SUMMARY

The present thesis discusses only problems concerning incompressible and inviscid fluids.

There are various techniques used to solve hydrodynamical problems in which conformal mapping plays an important part. In Chapter II two problem have been separately discussed. The first deals with an inviscid incompres fluid escaping in the form of a jet from an infinite chamber through a slit impinging normally on a wall. The second deals with the flow through a fin Borda mouthpiece. Here the inviscid incompressible fluid flows out of the reservoir through the mouthpiece to form a jet which is bounded by the free streamlines. These two problems were examined by C. A. Hachemeister and H. C. Levy respectively in the Quarterly of Applied Mathematics, Vol. 17, 1959, pp. 299-304 and Journal of Applied Mathematics and Physics (ZAMP) II, 1960, pp. 152-156. It is shown that from a mathematical standpoint Levy an Hachemeister were dealing with the same problem and consequently these two problems are included in one chapter. In Chapter III, the work has been extended by combining together the main physical features of the two proble of Chapter II. The new problem has then been solved by Schwarz-Christoffel transformations.

The transformation technique while mathematically very elegant suffers from a serious drawback for it is limited to a potential flow satisfying Laplace's equation. Thus this method cannot be used to solve problems of compressible fluids.

The hodograph technique not only solves the problems for incompressil fluids but has in other connections been adapted to solve problems involv: compressible fluids. The application of the hodograph method to incompres ible flow in this thesis may serve as an introduction to its use in the t: ment of compressible flow.

In Chapter IV first Levy's problem, discussed in Chapter II, has beer solved by hodograph methods. Then the Hachemeister problem has been solve In both these problems there is a "notched hodograph" which requires an extension of Mackie's work published in Proc. Edin. Math. Soc. II, 1958, p. 107. The notch in both the problems gives rise to a singular integral equation. Since the same singular integral equation is obtained for both problems we confirm here also that these two problems are mathematically identical. The singular integral equations have been solved analytically by an extension of method given by Mikhlin (Integral Equations by Mikhlin, Ch. III, pp. 131) and verified by comparison with results of Chapter II. In Chapter V, the problem discussed in Chapter III has been investigated by the hodograph method. There are now two "notches" instead of one. Due to these two notches two simultaneous integral equations are obtained. An analytical solution for these equations has not so far been found. The pattern of streamlines in this problem has some interest.

HODOGRAPH METHODS APPLIED TO SOLVE CERTAIN PROBLEMS ON THE FLOW OF JETS

being a THESIS presented by

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to the University of Glasgow in

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CONTENTS

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S. Rangachari,
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CHAPTER I

INTRODUCTION

Hydrodynamics is the study of fluid in motion. A fluid is a continuous medium or one that can be treated as such. Actual fluids fall into two categories, namely gases and liquids. A gas will untimately fill any closed space to which it has access and is therefore classified as a (highly) compressible fluid. All known liquids are to some extent compressible. For most purposes, it is, however, sufficient to regard liquids as incompressible fluids.

It is well known that in a two dimensional irrotational motion of an incompressible fluid there exist a potential function ϕ and a stream function ψ. These two can be combined together to give the complex potential w. w is an analytic function of z, and defined by f(z). By an analytic function f(z) we generally mean that f(z) is one-valued and satisfies the so-called Cauchy-Riemann equations.

If u and v are the velocity components in the direction of x and y axis of such a motion, then these are given by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

It is a simple matter to show that ϕ and ψ satisfy Laplace's equation and that

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0$$

or in polar coordinates q, θ

$$\frac{\partial^2 \psi}{\partial q^2} + \frac{1}{q}\frac{\partial \psi}{\partial q} + \frac{1}{q^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

It is usually preferable to solve hydrodynamical problems directly in the physical plane, that is to say in the plane of x and y. But there are problems, particularly those involving "free streamlines" on which the pressure and consequently the velocity remain constant, which cannot be easily tackled in the physical plane. In the first place the shape of these streamlines cannot be predetermined and the fact that the boundary condition on them is non-linear makes the problem more complicated. This difficulty is experienced especially with problems involving jets and wakes. These problems are sometimes easily solved in the hodograph plane where the boundary-value problem can be formulated and where the boundary conditions are linear.

The application of the hodograph method to problems in fluid dynamics dates back to the time of Helmholtz and Kirchoff. The underlying principle is simple. We change the independent variable of the governing differential equation to q and θ where q is the velocity and θ is the angle that the velocity vector makes with the positive direction of x-axis. Thus the physical plane is transformed into the plane of q and θ where, if u and v are the cartesian components of the velocity vector, then $u = q \cos \theta$ and $v = q \sin \theta$. This (q, θ) plane is known as the hodograph plane.

There are various techniques used to solve hydrodynamical problems in which conformal mapping plays an important part. If the solution for w in (say) t-plane is known and if the conformal transformation from t-plane to z-plane is known then the solution for w in the z-plane can be immediately obtained. Sometimes w passes through a series of transformations before it can be expressed in terms of z. Fortunately, if the boundaries are straight, the application of these transformations can be found by a single basic theorem viz. the Schwarz-Christoffel mapping theorem. In Chapter II two problems have been separately discussed. The first problem deals with an inviscid incompressible fluid escaping in the form of a jet through a slit of an infinite chamber and impinging normally on a wall, the side of the chamber being kept parallel to the wall. The second problem deals with the flow through a finite Borda mouthpiece. Here the inviscid incompressible fluid flows out of the reservoir through the mouthpiece to form a jet which is bounded by the free streamlines. These two problems were published by C. A. Hachemeister (1959) and H. C. Levy (1960) respectively in the Quarterly of Applied Mathematics, Vol. 17, 1959, pp. 299-304, and Journal of Applied Mathematics and Physics (ZAMP) II, pp. 152-156, 1960. It is shown that from a mathematical standpoint Levy and Hachemeister were dealing with the same problem and consequently these two problems are included in one chapter. Both these problems have been solved by Schwarz-Christoffel-transformations. In Chapter III the work has been extended by combining together the main features of the two problems of Chapter II. The new problem is solved by Schwarz-Christoffel transformations. The solution is then analysed to show that if the neck is withdrawn the solution reduces to Levy's solution and if the wall is withdrawn, it reduces to Hachemeister's. This is done as a check to verify that the new solution is correct.

The transformation technique while mathematically very elegant suffers from a serious drawback for it is limited to potential flow satisfying Laplace's equation. Thus this method cannot be used to solve problems of compressible fluids.

The hodograph technique not only solves the problems of incompressible fluids but may also in certain cases be adapted to solve the problems of compressible fluids and so an application of the hodograph method to incompressible flow serves as an easy and natural introduction to its use in the treatment of compressible flow.

The present thesis discusses only problems concerning incompressible and inviscid fluids.

It will be shown that the problems for incompressible flow can be solved directly in the hodograph plane. The mathematics used in applying this method will be seen to be quite straightforward compared to that of Schwarz-Christoffel.

Mackie (1958, Proc. Edin. Math. Soc. II, 107) has solved a number of basic problems of incompressible flow directly in the hodograph plane. He has shown how different techniques can be employed for different boundary-value problems in the hodograph plane. In one of these problems he comes across an integral equation giving thereby an indication that we may have to deal with integral equations to solve some of the boundary-value problems in the hodograph plane.

In Chapter IV, in the first place Levy's problem discussed in Chapter II has been solved by means of integral equations. Then the Bachemeister problem is solved. In both these problems we come across a "notched hodograph" which is an extension of Mackie's work. Mackie, in his paper referred to above has worked only with simple hodographs. In the problems discussed in this thesis we have two points on one streamline (one point being at infinity) in the physical plane where the velocity of the fluid is zero. Hence in between these two points we must have a point where the fluid attains a maximum velocity. When these points are plotted in the hodograph plane we obtain a "notched-hodograph". The "notch" in both the problems gives rise to a singular integral equation. Since we obtain the same singular integral equation for both the problems we confirm here also that these two problems are mathematically identical. These two singular integral equations have been solved analytically by an extension of method given by Mikhlin (Integral Equations by Mikhlin, Ch. III, pp. 131) and verified by comparison with the results of Chapter II.

In Chapter V, the problems discussed in Chapter II have been solved again by a hodograph method. Here we get two "notches" instead of one obtained in each of the previous examples. Due to these two notches we obtain two simultaneous integral equations. An analytical solution for these equations has not so far been obtained. The pattern of streamlines in this problem has some interest.

In what follows, the hodograph method applied to the problems discussed above brings into play the theory of singular Integral Equations and is a natural extension of the ideas put forward by Mackie in his paper published in 1958.

CHAPTER II

The method for finding the complex potential for problems of the type in which the fixed boundaries are rectilinear and the other boundaries are streamlines was first found by Helmholtz and Kirchhoff. This method comes from the facts that the direction of the velocity is constant on fixed boundaries and that the magnitude of the velocity is constant on free streamlines. In order to find the relationship between w and z Kirchhoff introduced the intermediate function G which is equal to $U \frac{dz}{dw}$ where U is the value of the velocity at infinity. ~~Since~~

Since, $\frac{dw}{dz} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} = u - iv = qe^{-i\theta}$, we obtain, $G = \frac{U}{q}e^{i\theta}$

where θ is the inclination of q to the x-axis. As θ is constant on a fixed boundary and q is constant on a free streamline the function $\Omega = \log G$ is introduced so that when the boundaries are transformed from the z-plane to the Ω-plane they are all straight lines.

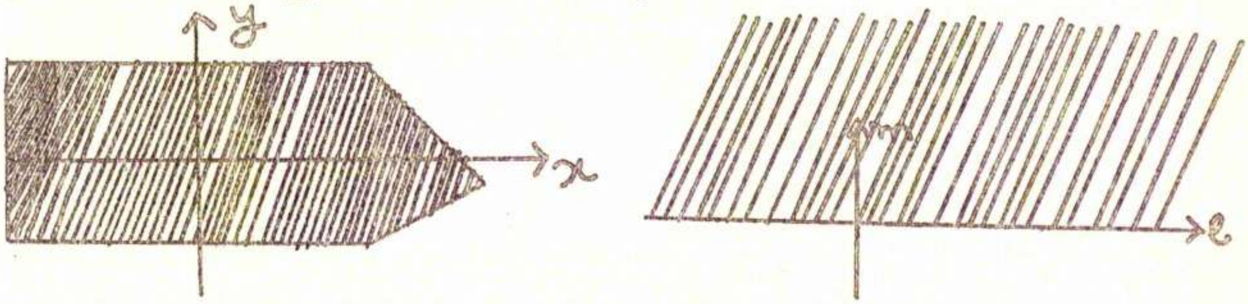
$$\Omega = \log G = \log \frac{U}{q} + i\theta.$$

The figure obtained in the Ω-plane is rectangular.

The second transformation is from the z-plane to the w-plane. Since $w = \phi + i\psi$, the figure on the w-plane is also rectangular. By means of the theorem of Schwarz and Christoffel it is possible to transform the rectangular figures in the Ω- and w-planes into the real axis of a fourth plane, called the t-plane. As corresponding points in the Ω- and w-planes are transformed into the same point on the t-plane, it is possible to find the relationships between w and t and Ω and t. The elimination of t gives the relationship between w and z. In the problems discussed in this chapter, it is found convenient to go into one more transformation, i.e. from the t-plane to the p-plane by choosing a suitable relation between t and p. Hence in addition to a relation between w and t and Ω and t a relation between w and p and Ω and p has also been obtained. The elimination of p then gives the complex potential.

THEOREM OF SCHWARZ AND CHRISTOFFEL

The theorem states that any polygon in the z-plane can be transformed into the real axis of the t-plane. Points which are inside the polygon are transformed into the upper half of the t-plane.



z-plane t-plane

The transformation is

$$\frac{dz}{dt} = K(t-l_1)^{\frac{\alpha_1}{\pi}-1}(t-l_2)^{\frac{\alpha_2}{\pi}-1}(t-l_3)^{\frac{\alpha_3}{\pi}-1}\cdots\cdots(t-l_r)^{\frac{\alpha_r}{\pi}-1}\cdots(t-l_n)^{\frac{\alpha_n}{\pi}-1}$$

where $t = l + im$, $\alpha_1, \alpha_2, \ldots \alpha_r \ldots \alpha_n$ are the internal angles of the polygon and $l_1, l_2, \ldots l_r \ldots l_n$ are the points on the l-axis which correspond to the corners of the polygon. K is a constant which may be complex.

If one of the values of ℓ_r is infinite, the factor corresponding to it is omitted from the equation of transformation and the angle α_r does not appear.

In hydrodynamical applications we shall be concerned only with simple polygons generally extending to infinity. Three of the numbers ℓ_1, ℓ_2 and ℓ_3 may be chosen arbitrarily to correspond to three of the vertices of a given polygon, the remainder must then be arranged so as to make the polygon of the right shape.

(a) Jet through a slit impinging on a wall

The flow out of a slit is a classical problem but the flow out of a slit against a wall which has a physical interest in connection with Hovercraft was examined by Levy (1960). It differs from the usual kind of problems as we shall see in having two stagnation points on one streamline.

An inviscid incompressible fluid is confined in an infinite chamber whose side is parallel to the wall distance h away. The chamber side has a slit of width d through which the fluid escapes (Figure 1).

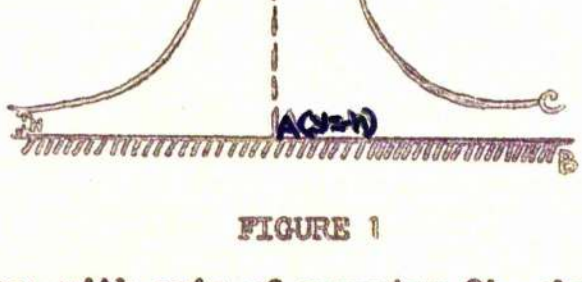


FIGURE 1

The physical plane; with axis of symmetry OA, chamber side DE, jet boundary DC, wall AB.

The solution is carried through by mapping the flow region in the plane of the complex velocity potential w and in the plane of the Helmholtz potential, Ω on an appropriate region of the plane of the auxiliary complex variable t.

Solution in details:

We need consider only the half of the flow field in $x \geq 0$ by symmetry.

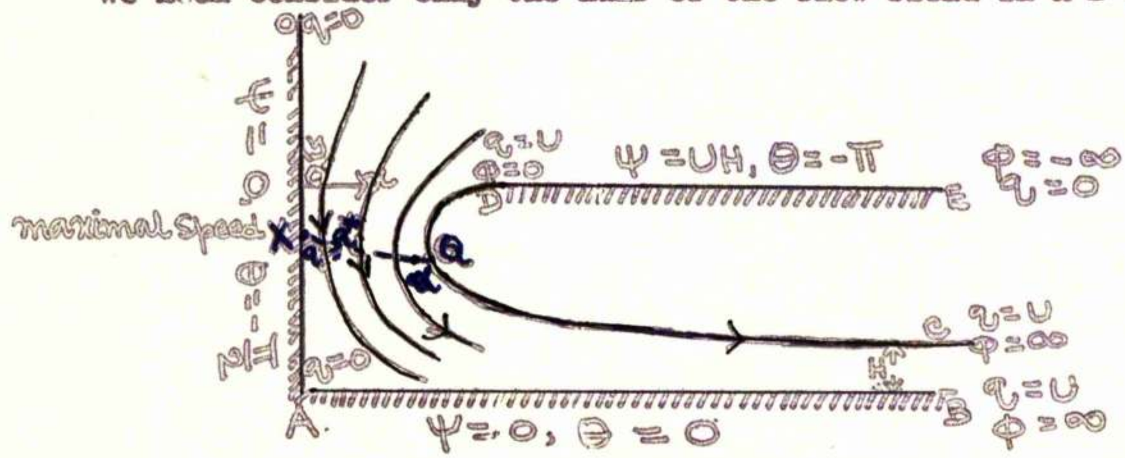


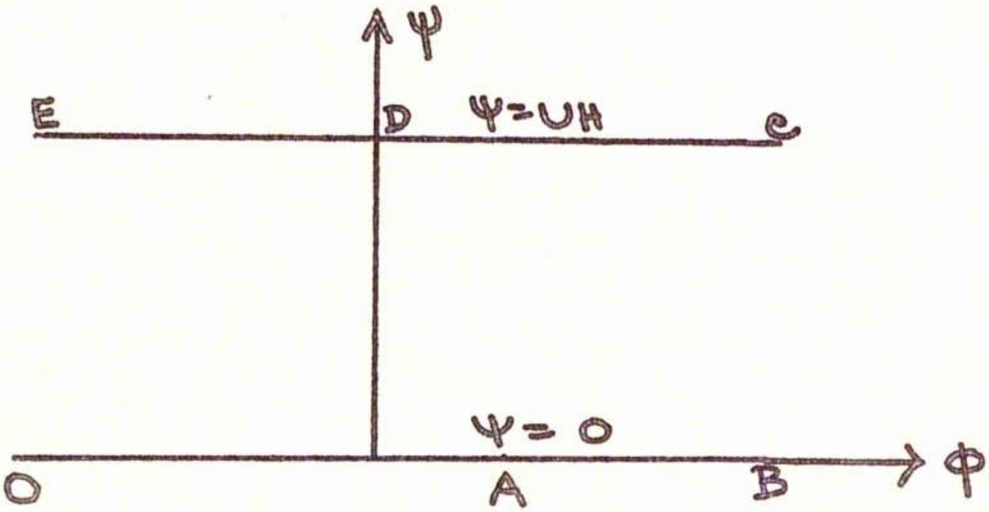
FIGURE 2

z-plane

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} > 0 \text{ near B}$$

Since the velocity is in the direction in which the velocity potential (ϕ) increases the value of ϕ at E is $-\infty$ and at C and B is $+\infty$. DC is the free streamline $\psi = U\,H$ where H is the height of C above B, and along the streamline OAB, $\psi = 0$.

The transformation $w = \phi + i\psi$ is now applied to the figure in the z-plane.



w-plane

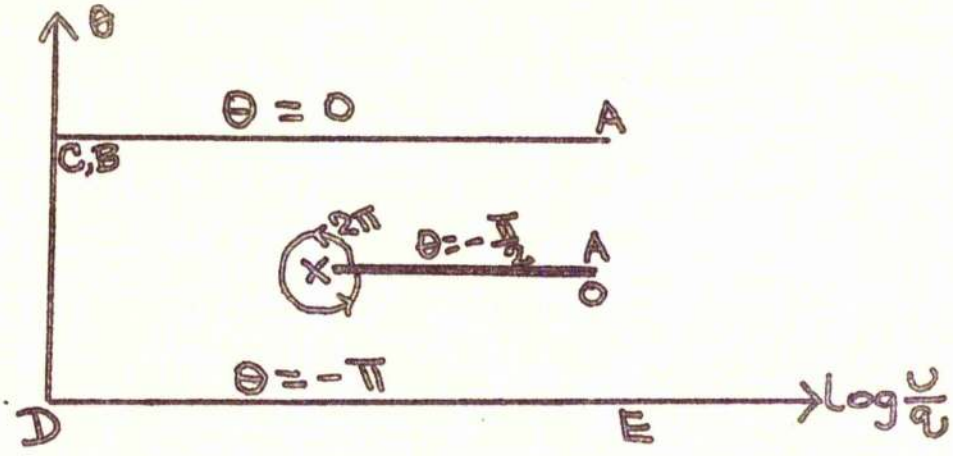
FIGURE 3

The second transformation $\Omega = \log \frac{U}{q} + i\theta$ is now applied to the figure in the z-plane.

From E to D the value of θ is constant at $\theta = -\pi$ while the value of q goes from zero at E to U at D.

From O to A the value of θ is constant at $\theta = -\pi/2$ while the value of q increases from $q = 0$ at O and then decreases and again becomes $q = 0$ at A. Since $q = 0$ both at O and A we must have a point X on OA which is a point of maximum speed. Let $q = q^{*}$ at X.

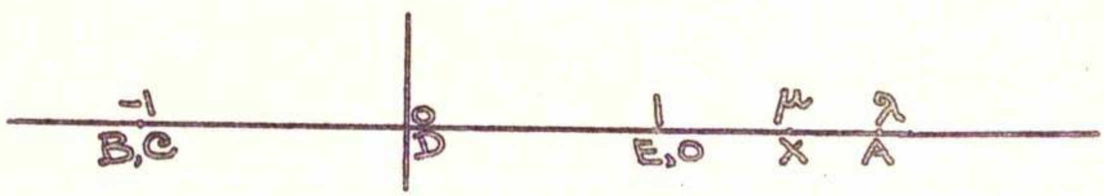
From A to B the value of θ is constant, $\theta = 0$, while the value of q goes from $q = 0$ at A to $q = U$ at B.



Ω-plane

FIGURE 4

In the mapping of the w- and Ω-planes onto the t-plane, the points (B,C), D, (E,O), X and A are made to correspond to $t = -1,\ 0,\ 1,\ \mu$ and λ respectively, where μ and λ are still to be determined.



t-plane

FIGURE 5

It will be convenient to introduce one more transformation, from the t-plane to a p-plane, by the substitution, $t = (1 + t)\frac{p^2}{2}$,

i.e.

