STUDIES IN THE THEORY OF DETONATION

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STUDIES IN THE THEORY OF DETONATION

Summary of Ph.D. Thesis by R. Welsh

The work of this thesis consists of two separate problems concerning the motion of detonation waves in inviscid, non-heat-conducting gases. A detonation wave is assumed to be a surface of discontinuity moving through the medium. On crossing this wave the gas particles instantaneously release a certain amount of heat energy. Whereas shock waves are only of the compressive type, waves involving such an energy release can be either compressive (detonations) or expansive (deflagrations).

Part I is concerned with the one-dimensional motion of such waves through a uniform gas contained in a straight tube, which is closed at the end where the waves are initiated. There are two accepted models for the motion of a combustible gas in this case. In the first a shock wave moves uniformly into the gas ahead of a deflagration wave which travels uniformly at subsonic speed relative to the stationary gas behind it. There is a region of uniform motion between the two waves. The second concerns sonic (Chapman-Jouguet) deflagrations and differs from the previous case in that there exists a point-centred, simple rarefaction wave immediately behind the deflagration. For a certain value of the shock speed of the latter system the speeds of the deflagration and shock coincide, resulting in a single detonation wave front.

In the present work the stability of these two systems is investigated by considering the effect on the flow of certain types of small disturbances. The subsonic model is found to be non-evolutionary, i.e., the problem of introducing specified small disturbances into the system has no unique solution. A self-generating, unstable solution is found and is calculated in the case of the shock being uniformly accelerated. The existence of such a solution suggests that the subsonic model is unstable.

The problem of small disturbances introduced into the Chapman-Jouguet model is shown to be evolutionary, provided the Chapman-Jouguet condition is relaxed for the disturbances. For two particular numerical cases it is shown that there is no self-generating solution in which the perturbation of the shock speed can be expressed as a power series in time, t. The solution for the perturbation on the system due to the gas being not quite at rest initially is found in terms of the initial velocity distribution along the tube.

Part II/

Part II is an investigation into the mathematical solution for a spherically or axially symmetric detonation front travelling into uniform gas in the direction towards the centre or axis of symmetry. It is known that there is no solution corresponding to a constant speed Chapman-Jouguet front for this case, unlike the expanding radially symmetric wave whose solution is analogous to the Chapman-Jouguet detonation in one dimension.

However, examination of Guderley's similarity solution for a radially symmetric contracting shock front, valid near the centre or axis of symmetry, suggests a method of solving the present problem. Guderley's solution shows that the shock accelerates towards the origin, where the solution is singular and the shock speed, particle velocity, and pressure are infinite. Since the addition of a heat release term across this front can only have a finite effect on the energy of the flow behind the front, it follows that the detonation problem can be considered as a small perturbation on the shock solution. This perturbation is of precisely the same form as the correction due to taking into account the sound speed of the stationary gas, which is neglected in Guderley's solution.

The equations of motion of this basic solution involving a shock front reduce to a single non-linear, first order, ordinary differential equation owing to the similarity assumption, which also permits the equations governing the perturbations to be written as a set of three simultaneous, linear, ordinary differential equations. The solution of the former single equation appears in the coefficients of the latter set of equations. Hence it is necessary to recompute Guderley's solution and his results are extended to higher values of γ. The assumption that the flow is regular on a certain characteristic, on which the flow may be singular, ensures that there is a unique solution to both the basic and detonation problems. The equations are integrated numerically using a specially devised method which makes use of the power series expansions in the vicinity of this characteristic, which becomes a point in terms of the redefined variables. The method used has no difficulty in dealing with the solution in the neighbourhood of this point, where the derivatives of three of the four variables, as given by the differential equations, are indeterminate (but are actually finite due to the regularity conditions).

CONTENTS

Introduction

The work of this thesis is concerned with the propagation of detonation waves through combustible gases.

Part I consists of an investigation into the stability of the subsonic and Chapman-Jouguet models (described in detail in Section 1) for the one-dimensional detonation of a uniform gas contained in an infinite straight tube, closed at the end where the detonation is initiated. The stability of these two families of uniformly expanding flows is considered by determining the effects due to certain types of small disturbances. In Section 2 the differential equations governing the propagation of disturbances through the systems are determined, the technique being that developed by Gundersen (4).

It is shown in Section 3(i) that the problem of small disturbances propagating through the flow of the subsonic model is non-evolutionary (i.e. has no unique solution). A self-generating, unstable solution, i.e. one arising of itself and not produced by any external agency, is found and evaluated in detail for the case of the shock wave in the flow being subject to a small, uniform acceleration. The calculations in this section were done on the Deuce computer at Glasgow University.

The investigation into the stability of the Chapman-Jouguet model in Section 4(i) shows that the problem of small disturbances introduced into the system is evolutionary and so has a unique solution, provided the Chapman-Jouguet condition is relaxed for the disturbances. In Section 4(iii) it is shown, for two particular numerical cases, that there is no self-generating solution in which the perturbation of the shock speed, expressed as a function of time t, can be expressed as a power series in t. The solution for the case in which the gas is initially approximately at rest is evaluated in Section 4(iv) in terms of the initial distribution of velocity in the tube.

In Part II the problem of a converging spherical detonation front moving into a uniform gas is investigated. The problem of spherically symmetric wave motions is introduced in Section 1 and certain relevant solutions are discussed. It is shown that the required solution for the detonation wave is a perturbation on the solution, obtained by Guderley (10), for a converging spherical shock wave.

The equations of motion and similarity assumptions are described in Section 2. A proof that there is no uniformly contracting solution is given here. The solution sought, like Guderley's solution for the shock wave, has the property that the front accelerates towards the centre of symmetry O, where its velocity is infinite, as are the velocity and pressure of the gas. Both solutions are valid near O. A decaying, expanding spherical shock wave is reflected from the centre of symmetry.

As well as taking into account the effects of the detonation the present solution can allow for a finite sound speed of the initial gas. In Section 3 the equations of motion and boundary conditions for the basic Guderley solution, and for the perturbation solution due to the detonation, are derived. The assumption of similarity reduces these equations to ordinary differential equations. The method used by Butler (6) to evaluate the Guderley solution is followed. As for the basic solution, it is necessary to assume that the flow is regular on a certain characteristic in order to have a unique solution to the problem. In terms of the redefined variables of the problem the path of this characteristic is a point. The condition of regularity gives boundary conditions on the flow at this point, which is a singular point of the ordinary differential equation for the basic flow and of one of the three equations governing the perturbations.

The equation for the basic flow has to be integrated in such a way that it satisfies a boundary condition at each end of the range (viz. the regularity condition at the characteristic and the conservation equations at the front) in order to find the path of the shock wave. This can only be done by trial and error as the differential equation is non-linear. The perturbation terms also have to satisfy boundary conditions at each end of the range but the problem is simplified in this case as the equations are linear. The problem is to find the path of the converging detonation wave.

The method by which the equations are integrated numerically is described in Section 4. This method is specially devised for this problem and is unaffected by the fact that the characteristic on which the flow has to be regular corresponds to a singular point of certain of the differential equations. The method makes use of the fact that the flow is regular at this point and so can be expanded as a power series in its neighbourhood.

The basic Guderley solution was evaluated on the Sirius Computer in the University of Strathclyde and for the solution for the detonation the Atlas Computer of the Science Research Council was used.

1. Detonations and Deflagrations in One Dimension

The subject of the propagation of reaction (combustion) fronts through gases was first studied experimentally around 1880 by a number of French physicists, chiefly Vieille, Mallard, Le Chatelier and Berthelot. On igniting one end of a column of gas contained in a uniform tube they found that a slow combustion wave, of velocity a few metres per second, was normally propagated through the gas. In certain cases, however, it was found that the flame accelerated very rapidly to velocities in the region of 2,000 metres per second. This latter type of process was called a detonation wave. The final steady velocity of the wave was found to depend upon only the chemical and physical nature of the gas.

A theoretical explanation of this phenomenon was put forward in 1899 by Chapman and independently by Jouguet in 1905. In this it was assumed that the chemical reaction takes place instantaneously, i.e., the reaction front can be considered to be a plane of discontinuity propagating through the gas, with burnt gas behind and unburnt gas ahead. The model is evidently very similar to that of a plane shock wave, the only differences being the release of heat energy by the gas particles and the change in the chemical nature of the gas.

A fuller treatment of the theory of detonation to be described here is given in Courant and Friedrichs (3) and by

Taylor and Tankin (5). The two models to be discussed are considered in detail by Adams and Pack (1), in particular with regard to the transition through the possible solutions to the final, stable Chapman-Jouguet detonation.

Consider such a wave involving chemical reaction moving through a column of gas contained in a straight tube. The reaction front can be brought to rest by means of a velocity transformation. It will be assumed that the gas is perfect, non-heat-conducting and inviscid and that its specific heats at constant volume and pressure c_v, c_p are constant. The pressure, density and particle velocity of the gas will be denoted by p, ρ, u respectively and $\gamma = c_p/c_v$. The suffices 0, 1 refer to the unburnt and burnt gas respectively.

$\gamma_1, p_1, \rho_1, u_1$ (burnt)	$\gamma_0, p_0, \rho_0, u_0$ (unburnt)

Since the front is considered to be stationary u_0, u_1 are in fact the particle velocities relative to the front.

Relations between the physical variables on either side of the front can be found, as for shock waves, by consideration of the conservation laws of mechanics. The equations of conservation of mass and momentum, which are identical to those for shock waves, are respectively

$$\rho_1 u_1 = \rho_0 u_0 \tag{1.1}$$

$$p_1 + \rho_1 u_1^2 = p_0 + \rho_0 u_0^2 \tag{1.2}$$

It will be assumed that the passage of unit mass of gas across the front gives rise to the release of a constant amount of energy, Q say. Thus the equation of conservation of energy can be written

$$\frac{1}{2}u_1^2 + \frac{\gamma_1}{\gamma_1 - 1}\cdot\frac{p_1}{\rho_1} = \frac{1}{2}u_0^2 + \frac{\gamma_0}{\gamma_0 - 1}\cdot\frac{p_0}{\rho_0} + Q \qquad (1.3)$$

From 1.1, 1.2 we deduce that

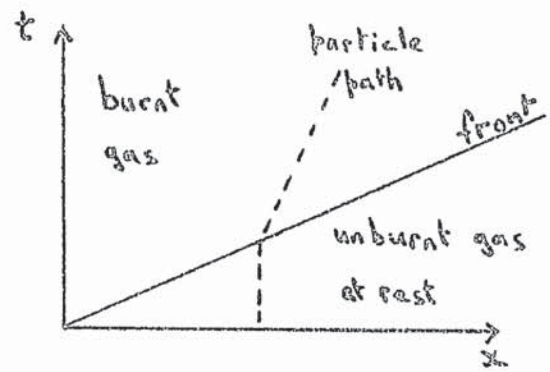
$$\frac{p_1 - p_0}{\frac{1}{\rho_1} - \frac{1}{\rho_0}} = -\rho_0^2 u_0^2 = -\rho_1^2 u_1^2 \qquad (1.4)$$

from which we can deduce further, as for shock waves, that pressure and density increase and decrease in the same direction. Whereas a shock wave can only increase the pressure of the gas into which it propagates, a wave front across which there is an energy release can either increase or decrease the pressure of the gas. Those fronts which cause an increase in pressure (compressive) are called detonations and the others (expansive) correspond to slow combustions and are called deflagrations.

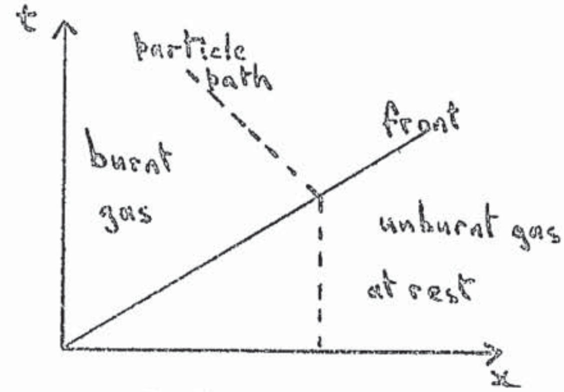
From 1.1, 1.2 we can derive the equation

$$\frac{p_1 - p_0}{u_1 - u_0} = -\rho_0 u_0 = -\rho_1 u_1 \qquad (1.5)$$

If the reaction front faces forwards then u_0, u_1 are in the negative direction. Thus for a detonation, i.e., $p_1 > p_0$, it follows that $u_1 > u_0$ and for a deflagration $u_1 < u_0$. If the gas ahead of the reaction front is stationary the representation in the x-t plane is as follows

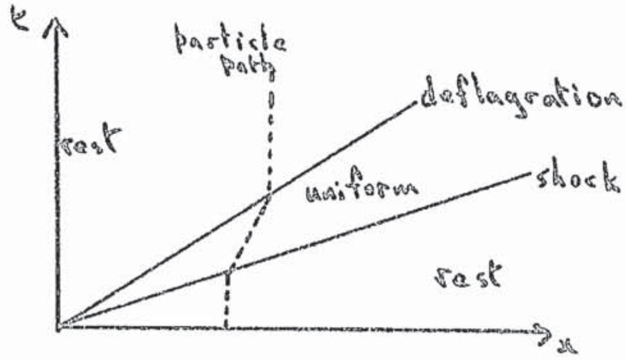


Detonation



Deflagration

Consider a deflagration wave starting at the closed end of a uniform, straight, semi-infinite tube. The gas behind this front is deflected towards the closed end of the tube. However a necessary boundary condition on the flow is that the particle velocity be zero at the closed end, $x = 0$. A consistent flow is obtained by introducing a shock wave moving ahead of the deflagration. Behind this shock the gas is deflected away from the closed end so that for an appropriate value of the shock speed the deflagration brings the gas to rest. The velocity of the shock wave is determined by the velocity of the deflagration. The system can be represented in the x-t plane as follows



This is the Zeldovitch - Doring - von Neumann model and has been accepted as the model for a deflagration in a closed tube since 1945.

Let the velocities of the deflagration (flame) and the shock in such a system be denoted by V, W respectively. The suffices 1, 0, 2 refer respectively to the region behind the flame, the region between the waves and the region ahead of the shock.

In the following c denotes sound speed and it is supposed that $\gamma_0 = \gamma_1 = \gamma$. The regions 2, 0 are identical chemically so that $\gamma_2 = \gamma$ also. The three conservation equations across the shock can be written in the form

$$\rho_0(W - u_0) = \rho_2 W \qquad (1.6)$$

$$p_0 + \rho_0(W - u_0)^2 = p_2 + \rho_2 W^2 \qquad (1.7a)$$

and 1.7a can be rewritten in terms of c, where $c^2 = \frac{\gamma p}{\rho}$, as

$$Wc_0^2 - \gamma u_0 W(W - u_0) = (W - u_0)\, c_2^2 \qquad (1.7b)$$

using 1.6

and

$$\frac{1}{2}(W - u_0)^2 + \frac{c_0^2}{\gamma - 1} = \frac{1}{2}W^2 + \frac{c_2^2}{\gamma - 1} \qquad (1.8)$$

The corresponding equations across the flame are the following with $u_1 = 0$. (The equations are here given for the case $u_1 \neq 0$, which will be required later)

$$\rho_1 (V - u_1) = \rho_0 (V - u_0) \tag{1.9}$$

$$p_1 + \rho_1 (V - u_1)^2 = p_0 + \rho_0 (V - u_0)^2 \tag{1.10a}$$

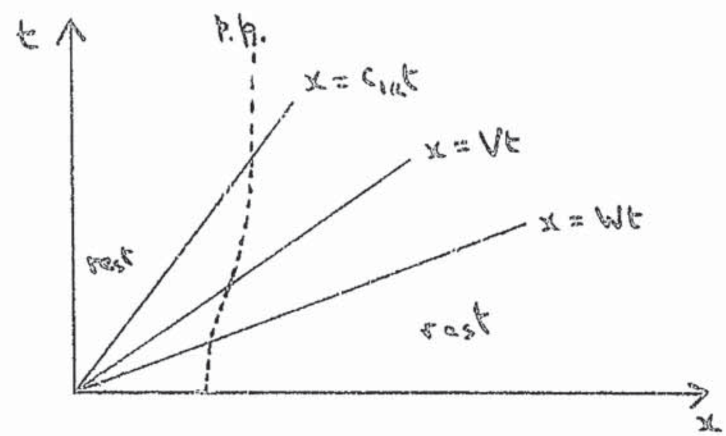
which may be written as

$$\begin{aligned}(V-u_0)\, c_1^2 + \gamma (V-u_0)(V-u_1)^2 \\ = (V-u_1) c_0^2 + \gamma (V-u_1)(V-u_0)^2\end{aligned} \tag{1.10b}$$

$$\tfrac{1}{2}(V-u_1)^2 + \frac{c_1^2}{\gamma-1} = \tfrac{1}{2}(V-u_0)^2 + \frac{c_0^2}{\gamma-1} + Q \tag{1.11}$$

With $u_1 = 0$ the equations 1.6 - 1.11 provide six equations determining the seven variables V, W, u_0, ρ_0, c_0, ρ_1 and c_1 in terms of the initial state of the gas, i.e., ρ_2 and c_2. There is thus one degree of freedom in the system. In particular the system is completely determined if either the shock or flame speed is prescribed and it is assumed that all possible systems are equally likely to occur.

This model, however, is only valid for subsonic flame speeds. It has been shown on thermodynamic grounds that it is impossible for the speed of the flame to exceed the sound speed of the burnt gas, so that the model does not give a true physical representation for flame speeds exceeding this value. Let us consider the case of the shock speed being such that it would require a supersonic flame. It will therefore be necessary to introduce a finite particle velocity u_1 immediately behind the flame such that $V - u_1 = c_1$. This is the Chapman-Jouguet condition, to be discussed later. In order to allow the value of u_1 to remain zero at $x = 0$ it will be necessary to introduce a simple rarefaction wave, centred at O, bringing the burnt gas to rest along the line $x = c_{1r}t$, where c_{1r} is the rest sound speed of the burnt gas. The representation of this (Chapman-Jouguet) model is as follows



Mathematically we have introduced a new variable u_1 into the system but the Chapman-Jouguet condition ($V - u_1 = c_1$) ensures that there is still only one degree of freedom, so that this model is also uniquely determined in terms of either V or W. The lower limit of the possible range of values of W is W^*. There is also an upper limit for the value of the shock speed,

W_m say, for which the speed of the flame equals that of the shock. In this case the paths of the flame and shock coincide and form a single detonation front.

The theory of Chapman-Jouguet detonations and deflagrations (that the speed of the front is sonic relative to the burnt gases) is given in detail in (3) and (5). From 1.4 we obtain

$$\frac{p_1 - p_0}{\frac{1}{\rho_1} - \frac{1}{\rho_0}} = -m^2 \qquad \text{(const)} \tag{1.12}$$

where $m = \rho_0 u_0 = \rho_1 u_1$

so that, if $\tau = \frac{1}{\rho}$, any final state (p_1, τ_1) must satisfy the condition

$$\frac{p_1 - p_0}{\tau_1 - \tau_0} < 0 \tag{1.13}$$

If we define the complete energy function $E(p, \tau)$ to be the sum of the energy of formation per unit mass g and the internal energy per unit mass e, where g is taken to be independent of p, τ but $e = e(p, \tau)$, then the form of the function $E(p, \tau)$ for the unburnt gas will differ from that for the burnt gas. Let the former be denoted by $E^{(0)}(p, \tau)$ and the latter by $E^{(1)}(p, \tau)$.

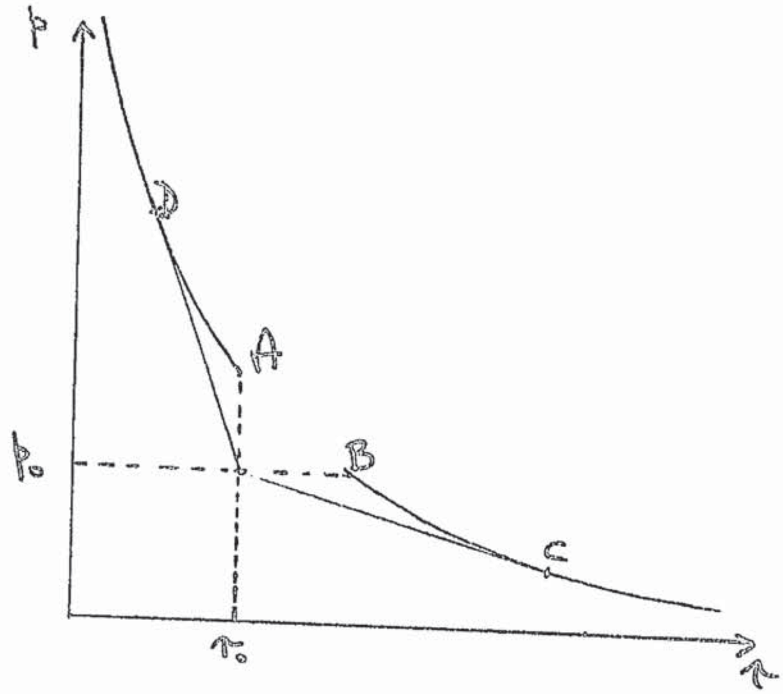
A more general form of the energy equation 1.3 is

$$E^{(1)}(p_1, \tau_1) + p_1\tau_1 + \tfrac{1}{2}u_1^2 = E^{(0)}(p_0, \tau_0) + p_0\tau_0 + \tfrac{1}{2}u_0^2 \tag{1.14}$$

By eliminating the velocities u_0, u_1 from 1.1, 1.2, 1.14, and dropping the suffix 1 for the burnt gas, we obtain the purely thermodynamic relation (the Hugoniot relation)

$$E^{(1)}(p,\tau) - E^{(0)}(p_0,\tau_0) + \tfrac{1}{2}(\tau-\tau_0)(p+p_0) = 0 \qquad (1.15)$$

This equation determines the family of possible final states (p, τ) for a gas in the initial state (p_0, τ_0). The states given by 1.15 give rise to a curve (the Hugoniot curve) in the (p, τ) plane, subject to the restriction (1.13).



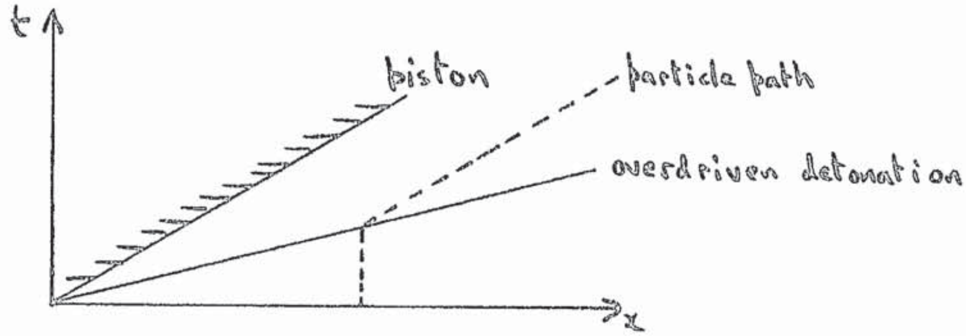
The upper branch of the curve corresponds to detonations and the lower branch to deflagrations. A line through (p_0, τ_0) intersecting the Hugoniot curve will, in general, intersect it at two distinct points, except for the lines through D, C which are tangential to the curve. The points D and C are the Chapman-

Jouguet points and can be shown to correspond to a sonic (Chapman-Jouguet) detonation and deflagration respectively. The state given by D is that final state for a detonation in which the entropy and $|u_0|$ each assume a minimum value, whereas C is the deflagration for which the entropy and $|u_0|$ each have a maximum value. Points above D correspond to 'strong' detonations, and points between A and D correspond to 'weak' detonations. Similarly points below C give rise to 'strong' deflagrations, and points between B and C give rise to 'weak' deflagrations.

The argument which excludes the possibility of strong deflagrations is given in Courant and Friedrichs. By considering the reaction to take place over a finite distance, with Σ denoting the fraction of burnt gas in the mixture at some control surface within the reaction, it is shown that at some stage in the reaction Σ would have to be greater than unity in the case of a strong deflagration. Thus it is concluded that strong deflagrations cannot occur in practice.

It is also shown in Courant and Friedrichs that the correct interpretation of a detonation wave is that of a shock front followed immediately by a deflagration. In particular a weak detonation consists of a shock followed by a strong deflagration. Thus weak detonations cannot occur in practice either.

A strong (overdriven) detonation can be produced by means of a piston moving uniformly into the burnt gases at a velocity greater than the theoretical Chapman-Jouguet fluid velocity immediately behind the front.



The work of part II of this thesis involves a converging spherical detonation wave which is overdriven, due, in this case, to the focussing effect as the origin is approached and the surface area of the front diminishes.

2. Equations of Motion

Before considering the effects of small disturbances on the systems described in Section 1 it will be necessary to derive the differential equations governing the propagation of these disturbances. The equations to be used are those due to Gundersen (4). It is assumed that there is a known solution for a basic, isentropic (i.e., entropy is constant throughout the fluid) one-dimensional flow, which is subject to small disturbances in the physical variables. Both the basic and perturbed flows must satisfy the equations for the one-dimensional motion of a gas, which is assumed to be perfect, inviscid, non-heat-conducting and have constant specific heats c_p, c_v. The equations governing the perturbations on the basic flow are obtained by linearising the equations of motion.

The fact that mass is conserved in the flow can be expressed by the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \qquad (2.1)$$

where t is time and x is the distance along the tube from the closed end.

The equation of momentum is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (2.2)$$

The third equation of motion is derived from the assumption that there is no dissipation due to heat conduction or viscosity.

Hence the specific entropy, s, remains constant along a particle path of the fluid

$$\text{i.e.} \quad \frac{Ds}{Dt} = \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0 \tag{2.3}$$

The equation of state for the gas can be expressed in the form

$$p = e^{\frac{s-s^{*}}{c_v}} \cdot \rho^{\gamma} \tag{2.4}$$

by suitable choice of units. s^{*} is the specific entropy of the gas at some reference state.

The four equations 2.1 - 2.4 serve to determine the four physical variables p, ρ, u, s which prescribe the state and motion of the gas completely. The sound speed of the gas c is given by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} \tag{2.5}$$

By use of 2.4 and 2.5 the equations 2.1 - 2.3 can be expressed in terms of u, c, s only. The equation 2.2 can be put in the form

$$u_t + u.u_x + k\,c.c_x - \frac{c^2}{\gamma(\gamma-1)c_v}\, s_x = 0 \tag{2.6}$$

where $k = \frac{2}{\gamma-1}$ and the suffices x, t denote partial derivatives. Equation 2.1 can be expressed as

$$c_t + u.c_x + \tfrac{1}{2}(\gamma-1)c.u_x = 0 \tag{2.7}$$

Mathematically the system is defined by the three differential equations 2.3, 2.6, 2.7 for u, c, s as functions of x, t. After solving these equations p, ρ can be found by using 2.4, 2.5.

The differential equations for u, c, s are

$$c_t + u.c_x + \tfrac{1}{2}(\gamma-1)\,c.u_x = 0 \qquad (2.7)$$

$$u_t + u.u_x + k\,c.c_x - \frac{c^2}{\gamma(\gamma-1)c_v}.s_x = 0 \qquad (2.6)$$

$$s_t + u.s_x = 0 \qquad (2.3)$$

Let the basic isentropic flow be given by

$$u = u_0(x, t)$$
$$c = c_0(x, t)$$
$$s = s_0 = \text{constant.}$$

Suppose that small disturbances are superimposed on this flow and that the perturbed flow is given by

$$u = u_0(x,t) + \bar{u}_0(x,t)$$
$$c = c_0(x,t) + \bar{c}_0(x,t)$$
$$s = s_0 + \bar{s}_0(x,t)$$

where $\bar{u}_0$, $\bar{c}_0$, $\bar{s}_0$ are of first order of smallness. Since u_0, c_0, s_0 is a solution of the equations of motion we can write

$$\left.\begin{aligned} c_{0t} + u_0.c_{0x} + \tfrac{1}{2}(\gamma-1)c_0.u_{0x} &= 0 \\ u_{0t} + u_0.u_{0x} + k\,c_0.c_{0x} &= 0 \\ s_0 &= \text{const.} \end{aligned}\right\} \qquad (2.8)$$

The perturbed flow must also satisfy the equations of motion so that

$$\bar{c}_{0t} + c_{0t} + (u_0+\bar{u}_0)(\bar{c}_{0x}+c_{0x}) + \tfrac{1}{2}(\gamma-1)(\bar{c}_0+c_0)(\bar{u}_{0x}+u_{0x}) = 0 \qquad (2.9)$$

$$\bar{u}_{0t}+u_{0t}+(\bar{u}_0+u_0)(\bar{u}_{0x}+u_{0x}) + k(\bar{c}_0+c_0)(\bar{c}_{0x}+c_{0x}) = \frac{(\bar{c}_0+c_0)^2}{\gamma(\gamma-1)c_v}\cdot\bar{s}_{0x} \qquad (2.10)$$

$$\bar{s}_{0t}+u_0\cdot\bar{s}_{0x} = 0 \qquad (2.11)$$

The first two of these equations can be simplified by using 2.8 and by eliminating all terms of second order of smallness, e.g., $\bar{u}_0\cdot\bar{c}_{0x}$ etc., to give

$$\bar{c}_{0t}+u_0\bar{c}_{0x}+\bar{u}_0 c_{0x}+\tfrac{1}{2}(\gamma-1)(\bar{c}_0\cdot u_{0x}+c_0\cdot\bar{u}_{0x}) = 0 \qquad (2.12)$$

$$\bar{u}_{0t}+\bar{u}_0\cdot u_{0x}+u_0\cdot\bar{u}_{0x}+k(\bar{c}_0 c_{0x}+c_0\bar{c}_{0x}) = \frac{c_0^2}{\gamma(\gamma-1)c_v}\cdot\bar{s}_{0x} \qquad (2.13)$$

Equation 2.11 expresses the fact that $\bar{s}_0(x,t)$ is constant along the particle paths of the basic flow i.e., on $\frac{dx}{dt} = u_0(x,t)$. Also, from equation 2.1, it follows that there exists a stream function ψ for the basic flow given by

$$\psi_x = \rho_0 \quad , \quad \psi_t = -\rho_0 u_0$$

and ψ = constant defines the particle paths of the flow. Thus $\bar{s}_0(x,t)$ can be written in the form

$$\bar{s}_0(x,t) = w(\psi)$$

for some function w. For convenience let us define the function

$H(x, t)$ by

$$c_0^2 \bar{\zeta}_{0x} = \rho_0 c_0^2 w'(\psi)$$
$$= \gamma(\gamma-1) c_v . H(x,t)$$

Let us introduce the Riemann invariants α, β of the basic flow given by

$$\tfrac{1}{2}(u_0 + k c_0) = \alpha , \quad \tfrac{1}{2}(-u_0 + k c_0) = \beta$$

and the corresponding functions $\bar{\alpha}$, $\bar{\beta}$ defined by

$$\tfrac{1}{2}(\bar{u}_0 + k \bar{c}_0) = \bar{\alpha} , \quad \tfrac{1}{2}(-\bar{u}_0 + k \bar{c}_0) = \bar{\beta}$$

The two combinations k (2.12) $\pm$ (2.13) then yield

$$\bar{\alpha}_t + (u_0 + c_0)\bar{\alpha}_x + \tfrac{1}{2}\left\{(\gamma+1)\bar{\alpha} + (\gamma-3)\bar{\beta}\right\}.\alpha_x = \tfrac{1}{2} H(x,t) \qquad (2.14)$$

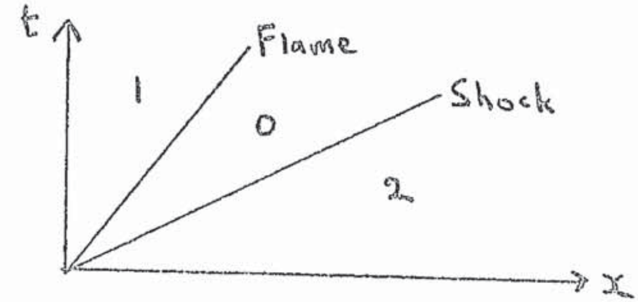
$$\bar{\beta}_t + (u_0 - c_0)\bar{\beta}_x + \tfrac{1}{2}\left\{(3-\gamma)\bar{\alpha} - (\gamma+1)\bar{\beta}\right\}.\beta_x = -\tfrac{1}{2} H(x,t) \qquad (2.15)$$

and this is the form of the perturbation equations to be used later.

