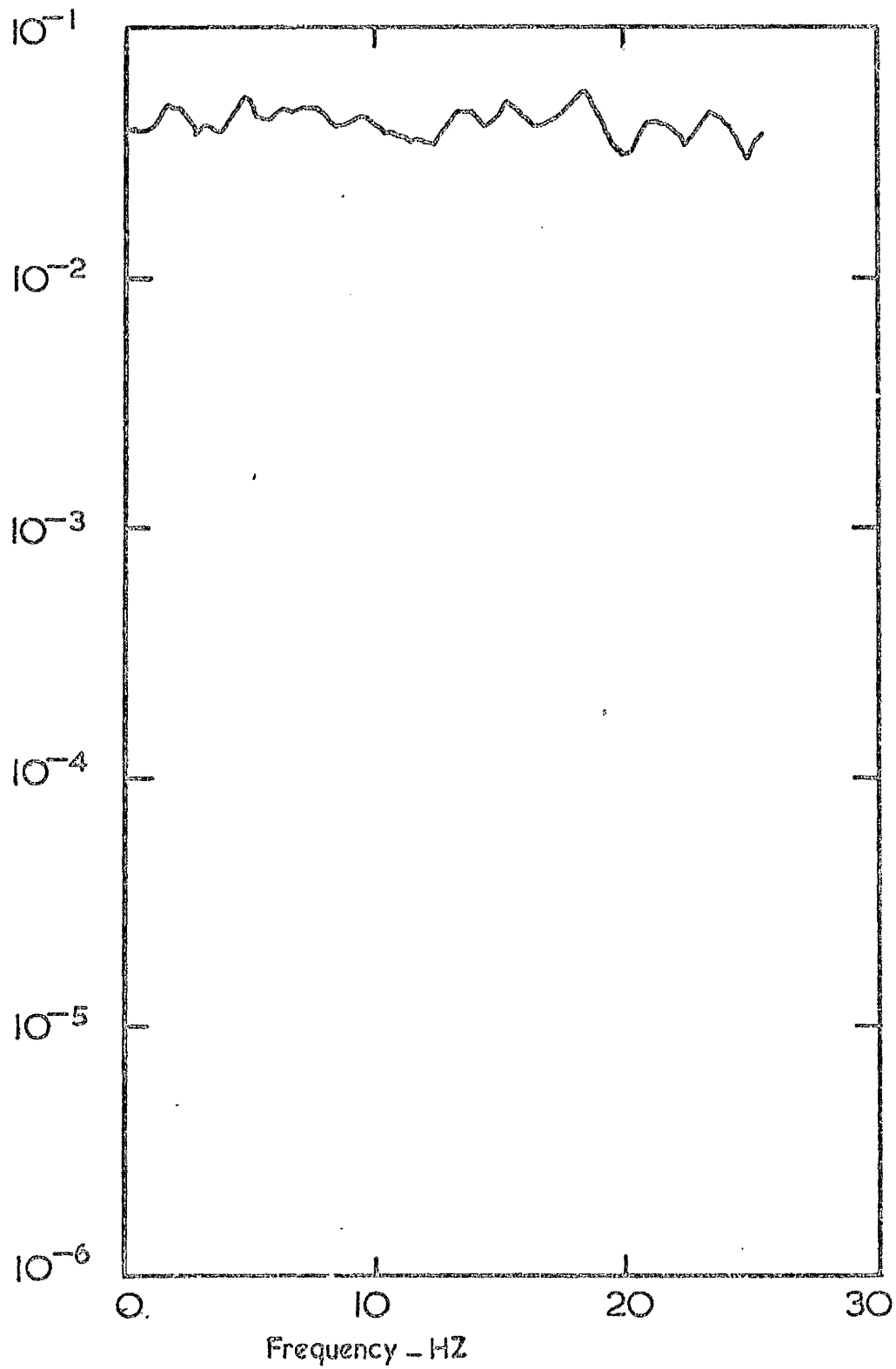
FIGURE 2

GAUSSIAN INPUT POWER SPECTRAL DENSITY

FIGURE 3
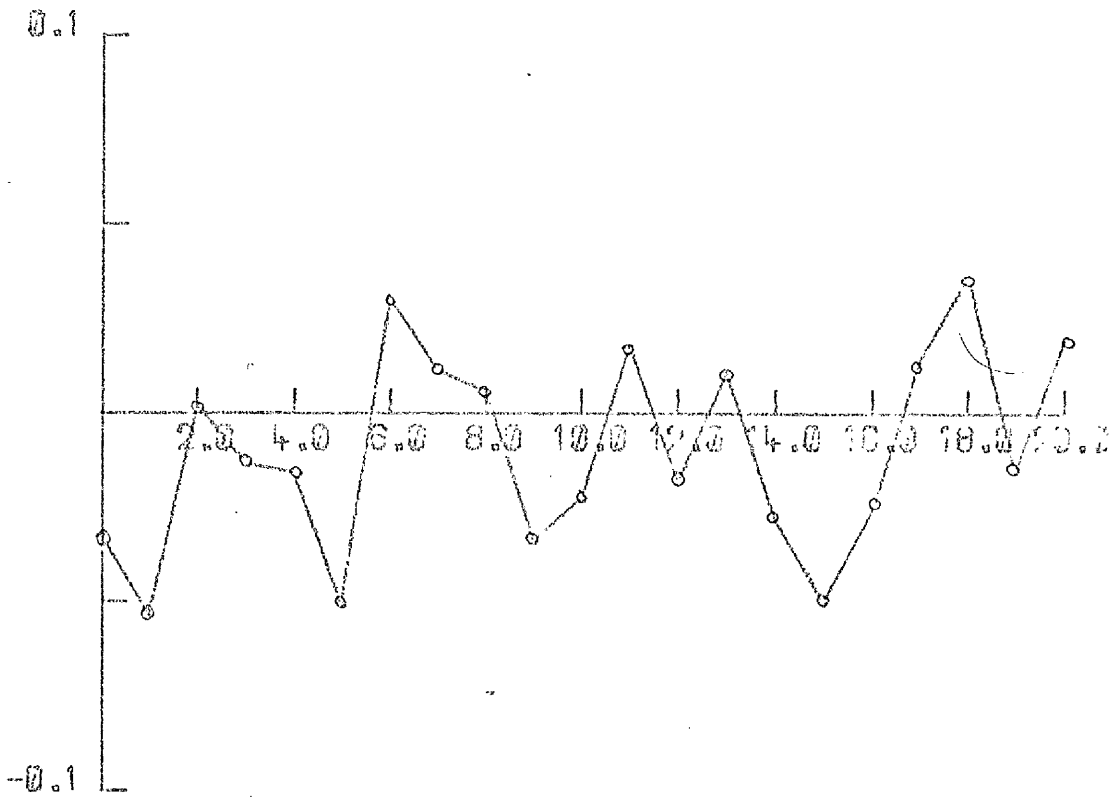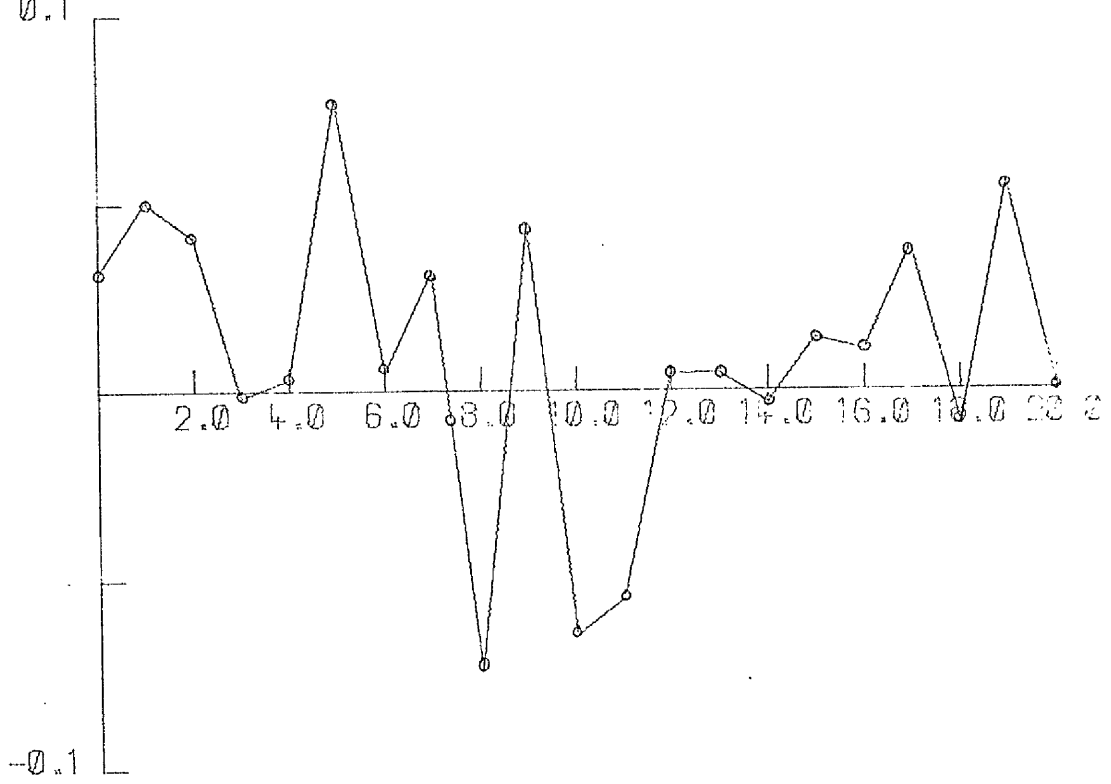
DATA SPACING UNITS   T2

GAUSSIAN (1000) NOISE — R(1,T2)

FIGURE 4
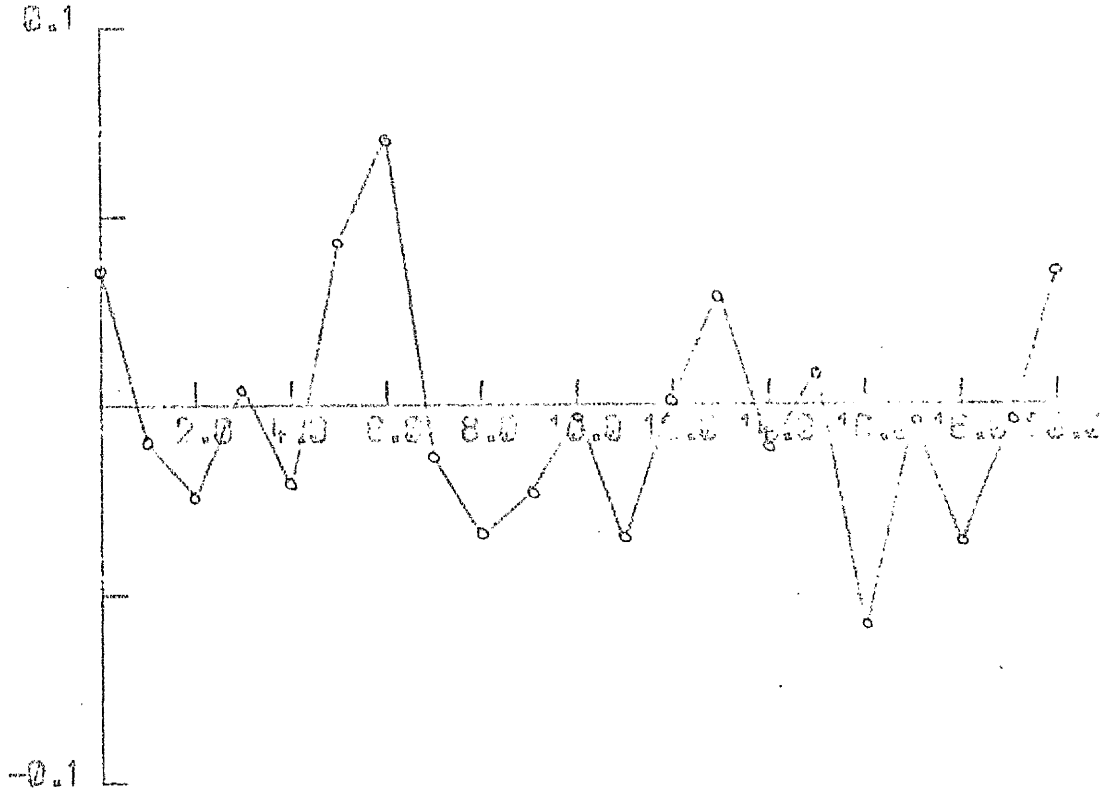
DATA SPACING UNITS T2

GAUSSIAN (1000) NOISE – R(10,T2)
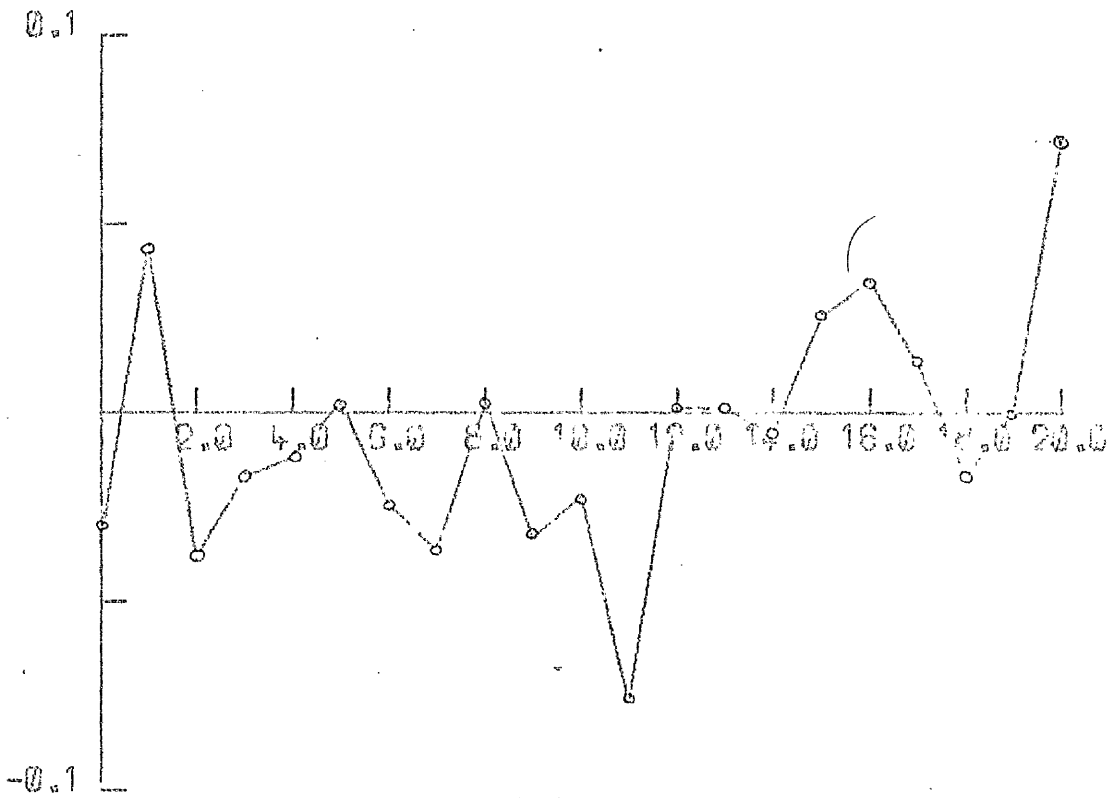
DATA SPACING UNITS T2

GAUSSIAN (1000) NOISE – R(20,T2)
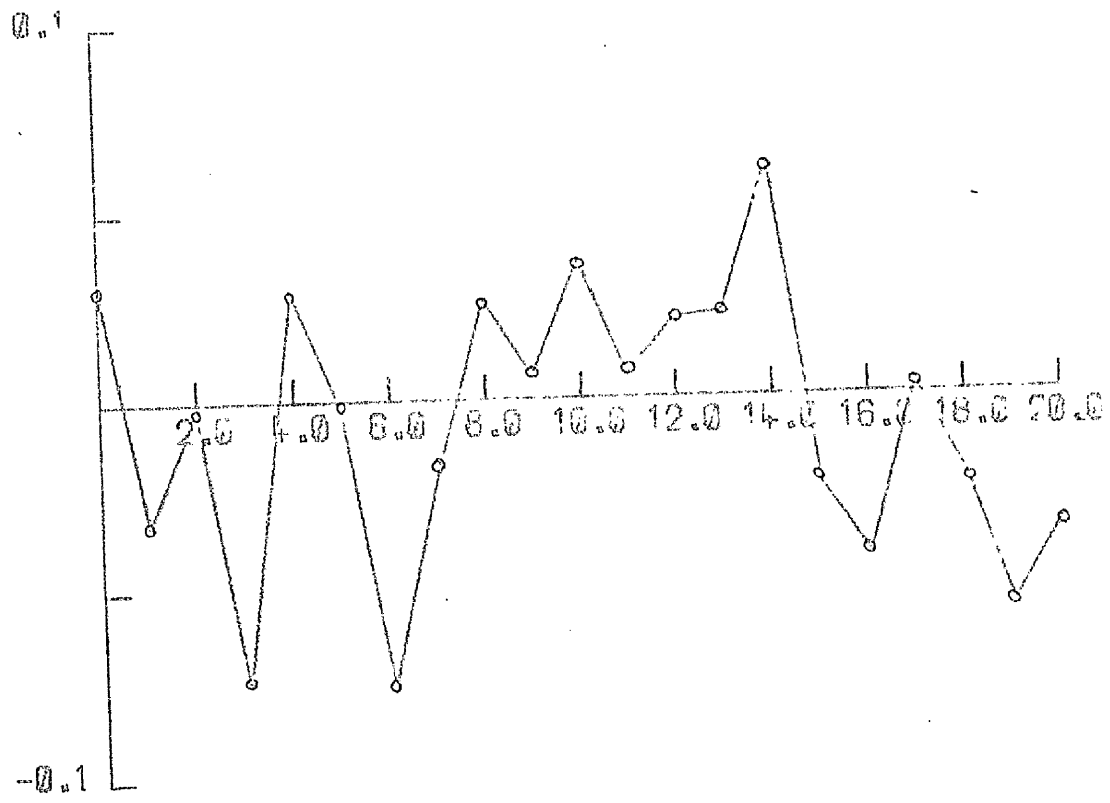
FIGURE 5

DATA SPACING UNITS T1

GAUSSIAN (1000) NOISE – R(T1,0)
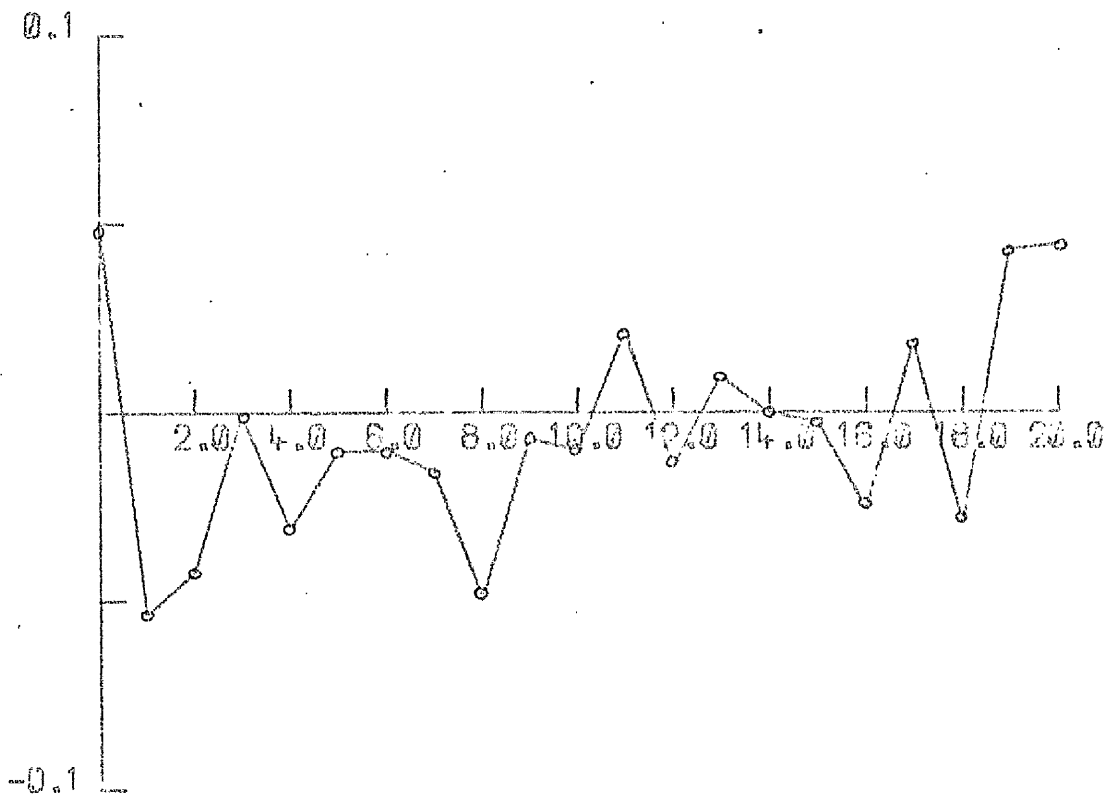
DATA SPACING UNITS T1

GAUSSIAN (1000) NOISE – R(T1,1)

FIGURE 6
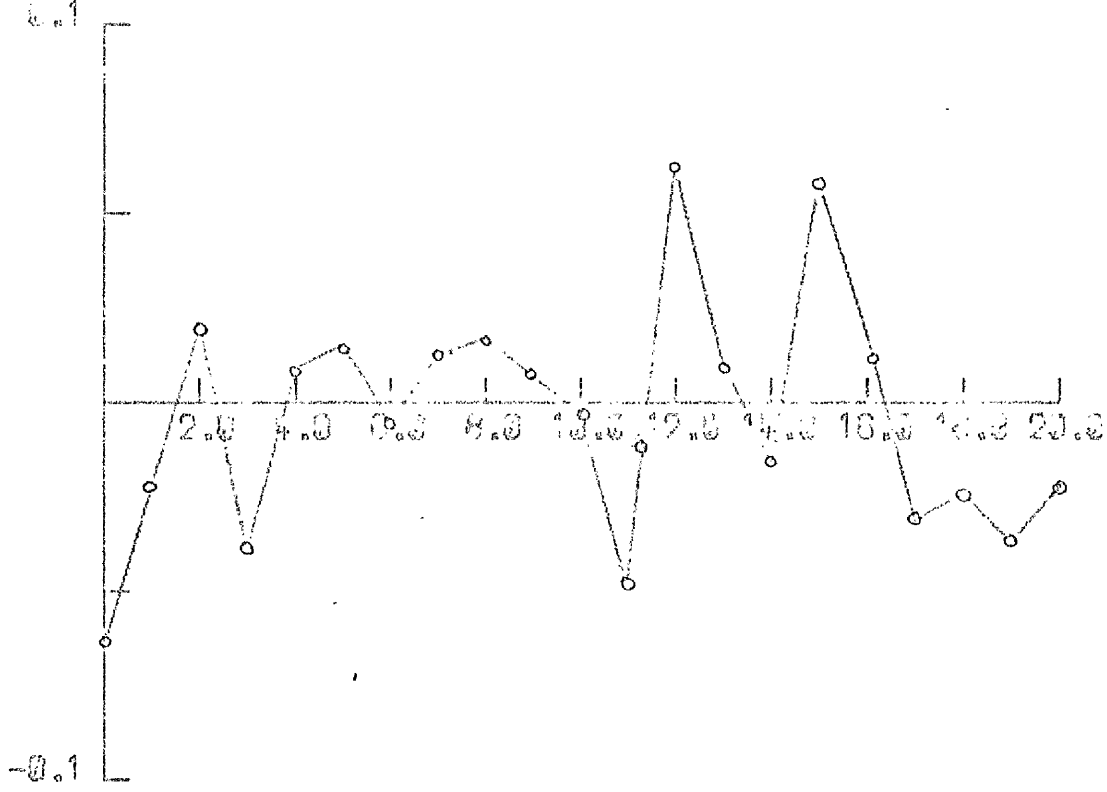
DATA SPACING UNITS T1

GAUSSIAN (1000) NOISE — R(T1,10)
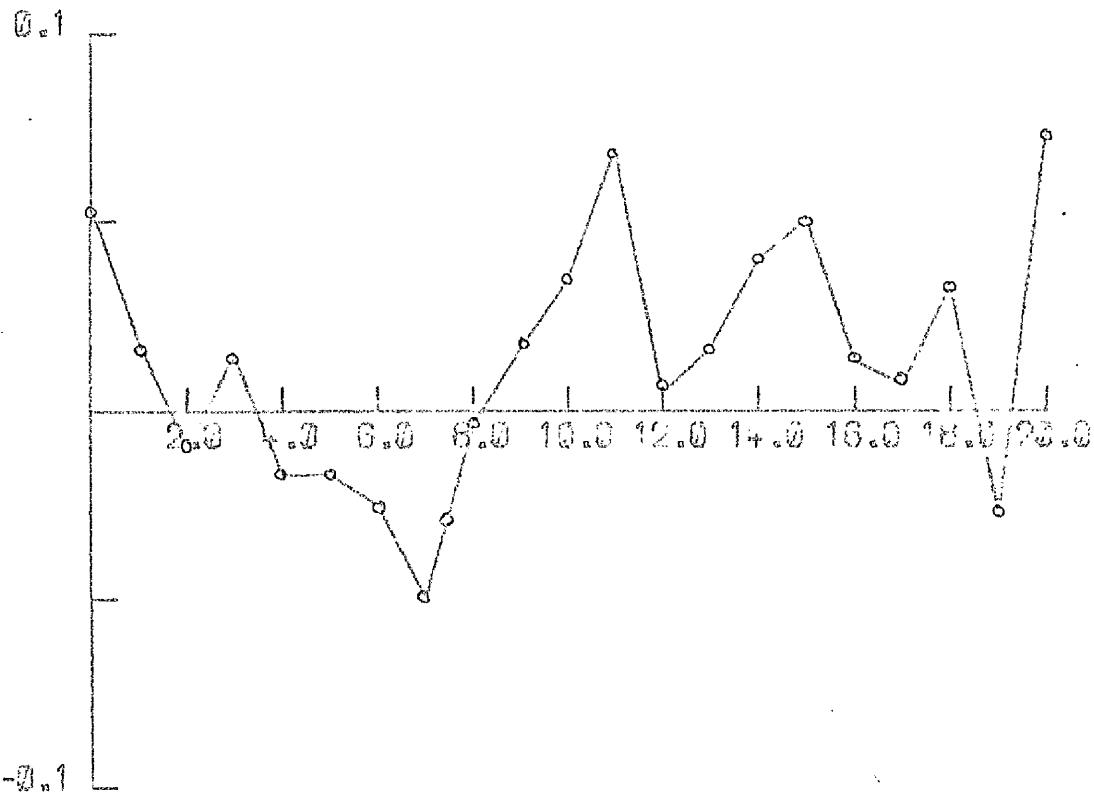
DATA SPACING UNITS T1

