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THE AIRGAP MAGNETIC FIELD IN ELECTRICAL MACHINES WITH SPECIAL REFERENCE TO THE EFFECT OF ECCENTRICITY

Thesis Presented for

the Degree of Doctor of Philosophy

By

Ove Kristoffer Gashus, B.Sc.

February, 1960

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Preface

The present work was initiated in 1953 by the rather tentative idea of investigating the presence of "even" harmonics in the air gap flux density wave of electrical machines. During these investigations, it became clear that, for the case of eccentric dissymmetry, which was seen as one of the main sources of "even" harmonics, little or no evidence of published work was to be found. This problem was then taken up as a subject for the present Thesis.

Experiments were carried out, and results were obtained which showed that eccentricity would account for up to very large percentage harmonic contents. An approximate theory was also established based on m.m.f. and permeance waves. This explained the phenomena in some cases, but was obviously wrong in other cases. For a rather long period of time, no progress was made due to the lack of a correct theory; the theory presented in this Thesis was only developed after the Author resumed the work two years ago. The theory may be regarded as the extension of the conventional m.m.f. theory to the case of eccentric rotors. A complete theory of constant-span (or equivalent) coil windings is presented as an introduction to the eccentric rotor theory, thus making the work self-contained.

/Originality

Originality is claimed for some of the extensions of the conventional m.m.f. theory, notably on the subject of unbalanced excitation and inductances and for the eccentric rotor theory.

The work was carried out in the Electrical Engineering Laboratories of the University from 1953 to 1959. For the first three years of this period, the Author was awarded a Research Grant and the James Watt Scholarship (two years) by the Faculty of Engineering.

The Author gratefully acknowledges the help and encouragement during the first year of this period of Dr. J.E. Parton (now Professor of Electrical Engineering at the University of Nottingham). He also wishes to thank Professor B. Hague for permission to use the facilities in the Laboratories and for helpful encouragement. Mr. W. Butler, who constructed the experimental machinery, is also thanked for his skilful and patient assistance.

List of Principal Symbols

B	-	Flux density
b_m, b_n	-	Fourier components of B
C_b, C_f	-	Winding Connection Factors
D	-	Air gap mean diameter
F	-	Winding factors
$F_{\alpha,m}$	-	Coil-span factor of mth harmonic
F_ρ	-	Distribution factor
F_γ	-	Skew factor
H	-	Air gap magnetic potential or "m.m.f. wave"
h_m, h_n	-	Fourier components of H
L	-	Inductance or inductance matrix
M	-	Mutual Inductance
N	-	Number of phases in a polyphase system
Q	-	Number of coils. (Repeatable section in Fractional-Slot theory).
R	-	Air gap mean radius
S	-	Sequence number or sequence transform matrix
T	-	Number of turns
T_c	-	per coil; T_{ph}: per phase
U	-	Unit Matrix
U_q	-	Unit column vector
X	-	Reactance

...

d	-	Integral number (in Fractional-Slot theory)

/e

e	-	2.71828 ..., base of Napierian logarithms
g	-	mean air gap length
k	-	Integral Number; Eccentricity ratio
l	-	Integral Number, (Number of identical groups in Fractional-Slot theory).
m,n	-	Harmonic orders in Fourier Expansions
p	-	Number of pairs of poles
q	-	Number of coils in sub-group. Integral Number. Order of Harmonics in Fourier Expansions.
r	-	Integral Number
s	-	Sequence Number
t	-	Time variable. Integral Number
u	-	Integral Number
v	-	Integral Number
x	-	Angular Measure in "Mechanical Radians"
y	-	$\frac{k}{1 + \sqrt{1 - k^2}}$, Common Function in the Eccentric Rotor Theory.

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α	-	Coil pitch
β	-	Phase spread
γ	-	Slot pitch
δ	-	Skew Angle
ε	-	Small Angle (Eccentric Rotor Theory)
θ	-	Angular Measure in "Electrical Radians"
λ_s	-	Characteristic roots of matrices

/μ_0

μ_o - Permeability of free space

Φ - Flux

ϕ - Phase angle

ψ - Flux Linkages

σ - Position or phase angle

ρ - Integral Number

ω - Angular Frequency.

CONTENTS.

1. Introduction

The air gap magnetic field is, as it were, the "working fluid" of electrical machines. An exact knowledge of the distribution of flux density in the air gap is the key both to the general analysis and to the design of such machinery. It has, therefore, been the subject of extensive enquiry and there exists a considerable amount of literature on the subject. There are, however, not many conclusive results to be found, and the subject is not likely to be exhausted for some time to come. The reasons usually given for this state of affairs are two; firstly, the complex geometry of the domains involved, secondly, the fact that the main part of the domains are occupied by iron having an extremely non-linear flux-magnetic intensity relation. Of these two factors, the second is by far the most important, since all field problems involving non-linear regions which at the same time have geometrically awkward boundaries are practically intractable by present methods of analysis.

For these reasons, the field in the air gap is usually evaluated by approximate analytical methods or, in some cases, by graphical methods. Again, there are two fundamentally different types of machines, namely, smooth-gap machines and salient-pole machines. Clearly, the first is much more amenable to analytical methods than the second. In fact, the second type is normally

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treated by methods which can only be justified on the grounds that they give reasonable results when the constants are empirically adjusted by comparison with experiment. This is certainly the case in the most popular of all theorems, namely the two-axis principle. In the smooth-gap machines there is more scope for exact analysis, and their fields have, in fact, been investigated by several workers[1,2,3]. But even in this simpler case, the exact analysis is probably of little practical value, since the analytical solution must be expressed in slowly convergent infinite series. Broadly speaking, the problem is to express the radial component of the flux density in the air gap as a function of the appropriate cylindrical polar co-ordinates and the currents present in the windings. The solution must be readily obtainable in Fourier Series to be of practical value; but the works seen by the Author have not been in such a form, and it may be reasonable not to expect them to be. Furthermore, the analytical solution breaks down completely when the iron parts are saturated. A moderately successful approximation can be obtained when the iron is considered infinitely permeable. This problem has been extensively treated by Buchholz[2], who has also tackled the problem of eccentricity in the air gap. However, the analytical treatment is always based on a simplified geometric model, and the accuracy obtained is always limited by this approximation.

The essence of these remarks is that the only theory that has

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1. See Bibliography, Page 140.

found general approval in Engineering circles is the simplest possible, namely, that of magnetomotive force waves and air gap permeance. This theory is based on the assumption of infinitely permeable iron and a simple rectilinear flux distribution. Given these two properties it is a simple matter to evaluate the flux distribution at any point in the air gap. The field problem becomes of secondary importance, and the theory is, in fact, more concerned with the distribution of the sources of the magnetic fields, namely the currents or winding coils. The theory of windings has become a concept which not always has an unmistakeable synonymity with the magnetic field in the air gap. This has provided a few loopholes in the reasoning and very often inconsistencies are apparent even in reputable texts. Thus Say[4] treats the m.m.f. distribution without reference to the actual flux distribution and applies the results to salient-pole machines without comment. It is easy to be left with the impression that m.m.f. has, in fact, a "distribution" independent of the flux. This misconception has also led to some erroneous statements about the distortion of the flux distribution due to an eccentric gap.

In this work, the m.m.f. theory of the common types of windings is developed along the lines first published by Arnold[5] at the beginning of the century. In his book, the principles of the method are clearly stated, but only simple types of windings are considered. The extension to a wider range of winding types is due

/to

to B. Hague[6] and A. Clayton[7]. The theory of the fractional-slot windings (which today have found a very wide application) has been less conclusively investigated. The earliest literature on the subject seems to be a paper by E.M. Tingley[8] in 1915. Tingley's analysis was extremely restricted, and the subject seems to have been void of mathematical treatment till 1927 when Q. Graham[9] presented a paper on the subject to the American Institute of Electrical Engineers. The paper was, however, more of a qualitative nature, and the first attempt to generalise the theory is due to Calvert[10]. In the discussion of this paper, Professor W.V. Lyon draws the distinction between "regular" and "irregular" balanced windings and states that only for the former type can general formulae be readily developed. The irregular ones "must each be considered as a separate problem". This is, in fact, the position today. However, there are two other notable contributions to the theory, by Malti and Herzog[11] and by M.M. Liwschitz[12,13,14] respectively. Although the latter is usually credited with the development of the so-called slot-star method, this is in fact, only a pictorial representation of the complex-number treatment developed by Malti and Herzog. Their paper[11] is the basis of the treatment presented in this work, and it contains a great deal of valuable information. Most recently there has appeared a paper in the Proceedings of the Institution of Electrical Engineers by Walker and Kerruish[19] which claims a simpler although rigorous treatment. Here some of the irregular windings are considered and distribution

/factors

factors are given for some of these windings. The complete theory is, however, still wanting.

Both the integral- and fractional-slot windings are usually treated specifically for the 3-phase narrow-spread case. In this Thesis the theory is generalized to any number of phases, and the harmonics due to symmetric as well as unsymmetric polyphase currents are analysed. This extension has revealed a number of points believed to be relatively unknown heretofore. The narrow-spread windings and pole-changing windings are included in an extremely general treatment involving symmetrical components and connection matrices. The important quantities known as air-gap inductances are thoroughly treated, and a critical exposition of the concept of synchronous reactances is given. The latter are shown to be special cases of the sequence reactances, which are the eigen values of the general air-gap inductance matrix.

The Author hopes to carry out further work along these lines in an effort to formalize the theory of linear machines, but this is outwith the scope of this Thesis. The theory of fractional-slot windings has been largely limited to balanced "regular" windings, while a special important type of irregular narrow-spread winding is treated in detail.

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The m.m.f. theory has in the past been successfully applied even to salient-pole machines, notwithstanding that in this case the very basis of the theory, a smooth constant-length gap, is not present. With this in mind, it occurred to the Author that the theory might equally well be applied to the case of an eccentric gap. The idea was to obtain the "permeance wave" of the eccentric gap and multiply this by the m.m.f. wave. The flux density was expected to follow. This simple method was pursued but led to a physically untenable result: the resulting flux-density function was not solenoidal. When more closely scrutinised, the method revealed the fact that the m.m.f. wave cannot in fact be evaluated before the flux distribution is known. The m.m.f. wave pertinent in the case of an eccentric gap is quite different from the conventional one obtained with constant gap lengths. The latter part of this Thesis is devoted to the theory of the eccentric gap phenomena, based on a modified m.m.f. wave. By this treatment the mathematical simplicity of the conventional m.m.f. theory is preserved, and readily applicable general formulae are obtained. The items considered are the waveform of the gap flux, the transverse force and the reactances respectively, so that together with the earlier sections, a fairly general treatment of the air gap phenomena is obtained. The results of the experimental investigations on an eccentric machine agree fairly well with the theory.

The analysis of the transverse pull on the rotor is quite general, and is applicable to any case in which the flux

/distribution

distribution is known. Specifically it is shown that the existence of transverse forces implies a very dense (at least partially dense) spectrum of harmonics. This result seems to have been largely overlooked in the literature known to the Author, although it is of prime importance. The resultant noise effect may also be assumed appreciable, and would merit further investigation.

2. The Theory of Integral-Slot Windings

In this section the analysis of the general polyphase winding will be presented. The treatment is substantially based on the work by Clayton, but it has been extended, and the mathematical procedure allows for greater generality, and relies to a lesser extent on graphical or physical arguments. More powerful methods of analysis are employed and several less known properties of these windings are discussed. The assumption of infinitely permeable iron is made throughout.

2.1 The Flux Distribution and Reactance of a Single Coil

The basic element in the analysis of integral slot as well as fractional-slot windings will in this work be taken as the approximate field distribution due to a single coil of arbitrary angular span α radians. (The angular measure is throughout taken in mechanical radians unless otherwise stated). The flux paths due to such a single coil are shown approximately in Fig. 1(a). It is assumed that the flux density is constant both inside and outside the coils and the fringing of the field in the neighbourhood of the coil is ignored. The convention is adopted of referring to the position of the centre of the coil by the angle σ relative to some arbitrary reference point on the stator.

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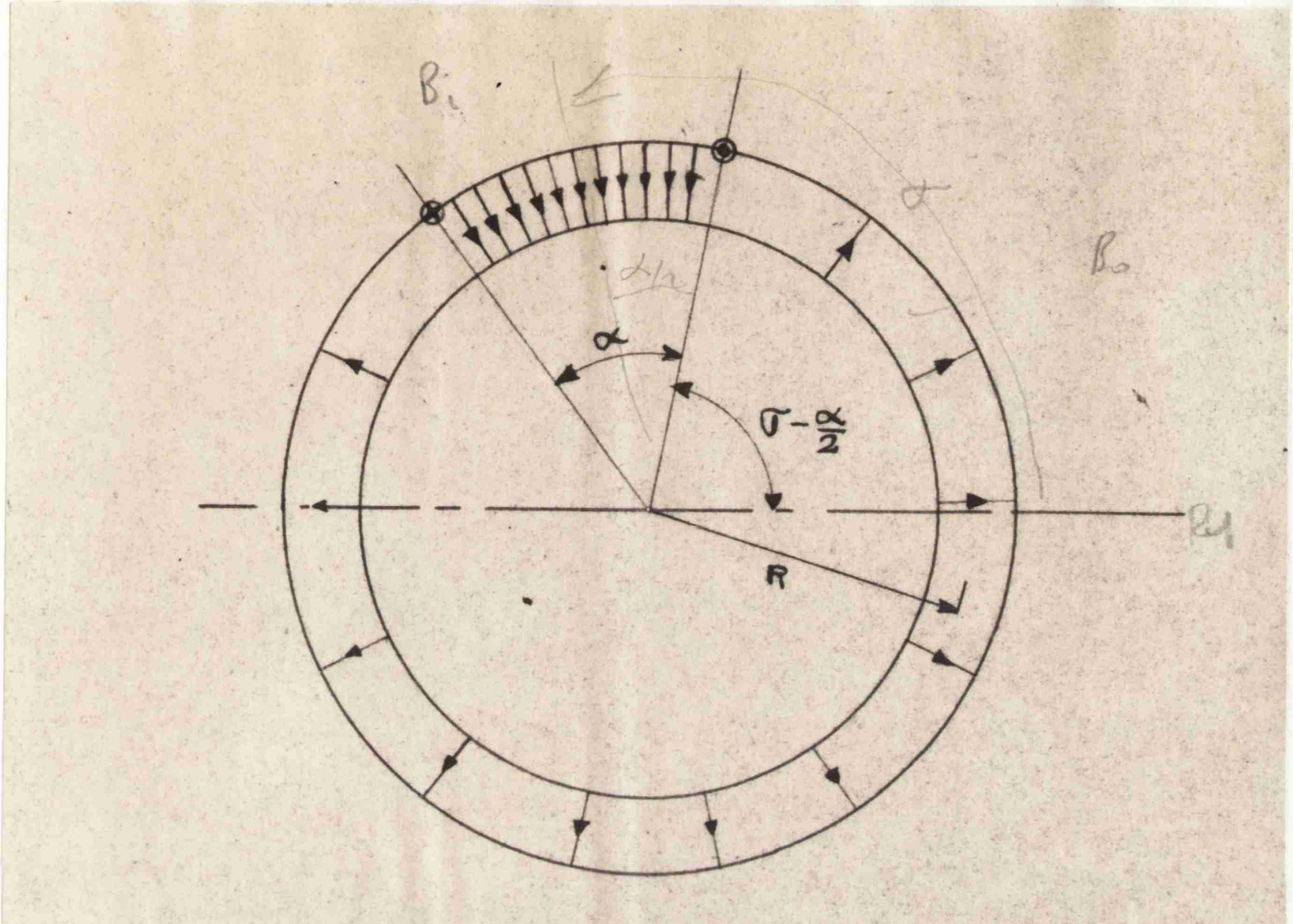

Fig 1a

The total flux produced by such a coil Φ_c, may be computed by the magnetic circuit law. We have,

$$\Phi_c = B_i L R \alpha = B_o L R (2\pi - \alpha) \qquad (1)$$

where B_i and B_o are the (constant) air gap flux densities inside and outside the coil respectively, L the axial length of the stator and rotor, and R is the mean gap radius.

Since the magnetic field is assumed to be zero apart from the gap, we may write the total reluctance of the magnetic circuit $\mathcal{R}$, as

$$\mathcal{R} = \frac{g}{\mu_o R L}\left(\frac{1}{\alpha} + \frac{1}{2\pi - \alpha}\right) = \frac{g}{\mu_o R L\left(1 - \frac{\alpha}{2\pi}\right)\alpha} \qquad (2)$$

/where

where g is the gap length.

Now if the coil has T_c turns and carries the current I, we have by putting $IT_c = \Phi_c \mathcal{R}$ and substituting from (1) and (2),

$$B_i = IT_c \frac{\mu_o}{g} \frac{2\pi - \alpha}{2\pi} \qquad (3(a))$$

$$B_o = IT_c \frac{\mu_o}{g} \frac{\alpha}{2\pi} \qquad (3(b))$$

We may now term $\frac{\mu_o}{g}$ the specific permeance of the air gap and the equations (3(a)) and (3(b)) may be interpreted as giving the "specific magnetic induction" (flux density) by multiplying the specific permeance of the gap by the magnetic potential across the air gap. The latter is the quantity which is usually referred to as the "m.m.f. wave", and in this case has the form shown in Fig. 1(b). It has the same waveform as the flux density function, and has the value $H_i = IT_c \frac{2\pi - \alpha}{2\pi}$ inside the coil, and the value $H_o = IT_c \frac{\alpha}{2\pi}$ outside the coil.

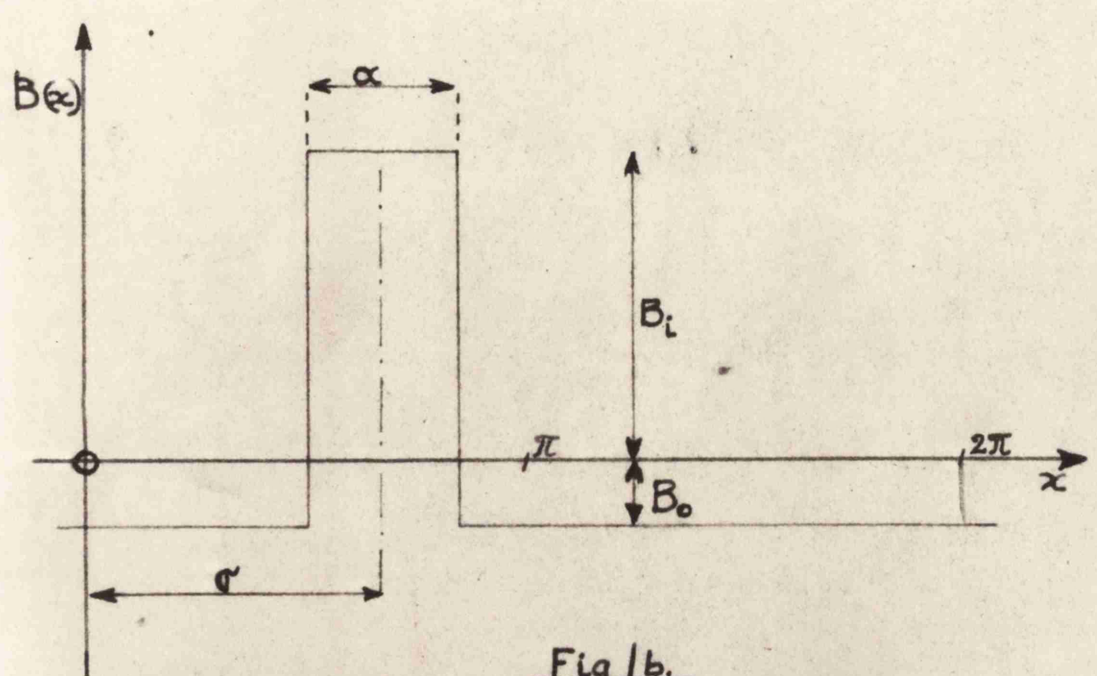

Fig 1b.

(The gap length g strictly should be taken as "the effective gap length". Since the normal machine has slotted stator and rotor surfaces, the reluctance of the above magnetic circuit is greater than that given by (2), due to the smaller effective cross-section. The necessary correction can easily be obtained from Carter's coefficient[18].

This must be understood to be a particular case, and no general principle is intended. The objections to the concept of an "m.m.f. wave" have largely been made due to lack of precise definition. The most irritating of the definitions are those which define m.m.f. waves as the integral of line currents without reference to the flux distribution. Although the result is correct, the definition is confusing since it leads the student to accepting the m.m.f. wave as a function of the currents alone while, in fact, it has no meaning until a flux distribution has been found in terms of which the magnetic potential can be evaluated. A blind application of the m.m.f. wave in the case of non-uniform air gaps will undoubtedly lead to erroneous results unless some modifications are made. An example of this is provided in Section 4.

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Although the present method of evaluating the flux distribution is clearly empirical, it is very nearly exact for the normal machine which has a small gap. Saturation of the iron will also upset the theory but, in that case, all known analytical methods also fail and cannot, therefore, in general claim any advantage.

If we accept the above basic assumption, we may represent the flux density function in the most convenient mathematical form and proceed by superposition to derive the flux resulting from any given group of coils and, finally, a whole distributed winding. This then provides enough information about reactances to perform any desired analysis of the machine. p12

In Fig. 1(b) the flux density is shown as a function of x, the angular displacement round the air gap. By the above interpretation it is identical in form with the magnetic potential function or the m.m.f. wave. The function is discontinuous at the slots containing the coils, but is easily represented by a Fourier Series. The shape is sometimes modified to a trapezium form in order to account for the fringing at the slots, but the advantage of this is of dubious value, since it does not allow for the slots at the interior and exterior of the coil which also produce a similar ripple in the flux wave. The tooth ripple frequency part cannot, therefore, be correctly represented by such an artifice, and the other harmonics are not drastically affected by the teeth.

The Fourier series representing the stepped function in Fig. 1(b) is given by

$$B(x) = \frac{2}{\pi}(B_i + B_o)\sum_{m=1} \frac{\sin m\,\alpha/2}{m} \cos m\,(x - \sigma) \qquad (4(a))$$

and using the relations (3) we have

$$B(x) = \frac{2}{\pi} IT_c \frac{\mu_o}{g} \sum_{m=1}^{\infty} \frac{1}{m} \sin m\,\alpha/2 \cos m(x - \sigma) \qquad (4(b))$$

It may be noted here that $\frac{2}{\pi} IT_c \frac{\mu_o}{g}$ is the average value of a sine function of amplitude $IT_c\,\mu_o/g$, i.e., the total m.m.f. of the coil multiplied by the specific permeance of the air gap. It is also the amplitude of the fundamental component in the series expansion, if $\alpha = \pi$. This is a convenient definition, and very easy to remember.

The inductance of the coil will now be considered. This has two distinct aspects, namely total inductance and harmonic inductances. In anticipation of the more general expressions which will follow later, these quantities will be considered for the single coil.

The total inductance is given by evaluating the total flux linkages per ampere, viz.

$$L = \frac{T_c}{I}\int_{-\alpha/2}^{\alpha/2} B(x)\,\frac{DL}{2}\,dx$$

Substituting from (4(b)) we have

$$L = \frac{T_c}{I}\int_{-\alpha/2}^{\alpha/2} \frac{2}{\pi} IT_c \frac{\mu_o}{g} \sum_{m=1}^{\infty} \frac{1}{m} \sin m\,\alpha/2 \cos m(x - \sigma)\frac{DL}{2}\,dx$$

which gives

$$L = \frac{2}{\pi}\,\frac{DL\mu_o}{g}\,T_c^2 \sum_{m=1}^{\infty} \left(\frac{1}{m}\sin m\,\alpha/2\right)^2 \qquad (5)$$

/Thus

Thus, the total inductance is seen to contain a series of components each due to the various harmonic components in the flux wave. The mth harmonic reactance can consequently be defined as

$$L_m = \frac{2}{\pi} \frac{DL\mu_o}{g} T_c^{\,2} \left(\frac{1}{m} \sin \frac{m\alpha}{2}\right)^2 \qquad (6)$$

This is proportional to the square of factor $\frac{1}{m} \sin m\,\alpha/2$, and subsequent analysis will show that the complete winding inductance has a similar form.

For convenience in the subsequent analysis the amplitudes will be referred to $\frac{2}{\pi} T_c \, \mu_o/g$ and we write this as

$$B_c = \frac{2}{\pi} T_c \frac{\mu_o}{g} \qquad (7)$$

Equation (4(b)) may then be written

$$B(x) = I\, B_c \sum_{m=1}^{\infty} \frac{1}{m} \sin m\,\alpha/2 \cos m\,(x - \sigma) \qquad (4(c))$$

or

$$B(x) = I\, B_c \sum_{m=1}^{\infty} \frac{1}{m} f_{\alpha,m} \cos m\,(x - \sigma) \qquad (4(d))$$

where $f_{\alpha,m}$ is the "coil span factor" $\sin m\,\alpha/2$. ✓

In the following it has been found advantageous to re-define the winding factors so that the coil span factor becomes

$$F_{\alpha,m} = \frac{1}{m} \sin m\,\alpha/2 \qquad (8)$$

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This gives greater conciseness to the formulae and is, therefore, an economy in the already rather lengthy formulae. The span factor is also already a function of m.

So far, the conductors (and slots) have been assumed to be parallel to the axis of the stator, but in many machines the slots are skewed, and a slightly different picture arises. Again by

/assuming

assuming that the flux densities inside and outside the coil are constant respectively it is clear that we may derive the values of these by the magnetic circuit law and will, in fact, be no different from the above. However, the flux density is now a function of the displacement along the axis of the rotor as well as the angular displacement round the gap. It is very undesirable, and in fact of little merit, to introduce a second variable in the flux density function, and the effect of skewing need only be considered when dealing with the mutual inductance of two coils (on different sides of the air gap). This is illustrated in Fig. 2(a) which shows two coils having a mutual skew angle δ. (It is not necessary to state which coil is skewed, but only that their sides make an angle δ with each other). The change in mutual flux linkages with the coil displacement is now clearly quite different when the coil sides cross and when they do not. The effect is precisely as if the flux density had a trapezoidal distribution and the coil sides were parallel (shown in Fig. 2(b)). It is, therefore, useful to represent this waveform in the Fourier Series form, viz.,

$$B(x) = \frac{2}{\pi}(B_i + B_o)\sum_{m=1}^{\infty} \frac{1}{m} \sin \tfrac{1}{2}m\alpha \cdot \frac{2}{m\delta} \sin \tfrac{1}{2}m\delta \cos m(x - \sigma) \qquad (9(a))$$

and take this as the flux density distribution whenever we deal with coils having a relative angle of skew δ. Again, the "skew factor" may be defined as

$$F_{\delta,m} = \frac{1}{m} \cdot \frac{2}{\delta} \sin \tfrac{1}{2}m\delta \qquad (10)$$

and by (7) we have

$$B(x) = IB_c \sum_{m=1}^{\infty} F_{\alpha,\gamma,m} \cos m\,(x - \sigma) \qquad (9(b))$$

where $F_{\alpha,\gamma,m} = F_{\alpha,m} \cdot F_{\gamma,m} = \frac{1}{m} \sin \tfrac{1}{2}m\alpha \cdot \frac{2}{m\delta} \sin \tfrac{1}{2}m\delta$. $F_{\gamma,m}$ is, of course, the conventional skew factor.

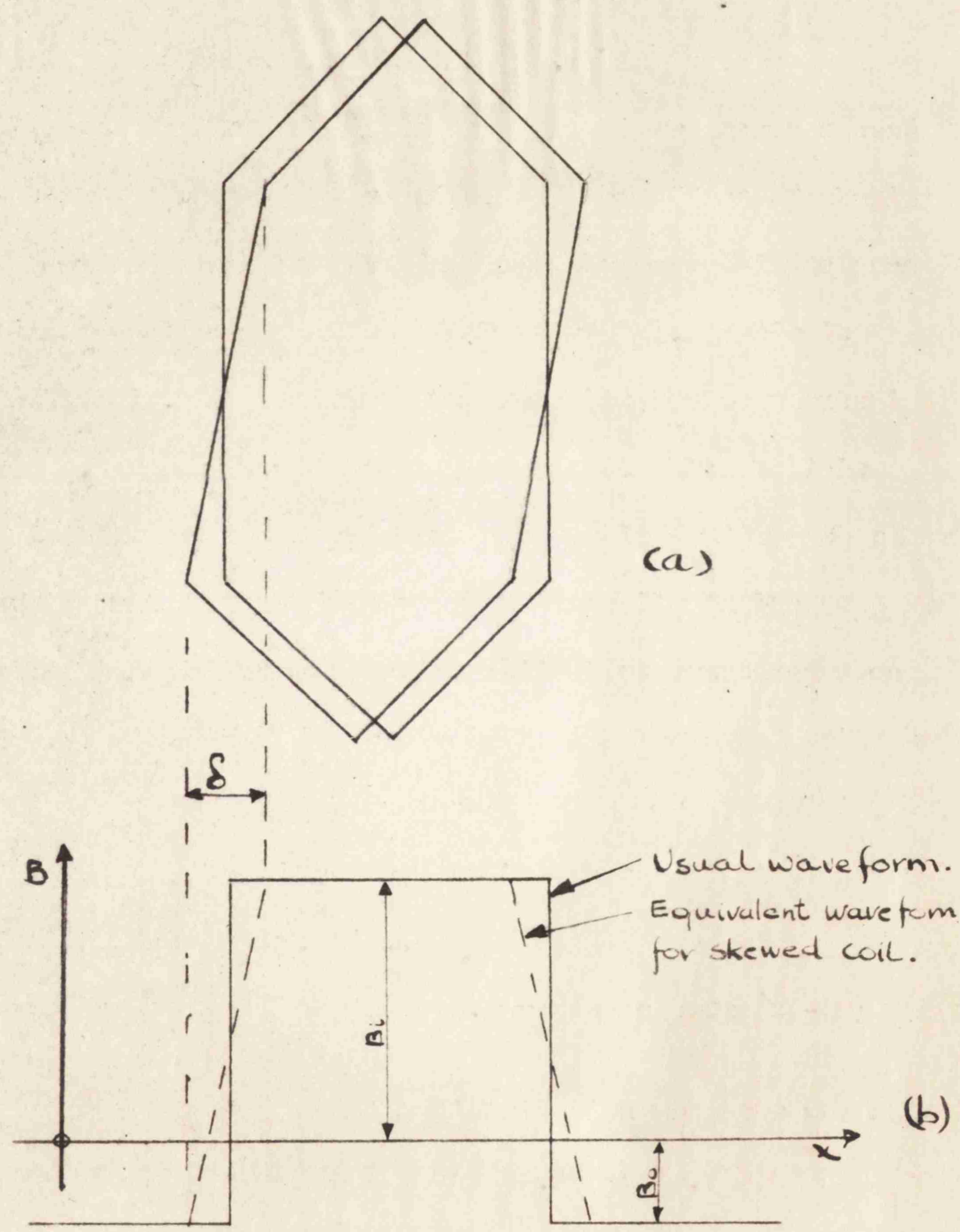

Fig 2.
Effect of skewing coil sides.

Equation (9(b)) will be the elementary building block in the further development of the theory. In general, I is a function of time, and B(x) therefore becomes B(x,t). Again if I is an harmonic function of time B(x,t) may be resolved into the familiar travelling waves.

The total field due to a number of coils excited by different currents presents in general a very complex picture, and only where there is a certain amount of symmetry in the space and time functions concerned can the resulting field be given a compact, useful mathematical form. However, by the technique of resolution into symmetrical components, it is often possible to obtain the most important parts of the functions involved, and make numerical calculations possible.

2.2 The Flux Distribution due to a Group of Equidistant Coils, similarly excited.

As the first step in the analysis of the flux distribution due to multiple-coil windings, we consider a group of q coils situated at equal intervals (e.g., one slot pitch apart).

The general problem is to sum the flux densities due to the individual coils at all points round the air gap. This is most conveniently done by summation of the series expansions of the individual functions. In the present case this gives rise to the series,

$$B(x) = IB_c \sum_{r=1}^{q} \sum_{m=1}^{\infty} F_{\alpha,\gamma,m} \cos m \left[x - (\sigma_1 + \overline{r-1}\gamma)\right]$$

where γ is the spacing between successive coils in the group. By performing the summation over r we obtain

$$B(x) = IB_c q \sum_{m=1}^{\infty} F_{\alpha,\gamma,m} \frac{\sin\frac{1}{2}mq\gamma}{q\sin\frac{1}{2}m\gamma} \cos m\,(x - \sigma) \qquad (11(a))$$

where $\sigma = \frac{1}{2}(\sigma_1 + (q-1)\gamma)$, i.e., the position of the centre of the group of q coils. Again we define

$$F_{\beta,m} = \frac{\sin\frac{1}{2}mq\gamma}{q\sin\frac{1}{2}m\gamma}$$

as the distribution factor, which is the value normally given for this. Also, the product $F_{\alpha,\gamma,m}\ F_{\beta,m}$ is contracted to $F_{\alpha,\beta,\gamma,m}$ or simply F_m. Since F_m does not contain $F_{\gamma,m}$ in all later applications, it must be clearly stated what F_m means, but the context should always suffice to make this clear. With this modification, equation (11(a)) becomes

$$B(x) = qIB_c \sum_{m=1}^{\infty} F_m \cos m\,(x - \sigma) \qquad (11(b))$$

In the case of a concentric group, the summation will be an arithmetical progression in α, the coil pitch, rather than the position angle σ, and we have to sum over q terms the expression

$$B(x) = IB_c \sum_{r=1}^{q} \sum_{m=1}^{\infty} \frac{1}{m} F_{\gamma,m} \sin\tfrac{1}{2}m\,(\alpha_1 + 2\,\overline{r-1}\,\gamma) \cos m\,(x - \sigma)$$

since the increase in the coil pitch in successive coils is 2γ. (The shortest coil having span α_1 radians). This yields

$$B(x) = qIB_c \sum_{m=1}^{\infty} \frac{1}{m} F_{\gamma,m} \frac{\sin\frac{1}{2}mq\gamma}{q\sin\frac{1}{2}m\gamma} \sin\tfrac{1}{2}m\,(\alpha_1 + 2\,\overline{q-1}\,\gamma)$$
$$\cos m\,(x - \sigma) \qquad (12)$$

/Here

Here again the distribution factor appears as before, and the factor $\frac{1}{m} \sin \frac{1}{2}m\ (\alpha_1 + 2\overline{q-1}\ \gamma)$ is the coil span factor of a coil of average span. Hence equation (12) can be considered as included in equation (11(b)) with the modification that the coil span factor must refer to the average span of the group.

It may be noted at this stage that a group of coils is not always arranged in such a simple manner as above, and summation may not be possible in the same way. Such cases arise in fractional-slot windings, which will be considered later.

2.3 The Flux Distribution due to a number of similarly-excited groups of coils.

So far, only basic elements of a complete winding have been considered. In this section a number of groups will be considered together and, in particular, the set forming a single-phase winding. The single-phase winding is here considered as a true single-phase winding, i.e., the limiting case of a so-called "wide spread" polyphase winding.

In the symmetrical single-phase winding there are p similar groups of coils spaced round the air gap at equal intervals of $\frac{2\pi}{p}$ radians. Such a winding will be seen to have p similar periods in the flux wave, and therefore has 2p fundamental "poles". Accordingly it is termed a 2p-pole winding. This single-phase winding has, of course, little practical importance, but it forms the basic element of the integral-slot polyphase windings and will, therefore, be considered in some detail.

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By the same procedure as applied above in deriving equation (11(b)) we have the field due to a single-phase winding,

$$B_{ph}(x) = qIB_c \sum_{\rho=1}^{p} \sum_{m=1}^{\infty} F_m \cos m \left(x - \sigma + \overline{\rho - 1}\,\frac{2\pi}{p}\right)$$

which after summation gives

$$B_{ph}(x) = qIB_c \sum_{m=1}^{\infty} F_m \,\frac{\sin m\pi}{\sin m\pi/p} \cos m\,(x - \sigma) \qquad (13(a))$$

The factor $\frac{\sin m\pi}{\sin m\pi/p}$ is another distribution factor for the winding, but it will here be referred to as the "connection factor" since it depends on the connections of the groups. In the following, the groups are assumed to be series connected (in the same sense), and the value of the connection factor is then simply p or zero according as m is a multiple of p or not. Thus only harmonic orders given by $m = np$ ($n = 1, 2, 3 \ldots$) can exist in the flux wave, and consequently the fundamental has period $2\pi/p$. It is now more convenient to change to "electrical" angular measure, defined by the equation $\theta = px$; and substituting $m = np$ and $x = \theta/p$ in (13(a)) gives

$$B_{ph}(x) = qIB_c \sum_{n=1}^{\infty} F_n \cos n\,(\theta - \sigma) \qquad (13(b))$$

where the arguments in F_n as well as σ are reckoned in "electrical radians"(1).

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(1) (The reason for the term "electrical radians" is, of course, that the e.m.f. induced in a conductor travelling in this field will have a repetitive period corresponding to a displacement given by $\theta = 2\pi$).

It should be noted that the field distribution due to this winding contains in general the complete spectrum of harmonics. The even harmonics are only completely suppressed if α, the coil span, is π electrical radians, since the coil-span factor is thus unity for all odd harmonics and zero for all even harmonics.

The extension of the above to polyphase windings will now be considered by the use of equation (13(b)). In this case, we have really already fixed the number of poles for the winding, and the treatment is therefore somewhat restricted. It will be shown later that if equation (11(b)) is used as the fundamental element, the treatment can be generalised to cover pole-changing windings as well as irregular windings. Since this part of the theory is original and is not required for the main trend of this Thesis, notes are only made on it in Appendix I.

2.4 The Flux Distribution due to an Integral-Slot Polyphase Winding.

We consider in this case firstly hemitropic windings, which in the N-phase case consist of N single-phase windings of the type analysed in the previous chapter. These are spaced out round the air gap at intervals of $2\pi/Np$ radians (for a 2p-pole winding) or $2\pi/N$ electrical radians in general. In general, these windings may carry currents which are not related in any way but will be assumed to have a common frequency. However, by the method of symmetrical component representation, they can always be reduced

/to

to N sets of currents, (N - 1) of which are sequential (i.e., having equal amplitude and an arithmetic progression in phase) and one set of which is the zero sequence set consisting of N equal co-phasal currents. These two types of excitation can be reasonably easily analysed and the flux distribution will be sought for these kinds of excitation. It will be termed sequence excitation, since this name covers the general case, and the names zero, positive and negative sequence is only sufficient to describe the 3-phase case.

2.4.1 Hemitropic Windings

The current in the rth phase of a symmetrical system of sequence S is given by

$$I_r = I_S \cos\left(\omega t - \overline{r-1}\, S \frac{2\pi}{N}\right) \qquad (14)$$

The flux density function for the same phase is given by

$$B_r(\theta) = I_r\, T_{ph}\, B_c \sum_{n=1}^{\infty} F_n \cos n\left(\theta - \sigma - \overline{r-1}\,\frac{2\pi}{N}\right)$$

where $T_{ph} = q\ T_c$ = Total number of turns per phase per pole.

Substituting for I_r we obtain

$$B_{rS}(\theta,t) = I_S\, T_{ph}\, B_c \sum_{n=1} \tfrac{1}{2}F_n \quad \cos\left(n\overline{\theta - \sigma} - \omega t - \overline{r-1}\;\overline{n-S}\,\frac{2\pi}{N}\right)$$

$$+ \cos\left(n\overline{\theta - \sigma} + \omega t - \overline{r-1}\;\overline{n+S}\,\frac{2\pi}{N}\right) \qquad (15(a))$$

and summing over all r to obtain the flux due to the complete winding,

$$B_S(\theta,t) = \sum_{r=1}^{N} B_{rS}(\theta,t) = I_S\, T_{ph}\, B_c \sum_{n=1} \tfrac{1}{2}F_n$$

$$C_f \cos\left(n\,\overline{\theta - \sigma} - \omega t - \overline{N-1}\;\overline{n-S}\,\frac{\pi}{N}\right) + C_b \cos\left(n\overline{\theta - \sigma} + \omega t\right.$$

$$\left. - \overline{N-1}\;\overline{n+S}\,\frac{\pi}{N}\right) \qquad (15(b))$$

where

$$C_{f_n} = \frac{\sin(n - S)\pi}{\sin(n - S)\pi/N} \qquad (16(a))$$

$$C_{b_n} = \frac{\sin(n + S)\pi}{\sin(n + S)\pi/N} \qquad (16(b))$$

C_{f_n} and C_{b_n} are of the same form as the group connection factors and will here be termed phase-sequence correction factors. (For the sufficient reason that they are functions of the number of phases and their sequence). Their value is zero in general, with the following exceptions:-

$$\left.\begin{array}{l} C_f = N \text{ when } n - S = kN \\ C_b = N \text{ when } n + S = kN \end{array}\right) \quad k = 0, 1, 2 \ldots.$$

The possible (non-zero) orders of harmonics are therefore contained in the formula

$$n = kN \pm S \qquad (16(c))$$

where the +ve denotes $C_f = N$, $C_b = 0$ and the -ve sign denotes $C_b = N$, $C_f = 0$.

