Singular Solutions and Symmetries of Ordinary Differential Equations

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ABSTRACT

The main objective of this thesis is to study singular solutions in the light of symmetries of ordinary differential equations. The practice of finding and using symmetries for integration and the use of differential forms in symmetry methods are introduced. Differential equations admitting singular solutions are introduced with emphasis on the Clairaut equation and its generalizations. These equations are investigated from the symmetry point of view with special attention to the role of singular solutions in symmetry reduction methods.

DECLARATION

I, Verena Guschelbauer, hereby certify that this dissertation has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

Signed: Verena Guschelbauer
Date: 18/06/07

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1. INTRODUCTION

The purpose of this document is to study singular solutions in the light of symmetries of ordinary differential equations. Symmetries are a key to solving differential equations. There are various methods for obtaining exact solutions of differential equations of a standard type. Most of these methods can be applied because of the underlying symmetry property of the differential equation and are in fact special cases of a few powerful symmetry methods. The merit of symmetry methods is that they can be applied to differential equations that are of unfamiliar type. Symmetries of a given differential equation are to be found and can then be used to construct exact solutions.

An introduction to the practice of finding and using symmetries for integration of differential equations is given by Stephani [21]. In this style the present document gives an introduction to point symmetries with an emphasis on applications. Symmetries are introduced by explaining the use of the infinitesimal generator of point transformations and giving a basic definition of point symmetries. Furthermore it is explained how symmetries of first, second and higher order differential equations can be found. Finally it is illustrated how symmetries can be utilized in integration strategies.

Another effective way of investigating differential equations and their symmetries as mentioned by Olver [18] is given by using the language of differential forms. For the study of differential geometry, topology and differential equations differential forms are a powerful tool [18]. An introduction to differential forms is given from basic definitions of differential forms as explained by Olver [18] and Flanders [11]. This is followed by explanations of the Lie derivative and closed and exact forms which are the key to symmetries as discussed by Olver [18].

With emphasis on illustrative applications and examples in the style of Harrison [13] it is explained how the language of differential forms can be used in the theory of symmetries. Simple examples, e.g. $y'' = 0$ have been used, since these are geometrically equivalent to more complicated examples. Some important concepts of symmetry methods can be explained with the aid of these simple equations.

The first integrals of $y'' = 0$ are $y - px$ and p with $y' = p$. Any relation between these integrals is given by $F(p, y - px) = 0$ for which a Clairaut equation, an equation admitting singular solutions, generally given by $y = px + f(p)$ is an example. Considering higher order differential equations $y^{(n)} = 0$ then the relation of their first integrals leads to more general Clairaut-type equations. Motivated by these findings the strategy for further research is to use the symmetries of

$y^{(n)} = 0$ to determine the symmetries of differential equations with singular solutions, in particular the Clairaut equation and its generalizations. This will give information on the role of singular solutions in symmetry methods.
This is followed by an introduction to differential equations with singular solutions. Special emphasis is on the known example of a Clairaut equation and its generalizations, given for example in Goursat [12]. It is explained how the solution of the generalized Clairaut equation can be obtained by the same method as Clairaut's equation [22].
This leads to an investigation of the Clairaut equation and generalized Clairaut equations from the symmetry point of view. Symmetries of the general and singular solutions of these equations are investigated. To obtain them the findings related to the symmetries of equations of type $y^{(n)} = 0$ have been used, e.g. symmetries for the Clairaut equation can be determined using the symmetries of $y'' = 0$. It is noted that symmetries of the general solution of a Clairaut equation are not equal to symmetries of the singular solution. This can be generalized for higher order equations $y^{(n)} = 0$ in relation to generalized Clairaut equations.
Furthermore, it is noted that the symmetries obtained lead to reductions from $y^{(n)} = 0$ to generalized Clairaut equations. This is followed by an investigation of how symmetries lead to a reduction of a given system of differential forms. It is studied in which way the reduced system inherits symmetries of the original system. This is investigated for systems of differential forms which are equivalent to second and third order differential equations. Finally the findings enable to investigate singular solutions in regard of symmetry methods for reduction.

2. INTRODUCTION TO POINT SYMMETRIES

This chapter gives an introduction to point symmetries by explaining the infinitesimal generator of point transformations and giving a basic definition of point symmetries. Furthermore it is explained how symmetries of first, second and higher order differential equations can be found. Finally it is illustrated how symmetries can be utilized in integration strategies.

2.1 Generators of point transformations

We start with the definition of a one-parameter group of point transformations, then consider its infinitesimal generator. This is followed by a consideration of a multiple-parameter group of point transformations and its generators. We define the normal form of an infinitesimal generator and explain the extension to derivatives of variables.

2.1.1 One-parameter groups of point transformations

Let $I \ni 0$ be an open subset of $\mathbb{R}$ and $\varepsilon \in I$ be an arbitrary parameter. A point transformation is defined as a map from points (x, y) into $(\tilde{x}, \tilde{y})$ as

$$\begin{aligned} \Phi_\varepsilon : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (\tilde{x}, \tilde{y}) \end{aligned}$$

where $\tilde{x} = \tilde{x}(x, y, \varepsilon)$, $\tilde{y} = \tilde{y}(x, y, \varepsilon)$. A set of point transformations form a local one-parameter group of point transformations G, if the group properties are satisfied, i.e.

- Closure property: For any elements Φ_{ε_1} and Φ_{ε_2} of G composition

  $$\Phi_{\varepsilon_1} \cdot \Phi_{\varepsilon_2} = \Phi_{\varepsilon_1 + \varepsilon_2}$$

  is an element of G.

