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Essays in Asset Price Bubbles

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Abstract

This thesis studies the field of asset price bubbles. It is comprised of three independent chapters. Each of these chapters either directly or indirectly analyse the existence or implications of asset price bubbles. The type of bubbles assumed in each of these chapters is consistent with rational expectations. Thus, the kind of price bubbles investigated here are known as rational bubbles in the literature. The following describes the three chapters.

Chapter 1: This chapter attempts to explain the recent US housing price bubble by developing a heterogeneous agent endowment economy asset pricing model with risky housing, endogenous collateral and defaults. Investment in housing is subject to an idiosyncratic risk and some mortgages are defaulted in equilibrium. We analytically derive the leverage or the endogenous loan to value ratio. This variable comes from a limited participation constraint in a one period mortgage contract with monitoring costs. Our results show that low values of housing investment risk produces a credit easing effect encouraging excess leverage and generates credit driven rational price bubbles in the housing good. Conversely, high values of housing investment risk produces a credit crunch characterized by tight borrowing constraints, low leverage and low house prices. Furthermore, the leverage ratio was found to be procyclical and the rate of defaults countercyclical consistent with empirical evidence.

Chapter 2: It is widely believed that financial assets have considerable persistence and are susceptible to bubbles. However, identification of this persistence and potential bubbles is not straightforward. This chapter tests for price bubbles in the United States housing market accounting for long memory and structural breaks. The intuition is that the presence of long memory negates price bubbles while the presence of breaks could artificially induce bubble behaviour. Hence,

we use procedures namely semi-parametric Whittle and parametric $ARFIMA$ procedures that are consistent for a variety of residual biases to estimate the value of the long memory parameter, d, of the log rent-price ratio. We find that the semi-parametric estimation procedures robust to non-normality and heteroskedasticity errors found far more bubble regions than parametric ones. A structural break was identified in the mean and trend of all the series which when accounted for removed bubble behaviour in a number of regions. Importantly, the United States housing market showed evidence for rational bubbles at both the aggregate and regional levels.

In the third and final chapter, we attempt to answer the following question: To what extend should individuals participate in the stock market and hold risky assets over their lifecycle? We answer this question by employing a lifecycle consumption-portfolio choice model with housing, labour income and time varying predictable returns where the agents are constrained in the level of their borrowing. We first analytically characterize and then numerically solve for the optimal asset allocation on the risky asset comparing the return predictability case with that of IID returns. We successfully resolve the puzzles and find equity holding and participation rates close to the data. We also find that return predictability substantially alter both the level of risky portfolio allocation and the rate of stock market participation. High factor (dividend-price ratio) realization and high persistence of factor process indicative of stock market bubbles raise the amount of wealth invested in risky assets and the level of stock market participation, respectively. Conversely, rare disasters were found to bring down these rates, the change being severe for investors in the later years of the life-cycle. Furthermore, investors following time varying returns (return predictability) hedged background risks significantly better than the IID ones.

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This thesis is dedicated to
my Parents

Declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Preface

The three chapters in this thesis studies the field of asset price bubbles. In the first chapter, we develop a theoretical model and show how housing price bubbles can emerge in equilibrium. The second chapter provides an empirical investigation into the timely identification of housing price bubbles. In the third chapter, we examine how price bubbles can influence household asset allocation and stock market participation rates. The common theme in all the three papers is asset price bubbles. We start our introduction by citing a historical example of an asset bubble.

Perhaps the earliest known example of an asset price bubble is the tulip bubble in Holland that started in 1634 and burst in February 1637. Amid the general fascination with rare species of tulips among the Dutch, prices on rare tulip bulbs rose, attracting the attention of speculators. Since the supply of rare bulbs was severely limited in the short run, and demand sky-rocketed due to the influx of speculators, prices rose rapidly amid heavy trading. At the bubble's peak, a single tulip bulb sold for an equivalent of $60,000 today. Following the Dutch tulip mania, there have been numerous episodes of boom-bust phenomena in asset markets. The last decade saw a bubble in the United States housing market. House prices peaked in early 2006, started to decline in 2006 and 2007, and reached historic lows in 2012. The collapse of this housing bubble was followed with widespread mortgage defaults and created a credit crisis which is widely considered to be the primary reason behind the 2007-09 economic recession, see Brunnermeier et al. (2012).

These events underscored the importance of financial frictions and their role in asset price volatility. The financial friction that played a large role in the recent crisis came from residential and commercial lending activities. A large part of the lending in current economies is secured through some form of collateral: residential and commercial mortgages are secured by the mortgaged property itself, corporate bonds are secured by the physical assets of the firm, and collateralized mortgage obligations and debt obligations and other similar instruments are secured by pools of loans that are in turn secured by physical property. The total of such collateralized lending is enormous: in 2007 at the peak of the housing bubble, the value of US residential mortgages alone was roughly $10 trillion, see Geanakoplos and Zame (2014). Not all of the assets of the borrower can be pledged as collateral. This can be due to several reasons. They can be either because some of the agents are not participating in the market or because the information is imperfect or because of institutional frictions such as limited commitment or enforcement, see Holmstrom and Tirole (2011). The fact that only a fraction of the assets can be collateralized implies intuitively that this fraction is an important determinant of borrower-lender dynamics. For a residential borrowing where the collateral is the stock of house owned by the borrower, this fraction is called the Loan to Value ratio. Changes in this loan to value ratio influences the availability of credit, default probabilities and asset prices.

In the first chapter, we attempt to understand these dynamics between the loan to value ratio, mortgage defaults and house prices. Importantly, we model how house price booms and busts can arise in equilibrium. The environment of our model includes an endowment economy with heterogenous agents, similar to Zhang (1997) and Rytchkov (2014). Agents differ in the level of their discount factor. We assume two types of agents or households, one type with high discount factor and the other with a low discount factor. Households with high discount factor are patient, called as Savers, and in equilibrium will lend to those with low discount factors, called as Borrowers. Both these household types derive utility from a durable housing good which is traded intertemporally. To finance the purchase of housing stocks, borrower households agree to a one period mortgage contract with the Savers. The stock of borrower household's housing is secured as the collateral. The structure of the mortgage contract we study is very close to that used by Bernanke and Gertler (1989), Carlstrom and Fuerst (1997) and Bernanke et al. (1999). We assume that the borrowers are hit by an idiosyncratic shock to their housing stock after the contract is agreed. Savers (Lenders) can observe these shocks only if they pay a monitoring cost. Those households who experienced a bad shock will find it optimal to default on their mortgages while the households who had a good shock will repay their loans with interest. Thus, in equilibrium, some mortgages will be defaulted on. The presence of defaults aids us in analytically deriving an expression for the loan to value ratio. This ratio is endogenous and depends on the deep parameters and variables which includes the stock of collateral, the monitoring costs, the realized shock etc. This ratio measures the net share of housing value that goes to the lenders as repayment. We restrict our analysis to finding the steady state equilibrium values.

The main contribution of this chapter is to produce a tractable way of analysing the impact of endogenous loan to value ratio and endogenous defaults that lead to equilibrium house price bubbles. The approach is simple, straightforward and can easily be extended to study more complicated dynamics. Our model thus extend the literature which assume that margin on the collateral is exogenous or rule out defaults, see Aiyagari and Gertler (1999), Coen-Pirani (2005), Santos and Woodford (1997), Miao (2014) and He et al. (2015). Furthermore, the few studies that do accommodate both defaults and endogenous margins usually assume highly complicated theoretical structures in the form of heterogenous beliefs and incomplete markets, see for example Geanakoplos (2003), Kubler and Schmedders (2012), Simsek (2013) and Brumm et al. (2015). These models, except for the two period case, cannot be studied analytically.

We find some important results. Firstly, the endogenous loan to value ratio and leverage was found to be procyclical in nature consistent with Geanakoplos and Zame (2014). High leverage is observed under high house prices and low leverage under low house prices. Secondly, low values of idiosyncratic risk of housing generates a credit-easing effect boosting excessive borrowing resulting in a rational price bubble in housing goods. Thirdly, high values of idiosyncratic risk causes a credit-crunch effect tightening the borrowing constraint restricting lending and thus lowering house prices. Fourth, the probability of default rises with declining house prices and increased uncertainty (risk). These results thus explain the foreclosure crisis observed after the burst of the housing bubble in the US wherein a substantial portion of mortgages were defaulted on.

Thus in chapter one, we show how endogenous collateral constraints with risk of defaults lead to housing price bubbles in equilibrium. The natural question then is how do we identify price bubbles from empirical data. This would then aid policy makers in designing a priori appropriate lending standards and prevent financial disasters from occurring.

In our second chapter, we undertake such a task and deal with the identification of house price bubbles in an accurate and timely manner. Our focus is on the United States housing market. Ever since the works of Blanchard (1979), Blanchard and Watson (1982a) and Diba and Grossman (1988), we know that identifying price bubbles involves monitoring for any deviation of the asset's price from its fundamentals. This is the theoretical definition of a rational bubble, see Brunnermeier and Oehmke (2013). As we focus on bubbles in the housing market, the fundamentals here would be the rents. The rents are kind of a payoff accrued to the homeowner and is generally considered in the literature as the real estate equivalent of stock market dividends, see Himmelberg et al. (2005). As the dividend-price ratio is a financial ratio for the stock market, we can equivalently consider the housing rent-price ratio as a financial ratio for the housing market, see Plazzi et al. (2010). The presence of a unit root in this rent-price ratio would mean that the housing price and its fundamentals, the rents, are not moving together. That is, they do not share a common trend implying the existence of a rational bubble in the housing market. As argued by Phillips and Yu (2011) the identification of price bubbles thus involves testing for the presence of a unit root in the rent-price ratio series.

The application of standard unit root tests to detect rational price bubbles have had mixed success, see Diba and Grossman (1987), Evans (1991b), Lamont (1998) and Horvath and Watson (2009). The primary reason for this is that these tests have low power in differentiating a unit root process with a near unit root process, see Diebold and Rudebusch (1991). If we represent d as a parameter that measures the persistence of a series, then $d = 1$ is a unit root process which implies the presence of rational bubbles but $d = 0.9$ is not a bubble process. Such processes with values of d close to one but not a unit root have a special property in that they are mean reverting in an extended period of time. This means that these series although divergent in the short run will eventually return to their mean, in our case the fundamentals, and thus rejects any bubble behaviour. These type of mean-reverting but persistent series are called long memory processes, originally identified by Granger and Joyeux (1980).

In this light, we use long memory models to identify house price bubbles. The use of these models involve estimating the memory parameter, d. The value of d indicates the persistence and thus the presence or absence of a rational bubble. We employ econometric tests in both the time domain, called parametric tests, and also the frequency domain, called semi-parametric tests, to test the null of a unit root $d = 1$ bubble process against a mean reverting $d < 1$ no bubble process. This is the first contribution in our chapter as existing literature have concentrated on using only one of these two methods. Furthermore, more often these methods have been applied to study stock price bubbles and not housing bubbles, see for example Koustas and Serletis (2005) and Cuñado et al. (2005).

It is well known ever since the work of Perron (1989) and Diebold and Inoue (2001) that the presence of structural breaks could artificially induce an unit root. In our context, structural breaks would mean a change in the fundamentals of the economy. Hence, accounting for such breaks is essential for an efficient identification of bubbles. We use standard break tests to identify any endogenous break in the mean and trend in the series and then demean and detrend it to arrive at an unbiased estimate of the persistence. This forms the second contribution of our chapter.

The third contribution involves the use of both regional metropolitan level data along with the aggregate data. This is motivated by the literature on housing supply elasticity. As elaborated by Green et al. (2005) and Levitin and Wachter (2012), housing price appreciation depends to a large extent on its supply elasticity and furthermore, with changing demographics and geography these elasticities differ widely. Our analysis thus identifies regional price bubbles along with their aggregate counterparts. The dataset we use covered the quarterly 31 year time period of 1982Q4-2013Q4.

Our results extend the literature in multiple dimensions. We first find that as expected the long memory methods provided better estimates compared to standard unit root tests. Secondly, the semi-parametric estimates of persistence were found to be more reliable than the parametric ones. These values were well above a unit root. Thirdly, when examined for endogenous breaks, one structural break was observed in all the series. This breakdate, around the year 2003, corresponded with the period of a change in the borrowing standards in the American credit market, see Glaeser et al. (2013) Once we adjusted for these breaks, all of the series gave comparatively lower values of persistence. However, one of the aggregate price series and 8 out of 12 regional series continued to exhibit unit root tendency. We concluded that there were price bubbles in the US housing market.

In the first and second chapters, we explicitly deal with asset price bubbles. While the first chapter explained how endogenous collateral constraints and mortgage defaults lead to credit driven housing price bubbles in equilibrium, the second chapter provided an empirical study on the timely identification of bubbles in housing markets. The presence of a bubble indicates some kind of inefficiency in the financial market, see Fama and French (1988, 1992). If markets were efficient, prices would never deviate from fundamentals and we would have no bubbles. Asset bubbles are not the only consequence of an inefficient financial market. Several puzzling phenomenon observed in the asset pricing literature can be attributed to these inefficiencies. The well known equity premium puzzle first studied by Mehra and Prescott (1985) is an example.

In the third chapter, we attempt to resolve two puzzles that are observed in asset markets that relates specifically to household portfolio choice. These are the stock market participation puzzle and the asset allocation puzzle. The 2007 Survey of Consumer Finances data reveal that only about 50% of US households invest in stocks, either directly or indirectly (via mutual funds in retirement and nonretirement accounts), and participation in European countries is even lower, see Bucks (2006) and Guiso et al. (2008). This is called the Stock Market Participation Puzzle. Furthermore, it has been observed in microeconomic panel income data by several economists that the few people who do participate in the stock market hold very little wealth in risky equities. This level of wealth in risky equities is found to follow a hump-shape through the investor's life-cycle, see Canner et al. (1997), Vissing-Jorgensen (2002) and Ameriks and Zeldes (2002). This is the Asset Allocation Puzzle.

Traditional models in portfolio allocation theory are at odds with these empirical facts. For instance, Merton (1969, 1971) and Samuelson (1969), consider a dynamic portfolio optimization problem in which investors maximize expected utility through their choice of risky and risk-free investments subject to a wealth constraint and obtain closed form solutions. Their theory predicts that the share invested in the risky asset is affected neither by the level of wealth nor by the consumption decision. In other words, an optimal investor should put 100% of his wealth in risky stocks, a counter-factual prediction. In fairness, the Merton and Samuelson results were derived under many restrictive assumptions, including power utility, independent and identically distributed (IID) returns on the risky and risk-free investments, the absence of market frictions, the absence of labour income or any durable goods.

In an attempt to reconcile the theory with the empirical facts, several authors have relaxed these assumptions. This has been achieved through incorporating labour income (Bodie et al. (1992), Benzoni et al. (2007)), generalizing preferences (Campbell and Viceira (1999), Gomes and Michaelidis (2005)), making intertemporal utility non-separable in a durable good such as housing (Grossman and Laroque (1990), Flavin and Yamashita (2011)) and analysing the effects of time variation in equity premium (Campbell et al. (2001), Michaelides and Zhang (2015)). Despite these advances, the puzzles still remain unresolved. One reason is that the focus on most of these studies incorporate only one or two dynamics, for example Gomes and Michaelidis (2005) generalizes preference and adds labour income but abstracts from durable housing and other factors. Importantly, with the exception of Campbell and Viceira (1999) and to an extent Viceira (2001) analytical studies are non-existent.

In the third chapter, we fill this gap in the literature and successfully resolve the two puzzles. We start with a reasonably rich model which is analytically solved to derive an expression for the optimal asset allocation in risky stocks. We then enrich the model in a life-cycle context and incorporate all relevant factors that influences stock market participation and life-cycle asset allocation puzzles.

The optimization problem is solved with numerical methods and policy functions are simulated.

Our analytical exercise gave valuable intuition in how and what factors determine the level of wealth invested in the risky asset when the household faces changes in the investment opportunity set, shocks to the labour income and shocks to durable housing prices. In this way we extend the seminal work of Campbell and Viceira (1999) to include durable goods and labour income. This is the first contribution of our chapter. The second contribution, as stated earlier, is in incorporating all relevant dynamics in a rich life-cycle model. Importantly, we extend the works of Cocco (2004) and Vestman (2012)'s life-cycle portfolio choice model which has both housing and risky labour income by including time varying returns, Epstein-Zin preferences, a bequest motive and uncertainty of death. Time varying returns implies that investors in our model can use a factor such as the log dividend-price ratio to predict future returns. Hence, this can also be called as return predictability.

Our main results can be summarized as follows. Firstly, our simulated data from the theoretical model gave levels of asset allocation and stock market participation rates which are very close to the estimated ones from the Survey of Consumer Finance dataset. In this way, we successfully resolve the stock market participation and the asset allocation puzzles, respectively. Secondly, we find that in the presence of housing both the stock market participation rate and the risky asset allocation share is found to be hump-shaped over the life-cycle consistent with empirical evidence, seeGuiso and Sodini (2013). This is consistent with other papers that include housing in the portfolio choice such as Cocco (2004), Yao and Zhang (2004) and Vestman (2012). We find that housing initiates a crowding out effect restricting younger liquidity constrained households from market participation and holding risky stocks.

The third result we find is that portfolio choice and market participation profiles are significantly different in the return predictability case relative to the IID case. We find that when returns are predictable from a factor such as the log dividend-price ratio, the optimal risky share of liquid wealth invested in the risky asset varies largely depending on the factor's: realization, persistence and volatility. High realizations and high persistence of the factor, specifically unit root or above, substantially shifts up both the risky equity allocated as well as the rate of stock market participation. Likewise, a huge dip in the factor or a high volatility suggesting a "rare disaster" in the economy brings down the liquid wealth invested in risky stocks. Unlike the bubble scenario, a rare disaster such as a market crash was found to have a heterogenous response over the life-cycle with older and retired households, over the age of 65, being more affected (adversely). In addition to this, investors under return predictability were able to hedge background risks, such as labour income or house price volatilities, better.

Chapter 1

Endogenous Collateral Constraints, Defaults and House Price Bubbles

1.1. Introduction

A variety of factors contributed to the global financial crisis of 2007-09. One such factor was the growing availability of subprime mortgage credit in the mid-2000s in the United States. Households were able to borrow higher multiples of income, with lower required downpayments. The onset of the crisis was characterized by a fall in house prices, an increase in mortgage defaults and home foreclosures. These events initially affected residential construction and the financial sector, but their negative effects spread quickly to other sectors of the economy. This crisis thus underlined the need for economic models to accommodate the financial sector.

Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) were the pioneers in emphasizing the role of the financial sector in a general equilibrium model. The role of the financial sector in these papers comes from the entrepreneurs (firms) need of external finance possibly to meet an investment opportunity. They find that the presence of agency costs mean that borrowing is limited and needs to be secured by some kind of a collateral. In general, the optimal lending contract would entail the entrepreneur to pledge a fraction of his assets as collateral. These studies show that when the collateral goes down in value, so would the amount that can be borrowed against it. Furthermore, the presence of financial frictions in the form of these collateral constraint result in amplification mechanisms whereby any real shocks to the economy (for instance, productivity) will get multiplied and propagated to other sectors of the economy.[1] Kiyotaki and Moore (1997) calls these credit cycles. A problem with this stream of thought, as stated by Geanakoplos (2003) is that it keeps the loan to value ratio constant. The loan to value ratio is the fraction of the asset that is collateralized (pledged). **Changes in the loan to value ratio has amplification effects on leverage, asset prices and the broader macroeconomy**.

We can make this statement clear by the following example. Consider the case of a homeowner (or hedge fund or a big investment bank) who takes out a loan using a house as collateral, he must negotiate not just the interest rate, but how much he can borrow. If the house costs \$100 and he borrows \$80 and pays \$20 in cash, we can say that the margin or downpayment is 20%, and the loan to value is \$80/\$100 = 80%. The leverage is the reciprocal of the margin, namely the ratio of the asset value to the cash needed to purchase it, or \$100/\$20 = 5. If you are leveraged five to one and the asset increases or decreases 1%, your wealth goes up or down 5%. In essence, the borrower is more sensitive to changes in housing wealth and prices, see Geanakoplos and Zame (2014).

We can formalize this example by specifying a mortgage contract where the loan to the borrowing agents is secured with the agent's asset (house) as collateral.

$$L_{t+1} \leq \psi P_{t+1} H_{t+1}, \quad \text{where} \quad 0 < \psi < 1 \tag{1.1}$$

where L_{t+1} is the loans offered to borrowers at time t, $P_{t+1}H_{t+1}$ is the value of the physical asset, the house, which is collateralized where P_{t+1} is the price of the house and H_{t+1} is the stock of house. Importantly, ψ here is defined as the Loan to Value Ratio (LTV). It is the fraction of the collateral that is actually pledged. If the borrower defaults, the lender gets this fraction of the collateral value. It follows then that $1 - \psi$ is the margin, $(1 - \psi)P_{t+1}H_{t+1}$ is the

[1]The key mechanism involves the link between "external finance premium", the difference between the cost of funds raised externally and the opportunity cost of funds internal to the firm, and the net worth of potential borrowers. With credit-market frictions present, and with the total amount of financing required held constant, standard models of lending with asymmetric information imply that the external finance premium depends inversely on borrowers' net worth. This inverse relationship arises because, when borrowers have little wealth to contribute to project financing, the potential divergence of interests between the borrower and the suppliers of external funds is greater, implying increased agency costs; in equilibrium, lenders should be compensated by higher agency costs through a larger premium. To the extent that borrowers' net worth is procyclical, the external finance premium will be countercyclical, enhancing the swings in borrowing and thus in investment, spending, and production, see Bernanke and Gertler (1989) and Bernanke et al. (1996, 1999).

downpayment and the leverage (ratio) is $\frac{1}{Margin} = \frac{1}{1-\psi}$. Here we have subsumed the interest rate on Loans within the variables.[2] ***The main objective of this chapter is in endogenizing the loan to value ratio*** ψ. Once the parameter ψ is endogenized, we then ***analyse its implications to asset pricing and default rates***. We use the terms margins, leverage, pledgeability parameter and loan to value ratios interchangeably throughout this chapter. As is clear from the above example, all of these concepts are contained in one variable, ψ.[3]

The underlying mechanism that generates the endogenous loan to value ratio is a one period mortgage contract agreed between a borrower and a lender. In our study, we assume an endowment economy model standard in the general equilibrium asset pricing literature such as Lucas Jr. (1978) and Zhang (1997). The economy is populated with two types of households who only differ in the discount factor. Households with a low discount factor are impatient and are hence called "Borrowers" and households with a high discount factor are patient and thus called "Savers" as in Rytchkov (2014). In equilibrium, the saver households will lend to the borrower households. These households have preferences defined over a durable housing good. This housing good is traded between the two households. Thus, the housing good provides both consumption services and also acts as an investment asset, see for example Cocco (2004) and Iacoviello (2004).

Borrowers use their houses as collateral for mortgages and experience idiosyncratic housing investment shocks. The realized shock is not observable to the lender. Lenders must pay a monitoring cost to observe borrower's realized housing return. This is the agency cost in the contract. Borrowers experiencing low realizations of the idiosyncratic shock default on their mortgages; Borrowers who repay their mortgages pay a state-contingent adjustable mortgage rate that is typically above the risk-free rate. The kind of mortgage contract that we discuss in this chapter is a one period mortgage contract. The contract is negotiated at the beginning of a period and resolved by the end of that same period. Borrowers who are unable to repay their loans, because of some kind of a bad realization of the shock, will default on their loans. In case of defaults, the borrower will lose

[2] Quadrini (2011), Miao and Wang (2012) and Miao et al. (2015) consider intratemporal loans meaning that no interest is charged by the lenders. Unlike these papers, in this chapter we model loans as intertemporal where the interest rate on loans are predetermined at the time of the contract agreement.

[3] In the context of our chapter, as the collateral here is the stock of housing, the appropriate term is the loan to value ratio. However, it is not unusual in the literature to call this parameter ψ as a margin, see for example Brumm et al. (2015).

his collateralized asset. In this way, our model as in Carlstrom and Fuerst (1997) **allows for endogenous defaults in equilibrium**.

The endogenous collateral constraint is derived from a limited participation constraint faced by the lenders in the optimal contract. For the structure of the contract, we follow Carlstrom and Fuerst (1997) and Bernanke et al. (1999) and is based on the costly state verification framework of Townsend (1979). Carlstrom and Fuerst (1997) and Bernanke et al. (1999) use the contract to understand the borrowing dynamics at the firm level where the collateral is the capital owned by the firm. We apply it to the households problem in our context where the collateral is the stock of house owned by the borrowers. We assume that the idiosyncratic shock follows a log normal distribution. Following Bernanke et al. (1999) the cumulative distribution function then gives us the probability or the rate of default. We derive the endogenous margin or loan to value ratio as a specific function of the underlying shock distribution. This ratio measures the net share of the housing value that goes to the lenders for repayment. This ratio depends on the realized level of the shock, the parameters that affect the equilibrium value of the shock, the monitoring cost of the lenders and also the borrower's housing stock. The implication here is that **the loan to value ratio is endogenous**.

The solution to the optimization problem faced by the borrower and saver households gives us the equilibrium steady state. We quantify the steady state values by an appropriate calibration of the parameters. We compare and contrast different steady state equilibrium that arises from different levels of risk. The risk is captured by the standard deviation of the idiosyncratic shock. We focus on three key variables, namely, the endogenous loan to value ratio (which also gives the leverage ratio), the probability of defaults and house prices. In what follows, we describe the main contributions and the results obtained from our study.

Contribution and Results. This chapter contributes to the literature in several ways. The main contribution of this chapter is in modelling a tractable method of analysing the impact of endogenous loan to value ratio and endogenous defaults that lead to equilibrium house price bubbles. We do this by formulating a borrowing contract secured by the level of housing stock held by the borrower. Constraints on limited participation added with the presence of idiosyncratic shocks mean that for a specific distribution assumption of the idiosycratic shock, the collateral constraint can be derived endogenously. This collateral constraint gives us the endogenous loan to value ratio which accommodates defaults.

The equilibrium can then be solved as a standard optimization problem. The equilibrium conditions are solved to understand the link between house price bubbles, collateral margins and default rates. Predominant literature in this field either assume that margin on the collateral is exogenous or rule out defaults, see Aiyagari and Gertler (1999), Coen-Pirani (2005), Santos and Woodford (1997), Miao (2014) and He et al. (2015). Furthermore, the few studies that do accommodate both defaults and endogenous margins usually assume highly complicated theoretical structures in the form of heterogenous beliefs and incomplete markets, see for example Geanakoplos (2003), Kubler and Schmedders (2012), Simsek (2013) and Brumm et al. (2015). These models, except for the two period case, cannot be studied analytically. Also, Models with heterogeneous beliefs (non-common priors) have the drawback that it is more difficult to conduct a thorough welfare analysis. It is not clear which beliefs should one assign to the social planner.

We find the following four results in this chapter. Firstly, the endogenous loan to value ratio was found to be procyclical in nature. As the loan to value and leverage are the same, we say that the endogenous leverage in our model is procyclical. High leverage is observed under high house prices and low leverage under low house prices. Secondly, low values of idiosyncratic risk of housing generates a credit-easing effect boosting excessive borrowing resulting in a rational price bubble in housing goods. Thirdly, high values of idiosyncratic risk causes a credit-crunch effect tightening the borrowing constraint restricting lending and thus lowering house prices. Fourth, the probability of default rises with declining house prices and increased uncertainty (risk). These results thus explain the foreclosure crisis observed after the burst of the housing bubble in the US wherein a substantial portion of mortgages were defaulted on.

Related Literature. Our chapter is close to the subject of housing bubbles, endogenous mortgage defaults and endogenous collateral constraints. We review some related papers, mainly concentrating on those set in a general equilibrium framework. Aiyagari and Gertler (1999), Coen-Pirani (2005) and Rytchkov (2014), are three studies that focus on the effects of collateral constraints on equilibrium asset prices (housing). While Aiyagari and Gertler (1999) finds that binding constraints lead to highly volatile asset prices, Coen-Pirani (2005) finds no effect of the constraint on prices. However, both these papers assume exogenous margins. Rytchkov (2014) considering a continuous time model with two types of agents endogenizes the margin by assuming it is a function of the state variable (consumption share) and finds that time varying margin constraints lead to an increase in the price of the risky asset. Rythckov's model, however, rules out any possibilty of default. Brumm et al. (2015) studies collateral requirements and asset pricing with default costs and endogenous margins in a rich dynamic context. Their results reveal that assets with different collateralizabililty possess different returns, there exists a collateral premium in these assets prices. Their model is much richer than ours but the approach in which they endogenize the margins is different. They follow the theory of Geanakoplos (2003) in doing this.[4]

He and Krishnamurthy (2013), Santos and Woodford (1997), Caballero and Krishnamurthy (2001), Hellwig and Lorenzoni (2009), Miao and Wang (2012) Jose A. Scheinkman (2013) and Miao and Wang (2014) are some papers that investigate the possibilty of bubbles arising from borrowing constraints. However, all of these assume exogenous margins and rule out defaults.

Our chapter is also close to the literature on mortgage defaults with endogenous loan to value ratios. In a recent paper, Campbell and Cocco (2015) study the mortgage default decision using a partial equilibrium theoretical model of a rational utility-maximizing household. They solve a dynamic model of a household that finances the purchase of a house with a mortgage, and must in each period decide how much to consume and whether to exercise options to default, prepay, or refinance the loan. They find that the level of negative home equity that triggers default depends on the extent to which households are borrowing constrained. As house prices decline, households with tightly binding borrowing constraints will default sooner than unconstrained households, because they value the immediate budget relief from default more highly relative to the longer-term costs. A higher LTV ratio (smaller down payment) was found to increase the

[4] Stein (1995) is an early work that considered an ad-hoc endogenous loan to value ratio in a static framework. An alternate interpretation to the margin parameter was given by Holmstrom and Tirole (2011) who assumed non-pledgeability in that firms (as well as consumers) can count on liquidating only part of their wealth whenever they need funds. Holmstrom and Tirole define shortage of inside liquidity as a scenario in which internal funds (profits) generated by the firm is not enough to meet its next period investment demands. This forces the firm to approach an external financial intermediary such as a bank for funds. They derive key insights regarding implications of aggregate shocks to financial market liquidity and explain underlying reasons behind the sub-prime crises. However, pledgeable income in their work cannot be directly compared to collateral in this chapter. Collateral can be different from pledgeable income in many contexts. The value of the assets backing up debt is often higher than the value of debt (the debt is over-collateralized). This may be because the underlying assets are risky and do not protect the investor's claim in all states of nature. Or it may be, because the value of collateral is worth less to the investor than it is to the borrowing firm. Note that even if the collateral is worth very little to the investor it can provide proper incentives for repayment of debt as long as the borrower prefers to repay the debt than lose his collateral and has the means to do so.

probability of negative home equity and mortgage default. The LTV ratios considered in this paper are exogenous and furthermore, the housing stock is fixed and doesn't change with time.

As far as the structure of the contract analysed in our model is concerned, it is very close to both Carlstrom and Fuerst (1997) and Bernanke et al. (1999). The key difference from these papers is that our model analyses a mortgage contract and thus the collateral is the housing stock owned by the borrowers. In this sense, Forlati and Lambertini (2011) comes close to our paper as they too apply the Bernanke type contract in a housing mortgage context. However, their study focuses on monetary policy shocks and its implications to risky mortgage defaults and is thus different from our analysis. One main difference from Kiyotaki and Moore (1997) is that the underlying contract in our model is based on the costly state verification model of Townsend (1979). In contrast, Kiyotaki and Moore (1997) build on the work of Hart and Moore (1994) and analyze the contracting in an environment in which there is ex-post renegotiation and inalienability of human capital. The consequence of such a contract is that borrowing is so tightly constrained by the level of the collateral that *default never occurs in equilibrium*. In contrast, our framework is similar to Bernanke and Gertler (1989) and Carlstrom and Fuerst (1997) in that *default is an equilibrium phenomenon*.

Roadmap. This chapter is organized as follows. The following §1.2 describes a selective review of the literature on rational bubbles, endogenous margins and financial constraints. The subsequent §1.3 describes in detail the theoretical model and defines the equilibrium. The results are reported in §1.4. Finally, §1.5 discusses the results and §1.6 concludes.

1.2. Literature Review

This section reviews the literature connected with the concepts explored in this chapter. Our chapter connects endogenous margins in collateral (borrowing) constraints with price bubbles. Hence, this literature review starts with the important studies that connect endogenous margins with price bubbles and later move towards the concept of a rational bubble and its formation from financial constraints.

1.2.1. Endogenous Margins and Asset Prices

The existing literature on endogenous margins and collateral constraints can be split into two. One that assumes heterogeneity, incomplete markets and defaults as in Geanakoplos (2003) and the second strand that uses the Value-at-Risk approach asin Brunnermeier and Pedersen (2008). In this section, we describe briefly the main studies in these fields.

The first major study on endogenous margins and its effect on asset prices was by Geanakoplos (2003). He finds that variation in leverage has a huge impact on the price of assets, contributing to economic bubbles and busts. The underlying assumption behind the Geanakoplos model is heterogenous valuation or beliefs and incomplete markets. Heterogenous valuations meaning that there is always a class of buyers for whom the asset is more valuable than it is for the rest of the public (standard economic theory, in contrast, assumes that asset prices reflect some fundamental value). Endogenous incomplete markets can arise when not all the assets are collateralizable.[5] These buyers are willing to pay more, perhaps because they are more optimistic, or they are more risk tolerant, or they simply like the assets more. If they can get their hands on more money through more highly leveraged borrowing (that is, getting a loan with less collateral), they will spend it on the assets and drive those prices up. If they lose wealth, or lose the ability to borrow, they will buy less, so the asset will fall into more pessimistic hands and be valued less. In the absence of intervention, **leverage becomes too high in boom times and too low in bad times**. As a result, in boom times asset prices are too high, and in crisis times they are too low. He calls this behaviour, **the leverage cycle**, see Geanakoplos (2010) Fostel and Geanakoplos (2008) and Simsek (2013) are some other studies that follow the same path and analyse leverage cycle and Aymanns and Farmer (2015) uses an agent based dynamic model to get quantitative results.

Brunnermeier and Pedersen (2008) builds a four period model in which the margin is endogenously determined by financiers who try to limit their counterparty credit risk.[6] They consider three groups of agents: customers and specula-

[5]This is a key assumption in his model that makes the margins endogenous. See for example Brumm et al. (2015) for a recent example.

[6]they do this by linking two liquidity concepts that determine the health of the asset market, funding liquidity and market liquidity. They define market liquidity as the ease with which the assets can be traded and funding liquidity as the ease with which investors or traders can obtain funding. They argue that both these liquidity concepts are interrelated. They define market liquidity as the difference between the transaction price and the fundamental value, and funding liquidity as speculators' scarcity (or shadow cost) of capital.

tors who trade assets and financier's who lend speculator's positions. Speculators face the constraint that the total margin on their positions $\mathbf{x_t}$ cannot exceed their capital W_t:

$$\sum_j (x_t^{j+} m_t^{j+} + x_t^{j-} m_t^{j-}) \leq W_t \tag{1.2}$$

where j is an index used to identify the security, $x_t^{j+} \geq 0$ and $x_t^{j-} \geq 0$ are the positive and negative parts of x_t^j, respectively, and m_t^{j+} and m_t^{j-} are the dollar margins on the long and short positions respectively. Speculators borrow from financiers who in turn set the margins such that their counterparty credit risk is minimized. The financier makes sure that the margins are big enough to cover their positions' π value at risk. For example, margins on a long positions m_t^{j+} is set such that:

$$\pi = Prob(-\Delta p_{t+1}^j > m_t^{j+} | \mathcal{F}_t) \tag{1.3}$$

the price drops (Δp_{t+1}^j) that exceed the margin m_t^{j+} will only happen with a probability π. The probability is exogenously chosen to be a small positive number close to zero such that the financier minimizes its risk from possible defaults by traders. The financier's margin depends on its information set $\mathcal{F}_t$. The financiers can be either perfectly informed in the sense that they know not just the fundamental value of the assets but also the aggregate shocks that hit the market or be imperfectly informed in that they only observe the asset's prices. The deviation of the asset's price from its fundamental value, $p_t^h - \nu_t^j$, is considered as a proxy measure for the market's *illiquidity*. When they assume that financiers are completely informed, they counterfactually conclude that the margins are decreasing in times of liquidity crises. Imposing information asymmetry between the financiers (lenders) and leveraged traders (speculators) makes the financiers unable to distinguish between fundamental shocks from liquidity shocks and thus tighten constraints (higher margins) producing a destabilizing effect on asset prices.

Both these studies although different in their approach produces qualitatively similar results. The approach of Geanakoplos (2003) finds that in a world where agents are heterogeneous and markets incomplete, **the ability to use an asset as a collateral (i.e., buying on margin) increases its price in equilibrium**. This happens because buying on margin makes it possible for a subset of agents who value the asset the most to determine its price. The increase in price represents a deviation from the Law of One Price (LOP), since two assets with the same payoff in all states of the world are priced differently. When assets can be used as collateral to borrow money, their prices not only reflect future cash flows but also their efficiency as liquidity providers. An identical result is obtained by Adrian Shin using the Brunnermeier and Pedersen (2008) Value at Risk approach. Essentially, they find that the price of any asset can be decomposed into two parts: its payoff value and its collateral value. The payoff value reflects the assets owner valuation of the future stream of payments, i.e. it is the value attached to the asset due to its investment role. But assets can also be used as collateral to borrow money. The collateral value reflects the asset owners valuation of this second role. This can theoretically create **deviations from Law of One Price** since two assets with identical payoffs can be priced differently if they have different collateral values.

An apparent weakness in the approach of Geanakoplos is that except for highly stylized two period versions of the model, the estimation and the results are not analytically tractable. Also, Basak and Shapiro (2001) analysing optimal consumption and wealth policies for a finitely lived agent finds some undesirable results when the Value-at-Risk is embedded in the optimizing framework. In particular, VaR risk managers incur larger losses than the non VaR counterparts in the most adverse states of the world.

This section explained how endogenous leverage (margins) can create asset price booms and busts. To further understand the concept of asset price movements, we introduce the concept of rational bubbles. The next section describes rational bubbles in a simple partial equilibrium framework and the following one reviews some of the literature that investigate the role of borrowing constraints and collateral in the creation of such bubbles in a general equilibrium framework.

1.2.2. Rational Bubbles

One of the first theoretical studies on rational bubbles was by Blanchard (1979) and Blanchard and Watson (1982a). Blanchard and Watson (1982a) using a partial equilibrium model finds that rationality of behaviour and expectations does not always imply that the market price of an asset be equal to its fundamental value. There can be rational deviations, in other words bubbles.

Blanchard and Watson characterize such a rational bubble using the efficient-market or the no arbitrage condition between stocks and a riskless asset. Let p_t be the price of a stock, d_t be the dividend, and r be the rate of return on the riskless assumed, constant over time. Then if risk neutral individuals arbitrage between stocks and the riskless asset, the expected rate of return on the stock, which is equal to the expected rate of capital gain plus the dividend-price ratio, must equal the riskless rate:

$$\frac{E[p_{t+1}|I_t] - p_t]}{p_t} + \frac{d_t}{p_t} = r \tag{1.4}$$

or by reorganizing

$$p_t = aE[p_{t+1}|I_t] + ad_t, \quad \text{where} \quad a = \frac{1}{1+r} < 1 \tag{1.5}$$

The coefficient a is the one-period discount factor and is less than one as long as long as the interest rate is positive: the price today depends on the expected price tomorrow but by less than one for one.

The linear difference equation (1.5) can be solved recursively by repeated substitution relying on the law of iterated expectations.[7] Solving the equation recursively up to time period T, we can arrive at many solutions depending on whether the transversality condition holds or not. The first solution is written as

$$p_t^* = \sum_{i=0}^{\infty} a^{i+1} E[d_{t+i}|I_t] \quad \text{if} \quad \lim_{T\to\infty} a_{T+1} E[p_{t+T+1}|I_t] = 0, \tag{1.6}$$

says that the price of the stock is the present discounted value of expected future dividends, the fundamentals. When the transversality condition do not hold, many other solutions are possible. One such solution can be written as

$$p_t = p_t^* + b_t, \quad E[b_{t+1}|I_t] = a^{-1} b_t \tag{1.7}$$

[7]The law of iterated expectations states that if Ω is an information set and ω is a subset of this information set, then for any variable x,

$$E[E[x|\Omega]|\omega] = E[x|\omega]$$

or, heuristically, if one has rational expectations and is asked how she would revise her expectation were she given more information, the answer must be that she is as likely to revise it up or down so that on average the revision wil be equal to zero. Applied to the information set I_t this implies in particular that

$$E[E[p|I_{t+1}]|I_t] = E[p|I_t]$$

where p_t^* is the solution from eq. (1.6). As long as $E[b_{t+1}|I_t] = a^{-1}b_t$, equation (1.7) is also a solution to the linear difference equation (1.5). Since a is less than one, b_t explodes in expected value, $\lim_{i \to \infty} E[b_{t+i}|I_t] = a^{-1}b_t \to +\infty$ if $b_t > 0$ and $-\infty$ if $b_t < 0$. As long as the process $b_t \neq 0$, the price of the stock will rationally deviate from its fundamentals. for this reason, Blanchard and Watson (1982a), calls b_t a *rational bubble*. Empirical testing for the possibilty of these bubbles were justified by Shiller (1983) and later West (1988). The theory of rational bubbles literature in a general equilibrium framework is extensive. It is beyond the scope of this chapter to review this. We provide a section in the Appendix that surveys some of the important works. The rational bubbles that we study in this chapter arises from borrowing constraints, these are essentially credit driven bubbles. In the next section we detail some of the studies in this area.

1.2.3. Rational Bubbles and Borrowing Constraints

Ever since the seminal work of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), recent papers in the rational bubble literature have incorporated financial constraints, specifically borrowing constraints in different forms, see Brunnermeier et al. (2012) and Miao (2014) for surveys of the rational bubble literature. Borrowing constraints are an important determinant of firm growth and survival and thus the general economy. Such constraints may arise in connection to the financing of investment opportunities faced by firms or temporary liquidity needs, like those needed to survive a recession, see Albuquerque and Hopenhayn (2004). We give some examples of such constraints, their implications to asset prices and limitations.

Suppose that an entrepreneur has an investment technology that produces one unit of output using one unit of investment. The entrepreneur can finance the investment I_t by his endowment w_t, one-period debt b_t, and a bubble asset B_t. A bubble asset is an asset that is intrinsically worthless but grows at the rate of the interest, see eq.(1.7). The debt has to be repaid the next period with interest. Furthermore, the lenders (such as banks) impose a borrowing constraint on the entrepreneur

$$(1 + r_{t+1})b_t \leq \lambda I_t, \quad \lambda \in (0, 1) \tag{1.8}$$

where r_{t+1} is the one period interest rate from time t to $t + 1$. This constraint says that the debt repayment in period $t + 1$ is limited by a fraction λ of the investment return I_t because the entrepreneur can only pledge this amount as collateral. Farhi and Tirole (2011) finds that in this case the investment satisfies

$$I_t = \frac{B_t + w_t}{1 - \lambda/(1 + r_{t+1})} \tag{1.9}$$

This equation says that the presence of a bubble $B_t > 0$ essentially raises the entrepreneur's net worth and hence investment. Thus **bubbles can crowd in investment**, rather than crowd out investment as in Diamond (1965) and Tirole (1985). Kocherlakota (2009) study another type of borrowing constraint. Kocherlakota examine a model economy in which capital re-allocation is critical. This re-allocation is accomplished via collateralized lending backed by land. However, land is scarce and so all entrepreneurs face borrowing constraints that bind infinitely often into the future. this constraint can be written as

$$(1 + r_{t+1})b_t \leq P_{t+1}L_t \tag{1.10}$$

where P_{t+1} is the land price in period $t+1$ and L_t represents the land holdings chosen in period t. These two ingredients imply that **equilibrium bubbles naturally emerge in the price of land, the collateral**. The resulting bubbles expand entrepreneurial borrowing capacity and generate more output, consumption, and welfare.

Kiyotaki and Moore (2012) consider a downpayment constraint given by:

$$b_t \leq \psi P_t L_t, \quad \psi \in (0, 1) \tag{1.11}$$

where P_tL_t represents the date t purchase price of the land and the fraction $1-\psi$ of the purchase value must be paid by the entrepreneur's net worth. As land is considered as an asset with no intrinsic value, this means that in the absence of a bubble $P_t = 0$ and hence there is no collateral for borrowing. The presence of a bubble in land prices makes $P_t > 0$ and thus the upper bound on the constraint gets relaxed. This is called **the "credit easing" effect of bubbles**. The bubble can help solve the collateral shortage problem. The movements of the bubble in land affect the borrowing capacity directly and thereby investments. One of the results we get in our analysis in this chapter is that once we endogenize the margin (ψ here) the credit easing effect creates the bubble and not the other way. A higher margin relaxes the liquidity shortage faced by the borrower.

In general, most of the literature that use financial constraints assume that it is of the form:

$$Loan_t \leq \xi.Collateral_t \tag{1.12}$$

meaning that the borrower can borrow up to a proportion $\xi \in (0,1)$ of the value of the collateral. The critical assumption here is that this parameter ξ, also called the pledgeability parameter (or the downpayment rate) is exogenous. The implication is that changes in financial conditions do not affect the level of the margin which is an unrealistic assumption. Some other papers in the literature that use such constraints include Santos and Woodford (1997), Caballero and Krishnamurthy (2001), Hellwig and Lorenzoni (2009), Miao et al. (2015), Miao and Wang (2012), Martin and Ventura (2012), Jose A. Scheinkman (2013) and Miao and Wang (2014).

Unlike the studies reviewed in this section, our model derives endogenous margins on collateral constraints (loan to value ratio), allows for defaults and price bubbles arise in equilibrium. As the underlying model and the plans and actions of all the agents are consistent with rational expectations, we call these bubbles as rational bubbles. Now that we have reviewed the relevant literature, we proceed to a detailed description of our theoretical model.

1.3. A Model of Endogenous Collateral and House Prices

Our model is built on a standard endowment economy asset pricing framework such as Lucas Jr. (1978) and Zhang (1997). It features a discrete time, infinite horizon economy with uncertainty. Time is denoted by $t = 0, 1, 2 \ldots \infty$ and continues forever. There is no production in the economy. The economy is populated with two types of households. One type of these households are called "Borrowers" and the second type are called as "Savers". We assume that these two types of households differ in their discount rates. This is inspired from several housing models in the literatures such as Iacoviello (2005), Monacelli (2009), Iacoviello and Neri (2010) and Forlati and Lambertini (2011). Households with high discount rates are patient and hence called "Savers" and households with low discount rates value current consumption more than future consumption and are hence called "Borrowers". All the households in a group are identical, in other words, our model has a representative borrower and a representative saver. Households who are impatient will borrow from households who are patient to smooth their consumption over time. Instead of assuming an exogenous borrowing constraint we derive it endogenously by explicitly modelling a one period mortgage contract between Savers (the Lenders) and Borrowers.

Following Iacoviello (2004) we assume that both the agents receive in each period some exogenous perishable endowment given by Y_t. They have preferences defined over only the durable housing. Aggregate housing is normalized to some constant and is in constant supply. However, shifts in housing demand across the two groups will affect housing prices as well as the allocation of housing between the borrowing and saving households.

1.3.1. Borrowers

We start with describing the preferences and constraints for the Borrower households. The discount factor for these Borrowing households denoted by β is lower than that of Savers, γ, that is $\beta < \gamma$.

The objective of the Borrowers is to maximize the following expected discounted utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(H_{t+1}^B), \quad 0 < \beta < 1, \tag{1.13}$$

where E_0 is the expectations formed at time zero, β is the household's subjective discount factor and $U(H_{t+1}^B)$ is an utility function which has its arguments the durable (housing) services denoted as H_{t+1}. Here we have assumed that the agent gets utility only from his stock of durable goods, housing, and does not value non durable consumption. This is primarily to maintain reasonable tractability in our analysis. With this simplification we can directly analyse the price of housing coming from the financial variables which we model later. It is assumed that the housing services that is carried over to period t (alternatively beginning of period t) are equal to H_{t+1}, see Forlati and Lambertini (2011). We assume that the utility function $U(.)$ is continuous, concave and strictly increasing. We use the superscript B to differentiate the borrower type households from savers. We have included housing as part of the household's utility function so that housing is not just an investment asset but also provides valuable utility services.

Following Cocco (2004) we define the utility function for the borrower households,

$$U(H_{t+1}^B) = \frac{(H_{t+1}^B)^{1-\sigma}}{1-\sigma} \tag{1.14}$$

as being a Constant Relative Risk Aversion (CRRA) functional form where as usual σ is the coefficient of relative risk aversion.

We assume that the stock of housing depreciates at the rate $\delta > 0$ every period and that the services derived from the house is quantitatively equivalent to its stock. At time t, the Borrowers face the following budget constraint:

$$P_t^H H_{t+1}^B + [1 - F_t(\overline{\omega}_t)](1 + R_{Z,t})L_t^B = L_{t+1}^B + Y_t^B + (1 - \delta)[1 - G_t(\overline{\omega_t})]P_t^H H_t^B \tag{1.15}$$

The left hand side of this equation (1.15) gives the financial expenses incurred by the household at time t. This includes the total value of housing stock chosen at time t, $P_t^H H_{t+1}^B$, where H_{t+1} is the total units of housing and P_t^H is the price of one unit of housing; and also payments on loans taken at period $t-1$ that have to repaid now $[1 - F_t(\overline{\omega}_t)](1 + R_{Z,t})L_t^B$. This loan repayment is the product of three components: the state contingent interest rate charged on the loans given by $R_{Z,t}$; the total amount of loans taken at time $t-1$, L_t^B; and also a fraction $[1 - F_t(\overline{\omega}_t)]$. To understand what this fraction means, we reiterate the argument that our model generates endogenous defaults in equilibrium. This means that not all the loans taken by the borrowers will be repaid. Some of it is defaulted. The value $[1 - F_t(\overline{\omega}_t)]$ indicates the fraction of loans that is repaid to lenders. The intuition behind this specific formulation and its construction will be explained later on.

The right hand side of the budget constraint (1.15) gives the total income available for the borrower households at time t. This includes the loans taken at time t, L_{t+1}^B; the endowment income Y_t^B and net housing value carried over from period $t-1$, $(1 - \delta)[1 - G_t(\overline{\omega_t})]P_t^H H_t^B$ where $\delta > 0$ is the depreciation rate on housing stock. The value $[1 - G_t(\overline{\omega_t})]$ is a fraction which indicates the left over stock of housing after the borrower's default in period $t-1$. Naturally, the term $G_t(\overline{\omega_t})$ indicates the fraction of the stock of Borrower's housing that was captured or seized by the lenders (Savers) as a consequence of default in period $t-1$. The interpretation and construction of ω, $F(.)$ and $G(.)$ and the terms of the one period mortgage contract are explained in the following paragraphs. For the specification of the contract, we follow Bernanke et al. (1999) and Forlati and Lambertini (2011).

We assume that each household consists of many members. The choice of housing investment H_{t+1} and state contingent interest rates on the loans next period depend on the contractual agreement of a representative member with the lender. Each member of the household, denoted by i, receives equal resources to purchase housing stock $H_{i,t+1}^B$. The total housing stock for the Borrowers would be then $\int_i H_{i,t+1}^B di = H_{t+1}^B$. The ith member finalizes the mortgage contract relating to the housing stock $H^B_{i,t+1}$. All the members in the household are ex-ante identical. Once the loan contract is agreed, the ith member experiences an idiosyncratic shock ω^i_{t+1} to his level of housing stock $H^B_{i,t+1}$. The ex-post housing value for this member is given by $\omega^i_{t+1}P^H_{t+1}H^B_{i,t+1}$. The implication here is that investments in housing is risky. The underlying mechanism behind the results in this chapter rely on constraints formed from the mortgage contract between the borrower and the lender.

The timing of this contract and the actions of each agents can be written as follows:

1. At time t, the Borrower household assigns an equal amount of resources to each of its i members.

2. These members purchase housing stock $H^B_{i,t+1}$ following the instructions of the household and manages it. The purchase of new housing stock is financed by a one period mortgage contract with a lender. The borrower receives a loan L_{t+1}.

3. Once the contract is agreed, this household member experiences an idiosyncratic shock ω^i_{t+1} on his stock of housing.

4. At time $t+1$, the lender pays a monitoring cost characterized by a proportion μ of the stock of the borrower's housing value. This monitoring cost ensures that the borrower will truthfully reveal his realized value of the shock.

5. The interest rate charged on the borrowers are now set so that it is state contingent (on the realised shock), $R_{Z,t+1}$.

6. The household member chooses its decision to default based on the ex-post realization of the shock on his housing value, $\omega^i_{t+1}(1-\delta)P^H_{t+1}H^B_{t+1}$ vis-a-vis the gross repayment to lenders, $(1+R_{Z,t+1})L^B_{t+1}$.

7. If the member defaults, he loses his pledged collateral (the stock of housing).

As in related contracting models of Bernanke and Gertler (1989), Bernanke et al. (1996), Carlstrom and Fuerst (1997) and Bernanke et al. (1999) we assume that the random variable w^i_{t+1} is independent and identically distributed (i.i.d) across members of the same household and log-normally distributed with a cumulative distribution function $F_{t+1}(\omega^i_{t+1})$. The mean and variance of $\ln \omega^i_{t+1}$ are deliberately chosen in a way that the $E_t(w^i_{t+1}) = 1$ for every time period

t. The key implication here is that although there are idiosyncratic shocks for the household members, the household in itself does not experience any risk, $E_t(\omega^i_{t+1} H^B_{i,t+1}) = H^B_{t+1}$. This important assumption will ensure that we do not need to keep track of each household's distribution of housing stocks across time and helps in making our model analytically tractable. Alternatively, we could assume shock not to individuals but to each households which is qualitatively similar to the Bewley (1987)-Aiyagari (1994) type models and would require the employment of numerical analysis.

As in Bernanke et al. (1999), we assume that the cumulative distribution function of the idiosyncratic shocks, F_{t+1}, is continuous and at least once differentiable. Furthermore, the hazard rate of the shock satisfies the following constaint

$$\frac{\partial \omega h(\omega)}{\partial \omega} > 0, \tag{1.16}$$

where $h(\omega) = \frac{\partial F(\omega)}{1-F(\omega)}$ is the hazard rate. The log normal distribution we assumed for the shock satisfies this restriction and is thus the primary motivation behind its use here. However, unlike the Bernanke et al. (1999) contract, we assume that housing investment riskiness can change over time. In other words, the cumulative distribution function of the shocks $F_{t+1}(\omega^i_{t+1})$ is time variant. We achieve this by letting the standard deviation $\sigma_{\omega,t}$ of $\ln \omega_t$ to follow an exogenous time varying process.

Once the idiosyncratic shocks are realized, the household member decides whether to repay his mortgage or default. If the member experiences good (high) realizations of shocks, he will repay the loan. However, if he faces bad (very low) shock realizations, he will find it optimal to default on his loans. The implication here is that the choice of default depends on the realized value of the shock ω_{t+1}. It is then natural to assume ***a cutoff or threshold value of the shock*** ω_{t+1} ***that will make the household indifferent between defaulting and non-defaulting***. Ever since the seminal work of Bernanke and Gertler (1989), several others have used this insight to analyse endogenous defaults in equilibrium: Arellano (2008), Mendoza and Yue (2012) and Chatterjee and Eyigungor (2012) are some examples of sovereign defaults.

The threshold value of the shock $\overline{\omega}_{t+1}$ is defined by[8]

$$\overline{\omega}_{t+1}(1-\delta)P^H_{t+1}H^B_{t+1} = (1+R_{Z,t+1})L^B_{t+1} \tag{1.17}$$

[8]This constraint can be considered as the housing equivalent of the Bernanke et al. (1999) constraint, see eq. (3.3) in their paper, imposed on borrowing firms. In their model, the borrowing firm pledges its capital as a collateral.

This can be considered as a no-default condition for the borrowers. The left hand side of this eq. (1.17) indicates the ex-post value of the collateral, that is, the value of the house after the shock has been realized. We have multiplied with $(1-\delta)$ in order to capture the depreciation that the house experiences between the time periods. The right hand side shows us the gross payment the borrower pays to the lender. The level of loans chosen at period t is given by L_{t+1} and the gross interest on borrowing is $(1+R_{Z,t+1})$. As we said before, the interest rate on borrowing is state contingent and set only after the state of nature, the shock has been realized.

From eq. (1.17) we can infer that when the realized shock falls at or above this threshold value $\omega_{t+1}^{i} \geq \overline{\omega_{t+1}}$, that is if $\omega_{t+1}^{i} \in [\overline{\omega}_{t+1}, \infty]$, the borrower will repay his loans. If on the other hand the realized shock falls below the threshold value $\omega_{t+1}^{i} < \overline{\omega}_{t+1}$, that is if $\omega_{t+1}^{i} \in [0, \overline{\omega_{t+1}})$, the borrower will find it optimal to default on his loans. In other words, the choice of default depends on the idiosyncratic shock realization.

Bernanke et al. (1999) derives eq. (1.17) from an optimal contract between borrowers and lenders in a costly state verification framework, first analyzed by Townsend (1979). In our context, the agency cost means that lenders do not observe the realized shock on the housing stocks faced by the borrowers. Therefore, the lender pays a monitoring cost that will induce the borrower to truthfully reveal his shock. The presence of monitoing costs thus removes the informational asymmetry and moral hazard problems. Before we detail the monitoring cost and its specification we discuss the costs of default to the borrowers.

In our model as in Campbell and Cocco (2015), the loans are nonrecourse. This means that although the loans (or mortgages) are secured by pledging the collateral (house), there is a restriction to the amount that can be collected by the lender in case the borrower defaults. In case of defaults, the household *members* lose their housing stocks to lenders. Nonrecourse loans means that the borrowers cannot be held personally liable for any differences that may arise between this collateral value and the actual loans that were given. The lender cannot force the borrower to pay from his future income. As loans in our model are essentially mortgages on houses, the assumption of non recourse debt is empirically consistent. For example, Crowe et al. (2013) says that non recourse debt explains the reality of subprime mortgage delinquencies which were at the core of the 2007-09 financial crises. Now that we have explained the cost of defaults to the borrowers, we proceed to the monitoring cost charged by the lenders.

Bernanke et al. (1999) assumes that the monitoring cost is equal to a fraction μ of the realized gross payoff to the defaulting firm's capital. We follow them and assume that the cost charged by the borrower is equal to a fraction of the housing value, $\mu\omega^i_{t+1}P^H_t H_{t+1}$. The implication of this cost is that defaults causes a decline in the stock of housing and thus its services. In other words, the attempt to monitor the project results in the destruction of $\mu\omega^i_{t+1}P^H_t H_{t+1}$ level of housing wealth, see Carlstrom and Fuerst (1997).

As far as the defaulting members of the household are concerned, we follow Forlati and Lambertini (2011) and assume that there is perfect risk sharing among household members so that consumption of non-durable housing goods and services are ex-post identical across all the members of the Borrower household. This means that Borrower household members are ex-post identical.

Now that we have explained the costs, benefits and actions of the borrowers in the contract we move on to the Lenders. For the contract to be feasible, the lenders need some incentive to participate. In the contract theory literature, this is explained using a *participation constraint*. The incentive for the lender in this contract is that they are guaranteed a pre-determined rate of return on the loans given. At time t the lenders make total loans L_{t+1} to Borrowers and demand the gross rate of return $(1 + R_{L,t})$. This rate of return is predetermined at t and non-state contingent. Hence, the time t participation constraint of lenders can be written as:

$$\begin{aligned}(1 + R_{L,t})L^B_{t+1} = \int_0^{\overline{\omega}_{t+1}} &\omega_{t+1}(1-\mu)(1-\delta)P^H_{t+1}H^B_{t+1}f_{t+1}(\omega)d\omega \\ &+ \int_{\overline{\omega}_{t+1}}^{\infty} (1 + R_{Z,t+1})L^B_{t+1}f_{t+1}(\omega)d\omega \qquad (1.18)\end{aligned}$$

where $f_t(\omega)$ is the probability density function of ω, which is time variant. The return on total loans, $(1 + R_{L,t})L^B_{t+1}$, supplied by the lenders is equal to the sum of two terms. The first term indicates the housing stock adjusted for monitoring costs and depreciation of defaulting Borrower members, $\int_0^{\overline{\omega}_{t+1}} \omega_{t+1}(1-\mu)(1-\delta)P^H_{t+1}H^B_{t+1}f_{t+1}(\omega)d\omega$. The second term shows the repayment of non-defaulting members, $\int_{\overline{\omega}_{t+1}}^{\infty}(1 + R_{Z,t+1})L^B_{t+1}f_{t+1}(\omega)d\omega$. After idiosyncratic shocks have realized, the threshold value $\overline{\omega}_{t+1}$ and the state-contingent mortgage rate $R_{Z,t+1}$ are determined so as to satisfy the participation constraint above. It is important to note here that the participation constraint holds state-by-state and not in expected terms. This means that an aggregate state that raises $\overline{\omega}_{t+1}$ and the rate of default on mortgages generates an increase in the adjustable rate $R_{Z,t+1}$ paid by non-defaulting members in order to satisfy the participation constraint eq. (1.18) in that state. This implies that periods characterized by rising default rates are also accompanied by rising interest rates in our model.