Since $(n \pm S)$ is always a multiple of N, it follows from (15(a)) that we may neglect the (constant) phase angle in (15(b)), and we may then write this equation as

$$B_S(\theta,t) = \frac{I_S qNB_c}{2} \sum_{k=1}^{\infty} F_{kN \pm S} \cos\left((kN \pm S)\, \overline{\theta - \sigma} \mp \omega t\right) \qquad (17)$$

where the signs are to be chosen as paired; i.e., +S and -ωt, and -S and +ωt respectively. This equation shows that in general, the flux wave is now composed of travelling waves of varying wavelength

/and

and with angular velocities of propagation given by $\pm \frac{\omega}{n}$ electrical radians per second. The harmonics having velocity of propagation $+ \frac{\omega}{n}$ are said to be forward travelling waves, moving in the positive direction of x. Similarly, the harmonics having velocity $- \frac{\omega}{n}$ are termed backward travelling waves. It will be noted that for a given sequence and harmonic order C_f and C_b cannot be non-zero simultaneously, except when $S = 0$. This means that only zero-sequence currents can produce pulsating or non-rotational fields in these windings. However, if two sequences are present simultaneously, it is possible to have both a forward and a backwards travelling wave of the same harmonic order, thus producing either a pulsating or elliptical wave.

This is a completely general statement of the fields produced by integral-slot windings, and it is easily seen that the windings produce in general all orders of harmonics with the exception of those eliminated by the winding factors. We contrast this with the common text-book theory, content with treating the balanced case. This special case is, of course, obtained by putting $S = 1$ in the above. For example, the normal 3-phase windings (narrow spread) are according to this treatment, 6-phase windings, and the possible harmonics are consequently $6k \pm 1$ under symmetrical, positive sequence excitation. (This winding will be considered in greater detail later).

In order to display this result, it is of interest to consider a 5-ph winding. This is not only of purely theoretical interest,

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because such windings do find occasional use in special machines. Tabulating the various cases we have:

S	n	Forward harmonics	Backwards harmonics
0	5k	5, 10, 15	5, 10, 15 ....
1	$5k \pm 1$	1, 6, 11	4, 9, 14 ....
2	$5k \pm 2$	2, 7, 12	3, 8, 13 ....
3	$5k \pm 3$	3, 8, 13	2, 7, 12 ....
4	$5k \pm 4$	4, 9, 14	1, 6, 11 ....

It will be noticed that the zero sequence excitation produces both a forward and backward rotating field, so that the resulting field is a pulsating one. If further the 1st and 4th sequences are both present, the resulting field is rotating but elliptical, and if they are in addition, equal in amplitude, the resulting field is pulsating. Similarly for the 2nd and 3rd sequences, but these produce different harmonics altogether. It may be noted that if n is allowed to take negative values, and its sign being interpreted as fixing the sense of rotation of the travelling waves, equation (16(c)) can be written

$$n = kN + S \quad ..... \qquad (16(d))$$

$$k = 0, \pm 1, \pm 2, \ ....$$

and equation (17) becomes accordingly:

$$B_S(\theta\ t) = \frac{I_S q N B_c}{2} \sum_{k=-\infty}^{\infty} F_{|kN+S|} \cos\left[\omega t - (kN + S)(\theta - \sigma)\right] \quad (17(a))$$

/It

It must be noted that $F_{|kN+S|}$ is evaluated for the numerical value of $(kN + S)$.

This convention was first adopted by F.T. Chapman[15] and later also used by R.Richter[16,17]. Though extremely useful as a compact notation, it has not been generally adopted, possibly due to the fact that a negative harmonic order does not have an obvious physical significance.

2.4.2 Hemisymmetric Windings

These windings are obtained from an N-phase (N-even) hemitropic winding by connecting in series opposition pairs of phases which are displaced by π electrical radians. The original N-phase winding can therefore be excited by an $N'-(=\frac{N}{2})$ phase system. If the latter is symmetrical, the field is obtainable by the above N-phase theory, but the general case needs separate consideration. We consider the field due to two N'-phase windings, series excited and differing in space-phase by π electrical radians. By equation (17) we have

$$B_S(\theta,t) = \tfrac{1}{2} I_S q N' B_c \sum_{k=1}^{\infty} F_{|kN'+S|} \Big[\cos\big(\overline{kN' \pm S}(\overline{\theta - \sigma}) \mp \omega t\big) - \cos\big(\overline{kN' \pm S}\ \overline{\theta - \sigma} - \pi \pm \omega t\big)\Big] \qquad (18)$$

and putting $(kN' \pm S) = n$ and contracting,

$$B_S(\theta,t) = I_S\, qN'\, B_c \sum_n F_n \sin^2 \frac{n\pi}{2} \cos\big(n\,\overline{\theta - \sigma} \mp \omega t\big) \qquad (18(a))$$

The additional factor $(\sin \frac{\pi n}{2})^2$ makes all even harmonics vanish, but it must be noted that n is now given by

$$n = kN' \pm S, \quad N' = N/2.$$

/Thus,

Thus, the winding, being fundamentally an N-phase winding, behaves as an N'-phase winding with the restriction that no even harmonics can be present.

Of particular interest is the case N' = 3. This is the commonly used 3-phase narrow spread winding. The table of harmonics is now,

S	n	Harmonics	
		Forward	Backward
0	3k	3, 9, 15	3, 9, 15 .....
1	3k ± 1	1, 7, 13	5, 11, 17 ....
2	3k ± 2	5, 11, 17 ...	1, 7, 13 .....

The presence of the triplen harmonics in both the forward and backward column indicates that these harmonics are pulsating. It is of interest to compare this table with that corresponding to the 6-phase winding from which it derives:

S	n	Harmonics	
		Forward	Backward
0	6k	6, 18, 24	6, 18, 24 ....
1	6k ± 1	1, 7, 13	5, 11, 17 ....
2	6k ± 2	2, 8, 14	4, 10, 16 ....
3	6k ± 3	3, 9, 15	3, 9, 15 .....
4	6k ± 4	4, 10, 16	2, 8, 14 .....
5	6k ± 5	5, 11, 17	1, 17, 13 ....

/From

From this table it is apparent that unless the even harmonics are eliminated by the winding factors, the even harmonics being of order 2 x triplen ones are also present as pulsating fluxes. Sequences 3, 1 and 5 correspond to sequences 0, 1 and 2 respectively in the corresponding 3-phase system, while the sequences 0, 2 and 4 which produce only even harmonics are eliminated by the connection factor. The connection factor arises in (18(a)) as a winding factor. It is possible to obtain this result without considering the winding distribution at all, apart from the fact that it is connected for a 3-phase winding. The interconnection of a 2N-phase winding to form an N-phase winding always removes the even numbered sequences in the 2N-phase system.

In Appendix I the theory of windings is treated in a novel way, by starting from the basic, primitive winding having N single coils. This winding basically is a 2-pole, hemitropic, N-phase winding. All other windings are obtained by <u>external</u> interconnections of these coils, and the theory of windings is therefore reduced to the study of the constraints introduced by the interconnections. (This is really a topological study, and may be of some importance and interest, but since it is not essential in the general scope of this Thesis and involves use of matrix algebra it has been deferred to the Appendices).

2.5 The Air-Gap Inductances of the Polyphase Windings

In this section the reactances due to the air-gap flux will be discussed. These are of fundamental importance in the analysis of all types of machines, but there is little evidence of this in most text-books. One reason seems to be that for a long period it was usual to perform the analysis of machines in terms of flux per pole and so derive induced e.m.f., both of the "transformer" and "rotational" type. This approach, although appealing to students at a lower level, is hopelessly restricted and of little value in advanced problems involving stability, etc., under transient conditions. Furthermore, the present Author would contend that if a little knowledge of differential equations is presupposed, the analysis in terms of reactances (or inductances) is by far the easier - if any degree of elegance is aimed at.

The inductances of the polyphase windings are fundamentally of three kinds. The first is the total inductance per phase, which normally would include the leakage inductance. No attempt will be made here to obtain the latter; the object is specifically to determine air-gap inductances. Secondly, there are the mutual inductances between phases, and thirdly the sequence inductances which will be shown to have a fundamental and important significance in the analysis. The sequence inductances will be shown to include what is normally termed the self-synchronous inductance per phase.

/This

This is, in fact, the inductance corresponding to the first sequence (or positive sequence).

2.5.1 Air-gap self inductance per phase

The total flux linkages in a distributed winding is clearly not the sum of the linkages produced by the single coils separately, since there is a large degree of mutual coupling between the coils, some of the fluxes being aiding - others opposing. The method adopted here for finding the total flux linkages per phase is to use the expression already obtained for the flux distribution, integrating this to find the flux linkages per coil and finally summing these linkages for all the coils. This method gives the total flux linkages as the sum of the harmonic flux linkages, which has the advantage of being more readily applicable to analysis.

The single-phase winding has the flux distribution given by

$$B_{ph}(\theta) = qIB_c \sum_{n=1}^{\infty} F_n \cos n(\theta - \gamma) \qquad (13(b))$$

In this case F_n corresponds to $F_{\alpha,\beta,n}$; the skew-factor is not to be included in this calculation. Clearly, the flux linkages for a single coil situated at the angle ξ relative to the reference axis ($\theta = 0$) is given by

$$\psi_c = \int_{\xi-\alpha/2}^{\xi+\frac{\alpha}{2}} B_{ph}(\theta)\ .d\theta\ .\ \frac{DLT_c}{2} \qquad (19)$$

where D is the mean gap diameter and L is the axial length of the gap. Evaluating we have,

$$\psi_c = \frac{DLT_c}{2p} \; qIB_c \sum_{n=1}^{\infty} \frac{F_n}{n} \Big[\sin n(\theta - \sigma)\Big]_{\xi - \alpha/2}^{\xi + \alpha/2}$$

$$\therefore \psi_c = \frac{DLT_c}{p} \cdot qIB_c \sum_{n=1}^{\infty} F_n \cdot \frac{1}{n} \sin \frac{n\alpha}{2} \cos n\,(\xi - \sigma) \qquad (19(a))$$

Again, for the group of q adjacent coils, this expression must be summed for q values of ξ, namely ξ_1, $\xi_1 + \gamma$, ..., $\xi_1 + \overline{q-1}\,\gamma$, the average value of which is σ. This gives

$$\psi_g = \frac{DLT_c}{p} \; q^2 IB_c \sum_{n=1}^{\infty} F_n^{\,2} \qquad (20)$$

In an integral-slot winding, the flux is symmetric about each group, and for p groups we have

$$\psi_{ph} = DLT_c \; q^2 \; IB_c \sum_{n=1}^{\infty} F_n^{\,2} \qquad (21(a))$$

giving the linkages per phase. Substituting for B_c and dividing by I we obtain

$$L_{ph} = \left(\frac{2\mu_o DL}{\pi g p^2} \sum_{n=1}^{\infty} F_n^{\,2} \right) \cdot T_{ph}^{\,2} \qquad (21(b))$$

where $T_{ph} = pqT_c$, the number of turns per phase. The reactance per phase is correspondingly,

$$X_{ph} = \left(\frac{4\mu_o DL}{g p^2} \, f \sum_{n=1}^{\infty} F_n^{\,2} \right) T_{ph}^{\,2} \qquad (21(c))$$

where f is the frequency in c/s.

The geometric factor $\frac{2\mu_o DL}{\pi \; g \; p^2} \sum_{n=1}^{\infty} F_n^{\,2}$ is of some interest. It gives at a glance the factors determining the inductance of the winding. Specifically it is seen to be proportional to the surface area of the mean gap and inversely proportional to the gap length and the square of the number of pole pairs. The factor containing the winding factors will in most cases be very nearly equal to $F_1^{\,2}$.

/Even

Even for a single coil winding, having full pitch coils, the series becomes

$$\sum_{n=1,3} \frac{1}{n^2} = 1 + 1/9 + 1/25 + \ldots..$$
$$= \frac{\pi^2}{8} \doteq 1.235$$

Thus even in the extreme case of single coil windings having full pitch coils, the fundamental component of flux accounts for 81% of the total inductance. In a distributed winding with chorded coils the figure would be above 97%. This is a clear indication of the degree of accuracy one may obtain with the simplified methods, based on the assumption of sinusoidal distribution of flux.

It may be noted that equations (21) are applicable to hemisymmetrical as well as hemitropic windings provided the winding factors for all even harmonics are put equal to zero.

2.5.2 The Mutual Inductance between Phases

For the evaluation of the mutual inductance we may again start with equation (13(b)), and consider the linkages produced in the βth phase per ampere in the αth phase.

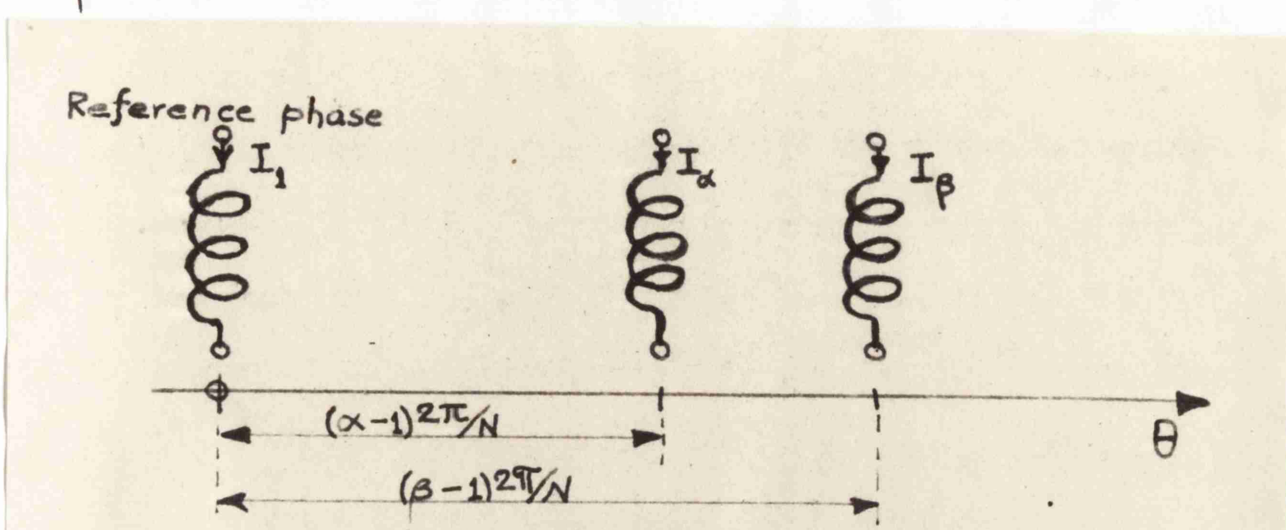

Fig3

Referring to Fig. 3, the flux distribution caused by a current I_α in the αth phase is given by

$$B_\alpha = I_\alpha \; qB_c \sum_n F_n \cos n \left(\theta - (\alpha - 1)\frac{2\pi}{N}\right) \qquad (22)$$

The total linkages produced in phase β is the sum of linkages in the individual coils. For a given coil (β_1) situated at ξ the linkages are

$$\psi_{\beta_1\alpha} = \int_{\xi - \alpha/2}^{\xi + \alpha/2} B_\alpha \, d\theta \cdot \frac{DLT_c}{2p} \qquad (23)$$

$$\therefore \psi_{\beta_1\alpha} = \frac{DLT_c}{2p} I_\alpha \; qB_c \sum_n F_n \frac{1}{n} \sin\frac{n}{2} \cos n \left(\theta + \xi - \overline{\alpha - 1}\,\frac{2\pi}{N}\right) \qquad (23(a))$$

and summing over all values of ξ within phase β, we obtain

$$\psi_{\beta\alpha} = DLT_c \; q^2 \; I_\alpha \; B_c \sum_n F_n^2 \cos n(\beta - \alpha)\frac{2\pi}{N} \qquad (23(b))$$

Again, substituting for B_c we obtain

$$\psi_{\beta\alpha} = \frac{2}{\pi}\frac{\mu_o}{g}\frac{DLT_{ph}^2}{p^2} I_\alpha \sum_n F_n^2 \cos n\,(\beta - \alpha)\frac{2\pi}{N} \qquad (23(c))$$

whence, the mutual inductance between phases α and β is given by

$$M_{\alpha\beta} = \frac{2}{\pi}\frac{\mu_o}{g}\frac{DLT_{ph}^2}{p^2} \sum_n F_n^2 \cos n\,(\beta - \alpha)\frac{2\pi}{N} \qquad (24)$$

The expression is symmetrical in α and β , and consequently $M_{\alpha\beta} = M_{\beta\alpha}$. Furthermore, if we put $\alpha = \beta$, (24) reduces to the self-inductance as given for equation (21(b)). The general equation, containing all the air-gap inductances can therefore be written

$$L_{\alpha\beta} = \sum_n L_{ph}^n \cos n\,(\alpha - \beta)\,\frac{2\pi}{N} \qquad (25)$$

where L_{ph}^n means the self-inductance due to the nth harmonic flux.

The above is easily extended to include the mutual

/inductance

inductance between stator and rotor coils. Since $(\alpha - \beta)\frac{2\pi}{N}$ is simply the relative angular displacement of the phases, we may in general replace this by $(\alpha_s - \beta_r)\frac{2\pi}{N} + xp$, giving the displacement at any time between the α_sth phase on the stator and the β_rth phase on the rotor, xp being the instantaneous displacement of the reference phases on the stator and rotor respectively. Further if the windings are different, so that their winding factors are different, this must also be taken into consideration. Denoting stator quantities and rotor quantities by subscripts 1 and 2 respectively, we have

$$M_{\alpha 1, \beta 2}, = \frac{2}{\pi}\frac{\mu_o}{g} \frac{DL}{p^2} T_{ph1} \cdot T_{ph2} \sum_n F_{n,1} F_{n,2} \times \cos n \left(\overline{\alpha_1 - \beta_2} \frac{2\pi}{N} + xp \right) \qquad (26)$$

This equation contains the most valuable of information as regards analysis and design of machines having smooth (non-salient) air gaps. It has, however, never come to the notice of the present Author, and is certainly not mentioned in any of the standard texts known to him. The torque produced in machines is directly proportional to $(M_{\alpha 1, \beta 2})_{max}$ and consequently, no complete analysis of smooth-gap machines can be without a mention of this quantity in one form or another.

2.5.3 The Sequence Inductances of Polyphase Windings

In order to illustrate the fundamental importance and usefulness of the quantities evaluated in the preceding paragraphs,

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in this section the analysis of polyphase machines will here be presented by the powerful mathematical methods of matrix algebra.

Again we are only concerned with smooth gap machines, so that the stator (or rotor) inductances are not functions of the rotor position, or of time. Considering firstly the winding on one side of the air gap only, or assuming that these are the only excited windings, we arrive at the following set of equations

$$
\begin{aligned}
V_1 &= \left[r + (l + L_{11})p\right] i_1 + L_{12} p i_2 + \ldots\ldots\ldots + L_{1N}\, p i_N \\
V_2 &= L_{21}\, p i_1 + \left[r + (l + L_{22})p\right] i_2 \ldots\ldots\ldots + L_{2N}\, p i_N \\
V_N &= L_{N1}\, p i_1 + L_{N2}\, p i_2 + \ldots\ldots\ldots\ldots + \left[r + (l + L_{NN})p\right] i_N
\end{aligned}
\qquad (27(a))
$$

where V_1, ---- V_N are the phase terminal voltages, i_1, ---- i_N the phase currents, r the resistance per phase, l the leakage inductance per phase and $L_{\alpha\beta}$ are the air gap inductances as defined by equation (25).

These equations can be written in matrix form as

$$V = \left\{(r + lp)U + Lp\right\} I \qquad (27(b))$$

or alternatively,

$$V_\alpha = \left\{(r + lp)\delta_{\alpha\beta} + L_{\alpha\beta}\, p\right\} i_\beta \qquad (27(c))$$

where U is the unit matrix, and $\delta_{\alpha\beta}$ is the Kronecker delta.

This is a system of differential equations which can be solved by ordinary, classical methods. However, the system can be reduced

/very

very considerably by a suitable change of variables. The most suitable choice cannot in general be determined by (27) alone, since there are more windings in the whole machine which all have to be considered in the complete analysis, but some fundamental considerations can be made.

The matrix L is symmetrical and real and, therefore, has all real latent roots. That is to say, the canonical form of L is real.

In order to reduce the equation (27) to diagonal form, we must find a (linear) transformation of variables given by

$$\begin{aligned} V &= SV_S \\ I &= SI_S \end{aligned} \qquad (28)$$

such that $S^{-1} LS = \Lambda$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \lambda_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \lambda_N \end{bmatrix}$$

i.e., Λ is a diagonal matrix. Substituting by (28) in (27(b)) then gives

$$V_S = \left\{(r + lp)U + \Lambda p\right\} I_S \qquad (29)$$

This equation is a set of N independent equations relating the 2N variables V_S and I_S by simple first order differential equations, and their solution is mainly trivial. Clearly, (29) constitutes a fundamental form of (27), and if easily obtained will serve to provide a much better insight into the nature of the solution.

The transform matrix S of (28) will now be constructed by

/the

the usual method. In the present case of L being the matrix given by equation (25) it will be apparent that the symmetrical component transform is applicable.

The matrix L is a circulant, that is to say it may be written in the form

$$L = \begin{bmatrix} l_1 & l_2 & l_3 & \text{------} & l_N \\ l_N & l_1 & l_2 & \text{------} & l_{N-1} \\ \\ l_2 & l_3 & l_4 & \text{------} & l_1 \end{bmatrix} \qquad (30(a))$$

where $l_\gamma = \sum_n L_{ph}{}^n \cos n\,(\gamma - 1)\,\frac{2\pi}{N}$, $\gamma = 1, 2 \ldots\ldots, N.$

Also, it is symmetric, whence $l_\gamma = l_{N - \gamma + 2}$, and we have

$$L = \begin{bmatrix} l_1 & l_2 & l_3 & \text{------} & l_N \\ l_2 & l_1 & l_2 & \text{------} & l_{N-1} \\ \\ l_N & l_{N-1} & l_{N-2} & \text{------} & l_1 \end{bmatrix} \qquad (30(b))$$

Both these properties of L are important in the evaluation of its latent roots. In the special case of N = 3, we observe that $l_2 = l_3$, so that all the elements off the principal diagonal are equal. This makes the transform matrix S less determinate, and in fact, many forms are possible. It is shown in Appendix II that the transform matrices which will diagonalise L are the symmetrical component matrix and a derived form respectively. For the case N = 3, the latter corresponds to the $\alpha\beta o$ component transform.

/The

The procedure for determining the latent roots of L, $(\lambda_1, \lambda_2, \ldots\ldots, \lambda_N)$ which will be denoted by L_S (S = 0, 1, 2 ..., N - 1) is given in Appendix II, and we may apply formula (II,10C), giving

$$L_S = \sum_{r=1}^{N} \left[\sum_n L_{ph}^{\ n} \cos n(r-1)\frac{2\pi}{N} \right] \cos S(r-1)\,\frac{2\pi}{N} \tag{31}$$

Thus inverting the order of summation,

$$\begin{aligned} L_S = \sum_n L_{ph}^{\ n}\, \tfrac{1}{2} \Bigg\{ & \frac{\sin(n+S)\pi}{\sin(n+S)\frac{\pi}{N}} \cos\,(n+S)(N-1)\frac{\pi}{N} \\ & + \frac{\sin(n-S)\pi}{\sin(n-S)\frac{\pi}{N}} \cos\,(n-S)(N-1)\frac{\pi}{N} \Bigg\} \end{aligned} \tag{32}$$

$$\therefore \quad L_S = \sum_{k=0,1,2\,\ldots} \frac{N}{2}\, L_{ph}^{\ kN\mp S} \tag{33}$$

The latent roots which are in fact the inductance terms in the equations (29) must naturally be termed sequence inductances. By (33) we see that these are made up of harmonic inductances; in fact, we may write

$$L_S^{\ n} = \frac{N}{2}\, L_{ph}^{\ n} \tag{34}$$

where, of course, n must satisfy the integral number equation

$$n = kN \mp S \qquad k = 0,\ 1,\ 2\ \ldots\ldots\ldots$$

We are now in a position to draw an equivalent circuit for the stator equations of a polyphase machine. In every sequence it is a <u>separate</u> RL circuit, where the inductance may be split into component parts as given by (33). These inductances are, of course, mutually coupled to the rotor coils, but if the rotor is also a symmetric winding, and the air gap is smooth, each is in fact coupled to only one equivalent RL circuit, thus giving rise to a complete equivalent circuit of the kind well known for induction motors. (The treatment

/of

of machine theory from this point of view offers a wide scope, and it is the Author's intention to pursue this work to develop a rigorous theory based on the above inductance calculations. It is hoped to remove many of the abstruse concepts regarding the inductances by showing how the various transformations are derived. A critical note on the subject of synchronous inductance is given in Appendix III, but space and the purpose of this Thesis does not allow any complete treatment of the subject here). Sec

3. The Theory of Fractional-Slot Windings

The fractional-slot windings have been known since the beginning of the century, and their advantages and disadvantages have been appreciated - although not fully understood - for a similar length of time. Their first application seems to have been in the construction of wave windings, and windings of very large machines. The first attempt to analyse these windings is apparently due to E.M. Tingley[8], who considered the possibility of combining unequal groups of coils in a lap winding. There was no general method of attack, but his viewpoint seems to have persisted right up to the present in certain quarters. The harmonic analysis of the field produced by these windings seems to date from 1927 in the form of Q. Graham's paper[9]. The mathematical treatment has been further developed by M.G. Malti and F. Herzog[11], and no substantial advance has been made since, although there are several publications on the subject. The bulk of this literature is devoted to 3-ph, narrow-spread windings, and normally written by designers - for designers. The treatment is, therefore, often lacking in freedom from unnecessary detail and in clarity. No attempt has been made to cover the general theory of polyphase windings, and full use of the properties of harmonic functions has not been made.

In the following a generalised analysis of polyphase fractional-slot windings is attempted. It is based on the

/fundamental

fundamental properties of harmonic functions, and brings out the mathematical origin of the slot-star method mentioned above. General formulae for distribution factors are derived and rigorous discussion of their field of application is presented. In the fractional-slot windings, the harmonic spectrum is in general denser than for integral-slot windings, but when properly designed may have a smaller overall harmonic content. Consequently, the analysis is also more involved than for integral-slot windings, although the writer is of the opinion that it has in the past been unnecessarily clouded by special, non-mathematical treatment. As an example, we may refer to Calvert's[10] extensive tables of distribution factors which gives the impression that practically no uniformity is obtainable in the treatment.

There is a slight modification possible in the arrangements of 2N'-phase, N'phase connected windings, and these will receive separate treatment.

3.1 Wide-Spread Fractional-Slot Windings

In the fractional-slot windings, the number of coils per phase is not a multiple of the number of pole pairs, that is, if Q is the number of slots, N the number of phases, and p the number of pole pairs, Q/N is not a multiple of p. Each phase has q' or q' + 1 coils per pole pair, where q' is the integral part of Q/Np.

Normally, the fraction $\frac{Q}{Np}$ may be reduced somewhat, i.e., Q and p may have some common factor, say ℓ. The winding may then

/be

be arranged in ℓ identical sections, or repeatable groups. Each of these groups is capable of producing a balanced polyphase system, and the analysis of any one such group suffices for the complete winding. However, some modifications are possible in some or all of these sections, so that they have to be analysed separately and finally combined in order to give the results for the complete winding. In any case, each of these groups forms a separate part of the winding and indeed in the analysis, and attention will firstly be directed to such a group.

We assume that $\frac{Q}{Np} = \frac{Q'\ell}{Np'\ell}$ so that the fraction $\frac{Q'}{Np'}$ contains no factor common in Q' and p'. Thus the irreducible group consists of Q' coils forming a balanced N-phase, 2p'-pole winding. A simple picture of this group is given in Fig. 4, indicating that $2\pi/\ell$ radians is spanned by the group. By this we mean that the top (or bottom) layer of the winding within this range is completely occupied by coils all belonging to the same section. It is immaterial whether we imagine the group position to be defined by the top layer or bottom layer, or indeed by the midpoints of the coils. The latter definition is most convenient for the analysis, and will be adhered to in this analysis.

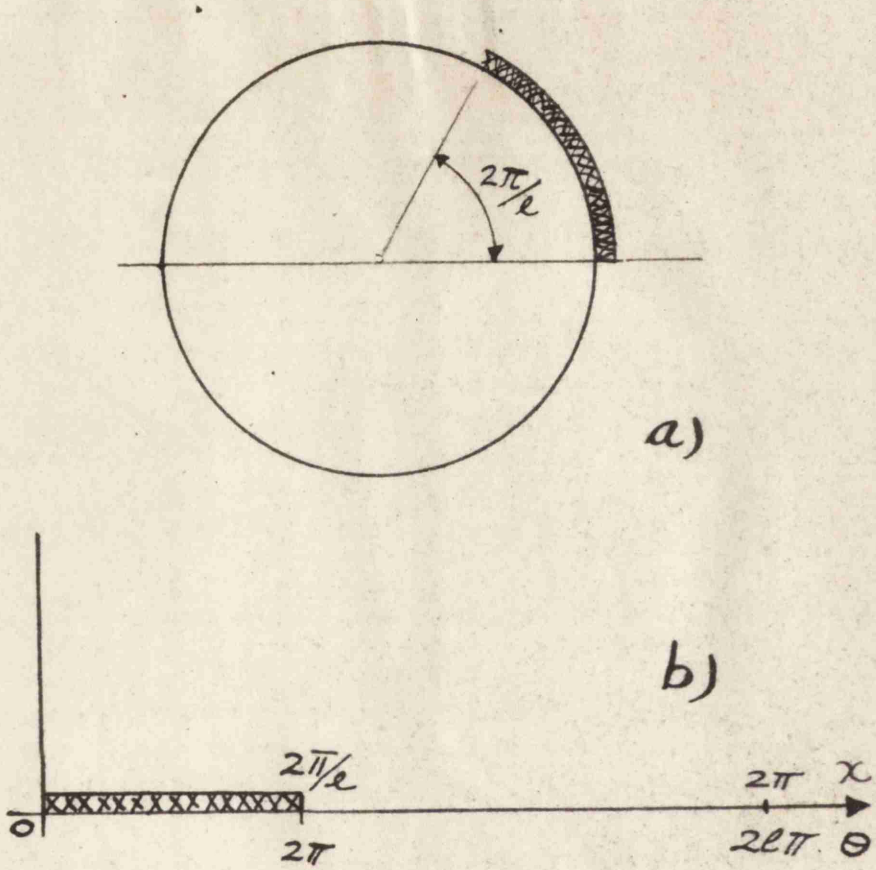


Fig 4.

The simplest type of winding is that where all the repeatable sections are identical since in this case, the width of the group must necessarily be the lowest possible period in the flux wave. Consequently, only harmonics being multiples of ℓ may exist, and we can treat each section as spanning the same $2\pi/\ell$ radians, rejecting all harmonics except those being multiples of ℓ. This will be clear in the subsequent development. When treating these types of windings, the primes on Q and p are dropped, and we consider Q coils forming an N-phase system with 2p poles over the interval $2\pi/\ell$ radians. Since in

/order

order to form a balanced system Q/N must be integral, Q contains the factor N, in fact $Q/N = q$, the number of coils per phase per section. Now Q and p contain no common factors, and p and N are therefore prime. In general, if p and N are not prime, the winding must either be integral slot or cannot be balanced.

(In this analysis no distinction is made between the number of coils and the number of slots, fully wound stampings only being considered. Sometimes windings are made where the number of poles is a multiple of the number of phases, and some unbalance may be accepted, or some slots may be left unwound. These windings are not included in this treatment).

Not therefore dealing with unsymmetrical windings

The basic function in the analysis is again taken to be that given by equation (10), but we shall use it in the form

$$B_m(x) = I\, B_c\, F_m\, e^{jm(x - \sigma)} \qquad (36)$$

and throughout consider the actual flux distribution to be the real part of $B_m(x)$. The exponential form has the added advantage of being in one-to-one correspondence with the slot-star method. The vector $e^{j\sigma}$ is easily recognised as defining the position of the midpoint of the coil, and consequently shifts the basic pattern given by

$$B_m(x) = \sum_m I\, B_c\, F_m\, e^{jmx}$$

forward by the angle σ.

If this winding consists of ℓ identical sections, there

/are

are ℓ similar groups per phase, and by equation (13(a)), only values of m being multiples of ℓ give non-zero values, thus writing $m = n\ell$, we have

$$B_{n\ell}(x) = \sum_{n} I \; B_c \; F_{n\ell} \, e^{jn\ell x} \tag{37}$$

Now if we write $\theta = \ell x$, being the electrical radian measure in this case, and again take the arguments of F_m as well as σ in the same measure, (36) becomes

$$B_n(x) = I \; B_c \; F_n \; e^{jn(\theta - \sigma)} \tag{36(a)}$$

The addition of $2\pi/l$ to x is equivalent to the addition of 2π to θ, so that (36(a)) can be simply multiplied by ℓ to give the sum of the fluxes due to the ℓ corresponding coils in the various repeatable groups. The analysis of a single group can be carried out as if it filled a complete 2π radians of a winding. If the sections are not identical, each section must be taken by itself, and the above restriction on the harmonic spectrum is not valid. The analysis then is considerably harder - as will be shown in due course.

Given equation (36(a)) and that the number of coils is Q, the distance between adjacent coils is given by $2\pi/Q = \gamma$. The position angle σ consequently increases by γ radians per coil. The summation of the contributions to the nth harmonic in the system is therefore of the form

$$B_n(\theta) = \sum_{r} I_r \; B_c \; F_n \; e^{jn\theta} \; e^{-jn(r-1)\gamma} \tag{36(c)}$$

/where

where r takes some of the values 1, 2, ..., Q. The next step is to consider the grouping of these Q coils into N phases, giving a predominant pth harmonic. ($F_p \doteq$ Unity is, of course, assumed; that is, the span is approximately one pole pitch). We notice that the number r may be taken to be the number of the coils, counted anticlockwise from a reference coil (or slot). It is well known that in order to obtain a balanced winding, the numbers r must form an arithmetical progression (modulus Q), i.e., for the first phase, r takes the values 1, 1 + d, 1 + 2d, ... $1 + (\frac{Q}{N} - 1)d$, for the next phase it takes the values

$$1 + \frac{Q}{N}d,\ 1 + (\frac{Q}{N} + 1)d,\ \ldots\ldots,\ 1 + (\frac{2Q}{N} - 1)d$$

and so on, (all numbers being reduced mod. Q). Now since the value of $e^{-jn(r-1)\frac{2\pi}{Q}}$ and $e^{-jn(r \bmod Q-1)\frac{2\pi}{Q}}$ are identical, the modulus consideration may be dropped here and the arithmetical progression substituted directly. This would not be permissible had we applied (36) directly, without stipulating that all sections are identical. In this case, the expression is in general not summable in closed form, as will be shown later. According to (36(c)) we may then write the flux distribution due to the pth phase as

$$B_n^{\rho}\ (\theta) = I_{\rho}\ B_c\ F_n\ e^{jn\theta} \sum_{t=(\rho-1)\frac{Q}{N}}^{\rho\frac{Q}{N}-1} e^{-jntd\frac{2\pi}{Q}} \qquad (38)$$

which is in summable form. It remains to determine d. This

/must

must be chosen so that the step d corresponds to approximately one pole pair pitch or a multiple thereof. Consequently we require $d \doteq k\frac{Q}{p}$ where k is any integer (less than p). Furthermore, in order that all the coils/pole should be in adjacent slots, it is required that $pd\frac{2\pi}{Q} = \frac{2\pi}{Q} \pm k.2\pi$ which is a sufficient condition. We can write this as

$d = \frac{1 \pm kQ}{p}$ where k is the smallest integer which makes the R.H.S. integral. This is equivalent to

$$d = \frac{kQ \pm 1}{p} \tag{39}$$

To test whether this in fact produces a balanced polyphase winding we substitute in (38), putting $n = p$.

$$B_p^{\rho}\ (\theta) = I_{\rho}\ B_c\ F_p \sum_{t=(\rho-1)\frac{Q}{N}}^{\rho\frac{Q}{N}-1} e^{-jptd\frac{2\pi}{Q}}$$

$$= I\ \ B_c\ F_p \sum_{t=(\rho-1)\frac{Q}{N}}^{\rho\frac{Q}{N}-1} e^{\mp jt\,2\pi/Q} \tag{40}$$

The $\mp$ sign to be chosen according to the $\pm$ sign in equation (39). In either case, the summations gives N equal magnitude vectors all having a difference in phase of $2\pi/N$, which is the required phase displacement, corresponding to a phase displacement of $2\pi/Np$ in θ.

The general summation gives

$$B_n^{\rho}\ (\theta) = I_{\rho}\ B_c\ F_n \sum_{t=0}^{\frac{Q}{N}-1} e^{-jntd\frac{2\pi}{Q}}$$

$$= I_{\rho}\ B_c\ F_n\ \frac{\sin nd\ \pi/N}{\sin nd\ \pi/Q}\ e^{-jnd(\frac{Q}{N}-1)\frac{\pi}{Q}} \tag{41}$$

/Thus

Thus, the distribution factor is emerging as

$$F_{\beta,n} = \frac{\sin n\, d\, Q/N\, \pi/Q}{Q/N \sin nd\, \pi/Q} \tag{42}$$

The distribution factor is of the same form as that of the integral-slot winding, and does in fact reduce to exactly the same form for values of n being multiples of p. Thus if $n = n'p$ we have

$$f_{\beta,n'} = \frac{\sin n'(kQ \pm 1)\frac{Q}{N} \cdot \pi/Q}{Q/N \sin n'(kQ \pm 1)\pi/Q}$$

$$\therefore \quad f_{\beta,n'} = \frac{\sin n'\, Q/N.\, \pi/Q}{Q/N \sin n'\, \pi/Q}(-1)^{\,n'k(Q/N - 1)} \tag{43}$$

From this equation it is evident that no harmonics which are multiples of N are possible unless n' is also a multiple of Q, and by virtue of the coilspan factor, these are also zero. Hence harmonic orders being multiples of the number of phases in the winding are generally eliminated.

The permeance harmonics due to the slotting of the stampings are of orders $Q \pm 1$ and $2Q \pm 1$ principally, i.e. $n' = (Q \pm 1)/p$ or $(2Q \pm 1)/p$. These are integral, that is to say they do exist if $k = 1$ or 2 respectively. Since $k < p$, it follows that unless p is larger than 2, one or both of the principal toothripples will still be present, and for effective reduction of the permeance harmonics, a larger number of poles per repeatable group would be desirable. These considerations do not, of course, apply to salient pole machines, where there is flux fringing at the pole shoes.

/In

In general, fractional-slot windings, having identical irreducible groups very often turn out to have undesirable sub-harmonics, that is harmonics of longer wave-length than the principal harmonic. Very often these harmonics are reduced by re-arranging successive groups so that the sub-harmonic in each group are altered in phase sufficiently to be cancelled out between the several groups.

For the purpose of analysis, to cover such cases, it is necessary to evaluate the complete flux density function produced by each irreducible group, and finally to sum these functions.

The irreducible groups must still be individually balanced polyphase systems, and their layout must again follow the law of arithmetical progression, now with the reduction modulo Q as an essential part.