GAUSSIAN (1000) NOISE — R(T1,20)
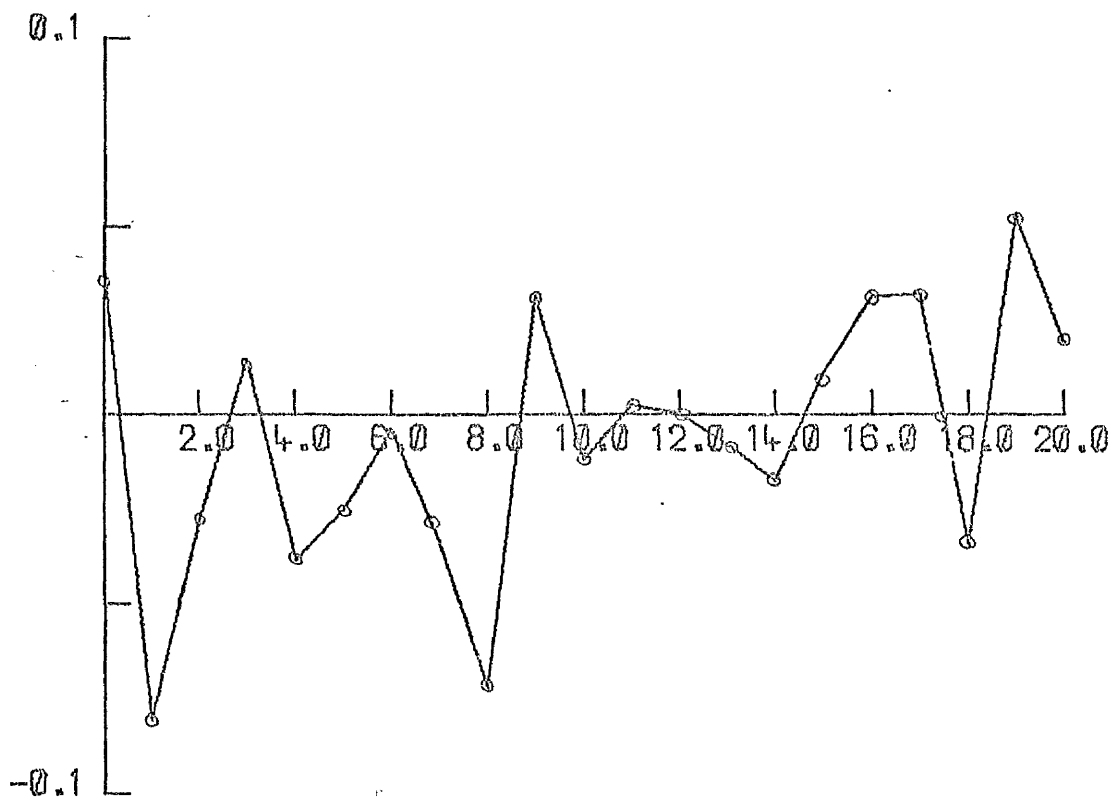
FIGURE 7

DATA SPACING UNITS   T1

GAUSSIAN WHITE NOISE(SET 2) - R(T1,1)

FIGURE 8
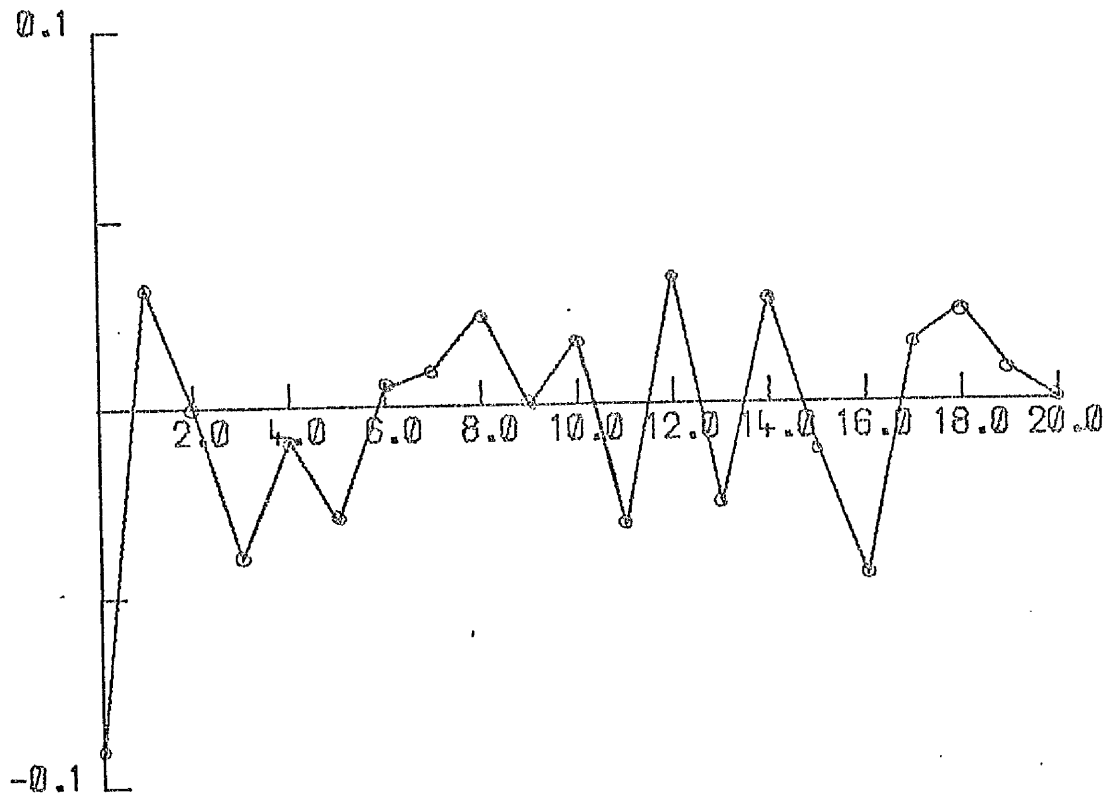
DATA SPACING UNITS T1

GAUSSIAN WHITE NOISE(SET 2) - R(T1,10)
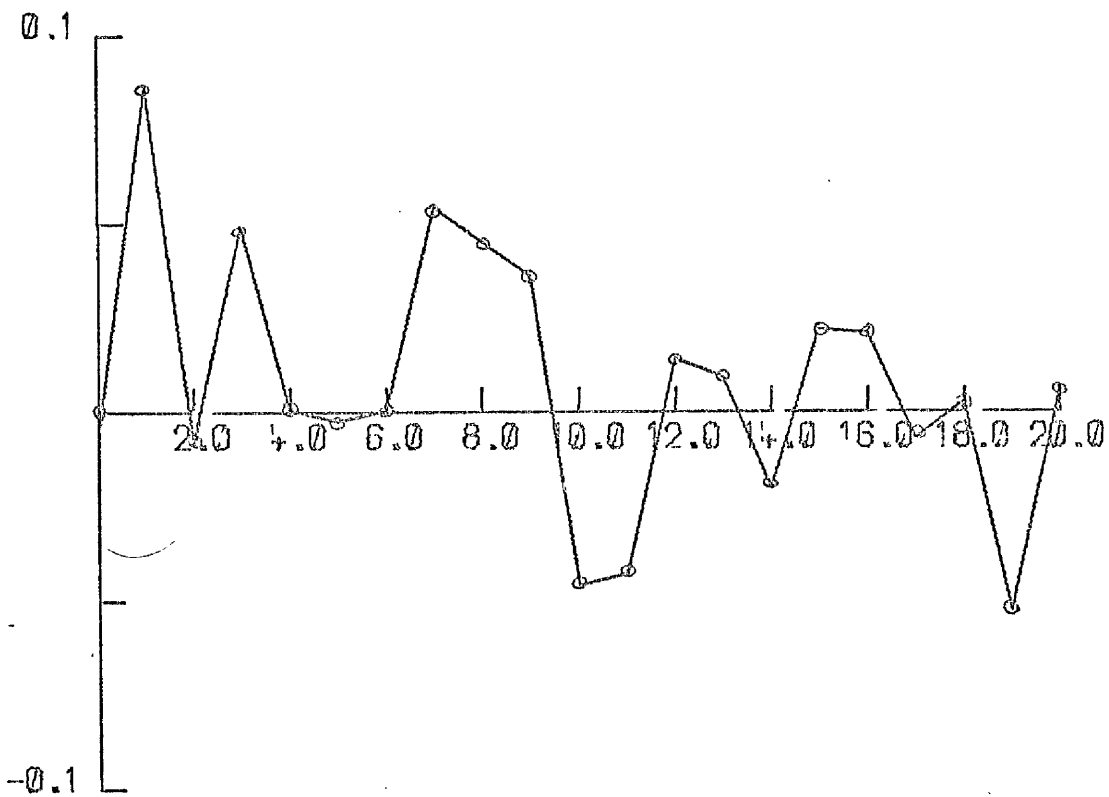
DATA SPACING UNITS T1

GAUSSIAN WHITE NOISE(SET 2) - R(T1,20)

FIGURE 9
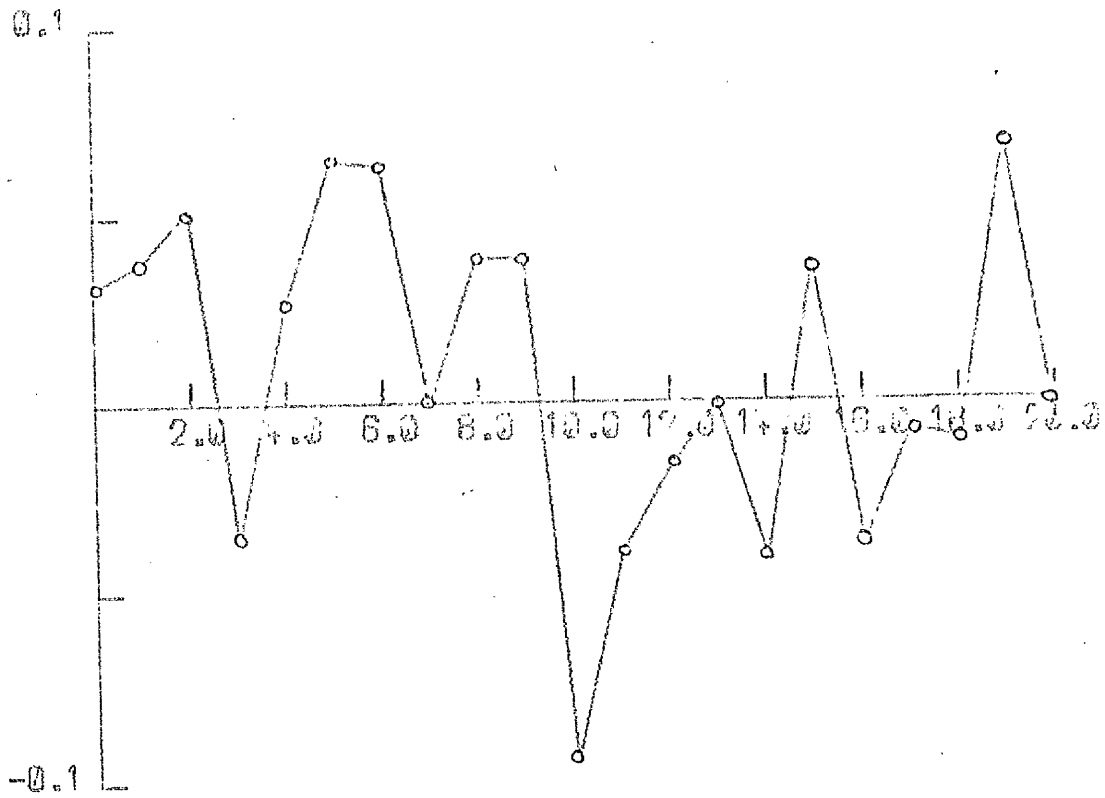
DATA SPACING UNITS T2

GAUSSIAN WHITE NOISE(SET 2) — R(0,T2)
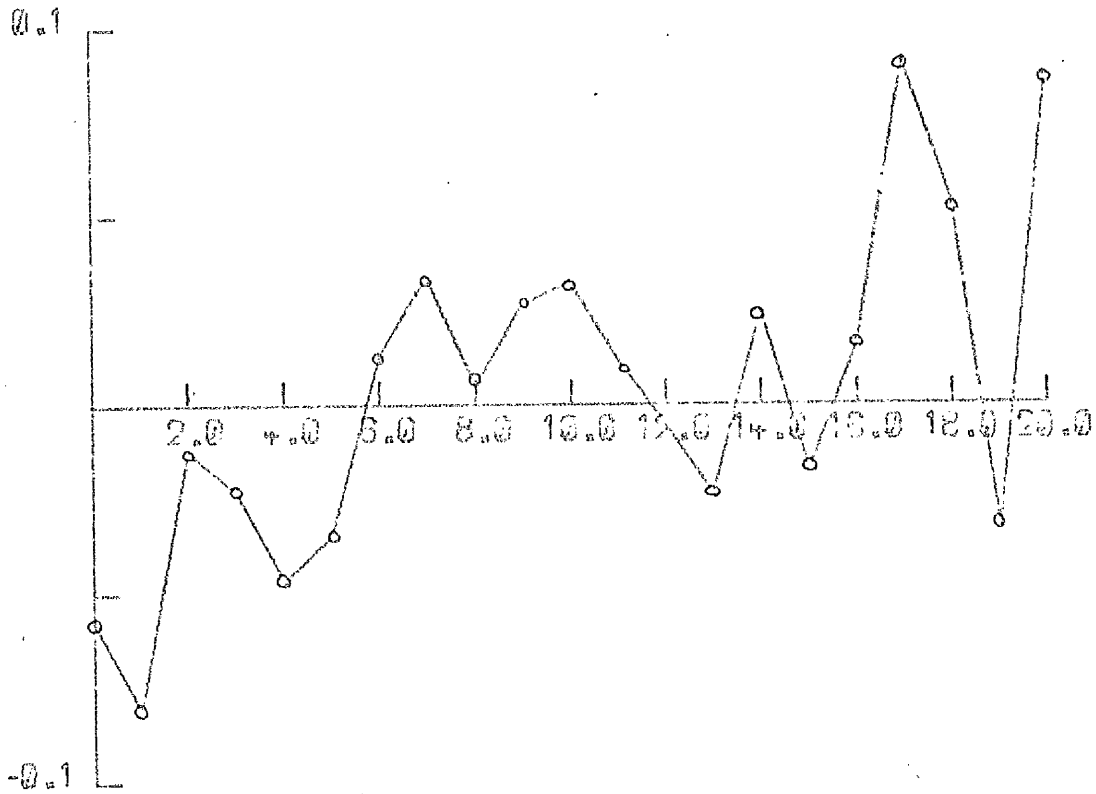
DATA SPACING UNITS T2

GAUSSIAN WHITE NOISE(SET 2) — R(1,T2)

FIGURE 10
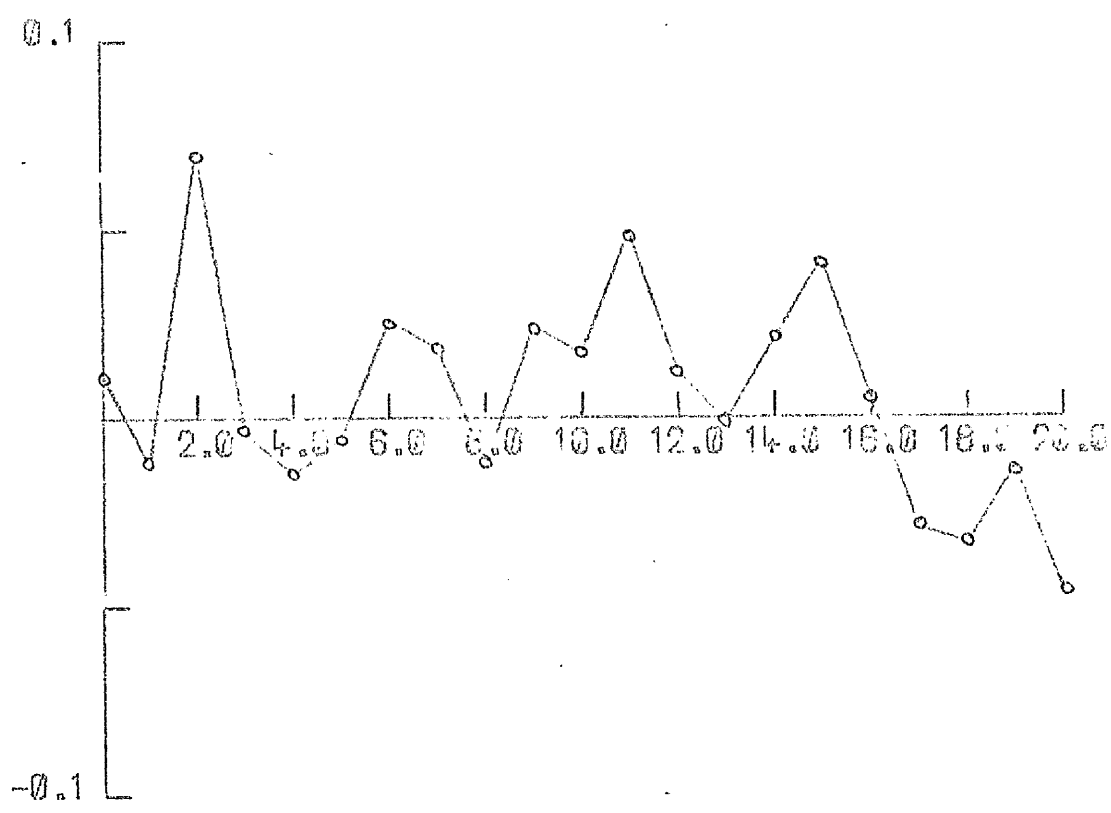
DATA SPACING UNITS T2

GAUSSIAN WHITE NOISE(SET 2) — R(10,T2)
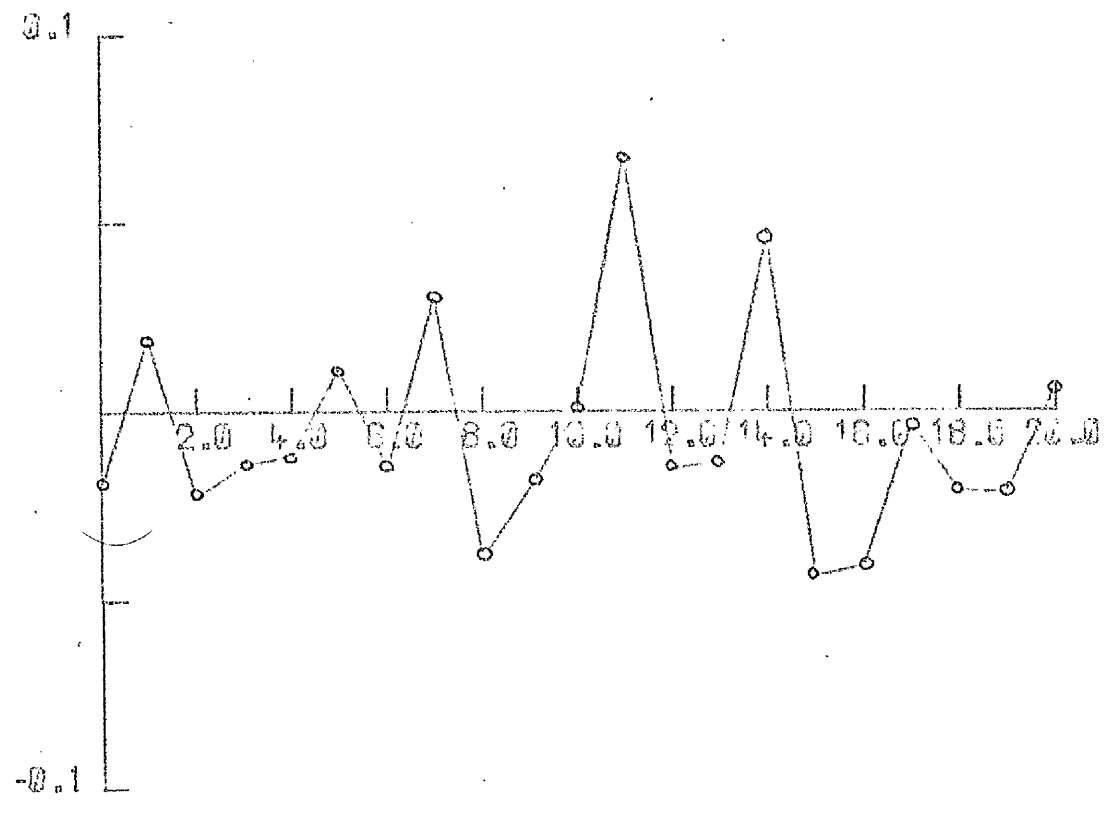
DATA SPACING UNITS T2

GAUSSIAN WHITE NOISE(SET 2) — R(20,T2)

FIGURE II
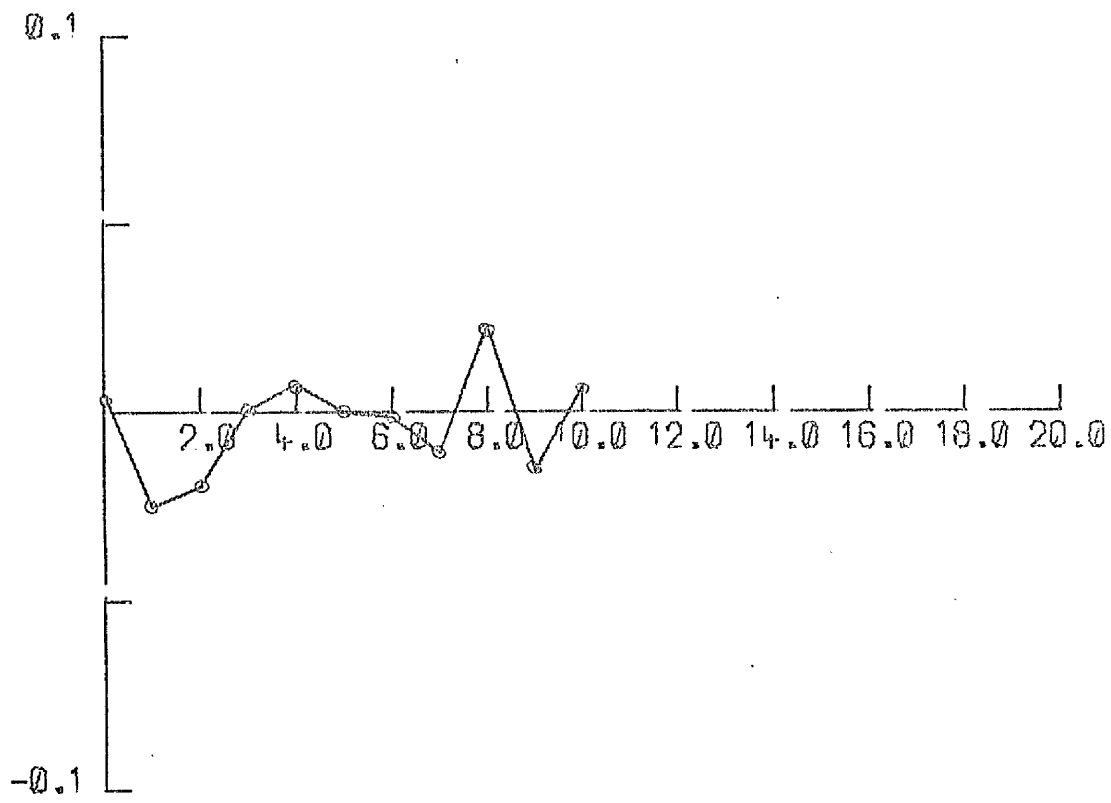
DATA SPACING UNITS   T1

GAUSSIAN (10,000) NOISE - R(T1,1)
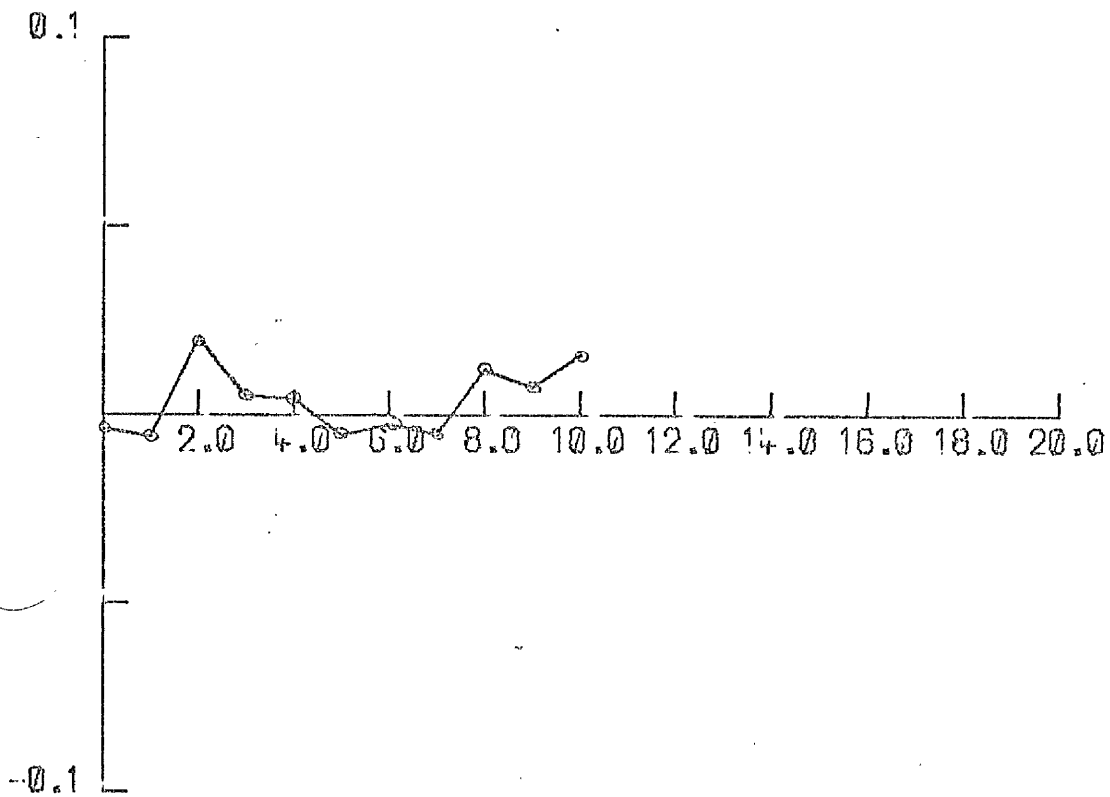
DATA SPACING UNITS   T1