For convenience, we express the expected value of the idiosyncratic shock conditional on the shock being less than or equal to the threshold value $\overline{\omega}_{t+1}$ as,

$$G_{t+1}(\overline{\omega}_{t+1}) = \int_0^{\overline{\omega}_{t+1}} \omega_{t+1} f_{t+1}(\omega) d\omega \tag{1.19}$$

where G_{t+1} is the expected value, ω_{t+1} is the shock and the rest $\int_0^{\overline{\omega}_{t+1}} f_{t+1}(\omega) d\omega$ indicates the probability of default which is nothing but the cumulative distribution function in the interval $[0, \overline{\omega}_{t+1}]$. Furthermore, we express the expected ***share*** of the housing value that goes to the lenders gross of the monitoring costs as,

$$\Gamma_{t+1}(\overline{\omega}_{t+1}) = \overline{\omega}_{t+1} \int_{\overline{\omega}_{t+1}}^{\infty} f_{t+1}(\omega) d\omega + G_{t+1}(\overline{\omega}_{t+1}) \tag{1.20}$$

Now we can rewrite the lender's participation constraint eq. (1.18) more compactly by substituting in eq. (1.19) and eq. (1.20) as:

$$(1 + R_{L,t}) L^B_{t+1} = [\Gamma_{t+1}(\overline{\omega}_{t+1}) - \mu G_{t+1}(\overline{\omega}_{t+1})](1 - \delta) P^H_{t+1} H^B_{t+1} \tag{1.21}$$

As this participation constraint of lenders arises out of a secured loan agreement between the lenders and the borrowers, it resembles the standard aggregate collateral constraints derived in models with Kiyotaki-Moore-like financial frictions. We can interpret the term

$$[\Gamma_{t+1}(\overline{\omega}_{t+1}) - \mu G_{t+1}(\overline{\omega}_{t+1})] \tag{1.22}$$

as the ***endogenous loan to value ratio or the endogenous margin or as the endogenous collateral pledgeable parameter***. This is the variable ψ that we mentioned in the introduction to this chapter, see eq. (1.1). This parameter for us, unlike most of the related literature (see Kiyotaki and Moore (1997), Jermann and Quadrini (2012) and Miao and Wang (2012) for example) is not a constant but an endogenous variable. It depends on several parameters such as the monitoring cost μ, the variance of the idiosyncratic shock to housing investment $\sigma^2_{\omega,t}$, the parameters affecting the equilibrium value of the shock, and also the Borrowers' housing stock (the asset that is collateralized).[9] As far as

[9] It has to be noted here that the monitoring cost is not the parameter μ, but μ of the housing value which means that as the housing value changes so will the monitoring cost.

the lenders are concerned, this endogenous Loan to Value ratio measures the net share of the housing value that he will receive as repayment in case the borrower defaults. It is important to note here that this ratio talks about the share of the collateral and usually lies in the interval $[0, 1]$.

If the borrower decides to default at time t, they are left with an amount of housing stock given by,

$$\int_{\overline{\omega}_{t+1}}^{\infty} \omega_{t+1}(1-\delta)P_t^H H_{t+1} f_{t+1}(\omega)d\omega = [1 - G_{t+1}(\overline{\omega}_{t+1})](1-\delta)P_t^H H_{t+1} \quad (1.23)$$

where we have used the assumption that $E_t(\omega_{t+1}) = 1$. We used the right hand side term in the borrowers budget constraint. We substitute the value of $R_{Z,t+1}$ from eq. (1.18) in the Borrowers budget constraint and rewrite this constraint in the form,

$$P_t^H H_{t+1}^B + (1 + R_{L,t-1})L_t^B = L_{t+1}^B + Y_t^B + (1-\delta)[1 - \mu G_t(\overline{\omega}_t)]P_t^H H_t^B \quad (1.24)$$

Now that we have all the relevant equations for the Borrower type. We can proceed to its optimization problem. The optimization problem for the Borrower households involves maximizing their intertemporal utility eq. (1.13) subject to the budget constraint eq. (1.24) and the participation constraint eq. (1.21). The choice variables are the stock of housing H_{t+1}^B, the loans L_{t+1}^B and the threshold level of idiosyncratic shock $\overline{\omega}_{t+1}$. We solve this by formulating a Lagrangian of the form:

$$\begin{aligned}\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ U(H_{t+1}^B) + \lambda_{BC,t}\Big(L_{t+1}^B + Y_t^B + (1-\delta)[1 - \mu G_t(\overline{\omega}_t)]P_t^H H_t^B \\ - P_t^H H_{t+1}^B - (1 + R_{L,t-1})L_t^B \Big) + \lambda_{PC,t}\Big([\Gamma_{t+1}(\overline{\omega}_{t+1}) - \mu G_{t+1}(\overline{\omega}_{t+1})](1-\delta)P_{t+1}^H H_{t+1}^B \\ - (1 + R_{L,t})L_{t+1}^B \Big) \Bigg\}\end{aligned} \quad (1.25)$$

where $\lambda_{BC,t}$ and $\lambda_{PC,t+1}$ are the Lagrangian multipliers on the borrowing constraint and the participation constraint respectively. For maximization, we take the first order conditions (F.O.C) with respect to the choice variables.

The F.O.C's with respect to the choice variables are given by,

$$\frac{\partial \mathcal{L}}{\partial H^B_{t+1}} = U'_{H^B_{t+1}} - \lambda_{BC,t} P^H_t + \beta(1-\delta) E_t \Big\{ [1 - \mu G_{t+1}(\overline{\omega}_{t+1})] P^H_{t+1} \lambda_{BC,t+1} + \lambda_{PC,t+1} P^H_{t+1} [\Gamma_{t+1}(\overline{\omega}_{t+1}) - \mu G_{t+1}(\overline{\omega}_{t+1})] \Big\} = 0 \quad (1.26)$$

$$\frac{\partial \mathcal{L}}{\partial L^B_{t+1}} = \lambda_{BC,t} - (1 + R_{L,t}) E_t \Big[\beta \lambda_{BC,t+1} + \lambda_{PC,t+1} \Big] = 0 \quad (1.27)$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\omega}_{t+1}} = -\beta \lambda_{BC,t+1} \mu G'_{t+1}(\overline{\omega}_{t+1}) + \lambda_{PC,t+1} \big[\Gamma'_{t+1}(\overline{\omega}_{t+1}) - \mu G'_{t+1}(\overline{\omega}_{t+1}) \big] = 0. \quad (1.28)$$

Here $U'_{H^B_{t+1}}$ is the derivative of the utility function with respect to housing stock. We cannot put expectations on the first order condition with respect to $\overline{\omega}_{t+1}$ as this equation will hold only state by state.

In this section, we started with specifying the preferences and constraints for the borrower households and derived an endogenous margin or loan to value ratio from a participation constraint of the lenders who agree to a one period mortgage contract with the borrowers. We then stated the optimization problem faced by the borrowers and derived the first order conditions for maximization. In the next section, we describe the problem faced by the saver households.

1.3.2. Savers

As we described in the introduction of this model, the second type of households called "Savers" are characterized by their high discount factors. These households are patient and act as lenders to the Borrower type of households. They maximize the expected discounted value of future utilities where the preferences are defined over both non-durable goods and durable housing services. We use the supercript S to differentiate the variables of savers from that of borrowers. The lifetime expected discounted utility for the savers is given by,

$$E_0 \sum_{t=0}^{\infty} \gamma^t U(H^S_{t+1}), \quad 0 < \beta < \gamma < 1, \quad (1.29)$$

where E_0 is the expectations formed at time zero, γ is the household's subjective discount factor and $U(H^S_{t+1})$ is the utility function which has its argument the durable (housing) services denoted as H^S_{t+1}. Savers just like borrowers receive an endowment given by Y^S_t. We assume that the utility function $U(.)$ is continuous, concave and strictly increasing. As before, housing is part of their utility function so that this durable good is not just an investment asset but also provides valuable utility services.

We define the utility function for the savers households similar to the borrowers,

$$U(H^S_{t+1}) = \frac{(H^S_{t+1})^{1-\sigma}}{1-\sigma} \tag{1.30}$$

as being composed of only the durable housing good. As before, σ is the coefficient of relative risk aversion which is the sames as that of borrowers implying that the heterogeneity in our economy comes only from the discount factor. As in the borrowers, we assume that the stock of housing depreciates at the same rate $\delta > 0$ every period. At time t, the savers face the following budget constraint:

$$P^H_t H^S_{t+1} + L^S_{t+1} = Y^S_t + (1-\delta)P^H_t H^S_t + (1 + R_{L,t-1})L^S_t. \tag{1.31}$$

The interpretation of these variables are the same as those of borrowers. The left hand side of this constraint shows the financial expenses incurred and the right hand side indicates the total income that the household holds at time t.

The optimization problem for the saver households involves maximizing the utility function eq. (1.29) subject to their budget constraint eq. (1.31) with respect to the two choice variables: housing stock H^S_{t+1} and loans L^S_{t+1}. We write the associated Lagrangian as follows:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \gamma^t \Bigg\{ U(H^S_{t+1}) + \lambda^S_{BC,t} \Big(Y^S_t + (1-\delta)P^H_t H^S_t + (1 + R_{L,t-1})L^S_t - P^H_t H^S_{t+1} - L^S_{t+1} \Big) \Bigg\} \tag{1.32}$$

where $\lambda^S_{BC,t}$ is the Lagrangian multiplier on the savers budget constraint. The first order conditions with respect to the choice variables can be written as,

$$\frac{\partial \mathcal{L}}{\partial H^S_{t+1}} = U'_{H^S_{t+1}} - \lambda^S_{BC,t} P^H_t + \gamma(1-\delta)E_t[\lambda^S_{BC,t+1} P^H_{t+1}] = 0 \tag{1.33}$$

$$\frac{\partial \mathcal{L}}{\partial L^S_{t+1}} = -\lambda^S_{BC,t} + \gamma(1 + R_{L,t})E_t[\lambda^S_{BC,t+1}] = 0 \tag{1.34}$$

Now that we have described the actions and problems of both the agents in the economy, we proceed to the definition of equilibrium.

1.3.3. The Equilibrium

The following definition characterizes the equilibrium in our economy.

DEFINITION 1: *An equilibrium for our endowment economy is a set of allocations, namely,*

$$\{H^j_t, H^j_{t+1}, L^j_t, L^j_{t+1}\}, \quad \textit{where} \quad j = B, S. \tag{1.35}$$

such that

1. *Each Borrower household in the economy maximizes its expected discounted lifetime utility subject to a stream of budget constraints and borrowing constraints summarized by equations (1.26,1.27 and 1.28).*
2. *Each Saver household in the economy maximizes its expected discounted lifetime utility subject to a stream of budget constraints summarized by the equations (1.33 and 1.34);*
3. *For each state of the world, the commodity markets clears:*

$$\begin{aligned} Y^S_t + Y^B_t = &\Big(H^B_{t+1} - (1-\delta)[1 - \mu G_t(\overline{\omega}_t)]H^B_t\Big) \\ &+\Big(H^S_{t+1} - (1-\delta)H^S_t\Big) \end{aligned} \tag{1.36}$$

   which says that the aggregate endowment in the economy is equal to the consumption of durable goods by the borrowing households plus the consumption of durable goods by the savers households. The consumption of durable housing at time period t is given by the net accumulation obtained by subtracting the initial stock of housing with the end of period housing stocks. These variables are given in the left and right hand sides of the households budget constraints eq. (1.24) and eq. (1.31), respectively. This equation takes into account the fact that a fraction of borrower's housing stock proportional to the monitoring costs $\mu G_t(\overline{\omega}_t)$ paid by the savers is effectively lost due to default. Thus, the net accumulation of housing accounts for both depreciation and defaults.

4. For each state of the world, the credit market clears:

$$L_t^B = L_t^S \tag{1.37}$$

which says that the net supply of loans is zero.

1.4. Results

To solve for the equilibrium results, we first need functional forms for the fraction of housing stock lost in case of defaults $G_{t+1}(\overline{\omega}_{t+1})$ and the expected share of housing value obtained by the lenders, $\Gamma_{t+1}(\overline{\omega}_{t+1})$. This would come from the distributional assumption on the idiosyncratic shock ω_{t+1}. We follow Bernanke et al. (1999) and assume that the shock follows a log normal distribution given by,

$$\ln(\omega_{t+1}) \sim N(-\frac{\sigma_{\omega,t+1}^2}{2}, \sigma_{\omega,t+1}^2) \tag{1.38}$$

where $-\frac{\sigma_{\omega,t+1}^2}{2}$ is the mean of the distribution denoted by $\mu_{\omega,t+1}$ and $\sigma_{\omega,t+1}^2$ is the variance of the distribution set such that $E_t(\omega_{t+1}) = 1$.[10] Also, to make housing investments risky we assume that $\sigma_{\omega,t+1} = \sigma_\omega + \epsilon_{\omega,t+1}$ where the error process $\epsilon_{\omega,t+1}$ is an AR(1) innovation given by:

$$\epsilon_{\omega,t+1} = \rho_\omega \epsilon_{\omega,t} + e_{t+1} \tag{1.39}$$

where ρ_ω is the persistence of the innovation and e_{t+1} is assumed to be a white noise process. This implies that the distribution functions $G(.)$ and $\Gamma(.)$ can be written as follows:

$$\begin{aligned} G_{t+1}(\overline{\omega}_{t+1}) &= \int_0^{\overline{\omega}_{t+1}} \omega_{t+1} f_{t+1}(\omega) d\omega \\ &= \exp\big(\mu_{\omega,t+1} + \frac{\sigma_{\omega,t+1}^2}{2}\big)\big[\frac{1}{2} + \frac{1}{2} erf\big(\frac{\ln(\overline{\omega}_{t+1}) - (\mu_{\omega,t+1} + \sigma_{\omega,t+1}^2)}{\sqrt{2}\sigma_{\omega,t+1}}\big)\big] \end{aligned} \tag{1.40}$$

[10] For a random variable X that follows the log normal distribution, the expected value is given as, $E(X) = \exp(mean + variance/2)$. Substituting values for the mean and the variance we can show that $E(X) = 1$.

$$\Gamma_{t+1}(\overline{\omega}_{t+1}) = \overline{\omega}_{t+1}\int_{\overline{\omega}_{t+1}}^{\infty} f_{t+1}(\omega)d\omega + G_{t+1}(\overline{\omega}_{t+1})$$
$$= \frac{\overline{\omega}_{t+1}}{2}\Big(1 - erf\big(\frac{\ln(\overline{\omega}_{t+1}) - \mu_{\omega,t+1}}{\sqrt{2}\sigma_{\omega,t+1}}\big)\Big) \tag{1.41}$$

where as we explained in the modelling section $f(.)$ is the probability distribution function and erf is the Gaussian error function. We calibrate the constant standard deviation of the shock σ_ω to estimate the above equations (1.40) and (1.41). The optimal endogenous leverage which is one of the key variables in our model is then obtained by these equations and the equilibrium conditions. Now that we have specified all the necessary equations, we proceed to calibrating our parameters.

The parameters values for our benchmark calibration are reported in Table 1.1. We follow Monacelli (2009) in choosing the values for the discount factors for Borrowers and Savers and the rate of depreciation for housing. The Saver's discount factor γ is set equal to 0.99 and Borrower's β is set equal to 0.98. We choose an annual depreciation rate for housing of 4 percentage points, implying that the parameter $\delta = 0.01$. The Saver discount factor pins down the steady-state interest rate at $R_L = 0.0101$ ($\frac{1}{1+R_L} = \gamma$) on a quarterly basis. This implies an annual interest rate equal of 4.1 percentage points.

Furthermore, following Carlstrom and Fuerst (1997) we set the standard deviation of idiosyncratic shock, $\sigma_\omega = 0.2$ and the persistence of this time varying volatility, $\rho_\omega = 0.983$. This implies that the shocks to the mortgage are highly persistent as is suggested by Campbell and Cocco (2015). We assume that the monitoring cost to be 12% of the house value consistent with other literature that have analysed the housing market in the United States, see Cagan (2006). Finally, the coefficient of relative risk aversion for the borrower and saver household preference is set to 2 following Cocco (2004) and the endowment income for the borrower and saver households are fixed at $Y^B = Y^S = 50$. There are no specific reasons for this endowment value, other values will produce qualitatively the same results

Table 1.1. Baseline Parameters

Description	Parameter	Value
Coefficient of Relative Risk Aversion	σ	2
Discount Factor of Borrower Households	β	0.98
Discount Factor of Saver Households	γ	0.99
Depreciation Rate of Housing Stock	δ	0.01
Standard deviation of Idiosyncratic Shocks	σ_ω	0.20
Monitoring Cost Proportion	μ	0.12
Persistence of Idiosyncratic Shocks	ρ_ω	0.983
Endowment of Borrowers	Y^B	50
Endowment of Savers	Y^S	50

Notes: This table reports the calibrated parameters for the benchmark model.

We have now specified the equilibrium characterizing equations and calibrated the parameters of the model. We proceed to estimate the steady state values for the three key variables in our model, the endogenous Loan to Value Ratio (LTV), the probability of default or the rate of default given by the cumulative distribution function $F(.)$ for an optimal shock $\overline{\omega}$ and the housing price, P^H. In principle, we can report the values for the stock of housing, leverage finance premium, loans etc. However, we concentrate only on the dynamics between the endogenous loan margin (LTV), the leverage ratio, default rates and house prices. To understand these dynamics we find steady state value for all the three variables under different values of the standard deviation σ_ω of the idiosyncratic uncertainty shock. These steady state values are reported in Table 1.2.

The first column in Table 1.2 describes the variables and the second to the fourth columns, the corresponding steady state values. Each column indicates a particular assumed value for the standard deviation of the idiosyncratic shock and the resulting steady state values. Hence, **each column represents an equilibrium** (steady state). We will infer about bubble behaviour in house prices by comparing these equilibria. The endogenous loan to value ratio is defined in eq. (1.22). For an optimal shock $\overline{\omega}$, we can obtain this ratio by considering the functional forms of $F(.)$ and $G(.)$ from eq.'s (1.40) and ((1.41)). Default rates are nothing but the cumulative distribution function $F(.)$. House prices are obtained by solving the first order conditions reported in the equilibrium definition.

Table 1.2. Steady State House Price, LTV and Default Rate

Variable	Steady State Values			
	$\sigma_\omega = 0$	$\sigma_\omega = 0.2$	$\sigma_\omega = 0.4$	$\sigma_\omega = 0.8$
Loan to Value Ratio	55.21%	47.18%	30.25%	19.18%
Leverage Ratio	2.23	1.89	1.433	1.237
Housing Price	4.01	3.11	2.99	2.10
Default Rate	0%	3.5%	8.7%	15%

Notes: This table reports the steady state values for the endogenous loan to value ratio, house price and the probability of default (default rate) for different values of σ_ω. The second column with $\sigma_\omega = 0.2$ corresponds is the benchmark calibration given in Table 1.1. Both the loan to value ratio and the default rate are expressed in percentages. The leverage ratio is given by $\frac{1}{1-LTV}$ where LTV is the loan to value ratio. Housing price is normalized by the aggregate endowment.

A direct result we can infer from Table 1.2 is that as σ_ω increases the endogenous loan to value ratio decreases and the default rate increases. As we described earlier, a higher standard deviation of the shock does not change the mean of our distribution (by construction), that is $E_t(\omega_{t+1}) = 1$. This means that any changes in the uncertainty can be called as a mean preserving spread of the distribution $F(.)$. Consequently, a rise in the standard deviation will keep the mean unchanged but will increase the skewness of the distribution of ω_t. As we followed Bernanke et al. (1999) and assumed a log-normal distribution, this means that $F(.)$ cannot take negative values implying that an increased skewness will result in the lower tail of the distribution becoming thicker. A thicker tail will hence imply a higher cumulative distribution function which of course means a higher rate of defaults on mortgages. In the presence of high mortgage defaults, the share of housing value (collateral) that the lender will receive will go down, that is the loan to value ratio will decline, see eq. (1.22).

A higher mortgage risk first impacts adversely the financial condition of the borrowers. The now worse borrower members of the household will default on their loans and lose the housing stock which was put up as collateral. In addition, as the loan to value ratio declines, the borrowers now experience a tightening in their credit arrangement. This reduces the capacity of the borrowers to take loans out of their stock of housing. Thus, the fact that the **loan to value ratio is endogenous means that the effect of uncertainty is multiplied**. Higher uncertainty now influences the economy through two channels, the collateralized asset and the endogenous margin (loan to value ratio).

We now examine the changes to the house prices. It is observable that there is a steady decline in house prices with increasing idiosyncratic risk. The first column reports results for the no shock case, $\sigma_\omega = 0$. This means that the borrowers now face no risk to their housing investments. The steady state house price (normalized by endowment) is found to be 4.01. With no risks to holding housing goods, there is an increased demand for housing. The fact that the Loan to Value ratio is very high, 55.1%, means that household borrower members can take bigger loans to finance their investments in housing. Naturally, the price of the house would then be high. Thus, there is **a "credit-easing" effect**. Relaxed borrowing conditions induce high asset prices. We call this steady state equilibrium as a **Bubbly Equilibrium**. This bubbly equilibrium is consistent with rational expectations but is driven by credit, hence the type of bubbles we find here are similar to the credit driven rational bubbles studied by Miao and Wang (2012). This credit easing effect works purely because the loan to value ratio here is endogenous. A fixed ratio would produce no such effects in the absence of any uncertainty.

Several studies such as Caballero and Krishnamurthy (2001), Miao and Wang (2012) and Miao (2014) for example in the literature talks of the credit easing effect caused by rational bubbles in the price of the collateralized asset. Unlike these papers where the loan to value ratio is exogenous, we have an endogenous margin. The implication here is that upward price movements arise directly from a shift in the endogenous loan to value ratio. Increased borrowing thus implies that the borrower households would invest more in housing stocks raising its equilibrium price.

The Bubbly equilibrium is thus characterized by high house prices, high loan to value ratios, no uncertainty and no defaults. High house prices and high loan to value ratios originate through the credit easing channel. The fact that high house prices can be obtained with no uncertainty and no defaults is an important result that justifies the use of endogenizing the loan to value ratio. This result thus extends those of Kocherlakota (2009) who found bubbly prices in the collateralized land only under the case of uncertainty.

An alternative way to explain this bubbly steady state equilibrium is to consider the leverage ratio. The values for the leverage ratio are reported in the second row. It is no coincidence that the bubbly equilibrium with the highest price of housing comes when the leverage ratio is also at its peak. This situation was observed in the recent sub-prime mortgage crisis when at the peak of the bubble, borrowers were excessively leveraged. In an important study, Corbae and Quintin (2015) notes that better access to loans with low down payments made it possible for more households to obtain the financing necessary to purchase a house. In our case, the high loan to value ratio means a low downpayment $(1 - LTV)$.

An upward change in risk lowers the steady state house price from a value of 4.01 when the idiosyncratic shock has a standard deviation of $\sigma_\omega = 0$ to a value of 2.1 when $\sigma_\omega = 0.8$. This can be explained as follows. First, a high risk in holding housing stock will lower the demand for this asset by households. Second, high risk is also associated with very tight borrowing margins. In other words, the loan to value ratio is very low (relatively). For instance, the steady state equilibrium value for LTV when $\sigma_\omega = 0.8$ is just 19.18%, a significant decline from the 55.21% value. This tight collateral margin will imply that borrowers are severely restricted in their borrowing capacity. This is called a **"credit crunch"** in the literature, see Brunnermeier and Sannikov (2014). An implication of such a credit crunch situation is that a significant amount of household members would find it optimal to default on their mortgage contract. A high rate of defaults by the borrowing household members means that these households have to downsize because of their lost collateral that was seized by the lenders. Borrowers will need to replenish their stock of housing and will demand more from the saver households. However, the tight margin constraints mean that there is in effect little demand for housing and thus prices go down. The credit crunch thus produces a steady state equilibrium characterized by low asset prices.

This credit crunch effect is what characterized the foreclosure crisis in US. The events that occurred in the US housing market can be described chronologically as follows. There was a fall in house prices and a rise in foreclosures since early 2006. House prices dipped around 2006-Q2 and then, except for a small rise in early 2007, fell continuously until 2009-Q2. At that point house prices stabilized for about a year, fell again for half a year and eventually began to rise. The rate of new foreclosures rose continuously between 2006-Q2 and 2008-Q4. Chatterjee and Eyigungor (2015) explains this foreclosure crisis in a general equilibrium model where homeowners experience three unanticipated shocks, one of which is in the stock of housing. Their analysis involves solving for the optimal policy functions. Our results here are at steady state levels but it can still provide us with the intuition behind this foreclosure crisis. As we see in our Table 1.2, the default rates rose steadily with increasing shocks. If we consider the no shock equilibrium representing the year 2006, the rest of the steady state values would then correspond to each subsequent year exhibiting falling house prices and increased foreclosures.

Analysing the last two rows of Table 1.2 we observe that as steady state house prices declined, the default rates increased. This evidence is consistent with what was observed in the United States during the last decade. For instance, Mayer et al. (2009) empirically documented this phenomenon and stated that roughly 1.7 million foreclosures were started in the first three quarters of 2008, an increase of 62 percent from the 1.1 million in the first three quarters of 2007. Garriga and Schlagenhauf (2009) for example use a general equilibrium model to analyse mortgage defaults and finds that falling house prices generates sizeable default rates at the aggregate level. Furthermore, in a recent paper Campbell and Cocco (2015) also finds that both adjustable rate mortgages and fixed rate mortgages experienced high default rates when there were a large decline in house price. The results of Campbell and Cocco (2015) were obtained using a partial equilibrium model where house prices were considered as exogenous. Our analysis here proves that even under market clearing endogenous prices, the results would still hold true.

In summary, our key results can be stated as follows. Firstly, increasing risk in housing stocks generates steady state equilibrium values characterized by high defaults, low prices and low LTV and leverage ratios, respectively. Secondly, the endogenous loan to value ratio amplifies any uncertainty present in the financial market. Thirdly, endogenous loan to value ratio leads to a credit-easing effect where the upper bound on borrowing is relaxed resulting in rational bubbly equilibrium in house prices. When hit with an idiosyncratic shock, endogenous LTV gets tightened restricting borrowing and producing a "credit crunch" effect bursting any bubble and lowering house prices.

Now that we have finished reporting and explaining the results, we proceed to a discussion section where we take a broader perspective of our results comparing it to other literature, explain the limitations of our analysis here and provide an alternative tractable ways to derive endogenous margins.

1.5. Discussion

One of the results in the previous section could appear to be counterfactual, specifically the relationship between equilibrium values for the LTV ratio and the default rate. It is observed that high LTV, in other words, low downpayment, was found to generate low default rates. For instance, when the standard deviation of the shock to housing stock is zero, the downpayment rate is 44.79% $(1 - LTV)$ while the default rate is zero. We can compare these values to the other extreme of $\sigma_\omega = 0.8$, where the downpayment rate now is 81% and the default rate to be 15%. These results suggests that households default when downpayment rates are very high which is not true. There are two reasons for this finding in our model.

Firstly, in our model the rational bubble equilibrium is given by the first column of Table 1.2. At this point, households have accumulated a lot of leverage but defaults have not yet started. Defaults occur when $\sigma_\omega \uparrow$ which lowers prices. This is well explained by Corbae and Quintin (2015) who using estimation as well as observing the Survey of Consumer Finances data finds that delinquencies and foreclosures start as soon as the bubble bursts and prices decline. The leverage is only accumulated during the boom phase. This boom phase is thus characterized by little or no defaults. Secondly, we have assumed that the shock follows a mean preserving spread. This was done to ensure that the expected mean of shock will always remain at unity, $E_t[\overline{\omega}_{t+1}] = 1$.

Our results also showed that an increase in house price is associated with an increase in leverage. That is, leverage is high during asset price booms and low during asset price busts. Hence, leverage in our model as in reality is procyclical.

As far as the modelling of the mortgage contract is concerned, we have used here a one-period debt contract similar to Bernanke et al. (1999) for tractability reasons. In reality, standard mortgages in the United States generally have a fixed 30-year term and about 70% of these mortgages have fixed rates, see Campbell and Cocco (2015). Moreover, subprime mortgages with nontraditional features (that is, adjustable rates) were at the heart of the recent crisis. Our model does not consider these alternative mortgage instruments and therefore cannot capture their role.

To introduce uncertainty, we considered only one idiosyncratic shock which applied to the stock of housing. A growing empirical literature, see for example **?** and **?** uses pre- and postcrisis mortgage data to study the importance of various shocks in households' decisions to default. This literature has documented that most defaults involve negative equity (loan value greater than the value of the collateral) but, at the same time, most households with negative equity choose not to foreclose. Most foreclosures thus appear to involve a combination of negative equity and other shocks.

In the recent crises, unlike the previous ones, the initial disruption started in the financial sector. It was not a shock in the real sector that was amplified through financial frictions but uncertainty which originated in the financial sector. One way to model this would be to add a shock to the endogenous leverage (LTV) itself. These could then be considered as financial shocks that arise independently from the real sector. For instance, Jermann and Quadrini (2012) assume that the pledgeability parameter as an independent stochastic process unrelated to market conditions. They interpret the parameter as the probability that the lender can recover the full value of the collateral and thus its complement as the probability that the recovery value is zero. They call exogenous shocks to the margin as "financial shocks".

We started this chapter by mentioning the research by Geanakoplos (2003) on leverage cycles. Our model in its present form cannot accommodate this behaviour. However, we can apply the theory developed by Gu et al. (2013) and generate leverage cycles. These cycles originate and propagate in the credit sector. These can be shown to exhibit deterministic, chaotic, and stochastic cycles. One attractive feature of these type of cycles are that they exist even when fundamentals are deterministic and time invariant. The key friction in this theory is limited commitment, meaning that borrowers (agent) have the option to renege on their contract and can divert funds from the lender (principal). In the appendix, we show how such leverage cycles can be modelled from an optimal contract between two firms.

In this chapter, we analysed the link between endogenous margins, collateral constraints and asset price bubbles. Our model was a departure from the standard asset pricing framework in that agents face constraints to the level of their borrowing. These constraints which we call collateral constraints can also be considered as rational debt constraints or solvency constraints. There are other ways in which we can depart from the frictionless asset market and endogenize the margins. First, is the consideration of exogenous incomplete markets in which there are not enough securities to insure against all possible states of nature, see the examples in Radner (1972) and Geanakoplos (1990). Second, there are models of liquidity constraints in which individual agents are restricted from borrowing as much as they wish in the credit market. Bewley (1987) is an example of this types of models. Third, there are also models of adverse selection and moral hazard, see Townsend (1979) and Prescott and Townsend (1984). All these types of models can be used to derive endogenous collateral constraints and thus the loan to value ratio.

1.6. Conclusion

In this chapter we modelled an endowment economy asset pricing model with heterogenous agents, endogenous loan to value ratios and endogenous defaults on mortgages to understand the boom and bust in house prices. The steady states of our model revealed several important results.

We find that credit driven rational bubbles can form in steady state house prices when a high leverage induces a credit easing effect that relaxes the borrowing constraint. Increased borrowing coupled with no uncertainty meant that house prices in this state were high. Conversely, a credit crunch situation evolved when the loan to value ratio declined following an increased risk. In this situation, the borrowing constraint tightened restricting the availability of loans and thus demand for housing resulting in lower asset prices. A direct consequence of these two effects was that the leverage ratio was high during asset price booms and low during asset price busts, that is pro-cyclical.

We restricted our analysis to just the steady states here. However, we could enrich this model and compute the optimal policy functions to get better insights on the dynamics between leverage, asset prices and defaults, see Brumm et al. (2015). Importantly, the fact that the leverage is endogenous implies that policy makers can regulate the collateral margin requirements and can thus prevent financial shocks from adversely affecting the economy. This can be achieved if they set margins (or the parameters that endogenizes the margin) such that they become counter-cyclical.

Incorporating money and price stickiness into this framework can help us in understanding how leverage frictions can influence of transmission of monetary policy. In this chapter we have assumed that the initial disruption is completely exogenous. There are authors such as Suarez and Sussman (1997) who have proposed models in which adverse selection could generate economic fluctuations even in the absence of exogenous shocks.

Appendix

1.A. Endogenous Collateral and Leverage Cycle

In this section we describe a model that endogenize the leverage and generates cycles. The model is kept deliberately simple so as to allow a transparent exposition of the mechanism. The key mechanism or the friction that drives leverage cycles is limited commitment. This limited commitment leads to endogenous collateral constraints. The model is described in the following paragraphs.

We consider an infinite horizon economy. Time is assumed to be discrete, denoted by t. Each period in the economy is assumed to have two subperiods. The economy is populated by two types of agents of equal measures, these are called Firms (or borrowers) and Households (or Lenders).

In each period t, there are two subperiods. These can also be considered as beginning of period t and end of period t. Both the agents have different needs in the two subperiods. In the first subperiod of t, the firms need investments in the form of capital which they consume to generate returns, denoted by y, in the second subperiod. Households are endowed with k units of consumption goods (can be considered as capital) at the beginning of every period. We assume that this good is non-storable and thus depreciates completely by the end of the period. Furthermore, households do not have investment opportunities meaning that they cannot use the k units of good to produce y. Thus, households value consuming y in the second subperiod more than holding on their endowment good k. This implies that there are gains to be had from trade.

The trade relationship between the firms and the households works as follows. The firms borrow k units of goods from the households in the first subperiod and promise, that is, sign a contract to deliver y to households in the second subperiod when the firms have finally realised the fruits of their investments.

The utility from this trade agreement is $U^F(k, y)$ for the firms and $U^H(y, k)$ for the households.[11] The ordering of the arguments inside the utility functions indicates the preference for each agents. Firms value more of k which is their consumption than y, that is, $\frac{\partial U^F(k,y)}{\partial k} > 0$ is strictly increasing and $\frac{\partial U^F(k,y)}{\partial y} < 0$ is strictly decreasing. Similarly, households value more of their consumption y than k, $\frac{\partial U^H(y,k)}{\partial y} > 0$ and $\frac{\partial U^H(y,k)}{\partial k} < 0$. We assume here that the utility is concave and twice differentiable. Furthermore the utility is non-negative for both the agents, $U^i \geq 0$, where $i = F, H$.

[11] The superscript F represents Firms and H represents the households.

The underlying mechanism that drives our result is the friction of limited commitment or enforcement. Once investment returns, y, are realized at the end of the period the firm has the incentive to renege from the contract and divert these funds y to their own benefit. If the firm behaves in this manner, it gets a payoff λy over and above its utility $U^F(k, y)$. Hence, λ is a parameter that indicates the temptation to renege. To prevent this from happening, we impose an additional constraint in the form

$$U^F(k, y + y') + \lambda y' \leq U^F(k, y), \quad \forall x, y \quad y' \geq 0 \tag{1.42}$$

This constraint says that the firms never finds it optimal ex ante to divert resources y' as he is better off not producing in the first place. This does not mean that the firm is not tempted to divert resources ex post after the production has taken place. The incentive to honour the obligation to the contract arises from the threat to exclude the firms from any future borrowing. This means that the firm will have an autarky (no trade) payoff of zero. Motivated by related limited commitment contract models in the literature such as Gu et al. (2013) we allow for imperfect monitoring. This means that if the firm defaults, there is a probability π that it will get caught. Consequently, with $1-\pi$ probability the lender will not be caught. Now that we have specified the environment and trading arrangements in the model, we proceed to explicitly state the contract variables and its associated constraints.

The contract between the firms and the households at time t is characterized by the pair of allocation (k_t, y_t). This contract specifies that the borrower (firm) gets k_t from the lender (household), and the borrower promises to deliver y_t to the lender. Consider V_t^F and V_t^H to be the value functions for the firms and households, respectively. The discount factor across periods is given by $\beta \in (0, 1)$. We assume that the discounting across subperiods is contained within the utility functions for each agent and do not model it explicitly. These value functions can be written as:

$$V_t^F = U^F(k_t, y_t) + \beta V_{t+1}^F, \tag{1.43}$$

$$V_t^H = U^H(y_t, k_t) + \beta V_{t+1}^H, \tag{1.44}$$

For a contract to be feasible, the lender must offer the agent a utility level that is at least as high as the utility level that the borrower obtains outside the relationship. These constraints are called as participation constraints in the literature. The firms outside opportunity level is normalized to zero. Thus, the participations constraints can be written in the form,

$$U^F(k_t, y_t) \geq 0 \tag{1.45}$$

and

$$U^H(y_t, k_t) \geq 0. \tag{1.46}$$

In addition to these constraints, there is an added constraint called the repayment constraint which gives our leverage limit. The repayment constraint applies only for the borrower (firm) and ensures that it is always optimal for the borrow to repay the loan at the end of the contract period. In our model, this constraint can be written as,

$$\lambda y_t + (1 - \pi)\beta V_{t+1}^F \leq \beta V_{t+1}^F \tag{1.47}$$

This equation says that the continuation value to remain in the contract βV_{t+1}^F is always greater than the value the firm will obtain if it reneges on the debt, that is defaults. In case of defaults, the firm gets the deviation payoff λy_t plus the continuation value obtained in case it was not caught $(1 - \pi)\beta V_{t+1}^F$.

The repayment constraint eq (1.47) can be rewritten in the form:

$$y_t \leq \frac{\beta\pi}{\lambda} V_{t+1}^F \tag{1.48}$$

Equation (1.48) says that the repayment y_t cannot exceed the discounted continuation value of the firm adjusted for monitoring costs and a fraction λ. Intuitively, this is the limit on the loans that the firm can take on. We call this the leverage limit. We can express eq. (1.48) as

$$y_t \leq \psi_t \tag{1.49}$$

where $\psi_t = \frac{\beta\pi}{\lambda} V_{t+1}^F$ is the endogenous limit on firms loans. In the lines of Alvarez and Jermann (2000) we can say that the equilibrium loan limit ϕ_t is defined such that the firms are indifferent between repaying ϕ_t and defaulting. For any feasible allocation, payoffs and thus ϕ_t should be bounded. Following Gu et al. (2013) we can define the equilibrium as follows:

DEFINITION 2: *An equilibrium is given by nonegative and bounded sequences of loan limits $\{\phi_t\}_{t=1}^{\infty}$ and contracts $\{k_t, y_t\}$ such that (i) $\forall t, (k_t, y_t)$ solves the conditions given y_t and (ii) ϕ_t solves the equilibrium conditions given $\{k_t, y_t\}_{t=1}^{\infty}$.*

The debt limit can be then expressed as a first order difference equation which can be solved and illustrated to depict leverage cycles.

1.B. Rational Bubbles in Overlapping Generations

Tirole (1982, 1985) extended the partial equilibrium framework of Blanchard and Watson (1982a) to a general equilibrium one. Tirole (1985) used Diamond (1965)'s two period overlapping generations model of capital accumulation in a production economy (inelastic labour), and gave necessary and sufficient conditions for the existence of a rational bubble. Tirole found that bubbles crowd out productive savings and cannot grow faster than the economy, their existence depends on a comparison between the interest rates and the growth rates of the economy. If the economy is dynamically efficient, the interest rate will exceed the growth of the economy, there cannot be a steady state with a positive valued bubble. However, if the economy is dynamically inefficient, there exists an equilibrium with a positive bubble value. In such a bubbly equilibrium, the growth rate of bubbles of is equal to the economy growth rate, which is equal to the interest rate.

Theoretical arguments can be made to rule out rational bubbles in a finite horizon framework through backward inductions. Since a bubble cannot grow from time T onwards, there cannot be a bubble of this size at time $T-1$, which rules out this bubble at $T-2$, etc. However, there is ample experimental evidence that individuals violate the backward induction principle. Most convincing are experiments on the centipede game, see Brunnermeier and Oehmke (2013).[12]

Furthermore, Allen et al. (1993) show that, when common knowledge is absent and short sale constraints bind, a bubble can exist for a finitely-lived asset. They describe an example in which the market price of a security can deviate above the present value of its dividends even though all the agents are rational and knows the dividends with certainty. The reason is that the agents' do not know each other's beliefs, in other words there is asymmetric information (i.e., there is a lack of common knowledge that was assumed in the previous backward-induction reasoning). Moreover, at the time or state when the bubble occurs, every agent is

[12] In this simple game, two players alternatively decide whether to continue or stop the game for a finite number of periods. On any move, a player is better off stopping the game than continuing if the other player stops immediately afterwards, but is worse o§ stopping than continuing if the other player continues afterwards. This game has only a single subgame perfect equilibrium that follows directly from backward induction reasoning. Each player's strategy is to stop the game whenever it is her turn to move. Hence, the first player should immediately stop the game and the game should never get off the ground. However, in experiments players continue to play the game - a violation of the backward induction principle

either short sale constrained or will be constrained at some possible contingency in the future. As beliefs are not common knowledge, even though all agents know that the price of the asset is over-valued, they all rationally believe that they will be able to sell the asset at a higher price to someone else before the true value is completely revealed.

Farhi and Tirole (2011) and Martin and Ventura (2012) introduce financial frictions to the Tirole (1985) model and show that bubbles can exist even though the equilibrium without bubbles is dynamically efficient. They show that dynamic efficiency and low interest rates are compatible in the presence of capital market imperfections.

Chapter 2

U.S. Housing Market Bubbles: A Long Memory Approach

2.1. Introduction

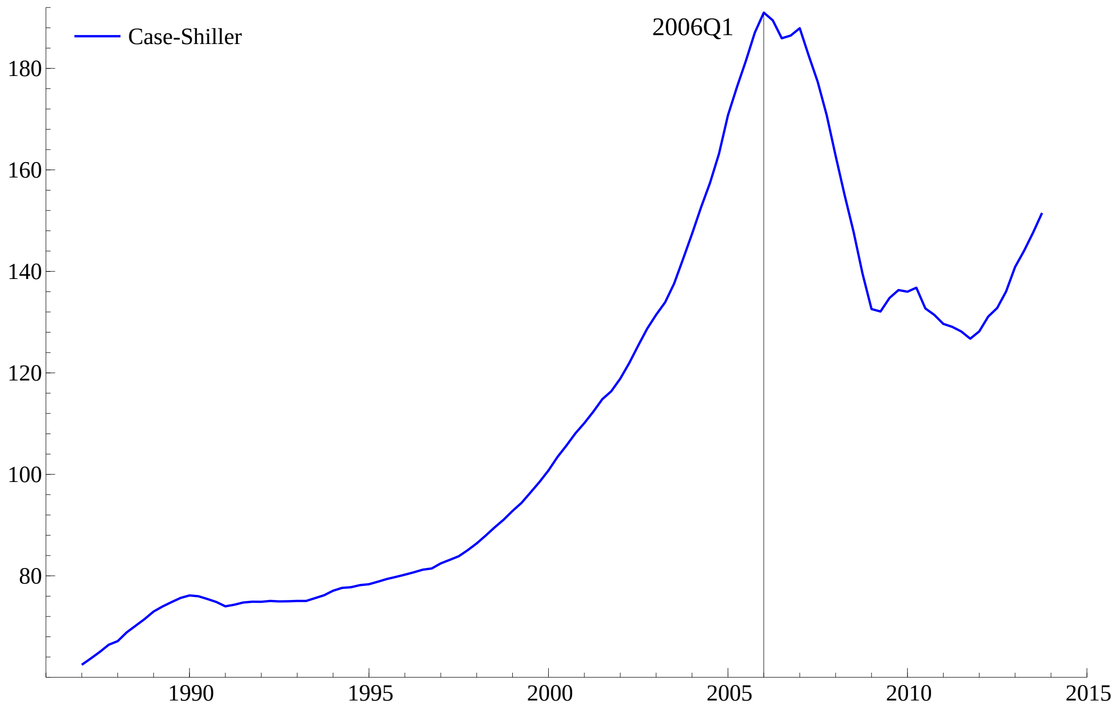
House prices in the United States rose rapidly between the early 1990's until the mid 2000's. Figure 2.1 graphically illustrates the time series behaviour of the Standard & Poor Case-Shiller House Price Index. The Index started its upward trend around 1996 reaching a peak in the first quarter of 2006. After the peak in 2006, they declined sharply and reached a low in 2012, see Case et al. (2012). Several explanations have been offered by scholars for this boom-bust episode such as misguided monetary policy; a global savings surplus; government policies encouraging affordable homeownership; irrational consumer expectations of rising housing prices; inelastic housing supply, mortgage securitization to name a few, see Levitin and Wachter (2012).

The 1997-2006 real house price appreciation prompted numerous economists and the national media to conclude that there was a bubble in the U.S. Housing Market. These proclamations arise from observing the largest crash in U.S. real estate market's history in 2007 that erased a significant portion of household wealth. Such a decline in household wealth has adverse macroeconomic effects, as already overextended consumers reduce spending to boost saving and improve their weakened financial position. In this context, a wide consensus among analysts and commentators has emerged on the importance of timely identification and understanding of a "housing bubble".[1] Consequently, there is a growing body

[1]The motivation for exploring the housing market is because of the large role that it had in the financial crisis. According to the Flow of Funds Accounts compiled by the Board of

of papers that examine for housing bubbles, see Abraham and Hendershott (1996), Higgins (1997), Himmelberg et al. (2005), Glaeser et al. (2008) and Phillips and Yu (2011) among others.[2]

Figure 2.1. S&P Case-Shiller House Price Index



Notes: Time series plot of the Standard & Poor Case-Shiller Real House Price Index in the time span 1987Q1-2013Q4. The peak of the price index, illustrated with a straight line, occurs in the first quarter of 2006.

In this chapter, we focus on rational bubbles in the U.S. Housing Market.[3] To begin with we have to address the question of defining an asset

Governors of the Federal Reserve System, households held about \$14.6 trillion in real estate at the end of March, 2003. By comparison, households held about \$12.8 trillion of corporate equities and mutual funds in January, 2000 - the peak of the stock market. Moreover, equity holdings are concentrated at the upper end of the wealth distribution, whereas housing is the major asset for most households, see Tracy and Schneider (2001). Examining booms and busts in home prices is thus important.

[2]See Mayer (2011) and Glaeser and Nathanson (2014) for a survey on the literature and Levitin and Wachter (2012) for possible reasons to the recent bubble in the United States.

[3]We often use the words Housing Market and housing prices throughout our narration. In the context of this chapter, they are one and the same. A bubble in the housing market means

price bubble. Despite the huge volume of literature on this topic, a consensus on the definition of an asset price bubble remains elusive. The most common and widely accepted definition relates an asset price bubble to any divergence from its fundamental price, see Blanchard and Watson (1982b), Diba and Grossman (1988) and Kindleberger and Aliber (2005). In case of the stock market, equity prices contain a rational bubble if investors are willing to pay more for the stock than they know is justified by the value of the discounted dividend stream i.e. the fundamentals, see Shiller (1989) and Gürkaynak (2008). This is because they expect to be able to sell it at an even higher price in the future, making the current high price an equilibrium price implying the existence of a rational bubble in the stock market.

As in the stock market, rational bubbles can occur in the housing market when there are deviations from the fundamental value of the house. An important fundamental value explaining house prices is the rental price, see Kivedal (2013). Hence, an investigation on its relationship with the house price, the house rent-price ratio, will give us valuable insights to the persistence and thus the possibility of bubbles in the housing market. The housing rent-price ratio is a financial ratio akin to the dividend-price ratio for the stock market. A low rent to price ratio indicates that the return on the housing asset for homeowners is low compared to other assets that they could hold and thus is unlikely to persist. For the return to rise to a level comparable with returns on competing assets, house prices would have to fall, see Hatzius (2002) and Case et al. (2005). The ratio of the Owners' Equivalent Rent Index from the Consumer Price Index (CPI) series to the House Price Index is often treated as the real estate equivalent of a dividend to price ratio for corporate equities, see Meese and Wallace (1994), Himmelberg et al. (2005), Kivedal (2013) and André et al. (2014).[4] In this chapter, we follow these papers and use the real rent to real house price ratio to detect rational bubbles in the housing market.

In addition to the theoretical problems of defining as asset bubble, researchers have found it a challenging task to empirically test for a bubble, see Gürkaynak (2008). One approach is to test for cointegration between dividends and stock prices. Cointegration implies that two or more series cannot drift apart indefinitely as they must satisfy a long run equilibrium condition. For example, a

a bubble in the housing prices and vice-versa.

[4]Gallin (2008) also used a long horizon regression approach and show that the rent to price ratio can accurately forecast housing prices and thus, lends empirical support to the use of this financial ratio as an indicator of valuation in the housing market.

cointegrating relationship between stock prices and dividends is inconsistent with rational bubbles, since stock prices and dividends are tied together in the long run. Another approach is to make use of the present value relation. The present value model says that the stock prices are equal to the sum of the expected discounted dividend sequence which is also called the fundamental price of the stock. Deviations from the present value model will imply that the stock market is not efficient, i.e. the existence of asset bubbles, see Koustas and Serletis (2005).

Univariate time series testing procedures such as unit root testing on present value model variables like the log dividend-price ratio is a possible way to test for price bubbles, see Diba and Grossman (1988) and Koustas and Serletis (2005). The presence of a unit root in the log dividend price ratio implies rational bubbles as in effect this means that stock prices and dividends do not share a common trend. In the context of the housing market, a rational bubble means that the housing prices and rents do not move together and consequently there is a unit root in the housing rent-price ratio. Phillips and Yu (2011) uses a sequential unit root test to date housing bubbles in the Case-Shiller log price to rent ratio. Some other studies that apply unit root tests on U.S. House prices (ignoring rents) include Meese and Wallace (1994), Meen (2002) and Canarella et al. (2011) to name a few.

Empirical papers which uses standard unit roots for bubble identification have had mixed success, see Lamont (1998) and Horvath and Watson (2009). This is mainly because of the low power of these integer order tests to reject the null of a unit root against the possibility of fractional roots.[5] For instance, standard unit root tests cannot distinguish between a unit root process and a near unit root process. Univariate processes that have persistence close to but not equal unity have a special property in that they have long memory. This means that the effect of a shock will last for an extended time period (hence the term "long memory") and will thus look like a bubble when in fact they are mean reverting. As such near unit root processes which revert to their mean in the long run can be mistakenly considered as possessing bubble behaviour. Hence, standard unit root tests are not adequate in testing for rational bubbles.

In this chapter, we use long memory models to investigate for the presence of housing bubbles. The long memory models, also known in the literature as Autoregessive Fractionally Integrated Moving Average processes ($ARFIMA$) stemmed from the seminal contribution made by Granger and Joyeux

[5] In addition to this, there is an enormous literature which argue that the presence of any structural breaks affects the performance of standard unit root tests, see Perron (1989, 1997).

(1980). There is evidence that long memory processes successfully model some economic and financial data, see Diebold and Rudebusch (1991), Hassler and Wolters (1995), Bhardwaj and Swanson (2006). The presence of long memory in the rent-price ratio will mean that even though there is a temporary deviation in house prices, they will eventually return to their fundamentals, i.e. rents. In other words, the presence of long memory negates bubble activity.

The presence of long memory in a series is detected by estimating the level of fractional integration, alternatively called the memory parameter, in the series. This parameter is denoted by d in this chapter. This is the key parameter of interest. An estimated value of d below 1 indicates mean-reversion, implying that exogenous shocks have temporary effects, while a value equal to or above 1 implies that exogenous shocks have permanent effects. The value of the integer d thus indicates the persistence of the series.[6] Intuitively, the value of d then shows the presence or absence of a bubble in the series. Long memory processes can also be non-stationary process when the value of d greater than 0.5. Non-stationarity in itself does not imply the presence of bubbles. As long as there is no unit root persistence, there will be no bubbles meaning that a non-stationary long memory process has no bubbles.

In this chapter, we first estimate the value of d and then test for the null hypothesis of a unit root $d = 1$ against that of a fractional unit root, $d < 1$. This is achieved by employing the Efficient Fractional Dicky Fuller Test of Lobato and Velasco (2007). Unlike standard unit root tests, this test is robust to the presence of fractional roots. If the null is not rejected, we conclude that there is a bubble in the rent-price ratio series.[7] Now that we have detailed the basic premise of our chapter, we state the key contributions that we make towards the existing literature.

This chapter makes three main contributions. The first contribution is in the use of long memory models to identify housing bubbles. To the best of our knowledge, there has been no study that use long memory models to test for bubble prevalence in the housing market. Existing papers, namely Koustas and Serletis (2005), Cuñado et al. (2005), Cuñado et al. (2012) and Kruse and Sibbertsen (2012), use long memory models to test for stock market bubbles. Inferences on

[6]Persistence measures the extent to which past economic shocks lead to permanent future changes. In a highly persistent series like a unit root process the effect of an economic shock will be permanent. In time series econometrics, persistence is usually tied to the value of the memory parameter, d.

[7]In the actual estimation procedure, we use the natural log of the rent-price aeries. This is motivated by our theoretical model where we derive an expression for the log rent-price ratio.

bubbles are based from the estimated value of persistence parameter, d, of the dividend-price ratio. The few papers, that test for persistence in the housing market, uses standard unit root and cointegration tests. For example, Gallin (2008) finds no evidence of cointegration between house prices and per capita income in the United States, whether using national level data from 1975 to 2002 or a panel of 95 metropolitan areas from 1978 to 2000. On the contrary, Gallin (2008) finds cointegration between U.S. house prices and rents over the period 1970 to 2005. Standard unit root tests, we discussed before have low power compared to the long memory tests which could explain these ambiguous results. Connected to the use of the long memory model is the type of procedure that is used to estimate the memory parameter d.

The literature on long memory estimation briefly classify the different methods to estimate d as either semi-parametric or parametric. Koustas and Serletis (2005) and Kruse and Sibbertsen (2012) use a parametric procedure of Sowell (1992) known as Exact Maximum Likelihood to test for bubbles in the S&P 500 Index. A major limitation of this approach is that it is not consistent when the time series follows a non-stationary process. Hence a unit root bubble process may be mistakenly rejected by this procedure. Also, parametric methods that are not specified correctly for non-normal or heteroskedastic errors will produce inefficient estimates of d and thus bubble presence. Standard inferencing using t or F tests would thus lead to erroneous conclusions. Furthermore, they suffer from small sample biases. To overcome these problems, we make use of Shimotsu and Phillips (2005) and Shimotsu (2009) semi-parametric methods that are both consistent under non-stationarity and also robust to heteroskedasticty. We compare these estimates with a parametric procedure by Beran (1995) called Non-Linear Least Squares which is also valid for non-stationary d.

The second contribution of our chapter is that we account for endogenous structural breaks when estimating the long memory parameter for the housing market. There is a huge volume of papers which argue that long memory can be spuriously induced by a structural break, see Diebold and Inoue (2001). Also, it is well known from works such as Perron (1989) that failure to allow for structural breaks in an intercept or trend can result in spuriously high estimates of the persistence parameter: Once one allows for changes over time in the mean, then deviations from this time-varying mean do not seem as persistent. Mayoral (2012) develops a time domain test of I(d) versus I(0) plus trends and/or breaks, and finds that the null of I(d) is not rejected in the U.S. inflation data. Choi and Zivot (2007), estimate the memory parameter, d, of an exchange rate forward discount series after adjusting for breaks in their mean. They find that the demeaned forward discount series produced significantly lower persistence values. Despite these findings, current papers on long memory and stock market bubbles do not account for possible structural breaks.[8] Additionally, with the exception of Barari et al. (2014) that test for unknown breaks in the aggregate Case-Shiller Index, estimation of structural breaks in the U.S. HPI's has been to the best of our knowledge non-existent. Moreover, the break test used by Barari et al. (2014) is not consistent for non-stationary series meaning that the number of breaks could be over estimated. Our work here fills this gap by using the Andrews (1993) and Andrews and Ploberger (1994) F statistic based structural tests to examine the presence of an endogenous break in the mean and trend of each series. Inference for break presence/absence was based on Hansen (2000)'s Fixed Regressor Bootstrap asymptotic p values that are consistent under non-stationary regressors and robust to heteroskedastic residuals. ***We estimate persistence after adjusting for potential breaks making our bubble analysis efficient. This ensures that the net persistence will account for time varying changes in fundamentals***. If we ignore these breaks, the gross persistence values will be inflated and will include both the bubble component as well as changes in fundamentals.

The third contribution is that we use both aggregate and regional level data in our empirical study. This is because the House Price Index (HPI) used can have dramatic ramifications on the assessment of whether a house price bubble exists, see McCarthy and Peach (2004). Furthermore, there is a plethora of research which suggests that an house price appreciation (depreciation) to a large extend depends on inelastic (elastic) housing supply, see Green et al. (2005) and Levitin and Wachter (2012) among others. The implication being that as housing supply elasticity differ from region to region so will the persistence of house prices. This would mean that national aggregated indices could hide possible bubbles which are regional in nature. Empirical papers, for example McCarthy and Peach (2004), Phillips and Yu (2011) and Nneji et al. (2013), that restrict their analysis to just the national indices could produce questionable findings. In this paper, we perform our analysis on not just the national indices but also on several metropolitan statistical areas. Additionally, we use a dataset ending in the

[8] Kruse and Sibbertsen (2012) does consider a structural break in their analysis by testing for changes in persistence. However, their test do not distinguish between mean-reverting and unit root persistence and consequently does not identify bubble behaviour.

last quarter of 2013 encompassing both the boom phase as well as the subsequent bust helping us in answering a key question of whether house prices have finally reverted to their fundamentals.

We can preview our results as follows. First, we find that the long memory models produced better estimates of the persistence when compared to standard unit root tests. Secondly, between the semi-parametric and parametric long memory estimation methods, it was found that the semi-parametric procedures gave reliable values of d. Based on these estimates we found that the persistence values were well above the unit root. This was true both for the aggregate and regional HPI's. Thirdly, we found one endogenous break in the mean and trend of each series. The break date coincided with the turnaround in the credit conditions in the borrowing market and was thus consistent with our a priori expectations. Finally, when we adjusted for these breaks by detrending and demeaning each series we found that the new persistence values (d) were significantly lower. Consequently, a few series now exhibited below unity persistence consistent with mean reverting long memory behaviour devoid of bubbles. Nevertheless, the aggregate Case-Shiller Index and 8 of 12 regional HPI's still indicated unit root bubble behaviour. We thus conclude that in the estimated time period of 1982Q4-2013Q4 the United States housing market shows evidence for rational price bubbles.

The rest of the chapter is organised as follows. In section 2.2 we provide a brief description of the present value relation of the Housing Market under rational expectations and introduces the notion of housing price bubbles. We then derive an expression for the log rent-price ratio. Section 2.3 outlines the long memory model and properties of the long memory parameter. Section 2.4 briefly goes through the structural break test that we implement, section 2.5 the different estimation methods. section 2.6 describes the dataset, section 2.7 the empirical results and finally section 2.8 concludes.

2.2. Theoretical Model of Housing Bubbles

2.2.1. Set-up

We begin by setting out some key concepts in modelling housing price bubbles. We start with a simple consumers' optimization problem to derive the basic asset pricing relationship assuming no arbitrage and rational expectations. Following Campbell and Shiller (1987, 1988), the expected discounted utility driven from consumption at time t, $u(c_t)$, is maximised in an endowment economy,

$$\max E_t \Big\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \Big\} \tag{2.1}$$

with discount rate β and Expectation E_t subject to the budget constraint:

$$c_t = y_t + (P_t + Rent_t)x_t - P_t x_{t+1} \tag{2.2}$$

where y_t is the endowment at time t, x_t is the asset, P_t is the after-payoff price of asset and $Rent_t$ is the payoff (dividend) received from the asset. In this chapter, we are looking at the asset class of the housing market and hence, P_t here is the real housing price and $Rent_t$ is the rent obtained from owning a house. We now proceed to the household's optimisation problem to derive the present value relation of the Housing Market.

