TWO NON-PERTURBATION TREATMENTS OF DOUBLE MESON-NUCLEON SCATTERING.

by

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CHAPTER I.

1.1 Introduction.

In recent years much experimental data has been collected on the properties of π-mesons and their interaction with nucleons. A great deal of theoretical work has been carried out trying to find some unified interpretation of these experimental results on the interaction of π-mesons and nucleons; however, no adequate theoretical treatment of the problem has yet been found. This is mainly due to the strength of this interaction which prevents its treatment as a small perturbation of the free particle states. Several non-perturbation types of approximation have been suggested for dealing with the problem and it is with the application of two of these which we shall be concerned.

It is well established from the experimental results that π-mesons have zero spin and odd intrinsic parity (a comprehensive review of these experiments and their interpretation is given in reference 1). They are therefore Bose particles and are described by a pseudoscalar field. Nucleons, on the other hand, are Fermi particles of spin $\frac{1}{2}$ and are described by the Dirac field. Of the many types of coupling possible between these fields, two have been studied most extensively; they are pseudoscalar (ps) coupling and pseudovector (pv) coupling and both have the simplifying feature that the interaction hamiltonian is linear in the meson field. It has been shown (30) that, as far as a perturbation treatment is valid, the theory with ps-coupling is renormalisable whereas that with pv-coupling is not, so that, in general, unambiguous finite results cannot be obtained using pv-coupling. We shall therefore concentrate our attention on the theory with ps-coupling and furthermore we shall take the interaction to be charge-symmetric (26) which maintains the charge independence of nuclear forces and ensures that the total isotopic spin is a constant of the motion.

Of the various approximation methods which have been suggested for solving the equations of this theory and which do not assume a weak coupling between the fields, two, which have met with some success, are those proposed by Tamm (35) and Dancoff (12) and by Cini and Fubini (9).

The method of Tamm and Dancoff - TD method - has been applied to the problem of meson-nucleon scattering by various authors (7),(16),(23),(28) but the only treatment which rigorously uses the charge symmetric pseudoscalar interaction hamiltonian is that by Dyson et al (16). They apply the method in lowest approximation and, neglecting all self-energy effects, calculate the elastic scattering phase shifts. Although the results which they obtain have no quantitative agreement with the phase shifts determined from the experimental results, the qualitative agreement is greatly increased over that obtained from a lowest order perturbation theory calculation. This leads one to feel that the basic ideas behind the method are sound and that, by carrying out a calculation to a higher order of approximation in the method, results would be obtained which would agree well with the corresponding experimental results.

The application of the method of Cini and Fubini - CF method - to meson-nucleon scattering has been carried out by Sartori and Wataghin (33). The lowest approximation has again been used and the nucleon treated non-relativistically so that a cut-off momentum has to be introduced to obtain finite results. By a suitable choice of the coupling constant and this cut-off, they obtain very good agreement with the experimental results for the important p-wave scattering phase shift - α_{33}- and rough qualitative agreement with the other experimental phase shifts.

The partial success of these two methods seems to indicate two things. Firstly, the pseudoscalar charge-symmetric interaction hamiltonian is not inconsistent with experiment and secondly, each of the two approximation methods has some degree of validity.

To investigate further the theory and also the validity of these approximation methods, it was thought useful to apply the theory, using these methods, to another process involving the interactions of mesons and nucleons. The process considered is that of meson production in meson-nucleon collisions i.e. $\pi + N \rightarrow \pi + \pi + N$ sometimes called double meson scattering. This process has a threshold energy when the rest energy of a meson is available in the centre-of-mass system of the original meson and nucleon; this occurs when the incident meson has a kinetic energy of just over 170 Mev in the laboratory system in which the initial nucleon is at rest. The production process is clearly closely connected to elastic meson-nucleon scattering, the two being competing processes when the incident meson has a kinetic energy greater that 170 Mev. By evaluating the cross section for double scattering using both the TD and CF approximation methods, further checks on the theory with ps-coupling and on the approximation methods are made available.

In the remaining part of this chapter, we shall discuss the experimental results which are available on doubling scattering and also the theoretical work which has already been done on the problem.

In Chapter II, the TD method and its application to double scattering is discussed. This work was originally undertaken using the boundary conditions discussed by Dyson (15) and the main conclusions concerning the presence of non-physical singularities were arrived at from the equations obtained with these conditions. However, since then a paper by Dyson and Dalitz (17) on meson-nucleon scattering has become available in preprint form. In this, mistakes are pointed out in the boundary conditions used by Dyson (15) and the correct conditions are formulated. Our equations have subsequently been modified by the use of these corrected boundary conditions and are now similar to the equations of Dyson and Dalitz. This modification of the boundary conditions simplifies the final set of equations but does not alter the final conclusions which had already been reached.

Chapter III is devoted to a discussion of the CF method and its application to the problem.

1.2 Experimental results on double scattering.

The first double scattering event to be observed (6) was found in a photographic plate which had been exposed to cosmic radiation at a high altitude; the incident meson had an energy of about 1 Bev and the produced mesons and proton had energies of about 375, 365 and 270 Mev respectively.

Fry (22) found another event of this type in a photographic emulsion which had been exposed to a laboratory produced beam of 220 Mev negative mesons. This appears to be the lowest energy at which double scattering has so far been observed.

Blau, Caulton and Smith (4),(5) have carried out quite an extensive investigation of the interactions of 500 Mev negative mesons with the nuclei of photographic emulsions. From their results, they estimate that the cross-section for the production of charged mesons from the collision of negative mesons and nucleons lies somewhere between 3.5 and 10 millibarns at this energy. They also find that in the centre-of-mass system, the produced mesons tend to come off in the backward direction and the nucleons in the forward direction.

Recently, experiments have been carried out using the 1.5 Bev negative meson beam of the Brookhaven cosmotron. The nuclear interactions of these mesons have been investigated using both photographic emulsions (11) and diffusion cloud chambers (18). At this high energy, as well as elastic and double meson scattering, various other processes are possible involving the production of larger numbers of π-mesons and also the production of heavy mesons. It is found that the total negative meson-nucleon interaction cross-section at this energy is about 35 millibarns whereas the elastic scattering cross-section is only 10 millibarns. Thus, in most of the collisions production of some type takes place. In those production events which were analysed in detail, it was found (18) that about 80% of the events resulted in double scattering, and the remaining 20% in the production of two extra mesons. Contrary to the results at 500 Mev, the final nucleons in the double scattering events were found to have a tendency to come off in the backward direction in the centre-of-mass system.

These are all the experimental results which are available at present. They do not give a very clear picture of the behaviour of the double scattering cross-section except in so far as it appears to rise from zero at threshold to some value larger than the elastic scattering cross-section around 1.5 Bev.

1.3 Previous theoretical work on double scattering.

A calculation has been carried out by Nelkin and Bethe (32) on the relative magnitudes of the double scattering and elastic scattering cross-sections. However, the details of the calculations are not clear as it has been published, so far, only in abstract form. They use the lowest order TD approximation and it would appear that they neglect all the contributions to their equations from self-energy terms. Making some approximations as to the smallness of the production process relative to the elastic scattering enables them to calculate the ratio of the production to the elastic scattering cross-sections; they find, among other numerical results which are not given, that this ratio is less than 1% at an incident meson energy of 400 Mev. In the discussion in Chapter II of the application of the TD method to the problem, it is shown that it is not valid to neglect the contributions from nucleon self-energy terms in the calculation of the double scattering cross-section. If Nelkin and Bethe have, as it appears, neglected self-energy terms in their equations, little significance can be attached to their results.

Two calculations have been carried out in each of which the angular distribution of the produced particle, obtained from various types of coupling of the meson and nucleon fields, are compared. Kovacs (27) has compared the angular correlation of the two emitted mesons for the two types of interaction when the meson-nucleon interaction is much stronger and much weaker than the meson-meson interaction. In the first case, the incident meson interacts directly with the nucleon which then emits two mesons; in the second case, the incident meson interacts with the virtual mesons surrounding the nucleon and two mesons are then produced by this mechanism. Kovacs has carried out a calculation at an incident meson energy of 1 Bev using scalar theory i.e. scalar mesons with scalar coupling. He finds in both cases that the angle between the two mesons tends to be small, this tendency being much stronger in the case of strong meson-meson interaction. Miyachi (31) has calculated the angular distribution of the π^+-meson relative to the incident π^--meson in the reaction $\pi^- + p \rightarrow \pi^+ + \pi^- + n$ at an incident meson energy of 210 Mev. He treats the nucleons non-relativistically and first of all compares the results of lowest order perturbation theory assuming ps-coupling and pv-coupling.

For ps-coupling, the π^+-meson tends to make a large angle θ with the direction of the incident meson whereas, for pv-coupling, the angular distribution is symmetrical about $\theta = \frac{\pi}{2}$ and is $(2+\cos^2\theta)$. As Miyachi points out, not much significance can be attached to these lowest order perturbation theory results, and he attempts to improve on them by considering the production as taking place in two steps. First of all a scattering takes place between the incident meson and nucleon which is followed by the nucleon emitting the second meson. For the first step, he uses the scattering matrix element calculated by Chew (7),(8) and assumes that the second step takes place through pv-coupling. The angular distribution obtained by this method is rather similar to that obtained from the lowest order pv-coupling calculation but more isotropic. At an energy of 210 Mev which is only 40 Mev above the threshold for production, one would expect that the mesons, having a very low energy, would be produced predominantly in s-states so that an almost isotropic distribution, as obtained by this calculation of Miyachi, seems likely to be correct.

The most extensive piece of theoretical work so far carried out on the problem has been done by d'Espagnat (19). He attempts to investigate as much of the general nature of the problem as he can without making use of any of the approximations of field theory. He does this by drawing an analogy between the processes of elastic and double meson-nucleon scattering and the resonance theory of nuclear reactions. By assuming that the production process is much smaller than the elastic scattering, which is certainly true near the production threshold, he is able to cast the formulae for the production and elastic scattering cross-sections into forms similar to the formulae which arise in the nuclear theory. For this analogy to be true, he finds that for those energies at which the elastic scattering goes through a resonance, the production cross-section must go through a maximum. From this he deduces that if the resonance in the elastic scattering at a certain energy is due to a resonance in a certain state of known angular momentum, parity and isotopic spin, then around this energy the production takes place predominantly through this same state. However, it is now almost certainly established that meson-nucleon scattering through the state of angular momentum $\frac{3}{2}$, even parity and isotopic spin $\frac{3}{2}$ has a resonance at about an incident meson energy of 190 Mev. d'Espagnat's results imply that, around this energy, the meson production takes place predominantly through this state; this is not consistent with the view that, around this energy which is only about 20 Mev above the production threshold, the mesons, having very low energies are produced almost wholly is s-states i.e. the production takes place through the state of angular momentum $\frac{1}{2}$ and even parity. This would seem to cast some doubt on the validity of drawing this analogy between meson-nucleon reactions and nuclear reactions.

Still making use of the assumption that production is much less probable than elastic scattering, d'Espagnat develops a relationship between the matrix element for production and the elastic scattering phase shifts, which he takes as known from experiment. However, to investigate the consequences of this relationship, he is forced to make some approximation restricting the types of intermediate states possible in the production process. By considering the contributions from the various terms, in which different intermediate states occur, of the lowest order perturbation theory matrix element for the production process $\pi^+ + p \rightarrow \pi^+ + \pi^+ + n$ and, by making use of the experimentally determined elastic scattering phase shifts, he maintains that the largest contribution to the matrix element for production comes from that part which corresponds to the following order of processes: scattering takes place between the incident meson and the proton and finally the proton emits an extra meson. Taking only this process into account, formulae for the differential and total production cross-sections in terms of the elastic scattering phase shifts are derived. The only result obtained by d'Espagnat from these formulae is the order of magnitude and energy dependence of the ratio of the production to the elastic scattering cross-sections. This ratio he compares with the same ratio calculated from the statistical theory of Fermi (20) which we shall discuss at the end of Chapter III. He finds that the two ratios agree in order of magnitude but that the ratio, determined from the statistical theory, increases more quickly with energy than does his.

CHAPTER II.

2.1 Formulation of the Tamm Dancoff method.

The approximation method as originally proposed by Tamm (35) and Dancoff (12) for solving problems involving interacting fields is based on the following idea. Suppose a system of interacting fields, which is described by the hamiltonian $H = H_0 + H'$ where H_0 is the hamiltonian of the free fields and H' the interaction hamiltonian, is in an eigenstate $|\Psi)$ of energy E i.e.

$$(H_0 + H')|\Psi) = E|\Psi) \tag{2.1}$$

Since H_0 is a hermitian operator, its eigenfunctions $|\Phi_n)$, $n = 0,1,2...$ form a complete orthonormal set, $|\Phi_n)$ being the state vector describing a state containing n free particles of total energy E_n — $H_0|\Phi_n) = E_n|\Phi_n)$. Thus, $|\Psi)$ can be expanded in terms of the $|\Phi_n)$ as

$$|\Psi) = \sum_n a(n)|\Phi_n) \tag{2.2}$$

where a(n) is the probability amplitude for finding the system containing the n free particles specified by $|\Phi_n)$ if it is first put into the state $|\Psi)$ and the interaction is then switched off.

Introducing (2.2) into (2.1), we obtain

$$(E - E_n)a(n) = \sum_m (\Phi_n|H'|\Phi_m)\, a(m) \tag{2.3}$$

which form an infinite set of coupled integral equations for the amplitudes a(n).

The TD method consists of approximating to this infinite set of equations by a finite set obtained by setting equal to zero all amplitudes for states containing more than N particles; the fundamental hypothesis behind this method is that, if N is large enough, the results will be insensitive to the value of N and will tend to some finite limit, which is the solution of the infinite set of equations (2.3), as N tends to infinity. (2.3) now becomes a finite set of (N+1) coupled integral equations for the amplitudes a(0), a(1)---a(N) which can, in principle, be solved rigorously.

However, making this approximation leads to a serious difficulty which is connected with the self-energy of the vacuum. Every state $|\Psi\rangle$ of the interacting fields contains a large number of particles which are continually being created and destroyed in the vacuum. Restricting the total number of particles to N sets up an artificial correlation between these vacuum fluctuations at points widely separated in space; this appears in the equations as a spurious effect of the vacuum fluctuations, which are, in general, badly divergent quantities, on the behaviour of the real particles. Dyson (14) has suggested the following modification of the TD method which overcomes this difficulty and which has other advantages over the original method which will be discussed later.

Let $|\bar{\Psi}_0\rangle$ be the vacuum state of the interacting fields with energy E i.e. $(H_0 + H')|\bar{\Psi}_0\rangle = E_0|\bar{\Psi}_0\rangle$ where E_0 is the lowest eigenvalue of H; let A(n) be the product of free particle annihilation operators for the particles specified by n and C(n') a similar product of free particle creation operators for the particles specified by n'. We now consider the quantity $\langle \bar{\Psi}_0 | C(n') A(n) | \bar{\Psi} \rangle$ in place of the original TD amplitude a(n). For the purpose of comparison, we note that, if $|\Phi_0\rangle$ is the vacuum state of the free fields, then $|\Phi_n\rangle = C(n)|\Phi_0\rangle$ so that from equation (2.2) the original TD amplitude can be expressed as

$$a(n) = \langle \Phi_n | \bar{\Psi} \rangle = \langle \Phi_0 | A(n) | \bar{\Psi} \rangle$$

Thus, in the amplitude a(n), the physical state $|\bar{\Psi}\rangle$ is described in terms of the bare particle state $|\Phi_0\rangle$, whereas, in the new amplitude $\langle \bar{\Psi}_0 | C(n') A(n) | \bar{\Psi} \rangle$, it is described in terms of the real state $|\bar{\Psi}_0\rangle$. $\langle \bar{\Psi}_0 | C(n') A(n) | \bar{\Psi} \rangle$ is interpreted as being the amplitude for finding n' free particles more and n free particles less in $|\bar{\Psi}\rangle$ than in $|\bar{\Psi}_0\rangle$. From the Schroedinger equations

$$H|\bar{\Psi}\rangle = E|\bar{\Psi}\rangle \quad \text{and} \quad H|\bar{\Psi}_0\rangle = E_0|\bar{\Psi}_0\rangle$$

it follows that

$$(\epsilon - E_n + E_{n'})\langle \bar{\Psi}_0 | C(n') A(n) | \bar{\Psi} \rangle = \langle \bar{\Psi}_0 | \left[C(n') A(n), H' \right] | \bar{\Psi} \rangle \qquad (2.4)$$

where E and E are the energies of the n and n' free particles respectively and $\epsilon = E - E_0$. The interaction hamiltonian H' can be expressed in terms of free particle creation and annihilation operators so that, after some manipulation, the commutator in (2.4) can be expressed as a sum of terms each of which is in normal order i.e. all creation operators lying to the left of all annihilation operators;

the right side is then expressed in terms of the new TD amplitudes. Thus, like (2.3), (2.4) is an infinite set of coupled integral equations which can be made finite by the TD approximation which we have already described.

In the case of the pseudoscalar charge-symmetric interaction hamiltonian $H' = ig\int d^3x\, \bar{\psi}(x)\gamma_5\tau_\alpha\phi_\alpha(x)\psi(x)$, where the notation is standard, the vacuum self-energy terms, which cause difficulty in (2.3), arise from the fact that, in (2.3) H' can create three particles at a point with two arbitrary momenta and subsequently annihilate these same three particles; the integration over the two free momenta leads to divergence difficulties. However, in (2.4) one of the particles created or annihilated by H' must belong to the set n or n'; this condition prevents the presence of any vacuum self-energy terms in the equations.

Equations (2.3) and (2.4) also differ in the fact that the energy appearing in (2.4) is the physically observable energy of the system since $|\Psi_0\rangle$ is the vacuum state; in (2.3), the energy E is not a physically meaningful quantity as what is observed is always an energy difference.

From now on, we shall work solely in terms of the modified TD method. It should be noted that nowhere in setting up equation (2.4), have we used the fact that $|\Psi_0\rangle$ is the vacuum state. Equation (2.4) holds for any "comparison state" $|\Psi_0\rangle$ and then ϵ is the energy difference between the states $|\bar{\Psi}\rangle$ and $|\Psi_0\rangle$.

2.2 The integral equation for elastic and double scattering.

We shall now set up the formalism for an investigation by the TD method of some of the consequences of assuming a charge-symmetric pseudoscalar interaction between the meson and nucleon fields.

The meson-nucleon system is described by the hamiltonian

$$H = H_0 + H' \tag{2.5}$$

where H_0 is the sum of the free meson and free nucleon field hamiltonians, and

$$H' = ig\int d^3x\, \bar{\Psi}(x)\gamma_5\tau_\alpha\phi_\alpha(x)\psi(x) \tag{2.6}$$

the repeated suffix α being summed over the values 1,2,3. The τ_α are the usual isotopic spin matrices. The $\phi_\alpha(x)$ are the meson field operators; $\phi_1(x)$ and $\phi_2(x)$ describe the charged mesons and $\phi_3(x)$ the neutral mesons. $\psi(x)$ is the nucleon field operator; it is an eight component spinor such that if $\tau^{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2)$

$$\tau^+\psi(x) = \psi_N(x) \quad \text{and} \quad \tau^-\psi(x) = \psi_P(x)$$

where $\psi_N(x)$ and $\psi_P(x)$ are respectively the neutron and proton field operators. Also

$$\tau_3\psi_N(x) = \psi_N(x) \quad \text{and} \quad \tau_3\psi_P(x) = -\psi_P(x)$$

$\bar{\Psi}(x) = \psi^+(x)\gamma_4$ where $\psi^+(x)$ is the hermitian conjugate of $\psi(x)$ and γ_4 and γ_5 are the usual Dirac matrices i.e. $\gamma_4 = \beta$ and $\gamma_5 = i\alpha_1\alpha_2\alpha_3$.

Working in the Schroedinger representation, we can expand the field variables as follows

$$\psi(x) = V^{-\frac{1}{2}}\sum_{p,u} b_u(p)\, u(p)\, e^{ip.x} \tag{2.7}$$

$$\psi^{+}(x) = V^{-\frac{1}{2}} \sum_{p,u} b_u^{*}(p)\, u^{+}(p)\, e^{-ip\cdot x} \tag{2.8}$$

$$\phi_\alpha(x) = \sum_{k} (2V\omega_k)^{-\frac{1}{2}} \left[a_\alpha(k) + a_\alpha^{*}(-k) \right] e^{ik\cdot x} \tag{2.9}$$

The momenta $\underline{p}$ and $\underline{k}$ are summed over the normal frequencies of the large volume V and the spinors u($\underline{p}$) are summed over the four spinors satisfying the equation

$$(\underline{\alpha}\cdot p + \beta M)\, u(p) = \pm E_p\, u(p) \tag{2.10}$$

where $E_p = \sqrt{p^2+M^2}$, M being the nucleon mass and the units being such that $\hbar = c = 1$. The spinors are normalised such that

$$u^{+}(p)\, u'(p) = \delta_{u,u'}\, \delta_{p,p'} \tag{2.11}$$

μ is the meson mass and $\omega_k = \sqrt{k^2+\mu^2}$.

$a_\alpha(k)$ is an operator which destroys a meson of momentum $\underline{k}$ in the charge state α and $a_\alpha^{*}(k)$ creates a similar particle. These operators satisfy the commutation rule

$$\left[a_\alpha(k), a_{\alpha'}^{*}(k') \right] = \delta_{\alpha\alpha'}\, \delta_{k,k'} \tag{2.12}$$

When u($\underline{p}$) is a positive energy spinor, $b_u(p)$ is an operator destroying nucleons described by u($\underline{p}$) and $b_u^{*}(p)$ is an operator creating similar particles; when u($\underline{p}$) is a negative energy spinor, $b_u(p)$ is an operator creating anti-nucleons described by u($\underline{p}$) and $b_u^{*}(p)$ destroys similar particles. These operators obey the anti-commutation rule

$$\left\{ b_u(p), b_{u'}^{*}(p') \right\} = \delta_{u,u'}\, \delta_{p,p'} \tag{2.13}$$

The meson and nucleon operators all commute with one another.

To make use of equation (2.4), we require an expression for H' in terms of creation and destruction operators; this can be obtained by introducing equations (2.7), (2.8) and (2.9) into (2.6). The integration over $\underline{x}$ and the summation over one of the momenta can be carried out, giving

$$H' = g \sum_{\substack{\underline{p},\underline{k} \\ u',u''}} \lambda_{\underline{k}} \left[a_\alpha(\underline{k}) + a_\alpha^*(-\underline{k}) \right] b^*_{u'}(\underline{p}+\underline{k})\, b_{u''}(\underline{p})\, \bar{u}'(\underline{p}+\underline{k})\, \gamma_5 \tau_\alpha u''(\underline{p}) \qquad (2.14)$$

where $\lambda_{\underline{k}} = i\,(2V\omega_{\underline{k}})^{-\frac{1}{2}}$

Let us first of all consider the two particle amplitude $\langle \bar{\Psi}_0 | b_u(-\underline{p})\, a_\alpha(\underline{p}) | \bar{\Psi} \rangle$. Equation (2.4) gives for this amplitude

$$(\epsilon - \omega_{\underline{p}} - \zeta_u(-\underline{p}) E_{\underline{p}}) \langle \bar{\Psi}_0 | b_u(-\underline{p})\, a_\alpha(\underline{p}) | \bar{\Psi} \rangle = \langle \bar{\Psi}_0 | \left[b_u(-\underline{p})\, a_\alpha(\underline{p}), H' \right] | \bar{\Psi} \rangle$$

where $\zeta_u(-\underline{p}) = +1$ for $(-\underline{p}\cdot\underline{\alpha} + \beta M)\, u(-\underline{p}) = +E_{\underline{p}}\, u(-\underline{p})$

-1 for $(-\underline{p}\cdot\underline{\alpha} + \beta M)\, u(-\underline{p}) = -E_{\underline{p}}\, u(-\underline{p})$

Using (2.14) and the commutation relations (2.12) and (2.13), the commutator in this equation can be written as a sum of normal products of the operators so that the right side is expressed in terms of TD amplitudes. We obtain

$$\begin{aligned}
(\epsilon - \omega_{\underline{p}} - \zeta_u(-\underline{p}) E_{\underline{p}}) \langle \bar{\Psi}_0 | b_u(-\underline{p})\, a_\alpha(\underline{p}) | \bar{\Psi} \rangle &= g \lambda_{\underline{p}}\, \theta_u^+(-\underline{p}) \sum_{u'} \left[\bar{u}(-\underline{p})\, \gamma_5 \tau_\alpha u'(0) \right] \langle \bar{\Psi}_0 | b_{u'}(0) | \bar{\Psi} \rangle \\
&+ g \sum_{\underline{k},u'} \lambda_{\underline{k}} \left[\bar{u}(-\underline{p})\, \gamma_5 \tau_\beta u'(-\underline{p}-\underline{k}) \right] \Big[\langle \bar{\Psi}_0 | b_{u'}(-\underline{p}-\underline{k})\, a_\beta(\underline{k})\, a_\alpha(\underline{p}) | \bar{\Psi} \rangle \\
&\qquad + \langle \bar{\Psi}_0 | a_\beta^*(-\underline{k})\, b_{u'}(-\underline{p}-\underline{k})\, a_\alpha(\underline{p}) | \bar{\Psi} \rangle \Big] \\
&+ g \lambda_{\underline{p}} \sum_{\underline{q},u',u''} \left[\bar{u}'(\underline{q}-\underline{p})\, \gamma_5 \tau_\alpha u''(\underline{q}) \right] \Big[\theta_u^+(-\underline{p})\, \theta_{u''}^+(\underline{q}) \langle \bar{\Psi}_0 | b^*_{u'}(\underline{q}-\underline{p})\, b_{u''}(\underline{q})\, b_u(-\underline{p}) | \bar{\Psi} \rangle \\
&\qquad + \theta_u^-(-\underline{p})\, \theta_{u''}^+(\underline{q}) \langle \bar{\Psi}_0 | b_u(-\underline{p})\, b^*_{u'}(\underline{q}-\underline{p})\, b_{u''}(\underline{q}) | \bar{\Psi} \rangle \\
&\qquad - \theta_u^+(-\underline{p})\, \theta_{u''}^-(\underline{q}) \langle \bar{\Psi}_0 | b_{u''}(\underline{q})\, b^*_{u'}(\underline{q}-\underline{p})\, b_u(-\underline{p}) | \bar{\Psi} \rangle \\
&\qquad - \theta_u^-(-\underline{p})\, \theta_{u''}^-(\underline{q}) \langle \bar{\Psi}_0 | b_u(-\underline{p})\, b_{u''}(\underline{q})\, b^*_{u'}(\underline{q}-\underline{p}) | \bar{\Psi} \rangle \Big]
\end{aligned} \qquad (2.15)$$

where $\theta_u^{\pm}(-p) = \frac{1}{2}(1 \pm \zeta_u(-p))$ and, in the derivation, use has been made of the identity $\bar{u}(p)\gamma_5 u(p) \equiv 0$.