GAUSSIAN (10,000) NOISE - R(T1,10)

FIGURE 12
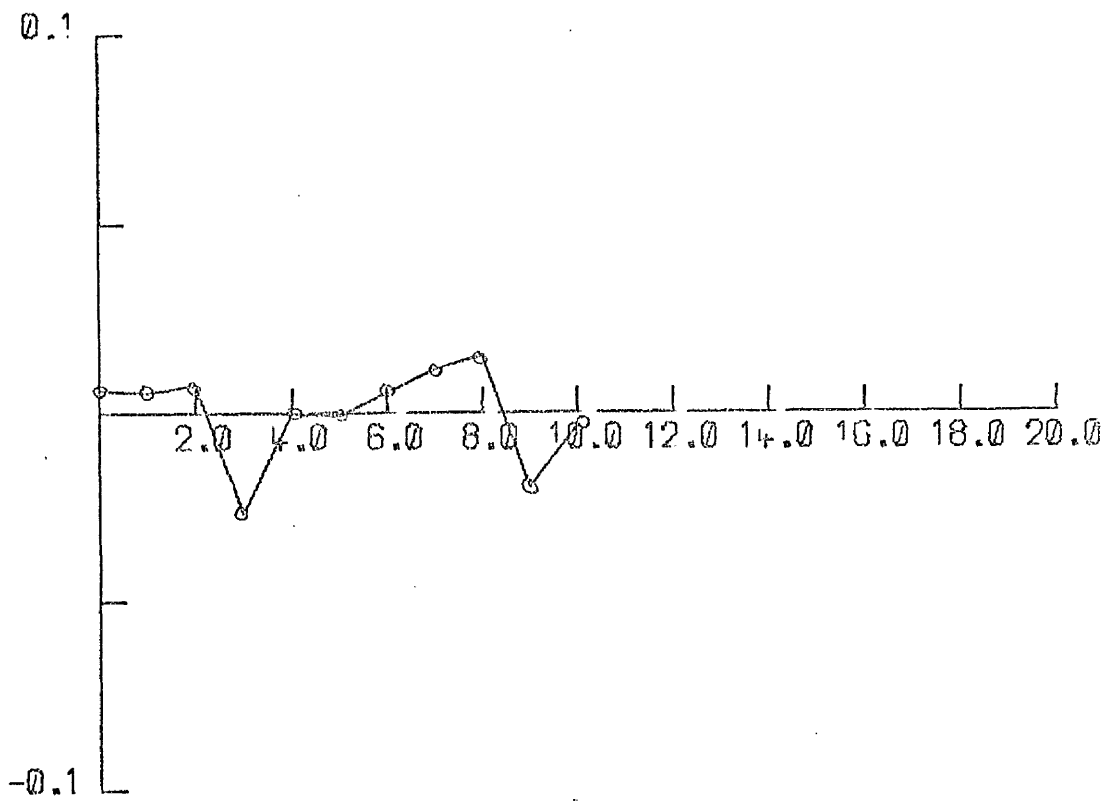
DATA SPACING UNITS T2

GAUSSIAN (10,000) NOISE - R(0,T2)
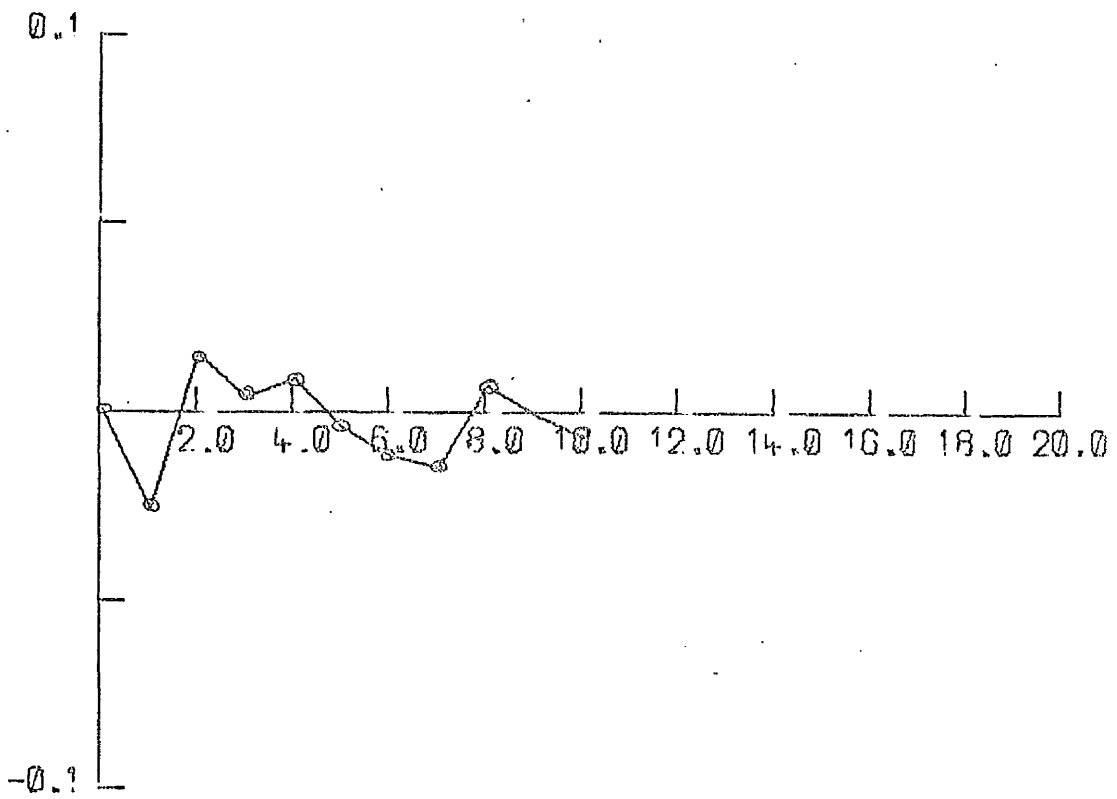
DATA SPACING UNITS T2

GAUSSIAN (10,000) NOISE - R(1,T2)

FIGURE 13
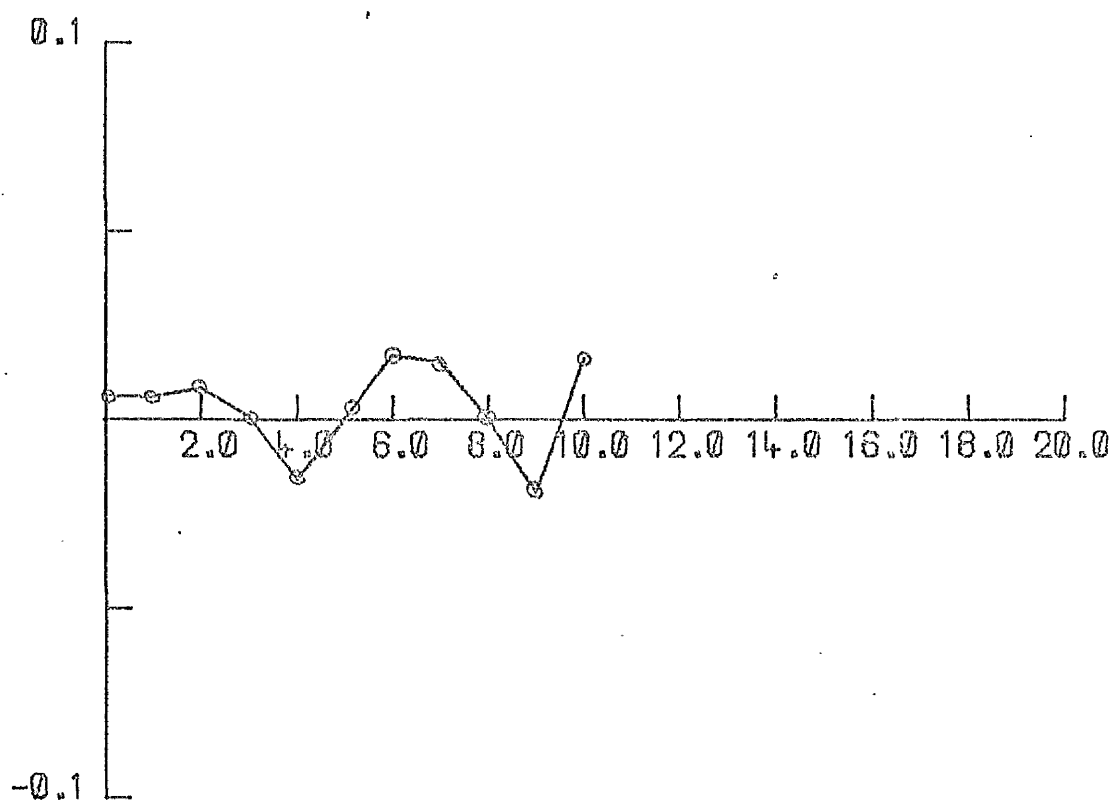
DATA SPACING UNITS T2

GAUSSIAN (10,000) NOISE - R(10,T2)

FIGURE 14
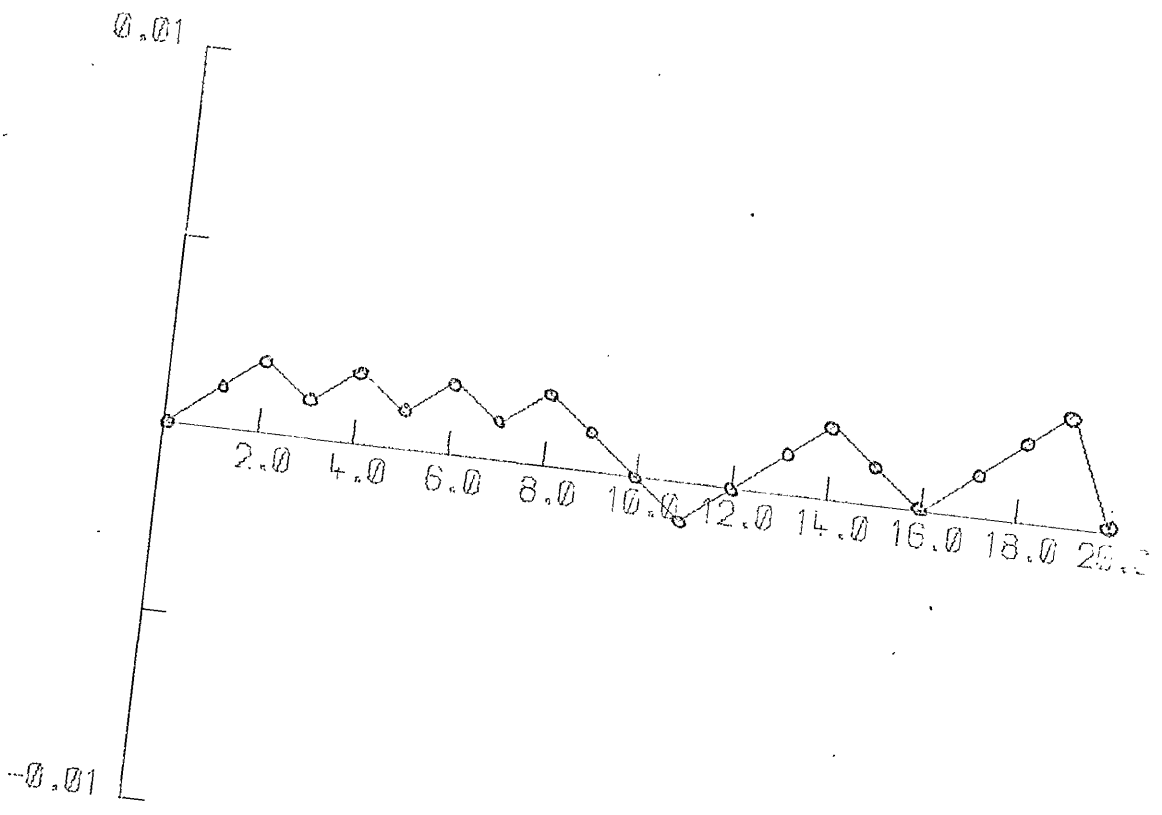
DATA SPACING UNITS T2

PSEUDO RANDOM BINARY – R(T1, 0)
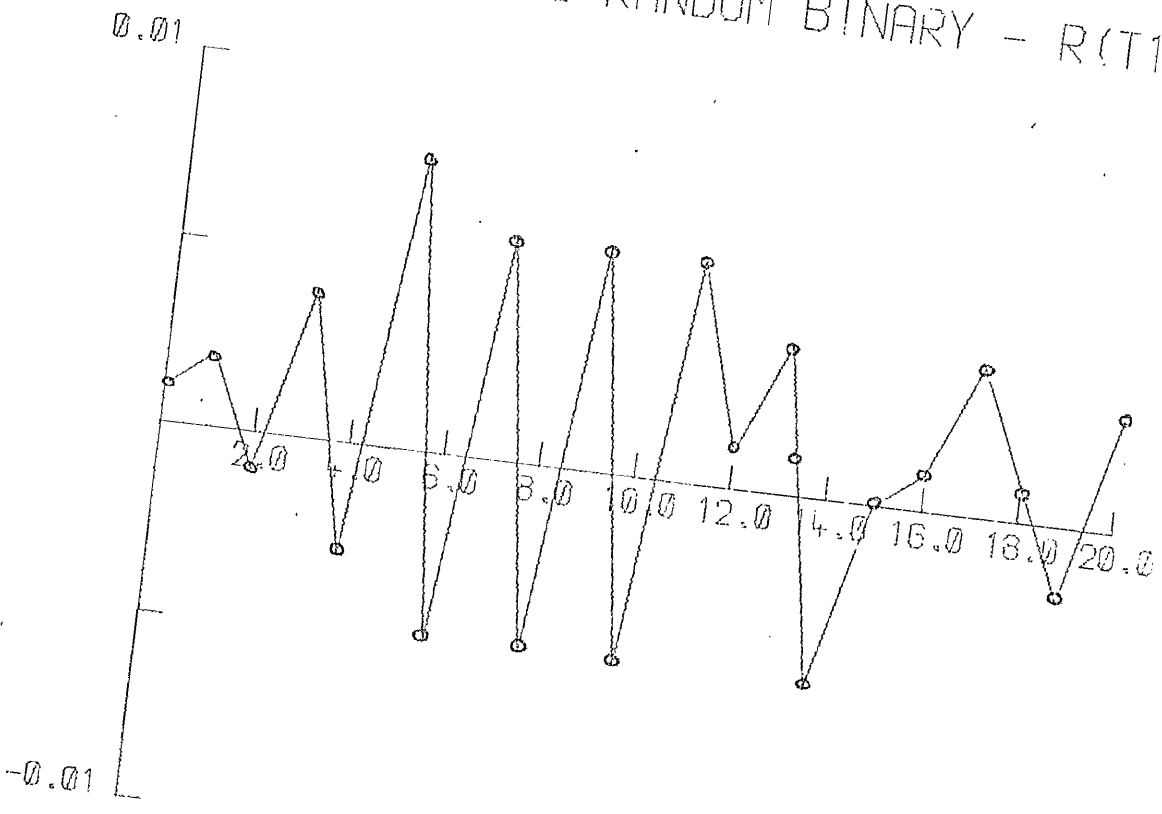
DATA SPACING UNITS T2

PSEUDO RANDOM BINARY – R(T1, 1)

FIGURE 15
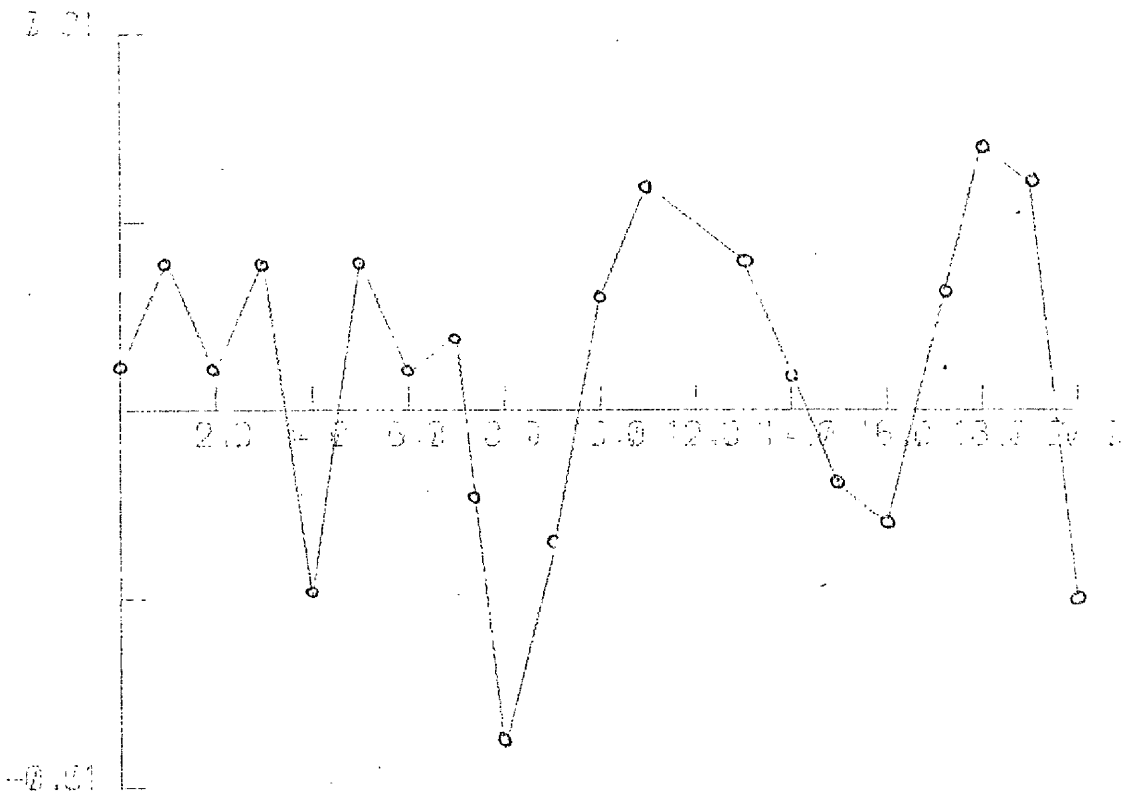
DATA SPACING UNITS T1

PSEUDO RANDOM BINARY — R(T1,5
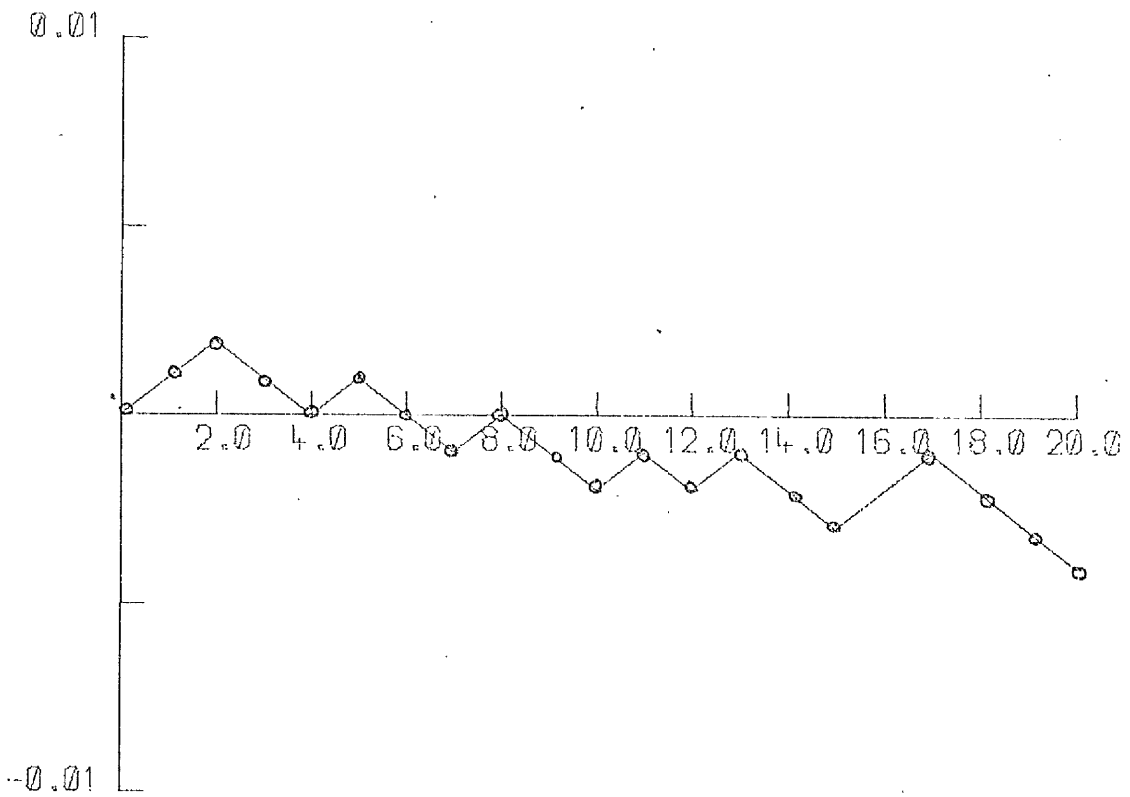
DATA SPACING UNITS T2

PSEUDO RANDOM BINARY — R(0,T2)

FIGURE 16
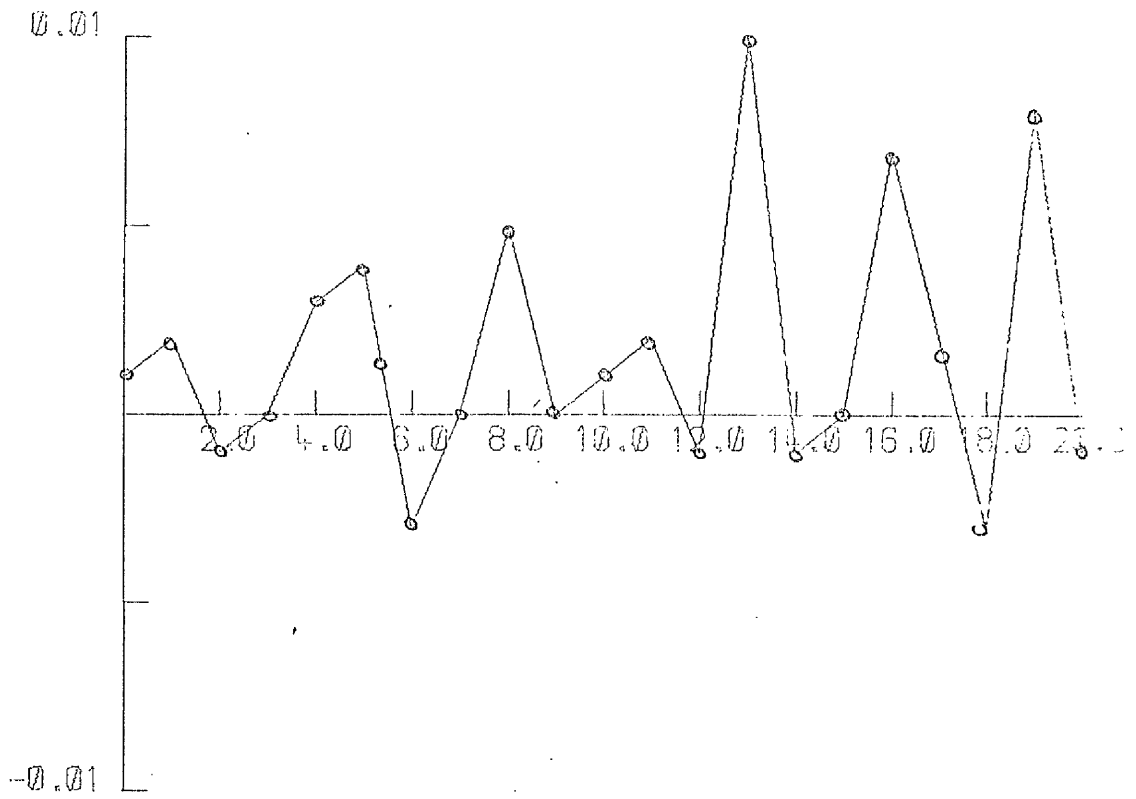
DATA SPACING UNITS T2

PSEUDO RANDOM BINARY - R(1,T2)
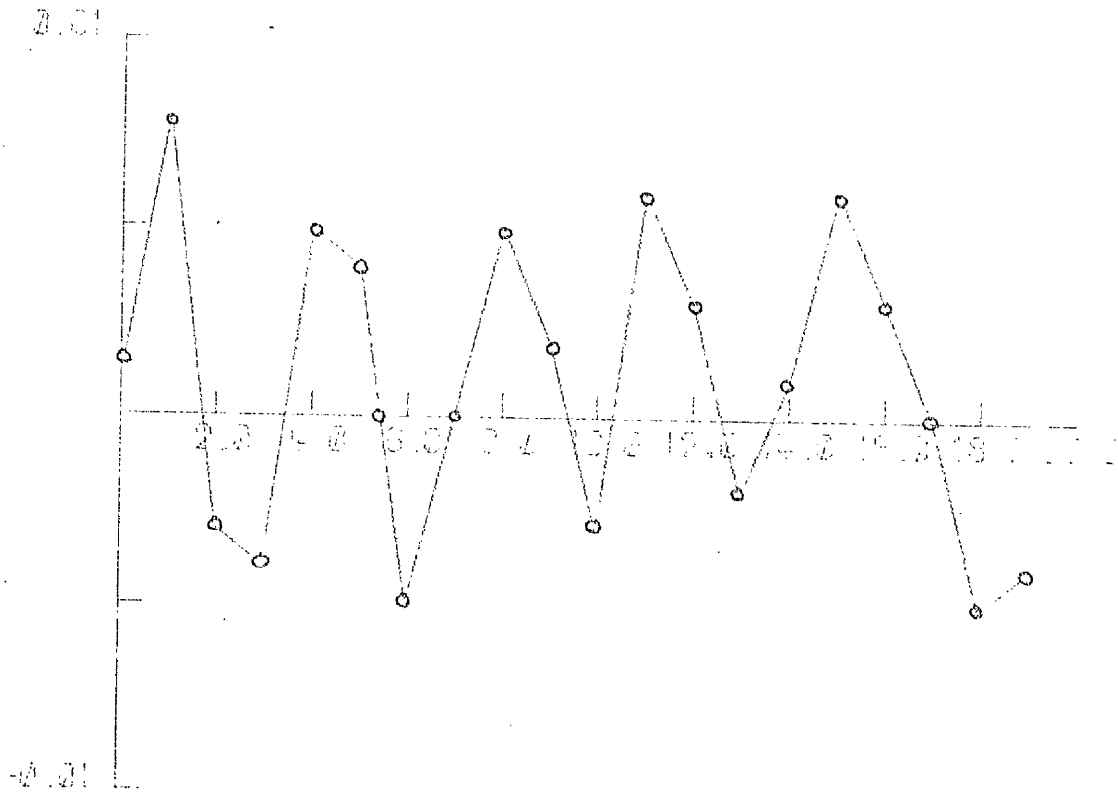
DATA SPACING UNITS T2