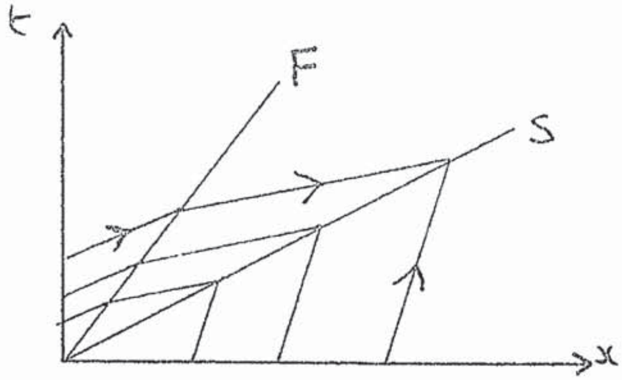
3. The Subsonic Model

(i) Stability and Uniqueness

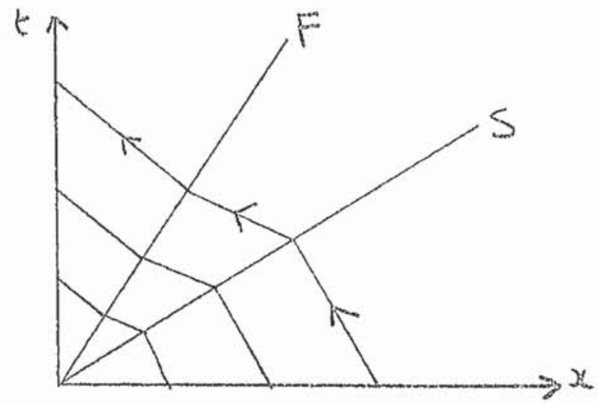
In order to discuss the stability of flows of this type let us consider the propagation of small disturbances throughout the system caused by a disturbance just ahead of the **shock.**



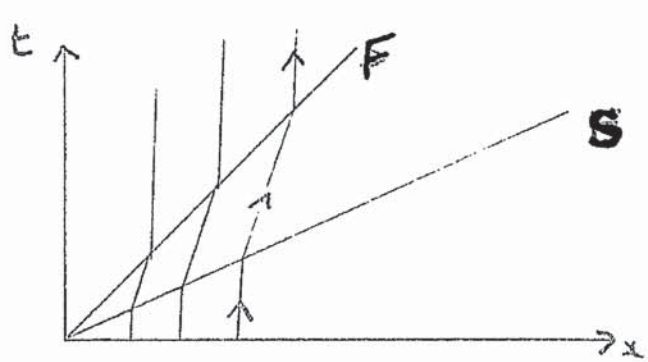
The positive and negative characteristics and particle paths of the three distinct regions are sketched roughly.



positive characteristics (C+)



negative characteristics (C-)



particle paths (Co)

Since u, c are constant in each region all the families are straight lines. The direction associated with each line is that of time increasing.

A disturbance immediately ahead of the shock gives rise to three distinct disturbances - the change in the shock speed and the two waves travelling along Co and C+ in the region 0 between the waves. Since the shock moves with supersonic speed relative to the region 0 no wave can be propagated from the shock along the positive characteristics in this region. To determine the three independent disturbances we have three equations due to the fact that the three conservation equations across the shock have to be satisfied when the disturbances are introduced. Hence we can regard these disturbances as known.

Consider now the effect of a disturbance arriving at the flame. Waves are propagated from the flame along Co and C- in the region 1 behind the flame and along the positive characteristic in the region ahead. Together with the displacement of the flame these make up four unknown disturbances. To determine these we have, as for the shock, the three conservation equations.

The condition that the tube is closed at $x = 0$ serves to determine the wave travelling along the positive characteristic in the region 1, which is the reflection at $x = 0$ of the wave travelling along the negative characteristic. We can thus neglect the fact that the tube is closed and also the reflected wave in considering the uniqueness of the solution due to a disturbance ahead of the shock.

Since there are seven degrees of freedom for the disturbed flow and only six equations determining them, it follows that there is no unique solution to the problem of a small disturbance ahead of the shock. Therefore the system is non-evolutionary (2).

This result can be arrived at by considering the effect of continuous small disturbances throughout the complete flow due to a given initial distribution of disturbances in the gas (i.e. on $t = 0$). It will be shown that the general solution for the perturbation of any region of flow contains a degree of freedom (i.e. an arbitrary function) corresponding to each characteristic along which disturbances can propagate.

For a given distribution of disturbances along $t = 0$ the solution for the region 2 can be considered as known. There are three degrees of freedom in the region 0 and another two corresponding to the displacements of the shock and the flame. If the tube is closed this gives one condition on three degrees of freedom in the region 1, reducing the number to two. If the tube is not closed at $x = 0$ then there are still only two degrees of freedom as disturbances cannot propagate along the positive characteristics in this case. In all there are seven unknowns - two in region 1, three in region 0, and the shock and flame disturbances - and there are only the six conservation equations to determine them in terms of the prescribed initial disturbances.

The fact that the system is non-evolutionary, i.e., that there is no unique solution to the problem of a given initial disturbance, is taken to mean that the system is unstable and cannot exist for any length of time in practice (2).

Since the problem is underprescribed by one equation let us seek a solution in which the region 2, ahead of the shock, is undisturbed, so that there are six equations governing the remaining seven variables. We can consider the problem to be that of finding the effect on the flow of a given, arbitrary, small displacement of the shock wave, i.e. of determining the six remaining unknowns in terms of the shock displacement. The solution for the disturbance problem, as posed in this fashion, is evidently self-generating and therefore unstable. The problem of continuous disturbances is complicated by the interplay of waves in the region 0 so that the value of any variable on the flame depends on the values of the shock displacement at three distinct instants (the three points of intersection with the shock of the three characteristics drawn through the point at the flame under consideration).

(ii) <u>The disturbance of the shock and the region between the waves</u>

It is to be assumed that no disturbances originate in the region 2 ahead of the shock. Hence this region remains at rest under the perturbation of the system, as disturbances originating at the shock cannot propagate into this region.

Consideration of the conservation equations across the shock, in perturbed form, gives three equations for the values of the disturbances immediately behind the shock in terms of the shock displacement. If the shock speed for the perturbed flow is $W_0 + \bar{\bar{W}}(t)$, where W_0 is the constant shock speed of the basic flow and $\bar{\bar{W}}(t)$ is of small order, then the boundary values of $\bar{u}_0$, $\bar{c}_0$, $\bar{s}_0$ on the shock are known in terms of $\bar{\bar{W}}(t)$. For the basic flow to exist at $t = 0$ it is necessary that $\bar{\bar{W}}(t) \longrightarrow 0$ as $t \longrightarrow 0$. It will be assumed that $\bar{\bar{W}}(t)$ is a continuous function so that no secondary shock waves are set up.

The perturbed form of the conservation equations across the shock are obtained by differentiating the basic equations 1.6, 1.7, 1.8. Elimination of Co between the equations 1.7, 1.8 gives

$$W_0^2 - \frac{\gamma+1}{2} W_0 u_0 - c_2^2 = 0 \tag{3.1}$$

which, on differentiation, gives

$$\begin{aligned} \bar{u}_0 &= \left(\frac{4}{\gamma+1} - \frac{u_0}{W_0}\right)\bar{W} \\ &= 2K_1\bar{W}, \text{ say.} \end{aligned} \tag{3.2}$$

Differentiation of 1.8 gives (using 3.2)

$$\bar{c}_0 = \frac{1}{kc_0}\left\{(W_0 - u_0)2K_1 + u_0\right\}\bar{W}$$

$$= (\gamma-1)K_2\bar{W}, \text{ say.} \tag{3.3}$$

From 1.6 we deduce

$$\frac{\bar{\rho}_0}{\rho_0} = 2\left\{\frac{u_0 - \frac{2}{\gamma+1}W_0}{W_0(u_0 - W_0)}\right\}\overline{W} \tag{3.4}$$

Since $c^2 = \gamma e^{\frac{S-S^*}{c_v}} \cdot \rho^{\gamma-1}$ we arrive at the following result for the entropy disturbance behind the shock

$$\bar{S}_0 = 2\left\{\frac{c_v}{c_0}(\gamma-1)K_2 - \frac{(\gamma-1)c_v}{W_0}\cdot\frac{\frac{2}{\gamma+1}W_0 - u_0}{W_0 - u_0}\right\}\overline{W} \tag{3.5}$$

$$= K_4\overline{W} \text{ say}$$

The pressure disturbance is given by

$$\frac{\bar{p}_0}{p_0} = 2\frac{\bar{c}_0}{c_0} + \frac{\bar{\rho}_0}{\rho_0} \tag{3.6}$$

Equations 3.2, 3.3, 3.5 give the boundary values of the perturbed flow in region 0 on the shock. Let us now find the general solution for the disturbances in this region. The governing equations, 2.14, 2.15, are

$$\bar{\alpha}_t + (u_0 + c_0)\bar{\alpha}_x + \tfrac{1}{2}\left\{(\gamma+1)\bar{\alpha} + (\gamma-3)\bar{\beta}\right\}\alpha_x = \tfrac{1}{2}H \tag{3.7}$$

$$\bar{\beta}_t + (u_0 - c_0)\bar{\beta}_x + \tfrac{1}{2}\left\{(3-\gamma)\bar{\alpha} - (\gamma+1)\bar{\beta}\right\}\beta_x = -\tfrac{1}{2}H \tag{3.8}$$

where u_0, c_0 are constant and $\alpha_x = \beta_x = 0$.

The function H is constant on the particle paths so that

$$H = H(x - u_0 t)$$

which can be written

$$\tfrac{1}{2}H(x - u_0 t) = \chi'(x - u_0 t), \text{ say.}$$

Thus 3.7, 3.8 become

$$\bar{\alpha}_t + (u_0 + c_0)\bar{\alpha}_x = \chi'(x - u_0 t) \tag{3.9}$$

$$\bar{\beta}_t + (u_0 - c_0)\bar{\beta}_x = -\chi'(x - u_0 t) \tag{3.10}$$

The characteristics of 3.9 are given by

$$dt = \frac{dx}{u_0 + c_0} = \frac{d\bar{\alpha}}{\chi'(x - u_0 t)}$$

Integration of the first pair gives

$$x - (u_0 + c_0)t = A_1, \text{ constant}$$

and the second pair can be written

$$dt = \frac{d\bar{\alpha}}{\chi'(A_1 + c_0 t)}$$

which gives

$$\bar{\alpha} = \frac{1}{c_0}\chi(A_1 + c_0 t) + A_2,$$

where A_2 is an arbitrary constant.

The general solution of equation 3.9 is therefore given by

$$\bar{\alpha} = F\{x - (u_0 + c_0)t\} + \frac{1}{c_0}\chi(x - u_0 t) \tag{3.11}$$

where F is an arbitrary function.

Similarly the general solution of 3.10 can be obtained in the form

$$\beta = G\left\{x-(u_0-c_0)t\right\}+\frac{1}{c_0}\chi(x-u_0t) \tag{3.12}$$

where G is an arbitrary function.
The entropy disturbance is given by

$$\frac{c_0^2}{2\gamma(\gamma-1)c_v}\cdot\bar{s}_0(x-u_0t)=\chi(x-u_0t) \tag{3.13}$$

The general solution for the perturbation of region 0 is given by 3.11, 3.12, 3.13. This contains three arbitrary functions F, G, χ which are constant along the positive characteristics, the negative characteristics and the particle paths respectively. These functions are determined by the conditions at the shock.

The path of the shock for the basic flow is

$$x = W_0 t$$

and for the perturbed flow it is

$$x = W_0 t + \int_0^t \bar{\bar{w}}(t)\,dt$$

Consider, for example, the value of the entropy perturbation $\bar{s}_0$ on the perturbed position of the front. This is

$$\bar{s}_0\left\{(W_0-u_0)t+\int_0^t \bar{\bar{w}}(t)\,dt\right\}$$
$$=\bar{s}_0\left\{(W_0-u_0)t\right\}+\bar{s}_0'.\int_0^t \bar{\bar{w}}(t)\,dt+\cdots$$

which has to be equated to $K_4\bar{\bar{W}}(t)$ from 3.5. Whereas the first term in the above expression is of first order smallness, the second is of second order and can therefore be neglected. In general the values on $x = W_0 t$ can be used for the boundary values on the perturbed path of the front.

Setting $x = W_0 t$ in 3.13 and using 3.5 gives

$$\mathcal{X}\left\{(W_0 - u_0)t\right\} = \frac{c_0^2 K_4}{2\gamma(\gamma-1)c_v}\overline{W}(t) = c_0 K_3 \overline{W}(t), \text{ say}$$

Hence

$$\mathcal{X}(x - u_0 t) = c_0 K_3 \overline{W}\left\{\frac{x - u_0 t}{W_0 - u_0}\right\} \qquad (3.14)$$

Similarly, setting $x = W_0 t$ in 3.11, 3.12 and using 3.2, 3.3 we obtain

$$(K_1 + K_2)\overline{W}(t) = F\left\{(W_0 - u_0 - c_0)t\right\} + K_3 \overline{W}(t)$$

$$(K_2 - K_1)\overline{W}(t) = G\left\{(W_0 - u_0 + c_0)t\right\} + K_3 \overline{W}(t)$$

and hence

$$F\left\{x - (u_0 + c_0)t\right\} = (K_1 + K_2 - K_3)\overline{W}\left\{\frac{x - (u_0 + c_0)t}{W_0 - (u_0 + c_0)}\right\}$$

$$G\left\{x - (u_0 - c_0)t\right\} = (K_2 - K_1 - K_3)\overline{W}\left\{\frac{x - (u_0 - c_0)t}{W_0 - (u_0 - c_0)}\right\}$$

After setting

$$R_1 = K_1 + K_2 - K_3$$

$$R_2 = K_1 - K_2 + K_3$$

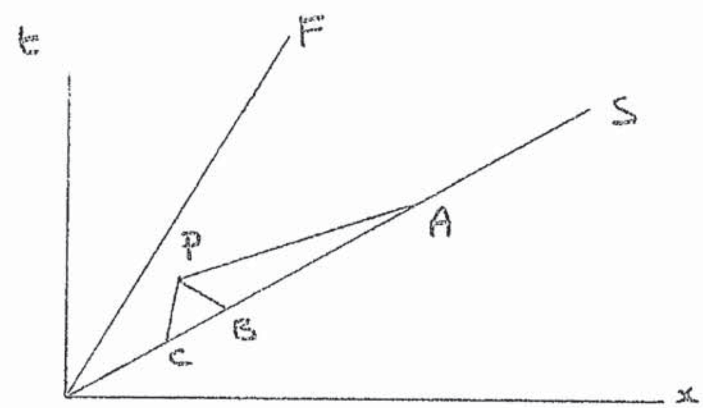
we can write the solution for the region 0 in the form

$$\bar{\alpha}_0 = k_1 \overline{W}\left\{\frac{x-(u_0+c_0)t}{W_0-(u_0+c_0)}\right\} + K_3 \overline{W}\left\{\frac{x-u_0 t}{W_0-u_0}\right\} \tag{3.15}$$

$$\bar{\beta}_0 = - k_2 \overline{W}\left\{\frac{x-(u_0-c_0)t}{W_0-(u_0-c_0)}\right\} + K_3 \overline{W}\left\{\frac{x-u_0 t}{W_0-u_0}\right\} \tag{3.16}$$

$$\bar{s}_0 = K_4 \overline{W}\left\{\frac{x-u_0 t}{W_0-u_0}\right\} \tag{3.17}$$

This solution can be considered geometrically as follows



P is any point in the region 0 and PA, PB, PC are the positive and negative characteristics and particle path through P. Then

$$\bar{\alpha}_0(P) = k_1 \overline{W}(A) + K_3 \overline{W}(C)$$

$$\bar{\beta}_0(P) = -k_2 \overline{W}(B) + K_3 \overline{W}(C)$$

$$\bar{s}_0(P) = K_4 \overline{W}(C)$$

The disturbances at P are seen to depend on the shock displacement at the three distinct points (or instants) A, B, C, due to the propagation of the disturbances along the three characteristics. It is observed that A is ahead of P in time.

This is not an inconsistency but is due to the fact that the disturbances are self-generating. Strictly speaking we cannot regard the disturbances as being caused by the displacement of the shock. Instead we must consider the perturbation of the system to be subject to certain conditions of the form 3.15, 3.16, 3.17. Nevertheless the solution can be expressed mathematically in terms of the function $\overline{W}(t)$.

This completes the solution for the region between the waves. The boundary values immediately ahead of the flame can be obtained by setting $x = V_0 t$ in this solution. These are

$$\bar{u}_0 = R_1 \overline{W}(k_1 t) + R_2 \overline{W}(k_2 t) \tag{3.18}$$

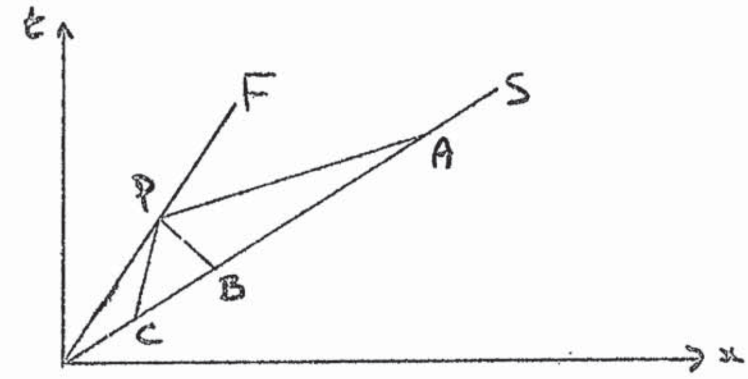
$$k \bar{c}_0 = R_1 \overline{W}(k_1 t) - R_2 \overline{W}(k_2 t) + 2 K_3 \overline{W}(k_3 t) \tag{3.19}$$

$$\bar{s}_0 = K_4 \overline{W}(k_3 t) \tag{3.20}$$

where

$$k_1 = \frac{V_0 - (u_0 + c_0)}{W_0 - (u_0 + c_0)}, \quad k_2 = \frac{V_0 - (u_0 - c_0)}{W_0 - (u_0 - c_0)}, \quad k_3 = \frac{V_0 - u_0}{W_0 - u_0}$$

and can be interpreted geometrically as follows



$$k_1 = \frac{t_A}{t_P}, \quad k_2 = \frac{t_B}{t_P}, \quad k_3 = \frac{t_C}{t_P}$$

Note that $k_1 > 1 > k_2 > k_3 > 0$.

Using the equation of state 2.4 we can evaluate the boundary value of the density disturbance ahead of the flame.

$$\frac{\bar{\rho}_0}{\rho_0} = k\frac{\bar{c}_0}{c_0} - \frac{\bar{s}_0}{c_v(\gamma-1)}$$

$$= \frac{1}{c_0}\left\{R_1\bar{W}(k_1t) - R_2\bar{W}(k_2t) - 2(\gamma-1)K_3\bar{W}(k_3t)\right\} \qquad (3.21)$$

(iii) <u>The disturbance of the flame and the region behind it</u>

The region 1 behind the flame is stationary in the basic flow so that the equations governing the propagation of disturbances throughout this region are

$$\bar{\alpha}_t + c_1\bar{\alpha}_x = \tfrac{1}{2}H(x)$$

$$\bar{\beta}_t - c_1\bar{\beta}_x = -\tfrac{1}{2}H(x)$$

$$\bar{s}_{1t} = 0 \qquad (c_1 \text{ constant})$$

where $c_1^2\bar{s}_{1x} = \gamma(\gamma-1)c_v H(x)$

$$= 2\gamma(\gamma-1)c_v \mathcal{K}_1'(x), \text{ say}$$

The general solution of these equations is

$$\bar{\alpha}_1 = F(c_1t - x) + \frac{1}{c_1}\mathcal{K}_1(x)$$

$$\bar{\beta}_1 = G(c_1t + x) + \frac{1}{c_1}\mathcal{K}_1(x)$$

$$\bar{s}_1 = \frac{2\gamma(\gamma-1)c_v}{c_1^2}\mathcal{K}_1(x)$$

The reflection condition at the closed end, i.e. that $\bar{u}_1 = \bar{\alpha}_1 - \bar{\beta}_1 = 0$ on $x = 0$, implies that $F \equiv G$ so that

$$\bar{\beta}_1 = F(c_1t+x) + \frac{1}{c_1}\chi_1(x)$$

Also

$$\frac{\bar{\rho}_1}{\rho_1} = k\frac{\bar{c}_1}{c_1} - \frac{\bar{s}_1}{c_v(\gamma-1)}$$

$$= \frac{1}{c_1}F(c_1t-x) + \frac{1}{c_1}F(c_1t+x) - \frac{2(\gamma-1)}{c_1^2}\chi_1(x)$$

The boundary values on the flame take the form

$$\bar{u}_1 = \bar{\alpha}_1 - \bar{\beta}_1 = F\{(c_1-V_0)t\} - F\{(c_1+V_0)t\} \qquad (3.22)$$

$$k\bar{c}_1 = \bar{\alpha}_1 + \bar{\beta}_1 = F\{(c_1-V_0)t\} + F\{(c_1+V_0)t\} + \frac{2}{c_1}\chi_1(V_0t) \qquad (3.23)$$

$$\frac{\bar{\rho}_1}{\rho_1} = \frac{1}{c_1}F\{(c_1-V_0)t\} + \frac{1}{c_1}F\{(c_1+V_0)t\} - \frac{2(\gamma-1)}{c_1^2}\chi_1(V_0t) \qquad (3.24)$$

These values for the flow behind the flame are related to the values ahead of the flame, derived previously, by the conservation equations in perturbed form, which are obtained by differentiating 1.9, 1.10, 1.11 and setting $u_1 = 0$.

$$-\frac{u_0}{V_0}\bar{V} + (V_0-u_0)\frac{\bar{\rho}_1}{\rho_1} - \left(1-\frac{u_0}{V_0}\right)\bar{u}_1 = (V_0-u_0)\frac{\bar{\rho}_0}{\rho_0} - \bar{u}_0 \qquad (3.25)$$

$$\frac{u_0}{V_0-u_0}\left\{c_0^2+\gamma(V_0-u_0)^2\right\}\bar{V} + 2(V_0-u_0)c_1\bar{c}_1 + \left\{c_0^2-\gamma(V_0^2-u_0^2)\right\}\bar{u}_1$$
$$= \left\{\frac{V_0}{V_0-u_0}c_0^2 - \gamma V_0(V_0-u_0)\right\}\bar{u}_0 + 2c_0V_0\bar{c}_0 \qquad (3.26)$$

$$u_0\bar{V} - V_0\bar{u}_1 + kc_1\bar{c}_1 = (u_0-V_0)\bar{u}_0 + kc_0\bar{c}_0 \qquad (3.27)$$

The terms on the right hand sides of these equations, $\bar{u}_0$, $\bar{c}_0$, $\frac{\bar{\rho}_0}{\rho_0}$, are given in terms of the function $\bar{W}$ by 3.18, 3.19, 3.21. Also $\bar{u}_1$, $\bar{c}_1$, $\frac{\bar{\rho}_1}{\rho_1}$ can be expressed in terms of the two functions F, χ_1, (3.22, 3.23, 3.24) so that the equations 3.25, 3.26, 3.27 can be regarded as three simultaneous linear algebraic equations for the functions $\bar{\bar{V}}$, F, χ_1, in terms of the function $\bar{\bar{W}}$. These equations contain the values of $\bar{\bar{V}}$ and χ_1 at the single points t and $V_0 t$ respectively. F appears at the two points $(c_1 - V_0)t$, $(c_1 + V_0)t$ and $\bar{\bar{W}}$ at $k_1 t$, $k_2 t$, $k_3 t$.

Elimination of $\bar{\bar{V}}$ from the simultaneous equations yields

$$V_0(V_0-u_0)\frac{\bar{\rho}_1}{\rho_1} - (2V_0-u_0)\bar{u}_1 + kc_1\bar{c}_1$$
$$= V_0(V_0-u_0)\frac{\bar{\rho}_0}{\rho_0} - (2V_0-u_0)\bar{u}_0 + kc_0\bar{c}_0 \tag{3.28}$$

$$c_0^2 V_0\frac{\bar{\rho}_1}{\rho_1} - \gamma u_0(V_0-u_0)\bar{u}_1 - kc_1(V_0-u_0)\bar{c}_1$$
$$= c_0^2 V_0\frac{\bar{\rho}_0}{\rho_0} - \gamma u_0(V_0-u_0)\bar{u}_0 - kc_0(V_0-\gamma u_0)\bar{c}_0 \tag{3.29}$$

We can now substitute for $\bar{u}_1$, $\frac{\bar{\rho}_1}{\rho_1}$, $\bar{c}_1$ in terms of F, χ_1 and then eliminate $\chi_1(V_0 t)$ to obtain the following equation (using 1.10b)

$$k_4 F(k_4 t)\left\{\frac{c_1}{\gamma V_0} - 1 + \frac{\gamma-1}{\gamma}\frac{c_0^2}{c_1(V_0-u_0)}\right\}$$
$$+ k_5 F(k_5 t)\left\{1 + \frac{c_1}{\gamma V_0} + \frac{\gamma-1}{\gamma}\frac{c_0^2}{c_1(V_0-u_0)}\right\}$$

$$= \frac{c_1}{\gamma(V_0-u_0)}\left\{c_0^2 + (V_0-u_0)^2\right\}\frac{\bar{\rho}_0}{\rho_0} - c_1\left\{2 + (\gamma-1)\frac{u_0 V_0}{c_1^2}\right\}\bar{u}_0$$
$$+ \frac{2c_0\bar{c}_0}{\gamma(V_0-u_0)c_1}\left\{c_1^2 + V_0(V_0-u_0)\right\},$$

(where $k_4 = c_1 - V_0$, $k_5 = c_1 + V_0$ so that $k_5 > k_4 > 0$)

which can be written as

$$B_1 F(k_4 t) + B_2 F(k_5 t) = E(t) \tag{3.30}$$

where $B_2 > B_1 > 0$ since $V_0 < c_1$ (using 1.10a).

Hence

$$-\frac{B_1^2}{B_2} F\left(\frac{k_4^2}{k_5} t\right) \quad - B_1 F(k_4 t) \qquad = -\frac{B_1}{B_2} E\left(\frac{k_4}{k_5} t\right)$$

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$$B_1\left(-\frac{B_1}{B_2}\right)^n F\left(\frac{k_4^{n+1}}{k_5^n} t\right) + B_1\left(-\frac{B_1}{B_2}\right)^{n-1} F\left(\frac{k_4^n}{k_5^{n-1}} t\right) = \left(-\frac{B_1}{B_2}\right)^n E\left(\frac{k_4^n}{k_5^n} t\right)$$

Summing these equations and taking the limit as $n \to \infty$ gives a remainder on the left hand side

$$\lim_{n \to \infty} \left\{ B_1\left(-\frac{B_1}{B_2}\right)^n \cdot F\left(\frac{k_4^{n+1}}{k_5^n} t\right) \right\}$$

$$= 0 \qquad \text{since } |B_1| < |B_2|,\ |k_4| < |k_5|$$

$$\text{and } \lim_{y \to 0} F(y) = 0.$$

Thus

$$B_2 F(k_5 t) = \sum_{r=0}^{\infty} \left(-\frac{B_1}{B_2}\right)^r E\left(\frac{k_4^r}{k_5^r} t\right)$$

so that

$$F(t) = \frac{1}{B_2} \sum_{r=0}^{\infty} \left(-\frac{B_1}{B_2}\right)^r E\left(\frac{k_4^r}{k_5^{r+1}} t\right).$$

This is the particular solution of 3.30 and in fact represents the general solution also. The complementary function, i.e. the solution of the reduced equation with $G(t) \equiv 0$, is obtained by setting $F(t) \propto t^n$, n constant, so that

$$B_1 k_4^n + B_2 k_5^n = 0$$

Since B_1, B_2, k_4, k_5 are all positive, this equation has no solution for n positive, which is essential in order that F(t) satisfies the conditions of the problem, i.e. that all disturbances are small and tend to zero as t tends to zero.

The solution for $\overline{V}$ is obtained by substituting for $\overline{u}_1, \overline{c}_1, \overline{\frac{\rho_1}{\rho_1}}$ into 3.25, 3.27 and eliminating $\mathcal{X}(Vt)$ between them, which gives

$$u_0\left\{\gamma-1-\frac{c_1^2}{V_0(V_0-u_0)}\right\}\overline{V} = 2\rho_0\overline{c}_0 + c_1^2\frac{\overline{\rho}_0}{\rho_0} - (V_0-u_0)\left\{\gamma-1+\left(\frac{c_1}{V_0-u_0}\right)^2\right\}\overline{u}_0$$

$$-k_4\left\{\gamma-1-\frac{c_1}{V_0}\right\}F(k_4t) - k_5\left\{\gamma-1+\frac{c_1}{V_0}\right\}F(k_5t)$$

$$= 2c_0\overline{c}_0 + c_1^2\frac{\overline{\rho}_0}{\rho_0} - (V_0-u_0)\left\{\gamma-1+\left(\frac{c_1}{V_0-u_0}\right)^2\right\}\overline{u}_0 - \frac{k_5}{B_2}\left\{\gamma-1+\frac{c_1}{V_0}\right\}E(t) \tag{3.31}$$

$$-\frac{1}{B_2}\left\{k_4\left(\gamma-1-\frac{c_1}{V_0}\right)-\frac{B_1}{B_2}k_5\left(\gamma-1+\frac{c_1}{V_0}\right)\right\}\cdot\sum_{\nu=1}^{\infty}\left(-\frac{B_1}{B_2}\right)^{\nu-1}E\left\{\left(\frac{k_4}{k_5}\right)^{\nu}t\right\}$$

where

$$E(t) = \left[\frac{c_0}{(V_0-u_0)c_1}\left\{V_0(V_0-u_0)+c_1^2-2c_0V_0\right\}+(\gamma+1)\frac{u_0V_0}{c_1}\right]R_1\overline{W}(k_1t)$$

$$+\frac{1}{c_1}\left[V_0(V_0-u_0)+c_1^2+2c_0V_0-(\gamma+1)\frac{u_0V_0}{c_0}(V_0-u_0)\right]R_2\overline{W}(k_2t)$$

$$+\frac{2(\gamma-1)u_0V_0(V_0-u_0)}{c_0c_1}K_3\overline{W}(k_3t),$$

$$\overline{u}_0 = R_1\overline{W}(k_1t) + R_2\overline{W}(k_2t),$$

$$k\overline{c}_0 = R_1\overline{W}(k_1t) - R_2\overline{W}(k_2t) + 2K_3\overline{W}(k_3t),$$

and

$$c_0\frac{\overline{\rho}_0}{\rho_0} = R_1\overline{W}(k_1t) - R_2\overline{W}(k_2t) - 2(\gamma-1)K_3\overline{W}(k_3t).$$

The equation 3.31 defines the displacement of the flame speed, $\bar{\bar{V}}$, in terms of the displacement of the shock speed, $\bar{\bar{W}}$. If the function $\bar{\bar{W}}(t)$ is chosen to be λt^n, where λ, n are constants and $n > 0$, then $V(t)$ is also proportional to t^n. In this case the infinite series is a geometric progression of common ratio $-\frac{B_1}{B_2} \cdot \left(\frac{k_4}{k_5}\right)^n$, which has a numerical value < 1. Thus the series is convergent and $\bar{\bar{V}}(t)$ is bounded if λ is finite. If $\bar{\bar{W}}(t)$ is chosen to be a periodic function, say $\bar{\bar{W}} = \lambda \sin wt$ (λ, w constants), then $E(t)$ is a linear combination of $\sin(k_1 wt)$, $\sin(k_2 wt)$, $\sin(k_3 wt)$, so that the infinite series is a linear combination of the three series

$$\sum_{r=0}^{\infty} \left(-\frac{B_1}{B_2}\right)^{r-1} \sin\left(\frac{k_4^r}{k_5^r} k_j wt\right), \qquad j = 1, 2, 3.$$

all of which are convergent since $|B_1| < |B_2|$. It is observed that if $\bar{\bar{W}}$ has frequency w then $\bar{\bar{V}}$ is a bounded combination of terms having frequencies

$$\left(\frac{k_4}{k_5}\right)^r k_j w \quad , \text{ where } r \text{ is a non-negative integer and } j = 1, 2, 3.$$

Since the problem is linear the solution due to $\bar{\bar{W}}(t)$ being a linear combination of oscillatory terms and terms of the form λt^n is simply the sum of the solutions due to each individual term in $\bar{\bar{W}}(t)$.