When

$$t = 0,\ p = 0$$

$$t = 1,\ p = 1$$

$$t = \lambda, (>1),\ p = \sqrt{\frac{2\lambda}{1+\lambda}} = \frac{1}{k}\text{(say)} > 1$$

$$t = 0^-, \; p = \sqrt{\frac{2(0^-)}{1+0^-}} \to \sqrt{-\frac{0}{1}} = i0$$

$$t = 0^+, \; p = 0$$

Hence, when

$$t = \left.\begin{matrix} 0^- \\ 0^+ \end{matrix}\right\}, \; p = \begin{matrix} i0 \\ 0 \end{matrix}$$

Similarly, when

$$t = \left.\begin{matrix} 1^+ \\ 1^- \end{matrix}\right\}, \; p = \left\{\begin{matrix} 1^+ \\ 1^- \end{matrix}\right.$$

Again, when

$$t = -1+\varepsilon, \; p = \sqrt{\frac{2(-1+\varepsilon)}{\varepsilon}} = \sqrt{-\frac{2}{\varepsilon}} \to i\infty \text{ as } \varepsilon \to 0$$

when $t = -1-\varepsilon, \; p = \sqrt{\frac{2(-1-\varepsilon)}{-\varepsilon}} = \sqrt{\frac{2}{\varepsilon}} \to \infty \text{ as } \varepsilon \to 0$

Hence, when

$$t = \left.\begin{matrix} -1^+ \\ -1^- \end{matrix}\right\}, \; p = \left\{\begin{matrix} i\infty \\ \infty \end{matrix}\right.$$

Hence the mapping of the p-plane is as follows:-

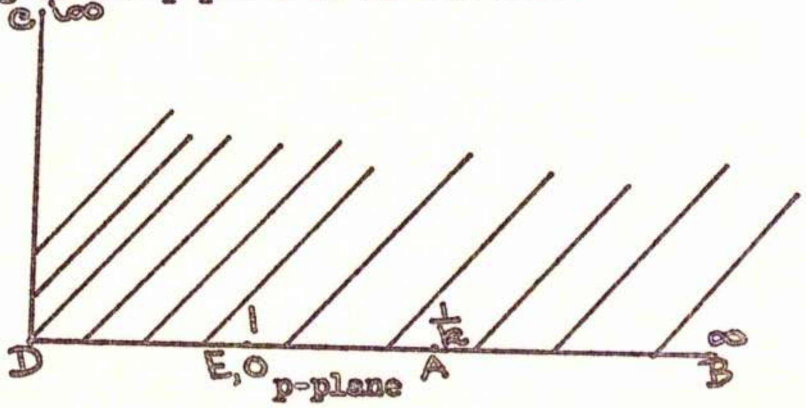


FIGURE 6

Therefore by the theorem of Schwarz and Christoffel, since the angles at EO and BC in the w-plane are zero, the relation between w and t is

$$\frac{d\omega}{dt} = \frac{\alpha}{(t-1)(t+1)} = \frac{\alpha}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right) \qquad (2.01)$$

$$\therefore \quad \omega = \frac{\alpha}{2}\operatorname{Log}\left(\frac{t-1}{t+1}\right) + \beta \qquad (2.02)$$

At A: $t > 1$, w is real;

$\therefore$ β is real if α is real.

At D: $t = 0$, $w = U H i$

$\therefore$ $\beta = 0$, and $UH = \frac{\alpha}{2}\pi$ i.e. $\alpha = \frac{2UH}{\pi}$

Hence, $w = \frac{UH}{\pi} \log \frac{t - 1}{t + 1}$

$$= \frac{UH}{\pi} \log (p^2 - 1) \ \ldots\ldots \qquad (2.03)$$

$$\text{since } t = \frac{1 + t}{2} p^2$$

The relationship between Ω and t by the theorem of Schwarz and Christoffel (since the angles at (B,C), D, (E,O), A and X are $\frac{\pi}{2}$, $\frac{\pi}{2}$, 0, 0 and 2π) is

$$\frac{d\Omega}{dt} = K \frac{(t-\mu)}{(t-\lambda)(1+t)^{\frac{1}{2}}t^{\frac{1}{2}}(t-1)} \qquad (2.04)$$

Since,

$$\left.\begin{aligned} t &= \frac{p^2}{2-p^2} \\ \frac{dt}{dp} &= \frac{4p}{(2-p^2)^2} \end{aligned}\right\} \quad \text{Now,} \quad \frac{d\Omega}{dp} = \frac{d\Omega}{dt}\cdot\frac{dt}{dp} \qquad (2.05)$$

Hence, substituting the values of t and $\frac{dt}{dp}$ in $\frac{d\Omega}{dp}$ we obtain by partial fractions

$$\frac{d\Omega}{dp} = \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right)\left\{\frac{1}{p-1} - \frac{1}{p+1}\right\} + \frac{K(\lambda-\mu)}{\sqrt{\lambda}(\lambda-1)}\left\{\frac{1}{p\sqrt{1+\lambda}-\sqrt{2\lambda}} - \frac{1}{p\sqrt{1+\lambda}+\sqrt{2\lambda}}\right\} \tag{2.06}$$

$$\therefore\ \Omega = \gamma + \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right)\mathrm{Log}\left(\frac{p-1}{p+1}\right) + \frac{K(\lambda-\mu)}{\sqrt{\lambda}(\lambda-1)\sqrt{1+\lambda}}\,\mathrm{Log}\left(\frac{p-\sqrt{\frac{2\lambda}{1+\lambda}}}{p+\sqrt{\frac{2\lambda}{1+\lambda}}}\right) \tag{2.07}$$

where γ is a constant of integration.

$$= \gamma + \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right)\mathrm{Log}\left(\frac{p-1}{p+1}\right) + \frac{K(\lambda-\mu)}{(\lambda-1)\sqrt{\lambda(1+\lambda)}}\log\left(\frac{kp-1}{kp+1}\right) \tag{2.08}$$

To evaluate γ, K, μ and λ we proceed as follows:-

At B, C, $\Omega = 0$, $|p|$ is infinite

$\therefore$ $\gamma = 0$.

Hence from (2.08) we obtain

$$\Omega = \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right)\mathrm{Log}\left(\frac{p-1}{p+1}\right) + \frac{K(\lambda-\mu)}{(\lambda-1)\sqrt{\lambda(1+\lambda)}}\mathrm{Log}\left(\frac{kp-1}{kp+1}\right) \tag{2.09}$$

Now, at D: $\Omega = -i\pi,\ t = 0,\ p = 0$

$$\therefore\quad -i\pi = \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right)i\pi + \frac{K(\lambda-\mu)}{\sqrt{\lambda(1+\lambda)}(\lambda-1)}\,i\pi$$

$$\therefore\quad -1 = \frac{K}{\sqrt{2}}\,\frac{1}{1-\lambda}\left\{(1-\mu) - \frac{\sqrt{2}(\lambda-\mu)}{\sqrt{\lambda(1+\lambda)}}\right\} \tag{2.10}$$

at $\left.\begin{matrix} O \\ E \end{matrix}\right\}, \; t = \begin{matrix} 1^+ \\ 1^- \end{matrix}, \; p = \begin{cases} 1^+ \\ 1^- \end{cases}, \; \Omega = \left.\begin{matrix} N - i\frac{\pi}{2} \\ N - i\pi \end{matrix}\right\} N \to \infty$

$\therefore$ from (2.09)
$$-i\frac{\pi}{2} = \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right) i\pi$$

i.e.
$$-\frac{1}{2} = \frac{K}{\sqrt{2}}\left(\frac{1-\mu}{1-\lambda}\right) \tag{2.11}$$

Hence from (2.10) and (2.11),

$$-\frac{1}{2} = -\frac{K(\lambda-\mu)}{(1-\lambda)\sqrt{\lambda(1+\lambda)}} \tag{2.12}$$

Again, dividing (2.11) by (2.12)

$$1 = -\frac{(1-\mu)\sqrt{\lambda(1+\lambda)}}{\sqrt{2}(\lambda-\mu)} \tag{2.13}$$

But,
$$\frac{2\lambda}{1+\lambda} = \frac{1}{k^2}$$

i.e.
$$\lambda = \frac{1}{2k^2-1} \tag{2.14}$$

Substituting the value of λ in (2.13) we obtain

$$\mu = \frac{k+1}{2k^2+k-1} \tag{2.15}$$

Hence at X,
$$p^2 = \frac{2\mu}{\mu+1} = \frac{1}{k} \quad \left(\text{i.e. } p = \frac{1}{\sqrt{k}}\right) \tag{2.16}$$

Again from (2.11) and (2.12) and (2.09)

$$\Omega = -\frac{1}{2}\log\left(\frac{p-1}{p+1}\right) - \frac{1}{2}\log\left(\frac{kp-1}{kp+1}\right)$$

i.e.
$$\Omega = -i\pi + \frac{1}{2}\log\left(\frac{1+p}{1-p}\right) + \frac{1}{2}\log\left(\frac{1+kp}{1-kp}\right) \tag{2.17}$$

where $k<1$

But, $$\Omega = \operatorname{Log} G = \log\left(\frac{v}{q e^{i\theta}}\right)$$

Hence from (2.17),

$$\operatorname{Log} G = -i\pi + \frac{1}{2}\operatorname{Log}\frac{(1+p)(1+kp)}{(1-p)(1-kp)}$$

which gives,

$$kp^2(G^2-1) - p(1+k)(1+G^2) + G^2 - 1 = 0$$

Since $G > 1$, writing $G' = \frac{1}{G}\ (<1)$, we obtain

$$kp^2(1-G'^2) - p(1+k)(1+G'^2) + (1-G'^2) = 0 \qquad (2.18)$$

Solving the quadratic, we get

$$p = \frac{\left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right) \pm \sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2 - \frac{4}{k}}}{2} \qquad (2.19)$$

Again from (2.18)

$$p^2 = \left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)p - \frac{1}{k}$$

which gives,

$$p^2 - 1 = \left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)p - \frac{1+k}{k}$$

Substituting the value of p from (2.19), we get

$$p^2 - 1 = \left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)\left[\left(\frac{1+k}{2k}\right)\left(\frac{1+G'^2}{1-G'^2}\right) \pm \frac{\sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2 - \frac{4}{k}}}{2}\right] - \frac{1+k}{k} \qquad (2.20)$$

at O, E and A, $G' = 0$ since $q = 0$ at these points.

But, at

$$\text{at} \left.\begin{matrix} O \\ E \end{matrix}\right\}, \quad p = \begin{cases} 1^+ \\ 1^- \end{cases}$$

$$\text{at } A, \quad p = \frac{1}{k}$$

Hence when $G = 0$, $$p = \begin{cases} 1^+, 1^- \\ \frac{1}{k} \end{cases} \tag{2.21}$$

So, when p is either 1^+ or 1^-, we obtain from (2.20)

$$0 = \frac{1+k}{k}\left[\frac{1-k}{2k} \pm \frac{1-k}{2k}\right]$$

L.H.S. = R.H.S. if we consider the negative sign.

Hence near O and E,

$$p^2-1 = \left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)\left[\left(\frac{1+k}{2k}\right)\left(\frac{1+G'^2}{1-G'^2}\right) - \frac{1}{2}\sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2 - \frac{4}{k}}\,\right] - \frac{1+k}{k} \tag{2.22}$$

Again, when $p = \frac{1}{k}$, as before we obtain from (2.20)

$$\left(\frac{1}{k}\right)^2 - 1 = \frac{1+k}{k}\left[\frac{1-k}{2k} \pm \frac{1-k}{2k}\right]$$

L.H.S. = R.H.S. if we consider the positive sign.

Thus near A,

$$p^2-1 = \left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)\left[\left(\frac{1+k}{2k}\right)\left(\frac{1+G'^2}{1-G'^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2 - \frac{4}{k}}\,\right] - \frac{1+k}{k} \tag{2.23}$$

Since $w = \frac{UH}{\pi} \log (p^2 - 1)$, the two values of w will be equal if the expression under the radical sign in (2.23) and (2.22) vanishes and this will be the point X.

Hence at X,

$$\left(\frac{1+k}{k}\right)^2\left(\frac{1+G^2}{1-G'^2}\right)^2 - \frac{4}{k} = 0 \tag{2.24}$$

But, at X, $G' = i\frac{q}{U}$ since $\theta = -\pi/2$.

Solving (2.24) we obtain

$$\frac{q}{U} = \frac{1-\sqrt{k}}{1+\sqrt{k}} \text{ or } \frac{1+\sqrt{k}}{1-\sqrt{k}}$$

The second value is inadmissible, since $\frac{q}{U} < 1$.

Hence, $$\frac{q}{U} = \frac{1-\sqrt{k}}{1+\sqrt{k}}$$

Let $q = q^*$ at X.

$$\therefore \quad \frac{q^*}{U} = \frac{1-\sqrt{k}}{1+\sqrt{k}} \tag{2.25}$$

Again, since $w = \frac{UH}{\pi} \log (p^2 - 1)$, we obtain from (2.23)

$$\omega = \frac{UH}{\pi} \operatorname{Log}\left[\left(\frac{1+k}{k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)\left\{\left(\frac{1+k}{2k}\right)\left(\frac{1+G'^2}{1-G'^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2 - \frac{4}{k}}\right\} - \frac{1+k}{k}\right]$$

i.e.

$$\omega = \frac{UH}{\pi} \operatorname{Log}\left[(1-k) + (1-k)G'^4 + 2(1+3k)G'^2 + (1+G'^2)\sqrt{(1-k)^2(1+G'^2)^2 + 16kG'^2}\right]$$

$$+ \text{ Real quantity} \tag{2.26}$$

R.H.S. should have a varying imaginary part giving ψ on $\theta = -\pi/2$ as we cross the jet from X to Q. This is possible if the expression under the radical sign in (2.26) is negative

i.e. if $$(1-k)^2(1+G'^2)^2 + 16kG'^2 < 0$$

i.e. if $$\frac{q}{U} > \frac{1-\sqrt{k}}{1+\sqrt{k}}, \text{ since } G' = i\frac{q}{U} \text{ on } \theta = -\frac{\pi}{2}$$

Comparing with (2.25) we find that this is true on the line $\theta = -\pi/2$ as we cross the jet from X to Q.

Let $\psi = \ell(q)$ express the variation of the stream function with speed q along the line on which $\theta = -\pi/2$ (shown by a dotted curve in figure 2). Hence from (2.26),

$$\ell(q) = \text{Im.p. of } \frac{UH}{\pi}\,\text{Log}\left[(1-k)+(1-k)\frac{q^4}{U^4}-2(1+3k)\frac{q^2}{U^2}+\left(1-\frac{q^2}{U^2}\right)\sqrt{(-1)\left\{16k\frac{q^2}{U^2}-(1-k)^2\left(1-\frac{q^2}{U^2}\right)^2\right\}}\right] \tag{2.27}$$

Now, we have to fix the sign of the square root of the negative quantity i.e. we have to see if the square root is $+i\alpha$ or $-i\alpha$ where

$$\alpha = 16k\frac{q^2}{U^2}-(1-k)^2\left(1-\frac{q^2}{U^2}\right)^2 > 0$$

Looking back to figure 2, we find that at Q, $q = U$ and $\theta = -\pi/2$. Hence $G' = i$.

Consider a point Q' very near to Q.

At Q', $G' = i\,(1-\epsilon)$ where ϵ is real > 0.

Considering the flow near A, we obtain from (2.23)

$$\omega = \frac{UH}{\pi}\log\left[\left(\frac{1+k}{2k}\right)\left(\frac{1+G'^2}{1-G'^2}\right)^2-1+\frac{1}{2}\left(\frac{1+G'^2}{1-G'^2}\right)\sqrt{\left(\frac{1+k}{k}\right)^2\left(\frac{1+G'^2}{1-G'^2}\right)^2-\frac{4}{k}}\,\right] + \frac{UH}{\pi}\log\left(\frac{1+k}{k}\right)$$

Hence, near Q'

$$\omega = \frac{UH}{\pi}\text{Log}\left[\left(\frac{1+k}{2k}\right)\left\{\frac{1-(1-\epsilon)^2}{1+(1-\epsilon)^2}\right\}^2-1+\frac{1}{2}\left\{\frac{1-(1-\epsilon)^2}{1+(1-\epsilon)^2}\right\}\sqrt{\left(\frac{1+k}{k}\right)^2\left\{\frac{1-(1-\epsilon)^2}{1+(1-\epsilon)^2}\right\}^2-\frac{4}{k}}\,\right] + \frac{UH}{\pi}\text{Log}\left(\frac{1+k}{k}\right)$$

$$= \frac{UH}{\pi}\text{Log}\left[O(\epsilon)-1+O(\epsilon)\sqrt{-\frac{1}{k}}\right]+\frac{UH}{\pi}\text{Log}\left(\frac{1+k}{k}\right)$$

Considering the square root of the negative quantity as $i\alpha'$ where $\alpha' = \frac{1}{k}$, we obtain, at Q'

$$\omega = \frac{UH}{\pi}\operatorname{Log}\left[-1 + o(\varepsilon) + i\,o(\varepsilon)\alpha'\right] + \frac{UH}{\pi}\operatorname{Log}\left(\frac{1+k}{k}\right)$$

$$= \frac{UH}{\pi}\operatorname{Log}(-1) + \frac{UH}{\pi}\operatorname{Log}\left[1 - o(\varepsilon) - i\,o(\varepsilon)\alpha'\right] + \frac{UH}{\pi}\operatorname{Log}\left(\frac{1+k}{k}\right)$$

Hence at Q'

$$\psi = UH - \frac{UH}{\pi}\beta, \text{ where } \beta = \tan^{-1}\left\{\frac{o(\varepsilon)\alpha'}{1-o(\varepsilon)}\right\}$$

i.e. $\psi < U\,H$, which is true as we cross the jet from Q to X.

This establishes that we must consider the square root of the negative quantity as $+\,i\alpha$ if we consider the expression of w near A. Similarly, it can be shown that if we consider the expression of w near E, then the choice of the square root of the negative quantity will be $-\,i\alpha$.

$\therefore$ from (2.27)

$$\ell(q) = \text{Im. p. of } \frac{UH}{\pi}\operatorname{Log}\left[(1-k) + (1-k)\frac{q^4}{U^4} - 2(1+3k)\frac{q^2}{U^2} + i\left(1-\frac{q^2}{U^2}\right)\sqrt{16k\frac{q^2}{U^2} - (1-k)^2\left(1-\frac{q^2}{U^2}\right)^2}\,\right]$$

$$= \frac{UH}{\pi}\tan^{-1}\left\{\frac{\left(1-\frac{q^2}{U^2}\right)\sqrt{16k\frac{q^2}{U^2} - (1-k)^2\left(1-\frac{q^2}{U^2}\right)^2}}{(1-k) - 2(1+3k)q^2/U^2 + (1-k)\,q^4/U^4}\right\} \qquad (2.28)$$

Let $M = \frac{q}{U}$, so that $M^* = \frac{q^*}{U}$

Hence,

$$\ell(q) = \begin{cases} \frac{UH}{\pi}\tan^{-1}\left\{\frac{(1-M^2)\sqrt{16kM^2 - (1-k)^2(1-M^2)^2}}{(1-k) - 2(1+3k)M^2 + (1-k)M^4}\right\} & \text{for } M > M^* \\ 0 \quad \text{for } M < M^* \end{cases} \qquad (2.29)$$

The curve on which $\psi = \ell(q)$ has been shown in the physical plane by the dotted line XQ and is such that it crosses all the stream lines on $\theta = -\pi/2$. in the hodograph plane.

$$\text{Let } L(M) = \frac{1}{\pi}\tan^{-1}\left\{\frac{(1-M^2)\sqrt{16kM^2-(1-k)^2(1-M^2)^2}}{(1-k)-2(1+3k)M^2+(1-k)M^4}\right\} \qquad (2.30)$$

so that $\quad \ell(q) = UHL(M)$

When $q = U$, $M = 1$

$\therefore \; L(1) = 1$

When $q = q^*$, $L(M^*) = 0$ since $\ell(q^*) = 0$ (surd is real)

Hence, $L(1) = 1, \; L(M^*) = 0$ (2.31)

Again, from (2.25) $\quad M^* = \dfrac{1-\sqrt{k}}{1+\sqrt{k}} \quad$ which gives

$$k = \left(\frac{1-M^*}{1+M^*}\right)^2 \qquad (2.32)$$

Substituting the value of k in the expression for L(M) we obtain a second form of L(M) as

$$L(M) = \frac{1}{\pi}\tan^{-1}\left\{\frac{(1-M^2)\sqrt{(M^2-M^{*2})(1-M^2M^{*2})}}{M^*(1+M^2)-2M^2(1+M^{*2})}\right\} \qquad (2.33)$$

Differentiating with respect to M we obtain

$$\frac{dL}{dM} = \frac{1}{\pi}\left\{\frac{M^*(1+M^2)^2+2M^2(1+M^{*2})}{M(1+M^2)\sqrt{(M^2-M^{*2})(1-M^2M^{*2})}}\right\} \qquad (2.34)$$

We shall have need to refer to the value of this derivative in later chapters.