The central part of the problem is the evaluation of

$$B_n^{\rho}(x) = I_{\rho}\, B_c\, F_m \sum_{t=(\rho-1)\frac{Q}{N}}^{\rho\frac{Q}{N}-1} e^{-jm(td)_Q \frac{2\pi}{\ell Q}} \qquad (44)$$

where we have retained Q as the number of slots per irreducible group and $(td)_Q$ means (td) mod. Q. The summation now gives non-zero results for all harmonics, in particular all those corresponding to m not being a multiple of ℓ.

/In

In order to simplify the following details, the sequence will be written as

$$z^{(td)_Q}, \ (t = 1, 2 \ldots\ldots, Q/N), \text{ where } z = e^{-jm2\pi/Q\ell}.$$

We now write $x = [Q/d]_i$ (i.e., x is the integral part of Q/d), and suppose

$$Q/N = ux + v, \text{ where } 0 \leqslant v < x, \ u \text{ and } v \text{ being integers.}$$

The sequence $(td)_Q$ may then be expressed by the sets

$$\begin{array}{llll}
1. \ldots & d, & 2d, \ \ldots\ldots\ldots\ldots\ldots & xd \\
2. \ldots & (x+1)d, & (x+2)d, \ \ldots\ldots\ldots, & (2x+h_2)d \\
3. \ldots & (2x+h_2+1)d, & (2x+h_2+2)d, \ \ldots\ldots\ldots, & (3x+h_3)d \\
 & & & \\
(u)..& ((u-1)x+h_{u-1}+1)d, & ((u-1)x+h_{u-1}+2)d, \ \ldots\ldots\ldots, & (ux+h_u)d \\
(u+1)..& (ux+h_u+1)d, & (ux+h_u+2)d, & (ux+v)d
\end{array}$$

where h_n is the principal remainder of $\left[\frac{nQ}{d}\right]_i - n\left[\frac{Q}{d}\right]_i$, that is

$$h_n = \left[n \left(\frac{Q}{d} - \left[\frac{Q}{d}\right]_i \right) \right]_i .$$

If $h_u \geqslant v$, the last (u + 1)th) row does not appear and the second last row will read

$$((u-1)x+h_{u-1}+1)d, \qquad \ldots\ldots\ldots, \quad (ux+v)d.$$

It will be noted that set 1 is in the range (1,Q), set 2 is in the range (Q + 1, 2Q) etc.

Thus by subtracting the appropriate multiples of Q in these sets, we obtain the required sequence $(td)_Q$.

We write

$$Q = xd + y$$

/whence

whence the sequence $(td)_Q$ can be represented by the sets

$$
\begin{array}{llll}
d\ , & 2d\ , \ldots, & xd & \ldots\ldots\ldots\ (1') \\
d - y, & 2d - y, \ldots, & (h_2 + x)d - y, & \ldots\ldots\ (2') \\
(h_2 + 1)\,d - 2y, & (h_2 + 2)d - 2y, \ldots, & (h_3 + x)d - y & \ldots\ldots\ (3')
\end{array}
$$

$$
\begin{array}{llll}
(h_{u-1}+1)d-(u-1)y, & (h_{u-1}+2)d-(u-1)y, \ldots, & (h_u + x)d-(u-1)y & \ldots\ldots\ (u') \\
(h_u + 1)d - uy, & (h_u + 2)d - uy, \ldots, & (vd - uy) & \ldots(u' + 1)
\end{array}
$$

The last row does not appear if $v \leqslant h_u$, and the second last row is then

$$(h_{u-1}+1)d-(u-1)y, \quad (h_{u-1}+2)d-(u-1)y, \ \ldots\ldots\ldots, \ vd-(u-1)y$$

By this arrangement the sequence $z^{(td)_Q}$ is summable in $(u+1)$ parts, since each of the sets (1') (2') (3') etc., now gives a geometric series.

Writing the series out in full, we obtain

$$
\begin{aligned}
\sum_{t=1}^{Q/N} z^{(td)Q} &= z^d + z^{2d} + \ \ldots\ldots\ldots + z^{xd} \\
&+ z^{-y}\,(z^d + z^{2d} + \ldots + z^{xd}) \\
&+ \ \ldots \\
&+ z^{xd-y}\,(z^d + z^{2d} + \ldots\ldots + z^{h_2 d}) \\
&+ z^{xd-2y}\,(z^d + z^{2d} + \ldots\ldots\ldots + z^{h_3 d}) \\
&+ \ \ldots \\
&+ z^{xd-\overline{u-1}y}\,(z^d + z^{2d} + \ldots + z^{h_u d}) \\
&+ z^{h_n d-uy}\,(z^d + z^{2d} + \ \ldots + z^{vd})
\end{aligned}
$$

Here, each of the brackets contain a geometric series, and by

/summation

summation of these we obtain

$$\sum_{t=1}^{Q/N} z^{(td)Q} = \frac{z^d\,(1 - z^{xd})}{1 - z^d}\,(1 + z^{-y} + \ldots + z^{h_{u-1}d - \overline{u-1}y})$$

$$+ \frac{1}{1 - z^d}\quad z^{\overline{x+1}d-y}\,(1 - z^{h_2 d}) + z^{\overline{x+1}d-2y}\,(1 - z^{h_3 d})$$

$$+ \ldots + z^{\overline{x+1}d-\overline{u-1}y}\,(1 - z^{h_u d})$$

$$+ \frac{z^{\overline{h_u+1}d-uy}}{1 - z^d}\,(1 - z^{vd})$$

$$= \frac{z^d - z^{\overline{x+1}d}\,(1 + z^y)}{1 - z^d}\,(z^{h_2 d-2y} + z^{h_3 d-3y} + \ldots + z^{h_{u-1}d - \overline{u-1}y})$$

$$+ \frac{z^d}{1 - z^d}\,(1 - z^{xd})(1-z^{-y}) + z^{xd-y}\,(1 - z^{-(u-1)y})/(1-z^{-y})$$

$$- z^{(h_u+x)d-(u-1)y} + z^{h_u d-uy}\,(1 - z^{vd})$$

finally,

$$\sum_{t=1}^{Q/N} z^{(td)Q} = \frac{z^d}{1 - z^d}\,\Big[(1 - z^{xd}\,(1 + z^y))\,S + (1 - z^{xd})(1-z^{-y})$$

$$+ z^{xd}\,(1 - z^{(u-1)y})/(z^y-1) + z^{h_u d-uy}\,(1 - z^{vd} - z^{xd+y})\Big] \qquad (45)$$

where

$$S = \sum_{n=2}^{u-1} z^{h_n d - ny}$$

It is evident that in the general case, this formula cannot be further reduced, and the general formula for the distribution factor of a particular group has little practical (or theoretical) value. The above expression has been derived and is presented here to indicate that in the case of irregular windings, a considerable amount of tedious arithmetic seems to be necessary to obtain the distribution factors.

/It

It may be argued that the present method of attack is too restricted - in as much as only one group is considered. If the winding as a whole, including several sections of repeatable type, but differing by a small rearrangement respectively, were to be treated in the one analysis, more powerful methods must be sought. A possible alternative method of analysis is suggested in Appendix I, although it is not carried as far as to include fractional-slot windings. The main idea is to formalize the addition processes by interpreting these as singular transformations of the currents in the various coils. The reason for this method of analysis is to be found in the close link between the separate space-harmonic functions and the time-harmonic functions - and their respective symmetrical components. By considering the complete N-coil winding as a balanced N-phase winding, and resolving the corresponding N currents into symmetrical (N-phase) components, clearly each space-harmonic as produced by each symmetrical component becomes trivial to evaluate. The problem is then to establish the various symmetrical components and add the contributions to each space-harmonic. The latter problem involves calculations of similar kind to those already undertaken, but it is felt that a much better organisation is obtained by this method, and that it may be developed to give methods of synthesis as well as analysis of any particular kind of winding, however irregular.

/The

The problem of evaluating the sum of the geometric progression $z^{(td)Q}$ can be stated in another form, which again illustrates the inherent difficulty. Instead of using congruences, the number (td) mod Q may be regarded as the principal value of $j\frac{Q}{2\pi}\log e^{-jtd\frac{2\pi}{Q}}$. Thus $e^{-jm\frac{2\pi}{Q\ell}}(td)_Q$ may be expressed as

$$e^{m/\ell \log e^{-jtd\frac{2\pi}{Q}}} = \left[e^{\log e^{-jtd\frac{2\pi}{Q}}}\right]^{m/\ell}$$

$$= (\cos td\frac{2\pi}{Q} - j\sin td\, 2\pi/Q)^{m/\ell} \qquad (46)$$

where it is understood that the bracket must be evaluated <u>before</u> raised to the power m/ℓ. Again if m/ℓ is integral, the order of the evaluation is immaterial. This is the justification of the simpler procedure applied for the regular windings.

3.2 Narrow-spread fractional-slot windings

As was the case in integral-slot windings, the narrow-spread windings can be considered as an even number, say 2N-phase winding, interconnected to form an N-phase winding. It is possible to obtain a balanced N-phase 2N-phase winding, and it is necessary to consider these separately. However, this theory includes the windings obtained from a balanced 2N-phase windings, and the latter need not be given any different treatment.

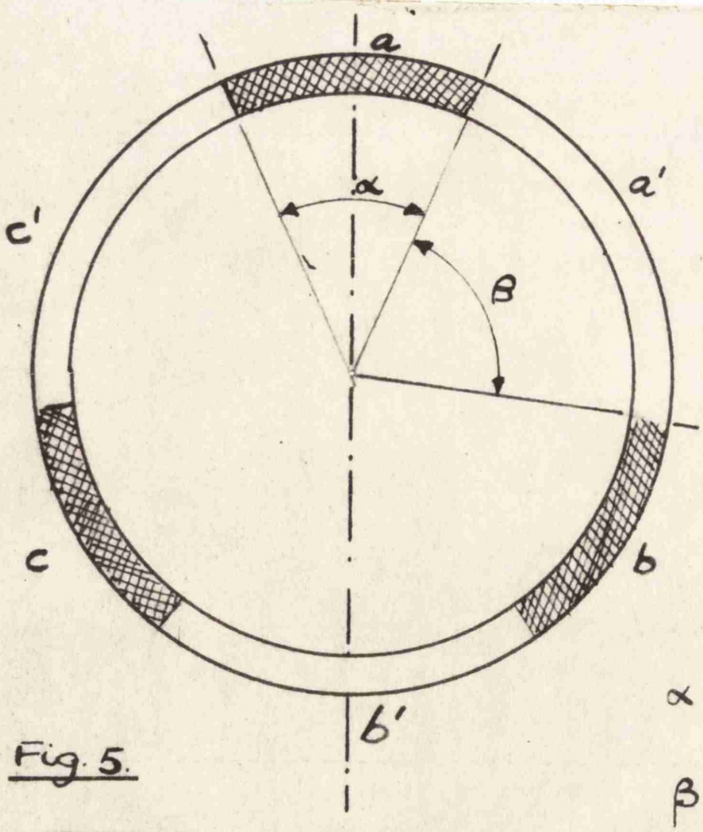
3.2.1 N-phase windings derived from an unbalanced 2N-phase winding

In an unbalanced 2N-phase winding of the type referred to

/here

here $Q/2N$ is not necessarily an integer, although Q/N must still be integral. If that is the case, it is possible to construct and accommodate in the given stampings two separate N-phase windings which may be series connected to form a single balanced N-phase winding. The two component windings have $(Q + x)/2N$ and $(Q - x)/2N$ coils per phase per repeatable section respectively. Here x is any integer in the range $1 \rightarrow (Q - 2Np)$ which makes the fractions integral respectively.

Now if $(Q \pm x)/2N$ is integral and $Q/N = q$ (an integer), it follows that $\frac{1}{2}(q \pm \frac{x}{N})$ is integral. Hence x is an odd or even multiple of N according as q is odd or even respectively.



$$\alpha = \frac{Q-x}{6Q} 2\pi$$

$$\beta = \frac{Q+x}{6Q} 2\pi$$

Fig. 5.

1. 3-ph. Winding $\frac{Q-x}{6}$ slots/phase

2. 3-ph Winding $\frac{Q+x}{6}$ slots/phase

C/L of a and b' are π radians apart.

The lay-out of such a winding is illustrated in Fig. 5. It is evident that such an arrangement would also be perfectly feasible with an integral-slot 2N-phase winding.

The repeatable group is again laid out according to the rule of arithmetic progression, but for either of the windings, the series does not go uniformly from one phase to the next, but jumps a number of steps. The summation of the harmonic fields must therefore be carried out for each separate phase accordingly. Thus for the first sub-winding we have,

$$\text{Phase 1:} \quad B_1^n(\theta) = I_1 B_c F_n \sum_{t=1}^{(Q+x)/2N} e^{-jntd\, 2\pi/Q} \qquad (47(a))$$

$$\text{Phase 2:} \quad B_2^n(\theta) = I_2 B_c F_n \sum_{t=Q/N+1}^{(3Q+x)/2N} e^{-jntd\, 2\pi/Q} \qquad (47(b))$$

$$\text{Phase } \rho: \quad B_\rho^n(\theta) = I_\rho B_c F_n \sum_{t=(\rho-1)\frac{Q}{N}+1}^{[(2\rho-1)Q+x]/2N} e^{-jntd\, 2\pi/Q} \qquad (47(c))$$

The second sub-winding is the complement to the first, and considering the phases succeeding phases 1, 2, ... , in the first respectively, denoting these by 1', 2', ... ', we have

$$\text{Phase 1':} \quad B_{1}'^{n}(\theta) = I_1 B_c F_n \sum_{t=(Q+x)/2N+1}^{Q/N} e^{-jntd\, 2\pi/Q} \qquad (48(a))$$

$$\text{Phase 2':} \quad B_{2}'^{n}(\theta) = I_2 B_c F_n \sum_{t=(3Q+x)/2N+1}^{2Q/N} e^{-jntd\, 2\pi/Q} \qquad (48(b))$$

$$\text{Phase } \rho': \quad B_{\rho}'^{n}(\theta) = I_\rho B_c F_n \sum_{t=[(2\rho-1)Q+x]/2N}^{Q/N} e^{-jntd\, 2\pi/Q} \qquad (48(c))$$

/The

The two sets are then interconnected, so that Phase 1 of the first set and Phase (N + 1)/2 in the second set are in series opposition. The total flux due to this combined winding is given by

$$B_I^n = B_1^n + B'^n_{(N+1)/2}$$

Equations (47(a)) and (47(ρ)) become on putting $\rho = (N + 1)/2$ and evaluating the series,

$$B_1^n\theta = I_I B_c F_n \frac{\sin nd \frac{Q + x}{2N} \pi/Q}{\sin nd\ \pi/Q} e^{-j\left(\frac{Q+x}{2N} + 1\right) nd\pi} \tag{49(a)}$$

and

$$B'^n_{(N+1)/2}(\theta) = I_I B_c F_n \frac{\sin nd \frac{Q - x}{2N} \frac{\pi}{Q}}{\sin nd\ \pi/Q}\ e^{-j\frac{(2N+1)Q+x}{2N} nd\pi} \tag{49(b)}$$

Upon addition we have

$$B_I^n(\theta) = I_I B_c F_n \frac{\sin nd \frac{Q+x}{2N} \frac{\pi}{Q} - e^{jnd\pi} \sin nd \frac{Q-x}{2N} \frac{\pi}{Q}}{\sin nd\ \pi/Q}$$

$$\times\ e^{-j\left(\frac{Q+x}{2N} + 1\right) nd\ \pi/Q} \tag{50}$$

The general distribution factor is therefore

$$F_{\beta,n} = \frac{\sin nd \frac{Q+x}{2N} \frac{\pi}{Q} - (-1)^{nd} \sin nd \frac{Q-x}{2N} \frac{\pi}{Q}}{Q/N \sin nd\ \pi/Q} \tag{51}$$

From this equation it is clear that if nd is an even number, we have

$$F_{\beta,n} = \frac{2\sin nd\ x\ \pi/2NQ.\cos nd\ \pi/2N}{Q/N \sin nd\ \pi/Q} \tag{52}$$

/and

and if nd is an odd number,

$$F_{\beta,n} = \frac{2 \cos ndx\ \pi/2NQ \sin nd\ \pi/2N}{Q/N \sin nd\ \pi/Q} \tag{53}$$

I doubt if they have been made up to date !!

These equations are of great importance in design of fractional-slot windings. Since x is a variable parameter restricted only to be a multiple of N, it may be chosen to minimize any given harmonic.

3.3 The field due to sequence excitation of fractional-slot windings.

The sequence excitation again gives rise to connection factors as in the case of integral-slot windings. These are however, different in this case and must be evaluated anew.

The general expression for the harmonic fields per phase must now be taken as given by equation (38),

$$B_n^{\rho}(\theta) = I_{\rho}\, B_c\, F_{\alpha,n}\, e^{jn\theta} \sum_{t=(\rho-1)\frac{Q}{N}}^{\frac{Q}{N}-1} e^{-jntd\frac{2\pi}{Q}} \tag{38}$$

and after summation we obtain

$$B_n^{\rho}(\theta) = I_{\rho}\, B_c F_{\alpha,n}\, e^{j\left[n\theta - \rho\, nd\frac{2\pi}{N} - \left(\frac{1}{N} + \frac{1}{Q}\right) nd\pi\right]} \tag{54(a)}$$

Taking the real part of this equation we have

$$B_n^{\rho}(\theta) = I_{\rho}\, B_c F_{\alpha,n} \cos\left(n\theta - \rho\, nd\ 2\pi/N - \phi_n\right) \tag{54(b)}$$

where ϕ_n is independent of ρ.

$$\text{Now}\quad I_{\rho} = \sum_{S=0}^{N-1} I_S \cos\left(\omega t - \overline{\rho-1}\, S\, \frac{2\pi}{N} - \phi_S\right) \tag{55}$$

and by substituting in (54(b)) and comparison with Section 2.4.1,

/it

it is evident that the connection factors are given by

$$C_{fn} = \frac{\sin(nd + S)\pi}{\sin(nd + S)\pi/N} \tag{56(a)}$$

and

$$C_{bn} = \frac{\sin(nd - S)\pi}{\sin(nd - S)\pi/N} \tag{56(b)}$$

Consequently the integral equation determining the harmonic spectrum for fractional-slot windings is

$$= \frac{k'N \pm S}{d} \tag{57}$$

$$k' = 0, 1, 2, \ldots..$$

This is a modification of equation (16) obtained for the integral-slot windings. If we write $d = \dfrac{kQ \pm 1}{p}$ we have

$$n = \frac{k'N \pm S}{kQ \pm 1}\, p \tag{58}$$

It is clear that n need not now be a multiple of p, and the spectrum may therefore be much denser than for a corresponding 2p-pole integral-slot winding. Of particular importance is the fact that there may often exist harmonics of the order $p \pm 1$. These harmonics are very likely to cause magnetic noise, and this problem has, in fact, been one of the major difficulties with these types of windings. In general, as will be shown later, the presence of adjacent harmonic orders (i.e., difference of unity in the order of harmonics) also will produce transverse forces on the rotor and may therefore have very serious effects.

Example 1

Suppose a stator having 75 slots is to be wound as a 3-phase, 20-pole machine.

We have

$$\frac{Q1}{1pN} = \frac{75}{10 \times 3} = \frac{15}{2 \times 3}$$

Hence $1 = 5$, $Q = 15$, $p = 2$; $Q/N = 5$

Thus $d = \dfrac{k15 \pm 1}{2} = 8$ (with $k = 1$)

The spectrum is therefore given by

$$n = \frac{3k' \mp S}{8} = \qquad (\alpha)$$

For positive sequence excitation,

$$n = \frac{3k' \mp 1}{8} = \qquad (\beta)$$

From (β) we have the series

$n = -1, +2, -4, +5, -7, +8, -10$, etc., where the signature refers to the sense of rotation of the harmonic fields. In terms of the electrical system, the harmonics are

$n = -\frac{1}{2}, 1, -2, +5/2, -7/2, 4, -5$, etc., showing that the spectrum consists of the usual integral-slot spectrum as given by

$$n = 1, 4, -5 \ldots$$

plus a similar spectrum of harmonics having twice the wave length and opposite sense of rotations given by

$$n = -\tfrac{1}{2}, -4/2, 5/2 \ldots.$$

The winding factor for the winding is given by

$$F_{\alpha,\beta,n} = \frac{\sin n\ 8.5.\frac{\pi}{5}}{5 \sin n8\ \frac{\pi}{5}} \sin \frac{n}{2}$$

/The

The coil span may be taken as $\left[\frac{Q}{2p}\right]_i \pi/Q$, which in this case gives

$$\alpha = 2\pi/5$$

Hence

$$F_{\alpha,\beta,n} = \frac{\sin n\, 8\pi/5}{5 \sin 8\pi/15} \sin n\, \pi/5$$

$$= \frac{\cos n \frac{37\pi}{15} - \cos n \frac{43\pi}{15}}{5/2 \sin n\, 8\pi/15}$$

This winding factor has been computed for the 8 lowest values of n to show the relative magnitudes of the harmonics

n	1	2	3	4	5	6	7	8
$F_{\alpha,\beta,n}$	0.103	0.790	0.000	0.250	0.000	0.000	-0.221	-0.221
$\frac{1}{n} F_{\alpha,\beta,n}$	0.103	0.395	0	0.075	0	0	-0.032	-0.0276

The lower line in the table gives the relative amplitudes of the harmonics. Clearly, the magnitude of the subharmonic ($n = 1$), is quite prohibitively large to make this a practical winding. However, the example shows quite clearly that the magnitudes of both subharmonics and fractional order harmonics may be large in these types of windings, and therefore demand careful design.

<u>Example 2</u>

Stampings having 306 slots, to be wound 3-ph, 28-pole, narrow spread.

$$1 = 2,\ Q = 153,\ p = 7,\ d = \frac{153 + 1}{7} = 22.$$

Each repeatable group (153 slots) will have two 3-ph windings

/occupying

occupying 153 + x and 153 - x slots respectively.

1st winding has $\frac{153 + x}{2 \times 3}$ slots/phase

2nd winding has $\frac{153 - x}{2 \times 3}$ slots/phase.

Thus x is any odd multiple of 3; the simplest case being x = 1.

The harmonic spectrum is defined by

$$n = \frac{k'N \pm S}{d} = \frac{k'3 \pm 1}{22} \quad (S = 1)$$

giving the spectrum

$$n = 1, -2, 4, -5, 7, -8, \ldots.$$

The winding coil pitch will be given by

$$\left[\frac{153}{2 \times 7}\right]_i = 10 \text{ slots} = \frac{7 \times 10}{153} 2\pi = 164^\circ \text{ el.deg.}$$

The distribution factor is given by

$$F_{\rho,n} = \frac{2 \sin n\, 22x\, \pi/6 \times 153 \cos n\, 22\pi/6}{51 \sin n\, 22\, \pi/153}$$

The value of x may be varied to obtain the optimum result as regards minimisation of unwanted harmonics. From the point of view of noise, the 8th harmonic will clearly be the most dangerous.

The optimum value of x is therefore given by

$$\frac{8 \times 22}{6 \times 153} x = 3k \quad \text{where k is any integer}$$

$\therefore \quad x = 15.65k$, from which x = 15 is the nearest integral solution.

The distribution factor for the 8th harmonic is consequently,

$$F_{\beta,8} = \frac{2 \sin 88 \times 5\pi/153 \times \cos 88\pi/3}{51 \sin 176\pi/153} = 0.0165$$

and for the principal harmonic we have

$$F_{\beta,7} = \frac{2 \sin 77 \times 5\pi/153 \cos 77\pi/3}{51 \sin 154\pi/153} \doteq 0.995$$

The dissymmetry has therefore not appreciably reduced the fundamental, and substantially reduced the most dangerous harmonic.

4. The Effects of a Parallel Eccentric Displacement of the Rotor

In the previous sections of this work, the theory of the air gap flux density distribution was reduced to what is more commonly known as the m.m.f. theory of windings by certain assumptions about the form of the gap flux. In this section, it will be shown that a similar method of attack is possible even when the rotor is eccentrically displaced. The only case considered is that of a parallel displacement, the skew displacement involves further severe difficulties. However, it is believed that if the skew is not very large, the distortion of the average flux density along the rotor is not very severe and does not, therefore, affect the induced e.m.f. The transverse forces resulting are, of course, also different in this case and give rise to a bending moment as well as a transverse force. The treatment of an actually eccentric rotor, i.e., where the non-uniformity of the air gap would move with the rotor would be quite different, but this case has not been thought of sufficient interest to be included here.

4.1 The Flux Density Distribution in a Smooth, Eccentric Air Gap

The problem of determining the field in an eccentric annular space is certainly tractable from an analytical point of view, since the solution is easily obtained from that of the

/concentric

concentric case by a well-known transformation[3]. Alternatively, the problem can be attacked by solving Laplace's Equation in terms of biaxal co-ordinates, which form the suitable co-ordinate system. These methods are appropriate when the permeability of the iron parts is considered infinite, so that the field in the iron is everywhere zero. In that case the outer boundary of the rotor are both immaterial. However, when the permeability of the iron is finite, these boundaries become important in the determination of the field. Since these boundaries do not correspond to any of the co-ordinates being constant, the problem of determining the constants of integration is hopeless by known methods. The geometric model of the machine for which an analytical solution has been obtained[3] is therefore very different from the actual machine. Especially severe does this approximation seem in the case of large diameter machines, since these have usually also a small axial length-bore ratio. In either case, these analytical methods are of more academic than engineering interest, and it must be of some value to investigate the possibility of establishing simpler methods which will give adequate accuracy.

The following treatment is substantially based on the same principle as outlined in Section 2.1. Again, the permeability of the iron is assumed infinite, and the flux distribution is assumed to be governed by the same principle as in the concentric case, namely that the field lines are lines of shortest distance between

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the stator and rotor surfaces. By the magnetic circuit law it is then a simple matter to obtain the relation between flux density and the currents in the windings. Original work in 1953

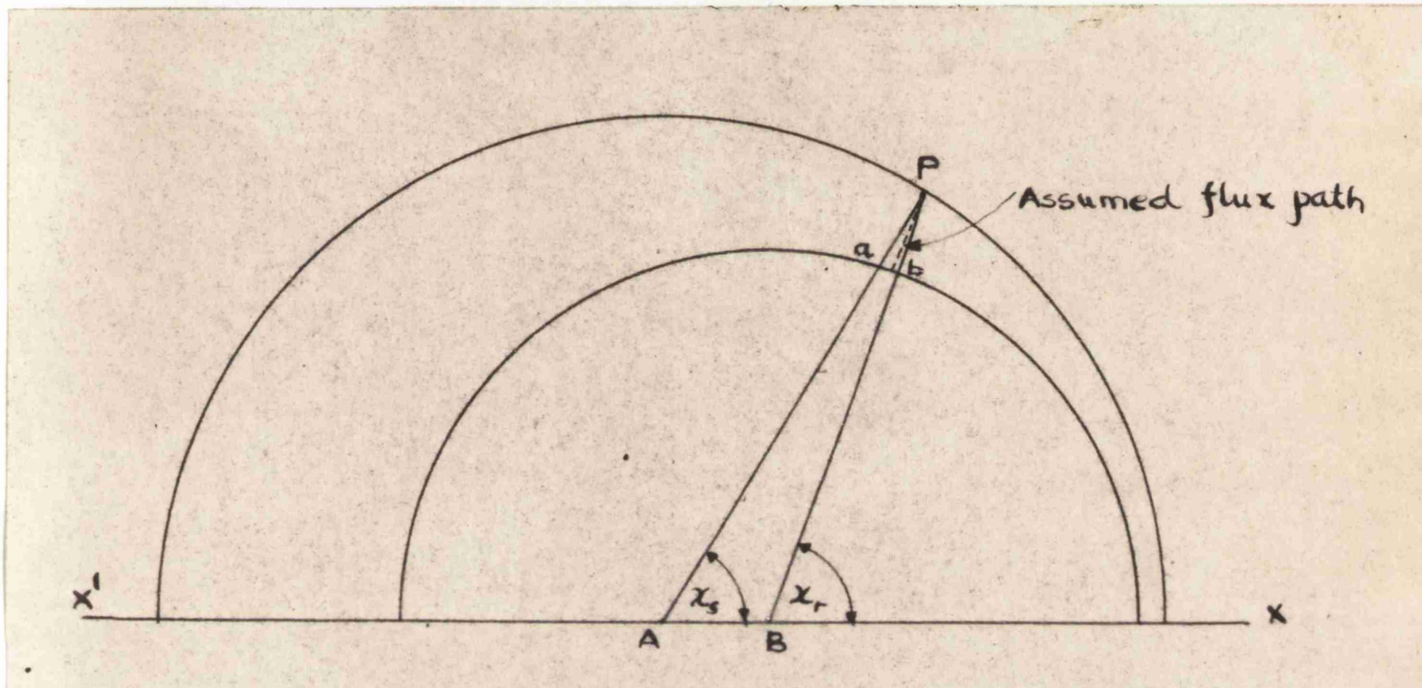

Fig 6.

The relevant geometric quantities are illustrated in Fig. 6. The plane containing the centre lines of the rotor and stator (X'X) is called the eccentricity plane, and all angular quantities are measured from this plane. It is now necessary to distinguish between stator and rotor angular measure; x_s and x_r in figure are clearly different when describing the same points in the stator (P). This difference which is called the eccentric angular anomaly, is in most cases negligible. The following considerations will serve to justify this statement.

From geometric considerations of Fig. 6, we have

$$AP \cos x_s = BP \cos x_r + AB$$

and $$AP \sin x_s = BP \sin x_r$$

$$\therefore \quad AP \cos x_s = AP \frac{\sin x_s}{\sin x_r} \cos x_r + AB$$

$$AP (\cos x_s \sin x_r - \sin x_s \cos x_r) = AB \sin x_r$$

$$\sin (x_r - x_s) = \frac{AB}{AP} \sin x_r = k' \sin x_r$$

$$x_r - x_s = \sin^{-1} (k' \sin x_r) \qquad (60)$$

k' is the ratio of the eccentric displacement to stator bore. If this quantity is not larger than 10^{-2}, (60) can be written, with good accuracy

$$x_r - x_s = k' \sin x_r \qquad (60(a))$$

The maximum anomaly is therefore of the order of 10^{-2} radians or 0.5 degrees. This may be of importance, but for practical cases, k' is much smaller than 10^{-2} and no correction is necessary.

The flux lines may be assumed to be straight line segments which in turn are supposed to emanate from P and bisect ab in Fig. 6. The length of this line segment is a function of the angular displacement x_s, the mean gap length g, and the eccentric displacement AB.

Again we have,

$$Pb = BP - Bb$$
$$= \left[AP^2 + AB^2 - 2AB.AP \cos x_s\right]^{\frac{1}{2}} - Bb$$

Normally, the displacement AB is small, say 1/100th of the bore radius AP, so that the square of this quantity may be

/ignored.

ignored.

$$\text{Thus } Pb = AP\left[1 - 2\,\frac{AB}{AP}\cos x_s\right]^{\frac{1}{2}} - Bb$$

Again, if the ratio $\frac{AB}{AP}$ is small, the root can be expanded binomially, so that

$$\begin{aligned} Pb &= AP - Bb - AB\cos x_s \\ &= g\,(1 - \frac{AB}{g}\cos x_s) \end{aligned}$$

where g (= $AP - Bb$) may be termed the mean gap length. It is in fact the mean of the maximum and minimum value of the variable gap length Pb, and corresponds to the constant gap length in the concentric case.

Further, the (constant) ratio $\frac{AB}{g}$ can be defined as the eccentricity (k), so that zero eccentricity means concentric rotor, while unit (or 100%) eccentricity corresponds to the extreme case when the rotor just touches the stator. Pb is the gap length at any point, and we write

$$Pb = g(1 - k\cos x_s) \tag{61}$$

where $k = AB/g$.

(It is not of any importance whether x_s or x_r is used in this formula, the error being of the order $k'k$ or substantially less than one per cent. Furthermore, the difference in length of Pa and Pb is insignificant, and the length of the flux path is thus very nearly equal to Pb).

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In the following the angular displacement x_s only will be used, and the subscript will be dropped, and x, unless otherwise stated, will mean x_s.

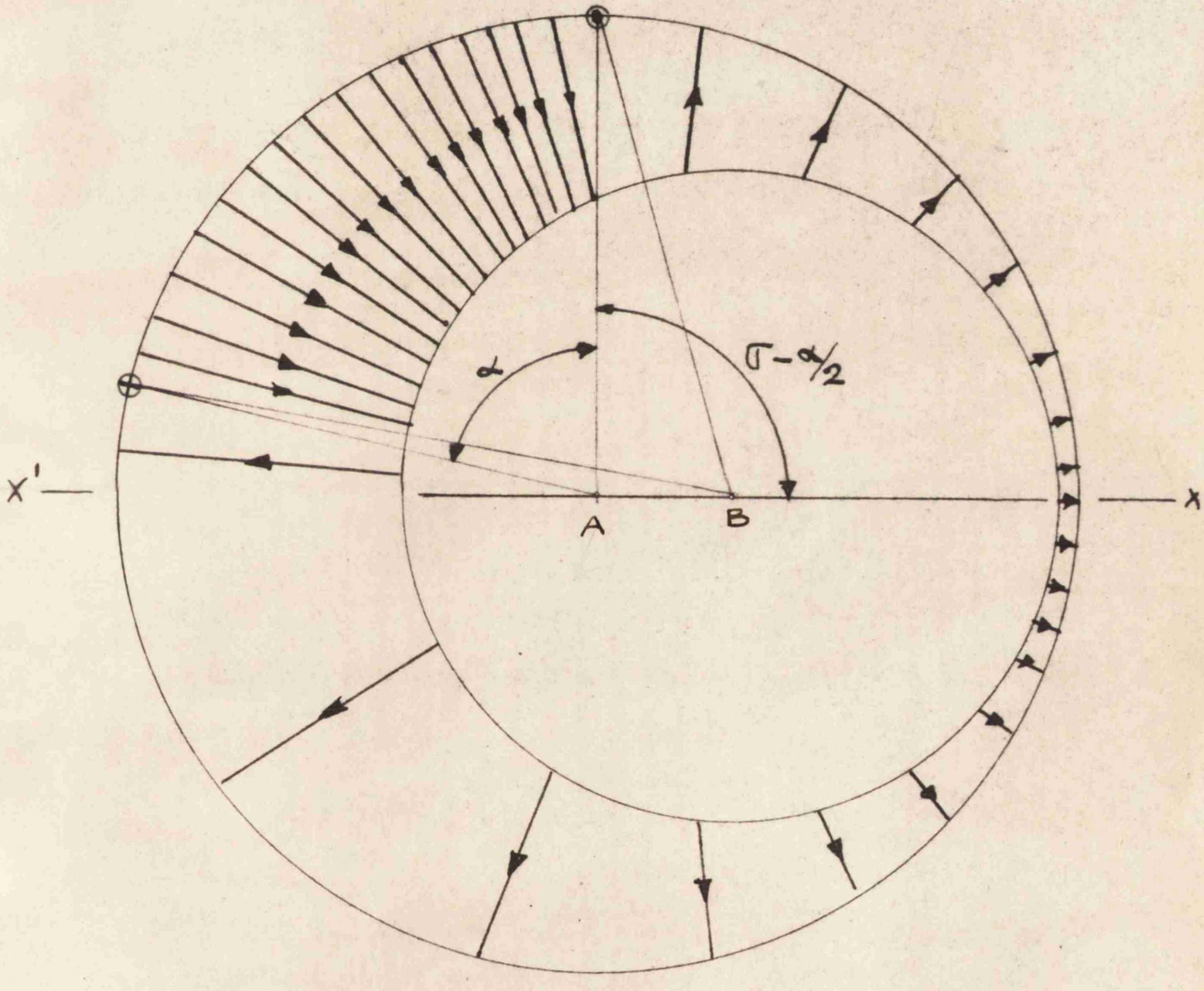


Fig 7.

In Fig. 7 the flux distribution due to a single coil is shown. The assumption is that the flux lines are radial lines drawn from the midpoint of AB, being of length $g(1 - k \cos x)$.

/This

This is the extent of the assumption, and the rest of the analysis is exact. It is clear that this assumption is not valid if the iron is highly saturated, or if the gap is much longer than in normal machines.

Since there is no drop of magnetic potential in the iron, the magnetic potential drop across the gap is constant inside and outside the coil, and the magnetic potential or m.m.f. wave is again a rectangular wave as shown in Fig. 8(a).

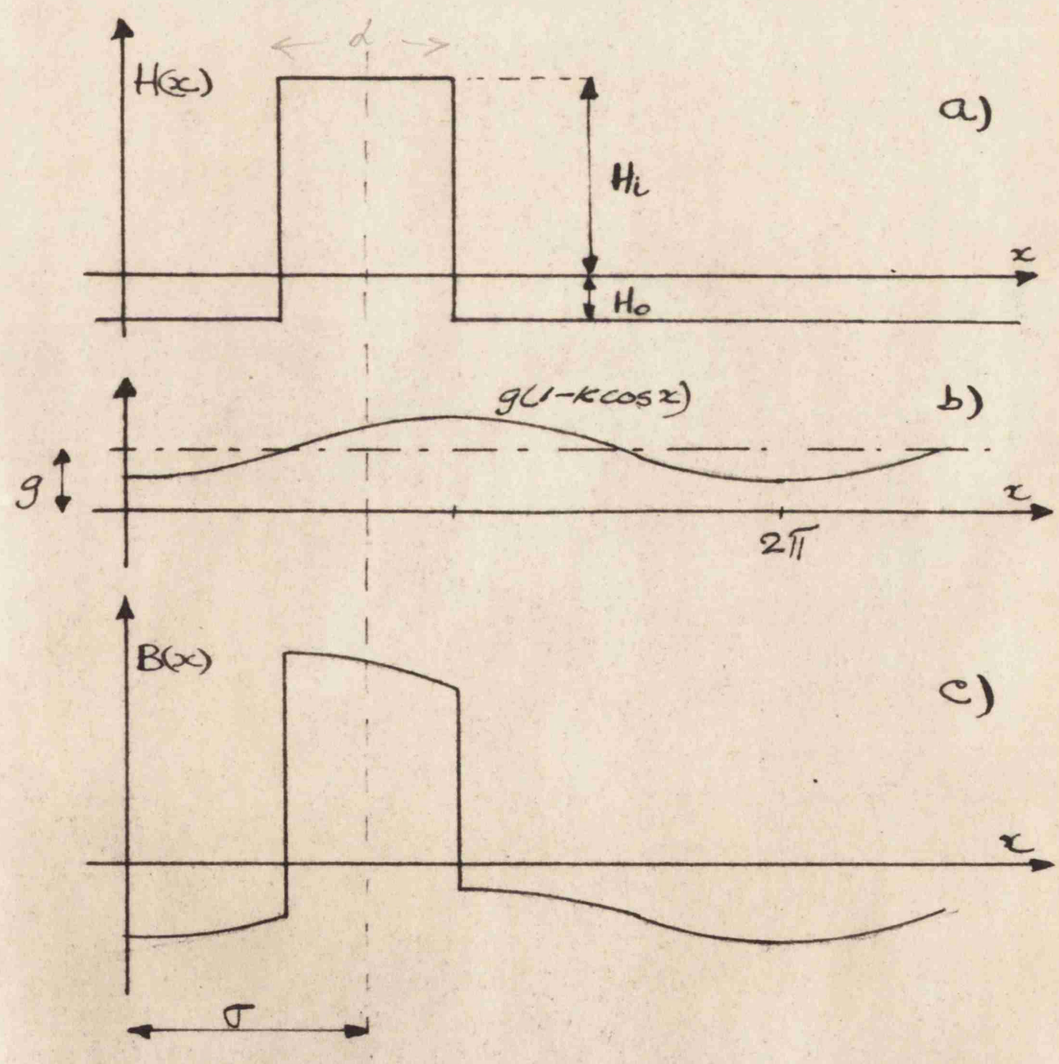


Fig 8.

a) MMF wave due to a single coil

b) Flux path length in the gap

c) Flux density wave due to a single coil in an eccentric gap.

