PION NUCLEON SCATTERING

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Part 1.

1. Introduction.

The study of the scattering of two particles is generally the first and most straightforward method of obtaining data on the interaction between the two fields. The scattering of π mesons from nucleons may be regarded as the most fundamental of the mesonic processes, and many physicists have worked on this problem, both theoretically and experimentally.

Experimental evidence indicates that the π meson has zero spin and that it has odd intrinsic parity i.e. the meson can be represented by a pseudoscalar operator or an odd wave function. The assumption of charge symmetry, in the pion-nucleon interaction, is reasonably well borne out by experiment, while the postulate of charge independence is compatible with all results to date. Stanghellini (1958) attempts to give a quantitative statement on the degree of deviation from charge independence, and finds that it is valid within the error of the experimental results which he uses. Thus isotopic spin can be treated as a good quantum number in a theoretical approach to the scattering process.

Total angular momentum is also conserved, and partial wave analysis of experimental results, up to the region round 250 MeV laboratory energy, have shown that S and P waves are sufficient to fit the data. At greater energies higher waves are found to occur, eg. as reported by Goodwin, Kenney and Perez-Mendez (1959).

P and S wave results thus obtained from experimental data are given by Orear (1955, 1956) and Pontecorvo (1959).

If we include the old assumption that the nucleon absorbs or emits pions singly, then the two interaction Hamiltonians that we have, with simplest local coupling, are:-

$$H_{I_{ps}} = i g \int \bar{\psi}(\underset{\sim}{x}) \gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}(\underset{\sim}{x}) \psi(\underset{\sim}{x}) \, d\underset{\sim}{x} \tag{1.1}$$

direct coupling in pseudoscalar theory, and

$$H_{I_{pv}} = i \, g/_{2M} \int \bar{\psi}(\underset{\sim}{x}) \gamma_5 \gamma_\mu \tau_i \frac{\partial \phi_i(\underset{\sim}{x})}{\partial x_\mu} \psi(\underset{\sim}{x}) \, d\underset{\sim}{x} \tag{1.2}$$

derivative coupling in pseudovector theory. ϕ_i are the three real components of the free meson field, and ψ is the free nucleon field in the usual notation.

The two Hamiltonians can be shown to be 'equivalent', see Section 2, to first order in the coupling constant.

However it is found that the pseudovector theory is not finite after renormalisation, following Dyson's definitions, and hence the direct coupling is generally preferred.

Because of the difficulty of carrying out accurate calculations with the pseudoscalar interaction, one cannot usually state to what extent the predictions of a theory are due to the method of approximation used. To give some insight on the problem simpler interactions and certain 'models' have been examined. Sometimes these formulations have exact solutions, but in general they are approximate methods which can be analysed and compared.

Early theoretical work revealed the inadequacy of perturbation theory, and as there are serious criticisms of the strong coupling method, (see Bethe and de Hoffman (1956)) any realistic approach to the scattering problem should not make use of these coupling constant limits.

2. The Foldy-Dyson Transformation.

In order to investigate the structure of the linear pseudoscalar Hamiltonian we can apply a Foldy-Dyson transformation, i.e. a unitary transformation of the Hamiltonian of the form

$$H \rightarrow H' = e^{iS} H e^{-iS} \qquad 2.1$$

The form of S is assumed to be

$$S = \int \psi^*(\underset{\sim}{x}) \, s \, \psi(\underset{\sim}{x}) \, d\underset{\sim}{x} \qquad 2.2$$

and s is determined by the requirement that in the new representation no pseudoscalar term should appear. The transformed Hamiltonian is highly non-linear in the meson field and fairly complex. The different terms are then simplified by expanding in powers of g, using some approximation.

As an example Berger, Foldy and Osborn (1952) have

$$S = \frac{1}{2} \int \psi^*(\underset{\sim}{x}) \gamma_5 \frac{\tau_i \phi_i}{\kappa} \tan^{-1}\left[\frac{g\kappa}{M}\right] \psi(\underset{\sim}{x}) \, d\underset{\sim}{x} \qquad 2.3$$

where $\kappa = (\sum_i \phi_i \phi_i)^{1/2}$ and for their expansion they assume $g\kappa/M$ small.

After expansion it is seen that the Hamiltonian terms are essentially arranged in powers of μ/M, where μ is the meson mass and M the nucleon mass.

In lowest orders the interaction becomes, in this representation,

$$H_I = \frac{g}{2M}\int \psi^{*}(\underset{\sim}{x})\left(\sigma_i \nabla_i\, \underset{\sim}{\phi}(\underset{\sim}{x})\cdot\underset{\sim}{\tau}\right)\psi(\underset{\sim}{x})\, d\underset{\sim}{x} + \frac{g^{2}}{2M}\int \bar{\psi}(\underset{\sim}{x})\psi(\underset{\sim}{x})\,\underset{\sim}{\phi}^{2}(\underset{\sim}{x})\, d\underset{\sim}{x}$$
$$+ \left(\frac{g}{2M}\right)^{2}\int \psi^{*}(\underset{\sim}{x})\,\underset{\sim}{\tau}\cdot\left(\underset{\sim}{\phi}(\underset{\sim}{x})\times\underset{\sim}{\pi}(\underset{\sim}{x})\right)\psi(\underset{\sim}{x})\, d\underset{\sim}{x} \qquad 2.4$$

The first term yields mainly P wave scattering, the other two S wave. However we have not really separated the Hamiltonian in angular dependence by the transformation since the neglected higher order terms give a coupling between the different waves.

A possible approach would be to treat these neglected terms as perturbations to Hamiltonian 2.4. This would not be very productive in the S wave case, however, due to the uncertainty of the size of the higher order terms, as is shown by the poor comparison with experiment of the S wave solutions obtained from 2.4 alone.

3. Fixed Source Theory.

One of the most important methods based on 2.4 is due to Chew (1954). In his well known 'static' approximation the nucleon is treated as an infinitely heavy source of mesons. He evaluates P wave scattering only and takes as his Hamiltonian

$$H_I = \frac{g}{2M} \tau_\lambda \int \rho(\underset{\sim}{r})\, \underset{\sim}{\sigma} \cdot \underset{\sim}{\nabla} \phi_\lambda(\underset{\sim}{r})\, d\underset{\sim}{r} \quad ; \qquad 3.1$$

$\rho(\underset{\sim}{r})$ is the nucleon source function normalised as $\int \rho(\underset{\sim}{r}) d\underset{\sim}{r} = 1$, and the nucleon's only degree of freedom is its isotopic spin.

This simplifies the mathematics as there is no recoil and nucleon pairs are not allowed. As has been pointed out, Chew requires a cut off to obtain convergence because of the gradient coupling. The cut off in momentum space can be said to have two physical equivalences in Chew's theory - namely -

1) It helps to replace the missing 'damping' effect of nucleon recoil, and
2) It represents possible non-locality of the interaction, i.e. if the meson nucleon interaction requires three fields to specify it completely, we could regard the two fields with cut off as

an approximation to the correct theory.

Using a Tamm-Dancoff type of method to obtain an integral equation for the scattering wave function, Chew then applies one of Schwinger's variational principles (Chew 1954a) to solve the equation. He obtains good results for the P wave phase shifts up to 200 MeV.

After this success some work by Drell, Friedman and Zachariasen (1956) was done on the application of static theory to the S wave case. The basic S wave Hamiltonian is

$$H_s = \frac{g^2}{2M}\int \bar{\psi}(\underset{\sim}{x})\psi(\underset{\sim}{x})\,\underset{\sim}{\phi}^2(\underset{\sim}{x})\,d\underset{\sim}{x} + \left(\frac{g}{2M}\right)^2\int \psi^*(\underset{\sim}{x})\,\underset{\sim}{\tau}.\left(\underset{\sim}{\phi}(\underset{\sim}{x})\times\underset{\sim}{\pi}(\underset{\sim}{x})\right)\psi(\underset{\sim}{x})\,d\underset{\sim}{x} \qquad 3.2$$

The ϕ^2 term, which is the lowest order, is the 'repulsive core' term and does not lead to the observed isotopic splitting of the two S wave phase shifts, δ_1 and δ_3, and so we have to add higher order terms. The next term is the $\underset{\sim}{\tau}.(\underset{\sim}{\phi}\times\underset{\sim}{\pi})$ one, and this couples the isotopic spins of the nucleon and the meson. It was hoped that these two terms might be sufficient, but it has been shown by Akiba and Sawada (1954) that only if the term is multiplied by an independent coefficient, α, and the ϕ^2 term reduced slightly, then the

S wave scattering can be given by the altered Hamiltonian. (α = 1 in the weak coupling limit, and 2 in the strong coupling limit).

Drell et al. take a separable source Hamiltonian

$$H_s = \lambda_0^0 \bar{\underset{\sim}{\phi}} \cdot \bar{\underset{\sim}{\phi}} + \lambda_0 \underset{\sim}{\tau} \cdot (\bar{\underset{\sim}{\phi}} \times \bar{\underset{\sim}{\pi}}) \qquad 3.3$$

where $\bar{\phi} = \int \phi(\underset{\sim}{x}) \rho(\underset{\sim}{x})\, d\underset{\sim}{x}$ and they treat λ_0^0 and λ_0 as separate parameters. Using an S-matrix formalism similar to that of Chew and Low (1956) they obtain good agreement with experiments for λ_0^0 reduced by a factor $\mu/2M$ from its transformation value. However, since we have to alter a coefficient which is fixed relative to the coupling constant, g, by the Foldy-Dyson transformation, this indicates that we cannot neglect the higher order terms.

This is borne out by Sartori and Wataghin (1954) who state that the use of Hamiltonian 3.2 is inconsistent after the lowest order, i.e. higher order scattering graphs of 3.2 are approximately of the same magnitude as the lowest order graphs of the neglected terms.

Another criticism of this type of approach is the neglect of recoil particularly in the S wave case. Fonda and Reina (1956) have attempted to add recoil to Chew's theory for the P wave. Also taking the Hamiltonian 2.4, but with the 'λ' parameters of Drell et al. for the S wave parts, they calculate the S wave phase shifts by means of the Tamm Dancoff method. They repeat the calculation neglecting recoil and compare the results. They find that δ_{33} is considerably smaller than Chew's result when recoil is included, but by decreasing the coupling constant the phase shift can be made to fit, though not quite so well as before. There is a notable difference in the S wave phase shifts, between the calculated values and those of Drell. Thus as far as the method and approximations used can be trusted to guide us, we may say that the inclusion of recoil has a very marked effect on δ_1 and δ_3 .

It is seen that the effect of the nucleons' momentum increases with increasing scattering energy, as one would expect.

Lomon (1956) claimed to have diagonalised a separable source version of the Hamiltonian 3.2, but later papers - Kobayashi and Klein (1958) and Bassetti (1958), have cast some doubt on this. Lomon's argument is not clear at some points, and by a stated approximation

Kobayashi obtains the same result, which is thus seen to be exact only in the classical limit. The results have the correct isotopic splitting, but the magnitudes are only fair.

Summing up it can be stated that the Hamiltonian 3.2 can only give us an indication of how the P and S phase shifts arise.

Keeping in mind the serious failing of the S wave calculations it is logical that the phases must be evaluated using the full Hamiltonian 1.1.

Levy and Marshak (1954) apply the lowest order Tamm-Dancoff method to the scattering, treating the nucleon as an extended source and using Hamiltonian 1.1. An approximate calculation yields a reasonable δ_3 but a bad δ_1, the latter being very cut off dependent. The authors hoped that a treatment including renormalisation would improve both the results but mainly the isotopic spin 1/2 value. Levy (1954, 1955) uses a covariant treatment and a renormalisation procedure which is dependent on the method of solving the scattering equations. For $g^2/4\pi = 7.5$ he gets a good fit to the old S wave data, i.e. where δ_1 was thought to go negative at about 170 MeV. However the result is extremely dependent on the coupling constant.

Sartori and Wataghin (1954) write down the two lowest orders of scattering in a covariant manner, and then take a non-relativistic limit for the S waves. They use the Deser, Thirring and Goldberger (1954) prescription for charge renormalisation and apply the variational principle of Cini and Fubini (1954) in the first approximation. They find the correct signs for δ_1 and δ_3 but the magnitudes are far too large.

In an earlier paper (Sartori et al. 1954a) the same authors applied the Cini-Fubini to the Hamiltonian used by Chew for the P wave phase shifts, and achieve very similar successful results.

4. The Tamm-Dancoff Method.

As an alternative to perturbation theory the old Tamm-Dancoff, (O.T.D.), method has been widely used, although it has some serious drawbacks. Essentially the method is that the state vector of a system of particles is expanded in terms of states corresponding to different numbers of free field creation operators acting on the bare vacuum. Provided that the states satisfy the usual conservation laws, charge, baryon number etc., an expansion coefficient can then be taken as the probability for finding the system in the state corresponding to the given number of bare particles.

Using the interaction Hamiltonian, an infinite set of coupled equations for the amplitudes are obtained, and the approximation consists of taking only a finite number of the amplitudes and hence a certain set of these equations. The neglected amplitudes are assumed small compared to those retained.

A series of papers applying the O.T.D. formalism to the full pseudoscalar γ_5 coupling was instigated by Dyson et al. (1954). Using only the set of amplitudes coupled, by the interaction Hamiltonian, to the amplitude for one nucleon with one meson, they obtain a single integral equation for their scattering wave function. Neglecting all renormalisation terms they use semi-numerical procedures and obtain rough qualitative results for the two phase shifts δ_3 and δ_{33} .

Kalos and Dalitz (1955) recalculate these results using more accurate numerical techniques. They also examine the effect of omitting nucleon pair transitions, and vary the coupling constant to give the best fit. In general, their results are a little better than Dyson's. It is found that the pair effect contributes nearly all of the $S_{3/2}$ phase shift, but a very low value of the coupling constant is needed if the result was to fit experiment. Tanaka (1957) attempts a partial renormalisation programme using part of a method due to Cini (1953). He obtains, as one might hope, a much better agreement with experiment for the phase shifts, and requires more reasonable values for the coupling constant.

One of the failings of the O.T.D. treatments, is the effect of the vacuum. If the number of particles

in the amplitudes is limited such that a vacuum graph is included, then a spurious vacuum effect arises. This is due to the fact that only a certain number of amplitudes are coupled to the vacuum, although physically an infinite number of virtual particles occur with each amplitude. To overcome this difficulty the new Tamm-Dancoff,(N.T.D.),Method can be used. For this we replace the bare vacuum in the O.T.D. by the physical vacuum. The energy of a state is now measured relative to the energy of the real vacuum, and vacuum self energy effects are removed.
Dalitz and Dyson (1955) set up the scattering equation in lowest order N.T.D, and examine the renormalisation effects. Owing to the occurrence of a non-physical pole and ambiguous vertex renormalisation, no numerical results are obtained.

Visscher (1954) attempted to evaluate the effects of the self energy and renormalisation terms by means of Cini's covariant formulation of the N.T.D. Method, (Cini 1953). However after renormalisation he found that, when he combined the finite remainders into an effective coupling constant, an unphysical pole appeared in the new, momentum dependent, coupling constant. This prevented any useful numerical predictions about the phase shifts.

Examinations of the validity of the Tamm-Dancoff procedures have been made by many authors. Morpurgo and Touschek (1953) apply the O.T.D. to Wentzel's pair theory, and from a comparison with the exact results, the O.T.D. results appear to be only a qualitative approximation.

We find that there are three main criticisms of the T.D. theories:-

1) The doubt concerning the convergence of the neglected amplitudes.
2) The lack of 'crossing' symmetry, and
3) The lack of unambiguous renormalisation procedures.

1. <u>Amplitude Convergence</u>.

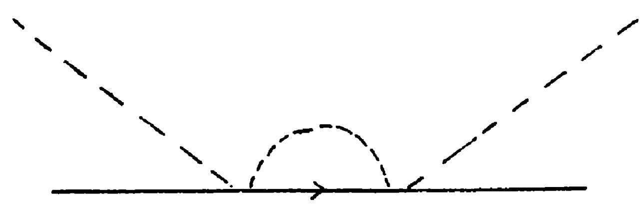
The O.T.D. and N.T.D. have been applied to the soluble problem of the anharmonic oscillator by K. Symanzig (Dalitz et al. 1955). He found that the O.T.D. terms diminished reasonably, while the N.T.D. amplitudes could even be exponentially increasing.

It is seen, however, that for terms involving large numbers of virtual particles the amplitudes are associated with denominators consisting essentially of the sum of the energies of the particles. Other things being equal these denominators will give smaller values for these 'high order' amplitudes. This would be in accord with the physical picture which we can obtain from the uncertainty principle, i.e. that the greater the total energy of an intermediate state, the less time the system will spend in that state.

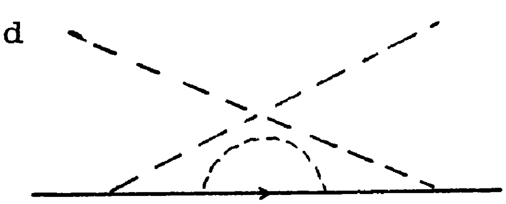
2. Crossing Symmetry.

The fact that mesons obey Bose statistics gives us a symmetry condition on the meson-nucleon scattering amplitude. This is the well known 'crossing' symmetry, and it indicates that any good scattering theory should always contain pairs of scattering graphs where the external meson lines are crossed and uncrossed.

That is any particular diagram eg.



should always be accompanied by its companion,


.

A corresponding symmetry for the nucleons is examined by Feldman and Matthews (1956), and they find that its effect is just equivalent to the meson symmetry effect provided that the results are taken on the energy shell.

It is at once seen that a Tamm-Dancoff Method which limits the number of particles allowed in the amplitudes will not satisfy this symmetry requirement. Gell-Mann and Goldberger (1954) use crossing symmetry to state a condition on the two S wave phase shifts, viz. there is no difference between the two S states at zero momentum, provided that $\mu/M \to 0$. This is so because the Isotopic spin dependent parts of a pair of crossed and uncrossed graphs tend to cancel. This means that a Tamm-Dancoff approximation should give a spurious large isotopic spin separation. However μ/M is not so vanishingly small and the experimental behaviour of δ_1 and δ_3 is not well known in such low energy regions.

Martin (1956) evaluates the S phases starting from Levy's covariant treatment of the Bethe Salpeter equation. He compares his results with fourth order perturbation theory (Wyld 1954) and also the work of

Kalos and Dalitz, and gets general agreement. Due to his approximations the only definite conclusion he can come to is the importance of including all the appropriate crossed graphs in a Tamm-Dancoff treatment.

3. <u>Renormalisation</u>.

Dyson (1953) proposed a plausible subtraction process for the second order nucleon self energy, and examined Cini's covariant renormalisation programme for both TD methods. He finds that the Cini renormalisation is only consistent and finite for the N.T.D. and that it is then equivalent to a 'double' application of his subtraction process.

Taylor (1954) and Dyson (1953a) both discuss the connections between the three dimensional O.T.D. and N.T.D. theories and the four dimensional Bethe-Salpeter equation. Approximations and the difficulties of renormalisation, especially in orders higher than the second, are examined. The covariant Bethe-Salpeter equation has not proved very useful due, primarily, to the difficulty of interpreting the meaning of the wave function. The more useful renormalisation programmes are dubious because of their utilisation of the Levy-Klein expansion.

Although the problem of removing infinities by renormalisation has not been solved as consistently as in perturbation theory, if we keep in mind the successes of the method, which cannot all be fortuitous, we may say that the Tamm-Dancoff can be a useful, though not generally very exact, method.

5. Intermediate Coupling Theory.

As an alternative to the Tamm-Dancoff approximations we have Intermediate Coupling Techniques, which generally make use of a variational principle. The usual test of such methods is the examination of the validity of the predictions in the two limits of weak and strong coupling, together with, of course, a comparison with experimental results.

Tomonaga (1947) first formulated an Intermediate Coupling approximation for meson theory using a Hartree-Fock calculation. An infinite number of virtual mesons is allowed bound to the nucleon, but they are assumed to be in a finite number of orbital states. The scattering of a free meson from the physical nucleon is pictured as the absorption of the incident meson into the nucleon's field and the emission of a 'scattered' meson from the bound states. Early work was applied only to static problems such as the calculations of self energies and magnetic moments with recoil neglected.

For pseudoscalar coupling Matthews and Salam (1952) consider the case where one nucleon pair is allowed in the nucleon's field, and compare it to the situation with no pairs allowed. A cut off at nucleon mass is used instead of renormalisation, and recoil is added nonrelatavistically.

The case where only processes involving nucleon pairs are allowed is examined by Moorhouse (1953), who investigates a method for subtracting the vacuum effects which can occur when pairs are included.

As an approach to the full scattering problem the case of a static nucleon scattering a charged scalar meson was examined by several authors.

Ito, Miyamoto and Watanabe (1955) tackle the scattering from a fixed nucleon in pseudoscalar theory. However with increasing complexity they have to make several assumptions, and allow not more than three bound mesons and only one unbound meson in their fields. The normal meson field operator is split into two parts: essentially a free meson field part orthogonal to the source function, and the bound part proportional to the source function. They obtain a δ_{33} resonance similar to that of Chew but the other P wave shifts are exceedingly large although negative.

Friedman, Lee and Christian (1955) examine the same problem. They reduce their Hamiltonian by taking the momentum dependence out from the particle creation and annihilation operators, and determine the physical nucleon by Tomonaga's method. The scattering is evaluated by using the state of a bare meson with a real nucleon as a trial function for a Ritz variational principle. By requiring that the theoretical δ_{33} curve pass through two experimental points, they determine the un-renormalised coupling constant and the value of the cut off. A reasonable fit for δ_{33} is obtained, and the other P phases are satisfactorily small, δ_{13} and δ_{31} being negative and equal. Although δ_{11} was found to be positive for the equivalent renormalised coupling constant, $f_r^2 = 0.105$, the authors state that the sign would alter for a smaller coupling constant.

This type of Intermediate Coupling Theory can give a fairly good fit to the main P wave phase shift. As we have seen nucleon pairs play a large part in the S wave scattering, and so, if intermediate coupling methods were to have a chance to give the S scattering, we would require to include virtual pairs in the nucleon field. This would yield more complications in the treatment and it is debatable as to whether it would be sufficient.

6. Exact Theories.

Theories with Hamiltonians which allow the scattering problem to be solved exactly are of great interest since we know that the results will have a strong connection with the formulation.

1) In his well known paper Lee (1954) studies the reaction

$$V \Leftrightarrow N + \theta \tag{6.1}$$

where he takes N and V to be neutral fermion fields and θ a scalar boson field. Taking the Hamiltonian for 6.1 to be the only interaction present he examines the two scattering systems

$$\begin{aligned} N + \theta &\rightarrow N + \theta \\ V + \theta &\rightarrow V + \theta \end{aligned} \tag{6.2}$$

Lee also investigates the scattering of neutral and charged scalar mesons from a fixed nucleon.

Many adaptations of and extensions to the Lee Model have been proposed by later workers. Haber-Schaim and Thirring (1955) firstly add recoil to the Lee model and secondly allow also the reaction $N \rightarrow W + \theta$ where W is a third type of nucleon, θ are symmetric scalar mesons and all the nucleons are fixed sources.

The Ruijgrok and Van Hove extension allows successive emission or absorption of an unlimited number of mesons from an infinitely heavy nucleon. The nucleon is given some number, $n > 1$, of internal states, and the model is given by

$$V_r \rightarrow V_{r+1} + \theta \qquad , \qquad 6.3$$

where

$$V_{s+n} = V_s \quad .$$

This model is exactly renormalisable, and Ruijgrok (1958) discusses a reformulation, in terms of physical particle definitions, which leads to a convergent theory.

The inclusion of pairs was proposed by Goldstein (1958) by allowing the θ particle transition into a nucleon anti-nucleon pair of a third type of nucleon field χ .

$$\begin{aligned} V &\Leftrightarrow N + \theta \\ \theta &\Leftrightarrow \chi + \bar{\chi} \end{aligned} \qquad 6.4$$

The sub-case of $\chi = N$ was also investigated.

Because of their simplicity these examples have been generally used to examine the mathematical structure of the renormalisation programmes for mass, vertex and coupling constant. The occurrence of the well known 'ghost' states in this type of theory has been looked into by Källén (1957) and others.

The scattering results and their dependence on renormalisation are accessible for analysis in these models. At present, however, models of the Lee type are not realistic pictures of the physical pion nucleon scattering problem.

2) Bosco and Stroffolini (1955), with a method reminiscent of the later Goldstein paper, attack the S wave scattering problem only. For their Hamiltonian they take that part of the relativistic γ_5 Hamiltonian which corresponds to the equation

$$\pi \Longleftrightarrow N + \bar{N} \qquad 6.5$$

They also neglect the vacuum reaction

$$\text{VACUUM} \Longleftrightarrow \pi + N + \bar{N} \qquad , \quad 6.6$$

The 'physical' meson is represented by a state of one 'bare' meson plus a state of a 'bare' nucleon pair. The 'bare' particles are those occurring in equations 6.5 and 6.6 , and the second state is merely the nucleon-anti-nucleon cloud of the meson. A counter term is included for mass renormalisation, which they fix by imposing a condition on the kinetic energy of the physical meson. Coupling constant renormalisation is by a normal method and the equation has a simple form in their model. Recoil is neglected, and a cut off taken at some fraction of the nucleon mass. The two resulting equations are solved by successive approximations for pairs of cut off and renormalised coupling constant values. For a scattering state of a bare nucleon together with a real meson, plus an interaction state of two bare nucleons with an anti-nucleon they solve the scattering equations exactly. For $g^2/4\pi = 1$ a good fit is achieved with **Orear's** two S wave phase shifts, although the energy dependence is not quite correct.