It is observed that the infinite sum in 3.31 is finite provided the function $E(t)$ is bounded, by comparison with a multiple of the geometric series $\sum_{\sigma=1}^{\infty}\left(-\frac{B_1}{B_2}\right)^{\sigma-1}$. This means that a bounded displacement $\overline{W}$ will always produce a bounded displacement $\overline{V}$. However, a solution proportional to a positive power of t is unbounded as $t \to \infty$, although our solution would no longer be valid. Thus the existence of such solutions shows that the system is unstable.

(iv) <u>The case of a uniformly accelerated shock</u>

Consider the particular case in which the shock accelerates uniformly, i.e. $\overline{W}(t) = \lambda t$ (λ constant) and the path of the shock is $x = W_0 t + \frac{1}{2}\lambda t^2$. The solution of the simultaneous linear equations 3.25, 3.26, 3.27, gives the following values for $\overline{u_1}$, $\overline{V}$

$$\frac{\overline{V}}{\lambda} = \frac{A_3 A_5 - A_2 A_6}{A_1 A_5 - A_2 A_4} \qquad (3.32)$$

$$\frac{\overline{u_1}}{\lambda V} = \frac{A_3 A_4 - A_1 A_6}{A_1 A_5 - A_2 A_4} \qquad (3.33)$$

where $A_1 = u_0$

$$A_2 = c_0^2 - (\gamma+1)u_0(V_0-u_0) + V_0^2 - u_0^2$$

$$A_3 = (2u_0-V_0)(k_1R_1+k_2R_2) + \frac{\gamma-1}{\gamma}c_0(k_1R_1-k_2R_2+2k_3K_3)$$
$$+ \frac{(V_0-u_0)c_1^2}{\gamma V_0 c_0}\left\{k_1R_1-k_2R_2-2(\gamma-1)k_3K_3\right\}$$

$$A_4 = \frac{-u_0}{V_0} + (\gamma-1)\frac{u_0(V_0-u_0)}{c_1^2}$$

$$A_5 = 2(V_0-u_0) + (\gamma-1)(V_0-u_0)\left(1-\frac{V_0^2}{c_1^2}\right)$$

$$A_6 = -\left\{1+(\gamma-1)\left(\frac{V_0-u_0}{c_1}\right)^2\right\}(k_1R_1+k_2R_2) + (\gamma-1)\frac{(V_0-u_0)c_0}{c_1^2}\left\{k_1R_1-k_2R_2+2k_3K_3\right\}$$
$$+ \frac{V_0-u_0}{c_0}\left\{k_1R_1-k_2R_2-2(\gamma-1)k_3K_3\right\}.$$

The solution for $\frac{\overline{V}}{\lambda}$, $\frac{\overline{u_1}}{\lambda}$ has been tabulated over a range of values of W for one set of values of the basic parameters (i.e. γ, Q, p_2, ρ_2).

For the sake of comparison we can consider the problem neglecting the fact that disturbances are propagated at certain speeds along the characteristics, assuming instead that any disturbances are propagated instantaneously. This is equivalent to assuming that any algebraic relations, involving the physical variables in the separate regions of flow, which hold for the basic flow, also hold for the perturbed values. As for the disturbed system, there is one degree of freedom in the basic flow, e.g. W_0, so that if the shock speed changes to $W_0 + \overline{W}$ then the disturbances in the flow are found simply by differentiating the six conservation equations relating the flows in each region.

As previously, the values behind the shock are

$$\overline{u}_0 = 2K_1\overline{W}$$
$$\overline{c}_0 = (\gamma-1)K_2\overline{W}$$
$$\frac{\overline{\rho}_0}{\rho_0} = 2\,\frac{u_0 - \frac{3}{\gamma+1}W_0}{W_0(u_0 - W_0)}\cdot\overline{W}$$

and in this case these equations represent the values of $\overline{u}_0$, $\overline{c}_0$, $\frac{\overline{\rho}_0}{\rho_0}$ at any point in the region 0.

Elimination of c_1^2 between 1.10, 1.11 with $u_1 = 0$ gives

$$(\gamma-1)\frac{Q}{u_0} + V + \frac{1}{k}u_0 = \frac{c_0^2}{V_0 - u_0}$$

which can be differentiated to give $\overline{V}$ in terms of $\overline{u}_0, \overline{c}_0$ and hence in terms of $\overline{W}$. This results in the equation

$$\frac{\overline{V}}{\overline{W}} = \frac{2(V_0-u_0)^2}{(V_0-u_0)^2+c_0^2}\left[\left\{\frac{c_0^2}{(V_0-u_0)^2}+(\gamma-1)\,\frac{Q-\frac{1}{2}u_0^2}{u_0^2}\right\}K_1+\frac{(\gamma-1)\,c_0}{V_0-u_0}K_2\right] \tag{3.34}$$

$= \lambda_1$ say.

If the path of the shock is

$$x = W_0 t + \int_0^t \overline{W}(t)\,dt$$

then the path of the flame is

$$x = V_0 t + \int_0^t \overline{V}(t)\,dt$$

$$= V_0 t + \lambda_1\int_0^t \overline{W}(t)\,dt$$

Consider the value of λ_1 in the case of a limitingly weak shock. The unit of velocity of the system is taken to be the sound speed of the initial gas i.e. the region 2. We have the following approximations for u_0, c_0, V_0, W_0

$$W_0 \sim 1,\quad c_0 \sim 1,\quad V_0 \ll 1,\quad u_0 \ll 1$$

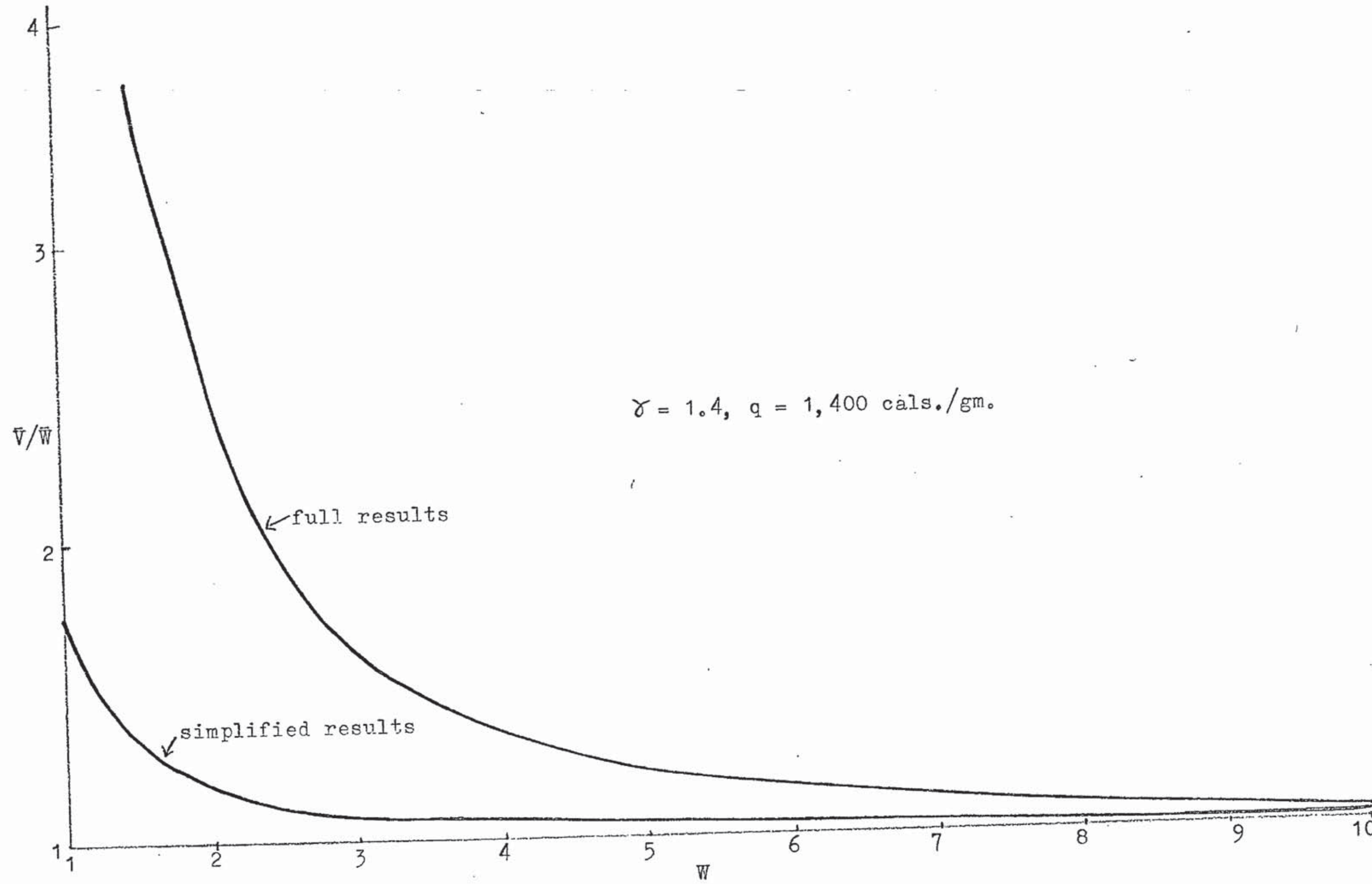
and

$$V_0 - u_0 \sim \frac{u_0}{(\gamma-1)Q}$$

so that

$$\lambda_1 \sim 2(V_0-u_0)^2\left\{\frac{c_0^2}{(V_0-u_0)^2}+\frac{(\gamma-1)Q}{u_0^2}\right\}K_1$$

$$= \frac{4}{\gamma+1}\left\{1+\frac{1}{(\gamma-1)Q}\right\} \tag{3.35}$$



γ = 1.4, q = 1,400 cals./gm.
full results
simplified results
$\bar{V}/\bar{W}$
W
1
2
3
4
5
6
7
8
9
10

FIG. 3.1

The approximations for a limitingly strong shock are

$$u_0 \sim \frac{2}{\gamma+1} W_0 \quad , \quad V_0 \sim W_0 \quad , \quad c_0^2 \sim \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} W_0$$

$$K_1 \sim \frac{2}{\gamma+1} \quad , \quad K_2 \sim \frac{2\gamma}{c_0(\gamma+1)^2}$$

so that $\lambda_1 \sim 1$.

Calculations have been made for this 'simplified' solution and the previous 'wave' solution. The values assigned to the parameters of the problem were

$\gamma = 1.4$

$c_2 = 1$ (by definition)

$\rho_2 = 1$ gm./litre

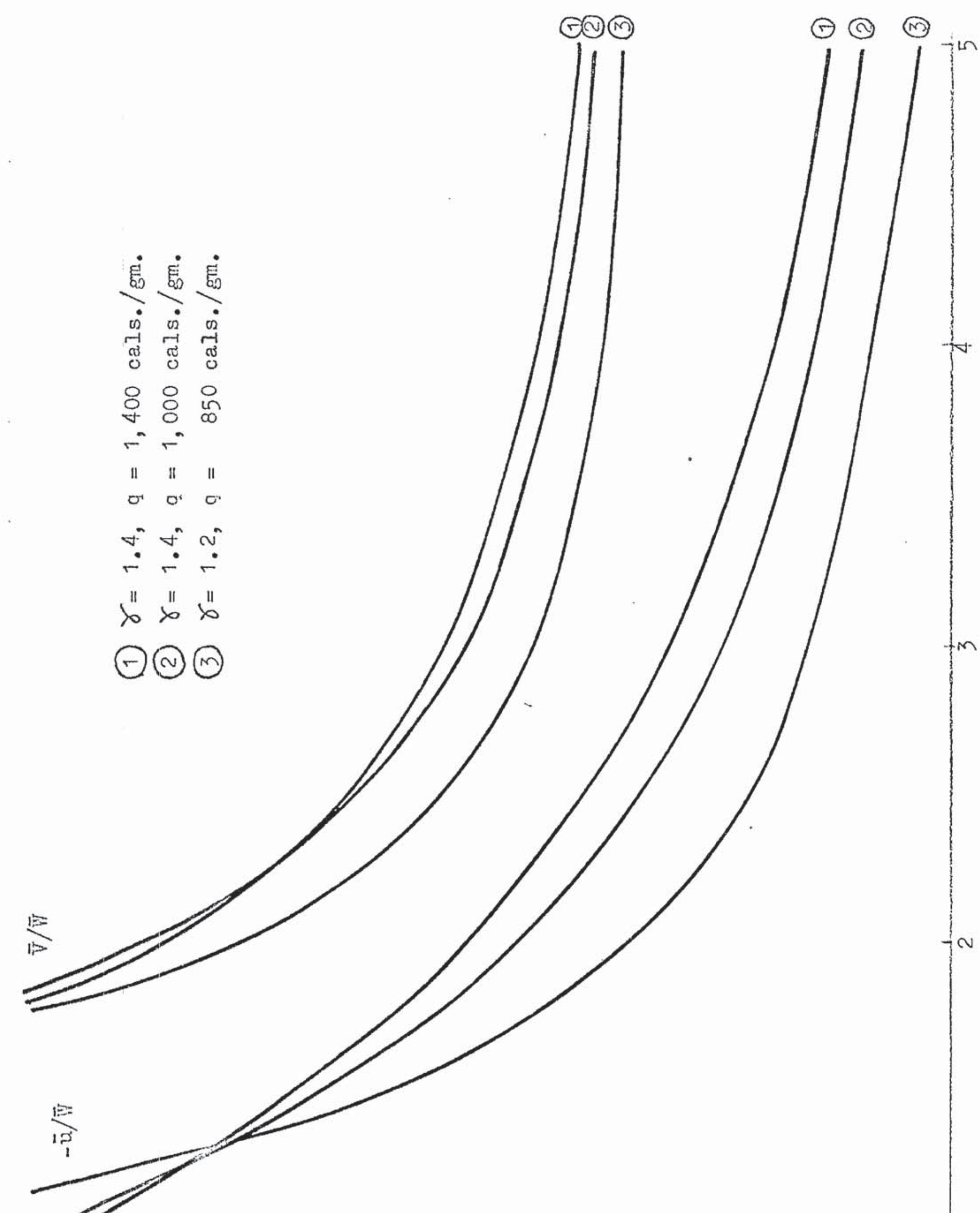
$p_2 = 1$ atmosphere

$Q = 1,400$ calories/gm.

$= 41.8$ on the scale defined by $c_2 = 1$.

With these values the upper limit, W_0^* for the shock speed for these subsonic flows takes the value 4.69. The range of values of W_0 used in the calculations was $1 \leq W_0 \leq 10$.

The two solutions are plotted on the graph Fig. 3.1. It is observed that both solutions for $\frac{\overline{V}}{\overline{W}}$ decrease steadily as W_0 is increased and tend asymptotically to the value unity at $W_0 = \infty$. The value given by the full theory is always greater than that due to the simplified theory, particularly for lower speed flows.



① $\gamma = 1.4$, $q = 1,400$ cals./gm.
② $\gamma = 1.4$, $q = 1,000$ cals./gm.
③ $\gamma = 1.2$, $q = 850$ cals./gm.
$\bar{V}/\bar{W}$
$-\bar{u}/\bar{W}$
W
0
1
2
3
4
5

FIG. 3.2

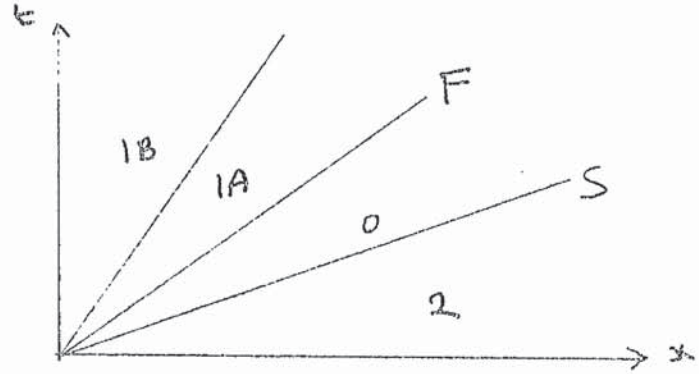
The limiting values at $W = 1$ are ∞ and 1.77 (given by 3.35) respectively.

The graph Fig. 3.2 shows the results of the full theory for three combinations of γ, Q. The values of $\frac{\bar{\bar{V}}}{\bar{\bar{W}}}$, $\frac{\bar{u}}{\bar{\bar{W}}}$ are plotted. The former is always greater than unity and has the limiting values ∞ and 1 at the lower and upper end of the scale, whereas $\frac{\bar{u}}{\bar{\bar{W}}}$ is always negative and has a finite limit at $W = 1$ and the limit 0 at $W = \infty$.

4. The Chapman-Jouguet Model

(i) Stability and Uniqueness

The representation of the Chapman-Jouguet model in the x-t plane is as follows.



The regions of flow are

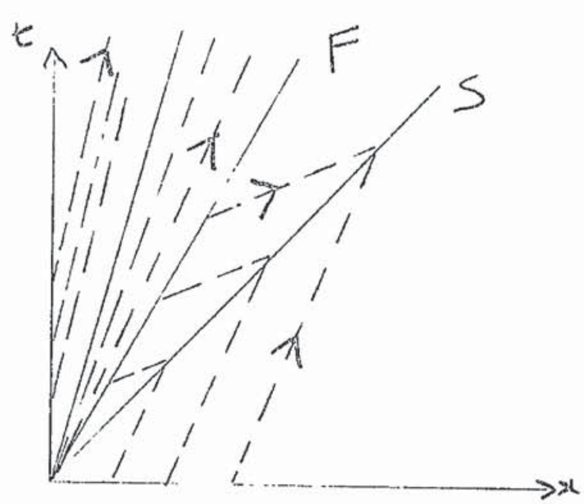
2 - state of rest ahead of the shock

0 - uniform motion between the wave fronts

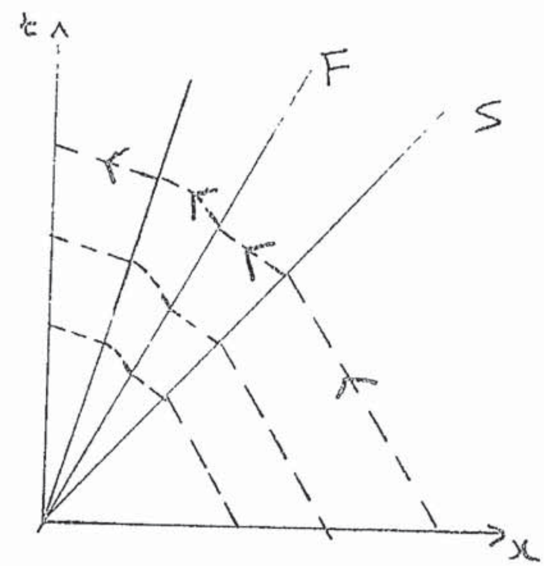
1A - simple wave behind the deflagration

1B - region of rest.

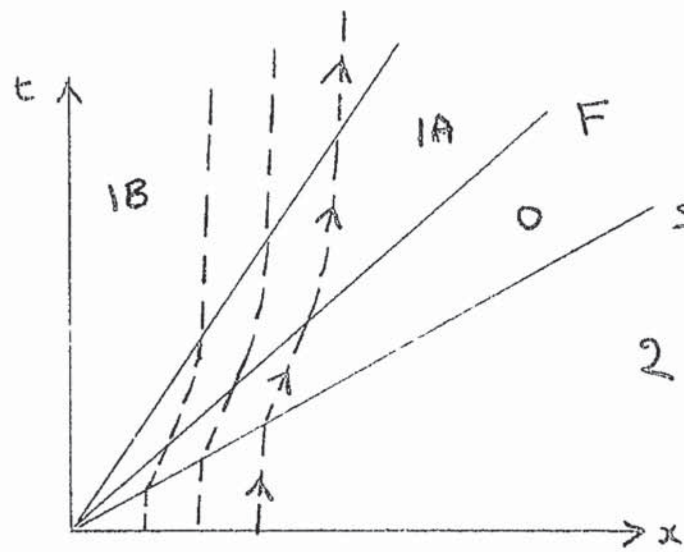
The positive and negative characteristics and particle paths are sketched



positive characteristics (C+)



negative characteristics (C-)



particle paths (Co)

The direction is that of time increasing.

In order to consider the stability of the system consider the effect of a disturbance propagating through the region 2 when it meets the shock. This results in two outgoing waves (C- and Co in region 0) as well as the displacement of the shock. The effect of a disturbance at the flame is to initiate three waves (C+ in region 0 and C-, Co in region 1A) as well as to displace the flame. On the last characteristic of the simple wave, i.e. the line separating 1A and 1B, two waves are propagated (C-, Co in region 1B). Since this is a sonic line the fact that the tube is closed does not affect the problem as waves reflected at the end will not catch up with this line. Thus there are 7 unknown waves and the 2 unknown displacements of the wave fronts. The equations governing the basic flow are ten in number composed of

1. the three conservation equations across the shock
2. the three conservation equations across the flame

3. the three equations demanding continuity between the regions 1A and 1B

4. the Chapman-Jouguet condition.

If these four sets of equations were applied to the perturbation of the system the problem would be overspecified. Since none of the equations 1, 2, 3 above can be relaxed without violating the laws of mechanics, let us relax the Chapman-Jouguet condition for the disturbances. The system is now evolutionary, i.e. we have nine equations to determine the nine unknowns appearing in the problem.

The same result is obtained by considering the effect of continuous disturbances ahead of the shock. The general solution of the region 0 has three degrees of freedom (as for the subsonic model) each corresponding to an arbitrary function. The simple wave, region 1A, has only two as the positive characteristics meet at O. The disturbances propagating along these characteristics are therefore constant, and in fact zero since all disturbances are zero at O. The region 2 has two degrees of freedom in its solution, the disturbances along the positive characteristics being zero if the tube is not closed and determined by the reflection condition at $x = 0$ if the end is closed. In either case they do not appear in the boundary values on the limiting sonic line and so cannot affect the solution ahead of this line. Therefore the problem has seven unknowns (functions) together with the two unknown wave front displacements, agreeing with the result of the previous paragraph.

The problem arises as to the effect on the system of an arbitrary distribution of disturbances in the region 2. The state of this region at $t = 0$ can be defined as a perturbation on the basic state of a uniform region at rest. However it will be of much interest to find out whether or not there exist any unstable, self-generating solutions of the type found for the subsonic model. In particular there may be some value of n for which such an unstable solution exists with $\overline{\overline{W}} = \lambda t^{n}$. This is in effect the complementary function of the solution. This problem will be investigated before the more general problem of a given disturbance in the region 2. To evaluate this latter solution disturbances in region 2 will have to be introduced so that the boundary conditions across the shock, and hence the solution obtained for the region 0 in the previous section, will have to be modified.

(ii) The disturbance of the simple wave

Let us consider the perturbation of the simple wave, which is the same for both problems.

From the theory of characteristics we obtain the following well known results for the basic flow in the wave (which is isentropic) $\beta = -\frac{1}{2}u_1 + \frac{c_1}{\gamma - 1}$ = constant on the lines $\dot{x} = u_1 - c_1$

i.e. the negative characteristics.

Since the characteristics $\dot{x} = u_1 - c_1$ come from the uniform region 1B it follows that

$$\beta = \frac{c_{1R}}{\gamma-1} = \beta_0 , \quad \text{say (constant)}$$

where c_{1R} is the rest sound speed of the simple wave so that $x = c_{1R}t$ is the limiting sonic line.

Also

$$\alpha = \tfrac{1}{2}u_1 + \frac{c_1}{\gamma-1} = \text{const} \quad \text{on } \dot{x} = u_1 + c_1$$

Therefore u_1, c_1 are constant along the positive characteristics, $\dot{x} = u_1 + c$, which are straight lines through the origin, the equations being $\frac{x}{t} = u_1 + c_1$. Thus we can find u_1, c_1, α as functions of x, t

$$u_1 = \frac{2}{\gamma+1}\left\{\frac{x}{t} - (\gamma-1)\beta_0\right\}$$

$$c_1 = \frac{\gamma-1}{\gamma+1}\left\{\frac{x}{t} + 2\beta_0\right\}$$

$$\alpha = \frac{3-\gamma}{\gamma+1}\beta_0 + \frac{2}{\gamma+1}\frac{x}{t}$$

Hence we can write the equations governing the perturbation of the simple wave, i.e. 2.14, 2.15, as

$$t\bar{\alpha}_t + x\bar{\alpha}_x + \left\{\bar{\alpha} + \frac{\gamma-3}{\gamma+1}\bar{\beta}\right\} = \tfrac{1}{2}t.H \qquad (4.1)$$

$$t\bar{\beta}_t + t(u_1 - c_1)\bar{\beta}_x = -\tfrac{1}{2}t.H \qquad (4.2)$$

One set of characteristics of 4.2 is given by

$$dt = \frac{dx}{u_1 - c_1}$$

i.e. the negative, curvilinear characteristics of the simple wave. This equation is

$$\dot{x} = \frac{3-\gamma}{\gamma+1}\frac{x}{t} - 4\frac{\gamma-1}{\gamma+1}\beta_0$$

which has the solution

$$\frac{\gamma-1}{\gamma+1}(x+2\beta_0 t) = c_1 t = \text{const} \times t^{\frac{3-\gamma}{\gamma+1}}$$

so that one form of these characteristics is

$$y^{\mu} = c_1 t^{2\mu} \quad = \text{constant, which defines } y \tag{4.3}$$

where $\mu = \frac{\gamma-1}{\gamma+1}$

and since $\rho_1 \propto c_1^k$ an alternative form is

$$\rho_1 c_1 t^2 = \text{constant.}$$

Before solving the equations 4.1, 4.2, it is necessary to find the equations of the particle paths in order to find the form of the function $H(x, t)$. These lines are defined by

$$\dot{x} = u_1$$

$$\text{i.e.} \quad \dot{x} = \frac{2}{\gamma+1}\left\{\frac{x}{t} - (\gamma-1)\beta_0\right\}$$

which has the solution

$$x + 2\beta_0 t = \text{const} \times t^{\frac{2}{\gamma+1}}$$

or $\qquad c_1 t^{\mu} = \text{const}$

or $\qquad \rho_1 c_1 t = \text{const.}$

Therefore the stream function ψ can be written as a function of either of the latter two variables given above. For convenience let us write

$$H(x,t) = 2\frac{c_1}{t} W'(c_1^2 t^{2\mu})$$

$$= 2\frac{c_1}{t} W'(c_1 y^{\mu}) \tag{4.4}$$

where W is an arbitrary function.

The equation 4.2 can now be solved for $\bar{\beta}$. One set of characteristics of this equation is the set of curvilinear characteristics of the simple wave, so that we can use the integral y = const to integrate the equation

$$\begin{aligned}\bar{\beta}_t &= -\tfrac{1}{2} H(x,t) \\ &= -\frac{c_1}{t} w'(c_1 y^{\mu}) \\ &= -y^{\mu}.t^{-2\mu-1} w'(y^{2\mu} t^{-2\mu})\end{aligned}$$

Therefore $$\bar{\beta} = \frac{1}{2\mu y^{\mu}} w(c_1 y^{\mu})$$

is a particular integral of 4.2 and the general solution is

$$\bar{\beta} = \frac{1}{2\mu y^{\mu}} w(c_1 y^{\mu}) \quad + \text{ some function of } y$$

$$= \frac{1}{2\mu y^{\mu}} w(c_1 y^{\mu}) + 2y^{\frac{1}{2}} F_0'(y) \qquad (4.5)$$

say, where F_0 is an arbitrary differentiable function.

The solution 4.5 can now be used to solve 4.1 for $\bar{\alpha}$. One set of characteristics of 4.1 is the set of rectilinear characteristics of the simple wave

i.e. $\frac{x}{t}$ = const.

or c_1 = const.

Therefore we can integrate the following equation for $\bar{\alpha}$ treating c_1 as a constant

$$\bar{\alpha}_t = \tfrac{1}{2} H(x,t) - \tfrac{1}{t}\bar{\alpha} + \frac{3-\gamma}{\gamma+1}\frac{\bar{\beta}}{t}$$

i.e.

$$\frac{d}{dt}(\bar{\alpha}t) = c_1 w'(c_1^2 t^{2\mu}) + \frac{3-\gamma}{\gamma+1}\left\{\frac{1}{2\mu c_1 t^{2\mu}} w(c_1 t^{2\mu}) + 2c_1^{\frac{1}{2\mu}} t F_0'(c_1^{\frac{1}{\mu}} t^2)\right\}$$

and hence

$$\bar{\alpha} = \frac{1}{2\mu y^{\mu}} w(c_1 y^{\mu}) + \frac{3-\gamma}{\gamma+1} y^{\frac{1}{2}} F_0'(y) \qquad (4.6)$$

This is a particular solution. To obtain the general solution for $\bar{\alpha}$ an additional term, constant on the rectilinear characteristics of the wave (e.g. $G(\frac{x}{t})$ for an arbitrary function G), must be added to 4.6. Since the characteristics $\frac{x}{t}$ = constant meet at O, G must be constant and in fact zero, since $G(\frac{x}{t}) \longrightarrow 0$ as x, t both $\longrightarrow 0$ along any path. Thus $G(\frac{x}{t}) \equiv 0$ and $\bar{\alpha}$, $\bar{\beta}$ are given by 4.5, 4.6.

The entropy disturbance $\bar{s}_1$ can be found in terms of w as follows. If

$$\bar{s}_1 = w_0(\psi) = w_0(c_1^2 t^{2\mu}) \text{ , say}$$

then the condition

$$c_1^2 \bar{s}_{1x} = \gamma(\gamma-1) c_v H(x,t)$$

implies that

$$c_1^2 w_0'(c_1^2 t^{2\mu}) . 2c_1 t^{2\mu} c_{1x} = \gamma(\gamma-1) c_v \frac{2c_1}{t} w'(c_1^2 t^{2\mu})$$

so that

$$w_0(c_1^2 t^{2\mu}) = \gamma(\gamma+1) c_v \left\{ \frac{w(c_1^2 t^{2\mu})}{c_1^2 t^{2\mu}} + \int_{z_0}^{c_1^2 t^{2\mu}} \frac{w(z)}{z^2}\, dz \right\}$$

where z_0 is an arbitrary constant

and hence

$$\bar{s}_1 = \gamma(\gamma+1) c_v \left\{ \frac{w(c_1 y^{\mu})}{c_1 y^{\mu}} + \int_{z_0}^{c_1 y^{\mu}} \frac{w(z)}{z^2}\, dz \right\} \qquad (4.7)$$

The equations 4.5, 4.6, 4.7 determine the perturbed flow in the simple wave in terms of the two functions F, w.

(iii) <u>Self-Generating Solutions</u>

Let us now consider the problem of the existence of self-generating solutions i.e. solutions in which the region ahead of the shock is unaffected by the perturbation of the system.