Equation (2.15) is one of the infinite set of coupled integral equations obtained from (2.4); the other equations are got by applying (2.4) to the amplitudes appearing on the right side of (2.15). This introduces amplitudes for states with larger numbers of particles which in turn lead to more integral equations.

Symbolically, the structure of this set of integral equations can be seen as follows. We can write (2.15) as

$$(2) \sim (1) + I(3)$$

where (n) represents an n-particle amplitude and I implies an integration or summation over some variable. (2.4) gives

$$(1) \sim I(2) \quad \text{and} \quad (3) \sim (2) + I(4)$$

and, in general

$$(n) \sim (n-1) + I(n+1)$$

The TD approximation is obtained by assuming that $(m) = 0$ for $m > N$ where N is some chosen number of particles.

For $N = 2$, the system of equations reduces to

$$(1) \sim I(2) \quad \text{and} \quad (2) \sim (1)$$

which, by substitution, gives a single equation for the one particle amplitude.

For $N = 3$, we have

$$(1) \sim I(2)\,, \quad (2) \sim (1) + I(3) \quad \text{and} \quad (3) \sim (2)$$

which, by substitution, leads to a single integral equation $(2) - I(2)$ for the two particle amplitude.

Similar considerations show that, for $N = 4$, the system reduces to two coupled integral equations for the two and three particle amplitudes and, as N increases, the number of coupled equations in the final set increases.

For the problem of meson production in meson-nucleon collisions, we must clearly choose $N \geq 3$ since the final state is a three particle state. We shall in fact take $N = 3$ as a first approximation to the problem so that the equations can be examined without the complications introduced by considering higher orders of approximation. We therefore proceed to apply equation (2.4) to the amplitudes appearing on the right side of (2.15) with the restriction that all amplitudes for states containing four or more particles are zero.

For the one nucleon amplitude, we obtain

$$(\epsilon - \mathfrak{I}_{u'}(0)M)(\Psi_0 | b_{u'}(0) | \Psi) = g \sum_{\underline{k}, u''} \lambda_{\underline{k}} \left[\bar{u}'(0) \gamma_5 \tau_\beta u''(-\underline{k}) \right] \Big[(\Psi_0 | b_{u''}(-\underline{k}) a_\beta(\underline{k}) | \Psi)$$
$$+ (\Psi_0 | a^{+}_\beta(-\underline{k}) b_{u''}(-\underline{k}) | \Psi) \Big] \qquad (2.16)$$

For the two meson one nucleon amplitudes we obtain

$$(\epsilon - \omega_{\underline{p}} - \omega_{\underline{k}} - \mathfrak{I}_{u'}(-\underline{p}-\underline{k}) E_{\underline{p}+\underline{k}})(\Psi_0 | b_{u'}(-\underline{p}-\underline{k}) a_\beta(\underline{k}) a_\alpha(\underline{p}) | \Psi)$$
$$= g\, \theta^{+}_{u'}(-\underline{p}-\underline{k}) \sum_{u''} \lambda_{\underline{p}} \left[\bar{u}'(-\underline{p}-\underline{k}) \gamma_5 \tau_\alpha u''(-\underline{k}) \right] (\Psi_0 | b_{u''}(-\underline{k}) a_\beta(\underline{k}) | \Psi)$$
$$+ g\, \theta^{+}_{u'}(-\underline{p}-\underline{k}) \sum_{u''} \lambda_{\underline{k}} \left[\bar{u}'(-\underline{p}-\underline{k}) \gamma_5 \tau_\beta u''(-\underline{p}) \right] (\Psi_0 | b_{u''}(-\underline{p}) a_\alpha(\underline{p}) | \Psi) \qquad (2.17)$$

and $(\epsilon - \omega_p + \omega_k - \zeta_{u'}(-p-k) E_{p+k}) \langle \Psi_0 | a_\beta^*(-k) b_{u'}(-p-k) a_\alpha(p) | \Psi \rangle$

$$= g\, \theta_{u'}^-(-p-k) \sum_{u''}' \lambda_k \left[\bar{u}'(-p-k) \gamma_5 \tau_\beta u''(-p) \right] \langle \Psi_0 | b_{u''}(-p) a_\alpha(p) | \Psi \rangle$$

$$+ g\, \theta_{u'}^+(-p-k) \sum_{u''}' \lambda_p \left[\bar{u}'(-p-k) \gamma_5 \tau_\alpha u''(-k) \right] \langle \Psi_0 | a_\beta^*(-k) b_{u''}(-k) | \Psi \rangle \qquad (2.18)$$

Finally, we consider the three nucleon amplitudes. When $u(-p)$ and $u''(q)$ are both positive energy spinors

$(\epsilon - E_q - E_p + \zeta_{u'}(q-p) E_{q-p}) \langle \Psi_0 | b_{u'}^*(q-p) b_{u''}(q) b_u(-p) | \Psi \rangle$

$$= g \lambda_p\, \theta_{u'}^-(q-p) \left[\bar{u}''(q) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_u(-p) a_\beta(p) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(-p) b_u(-p) | \Psi \rangle \right]$$

$$- g \lambda_q\, \theta_{u'}^-(q-p) \left[\bar{u}(-p) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_{u''}(q) a_\beta(-q) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(q) b_{u''}(q) | \Psi \rangle \right] \qquad (2.19)$$

When $u(-p)$ is a negative energy spinor and $u''(q)$ a positive energy spinor

$(\epsilon - E_q + E_p + \zeta_{u'}(q-p) E_{q-p}) \langle \Psi_0 | b_u(-p) b_{u'}^*(q-p) b_{u''}(q) | \Psi \rangle$

$$= g \lambda_q\, \theta_{u'}^+(q-p) \left[\bar{u}(-p) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_{u''}(q) a_\beta(-q) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(q) b_{u''}(q) | \Psi \rangle \right]$$

$$+ g \lambda_p\, \theta_{u'}^-(q-p) \left[\bar{u}''(q) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_u(-p) a_\beta(p) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(-p) b_u(-p) | \Psi \rangle \right] \qquad (2.20)$$

When $u(-p)$ is a positive energy spinor and $u''(q)$ a negative energy spinor

$(\epsilon + E_q - E_p + \zeta_{u'}(q-p) E_{q-p}) \langle \Psi_0 | b_{u''}(q) b_{u'}^*(q-p) b_u(-p) | \Psi \rangle$

$$= g \lambda_p\, \theta_{u'}^+(q-p) \left[\bar{u}''(q) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_u(-p) a_\beta(p) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(-p) b_u(-p) | \Psi \rangle \right]$$

$$+ g \lambda_q\, \theta_{u'}^-(q-p) \left[\bar{u}(-p) \gamma_5 \tau_\beta u'(q-p) \right] \left[\langle \Psi_0 | b_{u''}(q) a_\beta(-q) | \Psi \rangle + \langle \Psi_0 | a_\beta^*(q) b_{u''}(q) | \Psi \rangle \right] \qquad (2.21)$$

When both u(-p) and u"(q) are negative energy spinors

$$\begin{aligned}
&\left(\epsilon + E_{\underline{q}} + E_{\underline{p}} + \zeta_{u'}(\underline{q}-\underline{p})E_{\underline{q}-\underline{p}}\right)\langle\Psi_0|b_u(-\underline{p})\,b_{u''}(\underline{q})\,b^*_{u'}(\underline{q}-\underline{p})|\Psi\rangle \\
&= g\lambda_{\underline{p}}\,\theta^+_{u'}(\underline{q}-\underline{p})\left[\bar{u}''(\underline{q})\gamma_5\tau_\rho u'(\underline{q}-\underline{p})\right]\left[\langle\Psi_0|b_u(-\underline{p})a_\rho(\underline{p})|\Psi\rangle + \langle\Psi_0|a^*_\rho(-\underline{p})\,b_u(-\underline{p})|\Psi\rangle\right] \\
&\quad - g\lambda_{\underline{q}}\,\theta^+_{u'}(\underline{q}-\underline{p})\left[\bar{u}(-\underline{p})\gamma_5\tau_\rho u'(\underline{q}-\underline{p})\right]\left[\langle\Psi_0|b_{u''}(\underline{q})a_\rho(-\underline{q})|\Psi\rangle + \langle\Psi_0|a^*_\rho(\underline{q})\,b_{u''}(\underline{q})|\Psi\rangle\right]
\end{aligned} \tag{2.22}$$

The equations (2.16) - (2.22) can now be used to eliminate the amplitude appearing on the right side of equation (2.15) so that an equation is obtained containing only two particle amplitudes. However, before this can be done, the behaviour of these amplitudes, which are to be eliminated, must be investigated; in particular, their behaviour and interpretation must be studied at those values of the momentum variables for which the energy factors multiplying them in equations (2.16) - (2.22) are zero.

In equation (2.18), when $u'(-\underline{p}-\underline{k})$ is a positive energy spinor, there is a possibility that the amplitude $\langle\Psi_0|a^*_\rho(-\underline{k})\,b_{u'}(-\underline{p}-\underline{k})\,a_\alpha(\underline{p})|\Psi\rangle$ has a singularity, since there exist values of $\underline{p}$ and $\underline{k}$ for which

$$\epsilon - \omega_{\underline{p}} + \omega_{\underline{k}} - E_{\underline{p}+\underline{k}} = 0 \tag{2.23}$$

Such a singularity corresponds to the existence of real free particles whose momenta are those given by the solutions of equation (2.23), since for these momenta, if the right side of (2.18) is non-zero, the coordinate space transform of the amplitude is finite at infinity. In this case, the singularity corresponds to the existence of a real meson and nucleon in $|\Psi)$ and a real meson in the "comparison state" $|\Psi_0)$. However, the condition that the "comparison state" $|\Psi_0)$ should be the vacuum state does not allow the presence of a real particle in $|\Psi_0)$. Dyson (15) originally maintained that, to avoid the existence of a real particle in $|\Psi_0)$, the amplitude should have no δ-function singularities so that, in integrals, the energy denominator $(\epsilon-\omega_p+\omega_k-\epsilon_{p+k})^{-1}$ obtained on dividing (2,18) by this energy term, should be evaluated as a principal value. However, as has been pointed out by Dyson and Dalitz (17), this is not correct, since, on transforming to coordinate space, the principal value permits the existence of free particles described by standing waves at infinity. They point out that to ensure that a real particle never exists in $|\Psi_0)$, the amplitude $(\Psi_0|a^*_\beta(-k)\,b_{u'}(-p-k)\,a_\alpha(p)|\Psi)$ must be finite for all values of p and k. Thus, the right side of (2.18) must be zero for all p and k satisfying equation (2.23) when u'(-p-k) is a positive energy spinor, i.e.

$$g\lambda_p\,\bar{u}'(-p-k)\gamma_5\tau_\alpha\sum_{u''}{}' u''(-k)(\Psi_0|a^*_\beta(-k)\,b_{u''}(-k)|\Psi) = 0$$

for the infinite number of values of the momenta p and k satisfying (2.23) when u'(-p-k) is a positive energy spinor. Hence

$$g\lambda_p\gamma_5\tau_\alpha\sum_{u''}{}' u''(-k)(\Psi_0|a^*_\beta(-k)\,b_{u''}(-k)|\Psi) = 0$$

and since g and λ_p are non-zero, γ_5 and τ_α are nonsingular matrices and the spinors u"(-$\underset{\sim}{k}$) form a complete set, we obtain the condition that

$$\langle\Psi_0|a^*_\beta(-\underset{\sim}{k})b_{u''}(-\underset{\sim}{k})|\Psi\rangle = 0 \qquad (2.24)$$

for all k, β and u".

A similar argument must be applied to equation (2.20) when u'($\underset{\sim}{q}$-$\underset{\sim}{p}$) is a negative energy spinor since there exist values of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ for which

$$\epsilon - E_{\underset{\sim}{q}} + E_{\underset{\sim}{p}} - E_{\underset{\sim}{q}-\underset{\sim}{p}} = 0$$

This yields the condition that

$$\langle\Psi_0|b_u(-\underset{\sim}{p})a_\beta(\underset{\sim}{p})|\Psi\rangle = 0 \qquad (2.25)$$

for all β and $\underset{\sim}{p}$ when u is a negative energy spinor.

From (2.24) and (2.25) it follows that, to satisfy the condition that $|\Psi_0\rangle$ is the vacuum state of the system, of the four types of two particle amplitudes appearing in the equations viz. $\langle\Psi_0|b_u(-\underset{\sim}{p})a_\alpha(\underset{\sim}{p})|\Psi\rangle$ and $\langle\Psi_0|a^*_\alpha(-\underset{\sim}{p})b_u(-\underset{\sim}{p})|\Psi\rangle$ for u(-$\underset{\sim}{p}$) a positive or negative energy spinor, all must be identically zero for all $\underset{\sim}{p}$ and α except $\langle\Psi_0|b_u(-\underset{\sim}{p})a_\alpha(\underset{\sim}{p})|\Psi\rangle$ when u(-$\underset{\sim}{p}$) is a positive energy spinor. Thus, the only two particle amplitude appearing in the equations is that describing the presence of a meson and a nucleon in the physical state $|\Psi\rangle$.

From equation (2.17), it is seen that the amplitude $\langle\Psi_0|b_u(-\underset{\sim}{p}-\underset{\sim}{k})a_\beta(\underset{\sim}{k})a_\alpha(\underset{\sim}{p})|\Psi\rangle$ may have a singularity when $\epsilon > M+2\mu$ at the values of $\underset{\sim}{p}$ and $\underset{\sim}{k}$ which satisfy

$$\epsilon - \omega_{\underset{\sim}{p}} - \omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}+\underset{\sim}{k}} = 0$$

Such a singularity corresponds to the presence of two real mesons and one nucleon in the state $|\Psi\rangle$ which, in the process of double scattering, is the final state in which we are interested. In this final state, the particles should be represented as outgoing waves. This is achieved if the singularity in the amplitude due to the factor $(\epsilon - \omega_p - \omega_k - E_{p+k})^{-1}$ is avoided in integrations by adding a small positive imaginary quantity to this energy denominator i.e. we replace $(\epsilon - \omega_p - \omega_k - E_{p+k})^{-1}$ by

$$(\epsilon - \omega_p - \omega_k - E_{p+k} + i\eta)^{-1} = P\frac{1}{\epsilon - \omega_p - \omega_k - E_{p+k}} - i\pi\,\delta(\epsilon - \omega_p - \omega_k - E_{p+k})$$

where P means that a principal value is to be taken in the appropriate integrals. We let $\eta \to 0$ after all integrals have been performed.

In a similar way by taking the energy denominator obtained in (2.19) as

$$(\epsilon - E_q - E_p - E_{q-p} + i\eta)^{-1}$$

we ensure that, when $\epsilon > 3M$, the two real nucleons and anti-nucleon, which can be present in $|\Psi\rangle$ when $\underline{p}$ and $\underline{q}$ have values satisfying

$$\epsilon - E_q - E_p - E_{q-p} = 0$$

are represented by outgoing waves.

We are now in a position to make use of equations (2.16) - (2.22) along with the conditions (2.24) and (2.25) to eliminate from equation (2.15) all the amplitudes appearing on the right side. We let the normalisation volume $V \to \infty$ so that $\lambda_k \to i(2\omega_k)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}$ and the summations over the momenta become integrals over all momentum space. Having carried out all the eliminations, we obtain the following integral equation for the amplitude

$$(\epsilon - \omega_p - E_p)\langle\bar{\Psi}_0|\, b_{u_+}(-p)\, a_\alpha(p)\,|\bar{\Psi}\rangle =$$

$$g^2 \sum_{u'_+}{}' \int d^3k\; u^+_+(-p)\left[A(p,k)\tau_\beta\tau_\alpha + B(p,k)\tau_\alpha\tau_\beta\right] u'_+(-k)\,(\bar{\Psi}_0|\, b_{u'_+}(-k)\, a_\beta(k)\,|\bar{\Psi})$$

$$+ \sum_{u'_+}{}' \bar{u}_+(-p)\,\Omega(p,\epsilon)\, u'_+(-p)\,\langle\bar{\Psi}_0|\, b_{u'_+}(-p)\, a_\alpha(p)\,|\bar{\Psi}\rangle$$

$$+ \Pi_{\alpha\beta}(p,\epsilon)\,\langle\bar{\Psi}_0|\, b_{u_+}(-p)\, a_\beta(p)\,|\bar{\Psi}\rangle \qquad (2.26)$$

where the subscript + on the spinors implies a positive energy spinor and the various functions appearing in (2.26) are defined as follows

$$A(p,k) = \lambda_p \lambda_k \gamma_4\gamma_5\left[\frac{\Lambda^+(-p-k)}{\epsilon - \omega_p - \omega_k - E_{p+k} + i\eta} - \frac{\Lambda^-(-p-k)}{\epsilon - E_p - E_k - E_{p+k} + i\eta}\right]\gamma_5 \qquad (2.27)$$

$$B(p,k) = \lambda_p \lambda_k \gamma_4\gamma_5\left[\frac{\Lambda^+(0)}{\epsilon - M} + \frac{\Lambda^-(0)}{\epsilon + M}\right]\gamma_5 \qquad (2.28)$$

$$\Omega(p,\epsilon) = 3g^2\int d^3k\, \lambda_k^2\, \gamma_5\left[\frac{\Lambda^+(-p-k)}{\epsilon - \omega_p - \omega_k - E_{p+k} + i\eta} + \frac{\Lambda^-(-p-k)}{\epsilon - \omega_p + \omega_k + E_{p+k}}\right]\gamma_5 \qquad (2.29)$$

$$\Pi_{\alpha\beta}(p,\epsilon) = g^2\int d^3k\,\lambda_p^2\, Sp\left[\gamma_5\tau_\alpha\Lambda^+(-k)\gamma_5\tau_\beta\Lambda^-(-p-k)\right]$$

$$\times\left[\frac{1}{\epsilon - E_p - E_k - E_{p+k} + i\eta} - \frac{1}{\epsilon - E_p + E_k + E_{p+k}}\right] \qquad (2.30)$$

$$\text{with} \quad \Lambda^\pm(p) = \sum_u u(p)\,\bar{u}(p)\,\theta_u^\pm(p) = \frac{E_p \pm (\alpha\cdot p + \beta M)}{2E_p}\,\beta \qquad (2.31)$$

It is helpful to represent the various interaction terms in equation (2.26) by time ordered graphs which are shown in figure (2.1). The term in A(p,k) is represented

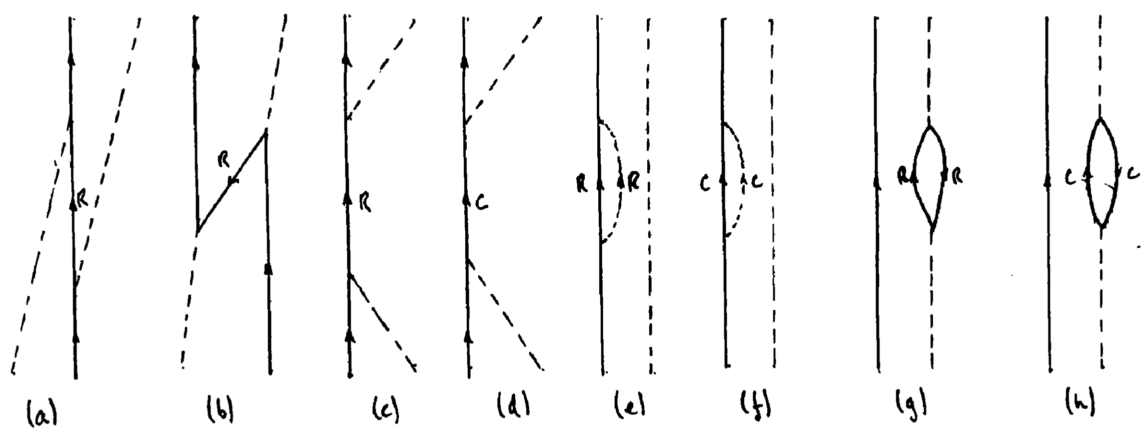

Fig.(1.1). The labels R and C on the intermediate state particles indicate respectively whether the particles are present in the real physical state $|\Psi\rangle$ or in the comparison state $|\Psi_0\rangle$.

by the graphs (a) and (b) and the term in B(p,k) by the graphs (c) and (d); these correspond to the scattering of the meson by the nucleon. The term in $\Omega(p,\epsilon)$ is a nucleon self-energy term and, is represented by graphs (e) and (f); graphs (g) and (h) represent the term in $\Pi_{\alpha\beta}(p,\epsilon)$ which is a meson self-energy term. No vacuum self-energy terms appear in the equation.

If the calculations leading to equation (2.26) are performed using the original formulation of the TD method, an equation is obtained for the one meson one nucleon amplitude which, although rather similar in structure to equation (2.26), has several important differences. The terms in equation (2.26) in which the particles propagate through the real physical state are the same in both equations i.e. the terms represented by graphs (a), (b), (c), (e) and (g). In the term corresponding to graph (d), $(\epsilon+M)^{-1}$ is replaced by $(\epsilon - E_p - M - E_k - \omega_p - \omega_k)^{-1}$. In the nucleon self-energy term corresponding to graph (f), $(\epsilon - \omega_p + \omega_k + E_{p+k})^{-1}$ is replaced by $(\epsilon + 2E_p - E_{p+k} - \omega_p - \omega_k)^{-1}$ and, in the meson self-energy term corresponding to graph (h), $(\epsilon - E_p + E_k + E_{p+k})^{-1}$ is replaced by $(\epsilon - E_p - E_k - E_{p+k} - 2\omega_p)^{-1}$. In addition to these differences, there exists a vacuum self-energy term

$$g^2 \int d^3q \int d^3k \, \lambda^2_{q+k} \frac{Sp\left[\gamma_5 \tau_\rho \Lambda^+(q) \gamma_5 \tau_\rho \Lambda^-(k)\right]}{\epsilon - E_p - \omega_p - E_q - E_k - \omega_{q+k}}$$

which is strongly divergent and cannot be made finite by any renormalisation proceedure. This is the type of term which was discussed earlier and which led to the modified formulation of the method which we are using.

2.3 Discussion of self-energy terms.

Equation (2.26), as it stands, contains divergent contributions from the meson and nucleon self-energy terms. Little use can be made of the equation until these divergences are removed. In the following discussion, we shall attempt to carry out this removal by a renormalisation proceedure.

To carry out a renormalisation programme in an unambiguous manner, it is well known that it must be done by a covariant proceedure. The first step is therefore to attempt to put the meson and nucleon self-energy terms into covariant forms. That this can be done was first shown by Visscher (36). An alternative proof is given in Appendix I. The result is

$$\Omega(\underline{p},\epsilon) = -\frac{3g^2}{(2\pi)^4}\int d^4k\,\gamma_5 S_F(P-k)\gamma_5\Delta_F(k) \equiv \Omega(P.\gamma) \tag{2.32}$$

and

$$\Pi_{\alpha\beta}(\underline{k},\epsilon) = -\frac{ig^2}{(2\pi)^4}\delta_{\alpha\beta}\frac{1}{\omega_k}\int d^4k\,Sp\left[\gamma_5 S_F(k)\gamma_5 S_F(Q+k)\right] \equiv \frac{1}{2\omega_k}\delta_{\alpha\beta}\Pi(Q^2) \tag{2.33}$$

where P is the 4-vector $(-\underline{k},i(\epsilon-\omega_k))$ and Q the 4-vector $(-\underline{k},i(\epsilon-E_k))$. S_F and Δ_F are the usual Feynman functions, which appear in covariant theory, defined by

$$S_F(q) = \frac{q\cdot\gamma + iM}{q^2+M^2-i\eta} \quad \text{and} \quad \Delta_F(q) = \frac{1}{q^2+\mu^2-i\eta}$$

and $q\cdot\gamma = \sum_{i=1}^{4} q_i\gamma_i$. The expressions $\Omega(P.\gamma)$ and $\Pi(Q^2)$, apart from constant factors, are respectively the second order nucleon and meson self-energy terms of covariant perturbation theory.

