INTERCHANGE INSTABILITIES IN A PLASMA

by

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CHAPTER I

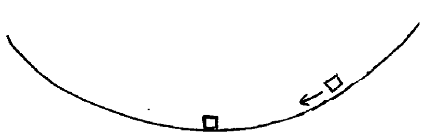
Survey of Hydromagnetic Stability Theory

§1. Introduction

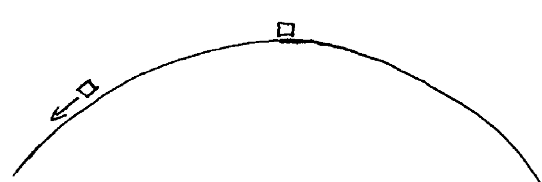
Hydromagnetic stability theory can be approached by first considering the stability of simple mechanical systems and extending the methods established in these problems to deal with the more complicated question of plasma stability. This approach will be employed here to derive the Princeton Energy Principle (2), probably the most useful and widely used instrument in the investigation of the stability of static equilibria of plasmas.

There are two methods of studying the stability of a mechanical system, which can be illustrated by the one dimensional problem of a particle in a potential well. These two methods correspond to the normal mode, and energy methods of which only the latter will be discussed here because of its wider applicability.

The requirement for equilibrium in a potential well is $\frac{dV(z)}{dz} = 0$, and this can occur in at least two ways.



1 (a)



1 (b)

If the particle is displaced from its position of equilibrium it experiences a force

1.1
$$F = - \frac{dV(z_0 + \delta z)}{dz} \simeq - \frac{d^2V}{dz^2} \cdot \delta z$$

If F has the same sign as δz the system is unstable (as shown in 1(b)), but if F has the opposite sign from δz, i.e. if $\frac{d^2V}{dz^2}$ is positive, the system is stable (as in 1(a)). To study the equilibrium stability of a particle in a potential, $V(z)$, the point of equilibrium z_0 is found; then the equation of motion of the particle when displaced by a small amount is written down, expanding the potential in powers of δz and keeping only the linear term.

1.2
$$m \frac{d^2}{dt^2}(\delta z) = \frac{\partial^2 V(z_0)}{\partial z^2} \cdot \delta z$$

Since $\frac{\partial^2 V(z_0)}{\partial z^2}$ is independent of δz this equation has solution

$\delta z \sim e^{\Omega t}$ where

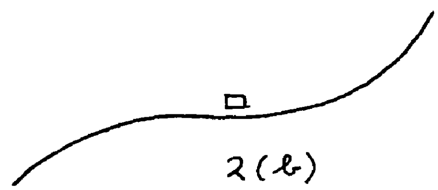
1.3
$$\Omega^2 = - \frac{1}{m} \frac{\partial^2 V(z_0)}{\partial z^2}$$

and the solutions are either oscillating, $\Omega^2 < 0$, or exponentially growing (and decaying), $\Omega^2 > 0$.

Hence the problem can be investigated by: (a) solving the linearized equation of motion with a time dependence $\exp \Omega t$, and determining the sign of Ω^2, or

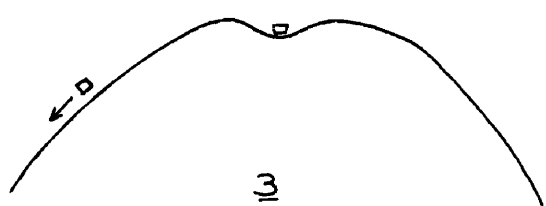
(b) investigating the sign of $\frac{\partial^2 V(z_0)}{\partial z^2}$. Of these two methods the first is generally applicable but difficult to carry out, while the second can be applied only when the forces are derivable from a potential. Both methods are limited by linearization, and are unable to distinguish between an extended flat potential where there is neutral stability (2a), and a point of inflection which is unstable (2b).

2(a)



2(b)

Furthermore this theory considers only local behaviour, so that a small potential well on a large barrier (see 3), although unstable to finite perturbations appears to be stable in the theory.



3

If a mechanical system has many degrees of freedom, 1.2 must be replaced by

$$1.4 \quad m \frac{d^2}{dt^2}(\delta q_i) = -\sum_j \frac{\partial^2 V}{\partial q_i \, \partial q_j} \delta q_j$$

$$= \sum_j a_{ij} \, \delta q_j$$

which is the equation of motion for a disturbance, δq_i, in any one of the co-ordinates, and which involves all the displacements. To solve the equations of motion it is necessary to introduce the normal co-ordinates

1.5 $$\delta Q_j = \sum_i b_{ij}\, \delta q_i$$

and such that equation 1.4 reduces to

1.6 $$\delta \ddot{Q}_j = A_{jj}\, \delta Q_j \qquad \text{(no summation implied)}$$

Taking the scalar product of 1.4 with $\delta q_i'$ (where the prime denotes differentiation with respect to time), and integrating with respect to time gives the result.

1.7 $$\sum_i \tfrac{1}{2} m (\delta q_i')^2 = \sum_{ij} \int \delta q_i'\, a_{ij}\, \delta q_j \, dt$$

$$= \tfrac{1}{2} \sum_{ij} \int dt \left\{ \frac{a_{ij} + a_{ji}}{2} (\delta q_i' \delta q_j + \delta q_j' \delta q_i) + \frac{a_{ij} - a_{ji}}{2} (\delta q_i' \delta q_j - \delta q_j' \delta q_i) \right\}$$

1.8 $$= \tfrac{1}{2} \sum_{ij} \delta q_i\, a_{ij}\, \delta q_j$$

since $$a_{ij} = -\frac{\partial^2 V}{\partial q_i \partial q_j} = -\frac{\partial^2 V}{\partial q_j \partial q_i} = a_{ji}$$

Note that written in vector form equation 1.8 is

1.9 $$\tfrac{1}{2} m (\delta \underset{\sim}{q}')^2 = \tfrac{1}{2} \delta \underset{\sim}{q} \cdot \underset{\sim}{A}(\delta \underset{\sim}{q})$$

and that its reduction to this form from 1.7

1.10 $$\tfrac{1}{2} m (\delta \underset{\sim}{q}')^2 = \int dt \left\{ \delta \underset{\sim}{q}' \cdot \underset{\sim}{A}(\delta \underset{\sim}{q}) \right\}$$

depends on the self adjointness of the operator $\underset{\sim}{A}$, i.e. for any two vectors in $\delta \underset{\sim}{q}$ space

1.11 $$\underset{\sim}{x} \cdot \underset{\sim}{A}(\underset{\sim}{y}) = \underset{\sim}{y} \cdot \underset{\sim}{A}(\underset{\sim}{x})$$

Hence from 1.8 the kinetic energy will increase if the quantity $\frac{1}{2}\sum_{ij} a_{ij}\,\delta q_i\,\delta q_j$ is positive for any choice of arbitrary δq_i, and this quantity is just the second order variation of the potential energy about equilibrium

$$-\delta W = -\left\{V(q_0+\delta q) - V(q_0)\right\}$$

$$= -\left\{V(q_0) + \sum_i \delta q_i \frac{\partial V(q_0)}{\partial q_i} + \frac{1}{2!}\sum_{ij} \delta q_i\,\delta q_j \frac{\partial^2 V}{\partial q_i \partial q_j}(q_0) - V(q_0)\right\}$$

1.12 $$= +\frac{1}{2}\sum_{ij} a_{ij}\,\delta q_i\,\delta q_j$$

Hence if $\left\{V(q_0+\delta q) - V(q_0)\right\}$ is positive the kinetic energy must be decreased by such a perturbation, and the system will be stable if $\left\{V(q_0+\delta q) - V(q_0)\right\}$ is positive for all possible choices of $\delta \underset{\sim}{q}$.

For a continuous system, such as hydrodynamics, the displacement, instead of being a vector function of time $\delta \underset{\sim}{q}(t)$, is a function of space and time $\underset{\sim}{\xi}(\underset{\sim}{x}, t)$, and the set of linear equations 1.4 must be replaced by a set of partial differential equations with suitable boundary conditions, the normal co-ordinates being eigenfunctions of this system while the Ω are eigenvalues. If the energy method is used the variation in energy $\delta W(\underset{\sim}{\xi})$ in general depends on the form of the functions $\underset{\sim}{\xi}$ and must be shown positive definite for all functions $\underset{\sim}{\xi}$ to establish stability.

§2. Energy Principle in Idealized Magneto - Hydrodynamics

Throughout this section the system considered will be supposed to be described by the idealized hydromagnetic equations. Written in unrationalized Gaussian units these are:

2.1 $$\rho \frac{d\underset{\sim}{v}}{dt} = -\operatorname{grad} p + \frac{\underset{\sim}{j} \times \underset{\sim}{B}}{c}$$

2.2 $$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0$$

2.3 $$\frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt}$$

2.4 $$\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} = 0$$

2.5 $$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{j}$$

2.6 $$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

2.7 $$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$$

Of these, equations 2.1 - 2.3 are the hydrodynamic equations, and can be derived from the Boltzmann Equation, on the assumption that the collision term dominates, by expanding in powers of the mean free path. It is therefore immediately assumed that the mean free path of the particles in the plasma is small compared to all other distances in the problem. Equation 2.4 expresses the infinite conductivity of the plasma, and is a particular case of the generalized Ohms Law obtained by Spitzer (1). Equations 2.5 - 2.7 are Maxwell's equations with displacement currents neglected. In section §4 an account will be given of attempts to remove from the theory the assumptions of small mean free path, and of scalar pressure.

To investigate the stability of a system satisfying 2.1 - 2.7 about its equilibrium, these equations are considered in terms of $\underset{\sim}{\xi}(\underset{\sim}{r}_0, t)$, the displacement that has been experienced in a time t by a fluid element originally at $\underset{\sim}{r}_0$, and are linearized by neglecting terms of second order in $\underset{\sim}{\xi}$. Thus equation 2.2 becomes

2.8 $$\frac{d\rho}{dt} + \rho_0 \underset{\sim}{\nabla} \cdot \frac{\partial \underset{\sim}{\xi}}{\partial t} = 0$$

and 2.3 becomes

2.9 $$\frac{\partial p_1}{\partial t} + \frac{\partial \underset{\sim}{\xi}}{\partial t} \cdot \underset{\sim}{\nabla} p_0 = \frac{\gamma p_0}{\rho_0}\left\{-\rho_0 \underset{\sim}{\nabla} \cdot \frac{\partial \underset{\sim}{\xi}}{\partial t}\right\}; \therefore p_1(\underset{\sim}{r}, t) = -\left\{\gamma p_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + (\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}) p_0\right\}$$

While the magnetic field satisfies

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} = \underset{\sim}{\nabla} \times (\underset{\sim}{v} \times \underset{\sim}{B})$$

2.10 $$\underset{\sim}{B}_1(\underset{\sim}{r}, t) = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)$$

With these relations substituted into 2.1 the linearized equation of motion is

2.11 $$\rho_0 \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi})$$

where

2.12 $$\underset{\sim}{F}(\underset{\sim}{\xi}) = \underset{\sim}{\nabla}\left\{\gamma p_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + (\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})p_0\right\} + \frac{1}{4\pi}\left\{(\underset{\sim}{\nabla}\times\underset{\sim}{Q})\times B_0 + (\underset{\sim}{\nabla}\times\underset{\sim}{B_0})\times\underset{\sim}{Q}\right\}$$

with

2.13 $$\underset{\sim}{Q}(\underset{\sim}{\xi}) = \underset{\sim}{\nabla}\times(\underset{\sim}{\xi}\times\underset{\sim}{B_0})$$

If the scalar product of 2.11 is taken with $\dot{\underset{\sim}{\xi}}$ $\left(\equiv \frac{\partial \underset{\sim}{\xi}}{\partial t}\right)$ and the result is integrated with respect to time and throughout space, the following result is obtained.

2.14 $$\frac{1}{2}\int \rho_0 \, \dot{\underset{\sim}{\xi}}^2 \, d\tau = \int dt \int d\tau \, \dot{\underset{\sim}{\xi}} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

This equation is the counterpart of equation 1.7 of the previous section and just as in that case the operator $\underset{\sim}{A}$ was shown to be self adjoint, so it has been proved (Bernstein et al (2)) that the operator $\underline{F}$ has this property

2.15 $$\int d\tau \left\{\underset{\sim}{\eta}\cdot\underset{\sim}{F}(\underset{\sim}{\xi}) - \underset{\sim}{\xi}\cdot\underset{\sim}{F}(\underset{\sim}{\eta})\right\} = 0$$

The demonstration of this fact depends on the existence of an energy integral for equations 2.1 - 2.7., viz

2.16 $$U = \int d\tau \left\{\frac{1}{2}\rho \underset{\sim}{v}^2 + \frac{\underset{\sim}{B}^2}{8\pi} + \frac{p}{\gamma - 1}\right\}$$

such that when the potential energy terms are expanded in $\underset{\sim}{\xi}$, the change in the potential energy is a quadratic form $\delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi})$ not involving $\dot{\underset{\sim}{\xi}}$, while the kinetic energy is just

2.17 $$K(\dot{\underset{\sim}{\xi}}, \dot{\underset{\sim}{\xi}}) = \frac{1}{2} \int \rho_0 \, \dot{\underset{\sim}{\xi}}^2 \, d\tau$$

Then using 2.14 and the fact that

$$K(\dot{\underset{\sim}{\xi}}, \dot{\underset{\sim}{\xi}}) + \delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi})$$

is constant,

$$\dot{K} = \int d\tau \, \dot{\underset{\sim}{\xi}} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

$$= - \delta \dot{W}$$

2.18 $$= - \delta W(\dot{\underset{\sim}{\xi}}, \underset{\sim}{\xi}) - \delta W(\underset{\sim}{\xi}, \dot{\underset{\sim}{\xi}})$$

From which 2.15 follows since $\dot{\underset{\sim}{\xi}}$ satisfies the same boundary conditions as $\underset{\sim}{\xi}$ and can therefore be chosen equal to any arbitrary displacement $\underset{\sim}{\eta}$. Hence, if $\dot{\underset{\sim}{\xi}}$ is replaced by $\underset{\sim}{\xi}$ in 2.18

2.19 $$\delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi}) = -\frac{1}{2} \int d\tau \, \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

It was further shown that the system can be unstable if and only if there exists an $\underset{\sim}{\xi}$ which makes δW negative.

After integration by parts, use of vector identities, and of the boundary condition

2.20 $$\underset{\sim}{n} \cdot \underset{\sim}{B} = 0$$

equation 2.19 can be written in the form, 2.21 $$\delta W = \frac{1}{2}\int d\tau \left\{ \frac{\underset{\approx}{Q}^2}{4\pi} - \underset{\sim}{J}\cdot\underline{\underset{\approx}{Q}\times\underset{\sim}{\xi}}_{\,c} + \gamma p_0 (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^2 + (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})p_0 \right\}$$

$$-\frac{1}{2}\int d\sigma \; \underset{\sim}{n}\cdot\underset{\sim}{\xi} \left\{ \gamma p_0 (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}) + (\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})p_0 - \frac{\underset{\sim}{B}_0\cdot\underset{\approx}{Q}}{4\pi} \right\}$$

If the plasma is bounded by a rigid conducting wall the boundary condition on this surface is $\underset{\sim}{n}\cdot\underset{\sim}{\xi} = 0$ so that the surface integral in 2.21 vanishes. Otherwise this integral can be split into a volume integration throughout the surrounding vacuum, and a surface integral over the plasma-vacuum interface. Then

2.22 $$\delta W = \delta W_p + \delta W_s + \delta W_v$$

2.23 $$\delta W_p = \frac{1}{2}\int_p d\tau \left\{ \frac{\underset{\approx}{Q}^2}{4\pi} - \underset{\sim}{J}\cdot\underline{\underset{\approx}{Q}\times\underset{\sim}{\xi}}_{\,c} + \gamma p_0 (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^2 + (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})p_0 \right\}$$

2.24 $$\delta W_v = \frac{1}{8\pi}\int_v d\tau \, (\underset{\sim}{\nabla}\times\underset{\sim}{A})^2$$

where $\underset{\sim}{A}$ is the vacuum vector potential.

2.25 $$\delta W_s = \frac{1}{2}\int d\sigma \, (\underset{\sim}{n}\cdot\underset{\sim}{\xi})^2 \, \underset{\sim}{n}\cdot\left[\!\left[\underset{\sim}{\nabla}\left(p + \frac{B^2}{8\pi}\right) \right]\!\right]$$

where $[\![X]\!]$ represents the increase in the quantity X on crossing the boundary.

Finally Bernstein et al have shown that the displacement $\underset{\sim}{\xi}$ considered need not satisfy all of the normal boundary conditions. The $\underset{\sim}{\xi}$'s considered need only satisfy the following conditions;

2.26 $$(\underset{\sim}{n}\times\underset{\sim}{A}) = -(\underset{\sim}{n}\cdot\underset{\sim}{\xi})\,\hat{\underset{\sim}{B}}_0$$

at a plasma vacuum interface ($\hat{\underset{\sim}{B}}_o$ is the vacuum field), and

2.27 $\underset{\sim}{n} \cdot \underset{\sim}{\xi} = 0$

at a rigid perfectly conducting surface. It was shown that if an $\underset{\sim}{\xi}$, satisfying only 2.26 and 2.27, can be found which makes δW negative, then there also exists an $\underset{\sim}{\xi}$ satisfying the full boundary conditions which makes δW negative. The great usefulness of the energy principle in stability problems is due to this fact.

The work of the Princeton Group (2) was based on an earlier energy principle, introduced by Lundquist (3) , in which the variation of the magnetic energy integral $\int \frac{B^2}{8\pi} d\tau$ was calculated in terms of the displacement vector $\underset{\sim}{\xi}$, using the fact that the assumption of perfect conductivity in 2.4 implies a 'frozen in field'. For if any area is considered in the plasma the flux through this area is $\int \underset{\sim}{B} \cdot d\underset{\sim}{S}$ and if the time derivative of this is taken moving with the plasma, then

2.28 $$\frac{\partial}{\partial t} \int \underset{\sim}{B} \cdot d\underset{\sim}{S} = - c \int (\underset{\sim}{\nabla} \times \underset{\sim}{E}^*) \cdot d\underset{\sim}{S}$$

where $\underset{\sim}{E}^* = \underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} = 0$

Hence the flux thro' any area fixed in the plasma remains constant.

The thermodynamic energy $\int \frac{p}{\gamma - 1} d\tau$ did not arise in this case because

2.29 $$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

was used as the equation of state instead of 2.3 i.e. an incompressible plasma was assumed.

§3. Applications of the Energy Principle.

The energy principle (2.21) derived above has been applied to various problems of more or less simple geometry. For problems with cylindrical symmetry the perturbations may be analysed as

$$3.1 \quad \underset{\sim}{\xi}(r) = \sum_{m,k} \xi_{mk}(r) \; \exp[i(m\theta + kz)]$$

where r, θ, z are cylindrical co-ordinates, and the Fourier components handled independently. The Euler equations for δW are algebraic and minimize δW for ξ_θ and ξ_z. When the minimization has been carried out δW may be written in terms of ξ_r alone.

$$3.2 \quad \delta W = \int r\,dr \left\{ \frac{(r f \xi' + g\xi)^2}{m^2 + k^2 r^2} + (f^2 - h)\xi^2 \right\}$$

where $f = k B_z + \frac{m}{r} B_\theta$

$g = k B_z - \frac{m}{r} B_\theta$

$h = \frac{2 B_\theta}{r^2} \frac{d}{dr}(r B_\theta)$

$\xi \equiv \xi_r$

and the prime denotes differentiation w.r.t r. Clearly δW can be negative only if h is positive, i.e. only if B_θ falls off less rapidly than $\frac{1}{r}$. Hence a sufficient condition for stability is

3.3 $$B_\theta \frac{d}{dr}(r B_\theta) < 0$$

and this can be achieved in the hard core or inverse pinch, in which a solid rod carries an axial current through the centre of the plasma. The current in the plasma is in the opposite direction to that in the conductor. Working from the energy integral 3.2 several authors including Laing (4) and Suydam (5) were able to find sufficient conditions for stability. For example such a condition obtained by Laing is

3.4 $$4\pi r p' > \frac{2}{3} B_\theta^2$$

to be satisfied throughout the plasma

Another is

3.5 $$\left.\begin{array}{ll} 4\pi r p' > \frac{2}{3} B_\theta^2 & r < r_0 \\ B_\theta = 0 & r \geqslant r_0 \end{array}\right\} \text{ For some } r_0$$

The most important single characteristic of these stable configurations is that the total current carried by the discharge is zero. Working in the other direction, Rosenbluth (6), by using specially chosen trial functions for the radial perturbation ξ_r, was able to show that there exist configurations, arbitrarily close to the stabilized pinch configurations which are unstable.