- Associative property: For any elements Φ_{ε_1}, Φ_{ε_2}, Φ_{ε_3} of G it is true that

  $$\Phi_{\varepsilon_1} \cdot (\Phi_{\varepsilon_2} \cdot \Phi_{\varepsilon_3}) = (\Phi_{\varepsilon_1} \cdot \Phi_{\varepsilon_2}) \cdot \Phi_{\varepsilon_3}.$$

- Identity element: There exists an identity element, which without loss of generality corresponds to $\varepsilon = 0$, such that for any element Φ_{ε_1} of G

$$\Phi_0 \cdot \Phi_{\varepsilon_1} = \Phi_{\varepsilon_1} \cdot \Phi_0 = \Phi_{\varepsilon_1}.$$

- Inverse element: For any element Φ_{ε_1} of G there exists an inverse element $\Phi_{-\varepsilon_1}$ in G such that

$$\Phi_{\varepsilon_1} \cdot \Phi_{-\varepsilon_1} = \Phi_0$$

2.1.2 Infinitesimal Generator

The action of the one-parameter group of transformations can be interpreted as motion in the (x, y) plane. An initial point (x_0, y_0) generates a curve as the parameter ε varies. For each considered point we obtain a curve which represents the action of the group which is shown in figure 2.1. The action

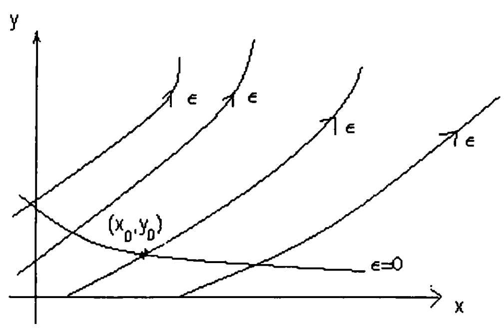

Fig. 2.1: Action of the group represented by curves.

of the transformation group is represented by a set of curves, each of which is characterized by its tangent vectors as shown in figure 2.2. Considering the field of tangent vectors X the transformation group which takes a point (x, y) to $(\tilde{x}, \tilde{y})$ can then be expressed by an infinitesimal transformation [21], [19]

$$\begin{aligned} \tilde{x} &= x + \xi(x, y)\varepsilon + \ldots \\ \tilde{y} &= y + \eta(x, y)\varepsilon + \ldots . \end{aligned}$$

The infinitesimals $\xi(x, y)$ and $\eta(x, y)$ are defined by

$$\begin{aligned} \xi(x, y) &= \frac{\partial \tilde{x}}{\partial \varepsilon}|_{\varepsilon=0} \\ \eta(x, y) &= \frac{\partial \tilde{y}}{\partial \varepsilon}|_{\varepsilon=0}. \end{aligned}$$

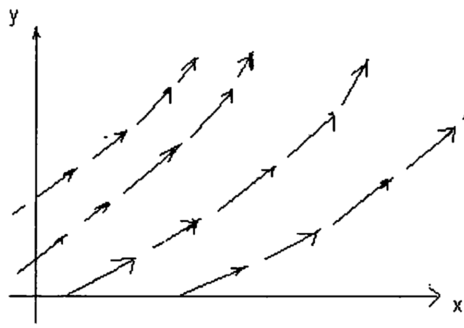

Fig. 2.2: Action of the group characterized by tangent vectors.

With these functions an infinitesimal generator can be given as

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

This infinitesimal generator determines the infinitesimal transformation and also the one-parameter group of transformations [21], [4], [5].

2.1.3 Generator of multiple-parameter groups of point transformations

Point transformations that depend on more than one parameter can be written as

$$\Phi_{\varepsilon_n} = (\tilde{x}(x,y,\varepsilon_n), \tilde{y}(x,y,\varepsilon_n))$$

where $n = 1, ..., N$. For each n there exists a generator X_n. These generators can be linearly combined to obtain

$$X = \sum_{n=1}^{N} a_n X_n$$

where the a_n are constant coefficients.

Example: The general projective transformations on the (x,y) plane can be represented by a 3×3 matrix

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or

$$\tilde{x} = \frac{ax + by + c}{gx + hy + i}$$
$$\tilde{y} = \frac{dx + ey + f}{gx + hy + i}$$

where

$$\tilde{x} = \frac{\tilde{u}}{\tilde{w}}, \quad \tilde{y} = \frac{\tilde{v}}{\tilde{w}}, \quad x = \frac{u}{w}, \quad x = \frac{u}{w}.$$

The infinitesimal generators of the projective transformations in $\mathbb{R}^2$ are given by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x} \\
X_2 &= \frac{\partial}{\partial y} \\
X_3 &= x\frac{\partial}{\partial x} \\
X_4 &= y\frac{\partial}{\partial x} \\
X_5 &= x\frac{\partial}{\partial y} \\
X_6 &= y\frac{\partial}{\partial y} \\
X_7 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \\
X_8 &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.
\end{aligned}$$

As stated above these generators can be combined to obtain an eight-parameter generator

$$X = (a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2)\frac{\partial}{\partial x} + (a_6 + a_7 x + a_8 y + a_5 xy + a_4 y^2)\frac{\partial}{\partial y}$$

where a_1, a_2, ..., a_8 are constants.