Due to the different energy denominators appearing in the self-energy terms in the equation obtained using the original formulation of the TD method, these terms cannot be put into covariant form. This is a further disadvantage of the original method since the divergences in the equation obtained using this method cannot be removed in any consistent and unambiguous manner.

From covariant theory we know that $\Omega(P.\gamma)$ can be written in the form (30)

$$\Omega(P.\gamma) = A + (P.\gamma - iM)B + S_N(P.\gamma)$$

where $A = \Omega(iM)$ and $B = \left[\frac{\partial \Omega(P.\gamma)}{\partial(P.\gamma)}\right]_{P.\gamma = iM}$ are infinite constants and $S_N(P.\gamma)$ is a divergence free integral which, as shown by Visscher (36), can be written as

$$S_N(P.\gamma) = -\frac{3ig^2}{16\pi^2}(P.\gamma - iM)^2\int_0^1 dx\int_0^1 dy\,\frac{x(1-x)}{\Lambda_0^2 + (P^2+M^2)(1-x)xy}$$
$$\times\left[(P.\gamma + iM)\left(1 - x - \frac{2Mx^2y(1-x)}{\Lambda_0^2}\right) - iMx\right] \qquad (2.34)$$

where $\Lambda_0^2 = M^2x^2 + \mu^2(1-x) - i\eta$. Hence, making use of (2.11), the nucleon self-energy term in (2.26) becomes

$$\sum_{u_4'} \bar{u}_+(-\underline{p})\,\Omega(\underline{p}.\epsilon)\,u_+'(-\underline{p})\langle\bar{\Psi}_0|b_{u_4'}(-\underline{p})a_\alpha(\underline{p})|\bar{\Psi}\rangle$$

$$= \left[\frac{M}{E_p}A + (\epsilon - \omega_p - E_p)B' + (\epsilon - \omega_p - E_p)R_N(p,\epsilon)\right]\langle\bar{\Psi}_0|b_{u_4}(-\underline{p})a_\alpha(\underline{p})|\Psi\rangle \quad (2.35)$$

where

$$R_N(p,\epsilon) = -\frac{3g^2}{16\pi^2}(\epsilon - \omega_p - E_p)\int_0^1 dx\int_0^1 dy\,\frac{x(1-x)}{\Lambda_0^2 + (P^2+M^2)(1-x)xy}$$
$$\times\left[(\epsilon - \omega_p + E_p)\left(1 - x - \frac{2Mx^2y(1-x)}{\Lambda_0^2}\right) + Mx\right] \qquad (2.36)$$

and B' = iB is a real quantity since it is clear from (2.29) and (2.31) that $\bar{u}(-p)\,\Omega(p,\epsilon)\,u(-p)$ is real.

The divergent quantities A and B' can now be removed by a renormalisation of the nucleon mass and the coupling constant g. To do this, the nucleon self-energy term in the form (2.35) is taken to the left side of equation (2.26). The term in A can be absorbed as a correction to the nucleon mass; this is the same as saying that this term cancels the nucleon mass renormalisation counter term which would be present if it had been explicitly used in the original hamiltonian of the system. By defining a renormalised coupling constant $G^2 = g^2(1-B')^{-1}$, the second term is absorbed as a coupling constant renormalisation, so that equation (2.26) now becomes

$$(\epsilon - \omega_p - E_p)\left(1 - G^2 R_N(p,\epsilon)\right)\langle\Psi_0|\, b_{u_+}(-p)\, a_\alpha(p)|\Psi\rangle =$$

$$G^2 \sum_{u'_+}' \int d^3k\, u_+^{+}(-p)\left[A(p,k)\tau_\beta\tau_\alpha + B(p,k)\tau_\alpha\tau_\beta\right] u'_+(-k)\,\langle\Psi_0|\, b_{u'_+}(-k)\, a_\beta(k)|\Psi\rangle$$

$$+ (1-B')^{-1}\, \Pi_{\alpha\beta}(p,\epsilon)\,\langle\Psi_0|\, b_{u_+}(-p)\, a_\beta(p)|\Psi\rangle \qquad (2.26')$$

We have thus been able to extract from the nucleon self-energy term a well-defined finite part and to interpret the remaining infinite parts as mass and coupling constant renormalisations.

In the case of the meson self-energy term, we can write (30)

$$\Pi(Q^2) = C + (Q^2 + \mu^2)D + S_M(Q^2)$$

where $C = \Pi(-\mu^2)$ and $D = \left[\frac{\partial \Pi(Q^2)}{\partial Q^2}\right]_{Q^2=-\mu^2}$ are infinite constants and $S_M(Q^2)$ is a divergence free integral which has been discussed by Visscher (36) who shows that it can be expressed as

$$S_M(Q^2) = \frac{g^2}{16\pi^2}(Q^2+\mu^2)^2\int_0^1 dx\int_0^1 dy\, \frac{x(1-x)}{M^2 - x(1-x)\mu^2 - i\eta}\left[\frac{Q^2(1-x)xy}{(1-x)xy(Q^2+\mu^2) + M^2 - x(1-x)\mu^2 - i\eta} - 1\right] \quad (2.37)$$

Thus, the meson self-energy term in (2.26) becomes

$$\Pi_{\alpha\beta}(p,\epsilon)(\bar{\Psi}_0|b_{u_+}(-p)a_\beta(p)|\bar{\Psi}) = \left[\frac{C}{2\omega_p} + \frac{D}{2\omega_p}(Q^2+\mu^2) + (\epsilon-\omega_p-E_p)R_M(p,\epsilon)\right](\bar{\Psi}_0|b_{u_+}(-p)a_\alpha(p)|\bar{\Psi}) \quad (2.38)$$

$$\text{where} \quad R_M(p,\epsilon) = \frac{1}{2\omega_p(\epsilon-\omega_p-E_p)}\, S_M(Q^2) \quad (2.39)$$

$$\text{and} \quad (Q^2+\mu^2) = -(\epsilon-\omega_p-E_p)(\epsilon+\omega_p-E_p)$$

Now, as has been pointed out by Dyson and Dalitz (17), to be able to remove divergent quantities V and W, say, from our equation by mass and charge renormalisations, it is necessary, as is clear from the discussion of the nucleon self-energy term, that they should appear in the form $V + W(\epsilon-\omega_p-E_p)$. In the meson self-energy term, the divergent quantities C and D do not appear in this form but in the form $\frac{C}{2\omega_p} - \frac{D}{2\omega_p}(\epsilon-\omega_p-E_p)(\epsilon+\omega_p-E_p)$. This is due essentially to the fact that the meson field is a Bose field. It is clear that the substraction from $\Pi(Q^2)$ of an expression of the form $V + W(\epsilon-\omega_p-E_p)$, with V and W divergent quantities, cannot produce a finite result. Thus, although we can obtain from the meson self-energy term a finite expression defined in an unambiguous manner through the covariant form of this term, we are not able to interpret the remaining divergent parts in terms of renormalisation of mass and coupling constant. This is rather a discouraging feature of the TD method but we propose to overcome this difficulty, for the present, by simply omitting the divergent terms C and D. Equation (2.26) then takes a form which is free from any terms containing explicit divergences

$$(\epsilon - \omega_p - E_p)\left(1 - G^2 R_N(p,\epsilon) - G^2 R_M(p,\epsilon)\right)\langle \Psi_0 | b_{u_+}(-p)\, a_\alpha(p) | \Psi \rangle$$

$$= G^2 \sum_{u'_+} \int d^3k\, u^+(-p)\left[A(p,k)\tau_\beta\tau_\alpha + B(p,k)\tau_\alpha\tau_\beta\right] u'_+(-k)\langle \Psi_0 | b_{u'_+}(-k)\, a_\beta(k) | \Psi \rangle \qquad (2.26'')$$

2.4 The modified integral equation.

We shall now discuss the equation (2.26") from which all explicit divergences have been removed. To do this, we introduce the quantity $\chi_\alpha(p)$ defined by

$$\chi_\alpha(p) = \sum_{u_+} \langle \Psi_0 | b_{u_+}(-p)\, a_\alpha(p) | \Psi \rangle\, u_+(-p) \qquad (2.40)$$

Since we have taken $|\Psi_0\rangle$ to be the true vacuum state of the system, $\chi_\alpha(p)$ is the probability amplitude for finding in the state $|\Psi\rangle$ a meson of momentum $\underset{\sim}{p}$ in the charge state α and a nucleon of momentum $-\underset{\sim}{p}$ described by the spinor $u(-\underset{\sim}{p})$.

Since the state of one meson and one nucleon has isotopic spin $\frac{1}{2}$ or $\frac{3}{2}$, $\chi_\alpha(p)$ can be written as a linear combinations of two functions $\chi_{\frac{1}{2}}(p)$ and $\chi_{\frac{3}{2}}(p)$, $\chi_{\frac{1}{2}}(p)$ being the probability amplitude for finding the meson and nucleon in a state of isotopic spin $\frac{1}{2}$ and $\chi_{\frac{3}{2}}(p)$ the amplitude for finding them in a state of isotopic spin $\frac{3}{2}$. Now, it is easy to show as has been done by Dyson et al. (16), that the isotopic spin operator $Q_{\alpha\beta} = \tau_\alpha\tau_\beta$ has eigenvalues $Q_{\frac{1}{2}} = 3$ and $Q_{\frac{3}{2}} = 0$ corresponding respectively to the states of isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ that $Q'_{\alpha\beta} = \tau_\beta\tau_\alpha$ has eigenvalues $Q'_{\frac{1}{2}} = -1$ and $Q'_{\frac{3}{2}} = 2$. Using this property of the operators $Q_{\alpha\beta}$ and $Q'_{\alpha\beta}$ and the orthogonality of the eigenfunctions corresponding to different eigenvalues of the isotopic spin, equation (2.26″) becomes

$$(\epsilon - \omega_p - E_p)\Big(1 - G^2 R_N(p,\epsilon) - G^2 R_M(p,\epsilon)\Big)\chi_I(p)$$

$$= G^2\int d^3k\left[A(p,k)Q'_I + B(p,k)Q_I\right]\chi_I(k) \tag{2.41}$$

where $I = \frac{1}{2}$ or $\frac{3}{2}$ is the total isotopic spin of the system. Defining the functions $F_I(p)$ and $R(p,\epsilon)$ by

$$F_I(p) = G^2\int d^3k\left[A(p,k)Q'_I + B(p,k)Q_I\right]\chi_I(k) \tag{2.42}$$

$$R(p,\epsilon) = R_N(p,\epsilon) + R_M(p,\epsilon) \tag{2.43}$$

we can put (2.41) into the form

$$\chi_I(p) = L_I(p)\,\delta\Big[(\epsilon-\omega_p-E_p)(1-G^2R(p,\epsilon))\Big] + P\,\frac{F_I(p)}{(\epsilon-\omega_p-E_p)(1-G^2R(p,\epsilon))} \qquad (2.44)$$

where $L_I(p)$ is some function of $\underset{\sim}{p}$ and P implies that, in any integration over $\underset{\sim}{p}$, a principal value is to be taken.

If we now define

$$\mathcal{Y}^M_{Jl\frac{1}{2}}(n_p, m) = \sum_s C^{l\ \frac{1}{2}\ J}_{M-s\ s}\ Y^{M-s}_l(n_p)\ Y^s_{\frac{1}{2}}(m) \qquad (2.45)$$

where $C^{l\ \frac{1}{2}\ J}_{M-s\ s}$ is a Clebsch Gordon coefficient ($C^{a\ b\ c}_{\alpha\ \beta} \equiv C_{ab}(c, \alpha+\beta; \alpha, \beta)$ of reference 3), $Y^{M-s}_l(n_p)$ is a spherical harmonic with n_p a unit vector in the direction of the momentum vector $\underset{\sim}{p}$ and $Y^s_{\frac{1}{2}}(m)$ is the probability amplitude for finding the nucleon with z-component of spin s where m is the z-component in the chosen z-direction i.e. $Y^s_{\frac{1}{2}}(m) = \delta_{sm}$, then $\mathcal{Y}^M_{Jl\frac{1}{2}}(n_p, m)$ is an eigenfunction of total angular momentum J, z-component M and orbital angular momentum of the meson and nucleon l i.e. of parity $(-1)^{l+1}$. These eigenfunctions form a complete orthonormal set so that $\chi_I(p)$ can be expanded as

$$\chi_I(p) = \sum_{JlM} \chi^M_{IJl}(p)\ \mathcal{Y}^M_{Jl\frac{1}{2}}(n_p, m) \qquad (2.46)$$

Now, the wave function in coordinate space $\psi_I(r)$ for one meson and one nucleon in a state of isotopic spin I is the Fourier transform of $\chi_I(p)$ so that.

$$\psi_I(r) = \int d^3p\ \chi_I(p)\, e^{i p\cdot r} = \int d^3p\ \chi_I(p) \sum_l i^l \sqrt{4\pi(2l+1)}\ j_l(pr)\ Y^0_l(n_p\cdot n_r) \qquad (2.47)$$

where $j_l(pr)$ is the spherical Bessel function of order l, (34).

The expansion (2.46) allows the angular integrations in (2.47) to be carried out, yielding

$$\psi_I(\underline{r}) = \sum_{JMl} \int_0^\infty p^2 dp\, \chi^M_{IJl}(p)\, \mathcal{Y}^M_{Jl\frac{1}{2}}(n_r, m)\, 4\pi i^l j_l(pr)$$

For large values of r, $j_l(pr)$ behaves as

$$j_l(pr) \sim \frac{1}{pr} \sin\left(pr - l\frac{\pi}{2}\right)$$

so that $\psi_I(\underline{r})$ behaves asymptotically as

$$\psi_I(\underline{r}) \sim \frac{2\pi}{ir} \sum_{JMl} \int_0^\infty p\, dp\, \chi^M_{IJl}(p)\left[e^{ipr} - (-1)^l e^{-ipr}\right] \mathcal{Y}^M_{Jl\frac{1}{2}}(n_r, m) \qquad (2.48)$$

By expanding the functions $L_I(\underline{p})$ and $F_I(\underline{p})$ in terms of the $\mathcal{Y}^M_{Jl\frac{1}{2}}(n_p, m)$ as has been done in the case of $\chi_I(\underline{p})$ in (2.46), and using the orthonormality of these angular momentum eigenfunctions, (2.44) yields

$$\chi^M_{IJl}(p) = L^M_{IJl}(p)\,\delta\left[(\epsilon - \omega_p - E_p)(1 - G^2 R(p,\epsilon))\right] + \mathcal{P}\frac{F^M_{IJl}(p)}{(\epsilon - \omega_p - E_p)(1 - G^2 R(p,\epsilon))} \qquad (2.49)$$

Inserting the expression (2.49) into (2.48), the integrals can be performed to give an asymptotic form for $\psi_I(\underline{r})$. Assuming for the present that the factor $(1 - G^2 R(p,\epsilon))$ is always non-zero, as might be expected, we obtain the following asymptotic expression

$$\psi_I(\underline{r}) \sim \frac{2\pi}{ir} \frac{E_{p_0}\omega_{p_0}}{\epsilon} \sum_{JMl} \Big\{ \left[L^M_{IJl}(p_0) - i\pi F^M_{IJl}(p_0)\right] e^{ip_0 r}$$
$$- (-1)^l \left[L^M_{IJl}(p_0) + i\pi F^M_{IJl}(p_0)\right] e^{-ip_0 r} \Big\} \mathcal{Y}^M_{Jl\frac{1}{2}}(n_r, m) \qquad (2.50)$$

where p_0 is defined by $\epsilon - \omega_{p_0} - \epsilon_{p_0} = 0$ and so by (2.36), (2.39) and (2.43) $R(p_0, \epsilon) = 0$.

Now, by the definition of scattering phase shifts, the coefficient of $\mathcal{Y}^M_{JL\frac{1}{2}}(n_r, m)$ in the expansion of the wave function of the two particle state behaves, for large values of r, as

$$e^{2i\delta_{IJl}(p_0)} e^{ip_0 r} - (-1)^l e^{-ip_0 r}$$

where p_0 is the magnitude of the relative momentum of the meson and nucleon and, $\delta_{IJl}(p_0)$ is the phase shift for scattering through a state of angular momentum J, isotopic spin I and parity $(-1)^{l+1}$. It follows from this that the phase shifts for meson-nucleon scattering are given in terms of the functions appearing in our equations by

$$e^{2i\delta_{IJl}(p_0)} = \frac{L^M_{IJl}(p_0) - i\pi F^M_{IJl}(p_0)}{L^M_{IJl}(p_0) + i\pi F^M_{IJl}(p_0)}$$

i.e. $$\tan \delta_{IJl}(p_0) = -\pi \frac{F^M_{IJl}(p_0)}{L^M_{IJl}(p_0)} \qquad (2.51)$$

which shows that the functions $F^M_{IJl}(p_0)$, $L^M_{IJl}(p_0)$ and so $\chi^M_{IJl}(p_0)$ are independent of M. Henceforth, we shall omit the superscript M on these functions.

Making use of the fact that angular momentum and the parity are conserved in all interactions, we can define two functions $A_{Jl}(p,k)$ and $B_{Jl}(p,k)$ by

$$\sum_{m,m'} \int d\Omega_p \int d\Omega_k \, \mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_p, m) \, A(p,k) \, \mathcal{Y}^{M'}_{J'l'\frac{1}{2}}(n_k, m') = A_{Jl}(p,k) \, \delta_{JJ'} \delta_{MM'} \delta_{ll'}$$

$$(2.52)$$

and $$\sum_{m,m'} \int d\Omega_p \int d\Omega_k \, \mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_p, m) \, B(p,k) \, \mathcal{Y}^{M'}_{J'l'\frac{1}{2}}(n_k, m') = B_{Jl}(p,k) \, \delta_{JJ'} \delta_{MM'} \delta_{ll'}$$

Inserting these definitions (2.52) into equation (2.44), this equation, after a little algebra, yields the integral equation

$$\Delta_{IJl}(p) = G^2 \frac{E_{p_0} \omega_{p_0}}{\epsilon} \Big[A_{Jl}(p,p_0) \, Q'_I + B_{Jl}(p,p_0) \, Q_I \Big]$$

$$+ G^2 P \int_0^\infty k^2 dk \, \frac{A_{Jl}(p,k) \, Q'_I + B_{Jl}(p,k) \, Q_I}{(\epsilon - \omega_k - E_k)(1 - G^2 R(k,\epsilon))} \qquad (2.53)$$

where $\Delta_{IJl}(p) = F_{IJl}(p) / L_{IJl}(p)$ so that the solution on the energy shell i.e. for $p = p_0$ of this single variable integral equation immediately gives the scattering phase shifts since, by equation (2.51), it follows that

$$\tan \delta_{IJl}(p_0) = -\pi \, \Delta_{IJl}(p_0) \qquad (2.54)$$

We note that, although the equation (2.53) determines the function $\Delta_{IJl}(p)$ for all values of p, it is only the value on the energy shell i.e. $\Delta_{IJl}(p_0)$ which has any direct physical significance.

We now consider the situation when the total energy of the system $\epsilon > M + 2\mu$. It is clear from the definitions (2.27), (2.29) and (2.36) of $A(p,k)$, $\Omega(p,\epsilon)$ and $R_k(p,\epsilon)$ that, when this situation obtains, the function $\Delta_{IJl}(p)$ is complex; the imaginary part comes from contributions to the integrals in (2.53) from terms in $A(p,k)$ and $\Omega(p,\epsilon)$ with the factor

$-i\pi\delta(\epsilon-\omega_p-\omega_k-E_{p+k})$, this factor being implied by the presence of the small positive imaginary part, $i\eta$, in the definitions of these functions. The phase shift has therefore a non-zero imaginary part, when $\epsilon > M+2\mu$, and this leads to the existence of inelastic as well as elastic meson-nucleon scattering. Now, the δ-function singularities, of which the imaginary part is composed, correspond to the existence of two real mesons and one real nucleon in the system. It follows that the inelastic scattering cross-section, given by the complex phase shift, is the cross-section for the production of a meson in a meson-nucleon collision. Thus from the solution of equation (2.53), we can obtain not only the elastic meson-nucleon scattering cross-section but also, for energies above the production threshold i.e. for $\epsilon > M+2\mu$ the total corss-section for the production of a meson in the collision.

Let us now consider the solution on the energy shell of equation (2.53) as a power series in G^2. This is obtained by iterating the equation and expanding the factor $(1-G^2R(k,\epsilon))^{-1}$, giving

$$
\begin{aligned}
\Delta_{IJl}(p_0) = {} & G^2\frac{E_{p_0}\omega_{p_0}}{\epsilon}\left[A_{Jl}(p_0,p_0)\alpha'_I + B_{Jl}(p_0,p_0)\alpha_I\right] \\
& + G^4\frac{E_{p_0}\omega_{p_0}}{\epsilon}P\int_0^\infty k^2dk\left[A_{Jl}(p_0,k)\alpha'_I + B_{Jl}(p_0,k)\alpha_I\right]\frac{1}{\epsilon-\omega_k-E_k}\left[A_{Jl}(k,p_0)\alpha'_I + B_{Jl}(k,p_0)\alpha_I\right] \\
& + G^6\frac{E_{p_0}\omega_{p_0}}{\epsilon}P\int_0^\infty k^2dk\Big\{\left[A_{Jl}(p_0,k)\alpha'_I + B_{Jl}(p_0,k)\alpha_I\right]\frac{R(k,\epsilon)}{\epsilon-\omega_k-E_k}\left[A_{Jl}(k,p_0)\alpha'_I + B_{Jl}(k,p_0)\alpha_I\right] \\
& + P\int_0^\infty q^2dq\left[A_{Jl}(p_0,k)\alpha'_I + B_{Jl}(p_0,k)\alpha_I\right]\frac{1}{\epsilon-\omega_k-E_k}\left[A_{Jl}(k,q)\alpha'_I + B_{Jl}(k,q)\alpha_I\right]\frac{1}{\epsilon-\omega_q-E_q}\left[A_{Jl}(q,p_0)\alpha'_I + B_{Jl}(q,p_0)\alpha_I\right]\Big\} \\
& + \text{higher order terms} \qquad (2.55)
\end{aligned}
$$

Each term in this expansion can be represented by a set of time ordered graphs of the appropriate order in G.