(b) Flow through a necked slit

We now illustrate another problem 'The Generalised Borda's mouth piece' which is physically apparently quite different from the previous example although the solution may be carried through in the same way. Here the inviscid incompressible fluid is confined in a semi-infinite reservoir which is bounded by the semi-infinite walls $A'B'C'$ and ABC having a gap of width $2r$ between them. Two walls $D'C'$ and DC of length 'a' have been projected into this semi-infinite reservoir forming a neck round the gap through which liquid escapes, thus forming a jet bounded by the free streamlines $D'E'$ and DE.

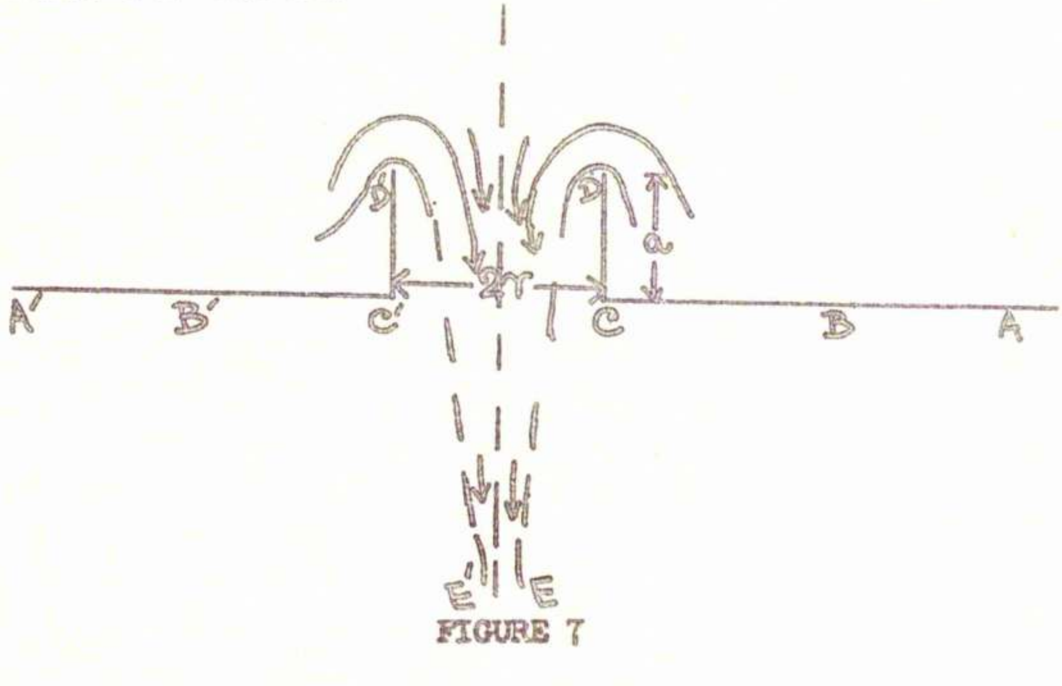


FIGURE 7

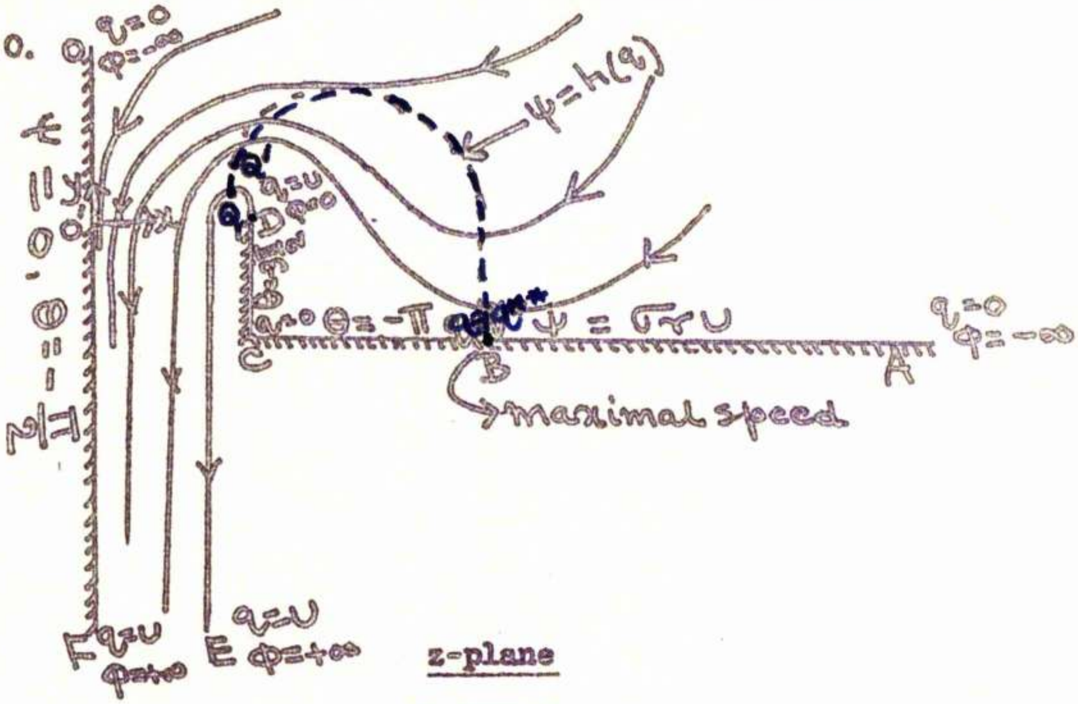
PHYSICAL PLANE

The jet contracts to the width $2\sigma r$ at $E'E$ far from the mouthpiece where the speed of the fluid is uniform and of value U. (σ being the coefficient of contraction.) The total efflux from the reservoir is therefore $2\sigma rU$. The speed of the fluid along the free streamlines $D'E'$

and DE is U. Along the wet side and reservoir walls A'B'C'D' and ABCD the surface speed varies. It is zero at the corners C' and C and at the infinite points A' and A. At two places B' and B the surface speed has a maximum. This follows because B is between A and C where the surface speed is zero.

Solution:

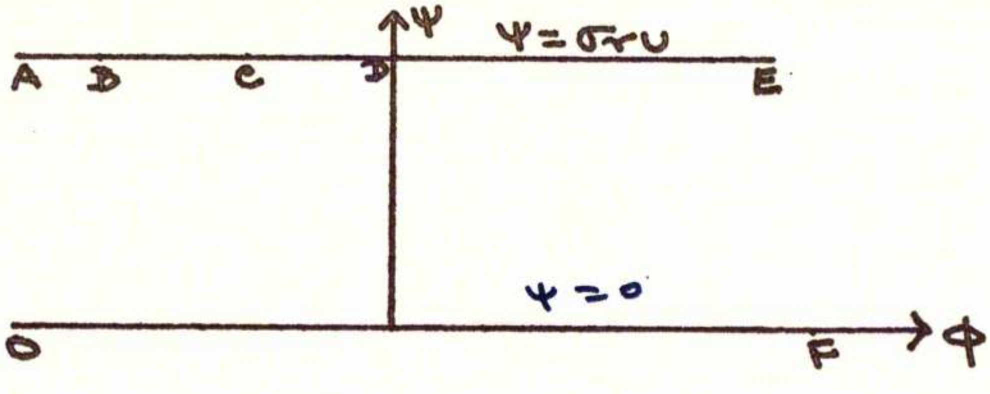
Because of symmetry we need consider only one half of the flow field in $x \geq 0$.



z-plane

FIGURE 8

The value of ϕ at A and O is $-\infty$ and that at F and E $+\infty$. DE is the free streamline $\psi = \sigma rU$ and OF is the streamline $\psi = 0$. The transformation $w = \phi + i\psi$ is now applied to the figure in the z-plane.



w-plane

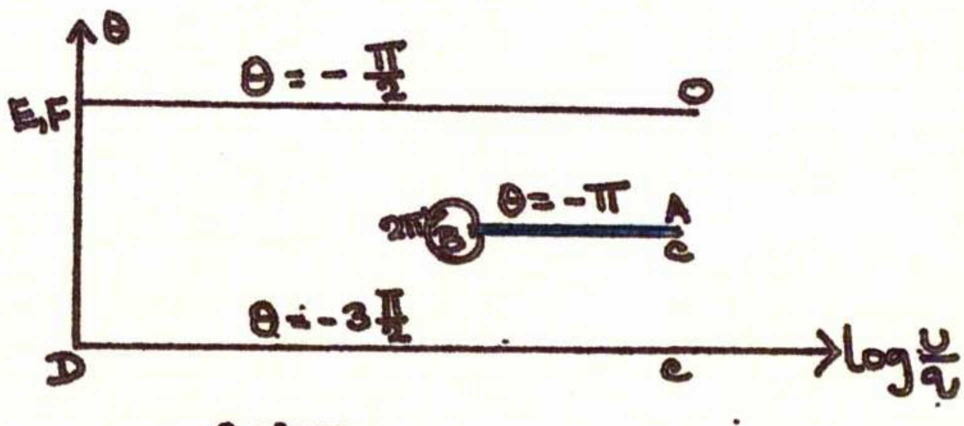
FIGURE 9

The second transformation $\Omega = \log \frac{U}{q} + i\theta$ is now applied to the figure in the z-plane.

From A to C the value of θ is constant at $-\pi$ while the value of q goes $q = 0$ at A to $q = q^{**}$ at B and then again falls to $q = 0$ at C.

From C to D θ is constant at $\theta = -3\frac{\pi}{2}$ while q goes from $q = 0$ to $q = U$ at D.

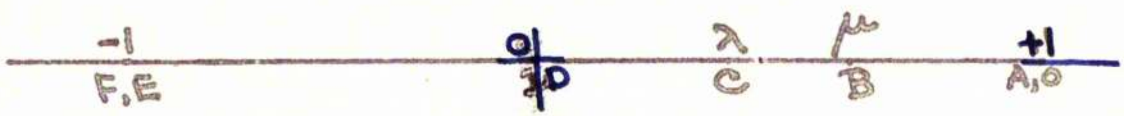
From O to F θ is constant at $\theta = -\pi/2$ while q goes from $q = 0$ at O to $q = U$ at F.



Ω-plane

FIGURE 10

In the mapping of the w- and Ω-planes onto the t-plane the points (F,E), D, C, B, (A,O) are made to correspond to -1, 0, λ, μ and +1 respectively where λ and μ are still to be determined.



t-plane

FIGURE 11

Here also for convenience we introduce one more transformation from the t-plane to a p-plane, by the substitution, $t = (1 + t)\frac{p^2}{2}$ and as before we get all the points of the t-plane in the first quadrant of the p-plane. The mapping is as follows:-

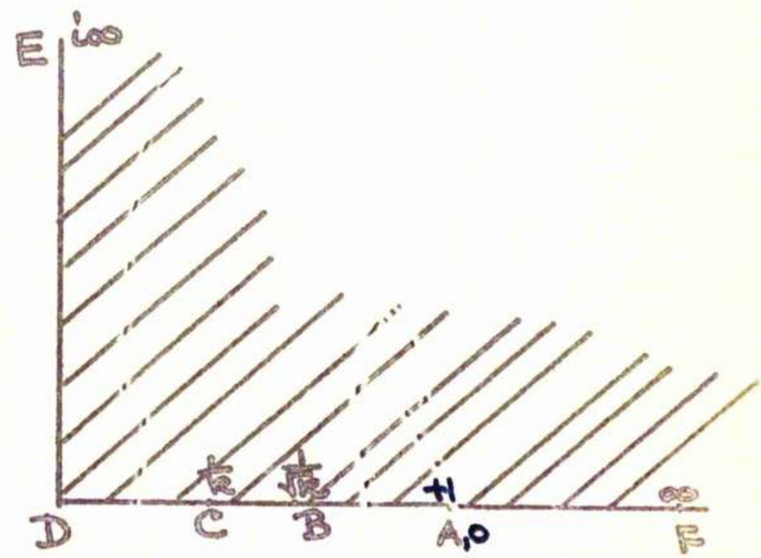
at C, $t = \lambda$

$$\therefore p = \sqrt{\frac{2\lambda}{1+\lambda}} = \left(\frac{1}{k}\text{ say}\right) < 1$$

since $\lambda < 1$

at B, $t = \mu$

$$\therefore p = \sqrt{\frac{2\mu}{1+\mu}} = \frac{1}{\sqrt{k}} \text{ (from 2.16)}$$



p-plane

FIGURE 12

The transformation from the t-plane to the p-plane has been explained in details in the previous example.

If we now compare this problem with that of Levy's we find that these two problems differ only in the pattern of streamlines and the boundary conditions. The streamline pattern has been shown in Figure 8. Applying the boundary conditions and proceeding exactly in the same way as in the previous example we obtain in place of (2.17)

$$\Omega = -\frac{3}{2}\pi i + \frac{1}{2}\log\left(\frac{1+p}{1-p}\right) + \frac{1}{2}\log\left(\frac{1+kp}{1-kp}\right) \tag{2.35}$$

and in place of (2.29),

$$h(q) = \begin{cases} \sigma\gamma U - \frac{\sigma\gamma U}{\pi}\tan^{-1}\left(\frac{(1-M^2)\sqrt{16kM^2-(k-1)^2(1-M^2)^2}}{(k-1)+2(1+3k)M^2+(k-1)M^4}\right) & \text{for } M > t^* \\ \sigma\gamma U \quad \text{for } M < t^* \text{ where } t^* = \frac{q^{**}}{U} < 1 \end{cases} \tag{2.36}$$

$\psi = h(q)$ being the curve on which $\theta = -\pi$ (shown by the dotted curve in Figure 8).

Let $$H(M) = -\frac{1}{\pi}\tan^{-1}\left\{\frac{(1-M^2)\sqrt{16kM^2-(k-1)^2(1-M^2)^2}}{(k-1)+2(1+3k)M^2+(k-1)M^4}\right\} \tag{2.37}$$

So that,

$$\begin{aligned} h(q) &= \sigma\gamma U + \sigma\gamma U H(M) \\ &= \sigma\gamma U[1+H(M)] \end{aligned} \tag{2.38}$$

From (2.36),
$$\left.\begin{aligned} h(U) &= \sigma_\gamma U \\ \text{also,}\quad h(q^{**}) &= \sigma_\gamma U \end{aligned}\right\} \tag{2.39}$$

Hence, from (2.38) and (2.39)

$$H(1) = H(t^*) = 0 \tag{2.40}$$

Also, in place of (2.32) we obtain

$$k = \left(\frac{1+t^*}{1-t^*}\right)^2 \tag{2.41}$$

Substituting the value of k in (2.37) we obtain the second form of H(M) as

$$H(M) = -\frac{1}{\pi}\tan^{-1}\left\{\frac{(1-M^2)\sqrt{(M^2-t^{*2})(1-M^2t^{*2})}}{t^*(1+M^2)^2+2M^2(1+t^{*2})}\right\} \tag{2.42}$$

Also,
$$\frac{dH}{dM} = -\frac{1}{\pi}\left\{\frac{t^*(1+M^2)^2-2M^2(1+t^{*2})}{M(1+M^2)\sqrt{(M^2-t^{*2})(1-M^2t^{*2})}}\right\} \tag{2.43}$$

CHAPTER III

FLOW THROUGH A NECKED SLIT IMPINGING ON A WALL

In Chapter II, we have discussed two problems separately, namely the flow through a slit impinging on a wall and a flow through a necked slit. In this chapter, we will combine them together and then solve it as one problem. The inviscid incompressible fluid is confined in a semi-infinite reservoir which is bounded by sides F'R'S' and FRS parallel to a wall B'AB distance h away. The chamber side has a slit of width 2r. Two walls D'F' and DF of length 'a' have been projected into this semi-infinite reservoir forming a neck round the slit through which liquid escapes.

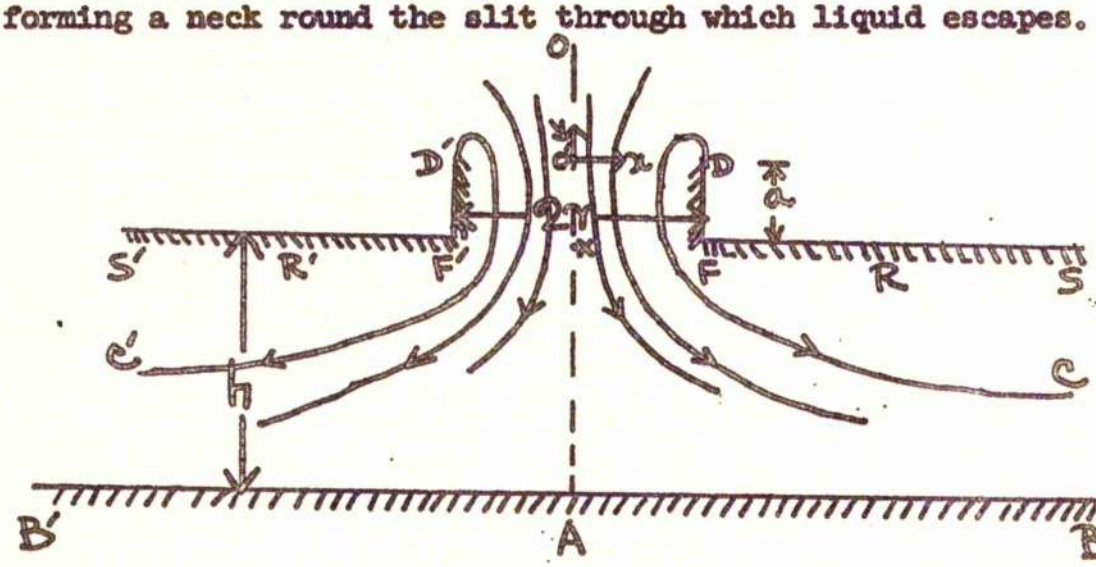


FIGURE 1

Physical plane; with axis of symmetry OA, chamber sides F'S' and FS. Jet boundary D'C' and DC.

The solution is carried through by mapping the flow region in the plane of the complex velocity potential w and in the plane of Helmholtz potential Ω , on an appropriate region of the plane of the auxiliary complex variable t and p. Because of symmetry we need consider only the half of the flow field in x ≥ 0.