Whether the tube is closed or not, the boundary values in the region 1B at $x = c_{1r}t$ are

$$\bar{u}_1 = -F(2c_{1R}t)$$

$$h\bar{c}_1 = F(2c_{1R}t) + \frac{2}{c_{1R}}\chi(c_{1R}t)$$

$$\bar{s}_1 = \frac{2\gamma(\gamma-1)c_v}{c_{1R}^2}\chi(c_{1R}t)$$

These values have to be matched with the values on $x = c_{1R}t$ in the perturbation of the simple wave as the flow is continuous across this line. Equating the two sets of values gives

$$-F(2c_{1R}t) = \frac{3-\gamma}{\gamma+1}y_1^{-\frac{1}{2}}F_0(y_1) - 2y_1^{\frac{1}{2}}F_0'(y_1)$$

$$F(2c_{1R}t) + \frac{2}{c_{1R}}\chi(c_{1R}t) = \frac{1}{\mu y_1^{\mu}}w(c_{1R}y_1^{\mu}) + \frac{3-\gamma}{\gamma+1}y_1^{-\frac{1}{2}}F_0(y_1) + 2y_1^{\frac{1}{2}}F_0'(y_1)$$

$$\chi(c_{1R}t) = \frac{c_{1R}^2}{2\mu}\left\{\frac{w(c_{1R}y_1^{\mu})}{c_{1R}y_1^{\mu}} + \int_{z_0}^{c_{1R}y_1^{\mu}}\frac{w(z)}{z^2}\,dz\right\}$$

where $y_1 = c_{1R}^{\frac{1}{\mu}}.t^2$

These conditions can be satisfied by F, χ provided F_0, w satisfy the equation

$$\frac{3-\gamma}{\gamma+1}y_1^{-\frac{1}{2}}F_0(y_1) = \frac{c_{1R}}{2\mu}\int_{z_0}^{c_{1R}y_1^{\mu}}\frac{w(z)}{z^2}\,dz$$

so that in general

$$\frac{3-\gamma}{\gamma+1}y^{-\frac{1}{2}}F_0(y) = \frac{c_1}{2\mu}\int_{z_0}^{c_1y^{\mu}}\frac{w(z)}{z^2}\,dz \tag{4.8}$$

This means that the perturbed flow in the simple wave can be expressed in terms of a single function, e.g. F_o, so that the three conservation equations across the flame front relate F_o, $\overline{V}$ to the flow ahead, which is known in terms of the shock disturbance $\overline{W}(t)$.

Let us set $\overline{W}(t) = \lambda t^n$, λ and n constants with $n > 0$, and examine whether or not we can find a solution which satisfies all the conditions of the problem.

Let us write

$$F_o(y) = h_4\, c_{1F}^{-\frac{n}{2\mu}} \cdot y^{\frac{n+1}{2}}$$

$$\int_{z_0}^{c_1 y^{\mu}} \frac{w(z)}{z^2}\, dz = \frac{2\mu}{n}\, c_{1F}^{-\frac{n}{\mu}} \cdot c_1^{\frac{n}{2\mu}}\, y^{\frac{n}{2}}$$

where c_{1F} is the value of c_1 on the flame and z_0 is chosen to be zero.

Then the values on the flame are

$$\overline{u}_1 = -(n+2\mu) h_4 t^n$$

$$k\overline{c}_1 = \left\{ \left(n + \tfrac{4}{\gamma+1}\right) h_4 + \frac{c_{1F}}{\mu} \cdot h_5 \right\} t^n$$

$$\overline{s}_1 = \gamma(\gamma+1)\, c_v\, h_5 \left(1 + \frac{2\mu}{n}\right) t^n$$

$$\frac{\overline{\rho}_1}{\rho_1} = \left\{ \left(n + \frac{4}{\gamma+1}\right) \frac{h_4}{c_{1F}} - \left(\frac{2\gamma}{n} + \gamma + 1\right) h_5 \right\} t^n$$

and the condition 4.8 implies that

$$h_5 = \frac{3-\gamma}{\gamma+1} \cdot \frac{n}{c_{1R}}\, \delta^{\frac{n}{2\mu}} \cdot h_4$$

where $\delta = \dfrac{c_{1F}}{c_{1R}} > 1$

since $-\frac{1}{2} u_{1F} + \dfrac{c_{1F}}{\gamma-1} = \dfrac{c_{1R}}{\gamma-1}$

Hence the boundary values can be written

$$\overline{u}_1 = -(n+2\mu)h_4 t^n$$

$$k\overline{c}_1 = \left\{n + \frac{4}{\gamma+1} + \frac{3-\gamma}{\gamma+1} n . \delta^{\frac{n}{2\mu}+1}\right\} h_4 t^n$$

$$\frac{\overline{\rho}_1}{\rho_1} = \frac{1}{c_{1F}} \left\{n + \frac{4}{\gamma+1} - \frac{2\gamma(3-\gamma)}{\gamma+1} \delta^{\frac{n}{2\mu}+1} - n(3-\gamma)\delta^{\frac{n}{2\mu}+1}\right\} h_4 t^n$$

For convenience let us write

$$\overline{V}(u_0 - u_1) = h_6 t^n$$

After substituting for $\overline{u}_0$, $\overline{c}_0$, $\frac{\overline{\rho}_0}{\rho_0}$ in terms of $\overline{W}(=\lambda t^n)$ we can write the conservation equations across the flame in perturbed form as

$$(V_0 - u_0)\frac{\overline{\rho}_1}{\rho_1} + \frac{u_1 - u_0}{c_1}\overline{V} - \frac{V_0 - u_0}{c_1}\overline{u}_1 = R_6 \lambda t^n$$

$$\frac{u_0 - u_1}{V_0 - u_0}\overline{V} - \mu\,\overline{u}_1 + \frac{2}{\gamma+1}\overline{c}_1 = R_7 \lambda t^n$$

$$(u_0 - u_1)\overline{V} - c_1 \overline{u}_1 + k c_1 \overline{c}_1 = R_8 \lambda t^n$$

where

$$R_6 = (V_0 - u_0)R_5 - R_3$$

$$R_7 = \frac{1}{\gamma+1}\left\{\frac{c_0^2}{(V_0-u_0)^2}R_3 + \frac{c_0}{V_0 - u_0}(\gamma-1)R_4\right\}$$

$$R_8 = -(V_0 - u_0)R_3 + c_0 R_4$$

$$R_3 = R_1 k_1^n + R_2 k_2^n$$

$$R_4 = R_1 k_1^n - R_2 k_2^n + 2K_3 k_3^n$$

$$c_0 R_5 = R_1 k_1^n - R_2 k_2^n - 2(\gamma-1)K_3 k_3^n$$

Thus h_4, h_6 have to satisfy the equations

$$\left\{n+1 - \frac{\gamma(3-\gamma)}{\gamma+1}\delta^{\frac{n}{2\mu}+1} - \frac{1}{2}n(3-\gamma)\delta^{\frac{n}{2\mu}+1}\right\}.2h_4 - \frac{1}{V_0 - u_0}h_6$$

$$= \frac{c_{1F}}{V_0 - u_0} R_6 \lambda \qquad (4.9)$$

$$\left\{n+1+\frac{3-\gamma}{\gamma+1}\,\frac{n}{2}\,\delta^{\frac{n}{2\mu}+1}\right\}\cdot 2h_4+\frac{h_6}{\mu(V_0-u_0)} = \frac{R_7}{\mu}\lambda \tag{4.10}$$

$$\left\{n+1+\frac{3-\gamma}{\gamma+1}\,\frac{n}{2}\,\delta^{\frac{n}{2\mu}+1}\right\}\cdot 2h_4+\frac{1}{c_{1F}}h_6 = \frac{R_8}{c_{1F}}\lambda \tag{4.11}$$

From these three equations h_4, h_6 can be eliminated resulting in an equation for n. From 4.10, 4.11

$$h_6 = \frac{V_0-u_0}{c_{1F}-\mu(V_0-u_0)}\left(c_{1F}R_7-\mu R_8\right)\lambda$$

$$= R_9\lambda, \text{ say} \tag{4.12}$$

Substituting this value for h_6 into 4.9, 4.10 and dividing the resulting equations gives

$$\frac{n+1-\frac{\gamma(3-\gamma)}{\gamma+1}\delta^{\frac{n}{2\mu}+1}-\frac{1}{2}n(3-\gamma)\delta^{\frac{n}{2\mu}+1}}{n+1+\frac{3-\gamma}{\gamma-1}\,\frac{n}{2}\,\delta^{\frac{n}{2\mu}+1}} = \frac{c_{1F}}{V_0-u_0}\cdot\frac{R_9+c_{1F}R_6}{R_8-R_9} \tag{4.13}$$

provided $Y(n) = n+1+\frac{3-\gamma}{\gamma-1}\,\frac{n}{2}\,\delta^{\frac{n}{2\mu}+1} \neq 0$,

but $Y(0)=1$ and $Y(n)$ is an increasing function of n so that $Y(n)=0$ has no positive root.

For such a solution to have physical significance it is necessary that $n > 0$. Equation 4.13 can be written

$$\left\{n+1-\frac{\gamma(3-\gamma)}{\gamma+1}\delta^{\frac{n}{2\mu}+1}-\frac{1}{2}n(3-\gamma)\delta^{\frac{n}{2\mu}+1}\right\}\cdot\left\{k_1^nC_1+k_2^nC_2+k_3^nC_3\right\}$$

$$=\left\{n+1+\frac{3-\gamma}{\gamma-1}\,\frac{n}{2}\,\delta^{\frac{n}{2\mu}+1}\right\}\cdot\left\{k_1^nD_1+k_2^nD_2+k_3^nD_3\right\} \tag{4.14}$$

where $C_1 = -(c_0 - V_0 + u_0)^2 R_1$

$$C_2 = -(c_0 + V_0 - u_0)^2 R_2$$

$$C_3 = 4c_0(V_0 - u_0) K_3$$

$$D_1 = (V_0 - u_0 - c_0)\left\{(\gamma-1)(V_0 - u_0) + \frac{c_{1F}}{c_0}(V_0 - u_0 - \gamma c_0)\right\} R_1$$

$$D_2 = (V_0 - u_0 + c_0)\left\{(\gamma-1)(V_0 - u_0) - \frac{c_{1F}}{c_0}(V_0 - u_0 + \gamma c_0)\right\} R_2$$

$$D_3 = -2(\gamma-1)(V_0 - u_0)\left\{c_0 + c_{1F}\frac{V_0 - u_0}{c_0}\right\} K_3$$

By inspection 4.14 has a root $n = -2\mu\ (< 0)$ having no physical significance.

For the case $\gamma = 3$ the equation 4.14 simplifies to give

$$n = -1$$

or

$$(C_1 - D_1)k_1^n + (C_2 - D_2)k_2^n + (C_3 - D_3)k_3^n = 0 \qquad (4.15)$$

For the upper limit of such flows, i.e. $W = W_m$ and $V = W$, we have $k_1 = k_2 = k_3 = 1$, so that 4.15 is simply the condition

$$C_1 - D_1 + C_2 - D_2 + C_3 - D_3 = 0$$

The left hand side of this expression was evaluated for the case

Q = 1,400 calories/gm.

p_2 = 1 atmosphere

ρ_2 = 1 gm./litre

c_2 = 1 by definition,

(for which $W^* = 8.87$, $W_m = 17.58$),

and found to have the value 1.01×10^4.

For the case $W = W^{\text{II}}$, and the above data, 4.15 becomes

$$1.44\times10^3\,(1.49)^n + 0.99\times10^2\,(0.87)^n - 0.89\times10^3\,(0.65)^n = 0$$

which has no solution with $n > 0$ as the left hand side is positive for $n = 0$ and is increasing for $n \geqslant 0$.

Thus we conclude that there are no self-generating solutions with $\overline{W} = \lambda t^n$ for the two numerical cases investigated. Hence, since the problem is linear, there is no solution when $\overline{W}$ is expressible as a power series in t.

(iv) <u>The evolutionary solution</u>

Let us now consider the problem of the disturbance of the system due to a specific external cause. For example we can consider the initial state of the gas, i.e. the region 2, to be approximately uniform with a distribution of velocity, $U(x)$ say, along the tube ($U(0) = 0$), the initial values of entropy and sound speed being constant.

The solution in the region 2, ahead of the shock, due to this initial distribution of disturbance is

$$\bar{u}_2 = \tfrac{1}{2}U(x-t) + \tfrac{1}{2}U(x+t)$$

$$k\bar{c}_2 = \tfrac{1}{2}U(x-t) - \tfrac{1}{2}U(x+t)$$

$$\bar{s}_2 = 0$$

and $\dfrac{\bar{\rho}_2}{\rho_2} = \dfrac{k\bar{c}_2}{c_2}$, where $c_2 = 1$,

and the values on the shock, $x = W_0 t$, are

$$\bar{u}_2 = \tfrac{1}{2}U\{(W_0-1)t\} + \tfrac{1}{2}U\{(W_0+1)t\} \tag{4.16}$$

$$k\bar{c}_2 = \tfrac{1}{2}\,\mho\left\{(W_0-1)t\right\} - \tfrac{1}{2}\,\mho\left\{(W_0+1)t\right\} \tag{4.17}$$

$$\frac{\bar{\rho}_2}{\rho_2} = \tfrac{1}{2}\,\mho\left\{(W_0-1)t\right\} - \tfrac{1}{2}\,\mho\left\{(W_0+1)t\right\} \tag{4.18}$$

Before differentiating the conservation equations across the shock (with $\bar{u}_2$, $\bar{c}_2$, $\bar{\rho}_2$ non-zero) we can derive the following general relationship from the conservation equations (1.6, 1.7, 1.8)

$$c_2^2 + (W_0 - u_2)\left\{-W_0 + \frac{\gamma+1}{2}u_0 - \frac{\gamma-1}{2}u_2\right\} = 0$$

which, on differentiation, gives

$$\bar{u}_0 = \frac{-4}{\gamma+1}\,\frac{\bar{c}_2}{W_0} + \left\{\frac{4}{\gamma+1} - \frac{u_0}{W_0}\right\}\overline{W} + \left\{\frac{\gamma-3}{\gamma+1} + \frac{u_0}{W_0}\right\}\bar{u}_2$$

since $c_2 = 1,\ u_2 = 0.$

Also, from the equation

$$k c_0^2 + (W_0 - u_0)^2 = k c_2^2 + (W_0 - u_2)^2$$

we deduce that

$$c_0\bar{c}_0 = \frac{\gamma-1}{2}u_0\overline{W} + \bar{c}_2 - \frac{\gamma-1}{2}W_0\bar{u}_2 + \frac{\gamma-1}{2}(W_0-u_0)\bar{u}_0$$

The density perturbation is given by

$$\frac{\bar{\rho}_0}{\rho_0} = \frac{\bar{\rho}_2}{\rho_2} + \frac{\bar{u}_0 - \overline{W}}{W_0 - u_0} + \frac{\overline{W} - \bar{u}_2}{W_0}$$

These boundary values, behind the shock, can be expressed in terms of the function $\mho$ using 4.16, 4.17, 4.18

$$\bar{u}_0 = 2K_1\overline{W}(t) + 2T_3\,\mho\left\{(W_0+1)t\right\} + 2T_4\left\{(W_0-1)t\right\} \tag{4.19}$$

$$k\overline{\tau}_0 = 2K_2\overline{W}(t) + 2T_5\overline{U}\{(W_0+1)t\} + 2T_6\overline{U}\{(W_0-1)t\} \qquad (4.20)$$

$$\begin{aligned}\overline{s}_0 &= 2\frac{c_v}{c_0}\overline{c}_0 + (\gamma-1)c_v\frac{\overline{\rho}_0}{\rho_0} \\ &= K_4\overline{W}(t) + T_1'\overline{U}\{(W_0+1)t\} + T_2'\overline{U}\{(W_0-1)t\}\end{aligned} \qquad (4.21)$$

where

$$2T_3 = \frac{\gamma-3}{2(\gamma+1)} + \frac{u_0+2\mu}{W_0}$$

$$2T_4 = \frac{\gamma-3}{2(\gamma+1)} + \frac{u_0-2\mu}{W_0}$$

$$2c_0T_5 = -\tfrac{1}{2} - \tfrac{1}{2}W_0 + (W_0-u_0)2T_3$$

$$2c_0T_6 = \tfrac{1}{2} - \tfrac{1}{2}W_0 + (W_0-u_0)2T_4$$

$$T_1' = \frac{4}{c_0}T_5 + 1 - \frac{4}{W_0-u_0}T_3 + \frac{1}{W_0}$$

$$T_2' = \frac{4}{c_0}T_6 - 1 - \frac{4}{W_0-u_0}T_4 + \frac{1}{W_0}$$

The boundary values 4.19, 4.20, 4.21 serve to determine the solution in the region 0 in terms of the functions $\overline{U}$, $\overline{W}$. Proceeding as previously we can obtain the boundary values in this region, at the flame, as

$$\begin{aligned}\overline{u}_0 = (K_1+K_2-K_3)\overline{W}(k_1t) - (K_2-K_1-K_3)\overline{W}(k_2t) \\ + (T_5+T_3-T_1)\overline{U}\{(W_0+1)k_1t\} - (T_5-T_3-T_1)\overline{U}\{(W_0+1)k_2t\} \\ + (T_6+T_4-T_2)\overline{U}\{(W_0-1)k_1t\} - (T_6-T_4-T_2)\overline{U}\{(W_0-1)k_2t\}\end{aligned}$$

$$k\bar{c}_0 = (k_1+k_2-k_3)\overline{W}(k_1t) + (k_2-k_1-k_3)\overline{W}(k_2t) + 2k_3\overline{W}(k_3t)$$
$$+ (T_5+T_3-T_1)\,\overline{U}\{(W_0+1)k_1t\} + (T_5-T_3-T_1)\,\overline{U}\{(W_0+1)k_2t\}$$
$$+ 2T_1\,\overline{U}\{(W_0+1)k_3t\} + (T_6+T_4-T_2)\,\overline{U}\{(W_0-1)k_1t\}$$
$$+ (T_6-T_4-T_2)\,\overline{U}\{(W_0-1)k_2t\} + 2T_2\overline{U}\{(W_0-1)k_3t\}$$

$$\bar{s}_0 = \frac{2\gamma(\gamma-1)c_v}{c_0}\left[K_3\overline{W}(k_3t) + T_1\overline{U}\{(W_0+1)k_3t\} + T_2\overline{U}\{(W_0-1)k_3t\}\right]$$

$$\frac{\bar{\rho}_0}{\rho_0} = k\,\frac{\bar{c}_0}{c_0} - \frac{\bar{s}_0}{c_v(\gamma-1)}$$

where
$$T_1' = \frac{2\gamma(\gamma-1)c_v}{c_0}T_1$$
$$T_2' = \frac{2\gamma(\gamma-1)c_v}{c_0}T_2$$

These are the boundary values just ahead of the flame, in terms of the unknown function $\overline{W}(t)$ and the given function $\overline{U}(t)$. We have to relate these to the values just behind the flame. Applying the equation 4.8 to the solution of the simple wave region, we can write the latter in terms of the single function w, as follows

$$\bar{\alpha}_1 = \frac{1}{2\mu y^{\mu}}\,w(c_1y^{\mu}) + \frac{c_1}{2\mu}\int_{z_0}^{c_1y^{\mu}}\frac{w(z)}{z^2}\,dz$$

$$\bar{\beta}_1 = \frac{1}{2\mu}\,\frac{\gamma+1}{3-\gamma}\left\{\frac{1}{y^{\mu}}\,w(c_1y^{\mu}) + c_1\int_{z_0}^{c_1y^{\mu}}\frac{w(z)}{z^2}\,dz\right\}$$

$$\bar{s}_1 = \frac{\gamma(\gamma+1)c_v}{c_1}\left\{\frac{1}{y^{\mu}}\,w(c_1y^{\mu}) + c_1\int_{z_0}^{c_1y^{\mu}}\frac{w(z)}{z^2}\,dz\right\}$$

so that the boundary values at the flame can be written

$$\overline{u}_1 = -S(t)$$

$$\overline{c}_1 = S(t)$$

$$\frac{\overline{\rho}_1}{\rho_1} = \frac{\gamma-2}{c_{1F}}\, S(t)$$

$$\text{where } S(t) = \frac{\gamma+1}{3-\gamma}\left\{\frac{W(c_{1F}\,y_1^N)}{y_1^N} + c_{1F}\int_{z_0}^{c_{1F}y_1^N}\frac{W(z)}{z^2}\,dz\right\}$$

The three conservation equations across the flame will serve to determine S, $\overline{\overline{V}}$, $\overline{\overline{W}}$ in terms of $\overline{U}$. These are

$$(V_0-u_0)\frac{\overline{\rho}_1}{\rho_1} + \frac{u_1-u_0}{c_1}\overline{\overline{V}} - \frac{V_0-u_0}{c_1}\overline{u}_1 = (V_0-u_0)\frac{\overline{\rho}_0}{\rho_0} - \overline{u}_0 \qquad (4.22)$$

$$\frac{u_0-u_1}{V_0-u_0}\overline{V} - \mu\,\overline{u}_1 + \frac{2}{\gamma+1}\overline{c}_1 = \frac{\overline{u}_0}{\gamma+1}\left\{\frac{c_0^2}{(V_0-u_0)^2}-\gamma\right\} + \frac{2}{\gamma+1}\frac{c_0}{V_0-u_0}\overline{c}_0 \qquad (4.23)$$

$$(u_0-u_1)\overline{V} - c_0\overline{u}_1 + k c_0\overline{c}_1 = -(V_0-u_0)\overline{u}_0 + k c_0\overline{c}_0 \qquad (4.24)$$

where the suffix 1 signifies the value at the flame.

In these six equations the values of the functions S, $\overline{\overline{V}}$ at the single instant t appear. $\overline{U}$ is evaluated at the six distinct points $(W \pm 1)\, k_j t$ $(j = 1, 2, 3)$, but is known. The unknown function $\overline{\overline{W}}$ is evaluated at the three distinct points $k_j t$ $(j = 1, 2, 3)$.

Expressed in terms of $\overline{\overline{V}}$, S the left hand sides of 4.22, 4.23, 4.24, are respectively

$$\frac{u_1-u_0}{c_1}\overline{\overline{V}} + (\gamma-1)\,\frac{V_0-u_0}{c_1}\,S,$$

$$\frac{u_0-u_1}{V_0-u_0}\overline{\overline{V}} + S,$$

$$(u_0-u_1)\overline{\overline{V}} + \frac{c_1}{\mu}S.$$

Elimination of $\overline{\overline{V}}$ between 4.22 and 4.24, and 4.22 and 4.23 gives respectively

$$\left\{\frac{c_1}{\mu}+(\gamma-1)(V_0-u_0)\right\}S(t)$$
$$= c_1(V_0-u_0)\frac{\overline{\rho_0}}{\rho_0} - c_1\overline{u_0} - (V_0-u_0)\overline{u_0} + kc_0\overline{c_0} \tag{4.25}$$

$$S(t) = \frac{c_1}{\gamma}\frac{\overline{\rho_0}}{\rho_0} - \frac{2}{\gamma+1}\overline{u_0} + \frac{2}{\gamma(\gamma+1)}\frac{c_0}{V_0-u_0}\overline{c_0} \tag{4.26}$$

since $(\gamma+1)(V_0-u_0)c_1 = c_0^2+\gamma(V_0-u_0)^2$

from 1.10s with $V_0-u_1 = c_1$

so that the equation relating $\overline{\overline{W}}$ to $\overline{U}$, obtained by elimination of S(t) between 4.25, 4.26, is

$$\frac{c_1}{\gamma}\left\{\frac{c_1}{\mu}-V_0+u_0\right\}\frac{\overline{\rho_0}}{\rho_0} + \frac{3-\gamma}{\gamma+1}\left\{V_0-u_0-\frac{c_1}{\mu}\right\}\overline{u_0}$$
$$+c_0\left\{\frac{k}{\gamma}\frac{c_1}{V_0-u_0}+\left(\frac{2\mu}{\gamma}-k\right)\right\}\overline{c_0} = 0 \tag{4.27}$$

or $c_0A_1\frac{\overline{\rho_0}}{\rho_0} + A_2\overline{u_0} + kA_3\overline{c_0} = 0$, say.

In terms of $\overline{\overline{W}}$, $\overline{U}$, 4.27 is

$$X_1 \overline{\overline{W}}(k_1 t) + X_2 \overline{\overline{W}}(k_2 t) + X_3 \overline{\overline{W}}(k_3 t)$$

$$= Y_1 \overline{U}\{(W_0+1)k_1 t\} + Y_2 \overline{U}\{(W_0+1)k_2 t\} + Y_3 \overline{U}\{(W_0+1)k_3 t\} \quad (4.28)$$

$$+ Z_1 \overline{U}\{(W_0-1)k_1 t\} + Z_2 \overline{U}\{(W_0-1)k_2 t\} + Z_3 \overline{U}\{(W_0-1)k_3 t\}$$

where

$$\frac{X_1}{K_3-K_1-K_2} = \frac{Y_1}{T_5+T_3-T_1} = \frac{Z_1}{T_6+T_4-T_2} = A_1+A_2+A_3 ,$$

$$\frac{X_2}{K_3+K_1-K_2} = \frac{Y_2}{T_5-T_3-T_1} = \frac{Z_2}{T_6-T_4-T_2} = A_1-A_2+A_3 ,$$

$$-\frac{X_3}{K_3} = \frac{Y_3}{T_1} = \frac{Z_3}{T_2} = 2\{A_3-(\gamma-1)A_1\} .$$

The solution of 4.28, written now as

$$X_1 \overline{\overline{W}}(k_1 t) + X_2 \overline{\overline{W}}(k_2 t) + X_3 \overline{\overline{W}}(k_3 t) = E(t)$$

for the function $\overline{\overline{W}}$ can be obtained, as for the corresponding equation for the previous section with two terms on the L.H.S.

The series solution is

$$\overline{\overline{W}}(t) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} \frac{(p+q)!}{p!\,q!} \frac{X_2^p X_3^q}{X_1^{p+q+1}} \cdot E\left(\frac{k_2^p k_3^q t}{k_1^{p+q+1}}\right) \quad (4.29)$$

which has been shown (4) to be the unique solution of 4.28 and to be uniformly convergent provided $|X_2| + |X_3| < |X_1|$.

REFERENCES

1. Adams, G. K. and Pack, D. C., Seventh Symposium (International) on Combustion, pp 812-819, 1958.

2. Anderson, J. E., Magnetohydrodynamic Shock Waves, pp 32-33, M.I.T. Press, 1963.

3. Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, pp 204-235, Interscience, New York, 1948.

4. Gundersen, G., J. Fluid Mechanics, vol.3, pp 553-581, 1958.

5. Taylor, G. I. and Tankin, R. S., pp 622-656 of High Speed Aerodynamics and Jet Propulsion, vol.III - Fundamentals of Gasdynamics (edited by H. W. Emmons), Oxford University Press, 1958.

INDEX TO PART II

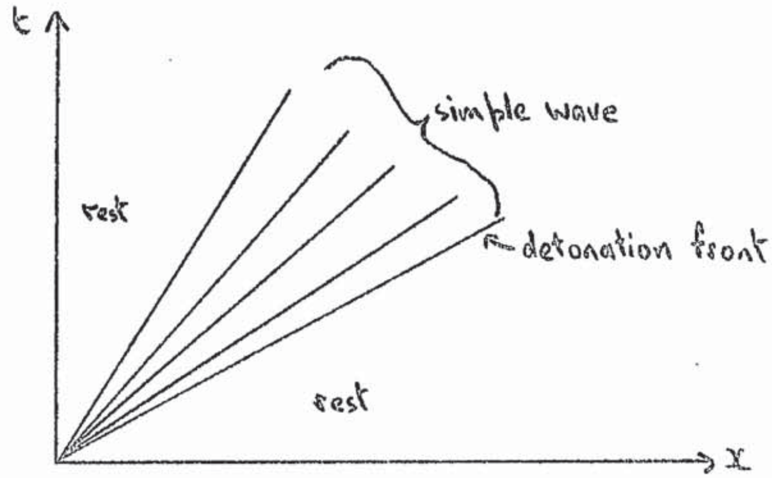
1. The Converging Detonation Wave

The problem to be considered here is that of a spherically (or axially) symmetric flow in which there is a spherical (or cylindrical) detonation front, moving inwards towards the centre (or axis) of symmetry. The wave is assumed to have been initiated by detonating the gas, initially uniform and at rest, simultaneously at all points on a spherical (or cylindrical) surface, so that the gas ahead of the wave front is at rest. It has been shown by Stanyukovich (24, pp 526-527) and by Selberg (23) that there is no solution involving a uniformly contracting, Chapman-Jouguet detonation front.

In order to investigate the present problem it will be necessary to examine certain types of solutions, involving shock fronts and detonation fronts, which are either plane or spherically ar axially symmetric.

There is a well known class of solutions for the one-dimensional, linear wave motion of an inviscid, non-heat-conducting gas in which all physical quantities are functions of the single variable x/t, where t is time and x is the linear coordinate in the direction in which variations may occur. Since all quantities depend only on x/t the flow has always the same pattern on a scale which expands uniformly as time increases. Although such a flow is unsteady the fact that all variables depend on only x/t simplifies the mathematical solution of the equations governing the flow, which can be reduced from partial differential equations in x, t to ordinary differential equations in the single variable x/t. For a general unsteady flow there is no such simplification. The particular type of solution in which the equations of motion reduce to ordinary differential equations is called a similarity solution. In the case just mentioned x/t is called the similarity variable. A more general similarity solution in which the similarity variable is of the form x/t^{α}, where α is a constant, will be discussed later. In order to decide whether the solution of a given problem is a similarity solution, one must investigate the dimensions of the basic parameters which determine the solution. This will be discussed fully in Section 2.

An example of a linear wave motion in which the similarity variable is x/t is the Chapman-Jouguet detonation wave starting at the closed end of a uniform tube filled initially with a uniform combustible gas. The detonation wave is considered to be a plane of discontinuity, moving with uniform velocity into the unreacted stationary gas. The front gives a forward velocity to the gas particles as they cross it. This velocity is reduced to zero by means of a simple rarefaction wave, along which the flow variables are constant on lines x/t = constant, and this wave terminates on the line x/t = the rest sound speed of the burnt gas.



One can also seek similarity solutions of the equations of motion for a flow which is not linear but has spherical or axial symmetry. In this case the physical quantities depend only on the radial coordinate R and the time t. It is natural to look for solutions of this type analogous to linear similarity flows. The spherical analogy of the linear Chapman-Jouguet detonation is the detonation of a uniform combustible gas initiated at a point within the gas. The solution to this problem has been obtained by Taylor (25) and is a similarity solution with similarity variable R/t. A spherical Chapman-Jouguet detonation front expands uniformly outwards into the unreacted stationary gas. There is a uniformly expanding core of stationary gas and the fluid velocity increases steadily from zero at the surface of the core up to the front.

A simple, one-dimensional, uniform similarity flow is that caused by a solid piston moving with uniform velocity into a column of uniform gas at rest contained in a uniform straight tube. The flow pattern consists of a shock wave travelling ahead of the piston into the stationary gas, with the region between the shock and the piston moving with the velocity of the piston. The spherical analogy, a uniformly expanding spherical piston, has been solved by Taylor (26). A spherical shock front moves with constant speed ahead of the piston, the gas ahead of the front being at rest. The velocity of the region between the shock and the piston increases steadily from the piston to the shock. Again the similarity variable is R/t.

For any similarity solution with similarity variable R/t the equation of a discontinuity must be of the form $R/t =$ constant, so that the velocity of the front, $\frac{d\lambda}{dt}$ (where the equation of the front is $R = \lambda$), must be constant. If the front moves into a uniform medium at rest its strength cannot alter i.e. the jumps in the physical variables across the front do not vary with time. Thus such a solution, in which the flow is adiabatic, must represent an isentropic flow apart from entropy jumps across discontinuities.

A more interesting, non-isentropic similarity solution is that of the instantaneous release of a large amount of energy at a point in a uniform, non-reacting gas. This is Taylor's point-explosion solution (27). A spherical shock wave expands from this point, which is the centre of symmetry. Since the amount of energy available to drive this shock is precisely equal to the energy liberated initially and so must be fixed (unlike the spherical piston), the front decays as it progresses through an increasing amount of the medium. The similarity variable in this case is $R/t^{\frac{2}{5}}$.

The solution involving a contracting spherical (or cylindrical) shock front, obtained by Guderley (10), suggests a method of solving the present problem of a contracting detonation front. This solution shows that the shock front accelerates as it approaches the origin, where the speed of the shock is infinite. As the shock wave progresses its surface area diminishes and the amount of gas passing through the front per unit displacement of the front decreases as λ decreases and tends to 0 at $R = 0$. The similarity hypothesis used to obtain the solution is based on the fact that the shock is strong, i.e. that λ is small relative to the initial radius of the front, so that the solution is independent of the method of initiation. The assumption that the shock is strong means that the values of the physical variables immediately behind the front depend only on the shock speed and the initial density of the medium. This forms the basis of the similarity hypothesis.