To simplify the arguments that follow we can, without any loss of generality in the final result, consider a special case of our problem; we consider the scattering of a positive meson by a proton allowing only positive mesons in the intermediate states and neglecting pair production. The graphs representing the terms in the expansion (2.55), for this special case, are shown up to order G^6 in figure (2.2). Graphs (1) and (2) correspond respectively

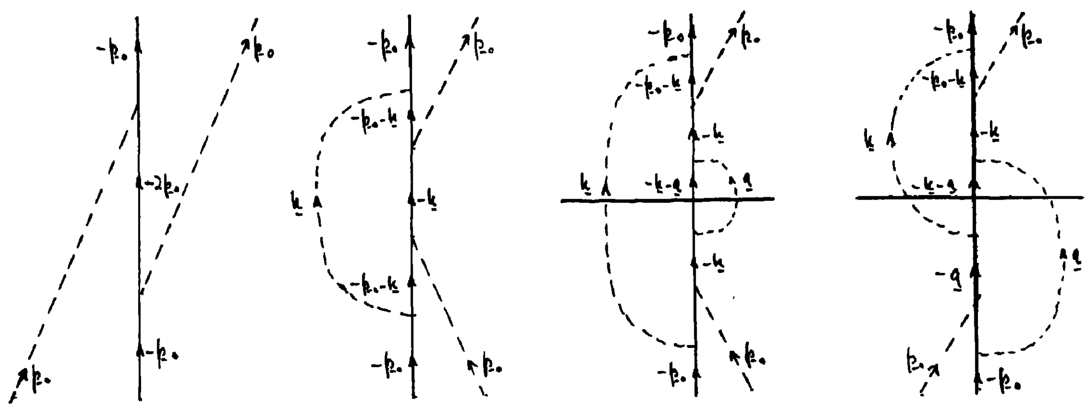

Fig. (2.2)

to the first and second terms in the expansion; the contributions from both these graphs are real. In graphs (3) and (4) which correspond to the two terms of order G^6 energy can be conserved, i.e. $\epsilon - \omega_q - \omega_k - E_{q+k} = 0$, in the "barred state" shown in the figure; when this is the case the terms in $-i\pi\,\delta(\epsilon - \omega_q - \omega_k - E_{q+k})$ contribute imaginary parts to the terms in (2.55). Thus, using (2.54), we can write

$$\tan\delta \approx aG^2 + bG^4 + (c+id)G^6 + \text{-----}$$

where a,b,c and d are all real quantities. From this, it follows that the elastic, σ_{el} , and production, σ_{prod} , cross-sections are given by

$$\sigma_{el} \sim \left|1 - e^{2i\delta}\right|^2 \sim 4a^2G^4 + \text{ higher order terms}$$

and $\quad \sigma_{prod} \sim 1 - \left|e^{2i\delta}\right|^2 \sim 4d\,G^6 + \text{ higher order terms}$

Now, the lowest production matrix element $(\underline{q},\underline{k}|M|\underline{p}_0)$ is the sum of the matrix elements $(\underline{q},\underline{k}|M_5|\underline{p}_0)$ and $(\underline{q},\underline{k}|M_6|\underline{p}_0)$ which are represented by the graphs (5) and (6) shown in figure (2.3). In terms of these matrix

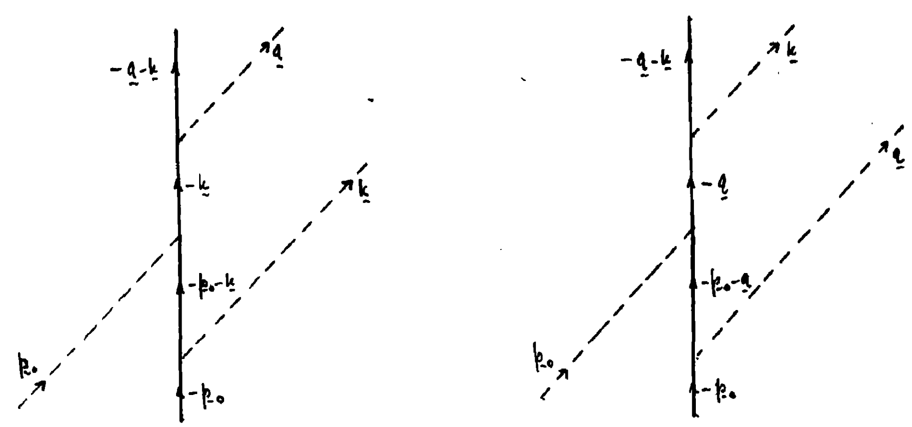

Fig. (2.3)

elements, the production cross-section is given by

$$\sigma_{prod} \sim \int d^3q \int d^3k \left|(\underline{q},\underline{k}|M|\underline{p}_0)\right|^2 \delta(\epsilon - \omega_q - \omega_k - E_{\underline{q}+\underline{k}})$$

i.e. $$\sigma_{prod} \sim \int d^3q \int d^3k \Big[\left|(\underline{q},\underline{k}|M_5|\underline{p}_0)\right|^2 + \left|(\underline{q},\underline{k}|M_6|\underline{p}_0)\right|^2 + (\underline{q},\underline{k}|M_5|\underline{p}_0)^*(\underline{q},\underline{k}|M_6|\underline{p}_0)$$
$$+ (\underline{q},\underline{k}|M_6|\underline{p}_0)^*(\underline{q},\underline{k}|M_5|\underline{p}_0)\Big]\,\delta(\epsilon - \omega_q - \omega_k - E_{\underline{q}+\underline{k}}) \qquad (2.56)$$

We see that, when energy is conserved in the "barred states" in graphs (3) and (4), these graphs are graphical representations of this integral, since graph (3) can be obtained by placing two graphs like (5) or (6) end to end and joining the meson and nucleon lines of the same momenta in the three particle state and (4) can be obtained by joining graphs (5) and (6) in a similar way. Thus, graph (3) corresponds to the first two terms in (2.56) and graph (4) to the last two interference terms. Thus, for the calculation of the meson production cross-section, it is essential to retain in our equations the contribution from the nucleon self-energy term as it would clearly give a meaningless result to neglect this contribution and consequently leave only the interference terms in expression (2.56) for the cross-section. This conclusion is clearly not altered by considering higher order terms in the expansion (2.55) or by removing the restrictions which were imposed on the problem earlier, in order to simplify the discussion.

However, as we shall see, the presence of this nucleon self-energy term in our equation leads to a difficulty which has prevented the making of any detailed calculations on it.

This difficulty arises in the following way. Our arguments up to this point have been based on the seemingly valid assumption that the factor $(1 - G^2 R(p, \epsilon))$ is never zero. This is not correct as has been shown by Visscher (36) and by Dyson and Dalitz (17) who have studied the behaviour of the functions $R_N(p,\epsilon)$ and $R_M(p,\epsilon)$ and find, that for very large values of p, they behave as follows

$$R_N(p,\epsilon) \sim \frac{3}{32\pi^2}\log\frac{p}{M} \quad \text{and} \quad R_M(p,\epsilon) \sim \frac{1}{32\pi^2}\frac{\epsilon}{p}\log\frac{p}{M}$$

Thus, for very large p, the nucleon self-energy dominates in the factor $(1-G^2R(p,\epsilon))$ so that this factor is very large and negative. But, by definition, $R_N(p_0,\epsilon) = R_M(p_0,\epsilon) = 0$ so that $(1-G^2R(p_0,\epsilon))$ is positive and equal to unity. Hence there exists some value p_0' of p, where $p_0' > p$, for which $(1-G^2R(p_0',\epsilon)) = 0$. Dyson and Dalitz find that for $G^2/4\pi = 14$, $p_0' = 1.3M$. Account must be taken of this zero of $(1-G^2R(p,\epsilon))$ in the analysis leading to equation (2.50) and this leads to the presence in the co-ordinate space wave function of the meson-nucleon system of terms in $e^{\pm i p_0' r}$ for large values of r. This implies the presence of particles of momentum $\underline{p}_0' > \underline{p}$ at infinity. However, the total energy of the system is $\epsilon = E_{p_0} + \omega_{p_0}$ so that, in order to conserve energy, a state containing particles of momentum $\underline{p}_0'$ could exist only if one of these particles, for example, had a rest mass smaller than the nucleon rest mass; such a particle would be the result of the formation of some type of bound system with a rest mass much smaller than M. There is no knowledge of the existence of such a system and we must regard this singularity at $p = p_0'$ as being completely unphysical and its presence being due somehow to the approximations made in applying the TD method. We are thus prevented from carrying out any calculations of elastic or double scattering cross-sections based on equation (2.53). It does not seem likely that these difficulties would disappear in a higher order TD type of calculation and so, this method for the calculation of the double scattering cross-section has had to be abandoned.

CHAPTER III.

3.1 Discussion of the Cini Fubini method.

The method proposed by Cini and Fubini (9) makes use of a variational proceedure to obtain a sequence of approximate solutions for the fundamental equations describing the system. We shall first of all discuss the development of the method, which is based on the formulation of scattering theory due to Lippmann and Schwinger (29).

We shall work in the interaction representation in which the system, at time t, is characterised by the state vector $|t)$ which satisfies the euqation

$$i\frac{\partial}{\partial t}|t) = H'(t)|t) \tag{3.1}$$

where we again use units with $\hbar = c = 1$ and H'(t) is that part of the hamiltonian of the system describing the interaction of the various parts.

The development of the system from some time in the remote past to some time in the remote future is described by a unitary matrix S such that

$$|\infty) = S|-\infty) \tag{3.2}$$

A knowledge of this S-matrix provides us with any information we may require regarding the development of the system over a very large interval of time.

In order to determine the S-matrix, we shall make use of the reaction matrix K which is introduced by Lippmann and

Schwinger and is connected to the S-matrix by

$$S = \frac{1 - \frac{i}{2}K}{1 + \frac{i}{2}K} \tag{3.3}$$

K is a hermitian matrix so that the unitarity of the S-matrix is maintained by (3.3) and, as shown by Lippmann and Schwinger, K is defined by the equations

$$K = \int_{-\infty}^{\infty} H'(t)V(t)\,dt \tag{3.4}$$

where
$$V(t) = 1 - i\int_{-\infty}^{\infty} H'(t')V(t')\,\epsilon(t-t')\,dt' \tag{3.5}$$

with
$$\begin{aligned} \epsilon(t-t') &= \tfrac{1}{2} \quad , \quad t > t' \\ &= -\tfrac{1}{2} \quad , \quad t < t' \end{aligned} \tag{3.6}$$

They also define an operator K' by

$$\begin{aligned} K' = \int_{-\infty}^{\infty} \Big\{ & H'(t)V(t) + V^{+}(t)H'(t) - V^{+}(t)H'(t)V(t) \\ & - i\int_{-\infty}^{\infty} dt'\, V^{+}(t)H'(t)\epsilon(t-t')H'(t')V(t') \Big\}\,dt \end{aligned} \tag{3.7}$$

where a dagger denotes the hermitian conjugate. It follows from (3.7) that K' is a hermitian operator for arbitrary V(t).

Now if $\delta K'$ is the variation in K' due to small arbitrary variations of V(t) and $V^{+}(t)$, then

$$\begin{aligned} \delta K' = & -\int_{-\infty}^{\infty} \delta V^{+}(t)H'(t)\Big[V(t) - 1 + i\int_{-\infty}^{\infty} H'(t')V(t')\epsilon(t-t')\,dt'\Big]dt \\ & - \int_{-\infty}^{\infty} dt\Big[V^{+}(t) - 1 - i\int_{-\infty}^{\infty} V^{+}(t')H'(t')\epsilon(t-t')\,dt'\Big]H'(t)\,\delta V(t) \end{aligned} \tag{3.8}$$

so that $\delta K' = 0$ if and only if $V(t)$ and $V^+(t)$ are solutions of equation (3.5). Also, when (3.5) is satisfied

$$K' = \int_{-\infty}^{\infty} H'(t) V(t) \, dt = K \tag{3.9}$$

Thus, $\delta K' = 0$ gives a variational principle for equation (3.5) and the stationary value of K' is the K-matrix whose hermitian property is maintained by this principle.

Cini and Fubini use this variational principle to determine a stationary value for the K-matrix for a certain type of trial operator V(t) which they choose in the following way.

H'(t) is, in general, proportional to some coupling constant g so that iteration of the integral equation (3.5) for V(t) yields a power series in g

$$V(t) = 1 + V_1(t) + V_2(t) + \cdots + V_i(t) + \cdots$$

$$\text{where} \quad V_i(t) = -i \int_{-\infty}^{\infty} H'(t') V_{i-1}(t') \, \epsilon(t-t') \tag{3.10}$$

and the suffix denotes the power of g in each term. It follows that

$$K = K_1 + K_2 + K_3 + \cdots + K_i + \cdots$$

$$\text{where} \quad K_i = \int_{-\infty}^{\infty} H'(t) V_{i-1}(t) \, dt \tag{3.11}$$

Formally, the infinite series (3.10) is an exact solution of equation (3.5) for V(t). As a trial operator for the variational principle we cut this series off at its n'th term and multiply each term of this finite series by a time independent operator. Thus, we introduce into (3.7) the trial operator

$$V(t) = \Lambda_1 + V_1(t)\Lambda_2 + \cdots + V_{n-1}(t)\Lambda_n \tag{3.12}$$

where the Λ_i are time independent operators whose forms are left undetermined at present. (3.7) becomes

$$K' = \sum_{i=1}^{n}\int_{-\infty}^{\infty} dt \Big\{ H'(t)V_{i-1}(t)\Lambda_i + \Lambda_i^+ V_{i-1}^+(t)H'(t) - \sum_{k=1}^{n}\Lambda_i^+ V_{i-1}^+(t)H'(t)V_{k-1}(t)\Lambda_k$$

$$- i\sum_{k=1}^{n}\int_{-\infty}^{\infty} dt'\, \Lambda_i^+ V_{i-1}^+(t)H'(t)\,\epsilon(t-t')\,H'(t')V_{k-1}(t')\Lambda_k \Big\}$$

which, with the help of (3.10) and (3.11), reduces to

$$K' = \sum_{i=1}^{n}\Big[K_i\Lambda_i + \Lambda_i^+ K_i - \sum_{k=1}^{n}\Lambda_i^+ (K_{i+k-1} - K_{i+k})\Lambda_k \Big] \tag{3.13}$$

The n'th order approximation, $K^{(n)}$, to the K-matrix is obtained by putting into equation (3.13) those values $\Lambda_j^{(n)}$ of the Λ_j for which $\delta K' = 0$. Thus, the $\Lambda_j^{(n)}$ are given by

$$K_i = \sum_{k=1}^{n}(K_{i+k-1} - K_{i+k})\Lambda_k^{(n)} \quad , \quad 1 \le i \le n \tag{3.14}$$

and then

$$K^{(n)} = \sum_{i=1}^{n} K_i \Lambda_i^{(n)} \tag{3.15}$$

The (n+1) equations given by (3.14) and (3.15) can be put more concisely as

$$K^{(n)} = \sum_{k=1}^{i} K_k + \sum_{k=1}^{n} K_{k+i} \Lambda_k^{(n)} \quad , \quad 0 \leq i \leq n \tag{3.16}$$

These equations (3.16) now give a sequence of approximations $K^{(1)}, K^{(2)}, K^{(3)}$--- to the K-matrix, each of which is hermitian.

Equation (3.3) can now be used to define a sequence of approximations $S^{(1)}, S^{(2)}, S^{(3)}$--- to the S-matrix, the n'th approximation $S^{(n)}$ being given by

$$S^{(n)} = \frac{1 - \frac{i}{2} K^{(n)}}{1 + \frac{i}{2} K^{(n)}} \tag{3.17}$$

Since $K^{(n)}$ is hermitian for all values of n, (3.17) ensures that the approximations $S^{(1)}, S^{(2)}, S^{(3)}$--- to the S-matrix are all unitary.

Now the S-matrix can be written as a power series in g

$$S = 1 + S_1 + S_2 + \cdots + S_r + \cdots \tag{3.18}$$

where, as shown by Dyson (13),

$$S_r = \frac{(-i)^r}{r!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_r \, P\left[H'(t_1) H'(t_2) \cdots H'(t_r)\right] \tag{3.19}$$

P ordering the operators chronologically from right to left. It therefore follows from (3.3), (3,11) and (3.18) that

$$2i S_r = \sum_{k=1}^{r-1} K_k S_{r-k} + 2K_r \quad , \quad r \geq 1 \tag{3.20}$$

Making use of (3.16) and (3.20), Cini and Fubini show, after some algebraic manipulations, that

$$S^{(n)} = \sum_{k=0}^{i} S_k + \sum_{k=1}^{n} S_{k+i} \Gamma_k^{(n)} \quad , \quad 0 \leq i \leq n \tag{3.21}$$

This set of equations determines the operators $\Gamma_k^{(n)}$ as the same functions of the S_i as the $\Lambda_k^{(n)}$ are of the K_i.

It is worthwhile noting that, although the set of equations (3.21) for $S^{(n)}$ do not have an explicit dependence on the K-matrix, the introduction of the K-matrix was necessary as an intermediate step in order to obtain a variational principle which maintained the unitarity of the S-matrix. Lippmann and Schwinger in fact derive a direct variational principle for the S-matrix but, as they show, it does not ensure that the resulting expression for the S-matrix is unitary.

The S_j in equation (3.21) can be evaluated using the covariant techniques of Feynman (21) and Dyson (13). However, they contain certain divergences which we should like to remove by some renormalisation proceedure. Cini and Fubini define such a proceedure by introducing a new set of (n+1) operators $\Delta_l^{(n)}$ such that

$$\Gamma_k^{(n)} = \sum_{l=k}^{n} \Delta_l^{(n)} \quad \text{and} \quad 1 = \sum_{l=0}^{n} \Delta_l^{(n)} \tag{3.22}$$

With these operators, equations (3.21) become

$$S^{(n)} = \sum_{l=0}^{n} \sigma_{i+1} \Delta_l^{(n)} \quad , \quad 0 \leq i \leq n \tag{3.23}$$

where
$$\sigma_i = \sum_{l=0}^{i} S_l \tag{3.24}$$

The divergences can now be removed from the σ_i by the standard methods of mass and coupling constant renormalisation discussed by Dyson (13) and Matthews (30). As is pointed out by Cini and Fubini, this prescription for renormalisation is not wholly satisfactory since the renormalised constants introduced into the equations are defined in terms of power series in the coupling constant. This does not fit well into the general spirit of the method which attempts to avoid the use of the normal perturbation method of defining quantities in this way. Another unsatisfactory feature of this prescription is that, in a given approximation $S^{(n)}$, approximations to the renormalised constants appear in the various σ_i to different orders in the coupling constant. In spite of these unsatisfactory features, Cini and Fubini use this prescription as the only unambiguous and consistent one available.

If σ_i' and S_i' are the finite parts of σ_i and S_i defined by this prescription, it follows from a reversal of the arguments used to obtain equations (3.23) and (3.24) that

$$S'^{(n)} = \sum_{k=0}^{i} S_k' + \sum_{k=1}^{n} S_{k+i}' \Gamma_k^{(n)} \quad , \quad 0 \leq i \leq n \tag{3.25}$$

Now, since the renormalised power series for S' i.e.

$$S' = 1 + S_1' + S_2' + \cdots + S_r' + \cdots \tag{3.26}$$

is unitary, then

$$\sum_{r=0}^{k} S_r' S_{k-r}'^{+} = 0 \quad , \quad k > 0 \tag{3.27}$$

This relationship (3.27) guarantees that the renormalised

$S'^{(n)}$ given by (3.25) is unitary.

For this method, which we have just developed, of obtaining a sequence of approximations to the S-matrix to have any validity, it is necessary that the power series expansions (3.10),(3.11) and (3.18) of the various operators used should be convergent or, at least, asymptotic for sufficiently small values of the coupling constant g. From analogy with electrodynamics, it would appear that this condition is satisfied.

Cini and Fubini have investigated a special case when the power series are divergent for $g = G$ where G is the actual meson-nucleon coupling constant, the divergence being due to the existence of a finite number of poles in the complex g-plane for $|g| < G$; these poles are closely related to the existence of isobaric states of the meson-nucleon system which lead to resonances in the cross-sections. They find that, although the power series diverge, the sequence of approximations $S^{(1)}, S^{(2)}, S^{(3)}$--- converges in this case. Thus, the method is applicable in this case where perturbation theory is not valid.

Cini, Morpurgo and Touschek (10) have applied the method to the Wentzel pair theory neglecting nucleon recoil. This theory can be solved exactly and they find that the method gives the exact solution in lowest approximation, a result which is not altered by going to a higher approximation.

Finally, it is useful to note that the S_j and K_j appearing in our equations have matrix elements only between states of equal energy. This property, as will be seen later, considerably helps in the solution of the operator equations for the $S^{(n)}$.

3.2 The equations for elastic and double scattering.

We now apply this method of Cini and Fubini to the problems of elastic and double meson-nucleon scattering. To do this, we use the lowest order approximation to the S-matrix which is $S^{(3)} - S^{(1)} = S_1 = 0$.

Making use of equations (3.21) for $n = 2$, we obtain

$$S^{(2)} = 1 + S_2 \rho_2^{(2)}$$

$$S^{(2)} = 1 + S_2 \rho_1^{(2)} + S_3 \rho_2^{(2)}$$

$$S^{(2)} = 1 + S_2 + S_3 \rho_1^{(2)} + S_4 \rho_2^{(2)}$$

which yield

$$S^{(2)} = 1 + S_2 (S_2 - S_3 - S_4 + S_3 S_2^{-1} S_3)^{-1} S_2 \qquad (3.28)$$

Similarly, equations (3.16) yield

$$K^{(2)} = K_2 (K_2 - K_3 - K_4 + K_3 K_2^{-1} K_3)^{-1} K_2 \qquad (3.29)$$

Defining the matrix T by the equation

$$S^{(2)} = 1 + T \qquad (3.30)$$

equation (3.28) becomes

$$T = S_2 + (P + Q - Q^2)\,T \tag{3.31}$$

where $$P = S_4 S_2^{-1} \text{ and } Q = S_3 S_2^{-1} \tag{3.32}$$

We now consider a meson-nucleon system whose total energy ϵ in the centre-of-mass system is such that $\epsilon < M+3\mu$; there are only two real states possible viz. a state of one meson and one nucleon and a state of two mesons and one nucleon. In the centre-of-mass system, the two particle state can be specified completely by the three components of the relative momentum of the two particles, the z-component of the nucleon spin and the charges of the meson and the nucleon i.e. by six quantities which we shall denote collectively by a Roman letter so that $|a)$ is the state vector of a two particle state. The three particle state requires ten quantities to characterise it completely; these will be denoted collectively by a Greek letter and they can be chosen as the linear momenta of the two mesons, the z-component of the nucleon spin and the charges of the three particles. The state vector $|\alpha)$ therefore describes a state of two mesons and one nucleon.

From the structure of the meson-nucleon interaction hamiltonian, $H'(t)$, which is given by (2.6) and is now time dependent since we are working in the interaction representation, it is clear that $H'(t)$ has matrix elements only between states differing by one meson and by zero or two nucleons. Hence, from equation (3.19), it follows that matrix elements of S_{2n} exist only between states differing by an even number of particles and matrix elements of S_{2n+1} exist only between states differing by an odd number of particles. Therefore

$$(\alpha|S_2|a) = (a|S_2|\alpha) = (\alpha|S_4|a) = (a|S_4|\alpha) = (a|S_3|b) = 0 \qquad (3.33)$$

so that, by (3.32)

$$(\alpha|P|a) = (a|P|\alpha) = (a|Q|b) = 0 \qquad (3.34)$$

Now the S-matrix has matrix elements only between states of the same energy and the same total linear momentum. Working in the centre-of-mass system guarantees that the total linear momentum is conserved between states. The conservation of energy can be explicitly introduced into the equations by defining matrices $\underset{\sim}{S}_i$ such that

$$(1|S_i|2) = \delta(E_1 - E_2)(1|\underset{\sim}{S}_i|2) \qquad (3.35)$$

where E_1 and E_2 are the total energies of the states defined by $|1)$ and $|2)$. It follows from (3.30) and (3.32) that the matrices T,P and Q also have matrix elements only between states of equal energies, so that matrices $\underset{\sim}{T}$,$\underset{\sim}{P}$ and $\underset{\sim}{Q}$ can be defined in the same way as $\underset{\sim}{S}_i$ in (3.35).

If we now take matrix elements of equation (3.31), we obtain, using (3.33),(3.34) and (3.35)

$$(a|\underline{T}|b) = (a|\underline{S}_2|b) + \sum_{\alpha}{}' (a|\underline{Q}|\alpha)\,\delta(E_a-E_\alpha)(\alpha|\underline{T}|b)$$
$$+ \sum_{c}{}' \Big[(a|\underline{P}|c) - \sum_{\alpha}{}' (a|\underline{Q}|\alpha)\,\delta(E_a-E_\alpha)(\alpha|\underline{Q}|c)\Big]\,\delta(E_a-E_c)(c|\underline{T}|b) \tag{3.36}$$

and,

$$(\alpha|\underline{T}|a) = \sum_{\beta}{}' \Big[(\alpha|\underline{P}|\beta) - \sum_{b}{}' (\alpha|\underline{Q}|b)\,\delta(E_\alpha-E_b)(b|\underline{Q}|\beta)\Big]\cdot\delta(E_\alpha-E_\beta)(\beta|\underline{T}|a)$$
$$+ \sum_{b}{}' (\alpha|\underline{Q}|b)\,\delta(E_\alpha-E_b)(b|\underline{T}|a) \tag{3.37}$$

where the summations mean that all variables specifying the state have to be summed over all possible values and the various matrix elements of the matrices $\underline{P}$ and $\underline{Q}$ are given by

$$(a|\underline{S}_4|b) = \sum_{c}{}' (a|\underline{P}|c)\,\delta(E_a-E_c)(c|\underline{S}_2|b) \tag{3.38}$$

$$(\alpha|\underline{S}_4|\beta) = \sum_{\gamma}{}' (\alpha|\underline{P}|\gamma)\,\delta(E_\alpha-E_\gamma)(\gamma|\underline{S}_2|\beta) \tag{3.39}$$

$$(\alpha|\underline{S}_3|a) = \sum_{b}{}' (\alpha|\underline{Q}|b)\,\delta(E_\alpha-E_b)(b|\underline{S}_2|a) \tag{3.40}$$

and
$$(a|\underline{S}_3|\alpha) = \sum_{\beta}{}' (a|\underline{Q}|\beta)\,\delta(E_a-E_\beta)(\beta|\underline{S}_2|\alpha) \tag{3.41}$$

These equations can be simplified by constructing the eigenfunctions of the various matrices $\underline{P}, \underline{Q}, \underline{T}$ and $\underline{S}_i$ appearing in them. To do this, we note that the matrices $\underline{P}$ and $\underline{Q}$ are functions of the $\underline{S}_i$ which are defined as functions of the hermitian matrices $\underline{K}_j$ by equations (3.20); it follows that the matrices $\underline{P}, \underline{Q}, \underline{T}$ and $\underline{S}_i$ have the same eigenfunctions as the matrices $\underline{K}_j$. We therefore consider the eigenvalue equation for $\underline{K}_k$, some typical one of the matrices $\underline{K}_j$,

$$\sum_a (b|\underset{\sim}{K}_k|a) f_a^{AA'} = K_k'^A f_b^{AA'} \tag{3.42}$$

where $K_k'^A$ is the eigenvalue of $\underset{\sim}{K}_k$ corresponding to the eigenfunction $f_a^{AA'}$, A' denoting any degeneracy which may exist. Since $\underset{\sim}{K}_k$ is hermitian, $K_k'^A$ is real and the $f_a^{AA'}$ can be arranged to form a complete orthonormal set so that

$$\sum_a f_a^{AA'*} f_a^{BB'} = \delta_{AB}\,\delta_{A'B'} \tag{3.43}$$

and, by the property of closure,

$$\sum_{AA'} f_a^{AA'*} f_b^{AA'} = \delta_{ab} \tag{3.44}$$

Now, since $\underset{\sim}{K}_k$ is invariant under rotation in coordinate space and in isotopic spin space, it follows that A corresponds to the total angular momentum J and the total isotopic spin I of the system and, A' to the z-component M of J and the z-component i of I. Thus, the eigenfunction $f_a^{AA'}$ must be a linear combination of products of eigenfunctions of the total angular momentum and of the total isotopic spin of the system corresponding to the appropriate values of these quantities.