2.2.2. Present Value Model of Housing Market

To solve the optimisation, we begin by substituting the constraint (2.2) into the objective function (2.1),

$$\max_{x_{t+1}} E_0 \Big\{ \sum_{t=0}^{\infty} \beta^t u(y_t + (P_t + Rent_t)x_t - P_t x_{t+1}) \Big\} \tag{2.3}$$

This is solved by the use of a Bellman equation which can be written as,

$$V(x_t, y_t, Rent_t) = \max_{x_{t+1}} \Big\{ u\Big[y_t + (P_t + Rent_t)x_t - P_t x_{t+1} \Big] + \beta E_t[V(x_{t+1}, y_{t+1}, Rent_{t+1})] \Big\}, \tag{2.4}$$

where $V(.)$ is the value function. The first order condition of the Bellman equation with respect to x_{t+1} is given by,

$$\frac{\partial V(x_t, y_t, Rent_t)}{\partial x_{t+1}} = u'(c_t)(-P_t) + \beta E_t \Big[\frac{\partial V(x_{t+1}, y_{t+1}, Rent_{t+1})}{\partial x_{t+1}} \Big] = 0 \tag{2.5}$$

The first derivative of the Bellman equation with respect to x_t:

$$\frac{\partial V(x_t, y_t, Rent_t)}{\partial x_t} = u'(c_t)(P_t + Rent_t) \tag{2.6}$$

Taking eq. (2.6) one-period ahead,

$$\frac{\partial V(x_{t+1}, y_{t+1}, Rent_{t+1})}{\partial x_{t+1}} = u'(c_{t+1})(P_{t+1} + Rent_{t+1}) \tag{2.7}$$

Combining eq (2.5) and eq (2.7) gives us the Euler equation:

$$u'(c_t)P_t = \beta E_t[u'(c_{t+1})(P_{t+1} + Rent_{t+1})] \tag{2.8}$$

which simply says that given house prices P_t and rents $Rent_t$, agents will find it optimal to increase their demand of the asset if the expected future gains to doing so are greater than the costs. It is generally assumed for asset pricing purposes that the utility function is linear, which implies constant marginal utility and risk neutrality, see Cochrane (2007). Hence,

$$E_t\Big[\frac{u'(c_{t+1})}{u'(c_t)}\Big] = 1 \tag{2.9}$$

and eq (2.8) solves to

$$P_t = \beta E_t(P_{t+1} + Rent_{t+1}) \tag{2.10}$$

Assuming further the existence of a riskless bond available in zero net supply with one period net interest rate, R (where $\beta = 1/1 + R$), no arbitrage implies

$$P_t = \frac{1}{1+R}E_t(P_{t+1} + Rent_{t+1}) \tag{2.11}$$

This is the Present Value Model of house prices which forms the basis for most asset pricing tests, see West (1987), Diba and Grossman (1988) and Evans (1991a). In eq (2.11) $0 < 1/(1+R) < 1$ is the discount factor.[9] Solving eq (2.11) forward j periods yields

$$P_t = \Big(\frac{1}{1+R}\Big)^j E_t[P_{t+i}] + \sum_{i=1}^{m} E_t[\Big(\frac{1}{1+R}\Big)^i Rent_{t+i}] \tag{2.12}$$

Assuming that the expected discounted value of the house in the indefinite future converges to zero:[10]

$$\lim_{k\to\infty}\Big(\frac{1}{1+R}\Big)^j E_t[P_{t+i}] = 0 \tag{2.13}$$

This allows us to obtain the fundamental value of the house, as the expected present value of future rents:

$$F_t = \Big(\frac{1}{1+R}\Big)\sum_{i=1}^{\infty} E_t[Rent_{t+i}]. \tag{2.14}$$

[9]Here we have treated discount rate as equal to returns. Alternatively, discount rates can be considered as a constant, return plus risk premia or consumption based Cochrane (1992). However, testing for bubbles with these specifications is not straightforward.

[10]The transversality condition is not testable in finite samples, so this assumption is held with caution, see Cochrane (1992) and Diba and Grossman (1988).

in (2.14), F_t is the fundamental value of the house. This is equal to P_t as long as there are constant returns. Abandoning the convergence assumption leads to an infinite number of solutions any one of which can be written in the form

$$P_t = F_t + B_t, \qquad\qquad B_t = \left(\frac{1}{1+R}\right) E_t[B_{t+1}] \tag{2.15}$$

The second term B_t is the "Asset Price Bubble". Now that we have shown how asset price bubbles can exist using a simple present value model derived from a consumption utility function, we now extend this constant discount rate model to a time varying one and then subsequently derive the expression for the log rent-price ratio. This is detailed in the following section.

2.2.3. The Rent-Price Ratio and Housing Price Bubble

By allowing the discount rate to vary over time, we are in turn implying that the housing returns are also stochastic. Furthermore, if the housing returns are stochastic, the expected present value is a nonlinear function of rents and housing prices. This section derives and discuss the log linear approximation of the housing rent-price ratio. Following the seminal paper by Campbell and Shiller (1987), the log linear dividend-price ratio relation has become one of the central equations in empirical finance research, particularly those on asset price bubbles.[11] Eq (2.11) can be rearranged with stochastic R_{t+1} to express the one period gross housing return from time t to $t+1$ as,

$$R_{t+1} = \frac{P_{t+1} + Rent_{t+1} - P_t}{P_t} \tag{2.16}$$

Rearranging eq (2.16) as follows:

$$1 + R_{t+1} = \frac{P_{t+1} + Rent_{t+1}}{P_t} \tag{2.17}$$

Taking the natural logarithm on both sides of eq (2.17),

$$r_{t+1} = \ln(1 + R_{t+1}) \tag{2.18}$$

$$= \ln(P_{t+1} + Rent_{t+1}) - \ln(P_t) \tag{2.19}$$

$$= \ln\left[\left(1 + \frac{Rent_{t+1}}{P_{t+1}}\right) P_{t+1}\right] - \ln(P_t) \tag{2.20}$$

$$= \ln[1 + e^{\delta_{t+1}}] + p_{t+1} - p_t. \tag{2.21}$$

[11]For example Campbell and Shiller (2001),Koustas and Serletis (2005), Cuñado et al. (2005, 2012) apply the model for stock markets and Campbell et al. (2009) and Ambrose et al. (2013) for the Housing Market.

The last step is obtained by defining

$$\delta_{t+1} = \ln \frac{Rent_{t+1}}{P_{t+1}} \tag{2.22}$$

where δ_{t+1} is the log rent-price ratio ($rent_t - p_t$) and e is the exponential function. Lower case letters, p_t and $rent_t$, represent the natural logs of real housing prices and real rents, respectively. The first term in (2.21) is non-linear in the log rent price ratio.

$$f(\delta_{t+1}) = \ln[1 + e^{\delta_{t+1}}] \tag{2.23}$$

The first order Taylor approximation of this term, similar approach in Engsted et al. (2012), is as follows:

$$f(\delta_{t+1}) = f(\hat{\delta}) + \left[\left(\frac{1}{1+e^{\hat{\delta}}}\right)e^{\hat{\delta}}(\delta_{t+1} - \hat{\delta})\right] \tag{2.24}$$

where $\hat{\delta}$ is the point around which the linearization is done.[12] Setting $\rho = (1+e^{\hat{\delta}})^{-1}$ implies,

$$(1 + e^{\hat{\delta}}) = \frac{1}{\rho} \tag{2.25}$$

and also,

$$e^{\hat{\delta}} = \frac{1-\rho}{\rho} \tag{2.26}$$

Combining eq's (2.24), (2.25) and (2.26) we get:

$$f(\delta_{t+1}) = \ln\left(\frac{1}{\rho}\right) + (1-\rho)\delta_{t+1} - (1-\rho)\ln\left(\frac{1-\rho}{\rho}\right) \tag{2.27}$$

Substituting eq (2.27) into (2.23) and the result in (2.21) we obtain an expression for the stochastic logged interest rate

$$r_{t+1} = k + \rho p_{t+1} + (1-\rho)rent_{t+1} - p_t \tag{2.28}$$

where k is a constant given by $k = \ln(\frac{1}{\rho}) - (1-\rho)\ln(\frac{1-\rho}{\rho})$. Adding and subtracting $rent_t$ in (2.28), we have

$$r_{t+1} = k + \delta_t - \rho\delta_{t+1} + \Delta rent_{t+1} \tag{2.29}$$

[12] $\hat{\delta}$ here is the unconditional mean of the rent-price ratio which is the usual norm in literature, see Engsted et al. (2012) and also Cochrane (2007).

Eq. (2.29) can be interpreted as a linear forward difference equation in δ_t:

$$\delta_t = -k + \rho\delta_{t+1} + r_{t+1} - \Delta rent_{t+1} \tag{2.30}$$

We solve the above expression by forward recursive substitution method assuming that the transversality condition holds. Imposing the no rational bubble condition (or the transversality condition),

$$\lim_{j\to\infty} \rho^j \delta_{t+j} = 0 \tag{2.31}$$

we get:

$$\delta_t \approx \sum_{j=0}^{\infty} \rho^j [r_{t+j+1} - \Delta rent_{t+j+1}] - \frac{k}{1-\rho}. \tag{2.32}$$

We can consider eq. (2.32) as an ex ante relationship. Taking expectations on both sides conditional on the information available at time t,

$$\delta_t = \sum_{j=0}^{\infty} \rho^j [E_t r_{t+j+1} - E_t \Delta rent_{t+j+1}] - \frac{k}{1-\rho}. \tag{2.33}$$

Eq. (2.33) states that the log rent-price ratio, δ_t can be written as the discounted sum of all future log returns minus the discounted sum of all future log rent changes less a constant term. The above expression also implies that if the log returns and the log rent changes are stationary stochastic process, then the log rent-price ratio is a stationary stochastic process under the transversality condition Craine (1993). On the contrary, the presence of a unit root in δ_t is consistent with asset price bubbles in the log rent price ratio. This is because the presence of a unit root will imply the lack of a cointegrating relationship between rents and price.

In summary, housing bubble presence depends on the persistence of the log rent-price ratio. In time series econometrics, we measure the persistence of a series by its autocorrelation function which is dependent on the value of the memory parameter d. Formal econometric tests for distinguishing between $d = 0$ a stationary non-bubble process and $d = 1$ a unit root bubble process exist in the literature, see, for example, Dickey and Fuller (1979) and its various extensions. However, the jump from $d = 0$ to $d = 1$ is often too extreme in the sense that you are not considering fractional values of d. These fractional roots have the interesting property that they possess long memory but eventually return to their mean. In this chapter, we test for bubbles in the Housing Market by differentiating between long memory processes and unit root process. By their definition, rational bubbles are highly persistent and their effect is permanent implying unit root behaviour. If the tested data exhibits unit root type behaviour then we conclude that it has a bubble. In contrast, if the series possesses long memory then we conclude that it *does not* contain a bubble. This is only one of the reasons that motivates us to use long memory models. The second reason is related to aggregation. Granger and Joyeux (1980) has shown that time and cross-sectional aggregation can generate long memory in aggregate processes. Testing for bubbles is done on house price indexes which are aggregates of several regional ones. We employ both aggregate and regional series in our study to prevent any biases in the testing for bubbles. The third reason is related to structural breaks. Granger and Hyung (2004) and Diebold and Inoue (2001) have shown that processes with certain kind of structural changes in mean appear indistinguishable from long-memory processes. Given the significant shocks that have beset the world economy over the past three decades, as well as the likelihood of structural change occurring over this period, a measure that allows for such change is clearly desirable.

2.3. Fractionally Integrated Processes - Long Memory Models

In this section, we introduce the econometric methodology crucial to our study on housing price bubbles and describe the distinction between long memory and bubbles. We start with a brief introduction on long memory processes, then move on to fractionally integrated processes $I(d)$ and discuss when and why fractionally integrated processes possess "long memory" and the ramifications on asset price bubbles. The presence of long memory in a time series can be defined in terms of its spectral density function or the autocorrelation function, see Robinson (1994).

Let x_t, $t = 0, \pm 1 ...$ be a time series indexed with time t. It is covariance stationary if the mean, $E(x_t) = \mu$, and the covariance, $Cov(x_t, x_{t+j}) = \gamma(j)$, do not depend on t. The spectral density of x_t is formally given by,

$$f(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) e^{-ij\lambda}, -\pi \leq \lambda \leq \pi \tag{2.34}$$

The series x_t is then, said to have "long memory" if

$$f(0) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma(j) = 0 \tag{2.35}$$

so that $f(\lambda)$ has a "pole" at frequency zero. Another way to look at long memory models is in connection with persistence in autocorrelations. The extent of the persistence is consistent with an essentially stationary process, but where the autocorrelations takes far longer to decay than the exponential rate associated with the $ARMA$ class. The phenomenon has been noted in different data sets by Hurst (1951) among others. When viewed as a time series realisation of a stochastic process, the autocorrelation function exhibits persistence that is neither consistent with an $I(1)$ process nor an $I(0)$ process. As the correlations decay to zero very slowly, they are not summable i.e.,

$$\sum_{k=-\infty}^{\infty} |\varrho(k)| = \infty \tag{2.36}$$

The intuitive interpretation is that the process has "long memory", see Baillie (1996). This is in contrast to "short memory" processes where the correlations decay quickly to zero such that,

$$\sum_{k=-\infty}^{\infty} |\varrho(k)| < \infty \tag{2.37}$$

For example, in an $ARMA$ process, the asymptotic decay of the correlations is exponential in the sense that there is an upper bound

$$|\varrho(k)| \leq ba^k \tag{2.38}$$

where $0 < b < \infty$, $0 < a < 1$ are constants.

The fractional integrating process $I(d)$, d is a fractional number, can be regarded as a halfway house between the $I(0)$ and $I(1)$ paradigms. Formally, *a time series is defined as integrated of order d, denoted as $I(d)$, when applying the differencing operator $(1-L)^d$ renders it a stationary, invertible autoregressive moving average ($ARMA$) process. When d is not an integer, the series is said to be fractionally integrated.*

In this case the series is represented by an Autoregressive Fractionally Integrated Moving Average ($ARFIMA$) model,

$$\Phi(L)(1-L)^d(y_t - \mu) = \Theta(L)\epsilon_t \tag{2.39}$$

where L is the lag operator, $L^k y_t = y_{t-k}$, and $\Phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$, $\Theta(L) = 1+\sum_{i=1}^{q} \theta_i L^i$, respectively, represent stationary autoregressive and moving average components, see Granger and Joyeux (1980) and Robinson (2003) and others. Further, ϵ_t has an unconditional $N(0, \sigma^2)$ distribution, and d can take non-integer values. The fractional differencing operator is defined by its binomial expansion

$$(1 - L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)} L^j \tag{2.40}$$

where $\Gamma(.)$ is the Gamma function.

$$(1 - L)^d = 1 - dL - \frac{d(1 - d)}{2!} L^2 - \frac{d(1 - d)(2 - d)}{3!} L^3 - ..... \tag{2.41}$$

In general, an $ARFIMA(p, d, q)$ model can be represented in the following form,

$$\Big(1 - \sum_{i=1}^{p} \phi_i L^i\Big)(1 - L)^d(y_t) = \Big(1 + \sum_{i=1}^{q} \theta_i L^i\Big)\epsilon_t. \tag{2.42}$$

When the lag order is zero, $ARFIMA(p, 0, q)$ follows an $ARMA(p, q)$ process with constant mean and variance over time. When $d = 1$, $ARFIMA(p, 1, q)$ is a non stationary process containing a unit root $ARIMA(p, 1, q)$. The effects of each shock persist and accumulate over periods of time, these integrated process are not mean reverting.

The ε_t term in eq (2.42) is the innovation in the process. Normally, this term is assumed to have a constant variance throughout. However, there is considerable empirical evidence that both U.S. aggregate and metropolitan housing prices exhibit time changing variance, see Crawford and Fratantoni (2003), Miller and Peng (2006) and Miles (2008).[13] Furthermore, Cont (2005) shows that investor inertia (due to high transaction costs and tax considerations) can cause volatility clustering. As a result, volatility has to be parameterized to reflect time varying effects.

Engle (1982) defined an ARCH process for ε_t of the form $\varepsilon_t = \eta_t \sigma_t$, where η_t is an independently and identically distributed ($i.i.d.$) process with $E(\eta_t) = 0$ and $Var(\eta_t) = 1$. Clearly, ε_t is serially uncorrelated with a mean equal to zero, but its

[13] Miller and Peng (2006) use a VAR model and calculates the impulse response functions to evaluate the effects of volatility shocks to several fundamental housing variables and Crawford and Fratantoni (2003) assess the efficiency of different time series models in forecasting house prices and find that $GARCH$ type models perform better.

conditional variance equals σ_t^2 and may change over time. This can be modelled by Autoregressive Conditional Heteroskedasticity Methods ($ARCH$). A variety of $ARCH$ models exist in the literature the main difference among them being the functional form of σ_t^2. The conditional variance in the Engle (1982) formulation is a distributed lag of past squared innovations.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2, \tag{2.43}$$

The $ARCH$ model can describe volatility clustering. The conditional variance of ε_t is an increasing function of the shock that occurred in period $t-1$. A large absolute value in ε_{t-1} implies that σ_t^2 and ε_t (in absolute value) are expected to be large. As a way of modelling persistent movements in volatility without estimating a very large number of coefficients in a high order $ARCH$ process, Bollerslev (1986) suggested the Generalized $ARCH$ ($GARCH$) model

$$\sigma_t^2 = \omega + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2. \tag{2.44}$$

There are some limitations to this $GARCH$ model. The non-negativity conditions on the conditional variance may be violated by the estimated method, since the coefficients of model probably are negative. Furthermore, $GARCH$ does not allow for any direct feedback between the conditional variance and the conditional mean.

For these reasons, we make use of the asymmetric exponential $GARCH$ model of Nelson (2009). These are absolute value $GARCH$ models and can accommodate effects of both positive and negative shocks. The $EGARCH$ models is expressed as follows

$$\ln(\sigma_t^2) = \omega + [1 - \beta(L)]^{-1}[1 + \alpha(L)]g(\eta_{t-1}) \tag{2.45}$$

Since the $\ln(\sigma_t^2)$ is modelled, the significant advantage of $EGARCH$ models is that even if the parameters are negative, σ_t^2 will be positive.

$$g(\eta_{t-1}) = \gamma_1 \eta_t + \gamma_2[|\eta_t| - E|\eta_t|] \tag{2.46}$$

The α parameter represents a magnitude effect or the symmetric effect of the model, the $GARCH$ effect and β measures the persistence in conditional volatility. When β is relatively large, volatility takes a long time to die out following a crisis in the market. The γ parameter measures the asymmetry or the leverage effect, the parameter of importance so that the $EGARCH$ model allows for testing of asymmetries. If $\gamma = 0$, then the model is symmetric. When $\gamma < 0$, then positive shocks (good news) generate less volatility than negative shocks (bad news). When $\gamma > 0$, it implies that positive innovations are more destabilizing than negative innovations. $E|\eta_t|$ depends on the assumptions made regarding the unconditional distribution of η_t. For a normal distribution which we assume in our analysis, $E|\eta_t| = \sqrt{2/\pi}$.

In this chapter, we couple the $EGARCH(1,1)$ model with the $ARFIMA$ model and then parametrically estimate it by the Quasi-Maximum Likelihood method on the log rent-price data. The purpose of this estimation exercise is to find the value of the persistence of the log rent-price ratio. The next section describes the implication for bubble prevalence under different values of the estimated persistence, d.

2.3.1. Persistence and Bubbles

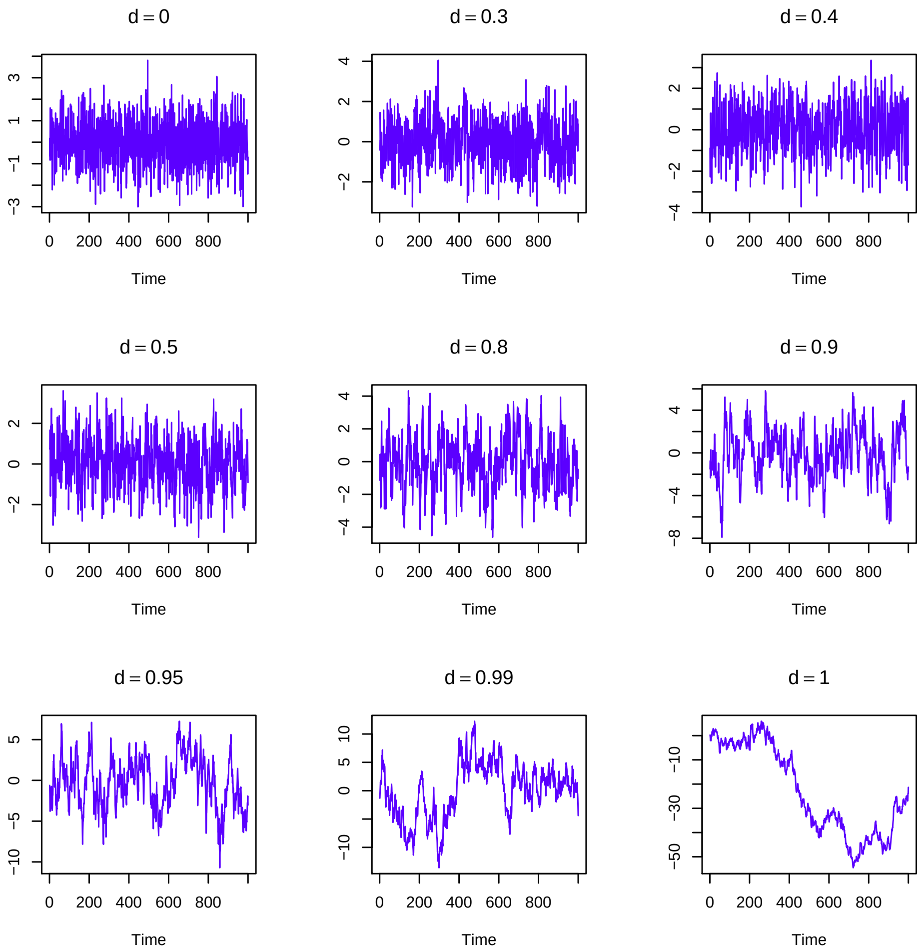
Bubble behaviour of the Housing Market is tested using the estimated values of the fractional integrating parameter, d, of the log rent-price ratio. Table 2.1 summarizes bubble analysis for different values of d. Fractionally integrated processes possess long memory when d lies between zero and one. They are also mean reverting in this interval, which means that in the case of the housing market, housing prices will return to its fundamentals ruling out the presence of price bubbles.

Table 2.1. Bubble Analysis

Order of integration(d) of δ_t	Analysis*
$-1/2 < d < 0$	anti-persistent and no bubbles
$d = 0$	stationary process and no bubbles
$0 < d < 1/2$	covariance-stationary
	and mean reverting implying no rational bubbles
$1/2 < d < 1$	non-stationary and possess long memory
	but also mean reverting implying no rational bubbles
$d \geq 1$	non-stationary explosive process, no mean-reversion implying rational bubbles

Figure 1 plots simulated $AR(1)$ process for different values of d. It is clearly seen that as d approaches 1, the series mimics the slope of an asset price bubble. This is especially true when d=0.99 very close to a unit toot process but possess the property of mean reversion and hence cannot be called a bubble. Standard empirical methods that look for a unit root in financial ratios do not consider the possibility of fractional roots and thus could mistake a mean reverting process as a bubble.

Figure 2.1. Simulations of $I(d)$ processes



Notes: This figure graphically illustrate Autoregressive (AR(1)) simulations of $I(0 \leq d \leq 1)$ processes. The top row traces paths for different covariance-stationary processes. The second and third row depict changes in the persistence as d approach unity.

The analysis here assumed no structural breaks. The presence of structural breaks in the time series distorts an efficient estimation of the integrating factor d, see Diebold and Inoue (2001). We implement structural change tests to account for this problem. The next section details a brief discussion on structural change literature and how it affects long memory estimation.

2.4. Structural Changes versus Long Memory

The problem of detecting structural changes in linear regression relationships has been an important topic in statistical and econometric research, see Hansen (1996) for a discussion. Existing literature on long memory and asset price bubbles do not account for structural breaks, see Cuñado et al. (2005), Koustas and Serletis (2005) and Cuñado et al. (2012). However, many studies indicate that the time series with structural breaks can induce a strong persistence in the autocorrelation function and hence generate "spurious" long memory, see Diebold and Inoue (2001), Granger and Hyung (2004), Perron and Qu (2007). Perron and Qu (2007) show how a stationary short memory process with level shifts can generate spurious long memory. Kruse and Sibbertsen (2012) considered a range of stable shifts and a change in persistence in several simulated experiments and simulation results confirm theoretical arguments which suggest that spurious evidence for long memory can easily be found.

Since the work of Quandt (1960), several methodologies (i.e. Andrews and Fair (1988), Perron (1989), Bai and Perron (1998, 2003) etc.) have been suggested to test for possible known or unknown single or multiple structural changes. However, a limitation of a majority of these tests is that the distribution theory used for these tests is primarily asymptotic and has been derived under the maintained assumption that the regressors are stationary. This excludes structural change in the marginal distribution of the regressors. As a result, these tests technically cannot discriminate between structural change in the conditional and marginal distributions.

In this chapter, we make use of Hansen (2000)'s 'fixed regressor bootstrap' method that is consistent for non-stationary regressors, achieves first-order asymptotic distribution and allows for arbitrary structural change in the regressors and accommodates heteroskedastic error processes. In the following paragraphs, we briefly describe this break methodology in the context of our chapter.

Consider the following linear regression with m breaks ($m+1$ regimes):

$$y_{ni} = \mu_{ni} + \psi_{ni} + u_{ni}, \tag{2.47}$$

where $i = 1, \ldots, n,$, $y_{ni} = \delta_{ni}$ is the log rent-price ratio, μ is the intercept or the mean and ψ is the trend. Testing for structural change in the mean and trend is all about checking whether or not μ and ψ is constant. Hansen (2000) finds that the distributions are different when the regressors are non-stationary and that the size and power distortions can be quite large. The structural change in μ (or ψ) can take the form

$$\mu_{ni} = \begin{cases} \mu & i < t_0, \\ \mu + \theta_n & i \geq t_0. \end{cases} \tag{2.48}$$

The parameter $t_0 \in [t_1, t_2]$ indexes the relative timing of the structural shift, and θ_n indexes the magnitude of the shift. Essentially, structural breaks are tested by the null- $H_0 : \theta_n = 0$ against $H_1 : \theta_n \neq 0$.

Hansen (2000) assumes that θ_n takes the form

$$\theta_n = \zeta\sigma/\sqrt{n} \tag{2.49}$$

with ζ fixed as $n \to \infty$. The parameter ζ indexes the degree of structural change under the local alternative $H_1 : \theta_n \neq 0$. We denote the ordinary least squares (OLS) estimators as $\hat{\mu}$ and $\hat{\psi}$, the residuals as $\hat{u}$ and the variance as $\hat{\sigma}^2 = (n - m)^{-1} \sum_{i=1}^{n} \hat{u}_i^2$. Under the alternative $H_1 : \theta_n \neq 0$, the model can be written as

$$y_{ni} = \mu + \theta_n I(i \geq t_0) + \psi + \theta_n I(i \geq t_0) + u_{ni} \tag{2.50}$$

For any fixed t, eq. (2.50) can be estimated by OLS, yielding estimates $(\hat{\mu}_t, \hat{\psi}_t, \hat{\theta}_t)$, residuals $\hat{u}_{it}$ and variance estimates $\hat{\sigma}_t^2 = (n - 2m)^{-1} \sum_{i=1}^{n} \hat{u}_{it}^2$. Let $\hat{t} = \arg\min \hat{\sigma}_t^2$ denote the least squares estimate of the breakdate and set $\tilde{\mu} = \hat{\mu}_{\hat{t}}$, $\tilde{\psi} = \hat{\psi}_{\hat{t}}$ and $\tilde{u}_i = \hat{u}_{i\hat{t}}$.

The standard test for H_0 against H_1 for known t (eg., Chow (1960)) is the Wald statistic:

$$F_t = \frac{(n - m)\hat{\sigma}^2 - (n - 2m)\hat{\sigma}_t^2}{\hat{\sigma}_t^2} \tag{2.51}$$

When the true changepoint t_0 is unknown, Quandt (1960) proposed the likelihood ratio test which is equivalent to

$$\sup F_n = sup_t F_t, \tag{2.52}$$

where the supremum is taken over $t \in (t_1, t_2)$. Andrews and Ploberger (1994) suggested an exponentially weighted Wald test

$$ExpF_n = \ln \int \exp(F_t/2) dw(t), \tag{2.53}$$

and the average F test

$$AveF_n = \int_t F_t dw(t), \tag{2.54}$$

where w is a measure putting weight $1/(t_2 - t_1)$ on each integer t in the interval $[t_1, t_2]$. Andrews and Ploberger (1994) and others assume that the regressors are stationary which as illustrated by Hansen (2000) affects the asymptotic distributions of the test statistics in complicated ways. Hansen (2000) advocates an alternative bootstrap distribution called 'Fixed Regressor Bootstrap'.

The bootstrap procedure for the $SupF$ test is discussed below. There are two forms of fixed regressor bootstrap, one appropriate if the error u_{ni} is (1.) homoskedastic and under (2.) heteroskedasticity. For the homoskedastic bootstrap, let $\{y_{ni} : i = 1, \ldots, n\}$ be a random sample from the $N(0, 1)$ distribution. Regress $y_{ni}(b)$ on μ_{ni} and ψ_{ni} to get the residual variance $\hat{\sigma}^2(b)$ and regress $y_{ni}(b)$ on μ_{ni}, $\mu_{ni}I(i \leq t)$, ψ_{ni} and $\psi_{ni}I(i \leq t)$ to get the residual variance $\hat{\sigma}_t^2(b)$ and Wald sequence

$$F_t(b) = \frac{(n-m)\hat{\sigma}^2(b) - (n-2m)\hat{\sigma}_t^2(b)}{\hat{\sigma}_t^2(b)} \tag{2.55}$$

The bootstrap test statistic is $SupF_n(b) = sup_{t_1 \leq t \leq t_2} F_t(b)$. The bootstrap p-value $p_n = 1 - G_n(SupF_n)$, where $G_n(x) = P(SupF_n(b) \leq x|J_n)$ denote the conditional distribution function of $SupF_n(b)$ and x is the regressors (μ and ψ). The bootstrap test rejects H_0 when p_n is small. We can allow for heteroskedastic errors by making a small modification. Set $y_{ni}^h(b) = z_i(b)\tilde{u}_i$, where $\{z_i(b) : i = 1, \ldots, n\}$ is an iid N(0,1) sample. The heteroskedasticity corrected p-value is then $p_n^h = 1 - G_n^h(SupF_n)$ where G_n^h is the modified conditional distribution.

In this chapter, we implement the three break tests based on the F statistic ($SupF$, $ExpF$ and $AveF$). We then use Hansen (2000)'s methodology to compute the heteroskedasticity corrected bootstrap p-values to make inference on presence of structural changes. The results are reported in §2.7.4. Now that we have detailed persistence characteristics on bubble behaviour and the implications of structural breaks on it, we proceed to a review of the methods that we employ to estimate the long memory persistence of the log rent-price ratios in the U.S. Housing Market.

2.5. Long Memory Estimation

The literature on estimating long memory models is extensive with a wide range of methods, see Li and Mcleod (1986), Hassler (1993) and Taqqu et al. (1995). In general, the estimators of the fractional order of integration, d, can be categorized into two groups - semi-parametric and parametric methods. ‘Semi-parametric’ methods do not require the modelling of a complete set of parameters, we are only interested in d. If a complete model is specified, such as an $ARFIMA$ we term the estimation ‘parametric’.[14] The main disadvantages of parametric methods are that they are computationally expensive and are subject to misspecification. On the other hand, semi-parametric models consider d as the most important parameter of interest and it is robust to mis-specification. A correctly specified ’parametric’ model aids us in analysing both the short run and the long run memory of the series whereas the semi-parametric procedures only concentrate on the long run persistence. However, we are only interested in the long run persistence d in this paper. Hence, we believe the semi-parametric methods which do not require the modelling of the short run dynamics and thus free of any specification errors should perform better. Our results in §2.7.4 validates this argument. Nevertheless, we implement both these estimation procedures, these are explained in brief in the following two sections.

2.5.0.1. Parametric Estimation Methods

Parametric $ARFIMA$ modelling can capture the long term persistence through the order of integration and also the short term persistence through the $ARMA$ process.

This chapter makes use of a full parametric estimation method, namely the Non-Linear Least Squares Estimation method of Beran (1995) which unlike Sowell (1992)’s Exact Maximum Likelihood (often used in the literature), is consistent in the non-stationary region as well. Beran (1995) developed an approximate maximum likelihood estimator based on minimising the sum of squared naive residuals, which is also applicable for non-stationary $ARFIMA$ processes with $d > 0.5$. The approximate log likelihood known as Non-linear Least Squares (NLS) is given by

$$\log L_A(d, \phi, \theta, \beta) = c - \frac{1}{2} \log \frac{1}{T-k} \sum_{t=2}^{T} \tilde{e}_t^2, \tag{2.56}$$

[14]Parametric methods can be in both the time domain and the frequency domain. Here, we concentrate only on time domain parametric methods for an effective comparison with the frequency domain semi-parametric ones.

where $\tilde{e}_t$ are the one-step prediction errors from the naive predictions defined near the $AR(\infty)$ representation of z_t

$$z_t = \sum_{j=1}^{\infty} \pi_j z_{t-j} + \varepsilon_t, \tag{2.57}$$

where $z_t = x_t - \mu_t$. The results for the Beran (1995) estimation procedure for the U.S. housing market is reported in §2.7.2. A major drawback of the parametric estimation is that the computed values are highly biased under misspecification. Presence of non-normal/$ARCH$/autocorrelation errors will lead to inefficient estimates. Importantly, for small samples most of the persistence would be concentrated in the $ARMA$ part and thus the d value would be significantly over-differenced. This motivates our use of the semi-parametric procedures.

2.5.0.2. Semi-parametric Estimation Methods

We use three semi-parametric estimation methods to evaluate the memory parameter, all of which are robust to both conditional heteroskedasticity and non-normality. We start with the Local Whittle Estimate (LWE) developed by Kuensch (1987) and Robinson (1995). It starts with the following Gaussian objective function, defined in terms of the parameters d and G

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\Big[\log(G\lambda_j^{-2d}) + \frac{\lambda_j^{2d}}{G} I_x(\lambda_j)\Big], \tag{2.58}$$

where the parameter m , usually referred to as the truncation point or the window bandwidth, is a function of n (the sample size), chosen such that as $n \to \infty$, $m/n \to 0$. Also, $I_x(\lambda_j)$ is the periodogram of X_t evaluated at the fundamental frequencies. Also, X_t is a fractional process with order d.

The local whittle procedure estimates G and d by minimising $Q_m(G,d)$, so that

$$(\hat{G},\hat{d}) = \underset{G\in(0,\infty),d\in[\Delta_1,\Delta_2]}{\arg\min} Q_m(G,d), \tag{2.59}$$

where Δ_1 and Δ_2 are numbers such that $-1/2 < \Delta_1 < \Delta_2 < \infty$.[15] Henceforth, we denote the Local Whittle estimation of d as $\hat{d}_{LWE}$. Shimotsu and Phillips (2004) finds that this Local Whittle estimator is not reliable when the value of d

[15]Robinson (1995) showed that $\sqrt{m}(\hat{d}-d_0) \to_d N(0,1/4)$ as $n \to \infty$ under certain conditions. Here d_0 is the true value of the d parameter.

is in the non-stationary zone ($d > 1/2$). The asymptotic theory is discontinuous at $d = 3/4$, at $d = 1$ and not consistent beyond unity. Although data differencing and tapering have been recommended (Velasco (1999) and Hurvich and Chen (2000)), these approaches do have some disadvantages, such as the need to determine the appropriate order of differencing and the effects of tapering on data trajectory and asymptotic variance.

Shimotsu and Phillips (2005) proposed an Exact Local Whittle Estimation procedure that does not rely on tapering or differencing pre-filters and which is consistent when $d \geq 1/2$. If for the fractional process X_t

$$(1 - L)^d X_t = u_t \mathbf{1}\{t \geq 1\}, \qquad t = 0, \pm 1, \pm 2 \ldots \tag{2.60}$$

where u_t is an $I(0)$ process with mean 0 and $\mathbf{1}\{.\}$ is an indicator function. If the spectral density of X_t is given by $f_u(\lambda_j) \sim G$ and $I_{\Delta^d X}$ is the periodogram of d-th difference of X_t, the fractional order d is estimated by

$$\hat{d}_{ELW} = \arg\min_{d \in [\Delta_1, \Delta_2]} R(d), \tag{2.61}$$

where

$$R(d) = \log \hat{G}(d) - 2d \frac{1}{m} \sum_{j=1}^{m} \log \lambda_j \tag{2.62}$$

and

$$\hat{G}(d) = \frac{1}{m} \sum_{j=1}^{m} I_{\Delta^d X}(\lambda_j) \tag{2.63}$$

$\hat{d}_{ELW}$ is called the Exact Local Whittle (ELW) Estimator of d.

Most economic and financial series is modelled with a mean and a time trend. In fact, the rent-price series we analyse in this paper does exhibit a polynomial trend in its trajectory. Shimotsu (2009) extended the ELW estimator to accommodate an unknown mean and a polynomial time trend. When the data X_t are generated by

$$X_t = \mu_0 + X_t^0; \qquad X_t^0 = (1 - L)^{-d_0} u_t \mathbf{1}\{t \geq 1\} \tag{2.64}$$

where μ_0 is a nonrandom unknown finite number. Shimotsu (2009) prescribes estimating the unknown mean, μ_0, as a linear combination of the sample mean, $\bar{X}$, and the first observation, X_1:

$$\tilde{\mu}(d) = w(d)\bar{X} + (1 - w(d))X_1, \tag{2.65}$$

where $w(d)$ is a twice continuously differentiable weight function such that $w(d) = 1$ for $d \leq 1/2$ and $w(d) = 0$ for $d \geq 3/4$.[16] With this estimate of μ_0, consider the modified ELW objective function:

$$R_F(d) = \log \hat{G}_F(d) - 2d\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j, \quad \hat{G}_F(d) = \frac{1}{m}\sum_{j=1}^{m} I_{\Delta^d(x-\tilde{\mu}(d))}(\lambda_j). \quad (2.66)$$

d is then estimated by applying a 2-step procedure on this objective function under the assumption that $f(\lambda)$ is bounded for $\lambda \in [0, \pi]$. They call this the 2-step Exact Local Whittle Estimator, $\hat{d}_{2ELW}$, which is given by

$$\hat{d}_{2ELW} = \hat{d}_T - R'_F(\hat{d}_T)/R''_F(\hat{d}_T), \quad (2.67)$$

where $\hat{d}_T$, known as the first stage estimator, is a tapered local Whittle estimator of Velasco (1999) and Hurvich and Chen (2000). R_F is the modified objective function defined in eq. (2.66). $\hat{d}_{2ELW}$ is asymptotically normal and $\sqrt{m}$ consistent. This two-step ELW estimator can be extended to the cases where the data has a polynomial time trend in addition to an unknown mean:

$$X_t = \mu_0 + \beta_1 t + \beta_2 t^2 + \ldots + \beta_k t^k + X_t^0; \quad X_t^0 = (1-L)^{-d_0} u_t \mathbf{1}\{t \geq 1\}. \quad (2.68)$$

d can be estimated by regressing X_t on $(1, t, \ldots, t^k)$ and then applying the two-step estimation to the residuals $\hat{X}_t$. In this chapter, we consider both an unknown mean and a polynomial trend and the d estimate is denoted as $\hat{d}_{2ELW}$. Both the ELW and the 2ELW inherit the desirable properties of the local whittle procedures in that they are robust to normal errors and conditional heteroskedasticity.

However, one problem in computing the Whittle estimators concerns with the choice of bandwidth m in finite samples. Hence, we follow Kumar and Okimoto (2007) and choose m based on simulations. We simulate $Y_t = (1-L)^{-d}\epsilon_t$, where ϵ_t is a Gaussian white noise process, with sample size 125.[17] The optimal m is the one which minimizes the sample Mean Squared Error (MSE) for several choices of m, that is $m = \{n^{0.60}, n^{0.65}, n^{0.70}, n^{0.75}, n^{0.80}\}$. Based on the simulation results, we select $m = 0.75$ for both 2ELW and ELW and $m = 0.80$ for the LWE. The simulation results also showed that the 2-step ELW estimator gave consistent estimates which were closer to the true value of d. The results for the semi-parametric procedures are reported in §2.7.3

[16] An example of $w(d)$ for $d \in [1/2, 3/4]$ is $(1/2)[1 + cos(4\pi d)]$.

[17] This is the sample size of the data that we use later on in our empirical section.

To summarize our key methods so far, we shall seek to apply non-stationary consistent parametric $ARFIMA$ and semi-parametric Whittle methods to estimate the order of integration d. These d values are examined for asset bubbles. We also examine whether endogenous breaks induce any bubbles.

2.6. Rent-Price Data

Having described a theoretical framework that defines housing bubbles and an econometric framework to test for the existence of these bubbles, we now introduce the dataset. We use a log normalized house rent to house price ratio of three national HPI's and 12 Metropolitan Statistical Areas (MSA). The dataset spans the quarterly 31 years time period 1982Q4-2013Q4. The fact that we use an updated dataset ensures that our analysis incorporates both the upswing as well as the subsequent collapse in U.S. house prices.

The aggregate level price of residential housing is measured by the HPI (House Price Indexes) published by the Federal Housing and Finance Agency (FHFA), the Standard & Poor 500 Case-Shiller Index (Case-Shiller) and the United States Bureau of Census (Census). Both the FHFA and the Case-Shiller measure house prices as changes to the price of owner occupied housing by reviewing repeat mortgage transactions on single-family properties whose mortgages have been purchased or securitized by Fannie Mae or Freddie Mac, see Calhoun (1996). The US Bureau of Census produces a constant-quality HPI which employs the hedonic methodology adjusting for several physical attributes.[18]

The FHFA also publishes regional HPI data for MSA's belonging to the four census regions :- Midwest, Northeast, South and West. In this chapter, to augment the aggregate level analysis we use HPI data of 12 MSA's, three each from the four census divisions.

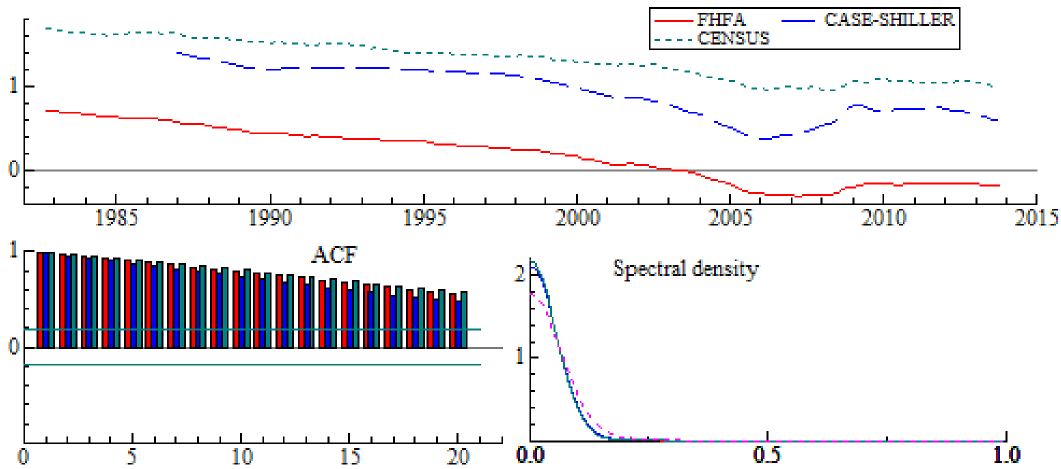
[18]Adjusted to attributes such as geographical location, lot size, number of bedrooms, number of bathrooms, type of parking facility, construction method, types of heating and air-conditioning systems etc. As such this index is a more reliable estimate of changes in housing prices than either the Case-Shiller or the FHFA. The Census index is also superior to other home price indexes such as the median price of existing homes sold by the National Association of Realtors (different from the NAR affordability index used here) and the median price of new homes sold published monthly by the Bureau of the Census of the United States Department of Commerce due to three main reasons. Firstly these two indices are not seasonally adjusted despite apparent seasonality; secondly there is high volatility in the short run as regional and product mix of sales varies from month to month; and thirdly as the underlying price data reflect only recent sales, the series may not accurately demonstrate housing stock values, see McCarthy and Peach (2004).

The rental data is extracted from the Owner's Equivalent Rent of Primary Residence belonging to the Consumer Price Index (CPI) market basket published by the Bureau of Labor Statistics (BLS) following Ayuso and Restoy (2006), Clayton (1996) and Mankiw and Weil (1989) among others. This index is based on the following question that the Consumer Expenditure Survey asks of consumers who own their primary residence: "If someone were to rent your home today, how much do you think it would rent for monthly, unfurnished and without utilities?". The ratio of this rental index and the HPI indexes gives us the nominal rental price ratio which is then deflated to real values by the region specific Consumer Price Index Excluding Shelter as in Abraham and Hendershott (1996). Both the HPI's and the rental data are also seasonally adjusted using the X-12 Census procedure.

There are several papers in the literature that use similar HPI and Rental Indexes to look for asset price deviations in the United States Housing Market. Abraham and Hendershott (1996) used repeat sales Fannie Mae and Freddie Mac index to assess contributions of several macroeconomic factors to the appreciation in real house prices in 30 MSA's belonging to the four census regions. Ayuso and Restoy (2006) use an FHFA quarterly dataset spanning 1987Q1 - 2003Q2 and a CPI-Rent data from BLS to examine for overvaluation of real housing prices in relation to rents. Barros et al. (2012) also use quarterly national and state level FHFA HPI data from 1975:1 till 2010:7 to check for fractional co-movement between regional and the national indexes. Campbell et al. (2009) make use of FHFA national and 23 MSA level HPI and rental data to decompose the variability of the rent-price ratio.

Figure 2.1 plots the national log rent-price ratios (δ_t) of FHFA, Case-Shiller and Census. When house prices are high relative to housing rents (i.e. fundamental values), the rent-price ratio is low. Thus, it can be concluded that a rent-price ratio far below its historical average indicates that asset prices have increased beyond fundamental values suggesting a possible bubble in the housing market. The non-availability of high frequency data for rental prices is the primary reason behind using a quarterly level analysis. It goes without saying that volatility effects (GARCH) will not be as prevalent as in the monthly series. However, housing unlike other assets such as stocks are illiquid because of the high transaction and labour costs involved. This coupled with the fact that buying and selling are done at irregular intervals imposes heteroskedasticity in the price indexes by construction. This is one of the reasons why we find substantial GARCH effects when using parametric $ARFIMA - GARCH$ estimation.

Figure 2.1. National Rent-Price Ratio's



Notes: This figure provides a graphical illustration of the log rent-price ratio of aggregate FHFA, Case-Shiller and Census. The general trend is downward in the log rent-price ratio associated with the upward trend in house prices.The second row plots the autocorrelation functions and the spectral densities.

On visual inspection, it is observable that the δ_t's of FHFA, Case-Shiller and Census follow a similar trend in that they gradually decline to a historical low in the 2006-2007 period (suggesting a bubble phase) followed by a gradual appreciation. This is in agreement with Krainer and Wei (2004) who calculate US FHFA house price to CPI rent ratio and find that house prices have been rising faster than implied rental values during the period of 1997-2004.

The autocorrelation function, second row in Figure (2.1), shows a slow decline with increasing lags and the spectral density plot has an upper bound at zero frequency both suggesting long memory in the national indices.[19] All the 12 Metropolitan Statistical Areas plotted in Figure (2.B.1) in the Appendix B follow a similar pattern in that they appear to be correlated with each other agreeing with Case et al. (1991) who demonstrate that house prices in the four regions of the United States are serially correlated.

[19]The spectral density of a covariance stationary process can be written as $f(\lambda) \sim G\lambda^{-2d}$ as $\lambda \to 0^+$ where d as usual is the memory parameter, $d \in (0,1)$ and $G \in (0,\infty)$. When $d = 1/2$, $f(\lambda)$ approaches to a positive constant at zero frequency, $d \in (0,1/2)$ it approaches to zero and when $d \in (1/2,1)$ it tends to infinity, see Robinson (1995).

Table 2.1. Descriptive Statistics

Housing Market	Mean	Min	Max	Std Dev	Skewness	Kurtosis	Jarque-Bera	Prob.	N
FHFA	0.182	-0.308	0.710	0.314	-0.056	1.671	9.266	0.019**	125
Case-Shiller	0.920	0.367	1.384	0.294	-0.280	1.728	8.691	0.012**	108
Census	1.309	0.941	1.681	0.230	-0.085	1.624	10.018	0.016**	125
Midwest									
Chicago	0.887	0.348	1.522	0.333	0.169	2.029	5.510	0.050*	125
Cleveland	0.818	0.449	1.372	0.291	0.512	1.891	11.872	0.011**	125
Detroit	0.794	0.275	1.496	0.365	0.373	1.843	9.869	0.016**	125
Northeast									
Boston	0.728	0.172	1.660	0.399	0.218	2.123	5.003	0.060*	125
New York	0.820	0.226	1.646	0.373	0.018	2.112	4.113	0.087*	125
Philadelphia	0.845	0.251	1.518	0.375	-0.204	1.829	8.002	0.025**	125
South									
Atlanta	0.715	0.261	1.248	0.307	0.041	1.590	10.396	0.015**	125
Dallas	0.600	0.206	0.926	0.227	-0.384	1.610	13.131	0.009***	125
Houston	0.442	0.018	0.799	0.245	-0.348	1.674	11.676	0.011**	125
West									
Los Angeles	0.670	-0.123	1.346	0.421	-0.113	2.059	4.879	0.063*	125
San Francisco	0.756	0.021	1.600	0.475	0.159	1.956	6.199	0.041**	125
Seattle	0.787	0.073	1.541	0.439	0.191	1.942	6.592	0.036**	125

Notes: These are descriptive for statistics on the log rent-price ratio for th complete set of dats aeries that we described in this section. The dataset spans the quarterly time period 1982Q4-2013Q4 (N=125) except Case-Shiller which starts in 1987Q1 (N=108). The Jarque-Bera statistic has a joint hypothesis that the skewness and the excess kurtosis is zero, i.e. null that the data is normally distributed. '***', '**' and '*' indicate rejection of the null at the 1%, 5% and 10% levels respectively. Deviations from the Normal distribution (fat tails) indicates the possibility of swings in the data.

The descriptive statistics for both aggregate (national) and regional rent-price ratio's are recorded in Table 2.1. Null of normality is uniformly rejected across the entire dataset indicating the presence of fat tails in the distribution, a characteristic generally observed in most of the financial time series data. Deviations from normality also imply that there are possible swings in the data or in other words, an asset price bubble. We also observe high standard deviation in the Western region comprising Los Angeles, San Francisco and Seattle. Case and Shiller (2003) find comparatively higher growth in house prices in the West Coast of the United States. Malpezzi (1996); Malpezzi and Wachter (2005) among others believe that the high magnitude of price appreciation in such regions is due to the metropolitan housing market's supply inelasticity. To further understand the time series properties of the US housing market and to infer about persistence and bubbles, we proceed to implement the tests discussed in section 2.5.

2.7. Empirical Results

This section reports and discusses the core empirical results obtained in this chapter on the existence of bubbles in the U.S. Housing Market. We start with some preliminary unit root tests to set the scene. Given the limitations of standard unit root testing we then progress to parametric and semi-parametric estimation of the fractional order of integration (d) parameter. Before we conclude we test whether the estimates of d are from a true long memory process or induced by a structural break.

2.7.1. *Evidence for Bubbles - Unit Root Tests*

Table 2.1. Unit Root Results

Housing Market	**Unit Root Tests**			Conclusion
	ADF	DF-GLS	KPSS	
FHFA	-2.202	-2.350	0.090	$I(0)/I(1)$
Case-Shiller	-2.455	-2.333	0.104	$I(0)/I(1)$
Census	-2.471	-2.524	0.088	$I(0)/I(1)$
Midwest				
Chicago	-2.398	-2.219	0.114	$I(0)/I(1)$
Cleveland	-0.674	-1.499	0.229***	$I(1)$
Detroit	-1.622	-1.691	0.213**	$I(1)$
Northeast				
Boston	-3.034	-2.965*	0.078	$I(0)/I(1)$
New York	-2.458	-2.341	0.083	$I(0)/I(1)$
Philadelphia	-1.741	-2.116	0.095	$I(0)/I(1)$
South				
Atlanta	-2.288	-2.315	0.120*	$I(1)$
Dallas	-2.936	-1.813	0.201**	$I(1)$
Houston	-1.258	-1.618	0.163**	$I(1)$
West				
Los Angeles	-1.710	-2.046	0.072	$I(0)/I(1)$
San Francisco	-0.553	-1.219	0.081	$I(0)/I(1)$
Seattle	0.288	-1.188	0.115	$I(0)/I(1)$
Crit. Value				
1%	-4.042	-3.565	0.216	
5%	-3.450	-3.018	0.146	
10%	-3.150	-2.728	0.119	

Notes: This table reports the unit root test results for the log rent-price ratios of the three national indexes (i.e. FHFA, Case-Shiller and Census) and the 12 regional MSA's. The sample spans the time period 1982Q4-2013Q4, except Case-Shiller which is 1987Q1-2013Q4. Critical values (1%, 5% and 10%) for the tests is given in the last three rows. All the unit root test statistics reported assume the presence of an intercept and a trend. p-values/Critical Values for DF-GLS, KPSS is obtained from MacKinnon(1996) and Kwiatoski-Phillips-Schmidt-Shin (1992). The null hypothesis for all the tests except KPSS is that δ_t has a unit root. The null hypothesis for KPSS is that the series is stationary. Lags are fixed at 12. For the KPSS test, the Newey West automatic method using Bartlett kernel selects the bandwidth. "*","**" and"***" indicate rejection of the null hypothesis at the 10%, 5% and 1% levels respectively. The last column gives the resultant conclusion of the unit root tests, specifically says whether the series is stationary $I(0)$, a unit root bubble process, $I(1)$ or $I(0)/I(1)$ when the tests contradict each other.

Integer order tests are implemented to examine the stationarity, $I(0)$, or non-stationarity, $I(1)$, of the log rent-price ratio (δ_t). The presence of a unit root, $I(1)$, in δ_t is consistent with the presence of housing price bubbles.

Table 2.1 reports the unit root results for the Housing Market. Column 1 defines the housing series estimated and columns 2-4 report test statistics of all the different tests implemented. We use two tests, namely the Augmented Dicky-Fuller (ADF) and the GLS-detrended Dicky-Fuller Elliot et al. (1996). Both these tests have a null hypothesis that the rent to price ratio has a unit root. The way in which classical hypothesis testing is carried out ensures that the null hypothesis is hard to reject. Kwiatkowski et al. (1992) argue that such unit-root tests often fail to reject a unit root because they have low power against relevant alternatives, such as, fractionally integrated series. They propose tests, known as KPSS tests, with the null hypothesis of stationarity against the alternative of a unit root. They argue that such tests should complement unit-root tests and that by testing both the unit-root and the stationarity hypotheses, one can distinguish between series that appear to be integrated, series that appear to be stationary, and series that are not very informative about whether or not they are stationary or have a unit root. The KPSS test statistics are reported in column 4 of Table 2.1. The lags for all these tests are fixed at 12 and the tests employ the presence of an intercept and a trend in the test regression.

The results in Table 2.1 aid us in presenting a preliminary empirical evidence on the presence of price bubbles in the U.S. Housing Market. The last column in the Table shows the conclusions we draw from the three tests. There are three cases we have to consider here. First, the case of unit root and bubbles - $I(1)$. This occurs when both the ADF and the DF-GLS test were unable to reject the null of a unit root and the KPSS test rejected the null of stationarity, $I(0)$. We find that 5 out of 15 series showed this behaviour.

Second, the case of no bubbles $I(0)$ when the ADF and the DF-GLS test reject the null of a unit root and the KPSS test do not reject the null of stationarity. We observe that none of the series exhibited this behaviour.

Third, the case of $I(0)/I(1)$ when the tests contradict each other. In this scenario, no conclusion can be drawn about bubbles from the tests. We will explain this with an example. Take the case of the aggregate FHFA series. The ADF and the DF-GLS both were unable to reject the unit root which should mean that the KPSS test will reject the null of stationarity. However, the KPSS test do not reject its null. In this case, we conclude that the series could be either an $I(0)/I(1)$. The majority, 10 out of 15 series, fall in this category. These series which include all the aggregate ones may possess long memory and mean reversion implying no bubbles or unit root or explosive indicating bubbles.

It is this inability of these standard unit root tests that validate our use of the fractional order methods. In the following sections, we make use of both parametric and semi-parametric estimation methods and also consider structural breaks to arrive at an efficient method to analyse housing bubbles.

We discuss the results for the regional series in more detail here. The results for the FHFA Regional MSA's are mixed. All the West and the Northeast regions as well as Chicago from the Midwest follow the national indexes in that they could be mean-reverting ($I(0)/I(1)$). However, Cleveland and Detroit from the Midwest; and Atlanta, Dallas and Houston from the South; show unit root tendency.[20] Most of the current literature that analyse the time series properties of U.S. Housing Indexes do not consider the ratio of rental to housing price series and tests for rational bubbles. They concentrate on merely the persistence of the house prices. Hence, an effective comparison with our results is not possible. Nevertheless, they uniformly find unit root persistence in the U.S. HPI's. For instance Canarella et al. (2011) who uses the DF-GLS test and finds overwhelming evidence to the presence of a unit root in the Case-Shiller MSA's for the monthly 23 year time period, January 1987-April 2009. Using quarterly data from 1975 to 1996 from the 50 US states, Muñoz (2004) also finds unit roots in house price changes, using the Dickey-Fuller Generalised Least Squares (DF–GLS) test. Meen (2002) compares the time-series behaviour of house prices in the US and the UK. Using quarterly data from 1976 to 1999 for the US and from 1969 to 1999 for the UK, Meen (2002) conducts both Augmented Dickey–Fuller (ADF) and Phillips–Perron (PP) unit-root tests on the level of house prices. He finds that in both countries house prices follow a difference stationary process. That is, house prices are $I(1)$.

It is clear from the results in Table 2.1 that the standard unit root tests do not provide a clear analysis about the exact persistence of the rent-price ratio's primarily because these test are too restrictive in that they consider only two possible values of d in the real space $d \in [0, 1]$. A recent unit root test by Cavaliere and Xu (2014) is widely applicable in series which are naturally bounded. THis test unlike the conventional ADF tests do not over-reject the null hypothesis. Nominal interest rates, unemployment rates and target zone exchange rates are examples of such bounded series. However, the use of this test in a model designed to detect bubbles which are marked by deviations from any bounds based on fundamentals is not appropriates. Nevertheless, this test complements the unit root tests used here. Furthermore, much research argues that the presence of structural breaks distorts the validity of standard unit root tests (Perron (1989, 1997)). This motivates our use of both parametric and semi-parametric methods for estimating the actual persistence or memory parameter (d). Koustas and Serletis (2005) also finds that unit root tests like ADF have low power in detecting

[20] Firstly, the KPSS test rejects the null of stationarity and secondly, both the ADF and the DF-GLS do not reject the null of a unit root.

asset price bubbles in the dividend-price ratio and advocates the use of $ARFIMA$ based parametric estimation methods. The following two sections describe their results.

2.7.2. Evidence for Bubbles - Parametric Estimation

The use of a full parametric estimation in the time domain, makes it possible to analyse both the short run (using ARMA) and the long run persistence (using d) simultaneously. Having two null hypothesis of $d = 1$ (unit root) or $d = 1$ (stationarity) allows our long memory methods to encompass both the ADF and KPSS tests in the previous section.

Existing literature that use maximum likelihood methods to estimate long memory in the time domain such as Koustas and Serletis (2005) and Kruse and Sibbertsen (2012) apply the Exact Maximum Likelihood method of Sowell (1992) on the dividend-price ratio in the U.S. Stock Market. A major drawback of this approach is that the Exact maximum Likelihood gives valid estimates of d only for stationary $ARFIMA$ process i.e., $d < 1/2$. At every step of the optimisation procedure, d is forced to take stationary values and hence, the resulting value of d is an over-differenced estimate. An additional problem in using a full parametric approach is that the precision with which the memory parameter is estimated hinges on the correct specification of the model, see Hauser et al. (1999). Thus, it is imperative to know the underlying data generating process before applying likelihood methods.

In this chapter, to circumvent these problems, we make use of the the Non-Linear Least Squares estimate (NLS) proposed by Beran (1995) which is consistent for $d > 1/2$ and we select the optimal model for the $ARMA$ part using the Bayesian Information Criterion (BIC). NLS given by eq. 2.56 in §2.5.2.5.0.1 is estimated on the following $ARFIMA$ equation for each time series,

$$\Big(1 - \sum_{i=1}^{p} \phi_i\Big)(1 - L)^d(\delta_t - \mu) = (1 + \sum_{i=1}^{q} \theta_i L^i)\varepsilon_t, \tag{2.69}$$

where p and q are the optimal AR and MA lags selected by BIC. The results of this estimation for the Housing Market dataset is reported in Tables 2.2 and 2.3.