2.1.4 *Normal form of a generator*

For any infinitesimal generator there exist coordinates (s, t), called canonical coordinates, such that it can be transformed to

$$X = \frac{\partial}{\partial s}$$

which is called the normal form. In these coordinates the generator acts like a translation. The visualization in the (x, y) plane shows that the curves which represent the action of the transformation group are straight lines in coordinates (s, t) as sketched in figure 2.3. The transformation can be given as

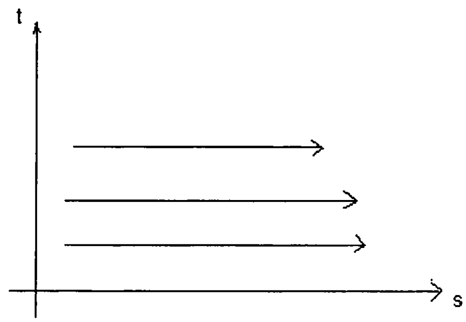

Fig. 2.3: Action of the transformation group in coordinates (s, t).

$$\begin{aligned}\tilde{t} &= t \\ \tilde{s} &= s + \varepsilon.\end{aligned}$$

To find these coordinates we start with the generator given in the form

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

The new coordinates s and t should satisfy

$$\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} = (Xs)\frac{\partial}{\partial s} + (Xt)\frac{\partial}{\partial t} = \frac{\partial}{\partial s}.$$

This equation leads to a system of differential equations to determine s and t

$$\begin{aligned}\xi\frac{\partial s}{\partial x} + \eta\frac{\partial s}{\partial y} &= 1 \\ \xi\frac{\partial t}{\partial x} + \eta\frac{\partial t}{\partial y} &= 0.\end{aligned}$$

Example: Given the generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The required coordinates (s, t) have to satisfy above stated conditions which lead to the system of differential equations

$$x\frac{\partial s}{\partial x} + y\frac{\partial s}{\partial y} = 1 \tag{2.1}$$

$$x\frac{\partial t}{\partial x} + y\frac{\partial t}{\partial y} = 0. \tag{2.2}$$

From (2.2) we get that t can be written as a function of y/x such as $\varphi(y/x)$, and (2.1) leads to $s = \frac{1}{2}\ln(xy) + \psi(y/x)$.

2.1.5 Extension of generators

Before applying to a differential equation a point transformation has to be extended to its derivatives which is done by

$$\begin{aligned}\tilde{y}' &= \frac{d\tilde{y}}{d\tilde{x}} = \frac{y'(\partial\tilde{y}/\partial y) + (\partial\tilde{y}/\partial x)}{y'(\partial\tilde{x}/\partial y) + (\partial\tilde{x}/\partial x)} = \tilde{y}'(x, y, y', \varepsilon)\\ \tilde{y}'' &= \frac{d\tilde{y}'}{d\tilde{x}} = \tilde{y}''(x, y, y', y'', \varepsilon) \;\; \ldots .\end{aligned}$$

The extension or prolongation of the point transformation to the derivatives leads to the extension of the infinitesimal generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

up to the nth derivative

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'} + \ldots + \eta^{(n)}\frac{\partial}{\partial y^{(n)}}.$$

To obtain the η', ..., $\eta^{(n)}$ we consider the previous representation of the action of the one-parameter group of transformations.

$$\begin{aligned}\tilde{x} &= x + \xi(x, y)\varepsilon + \ldots\\ \tilde{y} &= y + \eta(x, y)\varepsilon + \ldots .\end{aligned}$$

We can extend to the derivatives

$$\begin{aligned}\tilde{y}' &= y' + \eta'(x, y, y')\varepsilon + \ldots\\ &\ldots\\ \tilde{y}^{(n)} &= y^{(n)} + \eta^{(n)}(x, y, y', \ldots, y^{(n)})\varepsilon + \ldots .\end{aligned}$$

The η, η', ..., $\eta^{(n)}$ are defined by

$$\begin{aligned} \eta' &= \frac{\partial \tilde{y}'}{\partial \varepsilon}|_{\varepsilon=0} \\ \dots \\ \eta^{(n)} &= \frac{\partial \tilde{y}^{(n)}}{\partial \varepsilon}|_{\varepsilon=0}. \end{aligned}$$

Therefore we can write

$$\begin{aligned} \frac{d\tilde{y}}{d\tilde{x}} &= \frac{dy + \varepsilon d\eta + \dots}{dx + \varepsilon d\xi + \dots} = y' + \varepsilon(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}) + \dots \\ \frac{d\tilde{y}^{(n-1)}}{d\tilde{x}} &= \frac{dy^{(n-1)} + \varepsilon d\eta^{(n-1)} + \dots}{dx + \varepsilon d\xi + \dots} = y^{(n)} + \varepsilon(\frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}) + \dots . \end{aligned}$$

The η, η', ..., $\eta^{(n)}$ are obtained as

$$\begin{aligned} \eta' &= \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \\ \eta^{(n)} &= \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}. \end{aligned}$$

The expression for $\eta^{(n)}$ can be rewritten as

$$\begin{aligned} \eta^{(n)} &= \frac{d^n\eta}{dx^n} - y'\frac{d^n\xi}{dx^n} \\ &= \frac{d^n\eta}{dx^n} - \frac{d^n y'}{dx^n}\xi - y'\frac{d^n\xi}{dx^n} + y^{(n+1)}\xi \end{aligned}$$

and the formula for the extension can be written as

$$\eta^{(n)} = \frac{d^n}{dx^n}(\eta - y'\xi) + y^{(n+1)}\xi$$

which is independent of $y^{(n+1)}$.