The Euler equation of 3.2 is

3.6 $$\xi'' + P\xi' + Q\xi = 0$$

where P,Q are functions of f,g,h,k,m,r. In investigating this problem, Suydam (7) observed that the most dangerous instabilities were those 'fluted' displacements which interchange $\underset{\sim}{B}$ lines without bending them. The $\underset{\sim}{B}$ lines describe a set of spirals of pitch $\mu = \frac{B_\theta}{r B_z}$ which in general varies from layer to layer. The level lines of the $\underset{\sim}{\xi}$ field on the other hand, describe a set of spirals of pitch $-k/m$ which is constant throughout the plasma. Hence if these two sets of spirals match over a finite region of space then a displacement is possible which does not bend $\underset{\sim}{B}$ lines. Even if μ' is never zero it is still possible to chose k so that the two spiral systems match at one particular radius, and in this case displacements are possible which bend $\underset{\sim}{B}$ lines very little in the neighbourhood of this radius. It was therefore assumed that the worst possible choice of k,m is such that

3.7 $$f = kB_z + \frac{m}{r} B_\theta$$

vanishes at some point in the plasma,

r = a say.

Then at r = a, f = o and equation 3.6 has in fact a regular singularity. The solutions of the Euler equation can therefore be written as

$$\xi = (r-a)^{\nu} \left\{ \text{Power series in } (r-a) \right\}$$

where ν is a root of the indicial equation

$$\nu^2 + \nu + M = 0$$

where

$$M^2 \equiv - \left[\frac{8\pi p'}{r B_z^2} \left(\frac{\mu}{\mu'} \right)^2 \right]_{r=a}$$

It is possible to show that, for this particular solution of the Euler equation, if the roots of the indicial equation are complex, then the system is unstable, and if the roots are real the system is stable. Thus the following necessary condition must be satisfied throughout the system for stability

3.8 $$\left(\frac{\mu'}{\mu} \right)^2 + \frac{32\pi p'}{r B_z^2} > 0$$

This condition will be sufficient also, if the Euler equation 3.6 is in fact minimizing, and if the physical arguments leading to the particular choice of k do correspond to actual minimization with respect to k.

If the pitch μ is constant throughout the plasma the local instabilities described by 3.8 are replaced by convective instabilities in which whole flux tubes are interchanged the energy being derived from the expansion of the gas. A necessary condition for stability ($\mu'=o$) is

$$3.9 \quad B_\theta \frac{d}{dr}\left(\frac{B_\theta}{r}\right) + \frac{2B_\theta^4}{r^2(B_\theta^2 + B_z^2 + 4\pi\gamma p)} < o$$

More recently a necessary and sufficient condition has been given (Newcomb (11), Suydam (5)) for stability of a cylindrical plasma. It may be stated as follows:

A necessary and sufficient condition for plasma stability is that the solutions of the Euler equation of the energy integral 3.2 shall have no zeros between the singular points of the equation.

Just prior to the publication of the energy principle (2), Rosenbluth and Longmire (8) considered the problem of stability in a mirror machine with longitudinal and radial magnetic fields. In the limit of low pressure the following stability criterion was obtained

$$3.10 \quad \int \frac{dl}{R\, r\, B^2} > o$$

where R is the radius of curvature of the field line along which integration is carried, and r is the distance from the axis. The method used was to calculate the second order variation in the thermodynamic energy integral $\int \frac{p}{\gamma-1} d\tau$ due to the flute type of interchange. The assumptions made in this approach, and the limitations of the method will be discussed more fully in Chapter 2 of this paper in which the methods of Rosenbluth and Longmire are extended to the non-zero pressure case. In the same paper (8) a different approach to the same problem was based on a consideration of the individual particle orbits. Using two adiabatic invariants μ (magnetic moment of the orbiting particles), and $\int v_{\parallel} dl$ (where $v_{\parallel}$ is the component of velocity along the field line) the first order energy change was calculated for a single particle when its field line is displaced from its equilibrium position, and this energy change summed over all the particles tied to that line. The energy change obtained in this way was shown to be proportional to $\int \frac{p_{\parallel} + p_{\perp}}{R_{\tau} B^2} dl$ where $p_{\parallel}$ and $p_{\perp}$ are the components of the pressure p along and perpendicular to the field lines. Thus a criterion for surface stability can be stated as

3.11 $$\int \frac{p_{\parallel} + p_{\perp}}{R_{\tau} B^2} dl > 0$$

which reduces to 3.10 for scalar pressure.

In the original paper (2) the energy principle was also applied to the problem of stability in a mirror machine, with no field in the θ direction. Co-ordinates (ψ, θ, χ) were used such that the ψ direction is perpendicular to the field, and the χ direction along the field.

The perturbations ξ were Fourier analysed with respect to θ as follows

$$\xi_\psi = \sum_{m=0}^{\infty} \frac{X_m(\psi,\chi)}{r B} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\theta$$

3.12 $$\xi_\theta = \sum_{m=1}^{\infty} \frac{r Y_m(\psi,\chi)}{m} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\theta + \xi_\theta^0(\psi,\chi)$$

$$\xi_\chi = \sum_{m=0}^{\infty} B Z_m(\psi,\chi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\theta$$

Upon integration of δW with respect to θ the cross terms of the double product vanish so that

3.13 $$\delta W = \delta W_0 + 2\sum_{m=1}^{\infty} \delta W_m$$

The δW_m have to be treated separately, but the situation is simplified by the fact that if for $m \geq 1$ δW_m can be made negative by some choice of ξ so also can δW_{m+1}, so that in practice only the cases $m = \infty$, $m = 0$ need be investigated. Now each mode corresponds to a flute type of interchange considered by Rosenbluth and Longmire so that it appears that 3.10 may be a more general result than seemed likely.

If δW_m is considered with $m = \infty$ it is found that it can be minimized algebraically with respect to Y and can then be minimized with respect to Z . At this stage only one term in δW_∞ can possibly be negative and this is

$$\left\{ -2p' \int \frac{X^2\, dl}{R r B^2} \right\} \qquad \text{where } p' = \frac{1}{rB} \frac{\partial p}{\partial \psi}$$

Hence a sufficient condition for stability is $R > 0$ i.e. the field is everywhere convex to the region of higher pressure. A necessary condition for stability is obtained by assuming X independent of χ . This is

3.14 $$\left\{ \int \frac{dl}{R r B^2} \right\} \left\{ 2 \int \frac{dl}{R r B^2} - 4\pi p' \int \frac{dl}{B^3} - \frac{p'}{\gamma p} \int \frac{dl}{B} \right\} > 0$$

which, if $p' < 0$, can be satisfied for

3.15 $$\int \frac{dl}{R r B^2} > 0$$

or

3.16 $$\int \frac{dl}{R r B^2} < 2\pi p' \int \frac{dl}{B^3} + \frac{p'}{2\gamma p} \int \frac{dl}{B}$$

The Euler equation for minimization with respect to X cannot be solved exactly, but stability criteria can be obtained in certain limiting cases. Two such cases are (a) $\frac{B^2}{8\pi} \gg p$, i.e. magnetic pressure very much greater than material pressure, and (b) if the surface Ψ = constant (the field lines of the mirror machine lie in these surfaces) under consideration lies close to a cylinder. In both these cases the stability criterion obtained is 3.15 with 3.16 as the alternative in case (b) and

$$3.17 \quad \int \frac{dl}{R_r B^2} < \frac{p'}{2\gamma p} \int \frac{dl}{B}$$

as the alternative in case (a)

Thus the condition 3.10 originally derived for stability against the flute type of interchange in the low pressure limit turns out to be a necessary condition for plasma stability against all modes ($m \neq 0$), and in certain limiting cases turns out to be a sufficient condition also.

Any contained hydromagnetic equilibrium is topologically toroidal, and some of the results obtained for cylindrical systems have been extended to toroidal systems. Kadomtsev (9), for example, has also shown how to obtain an analogue of Suydam's criterion which is applicable to general toroidal configurations. The form of the necessary condition is

$$(\mu')^2 + 32\pi p' A > 0 \tag{3.18}$$

where, if ψ, χ are the longitudinal and azimuthal fluxes, I,J are the longitudinal and azimuthal currents, and Ω is the volume of the torus, then $\mu = \frac{d\chi}{d\psi}$ represents the number of turns of the line of force along the small perimeter in one revolution along the toroid, A is a rather complicated function of I, J, μ, Ω, and the prime denotes differentiation with respect to ψ. The method used to obtain this result is to Fourier analyse the components of the perturbation perpendicular to the field as follows,

$$\xi = \eta(\psi) \exp\left\{2\pi i(m\theta + n\zeta)\right\}$$

and to choose $\mu m + n = o$. This corresponds to the choice of $k B_z + \frac{m}{r} B_\theta = 0$ in the Suydam criterion to match the spirals of the two fields, $\underset{\sim}{B}$ and $\underset{\sim}{\xi}$.

Mercier (10) has obtained another analogue of Suydam's Criterion which is a necessary condition for stability in an axisymmetric torus. This criterion is more stringent than Kadomtsev's which can of course also be applied to the case of axial symmetry.

It is possible that a toroidal version of Newcomb's necessary and sufficient condition also exists. This would depend on whether there exist solutions of the Euler equation which leave an entire magnetic surface unperturbed. Sufficient conditions for the stability of toroidal configurations have been given by Mercier (10) and Suydam (12).

Few explicit forms for toroidal equilibria are known, as they can only be obtained by the solution of a non-linear partial differential equation, and as a consequence of this few detailed applications of the stability criteria have been made. One such application by Lüst, Suydam, Richtmeyer, Rotenberg and Levy (13) to a toroidal analogue of the Stabilized Pinch shows that the toroidal results are similar to the cylindrical results if the aspect ratio is not too small.

In spite of the many assumptions of ideal hydromagnetics there is a moderate agreement between the theoretical predictions and the experimental results. It should, however, be noted that many types of instability are almost independent of the particular properties assigned to the fluid; for example, the wriggling of a constricted discharge is an instability which requires only that the plasma be a flexible conductor, hence it is in no way remarkable that the theory should predict it. Attempts to investigate the detailed consequences of hydromagnetic stability theory have shown but little success. Magnetic probe measurements in a rapidly collapsing pinch showed evidence of fluctuations appearing most strongly in regions where the Suydam criterion was violated. On the other hand, attempts to produce the stable unpinch predicted by 3.2 have not been successful.

§4. Non-Idealized Hydromagnetics

Validity of the Hydrodynamic Equations

The standard equations of hydrodynamics depend, as was noted in §2, on the assumption of small mean free path in the fluid. In general this may not be a valid assumption, and in particular in a low density plasma will not hold. An attempt was therefore made (Chew, Goldberger, Low (14)) to derive a set of magneto-hydrodynamic equations from the Boltzmann equation, using some other localising property than collisions.

In a plasma the individual ions and electrons follow helical paths, orbiting around their guiding centres, which in turn follow field lines and, assuming that the radius of gyration ($\sim v_\perp \frac{mc}{eB}$) is small compared to other lengths in the system (such as $B/\nabla B$), it was this property which was used. The Boltzmann equation with no collisions was written down, and expanded in powers of $\frac{m}{e}$ (equivalent to the radius of gyration)

4.1 $$\frac{\partial f}{\partial t} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) f + \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right) \cdot \operatorname{grad}_{v} f = 0$$

where e is the ion charge, m the ion mass, f the Boltzmann function, and grad_v denotes gradient in velocity space. In powers of $\frac{m}{e}$ f is $f = f_0 + f_1 + f_2 + \ldots\ldots$

Then the series of equations obtained from 4.1 is

4.2 $$\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right)\cdot \text{grad}_v f_0 = 0$$

4.3 $$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\right)f_0 + \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right)\cdot \text{grad}_v f_1 = 0$$

$\cdots\cdots\cdots\cdots\cdots\cdots$

4.2 was solved by assuming that $\underset{\sim}{E}$ is perpendicular to $\underset{\sim}{B}$, a result which can be qualitatively justified by observing that the high mobility of electrons along the field lines might be expected to prevent any component of the electric field $\underset{\sim}{E}$ from building up in this direction.

4π }
c }
to be added to various terms

The first two moments of 4.3 were taken in the usual way giving the following macroscopic equations,

4.4 $$\frac{\partial\rho_0}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho_0\underset{\sim}{u}_0) = 0$$

4.5 $$\rho_0\frac{d\underset{\sim}{u}_0}{dt} = -\underset{\sim}{\nabla}\cdot\overleftrightarrow{P_0} + (\underset{\sim}{\nabla}\times\underset{\sim}{B})\times\underset{\sim}{B} + \left[\frac{\partial}{\partial t}(\underset{\sim}{u}_0\times\underset{\sim}{B})\right]\times\underset{\sim}{B} + (\underset{\sim}{u}_0\times\underset{\sim}{B})\,\underset{\sim}{\nabla}\cdot(\underset{\sim}{u}_0\times\underset{\sim}{B})$$

where $\rho_0 = m\int f_0\,d\underset{\sim}{v}$

$$\rho_0\underset{\sim}{u}_0 = m\int \underset{\sim}{v} f_0\,d\underset{\sim}{v} \quad ; \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}_0\cdot\underset{\sim}{\nabla}$$

and where use has been made of Maxwell's Equations and of the infinite conductivity equation

$$4.6 \quad E + \frac{u_0 \times B}{c} = 0$$

in order to eliminate f_1 in favour of a macroscopic quantity j_1 defined by

$$4.7 \quad j_1 = \nabla \times B + \frac{\partial}{\partial t}(u_0 \times B) + u_0 \, \nabla \cdot (u_0 \times B)$$

h 4π
h c.

$\overleftrightarrow{P_0}$ is a pressure tensor defined by

$$\overleftrightarrow{P_0} = m \int dv \, (v - u_0)(v - u_0) \, f_0$$

and is restricted because of the nature of the solution of 4.2 to be of the form

$$4.8 \quad \overleftrightarrow{P_0} = p_{\parallel} e e + p_{\perp} (\overleftrightarrow{1} - e e)$$

where e is the unit vector along the magnetic field and $\overleftrightarrow{1}$ the unit dyadic.
The above equations (4.4 and 4.5) correspond to the continuity equation, and the equation of motion (2.2 and 2.1 respectively) of the idealized case; 4.6 is the infinite conductivity equation (2.4), and Maxwell's Equations hold with j defined by 4.7. Hence only an equation of state is missing from the set. In order to complete this set of equations it was therefore

necessary to take the third moment of 4.3, and this yielded the following two equations.

4.9 $$\frac{dp_{\parallel}}{dt} = -p_{\parallel}\,\nabla\cdot\underset{\sim}{u}_{o} - 2p_{\parallel}\,\underset{\sim}{e}\cdot(\underset{\sim}{e}\cdot\nabla)\,\underset{\sim}{u}_{o} - \nabla\cdot\underset{\sim}{e}\,(q_{\parallel}+q_{\perp}) - 2\underset{\sim}{e}\cdot\nabla q_{\perp}$$

4.10 $$\frac{dp_{\perp}}{dt} = -2p_{\perp}\,\nabla\cdot\underset{\sim}{u}_{o} + p_{\perp}\,\underset{\sim}{e}\cdot(\underset{\sim}{e}\cdot\nabla)\underset{\sim}{u}_{o} - \nabla\cdot(q_{\perp}\underset{\sim}{e}) - q_{\perp}\,\nabla\cdot\underset{\sim}{n}$$

where the quantities $q_{\parallel}$ and $q_{\perp}$ are the components of the pressure-transport tensor which need not, in general, be zero. To determine $q_{\parallel}$ and $q_{\perp}$ a new equation is required, and this will bring in a fourth moment of the Boltzmann equation. Only if the terms containing $q_{\parallel}$ and $q_{\perp}$ are small compared to other terms in 4.9 and 4.10, and can be neglected, do genuine hydromagnetic equations of practical value emerge from the theory. If this is the case (i.e. $q_{\parallel}$, $q_{\perp}$ terms are small) then 4.9 and 4.10 reduce to a pair of equations of state agreeing with those found by the Princeton group (2) in extending the energy principle to cope with non-scalar pressure. These equations can be written in the following form,

4.11 $$\frac{d}{dt}\left(\frac{p_{\perp}}{\rho B}\right) = 0$$

4.12 $$\frac{d}{dt}\left(\frac{p_{\parallel}B^{2}}{\rho^{3}}\right) = 0$$

and complete the set 4.4 - 4.7.

These are the 'double adiabatic' equations of the Chew, Goldberger, Low (C-G-L) approximation which replace the single adiabatic equation of the magnetohydrodynamic (M-H) approximation,

$$2.3 \quad \frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

It was also shown that the integral

$$4.13 \quad \int \left\{ \tfrac{1}{2}\rho\, \underset{\sim}{u}_0^2 + p_\perp + \tfrac{1}{2} p_\parallel + \frac{B^2}{8\pi} + \frac{(\underset{\sim}{u}_0 \times \underset{\sim}{B})^2}{8\pi c^2} \right\} d\tau = U$$

is conserved by these equations and can be taken to be the energy of the system, thus forming a basis for the work of the Princeton group. The equations of state found by them (2)

$$4.14 \quad \frac{\delta p_\parallel}{p_\parallel} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} - 2\underset{\sim}{e}\cdot(\underset{\sim}{e}\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}$$

$$4.15 \quad \frac{\delta p_\perp}{p_\perp} = -2\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} + \underset{\sim}{e}\cdot(\underset{\sim}{e}\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}$$

were used to calculate the second order energy variation in terms of $\underset{\sim}{\xi}$ in the same way as for scalar pressure. The result obtained was:-

$$4.16 \quad \delta W_{DA} = \delta W_v + \delta W_s' + \delta W_p'$$

where

4.17 $$\delta W_s' = \frac{1}{2}\int d\sigma \left\{ (\underset{\sim}{n}\cdot\underset{\sim}{\xi})^2 \underset{\sim}{n}\cdot \left[\!\left[\underset{\sim}{\nabla}\left(p_\perp + \frac{B^2}{8\pi}\right)\right]\!\right] + \underset{\sim}{e}\cdot\underset{\sim}{\xi}\,(p_\parallel - p_\perp)\,\underset{\sim}{n}\cdot\left[\underset{\sim}{e}(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}) - (\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})\underset{\sim}{e}\right]\right\}$$

4.18 $$\begin{aligned}\delta W_p' = \frac{1}{2}\int d\tau \Big\{ & \frac{Q^2}{4\pi} - \underset{\sim}{J}\cdot\underset{\sim}{Q}\times\underset{\sim}{\xi} + \frac{5}{3}p_\perp(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})^2 + (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}p_\perp) \\ & + \frac{1}{3}p_\perp\left[\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} - 3q\right]^2 + q\,\underset{\sim}{\nabla}\cdot\left[\underset{\sim}{\xi}\,(p_\parallel - p_\perp)\right] \\ & + (p_\perp - p_\parallel)\Big[\underset{\sim}{e}\cdot\underset{\sim}{\nabla}\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\underset{\sim}{\xi}\cdot\underset{\sim}{e} - \underset{\sim}{\xi}\cdot(\underset{\sim}{\nabla}\underset{\sim}{e})\cdot(\underset{\sim}{\nabla}\underset{\sim}{\xi})\cdot\underset{\sim}{e} - 4q^2 + \underset{\sim}{e}\cdot(\underset{\sim}{\nabla}\underset{\sim}{\xi})\cdot(\underset{\sim}{e}\cdot\underset{\sim}{\nabla}\underset{\sim}{\xi}) \\ & - \underset{\sim}{\xi}\cdot(\underset{\sim}{\nabla}\underset{\sim}{e})\cdot(\underset{\sim}{e}\cdot\underset{\sim}{\nabla}\underset{\sim}{\xi})\Big]\Big\}\end{aligned}$$

where $q = \underset{\sim}{e}\cdot(\underset{\sim}{\nabla}\underset{\sim}{\xi})\cdot\underset{\sim}{e}$

It is possible that in some cases collisions might be sufficiently frequent to produce an isotropic pressure at equilibrium, but not frequent enough to maintain the isotropic pressure during an oscillation or instability time. Under these conditions the pressure will be determined during the motion by 4.14 and 4.15 with $p_\parallel = p_\perp = p$. In such a case the integrals in 4.16 reduce as follows

$$\delta W_s' = \delta W_s$$

4.19 $$\delta W_p' = \delta W_p + \frac{1}{2}\int d\tau \left\{ \frac{p}{3}(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi} - 3q)^2 \right\}$$

Hence if 2.22 is denoted by δW_{MH}

4.20 $$\delta W_{DA} \geq \delta W_{MH} \qquad \text{for each } \underset{\sim}{\xi}$$

where $\gamma = 5/3$ is assumed.