The two parameters that are of interest to us are d, the long memory parameter, and ρ the sum of autoregresive coefficients (SARC), $\rho = \sum_{i=1}^{p} \phi_i$ - a conventional measure of short run persistence in time series literature proposed by Andrews and yuan Chen (1994). Tests for bubbles are based on imposing linear restrictions, $d = 1$ and $d = 0$, respectively. We also test for $\rho = 1$, that is unit root in the short run. However, we are only concentrated on the long run behaviour of the log rent-price ratio's we conclude for bubble behaviour only when there is a unit root in d. Hence, the linear restrictions helps in examing for the stationarity or unit behaviour of the rent-price ratios and whether these are rational bubbles. Shaded columns indicate the series in which the null of a unit root in the long memory ($d = 1$) cannot be rejected and thus exhibits bubbles.

The first column of these tables shows the parameters, residual tests and linear restrictions while the other columns report the corresponding estimated values with standard error in parentheses. In addition to computing the parameters, we also implement residual tests namely, for normal, $ARCH$ and autocorrelation errors. The reported p-values less than 0.05 indicate rejection of the null and implies the existence of these errors at the 5% significance level. The gray shaded columns indicate series with bubble behaviour.

Table 2.2. Parametric ARFIMA Estimation - National δ_t's

Est	FHFA	Case-Shiller	Census
	$AR(4)MA(0)$	$AR(4)MA(0)$	$AR(3)MA(0)$
$\hat{d}$	1.290 (0.256)	1.276 (0.359)	0.074 (0.083)
$\hat{\phi}_1$	0.215 (0.261)	0.596 (0.342)	0.914 (0.118)
$\hat{\phi}_2$	-0.175 (0.103)	-0.209 (0.109)	0.041 (0.097)
$\hat{\phi}_3$	0.262 (0.134)	0.090 (0.069)	0.028 (0.031)
$\hat{\phi}_4$	0.215(0.088)	0.117 (0.056)	
μ	0.672 (0.139)	0.488 (0.899)	0.787 (0.350)
ρ	0.517	0.794	0.983
Residual Tests			
Normality	0.000***	0.000***	0.002***
ARCH	0.044**	0.077*	0.102
Autocorrelation	0.216	0.960	0.542
Linear Restr.			
$d = 0$	0.000***	0.000***	0.370
$d = 1$	0.257	0.442	0.000***
$\rho = 1$	0.000***	0.000***	0.471

Notes: This table reports the Non-Linear Least Squares parameter estimates of the log rent-price ratio's corresponding to the three national HPI's. The sample spans the quarterly time period 1982Q4-2013Q4 ($N = 125$), except Case-Shiller which is 1987Q1-2013Q4. d is the fractional integration parameter, ϕ_i's are estimated AR parameters of p order and μ is the constant mean used in the regression.'*', '**' and '***' indicate rejection of the null at the 10%, 5% and 1% levels respectively. Numbers in parentheses are standard errors of the estimated parameters. The null hypothesis in the residual tests are normality, no ARCH effects and no autocorrelation. The unit root hypothesis is tested by the linear restriction $d = 1$ while the stationarity hypothesis is tested by $d = 0$. The $ARCH$ effects are analysed using Engle's ARCH 1-1 (F stats) and the autocorrelation using Portmanteau (χ^2 stats). The Linear Restrictions use a χ^2 test statistic with one degree of freedom. Shaded columns indicate the possibility of unit roots in d and thus housing price bubbles. Estimation is done using the $arfima$ package in OxMetrics, see Doornik and Ooms (2003). ρ is the sum of autoregressive coefficients, for a stationary process $-1 < \rho < 1$.

Both the Case-Shiller and the FHFA report long memory estimates ($\hat{d}$) greater than 1 suggesting explosive roots and bubble behaviour. This is confirmed by the Linear Restrictions on d, the null of stationarity ($d = 0$) is rejected while that of a unit root ($d = 1$) is not. These long memory tests thus improve upon the ADF and KPSS tests as the reported results are much clearer and is hence a consistent estimator for bubble testing.

The Census has a d value close to zero suggesting covariance-stationarity. Also, the null of a unit root is rejected while that of stationarity is not negating bubble behaviour. Unlike for FHFA and Case-Shiller the SARC ($\rho = 1$) here is non-stationary suggesting that while in the long run Census reverts to its mean there are significant bubble like deviations in the short term. This contrasting behaviour is expected as the Census unlike the FHFA and the Case-Shiller is a constant quality index meaning that any improvements to an owner occupied housing will reflect in its price. McCarthy and Peach (2004) also find that the Census unlike other HPI's reverts to its mean in the long run. The unit root persistence in the short run is probably due to the housing bubble that was observed in the 2006-07 period. It is also important that we examine whether the residuals are sensible for estimation in Table 2.2.

Residual Tests suggest the presence of non-normality and heteroskedasticity in all the aggregate series.[21] The presence of $ARCH$ errors leads us to question the usefulness of the $ARFIMA$ approach but we control for these errors later on.

We now turn to the disaggregate data given by the 12 metropolitan areas. Table 2.3 describe parametric estimates for 12 such Metropolitan Statistical Areas,i.e. three MSA's each from the four census regions:- Midwest, Northeast, West and South. In contrast to the aggregate δ_t's, all 12 regional MSA's with the exception of New York report d values less than unity. Inferences for unit root bubble behaviour are made from the p-values reported for the linear restrictions. As described earlier, a rejection of stationarity ($d = 0$) and an inability to reject the unit root null ($d = 1$) would imply the possibility of bubbles. These regions are shaded in grey. These include two MSA's from the Northeast (Boston, New York) and all three West areas (Los Angeles, San Francisco and Seattle). The non-shaded series comprising 2/3 from Midwest (Cleveland, Detroit), 1/3 from Northeast (Philadelphia) and 3/3 from the South (Atlanta, Dallas, Houston) shows unit root behaviour in the short run i.e. ρ=1 is not rejected. We can imply that these series probably contained bubble episodes but are now reverting to their long run mean.

Of interest is the aggregate FHFA series showing a persistence of $\hat{d} = 1.290$ much greater than any of the regional ones. This is suggestive of an aggregate bias in the construction of the FHFA House Price Index.

The residual tests in Table 2.3 for the disaggregate data again reported normal and ARCH errors in most of the series implying that we have to model the variance. The presence of heteroskedastic errors in repeats sales indexes (FHFA, Case-Shiller) as argued by Goodman and Thibodeau (1998) has to do with the timing of house sales. Goodman and Thibodeau (1998) examine whether the

[21] The presence of heteroskedasticity can bring severe problems. Under classical assumptions, ordinary least squares regression procedure (OLS) gives best linear unbiased estimators (BLUE). However, with heteroscedasticity, OLS estimators are unbiased but not best, i.e. they are not minimum variance. Additionally, the variance calculated by standard OLS procedures are biased and this implies that the standard tests (t, F, etc.) are unreliable, see White (1982)

Table 2.3. Parametric ARFIMA Estimation - FHFA Regional δ_t's

Est	**Midwest**			**Northeast**			**South**			West		
	Chicago $AR(3)MA(0)$	Cleveland $AR(2)MA(0)$	Detroit $AR(1)MA(1)$	Boston $AR(1)MA(1)$	New York $AR(2)MA(2)$	Philadelphia $AR(4)MA(0)$	Atlanta $AR(4)MA(0)$	Dallas $AR(2)MA(1)$	Houston $AR(2)MA(0)$	Los Angeles $AR(4)MA(0)$	San Francisco $AR(2)MA(0)$	Seattle $AR(2)MA(0)$
$\hat{d}$	0.764 (0.137)	0.088 (0.041)	0.577 (0.179)	0.996 (0.046)	1.000 (0.000)	0.685 (0.168)	0.517 (0.269)	0.102 (0.124)	0.223 (0.175)	0.979 (0.126)	0.830 (0.289)	0.648 (0.260)
$\hat{\phi}_1$	0.626 (0.137)	0.908 (0.028)	0.988 (0.021)	0.933 (0.040)	0.415 (0.118)	0.451 (0.174)	0.639 (0.300)	0.934 (0.080)	0.763 (0.158)	0.404 (0.147)	0.504 (0.264)	0.575 (0.264)
$\hat{\phi}_2$	-0.108 (0.102)	0.068 (0.022)			0.467 (0.122)	0.139 (0.126)	0.090 (0.114)	0.070 (0.082)	0.197 (0.127)	0.413 (0.104)	0.238 (0.150)	0.358 (0.174)
$\hat{\phi}_3$	0.423 (0.083)					0.356 (0.141)	0.345 (0.108)			-0.370 (0.103)		
$\hat{\phi}_4$						0.004 (0.234)	-0.082 (0.201)			-0.031 (0.039)		
MA_1			-0.358 (0.191)	-0.594 (0.084)	-0.074 (0.140)			-0.136 (0.114)				
MA_2					-0.257 (0.109)							
μ	1.529 (0.160)	0.369 (0.125)	0.306 (0.179)	0.732 (9.611)	-12.256 (12.65)	1.609 (0.252)	1.271 (0.106)	1.405 (0.476)	0.207 (0.219)	0.743 (4.272)	-0.521 (2.155)	1.410 (0.391)
ρ	0.941	0.976	0.988	0.933	0.892	0.950	1.074	1.004	0.960	0.413	0.742	0.933
Residual Tests												
Normality	0.001***	0.107	0.134	0.059*	0.093	0.000***	0.010**	0.000***	0.000***	0.000***	0.000***	0.000***
ARCH 1-1	0.807	0.334	0.000***	0.833	0.001***	0.875	0.134	0.661	0.884	0.583	0.954	0.964
Autocorrelation	0.656	0.732	0.0002***	0.333	0.168	0.997	0.172	0.060*	0.927	0.768	0.986	0.998
Linear Restr.												
$d=0$ [p-**value**]	0.000***	0.032**	0.000***	0.000***	0.000***	0.000***	0.054*	0.410	0.203	0.000***	0.004***	0.012**
$d=1$ [p-**value**]	0.086*	0.000***	0.018**	0.933	0.392	0.062*	0.073*	0.000***	0.000***	0.868	0.557	0.177
$\rho=1$	0.000***	0.201	0.584	0.098*	0.000***	0.147	0.230	0.421	0.134	0.000***	0.000***	0.000***

Notes: This table reports the Non-Linear Least Squares (NLS) parameter estimates for the 12 metropolitan areas belonging to the four census regions. The sample spans the quarterly time period 1982Q4-2013Q4 ($N = 125$). 2.56 is the equation for NLS and it is estimated on eq 2.69 for each region. d is the fractional integration parameter, ϕ_i's are estimated AR parameters of p order and μ is the constant mean used in the regression. Evaluation is done on the natural logs of real rent to price ratios of each market concurrent with the literature. '***', '**' and '*' indicate rejection of the null hypothesis at the 1%, 5% and 10% levels respectively. Numbers in parentheses are standard errors of the estimated parameters, p-values less than 0.05 indicate rejection of the null hypothesis at the 5% level. The null hypothesis in the residual tests are normality, no ARCH effects and no autocorrelation. The unit root hypothesis is tested by the restriction $d = 1$ while the stationarity hypothesis is tested by $d = 0$. The $ARCH$ effects are analysed using Engle's ARCH 1-1 (F stats) and the autocorrelation using Portmanteau (χ^2 stats). The Linear Restrictions use a χ^2 test statistic with one degree of freedom. Shaded columns indicate the possibility of bubbles in that region. ρ is the sum of autoregressive coefficients and indicates short memory of the series.

likelihood that houses have different-vintage renovations can contribute to heteroskedasticity. For example, owner occupied homes tend to be improved (by the seller or the buyer or both) at the time of sale. In their empirical analysis, they hypothesize that the shorter the time between sales, the less extensive the undocumented improvements are likely to be, and hence the more accurate the predictions of house prices and of subsequent appreciation rates can be. They report that the interval between two sales contributes significantly to the size of residual variance. Fletcher et al. (2000) extends the work of Goodman and Thibodeau (1998) and finds that heteroskedasticity is present in hedonic based house price indexes (Census). Their study concluded that the appearance of non-constant residual variance was not because of outliers in the data but due to heterogeneity of houses i.e. the variance of the disturbance term differed between types of property (detached, semi-detached, terraced) and the age of the property. These studies imply that heteroskedasticity is an inherent part of the house price indexes and hence, has to be accounted for in the estimation procedure.

To address the problem of $ARCH$ errors in the residuals, we include an $EGARCH$ term (see Bollerslev and Ole Mikkelsen (1996)) in the $ARFIMA$ model (2.69). Maximum Likelihood Estimation is done on an $ARFIMA(p, d, q)-EGARCH(1, 1)$ process.[22] The addition of the $EGARCH$ process implies the estimation of (2.69) along with the following equations:

$$\sigma_t^2 = \omega + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2. \tag{2.70}$$

$$\ln(\sigma_t^2) = \omega + [1 - \beta(L)]^{-1}[1 + \alpha(L)]g(\eta_{t-1}) \tag{2.71}$$

where

$$g(\eta_{t-1}) = \gamma_1 \eta_t + \gamma_2[|\eta_t| - E|\eta_t|] \tag{2.72}$$

We assume the innovations η_t to be normal so that $E|\eta_t| = \sqrt{2/\pi}$.

[22]Estimation of ARFIMA-EGARCH is done using the Simulated Annealing algorithm for optimizing non-smooth functions with possible multiple local maxima. Starting from an initial point, the algorithm takes a step and the function is evaluated. When minimizing a function, any downhill step is accepted and the process repeats from this new point. An uphill step may be accepted. Thus, it can escape from local optima. This uphill decision is made by the Metropolis criteria. As the optimization process proceeds, the length of the steps decline and the algorithm closes in on the global optimum. Since the algorithm makes very few assumptions regarding the function to be optimized, it is quite robust with respect to non-quadratic surfaces. Several initial values are tested to obtain strong convergence.

Tables 2.4 and 2.5 report likelihood estimates of the parameters for the national and the regional log rent-price ratio's when the variance follow an $EGARCH(1,1)$ process. In general, the estimated long memory is significantly different to those of the pure ARFIMA estimates (Tables 2.2 and 2.3). The magnitude and sign of the difference is largely dependent on normality of the residuals rather than heteroskedasticity. Efficient estimation of maximum likelihood relies on the assumption of Normal Distribution, in other words the series is linear. The pure $ARFIMA$ results (Tables 2.2 and 2.3) indicated non-normal residuals and hence those estimates are not efficient i.e. unit root bubble testing which relies on linear restrictions on d (F stat.) could be unreliable. The addition of the $EGARCH$ term in the estimation process ensured that the residuals are normal and free of $ARCH$ errors. However, the residuals in most series do indicate serial correlation implying the results are subject to this caveat.

Table 2.4. Parametric ARFIMA-EGARCH Estimation - National δ_t's

Est	FHFA	Case-Shiller	Census
	$AR(4)MA(0)$	$AR(4)MA(0)$	$AR(3)MA(0)$
$\hat{d}$	0.692 (0.000)	1.591 (0.250)	0.484 (0.216)
$\hat{\phi}_1$	0.617 (0.001)	0.483 (0.392)	0.297 (0.273)
$\hat{\phi}_2$	0.091 (0.001)	-0.043 (0.581)	0.406 (0.068)
$\hat{\phi}_3$	0.111 (0.001)	0.301 (0.346)	0.277 (0.222)
$\hat{\phi}_4$	0.193 (0.000)	0.259 (0.168)	
$\hat{\mu}$	2.907 (0.000)	11.815 (4.765)	0.372 (2.919)
$\hat{\omega}$	-9.538 (0.165)	-8.817 (0.262)	-8.345 (0.206)
$\hat{\alpha_1}$	0.225 (0.155)	1.247 (0.809)	1.041 (0.259)
$\hat{\beta_1}$	-0.724 (0.028)	0.145 (0.702)	-0.719 (0.092)
$\hat{\gamma_1}$	0.279 (0.120)	0.004 (0.304)	0.343 (0.149)
$\hat{\gamma_2}$	0.744 (0.146)	0.700 (0.613)	0.679 (0.528)
ρ	1.042	1.000	0.980
Residual Tests			
Normality	0.973	0.364	0.108
ARCH	0.418	0.611	0.954
Autocorrelation	0.000***	0.007***	0.120
Linear Restr.			
$d = 0$	0.000***	0.000***	0.025**
$d = 1$	0.000***	0.715	0.017**
$\rho = 1$	0.255	0.450	0.100

Notes: This table reports Maximum Likelihood Estimates of the $ARFIMA-EGARCH$ model on the national δ_t's. The sample spans the quarterly time period 1982Q4-2013Q4 ($N = 125$), except Case-Shiller which is 1987Q1-2013Q4. d is the fractional integration parameter, ϕ_i's are estimated AR parameters of p order, μ and ω are the constant mean and variance respectively. Numbers in parentheses are standard errors of the estimated parameters. The null hypothesis in the residual tests are normality, no ARCH effects and no autocorrelation. '***', '**' and '*' indicates rejection of the null at the 1%, 5% and 10% levels, respectively. The unit root hypothesis is tested by the linear restriction $d = 1$ while the stationarity hypothesis is tested by $d = 0$. The $ARCH$ effects are analysed using Engle's ARCH 1-1 (F stats) and the autocorrelation using Portmanteau (χ^2 stats). The Linear Restrictions use a χ^2 test statistic with one degree of freedom. Shaded columns indicate the possibility of housing bubbles. Estimation is done using the $arfima$ package in OxMetrics, see Doornik and Ooms (2003). ρ is the sum of autoregressive coefficients, for a stationary process $-1 < \rho < 1$.

In contrast to the pure $ARFIMA$ results, only Case-Shiller in Table 2.4 amongst the aggregate δ_t's exhibits unit root in long memory (and also in short memory). FHFA is observed to have stationary d although there is a strong non-stationarity in the short memory component highlighted by the high value of ρ (1.042). Among the 12 MSA's in Table 2.5 only three (New York, Los Angeles and Seattle) exhibit long run unit root persistence ($d = 1$ is not rejected). The rest of the nine MSA's are stationary in d but possess unit root in the short memory AR component i.e. $\rho = 1$ is not rejected. In general, we can deduce that accounting for heteroskedasticity reduces the number of series which exhibited

Table 2.5. Parametric ARFIMA-EGARCH Estimation - FHFA Regional δ_t's

Est	**Midwest**			**Northeast**			**South**			West		
	Chicago $AR(3)MA(0)$	Cleveland $AR(2)MA(0)$	Detroit $AR(1)MA(1)$	Boston $AR(1)MA(1)$	New York $AR(2)MA(2)$	Philadelphia $AR(4)MA(0)$	Atlanta $AR(4)MA(0)$	Dallas $AR(2)MA(1)$	Houston $AR(2)MA(0)$	Los Angeles $AR(4)MA(0)$	San Francisco $AR(2)MA(0)$	Seattle $AR(2)MA(0)$
$\hat{d}$	0.841 (0.172)	0.378 (0.076)	0.276 (0.067)	0.295 (0.073)	1.039 (0.119)	0.667 (0.078)	0.334 (0.005)	0.267 (0.466)	0.244 (0.000)	1.175 (0.062)	0.561 (0.001)	1.345 (0.002)
$\hat{\phi}_1$	0.579 (0.145)	0.549 (0.006)	0.995 (0.007)	0.969 (0.011)	0.598 (0.002)	0.499 (0.079)	0.713 (0.040)	1.073 (0.209)	0.848 (0.001)	0.324 (0.084)	1.284 (0.007)	0.093 (0.039)
$\hat{\phi}_2$	-0.159 (0.131)	0.448 (0.006)			0.278 (0.002)	-0.037 (0.097)	0.074 (0.005)	-0.073 (0.210)	0.105 (0.001)	0.282 (0.080)	-0.312 (0.007)	0.261 (0.000)
$\hat{\phi}_3$	0.486 (0.118)					0.402 (0.076)	0.278 (0.038)			0.191 (0.043)		
$\hat{\phi}_4$						0.088 (0.085)	-0.066 (0.063)			-0.275 (0.116)		
MA_1					-0.444(0.119)			-0.449 (0.358)				
MA_2					0.100 (0.082)							
μ	1.175 (0.293)	-4.695 (0.076)	-1.812 (3.935)	0.290 (0.295)	0.664 (0.000)	3.209 (2.102)	41.517 (0.006)	10.049 (6.263)	0.206 (0.000)	1.375 (0.013)	-0.523 (0.001)	1.424 (0.023)
$\hat{\omega}$	-8.724 (0.151)	-8.750 (0.226)	-8.378 (0.225)	-8.148 (0.119)	-8.347 (0.159)	-7.942 (0.225)	-8.856 (0.156)	-8.424 (0.174)	-7.570 (0.553)	-8.272 (0.239)	-7.666 (0.271)	-8.321 (0.272)
$\hat{\alpha}_1$	2.934 (4.490)	0.729 (0.792)	2.234 (1.573)	-0.143 (0.648)	1.076 (0.287)	2.050 (0.594)	-0.002 (0.222)	-0.035 (1.463)	-0.702 (0.174)	0.785 (0.267)	0.700 (0.100)	0.823 (0.078)
$\hat{\beta}_1$	-0.515 (0.278)	0.455 (0.172)	0.352 (0.137)	0.103 (0.090)	-0.747 (0.046)	-0.823 (0.069)	-0.459 (0.083)	-0.530 (0.119)	0.786 (0.113)	-0.427 (0.361)	-0.421 (0.085)	-0.757 (0.066)
$\hat{\theta}_1$	-0.037 (0.044)	-0.181 (0.094)	0.071 (0.074)	-0.146 (0.209)	0.054 (0.102)	-0.389 (0.236)	0.274 (0.089)	0.152 (0.183)	-1.011 (0.376)	-0.098 (0.173)	-0.369 (0.204)	-0.332 (0.102)
$\hat{\theta}_2$	0.167 (0.319)	0.633 (0.379)	0.296 (0.203)	-0.363 (0.333)	0.715 (0.194)	0.229 (0.088)	0.628 (0.299)	0.136 (0.326)	1.337 (0.792)	1.675 (0.571)	1.489 (0.269)	1.419 (0.244)
ρ	0.906	0.997	0.995	0.969	0.876	0.952	1.002	1.000	0.953	0.522	0.972	0.354
Residual Tests												
Normality	0.660	0.305	0.127	0.258	0.145	0.000***	0.291	0.013**	0.000***	0.011**	0.045**	0.666
ARCH	0.565	0.418	0.937	0.989	0.971	0.840	0.766	0.982	0.861	0.289	0.808	0.917
Autocorrelation	0.368	0.017**	0.008***	0.105	0.001***	0.225	0.028**	0.560	0.560	0.062	0.053*	0.014**
Linear Restr.												
$d=0$ [**p-value**]	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***	0.000***
$d=1$ [**p-value**]	0.086*	0.000***	0.000***	0.000***	0.742	0.000***	0.000***	0.000***	0.000***	0.696	0.000***	0.152
$\rho=1$	0.102	0.225	0.510	0.250	0.001***	0.104	0.525	0.725	0.101	0.000***	0.108	0.000***

Notes: This table reports Maximum Likelihood Estimates of the $ARFIMA-EGARCH$ model on the national δ_t's. The sample spans the quarterly time period 1982Q4-2013Q4 ($N=125$), except Case-Shiller which is 1987Q1-2013Q4. d is the fractional integration parameter, ϕ_i's are estimated AR parameters of p order, μ and ω are the constant mean and variance respectively. Numbers in parentheses are standard errors of the estimated parameters. The null hypothesis in the residual tests are normality, no ARCH effects and no autocorrelation. '***', '**' and '*' indicates rejection of the null at the 1%, 5% and 10% levels, respectively. The unit root hypothesis is tested by the linear restriction $d=1$ while the stationarity hypothesis is tested by $d=0$. The $ARCH$ effects are analysed using Engle's ARCH 1-1 (F stats) and the autocorrelation using Portmanteau (χ^2 stats). The Linear Restrictions use a χ^2 test statistic with one degree of freedom. Shaded columns indicate the possibility of housing bubbles. Estimation is done using the $arfima$ package in OxMetrics, see **?**. ρ is the sum of autoregressive coefficients, for a stationary process $-1<\rho<1$.

bubble behaviour. Specifically, the aggregate FHFA and the regional MSA's of Boston and San Francisco no longer show unit root persistence in the long memory. This explicitly depicts the effects that non-normal and $ARCH$ errors has on the d value. Parametric methods imply there is no considerable evidence of bubbles in housing prices.

Hypothesis tests based on asymptotic theory can be misleading when you have a small finite number of observations, 125. The likelihood based parametric $ARFIMA$ estimates could be biased and confidence levels for Wald tests may deviate significantly from the normal levels. One way to address this issue is to use parametric bootstrap methods to investigate the comparative performance of the estimates in a similar sample. Parametric bootstrap inference can be used to test for any value of d. For example in related literature on parametric long memory estimation, Koustas and Serletis (2005) generate 1000 pseudo-samples by drawing from completely specified ARFIMA data generating processes with independent normal errors. For each pseudo-sample they compute parameter estimates and associated t-values for tests on the true data generating process.

Although the use of parametric procedure separates the long run persistence from short run dynamics, the results as we found are sensitive to both normality and conditional heteroskedasticity. This motivates our use of semi-parametric methods in the frequency domain that are robust to both these errors.

2.7.3. Evidence for Bubbles - Semi-parametric Estimation

We described in detail three semi-parametric methods to estimate the value of the long memory parameter d, namely the Local Whittle Estimator, $\hat{d}_{LWE}$, the Exact Local Whittle Estimator $\hat{d}_{ELW}$ and the 2-step Exact Local Whittle Estimator, $\hat{d}_{2ELW}$. While $\hat{d}_{LWE}$ is not consistent when $d > 0.5$, the $\hat{d}_{ELW}$ and $\hat{d}_{2ELW}$ in contrast provide good estimates even in the non-stationary region. Importantly, all the three Whittle based estimation methods are robust to conditional heteroskedasticity and non-normality, see Robinson and Henry (1999), Henry (2001) and Nielsen and Frederiksen (2005). However, a drawback of using these semi-parametric procedures is that the estimate depends on the value of the bandwidth parameter or Fourier frequency m, see Baillie (1996).[23]

[23] Semi-parametric estimates have a slower rate of convergence than parametric ones but better robustness properties. However, parametric estimates are consistent under short samples, see Robinson and Henry (1999).

To address this issue, we follow Kumar and Okimoto (2007) and choose m based on simulations. We simulate $Y_t = (1-L)^{-d}\epsilon_t$, where ϵ_t is a Gaussian white noise process, with sample size 125.[24] The optimal m is the one which minimizes the sample Mean Squared Error for several choices of m, that is $m = \{n^{0.60}, n^{0.65}, n^{0.70}, n^{0.75}, n^{0.80}\}$. Based on the simulation results, we select $m = 0.75$ for both 2ELW and ELW and $m = 0.80$ for the LWE. The simulation results also showed that the 2-step ELW estimator gave consistent estimates which were closer to the true value of d. Armed with the optimal m, we estimate the memory parameter using the three Whittle procedures. The results are reported in Table 2.6. We include the parametric results for comparison.

In general, it is seen in Table 2.6 that the Exact Local Whittle methods, $\hat{d}_{ELW}$ and $\hat{d}_{2ELW}$, report a higher persistence value compared to the Local Whittle procedure. This is expected as the Local Whittle Estimate, is less preferred and $\hat{d}_{LWE}$, gives over-differenced values when the series is non-stationary, see Shimotsu and Phillips (2005).

Unit root testing on the frequency domain Whittle estimates is implemented by the application of the Efficient Fractional Dicky-Fuller Test (EFDF) of Lobato and Velasco (2007). The EFDF tests the null of a unit root ($d = 1$) against the alternative of a fractional root ($d = \hat{d} < 1$). The numbers in bold indicate that the null of a unit root was not rejected.

It is observed that only three aggregate or disaggregate series exhibit unit root behaviour consistent with price bubbles. These are the log rent-price ratio's of the aggregate Census and the Southern MSA's of Dallas and Houston. This is despite the fact that the 2-step ELW estimates are higher than unity for most of the series. This is because the EFDF that examines unit root behaviour can only accommodate stationary values of d in the alternative, i.e. $d < 1$. To the best of our knowledge, right-tailed unit root tests in the frequency domain do not exist in the literature. In order to overcome this limitation, we impose a Linear Restriction of $d = 1.5$, i.e. an explosive root in the frequency domain. $I(d > 1)$ in column 5 indicates that a null of an explosive root was not rejected or in other words a bubble exists.

[24] This is the sample size of our dataset.

Table 2.6. Semi-parametric and Parametric Estimates of National and Regional δ_t's

Housing Market	**Semi-Parametric Estimates**				**Parametric Estimates**				
	$\hat{d}_{LWE}$	$\hat{d}_{ELW}$	$\hat{d}_{2ELW}$	Conclusion	$\hat{d}_{ARFIMA}$	ρ_{ARFIMA}	$\hat{d}_{ARFIMA-EGARCH}$	$\rho_{ARFIMA-EGARCH}$	Conclusion
FHFA	0.965	1.077	1.368	$I(d>1)$	**1.290**	0.517	0.692	**1.042**	$I(0)$
Case-Shiller	0.951	1.059	1.615	$I(d>1)$	**1.276**	0.794	**1.591**	**1.000**	$I(1)$
Census	**0.924**	**1.031**	**1.192**	$I(1)$	0.074	**0.983**	0.484	**0.980**	$I(0)$
Midwest									
Chicago	0.962	1.039	1.343	$I(d>1)$	**0.764**	0.941	**0.841**	0.906	$I(1)$
Cleveland	0.955	1.034	1.195	$I(d>1)$	0.088	**0.976**	0.378	**0.997**	$I(0)$
Detroit	0.930	1.037	1.309	$I(d>1)$	0.577	**0.988**	0.276	**0.995**	$I(0)$
Northeast									
Boston	0.934	1.034	1.487	$I(d>1)$	**0.996**	0.933	0.295	**0.969**	$I(0)$
New York	0.989	1.050	1.568	$I(d>1)$	**1.000**	0.892	**1.039**	0.876	$I(1)$
Philadelphia	1.096	1.118	1.376	$I(d>1)$	0.685	**0.950**	0.667	**0.952**	$I(0)$
South									
Atlanta	0.960	1.042	1.244	$I(d>1)$	0.517	**1.074**	0.334	**1.002**	$I(0)$
Dallas	**0.941**	**1.053**	**1.074**	$I(1)$	0.102	**1.004**	0.267	**1.000**	$I(0)$
Houston	**1.069**	**1.203**	**1.221**	$I(1)$	0.223	**0.960**	0.244	**0.953**	$I(0)$
West									
Los Angeles	1.086	1.161	1.690	$I(d>1)$	**0.979**	0.413	**1.175**	0.522	$I(1)$
San Francisco	1.120	1.147	1.577	$I(d>1)$	**0.830**	0.742	0.567	**0.972**	$I(0)$
Seattle	1.139	1.177	1.458	$I(d>1)$	**0.648**	0.933	**1.345**	0.354	$I(1)$

Notes: This table reports semi-parametric estimates of d for the log rent-price ratios of the three national HPI's and 12 MSA's. The sample spans the quarterly time period 1982Q4-2013Q4 except for Case-Shiller which is 1987Q1-2013Q4. The optimal frequency, m, selected by simulations is $n^{0.80}$ for LWE and $n^{0.75}$ for ELW and 2ELW (de-trended), where n is the sample size i.e. 125. The asymptotic standard error for LWE is 0.072 and for ELW and 2ELW is 0.081. Unit root test for the semi-parametric whittle estimates is done by implementing the Efficient Dicky-Fuller Test (EFDF). This tests for the null of a unit root ($d = 1$) against the alternative of fractional roots ($d = \hat{d} < 1$). For the parametric procedure, Linear Restrictions on d and ρ acts as unit root tests. Numbers in bold indicate that the null of a unit root cannot be rejected. Parametric estimates are extracted from Tables 2.2, 2.3, 2.4 and 2.5.

We can summarize that the log rent-price ratio's of all three aggregate and 12 MSA's follow a process consistent with housing bubbles. *These results contrast strongly with the parametric test.* Barros et al. (2012) estimate Whittle and log periodogram estimates on regional FHFA HPI's of several U.S. States and find long memory values of $d > 1$ in most of them. They also implement semi-parametric Whittle estimate on FHFA national HPI in the quarterly span of 1975:1-2010:7 and get a value of $\hat{d}_{LWE} = 1.500$ and a parametric value of 1.478. They reject the null of a unit root. However, they did not consider either explosive alternatives or the rental series in their analysis.

The three MSA's from the Northeast region (Boston, New York and Philadelphia) on average reported higher than unity persistence in the long memory based on the $2ELW$ method (and also the parametric $EGARCH$ one for New York). In general there is widespread consensus in the literature (see Case and Shiller

(2003); Case et al. (2012)) and anecdotal evidence for self-fulfilling price expectations or housing bubbles in the Boston and New York metropolises.

One reason for such high persistence in these cities, as argued by Gyourko et al. (2013), is that the marginal home buyers in "superstar" cities are high income household who have moved from other parts of the city. This pattern would imply that the median homes in such cities are purchased by new residents whose income exceeds that of the median income. Furthermore, our result provide empirical validity to arguments by Green et al. (2005) and others who hypothesize that house price appreciation depends largely on elasticity of housing supply. They compute supply elasticities for 45 MSA's and find that densely populated regions like New York and Los Angeles have highly inelastic housing supply. This explains why we obtain higher than average d values in the Northeast and West MSA's.

Columns 6-9 in Table 2.6 report the parametric results extracted from the previous section. Comparisons can be made using the two most reliable estimators, the semi-parametric $\hat{d}_{2ELW}$ and the parametric $\hat{d}_{ARFIMA-EGARCH}$. It is evident that the estimated values of long memory by the 2-step Whittle is significantly larger than the $ARFIMA - EGARCH$ one. Most of the persistence for the parametric procedure is concentrated in the AR part reflected by near unity values of ρ. The Census series is a perfect example for this phenomenon. Here the 2-step ELW gave a unit root long memory value of 1.192 while the $ARFIMA-EGARCH$ procedure estimated d as 0.484 which is stationary. This is primarily because of the short sample size we use, for small samples most of the persistence will be carried by the short run $ARMA$ components resulting in low values of d. The persistence in the short run indicated by ρ for the Census was 0.980 with a unit root. Furthermore, our results from the parametric $ARFIMA-EGARCH$ model is not completely reliable as the residual tests did indicate normal and autocorrelation errors in some of the series. Nevertheless, we caveat both the parametric and the non-parametric results to the presence of structural breaks which if present can induce spurious long memory. It is imperative that we examine for any endogenous breaks and account for it in our estimation.

2.7.4. Evidence for Bubbles - Structural Changes

So far we have identified evidence of rational bubbles in house prices using our most reliable method, namely, the semi-parametric tests. Structural breaks in time series generate slowly decaying autocorrelations and can thus generate spurious long memory behaviour. In this section, we test for the presence of abrupt breaks in the mean and trend of the log rent-price ratio's. We then estimate the d parameter after adjusting for these breaks.

2.7.4.1. Estimating Break in Mean and Trend

To complement the ocular evidence presented in the plots of the national and regional rent-price ratios (Figure 2.1 and Figure 2.B.1) we implement three tests, namely the *SupF* test of Andrews (1993) and the *ExpF* and *AveF* of Andrews and Ploberger (1994) (refer eq. (2.52), (2.53) and (2.54) from §2.5.1) to test for possible breaks in the mean and trend of each series. Inference for presence or absence of breaks is based on Hansen (2000)'s bootstrap heteroskedasticity corrected p-values. We limit our analysis to one break considering the relatively short sample size we are estimating on.

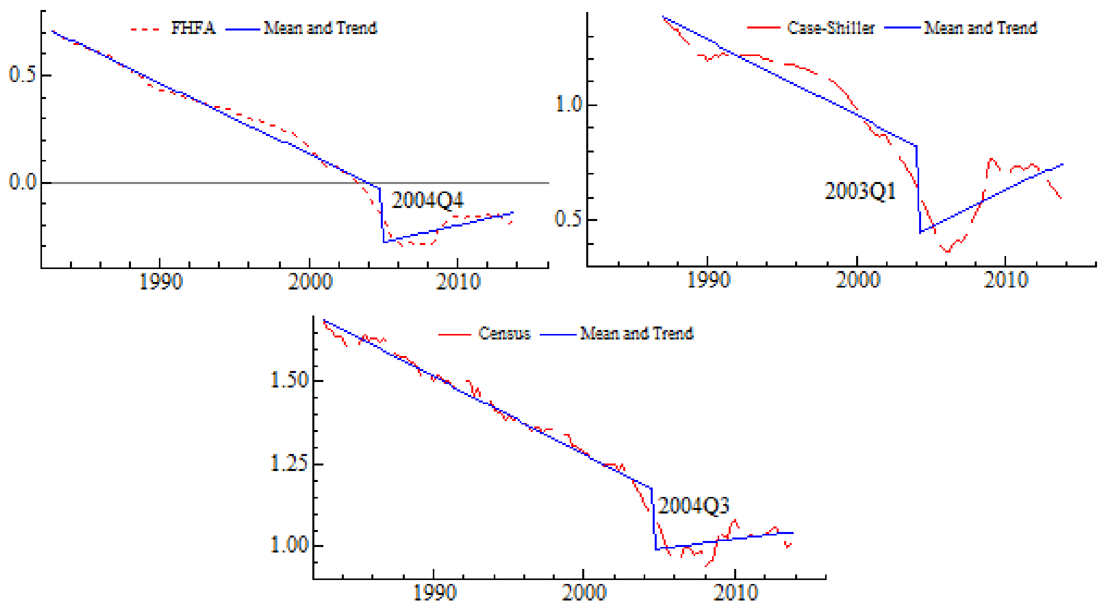
Table 2.7. Structural Change Tests on National and Regional δ_t's

Housing Market	*SupF*			*ExpF*			*AveF*			Breakdate
	Test Stat.	Bootstrap p	Hetero- p	Test Stat.	Bootstrap p	Hetero- p	Test Stat.	Bootstrap p	Hetero- p	
FHFA	488.186	0.000***	0.000***	239.635	0.000***	0.000***	83.092	0.000***	0.000***	2004Q4
Case-Shiller	210.574	0.000***	0.000***	101.105	0.000***	0.000***	64.012	0.000***	0.000***	2003Q1
Census	348.712	0.000***	0.000***	170.718	0.000***	0.000***	60.832	0.000***	0.000***	2004Q3
Midwest										
Chicago	466.886	0.000***	0.000***	229.448	0.000***	0.000***	117.522	0.000***	0.000***	2003Q2
Cleveland	795.594	0.000***	0.000***	394.258	0.000***	0.000***	350.387	0.000***	0.000***	2003Q2
Detroit	1317.230	0.000***	0.000***	654.215	0.000***	0.000***	515.738	0.000***	0.000***	2003Q1
Northeast										
Boston	65.951	0.000***	0.001***	29.831	0.000***	0.001***	33.870	0.000***	0.000***	1989Q4
New York	90.028	0.000***	0.000***	41.159	0.000***	0.000***	34.680	0.000***	0.000***	2004Q4
Philadelphia	158.642	0.000***	0.000***	75.380	0.000***	0.000***	40.127	0.000***	0.000***	2004Q4
South										
Atlanta	564.129	0.000***	0.000***	277.694	0.000***	0.000***	132.966	0.000***	0.000***	2003Q4
Dallas	255.068	0.000***	0.000***	123.601	0.000***	0.000***	152.733	0.000***	0.000***	1999Q4
Houston	100.265	0.000***	0.000***	46.521	0.000***	0.000***	64.974	0.000***	0.000***	2004Q1
West										
Los Angeles	161.429	0.000***	0.000***	76.369	0.000***	0.000***	37.796	0.000***	0.000***	2005Q1
San Francisco	128.458	0.000***	0.000***	60.427	0.000***	0.000***	41.437	0.000***	0.000***	2004Q4
Seattle	302.679	0.000***	0.000***	147.719	0.000***	0.000***	85.516	0.000***	0.000***	2004Q4

Notes: This table reports the test statistics of the *SupF* test of Andrews (1993) and the *ExpF* and *AveF* of Andrews and Ploberger (1994) (refer eq. (2.52), (2.53) and (2.54) from §2.5.1) and Hansen (2000) bootstrap p values under homoskedastic (Bootstrap p in the table)and heteroskedastic (Hetero- p in the table) residuals. All the three tests examine the null of no breaks against the alternative of breaks in the mean and trend. The trimming parameter is set at 0.15 which means the starting index for break search is 18.

Table 2.7 reports the test statistics of the three break tests ($SupF$, $AveF$ and $ExpF$) along with Hansen (2000)'s bootstrap p values under homoskedastic and heteroskedastic residual processes. All the three tests examine the null of no change against the alternative of a structural change in the mean (μ) and trend (ψ) of the log rent-price ratio's. The reported p values uniformly reject the null hypothesis at the 1% level strongly implying the presence of a structural break in the mean and trend of all the log rent-price series. The estimated breakdate for the national series is illustrated in Fig. 2.1.

Figure 2.1. Estimated Breakdates of National Rent-Price Ratio's



Notes: This figure provides a graphical illustration of the estimated breaks in mean and trend of the log rent-price ratio's of FHFA, Case-Shiller and Census aggregate series. The break date of each series is given in text in the corresponding plots.

A brief review of the estimated breakdates reveal that in general they are centered around 2003Q1-2004Q4. On visual inspection it is clear that there is a marked change in the trajectory of the log rent-price ratio's between 1982Q4-2003Q1 and then on. This is consistent with a sudden upsurge followed by downturn in housing prices highlighted by a U shape in the δ_t trajectory from 2004Q4. Our structural change test revealed one endogenous break in all of the series. The breakdate was found to lie in general in the 2003-04 time period.

This breakdate suggests a shift in household beliefs. For example, Piazzesi and Schneider (2009) using Michigan Consumer Survey data finds that the U.S. housing boom had two distinct stages. In the first stage, during 2002-03, about 72% of households cited favourable credit conditions and believed that the time for buying a house was good. From 2004, in the second stage, houses were considered too expensive but the number of agents who were optimistic about future price increased from 10% in 2003Q4 to over 20% by 2005Q2. Our results thus empirically validate these arguments.

From the breakdates for the regional MSA's, it is apparent that they mirror the national series i.e. the breakdates found are in and around the 2004-05 time period. The exception to this was Boston (1989Q4) and Dallas (1999Q4). This divergent behaviour we believe has more to do with regional rental changes rather than house prices. Severe regulations cap the rent you can charge on residential households in several MSA's in the United States which includes the Boston and Dallas metropolises.

A recent forecasting paper by Barari et al. (2014) found four structural breaks in the aggregate Case-Shiller Index (1991:1-2009:12) by applying the Bai and Perron (1998, 2003) procedure. It is well documented that the Bai and Perron (1998) procedure could overestimate the number of breaks when the regressors are non-stationary. Canarella et al. (2011) applied the Lumsdaine and Papell (1997) and the Lee and Strazicich (2001) tests to the Case-Shiller 10 city regional HPI's and found two breaks in the intercept and trend for all the 10 metro areas. Specifically the tests indicated that the second breakdate for the most of the metro areas (Chicago, Los Angeles, San Francisco, New York) was found to occur around 2005-2006 time period concurring with our results. Both these papers however do not take into account the effects of the rental series. Nevertheless, Nneji et al. (2013) use a Markov switching approach to the aggregate FHFA price-rent ratio in the period 1960-2011 and finds that the housing market switched from a low price-rent ratio to a high price-rent ratio around the year 2000.

Now that we have identified breaks we proceed to a robust estimation of long memory on the de-meaned and de-trended series, i.e. $\delta_t - \hat{\mu} - \hat{\psi}$. The next section reports and discusses these results.

2.7.4.2. Long Memory Estimation on De-meaned and De-trended δ_t's

Table 2.8. Long Memory on De-meaned and De-trended δ_t

Housing Market	Semi-Parametric Estimates				Parametric Estimates				
	$\hat{d}_{LWE}$	$\hat{d}_{ELW}$	$\hat{d}_{2ELW}$	Conclusion	$\hat{d}_{ARFIMA}$	ρ	$\hat{d}_{ARFIMA-EGARCH}$	$\rho_{ARFIMA-EGARCH}$	Conclusion
FHFA	0.646	0.662	0.662	$I(0)$	-0.107	**0.834**	0.687	0.337	$I(0)$
Case-Shiller	**0.864**	**1.078**	**1.079**	$I(1)$	0.185	**0.815**	0.738	0.645	$I(0)$
Census	0.585	0.645	0.652	$I(0)$	0.037	0.645	-0.184	0.743	$I(0)$
Midwest									
Chicago	**0.772**	**0.906**	**0.930**	$I(1)$	0.016	**0.874**	**0.803**	0.172	$I(1)$
Cleveland	**0.880**	**1.007**	**0.991**	$I(1)$	0.046	**0.909**	0.134	0.911	$I(0)$
Detroit	**0.900**	**1.137**	**1.123**	$I(1)$	0.105	**0.960**	0.207	0.811	$I(0)$
Northeast									
Boston	**0.965**	**1.089**	**1.092**	$I(1)$	0.149	**0.929**	**1.233**	-0.101	$I(1)$
New York	**0.978**	**1.055**	**1.136**	$I(1)$	0.104	**0.945**	0.106	0.919	$I(0)$
Philadelphia	**0.768**	**1.090**	**1.032**	$I(1)$	**0.620**	0.329	0.192	0.772	$I(0)$
South									
Atlanta	0.668	0.692	0.691	$I(0)$	0.412	0.377	0.637	0.278	$I(0)$
Dallas	0.840	0.907	0.915	$I(0)$	-0.060	**0.900**	-0.276	0.883	$I(0)$
Houston	**0.673**	**0.905**	**0.940**	$I(1)$	0.021	**0.841**	-0.152	0.894	$I(0)$
West									
Los Angeles	**1.008**	**1.140**	**1.132**	$I(1)$	0.130	**0.877**	0.666	0.977	$I(0)$
San Francisco	**0.910**	**1.082**	**1.060**	$I(1)$	0.103	**0.895**	0.122	0.907	$I(0)$
Seattle	**0.770**	**0.949**	**0.972**	$I(1)$	0.118	**0.823**	0.746	0.837	$I(0)$

Notes: Evaluation is done on the residual equation $y_t - \hat{\mu} - \hat{\psi}$. Detailed description of the detrending and demeaning procedure is described in section 2.4. The optimal frequency, m, selected by simulations is $n^{0.80}$ for LWE $n^{0.75}$ for ELW and 2ELW, n is the sample size i.e. 125. The asymptotic standard error for LWE is 0.072 and for ELW and 2ELW is 0.081. For the parametric estimation, $AR(1)MA(0)$ is the optimal model selected by BIC. The residual tests indicated the presence of normal errors in all the series. Addition of the $EGARCH$ process removed $ARCH$ and autocorrelation errors but not the normal errors. Unit root tests for semi-parametric methods is done by applying the EFDF test and for the parametric method by applying Linear Restrictions, that is $d = 1$ and $\rho = 1$. Numbers in bold indicate that the null of a unit root was not rejected at the 5% level. The test statistics and critical values of the EFDF test is reported in Table 2.C.1.

Table 2.8 reports semi-parametric (i.e. $\hat{d}_{LWE}$, $\hat{d}_{ELW}$ and $\hat{d}_{2ELW}$) and parametric ($ARFIMA$ and $ARFIMA - EGARCH$) persistence estimates for each series after adjusting for structural breaks. Estimation is done on the filtered demeaned and detrended series, $\delta_t - \hat{\mu} - \hat{\psi}$ where is the $\hat{\mu}$ mean and $\hat{\psi}$ is the trend. As expected, the $\hat{d}$ values for the de-trended series are much lower validating the arguments by Diebold and Inoue (2001) and Granger and Hyung (2004) that the presence of level or trend shifts could spuriously generate long memory.

Unit root tests for the semi-parametric estimates are conducted using the Efficient Fractional Dicky-Fuller (EFDF) tests of Lobato and Velasco (2007). The test statistics and critical values for this test is reported in Table 2.C.1 in the Appendix. Linear restrictions of $d = 1$ and $\rho = 1$ effectively tests for the null of a unit root in the long run and in the short run AR process in the case of the parametric estimation. Bold numbers in Table 2.8 indicate regions where the null of a unit root was not rejected at the 5% level. Columns 5 and 10 in the Table describe the conclusions we can draw from analysing the two most reliable estimators, the 2-step ELW and the $ARFIMA - EGARCH$.

We find that the semi-parametric procedure performs better than the parametric one. The residual tests of the $ARFIMA-EGARCH$ procedure indicated the presence of non-normality which implies that the likelihood estimates that we recorded in Table 2.8 are inefficient and unit root tests, $d = 1$, unreliable. In fact, parametric estimation on some series (the aggregate Census, South MSA's of Dallas and Houston) produced negative values of d i.e. anti-persistence. Granger and Hyung (2004) also find negative or over differenced d values on a de-meaned Standard & Poor 500 stock returns series. This is probably due to the existence of some form of nonlinearity, such as a smooth transition, a nonlinear trend, etc. in the series which is not adequately captured by the linear $ARFIMA$ model. The semi-parametric methods on the other hand are robust to non-normal and heteroskedastic errors. As discussed earlier in §2.7.2, the studies of Goodman and Thibodeau (1998) and Fletcher et al. (2000) prove that both repeat sales and hedonic based house price indices suffer from heteroskedasticity. The addition of the $EGARCH$ term takes care of the heteroskedasticity problem. However, the issue of non-normal residuals still remain unresolved.

Furthermore, data limitation of the rental series meant that we used a small sample size in our analysis, $T = 125$. In the parametric procedures, the $ARMA$ part captures most of the persistence resulting in low d values. There are additional problems we encounter during the optimization of the $ARFIMA - EGARCH$ model. We use the Simulated Annealing Algorithm for optimization which requires specifying starting values. We simulated different values to get strong convergence. However, despite these efforts some series only converged weakly.

While the semi-parametric procedure detected unit root persistence consistent with bubbles in two national (Case-Shiller and the NAR Housing Affordability Index) and 10 regional series, the parametric procedure identified bubble behaviour in only two regional MSA's (Chicago and Boston). None of the aggregate indexes showed any unit root tendencies under the parametric method.

Comparing the three national indexes (FHFA, Case-Shiller and Census) we arrive at two main conclusions.[25] Firstly, we find substantial heterogeneity in the persistence of the aggregate HPI's. The semi-parametric estimates reveal that the Case-Shiller HPI, unlike the FHFA and the Census HPI's, follows a unit root process consistent with housing bubbles. The most reliable estimated long memory value of $d = 1.079$ for Case-Shiller in Table 2.8 suggests an explosive process agreeing with Phillips and Yu (2011) who implemented a sequential right-tailed unit root test on Case-Shiller price-rent ratio and rejected the null of a unit root against an explosive alternative consistent with housing bubbles. Although the $ARFIMA - EGARCH$ model rejects a unit root here, the computed d value (0.738) is the highest among all the three aggregate HPI's. This contrasting behaviour between Case-Shiller and the other two aggregate series (FHFA and Census), we believe, lies in the way the three HPI's are constructed. Although both FHFA and Case-Shiller use a weighted repeat sales methodology, the HPI of FHFA is based only on homes sales with conforming home mortgages (loans less than $417,000), which eliminates a fair percentage of real estate transactions. Case-Shiller looks at all home sales, regardless of the mortgage amount. Furthermore, the S&P Case-Shiller Indexes are value-weighted, meaning that price trends for more expensive homes have greater influence on estimated price changes than other homes. FHFA, on the other hand, weights price trends equally for all properties. The geographic coverage of the indexes also differs. The S&P Case-Shiller National Home Price Index, for example, does not have valuation data from 13 states. FHFA's aggregate index is calculated using data from all the states.

Secondly, among all the three aggregate HPI's, the estimated long memory persistence, by both d_{2ELW} and $d_{ARFIMA-EGARCH}$, are lowest in the Census Index. This mirrors the result in McCarthy and Peach (2004) who also found that the price deviations from fundamentals in the Census HPI is much lower than either FHFA or Case-Shiller as the Census is a constant-quality index. That is, both the FHFA and the Case-Shiller Index do not take into account changes in the physical characteristics of homes and so does not control for depreciation or additions and alterations between sale dates that could have changed the quality and thus price of the house. In essence, Census accounts for the heterogeneity among

[25] Although we are estimating on the rent-price ratio's we can make effective comparisons between the three national HPI's as the national rental series (in the numerator) is the same. That is, any appreciation or depreciation differences between the three δ_t's has more to do with the particular HPI variations than any aggregate rental changes.

houses.[26] Furthermore, both Case-Shiller and FHFA use a "repeat sales" methodology to examine house price changes which has its own important caveats. Most importantly, the index is based only on the sample of homes that have sold at least twice (hence the term "repeat sales"), a fact which serves to exclude all new construction (which can account for more than 10% of real estate transactions). For these and other reasons, McCarthy and Peach (2004) argue that the Census series is the most appropriate among all the three HPI's to infer on house price appreciation relative to fundamentals such as rents in the United States. Case et al. (1991) compares the hedonic methodology with the repeat sales one and finds that the hedonic based HPI possess lower bias and inefficiency problems.

As the log rent-price ratio of the Census Index in the 31 year time span 1982Q4-2013Q4 is found to follow a stationary mean-reverting long memory process, we can say that in the national level the United States is devoid of bubble behaviour i.e. the housing market is efficient agreeing with Capozza and Seguin (1996), Linneman (1986) and Meese and Wallace (1994) who observe that in the long run the aggregate United States Housing Market is efficient in that they follow the present value relation. They believe that high transaction costs, measurement errors and failure to account for a risk premium in the homeowner cost of capital can cause a short run violation of rationality but in the long run, the U.S. housing markets are bubble free. The presence of a unit root in the NAR Housing Affordability Index implies that housing is now more affordable to single family households earning the median income thus validating our no bubble result.

The FHFA regional rent-price ratios,δ_t's, in general contrasted their aggregate counterpart such that 10 out of 12 metro areas exhibited unit root long memory persistence consistent with housing price bubbles. To our knowledge time series analysis of FHFA MSA's is absent. A recent paper by Canarella et al. (2011) applied structural break and non-linearity adjusted unit root tests to the Case-Shiller 10 city regional index. They found that although tests indicated that structural breaks existed in the rate of capital gain from the sale of houses during the early 1990s and the first half of the 2000's yet they were unable to reject the null of a unit root in most of the metropolitan regions agreeing with our results. Atlanta and Dallas metro areas from the South in contrast to the other 10 MSA's

[26]No two houses are the same. At the very least, they differ in location. They may differ in neighborhood, city, or metro area. Even the difference of a few hundred feet can have an appreciable price effect. Obviously, so too will other attributes, both locational and physical. The combination of these characteristic attributes can be considered as the house's "quality", see Rappaport (2007).

rejected the null of a unit root in d. They thus paralleled their aggregate FHFA counterpart and negated any bubble activity. This can be attributed to the low rentals in the South region as argued by Thibodeau (1995) who constructed a constant quality hedonic house price index for several MSA's. The study revealed that in general the shelter rental estimates for the Northeast and the West regions were about 2.9 to 4.6 times those of the South region.

2.8. Discussion

We began our empirical analysis with standard unit root tests i.e. ADF, DF-GLS and KPSS, which gave ambiguous evidence to the presence or absence of a unit root in most of the series. This was primarily due to the fact that these tests are too restrictive such that they consider just two values of d in the real space [0,1] i.e. they ignored the possibility of fractional roots. We thus, proceeded to estimate the fractional value of d using two methods: Parametric $ARFIMA$ in the time domain and Semi-Parametric Whittle estimates in the frequency domain. Both the methodologies applied procedures that were consistent in the non-stationary region ($d > 1/2$). We found that the ***long memory procedures allowed us to identify the persistence more accurately***. Among the two long memory methods, the ***semi-parametric procedure was more reliable*** and produced better estimates than the parametric $ARFIMA$ method.

The parametric pure $ARFIMA$ estimates found bubble behaviour in the log rent-price ratio's of two aggregate HPI's (FHFA, Case-Shiller) and four FHFA regional MSA's (2/3 from Northeast and 3/3 from the West regions). The addition of the asymmetric $EGARCH$ removed non-normal and heteroskedatic residual errors and concluded bubble processes in fewer regions (aggregate Case-Shiller HPI, 1/3 from Northeast and 2/3 from the West regions). This implied that the presence of these residual errors does impact the value of long memory persistence.

Semi-parametric estimates which do not require the modelling of the short run dynamics and are hence robust to these errors reported significantly higher values of d and implied bubble behaviour in all the three national and 12 regional series. A comparison between the two methodologies revealed that the parametric procedures captured most of the persistence in the short run $ARMA$ component i.e. the null of a unit root was not rejected in the Sum of Autoregressive Coefficients (ρ). This we believe is primarily due to the relatively short sample size we use, data limitations of the rental series preclude us from further investigation. Hence, we concluded that *the semi-parametric procedure is a superior methodology* and thus more reliable as far as our analysis is concerned. Thus our results argue for employing semi-parametric methods against parametric time domain ones.[27]

Although linear methods are widely used when testing for long memory, it is documented in the literature that presence of structural breaks in the levels or trends could slow down the d convergence generating "spurious" long memory. In light of this, we tested for endogenous breaks using three tests based on the Quandt (1960)'s F statistic ($SupF$, $ExpF$ and $AveF$). To infer for presence of breaks we computed Hansen (2000)'s "Fixed Regressor Bootstrap" asymptotic p-values that are consistent under non-stationary regressors and heteroskedastic residuals.

We found one endogenous structural break in the rent-price series. This estimated breakdate for the mean and trend of the log rent-price ratio's in general occurred in the 2003/04 time period. This validates arguments made in the literature about change in household beliefs characterizing this period. For instance, in a study of data from the Michigan Survey of Consumers, Piazzesi and Schneider (2009) report that "starting in 2004, more and more households became optimistic after having watched house prices increase for several years."[28] ***The presence of a break means a shift in the fundamentals of the housing market, as we are examining for rational bubbles, that is deviations from fundamentals, we cannot neglect this break***. Accounting for this break will ensure that the net persistence of the series will contain only deviations that arise from non-fundamental factors or bubbles. We do this by demeaning and detrending each of the series.

[27] In contrast, Baillie and Kapetanios (2007) argues that a correctly specified parametric model is superior to other alternative procedures. The small sample size we use implies that the maximum likelihood parametric methodology captures most of the persistence in the short run $ARMA$ part resulting in a biased low value of d. Furthermore, Goodman and Thibodeau (1998) and Fletcher et al. (2000) argue that the House Price Indexes inherently contain heteroskedasticity due to several factors such as the differences in the timing of house sales. We find that this is true from our results.

[28] Cerqueti and Costantini (2011) present empirical evidence of the bubbles phenomena in the international stock markets over the period 1992:1- 2010:6 for a panel of 18 OECD countries. They use a similar theoretical model like ours, namely the log–linear present-value model of Campbell and Shiller (1988), and investigate the presence of rational bubbles in the log dividend-price ratio and total returns. They use panel unit root and cointegration methodology, and allow for multiple endogenous structural breaks in the individual series. Their procedure regarding structural breaks in the dividends and returns data shows that breaks occur around the same dates for most of the countries, in particular around the "tech-stock" bubbles period.

Long memory was then estimated on the new filtered series. The ex-post persistence values were significantly lower validating our approach. When applied to the demeaned and detrended series, the semi-parametric method found unit root process consistent with housing price bubbles in just one national HPI (Case-Shiller). Nevertheless, 10 out of 12 FHFA MSA's continued to exhibit bubble type behaviour. As before, these conclusions differed drastically from the parametric procedures. The $ARFIMA - EGARCH$ model in contrast found $I(1)$ type behaviour in only two MSA's (Chicago and Boston). Considering the small sample bias and the presence of residual errors, **we infer for bubble behaviour from the semi-parametric Whittle estimators**, specifically the 2-step Exact Local Whittle of Shimotsu (2009).

Among the three aggregate HPI's only the value weighted repeat sales S&P Case-Shiller Index indicated bubble behaviour. FHFA which is also constructed based on a repeat sales weighted procedure was devoid of bubbles, so was the Constant Quality House Price Index from Census. The following question arises - "Which of these three national HPI's best describes aggregate housing price trend of U.S.?" The answer to this depends, as argued by Rappaport (2007), on one's purpose.

A bubble type behaviour in the value weighted aggregate Case-Shiller Index suggests price rises in expensive houses in big metropolises. Mean-reverting tendency in the FHFA Index describes that in general the aggregate value of the household has returned to mean levels. The Census Index which provides little evidence of housing bubbles says that the residential construction sector is healthy.

Our results for the FHFA regional MSA's indicated that persistence was on average higher in the Northeast and the West regions. The metropolitan areas like New York (Northeast), Los Angeles and San Francisco (West) are densely populated possessing highly inelastic housing supply and thus we empirically validate theoretical arguments put forward by Green et al. (2005), Glaeser et al. (2008), Glaeser et al. (2012) and others that supply elasticity and price growth is positively correlated. A further implication is the possibility of aggregation bias. In contrast, we found that the aggregate series exhibited a lower persistence than other disaggregate ones. This indicates that some region specific factors average out in the aggregation, i.e. **no aggregation bias**. This agrees with Campbell et al. (2009) who also find that the FHFA rent-price ratio is 30% more volatile in the regional level than the national level suggesting that some region-specific factors average out in the aggregate. This is because regional housing markets may not always respond at the same time to a common economic shock as regional sensitivities to demand and supply varies due to differences in area specific factors such as migration patterns, per capita income, availability of mortgage, labour mobility, demographics, degree of urbanization, rental housing market etc see, Malpezzi (1996) and Barros et al. (2012).