2.2 Symmetry definition

A given point transformation

$$\Phi_\varepsilon(x, y) = (\tilde{x}(x, y, \varepsilon), \tilde{y}(x, y, \varepsilon))$$

is a symmetry transformation of an ordinary differential equation

$$H(x, y, y', \dots, y^{(n)}) = 0$$

if the form of the differential equation does not change, i.e. whenever $y = f(x)$ is a solution, so is $\tilde{y} = f(\tilde{x})$ where $(\tilde{x}, \tilde{y}) = \Phi_\varepsilon(x, y)$. The infinitesimal generator X forms an algebra of symmetries for the ordinary differential equation $H(x, y, y', ..., y^{(n)}) = 0$ if both

$$XH(x, y, y', ..., y^{(n)}) = 0$$

and

$$H(x, y, y', ..., y^{(n)}) = 0$$

hold. If the highest derivative of the differential equation can be isolated such that

$$y^{(n)} = \omega(x, y, y', ..., y^{(n-1)})$$

then $H(x, y, y', ..., y^{(n)}) = 0$ can also be written in form of a linear differential operator

$$Af = (\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial x} + ... + \omega \frac{\partial}{\partial y^{(n-1)}})f = 0$$

where $f(x, y, y', y'', ..., y^{(n-1)}) = \text{const.}$ is an integral. The equations

$$Af = 0$$

and

$$y^{(n)} = \omega$$

are equivalent. A is called the characteristic vector field of the equation. The generator X is a symmetry if

$$Af = 0 \Rightarrow A(Xf) = 0$$

holds. Since $XAf = 0$ we can write

$$[X, A]f = (XA - AX)f = 0\,.$$

Since $[X, A]f = 0$ and $Af = 0$ have the same solutions, they may differ only by a factor of the form $\lambda = \lambda(x, y, y', ..., y^{(n-1)})$ Therefore the symmetry definition can be written as

$$[X, A] = \lambda A.$$

By using again the condition $XH = 0$ and taking $H = y^{(n)} - \omega$ we get another useful formulation of the symmetry definition. Since the extended generator has the form

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} + ... + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}$$

the equation $X\omega = Xy^{(n)}$ leads to the symmetry definition

$$X\omega = \eta^{(n)}.$$

2.3 Finding point symmetries

This section gives an introduction to how point symmetries of a given differential equation can be found. This is explained and illustrated by examples more specifically for first and second order differential equations. A more general outline is given for finding point symmetries of higher order differential equations.

2.3.1 Lie point symmetries of First Order Differential Equations

A first order differential equation given by

$$y' = \omega(x, y)$$

has a symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

if the symmetry condition $X\omega = \eta^{(n)}$ as mentioned in the previous section is satisfied, i.e.

$$X\omega = (\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y})\omega = \eta'$$

must hold. This condition can be written in terms of ξ and η as

$$\xi\omega_{,x} + \xi_{,x}\omega + \xi_{,y}\omega^2 = \eta_{,x} + \eta_{,y}\omega - \eta\omega_{,y}$$

[21], [9]. In fact the number of solutions for the above equation is infinite, however, there is no systematic method to find them. Symmetries of simple structure can often be found by inspecting a given differential equation.

Example: Given the first order differential equation

$$y' = xy\,.$$

The symmetry condition is here

$$\xi y + \xi_{,x}xy + \xi_{,y}x^2y^2 = \eta_{,x} + \eta_{,y}xy - \eta x.$$

This equation is satisfied for example for $\xi = 1/x$ and $\eta = 0$ which correspond to the symmetry generator

$$X = \frac{1}{x}\frac{\partial}{\partial x}.$$

2.3.2 Lie point symmetries of Second Order Differential Equations

A second order differential equation given by

$$y'' = \omega(x, y, y')$$

has a symmetry

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta'(x,y,y')\frac{\partial}{\partial y'}$$

provided that the following condition is satisfied

$$X\omega = (\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'})\omega = \eta''.$$

Written in terms of ξ and η this condition becomes

$$\omega(\eta_{,y} - 2\eta_{,x} - 3y'\eta_{,y}) - \omega_{,x}\xi - \omega_{,y}\eta - \omega_{,y'}[\eta_{,x} + y'(\eta_{,y} - \xi_{,x}) - y'^2\xi_{,y}] +$$
$$\eta_{,xx} + y'(2\eta_{,xy} - \xi_{,xx}) + y'^2(\eta_{,yy} - 2\xi_{,xy}) - y'^3\xi_{,yy} = 0$$

[21], [10].

Example: Given the second order differential equation

$$y'' = 0$$

then the condition for symmetries is

$$\eta_{,xx} + y'(2\eta_{,xy} - \xi_{,xx}) + y'^2(\eta_{,yy} - 2\xi_{,xy}) - y'^3\xi_{,yy} = 0$$

from which follows that $\xi_{,yy} = 0$ and $\eta_{,xx} = 0$. Hence ξ and η have the form

$$\xi = \alpha(x)y + \beta(x)$$
$$\eta = \gamma(y)x + \delta(y).$$

Equating to zero the coefficients of y'^2 and y' gives $\eta_{,yy} - 2\xi_{,xy} = 0$ and $\xi_{,xx} - 2\eta_{,xy} = 0$ which leads to

$$\gamma''(y)x + \delta''(y) - 2\alpha'(x) = 0$$
$$\alpha''(x)y + \beta''(x) - 2\gamma'(y) = 0$$

from which we obtain $\gamma''(y) = 0$ and $\alpha''(x) = 0$. Therefore we get

$$\begin{aligned}\alpha &= c_1x + c_2\\ \beta &= c_3x^2 + c_5x + c_6\\ \gamma &= c_3y + c_4\\ \delta &= c_1y^2 + c_7y + c_8\end{aligned}$$

resulting in the eight-parameter generator of the projective transformations in $\mathbb{R}^2$ as mentioned in a previous section

$$X = (a_1 + a_2x + a_3y + a_4xy + a_5x^2)\frac{\partial}{\partial x} + (a_6 + a_7x + a_8y + a_5xy + a_4y^2)\frac{\partial}{\partial y}.$$