Now at any point in the air gap

$$(Pb)\,.\,B(x) \;=\; \mu_o\, H(x) \tag{62}$$

where H(x) is the magnetic potential and B(x) the flux density; thus substituting for Pb we have

$$B(x) \;=\; \frac{\mu_o\, H(x)}{g(1 - k\cos x)} \tag{63}$$

This is illustrated in Fig. 8(c).

The second condition we may impose is that B is solenoidal. Thus integrating the flux density over the stator surface, we have

$$\int_0^{2\pi} B(x)\,dx = 0, \text{ which follows from div } \underline{B} = 0$$

Pointed out by me in 1955

whence,

$$\int_{\sigma-\frac{\alpha}{2}}^{\sigma+\frac{\alpha}{2}} \frac{\mu_o H_i}{g(1 - k\cos x)}\,dx = \int_{\sigma+\alpha/2}^{\sigma-\frac{\alpha}{2}+2\pi} \frac{\mu_o H_o}{g(1 - k\cos x)}\,dx$$

20

and integrating we have,

$$\int \frac{dx}{1 - k\cos x} = \frac{2}{\sqrt{1-k^2}} \tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan\tfrac{1}{2}x$$

and inserting the limits we obtain the relation

$$H_i\left[\tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan\left(\tfrac{1}{2}\sigma + 1/4\alpha\right) - \tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan\left(\tfrac{1}{2}\sigma - 1/4\alpha\right)\right]$$

$$= H_o\left[\tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan\left(\tfrac{1}{2}\sigma - 1/4\alpha + \pi\right) - \tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan\left(\tfrac{1}{2}\sigma + 1/4\alpha\right)\right]$$

Because of the multiple values of the inverse tangent we may write this in the form,

$$\frac{H_i}{H_o} = \frac{k_1\pi - a}{k_2\pi + a} \tag{64(a)}$$

/where

where k_1, k_2 are integers yet to be determined, and a is the principal value of

$$2\left(\tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan(\tfrac{1}{2}\sigma + 1/4\alpha) - \tan^{-1}\sqrt{\frac{1+k}{1-k}}\tan(\tfrac{1}{2}\sigma + 1/4\alpha)\right)$$

By a further reduction we have

$$a = 2\tan^{-1}\frac{\sqrt{1-k^2}\left(\tan\tfrac{1}{2}\alpha\right)}{1 - k\cos\sigma} \tag{65}$$

from which it is immediately apparent that

$$\lim_{k \to 0} a = \alpha \qquad \text{and}$$

$$\lim_{k \to 0} \frac{H_i}{H_o} = \frac{k_1\pi - \alpha}{k_2\pi + \alpha}$$

Now it is necessary that the latter limit must conform with the corresponding ratio in equations (3(a)) and 3(b)), whence we have $k_1 = 2$, $k_2 = 0$ and finally,

$$\frac{H_i}{H_o} = \frac{2\pi - a}{a} \tag{64(b)}$$

This expression is of the same form as that obtained in the concentric case, but the angle α is replaced by a, as given by equation (65).

Again, by the magnetic circuit law,

$$H_i + H_o = I_c T_c \tag{66}$$

and combining with (64(b)) we have

$$H_i = I_c T_c \left(1 - \frac{a}{2\pi}\right) \tag{67(a)}$$

$$H_o = I_c T_c \cdot \frac{a}{2\pi} \tag{67(b)}$$

/The

The Fourier series corresponding to the m.m.f. wave is therefore given by

$$H(x) = I_c T_c \left[(\alpha - a) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin m\frac{\alpha}{2} \cos m(x - \sigma) \right] \qquad (68)$$

which can be written

$$H(x) = h_o + \sum_{m=1}^{\infty} h_m \cos m(x - \sigma) \qquad (69)$$

The m.m.f. wave for the eccentric gap is identical with the m.m.f. wave for the concentric case with the exception of the constant h_o.

The flux density function follows from equation (63)

$$B(x) = \frac{\mu_o}{g(1 - k\cos x)} \left[h_o + \sum_{m=1}^{\infty} h_m \cos m(x - \sigma) \right] \qquad (70)$$

since h_o and h_m are known functions of α and σ it is now possible to expand B(x) in a series of the form

$$B(x) = \sum_{q=1}^{\infty} (b_q^c \cos qx + b_q^s \sin qx) \qquad (71)$$

By the usual procedure,

$$b_q^c = \frac{1}{\pi} \cdot \int_0^{2\pi} B(x) \cos qx \, dx \qquad (72)$$

$$b_q^s = \frac{1}{\pi} \int_0^{2\pi} B(x) \sin qx \, dx \qquad (73)$$

These integrations are possible, but require an extensive manipulation, and it has been found much easier to expand the factor $1/1\text{-}k\cos x$ by the binomial theorem, and convert the powers of $\cos x$ into equivalent multiple angle series. Then this series

/is

is multiplied by the bracket in (70) and the coefficients of like harmonic terms finally added. The procedure is rather tedious, and the details are given in Appendix V. By the appropriate results found there we have

$$B(x) = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \quad h_o \left(1 + 2 \sum_{q=1}^{\infty} y^q \cos qx\right)$$

$$+ \sum_{m=1}^{\infty} h_m \left[\cos m\sigma \left(y^m + \sum_{q=1}^{\infty} (y^{|q-m|} + y^{q+m}) \cos qx\right)\right.$$

$$\left. + \sin m\sigma \left(\sum_{q=1}^{\infty} (y^{|q-m|} - y^{q+m}) \sin qx\right)\right] \tag{74}$$

where $y = \frac{k}{1 + (1 - k^2)^{\frac{1}{2}}}$ and $|q - m|$ means the numerical value of $(q - m)$.

Since $B(x)$ cannot contain any constant term, it follows that

$$h_o + \sum_{m=1}^{\infty} h_m y^m \cos m\sigma = 0 \tag{75}$$

whence

$$h_o = -\sum_{m=1}^{\infty} h_m y^m \cos m\sigma \tag{75(a)}$$

Substituting this value in (74), there results

$$B(x) = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_{m=1}^{\infty} h_m \left[\cos m\sigma \sum_{q=1}^{\infty} (y^{|q-m|} - y^{q+m}) \cos qx\right.$$

$$\left. + \sin m\sigma \sum_{q=1} (y^{|q-m|} - y^{q+m}) \sin qx\right] \tag{76}$$

and finally, we have

$$B(x) = \frac{\mu_o}{g(1-k^2)^{\frac{1}{2}}} \sum_{q=1}^{\infty} \sum_{m=1}^{\infty} h_m \left(y^{|q-m|} - y^{q+m}\right) \cos(qx - m\sigma) \quad (77)$$

This is a very convenient, and surprisingly simple expression, where the harmonic terms are simple functions of the harmonic amplitudes in the conventional m.m.f. wave.

From equation (77) it is clear that the formula is equally valid whether we interpret $\sum_m h_m \cos m(x - \sigma)$ as the m.m.f. wave due to a single coil or a complete winding, since additions of terms due to several coils will involve $m\sigma$ only. The result will be the same in whatever order m.m.fs. or fluxes are added. This reveals the great power of application of equation (77).

In order to find the amplitude of a given harmonic in the flux wave it is advantageous to consider (77) in the form

$$B(x) = \frac{\mu_o}{g(1-k^2)^{\frac{1}{2}}} \sum_{q=1}^{\infty} \left(b_q^c \cos qx + b_q^s \sin qx\right) \quad (78)$$

where

$$b_q^c = \sum_m^{\infty} h_m \left(y^{|q-m|} - y^{q+m}\right) \cos m\sigma \quad (79)$$

and

$$b_q^s = \sum_m^{\infty} h_m \left(y^{|q-m|} - y^{q+m}\right) \sin m\sigma \quad (80)$$

The amplitude of the qth harmonic is then given by

$$\frac{\mu_o}{g(1-k^2)^{\frac{1}{2}}} \left[(b_q^c)^2 + (b_q^s)^2\right]^{\frac{1}{2}} = b_q \quad (81)$$

By (79) and (80), putting $h_m \left(y^{|q-m|} - y^{q+m}\right) = H_{m,q}$

/we

we have,

$$b_q = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left[\left(\sum_m^{\infty} H_{m,q} \cos m\sigma \right)^2 + \left(\sum_m^{\infty} H_{m,q} \sin m\sigma \right)^2 \right]^{\frac{1}{2}}$$

$$\therefore \quad b_q = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left[\sum_m^{\infty} \left(H_{m,q}^2 + 2 \sum_{n=1}^{\infty} H_{m,q} H_{m+n,q} \cos n\sigma \right) \right]^{\frac{1}{2}} \qquad (82)$$

This function is bounded, the upper boundary being

$$b_{q\max} = \frac{\mu_o}{g(1 - k^2)} \sum_m^{\infty} H_{m,q} \qquad (83)$$

The lower boundary is more difficult to find, and depends on the successive values of the coefficients H_m. The most useful expression is probably that giving the r.m.s. value of b_q as a function of σ. This is

$$b_{q(rms)} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left[\sum_m^{\infty} H_{m,q}^2 \right]^{\frac{1}{2}} \qquad (84)$$

As a rule, there will be one or two dominant terms in the sequence H_m, namely those corresponding to h_p, the fundamental or principal harmonic in the m.m.f. wave, and that corresponding to $m = q$, if h_q is non-zero. The latter case gives a maximum of the function $(y^{|q-m|} - y^{q-m})$. It may very often be sufficient to take these two terms into account, and the evaluation of b_q may be performed very quickly.

The value of $b_{q\max}$ is given by

$$b_{q\max} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_m^{\infty} h_m \left(y^{|q-m|} - y^{q+m} \right) \qquad (85)$$

If $q \leqslant m$ for all values of m we have the simpler form

$$b_{q\max} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left(1 - y^{2q}\right) \sum_m^{\infty} h_m \, y^{m-q} \qquad (85(a))$$

/While

While if $q > m$ for some values of m, the summation has two different ranges, viz.

$$b_{q\max} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left\{ \sum_{m=1}^{q-1} h_m \, y^{q-m} \, (1 - y^{2m}) \right.$$

$$\left. + (1 - y^{2q}) \sum_{m=q}^{\infty} h_m \, y^{m-q} \right] \qquad (85(b))$$

Similarly, the formula for $b_{q(rms)}$ reduces to

$$b_{q(rms)} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} (1 - y^{2q}) \left[\sum_{m}^{\infty} h_m^{\,2} \, y^{2m-q} \right]^{\frac{1}{2}} \qquad (86(a))$$

where $q \leqslant m$ for all values of m; and in the more general case where $m < q$ for some values of m,

$$b_{q(rms)} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left\{ \sum_{m=1}^{q-1} h_m^{\,2} \, y^{2q-m} \, (1 - y^{2m})^2 \right.$$

$$\left. + (1 - y^{2q})^2 \sum_{m=q}^{\infty} h_m^2 \, y^{2m-q} \right\}^{\frac{1}{2}} \qquad (86(b))$$

To facilitate the computation of harmonics with the aid of these formulae, the values of $(1 - k^2)^{-\frac{1}{2}}$ and y^n are calculated and tabulated for values of $k \subset 0.05 - 0.95$ to an accuracy of four decimal places (Table 1, page 102).

The simplest case arises when the m.m.f. wave is nearly sinusoidal. In these cases it will be sufficient to consider the fundamental m.m.f. plus the harmonics of order $q - 1$, q and $q + 1$. Thus, we have by (85)

$$b_{q_{max}} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \Big[h_p \left(y^{|q-p|} - y^{q+p} \right)$$

$$+ h_{q-1} \left(y - y^{2q-1} \right)$$

$$+ h_q \left(1 - y^{2q} \right) + h_{q+1} \left(y - y^{2q+1} \right) \Big] \qquad (87)$$

where the term in h_p must be ignored if $p = q$, or $p = q \pm 1$. Also, since h_m rarely contains the full spectrum, some of these factors are normally zero. In fact for all sub-harmonics, i.e. $q < p$, h_{q-1}, h_q and h_{q+1} all are zero, and the expression contains only the fundamental m.m.f. coefficient. This is interpreted as the not surprising result that the sub-harmonics are only functions of the fundamental component of the m.m.f. wave.

Finally, if the m.m.f. wave is purely sinusoidal, the value of $b_{q_{max}}$ and $b_{q(rms)}$ are both given by the expression

$$b_q = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \; h_p \left(y^{|q-p|} - y^{q+p} \right) \qquad (88)$$

It is clear that in this case, the severe distortion of the field at smaller values of eccentricity is entirely due to the introduction of harmonics of the orders $p - 1$ and $p + 1$. Their amplitudes are

$$b_{p+1} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \; h_p \, y \left(1 - y^{2p} \right) \qquad (89)$$

and

$$b_{p-1} = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \; h_p \, y \left(1 - y^{2p-1} \right) \qquad (90)$$

These are both of the order yb_p. Therefore, the function y

/gives

gives a quick qualitative measure of the field harmonics introduced by the eccentricity.

In practice it is often required to have the relative magnitudes of the harmonics, rather than their actual magnitudes. Thus b_q is required as a fraction of b_p. In the case of a purely sinusoidal m.m.f. wave, we have

$$\frac{b_q}{b_p} = \frac{y^{|p-q|} - y^{p+q}}{1 - y^{2p}} \tag{91}$$

4.2 The Flux Density at the Rotor Surface

In most cases, it is accurate enough to take the flux density function evaluated in Section 3.1 as the same at stator and rotor surface. The kind of correction to be made if the gap and eccentricity are abnormally large will now be investigated.

The difference in the flux function is due to two causes, (a) there is a compression of the tubes of flux due to the smaller diameter of the rotor, and (b) due to the eccentric angular anomaly.

These two effects can be seen in Fig. 9. The same amount of flux crosses the two segments QP and qp, being of different lengths, and the mean angular position of these segments referred to the angular measures x_s and x_r differ by an angle $\varepsilon + k' \sin x_r$. The additional anomaly ε is proportional to $k' \sin x_r$ and much smaller than this quantity, and will be neglected in the following.

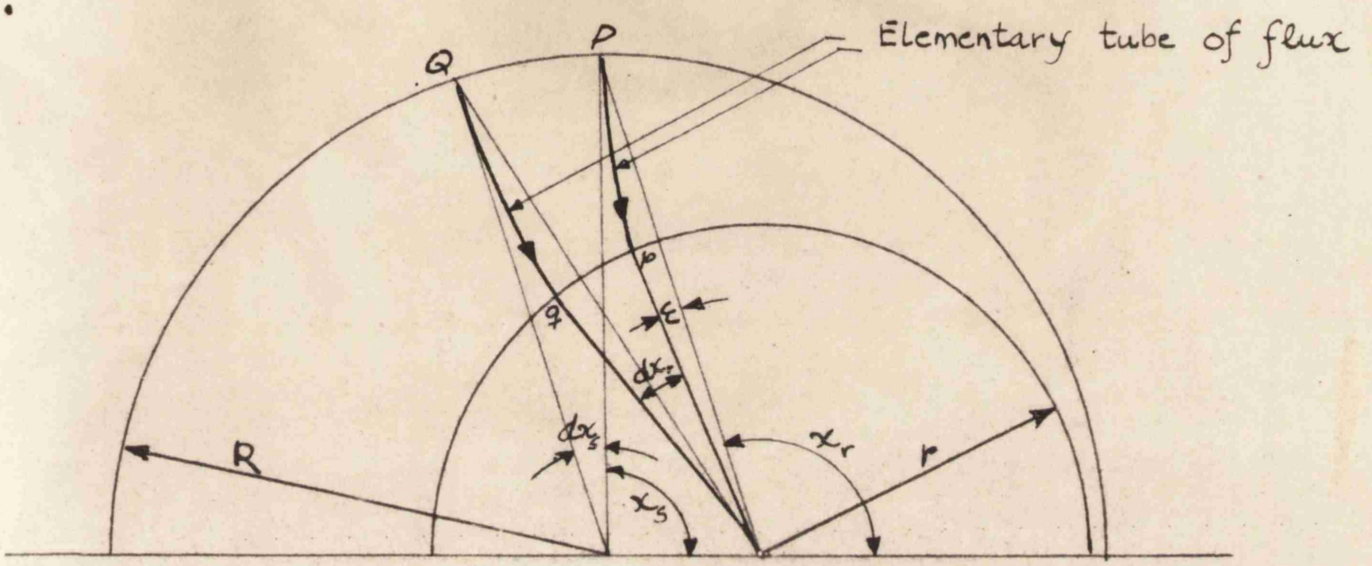


Fig 9.

From this Figure, the following relation is obvious;

$$B(x_s) \cdot R\, dx_s = B(x_r) \cdot r\, dx_r \tag{92}$$

where R is the radius of the stator bore and r is the radius of the rotor bore.

Therefore,

$$B(x_r) = \frac{R}{r} B(x_s) \frac{dx_s}{dx_r} \tag{92(a)}$$

Putting $x_s = x_r - k' \sin x_r$ in the L.H.S. of (88(a)), and using (78) we have,

$$B(x_r) = \frac{R}{r} \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \frac{(1 - k'\cos x_r)}{} \sum_{q=1}^{\infty} \Big[b_q^c \cos q(x_r - k' \sin x_r)$$

$$+ b_q^s \sin q\,(x_r - k' \sin x_r) \Big] \tag{93}$$

The subscript r may now be dropped; since $k'\sin x$ is of the order of 0.02, i.e., corresponding to 1^o, the value of $\cos(k' \sin x)$ differs from unity by about 0.002 only, and we therefore ignore this.

/Accordingly,

Accordingly, (93) becomes

$$B_r(x) = \frac{R}{r}\frac{\mu_o}{g(1-k^2)^{\frac{1}{2}}}(1 - k'\cos x)\sum_{q=1}^{\infty}\Big\{b_q^c(\cos qx + k'\sin qx \sin x)$$

$$+ b_q^s(\sin qx - k'\cos qx \sin x)\Big\} \qquad (94)$$

In order to obtain (89(a)) in the same form as (78), we expand this in Fourier Series form,

$$B_r(x) = \frac{\mu_o}{g(1-k^2)^{\frac{1}{2}}}\sum_{q=1}^{\infty} B_q^c \cos qx + B_q^s \sin qx) \qquad (95)$$

where by the usual procedure we obtain

$$B_q^c = \frac{1}{\pi}\int_o^{2\pi}(1 - k'\cos x)\sum_{n=1}^{\infty}\Big[b_n^c(\cos nx \cos qx$$

$$+ 2k'\sin nx \sin x \cos qx)$$

$$+ b_n^s(\sin nx \cos qx - 2k'\sin nx \sin x \cos qx)\Big] dx$$

Hence,

$$B_q^c = b_{2-q}^c \cdot \frac{k'^2}{2} - b_{q-1}^c\frac{3k'}{2} + b_q^c(1 - \frac{3k'^2}{2}) + b_{q+1}^c\frac{k'}{2}$$

$$-(b_{q+2}^c + b_{q-2}^c)\frac{k'^2}{2} - b_{2-q}^s\frac{k'^2}{2} + (b_{q-1}^s - b_{q+1}^s)(k' + \frac{k'^2}{2}) \qquad (96)$$

where the term b_{2-q} only occurs when $q = 1$, i.e., it is always b_1.

Neglecting terms involving k'^2, we have

$$B_q^c = b_q^c - k'(3/2\, b_{q-1}^c - \tfrac{1}{2}b_{q+1}^c) + (b_{q-1}^s - b_{q+1}^s)k' \qquad (97)$$

There will be a similar expression for B_q^s. It is apparent that considerable labour will be involved in evaluating a larger number of these coefficients, and this refinement of the theory

/leads

leads exactly to the same computational difficulties which arise in the analytical solutions. Since only practically usable formulae are aimed at in this work, this extension is not carried any further.

4.3 The Rotating Fields in the Eccentric Gap

In polyphase windings, by far the most important part of the m.m.f. wave consists of travelling waves, and the resultant flux density wave will also consist of travelling waves.

If the normal m.m.f. wave of the winding is of the form

$$H(x,t) = \sum_m h_m \cos\left(m\,(x - \Gamma_m) \pm \omega t\right)$$

we may replace $m\Gamma$ in equation (77) by $(m\,\Gamma_m \pm \omega t)$. Consequently, according to (78) etc., we can write the flux density wave as

$$B(x,t) = \sum_{q=1}^{\infty} \left|\left(b_q^f \cos\,(qx - \omega t) + b_q^b \cos\,(qx + \omega t)\right)\right. \tag{98}$$

where,

$$b_q^f = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left[\sum_{m_f} (H^2_{m,q} + 2 \sum_{n=1}^{\infty} H_{m+n,q}\, H_{m,q} \cos n\,\Gamma_m)\right]^{\frac{1}{2}} \tag{99(a)}$$

$$b_q^b = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \left[\sum_{m_b} (H^2_{m,q} + 2 \sum_{n=1}^{\infty} H_{m+n,q}\, H_{m,q} \cos n\,\Gamma_m)\right]^{\frac{1}{2}} \tag{99(b)}$$

and $\sum_{m_f}$ and $\sum_{m_b}$ means summation with respect to values of m corresponding to forward and backward rotating fields respectively.

It is clear from the above equations that the travelling

/waves

waves in the flux wave tend to be elliptic, and the complete picture of the flux wave is extremely complex. A general discussion of their nature seems of little use, but the above equations are sufficient to evaluate all the harmonic fields in any given case.

There is, however, one case which may be dealt with in detail, namely that where the m.m.f. wave is a pure sine wave.

Then we have the form of $H(x,t)$ as

$$H(x,t) = h_p \cos (px - \omega t) \tag{100}$$

and consequently,

$$B(x,t) = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_{q=1}^{\infty} h_p \left(y^{|q-p|} - y^{q+p}\right) \cos (qx - \omega t) \tag{101}$$

In this case it turns out that all the harmonics rotate in the same sense, at angular speeds given by ω/q rad/sec. A curious, and unexpected, consequence of this is that even a sinusoidal m.m.f. wave may produce fields moving at speeds in excess of synchronism.

4.4 The Transverse Pull on the Rotor

The transverse pull resulting from the distortion of the magnetic field due to eccentricity will now be considered. The method adopted is that of evaluating the stresses at the surface of the rotor and then integrating over the whole surface to find the resulting force.

/The

The boundary stresses follow from Maxwell's hypothesis of electric and magnetic stresses. We are here concerned only with the magnetic field stress.

In a magnetic field, the energy density is given by the expression,

$$W_m = \frac{1}{2} HB \text{ Joules/m}^3$$

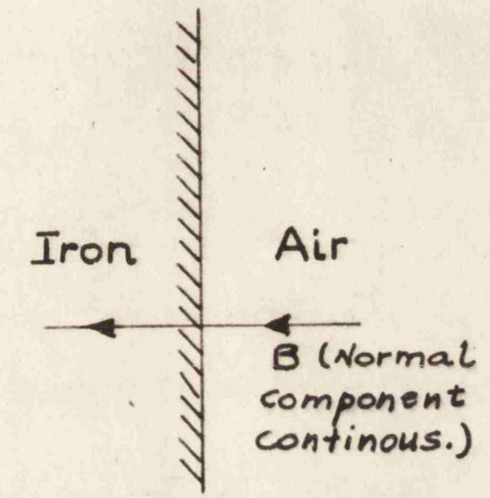
Energy density can also be expressed as

$$W_m \quad \frac{N - m}{m^3} = W_m \frac{N}{m^2}$$

which has the dimension of stress (force/unit area). If the field is uniform, Maxwell's hypothesis assumes that the region where such an energy density exists is subjected to a tension W_m in the direction of B (or H) and a compression W_m perpendicular to B (or H). At boundaries (where the normal component of B must be continuous in the absence of sources) the value of H may change if the permeability is changed, and there is consequently a change in the value of W_m. This change is interpreted as the mechanical stress on the boundary. This is illustrated in Fig. 10. Since in the case of infinitely permeable iron, the Maxwell stress (normal) in the iron must be zero, and the mechanical stress on the iron surface is therefore equal to the Maxwell stress in the air. By putting $H = B/\mu_o$ for the air gap, we obtain the equation

$$f = B^2/2\mu_o \quad N/m^2 \qquad (102)$$

for the stress on the iron surface.



Field Conditions:

Iron	Air
$H = 0$	$H = B/\mu_0$

Stress Conditions:

Iron	Air
$f_\tau = 0$	$f_\tau = B^2/2\mu_0$
$f_\sigma = B^2/2\mu_0$	$f_\sigma = 0$

f_τ - Maxwell stress; f_σ - mechanical stress

Fig 10.

The equilibrium of stress at boundaries.

The phenomena can also be explained by considering the magnetostatic equivalent of the system. The iron is then assumed to consist of magnetic dipoles which, under the given conditions, are completely aligned with the field. Inside the iron, the net field is then zero, since the dipoles neutralise each other, and there appears free poles only at the surface of the iron/air boundaries. (Fig. 11).

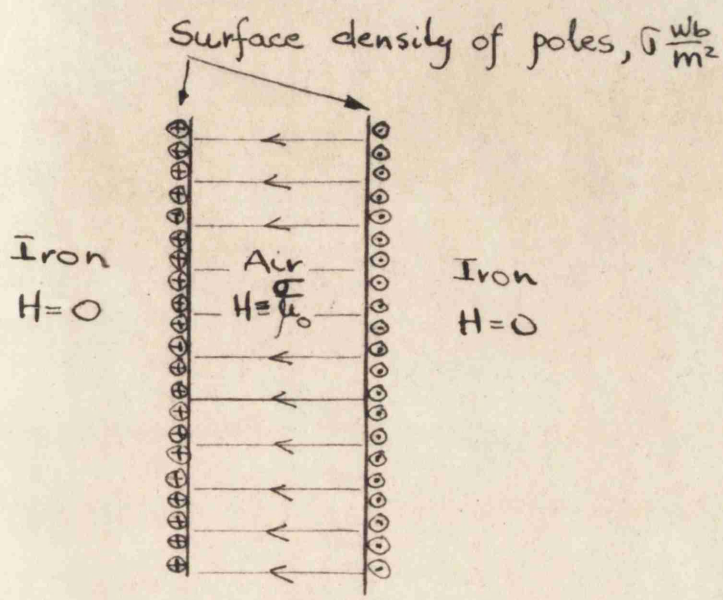


Fig 11.

Magnetostatic equivalent of fig.10.

A concentration of poles corresponding to an air gap flux density B is in this case given by $\sigma = B$ Wb/m^2. Again, the force per unit area of these surface distributions of poles is H, where H is the field due to one of the surface distributions only. For an infinite plane surface this is given by $H = \sigma/2\mu_0$, whence the stress is given by $\sigma^2/2\mu_0$ N/m^2. Putting $\sigma = B$ we obtain again equation (102).

Formula (102) is directly applicable to the present case provided the assumption is made that the flux is everywhere perpendicular to the iron surface. Since the force on the rotor must be equal and opposite to the force on the stator, we will evaluate the latter, since our formulae for the flux distribution are strictly evaluated for the stator surface only.

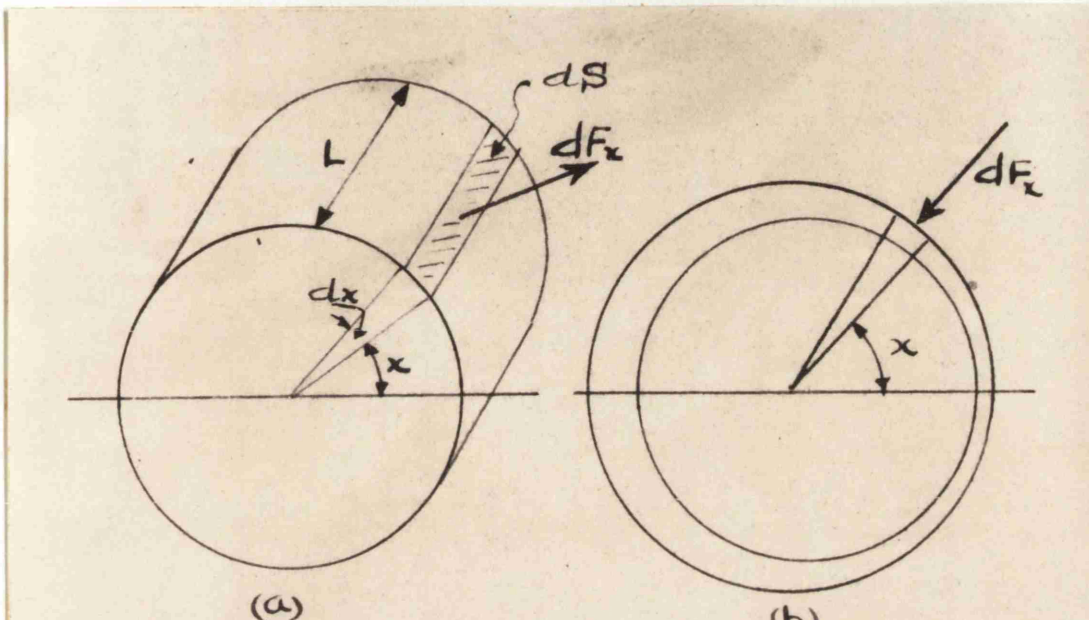


Fig 12.

Consider an elementary area dS (Fig. 12(a)) given by

$$dS = LRdx$$

where R is stator bore radius and L the axial length. The force dF_x on dS is given by

$$dF_x = \frac{1}{2\mu_o} B^2(x)\ LRdx$$

It will be convenient to resolve this force along the planes $x = 0$ (i.e., the eccentric plane) and $x = \pi/2$ respectively. These components are given by

$$dF_o = \frac{LR}{2\mu_o} B^2(x)\ \cos x\ dx \qquad \text{and}$$

$$dF_{\pi/2} = \frac{LR}{2\mu_o} B^2(x)\ \sin x\ dx$$

respectively. The total forces on the stator along these directions are therefore

$$F_o = \frac{IR}{2\mu_o} \int_o^{2\pi} B^2(x) \cos x \, dx \qquad (103)$$

and

$$F_{\pi/2} = \frac{IR}{2\mu_o} \int_o^{2\pi} B^2(x) \sin x \, dx \qquad (104)$$

The total force is finally given by

$$F = (F_o^2 + F_{\pi/2}^2)^{\frac{1}{2}} \text{ Newtons} \qquad (105)$$

The form of B(x) most suitable for carrying out the integration is given by (78), written in the form

$$B(x) = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_{q=1}^{\infty} \Big[\big(\sum_m H_{m,q} \cos m\sigma \big) \cos qx$$

$$+ \big(\sum_m H_{m,q} \sin m\sigma \big) \sin qx \Big] \qquad (106)$$

If, for the moment we put

$$X_q = \sum_m H_{m,q} \cos m\sigma \qquad (107)$$

and

$$Y_q = \sum_m H_{m,q} \sin m\sigma \qquad (108)$$

we obtain after substitution in (106) and squaring,

$$B^2(x) = \frac{\mu_o^2}{g^2(1-k^2)} \sum_{q=1}^{\infty} \Big[X_q^2 + Y_q^2 + X_q Y_q \sin 2qx$$

$$+ \sum_{n=1}^{\infty} \Big(X_q\, X_{q+n}\, (\cos \overline{2q+n}x + \cos nx)$$

$$+ Y_q\, Y_{q+n}\, (\cos nx - \cos \overline{2q+n}x)$$

$$+ X_q\, Y_{q+n}\, (\sin \overline{2q+n}x + \sin nx)$$

$$+ Y_q\, X_{q+n}\, (\sin \overline{2q+n}x - \sin nx)\Big)\Big] \qquad (109)$$

Substituting this in the force integrals and noting that the only part of (109) that will contribute to these integrals are the terms involving cosx and sinx, we have

$$F_o = \frac{lR}{2} \frac{\mu_o}{g^2(1-k^2)} \int_o^{2\pi} (X_q\, X_{q+1} + Y_q\, Y_{q+1}) \cos^2 x \, dx$$

$$F_{\pi/2} = \frac{lR}{2} \frac{\mu_o}{g(1-k^2)} \int_o^{2\pi} (X_q\, Y_{q+1} - Y_q\, X_{q+1}) \sin^2 x \, dx$$

whence, finally

$$F_o = \frac{lR\pi}{2} \frac{\mu_o}{g^2(1-k^2)} \sum_{q=1}^{\infty} (X_q\, X_{q+1} + Y_q\, Y_{q+1}) \qquad (110)$$

$$F_{\pi/2} = \frac{lR\pi}{2} \frac{\mu_o}{g^2(1-k^2)} \sum_{q=1}^{\infty} (X_q\, Y_{q+1} - Y_q\, X_{q+1}) \qquad (111)$$

Equations (110) and (111) are quite general and may serve to calculate the force in any case where the flux density distribution is known, not only in the case of eccentric rotors. They reveal the very important fact that only harmonics whose order (in the mechanical system) differ by unity contribute to the resultant force. Consequently, there can be no transverse force due to odd or even harmonics

/alone,

alone, but occurs only if these are simultaneously present. This mechanism is illustrated in Fig. 13(a) and (b). In these diagrams the harmonics of the flux density are indicated in polar plots, the direction of the fluxes labelled by North and South polarities respectively. The "poles" interact, since like poles will produce high flux density, while opposite poles will produce lower flux density. The difference in surface stress can be interpreted as outward and inward forces as shown. In Fig. 13(a) the fundamental is shown together with the second harmonic. The difference forces have a resultant as shown. In Fig. 13(b) the fundamental is shown together with the third harmonic. In this case it is clear that the difference forces cancel out in pairs, and that no transverse force results.

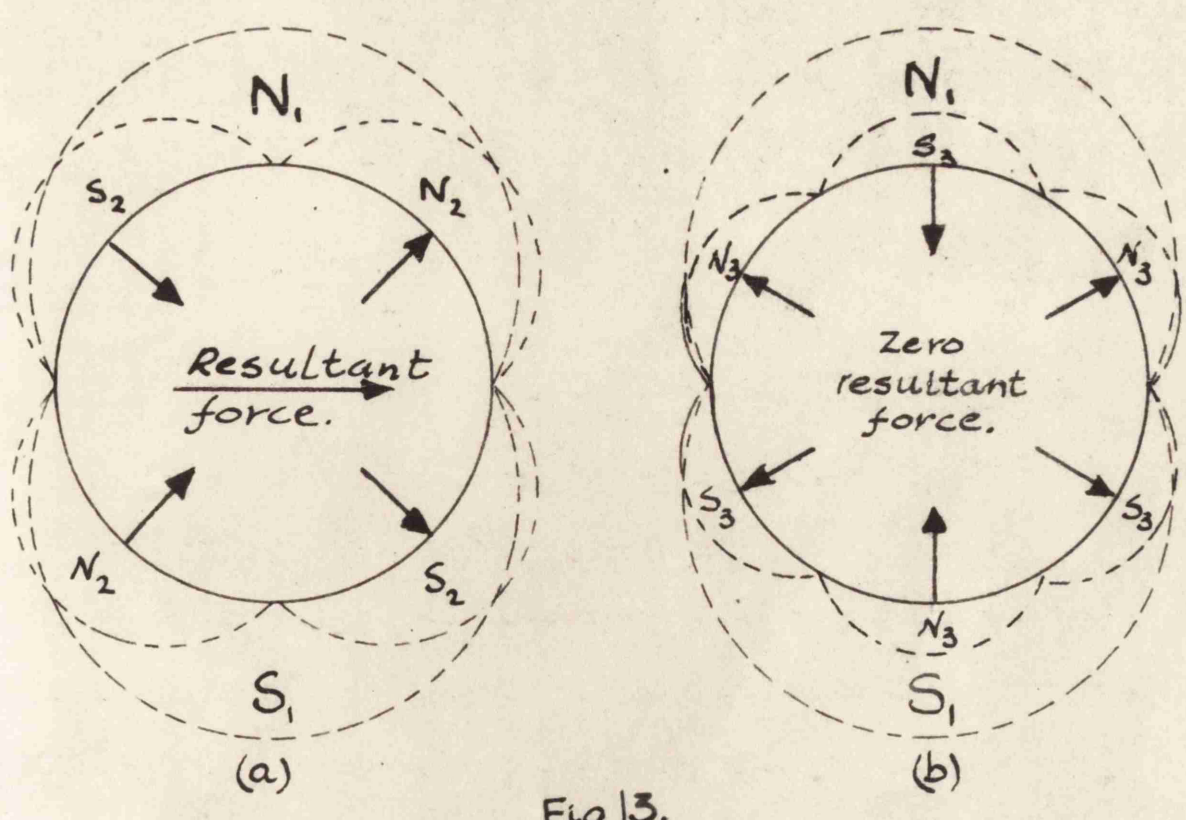


Fig 13.

This conclusion has one immediate practical application. It reveals that a 2-pole, widespread winding having short-pitched coils will give rise to transverse forces and is therefore, to be avoided in practice. In the multipolar winding which has at least two identical groups, the consideration does not apply, since the harmonics are then all multiples of some number greater than unity; which implies that they differ in order by more than one.

For the eccentric case, the harmonic spectrum is in general complete, and transverse forces will always be present. The evaluation of (110) and (111) for the values of X_q and Y_q given by (107) and (108) will now be carried out. The values of the forces depend on expressions of the general type

$$\sum_{q=1} \sum_{m} H_{m,q} \begin{pmatrix} \cos m\sigma \\ \sin m\sigma \end{pmatrix} H_{m,q+1} \begin{pmatrix} \cos m\sigma \\ \sin m\sigma \end{pmatrix}$$

Inverting the order of summation gives

$$\sum_{m} S_m \begin{pmatrix} \cos m\sigma \\ \sin m\sigma \end{pmatrix} \begin{pmatrix} \cos m\sigma \\ \sin m\sigma \end{pmatrix}$$

where

$$S_m = \sum_{q=1} H_{m,q} \; H_{m,q+1}$$

Inserting the values for $H_{m,q}$ and $H_{m,q+1}$ as defined on Page 77, we have

$$S_m = \sum_{q=1}^{\infty} H_m^2 \left(y^{|q-m|} - y^{q+m}\right)\left(y^{|q+1-m|} - y^{q+m+1}\right)$$

$$= h_m^2 \sum_{q=1}^{m-1} y^{2m} \left(y^{-(2q+1)} - y^{2q+1} - y^{-1} - y\right) + \sum_{q=m}^{\infty} \left(y^{-m} - y^{m}\right)^2 y^{2q+1}$$

$$\therefore \; S_m = h_m^2 \left[\frac{2y(1 - y^{2m})}{1 - y^2} - my^{2m-1}\left(1 + y^2\right)\right] \qquad (112)$$

Thus we obtain the following identities,

$$\sum_{q=1}^{\infty} (X_q X_{q+1} + Y_q Y_{q+1}) = \sum_m S_m (\cos^2 m\sigma + \sin^2 m\sigma)$$

$$= \sum_m S_m$$

$$\sum_{q=1}^{\infty} (X_q Y_{q+1} - Y_q X_{q+1}) = \sum_m S_m (\cos m\sigma \sin m\sigma - \sin m\sigma \cos m\sigma)$$

$$= 0$$

Whence finally, we obtain the force equations,

$$F_o = \frac{IR\pi}{2} \frac{\mu_o}{g^2(1 - k^2)^{\frac{1}{2}}} \sum_m h_m^2 \left[\frac{2y(1 - y^{2m})}{1 - y^2} - my^{2m-1}(1 + y^2)\right] \quad (113(a))$$

$$F_{\pi/2} = 0 \quad (113(b))$$

The theory therefore shows that the direction of the transverse force is always in the plane of the eccentric displacement, no matter what the disposition of the winding m.m.f. This means that the transverse force arising from a revolving field is constant. Apart from the increase (or decrease) in the bearing pressure, the mechanical effect of such a force would be small. However, for pulsating fields or elliptic fields there would be a pulsating transverse force of frequency twice that of the exciting currents. Since this force is due to a two-pole, sinusoidally-distributed "force wave", the noise effect can be obtained fairly easily with the aid of formulae developed by F.W. Carter[18]. (No data is available to the Author of machines exhibiting magnetic noise, and the possibility of attributing noisiness to eccentricity

/has

therefore not been investigated. It is hoped that further work along these lines may be undertaken in the future).