Our results of high persistence in the rent-price ratio agrees with that of André et al. (2014) who investigated the persistence of housing price-to-income and price-to-rent ratios in 16 OECD countries over a 40-year period, using a fractional integration framework. They find that these ratios tend to fluctuate around a stable level over the very long term, they are generally not found to be mean-reverting over the sample. They find that the order of integration of price to income and price to rent ratios are above unity for most countries, and thus exogenous shocks to these ratios will be permanent. Moreover, the integration order is in most cases significantly higher than 1, suggesting that shocks are in fact amplified. However, a drawback of their approach is that even though they use the semi-parametric method for estimation they use Whittle methods which are not powerful in case of non-stationarity. Furthermore, they do not find any evidence for structural breaks.

It can be argued that our investigation of bubble presence examined the diverge of prices from only one fundamental factor, the rents. However, there are several other fundamentals such as interest rates, construction costs, labour costs, geographical location etc that influence the movement of house prices. In a recent paper, Kivedal (2013) finds that our analysis is consistent as far as the recent boom-bust episode in the U.S. is concerned. Kivedal (2013) shows that there is an explosive root in house prices, while the rental price does not contain explosive elements. This implies bubble behaviour consistent with our results. This also holds in the case where the net rental price is used, indicating that the declining interest rate in the period before the subprime financial crisis is not a strong enough effect to explain the large increase in the house price that exceeds the increase in the rental price.

In an important paper, Evans (1991a) addresses the inability of standard unit root tests in detecting a special class of rational price bubbles which are positive, explosive and periodically collapsing. Standard unit root tests incorrectly reject the null of a unit root when such bubbles are present in the data. Phillips and Yu (2011) developed a recursive unit root testing procedure (PSY) that accommodates such periodically collapsing bubbles, these test for the null of a unit root ($d = 1$) against an explosive root ($d > 1$). These right tailed tests do not consider the possibility of either a structural break or long memory. The methodology used in this chapter can be easily extended to consider such collapsing bubbles. Appendix 2.A simulates these types of bubbles and reports our results. We find two key results. Firstly, the value of long memory, d, rises as the probability of a bubble to collapse increases and Secondly, a Wald Test that looks for a change in d values can successfully detect these types of bubbles.

2.9. Conclusion

In this chapter we use long memory models to estimate the persistence of the log rent-price ratio's in the national and regional House Price Indexes in the United States and thereby investigate the presence of price bubbles in the housing market. We based our analysis of bubble presence (absence) depending on unit root (mean-reverting) persistence. Essentially, we test for the null of a unit root, I(1), against the alternative of stationarity/mean-reversion, I($d < 1$). We analysed a quarterly dataset that spanned the 31 year time period 1982Q4-2013Q4 encompassing both the observed upturns and downturns in US housing prices.

Our results revealed that the semi-parametric procedures which are robust to contaminations in the form of short term correlation, normality and heteroskedasticity found bubble behaviour in far more regional markets than when using the parametric procedure. Results for bubble identification were different for different House Price Indexes. While the Census and the FHFA series showed no unit root bubble behaviour, the Case-Shiller Index did. We also found that the regional indexes were far more volatile than the aggregate ones. Finally, we found an endogenous break in all the series. This breakdate coincided with the turn of credit market conditions in the United States. When we adjust the series for this break, we find significantly lower persistence. In summary, we conclude that there is strong evidence for the presence of housing bubbles in the U.S. housing market. Furthermore, more of the regional series exhibited bubble type behaviour than the national ones.

This study opens up several possible areas for future investigation. The chief one relates to the time varying characteristic of long memory. To the best of our knowledge, appropriate methods to estimate a changing d parameter robust to structural breaks is not available. Roueff and von Sachs (2011) construct a semi-parametric procedure to estimate d when it is time varying. However, the procedure is only consistent for stationary processes. Efficient estimation of the d

parameter free of short run contaminations will help in the timely identification of bubbles which has wide scale advantages. The accurate estimation of persistence will directly assist policy makers in the United States housing industry to make optimal decisions. In fact, when realtor authorities have a priori knowledge of the persistence on housing prices, they can design appropriate housing strategies to adjust persistence in house prices benefiting urban consumers.

Appendix

2.A. Robustness - Periodically Collapsing Bubbles

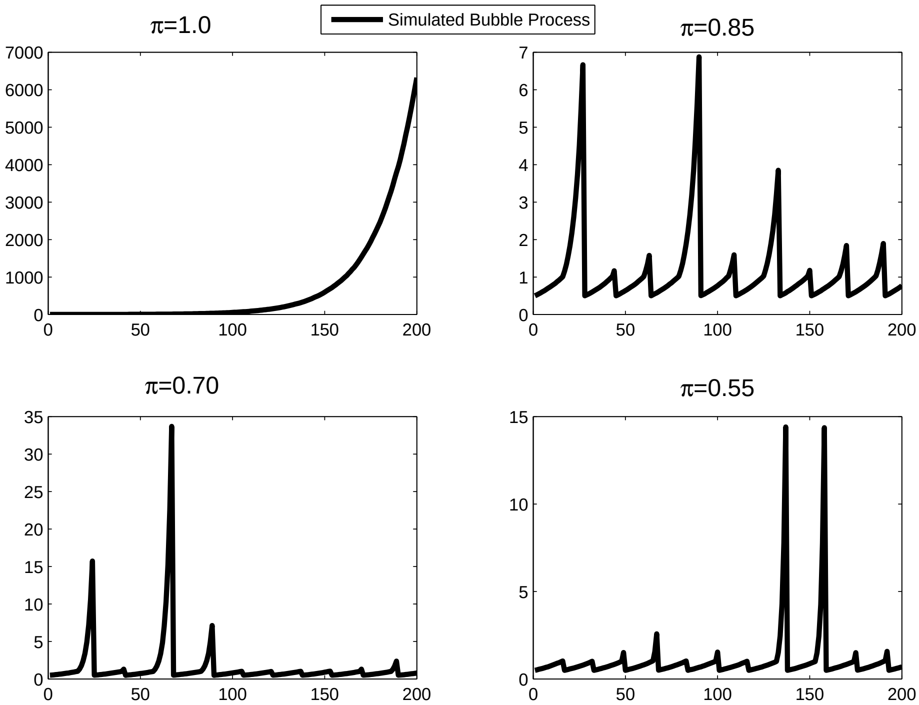
In this section, we see whether the semi-parametric long memory methods used in this chapter can be used to detect a special type of bubbles introduced by Evans (1991a). These are a class of positive and explosive periodically collapsing bubbles which are consistent with rational expectations. They take the form:

$$B_{t+1} = \begin{cases} (1+r)B_t u_{t+1} & \text{if } B_t \leq \alpha \\ \left[\omega + \frac{(1+r)}{\pi}\theta_{t+1}(B_t - \frac{1}{1+R}\omega)\right] u_{t+1} & \text{if } B_t > \alpha \end{cases} \tag{2.73}$$

where ω and α are positive parameters with $0 < \omega < (1+r)\alpha$, u_{t+1} is an exogenous i.i.d positive random variable with $E_t(u_{t+1}) = 1$, θ_{t+1} is an i.i.d Bernoulli process which takes the value 1 with probability π and 0 with probability $1-\pi$. As long as $B_t \leq \alpha$, the bubble grows at mean rate $1+r$ and when $B_t > \alpha$ the bubble bursts into a phase in which it grows at the faster rate $(1+r)/\pi$ as long as the eruption continues. The bubble collapses with probability $1-\pi$ per period and eventually falls to a mean positive value of ω, from which the process begins again.

These bubbles appear to be stationary when unit root tests are applied even though they contain explosive roots, except in the case where the probability of collapse is very close to zero. We simulate periodically collapsing bubbles for different values of π and then estimate the long memory parameter for these processes. Figure 2.A.1 plots these simulated bubble processes. The following table describes the results for long memory estimation.

Figure 2.A.1. Simulations of Periodically Collapsing Bubbles



Notes: This figure depicts simulated Evans (1991a) periodically collapsing bubbles at four different probabilities of bubble collapse $(1-\pi)$. The bubles were simulated using r=0.05, α=1, $\omega = 0.5$, initial B_t=ω, $u_{t+1} = exp(y_\tau - \tau^2/2)$ where $y_\tau \sim N(0, \tau^2)$ and τ=0.05.

Table 2.A.1. Long Memory Estimation of Simulated PCB's

	$m = 40$			$m = 60$			$m = 80$		
π	d_{LWE}	d_{2ELW}	$d_{2ELWdet}$	d_{LWE}	d_{2ELW}	$d_{2ELWdet}$	d_{LWE}	d_{2ELW}	$d_{2ELWdet}$
1.0	0.906	2.287	2.273	0.905	2.163	2.176	0.885	2.189	2.206
	(0.079)	(0.079)	(0.079)	(0.064)	(0.064)	(0.064)	(0.055)	(0.055)	(0.055)
0.85	0.539	0.590	0.607	0.624	0.714	0.714	0.672	0.836	0.835
	(0.079)	(0.079)	(0.079)	(0.064)	(0.064)	(0.064)	(0.055)	(0.055)	(0.055)
0.70	0.369	0.397	0.385	0.485	0.563	0.598	0.530	0.678	0.679
	(0.079)	(0.079)	(0.079)	(0.064)	(0.064)	(0.064)	(0.055)	(0.055)	(0.055)
0.55	0.161	0.180	0.155	0.278	0.338	0.331	0.357	0.487	0.485
	(0.079)	(0.079)	(0.079)	(0.064)	(0.064)	(0.064)	(0.055)	(0.055)	(0.055)

Notes: This table reports the semi-parametric estimates of long memory for the simulated periodically collapsing bubbles. We use three different bandwidths, m, and four different probabilities of bubble collapse, $1-\pi$. n is the sample size which is 200. d_{LWE} is the Local Whittle estimator, d_{2ELW} and $d_{2ELWdet}$ are 2-step Exact Local Whittles estimators without and with de-trending. The asymptotic standard errors for the estimates are given in parenthesis.

It is apparent from Table 2.A.1 that the estimated value of long memory d decreases as the probability of bubble collapse π increases. Presence of periodically collapsing bubbles will thus lead to erroneous conclusions about bubble behaviour. A stationary mean reverting series could thus potentially contain bubbles. The Shimotsu (2006) can be used in the frequency domain to detect these types of bubbles as described below. Table 2.A.2 show that as the bandwidth window (m) and the number of subsamples (b) increases, the Wald statistic gets better in detecting periodically collapsing bubbles. Simulations help us in using this test as an effective way to test for periodically collapsing bubbles in the U.S. Housing Market. On testing in our dataset, we did not find periodically collapsing bubble phenomenon and thus we omit the discussion in our text.

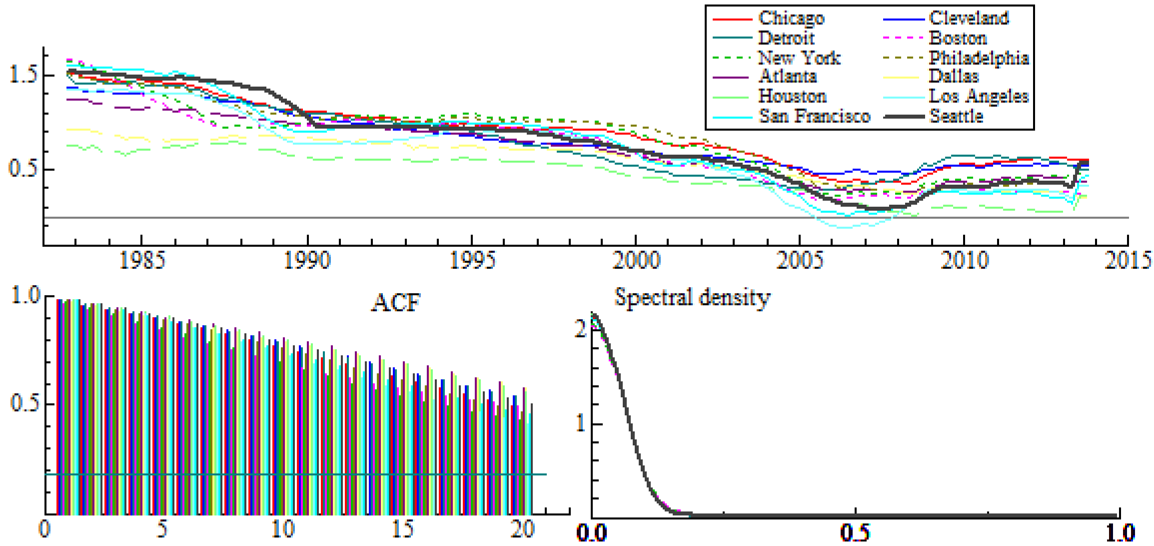
Table 2.A.2. Detecting periodically collapsing bubbles

π	m	$\hat{d}$	d		W_c	
			b=10	b=20	b=10	b=20
1.0	40	2.217	1.412	1.920	0.020	13.018
0.85	40	0.590	1.028	1.535	13.568	8.675
0.70	40	0.397	1.181	1.739	29.191***	15.199
0.55	40	0.180	0.713	1.468	3.421	17.210
1.0	60	2.163	1.438	1.515	0.034	8.213
0.85	60	0.714	1.103	1.791	22.001***	46.370***
0.70	60	0.563	1.189	1.907	48.614***	110.934***
0.55	60	0.338	0.817	1.459	7.291	68.467***
1.0	80	2.189	1.487	1.600	0.069	12.334
0.85	80	0.836	1.185	2.216	29.384***	154.236***
0.70	80	0.678	1.289	2.563	62.184***	424.969***
0.55	80	0.487	0.900	2.262	11.824	501.339***

Notes: The W_c statistic tests the null that the memory parameter has remained constant throughout the subsamples and is χ^2 distributed with $b-1$ degrees of freedom where b is the number of subsamples. *** indicates rejection of the null at the 5% level indicating the presence of periodically collapsing bubbles.

2.B. Rent-Price Ratio - FHFA Regional

Figure 2.B.1. Regional



Notes: This figure provides a graphical illustration of the log rent-price ratio of 12 FHFA Regional MSA's. The second row plots the autocorrelation functions and the spectral densities.

2.C. Efficient Fractional Dicky-Fuller Test

Table 2.C.1. EFDF Test on break adjusted semi-parametric estimates

Housing Market	Break adjusted (EFDF)		
	$\hat{d}_{LWE}$	$\hat{d}_{ELW}$	$\hat{d}_{2ELW}$
FHFA	-2.880***	-3.005***	-3.005***
Case-Shiller	0.146	0.183	0.184
Census	-3.085***	-3.266***	-3.248***
NAR	0.970	1.041	1.056
Midwest			
Chicago	-1.406	-1.348	-1.317
Cleveland	-0.067	-0.021	-0.037
Detroit	0.779	0.779	0.777
Northeast			
Boston	0.895	0.928	0.931
New York	0.225	0.230	0.290
Philadelphia	-0.866	-0.798	-0.855
South			
Atlanta	-2.874***	-3.001***	-3.002***
Dallas	-2.111**	-2.111**	-2.186**
Houston	-1.100	-1.071	-1.033
West			
Los Angeles	0.211	0.269	0.261
San Francisco	0.129	0.139	0.119
Seattle	-0.438	-0.453	-0.419
Crit. Value			
1%	-2.551	-2.551	-2.551
5%	-1.904	-1.904	-1.904
10%	-1.564	-1.564	-1.564

Notes: This table reports the Efficient Fractional Dicky-Fuller test statistics on the demeaned and detrended log rent-price ratio's of the four national and the 12 regional MSA's. The sample spans the quarterly time period 1982Q4-2013Q4, except Case-Shiller which is 1987Q1-2013Q4. The EFDF tests for the null of a unit root ($d = 1$) against fractional roots i.e. $d = \hat{d} < 1$, where $\hat{d}$ is one of the three semi-parametric estimates ($d_{\hat{LWE}}$, $d_{\hat{ELW}}$ and $d_{2\hat{ELW}}$). Critical values (1%, 5% and 10%) for the tests is given in the last three rows. Critical values for Case-Shiller are -2.527, -1.876 and -1.533 at the 1%, 5% and 10 % levels respectively. '***' indicate rejection of the null of a unit root at the 1% level.

Chapter 3

Optimal Life-Cycle Asset Allocation with Return Predictability, Risky Housing and Non-Tradable Labour Income

3.1. Introduction

Financial advisors and much of the academic literature argue that young investors should place most of their savings in stocks, which historically have paid a high risk premium relative to US Treasury securities, and switch to less risky assets as they age. For instance, Malkiel (1996) recommends putting a percentage of assets equal to the number 100 minus an investor's age in a well-diversified portfolio of stocks.

However, ***low stock market participation rates and moderate equity holdings for stock market participants are observed in US data***. In this chapter, the key variable of interest is the proportion of assets held in risky assets and is denoted by α_t. The 2007 Survey of Consumer Finance (SCF) shows that only 55.3% of US households have direct or indirect holdings of risky assets. This low stock market participation by households despite high expected returns is called the **STOCK MARKET PARTICIPATION PUZZLE**. Furthermore, data from the Panel Study of Income Dynamics (PSID) for the 1968-2007 period show that the median household direct risky asset holdings and indirect risky asset holdings are zero. Moreover, lifecycle risky asset holdings are "hump shaped." Young investors typically hold very little stock, progressively increase their risky assets holdings as they age, and decrease their exposure to stock market risk when they approach retirement, see Vissing-Jorgensen (2002), Ameriks and Zeldes (2002), Alan (2006) and Campbell (2006). Canner et al. (1997) calls this the **ASSET ALLOCATION PUZZLE**.

Interest in these puzzles are not confined to stock brokers. Extensive privatisation in countries ranging from the financially developed, such as the United States, to emerging market economies hinges on developing and maintaining a broad base of stockholders. While initial participation is encouraged by extensive advertising or by enthusiasm for market structures, the sources of the reluctance to hold stocks in a financially mature country such as the United States or the United Kingdom are puzzling, see Haliassos and Bertaut (1995), Poterba (2002). Stockholding was shown by Mankiw and Zeldes (1991) to have implications for the widely researched 'equity premium puzzle', see Mehra and Prescott (1985), confining attention to stockholders lowers the risk aversion implied by the equity premium.[1]

Several explanations for the observed limited stock market participation have been offered in the literature. Haliassos and Bertaut (1995) find theoretical evidence in an expected utility maximisation framework that presence of short-sales constraints and business cycle risks can deter stockholding. Hong et al. (2004) empirically analyse data from the Health and Retirement Study, and find that social household - those who interact with their neighbour or attend church - are substantially more likely to invest than non social households controlling for wealth, education, race and risk tolerance. van Rooij et al. (2011) finds using survey data that respondents who displayed reasonable levels of financial literacy in terms of grasping concepts such as interest compounding, inflation and time value of money hold stocks in their portfolio implying that financial literacy affects stockholding. It is conventional in the literature to club all these factors such as social interaction, financial literacy etc., in the form of a fixed entry cost that deters equity market participation, see Haliassos and Michaelides (2003), Guiso et al. (2003) and Alan (2006) among others. In this chapter, we follow these papers and consider an exogenous fixed cost of stock market participation.

[1]The equity premium puzzle is a phenomenon that describes the anomalously higher historical real returns of stocks over government bonds. The equity premium, which is defined as equity returns less bond returns, has been about 6% on average for the past century. It is supposed to reflect the relative risk of stocks compared to "risk-free" government bonds, but the puzzle arises because this unexpectedly large percentage implies a suspiciously high level of risk aversion among investors.

The choice of whether or not the investor pays the fixed cost and participates in the stock market depends on his level of wealth which varies with the investor's age. The Survey of Consumer Finances data on household portfolios reveal that, portfolio share devoted to risky assets has a hump shaped profile with respect to age (Campbell (2006), Flavin and Yamashita (2011)). That is, as households accumulate wealth they tend to invest an increasing fraction of their wealth in risky assets. In contrast, conventional wisdom maintains that for reasonable levels of risk aversion, young agents should place a large proportion of their wealth into the market portfolio, and this proportion should decline as the agent nears retirement, see Davis and Willen (2013).

Both empirical observation and conventional wisdom seem at odds with the academic literature. Early and enduring theoretical contributions include Merton (1969, 1971), and Samuelson (1969). Merton (1969, 1971) considers a dynamic portfolio optimization problem in which investors maximize expected utility through their choice of risky and risk-free investments, subject to a wealth constraint. Closed form solutions for optimal portfolio shares are obtained using dynamic programming arguments when returns are generated by a Brownian motion process, and for hyperbolic absolute risk aversion (HARA) utility functions, a class that includes constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA). One important result that emerges from Merton's analysis is a *two-fund separation theorem*. It states that given n assets with log-normally distributed prices, there exists a unique pair of "mutual funds" consisting of a linear combination of the assets, such that independent of preferences, wealth distribution, or time horizon, investors will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets. This reduces the analysis of many assets to a two-asset case. With CRRA utility, and one risky and one risk-free asset representing the two funds, the theory has the testable property that the share invested in the risky asset is affected neither by the level of wealth nor by the consumption decision, see Curcuru et al. (2004).

These early studies conclude counter-factually that a long-lived agent should hold a constant fraction of his wealth in the risky asset throughout his life. When calibrated to historical values of the equity premium and stock market return volatility, these models predict that the appropriate proportion of wealth placed in the risky asset is large, sometimes higher than 100%. These models generate little heterogeneity in stock market participation even if there is significant variation in risk aversion across agents. These results are also derived under many restrictive assumptions, including power utility, independent and identically distributed (IID) returns on the risky and risk-free investments, the absence of market frictions, the absence of labour income etc., see Benzoni et al. (2007).

In an attempt to reconcile theory and observation, many of the restrictive assumptions underlying the Merton (1969) and Samuelson (1969) results have been progressively relaxed. This has been achieved through incorporating labour income (Bodie et al. (1992), Benzoni et al. (2007)), generalizing preferences (Campbell and Viceira (1999), Gomes and Michaelidis (2005)), making intertemporal utility non-separable in a durable good such as housing (Grossman and Laroque (1990), Flavin and Yamashita (2011)) and analysing the effects of time variation in equity premium (Campbell et al. (2001)).

Although each of these dynamics have been used independently in several studies, to the best of our knowledge, none of them incorporates all the dynamics. Furthermore, analytical expressions for α_t in a substantially realistic model in discrete time setup is more or less non-existent. In this chapter, we do both. First, we use a reasonably stylized model to derive an expression for risky asset demand, α_t. In the second section, we extend this model to incorporate all the above discussed features in a life-cycle context. The next subsection describes the main contributions and key results that we obtain.

3.1.1. Contribution and Results

We have contributions in both the analytical and the numerical sections of this chapter. We start with a moderately stylized model in Section III abstracting from life-cycle dynamics but still having time varying returns, a risky durable good and uncertain labour income with Epstein-Zin preferences. We then analytically characterize the optimal risky asset demand. This approach is closer to Campbell and Viceira (1999), Viceira (2001) and Yogo (2006) in that we derive approximate log linearized Euler equations and budget constraints. These equations incorporate a risky labour income and are potentially useful for empirical research, particularly in explaining the cross-sectional variation of asset returns. Importantly, we express the optimal risky equity demand as the sum of two components, a myopic demand and an intertemporal hedging demand. Our analytical characterization provides valuable intuition to the factors that determine the level of wealth invested in the risky asset when the household faces changes in the investment opportunity set, shocks to the labour income and shocks to durable housing prices. In this way we extend the seminal work of Campbell and Viceira (1999) to include durable goods and labour income.

In Section IV we extend the stylized model by incorporating short-sales and borrowing constraints, calibrated hump-shaped labour income, and a risky stochastic house price process in a life-cycle context. Essentially we extend Cocco (2004) and Vestman (2012)'s life-cycle portfolio choice model which has both housing and risky labour income by including (i.) time varying returns, (ii.) Epstein-Zin preferences, (iii.) a bequest motive and uncertainty of death. Time varying returns implies that investors in our model can use a factor such as the log dividend-price ratio (log dividend yield) to predict expected excess returns and can devise strategies in response to changing opportunities. We then numerically solve it to understand the evolution of risky asset demand, α_t, and the level of stock market participation over the life-cycle. Our results provide valuable insights to the resolution of these puzzles and other portfolio problems. Our results can be summarized as follows.

Firstly, we find that in the presence of housing both the stock market participation rate and the risky asset allocation share is found to be hump-shaped over the life-cycle consistent with empirical evidence, see Attanasio et al. (2012) and Guiso and Sodini (2013). Thus, consistent with other models that include housing such as Cocco (2004), Yao and Zhang (2004), Li and Yao (2007) and Vestman (2012), housing initiates a crowding out effect restricting younger liquidity constrained households from market participation and equity market investments.

Secondly, we find that both risky asset allocation as well as the stock market participation rate is extremely sensitive to factor realization, factor volatilities and the persistence of the factor process. We find that both, a high factor realization and a high factor persistence are positively related to the equity allocation. By factor, we mean the dividend-price ratio. In other words, unit root persistence in the factor process indicative of stock market frenzies such as bubbles can generate substantially high levels of equity demand and market participation. Furthermore, a huge drop in realized levels of the return predicting factor and high volatility in the factor produces a subsequent fall in risky asset allocation. The drop was much larger in the later years of the life-cycle (65-100). These suggest a "rare disaster" in the economy. Our results thus extend the disaster literature, Barro (2006, 2009) and Wachter (2013), to understand household asset allocation. We also find that investors can hedge background risks such as labour income and house prices better under return predictability compared to the IID case. Intuitively, this means that substantial welfare losses arising from investment mistakes can be avoided, see Calvet et al. (2009) and von Gaudecker (2015).

Thirdly, we resolve both the stock market participation and the asset allocation puzzles. Our simulated results predict levels of asset allocation and stock market participation rates which are very close to the estimated ones from the Survey of Consumer Finance dataset. Importantly, our results arise without resorting to preference heterogeneity which is the case with Gomes and Michaelidis (2005) and Vestman (2012). In other words, a moderate level of risk aversion and a moderate level of elasticity of substitution can successfully replicate the observed participation and equity shares.

The rest of the chapter is organized as follows. In §3.2 we describe the literature review, in §3.3 we detail a reasonably stylized model of discrete time portfolio choice and §3.4 describes the corresponding analytical characterization for the Euler equations, budget constraints and importantly the optimal risky asset demand. §3.5 extends this model to a richer life-cycle one which is empirically calibrated and the simulated results are then plotted, tabulated and described in §3.6. Finally, §3.7 concludes.

3.2. Literature Review

In this section, we briefly review the literature on each of the modifications that are critical in our model on household portfolio choice. These include labour income, risky housing, return predictability and recursive preferences.

3.2.1. Labour Income and Portfolio Choice

A crucial element one needs to consider when discussing portfolio choice over the life-cycle is labour income and the risk associated with it. For many agents, the human capital (i.e., the certainty-equivalent present value) tied up in terms of future wages dwarfs their financial wealth. As such, it is intuitive that the optimal portfolio choice that takes labour income into account may generate significantly different predictions.

Benzoni et al. (2007), discusses the role of labour income risk in explaining lifecycle asset allocation decisions. They argue that the correlation between shocks to stock market returns and wages is an increasing function of the investment horizon. For a young investor, this effect generates a large positive correlation between stock returns and the unobservable return to human capital. That is, the present value of future labour income flows acquires features identical to stocks in that returns can be volatile and unpredictable. However, older investors, who have shorter times to retirement, are much less exposed to long-run labour income risk. Hence, their remaining human capital becomes more identical to bonds in that returns are stable and highly predictable, see also Heaton and Lucas (1997) and Viceira (2001). Together, these effects create a hump-shaped optimal portfolio decision over the investor's lifecycle, consistent with empirical observation. In conclusion, the level and risk of labour income risk varies with age over the lifecycle and portfolio choice for an investor depends on these changes. We thus consider labour income, that is calibrated to capture the hump-shape, in our lifecycle asset allocation model.

Recent literature on portfolio selection in the lifecycle context considers labour income risks. However most of these papers do not explicitly account for housing, see for example Dammon (2001), Ameriks and Zeldes (2002), Campbell (2006), Cocco et al. (2004), Gomes and Michaelidis (2005) and Davis et al. (2006) among others. For most households, a house is the single most important consumption good, appearing, as an argument of the utility function and at the same time, the dominant asset in the portfolio. On average over 1952-2013 in the US, housing wealth accounts for 35% of household assets and 40% of household net worth (assets minus liabilities), while home equity (housing wealth minus mortgage debt) is 23% of assets and 26% of net worth. Furthermore, two-thirds of all households in the U.S. own their home and for most home-owning households, housing accounts for a substantial portion of total wealth, see Davis and Nieuwerburgh (2014). In the following section we review the literature that accommodate housing in making portfolio choice decisions.

3.2.2. The Role of Risky Housing in Portfolio Allocation

As argued by Corradin et al. (2014) and Davis and Nieuwerburgh (2014), there are several housing specific characteristics that make portfolio allocation decisions nontrivial. First, housing is illiquid in the sense that changing the quantity of housing may take time and/or require incurring substantial transaction costs. Therefore, homeowners would optimally want to rebalance their housing position less frequently than other investment assets. Second, house is an indivisible good, that is, a limited assortment of types and sizes are available for purchase at any time which in general includes a minimum size. Third, home ownership and housing consumption are generally intimately related. Most households own only one home and live in the house they own. Fourth, housing has an investment dimension in that households can use it as a collateral against which they can borrow. Investment in housing is much more leveraged than investments in other financial assets and the value of owned housing limits the amount of leverage in households' portfolios. Finally, house prices move with business cycles and exhibit both persistence and volatility making them a risky investment.

Despite these implications of housing on optimal portfolio choice, most papers do not consider an individual's investment in a home (i.e., a durable consumptions good). Grossman and Laroque (1990) present the first exception as they develop a theoretical model with a single illiquid durable consumption good (e.g., a house) from which an infinitely lived investor derives utility. The illiquidity derives from the fact that transaction costs are born when the good (house) is sold. In addition to the durable good the individual can invest in a risk free asset and a set of risky financial assets. At each time, the individual must decide whether to acquire a larger (smaller) house and how to allocate his or her remaining wealth among financial assets. Grossman and Laroque show that it is optimal for the individual to wait for large increases (decreases) in wealth to raise (reduce) their consumption of the durable consumption good. In addition, they conclude that transaction costs cause the individual to allocate a smaller portion of their financial wealth to risky assets than would occur if the individual could adjust homeownership continuously. A drawback of their analysis is that only the durable good is considered in the utility function, ignoring completely nondurable consumption, implying that the potential spillover effects on nondurable consumption or the implications for portfolio allocation of housing risk arising from variation in the relative price of housing.

In a paper designed to explain the equity premium puzzle Chetty and Szeidl (2007) show, in a two good model, one of which is a durable consumption good, that a "consumption commitment" (e.g., for a house), will result in individuals acting as if they are more risk averse. These authors conclude that their model can fully resolve the equity premium puzzle. Alternatively Piazzesi et al. (2007) and Yogo (2006) consider the effect of "composition risk" on asset pricing.[2] Here

[2] The standard Consumption-Capital Asset Pricing Model focuses on consumption risk, which relates changes in the conditional distribution of a single factor, the aggregate consumption growth, to asset prices. However, consumption-savings decisions depend not only on the uncertain overall size of future consumption bundles, but also on their uncertain composition, for example, between housing and other consumption. Composition risk is an added risk which relates changes in asset prices also to changes in expenditure shares of housing relative to other non-durable goods.

the individual's utility function is not separable between the consumption of a durable good and a non-durable good. These authors find that composition risk, variations in the consumption of the durable good relative to the consumption of other goods can help explain time variations in the equity premium. The presence of such composition risk makes investors highly risk averse boosting their precautionary savings motive and thus shifts down risky asset allocation. This effect is particularly severe in recessions.[3]

Flavin and Yamashita (2002); Flavin and Nakagawa (2008) study the impact of the portfolio constraint imposed by the consumption demand for housing on an individual's optimal holdings of financial assets. In addition to a house, the individual can invest in T-Bills, T-Bonds, stocks, and borrow through a mortgage loan. They use PSID data to explore the life cycle impact of the "housing constraint" (as reflected by the ratio of housing to net worth) on the individual's optimal holding of financial assets. Flavin and Yamashita (2002) use mean-variance analysis to characterize optimal portfolios of financial asset over the life cycle. They conclude that an exogenous increase in the value of the house owned results in a relatively large shift from equities to bonds in a mean-variance optimal portfolio. All these papers immensely contributed to our understanding of the role of the durable good in portfolio choice decision, but they did not consider life-cycle dynamics.

In an important paper, Cocco (2004) developed an empirically parameterized model of consumption and portfolio choice when there is an illiquid durable consumption good (a house) *in a life-cycle setting*. In his paper the individual purchases a home for the consumption services it provides. The individual has a stochastic income and can invest in two financial assets: a risky stock and riskless Treasury bills. Cocco uses this portfolio optimization model to predict the cross-sectional pattern of variation in the composition of wealth by age and net worth.

In a similar life-cycle environment, Yao and Zhang (2004) investigate the optimal portfolio decisions of an individual who can obtain housing services from renting or by buying a home. They investigate the decision as to how an individual should obtain these services (i.e., rent or buy) and the implications of this

[3] During recessions, because investors expect higher future consumption, they try to sell stocks today to increase current consumption. This intertemporal substitution mechanism drives stock prices down in bad times. Investors' concern with composition risk implies that recessions are perceived as particularly severe when the share of housing consumption is low. That is, a new intertemporal substitution mechanism increases the downward pressure on stock prices in severe recessions.

decision on investment choices. In their model the expected real rate of home value appreciation is assumed to be zero. Yao and Zhang find that homeownership has an important impact on the individual's portfolio choice; specifically homeowners substitute home equity for risky stocks. These authors find that, over the life-cycle, the policy of always renting or buying a home can results in large losses in welfare, with the largest being born by individuals with substantial net wealth who are constrained to rent or older individuals with very little net worth who are constrained to buy.

Yogo (2009) develops a life-cycle model in which a household faces stochastic health depreciation, which affects its marginal utility of consumption and life expectancy, to analyse the portfolio choice in retirement. The household receives retirement income including Social Security and chooses consumption, health expenditure, and allocates wealth between bonds, stocks, and housing to maximize its lifetime utility that includes a bequest motive. Yogo finds that in addition to the housing risk, health expenditure also significantly affect the level of risky stock allocation- households are more likely to invest in stocks when they are healthy. A limitation in this paper, acknowledged by Yogo, is that the analysis is restricted to the retirement phase and ignores the working period of the household.

In a recent empirical paper, Chetty and Szeidl (2014) distinguish between home equity wealth and mortgage debt, as they have opposite signed effects on portfolio choice. They find that increases in mortgage debt reduce stock holding significantly, whereas increases in home equity wealth raise stock holding. In addition, they provide evidence that higher housing investment substantially reduces the amount that households invest in risky stocks.

Although these papers provide valuable insights to the issue of durable goods impact on risky asset allocation over the life-cycle, they ignore a very important stylized fact in the financial economics literature, which is that stock returns are time-varing and is predictable through financial factors such as the log dividend-price ratios.

3.2.3. The Role of Return Predictability and Rare Disasters in Portfolio Allocation

A large body of empirical literature has documented the long-term predictability of asset returns and the linkages between wealth and other macroeconomic variables. An important reason for the interest in this relation is that expected excess returns on assets appear to vary with the business cycle. For instance, Chen (1991) studies the relation between changes in the financial investments opportunity set and the macroeconomy. Chen (1991) finds that state variables such as the dividend-price ratio, the default premium, the term premium etc are good indicators of recent and future economic growth. Importantly, he finds that these variables are positively correlated with expected excess return and future economic growth; and negatively correlated with recent economic growth. The counter-cyclicality of risk premium is found to hold even when post 1990 stock market data is considered, see Henkel et al. (2011).

Different explanations have been offered for this empirical result, namely: inefficiencies of financial markets (Fama and French (1988, 1992) and Fama (1998)); the rational response of agents to time-varying investment opportunities driven by variation in risk aversion Campbell and Cochrane (2000) or in the joint distribution of consumption and asset returns.

One area where return predictability has profound implications is asset allocation. For long-term investors the static Markowitz Mean-Variance model will only be suitable under very strict assumptions, one of them being that investment opportunities are constant over time, meaning that returns are unpredictable. If this is not the case, long-term investors can benefit from the return predictability, both in the form of market-timing and in the form of intertemporal hedging of future return risk. Neither of these effects are captured by the static Mean-Variance model.

Lynch (2001) assesses the impact of return predictability on portfolio choice for a multi-period investor by characterizing the intertemporal hedging demand in a continuous time setting. Lynch finds that parameters such as the persistence of the return predicting process can have a large impact on the optimal risky share of asset allocation. He attributes the variation in the risky share to hedging motives. However, his model is highly stylized and abstracts from life-cycle dynamics, labour income, any durable good or short sales constraints. Nevertheless, these results are consistent with what we find.

There have been a few recent papers which argue how "rare disasters" in the economy can resolve several puzzles in the finance literature including but not limited to the equity premium puzzle (Mehra and Prescott (1985)), the risk free rate puzzle (Weil (1990)) and the excess volatility puzzle (Shiller (1981)).[4] This

[4]The risk free rate puzzle emerges out of the equity premium puzzle: why are the risk free rates so low if the agents are so averse to intertemporal substitution. The excess volatility puzzle is the stylized fact that volatility of dividends (fundamentals) cannot explain the much larger volatility in stock returns.

strand of literature can be traced back to Rietz (1988) who models the possibility of a low probability depression like state and shows how such a state can explain these puzzles. The motivation is that Risk-averse equity owners demand a high return to compensate for the extreme losses they may incur during an unlikely, but severe, market crash. To the extent that equity returns have been high with no crashes equity owners have been compensated for the crashes that happened not to occur.

An open question has therefore been whether the risk is sufficiently high, and the rare disaster adequately severe, to quantitatively explain the equity premium. Recently, Barro (2006) revitalized this literature by analysing 20th century disasters using GDP and stock market data for 35 countries and showed that it is possible to explain the high equity premium when the disaster probability is set at roughly 2% per year. The framework of his model is based on Lucas' representative-agent, fruit-tree model of asset pricing with exogenous, stochastic production with tractable elements of closed economy and complete markets. The investor is allowed to hold two assets, one of which is risky and the other riskless. At every date, the agent faces a constant exogenous probability of disaster risk, and an associated size of this collapse. These parameters act as determinants in the analytical closed form solutions of Barro's optimal expected risky premium and risk free return.Since Barro (2006), several papers have come out and have been successful in explaining several asset market puzzles such as the excess stock return volatility (Wachter (2013))

If rare economic disasters can solve the pricing puzzles, intuitively they should also explain the observed household portfolio holdings (quantity) and/or the limited rates of equity market participation. In other words, perceived risk associated with a disaster in stock markets should be revealed in household portfolios. However, such endeavours have been by and large unsuccessful.

For example Alan (2012) examines whether such rare economic disasters as argued by Barro (2006) can explain the asset allocation and stock market participation puzzles. Alan (2012) finds that it is difficult to reconcile the results of the calibrated model with observed levels of limited asset allocation and participation rates unless an implausible level of labour market stress is assumed at the time of the disaster.

In a related exercise Fagereng et al. (2013) develop and numerically simulate the standard life-cycle model of portfolio allocation incorporating labour income risks and IID investment opportunity sets adding a small subjective probability of a large loss when investing in stocks (a "disaster" event) where the parameters are calibrated to Norwegian Household Panel Data. Their study predicts a joint pattern and level of participation and the risky asset share over the life cycle similar to the one observed in the data, with early rebalancing of the risky share before retirement. However, the stock market participation rate is found to be, counter-factually, 100% for most part of the agent's life.

Michaelides and Zhang (2015) who solve for optimal portfolio choice and consumption in a standard life-cycle model *without housing* but with recursive preferences and undiversable labour income risk and importantly accommodating a *predictable time varying equity premium*. They find that in the presence of return predictability ignoring market information can lead to substantial welfare losses. In this chapter we model return predictability following Michaelides and Zhang (2015) but we do not focus on welfare analysis.

Some recent papers have investigated the impact of *return predictability in house prices* on optimal portfolio choice. For instance, Fischer and Stamos (2013) study the decisions of households facing time varying expected growth rates in house prices and show that homeownership rates, as well as the sizes of housing and mortgages, increase during good periods of housing market cycles. Their results do not point to a statistically significant impact of the regime of housing market cycles on stock holding. However, Corradin et al. (2014) find that the share of wealth invested in risky assets is lower during periods of high expected growth in house prices and that the decrease in risky portfolio holdings for households moving to a more valuable house is greater in high-growth periods. Unlike these papers, we do not model return predictabiity in house prices but assume that excess stock returns are predictable. There is considerable empirical evidence that house prices and stock prices are uncorrelated and that return predictability in stock prices crucially affects risky portfolio choice.

3.2.4. Role of recursive preferences in portfolio choice

An often made assumption in several portfolio choice models is homogeneity in preferences. In other words, investors are assumed to have Constant Relative Risk Aversion (CRRA). This implies that as agents become more risk averse, they simultaneously become more intolerant of intertemporal variation in consumption. Consequently, higher risk aversion results in higher predicted levels of savings. The importance of the equity premium relative to the fixed participation cost increases with the level of savings. For some parameters, more risk-averse agents are therefore, counter-intuitively, more likely to participate in the stock market, see Curcuru et al. (2004).

This counter-factual prediction arises because of the shortcomings of the CRRA utility function - the coefficient of relative risk aversion and the elasticity of intertemporal substitution are represented in just one parameter. In some sense, this is consistent with the way the risk is modelled in expected utility framework: uncertainty is the expansion of the decision making scenario to a multiplicity of states of nature. Total utility is the expected value of optimal decision making in each of these states. Thus, there is no difference between time and states of nature. Time is just another subindex to identify states of the world. However, households seem to regard time and uncertainty as essentially different phenomena, see Weil (1990). It is natural then to seek a representation of preferences that can treat these two components of reality separately. This has been addressed by Epstein (1988), who axiomatically worked on non-expected utility and came up with a non-expected utility function representation for a preference relation that considers time and states of nature as more than just two indices of the state of the world.

The advantage of such a separation has been highlighted by the empirical literature on the behavior of asset returns and consumption over time. Expected utility, representative agent, optimizing models have not performed well empirically, Hansen and Singleton (1983) and Mehra and Prescott (1985). One possible explanation for this poor performance is the above noted inflexibility of the expected utility specification, see Epstein and Zin (1989).

Svensson (1989) analysing the portfolio choice problem in a non-stochastic environment conclude that the optimal portfolio choice depends *only* on the risk aversion parameter but not on the intertemporal elasticity of substitution. Similarly, Weil (1990) assuming independent and identically distributed (IID) interest rates over time finds that asset allocations are myopic and does not include a component to hedge against intertemporal changes in the investment opportunity set. However, Campbell and Viceira (1999), Chacko and Viceira (2005) and many others who allow for non-IID stochastic investment find that both these parameters matter in determining optimal consumption-portfolio choice decisions.

Campbell and Viceira (1999) solves analytically the optimal portfolio choice assuming Epstein-Zin-Weil preferences, however, ignoring life-cycle labour income dynamics or the presence of a risky durable housing good. In the first section of this chapter, we extend their work by including both risky labour income and a durable housing good and then derive an approximate analytical characterization of the optimal risky portfolio choice.

Gomes and Michaelidis (2005) numerically solves a realistically calibrated life-cycle model incorporating recursive preferences and attempts to resolve the asset allocation and the stock market participation puzzles. Vestman (2012) extend Gomes and Michaelidis (2005) model to include housing (without return predictability) and analyse the owning versus renting decisions of households and its impact on stock market participation. They find that the life-cycle model, when calibrated to Swedish household level data, predicts lower market participation for renters relative to homeowners. Importantly, both these papers find that heterogeneity in preferences can explain the allocation and participation puzzles.

3.3. A Dynamic Model of Consumption, Housing and Portfolio Choice

In this section, we describe a stylized model of consumption and asset allocation. We then derive Euler equations which are log-linearized to derive an expression for the optimal risky asset allocation, α_t, the key variable of interest.

3.3.1. Assumptions on Investor Preferences

We consider a partial equilibrium problem in which the investor's preferences are described by the recursive utility proposed by Epstein and Zin (1989, 1991) and Weil (1990). These preferences allow us to disentangle the relative risk aversion and the elasticity of substitution parameters. As these preferences are vital to our analysis, we explain them in detail in the following paragraphs.

To arrive at the recursive preferences we start with the standard expected utility time separable preferences defined as the expected discounted sum of utilities derived from the consumption of non-durable goods (C_t) and housing (H_t) services:

$$V_t = E_t \sum_{s=0}^{\infty} \beta^{s-t} U(C_{t+s}, H_{t+s}) \tag{3.1}$$

where $U(.)$ is the concave, increasing and twice continuously differentiable per-period utility function, E_t denotes the expectations at time t and β is the time preference rate. This value function, V_t, can be defined recursively as

$$V_t = U(C_t, H_t) + \beta E_t V_{t+1} \tag{3.2}$$

Scaling with $(1-\beta)$ [5]

$$V_t = (1-\beta)U(C_t, H_t) + \beta E_t V_{t+1} \tag{3.3}$$

Epstein-Zin-Weil generalize this value function and express it recursively over current (deterministic) consumption and a certainty equivalent $\mu_t(V_{t+1})$ over tomorrow's utility

$$V_t = W(U(C_t, H_t), \mu_t(V_{t+1})) \tag{3.4}$$

where W is an aggregator and the Certainty Equivalent part is defined as:

$$\mu_t(V_{t+1}) = G^{-1}(E_t G(V_{t+1}))$$

with W and G increasing and concave. $\mu_t(V_{t+1}) = V_{t+1}$ if there is no uncertainty on V_{t+1} (future consumption). The more concave G is and the more uncertain V_{t+1} is, the lower is $\mu_t(V_{t+1})$. We follow most of the related literature (Campbell (1993), Campbell and Viceira (1999) for instance) and consider a Constant Elasticity of Substitution (CES) form for the aggregator W and a power functional form for G as

$$W(c,z) = [c^\zeta + \beta z^\zeta]^{1/\zeta}, \quad 0 \neq \zeta < 1,\ 0 < \beta < 1 \tag{3.5}$$

$$G(x) = \frac{x^{1-\Psi}}{1-\Psi}, \quad \Psi > 0 \tag{3.6}$$

with elasticity of substitution $\psi = (1-\zeta)^{-1}$. Thus, ζ is a parameter that is understood to reflect substitutability and Ψ is the relative risk aversion coefficient. Expressing the recursive utility in these functional forms results in the utility function we make use of in this chapter:

$$V_t = \left\{ (1-\beta)u(C_t, H_t)^{1-\frac{1}{\psi}} + \beta E_t \Big[V_{t+1}^{1-\gamma} \Big]^{1/\kappa} \right\}^{\frac{1}{1-1/\psi}} \tag{3.7}$$

where $\kappa = (1-\gamma)/(1-1/\psi)$ specifies the preferences for the timing of the resolution of the uncertainty. If $\kappa < 1$ agents prefer earlier resolution of uncertainty and late

[5]Homogeneity of the recursive preferences implies that they are scale invariant, that is, the order of preferences do not change.

resolution otherwise. This reduces to the standard nonseperable expected utility form as in Ogaki and Reinhart (1998) when $\psi = \gamma^{-1}$. We further assume that the intratemporal utility follows the constant elasticity of substitution form:

$$u(C_t, H_t) = \begin{cases} [\delta C_t^{1-1/\rho} + (1-\delta)H_t^{1-1/\rho}]^{\frac{1}{1-1/\rho}} & \text{if } \rho \neq 1 \\ C_t^{\delta} H_t^{1-\delta} & \text{if } \rho = 1. \end{cases} \tag{3.8}$$

where $\delta \in (0, 1)$ measures the relative importance of housing to non-durable goods consumption and $\rho \geq 0$ is the intratemporal elasticity of substitution. For high values of ρ, agents are willing to substitute the two goods within each period. The two goods become perfect substitutes as $\rho \to \infty$ and perfect complements as $\rho \to 0$. Taking the limit as $\rho \to 1$ yields the Cobb-Douglas specification. Equations (3.7) and (3.8) describe the inter and intratemporal utility functions, respectively, that describe the preferences for the consumers in our model. We have derived these equations to ensure that the risk aversion parameter is separated from the elasticity of substitution parameter.[6]

3.3.2. Assumptions on Labour Income and Human Capital

For most people, labour wealth, that is the present value of future wages, also known as human capital, dwarfs financial wealth. Furthermore, unlike other assets human capital cannot be traded. We assume that labour is supplied inelastically and in the context of our work, it is exogenous.

We follow Viceira (2001) and assume that there are two states for labour income that occur with constant probabilities, employment and retirement. The employment state occurs with probability π_e wherein the investor receives a realization of the income process. The retirement state occurs with probability $\pi^r = 1 - \pi_e$ with $0 < \pi^r < 1$, and it is irreversible: If this state occurs, labour income is set to zero forever. After retirement, the individual faces each period a constant probability of death π^d. Blanchard (1985) and Gertler (1999) have used this probabilistic device to understand horizon effects on decision making, while preserving analytical advantages of an infinite-horizon model. In section §3.4 we consider a more realistic but less tractable life-cycle model with a finite horizon.

In the employment state, labour income is subject to permanent, multiplicative shocks. We model the labour income as in Carroll and Samwick (1997):

$$Y_t = Y_{t-1} \exp(g + \xi_t) \tag{3.9}$$

[6]In the Appendix, we give a detailed explanation of the recursive equation used here.

where $\xi_t \sim NIID(0, \sigma_\xi^2)$. This equation says that the Labour Income at time t, Y_t, is expressed as the product of last period, $t-1$, income Y_{t-1} and the exponent of a mean growth in income term g added to a stochastic component ξ_t. An equivalent way for expressing eq. (3.9) is by taking logs on both sides which would gives us an $AR(1)$ equation with a drift component. We define permanent income in the form of Carroll (1997) as the level of capital income the household would have received in the absence of any transitory shocks. Empirical evidence reveals that the labour income is subject to both transitory and persistent shocks. To make our model analytically tractable and also motivated by Viceira (2001) that transitory shocks have little impact on portfolio allocation, we abstract from the use of these type of shocks. Nevertheless, we do consider these shocks in the numerical section.

It is interesting to note that to the best of our knowledge, existing literature that work with Epstein and Zin (1989, 1991) preferences and aim for deriving analytical solutions do not explicitly consider labour income. This is surprising considering Epstein and Zin (1991) themselves remarked that although " ... a term measuring labor income is not present in our wealth constraint. If labor income is nonstochastic and there is a riskless asset, then the sequence of incomes can be discounted back to period 0 and treated as part of the initial endowment. If labor income is stochastic, then the wealth return form is still applicable provided that the wealth measure is reinterpreted ..." We follow their advice and reinterpret wealth when deriving the Euler equations noting that the two states of nature for income, that is employment and retirement, will now translate to two states of nature for the new wealth. However, when log linearizing the budget constraint, we explicitly decompose the wealth into financial wealth, labour income and housing wealth to understand the effects of each on optimal portfolio choice and consumption.[7]

3.3.3. Assumptions on Housing

We assume a correspondence between the size of the house the investor owns and the consumption benefits (flow of service) derived from it. We also assume that the investor owns the house, ignoring rental occupied housing. At any time period t the investor owns H_t units of the durable housing good. The size and

[7] Campbell (1996) and Jagannathan and Wang (1996) consider "tradable" income as dividends of human capital. They then modify the gross portfolio return as a weighted linear combination of financial wealth and human wealth.

quality of the house is dynamic in that it depreciates at the rate $\nu \in (0, 1]$ in each period. After depreciation, the household chooses housing expenditure EX_t, which can be negative in the case of downsizing. Following Yogo (2006) the housing accumulation follows

$$H_t = (1 - \nu)H_{t-1} + EX_t \tag{3.10}$$

The price of housing fluctuates over time. The price of other consumption goods (the numeraire) is fixed and normalized to one and consider P_t^H to denote the real price of house. This real house price would be used later when we describe the budget constraint.

3.3.4. Assumptions on Investment Opportunities

There are two financial assets that the investor holds, namely a risky stock and a riskless bond. The household can freely trade in both the assets without incurring any transaction costs. The gross return on the portfolio that the investor yields from period t to $t + 1$ is given by:

$$R_{p,t+1} = R_f + \alpha_{t+1}(R_{1,t+1} - R_f) \tag{3.11}$$

This equation says that the return on the portfolio, $R_{p,t+1}$, is the sum of the return from the risk free asset, R_f and the total expected excess return on the risky asset. The variable α_t is the portfolio weight on the risky asset.[8] Following Campbell and Viceira (1999), the expected excess return on the risky asset, $R_{1,t+1} - R_f$, can be expressed in two different forms. First, they can be IID and not predictable,

$$r_{t+1} - r_f = \mu_S + \epsilon_{t+1}^S \tag{3.12}$$

where $R_{1,t+1} = exp(r_{1,t+1})$, $R_f = exp(r_f)$ and α_t is the proportion of total wealth invested in the risky asset at time t. Second, they can also be time varying and predictable with a single factor, f_t, that can predict future excess returns as in Pástor and Stambaugh (2012) or Michaelides and Zhang (2015):

$$r_{t+1} - r_f = f_t + z_{t+1} \tag{3.13}$$

where

$$f_{t+1} = \mu_S + \phi(f_t - \mu_S) + \epsilon_{t+1}^S \tag{3.14}$$

[8] Here we have suppressed the expectation term for convenience.

Here ϵ^S_{t+1} and z_{t+1}, the two innovations to excess returns are assumed to be i.i.d normal random variables with mean zero and variance σ^2_S and σ^2_z. The factor f_t can be considered as the log dividend-price ratio. Eq. 3.12 is when the returns are IID and the rest two arises only when they are predictable. One of the key contributions of this chapter is in comparing the values for risky asset allocation α_t, that is the weight on risky asset, when the returns are predictable eq. (3.13) against when they are not eq. (3.12).

3.3.5. The Inter-Temporal Optimization Problem

The investor's optimization problem involves maximizing the utility function subject to the intertemporal budget constraint which is constructed as follows. At every period t the household enters with financial wealth W_t and stock of durable housing H_t. The household then receives labour income Y_t, this combined wealth is used to meet consumption C_t and housing expenditure EX_t at the price P^H_t. The wealth remaining after these expenditures, the savings, is allocated between risky stocks and riskless bonds. The flow of wealth from t to $t+1$ is written as:

$$W_{t+1} = (W_t + Y_t - C_t + P^H_t(H_t - EX_t))(R_{p,t+1}) \tag{3.15}$$

where it is to be noted that W_{t+1} is the financial wealth at time $t+1$. The total wealth will include both the financial welath as well as the housing wealth. As we care mainly about the optimal risky asset allocation we do not model the wealth explicitly. We leave this task to the numerical §3.4 of this chapter. Here $R_{p,t+1}$ is the one period return on wealth from time t to time $t+1$ and is given as before by

$$R_{p,t+1} = R_f + \alpha_{t+1}(R_{1,t+1} - R_f) \tag{3.16}$$

For the time being, we do not consider adjustment costs to housing. Following Cuoco and Liu (2000), Bansal and Yaron (2004), Yogo (2006, 2009) and João F. Gomes et al. (2009) we define the intertemporal marginal rate of substitution (IMRS) as

$$M_{t+1} = \left[\beta\Big(\frac{C_{t+1}}{C_t}\Big)^{-1/\psi}\Big(\frac{v(H_{t+1}/C_{t+1})}{v(H_t/C_t)}\Big)^{1/\rho-1/\psi} R^{1-1/\kappa}_{p,t+1}\right]^{\kappa} \tag{3.17}$$

where

$$v\Big(\frac{H_t}{C_t}\Big) = \left[1 - \delta + \delta\Big(\frac{H_t}{C_t}\Big)^{1-1/\rho}\right]^{1/(1-1/\rho)} \tag{3.18}$$

M_{t+1}, also known as the Stochastic Discount Factor (SDF) or the pricing kernel, is the discounted ratio of marginal utility tomorrow to marginal utility today.[9]. The novelty of Epstein-Zin preferences is that the pricing kernel (SDF) for each individual asset depends not only on the present and future consumption but also on the household's total market return, see Vissing-Jorgensen and Attanasio (2003).

Depending on the realized state for labour income Y_t, there are two sets of first order conditions for this intertemporal optimization problem. We derive the Euler equations in the Appendix. The Euler equation for the employment state is written as,

$$1 = E_t \Bigg\{ \Bigg(\Bigg[\pi^e \beta^e \Big(\frac{C^e_{t+1}}{C^e_t} \Big)^{-1/\psi} \frac{v(H^e_{t+1}/C^e_{t+1})}{v(H^e_t/C^e_t)}^{1/\rho - 1/\psi} R^{1-1/\kappa}_{p,t+1} \Bigg]^{\kappa} + \tag{3.19}$$

$$\Bigg[(1-\pi^e) \beta^r \Big(\frac{C^r_{t+1}}{C^e_t} \Big)^{-1/\psi} \frac{v(H^r_{t+1}/C^r_{t+1})}{v(H^e_t/C^e_t)}^{1/\rho - 1/\psi} R^{1-1/\kappa}_{p,t+1} \Bigg]^{\kappa} \Bigg) R_{i,t+1} \Bigg\} \tag{3.20}$$

and the retired state as

$$1 = E_t \Bigg\{ \Bigg[\beta^r \Big(\frac{C^r_{t+1}}{C^r_t} \Big)^{-1/\psi} \frac{v(H^r_{t+1}/C^r_{t+1})}{v(H^r_t/C^r_t)}^{1/\rho - 1/\psi} R^{1-1/\kappa}_{p,t+1} \Bigg]^{\kappa} R_{i,t+1} \Bigg\} \tag{3.21}$$

where $\beta^e = \beta$ and $\beta^r = (1-\pi^d)\beta$ and e, r represents variables in the employment and retirement states. Both these equations hold irrespective of the number of tradable assets available. In this chapter, i denotes the riskless bond, the risky security or the investor's portfolio p. When $i = p$, the Euler equations reduce to[10]

$$1 = E_t \Bigg\{ \Bigg(\Bigg[\pi^e \beta^e \Big(\frac{C^e_{t+1}}{C^e_t} \Big)^{-1/\psi} \frac{v(H^e_{t+1}/C^e_{t+1})}{v(H^e_t/C^e_t)}^{1/\rho - 1/\psi} \Bigg]^{\kappa} +$$

$$\Bigg[(1-\pi^e) \beta^r \Big(\frac{C^r_{t+1}}{C^e_t} \Big)^{-1/\psi} \frac{v(H^r_{t+1}/C^r_{t+1})}{v(H^e_t/C^e_t)}^{1/\rho - 1/\psi} \Bigg]^{\kappa} \Bigg) R^{\kappa}_{p,t+1} \Bigg\} \tag{3.22}$$

and

$$1 = E_t \Bigg\{ \Bigg[\beta^r \Big(\frac{C^r_{t+1}}{C^r_t} \Big)^{-1/\psi} \frac{v(H^r_{t+1}/C^r_{t+1})}{v(H^r_t/C^r_t)}^{1/\rho - 1/\psi} \Bigg]^{\kappa} R^{\kappa}_{p,t+1} \Bigg\} \tag{3.23}$$

[9]The derivation of the stochastic discount factor and the Euler equation is given in the Appendix

[10]The superscript in consumption C here refers to the employment state. When we begin our log linearisation we remove this superscript and analyse the two cases separately. The expectations in those equations are subsumed with an upper script e which refers to expectations and not employment.

In the absence of arbitrage and under complete markets, there exists a strictly positive stochastic discount factor, M_{t+1}, which satisfies equation's (3.22) and (3.23) for any number of tradable assets (see Campbell (2000)).[11] These Euler equations (3.22) and (3.23) reveal that when the utility is not additively separable in the non-durable and durable consumption goods, marginal utility has an extra multiplicative term $v(H/C)^{\kappa(\frac{1}{\rho}-\frac{1}{\psi})}$. The effect of the expediture share of the durable to the non-durable good (H/C) on marginal utility depends on the relative magnitudes of ψ and ρ. When $\psi = \rho$, i.e. when utility is additively separable in durable and non-durable consumption goods, the marginal utility is independent of the durable consumption. If $\psi < \rho$, then for a given level of non-durable consumption, marginal utility decreases in the ratio of stock of durables to non-durables (see Yogo (2006)).