This paper is interesting because of the good results which such a simple model yields. The importance of pair effects in the S wave phase shifts is once again stressed. The small coupling constant and the strong cut off seem to cut down the effect of the ϕ^2 Hamiltonian term which is known to give too large results for the S wave scattering. It might be hoped that the coupling constant could be increased by extending the model.

7. Dispersion Relations.

In recent years, the Dispersion Relation method has proved to be an important tool in theoretical physics. It has been well used in the analysis of meson phenomena, in particular pion nucleon scattering and pion photoproduction at a nucleon.

The basic assumptions of the theory are the Unitarity and Lorentz Invariance of the scattering matrix, and the Principle of Causality. Essentially a relation is established between the Hermitian and Antihermitian parts of the scattering amplitude, S. Writing S = D + iA, where D and A are both Hermitian, the equation has the form

$$\mathcal{D}(E) = \mathcal{B}(E) + \frac{(E - E_0)^{n+1}}{\pi} \mathcal{P}\int \frac{A(E')\, dE'}{(E'-E)(E'-E_0)^{n+1}} + \sum_{r=0}^{n} C_r E^r, \qquad 7.1$$

where $\mathcal{B}(E)$ are the residues of isolated poles contained in the field of integration, and $\mathcal{P}$ is the principle value. $A(E)$ is assumed not to diverge faster than E^n at infinity. C_r are undetermined constants resulting from a Cauchy integration, and are removed by subtraction procedures. In most reasonable treatments n is assumed = 0 and only one subtraction is required. For values of $n > 0$ the numerical work is prohibitive, and although the Dispersion Relation is more accurate for higher values, $n = 0$ is sufficient in many cases.

For applications the S matrix is separated out into its angular momentum and isotopic spin components, and relations of the form 7.1 are obtained in terms of these components.

It is found that it is particularly easy to examine the case of forward scattering, (Bogoliubov, 1959), since by the 'optical theorem' we have

$$A = \mathcal{I}_m f = c\,|\underset{\sim}{p}|\,\sigma \qquad 7.2$$

where f is the forward scattering amplitude,

σ the total cross section for the process considered,

$\underset{\sim}{p}$ is the momentum of the incident particle, and

c is a constant depending on the units used.

Thus we can put total cross section data into the integral of equation 7.1, perform the integration and find $\mathcal{R}_e f$.

We have the well known results from general scattering theory,

$$\frac{d\,\sigma(\theta)}{d\,\Omega} = K|f(0)|^2 = K\left[(\mathcal{R}_e f(0))^2 + (\mathcal{I}_m f(0))^2\right] \qquad 7.3$$

$$\frac{d\sigma(\theta)}{d\Omega} = \sum_n L_n P_n(\theta) \tag{7.4}$$

where K is again a constant depending on the units,

θ is the angle of scattering (equals 0 for forward scattering), and

L_n is a known function of the phase shifts for the scattering process.

Equations 7.3 and 7.4 may be used to resolve the ambiguity between the sets of phase shifts in pion nucleon scattering, as is done by Anderson, Davidon and Kruse (1955).

For nonforward scattering some other method of evaluating the dispersion integrals must be found. As an example see the paper by Chew mentioned below. Most of the papers, which have been published, give the formulation of the Dispersion Relations and discuss their validity and applicability. However, some authors have calculated pion nucleon scattering results which they compare with experiment.

Anderson et al. (1955) evaluate the Dispersion Relations of Goldberger, Miyazawa and Oehme (1955) for the scattering of π^+ and π^- from protons. A good fit with the experimental P phase shifts is obtained, for a suitable coupling constant, by using the S scattering lengths of Orear (1954) to evaluate the zero momentum forward scattering amplitudes.

Formula for the low energy phase shifts have been derived by Chew, Goldberger, Low and Nambu (1957), who assume that the δ_{33} resonance gives the only contribution to the dispersion integrals. The P wave phase shifts obtained are very similar to the results of 'static theory', but the results are only considered valid for the low energy cross sections used.

Finally Gilbert (1957) with a new, more convergent, form of the Dispersion Relations, evaluates the pion nucleon coupling constant. Assuming the S waves small, he obtains a good fit for the S scattering lengths by using the determined coupling constant and integrals over the P wave resonances.

An important development has been formulated by Mandelstam (1958) who uses new relatavistic Dispersion Relations as the basis of his theory instead of the usual field theory equations. An interesting step is his inclusion of the π-π interaction, but as yet he has achieved no numerical predictions, although his treatment has been verified in fourth order perturbation theory.

Summarising this section, we see that, although Dispersion Relations have given new correlations between the pion nucleon scattering data, they are unable to predict individual results dependent, say, on only the coupling constant.

8. The Inclusion of Further Interactions.

As we have seen numerous models have had fair success in reproducing the P wave phase shifts, notably the $P_{1/2 \cdot 3/2}$. These models however even if applicable to the S wave case, fail to yield S phases of a comparable accuracy to that of the P phase. To overcome this, it was suggested that there might be present another interaction, which, only in the S wave case, gave results of the same order of magnitude as the meson nucleon interaction.

Matthews and Edwards (1957), with a rough calculation, attempted to add strange particle effects to the ϕ^2 Hamiltonian term from the Foldy Dyson transformation. They allow the three reactions

$$\pi \Leftrightarrow K + K \quad , \qquad N \Leftrightarrow \Lambda + K \quad ,$$
$$N \Leftrightarrow \Sigma + K \quad , \qquad\qquad 8.1$$

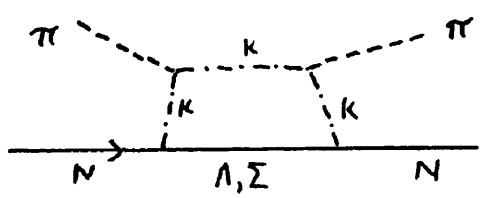
all in direct local coupling, but they obtain only a small effect.

A better treatment by Langer (1957) uses pseudo-scalar theory. He neglects the $\pi - K$ interaction and the Λ and Ξ particles. The added $\pi - \Sigma$ interaction is taken equal to the $\pi - N$ interaction. Neglecting the fact that the K mesons form a doublet in isotopic spin space, he further adds

$$-g_s\left[(\bar{\psi}_N \gamma_5 \underset{\sim}{\tau}\cdot\underset{\sim}{\phi}^K \psi_\Sigma) + (\bar{\psi}_\Sigma \underset{\sim}{\phi}^K\cdot\underset{\sim}{\tau}\, \psi_N)\right] \qquad 8.2$$

to his Lagrangian density. Making simplifying assumptions, in what we might hope was a reasonable order of magnitude calculation, Langer obtains the correct isotopic splitting, but the magnitude of his S phase shifts are a little small.

Budini and Fonda (1957) examine the case where the pion interacts with the nucleon through an intermediate K meson field: the lowest order graph being



They find that the calculated results tend to cancel the scattering contributions from the repulsive ϕ^2 term. By requiring that this reduction is approximately that needed to fit experiment, they obtain numerical relations between the various coupling constants involved. Though the calculation is an approximate one, the values found for the coupling constants do not disagree with what the known experimental data allow.

Mitra and Dyson (1953) suggested using a long distance attractive π-π interaction. The incident meson could then be envisaged as interacting with the virtual mesons in the field of the nucleon. Ross (1954) assumed a potential for this interaction of the form $V e^{-r/a}$ and examined the effects for different values of the depth and the range. The results are discouraging although mainly inconclusive due to the calculations being fitted to old S wave data.

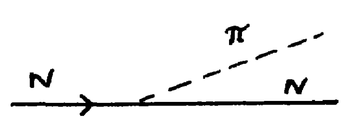
9. Conclusions.

The S wave scattering of π mesons from nucleons is the outstanding unsolved problem in pion physics today. It raises the fundamental question - Does the pseudoscalar theory adequately represent the meson nucleon interaction, or does the theory require to be non-local, i.e. are other interactions present?

A necessary condition for the sufficiency of the Hamiltonian $ig\int\bar{\psi}\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\,\psi\,d\underset{\sim}{x}$ is that it should allow the S and P wave phase shifts to be calculated simultaneously by the one procedure.

Of the proposed extra interactions, the π-π one is thought to be important only in the region around zero kinetic energy. For the Strange Particle cases, it may be a little optimistic to hope that they have only a small effect. However we have the experimental evidence on the difficulty of creating strange particles. Even at high energies the cross section for strange particle production in a pion nucleon collision is only a few percent of that for pion production.

In P waves the important reaction is



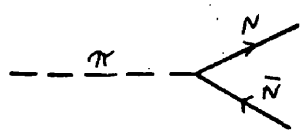
9.1

the strange particle analogue being,

9.2

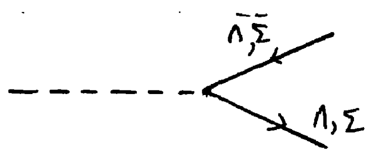
The coupling constant in 9.1 is thought to be more than three times that of 9.2, and the total mass of the intermediate state in 9.2 is about four pion masses greater than the mass in 9.1.

For the S wave case, we have



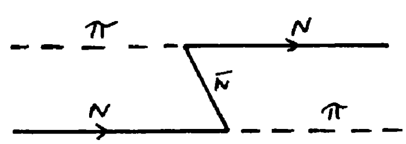
9.3

and



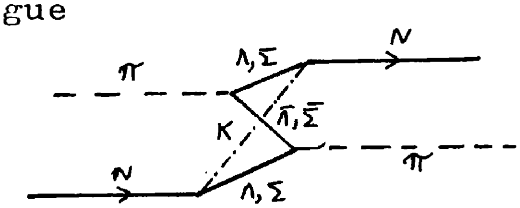
9.4

These two are equivalent except for the masses if we assume the usual coupling symmetry. However in a scattering graph we find 9.2 and 9.4 associated, e.g.



9.5

has the analogue



9.6

Hence we might hope that virtual intermediate states of strange particles are not essential to the theory.

Also the Dispersion Relations have given us correlations between the experimental S and P phases, and this can be taken as a strong hint that the meson and the nucleon form a 'closed system', to a good approximation, for the scattering process.

Weighing up the evidence, it seems worthwhile to try to evaluate the S and P phases using the relativistic γ_5 Hamiltonian and some model which is applicable to both S and P wave scattering.

Our model, as can be seen in Part II of this thesis, developed mainly from the work of Bosco and Stroffolini and also of Friedman, Lee and Christian.

Part 2.

Wave functions for physical nucleons and mesons are defined in terms of bare field operators. Renormalisation is examined for the coupling constant and the physical particle masses, a smooth cut-off being used on divergent integrals. In terms of these physical particles, the scattering is determined using a Raleigh-Ritz Variational Principle. The scattering phase shifts, δ_1 , δ_3 , and δ_{33} are found for different groups of scattering graphs.

10. Preliminary Definitions and Notation.

For the interaction of the meson and nucleon fields we take a total Hamiltonian* of the form

$$H = H_M + H_N + H_I + \text{ Renormalisation counter terms} \qquad 10.1$$

The free nucleon Hamiltonian

$$H_N = \int d\underset{\sim}{x}\; \psi^*(\underset{\sim}{x}) \left[-i \underset{\sim}{\alpha} \cdot \underset{\sim}{\nabla} + \beta M \right] \psi(\underset{\sim}{x}) \quad , \qquad 10.2$$

the free meson Hamiltonian

$$H_M = \tfrac{1}{2} \int d\underset{\sim}{x} \left\{ \pi_\alpha^2(\underset{\sim}{x}) + (\nabla \phi_\alpha(\underset{\sim}{x}))^2 + \phi_\alpha^2(\underset{\sim}{x}) \right\} , \qquad 10.3$$

and the relatavistic γ_5 interaction

$$H_I = i g \int d\underset{\sim}{x}\; \psi^*(\underset{\sim}{x})\, \beta \gamma_5\, \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}(\underset{\sim}{x})\, \psi(\underset{\sim}{x}) \quad . \qquad 10.4$$

* The system of units used is $\hbar = c = \mu = 1$. See Appendix 1.

The Dirac matrices α, β, and γ, are defined in Appendix 2. We expand the meson and nucleon field operators in momentum space as follows:

$$\psi(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d\underset{\sim}{p} \sum_{u} b_{\underset{\sim}{p}u}\, u_{\underset{\sim}{p}}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} , \qquad 10.5$$

$$\psi^{*}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d\underset{\sim}{p} \sum_{u} b^{*}_{\underset{\sim}{p}u}\, u^{*}_{\underset{\sim}{p}}\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} , \qquad 10.6$$

$$\phi_{\alpha}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d\underset{\sim}{k}\, (2\omega_{\underset{\sim}{k}})^{-1/2} \left[a_{\underset{\sim}{k}\alpha} + a^{*}_{-\underset{\sim}{k}\alpha} \right] e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} , \qquad 10.7$$

and the conjugate momentum to ϕ_{α} ,

$$\pi_{\alpha}(\underset{\sim}{x}) = (2\pi)^{-3/2} \int d\underset{\sim}{k}\; i \left(\frac{\omega_{\underset{\sim}{k}}}{2} \right)^{1/2} \left[a_{\underset{\sim}{k}\alpha} - a^{*}_{-\underset{\sim}{k}\alpha} \right] e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}} . \qquad 10.8$$

The spinor u can describe four possible states of spin and energy for the nucleon. ϕ_1 and ϕ_2 are the real Hermitian components of the complex charged π-meson field, ϕ_3 represents the neutral pion field; $\alpha = 1, 2$ or 3.

The spinor $u_{\underset{\sim}{p}\pm}$, for positive and negative energies respectively, satisfies the equation

$$\left[\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta M \right] u_{\underset{\sim}{p}\pm} = \pm E_{\underset{\sim}{p}}\, u_{\underset{\sim}{p}\pm} , \qquad 10.9$$

and are normalised by $u^{*}u = 1$ 10.10

The energies $E_{\underset{\sim}{p}}$ and $\omega_{\underset{\sim}{k}}$ are given by

$$E_{\underset{\sim}{p}} = (M^2 + \underset{\sim}{p}^2)^{1/2} \qquad 10.11$$

$$\omega_{\underset{\sim}{k}} = (\mu^2 + \underset{\sim}{k}^2)^{1/2}$$

where M and μ are the nuclear and meson masses respectively.

Projection operators $\Lambda^{\pm}(\underset{\sim}{p})$ are defined as:

$$\Lambda^{+}(\underset{\sim}{p}) = \sum_{u^{+}} u_{\underset{\sim}{p}} u_{\underset{\sim}{p}}^{*} \quad , \quad \Lambda^{-}(\underset{\sim}{p}) = \sum_{u^{-}} u_{\underset{\sim}{p}} u_{\underset{\sim}{p}}^{*} \qquad 10.12$$

or in the more convenient form

$$\Lambda^{\pm}(\underset{\sim}{p}) = \left[E_{\underset{\sim}{p}} \pm (\underset{\sim}{\alpha}.\underset{\sim}{p} + \beta M)\right] \Big/ 2E_{\underset{\sim}{p}} \qquad 10.13$$

Writing the Hamiltonians in the conventional operator form in momentum space we have

$$H_N = \int E_{\underset{\sim}{p}}\, b^{*}_{\underset{\sim}{p}u}\, b_{\underset{\sim}{p}u}\, d\underset{\sim}{p} \qquad 10.14$$

$$H_M = \int \omega_{\underset{\sim}{k}}\, a^{*}_{\underset{\sim}{k}\alpha}\, a_{\underset{\sim}{k}\alpha}\, d\underset{\sim}{k} \qquad 10.15$$

$$H_I = \frac{g}{(2\pi)^{3/2}} \iint \frac{d\underset{\sim}{p}\, d\underset{\sim}{k}}{(2\omega_{\underset{\sim}{k}})^{1/2}} \sum_{\substack{u,v\\ \alpha}} (v^{*}\gamma\tau_{\alpha} u)\left[a^{*}_{\underset{\sim}{k}\alpha} + a_{-\underset{\sim}{k}\alpha}\right] b^{*}_{\underset{\sim}{p}+\underset{\sim}{k},v}\, b_{\underset{\sim}{p},u} \qquad 10.16$$

γ is used to denote $i\beta\gamma_5$

If u is a positive energy spinor $b^{*}_{\underset{\sim}{p}u} = c^{*}_{\underset{\sim}{p}u}$ and creates a nucleon of momentum $\underset{\sim}{p}$, whereas if u is a negative energy spinor

$$b^{*}_{\underset{\sim}{p}u} = d_{\underset{\sim}{p}u}$$

and annihilates an anti nucleon of momentum $-\underset{\sim}{p}$.

Also $\qquad b_{\underset{\sim}{p}u_{+}} = c_{\underset{\sim}{p}u_{+}}$

and $\qquad b_{\underset{\sim}{p}u_{-}} = d^{*}_{\underset{\sim}{p}u_{-}}$

The commutation or anti-commutation rules satisfied by the operators a, c, and d, are as follows

$$\left[a_{\underset{\sim}{k}\alpha}, a^*_{\underset{\sim}{k}'\alpha'} \right] = \delta(\underset{\sim}{k} - \underset{\sim}{k}')\, \delta_{\alpha\alpha'}$$

$$\left[c_{\underset{\sim}{p}u}, c^*_{\underset{\sim}{p}'v} \right]_+ = \delta(\underset{\sim}{p} - \underset{\sim}{p}')\, \delta_{uv} \qquad 10.17$$

$$\left[d^*_{\underset{\sim}{p}u}, d_{\underset{\sim}{p}'v} \right]_+ = \delta(\underset{\sim}{p} - \underset{\sim}{p}')\, \delta_{uv}$$

Any other anti-commutator of the c and d operators is zero, and likewise any other commutation of the a operator.

Any commutation of the a operator with either the c or the d operators is also zero.

This means that we can treat the particles represented by a, c, and d, as separate independent particles.

The renormalisation terms mentioned in equation 10.1 are taken as

$$- \int \delta E(\underset{\sim}{k})\, b^*_{\underset{\sim}{k}u}\, b_{\underset{\sim}{k}u}\, d\underset{\sim}{k} \qquad 10.18$$

$$- \int \delta\, \omega(\underset{\sim}{k})\, a^*_{\underset{\sim}{k}\alpha}\, a_{\underset{\sim}{k}\alpha}\, d\underset{\sim}{k} \quad .$$

and they will be discussed in sections 11 and 12.2. At present we add them in to H_N and H_M of equations 10.14 and 10.15, thus forming

$$H_N' = \int E_{(\underset{\sim}{p})} \, b^{x}_{\underset{\sim}{p}u} \, b_{\underset{\sim}{p}u} \, d\underset{\sim}{p} \, , \tag{10.19}$$

$$H_M' = \int \omega_{(\underset{\sim}{k})} \, a^{x}_{\underset{\sim}{k}\alpha} \, a_{\underset{\sim}{k}\alpha} \, d\underset{\sim}{k} \, , \tag{10.20}$$

with H_I as before.

We note that

$$\begin{aligned} E_{(\underset{\sim}{p})} + \delta E_{(\underset{\sim}{p})} &= E_{\underset{\sim}{p}} \, , \\ \omega_{(\underset{\sim}{k})} + \delta \omega_{(\underset{\sim}{k})} &= \omega_{\underset{\sim}{k}} \, , \end{aligned} \tag{10.21}$$

and the total Hamiltonian H is now

$$H = H_N' + H_M' + H_I \, . \tag{10.22}$$

11. Formulation of the Physical Particles.

As was seen in Part I, some scattering models are of interest because of their applicability to some part of the scattering problem. We now set up trial wave functions for a physical nucleon and a physical pion. The model incorporates the main points, which the earlier models emphasise, but it is limited in such a way that the theory does not become too complex. Thus we hope that a good numerical calculation may be made for the resulting phase shifts.

An attempt was first made using the real vacuum, i.e. that which occurs in the New Tamm-Dancoff theory, but this soon proved very cumbersome and the bare vacuum Ψ_0 was used instead. Ψ_0 contains no particles or anti-particles, i.e. negative energy states of the nucleon are all filled.

The Physical Nucleon.

The trial wave function we take for the real nucleon is

$$N^*_{\underset{\sim}{k},u} \Psi_0 = \Big[f_1(\underset{\sim}{k}) C^*_{\underset{\sim}{k}u} + \int d\underset{\sim}{m} \sum_{v,\alpha} f_2(\underset{\sim}{k}-\underset{\sim}{m}, \underset{\sim}{m})\, C^*_{\underset{\sim}{k}-\underset{\sim}{m}\,v}\, a^*_{\underset{\sim}{m}\,\alpha} \qquad 11.1$$

$$+ \frac{1}{2} \int d\underset{\sim}{k}'\, d\underset{\sim}{p}'\, d\underset{\sim}{p} \sum_{\substack{u',v' \\ V,\beta}} f_3(\underset{\sim}{k}', \underset{\sim}{p}, \underset{\sim}{p}+\underset{\sim}{p}'+\underset{\sim}{k}'-\underset{\sim}{k}, \underset{\sim}{p}')\, C^*_{\underset{\sim}{k}'u'}\, C^*_{\underset{\sim}{p}v'}\, d^*_{\underset{\sim}{p}+\underset{\sim}{p}'+\underset{\sim}{k}'-\underset{\sim}{k}\,V}\, a^*_{\underset{\sim}{p}'\beta} \Big] \Psi_0$$

Summations over state indices are limited by conservation laws. For convenience, particle state indices are not written in the functions f_i , but are understood to be implicitly in with the momenta.

We require that the renormalisation terms 10.18 give the physical nucleon state an energy eigen value $E_{\underset{\sim}{k}}$ i.e. the experimental energy $(M^2+\underset{\sim}{k}^2)^{1/2}$. Then we determine the functions f_i from the Raleigh-Ritz Variational Principle using the Hamiltonian 10.22,

$$\delta(\Psi_o N_{\underset{\sim}{k}u} | H - E_{\underset{\sim}{k}} | N^*_{\underset{\sim}{k}u} \Psi_o) = 0 \qquad 11.2$$

together with the normalisation condition

$$(\Psi_o N_{\underset{\sim}{k}u} | N^*_{\underset{\sim}{k}u} \Psi_o) = 1 \qquad 11.3$$

We assume the usual bare vacuum properties

$$(\Psi_o | \Psi_o) = 1 \qquad \text{and} \qquad A\Psi_o = 0 \qquad 11.4$$

where A is any particle annihilation operator:

From 11.2 taking variations with respect to $f_1, f_2, + f_3$ we have:

$$(E_{\mathbf{k}} - E_{(\mathbf{k})}) f_1(\mathbf{k}) = \frac{g}{(2\pi)^{1/2}} \int \frac{d\mathbf{m}}{(2\omega_{\mathbf{m}})^{1/2}} \sum_{v,\beta} (u^*_{\mathbf{k}} \gamma \tau_\beta v_{\mathbf{k}-\mathbf{m}}) f_2(\mathbf{k}-\mathbf{m}, \mathbf{m})$$

$$+ \frac{g}{(2\pi)^{3/2}} \int \frac{d\mathbf{m}\, d\mathbf{q}}{(2\omega_{\mathbf{m}})^{1/2}} \sum_{\substack{V, u' \\ \beta}} (V^*_{\mathbf{q}+\mathbf{m}} \gamma \tau_\beta u'_{\mathbf{q}}) f_3(\mathbf{k}, \mathbf{q}, \mathbf{q}+\mathbf{m}, \mathbf{m}) \qquad 11.5$$

11.6

$$(E_{\mathbf{k}} - E_{(\mathbf{k}-\mathbf{m})} - \omega_{(\mathbf{m})}) f_2(\mathbf{k}-\mathbf{m}, \mathbf{m}) = \frac{g}{(2\pi)^{1/2}} \frac{(v^*_{\mathbf{k}-\mathbf{m}} \gamma \tau_\beta u_{\mathbf{k}})}{(2\omega_{\mathbf{m}})^{1/2}} f_1(\mathbf{k}) .$$

$$(E_{\mathbf{k}} - E_{(\mathbf{k}')} - E_{(\mathbf{p})} - E_{(\mathbf{k}'+\mathbf{p}+\mathbf{p}'-\mathbf{k})} - \omega_{(\mathbf{p}')}) f_3(\mathbf{k}', \mathbf{p}, \mathbf{k}'+\mathbf{p}+\mathbf{p}'-\mathbf{k}, \mathbf{p}') =$$

$$\frac{g}{(2\pi)^{1/2}} (2\omega_{\mathbf{p}'})^{-1/2} f_1(\mathbf{k}) \Big[(v'^*_{\mathbf{p}} \gamma \tau_\beta V_{\mathbf{k}'+\mathbf{p}+\mathbf{p}'-\mathbf{k}}) \, \delta(\mathbf{k}'-\mathbf{k}) \, \delta_{uu'} \qquad 11.7$$

$$- (u'^*_{\mathbf{k}'} \gamma \tau_\beta V_{\mathbf{k}'+\mathbf{p}'+\mathbf{p}-\mathbf{k}}) \, \delta(\mathbf{p}-\mathbf{k}) \, \delta_{v'u} \Big]$$

Using equations 11.6 and 11.7 we find that the normalisation condition 11.3 gives us

$$f_1^{-2}(\mathbf{k}) = 1 + \frac{g^2}{(2\pi)^3} \int \frac{d\mathbf{m}}{2\omega_m} \sum_{v,\beta} \frac{(u^*_{\mathbf{k}} \gamma \tau_\beta w_{\mathbf{k}-\mathbf{m}})(w^*_{\mathbf{k}-\mathbf{m}} \gamma \tau_\beta u_{\mathbf{k}})}{(E_{\mathbf{k}} - E_{(\mathbf{k}-\mathbf{m})} - \omega_{(\mathbf{m})})^2} \qquad 11.8$$

$$+ \frac{g^2}{(2\pi)^3} \int \frac{d\mathbf{m}\, d\mathbf{q}}{2\omega_m} \sum_{\substack{V, w \\ \beta}} \frac{(V^*_{\mathbf{q}+\mathbf{m}} \gamma \tau_\beta w_{\mathbf{q}})(w^*_{\mathbf{q}} \gamma \tau_\beta V_{\mathbf{q}+\mathbf{m}})(1 - \delta(\mathbf{q}-\mathbf{k})\, \delta_{uw})}{(E_{\mathbf{k}} - E_{(\mathbf{k})} - E_{(\mathbf{q})} - E_{(\mathbf{q}+\mathbf{m})} - \omega_{(\mathbf{m})})^2}$$

This equation can be solved for $f_1(\underset{\sim}{k})$ and hence, via substitution in equations 11.6 and 11.7, we can find f_2 and f_3. Equation 11.5 simply leads to an equation for $E(\underset{\sim}{k})$ and this will be examined in the section 13. .