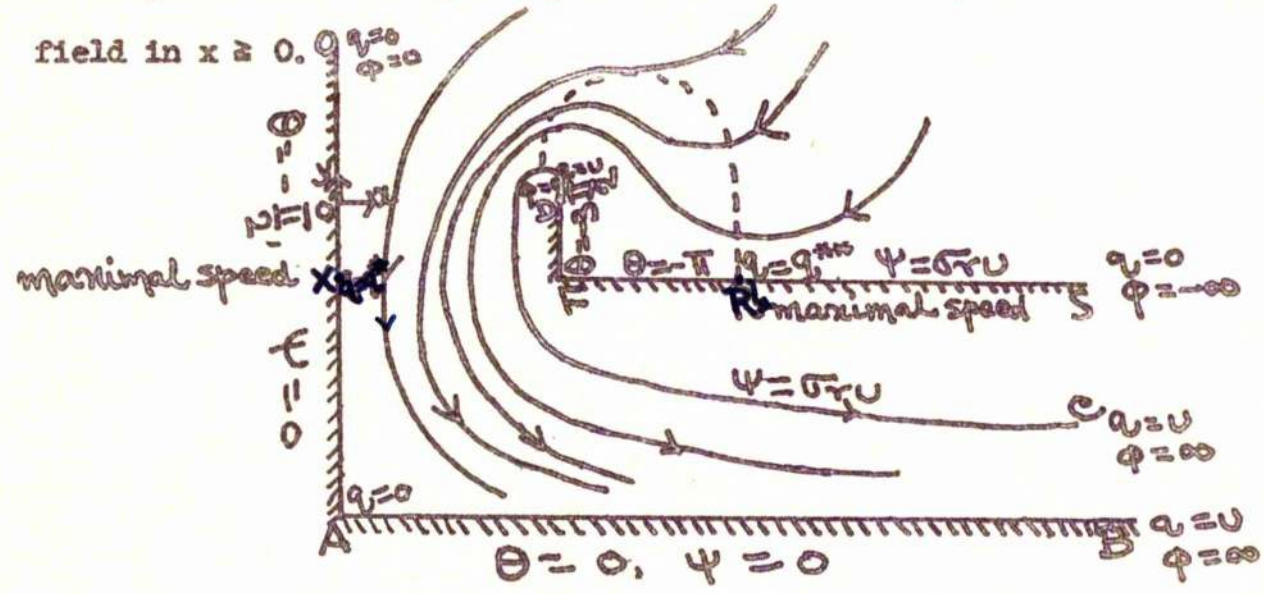
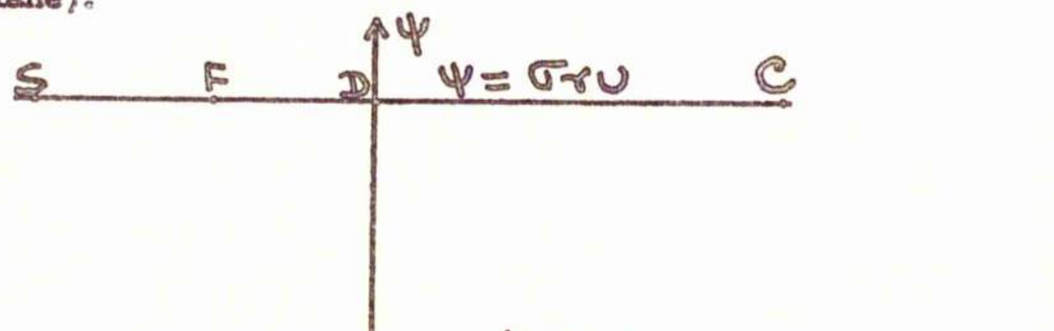


FIGURE 2

z-plane

Since the velocity is in the direction in which the velocity potential (ϕ) increases, the value of ϕ at C and B is $+\infty$ and at O and S it is $-\infty$. DC is the free streamline $\psi = \sigma r U$ and along the streamline OAB $\psi = 0$. The transformation $w = \phi + i\psi$ is now applied to the figure in the z-plane (physical plane).



w-plane

FIGURE 3

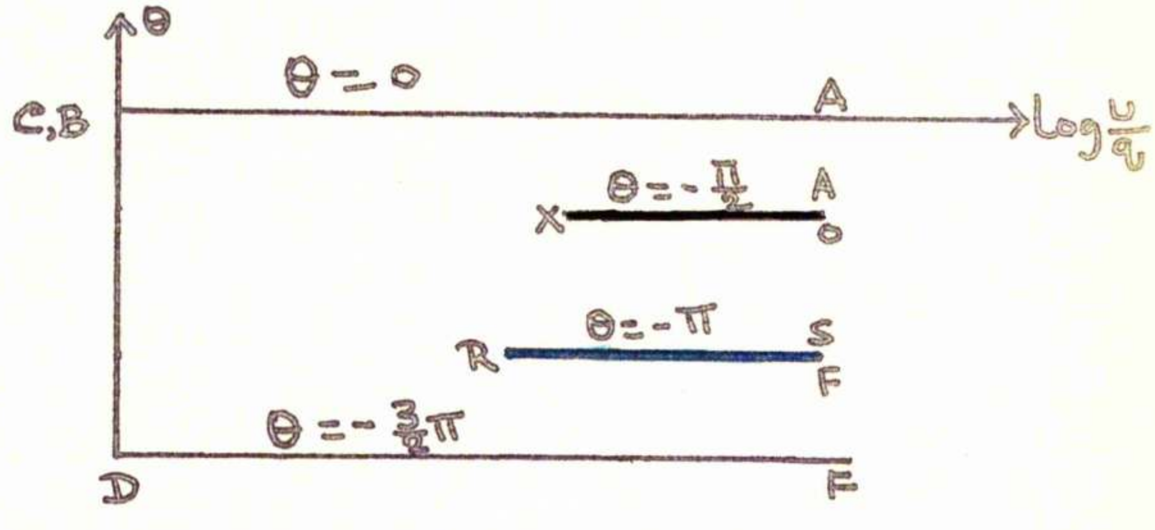
The second transformation $\Omega = \log \frac{U}{q} + i\theta$ is now applied to the figure in the z-plane.

From S to F the value of θ is constant at $\theta = -\pi$ while the value of q increases from $q = 0$ at S to $q = q^{**}$ at R and then decreases to $q = 0$ at F.

From F to D, θ is constant at $\theta = -3\frac{\pi}{2}$ while q goes from $q = 0$ at F to $q = U$ at D.

From O to A the value of θ is constant at $\theta = -\pi/2$ while the value of q first increases from $q = 0$ at O to $q = q^{*}$ at X and then decreases to $q = 0$ at A.

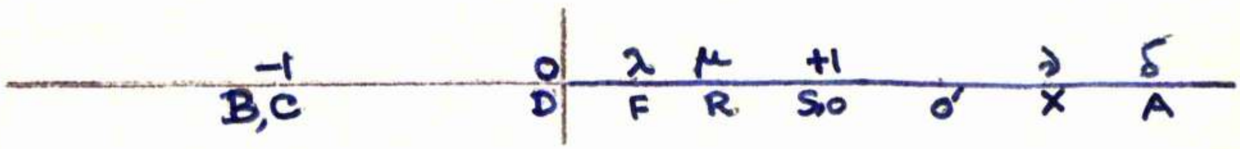
From A to B the value of θ is constant at $\theta = 0$ while the value of q goes from $q = 0$ at A to $q = U$ at B.



Ω-plane

FIGURE 4

The mapping of w and Ω planes into the t-planes, the points (B,C), D, F, R, (S,O), X and A correspond to -1, 0, λ, μ, $+1$, ν and δ respectively.



t-plane

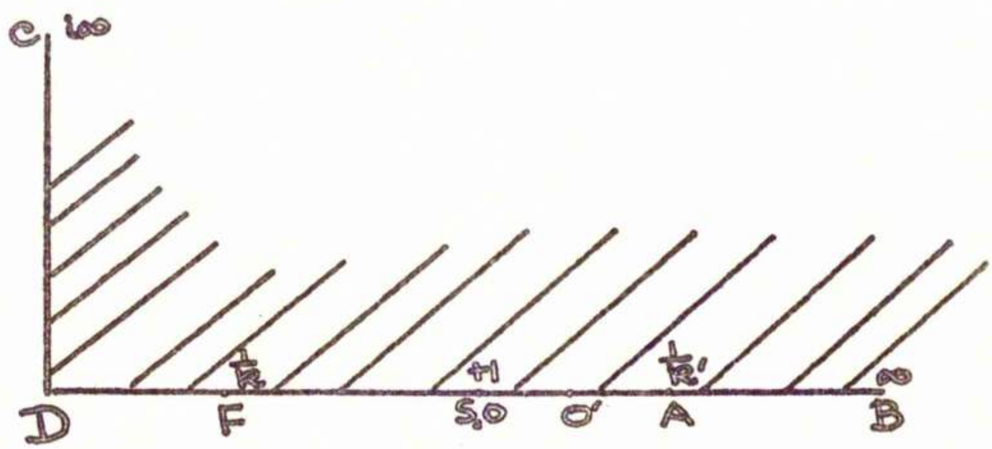
FIGURE 5

Let $p=\sqrt{\frac{2t}{1+t}}$; when $t=\lambda$, $p=\sqrt{\frac{2\lambda}{1+\lambda}}=\frac{1}{k}$ (say), when

$$t=\delta,\ p=\sqrt{\frac{2\delta}{1+\delta}}=\frac{1}{k'}\ \text{(say)}$$

$$k > 1 \text{ and } k' < 1$$

The transformation of the t-plane to the p-plane by this substitution has already been discussed in Chapter II. The points B, C, D, F, S, O, and A are mapped into the p-plane at points ∞, $i\infty$, 0, $\frac{1}{k}$ (< 1), 1, 1, and $\frac{1}{k'}$, respectively. i.e. all points in the t-plane are mapped in the first quadrant of the p-plane.



p-plane

FIGURE 6

Since the angles at (B,C) and (S,O) are zero, the relation of w and t by the theorem of Schwarz-Christoffel is

$$\frac{dw}{dt} = L\frac{1}{(t+1)(t-1)} = \frac{L}{2}\left\{\frac{1}{t-1} - \frac{1}{t+1}\right\} \tag{3.01}$$

Hence on integration, $\quad w = \frac{L}{2}\log\left(\frac{t-1}{t+1}\right) + M$

Applying conditions at B and it is easily seen that $M = 0$, $L = \frac{2\sigma\gamma U}{\pi}$

Hence, $$w = \frac{\sigma\gamma U}{\pi}\log\left(\frac{t-1}{t+1}\right) \tag{3.02}$$

$$= \frac{\sigma\gamma U}{\pi}\log(p^2-1) \tag{3.03}$$

The relationship between Ω and t is

$$\frac{d\Omega}{dt} = K\frac{(t-\mu)(t-\nu)}{(t+1)^{\frac{1}{2}}t^{\frac{1}{2}}(t-\lambda)(t-\delta)(t-1)} \tag{3.04}$$

since the angles at (B,C), D, F, R, (S,O), X and A in the Ω-plane are $\pi/2$, $\pi/2$, 0, 2π, 0, 2π and 0 respectively.

Since, $t = \frac{p^2}{2-p^2}$, $\frac{dt}{dp} = \frac{4p}{(2-p^2)^2}$, $\frac{d\Omega}{dp} = \frac{d\Omega}{dt}\frac{dt}{dp}$

we obtain

$$\frac{d\Omega}{dp} = K\sqrt{2}\,\frac{(1+\mu)(1+\nu)}{(1+\lambda)(1+\delta)}\,\frac{\left(p^2-\frac{2\mu}{1+\mu}\right)\left(p^2-\frac{2\nu}{1+\nu}\right)}{(p^2-1)\left(p^2-\frac{2\lambda}{1+\lambda}\right)\left(p^2-\frac{2\delta}{1+\delta}\right)}$$

and, by partial fractions,

$$\frac{d\Omega}{dp} = \frac{K}{\sqrt{2}}\frac{(1-\mu)(1-\nu)}{(1-\lambda)(1-\delta)}\left(\frac{1}{p-1} - \frac{1}{p+1}\right) + \frac{K}{\sqrt{2\lambda}}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-1)(\lambda-\delta)}\left(\frac{1}{p\sqrt{1+\lambda}-\sqrt{2\lambda}} - \frac{1}{p\sqrt{1+\lambda}+\sqrt{2\lambda}}\right)$$
$$+ \frac{K}{\sqrt{2\delta}}\frac{(\delta-\mu)(\delta-\nu)}{(\delta-1)(\delta-\lambda)}\left(\frac{1}{p\sqrt{1+\delta}-\sqrt{2\delta}} - \frac{1}{p\sqrt{1+\delta}+\sqrt{2\delta}}\right) \quad (3.05)$$

whence,

$$\Omega = \gamma + \frac{K}{\sqrt{2}}\frac{(1-\mu)(1-\nu)}{(1-\lambda)(1-\delta)}\log\left(\frac{p-1}{p+1}\right) + \frac{K}{\sqrt{2\lambda(1+\lambda)}}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-1)(\lambda-\delta)}\log\left(\frac{p-\sqrt{\frac{2\lambda}{1+\lambda}}}{p+\sqrt{\frac{2\lambda}{1+\lambda}}}\right)$$
$$+ \frac{K}{\sqrt{2\delta(1+\delta)}}\frac{(\delta-\mu)(\delta-\nu)}{(\delta-1)(\delta-\lambda)}\log\left(\frac{p-\sqrt{\frac{2\delta}{1+\delta}}}{p+\sqrt{2\delta/1+\delta}}\right) \quad (3.06)$$

There are six unknowns (γ, K, λ, μ, ν and δ). We have to evaluate them.

At B, C: $\Omega = 0, \quad t = \begin{cases} -1^- \\ -1^+ \end{cases}, \quad p = \begin{cases} \infty \\ i\infty \end{cases}$

Hence, it can be easily proved that $\gamma = 0$.

$$\Omega = \frac{K}{\sqrt{2}}\frac{(1-\mu)(1-\nu)}{(1-\lambda)(1-\delta)}\log\left(\frac{p-1}{p+1}\right) + \frac{K}{\sqrt{2\lambda(1+\lambda)}}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-1)(\lambda-\delta)}\log\left(\frac{kp-1}{kp+1}\right)$$
$$+ \frac{K}{\sqrt{2\delta(1+\delta)}}\frac{(\delta-\mu)(\delta-\nu)}{(\delta-1)(\delta-\lambda)}\log\left(\frac{k'p-1}{k'p+1}\right) \quad (3.07)$$

where $\frac{1}{k} = \sqrt{\frac{2\lambda}{1+\lambda}}\ (<1), \quad \frac{1}{k'} = \sqrt{\frac{2\delta}{1+\delta}}\ (>1)$

At D: $\Omega = -3\frac{\pi}{2}i, \quad t = 0, \quad p = 0$

$\therefore$ from (3.07)

$$-\frac{3}{2}=\frac{K}{\sqrt{2}}\frac{(1-\mu)(1-\nu)}{(1-\lambda)(1-\delta)}+\frac{K}{\sqrt{\lambda(1+\lambda)}}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-1)(\lambda-\delta)}+\frac{K}{\sqrt{\delta(1+\delta)}}\frac{(\delta-\mu)(\delta-\nu)}{(\delta-1)(\delta-\lambda)} \quad (3.08)$$

At O, S: $\displaystyle \Omega \underset{H\to\infty}{=} \begin{cases} H-i\frac{\pi}{2} \\ H-i\pi \end{cases}, \quad t=\begin{cases} 1^+ \\ 1^- \end{cases}, \quad p=\begin{cases} 1^+ \\ 1^- \end{cases}$

Applying these two conditions in (3.07), it is easily seen that

$$-\frac{1}{2}=\frac{K}{\sqrt{\delta(1+\delta)}}\frac{(\delta-\mu)(\delta-\nu)}{(\delta-1)(\delta-\lambda)} \quad (3.09)$$

and,
$$-\frac{1}{2}=\frac{K}{\sqrt{2}}\frac{(1-\mu)(1-\nu)}{(1-\lambda)(1-\delta)} \quad (3.10)$$

Hence from (3.08),(3.09) and (3.10) we obtain

$$-\frac{1}{2}=\frac{K}{\sqrt{\lambda(1+\lambda)}}\frac{(\lambda-\mu)(\lambda-\nu)}{(\lambda-1)(\lambda-\delta)} \quad (3.11)$$

Substituting the values of (3.09), (3.10), (3.11) in (3.07) we obtain,

$$\begin{aligned}\Omega &= -\frac{1}{2}\log\left(\frac{p-1}{p+1}\right)-\frac{1}{2}\log\left(\frac{kp-1}{kp+1}\right)-\frac{1}{2}\log\left(\frac{k'p-1}{k'p+1}\right) \\ &= \frac{1}{2}\log\left(\frac{1+p}{p-1}\right)+\frac{1}{2}\log\left(\frac{kp+1}{kp-1}\right)+\frac{1}{2}\log\left(\frac{k'p+1}{k'p-1}\right)\end{aligned} \quad (3.12)$$

Allowing for the actual values taken by Ω we can write this as

$$\Omega = -\frac{3}{2}\pi i + \frac{1}{2}\log\left(\frac{1+p}{1-p}\right) + \frac{1}{2}\log\left(\frac{1+kp}{1-kp}\right) + \frac{1}{2}\log\left(\frac{1+k'p}{1-k'p}\right) \quad (3.13)$$

We now proceed to derive from this result the particular case obtained in the previous chapter.

From (3.12), $$e^{\Omega} = i\left\{\left(\frac{1+p}{1-p}\right)\left(\frac{1+kp}{1-kp}\right)\left(\frac{1+k'p}{1-k'p}\right)\right\}^{\frac{1}{2}} \quad (3.14)$$

Again since,

$$\Omega = \log\left(\frac{v}{q}\right) + i\theta = \log\left(\frac{v}{qe^{-i\theta}}\right)$$

$$= \log\left(v\frac{dz}{dw}\right) = \log\left(v\frac{dz}{dp}\cdot\frac{dp}{dw}\right)$$

$$= \log\left(v\frac{dz}{dp}\cdot\frac{dp}{dt}\cdot\frac{dt}{dw}\right)$$

we obtain

$$e^{\Omega} = v\frac{dz}{dp}\frac{dp}{dt}\frac{dt}{dw} = v\frac{dz}{dp}\frac{(2-p^2)^2}{4p}\pi\cdot\frac{t^2-1}{2\sigma v}$$

Hence substituting t in terms of p,

$$e^{\Omega} = \frac{\pi}{2\sigma\gamma}\left(\frac{p^2-1}{p}\right)\frac{dz}{dp} \quad (3.15)$$

$\therefore$ from (3.14), $$dz = \frac{2\sigma\gamma}{\pi}\frac{p}{p^2-1}e^{\Omega}dp$$

Substituting e^{Ω} from (3.13) and integrating z from D to F we obtain

$$\int_D^F dz = -i\,\frac{2\sigma r}{\pi}\int_0^{\frac{1}{k}}\left(\frac{p}{1-p^2}\right)\left(\frac{1+p}{1-p}\right)^{\frac{1}{2}}\left(\frac{1+kp}{1-kp}\right)^{\frac{1}{2}}\left(\frac{1+k'p}{1-k'p}\right)^{\frac{1}{2}}dp \qquad (3.16)$$

i.e.

$$\int_r^{r-ia} dz = -i\,\frac{2\sigma r}{\pi}\int_0^{\frac{1}{k}}\left(\frac{p}{1-p^2}\right)\left(\frac{1+p}{1-p}\right)^{\frac{1}{2}}\left(\frac{1+kp}{1-kp}\right)^{\frac{1}{2}}\left(\frac{1+k'p}{1-k'p}\right)^{\frac{1}{2}}dp$$

Hence on integration

$$a = \frac{2\sigma r}{\pi}\int_0^{\frac{1}{k}}\left(\frac{p}{1-p^2}\right)\left(\frac{1+p}{1-p}\right)^{\frac{1}{2}}\left(\frac{1+kp}{1-kp}\right)^{\frac{1}{2}}\left(\frac{1+k'p}{1-k'p}\right)^{\frac{1}{2}}dp \qquad (3.17)$$

a = 0 requires the upper limit of the integral to vanish, i.e. k is infinite. And when k is infinite, we obtain from (3.13)

$$\Omega = -i\pi + \tfrac{1}{2}\log\left(\frac{1+p}{1-p}\right) + \tfrac{1}{2}\log\left(\frac{1+k'p}{1-k'p}\right)$$

which verifies the solution of the first problem of Chapter II (see 2.17).

Again, consider a point $O' = p^*$ in the p-plane. Here $p > 1$ and also $p > \frac{1}{k}$ (i.e. $kp > 1$) and $p < \frac{1}{k'}$, (i.e. $k'p < 1$).

Hence from (3.12) we obtain,

$$\Omega = -i\frac{\pi}{2} + \tfrac{1}{2}\log\left(\frac{1+p}{p-1}\right) + \tfrac{1}{2}\log\left(\frac{kp+1}{kp-1}\right) + \tfrac{1}{2}\log\left(\frac{1+k'p}{1-k'p}\right)$$

i.e.
$$e^{\Omega} = -i\left(\frac{p+1}{p-1}\right)^{\frac{1}{2}}\left(\frac{kp+1}{kp-1}\right)^{\frac{1}{2}}\left(\frac{k'p+1}{1-k'p}\right)^{\frac{1}{2}} \qquad (3.18)$$

Hence, as before,

$$\int_{O'}^{A} dz = i\,\frac{2\sigma r}{\pi}\int_{p^*}^{\frac{1}{k'}}\left(\frac{p}{1-p^2}\right)\left(\frac{1+p}{p-1}\right)^{\frac{1}{2}}\left(\frac{kp+1}{kp-1}\right)^{\frac{1}{2}}\left(\frac{1+k'p}{1-k'p}\right)^{\frac{1}{2}}dp$$

Since O' is the origin and A is $\left[0,\ -i(h+a)\right]$ in the physical plane we obtain on integration,

$$h+a=\frac{2\sigma_r}{\pi}\int_{p^*}^{\frac{1}{k'}}\left(\frac{p}{1-p^2}\right)\left(\frac{1+p}{p-1}\right)^{\frac{1}{2}}\left(\frac{kp+1}{kp-1}\right)^{\frac{1}{2}}\left(\frac{1+k'p}{1-k'p}\right)^{\frac{1}{2}}dp$$

when $h \to \infty$, R.H.S. should tend to infinity which means that $\frac{1}{k'} \to \infty$

i.e. $k' \to 0$.

But, $\quad \frac{1}{k'}=\sqrt{\frac{2\delta}{1+\delta}}$

Hence, if $k' \to 0$, $\quad 1+\delta=0$

i.e. $\delta=-1$

i.e. the point A coincides with B and C.

Hence when $k'=0$, we obtain from (3.13)

$$\Omega=-\frac{3}{2}\pi i+\frac{1}{2}\log\left(\frac{1+p}{1-p}\right)+\frac{1}{2}\log\left(\frac{1+kp}{1-kp}\right) \qquad \text{c.f. (2.35)}$$

Hence it has been verified that if the neck is withdrawn, the problem reduces to Levy's and if the wall is withdrawn, it reduces to Hachemeister's. This verification also establishes the correctness of the result of this new problem.

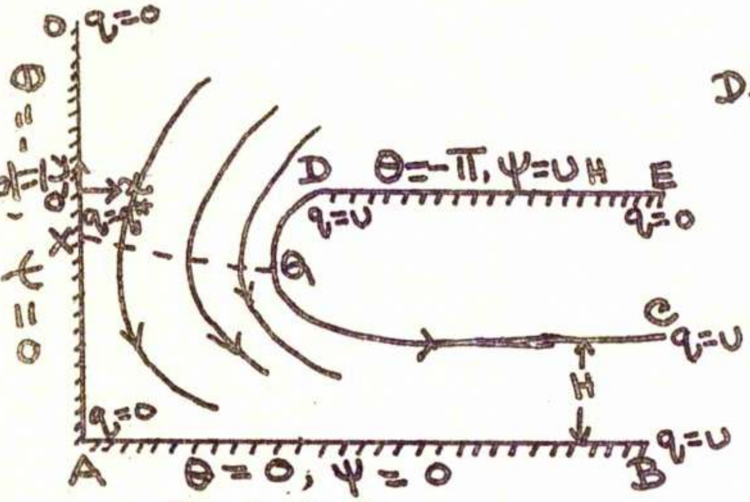
We can find out the complex potential w from (3.13) and (3.03) in the same way as has been obtained in Chapter II. But since the form of p is very complicated in this case it is found desirable to close this chapter at this stage.