PSEUDO RANDOM BINARY - R(5,T2)

FIGURE 17

P(t)

Constant parameter
linear system

$W(-\tau)$ $\alpha(i\xi)$

X(t)

FIGURE 18

EXCITATION HISTORY

```
┌──────────────┐        ┌──────────────┐        ┌──────────────┐
│  Compute     │        │ Using impul- │        │ Combine      │
│  higher      │───────▶│ sive recep-  │───────▶│ moments      │
│  product     │        │ tance derive │        │ to give      │
│  moment      │        │ response     │        │ response     │
│  functions   │        │ moment       │        │ characteristic│
└──────────────┘        │ functions    │        │ function     │
                        └──────────────┘        └──────┬───────┘
                                                       │
┌──────────────┐        ┌──────────────┐        ┌──────▼───────┐
│  Fourier     │        │ Derive       │        │ Derive       │
│  Transform   │───────▶│ spectral     │───────▶│ response     │
│  to give     │        │ density      │        │ product      │
│  spectral    │        │ functions    │        │ moment       │
│  densities   │        │ of response  │        │ functions    │
│  of          │        └──────────────┘        └──────────────┘
│  excitation  │
└──────────────┘
```

```
                        ┌──────────────┐
                        │ Fourier      │
                        │ Inversion    │
                        │ of charac-   │
                        │ teristic     │
                        │ function to  │
                        │ give         │
                        │ probability  │
                        │ distributions│
                        └──────┬───────┘
                               │
                               ▼
                     RESPONSE DISTRIBUTION
```

STEPS IN THE COMPUTATION OF RESPONSE DISTRIBUTIONS

FIGURE 19
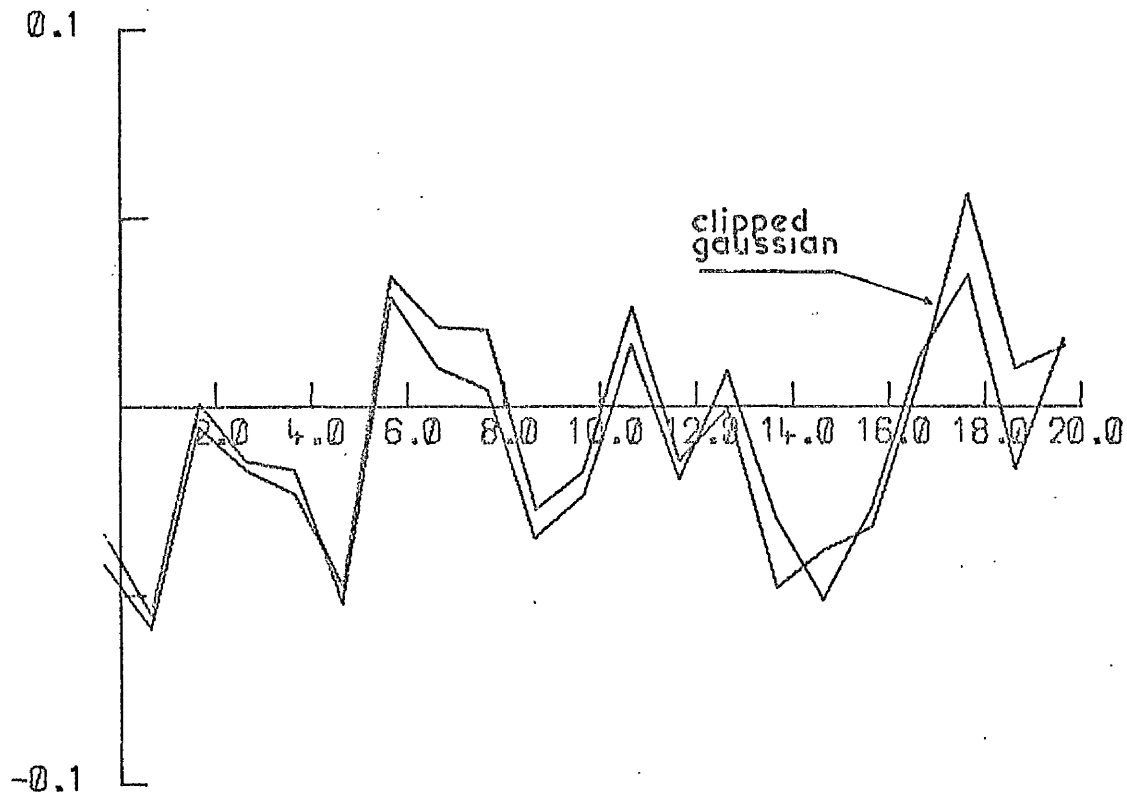
clipped
gaussian

DATA SPACING UNITS T2

GAUSSIAN NOISE + 1.5 SIGMA CLIPPED GAUSSIAN — R(1,T2

FIGURE 20
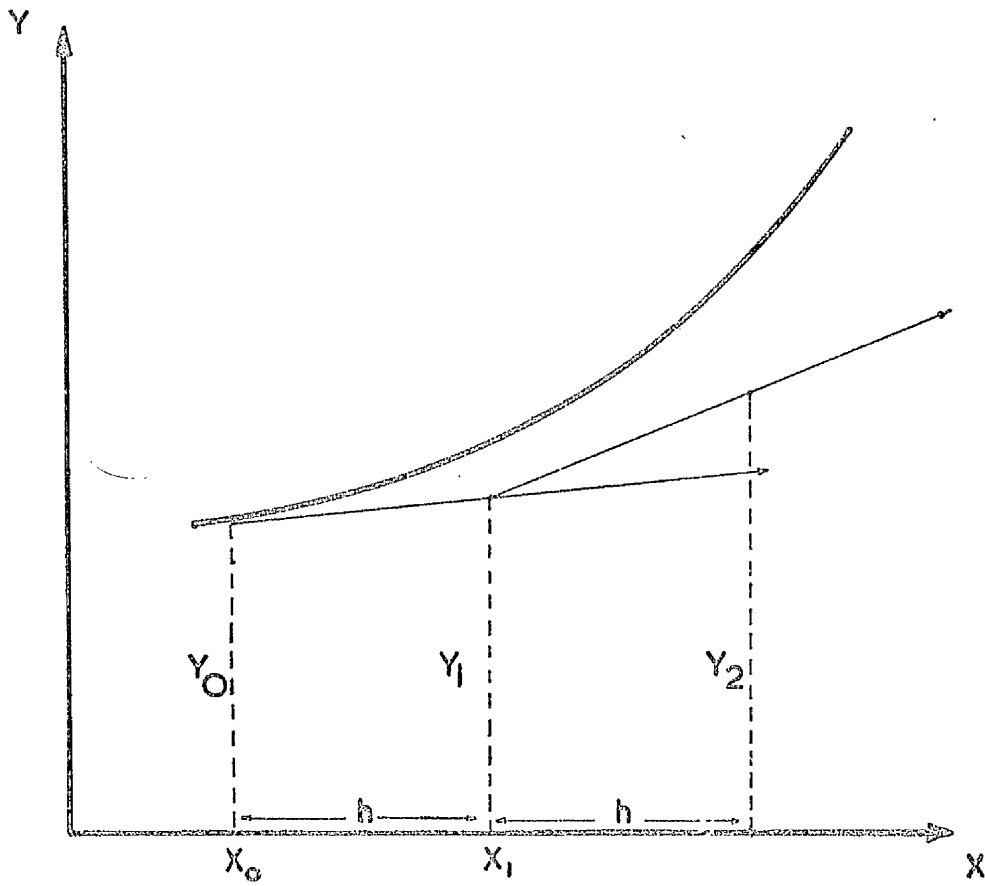
EULER'S METHOD

FIGURE 21
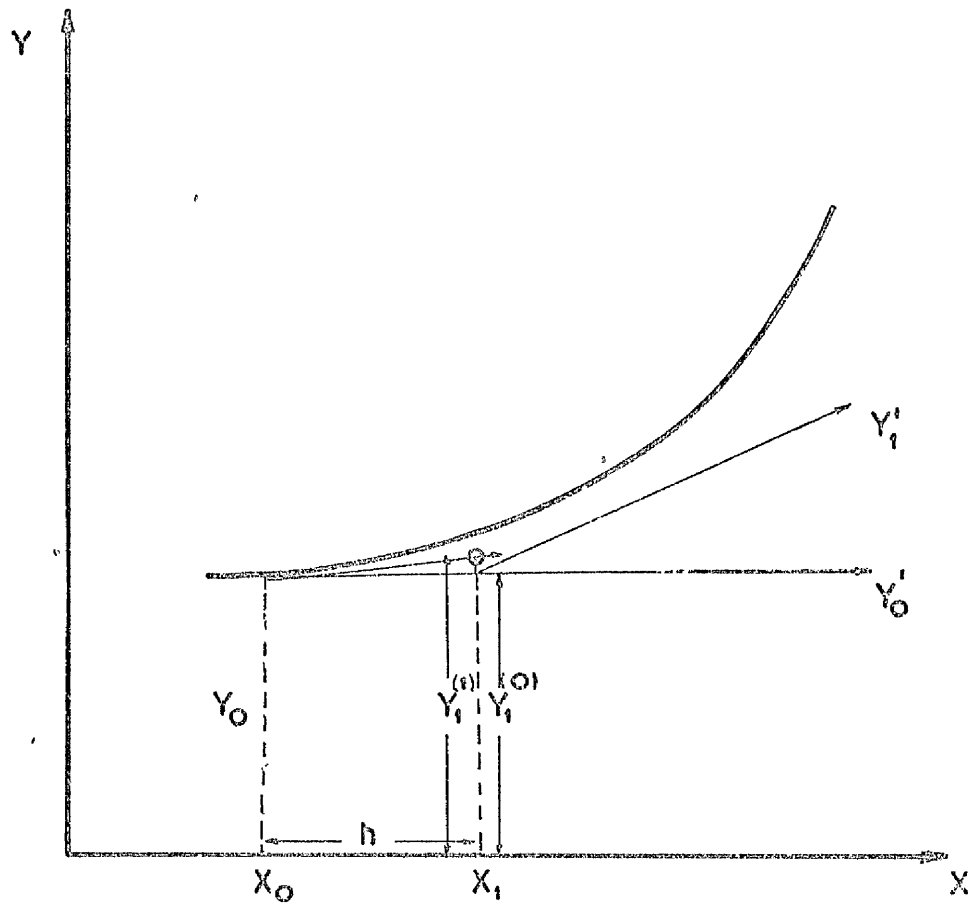
IMPROVED EULER METHOD

FIGURE 22

MINIMUM   DISPL VERSUS TIME
=1.3146E=02

TIME	DISPL	I
0.0	0.0	
2.0000E=02	=2.4148E=05	
4.0000E=02	=1.6385E=04	
6.0000E=02	=4.3271E=04	
6.0000E=02	=8.9190E=04	
1.0000E=01	=1.5685E=03	
1.2000E=01	=2.1250E=03	
1.4002E=01	=2.7613E=03	
1.6000E=01	=3.5861E=03	
1.8000E=01	=4.5288E=03	
2.0000E=01	=5.5053E=03	
2.2000E=01	=6.3356E=03	
2.4000E=01	=7.0795E=03	
2.6000E=01	=8.1982E=03	
2.8000E=01	=9.3973E=03	
3.0000E=01	=1.0384E=02	
3.2000E=01	=1.1312E=02	
3.4000E=01	=1.2318E=02	
3.6000E=01	=1.2804E=02	
3.8000E=01	=1.3066E=02	
4.0000E=01	=1.3132E=02	
4.2000E=01	=1.2849E=02	
4.4000E=01	=1.2331E=02	
4.6000E=01	=1.1459E=02	
4.8000E=01	=1.0410E=02	
5.0000E=01	=9.3158E=03	
5.2000E=01	=8.4406E=03	
5.4000E=01	=7.3751E=03	
5.6000E=01	=5.8568E=03	
5.8000E=01	=4.3139E=03	
6.0000E=01	=2.8135E=03	
6.2000E=01	=1.8019E=03	
6.4000E=01	=9.2789E=04	
6.6000E=01	=3.1462E=04	
6.8000E=01	=9.3162E=05	
7.0000E=01	=1.4536E=04	
7.2000E=01	=4.9367E=04	
7.4000E=01	=9.6291E=04	
7.6000E=01	=1.8298E=03	
7.8000E=01	=2.7665E=03	
8.0000E=01	=3.6693E=03	
8.2000E=01	=4.4167E=03	
8.4000E=01	=5.0240E=03	
8.6000E=01	=6.0154E=03	
8.8000E=01	=7.2660E=03	
9.0000E=01	=8.5331E=03	
9.2000E=01	=9.6347E=03	
9.4000E=01	=1.0213E=02	
9.6000E=01	=1.0367E=02	
9.8000E=01	=1.0133E=02	
1.0000E 00	=9.6190E=03	

FIGURE 23

X(t)

response

mass m

stiffness
k

damping c

F(t)

excitation
force
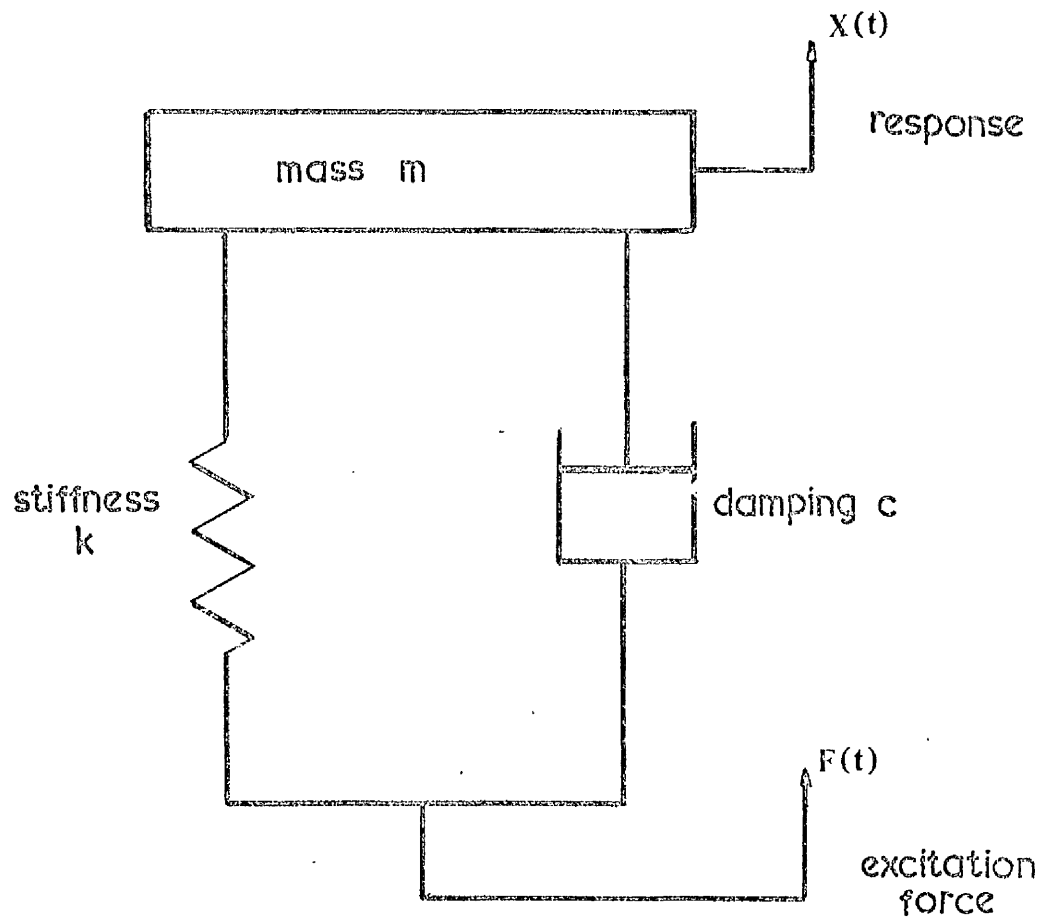
THE TEST SYSTEM

FIGURE 24
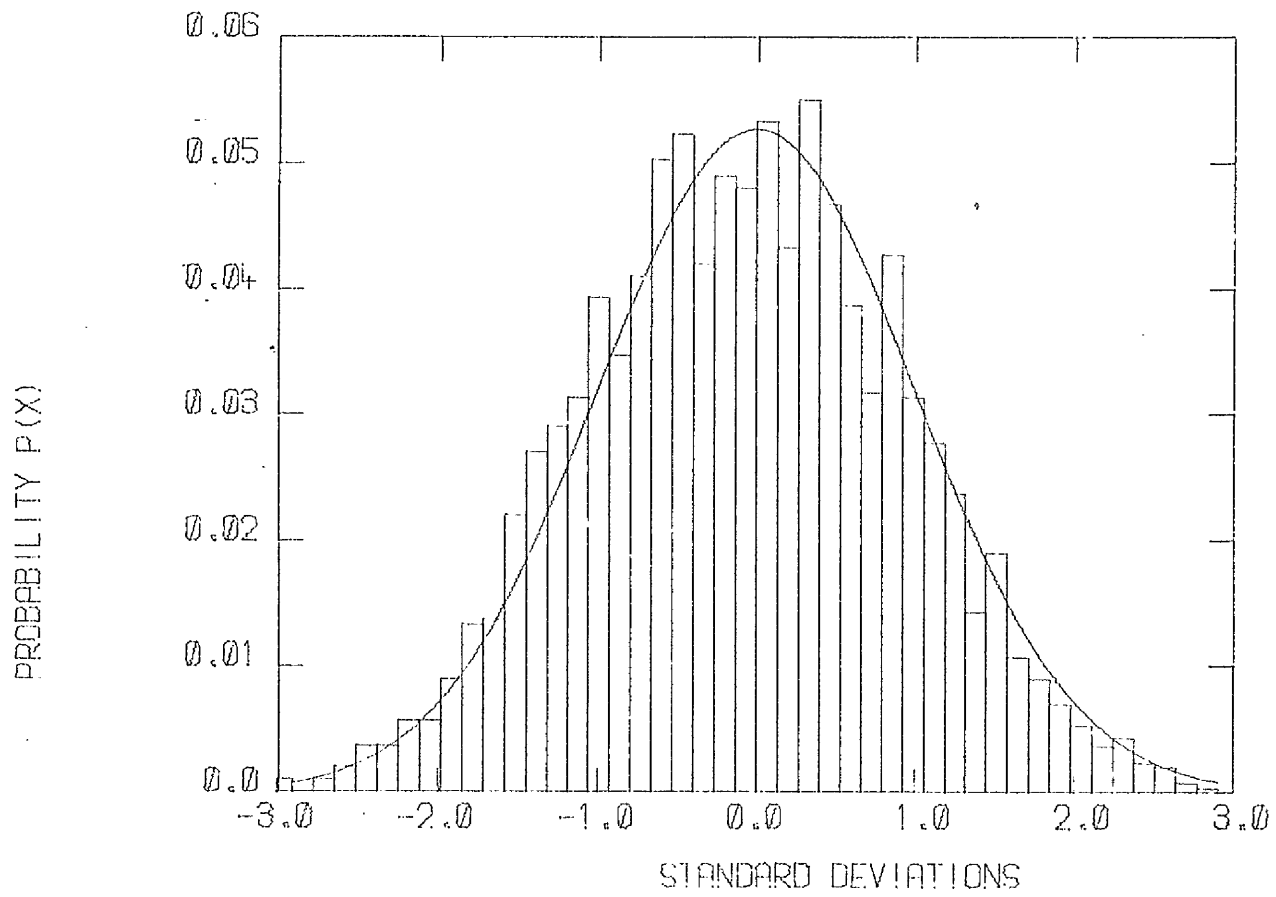
GAUSSIAN WHITE NOISE INPUT

FIGURE 25
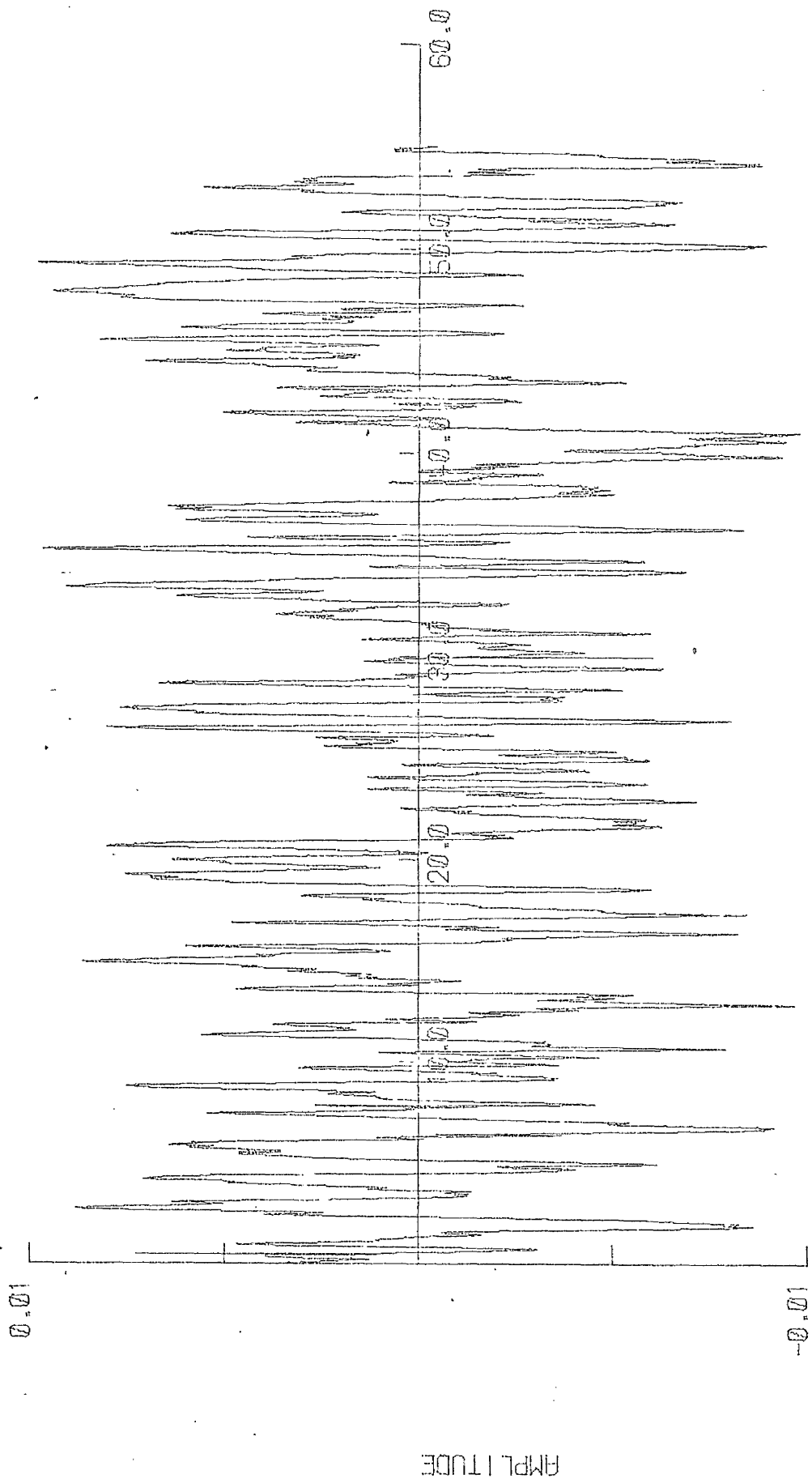
TIME (SECS)