The Physical Meson.

This proceeds exactly parallel to the nucleon case. The trial wave function is

$$\pi^*_{\underset{\sim}{k}\alpha}\Psi_0 = \Big[f_4(\underset{\sim}{k}) a^*_{\underset{\sim}{k}\alpha} + \int d\underset{\sim}{p} \sum_{V,\omega} f_5(\underset{\sim}{p}, -\underset{\sim}{k}+\underset{\sim}{p})\, c^*_{\underset{\sim}{p},\omega}\, d^*_{\underset{\sim}{p}-\underset{\sim}{k},V}$$

$$+\frac{1}{2}\int d\underset{\sim}{p}\, d\underset{\sim}{p}'\, d\underset{\sim}{q} \sum_{\substack{V,\omega\\ \beta,\gamma}} f_6(\underset{\sim}{p}, \underset{\sim}{p}+\underset{\sim}{p}'+\underset{\sim}{q}-\underset{\sim}{k}, \underset{\sim}{p}', \underset{\sim}{q})\, c^*_{\underset{\sim}{p}\omega}\, d^*_{\underset{\sim}{p}+\underset{\sim}{p}'+\underset{\sim}{q}-\underset{\sim}{k},V}\, a^*_{\underset{\sim}{p}'\gamma}\, a^*_{\underset{\sim}{q}\beta}\Big]\Psi_0 \qquad 11.9$$

notation as before. Correspondingly we also have, for $\omega_{\underset{\sim}{k}}$ the experimental meson energy $(\mu^2+\underset{\sim}{k}^2)^{1/2}$.

$$\delta(\Psi_0 \pi_{\underset{\sim}{k}\alpha} | H - \omega_{\underset{\sim}{k}} | \pi^*_{\underset{\sim}{k}\alpha}\Psi_0) = 0 \qquad 11.10$$

and

$$(\Psi_0 \pi_{\underset{\sim}{k}\alpha} | \pi^*_{\underset{\sim}{k}\alpha}\Psi_0) = 1 \qquad 11.11$$

The three equations for the amplitudes f_4, f_5, and f_6, are

$$(\omega_{\underset{\sim}{k}} - \omega(\underset{\sim}{k}))f_4(\underset{\sim}{k}) = \frac{g}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{p}}{(2\omega_{\underset{\sim}{p}})^{1/2}} \sum_{V,\omega^+} (V^*_{\underset{\sim}{p}-\underset{\sim}{k}} \gamma \tau_\alpha \omega_{\underset{\sim}{p}}) f_5(\underset{\sim}{p}, \underset{\sim}{p}-\underset{\sim}{k})$$

$$+ \frac{g}{(2\pi)^{3/2}} \int \frac{d\underset{\sim}{p}\, d\underset{\sim}{q}}{(2\omega_{\underset{\sim}{q}})^{1/2}} \sum_{\substack{V,\omega^+\\ \beta}} (V^*_{\underset{\sim}{q}+\underset{\sim}{p}} \gamma \tau_\beta \omega_{\underset{\sim}{p}}) f_6(\underset{\sim}{p}, \underset{\sim}{q}+\underset{\sim}{k}, \underset{\sim}{q}, \underset{\sim}{k}) \qquad 11.12$$

$$(\omega_{\underset{\sim}{k}} - E(\underset{\sim}{p}-\underset{\sim}{k}) - E(\underset{\sim}{p}))f_5(\underset{\sim}{p}, \underset{\sim}{p}-\underset{\sim}{k}) = \frac{g}{(2\pi)^{3/2}} \frac{(\omega^*_{\underset{\sim}{p}} \gamma \tau_\alpha V_{\underset{\sim}{p}-\underset{\sim}{k}})}{(2\omega_{\underset{\sim}{k}})^{1/2}} f_4(\underset{\sim}{k}) \qquad 11.13$$

$$(\omega_{\underset{\sim}{k}} - \omega(\underset{\sim}{k}') - \omega(\underset{\sim}{q}) - E(\underset{\sim}{k}'+\underset{\sim}{q}+\underset{\sim}{p}-\underset{\sim}{k}) - E(\underset{\sim}{p}))f_6(\underset{\sim}{p}, \underset{\sim}{k}'+\underset{\sim}{q}+\underset{\sim}{p}-\underset{\sim}{k}, \underset{\sim}{q}, \underset{\sim}{k}') =$$

$$\frac{g}{(2\pi)^{3/2}} f_4(\underset{\sim}{k}) \Bigg[\frac{(\omega^*_{\underset{\sim}{p}} \gamma \tau_\gamma V_{\underset{\sim}{k}'+\underset{\sim}{q}+\underset{\sim}{p}-\underset{\sim}{k}})}{(2\omega_{\underset{\sim}{k}'})^{1/2}} \delta(\underset{\sim}{q}-\underset{\sim}{k})\delta_{\alpha\beta} \qquad 11.14$$

$$+ \frac{(\omega^*_{\underset{\sim}{p}} \gamma \tau_\beta V_{\underset{\sim}{k}'+\underset{\sim}{q}+\underset{\sim}{p}-\underset{\sim}{k}})}{(2\omega_{\underset{\sim}{q}})^{1/2}} \delta(\underset{\sim}{k}'-\underset{\sim}{k})\delta_{\alpha\gamma} \Bigg]$$

As before the normalisation condition 11.11 yields

$$f_4^{-2}(\underset{\sim}{k}) = 1 + \frac{g^2}{(2\pi)^3} \int \frac{d\underset{\sim}{p}}{(2\omega_{\underset{\sim}{p}})} \sum_{V,\omega^+} \frac{(V^*_{\underset{\sim}{p}-\underset{\sim}{k}} \gamma \tau_\alpha \omega_{\underset{\sim}{p}})(\omega^*_{\underset{\sim}{p}} \gamma \tau_\alpha V_{\underset{\sim}{p}-\underset{\sim}{k}})}{(\omega_{\underset{\sim}{k}} - E(\underset{\sim}{p}-\underset{\sim}{k}) - E(\underset{\sim}{p}))^2} \qquad 11.15$$

$$+ \frac{g^2}{(2\pi)^3} \int \frac{d\underset{\sim}{p}\, d\underset{\sim}{q}}{(2\omega_{\underset{\sim}{p}})} \sum_{\substack{V,\omega^+\\ \beta}} \frac{(V^*_{\underset{\sim}{p}+\underset{\sim}{q}} \gamma \tau_\beta \omega_{\underset{\sim}{q}})(\omega^*_{\underset{\sim}{q}} \gamma \tau_\beta V_{\underset{\sim}{p}+\underset{\sim}{q}})(1 + \delta(\underset{\sim}{p}-\underset{\sim}{k})\delta_{\alpha\beta})}{(\omega_{\underset{\sim}{k}} - \omega(\underset{\sim}{k}) - \omega(\underset{\sim}{p}) - E(\underset{\sim}{p}+\underset{\sim}{q}) - E(\underset{\sim}{q}))^2}$$

Similarly to the nucleon situation we can obtain f_4, f_5, and f_6, and equation 11.12 reduces to an equation for $\omega(\underset{\sim}{k})$

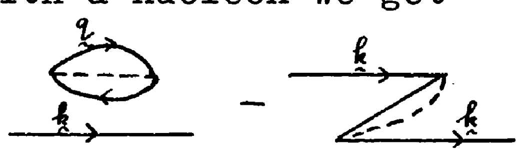
We note here that if more probability amplitudes f_i were added then solving for the two sets of amplitudes would be much more difficult as coupled integral equations would arise. In our trial function f_2 and f_3 are coupled only to f_1, since we are using the interaction Hamiltonian 10.16, and similarly for f_5, f_6, and f_4.

12. Vacuum Subtraction.

Because we use the bare vacuum, our trial wave functions have the same vacuum effect trouble as occurs in the Old Tamm-Dancoff formulation of the scattering by Dyson et al. (1954).

We see that only the two amplitudes f_3 and f_6 couple the vacuum fluctuations to the systems. These two states can contain the closed loop,  , together with a bare particle, and this loop is the lowest order vacuum diagram which is allowed by the interaction Hamiltonian. In any evaluation we find divergent contributions from terms containing this loop, and hence we must subtract it out.

It is found, e.g. see equation 13.2, that the state, with amplitude f_3 , gives results in accordance with the Pauli Exclusion Principle. When the loop occurs with a nucleon we get



i.e. unrestricted pairs formed in the nucleon's cloud minus the case with two identical nucleons.

The meson case with f_6 is similar except that the second term is added, not subtracted, due to the meson's Bose statistics.

In the case of the weak coupling constant any graphs including the vacuum closed loop are dropped, since they are exactly equal to the expressions resulting from a similar calculation for the vacuum alone; see Matthews and Salam (1952) and Moorhouse (1953). For a realistic coupling constant this would be only approximate and we should take for the expressions in our model

'Vacuum with particle present - Vacuum alone.'

This, however, would entail quite a large amount of work, and would necessitate the formulation of a physical vacuum. For the sake of simplicity throughout the problem, we neglect any graph which includes an unconnected vacuum loop. It is to be hoped that this consistent approximation does not have any serious effect on our results. This procedure is the same as the exact weak coupling method mentioned above.

13. Energy Values for Renormalisation Terms.

In this section we endeavour to find values for the energies $\delta E(\underset{\sim}{k})$ and $\delta \omega(\underset{\sim}{k})$ of the renormalisation terms 10.18. As we see from 10.21, it is equivalent to find $E(\underset{\sim}{k})$ and $\omega(\underset{\sim}{k})$. Due to our method of renormalisation we find that we have to solve coupled integral equations, and the numerical work is made much easier if we assume that we can write the energies in the following form, to a good approximation,

$$\begin{aligned} E(\underset{\sim}{k}) &= (M_0^2 + \underset{\sim}{k}^2)^{1/2} \\ \omega(\underset{\sim}{k}) &= (\mu_0^2 + \underset{\sim}{k}^2)^{1/2} \end{aligned} \qquad 13.1$$

In a relatavistic covariant theory this would, of course, be exact. We note from the equations 13.2 that as the coupling constant $g \to 0$ then $E(\underset{\sim}{k}) \to E_{\underset{\sim}{k}}$ and $\omega(\underset{\sim}{k}) \to \omega_{\underset{\sim}{k}}$ and so with the above assumption we have $M_0 \to M$ and $\mu_0 \to \mu$. The problem thus reduces to finding the masses M_0 and μ_0 , which for convenience we shall call the 'bare' masses.

Substituting equations 11.6 and 11.7 into 11.5

and cancelling $f_1(\underset{\sim}{k})$ throughout, we obtain

$$E_{\underset{\sim}{k}} - E(\underset{\sim}{k}) = +\frac{g^2}{(2\pi)^3}\cdot 3 \int \frac{d\underset{\sim}{p}}{2\omega_{\underset{\sim}{p}}} \frac{(u^*_{\underset{\sim}{k}}\,\gamma\,\Lambda^+(\underset{\sim}{k}-\underset{\sim}{p})\,\gamma\,u_{\underset{\sim}{k}})}{(E_{\underset{\sim}{k}} - \omega(\underset{\sim}{p}) - E(\underset{\sim}{k}-\underset{\sim}{p}))}$$

$$- \frac{g^2}{(2\pi)^3}\, 3 \int \frac{d\underset{\sim}{p}}{2\omega_{\underset{\sim}{p}}} \cdot \frac{(u^*_{\underset{\sim}{k}}\,\gamma\,\Lambda^-(\underset{\sim}{k}+\underset{\sim}{p})\,\gamma\,u_{\underset{\sim}{k}})}{(E_{\underset{\sim}{k}} - 2E(\underset{\sim}{k}) - E(-\underset{\sim}{k}+\underset{\sim}{p}) - \omega(\underset{\sim}{p}))} \qquad 13.2$$

$$+ \frac{g^2}{(2\pi)^3}\, 6 \int \frac{d\underset{\sim}{q}\, d\underset{\sim}{p}}{2\omega_{\underset{\sim}{p}}} \cdot \frac{T_r\,(\Lambda^-(\underset{\sim}{q}+\underset{\sim}{p})\,\gamma\,\Lambda^+(\underset{\sim}{q})\,\gamma)}{(E_{\underset{\sim}{k}} - E(\underset{\sim}{k}) - E(\underset{\sim}{q}+\underset{\sim}{k}) - E(\underset{\sim}{q}) - \omega(\underset{\sim}{p}))}$$

T_r is an operator which implies that the trace has to be taken of the expression which follows. The τ matrices are the Pauli spin matrices and their properties used are

$$\sum_i \tau_i \tau_i = 3$$

$$\text{Trace}\left(\sum_i \tau_i \tau_i\right) = 6 \qquad 13.3$$

In an exactly similar manner we obtain from equation 11.12

$$\omega_{\underset{\sim}{k}} - \omega(\underset{\sim}{k}) = \frac{g^2}{(2\pi)^3}\, 2 \int \frac{d\underset{\sim}{p}}{2\omega_{\underset{\sim}{k}}} \cdot \frac{T_r\,(\Lambda^-(\underset{\sim}{p})\,\gamma\,\Lambda^+(\underset{\sim}{k}+\underset{\sim}{p})\,\gamma}{(\omega_{\underset{\sim}{k}} - E(\underset{\sim}{k}+\underset{\sim}{p}) - E(\underset{\sim}{p}))}$$

$$+\frac{g^2}{(2\pi)^3}\, 2\int \frac{d\underset{\sim}{p}}{2\omega_{\underset{\sim}{k}}} \cdot \frac{T_r\left(\Lambda^-(\underset{\sim}{p})\,\gamma\,\Lambda^+(\underset{\sim}{p}-\underset{\sim}{k})\,\gamma\right)}{\left(\omega_{\underset{\sim}{k}} - 2\omega_{(\underset{\sim}{k})} - E_{(\underset{\sim}{p})} - E_{(\underset{\sim}{p}-\underset{\sim}{k})}\right)}$$

$$+\frac{g^2}{(2\pi)^3}\, 6\int \frac{d\underset{\sim}{p}\, d\underset{\sim}{q}\;\; T_r\left(\Lambda^-(\underset{\sim}{p})\,\gamma\,\Lambda^+(\underset{\sim}{p}-\underset{\sim}{q})\,\gamma\right)}{2\omega_{\underset{\sim}{q}}\left(\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} - E_{(\underset{\sim}{p})} - E_{(\underset{\sim}{p}-\underset{\sim}{q})} - \omega_{(\underset{\sim}{q})}\right)} \qquad 13.3$$

We use the following property of the projection operator and the matrix γ which equals $i\beta\gamma_5$,

$$\gamma\,\Lambda^{\pm}(\underset{\sim}{k})\,\gamma = \Lambda^{\mp}(\underset{\sim}{k}) \qquad 13.4$$

This can be easily proved by writing the matrices as Dirac γ matrices and anti-commuting them as usual.

We average over the spinor $u_{\underset{\sim}{k}}$ inequation 13.2 and evaluate the traces to obtain.

$$E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} = \frac{g^2}{(2\pi)^3}\, 3\int \frac{d\underset{\sim}{q}\left(E_{\underset{\sim}{q}}E_{\underset{\sim}{k}} - M^2 - \underset{\sim}{q}\cdot\underset{\sim}{k}\right)}{2\omega_{\underset{\sim}{k}-\underset{\sim}{q}}\; 2E_{\underset{\sim}{q}}E_{\underset{\sim}{k}}\left(E_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k}-\underset{\sim}{q})} - E_{(\underset{\sim}{q})}\right)}$$

$$-\frac{g^2}{(2\pi)^3}\, 3\int \frac{d\underset{\sim}{q}\left(E_{\underset{\sim}{q}}E_{\underset{\sim}{k}} + M^2 - \underset{\sim}{q}\cdot\underset{\sim}{k}\right)}{2\omega_{\underset{\sim}{k}+\underset{\sim}{q}}\, 2E_{\underset{\sim}{q}}E_{\underset{\sim}{k}}\left(E_{\underset{\sim}{k}} - 2E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{q})} - \omega_{(\underset{\sim}{k}+\underset{\sim}{q})}\right)} \qquad 13.5$$

$$+\frac{g^2}{(2\pi)^3}\, 6\int \frac{d\underset{\sim}{q}\, d\underset{\sim}{p}\left(E_{\underset{\sim}{q}}E_{\underset{\sim}{p}} + M^2 - \underset{\sim}{q}\cdot\underset{\sim}{p}\right)}{2\omega_{\underset{\sim}{q}+\underset{\sim}{p}}\; E_{\underset{\sim}{q}}E_{\underset{\sim}{p}}\left(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{q})} - E_{(\underset{\sim}{p})} - \omega_{(\underset{\sim}{q}+\underset{\sim}{p})}\right)}$$

and

$$\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} = \frac{g^2}{(2\pi)^3} \cdot 2 \cdot \int \frac{d\underset{\sim}{q}\ (E_{\underset{\sim}{q}} E_{\underset{\sim}{k}+\underset{\sim}{q}} + (\underset{\sim}{k}+\underset{\sim}{q}) \cdot \underset{\sim}{q} + M^2)}{2\omega_{\underset{\sim}{k}}\ E_{\underset{\sim}{q}} E_{\underset{\sim}{k}+\underset{\sim}{q}} \ (\omega_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - E_{(\underset{\sim}{q})})} \qquad 13.6$$

$$+ \frac{g^2}{(2\pi)^3} \cdot 2 \cdot \int \frac{d\underset{\sim}{q}\ (E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} + (\underset{\sim}{k}+\underset{\sim}{q}) \cdot \underset{\sim}{q} + M^2)}{2\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} \ (\omega_{\underset{\sim}{k}} - 2\omega_{(\underset{\sim}{k})} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - E_{(\underset{\sim}{q})})}$$

$$+ \frac{g^2}{(2\pi)^3} \cdot 6 \cdot \int \frac{d\underset{\sim}{q}\, d\underset{\sim}{p}\ (E_{\underset{\sim}{p}} E_{\underset{\sim}{q}} - \underset{\sim}{p} \cdot \underset{\sim}{q} + M^2)}{2\omega_{\underset{\sim}{p}+\underset{\sim}{q}}\ E_{\underset{\sim}{p}} E_{\underset{\sim}{q}} (\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} - \omega_{(\underset{\sim}{p}+\underset{\sim}{q})} - E_{(\underset{\sim}{p})} - E_{(\underset{\sim}{q})})}$$

We introduce a smooth cut-off function of the form $\frac{A^2}{A^2+\underset{\sim}{P}^2}$ for each nucleon line at each vertex. $\underset{\sim}{P}$ is the momentum value of the nucleon. This is amply sufficient to ensure convergence in all the integrals.

According to our proposed vacuum subtraction, we now omit the third term on the right hand side of equations 13.5 and 13.6. If we now put $\underset{\sim}{k} = 0$ we arrive at the final integral equations for the bare masses M_0 and μ_0.

$$M_0 = M$$

$$- \frac{3g^2}{8\pi^2} \int \frac{q^2 dq \left[A^2 / A^2 + q^2 \right]^2 (E_q - M)}{\omega_q \, E_q \, (M - \omega(q) - E(q))}$$ 13.7

$$+ \frac{3g^2}{8\pi^2} \int \frac{q^2 dq \left[A^2 / A^2 + q^2 \right]^2 (E_q + M)}{\omega_q \, E_q \, (M - 2|M_0| - E(q) - \omega(q))}$$

and

$$\mu_0 = 1$$

$$- \frac{g^2}{\pi^2} \int \frac{q^2 dq \left[A^2 / A^2 + q^2 \right]^4}{(1 - 2 E(q))}$$ 13.8

$$- \frac{g^2}{\pi^2} \int \frac{q^2 dq \left[A^2 / A^2 + q^2 \right]^4}{(1 - 2|\mu_0| - 2 E(q))}$$

Here we have used

$$\int d\underset{\sim}{q} = \int q^2 dq \; d\cos\theta \; d\phi$$ 13.9

$$= 4\pi \int q^2 dq .$$

13.1 Calculation and Results.

As we see the equations 13.7 and 13.8 are non-linear since the denominators are dependent on the bare masses and cannot be expanded. The most convenient method of solution is to use initial trial values for the bare masses in the integrals and to reiterate the coupled equations.

A programme for the D.E.U.C.E. computer was written to do this. An interpretive scheme, Tabular Interpretive Programme (T.I.P.) deals mainly with data in the form of columns, and since programmes can be written fairly quickly in this scheme, it is used throughout this work.

The programme reiterated and punched out the current values of M_0 and μ_0 every 45 seconds. The numerical data used is as follows:-

The parameter A for the cut-off was taken equal to M the nucleon mass:

The integrations were performed numerically using a three point Simpsons Rule with weights:

$$W_i = \tfrac{1}{3}(1, 4, 1) \times \text{ Interval between points:}$$

The range of pivoted points was

$$0(\cdot 5)7(1)15(4)39 \quad ,$$

29 points in all.

The bare masses were evaluated for different values of the coupling constant and the results obtained are contained in figures 13.1 and 13.2

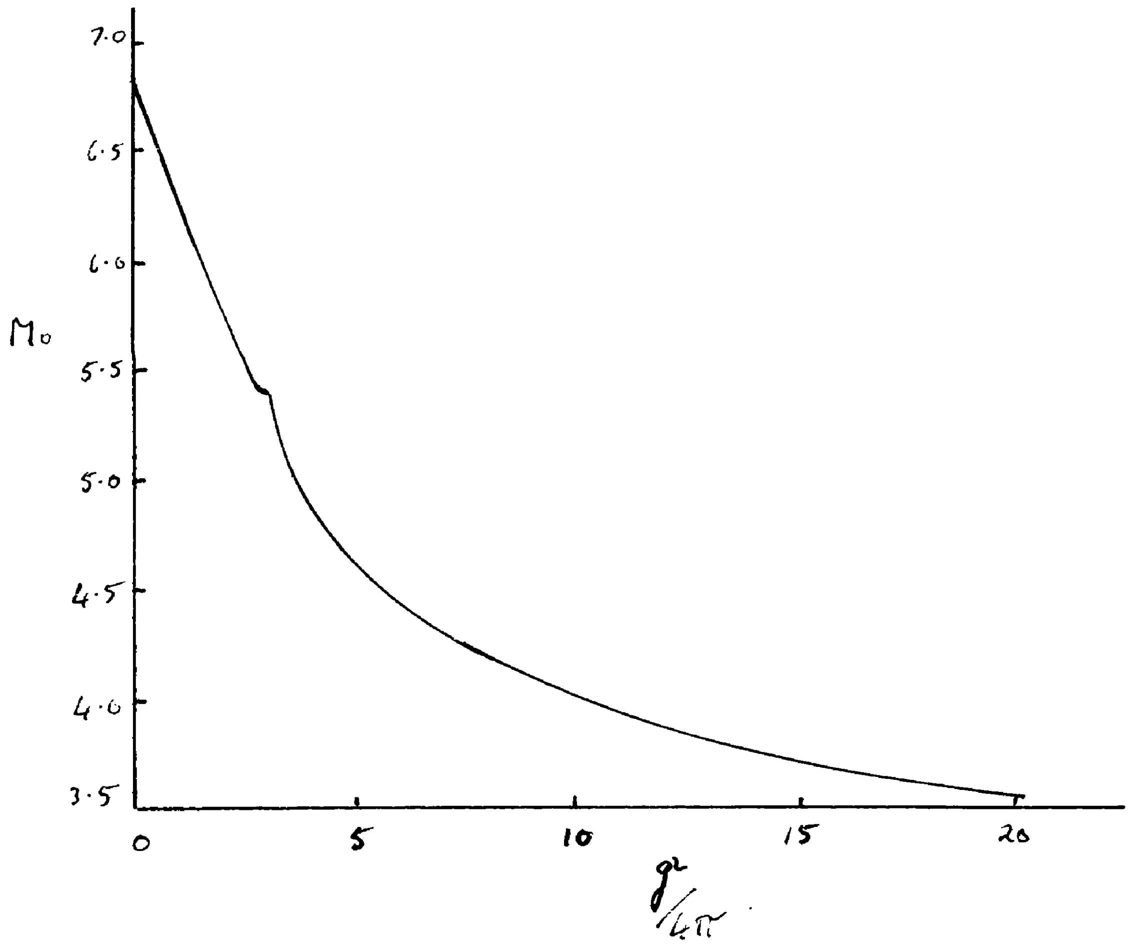


Fig. 13.1. Value of M_0 as a function of $g^2/4\pi$.