If $\underset{\sim}{n}_a$ is a unit vector in the direction of $\underset{\sim}{k}_a$, the relative linear momentum of the meson and nucleon in the two particle state $|a)$, then $\mathcal{Y}^M_{J l \frac{1}{2}}(\underset{\sim}{n}_a, m_a)$, as defined in (2.45), is an eigenfunction corresponding to a total angular momentum J with z-component M and $l = J \pm \frac{1}{2}$ is the relative orbital angular momentum of the two particles.

Similarly, the eigenfunction corresponding to a total isotopic spin I with z-component i is given by

$$x^{i}_{I1\frac{1}{2}}(t_a,t_a') = \sum_{z} C^{1\ \frac{1}{2}\ \ I}_{i-z\ z}\ Y^{i-z}_{1}(t_a)\, Y^{z}_{\frac{1}{2}}(t_a') \tag{3.45}$$

where t_a and t_a' are the z-components of the meson and nucleon isotopic spins in some chosen z-direction so that

$$Y^{i-z}_{1}(t_a) = \delta_{t_a,i-z} \quad \text{and} \quad Y^{z}_{\frac{1}{2}}(t_a') = \delta_{t_a',z} \tag{3.46}$$

Thus,

$$f^{AA'}_{a} \equiv f^{JI,Mi}_{a} = \sum_{l=J-\frac{1}{2}}^{J+\frac{1}{2}} F_{lJI}(k_a)\, \mathcal{Y}^{M}_{Jl\frac{1}{2}}(n_a,m_a)\, x^{i}_{I1\frac{1}{2}}(t_a,t_a') \tag{3.47}$$

where $F_{lJI}(k_a)$ is some function of k_a.

Now, multiplying both sides of (3.42) by $f^{AA'*}_{c}$ and summing over A and A', we obtain, with the use of (3.44),

$$(b|\underset{\sim}{K}_k|c) = \sum_{AA'} f^{AA'*}_{c}\, K'^{A}_{k}\, f^{AA'}_{b} \tag{3.48}$$

so that

$$(b|\underset{\sim}{K}_k|a) = \sum_{JIMi}\sum_{ll'} \mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_a,m_a)\, x^{i*}_{I1\frac{1}{2}}(t_a,t_a')\, F^{*}_{lJI}(k_a)\, K'^{JI}_{k}\, F_{l'JI}(k_b)$$
$$\times\, \mathcal{Y}^{M}_{Jl'\frac{1}{2}}(n_b,m_b)\, x^{i}_{I1\frac{1}{2}}(t_b,t_b') \tag{3.49}$$

However, $\underset{\sim}{K}_k$ has matrix elements only between states of equal energy so that $E_{k_a}+\omega_{k_a} = E_{k_b}+\omega_{k_b}$ i.e. $k_a = k_b$. Also, $\underset{\sim}{K}_k$ is invariant under a reversal of the coordinate axes, so that it has matrix elements only between states of the same parity. Thus, in (3.49) $(-1)^{l} = (-1)^{l'}$; but $l = J\pm\frac{1}{2}$ and $l' = J\pm\frac{1}{2}$, so that $l = l'$. Putting

$$K^{I}_{klJ}(k_a;k_a) = F^{*}_{lJI}(k_a)\,K'^{JI}_{k}\,F_{lJI}(k_a) \qquad (3.50)$$

we obtain,

$$(b|\underline{K}_k|a) = \sum_{JIl} K^{I}_{klJ}(k_a;k_a) \sum_{Mi} \mathcal{Y}^{M}_{Jl\frac{1}{2}}(n_b,m_b)\,\mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_a,m_a)\,\chi^{i}_{I1\frac{1}{2}}(t_b,t'_b)\,\chi^{i*}_{I1\frac{1}{2}}(t_a,t'_a) \qquad (3.51)$$

Since K'^{JI}_{k} is real, it follows from (3.50) that $K^{I}_{klJ}(k_a;k_a)$ is a real function of k_a.

If the matrix $\underline{R}$ represents any of the matrices $\underline{P}, \underline{Q}, \underline{T}$ or $\underline{S}_i$ appearing in our equations, then, as has already been discussed, $\underline{R}$ has the same eigenfunctions as the $\underline{K}_j$ and so can be expanded like $\underline{K}_k$ giving,

$$(b|\underline{R}|a) = \sum_{JIl} R^{I}_{lJ}(k_a;k_a) \sum_{Mi} \mathcal{Y}^{M}_{Jl\frac{1}{2}}(n_b,m_b)\,\mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_a,m_a)\,\chi^{i}_{I1\frac{1}{2}}(t_b,t'_b)\,\chi^{i*}_{I1\frac{1}{2}}(t_a,t'_a) \qquad (3.52)$$

However, since $\underline{R}$ is not a hermitian operator, $R^{I}_{lJ}(k_a;k_a)$ is not a real function of k_a.

For a three particle state, we define

$$Y^{JM}_{l_1l_2(L)\frac{1}{2}}(n'_\alpha,n''_\alpha,m_\alpha) = \sum_{s,m} C^{L\ \frac{1}{2}\ J}_{M-s\ s}\, C^{l_1\ l_2\ L}_{M-s-m\ m}\, Y^{M-s-m}_{l_1}(n'_\alpha)\, Y^{m}_{l_2}(n''_\alpha)\, Y^{s}_{\frac{1}{2}}(m_\alpha) \qquad (3.53)$$

where $\underline{n}'_\alpha$ and $\underline{n}''_\alpha$ are unit vectors in the directions of $\underline{k}'_\alpha$ and $\underline{k}''_\alpha$, the linear momenta of the two mesons. $Y^{JM}_{l_1l_2(L)\frac{1}{2}}(n'_\alpha,n''_\alpha,m_\alpha)$ is an eigenfunction corresponding to a total angular momentum J with z-component M for which the mesons have orbital angular momenta l_1 and l_2 combining together to give an angular momentum L.

We also define

$$X^{Ii}_{11(j)\frac{1}{2}}(t'_\alpha, t''_\alpha, t_\alpha) = \sum_{\kappa z} C^{j\ \ \frac{1}{2}\ \ I}_{i-\kappa\ \ \kappa} \, C^{1\ \ \ 1\ \ j}_{i-\kappa-z\ \ z} \, Y_1^{i-\kappa-z}(t'_\alpha) \, Y_1^{z}(t''_\alpha) \, Y_{\frac{1}{2}}^{\kappa}(t_\alpha) \qquad (3.54)$$

which is an eigenfunction of total isotopic spin I with z-component i for which the meson pair has isotopic spin j.

With these definitions (3.53) and (3.54) and making use of the same arguments concerning the rotational invariance of the system in coordinate space and in isotopic spin space, as were used in deducing (3.52), we obtain the following expansions

$$(\beta|R|a) = \sum_{\substack{Jll_1l_2L \\ Ij}} R^{Ij}_{lJl_1l_2(L)}(k'_\beta, k''_\beta; k_a) \sum_{Mi} \mathcal{Y}^{JM}_{l_1l_2(L)\frac{1}{2}}(n'_\beta, n''_\beta, m_\beta) \, \mathcal{Y}^{M*}_{Jl\frac{1}{2}}(n_a, m_a)$$
$$\times X^{Ii}_{11(j)\frac{1}{2}}(t'_\beta, t''_\beta, t_\beta) \, \chi^{i*}_{I1\frac{1}{2}}(t_a, t'_a) \qquad (3.55)$$

$$(a|R|\beta) = \sum_{\substack{Jll_1l_2L \\ Ij}} R^{Ij}_{lJl_1l_2(L)}(k_a; k'_\beta, k''_\beta) \sum_{Mi} \mathcal{Y}^{M}_{Jl\frac{1}{2}}(n_a, m_a) \, \mathcal{Y}^{JM*}_{l_1l_2(L)\frac{1}{2}}(n'_\beta, n''_\beta, m_\beta)$$
$$\times \chi^{i}_{I1\frac{1}{2}}(t_a, t'_a) \, X^{Ii*}_{11(j)\frac{1}{2}}(t'_\beta, t''_\beta, t_\beta) \qquad (3.56)$$

and

$$(\beta|R|\alpha) = \sum_{\substack{Jl_1l_2L \\ l'_1l'_2L' \\ Ijj'}} R^{Ijj'}_{l_1l_2(L)J(L')l'_1l'_2}(k'_\beta, k''_\beta; k'_\alpha, k''_\alpha) \sum_{Mi} \mathcal{Y}^{JM}_{l_1l_2(L)\frac{1}{2}}(n'_\beta, n''_\beta, m_\beta)$$
$$\times \mathcal{Y}^{JM*}_{l'_1l'_2(L')\frac{1}{2}}(n'_\alpha, n''_\alpha, m_\alpha) \, X^{Ii}_{11(j)\frac{1}{2}}(t'_\beta, t''_\beta, t_\beta) \, X^{Ii*}_{11(j')\frac{1}{2}}(t'_\alpha, t''_\alpha, t_\alpha) \qquad (3.57)$$

Remembering that the meson has odd intrinsic parity, conservation of parity restricts the summations in (3.55) and (3.56) to values of l_1, l_2 and l such that $(l_1 + l_2 - l)$ is odd and in (3.57) to values of l_1, l_2, l'_1 and l'_2 such that $(l_1 + l_2 - l'_1 - l'_2)$ is even.

(3.52), (3.55), (3.56) and (3.57) provide us with expansions of the matrix elements of the operators appearing in our equations between states specified by linear momenta etc. in terms of matrix elements of the same operators between states specified by angular momenta, parity and isotopic spin. We shall use these expansions to help simplify the equations (3.36) - (3.41).

Meanwhile, it is worth noting that, as far as isotopic spin dependence is concerned, any production matrix element for the production of a meson in a meson-nucleon collision, i.e. a matrix element of the type $(\beta|R|\alpha)$, can be expressed in terms of four independent matrix elements R^{Ij} where for $I = \frac{3}{2}$, $j = 1$ or 2 and for $I = \frac{1}{2}$, $j = 0$ or 1. This is a result of the conservation of isotopic spin which reduces the total number of independent processes that are possible. This is the same situation as arises in the problem of meson production in nucleon-nucleon collisions where as shown by Watson and Brueckner (37), all the production matrix elements can be expressed in terms of three independent quantities.

Inserting the expansions (3.52) and (3.55) into equation (3.40) we obtain

$$\sum_{\substack{J\ell\ell_1\ell_2 L\\ IjMi}} Y^{JM}_{\ell_1\ell_2(L)\frac{1}{2}}(\underline{n}'_\alpha,\underline{n}''_\alpha,m_\alpha)\, X^{Ii}_{1(j)\frac{1}{2}}(t'_\alpha,t''_\alpha,t_\alpha)\, S^{Ii}_{J(J\ell_1\ell_2(L)}(k'_\alpha,k''_\alpha;k_\alpha)\, \mathcal{Y}^{M*}_{J\ell\frac{1}{2}}(\underline{n}_\alpha,m_\alpha)\, \chi^{i*}_{I1\frac{1}{2}}(t_\alpha,t'_\alpha)$$

$$= \sum_{\substack{JM\ \ell_1\ell_2 L\\ J'M'\ell'\ell''\\ IjiI'i'}} \sum_{\substack{t_b t'_b\\ m_b}} \int d^3k_b\, Y^{JM}_{\ell_1\ell_2(L)\frac{1}{2}}(\underline{n}'_\alpha,\underline{n}''_\alpha,m_\alpha)\, X^{Ii}_{1(j)\frac{1}{2}}(t'_\alpha,t''_\alpha,t_\alpha)\, Q^{Ij}_{\ell_1\ell_2(L)J\ell''}(k'_\alpha,k''_\alpha;k_b)$$

$$\times\ \mathcal{Y}^{M*}_{J\ell''\frac{1}{2}}(\underline{n}_b,m_b)\, \chi^{i*}_{I1\frac{1}{2}}(t_b,t'_b)\, \delta(E-\omega_{k_b}-E_{k_b})\, \mathcal{Y}^{M'}_{J'\ell'\frac{1}{2}}(\underline{n}_b,m_b)\, \chi^{i'}_{I'1\frac{1}{2}}(t_b,t'_b)$$

$$\times\ S^{I'}_{2\ell'J'}(k_a;k_a)\, \mathcal{Y}^{M'*}_{J'\ell'\frac{1}{2}}(\underline{n}_a,m_a)\, \chi^{i'*}_{I'1\frac{1}{2}}(t_a,t'_a) \qquad (3.58)$$

where $E_{k_b} = \sqrt{k_b^2 + M^2}$ and $\omega_{k_b} = \sqrt{k_b^2 + \mu^2}$.

Using the orthonormality properties

$$\sum_m \int d\Omega\, \mathcal{Y}^{M*}_{J\ell\frac{1}{2}}(\underline{n},m)\, \mathcal{Y}^{M'}_{J'\ell'\frac{1}{2}}(\underline{n},m) = \delta_{JJ'}\,\delta_{MM'}\,\delta_{\ell\ell'} \tag{3.59}$$

$$\sum_m \int d\Omega \int d\Omega'\, Y^{JM*}_{\ell_1\ell_2(L)\frac{1}{2}}(\underline{n},\underline{n}',m)\, Y^{J'M'}_{\ell_1'\ell_2'(L')\frac{1}{2}}(\underline{n},\underline{n}',m) = \delta_{JJ'}\,\delta_{MM'}\,\delta_{\ell_1\ell_1'}\,\delta_{\ell_2\ell_2'}\,\delta_{LL'} \tag{3.60}$$

$$\sum_{t_a t_a'} \chi^{i*}_{I1\frac{1}{2}}(t_a,t_a')\, \chi^{i'}_{I'1\frac{1}{2}}(t_a,t_a') = \delta_{II'}\,\delta_{ii'} \tag{3.61}$$

and

$$\sum_{t_a t_a' t_a''} X^{Ii*}_{11(j)\frac{1}{2}}(t_a',t_a'',t_a)\, X^{I'i'}_{11(j')\frac{1}{2}}(t_a',t_a'',t_a) = \delta_{II'}\,\delta_{ii'}\,\delta_{jj'} \tag{3.62}$$

the integration and all the summations in (3.58) can be carried out, giving

$$S^{Ij}_{3\ell J\ell_1\ell_2(L)}(k_a',k_a'';k_a) = Q^{Ij}_{\ell J\ell_1\ell_2(L)}(k_a',k_a'';k_a)\,\rho_a\, S^{I}_{2\ell J}(k_a;k_a) \tag{3.40'}$$

where

$$\rho_a = \frac{k_a E_{k_a}\omega_{k_a}}{E_{k_a}+\omega_{k_a}} = \frac{1}{\epsilon} k_a E_{k_a}\omega_{k_a}. \tag{3.63}$$

Similarly, using expansion (3.52) in equation (3.38) gives

$$S^{I}_{4\ell J}(k_a;k_a) = P^{I}_{\ell J}(k_a;k_a)\,\rho_a\, S^{I}_{2\ell J}(k_a;k_a) \tag{3.38'}$$

The expansions (3.52), (3.55) and (3.56) allow us to carry out all the integrations and summations over the variables in a two particle intermediate state. The situation is more complicated in the case of a three particle intermediate state where, as we shall see presently the expansions (3.55) - (3.57) do not allow us to carry out all the summations and integrations.

However, let us suppose for the moment, that $\epsilon < M+2\mu$ so that the system cannot exist in a real three particle state. The equations (3.36) - (3.41) reduce to

$$(a|T|b) = (a|S_2|b) + \sum_c (a|\rho|c)\,\delta(E_a - E_c)(c|T|b) \tag{3.64}$$

where $$(a|S_4|b) = \sum_c (a|\rho|c)\,\delta(E_a - E_c)(c|S_2|b) \tag{3.65}$$

Using the same arguments and expansions as were used in deducing (3.38') and (3.40'), (3.64) yields,

$$T^I_{13}(k_a;k_a) = S^I_{213}(k_a;k_a) + P^I_{13}(k_a;k_a)\,\rho_a\,T^I_{13}(k_a;k_a) \tag{3.64'}$$

which, with (3.38'), gives

$$T^I_{13}(k_a;k_a) = \frac{S^{I^2}_{213}(k_a;k_a)}{S^I_{213}(k_a;k_a) - S^I_{413}(k_a;k_a)} \tag{3.66}$$

Thus, for the simple two-body problem, at energies below the threshold for the production of any new particles, the operator equations, deduced from the formalism of Cini and Fubini, reduce to simple algebraic equations from which numerical results can easily be calculated. It is clear that this simplification does not depend on the fact that we have carried through the calculations only in the lowest order of approximation of the CF method so that to any order of approximation, the two-body equations reduce to a set of algebraic equations. However, in higher orders of approximation, difficulties arise in the evaluation of the higher order S-matrix elements occurring in the equations; indeed, in this lowest approximation, which we are considering, an analytic form for the matrix element of S has not yet been obtained in pseudoscalar theory with ps-coupling.

The expression for the reaction matrix, K, analogous to (3.66)

$$K^{I}_{lJ}(k_a;k_a) = \frac{K^{I^2}_{1lJ}(k_a;k_a)}{K^{I}_{2lJ}(k_a;k_a) - K^{I}_{4lJ}(k_a;k_a)} \tag{3.67}$$

has been used by Sartori and Wataghin (33) to calculate the meson-nucleon elastic scattering phase shifts. They do not make a complete relativistic calculation but make the following non-relativistic approximations. For the calculation of the p-wave phase shifts, they use the interaction hamiltonian

$$H' = \frac{g}{2M}\int d^3x\, \rho(x)\, \underline{\sigma}.\underline{\nabla}\, \tau_\alpha \phi_\alpha(x) \tag{3.68}$$

which is the hamiltonian used by Chew et al. (7),(8) and which can be deduced from the complete ps(ps) hamiltonian (2.6) by considering only positive energy states and neglecting nucleon recoil. $\rho(x)$ is the nucleon source density and the σ_i are the usual Pauli matrices. The neglect of nucleon recoil must be compensated for by introducing in momentum integrals a cut-off which is treated as an arbitrary parameter of the theory. By choosing suitable values for this cut-off and for the coupling constant, g, which is the only other parameter in the theory, Sartori and Wataghin were able to establish good agreement between their results for the p-wave phase shift for scattering through a state of total angular momentum $\frac{3}{2}$ and isotopic spin $\frac{3}{2}$ and the experimental results as deduced by Glicksman (24). Rough agreement was obtained with the other experimental phase shifts. The values of the two parameters used, correspond fairly closely to the values used by Chew in his calculations using the TD method.

For the calculation of the s-wave phase shifts, the matrix elements are evaluated using the complete relativistic hamiltonian (2.6) and then a non-relativistic approximation is taken by expanding out the nucleon energy in terms of the nucleon momentum and retaining only the two lowest order terms in an expression. Using the same values of the coupling constant and the cut-off momentum as were used in the p-wave calculations, they find that the s-wave phase shifts agree qualitatively with the experimental values for incident meson energies above 30 Mev. Below 30 Mev, the phase shift for scattering through a state of isotopic spin $\frac{1}{2}$ goes through a resonance; this resonance is not observed experimentally and Sartori and Wataghin attribute its presence to the fact that, in lowest approximation, the trial operator used in the CF method is too simple to account for all the details of the full solution.

We now return to the more general case of our original problem when $\epsilon < M+3\mu$ and the production of a meson becomes possible. We must now consider terms in our equations in which there is a sum over the variables of a three particle state.

Inserting expansions (3.55) - (3.57) into equation (3.41), we obtain

$$\sum_{\substack{J l l_1 l_2 L \\ I j \; M i}}' \mathcal{Y}^{M}_{J l \frac{1}{2}}(\underline{n}_\alpha, m_\alpha)\, x^{i}_{I l \frac{1}{2}}(t_\alpha, t_\alpha')\, S^{Ij}_{J l J l_1 l_2 (L)}(k_\alpha; k_\alpha', k_\alpha'')\, \mathcal{Y}^{JM*}_{l_1 l_2 (L) \frac{1}{2}}(\underline{n}_\alpha', \underline{n}_\alpha'', m_\alpha)\, X^{Ii*}_{11(j)\frac{1}{2}}(t_\alpha', t_\alpha'', t_\alpha)$$

$$= \sum_{\substack{J M l l_1 l_2 L \\ J' M' l_1' l_2' L' \\ l_1'' l_2'' L'' \\ I i I' i' j j' j''}}' \sum_{\substack{t_\beta t_\beta' t_\beta'' \\ m_\beta}} \int d^3k_\beta' \int d^3k_\beta''\, \mathcal{Y}^{M}_{J l \frac{1}{2}}(\underline{n}_\alpha, m_\alpha)\, x^{i}_{I l \frac{1}{2}}(t_\alpha, t_\alpha')\, Q^{Ij''}_{l J l_1'' l_2'' (L'')}(k_\alpha; k_\beta', k_\beta'')$$

$$\times\, \mathcal{Y}^{JM*}_{l_1'' l_2'' (L'')\frac{1}{2}}(\underline{n}_\beta', \underline{n}_\beta'', m_\beta) \cdot X^{Ii*}_{11(j'')\frac{1}{2}}(t_\beta', t_\beta'', t_\beta)\, \delta(\epsilon - \omega_{k_\beta'} - \omega_{k_\beta''} - E_{\underline{k}_\beta' + \underline{k}_\beta''})\, \mathcal{Y}^{J'M'}_{l_1' l_2' (L')\frac{1}{2}}(\underline{n}_\beta', \underline{n}_\beta'', m_\beta)$$

$$\times\, X^{I'i'}_{11(j')\frac{1}{2}}(t_\beta', t_\beta'', t_\beta)\, S^{I j' j}_{J l_1' l_2' (L') J (L) l_1 l_2}(k_\beta', k_\beta''; k_\alpha', k_\alpha'')\, \mathcal{Y}^{J'M'*}_{l_1 l_2 (L)\frac{1}{2}}(\underline{n}_\alpha', \underline{n}_\alpha'', m_\alpha)\, X^{I'i'*}_{11(j)\frac{1}{2}}(t_\alpha', t_\alpha'', t_\alpha)$$

(3.69)

However, the second order S-matrix element between two three particle states of the type $(\underline{k}, \underline{k}' | S_2 | \underline{k}, \underline{k}')$ is non-zero only if the momentum of one of the mesons in the initial state is the same as that of a similarly charged meson in the final state; this is easily seen from the Feynman graphs representing the second order matrix element - one of the meson lines must go straight through. Thus

$$(\underline{k}, \underline{k}' | S_2 | \underline{k}, \underline{k}') = (\underline{k}, \underline{k}' | S_2 | \underline{k}, \underline{k}')\, \delta(\underline{k}' - \underline{k}') \qquad (3.70)$$

Using (3.70) along with (3.61) and (3.62) in (3.69), we obtain

$$\sum_{\substack{J\lambda l_1 l_2 L\\M}} \mathcal{Y}^{M}_{J\lambda\frac{1}{2}}(n_a, m_a)\, S^{Ij}_{J\lambda J l_1 l_2 (L)}(k_a; k_\alpha', k_\alpha'')\, \mathcal{Y}^{JM*}_{l_1 l_2 (L)\frac{1}{2}}(n_\alpha', n_\alpha'', m_\alpha)$$

$$= \sum_{\substack{JM l_1 l_2 L\\ J'M' l_1' l_2' L'\\ l_1'' l_2'' L'' j'}} \sum_{m_\beta}{}' \int d^3k_\beta'\, \mathcal{Y}^{M}_{J\lambda\frac{1}{2}}(n_a, m_a)\, Q^{Ij'}_{\lambda J l_1'' l_2''(L'')}(k_a; k_\beta', k_\alpha'')\, \mathcal{Y}^{JM*}_{l_1'' l_2''(L'')\frac{1}{2}}(n_\beta', n_\alpha'', m_\beta)$$

$$\times\, \delta\left(\epsilon - \omega_{k_\beta'} - \omega_{k_\alpha''} - E_{k_\beta'+k_\alpha''}\right) \mathcal{Y}^{J'M'}_{l_1' l_2'(L')\frac{1}{2}}(n_\beta', n_\alpha'', m_\beta)$$

$$\times\, S^{Ij'j}_{J l_1' l_2'(L') J'(L) l_1 l_2}(k_\beta', k_\alpha''; k_\alpha', k_\alpha'')\, \mathcal{Y}^{J'M'*}_{l_1 l_2 (L)\frac{1}{2}}(n_\alpha', n_\alpha'', m_\alpha) \qquad (3.71)$$