CHAPTER IV

JET THROUGH A SLIT IMPINGING ON A WALL AND A FLOW THROUGH A NECKED SLIT BY A HODOGRAPH METHOD

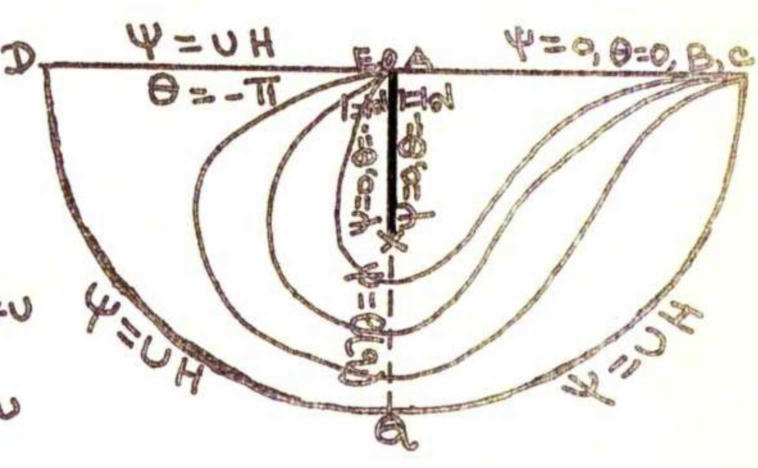
1. Jet through a slit impinging on a wall

The physical and hodograph planes are shown in Figures 1 and 2. It is sufficient to consider only one half of the plane because of symmetry.



Physical plane

FIGURE 1



Hodograph plane

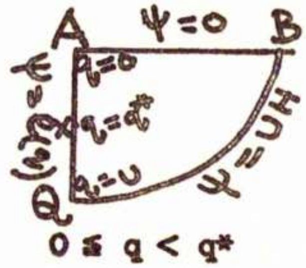
FIGURE 2

The flow is governed by the equation

$$\frac{\partial^2 \psi}{\partial q^2} + \frac{1}{q}\frac{\partial \psi}{\partial q} + \frac{1}{q^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad (4.01)$$

Our problem may formally be stated as the search of the solution of equation (4.01) subject to the boundary conditions for the portion ABCQ and AQDE. The boundary conditions in the hodograph plane are $\psi = 0$ on AB ($\theta = 0$), $\psi = U\,H$ on ED ($\theta = -\pi$) and $\psi = UH$ on DQC where U is the velocity of the jet at infinity downstream and H is the height of B above C and $\psi = 0$ on OXA $(\theta = -\pi/2)$ $\psi = f(q)$ on XQ $(\theta = -\pi/2)$.

We will first consider the portion ABCQ with the following boundary conditions imposed on it.



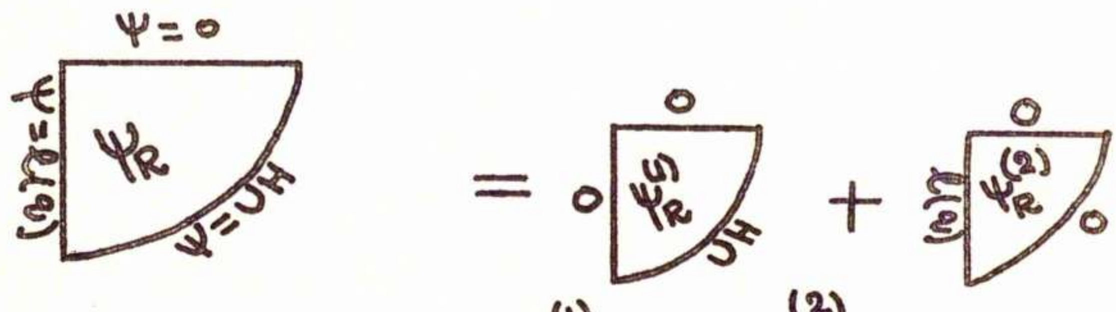
That $\psi = 0$ on $\theta = 0$

$$\psi = UH, \quad (-\pi/2 < \theta < 0) \tag{4.02}$$

$$\psi = f(q) \text{ on } \theta = -\pi/2 \text{ with } f(q) = \begin{cases} 0, & 0 \leq q < q^* \\ f(q) & q^* < q < U \end{cases}$$

q^* being the velocity of the fluid at X.

In practice it is found convenient to obtain the solution into two parts which are superimposed to give the final result. In the first part we find the solution for $\psi = 0$ on AB, $\psi = 0$ on AQ and $\psi = UH$ on BQ and in the second part we find the solution for which $\psi = 0$ on AB, $\psi = 0$ on BQ and $\psi = f(q)$ on AQ with $f(q) = 0$ on AX. These two boundary value problems will be treated separately. We will call the combined solution ψ_R i.e. the value of ψ for the right hand side of the hodograph plane.



$$\psi_R = \psi_R^{(1)} + \psi_R^{(2)} \tag{4.03}$$

Solution for $\psi_R^{(1)}$

We require to find a function $\psi(q,\theta)$ which satisfies (4.01) and the following boundary conditions:-

(a) $\psi = 0$ on $\theta = 0$

(b) $\psi = 0$ on $\theta = -\frac{\pi}{2}$

(c) $\psi = UH$ when $q = U$ $\left(-\frac{\pi}{2} < \theta < 0\right)$

The most general solution of (4.01) is given by

$$\psi = \left[Aq^{n} + Bq^{-n}\right]\sin(n\theta + \varepsilon) \tag{4.04}$$

If it is to satisfy (a) and (b) then (4.04) must be of the form

$\psi = A\, q^{n}\sin n\theta$.

But, $\psi = 0$ when $\theta = -\pi/2$ suggests that n must be an even integer. Let $n = 2K$.

Hence, $\psi = A\, q^{2K}\sin 2K\theta$

Hence the most general solution of (4.01) satisfying (a) and (b) is

$$\psi = \sum_{1}^{\infty} A q^{2K} \sin 2K\theta \tag{4.05}$$

Again condition (c) requires that $\psi = UH$ when $q = U$.

$$\therefore\ UH = \sum A U^{2K} \sin 2K\theta \tag{4.06}$$

By the theory of Fourier sine series

$$\int_{-\pi/2}^{0} UH \sin 2K\theta = \int_{-\frac{\pi}{2}}^{0} A U^{2K} \sin^2 2K\theta\, d\theta$$

which gives, $A = 0$ if K is even

$= -\frac{4UH}{\pi}\frac{1}{U^{2K}}$ when K is odd

Let $K = 2n + 1$ where n is an integer.

From (4.05) $$\psi = \sum_{n=0}^{\infty} \frac{4UH}{\pi}\frac{1}{2n+1}\left(\frac{q}{U}\right)^{4n+2}\sin(4n+2)\theta \qquad (4.07)$$

i.e. $$\psi_R^{(1)} = -\frac{4UH}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{q}{U}\right)^{4n+2}\sin(4n+2)\theta$$

But, $$\sum_{0}^{\infty}\frac{1}{2n+1}\left(\frac{q}{U}\right)^{4n+2}\sin(4n+2)\theta$$
$$= \text{Im. part, of } \sum_{0}^{\infty}\frac{1}{2n+1}\left(\frac{q}{U}\right)^{4n+2} e^{i(4n+2)\theta}$$
$$= \frac{1}{2}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\} \text{ where } q = UM \qquad (4.08)$$

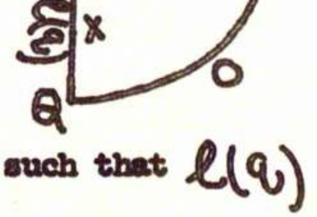
Substituting this value in (4.07) we obtain,

$$\psi_R^{(1)} = -\frac{2UH}{\pi}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\} \qquad (4.09)$$

Solution for $\psi_R^{(2)}$

In this problem we seek the solution of (4.01) with the following boundary conditions imposed upon it:-

$\psi = 0$, on $\theta = 0$



$\psi = 0$, $-\frac{\pi}{2} < \theta < 0$

$\psi = \ell(q)$ on $\theta = -\frac{\pi}{2}$ such that $\ell(q) = \begin{cases} 0 \text{ on } AX \\ \neq 0 \text{ on } XQ \end{cases}$

Finite Fourier Transforms are used to transform (4.01) with respect to θ. The resulting ordinary differential equation in q is solved by finding the Green's Function to give the transform of the unknown function $\psi(q,\theta)$. The required solution of (4.01) is finally obtained from the inverse transform.

Define,
$$\Psi(q,n) = \int_{-\frac{\pi}{2}}^{0} \psi(q,\theta)\sin 2n\theta \, d\theta \qquad (4.10)$$

so that
$$\psi(q,\theta) = \frac{4}{\pi}\sum \Psi(q,n)\sin 2n\theta \qquad (4.11)$$

Multiplying (4.01) by sin $2n\theta$ and integrating with respect to θ from $\theta = -\pi/2$ to $\theta = 0$ we obtain,

$$\int_{-\frac{\pi}{2}}^{0} \frac{\partial^2\psi}{\partial q^2}\sin 2n\theta \, d\theta + \int_{-\frac{\pi}{2}}^{0} \frac{1}{q}\frac{\partial\psi}{\partial q}\sin 2n\theta \, d\theta + \int_{-\frac{\pi}{2}}^{0} \frac{1}{q^2}\frac{\partial^2\psi}{\partial\theta^2}\sin 2n\theta \, d\theta = 0$$

Hence, we obtain the ordinary differential equation

$$\frac{d^2\Psi}{dq^2} + \frac{1}{q}\frac{d\Psi}{dq} - \frac{4n^2}{q^2}\Psi = -(-1)^n\frac{2n}{q^2}\ell(q)$$

which can be written in the form

$$\frac{d}{dq}\left[q\frac{d\Psi}{dq}\right] - \frac{4n^2}{q}\Psi = -(-1)^n\frac{2n}{q}\ell(q) \qquad (4.12)$$

The required Green's function is the solution $G(q\ q')$ of the equation

$$\frac{d}{dq}\left[q\frac{dG}{dq}\right] - \frac{4n^2}{q}G = \delta(q-q') \qquad (4.13)$$

which (i) is finite at q = 0

(ii) assumes the value zero on q = U

(iii) is continuous on q = q'

In (4.13), $\delta(q - q')$ is the Dirac Delta Function.

The most general solution of (4.13) which satisfies the condition (i) and (ii) is

$$G(q,q') = \begin{cases} A q^{2n} & q < q' \\ B\left[\left(\frac{q}{U}\right)^{2n} - \left(\frac{U}{q}\right)^{2n}\right] & q' < q \end{cases} \tag{4.14}$$

where A and B are constants.

We now choose A and B to fulfil (iii).

This requires that

$$A q'^{2n} = B\left[\left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n}\right] \tag{4.15}$$

Also on integration of (4.13) from q = q' - 0 to q = q' + 0, the continuity of G requires that

$$\left[q\frac{dG}{dq}\right]_{q'-0}^{q'+0} = 1$$

i.e.
$$G_2' - G_1' = \frac{1}{q'} \tag{4.16}$$

Hence from (4.14) and (4.16) we obtain

$$B\left[2n\left(\frac{q'}{U}\right)^{2n-1}\left(\frac{1}{U}\right) - 2n\left(\frac{U}{q'}\right)^{2n-1}\left(-\frac{U}{q'^2}\right)\right] - 2nAq'^{2n-1} = \frac{1}{q'} \tag{4.17}$$

Hence solving (4.17) and (4.16) we obtain

$$A = \frac{1}{4n}\frac{1}{U^{2n}}\left[\left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n}\right] \tag{4.18}$$

$$B = \frac{1}{4n}\left(\frac{q'}{U}\right)^{2n}$$

Substituting the values of A and B from (4.18) in (4.14) we obtain the required Green's function of (4.12) namely,

$$G(q,q') = \begin{cases} \frac{1}{4n}\left(\frac{q}{U}\right)^{2n}\left[\left(\frac{q'}{U}\right)^{2n}-\left(\frac{U}{q'}\right)^{2n}\right] & q<q' \\ \frac{1}{4n}\left(\frac{q'}{U}\right)^{2n}\left[\left(\frac{q}{U}\right)^{2n}-\left(\frac{U}{q}\right)^{2n}\right] & q'<q \end{cases} \qquad (4.19)$$

If we multiply (4.12) by $G(q,q')$ and (4.13) by $\Psi(q)$ and subtract, then on integration with respect to q from q = 0 to q = U, we have after making use of boundary values of G and Ψ

$$\Psi(q') = (-1)\int_0^U (-1)^n \frac{2n}{q}\,\ell(q)\,G(q\,q')\,dq' \qquad (4.20)$$

Thus on interchanging q and q' we may obtain the required solution of (4.12) by evaluating the integral

$$\Psi(q) = (-1)\int_0^U (-1)^n \frac{2n}{q'}\,\ell(q')\,G(q\,q')\,dq' \qquad (4.21)$$

We have reached a stage where we must specify $\ell(q')$ more fully, we have to consider the following situations for $\ell(q')$

$$\ell(q) = \begin{cases} 0 & \text{when } q<q^* \\ \neq 0 & \text{when } q^*<q \end{cases} \qquad (4.22)$$

Also we have the following range of integration:-

$$\begin{array}{ll} \text{(a)} & 0<q'<q^*<U \\ \text{(b)} & 0<q^*<q'<U \end{array} \qquad (4.23)$$

So, from (4.21) and from (4.23) when $q < q^*$

$$\Psi(q) = -\int_{q^*}^{U} (-1)^n \frac{2n}{q'} \ell(q') \frac{1}{4n} \left(\frac{q}{U}\right)^{2n} \left[\left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n}\right] dq' \tag{4.24}$$

and, when $q > q^*$

$$\Psi(q) = -\int_{q^*}^{q} \frac{(-1)^n}{2} \frac{\ell(q')}{q'} \left(\frac{q'}{U}\right)^{2n} \left[\left(\frac{q}{U}\right)^{2n} - \left(\frac{U}{q}\right)^{2n}\right] dq' - \int_{q}^{U} \frac{(-1)^n}{2} \frac{\ell(q')}{q'} \left(\frac{q}{U}\right)^{2n} \left[\left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n}\right] dq' \tag{4.25}$$

Since we are not interested when $q < q^*$, we will consider the case when $q > q^*$.

Hence from (4.11) and (4.25) we obtain

$$\psi_R^{(2)} = \psi(q, \theta) = \begin{array}{l} -\dfrac{4}{\pi} \displaystyle\sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{2} \left[\left(\frac{q}{U}\right)^{2n} - \left(\frac{U}{q}\right)^{2n}\right] \int_{q^*}^{q} \left(\frac{q'}{U}\right)^{2n} \frac{\ell(q')}{q'} dq' \right\} \sin 2n\theta \\ -\dfrac{4}{\pi} \displaystyle\sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{2} \left(\frac{q}{U}\right)^{2n} \int_{q}^{U} \left[\left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n}\right] \frac{\ell(q')}{q'} dq' \right\} \sin 2n\theta \end{array} \tag{4.26}$$

It can be easily proved that the infinite series in (4.26) are uniformly convergent. Hence changing the order of integration and summation we obtain from (4.26)

$$\psi_R^{(2)} = \begin{array}{l} -\dfrac{4}{\pi} \displaystyle\int_{q^*}^{q} \frac{\ell(q')}{2q'} \left[\sum_{0}^{\infty} \left(\frac{q'}{U}\right)^{2n} \sin 2n\theta (-1)^n \left\{ \left(\frac{q}{U}\right)^{2n} - \left(\frac{U}{q}\right)^{2n} \right\} \right] dq' \\ -\dfrac{4}{\pi} \displaystyle\int_{q}^{U} \frac{\ell(q')}{2q'} \left[\sum_{0}^{\infty} (-1)^n \sin 2n\theta \left(\frac{q}{U}\right)^{2n} \left\{ \left(\frac{q'}{U}\right)^{2n} - \left(\frac{U}{q'}\right)^{2n} \right\} \right] dq' \end{array} \tag{4.27}$$

But

$$\sum_0^\infty \left(\frac{q'}{U}\right)^{2n}(-1)^n \sin 2n\theta\left\{\left(\frac{q}{U}\right)^{2n}-\left(\frac{U}{q}\right)^{2n}\right\}$$

$$= \text{Im. p. of } \sum_0^\infty (-1)^n e^{i2n\theta}\left\{\left(\frac{qq'}{U^2}\right)^{2n}-\left(\frac{q'}{q}\right)^{2n}\right\}$$

$$= \left[\frac{q^2q'^2}{q^4+2q^2q'^2\cos 2\theta+q'^4}-\frac{q^2q'^2U^4}{U^8+2q^2q'^2U^4\cos 2\theta+q^4q'^4}\right]\sin 2\theta \tag{4.28}$$

Similarly,

$$\sum_0^\infty (-1)^n \sin 2n\theta\left(\frac{q}{U}\right)^{2n}\left\{\left(\frac{q'}{U}\right)^{2n}-\left(\frac{U}{q'}\right)^{2n}\right\}$$

$$= \left[\frac{q^2q'^2}{q^4+2q^2q'^2\cos 2\theta+q'^4}-\frac{q^2q'^2U^4}{U^8+2q^2q'^2U^4\cos 2\theta+q^4q'^4}\right]\sin 2\theta \tag{4.29}$$

Substituting the values of (4.28) and (4.29) and writing $M = \frac{q}{U}$, $M' = \frac{q'}{U}$ and $M^* = \frac{q^*}{U}$ we obtain from (4.27)

$$\psi_R^{(2)} = -\frac{2}{\pi}\int_{M^*}^1 \ell(M')\frac{(M^2\sin 2\theta)M'dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4}+\frac{2}{\pi}\int_{M^*}^1 \ell(M')\frac{(M^2\sin 2\theta)M'dM'}{M^4M'^4+2M^2M'^2\cos 2\theta+1} \tag{4.30}$$

Again since, $\psi_R = \psi_R^{(1)} + \psi_R^{(2)}$ if we set $\ell(M) = UHL(M)$ we obtain from (4.09) and (4.30)

$$\psi_R = -\frac{2UH}{\pi}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\}-\frac{2UH}{\pi}\int_{M^*}^1 L(M')\frac{(M^2\sin 2\theta)M'dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4} \tag{4.31}$$

$$+\frac{2UH}{\pi}\int_{M^*}^1 L(M')\frac{(M^2\sin 2\theta)M'dM'}{1+2M^2M'^2\cos 2\theta+M^4M'^4}$$

But,

$$\int_{M^*}^1 L(M')\frac{(M^2\sin 2\theta)M'dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4} = \int_{M^*}^1 L(M')\frac{(M^2\sin 2\theta)M'dM'}{(M^2\cos 2\theta+M'^2)^2+(M^2\sin 2\theta)^2}$$

Integrating by parts and remembering that we are working in the <u>fourth quadrant</u>, we obtain

$$\int_{M^*}^{1} L(M') \frac{(M^2 \sin 2\theta) M' \, dM'}{M^4 + 2M^2 M'^2 \cos 2\theta + M'^4}$$

$$= \left[-\frac{1}{2} L(M') \tan^{-1}\left\{\frac{M'^2 + M^2\cos 2\theta}{-M^2 \sin 2\theta}\right\}\right]_{M^*}^{1} + \frac{1}{2}\int_{M^*}^{1} \frac{dL}{dM'} \tan^{-1}\left\{\frac{M'^2+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} dM' \qquad (4.32)$$

$$= -\frac{1}{2} \tan^{-1}\left\{\frac{1+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} + \frac{1}{2}\int_{M^*}^{1} \frac{dL}{dM'}\tan^{-1}\left\{\frac{M'^2+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} dM'$$

Similarly,

$$\int_{M^*}^{1} L(M') \frac{(M^2 \sin 2\theta) M' \, dM'}{1 + 2M^2 M'^2 \cos 2\theta + M^4 M'^4}$$

$$= -\frac{1}{2} \tan^{-1}\left\{\frac{M^2+\cos 2\theta}{-\sin 2\theta}\right\} + \frac{1}{2}\int_{M^*}^{1} \frac{dL}{dM'}\tan^{-1}\left\{\frac{M^2 M'^2+\cos 2\theta}{-\sin 2\theta}\right\} dM' \qquad (4.33)$$

Hence substituting the values from (4.32) and (4.33) in (4.31) we get

$$\psi_R = \begin{array}{l} -\frac{2UH}{\pi}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\} + \frac{UH}{\pi}\tan^{-1}\left\{\frac{1+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} \\ -\frac{UH}{\pi}\tan^{-1}\left\{\frac{M^2+\cos 2\theta}{-\sin 2\theta}\right\} - \frac{UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M'^2+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} dM' \\ \qquad + \frac{UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M^2M'^2+\cos 2\theta}{-\sin 2\theta}\right\} dM' \end{array} \qquad (4.34)$$

We shall later need the partial derivative of this expression with respect to θ.