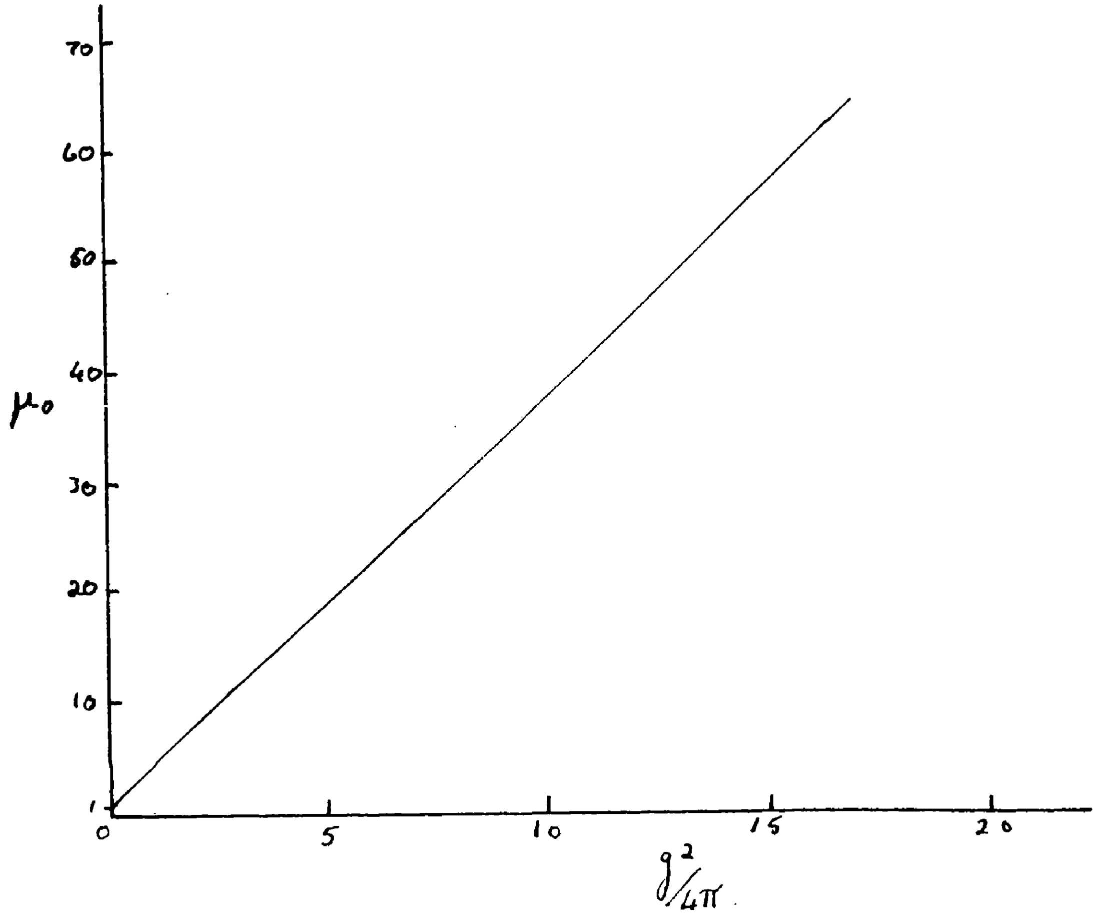


Figure 13.2. Value of μ_0 as a function of $g^2/4\pi$

In figure 13.1 the top section of the curve is fairly linear until $g^2/4\pi \sim 2.6$ where it curves to meet the lower section. For points around the juncture the masses were evaluated, and a discontinuity in gradient was found.

In figure 13.2, the results are very nearly linear, and the large values are mainly the result of the comparatively small denominator $[1 - \lambda E(\underset{\sim}{q})]$ in the first integral of 13.8.

For the purposes of comparison, the results in Table 13.1 were also evaluated for a coupling constant given by $g^2/4\pi = 15$ and initial trial values M and μ.

	M_0	μ_0
Theory as above.	3.734	57.41
Theory as above but equations uncoupled.	-	42.53
Theory not with a smooth cut-off, but with a sharp cut off at 5.6.	5.118	97.84

Table 13.1. Mass values for comparison at $g^2/4\pi = 15$

When the equations are uncoupled no value of M_0 was found for trial values of M and μ because in this case there is a pole in the first integrand in equation 13.7.

For a sharp cut-off the high value of M_0 occurs because the contribution from the first integral term of equation 13.7 is not cut down at higher momenta, and so it cancels more of the second integral term, which gives negative results.

14. The Normalisation Functions.

We shall call the amplitudes f_1 and f_4 the normalisation functions, they are determined by the normalisation conditions on the physical particles. Using the same procedures as were used in section 13, we obtain the following equations, for f_1 and f_4, from equations 11.8 and 11.15 respectively.

$$
\begin{aligned}
f_1^{-2}(\underset{\sim}{k}) = 1 &+ \frac{3g^2}{(2\pi)^3}\int \frac{d\Omega\, q^2 dq\, (E_{\underset{\sim}{k}} E_{\underset{\sim}{q}} - M^2 + \underset{\sim}{k}\cdot\underset{\sim}{q})}{4\,\omega_{\underset{\sim}{q}+\underset{\sim}{k}} E_{\underset{\sim}{k}} E_{\underset{\sim}{q}} \left(E_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{q}+\underset{\sim}{k})} - E_{(\underset{\sim}{q})}\right)^2} \\
&- \frac{3g^2}{(2\pi)^3}\int \frac{d\Omega\, q^2 dq\, (E_{\underset{\sim}{k}} E_{\underset{\sim}{q}} + M^2 - \underset{\sim}{k}\cdot\underset{\sim}{q})}{4\,\omega_{\underset{\sim}{q}+\underset{\sim}{k}} E_{\underset{\sim}{k}} E_{\underset{\sim}{q}} \left(E_{\underset{\sim}{k}} - 2E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{q})} - \omega_{(\underset{\sim}{q}+\underset{\sim}{k})}\right)^2} \\
&+ \frac{6g^2}{(2\pi)^3}\int \frac{d\underset{\sim}{p}\, d\underset{\sim}{q}\, (E_{\underset{\sim}{p}} E_{\underset{\sim}{q}} + M^2 - \underset{\sim}{q}\cdot\underset{\sim}{p})}{2\,\omega_{\underset{\sim}{q}+\underset{\sim}{p}} E_{\underset{\sim}{p}} E_{\underset{\sim}{q}} \left(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{p})} - E_{(\underset{\sim}{q})} - \omega_{(\underset{\sim}{p}+\underset{\sim}{q})}\right)^2}
\end{aligned}
\qquad 14.1
$$

$$
\begin{aligned}
f_4^{-2}(\underset{\sim}{k}) = 1 &+ \frac{2g^2}{(2\pi)^3}\int \frac{d\Omega\, q^2 dq\, (E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} + E_{\underset{\sim}{q}}^2 + \underset{\sim}{k}\cdot\underset{\sim}{q})}{2\,\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} \left(\omega_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - E_{(\underset{\sim}{q})}\right)^2} \\
&+ \frac{2g^2}{(2\pi)^3}\int \frac{d\Omega\, q^2 dq\, (E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} + E_{\underset{\sim}{q}}^2 + \underset{\sim}{k}\cdot\underset{\sim}{q})}{2\,\omega_{\underset{\sim}{k}} E_{\underset{\sim}{k}+\underset{\sim}{q}} E_{\underset{\sim}{q}} \left(\omega_{\underset{\sim}{k}} - 2\omega_{(\underset{\sim}{k})} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - E_{(\underset{\sim}{q})}\right)^2} \\
&+ \frac{6g^2}{(2\pi)^3}\int \frac{d\underset{\sim}{q}\, d\underset{\sim}{p}\, (E_{\underset{\sim}{p}+\underset{\sim}{q}} E_{\underset{\sim}{q}} + E_{\underset{\sim}{q}}^2 + \underset{\sim}{p}\cdot\underset{\sim}{q})}{2\,\omega_{\underset{\sim}{p}} E_{\underset{\sim}{p}+\underset{\sim}{q}} E_{\underset{\sim}{q}} \left(\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} - \omega_{(\underset{\sim}{p})} - E_{(\underset{\sim}{p}+\underset{\sim}{q})} - E_{(\underset{\sim}{q})}\right)^2}
\end{aligned}
\qquad 14.2
$$

As we shall see later, these functions appear in the scattering kernal and are quite important in determining the $P_{3/2 \cdot 3/2}$ resonance. The last term in each equation is the vacuum contribution which we neglect. The same cut off as before is added for each spinor which appeared in the equations. An approximation is made here for particles of momentum $(\underset{\sim}{k} + \underset{\sim}{q})$ say. Instead of the cut off $A^2/A^2 + (\underset{\sim}{k} + \underset{\sim}{q})^2$ we take $A^2/A^2 + k^2 + q^2$ so that the cut off functions are not angular dependent.

As can be seen from the equations there are two integrations to be performed numerically. For the angular integration we take the 11 pivotal values for the cosine of the angle, $-1\ (\cdot 2)\ +1$ and use a three point Simpson's rule with weights $w_i = 1/3\,(1,4,1)\,0\cdot 2$. All the other numerical data is as in section 13.1.

A T.I.P. programme was written to find $f_1(\underset{\sim}{k})$ and $f_4(\underset{\sim}{k})$. Due to the double integration, about 6 minutes 8 seconds are required to evaluate and punch out one value each of $f_1(\underset{\sim}{k})$ and $f_4(\underset{\sim}{k})$.

Two cases were evaluated with the following parameters:-

1) $g^2/4\pi = 15$ with $M_0 = M$ and $\mu_0 = \mu$.

2) $g^2/4\pi = 15$ with the values of M_0 and μ_0 found in section 13.1 for $g^2/4\pi = 15$.

The values found are shown in figures 14.1 and 14.2.

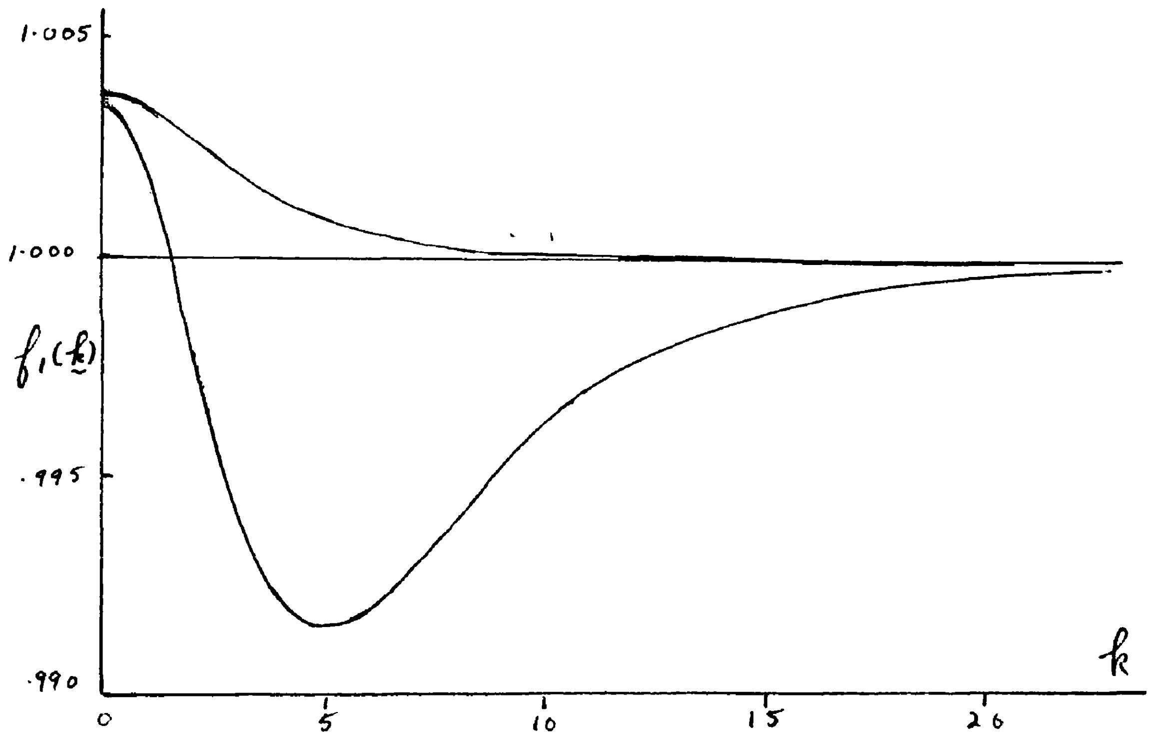


Figure 14.1 $f_1(\underset{\sim}{k})$ as a function of k.

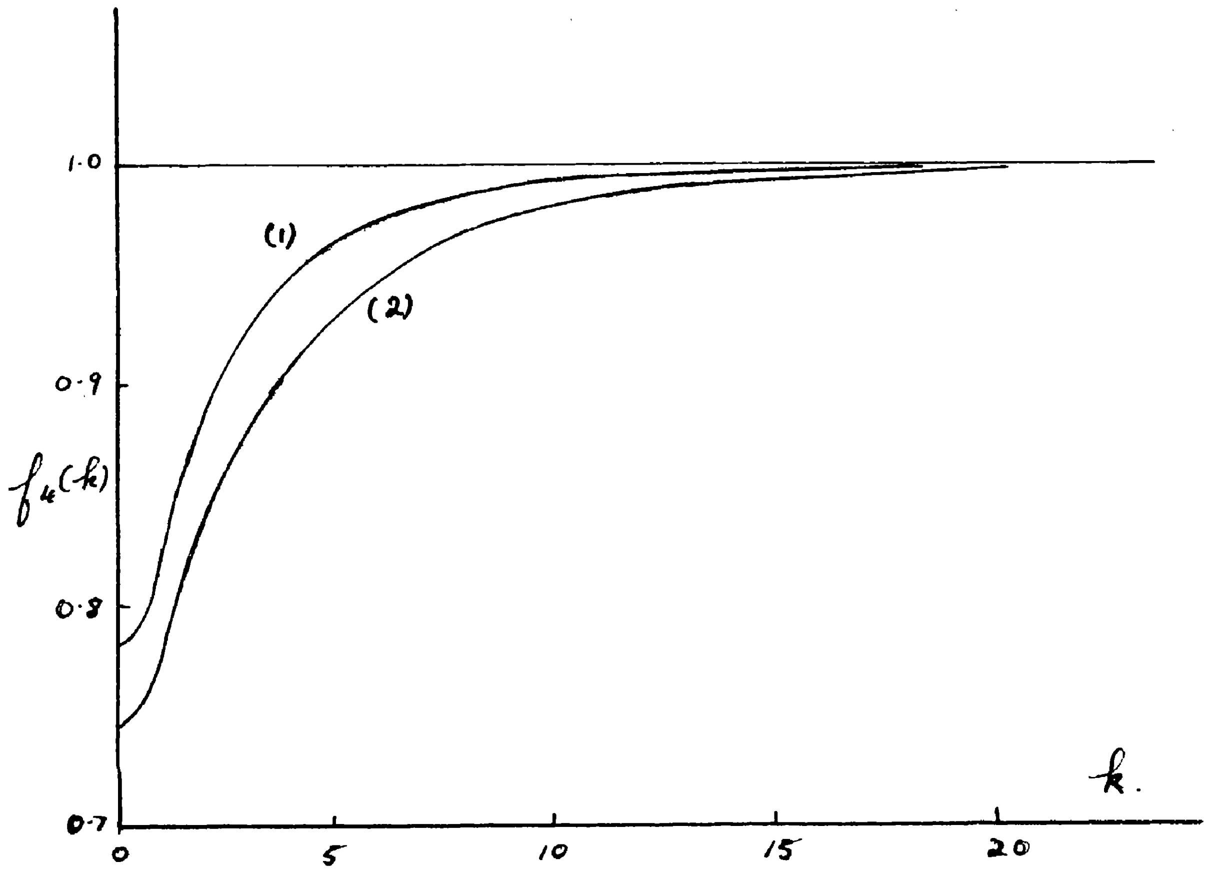


Figure 14.2 $f_4(k)$ as a function of k.

For case (1) the values for another coupling constant, g say, are found from the following rule.

$$f_g(\underset{\sim}{k}) = \left[\left(f_{15}^{-2}(\underset{\sim}{k}) - 1 \right) \frac{g^2}{60\pi} - 1 \right]^{-1/2} \qquad 14.3$$

For the range of coupling constants in which we are interested, the accuracy of the computer is sufficient for this rule to be used, even when the normalisation functions are quite close to unity. When the accuracy breaks down the error introduced can be neglected compared to unity.

15. Coupling Constant Renormalisation.

The coupling constant, g , which we have been using is the coupling strength between the 'bare' particles, and therefore it is not experimentally observable. In order to evaluate the scattering phase shifts etc., in our theory, we require to know the value of g which corresponds to the renormalised coupling constant, g_r , which is generally taken as $g_r^2/4\pi \simeq 15$.

In electrodynamics there are several different requirements which lead to the unique charge. However, as Källén (1954) has pointed out, the renormalised coupling constant is not uniquely defined in meson theory. Two very important papers on this topic are Kroll and Ruderman (1954) and Deser, Thirring and Goldberger (1954). They approach the coupling constant renormalisation problem from different definitions and they arrive at different results. The Deser et al. prescription leads to a small coupling constant, and, if the method is to be believed, severe doubt is cast on the validity of pseudoscalar coupling theory.

Since the Kroll and Ruderman requirements have certain similarities in form to the renormalisation in electrodynamics, it has been used by a number of authors examining the meson renormalisation problem.

With our model we take the following equation as the definition of coupling constant renormalisation.

$$g_r = \mathcal{L}_{k \to 0} \mathcal{L}_{k' \to k} g \frac{(\Psi_0 N_{\underset{\sim}{k}'} | \beta \gamma_5 \tau_\lambda | N^*_{\underset{\sim}{k}} \Psi_0)}{(\Psi_0 C_{\underset{\sim}{k}'} | \beta \gamma_5 \tau_\lambda | C^*_{\underset{\sim}{k}} \Psi_0)} f_4(\underset{\sim}{k}' - \underset{\sim}{k}') \qquad 15.1$$

This is the Watson-Lepore definition as used in Nuclear Physics, and is similar to the one in Chew's static theory. We note the resemblance to the definition of charge renormalisation in quantum electrodynamics by means of the formalism of Dyson (1949).

We can write in Dyson's notation

$$g_r = Z_1^{-1} Z_2 Z_3^{1/2} g \qquad 15.2$$

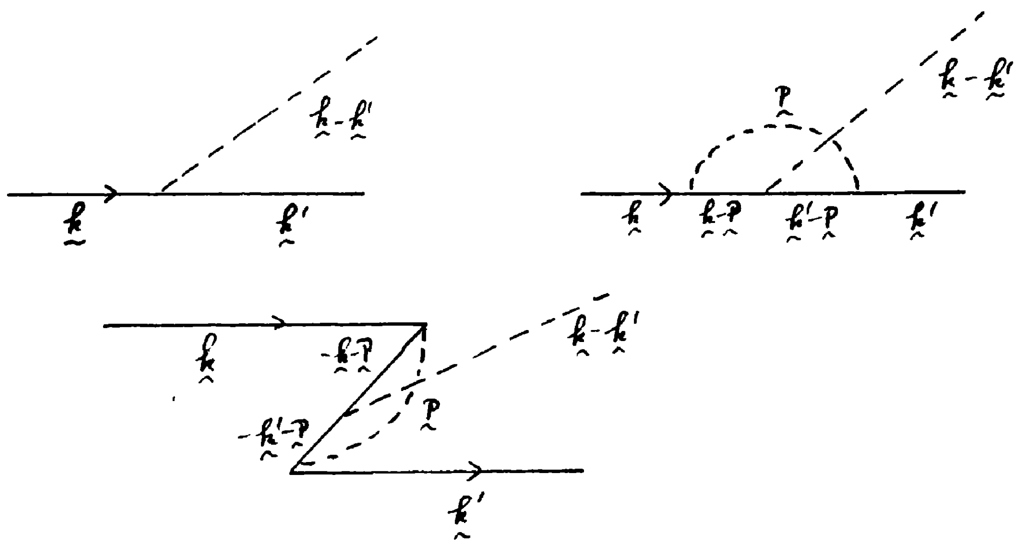
where
$$Z_3^{1/2} = \mathcal{L}_{k \to 0} f_4(\underset{\sim}{k})$$

$$Z_2^{1/2} = \mathcal{L}_{k \to 0} f_1(\underset{\sim}{k}) \qquad 15.3$$

and Z_1^{-1} is the ratio of the renormalised vertex operator to the unrenormalised vertex operator.

It is noted that we are investigating the interaction of a nucleon with a zero energy, zero momentum meson.

In our first examination of equation 15.1 we include in the numerator only the three graphs following;



This leads us to the equation

$$Z_1^{-1} = 1 + \mathcal{L}_{k\to 0}\, \mathcal{L}_{k'\to k}\, \frac{g^2}{(2\pi)^3} \sum_{\substack{\text{INT}\\ \text{STATES}\\ \text{ETC}}} \int \frac{dq}{2\omega_q} \times \tag{15.4}$$

$$\Bigg[\frac{(u^*_{k'}\gamma\tau_i u_{k'-q})(u^*_{k'-q}|\beta\gamma_5\tau_\lambda|u_{k-q})(u^*_{k-q}\gamma\tau_i u_k)}{(E_k - \omega_{(q)} - E_{(k-q)})(E_{k'} - \omega_{(q)} - E_{(k'-q)})\,(u^*_{k'}|\beta\gamma_5\tau_\lambda|u_k)}$$

$$+ \frac{(u^*_{k'}\gamma\tau_i V_{k'+q})(V^*_{k'+q}|\beta\gamma_5\tau_\lambda|V_{k+q})(V^*_{k+q}\gamma\tau_i u_k)}{(E_k - E_{(k)} - E_{(k')} - E_{(k'+q)} - \omega_{q})(E_{k'} - E_{(k')} - E_{(k)} - E_{(k+q)} - \omega_{(q)}) \times (u^*_{k'}|\beta\gamma_5\tau_\lambda|u_k)} \Bigg]$$

Using

$$\sum_i \tau_i \, \tau_\lambda \, \tau_i = -\tau_\lambda$$

we obtain

$$Z_1^{-1} = 1 - \mathcal{L}_{k\to 0} \mathcal{L}_{k'\to k} \frac{g^2}{(2\pi)^3} \int \frac{d\underset{\sim}{q}}{2\omega_q} \times$$

$$\times \Bigg[\frac{(u^*_{k'} \Lambda^-(\underset{\sim}{k}'+\underset{\sim}{q}) \beta \gamma_5 \Lambda^-(\underset{\sim}{k}+\underset{\sim}{q}) u_k)}{(E_k - \omega_{(q)} - E_{(k+q)})(E_{k'} - \omega_{(q)} - E_{(k'+q)}) (u^*_{k'} \beta \gamma_5 u_k)} \tag{15.5}$$

$$+ \frac{(u^*_{k'} \Lambda^+(\underset{\sim}{k}'+\underset{\sim}{q}) \beta \gamma_5 \Lambda^+(\underset{\sim}{k}+\underset{\sim}{q}) u_k)}{(E_k - E_{(k)} - E_{(k')} - E_{(k'+q)} - \omega_{(q)})(E_{k'} - E_{(k')} - E_{(k)} - E_{(k+q)} - \omega_{(q)})(u^*_{k'} \beta \gamma_5 u_k)} \Bigg]$$

Hence by the usual techniques we arrive at

$$Z_1^{-1} = 1$$

$$- \frac{g^2}{8\pi^2} \int \frac{q^2\, dq}{\omega_q E_q^3} \left\{ \frac{M^2(E_q - M) - \tfrac{2}{3} q^2 M}{(M - \omega_{(q)} - E_{(q)})^2} \right. \tag{15.6}$$

$$\left. + \frac{M^2(E_q + M) + \tfrac{2}{3} q^2 M}{(M - 2M_0 - \omega_{(q)} - E_{(q)})^2} \right\}$$

From our rule for the cut off we add the function

$$\left[\frac{A^2}{A^2 + \underset{\sim}{q}^2} \right]^4$$

to the integrands. We note from the previous section that $f_1(0)$ and $f_4(0)$ are simple to evaluate. For convenience then we calculated g_r, equal to $Z_1^{-1} Z_2 Z_3^{1/2} g$, in the one programme.

15.1 Results.

The T.I.P. programme, which was written, evaluated g_r for triads of values (g, M_0, μ_0) The time required being about 60 seconds for the first set and 35 seconds thereafter due to an inner loop in the programme.

The pivoted points and integration weights were as before, and again $A = M$.

The results are shown in figure 15.1, for the two cases:-

(1) $M_0 = M$ and $\mu_0 = \mu$

(2) M_0 and μ_0 the masses corresponding to the coupling constant in the triad.

As we see from figure 15.1 there is no real g corresponding to $g_r^2/4\pi = 15$ in either case (1) or (2). In an attempt to overcome this we could include further graphs in equation 15.4.