2.3.3 Lie point symmetries of Higher Order Differential Equations

To find symmetries of higher order differential equations one proceeds similarly to the case of first or second order differential equations. It can be stated that an nth order linear differential equation has at least an n-parameter group of Lie point symmetries. However with increasing n the method described for second-order differential equations becomes more and more complicated. Therefore it can be easier to try to obtain simple structured symmetries by inspecting the given differential equation. Symmetries most often consist of the forms $\partial/\partial x$, $x\partial/\partial x$, $\partial/\partial y$, $y\partial/\partial y$ and $\alpha x\partial/\partial x + \beta y\partial/\partial y$.

2.4 Integration strategies

Symmetries of a differential equation can be utilized for finding its solutions. This section explains how symmetries can be used to solve first order differential equations. Moreover it is explained how symmetries can be used to achieve a reduction of a differential equation.

2.4.1 Solving First Order Differential Equations

A first order differential equation is given by

$$y' = \omega(x, y)$$

or

$$Af = (\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y})f = 0.$$

Its symmetry generator is

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

The solution can be given using a line integral

$$\varphi(x, y) = \int \frac{dy - \omega dx}{\eta - \xi\omega} = \varphi_0 = \text{constant}$$

since

$$A\varphi = -\frac{\omega}{\eta - \xi\omega} + \omega\frac{1}{\eta - \xi\omega} = 0.$$

Example: Given the differential equation

$$y' = -\frac{1}{x + e^{-y}}$$

and the symmetry generator

$$X = -x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Using the statement above, the solution can be obtained by

$$\begin{aligned}\varphi(x, y) &= \int \frac{dy - (-\frac{1}{x+e^{-y}})dx}{1 - (-x)(-\frac{1}{x+e^{-y}})} \\ &= \int e^y dx + (xe^y + 1)dy \\ &= xe^y + y\end{aligned}$$

[21], [15].

2.4.2 Reduction of Higher Order Differential Equations

An nth order differential equation given by

$$y^{(n)} = \omega(x, y, y', ..., y^{(n-1)})$$

or

$$Af = (\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + ... + \omega\frac{\partial}{\partial y^{(n-1)}})f = 0$$

can be reduced to a differential equation of order $n - 1$ by using a symmetry. Suppose a symmetry generator is given in normal form

$$X = \frac{\partial}{\partial s}.$$

The differential equation can then be transformed to

$$s^{(n)} = \tilde{\omega}(t, s, s', ..., s^{(n-1)})$$

and the symmetry condition now reads

$$X\tilde{\omega} = \frac{\partial}{\partial s}\tilde{\omega} = 0.$$

This shows however that $\tilde{\omega}$ does not depend on s and is therefore

$$s^{(n)} = \tilde{\omega}(t, s', ..., s^{(n-1)})$$

a differential equation of order $n-1$ for s' [21], [1]. This procedure may be repeated whenever we have a symmetry of the reduced equation, finally resulting in a first order differential equation, though there is no guarantee that any reduced equation will have such a symmetry.

Example: Given the differential equation

$$y'' = x^3 y^2$$

and the symmetry generator

$$X = x\frac{\partial}{\partial x} - 5y\frac{\partial}{\partial y}.$$

The generator can be transformed to its normal form using $t = yx^5$ and $s = \ln x$ which brings the differential equation to the form

$$s'' = -t^2 s'^3 + 30ts'^3 - 11s'^2$$

which is a first order differential equation

$$u' = -v^2u^3 + 30vu^3 - 11u^2$$

for $v = t$ and $u = s'$ [21].

2.4.3 Integration Strategies for Second Order Differential Equations

As shown in the previous section a first order differential equation with a symmetry can be solved by using a line integral. This section explains integration strategies for a second order differential equation given by

$$y'' = \omega(x, y, y')$$

or

$$Af = (\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial y'})f(x, y, y') = 0.$$

If it has a two-parameter group of transformations, then the two generators can be used to solve it in different ways. Two integration strategies for second order differential equations are illustrated as follows.

First integration strategy

One integration strategy is to transform the given generators of a second order differential equation to a simple form. This procedure has several cases.
Given the differential equation and its symmetry generators

$$X_1 \quad = \quad \xi_1\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial y}$$

$$X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$$

and

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}.$$

There are four cases to be distinguished:

- $[X_1, X_2] = 0$ and $\Delta \neq 0$
  The generators can be transformed to $X_1 = \frac{\partial}{\partial s}$ and $X_2 = \frac{\partial}{\partial t}$ and the differential equation becomes $s'' = \tilde{\omega}(s')$.
- $[X_1, X_2] = 0$ and $\Delta = 0$
  The generators become $X_1 = \frac{\partial}{\partial s}$ and $X_2 = t\frac{\partial}{\partial s}$ and the differential equation becomes $s'' = \tilde{\omega}(t)$.
- $[X_1, X_2] = X_1$ and $\Delta \neq 0$
  The generators can be transformed to $X_1 = \frac{\partial}{\partial s}$ and $X_2 = t\frac{\partial}{\partial t} + t\frac{\partial}{\partial s}$ where the differential equation will be $s'' = \frac{\tilde{\omega}(s')}{t}$.
- $[X_1, X_2] = X_1$ and $\Delta = 0$
  The generators become $X_1 = \frac{\partial}{\partial s}$ and $X_2 = s\frac{\partial}{\partial s}$ with $s'' = s'\tilde{\omega}(t)$.