Differentiating with respect to θ we obtain,

$$\frac{\partial \psi_R}{\partial \theta} = \begin{array}{l} -\frac{8UH}{\pi}\frac{M^2(1-M^4)\cos 2\theta}{(1-M^4)^2+4M^4\sin^2 2\theta} - \frac{2UH}{\pi}\frac{(1-M^4)}{1+2M^2\cos 2\theta+M^4} \\ -\frac{2UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{M^2(M^2+M'^2\cos 2\theta)\,dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4} \\ \qquad + \frac{2UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{(1+M^2M'^2\cos 2\theta)\,dM'}{1+2M^2M'^2\cos 2\theta+M^4M'^4} \end{array} \qquad (4.35)$$

We will now proceed to work with the left half of the hodograph plane and obtain similar expressions for ψ_L and its derivative.

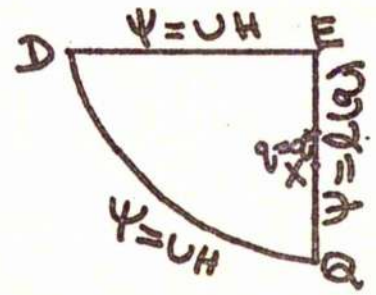
<u>Solution for the left half of the hodograph plane</u>

For the left half of the hodograph plane we have the following boundary conditions:-

$\psi = UH$ on $\theta = -\pi$

$\psi = \ell(q)$ on $\theta = -\pi/2$

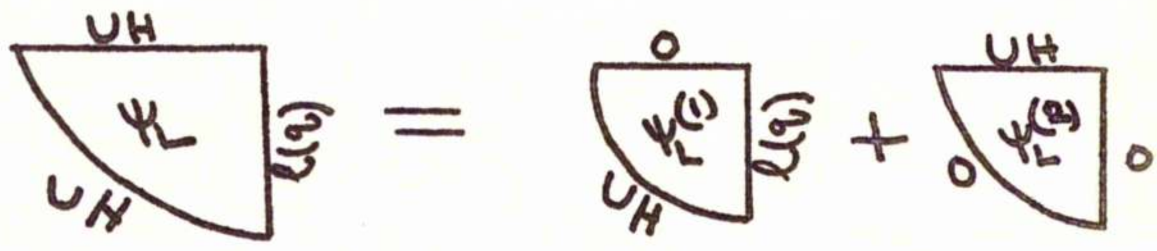
and $\psi = UH$ when $-\pi < \theta < -\pi/2$



(4.36)

Here also it is found convenient to obtain the solution into two parts which are then superimposed to give the final result. In the first part we find a solution for which $\psi = 0$ on DE, $\psi = \ell(q)$ on EQ and $\psi = UH$ on DQ. In the second part we seek a solution for which $\psi = UH$ on DE, $\psi = 0$ on EQ and $\psi = 0$ on DQ.

These two boundary value problems will be treated separately. We will call the combined solution as ψ_L i.e. the value of ψ for the left half of the hodograph plane.



$$\psi_L = \psi_L^{(1)} + \psi_L^{(2)} \qquad (4.37)$$

Solution of $\psi_L^{(1)}$

Here the boundary conditions of ψ are as follows:-

$$\psi = 0 \quad \text{on} \quad \theta = -\pi$$
$$\psi = \ell(q) \text{ on } \theta = -\pi/2 \tag{4.38}$$
$$\text{and} \quad \psi = UH \quad \text{when } -\pi < \theta < -\pi/2$$

But we have already solved the problem for which ψ satisfies the following conditions (see the solution of ψ_R i.e. (4.02))

$$\psi = 0 \quad \text{on} \quad \theta = 0$$
$$\psi = \ell(q) \text{ on } \theta = -\pi/2 \tag{4.39}$$
$$\text{and} \quad \psi = UH, \quad -\pi/2 < \theta < 0$$

If we replace θ by $-\pi - \theta$ in (4.39) we obtain (4.38). Thus, if we replace θ by $-\pi - \theta$ in the solution of ψ_R we will obtain the solution of $\psi_L^{(1)}$.

Hence from (4.34)

$$\psi_L^{(1)} = \begin{aligned}[t] & \frac{2UH}{\pi}\tan^{-1}\left\{\frac{2\mu^2\sin 2\theta}{1-\mu^4}\right\} + \frac{UH}{\pi}\tan^{-1}\left\{\frac{1+\mu^2\cos 2\theta}{\mu^2\sin 2\theta}\right\} - \frac{UH}{\pi}\tan^{-1}\left\{\frac{\mu^2+\cos 2\theta}{\sin 2\theta}\right\} \\ & - \frac{UH}{\pi}\int_{\mu^*}^{1}\frac{dL}{d\mu'}\tan^{-1}\left\{\frac{\mu'^2+\mu^2\cos 2\theta}{\mu^2\sin 2\theta}\right\}d\mu' + \frac{UH}{\pi}\int_{\mu^*}^{1}\frac{dL}{d\mu'}\tan^{-1}\left\{\frac{\mu^2\mu'^2+\cos 2\theta}{\sin 2\theta}\right\}d\mu' \end{aligned} \tag{4.40}$$

Solution of $\psi_L^{(2)}$

Here we have the following boundary conditions:-

(i) $\psi = UH$ on $\theta = -\pi$

(ii) $\psi = 0$ on $\theta = -\pi/2$

(iii) $\psi = 0$ when $q = U$ $(-\pi < \theta < -\pi/2)$

The most general solution satisfying (i) and (ii) is

$$\psi = -\frac{2UH}{\pi}\left(\theta+\frac{\pi}{2}\right)+\sum a_n q^{2n}\sin 2n\theta \qquad (4.41)$$

The Condition (iii) requires that $\psi = 0$ when $q = U$

whence, $$\frac{2UH}{\pi}\left(\theta+\frac{\pi}{2}\right)=\sum a_n U^{2n}\sin 2n\theta$$

By the theory of Fourier Sine Series we obtain,

$$a_n = -\frac{2UH}{\pi n}\frac{1}{U^{2n}}$$

Hence from (4.41) we get

$$\psi_L^{(2)} = -\frac{2UH}{\pi}\left(\theta+\frac{\pi}{2}\right)-\frac{2UH}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{q}{U}\right)^{2n}\sin 2n\theta$$

But, it can be easily shown that

$$\sum\frac{1}{n}\left(\frac{q}{U}\right)^{2n}\sin 2n\theta = \tan^{-1}\left\{\frac{M^2\sin 2\theta}{1-M^2\cos 2\theta}\right\}$$ where, as before, $\frac{q}{U}=M$

Hence,

$$\psi_L^{(2)} = -\frac{2UH}{\pi}\left(\theta+\frac{\pi}{2}\right)-\frac{2UH}{\pi}\tan^{-1}\left\{\frac{M^2\sin 2\theta}{1-M^2\cos 2\theta}\right\} \qquad (4.42)$$

Hence from (4.37), (4.40) and (4.42) we obtain

$$\frac{2UH}{\pi}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\}+\frac{UH}{\pi}\tan^{-1}\left\{\frac{1+M^2\cos 2\theta}{M^2\sin 2\theta}\right\}$$

$$\psi_L = -\frac{2UH}{\pi}\left(\theta+\frac{\pi}{2}\right)-\frac{UH}{\pi}\tan^{-1}\left\{\frac{M^2+\cos 2\theta}{\cancel{1-}M^2\sin 2\theta}\right\}$$
$$-\frac{2UH}{\pi}\tan^{-1}\left\{\frac{M^2\sin 2\theta}{1-M^2\cos 2\theta}\right\}-\frac{UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M'^2+M^2\cos 2\theta}{M^2\sin 2\theta}\right\}dM' \qquad (4.43)$$
$$+\frac{UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M^2M'^2+\cos 2\theta}{\sin 2\theta}\right\}dM'$$

Differentiating with respect to θ we obtain,

$$
\begin{aligned}
\frac{\partial\psi_L}{\partial\theta} = & -\frac{2UH}{\pi} + \frac{8UH}{\pi}\frac{M^2(1-M^4)\cos 2\theta}{(1-M^4)^2+4M^2\sin^2 2\theta} - \frac{2UH}{\pi}\frac{M^2(M^2+\cos 2\theta)}{1+2M^2\cos 2\theta+M^4} \\
& + \frac{2UH}{\pi}\frac{1+M^2\cos 2\theta}{1+2M^2\cos 2\theta+M^4} - \frac{4UH}{\pi}\frac{M^2(\cos 2\theta - M^2)}{1-2M^2\cos 2\theta+M^4} \\
& + \frac{2UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{M^2(M^2+M'^2\cos 2\theta)\,dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4} - \frac{2UH}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{1+M^2M'^2\cos 2\theta}{1+2M^2M'^2\cos 2\theta+M^4M'^4}\,dM'
\end{aligned}
\qquad (4.44)
$$

At this stage we shall state that since ψ satisfies Laplace's equation, its partial derivatives must be continuous across the line XQ in the hodograph plane and on this line we must have

$$
\left(\frac{\partial\psi_R}{\partial\theta}\right)_{\theta=-\frac{\pi}{2}} = \left(\frac{\partial\psi_L}{\partial\theta}\right)_{\theta=-\frac{\pi}{2}} \qquad (4.45)
$$

The first integral on the right hand side of each of (4.35) and (4.44) is singular when $\theta = -\pi/2$. To avoid this singularity we will deform the contour of this integral into the complex M' plane before letting θ tend to $-\pi/2$. Then we will make use of the result given by (4.45).

Thus to find,

$$
I = \lim_{\theta\to-\frac{\pi}{2}}\int_{M^*}^{1}\frac{dL}{dM'}\frac{M^2(M^2+M'^2\cos 2\theta)\,dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4}
$$

we will first consider the expression

$$
\begin{aligned}
E = \frac{M^2(M^2+M'^2\cos 2\theta)}{M^4+2M^2M'^2\cos 2\theta+M'^4} &= \frac{1}{2}\frac{M^2e^{-i2\theta}}{M'^2+M^2e^{-i2\theta}} + \frac{1}{2}\frac{M^2e^{i2\theta}}{M'^2+M^2e^{i2\theta}} \\
&= \frac{1}{2}\frac{t^2}{M'^2+t^2} + \frac{1}{2}\frac{\bar{t}^2}{M'^2+\bar{t}^2} \quad \text{where } t = Me^{-i\theta},\ \bar{t} = Me^{i\theta} \\
&= -\frac{1}{2}\frac{T^2}{M'^2-T^2} - \frac{1}{2}\frac{\bar{T}^2}{M'^2-\bar{T}^2} \quad \text{where } T = -it,\ i\bar{t} = \bar{T}
\end{aligned}
$$

Hence,

$$I = \lim_{\theta \to -\frac{\pi}{2}} -\frac{1}{2}\int_{M^*}^{1} \frac{dL}{dM'} \left\{ \frac{T^2}{M'^2 - T^2} + \frac{\bar{T}^2}{M'^2 - \bar{T}^2} \right\} dM'$$

$$= -\frac{1}{2}\left[I_1 + I_2 \right] \qquad (4.46)$$

where

$$I_1 = \lim_{\theta \to -\pi/2} \int_{M^*}^{1} \frac{dL}{dM'} \frac{T^2}{M'^2 - T^2} dM'$$

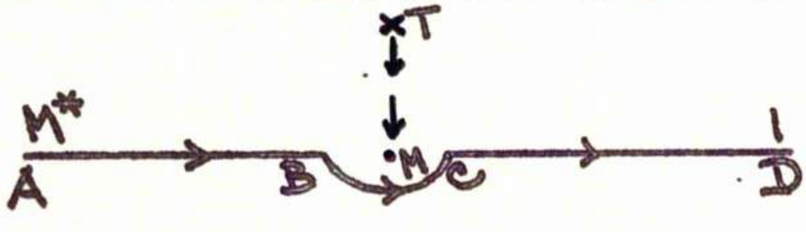
$$I_2 = \lim_{\theta \to -\frac{\pi}{2}} \int_{M^*}^{1} \frac{dL}{dM'} \frac{\bar{T}^2}{M'^2 - \bar{T}^2} dM'$$

First we will consider the integral I_1.

when $\theta \to -\frac{\pi}{2}$, $e^{-i\theta} \to i$, $e^{i\theta} \to -i$

and hence, $t \to Mi$, $\bar{t} \to -iM$; $T \to M$, $\bar{T} \to M$ (4.47)

The singularity of I_1 is at $M' = T$ ($= M$ where M is real). Let us deform the contour by a semi-circular indentation below the real axis when T approaches M from above. Thus the integral is regular on the new contour. This is quite clear if we look at the drawing in the M'-plane.



<u>M'-plane</u>

Thus,

$$I_1 = \lim_{\theta \to -\frac{\pi}{2}} \int_{M^*}^{1} \frac{dL}{dM'} \frac{T^2}{M'^2 - T^2} dM' = \lim_{\theta \to -\frac{\pi}{2}} \left[\int_A^B (\ \ ) dM' + \int_B^C (\ \ ) dM' + \int_C^D (\ \ ) dM' \right]$$

$$= \int_A^D \frac{dL}{dM'} \frac{M^2 dM'}{M'^2 - M^2} + \lim_{\theta \to -\frac{\pi}{2}} \int_B^C \frac{dL}{dM'} \frac{T^2}{M'^2 - T^2} dM'$$

since $T = M$ on the real axis.

$$I_1 = \oint_{M^*}^{1} \frac{dL}{dM'} \frac{M^2}{M'^2 - M^2} dM' + \frac{1}{2} \cdot 2\pi i \lim_{\theta \to -\frac{\pi}{2}} \text{[residue of the integral from B to C at } M' = T]$$

$\oint$ denotes the Cauchy Principal Value. The positive sign in the second term of the right hand side is due to the fact that here the sense of the contour is anti-clockwise and we multiply the value of the integral by $\frac{1}{2}$ since the contour here is a semi-circle.

Now residue of

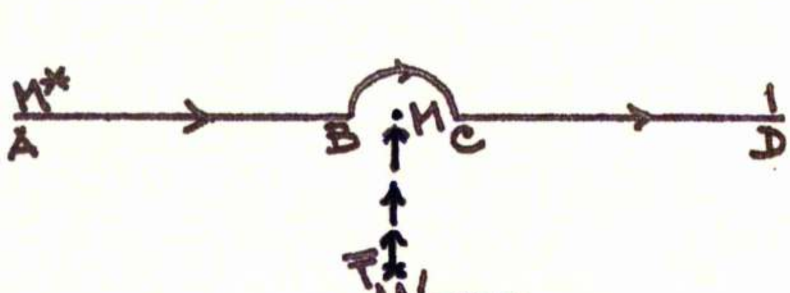
$$\left(\frac{dL}{dM'} \frac{T^2}{M'^2 - T^2}\right)_{M'=T} = \left(\frac{dL}{dM'}\right)_{M'=T} \frac{T^2}{2T} = \left(\frac{dL}{dM'}\right)_{M'=T} \cdot \frac{T}{2}$$

$$= \frac{M}{2} \left(\frac{dL}{dM'}\right)_{M'=T}, \text{ since } T = M \text{ when } \theta \to -\frac{\pi}{2}$$

Hence

$$I_1 = \oint_{M^*}^{1} \left(\frac{dL}{dM'}\right) \frac{M^2 dM'}{M'^2 - M^2} + \frac{\pi i}{2} M \left(\frac{dL}{dM'}\right)_{M'=T} \tag{4.48}$$

Similarly, to find, $I_2 = \lim_{\theta \to -\pi/2} \int_{M^*}^{1} \left(\frac{dL}{dM'}\right) \frac{\bar{T}^2}{M'^2 - \bar{T}^2} dM'$ we deform the contour by means of a semi-circular indentation above the real axis as $\bar{T}$ approaches M from below. The integral will now be regular on this new contour. This will be clear from the figure drawn in the M'-plane.



M'-plane

Thus,

$$I_2 = \lim_{\theta \to -\frac{\pi}{2}} \int_{M^*}^{1} \left(\frac{dL}{dM'}\right) \frac{\bar{T}^2}{M'^2 - \bar{T}^2} dM' = \lim_{\theta \to -\pi/2} \left[\int_A^B (\;) dM' + \int_B^C (\;) dM' + \int_C^D (\;) dM' \right]$$

$$= \lim_{\theta \to -\frac{\pi}{2}} \oint_A^D \left(\frac{dL}{dM'}\right) \frac{\bar{T}^2}{M'^2 - \bar{T}^2} dM' + \lim_{\theta \to -\frac{\pi}{2}} \int_B^C \left(\frac{dL}{dM'}\right) \frac{\bar{T}^2}{M'^2 - \bar{T}^2} dM'$$

$$I_2 = \oint_{M^*}^{1} \frac{dL}{dM'} \frac{M^2 dM'}{M'^2 - M^2} - \frac{1}{2} 2\pi i \lim_{\theta \to -\frac{\pi}{2}}$$ [residue of the integral from B to C at $M' = \bar{T}$]

since $\bar{T} = M$ when $\theta = -\pi/2$.

The negative sign in the second term of the right hand side of I_2 is due to the fact that here the sense of the contour is clock-wise. Proceeding in the same way as was done for I_1 we find

$$I_2 = \oint_{M^*}^{1} \left(\frac{dL}{dM'}\right) \frac{M^2}{M'^2 - M^2} dM' - \frac{\pi i}{2} M \left(\frac{dL}{dM'}\right)_{M' = \bar{T}} \qquad (4.49)$$

Since $T = \bar{T}$ on the real axis, we obtain from (4.46), (4.48) and (4.49)

$$I = - \int_{M^*}^{1} \left(\frac{dL}{dM'}\right) \frac{M^2}{M'^2 - M^2} dM' \qquad (4.50)$$

Now applying (4.45), we obtain from (4.35), (4.44) and (4.50)

$$\int_{\mu^*}^{1}\left(\frac{dL}{d\mu'}\right)\frac{\mu'^2 d\mu'}{(\mu'^2-\mu^2)(1-\mu^2\mu'^2)} = \frac{1}{2}\frac{1}{(1+\mu^2)^2} \tag{4.51}$$

This is a singular integral equation, the singularity being at $\mu' = \mu$ since $\mu^* < \mu < 1$. The integral on the left is now to be interpreted as the Cauchy Principal Value. To solve (4.51) we proceed as follows:-

Let $\mu'^2 = p$ and $\mu^2 = t$

Substituting these values in (4.51) we obtain

$$\int_{\mu^{*2}}^{1}\sqrt{p}\,\frac{dL}{d\mu'}\frac{dp}{(p-t)(1-pt)} = \frac{1}{(1+t)^2}$$

Again writing $\sqrt{p}\,\dfrac{dL}{d\mu'} = Q(p)$ and $\mu^{*2} = p^*$ (4.52)

we get

$$\int_{p^*}^{1} Q(p)\frac{dp}{(p-t)(1-pt)} = \frac{1}{(1+t)^2} \tag{4.53}$$

This is not a standard integral equation owing to an extra factor in the denominator in L.H.S. But (4.53) can be brought into a standard integral equation (sometimes called the 'aerofoil equation') by the following device:-

Equation (4.53) can be written as

$$\int_{p^*}^{1} Q(p)\frac{dp}{pt\left[\frac{1+t^2}{t}-\frac{1+p^2}{p}\right]} = \frac{1}{(1+t)^2}$$

or $$\int_{p^*}^{1} \frac{Q(p)}{p} \frac{dp}{\frac{1+t^2}{t} - \frac{1+p^2}{p}} = \frac{t}{(1+t)^2}$$

or $$\int_{p^*}^{1} \frac{Q(p)}{p} \frac{dp}{\frac{1+t^2}{t} - \frac{1+p^2}{p}} = \frac{1}{\frac{1+t^2}{2}+2} \tag{4.54}$$

Let $\frac{1+p^2}{p} = u$ and $\frac{1+t^2}{t} = \gamma$ (4.55)

i.e. $$p = \frac{u \pm \sqrt{u^2-4}}{2}$$

When $$p = p^*, \quad u = \frac{1+p^{*2}}{p^*} > 2$$
$$p = 1, \quad u = 2$$

Hence when p varies from p* to 1 i.e. when p varies from a value less than 1 to +1, u varies from a value greater than 2 to 2, i.e. when p increases, u decreases, i.e. $\frac{dp}{du}$ must be negative.