This is not surprising when it is observed that although 4.11 and 4.12 imply 2.3 the converse is not the case, i.e. there are additional constraints.

Also starting from the collisionless Boltzmann equation Watson (15) investigated static equilibria assuming small Larmor radius (ζ) and small $\beta = \frac{8\pi p}{B^2}$. By solving the Boltzmann equation using individual particle orbits, Watson showed that, to first order in ζ, the pressure tensor is indeed diagonal in the (ψ, θ, χ) system of co-ordinates (see page 20). It was further shown that if $\overleftrightarrow{Q}$ is the pressure-transport tensor, then to zero order in ζ

4.21 $$\overleftrightarrow{Q} = 0$$

Finally it was shown that

4.22 $$\frac{\partial p_{\parallel}}{\partial \chi} + \frac{p_{\perp} - p_{\parallel}}{B} \frac{\partial B}{\partial \chi} = 0$$

This last equation is just the χ component of the equation of motion for a static equilibrium assuming $\overleftrightarrow{P}$ diagonal.

Brueckner and Watson (16) using the same assumptions investigated the non-static case.

It was found that the pressure tensor was still diagonal, but that the heat flow (Pressure transport) tensor was not now zero. The C-G-L equations were derived in the form 4.9, 4.10 and it was noted that if there is no heat flow, and if the field lines are nearly straight, these equations can be written in the form

4.23 $$\frac{dp_{\parallel}}{dt} = -3p_{\parallel}\frac{\partial v_1}{\partial x_1} - p_{\parallel}\left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)$$

4.24 $$\frac{dp_{\perp}}{dt} = -2p_{\perp}\left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) - p_{\perp}\frac{\partial v_1}{\partial x_1}$$

where v_1, v_2, v_3 are the velocities, and x_1, x_2, x_3 the 'lengths' in the χ, ψ, θ directions respectively. These equations show clearly the decoupling effect of the collisionless theory. The form of these equations results from the fact that heat exchange is not possible between longitudinal and transverse degrees of freedom, i.e. parallel to and perpendicular to $\underline{B}$. The longitudinal compression has $\gamma = 3$ since it is one dimensional. Transverse compression has $\gamma = 2$ since it is two dimensional. The cross terms represent the effect of density changes only on the pressure. This is clearly shown if the two compressions are considered separately. Then for transverse compression:-

4.25 $$\frac{d}{dt}\left(\frac{p_\perp}{\rho^2}\right) = 0 \quad ; \quad \frac{d}{dt}\left(\frac{p_\parallel}{\rho}\right) = 0$$

while for longitudinal compression

4.26 $$\frac{d}{dt}\left(\frac{p_\parallel}{\rho^3}\right) = 0 \quad ; \quad \frac{d}{dt}\left(\frac{p_\perp}{\rho}\right) = 0$$

where in each case the first equation expresses the adiabatic behaviour of one component while the second equation expresses the constancy of the energy associated with the other component.

To eliminate $\underset{\sim}{E}'$ Brueckner and Watson introduced a variable $\underset{\sim}{\xi}$ defined by

4.27 $$\frac{\partial \underset{\sim}{\xi}}{\partial t} = c \, \frac{\underset{\sim}{E}' \times \underset{\sim}{B}_0}{B_0^2}$$

Where for any quantity Z

$Z = Z_0 + Z'$ where Z_0 is the static value of Z .

Then starting from the Maxwell equation

$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{J} + \frac{1}{c} \frac{\partial E}{\partial t}$,

which reduces to

4.28 $$\frac{K}{c} \frac{\partial \underset{\sim}{E}'}{\partial t} = \underset{\sim}{\nabla} \times \underset{\sim}{B}' - \frac{4\pi}{c} \underset{\sim}{J}'$$

when the static part is removed, the following integral was derived

$$\int \rho_0 \, \underset{\sim}{\xi} \cdot \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} \, d\tau = - \int \left\{ \frac{Q_1^2}{4\pi} - \underset{\sim}{J} \cdot \underset{\sim}{\xi} \times \underset{\sim}{Q}_1 - (\nabla \underset{\sim}{\xi})^2 : P_T' \right\} d\tau \tag{4.29}$$

define tensor

where $\underset{\sim}{Q}_1 = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)$

There is here a close similarity to the work of Bernstein et al (2), and 4.29 can be used to determine stability. Nevertheless a variational principle cannot be derived from these results for the following reason.

In the ideal hydromagnetic case it was possible to show that the right hand side of 4.28 was the negative of twice the potential energy, whereas in the present case this is not the case, and Brueckner and Watson were unable to replace 4.28 by an energy equation. As was observed above 4.29 can be used to test stability, for if the motion across the field lines is slow compared to thermal velocities, contours of constant density will remain parallel to magnetic field lines. This means that $\underset{\sim}{\xi} \cdot \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2}$ will have the same sign everywhere along a field line. Stability or instability will then occur according as $\underset{\sim}{\xi} \cdot \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} \lessgtr 0$. For any assumed $\underset{\sim}{\xi}$, the frequency Ω of the motion may be estimated from 4.29 by putting $\frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} = \Omega^2 \underset{\sim}{\xi}$, evaluating the integrals, and solving for Ω . Brueckner and Watson applied this technique to the Kruskal - Schwarzschild problem of gravitional instability and obtained the same result as Kruskal and Schwarzschild for the instability rate.

The bulk of the recent work on plasma stability has used the Boltzmann equation for its starting point, and so it was not surprising that the next important step should come from this direction. This step was in fact the publication of an energy principle derived from the collisionless Boltzmann equation with small Larmor radius. Kruskal and Oberman (17) and Rosenbluth and Rostoker (18) approaching the problem in different ways produced energy principles which, in the isotropic pressure case, are identical. Kruskal and Oberman start from the energy integral

$$4.30 \quad E_n = \int d\tau \frac{1}{8\pi} (\underset{\sim}{B}^2 + \underset{\sim}{E}^2) + \sum \iiint \frac{B}{v_\parallel} d\mu \, dw \, d\tau \left[mf \left(\frac{v_\parallel^2}{2} + \mu B + \frac{u_E^2}{2} \right) \right]$$

where f is the Boltzmann distribution function in $\underset{\sim}{x}$, $\underset{\sim}{v}$ space of a particular species of particles and the summation is over all species.

$$\underset{\sim}{B} = B\underset{\sim}{n} \quad ; \quad v_{\parallel} = \underset{\sim}{v}\cdot\underset{\sim}{n} \quad ; \quad \underset{\sim}{\alpha} = \frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B^2}$$

4.31 $$\underset{\sim}{v} = \underset{\sim}{\alpha} + \underset{\sim}{v_{\perp}} + v_{\parallel}\underset{\sim}{n} \quad ; \quad \mu = \frac{v_{\perp}^2}{2B}$$

$$\omega = \frac{v_{\parallel}^2}{2} + \mu B$$

The quantity $B/v_{\parallel}\, d\mu\, d\omega$ represents the volume element in velocity space. It is assumed that the boundaries present are such as to present no complications e.g. rigid and perfectly conducting walls with $\underset{\sim}{B}$ tangential. The properties depending on small Larmor radius, are as follows

(a) the magnetic moment μ of a particle is constant along the particle motion.

(b) f is rotationally symmetric in velocity space about a line parallel to B and passing through the point $\underset{\sim}{\alpha}$

(c) $\underset{\sim}{\alpha}$ is the common perpendicular drift velocity of all particles.

It is this last fact which permits the introduction into the formalism of a displacement vector $\underset{\sim}{\xi}\,(\underset{\sim}{x}, t)$ as in (16).

In the C-G-L approximation it was assumed that one particle species was of much lower mass than the other in order to satisfy the condition that $\underset{\sim}{\xi} \cdot \underset{\sim}{n}$ vanish. Kruskal and Oberman do not make this assumption, and find that $\underset{\sim}{\xi} \cdot \underset{\sim}{n}$ is in fact zero to lowest order in m/e, and consequently as in (14) that $\overleftrightarrow{P}$ is diagonal to this order in m/e in the (ψ, θ, χ) co-ordinate system.

The first order change in energy (4.30) was written down in terms of the fields $\underset{\sim}{B}$ and f, and the first order variations of these fields $\underset{\sim}{B}^*$ and f* where

$$\underset{\sim}{B}^* = (\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\underset{\sim}{\xi} - \underset{\sim}{B}(\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})$$

This first order energy change was then shown to be zero when the condition was made that all constants of the motion should have their equilibrium values. The second order variation of the energy was then written down in terms of f*, f** and the displacement $\underset{\sim}{\xi}$. Using the same condition that the constants of the motion should have their equilibrium values, f** the second order variation of f was eliminated, and the remaining quadratic form in $\underset{\sim}{\xi}$ and f* minimised, subject to this condition, with respect to f*. Thus a necessary and sufficient condition for stability is that the quadratic in $\underset{\sim}{\xi}$ obtained by these methods be positive. The resultant expression for δW may be expressed in terms of 4.18 (δW_{DA} within the plasma)

4.32 $$\delta W = \delta W_{DA} + I$$

where it can be shown by means of a Schwarz inequality that $I \leq 0$ and therefore that

4.33 $$\delta W \leq \delta W_{DA}$$

An important inequality can be obtained in the other direction when the equilibrium distribution functions (g) are isotropic. In this case $p_\perp = p_\parallel = p$ and

4.34 $$\delta W = \frac{1}{2}\int d\tau \left\{ \frac{\underset{\sim}{Q}^2}{4\pi} + \underset{\sim}{J}\cdot\underset{\sim}{\xi}\times\underset{\sim}{Q} + (\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi})(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})p \right\} + I_1$$

where

4.35 $$I_1 = \frac{15}{4}\int_0^{\frac{1}{B}} p\,dy \int d\tau\, B(1-yB)^{-\frac{1}{2}} \left\{ \int_T B\,d\tau\,(1-yB)^{-\frac{1}{2}} \left[\left[(1-\tfrac{3}{2}yB)\underset{\sim}{n}\underset{\sim}{n}:\nabla\underset{\sim}{\xi} + \tfrac{1}{2}yB\,\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}\right] \Big/ \int_T B\,d\tau\,(1-yB)^{-\frac{1}{2}} \right\}^2$$

with $y = \dfrac{2\mu B}{mv^2}$ and $\int_T d\tau = d\phi \int_T \dfrac{dl}{B(l)}$ is an integration over the volume of a flux tube T. This result was also obtained by Rosenbluth and Rostoker. Since the integrand in I_1 is positive a Schwarz inequality can be applied, and when this is done the $\mathcal{Y}$ integration can be carried out. This result is

4.36 $$\delta W \geq \delta W_{MH} = \frac{1}{2}\int d\tau\left\{\frac{Q^2}{4\pi} + J.\xi\times Q + (\nabla.\xi)(\xi.\nabla)p + \frac{5}{3}p\left(\overline{\nabla.\xi}\right)^2\right\}$$

where

$$\left(\overline{\nabla.\xi}\right) = \int_T d\tau(\nabla.\xi)\Big/\int_T d\tau = \int\frac{(\nabla.\xi)\,dl}{B(l)}\Big/\int\frac{dl}{B(l)}$$

Thus if the pressure is isotropic the following inequalities hold for each ξ

4.37 $$\delta W_{MH}(\xi) \leq \delta W(\xi) \leq \delta W_{DA}(\xi)$$

and if ξ_1, ξ_2, ξ_3 minimise δW_{MH}, δW and δW_{DA} respectively then

$$\delta W_{MH}(\xi_1) \leq \delta W_{MH}(\xi_2) \leq \delta W(\xi_2) \leq \delta W(\xi_3) \leq \delta W_{DA}(\xi_3)$$

so that stability in the M-H theory implies stability in the more refined theories. Finally Kruskal and Oberman considered the problem when collisions are no longer negligible and showed that

$$\delta W_{Coll} \geq \delta W_{MH}$$

Rosenbluth and Rostoker approached the problem by solving the collisionless linearized Boltzmann equation for the perturbed distribution function assuming an exponential time dependence

$$\delta f = f_1(\underset{\sim}{x}, \underset{\sim}{v})\, e^{pt}$$

In solving for δf it was assumed that the equilibrium distribution function $f(\underset{\sim}{x}, v_\perp, v_\parallel)$ was isotropic i.e. $f(\underset{\sim}{x}, v^2)$. A two component displacement vector $\delta\underset{\sim}{\xi} = \underset{\sim}{\xi}\, e^{pt}$ was introduced, defined as follows

4.38 $$\delta \underset{\sim}{E} = \frac{p}{c}\, \underset{\sim}{B} \times \delta\underset{\sim}{\xi} + \delta E_\parallel$$

the component of $\underset{\sim}{\xi}$ parallel to $\underset{\sim}{B}$ being set equal to zero. This is equivalent to the $\underset{\sim}{\xi}$ variable introduced by Brueckner and Watson (4.27). $\delta\underset{\sim}{B}$, $\delta\underset{\sim}{J}$ could then be expressed in terms of $\underset{\sim}{\xi}$ through $\delta\underset{\sim}{E}$ using the Maxwell equations. $\delta\underset{\sim}{E}$ satisfies the Maxwell equation.

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi}{c}\underset{\sim}{J} + \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t}$$

and starting from this equation it was shown that, for the marginal stability case, $p \longrightarrow o$

4.39 $$\underset{\sim}{B} \times \underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times (\underset{\sim}{B} \times \underset{\sim}{\xi}) - (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{\nabla} \times (\underset{\sim}{B} \times \underset{\sim}{\xi}) = 4\pi \underset{\sim}{\nabla} . \delta \overleftrightarrow{P}$$

where $\delta \overleftrightarrow{P}$ is the fluctuation in the tensor pressure. Now Rosenbluth and Rostoker were able to show that the pair of equations 4.39 (one for each $\underset{\sim}{\xi}$ component) can be derived from a variational principle, the appropriate functional being the energy change which results from the perturbation

4.40 $$W = \int \left\{ \frac{\delta \underset{\sim}{E} . \delta \underset{\sim}{J}}{2p} + \frac{|\delta B|^2}{8\pi} \right\} d\tau$$

When W is stated as a function of $\underset{\sim}{\xi}$ then the condition that W be a minimum with respect to arbitrary variations of $\underset{\sim}{\xi}$ turns out to be just that 4.39, the zero frequency equation is satisfied. Thus $W_{min} > 0$ is a necessary condition for stability, and it was possible to show that this was also a sufficient condition.

4.40 may be written as

$$W = W_o + W_1$$

where

4.41 $$W_o = \frac{1}{2} \int d\tau \left\{ \frac{Q^2}{4\pi} - \underset{\sim}{J} . \underset{\sim}{Q} \times \underset{\sim}{\xi} + (\underset{\sim}{\nabla} . \underset{\sim}{\xi})(\underset{\sim}{\xi} . \underset{\sim}{\nabla}) P \right\}$$

4.42 $$W_1 = \frac{1}{2}\int d\tau \left\{ (\delta P_{\parallel} - \delta P_{\perp}) \underset{\sim}{n} \cdot (\underset{\sim}{n} \cdot \underset{\sim}{\nabla}) \underset{\sim}{\xi} + \delta P_{\perp} \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} \right\}$$

Now $W_{MH} = W_0 + W_2$

where W_0 is formally the same as 4.41 and W_2 is

4.43 $$W_2 = \frac{\gamma}{2}\int P (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 d\tau$$

In this case $\underset{\sim}{\xi}$ is a three component variable, but W_0 is stationary with respect to variations of $\xi_{\parallel}$ and W_2 is minimised by choosing

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} = \int (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}_{\perp}) \frac{dl}{B} \Big/ \int \frac{dl}{B}$$

4.44 $$= (\overline{\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}})$$

as used by Kruskal and Oberman.

Using this value in 4.43 Rosenbluth and Rostoker showed that

4.45 $$W_2 \geqslant \frac{3\gamma}{5} W_1$$

and therefore that for a $\gamma = 5/3$ fluid

$$W \geqslant W_{MH}$$

in agreement with Kruskal and Oberman.

Equations 4.41 and 4.42 for W(when P is isotropic at equilibrium) were obtained identically by

Kruskal and Oberman. An upper bound was also found for W, namely

$$W \leqslant W_{DA}$$

where W_{DA} is the C-G-L energy change when $\xi_{\parallel}$ is taken to be zero. This inequality is less general than 4.33, since the equilibrium pressure is assumed isotropic.

In the final section of this paper the relationships among the M-H, C-G-L and R-R theories were considered. This was done by calculating W_1, W_2, W_u (the C-G-L energy change - W_o) in terms of the change in particle energy T when a field line is displaced in the plasma. The t$\int v_{\parallel}$invariants of the motion are μ(magnetic moment) and $\int v_{\parallel}\, dl$ (action integral). Then in terms of T and δT the integral W_1 is

$$W_1 = \frac{5}{4} \sum \frac{(\delta T)^2}{T}$$

where the summation is over all particles.

The integral W_2 of the M-H theory is

$$W_2 = \frac{5}{4} \sum \frac{\langle \delta T \rangle^2}{T}$$

where the brackets indicate an average with respect to μ over all particles of energy T on a line of force.

Similarly W_u can be written as

$$W_u = \frac{5}{4} \sum \frac{(\delta T_u)^2}{T}$$

where δT_u is the energy change predicted if the action integral $\int v_{\parallel}\, dl$ is replaced by $v_{\parallel}\, dl$ as the invariant. Thus the three theories correspond to (a) an average along each particle's orbit $\left\{ \int v_{\parallel}\, dl = \text{Constant} \right\}$, followed by a summation over all particles in the R-R case. (b) a double average in the M-H case (c) a double sum in the C-G-L case. In view of this the inequalities obtained by these authors are not surprising.

If the plasma equilibrium has anisotropic pressure the C-G-L equations predict the occurrence of new types of instabilities. Thus it has been shown (Chandrasekhar, Kaufman, and Watson (19)) that plane waves in an infinite homogeneous medium with a uniform magnetic field become unstable if either the parallel or perpendicular component of pressure becomes too large. The criteria for instability are

$$p_{\parallel} > p_{\perp} + B^2/4\pi$$

$$p_{\perp}^2 > 6 p_{\parallel} \left(p_{\perp} + B^2/8\pi \right)$$

The former corresponds to propagation parallel to the field lines. These are the 'firehose' instabilities and arise from the centrifugal accelerations experienced by particles streaming along curved field lines. The latter criterion corresponds to propagation perpendicular to the field lines. This is the 'mirror' instability and arises from the trapping of particles in regions of weak field. If the field is locally weak, particles are retarded by the acceleration $\mu \frac{\partial B}{\partial x}$, the local pressure increases and the magnetic field is still further weakened.

CHAPTER 11

Interchange Instabilities in a Plasma

§5. Derivation of Field Properties.