RESPONSE WITH ZETA = 0.5

FIGURE 26

AMPLITUDE
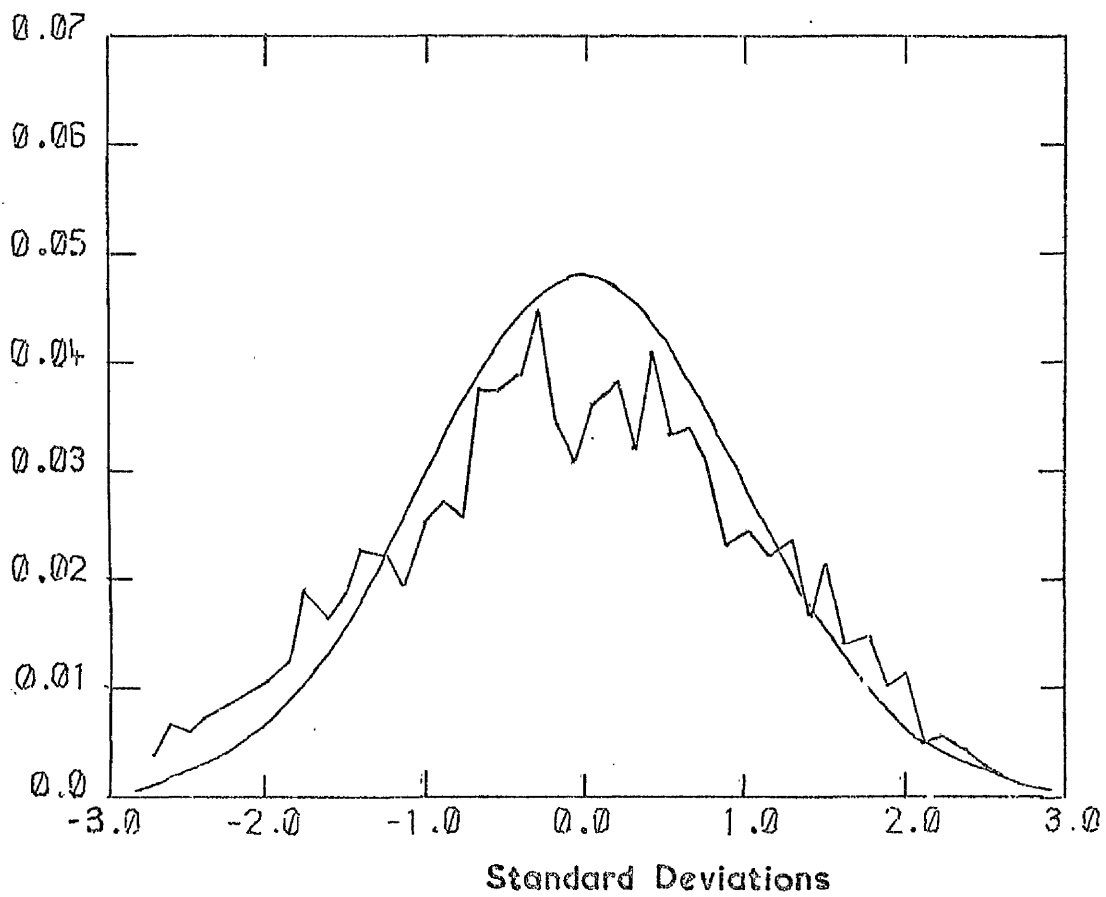
RESPONSE WITH ZETA = O.5
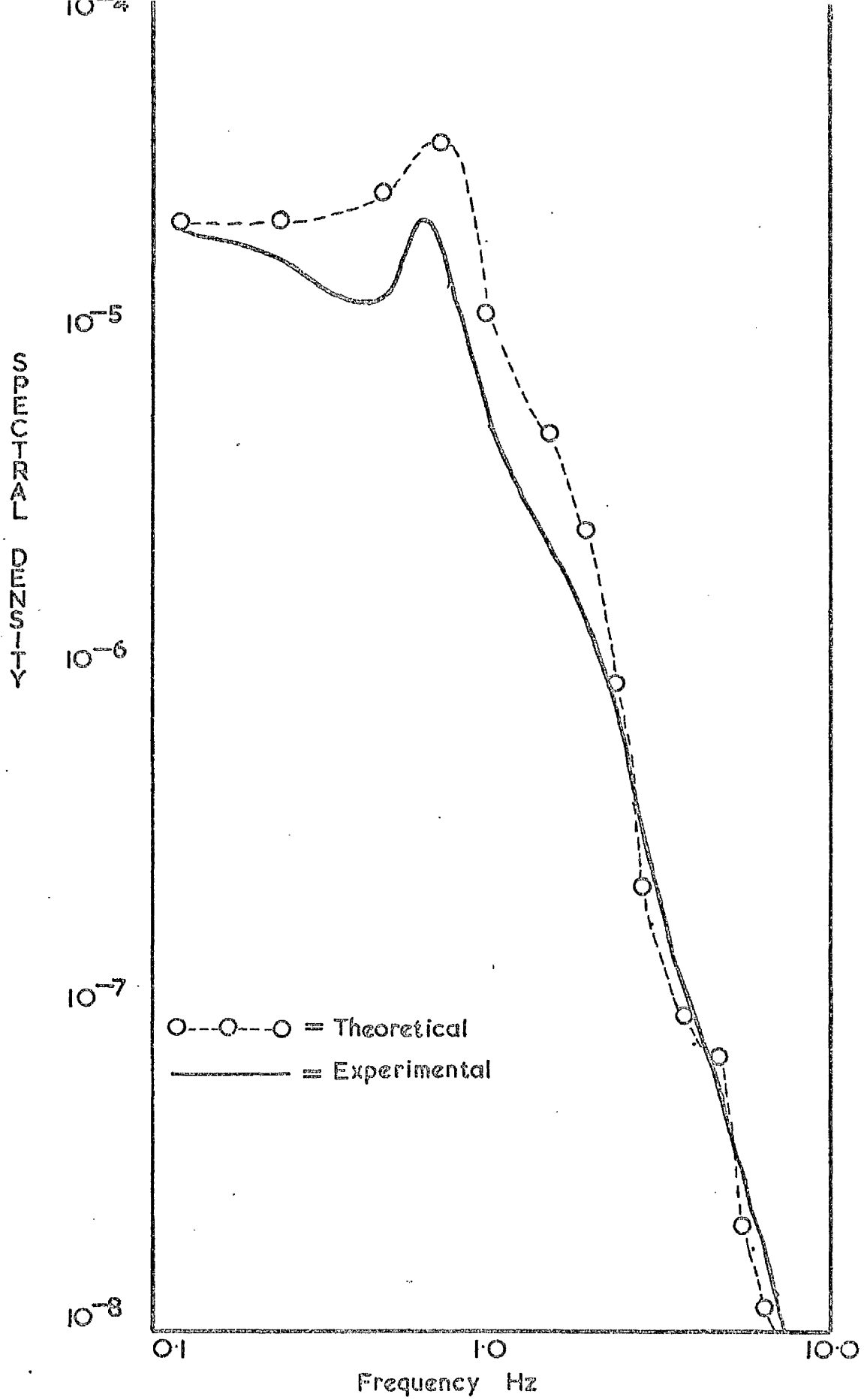
FIGURE 27

RESPONSE WITH ZETA = 0.5

FIGURE 28
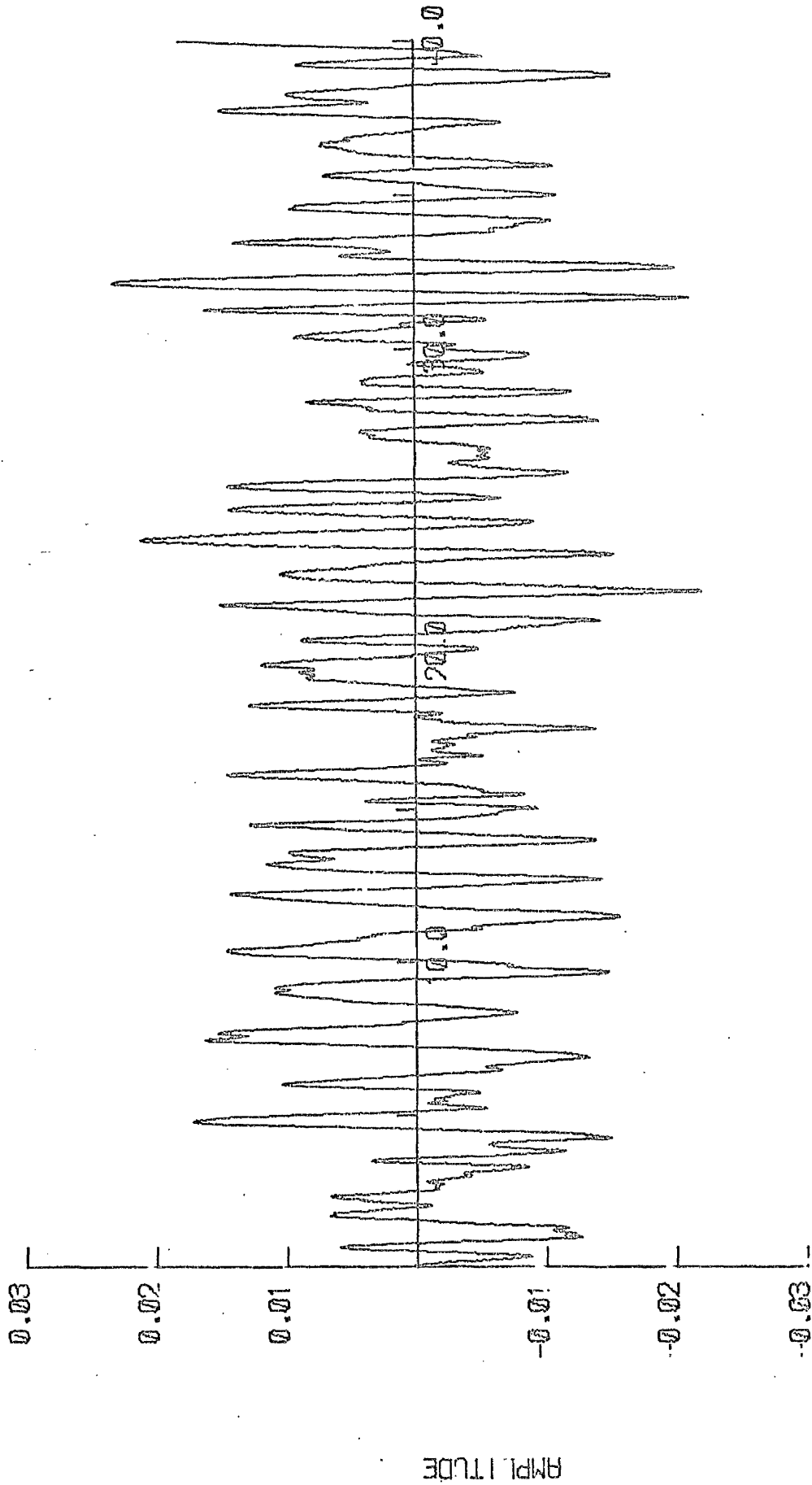
TIME (SECS)

RESPONSE WITH ZETA = 0.25

FIGURE 29
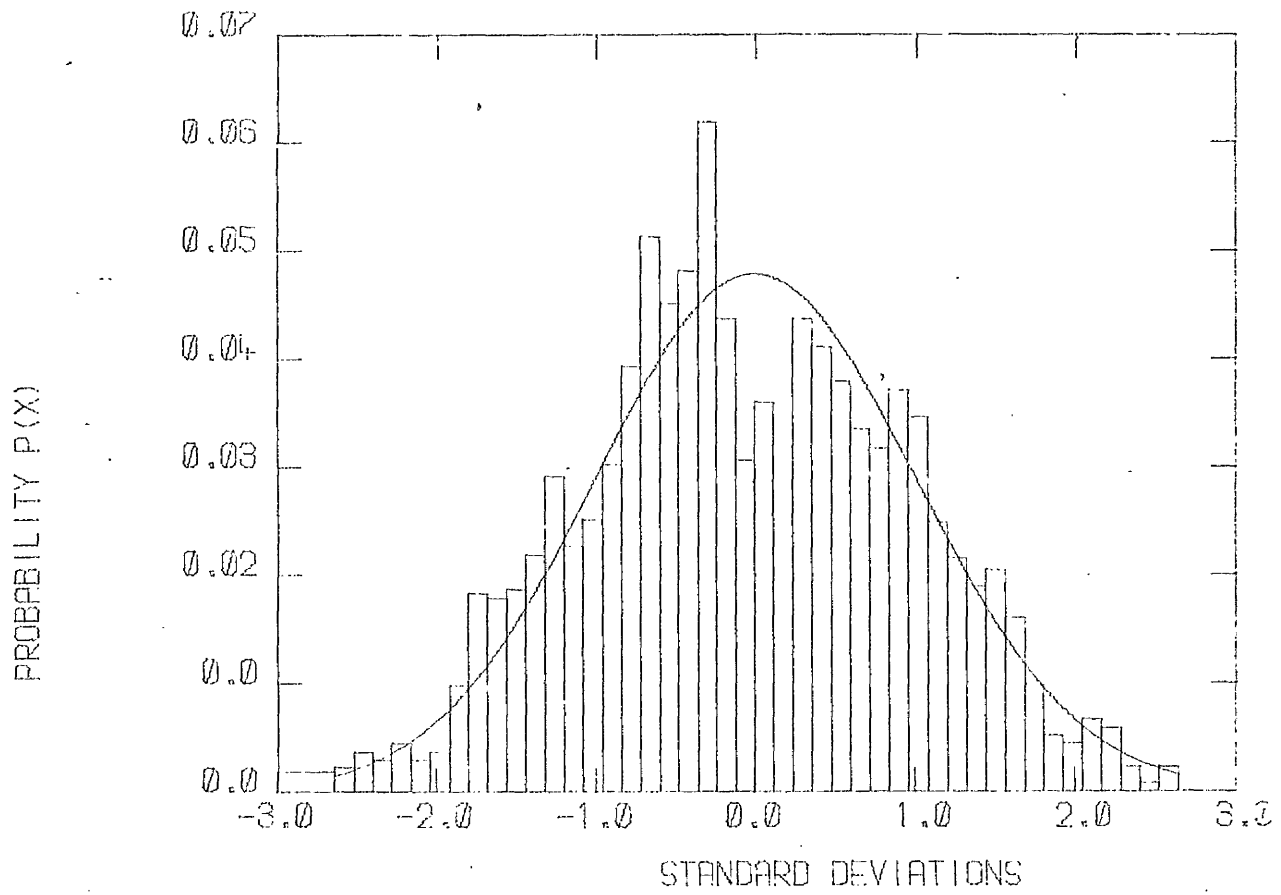
RESPONSE WITH ZETA = 0.25
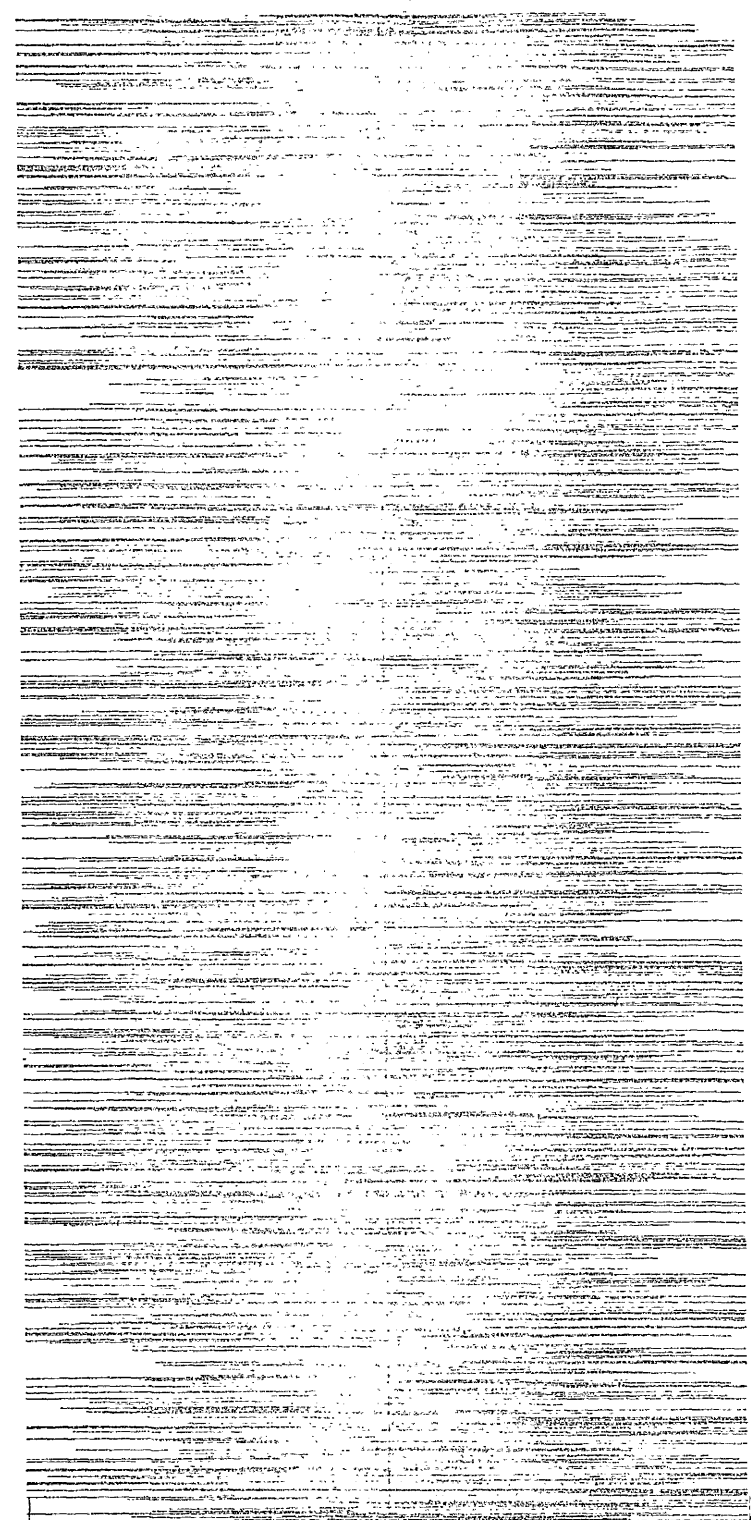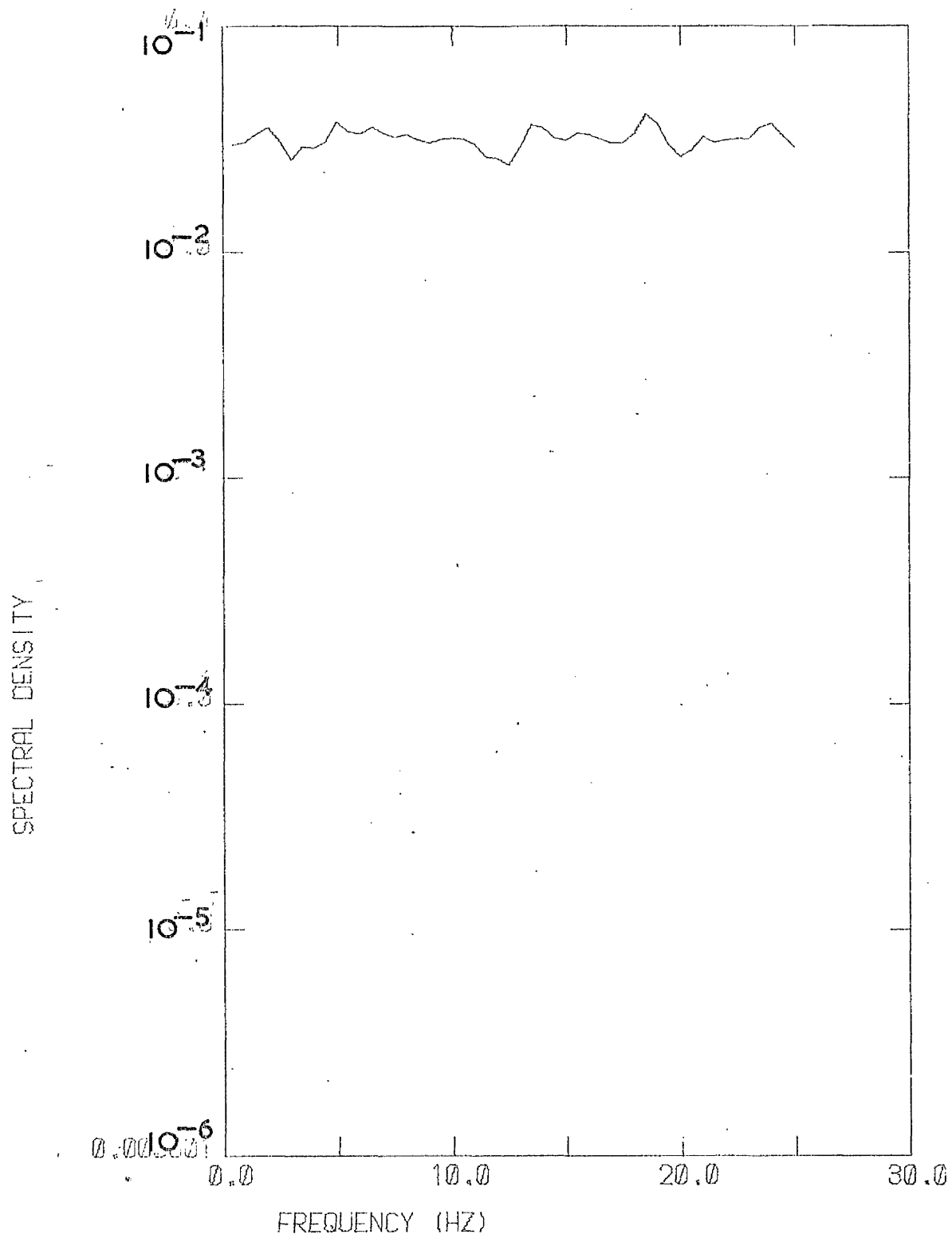
FIGURE 30
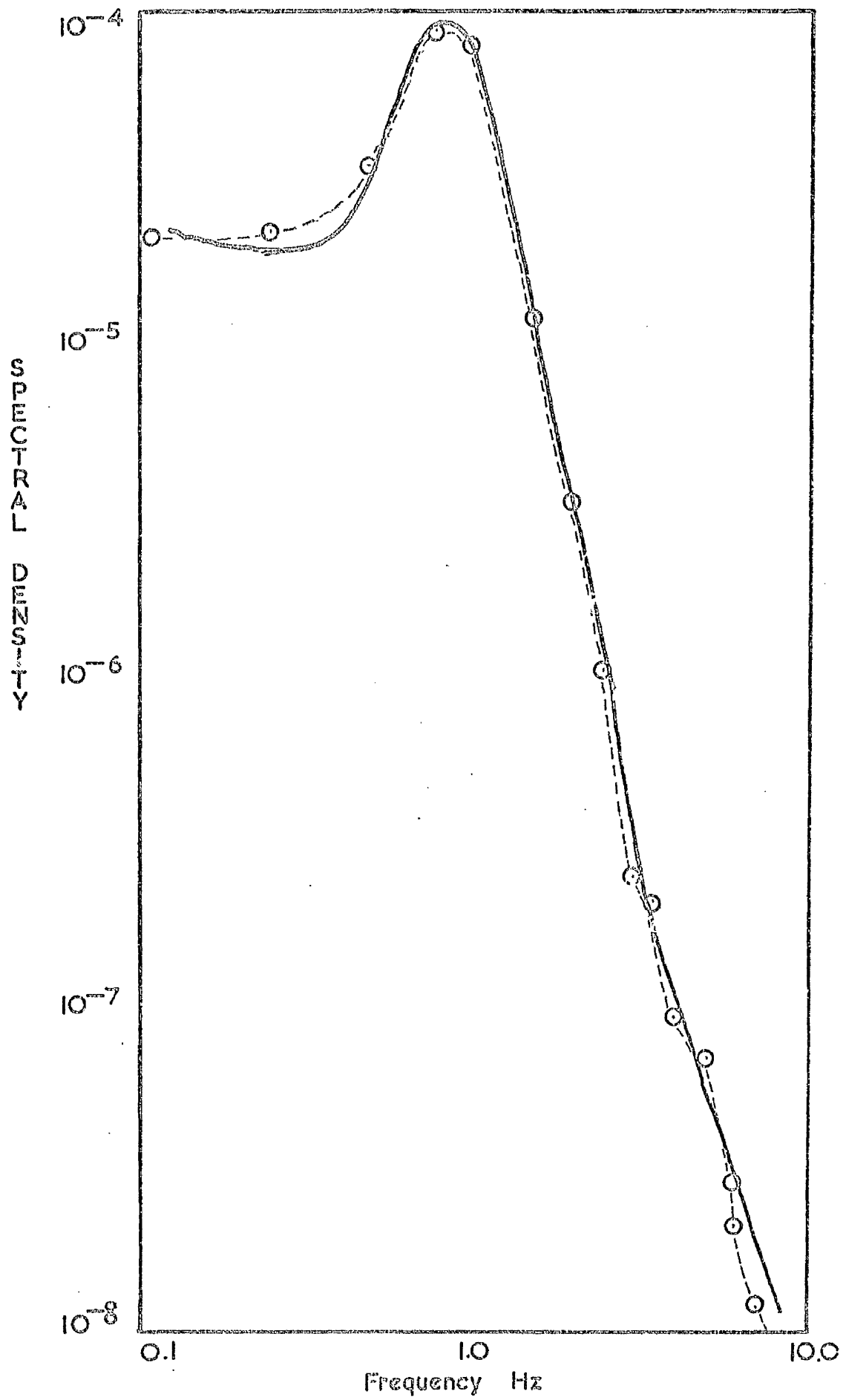
RESPONSE WITH ZETA = 0·25

FIGURE 31
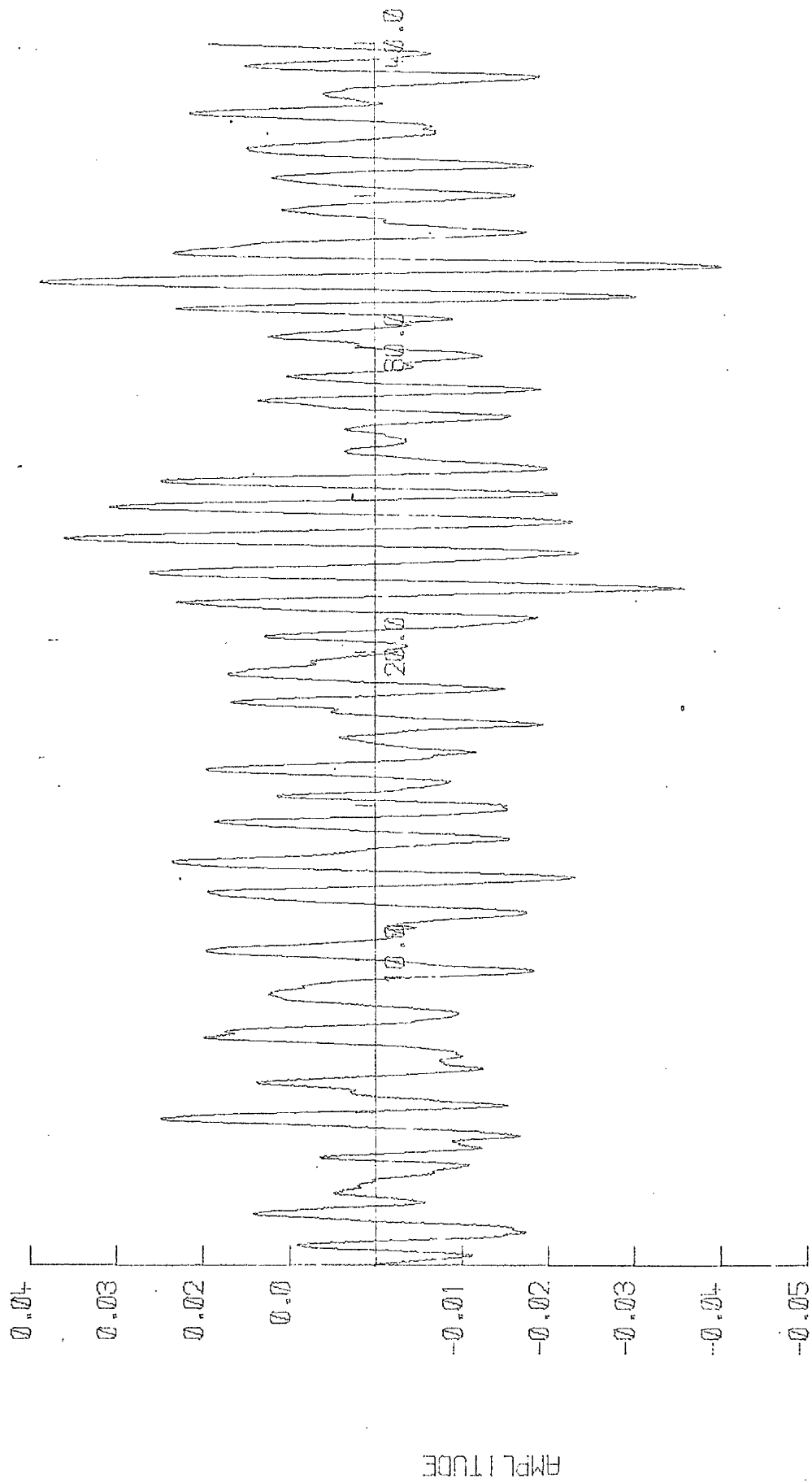
TIME (SECS)