Hence, we must take

$$p = \frac{u - \sqrt{u^2-4}}{2}$$

$$\therefore \frac{dp}{du} = -\frac{p}{\sqrt{u^2-4}} \tag{4.56}$$

i.e. $$\frac{dp}{p} = -\frac{du}{\sqrt{u^2-4}} \tag{4.57}$$

Also let, $Q(p) = G(u)$ (4.58)

Substituting the values from (4.55), (4.57) and (4.58) in (4.54) we obtain

$$\int_2^{\frac{1+p^{*2}}{p^*}} \frac{G(u)}{\sqrt{u^2-4}} \frac{du}{u-\gamma} = -\frac{1}{\gamma+2}$$

or

$$\int_2^{\frac{1+p^{*2}}{p^*}} \frac{K(u)}{u-\gamma} du = f(\gamma) \tag{4.59}$$

where $K(u) = \frac{G(u)}{\sqrt{u^2-4}}$ and $f(\gamma) = -\frac{1}{\gamma+2}$ (4.60)

Let $\frac{1+p^{*2}}{p^*} = \lambda$. Hence from (4.60) we obtain

$$\int_2^{\lambda} \frac{K(u)}{u-\gamma} du = f(\gamma) \tag{4.61}$$

This is a singular integral equation of the first kind. To solve this we refer to Integral Equations by Mikhlin, p. 131, Chapter III. To quote the result,

if, $$\frac{1}{\pi i}\int_\alpha^\beta \frac{\varphi(\xi)}{\xi-t} d\xi = f(t) \tag{4.62}$$

then
$$\varphi(t) = \frac{1}{\pi i\sqrt{(t-\alpha)(t-\beta)}}\int_\alpha^\beta \frac{\sqrt{(\xi-\alpha)(\xi-\beta)}}{\xi-t} f(\xi)d\xi + \frac{C}{\sqrt{(t-\alpha)(t-\beta)}}$$

where α and β are the beginning and end of the unclosed contour.

Comparing (4.61) and (4.62) we obtain

$$K(\gamma) = \frac{1}{\pi i\sqrt{(\gamma-2)(\gamma-\lambda)}}\int_2^\lambda \frac{\sqrt{(u-2)(u-\lambda)}}{u-\gamma} \frac{f(u)du}{\pi i} + \frac{C}{\sqrt{(\gamma-2)(\gamma-\lambda)}} \tag{4.63}$$

and since $\gamma < \lambda$ and $f(u) = -\frac{1}{u+2}$

we obtain,

$$k(r) = -\frac{1}{\pi^2(r+2)\sqrt{(r-2)(\lambda-r)}}\left[\int_2^\lambda \frac{\sqrt{(u-2)(\lambda-u)}}{u+2}du - \int_2^\lambda \frac{\sqrt{(u-2)(\lambda-u)}}{u-r}du\right] - \frac{ci}{\sqrt{(r-2)(\lambda-r)}} \tag{4.64}$$

By making substitution, $u-2=(\lambda-u)x^2$, we find

$$\int_2^\lambda \frac{\sqrt{(u-2)(\lambda-u)}}{u+2}du = \frac{\pi}{2}\left[6+\lambda-4\sqrt{\lambda+2}\right] \tag{4.65}$$

and by the same substitution and remembering that the second integral in (4.64) has a singularity at u = r we find from Cauchy principal value,

$$\int_2^\lambda \frac{\sqrt{(u-2)(\lambda-u)}}{u-r}du = \frac{\pi}{2}\left[\lambda+2-2r\right] \tag{4.66}$$

Hence from (4.64), (4.65) and (4.66) we obtain,

$$k(r) = -\frac{2+r-2\sqrt{\lambda+2}}{\pi(r+2)\sqrt{(r-2)(\lambda-r)}} - \frac{ci}{\sqrt{(r-2)(\lambda-r)}}$$

i.e.

$$k(u) = -\left[\frac{2+u-2\sqrt{\lambda+2}}{\pi(u+2)\sqrt{(u-2)(\lambda-u)}} + \frac{ci}{\sqrt{(u-2)(\lambda-u)}}\right]$$

From (4.60), we obtain

$$G(u) = -\left[\frac{2+u-2\sqrt{\lambda+2}+\pi ci(u+2)}{\pi\sqrt{(u+2)(\lambda-u)}}\right]$$

and from (4.55) and (4.58) we obtain

$$Q(p) = -\left[\frac{2+\frac{1+p^2}{p}-2\sqrt{\frac{1+p^{*2}}{p^*}+2}+\pi ci\left(\frac{1+p^2}{p}+2\right)}{\pi\sqrt{\frac{1+p^2}{p}+2}\sqrt{\frac{1+p^{*2}}{p^*}-\frac{1+p^2}{p}}}\right]$$

since $\lambda = \dfrac{1+p^{*2}}{p^*}$

$$= -\frac{\sqrt{p^*}(1+p)^2-2p(1+p^*)+\pi ci\sqrt{p^*}(1+p)^2}{\pi(1+p)\sqrt{(p-p^*)(1-pp^*)}}$$

Hence,

$$\frac{Q(p)}{\sqrt{p}} = -\frac{\sqrt{p^*}(1+p)^2 - 2p(1+p^*) + \alpha\sqrt{p^*}(1+p)^2}{\pi(1+p)\sqrt{p}\sqrt{(p-p^*)(1-pp^*)}}$$

where $\alpha = \pi c i$ (say) = constant

(4.67)

Now $$\frac{dL}{dp} = \frac{dL}{d\mu'}\frac{d\mu'}{dp} = \frac{Q(p)}{\sqrt{p}}\frac{1}{2\sqrt{p}} \text{ since } \mu'^2 = p$$

i.e. $$\frac{dL}{dp} = \frac{1}{2\sqrt{p}}\left[-\frac{\sqrt{p^*}(1+p)^2 - 2p(1+p^*) + \alpha\sqrt{p^*}(1+p)^2}{\pi(1+p)\sqrt{p}\sqrt{(p-p^*)(1-pp^*)}}\right]$$

Hence on integration,

$$L(p) = -\frac{1}{2\pi}\int\frac{(1+\alpha)\sqrt{p^*}(1+p)^2 - 2p(1+p^*)}{p(1+p)\sqrt{(p-p^*)(1-pp^*)}}dp + \gamma$$

where γ is a constant of integration

$$= -\frac{1}{2\pi}\int\left[(1+\alpha)\sqrt{p^*} + \frac{(1+\alpha)\sqrt{p^*}}{p} - \frac{2(1+p^*)}{1+p}\right]\frac{dp}{\sqrt{(p-p^*)(1-pp^*)}} + \gamma$$

Substituting, $p - p^* = (1 - pp^*)\,u^2$ and on integration we get

$$L(p) = \frac{2}{\pi}\tan^{-1}\left\{\sqrt{\frac{p-p^*}{1-pp^*}}\right\} - \frac{1+\alpha}{\pi}\tan^{-1}\left\{\frac{\sqrt{(p-p^*)(1-pp^*)}}{(1-p)\sqrt{p^*}}\right\} + \gamma \quad (4.68)$$

Again, since, $L(p^*) = 0$

and $L(1) = 1$

we find,

$$\alpha = -2\,,\ \gamma = 0$$

Hence $\quad C = -\frac{2}{\pi i}$ since $\alpha = \pi c i$

This is how the constant is evaluated. Substituting the value of α and γ in (4.69) we get

$$L(p) = \frac{2}{\pi}\tan^{-1}\left\{\sqrt{\frac{p-p^*}{1-pp^*}}\right\} + \frac{1}{\pi}\tan^{-1}\left\{\frac{\sqrt{(p-p^*)(1-pp^*)}}{(1-p)\sqrt{p^*}}\right\}$$

$$= \frac{1}{\pi}\tan^{-1}\left\{\frac{(1-p)\sqrt{(p-p^*)(1-pp^*)}}{\sqrt{p^*}(1+p)^2-2p(1+p^*)}\right\}$$

and since $p = M'^2$, $p^* = M^{*2}$, we obtain

$$L(M') = \frac{1}{\pi}\tan^{-1}\left\{\frac{(1-M'^2)\sqrt{(M'^2-M^{*2})(1-M'^2M^{*2})}}{M^*(1+M'^2)-2M'^2(1+M^{*2})}\right\} \qquad (4.69)$$

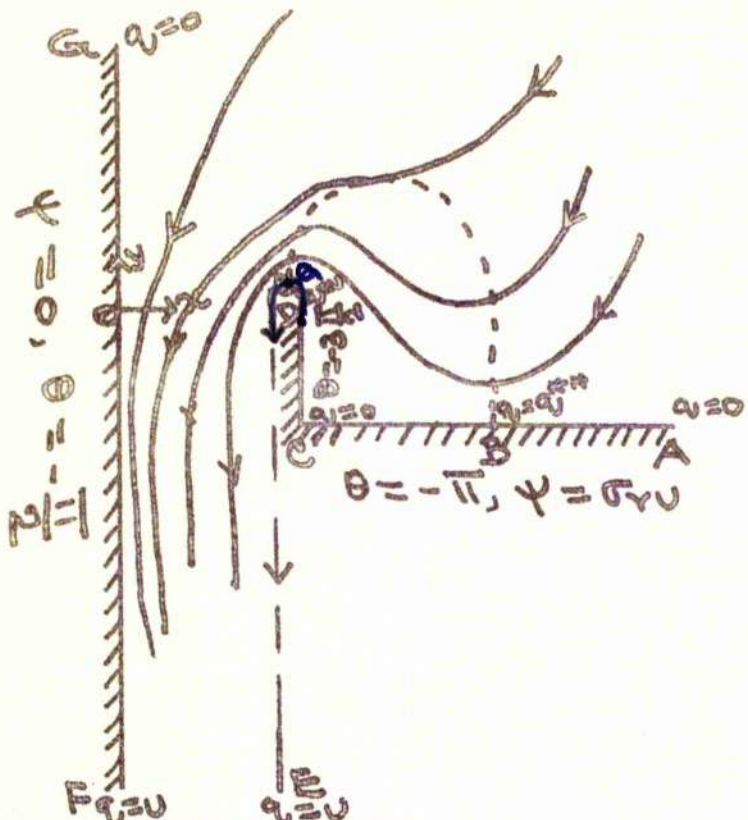
Also,

$$\frac{dL}{dM'} = \frac{Q(p)}{\sqrt{p}} = \frac{1}{\pi}\left[\frac{M^*(1+M'^2)^2+2M'^2(1+M^{*2})}{M'(1+M'^2)\sqrt{(M'^2-M^{*2})(1-M'^2M^{*2})}}\right] \qquad (4.70)$$

which agree with the result obtained by Schwarz-Christoffel method.

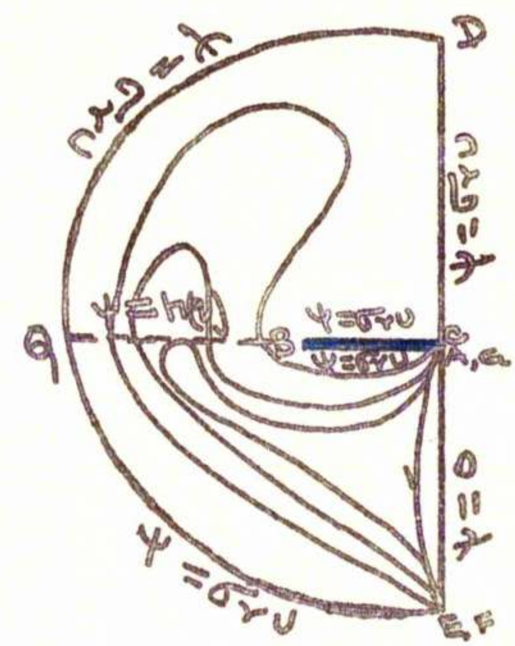
Flow through a necked slit

The physical and the hodograph planes are shown in Figures 3 and 4. Because of symmetry we consider only one half of the plane.



Physical plane

FIGURE 3



Hodograph plane

FIGURE 4

The physical plane has already been described in Chapter II. The boundary condition in the hodograph plane are $\psi = 0$ on GF $(\theta = -\pi/2)$, $\psi = \sigma r U$ on ABC $(\theta = -\pi)$, $\psi = \sigma r U$ on CD $(\theta = -3\pi/2)$, $\psi = \sigma r U$ on DQE where U is the velocity of the jet at infinity, $\psi = h(q)$ on BQ (locus $\theta = -\pi$).

The distinct features of the streamlines in this problem are that unlike Levy's problem, all the streamlines do not cross the line BQ. Moreover, the streamlines which cross the line BQ (locus $\theta = -\pi$), cross it twice. There is only one streamline which just touches the line BQ. Beyond this line, the other streamline neither touch nor cross the line BQ.

This problem though physically different from that of Levy's has been shown already to have the same mathematical character. Hence we will not solve this problem in details. We will show how to match the boundary conditions of this problem in the hodograph plane with Levy's and show that it must be derived from the same singular integral equation. The solutions may then be found directly from Levy's. We proceed to solve the problem as follows:-

For convenience, let $\sigma r U = K$ where K is a constant. The boundary conditions in the hodograph plane will be now $\psi = K$ on CD, DQE, AB and BC, $\psi = 0$ on GF, $\psi = h(q)$ on BQ. We will then take out K from each of the boundary values. This is allowed since K is constant. We will denote the source and sink by a dot and a cross. In fact, the following three diagrams represent mathematically equivalent flows.

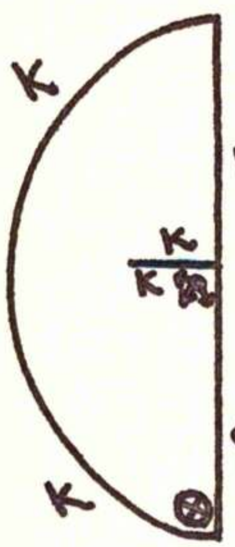


FIGURE 5

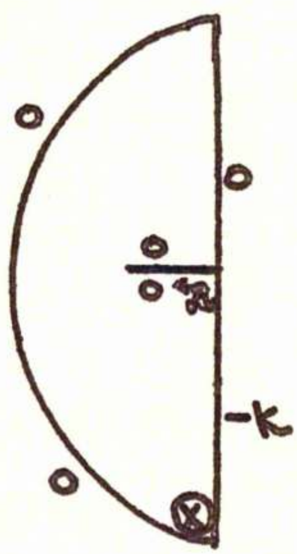


FIGURE 6

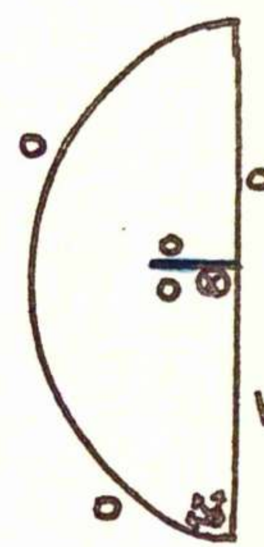


FIGURE 7

Figure 7 follows immediately from Figure 6. They differ only in the direction of flow of the fluid, since the source and sink in Figure 6 have been interchanged in Figure 7.

Again from Figure 7 we obtain,

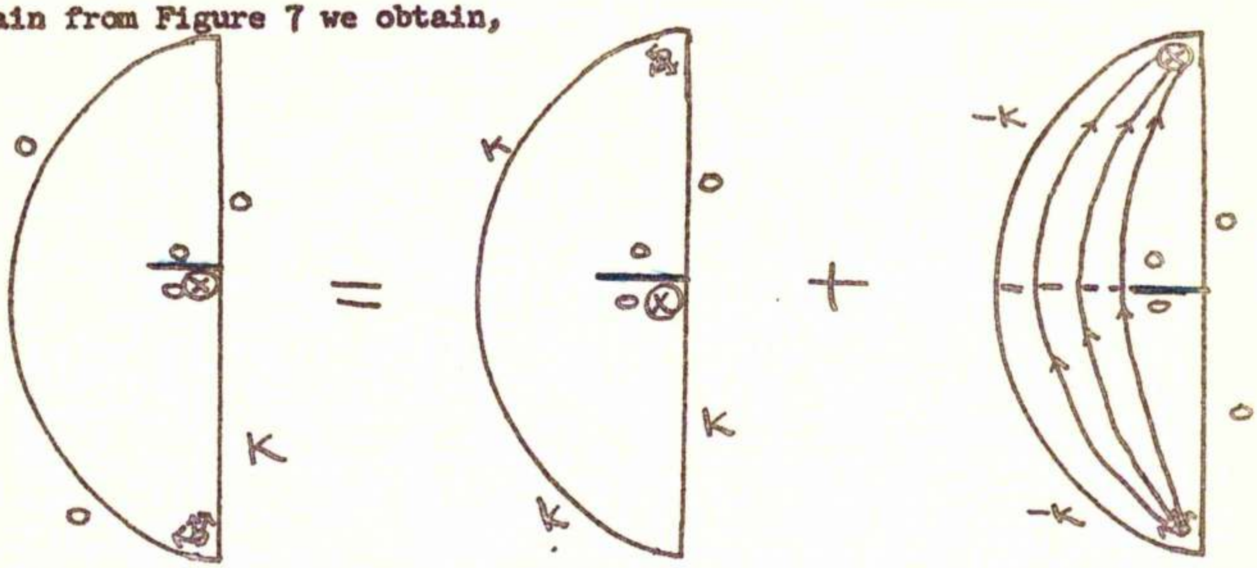


FIGURE 7 FIGURE 8 FIGURE 9

Figure 8 is exactly Levy's problem and has been solved previously. Figure 7 differs from Figure 8 by Figure 9. But Figure 9 represents a flow between source and sink symmetrical in the hodograph plane; hence the normal derivative $\frac{\partial\psi}{\partial\theta}$ is zero of the stream function on the dotted line (locus $\theta = -\pi$) in Figure 9. So we will get the same integral equation for this problem as has been obtained for Levy.

Thus, instead of (4.51) by setting $h(M) = \sigma r U\,[1 + H(M)]$ we obtain in this case

$$\int_{t^*}^{1} \frac{dH}{dM'} \frac{M'^2\,dM'}{(M'^2 - M^2)(1 - M^2 M'^2)} = \frac{1}{2}\frac{1}{(1+M^2)^2} \tag{4.71}$$

The solution is carried out exactly in the same way as was done in the previous example and since we have set $h(M) = \sigma r U\,[1 + H(M)]$, where $M = q/U$ we find by applying boundary conditions on $h(M)$ at Q (where $q = U$) and at B (where $q = q^{**}$) that $H(1) = H(t^*) = 0$, t^* being equal to $\frac{q^{**}}{U}$. These two values of H give $\alpha = \gamma = 0$ in (4.68). Hence from (4.68), instead of $L(p)$ we obtain in this case

$$H(p) = \frac{1}{\pi}\tan^{-1}\left\{-\frac{(1-p)\sqrt{(p-p^*)(1-pp^*)}}{(1+p)^2\sqrt{p^*}+2p(1+p^*)}\right\} \quad (4.72)$$

and since $p = M'^2$, $p^* = t^{*2}$, we obtain from (4.72)

$$H(M') = -\frac{1}{\pi}\left\{\tan^{-1}\frac{(1-M'^2)\sqrt{(M'^2-t^{*2})(1-M'^2t^{*2})}}{t^*(1+M'^2)^2+2M'^2(1+t^{*2})}\right\} \quad (4.73)$$

Similarly, we obtain from (4.67)

$$\frac{dH}{dM'} = \frac{Q(p)}{\sqrt{p}} = -\frac{1}{\pi}\left\{\frac{t^*(1+M'^2)^2-2M'^2(1+t^{*2})}{M'(1+M'^2)\sqrt{(M'^2-t^{*2})(1-M'^2t^{*2})}}\right\} \quad (4.74)$$

These two results also agree with those obtained by Schwarz-Christoffel method.

CHAPTER V

Flow through a necked slit impinging on a wall by a hodograph method

As in Chapter III, we combine the two problems of Chapter IV together and then solve the new problem by a hodograph method. For symmetry only one half of the plane will be considered.