The problem to be considered is that of the stability of the static equilibrium of a plasma in an axially symmetric magnetic field, which in the most general case will be $\underset{\sim}{B} = \left(B_r(r,z),\ B_\theta(r,z),\ B_z(r,z)\right)$ in cylindrical co-ordinates r, θ, z . The plasma is assumed to be an ideal non-viseous infinitely conducting fluid. In unrationalised Gaussian units the equations satisfied by the equilibrium configuration are as follows:-

5.1 $$\underset{\sim}{\nabla} p = \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c}$$

5.2 $$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{J}$$

5.3 $$\underset{\sim}{\nabla} . \underset{\sim}{B} = 0$$

5.4 $$\underset{\sim}{E} = 0 \ , \quad \underset{\sim}{v} = 0$$

where $\underline{J}$,$\underline{E}$,$\underline{B}$,$\underline{v}$ are the current density, electric and magnetic fields, and fluid velocity respectively.

From 5.1 and 5.2 it follows that

5.5 $$\underset{\sim}{\nabla} p = \frac{1}{4\pi}(\underset{\sim}{\nabla}\times\underset{\sim}{B})\times \underset{\sim}{B}$$

Using 5.3 to express $\underset{\sim}{B}$ as curl $\underset{\sim}{A}$

$$B_r = \frac{1}{r}\left\{\frac{\partial A_z}{\partial\theta} - \frac{\partial}{\partial z}(A_\theta r)\right\} = -\frac{1}{r}\frac{\partial\psi}{\partial z}$$

5.6 $$B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$B_z = \frac{1}{r}\left\{\frac{\partial}{\partial r}(A_\theta r) - \frac{\partial A_r}{\partial\theta}\right\} = \frac{1}{r}\frac{\partial\psi}{\partial r}$$

where ψ is defined to be $A_\theta r$, and $\frac{\partial A_z}{\partial\theta}$, $\frac{\partial A_r}{\partial\theta}$ vanish since B_θ is not a function of θ , i.e. there is axial symmetry. This definition of B automatically satisfies 5.3.

Taking the scalar product of B with 5.1 gives

$$(\underset{\sim}{B}.\underset{\sim}{\nabla})p = (\underset{\sim}{J}\times\underset{\sim}{B})\cdot\frac{\underset{\sim}{B}}{c} \equiv 0$$

i.e. $B_r\frac{\partial p}{\partial r} + B_z\frac{\partial p}{\partial z} \equiv 0$ since $\frac{\partial p}{\partial\theta} \equiv 0$

5.7 $$\therefore \frac{\partial p}{\partial z}\cdot\frac{\partial\psi}{\partial r} - \frac{\partial p}{\partial r}\cdot\frac{\partial\psi}{\partial z} \equiv \frac{\partial(p,\psi)}{\partial(r,z)} \equiv 0$$

Hence $p = p(\psi)$ is a function of ψ alone, i.e.

surfaces of constant ψ are also surfaces of constant p.

If $B_\theta = 0$, $A_r = A_z = 0$ is a suitable choice in 5.6 and

5.8 $$\underset{\sim}{B} = \left(-\frac{1}{r}\frac{\partial\psi}{\partial z},\ 0,\ \frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

If $B_\theta \neq 0$ we must consider 5.5, noting that axial symmetry requires $\frac{\partial p}{\partial\theta} \equiv 0$

The θ component of 5.5 gives

$$\left[(\underset{\sim}{\nabla}\times\underset{\sim}{B})\times B\right]_\theta \equiv 0$$

i.e.

$$\frac{B_z}{r}\frac{\partial}{\partial z}(r B_\theta) + \frac{B_r}{r}\frac{\partial}{\partial r}(r B_\theta) \equiv 0$$

5.9 $$\therefore \frac{\partial\psi}{\partial r}\cdot\frac{\partial(r B_\theta)}{\partial z} - \frac{\partial\psi}{\partial z}\cdot\frac{\partial(r B_\theta)}{\partial r} \equiv \frac{\partial(\psi,\ r B_\theta)}{\partial(r, z)} \equiv 0$$

which has the solution

5.10 $$B_\theta = \frac{f(\psi)}{r}$$

The r and z components of 5.5. yield the same result

$$4\pi\left(\frac{\partial p}{\partial r},\ 0,\ \frac{\partial p}{\partial z}\right) = \left((\underset{\sim}{\nabla}\times\underset{\sim}{B})_\theta B_z - B_\theta(\underset{\sim}{\nabla}\times\underset{\sim}{B})_z,\ 0,\ B_\theta(\underset{\sim}{\nabla}\times\underset{\sim}{B})_r - B_r(\underset{\sim}{\nabla}\times\underset{\sim}{B})_\theta\right)$$

$$\therefore 4\pi\frac{dp}{d\psi}\left(\frac{\partial\psi}{\partial r},\ 0,\ \frac{\partial\psi}{\partial z}\right) = \left\{\frac{(\underset{\sim}{\nabla}\times\underset{\sim}{B})_\theta}{r} - \frac{B_\theta}{r}\frac{\partial f}{\partial\psi}\right\}\left(\frac{\partial\psi}{\partial r},\ 0,\ \frac{\partial\psi}{\partial z}\right)$$

5.11 $$4\pi p' = \frac{(\nabla \times \mathbf{B})_\theta}{r} - \frac{ff'}{r^2}$$

For $B_\theta \equiv 0$ this reduces to

5.12 $$4\pi p' = \frac{(\nabla \times \mathbf{B})_\theta}{r}$$

Hence in general all the requirements of the field equations will be satisfied if the field is specified in terms of two arbitrary functions $\psi = \psi(r, z)$, and $f = f(\psi)$, defining B_r, B_θ, B_z uniquely.

The field lines are defined in r, θ, z space by the equations.

5.13 $$\frac{dr}{B_r} = \frac{dz}{B_z} = \frac{r\,d\theta}{B_\theta}$$

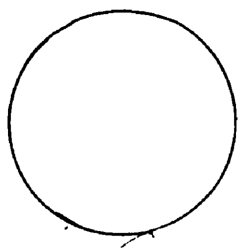
Hence

5.14 $$\frac{dr}{dz} = \left\{ -\frac{\partial\psi}{\partial z} \Big/ \frac{\partial\psi}{\partial r} \right\}$$

i.e. the field lines lie in the surfaces of constant ψ, and therefore of constant pressure. If $B_\theta \equiv 0$ the lines are planar, i.e. are two dimensional curves in r, z space.

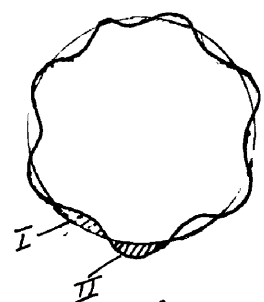
§6. Statement of the Method.

Rosenbluth and Longmire (8) have considered the stability of a plasma in the magnetic field $\underset{\sim}{B} = \left(B_r(r,z),\ 0,\ B_z(r,z)\right)$ against the fluted type of interchange. The method used in (8) will be stated and discussed in this section before it is extended to more general cases in the next and subsequent sections.



6a

Cross section of the unperturbed plasma



6b

Cross section of plasma with fluted perturbation.

The diagrams above show the nature of the perturbation considered, and it is clear from 6b that this perturbation is equivalent to the exchange of the material in regions I and II. If also perfect conductivity is assumed the field is 'frozen in' and exchange of material implies also exchange of magnetic flux. The equilibrium potential energy of the plasma is

$$6.1 \quad U = \int d\tau \left\{ \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right\}$$

and if this quantity should be decreased by an exchange of flux tubes the excess energy will go into kinetic energy of the perturbation causing instability. Thus Rosenbluth and Longmire examined the sign of ΔU , the change in the potential energy of the plasma, when two infinitesimal neighbouring flux tubes were interchanged. The problem was considered in the limit $\beta = \frac{8\pi p}{B^2} \ll 1$. In this limit the magnetic field is nearly identical with the vacuum magnetic field so that it was assumed that any distortion of the field would increase its energy. Hence only interchanges which left the magnetic energy unchanged were considered.

The magnetic energy in a flux tube is

6.2 $$E_m = \int \frac{B^2}{8\pi} dV = \int \frac{B^2}{8\pi} A \, d\ell$$

where ℓ is the co-ordinate along the flux tube and A is its cross-sectional area.

6.3 $BA = \phi =$ flux contained in the tube which is constant along the length of the tube.

Thus on combining 6.2 and 6.3

6.4 $$E_m = \frac{\phi^2}{8\pi} \int \frac{d\ell}{A}$$

Now one of the basic assumptions in (8), an assumption which will eventually be removed, was that the material of tube I occupies, after exchange, exactly the volume of tube II, and that the flux of tube I is contained in exactly the cross section of tube II. i.e. that the remainder of the plasma is undistorted by the interchange. Thus the change of magnetic energy on interchanging flux tubes I and II is

$$\Delta E_m = \frac{1}{8\pi}\left[\phi_1^2\int_2 \frac{dl}{A} + \phi_2^2\int_1 \frac{dl}{A} - \phi_1^2\int_1 \frac{dl}{A} - \phi_2^2\int_2 \frac{dl}{A}\right]$$

6.5 $$= \frac{1}{8\pi}\left(\phi_2^2 - \phi_1^2\right)\left\{\int_1 \frac{dl}{A} - \int_2 \frac{dl}{A}\right\}$$

Hence in general $\Delta E_m = 0 \Rightarrow \phi_2 = \phi_1$

The stability of the system was therefore to be determined by the sign of the change in material potential energy due to the interchange

6.6 $$\Delta E_p = \frac{1}{\gamma - 1}\left\{\int p_1' \, dV_1 + \int p_2' \, dV_2 - p_1 V_1 - p_2 V_2\right\}$$

where p_1' is the pressure in flux tube I when it contains the material from tube II, and similarly for p_2'. Here another assumption was made (again to be removed at a later stage of the paper) namely; that p_1' and p_2' are also constant along a field line.

The pressures p_1', p_2' were calculated assuming that the adiabatic equation

2.3 $$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

was satisfied.

Now this equation can be applied either locally or over a whole tube. If applied locally the result is

6.7 $$p_1' = p_2 \frac{(dv_2)^\gamma}{(dv_1)^\gamma} \quad ; \quad p_2' = p_1 \frac{(dv_1)^\gamma}{(dv_2)^\gamma}$$

in which case p_1', p_2' are not constant along the length of a field line.

If applied over a complete flux tube the result is

6.8 $$p_1' = p_2 \frac{\left\{\int dv_2\right\}^\gamma}{\left\{\int dv_1\right\}^\gamma} = p_2 \left(\frac{V_2}{V_1}\right)^\gamma \; ; \; p_2' = p_1 \frac{\left\{\int dv_1\right\}^\gamma}{\left\{\int dv_2\right\}^\gamma} = p_1 \left(\frac{V_1}{V_2}\right)^\gamma$$

which produces constant p_1', p_2'.

We would expect the local adiabatic law to hold if the instability time were short compared to the thermalisation time, and the constant pressure case to arise if the thermalisation time were short compared to the instability time. It will, however, be proved in a later section that the worst possible pressure distribution from the point of view of stability is, in fact, that of constant pressure, and in any case this assumption is consistent with the hydrodynamic assumption of frequent collission.

Using 6.8 and writing

6.9 $$\begin{cases} p_2 = p + \delta p & p_1 = p \\ V_2 = V + \delta V & V_1 = V \end{cases}$$

where δp, δV are infinitesimal, 6.6 becomes

$$\Delta E_p = \frac{1}{\gamma-1}\left\{pV\left(1+\frac{\delta p}{p}\right)\left(1+\frac{\delta V}{V}\right)^{\gamma} + pV\left(1+\frac{\delta V}{V}\right)^{1-\gamma} - pV - pV\left(1+\frac{\delta p}{p}\right)\left(1+\frac{\delta V}{V}\right)\right\}$$

$$= \frac{1}{\gamma-1}\left\{pV\left[1+\frac{\delta p}{p}+\gamma\frac{\delta V}{V}+\gamma\frac{\delta p}{p}\frac{\delta V}{V}+1+(1-\gamma)\frac{\delta V}{V}+\frac{(1-\gamma)(-\gamma)}{2}\left(\frac{\delta V}{V}\right)^2\right]\right.$$

$$\left. - pV\left[1+1+\frac{\delta p}{p}+\frac{\delta V}{V}+\frac{\delta p}{p}\cdot\frac{\delta V}{V}\right]\right\}$$

$$= \frac{pV}{\gamma-1}\left\{(\gamma-1)\frac{\delta p}{p}\frac{\delta V}{V} + \gamma(\gamma-1)\left(\frac{\delta V}{V}\right)^2\right\}$$

6.10 $$\therefore \Delta E_p = \delta V\left\{\delta p + \gamma p\frac{\delta V}{V}\right\}$$

In general p must decrease towards the outer limits of the plasma where it must tend to zero. Hence $\delta p/p \longrightarrow -\infty$ in this region, and $\left(\delta p/p + \gamma\,\delta V/V\right)$ will certainly be less than zero.

The stability condition $\Delta E_p > 0$ then reduces to

6.11 $$\delta\left\{\int A\,dl\right\} < 0$$

i.e. $\Phi\,\delta\int\frac{dl}{B} < 0$ since $\Phi_2 = \Phi_1$

Applying this criterion, with the additional assumption curl $\underset{\sim}{B} = 0$ in the low pressure region considered, to a mirror machine with magnetic field B = $\underset{\sim}{B} = \left(B_r(r,z),\ 0,\ B_z(r,z) \right)$ Rosenbluth and Longmire obtained

$$3.10 \quad \int \frac{d\ell}{R_r B^2} > 0$$

for stability.

It must be noted here that the perturbation used might not be physically realistic, i.e. in terms of the energy principle, it is possible that no $\underset{\sim}{\xi}$ satisfying the hydromagnetic equations, exists which describes the motion visualized in this theory.

§7. Generalisations to Finite Pressure Plasma.

In order to investigate stability within the plasma, as distinct from near the plasma boundaries, it is necessary to remove the condition $p \to 0$.

The consequences of this are as follows:-

a) curl $\underset{\sim}{B}$ is no longer zero but must be obtained from 5.5.

b) both factors of $\Delta E_p = p\,\delta V\left(\frac{\delta p}{p} + \gamma \frac{\delta V}{V}\right)$ must be evaluated.

c) ΔE_m cannot now be ignored since $\underset{\sim}{B}$ no longer approximates to a vacuum field.

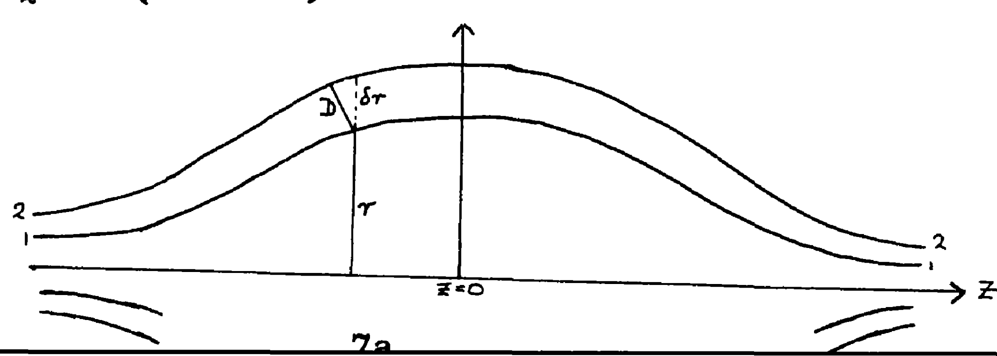
Restrictions which are not relaxed in this section are those assuming p_2' and p_1' constant along a flux tube, and assuming the volumes and cross-sections of the two flux tubes undisturbed by the interchange.

The requirement for stability is now

7.1 $$\Delta E = \delta V\left(\delta p + \gamma p \frac{\delta V}{V}\right) + \frac{1}{8\pi}\left(\phi_2^2 - \phi_1^2\right)\left\{\int \frac{dl_1}{A_1} - \int \frac{dl_2}{A_2}\right\}$$

This expression will be evaluated for a mirror machine with magnetic field

7.2 $$\underset{\sim}{B} = \left(B_r(r,z),\ 0,\ B_z(r,z)\right)$$



7a

Diagram 7a shows the planar field lines of a mirror machine with field given by 7.2. Before evaluating 7.1. we introduce the variables ξ , η and χ defined by

7.3 $$\xi = \psi_2 - \psi_1 = \delta\psi$$

7.4 $$\eta = \frac{\phi_2 - \phi_1}{\phi_1} = \frac{\delta\phi}{\phi}$$

From the diagram the constant flux through the annulus between the surfaces of constant ψ containing the lines 1 and 2 is

7.5 $$\chi = 2\pi r \, BD = 2\pi r \, B_z \, \delta r$$

Then

7.6 $$\delta p = \frac{dp}{d\psi}\delta\psi = p'\xi$$

$$\delta p = \frac{\partial p}{\partial r}\,\delta r$$

$$= \frac{\partial p}{\partial r} \cdot \frac{\chi}{2\pi} \cdot \frac{1}{r B_z}$$

$$= p' \frac{\chi}{2\pi}$$

7.7 Hence $\frac{\chi}{2\pi} = \xi$

Obviously ξ, η, χ are all constant along a field line since they depend on ψ or ϕ only

Using these results we will now evaluate the change in the magnetic energy

$$\Delta E_m = \frac{1}{8\pi}(\phi_2^2 - \phi_1^2)\left\{\int \frac{dl_1}{A_1} - \int \frac{dl_2}{A_2}\right\}$$

$$= -\frac{2\phi\,\delta\phi}{8\pi}\;\delta\left\{\frac{1}{\phi}\int B\,dl\right\}$$

$$= -\frac{\phi\,\delta\phi}{4\pi}\left\{\frac{1}{\phi}\,\delta\int B\,dl - \frac{\delta\phi}{\phi^2}\int B.dl\right\}$$

$$= \frac{\delta\phi}{4\pi}\left\{\eta\int B\,dl - \delta\int B\,dl\right\}$$

$$\delta\int B\,dl = \int_2 B\,dl - \int_1 B\,dl$$

$$= -\int(\underset{\sim}{\nabla}\times\underset{\sim}{B}).\,d\underset{\sim}{S}$$

by connecting the field lines at the ends to make a closed contour, and using Stokes Theorem

$$= -\int(\underset{\sim}{\nabla}\times\underset{\sim}{B})_\theta\, D\,dl$$

where D is the perpendicular distance between the two tubes and is defined by 7.5

D = $\xi / r B$. Using also 5.12.