Example: Given the second order differential equation

$$y'' = \frac{3}{2}\frac{y'^2}{y}$$

with the symmetries

$$X_1 = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'}$$
$$X_2 = \frac{\partial}{\partial x}.$$

The generators X_1 and X_2 commute and Δ does not vanish

$$[X_1, X_2] = 0, \qquad \Delta = -y.$$

This is the first case which is described above from which follows that there are coordinates $s(x, y)$ and $t(x, y)$ for which the generators take the normal form and the differential equation can be written as $s'' = \tilde{\omega}(s')$. It is known that these coordinates must satisfy

$$X_1 t = 0 \quad X_2 t = 1$$

and

$$X_1 s = 1 \quad X_2 s = 0 \ .$$

Therefore s and t can be obtained as

$$t = \int \frac{-\eta_1 dx + \xi_1 dy}{\Delta} = x$$

$$s = \int \frac{\eta_2 dx - \xi_2 dy}{\Delta} = \ln y.$$

This transforms the given equation into

$$s'' = \frac{1}{2} s'^2$$

which is of the form $s'' = \tilde{\omega}(s')$ as stated above.

Second integration strategy

A second integration strategy is given by transforming the given generators into normal forms in the space of first integrals. If a second order differential equation is given as

$$Af = (\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \omega(x, y, y') \frac{\partial}{\partial y'}) f = 0$$

then a first integral is given by

$$\varphi(x, y, y') = \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ \xi_1 & \eta_1 & \eta_1' \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} 1 & y' & \omega \\ \xi_1 & \eta_1 & \eta_1' \\ \xi_2 & \eta_2 & \eta_2' \end{vmatrix} \neq 0$$

since

$$A\varphi = 0$$

is satisfied. Using φ as a new variable instead of y' will lead to the differential equation $A\psi = (\frac{\partial}{\partial x} + y'(x, y, \varphi)\frac{\partial}{\partial y})$ and the solution ψ can be obtained by

$$\psi(x, y, \varphi) = \int \frac{dy - y'(x, y, \varphi) dx}{\eta_1 - \xi_1 y'}.$$

This solution is given in terms of x,y and φ and can be set to a constant to get $y = y(x, \varphi_0, \psi_0)$. If the symmetry generators satisfy $[X_1, X_2] = 0$ then another first integral would be given by

$$\psi(x,y,y') = \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ \xi_2 & \eta_2 & \eta_2' \end{vmatrix}}{\Delta}.$$

Example: Given the differential equation

$$y'' = \frac{3}{2}\frac{y'^2}{y}.$$

We know that the symmetries $X_1 = y\frac{\partial}{\partial y} + \frac{\partial}{\partial y'}$ and $X_2 = \frac{\partial}{\partial x}$ commute. For the second integration strategy we need Δ as described above

$$\Delta = \begin{vmatrix} 1 & y' & \frac{3}{2}\frac{y'^2}{y} \\ 0 & y & y' \\ 1 & 0 & 0 \end{vmatrix} = -\frac{1}{2}y'^2.$$

Then we get the solution φ from

$$\begin{aligned} \varphi &= \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & y' & \frac{3}{2}\frac{y'^2}{y} \\ 0 & y & y' \end{vmatrix}}{\Delta} \\ &= \int (1dx + 2\frac{1}{y'} - 2\frac{y}{y'^2}dy') \\ &= x + 4\frac{y}{y'} \end{aligned}$$

and ψ from

$$\begin{aligned} \psi(x,y,y') &= \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & y' & \frac{3}{2}\frac{y'^2}{y} \\ 1 & 0 & 0 \end{vmatrix}}{\Delta} \\ &= \int (-3\frac{1}{y}dy + 2\frac{1}{y'}dy') \\ &= -3\ln y + 2\ln y'. \end{aligned}$$

Taking φ and ψ as constants leads to the solution

$$y = \frac{a}{(x+b)^2}.$$

2.4.4 Integration Strategy for Second Order Differential equations with more than two Point Symmetries

A second order differential equation can have up to eight Lie point symmetries and if there is a two-parameter group of transformations the solution can be found via line integrals as shown in the previous section. A differential equation may admit a group of symmetries that does not contain a a two-parameter group of transformations. This is the case if the Lie group has the commutators $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$. Then another strategy should be used. Since there is a linear relation

$$\mu_1 X_1 + \mu_2 X_2 + \mu_3 X_3 + \nu A = 0$$

the equation can be written as

$$X_1 = \varphi(x, y, y')X_2 + \psi(x, y, y')X_3 + \nu(x, y, y')A$$

where φ and ψ are first integrals which satisfy $A\varphi = 0$ and $A\psi = 0$.

2.5 Summary

In this chapter we have given the definition and properties of point symmetries. Methods how to find symmetries of a given differential equation have been explained and illustrated with examples. Finally it has been shown how symmetries can be used in integration strategies. The use of symmetries for the integration of differential equations has the benefit that they make it possible to have a systematic approach. Therefore, symmetries can also be used in computerized integration procedures.

3. INTRODUCTION TO DIFFERENTIAL FORMS

Another way of investigating differential equations and their symmetries is given by using the language of forms. This chapter gives an introduction to differential forms from basic definitions. The exterior derivative and closed and exact forms are explained. Although the theory can be developed quite generally we will restrict definitions and examples to low values of the degree of the form, because of the applications we have in mind.