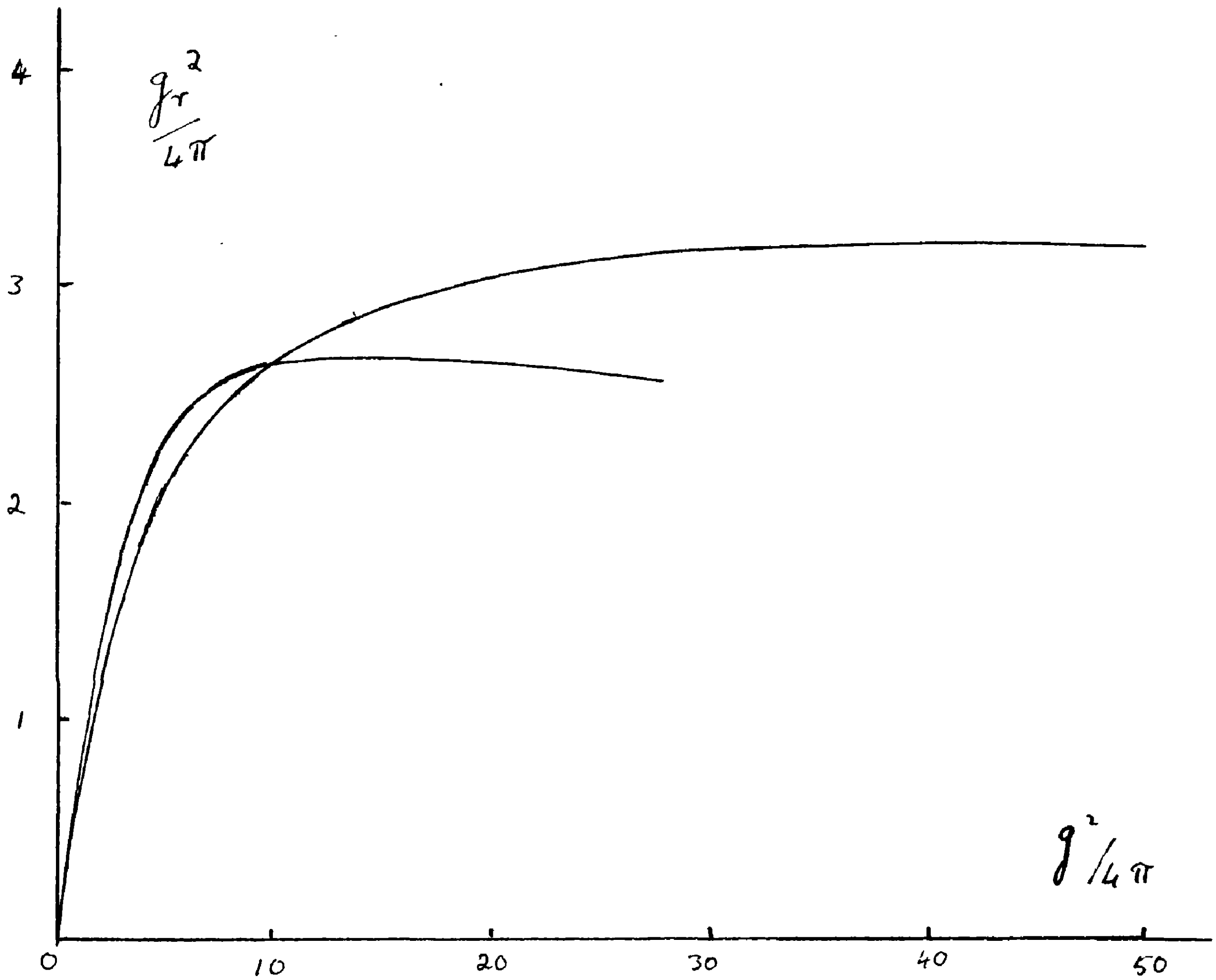
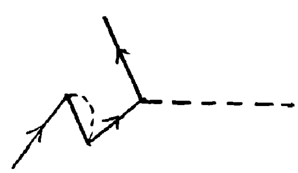


Figure 15.1 g_r as a function of (g, M_0, μ_0)

A possible graph being  which could occur dúe to our vacuum subtraction. However at present we decide to determine g by fitting the scattering $P_{3/2 \cdot 3/2}$ phase shift to the experimental results.

16. The Scattering State.

In order to describe the scattering of a physical meson from a physical nucleon, we use a trial function and a variational principle for the determination of the phase shifts.

We define the scattering wave function in the centre of mass system as

$$
\begin{aligned}
\Psi_s = & \sum_{\alpha, i} \int d\underset{\sim}{k}\, \chi(\underset{\sim}{k})\, C_{i\alpha}\, N^{*}_{\underset{\sim}{k}\alpha}\, \pi^{*}_{-\underset{\sim}{k}i}\, \Psi_0 \\
& + \sum_{\alpha} \theta_\alpha\, N^{*}_{\underset{\sim}{0}\alpha}\, \Psi_0 \qquad\qquad 16.1 \\
& + \sum_{\alpha,\beta,\gamma} \int d\underset{\sim}{k}\, d\underset{\sim}{q}\, \phi(\underset{\sim}{k},\underset{\sim}{q},\underset{\sim}{k}+\underset{\sim}{q})\, c^{*}_{\underset{\sim}{k}\alpha}\, c^{*}_{\underset{\sim}{q}\beta}\, d^{*}_{\underset{\sim}{k}+\underset{\sim}{q}\,\gamma}\, \Psi_0 \\
& + \sum_{i,j,\alpha} \int d\underset{\sim}{p}\, d\underset{\sim}{q}\, \eta(\underset{\sim}{p},\underset{\sim}{q},-\underset{\sim}{p}-\underset{\sim}{q})\, a^{*}_{\underset{\sim}{p}i}\, a^{*}_{\underset{\sim}{q}j}\, c^{*}_{-\underset{\sim}{p}-\underset{\sim}{q}\,\alpha}\, \Psi_0 \\
& + \sum_{\substack{\alpha,\beta,\gamma \\ i,j}} \int d\underset{\sim}{k}\, d\underset{\sim}{q}\, d\underset{\sim}{p}\, d\underset{\sim}{s}\, K(\underset{\sim}{k},\underset{\sim}{q},\underset{\sim}{k}+\underset{\sim}{q}+\underset{\sim}{p}+\underset{\sim}{s},\underset{\sim}{p},\underset{\sim}{s})\,. \\
& \qquad .\, c^{*}_{\underset{\sim}{k}\alpha}\, c^{*}_{\underset{\sim}{q}\beta}\, d^{*}_{\underset{\sim}{k}+\underset{\sim}{q}+\underset{\sim}{p}+\underset{\sim}{s}\,\gamma}\, a^{*}_{\underset{\sim}{p}i}\, a^{*}_{\underset{\sim}{s}j}\, \Psi_0
\end{aligned}
$$

This trial function is for a state of particular total angular momentum , J, and total isotopic spin, I. The $C_{i\alpha}$ are the appropriate numerical factors required to construct this I, J state from a nucleon in state α and a meson in state i. Similar constants are to be understood in the last three terms, and from here on we will also explicitly omit the $C_{i\alpha}$ for convenience. The last three terms we shall call the 'interaction' terms.

The scattering wave function $\chi(k)$ describes the pi-meson scattering for the given I, J state. The functions χ, ϕ, η, and K, together with the constant θ are determined by the variational procedure

$$\delta(\Psi_s \mid H - E - \omega \mid \Psi_s) = 0 \tag{16.2}$$

where E and ω are the incident nucleon and meson energies respectively. We use the total Hamiltonian of equation 1.22 with $H_0 = H_n' + H_N'$.

The θ_α term only occurs for a $P_{1/2 \cdot 1/2}$ state, and taking the variation 16.2 with respect to θ_α gives us the requirement that the scattering state Ψ_s should be orthogonal to the physical nucleon state.

We now take the variations with respect to the functions as noted. For convenience we write the results in a 'shorthand' notation, in which we omit state indices and momenta, and use the Bra and Ket notation.

Variations with respect to χ

$$
\begin{aligned}
-\langle \pi N | H-E | \int \chi \pi N \rangle = \\
& \langle \pi N | H-E | \int \phi c c d \rangle \\
& + \langle \pi N | H-E | \theta N \rangle \\
& + \langle \pi N | H-E | \int \eta \, c a a \rangle \\
& + \langle \pi N | H-E | \int K \, c c d a a \rangle
\end{aligned}
\tag{16.3}
$$

Variations with respect to θ

$$
\theta = - \langle N | \int \chi N \pi \rangle \tag{16.4}
$$

Variations with respect to ϕ

$$
\begin{aligned}
-\langle d c c | H_0 - E | \int \phi \, c c d \rangle = \\
& \langle d c c | H_I | \theta N \rangle \\
& + \langle d c c | H-E | \int \chi N \pi \rangle
\end{aligned}
\tag{16.5}
$$

Variations with respect to η

$$- \langle c\,a\,a \,|\, H_0 - E \,|\, \int \eta \; a\,a\,c \rangle = \langle c\,a\,a \,|\, H - E \,|\, \int \chi N \pi \rangle + \langle c\,a\,a \,|\, H_I \,|\, \theta N \rangle \qquad 16.6$$

Variations with respect to κ

$$- \langle a\,a\,d\,c\,c \,|\, H_0 - E \,|\, \int \kappa \; c\,c\,d\,a\,a \rangle = \langle a\,a\,d\,c\,c \,|\, H - E \,|\, \int \chi N \pi \rangle + \langle a\,a\,d\,c\,c \,|\, H_I \,|\, \theta N \rangle \qquad 16.7$$

We see that, except in the $P_{\frac{1}{2}\cdot\frac{1}{2}}$ case, the 'interaction' terms each couple back only to the scattering function χ . The $P_{\frac{1}{2}\cdot\frac{1}{2}}$ case is neglected at present.

In equations 16.4 to 16.7 we can now evaluate the matrix elements. We write the Hamiltonians H_0 and H_I of equation 1.22 explicitly in terms of annihilation and creation operators, and similarly with the physical particle wave functions N and π We then commute, or anti-commute as appropriate, the creation operators to the left. Using the properties of the vacuum, we can evaluate the matrix elements in terms of the probability amplitudes of the real particles and functions of the form $u^{*}\beta\gamma_5\tau_\nu v$, where u and v are spinors. This type of expression has been seen previously in earlier sections.

Thus we have found the functions ϕ, η, and κ, of the interaction terms. On substitution back into equation 16.3 a single integral equation is obtained for χ.

The algebra indicated above has been performed, but will not be reproduced here, not only because it is extremely lengthy, but also because we mainly wish to examine the results from the first term in equation 16.1.

It might be hoped that this first term yields a large proportion of the scattering, and the evaluation of the scattering due to this term is the main aim of this thesis.

17. Integral Equation Formation.

The equation which we have to solve is

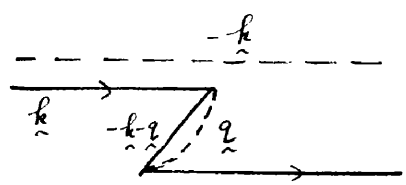
$$\left(\Psi_0 \, \pi_{-k j} \, N_{k \beta} \,\middle|\, H - E - \omega \,\middle|\, \sum_{i,\alpha} \int d s \, \chi(s) \, N^*_{s \alpha} \, \pi^*_{-s i} \, \Psi_0 \right) = 0 \qquad 17.1$$

If we write the particle creation and annihilation operators in the equation explicitly and reduce the matrix elements in the manner described, we find two types of terms. The diagonal terms have $s = -k$; the scattering terms are integrated over s , and form the kernel of the integral equation.

17.1 The Diagonal Terms.

We will examine diagonal terms from 17.1 which correspond to the three graphs.

and



For convenience and compactness in writing down terms from 17.1 we will omit Ψ_0 and $\sum_{i\alpha} \int d s \, \chi(s)$.

From the term

$$\left(f_4(\underset{\sim}{k})\, a_{-\underset{\sim}{k} j}\; f_1(\underset{\sim}{k})\, c_{\underset{\sim}{k}\beta} \,\middle|\, H_0 - E - \omega \,\middle|\, f_1(\underset{\sim}{s})\, c^*_{\underset{\sim}{s}\alpha}\; f_4(\underset{\sim}{s})\, a^*_{-\underset{\sim}{s} i} \right)$$

we get the contribution for $\underset{\sim}{s} = \underset{\sim}{k}$

$$+ f_1^2(\underset{\sim}{k})\, f_4^2(\underset{\sim}{k}) \left[E(\underset{\sim}{k}) + \omega(\underset{\sim}{k}) - E - \omega \right] \chi(\underset{\sim}{k}) \qquad 17.2$$

Similarly

$$\left(f_4(\underset{\sim}{k})\, a_{-\underset{\sim}{k} j} \int d\underset{\sim}{q}\, f_2(-\underset{\sim}{q}, \underset{\sim}{k}+\underset{\sim}{q})\, c_{-\underset{\sim}{q}}\; a_{\underset{\sim}{k}+\underset{\sim}{q}} \,\middle|\, H_I \,\middle|\, f_1(\underset{\sim}{s})\, c^*_{\underset{\sim}{s}\alpha}\; f_4(\underset{\sim}{s})\, a^*_{-\underset{\sim}{s} i} \right)$$

and its complex conjugate, give us

$$+ \frac{g^2}{(2\pi)^3} f_1^2(\underset{\sim}{k}) f_4^2(\underset{\sim}{k}) \int \frac{2\, d\underset{\sim}{q}}{\left(E_{\underset{\sim}{k}} - \omega(\underset{\sim}{q}) - E(\underset{\sim}{k}+\underset{\sim}{q})\right)} \left(u^*_{\underset{\sim}{k}}\, \Lambda^-(\underset{\sim}{k}+\underset{\sim}{q})\, u_{\underset{\sim}{k}}\right) \chi(\underset{\sim}{k}) \qquad 17.3$$

From the term

$$\left(f_4(\underset{\sim}{k})\, a_{-\underset{\sim}{k} j} \int d\underset{\sim}{q}\, f_2(-\underset{\sim}{q}, \underset{\sim}{k}+\underset{\sim}{q})\, c_{\underset{\sim}{q}}\; a_{\underset{\sim}{k}+\underset{\sim}{q}} \,\middle|\, H_0 - E - \omega \,\middle|\, \cdot \right.$$

$$\left. \int d\underset{\sim}{q}'\, f_2(-\underset{\sim}{q}', \underset{\sim}{s}+\underset{\sim}{q}')\, a^*_{\underset{\sim}{s}+\underset{\sim}{q}'}\; c^*_{-\underset{\sim}{q}'}\; f_4(\underset{\sim}{s})\, a^*_{-\underset{\sim}{s} i} \right)$$

we get the diagonal term

$$+ \frac{g^2}{(2\pi)^3} f_1^2(\underset{\sim}{k}) f_4^2(\underset{\sim}{k}) \int \frac{\left(E(\underset{\sim}{k}+\underset{\sim}{q}) + \omega(\underset{\sim}{k}) + \omega(\underset{\sim}{q}) - E - \omega\right)}{\left(E_{\underset{\sim}{k}} - \omega(\underset{\sim}{q}) - E(\underset{\sim}{k}+\underset{\sim}{q})\right)^2}\, d\underset{\sim}{q} \left(u^*_{\underset{\sim}{k}}\, \Lambda^-(\underset{\sim}{k}+\underset{\sim}{q})\, u_{\underset{\sim}{k}}\right) \chi(\underset{\sim}{k}) \qquad 17.4$$

Also

$$\left(f_4(\underset{\sim}{k})\, a_{-\underset{\sim}{k}j} \int d\underset{\sim}{k}'\, d\underset{\sim}{q}\, d\underset{\sim}{q}'\, f_3(\underset{\sim}{k}', \underset{\sim}{q}, \underset{\sim}{k}'+\underset{\sim}{q}+\underset{\sim}{q}'-\underset{\sim}{k}, \underset{\sim}{q}')\, a_{\underset{\sim}{q}'}\, d_{\underset{\sim}{k}'+\underset{\sim}{q}'+\underset{\sim}{q}-\underset{\sim}{k}}\, c_{\underset{\sim}{q}}\, c_{\underset{\sim}{k}'} \right| \times$$
$$\left| H_I \right| f_1(\underset{\sim}{s})\, c^*_{\underset{\sim}{s}\alpha}\, f_4(\underset{\sim}{s})\, a^*_{-\underset{\sim}{s}i} \Big)$$

and its complex conjugate give us

$$-\frac{g^2}{(2\pi)^3} f_1^2(\underset{\sim}{k}) f_4^2(\underset{\sim}{k}) \int \frac{2\, d\underset{\sim}{q}}{\left(E_{\underset{\sim}{k}} - 2E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - \omega_{(\underset{\sim}{q})}\right)} \left(u^*_{\underset{\sim}{k}}\, \Lambda^+(\underset{\sim}{k}+\underset{\sim}{q})\, u_{\underset{\sim}{k}}\right) \chi(\underset{\sim}{k}) \qquad 17.5$$

Finally one of the contributions from

$$\left(f_4(\underset{\sim}{k})\, a_{-\underset{\sim}{k}j} \int f_3\, a\, d\, c\, c \right| H_0 - E - \omega \left| \int f_3\, c^* c^* d^* a^* f_4(\underset{\sim}{s})\, a_{-\underset{\sim}{s}i} \right)$$

is

$$-\frac{g^2}{(2\pi)^3} f_1^2(\underset{\sim}{k}) f_4^2(\underset{\sim}{k}) \int \frac{\left(E_{(\underset{\sim}{k}+\underset{\sim}{q})} + 2E_{(\underset{\sim}{k})} + \omega_{(\underset{\sim}{k})} + \omega_{(\underset{\sim}{q})} - E - \omega\right) d\underset{\sim}{q}}{\left(E_{\underset{\sim}{k}} - 2E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - \omega_{(\underset{\sim}{q})}\right)^2} \times \left(u^*_{\underset{\sim}{k}} \Lambda^+(\underset{\sim}{k}+\underset{\sim}{q}) u_{\underset{\sim}{k}}\right) \chi(\underset{\sim}{k}) \qquad 17.6$$

We add 17.2, .4, .5 and .6 together and find

$$+ f_1^2(\underset{\sim}{k}) f_4^2(\underset{\sim}{k})\, \chi(\underset{\sim}{k}) \Big\{ \left[\omega_{(\underset{\sim}{k})} + E_{(\underset{\sim}{k})} - E - \omega\right] +$$

$$+ \frac{g^2}{(2\pi)^3} \int \left[\frac{E_{\underset{\sim}{k}} + \omega_{(\underset{\sim}{k})} - E - \omega}{\left(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - \omega_{(\underset{\sim}{q})}\right)^2} + \frac{1}{\left(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{q})} - \omega_{(\underset{\sim}{q})}\right)} \right] \times \qquad 17.7$$

$$\times \left(u^*_{\underset{\sim}{k}}\, \Lambda^-(\underset{\sim}{k}+\underset{\sim}{q})\, u_{\underset{\sim}{k}}\right) d\underset{\sim}{q}$$

$$-\frac{g^2}{(2\pi)^3}\int\left[\frac{E_{\underset{\sim}{k}}+\omega(\underset{\sim}{k})-E-\omega}{(E_{\underset{\sim}{k}}-2E(\underset{\sim}{k})-E(\underset{\sim}{k}+\underset{\sim}{q})-\omega(\underset{\sim}{q}))^2}+\frac{1}{(E_{\underset{\sim}{k}}-2E(\underset{\sim}{k})-E(\underset{\sim}{k}+\underset{\sim}{q})-\omega(\underset{\sim}{q})}\right]\cdot$$

$$\left.\times\left(u^*_{\underset{\sim}{k}}\Lambda^+(\underset{\sim}{k}+\underset{\sim}{q})u_{\underset{\sim}{k}}\right)d\underset{\sim}{q}\right\}$$

We see that the second term in each square bracket can be grouped together, according to equation 13.2 to form $E_{\underset{\sim}{k}}$. Taking the common factor $(E_{\underset{\sim}{k}}+\omega(\underset{\sim}{k})-E-\omega)$ out the remainder we find forms the normalisation equation 11.8 and the final result is

$$f_4^2(\underset{\sim}{k})\ (E_{\underset{\sim}{k}}+\omega(\underset{\sim}{k})-E-\omega)\ \chi(\underset{\sim}{k}) \tag{17.8}$$

We can now add up a similar set of graphs except that the meson line contains a self energy part. It is easily seen that the total contribution of all such graphs is merely

$$(E_{\underset{\sim}{k}}+\omega_{\underset{\sim}{k}}-E-\omega)\ \chi(\underset{\sim}{k}) \tag{17.9}$$

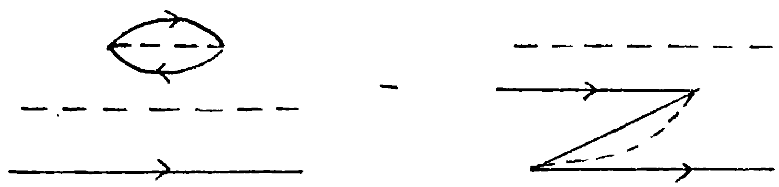
There are, however, two groups of diagonal terms which do not contribute in the above fashion.

The first arises from the fact that the wave function $N^*\pi^*\underset{\sim}{\Psi}_0$ contains two states, i.e.

$$f_1\, c^* f_6\ a^* a^* c^* d^*\ \Psi_0\quad,\quad f_3\ c^* c^* d^* a^*\ f_4\ a^*\Psi_0$$

which are not orthogonal to each other.

The second set occurs because we have two real particles. The presence of the second particle affects the vacuum associated with the first particle, eg. if a vacuum loop occurs with the meson,  , then, since there is another nucleon, the Pauli Principle gives us



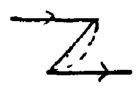
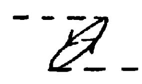
We note that both sets of extra terms result from vacuum effects. Essentially a 'transference' of vacuum effects takes place from one particle to the other either in the probability amplitudes or in the Feynman graph for the event.

Now it is essential that equation 17.9 should stay as it is with a zero on the 'energy shell', i.e. when $\underset{\sim}{k}$ equals the incident momentum. The vacuum subtraction proposed in section 12 has to be extended to remove these further vacuum troubles.

We find in the algebraic results that the following four expressions can appear in the denominators

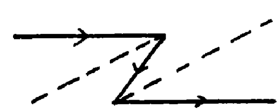
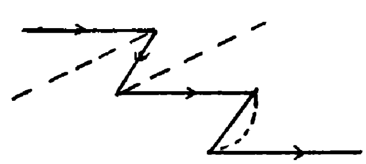
$$\left.\begin{array}{l} E_{\underset{\sim}{k}} - E(\underset{\sim}{k}) - E(\underset{\sim}{x}) - E(\underset{\sim}{x}+\underset{\sim}{y}) - \omega(\underset{\sim}{y}) \\ \omega_{\underset{\sim}{k}} - \omega(\underset{\sim}{k}) - E(\underset{\sim}{x}) - E(\underset{\sim}{x}+\underset{\sim}{y}) - \omega(\underset{\sim}{y}) \end{array}\right\} \qquad 17.10$$

$$\left.\begin{array}{l} E_{\underset{\sim}{k}} - 2E(\underset{\sim}{k}) - E(\underset{\sim}{k}+\underset{\sim}{y}) - \omega(\underset{\sim}{y}) \\ \omega_{\underset{\sim}{k}} - 2\omega(\underset{\sim}{k}) - E(\underset{\sim}{k}+\underset{\sim}{y}) - E(\underset{\sim}{y}) \end{array}\right\} \qquad 17.11$$

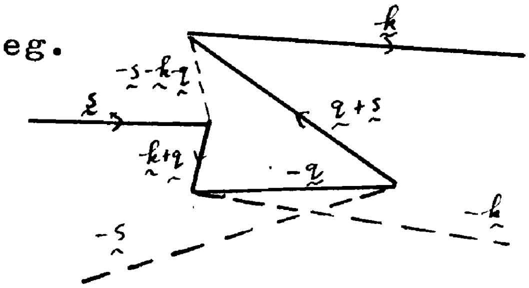
The last two we find are associated with the self energy graphs  and  respectively. The first two are associated with unconnected vacuum loops and the two groups mentioned above. We therefore determine to neglect any non-scattering term in which the expressions 17.10 appear as part of the denominator.

17.2 The Scattering Terms.

The kernel of the integral equation is made up of a large number of scattering terms which fall into three classifications.

1) Second order terms eg. 
2) Fourth order terms consisting of a second order term and a self energy part eg. 

3) True fourth order terms eg.



As examples the derivation of one second and one fourth order graph will be shown in full.

<u>Second Order Term</u>.