7.8 $$\delta \int B\, dl = -4\pi p' \xi \int \frac{dl}{B}$$

7.9 $$\therefore \frac{\Delta E_m}{\phi} = \frac{\eta}{4\pi} \left\{ \eta \int B\, dl + 4\pi p' \xi \int \frac{dl}{B} \right\}$$

The change in the material energy is

$$\Delta E_p = \delta V \left(\delta p + \gamma p \frac{\delta V}{V} \right)$$

where $\delta p = p'\xi$, and $V = \phi \int \frac{dl}{B}$

$$\delta V = \delta \left\{ \phi \int \frac{dl}{B} \right\}$$

7.10 $$= \delta\phi \int \frac{dl}{B} + \phi\, \delta \int \frac{dl}{B}$$

$$\delta \int \frac{dl}{B} = \delta \int \frac{\underset{\sim}{B} \cdot \underset{\sim}{dl}}{B^2}$$

$$= -\xi \int \left(\underset{\sim}{\nabla} \times \frac{\underset{\sim}{B}}{B^2} \right)_\theta \frac{dl}{rB}$$

7.11 $$= -\xi \int \frac{(\underset{\sim}{\nabla} \times \underset{\sim}{B})_\theta\, dl}{r B^3} - \xi \int \left\{ B_r \frac{\partial}{\partial z}\left(\frac{1}{B^2}\right) - B_z \frac{\partial}{\partial r}\left(\frac{1}{B^2}\right) \right\} \frac{dl}{rB}$$

The integrand of the second integral in this expression must now be evaluated in terms of the field variables B, r and R the radius of curvature of the field line in r, z space.

$$B_r \frac{\partial}{\partial z}\left(\frac{1}{B^2}\right) - B_z \frac{\partial}{\partial r}\left(\frac{1}{B^2}\right) = -\frac{1}{B^4}\left\{ B_r \frac{\partial}{\partial z} - B_z \frac{\partial}{\partial r} \right\} B^2$$

$$= -\frac{2}{B^4}\left\{ B_r \left(B_r \frac{\partial B_r}{\partial z} + B_z \frac{\partial B_z}{\partial z} \right) - B_z \left(B_r \frac{\partial B_r}{\partial r} + B_z \frac{\partial B_z}{\partial r} \right) \right\}$$

7.12 $$= \frac{2}{B^4}\left\{ B_z^2 \frac{\partial B_z}{\partial r} + B_r B_z \left(\frac{\partial B_r}{\partial r} - \frac{\partial B_z}{\partial z} \right) - B_r^2 \frac{\partial B_r}{\partial z} \right\}$$

Now if R is the radius of curvature of the field line (flux tube) along which integration is carried out

7.13 $$\frac{1}{R} = \frac{d^2 r}{dz^2} \Big/ \left\{ 1 + \left(\frac{dr}{dz}\right)^2 \right\}^{3/2}$$

thus $$\frac{1}{R} = \frac{d}{dz}\left(\frac{B_r}{B_z}\right) \times \frac{B_z^3}{B^3} \qquad \text{From } 5.13$$

$$= \frac{B_z^3}{B^3}\left\{ \frac{\partial}{\partial z}\left(\frac{B_r}{B_z}\right) + \frac{B_r}{B_z}\frac{\partial}{\partial r}\left(\frac{B_r}{B_z}\right) \right\}$$

$$= \frac{B_z^3}{B^3}\left\{ \frac{1}{B_z}\frac{\partial B_r}{\partial z} - \frac{B_r}{B_z^2}\frac{\partial B_z}{\partial z} + \frac{B_r}{B_z^2}\frac{\partial B_r}{\partial r} - \frac{B_r^2}{B_z^3}\frac{\partial B_z}{\partial r} \right\}$$

$$\therefore \frac{1}{R} = \frac{1}{B^3}\left\{ B_z^2 \frac{\partial B_r}{\partial z} + B_r B_z \left(\frac{\partial B_r}{\partial r} - \frac{\partial B_z}{\partial z} \right) - B_r^2 \frac{\partial B_z}{\partial r} \right\}$$

Comparing this with 7.12 it is clear that

7.14 $$\left(B_r \frac{\partial}{\partial z} - B_z \frac{\partial}{\partial r}\right)\frac{1}{B^2} \equiv \frac{2}{RB} - \frac{2(\nabla \times B)_\theta}{B^2}$$

Substituting this result in 7.11 gives

7.15 $$\delta \int \frac{dl}{B} = 4\pi p' \xi \int \frac{dl}{B^3} - 2\xi \int \frac{dl}{R_r B^2}$$

and this in turn gives the following result for δV

7.16 $$\frac{\delta V}{\phi} = \eta \int \frac{dl}{B} + 4\pi p' \xi \int \frac{dl}{B^3} - 2\xi \int \frac{dl}{R_r B^2}$$

Writing $\int \frac{dl}{B} = K$; $\int B\,dl = J$;

$$4\pi p' \int \frac{dl}{B^3} - 2 \int \frac{dl}{R_r B^2} = I$$

so that $\frac{\delta V}{\phi} = \eta K + \xi I$, the expression for ΔE_p can now be evaluated

7.17 $$\frac{\Delta E_p}{\phi} = (\eta K + \xi I)\left(p'\xi + \gamma p \frac{(\eta K + \xi I)}{K}\right)$$

Combining 7.17 and 7.9 the complete energy change may be written as

$$\frac{\eta^2}{4\pi} J + \eta \xi p' K + \eta \xi p' K + \xi^2 p' I + \frac{\gamma p}{K}(\eta K + \xi I)^2 = \frac{\Delta E}{\phi}$$

7.18 $$\frac{\Delta E}{\phi} = \eta^2\left[\frac{J}{4\pi} + \gamma p K\right] + 2\eta\xi\left[p'K + \gamma p I\right] + \xi^2\left[p'K + \gamma p I\right]\frac{I}{K}$$

i.e. the second order energy change is a quadratic form $\eta^2 a + 2\eta\xi\, b + \xi^2 c = \Gamma(\eta,\xi)$ in the variables η, ξ. A necessary condition for stability is, therefore, that $\Gamma(\eta,\xi)$ be positive definite, otherwise there exist values of $\delta\psi$ $(\equiv \xi)$ and $\delta\phi$ $(\equiv \eta\phi)$ which make ΔE negative. The condition for positive definiteness is (i) $a > 0$; and (ii) $ac - b^2 > 0$

(i) is automatically satisfied since the integrals J and K are positive. The condition to be satisfied for stability is therefore (ii)

$$\left(\frac{IJ}{4\pi K} + \gamma p I\right)\left(p'K + \gamma p I\right) - \left(p'K + \gamma p I\right)^2 > 0$$

7.19 *i.e.* $$\left(p'K + \gamma p I\right)\left(\frac{IJ}{4\pi K} - p'K\right) > 0$$

This inequality may be satisfied in two ways as follows:-

(i) $\gamma p I > -p'K$ and $I > 4\pi p' \frac{K^2}{J}$

or

(ii) $\gamma p I < -p'K$ and $I < 4\pi p' \frac{K^2}{J}$

Taking $p' < 0$ these reduce to

(1) $\gamma p I > -p'K$

or

(11) $I < 4\pi p' \frac{K^2}{J}$

Written explicitly as integrals these are

7.20 $$\int \frac{dl}{R_r B^2} < 2\pi p' \int \frac{dl}{B^3} + \frac{p'}{2\gamma p} \int \frac{dl}{B}$$

or

7.21 $$\int \frac{dl}{R_r B^2} > 2\pi p' \left\{ \int \frac{dl}{B^3} - \frac{\left(\int \frac{dl}{B} \right)^2}{\int B\, dl} \right\}$$

Now 7.20 is precisely the inequality 3.16 obtained by Bernstein et al (2), and 7.21 only differs from 3.15 by a Schwarz inequality. 7.21 is in fact a less stringent condition than 3.15, and this is not surprising in view of the restricted nature of the perturbation considered in this theory.

§8. More General Perturbations.

In this section we continue to be concerned with a mirror machine type of magnetic field in which there is no θ component. As in the previous sections the system will be assumed to obey the ideal hydromagnetic equations with scalar pressure but an attempt will be made to consider more general perturbations. It has been assumed up till now that the pressure in the flux tubes I and II is constant after the exchange has taken place. The consequences of relaxing this condition will be investigated, as will be the consequences of allowing the volumes, and cross-sectional areas of flux tubes I and II after the exchange to differ from their values before the exchange.

Consider, first, the change in material energy

6.6 $$\Delta E_p = \frac{1}{\gamma-1}\left\{\int p_1' A_1 dl_1 + \int p_2' A_2 dl_2 - p_1 V_1 - p_2 V_2\right\}$$

Assuming that the pressure obeys a local adiabatic law during the exchange,

$$(p_1')^{\frac{1}{\gamma}} d\tau_1' = p_2^{\frac{1}{\gamma}} d\tau_2$$

$$(p_2')^{\frac{1}{\gamma}} d\tau_2' = p_1^{1/\gamma} d\tau_1$$

and integrating these equations along the flux tubes we obtain the following constraint equations

8.1 $$\int (p_1')^{\frac{1}{\gamma}} A_1 \, dl_1 = p_2^{1/\gamma} V_2$$

8.2 $$\int (p_2')^{\frac{1}{\gamma}} A_2 \, dl_2 = p_1^{\frac{1}{\gamma}} V_1$$

Now suppose

8.3 $$p_1' = p_2 \left(\frac{V_2}{V_1}\right)^{\gamma} + p^{*} + p^{**}$$

8.4 $$p_2' = p_1 \left(\frac{V_1}{V_2}\right)^{\gamma} + p^{\dagger} + p^{\dagger\dagger}$$

where p^{*}, $p^{\dagger}$ are the first order infinitesimal fluctuations of the pressure, and p^{**}, $p^{\dagger\dagger}$ are the second order fluctuations, and all four are functions of l ; e.g. $p^{*} = p^{*}(l)$. On substituting 8.3, 8.4 into 8.1 and 8.2 equations are obtained relating p^{**} to p^{*} and $p^{\dagger\dagger}$ to $p^{\dagger}$. Using these relations we will eliminate p^{**} and $p^{\dagger\dagger}$, making it possible to minimise algebraically for p^{*}, $p^{\dagger}$.

8.1 and 8.2 reduce as follows

$$\int (p_1')^{\frac{1}{\gamma}} A_1 \, dl_1 = \int p_2^{\frac{1}{\gamma}} \frac{V_2}{V_1} \left\{ 1 + \frac{p^{*}}{p_2}\left(\frac{V_1}{V_2}\right)^{\gamma} + \frac{p^{**}}{p_2}\left(\frac{V_1}{V_2}\right)^{\gamma} \right\}^{\frac{1}{\gamma}} A_1 \, dl_1$$

$$= p_2^{\frac{1}{\gamma}} V_2 + \int p_2^{\frac{1}{\gamma}} \frac{V_2}{V_1} \frac{1}{\gamma} \left\{ \left(\frac{V_1}{V_2}\right)^{\gamma} + \frac{(1-\gamma)}{2\gamma}\left(\frac{V_1}{V_2}\right)^{2\gamma}\left(\frac{p^{*}+p^{**}}{p_2}\right) \right\} \left(\frac{p^{*}+p^{**}}{p_2}\right) A_1 \, dl_1$$

But $\int (p_1')^{\frac{1}{\gamma}} A_1 \, dl_1 = p_2^{\frac{1}{\gamma}} V_2 \qquad (8.1)$

Hence

$$\frac{p_2^{\frac{1}{\gamma}}\left(\frac{V_1}{V_2}\right)^{\gamma-1}}{\gamma p_2} \int \left\{ p^* + p^{**} + \left(\frac{V_1}{V_2}\right)^{\gamma} \frac{(1-\gamma)}{2\gamma} \frac{(p^*)^2}{2 p_2} \right\} A_1 dl_1 = 0$$

8.5 i.e. $\int p^* A_1 \, dl_1 = 0$

8.6 $\int p^{**} A_1 \, dl_1 = \left(\frac{V_1}{V_2}\right)^{\gamma} \frac{\gamma-1}{2\gamma p_2} \int (p^*)^2 A_1 \, dl_1$

In exactly similar fashion

8.7 $\int p^{+} A_2 \, dl_2 = 0$

8.8 $\int p^{++} A_2 \, dl_2 = \left(\frac{V_2}{V_1}\right)^{\gamma} \frac{\gamma-1}{2\gamma p_1} \int (p^{+})^2 A_2 \, dl_2$

On substituting 8.3 and 8.4 into 6.6 this equation becomes

$$\Delta E_p = \frac{1}{\gamma-1} \left\{ \int p_2 \left(\frac{V_2}{V_1}\right)^{\gamma} A_1 dl_1 + \int p_1 \left(\frac{V_1}{V_2}\right)^{\gamma} A_2 dl_2 - p_1 V_1 - p_2 V_2 \right\}$$

$$+ \frac{1}{\gamma-1} \left\{ \int (p^* + p^{**}) A_1 dl_1 + \int (p^{+} + p^{++}) A_2 dl_2 \right\}$$

$$\therefore \Delta E_p = \frac{\gamma-1}{\gamma-1} \delta V \left(\delta p + \gamma p \frac{\delta V}{V}\right) + \frac{\gamma-1}{(\gamma-1) 2\gamma p} \int \left\{ (p^*)^2 + (p^{+})^2 \right\} A \, dl$$

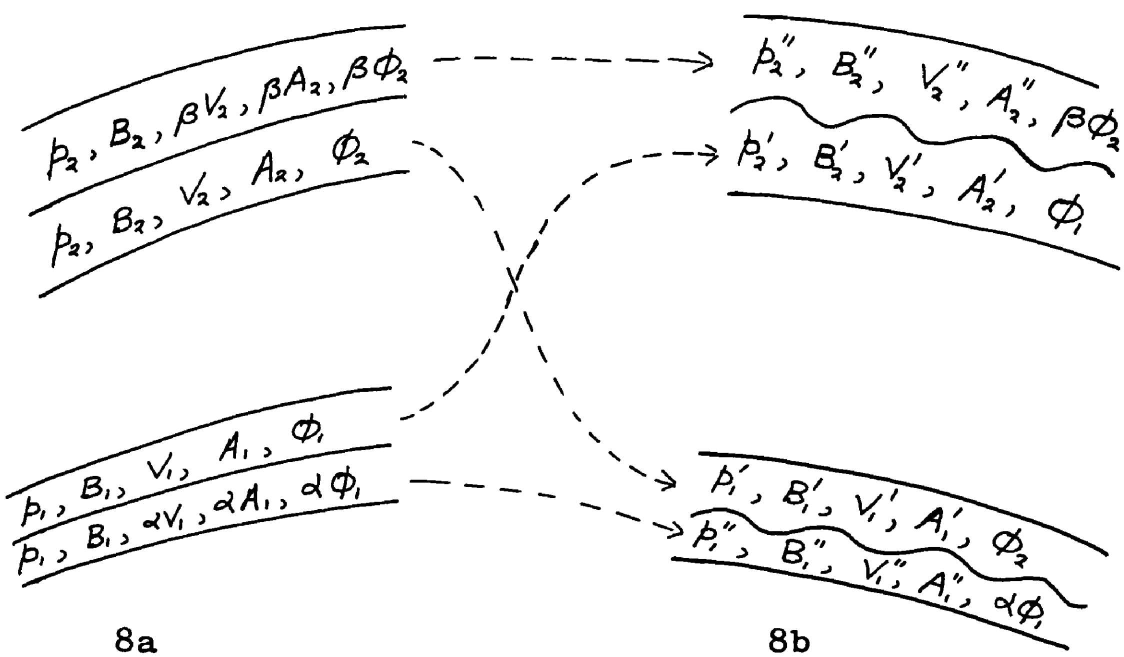
after using 6.10 to obtain the first term, 8.5 and 8.7 to eliminate the first order part of the second term and noting that $\left(V_2/V_1\right)^{\gamma}$ and $\left(V_1/V_2\right)^{\gamma} \cdot p_1/p_2$ are 1 to zero order. Hence

$$8.9. \quad \Delta E_p = \left(\Delta E_p\right)_1 + \frac{1}{\gamma p}\int (\Delta p)^2 A\, d\ell$$

where $(\Delta p)^2 = \dfrac{(p^*)^2 + (p^\dagger)^2}{2}$, and $(\Delta E_p)_1$ is given by 6.10

Thus 8.9 is minimised by choosing $\Delta p = 0$ i.e. ΔE has its minimum value when p_1' and p_2' are constant along their respective flux tubes.

Finally, in this section, a new perturbation will be considered which may be represented diagramatically as shown below.



8a

Unperturbed

8b

Perturbed

Here the change of magnetic and material energy due to this perturbation may be written down as eight energy integrals along the four perturbed and four unperturbed flux tubes. The original flux tubes I and II are now no longer constrained to have the same dimensions after the exchange as before, and to cope with the resulting distortion of field lines and plasma outside these tubes the concept of neighbouring flux tubes III and IV has been introduced. These tubes may be regarded as enveloping the original tubes I and II. The method will consist of minimizing the energy change with respect to the cross-sectional area (on which the magnetic energy depends) and the volume (on which the material energy depends) subject to the condition that the plasma external to the four flux tubes remains unperturbed, and to the constraint equations arising from the purely geometrical relationship between the volume and cross sectional area of the perturbed flux tubes. When this has been done the final expression still depends on β and α , quantities measuring the relative magnitude of the enveloping flux tubes, in the form $\frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1}$ and the expression can then be minimized by maximising this factor, i.e. by choosing $\alpha \rightarrow \infty$, $\beta \rightarrow \infty$ which is the condition we would expect on intuitive grounds since this corresponds to the perturbations external to tubes I and II being absorbed in a relatively infinite medium. There now arises the question as to what orders of magnitude the variables must be. In particular if α and β are to become infinite some of the procedure of changing volume integrals over flux tubes to line integrals along field lines must now be examined more closely. Since the answer to this question is of fundamental importance to the present theory it will be discussed in full in a separate section (§9). In the meantime it will be assumed that the orders of magnitude of the fluxes ϕ_1, ϕ_2, $\alpha\phi_1$, $\beta\phi_2$ are such as to make the following work valid.

Before writing down explicit expressions for ΔE_m and ΔE_p the constraints will be considered. Conservation of flux requires that:-

8.10 $$B_1' A_1' = B_2 A_2 \quad ; \quad B_2' A_2' = B_1 A_1$$

8.11 $$B_1'' A_1'' = \alpha B_1 A_1 \quad ; \quad B_2'' A_2'' = \beta B_2 A_2$$

Conservation of total cross-section requires that:-

8.12 $A_1(1+\alpha) = A_1'' + A_1'$; $A_2(1+\beta) = A_2'' + A_2'$

Conservation of total volume requires that:-

8.13 $V_1(1+\alpha) = V_1'' + V_1'$; $V_2(1+\beta) = V_2'' + V_2'$

Then if

8.14 $A_1' = A_1(1+f)$; $A_2' = A_2(1+g)$

where f and g are infinitesimal functions of ℓ, substitution into 8.12 gives

8.15 $A_1'' = A_1(\alpha - f)$; $A_2'' = A_2(\beta - g)$

and if

8.16 $V_1' = V_1(1+h)$; $V_2' = V_2(1+k)$

where h and k are infinitesimal constants, substitution of 8.16 into 8.13 gives

8.17 $V_1'' = V_1(\alpha - h)$; $V_2'' = V_2(\beta - k)$

The geometrical constraints are

$$V_1' = \int A_1' \, dl_1 \qquad ; \quad V_2' = \int A_2' \, dl_2$$

i.e. $V_1(1+k) = \int A_1(1+f)\,dl_1 \quad ; \quad V_2(1+k) = \int A_2(1+g)\,dl_2$

8.18 i.e. $kV_1 = \int f A_1 \, dl_1 \qquad ; \qquad kV_2 = \int g \, A_2 \, dl_2$

Expanding f,g,h,k in powers of ξ gives

8.19 $f = f_1 + f_2 + f_3 + \ldots\ldots\ldots$

and similarly for g,h,k.