RESPONSE WITH ZETA = 0.1

FIGURE 32
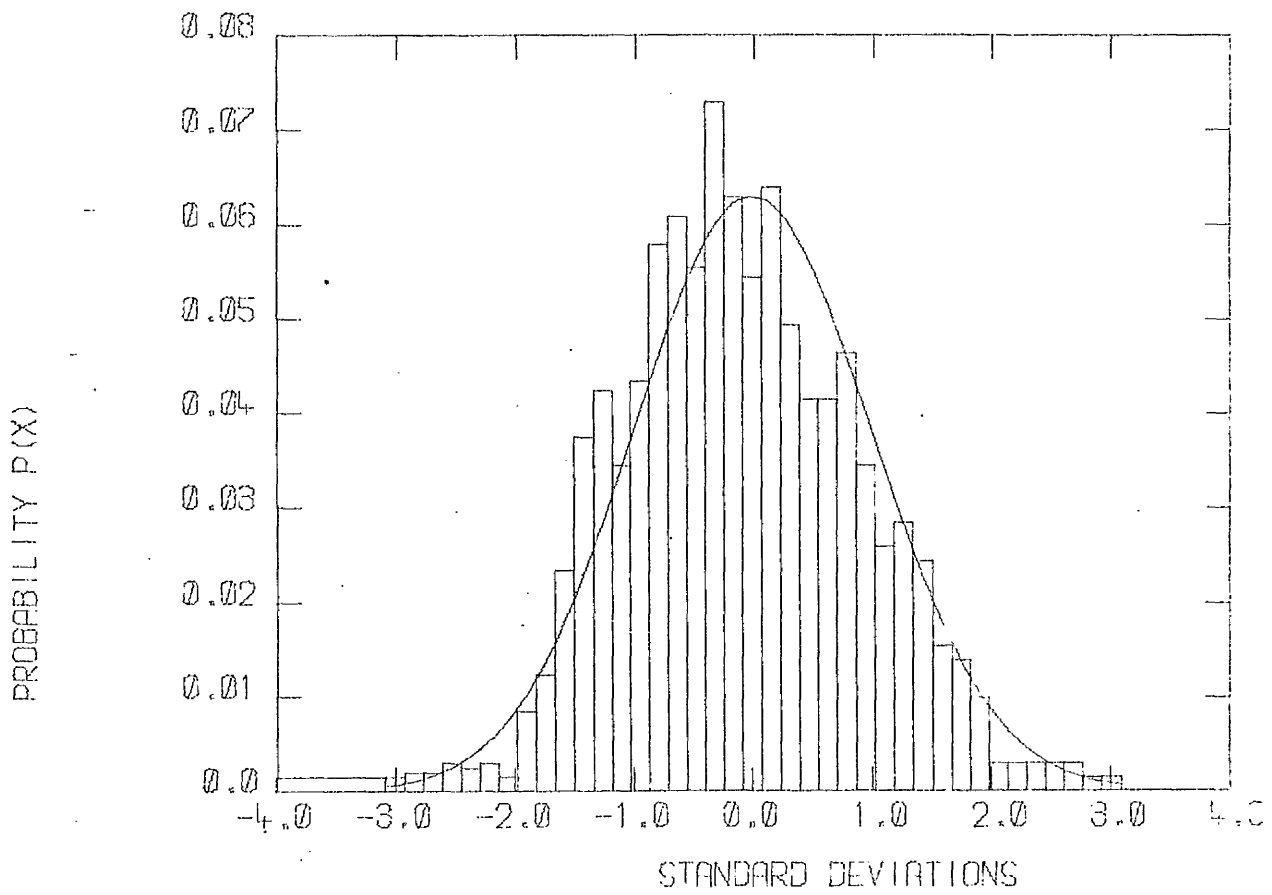
RESPONSE WITH ZETA = 0.1

FIGURE 33
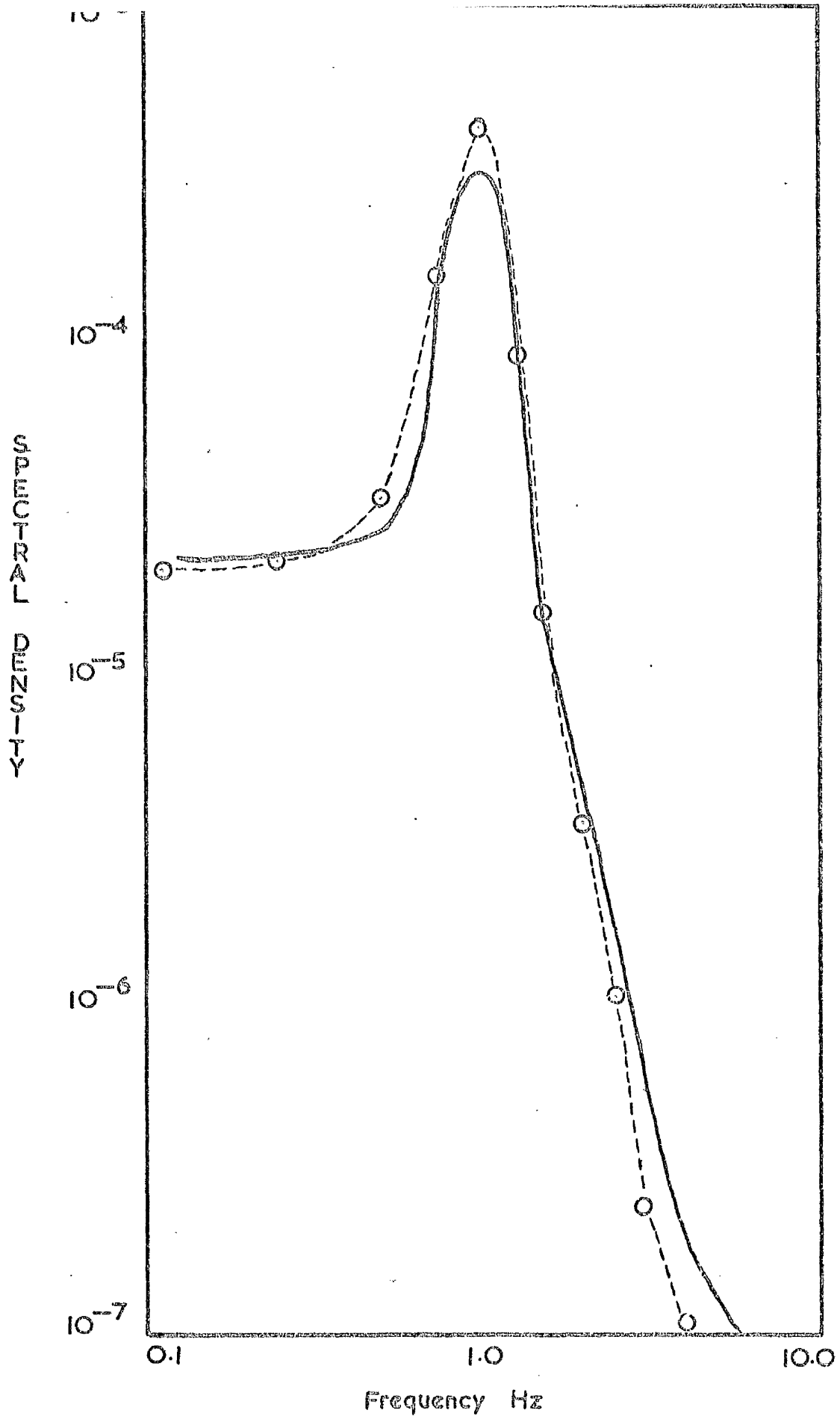
RESPONSE WITH ZETA = O·1
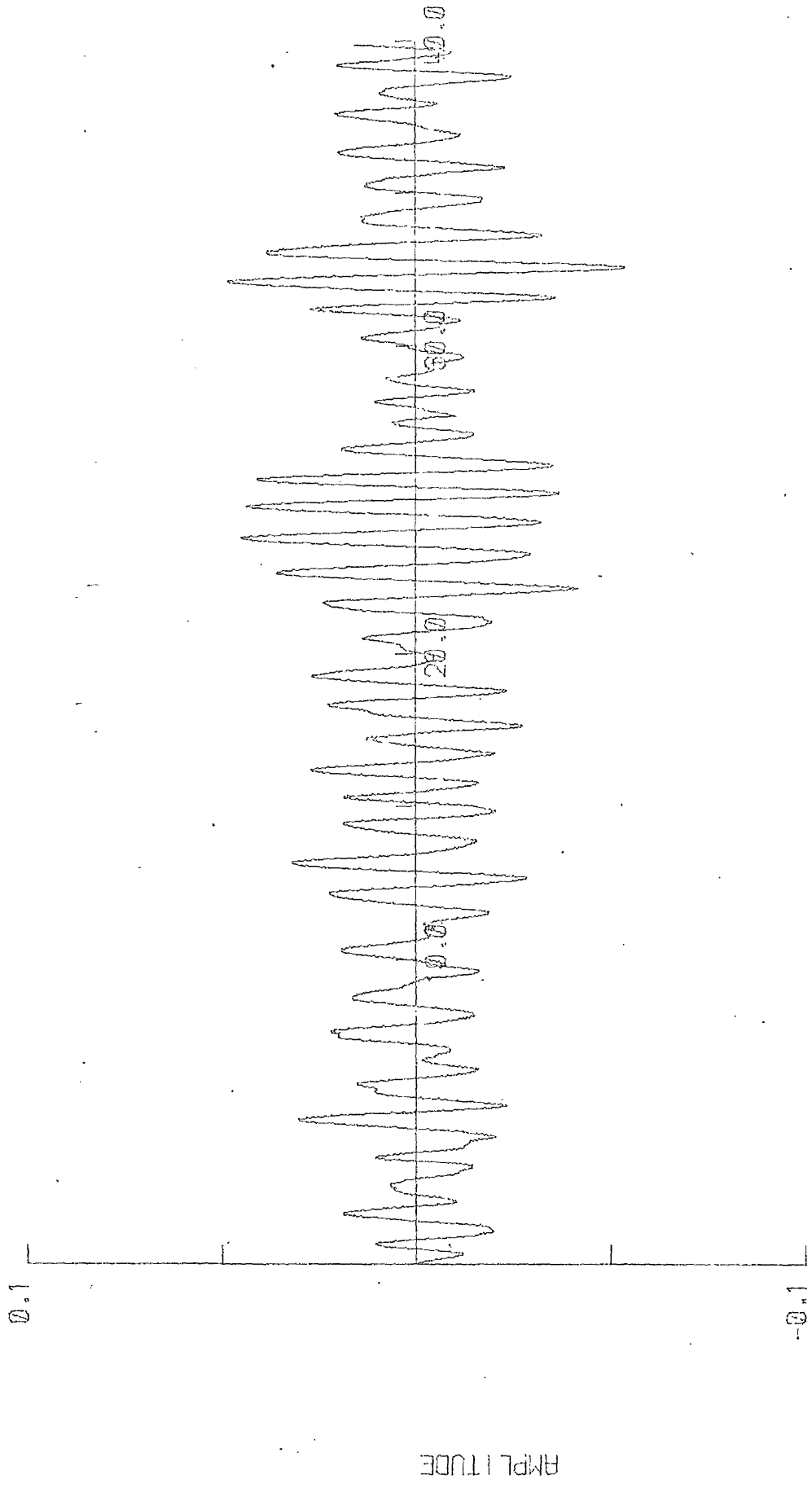
FIGURE 34

TIME (SECS)

RESPONSE WITH ZETA = 0.05

FIGURE 35
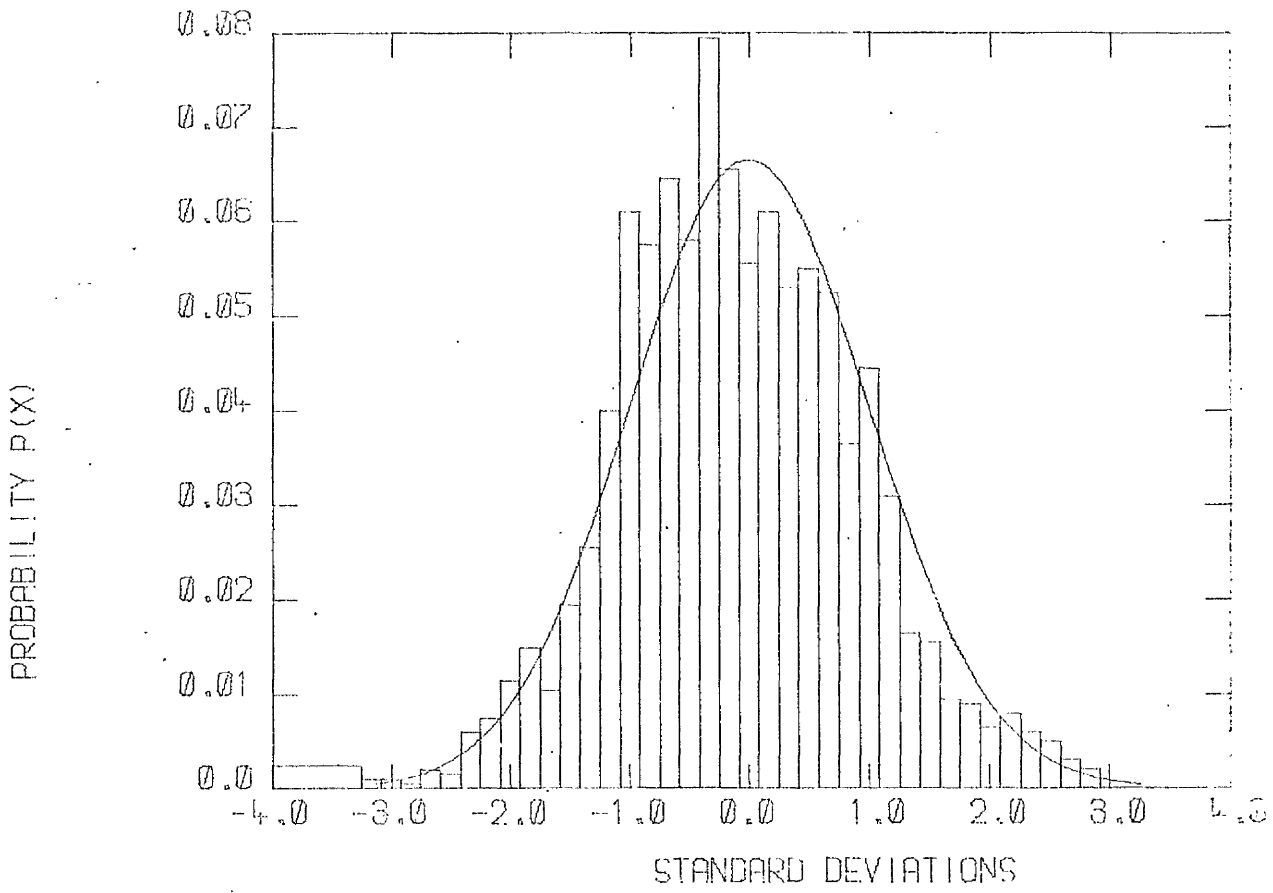
RESPONSE WITH ZETA = 0.05

FIGURE 36
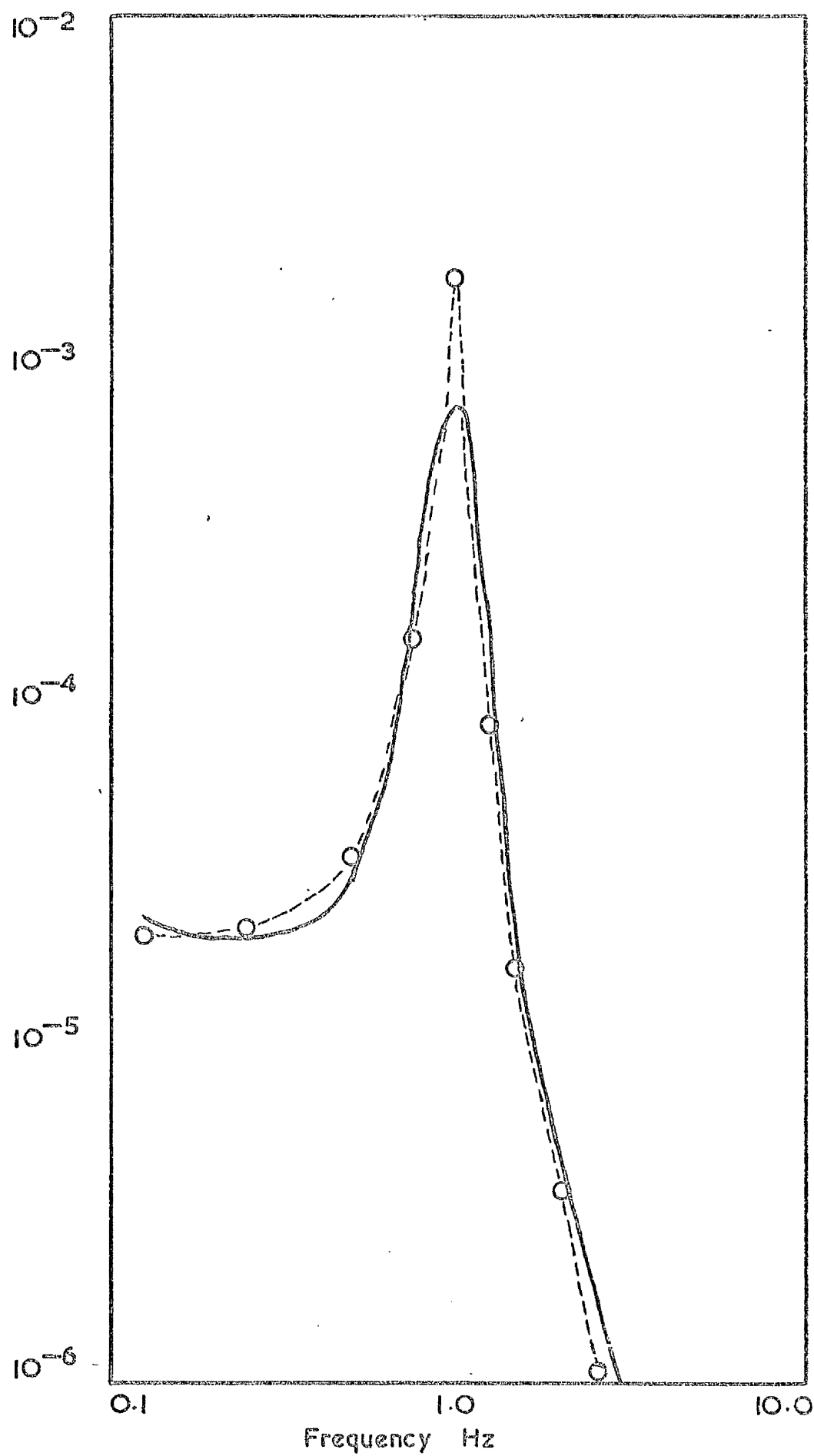
RESPONSE WITH ZETA = 0.05

FIGURE 37
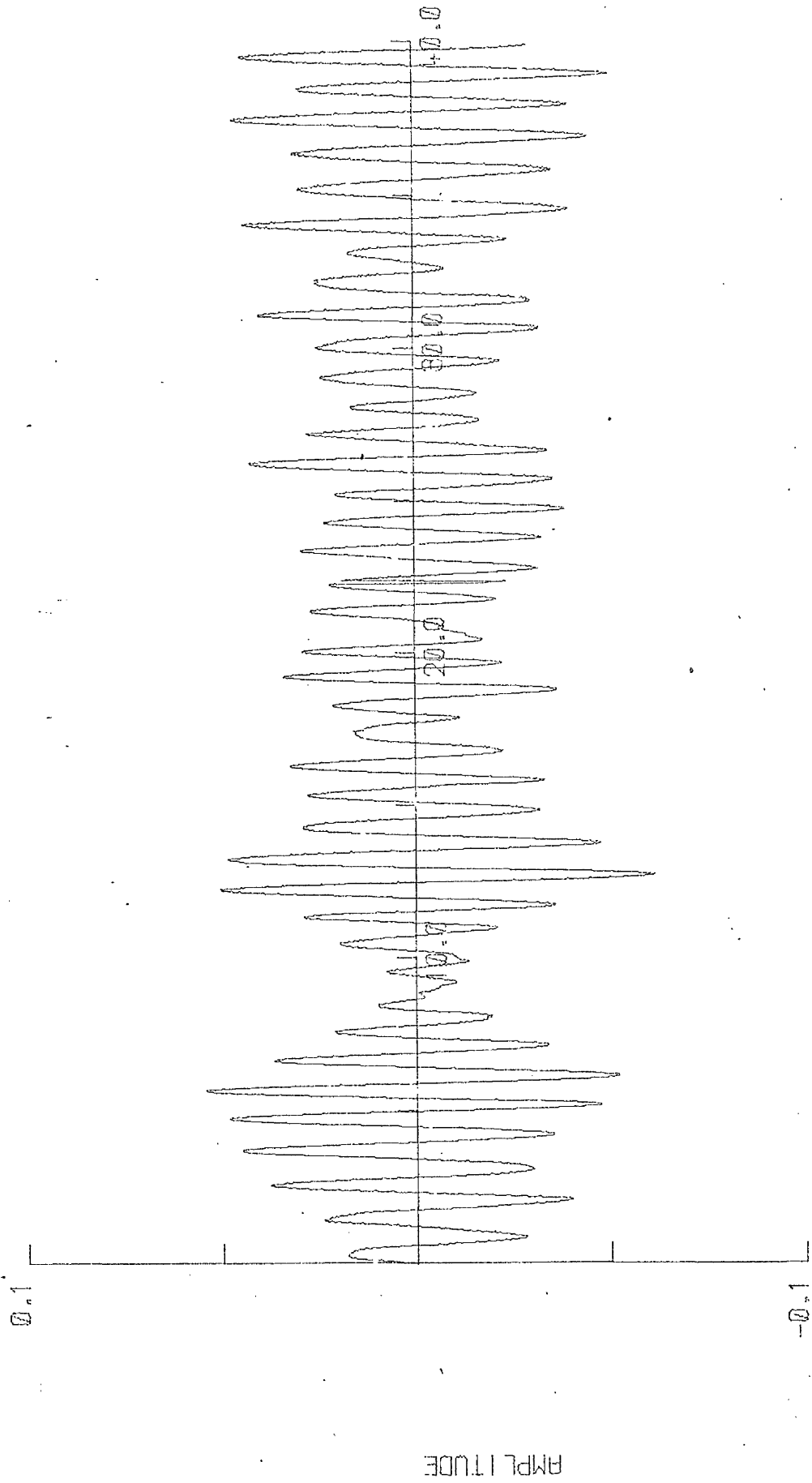
TIME (SECS)