We will consider one of the terms from 17.1, which leads to a second order scattering graph.

$$\left(\Psi_0 \int f_s(q, q+k)\, d_{q+k\,\nu}\, c_{q\,\sigma}\, dq \;\; f_1(k)\, c_{k\,u} \,\middle|\, H_0 - E - \omega \,\middle|\, . \right. \qquad 17.12$$

$$\left. .\; f_1(s)\, c^*_{s\,\alpha} \int f_s(q', q'+s)\, c^*_{q'\sigma'}\, d^*_{q'+s\,\nu'}\, dq'\, \Psi_0 \right)$$

The interaction Hamiltonian gives no contribution here, but $H_0 - E - \omega$ gives

$$\left(E_{(k)} + E_{(q)} + E_{(q+k)} - E - \omega \right) \qquad 17.13$$

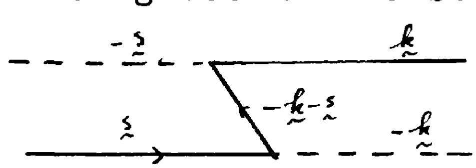
On anticommuting the operators we find the following delta functions to occur,

$$\delta(q + k - q' - s)$$

together with

$$\delta(k - s)\, \delta(q - q') \qquad \text{or} \qquad -\delta(k - q')\, \delta(q - s)$$

The second of these gives us the scattering graph



Substituting for f_5 and summing over the intermediate antinucleon state we get

$$-\frac{g^2}{(2\pi)^3} f_1(k) f_4(k) \int d\underset{\sim}{s}\ (u_{\underset{\sim}{k}}^{*} \gamma \Lambda^{-}(\underset{\sim}{k}+\underset{\sim}{s}) \gamma u_{\underset{\sim}{s}}) \tau_i \tau_j\ f_1(\underset{\sim}{s}) f_4(\underset{\sim}{s}) \times$$

$$\times \frac{(E_{(\underset{\sim}{k})} + E_{(\underset{\sim}{s})} + E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E - \omega)\ \chi(\underset{\sim}{s})}{(4\omega_{\underset{\sim}{k}}\omega_{\underset{\sim}{s}})^{1/2}(\omega_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E_{(\underset{\sim}{s})})(\omega_{\underset{\sim}{s}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E_{(\underset{\sim}{k})})} \qquad 17.14$$

<u>Fourth Order Term</u>.

We will consider another term of 17.1, for which H_I gives the only matrix element.

$$(\Psi_0 f_4(\underset{\sim}{k}) a_{\underset{\sim}{k}j} \int f_2(-\underset{\sim}{q}, \underset{\sim}{k}+\underset{\sim}{q})\, a_{\underset{\sim}{k}+\underset{\sim}{q}}\, c_{-\underset{\sim}{q}}\, d\underset{\sim}{q}\ | H_I | \times \qquad 17.15$$

$$\times \int f_2(-\underset{\sim}{q}', \underset{\sim}{q}'+\underset{\sim}{s})\, c_{\underset{\sim}{q}'}^{*}\, a_{\underset{\sim}{q}'+\underset{\sim}{s}}^{*}\, d\underset{\sim}{q}' \int f_5(\underset{\sim}{m}, \underset{\sim}{m}+\underset{\sim}{s})\, c_{\underset{\sim}{m}}^{*}\, d_{\underset{\sim}{m}+\underset{\sim}{s}}^{*}\, d\underset{\sim}{m}\, \Psi_0)$$

The component of H_I used is

$$H_I = g \int \frac{(V_{\underset{\sim}{p}+\underset{\sim}{n}}^{-*} \gamma\, \tau_\alpha\, U_{\underset{\sim}{p}}^{+})}{(2\omega_{\underset{\sim}{n}})^{1/2}}\ d_{\underset{\sim}{p}+\underset{\sim}{n}}\, c_{\underset{\sim}{p}}\, a_{-\underset{\sim}{n}}^{*}\ d\underset{\sim}{p}\, d\underset{\sim}{n}$$

On commuting the meson operators we get the delta functions

$$\delta(\underset{\sim}{k}+\underset{\sim}{q}+n)\ \delta(-\underset{\sim}{k}+\underset{\sim}{q}'+\underset{\sim}{s})$$

or

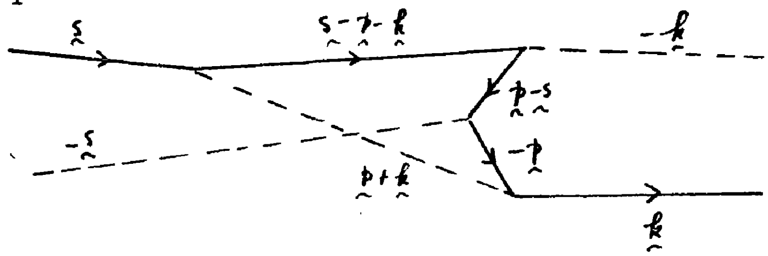
$$\delta(\underset{\sim}{k}+\underset{\sim}{q}-\underset{\sim}{q}'-\underset{\sim}{s})\ \delta(\underset{\sim}{k}-\underset{\sim}{n})$$

On anticommuting the nucleon operators we find

$-\delta(\underset{\sim}{m}+\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{n})$ together with

$$-\delta(\underset{\sim}{q}-\underset{\sim}{q}')\,\delta(\underset{\sim}{p}-\underset{\sim}{m}) \quad \text{or} \quad +\delta(\underset{\sim}{q}+\underset{\sim}{m})\,\delta(\underset{\sim}{p}+\underset{\sim}{q}')$$

Using the second of both alternatives we get the scattering graph



Substituting for f_2 and f_6, and summing over the intermediate nucleon and antinucleon states we find the expression

$$-\frac{g^4}{(2\pi)^6} f_1(k) f_4(k) \int \frac{ds\, dp\, (u_k^* \gamma \Lambda^+(-p) \gamma \Lambda^-(s-p) \gamma \Lambda^+(s-p-k) \gamma u_s)}{2\omega_{p+k} (4\omega_s \omega_k)^{1/2}} \cdot \frac{\tau_\alpha \tau_i \tau_f \tau_\alpha \; f_1(s) \; f_4(s) \; \chi(s)}{(E_s - E_{(s-p-k)} - \omega_{(p+k)})(\omega_k - E_{(s-p-k)} - E_{(p-s)})(E_k - E_{(p)} - \omega_{(p+k)})} \qquad 17.16$$

Anticommuting the τ_α matrices and summing over α we find

$$\tau_\alpha \tau_i \tau_f \tau_\alpha = 2\tau_f \tau_i + \tau_i \tau_f$$

In equations 17.14 and 17.17 we remove the matrix γ by using the relation $\gamma \Lambda^{\pm} \gamma = \Lambda^{\mp}$ as before.

18. <u>Reduction of Equation for Numerical Integration</u>.

We have now obtained an integral equation of the form

$$(E_{\underset{\sim}{k}} + \omega_{\underset{\sim}{k}} - E - \omega)\chi(\underset{\sim}{k}) = \int d\underset{\sim}{s}\, u^*_{\underset{\sim}{k}}\, \mathcal{P}(\underset{\sim}{k},\underset{\sim}{s})\, u_{\underset{\sim}{s}}\, \mathcal{I}_{if}\, \chi(\underset{\sim}{s}) \qquad 18.1$$

where $\mathcal{P}(\underset{\sim}{k},\underset{\sim}{s})$ contains projection operators, and $\mathcal{I}_{if}$ stands for the eigen-value of the nucleon isotopic spin operators for each term in the kernel, as explained in appendix 3.

We now re-define $\chi(\underset{\sim}{k})$ thus

$$\sum_{u^+} u_{\underset{\sim}{k}}\, \chi(\underset{\sim}{k}) = \chi'(\underset{\sim}{k})$$

and we drop the prime. Thus we have

$$(E_{\underset{\sim}{k}} + \omega_{\underset{\sim}{k}} - E - \omega)\chi(\underset{\sim}{k}) = \int d\underset{\sim}{s}\, \Lambda^+(\underset{\sim}{k})\, \mathcal{P}(\underset{\sim}{k},\underset{\sim}{s})\, \mathcal{I}_{if}\, \chi(\underset{\sim}{s}) \qquad 18.2$$

The $\chi(\underset{\sim}{k})$ now defined is obviously not of definite parity. However $\chi(\underset{\sim}{k})$ is defined for a nucleon state of positive energy, and we may write the spinor function $\chi(\underset{\sim}{k}) = \begin{pmatrix} \chi_1(\underset{\sim}{k}) \\ \chi_2(\underset{\sim}{k}) \end{pmatrix}$, where $\chi_1(\underset{\sim}{k})$ and $\chi_2(\underset{\sim}{k})$ are the large and small two component spinors respectively.

Eliminating χ_2 by the usual methods we may write

$$\chi(\underset{\sim}{k}) = \left(1 - \frac{\gamma_5(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})}{E_{\underset{\sim}{k}} + M}\right)\begin{pmatrix}\chi_1(\underset{\sim}{k}) \\ 0\end{pmatrix} \tag{18.3}$$

Since $\underset{\sim}{\sigma}\cdot\underset{\sim}{k}$ is invariant to rotations in co-ordinate space, $\chi_1(\underset{\sim}{k})$ has the same total angular momentum as $\chi(\underset{\sim}{k})$. $\chi_1(\underset{\sim}{k})$ has also a definite parity. We now wish, therefore, to rewrite equation 18.2 in terms of χ_1 To do this we must obviously pick out the groups of matrices with diagonal elements only, i.e. matrices such that the large and small components are not mixed. Such matrices are eg. unity, γ_5^2, β , or any of their products. The σ matrices have only two components and of course are not affected by this elimination.

From appendix 2, we see that $\gamma_5^2 = 1$ and $\beta = 1$ for the large component.

The matrix dependent part of the second order term 17.14

$$(E_{\underset{\sim}{k}} + \underset{\sim}{\alpha}\cdot\underset{\sim}{k} + \beta M)(E_{\underset{\sim}{k}+\underset{\sim}{s}} + \underset{\sim}{\alpha}\cdot(\underset{\sim}{k}+\underset{\sim}{s}) + \beta M)\left(1 - \frac{\gamma_5(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{E_{\underset{\sim}{s}} + M}\right),$$

becomes

$$\Big\{(E_{\underset{\sim}{k}}+M)(E_{\underset{\sim}{k}+\underset{\sim}{s}}+M) + \frac{(E_{\underset{\sim}{k}}+M)\ \underset{\sim}{\sigma}\cdot(\underset{\sim}{k}+\underset{\sim}{s})\ \underset{\sim}{\sigma}\cdot\underset{\sim}{s}}{E_{\underset{\sim}{s}}+M} + \underset{\sim}{\sigma}\cdot\underset{\sim}{k}\ \underset{\sim}{\sigma}\cdot(\underset{\sim}{s}+\underset{\sim}{k})$$

$$+ \frac{(E_{\underset{\sim}{k}+\underset{\sim}{s}}-M)\ \underset{\sim}{\sigma}\cdot\underset{\sim}{k}\ \underset{\sim}{\sigma}\cdot\underset{\sim}{s}}{E_{\underset{\sim}{s}}+M}\Big\}$$

$$= \Big[(E_{\underset{\sim}{k}}+M)(E_{\underset{\sim}{k}+\underset{\sim}{s}}+E_{\underset{\sim}{s}}+E_{\underset{\sim}{k}}-M)$$

$$+ (E_{\underset{\sim}{k}}+E_{\underset{\sim}{k}+\underset{\sim}{s}}+E_{\underset{\sim}{s}}+M)\frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{k}\ \underset{\sim}{\sigma}\cdot\underset{\sim}{s}}{E_{\underset{\sim}{s}}+M}\Big] \qquad 18.4$$

Using $(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{k}) = \underset{\sim}{k}^2 = E_{\underset{\sim}{k}}^2 - M^2$, $\underset{\sim}{\alpha} = -\gamma_5\underset{\sim}{\sigma}$ and the fact that the α and β matrices anticommute. Calling the expression 18.4 $B(\underset{\sim}{k},\underset{\sim}{s})$ the second order term we are considering is now

$$-\frac{g^2}{(2\pi)^3} f_1(\underset{\sim}{k}) f_4(\underset{\sim}{k}) \int d\underset{\sim}{s}\, B(\underset{\sim}{k},\underset{\sim}{s}) \left(E_{(\underset{\sim}{k})}+E_{(\underset{\sim}{s})}+E_{(\underset{\sim}{k}+\underset{\sim}{s})}-E-\omega\right)\times$$

$$\times \frac{I_f\ f_1(\underset{\sim}{s}) f_4(\underset{\sim}{s})\ \chi(\underset{\sim}{s})}{(4\omega_{\underset{\sim}{s}}\omega_{\underset{\sim}{k}})^{1/2}(\omega_{\underset{\sim}{s}}-E_{(\underset{\sim}{k}+\underset{\sim}{s})}-E_{(\underset{\sim}{k})})(\omega_{\underset{\sim}{k}}-E_{(\underset{\sim}{k}+\underset{\sim}{s})}-E_{(\underset{\sim}{s})})} \qquad 18.5$$

where $I_f = \binom{-1}{2}$ for this term, in the notation of appendix 3.

The method for the fourth order terms can be exactly the same as that for the second order terms. However this is tedious due to the extra two projection operators, as in equation 17.16.

The work can be cut down if we use the properties of the projection operators. We examine 17.16 before the projection operators are introduced. We have the expression

$$\left(V^{*}_{\underset{\sim}{s}-\underset{\sim}{p}}\ \gamma\ u^{+}_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} \right)\left(u^{+*}_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}}\ \gamma\ u_{\underset{\sim}{s}} \right)$$

appearing, summed over intermediate states.

This equals $\quad V^{*}_{\underset{\sim}{s}-\underset{\sim}{p}}\ \Lambda^{-}(\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k})\, u_{\underset{\sim}{s}}$

$$= V^{*}_{\underset{\sim}{s}-\underset{\sim}{p}} \frac{\left(E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} - \underset{\sim}{\alpha}.(\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}) - \beta M \right)}{2 E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}}} u_{\underset{\sim}{s}}$$

$$= V^{*}_{\underset{\sim}{s}-\underset{\sim}{p}} \frac{\left(E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} + E_{\underset{\sim}{s}-\underset{\sim}{p}} + \underset{\sim}{\alpha}\cdot\underset{\sim}{k} \right)}{2 E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}}} u_{\underset{\sim}{s}}$$

We use this property, and also anticommute the α matrices in order to use eg. $\underset{\sim}{\alpha}.\underset{\sim}{s}\ u_{\underset{\sim}{s}} = (E_{\underset{\sim}{s}} - \beta M)\, u_{\underset{\sim}{s}}$

We note that

$$(\underset{\sim}{\alpha}\cdot\underset{\sim}{k})(\underset{\sim}{\alpha}\cdot\underset{\sim}{s}) = -(\underset{\sim}{\alpha}\cdot\underset{\sim}{s})(\underset{\sim}{\alpha}\cdot\underset{\sim}{k}) + 2(\underset{\sim}{k}\cdot\underset{\sim}{s}) .$$

Hence we find that we have to reduce the expression

$$\left(E_{\underset{\sim}{k}} + \beta M - \gamma_5\, \underset{\sim}{\sigma}\cdot\underset{\sim}{k} \right)\left(A + \beta B + C\,\gamma_5\, \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \right)\left(1 - \frac{\gamma_5(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{E_{\underset{\sim}{s}} + M} \right)$$

which is as in the second order term and quickly done. We find that

$$A = (E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} + E_{\underset{\sim}{s}-\underset{\sim}{p}})\Big[E_{\underset{\sim}{s}-\underset{\sim}{p}}(E_{\underset{\sim}{p}} + E_{\underset{\sim}{s}-\underset{\sim}{p}} + E_{\underset{\sim}{s}}) - E_{\underset{\sim}{s}}(E_{\underset{\sim}{p}} + E_{\underset{\sim}{s}-\underset{\sim}{p}}) + 2\underset{\sim}{s}.\underset{\sim}{p} - E_{\underset{\sim}{s}}^2\Big]$$

$$+ (E_{\underset{\sim}{p}} + E_{\underset{\sim}{s}-\underset{\sim}{p}})\Big[E_{\underset{\sim}{s}}E_{\underset{\sim}{k}} - 2(\underset{\sim}{s}-\underset{\sim}{p}).\underset{\sim}{k} + E_{\underset{\sim}{k}}E_{\underset{\sim}{s}-\underset{\sim}{p}}\Big]$$

$$+ (M^2 + E_{\underset{\sim}{k}}E_{\underset{\sim}{s}} + 2\underset{\sim}{s}.\underset{\sim}{k})E_{\underset{\sim}{s}-\underset{\sim}{p}} - E_{\underset{\sim}{k}}(E_{\underset{\sim}{s}}^2 - 2\underset{\sim}{s}.\underset{\sim}{p})$$

$$B = -M\Big[E_{\underset{\sim}{s}-\underset{\sim}{p}}(E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} + 2E_{\underset{\sim}{s}-\underset{\sim}{p}} + E_{\underset{\sim}{p}} - E_{\underset{\sim}{k}} - E_{\underset{\sim}{s}})$$

$$+ E_{\underset{\sim}{s}}(E_{\underset{\sim}{p}} - E_{\underset{\sim}{k}} - E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}}) - 2(\underset{\sim}{s}-\underset{\sim}{p}).\underset{\sim}{k}\Big]$$

and

$$C = \Big[(E_{\underset{\sim}{s}} - E_{\underset{\sim}{k}} + E_{\underset{\sim}{p}} - E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}})M\Big]\beta$$

$$+\Big[(E_{\underset{\sim}{s}-\underset{\sim}{p}-\underset{\sim}{k}} + E_{\underset{\sim}{s}-\underset{\sim}{p}})(E_{\underset{\sim}{p}} + E_{\underset{\sim}{s}-\underset{\sim}{p}} - E_{\underset{\sim}{s}}) - E_{\underset{\sim}{k}}(E_{\underset{\sim}{p}} + E_{\underset{\sim}{s}-\underset{\sim}{p}} + E_{\underset{\sim}{s}}) + M^2 + 2\underset{\sim}{s}.\underset{\sim}{k}\Big]$$

The integral equation is now of the form

$$(E_{\underset{\sim}{k}} + \omega_{\underset{\sim}{k}} - E - \omega)\,\chi_1(\underset{\sim}{k}) = \int d\Omega\, s^2 ds\, f_{if}\, K(\underset{\sim}{k},\underset{\sim}{s})\,\chi_1(\underset{\sim}{s})$$

We would now like to perform the angular integration.

$K(\underset{\sim}{k},\underset{\sim}{s})$ is angular dependent through terms eg. $(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})$ and $E_{\underset{\sim}{k}+\underset{\sim}{s}}$. As we have defined $\chi_1(\underset{\sim}{k})$, it is an eigen function of the total angular momentum. We will henceforth drop the subscript on $\chi_1(\underset{\sim}{k})$

We define two operators S_n and R_n by

$$S_n \, \chi(\underset{\sim}{s}) = \frac{1}{4\pi} \int d\Omega_s \, P_n(\alpha) \, \chi(\underset{\sim}{s}) \tag{18.6}$$

$$R_n \, \chi(\underset{\sim}{s}) = \frac{1}{4\pi} \int d\Omega_s \, \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{|\underset{\sim}{k}|\,|\underset{\sim}{s}|} \, P_n(\alpha) \, \chi(\underset{\sim}{s}) \tag{18.7}$$

where $\cos^{-1}\alpha$ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{s}$, and P_n are the Legendre Polynomials.

It is shown in appendix 4 that

$$S_n \, \chi(\underset{\sim}{s}) = \frac{1}{2n+1} \, \delta_{n\ell} \, \chi(s)$$

where ℓ is the orbital angular momentum of the meson in the state being considered. It is also shown that $R_n \chi(\underset{\sim}{s}) = \frac{1}{2n+1} \, \delta_{n,\ell\pm1} \, \chi(s)$ when the total angular momentum $j = \ell \pm 1/2$.

Hence we have the following table,

Angular Momentum of State	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$
Corresponding S operator	S_0	S_1	S_1
Corresponding R operator	R_1	R_0	R_2

Table 18.1 S and R operators for angular momentum states.

We expand the angular dependent terms of the kernel in Legendre Polynomials, eg.

$$\left[E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - \omega_{(\underset{\sim}{s})}\right]^{-1} = \sum_{0}^{\infty} A_n P_n(\alpha)$$

The derivation of such coefficients as A_n is given in appendix 5.

Thus, we can now evaluate the solid angle integration. For example

$$\int d\Omega_s \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - \omega_{(\underset{\sim}{s})})} \chi(\underset{\sim}{s}) = 4\pi\, |\underset{\sim}{k}||\underset{\sim}{s}| \sum A_n R_n \chi(s) \qquad 18.8$$

The value of n is chosen from Table 18.1 to correspond to the orbital angular momentum of the state which we are examining.

We have now obtained an integral equation in one variable, i.e. the magnitude of the meson momentum in the centre of mass system. This equation can now be solved numerically.

18.1 Derivation of the Scattering Amplitude and Phase Shift.

It is well known that the asymptotic behaviour of a co-ordinate space wave function is determined by the singularities in the corresponding momentum space wave function.

Our equation for $\chi(k)$ is of the form

$$(E_k + \omega_k - E - \omega)\chi(k) = -\int L(k,s)\,\chi(s)\,ds \qquad 18.9$$

$\chi(k)$ has a singularity at $k =$ incident momentum, and so we can write the solution for $\chi(k)$ in the form

$$\chi(k) = c\,\delta(E_k + \omega_k - E - \omega) - \frac{1}{E_k + \omega_k - E - \omega}\int L(k,s)\,\chi(s)\,ds \qquad 18.10$$

We can let the normalisation constant $c = 1$ since the normalisation of $\chi(k)$ has not been determined. According to the usual scattering theory we define the non singular scattering amplitude $f(k)$ by

$$\chi(k) = \delta(E_k + \omega_k - E - \omega) + \mathcal{P}\,\frac{f(k)}{E_k + \omega_k - E - \omega} \qquad 18.11$$

where we choose to take the principal value, $\mathcal{P}$, which gives real solutions and simplifies the calculations.

Substituting 18.11 into 18.9 we have

$$f(k) = -\int L(k,s)\left\{\delta(E_s+\omega_s-E-\omega) + \mathcal{P}\frac{1}{E_s+\omega_s-E-\omega}\, f(s)\right\} ds \tag{18.12}$$

We find the Born approximation is

$$f_B(k) = -\frac{L(k,p)\, E_p\, \omega_p}{p\,(E_p+\omega_p)} \tag{18.13}$$

where $\underset{\sim}{p}$ is the incident momentum. The final equation is

$$f(k) = f_B(k) - \mathcal{P}\int \frac{L(k,s)\; f(s)}{E_s+\omega_s-E_p-\omega_p}\, ds \tag{18.14}$$

From appendix 6 we see that the phase shift δ is given by

$$\tan\delta = \pi\, f(p) \tag{18.15}$$

for the state which we are considering with total isospin and angular momentum I, J .

19. Numerical Work and Programmes.

The one variable integral equation now obtained includes in the kernel all the second and fourth order terms. We would like to investigate the effects of the three different groups of scattering graphs, and so we first evaluate the second order terms.

In order to show explicitly the angular dependence of these terms we start from the equation

$$(E_{\underset{\sim}{k}} + \omega_{\underset{\sim}{k}} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}})\, \chi(\underset{\sim}{k}) = - \int K(\underset{\sim}{k}, \underset{\sim}{s})\, \chi(\underset{\sim}{s})\, d\underset{\sim}{s} \qquad 19.1$$

where $\underset{\sim}{p}$ is the incident momentum in the centre of mass frame.