The angular integrations in (3.71) can be performed as is shown in Appendix II and the equation becomes

$$S^{Ij}_{J\lambda J l_1 l_2 (L)}(k_a; k_\alpha', k_\alpha'') = \sum_{\substack{l_1' l_2' L'\\ l_1'' l_2'' L''\\ J' \mu\, j'}} \int dk_\beta'\, Q^{Ij'}_{\lambda J l_1'' l_2''(L'')}(k_a; k_\beta', k_\alpha'')\, F_\mu\left(l_1 l_2 (L)\, l_1' l_2' (L')\, l_1'' l_2'' (L'')\, J J' \lambda_2 L\right)$$
$$\times\, G_\mu(k_\beta', k_\alpha'')\, S^{Ij'j}_{J l_1' l_2'(L') J'(L) l_1 l_2}(k_\beta', k_\alpha''; k_\alpha', k_\alpha'') \qquad (3.41')$$

where

$$F_\mu\left(l_1 l_2 (L)\, l_1' l_2' (L')\, l_1'' l_2'' (L'')\, J J' \lambda_2 L\right) = (-1)^{J+J'+l_2'+l_1''+l_2''+L'}\, \frac{1}{4\pi}\, (2J'+1)(2L+1)$$

$$\times \sqrt{\left\{\tfrac{1}{4\pi}(2l_2+1)(2l_1'+1)(2l_2'+1)(2l_2''+1)(2L'+1)(2L''+1)(2\mu+1)\right\}}\; C^{l_1'\ \mu\ l_1''}_{0\ \ 0\ \ 0}$$

$$\times \sum_{\alpha\rho}{}' \sqrt{(2\alpha+1)(2\rho+1)}\; C^{l_2''\ \mu\ \rho}_{0\ \ 0\ \ 0}\; C^{\rho\ \ l_2'\ \alpha}_{0\ \ 0\ \ 0}\; C^{\alpha\ l_2''\ l_2}_{0\ \ 0\ \ 0}\; W(l_1'\mu L l_2; l_1\rho)$$

$$\times\, W(L' l_2' L\rho; l_1'\alpha)\, W(J'\tfrac{1}{2}\alpha L; L'J)\, W(\alpha l_2' L l_1''; l_2 L'')\, W(\alpha L'' J \tfrac{1}{2}; L J') \qquad (3.72)$$

W(abcd;ef) is a Racah coefficient (2) and

$$\left.\begin{aligned} G_\mu(k_\beta'; k_\alpha'') &= 2\pi \frac{k_\beta'}{k_\alpha''}\left(\epsilon - \omega_{k_\beta'} - \omega_{k_\alpha''}\right) Y^{0*}_{\mu}(\cos\theta_0), \quad k_\alpha'' \neq 0 \\ &= \sqrt{4\pi}\, k_\beta' \frac{E_{k_\beta'}\,\omega_{k_\beta'}}{E_{k_\beta'}+\omega_{k_\beta'}}\, \delta_{\mu 0}\, \delta(k_\beta' - k_{\beta_0}'), \quad k_\alpha = 0 \end{aligned}\right\} \qquad (3.73)$$

where $\epsilon - \mu - \omega_{k'_{\beta_0}} - E_{k'_{\beta_0}} = 0$ defines k'_{β_0} and $\epsilon - \omega_{k'_\beta} - \omega_{k''_\alpha} - \sqrt{k'^2_\beta + k''^2_\alpha + M^2 + 2k'_\beta k''_\alpha \cos\theta_0} = 0$ defines $\cos\theta_0$; The values of k'_β and k''_α are restricted to those values for which $-1 \leq \cos\theta_0 \leq 1$ The limits of this restriction can be seen by plotting k''_α against k'_β for various values of $\cos\theta_0$; This plot is

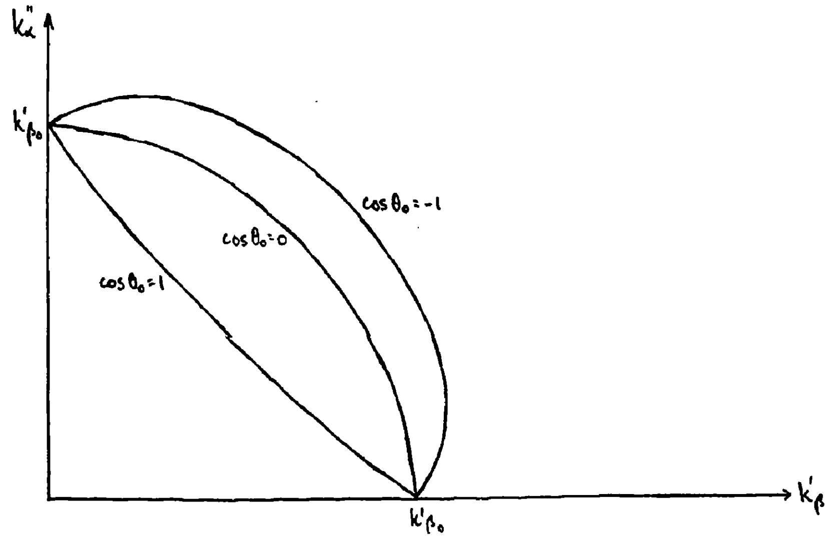

Fig. (3.1)

sketched in figure (3.1). Values of k'_β and k''_α are allowed only if the point in the figure corresponding to a given pair of values lies within or on the boundary of the area defined by the curve for $\cos\theta_0 = \pm 1$. This area is zero for $\epsilon = M + 2\mu$, i.e. at the meson production threshold, and increases as increases above this value.

Applying the same arguments to equation (3.39) as we have just applied to (3.41), we obtain

$$S^{Ijj'}_{4l_1l_2(L)J(L')l_1'l_2'}(k'_j, k''_j; k'_\alpha, k''_\alpha) = \sum_{\substack{l_1''l_2''L'' \\ l_1'''l_2'''L''' \\ J'\mu\, j''}} \int dk'_\beta \, P^{Ijj''}_{l_1l_2(L)J(L'')l_1''l_2''}(k'_\gamma, k''_\gamma; k'_\beta, k''_\alpha)$$
$$\times F_\mu\left(l_1'l_2'(L')\, l_1'''l_2'''(L''')\, l_1''l_2''(L'')\, JJ'l_2'L'\right) G_\mu(k'_\beta, k''_\alpha)$$
$$\times S^{Ij''j'}_{2l_1'''l_2'''(L''')J'(L')l_1'l_2'}(k'_\beta, k''_\alpha; k'_\alpha, k''_\alpha) \qquad (3.39')$$

with the same definitions and limits of integration as apply to (3.41').

In equations (3.39') and (3.41'), the three-dimensional integrations have been reduced to one integration over a single momentum variable, but the expressions are still considerably complicated by the large number of summations over angular momentum indices. The second and third order S-matrix elements appearing in (3.39') and (3.41') can be calculated fairly easily using the covariant techniques of Feynman (21) and Dyson (13). The fourth order matrix element in (3.39') is the sum of matrix elements corresponding to a number of Feynman graphs; these graphs divide into two sets. In the first set, the four mesons of the initial and final states all interact with the nucleon; these graphs do not present any difficulty in their evaluation. The second set consists of graphs in which one of the mesons does not interact with the nucleon at all; the matrix elements corresponding to these graphs present considerable difficulty since, even after those, which are formally infinite, have been made finite by a renormalisation, some of the integrals in terms of which the matrix elements are expressed, cannot be performed analytically for arbitrary values of the momenta of the particles.

Assuming a knowledge of all the S-matrix elements appearing in equations (3.39') and (3.41'), finding a solution of these equations for the matrix elements of the matrices

P and Q is a complicated procedure due to the large number of summations to be carried out over angular momentum indices. Having solved these equations, the integral equations (3.36) and (3.37), in which the matrix elements of P and Q appear as kernels, have still to be solved for the T-matrix elements before any cross-sections can be calculated. This is a long and complicated programme which we have made no attempt to carry through. Instead, we have made the following approximation, for which, however, we have been able to find no real justification at present.

In equation (3.37), the terms on the right side involving $(\beta|T|a)$ are assumed to be small compared to the other terms on this side of the equation. We neglect these terms and, consequently, the integral equation becomes the simple equation

$$(\alpha|T|a) = \sum_b (\alpha|Q|b)\,\delta(E_\alpha - E_b)(b|T|a) \qquad (3.74)$$

which, after using the expansions (3.52) and (3.55) of the matrix elements and equation (3.40'), yields

$$T^{I_j}_{\lambda J l_1 l_2 (L)}(k'_\alpha, k''_\alpha; k_a) = \frac{S^{I_j}_{3\lambda J l_1 l_2 (L)}(k'_\alpha, k''_\alpha; k_a)}{S^{I}_{2\lambda J}(k_a; k_a)}\, T^{I}_{\lambda J}(k_a; k_a) \qquad (3.75)$$

Some justification for our approximation may come from the fact that, close to the production threshold, the production cross-section is much smaller that the elastic scattering cross-section. This may imply that the terms in (3.37) involving the production matrix element $(\beta|T|\alpha)$ are much smaller than those involving the elastic scattering matrix element. However, too much reliance cannot be put on this argument, since the smallness of the cross-section for a process near its threshold energy is, in general, due to the small density of final states factor and not to any smallness of the matrix element.

For energies below the production threshold, the elastic scattering phase shift, $\delta^{I}_{lJ}(k_\alpha)$, for scattering through a state of angular momentum J, parity $(-1)^{l+1}$ and isotopic spin I is related to the T and K matrix elements by

$$e^{2i\delta^{I}_{lJ}(k_\alpha)} - 1 = \rho_\alpha T^{I}_{lJ}(k_\alpha;k_\alpha) \qquad (3.76)$$

and

$$2\tan\delta^{I}_{lJ}(k_\alpha) = -\rho_\alpha K^{I}_{lJ}(k_\alpha;k_\alpha) \qquad (3.77)$$

For energies above threshold, inelastic scattering becomes possible and the phase shifts become complex. However, for energies lying above but close to the threshold, the inelastic i.e. the production cross-section is much smaller than the elastic scattering cross-section so that, in calculations on the elastic scattering process, the effect of the competing production process can be neglected. Thus, in this energy region near threshold, experimental data on elastic meson-nucleon scattering can be analysed on the assumption that the scattering phase shifts are real. Equations (3.76) and (3.77) can then be used to find experimental behaviour of $T^{I}_{\ell s}(k_a;k_a)$ and $K^{I}_{\ell s}(k_a;k_a)$ as functions of the energy.

Much work has been carried out on the phase shift analysis of the experimental results on meson-nucleon scattering, the most recent and extensive of which is the work of de Hoffmann et al. (25); the experimental data has been analysed up to an incident meson energy in the laboratory system of 217 Mev which is about 45 Mev above the production threshold. The production process has been completely neglected in their analysis. We shall use their results in equation (3.76) to calculate the values of $T^{I}_{\ell s}(k_a;k_a)$ which we shall use in equation (3.75). A calculation of the second and third order S-matrix elements then allows us to find the values of $T^{I_i}_{\ell s j,\lambda_2(L)}(k_a',k_a'';k_a)$ from which the double scattering cross-section may be calculated.

3.3 Calculation and discussion of double scattering cross-sections.

(A) Formulae for double scattering cross-section.

We now consider in detail one particular double scattering process; this we choose to be the production of a positive meson in the collision of a positive meson and a proton

$$\pi^+ + p \longrightarrow \pi^+ + \pi^+ + n \tag{3.78}$$

To calculate the matrix element for a particular production process, it is convenient to use the original isotopic spin representation in which a state is specified by the z-components of the isotopic spin of the individual particles. Instead of (3.52) and (3.55), we have

$$(b|\underset{\sim}{R}|a) = \sum_{lJ}(t_b,t_b'|R_{lJ}(k_a;k_a)|t_a,t_a')\sum_M \mathcal{Y}^M_{Jl\frac{1}{2}}(\underset{\sim}{n}_b,m_b)\,\mathcal{Y}^{M*}_{Jl\frac{1}{2}}(\underset{\sim}{n}_a,m_a) \tag{3.79}$$

and

$$(\beta|\underset{\sim}{R}|a) = \sum_{lJl_1l_2L}{}'(t_\beta',t_\beta'',t_\beta|R_{lJl_1l_2(L)}(k_\beta',k_\beta'';k_a)|t_a,t_a')\sum_M{}' \mathcal{Y}^{JM}_{l_1l_2(L)\frac{1}{2}}(\underset{\sim}{n}_\beta',\underset{\sim}{n}_\beta'',m_\beta)\,\mathcal{Y}^{M*}_{Jl\frac{1}{2}}(\underset{\sim}{n}_a,m_a) \tag{3.80}$$

so that, in our approximation, equation (3.37) gives

$$(t_1',t_2',t_3'|T_{lJl_1l_2(L)}(p,k;q)|t_1,t_2) = \sum_{t_1''t_2''}{}'(t_1',t_2',t_3'|Q_{lJl_1l_2(L)}(p,k;q)|t_1''t_2'')\rho_q(t_1'',t_2''|T_{lJ}(q;q)|t_1,t_2) \tag{3.81}$$

where

$$\sum_{t_1''t_2''}{}'(t_1',t_2',t_3'|Q_{lJl_1l_2(L)}(p,k;q)|t_1''t_2'')\rho_q(t_1'',t_2''|S_{2lJ}(q;q)|t_1,t_2) = (t_1',t_2',t_3'|S_{3lJl_1l_2(L)}(p,k;q)|t_1,t_2) \tag{3.82}$$

For the production process (3.78), $t_1'=t_2'=-1$, $t_3'=+\frac{1}{2}$, $t_1=-1$ and $t_2=-\frac{1}{2}$; it follows from charge conservation that $t_1''=-1$ and $t_2''=-\frac{1}{2}$ so that (3.82) gives

$$(-1,-1,\tfrac{1}{2}|Q_{lJl_1l_2(L)}(p,k;q)|-1,-\tfrac{1}{2}) = \frac{(-1,-1,\frac{1}{2}|S_{3lJl_1l_2(L)}(p,k;q)|-1,-\frac{1}{2})}{\rho_q\,(-1,-\frac{1}{2}|S_{2lJ}(q;q)|-1,-\frac{1}{2})} \tag{3.83}$$

and, with (3.81), this yields

$$(-1,-1,\tfrac{1}{2}|T_{lJl_1l_2(L)}(p,k;q)|-1,-\tfrac{1}{2}) = \frac{(-1,-1,\frac{1}{2}|S_{3lJl_1l_2(L)}(p,k;q)|-1,-\frac{1}{2})}{(-1,-\frac{1}{2}|S_{2lJ}(q;q)|-1,-\frac{1}{2})}(-1,-\tfrac{1}{2}|T_{lJ}(q;q)|-1,-\tfrac{1}{2}) \tag{3.84}$$

For other production processes where the final state can be produced from two different initial states e.g.

$$\left.\begin{array}{l}\pi^- + p \\ \pi^0 + n\end{array}\right\} \rightarrow \pi^- + \pi^0 + p$$

in place of equation (3.83), we have two simultaneous equations for the two Q-matrix elements which appear in equation (3.81).

It follows from (3.79) and (3.52) that

$$(t_1'', t_2'' | T_{\lambda J}(q;q) | t_1, t_2) = \sum_{Ii} T^I_{\lambda J}(q;q)\, x^i_{I\frac{1}{2}}(t_1'', t_2'')\, x^{i*}_{I\frac{1}{2}}(t_1, t_2)$$

$$= \sum_I T^I_{\lambda J}(q;q)\, C^{1\ \frac{1}{2}\ I}_{t_1''\ t_2''}\, C^{1\ \frac{1}{2}\ I}_{t_1\ t_2}\, \delta_{t_1+t_2,\, t_1''+t_2''} \qquad (3.85)$$

where use has been made of the definitions (3.45) and (3.46). Thus, with the help of (3.76), any matrix element $(t_1'', t_2'' | T_{\lambda J}(q;q) | t_1, t_2)$ can be expressed in terms of the phase shifts for scattering.

For the process (3.78) we have $C^{1\ \frac{1}{2}\ I}_{-1\ -\frac{1}{2}} = \delta_{I,\frac{3}{2}}$ so that

$$(-1, -\tfrac{1}{2} | T_{\lambda J}(q;q) | -1, -\tfrac{1}{2}) = T^{\frac{3}{2}}_{\lambda J}(q;q) = \frac{2i}{\rho_q} e^{i\delta^{\frac{3}{2}}_{\lambda J}(q)} \sin \delta^{\frac{3}{2}}_{\lambda J}(q) \qquad (3.86)$$

Now, if $(\underline{p}^{t_1'}, \underline{k}^{t_2'}, (-\underline{p}-\underline{k})^{t_3'}_{m'} | \underline{T} | \underline{q}^{t_1}, -\underline{q}^{t_2}_m)$ is the T-matrix element for the production of two mesons of momenta $\underline{p}$ and $\underline{k}$ with z-components of isotopic spin t_1' and t_2' and a nucleon of momentum $(-\underline{p}-\underline{k})$ with z-component of spin m' and of isotopic spin t_3' from a state of one meson and one nucleon whose momenta are respectively $\underline{q}$ and $-\underline{q}$ and whose z-components of isotopic spin are t_1 and t_2, the nucleon having z-component of spin m, then, as is shown in Appendix III, the total production cross-section, $(t_1', t_2', t_3' | \sigma(\epsilon) | t_1, t_2)$, where the z-components of the isotopic spins of the particles are still specified, is given by

$$(t_1', t_2', t_3' | \sigma(\epsilon) | t_1, t_2) = (2\pi)^2 \frac{E_q \omega_q}{q \epsilon} \int d^3p \int d^3k \, \frac{1}{2} \sum_{m, m'} \left| \left(p^{t_1'}, k^{t_2'}, (-p-k)^{t_3'}_{m'} | T | q^{t_1}, -q^{t_2}_m \right) \right|^2$$
$$\times \, \delta(\epsilon - \omega_p - \omega_k - E_{p+k}) \qquad (3.87)$$

where $\epsilon = E_q + \omega_q$. Making use of (3.80), we can write

$$\left(p^{t_1'}, k^{t_2'}, (-p-k)^{t_3'}_{m'} | T | q^{t_1}, q^{t_2}_m \right) = \sum_{lJl_1l_2L} (t_1', t_2', t_3' | T_{lJl_1l_2(L)}(p, k; q) | t_1, t_2)$$
$$\times \sum_M Y^{JM}_{l_1 l_2 (L) l \frac{1}{2}}(n_p, n_k, m') \, \mathcal{Y}^{M*}_{J l \frac{1}{2}}(n_q, m) \qquad (3.88)$$

Since the experimental results on meson-nucleon scattering have been analysed to give scattering phase shifts only up to an incident meson energy about 45 Mev above the meson production threshold in the laboratory system, we shall be concerned with the production only at energies close to threshold; the final state particles will therefore have low kinetic energies and will consequently be produced almost entirely in s-states. The terms in (3.88) in which $l_1 = l_2 = 0$ are therefore much larger than the other terms in the expansion at these energies, so that those terms for which $l_1 \neq 0$ and $l_2 \neq 0$ can be neglected completely. When $l_1 = l_2 = 0$, $L = 0$ and $J = \frac{1}{2}$ and it follows from conservation of parity that $l = 1$; thus, in (3.88) we retain only terms in $T_{1\frac{1}{2}00(0)}(p, k; q)$.

Inserting (3.88) into (3.87), the summations over the spins are easily carried out giving

$$(t_1', t_2', t_3' | \sigma(\epsilon) | t_1, t_2) = \frac{E_q \omega_q}{16\pi \epsilon q} \int d^3p \int d^3k \left| (t_1', t_2', t_3' | T_{1\frac{1}{2}00(0)}(p,k;q) | t_1, t_2) \right|^2 \times \delta(\epsilon - \omega_p - \omega_k - E_{p+k}) \qquad (3.89)$$

for the total production cross-section at energies close to threshold.

Henceforth, we shall be concerned only with the process (3.78) and, as no ambiguity will arise, we shall omit the isotopic spin variables and denote the cross-section by $\sigma(\epsilon)$ Then,

$$\sigma(\epsilon) = \frac{E_q \omega_q}{16\pi \epsilon q} \int d^3p \int d^3k \left| (-1,-1,\tfrac{1}{2} | T_{1\frac{1}{2}00(0)}(p,k;q) | -1,-\tfrac{1}{2}) \right|^2 \delta(\epsilon - \omega_p - \omega_k - E_{p+k}) \qquad (3.90)$$

If we put $\underset{\sim}{T} = \underset{\sim}{S}_3$, we obtain the lowest order perturbation theory result, $\sigma_p(\epsilon)$, for the cross-section:

$$\sigma_p(\epsilon) = \frac{E_q \omega_q}{16\pi \epsilon q} \int d^3p \int d^3k \left| (-1,-1,\tfrac{1}{2} | S_{3;1\frac{1}{2}00(0)}(p,k;q) | -1,-\tfrac{1}{2}) \right|^2 \delta(\epsilon - \omega_p - \omega_k - E_{p+k}) \qquad (3.91)$$

Using (3.84) and (3.86), we obtain the cross-section , $\sigma_{CF}(\epsilon)$, derived from the CF method:

$$\sigma_{CF}(\epsilon) = \left(\frac{2\epsilon}{q E_q \omega_q} \right)^2 \frac{\sin^2 \delta^{\frac{1}{2}}_{1\frac{1}{2}}(q)}{\left| (-1,-\tfrac{1}{2} | S_{2;\frac{1}{2}}(q;q) | -1,-\tfrac{1}{2}) \right|^2} \sigma_p(\epsilon) \qquad (3.92)$$

(B) Second and third order S-matrix elements.

The only Feynman graph contributing to the matrix element $(-1,-\tfrac{1}{2} | S_{2;\frac{1}{2}}(q;q) | -1,-\tfrac{1}{2})$ is shown in figure (3.2) where capital letters refer to four-vectors. Using the interaction hamiltonian (2.6), the matrix element corresponding to this graph is

$$\left(K'^{-1}, Q'^{-\frac{1}{2}} \middle| S_2 \middle| K^{-1}, Q^{-\frac{1}{2}}\right) = -2g^2(2\pi)^4\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\right]^4 \frac{M}{\sqrt{Q'_0 Q_0}} \frac{1}{2\sqrt{K'_0 K_0}} \delta_4(Q+K-Q'-K')$$

$$\times\ \bar{u}(Q')\,\gamma_5\, \frac{(Q-K').\gamma + iM}{(Q-K')^2+M^2}\,\gamma_5\, u(Q) \qquad (3.93)$$

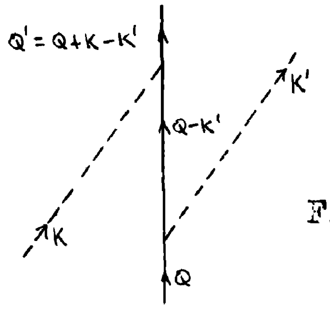

Fig. (3.2)

where standard notation has been used and the spinors are normalised so that $\bar{u}(Q)u(Q) = 1$.