The numerical calculation of the eccentric force will be facilitated by the values of the function

$$E_1(k,m) = \frac{1}{(1 - k^2)}\left[\frac{2y(1 - y^{2m})}{(1 - y^2)} - my^{2m-1}(1 + y^2)\right] \quad (114)$$

given in Table 2 (page 103).

In most cases, it will be found that unless the m.m.f. wave form is unusually crude, the transverse force depends almost entirely on the fundamental. Thus even if the m.m.f. wave is a square wave - i.e. of the form

$$H(x) = h_p\left(\cos px + \frac{1}{3}\cos 3px + \frac{1}{5}\cos 5px + \ldots\right)$$

the force is about 87% due to the fundamental. (About 80% in the case $p = 1$). For the much better waveshapes commonly encountered, the force is about 97% due to the fundamental.

Thus a practical formula for the transverse force can be given as

$$F_t = \frac{LR\pi}{2}\,\frac{\mu_o}{g^2(1 - k^2)}\,h_p^2\left[\frac{2y(1 - y^{2p})}{1 - y^2} - py^{2p-1}(1 + y^2)\right] (115)$$

The accuracy of this formula has not as yet been subjected to any test. The measurement of transverse force required some rather elaborate apparatus to be set up, and to furnish a proper check on the formula, windings of different numbers of poles would need to be used. In view of these mechanical

/difficulties

difficulties this work has been deferred at the moment.

In older works on the magnetic pull, the force is usually quoted in terms of the average flux density. This form is easily obtained from (115) where it is assumed that the flux wave harmonics are given by equation (88). If we put $b_{eff} = \frac{1}{\sqrt{2}} b_p$ we have

$$b_{eff} = \frac{1}{\sqrt{2}} \frac{\mu_o}{g} \frac{(1 - y^{2p})}{(1 - k^2)^{\frac{1}{2}}} h_p \qquad (116)$$

and substituting the value of h_p given by this equation in (115), we obtain

$$F_t = \frac{LR\pi}{\mu_o} b^2_{eff} \frac{1}{(1 - y^{2p})^2} \left[\frac{2y(1 - y^{2p})}{1 - y^2} - py^{2p-1} (1 + y^2)\right] \qquad (117)$$

Rearranging to obtain the most convenient form for computation, we have

$$F_t = \frac{DL\pi}{2\mu_o} b^2_{eff} \frac{2y - (p + 2y^2 - py^4)y^{2p-1}}{(1 - y^{2p})^2 (1 - y^2)} \qquad (118)$$

This expression, when written in the form

$$F_t = \frac{A}{2\mu_o} b^2_{eff} E_2(y,p) \qquad (119)$$

where A is the total surface area, gives the transverse force as a fraction of the total surface force acting on the stator and rotor surface, the flux density being assumed sinusoidal.

For larger values of p, and small values of y, it is evident that (119) may be approximately represented by

$$F_t = \frac{A}{\mu_o} b^2_{eff} \frac{y}{1 - y^2} \qquad (120)$$

/and

and substituting $y = k/1 + \sqrt{1 - k^2}$, we have

$$F_t = \frac{A}{2\mu_o} b_{eff}^2 \frac{k}{1 - k^2/1 + \sqrt{1 - k^2}} \qquad (120(a))$$

and by a further approximation,

$$F_t = \frac{A}{2\mu_o} b_{eff}^2 k \qquad (121)$$

These equations may now be compared with the existing literature on the subject. There was considerable attention paid to the subject of magnetic pull in the first decade of this century, and a summary of the several articles which appeared during this interval are given by Gray and Pertch[21]. The above derivation, although quite different in theory, contains all the versions found in this paper. Thus (121) is Behrend's Formula, (120(a)) is very nearly Sumec's Formula, and Fisher-Hinnen's and Knowlton's Formulae are, in fact, of the same form as (119) using only the linear part of the function, i.e. $k < 0.5$. The dependence of $f(y,p)$ on p is, however, quite different in these formulae. The formula due to Rey, is also of the same form as (119), but does not include p.

The values of $E_2(k,p)$ have been computed and are given in Table 3. The principal curves are plotted in Fig. 14, which also gives a comparison with Rey's Formula.

4.5 Inductances of Windings with Eccentric Air Gaps

The flux linkages produced in the windings when the flux distribution is distorted by eccentricity must be

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obtained by integrating the flux density over the area spanned by each coil, and summing for all coils. In the case of unsymmetric windings, this would have to be done in single steps, and the calculation in that case is exceedingly tedious. However, the majority of windings are symmetric, and if further all coils per phase are connected in series, there are several short cuts possible. Only these types are considered here.

Consider a symmetric 2p-pole winding which gives rise to the conventional m.m.f. function

$$H(x) = \sum_n h_n \cos np\,(x - \sigma') \tag{122}$$

where it is to be noted that h_n is proportional to F_n, the winding factor of the nth harmonic in the electrical system of reference. For the resulting flux density function, we may refer to equation (77). Thus

$$B(x) = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_q \sum_n h_n \left(y^{|q-np|} - y^{q+np}\right) \times \cos\,(qx - np\,\sigma') \tag{123}$$

Now in the type of winding assumed, it is evident that only harmonic orders of np, namely those of the m.m.f. wave, can produce effective linkages. Furthermore, by writing B(x) in the form

$$B(x) = U_q \cos q\,(x - \sigma') + V_q \sin q\,(x - \sigma') \tag{124}$$

it is evident that the sine terms do not contribute to the self-flux linkages. Equating the expressions given by (123) and

/(124) we

(124) we have

$$U_q = \frac{\mu_o}{g(1 - k^2)^{\frac{1}{2}}} \sum_n h_n \left(y^{|q-np|} - y^{q+n}\right) \cos(q - np)\sigma \qquad (125)$$

The ratio $U_q/U_q^o = \lambda_{e,q}$, that is the ratio of the value of U_q to its value in the concentric case can therefore be expressed as

$$\lambda_{e,q} = \frac{1}{(1 - k^2)^{\frac{1}{2}} F_{q/p}} \sum_n F_n \left(y^{|q-np|} - y^{q+np}\right) \cos(q - np)\sigma \qquad (126)$$

Now since the effective flux density has been increased in the ratio $\lambda_{e,q}$:1 it follows also that the inductance corresponding to the same (qth) harmonic must be increased in the same ratio. By (21) and (126) we have then

$$L_{ph}^{q/p} = \frac{2\mu_o DL}{\pi g p^2 (1 - k^2)^{\frac{1}{2}}} F_{q/p} \sum_n F_n \left(y^{|q-np|} - y^{q+np}\right) \cos(q - np) \qquad (127)$$

and the total inductance

$$L_{ph} = \sum_{q/p=1,..} L_{ph}^{q/p} \qquad (128)$$

where q/p takes the values corresponding to those of n in (122).

The notation q/p as the order of harmonic is clumsy but necessary, if the advantage of having the m.m.f. given in terms of electrical order of harmonics is to be preserved.

Since the value of L_{ph} is now a function of σ it may be deduced that there will in general be unbalance in the phase

/inductances

inductances in a polyphase winding. The magnitude of unbalance, or even the kind of unbalance depends on the values of F_n. If the m.m.f. is very nearly sinusoidal, $F_n \doteq 0$, $n \neq 1$ and we then have

$$L_{ph} \doteq \frac{2\mu_o}{\pi g p^2 (1 - k^2)^{\frac{1}{2}}} F_1^2 (1 - y^2) \tag{129}$$

which is independent of σ and no phase unbalance arises. Thus the purer the waveform of the m.m.f. wave, the less unbalance due to eccentricity.

The effect of unbalance in the phase reactance will bring about a corresponding unbalance in the exciting currents when excited from a balanced voltage supply. The effect is therefore, in general to produce unwanted sequence currents and corresponding losses.

The effect on the mutual inductance between phases may be similarly obtained. Suppose the mutual inductance between two phases disposed at $\sigma + \alpha\frac{2\pi}{Np}$ and $\sigma + \beta\frac{2\pi}{Np}$ respectively. That is according to Fig. 3, with the exception that σ is now taken as the origin (for obvious reasons). The flux produced by phase is again

$$B(x) = \frac{\mu_o}{g(1 - k^2)} \sum_q \sum_n h_n(y^{|q-np|} - y^{q+m}) \cos\left[qx - np(r + \frac{2\pi}{Np})\right] \tag{130}$$

and by equating this to the expression

$$B(x) = U_q \cos q\,(x - \sigma - \beta\frac{2\pi}{Np}) + V_q \sin q\,(x - \sigma - \beta\frac{2\pi}{Np}) \tag{131}$$

where the cosine terms only are effective in producing linkages

/with

with phase β, we obtain the corresponding flux linkages, and finally the mutual inductance, which is

$$M_{\alpha\beta}^{q/p} = \frac{2\mu_0 DL}{\pi g p^2 (1 - k^2)^{\frac{1}{2}}} F_{q/p} \sum_n F_n \left(y^{|q-m|} - y^{q+m} \right) \cos\left[\overline{q - m}\, \sigma - q(\alpha - \beta)\frac{2\pi}{Np} \right] \quad (132)$$

whence

$$M_{\alpha\beta} = \sum_{q/p} M^{q/p} \quad (133)$$

Equation (132) can be seen to be a generalisation of (24) to which it reduces when $y = 0$.

The mutual inductances between rotor and stator windings are differently affected, since the winding factors may now be quite different for the two windings. In fact, it may happen that the secondary winding can have effective linkage with the harmonics introduced by the eccentricity, and there does then exist a mutual coupling between the windings due to these harmonics. However, if this is the case, it is non-reciprocal because such harmonics cannot be linked by the primary winding. Such a non-reciprocal mutual inductance cannot transfer any power, and these harmonics are in fact, reduced to a very small value by the secondary induced currents. Thus squirrel-cage motors may be assumed to be free from these harmonics almost entirely, and in fact, incur very small losses due to them. The compensating rotor currents may be computed from the value of the respective harmonic flux density amplitudes. If the

/speed

speed of the rotor happens to coincide with the harmonic travelling wave in question, however, no rotor currents will be induced and the harmonic will not be compensated, but this will be a very rare occurrence. Since the major harmonics introduced by the eccentricity are of orders adjacent to that of the fundamental, i.e., have $2(p \pm 1)$ harmonic poles, their synchronous speeds will be $(1 \pm 1/p)$ times the fundamental synchronous speed respectively. It would be theoretically possible for a multi-polar induction motor to run at this speed, ($(1-1/p)$ times synchronous speed), and the harmonic might then be more prominent. In turbo-alternators which are not equipped with damper cage windings, however, the eccentricity harmonics would always be present.

TABLE 1 - The Functions $(1 - k^2)^{-\frac{1}{2}}$; y^n

k	$(1-k^2)^{-\frac{1}{2}}$	y	y^2	y^3	y^4	y^5	y^6	y^7	y^8	y^9	y^{10}	y^{11}	y^{12}	y^{13}	$\bar{y}^{14}$
0.05	1.00125	0.0250	0.0006												
0.10	1.00504	0.0501	0.0025	0.0001											
0.15	1.01144	0.0726	0.0053	0.0005											
0.20	1.02062	0.1010	0.0102	0.0010	0.0001										
0.25	1.03280	0.1270	0.0161	0.0021	0.0003										
0.30	1.04828	0.1535	0.0236	0.0036	0.0006	0.0001									
0.35	1.06752	0.1807	0.0327	0.0059	0.0011	0.0002									
0.40	1.09109	0.2087	0.0436	0.0091	0.0019	0.0004	0.0001								
0.45	1.11979	0.2377	0.0565	0.0134	0.0032	0.0008	0.0002								
0.50	1.15470	0.2679	0.0715	0.0191	0.0051	0.0014	0.0004	0.0001							
0.55	1.19737	0.2997	0.0898	0.0269	0.0081	0.0024	0.0007	0.0002	0.0001						
0.60	1.25000	0.3333	0.1111	0.0370	0.0123	0.0041	0.0014	0.0005	0.0002						
0.65	1.31590	0.3693	0.1364	0.0504	0.0186	0.0069	0.0025	0.0009	0.0004	0.0001					
0.70	1.40028	0.4084	0.1668	0.0681	0.0278	0.0113	0.0046	0.0019	0.0008	0.0003	0.0001				
0.75	1.51186	0.4647	0.2160	0.0981	0.0467	0.0217	0.0101	0.0047	0.0021	0.0010	0.0005	0.0002	0.0001		
0.80	1.66667	0.5	0.2500	0.1250	0.0625	0.0313	0.0156	0.0018	0.0039	0.0020	0.0010	0.0005	0.0002	0.0001	
0.85	1.89832	0.5567	0.3099	0.1726	0.0961	0.0548	0.0298	0.0166	0.0092	0.0052	0.0027	0.0016	0.0005	0.0002	0.0001
0.90	2.29416	0.6268	0.3929	0.2462	0.1543	0.0967	0.0606	0.0380	0.0238	0.0149	0.0094	0.0059	0.0037	0.0023	0.0014
0.95	3.20256	0.7239	0.5241	0.3794	0.2747	0.1989	0.1440	0.1042	0.0754	0.0546	0.0395	0.0287	0.0208	0.0150	0.0108

TABLE 2

$$E_1\ (k,p) = (2y - (p + 2y^2 - py^4)\ y^{2p - 1})/(1 - k^2)(1 - y^2)$$

k \ p	1	2	3	4	5	
0.1	0.00505	0.1013	0.1015	0.1015	0.1015	0.1015
0.2	.1042	.2105	.2126	.2126	.2126	.2126
0.3	.1648	.3374	.3445	.3455	.3455	.3455
0.4	.2376	.4960	.5151	.5199	.5199	.5199
0.5	.3305	.7105	.7475	.7695	.7695	.7695
0.6	.4630	1.0290	1.1494	1.1682	1.1717	1.1717
0.7	.6672	1.5839	1.8388	1.9030	1.9188	1.9223
0.8	1.0418	2.6037	3.3205	3.5807	3.6815	3.7037
0.9	2.0043	5.5837	8.0850	9.5004	10.2619	10.8725

TABLE 3

$$E_2\ (k,p) = (2y - (p + 2y^2 - py^4)\ y^{2p-1})/(1 - y^{2p})^2\ (1 - y^2)$$

k \ p	1	2	3	4	5	
0.1	0.0503	0.1003	0.1005	0.1005	0.1005	0.1005
0.2	0.1021	.2021	.2041	.2041	.2041	.2041
0.3	0.1573	.3072	.3135	.3135	.3135	.3135
0.4	0.2182	.4174	.4327	.4364	.4364	.4364
0.5	0.2874	.5362	.5608	.5770	.5770	.5770
0.6	0.3756	.6668	.7366	.7477	.7499	.7500
0.7	0.4901	.8308	.9420	.9707	.9786	.9803
0.8	0.6674	.9981	1.2143	1.2995	1.3266	1.3333
0.9	1.0335	1.6270	1.6346	1.8481	1.9674	2.0649

The function $\left\{2y-(p+2y^2-py^4)y^{2p-1}\right\}/(1-y^{2p})^2(1-y^2)$

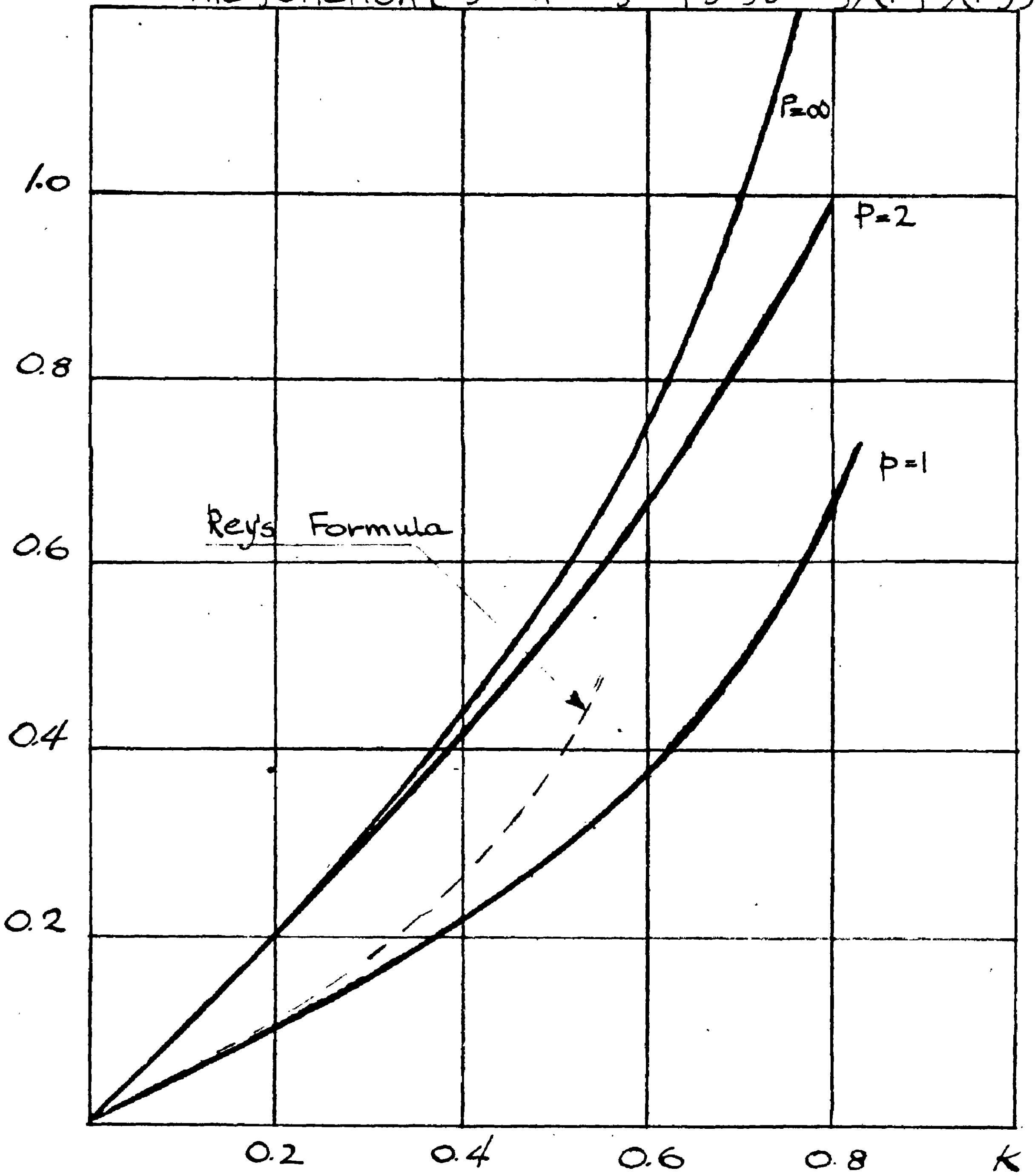


Fig 14.

5. Experimental Work

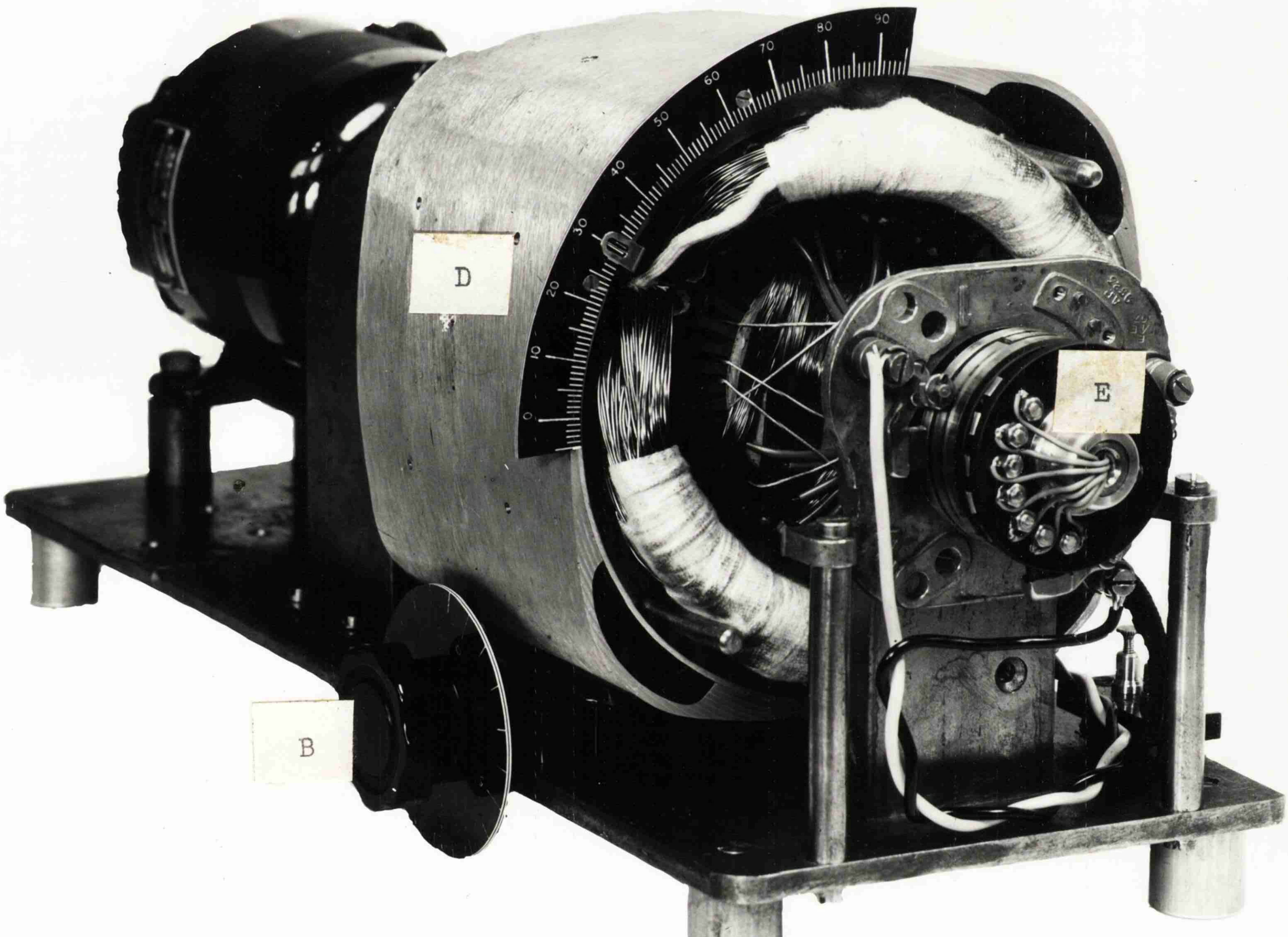
The waveform of the flux density distribution in machines can normally be obtained fairly easily. The measurements of actual eccentric displacement of the rotor may, on the other hand, be nearly impossible. Furthermore, with the normal end-shield construction, the air gap eccentricity cannot be varied, and any standard machine would be of little use for this purpose. A special machine was therefore constructed. This machine and the tests performed on it are described in Section 5.1 below. As a practical matter, the air gap was made relatively large and, consequently, high flux densities were difficult to obtain. Although this is no drawback for testing the theoretical values found in the previous Section, no indication of the effects of saturation could be obtained by it. A standard type of machine was therefore adapted for this purpose, the details of the machine and tests are given in Section 5.2 below. Small 5 Hp

5.1 Investigation on a variable eccentricity machine

The main results required in the eccentric rotor problem are the functions relating the harmonic amplitudes in the flux wave, the independent variable being the eccentricity. The machine described in this Section was specially constructed to allow such measurements to be made.

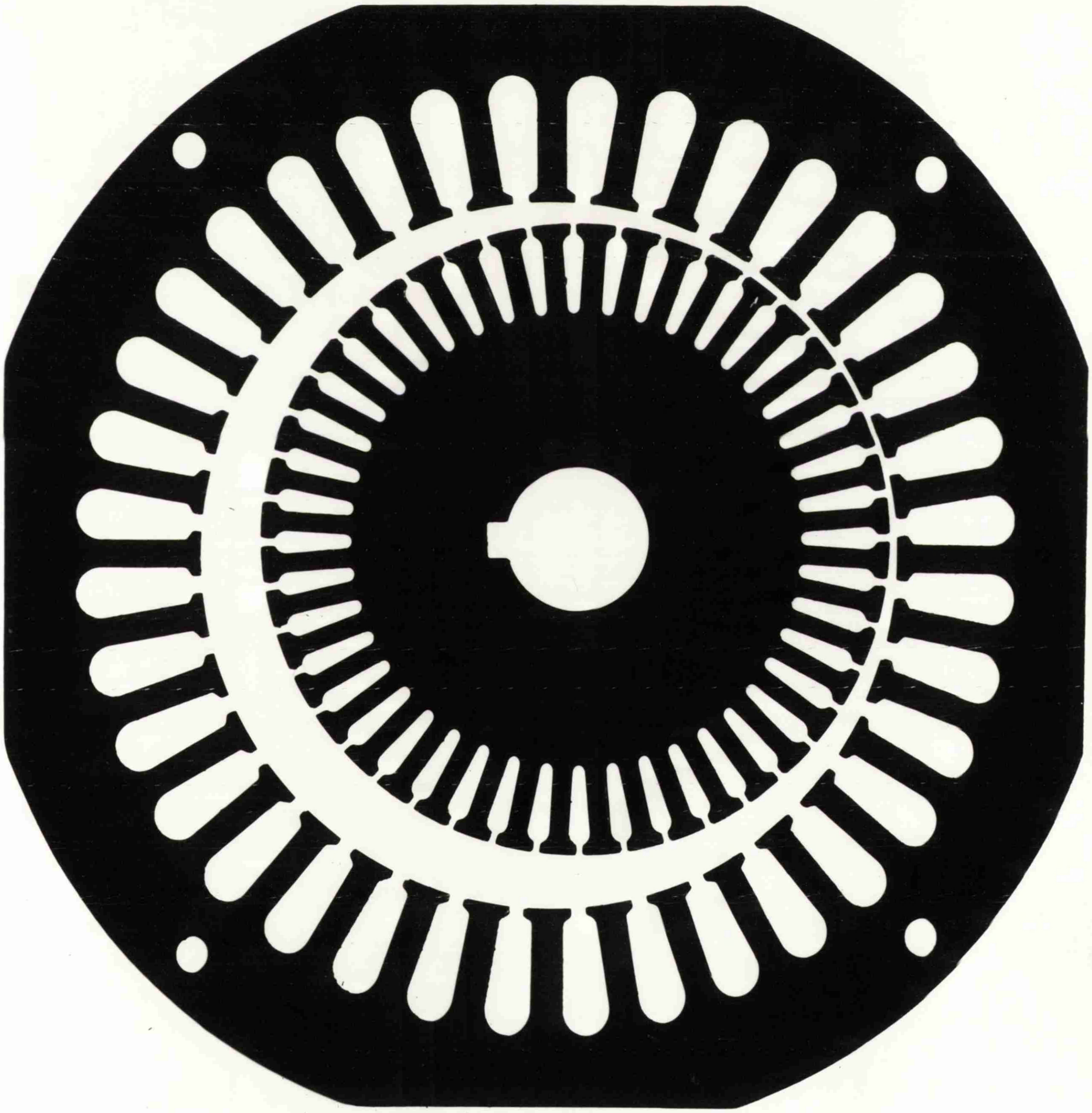
Since the actual lateral displacement of the rotor is only a fraction of the nominal gap length, being usually of the

/order


D
E
B

order of 20 - 30 mils, the exact measurement of this displacement offers considerable difficulty. In the machine constructed, the air gap was made abnormally long (3/16") so that the actual maximum displacement involved corresponds to 3/8" or 200 mils approximately. By a micrometer movement, this displacement could be measured fairly accurately. Also in the approximate theory, the percentage increase in harmonic content is only a function of the dimension less ratio k, and the relative magnitudes are therefore not affected. (However, this approximation depends on the smallness of the ratio of the displacement to mean gap radius, and therefore may be unreliable. The error is assessed below). Further, in order to reduce calculations to a minimum, a two-pole exciting winding giving a nearly sinusoidal m.m.f. was used. This was obtained by suitably grading the number of turns of the successive coils.

The photograph (Plate 1) shows the assembled machine, mounted on a base plate and coupled to its driving motor A. The rotor runs in bearings fixed in brackets which are secured to the base plate and is quite separate from the stator. The stator housing D is fixed to a bevel edge plate which is held in position at the base plate by two guide strips. The bevel edge plate can be made to move in the guide strips by a screwed rod. This forms a micrometer type of movement, and the



PIATE II

displacement may be read on the graduated disc attached to the end of the screwed rod, (B on the photograph).

The stator movement can be determined to an accuracy of the order of 0.001", and allows the eccentricity (k) to be read with an accuracy of one per cent.

The stator stampings were machined so that they can be rotated inside the housing itself. This allows the winding to be moved relatively to the eccentric plane without altering the eccentricity. Thus independent control of the variables k and φ is possible.

Plate II shows a contact print made of the actual stampings used for the machine, and gives an exact picture of the dimensions involved.

The rotor is wound with a number of search coils of span varying from full pitch to half pitch. The ends of the coils are brought through the drilled shaft to a selector switch (E on the photograph) at the end of the shaft. This allows any of the search coils to be connected to the sliprings. The wide range of pitch of the search coils allows all harmonics to be measured with small attenuation.

The driving motor as shown in Plate I is a small

/universal

universal type commutator motor. Since the speed obtained with this motor is not very stable, this was eventually replaced by a squirrel-cage motor with "poles" milled out in the rotor so that it would run synchronously at a speed of 1500 rev/min. This provided a very satisfactory stable frequency output.

Particulars of the Machine

Stator: 36 slots, 3 5/8" bore.

Wound 2-pole, split concentric; 7 coils per pole. The lay-out of the winding is shown in Fig. 15.

The grading of the number of turns of the coils gives the m.m.f. curve shown in Fig. 16. The fundamental component represents the major part of total m.m.f. apart from the higher order harmonics due to the slotting.

Rotor: 44 slots, 3 1/4" diameter.

Wound with 7 search coils, each 75 turns, having the following pitches:

Coil No.	Span: Slot Pitches	Span: Degrees
1	22	180°
2	20	164°
3	18	147°
4	16	131°
5	14	115°
6	12	98°
7	11	90°

Nominal air gap - 3/16", 187.5 mils.

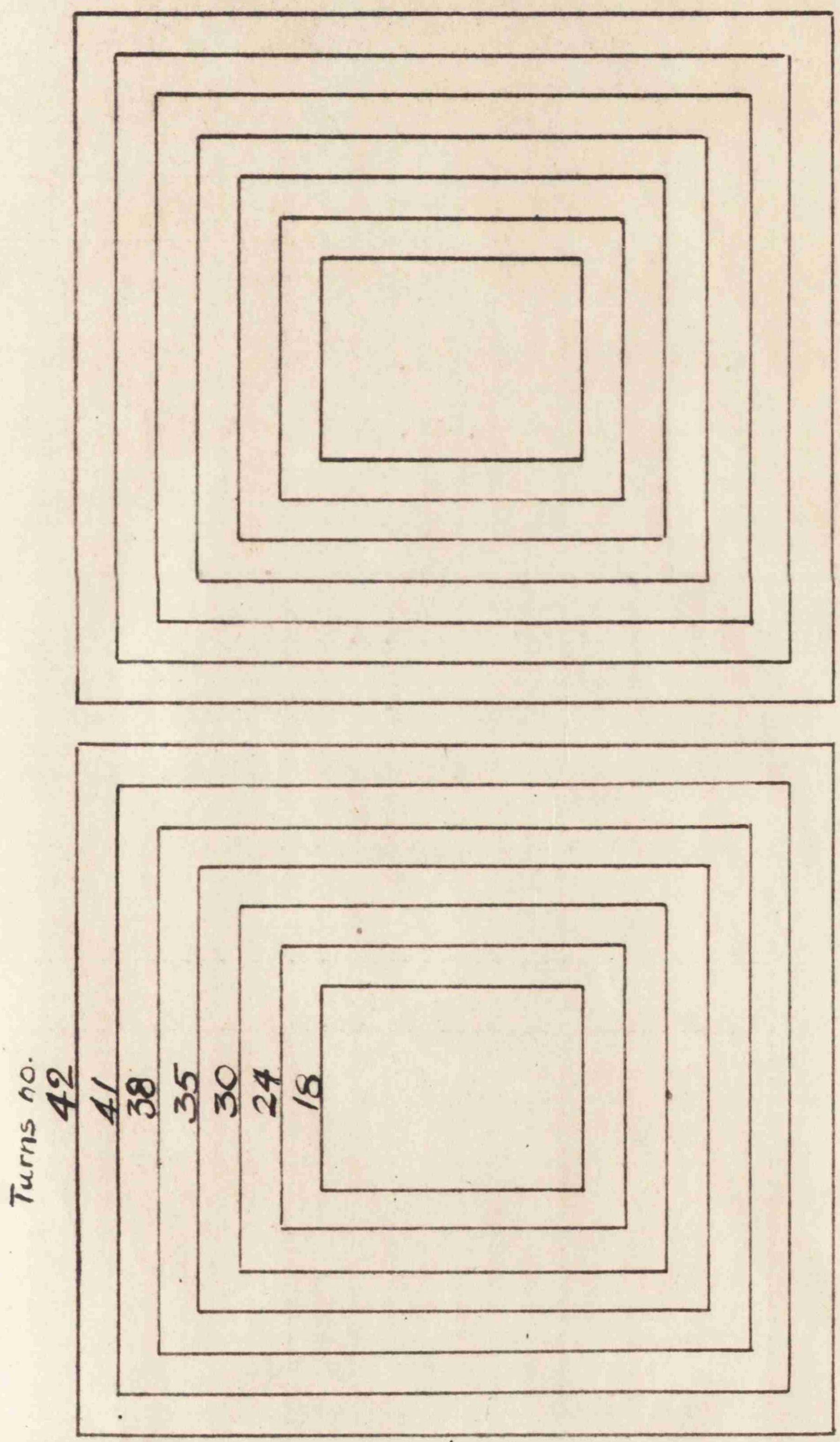


Fig 15.
Lay-out of stator winding

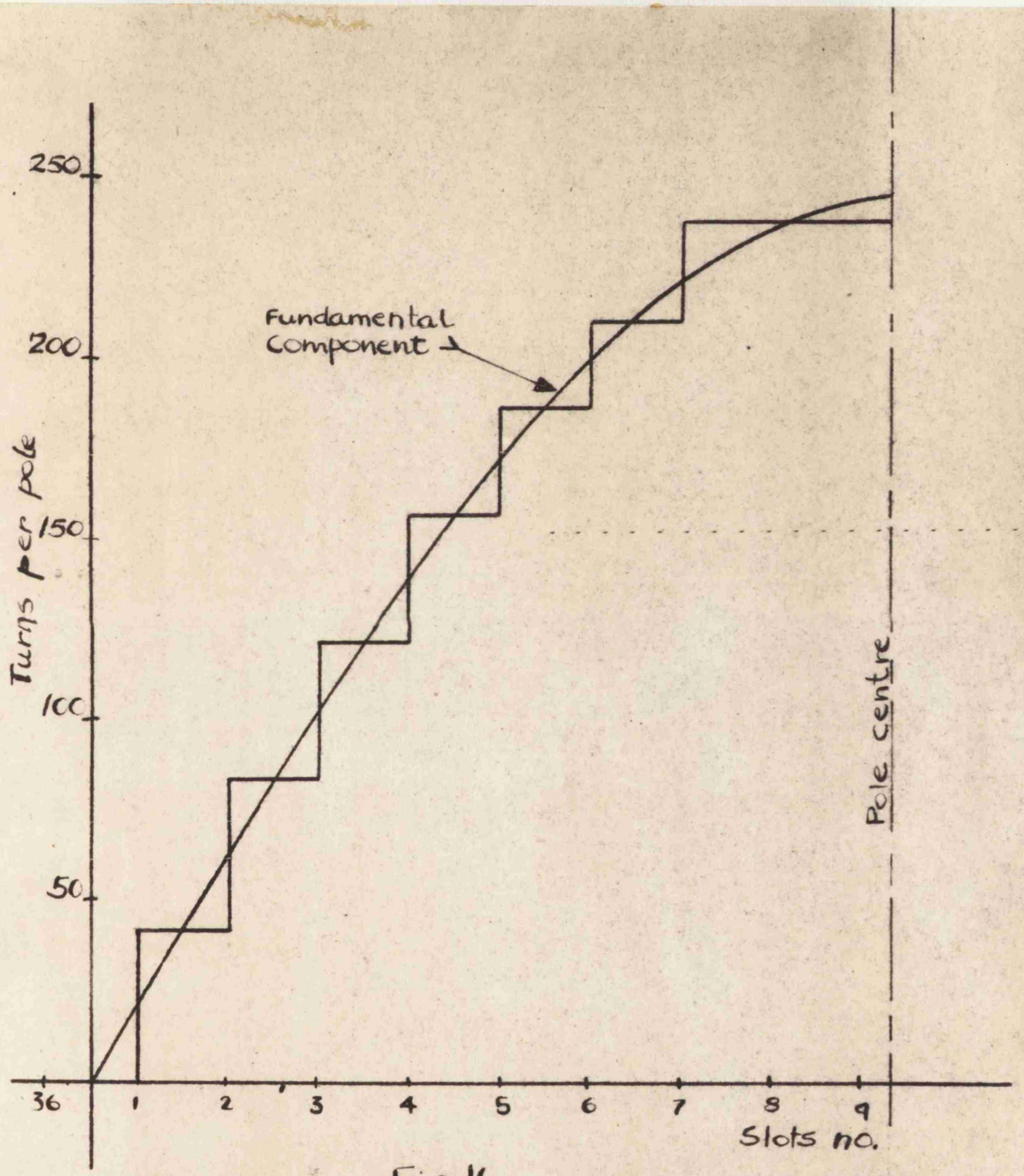


Fig 16.

MMF wave form of stator winding

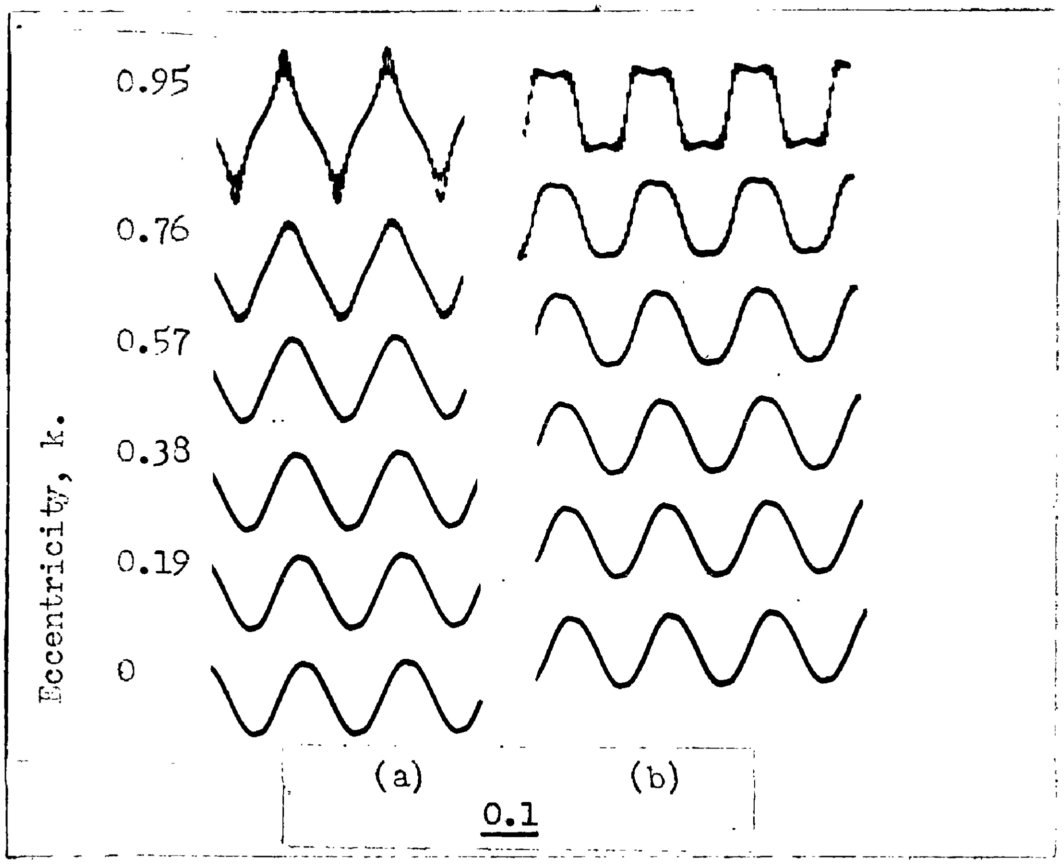
Method of Measurements

The search coil voltages have wave forms varying from almost pure sine form to the very complex form containing the complete lower spectrum in the case of extreme eccentricity.