The kernel consisting of all the second order contributions is

$$K(\underset{\sim}{k}, \underset{\sim}{s}) =$$

$$+ \frac{g^2}{(2\pi)^3} f_1(\underset{\sim}{k}) f_4(\underset{\sim}{k}) \binom{-1}{2} \int \frac{d\underset{\sim}{s}\, f_1(\underset{\sim}{s})\, f_4(\underset{\sim}{s})\, \chi(\underset{\sim}{s})}{8\,(\omega_{\underset{\sim}{s}}\, \omega_{\underset{\sim}{k}})^{1/2}\, E_{\underset{\sim}{k}}\, E_{\underset{\sim}{k}+\underset{\sim}{s}}} \cdot \qquad 19.2$$

$$\cdot \left\{ (E_{\underset{\sim}{k}} + M)(E_{\underset{\sim}{k}+\underset{\sim}{s}} - M) - (\underset{\sim}{\sigma}.\underset{\sim}{k})(\underset{\sim}{\sigma}.\underset{\sim}{k}+\underset{\sim}{s}) + \frac{(E_{\underset{\sim}{k}+\underset{\sim}{s}} + M)(\underset{\sim}{\sigma}.\underset{\sim}{k})(\underset{\sim}{\sigma}.\underset{\sim}{s}) - (E_{\underset{\sim}{k}} + M)(\underset{\sim}{\sigma}.\underset{\sim}{k}+\underset{\sim}{s})(\underset{\sim}{\sigma}.\underset{\sim}{s})}{E_{\underset{\sim}{s}} + M} \right\}$$

$$\cdot \left[\frac{E_{\underset{\sim}{k}} + E_{\underset{\sim}{s}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}}{(E_{\underset{\sim}{s}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - \omega_{(\underset{\sim}{k})})(E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - \omega_{(\underset{\sim}{s})})} \right]$$

$$-\frac{g^2}{(2\pi)^3} f_1(\underset{\sim}{k}) f_4(\underset{\sim}{k}) \begin{pmatrix} -1 \\ 2 \end{pmatrix} \int \frac{d\underset{\sim}{s}\; f_1(\underset{\sim}{s})\, f_4(\underset{\sim}{s})\, \chi(\underset{\sim}{s})}{8\,(\omega_{\underset{\sim}{s}}\, \omega_{\underset{\sim}{k}})^{1/2} E_{\underset{\sim}{k}}\, E_{\underset{\sim}{k}+\underset{\sim}{s}}} \times$$

$$\cdot \left\{ (E_{\underset{\sim}{k}}+M)(E_{\underset{\sim}{k}} + E_{\underset{\sim}{k}+\underset{\sim}{s}} + E_{\underset{\sim}{s}} + M) + (E_{\underset{\sim}{k}} + E_{\underset{\sim}{k}+\underset{\sim}{s}} + E_{\underset{\sim}{s}} + M) \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{E_{\underset{\sim}{s}}+M} \right\}$$

$$\cdot \left[\frac{\omega_{\underset{\sim}{k}} + \omega_{\underset{\sim}{s}} - E(\underset{\sim}{k}+\underset{\sim}{s}) - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}}{(\omega_{\underset{\sim}{k}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E_{(\underset{\sim}{s})})\,(\omega_{\underset{\sim}{s}} - E_{(\underset{\sim}{k}+\underset{\sim}{s})} - E_{(\underset{\sim}{k})})} \right]$$

$$-\frac{g^2}{(2\pi)^3} f_1(\underset{\sim}{k}) f_4(\underset{\sim}{k}) \begin{pmatrix} 3 \\ 0 \end{pmatrix} \int \frac{d\underset{\sim}{s}\; f_1(\underset{\sim}{s})\, f_4(\underset{\sim}{s})\, \chi(\underset{\sim}{s}) . 2(E_{\underset{\sim}{k}}+M)}{8\,(\omega_{\underset{\sim}{s}}\, \omega_{\underset{\sim}{k}})^{1/2}\; E_{\underset{\sim}{k}}}$$

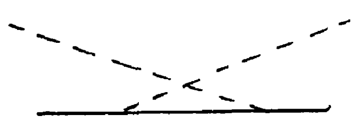
$$\Big\{ (E_{\underset{\sim}{k}} + \omega_{(\underset{\sim}{k})} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}) \left[E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{s})} - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{s})}\right]^{-1} \left[\omega_{\underset{\sim}{s}} - \omega_{(\underset{\sim}{s})} - E(\underset{\sim}{k}) - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{k})}\right]^{-1}$$

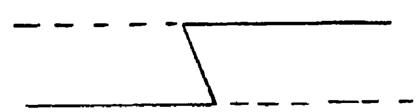
$$+ (\omega_{\underset{\sim}{k}} + E_{(\underset{\sim}{k})} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}) \left[\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} - E(\underset{\sim}{s}) - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{s})}\right]^{-1} \left[E_{\underset{\sim}{s}} - E_{(\underset{\sim}{s})} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{k})}\right]^{-1}$$

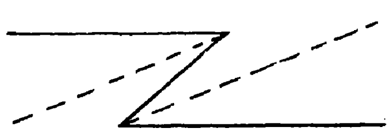
$$+ (E_{\underset{\sim}{s}} + \omega_{(\underset{\sim}{s})} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}) \left[E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{s})} - E_{(0)} - \omega_{(\underset{\sim}{s})}\right]^{-1} \left[E_{\underset{\sim}{k}} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{s})} - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{s})}\right]^{-1}$$

$$+ (\omega_{\underset{\sim}{s}} + E_{(\underset{\sim}{s})} - E_{\underset{\sim}{p}} - \omega_{\underset{\sim}{p}}) \left[\omega_{\underset{\sim}{k}} - \omega_{(\underset{\sim}{k})} - \omega_{(\underset{\sim}{s})} - E_{(\underset{\sim}{0})} - E_{(\underset{\sim}{s})}\right]^{-1} \left[\omega_{\underset{\sim}{s}} - \omega_{(\underset{\sim}{s})} - E_{(\underset{\sim}{k})} - E_{(\underset{\sim}{0})} - \omega_{(\underset{\sim}{k})}\right]^{-1} \Big\}$$

These three terms correspond respectively to

(A) 

(B) 

and (C) 

Graph (C) as we note has no angular dependence and the isotopic spin eigen values are $\binom{3}{0}$. Thus (C) only contributes in the $S_{\frac{1}{2}\cdot\frac{1}{2}}$ case.

The numerator of graph (A) has the form

$$E_{\underset{\sim}{k}+\underset{\sim}{s}}\left(E_{\underset{\sim}{k}}+M+\frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{E_{\underset{\sim}{s}}+M}\right. + \left[(E_{\underset{\sim}{k}}+M)(M-E_{\underset{\sim}{k}}-E_{\underset{\sim}{s}})+\frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})(-M-E_{\underset{\sim}{k}}-E_{\underset{\sim}{s}})}{E_{\underset{\sim}{s}}+M}\right] \qquad 19.3$$

while the square bracket may be written

$$\frac{E_{\underset{\sim}{s}}+\omega_{(\underset{\sim}{s})}-E_{\underset{\sim}{t}}-\omega_{\underset{\sim}{t}}}{(E_{\underset{\sim}{k}}-E_{(\underset{\sim}{k}+\underset{\sim}{s})}-\omega_{(\underset{\sim}{s})})(E_{\underset{\sim}{s}}+\omega_{(\underset{\sim}{s})}-E_{\underset{\sim}{k}}-\omega_{(\underset{\sim}{k})})} - \frac{E_{\underset{\sim}{k}}+\omega_{(\underset{\sim}{k})}-E_{\underset{\sim}{t}}-\omega_{\underset{\sim}{t}}}{(E_{\underset{\sim}{s}}-E_{(\underset{\sim}{k}+\underset{\sim}{s})}-\omega_{(\underset{\sim}{k})})(E_{\underset{\sim}{s}}+\omega_{(\underset{\sim}{s})}-E_{\underset{\sim}{k}}-\omega_{(\underset{\sim}{k})})} \qquad 19.4$$

We thus have the two types of angular dependent terms

$$\left[B-E_{(\underset{\sim}{k}+\underset{\sim}{s})}\right]^{-1} \qquad \text{and} \qquad \left[E_{\underset{\sim}{k}+\underset{\sim}{s}}\left(B-E_{(\underset{\sim}{k}+\underset{\sim}{s})}\right)\right]^{-1}$$

and the method of finding the expansion coefficients is explained in appendix 5. We find that graph (B) yields similar terms.

The partial fractions, 19.4, can only be taken for $\underset{\sim}{s} \neq \underset{\sim}{k}$, the off - diagonal terms. Thus on the diagonal we have to evaluate expansion co-efficients for terms of the form

$$\left[B - E_{(\underset{\sim}{k}+\underset{\sim}{k})} \right]^{-2} \quad \text{and} \quad \left[E_{\underset{\sim}{k}+\underset{\sim}{k}'} \left(B - E_{(\underset{\sim}{k}'+\underset{\sim}{k})}\right)^{2} \right]^{-1}$$

where $|\underset{\sim}{k}'| = |\underset{\sim}{k}|$

Kernel Programmes.

A programme was written for the kernel 19.2 in the T.I.P. scheme used previously. Each row of the kernel matrix was evaluated as a column in T.I.P., and the normalisation functions of section 14 were used.

The programme evaluated the expansion co-efficients as in appendix 5 and all the angular integration contributions, eg. equation 18.8, were summed for each scattering state according to Table 18.1.

However the $k=0$, $S=0$, and $k=S$, positions were given dummy numbers in each T.I.P. column for the following reasons.

For $k=0$ there is no angular dependence in the kernel and equation 19.1 gives only S wave contributions. If $k=0$, or $S=0$, then the appendix 5 method cannot be used since r^{-1} appears.

However for $S=0$ the kernel terms also $=0$. The main programme was altered to yield the $k=0$ contributions.

The reason for the $k=S$ trouble was given previously, and a separate programme was used to find the diagonal contributions.

A final programme placed the correct $k=0$ and $k=S$ elements and, adding the three terms according to their isotopic spin eigen values, punched out rows of the kernel of the equation

$$(E_k + \omega_k - E_p - \omega_p)\,\chi(k) = -\int L(k,s)\,\chi(s)\,ds \quad . \qquad 19.5$$

The numerical data for these programmes will be quoted with the results in the next section. The pivoted points used were $0(\cdot 5)7(1)15(4)39$, one kernel being evaluated for each scattering energy. The total average time required to form each row of this kernel was about 9.8 minutes.

We proceed now to the numerical evaluation of section 18.1.

Scattering Amplitude Programme.

We have now to perform a principal value integration as in equation 18.14. There are several possible methods, but we make use of a straightforward subtraction procedure. We have

$$\mathcal{P}\int \frac{L(k,s)\, f(s)\, ds}{E_s + \omega_s - E_p - \omega_p} = \mathcal{P}\int\left\{\frac{L(k,s)\, f(s)}{(E_s+\omega_s-E_p-\omega_p)} - \frac{L(k,p)\, f(p)\, 2E_p\omega_p}{(s^2-p^2)\,(E_p+\omega_p)}\right\} ds \qquad 19.6$$

where we use the result

$$\mathcal{P}\int \frac{ds}{s^2-p^2} = 0 \qquad 19.7$$

We are going to solve the integral equation by rewriting it as a set of simultaneous linear equations in the scattering amplitude values at the pivotal points. Thus we have the equation

$$f(k_i) = -\frac{L(k_i,p)\, E_p\,\omega_p}{p\,(E_p+\omega_p)} - \sum_j \frac{L(k_i,k_j)\, W_j\, f(k_j)}{E_{k_j}+\omega_{k_j}-E_p-\omega_p} + \sum_j \frac{W_j}{k_j^2-p^2}\, \frac{2E_p\,\omega_p\, L(k_i,p)}{E_p+\omega_p}\, f(p) \qquad 19.8$$

Since we are trying to find $f(p)$ we must at this stage express $f(p)$ in terms of the unknowns $f(k_j)$

For this we use the quadratic interpolation rule

$$f(p) = \frac{(p-k_2)(p-k_3)}{(k_1-k_2)(k_1-k_3)} f(k_1)$$

$$+ \frac{(p-k_1)(p-k_3)}{(k_2-k_1)(k_2-k_3)} f(k_2) + \frac{(p-k_1)(p-k_2)}{(k_3-k_1)(k_3-k_2)} f(k_3)$$

$$= \alpha_1 f(k_1) + \alpha_2 f(k_2) + \alpha_3 f(k_3) \qquad 19.9$$

k_1, k_2, k_3 are three consecutive pivotal points including p in their range.

We thus write the last term of equation 19.8 as

$$\sum_j \frac{W_j}{k_j^2 - p^2} \cdot \frac{2 E_p \omega_p L(k_i, p)}{E_p + \omega_p} \sum_\ell \alpha_\ell f(k_\ell)$$

where α_ℓ is only non-zero for three of the pivotal points k_ℓ. The final equation for programming is thus:-

$$\sum_j \left[\frac{L(k_i, k_j) W_j}{E_{k_j} + \omega_{k_j} - E_p - \omega_p} - \sum_\ell \frac{W_\ell}{k_\ell^2 - p^2} \frac{2 E_p \omega_p L(k_i, p)}{(E_p + \omega_p)} \alpha_j + \delta_{ij} \right] f(k_j)$$

$$= - \frac{L(k_i, p) E_p \omega_p}{p (E_p + \omega_p)} \qquad 19.10$$

We will obtain results for scattering momenta not equal to any pivotal point. In the subtraction procedure 19.6 we hope that any errors, which arise from the numerical treatment, will tend to occur to the same extent in both terms and so cancel. The programme for equation 19.10 was written with this point in mind.

We can write equation 19.10 in the matrix form

$$A_{ij} f_j = C_i$$

A T.I.P. scheme programme was written to evaluate the square matrix A_{ij} and the column C_i. The matrix $[A_{ij}, C_i]$ was formed row by row and punched out using a special punching subroutine. These rows were then grouped as a 29 x 30 matrix for input to the last programme. The total time required for these two stages was found to be about 11 minutes.

The last programme was a standard library programme for DEUCE written in basic, LEO 6/1, which evaluated the simultaneous equations and punched out the scattering amplitude.

The sections of this programme are:-
Binary input for data, Reduce equations, Back substitute, and Punch out in decimal. The method used is pivotal condensation and the time required is 3 minutes 20 seconds for our equations.

20. Scattering Results for Second Order Terms.

For the second order terms as described, it was decided to evaluate the phase shifts using $M_0 = M$ and $\mu_0 = \mu$.

This makes the examination of the 'g' dependence of the phase shifts much easier and quicker, since M_0 and μ_0 are functions of g.

It was noted that the integrand of equation 19.5 is convergent as it stands, with the normalisation functions of section 14. The first results were then calculated without cut off factors. The $S_{1/2 \cdot 1/2}$ state scattering kernel showed the weakest convergence inside the pivotal point range.

The usual 3 point Simpson's rule was used. For $g^2/4\pi = 15$ the following results are tabulated for the phase shifts at four scattering momenta. The 0.1 results are very approximate due to 'peaking' of the scattering amplitude between 0 and 0.5. In the table, 20.1, the upper number in each section is the Born result.

As we see the momentum value 1.68 corresponds to the resonance peak in the $P_{3/2 \cdot 3/2}$ scattering. We would

MOMENTUM IN CENTRE OF MASS SYSTEM	K.E. IN LAB. SYSTEM (MeV)	PHASE SHIFTS (DEGREES)		
		$S_{1/2}$	$S_{3/2}$	$P_{3/2,3/2}$
0·1	0·907	+3°38′ −2°28′	−6°38′ −2°	~ +1′ +2′
0·77	49·4	+27°23′ −16°37′	−45°6′ −9°2′	+1°25′ +1°45′
1·126	98·5	+38°27′ −24°56′	−57°50′ −14°15′	+3°54′ +5°37′
1·68	197·3	+50°44′ −38°20′	−67°53′ −19°2′	+10° +15°50′

Table 20.1 Born / Total phase shifts for $g^2/4\pi = 15$, with no cut off.

like to know if a resonance occurs at this energy and if so at what value of the coupling constant. Figure 20.1 shows the variation with coupling constant obtained. It must be remembered that f_1 and f_4 are functions of g. For comparison f_1 and f_4 were kept fixed at their value for $g^2/4\pi = 15$, and the coupling constant varied. The peak was found to occur at $g^2/4\pi \simeq 41{\cdot}8$, as compared with the correct value of $g^2/4\pi \simeq 57$.

This very large coupling constant is, of course, the unrenormalised coupling constant, and would be reduced by renormalisation*.

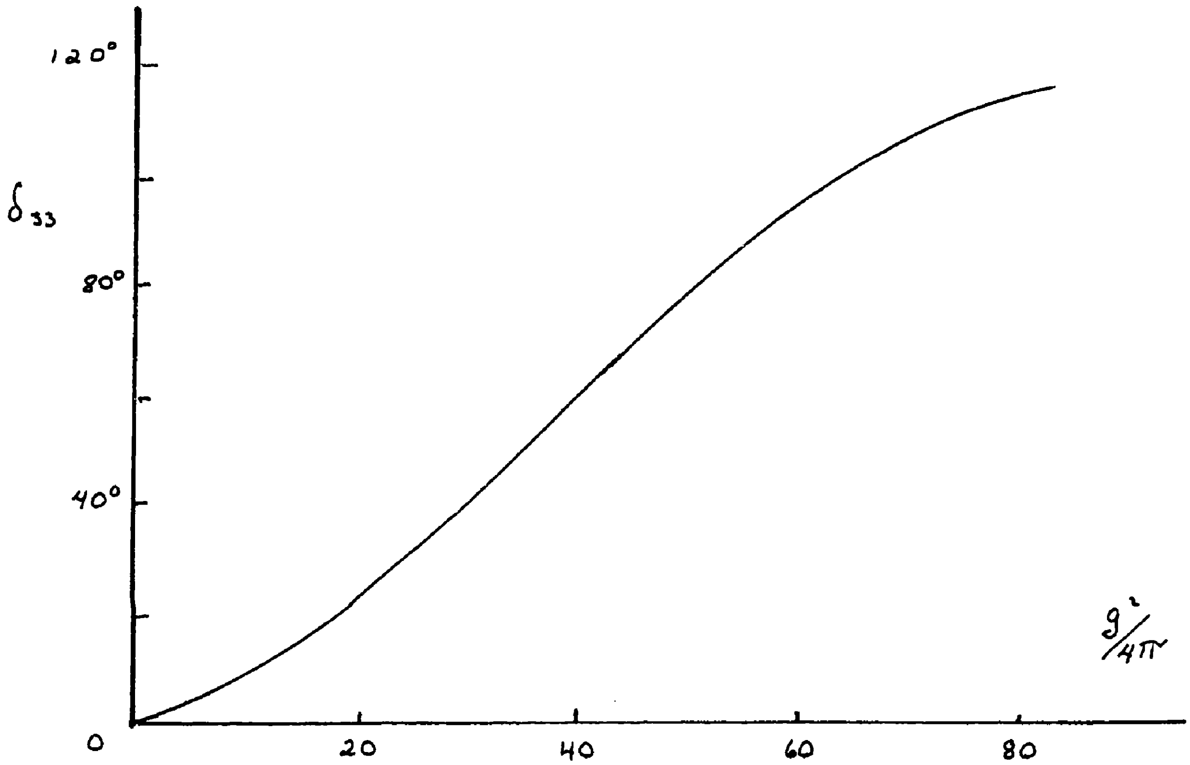


Figure 20.1 Variation of the 1.68 $P_{3/2,3/2}$ phase shift with $g^2/4\pi$

Using the resonance coupling constant thus determined the results in table 20.2 were evaluated.

These coupling constant variations were performed by changing the coupling constant in the intermediate results for $g^2/4\pi = 15$. This was done by means of a small auxiliary programme. The total extra time required to obtain each phase shift by this method was about

* This reduction is very strong in our model. See section 13.1

20 minutes.

	$S_{1/2}$	$S_{3/2}$	$P_{3/2\cdot 3/2}$
Born Phase Shift	$+70°21'$	$-79°50'$	$+22°9'$
Total Phase Shift	$-32°6'$	-	$+\tan^{-1}(8440)$

Table 20.2 1.68 phase shifts for the resonance coupling constant, $\frac{g^2_{RES}}{4\pi} \simeq 57$.

The missing result in table 20.2 was not obtained due to failures in the last programme Leo 6/1. The failures indicated that the matrix was ill-conditioned for this phase shift. This question is held over for further investigation.

In other scattering models it has been found that the S phases require a small coupling constant to fit them to experiment. The behaviour of the 0.77, S phase shifts was found for coupling constants between 0 and 15, and the results are shown in figures 20.2 and 20.3.

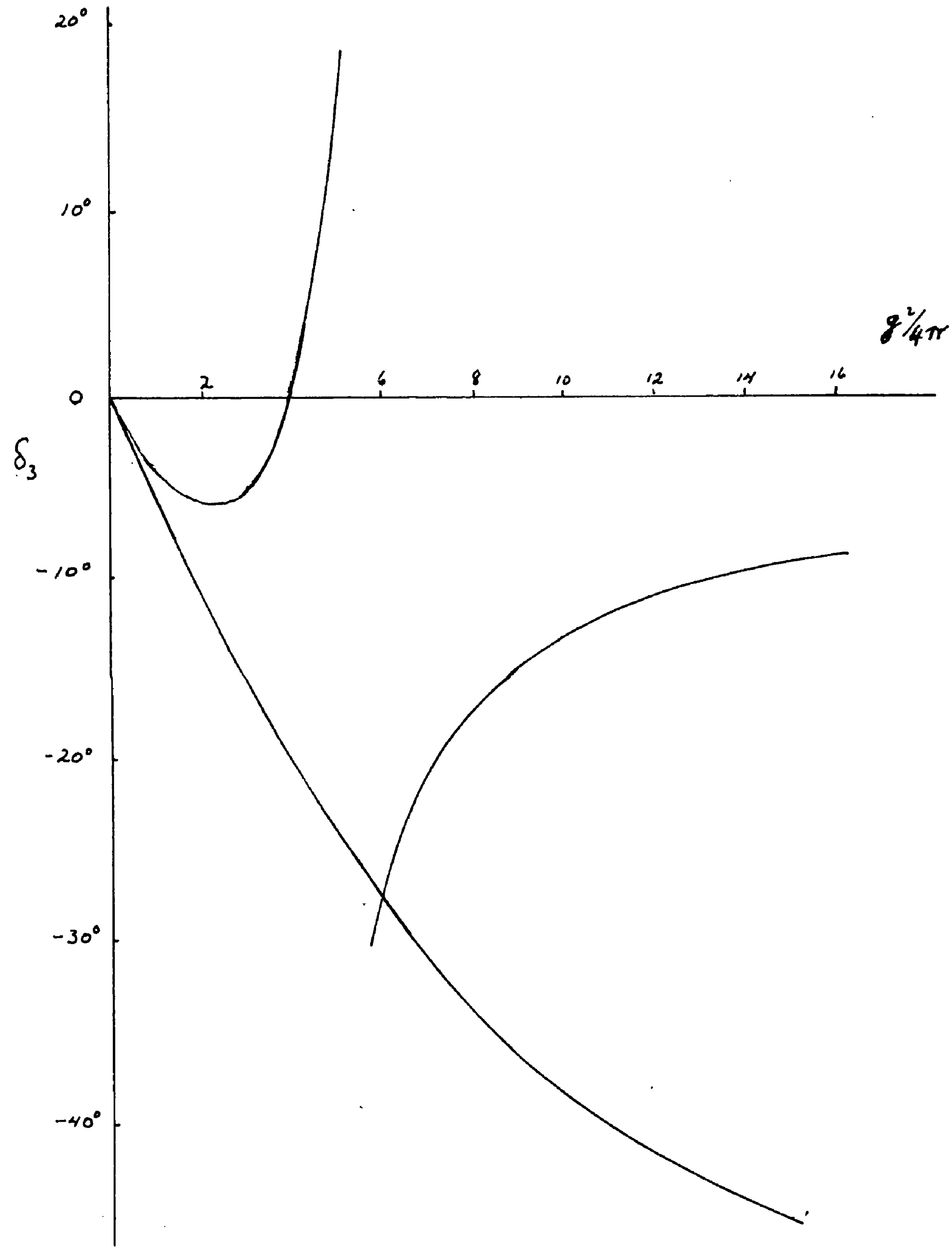


Figure 20.2 0.77, $S_{3/2}$ phase shift variation with $g^2/4\pi$

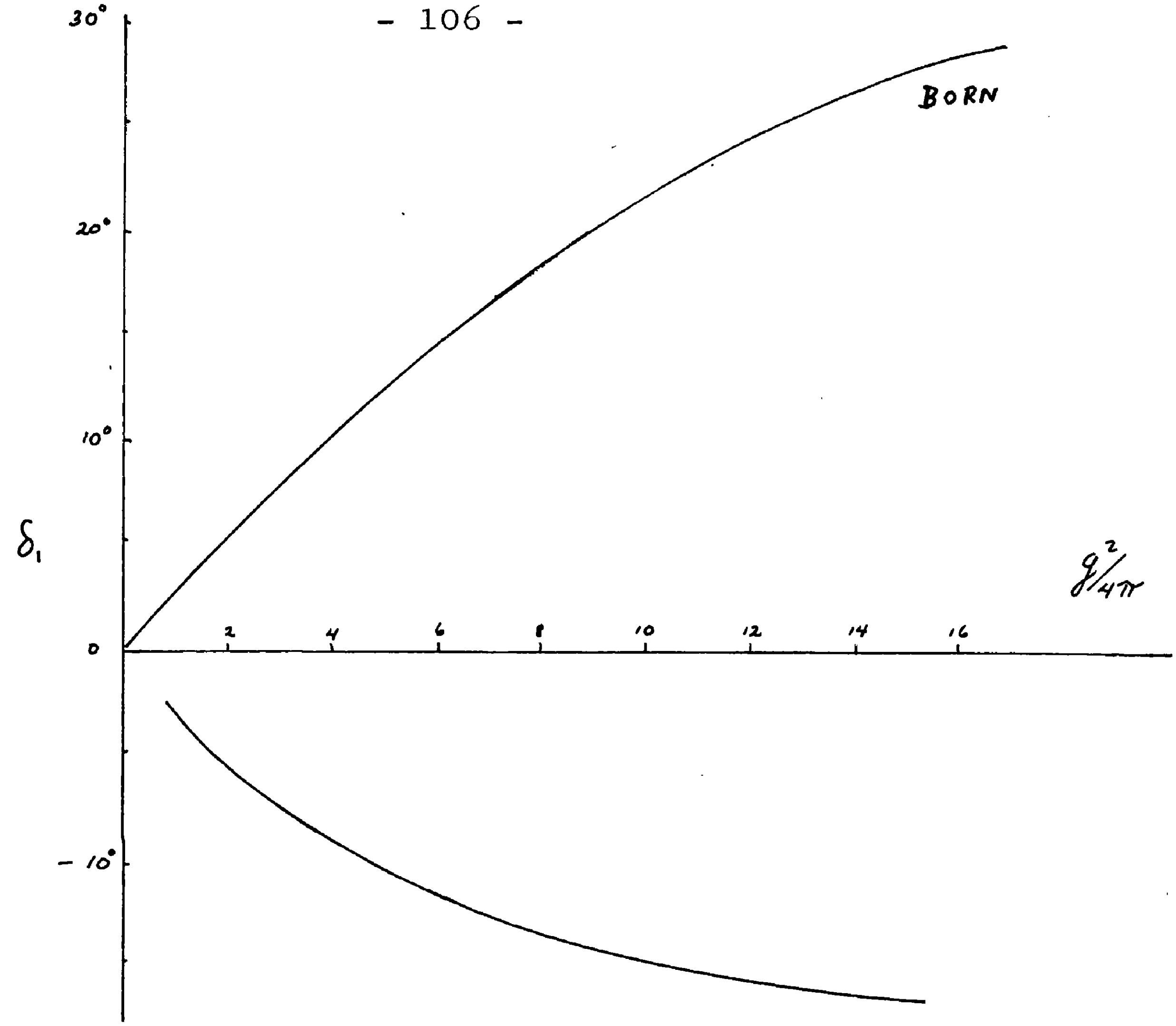


Figure 20.3. 0.77, $S_{1/2}$ phase shift variation with $g^2/4\pi$.