If one considers a detonation front in place of a shock front then there is an additional amount of energy Q, the heat energy released per unit mass of the gas and assumed constant, available for each unit mass of gas behind the front. However the speed of the shock front and the kinetic and interal energies of the gas immediately behind the front are very large and tend to infinity as λ tends to 0. Thus the effect of introducing the finite amount of energy Q into the system must be of small order and the effects due to Q can be obtained as perturbations on the basic Guderley solution. It will also be shown that the effect of taking into account the sound speed of the stationary gas in the Rankine-Hugoniot conservation relations across the front, neglected in the Guderley solution, does in fact give rise to perturbation terms of precisely the same form as those due to Q. Hence the solution to be developed here will be capable of determining the result of both of these effects, together or in isolation. The problem is in fact linear in the two separate effects of heat release and finite sound speed.

In the Guderley solution the shock speed, U_s^*, has magnitude proportional to $\lambda^{1-\frac{1}{\alpha}}$ where α is a constant and $0 < \alpha < 1$ since $U_s^* \longrightarrow \infty$ as $\lambda \longrightarrow 0$. Also u_s^*, c_s^*, the particle velocity and sound speed of the gas immediately behind the shock front, have magnitudes proportional to $\lambda^{1-\frac{1}{\alpha}}$. Thus the kinetic and internal energies per unit mass, which contain u_s^{*2}, c_s^{*2}, are of order $\lambda^{2-2/\alpha}$. We are to assume that the effects of the detonation are of relatively small order so let us write u_D^*, c_D^*, the values of u, c immediately behind the detonation front, in the form

$$u_D^* = u_s^* + \bar{u}\,\lambda^a$$

$$c_D^* = c_s^* + \bar{c}\,\lambda^b, \text{ where } \bar{u}, \bar{c} \text{ are finite,}$$

then the additional terms in the energies per unit mass are of the form $\bar{u}\, u_s^* \lambda^a$, $\bar{c}\, c_s^* \lambda^b$. Since these are due directly to the heat release Q, which is finite, they must be of finite order. Thus

$$a = b = -1 + \frac{1}{\alpha}$$

since u_s^*, c_s^* are of order $\lambda^{1-\frac{1}{\alpha}}$, and the perturbations are of order $\lambda^{-2+\frac{2}{\alpha}}$ relative to the Guderley solution. This result will be shown later to be consistent with that obtained from examination of the jump relations across the front.

In the present problem energy considerations can tell us only the form of the solution. In the problem of a point-explosion the conservation of energy shows that the similarity variable is $R/t^{\frac{2}{5}}$. In this latter case the motion is confined to the region $0 \leq R \leq \lambda$ so that an explicit form of the conservation of energy can be obtained. In the present problem this is not the case as the motion is confined to the region $R \geq \lambda$, and the solution is invalid at large values of R.

The speed of the shock in the Guderley solution is given by

$$U_s^* = -\lambda^{1-\frac{1}{\alpha}}$$

Thus it is to be expected that the speed of the detonation front will be of the form

$$U^* = -\lambda^{1-\frac{1}{\alpha}}\left(1 + \beta\,\lambda^{-2+\frac{2}{\alpha}}\right), \text{ where } \beta \text{ is a constant.}$$

The main interest is to evaluate the parameter $\dot{a}$ for the shock front and the parameter β determining the correction for the detonation speed. Whereas the parameter $\dot{a}$ for the point-explosion can be evaluated simply by consideration of conservation of energy, in the case of the Guderley solution it is necessary to integrate the governing differential equations in order to evaluate $\dot{a}$. In place of the energy equation one makes use of the assumption that the flow is regular on a certain characteristic. Evaluation of $\dot{a}$ is effected by integration of the equations of motion, which can be reduced to a single ordinary differential equation, and by making the solution satisfy the regularity condition at one end of the range and the jump relations across the front at the other. This can only be done by trial and error. The results given by Butler (6) for the six cases $\gamma = 1.2,\ 1.4,\ 5/3$ for spherical and cylindrical symmetry have been recomputed and extended to the case $\gamma = 3$, corresponding to the motion of the products of a detonation. It is necessary to tabulate the Guderley solution over the range of integration in order to evaluate the solution for the perturbations as the former appears in the coefficients of the differential equations governing the latter.

In order to evaluate β it is necessary to integrate the three simultaneous ordinary differential equations for the perturbations. As for the shock, the solution for the perturbations must satisfy the regularity condition and the jump relations simultaneously. It is necessary to take into account the displacements of the paths of the front and the characteristic in considering the boundary conditions there. It is found that these give three conditions at the front in terms of the unknown β, and three at the characteristic in terms of the displacement of this characteristic and one other unknown parameter. As these are the only conditions to be satisfied the problem is soluble. The evaluation of β is greatly simplified by the fact that the equations governing the perturbations are linear. Having found any two linearly independent solutions of these equations the appropriate solution can be readily found as a combination of these two.

When the detonation front reaches the point $R = 0$ it will be reflected as a shock wave, if the reaction has been completed.

The section dealing with the equations of motion and boundary conditions for the detonation problem will follow the method used by Butler for the Guderley solution. The same transformations of the physical variables will be used. Substitution of the form of the solution for the physical variables into the equations of motion gives the required ordinary differential equations. These appear as coefficients of increasing powers of R in the expansion of the equations of motion, the leading term giving the equation for the Guderley solution and the second giving the equations to be satisfied by the perturbations.

Several authors have studied related similarity solutions and corrections to those solutions since Taylor's solution to the spherical piston problem. This appears to be the first solution of this type, followed by the same author's expanding Chapman-Jouguet detonation front and point-explosion solutions. The exact solution of the point-explosion was given by J. L. Taylor (28) from energy considerations. Sakurai (19, 20) has found the solution for the second term in the expansion of the point-explosion solution, of order U^{n-2} due to allowing the rest sound speed to be finite. In effect this extends the solution to higher values of R, where the shock is less strong and the rest sound speed is no longer negligible (as, of course, the present work does for the Guderley solution when only the effect of the sound speed is considered). Korobeinikov and Ryazanov (16) have considered a point-explosion in which the initial density varies as a power of R, also taking the initial sound speed into account.

Hafele (11) has found a similarity solution for a plane shock, decaying in strength due to the absence of any reinforcement. This is effectively the Guderley solution for the linear, one-dimensional case. Jones (15) has obtained a similarity solution for a plane shock, taking the initial sound speed into account, propagating through a region in which the density varies as a power of x (the linear coordinate), not varying more rapidly than x^{-3}.

The problem of a plane shock wave arriving at a vacuum has been studied by Sakurai (21), who finds a similarity solution, the method being very similar to that of the Guderley solution. The shock speed varies as a power of x, the power being evaluated by means of a regularity condition.

The collapse of an empty cavity in water (cavitation) would appear to be a problem similar to the converging shock wave. The former problem has been investigated by Hunter (13, 14), who integrates the governing equations numerically for the initial stages and continues this solution for the latter stages of the collapse, when R is small, by means of a similarity solution. In this case also the origin is a singularity, and again the exponent in the formula for the speed of the front is found by means of a regularity condition. The similarity solution in this case differs from that for the converging shock in that it is isentropic (i.e. entropy is uniform throughout the fluid) and the boundary conditions at the front are dissimilar. The similarity hypothesis in this case requires that the density at the front is zero. Holt and Schwartz (12) have found a correction to Hunter's solution, of order U^{n-2} by allowing the density to be finite.

Perry and Kantrowitz (18) have made experimental studies of converging cylindrical shock waves. They succeeded in forming shocks of moderate strength (Mach number 1.7) and found that the stability decreased rapidly with increasing strength.

The numerical solution for a converging cylindrical shock front has been performed by Payne (17), whose results are in agreement with the Guderley similarity solution for the final stages of collapse.

A discussion of spherical waves (called progressing waves) is given by Courant and Friedrichs (9). The theory of similarity solutions is given by Sedov (22), which also contains the solutions to several of the problems mentioned previously. The Guderley solution is also given by Stanyukovich (24, pp 521-528).

2. Equations of Motion and Similarity

Let us consider the spherically (or axially) symmetric motion of a perfect, inviscid, non-heat-conducting gas, having constant specific heats at constant volume and pressure, c_v, c_p. The variables will be denoted as follows

- u - particle velocity (in the outward radial direction)
- p - pressure
- ρ - density
- c - speed of sound
- s - specific entropy
- t - time
- R - radial distance (measured from the centre of symmetry)
- $\gamma = c_p/c_v$.

The equations governing the symmetric motion of such a gas are

$$\frac{\partial \rho^*}{\partial t} + u^* \frac{\partial \rho^*}{\partial R} + \rho^* \frac{\partial u^*}{\partial R} + j \frac{u^* \rho^*}{R} = 0 \qquad (2.1)$$

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial R} + \frac{1}{\rho^*} \frac{\partial p^*}{\partial R} = 0 \qquad (2.2)$$

$$\frac{\partial s^*}{\partial t} + u^* \frac{\partial s^*}{\partial R} = 0 \qquad (2.3)$$

where $*$ denotes a physical quantity and j has the value 2 for spherical symmetry. The corresponding equations for axially symmetric flow are the above with $j = 1$. Equation 2.1 is the conservation of mass, 2.2 the equation of momentum, and 2.3

expresses the fact that s^* is constant for any particle of the gas i.e. the fact that the gas is non-heat-conducting and inviscid.

These equations can be written more conveniently in characteristic form

$$\frac{\partial}{\partial t}\left(u^* \pm k c^*\right) + \left(u^* \pm c^*\right)\frac{\partial}{\partial R}\left(u^* \pm k c^*\right)$$

$$= \mp j\,\frac{u^* c^*}{R} + \frac{1}{\gamma} c^{*2}\frac{\partial \phi^*}{\partial R} \qquad (2.4)$$

$$\frac{\partial \phi^*}{\partial t} + u^*\,\frac{\partial \phi^*}{\partial R} = 0 \qquad (2.5)$$

where $k = \frac{2}{\gamma - 1}$ and c^* is the local sound speed defined by

$$c^{*2} = \left(\frac{\partial p^*}{\partial \rho^*}\right)_{s^*} = \frac{\gamma p^*}{\rho^*}$$

The quantity ϕ^* is a measure of the specific entropy s^* and is defined by

$$\phi^* = \log\left\{\frac{c^{*\frac{2\gamma}{\gamma-1}}}{p^*}\right\}$$

Suppose that U^* is the velocity of a wave front $R = \lambda(t)$, along which the variables u^*, c^*, ϕ^* are given in terms of U^* and M, where M is some quantity whose dimensions contain that of mass, say density. This is the case if the wave front is a strong shock wave (i.e. the sound speed and particle velocity ahead of the front are negligible in comparison to the speed of the front). For dimensional reasons, the boundary values of u^*, c^*, p^* at this front must be of the form

$$u^* = B_1 U^*$$
$$c^* = B_2 U^*$$
$$p^* = B_3 M U^{*2}$$

where B_1, B_2, B_3 are dimensionless constants.

Hence, the value of φ^* at the front is given by

$$\phi^* = k \log U^* + \phi_0^*$$

where ϕ_0^* is the value of ϕ^* at some reference state.

The flow is specified by the two quantities U^*, M, at least in the domain of dependence of this wave front. Since U^* has the dimensions of velocity, it follows that its dependence on R, λ must be of the form

$U^* = \alpha \frac{\lambda}{t}$, where α is a dimensionless constant since $\frac{\lambda}{t}$ is the only quantity having the dimensions of velocity which can be formed from λ, t, M. At any point on the front

$$U^* = \frac{d\lambda}{dt} = \alpha \frac{\lambda}{t} .$$

Hence the equation of the front is

$t = A\lambda^{\frac{1}{\alpha}}$, where A is an arbitrary dimensionless constant. Let us fix the length scale by setting $A = \alpha$ so that the front is

$$\frac{t}{\alpha\lambda^{\frac{1}{\alpha}}} = 1$$

and its velocity is

$$U^* = \lambda^{1 - \frac{1}{\alpha}} .$$

Hence the boundary values can be written in the form

$$u^* = B_1 \lambda^{1-\frac{1}{\alpha}}$$

$$c^* = B_2 \lambda^{1-\frac{1}{\alpha}}$$

$$\phi^* = k(1-\tfrac{1}{\alpha}) \log \lambda + \phi_0^*$$

If we let $\xi = \frac{t}{\alpha R^{\frac{1}{\alpha}}}$ then, since no new dimensional parameters can appear in the general values of u^*, c^*, ϕ^*, it follows that these variables must be of the form

$$u^* = u(\xi) R^{1-\frac{1}{\alpha}}$$

$$c^* = c(\xi) R^{1-\frac{1}{\alpha}}$$

$$\phi^* = k(1-\tfrac{1}{\alpha}) \log R + \phi(\xi) + \phi_0^*$$

where $u(1) = B_1$, $c(1) = B_2$, $\phi(1) = 0$.

These values can now be substituted into the equations of motion 2.4, 2.5 to give

$$\left\{1 - \xi(u \pm c)\right\} \frac{d}{d\xi}(u \pm kc)$$
$$= (1-\alpha)(u \pm c)(u \pm kc) \mp j\alpha uc - \frac{c^2}{\gamma}\left\{\xi \cdot \frac{d\phi}{d\xi} + k(1-\alpha)\right\} \quad (2.6)$$

$$\frac{d\phi}{d\xi} = \frac{k(1-\alpha)u}{1-\xi u} \quad (2.7)$$

from which $\frac{d\phi}{d\xi}$ can be eliminated to give the two equations

$$\left\{1 - \xi(u \pm c)\right\} \frac{d}{d\xi}(u \pm kc)$$
$$= (1-\alpha)(u \pm c)(u \pm kc) \mp j\alpha uc - \frac{k(1-\alpha)}{\gamma} \cdot \frac{c^2}{1-\xi u} \quad (2.8)$$

which can be simplified by means of transforming to the variables $r(\xi)$, $s(\xi)$, defined by

$$r = u\xi \quad , \quad s = c\xi$$

to produce the equations

$$\xi(1 - r \mp s)\frac{d}{d\xi}(r \pm ks) = \frac{B_\pm}{r-1} \tag{2.9}$$

where

$$B_\pm = (r-1)\left\{1 - \alpha(r \pm s)\right\}(r \pm ks) \mp j\alpha(r-1)rs + \frac{k(1-\alpha)}{\gamma}s^2$$

from which we obtain

$$\left.\begin{aligned} 2D\xi\frac{dr}{d\xi} &= (1-r+s)B_+ + (1-r-s)B_- \\ 2kD\xi\frac{ds}{d\xi} &= (1-r+s)B_+ - (1-r-s)B_- \end{aligned}\right\} \tag{2.10}$$

where $D = (r-1)(1-r+s)(1-r-s)$

The variable ξ can now be eliminated to give the following equation for $r = r(s)$

$$\frac{1}{k}\frac{dr}{ds} = \frac{(1-r+s)B_+ + (1-r-s)B_-}{(1-r+s)B_+ - (1-r-s)B_-} \tag{2.11}$$

The time $t = 0$ will always be taken to be the instant at which the wave front is at the origin. Thus $t > 0$ corresponds to expanding waves and $t < 0$ to contracting waves. Expanding waves have positive values of s and contracting ones have negative values.

Consider now the special case of $\alpha = 1$, for which the similarity variable is R/t so that the flow is uniform and isentropic, as mentioned in section 1. Thus

$$\xi = \frac{t}{R} = \frac{1}{U^*}$$

and the solution can be written in the form

$$u^* = u\left(\frac{t}{R}\right)$$

$$c^* = c\left(\frac{t}{R}\right)$$

$$\phi^* = \phi\left(\frac{t}{R}\right) + \phi_0^*$$

The fact that the flow is isentropic can be seen from 2.7 which reduces to

$$\frac{d\phi}{d\xi} = 0 \text{, if } \alpha = 1.$$

Thus $\phi = 0$. The equation 2.11 with $\alpha = 1$ reduces to

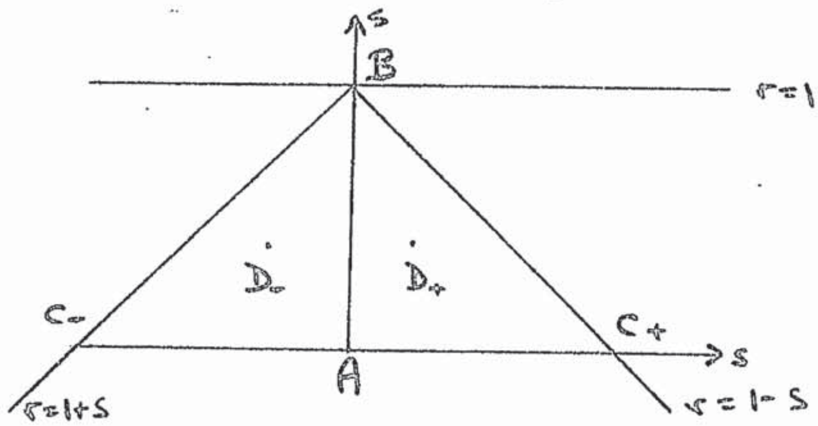
$$\frac{dv}{ds} = \frac{v}{s} \cdot \frac{(1-v)^2 - s^2(1+j)}{(1-v)(1-v-\frac{j}{k}v) - s^2} \qquad (2.12)$$

Stanyukovich (24, pp 526-527) shows that no solutions of this form can be extended from the point corresponding to a Chapman-Jouguet detonation in the contracting case. Selberg (23) shows that neither a Chapman-Jouguet nor an overdriven uniform contracting detonation is possible by considering the expansion of the solution away from the front. The argument given here for the non-existence of a uniform converging front clarifies the above two explanations, and involves examination of the integral curves of equation 2.12. These curves are given on page 426 of Courant and Friedrichs (9).

The equation has six singular points

$$A(0,0),\ B(1,0),\ C_{\pm}(0,\pm 1),\ D_{\pm}\left(\frac{1}{1+\frac{1+j}{k}}\ ,\ \frac{\pm 1}{\sqrt{1+j}\left(1+\frac{k}{1+j}\right)}\right).$$

Since $k = \frac{2}{\gamma-1} > 0$ for $\gamma > 1$ and $j = 1$ or 2, $D_{\pm}$ lie within the triangular region between $D = 0$ and $r = 0$.



Since D changes sign on crossing the lines $r = 1 \pm s$ it follows that the direction of the solution curves (i.e. the direction of increasing t for a given R) is reversed on crossing either of these lines. In general a point in the r, s plane corresponds to a line R/t = constant in the physical R, t plane and an arc of a solution curve in the r, s plane corresponds to the complete physical solution for some specific flow.

The jump relations across a discontinuity in the flow determine the values of the physical variables immediately behind the wave, and so a discontinuity of specified strength gives a point P in the r, s plane, this point corresponding to the path of the wave. The solution for the flow behind such a discontinuity corresponds to a solution curve of 2.12 starting at P and leaving P in the direction of increasing t. Let us consider the jump relations across a discontinuity, whose width is assumed to be negligible in comparison with R, so that the relations across it are precisely the equations across a plane front. Suppose there is an instantaneous reaction in the gas as the front passes through and this releases a constant amount of energy Q per unit mass. This corresponds to a detonation wave and the special case Q = 0 corresponds to a shock wave.

Let U^* be the speed of the front and the subscripts 0, 1 refer to the values immediately ahead of and behind the front respectively. The jump relations are the conservation equations of mass, momentum, and energy and are

$$\rho_1^*(u_1^* - U^*) = \rho_0^*(u_0^* - U^*)$$

$$p_1^* + \rho_1^*(u_1^* - U^*)^2 = p_0^* + \rho_0^*(u_0^* - U^*)^2$$

$$\frac{1}{2}(u_1^* - U^*)^2 + \frac{\gamma}{\gamma-1}\frac{p_1^*}{\rho_1^*} = \frac{1}{2}(u_0^* - U^*)^2 + \frac{\gamma}{\gamma-1}\frac{p_0^*}{\rho_0^*} + Q \qquad (2.13)$$

If the gas ahead of the front is at rest then $u_0^* = 0$. Replacing $\frac{\gamma p^*}{\rho^*}$ by c^{*2} and eliminating ρ_0^*, ρ_1^* from these equations gives

$$\mu^2(u_1^* - U^*)^2 + (1-\mu^2)c_1^2 = \mu^2 U^{*2} + (1-\mu^2)c_0^{*2} + 2\mu^2 Q$$

$$c_0^{*2} = U^*\left\{\frac{c_1^{*2}}{U^* - u_1^*} - \gamma u_1^*\right\}, \text{ where } \mu^2 = \frac{\gamma-1}{\gamma+1} > 0$$

s.ℓ.	shock locus
----	detonation locus
S_1	infinite shock
D_1	infinite detonation
D_2	Chapman-Jouguet detonation

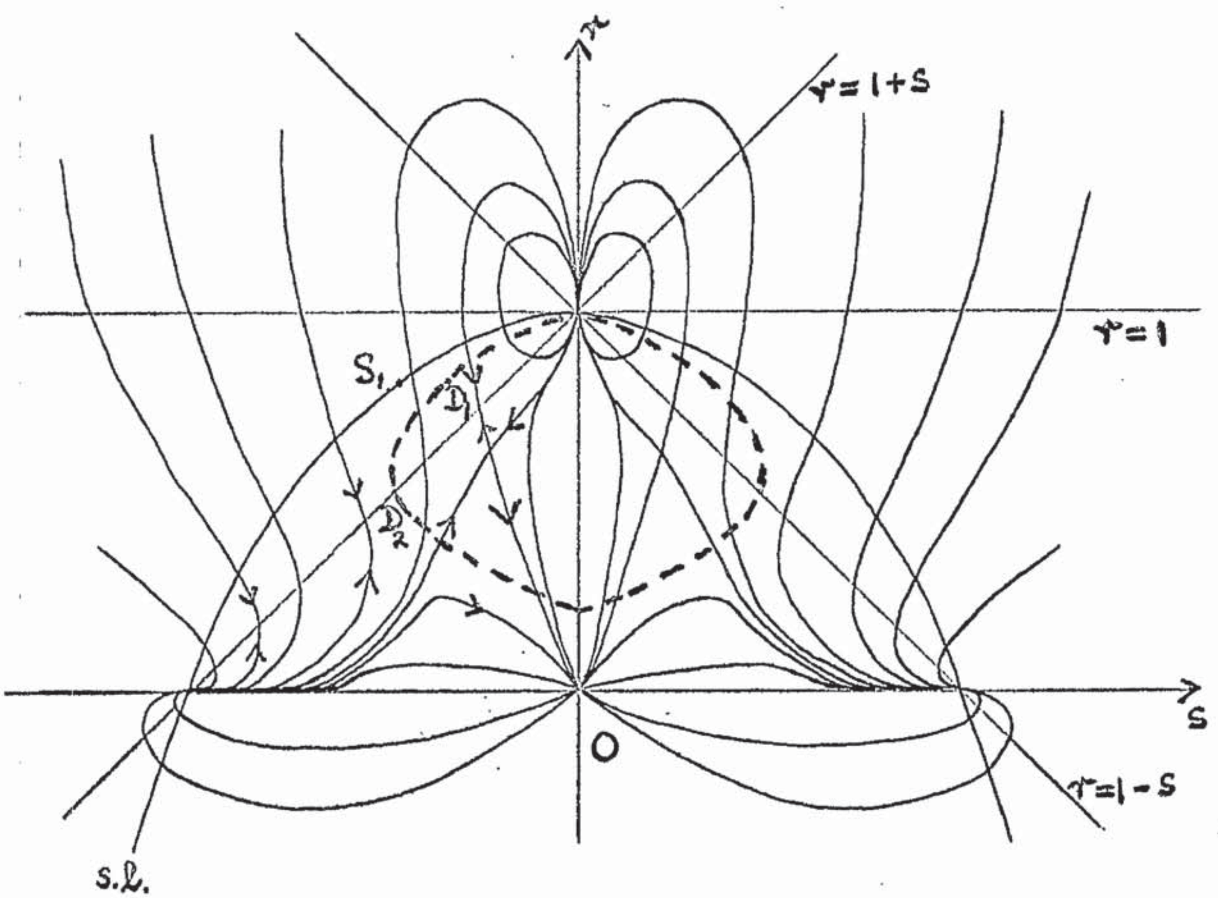


FIG. 2.1

from which c_0^{*2} can be eliminated to give the following equation for c_1^{*2}

$$c_1^{*2} = \left(U_1^* - u_1^* \right) \left\{ \frac{\gamma-1}{2} u_1^* + U^* - (\gamma-1) \frac{Q}{u_1^*} \right\}$$

In terms of the variables r, s this is

$$s^2 = (1-r) \left\{ \frac{\gamma-1}{2} r + 1 - \frac{(\gamma-1)Q}{r\, U^{*2}} \right\} \tag{2.14}$$

and for a shock

$$s^2 = (1-r) \left\{ \frac{\gamma-1}{2} r + 1 \right\} \tag{2.15}$$

Since $r > 0$ it is observed that, for a given value of r, the corresponding value of s for a detonation is numerically less than that for a shock. The 'shock locus' 2.15 is an ellipse.

The solution curves for 2.12 are sketched in Fig. 2.1. They do not differ in character whatever value be assigned to γ and whether j = 1 or 2. The 'shock locus' (s.l.) 2.15, and 'detonation locus' 2.14, are exhibited. These curves can only represent such waves for values of r not greater than $\frac{2}{\gamma+1}$, which corresponds to a shock or detonation of infinite strength. The direction associated with the solution curves is that of time increasing. Positive values of s correspond to expanding fronts and negative values to contracting fronts. The lines $r = 1 \pm s$ correspond to sonic (Chapman-Jouguet) paths.

Since no solution curves leave the line $r = 1 + s$, $s < 0$, it follows that there is no contracting solution of the R/t type for a Chapman-Jouguet detonation front. The part of the 'detonation locus' corresponding to overdriven fronts lies between the Chapman-Jouguet point and $r = \frac{2}{\gamma + 1}$. The solution curves starting at points on this arc run into the sonic line $r = 1 + s$, where the particle velocity is not zero, which is not physically possible. Hence there are no overdriven solutions.

3. The Equations and Boundary Conditions for the Detonation Solution

It was shown previously that the effect of allowing a release of heat energy across a converging spherical shock wave is to introduce perturbation terms of order $\lambda^{-2+\frac{2}{\gamma}}$ in the boundary values at the shock and in the speed of the front. It will now be shown that this is consistent with the conservation equations across such a detonation front subject to the condition that U^* is large in comparison with both the initial sound speed of the gas and $Q^{\frac{1}{2}}$.

The solution of the conservation equations 2.13 determining the jumps across the front for u^* is

$$u^* = \frac{U^{*2} - c_0^{*2} + \sqrt{(U^{*2} - c_0^{*2})^2 - 2(\gamma^2-1)Q\,U^{*2}}}{(\gamma+1)\,U^*}$$

and since $U^* \gg c_0^*$, $Q^{\frac{1}{2}}$ this can be written

$$u^* = \frac{2}{\gamma+1}\,U^* - K.\,U^{*-1} \qquad (3.1)$$

where terms of order U^{*-3} have been neglected

and $K = \frac{2}{\gamma+1}\,c_0^{*2} + (\gamma-1)Q$

We derive similar expressions for c^*, φ^*

$$\pm c^* = E\,U^* + E'\,U^{*-1} \qquad (3.2)$$

where

$$E' = \frac{6\gamma-\gamma^2-1}{2(\gamma+1)\sqrt{2\gamma(\gamma-1)}}\,c_0^{*2} + \frac{1}{4}\sqrt{2\gamma(\gamma-1)}\,(3-\gamma)\,Q$$

$$E = \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1}.$$

and the positive sign refers to an expanding wave front and the negative sign to a contracting one.

$$\phi^* = k \log(\pm U^*) + H_0 U^{*^{-2}} + \phi_0^* \qquad (3.3)$$

where $$H_0 = \frac{\gamma+1}{2(\gamma-1)} \left\{ (\gamma+1)Q + \frac{3\gamma-1}{\gamma(\gamma-1)} c_0^{*^2} \right\}$$

The equations corresponding to a strong shock wave are 3.1, 3.2, 3.3 without the terms containing Q, $c_0^{*^2}$. It is observed that the correction terms introduced into these boundary values are all of order U^{*-2}, and since $U^* = -\lambda^{1-\frac{1}{\alpha}}$ (taking the incoming wave), these correction terms are of order $\lambda^{-2+\frac{2}{\alpha}}$ By taking the appropriate values of K, E', H_0 we can find the effects on the solution of introducing Q and $c_0^{*^2}$, either separately or combined. These boundary values also show that if γ is nearly unity, i.e. the gas has a very high internal energy, then the effect of Q is negligible in comparison with that of $c_0^{*^2}$.

The velocity for a contracting spherical shock wave is

$$U^* = -\lambda^{1-\frac{1}{\alpha}}, \qquad \text{where } 0 < \alpha < 1,$$

so let us write the equation of the converging detonation wave as

$$U^* = -\lambda^{1-\frac{1}{\alpha}}\left(1+\beta\lambda^{-2+\frac{2}{\alpha}}\right)$$

where β is a constant depending on $\gamma, j, Q, c_0^{*^2}$

The boundary values behind the front can be written

$$u^* = -\frac{2}{\gamma+1}\lambda^{1-\frac{1}{\alpha}} + \left(k - \frac{2\beta}{\gamma+1}\right)\lambda^{-1+\frac{1}{\alpha}}$$

$$c^* = E\lambda^{1-\frac{1}{\alpha}} + (E' + \beta E)\lambda^{-1+\frac{1}{\alpha}}$$

$$\phi^* = k\left(1-\frac{1}{\alpha}\right)\log\lambda + (H_0 + k\beta)\lambda^{-2+\frac{2}{\alpha}} + \phi_0^*$$

where terms of relative order $\lambda^{-4+\frac{4}{\alpha}}$ have been neglected.

The general solution contains no new dimensional parameters and can be written in the form

$$\left.\begin{aligned} u^* &= u(\xi)R^{1-\frac{1}{\alpha}} + \bar{u}(\xi)R^{-1+\frac{1}{\alpha}} \\ c^* &= c(\xi)R^{1-\frac{1}{\alpha}} + z(\xi)R^{-1+\frac{1}{\alpha}} \\ \phi^* &= k\left(1-\frac{1}{\alpha}\right)\log R + \phi(\xi) + F(\xi)R^{-2+\frac{2}{\alpha}} \end{aligned}\right\} \qquad (3.4)$$

and in this case $\xi \leq 0$.

The functions u, c ϕ determine the Guderley solution and the functions $\bar{u}$, $\bar{c}$, F determine the corrections for the detonation. We can now determine the boundary values of these six functions at the front, taking into account the fact that the path of the front is now slightly displaced from the shock path. The equation of the shock path is $\xi = -1$ so that the equation of the detonation path is

$$\xi = -1\left(1 + \beta_1\lambda^{-2+\frac{2}{\alpha}}\right)$$

where $\beta_1 = \frac{-\beta}{3-2\alpha}$

so the detonation is

$$\xi = -1 + \frac{\beta}{3-2\alpha}\lambda^{-2+\frac{2}{\alpha}} \qquad (3.5)$$

The boundary values at $\xi = -1$ can now be found by substituting 3.5 into 3.4 to give the values of u^*, c^*, φ^* there. This gives

$$\left.\begin{aligned} u^* &= u(-1)\lambda^{1-\frac{1}{\alpha}} + \left\{\frac{\beta}{3-2\alpha}u'(-1) + \bar{u}(-1)\right\}\lambda^{-1+\frac{1}{\alpha}} \\ c^* &= c(-1)\lambda^{1-\frac{1}{\alpha}} + \left\{\frac{\beta}{3-2\alpha}c'(-1) + \bar{c}(-1)\right\}\lambda^{-1+\frac{1}{\alpha}} \\ \phi^* &= k\left(1-\frac{1}{\alpha}\right)\log R + \phi(-1) + \left\{F(-1) + \frac{\beta}{3-2\alpha}\phi'(-1)\right\}\lambda^{-2+\frac{1}{\alpha}} \end{aligned}\right\} \quad (3.6)$$

from which we deduce

$$\begin{aligned} u(-1) &= \frac{-2}{\gamma+1} \\ c(-1) &= E \\ \phi(-1) &= \phi_0^* \end{aligned} \qquad (3.7)$$

and

$$\left.\begin{aligned} \bar{u}(-1) &= K - \frac{2\beta}{\gamma+1} - \frac{\beta}{3-2\alpha}u'(-1) \\ \bar{c}(-1) &= E' + \beta E - \frac{\beta}{3-2\alpha}c'(-1) \\ F(-1) &= H_0 + k\beta - \frac{\beta}{3-2\alpha}\phi'(-1) \end{aligned}\right\} \quad (3.8)$$

The derivative terms appearing in 3.8 may be evaluated straightforwardly using the equations of motion 2.8 and the boundary values 3.7.