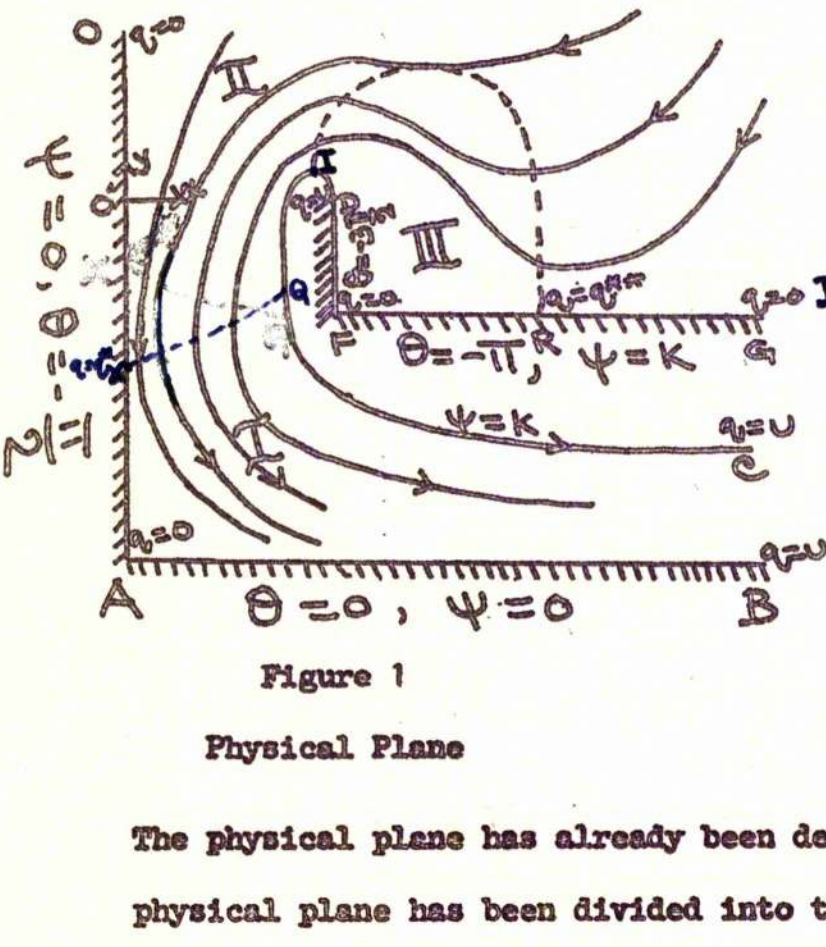


Figure 1

Physical Plane

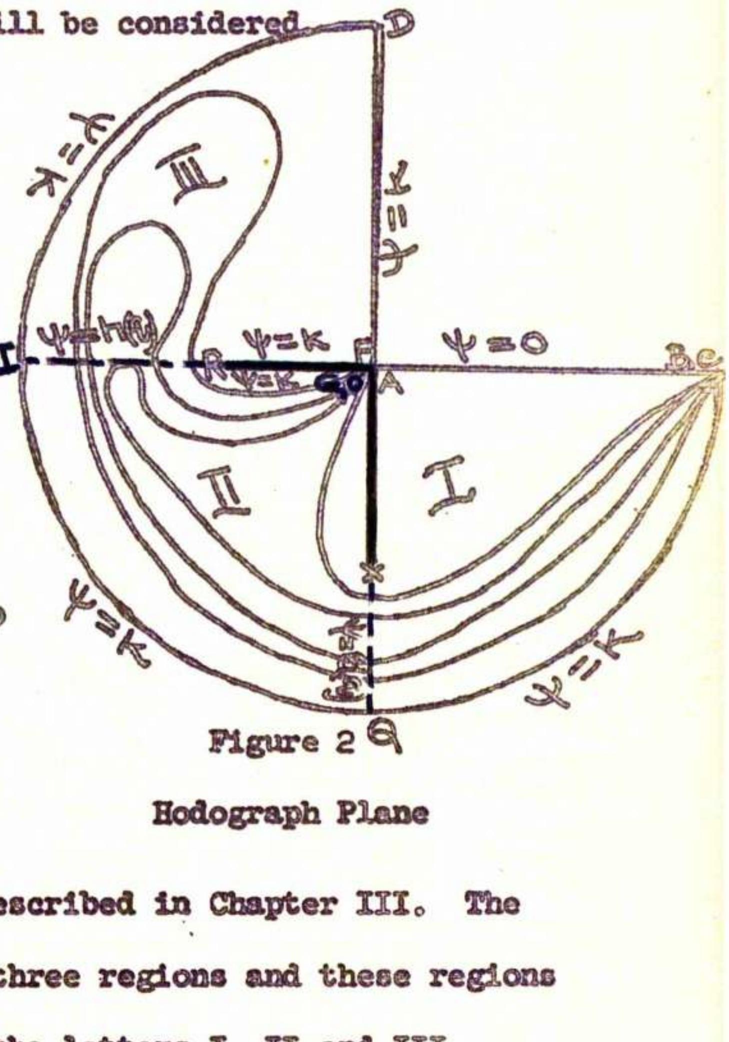


Figure 2

Hodograph Plane

The physical plane has already been described in Chapter III. The physical plane has been divided into three regions and these regions in the hodograph plane are marked by the letters I, II and III.

The boundary conditions in the hodograph plane are $\psi = 0$ on AX $(\theta = -\pi/2)$ and AB $(\theta = -\pi)$ $\psi = K$ (where K is a constant $=$ UH) on GRF $(\theta = -\pi)$ and FD $(\theta = -3\pi/2)$, $\psi = K$ on DIQC, DIQC being the free streamline, $\psi = h(q)$ on RI (locus $\theta = -\pi$) and $\psi = \ell(q)$ on XQ (locus $\theta = -\pi/2$).

Considering the portion ABCQ in the hodograph plane, we find that,

$$\psi = 0 \quad \text{on } \theta = 0$$

$$\psi = \ell(q) \text{ on } \theta = -\pi/2 \text{ such that } \ell(q) = 0 \text{ on AX}$$

$$\psi = K \text{ on QC } (-\pi/2 < \theta < 0).$$

Hence comparing with (4.02) we obtain from (4.37)

$$\psi_R = \begin{array}{l} -\frac{2K}{\pi}\tan^{-1}\left\{\frac{2M^2\sin 2\theta}{1-M^4}\right\} + \frac{K}{\pi}\tan^{-1}\left\{\frac{1+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\} \\ -\frac{K}{\pi}\tan^{-1}\left\{\frac{M^2+\cos 2\theta}{-\sin 2\theta}\right\} - \frac{K}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M'^2+M^2\cos 2\theta}{-M^2\sin 2\theta}\right\}dM' \\ +\frac{K}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\tan^{-1}\left\{\frac{M^2M'^2+\cos 2\theta}{-\sin 2\theta}\right\}dM' \end{array} \qquad (5.01)$$

ψ_R being the value of ψ on the right hand side of the line AQ.

Differentiating with respect to θ, we get

$$\frac{\partial \psi_R}{\partial \theta} = \begin{array}{l} -\frac{8K}{\pi}\frac{M^2(1-M^4)\cos 2\theta}{(1-M^4)^2+(2M^2\sin 2\theta)^2} + \frac{2K}{\pi}\frac{M^2(M^2+\cos 2\theta)}{1+2M^2\cos 2\theta+M^4} \\ -\frac{2K}{\pi}\frac{1+M^2\cos 2\theta}{1+2M^2\cos 2\theta+M^4} - \frac{2K}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{M^2(M^2+M'^2\cos 2\theta)dM'}{M^4+2M^2M'^2\cos 2\theta+M'^4} \\ +\frac{2K}{\pi}\int_{M^*}^{1}\frac{dL}{dM'}\frac{(1+M^2M'^2\cos 2\theta)dM'}{1+2M^2M'^2\cos 2\theta+M^4M'^4} \end{array} \qquad (5.02)$$

For the portion AQI, we have the following boundary conditions:-

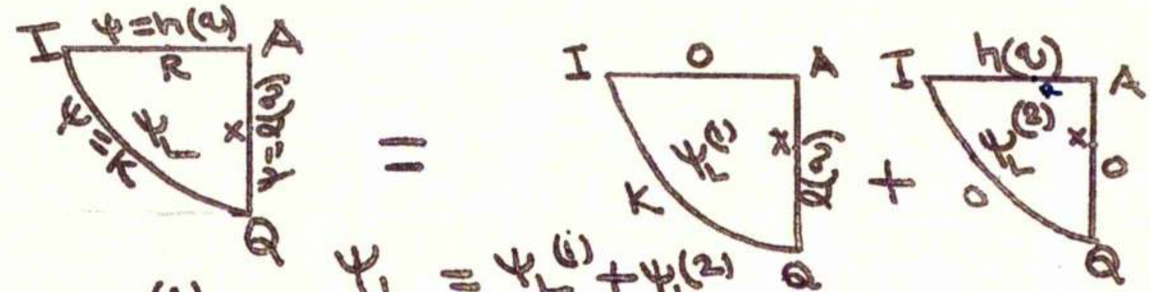
$$\psi = h(q) \text{ on AI } (\theta = -\pi) \text{ such that } h(q) = K \text{ on AR}$$

$$\psi = K \text{ on IQ } (-\pi < \theta < -\pi/2)$$

$$\psi = \ell(q) \text{ on AQ such that } \ell(q) = 0 \text{ on AX}$$

It is found convenient to obtain the solution in two parts which are then

superimposed to give the solution of the entire part. We will call these two parts as $\psi_L^{(1)}$ and $\psi_L^{(2)}$ and the combined solution as ψ_L.



$$\psi_L = \psi_L^{(1)} + \psi_L^{(2)} \qquad (5.03)$$

Solution for $\psi_L^{(1)}$

Here, $\psi = 0$ on $\theta = -\pi$

$\psi = K$ $-\pi < \theta < -\pi/2$

and $\psi = \ell(q)$ on $\theta = -\pi/2$ with $\ell(q) = 0$ on AX

Hence replacing θ by $-\pi-\theta$ in (4.37) we obtain,

$$\psi_L^{(1)} = \frac{2K}{\pi}\tan^{-1}\left\{\frac{2r^2\sin 2\theta}{1-r^4}\right\} + \frac{K}{\pi}\tan^{-1}\left\{\frac{1+r^2\cos 2\theta}{r^2\sin 2\theta}\right\} - \frac{K}{\pi}\tan^{-1}\left\{\frac{r^2+\cos 2\theta}{\sin 2\theta}\right\} - \frac{K}{\pi}\int_{r^*}^{1}\frac{d\ell}{dr'}\tan^{-1}\left\{\frac{r'^2+r^2\cos 2\theta}{r^2\sin 2\theta}\right\}dr' + \frac{K}{\pi}\int_{r^*}^{1}\frac{d\ell}{dr'}\tan^{-1}\left\{\frac{r r'^2+\cos 2\theta}{\sin 2\theta}\right\}dr' \qquad (5.04)$$

Solution for $\psi_L^{(2)}$

Here, $\psi = h(q)$ on $\theta = -\pi$ with $\psi = K$ on AR

$\psi = 0$ on $\theta = -\pi/2$

and $\psi = 0$ $-\pi < \theta < -\pi/2$

Hence replacing θ by $\theta + \pi/2$ in (4.33) and remembering that $h(q) \neq 0$ on AR ($q \ll q^{**}$) we obtain from (4.24), (4.27) and (4.33)

$$\psi_L^{(2)} = \frac{K}{\pi}\tan^{-1}\left\{\frac{t^{*2}(1-r^4)\sin 2\theta}{r^2(1+t^{*4}) - t^{*2}(1+r^4)\cos 2\theta}\right\} + \frac{2}{\pi}\int_{t^*}^{1} h(r')\frac{(r^2\sin 2\theta)r'\,dr'}{r^4-2r^2r'^2\cos 2\theta + r'^4} - \frac{2}{\pi}\int_{t^*}^{1} h(r')\frac{(r^2\sin 2\theta)r'\,dr'}{1-2r^2r'^2\cos 2\theta + r^4r'^4}$$

where $t^* = q^{**}/q$

But $h(M) = [1 + H(M)]$ where $H(1) = H(t^*) = 0$.

Substituting the value of $h(M)$ and integrating by parts we obtain from (5.04)

$$\psi_L^{(2)} = \begin{aligned}[t] & \frac{K}{\pi}\tan^{-1}\left\{\frac{1-r^2\cos 2\theta}{r^2\sin 2\theta}\right\} - \frac{K}{\pi}\tan^{-1}\left\{\frac{r^2-\cos 2\theta}{\sin 2\theta}\right\} \\ & + \frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dH'}\tan^{-1}\left\{\frac{r^2 r'^2-\cos 2\theta}{\sin 2\theta}\right\}dH' - \frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dH'}\tan^{-1}\left\{\frac{r'^2-r^2\cos 2\theta}{r^2\sin 2\theta}\right\}dH' \end{aligned} \tag{5.05}$$

Hence from (5.03), (5.04) and (5.05) we obtain

$$\psi_L = \begin{aligned}[t] & -\frac{K}{\pi}\int_{M^*}^{1}\frac{dL}{dH'}\tan^{-1}\left\{\frac{r'^2+r^2\cos 2\theta}{H^2\sin 2\theta}\right\}dH' + \frac{K}{\pi}\int_{M^*}^{1}\frac{dL}{dH'}\tan^{-1}\left\{\frac{r^2 r'^2+\cos 2\theta}{\sin 2\theta}\right\}dH' \\ & -\frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dH'}\tan^{-1}\left\{\frac{r'^2-r^2\cos 2\theta}{r^2\sin 2\theta}\right\}dr' + \frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dH'}\tan^{-1}\left\{\frac{r^2 r'^2-\cos 2\theta}{\sin 2\theta}\right\}dr' \end{aligned} \tag{5.06}$$

Differentiating with respect to θ we get

$$\frac{\partial\psi_L}{\partial\theta} = \begin{aligned}[t] & \frac{2K}{\pi}\int_{M^*}^{1}\frac{dL}{dr'}\frac{r^2(r^2+r'^2\cos 2\theta)\,dr'}{r^4+2r^2r'^2\cos 2\theta+r'^4} - \frac{2K}{\pi}\int_{r^*}^{1}\frac{dL}{dH'}\frac{(1+r^2r'^2\cos 2\theta)\,dr'}{1+2r^2r'^2\cos 2\theta+r^4r'^4} \\ & -\frac{2K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\frac{r^2(r^2-r'^2\cos 2\theta)\,dr'}{r^4-2r^2r'^2\cos 2\theta+r'^4} + \frac{2K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\frac{(1-r^2r'^2\cos 2\theta)\,dH'}{1-2r^2r'^2\cos 2\theta+r^4r'^4} \end{aligned} \tag{5.07}$$

For the portion GDI, we have the following boundary conditions:-

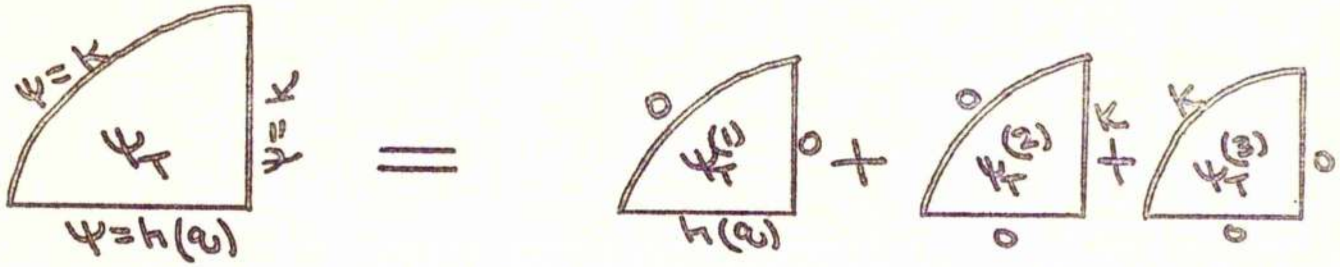
$\psi = h(q)$ on GI with $h(q) = K$ on GR $(\theta = -\pi)$

$\psi = K$ on FD $(\theta = -3\pi/2)$

$\psi = K$ on DI $(-3\pi/2 < \theta < -\pi)$

It is convenient to obtain the solution in three parts which are then superimposed to give the final result. We call the entire solution ψ_T i.e. the solution of ψ on the top of the line GI.

Thus



$$\text{i.e.}\quad \psi_T = \psi_T^{(1)} + \psi_T^{(2)} + \psi_T^{(3)} \tag{5.08}$$

Replacing θ by $-\theta - 2\pi$ in (5.05) we obtain

$$\psi_T^{(1)} = \begin{array}{l} \frac{K}{\pi}\tan^{-1}\left\{\frac{1-r^2\cos 2\theta}{-r^2\sin 2\theta}\right\} - \frac{K}{\pi}\tan^{-1}\left\{\frac{r^2-\cos 2\theta}{-\sin 2\theta}\right\} \\ -\frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\tan^{-1}\left\{\frac{r'^2-r^2\cos 2\theta}{-r^2\sin 2\theta}\right\}dr' + \frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\tan^{-1}\left\{\frac{r^2r'^2-\cos 2\theta}{-\sin 2\theta}\right\}dr' \end{array} \tag{5.09}$$

Replacing θ by $\theta + \pi/2$ in (4.46), we get

$$\psi_T^{(2)} = -\frac{2K}{\pi}(\theta+\pi) + \frac{2K}{\pi}\tan^{-1}\left\{\frac{r^2\sin 2\theta}{1+r^2\cos 2\theta}\right\} \tag{5.10}$$

Again replacing θ by $\theta + \pi$ in (4.09) we get

$$\psi_T^{(3)} = -\frac{2K}{\pi}\tan^{-1}\left\{\frac{2r^2\sin 2\theta}{1-r^4}\right\} \tag{5.11}$$

Hence from (5.08), (5.09), (5.10) and (5.11) we obtain

$$\psi_T = \begin{array}{l} -\frac{2K}{\pi}(\theta+\pi) - \frac{K}{\pi}\tan^{-1}\left\{\frac{(1-r^4)\sin 2\theta}{2r^2-(1+r^4)\cos 2\theta}\right\} \\ +\frac{2K}{\pi}\tan^{-1}\left\{\frac{r^2\sin 2\theta}{1+r^2\cos 2\theta}\right\} - \frac{2K}{\pi}\tan^{-1}\left\{\frac{2r^2\sin 2\theta}{1-r^4}\right\} \\ -\frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\tan^{-1}\left\{\frac{r'^2-r^2\cos 2\theta}{-r^2\sin 2\theta}\right\}dr' + \frac{K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\tan^{-1}\left\{\frac{r^2r'^2-\cos 2\theta}{-\sin 2\theta}\right\}dr' \end{array} \tag{5.12}$$

Differentiating with respect to θ we get

$$
\begin{aligned}
\frac{\partial\psi_T}{\partial\theta} = & -\frac{2K}{\pi} - \frac{2K}{\pi}\,\frac{(1-r^4)\cos 2\theta\left[2r^2-(1+r^4)\cos 2\theta\right]-(1-r^8)\sin^2 2\theta}{\left[2r^2-(1+r^4)\cos 2\theta\right]^2+\left[(1-r^4)\sin 2\theta\right]^2} \\
& + \frac{4K}{\pi}\,\frac{r^2(r^2+\cos 2\theta)}{1+2r^2\cos 2\theta+r^4} - \frac{8K}{\pi}\,\frac{r^2(1-r^4)\cos 2\theta}{(1-r^4)^2+4r^4\sin^2 2\theta} \\
& + \frac{2K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\,\frac{r^2(r^2-r'^2\cos 2\theta)\,dr'}{r^4-2r^2r'^2\cos 2\theta+r'^4} - \frac{2K}{\pi}\int_{t^*}^{1}\frac{dH}{dr'}\,\frac{(1-r^2r'^2\cos 2\theta)\,dr'}{1-2r^2r'^2\cos 2\theta+r^4r'^4}
\end{aligned}
\qquad (5.13)
$$

Since $\left(\frac{\partial\psi_R}{\partial\theta}\right)_{\theta=-\frac{\pi}{2}} = \left(\frac{\partial\psi_L}{\partial\theta}\right)_{\theta=-\frac{\pi}{2}}$ we obtain from (5.02)

and (5.07)

$$
2\int_{r^*}^{1}\frac{dL}{dr'}\,\frac{r'^2\,dr'}{(r^2-r'^2)(1-r^2r'^2)} + \int_{t^*}^{1}\frac{dH}{dr'}\,\frac{r'^2\,dr'}{(r^2+r'^2)(1+r^2r'^2)} = -\frac{1}{(1+r^2)^2} \qquad (5.14)
$$

Again since, $\left(\frac{\partial\psi_L}{\partial\theta}\right)_{\theta=-\pi} = \left(\frac{\partial\psi_T}{\partial\theta}\right)_{\theta=-\pi}$ we obtain from (5.07)

and (5.13)

$$
\int_{r^*}\frac{dL}{dr'}\,\frac{r'^2\,dr'}{(r^2+r'^2)(1+r^2r'^2)} - 2\int_{t^*}^{1}\frac{dH}{dr'}\,\frac{r'^2\,dr'}{(r'^2-r^2)(1-r^2r'^2)} = 0 \qquad (5.15)
$$

It is seen that when $\theta = -\pi/2$, the first integral in the right hand sides of (5.02) and (5.07) is singular and when $\theta = -\pi$, the third integral in the right hand side of (5.07) and the (same) first integral on the right hand side of (5.13) is singular. But it has been shown in Chapter IV how these integrals are reduced to Cauchy Principal values (vide 4.53, 4.54 and 4.55). Hence the singular integrals given by (5.14) and (5.15) are now to be interpreted as Cauchy Principal values.

Proceeding in the same way as in Chapter IV these two singular integral equations are reduced to,

$$2\int_2^{\alpha^*} k_1(u)\,\frac{du}{u-\gamma} + \int_2^{\beta^*} k_2(u)\,\frac{du}{u+\gamma} = -\frac{2}{\gamma+2}$$

and

$$\int_2^{\alpha^*} k_1(u)\,\frac{du}{u+\gamma} + 2\int_2^{\beta^*} k_2(u)\,\frac{du}{u-\gamma} = 0 \qquad (5.16)$$

where $k_1(u)$ and $k_2(u)$ etc, have been derived in the same way as in Chapter IV. Analytical methods of solution of these simultaneous integral equations have not yet been developed, but would provide the subject of further work in this field.

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