/The

The waveforms to be analysed are displayed in the oscillograms 0.1 - 0.4. Clearly for the wide range of measurements a quick reading instrument was necessary. The only instrumentation available at the time of construction was a wattmeter, which was used in conjunction with a standard L.F. Oscillator. This was very unsatisfactory at the lowest and highest frequencies, due in the first case to the difficulty of obtaining a completely pure 25-c/s current, and in the second case low amplitude of the high frequency components prevented accurate measurements. The Author was subsequently provided with a Muirhead-Pametrada Wave Analyser, which proved very suitable. This instrument, which is capable of discrimination of the order of 75 db in the octave, should be expected to give reliable results for at least the lower order harmonics.

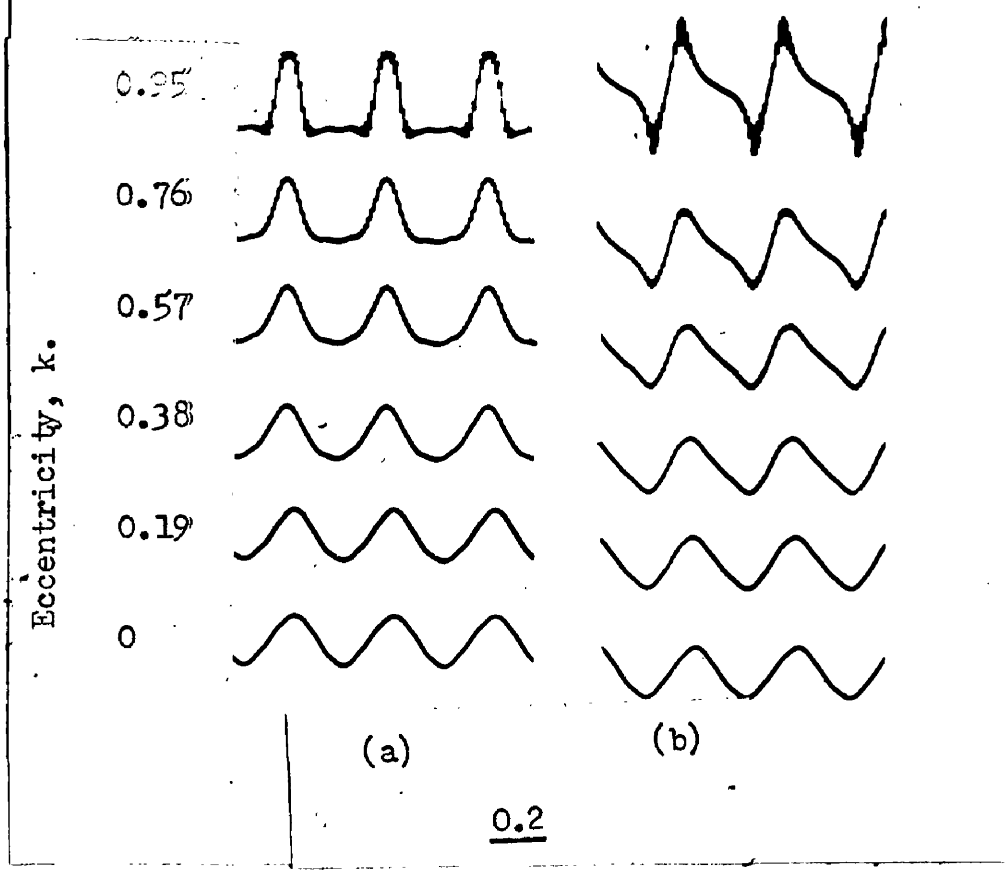
The oscillographic work was done with a Cossor 1035 Oscillograph with camera. The following oscillograms illustrate very clearly the effects of making the rotor eccentric and support the theory and measurements.



The above oscillograms were taken of the voltage induced in a full pitch search coil. Since the coilspan factor for all the odd order harmonics is unity and zero for all the even order ones, only odd harmonics are present in these waveforms. The eccentricity was raised in steps up to 0.95, and the waveforms show an appreciable distortion due to the relatively rapid increase of the higher order harmonics at extreme eccentricity. Apparently, little distortion can be detected for eccentricities up to 0.5. As may be expected at the highest eccentricity, a pronounced tooth ripple is introduced due to the lack of fringing in the very short parts of the gap.

(a) Eccentric angle $\sigma = 0$

(b) " " $\sigma = \pi/2$.

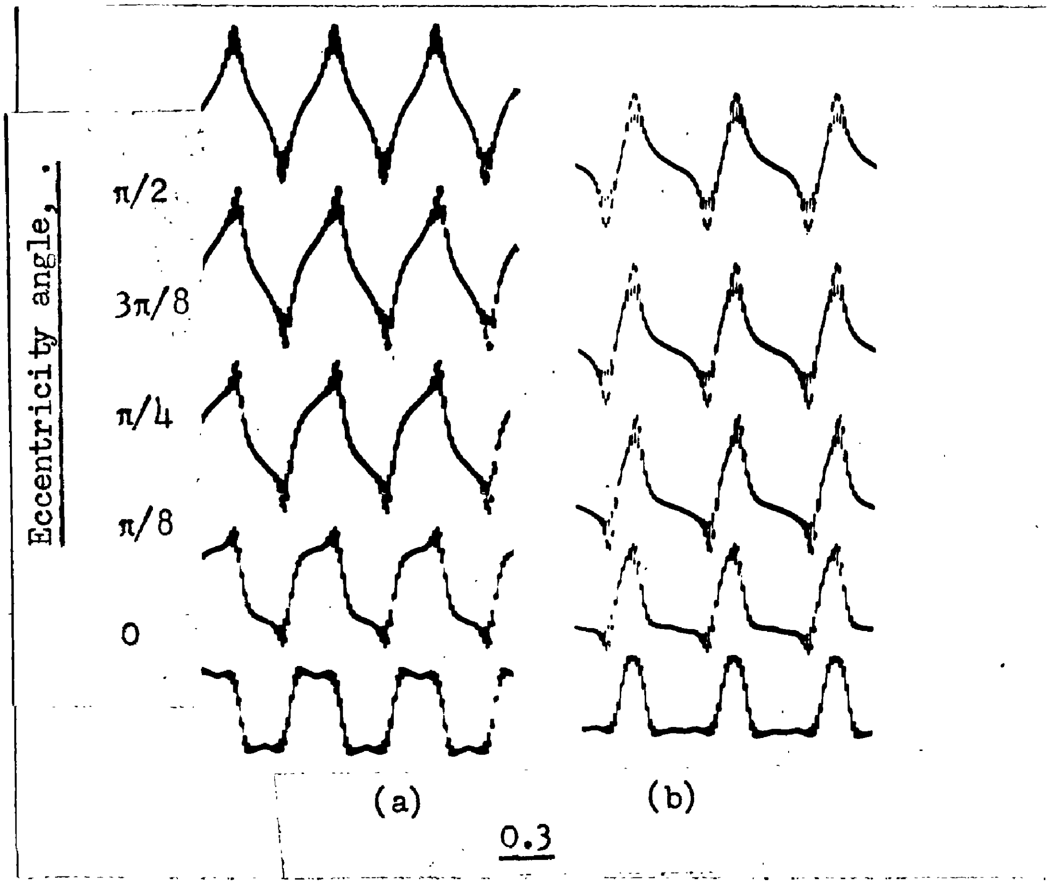


These oscillograms are taken of the voltage induced in the search coil having $\pi/2$ pitch. The coil span factor for the odd harmonics are here all 0.7071, while all the even harmonics being an odd multiple of 2 have unity coil span factor; the rest of the even harmonics have zero coil span factor.

The oscillograms show clearly the familiar skew symmetric waves due to the presence of even harmonics, especially the 2nd harmonic. Again the increase in tooth-ripples is very noticeable at extreme eccentricity.

(a) Eccentric angle, $\sigma = 0$

(b) Eccentric angle, $\sigma = \pi/2$

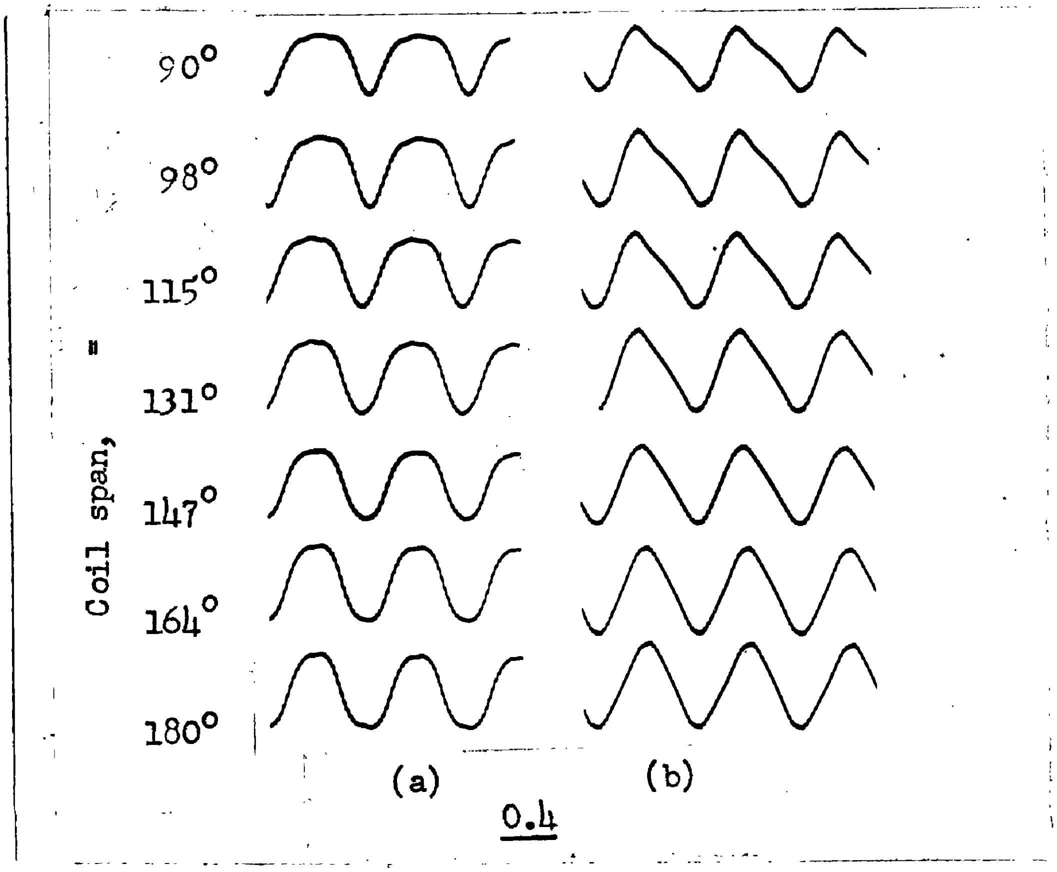


These oscillograms display the effect of varying the eccentric angle. In (a), a full pitch search coil is used and in (b) a half pitched coil is used. Since the machine has a sinusoidal m.m.f. wave, the flux function is (Equation (77)):

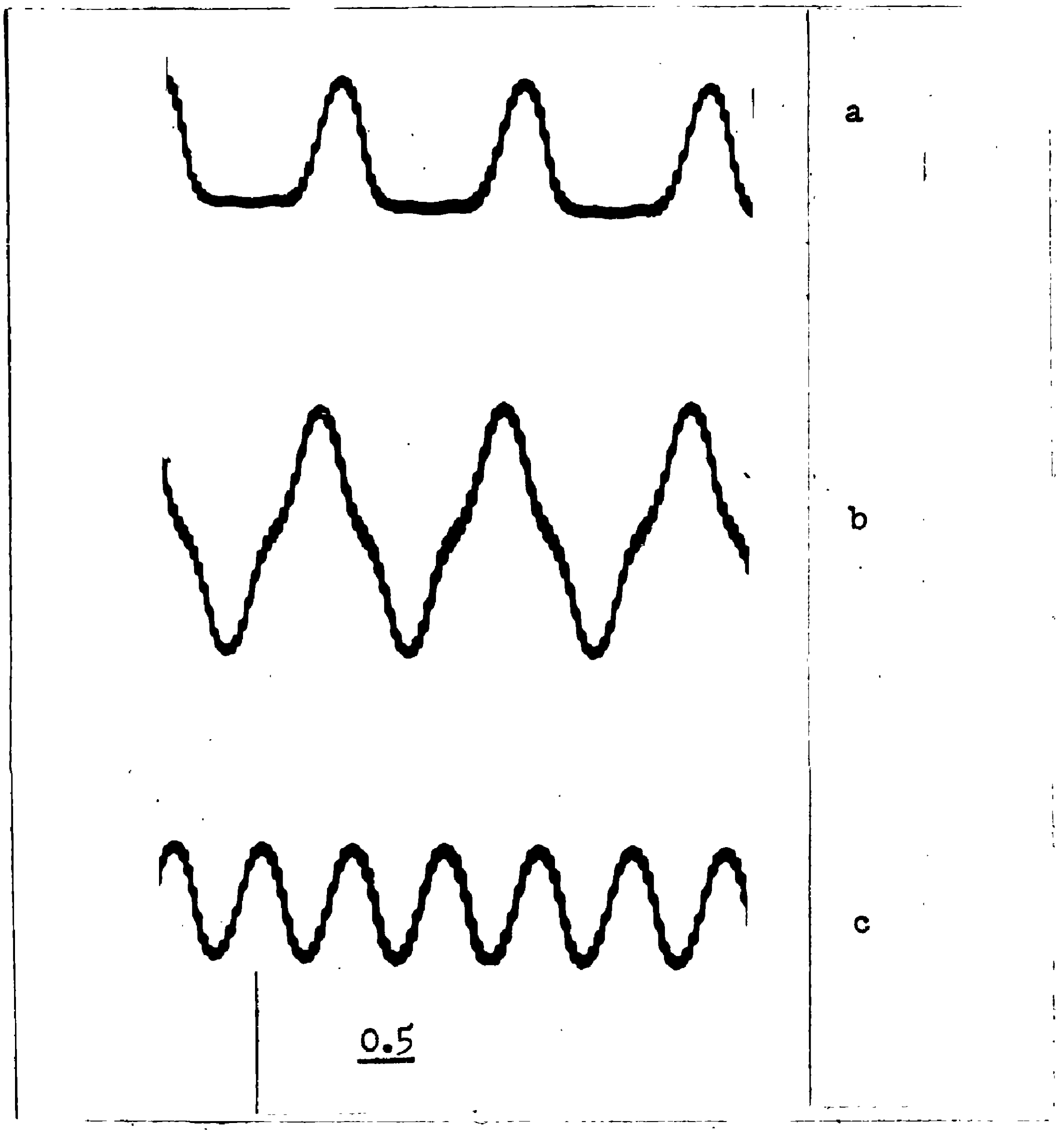
$$B(x) = \text{constant} \times \sum_q Y_q \,(\cos\sigma \cos qx + \sin\sigma \sin qx)$$

With $\sigma = 0$, this series has cos terms only and when $\sigma = \pi/2$ there are only sine terms. By inspection of the oscillograms, noting that the main harmonics are the 2nd and 3rd in (b) and (a) respectively, the correctness of the formula on this point may be verified.

The eccentricity was held constant for all of these wave forms, and the magnitudes of the harmonics are therefore expected to be constant.



For the above oscillograms, the eccentricity was held constant at 0.6, and the waveforms in the various search coils displayed. The set (a) was taken with eccentric angle $\pi/2$, and (b) with eccentric angle 0. These waveforms merely illustrate the complete set of measurements possible with the search coils.



These oscillograms illustrate the principle of 1:2 ratio pole changing, but not in the usual sense. Two search coils, having half-pitch span, have their individual induced voltage wave forms illustrated in (a). These coils are displaced 180° on the rotor, and by series opposition connection all the odd harmonics are added, while all the even harmonics are cancelled. The resulting wave form is shown in (b). When they are connected in series aiding, all the odd harmonics cancel, and all the even harmonics add. This is shown in (c). (Mostly 2nd harmonic).

<u>Measurement of the harmonic contents in the flux wave in the eccentric gap</u>

The flux density wave can be assumed to be of the form

$$B(x) = \sum_{n=1} b_n \cos(nx + \phi_n)$$

The voltage induced in a search coil of span α radians is therefore given by

$$V = \frac{d}{dt} \; TA \int_{\theta - \alpha/2}^{\theta + \alpha/2} b_n \cos(nx + \phi_n)dx$$

where T is the number of turns in the coil; $A = \frac{DL}{2}$.

$$\therefore \; V = \frac{d}{dt} \, TDL - \sum \frac{b_n}{n} \sin \frac{n\alpha}{2} \sin(n\theta + \phi_n)$$

Again, if the rotor carrying the search coil is driven at the constant angular speed ω, $\theta = \omega t$, whence

$$V = TDL\omega \sum b_n \sin \frac{n\alpha}{2} \cos\overline{n\omega t + \phi_n}$$

$$= k \sum v_n \cos \overline{n\omega t + \phi_n}$$

Thus the nth harmonic in the induced voltage is directly proportional to the nth harmonic in the flux wave, and

$$\frac{b_n}{b_1} = \frac{v_n \sin \alpha/2}{v_1 \sin n \, \alpha/2}$$

From this relation, the relative amplitudes of the harmonics in the flux wave are readily obtained from the corresponding ratio of the voltage harmonics.

/For

For examination of the variation in the harmonic content of the flux wave in the constructed machine, the wave analyser was connected to the rotor brushes with search coil No. 3 connected to the sliprings. The field was excited to 1A d.c., which gives a maximum average flux density of 0.3 Wb/m^2. This flux density is sufficiently low to exclude any effect due to saturation of the teeth. ✓ good

Important The rotor was then centred by tuning the analyser to the second harmonic and moving the stator till a minimum reading was obtained. Since the analyser will record voltages of less than 1mV, this proved a very satisfactory procedure. The fundamental component of the search coil voltage had an amplitude of the order of 2V, and changes in the 2nd harmonic of the order of 1mV could be detected. This corresponds to movements of the rotor less than 0.001" and the method therefore allows centering to this degree of accuracy.

The analyser was then tuned to the fundamental (25 c/s) and the readings noted for 10 different (eccentric) positions of the rotor. The rotor was again centred and the procedure repeated for the 2nd harmonic, and again for the higher harmonics. The results are tabulated below, showing immediately below each reading the corresponding ratio $\frac{v_n}{v_1}\frac{\sin \alpha/2}{\sin n\alpha/2}$. ? explain

p 122 et seq

/It

It should be noticed that the readings corresponding to 0.99 eccentricity must necessarily be somewhat approximate. The readings were taken with the rotor nearly touching the stator, but there must clearly be some air gap left.

doesn't K=.99 provide a better?

Presumably there may also be a slight angle between the centre lines of the rotor and stator, since the mechanical construction is not easy. This would mean that for 0.99 some smaller figure should be substituted, and in fact all the eccentricities should be reduced in the same proportion.

The test was carried out for eccentric angles of 0 and $\pi/2$ respectively, and both results are tabulated. According to the theory developed above, in the case of a purely sinusoidal m.m.f. waveform, no difference should be expected. In fact, the two results differ little, but the amount seemed to be too large to be due to any fault in the predicted value, and a test was carried out to find the exact nature of the variation. The variation was finally found to be due to two causes:-

1. The stator stampings had not been completely concentrically machined, so that the rotation of these in the housing slightly altered the amount of eccentricity.

2. The analyser was slightly phase sensitive, and the change of phase incurred by turning the stator altered the reading, sometimes appreciably.

/The

The first of these causes was easily detected by the centring procedure outlined above. The discrepancy was about 0.003", corresponding to a variation of 0.03 in the eccentricity. ?

Good point

The second of the causes, being slightly on the incredulous side, was only detected by observing the search coil voltage in the concentric position on an ordinary sensitive voltmeter simultaneously with the analyser. The waveform is almost purely sinusoidal, and the two instruments would be expected to give identical readings. When turning the stator, the voltmeter reading remained practically steady, while the analyser reading varied ~~with~~ by a small amount corresponding to the difference in the above tests. The reason for this faulty reading of the analyser is believed to be due to the fact that the voltage observed happened to have a frequency of 25 c/s, being a submultiple of the supply frequency. Some slight amount of hum present in the amplifiers may interfere with frequencies which correspond to the hum frequencies.

Wrong word?

by

Since no better instrument for wave analysis was available, the given results were taken to average to the correct values.

Only thing to do.

For the two-pole machine under test, and assuming that the m.m.f. wave form is sinusoidal, the flux density wave form is given by equation (77) as

$$B(x) = b_{10} \sum_{q=1}^{\infty} y^{q-1} \frac{1 - y^2}{(1 - k^2)^{\frac{1}{2}}} \cos q \overline{x - \sigma}$$

where b_{10} is the amplitude of the fundamental component in the concentric position. The function $y^{q-1} (1 - y^2)/(1 - k^2)^{\frac{1}{2}}$ is tabulated in Table 4, and shown graphically for the four lowest values of q in Fig. 17. The results of tests 1 and 2 are shown alongside these functions in Figs. 18 - 20. The agreement must be admitted to be fair and if the argument for reading the eccentricity actually lower than that given is accepted, the agreement is still better. Clearly, the effect of the third harmonic actually present in the m.m.f. wave is appreciable in the resultant. Here the initial third harmonic (which increases as $1 - y^2/(1 - k^2)^{\frac{1}{2}}$) opposes that introduced by the distortion of the fundamental (which increases as $y^2(1 - y^2)/(1 - k^2)^{\frac{1}{2}}$). The net effect is a minimum at 0.35 eccentricity.

Goes no further than 4th

Agree

Test 1. $\sigma = 0$. Search Coil No. 3. $I_f = 1A$

n \ k	0	.15	.30	.45	.60	.75	.85	.90	.95	.99
1	1,800 100	1,800 100+	1,830 101.5	1,880 104	1,950 109.1	2,050 114	2,180 121	2,250 125	2,370 131.5	2,550 141.5
2	12 1.2	58 5.7	125 12.2	195 19.1	283 27.6	415 40.5	538 52.5	620 60.6	715 70	925 90.5
3	32 2.7	35 3.0	48.5 4.1	77.5 6.6	126 10.7	220 18.6	335 27.5	430 36.4	540 45.8	850 72
4	2 0.1	2.75 0.2	6 0.4	15 0.9	39 2.4	110 6.7	203 12.5	292 17.8	450 27.5	780 47.6
5	4.5 1.8	4.5 1.8	4.3 1.7	4.0 1.6	3.2 1.3	4.75 1.9	18.8 9	24.5 9.6	54 21.1	100 39.1
6	1.2 0.1	1.75 0.1	3.0 0.2	5.5 0.3	11 0.6	27 1.5	64 3.6	107 6.1	230 13.0	450 25.5
7	6.7 0.9	6.7 0.9	6.7 0.9	6.7 0.9	7.0 1.0	10 1.3	18.3 2.5	29 3.9	69 9.2	150 20
8	-	-	-	2 0.1	3 0.2	3.75 0.3	9 0.7	20 1.5	51 3.7	125 9.1

Test 1. $\sigma = 0$. Search Coil No. 3. I_f = 1A (Continued)

n \ k	0 15 30 45	60	75	85	90	95	99
9	(Upper figure in each pair is the actual reading of the analyser,	4.2 0.3	3.8 0.3	3.8 0.3	9.5 0.6	29 1.9	93 6.1
10	(in mV). The lower figure is	0.6 0.1	0.75 0.1	1.5 0.3	2.9 0.6	8.0 1.6	45 8.9
11	the corresponding percentage of the reading for the fundamental	3.6 0.2	3.75 0.2	4.75 0.3	8.2 0.4	20 1.1	95 5.3
12	in the concentric position,	0.3 0.1	0.5 0.1	0.8 0.2	1.5 0.3	4.4 0.9	32 6.3
13	corrected for coil span attenuation).	1.0 0.1	1.5 0.1	2.0 0.1	3.1 0.2	8.3 0.6	58 3.8

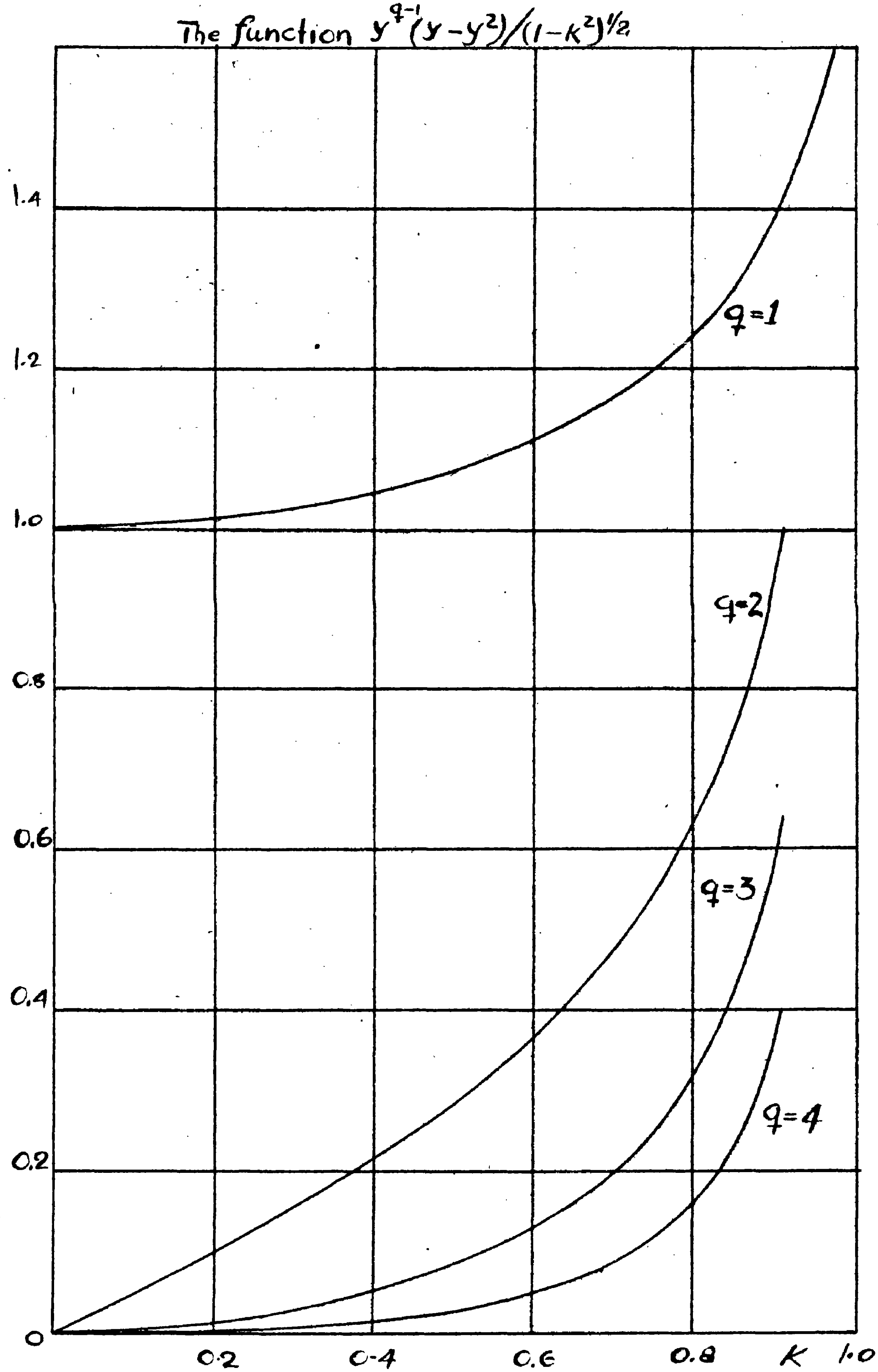
Test 2. $\sigma = \frac{\pi}{2}$. Search Coil No. 3. I_f = 1A

k	0	15	30	45	60	75	85	90	95	99
1	2,080 100	2,100 101	2,110 101.5	2,180 105	2,260 108.8	2,480 119.2	2,700 129.8	2,850 137	3,150 152	3,500 168
2	23.1 2.0	62.5 5.3	125 10.6	200 17.0	300 25.5	435 37	600 57	730 62	900 76.5	1150 97.8
3	31 2.2	26 1.8	11 0.8	22 1.5	72.5 5.1	180 12.7	330 23.2	440 30.9	660 46.4	860 60.5
4	2.3 0.1	2.4 0.1	5.1 0.3	2.5 0.1	18 0.9	75 3.8	183 9.3	290 14.7	470 23.8	750 38.0
5	4.7 1.5	4.7 1.5	5.0 1.6	5.2 1.7	4.5 1.5	3.0 1.0	13 4.2	27 8.8	60 19.5	110 35.7
6	1.8 -	5.2 0.2	9.0 0.4	13.5 0.6	17.0 0.8	14 0.76	13.5 0.6	57 2.8	160 7.4	330 15.3
7	6.7 0.7	6.7 0.7	7.0 0.8	7.7 0.9	8.7 1.0	9.6 1.1	5.5 0.6	3.0 0.3	33 3.7	97 10.8
8	-	-	2.5 0.2	4.6 0.3	7.5 0.5	11.5 0.7	10 0.6	5 0.3	39 2.4	110 6.7

TABLE 4

The function $y^{q-1} (1 - y^2)/(1 - k^2)^{\frac{1}{2}} \times 100$

q \ k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95
1	100	100.3	101.0	102.4	104.4	107.2	111.1	116.7	124.5	131.0	139.3	152.4
2		5.0	10.2	15.7	21.4	28.7	37.0	47.5	62.3	72.9	87.3	110.3
3.		0.3	1.0	2.4	4.5	7.7	12.3	19.4	31.1	40.6	54.7	79.9
4				0.4	0.9	2.0	4.1	7.9	15.6	22.7	34.3	57.8
5				0.1	0.2	0.6	1.4	3.2	7.8	12.6	21.3	41.9
6						0.2	0.5	1.3	3.9	7.2	13.5	30.3
7							0.2	0.5	1.9	3.9	8.4	22.7
8							0.1	0.2	1.0	2.2	5.3	16.4
9								0.1	0.5	1.2	3.3	11.9
10									0.3	0.7	2.1	8.6
11									0.1	0.4	1.3	6.2
12									0.1	0.3	0.8	4.5



The function $y^{q-1}(y-y^2)/(1-k^2)^{1/2}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
K
1.0
q=1
q=2
q=3
q=4

Fig 17.

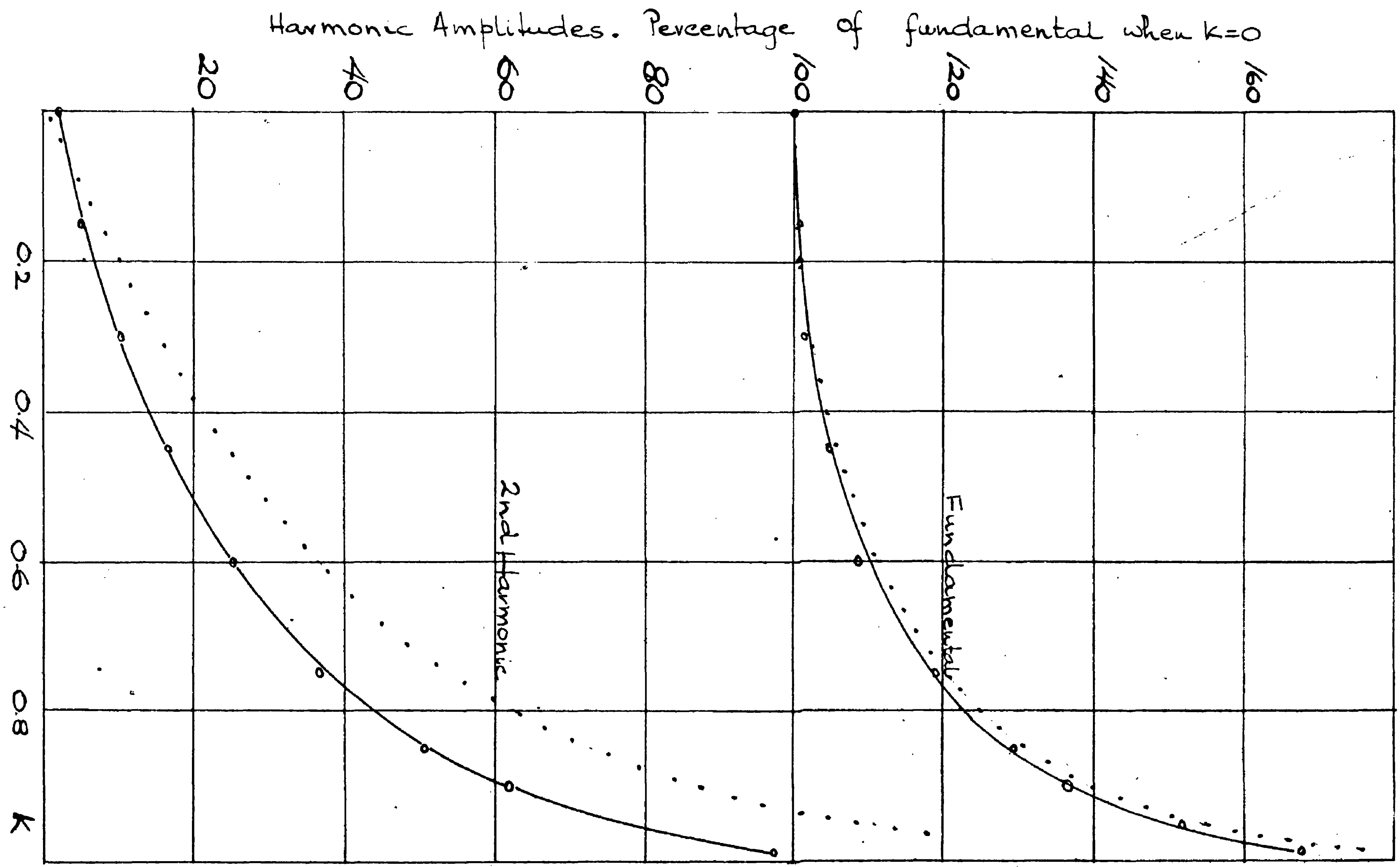


Fig. 18. o—o—o Exp. values
· · · · Value predicted by the formula $b_q = b_{10}\, y^{q-1}(1-y^2)/(1-k^2)^{1/2}$

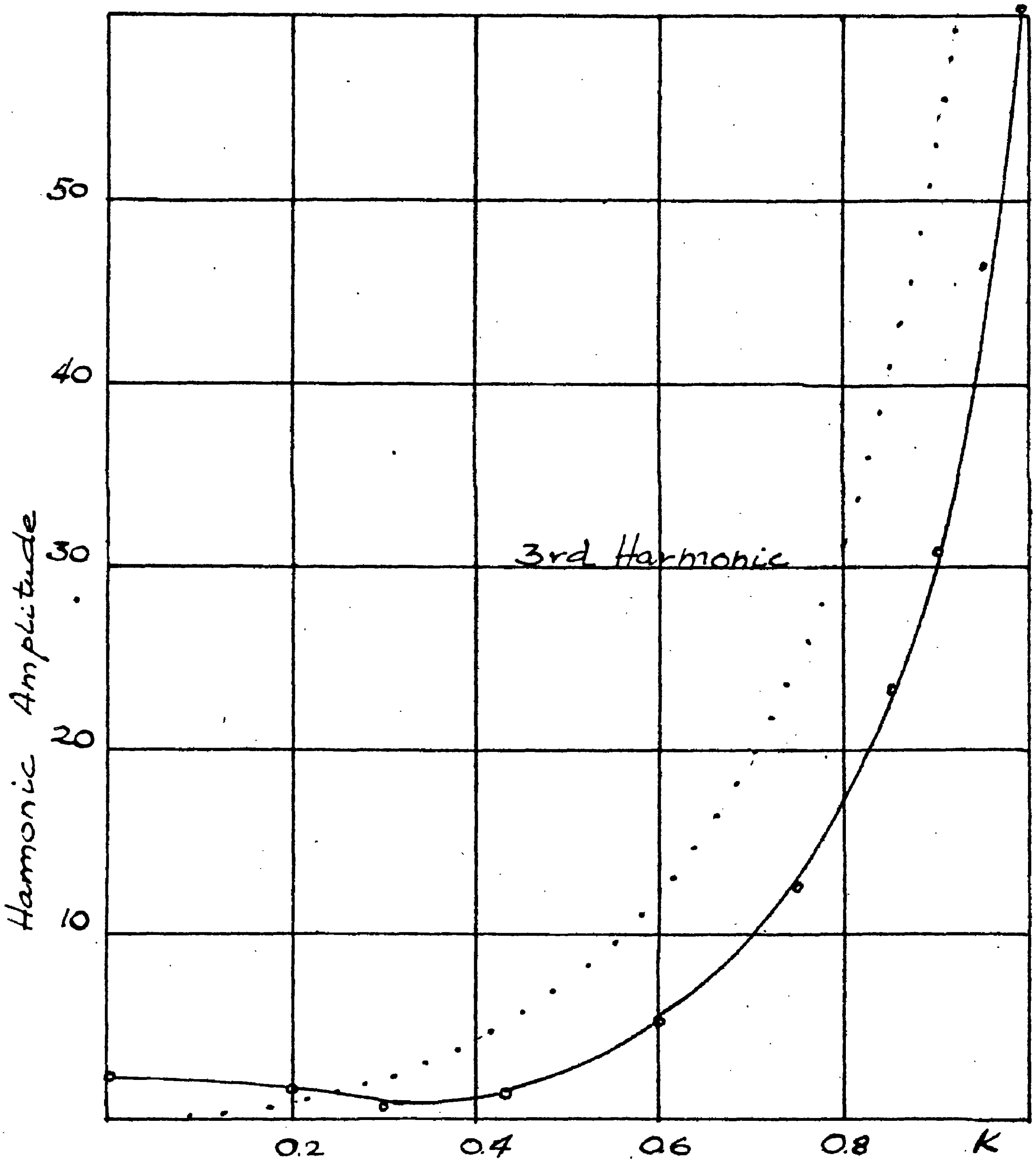


Fig 19.