Using 6.4 ΔE_m is

8.20
$$8\pi\Delta E_m = \left\{\phi_1^2\int\frac{dl_2}{A_2'} + \phi_2^2\int\frac{dl_1}{A_1'} - \phi_1^2\int\frac{dl_1}{A_1} - \phi_2^2\int\frac{dl_2}{A_2}\right\}$$
$$+\left\{(\beta\phi_2)^2\int\frac{dl_2}{A_2''} + (\alpha\phi_1)^2\int\frac{dl_1}{A_1''} - (\beta\phi_2)^2\int\frac{dl_2}{\beta A_2} - (\alpha\phi_1)^2\int\frac{dl_1}{\alpha A_1}\right\}$$

i.e.
$$8\pi\Delta E_m = \left\{\phi_1^2\int\frac{dl_2}{A_2}(1+g)^{-1} + \phi_2^2\int\frac{dl_1}{A_1}(1+f)^{-1} - \phi_1^2\int\frac{dl_1}{A_1} - \phi_2^2\int\frac{dl_2}{A_2}\right\}$$
$$+\left\{(\beta\phi_2)^2\int\frac{dl_2}{\beta A_2}\left(1-\frac{g}{\beta}\right)^{-1} + (\alpha\phi_1)^2\int\frac{dl_1}{\alpha A_1}\left(1-\frac{f}{\alpha}\right)^{-1} - (\beta\phi_2)^2\int\frac{dl_2}{\beta A_2} - (\alpha\phi_1)^2\int\frac{dl_1}{\alpha A_1}\right\}$$

where use has been made of 8.14 and 8.15

i.e.

$$8\pi\,\Delta E_m = (\phi_2^2-\phi_1^2)\left\{\int\frac{dl_1}{A_1}-\int\frac{dl_2}{A_2}\right\}+\phi_1^2\int\frac{dl_2}{A_2}\left(-g_1-g_2+g_1^2\right)$$

$$+\phi_2^2\int\frac{dl_1}{A_1}\left(-f_1-f_2+f_1^2\right)+(\beta\phi_2)^2\int\frac{dl_2}{\beta A_2}\left(\frac{g_1}{\beta}+\frac{g_2}{\beta}+\frac{g_1^2}{\beta^2}\right)$$

$$+(\alpha\phi_1)^2\int\frac{dl_1}{\alpha A_1}\left(\frac{f_1}{\alpha}+\frac{f_2}{\alpha}+\frac{f_1^2}{\alpha^2}\right)$$

i.e.

$$8\pi\,\Delta E_m = 8\pi(\Delta E_m)_1+\int\frac{dl_2}{A_2}\left\{(g_1^2-g_1-g_2)\phi_1^2+\phi_2^2\left(\frac{g_1^2}{\beta}+g_1+g_2\right)\right\}$$

$$+\int\frac{dl_1}{A_1}\left\{\phi_2^2(f_1^2-f_1-f_2)+\phi_1^2\left(\frac{f_1^2}{\alpha}+f_1+f_2\right)\right\}$$

$$= 8\pi(\Delta E_m)_1+\phi\,F(g_1, g_2, f_1, f_2)$$

where $(\Delta E_m)_1$ is given by 6.5, and F written to second order in ξ is

$$\phi F = \int\frac{dl}{A}\left\{\phi^2 f_1^2\left(1+\frac{1}{\alpha}\right)+\phi^2 g_1^2\left(1+\frac{1}{\beta}\right)+(\phi_2^2-\phi_1^2)(g_1-f_1)\right\}$$

Hence

8.21 $$\frac{\Delta E_m}{\phi} = \frac{(\Delta E_m)_1}{\phi}+\frac{F(f,g)}{8\pi}$$

where

$$8.22 \quad F = \int B\,dl \left\{ \frac{\alpha+1}{\alpha} f_1^2 + \frac{\beta+1}{\beta} g_1^2 + 2\zeta (g_1 - f_1) \right\}$$

In evaluating ΔE_p it will be assumed that the pressures in the perturbed flux tubes are constant along the lengths of these tubes, an assumption which is justified by the remarks at the beginning of this section

$$8.23 \quad (\gamma - 1)\Delta E_p = \left(p_2' V_2' + p_1' V_1' - p_1 V_1 - p_2 V_2 \right)$$
$$+ \left(p_2'' V_2'' + p_1'' V_1'' - \alpha p_1 V_1 - \beta p_2 V_2 \right)$$

The adiabatic equation 2.3 is used to evaluate p_1', p_2', p_1'', p_2''. Thus:-

$$8.24 \quad p_1' = \left(\frac{V_2}{V_1'}\right)^{\gamma} p_2 \quad ; \quad p_2' = \left(\frac{V_1}{V_2'}\right)^{\gamma} p_1$$

$$8.25 \quad p_1'' = \left(\frac{\alpha V_1}{V_1''}\right)^{\gamma} p_1 \quad ; \quad p_2'' = \left(\frac{\beta V_2}{V_2''}\right)^{\gamma} p_2$$

After substitution of these results, 8.23 becomes

After substitution of these results, 8.23 becomes

$$(\gamma-1)\Delta E_p = \left\{ p_1\left(\frac{V_1}{V_2'}\right)^{\gamma-1} V_1 + p_2\left(\frac{V_2}{V_1'}\right)^{\gamma-1} V_2 - p_1 V_1 - p_2 V_2 \right\}$$

$$+\left\{ p_2\,\beta V_2\left(\frac{\beta V_2}{V_2''}\right)^{\gamma-1} + p_1\alpha V_1\left(\frac{\alpha V_1}{V_1''}\right)^{\gamma-1} - p_1\alpha V_1 - p_2\beta V_2 \right\}$$

and on substitution of 8.16, 8.17 into it this last equation becomes

$$(\gamma-1)\Delta E_p = \left\{ p_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}(1+k)^{1-\gamma} + p_2V_2\left(\frac{V_2}{V_1}\right)^{\gamma-1}(1+k)^{1-\gamma} - p_1V_1 - p_2V_2 \right\}$$

$$+\left\{ p_2\beta V_2\left(1-\frac{k}{\beta}\right)^{1-\gamma} + p_1\alpha V_1\left(1-\frac{k}{\alpha}\right)^{1-\gamma} - p_1\alpha V_1 - p_2\beta V_2 \right\}$$

$$= (\gamma-1)(\Delta E_p)_1 + p_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}\left\{(1-\gamma)(k_1+k_2) + \frac{\gamma(\gamma-1)}{2}k_1^2\right\}$$

$$+ p_2V_2\left(\frac{V_2}{V_1}\right)^{\gamma-1}\left\{(1-\gamma)(k_1+k_2) + \frac{\gamma(\gamma-1)}{2}k_1^2\right\}$$

$$+ p_2\beta V_2\left\{\frac{\gamma-1}{\beta}(k_1+k_2) + \frac{\gamma(\gamma-1)}{2\beta^2}k_1^2\right\}$$

$$+ p_1\alpha V_1\left\{\frac{\gamma-1}{\alpha}(h_1+h_2) + \frac{\gamma(\gamma-1)}{2\alpha^2}h_1^2\right\}$$

i.e.

$$(\gamma-1)\Delta E_p = (\gamma-1)(\Delta E_p)_1 + \phi(\gamma-1)\,G(k_1, k_2, h_1, h_2)$$

where $(\Delta E_p)_1$ is given by 6.10, and G, written to second order in ξ is given by

$$\phi G = pV\frac{\gamma}{2}\left\{\frac{\beta+1}{\beta}k_1^2 + \frac{\alpha+1}{\alpha}h_1^2\right\} + k_1\left\{p_2V_2 - p_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}\right\}$$

$$+ \; h_1\left\{p_1V_1 - p_2V_2\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right\}$$

$$= \frac{\gamma pV}{2}\left\{\frac{\beta+1}{\beta}k_1^2 + \frac{\alpha+1}{\alpha}h_1^2\right\}$$

$$+ \; (k_1 - h_1)\,pV\left\{\frac{\delta p}{p} + \frac{\delta V}{V} - (1-\gamma)\frac{\delta V}{V}\right\}$$

i.e.

8.26 $$G(h, k) = \frac{\gamma pK}{2}\left\{\frac{\alpha+1}{\alpha}h_1^2 + \frac{\beta+1}{\beta}k_1^2\right\} + pK(k_1 - h_1)\left(\frac{\delta p}{p} + \gamma\frac{\delta V}{V}\right)$$

and

8.27 $$\frac{\Delta E_p}{\phi} = \frac{(\Delta E_p)_1}{\phi} + G(h, k)$$

Adding 8.21 and 8.27, and dropping the subscript 1 on h,k,f and g

8.28 $$\frac{\Delta E}{\phi} = \frac{(\Delta E)_1}{\phi} + G(h, k) + \frac{F(f, g)}{8\pi}$$

On dividing 8.18 by ϕ and considering only first order components the constraint equations may be written as

8.29 $$h K = \int f \frac{dl}{B} \quad ; \quad k K = \int g \frac{dl}{B}$$

We now proceed to minimise ΔE with respect to f and g subject to the constraint equations 8.29. This is equivalent to minimising $F(f, g)$. The Euler-Lagrange equations are

$$B \left\{ 2f \frac{\alpha+1}{\alpha} - 2\zeta \right\} + \frac{2\lambda}{B} = 0$$

$$B \left\{ 2g \frac{\beta+1}{\beta} + 2\zeta \right\} + \frac{2\mu}{B} = 0$$

8.30 $$\therefore \; f = \frac{\alpha}{\alpha+1} \left\{ \zeta - \frac{\lambda}{B^2} \right\}$$

8.31 $$g = \frac{-\beta}{\beta+1} \left\{ \zeta + \frac{\mu}{B^2} \right\}$$

Substituting back into 8.28 to evaluate λ and μ we get (λ, μ are Lagrange multipliers)

$$h K = \frac{\alpha}{\alpha+1} \left[\zeta K - \lambda \int \frac{dl}{B^3} \right]$$

$$k K = \frac{-\beta}{\beta+1} \left[\zeta K + \mu \int \frac{dl}{B^3} \right]$$

On writing $\int \frac{dl}{B^3} = H$ these equations become

$$\lambda = \frac{K}{H} \left(\zeta - \frac{\alpha+1}{\alpha} h \right)$$

$$\mu = -\frac{K}{H} \left(\zeta + \frac{\beta+1}{\beta} k \right)$$

On substituting these results back into 8.22

F becomes

$$F_m = \int B\,dl \left\{ \frac{\alpha}{\alpha+1}\left(\zeta^2 - \frac{2\zeta\lambda}{B^2} + \frac{\lambda^2}{B^4}\right) + \frac{\beta}{\beta+1}\left(\zeta^2 + \frac{2\zeta\mu}{B^2} + \frac{\mu^2}{B^4}\right) + \frac{\alpha}{\alpha+1}\left(\frac{2\zeta\lambda}{B^2} - 2\zeta^2\right) - \frac{\beta}{\beta+1}\left(2\zeta^2 + \frac{2\zeta\mu}{B^2}\right)\right\}$$

i.e.

$$F_m = -\zeta^2\left[\frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1}\right]\int B\,dl + \left\{\frac{\alpha}{\alpha+1}\lambda^2 + \frac{\beta}{\beta+1}\mu^2\right\}\int\frac{dl}{B^3}$$

$$= -\zeta^2\Omega J + \frac{\alpha}{\alpha+1}\frac{K^2}{H}\left(\zeta^2 - 2\zeta h\frac{\alpha+1}{\alpha} + \left(\frac{\alpha+1}{\alpha}\right)^2 h^2\right)$$

$$+ \frac{\beta}{\beta+1}\frac{K^2}{H}\left(\zeta^2 + 2\zeta k\frac{\beta+1}{\beta} + \left(\frac{\beta+1}{\beta}\right)^2 k^2\right)$$

where $\Omega = \frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1}$, and F_m is the minimised value of F.

8.32 $\therefore \quad F_m = \zeta^2\Omega\left(\frac{K^2}{H} - J\right) + 2\zeta\frac{K^2}{H}(k - h) + \frac{K^2}{H}\left\{\frac{\alpha+1}{\alpha}h^2 + \frac{\beta+1}{\beta}k^2\right\}$

ΔE is now a function of h, k, α, β

$$\frac{\Delta E}{\phi} = \frac{(\Delta E)_1}{\phi} + G(h, k) + \frac{F_m(h, k)}{8\pi}$$

The next step is therefore to minimise $G + \frac{F_m}{8\pi}$ with respect to the pair h, k.

The Euler equations are

$$2\eta\frac{K^2}{H} + 2\frac{\beta+1}{\beta}\frac{K^2}{H}k + 8\pi\left\{\gamma p K\left(\frac{\beta+1}{\beta}\right)k + K\left(\delta p + \gamma p\frac{\delta V}{V}\right)\right\} = 0$$

$$\therefore \frac{\beta+1}{\beta}k\left\{\frac{K}{H} + 4\pi\gamma p\right\} + \left\{\eta\frac{K}{H} + 4\pi\left(\delta p + \gamma p\frac{\delta V}{V}\right)\right\} = 0$$

8.33 $$\therefore k = -\frac{\beta}{\beta+1}\,\frac{\eta K + 4\pi H\left(\delta p + \gamma p\frac{\delta V}{V}\right)}{K + 4\pi\gamma p H}$$

and

$$-2\eta\frac{K^2}{H} + 2\frac{\alpha+1}{\alpha}\frac{K^2}{H}h + 8\pi\left\{\gamma p K\left(\frac{\alpha+1}{\alpha}\right)h - K\left(\delta p + \gamma p\frac{\delta V}{V}\right)\right\} = 0$$

$$\therefore \frac{\alpha+1}{\alpha}h\left\{\frac{K}{H} + 4\pi\gamma p\right\} = \eta\frac{K}{H} + 4\pi\left(\delta p + \gamma p\frac{\delta V}{V}\right)$$

8.34 $$\therefore h = \frac{\alpha}{\alpha+1}\,\frac{\eta K + 4\pi H\left(\delta p + \gamma p\frac{\delta V}{V}\right)}{K + 4\pi\gamma p H}$$

8.35 $$G + \frac{F_m}{8\pi} = \frac{\alpha+1}{\alpha}h^2\left\{\frac{\gamma p K}{2} + \frac{K^2}{8\pi H}\right\} + \frac{\beta+1}{\beta}k^2\left\{\frac{\gamma p K}{2} + \frac{K^2}{8\pi H}\right\}$$

$$+ (k-h)\left\{\eta\frac{K^2}{4\pi H} + K\left(\delta p + \gamma p\frac{\delta V}{V}\right)\right\} + \frac{\Omega\eta^2}{8\pi}\left[\frac{K^2}{H} - J\right]$$

Then writing $$C = \frac{\Omega\eta^2}{8\pi}\left[\frac{K^2}{H} - J\right]$$

8.36 $$Q = \eta K + 4\pi H\left(\delta p + \gamma p\frac{\delta V}{V}\right)$$

and

$$S = K + 4\pi\gamma p H$$

and substituting 8.33 and 8.34 into 8.35 this equation becomes

$$\left(G + \frac{F_m}{8\pi}\right)_m = \left\{ \frac{\gamma p K}{2} + \frac{K^2}{8\pi H} \right\} \frac{Q}{S}(h - k) + \frac{KQ}{4\pi H}(k - h) + c$$

$$= \frac{1}{2} \frac{KQ}{4\pi H}(k - h) + c$$

8.37 $$= -\frac{\Omega}{2} \frac{KQ^2}{4\pi HS} + c$$

where the suffix m denotes that $(G + F_m/8\pi)$ has now been minimised with respect to h,k.

Using 7.16 we may write down Q as

$$Q = \eta K + 4\pi p' H \xi + 4\pi \gamma p H \frac{(\eta K + \xi I)}{K}$$

$$= \eta(K + 4\pi\gamma p H) + \xi \frac{4\pi H}{K}(p'K + \gamma p I)$$

8.38 $$= \eta S + \frac{4\pi H}{K} \xi T$$

where $T = p'K + \gamma p I$

Substituting 8.38 in 8.37 gives

$$(G + F_m/8\pi)_m = -\frac{\Omega}{2} \frac{K}{4\pi HS} \left\{ \eta^2 S^2 + \frac{8\pi H}{K} \eta \xi T S + \xi^2 \left(\frac{4\pi H T}{K}\right)^2 \right\} + c$$

$$= -\frac{\Omega}{2} \left\{ \frac{KS}{4\pi H} \eta^2 + 2\eta \xi T + \xi^2 \frac{4\pi H}{K} \frac{T^2}{S} \right\} + C$$

$$\therefore \left(G + \frac{F_m}{8\pi}\right)_m = -\frac{\Omega}{2}\left\{\eta^2\left(\frac{K^2}{4\pi H} + \gamma p K\right) + 2\eta\xi\left(p'K + \gamma p I\right) + \xi^2 \frac{4\pi H}{K} \frac{\left(p'K + \gamma p I\right)^2}{K + 4\pi\gamma p H}\right\} + \frac{\Omega}{2}\eta^2\left[\frac{K^2}{4\pi H} - \frac{J}{4\pi}\right]$$

i.e.

8.39 $$\left(G + \frac{F_m}{8\pi}\right)_m = -\frac{\Omega}{2}\left\{\eta^2\left(\frac{J}{4\pi} + \gamma p K\right) + 2\eta\xi\left(p'K + \gamma p I\right) + \xi^2 \frac{4\pi H}{K}\frac{\left(p'K + \gamma p I\right)^2}{K + 4\pi\gamma p H}\right\}$$

Returning to 8.37 it is clear, since K,H,S are all positive and since C is negative, that this expression is minimised by maximising $\Omega = \left[\frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1}\right]$. This is done by allowing α, β to take unboundedly large values. When this is done $\Omega \to 2$. The final expression for ΔE is therefore

$$\frac{\Delta E}{\phi} = \frac{(\Delta E)_1}{\phi} + \left(G + \frac{F_m}{8\pi}\right)_m$$

with $\frac{(\Delta E)_1}{\phi}$ given by 7.18, and this is

8.40 $$\frac{\Delta E}{\phi} = \eta^2\left[\frac{J}{4\pi} + \gamma p K\right] + 2\eta\xi\left[p'K + \gamma p I\right] + \xi^2\left[p'K + \gamma p I\right]\frac{I}{K} - \eta^2\left[\frac{J}{4\pi} + \gamma p K\right] - 2\eta\xi\left[p'K + \gamma p I\right] - \xi^2\left[p'K + \gamma p I\right]^2 \frac{4\pi H}{K(K + 4\pi\gamma p H)}$$

i.e.

$$\frac{\Delta E}{\phi} = \xi^2 \frac{(p'K + \gamma p I)}{K} \left\{ I - \frac{4\pi H\,(p'K + \gamma p I)}{K + 4\pi\gamma p H} \right\}$$

$$= \xi^2 \frac{(p'K + \gamma p I)}{K} \cdot \frac{(IK - 4\pi p' H K)}{K + 4\pi\gamma p H}$$

8.41 $$= \xi^2 \frac{(p'K + \gamma p I)(I - 4\pi p' H)}{K + 4\pi\gamma p H}$$

As with 7.19 this will be positive if either of two conditions is satisfied

8.42 $$\int \frac{dl}{R r B^2} > 0$$

or

8.43 $$\int \frac{dl}{R r B^2} < 2\pi p' \int \frac{dl}{B^3} + \frac{p'}{2\gamma p} \int \frac{dl}{B}$$

This is precisely the result obtained by Bernstein et al (2) using the energy principle. In fact 8.41 is identical to their expression with X for ξ where $X/rB = \xi_\psi$ the ψ component of the displacement. In 8.41 $\xi/rB = D$ the perpendicular displacement of the flux tube.

§9. Validity of approximations.

The crux of the theory in earlier sections is the assumption that the volume integrations over flux tubes can be replaced by line integrals along lines of force. This obviously requires that certain variables be infinitesimal, and we will now investigate the detailed consequences of these assumptions.

In §7 this substitution of line integrals for volume integrals was, up to a point, rigorously possible, and it automatically defined the lines of magnetic field ℓ_1, and ℓ_2 as follows.

$$\int_i dV = \int\int d\ell_i \, dA \qquad (i = 1, 2)$$

$$= \int d\ell_i \int_0^{A_i(\ell_i)} dA$$

9.1 $$= \int A_i \, d\ell_i$$

The flux contained in a tube is

9.2 $$\phi_i = \int_0^{A_i} B \, dA$$

and the magnetic energy integral is

$$\int_i B^2 dV = \int dl_i \int_0^{A_i} B^2 dA$$

and since $d\phi = B\,dA$ this is

9.3 $$\int B^2 dV = \int dl_i \int_0^{\phi_i} B\,d\phi$$

Equations 9.1 - 9.3 are all exact equalities, but to handle them more easily the following approximation was made.

If $B = B(\lambda, l_\lambda)$ where λ labels the field lines in a tube then B was replaced by $B_i = B(i, l_i)$ $(i = 1, 2)$ which is its value at a particular point in the cross section A_i (viz where l_i cuts A_i). Then 9.1 - 9.3 reduce to

9.4 $$\int_i dV = \int \frac{\phi_i\, dl_i}{B_i}$$

since

9.5 $$\phi_i = B_i \int_0^{A_i} dA = B_i A_i$$

$$\int_i B^2 dV = \int_0^{\phi_i} d\phi \int B_i\, dl_i$$

i.e

9.6 $$\int_{\lambda} B^2 dV = \Phi_{\lambda} \int B_{\lambda} \, dl_{\lambda}$$

which are precisely the equations used in §7 and §8.