The Euler equations (3.22) and (3.23) represent equilibrium conditions for a consumer who holds risky assets and wishes to smooth consumption over time. Although we have refrained from using borrowing constraints or transaction costs, we stress that the corresponding Euler equations under the presence of such market imperfections will be similar as long as the constraints are not binding between two given time periods, see Attanasio and Weber (2010).

3.4. An Approximate Analytical Characterization

Exact closed form solutions for this optimization problem do not exist unless we restrict our analysis to the retirement state and assume that the investment opportunity set is constant, see Merton (1973). Thus, we are left with either solving it numerically or approximating the non-linear equations with their percentage deviations from the steady state, that is log-linearizing. We pursue both these methods. Firstly, we find an approximate analytical solution in the lines of Campbell (1993), Campbell and Viceira (1999) and Viceira (2001). We then extend this

[11] The more familiar notation of these Euler equations is the form

$$E_t[M_{t+1}R_{i,t+1}] = 1, \quad i = f, p$$

meaning that there are state prices, positive discount factors one for each state and date such that the state price of any asset is merely the state price weighted sum of future payoffs. This is the basis for all modern asset pricing models, each one with a specific form of SDF, see Bansal and Yaron (2004) literature to a more realistic life-cycle model with calibrated exogenous variables, add collateral constraints on housing etc which is then solved numerically.

We derive the approximate analytical solution building on the method proposed by Campbell (1993). Firstly, we log-linearize the Euler equations and the budget constraints around the stationary steady state. We approximate the Euler equation using a second order expansion to capture the second moment effects such as precautionary savings. [12] Then, we characterize the properties of, α_t, the optimal savings allocation on the risky asset in terms of its determinants.

3.4.1. Log Linearized Euler Equations

The first step in the solution method is to log linearize the Euler equations for the employment eq. (3.22) and the retirement eq. (3.23) states. Appendix (3.C) details our approach. The log linear Euler equation for the retirement state is derived as

$$0 \approx \Bigg(E_t\Big[\kappa \ln \beta^r - \frac{\kappa}{\psi}(c^r_{t+1} - c^r_t) - \frac{\kappa\delta}{\psi}(h^r_{t+1} - h^r_t) + \frac{\kappa\delta}{\psi}(c^r_{t+1} - c^r_t) + \kappa r_{p,t+1}\Big] \\ + \frac{1}{2} Var_t\big(\kappa r_{p,t+1} - \frac{\kappa}{\psi}(c^r_{t+1} - c^r_t) - \frac{\kappa\delta}{\psi}(h^r_{t+1} - h^r_t) + \frac{\kappa\delta}{\psi}(c^r_{t+1} - c^r_t)\big)\Bigg) \tag{3.24}$$

where lower case letters denote variables in logs. This equation implies that there is a linear relationship between expected log consumption growth and the expected log return on wealth. The retirement state is characterized by no labour income. Thus, this equation can be considered an extension of the Campbell and Viceira (1999) log Euler equation with a durable consumption good. As long as the return on wealth and consumption (both durable and non-durable) growth is conditionally log normal, this equation will hold exactly. Our assumptions on return on wealth makes it conditionally log-normal.[13]

[12] Precautionary savings comes from σ^2, the volatility of consumption. When consumption is more volatile, consumers are more worried about the low consumption states than they are pleased by the high consumption states. Therefore, people want to save more bringing down interest rates. A measure of precautionary savings requires that the utility be third differentiable, that is to say, Linear-Quadratic preferences exhibit no prudence, see Attanasio and Weber (2010).

[13] If $X \sim LN$,X is lognormal, such that $\log X$ follows a normal distribution with $\log X = x \sim N(\mu, \sigma^2)$. Then, $EX = \exp(\mu + \sigma^2/2)$ and $\log EX = E \log X + \frac{1}{2} Var(log X)$ .The approximation $\ln E \exp(z) \approx Ez + \sigma_z^2/2$ is exact when z is a normal random variable.

Unlike retirement, while employed the investor faces the risk of being retired in the next state. Hence, optimal inter-temporal consumption now is weighed for the possible states of nature. The log linear Euler equation for the employment state, given by,

$$1 \approx \sum_{s=e,r} \pi_s^{\kappa} \Bigg(E_t \Big[1 + \kappa \ln \beta^s - \frac{\kappa}{\psi}(c_{t+1}^s - c_t^e) - \frac{\kappa\delta}{\psi}(h_{t+1}^s - h_t^e) + \frac{\kappa\delta}{\psi}(c_{t+1}^s - c_t^e) + \kappa r_{p,t+1} \Big] \\ + \frac{1}{2} Var_t \big(\kappa r_{p,t+1} - \frac{\kappa}{\psi}(c_{t+1}^s - c_t^e) - \frac{\kappa\delta}{\psi}(h_{t+1}^s - h_t^e) + \frac{\kappa\delta}{\psi}(c_{t+1}^s - c_t^e) \big) \Bigg) \tag{3.25}$$

is equal to the probability weighted sum of the log-linear Euler equations for both states of nature of the labour income process, the employment and the retirement.

A crucial assumption we made while deriving both these Euler equations is that the intra-period felicity follows a Cobb-Douglas form with intra-temporal elasticity of substitution $\rho = 1$. This assumption is primarily to maintain tractability. Also, empirical estimates such as by Ogaki and Reinhart (1998) and by Yogo (2006) find ρ to be very close to 1. Theoretically, if $\rho \neq 1$, this introduces another state variable H_t/C_t to the model, and the share of durable consumption varies with time. Furthermore, as Yang (2011) elaborates, when ρ is not very different from one, this generates only small temporal variations in the quantities of interest.

If we subtract the loglinear Euler equation for the riskless asset ($i = f$) from the loglinear Euler equation for the risky asset ($i = S$), for the retirement case, we find that:[14]

$$E_t r_{i,t+1} - r_f + \frac{1}{2} Var_t[r_{i,t+1}] = \frac{\kappa}{\psi}(1-\delta) cov(r_{i,t+1}, \Delta c_{t+1}) + \frac{\kappa\delta}{\psi} cov(r_{i,t+1}, \Delta h_{t+1}) + (1-\kappa) cov(r_{i,t+1}, r_{p,t+1}) \tag{3.26}$$

This equation says that the expected excess log return on the risky asset is determined by its own variance and by a weighted combination of three covariances. The first covariance is between log return on the risky asset and consumption growth with weight $\frac{\kappa}{\psi}(1-\delta)$, the second covariance is between durable consumption growth (housing) and log return on the risky asset with weight $\frac{\kappa\delta}{\psi}$ and the third one is between log returns on the risky asset and log return on the portfolio with weight $(1-\kappa)$. Equation (3.26) is the starting point of our analysis of optimal portfolio choice.

Following Campbell (1993, 1996) and Viceira (2001) we also log-linearise the budget constraint for both states around the mean consumption to income ratio, the wealth to income ratio and the housing wealth to income ratios. For the employment state we obtain,

$$w^e_{t+1} - y_{t+1} = k^e + \rho^e_w(w^e_t - y_t) - \rho^e_c(c^e_t - y_t) + \rho^e_h(p^h_t + \ln(1-\nu) + h^e_{t-1} - y_t) - \Delta y_{t+1} + r^e_{p,t+1} \tag{3.27}$$

where we have suppressed the Expectations operator in the superscript e. The details of the derivation is given in Appendix. The log-linearisation constants k^e and the ρ's are endogenous in that they depend on the average log consumption to income, log wealth to income and log housing wealth to income ratios. The approximation should hold exactly when these ratios are constant. In the retirement

[14]For a general risky asset, $i = S$, we use log-linearized forms of the Euler equation (3.21) meaning that now we have an extra term in both the expectation and variance operators in eq's. (3.24) and (3.25) given by $(\kappa-1)r_{i,t+1}$. Furthermore, while deriving this equation we make use of the following properties for variance and covariance

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) + 2cov(X,Y) \\ Var(X+a) &= Var(X) \\ cov(X, Y+Z) &= cov(X,Y) + cov(X,Z) \\ cov(X, Y+a) &= cov(X,Y) \\ cov(X,a) &= 0, \end{aligned}$$

where X, Y, Z are random variables and a is a constant. With these results, the derivation is straight forward and is hence omitted.

state, there is no labour income ($Y_t = 0$), thus the log linear budget constraint simplifies to

$$w_{t+1}^r - w_t = k^r + \rho_h^r(p_t^h + \ln(1-\nu) + h_{t-1}^r - w_t) - \rho_c^r(c_t^r - w_t) + r_{p,t+1}^r \quad (3.28)$$

where k^r and the ρ's are again log-linearisation constants interpreted as before. The log-linearised budget constraints took the return on the wealth portfolio as given, and does not relate it to the returns on individual assets. We can approximate the log portfolio return on wealth as [15]

$$r_{p,t+1} - r_f = \alpha_t(r_{t+1} - r_f) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 \quad (3.29)$$

where α_t is the vector of risky asset weights, ${\sigma_t}^2$ is the vector containing the diagonal elements of Σ_t.

We can now use these log-linearised equations to characterise the investor's approximate optimal portfolio choice policy in each state of the labour income. As once the retirement state occurs, the investor cannot revert back to being employed, the optimal rules in this states are independent of those in the employment state. On the contrary, in the employment state, the investor must take into account the off-chance of being retired in the near future when deciding on asset allocation.

3.4.2. Characterizing the Optimal Portfolio Choice

In the last section, we derived the log excess return on the risky asset for the retirement state, eq. (3.26), as

$$\begin{aligned} E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma_{it}^2 = \frac{\kappa}{\psi}(1-\delta)\sigma_{r_{i,t+1},\Delta c_{t+1}} + \frac{\kappa\delta}{\psi}\sigma_{r_{i,t+1},\Delta h_{t+1}} \\ +(1-\kappa)\sigma_{r_{i,t+1},r_{p,t+1}} \end{aligned} \quad (3.30)$$

where $Var_t[r_{i,t+1}] = \sigma_{it}^2$, $cov_t(r_{i,t+1}, \Delta c_{t+1}) = \sigma_{r_{i,t+1},\Delta c_{t+1}}$, $cov_t(r_{i,t+1}, \Delta h_{t+1}) = \sigma_{r_{i,t+1},\Delta h_{t+1}}$ and $cov_t(r_{i,t+1}, r_{p,t+1}) = \sigma_{r_{i,t+1},r_{p,t+1}}$. As in the retirement state, with no labour income, our strategy, following Campbell and Viceira (1999), is to characterize the covariance terms as functions of the exogenous risky asset return

[15] The derivation of log approximation to portfolio return is standard in the literature, see for example Campbell and Viceira (2002).

and the stationary consumption-wealth ratio. The covariance between log risky asset return and non-durable consumption growth is written as,

$$\begin{aligned}
\sigma_{r_{i,t+1},\Delta c_{t+1}} &= cov_t(r_{i,t+1},\Delta c_{t+1}) \\
&= cov_t(r_{i,t+1},(c_{t+1}-w_{t+1})-(c_t-w_t)+\Delta w_{t+1}) \\
&= cov_t(r_{i,t+1},(c_{t+1}-w_{t+1}))-cov_t(r_{i,t+1},c_t-w_t)+cov(r_{i,t+1},\Delta w_{t+1}) \\
&= cov_t(r_{i,t+1},(c_{t+1}-w_{t+1}))+cov_t(r_{i,t+1},\Delta w_{t+1}) \\
&= \sigma_{r_{i,t+1},(c_{t+1}-w_{t+1})}+cov_t(r_{i,t+1},r_{p,t+1}) \\
&= \sigma_{r_{i,t+1},(c_{t+1}-w_{t+1})}+\alpha_{it}var_t(r_{i,t+1}) \\
&= \sigma_{r_{i,t+1},(c_{t+1}-w_{t+1})}+\alpha_{it}\sigma^2_{it} \qquad (3.31)
\end{aligned}$$

where the second equality is trivial algebra, the third uses properties of the covariance operator for random variables, and the rest follows from substituting values for Δw_{t+1} and $r_{p,t+1}$ from the retirement state log linearised equations (3.28) and (3.29). In addition to these we also use the fact that $cov(x_{t+1},z_t)=0$, see Campbell and Viceira (1999). In similar fashion, the covariance between log risky asset return and durable (housing) consumption growth is derived as

$$\sigma_{r_{i,t+1},\Delta h_{t+1}} = \sigma_{r_{i,t+1},(h_{t+1}-w_{t+1})}+\alpha_{it}\sigma^2_{it} \qquad (3.32)$$

and finally the covariance between log risky asset return is a direct implication of equation (3.29):

$$\begin{aligned}
\sigma_{r_{i,t+1},r_{p,t+1}} &= cov_t(r_{i,t+1},r_f+\alpha_t'(r_{t+1}-r_f)+\frac{1}{2}\alpha_t'\sigma^2_t-\frac{1}{2}\alpha_t'\Sigma_t\alpha_t) \\
&= \alpha_{it}\sigma^2_{it} \qquad (3.33)
\end{aligned}$$

Now that we have characterized the three covariance terms, we substitute these terms into (3.30) to get

$$\begin{aligned}
E_t r_{i,t+1}-r_f+\frac{1}{2}\sigma^2_{it} &= \frac{\kappa}{\psi}(1-\delta)(\sigma_{r_{i,t+1},(c_{t+1}-w_{t+1})}+\alpha_{it}\sigma^2_{it}) \\
&+\frac{\kappa\delta}{\psi}(\sigma_{r_{i,t+1},(h_{t+1}-w_{t+1})}+\alpha_{it}\sigma^2_{it})+(1-\kappa)\alpha_{it}\sigma^2_{it} \qquad (3.34)
\end{aligned}$$

which can be rearranged using the fact that the parameter specifying the timing of the resolution of uncertainty is $\kappa=(1-\gamma)/(1-\frac{1}{\psi})$. We substitute to get the optimal portfolio allocation on the risky assets for the retirement state. The employment state follows the same procedure. These results are described in the following Proposition.

PROPOSITION 1: ***The optimal portfolio share of risky assets for the retirement state is*** [16]

$$\alpha^r_{it} = \frac{1}{\gamma}\frac{E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma^2_{it}}{\sigma^2_{it}} + \Big(\frac{1}{1-\psi}\Big)\Big(\frac{1-\gamma}{\gamma}\Big)\left(\frac{(1-\delta)\sigma_{i,(c_{t+1}-w_{t+1})} + \delta\sigma_{i,(h_{t+1}-w_{t+1})}}{\sigma^2_{it}}\right) \tag{3.35}$$

$$= \alpha^{MD_r}_{it} + \alpha^{HD_r}_{it}. \tag{3.36}$$

where $Var_t(r_{i,t+1}) = \sigma^2_{it}$, $cov(r_{i,t+1}, c_{t+1} - w_{t+1}) = \sigma_{i,c_{t+1}-w_{t+1}}$ *and* $cov(r_{i,t+1}, h_{t+1} - w_{t+1}) = \sigma_{i,h_{t+1}-w_{t+1}}$.

and for the employment state is

$$\alpha^e_{it} = \frac{1}{\gamma}\frac{E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma^2_{it}}{\sigma^2_{it}} + \Big(\frac{1}{1-\psi}\Big)\Big(\frac{1-\gamma}{\gamma}\Big)\left(\sum_{s=e,r}\pi^{\kappa}_s\Big[\frac{(1-\delta)\sigma^r_{i,(c^s_{t+1}-w_{t+1})} + \delta\sigma^r_{i,(h^s_{t+1}-w_{t+1})}}{\sigma^2_{it}}\Big]\right) \tag{3.37}$$

$$= \alpha^{MD_e}_{it} + \alpha^{HD_e}_{it}. \tag{3.38}$$

where $Var_t(r_{i,t+1}) = \sigma^2_{it}$, $cov(r_{i,t+1}, c^s_{t+1} - w_{t+1}) = \sigma_{i,c^s_{t+1}-w_{t+1}}$ *and* $cov(r_{i,t+1}, h^s_{t+1} - w_{t+1}) = \sigma_{i,h^s_{t+1}-w_{t+1}}$.

Proof : See Appendix (3.E).

The first equation characterizes the optimal portfolio choice for the risky asset in the retirement state when there is no labour income and the second one with labour income. These equations have two parts. The first part, α^{MD}_{it} captures any asset demand induced completely from the current risk premium adjusted for Jensen's inequality by adding one half the own variance, called the "myopic demand" of risky asset. The myopic demand corresponds to the single-period

[16]Equation (3.36) gives us valuable information to the determinants of optimal risky asset allocation, however, it is not a complete solution of the model because the current optimal portfolio allocation is a function of future portfolio and consumption decisions which are endogenous in our model. This dependence on future consumption and portfolio decisions operates through the conditional covariances. The conditional covariances depends on the log non-durable consumption to wealth ratio and log durable housing consumption to wealth ratio. These equations can be solved forward and expressed in terms of expectations of future consumption and portfolio returns, see Campbell (1993) equation (3.9).To solve for an exact solution to optimal consumption and portfolio policies, the method of Campbell and Viceira (1999) can be applied to guess a functional form for these policies and identify the parameters using the technique of undetermined coefficients. This analysis is beyond the scope of this chapter and is left for future work. Instead we characterize the solution as we are only interested in the economic intuition behind these solutions. The quantitative analysis is left for the numerical section.

demand for an asset, when there are no changes in the investment opportunity set, as in the traditional single-period portfolio choice problems. This myopic component is directly proportional to the risk premium, $E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma_{it}^2$, and inversely proportional to the investor's risk aversion, γ.

The second term, α_{it}^{HD} describes Merton (1969, 1973)'s "inter-temporal hedging demand". The hedge demand corresponds to the additional demand for an asset, when the changes in the investment opportunity set are incorporated in the portfolio choice problem, as in the multi-period portfolio choice problem of Merton (1973). This component arises when the investor seeks to hedge against future shocks to the investment opportunity set. As investment opportunities are varying over time, long-term investors care about shocks to investment opportunities. In other words, the productivity of wealth also matters and not just the wealth itself.

Samuelson (1969) and Merton (1971) state conditions under which a long term investor finds it optimal to act myopically, choosing the same portfolio as a short term investor. These include power utility and IID returns. Power utility (also logarithmic utility) implies constant relative risk aversion nullifying our model of recursive preferences. Next, if returns are IID no new information arrives between one period and the next, so there is no reason for the portfolio choice to change inter-temporally. Thus, both conditions imply that there are no changes in time over investment opportunities that might induce changes in consumption (durable and non-durable) relative to wealth. Campbell and Viceira (2001) equate these conditions to a constant consumption-wealth ratio meaning that

$$\begin{aligned}\sigma_{i,c_{t+1}-w_{t+1}} &= 0 \\ \sigma_{i,h_{t+1}-w_{t+1}} &= 0.\end{aligned} \tag{3.39}$$

Thus, we are left with just the myopic part of risky asset demand,

$$\alpha_{it}^{r} = \alpha_{it}^{MD_r} = \frac{1}{\gamma}\frac{E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma_{it}^2}{\sigma_{it}^2} \tag{3.40}$$

which is exactly the result of Viceira (2001) for the retirement state. This equation states that optimal portfolio choice is independent of the level of wealth and is only optimized over the mean and variance of the risky return. The presence of a durable good makes no difference to the portfolio rule. In contrast, our model specifies time varying investment opportunity sets, as expected returns are state dependent, and hence the hedging component is non-zero meaning that the presence of the durable good does influence the proportion of wealth invested in the risky asset. Unfortunately as we do not have a complete analytical solution we cannot exactly pin-point the way in which the durable housing good impacts α_t. The only point we make is that housing forms a kind of background risk for the investor meaning that it is undiversifiable and hence should, all else constant, bring down α_t.

Nevertheless, three important results can be derived from the Proposition. Firstly, we find that the relative risk aversion parameter γ is inversely related to α_t and the elasticity of intertemporal substitution ψ is directly related to α_t. Both these parameters are thus found to have opposite affects on the optimal risky equity demanded. The fact that increasing risk aversion decreases risky asset demand is universal throughout the literature, see Campbell and Viceira (1999), Barberis (2000), Campbell (2006) etc. This is true even in the presence of housing, for example Flavin and Yamashita (2002) find decreasing amount of wealth invested in risky stocks or housing with increasing risk aversion in their quantitative analysis using PSID data. However, existing literature is conflicted regarding the effect of the EIS parameter on α_t. For example Vissing-Jorgensen (2002), Gomes and Michaelidis (2005) and Gârleanu and Panageas (2015) find that higher EIS motivates more consumption smoothing and thereby higher savings and risky asset accumulation. However, Vestman (2012) predicts using a lifecycle portfolio choice model (with housing) that higher EIS lowers risky equity demand. In this chapter consistent with our analytical prediction, Proposition 1, our numerical model also shows the same positive relationship between EIS and α_t.

A second result that we can derive from Proposition 1 is that *in the absence of any correlation between consumption, house prices, labour income or risky returns, the optimal portfolio share of savings in the risky asset simplifies to the myopic demand*. In other words there is no hedging component. If we set all the correlations or covariances to zero, we get $\alpha_{it} = \alpha_{it}^{MD}$. This proposition becomes very valuable in our numerical analysis where for the benchmark model we set all correlations to zero. We then impose empirically calibrated covariances or correlations of labour income, housing prices etc with returns so that we can quantify the hedging demand. It has to be noted that the absence of hedging motives does not imply that the risky asset demand is fully myopic because investors, specially when returns are mean reverting, can accrue huge wealth by timing the market. That is, optimal strategies contains some planning for the future.

A third and final result that we get from Proposition 1 is that risky asset demand under time varying returns or return predictability can be substantially different from the IID case. Importantly, if the factor predicting returns is high so will be the expected excess returns and hence a higher risky asset is demanded, refer eq. 3.12 and the subsequent ones. Furthermore, a higher persistence ϕ of the return predicting factor also results in an increased share of the risky equity demanded. A unit root persistence is suggestive of a market bubble implying under such speculative markets α_t goes up. Eraker et al. (2003), Broadie et al. (2007) and Elkamhi and Stefanova (2015) to name a few finds strong evidence for jumps in returns during the periods of stock market bubbles. A very low realization of the factor can arise due to a market crash, alternatively termed as "rare disaster", when there is a huge drop in all macroeconomic variables. Hence, α_t is found to be procyclical and as factor processes follow a long run mean growth rate so will the risky equity demanded. This explains why some recent empirical studies such as Guiso and Sodini (2013) find that the level of wealth invested in risky equities has been steadily increasing.

The proof for these three results are fairly obvious from Proposition 1 and requires no algebraic work, hence omitted from discussion. The analysis in this section abstracted from realistic arguments such as the presence of transaction costs, borrowing constraints, etc. Furthermore, the proposition 1 was more intuitive, the arguments were not based on an exact solution of either consumption or portfolio choice.

3.5. A Life-Cycle Exercise - Quantitative Results

In the last section we derived an analytical expressions for the risky asset allocation demand, α_t, for the two cases of labour income: employment and retirement. Proposition 1 describes the derived expression for α_t. This Proposition gives two important results. Firstly, the risky equity demanded is inversely proportional to the risk aversion and directly related to the elasticity of intertemporal substitution. Secondly, the risky equity demanded is significantly different when returns are predictable (time varying) compared to the IID case. Importantly for high factors (such as the dividend-price ratio) the investor increases his holdings of the risky asset, $\alpha_t \uparrow$.

The analysis though insightful abstracted from realistic labour income dynamics. In this section, we extend the stylized model to a realistic life-cycle one with short-sales and borrowing constraints, bequest motive, uncertain survival probability and importantly an empirically calibrated labour income process. We then calibrate the parameters and simulate it to get quantitative results. We also check how our numerical results differ from our analytical characterization in the previous section.

3.5.1. The Life-Cycle Model

3.5.1.1. Modified Preferences

We extend the preferences to a discrete time life-cycle model where t denotes adult age and is given by effective age minus 19. Each period corresponds to 1 year and agents live for a maximum of T periods ($T = 81$). The probability that a consumer or investor is alive at time $t+1$ conditional on being alive at time t is denoted by p_t. This p should not be confused with the price of housing used in the last section. We will use the superscript h when we refer to housing prices. The presence of conditional survival probabilities means there is now uncertainty regarding mortality and hence, this induces the "precautionary savings" motive of the investor.

In each period t, a household as before, derives utility from consuming non-durable goods C_t and durable housing H_t. The household's modified preferences are defined by:

$$V_t = \left\{ (1-\beta p_t)u(C_t, H_t)^{1-\frac{1}{\psi}} + \beta E_t \Big[p_t V_{t+1}^{1-\gamma} + (1-p_t) b \frac{(W_{t+1}/b)^{1-\gamma}}{1-\rho} \Big]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}} \tag{3.41}$$

where β is the time discount factor, ψ is the Elasticity of Inter-temporal Substitution, γ is the coefficient of Relative Risk Aversion, b determines the strength of the bequest motive and W_{t+1} denotes the wealth at time $t+1$. β denotes the time preference, or impatience, inducing conditional survival probabilities multiplicatively with β implies that consumers get more and more impatient with age. A bequest motive is a reason for why households do not run down their wealth faster during retirement. The terminal condition for the recursive equation is expressed in terms of the bequest motive and terminal wealth as:

$$V_{T+1} = b \frac{(W_{T+1}/b)^{1-\rho}}{1-\rho} \tag{3.42}$$

The intra-period consumption aggregator as before takes a constant elasticity of substitution (CES) form, see eq. (3.8), reproduced here for convenience.

$$u(C_t, H_t) = \begin{cases} [\delta C_t^{1-1/\rho} + (1-\delta) H_t^{1-1/\rho}]^{\frac{1}{1-1/\rho}} & \text{if } \rho \neq 1 \\ C_t^{\delta} H_t^{1-\delta} & \text{if } \rho = 1. \end{cases} \tag{3.43}$$

where $\delta \in (0,1)$ measures the relative importance of housing to non-durable goods consumption and $\rho \geq 0$ is the intratemporal elasticity of substitution.

3.5.1.2. Modified Labour Income Risk

Empirically, it has been found that the flow of labour income is well represented as a sum of three components: an aggregate component that is subject to economy-wide fluctuations; an idiosyncratic component, which captures individual specific shocks; and a deterministic component due to lifecycle predictability in wages, see Bodie et al. (1992), Carroll and Samwick (1997) and Gourinchas and Parker (2002). These three components combine to create a hump-shaped deterministic lifecycle labour income profile, wages increase with age when workers are young and then decline when they approach retirement.

We extent our stylized model in §2.2 to now include both persistent and transitory shocks. Furthermore, instead of having exogenous retirement probability, we fix the retirement date at K corresponding to actual age 65. The investor j works for the first K periods of his life supplying labour inelastically in each period and receive stochastic labour income Y_{jt} against which he cannot borrow. The investor j's age t labour income before and after retirement is exogenously given by:[17]

$$\log(Y_{jt}) = \begin{cases} f(t, Z_{jt}) + \nu_t + \omega_{jt} & \text{for } t \leq K \\ \lambda f(K, Z_{jK}) & \text{for } t > K, \end{cases} \tag{3.44}$$

where $f(t, Z_{jt})$ is a deterministic component of age t:

$$f(t, Z_{jt}) = \beta_0 + \beta_1 age_{jt} + \beta_2 age_{jt}^2/10 + \beta_3 age_{jt}^3/100 \tag{3.45}$$

Thus, the prior retirement log income is the sum of a deterministic component that can be calibrated to capture the hump shaped earnings over the life-cycle and

[17] In reality labour income is not exogenous, individuals must decide how many hours to work and how much effort to put on the job, decisions that will influence the amount of labor income received. By having exogenous labour income, we rule out the possibility that an individual who has had a bad portfolio return can work more hours to compensate for it

two random components, one transitory and one persistent. Also, ν_t represents the aggregate component and ω_{jt} captures idiosyncratic shocks. Following the literature, we assume that the idiosyncratic labour income risk ω_{jt} is an IID normally distributed random variable - $\omega_{jt} \sim N(0, \sigma_\omega^2)$. Furthermore, the aggregate shock ν_t follows a random walk:

$$\nu_t = \nu_{t-1} + \epsilon_t^\nu \tag{3.46}$$

where ϵ_t^ν is IID $N(0, \sigma_\nu^2)$. Retirement is assumed to be exogenous and deterministic with all households retiring in time period K, corresponding to age 65 ($K = 46$). Following Gomes and Michaelidis (2005) retirement income is modelled as a constant fraction $\lambda \in [0, 1]$ of permanent labour income in the last working year:

$$\log(Y_{jt}) = \log\lambda + f(K, Z_{jK}) \tag{3.47}$$

3.5.1.3. Modified Illiquid Housing with Constraints

In our analytical section, we had several restrictive assumptions on the durable good, housing. We refrained from any transaction costs which made the house a liquid asset. In reality, the household has to decide every period if he should move or not and faces substantial costs when doing so. This can be endogenous or even exogenous depending on labour income or family specific shocks. We start by assuming that:

$$H_t \geq H_{min} \quad \forall t \tag{3.48}$$

where H_{min} is the minimum house size. This constraint basically takes care of the indivisibility property of the house. As in Yao and Zhang (2004) and Hu (2005), we assume that in each period t, with probability π_h the household is forced to sell the house and buy an other one. Cocco (2004) calls this an "involuntary move". With probabililty $1 - \pi_h$ the household is not forced to move, but may still do so if that is optimal. We use a state variable $InvMove_t$ to capture involuntary house trades which takes the value of 1 if at t the household is in a state where it if forced to move, and zero otherwise.

The house sale is associated with a monetary cost equal to a proportion Λ of the house value and instead of assuming that the house depreciates every period, we assume that there is an annual maintenance cost equal to a proportion mc_h of the house value.

The price of housing fluctuates over time. The price of other goods consumption (the numeraire) is fixed and normalized to one. Let p_t^h denote the real log price of house. We follow Campbell and Cocco (2003) and assume that the real house price growth is given by

$$\Delta p_t^h = \mu_h + \epsilon_t^h \tag{3.49}$$

a constant μ_h and an i.i.d normally distributed shock with mean zero and variance σ_h^2.

3.5.1.4. Financial Assets and Credit Markets

We improve on our analytical modelling section with the addition of a fixed cost of equity market participation and the existence of a mortgage. As before, we assume that there are two assets that the household can invest: a riskless asset with gross real return $R_f = \exp[r_f]$, which we call Bonds, and a risky asset with gross real return $R_t = \exp[r_t]$, which we call Stocks. The existing literature that analyse optimal portfolio choice with housing assume a constant opportunity set or in other words I.I.D returns. In contrast, we consider two cases for the log excess return on the risky asset. The excess log return can be either an IID process as in Cocco (2004), Yao and Zhang (2004) or Hu (2005):[18]

$$r_{t+1} - r_f = \mu_S + \epsilon_{t+1}^S \tag{3.50}$$

or time varying with a single factor, f_t, that can predict future excess returns as in Pástor and Stambaugh (2012) or Michaelides and Zhang (2015):

$$r_{t+1} - r_f = f_t + z_{t+1} \tag{3.51}$$

where

$$f_{t+1} = \mu_S + \phi(f_t - \mu_S) + \epsilon_{t+1}^S \tag{3.52}$$

Here ϵ_{t+1}^S and z_{t+1}, the two innovations to excess returns are assumed to be IID normal random variables with mean zero and variance σ_S^2 and σ_z^2. The factor f_t can capture the widely documented mean-reversion aspect of stock market returns, see Campbell and Viceira (1999). We calibrate this factor as the log dividend-price ratio. The dollar amount the investor has in Treasury Bills and Stocks are represented as B_t and S_t respectively. We follow Cocco (2004) and

[18]This is the same equation we use in the analytical section, see eq. (3.12).

Cocco et al. (2004) and assume that the investor cannot short-sell either of these assets so that:

$$S_t \geq 0, \tag{3.53}$$

$$B_t \geq 0 \quad \forall\, t \tag{3.54}$$

An implication of these constraints is that the household cannot lever up using future labour income or retirement wealth to invest in the stock market. It also means that the allocation of wealth to both stocks and bonds remain non-negative at all dates.[19] Furthermore, there is a fixed cost involved in equity market participation. This fixed cost can be considered as the cost of opening a brokerage account, understanding how the market works, the cost of financial literacy etcetera. Alan (2006) estimate this cost as 2% of annualized labour income while Gomes and Michaelidis (2005) calibrate this fixed cost as 6% . Following these, we assume that the investor incurs a fixed one time cost which is a proportion F of the permanent component of labour income.

In addition to the two financial assets, the investor who is also a homeowner can borrow against the value of the house, which we call mortgage, at a gross real fixed rate of R_D. The dollar amount the investor owes in mortgage at date t is denoted as D_t. Following Cocco (2004), the investor can borrow up to the house value minus a down-payment, which is assumed to be a proportion d of the value of the house so that:

$$D_t \leq (1-d)P_t H_t, \quad \forall\, t \tag{3.55}$$

3.5.1.5. The Household's Modified Optimization Problem

The investor maximizes

$$\max_{(C_t, H_t, D_t, FC_t, \triangle_t)_{t=1}^{T}} E(V_0) \tag{3.56}$$

where V_0 is given by eqn's 3.41, 3.42, 3.43 and is subject to the Labour Income Constraints 3.44, 3.45, 3.46 and 3.47, the housing constraint 3.48, the financial assets constraints 3.50, 3.51, 3.52, 3.53, 3.54, 3.55 and the budget constraints

[19] Benzoni et al. (2007) find that the presence of short-sale constraints can limit equity market participation, specially young and poor investors who have a high incentive to short stocks.

expressed in terms of cash on hand X_t:[20]

$$S_t + B_t = \begin{cases} X_t - C_t - FC_tFY_t - mc_hP_tH_{t-1} + D_t, & \forall t \\ X_t - C_t - FC_tFY_t - mc_hP_tH_{t-1} + D_t + (1-\Lambda)P_tH_{t-1} - P_tH_t, & \forall t \end{cases} \tag{3.57}$$

where the first one is the no house trade case and the second one expresses the house trade case. Here, we define cash in hand, X_t, along the lines of Deaton (1991) and Carroll (1997), as the sum of liquid or financial wealth and labour income:

$$X_t = (R_tS_{t-1} + R_fB_{t-1} - R_DD_{t-1}) + Y_t \tag{3.58}$$

Here, α_t denotes the level of wealth invested in risky stocks over stocks plus bonds at any time t. This is akin to α_t in our analytical section. Also, FC_t is an indicator variable which is allowed to take the value of 1 if the investor chooses to pay the fixed cost and zero otherwise. Wealth at date $T+1$, the bequeathed wealth after the terminal period T, is given by

$$W_{T+1} = X_{T+1} - mc_hH_TP_{T+1} + (1-\Lambda)H_TP_{T+1} \tag{3.59}$$

This is the bequeathed wealth which is equal to financial wealth plus housing wealth net of debt outstanding. The numerical solution method we use is standard Value function Iteration and is explained in the Appendix (3.G).

3.5.2. Calibration

The parameters for our benchmark model are reported in Table 3.1. We follow the literature and set standard values for the preference parameters as $\gamma = 5$ (risk aversion), $\psi = 0.2$ (EIS), $\beta = 0.96$ (discount factor) and $\varepsilon = 0.10$ (IES). These values for γ and ψ imply that we are in fact assuming $\gamma = 1/\psi$ or in other words CRRA preferences. This is mainly to make our results comparable to the literature. We do change these values later on to test for the sensitivity of our results for these parameters.

The investor dies with probability 1 at age 100. Prior to this age, conditional survival probabilities for other ages have been taken form the National Center for Health Statistics as in Winter et al. (2012). Following Gomes and Michaelidis (2005), we set the bequest motive at 2.5. We also present sensitivity analysis for this parameter. Table 3.1 reports all the calibrated parameters of our model.

[20] For a detailed discussion on how we construct these budget constraints, refer to Appendix (3.F).

Table 3.1. Baseline Parameters

Description	Parameter	Value
Coefficient of Relative Risk Aversion	γ	5
Elasticity of Intertemporal Substitution	ψ	0.2
Time Discount Factor	β	0.96
Intratemporal Elasticity of Substitution	ρ	0.10
Preference for Housing	$1 - \delta$	0.10
Minimum House Size	H_{min}	0.20
Maintenance Cost	mc_h	0.01
Transaction Cost	Λ	0.08
Probability of Involuntary Move	π_h	0.03
Mean Growth Rate of House Prices	μ_h	0.012
Standard deviation of house price shocks	σ_h	0.048
Bequest Motive	b	2.5
Downpayment	d	0.15
Fixed cost of equity participation	F	0.025
Riskless rate	R_f	1.02
Mortgage rate	R_D	1.04
Mean Stock return	μ_S	1.06
Persistence of excess stock return	ϕ	0.9
Std of stock returns	σ_z	0.18
Corr. between factor and the return innovation	ρ_{z,ϵ^S}	-0.6
Innovation to the factor	σ_{ϵ^S}	0.007

Notes: This table reports the parameters for the benchmark model. The minimum house size and fixed costs are expressed as proportions to labour income.

For labour income we follow Cocco et al. (2004), who estimate the deterministic and permanent components of the labour income process using PSID data. As age profiles differ in shape across education groups, they control for education by splitting the sample into three groups: those without high school education; a second group with high school education but no college degree; and finally college graduates. Estimated parameters for the labour income process are reported in Table (3.2). The large estimate for the replacement ratio (λ) during retirement (93% for the college educated) arises from using both social security and private pension accounts to estimate the benefits in the PSID data and is consistent with not explicitly modelling the tax-deferred saving through retirement accounts. The labour income process is deflated with the Consumer Price Index to account for inflationary effects.

Table 3.2. Labour Income Process : Coefficients in the Age Polynomial

	No High School	High School	College
Constant	-2.1361	-2.1700	-4.3148
Age	0.1684	0.1682	0.3194
$\text{Age}^2/100$	-0.0353	-0.0323	-0.0577
$\text{Age}^3/100$	0.0023	0.0020	0.0033
λ	0.88983	0.68218	0.938873
σ_ω	0.136	0.131	0.133
σ_ν	0.019	0.019	0.019

Notes: These are values estimated by Cocco et al. (2004) for a third order age polynomial, λ is the constant fraction for the retirement period. σ_ω and σ_ν are the standard deviations of the idiosyncratic and aggregate shocks, respectively. The coefficients and the standard deviations were estimated by fitting eq.'s 3.44 - 3.47 on the Panel Study of Income Dynamics dataset.

We calibrate several parameters related to housing. We follow Cocco (2004) and set it at 15%.[21] Regarding the house price process in eq. (3.49), we need to estimate parameters of the random walk with drift process. One approach could be as in Cocco (2004) who uses self-assessed house values from the PSID data from 1970-1992 to construct a House Price Index. However, PSID data on house prices suffer from measurement errors as they are self-assessed values. Furthermore, PSID surveys are conducted only every two years. In light of these arguments, Pelletier and Tunc (2015) use the Case-Shiller Index. This is again not an ideal procedure as the repeat-sales Case-Shiller Index do not control for changes in quality of the house and gives a higher weightage to expensive houses.

To address these concerns, we use the seasonally adjusted constant quality house price index brought out by the U.S. Bureau of Census for the years 1970 - 2014. We first deflate the house prices using the Consumer Price Index to get real prices and then estimate for values of the real growth rate μ_h and the standard deviation σ_h. Our estimates reveal that $\mu_h = 0.012$ and $\sigma_h = 0.048$, we set them accordingly.

Regarding the financial assets and credit market parameters, we set the mean equity premium μ_S at 4.00%, a level considered reasonable by Mehra and Prescott (1985); Mehra (2012). The risk free rate is fixed at 2% and the annual mortgage rate is set at 4% following Campbell and Cocco (2015). As a proxy for a factor to predict stock returns we use the log dividend yield. This is widely considered as

[21] Yao and Zhang (2004) and Hu (2005) set the required downpayment rate at 20% of the house value.

one of the most important predictors of stock returns based on OLS regressions over long horizon data, see Campbell and Shiller (1988).

For the rest of the return predictability parameters, we follow Pástor and Stambaugh (2012) and Michaelides and Zhang (2015) who use an annual frequency CRSP stock return data to estimate time varying excess return variables. They find a high persistence parameter of $\phi = 0.9$ and the unconditional stock market volatility, σ_z given by the unconditional standard deviation of stock returns is found to be equal to 0.18. The correlation between the factor and the return innovation, ρ_{z,ϵ^S}, is an important parameter as this along with factor process determines whether the stock returns exhibit the attractive property of *mean reversion*. A negative correlation coupled with an AR(1) factor process implies mean reversion, see Choe et al. (2007). Pástor and Stambaugh (2012) estimate this parameter to be -0.6. Furthermore, the factor innovation is seen to be smooth in the literature and we set it as $\sigma_{\epsilon^S} = 0.007$ for the baseline model. When we report our results we compare this values with the case of a more volatile factor innovation ($\sigma_{\epsilon^S} = 0.017$).

3.6. Life-Cycle Portfolio Choice - Simulated Results

In the following subsections, we discuss our simulated results for the optimal portfolio choice over the life-cycle from the policy functions obtained from our Value Function Iteration procedure. We start with the case of IID returns, followed by return predictabiity. We finish this section with a discussion on the reported levels of stock market participation with and without housing and quantify the impact of preference heterogeneity.

3.6.1. Asset Allocation over the Life-Cycle - IID Returns

The theoretical and the empirical literature provide counterfactual evidence regarding the portfolio share invested in stocks vis-a-vis age. The theoretical literature predicts that as long as labour income is uncorrelated with risky stocks the proportion of wealth invested in stocks is decreasing over life (see Heaton and Lucas (1997) and Cocco et al. (2004)) while the empirical literature has found that this risky share is actually increasing over life-cycle, see Ameriks and Zeldes (2002). Our results reported in Table 3.1, assuming that the excess returns on the risky asset is independent and identically distributed, IID, complement those of Cocco (2004) in that we provide an explanation to this puzzle.

Table 3.1. Portfolio shares by age predicted by the model - I.I.D Returns

	Liquid Assets				Financial Assets				Total Assets			
Asset	<35	35-50	50-65	$\geq$ 65	<35	35-50	50-65	$\geq$ 65	<35	35-50	50-65	$\geq$ 65
Stocks	0.023	0.348	0.612	0.740	0.009	0.075	0.310	0.102	0.004	0.035	0.121	0.088
Bonds	0.977	0.652	0.388	0.260	0.035	0.035	0.045	0.020	0.005	0.005	0.020	0.009
Liquid Assets	1.000	1.000	1.000	1.000	0.044	0.110	0.355	0.122	0.009	0.040	0.141	0.097
House Value					0.956	0.890	0.645	0.878	0.145	0.168	0.304	0.820
Financial Assets					1.000	1.000	1.000	1.000	0.154	0.208	0.445	0.917
Human Capital									0.846	0.792	0.555	0.083
Total Assets									1.000	1.000	1.000	1.000
Debt					0.702	0.502	0.345	0.625	0.101	0.099	0.188	0.488
Stock Mkt. P.	0.025	0.245	0.680	0.800								

Notes: This table reports mean portfolio shares of various assets relative to liquid assets, financial assets and total assets when our model was simulated under the calibrations specified in Table (3.1) for 10,000 agents and when the returns are assumed to be IID. We report the results only for the college educated income group. Qualitatively similar results were obtained for the other two groups. We follow Cocco (2004) in defining the composition of the various asset classes, consistent with our model specification. Hence, Liquid Assets are defined as the sum of risky stocks and riskless treasury Bills. Financial assets are liquid assets plus house value and Total Assets are financial assets plus human capital. Debt is reported relative to financial assets and total assets. Stock Market Participation (Stock Mkt. P. in the Table) is the proportion of investors who decide to participate in the stock market. The investors are categorized by 15 year age groups. These results assume no correlation between labour income to house prices or risky stock returns.

The first four columns in Table 3.1 reports the change in the share of wealth invested in the risky stocks and the riskless bonds predicted by the model. It is clear that our model which incorporate housing predicts an increasing share of stock investments, starting from 2.3% in the <35 age group to almost 74% in the retirement period ($\geq$ 65). The presence of an illiquid asset in the form of housing implies that in the early part of adult life, the investor is liquidity constrained (depicted by the high level of debt) and thus chooses not to pay the fixed cost for participating in the stock market. Inspecting values for the stock market participation rate, which is the proportion of households who participate in equity markets, it is clear that only 2.5% of all investors enter the stock markets in the early life period. In fact, for this 20-35 lower age group liquid assets form only 4.44% of financial assets, all the rest of the wealth is held as real estate (housing).

With rising age, the level of other asset holdings in the form of labour income and house value becomes large enough to entice more and more investors to pay the fixed cost. Thus the market participation rate increases and so does the level of wealth invested in risky stocks. Investors accumulate enough wealth to both afford their mortgage and also to partake in risky investments. This explains why old investors are more willing to accept risk in their portfolio of liquid assets since future consumption is less correlated with the return on the liquid assets portfolio. It is noteworthy that a large proportion of wealth relative to financial assets is held in housing and this varies between 95% in the 20-35 age group to 64% in the 50-65 age group. This result confirms almost all existing studies that a majority of wealth of households is held in the form of real estate, see Mian and Sufi (2009).

The last four columns report the level of assets relative to total assets, where total assets comprises both financial wealth and human capital. We define human capital following Heaton and Lucas (2000) as the expected value of future labour income discounted at the annual interest rate of 5%. It is apparent that human capital, equivalently labour income, is an important determinant of wealth at all ages. Particularly for the young investors, in the 20-35 category for whom 84% of total assets comprises human capital. As investor's age, there is decreasing importance of labour income which explains the increasing share of wealth in housing relative to total assets.

The reported results in Table 3.1 are identical to those of Cocco (2004) which was expected as our model is built on his and we also used similar calibrations. Although Cocco (2004) uses CRRA preferences, our benchmark calibration values implicity meant that the relative risk aversion is equal to the reciprocal of the relative risk aversion.

One of the main contributions of this chapter over related literature is jointly modelling time-varying risky stock returns in the presence of non-diversifiable labour income risk when the investor is liquidity constrained from investment in risky housing. Table 3.1 reported our simulated results when the log returns on the risky asset is I.I.D. Now we proceed to the case of time varying returns (return predictability).

3.6.2. Asset Allocation Over the Life-Cycle - Time Varying Returns

As the effects of time varying equity premium or return predictability on risky portfolio choice is best seen through a graphical illustration, we plot the results rather than tabulating it. We modelled the return predictability through the log dividend-price ratio, see eq.'s (3.51) and (3.52). The expected excess return on risky stocks is positively correlated with this dividend-price ratio. Hence, high dividend-price ratio implies high stock returns and as we learned from Proposition

1 this implies a higher level of risky asset demanded. This effect is exactly what we observe in our numerical simulation results validating our analytical characterization. Figure 3.1 plots the risky asset allocation over the life-cycle under two different realizations of the log dividend-price ratio. To better understand our results, we also plot the IID case.

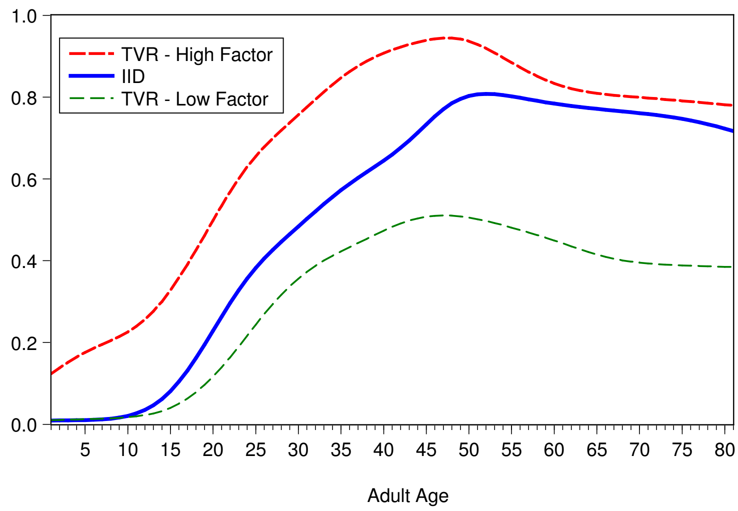


Figure 3.1. Risky Asset Allocation (α_t) over Life-Cycle: IID vs Time Varying Expected Returns (TVR) Under Different Factor Realisations. This figure plots the simulated mean asset allocation for three cases. The constant investment opportunity set or the IID case and for two different realisations when the expected excess returns on the risky stocks are time varying - a high factor realisation and a low factor realisation. The dashed plots are two extreme possibilities of the log dividend-yield. The horizontal axis describes the adult age which is actual age - 19 and the vertical axis represents the share of wealth in stocks ($\alpha_t \in [0, 1]$). Results are calculated by taking the average value over 10,000 simulated investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes.

The IID case in Figure 3.1 is a graphical illustration to the tabulated results in Table 3.1. As reported in the Table, mean share of wealth invested in risky stocks is very low, close to zero, in the first part of the life-cycle as households are liquidity constrained for two reasons: (i) from investing in the house and (ii) low earnings due to higher time discounting of labour income. As the investors grow older (and richer with increasing human capital) and accumulate wealth, they start investing more and more into the risky asset expecting high returns. This is also motivated by the fact that in midlife, saving for retirement becomes a crucial determinant of the agent's behaviour. The risky asset investment reaches its peak around the age of 65 when the investor retires. Here around 80% of the total wealth is being invested in the risky asset. This hump-shaped profile of α_t that we predict is consistent with the findings of other lifecycle portfolio models (IID) with housing such as Cocco (2004), Yao and Zhang (2004), Davis et al. (2006) and Vestman (2012) among others. Davis et al. (2006) do not include housing in their analysis but as our tabulated results showed, housing is a deterrent only in the initial years of an agent's life.

Retirement brings with it a fall in labour income, and hence a fall in wealth allocated to financial assets. The optimal allocation is then determined by the speed of the fall in wealth. This depends both on the discount factor which is adjusted for mortality risk (conditional survival probability) and the strength of the bequest motive. It is noticeable that the pace of decline in risky investment during the last years (ages 65-100) is slow which is primarily because agents have a motive to bequeath their wealth when they die. This is why unlike models without the bequest motive, see Davis et al. (2006) and Cocco et al. (2004) for example, we do not have a steep negative slope in the retirement period.

The two dashed lines represent the time varying returns cases. When returns are independent and identically distributed (IID), there is no added information between periods or in other words the future looks exactly like the past. However in reality, information changes all the time and the investors can use this to re-optimize before the end of their investment horizon. Time varying expected returns is equivalent to return predictability, that is $\phi \neq 0$. In our calibration, we followed the literature (see Campbell and Shiller (1988)) and chose the log dividend-price ratio as a factor that can be used to predict future returns.

The two time varying return curves indicate two different realizations, high and low, of the return predictable factor- the log dividend-price ratio. This factor describes the investment opportunity set. When the factor realizations are high, there is a favourable investment environment for the household. Equivalently, when the factor realizations are high, the expected excess returns on the risky asset is also high.[22] Higher expected excess returns does not necessarily mean high α_t. This is easily seen through our analytical characterization, see Lemma (1), where we split the optimal share of asset allocation into the expected excess return (myopic) component and an intertemporal hedging component. Thus,

[22]This is not always true. Despite the overwhelming literature that argue for return predictability, some recent studies such as Goyal and Welch (2008) question their poor out-of-sample predictions.

there can be a scenario where the investor decides to increase his investment in the risky asset to take advantage of the higher returns, especially, in the initial years and thus $\uparrow \alpha_t$. The investor can also realize that with the higher expected returns, he needs less money to achieve the same dollar return and thus $\downarrow \alpha_t$. Essentially, change in α_t depends on the hedging and the market timing motive of the investor.

The reported results in Figure 3.1 are for our benchmark calibrations where we abstracted from any type of correlations. Thus, there are no intertemporal hedging motives, refer Lemma 1. However the investor can aggressively time the market. As the two TVR cases significantly departures from the IID line, we can say that our model predicts aggressive timing strategies by the investors concurrent with the results of Michaelides and Zhang (2015). However, unlike Michaelides and Zhang (2015) we have included housing which provides an additional wealth component and also a risk component. These components mean that the divergence from the IID case is much larger.

Higher factor realizations mean higher expected stock returns and an almost parallel upward shift in the mean share of liquid wealth allocated to stocks. The difference is much more pronounced in the initial stages of the life-cycle. Between the ages of 20-35 whereas the IID returns predict only 2.3% allocation of liquid wealth in stocks, under high expected returns it is almost 10 times that- averaging at well over 20%. The intuition behind this is that although the presence of housing makes agents liquidity constraint, the attraction of high expected excess returns entices household to pay the fixed cost and venture into equity markets. This is true as long as the return on equity (R_{t+1}) is significantly higher than the mortgage rate (R_D). Our benchmark calibration ensures that this is always true. Also, the equity market participation rates, as we later find, is also much higher even among the constrained young households.

Similarly, if investors expect very low risky returns, the share of wealth invested in stocks drops down substantially. However, it is observable that the fall in α_t is strong in the retirement ages of 65-100. This is expected as households in this stage of their life would rather invest in riskless bonds as they anticipate the returns on stocks to be very near to that of bonds but without any risk, $E_t r_{t+1} - r_f \approxeq 0$. There is no reward for holding the risky asset. Furthermore, as Michaelides and Zhang (2015) argue factors such as the log dividend yield are highly persistent and thus it takes a substantial amount of time for a change (in α_t) to happen and when it does happen like in the retirement period the portfolio moves relatively quickly.

It is noteworthy that the speed at which investors run down their wealth after retirement (65-100) is marginally higher in both the time varying return processes. This is because investors are more sensitive to uncertainty in the later years of their life. The presence of mortality risk coupled with extra risk from the factor (dividend-price ratio) process and the absence of any hedging motive imply that investors would desire to divest his liquid wealth from risky stocks to riskless bonds. In the last years of the agent's life, the bequest motive kicks in and the slope stabilizes.

An important contribution that we can make from Figure 3.1 is that the impact of a low factor realization on α_t is much more pronounced than a high factor one. The huge drop in the financial factor is indicative of a market crash or in the words of Rietz (1988) a "rare disaster". The recent financial crisis is one example. Barro (2006, 2009) and others have been successful in explaining several asset market puzzles by incorporating such a disaster. Our result here contribute to this literature in that we observe that these rare events can also explain the limited risky asset allocation puzzle. As we see, a very low factor suggestive of a disaster event results in a significant decline in risky assets demanded and, as we later find out, in the rate of stock market participation. As Figure 3.1 shows, there is also marked heterogeneity in the α_t profile. In the possibility of a disaster risk, old and retired households rebalance their portfolio more towards risk free assets than young and employed ones. Retired households have low levels of human capital, fast depreciating, and thus prefer risk free investment.

It has to be emphasized here that we do not explicitly model the probability of an event risk as is the norm in the disaster literature. Instead the probability is implicitly contained in the discretization process. The continuous time returns process in eq. 3.51, for computation, was discretized into 15 Markov states of nature. Each state has a probability attached with it. An extremely high or extremely low factor states of nature are associated with low probabilities. Hence, these probabilities can be considered as indicative of an event risk. These are qualitatively equivalent to assuming that each date the investor faces a constant exogenous probability of a drop in returns.

An additional way to assess the impact of rare events such as a market crash on life-cycle asset allocation is through the volatility of the factor process. High volatility is an observed phenomenon in such disasters, see Wachter (2013). A high volatility in the log dividend yield process makes investment in stocks riskier. The expected excess returns for low factor realizations, a disaster, could be negative.

A rational investor should thus rebalance his portfolio towards riskless bonds or even not participate in the stock market at all.

Our simulated results reveals that α_t is sensitive to the conditional volatility of the factor process and are thus consistent with the above reasoning. Figure 3.2 shows the effect of different factor volatilities relative to the IID model. The deviation from the IID model is found to be larger when the factor is perceived to be more volatile ($\sigma_{\epsilon^S} = 0.017$) than for the benchmark moderate volatility ($\sigma_{\epsilon^S} = 0.007$).

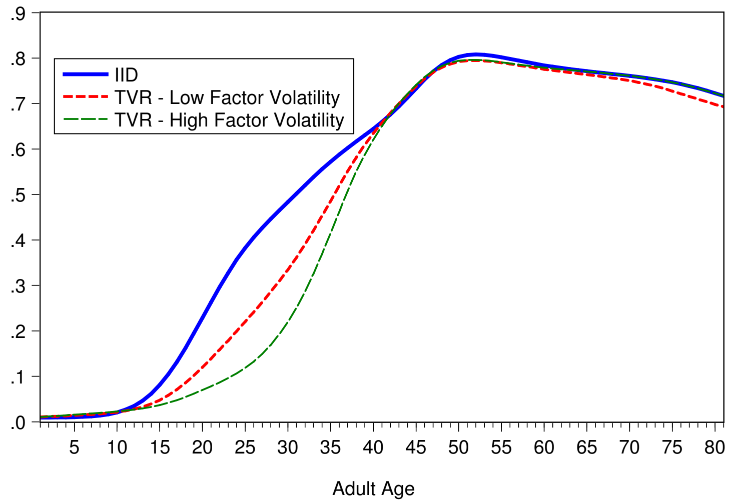


Figure 3.2. Risky Asset Allocation (α_t) over Life-Cycle: IID vs Time Varying Expected Returns (TVR) Under Different Factor Volatilities. This figure plots the simulated mean asset allocation for three cases. The constant investment opportunity set or the IID case and for two different factor volatilities, high ($\sigma_{\epsilon^S} = 0.017$) and low (Benchmark $\sigma_{\epsilon^S} = 0.007$), when the expected excess returns are time varying. The horizontal axis describes the adult age which is actual age - 19 and the vertical axis represents the share of wealth in stocks ($\alpha_t \in [0, 1]$). Results are calculated by taking the average value over 10,000 investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes.

A high factor volatility suggestive of a rare disaster in the economy shifts down the risky asset allocation curve. Importantly, the response in α_t to a higher risk in stocks are heterogenous over the life-cycle. Unlike the factor realizations scenario, see Figure 3.1, the effect on young investors is insignificant, see Figure 3.2. Until the age of 30, investment in risky assets is almost the same. The effect is more visible in the middle age group, specifically between the ages of 35-55.

Retirement ages and then on until death also do not exhibit any marked changes vis-a-vis the IID case. This is because when faced with higher volatilities most younger workers do not react much since they have several decades to adjust before they retire but as they get older they realise that they have little time to adjust their wealth. Hence, they aggressively hedge such risks through bonds.

Our results agree with that of Chai et al. (2011) who estimates a life-cycle model with return predictability, flexible labour income and finds that financial risk/stock market shocks have little or no effect on young households but the older and retired households invest more in equities. However, Michaelides and Zhang (2015) finds that the average share of wealth in risky assets is substantially lower among the young households compared to the older ones. This is mainly because of the absence of housing in their model. Their simulated α_t counterfactually predicts almost a 100% allocation in stocks in the 20-35 age group. For expositional clarity, we plot the change in financial wealth over the life-cycle under high factor volatilities or disaster risks.

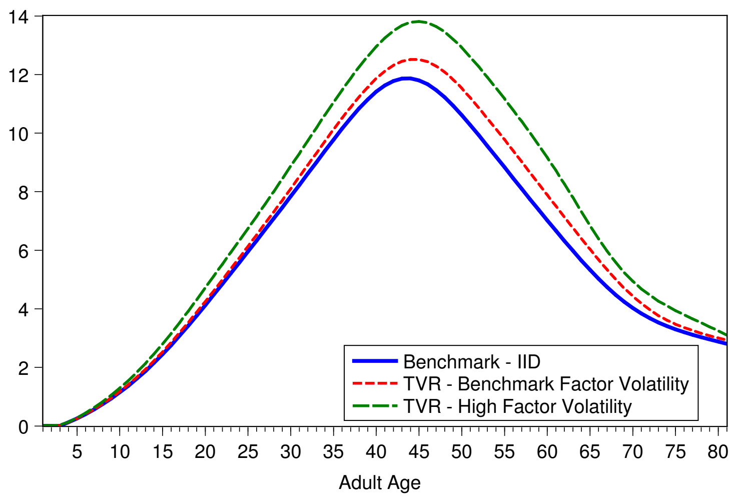


Figure 3.3. Mean Normalized Financial Wealth over Life-Cycle: IID vs Time Varying Expected Returns under different factor risks. This figure plots the simulated mean financial wealth normalized by the permanent component of labour income (total labour income net of transitive component) for heterogenous households over the lifecycle under the baseline parameters and assuming that there is a fixed cost of equity participation. The dashed lines represent the TVR cases when the volatility of the factor process is 0.007 and 0.017. The vertical axis displays the normalized wealth and the horizontal axis the adult age. Results are calculated by taking the average value over 10,000 investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes

Figure 3.3 plots the life-cycle financial wealth normalized by the permanent component of labour income for different factor volatilities. The continuous line represents the IID case with no factor volatility while the dashed lines represent a small (0.007) and a high (0.017) volatility case, respectively.