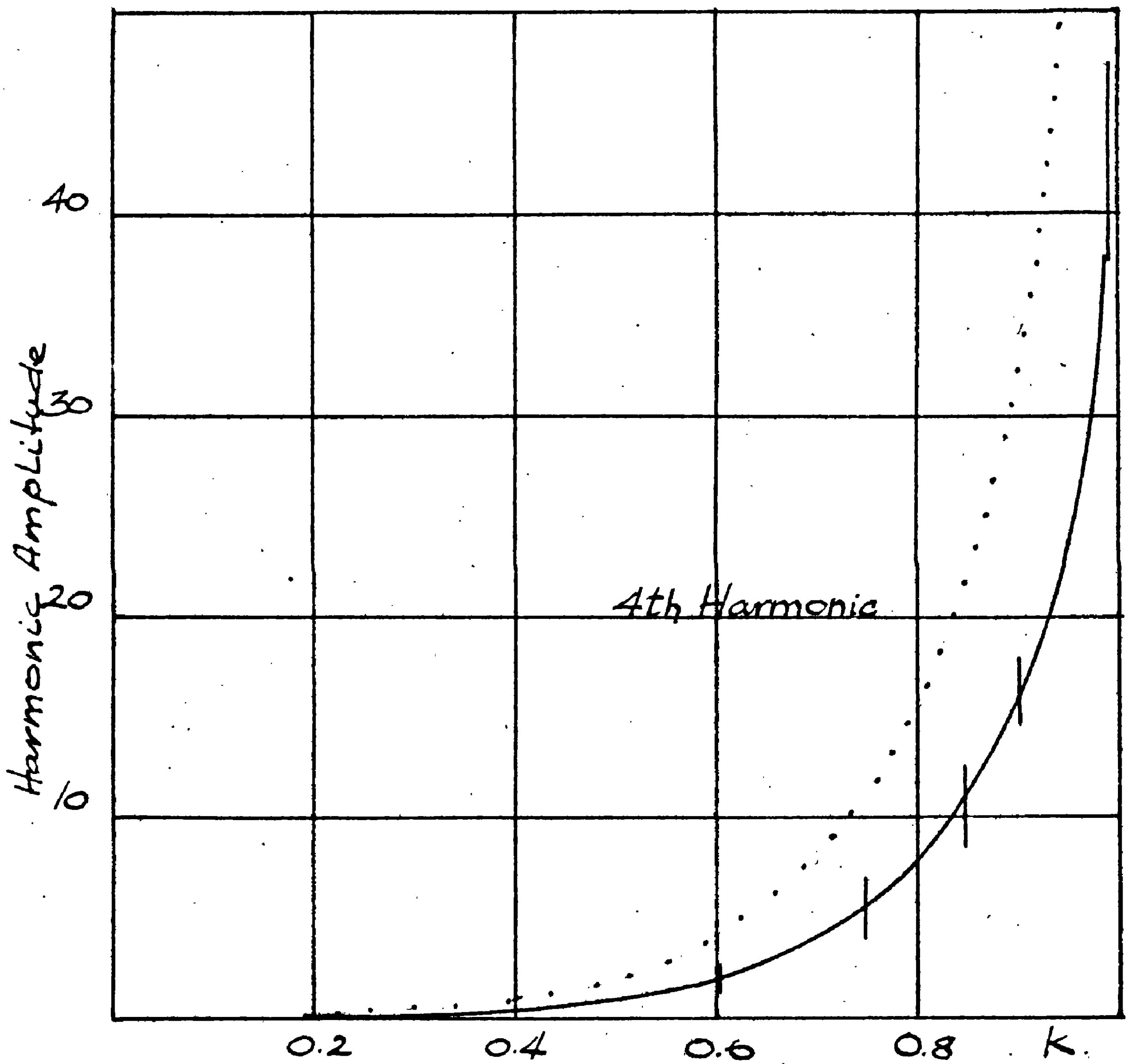


Fig. 20.

5.2 Investigations on a standard machine

For a complete analysis of the air gap field under various kinds of excitation and load conditions very extensive measurements would be required, and some of these measurements would require very elaborate apparatus. In the present case, only two crucial tests were performed in order to test the validity of some of the results predicted in the previous theoretical work. In the first case, it is shown that in a rotating field, the harmonics due to eccentricity behave in the manner described in Section 4.3, and in the second case the effect of saturation is considered. No artificial eccentricity was introduced in this machine, and the results may therefore be taken as typical of a randomly chosen production-type machine.

Details of test machine

Induction Motor, 2 H.P., (Crompton-Parkinson Ltd.)

Stator Bore $4\frac{1}{2}$"

Air gap 1/64" (approximately)

Stator Winding: 24 coils, 5/6 full pitch, 4 poles.

Rotor Windings: Search coils, each 10 turns, various pitch. One 20-turn coil wound Gramme type to eliminate pitch factors. Any of these search coils may be connected to sliprings through a switch at the end of the shaft.

/The

The machine was coupled to a large d.c. machine by a V-belt drive. This provides a steady, variable-speed drive essential for these tests.

5.2.1 Travelling-wave harmonics

In Section 4.3 it was shown that any winding where the principal part of the m.m.f. wave is sinusoidal produces a complete spectrum of harmonics due to eccentricity which all travel in the same direction with a synchronous speed ω/m rad/sec, ω being the frequency of the exciting currents and m the order of harmonics.

The winding employed for a test of this statement was a narrow-spread, 4-pole, 3-phase winding. The spectrum of m.m.f. harmonics is suitable, because it contains only a very small 5th and 7th harmonic, and these are the only components that will affect the lower order eccentricity harmonics.

The travelling harmonic waves may be detected by the rotor search coil and distinguished by analysis of the search coil induced voltage when the rotor is driven at any constant speed. The induced voltage is, however, of a very complex waveform, since the component frequencies do not retain their integral frequency relationship when the rotor speed is altered. The slip frequencies corresponding to the m.m.f.

harmonic of the given winding is shown as a function of rotor speed in Fig. 21. N_S is the synchronous speed of the fundamental. At stand-still, the induced voltage frequency is 50 c/s, this being the mains frequency, while at any other speed there are various frequency components. From the point of view of measurement of these harmonics, it is important that none of the components coincide. The low-speed region is seen to be useless for an analyser with a finite bandwidth. Furthermore, since the fundamental is so much larger than the harmonics, the regions where this coincides with the harmonics

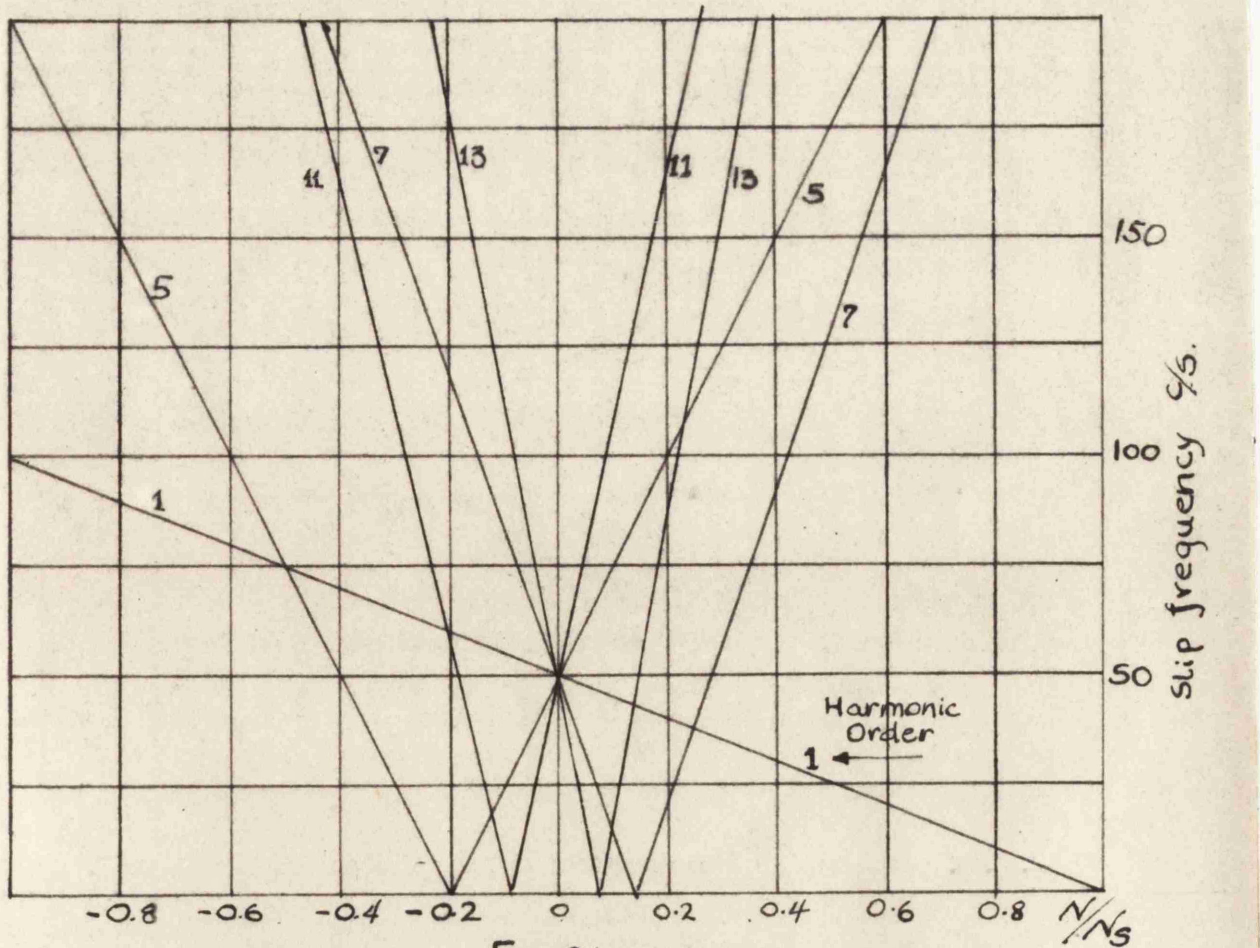


Fig 21.

Slip frequency chart.

The eccentricity harmonics due to the fundamental component of m.m.f. is shown in Fig. 22. The harmonic spectrum is here labelled according to the number of pole-pairs for each harmonic, the corresponding m.m.f. harmonics are labelled I, V, VII (corresponding to Fig. 21). The only satisfactory region for measurements can be seen to be at $N \leqslant -0.8N_s$, i.e., when the rotor is driven <u>against</u> the main field at approximately synchronous speed.

Good point

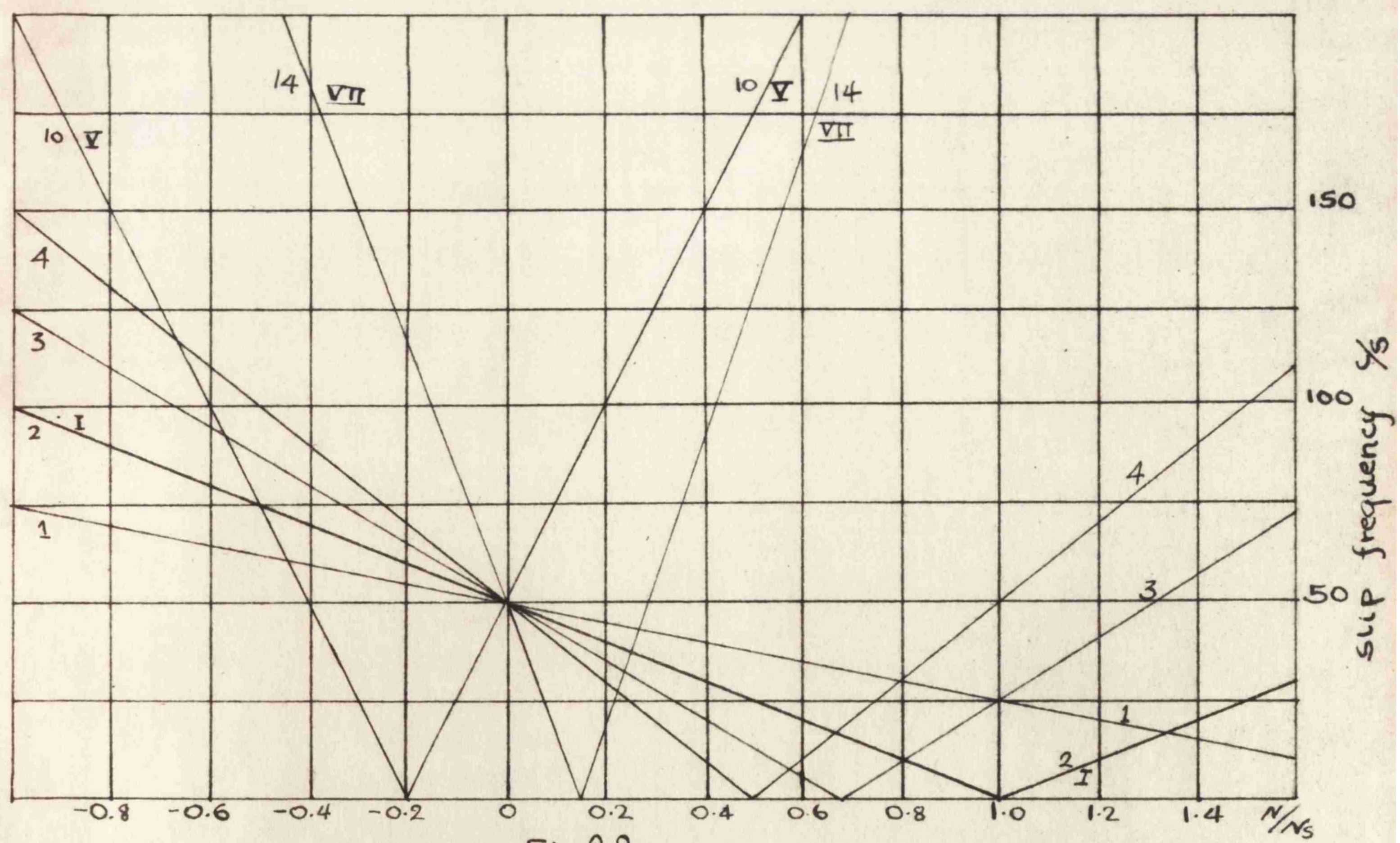


Fig 22

Slip frequency chart

Roman numbers indicate harmonic orders referred to the electrical system.

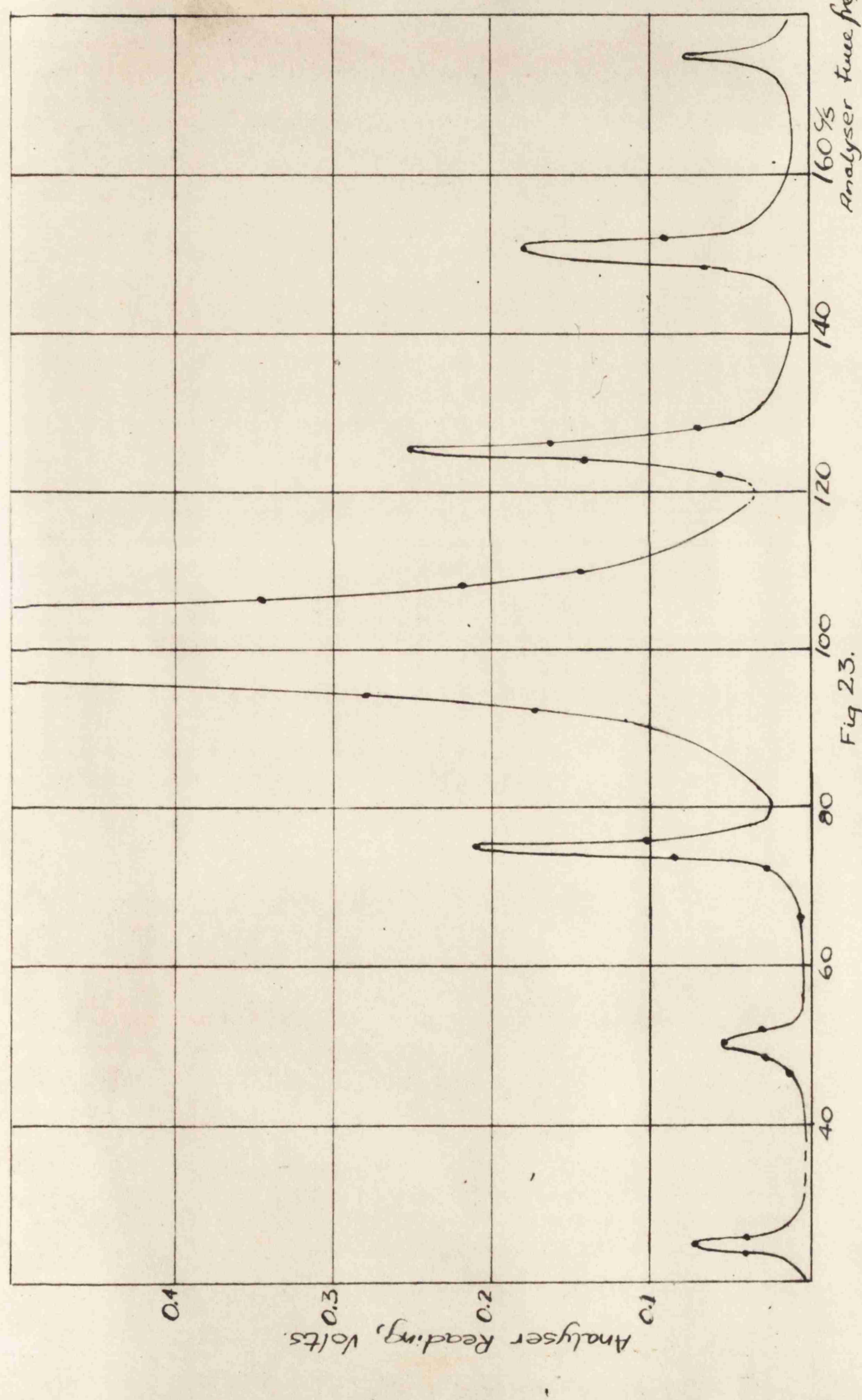


Fig 23.
Frequency spectrum of travelling field.
$N = -N_3$

The voltage induced in the Gramme coil was first analysed at the constant speed $N = -N_S$. The response of the analyser is shown in Fig. 23. This clearly indicates that the spectrum does indeed include the frequencies predicted. The 75-c/s, 125-c/s and the 150-c/s components which correspond to the 1st, 3rd and 4th harmonics respectively are all present. To dispel any doubt that these are in fact constant-magnitude rotating fluxes, their presence must be recorded at several speeds. The frequency at any given speed was deduced from Fig. 22 and the magnitude measured by the analyser. The results are shown in Fig. 24. The points are rather scattered, due to the difficulty in measurements, but they indicate that the magnitude is proportional to the frequency. This shows that the corresponding rotating fluxes have constant magnitude.

The large peak in Fig. 23 which corresponds to the principal harmonic (the fundamental in the m.m.f. wave) indicates also the limitation of the amplifier. Clearly, the bandwidth required for this large harmonic is about $\pm$ 10 c/s at 100 c/s and in general, it requires $\pm$ 10% of the frequency.

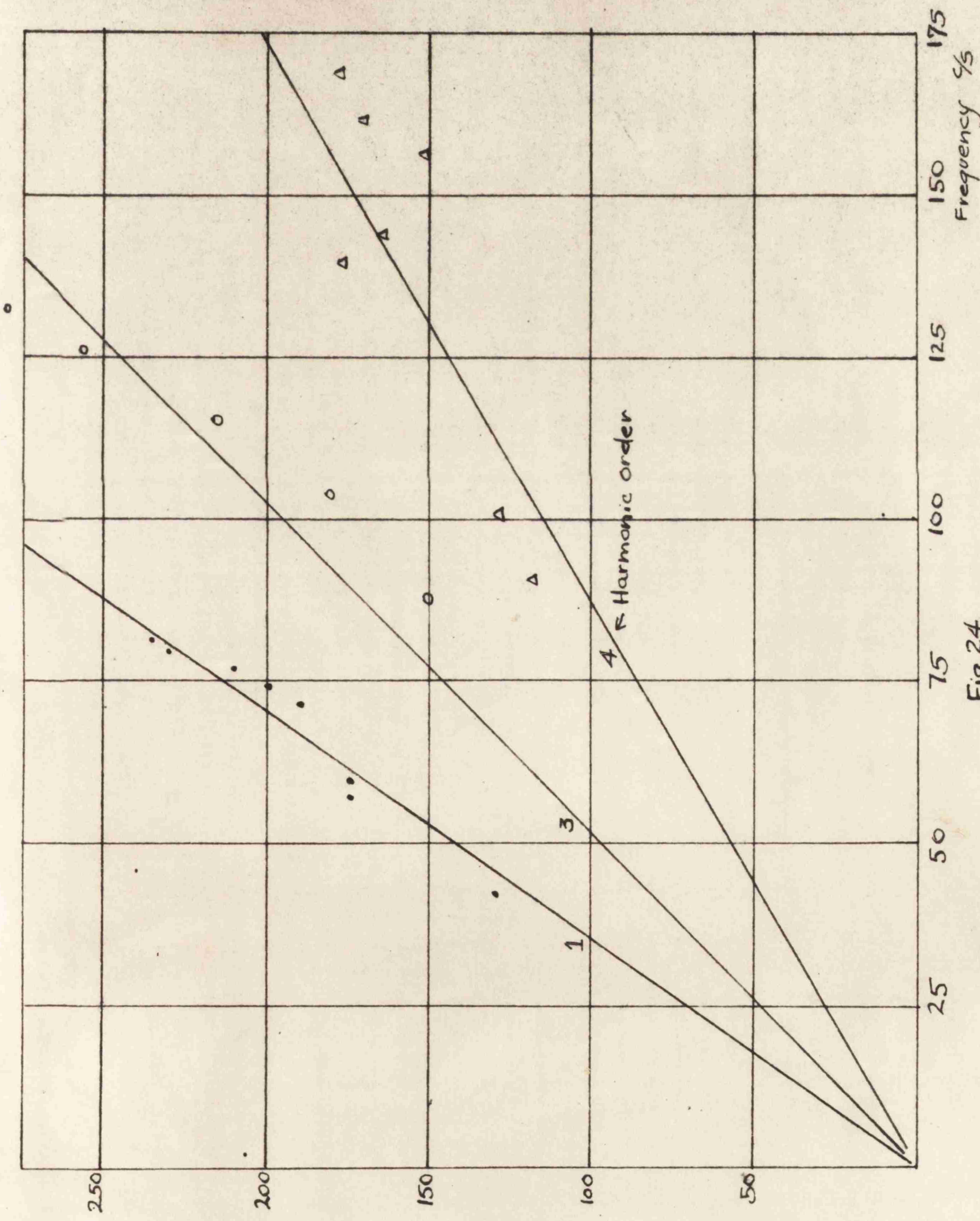
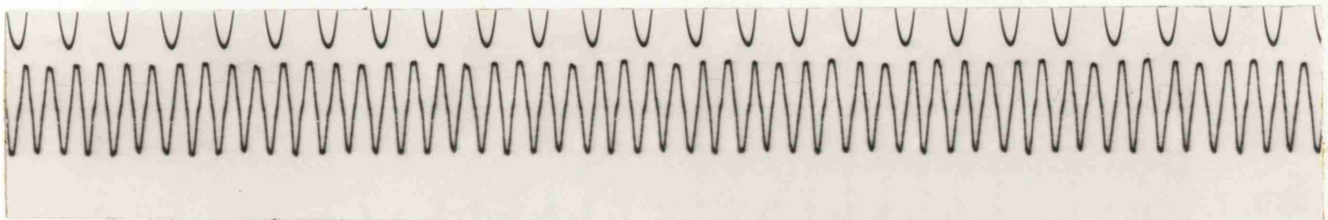

25
50
75
100
125
150
175
Frequency c/s
250
200
150
100
50
1
3
4
Harmonic order

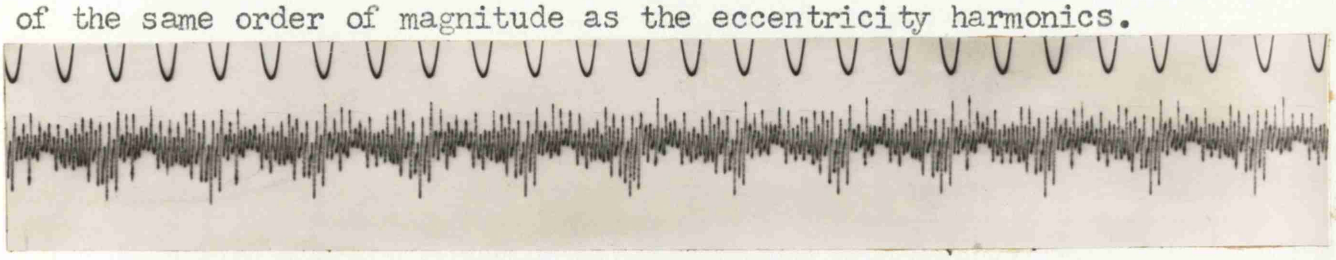
Fig 24.

The waveform analysed in Fig. 23 is shown in Oscillogram O.6. The timing wave is 50 c/s. Clearly, the percentage harmonic content is very small, but variations in the waveform corresponding to lower frequencies can just be detected.

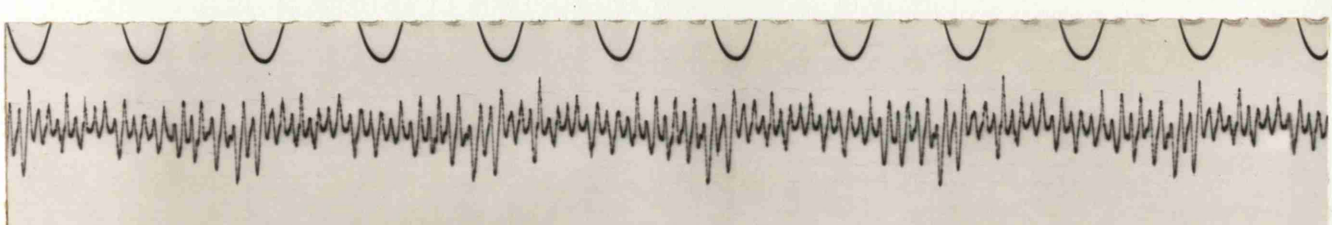


O.6

Oscillograms showing the waveform of the search coil induced voltage when driven at synchronous speed forward are also shown. O.7 and O.8 represent the same waveform, but recorded with different film speeds. The timing wave is again 50 c/s, and a strong 25 c/s is now apparent. This is again consistent with Fig. 22, which shows that such a component may be due to either of the two lowest harmonics due to eccentricity, (1 or 3). In this case the tooth-ripple harmonics are clearly of the same order of magnitude as the eccentricity harmonics.



O.7



O.8

5.2.2 Effect of saturation

One phase of the stator winding was in this test excited by d.c. current. The rotor was driven at a constant speed of 1200 rev/min, and the Gramme search coil e.m.f. was again analysed at varying excitation. The results are plotted in Fig. 25. The logarithmic plot allows a comparison between all the harmonics which are, of course, of widely varying amplitudes. The harmonic orders refer again to the actual number of "harmonic pole-pairs", and the harmonic orders of the corresponding m.m.f. wave are added in Roman numerals (where they exist).

The results are quite striking. The harmonics which are due to the m.m.f. wave directly behave as would be expected in a saturating magnetic circuit, showing the familiar "knee", and then flattening out. The harmonics due to eccentricity, however, show in fact a definite maximum in the region of the knee and then decrease considerably. The explanation of this curious result is not at all obvious, and no theoretical treatment will be attempted here. The effect is, however, in full agreement with the known behaviour of the transverse pull. This is known to have a maximum somewhere in the region of the knee, and this certainly accords with these results.

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It must be admitted, therefore, that the formulae evaluated for the flux density distribution in Section 4 must

/be

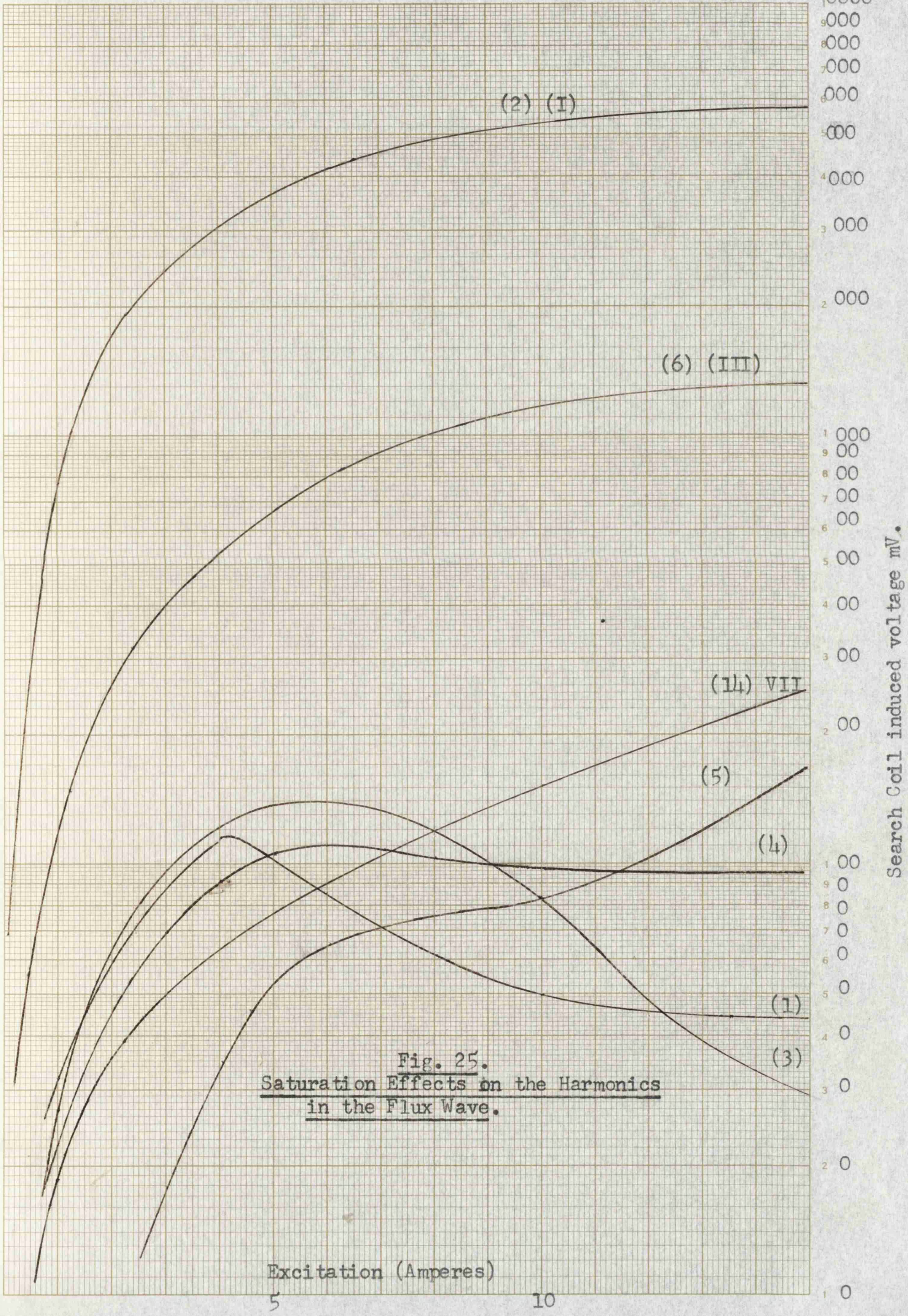

Fig. 25.
Saturation Effects on the Harmonics in the Flux Wave.

be used with due caution, and that they cannot be expected to hold for high flux densities.

This is another instance of the quite unpredictable behaviour of non-linear media and indicates that a very interesting problem is at hand. A possible method of attack on this problem is to assume that the saturation in any case is limited to the teeth, and consider the toothed region as polarisable only in the radial direction. This would still leave two infinitely permeable boundary regions, the stator and rotor cores respectively, and the problem might thus again be tackled by analytical methods. Good line

6. Conclusions

The first three Sections of this Thesis do not break new ground in so far that the problem has already been analysed. It has been found, however, that the extension of the conventional symmetric analysis to the general unsymmetric case has not added much to the complexity of the situation, and has led to a rather simple generalisation of the known formulae. The most striking of these may be replacement of the well-known expression,

$$n = kN \pm 1 \quad \text{by} \quad n = kN \pm S \qquad k = 0, 1, 2, 3 \ldots\ldots$$

for the harmonic spectrum of an integral-slot winding, and a similar expression for the fractional-slot case. Also, a complete air gap inductance matrix has been derived and shown to have a simple canonical form. The matrix reduces to diagonal form when the variables are symmetrical components (or $\alpha\beta 0$ components), and the inherent simplest reference frame is shown to be that of symmetrical components.

The treatment of the winding theory was originally intended to be only the introduction to the work on the eccentric rotor problem but while writing this, the need for a more general and powerful method of analysis became obvious ~~to the Author.~~ These requirements may be satisfied by some kind of topological analysis in terms of the simple connection matrices. These ideas are suggested in Appendix I, but considerably more analysis would be needed to establish a

/coherent

coherent theory. The field is, in the Author's opinion, full of interest and promise, and would warrant further study. It is of special interest to be able to synthesize windings from an optimum waveform point of view, but the methods which have been developed to date are only trial and error based. This is especially true of fractional-slot windings. Synthesis normally requires more fundamental techniques, and it is hoped in the near future to tackle this problem from the point of view suggested in Appendix I.

In carrying over the m.m.f. methods of flux density calculation to the case of eccentric rotors, a fundamental difference is encountered. The m.m.f. wave cannot any longer be entirely determined by the current distribution, but must contain an additional term arising from the eccentricity. The value of this has been evaluated, and the complete flux density function has been determined as a function of the eccentricity. The results, which are supported by the experimental results, may be obtained very quickly from the prepared tables. These results may be contrasted with similar results obtained by the more exact methods of field analysis.[3] If computed results are required, the latter methods would require the aid of a Computer and perhaps, in the end, fall down on the basic simplifications. The formulae evaluated in this Thesis are not formidable even to slide-rule workers.

The effect of eccentricity on the rotating fields is, in most cases, similar to the introduction of unbalance in excitation, i.e., elliptic fields will arise. Also the eccentricity harmonics which will mainly be produced by the fundamental component in the current distribution (or m.m.f. wave) are shown to rotate in the same sense, and sometimes faster than the normal m.m.f. wave. This effect has also been shown by experiment.

In most cases, the change in the total inductance of the winding with small eccentric displacements will not be significant, but it is possible that the relative magnitudes of the harmonic inductances may be upset. A peculiar non-reciprocal mutual inductance also occurs, which accounts for the elimination of certain eccentricity harmonics.

The transverse force has been shown to depend almost entirely on the fundamental component of m.m.f. and is considerably lower for the two-pole machine than the multipolar machines. This is due to the fact that for a 2p-pole winding, the most important eccentricity harmonics have 2(p - 1) and 2(p + 1) "poles" respectively. If p = 1 only the latter exists, and consequently the effect is smaller. An interesting problem arising from the general analysis is how far the eccentricity harmonics may be responsible for "magnetic" noise. Investigations in this respect have not

/been

been made, but might well prove interesting and fruitful. A procedure which would quickly determine this for any given machine could be devised fairly simply and might be an interesting project.

The eccentric transverse force has long been known to be considerably affected by saturation in the teeth, and this also holds for the flux distribution. In fact, it can be argued physically that the lower the saturation point, the smaller will be the effects of eccentricity. (Taking the extreme example of no iron present, the effect is clearly not present at all). The transverse force is known to be a maximum when the flux density corresponds to a point somewhere on the "knee" of the B/H curve, and in terms of the formulae evaluated above, we may deduce that the eccentricity harmonics have their relatively largest amplitudes at this point. This has been confirmed by experiment, and the results clearly indicate the limitations of the linear analysis. The non-linear analysis may bring to light further interesting phenomena, but will require considerable effort and ingenuity.

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APPENDIX I

The generalised theory of constant span polyphase Windings

A statement of a general method of analysing windings will now be given. Since special types of windings are of a great variety, no complete theory in the sense of pursuing every case to a final answer will be attempted here, but the frame of reference into which all windings must fit will be given. Such a frame is the symmetrical component system.

The common property of all windings having constant span can be stated as follows: N coils, all producing similar flux distributions are symmetrically disposed round the air gap. Such a system forms an N-phase, 2-pole winding. Since N, the total number of coils is large, it is normally excited by a polyphase system of $N' = N/q$ phases, N' being in general a much smaller number than N. The N coils are, therefore, connected in groups of q coils, all carrying the same amount. These connections can be viewed as constraints, and their effect is in fact, to reduce the possible sequence numbers in the original N-phase winding to N', at least if the connections are also symmetrical.

The N-phase basic winding, which may be called the primitive winding, gives a flux distribution

$$B(x,t) = \sum_{r=1}^{N} \hat{I}_r B_c \sum_{m=1} F_{\alpha,m} \cos m (x - \sigma_r) \qquad (I.1)$$

where the N currents I_r have not been given any particular inter-relations apart from having common frequency.

Consequently they may be resolved into the N component systems given by their symmetrical components. For any given sequence, the resulting field is given by

$$B_s(x,t) = \sum_{r=1}^{N} B_c \hat{I}_s \cos(\omega t - \overline{r-1}\, s\tfrac{2\pi}{N}) \sum_{m=1}^{\infty} F_{\alpha,m} \cos m(x - \overline{r-1}\tfrac{2\pi}{N})$$

$$= B_c \hat{I}_s \sum_{m=1}^{\infty} \frac{F_{\alpha,m}}{2} (C_f \cos(\omega t - mx) + C_b \cos(\omega t + mx))$$

where $C_f = \dfrac{\sin(s - m)\pi}{\sin(s - m)\pi/N}$ (I.2)

$C_b = \dfrac{\sin(s + m)\pi}{\sin \pi/N}$

so that (120) can be written

$$B_s(x,t) = B_c \hat{I}_s \sum_{m=-\infty}^{+\infty} F_{\alpha,|m|} \frac{\sin(s + m)\pi}{\sin(s + m)\pi/N} \cos(\omega t + mx) \qquad (I.3)$$

This shows again that harmonics of the orders

$s + m = kN \qquad k = 0, \pm 1, \pm 2, \ldots\ldots$

i.e. $m = kN - s$ (I.4)

are possible.

It is apparent that in general, any value of m is possible, provided s can take all values from 1 to N - 1, the harmonic fields being travelling waves rotating in the forward or backward

/directions,

directions, or being pulsating, depending on the sequence numbers. This result is only a mathematical statement of the physically obvious result that a winding can be made to have any number of pole pairs. The windings which are symmetrically connected to form a 2p-pole winding will have a constraint on the sequence numbers so that the lowest possible value of m is p. That is, p = s or N - s, whichever is the lower. Clearly in the 2-pole case, s = 0, 1 .... therefore m = -1, for s = 1 etc.

In order to find the constraint of the sequence numbers resulting from a given interconnection of the coils, it is necessary to consider the transform relating the phase currents and their symmetrical components. This is in matrix form

$$\begin{bmatrix} I_1 \\ I_2 \\ \\ I_N \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \cdots\cdots & s_{1N} \\ s_{21} & s_{22} & \cdots\cdots & s_{2N} \\ \\ s_{N1} & \cdots\cdots & & s_{NN} \end{bmatrix} \begin{bmatrix} I'_0 \\ I'_1 \\ \\ I'_{N-1} \end{bmatrix} \qquad (I.5)$$

where the primes denote symmetrical components. The matrix $(s_{\alpha\beta})$ is defined by

$$s_{\alpha\beta} = e^{-j(\alpha - 1)(\beta - 1)\frac{2\pi}{N}} \qquad (I.6)$$

its basic element being $e^{-j2\pi/N}$, (i.e., the conjugate of the principal Nth root of unity).

/We

We can express (I.5) in index notation by the relation

$$I_\alpha = e^{-j(\alpha-1)(\beta-1)\frac{2\pi}{N}} I'_\beta \qquad (I.7)$$

where I_α are the phase currents, and I'_β their symmetrical components. β is related to the sequence number by $\beta = s + 1$. The matrix $\frac{1}{\sqrt{N}}(s_{\alpha\beta})$ is unitary, and its inverse ($(s_{\alpha\beta})$ being symmetric) is consequently $\frac{1}{N}(s^*_{\alpha\beta})$. The inverse of $(s_{\alpha\beta})$ follows from the relation

$$\frac{1}{\sqrt{N}}(s_{\alpha\beta}) \cdot \frac{1}{\sqrt{N}}(s^*_{\alpha\beta}) = E \quad \text{(unit matrix)}$$

$$\therefore (s_{\alpha\beta})^{-1} = \frac{1}{N}(s^*_{\alpha\beta}) \qquad (I.8)$$

The inverse of the transformation of (I.7) is then

$$I'_\beta = \frac{1}{N} e^{j(\alpha-1)(\beta-1)\frac{2\pi}{N}} I_\alpha \qquad (I.9)$$

or, since $\beta - 1 = s$,

$$I'_s = \frac{1}{N} e^{j(\alpha-1)s\frac{2\pi}{N}} I_\alpha \qquad (I.10)$$

After an interconnection of some of the coils, I_α will be related to N' new currents, say I_γ, through a singular transform

$$I_\alpha = (C_{\alpha\gamma}) I_\gamma \qquad (I.11)$$

The matrix $C_{\alpha\gamma}$ will have only unit elements (± 1) in general, if the connections are all series connections. Such a matrix is termed (by Kron, G.) a connection matrix. It is singular, since it has more rows than columns. Substituting in (27(a))

/we

we have

$$I'_s = \frac{1}{N} e^{j(\alpha - 1)s \frac{2\pi}{N}} C_{\alpha\gamma} I_\gamma \qquad (I.12)$$

Contracting with respect to α gives a new matrix, say R, such that

$$I'_s = \frac{1}{N} \left(r_{s\gamma} \right) I_\gamma \qquad (I.13)$$

The matrix R has in general s non-zero rows, so that all possible sequences are again obtainable. However, if the connections are symmetrical in some ways, several rows may be found to be zero. Thus only some of the original N sequence numbers will now be found possible.