Consider the equations of motion of the system. The equations which u^*, c^*, ϕ^* must satisfy are the partial differential equations 2.1, 2.2, 2.3 for the spherically symmetric flow of an inviscid, non-heat-conducting perfect gas.

These were put in characteristic form as 2.4, 2.5

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left(u^* \pm k c^*\right) + \left(u^* \pm c^*\right)\frac{\partial}{\partial R}\left(u^* \pm k c^*\right) \\ &\qquad = \mp \frac{j\, u^* c^*}{R} + \frac{c^{*2}}{\gamma}\frac{\partial \phi^*}{\partial R} \\ &\frac{\partial \phi^*}{\partial t} + u^* \frac{\partial \phi^*}{\partial R} = 0 \end{aligned}\right\} \qquad (3.9)$$

The governing equations for the six variables $u, c, \phi, \bar{u}, \bar{c}, F$ can now be derived by substituting u^*, c^*, ϕ^* from 3.4, into 3.9. This results in power series in R, the power of R increasing successively by $-2 + \frac{2}{\alpha}$, which is positive. Setting the leading coefficients zero in each equation results in the differential equations for the shock flow. The second coefficients give rise to the linear diffenential equations which $\bar{u}, \bar{c}, F$ must satisfy. These are

$$\begin{aligned} &\left\{1 - \xi(u \pm c)\right\}\left(u' \pm k c'\right) \\ &= (1-\alpha)(u \pm c)(u \pm k c) \mp j \alpha u c - \frac{c^2}{\gamma}\left\{\xi \phi' + k(1-\alpha)\right\} \end{aligned} \qquad (3.10)$$

$$\phi' = \frac{k(1-\alpha) u}{1 - \xi u} \quad , \left(\text{where } ' \equiv \frac{d}{d\xi} \right) \qquad (3.11)$$

which coincide with the equations 2.6, 2.7 derived previously, and

$$\begin{aligned} &\left\{1 - \xi(u \pm c)\right\}\left(\bar{u}' \pm k \bar{c}'\right) + (1-\alpha)\left\{(u \pm c)(\bar{u} \pm k\bar{c}) - (u \pm kc)(\bar{u} \pm \bar{c})\right\} \\ &- \xi(\bar{u} \pm \bar{c})(u' \pm k c') \end{aligned} \qquad (3.12)$$

$$= \mp j\alpha(u\bar{c} + \bar{u}c) - \frac{2c\bar{c}}{\gamma}\left\{k(1-\alpha) + \xi\phi'\right\} - \frac{c^2}{\gamma}\left\{\xi F' - 2(1-\alpha)F\right\}$$

$$(1-\xi u)F' + 2(1-\alpha)uF = k(1-\alpha)\bar{u} + \xi\bar{u}\,\phi' \qquad (3.13)$$

as previously, we can eliminate ϕ' between 3.10, 3.11 to get

$$\{1-\xi(u\pm c)\}(u'\pm kc')$$
$$= (1-\alpha)(u\pm c)(u\pm kc) \mp j\alpha uc - \frac{k(1-\alpha)}{\gamma}\frac{c^2}{1-\xi u} \qquad (3.14)$$

To solve the detonation problem it is necessary to integrate 3.12, 3.13 after eliminating ϕ', u', c', using 3.10, 3.11, and solving 3.14 for u, c. The problem is specified by the five differential equations, 3.14, 3.12, 3.13, together with the appropriate boundary values. The boundary values for $\ddot{u}$, $\ddot{c}$, F at $\xi = -1$ are given by 3.8 in terms of the unknown parameter β, and those for u, c are given by 3.7. The parameter α which appears in equations 3.14 is as yet undetermined. Before we can find a unique solution we must obtain one extra boundary condition for the shock solution and one for the detonation.

Before considering the form of the remaining boundary conditions let us rewrite the problem in terms of the variables $\tau(\xi)$, $s(\xi)$, $\bar{\tau}(\xi)$, $\bar{s}(\xi)$ defined by

$$\tau = u\xi \quad , \quad s = c\xi$$
$$\bar{\tau} = \bar{u}\xi \quad , \quad \bar{s} = \bar{c}\xi$$

The differential equations for $\tau, s, \bar{\tau}, \bar{s}, F$ obtained from 3.14, 3.12, 3.13 are

$$\xi(1-r\mp s)(r'\pm ks') = \frac{B_\pm}{r-1} \tag{3.15}$$

$$\xi(1-r\mp s)(\bar{r}'\pm k\bar{s}') = \frac{A_\pm}{(r-1)^2(1-r\mp s)} \tag{3.16}$$

$$\xi(1-r)F' + 2(1-\alpha)rF = \frac{k(1-\alpha)}{1-r}\bar{r} \tag{3.17}$$

where

$$B_\pm = (r-1)\{1-\alpha(r\pm s)\}(r\pm ks) \mp j\alpha rs(r-1) + \frac{k(1-\alpha)}{\gamma}s^2$$

$$\frac{A_\pm}{(r-1)^2(1-r\mp s)} = (1-r\mp s)(\bar{r}\pm k\bar{s}) + (\bar{r}\pm\bar{s})\left\{\frac{B_\pm}{(r-1)(1-r\mp s)} - r\mp ks\right\}$$
$$\mp(1-\alpha)(k-1)(r\bar{s}-\bar{r}s) \mp j\alpha(r\bar{s}+\bar{r}s)$$
$$+\frac{2k(1-\alpha)}{\gamma}s\bar{s} - \frac{1-\alpha}{\gamma}\frac{s^2}{(1-r)^2}\left\{k\bar{r}+2(r-1)F\right\},$$

where r', s', ϕ', F' have been eliminated from $A_\pm$ using 3.15, 3.11, 3.13.

From the five equations 3.15, 3.16, 3.17 we can eliminate ξ to reduce the problem to four equations for r, $\bar{r}$, $\bar{s}$, F considered as functions of s.

$$(1-r-s)\left(\frac{dr}{ds}+k\right)B_- = (1-r+s)\left(\frac{dr}{ds}-k\right)B_+ \tag{3.18}$$

$$(r-1)(1-r-s)\left(\frac{d\bar{r}}{ds}+k\frac{d\bar{s}}{ds}\right)B_+ = A_+\left(\frac{dr}{ds}+k\right) \tag{3.19}$$

$$(r-1)(1-r+s)\left(\frac{d\bar{r}}{ds}-k\frac{d\bar{s}}{ds}\right)B_- = A_-\left(\frac{dr}{ds}-k\right) \tag{3.20}$$

$$(r-1)\frac{dF}{ds}B_+ = (1-\alpha)(1-r-s)\left(\frac{dr}{ds}+k\right)\left\{2r(r-1)F+k\bar{r}\right\} \tag{3.21}$$

The boundary values of r, $\bar{r}$, $\bar{s}$, F at $\xi = -1$, i.e. $s = -E$, can be deduced from 3.7, 3.8. After replacing u', c', ψ' in 3.8 by means of the equations 3.10, 3.12 we get the boundary values to be

$$r(-E) = \frac{2}{\gamma+1} \tag{3.22}$$

$$\left.\begin{aligned} \bar{r}(-E) &= \frac{2\beta}{(\gamma+1)(3-2\alpha)}\left\{\frac{2j\alpha\gamma}{\gamma+1} + 2\alpha - 1\right\} - K \\ \bar{s}(-E) &= \frac{-\beta E}{3-2\alpha}\left\{\frac{j\alpha(\gamma-1)}{\gamma+1} + \alpha + k - k\alpha\right\} - E' \\ F(-E) &= H_0 + k\beta\left\{1 + \frac{k(1-\alpha)}{3-2\alpha}\right\} \end{aligned}\right\} \tag{3.23}$$

The function $r(s)$, which corresponds to the shock solution, has to satisfy the differential equation 3.18 and the single boundary condition 3.22. Since 3.22 contains the unknown parameter α one other boundary condition is required. The correction terms for the detonation $\bar{r}(s)$, $\bar{s}(s)$, $F(s)$ have to satisfy the differential equations 3.19, 3.20, 3.21 and the boundary conditions 3.23, containing β. One extra condition is also required in order to evaluate β.

The two remaining conditions can be determined by examining the lines along which the basic solution of the shock problem, determined by equations 3.14, may be singular. There are four such lines

(i) $$1-(u\pm c)\xi = 0$$

(ii) $$1-u\xi = 0$$

(iii) $$\xi = -\infty$$

In the R, t plane these lines have equations of the form ξ = constant, i.e. they must pass through the origin. On these lines

$$\frac{t}{\alpha R^{\frac{1}{\alpha}}} = \text{const}$$

so that
$$\frac{dR}{dt} = \frac{\alpha R}{t} = \frac{R^{1-\frac{1}{\alpha}}}{\xi}$$

and for (i),
$$\frac{dR}{dt} = (u\pm c)R^{1-\frac{1}{\alpha}} = u^*\pm c^*$$

i.e. the positive and negative characteristics in the R, t plane passing through the origin. Similarly (ii) is the particle path through the origin and (iii) is R = 0, $t \leq 0$.

The flow is thus seen to have possible singularities on the two characteristics and particle path which are at R = 0 at the same instant (i.e. t = 0) as the wave front, and also on the t-axis. It is assumed that the solution we are seeking, which is valid for small values of R, is independent of the conditions at large values of R, i.e. is independent of the method of initiation. As the front approaches the origin the disturbances it produces become unbounded, so that any initial effect will eventually be overshadowed by the effects of the shock. Therefore we shall require that the solution be regular on the limiting negative characteristic (l.n.c.) through the origin. This enables us to evaluate the solution for $t < 0$. There is no positive characteristic through 0 for $t < 0$ as the flow is supersonic ahead of the front.

There is a positive characteristic through 0 in the region $t > 0$. This has been shown (6) to be a regression edge, which points to the fact that the front is reflected outwards from 0 as a discontinuity. If it is assumed that all the available chemical energy in the medium is released by the incoming wave then the reflected expanding front must be a shock wave.

The particle path through 0 is the t-axis. The solution will be singular on the t-axis, i.e. at the centre of symmetry, where the speed of the front, which is proportional to $\lambda^{1-\frac{1}{\alpha}}$ where $0 < \alpha < 1$, becomes infinite. As we have assumed that the effect of the detonation is of small order relative to the Guderley solution, the above argument for regularity on the l.n.c. holds for the solution for the detonation wave. It will be seen that this condition of regularity gives the extra conditions necessary for specifying a unique solution to the complete problem.

Since u^*, c^*, ϕ^* are regular functions of R, t, the functions $u, c, \phi, \bar{u}, \bar{c}, F$ must be regular in ξ so that $\sigma, \bar{\sigma}, \bar{s}, F$ are regular functions of s. If s_0 is the value of s on the l.n.c. then $\sigma, \bar{\sigma}, \bar{s}, F$ can be expanded as a power series about this point. Let the expansions be

$$\sigma = \sigma_0 + \sigma_1 (s - s_0) + \quad - - \quad - \cdot + \sigma_n (s - s_0)^n + - - -$$

$$\bar{\sigma} = \bar{\sigma}_0 + \bar{\sigma}_1 (s - s_0) + \quad - - \quad - - + \bar{\sigma}_n (s - s_0)^n + - - -$$

$$\bar{s} = \bar{s}_0 + \bar{s}_1 (s - s_0) + \quad - - \quad - - + \bar{s}_n (s - s_0)^n + - - -$$

$$F = F_0 + F_1 (s - s_0) + \quad - - \quad - - + F_n (s - s_0)^n + - - -$$

Let the equation of the l.n.c. be $\xi = \xi_1 \ (< 0)$ for the shock flow. This path will be slightly disturbed for the detonation wave so in this case let the path be

$$\xi = \xi_1 \left(1 + \delta R^{-2 + \frac{2}{\alpha}}\right)$$

so that $t = \alpha \xi_1 R^{\frac{1}{\alpha}} \left(1 + \delta . R^{-2 + \frac{2}{\alpha}}\right)$ on this line

and $$\frac{dR}{dt} = \frac{1}{\xi_1} R^{1 - \frac{1}{\alpha}} - \frac{3 - 2\alpha}{\xi_1} \delta R^{-1 + \frac{1}{\alpha}} \qquad (3.24)$$

on the l.n.c.

The general forms of u^*, c^*, ϕ^* are

$$u^* = u(\xi) R^{1 - \frac{1}{\alpha}} + \bar{u}(\xi) R^{-1 + \frac{1}{\alpha}}$$

$$c^* = c(\xi) R^{1 - \frac{1}{\alpha}} + z(\xi) R^{-1 + \frac{1}{\alpha}}$$

$$\phi^* = k\left(1 - \frac{1}{\alpha}\right) \log R + \phi(\xi) + F(\xi) R^{-2 + \frac{2}{\alpha}}$$

so that the values on the l.n.c., $\xi = \xi_1 \left(1 + \delta R^{-2 + \frac{2}{\alpha}}\right)$, are

$$u^* = u(\xi_1)R^{1-\frac{1}{\alpha}} + \left\{ \left(\frac{du}{d\xi}\right)_{\xi=\xi_1} \cdot \xi_1 \delta + \bar{u}(\xi_1) \right\} R^{-1+\frac{1}{\alpha}}$$

$$c^* = c(\xi_1)R^{1-\frac{1}{\alpha}} + \left\{ \left(\frac{dc}{d\xi}\right)_{\xi=\xi_1} \cdot \xi_1 \delta + \bar{c}(\xi_1) \right\} R^{-1+\frac{1}{\alpha}} \qquad (3.25)$$

$$\phi^* = k\left(1-\tfrac{1}{\alpha}\right)\log R + \phi(\xi_1) + \left\{ \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1} \cdot \xi_1 \delta + F(\xi_1) \right\} R^{-2+\frac{2}{\alpha}}$$

On this l.n.c.

$$\frac{dR}{dt} = u^* - c^*$$

so that, using 3.24,

$$\frac{1}{\xi_1} R^{1-\frac{1}{\alpha}} - \frac{3-2\alpha}{\xi_1}\,\delta R^{-1+\frac{1}{\alpha}}$$

$$= (u_1 - c_1)R^{1-\frac{1}{\alpha}} + \left\{ \xi_1 \delta (u_1' - c_1') + (\bar{u}_1 - \bar{c}_1) \right\} R^{-1+\frac{1}{\alpha}}$$

where $u_1 = u(\xi_1)$, $u_1' = \left(\frac{du}{d\xi}\right)_{\xi=\xi_1}$ etc.

and equating coefficients of R in this equation gives

$$\xi_1(u_1 - c_1) = 1$$

so that $v_0 - s_0 = 1$ on the l.n.c. $\qquad (3.26)$

and $\quad \xi_1^2 \delta(u_1' - c_1') + \xi_1(\bar{u}_1 - \bar{c}_1) + (3-2\alpha)\delta = 0 \qquad (3.27)$

But $\quad \xi_1^2 u_1' = \xi_1^2 \left(-\frac{v}{\xi^2} + \frac{1}{\xi}\frac{dv}{d\xi} \right)_{\xi=\xi_1}$

$$= -v(\xi_0) + \xi_1 \left(\frac{dv}{d\xi}\right)_1 = -v_0 + v_1 \xi_1 \left(\frac{ds}{d\xi}\right)_1$$

and from 3.15 +

$$-2 s_0 \xi_1 \left(\frac{d\sigma}{ds} + k\right)_1 \cdot \left(\frac{ds}{d\xi}\right)_1 = \frac{B_{0+}}{s_0} ,$$

$$\text{where } B_{0+} = B_+(\sigma_0, s_0)$$

Hence

$$\xi_1^2 u_1' = -\sigma_0 - \frac{\sigma_1 B_{0+}}{2 s_0^2(\sigma_1 + k)}$$

and similarly

$$\xi_1^2 c_1' = -s_0 - \frac{B_{0+}}{2 s_0^2(\sigma_1 + k)}$$

Inserting these into 3.27 gives

$$\bar{\sigma}_0 = \bar{s}_0 + 2(\alpha - 1)\delta + \frac{(\sigma_1 - 1) B_{0+}}{2 s_0^2(\sigma_1 + k)} \cdot \delta \qquad (3.28)$$

We thus have two equations 3.26, 3.28 relating $s_0, \sigma_0, \bar{\sigma}_0, \bar{s}_0$
However the l.n.c. is a negative characteristic so that the variables on it must satisfy the characteristic condition which is

$$d(u^* - k c^*) = \frac{j c^* u^*}{R} dt - \frac{c^*}{\gamma} d\phi^* \qquad (3.29)$$

where u^*, c^*, ϕ^* are given by 3.25.
When expanded this condition has terms of order $R^{-\frac{1}{\alpha}}, R^{-2+\frac{1}{\alpha}}, R^{-1+\frac{3}{\alpha}}, \ldots$
Setting the two leading coefficients zero gives

$$\left(1-\frac{1}{\alpha}\right)(\sigma_0 - k s_0) = j \sigma_0 s_0 - \frac{k}{\gamma}\left(1-\frac{1}{\alpha}\right) s_0 \tag{3.30}$$

$$\left\{\bar{s}_0 - \frac{\delta B_{0+}}{2 s_0^2(\sigma_1 + k)}\right\} \cdot \left\{\frac{j\alpha}{\alpha-1}(\sigma_0 + s_0) + 1 - \frac{2}{\gamma\mu}\right\}$$
$$+ \delta s_0 \left\{2j\alpha - 1 + \frac{2}{\gamma} + \frac{2k\alpha}{\gamma} + \frac{j\alpha(2\alpha-1)}{1-\alpha}\sigma_0\right\}$$
$$+ \delta\left\{2\alpha - 3 - \frac{2k(1-\alpha)}{\gamma}\right\} + \frac{2}{\gamma} s_0 F_0 = 0. \tag{3.31}$$

The four equations 3.26, 3.28, 3.30, 3.31 serve to determine $s_0, \sigma_0, \bar{\sigma}_0, \bar{s}_0$ in terms of F_0, δ. The last two of these equations could have been derived by examination of the differential equations 3.15, 3.16. On this characteristic $\sigma = 1 + s$ and the solution has to be regular. Since the left hand side of 3.15 vanishes at $s = s_0$ the right hand side must vanish also, for $\frac{d\sigma}{d\xi}, \frac{ds}{d\xi}$ to remain finite at $s = s_0$. Hence $B_- = 0$ at $s = s_0$, which is precisely equation 3.30. If we write

$$A_- = A_{0-} + A_{1-}(s-s_0) + \cdots + A_{n-}(s-s_0)^n + \cdots$$

then the derivatives $\frac{d\bar{\sigma}}{d\xi}, \frac{d\bar{s}}{d\xi}$ will be finite at $s = s_0$ provided $A_{0-} = 0$ and $A_{1-} = 0$. The first of these follows from 3.30, and after some algebra it can be shown that the second, $A_{1-} = 0$, is identical to 3.31. If r_0 is replaced by $1 + s_0$ in equation 3.30 the resulting equation for s_0 is a quadratic and so in general there are two possible combinations $(\sigma_{0+}, s_{0+}), (\sigma_{0-}, s_{0-})$.

It can now be seen that these four conditions for the boundary values at $s = s_0$, together with the governing differential equations and the boundary conditions 3.22, 3.23 determine the solution uniquely. To solve for $r = r(s)$ the equation 3.20, containing α, must be integrated subject to the boundary values $\sigma(s_0) = \sigma_0$, $\sigma(-E) = \frac{2}{\gamma+1}$. In this way we can evaluate α and the appropriate solution for $r = r(s)$. The detonation correction terms $\bar{\sigma}$, $\bar{s}$, F are determined by the three linear differential equations 3.19, 3.20, 3.21 subject to the boundary values 3.23 at $s = -E$, in terms of the unknown parameter β, and the boundary values 3.28, 3.31 at $s = s_0$, in terms of the unknown parameters δ, F_0.

The Whitham Simplified Analysis

Before considering the results of the complete numerical solution of the problem as already described, it will be of interest to examine the solution obtained by application of the much simplified analysis, in the form given by Whitham (29), for the motion of non-stationary shocks.

The problem of a converging spherical or cylindrical wave front is evidently mathematically equivalent to the problem of a similar wave front travelling along a tube in the shape of a cone or an infinite wedge respectively, neglecting the effects of heat conduction and viscosity at the walls of the tube. The motion of shocks in tubes of slowly varying cross-section has been studied by Chester (7) and Chisnell (8). The latter has applied his results to the problem of converging shocks and compared his results with those for the similarity solution obtained by Butler (6) for the final stages when the shock is very strong. As the area of the tube varies as R^{-1} (cylindrical) and as R^{-2} (spherical) the cross-sectional area is varying very rapidly in the final stages. In spite of this the results of the simplified analysis are extremely close to those given by the similarity solution (at worst $\frac{1}{2}\%$ difference in the evaluation of α, and much less than that in most cases).

It has been shown by Whitham (29) that the approximate results of Chester and Chisnell can be obtained by a very simple method, the one to be used here. This is to apply the characteristic condition, which has to be satisfied by the flow variables behind the shock, to the actual boundary values immediately behind it. Whitham has applied his theory to the problem of converging shocks and obtained the same numerical results as found by Chisnell. This analysis is extremely simple and can be applied to the present problem for the evaluation of β, the parameter determining the correction to the speed of the front.

The appropriate characteristic condition was derived previously (3.29) as

$$d(u^* - kc^*) = j u^* c^* R^{-1} dt - \frac{1}{\gamma} c^* d\phi^*$$

According to the simplified theory this equation has to be satisfied by the boundary values at the front i.e.

$$u^* = -\frac{2}{\gamma+1} R^{1-\frac{1}{\alpha}} + \left(K - \frac{2\beta}{\gamma+1}\right) R^{-1+\frac{1}{\alpha}}$$

$$c^* = E R^{1-\frac{1}{\alpha}} + (E' + \beta E) R^{-1+\frac{1}{\alpha}}$$

$$\phi^* = k\left(1 - \frac{1}{\alpha}\right) \log R + (H_0 + k\beta) R^{-2+\frac{2}{\alpha}}$$

On substituting these values into the characteristic condition and using the fact that $\frac{dR}{dt} = u^* - c^*$, we obtain a polynomial in powers of $R^{2-\frac{2}{\alpha}}$, which has to be identically zero. Equating the coefficient of the highest order term to zero gives the following formula for α

$$\frac{1}{\alpha} - 1 = \frac{\frac{jE}{\gamma+1}}{\left(\frac{2}{\gamma+1} + E\right)\left(\frac{1}{\gamma+1} + \frac{E}{\gamma}\right)}$$

which is identical to the formula 11 given by Whitham (29). On setting the coefficient of the next highest term zero we obtain the following equation for β, after simplifying by using the above formula for α

$$\beta = \frac{E^2 K - \frac{4}{(\gamma+1)^2} E'}{\frac{4}{\gamma+1} E\left(\frac{2}{\gamma+1} + E\right)} + \frac{\frac{2}{\gamma} E H_0 + K - \frac{2}{\gamma_N} E'}{4\left(\frac{1}{\gamma+1} + \frac{1}{\gamma} E\right)}$$

It is of interest to note that this theory predicts that, for a given value of γ, $\frac{1}{\alpha} - 1$ is proportional to j, which is found to be very nearly the case for the similarity solution. It also gives values for β which are independent of j. There is no reason to expect the values of β given by the similarity solution to be completely independent of j. The motion of the shock is largely governed by the focussing effect whereas the effect of the addition of a uniform heat release is purely a volume effect. The present simplification does not give the correct wave propagation behind the front as the characteristic condition is incorrectly applied. The numerical results will be given later along with those of the complete analysis.

4. The Method of Solution

The mathematical problem relating to the basic shock wave solution can be written as

$$\frac{1}{k}\frac{dr}{ds} = \frac{(1-r+s)B_+ + (1-r-s)B_-}{(1-r+s)B_+ - (1-r-s)B_-} \tag{4.1}$$

with

$$\left.\begin{aligned} r(-E) &= \frac{2}{\gamma+1} \\ r(s_0) &= 1+s_0 \end{aligned}\right\} \tag{4.2}$$

where $B_\pm$ and s_0 depend on a, and s_0 has one of two possible values. To evaluate a we must integrate the equation 4.1 between $s = -E$ and $s = s_0$ in such a way that the two conditions 4.2 are satisfied. Since $r = 1 + s$ and $B_- = 0$ at $s = s_0$ both the numerator and denominator vanish at $s = s_0$, but, since $r(s)$ is assumed to be regular at this point, $\frac{dr}{ds}$ is in fact finite for the required solution. However, the fact that the numerator and denominator vanish simultaneously at $s = s_0$ may give rise to difficulties in the integration of 4.1. The method used to perform this integration, to be described later, effectively generates the power series for $r(s)$ about the point $s = s_0$.

It was noted that the equation determining s_0 for given values of a, γ, j, is a quadratic so that there are two possible values for s_0. When the power series for $r(s)$

$$r = r_0 + r_1(s-s_0) + \cdots + r_n(s-s_0)^n + \cdots$$

is substituted into the differential equation 4.1 and the coefficients of the individual powers of $s - s_0$ are equated to zero, recurrence relations are obtained for the coefficients r_i. The first two of these are

$$B_{0-} = 0$$

$$2s_0(r_1+k)B_{1-} + (1-r_1)(r_1-k)B_{0+} = 0$$

$$\text{where } B_{\pm} = B_{0\pm} + B_{1\pm}(s-s_0) + - - - -$$

The equation $B_{0-} = 0$ is equivalent to 3.30, and the second of the above equations serves to determine r_1 in terms of r_0, s_0 and is a quadratic in r_1. (B_{0+} contains r_0 only and B_{1-} contains r_1 linearly and r_0). The coefficient of $(s-s_0)^n$ equated to zero gives

$$-2s_0(r_1+k)B_{n-} - 2s_0 n r_n B_{1-} + r_n(r_1-k)B_{0+} - (1-r_1)n r_n B_{0+}$$

$$+ \text{ terms in } r_{n-1}, r_{n-2} - - - -$$

$$= 0 \quad \text{for } n \geqslant 2$$

where B_{n-} contains r_n, r_{n-1}, - - - - - - only and is linear in r_n.

Thus if r_0, r_1 are determined, the remaining coefficients r_i are uniquely defined in terms of these two. For a given a there are four possible solutions satisfying the given boundary conditions on the l.n.c.

If one selects a value of a, between 0 and 1, and chooses one of the four possible solutions and then integrates 4.1 as far as the shock point, i.e. $s = -E$, the value obtained for

$r(-E)$ will in general differ from the required value. Suppose that $r(-E)$ exceeds $\frac{2}{\gamma+1}$ by an amount d, the discrepancy (a function of α). The problem is to find some value of α for which one of the four solutions gives the value 0 for d.

Having obtained the appropriate value of α, for which $d = 0$, and tabulated the solution for $r = r(s)$ over the range $-E$ to s_0, we can now consider the solution for the detonation correction terms $\bar{r}(s)$, $\bar{s}(s)$, $F(s)$. The differential equations which these functions must satisfy are 3.19, 20, 21 where all the coefficients are now known. In this case the unknown parameters appear only in the boundary conditions. The boundary values at $s = -E$ are given by 3.23 in terms of β and also the values of $\gamma, \lambda, c_0^2, Q$ which are to be assigned. Also the boundary values $\bar{r}_0$, $\bar{s}_0$, at $s = s_0$, are given by 3.28, 3.21 in terms of the unknown parameters δ, F_0. The required solution has to satisfy the differential equations 3.19, 20, 21 and these six boundary conditions (i.e. including $(F)_0 = F_0$) containing the three unknowns β, δ, F_0. If we obtain two independent solutions of the differential equations, each corresponding to a separate choice of the combination δ, F_0 and neither in general satisfying the appropriate conditions at $s = -E$, the required solution to the problem can be found by taking the linear combination of these two solutions which does satisfy the conditions at $s = -E$. Thus we can find the value of β so that the speed of the incoming front U^{*}, which is given by

$$U^{*} = -\lambda^{1-\frac{1}{\alpha}}\left(1 + \beta\lambda^{-2+\frac{3}{\alpha}}\right)$$

is known in terms of the distance λ from the centre of symmetry.

Let $\delta^{(0)}$, $F_O^{(0)}$, and $\delta^{(1)}$, $F_O^{(1)}$ be the two pairs of values taken for δ, F_O and let $\bar{r}_O^{(0)}$, $\bar{r}_H^{(0)}$ and $\bar{r}_O^{(1)}$, $\bar{r}_H^{(1)}$ be the values at $s = s_O$, $-E$ of the two solutions so obtained. From these we wish to calculate the appropriate values of δ, F_O, β. Consider the solution formed by taking X times the (0) solution and Y times the (1) solution. Then this combination has to satisfy 3.23 which we can write as

$$\left.\begin{aligned} \bar{r}_H &= A_1\beta - K \\ \bar{s}_H &= A_2\beta - E' \\ F_H &= A_3\beta + H_0 \end{aligned}\right\} \tag{4.3}$$

so that we require

$$\left.\begin{aligned} X\bar{r}_H^{(0)} + Y\bar{r}_H^{(1)} &= A_1\beta - K \\ X\bar{s}_H^{(0)} + Y\bar{s}_H^{(1)} &= A_2\beta - E' \\ XF_H^{(0)} + YF_H^{(1)} &= A_3\beta + H_0 \end{aligned}\right\} \tag{4.4}$$

which can be readily solved for X, Y, β. The correct values of δ, F_O can now be found from

$$\left.\begin{aligned} F_0 &= XF_0^{(0)} + YF_0^{(1)} \\ \delta &= \frac{X\left(\bar{r}_0^{(0)} + \bar{s}_0^{(0)}\right) + Y\left(\bar{r}_0^{(1)} + \bar{s}_0^{(1)}\right)}{2(\alpha-1) + \dfrac{(\sigma_1 - 1)B_{0+}}{2s_0^2(\sigma_1 + k)}} \end{aligned}\right\} \tag{4.5}$$

The Numerical Solution

Consider the evaluation of e and the tabulation of $r = r(s)$. The integration of 4.1 will be performed by making use of the power series expansion for $r(s)$ and by developing the solution by means of an iterative method which effectively takes into account an extra term in this series at each iteration. Although the expansion about the point $s = s_o$ is used to evaluate the solution at all points of the subdivision, for all cases the range of integration ($-E$ to s_o) is never greater than 0.2.

The equation for $r = r(s)$ can be written as

$$f(r, r', s) \equiv (1-r-s)(r'+k)B_- - (1-r+s)(r'-k)B_+ \\ = 0 \tag{4.6}$$

and the solution can be expanded about the point $s = s_o$ as

$$r = r_o + r_1(s-s_o) + \cdots + r_n(s-s_o)^n + \cdots$$

On substituting this series into 4.6 we obtain the power series

$$f_o + f_1(s-s_o) + \cdots + f_n(s-s_o)^n + \cdots$$

say, where $f_n = 0$ for all n

and $f_o = 0$ determines r_o,

$f_1 = 0$ determines r_1, etc.

By solving the equations $f_n = 0$ for r_n we can develop the series solution for $r(s)$, in theory. However the algebra involved is prohibitive, particularly as the solution of each equation $f_n = 0$ is a separate procedure. For this reason it was thought necessary to develop an iterative procedure for developing the solution. In order to do this we seek approximations R_n, R_n' to σ, σ' at the nth stage of the iteration, with R_n, R_n' equal to the series expansions for σ, σ' respectively as far as the term involving σ_n (i.e. the term in $(s-s_0)^n$ for R_n and $(s-s_0)^{n-1}$ for R_n'). By means of the appropriate iterative procedure each successive approximation R_n, R_n' may be formed and the solution may be evaluated to any given accuracy, by taking into account as many terms in the series expansions as are found to be necessary.