3.1 Definition of Differential Forms

This section gives a definition of differential forms by first defining alternating p-forms. Contraction and wedge-product are explained, leading to the definition of differential forms.

3.1.1 Alternating p-forms

Let V be an n-dimensional vector space over $\mathbb{R}$. A map

$$\eta : V^p \mapsto \mathbb{R}$$

is then called alternating p-form if it has the following properties [8], [11]:

- η is p-linear, that is linear in each argument
- η is alternating, that is for $1 \leq i < j \leq p$ it satisfies $\eta(v_1, ..., v_i, ..., v_j, ..., v_p) = -\eta(v_1, ..., v_{i-1}, v_j, v_{i+1}, ..., v_{j-1}, v_i, v_{j+1}, ..., v_p)$.

A 1-form $\eta : V \mapsto \mathbb{R}$ satisfies the conditions for a general linear map. For all u and v of V and all a of $\mathbb{R}$

- $\eta(u + v) = \eta(u) + \eta(v)$
- $\eta(a\, v) = a\, \eta(v)$.

The set of all linear maps $\eta : V \mapsto \mathbb{R}$ is called the dual vector space to the vector space V and denoted V^*.

3.1.2 Contraction of a differential form

The contraction of a differential form is an operator denoted by $\rfloor$. If η is a p-form and $v \in V$, then $v\rfloor\eta$ is a $(p-1)$-form defined by

$$(v\rfloor\eta)(v_1, v_2, ..., v_{(p-1)}) = \eta(v, v_1, v_2, ..., v_p).$$

Examples:

- η is a one-form: $v\rfloor\eta = \eta(v)$
- η is a two-form: $(v\rfloor\eta)(v_1) = \eta(v, v_1)$
- η is a three-form: $(v\rfloor\eta)(v_1, v_2) = \eta(v, v_1, v_2)$

3.1.3 The wedge-product $\wedge$ of alternating p-forms

Let V be a real vector space and let η be a p-form and ω a q-form. Then the wedge product of η and ω defines an alternating $(p+q)$-form.

Examples:

- for $p = 0$: let $\eta = c$, then
$$\eta \wedge \omega = c\omega$$
- for $p = 1$ and $q = 1$:
$$(\eta \wedge \omega)(v_1, v_2) = \eta(v_1)\omega(v_2) - \eta(v_2)\omega(v_1)$$
- for $p = 2$ and $q = 1$:
$$(\eta \wedge \omega)(v_1, v_2, v_3) = \eta(v_1)\omega(v_2, v_3) - \eta(v_2)\omega(v_1, v_3) + \eta(v_3)\omega(v_1, v_2)$$
- for $p = 2$ and $q = 2$:
$$\begin{aligned}(\eta \wedge \omega)(v_1, v_2, v_3, v_4) \quad = \quad & \eta(v_1, v_2)\omega(v_3, v_4) + \eta(v_3, v_4)\omega(v_1, v_2) - \eta(v_1, v_3)\omega(v_2, v_4) \\ & -\eta(v_2, v_4)\omega(v_1, v_3) + \eta(v_1, v_4)\omega(v_2, v_3) + \eta(v_2, v_3)\omega(v_1, v_4)\end{aligned}$$

Let $e_1, ..., e_n$ be a basis of the n-dimensional real vector space V. Then a basis for the dual vector space V^* is given by $\delta^1, ..., \delta^n$ where $\delta^i(e_j) = \delta_{ij}$.
The alternating p-forms $\delta^{i_1} \wedge ... \wedge \delta^{i_p}$ with $1 \leq i_1 < ... < i_p \leq n$ are a basis of the space of p-forms on V and each alternating p-form can be written as

$$\eta = \sum_{1 \leq i_1 < ... < i_p \leq n} a_{i_1 ... i_p} \delta^{i_1} \wedge ... \wedge \delta^{i_p}, \qquad a_{i_1 ... i_p} \epsilon \mathbb{R}$$

with $a_{i_1 ... i_p} = \eta(e_{i_1}, ..., e_{i_p})$.

3.1.4 Differential forms on an n-dimensional manifold

A differential form of degree p or p-form η on the n-dimensional manifold M is given by the map

$$\eta : M \to \Gamma(\bigcup_{x \in M} (T_x^* M))$$

see 18], [14]. Each $x \in M$ has an alternating p-form

$$\eta_x : (T_x M)^p \to \mathbb{R}.$$

Let P be a point in R^n. Then a differential form at P is an expression

$$\sum a_H dx^{h_1} \wedge dx^{h_2} \wedge ... \wedge dx^{h_p}$$

where the a_H are constants and H is the multi index $h_1, ..., h_p$.
Let U be an open subset of R^n. A p-form on U is obtained by choosing a p-form at each point of U. Hence a p-form θ has the representation

$$\theta = \sum a_H(x_1, ..., x_n) dx^{h_1} \wedge dx^{h_2} \wedge ... \wedge dx^{h_p}$$

where $a_H(x_1, ..., x_n)$ are smooth functions on U.

The wedge-product on differential forms

Let M be an n-dimensional manifold. Then the wedge product of differential forms on M is the map

$$\wedge : \bigwedge^{p} \times \bigwedge^{q} \to \bigwedge^{p+q}.$$

It is defined for each point $x \in M$ by

$$(\eta \wedge \omega)_x = \eta_x \wedge \omega_x.$$

For each real vector space V the direct sum

$$\bigwedge = \bigoplus_{p=0}^{\infty} \bigwedge^{p}$$

is a graded algebra under the wedge product $\wedge$. The wedge product is

- associative: $(\eta \wedge \omega) \wedge \xi = \eta \wedge (\omega \wedge \xi)$
- anti commutative: $\eta \wedge \omega = (-1)^{p \cdot q} \omega \wedge \eta$ for η from $\bigwedge^q$ and ω from $\bigwedge^p$.