It is apparent from Figure 3.3 that the financial wealth which is the sum of liquid wealth and housing wealth increases with increasing dividend-price volatility. This is despite the fact that the share of wealth invested in risky stocks has gone down, see Figure 3.2. The logical explanation in this circumstance is that the housing wealth rose enough to more than make up for the decline in liquid wealth. This means that households anticipating a market crash hedged this risk by the illiquid housing asset. This result although agreeing with Michaelides and Zhang (2015) is not consistent with empirical studies.

For instance, Barro (2006, 2009) and Wachter (2013) associate a market crash or a rare disaster with almost 25% drop in consumption with equivalent levels in household wealth. There are several reasons for our counter-factual result. Chief among them is the assumption we made in our calibration that the stock and housing markets are not correlated. This implies that as long as households can afford to pay the transaction costs they are freely able to hedge any such risks through the housing asset. In reality, housing is highly procyclical meaning that disasters will almost certainly be accompanied by a drop in house prices nullifying any sort of hedging strategies by the investors.

Nevertheless, the hump shaped profile of life-cycle financial wealth is consistent with recent empirical work on micro data (Survey of Consumer Finances). For instance, Fernández-Villaverde and Krueger (2011) finds that households keep on accumulating wealth from the beginning of their lives until retirement, at which point they start running down their wealth.

It is a stylized fact that the log dividend-yield is highly persistent. A value of zero persistence in the factor, $\phi = 0$, implies no influence of the factor process or in other words no return predictability or simply IID. As the value of the persistence rises, so does the ability to predict returns. Our benchmark specification for ϕ was set at 0.9. This value is representative of mean reversion in returns. Our benchmark calibration also set a negative correlation between innovations to the stock return and the factor, log dividend-price ratio. A direct implication of this parametrization, along with the assumed AR(1) process for the factor, is that stock returns exhibit the property of *mean reversion*. Mean reversion in stock returns reduces long-term risk relative to short-term risk, that is, stocks are less risky in the long run. Hence, mean reversion makes stocks attractive to a life-cycle investor.

Highly persistent stock returns (i.e. the persistence parameter is greater than or equal to one, $\phi \geq 1$) will generate abnormal returns for the investor. Such an explosive process is indicative of an "asset price bubble". Although there has been several studies that examine the impact of mean reversion on risky asset allocation, cases of bubbles have never been examined. Intuitively if ϕ affects the risky returns, it is natural then that the risky asset allocation decision is affected as well. To test this, we compute the optimal risky portfolio shares for a grid of persistence values. The results are graphically illustrated in Figure 3.4.

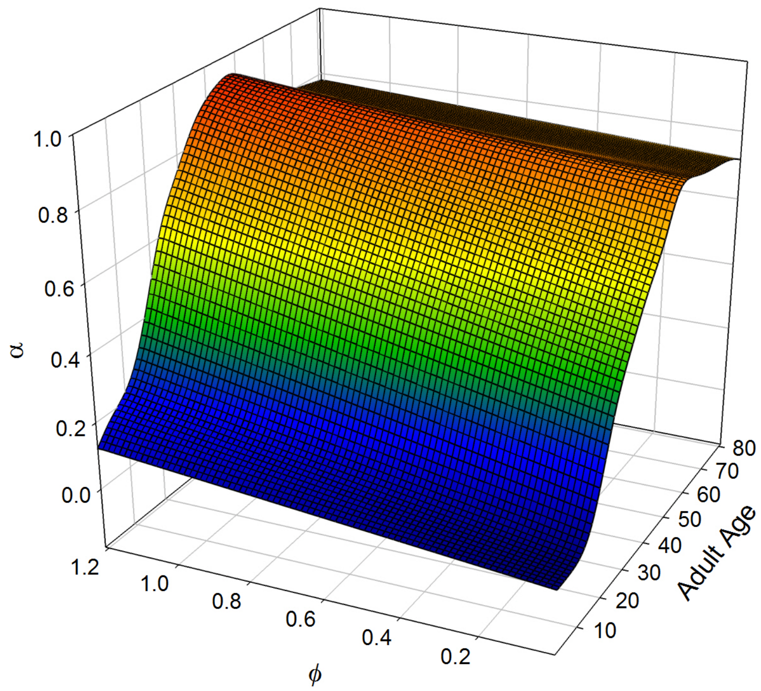


Figure 3.4. Risky Asset Allocation (α_t) over Life-Cycle and over Persistence of the Factor Process. This figure plots the simulated mean asset allocation for the life-cycle when the persistence, ϕ, of the log dividend-price ratio vary from zero corresponding to the IID case to an explosive process, $\phi = 1.2$. The x-axis describes the adult age which is actual age - 19, the y-axis shows the range of values for ϕ and the z-axis represents the share of stocks in wealth ($\alpha_t \in [0, 1]$). Results are calculated by taking the average value over 10,000 investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes. To get a smoothed continuous curve, we first evaluated α_t for a grid of values for the persistence parameter which was then plotted.

It is clear from Figure 3.4 that as the persistence parameter, ϕ, has a monotonic increasing relationship with risky asset allocation. This holds even for young liquidity constrained investors. For example, when $\phi = 0$ returns are IID and the level of α_t is almost 2.5% at the age of 25. However at the same age when $\phi = 1.2$, risky asset share is close to 18%. As we discussed before a value of $\phi \geq 1$ indicates the possibility of a bubble in the stock market. Our results predict that if investors have self-fulfilling price expectations they would invest more in risky assets even if it entails paying a substantial amount to participate in the equity markets.

We will examine this result by parts. First we will talk about the non-bubble case, $\phi < 1$. When the coefficient of the factor is positive and strictly less than unity, stock returns are mean reverting. Mean reversion boosts demand for risky assets. This is consistent with our predictions in Figure 3.4. It also agrees with several important studies in the literature. For example, Campbell and Viceira (1999) consider an infinitely lived investor with Epstein-Zin utility defined over consumption. They estimate the parameters of the model from post-war U.S. data and find that the estimated mean reversion dramatically increases the average optimal equity allocation of a conservative long-term investor. Lynch (2001) also find that the predictive persistence parameter, ϕ, has a large impact on the optimal risky asset allocation. For example, he predicts that varying the parameter from 0.85 to 0.96 raises the average share of wealth allocated to risky assets by 6%. Both these papers are highly stylized and unlike ours do not consider non-tradable labour income or risky housing and is not designed to explain life-cycle dynamics.

More recently Benzoni et al. (2007) uses a rich life-cycle portfolio choice model assuming a cointegration between aggregate labour income and stock market returns (dividends) and also considering return predictability find that young agents should short the risky stock. That is, the young agent chooses to sell the market portfolio short to hedge the risk associated with her human capital position. They also find that as they decrease ϕ to zero, results approach the IID case. In the presence of short sales constraints, agents do not participate in the stock market at all. However, at even small increases in ϕ such as 0.08 to 0.10, young agents (20) invest all of their wealth in risky stocks due to the hedging demand contrasting empirical evidence. This is mainly because of the lack of an illiquid good that can restrict consumption smoothing abilities.

In essence, our prediction of increasing wealth allocated to risky stocks (α_t) with increasing persistence of the return predicting factor (ϕ) for the no-bubble mean reverting case is consistent with the literature. However, the case of unit root $\phi = 1$ and explosive roots $\phi > 1$ is not clear. When the factor process follows a random walk, returns are not predictable implying all else constant the share of risky assets should go down. Our results contradict this finding. We find that α_t is strictly increasing even at such non-stationary values. This increase is apparent uniformly at all ages of the agent's life-cycle. This is mainly because we find an upward drift in the level of households participating in equity markets as soon as ϕ reaches unity. In other words, the arrival of bubble brings with it an increased participation in equity markets, see Figure 3.7. As more investors participate the average risky share of liquid wealth invested in stocks goes up. This is a realistic argument as it is observed in general that stock market bubbles are characterized by a large volume of trades and increased participation, see Basak and Makarov (2014).

High persistence in the log dividend-yield is not necessarily indicative of a stock market bubble. For example, it is widely documented that abrupt changes in the form of structural breaks or regime shifts can induce non-stationarity in the persistence parameter. However, we argue that even if that is the case, our results would still hold. Firstly, the presence of high quality structural break tests have only been recently available in the public domain. They have not been available long enough to allow an investor to utilize these new techniques in real time to search for structural breaks to rebalance their portfolio allocation decisions. Secondly, the identification of structural breaks is a purely statistical exercise mostly unrelated to predictable or observable economic events. For these reasons, we can safely justify our result that on an average, *households invest more of their liquid wealth in risky stocks in the presence of a bubble.*

Of interest is the fact that we do not observe any shift of wealth from stocks to housing when we analysed the wealth profiles (not reported). The so called "wealth effect" hence is absent in a stock market boom unlike a housing market boom. Fischer and Stamos (2013) argues based on impulse response analysis that a housing market boom raises the value of the home in which the household lives and substantially increases its housing wealth. Both owners and renters decrease their equity and bond holdings as an immediate reaction to a housing market boom. In the long run, however, both bond and equity holdings are higher, due to the positive wealth effect resulting from the housing market boom. Motivated by their study, we analyse the impact of several parameters related to housing in forming the optimal asset allocation. We do this exercise in the next subsection.

3.6.3. *Asset Allocation over the Life-Cycle - House Price Risk and Hedging Motives*

To analyse and measure the effects of housing on portfolio allocation, we compare the results of the benchmark model which incorporates housing with a model excluding housing. The no housing model includes all the features of the existing one discarding everything related to housing: there is no durable good in the utility function ($\delta = 1$) and no collateral in the investment set ($D_t = 0$, $p_t^h = 0$). Essentially, this model simplifies to the Gomes and Michaelidis (2005) one.

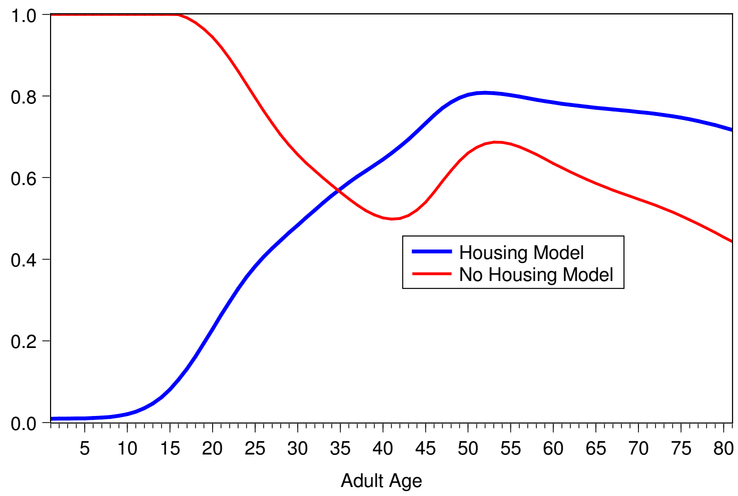


Figure 3.5. Risky Asset Allocation (α_t) over Life-Cycle: Housing versus No Housing Model. This figure plots the simulated mean asset allocation for the benchmark model with housing against a no housing model. The horizontal axis describes the adult age which is actual age - 19 and the vertical axis represents the share of wealth in stocks ($\alpha_t \in [0, 1]$). Results are calculated by taking the average value over 10,000 investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes.

It is clear from Figure 3.5 that the life-cycle profile of risky equity demanded is substantially different in both cases. The difference is marked in the initial years of the household. In the absence of housing, agents are no more liquidity constrained. Thus, household invests their entire liquid wealth (100%) in risky stocks to take advantage of the high equity premium. Around the age of 35 - 40, as the value of human capital starts decreasing and financial wealth grows, the household starts rebalancing its portfolio towards riskless bonds. This explains the downward slope between the age of 35-65. In contrast, the housing model shows an upward sloping α_t curve. This is largely because the presence of housing increases the total wealth that the investor holds. The housing wealth offsets changes in labour income. At the age of retirement, the portfolio share invested in risky stocks in the presence of housing is significantly higher at 80% compared to 62% without housing. The intuition is that as soon as the agent retires, around adult age 46, the household starts dissaving. If the investor owns a house, he has more savings that can be invested in stocks.

The no housing model, consistent with Cocco et al. (2004), Gomes and Michaelidis (2005), Benzoni et al. (2007), Michaelides and Zhang (2015), predicts very large portfolio risky shares for young households. This prediction is counterfactual. For instance, in the last wave of the Survey of Consumer Finances (SCF) only 12% of participating young households have a share of risky assets that exceed 80%, see Guiso and Sodini (2013). The predictions of our benchmark housing model, hence, are more in line with empirical results. Housing initiates a *crowding out effect* keeping liquid assets low and young agents from investing in risky stocks. As we have seen that housing plays a major role in constructing the life-cycle profile of risky portfolio shares, it is intuitive that shocks to housing prices should affect the level of α_t. Furthermore, there is substantial evidence that house prices are correlated with aggregate labour income shocks. These are investigated and the simulated results are plotted in Figure 3.6.

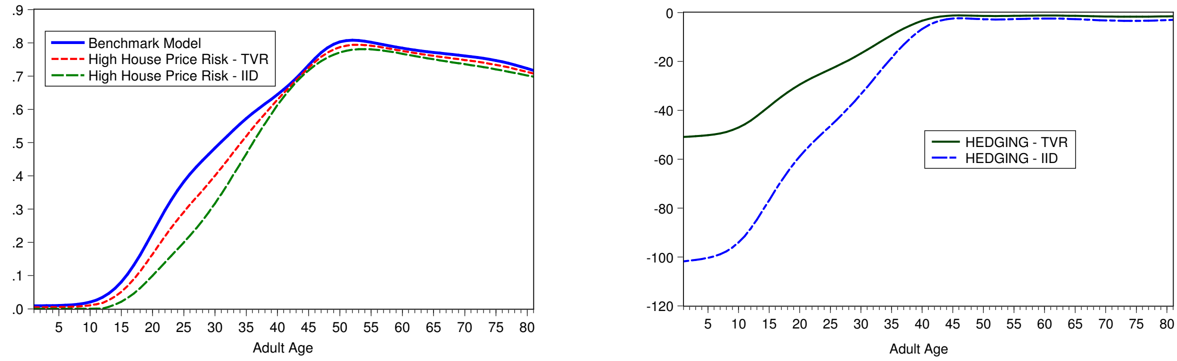


Figure 3.6. House Price Risk on α_t over the Life-Cycle: IID versus TVR. This figure illustrates the change in the level of risky equity demanded over the life-cycle from the Benchmark model when the returns are (i.) IID and when they are (ii.) time varying and return predictable. To initiate the change in α_t we jointly assume that the aggregate labour income shocks are positively correlated with the log house prices, $\rho_{\nu,h} = 0.553$, and that there is an increasing in house price volatility from $\sigma_h = 0.048$, the Benchmark specification, to 0.14. The right graph plots the intertemporal hedging demand relative to the Benchmark model. The hedging demand is computed in terms of percentage deviations, $hedge = 100 * \frac{\alpha' - \alpha_{Benchmark}}{\alpha_{Benchmark}}$ where α' is the new portfolio risky share. The horizontal axis describes the adult age which is actual age - 19 and the vertical axis represents the proportion of liquid wealth in stocks.

To understand the impact of house price risk on risky asset allocation, α_t, we first impose a positive correlation between aggregate labour income shocks and log house price, a realistic assumption which we ignored in our benchmark calibration.[23] The correlation between income shocks and house prices is set at a value of $\rho_{\nu,h} = 0.553$ consistent with other studies, see Cocco (2004). Next we increase the level of house price volatility, σ_h from the benchmark value of 0.048 to 0.14. We examine the change in α_t when returns are (i.) IID and when they are (ii.) time varying and predictable (TVR). The continuous line in the left panel of Figure 3.6 represents the Benchmark case and the other dashed lines represent the new risky share when returns follow the two different processes.

Two results can be derived from the above figure. Firstly, under higher riskiness in housing, the α_t curve shifts down and thus *the risky portfolio share is lower throughout the life of the investor*. Secondly, the change in α_t is higher (more negative) when the returns are IID than when they are time-varying meaning that *investors hedge the housing risk better under the TVR case.*

[23]We emphasize here that imposing a positive correlation between labour income and house prices brings no qualitative change to any of the results that we discussed in the preceding sections.

As characterized in Proposition 1, the absence of any correlation between any of the variables implies that there are no hedging effects. Hence, the two dashed curves, which are in the presence of correlations can be called as the hedging effect, the benchmark model being the myopic one. The right panel of the above figure depicts the intertemporal hedging demand, which is nothing but the percentage deviation from the benchmark model.

The divergence (zero represents the benchmark model) is very high when the investor is young, close to 100% in absolute terms at the age of 20. There are two forces at play here. The increased volatility means that investment in housing is now risky. The positive correlation with labour income shocks means that housing is no more a good hedge against labour income risk. Both these effects combine to force investors to tilt their financial portfolio toward liquid financial assets, in the form of bonds, and away from the risky illiquid housing bringing down α_t. As these shocks are positively correlated with labour income, the effect lasts until retirement. This explains why the divergence is close to zero and remains that way from the age of 65.

The results for the IID case is consistent with other papers in the literature. For instance, Cocco (2004) finds using data simulated from a similar model that the portfolio share of stocks relative to financial assets is 13% lower for young households and 9% for older ones. Curcuru et al. (2004) and Kullmann and Siegel (2005) use regression models to investigate the role of housing wealth for both stock market participation and equity shares conditional on participation. Curcuru et al. (2004) report a negative effect of the house value to financial wealth on participation. Kullmann and Siegel (2005) finds that housing price risk is associated with lower stock market participation and, conditional on participation, lower equity investments.

The more interesting and a contributory result that we get is the variation between IID returns and time varying returns (TVR). A look at the right hand panel of Figure 3.6 reveals that the deviations from the benchmark model is substantially lower when returns are predictable compared to the case of no predictability. In fact, the hedging demand for the IID case is almost double (negative) at all ages until retirement. The fact that returns are predictable, under the TVR case, means that investors can strategize and time the market. Market timing implies that substantial gains can be made by investing in risky stocks. Hence, despite the rise in background risks (labour income) investors are able to hedge these risks so that the fall in risky portfolio shares is limited. However, when the returns are IID no such predictability can happen.

Cocco et al. (2004) and Calvet et al. (2009) finds that substantial welfare losses are incurred by households who move away from stocks. Hence, a normative implication of our result is that if households can reasonably predict returns by following a factor such as the log dividend yield, they can safeguard themselves from cycles (risks) prevalent in the housing market.

3.6.4. The Stock Market Participation Rates Over the Life-Cycle

In this section, we attempt to shed light on the limited equity market participation puzzle. The predicted levels of participation over the life-cycle and over different levels of the factor persistence, ϕ, is plotted in Figure 3.7.

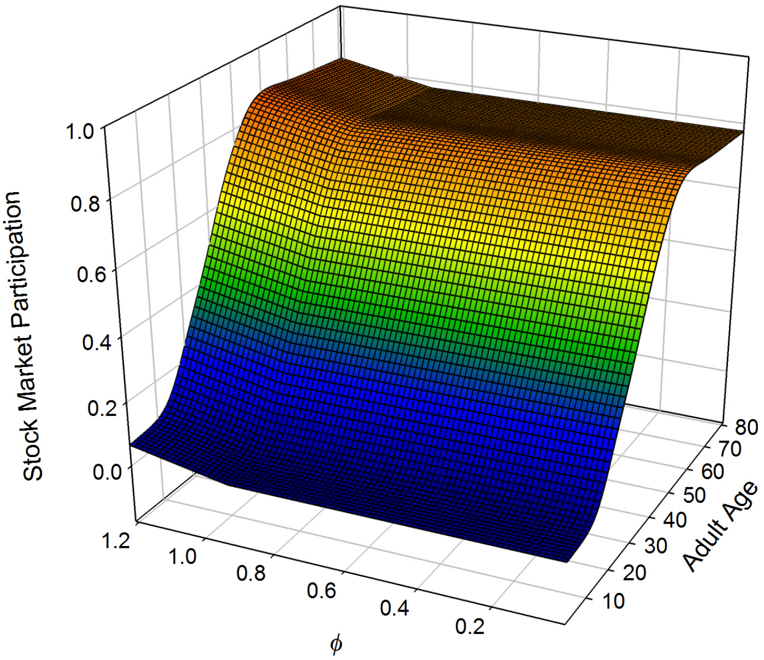


Figure 3.7. Stock Market Participation over the Life-Cycle and over Factor Persistence. This figure illustrates the proportion of households that participate in the equity market over the life-cycle and over a grid of values for the persistence of the log dividend-price ratio. The case of $\phi = 0$ represent the Benchmark model with IID stock returns; when $0 < \phi < 1$, returns are mean reverting and time varying predictable; and unit root and above indicate a possible bubble in the stock market. The x-axis describes the adult age which is actual age - 19, the y-axis describes the persistence of the dividend-price ratio and the z-axis represents the proportion of households that participate in the equity market in the range - $[0, 1]$.. Results are calculated by taking the average value over 10,000 investors, using the derived optimal policy functions and for each investor simulating a different path for the exogenous stochastic processes.

Participation rates are monotonically increasing throughout the life-cycle. It is no coincidence that the shape of the curve is similar to the risky equity share, α_t plotted in Figure 3.4. Several related studies, Gomes and Michaelidis (2005) and Guiso and Sodini (2013), have found that the participation decision is an increasing function of wealth and the optimal share of wealth invested in the risky assets. The decision to invest in equity markets involves paying a fixed cost of participation. Thus, it is natural to find that only 52.8%, on an average, of households decide to venture into stock markets. The rate of participation is very low in the beginning of the life-cycle, increasing through the middle ages and reaching its peak around the age of retirement. From then on until death, households deem participation too risky considering the low level of human capital and thus there is a slow but steady decline. This result is consistent with the literature that model housing in life-cycle portfolio allocation, see Cocco (2004) and Vestman (2012).

An interesting contribution that we derive from Figure 3.7 is when the persistence of the factor, ϕ is at or above a unit root. This imply an explosive process for the dividend-price ratio indicating a "major event" such as a bubble. When such an event happens, there is a marked upward shift in the participation rate even at the initial stages of the life-cycle. Young households despite being liquidity constrained from investing in the house still pay the fixed cost of market participation enticed with an abnormally high expected returns to risky stocks. The money illusion aspect associated with market frenzies, as argued by Brunnermeier and Julliard (2007), exacerbates this phenomena. This also explains why in Figure 3.4 we found a big upward shift when the persistence transitioned from a mean reverting to a unit root process.

Major market events, such as the Dot-Com bubble of 1995-1999, will draw immediate attention from all investors, wherein agents who can observe the time-varying investment opportunities can strategize (buy and hold for example) and stand to benefit relative to IID investors. Thus, fixed information costs of participation can effectively be diluted. In effect, this means that households following an IID process are committing a financial mistake (Calvet et al. (2009) and von Gaudecker (2015)) and are losing out on substantial wealth gains

A unit root is a permanent effect. Thus, our results give a theoretical explanation to the fact that several empirical studies based on SCF data have found that participation rates have been steadily increasing over the years. For example, Poterba (2002) argue that baby boomers are participating more heavily in the stock market. Furthermore, Ameriks and Zeldes (2002), Curcuru et al. (2004) and Guiso and Sodini (2013) attribute this steady increase to changes in expected returns, steady growth of stock market and low cost mutual funds.

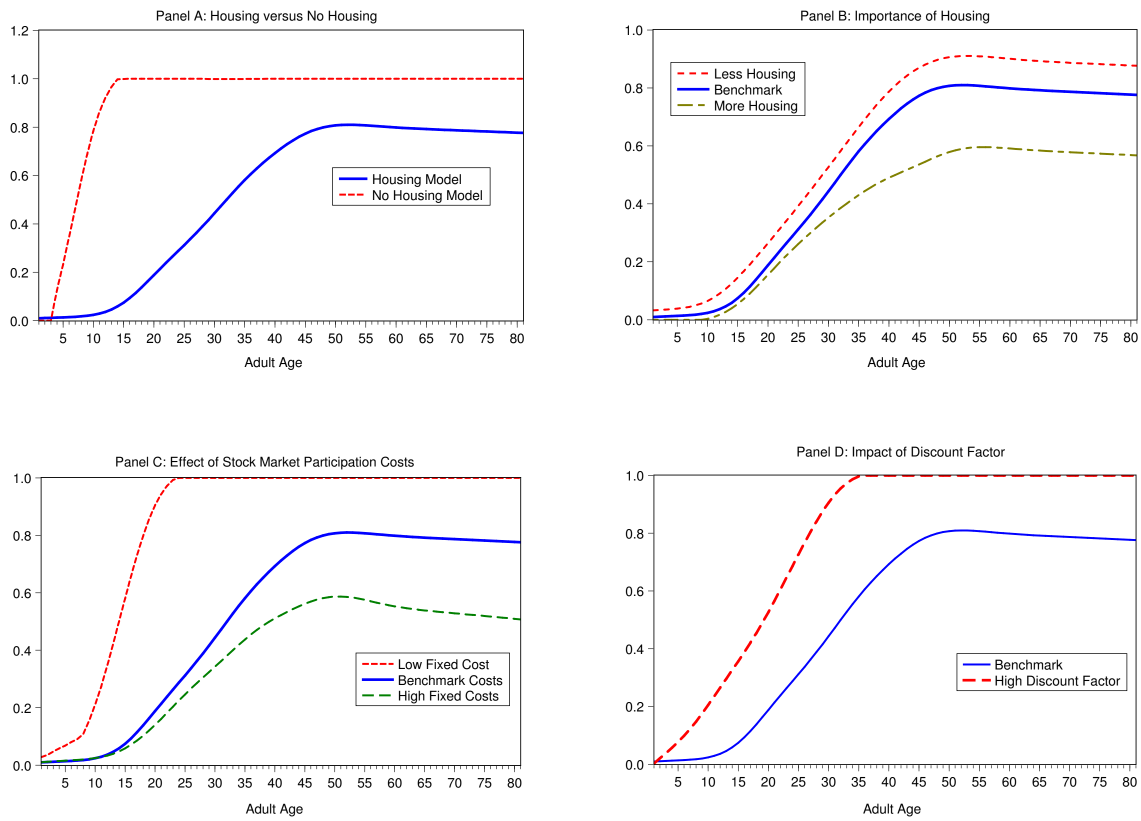


Figure 3.8. Stock Market Participation over the Life-Cycle: Comparative Statics. This figure illustrates a comparative statics exercise for the equity market participation rate for households over the lifecycle. Panel A compares the benchmark model incorporating risky housing with a no-housing model. Panel B shows how the participation rate varies as the importance of housing represented by the parameter 1 - δ is increased from the benchmark value 0.1 to 0.15 and when it is lowered to 0.05. Panel C shows the impact of changes in the fixed cost of equity market participation (F), low 1.5% and high 3.5% relative to the Benchmark F of 2.5%. Panel D plots the impact of a high discount factor, $\beta = 0.99$. The horizontal axis describes the adult age which is actual age - 19 and the vertical axis represents the level of stock market participation. The continuous line in all the four panels illustrate the Benchmark calibrations.

Figure 3.8 illustrates the level of market participation when we change several underlying parameters of the model. We compare the changes relative to the Benchmark Case.

Panel A and B illustrate the importance of housing wealth for agents. As we expected and in line with our results in Figure 3.5, we find that when households are no longer constrained by investment in housing (Panel A) there is 100% participation in equity markets at a relatively young age of 35. In contrast, the presence of housing prevents a substantial portion of agents from investing in equities. The effect is more apparent in Panel B of the figure where we both increased and decreased the importance of housing (1 - δ) to 0.15 and 0.05, respectively. More households participated under lower housing and vice-a-versa. Interestingly, the equivalent change in participation was more with higher housing than when it was lowered.

For our benchmark calibration, we imposed a fixed cost of equity participation, F, which was exogenously set as a proportion of the permanent component of labour income following Gomes and Michaelidis (2005). This proportion, 2.5% of households annual income, reflects both the monetary cost associated with the initial investment in the stock market as well as the opportunity cost involved in obtaining the necessary information for making such an investment. If the average household has an annual labour income of \$35,000, then this fixed cost would come up to about \$875. It is intuitive that any change in this fixed cost would affect the participation decision in the opposite direction. Consequently, we find in Panel C that a high F (3.5%) brings down the rate of participation and a low F (1.5%) shifts up the rate.

Panel D, shows the effect of raising the discount factor β from the benchmark value of 0.96 to 0.99. A high discount factor means that investors save more and are more willing to pay the costs of equity participation agreeing with Cocco (2004).

3.6.5. Preference Heterogeneity and the Puzzles

In the introduction and the modelling sections, we stressed the importance of preference heterogeneity. In other words how the CRRA preference assumption that tie both the risk aversion and the intertemporal substitution parameters to be reciprocals of each other is too restrictive. In our analytical characterization in Lemma 1, we found that both EIS and RRA affect the level of risky asset allocation in opposite directions. α_t was found to be inversely proportional to risk aversion and directly proportional to the EIS (inverse with a negative sign). Our benchmark preference values were $\rho = 5$ and $\psi = 0.2$, thus we implicitly assumed CRRA preferences. We now keep one of the parameter constant and change the other to see how these influence the asset allocation and participation decisions. The results are reported in Table 3.2.

Table 3.2. Stock Market Participation and Stock Holdings

	Stock Mkt. Participation (%)	α (%)
Survey of Consumer Finances	51.94%	54.76%
Benchmark Model ($\rho = 5$ and $\psi = 0.2$)	52.8%	50.8%
Model (Case I - $\rho = 5$ and $\psi = 0.3$)	54.2%	54.1%
Model (Case II - $\rho = 7$ and $\psi = 0.2$)	50.2%	48.9%

Notes: This table reports the stock market participation rates and the risky stock holdings (α), in percentages. These are obtained by taking the average value from all the simulated agents. The first row reports the statistics estimated by Gomes and Michaelidis (2005) based on the 2001 wave of the Survey of Consumer Finances. The second row reports the simulated results for our Benchmark model. The third and fourth rows report values when we change the benchmark preference parameters, the coefficient of relative risk aversion ρ and the elasticity of intertemporal substitution, ψ.

The first row of Table 3.2 reports estimated levels of stockholding and participation from the 2001 wave of the Survey of Consumer Finances (SCF), reproduced from Gomes and Michaelidis (2005). Validating our approach, our benchmark model with homogeneous preferences, second row in the Table, predicts an average stock market participation rate of 52.8% and conditional on participation an average risky stock holdings share of 50.8% which matches very well the Survey data.

This finding contrasts with both the theoretical and empirical literature that argue for preference heterogeneity. Using microeconomic data, Vissing-Jorgensen (2002) estimate the level of risk aversion and elasticity of intertemporal substitution accounting for limited market participation. They find that loosening the link between EIS and risk aversion and offer risk aversion estimates for stockholders at around 5-10 and EIS estimates around 1. In a partial equilibrium life-cycle portfolio choice model much like ours with recursive preferences, but without housing, Gomes and Michaelidis (2005) simultaneously matches stock market participation rate and risky asset allocation conditional on participation *when they allow for preference heterogeneity*. By considering a 50% split between investors with low risk aversion ($\rho = 1.2$) and low EIS ($\psi = 0.2$); and investors with moderate risk aversion ($\rho = 5$) and moderate EIS ($\psi = 0.5$), they predict a participation rate of 52.1% and an equity share of 54.5% in line with the empirical evidence (row 1).

As our benchmark results showed, we do not need such heterogeneity in preferences (Gârleanu and Panageas (2015)) to match the data. In fact when we assume preference heterogeneity, as the third and fourth rows of the Table 3.2 indicate, the levels of asset holdings in particular are found to be much worse. The third row of Table 3.2 reports simulated results when 50% of investors followed the Benchmark specification and the rest 50% had the same risk aversion ($\rho = 5$) but a higher EIS ($\psi = 0.3$). We find that the participation rate as well as the share of equity holdings went up. This is expected as consistent with our analytical characterization, see Lemma 1, higher levels of EIS imply more desire to smooth consumption and the investors accumulate more wealth. This means increased participation and increased equity share. Conversely, when we consider a 50% separation between benchmark calibrated agents and a 50% with higher risk averion ($\rho = 7$) but the same EIS we find higher participation and equity allocation. These results are consistent with Vestman (2012) who incorporates housing in the portfolio allocation model and consequently find lower participation and asset allocation with higher EIS (0.33) and higher participation with lower ρ.

It is to be noted that the presence of housing amplifies the effective risk aversion for the investor. This effect has recently been documented by Brunnermeier and Nagel (2008) and Zanetti (2014). This explains why our model gives a relatively lower stock holding value compared to the SCF data, 50.8% against 54.76%.

In our analysis in Table 3.2, we used SCF data from the 2001 wave while we incorporated recent data when estimating the moments of the house price process. This is not an ideal approach when there is substantial evidence that both participation rates and stock holdings have been rising throughout the years, see Guiso and Sodini (2013). However, the fact that recent studies have been concentrated in Scandinavian countries and not the United States forces us to use the 2001 data.

3.7. Discussion

To summarize the results section, we find five main results. Firstly we were able to find a hump-shaped profile of lifecycle risky asset allocation concurrent with the empirical literature. Secondly, we found that the risky asset share is highly sensitive to factor realizations when returns are time varying and predictable. High factor realizations or high dividend-yield shifts up the α_t curve at all ages. Extremely low factor realizations or very high factor volatility can indicate a rare disaster in the sense of Barro (2006, 2009) and this adversely affects the level of equity demanded. Thirdly, a unit root or explosive roots in the dividend-yield process indicative of a bubble significantly increases both the participation rate as well as the equity share conditional on participation. Fourthly, if investors have the option of return predictability they can hedge risks associated with housing cycles better compared to no predictability- the IID case. Finally, the presence of housing can resolve both the limited asset allocation and the limited participation puzzles without resorting to preference heterogeneity.

Fernández-Villaverde and Krueger (2011) demonstrate that consumer durables are crucial to explain the life cycle profiles of consumption and savings. Households begin their economic life without a stock of durables and they are precluded from building this stock immediately because of the presence of limited intertemporal markets. As a consequence, during the first part of their life cycle, households are forced to progressively accumulate durables and compromise on their consumption of nondurables and accumulation of financial assets. This phenomenon can explain why we observe that empirical life cycle consumption profiles, both of durables and nondurables, are hump shaped, even after controlling for demographics characteristics and why most households do not hold any substantial financial wealth until they enter into their forties.[24]

Before we conclude, some important caveats are worth mentioning. In our theoretical model we made two assumptions that can alter some of the major results. Firstly, we imposed short sales constraints on the investor meaning in that in the presence of rare disaster in the economy, investors/market participants are unable to hedge these risks by shorting the risky asset. For example Munk and Sørensen (2010) finds that in the absence of short-sales constraints, in the very early years (20-25), stocks are so attractive that the investor typically has 100% in stocks, but after a few years the long-term bond enters due to its hedging qualities. Secondly, we imposed the condition that every household has to own a house. This is why the participation rates are very low in the 20-25 age group. This forced homeownership meant that younger and poorer households have no option to rent. This explains why Yao and Zhang (2004) and Li and Yao (2007) find relatively higher levels of participation and risky asset allocation in the early years. Thirdly, several studies have found that the parameters in the time varying return predictable process is highly uncertain. In an influential paper, Stambaugh (1999) finds that the optimal buy-and-hold stock allocation can be higher at low values of the current dividend yield than at high values, even though the long-horizon stock return has a lower mean at the low dividend yield and can have at least as high a variance. This result can be traced to skewness in long-horizon stock returns arising from uncertainty about parameters, particularly the autoregressive coefficient of dividend yield. The skewness in the predictive distribution of returns is positive at low dividend yields and negative at high yields, and the effect of this

[24]The original version of this article was published as a working paper in 2001 and was the harbinger of multiple papers in this literature.

skewness can be strong enough to produce a textitnegative association between the optimal stock allocation and dividend yield.

3.8. Conclusion

In this chapter, we combine two important streams of literature in financial economics, the return predictability of stocks and the portfolio choice in the presence of risky illiquid housing, to examine patterns of risky asset allocation and stock market participation through the life-cycle. We successfully resolve two important puzzles observed in household data, the limited stock allocation puzzle and the limited equity market participation puzzle. We also looked at how these puzzles behave when there is a rare event in financial markets such as a bubble or when there is a disaster such as the recent economic crisis. Our results contribute to the literature in multiple ways.

The models closer to ours in the literature are Campbell and Viceira (1999), Cocco (2004), Gomes and Michaelidis (2005) and Michaelides and Zhang (2015). Campbell and Viceira (1999) solves for optimal α_t analytically but assumes a highly stylized model with no labour income or durable housing. In the analytical section of this chapter, we extended their work to include both housing and labour income. As in Gomes and Michaelidis (2005) we have recursive preferences and a calibrated labour income process in a lifecycle context to which we add the durable housing good. Cocco (2004) is perhaps the closest model to ours in that it models housing both as part of the utility function and allows collateralized borrowing. However, Cocco assumes that the returns are IID. We nest the IID as special case of expected returns which can be time varying and predictable. Michaelides and Zhang (2015) analyses return predictability in a lifecycle context but without housing.

The main result we find is that portfolio choice and market participation profiles are much different in the return predictability case relative to the IID case. We find the when returns are predictable from a factor such as the log dividend-price ratio, the optimal risky share of liquid wealth invested in the risky asset varies largely depending on the factor's: realization, persistence and volatility. High realizations and high persistence of the factor, specifically unit root or above, substantially shifts up both the risky equity allocated as well as the rate of stock market participation. Likewise, a huge dip in the factor or a high volatility suggesting a "rare disaster" in the economy brings down the liquid wealth invested in risky stocks. Unlike the bubble scenario, a rare disaster such as a market crash was found to have a heterogenous response over the life-cycle with older and retired households (65-100) being more affected (adversely).

Furthermore, investors are able to hedge background risks, such as labour income or house price volatilities, better when they can predict expected excess returns implying that substantial welfare losses can be avoided vis-a-vis the IID case. Finally, the presence of housing predicted a hump-shaped profile for risky asset allocation and participation with simulated rates very close to the Survey of Consumer Finances estimates without any preference heterogeneity.

There are several interesting extensions to our model which are worth pursuing. We only considered three financial assets, ignoring assets such as cash, which also acts as a medium of exchange and with it comes the risk of interest rate changes. Another possible area of future research is incorporating ideas of parameter or model uncertainty in the time varying regression framework, see Michaelides and Zhang (2015).

Appendix

3.A. The Euler Equations

We follow Bansal and Yaron (2004) and simplify the portfolio choice problem through a change of variables in order to utilize the arguments made by Epstein and Zin (1991). Each period the household invests available wealth after the labour income is realized in the financial assets, the households savings in period t is :

$$\sum_{i=0}^{2} B_{it} = W_t + Y_t - C_t - P_t^H EX_t \tag{3.60}$$

Given that, the intertemporal budget constraint can be expressed as

$$W_{t+1} = \sum_{i=0}^{2} B_{it} R_{i,t+1} \tag{3.61}$$

where B_{it} represents all the financial assets that the household owns at time t. The Euler equations that we derive hold for any number of assets that the investor owns, however for tractability, we only consider two financial assets. The risky stocks and the riskless bonds who have a time varying equity premium. Housing in our model is considered as a liquid asset which can be costlessly traded without incurring any transaction cost. Defining the gross rate of return from housing from period t to $t+1$ after accounting for depreciation ν as

$$R_{H,t+1} = \frac{(1-\nu)P_{t+1}^H}{P_t^H}, \tag{3.62}$$

$$B_{3,t} = P_t^H H_t, \tag{3.63}$$

$$\tilde{W}_t = W_t + Y_t + (1-\mu)P_{t+1}^H H_t. \tag{3.64}$$

where $\tilde{W}_t$ is the redefined wealth, comprising the financial wealth, labour income and the housing wealth.[25] The fact that we include labour income inside the new wealth variable implies that the two states of nature for inome process will now

[25]It has to be noted that $R_{H,t+1}$ is the gross return on housing which is given by

$$R_{H,t+1} = 1 + \text{simple return}, \quad \text{where}$$

$$\text{Simple return} = \frac{P_{t+1}(1-\nu) - P_t}{P_t} = \frac{P_{t+1}(1-\nu)}{P_t} - 1$$

translate into two states for the wealth process, one for retirement and the other for employment. The intraperiod budget constraint can be defined as

$$\sum_{i=0}^{3} B_{it} = W_t + Y_t - C_t + P_t^H(H_t - EX_t) = \tilde{W}_t - C_t \tag{3.65}$$

and the intertemporal flow of wealth is

$$W_{t+1} = \sum_{i=0}^{3} B_{it} R_{i,t+1}. \tag{3.66}$$

Then following Yogo (2006) when the portfolio weight $\alpha_{it} = B_{it}/(\tilde{W}_t - C_t)$, we end up with the equation in the wealth return form that we follow in the main text:

$$W_{t+1} = (\tilde{W}_t - C_t) \sum_{i=0}^{3} \alpha_{it} R_{i,t+1} \tag{3.67}$$

The Bellman equation for the household's intertemporal optimization problem in the employment state takes the form

$$V_t^e(W) = \max_{C^e, \alpha_1^e, \alpha_2^e, \alpha_3^e} \left\{ (1-\beta) u(C_t, H_t)^{1-\frac{1}{\psi}} + \beta E_t \left[V_{t+1}^{1-\gamma} \right]^{1/\kappa} \right\}^{\frac{1}{1-1/\psi}} \tag{3.68}$$

where the continuation value V_{t+1} is a weighted sum over the probability of employment π^e and retirement $(1-\pi^e)$. When the agent is employed, he faces uncertainty in the next period.

$$V_{t+1}(W_{t+1}) = \pi^e V_{t+1}^e(W_{t+1}) + (1-\pi^e) V_{T+1}^r(W_{t+1}) \tag{3.69}$$

As in Epstein and Zin (1991), the homogeneity of the value function imply that the optimal value can be written as a function of only wealth

$$V_t^e(\tilde{W}_t) = \phi_t \tilde{W}_t^e \tag{3.70}$$

Since $P_t^H H_t = \alpha_{3,t}(\tilde{W}_t - C_t)$, H_t can be substituted out of intraperiod utility as

$$u(C_t, H_t) = C_t \left[1 - \delta + \delta \left(\frac{\alpha_{3,t}(\tilde{W}_t/C_t - 1)}{P_t} \right)^{1-1/\rho} \right] \tag{3.71}$$

$$= \tilde{v}\left(\frac{\tilde{W}_t}{C_t}, \alpha_{3,t} \right) \tag{3.72}$$

We simplify the notation $\tilde{v}\Big(\frac{\tilde{W}_t}{C_t}, \alpha_{3,t}\Big) \equiv \tilde{V}_t$ and let $\Theta = \tilde{v}_t^{1-1/\rho}$, $\Xi \equiv \tilde{W}_t/C_t - 1$. Using the intertemporal budget constraint (3.67) the first order condition of the Bellman equation (3.68) w.r.t consumption can be written as

$$(1-\beta)(C_t\tilde{v}_t)^{-1/\psi}\Bigg(\tilde{v}_t - \vartheta^{\frac{1}{1-1/\rho}-1}\Xi^{-\frac{1}{\rho}}\delta\big(\frac{\alpha_{3,t}}{P_t^H}\big)^{1-\frac{1}{\rho}}\big(\frac{\tilde{W}_t}{C_t}\big)\Bigg) - \beta(\tilde{W}_t - C_t^*)^{-\frac{1}{\psi}}E_t[\phi_{t+1}^{1-\gamma}]^{\frac{1}{\kappa}} = 0. \tag{3.73}$$

Let $E_t[\phi_{t+1}^{1-\gamma}R_{p,t+1}^{1-\gamma}]^{\frac{1}{\kappa}} \equiv \mu^*$. Noticing that

$$\Xi^{-\frac{1}{\rho}}\delta\Big(\frac{\alpha_{3,t}}{P_t^H}\Big)^{1-\frac{1}{\rho}}\big(\frac{\tilde{W}_t}{C_t^*}\big) = \frac{\tilde{v}_t^{\,1-\frac{1}{\rho}-(1-\delta)}}{\frac{\tilde{W}_t}{C_t^*}-1} \tag{3.74}$$

and $\vartheta^{1/(1-1/\rho)-1} \equiv \tilde{v}_t^{\,-1/\rho}$ yields

$$(1-\beta)(1-\delta)(C_t^*)^{-\frac{1}{\psi}}\tilde{v}_t^{\,\frac{1}{\rho}-\frac{1}{\psi}}\tilde{W}_t - (1-\beta)(C_t^*)^{1-\frac{1}{\psi}}\tilde{v}_t^{\,1-\frac{1}{\psi}} = \beta(\tilde{W}_t - C_t^*)^{1-\frac{1}{\psi}}\mu^* \tag{3.75}$$

Substituting optimal consumption C_t^* to the Bellman equation (3.68) and using (3.70) we get

$$\phi_t\tilde{W}_t = \Bigg[(1-\beta)(C_t^*\tilde{v}_t)^{1-\frac{1}{\psi}} + \beta(\tilde{W}_t - C_t^*)^{1-\frac{1}{\psi}}\mu^*\Bigg]^{\frac{1}{1-1/\psi}} \tag{3.76}$$

Plugging (3.75) to (3.76) gives

$$\phi_t = \Bigg[(1-\beta)(1-\delta)\tilde{v}_t^{\,\frac{1}{\rho}-\frac{1}{\psi}}\Bigg]^{\frac{1}{1-1/\psi}}\Big(\frac{C_t^*}{W_t}\Big)^{\frac{1}{1-\psi}} \tag{3.77}$$

Yogo (2006) shows that using the arguments made by Epstein and Zin (1991) one gets the set of FOC's w.r.t C_t and portfolio choice $\alpha_{i,t}$ for $i = 1, \ldots, 3$

$$E_t[M_{t+1}^*R_{p,t+1}^*] = \Big(1 - \frac{\alpha_{3,t}u_H}{P_t^H u_C}\Big)^{\kappa}, \tag{3.78}$$

$$E_t[M_{t+1}^*(R_{i,t+1} - R_{f,t+1})] = 0, \quad i = 1, \ldots, 3 \tag{3.79}$$

$$E_t[M_{t+1}^*(R_{f,t+1} - R_{3,t+1})] = \frac{u_H}{P_t^H u_C}\Big(1 - \frac{\alpha_{3,t}u_H}{P_t^H u_C}\big)^{\kappa-1}. \tag{3.80}$$

which together imply that

$$E_t[M_{t+1}^*R_{i,t+1}] = \Big(1 - \frac{\alpha_{3,t}u_H}{P_t^H u_C}\Big)^{\kappa-1}. \tag{3.81}$$

After normalization, we get the standard Euler equations in the text. The case of retirement and employment affects only the discount variable and hence it is straightforward. For the employment state, we write the expectations as a probability weighted average of the two values while under retirement there is no uncertainty.

3.B. Intratemporal Optimization

Following Yogo (2006) and Bednarek (2014) the marginal rate of substitution between the durable and the non-durable housing consumption good is

$$\frac{u_H}{u_C} = \frac{\delta}{1-\delta}\Big(\frac{H}{c}\Big)^{-1/\rho}. \tag{3.82}$$

where u_C is the marginal utility with respect to consumption and u_H with Housing. The optimal consumption of the durable housing requires an intratemporal first order condition in the form

$$\frac{u_{H_t}}{u_{C_t}} = P_t^H - (1-\nu)E_t[M_{t+1}P_{t+1}^H] = Q_t \tag{3.83}$$

Q_t here is interpreted as the user cost of the service flow for the housing good. As the durable good in our model is a house, the user cost is nothing but the rent. This equation simply says that the marginal rate of substitution between the durable good and nondurable good consumption goods must equal the relative price of the durable good. When the depreciation rate $\nu = 1$ and the intratemporal substitution $\rho = 1$, this equation reduces to $\delta/(1-\delta) = PH/C$, meaning that δ can be interpreted as the expenditure share of the durable good.

3.C. Log Linear Euler Equations

We first derive the log linearized version of the highly non-linear Euler equations for the retirement state. We can write equation (3.22) as

$$\begin{aligned}1 = (\pi^e)^\kappa E_t\Bigg[&\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}\ln(C_{t+1}^e/C_t^e) + (\frac{1}{\rho}-\frac{1}{\psi})\ln(v(C_{t+1}^e, H_{t+1}^e)/v(C_t^e, H_t^e)) + \ln R_{i,t+1}\big)\Big\}\Bigg] \\ +(1-\pi^e)^\kappa E_t\Bigg[&\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}\ln(C_{t+1}^r/C_t^e) + (\frac{1}{\rho}-\frac{1}{\psi})\ln(v(C_{t+1}^r, H_{t+1}^r)/v(C_t^e, H_t^e)) + \ln R_{i,t+1}\big)\Big\}\Bigg]\end{aligned} \tag{3.84}$$

Denoting log variables with lower case letters

$$1 = (\pi^e)^\kappa E_t\Bigg[\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}(c^e_{t+1} - c^e_t) + (\frac{1}{\rho} - \frac{1}{\psi})(v(c^e_{t+1}, h^e_{t+1}) - v(c^e_t, h^e_t)) + r_{i,t+1}\big)\Big\}\Bigg]$$
$$+(1-\pi^e)^\kappa E_t\Bigg[\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}(c^r_{t+1} - c^e_t) + (\frac{1}{\rho} - \frac{1}{\psi})(v(c^r_{t+1}, h^r_{t+1}) - v(c^e_t, h^e_t)) + r_{i,t+1}\big)\Big\}\Bigg] \tag{3.85}$$

Assuming intraperiod utility as a Cobb-Douglas form with $\rho = 1$,

$$u(C_t, H_t) = C_t^\delta H_t^{1-\delta} \tag{3.86}$$
$$= C_t\Big(\frac{H_t}{C_t}\Big)^\delta \tag{3.87}$$
$$= C_t v(C_t, H_t) \tag{3.88}$$

implying

$$v_t(C_t, H_t) = \Big(\frac{H_t}{C_t}\Big)^\delta \tag{3.89}$$

Taking logs,

$$\ln(v_t) = \delta(h_t - c_t) \tag{3.90}$$

Plugging this into the Euler equation we get

$$1 = (\pi^e)^\kappa E_t\Bigg[\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}(c^e_{t+1} - c^e_t) - \frac{\delta}{\psi}((h^e_{t+1} - h^e_t) + (c^e_{t+1} - c^e_t)) + r_{i,t+1}\big)\Big\}\Bigg]$$
$$+(1-\pi^e)^\kappa E_t\Bigg[\exp\Big\{\kappa\big(\ln\beta^e - \frac{1}{\psi}(c^r_{t+1} - c^e_t) - \frac{\delta}{\psi}((h^r_{t+1} - h^e_t) + (c^r_{t+1} - c^e_t)) + r_{i,t+1}\big)\Big\}\Bigg] \tag{3.91}$$

Following Viceira (2001), we can write this Euler equation in two variables

$$1 = \pi^\kappa E_t[\exp\{x_{t+1}\}] + (1-\pi)^\kappa E_t[\exp\{y_{t+1}\}] \tag{3.92}$$

where x_{t+1} is the first term and y_{t+1} is the second term on the right hand side. Taking a second order Taylor expansion of $\exp\{x_{t+1}\}$ and $\exp\{y_{t+1}\}$ around $\bar{x}_t =$

$E_t[x_{t+1}]$ and $\bar{y}_t = E_t[y_{t+1}]$, we get

$$1 \approx \pi^{\kappa} E_t\Bigg[\exp\{\bar{x}_t\}\Big(1 + (x_{t+1} - \bar{x}_t) + \frac{1}{2}(x_{t+1} - \bar{x}_t)^2\Big)\Bigg]$$

$$+(1-\pi)^{\kappa} E_t\Bigg[\exp\{\bar{y}_t\}\Big(1 + (y_{t+1} - \bar{y}_t) + \frac{1}{2}(y_{t+1} - \bar{y}_t)^2\Big)\Bigg] \tag{3.93}$$

$$\approx \pi \exp\{\bar{x}_t\}\Big(1 + \frac{1}{2}Var_t(x_{t+1})\Big) + (1-\pi)\exp\{\bar{y}_t\}\Big(1 + \frac{1}{2}Var_t(y_{t+1})\Big) \tag{3.94}$$

A first order Taylor expansion around zero, as in Viceira (2001), gives the result

$$1 \approx \pi^{\kappa}\Big(1 + \bar{x}_t + \frac{1}{2}Var_t(x_{t+1})\Big) + (1-\pi)\Big(1 + \bar{y}_t + \frac{1}{2}Var_t(y_{t+1})\Big) \tag{3.95}$$

Substituting the values of $x_t, x_{t+1}, y_t, y_{t+1}$ to eq. (3.91) we get the log linear euler equation for the employment state

$$1 \approx \sum_{s=e,r} \pi_s^{\kappa}\Bigg(E_t\Big[1 + \kappa \ln \beta^s - \frac{\kappa}{\psi}(c_{t+1}^s - c_t^e) - \frac{\kappa\delta}{\psi}(h_{t+1}^s - h_t^e) + \frac{\kappa\delta}{\psi}(c_{t+1}^s - c_t^e) + \kappa r_{i,t+1}\Big]$$
$$+\frac{1}{2}Var_t\big(\kappa r_{i,t+1} - \frac{\kappa}{\psi}(c_{t+1}^s - c_t^e) - \frac{\kappa\delta}{\psi}(h_{t+1}^s - h_t^e) + \frac{\kappa\delta}{\psi}(c_{t+1}^s - c_t^e)\big)\Bigg) \tag{3.96}$$

for $i = p$. The corresponding equation for the retirement state is straightforward when we notice that under the retirement state the labour income is zero and furthermore there is no uncertainty, that is, it is an irreversible state. The log linear euler equation for the retirement state is thus

$$0 \approx \Bigg(E_t\Big[\kappa \ln \beta^r - \frac{\kappa}{\psi}(c_{t+1}^r - c_t^r) - \frac{\kappa\delta}{\psi}(h_{t+1}^r - h_t^r) + \frac{\kappa\delta}{\psi}(c_{t+1}^r - c_t^r) + \kappa r_{i,t+1}\Big]$$
$$+\frac{1}{2}Var_t\big(\kappa r_{i,t+1} - \frac{\kappa}{\psi}(c_{t+1}^r - c_t^r) - \frac{\kappa\delta}{\psi}(h_{t+1}^r - h_t^r) + \frac{\kappa\delta}{\psi}(c_{t+1}^r - c_t^r)\big)\Bigg) \tag{3.97}$$

3.D. Log-Linear Budget Constraint

The wealth-return intertemporal budget constraint for the employment state is given by

$$W_{t+1} = \Big(W_t + Y_t - C_t + P_t^H(H_t - EX_t)\Big)R_{p,t+1} \tag{3.98}$$

Following Campbell (1993, 1996) and Viceira (2001) we start the log-linearization by dividing with the labour income and express it as

$$\frac{W_{t+1}}{Y_{t+1}} = \Big(1 + \frac{W_t}{Y_t} - \frac{C_t}{Y_t} + \frac{P_t^H(H_t - EX_t)}{Y_t}\Big)\Big(\frac{Y_t}{Y_{t+1}}\Big)R_{p,t+1} \tag{3.99}$$

As $P_t^H(H_t - EX_t)$ is nothing but the net housing wealth after accounting for any expediture (EX_t), we denote this by a new variable W_t^H.Taking logs

$$w_{t+1} - y_{t+1} = \ln\Big(1 + \exp(w_t - y_t) - \exp(c_t - y_t) + \exp(w_t^h - y_t)\Big) - \Delta y_{t+1} + r_{p,t+1} \tag{3.100}$$

where lower case letters as usual denote log variables and Δ is the first difference operator. The first term in the right hand side, $\ln(1+\exp(w_t - y_t) - \exp(c_t - y_t) + \exp(w_t^h - y_t))$, is non-linear. We linearise this term by taking a first order Taylor expansion around the stationary log consumption-income, log wealth-income and log housing wealth-income ratios. That is, the equation

$$\ln(1 + \exp(w_t - y_t) - \exp(c_t - y_t) + \exp(w_t^h) - y_t)) \tag{3.101}$$

is linearized around $(w_t - y_t) = E[w_t - y_t]$, $(c_t - y_t) = E[c_t - y_t]$ and $w_t^h - y_t = E[w_t^h - y_t]$. For simplicity, we represent $m_t = w_t - y_t$, $n_t = c_t - y_t$ and $o_t = w_t^h - y_t$ and then log-linearize around $\bar{m}_t = E[m_t]$, $\bar{n}_t = E[n_t]$ and $\bar{o}_t = E[o_t]$ we approximate the non-linear equation (3.101) as

$$\begin{aligned} \ln(1 + \exp(m_t) - \exp(n_t) + \exp(o_t)) &\approx \ln\big(1 + \exp(\bar{m}_t) - \exp(\bar{n}_t) + \exp(\bar{z}_t)\big) \\ &+ \quad \rho_w(m_t - \bar{m}_t) - \rho_c(n_t - \bar{n}_t) + \rho_h(o_t - \bar{o}_t) \end{aligned} \tag{3.102}$$

where

$$\rho_w = \frac{\exp(\bar{m}_t)}{1 + \exp(\bar{m}_t) - \exp(\bar{n}_t) + \exp(\bar{o}_t)}, \tag{3.103}$$

$$\rho_c = \frac{\exp(\bar{n}_t)}{1 + \exp(\bar{m}_t) - \exp(\bar{n}_t) + \exp(\bar{o}_t)}, \quad \text{and} \tag{3.104}$$

$$\rho_h = \frac{\exp(\bar{o}_t)}{1 + \exp(\bar{m}_t) - \exp(\bar{n}_t) + \exp(\bar{o}_t)} \tag{3.105}$$

Plugging this log linearized equation in (3.100) and substituting the values for m_t, n_t and o_t we get,

$$w_{t+1}^e - y_{t+1} = k^e + \rho_w^e(w_t^e - y_t) - \rho_c^e(c_t^e - y_t) + \rho_h^e(w_t^{h_e} - y_t) - \Delta y_{t+1} + r_{p,t+1}^e \tag{3.106}$$

where we have added the superscript e to differentiate from the retirement state. The constant k^e is given by

$$k^e = \ln(1 + \exp(\bar{m}_t) - \exp(\bar{n}_t) + \exp(\bar{o}_t)) - \rho_w \bar{m}_t + \rho_c \bar{n}_t - \rho_h \bar{o}_t \tag{3.107}$$

We also have

$$H_t = (1-\nu)H_{t-1} + EX_t \tag{3.108}$$

that is

$$H_t - EX_t = (1-\nu)H_{t-1} \tag{3.109}$$

implying

$$P_t^H(H_t - EX_t) = P_t^H(1-\nu)H_{t-1} \tag{3.110}$$

In logs

$$\ln(P_t^H(H_t - EX_t)) = p_t^H + \ln(1-\nu) + h_{t-1} \tag{3.111}$$

or

$$w_t^h = p_t^h + \ln(1-\nu) + h_{t-1} \tag{3.112}$$

Plugging this into eq. (3.106) we get the log linear intertemporal budget constraint for the employment state,

$$w_{t+1}^e - y_{t+1} = k^e + \rho_w^e(w_t^e - y_t) - \rho_c^e(c_t^e - y_t) + \rho_h^e(p_t^h + \ln(1-\nu) + h_{t-1}^e - y_t) - \Delta y_{t+1} + r_{p,t+1}^e \tag{3.113}$$

For the retirement state, there is no labour income, hence,

$$W_{t+1} = \Big(W_t - C_t + W_t^H\Big)R_{p,t+1} \tag{3.114}$$

Dividing through W_t,

$$\frac{W_{t+1}}{W_t} = \Big(1 - \frac{C_t}{W_t} + \frac{W_t^H}{W_t}\Big)R_{p,t+1} \tag{3.115}$$

The rest of the derivation is in similar fashion as the employment state (above). Taking logs and then linearizing around stationary log endowment wealth to housing wealth ratio, $w_t^h - w_t = E[w_t^h - w_t]$ and log consumption to wealth ratio,

$c_t - w_t = E[c_t - w_t]$ we arrive at the log linearized intertemporal budget constraint for the retirement state,

$$w^r_{t+1} - w_t = k^r + \rho^r_h(w^{h_r}_t - w_t) - \rho^r_c(c^r_t - w_t) + r^r_{p,t+1} \tag{3.116}$$

where we have added the superscript r to indicate retirement state. Expressing housing wealth in price form,

$$w^r_{t+1} - w_t = k^r + \rho^r_h(p^h_t + \ln(1-\nu) + h^r_{t-1} - w_t) - \rho^r_c(c^r_t - w_t) + r^r_{p,t+1} \tag{3.117}$$

Here the constant k^r is

$$k^r = \ln(1 + \exp(E[w^{h_r}_t - w_t]) - \exp(E[c^r_t - w_t])) - \rho^r_H E[w^h_t - w_t] + \rho^r_c E[c^r_t - w_t] \tag{3.118}$$

and

$$\rho^r_h = \frac{\exp(E[w^{h_r}_t - w_t])}{1 + \exp(E[w^{h_r}_t - w_t]) - \exp(E[c^r_t - w_t])}, \tag{3.119}$$

$$\rho^r_c = \frac{\exp(E[c^r_t - w_t])}{1 + \exp(E[w^{h_r}_t - w_t]) - \exp(E[c^r_t - w_t])}. \tag{3.120}$$

3.E. Optimal Portfolio Choice in the Employment State

The result for the retirement state follows directly from the discussion in the text. In what we follows we detail the derivation for the employment state. In the employment state, the log Euler equation for a general risky asset (stocks or house) is given by

$$1 = \sum_{s=e,r} \pi^\kappa_s \Bigg(E_t\Big[1 + \kappa \ln \beta^s + \frac{\kappa}{\psi}\Delta c^s_{t+1}(\delta - 1) - \frac{\kappa\delta}{\psi}\Delta h^s_{t+1} + (\kappa - 1) r_{p,t+1} + r_{i,t+1}\Big] + \frac{1}{2} Var_t\big(r_{i,t+1} + (\kappa - 1) r_{p,t+1} + \frac{\kappa}{\psi}\Delta c^s_{t+1}(\delta - 1) - \frac{\kappa\delta}{\psi}\Delta h^s_{t+1}\big)\Bigg)$$

and for a risk free asset $i = f$ is:

$$1 = \sum_{s=e,r} \pi^\kappa_s \Bigg(E_t\Big[1 + \kappa \ln \beta^s + \frac{\kappa}{\psi}\Delta c^s_{t+1}(\delta - 1) - \frac{\kappa\delta}{\psi}\Delta h^s_{t+1} + (\kappa - 1) r_{p,t+1} + r_f\Big] + \frac{1}{2} Var_t\big((\kappa - 1) r_{p,t+1} + \frac{\kappa}{\psi}\Delta c^s_{t+1}(\delta - 1) - \frac{\kappa\delta}{\psi}\Delta h^s_{t+1}\big)\Bigg)$$

Subtracting the second equation from the first and simplifying we get,

$$E_t[r_{i,t+1}] - r_f + \sum_{s=e,r} \pi_s^\kappa \frac{1}{2} Var_t(r_{i,t+1}) = \sum_{s=e,r} \pi_s^\kappa \Big((1-\kappa) cov(r_{i,t+1}, r_{p,t+1}) \quad + \\ \frac{\kappa}{\psi}(1-\delta) cov(r_{i,t+1}, \Delta c_{t+1}^s) + \kappa \frac{\delta}{\psi} cov(r_{i,t+1}, \delta h_{t+1}^s) \Big) \tag{3.121}$$

We make an assumption that $\pi_e^\kappa + (1-\pi_e)^\kappa \approxeq 1$. This approximation is exact under CRRA preferences , that is when $\kappa = 1$. Equation (3.121) simplifies to

$$E_t[r_{i,t+1}] - r_f + \frac{1}{2} Var_t(r_{i,t+1}) = (1-\kappa) cov(r_{i,t+1}, r_{p,t+1}) \\ + \sum_{s=e,r} \pi_s^\kappa \Big(\quad \frac{\kappa}{\psi}(1-\delta) cov(r_{i,t+1}, \Delta c_{t+1}^s) + \kappa \frac{\delta}{\psi} cov(r_{i,t+1}, \delta h_{t+1}^s) \Big) \tag{3.122}$$

We derived expressions for the three covariance terms as functions of wealth and portfolio choice for the retirement state:

$$\begin{aligned} cov(r_{i,t+1}, r_{p,t+1}) &= \alpha_{it}^e \sigma_{it}^2 \\ cov(r_{i,t+1}, \Delta c_{t+1}^s) &= cov(r_{i,t+1}, c_{t+1}^s - w_{t+1}) + \alpha_{it}^e \sigma_{it}^2 \\ cov(r_{i,t+1}, \Delta h_{t+1}^s) &= cov(r_{i,t+1}, h_{t+1}^s - w_{t+1}) + \alpha_{it}^e \sigma_{it}^2 \end{aligned}$$

These will hold for the employment state as well. Substituting these values and rearranging for α_{it}^e using the fact that $\kappa = \frac{1-\gamma}{1-\frac{1}{\psi}}$:

$$\alpha_{it}^e = \frac{1}{\gamma} \frac{E_t r_{i,t+1} - r_f + \frac{1}{2}\sigma_{it}^2}{\sigma_{it}^2} + \Big(\frac{1}{1-\psi}\Big)\Big(\frac{1-\gamma}{\gamma}\Big) \left(\sum_{s=e,r} \pi_s^\kappa \Big[\frac{(1-\delta)\sigma_{i,(c_{t+1}^s - w_{t+1})} + \delta \sigma_{i,(h_{t+1}^s - w_{t+1})}}{\sigma_{it}^2} \Big] \right) \tag{3.123}$$

where $Var_t(r_{i,t+1}) = \sigma_{it}^2$, $cov(r_{i,t+1}, r_{p,t+1}) = \sigma_{r_{i,t+1}, r_{p,t+1}}$, $cov(r_{i,t+1}, c_{t+1}^s - w_{t+1}) = \sigma_{i, c_{t+1}^s - w_{t+1}}$ and $cov(r_{i,t+1}, h_{t+1}^s - w_{t+1}) = \sigma_{i, h_{t+1}^s - w_{t+1}}$.