RESPONSE WITH ZETA = 0.02

FIGURE 38
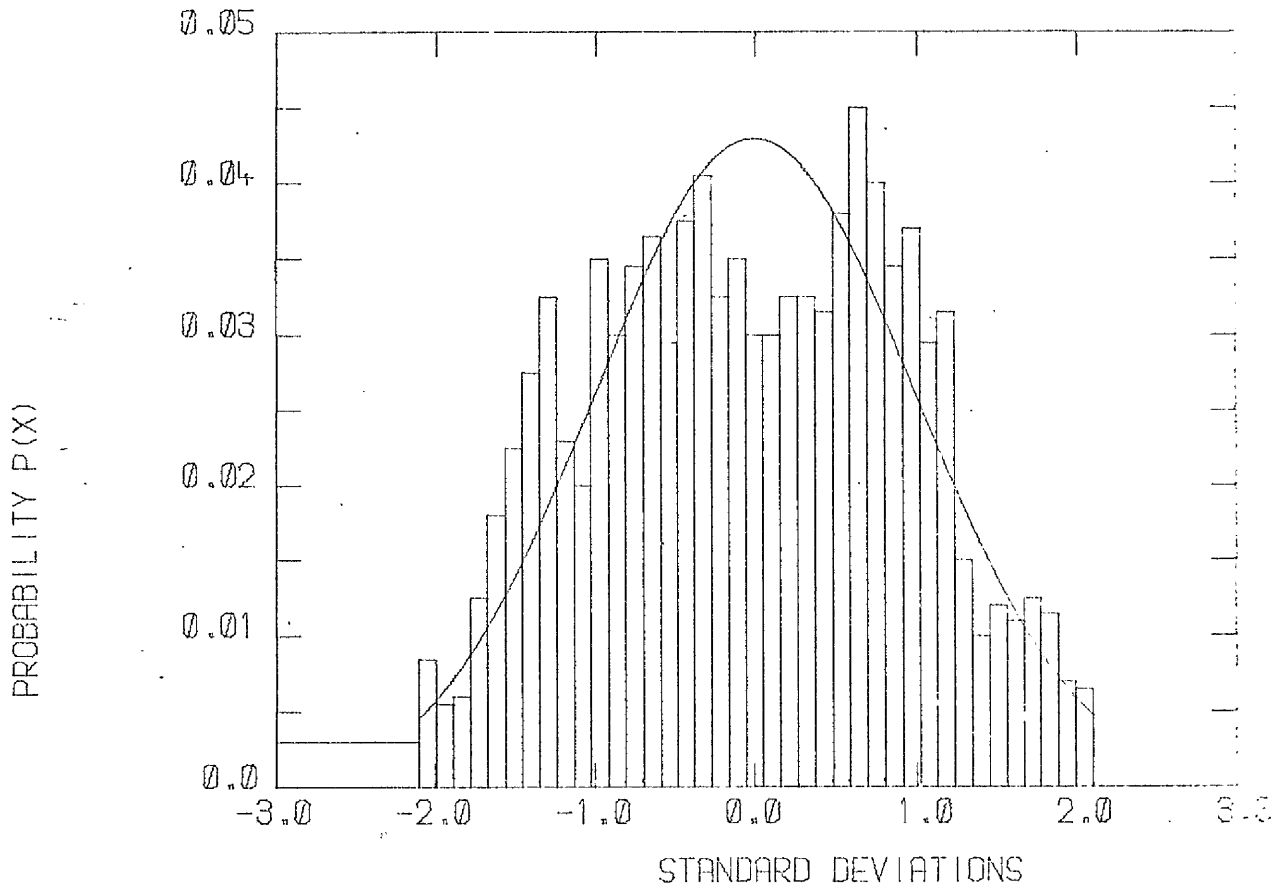
AMPLITUDE

RESPONSE WITH ZETA = 0.02

FIGURE 39

SPECTRAL DENSITY

Frequency Hz
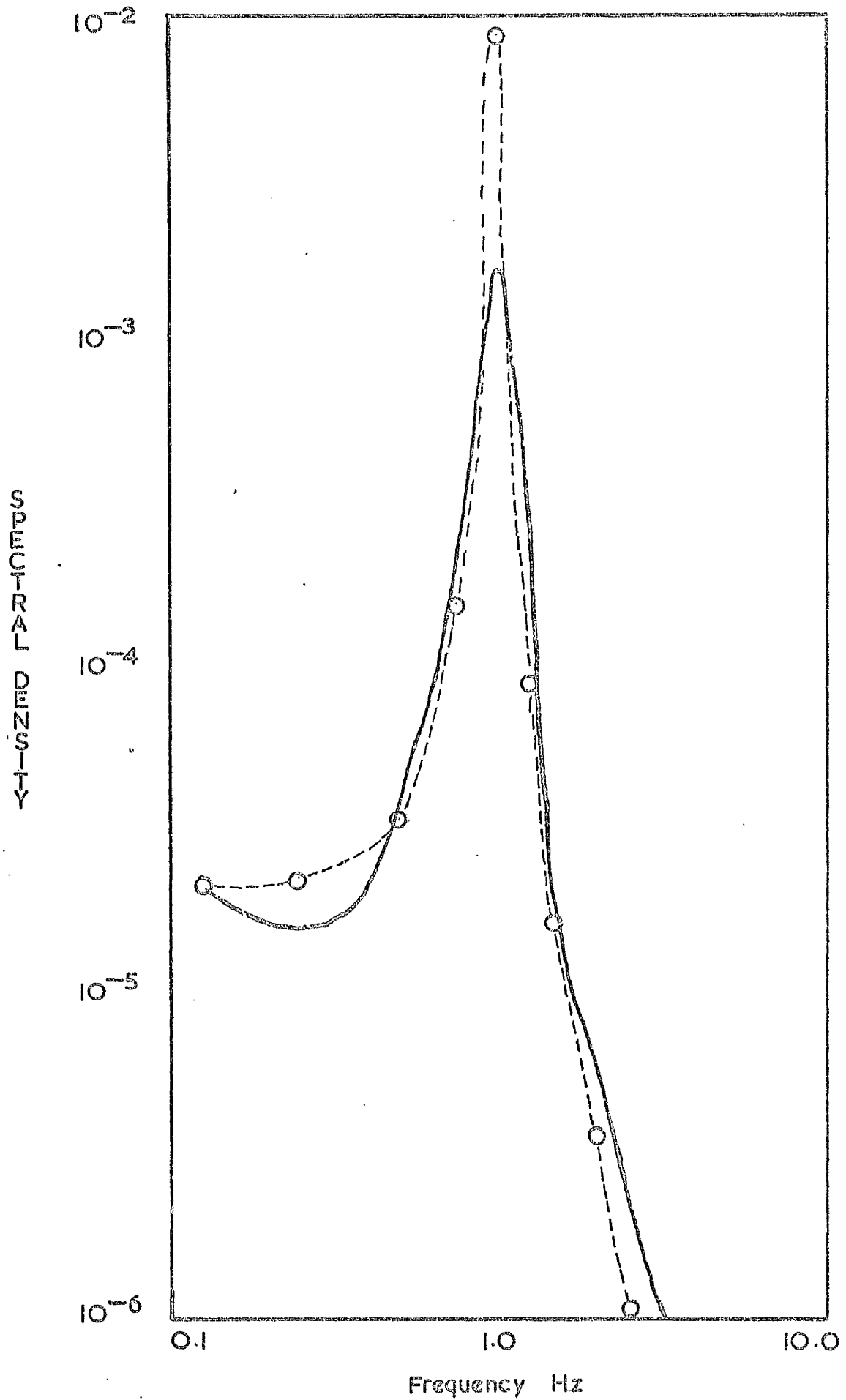
RESPONSE WITH ZETA = 0.02

FIGURE 40

GAUSSIAN INPUT CLIPPED AT 1.5 SIGMA

FIGURE 41

GAUSSIAN INPUT CLIPPED AT 1.5 SIGMA PSD
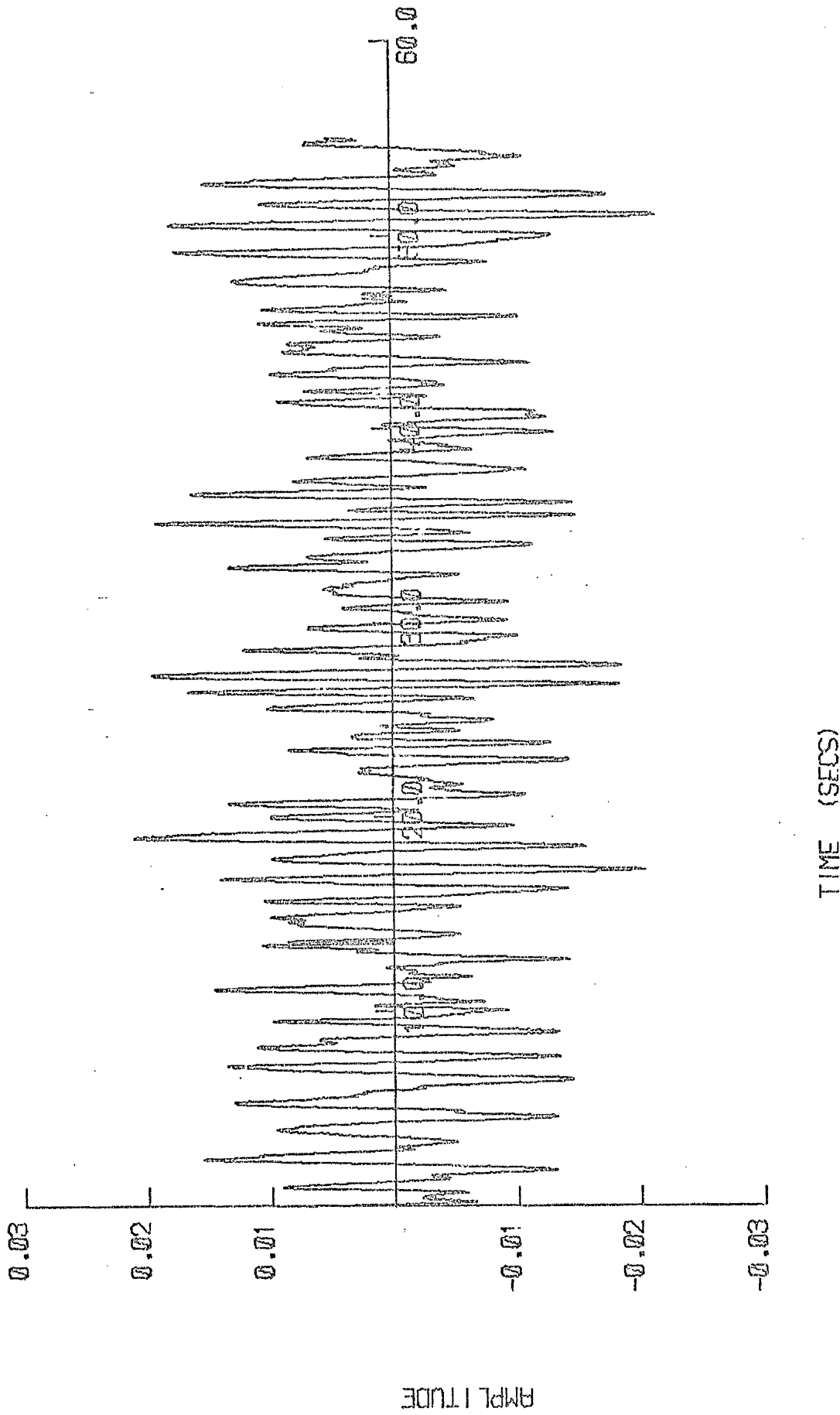
FIGURE 42

RESPONSE TO 1.5 S.D. CLIPPED GAUSSIAN ZETA=0.25

FIGURE 43
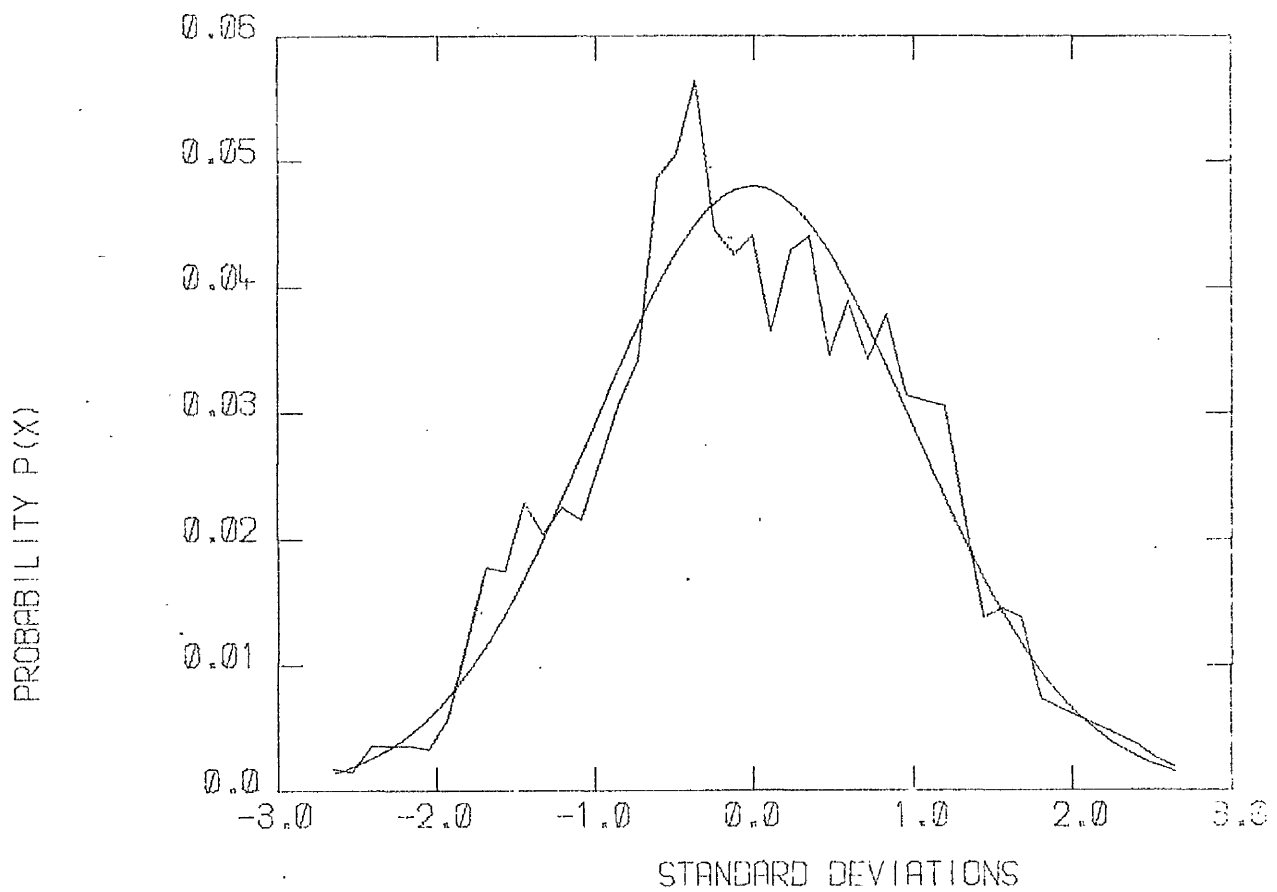
1.5 SIGMA CLIPPED — ZETA = 0.25

FIGURE 44
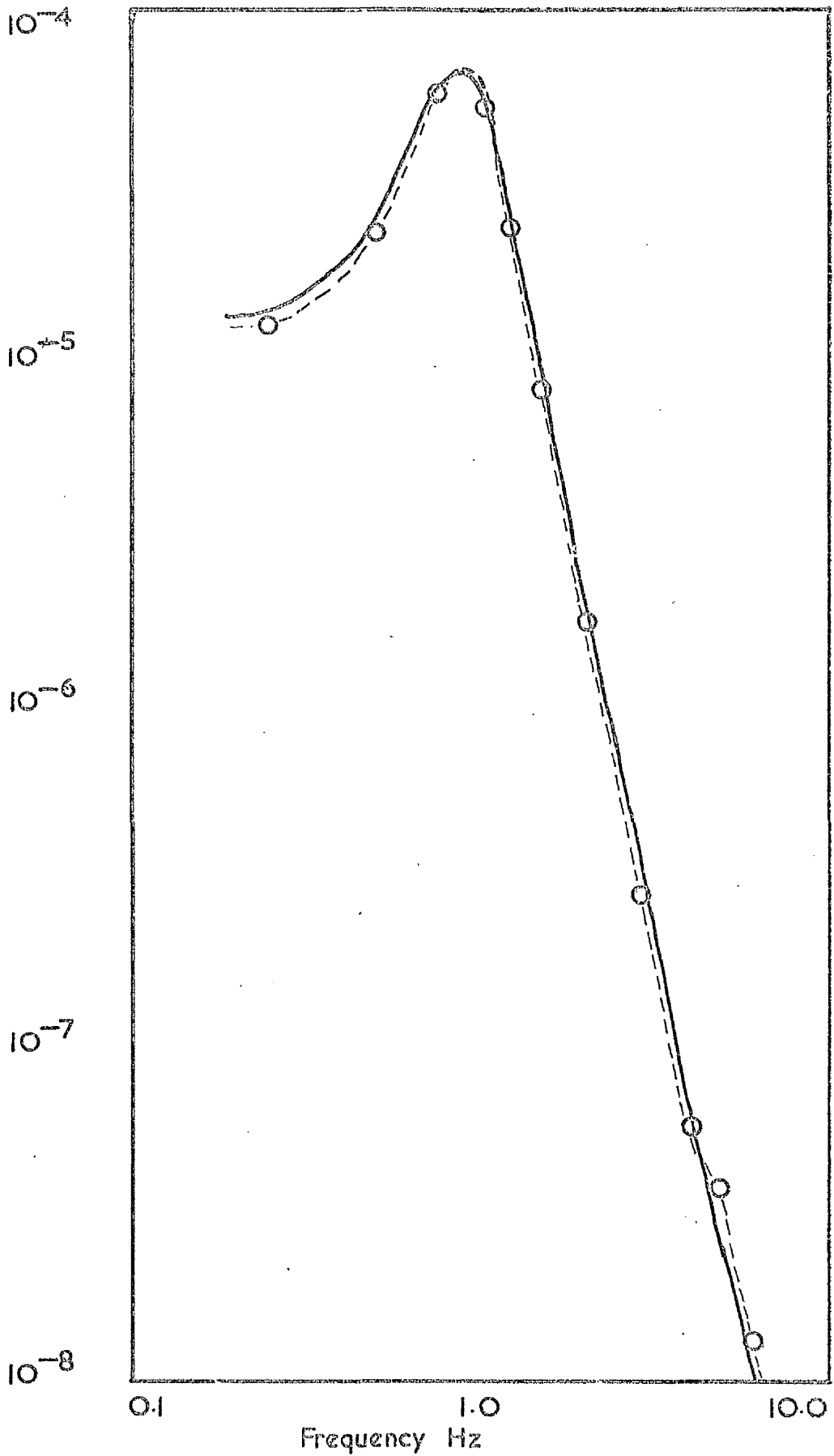
RESPONSE TO 1.5 SIGMA CLIPPED GAUSSIAN

ZETA = 0.25

FIGURE 45
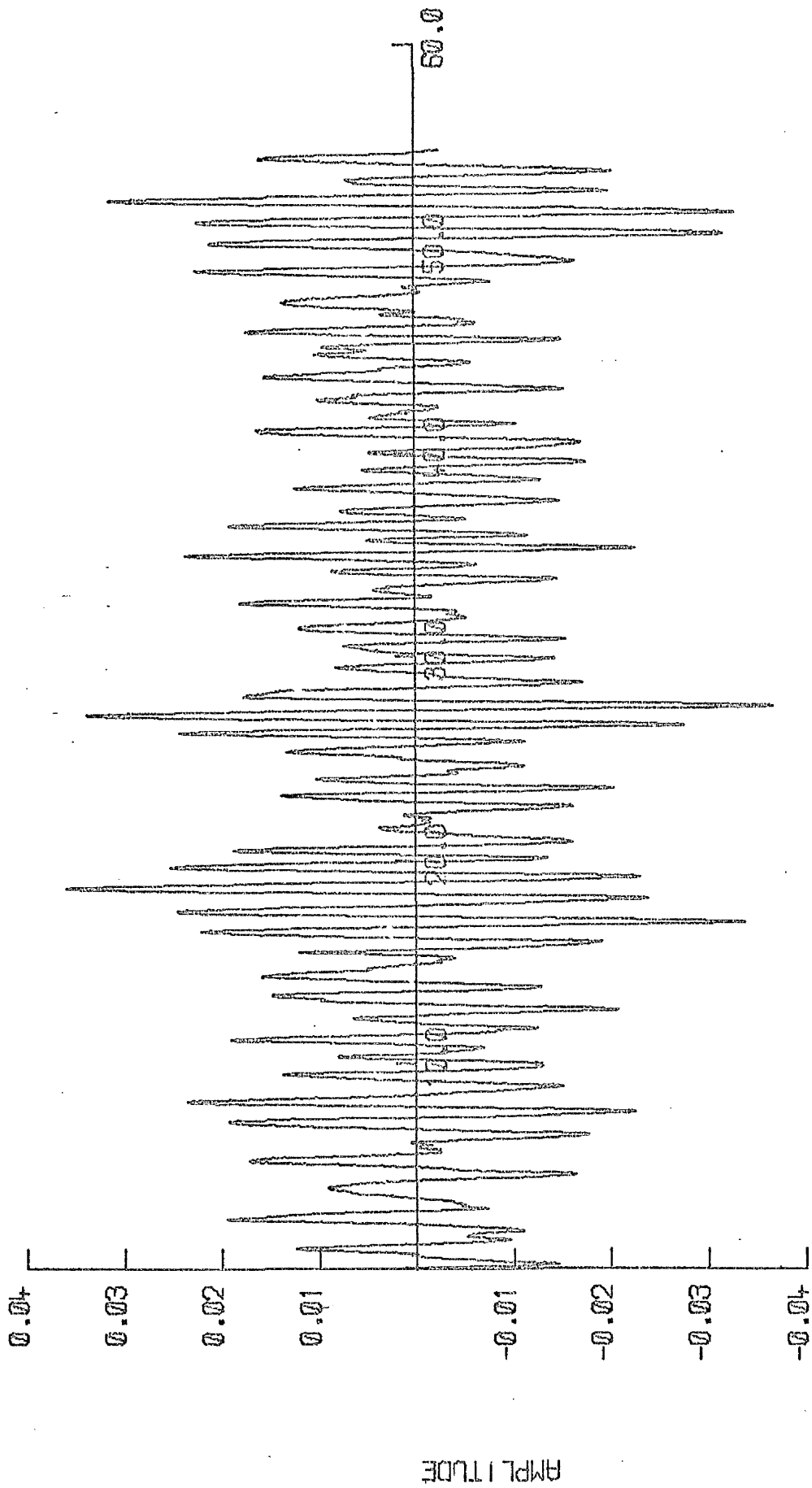
RESPONSE TO 1.5 S.D. CLIPPING ZETA=0.1

TIME (SECS)

FIGURE 46

AMPLITUDE
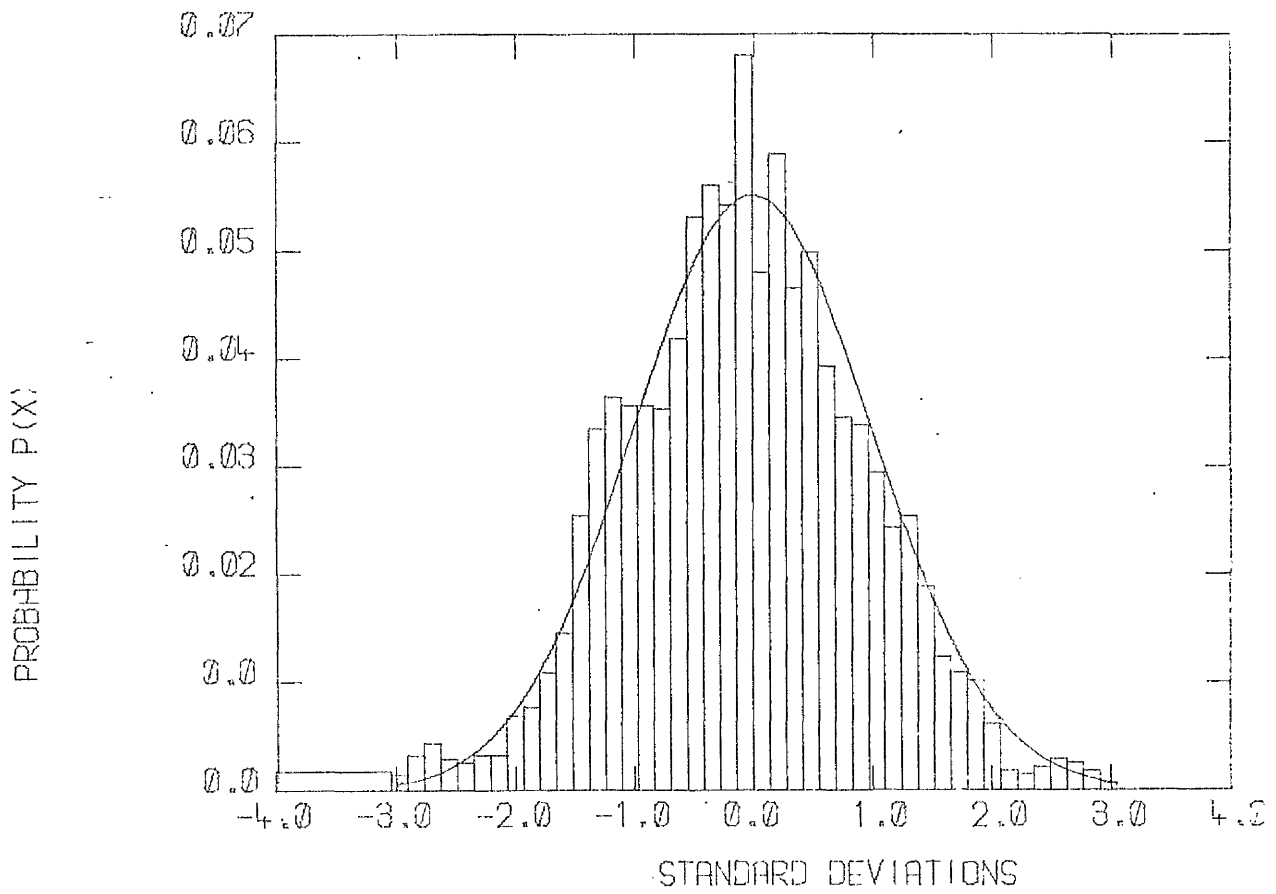
1.5 SIGMA CLIPPED — ZETA = 0.1

FIGURE 47
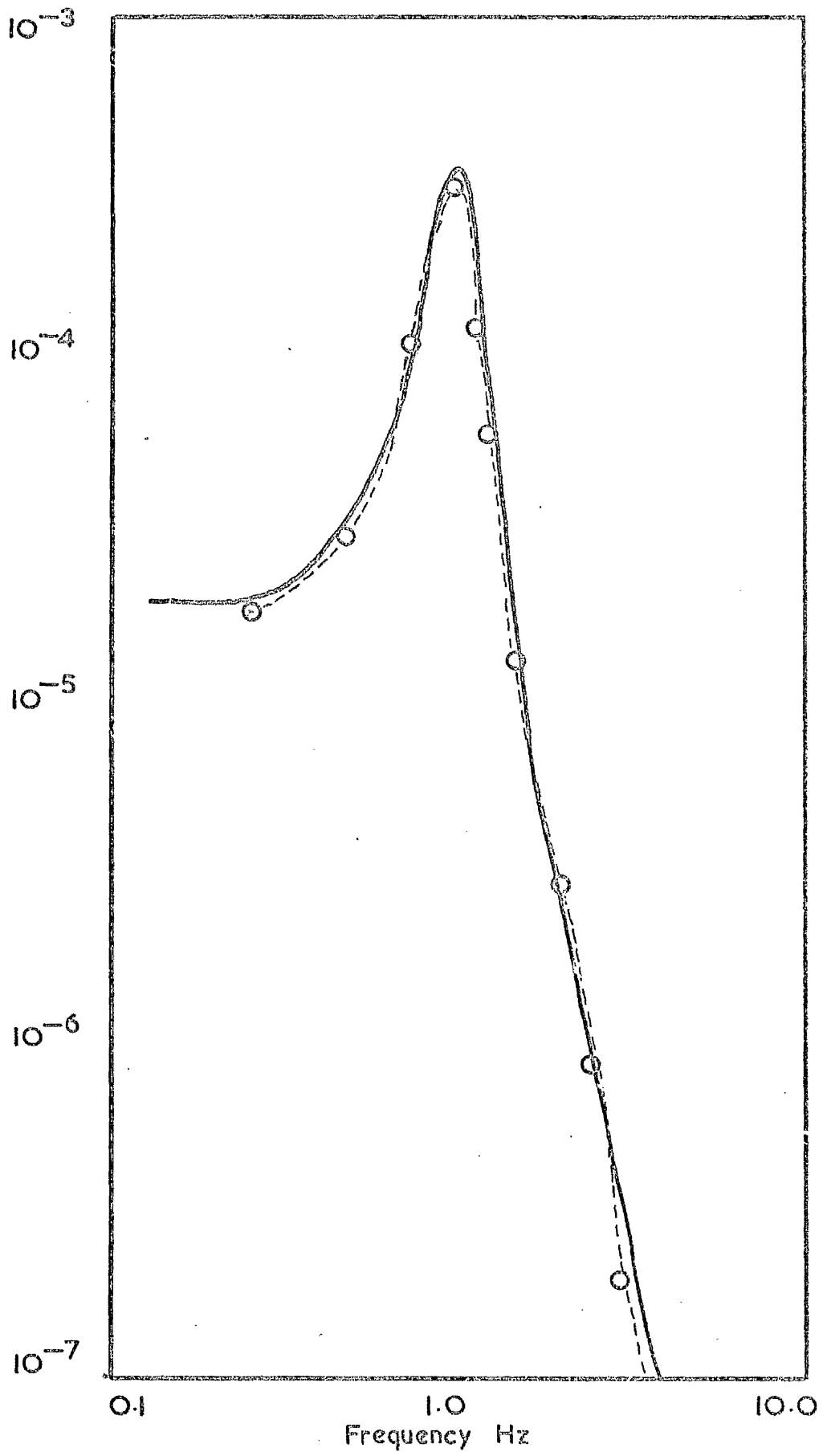
RESPONSE TO 1.5 SIGMA CLIPPED GAUSSIAN