Let R_1, R_1' be the initial approximations to σ, σ' defined by

$$R_1 = \sigma_0 + \sigma_1(s-s_0) \quad , \quad R_1' = \sigma_1$$

By solving $f_0 = 0$, $f_1 = 0$ for σ_0, σ_1 we can tabulate the functions R_1, R_1'. By substituting R_1, R_1' into 4.6 and examining the leading term we can derive a formula for R_2' which is the second approximation to σ' accurate as far as the term in $(s - s_0)$.

$$f(R_1, R_1', s) = (R_1-\sigma)\left(\frac{\partial f}{\partial \sigma}\right)_s + (R_1'-\sigma')\left(\frac{\partial f}{\partial \sigma'}\right)_s + \frac{1}{2}(R_1-\sigma)^2\left(\frac{\partial^2 f}{\partial \sigma^2}\right)_s$$

$$+ (R_1-\sigma)(R_1'-\sigma')\left(\frac{\partial^2 f}{\partial \sigma \partial \sigma'}\right)_s + \cdots \qquad \left(\text{where } \frac{\partial^2 f}{\partial \sigma'^2} \equiv 0, \ \left(\frac{\partial f}{\partial \sigma'}\right)_0 = 0\right)$$

$$= -\sigma_2(s-s_0)^2\left\{\left(\frac{\partial f}{\partial \sigma}\right)_0 + 2\left(\frac{\partial f}{\partial \sigma}\right)_1\right\}$$

$$+(s-s_0)^3\left\{-\sigma_3\left(\frac{\partial f}{\partial \sigma}\right)_0 - \sigma_2\left(\frac{\partial f}{\partial \sigma}\right)_1 - 2\sigma_2\left(\frac{\partial f}{\partial \sigma'}\right)_2 - 3\sigma_3\left(\frac{\partial f}{\partial \sigma'}\right)_1 + 2\sigma_2^2\left(\frac{\partial^2 f}{\partial \sigma \partial \sigma'}\right)_0\right\} + O\left((s-s_0)^4\right)$$

Let us denote $f(R_1, R_1', s)$ by $f(1)$ and define

$$R_2' = R_1' - \frac{2f(1)}{(s-s_0)\left\{\left(\frac{\partial f}{\partial r}\right)_0 + 2\left(\frac{\partial f}{\partial r'}\right)_1\right\}} \tag{4.7}$$

$$= \sigma_1 + 2\sigma_2(s-s_0) + 3\varepsilon_3(s-s_0)^2 + o\left((s-s_0)^3\right)$$

$$\text{where} \quad \varepsilon_3 = \frac{2}{3\left\{\left(\frac{\partial f}{\partial r}\right)_0 + 2\left(\frac{\partial f}{\partial r'}\right)_1\right\}} \cdot \left\{ \begin{aligned} &\sigma_3\left(\frac{\partial f}{\partial r}\right)_0 + 3\sigma_3\left(\frac{\partial f}{\partial r'}\right)_1 + \sigma_2\left(\frac{\partial f}{\partial r}\right)_1 \\ &+ 2\sigma_2\left(\frac{\partial f}{\partial r'}\right)_2 - 2\sigma_2^2\left(\frac{\partial^2 f}{\partial r\,\partial r'}\right)_0 \end{aligned} \right\}$$

and ε_3 is independent of s.

Thus we can use 4.7 to tabulate R_2' at each point of the range and R_2 can now be evaluated by integrating this table numerically, to give

$$R_2 = \sigma_0 + \sigma_1(s-s_0) + \sigma_2(s-s_0)^2 + \varepsilon_3(s-s_0)^3 + o\left((s-s_0)^4\right)$$

provided the integration has not introduced errors of order $(s-s_0)^3$. The process can be repeated indefinitely and, in general

$$R_n' = \sigma_1 + 2\sigma_2(s-s_0) + \cdots + n\sigma_n(s-s_0)^{n-1} + (n+1)\varepsilon_{n+1}(s-s_0)^n + O\left((s-s_0)^{n+1}\right)$$

$$R_n = \sigma_0 + \sigma_1(s-s_0) + \cdots + \sigma_n(s-s_0)^n + \varepsilon_{n+1}(s-s_0)^{n+1} + o\left((s-s_0)^{n+2}\right)$$

$$\text{where} \quad \varepsilon_{n+1} = \frac{n-1}{n\left\{\left(\frac{\partial f}{\partial r}\right)_0 + (n+1)\left(\frac{\partial f}{\partial r'}\right)_0\right\}} \cdot \left\{ \begin{aligned} &\sigma_{n+1}\left(\frac{\partial f}{\partial r}\right)_0 + (n+1)\sigma_{n+1}\left(\frac{\partial f}{\partial r'}\right)_0 \\ &+ \sigma_n\left(\frac{\partial f}{\partial r}\right)_1 + n\sigma_n\left(\frac{\partial f}{\partial r'}\right)_2 \end{aligned} \right\}$$

The iteration formula for R'_{n+1} is

$$R'_{n+1} = R'_n - \frac{n\, f(n)}{(s-s_0)\left\{\left(\frac{\partial f}{\partial \sigma}\right)_0 + (n+1)\left(\frac{\partial f}{\partial \sigma}\right)_1\right\}} \tag{4.8}$$

To integrate the table of values of R'_n we form

$$R_n(s_i) = R_n(s_{i-1}) + \int_{s_{i-1}}^{s_i} R'_n \, . \, ds$$

$$\text{where } R_n(s_0) = r_0$$

It will be most convenient to use an integration formula which involves the values of R'_n only at points of the initial subdivision. In all the calculations the following four-point formula was used for the integration

$$\int_{s_{i-1}}^{s_i} r' \, ds = \frac{h}{24}\left\{-r'_{i-2} + 13 r'_{i-1} + 13 r'_i - r'_{i+1}\right\} \tag{4.9}$$

where h is the step width and $r'_i = r'(s_i)$. This formula has symmetry in the points of evaluation and requires one extrapolation at each end of the range, since r_0 is known. The relative error in 4.9, due to truncation, is given by

$$\frac{11}{720} h^4 \left(\frac{d^4 r}{ds^4}\right)_{s_j} \quad \text{where } s_{i-2} < s_j < s_{i+1} \; .$$

With 5 subdivisions over a range of 0.2 in s, h has the value 0.04, so that

$$\frac{11}{720} h^4 \doteqdot 4.10^{-8}$$

It is essential that the accuracy of the integration formula 4.9 be such that any errors introduced into R_n, by use of this formula, are of order not greater than $(s-s_0)^{n+2}$. This is arranged by making h sufficiently small, and in practice it was found that five subdivisions were sufficient.

For the extrapolation at the ends of the range the formula used was the four-point formula, corresponding to the integration formula 4.8, in order that the errors in extrapolation and integration be consistent. This formula is

$$r'_{i+1} = 4r'_i - 6r'_{i-1} + 4r'_{i-2} - r'_{i-3} \tag{4.10}$$

The method entails only a small amount of algebra and, being iterative, is ideally suited for programming on a computer. It is self-checking to the extent that it can only converge if the successive approximations satisfy the equation more accurately at each stage, and it can be easily checked that the resulting tabulations for r, r' satisfy the equation by examining the residual, $\xi(n)$, at each stage. Since the range of integration is very small there is no difficulty in using the expansion of the solution about a single point. Perhaps the most important advantage is that no difficulty is encountered at the starting point $s = s_0$, where $\frac{dx}{ds}$ is indeterminate for direct substitution in the differential equation.

The resulting solution for $r(s)$, corresponding to the correct value of α (determined by trial and error), can be stored in the computer to be used in the evaluation of $\bar{r}$, $\bar{s}$, F.

The equations governing the perturbation quantities $\bar{r}$, $\bar{s}$, F are

$$\left.\begin{aligned} L(s) &\equiv (r-1)(1-r-s)B_+\cdot(\bar{r}'+k\bar{s}') - A_+\cdot(r'+k) = 0 \\ M(s) &\equiv (r-1)(1-r+s)B_-(\bar{r}'-k\bar{s}') - A_-\cdot(r'-k) = 0 \\ N(s) &\equiv (r-1)B_+F' - (1-\alpha)(1-r-s)(r'+k)\left\{2r(r-1)F + k\bar{r}\right\} = 0 \end{aligned}\right\} \quad (4.11)$$

where $r = r(s)$ is now known, $B_\pm$ depend on r, s only, and $A_\pm$ are polynomial combinations of $\bar{r}$, $\bar{s}$, F, r, s and are linear in $\bar{r}$, $\bar{s}$, F.

The method to be employed in the integration of 4.11 is simply an extension of that used to evaluate $r(s)$. Having selected any two values for $\bar{s}$, F, we start at $s = s_0$ and satisfy the regularity conditions 3.28, 31. The equations 4.11 are to be integrated from $s = s_0$ to $s = -E$. Let the power series expansions for $\bar{r}$, $\bar{s}$, F be

$$\bar{r} = \bar{r}_0 + \bar{r}_1(s-s_0) + \cdots + \bar{r}_n(s-s_0)^n + \cdots$$

$$\bar{s} = \bar{s}_0 + \bar{s}_1(s-s_0) + \cdots + \bar{s}_n(s-s_0)^n + \cdots$$

$$F = F_0 + F_1(s-s_0) + \cdots + F_n(s-s_0)^n + \cdots$$

and let

$$\bar{R}_n = \bar{r}_0 + \bar{r}_1(s-s_0) + \cdots + \bar{r}_n(s-s_0)^n + \eta_{n+1}(s-s_0)^{n+1} + o\left((s-s_0)^{n+2}\right)$$

$$\bar{R}_n' = \bar{r}_1 + 2\bar{r}_2(s-s_0) + \cdots + n\bar{r}_n(s-s_0)^{n-1} + (n+1)\eta_{n+1}(s-s_0)^n + o\left((s-s_0)^n\right)$$

$$\bar{S}_n = \bar{s}_0 + \bar{s}_1(s-s_0) + \cdots + \bar{s}_n(s-s_0)^n + \theta_{n+1}(s-s_0)^{n+1} + o\left((s-s_0)^{n+2}\right)$$

$$\bar{S}_n' = \bar{s}_1 + 2\bar{s}_2(s-s_0) + \cdots + n\bar{s}_n(s-s_0)^{n-1} + (n+1)\theta_{n+1}(s-s_0)^n + o\left((s-s_0)^{n+1}\right)$$

$$E_n = F_0 + F_1(s-s_0) + \cdots + F_n(s-s_0)^n + \zeta_{n+1}(s-s_0)^{n+1} + o\left((s-s_0)^{n+2}\right)$$

$$E_n' = F_1 + 2F_2(s-s_0) + \cdots + nF_n(s-s_0)^{n-1} + (n+1)\zeta_{n+1}(s-s_0)^n + o\left((s-s_0)^{n+1}\right)$$

The solution can be derived in theory by expanding $L(\bar{r}, \bar{r}', s, \bar{s}', F, s)$ as a power series in $(s - s_0)$.

$$L_0 + L_1(s-s_0) + \cdots + L_n(s-s_0)^n + \cdots$$

and setting $L_i = 0$ for all i, together with the same operation on M, N. $L_0 = 0$ gives

$$\bar{r}_1 + k\bar{s}_1 = \frac{-A_{0+}(r_1 + k)}{2s_0^2 \cdot B_{0+}} \tag{4.12}$$

and $M_2 = 0$ ($M_0 = M_1 = 0$ by reason of the regularity condition) gives

$$\bar{r}_1 - k\bar{s}_1 = \frac{A_{2-}(r_1 - k)}{s_0(1-r_1)B_{1-}} \tag{4.13}$$

where A_{2-} is linear in $\bar{r}_1$, $\bar{s}_1$

and, finally, $N_0 = 0$ gives

$$B_{0+}F_1 = -2(1-\alpha)(r_1+k)(2r_0s_0F_0 + k\bar{r}_0) \tag{4.14}$$

Thus we can evaluate $\bar{r}_1, \bar{s}_1, F_0$ corresponding to the particular choice of s, F_0, and tabulate the initial approximations

$$\bar{R}_1 = \bar{r}_0 + \bar{r}_1(s-s_0), \quad \bar{R}_1' = \bar{r}_1 \qquad \text{etc.}$$

so that initially $\eta_2 = \theta_2 = \zeta_2 = 0$.

Let us consider the iteration formulae in general. Suppose that the residuals $L(n)$, $M(n)$, $N(n)$ have been obtained by substitution of $\bar{R}_n$, $\bar{R}_n'$ etc into L, M, N. These residuals can be expanded in powers of $(s-s_0)$

$$L(n) = (n+1)(s-s_0)^n \left\{ (\eta_{n+1} - \bar{r}_{n+1})\left(\frac{\partial L}{\partial \bar{r}'}\right)_0 + (\theta_{n+1} - \bar{s}_{n+1})\left(\frac{\partial L}{\partial \bar{s}'}\right)_0 \right\}$$

$$+ o\left((s-s_0)^{n+1}\right),$$

$$M(n) = (s-s_0)^{n+2} \cdot \left[\begin{aligned} & (\eta_{n+1} - \bar{r}_{n+1}) \left\{ \left(\frac{\partial M}{\partial r}\right)_1 + (n+1) \left(\frac{\partial M}{\partial r'}\right)_2 \right\} \\ & + (\theta_{n+1} - \bar{s}_{n+1}) \left\{ \left(\frac{\partial M}{\partial s}\right)_1 + (n+1) \left(\frac{\partial M}{\partial s'}\right)_2 \right\} \\ & + (\zeta_{n+1} - F_{n+1}) \left(\frac{\partial M}{\partial F}\right)_1 \end{aligned} \right]$$

$$+ \, o\left((s-s_0)^{n+3}\right),$$

and

$$N(n) = (n+1)(s-s_0)^n (\zeta_{n+1} - F_{n+1}) \left(\frac{\partial N}{\partial F'}\right)_0 + o\left((s-s_0)^{n+1}\right)$$

where, for example, $\left(\frac{\partial M}{\partial r}\right)_1$ is the coefficient of $(s-s_0)$ in the expansion of $\frac{\partial M}{\partial r}$ and is also the first non-zero coefficient.

Thus, neglecting all terms except the first in each of these three expansions, we arrive at the following iteration formulae for $\bar{R}'_{n+1}$, $\bar{S}'_{n+1}$, E'_{n+1} in terms of $L(n)$, $M(n)$, $N(n)$

$$E'_{n+1} = E'_n - \frac{N(n)}{\left(\frac{\partial N}{\partial F'}\right)_0} \qquad (4.15)$$

$$\bar{R}'_{n+1} = R'_n - P \cdot \left[\begin{aligned} & \left\{ \left(\frac{\partial M}{\partial s}\right)_1 + (n+1) \left(\frac{\partial M}{\partial s'}\right)_2 \right\} L(n) \\ & - (n+1) \left(\frac{\partial L}{\partial s'}\right)_0 M(n) (s-s_0)^{-2} \\ & + \frac{N(n) \left(\frac{\partial M}{\partial F}\right)_1 \left(\frac{\partial L}{\partial s'}\right)_0}{\left(\frac{\partial N}{\partial F'}\right)_0} \end{aligned} \right] \qquad (4.16)$$

$$\bar{\zeta}'_{n+1} = \zeta'_n - P \cdot \left[\begin{array}{l} (n+1)\left(\dfrac{\partial L}{\partial \bar{\zeta}'}\right)_0 M(n)(s-s_0)^{-2} \\ -\left\{\left(\dfrac{\partial M}{\partial \bar{\zeta}}\right)_1 + (n+1)\left(\dfrac{\partial M}{\partial \bar{\zeta}'}\right)_2\right\} L(n) \\ -\dfrac{\left(\dfrac{\partial N}{\partial F'}\right)_1 \left(\dfrac{\partial L}{\partial \bar{\zeta}'}\right)_1}{\left(\dfrac{\partial N}{\partial F'}\right)_0} N(n) \end{array} \right] \qquad (4.17)$$

where

$$P^{-1} = \left(\frac{\partial L}{\partial \bar{\zeta}'}\right)_0 \cdot \left\{\left(\frac{\partial M}{\partial \bar{\zeta}}\right)_1 + (n+1)\left(\frac{\partial M}{\partial \zeta'}\right)_2\right\}$$

$$- \left(\frac{\partial L}{\partial \zeta'}\right)_0 \cdot \left\{\left(\frac{\partial M}{\partial \bar{\zeta}}\right)_1 + (n+1)\left(\frac{\partial M}{\partial \bar{\zeta}'}\right)_2\right\} .$$

5. Results

The integration of 4.1 to evaluate the solution for $\tau = \tau(s)$, for any given value of α, was performed by means of a program, shown in outline in Fig. 5.1, written in Sirius Autocode. Having selected the values of γ, j, the solution can be obtained for any particular value of α and, by trial and error we can find the value of α for which d = 0, to the required accuracy. Having fixed γ, j, α one of the four solutions (corresponding to $s_{0\pm}$, $r_{1\pm}$) has to be selected. In practice it was found that one and only one of these four solutions could be made to satisfy the condition at the shock (s = -E) as well as at the l.n.c. ($s = s_0$).

The solutions obtained by Butler for the six cases γ = 6/5, 7/5, 5/3 with j = 1, 2 were calculated, in each case the same solution 'branch' (s_{0-}, r_{1-}) was found to be appropriate. The solution for the case γ = 3, corresponding to motion of products of a detonation, was sought. It was found that the branch selected above could not be made to satisfy d = 0. However a solution was found by integrating along the branch s_{0+}, r_{1-}. For this reason it was thought necessary to investigate the integral curves of the differential equation 4.1 with a view to examining the manner in which the changeover takes place and also the uniqueness and existence of the solution, particularly in the region of the changeover. The integral curves for the case γ = 1.4,

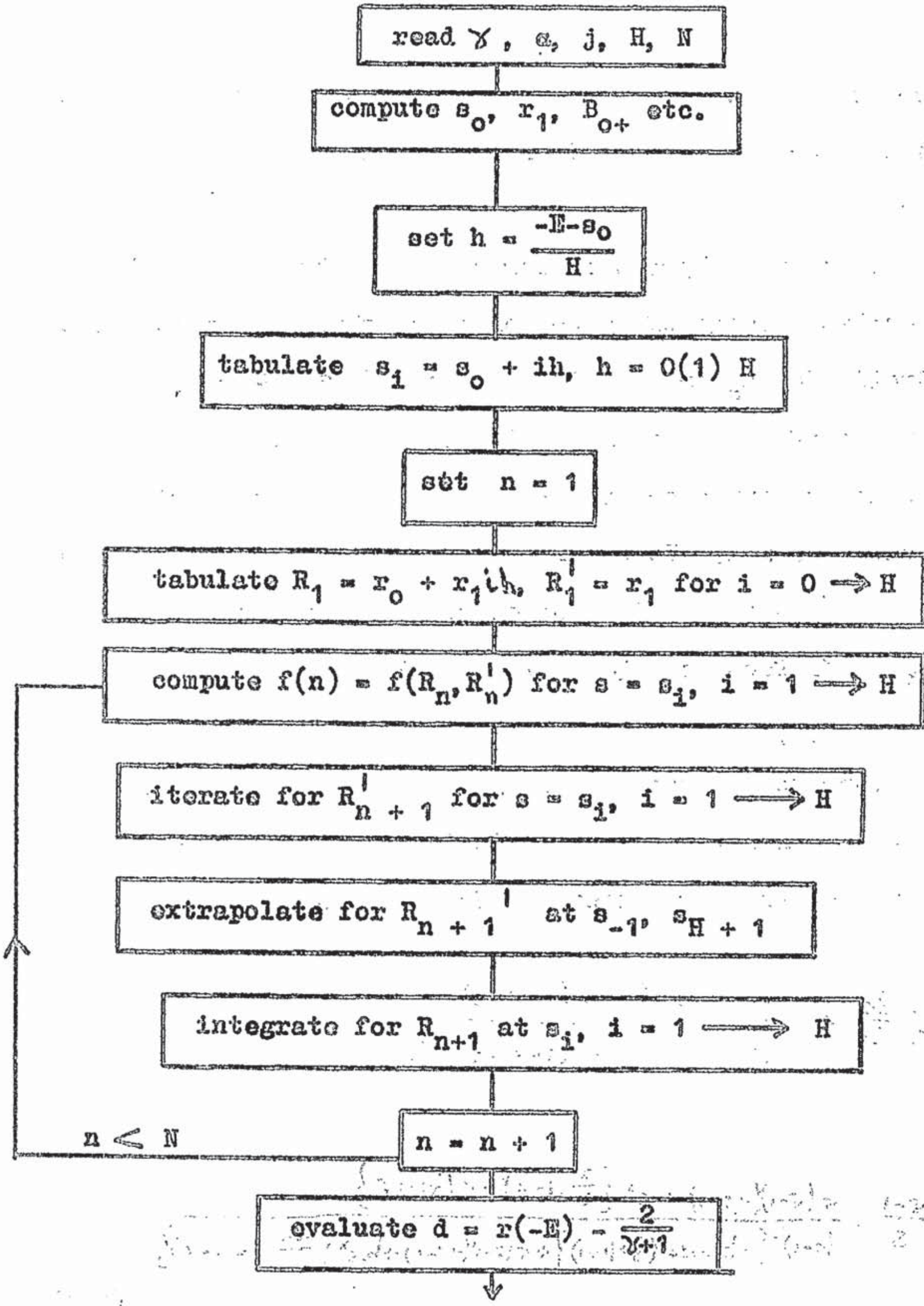

read γ, a, j, H, N
compute s_o, r_1, B_{o+} etc.
set $h = \frac{-E-s_o}{H}$
tabulate $s_i = s_o + ih$, $h = 0(1)\ H$
set $n = 1$
tabulate $R_1 = r_o + r_1 ih$, $R_1' = r_1$ for $i = 0 \Rightarrow H$
compute $f(n) = f(R_n, R_n')$ for $s = s_i$, $i = 1 \Rightarrow H$
iterate for R'_{n+1} for $s = s_i$, $i = 1 \Longrightarrow H$
extrapolate for R_{n+1}' at s_{-1}, s_{H+1}
integrate for R_{n+1} at s_i, $i = 1 \Longrightarrow H$
$n < N$
$n = n + 1$
evaluate $d = r(-E) - \frac{2}{\gamma+1}$

FIG. 5.1

$j = 2$ are given by Guderley. The two cases selected here, corresponding to distinct branches for the actual solution with the correct value of α, are $\gamma = \frac{5}{3}$, $j = 2$ and $\gamma = 3$, $j = 2$.

Let us note here that for a given value of γ there is a prohibited range of values of α for which the roots of the quadratic for s_0 are imaginary, $\alpha_1 < \alpha < \alpha_2$, say. The value of α_1 is never greater than about 0.05 for any of the cases examined and the range $0 < \alpha < \alpha_1$ never yields a correct solution. However, for $j = 2$, as γ increases from 1.2 to 5/3 the actual value of α, α_A, approaches the value of α_1. In fact for $\gamma = 5/3$, $j = 2$

$$\alpha_A = 0.688 \quad , \quad \alpha_1 = 0.687$$

It is thus to be expected that, if, for some value of γ, $\alpha_A = \alpha_1$, then this value of γ is in some sense critical.

The differential equation for $r(s)$ is

$$\frac{1}{k}\frac{dr}{ds} = \frac{(1-r+s)\,\beta_+ + (1-r-s)\,\beta_-}{(1-r+s)\,\beta_+ - (1-r-s)\,\beta_-}$$

$$= \frac{r-1}{s}\cdot\frac{r(r-1)(\alpha r-1) + s^2\left\{\frac{2}{\gamma}(1-\alpha) - \alpha(j+1)\,r\right\}}{(r-1)^2\left\{-k+\alpha r(j+k+1)\right\} + r(r-1)(\alpha r-1) + k s^2\left\{\frac{k\alpha}{\gamma} + \alpha(1-r)\right\}}$$

(5.1)

This equation has nine singular points in the s-r plane. There are three on the r-axis $P_4(0,0)$, $P_1(0,1)$, $(0,\frac{1}{2})$, and three in the region $s < 0$

$$P_2\ (s_{0+},\ 1+s_{0+})$$
$$P_3\ (s_{0-},\ 1+s_{0-})$$
$$P_5\ \left(S,\ \frac{k}{\alpha(j+k+1)}\right)$$

using Guderley's suffices. The remaining three points are the mirror images of P_2, P_3, P_5 in the s-axis and are in the region $s > 0$ (which corresponds to $\xi > 0$ and expanding waves and so can be ignored.). The shock point lies below $r = 1$ (the value of r here is $\frac{2}{\gamma+1}$) so that consideration of the integral curves can be restricted to the region $s < 0$, $0 < r < 1$. The quantities $s_{0\pm}$ are the two roots of the quadratic for s_0 and S is the negative solution for s of

$$s^2 = \frac{\sigma(\sigma-1)(\alpha\sigma-1)}{k\left\{\frac{k-\alpha}{\gamma}+\alpha(1-\sigma)\right\}}$$

where $$\sigma = \frac{k}{\alpha(j+k+1)}$$

In calculating the positions of the singularities of interest, i.e. P_1, P_2, P_3, P_4, P_5, it is observed that the point P_5 lies above the line $r = 1 + s$ for the case $\gamma = \frac{5}{3}$ and below it for the case $\gamma = 3$.

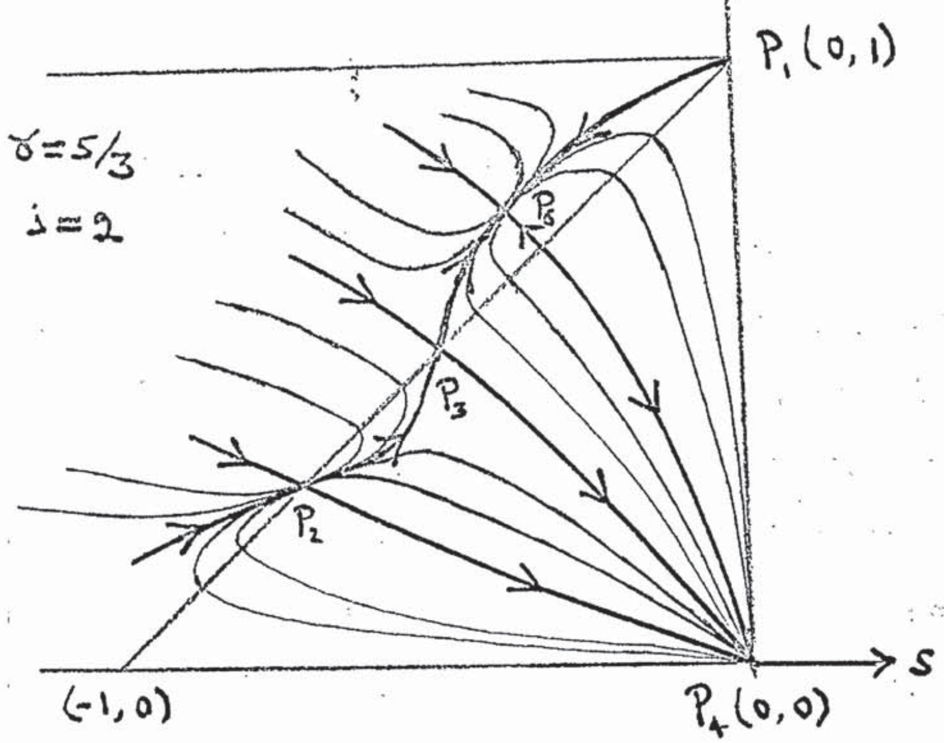


FIG. 5.2

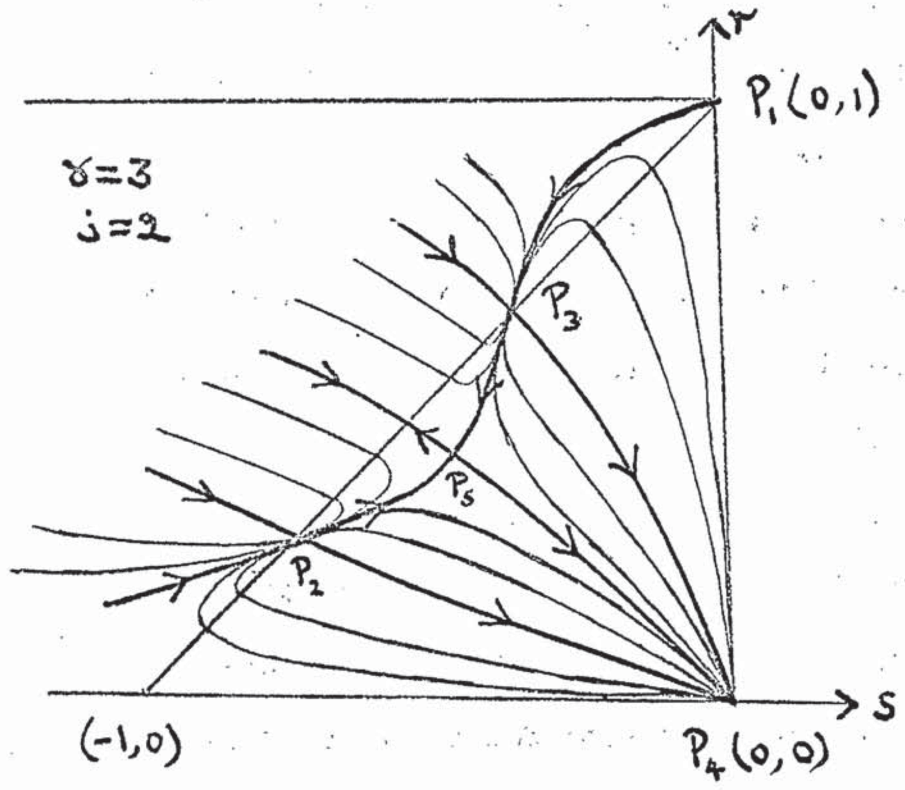


FIG. 5.3

Let us now consider the nature of these singular points. The points P_1, P_4 are respectively a saddle point and a degenerate node, irrespective of the values of γ, j, α.

For $\gamma = 5/3$, the points P_2, P_3 are respectively a node and a saddle point, and P_5 is a node. For these calculations the value of α taken was the one which was found to be the correct one (i.e. 0.688377). The integral curves can now be drawn in the region of interest and are sketched in Fig. 5.2 (not to scale as P_5 is in fact very close to the straight line $r = 1 + s$). The direction of the curves is that of time increasing. All curves, except the four limiting ones through P_2 and P_3, change direction on crossing the line $r = 1+s$. The shock point is $\left(\frac{-\sqrt{5}}{4}, \frac{3}{4}\right)$ and lies between the straight lines $r = 1 + s$, $r = 1$.

In order to solve the problem it is necessary to choose α such that one of the limiting integral curves, starting at P_2 or P_3, runs into the shock point, with time decreasing towards the shock. It is observed that the curve P_3 P_5 can be disregarded as time increases on it away from P_3.

The line $t = 0$ in the $R - t$ plane corresponds to the point $P_4(0, 0)$ in the s-r plane. It must be possible for the solution in the s-r plane to follow a singularity-free path from the shock point through P_2 or P_3, and then to P_4. Of the remaining three curves under consideration only two pass through P_4. There is no apparent difference in the nature of these two curves to suggest a preference of one to the exclusion of the other. In practice it was always found, for values of γ in this neighbourhood, that the curve through P_3 is the appropriate one.

The integral curves for the case $\gamma = 3$, $j = 2$ differ from the previous ones and are shown in Fig. 5.3. Here P_2 and P_3 are both nodes and P_5 is a saddle point, now below $r = 1 + s$. Two of the four curves under consideration run towards P_5 and so can be discounted. The remaining two are each the limiting curve through a node, not having the same slope at this point as all the remaining curves running towards the singularity. Again there is no obvious preference. The required curve is in fact the one through P_2. For no value of $\underset{\cdot}{a}$ was it found possible to make the one through P_3 pass through the shock point.