After some algebra, this yields

$$\left(\underline{q}'^{-1}, -\underline{q}'^{-\frac{1}{2}} \middle| \underline{S}_2 \middle| \underline{q}^{-1}, -\underline{q}^{-\frac{1}{2}}\right) = \sum_{Jl} \left(-1,-\tfrac{1}{2}\middle| S_{2lJ}(q;q) \middle| -1,-\tfrac{1}{2}\right) \sum_M \mathcal{Y}^M_{Jl\frac{1}{2}}(\underline{n}_{q'},m') \, \mathcal{Y}^{M*}_{Jl\frac{1}{2}}(\underline{n}_q,m) \qquad (3.94)$$

with

$$\left(-1,-\tfrac{1}{2}\middle| S_{2lJ}(q;q) \middle| -1,-\tfrac{1}{2}\right) = U_l(q) + V_{l+1}(q)\,\delta_{J,l+\frac{1}{2}} + V_{l-1}(q)\,\delta_{J,l-\frac{1}{2}} \qquad (3.95)$$

where

$$U_l(q) = 2\pi\sqrt{\frac{4\pi}{2l+1}} \int_{-1}^{1} d(\cos\theta)\, U(q,\cos\theta)\, Y_l^{0*}(\cos\theta) \qquad (3.96)$$

and

$$V_l(q) = 2\pi\sqrt{\frac{4\pi}{2l+1}} \int_{-1}^{1} d(\cos\theta)\, V(q,\cos\theta)\, Y_l^{0*}(\cos\theta) \qquad (3.97)$$

$$U(q,\cos\theta) = -\frac{i}{2\pi}\left(\frac{g^2}{4\pi}\right) \frac{E_q+M}{E_q\omega_q} \frac{E_q+\omega_q-M}{2E_q\omega_q-\mu^2+2q^2\cos\theta} \qquad (3.98)$$

and

$$V(q,\cos\theta) = -\frac{i}{2\pi}\left(\frac{g^2}{4\pi}\right) \frac{q^2}{E_q\omega_q(E_q+M)} \frac{E_q+\omega_q+M}{2E_q\omega_q-\mu^2+2q^2\cos\theta} \qquad (3.99)$$

Thus

$$\left(-1,-\tfrac{1}{2}\middle| S_{21\frac{1}{2}}(q;q) \middle| -1,-\tfrac{1}{2}\right) = U_1(q) + V_0(q) \qquad (3.100)$$

where

$$U_1(q) = -\frac{i}{2}\left(\frac{g^2}{4\pi}\right) \frac{(E_q+M)(E_q+\omega_q-M)}{q^2 E_q\omega_q}\left[2 - \frac{2E_q\omega_q-\mu^2}{2q^2}\ln\left(\frac{2E_q\omega_q-\mu^2+2q^2}{2E_q\omega_q-\mu^2-2q^2}\right)\right] \qquad (3.101)$$

and

$$V_0(q) = -\frac{i}{2}\left(\frac{g^2}{4\pi}\right) \frac{E_q+\omega_q+M}{E_q\omega_q(E_q+M)} \ln\left(\frac{2E_q\omega_q-\mu^2+2q^2}{2E_q\omega_q-\mu^2-2q^2}\right) \qquad (3.102)$$

Two Feynman graphs contribute to the matrix element $\left(-1,-1,\frac{1}{2}\middle|S_{3,\frac{1}{2}00(0)}(p,k;q)\middle|-1,-\frac{1}{2}\right)$; they are shown in figure (3.3). The

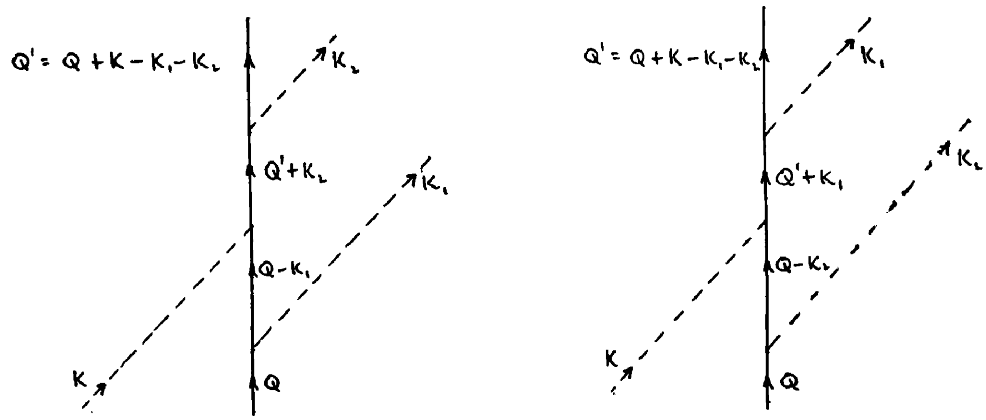

Fig. (3.3)

matrix element corresponding to these graphs is

$$\left(K_1^{-1}, K_2^{-1}, Q'^{+\frac{1}{2}}\middle|S_3\middle|K^{-1}, Q^{-\frac{1}{2}}\right) = (\sqrt{2}\,g)^3(2\pi)^4\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\right]^5 \frac{M}{\sqrt{Q_0'Q_0}}\,\frac{1}{\sqrt{2K_1}}\,\frac{1}{\sqrt{2K_2}}\,\frac{1}{\sqrt{2K}}\,\delta_4(K_1+K_2+Q'-K-Q)$$

$$\times\,\bar{u}(Q')\gamma_5\left[\frac{(Q'+K_2)\cdot\gamma+iM}{(Q'+K_2)^2+M^2}\,\gamma_5\,\frac{(Q-K_1)\cdot\gamma+iM}{(Q-K_1)^2+M^2}\right.$$

$$\left.+\,\frac{(Q'+K_1)\cdot\gamma+iM}{(Q'+K_1)^2+M^2}\,\gamma_5\,\frac{(Q-K_2)\cdot\gamma+iM}{(Q-K_2)^2+M^2}\right]\gamma_5 u(Q) \tag{3.103}$$

Since we consider only energies close to the production threshold, we can neglect the nucleon recoil in the factors $\frac{M}{Q_0'}$ and $\frac{1}{M+Q_0'}$, which appear in the matrix element, and replace them by 1 and $\frac{1}{2M}$ respectively. Straightforward calculations finally yield

$$(-1,-1,\tfrac{1}{2}|S_{3\frac{1}{2}00(0)}(p,k;q)|-1,-\tfrac{1}{2}) = -\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\left(\frac{g^2}{4\pi}\right)^{\frac{3}{2}}\sqrt{\frac{M}{E_q\,\omega_q\,\omega_p\,\omega_k}}\Big[U(p,k,q)F_0(p,q)$$

$$+U(k,p,q)F_0(k,q)+V(p,k)F_1(p,q)+V(k,p)F_1(k,q)\Big] \quad (3.104)$$

where $$U(p,k,q) = -\frac{q}{E_q+M}\left[\frac{1}{2M}(\omega_p+\omega_k)(\mu^2+\omega_p\omega_k)-\omega_p\omega_k\right] \quad (3.105)$$

$$V(p,k) = \frac{p}{2M}\left(\omega_p\omega_k-2M\omega_k-k^2\right) \quad (3.106)$$

and $$F_\ell(p,q) = 2\pi\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{E_q+M}{2M}}\;\frac{1}{2\epsilon(\omega_p-\omega_q)}\int_{-1}^{1}d(\cos\theta)\,\frac{Y_\ell^{0*}(\cos\theta)}{2E_q\omega_p-\mu^2+2pq\cos\theta} \quad (3.107)$$

so that,

$$F_0(p,q) = \frac{\pi}{2}\sqrt{\frac{E_q+M}{2M}}\;\frac{1}{\epsilon pq(\omega_p-\omega_q)}\ln\left(\frac{2E_q\omega_p-\mu^2+2pq}{2E_q\omega_p-\mu^2-2pq}\right) \quad (3.108)$$

and $$F_1(p,q) = \frac{\pi}{2}\sqrt{\frac{E_q+M}{2M}}\;\frac{1}{\epsilon pq(\omega_p-\omega_q)}\left[2-\frac{2E_q\omega_p-\mu^2}{2pq}\ln\left(\frac{2E_q\omega_p-\mu^2+2pq}{2E_q\omega_p-\mu^2-2pq}\right)\right] \quad (3.109)$$

(C) Cross-section in lowest order perturbation theory.

Equations (3.104) - (3.109) allow us to evaluate in (3.91). However, the integrals in (3.91) are complicated considerably by the fact that the δ-function in the integrand depends not only on the magnitudes of the momenta $\underline{p}$ and $\underline{k}$ but also on the angle θ_0 between these two vectors. In his work on meson production processes, Fermi (20) simplifies the integration by making use of the approximation that the meson mass is much smaller than the nucleon mass i.e. $\mu/M \ll 1$. This approximation allows one to neglect the recoil energy of the nucleon in the δ-function so that

E_{p+k} is effectively replaced by M; the δ-function is then independent of the angle θ_0. This approximation is made in analogy to the approximation usually made in β-decay theory where the recoil energy of the residual nucleus is neglected since the mass of the emitted electron is very much smaller than the mass of the nucleus.

The δ-function in (3.91) restricts the integral through (p,k,θ_0)-space to be over a certain surface. The form of this surface can be seen from figure (3.4) where sections

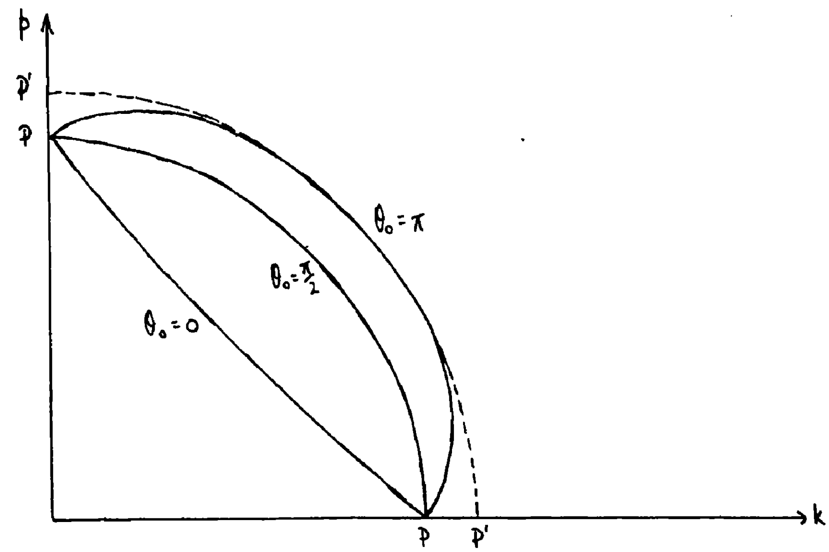

Fig. (3.4)

through the surface are sketched for various values of in full lines. P is defined by $\epsilon-\mu-\omega_P-E_P=0$. Fermi's approximation replaces this surface by one of constant section; its section is shown in figure (3.4) by the broken line; P' is defined by $\epsilon-\mu-\omega_{P'}-M=0$ so that P'> P.

We shall not make use of Fermi's approximation as it seems rather a drastic approximation to assume that when, in actual fact $\mu/M \simeq 0.15$. Instead, we proceed as follows.

For energies very close to threshold, all the final state particles can be treated non-relativistically, so that, putting $\epsilon = M+2\mu+T$, the δ-function in (3.91) becomes $\delta\left(T-\frac{p^2+k^2}{2\mu}-\frac{(p+k)^2}{2M}\right)$ and, consistent with this non-relativistic approximation, the kinetic energies of all the produced particles are neglected in $S_{31\frac{1}{2}00(0)}(p,k;q)$. Equation (3.91) becomes

$$\sigma_p(\epsilon) = \frac{E_q\omega_q}{16\pi\epsilon q}\left|(-1,-1,\tfrac{1}{2}|S_{31\frac{1}{2}00(0)}(0,0;q)|-1,-\tfrac{1}{2})\right|^2\int d^3p\int d^3k\,\delta\left(T-\frac{p^2+k^2}{2\mu}-\frac{(p+k)^2}{2M}\right) \qquad (3.110)$$

Making the transformation of variables $\underset{\sim}{p}+\underset{\sim}{k}=\underset{\sim}{a}$ and $\underset{\sim}{p}-\underset{\sim}{k}=\underset{\sim}{b}$, (3.110) becomes

$$\sigma_p(\epsilon) = \frac{1}{8}\,\frac{E_q\omega_q}{16\pi\epsilon q}\left|(-1,-1,\tfrac{1}{2}|S_{31\frac{1}{2}00(0)}(0,0;q)|-1,-\tfrac{1}{2})\right|^2\int d^3a\int d^3b\,\delta\left[T-\frac{b^2}{4\mu}-\left(\frac{1}{4\mu}+\frac{1}{2M}\right)a^2\right] \qquad (3.111)$$

The δ-function now depends only on the magnitudes of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The integrations are all elementary and can be performed giving

$$\sigma_p(\epsilon) = \frac{\pi^2}{4}\,\frac{E_q\omega_q}{\epsilon q}\left|(-1,-1,\tfrac{1}{2}|S_{31\frac{1}{2}00(0)}(0,0;q)|-1,-\tfrac{1}{2})\right|^2\left(\frac{\mu^2 M}{M+2\mu}\right)^{\frac{3}{2}}T^2 \qquad (3.112)$$

Taking $\mu = 140$ Mev and $M = 938$ Mev so that $\frac{M}{\mu} = 6.7$, the threshold for meson production occurs at an incident meson kinetic energy in the laboratory system of $E_\pi = 171$ Mev. The results of calculations made using equations (3.104) - (3.109) with (3.112) for various values of the incident meson kinetic energy, E_π, are given in Table I. Although values of $\sigma_p(\epsilon)$ are given up to $E_\pi = 200$ Mev, the non-relativistic approximation used in deriving (3.112) is not

E_π in Mev.	q/μ	ϵ/μ	$\sigma_p(\epsilon) \times \left(\frac{g^2}{4\pi}\right)^{-3} 10^{31}\,cm^{-2}$	
			non-rel. approx.	"mean surface" approx.
171	1.530	8.700	0	0
180	1.575	8.748	0.54	–
190	1.626	8.803	2.91	2.49
200	1.674	8.856	4.84	5.34
215	1.747	8.937	–	13.56

Table I.

valid as far above threshold as this.

At energies where the non-relativistic approximation is not valid, we replace the integration over the surface defined by $\epsilon-\omega_p-\omega_k-E_{p+k}=0$ by an integration over the surface defined by $\epsilon-\omega_p-\omega_k-W_{p+k}=0$ where $W_{p+k}=\sqrt{p^2+k^2+M^2}$. This is a surface whose section is given by the curve $\theta_0=\frac{\pi}{2}$ in figure (3.4) and is independent of θ_0. This surface can be regarded as being, in some way a mean of the variations with θ_0 of the original surface; it allows a certain amount of nucleon recoil to be taken into account. All the integrations in (3.91) can now be performed analytically with the exception of one, giving

$$\sigma_p(\epsilon) = \pi\,\frac{E_q\omega_q}{\epsilon q}\int_0^p k^2 dk\;\frac{p\,\omega_p W_{p+k}}{\omega_p+W_{p+k}}\left|\left(-1,-1,\tfrac{1}{2}\right|S_{31;\,00(0)}(p,k;q)\left|-1,-\tfrac{1}{2}\right)\right|^2 \qquad (3.113)$$

where, in the integrand, p is a function of k defined by $\epsilon - \omega_p - \omega_k - W_{p+k} = 0$ and, as before, P is defined by $\epsilon - \mu - \omega_P - E_P = 0$

It was not found possible to perform the integration in (3.113) analytically. It was therefore carried out numerically and, to the accuracy required, this was quite well accomplished by plotting the integrand as a function of k and finding the area enclosed by the curve and the limits of integration. This calculation was carried out for $E_\pi = 190$, 200 and 215 Mev, the results being shown in the last column of Table I.

This approximation will be fairly good when the values of the third order S-matrix element on the original surface do not vary much from the values on the "mean surface". For energies close to threshold, all the surfaces lie near to one another so that, providing the S-matrix is a reasonably well-behaved function of the momenta it will not vary much from one surface to another. This, along with the fair amount of agreement obtained between the result of this approximation and the result from the non-relativistic approximation at $E_\pi = 190$ Mev, where the non-relativistic approximation should be valid, provides some justification for the approximation.

A graph of the results given in Table I is shown in figure (3.5)

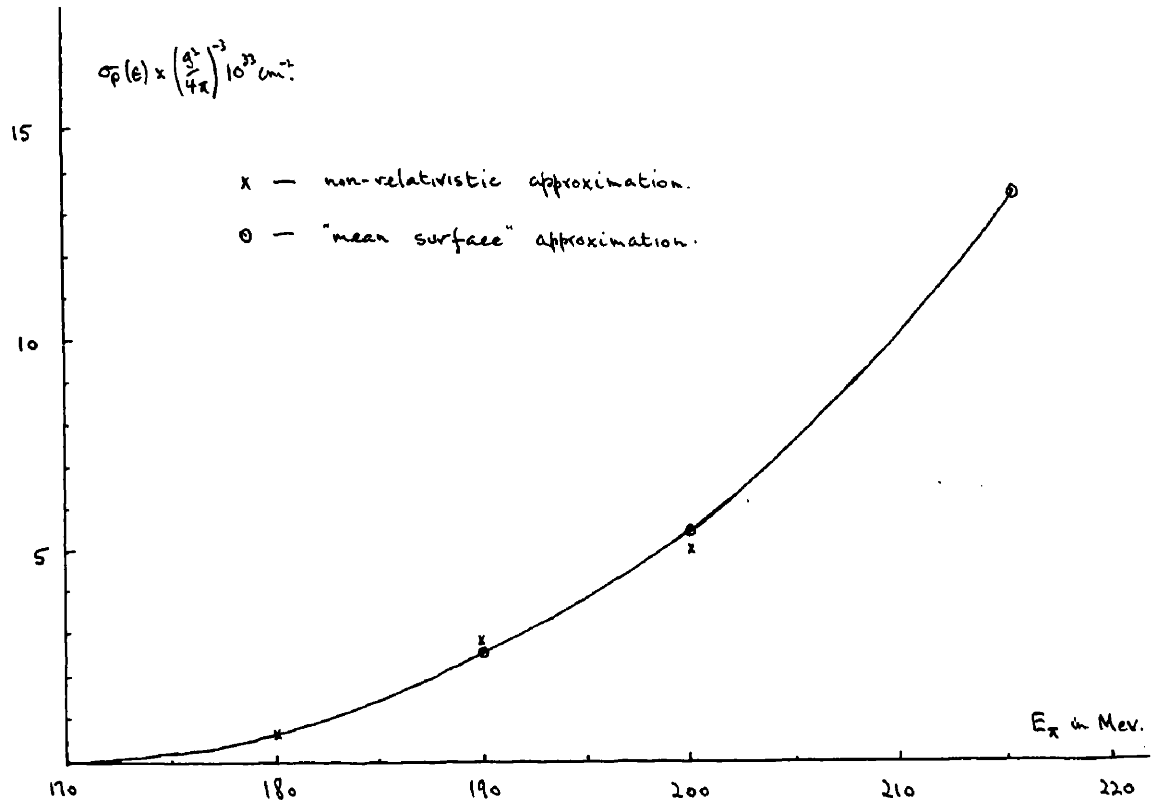

Fig. (3.5)

(D) Cross-section from Cini Fubini method.

We can now calculate the values of $\sigma_{cp}(\epsilon)$ for various values of E_π, by making use of equation (3.92). To do this we require a knowledge of $(-1,-\tfrac{1}{2}|S_{2\frac{1}{2}}(q;q)|-1,-\tfrac{1}{2})$ and $\delta^{\frac{3}{2}}_{1\frac{1}{2}}(q)$.

The values of the second order S-matrix element for different values of q can be evaluated from equations (3.100) - (3.102); these are given in the third column of Table II.

As is shown by de Hoffmann et al. (25), the phase shift analysis of the experimental data on meson-nucleon scattering does not lead to a unique set of phase shifts. However, on the grounds of the present theoretical predictions of the qualitative behaviour of the phase shifts and requiring that the phase shifts should be smooth functions of the energy of the system, they single out one solution which they believe is almost certain to be the correct one - this solution is the one obtained using interpolation (b) on Track I, in the notation of their paper. In this solution,

E_π in Mev.	$\delta^{3/2}_{1\frac{1}{2}}$	$(-1,-\frac{1}{2}\vert S_{21\frac{1}{2}}(q;q)\vert -1,-\frac{1}{2})\times i\left(\frac{g^2}{4\pi}\right)^{-1}10^3$	$\sigma_p(\epsilon)\times\left(\frac{g^2}{4\pi}\right)^{-3}10^{33}\,cm^{-2}$	$\sigma_{cf}(\epsilon)\times\left(\frac{g^2}{4\pi}\right)^{-4}10^{30}\,cm^{-2}$
171	-	-	0	0
180	$-6.6°$	5.993	0.54	0.15
190	$-8.6°$	6.097	2.49	1.01
200	$-10.6°$	6.190	5.34	2.86
215	$-13.6°$	6.318	13.56	10.41

Table II.

the phase shift $\delta^{3/2}_{1\frac{1}{2}}$ - α_{31} in their notation - is a linear function of the kinetic energy of the incident mesons in the laboratory system i.e. of E_π; this variation is shown in figure (3.6) and the required values of $\delta^{3/2}_{1\frac{1}{2}}$ are given in the second column of Table II.

In (3.92), the values of $\sigma_p(\epsilon)$ obtained from the "mean

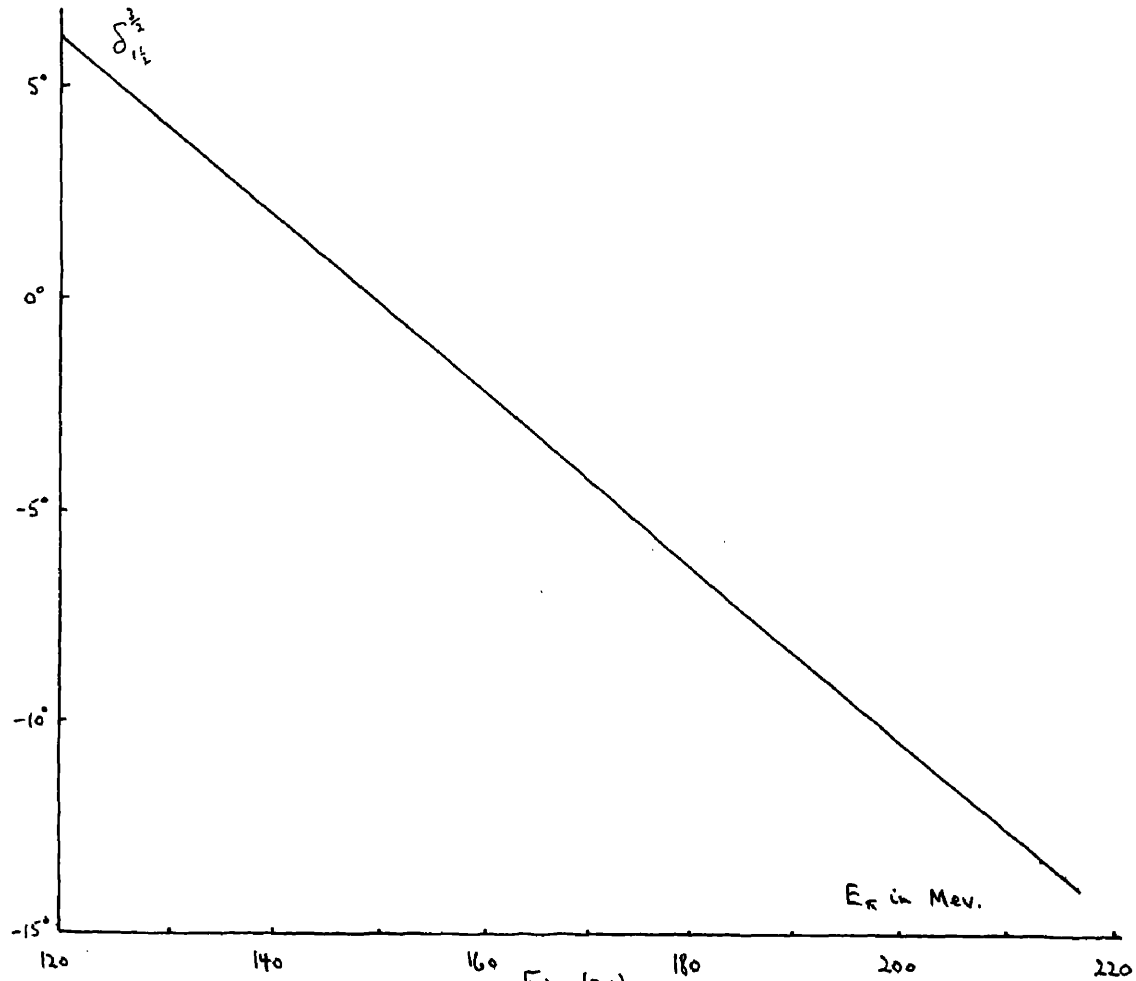
$\delta^{3/2}_{1\frac{1}{2}}$
5°
0°
-5°
-10°
-15°
E_π in Mev.
120
140
160
180
200
220

Fig. (3.6).

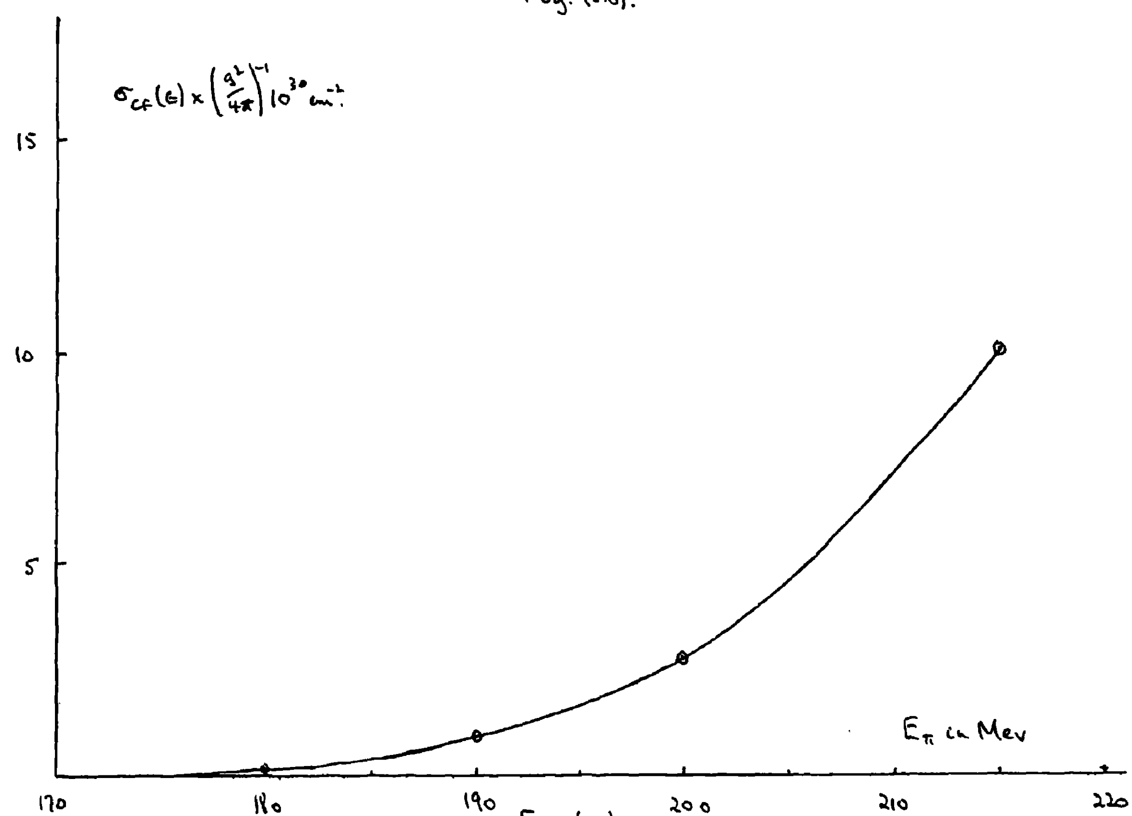
$\sigma_{cf}(\epsilon) \times \left(\frac{g^2}{4\pi}\right)^{-1} 10^{30}$ cm^{-2}.
15
10
5
E_π in Mev
170
180
190
200
210
220

Fig. (3.7).

surface" approximation were used at all energies except $E_{\pi} = 180$ Mev where the non-relativistic approximation result was used. The values of $\sigma_{cf}(\epsilon)$ obtained are given in the last column of Table II and the corresponding points plotted in figure (3.7).