ZETA = 0.1

FIGURE 48
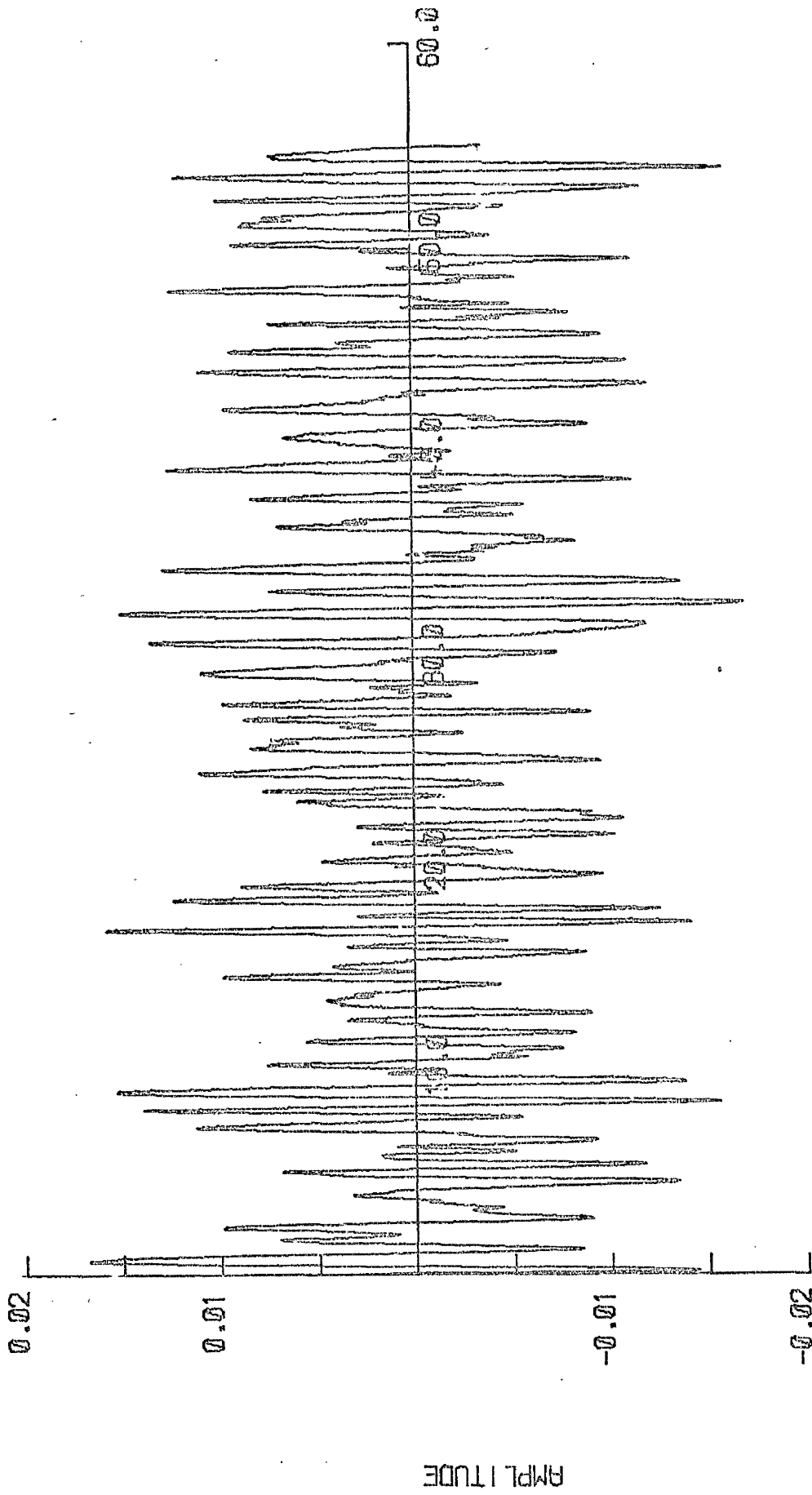
RESPONSE TO 1 SIGMA CLIPPED GAUSSIAN ZETA=0.25

FIGURE 49
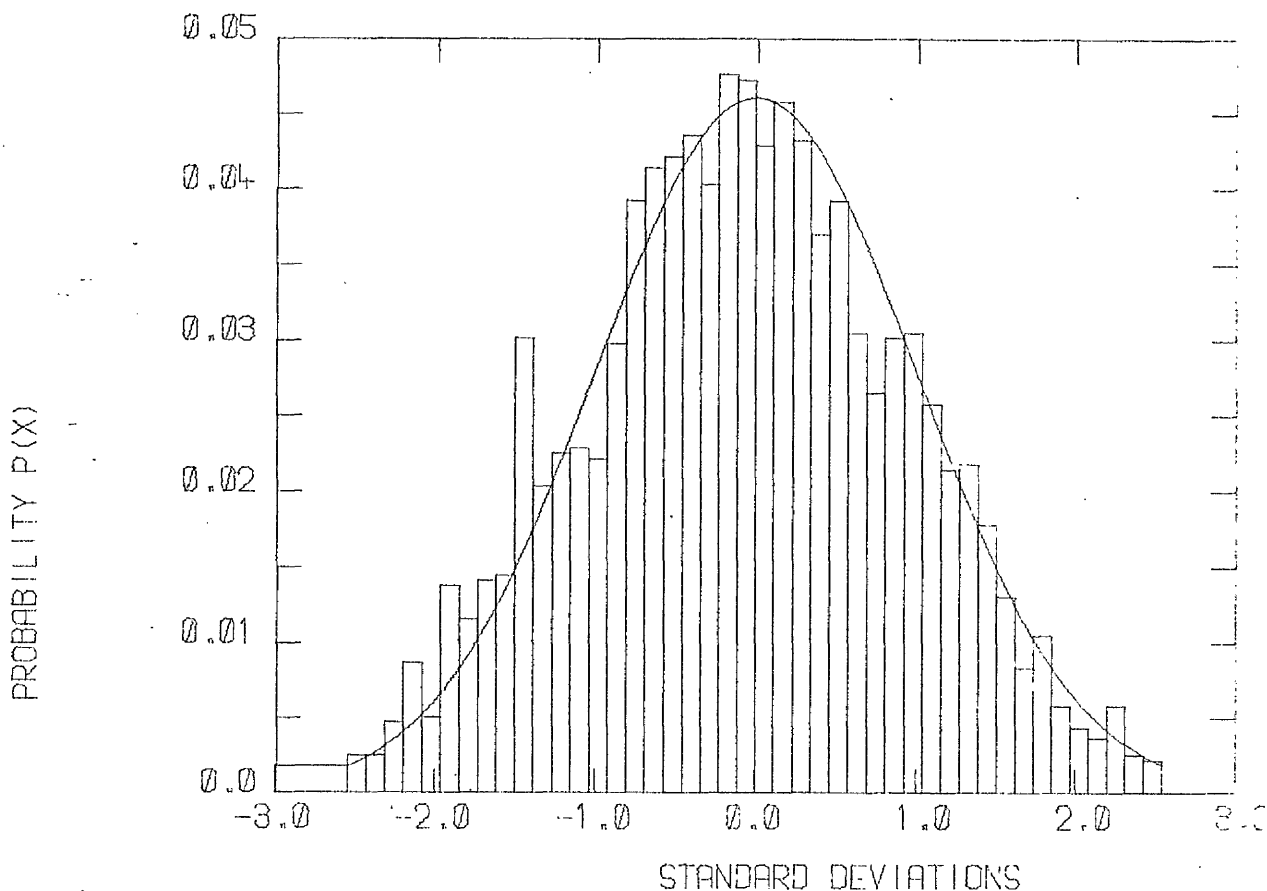
1 SIGMA CLIPPED GAUSSIAN ZETA=0.25

FIGURE 50
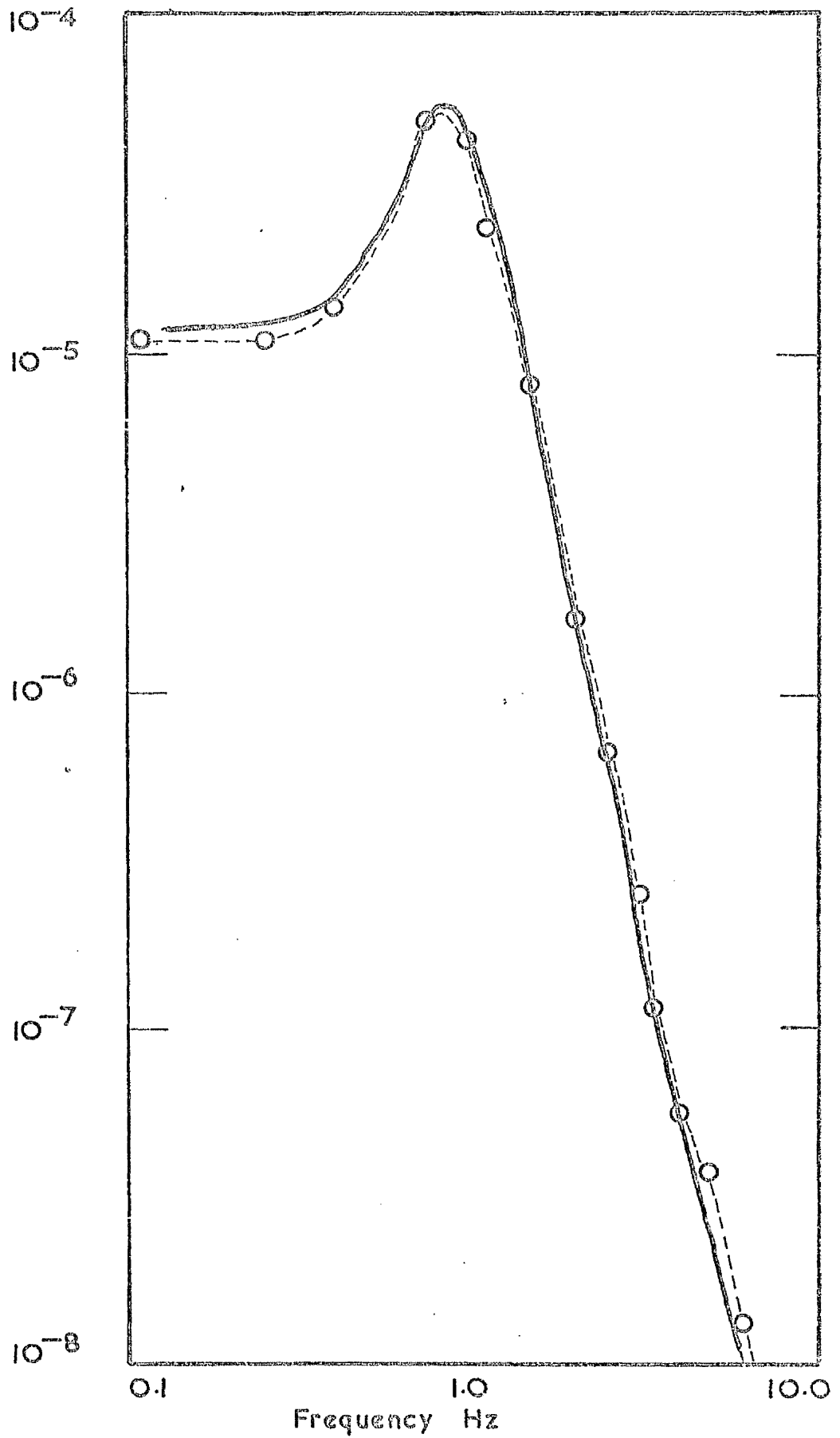
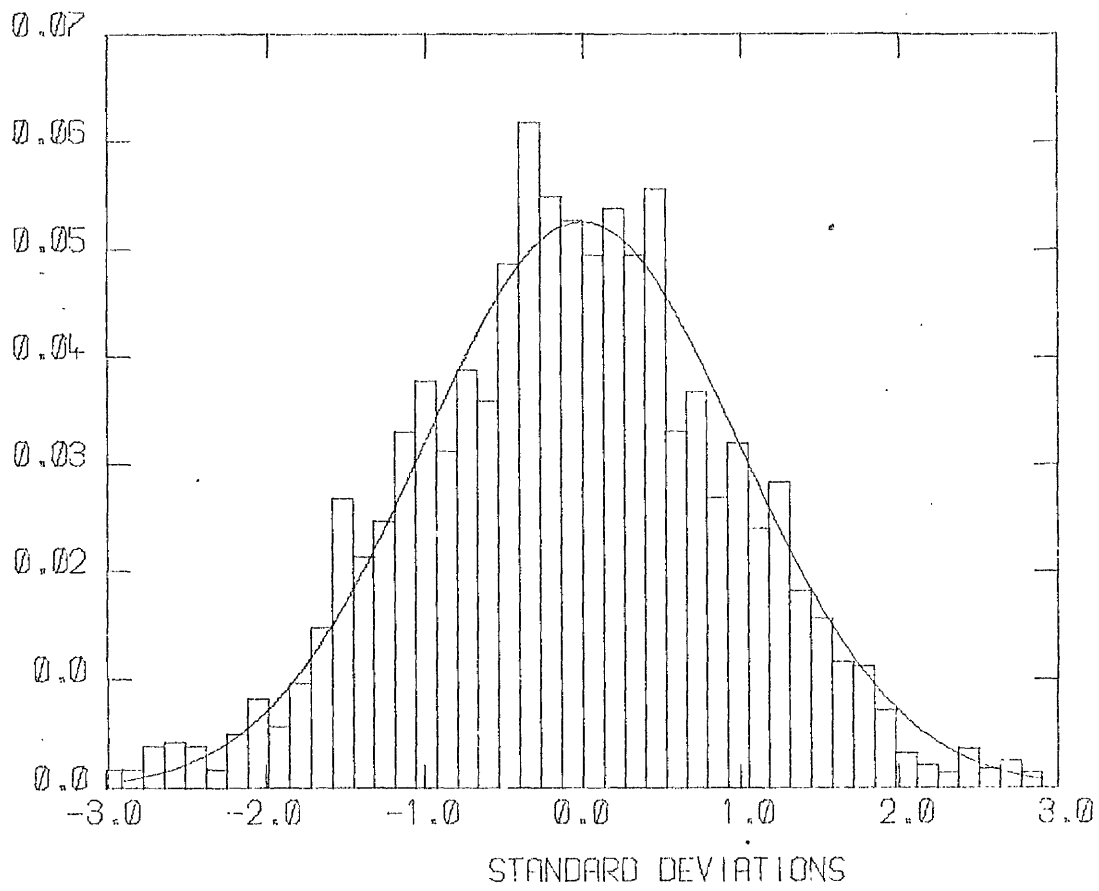RESPONSE TO 1 SIGMA CLIPPED GAUSSIAN

ZETA = 0.25

FIGURE 51
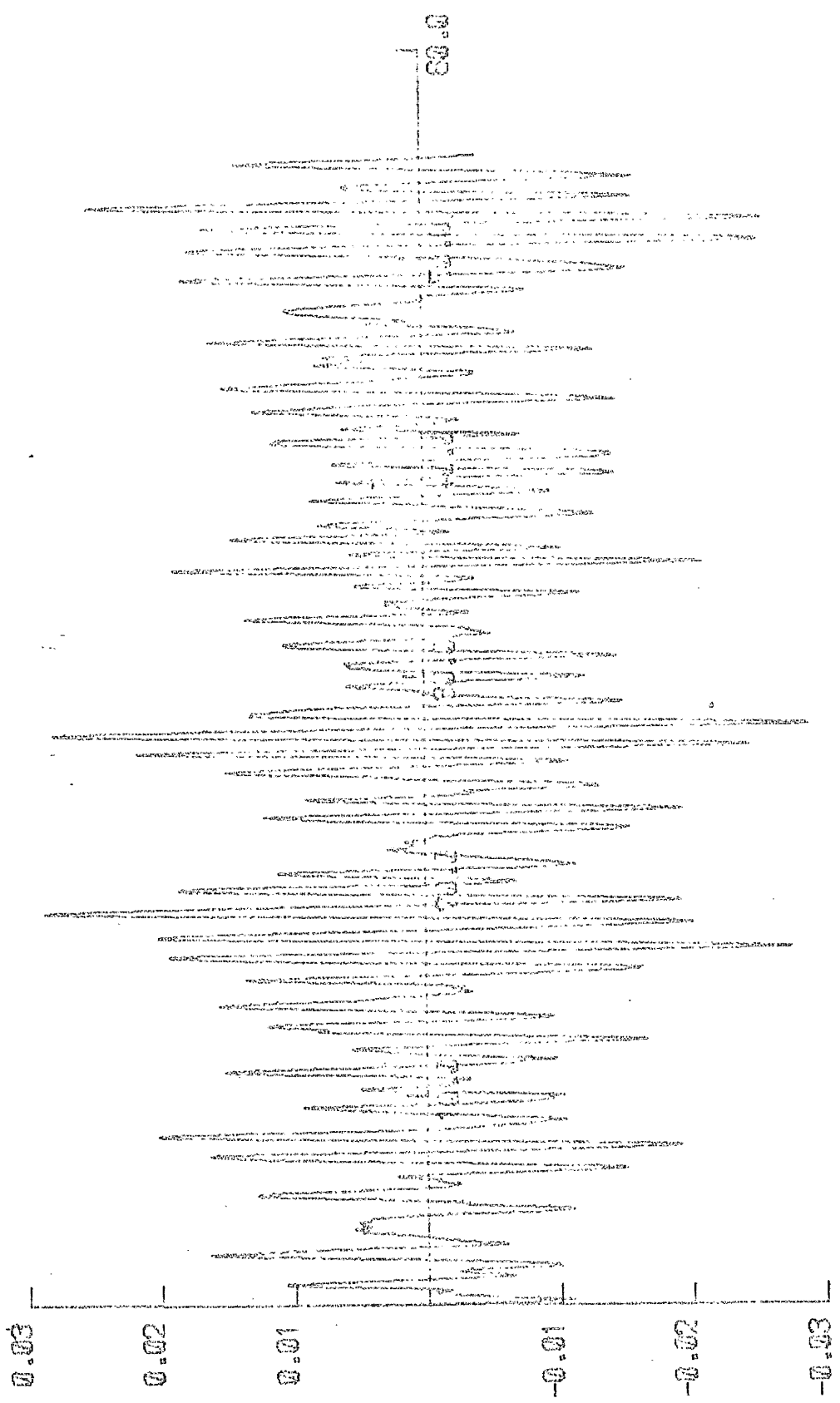
TIME (SECS)

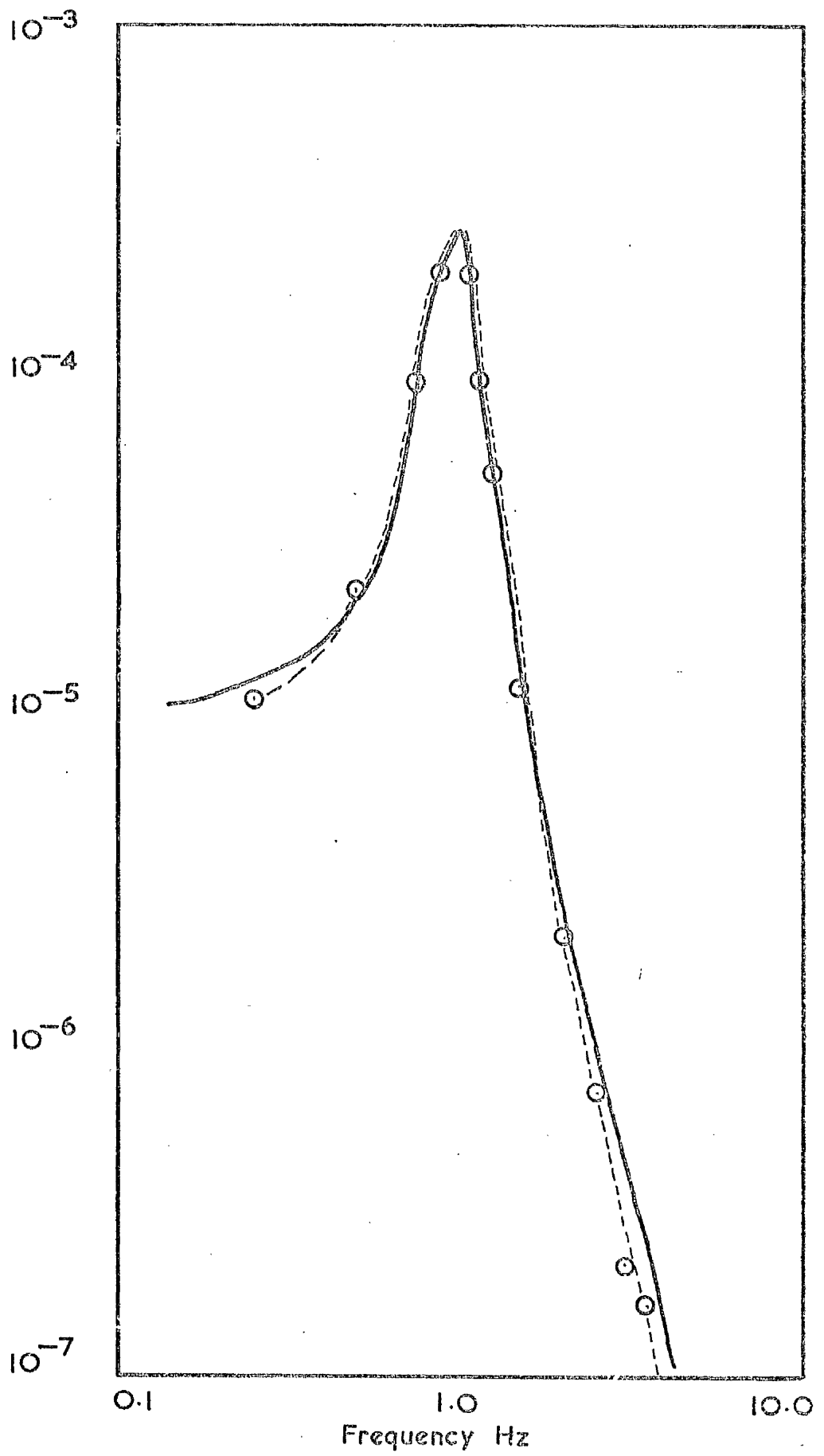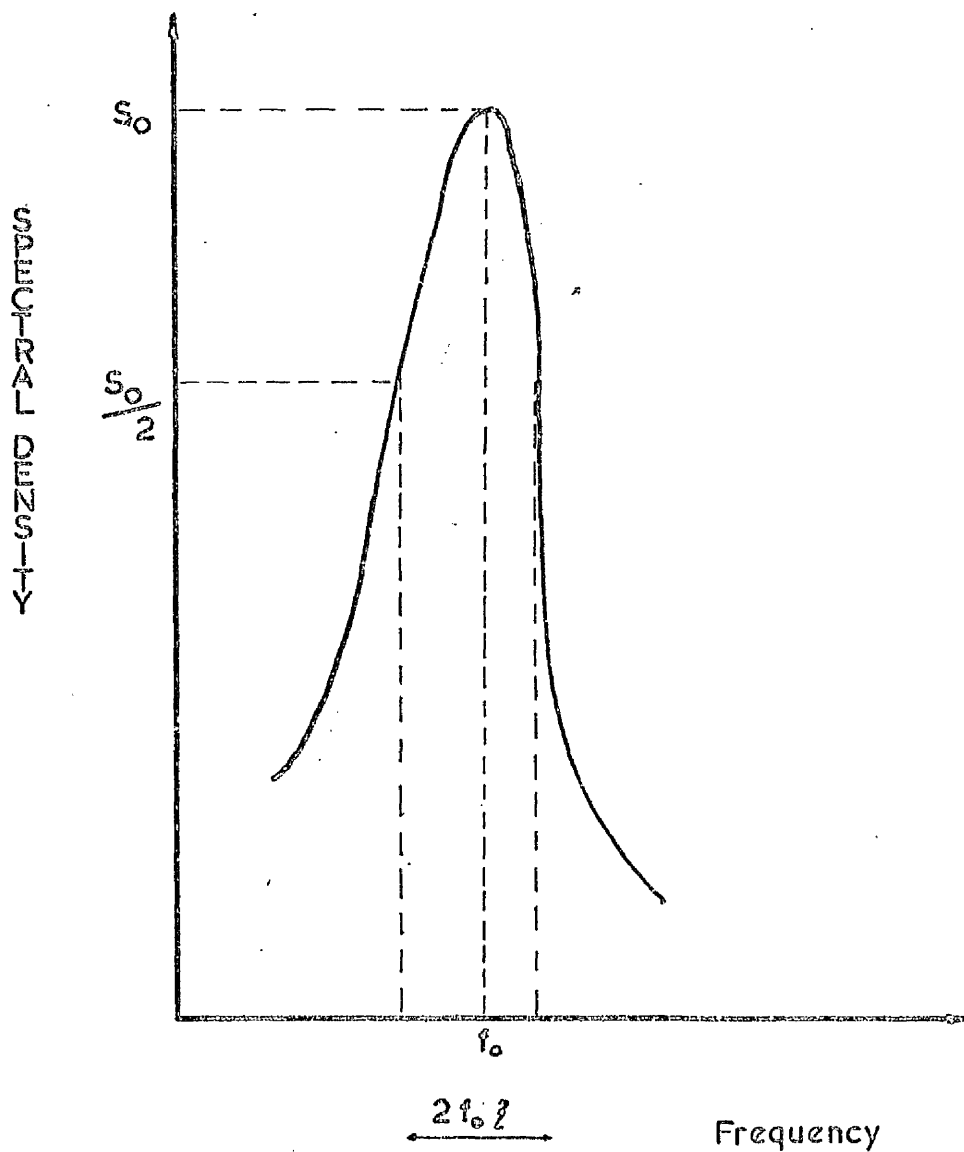RESPONSE TO 1 S.0. CLIPPED GAUSSIAN ZETA=0.1

FIGURE 52

1 SIGMA CLIPPED — ZETA = 0.1

FIGURE 53

RESPONSE TO I SIGMA CLIPPED GAUSSIAN

ZETA = 0·1

FIGURE 54

NARROW BAND RESPONSE

FIGURE 55