It is thus seen that investigation of the integral curves in the s - r plane does not in fact settle the issue of choice of solution or of the uniqueness and existence of the solution. It would seem that the question of choice of solution can only be settled numerically. As $\underset{\cdot}{a}$ is varied the positions of three singularities P_2, P_3, P_5 vary. It must be that the variations of their positions, as $\underset{\cdot}{a}$ is varied, are restricted in such a manner as only one of the possible curves can be made to pass through the shock point, for the solution to be unique for all cases. The discrepancy d was tabulated over a range of values

e	d (P_2 soln.)	d (P_3 soln.)	e	d (P_2 soln.)	d (P_3 soln.)
0.005	r_1 imag.	1.13	0.623	0.330	0.458
0.010	r_1 imag.	1.14	0.625	0.213	0.483
0.015	r_1 imag.	1.12	0.63	0.087	0.507
0.020	s_0 imag.	s_0 imag.	0.7	-0.280	r_1 imag.
0.6226	s_0 imag.	s_0 imag.	0.8	-0.412	r_1 imag.
0.6227	0.371	0.434	0.9	-0.473	r_1 imag.

Table 5.1

$\gamma = 1.865$			$\gamma = 1.875$		
e	$d(P_2)$	$d(P_3)$	e	$d(P_2)$	$d(P_3)$
0.674453	s_0 imag.	s_0 imag.	0.6738558	s_0 imag.	s_0 imag.
0.6744535	-0.00459	-0.00033	0.6738559	+0.00206	+0.0033
0.674454	-0.00557	+0.00066	0.6738560	+0.00132	+0.0039
0.674456	-0.00798	+0.00308	0.6738562	+0.00079	+0.0046
0.674460	-0.0109	+0.00606	0.67386	-0.00387	+0.0092
0.675	-0.0749	+0.0751	0.675	-0.1003	+0.118

Table 5.2

of $\dot{a}$ for $\gamma = 3$, $j = 2$ (in this case $\dot{a}_1 = 0.0185$, $\dot{a}_2 = 0.6226$) for the two integral curves, between which we must choose. The results given in table 5.1 show that only one solution is possible.

It can be seen from the two sets of integral curves that the changeover from one solution to the other takes place as follows. (We shall consider only the spherical case $j = 2$, and assume that the cylindrical case is similar). As γ is increased from 5/3 the roots of the quadratic for s_0 (using in each case the correct value of $\dot{a}$) become closer and closer together. For some critical value of γ, γ_c say, the roots are equal and P_2, P_3 coincide. In order for this transition to take place smoothly P_5 must also coincide with P_2, P_3 for $\gamma = \gamma_c$. As γ_c is approached, either from above or below, the region surrounding the inner singularity, which is a saddle in either case, must vanish. Thus for the critical case $\gamma = \gamma_c$ the three singularities P_2, P_3, P_5 merge into a single singularity, a node, so that a smooth transition takes place. For values of $\gamma < \gamma_c$ it would appear that the upper integral curve will provide a solution and a unique solution, and for $\gamma > \gamma_c$ the lower curve provides the required unique solution.

The value of γ_c was calculated by trial and error and found to be 1.87. The results are shown in table 5.2. For

γ	$\dot{a}$	$\dot{a}$ (Whitham)
1.2	0.757142	0.754021
1.4	0.717174	0.717288
5/3	0.688377	0.688654
3	0.636411	0.629542

Table 5.3(a) (spherical)

γ	$\dot{a}$	$\dot{a}$ (Whitham)
1.2	0.861163	0.859762
1.4	0.835323	0.835373
5/3	0.815625	0.816043
3	0.775667	0.772661

Table 5.3(b) (cylindrical)

For the case $\gamma = \gamma_c$ the roots of the quadratic for a_0 are equal and the correct value of α is α_1. In order to calculate γ_c it is necessary to evaluate α_1 for the particular estimate of γ_c and then do a brief tabulation of d for values of α slightly greater than α_1. Although the uniqueness of the solution in the region of the changeover cannot be proved by calculations of this sort, the results given in table 5.2. certainly suggest that the solution is always unique.

For the case $\gamma = 3$, $j = 1$ it was also found that the integral curve through P_2 was the appropriate one, so that there is some critical value of γ between 5/3 and 3, in the cylindrical case.

The values of α, in the eight selected cases, $\gamma = 6/5$, 7/5, 5/3, 3 with $j = 1, 2$, were calculated and are shown in tables 5.3. The results are in agreement with those obtained by Butler to the quoted accuracy of six decimal places, apart from the case $\gamma = 1.4$, $j = 2$, the difference in this case being in the fourth decimal place. The value given by Stanyukovich for $\gamma = 3$, $j = 1$ (i.e. 0.810) is in error by 0.034.

In the present work it was found that five subdivisions were sufficient for evaluating α to six decimal places. Approximately 8 iterations were required. The discrepancies of the tabulated solutions are all less than 10^{-5} in magnitude,

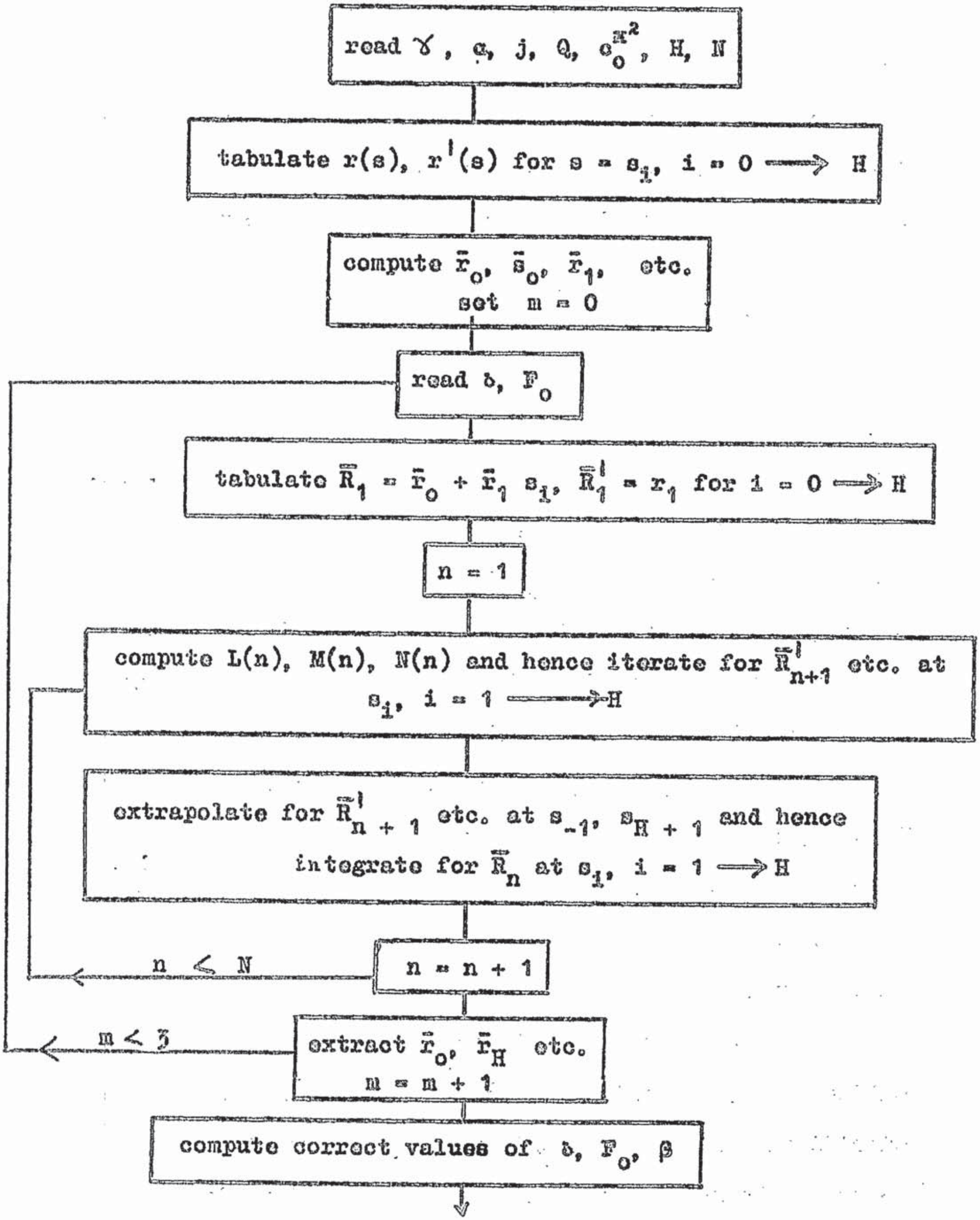

read γ, c, j, Q, $c_o^{x^2}$, H, N
tabulate $r(s)$, $r'(s)$ for $s = s_i$, $i = 0 \longrightarrow H$
compute $\bar{r}_o$, $\bar{s}_o$, $\bar{r}_1$, etc.
set $m = 0$
read δ, F_o
tabulate $\bar{\bar{R}}_1 = \bar{r}_o + \bar{r}_1 s_i$, $\bar{\bar{R}}_1' = r_1$ for $i = 0 \Longrightarrow H$
$n = 1$
compute $L(n)$, $M(n)$, $N(n)$ and hence iterate for $\bar{\bar{R}}_{n+1}'$ etc. at s_i, $i = 1 \longrightarrow H$
extrapolate for $\bar{\bar{R}}_{n+1}'$ etc. at s_{-1}, s_{H+1} and hence integrate for $\bar{\bar{R}}_n$ at s_i, $i = 1 \Longrightarrow H$
$n \leqslant N$
$n = n + 1$
$m < 3$
extract $\bar{r}_o$, $\bar{r}_H$ etc.
$m = m + 1$
compute correct values of δ, F_o, β

Fig. 5.4

and the values of β obtained by use of these data is taken to be accurate to four significant figures.

Having evaluated α and tabulated the solution for $r = r(s)$, let us consider the evaluation of β. To this end a program was written in Atlas Algol, the flow diagram being given in Fig. 5.4.

As mentioned previously, the linearity of the equations determining $\bar{r}$, $\bar{s}$, F means that β may be evaluated without recourse to trial and error, although boundary conditions have to be satisfied at each end (the front and the l.n.c.) of the range of integration. The procedure is to evaluate two independent solutions of the governing equations, corresponding to distinct choices of the combination δ, F_0. The correct values of β, δ, F_0 can now be calculated from these two solutions, as described in section 4, and the solution for $\bar{r}$, $\bar{s}$, F appropriate to the correct values of δ F_0, β may also be evaluated, if required. In all the calculations the selected combinations of δ, F_0 were (1, 1) and (1, -1), for which the two solutions were sufficiently independent.

The calculations in this section were performed with six subdivisions. In order to evaluate $\bar{r}_H$, $\bar{s}_H$, F_H correct to five significant figures, approximately 20-30 iterations were found to be necessary. As an example 24 iterations were used for the case $\gamma = 3$, $j = 2$, $Q = 1$, $c_0^{?} = 0$. In the evaluation of the solution with the correct values of δ, F_0, β (namely -1.5908, 10.763, 4.4760 respectively) 24 iterations were used. After 23 iterations the values at the end point, $s = -E$, of the range were

$$L = -1.85 \times 10^{-4},\ M = 5.76 \times 10^{-7},\ N = 1.89 \times 10^{-5},$$

$$\bar{r} = 0.8273897$$

$$\bar{s} = -3.6726337$$

$$F = 9.4182447$$

and after 24 iterations

$$L = -1.43 \times 10^{-4},\ M = 4.41 \times 10^{-7},\ N = 1.42 \times 10^{-5}$$

$$\bar{r} = 0.8273994$$

$$\bar{s} = -3.6726241$$

$$F = 9.4182420.$$

The errors in the derivatives are, of course, greater than those for the functions at any given stage, but the derivatives do not enter into the evaluation of β.

As well as checking that the solution converges we can verify that it does in fact converge to the values (at $s = -E$)

$$\bar{r} = A_1 \beta - K$$

$$\bar{s} = A_2 \beta - E'$$

$$F = A_3 \beta + H_0.$$

In the above case

$$A_1 \beta - K = 0.8274068$$

$$A_2 \beta - E' = -3.6726168$$

$$A_3 \beta + H_0 = 9.4182399.$$

γ	β(Whitham)	β	$\beta - \frac{\gamma+1}{2} K$	$\beta + \frac{E'}{E}$
1.2	-0.0482	-0.1009	-0.3209	0.8891
1.4	0.2158	0.2508	-0.2292	1.2108
5/3	0.5894	0.7737	-0.1152	1.6626
3	3.2679	4.4760	+0.4760	4.4760

Table 5.4(a) (spherical, $Q = 1$, $c_0^{x^2} = 0$)

γ	β(Whitham)	β	$\beta - \frac{\gamma+1}{2} K$	$\beta + \frac{E'}{E}$
1.2	-0.04816	-0.08199	-0.3020	0.9080
1.4	0.2158	0.2310	-0.2490	1.1910
5/3	0.5894	0.6692	-0.2995	1.4783
3	3.268	3.594	-0.4056	3.5944

Table 5.4(b) (cylindrical, $Q = 1$, $c_0^{x^2} = 0$)

γ	β(Whitham)	β	$\beta - \frac{\gamma+1}{2}K$	$\beta + \frac{E'}{E}$
1.2	-0.4730	-0.7047	-1.7047	4.2536
1.4	0.2172	0.3097	-0.6903	2.7382
5/3	0.4541	0.6942	-0.3058	2.0942
3	0.6667	1.0435	+0.0435	1.3769

Table 5.4(c) (spherical, $Q = 0$, $c_0^{N^2} = 1$)

γ	β(Whitham)	β	$\beta - \frac{\gamma+1}{2}K$	$\beta + \frac{E'}{E}$
1.2	-0.4730	-0.6211	-1.6212	4.3371
1.4	0.2172	0.2593	-0.7407	2.6879
5/3	0.4541	0.5587	-0.4413	1.9587
3	0.6667	0.7738	-0.2262	1.1071

Table 5.4(d) (cylindrical, $Q = 0$, $c_0^{N^2} = 1$)

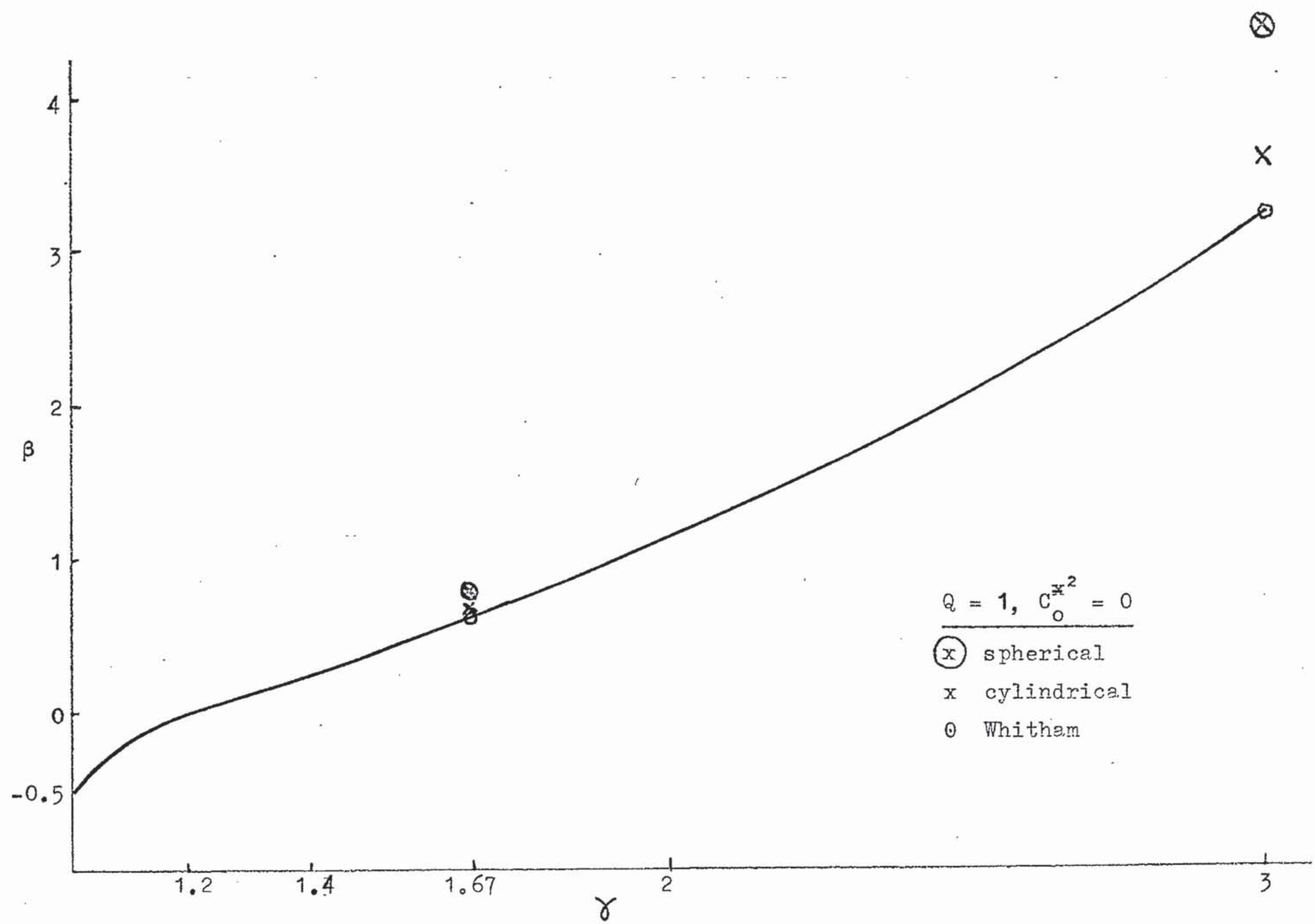
The present solution gives the correction terms to the Guderley solution due to the separate or combined effects of the heat release Q and allowing the sound speed c_0^* (neglected in the Guderley similarity solution) to be finite. However, since the quantities K, E', H_0 are linear combinations of Q, c_0^{*2}, it follows that the two effects combine linearly. Thus it is necessary to consider only two cases for given γ and j, e.g. the two solutions $Q = 1$, $c_0^{*2} = 0$ and $Q = 0$, $c_0^{*2} = 1$. For any given situation the actual solution can be evaluated by taking the appropriate linear combinations of these two solutions.

The value of β has been calculated for the 16 cases $\gamma = 6/5, 7/5, 5/3, 3$, with $j = 1, 2$, and $Q = 0$, $c_0^{*2} = 1$ and $Q = 1$, $c_0^{*2} = 0$. The results are given in tables 5.4, together with those for the simplified Whitham analysis. The fluid velocity and sound speed immediately behind the front are given by

$$u^* = \frac{-2}{\gamma+1} \lambda^{1-\frac{1}{\alpha}} \left\{ 1 + \left(\beta - \frac{\gamma+1}{2} K \right) \lambda^{-2+\frac{2}{\alpha}} \right\}$$

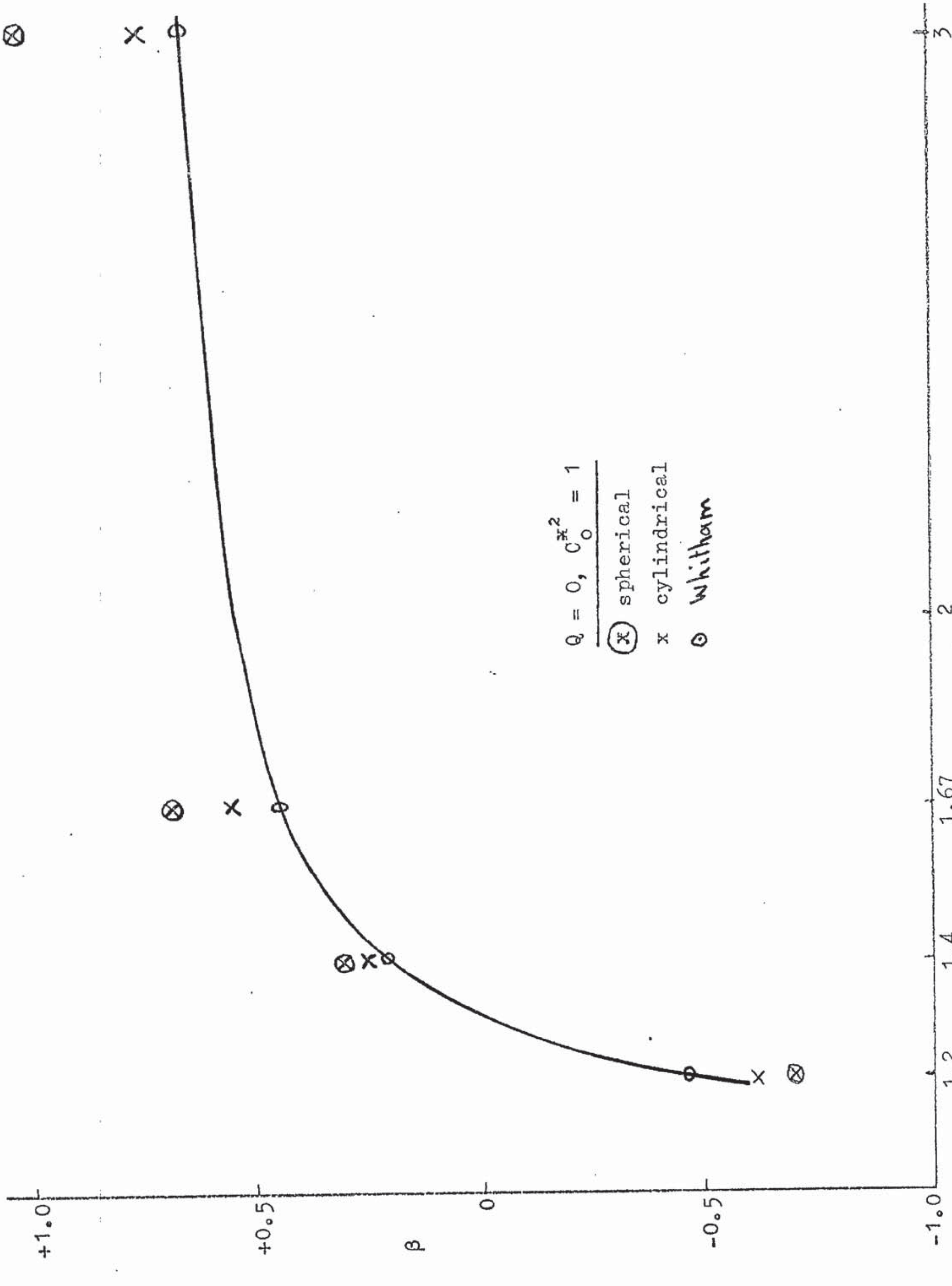
$$c^* = E \lambda^{1-\frac{1}{\alpha}} \left\{ 1 + \left(\beta + \frac{E'}{E} \right) \lambda^{-2+\frac{2}{\alpha}} \right\}$$

The coefficients $\beta - \frac{\gamma + 1}{2} K$, $\beta + \frac{E'}{E}$ are also tabulated.



4
3
2
1
0
-0.5
β
1.2
1.4
1.67
2
3
γ
Q = 1, C_o^{*2} = 0
x spherical
x cylindrical
θ Whitham

FIG. 5.5



+1.0
+0.5
0
-0.5
-1.0
β
1.2
1.4
1.67
2
3
γ
Q = 0, C*o² = 1
spherical
cylindrical
Whitham

FIG. 5.6

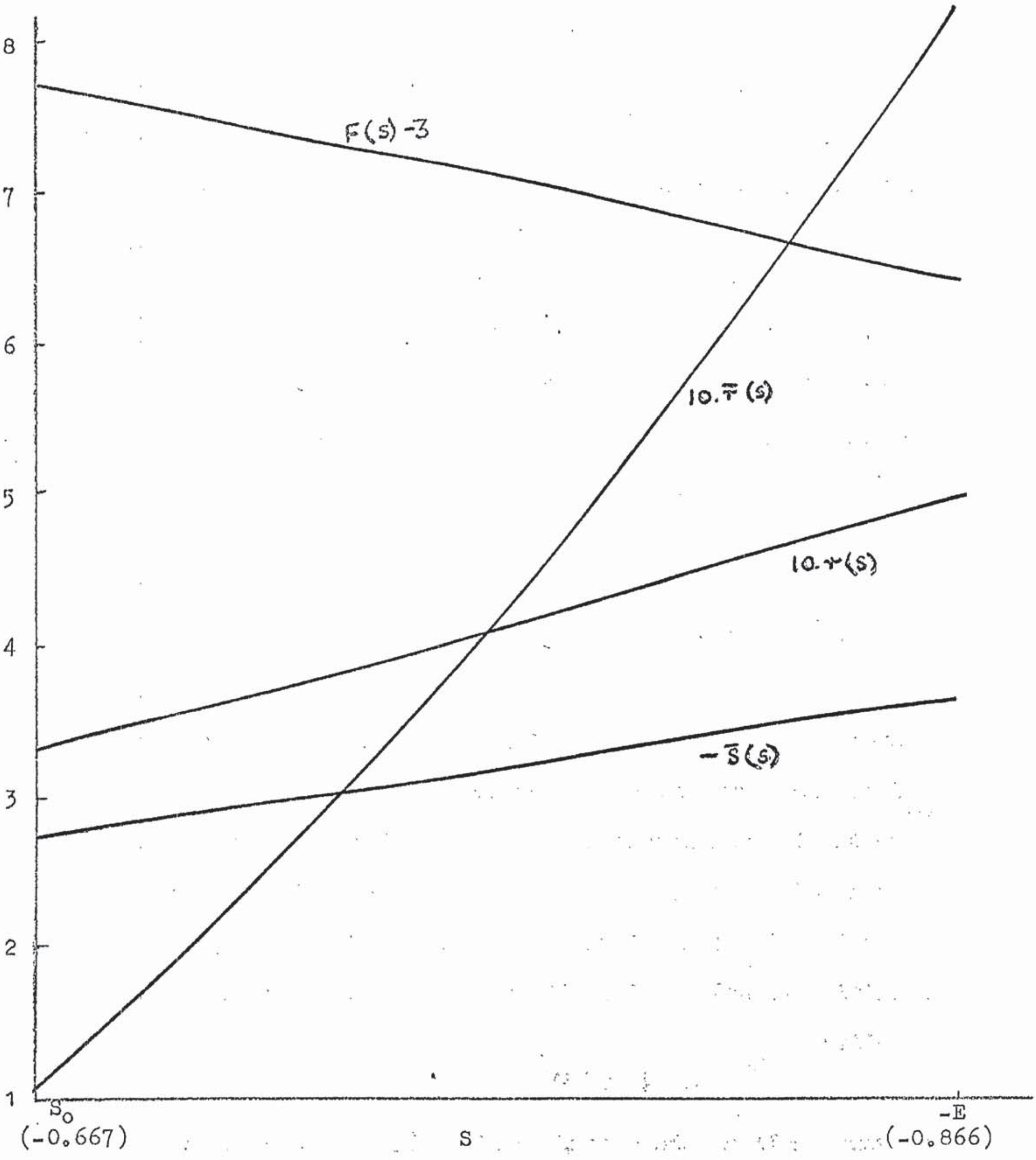


FIG. 5.7

A complete tabulation in the case of the simplified analysis can be found readily and a graph of β against γ is given in Figs. 5.5, 6, the discrete points being the full solution for comparison. In Fig. 5.5 the three separate values of β for γ = 1.2, 1.4 are indistinguishable on the scale of the graph.

The solutions for the functions $\bar{\bar{r}}$, $\bar{\bar{s}}$, F are not of particular interest and were calculated for the appropriate values of δ, F_0, β only in the case $\gamma = 3$, $j = 2$, $Q = 1$, $c_{\infty}^{*2} = 0$, to check that the method converged to the correct values of $\bar{\bar{r}}$, $\bar{\bar{s}}$, F. The graphs of r, $\bar{\bar{r}}$, $\bar{\bar{s}}$, F for this case are given in Fig. 5.7.

Discussion of Results

The results for α in tables 5.3, and the corresponding values obtained by the simplified analysis, have been given and compared by Whitham, apart from the case $\gamma = 3$. He investigates the reason for the remarkable accuracy of the simplified results. The shock is in fact not sufficiently close to being a negative characteristic for the appropriate condition to hold on it. The reason is, in fact, that the quantity

$$p^*_t + \rho^* c^* u^*_t$$

is very small at the shock, causing the characteristic condition to be nearly satisfied there.

For the additional case, $\gamma = 3$, studied here, the error of the approximate method is larger but is still not greater than about 1% . From the tables it is observed that the error in the approximate evaluation changes sign between $\gamma = 1.2$ and $\gamma = 1.4$ and does so again between $\gamma = \frac{5}{3}$ and $\gamma = 3$ (for both $j = 1$ and $j = 2$).

The simplified solution for β is much less accurate than that for a, as seen in tables 5.4. (Presumably the quantity $b^*_t + s^* c^* u^*_t$ is not especially small in this case.) The approximation that β is independent of j is not particularly good. In every case considered it is seen that the approximate value of β is numerically less than the cylindrical value which is, in turn, numerically less than the spherical value. In spite of the lack of accuracy of the approximate evaluation of β, the results obtained appear to follow very closely the behaviour of the correct values (for both $j = 1, 2$), as seen from the graphs 5.5, 6. The values of γ for which $\beta = 0$ were found, by the approximate method, to be $\gamma = 1.30$ for $Q = 0$, $c_0^{*2} = 1$ and $\gamma = 1.24$ for $Q = 1$, $c_0^{*2} = 0$. The fact that $\beta = 0$ for some particular value of γ means that for this value of γ the speed of the front is unaltered by taking into account either of the two effects (i.e. heat release and finite sound speed), although the flow behind is altered. For values of γ above this critical value the speed of the front is greater than that of the basic Guderley solution, and less than for values of γ less than the critical value.

The parameter determining the correction to the fluid velocity behind the front, $\beta - \frac{\gamma + 1}{2} K$, is seen to be negative in all cases except $\gamma = 3$, $j = 1$ or 2. The sound speed, on the other hand, is always greater than the value given by the Guderley solution since $\beta + {}^{E'}/E$ is always a positive quantity.

REFERENCES

6. Butler, D. S., unpublished Ministry of Supply report.

7. Chester, W., Phil.Mag., vol.45, pp 1293-1301, 1954.

8. Chisnell, R. F., J. Fluid Mechanics, vol.2, pp 286-298, 1957.

9. Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, pp 416-433, Interscience, New York, 1948.

10. Guderley, G., Luftfahrtforsch, vol.19, pp 302-312, 1942.

11. Hafele, W. von, Z. Natursforsch, vol.10A, pp 1006-1016, 1955.

12. Holt, M. and Schwartz, N., Phys. Fluids, vol.6, pp 521-525, 1963.

13. Hunter, C., J. Fluid Mechanics, vol.8, pp 241-263, 1960.

14. Hunter, C., J. Fluid Mechanics, vol.15, pp 289-305, 1963.

15. Jones, C. W., Proc. Roy. Soc. A, vol.228, pp 82-96, 1955.

16. Korobeinikov, V. P. and Riazanov, E. V., J. App. Math. and Mech., vol.23, pp 1066-1080, 1959.

17. Payne, R. B., J. Fluid Mechanics, vol.2, pp 185-200, 1957.

18. Perry, R. W. and Kantrowitz, K., J. App. Phys., vol.22, pp 878-886, 1951.

19. Sakurai, A., J. Phys. Soc. Japan, vol.8, pp 662-669, 1953.

20. Sakurai, A., J. Phys. Soc. Japan, vol.9, pp 256-266, 1954.

21. Sakurai, A., Comms. Pure App. Maths., vol.13, pp 353-370, 1960.

22. Sedov, L. I., Similarity and Dimensional Methods in Mechanics, (translation edited by M. Holt), Academic Press, New York, 1959.

23. Selberg, H. L., Det Kongelige Norske Videnskabers Selskabs Forhandlinger, vol.32, pp 134-138, 1959.

24. Stanyukovich, K. P., Unsteady Motion of Continuous Media (translation edited by M. Holt), Pergamon, 1960.

25. Taylor, G. I., Proc. Roy. Soc. A, vol.200, pp 235-247, 1950.

26. Taylor, G. I., Proc. Roy. Soc. A., vol.186, pp 273-292, 1946.

27. Taylor, G. I., Proc. Roy. Soc. A, vol.201, pp 159-174, 1950.

28. Taylor, J. L., Phil. Mag., vol.46, pp 374-375, 1955.

29. Whitham, G. B., J. Fluid Mechanics, vol.4, pp 337-360, 1958.

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