(E) Cross-section from statistical theory.

The ratio of the meson production cross-section to the elastic scattering cross-section has also been calculated using the statistical theory of Fermi (20). This theory is based on the following idea. When a meson and a nucleon collide, their kinetic energy is suddenly released into a small volume surrounding the point of collision, the size of this volume being governed by the extent of the meson field surrounding the nucleon. As a first approximation, this volume is taken as a sphere of radius μ^{-1}. However, this sphere suffers a Lorentz contraction due to the motion of the nucleon of momentum $\underline{q}$; we therefore consider the energy as being released into a volume $V = \frac{4\pi}{3} \frac{M}{\mu^3 E_q}$. Within this volume, mesons, nucleons and anti-nucleons are continually being created and destroyed. The theory assumes that the meson-nucleon interaction is so strong that statistical equilibrium is attained within this volume before it breaks up into a number of freely moving particles. The probability that the final state will contain a certain number of particles is therefore proportional to the probability that these particles are all simultaneously present in the small volume V. This theory depends on the existence of a very strong coupling between mesons and nucleons in contrast to the weak coupling required for the validity of perturbation theory.

In the collision of a meson and a nucleon, each of momentum q, the ratio of the meson production or double scattering cross-section, σ_D, to the elastic scattering cross-section, σ_{el}, is given by

$$\frac{\sigma_D}{\sigma_{el}} = \frac{S(3)}{S(2)} \tag{3.114}$$

where $$S(2) = \frac{V}{(2\pi)^3}\int d^3p\ \delta(\epsilon-\omega_p-E_p) = \frac{V}{2\pi^2}\ \frac{qE_q\omega_q}{\epsilon} \tag{3.115}$$

and $$S(3) = \frac{V^2}{(2\pi)^6}\int d^3p\int d^3k\ \delta(\epsilon-\omega_p-\omega_k-E_{p+k}) = \frac{V^2}{16\pi^3}\left(\frac{\mu^2 M}{M+2\mu}\right)^{\frac{3}{2}}T^2 \tag{3.116}$$

In evaluating the integral in (3.116), we have assumed that the total energy of the system is sufficiently near the production threshold that a non-relativistic approximation is valid as far as orders of magnitude are concerned.

In the collision of a positive meson and a proton, the probabilities that the final nucleon is a neutron or a proton are equal, so that with σ the cross-section for process (3.78)

$$\frac{\sigma}{\sigma_{el}} = \frac{1}{2}\,\frac{S(3)}{S(2)} \tag{3.117}$$

Various values of σ/σ_{el}, evaluated from equations (3.115) - (3.117), are given in the second column of Table III.

E_π in Mev.	$\sigma/\sigma_d \times 10^4$	$\sigma_{cf}/\sigma_d \times 10^4$
171	0	0
180	0.8	0.1
190	3.1	0.8
200	6.8	2.4
215	14.6	10.1

Table III.

In the last column of Table III, the values of σ_d used in the calculation of the ratio σ_{cf}/σ_d are those corresponding to the phase shifts already employed in the calculation of σ_{cf} (cf. figure 2 of reference 25). To obtain definite numerical results for this ratio, for comparison with the results of the Fermi theory, we have used the value $g^2/4\pi = 16$; this is the value found by Dyson et al. (16) in their treatment of meson-nucleon scattering by the TD method which gives the best fit with experimental results.

(F) Discussion of results.

The results in Table II show that the double scattering cross-section $\sigma_{cf}(t)$ is larger than the corresponding cross-section $\sigma_p(\epsilon)$, obtained from lowest order perturbation theory, by a factor of the order of $\left(\frac{g^2}{4\pi}\right)^{-2} 10^3$ which, for $\frac{g^2}{4\pi} \sim 10$, is of the order of 10. Also, for this value of the coupling constant, $\sigma_{cf}(\epsilon) \sim 10^{-2}$mb. at an incident meson energy of 200 Mev. Now, at this energy the experimentally determined cross-section for the elastic scattering of positive mesons on protons is of the order of 100 mb. so that our calculated double scattering cross-section is smaller than the experimentally determined elastic scattering cross-section by a factor of 10^{-4}. It does not appear possible that, with present experimental techniques, the double scattering could be detected and measured with any accuracy at these energies close to the threshold when the competing process of elastic scattering has such a relatively large cross-section. Indeed, as has been discussed in Chapter I, the lowest energy at which any measurements have been made on the double scattering process is 500 Mev. for the scattering of negative mesons on protons (3),(4); even at this energy it was found difficult to measure the double scattering cross-section with any accuracy so that its value was only fixed, by these experiments, to lie somewhere in the wide range of values 3.5 - 10mb.

From the results in Table III, we see that, as far as orders of magnitude are concerned, the ratios of the double to the elastic scattering cross-section, calculated from the two independent theories of Fermi and of Cini and Fubini, agree extremely well, for $\frac{g^2}{4\pi} \sim 10$, at energies close to threshold. d'Espagnat (19) obtains a similar agreement between the ratio calculated from his theory and that of the Fermi theory; however, he finds that, as the energy increased, the Fermi ratio increases more rapidly than does his. Our results seem to indicate a less rapid increase of the Fermi ratio compared to the ratio from the CF method. It should be noted however that the non-relativistic approximation, used in calculating the Fermi ratio, loses its validity as the energy increases so that the results calculated with it do not show the correct energy dependence of the ratio at these higher energies.

Due to the approximations which were made to overcome the difficulties encountered in carrying through a calculation, based on the CF method, for a system in which states containing more than two particles are possible, our results cannot be used as the basis for a discussion of the validity of the CF method. The difficulties associated with applying the method to any but the simple two-body problem, as has been done with some success by Sartori and Wataghin (33), are clear from our work.

APPENDICES.

Appendix I.

To establish the relationships between the nucleon and meson self-energy terms (2.29) and (2.30) and the covariant expressions given in (2.32) and (2.33), we carry out the k_0-integration in the covariant integrals.

$$\Omega(P.\gamma) = -\frac{3g^2}{(2\pi)^4}\int d^4k\, \gamma_5 S_F(P-k)\gamma_5 \Delta_F(k)$$

$$= -\frac{3g^2}{(2\pi)^4}\int d^4k\, \gamma_5 \left[\frac{H_{-\underline{p}-\underline{k}} + P_0 - k_0}{E^2_{\underline{p}+\underline{k}} - (P_0-k_0)^2 - i\eta}\right] \frac{i\beta}{\omega_k^2 - k_0^2 - i\eta}\gamma_5$$

where $H_{-\underline{p}-\underline{k}} = -\underline{\alpha}.(\underline{p}+\underline{k}) + \beta M$ and $P_0 = \epsilon - \omega_p$. Therefore

$$\Omega(P.\gamma) = -\frac{3g^2}{(2\pi)^4} 2\pi i \int d^3k \left\{\text{Sum of residues at } k_0 = -\omega_k + i\eta \text{ and } k_0 = P_0 - E_{\underline{p}+\underline{k}} + i\eta\right\}$$

$$= \frac{6\pi i g^2}{(2\pi)^4}\int d^3k\, \gamma_5\left[\frac{P_0 + \omega_k + H_{-\underline{p}-\underline{k}}}{2\omega_k(P_0+\omega_k+E_{\underline{p}+\underline{k}}-i\eta)(P_0+\omega_k-E_{\underline{p}+\underline{k}}+i\eta)}\right.$$

$$\left. + \frac{H_{-\underline{p}-\underline{k}} + E_{\underline{p}+\underline{k}}}{2E_{\underline{p}+\underline{k}}(P_0-\omega_k-E_{\underline{p}+\underline{k}}+i\eta)(P_0+\omega_k-E_{\underline{p}+\underline{k}}-i\eta)}\right] i\beta\gamma_5$$

$$= \frac{3\pi i g^2}{2(2\pi)^4}\int \frac{d^3k}{\omega_k}\gamma_5\left[\frac{E_{\underline{p}+\underline{k}} + H_{-\underline{p}-\underline{k}}}{E_{\underline{p}+\underline{k}}(P_0-\omega_k-E_{\underline{p}+\underline{k}}+i\eta)} + \frac{E_{\underline{p}+\underline{k}} - H_{-\underline{p}-\underline{k}}}{E_{\underline{p}+\underline{k}}(P_0+\omega_k+E_{\underline{p}+\underline{k}}-i\eta)}\right] i\beta\gamma_5$$

Thus,

$$\Omega(P.\gamma) = 3g^2\int d^3k\, \lambda_k^2\, \gamma_5\left[\frac{\Lambda^+(-\underline{p}-\underline{k})}{\epsilon-\omega_p-\omega_k-E_{\underline{p}+\underline{k}}+i\eta} + \frac{\Lambda^-(-\underline{p}-\underline{k})}{\epsilon-\omega_p+\omega_k+E_{\underline{p}+\underline{k}}-i\eta}\right]\gamma_5 = \Omega(\underline{p},\epsilon)$$

For the meson self-energy term, we have

$$\Pi(Q^2) = -\frac{ig^2}{\pi(2\pi)^3}\int d^4k\, Sp\left[\gamma_5 S_F(k)\gamma_5 S_F(Q+k)\right]$$

$$= -\frac{ig^2}{\pi(2\pi)^3}\int d^4k\, Sp\left[\frac{H_{-\underline{k}}+k_0}{E_k^2-k_0^2-i\eta}\, i\beta\gamma_5\, \frac{H_{-\underline{p}-\underline{k}}+Q_0+k_0}{E_{\underline{p}+\underline{k}}^2-(Q_0+k_0)^2-i\eta}\, i\beta\gamma_5\right]$$

where $Q_0 = \epsilon - E_p$. Therefore,

$$\Pi(Q^2) = -\frac{2g^2}{(2\pi)^3}\int d^3k\, Sp\left[\frac{(H_{-\underline{k}}+E_k)\, i\beta\gamma_5\, (H_{-\underline{p}-\underline{k}}+Q_0-E_k)\, i\beta\gamma_5}{2E_k(Q_0-E_k+E_{\underline{p}+\underline{k}}-i\eta)(Q_0-E_k-E_{\underline{p}+\underline{k}}+i\eta)}\right.$$

$$\left. + \frac{(H_{-\underline{k}}-Q_0-E_{\underline{p}+\underline{k}})\, i\beta\gamma_5\, (H_{-\underline{p}-\underline{k}}-E_{\underline{p}+\underline{k}})\, i\beta\gamma_5}{2E_{\underline{p}+\underline{k}}(Q_0-E_k+E_{\underline{p}+\underline{k}}+i\eta)(Q_0+E_k+E_{\underline{p}+\underline{k}}-i\eta)}\right]$$

After some algebra and using the relations $Sp\,\beta = Sp\,\alpha_i = 0,\ i=1,2,3$, we obtain

$$\Pi(Q^2) = -\frac{g^2}{(2\pi)^3}\int d^3k\, Sp\left[\gamma_5\Lambda^+(-\underline{k})\gamma_5\Lambda^-(-\underline{p}-\underline{k})\right]\left[\frac{1}{\epsilon-E_p-E_k-E_{\underline{p}+\underline{k}}+i\eta} - \frac{1}{\epsilon-E_p+E_k+E_{\underline{p}+\underline{k}}-i\eta}\right]$$

so that $\Pi_{\alpha\beta}(\underline{p},\epsilon) = \frac{1}{2\omega_p}\Pi(Q^2)\,\delta_{\alpha\beta}$.

Appendix II.

We wish to perform the following integral

$$I = \sum_m{}' \int d\Omega_p\, \mathcal{Y}^{JM*}_{l''l_1''(l'')_{\frac{1}{2}}}(\underline{n}_p,\underline{n}_k,m)\,\delta(\epsilon-\omega_p-\omega_k-E_{\underline{p}+\underline{k}})\,\mathcal{Y}^{J'M'}_{l'l_1'(l')_{\frac{1}{2}}}(\underline{n}_p,\underline{n}_k,m) \qquad (A.1)$$

For p and k $\neq$ 0, making use of the closure property of spherical harmonics, we can write

$$\delta\left(\epsilon-\omega_p-\omega_k-E_{p+k}\right) = \sum_{\mu} 2\pi \frac{\epsilon-\omega_p-\omega_k}{pk} Y_{\mu}^{0*}(\cos\theta_0)\, Y_{\mu}^{0}(\underline{n}_p\cdot\underline{n}_k) \tag{A.2}$$

where $\epsilon-\omega_p-\omega_k-\sqrt{p^2+k^2+M^2+2pk\cos\theta_0} = 0$ defines $\cos\theta_0$. Therefore,

$$I = 2\pi \frac{\epsilon-\omega_p-\omega_k}{pk} \sum_{\mu} Y_{\mu}^{0*}(\cos\theta_0) \sqrt{\frac{4\pi}{(2l+1)}} \sum_{m\nu}{}' \int d\Omega_p\, \mathcal{Y}^{JM*}_{l_1''l_2''(L'')\frac{1}{2}}(\underline{n}_p,\underline{n}_k,m)\, Y_{\mu}^{\nu*}(\underline{n}_k)$$
$$\times Y_{\mu}^{\nu}(\underline{n}_p)\, \mathcal{Y}^{J'M'}_{l_1'l_2'(L')\frac{1}{2}}(\underline{n}_p,\underline{n}_k,m) \tag{A.3}$$

Making use of (3.53), the orthonormality of the spherical harmonics, and the identity

$$Y_a^{\alpha}(\underline{n})\, Y_b^{\beta}(\underline{n}) = \sum_{c}{}' \sqrt{\frac{(2a+1)(2b+1)}{4\pi(2c+1)}}\; C^{a\;b\;c}_{0\;0}\; C^{a\;b\;c}_{\alpha\;\beta}\; Y_c^{\alpha+\beta}(\underline{n}) \tag{A.4}$$

we obtain,

$$I = 2\pi \frac{\epsilon-\omega_p-\omega_k}{pk} \sum_{\mu} Y_{\mu}^{0*}(\cos\theta_0) \sum_{\alpha\rho t x x'} (-1)^{x'} \frac{1}{4\pi} \sqrt{\frac{(2l_1'+1)(2l_2'+1)(2l_1''+1)}{(2\alpha+1)(2l_1''+1)}}\; C^{L''\;\frac{1}{2}\;J}_{M-t\;t}\; C^{l_1''\;l_2''\;L}_{M-t-x\;x}$$
$$\times C^{L'\;\frac{1}{2}\;J'}_{M'-t\;t}\; C^{l_1'\;l_2'\;L'}_{M'-t-x'\;x'}\; C^{l_2''\;\mu\;\rho}_{0\;0}\; C^{l_2''\;\mu\;\rho}_{x\;M-M'-x+x'}$$
$$\times C^{l_1'\;\mu\;l_1''}_{0\;0}\; C^{l_1'\;\mu\;l_1''}_{M'-t-x'\;M-M'-x+x'}\; C^{\rho\;l_2'\;\alpha}_{0\;0}\; C^{\rho\;l_2'\;\alpha}_{M-M'-x+x'\;-x'}\; Y_{\alpha}^{M-M'*}(\underline{n}_k) \tag{A.5}$$

By making use of the symmetry relations of the Clebsch Gordon coefficients (3) and of the property of the Racah coefficients (2)

$$C^{a\;f\;c}_{\alpha\;\gamma-\alpha}\, W(abcd;ef) = \frac{1}{\sqrt{(2e+1)(2f+1)}} \sum_{\beta}{}' C^{a\;b\;e}_{\alpha\;\beta}\; C^{e\;d\;c}_{\alpha+\beta\;\gamma-\alpha-\beta}\; C^{b\;d\;f}_{\beta\;\gamma-\alpha-\beta} \tag{A.6}$$

the summations over t, x and x' can be carried out, giving

$$I = 2\pi \frac{\epsilon-\omega_p-\omega_k}{pk} \sum_{\mu} (-1)^{L'+l_1'+l_2'+\mu+l_2''+2J} \frac{1}{4\pi} \sqrt{\left\{(2l_1'+1)(2l_2'+1)(2L'+1)(2l_2''+1)(2L''+1)(2J+1)(2\mu+1)\right\}}$$
$$\times C^{l_1'\;\mu\;l_1''}_{0\;0}\; Y_{\mu}^{0*}(\cos\theta_0)$$
$$\times \sum_{\alpha\rho} \sqrt{2\rho+1}\; C^{l_2''\;\mu\;\rho}_{0\;0}\; C^{\mu\;l_2'\;\alpha}_{0\;0}\; C^{\alpha\;J\;J'}_{M'-M\;M}\; W(l_1'\mu L l_2; l_1\rho)$$
$$\times W(L'l_2'L\rho; l_1'\alpha)\; W(J'\tfrac{1}{2}\alpha L; L'J)\; Y_{\alpha}^{M'-M}(\underline{n}_k) \tag{A.7}$$

With (A.4) and (A.6), this yields

$$I\,\mathcal{Y}^{J'M'*}_{l_1 l_2 (L) \frac{1}{2}}(\underline{n}_q, \underline{n}_k, m') = 2\pi \frac{\epsilon - \omega_p - \omega_k}{pk} \sum_{\mu\beta\gamma}{}' Y^{0*}_{\mu}(\cos\theta_0)\, F_\mu\left(l_1 l_2 (L)\, l_1' l_2' (L')\, l_1'' l_2'' (L'')\, J J' \beta\gamma\right)$$

$$\times\, \mathcal{Y}^{JM*}_{l_1 \beta (\gamma) \frac{1}{2}}(\underline{n}_q, \underline{n}_k, m') \qquad \text{(A.8)}$$

where

$$F_\mu\left(l_1 l_2 (L)\, l_1' l_2' (L')\, l_1'' l_2'' (L'')\, J J' \beta\gamma\right) = (-1)^{J+J'+l_2+l_2'+l_2''+L+L'+\beta+\gamma}\ \frac{1}{4\pi}(2J'+1)$$

$$\times \sqrt{\left\{\frac{1}{4\pi}(2l_2+1)(2l_1'+1)(2l_2'+1)(2l_2''+1)(2L+1)(2L'+1)(2L''+1)(2\gamma+1)(2\mu+1)\right\}}\; C^{l_1'\ \mu\ l_1''}_{0\ \ 0}$$

$$\times \sum_{\alpha\rho}{}' \sqrt{(2\alpha+1)(2\rho+1)}\; C^{l_2''\ \mu\ \rho}_{0\ \ 0}\; C^{\rho\ l_2'\ \alpha}_{0\ \ 0}\; C^{\alpha\ l_2''\ \beta}_{0\ \ 0}\; W(l_1'\mu L l_2; l_1\rho)$$

$$\times\, W(L' l_2' l\rho; l_1'\alpha)\, W(J' \tfrac{1}{2} \alpha L; L' J)\, W(\alpha l_1'' \gamma l_2''; \beta L'')\, W(\alpha L'' J \tfrac{1}{2}; \gamma J') \qquad \text{(A.9)}$$

Hence for $k \neq 0$,

$$\int p^2 dp\, I\,\mathcal{Y}^{J'M'*}_{l_1 l_2 (L) \frac{1}{2}}(\underline{n}_q, \underline{n}_k, m') = \sum_{\mu}{}' \int dp\, G_\mu(p,k)\, F_\mu\left(l_1 l_2 (L)\, l_1' l_2' (L')\, l_1'' l_2'' (L'')\, J J' \beta\gamma\right)$$

$$\times\, \mathcal{Y}^{JM*}_{l_1 \beta (\gamma) \frac{1}{2}}(\underline{n}_q, \underline{n}_k, m') \qquad \text{(A.10)}$$

where

$$G_\mu(p,k) = 2\pi \frac{p}{k}(\epsilon - \omega_p - \omega_k)\, Y^{0*}_\mu(\cos\theta_0) \qquad \text{(A.11)}$$

and the integral is over those values of p such that, for a given k, $-1 \leq \cos\theta_0 \leq 1$.

When $k = 0$,

$$\delta(\epsilon - \omega_p - \omega_k - E_{p+k}) = \frac{E_p\,\omega_p}{p(E_p+\omega_p)}\,\delta(p - p_0) \qquad \text{(A.12)}$$

where p_0 is defined by $\epsilon - \omega_{p_0} - \mu - E_{p_0} = 0$. Thus, for $k = 0$,

$$\delta(\epsilon - \omega_p - \omega_k - E_{p+k}) = \sqrt{4\pi}\, \frac{E_p \omega_p}{p(E_p+\omega_p)}\, \delta(p-p_o)\, Y_o^o(n_p \cdot n_k)$$

This leads to the formula (A.10) for the integral, where, in this case,

$$G_\mu(p,k) = \sqrt{4\pi}\, \frac{p\, E_p\, \omega_p}{E_p+\omega_p}\, \delta(p-p_o)\, \delta_{\mu,o} \;.$$

Appendix III.

The following proof is essentially that given by Dr. P.T. Matthews in an unpublished set of lecture notes on an "Introduction to Field Theory".

The total transition probability for a system of one meson of momentum $\underset{\sim}{q}$ and one nucleon of momentum $-\underset{\sim}{q}$ going into a state of two mesons and one nucleon is

$$\int d^3p \int d^3k \int d^3r\, |(\underset{\sim}{p},\underset{\sim}{k},\underset{\sim}{r}|S|\underset{\sim}{q},-\underset{\sim}{q})|^2 = \int d^3p \int d^3k \int d^3r\, |(\underset{\sim}{p},\underset{\sim}{k},\underset{\sim}{r}|T|\underset{\sim}{q},-\underset{\sim}{q})|^2 \left[\delta_4(P_i-P_f)\right]^2$$

by (3.30) and (3.35) where P_i and P_f are the total four-momenta of the initial and final states respectively. We suppress the spin and isotopic spin variables for the present. Now,

$$\left[\delta_4(P_i-P_f)\right]^2 = \delta_4(0)\, \delta_4(P_i-P_f)$$

and, for the energy component of $\delta_4(0)$, for example, we can write

$$\delta(0) = \lim_{T\to\infty} \frac{1}{2\pi}\left[\int_{-T/2}^{T/2} e^{iEt}\, dt\right]_{E=0} = \lim_{T\to\infty} \frac{T}{2\pi}$$

so that for $\delta_4(0)$, we have

$$\delta_4(0) = \lim_{\substack{T\to\infty \\ V\to\infty}} \frac{VT}{(2\pi)^4}$$

where VT is the total space-time volume considered.

It follows that the total transition probability per unit time per unit volume is

$$\frac{1}{(2\pi)^4}\int d^3p \int d^3k \, |(\underline{p}, \underline{k}, -\underline{p}-\underline{k}|T|\underline{q}, -\underline{q})|^2 \, \delta(\epsilon - \omega_p - \omega_k - E_{p+k})$$

where $\epsilon = E_q + \omega_q$ and the $\underline{r}$-integration has been carried out.

The wave functions, which we use to describe the mesons and nucleons, are normalised so that the density of mesons or nucleons is $\frac{1}{(2\pi)^3}$ per unit volume. Hence, the total transition probability per unit time per unit volume per unit density of incoming particles is

$$(2\pi)^2 \int d^3p \int d^3k \, |(\underline{p}, \underline{k}, -\underline{p}-\underline{k}|T|\underline{q}, -\underline{q})|^2 \, \delta(\epsilon - \omega_p - \omega_k - E_{p+k})$$

The total cross-section, $\sigma(\epsilon)$, which is the total transition probability per unit flux of incoming particles, is obtained from this by dividing by the magnitude of the relative velocity of the initial nucleon and meson system i.e. by $\left(\frac{q}{\omega_q} + \frac{q}{E_q}\right) = \frac{q\epsilon}{E_q\omega_q}$. Thus

$$\sigma(\epsilon) = (2\pi)^2 \frac{E_q\omega_q}{q\epsilon} \int d^3p \int d^3k \, |(\underline{p}, \underline{k}, -\underline{p}-\underline{k}|T|\underline{q}, -\underline{q})|^2 \, \delta(\epsilon - \omega_p - \omega_k - E_{p+k}).$$

Taking spin and isotopic spin into account, the total cross-section is obtained by averaging this expression over the initial spin states and summing in over the final spin states. This gives equation (3.87) of the text.

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