Now if ξ is taken to be the fundamental variable, it is clear that 9.4 and 9.6, which occur in the expression 7.1 for ΔE, must each be correct to zero order in ξ. i.e. we must have

9.7 $$\int_{\lambda} dV = \Phi_{\lambda} \int \frac{dl_{\lambda}}{B_{\lambda}} \left[1 + O(\xi)\right]$$

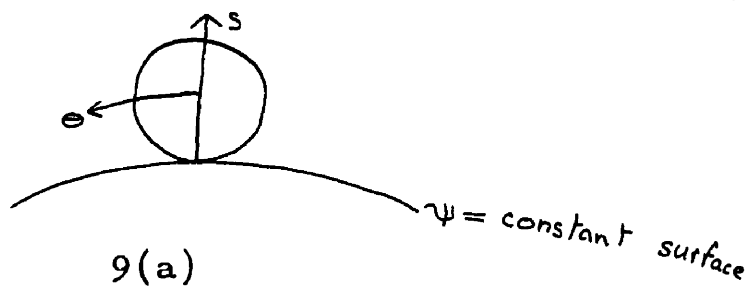
and

9.8 $$\int_{\lambda} B^2 dV = \Phi_{\lambda} \int B_{\lambda} \, dl_{\lambda} \left[1 + O(\xi)\right]$$

The question to be considered here is: what order of magnitude must Φ_{λ} be in order that 9.7 and 9.8 will be satisfied? We answer this by considering 9.1 - 9.3, and by assuming that ϕ is an infinitesimal, i.e.

9.9 $$\phi = O(\xi^n), \quad (n \geqslant 1)$$

To consider $\int dA$ it is convenient to introduce variables s, θ such that, ℓ, θ, S form a local system of cylindrical polars. S is perpendicular to the ψ = constant surface, and ℓ along the surface. Then $\int dA = \iint s\,d\theta\,ds$ as shown below



9(a)
Section of flux tube.

9.10 Then $\int_{\ell} B\,dA = \int d\theta \int_{f_1(\theta)}^{f_2(\theta)} B(s,\ell)\,s\,ds$

Expanding B in a Taylor series about $B_\ell = B(s_\ell, \ell_\ell)$ 9.10 becomes

$$\Phi_\ell = \int d\theta \int_{f_1(\theta)}^{f_2(\theta)} \left[B_\ell + \frac{\partial B_\ell}{\partial s}(s-s_\ell)\right] s\,ds$$

$$= B_\ell A_\ell + \frac{\partial B_\ell}{\partial s}\int d\theta \int_{f_1}^{f_2}(s-s_\ell)\,s\,ds$$

$$\leqslant B_\ell A_\ell + \left|\frac{\partial B_\ell}{\partial s}\max(s-s_\ell)A_\ell\right|$$

But $\max(s-s_\ell) = O(A_\ell) = O(\Phi_\ell)$

Hence

9.11 $\Phi_i = B_i A_i \left[1 + O(\Phi_i)\right]$

In similar fashion

$\int B^2 dA = B_i^2 A_i \left[1 + O(\Phi_i)\right]$. Thus 9.3 becomes

9.12 $\int_i B^2 dV = \int B_i^2 A_i \, dl_i \left[1 + O(\Phi_i)\right]$

and on inverting 9.11 to obtain

9.13 $B_i A_i = \Phi_i \left[1 + O(\Phi_i)\right]$

and using this in 9.12 this equation becomes

9.14 $\int_i B^2 dV = \Phi_i \int B_i \, dl_i \left[1 + O(\Phi_i)\right]$

Similarly 9.1 is

9.15 $\int_i dV = \Phi_i \int \frac{dl_i}{B_i} \left[1 + O(\Phi_i)\right]$

Hence 9.7 and 9.8 will immediately be satisfied if n is taken to be 1 in 9.9 i.e. if

9.16 $\Phi = O(\xi)$

Turning now to the actual perturbation as visualised in diagram 6b; if there are m flutes, then

$$9.17 \quad \phi_\alpha = O(\xi/m) \quad \begin{cases} = O(\xi) & \text{If } m \text{ is finite} \\ = O(\xi^n) & \text{If } m \text{ is infinite, in fact if } m = O(\xi^{1-n}) \quad (n>1) \end{cases}$$

In either case 9.16 is satisfied.

Thus the theory of §7 holds whether or not $m \to \infty$. In §8, however, the above remarks hold with ϕ_1, ϕ_2 replaced by $\alpha\phi_1$, $\beta\phi_2$ and hence according to 9.16 we must have

$$\alpha\phi_1 = O(\xi) \quad ; \quad \beta\phi_2 = O(\xi)$$

and therefore

$$9.18 \quad \phi_\alpha = O(\xi/\alpha_\alpha) \qquad (\alpha_1 = \alpha, \ \alpha_2 = \beta)$$

Thus if m is finite $\phi_\alpha = O(\xi)$, and according to 9.18 the approximations are only valid if $\phi_\alpha = O(\xi/\alpha_\alpha)$. Thus α_α may not be infinite. Only if m is infinite, $m = O(\xi^{-n})$, $\phi_\alpha = O(\xi^{n+1})$, may α_α be permitted to go to infinity. Finally we note that since $\zeta = O(\xi)$ the independent variable $\delta\phi = \zeta\phi$ must satisfy

9.19 $$\delta\phi = O(\xi\phi)$$

These results may be summarised as follows:-
In § 7 the theory applies to finite m modes as well as to $m = \infty$. In the finite m case the variables satisfy

$$\phi = O(\xi) \; ; \; \delta\phi = O(\xi^2) \; ; \; \zeta = O(\xi)$$

For $m = \infty$, if $m = O(\xi^{-n})$ the variables satisfy the following equations

$$\phi = O(\xi^{n+1}) \; ; \; \delta\phi = O(\xi^{n+2}) \; ; \zeta = O(\xi)$$

In §8, if α, β are allowed to be infinite the approximations are valid only for $m = \infty$ In this case if $m = O(\xi^{-n})$ the following equations hold

$$\phi_i = O(\xi^{n+1}) \; ; \; \delta\phi = O(\xi^{n+2}) \; ; \; \zeta = O(\xi)$$

and $\alpha_i = O(\xi^{-n})$, and thus $\alpha_i\phi_i = O(\xi)$

If m is finite the maximisation of

$\Omega = \frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1}$ must now be carried out subject to a constraint, viz:-

9.20 $\phi_i \leq K_1 \xi / \alpha_i$ (from 9.18)

where K_1 is a finite (but arbitrarily large) constant.

In fact, for the interchange considered

9.21 $\phi_i \leq \frac{K_2 \xi}{m}$ where K_2 is a finite constant

Hence the constraint is equivalent to

$$\frac{K_2}{m} \leq \frac{K_1}{\alpha_i}$$

i.e.

9.22 $\alpha_i \leq \frac{K_1}{K_2} m$

Then to maximise Ω, $\alpha = \beta = \frac{K_1}{K_2} m$, and

9.23 $\Omega_{max} = 2\left(1 + \frac{K_2}{K_1 m}\right)^{-1}$

where as noted above K_1 may be chosen as large as we like, still remaining finite. If this value of Ω max is used the result is

$$\frac{\Delta E}{\phi} = \left(\frac{\Delta E}{\phi}\right)_{m=\infty} + 2\left\{1 - \left(1 + \frac{K_2}{K_1 m}\right)^{-1}\right\}\left\{\begin{matrix}\text{positive}\\ \text{quantity}\end{matrix}\right\}$$

9.24
$$= \left(\frac{\Delta E}{\phi}\right)_{m=\infty} + \left\{\frac{K_2}{K_2 + m K_1}\right\}\left\{\begin{matrix}\text{positive}\\ \text{quantity}\end{matrix}\right\}$$

Hence if mode m is unstable so also is $m+1$, and the worst possible case is with $m \rightarrow \infty$.

This result was also obtained by Bernstein et al. (2).

§10. Inclusion of B_θ Field.

The theory of §7 and §8 can be applied to the case with magnetic field

$$\underset{\sim}{B}' = \left(B_r(r, z),\quad B_\theta(r, z),\quad B_z(r, z)\right) \tag{10.1}$$

whenever the exchange of flux tubes visualised in these sections is possible. This requires that two flux tubes having originally the same θ co-ordinate will continue to have the same θ co-ordinate along their lengths, and will not slip round the ψ = constant surfaces away from each other. In the general case the θ co-ordinate is given by $\int d\theta = \int \frac{d\theta}{dl}\, dl$. The condition under which the theory of §7 and §8 will be valid is therefore

$$\frac{\partial}{\partial\psi} \int \frac{d\theta}{dl}\, dl \equiv 0 \tag{10.2}$$

The lines of force of the field 10.1 are defined by

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{dz}{B_z} = \frac{dl}{B} = \frac{dl'}{B'} \tag{10.3}$$

where $\underset{\sim}{B} = \left(B_r(r, z), \quad 0, \quad B_z(r, z) \right)$, B' is given by 10.1 and dl, dl' are elements of length along $\underline{B}$ and $\underline{B}'$ respectively.

From 10.3

10.4 $$\frac{d\theta}{dl} = \frac{B_\theta}{rB} = \mu$$

The operators $\frac{\partial}{\partial\psi}$, and δ are related as follows

10.5 $$\delta = D\frac{\partial}{\partial n} = \delta\psi\frac{\partial}{\partial\psi}$$

where $\frac{\partial}{\partial n}$ is the derivative normal to the surface, and therefore

10.6 $$\frac{\partial}{\partial n} = \frac{B_z}{B}\frac{\partial}{\partial r} - \frac{B_r}{B}\frac{\partial}{\partial z}$$

Then the condition 10.2 is

$$\frac{1}{\xi}\,\delta\int\mu\, dl \equiv 0$$

where δ is given by 10.5, and this integral can be evaluated using the methods of §7. Thus

$$\int \left[\nabla \times \underset{\sim}{B} \frac{B_\theta}{r B^2} \right]_\theta \frac{\xi}{r B} \, dl = 0$$

$$\therefore \int \frac{dl}{r B} \left\{ \frac{\mu}{B} (\underset{\sim}{\nabla} \times \underset{\sim}{B})_\theta + \left(B_r \frac{\partial}{\partial z} - B_z \frac{\partial}{\partial r} \right) \frac{B_\theta}{r B^2} \right\} = 0$$

$$\therefore \int \frac{dl}{r B} \left\{ \frac{\mu}{B} (\underset{\sim}{\nabla} \times \underset{\sim}{B})_\theta - \frac{\partial \mu}{\partial n} + \frac{\mu}{B^2} \left(B_z \frac{\partial B}{\partial r} - B_r \frac{\partial B}{\partial z} \right) \right\} = 0$$

after using 10.5. When 7.14 is used to evaluate the last term of this integral as $\mu \left(\frac{1}{R} - \frac{(\underset{\sim}{\nabla} \times \underset{\sim}{B})_\theta}{B} \right)$, and $\partial \mu / \partial n$ is replaced by $r B \frac{\partial \mu}{\partial \psi}$, the condition for validity of the exchange reduces to

10.7 $$\int \mu \, dl \left\{ \frac{\mu'}{\mu} - \frac{1}{R r B} \right\} \equiv 0$$

where $\mu' = \frac{\partial \mu}{\partial \psi}$

In evaluating ΔE, equation 7.1 still holds, but 5.11 replaces 5.12 in the elimination of $(\underset{\sim}{\nabla} \times \underset{\sim}{B})_\theta$.

The four integrals involved in 7.1 are:-

$$\int \frac{dl}{B} \;;\quad \delta \int \frac{dl}{B} \;;\quad \int B \, dl \;;\quad \delta \int B \, dl$$ and these

integrals are replaced as follows

10.8 $$\int \frac{dl}{B} \rightarrow \int \frac{dl'}{B'} = \int \frac{dl}{B} = K$$

$$\delta\int \frac{dl}{B} \rightarrow \delta\int \frac{dl'}{B'} = \delta\int \frac{dl}{B}$$

$$= -2\xi\int \frac{dl}{R_r B^2} + \xi\int \frac{(\nabla \times \underset{\sim}{B})_\theta \, dl}{r B^3}$$

10.9 $$= -2\xi\int \frac{dl}{R_r B^2} + 4\pi p'\xi\int \frac{dl}{B^3} + ff'\xi\int \frac{dl}{r^2 B^3}$$

using 5.11.

$$\int B\,dl \rightarrow \int B'dl' = \int (B')^2 \frac{dl}{B}$$

$$= \int B\,dl + \int B_\theta^2 \frac{dl}{B}$$

10.10 $$= \int B\,dl + f^2\int \frac{dl}{r^2 B}$$

$$\delta\int B\,dl \rightarrow \delta\int B'dl' = \delta\int B\,dl + \delta\left\{ f^2\int \frac{dl}{r^2 B} \right\}$$

$$= -\xi\int \frac{(\nabla \times \underset{\sim}{B})_\theta \, dl}{r B} + 2ff'\xi\int \frac{dl}{r^2 B} + f^2\,\delta\int \frac{dl}{r^2 B}$$

since $\delta f = \frac{\partial f}{\partial \psi}\delta\psi = f'\xi$ .Then using 5.11

10.11 $$\delta\int B'dl' = -4\pi p'\xi\int \frac{dl}{B} + ff'\xi\int \frac{dl}{r^2 B} + f^2\,\delta\int \frac{dl}{r^2 B}$$

Stokes Theorem is used to evaluate the last term

$$\delta\int\frac{dl}{r^2B} = -\xi\int\frac{dl}{rB}\left[\nabla \times \frac{B}{r^2B^2}\right]_\theta$$

$$= -\xi\int\frac{dl}{r^3B^3}(\nabla\times B)_\theta - \xi\int\frac{dl}{rB}\left(B_r\frac{\partial}{\partial z} - B_z\frac{\partial}{\partial r}\right)\frac{1}{r^2B^2}$$

Then using 7.14 to evaluate the second term

$$\delta\int\frac{dl}{r^2B} = -\xi\int\frac{(\nabla\times B)_\theta\,dl}{r^3B^3} - 2\xi\int\frac{B_z\,dl}{r^4B^3} - 2\xi\int\frac{dl}{r^3B^2}\left(\frac{1}{R} - \frac{(\nabla\times B)_\theta}{B}\right)$$

10.12 $$= \xi\int\frac{(\nabla\times B)_\theta\,dl}{r^3B^3} - 2\xi\int\frac{B_z\,dl}{r^4B^3} - 2\xi\int\frac{dl}{Rr^3B^2}$$

On substitution of 10.12 into 10.11 and use of 5.11

10.13 $$\delta\int B'dl' = -4\pi p'\xi\int\frac{dl}{B} + ff'\xi\int\frac{dl}{r^2B} + 4\pi p'f^2\xi\int\frac{dl}{r^2B^3}$$

$$+f^3f'\xi\int\frac{dl}{r^4B^3} - 2f^2\xi\int\frac{B_z\,dl}{r^4B^3} - 2f^2\xi\int\frac{dl}{Rr^3B^2}$$

Using these equations the expression for ΔE obtained in 7.1 can now be evaluated. The various integrals involved will be denoted as follows

$$4\pi p'\int\frac{dl}{B^3} - 2\int\frac{dl}{RrB^2} = I \;;\; \int B\,dl = J \;;\; \int\frac{dl}{B} = K$$

10.14 $$\int\frac{dl}{r^2B} = L \;;\; \int\frac{dl}{r^2B^3} = M \;;\; \int\frac{dl}{r^4B^3} = N$$

$$\int\frac{B_z\,dl}{r^4B^3} + \int\frac{dl}{Rr^3B^2} = P$$

Then

$$\int \frac{dl'}{B'} = K \ ; \ \delta\int \frac{dl'}{B'} = \xi I + ff'\xi M \ ; \ \int B'dl' = J + f^2 L$$

10.15
$$\delta\int B'dl' = -4\pi p'\xi K + ff'\xi L + 4\pi p' f^2 \xi M + f^3 f' \xi N - 2f^2 \xi P$$

Then

6.10 $$\Delta E_p = \delta V\left(\delta p + \gamma p \frac{\delta V}{V}\right)$$ where

$$V = \phi\int \frac{dl'}{B'} = \phi K \ ; \ \delta V = \delta\phi . K + \phi\left(\xi I + ff'\xi M\right)$$

$$\therefore \frac{\Delta E_p}{\phi} = \left(\eta K + \xi I + \xi ff'M\right)\left\{p'\xi + \frac{\gamma p}{K}\left(\eta K + \xi I + \xi ff'M\right)\right\}$$

10.16
$$= \eta^2(\gamma p K) + \eta\xi\left(p'K + 2\gamma p(I + ff'M)\right) + \xi^2\frac{(I+ff'M)}{K}\left\{p'K + \gamma p\,(I + ff'M)\right\}$$

and from 6.5

$$\frac{\Delta E_m}{\phi} = \frac{\eta}{4\pi}\left\{\eta\int B'dl' - \delta\int B'dl'\right\}$$

$$= \frac{\eta}{4\pi}\left\{\eta J + \eta f^2 L + 4\pi p'\xi K - ff'\xi L - 4\pi p' f^2 \xi M - f^3 f' \xi N + 2f^2 \xi P\right\}$$

10.17
$$= \frac{\eta^2}{4\pi}\left\{J + f^2 L\right\} + \frac{\eta\xi}{4\pi}\left\{4\pi p'K - ff'L - 4\pi p' f^2 M - f^3 f' N + 2f^2 P\right\}$$

Then combining 10.16 and 10.17 the result is

10.18 $$\frac{(\Delta E)_1}{\phi} = \zeta^2\left\{\frac{J}{4\pi} + \frac{f^2L}{4\pi} + \gamma p K\right\} + 2\zeta\xi\left\{p'K + \gamma p(I + ff'M) - \frac{ff'L}{8\pi} - \frac{p'f^2M}{2} - \frac{f^3f'N}{8\pi} + \frac{f^2P}{4\pi}\right\}$$

$$+ \xi^2\frac{(I + ff'M)}{K}\left\{p'K + \gamma p(I + ff'M)\right\}$$

Section § 8 goes through as before with 8.37 giving the resultant negative term in $\Delta E/\phi$. With $\Omega = 2$ this is

8.37 $$\left(G + \frac{F_m}{8\pi}\right)_m = -\frac{KQ^2}{4\pi HS} + \frac{\zeta^2}{4\pi}\left[\frac{K^2}{H} - J - f^2L\right]$$

Where

$$H = \int\frac{dl'}{(B')^3} \quad ; \quad K = \int\frac{dl'}{B'} = \int\frac{dl}{B}$$

$$S = K + 4\pi\gamma pH \quad ; \quad Q = \zeta K + 4\pi H\left(\delta p + \gamma p\frac{\delta V}{V}\right)$$

i.e.

$$Q = \zeta K + 4\pi H\left[p'\xi + \frac{\gamma p}{K}\left(\zeta K + \xi(I + ff'M)\right)\right]$$

$$= \zeta(K + 4\pi\gamma pH) + \frac{4\pi H}{K}\xi\left[p'K + \gamma p(I + ff'M)\right]$$

Hence

$$\frac{KQ^2}{4\pi HS} = \zeta^2 \frac{K(K+4\pi\gamma p H)}{4\pi H} + 2\zeta\xi\left[p'K + \gamma p(I + ff'M)\right]$$

$$+ \frac{4\pi H}{K}\xi^2 \frac{\left[p'K + \gamma p(I+ff'M)\right]^2}{K + 4\pi\gamma p H}$$

Hence

10.19 $$\left(G + \frac{F_m}{8\pi}\right)_m = -\zeta^2\left[\frac{J}{4\pi} + \frac{f^2L}{4\pi} + \gamma p K\right]$$

$$-2\zeta\xi\left[p'K + \gamma p(I+ff'M)\right]$$

$$-\xi^2\frac{4\pi H}{K}\frac{\left[p'K+\gamma p(I+ff'M)\right]^2}{K+4\pi\gamma p H}$$

10.20 $$\therefore \left(\frac{\Delta E}{\Phi}\right)_{min} = 2\zeta\xi\left\{\frac{f^2P}{4\pi} - \frac{f^3f'N}{8\pi} - \frac{p'f^2M}{2} - \frac{ff'L}{8\pi}\right\}$$

$$+\xi^2\frac{\left\{p'K+\gamma p(I+ff'M)\right\}}{K}\left\{(I+ff'M) - \frac{4\pi H\left(p'K+\gamma p(I+ff'M)\right)}{K+4\pi\gamma p H}\right\}$$

on addition of 10.18 and 10.19.

This is a quadratic form in ζ , ξ which can always be made negative by suitable choices of ζ and ξ . It will now be shown that when condition 10.7 is applied to the above quadratic in ζ, ξ that the cross term in $\zeta\xi$ vanishes leaving an expression involving ξ^2 alone. This is of course precisely what happened in §8, but the co-efficient of ξ^2 now differs from the one obtained there. The indefiniteness of 10.20 may be taken as confirmation of the belief that the theory will fail to be valid if 10.7 is not satisfied.