As mentioned above, no attempt will be made here to establish the matrices $(R_{s\gamma})$ for the various classes of windings, but the Author hopes to resume this work at a later stage. It seems likely that pole-changing windings as well as fractional-slot windings could be profitably treated in this way. In order to show the suggestiveness and power of the method, a couple of examples are worked here.

Consider the connection of the coils into N' groups of q adjacent coils, i.e. N = qN'.

The corresponding connection matrix is of the form

$$
(C_{\alpha\gamma}) = \left[\begin{array}{l}
1,\ 0,\ 0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
1,\ 0,\ 0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
1,\ 0,\ 0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
0,\ 1,\ 0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
0,\ 1,\ 0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0,\ 1,\ 0,\ 0\ \cdots\cdots\cdots\cdots\ 0 \\
0,\ 0,\ 1,\ 0\ \cdots\cdots\cdots\cdots\ 0 \\
0,\ 0,\ 1,\ 0\ \cdots\cdots\cdots\cdots\ 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0,\ 0,\ 1,0\ \cdots\cdots\cdots\cdots\cdots\ 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots,0,1,0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots,0,1,0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots,0,1,0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots,0,0,1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots,0,0,1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\ 0,0,0,1
\end{array}\right] \qquad (I.14)
$$

or, more simply

$$
(C_{\alpha\gamma}) = \left[\begin{array}{lll}
u_q & \cdots & \cdots \\
\cdot\cdot & u_q & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
 & & u_q
\end{array}\right] \qquad (I.15)
$$

where u_q is a column vector containing only q units. (This kind of matrix is well known in the topological analysis of linear graphs. See for example Veblen: Analysis Situs, American Mathematical Society 1931, page 11).

The only non-zero elements are given by the relation

$$C_{\alpha\gamma} = 1 \quad \begin{cases} \alpha = (\gamma - 1)q + r \\ r = 1, 2, \ldots, q \end{cases}$$

Using this relation in (I.12) and contracting with respect to α, we obtain

$$I'_s = \frac{1}{N} \sum_{r=1}^{q} e^{j((\gamma - 1)q + r - 1)s\frac{2\pi}{N}} I_\gamma \qquad (I.16)$$

$$\therefore I'_s = \frac{q}{N} e^{j(q-1)s\frac{\pi}{N}} \frac{\sin qs\pi/N}{q\sin s\pi/N} e^{j(\gamma - 1)s\frac{2\pi}{N'}} I_\gamma \qquad (I.17)$$

The factor $e^{j(\gamma - 1)s\frac{2\pi}{N'}}$ will be recognised as the factor corresponding to the matrix in (I.10) and we have again the factor $\frac{\sin qs\pi/N}{q\sin s\pi/N}$ which will be shown to be a generalisation of the distribution factor. We note that $I'_s = 0$ if s is a multiple of N/q., i.e., (q - 1) sequences are eliminated by the interconnection.

Since m = kN + s, we may write the above factor as

$$\frac{\sin qs\pi/N}{q\sin s\pi/N} = \frac{\sin(mq-kNq)\pi/N}{q\sin(m-kN)\pi/N} = \frac{\sin mq\pi/N}{q\sin m\pi/N} (-1)^{kq - 1} \qquad (I.18)$$

which is numerically equal to the normal distribution factor. This equivalence shows that all the harmonics generated by a particular sequence excitation have equal distribution factors. This interesting feature may not, however, be of much practical use, since in general there are several sequences in the primitive set corresponding to a given sequence in the supply system.

/In

In order to obtain the most general expression for the N sequence currents in the integral-slot winding, a second constraint is necessary, corresponding to the interconnection of the N' groups into a 2p-pole, N"-phase winding. The connections are such that p groups (equidistant) are connected together in the same sense. Thus, in the column I_γ

$$\begin{aligned} I_1 &= I_{1+N''} = I_{1+2N''} = \ldots\ldots = I_{1+\overline{p-1}N''} \\ I_2 &= I_{2+N''} = I_{2+2N''} = \ldots\ldots \quad I_{2+\overline{p-1}N''} \end{aligned} \qquad (I.19)$$

The required connection matrix $(C_{\gamma\delta})$ is therefore given by

$$C_{\gamma\delta} = \begin{cases} 1 \ , & \gamma = \delta + kN'' \\ 0 \ , & \gamma = \delta + kN'' \end{cases} \qquad k = 0,\ 1,\ \ldots.,\ p-1$$

and substituting in equation (I.17) we have

$$I'_s = \frac{q}{N}\, e^{j\,\overline{q-1}\, s\frac{\pi}{N}}\ \frac{\sin qs\pi/N}{q\sin s\pi/N} \sum_{k=0}^{p-1} e^{j(\delta + kN'' - 1)s\frac{2\pi}{N}}\, I_\delta$$

and summing over all k there results

$$I'_s = \frac{q}{N}\, e^{j\,\overline{q-1}\, s\frac{\pi}{N}}\ \frac{\sin qs\pi/N}{\sin s\ \pi/N}\ \frac{\sin s\pi}{\sin s\pi/p}\ e^{j\overline{p-1}\,\frac{s\pi}{p}}\ e^{j\,\overline{\delta-1}\, s\frac{2\pi}{N'}}\, I_\delta$$

(I.20)

Here the factor $\frac{\sin qs\pi/N}{\sin s\ \pi/p} = \begin{cases} P, \text{ when } s = p \\ 0, \text{ when } s \neq p \end{cases}$

$\sigma = 0,\ 1,\ 2,\ \ldots.,\ N''q-1$

and the factor $\frac{\sin qs\pi/N}{\sin s\pi/N} = N$ except when s is a multiple of N/q. (or N"p).

Again by (I.4) we may now define the harmonic spectrum

/by

by the equation

$$m = (kN''q + \sigma)p \qquad (I.21)$$

where σ is not a multiple of N''.

Clearly there are q-1 sequence numbers excluded by the last statement, so that there are $N''q - 1 - (q - 1) = (N'' - 1)q$ possible values of σ in (I.21).

The distribution factors for the values of s given by σp becomes

$$f_{\beta,\sigma} = \frac{\sin q\sigma p\pi/N}{q\sin\sigma p\pi/N} = \frac{\sin\sigma\pi/N''}{q\sin\sigma\pi/N''q} \qquad (I.22)$$

This has an interesting consequence: since σ may only take $(N'' - 1)q$ different values there may only be $(N'' - 1)q$ distinct distribution factors in such a winding.

Substituting σp throughout (I.20) for s we have

$$I'_{\sigma p} = \frac{qp}{N}(-1)^{\overline{p-1}\sigma} e^{j\frac{q-1}{q}\sigma\frac{\pi}{N''}} \cdot f_{\beta,\sigma} e^{j\overline{\delta-1}\frac{2\pi}{N''}} I_{\delta} \qquad (I.23)$$

Again we may substitute for I_{δ} its symmetrical components, given by

$$I_{\delta} = e^{-j(\delta-1)s''\frac{2\pi}{N''}} I_{s''} \qquad (I.24)$$

so that

$$I'_{\sigma p} = \frac{qp}{N}(-1)^{(p-1)\sigma} e^{j\frac{q-1}{q}\frac{\pi}{N''}} f_{\beta\sigma} \sum_{\delta=1}^{N''} e^{-j(\delta-1)(s''-\sigma)\frac{2\pi}{N''}} I_{s''}$$

and summing over all δ we have

$$I'_{\sigma p} = \frac{qp}{N} (-1)^{\overline{p-1}\,\sigma} f_{\beta,\sigma} \frac{\sin(s''-\sigma)\pi}{\sin(s''-\sigma)\pi/N''} .e^{j\frac{q-1}{q}\sigma\frac{\pi}{N''}}$$

$$\times e^{-j(N'' - 1)(s'' - \sigma)\pi/N''} \quad I_{s''} \qquad (I.25)$$

Here the factor

$$\frac{\sin(s'' - \sigma)\pi}{\sin(s'' - \sigma)\pi/N''} = \begin{cases} N'', & s'' - \sigma = kN'' \\ 0, & s'' - \sigma \neq kN'' \end{cases} \quad k = 0, \pm 1 \ldots. \pm q - 1$$

For a given sequence s" in the N"-phase excitation system, the possible values of σ, determining the sequence numbers σp in the primitive system are therefore given by

$$\sigma = s'' + kN'', \quad k = 0, 1, \ldots.., q - 1$$

Since $s'' < N''$, this equation automatically excludes multiples of N" for σ except when s" is zero. The only possible value of σ for s" = 0 is therefore also 0. Other values of s" gives rise to q values of σ. A winding containing q coils per phase per pole has, therefore, only q distinct distribution factors appropriate to each type of sequence excitation. This conclusion is easily verified by examples. It may be that in this exposition these conclusions seem pointless, but it appears that this kind of information may be of great interest when dealing with more complex types of windings, and this kind of structural analysis of the windings ought at least to be carried out. The Author hopes to do this at a later stage.

/Another

Another example of this treatment is the relation between the fluxes produced by N'-phase windings derived from a primitive 2N'-phase windings. These windings have already been considered in Section 2.4.2., and it suffices to take up the analysis from Equation (17), where we may write N = 2N'. The connection matrix relating the 2N' currents in the primitive winding to the N' supply currents is

$$C_{\alpha\gamma} \quad \begin{array}{r} N'\left\{\vphantom{\begin{matrix}1\\0\\0\\.\\.\\\cdots\\\cdots\\0\\\cdots\end{matrix}}\right. \\ \vphantom{\begin{matrix}-1\\0\\.\\.\\.\\\cdots\\\cdots\\0\end{matrix}} \end{array} \left[\begin{matrix} 1 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 1 & 0 & \\ . & 0 & 0 & \text{etc.} \\ . & 0 & 1 & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ 0 & 0 & . & \\ \cdots & \cdots & \cdots & \\ -1 & 0 & - & \\ 0 & 0 & 0 & \\ . & -1 & 0 & \\ . & 0 & 0 & \\ . & 0 & -1 & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \end{matrix}\right] \qquad (I.26)$$

The scheme is contained in the statement

$$C_{\alpha\gamma} \quad \begin{cases} = 1 \text{ when } \alpha = (2\gamma - 1) \bmod N \\ = -1 \text{ when } \alpha = (2\gamma - 1 + N') \bmod N \end{cases}$$

Substituting this in (29) we have

$$I'_s = \frac{1}{N}\left[e^{j((2\gamma-1)\bmod N-1)s\frac{2\pi}{N}} - e^{j((2\gamma-1+N')\bmod N-1)s\frac{2\pi}{N}}\right] I_\gamma$$

Since the modulus N restriction only amounts to the subtraction

/of

of 2π or multiples from the argument we can relax this, and the expression becomes

$$I'_s = \frac{1}{N} \, e^{j\,\overline{2\gamma - 1}\, s\frac{2\pi}{N}} \left[1 - e^{jN's\,\frac{2\pi}{N}}\right] I_\gamma$$

$$\therefore \; I'_s = \frac{1}{N} \, e^{j(\gamma - 1)s\frac{2\pi}{N'}} \left[1 - e^{js\pi}\right] I_\gamma \qquad (I.27)$$

The factor $\left[1 - e^{js\pi}\right]$ vanishes for all even values of s. This eliminates the zero sequence together with the 2nd and 4th in the 6-phase, 3-phase connected winding in agreement with the results obtained in Section 2.4.2.

Were the coils connected in the same sense, we should have the factor $\left[1 + e^{js\pi}\right]$ and thereby eliminating all odd values of s, and we find that all the harmonics are now even, thus effectively doubling the number of poles in the winding. This covers the cases of 1 - 2 ratio pole-changing windings.

It is to be noted that the above algebra is only pertinent when N' is an odd number. If N' is even, the parent 2N'-phase winding cannot be connected in the same fashion, and with the exception of the 4-phase, 2-phase connected windings, these have not much interest.

APPENDIX II

The reduction to canonical form of circulant matrices

Consider the matrix L given by Equation (30), Section 2.5.3. We seek a matrix S such that $S^{-1}LS$ is a diagonal matrix. Since S must be non-singular, it follows that all the columns (or rows) of S must be linearly independent. It is convenient to use the index notation, and we write

$$S = S^{(\beta)} = \left\{S^{(1)},\ S^{(2)},\ \ldots\ldots,\ S^{(N)}\right\} \tag{II.1}$$

where $S^{(\beta)}$ are the column vectors of S.

Since $S^{-1}LS = \Lambda$, we have

$$LS = S\Lambda \tag{II.2}$$

or

$$LS^{(\beta)} = S^{(\beta)}\Lambda_\beta \tag{II.3}$$

where Λ_β means the βth row (or column) of Λ. Since Λ is diagonal, we have with

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}, \qquad \Lambda_\beta = \begin{bmatrix} \vdots \\ \lambda_\beta \\ \vdots \end{bmatrix}$$

Thus having found the values of λ_β, equations (II.3) suffices to determine the column vectors $S^{(\beta)}$.

By (II.3) it follows that

$$(L - U\lambda_\beta)S^{(\beta)} = 0 \tag{II.4}$$

where U is the unit matrix.

/Consequently,

Consequently, if there is a non-trivial solution for $S^{(\beta)}$,

$$\left| L - U\lambda_\beta \right| = 0 \qquad \text{(II.5)}$$

This is, of course, the characteristic equation of the matrix L, and λ_β is a latent root. The equation is of order N in λ_β and has therefore N roots, some of which may be equal or repeated. (In general, if there are repeated roots, it may not be possible to find N independent solutions for $S^{(\beta)}$, and the required non-singular transform may not exist. This eventuality is of considerable mathematical interest, but need not concern us here). We now drop the suffix β in λ_β and will endeavour to find the roots of the equation

$$\left| L - U\lambda \right| = 0 \qquad \text{(II.5(a))}$$

where L is the circulant matrix given by (30).

Thus we have

$$\begin{vmatrix} l_1 - \lambda, & l_2, & \ldots\ldots\ldots, & l_N \\ l_N, & l_1 - \lambda, & \ldots\ldots, & l_{N-1} \\ & & & \\ l_2, & l_3, & \ldots\ldots\ldots, & l_1 - \lambda \end{vmatrix} = 0 \qquad \text{(II.6)}$$

which again is a circulant determinant. This determinant can be factorised into linear factors by a rather elegant method as follows:-

Let ω_r be an Nth root of unity, so that $\omega_r = e^{j\frac{2\pi}{N}(r-1)}$ and construct the matrix

$$C_r = \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ \cdots & \omega_r & \cdots & \cdots \\ \cdots & \cdots & \omega_r^2 & \cdots \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ \cdots & \cdots & \cdots & \omega_r^{N-1} \end{bmatrix} \qquad \text{(II.7)}$$

Clearly, $$C_r^{-1} = \begin{bmatrix} 1 & & & & \\ & \omega_r^{-1} & & & \\ & & \omega_r^{-2} & & \\ & & & \ddots & \\ & & & & \omega_r^{1-N} \end{bmatrix} \qquad \text{(II.8)}$$

Furthermore $\left| C_r \; A \; C_r^{-1} \right| = \left| A \right|$, so that any factor of the R.H.S. is also a factor of the L.H.S. and conversely. We now construct

$$\left| C_r \; (L - \lambda U) \; C_r^{-1} \right| = \left| L - \lambda U \right| \qquad \text{(II.9)}$$

To facilitate the forming of the matrix product we make use of the index notation. Thus with

$$C_r \equiv \omega_r^{(\alpha - 1)} \delta_{\alpha\beta} \qquad (L - \lambda U) \equiv a_{\beta\gamma}$$

and $C_r^{-1} \equiv \delta_{\gamma\varepsilon} \omega_r^{(1-\varepsilon)}$, we have

$$C_r \; (L - \lambda U) C_r^{-1} \equiv \omega_r^{\alpha - 1} \delta_{\alpha\beta} \; a_{\beta\gamma} \delta_{\gamma\varepsilon} \; \omega^{(1-\varepsilon)}$$

$$\equiv \omega_r^{(\alpha - \varepsilon)} \; a_{\alpha\varepsilon}$$

Consequently, we have

$$\left| L - \lambda U \right| = \begin{vmatrix} l_1 - \lambda, & l_2 \omega_r^{N-1}, & \cdots\cdots, & l_N \; \omega_r \\ l_N \; \omega_r \; , & l_1 - \lambda \; , & \cdots\cdots, & l_{N-1} \; \omega_r^2 \\ \cdots\cdots & & & \\ \cdots\cdots & & & \\ \cdots\cdots & & & \\ l_2 \omega^{N-1}, & l_3 \omega^{N-2}, & \cdots\cdots, & l_1 - \lambda \end{vmatrix} \qquad \text{(II.9)}$$

where use is made of the fact that $\omega_r^m = \omega_r^{N+m}$ in order to make all the exponents of ω_r positive. It is now clear that the

/sum

sum of all the elements in each column are equal, whence this sum is a factor of the determinant; furthermore this is so whatever r we choose in the range, $1 \leqslant r \leqslant N$, consequently any of the N factors,

$$l_1 + \omega_r^{N-1} l_2 + \ldots\ldots + \omega_r l_N - \lambda\ , \quad r = 1, 2, \ldots, N$$

are factors of $|L - \lambda U|$. Finally, since these N factors are distinct, and $|L - \lambda U|$ is a polynomial of degree N in λ, these are all the factors. Consequently, the solution of (II.5) is

$$\lambda_\beta = \sum_{r=1}^{N} l_r\, \omega_\beta^{N-(r-1)} \qquad \text{(II.10)}$$

or

$$\lambda_\beta = \sum_{r=1}^{N} l_r\, e^{-j\frac{2\pi}{N}(\beta-1)(r-1)} \qquad \text{(II.10(a))}$$

Substituting in (II.4), we have

$$\left\{L - U \sum_{r=1}^{N} l_r\, e^{-j\frac{2\pi}{N}(\beta-1)(r-1)}\right\} S^{(\beta)} = 0 \qquad \text{(II.4(a))}$$

which when written out in full becomes

$$\begin{bmatrix} -\sum_{r=2}^{N} l_r \omega_\beta^{1-r}, & l_2, & \ldots\ldots\ldots, & l_N \\ l_N, & -\sum_{r=1}^{N} l_r \omega_\beta^{1-r}, & \ldots\ldots, & l_{N-1} \\ \ldots\ldots\ldots & & & \\ \ldots\ldots\ldots & & & \\ l_2, & l_3, & \ldots\ldots\ldots\ldots\ldots, & -\sum_{r=1}^{N} l_r \omega_\beta^{1-r} \end{bmatrix} \begin{bmatrix} S_1^{(\beta)} \\ S_2^{(\beta)} \\ \\ \\ S_N^{(\beta)} \end{bmatrix} = 0 \qquad \text{(II.11)}$$

A solution of these equations must satisfy the relations

$$S_1^{(\beta)} = \omega_\beta S_2^{(\beta)} = \omega_\beta^2 S_3^{(\beta)} = \ldots\ldots = \omega_\beta^{N-1} S_N^{(\beta)}$$

The simplest solution is given by

$$S^{(\beta)} = \begin{bmatrix} 1 \\ \omega_\beta^{-1} \\ \omega_\beta^{-2} \\ \cdots\cdots \\ \cdots\cdots \\ \omega_\beta^{-N+1} \end{bmatrix} \qquad \text{(II.12)}$$

These column vectors have modulus $\sqrt{N}$, and are quasi-unitary.

The transform matrix given by the N columns in (II.12)

($\beta = 1, 2, \ldots N$) is the matrix S, where

$$S_\alpha^\beta = \omega_\beta^{-(\alpha-1)} = e^{-j(\alpha-1)(\beta-1)\frac{2\pi}{N}}$$

which is, of course, the familiar symmetrical component transform. By its quasi-unitary property, the inverse is easily shown to be:

$$S^{-1} = \frac{1}{N} \quad e^{j(\alpha-1)(\beta-1)\frac{2\pi}{N}} \qquad \text{(II.14)}$$

The above procedure is quite general, and applies to all circulant matrices.

If the matrix in addition to circulant symmetry is also persymmetric (or simply symmetric), we have the addtional property that

$$l_r = l_{N-r+2}$$

It can be shown that by applying this constraint to (II.10(a))

/that

that the imaginary part vanishes; the proof is, however, a bit clumsy. A more general proof follows from the general theorem found in any book on Advanced Algebra that the latent roots of a real symmetric matrix are necessarily real.
(II.10(a)) thus becomes

$$\lambda_\beta = \sum_{r=1}^{N} l_r \cos \frac{2\pi}{N} (\beta - 1)(N - r + 1)$$

$$\therefore \lambda_\beta = \sum_{r=1}^{N} l_r \cos \frac{2\pi}{N} (\beta - 1)(r - 1) \qquad \text{(II.10(c))}$$

This equation is of importance in deriving the sequence inductances of a polyphase winding.

In the case of symmetric circulants, the unitary transformation of symmetrical components is still applicable and does, in fact, diagonalise the system. However, in the solution of (II.11) it is now apparent that both the real and imaginary parts of $S^{(\beta)}$ given by (II.12) are independent and, in fact, orthogonal solutions of (II.11). This would suggest that in this case a real basis can be found for the matrix L. But since there are 2(N - 1) pairs of real and imaginary (plus one unit column) corresponding to the set of N columns given by (II.12), some of these must be linearly dependent. The real parts are given by

$$\operatorname{Re} S_\alpha^\beta = \cos(\beta - 1)(\alpha - 1)\frac{2\pi}{N} \qquad \text{(II.15)}$$

and the imaginary parts

$$\operatorname{Im} S_\alpha^\beta = -\sin(\beta - 1)(\alpha - 1)\frac{2\pi}{N} \qquad \text{(II.16)}$$

/To

To find which of the real part columns are independent, we merely apply the orthogonality test:

$$S_\alpha^{\beta_1} \; S_\alpha^{\beta_2} = \sum_{\alpha=1}^{N} \cos(\beta_1 - 1)(\beta_2 - 1)\frac{2\pi}{N} \cos(\beta_2 - 1)(\alpha - 1)\frac{2\pi}{N}$$

$$= \frac{1}{2} \sum_{\alpha=1}^{N} \left(\cos(\beta_1 + \beta_2 - 2)(\alpha - 1)\frac{2\pi}{N} + \cos(\beta_1 - \beta_2)(\alpha - 1)\frac{2\pi}{N} \right)$$

$$= \frac{1}{2} \left\{ \frac{\sin(\beta_1 + \beta_2 - 2)\pi}{\sin(\beta_1 + \beta_2 - 2)\pi/N} \cos(\beta_1 + \beta_2 - 2)(N-1)\frac{\pi}{N} \right.$$

$$\left. + \frac{\sin(\beta_1 - \beta_2)\pi}{\sin(\beta_1 - \beta_2)\pi/N} \cos(\beta_1 - \beta_2)(N - 1)\frac{\pi}{N} \right\} \qquad \text{(II.17)}$$

Since $\beta \leq N$, it is easily seen that the second factor is non-zero only if $\beta_1 = \beta_2$, which is trivial, and the first factor is non-zero only if $\beta_1 + \beta_2 - 2 = N$. By utilising these criteria we have

$$S_\alpha^{\beta_1} \; S_\alpha^{\beta_2} = \tfrac{1}{2}N \cos(N - 1)\pi \text{ where } \beta_1 + \beta_2 = N + 2$$

and $S_\alpha^{\beta_1} \; S_\alpha^{\beta_2} = 0$ where $\beta_1 + \beta_2 \neq N + 2$.

Thus the columns determined by

$$\beta_1 + \beta_2 \neq N + 2 \qquad \text{(II.18)}$$

are linearly independent, and represent eigenvectors of the symmetrical circulant matrix.

By a similar test applied to the columns given by (II.16) we obtain the identical criterion.

/Clearly

Clearly then if the β first columns are considered, (β = 1, 2, ..., b) these are all linearly independent if $2b + 1 < N + 2$, i.e., $b < \left[\frac{N+1}{2}\right]_i$; where $\left[\frac{N+1}{2}\right]_i$ is the integral part of $\frac{N+1}{2}$. Thus the $\left[\frac{N+1}{2}\right]_i$ first of the columns given by (II.15) and (II.16) are linearly independent. It is, of course, equally valid to say that the set of columns corresponding to $\beta > \left[\frac{N+1}{2}\right]_i$ is also a set of linearly independent solutions. A very simple example of this principle is furnished by the problem of finding the possible eigenvector representations for the 3-phase inductance matrix. This is, of course, a symmetrical circulant, and the well known symmetrical co-ordinate axis determined by the transform matrix,

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

It is here obvious even without the help of the criterions mentioned above that the only real independent columns are given by

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \sqrt{3}/2 \\ 1 & -\frac{1}{2} & -\sqrt{3}/2 \end{bmatrix}$$

This is, of course, the transform matrix generally known as the Clarke transform or α, β, 0-sequence transform. Thus there cannot be any other transforms which will diagonalise the given inductance matrix, which accounts

/for

for the fact that no others have been suggested.

It is of interest to note that none of these transforms were originally developed by the method presented here. The present Author is of the opinion that this method is the only natural way of dealing with systems of this kind, and that the elementary algebra required is not beyond any graduate students. Consequently, it is advocated that matrix algebra be included in Engineering Mathematics courses to the benefit of all who will have occasion to deal with systems of differential equations. The powerful method of eigenvector representation such as the above should not be withheld from Engineers any longer, considering it has been a treasured tool of mathematical physicists for decades.

Plea — hardly in place here tho' I consider [illegible] [illegible]

Appendix III

A critical note on the use of synchronous reactance

The term synchronous reactance (or impedance) is normally associated with the positive sequence reactance per phase corresponding to the fundamental space component of flux. In practice, however, it appears that when this is stated as "synchronous reactance per phase", sufficient care is not always taken to distinguish between the quantities associated with the actual phase voltages and currents and the purely fictitious symmetrical components. In fact, the "reactances" corresponding to the symmetrical components are often inappropriately used in equations involving actual phase voltages and currents.

As an example we may quote two papers by Y.H. Ku[22,23], which illustrate this point. In one of the first papers[22] treating the synchronous machine by operational methods we find:

"The stator equations for the synchronous, round-rotor machine are

$$v_a = (R + Lp)i_a + p(M \cos\theta \, i_f)$$

$$v_b = (R + Lp)i_b + p(M\cos(\theta - 2\pi/3)i_f)$$

$$v_c = (R + Lp)i_c + p(M\cos(\theta - 4\pi/3)i_f)$$

where R is the resistance per phase, L the self-synchronous

/inductance

inductance per phase, and M the mutual inductance between the field and a stator phase when the axes of these are coincident".

We must reject these equations as false, since the basic equations are clearly,

$$v_a = (R + Lp)i_a + M_{ab}\, pi_b + M_{ac} pi_c + p(M \cos\theta\, i_f)$$

$$v_b = (M_{ba}\, i_a + (R + Lp)i_b + M_{bc}\, pi_c + p(M \cos(\theta - 2\pi/3)i_f)$$

$$v_c = (M_{ca}\, i_a + M_{cb}\, i_b + (R + Lp)i_c + p(M \cos(\theta - 4\pi/3)i_f)$$

where L is the total inductance per phase, M_{ab} the mutual inductance between phases a and b etc, and R and M are defined as above. By neglecting all the harmonics in the space distribution, we may write these (by):

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = (R + lp) \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + L_{ph} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} p \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + pM \begin{bmatrix} \cos\theta \\ \cos(\theta - 2\pi/3) \\ \cos(\theta - 4\pi/3) \end{bmatrix} i_f$$

where l is the leakage reactance per phase.

By no algebraic substitution is it possible to obtain the equations given by Y.H. Ku. However, using the symmetrical component transformation we obtain the simpler equations:

$$v_0 = (R + L_0 p)i_0$$

$$v_1 = (R + L_1 p)i_1 + 3/2\, Mpe^{j\theta}\ i_f$$

$$v_2 = (R + L_1 p)i_2 + 3/2\, Mpe^{-j\theta}\ i_f$$

/The

The constant speed case (synchronous speed) results in $\theta = \omega t + \delta$ where ω is the angular frequency of v_a v_b and v_c (and i_a i_b and i_c) and furthermore, if the terminal voltages are balanced, $v_0 = v_2 = 0$. We have then, assuming $v_1 = \sqrt{2}\, V_1 e^{j\omega t}$ and $i_1 = 2\sqrt{I_1} e^{j\omega t + \phi}$

$$V_1 = (R + j\omega L_1)\, I_1 e^{j\phi} + 3/2 \quad 2\, j\omega M e^{j\delta} \ i_f$$

or $$V_1 = (R + j\omega L_1)\, I_1 e^{j\phi} + E_1 e^{j\delta}$$

which is the familiar balanced synchronous operation equation.

We must therefore maintain that the quantity "self-synchronous inductance per phase" is a misnomer: it does in fact not exist as a "reactance per phase" at all, but in the positive (or negative) sequence equivalent network. It is an abstracted value, and should not be used in any but the abstracted equations.

It is obvious that the fact that the positive sequence components are proportional to the phase components, and identical in the reference phase for a balanced system accounts for the identification of "per phase" and symmetrical component reactances. There is, however, a clear difference between them, and this difference ought to be pointed out very strongly.

Similar remarks apply to the α, β, 0 sequence system. The α-component is equal to the a-phase quantity when there is complete symmetry, and is therefore often confused with

/phase

phase quantities.

Fortunately - or maybe unfortunately - the constants of the equations are always determined experimentally, so that the inconsistency is to some extent obviated. For example, Y.H. Ku's equations will work in practice for symmetric operation, since the factor 3/2 which is neglected is absorbed in M.

APPENDIX IV

General Expressions for $\cos^m x$ in terms of multiple angle agreements

Two cases arise in this expansion, namely odd and even values of m.

Consider firstly, the case where m is odd. Writing m as 2m-1 we have

$$\cos^{2m-1} x = \frac{1}{2^{2m-1}} (e^{ix} + e^{-ix})^{2m-1}$$

and expanding the bracket,

$$= \frac{1}{2^{2m-1}} \; {}^{2m-1}C_o \, e^{i(2m-1)x} + {}^{2m-1}C_1 \, e^{i(2m-3)x} +$$

$$+ {}^{2m-1}C_r \, e^{i(2m-2r-1)x} + \ldots + {}^{2m-1}C_{m-1} \, e^{ix} + {}^{2m-1}C_m \, e^{-ix} + \ldots$$

$$+ {}^{2m-1}C_{2m-r-1} \, e^{-i(2m-2r-1)x} + \ldots \; {}^{2m-1}C_{2m-1} \, e^{-i(2m-1)x}$$

$$\cos^{2m-1} x = \frac{2}{2^{2m-1}} \sum_{r=1}^{m} {}^{2m-1}C_{m-r} \cos \overline{2r-1}x \qquad \text{(IV.1)}$$

In the second case, where m is even, m is written as 2m, giving

$$\cos^{2m} x = \frac{1}{2^{2m}} (e^{ix} + e^{-ix})^{2m}$$

$$= \frac{1}{2^{2m}} ({}^{2m}C_o \, e^{i2mx} + {}^{2m}C_1 \, e^{i(2m-2)x} + \ldots.$$

$$+ {}^{2m}C_r \, e^{i(2m-2r)x} + \ldots + {}^{2m}C_{m-1} \, e^{i2x} + {}^{2m}C_m + {}^{2m}C_{m+1} \, e^{-i2x} +$$

$$..+ {}^{2m}C_{2m-r} \, e^{-i(2m-2r)x} + \ldots + {}^{2m}C_{2m} \, e^{-i2mx}$$

$$\therefore \cos^{2m} x = \frac{1}{2^{2m}} \left[{}^{2m}C_m + 2 \sum_{r=1}^{m} {}^{2m}C_{m-r} \cos 2rx \right] \qquad \text{(IV.2)}$$

APPENDIX V

The expansion of $\begin{pmatrix}\text{sinnx}\\ \text{cosnx}\end{pmatrix}(1 - k\cos x)^{-1}$ in terms of multiple angle arguments

The binomial expansion of $(1 - k\cos x)^{-1}$ gives,

$$(1 - k\cos x)^{-1} = 1 + k\cos x + k^2 \cos^2 x + \ldots\ldots\ldots$$

$$= 1 + \sum_{m=1}^{\infty} (k^{2m} \cos^{2m} x + k^{2m-1} \cos^{2m-1} x)$$

By the results of Appendix IV, this becomes

$$= 1 \sum_{m=1}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_m$$

$$+ 2 \sum_{m=1}^{\infty} \left(\frac{k}{2}\right)^{2m} \sum_{r=1}^{m} {}^{2m}C_{m-r} \cos 2rx$$

$$+ 2 \sum_{m=1}^{\infty} \left(\frac{k}{2}\right)^{2m-1} \sum_{r=1}^{m} {}^{2m-1}C_{m-r} \cos \overline{2r - 1}x$$

$$= \sum_{m=0}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_m + 2 \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_{m-r} \cos 2rx$$

$$+ 2 \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} \left(\frac{k}{2}\right)^{2m-1} {}^{2m-1}C_{m-r} \cos \overline{2r - 1}x \quad (V.1)$$

Now the m-summations may be carried out independent of r in the first term, and for any value of r in the cosine coefficients.

Thus

$$\sum_{m=0}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_m = \sum_{m=0}^{\infty} \left(\frac{k}{2}\right)^{2m} \frac{\Gamma(2m + 1)}{\Gamma(m + 1)\,\Gamma(m + 1)}$$

/Expanding

Expanding $\Gamma(2m + 1)$ by the gamma-function duplication formula gives

$$\sum_{m=o}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_m = \frac{1}{\sqrt{\pi}} \sum_{m=o}^{\infty} \frac{\Gamma(m+\frac{1}{2})\Gamma(m+1)}{\Gamma(m+1)\Gamma(m+1)} \; k^{2m}$$

$$= \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} \; F\left(\tfrac{1}{2}, 1, 1; k^2\right)$$

where $F(\frac{1}{2}, 1, 1; k^2)$ is the hypergeometric function of elements $\frac{1}{2}, 1, 1; k^2$. By a well known change of the elements this reduces to

$$(1 - k^2)^{-\frac{1}{2}} \; F\left(\tfrac{1}{2}, 0, 1; k^2\right) = (1 - k^2)^{-\frac{1}{2}}$$

Thus

$$\sum_{m=o}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_m = (1 - k^2)^{-\frac{1}{2}} \qquad (V.2)$$

Again,

$$\sum_{m=r}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_{m-r} = \left(\frac{k}{2}\right)^{2r} \sum_{m=o}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m+2r}C_m$$

$$= \left(\frac{k}{2}\right)^{2r} \sum_{m=o}^{\infty} \frac{\Gamma(2m + 2r + 1)}{\Gamma(m + 2r + 1)\,\Gamma(m + 1)} \left(\frac{k}{2}\right)^{2m}$$

$$= \left(\frac{k}{2}\right)^{2r} \sum_{m=o}^{\infty} \frac{\Gamma(m + r + \frac{1}{2})\,\Gamma(m + r + 1)\; 2^{2m+2r}}{\Gamma(m + 2r + 1)\,\Gamma(m + 1)\;\sqrt{\pi}} \left(\frac{k}{2}\right)^{2m}$$

$$= \left(\frac{k}{2}\right)^{2r} \frac{2^{2r}}{\sqrt{\pi}} \; \frac{\Gamma(r + \frac{1}{2})\,\Gamma(r + 1)}{\Gamma(2r + 1)} \; F\left(r+\tfrac{1}{2}, r+1, 2r+1; k^2\right)$$

$$= \left(\frac{k}{2}\right)^{2r} \frac{2^{2r}}{\sqrt{\pi}} \; \frac{\Gamma(r + \frac{1}{2})\,\Gamma(r + 1)\,\sqrt{\pi}}{\Gamma(r + \frac{1}{2})\,\Gamma(r + 1)\; 2^{2r}} \; (1-k^2)^{-\frac{1}{2}} \; F\left(r+\tfrac{1}{2}, r, 2r+1; k^2\right)$$

$$= \left(\frac{k}{2}\right)^{2r} (1 - k^2)^{-\frac{1}{2}} \; F\left(r, r+\tfrac{1}{2}, 2r + 1; k^2\right)$$

/Now

Now[24]

$$F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}; z) = \left(\frac{2}{1+\sqrt{1-z}}\right)^2 F\left(2\alpha, \alpha-\beta+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

Thus the above reduces to

$$= \left(\frac{k}{2}\right)^{2r} (1-k^2)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-k^2}}\right)^{2r} F\left(2r, 0, 2r+1; \frac{\sqrt{1-k^2}-1}{\sqrt{1-k^2}+1}\right)$$

and finally,

$$\sum_{m=r}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m}C_{m-r} = (1-k^2)^{-\frac{1}{2}} \left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2r} \qquad (V.3)$$

By similar process,

$$\sum_{m=r}^{\infty} \left(\frac{k}{2}\right)^{2m} {}^{2m-1}C_{m-r} = (1-k^2)^{-\frac{1}{2}} \left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2r-1} \qquad (V.4)$$

substituting V.2, 3, and 4 in V.1 gives

$$(1-k\cos x)^{-\frac{1}{2}} = (1-k^2)^{-\frac{1}{2}} \left(1 + 2\sum_{p=1}^{\infty} \left(\frac{k}{1+\sqrt{1-k^2}}\right)^p \cos px\right) \qquad (V.5)$$

$$= (1-k^2)^{-\frac{1}{2}} \left(1 + 2\sum_{p=1}^{\infty} y^p \cos px\right) \qquad (V.5(a))$$

where $y = \dfrac{k}{1+\sqrt{1-k^2}}$.

Multiplying (V.5(a)) by sin nx gives

$$\sin nx\,(1-k\cos x)^{-1}$$

$$= (1-k^2)^{-\frac{1}{2}} \left(\sin nx + \sum_{p=1}^{\infty} y^p \left(\sin(n+p)x + \sin(n-p)x\right)\right)$$

$$= (1 - k^2)^{-\frac{1}{2}} \; (\sin nx + \sum_{q=n+1}^{\infty} y^{q-n} \sin qx$$

$$+ \sum_{q=1}^{n-1} y^{n-q} \sin qx$$

$$- \sum_{q=1}^{\infty} y^{q+n} \sin qx$$

$$= (1 - k^2)^{-\frac{1}{2}} \left\{ \sin nx + \sum_{q=n+1}^{\infty} (y^{-n} - y^n)\, y^q \sin qx \right.$$

$$\left. + \sum_{q=1}^{n-1} \left(y^n (y^{-q} + y^q) \sin qx - y^{2n} \sin nx \right) \right\}$$

$$= (1 - k^2)^{-\frac{1}{2}} \left\{ \sum_{q=1}^{n} y^n (y^{-q} + y^q) \sin qx \right.$$

$$\left. + \sum_{q=n+1}^{\infty} (y^{-n} - y^n)\, y^q \sin qx \right\}$$

Thus,

$$\sin nx \, (1 - k \cos x)^{-1} = (1 - k^2)^{-\frac{1}{2}} \sum_{q=1}^{\infty} (y^{|q-n|} - y^{q+n}) \sin qx \qquad (V.6)$$

where $|q-n|$ is the numerical value of q-n.

Similarly, $\cos nx \, (1 - k\cos x)^{-1}$

$$= (1 - k^2)^{-\frac{1}{2}} \; (\cos nx + \sum_{p=1}^{\infty} y^p (\cos(n + p) x + \cos(n - p)x))$$

which by a process similar to the above gives

$$\cos nx(1-k\cos x)^{-1} = (1-k^2)^{-\frac{1}{2}} \; (y^n + \sum_{q=1}^{\infty} (y^{|q-n|} + y^{q+n}) \cos qx) \qquad (V.7).$$