3.F. Constructing the Budget Constraints

For any time period $t > 1$, the consumption C_t of other non-durable goods for the investor or the household can be written as:

$$C_t = \text{Financial Wealth}_t + \text{Labour Income}_t - \text{Expenses}_t - \text{Asset Allocation}_t \tag{3.124}$$

That is, at each period t the investor can consume what is left of once he allocates his income and wealth towards expenses and accumulating assets. The financial wealth at time t is the gross returns from the three assets, Stocks, Bills and Mortgage, he held in period $t-1$:

$$\text{Financial Wealth}_t = R_t S_{t-1} + R_f B_{t-1} - R_D D_{t-1} \tag{3.125}$$

Also,

$$\text{Labour Income}_t = Y_t \tag{3.126}$$

At every period the investor can choose to spend the fixed cost involved in equity market participation (if he has not done so before). We let FC_t take the value of 1 if the investor chooses to pay the fixed cost of equity market participation in period t and zero otherwise and F the monetary value involved in the participation. Furthermore, at every period the investor has to spend money on housing maintenance. The total expenses is, thus given by:

$$\text{Expenses}_t = FC_t F Y_t + \delta P_t H_{t-1} \tag{3.127}$$

The investor allocates the remaining wealth after expenses into the three financial assets:

$$\text{Asset Allocation}_t = S_t + B_t - D_t \tag{3.128}$$

Plugging the four expressions into the budget constraint:

$$\begin{aligned} C_t = {} & (R_t S_{t-1} + R_f B_{t-1} - R_D D_{t-1}) + Y_t - \\ & (FC_t F Y_t + mc_h P_t H_{t-1}) - (S_t + B_t - D_t) \end{aligned} \tag{3.129}$$

We define cash on hand, X_t, in the lines of Deaton (1991) and Carroll (1997), as the sum of liquid or financial wealth and Labour Income:

$$X_t = (R_t S_{t-1} + R_f B_{t-1} - R_D D_{t-1}) + Y_t \tag{3.130}$$

Expressing the budget constraint in terms of cash on hand and rearranging we get the date t inter-temporal budget constraint for the investor:

$$S_t + B_t = \begin{cases} X_t - C_t - FC_t F Y_t - \delta P_t H_{t-1} + D_t, & \forall t \\ X_t - C_t - FC_t F Y_t - \delta P_t H_{t-1} + D_t + (1-\Lambda) P_t H_{t-1} - P_t H_t, & \forall t \end{cases} \tag{3.131}$$

Also, consumption must be non-negative at all dates:

$$C_t \geq 0, \quad \forall\ t \tag{3.132}$$

and we define $\triangle$ as the proportion of liquid assets held in stocks over stocks plus bills:

$$\triangle_t = \frac{S_t}{S_t + B_t} \quad \forall\ t, \quad \triangle_t \in [0, 1] \tag{3.133}$$

Wealth at date $t + 1$ is given by

$$W_{T+1} = X_{T+1} - mc_h H_T P_{T+1} + (1 - \Lambda) H_T P_{T+1} \tag{3.134}$$

3.G. Numerical Solution

The life-cycle constrained optimisation problem cannot be solved analytically. The setup of the model with discrete choice of stock market investment, fixed costs and the presence of borrowing constraints imply that we cannot rely on the existence of smooth first order conditions that could otherwise have used to solve the model efficiently. Hence, we resort to Value Function Iteration, a robust method of optimization based on the Contraction Mapping Theorem.[26]

As this is a finite time optimization problem, a solution exists and is deduced by the Backward Induction Algorithm. At the terminal period, the value function reduces to the bequest function. Iterating each period backwards, we get the optimal policies for consumption, housing, debt and asset allocation. The stock market participation decision is made based on comparing the value functions conditional on having paid the fixed cost with the no fixed cost value function. Similarly, the decision to move house is based on choosing the action that gives the maximum value function conditional on no movement against movement.

Shocks to the equity premium, house prices and labour income were approximated using Gaussian-Hermite Quadrature. Optimization over different choices were implemented using grid search. We reduce the state-space dimensionality through standardizing by the permanent component of labour income for faster computation.

[26] According to the Contraction Mapping Theorem, there is an operator $T[.]$ that maps the value function into itself, $v = T[v]$. Under particular conditions, $T[.]$ has a unique fixed point, say $v*$, such that $v* = T[v*]$, and that a sequence of v's, $v_{n+1} = T[v_n]$, initiated at any v_0 converges to this fixed point if the state space is a complete metric space.

3.H. Recursive Utility

In this section we explore the construction of recursive utility used in this chapter and detail the implications for the temporal behaviour of consumption and portfolio allocation. Epstein and Zin (1989) analytically prove (i) the existence of recursive intertemporal utility functions, and (ii) the existence of optima to corresponding optimization problems.

Epstein and Zin (1989) follow Kreps and Porteus(1978) and define the consumption space in terms of temporal lotteries to model the way in which consumption uncertainty is resolved over time. Each temporal lottery d can be pictured as an infinite probability tree in which each branch corresponds to a deterministic consumption stream $y \in R_+^\infty$. The lottery d can be identified with a pair (c_0, m) where $c_0 \geq 0$ denotes the nonstochastic period 0 level of consumption and m, a probability measure over the set of $t = 1$ nodes in the tree, represents the uncertain future.

To understand the structure of recursive utility, consider V as a utility function defined as

$$V(c_0, c_1, \ldots) = W(c_0, V(c_1, c_2, \ldots)) \tag{3.135}$$

for some function W, termed as an aggregator as it combines current consumption and future utility to determine current utililty. In the presence of stochastic terms, future utility is random and thus it is natural to compute a certainty equivalent for random future utility and then to combine the certainty equivalent utility level with c_0 via an aggregator. The utility function V is *recursive* if it satisfies the following equation on its domain:

$$V(c_0, m) = W(c_0, \mu(V[m])) \tag{3.136}$$

for some increasing aggregator function $W : R_+^2 \to R_+$ and some certainty equivalent μ. For the existence of utility functions in the form of eq 3.136 requires that the aggregator W has the Constant Elasticity of Substitution form given by

$$W(c, z) = [c^\rho + \beta z^\rho]^{1/\rho}, \quad 0 \neq \rho < 1,\ 0 < \beta < 1 \tag{3.137}$$

with elasticity of substitution $\sigma = (1 - \rho)^{-1}$. Thus, ρ is a parameter that is understood to reflect substitutability. Assuming that the certainty equivalent form is in the Kreps and Porteus (1978) class,

$$\mu(p) = (E_p \tilde{x}^\alpha)^{1/\alpha}, \quad p \in M(R_+), \tag{3.138}$$

where $0 \neq \alpha < 1$ can be considered as the risk aversion parameter, consequently V follows

$$V(c, m) = [c_0^\rho + \beta(E_m V^\alpha(.))^{\rho/\alpha}]^{1/\rho} \tag{3.139}$$

Regarding the attitudes towards the timing of the resolution of uncertainty, given the functional form of eq. 3.139, early (late) resolution is preferred if $\alpha < (>)\rho$. it is noteworthy that the timing of this uncertainty differs between different classes of utility functions, the von-Nuemann - Morgenstern Expected Utility form gives indifference towards any kind of resolution of uncertainty.

Bibliography

Abraham, J. M. and Hendershott, P. H. (1996). Bubbles in Metropolitan Housing Markets. *Journal of Housing Research*, 7(2):191–207.

Aiyagari, S. and Gertler, M. (1999). "Overreaction" of Asset Prices in General Equilibrium. *Review of Economic Dynamics*, 2(1):3–35.

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics*, 109(3):659–684.

Alan, S. (2006). Entry costs and stock market participation over the life cycle. *Review of Economic Dynamics*, 9(4):588–611.

Alan, S. (2012). Do disaster expectations explain household portfolios? *Quantitative Economics*, 3(1):1–28.

Albuquerque, R. and Hopenhayn, H. A. (2004). Optimal lending contracts and firm dynamics. *Review of Economic Studies*, 71(2):285–315.

Allen, F., Morris, S., and Postlewaite, A. (1993). Finite bubbles with short sale constraints and asymmetric information. *Journal of Economic Theory*, 61(2):206–229.

Alvarez, F. and Jermann, U. J. (2000). Efficiency, Equilibrium, and Asset Pricing with Risk of Default. *Econometrica*, 68(4):775–797.

Ambrose, B. W., Eichholtz, P., and Lindenthal, T. (2013). House Prices and Fundamentals: 355 Years of Evidence. *Journal of Money, Credit and Banking*, 45(2-3):477–491.

Ameriks, J. and Zeldes, S. (2002). How do household portfolio shares vary with age. *TIAA-CREF working paper*, 112(2):6–12.

André, C., Gil-Alana, L. A., and Gupta, R. (2014). Testing for persistence in housing price-to-income and price-to-rent ratios in 16 OECD countries. *Applied Economics*, 46(18):2127–2138.

Andrews, D. W. K. (1993). Tests for Parameter Instability and Structural Change With Unknown Change Point. *Econometrica*, 61(4):821–856.

Andrews, D. W. K. and Fair, R. C. (1988). Inference in Nonlinear Econometric Models with Structural Change. *The Review of Economic Studies*, 55(4):615–639.

Andrews, D. W. K. and Ploberger, W. (1994). Optimal Tests when a Nuisance Parameter is Present Only Under the Alternative. *Econometrica*, 62(6):1383–1414.

Andrews, D. W. K. and yuan Chen, H. (1994). Approximately Median-Unbiased Estimation of Autoregressive Models. *Journal of Business & Economic Statistics*, 12(2):187–204.

Arellano, C. (2008). Default risk and income fluctuations in emerging economies. *American Economic Review*, 98(3):690–712.

Attanasio, O. P., Bottazzi, R., Low, H. W., Nesheim, L., and Wakefield, M. (2012). Modelling the demand for housing over the life cycle. *Review of Economic Dynamics*, 15:1–18.

Attanasio, O. P. and Weber, G. (2010). Consumption and Saving: Models of Intertemporal Allocation and Their Implications for Public Policy. *Journal of Economic Literature*, 48(3):693–751.

Aymanns, C. and Farmer, J. D. (2015). The dynamics of the leverage cycle. *Journal of Economic Dynamics and Control*, 50:155–179.

Ayuso, J. and Restoy, F. (2006). House prices and rents: An equilibrium asset pricing approach. *Journal of Empirical Finance*, 13(3):371–388.

Bai, J. and Perron, P. (1998). Estimating and Testing Linear Models with Multiple Structural Changes. *Econometrica*, 66(1):47–78.

Bai, J. and Perron, P. (2003). Computation and analysis of multiple structural change models. *Journal of Applied Econometrics*, 18(1):1–22.

Baillie, R. T. (1996). Long memory processes and fractional integration in econometrics. *Journal of Econometrics*, 73(1):5–59.

Baillie, R. T. and Kapetanios, G. (2007). Semi Parametric Estimation of Long Memory : The Holy Grail or a Poisoned Chalice ? In *International Symposium on Financial Econometrics*, pages 1–37.

Bansal, R. and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance*, 59(4):1481–1509.

Barari, M., Sarkar, N., Kundu, S., and Chowdhury, B. K. (2014). Forecasting House Prices in the United States with Multiple Structural Breaks. *International Econometric Review*, 6(1):1–23.

Barberis, N. (2000). Investing for the Long Run when Returns Are Predictable. *The Journal of Finance*, 55(1):225–264.

Barro, R. J. (2006). Rare Disasters and Asset Markets in the Twentieth Century. *The Quarterly Journal of Economics*, 121(3):823–866.

Barro, R. J. (2009). Rare Disasters, Asset Prices, and Welfare Costs. *American Economic Review*, 99(1):243–264.

Barros, C. P., Gil-Alana, L. a., and Payne, J. E. (2012). Comovements among U.S. state housing prices: Evidence from fractional cointegration. *Economic Modelling*, 29(3):936–942.

Basak, S. and Makarov, D. (2014). Strategic Asset Allocation in Money Management. *The Journal of Finance*, 69(1):179–217.

Basak, S. and Shapiro, A. (2001). Value-at-Risk-Based Risk Management. *The Review of Financial Studies*, 14(2):371–405.

Bednarek, Z. (2014). Long-Run Optimization Horizon, Durable Consumption Growth and Estimation of Risk Aversion. *Working Paper.*

Benzoni, L., Collin-Dufresne, P., and Goldstein, R. S. (2007). Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *The Journal of Finance*, 62(5):2123–2167.

Beran, J. (1995). Maximum likelihood estimation of the differencing parameter for invertible short and long memory autoregressive integrated moving average models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 57(4):659–672.

Bernanke, B. and Gertler, M. (1989). Agency Costs, Net Worth and Business Fluctuations. *The American Economic Review*, 79(1):14–31.

Bernanke, B., Gertler, M., and Gilchrist, S. (1996). The Financial Accelerator and the Flight to Quality. *Review of Economics and Statistics*, 78(1):1–15.

Bernanke, B. S., Gertler, M., and Gilchrist, S. (1999). The financial accelerator in a quantitative business cycle framework. In *Handbook of Macroeconomics*, volume 1, pages 1341–1393.

Bewley, T. (1987). Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers. Technical report, Cowles Foundation Paper 671, North Holland.

Bhardwaj, G. and Swanson, N. R. (2006). An empirical investigation of the usefulness of ARFIMA models for predicting macroeconomic and financial time series. *Journal of Econometrics*, 131(1-2):539–578.

Blanchard, O. and Watson, M. (1982a). Bubbles, Rational Expectations and Financial Markets. In Wachtel, P., editor, *Crises in the Economic and Financial Structure*. Lexington Books, Lexington, MA.

Blanchard, O. and Watson, M. (1982b). Bubbles, Rational Expectations and Financial Markets. Technical report, National Bureau of Economic Research, Cambridge, MA.

Blanchard, O. J. (1979). Speculative bubbles, crashes and rational expectations. *Economics letters*, 3(1979):387–389.

Blanchard, O. J. (1985). Debt, Deficits, and Finite Horizons.

Bodie, Z., Merton, R. C., and Samuelson, W. F. (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control*, 16(3-4):427–449.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3):307–327.

Bollerslev, T. and Ole Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, 73(1):151–184.

Broadie, M., Chernov, M., and Johannes, M. (2007). Model Specification and Risk Premia: Evidence from Futures Options. *The Journal of Finance*, 62(3):1453–1490.

Brumm, J., Grill, M., Kubler, F., and Schmedders, K. (2015). Collateral Requirements and Asset Prices. *International Economic Review*, 56(1):1–25.

Brunnermeier, M. K., Eisenbach, T. M., and Sannikov, Y. (2012). Macroeconomics with Financial Frictions: A Survey. *National Bureau of Economic Research Working Paper Series*, No. 18102.

Brunnermeier, M. K. and Julliard, C. (2007). Money Illusion and Housing Frenzies. *Review of Financial Studies*, 21(1):135–180.

Brunnermeier, M. K. and Nagel, S. (2008). Do wealth fluctuations generate time-varying risk aversion? Micro-evidence on individuals' asset allocation. *American Economic Review*, 98(3):713–736.

Brunnermeier, M. K. and Oehmke, M. (2013). Bubbles, Financial Crises, and Systemic Risk. In *Handbook of the Economics of Finance*, volume 2, chapter 18, pages 1221–1288.

Brunnermeier, M. K. and Pedersen, L. H. (2008). Market Liquidity and Funding Liquidity. *Review of Financial Studies*, 22(6):2201–2238.

Brunnermeier, M. K. and Sannikov, Y. (2014). A Macroeconomic Model with a Financial Sector. *The American Economic Review*, 104(2):379–421.

Bucks, B. K. (2006). Recent Changes in U.S. Family Finances: Evidence from the 2001 and 2004 Survey of Consumer Finances. *Federal Reserve Bulletin*, 95:A1–A56.

Caballero, R. J. and Krishnamurthy, A. (2001). International and domestic collateral constraints in a model of emerging market crises. *Journal of Monetary Economics*, 48(3):513–548.

Cagan, C. (2006). A Ripple, Not A Tidal Wave: Foreclosure Prevalence and Foreclosure Discount. *First American Real Estate Soultions*, pages 1–17.

Calhoun, C. A. (1996). OFHEO House Price Indexes: HPI Technical Description. Technical report, Office of Federal Housing Enterprice Oversight, Washington D.C.

Calvet, L. E., Campbell, J. Y., and Sodini, P. (2009). Measuring the Financial Sophistication of Households. *American Economic Review*, 99(2):393–398.

Campbell, J. and Shiller, R. (2001). Valuation Ratios and the Long-Run Stock Market Outlook: An Update. Technical report, National Bureau of Economic Research, Cambridge, MA.

Campbell, J. Y. (1993). Intertemporal Asset Pricing without Consumption Data. *The American Economic Review*, 83(3):487–512.

Campbell, J. Y. (1996). Understanding Risk and Return. *Journal of Political Economy*, 104(2):298–345.

Campbell, J. Y. (2000). Asset Pricing at the Millennium. *The Journal of Finance*, 55(4):1515–1567.

Campbell, J. Y. (2006). Household Finance. *Journal of Finance*, 61(4):1553–1604.

Campbell, J. Y., Cocco, J., Gomes, F., Maenhout, P. J., and Viceira, L. M. (2001). Stock Market Mean Reversion and the Optimal Equity Allocation of a Long-Lived Investor. *European Finance Review*, 5(3):269–292.

Campbell, J. Y. and Cocco, J. F. (2003). Household Risk Management and Optimal Mortgage Choice. *Quarterly Journal of Economics*, 118(4):1449–1494.

Campbell, J. Y. and Cocco, J. F. (2015). A Model of Mortgage Default. *The Journal of Finance*, 70(4):1495–1554.

Campbell, J. Y. and Cochrane, J. H. (2000). Explaining the Poor Performance of Consumption-based Asset Pricing Models. *The Journal of Finance*, 55(6):2863–2878.

Campbell, J. Y. and Shiller, R. J. (1987). Cointegration and Tests of Present Value Models. *Journal of Political Economy*, 95(5):1062–1088.

Campbell, J. Y. and Shiller, R. J. (1988). The dividend-price ratio and expectations of future dividends and discount factors. *Review of Financial Studies*, 1(3):195–228.

Campbell, J. Y. and Viceira, L. M. (1999). Consumption and Portfolio Decisions when Expected Returns are Time Varying. *The Quarterly Journal of Economics*, 114(2):433–495.

Campbell, J. Y. and Viceira, L. M. (2001). Who Should Buy Long-Term Bonds? *The American Economic Review*, 91(1):99–127.

Campbell, J. Y. and Viceira, L. M. (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press, Oxford, United Kingdom.

Campbell, S. D., Davis, M. A., Gallin, J., and Martin, R. F. (2009). What moves housing markets: A variance decomposition of the rent–price ratio. *Journal of Urban Economics*, 66(2):90–102.

Canarella, G., Miller, S., and Pollard, S. (2011). Unit Roots and Structural Change: An Application to US House Price Indices. *Urban Studies*, 49(4):757–776.

Canner, N., Mankiw, N. G., and Weil, D. N. (1997). An Asset Allocation Puzzle. *The American Economic Review*, 87(1):181–191.

Capozza, D. R. and Seguin, P. J. (1996). Expectations, efficiency, and euphoria in the housing market. *Regional Science and Urban Economics*, 26(3-4):369–386.

Carlstrom, C. T. and Fuerst, T. S. (1997). Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis. *American Economic Review*, 87(5):893–910.

Carroll, C. D. (1997). Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *The Quarterly Journal of Economics*, 112(1):1–55.

Carroll, C. D. and Samwick, A. A. (1997). The nature of precautionary wealth. *Journal of Monetary Economics*, 40(1):41–71.

Case, B., Pollakowski, H. O., and Wachter, S. M. (1991). On Choosing Among House Price Index Methodologies. *Real Estate Economics*, 19(3):286–307.

Case, K., Shiller, R., and Thompson, A. (2012). What Have They Been Thinking? Home Buyer Behavior in Hot and Cold Markets. Technical report, National Bureau of Economic Research, Cambridge, MA.

Case, K. E., Quigley, J. M., and Shiller, R. J. (2005). Comparing Wealth Effects: The Stock Market versus the Housing Market. *Advances in Macroeconomics*, 5(1):1235–1235.

Case, K. E. and Shiller, R. J. (2003). Is There a Bubble in the Housing Market? *Brookings Papers on Economic Activity*, 2003(2):299–342.

Cavaliere, G. and Xu, F. (2014). Testing for unit roots in bounded time series. *Journal of Econometrics*, 178(PART 2):259–272.

Cerqueti, R. and Costantini, M. (2011). Testing for rational bubbles in the presence of structural breaks: Evidence from nonstationary panels. *Journal of Banking & Finance*, 35(10):2598–2605.

Chacko, G. and Viceira, L. M. (2005). Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. *Review of Financial Studies*, 18(4):1369–1402.

Chai, J., Maurer, R., Mitchell, O. S., and Rogalla, R. (2011). Lifecycle Impacts of the Financial and Economic Crisis on Household Optimal Consumption, Portfolio Choice, and Labor Supply. *National Bureau of Economic Research Working Paper Series*, No. 17134.

Chatterjee, S. and Eyigungor, B. (2012). Maturity, indebtedness, and default risk. *American Economic Review*, 102(10):2674–2699.

Chatterjee, S. and Eyigungor, B. (2015). A quantitative analysis of the U.S. housing and mortgage markets and the foreclosure crisis. *Review of Economic Dynamics*, 18(2):165–184.

Chen, N.-f. (1991). Financial Investment Opportunities and the Macroeconomy. *The Journal of Finance*, 46(2):529–554.

Chetty, R. and Szeidl, A. (2007). Consumption Commitments and Risk Preferences. *The Quarterly Journal of Economics*, 122(2):831–877.

Chetty, R. and Szeidl, A. (2014). The Effect of Housing on Portfolio Choice. *National Bureau of Economic Research Working Paper Series*, No. 15998.

Choe, K.-I., Nam, K., and Vahid, F. (2007). Necessity of negative serial correlation for mean-reversion of stock prices. *The Quarterly Review of Economics and Finance*, 47(4):576–583.

Choi, K. and Zivot, E. (2007). Long memory and structural changes in the forward discount: An empirical investigation. *Journal of International Money and Finance*, 26(3):342–363.

Chow, G. C. (1960). Tests of Equality Between Sets of Coefficients in Two Linear Regressions. *Econometrica*, 28(3):591–605.

Clayton, J. (1996). Rational expectations, market fundamentals and housing price volatility. *Real Estate Economics*, 24(4):441–470.

Cocco, J. F. (2004). Portfolio Choice in the Presence of Housing. *Review of Financial Studies*, 18(2):535–567.

Cocco, J. F., Gomes, F. J., and Maenhout, P. J. (2004). Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies*, 18(2):491–533.

Cochrane, J. H. (1992). Explaining the variance of price-dividend ratios. *Review of Financial Studies*, 5(2):243–280.

Cochrane, J. H. (2007). The Dog That Did Not Bark: A Defense of Return Predictability. *Review of Financial Studies*, 21(4):1533–1575.

Coen-Pirani, D. (2005). Margin requirements and equilibrium asset prices. *Journal of Monetary Economics*, 52(2):449–475.

Cont, R. (2005). Volatility Clustering in Financial Markets: Empirical Facts and Agent-Based Models. *SSRN Electronic Journal*.

Corbae, D. and Quintin, E. (2015). Leverage and the Foreclosure Crisis. *Journal of Political Economy*, 123(1):1–65.

Corradin, S., Fillat, J. L., and Vergara-Alert, C. (2014). Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs. *Review of Financial Studies*, 27(3):823–880.

Craine, R. (1993). Rational bubbles. *Journal of Economic Dynamics and Control*, 17(5-6):829–846.

Crawford, G. W. and Fratantoni, M. C. (2003). Assessing the Forecasting Performance of Regime-Switching, ARIMA and GARCH Models of House Prices. *Real Estate Economics*, 31(2):223–243.

Crowe, C., Dell'Ariccia, G., Igan, D., and Rabanal, P. (2013). How to deal with real estate booms: Lessons from country experiences. *Journal of Financial Stability*, 9(3):300–319.

Cuñado, J., Gil-Alana, L., and de Gracia, F. P. (2005). A test for rational bubbles in the NASDAQ stock index: A fractionally integrated approach. *Journal of Banking & Finance*, 29(10):2633–2654.

Cuñado, J., Gil-Alana, L. A., and Pérez de Gracia, F. (2012). Testing for persistent deviations of stock prices to dividends in the Nasdaq index. *Physica A: Statistical Mechanics and its Applications*, 391(20):4675–4685.

Cuoco, D. and Liu, H. (2000). Optimal consumption of a divisible durable good. *Journal of Economic Dynamics and Control*, 24(4):561–613.

Curcuru, S., Heaton, J., Lucas, D., and Moore, D. (2004). Heterogeneity and Portfolio Choice: Theory and Evidence. In Hansen, L. P., editor, *The Handbook of Financial Econometrics*, page 12.

Dammon, R. M. (2001). Optimal Consumption and Investment with Capital Gains Taxes. *Review of Financial Studies*, 14(3):583–616.

Davis, M. A. and Nieuwerburgh, S. V. (2014). Housing, Finance and the Macroeconomy. *National Bureau of Economic Research Working Paper Series*, No. 20287.

Davis, S. J., Kubler, F., and Willen, P. (2006). Borrowing Costs and the Demand for Equity over the Life Cycle. *Review of Economics and Statistics*, 88(2):348–362.

Davis, S. J. and Willen, P. (2013). Occupation-Level Income Shocks and Asset Returns: Their Covariance and Implications for Portfolio Choice. *Quarterly Journal of Finance*, 03(04):111–135.

Deaton, A. (1991). Saving and Liquidity Constraints. *Econometrica*, 59(5):1221–1248.

Diamond, P. A. (1965). National Debt in a Neoclassical Growth Model. *The American Economic Review*, 55(5):1126–1150 CR – Copyright © 1965 American Eco.

Diba, B. T. and Grossman, H. I. (1987). On the Inception of Rational Bubbles. *Quarterly Journal of Economics*, 102(3):697–700.

Diba, B. T. and Grossman, H. I. (1988). Explosive Rational Bubbles in Stock Prices? *American Economic Review*, 78(3):520–530.

Dickey, D. A. and Fuller, W. A. (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root. *Journal of the American Statistical Association*, 74(366a):427–431.

Diebold, F. X. and Inoue, A. (2001). Long memory and regime switching. *Journal of Econometrics*, 105(1):131–159.

Diebold, F. X. and Rudebusch, G. D. (1991). On the power of Dickey-Fuller tests against fractional alternatives. *Economics Letters*, 35(2):155–160.

Doornik, J. a. and Ooms, M. (2003). Computational aspects of maximum likelihood estimation of autoregressive fractionally integrated moving average models. *Computational Statistics & Data Analysis*, 42(3):333–348.

Elkamhi, R. and Stefanova, D. (2015). Dynamic Hedging and Extreme Asset Co-movements. *Review of Financial Studies*, 28(3):743–790.

Elliot, G., Rothenberg, T. J., and Stock, J. H. (1996). Efficient Tests for an Autoregressive Unit Root. *Econometrica*, 64(4):813–836.

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, 50(4):987–1007.

Engsted, T., Pedersen, T. Q., and Tanggaard, C. (2012). The Log-Linear Return Approximation, Bubbles, and Predictability. *Journal of Financial and Quantitative Analysis*, 47(03):643–665.

Epstein, L. G. (1988). Risk aversion and asset prices. *Journal of Monetary Economics*, 22(2):179–192.

Epstein, L. G. and Zin, S. E. (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, 57(4):937–969.

Epstein, L. G. and Zin, S. E. (1991). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. *Journal of Political Economy*, 99(2):263–286.

Eraker, B., Johannes, M., and Polson, N. (2003). The Impact of Jumps in Volatility and Returns. *The Journal of Finance*, 58(3):1269–1300.

Evans, G. (1991a). Pitfalls in testing for explosive bubbles in asset prices. *The American Economic Review*, 81(4):922–930.

Evans, G. W. (1991b). Pitfalls in testing for explosive bubbles in asset prices. *The American Economic Review*, 81(4):922–930.

Fagereng, A., Gottlieb, C., and Guiso, L. (2013). Asset Market Participation and Portfolio Choice Over the Life Cycle. *CEPR Discussion Paper No. DP9691.*

Fama, E. F. (1998). Market efficiency, long-term returns, and behavioral finance1. *Journal of Financial Economics*, 49(3):283–306.

Fama, E. F. and French, K. R. (1988). Dividend yields and expected stock returns. *Journal of Financial Economics*, 22(1):3–25.

Fama, E. F. and French, K. R. (1992). The Cross-Section of Expected Stock Returns. *Journal of Finance*, 47(2):427–465.

Farhi, E. and Tirole, J. (2011). Bubbly Liquidity. *The Review of Economic Studies*, 79(2):678–706.

Fernández-Villaverde, J. and Krueger, D. (2011). Consumption and Saving over the Life-Cycle: How Important Are Consumer Durables? *Macroeconomic Dynamics*, 15(05):725–770.

Fischer, M. and Stamos, M. Z. (2013). Optimal life cycle portfolio choice with housing market cycles. *Review of Financial Studies*, 26(9):2311–2352.

Flavin, M. and Nakagawa, S. (2008). A Model of Housing in the Presence of Adjustment Costs: A Structural Interpretation of Habit Persistence. *The American Economic Review*, 98(1):474–495.

Flavin, M. and Yamashita, T. (2002). Owner-Occupied Housing and the Composition of the Household Portfolio. *The American Economic Review*, 92(1):345–362.

Flavin, M. and Yamashita, T. (2011). Owner-Occupied Housing: Life-Cycle Implications for the Household Portfolio. *The American Economic Review*, 101(3):609–614.

Fletcher, M., Gallimore, P., and Mangan, J. (2000). Heteroscedasticity in hedonic house price models. *Journal of Property Research*, 17(2):93–108.

Forlati, C. and Lambertini, L. (2011). Risky Mortgages in a DSGE Model. *International Journal of Central Banking*, 7(1):285–335.

Fostel, A. and Geanakoplos, J. (2008). Leverage Cycles and the Anxious Economy. *American Economic Review*, 98(4):1211–1244.

Gallin, J. (2008). The Long Run Relationship Between House Prices and Rents. *Real Estate Economics*, 36(4):635–658.

Gârleanu, N. and Panageas, S. (2015). Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing. *Journal of Political Economy*, 123(3):670–685.

Garriga, C. and Schlagenhauf, D. E. (2009). Home Equity, Foreclosures and Bailouts.

Geanakoplos, J. (1990). An introduction to general equilibrium with incomplete asset markets. *Journal of Mathematical Economics*, 19(1-2):1–38.

Geanakoplos, J. (2003). Liquidity, Default, and Crashes: Endogenous Contracts in General Equilibrium. In Dewatripont, M., Hansen, L. P., and Turnovsky, S. J., editors, *Advances in Economics and Econometrics: Theory and Applications*, pages 170–205. Cambridge University Press, Cambridge, UK.

Geanakoplos, J. (2010). The Leverage Cycle. In Acemoglu, D., Rogoff, K., and Woodford, M., editors, *NBER Macroeconomics Annual*, pages 1–65. University of Chicago Press.

Geanakoplos, J. and Zame, W. R. (2014). Collateral equilibrium, I: a basic framework. *Economic Theory*, 56(3):443–492.

Gertler, M. (1999). Government debt and social security in a life-cycle economy.

Glaeser, E. L., Gottlieb, J. D., and Gyourko, J. (2013). Can Cheap Credit Explain the Housing Boom? In *Housing and the Financial Crisis*, pages 301–360.

Glaeser, E. L., Gottlieb, J. D., and Tobio, K. (2012). Housing booms and city centers. In *American Economic Review*, volume 102, pages 127–133.

Glaeser, E. L., Gyourko, J., and Saiz, A. (2008). Housing supply and housing bubbles. *Journal of Urban Economics*, 64(2):198–217.

Glaeser, E. L. and Nathanson, C. G. (2014). Housing Bubbles. *National Bureau of Economic Research Working Paper Series*, No. 20426.

Gomes, F. and Michaelidis, A. (2005). Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence. *The Journal of Finance*, 60(2):869–904.

Goodman, A. C. and Thibodeau, T. G. (1998). Dwelling Age Heteroskedasticity in Repeat Sales House Price Equations. *Real Estate Economics*, 26(1):151–171.

Gourinchas, P.-O. and Parker, J. A. (2002). Consumption Over the Life Cycle. *Econometrica*, 70(1):47–89.

Granger, C. W. and Hyung, N. (2004). Occasional structural breaks and long memory with an application to the S&P 500 absolute stock returns. *Journal of Empirical Finance*, 11(3):399–421.

Granger, C. W. J. and Joyeux, R. (1980). An Introduction to Long Memory Time Series Models and Fractional Differencing. *Journal of Time Series Analysis*, 1(1):15–29.

Green, R. K., Malpezzi, S., and Mayo, S. K. (2005). Metropolitan-Specific Estimates of the Price Elasticity of Supply of Housing, and Their Sources. *American Economic Review*, 95(2):334–339.

Grossman, S. J. and Laroque, G. (1990). Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods. *Econometrica*, 58(1):25–51.

Gu, C., Mattesini, F., Monnet, C., and Wright, R. (2013). Endogenous Credit Cycles. *Journal of Political Economy*, 121(August):940–965.

Guiso, L., Haliassos, M., and Jappelli, T. (2003). Household stockholding in Europe: where do we stand and where do we go? *Economic Policy*, 18(36):123–170.

Guiso, L., Sapienza, P., and Zingales, L. (2008). Trusting the stock market. *Journal of Finance*, 63(6):2557–2600.

Guiso, L. and Sodini, P. (2013). Chapter 21 - Household Finance: An Emerging Field. In George M. Constantinides, M. H. and of the Economics of Finance, R. M. S. B. T. H., editors, *Handbook of the Economics of Finance*, volume Volume 2,, chapter 21, pages 1397–1532. Elsevier.

Gürkaynak, R. S. (2008). Econometric Tests of Asset Price Bubbles: Taking Stock. *Journal of Economic Surveys*, 22(1):166–186.

Gyourko, J., Mayer, C., and Sinai, T. (2013). Superstar Cities. *American Economic Journal: Economic Policy*, 5(4):167–199.

Haliassos, M. and Bertaut, C. C. (1995). Why do so Few Hold Stocks? *The Economic Journal*, 105(432):1110.

Haliassos, M. and Michaelides, A. (2003). Portfolio Choice and Liquidity Constraints*. *International Economic Review*, 44(1):143–177.

Hansen, B. E. (1996). Inference When a Nuisance Parameter Is Not Identified Under the Null Hypothesis. *Econometrica*, 64(2):413.

Hansen, B. E. (2000). Testing for structural change in conditional models. *Journal of Econometrics*, 97(1):93–115.

Hansen, L. P. and Singleton, K. J. (1983). Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns. *Journal of Political Economy*, 91(2):249–265.

Hart, O. and Moore, J. (1994). A Theory of Debt Based on the Inalienability of Human Capital. *The Quarterly Journal of Economics*, 109(4):841–879.

Hassler, U. (1993). Regression of Spectral Estimators with Fractionally Integrated Time Series. *Journal of Time Series Analysis*, 14(4):369–380.

Hassler, U. and Wolters, J. (1995). Long Memory in Inflation Rates: International Evidence. *Journal of Business & Economic Statistics*, 13(1):37–45.

Hatzius, J. (2002). *Housing and the US Consumer: Mortgaging the Economy's Future*. Goldman Sachs.

Hauser, M. A., Pötscher, B. M., and Reschenhofer, E. (1999). Measuring persistence in aggregate output: ARMA models, fractionally integrated ARMA models and nonparametric procedures. *Empirical Economics*, 24(2):243–269.

He, C., Wright, R., and Zhu, Y. (2015). Housing and liquidity. *Review of Economic Dynamics*, 18(3):435–455.

He, Z. and Krishnamurthy, A. (2013). Intermediary Asset Pricing. *American Economic Review*, 103(2):732–770.

Heaton, J. and Lucas, D. (1997). Market Frictions, Savings Behaviour and Portfolio Choice. *Macroeconomic Dynamics*, 1(01):76–101.

Heaton, J. and Lucas, D. (2000). Portfolio Choice in the Presence of Background Risk. *The Economic Journal*, 110(460):1–26.

Hellwig, C. and Lorenzoni, G. (2009). Bubbles and Self-Enforcing Debt. *Econometrica*, 77(4):1137–1164.

Henkel, S. J., Martin, J. S., and Nardari, F. (2011). Time-varying short-horizon predictability. *Journal of Financial Economics*, 99(3):560–580.

Henry, M. (2001). Robust Automatic Bandwidth for Long Memory. *Journal of Time Series Analysis*, 22(3):293–316.

Higgins, M. (1997). Asset market hangovers and economic growth: the OECD during 1984-93. *Oxford Review of Economic Policy*, 13(3):110–134.

Himmelberg, C., Mayer, C., and Sinai, T. (2005). Assessing High House Prices: Bubbles, Fundamentals and Misperceptions. *Journal of Economic Perspectives*, 19(4):67–92.

Holmstrom, B. and Tirole, J. (2011). *Inside and Outside Liquidity*. MIT Press.

Hong, H., Kubik, J. D., and Stein, J. C. (2004). Social Interaction and Stock-Market Participation. *The Journal of Finance*, 59(1):137–163.

Horvath, M. T. and Watson, M. W. (2009). Testing for Cointegration When Some of the Cointegrating Vectors are Prespecified. *Econometric Theory*, 11(05):984.

Hu, X. (2005). Portfolio choices for homeowners. *Journal of Urban Economics*, 58(1):114–136.

Hurst, H. E. (1951). {Long-term storage capacity of reservoirs}. *Trans. Amer. Soc. Civil Eng.*, 116:770–808.

Hurvich, C. M. and Chen, W. W. (2000). An Efficient Taper for Potentially Overdifferenced Long-memory Time Series. *Journal of Time Series Analysis*, 21(2):155–180.

Iacoviello, M. (2004). Consumption, house prices, and collateral constraints: A structural econometric analysis. *Journal of Housing Economics*, 13(4 SPEC.ISS.):304–320.

Iacoviello, M. (2005). House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle. *The American Economic Review*, 95(3):739–764.

Iacoviello, M. and Neri, S. (2010). Housing Market Spillovers: Evidence from an Estimated DSGE Model. *American Economic Journal: Macroeconomics*, 2(2):125–164.

Jagannathan, R. and Wang, Z. (1996). The Conditional CAPM and the Cross-Section of Expected Returns. *The Journal of Finance*, 51(1):3–53.

Jermann, U. and Quadrini, V. (2012). Macroeconomic Effects of Financial Shocks. *American Economic Review*, 102(1):238–271.

João F. Gomes, Leonid Kogan, and Motohiro Yogo (2009). Durability of Output and Expected Stock Returns. *Journal of Political Economy*, 117(5):941–986.

Jose A. Scheinkman, L. W. (2013). Borrowing Constraints and Aggregate Economic Activity. *Econometrica*, 54(1):23–45.

Kindleberger, C. P. and Aliber, R. Z. (2005). *Manias, panics, and crashes: a history of financial crises*. John Wiley & Sons, Hoboken, N.J.

Kivedal, B. K. (2013). Testing for rational bubbles in the US housing market. *Journal of Macroeconomics*, 38(PB):369–381.

Kiyotaki, N. and Moore, J. (1997). Credit Cycles.

Kiyotaki, N. and Moore, J. (2012). Liquidity, Business Cycles, and Monetary Policy. Technical report, National Bureau of Economic Research, Cambridge, MA.

Kocherlakota, N. R. (2009). Bursting Bubbles : Consequences and Cures.

Koustas, Z. and Serletis, A. (2005). Rational bubbles or persistent deviations from market fundamentals? *Journal of Banking & Finance*, 29(10):2523–2539.

Krainer, J. and Wei, C. (2004). House prices and fundamental value. *FRBSF Economic Letter*, (Oct 1).

Kruse, R. and Sibbertsen, P. (2012). Long memory and changing persistence. *Economics Letters*, 114(3):268–272.

Kubler, F. and Schmedders, K. (2012). Financial Innovation and Asset Price Volatility. *American Economic Review Papers and Proceedings*, 102(3):147–151.

Kuensch, H. R. (1987). Statistical aspects of self-similar processes. In *Proceedings of the first World Congress of the Bernoulli Society*, volume 1, pages 67–74. VNU Science Press Utrecht.

Kullmann, C. and Siegel, S. (2005). Real Estate and its Role in Household Portfolio Choice. *SSRN Electronic Journal*.

Kumar, M. S. and Okimoto, T. (2007). Dynamics of Persistence in International Inflation Rates. *Journal of Money, Credit and Banking*, 39(6):1457–1479.

Kwiatkowski, D., Phillips, P. C., Schmidt, P., and Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics*, 54(1-3):159–178.

Lamont, O. (1998). Earnings and Expected Returns. *The Journal of Finance*, 53(5):1563–1587.

Lee, J. and Strazicich, M. C. (2001). Break Point Estimation and Spurious Rejections With Endogenous Unit Root Tests. *Oxford Bulletin of Economics and Statistics*, 63(5):535–558.

Levitin, A. J. and Wachter, S. M. (2012). Explaining the Housing Bubble. *SSRN Electronic Journal*.

Li, W. and Yao, R. (2007). The Life-Cycle Effects of House Price Changes. *Journal of Money, Credit and Banking*, 39(6):1375–1409.

Li, W. K. and Mcleod, A. I. (1986). Fractional time series modelling. *Biometrika*, 73(1):217–221.

Linneman, P. (1986). An empirical test of the efficiency of the housing market. *Journal of Urban Economics*, 20(2):140–154.

Lobato, I. N. and Velasco, C. (2007). Efficient Wald Tests for Fractional Unit Roots. *Econometrica*, 75(2):575–589.

Lucas Jr., R. E. (1978). Asset Prices in an Exchange Economy. *Econometrica*, 46(6):1429–1445.

Lumsdaine, R. L. and Papell, D. H. (1997). Multiple Trend Breaks and the Unit-Root Hypothesis. *Review of Economics and Statistics*, 79(2):212–218.

Lynch, A. W. (2001). Portfolio choice and equity characteristics: characterizing the hedging demands induced by return predictability. *Journal of Financial Economics*, 62(1):67–130.

Malkiel, B. G. (1996). *A Random Walk Down Wall Street: Including a Life-Cycle Guide to Personal Investing.* WW Norton & Company, New York.

Malpezzi, S. (1996). Housing Prices , Externalities , and Regulation in U . S . Metropolitan Areas. *Journal of Housing Research*, 7(2):209–241.

Malpezzi, S. and Wachter, S. M. (2005). The role of speculation in real estate cycles. *Journal of Real Estate Literature*, 13(2):141–164.

Mankiw, N. and Weil, D. N. (1989). The baby boom, the baby bust, and the housing market. *Regional Science and Urban Economics*, 19(2):235–258.

Mankiw, N. and Zeldes, S. P. (1991). The consumption of stockholders and nonstockholders. *Journal of Financial Economics*, 29(1):97–112.

Martin, A. and Ventura, J. (2012). Economic Growth with Bubbles. *American Economic Review*, 102:3033–3058.

Mayer, C. (2011). Housing Bubbles: A Survey. *Annual Review of Economics*, 3(1):559–577.

Mayer, C., Pence, K., and Sherlund, S. M. (2009). The Rise in Mortgage Defaults. *Journal of Economic Perspectives*, 23(1):27–50.

Mayoral, L. (2012). Testing for Fractional Integration Versus Short Memory with Structural Breaks*. *Oxford Bulletin of Economics and Statistics*, 74(2):278–305.

McCarthy, J. and Peach, R. W. (2004). Are home prices the next bubble? *FRBNY Economic Policy Review*, 10(3):1–17.

Meen, G. (2002). The Time-Series Behavior of House Prices: A Transatlantic Divide? *Journal of Housing Economics*, 11(1):1–23.

Meese, R. and Wallace, N. (1994). Testing the Present Value Relation for Housing Prices: Should I Leave My House in San Francisco? *Journal of Urban Economics*, 35(3):245–266.

Mehra, R. (2012). Consumption-Based Asset Pricing Models. *Annual Review of Financial Economics*, 4(1):385–409.

Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*, 15(2):145–161.

Mendoza, E. G. and Yue, V. Z. (2012). A general equilibrium model of sovereign default and business cycles. *Quarterly Journal of Economics*, 127(2):889–946.

Merton, R. C. (1969). Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics*, 51(3):247–257 CR – Copyright © 1969 The MIT Press.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413.

Merton, R. C. (1973). An Intertemporal Capital Asset Pricing Model. *Econometrica*, 41(5):867–887.

Mian, A. and Sufi, A. (2009). The Consequences of Mortgage Credit Expansion: Evidence from the U.S. Mortgage Default Crisis. *Quarterly Journal of Economics*, 124(4):1449–1496.

Miao, J. (2014). Introduction to economic theory of bubbles. *Journal of Mathematical Economics*, 53(0):130–136.

Miao, J. and Wang, P. (2012). Bubbles and total factor productivity. In *American Economic Review*, volume 102, pages 82–87.

Miao, J. and Wang, P. (2014). Sectoral bubbles, misallocation, and endogenous growth. *Journal of Mathematical Economics*, 53:153–163.

Miao, J., Wang, P., and Zhou, J. (2015). Asset Bubbles, Collateral, and Policy Analysis. *Journal of Monetary Economics (Forthcoming)*.

Michaelides, A. and Zhang, Y. (2015). Stock Market Mean Reversion and Portfolio Choice Over the Life Cycle. *SSRN Electronic Journal*.

Miles, W. (2008). Volatility clustering in US home prices. *Journal of Real Estate Research*, 30(1):73–90.

Miller, N. and Peng, L. (2006). Exploring Metropolitan Housing Price Volatility. *The Journal of Real Estate Finance and Economics*, 33(1):5–18.

Monacelli, T. (2009). New Keynesian models, durable goods, and collateral constraints. *Journal of Monetary Economics*, 56(2):242–254.

Munk, C. and Sørensen, C. (2010). Dynamic asset allocation with stochastic income and interest rates. *Journal of Financial Economics*, 96(3):433–462.

Muñoz, S. (2004). Real effects of regional house prices: dynamic panel estimation with heterogeneity. In *Financial Markets Group*, London. London School of Economics and Political Science.

Nelson, D. B. (2009). Stationarity and Persistence in the GARCH(1,1) Model. *Econometric Theory*, 6(03):318.

Nielsen, M. Ø. and Frederiksen, P. H. (2005). Finite Sample Comparison of Parametric, Semiparametric, and Wavelet Estimators of Fractional Integration. *Econometric Reviews*, 24(4):405–443.

Nneji, O., Brooks, C., and Ward, C. (2013). Intrinsic and Rational Speculative Bubbles in the US Housing Market: 1960–2011. *Journal of Real Estate Research*, 31(2):121–151.

Ogaki, M. and Reinhart, C. M. (1998). Measuring Intertemporal Substitution: The Role of Durable Goods. *Journal of Political Economy*, 106(5):1078–1098.

Pástor, u. and Stambaugh, R. F. (2012). Are Stocks Really Less Volatile in the Long Run? *Journal of Finance*, 67(2):431–478.

Pelletier, D. and Tunc, C. (2015). Endogenous Life-Cycle Housing Investment and Portfolio Allocation. *Working Paper*, NCSU.

Perron, P. (1989). The Great Crash, the Oil Price Shock, and the Unit Root Hypothesis. *Econometrica*, 57(6):1361.

Perron, P. (1997). Further evidence on breaking trend functions in macroeconomic variables. *Journal of Econometrics*, 80(2):355–385.

Perron, P. and Qu, Z. (2007). An analytical evaluation of the log-periodogram estimate in the presence of level shifts. *Unpublished Manuscript, Department of Economics, Boston University.*

Phillips, P. C. B. and Yu, J. (2011). Dating the timeline of financial bubbles during the subprime crisis. *Quantitative Economics*, 2(3):455–491.

Piazzesi, M. and Schneider, M. (2009). Momentum Traders in the Housing Market: Survey Evidence and a Search Model. *American Economic Review*, 99(2):406–411.

Piazzesi, M., Schneider, M., and Tuzel, S. (2007). Housing, consumption and asset pricing. *Journal of Financial Economics*, 83(3):531–569.

Plazzi, A., Torous, W., and Valkanov, R. (2010). Expected returns and expected growth in rents of commercial real estate. *Review of Financial Studies*, 23(858):3469–3519.

Poterba, J. M. (2002). Chapter 17 – Taxation, Risk-Taking, and Household Portfolio Behavior. In *Handbook of Public Economics*, volume 3, pages 1109–1171.

Prescott, E. C. and Townsend, R. M. (1984). Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard. *Econometrica*, 52(1):21–45.

Quadrini, V. (2011). Financial Frictions in Macroeconomic Fluctuations. *Economic Quarterly*, 97(3):209–254.

Quandt, R. E. (1960). Tests of the Hypothesis That a Linear Regression System Obeys Two Separate Regimes. *Journal of the American Statistical Association*, 55(290):324–330.

Radner, R. (1972). Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets. *Econometrica*, 40(2):289–303.

Rappaport, J. (2007). A guide to aggregate house price measures. *ECONOMIC REVIEW-FEDERAL RESERVE BANK OF KANSAS CITY*, 92(2):41.

Rietz, T. A. (1988). The equity risk premium a solution. *Journal of Monetary Economics*, 22(1):117–131.

Robinson, P. M. (1994). Time series with strong dependence. In *Advances in Econometrics, Sixth World Congress*, volume 1, pages 47–95.

Robinson, P. M. (1995). Gaussian Semiparametric Estimation of Long Range Dependence. *The Annals of Statistics*, 23(5):1630–1661.

Robinson, P. M. (2003). *Time series with long memory.* Oxford University Press.

Robinson, P. M. and Henry, M. (1999). Long and short memory conditional heteroskedasticity in estimating the memory parameter of levels. *Econometric Theory*, 15(03):299–336.

Roueff, F. and von Sachs, R. (2011). Locally stationary long memory estimation. *Stochastic Processes and their Applications*, 121(4):813–844.

Rytchkov, O. (2014). Asset Pricing with Dynamic Margin Constraints. *The Journal of Finance*, 69(1):405–452.

Samuelson, P. A. (1969). Lifetime Portfolio Selection By Dynamic Stochastic Programming. *The Review of Economics and Statistics*, 51(3):239–246 CR – Copyright © 1969 The MIT Press.

Santos, M. S. and Woodford, M. (1997). Rational Asset Pricing Bubbles. *Econometrica*, 65(1):19–57.

Shiller, R. J. (1981). Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends? *The American Economic Review*, 71(3):421–436.

Shiller, R. J. (1983). Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends? Reply. *The American Economic Review*, 73(1):236–237 CR – Copyright © 1983 American Econom.

Shiller, R. J. (1989). The Efficiency of the Market for Single-Family Homes. *American Economic Review*, 79(1):125–137.

Shimotsu, K. (2006). Simple (but effective) tests of long memory versus structural breaks.

Shimotsu, K. (2009). EXACT LOCAL WHITTLE ESTIMATION OF FRACTIONAL INTEGRATION WITH UNKNOWN MEAN AND TIME TREND. *Econometric Theory*, 26(02):501.

Shimotsu, K. and Phillips, P. C. B. (2004). Local Whittle estimation in nonstationary and unit root cases. *The Annals of Statistics*, 32(2):656–692.

Shimotsu, K. and Phillips, P. C. B. (2005). Exact local Whittle estimation of fractional integration. *The Annals of Statistics*, 33(4):1890–1933.

Simsek, A. (2013). Belief Disagreements and Collateral Constraints. *Econometrica*, 81(1):1–53.

Sowell, F. (1992). Maximum likelihood estimation of stationary univariate fractionally integrated time series models. *Journal of Econometrics*, 53(1-3):165–188.

Stambaugh, R. F. (1999). Predictive regressions. *Journal of Financial Economics*, 54(3):375–421.

Stein, J. C. (1995). Prices and Trading Volume in the Housing Market: A Model with Down-Payment Effects. *The Quarterly Journal of Economics*, 110(2):379–406.

Suarez, J. and Sussman, O. (1997). Endogenous Cycles in a Stiglitz–Weiss Economy. *Journal of Economic Theory*, 76(1):47–71.

Svensson, L. E. O. (1989). Portfolio choice with non-expected utility in continuous time. *Economics Letters*, 30(4):313–317.

Taqqu, M. S., Vadim, T., and Willinger, W. (1995). ESTIMATORS FOR LONG-RANGE DEPENDENCE: AN EMPIRICAL STUDY. *Fractals*, 03(04):785–798.

Thibodeau, T. G. (1995). House Price Indices from the 1984 – 1992 MSA American Housing Surveys. *Journal of Housing Research*, 6(3):439–481.

Tirole, J. (1982). On the Possibility of Speculation under Rational Expectations. *Econometrica*, 50(5):1163–1181.

Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica*, 53:1499–1528.

Townsend, R. M. (1979). Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory*, 21(2):265–293.

Tracy, J. and Schneider, H. (2001). Stocks in the Household Portfolio : A Look Back at the 1990s. *Current Issues in Economics and Finance*, 7(4).

van Rooij, M., Lusardi, A., and Alessie, R. (2011). Financial literacy and stock market participation. *Journal of Financial Economics*, 101(2):449–472.

Velasco, C. (1999). Gaussian Semiparametric Estimation of Non-stationary Time Series. *Journal of Time Series Analysis*, 20(1):87–127.

Vestman, R. (2012). Limited Stock Market Participation Among Renters and Home Owners. *Working Paper, Stockholm University.*

Viceira, L. M. (2001). Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *The Journal of Finance*, 56(2):433–470.

Vissing-Jorgensen, A. (2002). Towards an Explanation of Household Portfolio Choice Heterogeneity: Nonfinancial Income and Participation Cost Structures. Technical report, National Bureau of Economic Research, Cambridge, MA.

Vissing-Jorgensen, A. and Attanasio, O. P. (2003). Stock-market participation, intertemporal substitution, and risk-aversion. *American Economic Review*, 93(2):383–391.

von Gaudecker, H.-M. (2015). How Does Household Portfolio Diversification Vary With Financial Sophistication and Advice? *Journal of Finance*, 70(2):489–507.

Wachter, J. A. (2013). Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility? *The Journal of Finance*, 68(3):987–1035.

Weil, P. (1990). Nonexpected Utility in Macroeconomics. *The Quarterly Journal of Economics*, 105(1):29–42.

West, K. D. (1987). A Specification Test for Speculative Bubbles. *Quarterly Journal of Economics*, 102(3):553.

West, K. D. (1988). Dividend Innovations and Stock Price Volatility. *Econometrica*, 56(1):37–61.

White, H. (1982). Maximum Likelihood Estimation of Misspecified Models. *Econometrica*, 50(1):1–25.

Winter, J. K., Schlafmann, K., and Rodepeter, R. (2012). Rules of Thumb in Life-cycle Saving Decisions. *Economic Journal*, 122(560):479–501.

Yang, W. (2011). Long-run risk in durable consumption. *Journal of Financial Economics*, 102(1):45–61.

Yao, R. and Zhang, H. H. (2004). Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *Review of Financial Studies*, 18(1):197–239.

Yogo, M. (2006). A Consumption-Based Explanation of Expected Stock Returns. *The Journal of Finance*, 61(2):539–580.

Yogo, M. (2009). Portfolio Choice in Retirement: Health Risk and the Demand for Annuities, Housing, and Risky Assets. *National Bureau of Economic Research Working Paper Series*, No. 15307.

Zanetti, F. (2014). Housing and relative risk aversion. *Economics Letters*, 123(1):23–25.

Zhang, H. H. (1997). Endogenous Borrowing Constraints with Incomplete Markets. *The Journal of Finance*, 52(5):2187–2209.