Finally for comparison phase shifts were evaluated, with the cut off terms included in the kernel. Table 20.3 and 20.4 give the results.

With the definition, 18.15, of the phase shifts, we have

$$\delta = \tan^{-1}[\pi f(p)] + n\pi ,$$

where n is a signed integer. For convenience the numerical results are quoted for $n = 0$.

Scattering Momentum	$S_{1/2}$	$S_{3/2}$	$P_{3/2\cdot 3/2}$
·77	−18°53′	−17°42′	+1°45′
1·68	−45°25′	−43°19′	+7°54′
1·68 (BORN)	+40°52′	−59°39′	+7°

Table 20.3 Cut off dependent phase shifts for $g^2/4\pi = 15$.

$g^2/4\pi$	15	30	50	80
δ_{33}	7°54′	13°56′	20°10′	26°23′

Table 20.4 1·68, $P_{3/2\cdot 3/2}$, cut off phase shift for different coupling constants.

21. Discussion of Results.

The $P_{1/2 \cdot 1/2}$ Phase Shift.

From Table 20.1 we see that as usual this state is the only one where the 'kernel' contributions do not cancel with the Born results.

As has been seen this model gives a resonance in the $P_{1/2 \cdot 1/2}$ scattering state. A large unrenormalised coupling constant is required to fit the 90° phase shift to the approximate experimental position.

For the case including cut off in the kernels, preliminary results are quoted. At present there is insufficient data to judge whether or not there is a resonance. The cut off is the arbitrary one which was assumed throughout the work.

The S Phase Shifts.

From Table 20.1 we see that the S phase shifts vary approximately linearly with the centre of mass momentum, for the range of points we have examined.

In this model, however, we have some notable differences from other models -

1) The Born results are of the correct sign while the total scattering results are both negative.

2) In the .77 momentum case at least, the $S_{3/2}$ phase shift has, for coupling constant variation, the peculiar behaviour which we see in Figure 20.2. The two sections of the graph have a phase difference of probably π .

For $g^2/4\pi = 5.1$ the Leo 6/1 programme failed and indicated that the numbers were becoming too large for the capacity of the computer. Hence we know that 5.1 is very close indeed to the $\delta = \pi/2$ position, and so the scattering amplitude value was tending to infinity.

The magnitudes of the S phase shifts for $g^2/4\pi = 15$ are both about a factor of 2 up on those of Orear (1956), and they increase with increasing coupling constant.

The effect of including the cut off in the S states can be seen by comparing tables 20.3 and 20.1. We see that for momenta 1.68 and .77 both the S phase shifts are increased, but that the $S_{3/2}$ increase is very much greater than that for the $S_{1/2}$. This is probably due to the cut off effect on the slow convergence of the kernels, which was mentioned before.

We see therefore that a cut off and a small coupling constant might yield the correct magnitudes for δ_1 and δ_3

but the sign of δ_1 would still be wrong.*

At the .77 momentum, we see that the Born results, at $g^2/4\pi \simeq 1.5$, approximately fit the S phase shifts of Orear (1955).

Numerical Procedure.

The choice of the pivotal points and the use of Simpson's rule for integration seem reasonably satisfactory. Except in the 0.1 case the numerical quantities which appear varied smoothly.

Programmes in the Tabular Interpretive Scheme, which was mainly used, were slow but easy to restart in the case of minor computer failures. Writing the programmes in this scheme was quick due to the simple form of the code words. However since data in the machine is block floated, i.e. the binary point is in the same position for each number in a column, inaccuracies could arise due to a large spread of numbers.

In the cut off the value of A, which we arbitrarily took as $A = M$, could perhaps be determined by requiring that in figure 15.1 a real value of g should exist for $\frac{g_r^2}{4\pi} = 15$.

* If the coupling constant is small enough the phase shifts will, of course, tend to the Born results.

Conclusions.

These phase shifts come from the second order terms only and we may say that the situation is quite promising.

If we examine table 20.1 we see that -

1) Any increase in the Born, or 'kernel', contribution to the phase shift will bring down the coupling constant required for resonance, in the $P_{3/2,3/2}$ case.
2) A fairly small increase in the 'kernel' contribution, relative to the Born, will bring the $S_{3/2}$ phase shift much nearer the experimental values.
3) The 'kernel' contributions have to be much smaller, in the $S_{1/2}$ case, so that the phase shift may remain positive but less than the Born result. The $S_{1/2}$ wave, however, is much more seriously affected by Renormalisation than the $S_{3/2}$, and in this case the fourth order terms including self energy graphs could have an important effect.

We thus see that a calculation including the fourth order and interaction terms could be profitable, from the point of view of fitting the phase shifts to the experimental results.

Acknowledgments.

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Appendix 1.

Mass Values.

We take the currently accepted values for the masses as follows:

Mass of charged pion	$= 273.27\ m_e$	$(\pm .12)$
" " neutral "	$= 264.27\ m_e$	$(\pm .3)$
" " proton	$= 1836.12\ m_e$	$(\pm .04)$
" " neutron	$= 1838.65\ m_e$	$(\pm .04)$

Since we are using a charge independent theory we take the weighted mean for the pion and the nucleon, and we find:

$$\mu = 270.27\ m_e$$

and $$M = 1837.38\ m_e$$

Hence we have $M = 6.79833\ \mu$ and we round this off to $\underline{M = 6.8\mu}$

Also $\underline{\mu = 138.10 \text{ MeV}}$.

Appendix 2.

The Dirac Matrices.

The three Pauli spin matrices , σ_i , are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and we denote $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by 1 .

As usual the α matrices are defined as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We define the Hermitian set of γ matrices as follows:

$$\gamma_j = i\beta\alpha_j \quad , \gamma_4 = \beta$$

Hence $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This set of γ matrices anti-commute.

For convenience, we denote $i\beta\gamma_5$ by γ.

Appendix 3.

Eigenvalues of the τ Operators.

The τ operators are the nucleon isotopic spin operators. We find that they occur as $\tau_i \tau_f$ and $\tau_f \tau_i$, where τ_i is associated with the incident meson and τ_f with the final meson. As the scattering state is an eigenfunction of isotopic spin, we wish to find the eigenvalues of these operators.

Consider the 3 x 3 matrix $Q_{\alpha\delta}$ with element $\alpha,\delta = \tau_\alpha \tau_\delta$

Then $\quad Q^2_{\alpha\delta} = Q_{\alpha\beta}\, Q_{\beta\delta} = \tau_\alpha \tau_\beta \tau_\beta \tau_\delta = 3\,\tau_\alpha \tau_\delta = 3\, Q_{\alpha\delta}$.

Then $Q_{\alpha\delta}$ has the eigenvalues 0 or 3.

Only for a state of isotopic spin $1/2$ may the incident meson be annihilated before the final meson is created.

Therefore $\quad \tau_f \tau_i = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad$ for isotopic spin $I = \begin{cases} 1/2 \\ 3/2 \end{cases}$

Since $\quad \tau_\alpha \tau_\beta + \tau_\beta \tau_\alpha = 2\,\delta_{\alpha\beta}$,

then $\quad \tau_i \tau_f = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad$ for isotopic spin $\underline{I} = \begin{cases} 1/2 \\ 3/2 \end{cases}$

In the text we will use $\mathcal{J}_{if}$. as a general representation of the isotopic spin eigenvalue for each term, i.e. it is not a common factor.

Appendix 4.

The S_n and R_n Operators for Angular Integration.

We examine the operators S_n and R_n defined by

$$S_n \chi(\underset{\sim}{s}) = \frac{1}{4\pi}\int d\Omega_s\, P_n(\alpha)\, \chi(\underset{\sim}{s})$$

$$R_n \chi(\underset{\sim}{s}) = \frac{1}{4\pi}\int d\Omega_s\, \frac{(\underset{\sim}{\sigma}\cdot\underset{\sim}{k})(\underset{\sim}{\sigma}\cdot\underset{\sim}{s})}{|\underset{\sim}{k}|\,|\underset{\sim}{s}|}\, P_n(\alpha)\, \chi(\underset{\sim}{s})$$

$\cos^{-1}\alpha$ is the angle between $\underset{\sim}{k}$ and $\underset{\sim}{s}$, and the integration is taken relative to $\underset{\sim}{k}$ as the Z axis.

We can write

$$\begin{aligned} S_n \chi(\underset{\sim}{s}) &= \frac{1}{4\pi}\int d\Omega_s \sqrt{\frac{4\pi}{2n+1}}\, Y_{n,0}(\alpha)\, \chi(\underset{\sim}{s}) \\ &= \frac{1}{4\pi}\int d\Omega_s\, \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y^{*}_{n,m}(\theta_s,\phi_s)\, Y_{n,m}(\theta_k,\phi_k)\, \chi(\underset{\sim}{s}) \end{aligned} \qquad \text{A.4.1}$$

Here we have used the well known addition theorem for spherical harmonies. Now $\chi(\underset{\sim}{s})$ is an eigenfunction of the total angular momentum for a state of a nucleon and a meson, eg. for the $P_{3/2 \cdot 3/2}$ state

$$\chi(\underset{\sim}{s}) = \chi(s)\, Y_{1/2\,1/2}\, Y_{11}(\theta_s,\phi_s)$$

in the usual notation. Using the orthonormality of the spherical harmonics it is easily seen that

$$S_n \chi(\underset{\sim}{s}) = \frac{1}{2n+1}\,\delta_{n\ell}\, \chi(s)$$

The $Y_{n,m}(\theta_k, \phi_k)$ left in the equation A.4.1, is also contained in the $\chi(\underset{\sim}{k})$ on the left hand side of our integral equation. On multiplying by $Y^*_{nm}(\theta_k, \phi_k)$ and $\int d\Omega_k$ we remove them.

To examine the R_n operator we let

$\bar{\chi}(\underset{\sim}{s}) = \frac{\underset{\sim}{\sigma}.\underset{\sim}{s}}{|\underset{\sim}{s}|} \chi(\underset{\sim}{s})$ Now although $\bar{\chi}(\underset{\sim}{s})$ and $\chi(\underset{\sim}{s})$ have the same total angular momentum, they are clearly of opposite parity. Thus if $\chi(\underset{\sim}{s})$ has $\ell = j \mp 1/2$ $\bar{\chi}(\underset{\sim}{s})$ has $\ell = j \pm 1/2$

As we could expect, we see that $\bar{\chi}(\underset{\sim}{s})$ is $\chi(\underset{\sim}{s})$ with nucleon 'spin flip' but the same total angular momentum.

We have $R_n \chi(\underset{\sim}{s}) = S_n \bar{\chi}(\underset{\sim}{s}) \frac{\underset{\sim}{\sigma}.\underset{\sim}{k}}{|\underset{\sim}{k}|}$, and using the results for the S_n operator we have

$$R_n \chi(\underset{\sim}{s}) = \frac{1}{2n+1} \delta_{n,\ell \pm 1} \chi(s)$$

for $j = \ell \pm 1/2$. To obtain this we also use the relation

$$(\underset{\sim}{\sigma}.\underset{\sim}{k})(\underset{\sim}{\sigma}.\underset{\sim}{k}) = \underset{\sim}{k}.\underset{\sim}{k} = |\underset{\sim}{k}|^2$$

which comes from the anticommutation properties of the σ's .

Appendix 5.

Angular Integration Expansion Coefficients.

We have to expand several angular dependent expressions in terms of Legendre Polynomials. As we see from Table 18.1, only the first three expansion coefficients are required.

The main reasons for this method of angular integration are three-fold. The method is quite neat and if the integration is performed analytically we then have faster programmes and also less inaccuracy in the numerical work because we don't have to perform a numerical double integration.

The method used is similar to that of Dyson (1954). The coefficients we require are given by an integral over α , the cosine of the angle between $\underset{\sim}{k}+\underset{\sim}{s}$, as we shall see later.

We change the variable from α to Z , where Z is $E_{\underset{\sim}{k}+\underset{\sim}{s}}$. The following relations are used

$$2ks\alpha = Z^2 - M^2 - k^2 - s^2$$

$$ks\, d\alpha = z\, dz$$

$$E_{(\underset{\sim}{k}+\underset{\sim}{s})} = (M_0^2 - M^2 + Z^2)^{1/2}$$

Let $ks = r$ and $M^2+k^2+s^2 = t$. Since $M \geqslant M_0$

let $\mathcal{B}^2 = M^2 - M_0^2$ and we have

$$2\tau\alpha = z^2 - t$$
$$\tau\, d\alpha = z\, dz$$
$$E_{(\underset{\sim}{k}+\underset{\sim}{s})} = (z^2 - \mathcal{B}^2)^{1/2}$$

The integration limits $\alpha = \begin{cases} +1 \\ -1 \end{cases}$ give $Z = \begin{cases} \sqrt{t + 2\tau} = Z_+ \\ \sqrt{t - 2\tau} = Z_- \end{cases}$

As examples we evaluate some general terms.

Let $$\left[A + E_{(\underset{\sim}{k}+\underset{\sim}{s})}\right]^{-1} = \sum y_n P_n(\alpha)$$

then $$y_n = \frac{2n+1}{4\pi} \int_{-1}^{+1} \frac{P_n(\alpha)\, d\alpha}{\left[A + E_{(\underset{\sim}{k}+\underset{\sim}{s})}\right]}$$

In the kernel, however, we have a factor 2π from the other angular integration due to symmetry, and we include this 2π here. Thus changing the variable $\alpha \Rightarrow z$, we have

$$y_n = \frac{n + 1/2}{\tau} \int_{Z_-}^{Z_+} \frac{P_n\left(\frac{z^2 - t}{2\tau}\right) z\, dz}{\left[A + \sqrt{z^2 - \mathcal{B}^2}\right]}$$

For this type of integrand it is now most convenient to change $z \Rightarrow y$ where $y = \sqrt{z^2 - \mathcal{B}^2}$

$$y\, dy = z\, dz$$ and $$z = \begin{cases} Z_+ \\ Z_- \end{cases} \rightarrow y = \begin{cases} \sqrt{Z_+^2 - \mathcal{B}^2} = Y_+ \\ \sqrt{Z_-^2 - \mathcal{B}^2} = Y_- \end{cases}$$

We have

$$Y_0 = \frac{1}{2r} \int_{Y_-}^{Y_+} \frac{Y}{A+Y}\, dY$$

$$= \frac{1}{2r} \left\{ Y - A \log (Y+A) \right\} \Big|_{Y_-}^{Y_+}$$

$$= \frac{1}{2r} \left[\sqrt{Z_+^2 - B^2} - \sqrt{Z_-^2 - B^2} + A \log \left\{ \frac{\sqrt{Z_-^2 - B^2} + A}{\sqrt{Z_+^2 - B^2} + A} \right\} \right]$$

Higher coefficients may be found easily in terms of lower coefficients and simple integrals over Y, and hence we will not explicitly evaluate them here. Let

$$\left[E_{\underset{\sim}{k}+\underset{\sim}{s}} \left(A + E_{(\underset{\sim}{k}+\underset{\sim}{s})} \right) \right]^{-1} = \sum_n W_n P_n(\alpha)$$

We cannot split the left hand side into partial fractions since generally $E_{\underset{\sim}{k}+\underset{\sim}{s}} \neq E_{(\underset{\sim}{k}+\underset{\sim}{s})}$.

In terms of the function Z,

$$W_n = \frac{n+1/2}{r} \int_{Z_-}^{Z_+} \frac{P_n\left(\frac{Z^2 - t}{2r}\right) dZ}{\left(A + \sqrt{Z^2 - B^2}\right)}$$

We now transform $Z \rightarrow X$ with $\quad Z = \frac{B(x^2+1)}{2x}$

$$X = \underline{Z + \sqrt{Z^2 - B^2}} \qquad \text{and} \qquad dZ = B \cdot \frac{x^2-1}{2x^2} \cdot dx$$

We let $X_{+,-}$ correspond to $Z_{+,-}$ and we get

$$W_0 = \frac{1}{2\tau} \int_{X_-}^{X_+} \frac{B\,(x^2-1)\;dx}{2x^2\left(A + B\frac{(x^2-1)}{2x}\right)}$$

$$= \frac{1}{2\tau} \int_{X_-}^{X_+} \left\{ \frac{1}{x} - \frac{2A}{(Bx^2 + 2AX - B)} \right\} dx$$

$$= \frac{1}{2\tau}\left[\log x - \frac{A}{\sqrt{B^2+A^2}} \log\left\{ \frac{By + A - \sqrt{B^2+A^2}}{By + A + \sqrt{B^2+A^2}} \right\}\right] \Bigg|_{X_-}^{X_+}$$

For $B = 0$ the above transformation is invalid but

$$W_n = \frac{n+1/2}{\tau} \int_{Z_-}^{Z_+} \frac{P_n\left(\frac{Z^2 - t}{2\tau}\right) dz}{(A + Z)}$$

and hence

$$W_0 = \frac{1}{2\tau} \log\left(\frac{A + Z_+}{A + Z_-} \right)$$

As in the Y_n case, W_1 and W_2 are easily determined. The only other two expressions which occur for the second order terms are

$$\left[A + E_{(\underset{\sim}{k}+\underset{\sim}{s})}\right]^2 = \sum_n Z_n P_n(\alpha)$$

and

$$\left[E_{\underset{\sim}{k}+\underset{\sim}{s}}\left(A + E(\underset{\sim}{k}+\underset{\sim}{s})\right)^2\right]^{-1} = \sum_n V_n P_n(\alpha)$$

These two expressions appear in the diagonal terms, and we evaluate them using the transformations

$$Z \Rightarrow Y \qquad , Y = \sqrt{Z^2 - B^2}$$

and

$$Z \Rightarrow X \qquad , X = \frac{Z + \sqrt{Z^2 - B^2}}{B}$$ respectively.

We have

$$Z_0 = \frac{1}{2\gamma}\left\{ \log(Y+A) + \frac{A}{A+Y} \right\} \Bigg|_{Y-}^{Y+}$$

and

$$V_0 = \frac{1}{\gamma}\left\{ \frac{AB^2 - A^2BX}{(A^2+B^2)(B^2X^2 + 2ABX - B^2)} + \frac{B^2}{2(A^2+B^2)^{3/2}} \log\left\{ \frac{BX + A - \sqrt{A^2+B^2}}{BX + A + \sqrt{A^2+B^2}} \right\} \right\} \Bigg|_{X-}^{X+}$$

As in the W_n case, for $B = 0$ we have the simpler result,

$$V_0 = \frac{1}{2\gamma}\left\{ \frac{1}{A+Z_-} - \frac{1}{A+Z_+} \right\}$$

We do not explicitly evaluate the higher coefficients for the same reason as before.

Appendix 6.

The Relation Between the Scattering Amplitude and the Phase Shift.

From general scattering theory we know that the scattering wave function in co-ordinate space has the asymptotic behaviour.

$$\chi(r) \sim \sin\left[pr-(\ell+\tfrac{1}{2})\pi/2\right] + \tan\delta_\ell \cos\left[pr-(\ell+\tfrac{1}{2})\pi/2\right]$$

for a state of angular momentum ℓ.

We now expand $\chi(r)$ in terms of spherical Bessel functions $j_\ell(kr)$ whose asymptotic behaviour is

$$j_\ell(kr) \sim \sin\left[kr-(\ell+\tfrac{1}{2})\pi/2\right] .$$

$$\chi(r) = \int \chi(k)\, j_\ell(kr)\, dk .$$

Solving for the expansion coefficients $\chi(k)$ and substituting the asymptotic expressions for $\chi(r)$ and $j_\ell(kr)$ we find

$$\chi(k) = \delta(k-p) + \frac{1}{\pi}\,\frac{\tan\delta_\ell}{k-p} \quad \text{+ terms finite at } k=p.$$

Equation 18.11 states

$$\chi(k) = \delta(E_k+\omega_k-E_p-\omega_p) + \mathcal{P}\,\frac{f(k)}{E_k+\omega_k-E_p-\omega_p}$$

From these two equations we find that

$$f(p) = \frac{1}{\pi}\tan\delta_\ell .$$

It is to be noted that we are not dealing with properly normalised wave functions. However as we only require the above equation, which is exact, we do not need to use the normalisation functions.

References.

Akiba, T. and Sawada, K. (1954), Prog. Theoret, Phys. 12, 94.

Anderson, H.L., Davidon, W.C., and Kruse, U.E. (1955), Phys. Rev. 100, 339.

Bassetti, M. (1958), Nuovo Cim., (10), 8, 361.

Berger, J.M., Foldy, L.L., and Osborn, R.K. (1952), Phys. Rev. 87, 1061.

Bethe, H.A. and de Hoffman, F. (1956), Mesons and Fields, Vol. II, section 38.

Bogoliubov, N.N., and Shirkov, D.V. (1959), Introduction to the Theory of Quantized Fields, Vol. III, section 46.

Bosco, B. and Stroffolini, R. (1955), Nuovo Cim., (10), 2, 433.

Budini, P. and Fonda, L. (1957) Nuovo Cim., (10), 5, 306.

Chew, G.F. (1954), Phys. Rev. 95, 1669.

Chew, G.F. (1954a), Phys. Rev. 93, 341.

Chew, G.F., Goldberger, M.L., Low, F.E., and Nambu, Y. (1957), Phys. Rev. 106, 1337.

Chew, G.F. and Low, F.E. (1956), Phys. Rev. 101, 1570.

Cini, M. (1953), Nuovo Cim., (9), 10, 526, 614.

Cini, M. and Fubini, S. (1954), Nuovo Cim., (9), 11, 142.

Dalitz, R.G. and Dyson, F.J. (1955), Phys. Rev. 99, 301.

Deser, S., Thirring, W.E. and Goldberger, M.L. (1954), Phys. Rev. 94, 711.

Drell, S.D., Friedman, M.H. and Zachariasen, F. (1956), Phys. Rev. 104, 236.

Dyson, F.J. (1949), Phys. Rev. 75, 1736.

Dyson, F.J. (1953), Phys. Rev. 91, 421.

Dyson, F.J. (1953a), Phys. Rev. 91, 1543.

Dyson, F.J. et al (1954), Phys. Rev. 95, 1644.

Feldman, G. and Matthews, P.T. (1956), Phys. Rev. 102, 1421.

Fonda, L. and Reina, I. (1956), Nuovo Cim., (10), 4, 1399.

Friedman, M.H., Lee, T.D., and Christian, R. (1955), Phys. Rev. 100, 1494.

Gell-Mann, M. and Goldberger, M.L. (1954), Proc. Rochester Conf. p.26 ff.

Gilbert, W. (1957), Phys. Rev. 108, 1078.

Goldberger, M.L., Miyazawa H. and Oehme, R. (1955), Phys. Rev. 99, 986.

Goldstein, J.S. (1958), Nuovo Cim., (10), 9, 504.

Goodwin, L.K., Kenney, R.W. and Perez-Mendez, V. (1959) Phys. Rev. Letters 3, 522.

Haber-Schaim, U. and Thirring, W. (1955), Nuovo Cim. (10), 2, 100.

Ito, D., Miyamoto, Y. and Watanabe, Y. (1955), Prog. Theoret. Phys. 13, 594.

Kallen, G. (1954), Nuovo Cim. (9), 12, 217.

Kallen, G. (1957) CERN, 57-43.

Kalos, M.H. and Dalitz, R.H. (1955), Phys. Rev. 100, 1515.

Kobayashi, T. and Klein, A. (1958), Nuovo Cim. (10), 8, 850.

Kroll, N.M. and Ruderman, M.A. (1954), Phys. Rev. 93, 233.

Langer, J.S. (1957), Nuovo Cim., (10), 6, 674.

Lee, T.D. (1954), Phys. Rev. 95, 1329.

Levy, M.M. (1954), Phys. Rev. 94, 460.

Levy, M.M. (1955), Phys. Rev. 98, 1470.

Levy, M.M. and Marshak, R.E. (1954), Nuovo Cim. (9), 11, 366.

Lomon, E.L. (1956), Nuovo Cim. (10), 4, 106.

Mandelstam, S. (1958), Phys. Rev. 112, 1344.

Martin, A. (1956), Nuovo Cim. (10), 4, 369.

Matthews, P.T. and Edwards, S.F. (1957), Phil. Mag. 8, 176.

Matthews, P.T. and Salam, A. (1952), Phys. Rev. 86, 715.

Mitra, A.N. and Dyson, F.J. (1953), Phys. Rev. 90, 372.

Moorhouse, R.G. (1953), Phys. Rev. 89, 958.

Morpurgo, G. and Touschek, B.F. (1953), Nuovo Cim. (9), 10, 1681.

Orear, J. (1954), Phys. Rev. 96, 176.

Orear, J. (1955), Phys. Rev. 100, 288.

Orear, J. (1956), Nuovo Cim. (10), 4, 856.

Pontecorvo, B. (1959), Kiev Conference Notes, 1959.

Ross, M. (1954), Phys. Rev. 95, 1687.

Ruijgrok, T.W. (1958), Physica, 24, 185.

Sartori, L. and Wataghin, V. (1954), Nuovo Cim. (9), 12, 260.

Sartori, L. and Wataghin, V. (1954a), Nuovo Cim. (9), 12, 145.

Stanghellini, A. (1958), Nuovo Cim. 10, 398.

Tanaka, K. (1957), Phys. Rev. 105, 1109.

Taylor, J.C. (1954), Phys. Rev. 95, 1313.

Tomonaga, S. (1947), Prog. Theoret. Phys. 2, 6.

Visscher, W.M. (1954), Phys. Rev. 96, 788.

Wyld, H.W. (1954), Phys. Rev. 96, 1661.