The co-efficient of the $z\xi$ term in 10.20 is

$$\frac{2}{8\pi}\left\{ 2f^2\int\frac{dl}{Rr^3B^2} + 2f^2\int\frac{B_z\,dl}{r^4\,B^3} - f^3f'\int\frac{dl}{r^4B^3} - ff'\int\frac{dl}{r^2B} - 4\pi p'f^2\int\frac{dl}{r^2B^3}\right\} \tag{10.21}$$

Now

$$\mu = \frac{B_\theta}{rB} = f/r^2B$$

$$\mu' = \frac{\partial\mu}{\partial\psi}$$

$$= \frac{\partial}{\partial\psi}\left(f/r^2B\right)$$

$$= \frac{f'}{r^2B} + \frac{f}{rB}\frac{\partial}{\partial n}\left(\frac{1}{r^2B}\right)$$

Where 10.5 has been used to replace $\frac{\partial}{\partial\psi}$ by $\frac{1}{rB}\;\frac{\partial}{\partial n}$ When 10.6 is used to express $\frac{\partial}{\partial n}$ the above equation becomes

$$\mu' = \frac{f'}{r^2B} + \frac{f}{rB^2}\left(B_z\frac{\partial}{\partial r} - B_r\frac{\partial}{\partial z}\right)\frac{1}{r^2B}$$

$$= \frac{f'}{r^2B} - \frac{2f\,B_z}{r^4B^3} + \frac{f}{r^3B^2}\left(B_z\frac{\partial}{\partial r} - B_r\frac{\partial}{\partial z}\right)\frac{1}{B}$$

Then using 7.14 to evaluate the last term

$$\mu' = \frac{f'}{r^2 B} - \frac{2 f B_z}{r^4 B^3} + \frac{f}{r^3 B^2}\left(\frac{(\nabla \times B)_\theta}{B} - \frac{1}{R}\right)$$

i.e.

10.22 $$\frac{f}{R r^3 B^2} = \frac{f'}{r^2 B} - \frac{2 f B_z}{r^4 B^3} + \frac{4\pi p' f}{r^2 B^3} + \frac{f^3 f'}{r^4 B^3} - \mu'$$

Then using this equation to eliminate all the terms except $f^2 \int \frac{dl}{R r^3 B^2}$ in 10.21. gives the following expression for the co-efficient of ξ in 10.20

10.23 $$\frac{1}{4\pi}\left\{ f^2 \int \frac{dl}{R r^3 B^2} - f \int \mu' \, dl \right\} = \frac{f}{4\pi}\left\{ \int \left(\frac{\mu}{R r B} - \mu'\right) dl \right\}$$

and according to 10.7 this is zero.

Thus 10.20 has reduced to

10.24 $$\left(\frac{\Delta E}{\phi}\right)_{min} = \xi^2 \frac{\{p' K + \gamma p (I + f f' M)\}\{I + f f' M - 4\pi p' H\}}{K + 4\pi \gamma p H}$$

As with 7.19 this expression will be positive if either of two inequalities is satisfied

$$I + f f' M < 4\pi p' H \qquad \text{or} \qquad I + f f' M > -\frac{p'}{\gamma p} K$$

i.e.

10.25 $$\int \frac{dl}{R r B^2} > 2\pi p' \int \frac{dl}{B^3} - 2\pi p' \int \frac{dl'}{(B')^3} + \frac{f f'}{2} \int \frac{dl}{r^2 B^3}$$

or

10.26 $$\int \frac{dl}{R r B^2} < 2\pi p' \int \frac{dl}{B^3} + \frac{p'}{2\gamma p} \int \frac{dl}{B} + \frac{f f'}{2} \int \frac{dl}{r^2 B^3}$$

10.25 may be rearranged as

$$\int \frac{dl}{R r B^2} > 2\pi p' \int \frac{dl}{B} \left(\frac{1}{B^2} - \frac{1}{B^2 + B_\theta^2} \right) + \frac{f f'}{2} \int \frac{dl}{r^2 B^3}$$

$$= 2\pi p' \int \frac{B_\theta^2 \, dl}{B^3 (B^2 + B_\theta^2)} + \frac{f f'}{2} \int \frac{dl}{r^2 B^3}$$

Since 10.21 is zero at all points in the plasma it follows that the integrands are zero, and therefore that

10.27 $$\frac{2f}{R r B} + \frac{2 f B_z}{r^2 B^2} - \frac{f^2 f'}{r^2 B^2} - f' - \frac{4\pi p' f}{B^2} = 0$$

$$\therefore f' \frac{B_\theta^2 + B^2}{B^2} = 2f \left\{ \frac{1}{R r B} + \frac{B_z}{r^2 B^2} - \frac{2\pi p'}{B^2} \right\}$$

Substituting this into 10.25 and 10.26 we get

$$\int \frac{dl}{R r B^2} > 2\pi p' \int \frac{B_\theta^2 \, dl}{B^3 (B^2 + B_\theta^2)} + \int \frac{B^2 f^2 \, dl}{r^2 B^3 (B^2 + B_\theta^2)} \left\{ \frac{1}{R r B} + \frac{B_z}{r^2 B^2} - \frac{2\pi p'}{B^2} \right\}$$

$$\therefore \int \frac{dl}{R r B^2} - \int \frac{dl}{R r B^2} \frac{B_\theta^2}{B^2 + B_\theta^2} > \int \frac{f^2 B_z \, dl}{r^4 B^3 (B_\theta^2 + B^2)}$$

i.e.

10.28 $$\int \frac{dl}{Rr(B^2+B_\theta^2)} > \int \frac{B_\theta^2 B_z \, dl}{r^2 B^3 (B^2+B_\theta^2)}$$

Applying 10.27 to 10.26 we get

$$\int \frac{dl}{Rr(B^2+B_\theta^2)} < 2\pi p' \int \frac{dl}{B^3} + \frac{p'}{2\gamma p}\int \frac{dl}{B} - 2\pi p' \int \frac{B_\theta^2 \, dl}{B^3(B^2+B_\theta^2)} + \int \frac{B_\theta^2 B_z \, dl}{r^2 B^3 (B^2+B_\theta^2)}$$

10.29 $$\int \frac{dl}{Rr(B^2+B_\theta^2)} < 2\pi p' \int \frac{dl}{B(B^2+B_\theta^2)} + \frac{p'}{2\gamma p}\int \frac{dl}{B} + \int \frac{B_\theta^2 B_z \, dl}{r^2 B^3 (B^2+B_\theta^2)}$$

A particular case of some interest is the cylindrical plasma with field $\underset{\sim}{B}' = (0, B_\theta(r), B_z(r))$, (no z dependence since $B_r = -\frac{1}{r}\frac{\partial \psi}{\partial r} \equiv 0$).

This can be obtained from 10.1 by allowing

$B_r \rightarrow 0$, in which case $R \rightarrow \infty$, $B \rightarrow B_z$, $\mu \rightarrow B_\theta / r B_z$, $\frac{\partial}{\partial n} \longrightarrow \frac{\partial}{\partial r}$ and therefore $\frac{\partial}{\partial \psi} \rightarrow \frac{1}{r B_z}\frac{\partial}{\partial r}$

5.11 now reads

$$-\frac{1}{r}\frac{dg}{dr} = 4\pi p' + \frac{f f'}{r^2}$$

where $g \equiv B_z$

i.e.

10.30 $$4\pi p' = -\frac{\dot{g}}{r} - \frac{f\dot{f}}{r^3 g}$$

where $\dot{f}, \dot{g}$ are derivatives with respect to r.

10.7 now reads

$\mu' = o$. and therefore $\dot{\mu} = o$

i.e.

10.31 $$\frac{\dot{f}}{f} - \frac{\dot{g}}{g} - \frac{2}{r} = 0$$

and the inequality 10.29 reduces to

$$0 < \frac{4\pi p'}{2}\left\{\frac{1}{g(g^2+f^2/r^2)} + \frac{1}{4\pi\gamma p g}\right\} + \frac{f^2}{r^4 g^2(g^2+f^2/r^2)}$$

eliminating p' by using 10.30 this reduces to

$$0 < -\frac{1}{rg^2}\left\{\dot{g}g + \frac{\dot{f}f}{r^2}\right\}\left\{\frac{4\pi\gamma p + g^2 + f^2/r^2}{4\pi\gamma p(g^2 + f^2/r^2)}\right\} + \frac{2f^2/r^2}{r^2 g^2(g^2+f^2/r^2)}$$

From 10.31

$$\dot{g} = g\left(\frac{\dot{f}}{f} - \frac{2}{r}\right) = \frac{gr^2}{f}\frac{d}{dr}\left(\frac{f}{r^2}\right)$$

$$\therefore \left\{\frac{g^2 r}{f}\frac{d}{dr}\left(\frac{f}{r^2}\right) + \frac{f}{r}\frac{d}{dr}\left(\frac{f}{r^2}\right) + \frac{2f^2}{r^4}\right\}\left\{\frac{4\pi\gamma p + g^2 + f^2/r^2}{4\pi\gamma p}\right\} - \frac{2f^2/r^2}{r^2} < 0$$

i.e.

$$\left\{ \frac{B_\theta^2 + B_z^2}{B_\theta} \frac{d}{dr}\left(\frac{B_\theta}{r}\right)\right\}\left\{\frac{4\pi\gamma p + B_\theta^2 + B_z^2}{4\pi\gamma p}\right\} + \frac{2B_\theta^2(B_\theta^2+B_z^2)}{r^2 \cdot 4\pi\gamma p} < 0$$

i.e.

10.32 $$B_\theta \frac{d}{dr}\left(\frac{B_\theta}{r}\right) + \frac{2B_\theta^4}{r^2(B_\theta^2+B_z^2+4\pi\gamma p)} < 0$$

This is identical with 3.9, a criterion which was obtained from the energy principle (2).

§11. Summary.

Section §5 consists of an investigation of the properties of the 'mirror machine' field with or without an azimuthal component. Functions $\psi = \psi(r, z)$ (the stream function), and $f = f(\psi)$ are introduced, which define the field uniquely and are constrained to satisfy 5.11. It is found that the lines of force of the magnetic field lie in the surfaces of constant ψ, and that the pressure is constant over these surfaces.

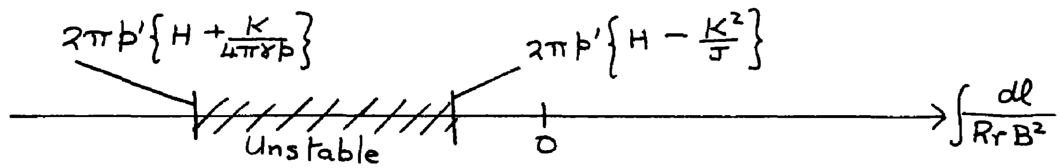
In section §6 the thermodynamic approach to stability in a plasma, used by Rosenbluth and Longmire, is introduced. For low pressure ($\beta \ll 1$) it is shown that the most dangerous perturbations of the plasma (as regards stability) are those which do not distort the magnetic field, and that the 'fluted ' interchange(6b) in which neighbouring tubes of equal flux are interchanged is such a perturbation. The change in potential energy of the plasma (6.1) is evaluated for such an interchange in 6.5 and 6.10, and the stability criterion obtained by Rosenbluth and Longmire stated as

3.10 $$\int \frac{dl}{R_r B^2} > 0$$

where the integration is along a line of force in the plasma, and R is the radius of curvature of that line.

In §7 the assumption of infinitesimal pressure ($\beta \ll 1$) is removed, and a stability criterion is established for the same perturbation as that discussed in §6, i.e. one for which neighbouring tubes of flux (not necessarily equal fluxes now) are interchanged, the magnetic field carrying with it the plasma, because of the 'frozen in' field which is a consequence of the assumption of infinite conductivity, while the remainder of the plasma external to these tubes remains undistorted. It is no longer certain that this is the most dangerous type of instability, so that although the criterion obtained is sufficient for stability against this particular interchange it may not be a sufficient condition for stability in general. Neither is the condition a necessary one unless it is assured that such an interchange is permitted by the equations of motion. Since, however, the results of section §8 are identical to those of Bernstein et al.(2) whose method permits only those perturbations which are physically possible (only displacement vectors $\underset{\sim}{\xi}(x, t)$ which satisfy the equations of motion), it appears that the interchanges visualised in §6 - §10 are indeed dynamically possible and therefore that all the criteria produced in these sections are, at least, necessary for stability. The conditions for instability obtained in §7 is

$$2\pi p'\left\{\int\frac{dl}{B^3} - \frac{\left[\int\frac{dl}{B}\right]^2}{\int B\,dl}\right\} > \int\frac{dl}{R r B^2} > 2\pi p'\left\{\int\frac{dl}{B^3} + \frac{1}{4\pi\gamma p}\int\frac{dl}{B}\right\}$$



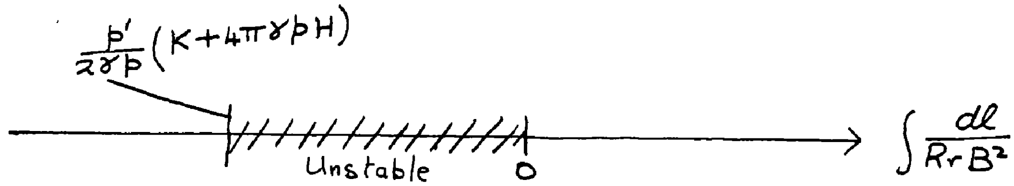
Section §8 consists of generalisations of the theory and methods of §7. The pressures in the perturbed tubes are allowed to vary along the lengths of these tubes, although still obeying the adiabatic law 2.3 locally. The energy change is calculated and found to be minimised by choosing constant pressures after interchange in the two tubes, and this is achieved by applying 2.3 over the complete volumes of the tubes. This is of course consistent with the hydrodynamic approach to the problem. Since the criteria of §7 are slightly less stringent than those obtained from the Princeton energy principle (2) it is clear that the perturbation visualised in §7 is not the most dangerous possible. The greatest restriction on the interchange in §7 is that the tubes are constrained to retain their shapes (i.e. volumes and cross-sections) so that the energy integral (6.1) over the total plasma volume may be replaced by two integrals along the infinitesimal tubes. This restriction is removed in § 8, and distortion of field and plasma external to the two original tubes (1 and 2) is contained within two neighbouring tubes (3 and 4), which are eventually allowed to become of relatively infinite volume. Plasma external to these four tubes must now remain undisturbed by the interchange, and this condition places constraints on the combined volumes and cross sections of the perturbed tubes (1 + 3) and (2 + 4), without constraining the original pair (1,2) directly. (see diagrams 8a, 8b).

The potential energy change, Δ E, can now be minimised with respect to variations of the volumes and cross-sections of tubes 1 and 2 subject to these constraints. This results in the original expression for Δ E (a quadratic form in η, ξ) being replaced by the expression obtained by Bernstein et al viz.:

$$\Delta E = (\tau B D)^2 \left\{ \frac{(I - 4\pi p' H)(p' K + \gamma p I)}{K + 4\pi \gamma p H} \right\}$$

where D is the displacement perpendicular to the field lines, and I,H,K are integrals along the field lines. This yields the following criterion for instability

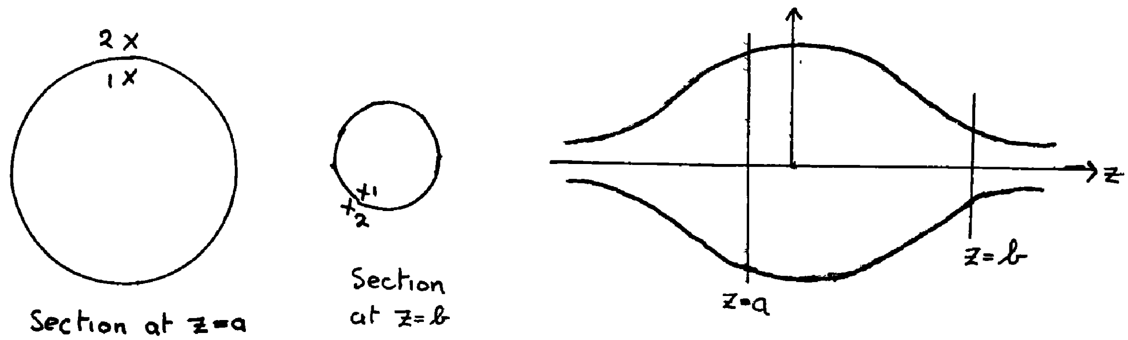
$$0 > \int \frac{d\ell}{R_r B^2} > 2\pi p' \left\{ \int \frac{d\ell}{B^3} + \frac{1}{4\pi\gamma p} \int \frac{d\ell}{B} \right\}$$



Section §9 is occupied by an investigation of the validity of the methods used in §6 - §8, and it is shown that (1) the theory of §6, §7 holds for 'fluted' interchanges of all modes ($m \neq 0$) and that (2) the theory of §8 holds for $m \to \infty$, and that for finite m

$$(\Delta E)_m \geqslant (\Delta E)_{m=\infty}$$

Finally, in §10 the methods of §8 are applied to a magnetic field with an azimuthal component. It is shown that the interchange is only possible when $\frac{\mu'}{\mu} = \frac{1}{R_r B}$ where $\mu = B_\theta / rB$ and $\mu' = \frac{d\mu}{d\psi}$, a condition arising from the requirement that two tubes, adjacent at some point in the plasma will continue to be adjacent along their lengths, i.e. that they have equal θ co-ordinates along their lengths



If this condition is not satisfied the interchange of complete flux tubes is no longer possible, so that stability may be enhanced by the addition of a B_θ field. For a field satisfying the additional condition $\mu'/\mu = 1/R_r B$ the criterion for instability is

$$\int \frac{B_\theta^2 B_z\, dl}{r^2 B^3 (B^2+B_\theta^2)} > \int \frac{dl}{R_r (B^2+B_\theta^2)} > 2\pi p' \left\{ \int \frac{dl}{B(B^2+B_\theta^2)} + \frac{1}{4\pi\gamma p}\int \frac{dl}{B} \right\} + \int \frac{B_\theta^2 B_z\, dl}{r^2 B^3 (B^2+B_\theta^2)}$$

For a cylindrical field $\underset{\sim}{B} = (0, B_\theta(r), B_z(r))$ the criterion for stability, obtained from the above is

$$B_\theta \frac{d}{dr}\left(\frac{B_\theta}{r}\right) + \frac{2 B_\theta^4}{r^2 (B_\theta^2 + B_z^2 + 4\pi\gamma p)} < 0$$

The additional condition to be satisfied in this case is $d\mu/dr = 0$. If this does not hold, Suydam's criterion applies.

Possible Extensions of These Methods.

The theory as presented here, although applied to a 'mirror machine' type of field cannot in fact describe a mirror machine, because in the hydrodynamic approach there is no containment along the lines of force (p = constant). The C-G-L equations, in spite of the shortcomings of the theory from which they arise, could be applied to the method used in this paper to give some information regarding the importance of the tensor pressure. This information should be the same as that which could be obtained from the Princeton Energy Principle (2) for tensor pressure since this contains the same assumption of zero heat flow along the lines of force as the C-G-L theory. The energy integral 6.1 would be replaced by $\int\left(\frac{B^2}{8\pi} + p_\perp + \frac{1}{2}p_\parallel\right)d\tau$ and the C-G-L equations, replacing the scalar pressure adiabatic law $\frac{d}{dt}\left(p/\rho^\gamma\right) = 0$ would be used to calculate the two components of the pressure ($p_\perp$, $p_\parallel$) in the flux tubes after interchange.

The only stability condition available at present showing the roles played by $p_\perp$ and $p_\parallel$ is that obtained by Rosenbluth and Longmire from a consideration of individual particle energies, and is applicable only to the boundaries of the plasma, being only a first order calculation . This condition is

$$\int \frac{p_\perp + p_\parallel}{R \tau B^2} d\ell > 0$$

which probably only represents half of the true stability condition. (c.f. 8.42 and 8.43)

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