Examples:

A differential 1-form on $\mathbb{R}^3$ is an expression

$$\alpha = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz.$$

A differential 2-form on $\mathbb{R}^3$ is an expression

$$\beta = F(x,y,z)dy \wedge dz + G(x,y,z)dz \wedge dx + H(x,y,z)dx \wedge dy.$$

Their wedge-product is the 3-form

$$\alpha \wedge \beta = (AF + BG + CH)dx \wedge dy \wedge dz.$$

3.2 Exterior Derivative

This section gives the definition of the exterior derivative, illustrated by examples.

3.2.1 Definition

The exterior derivative is a map

$$d : \bigwedge^{p} \to \bigwedge^{p+1}$$

[20]. It takes a p-form Θ to a $(p+1)$-form $d\Theta$. If θ_1 and θ_2 are p-form and q-form respectively

$$\begin{aligned} d(\theta_1 + \theta_2) &= d\theta_1 + d\theta_2 \\ d(\theta_1 \wedge \theta_2) &= d\theta_1 \wedge \theta_2 + (-1)^p \theta_1 \wedge d\theta_2. \end{aligned}$$

Examples:

The exterior derivative of a function $f(x,y,z)$ in $\mathbb{R}^3$ results in a 1-form and is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Given a 1-form $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$, then its derivative is a 2-form given by

$$d(Fdx+Gdy+Hdz) = (G_x - F_y)dx \wedge dy + (H_y - G_z)dy \wedge dz + (F_z - H_x)dz \wedge dx.$$

The exterior operator d applied to a 2-form results in a 3-form which is an expression $f(x,y,z)dx \wedge dy \wedge dz$, generated by

$$d(Fdy \wedge dz + Gdz \wedge dx + Hdx \wedge dy) = (F_x + G_y + H_z)dx \wedge dy \wedge dz.$$

For each p-form θ it is true that

$$d(d\theta) = 0.$$

For example if a 0-form is given by a real valued function $f(x, y, z)$ this means $d(df) = 0$, which can be shown by

$$d(df) = (f_{yx} - f_{xy})dx \wedge dy + (f_{zy} - f_{yz})dy \wedge dz + (f_{xz} - f_{zx})dz \wedge dx = 0.$$

Considering the example of a 1-form $\theta = Fdx + Gdy + Hdz$ then $d(d\theta) = 0$, because

$$d(d\theta) = [G_{xz} - F_{yz} + H_{yx} - G_{zx} + F_{zy} - H_{xy}]dx \wedge dy \wedge dz = 0.$$

3.2.2 Closed and exact forms

A closed form is a differential p-form α satisfying

$$d\alpha = 0.$$

An exact form is a differential form $(p+1)$-form α satisfying

$$\alpha = d\beta$$

for some p-form β. Hence, each exact form is closed, because

$$d\alpha = d(d\beta) = 0.$$

However, closed does not imply exact.

3.3 Differential Equations in the language of forms

A differential equation can be formulated by a set of differential forms, proceeding as follows. The derivatives of the dependent variables are to be defined as new variables. If the given differential equation is linear in the highest derivative, then it can be given as a set of quasi-linear first order differential equations from which relations between the differentials can be obtained. These relations will lead to the set of differential forms, representing the original differential equation as a differential ideal $I \lhd \bigwedge$ with $dI \subset I$.

Example: Consider an ordinary differential equation $y''' = s(x, y, y', y'')$. Then an equivalent set of first order equations is

$$\frac{dy}{dx} = p, \ \frac{dp}{dx} = \omega, \ \frac{d\omega}{dx} = s(x, y, p, \omega).$$

From the relations $dy = p\,dx$, $dp = \omega\,dx$ and $d\omega = s\,dx$ the equivalent system of 1-forms can be obtained as

$$\begin{aligned} \theta_1 &= dy - p\,dx \\ \theta_2 &= dp - \omega\,dx \end{aligned}$$

$$\theta_3 \quad = \quad d\omega - s\,dx.$$

For the differential equation $A = \partial_x + p\partial_y + \omega\partial_p + s\partial_\omega$ the above forms then satisfy

$$A\rfloor\theta_1 = A\rfloor\theta_2 = A\rfloor\theta_3 = 0.$$

This system is closed, because

$$\begin{aligned} d\theta_1 &= -dp \wedge dx = dx \wedge \theta_2 \\ d\theta_2 &= -d\omega \wedge dx = dx \wedge \theta_3 \\ d\theta_3 &= -ds \wedge dx = s_y dx \wedge \theta_1 + s_p dx \wedge \theta_2 + s_\omega dx \wedge \theta_3. \end{aligned}$$

3.4 Summary

In this chapter we have given an introduction to differential forms from basic definitions. The basic concepts of wedge product, exterior derivative and closed and exact forms have been explained. Differential forms are a powerful tool for the study of differential equations. Their utilization is illustrated in the following chapters.

4. DIFFERENTIAL FORMS AND SYMMETRIES

This chapter is an introduction to how the language of differential forms can be used in the theory of symmetries. The Lie Derivative will be defined. It will be explained how forms can be used to find symmetries. Furthermore there will be detailed illustrations regarding concrete systems of forms. Simple equations have been chosen because Clairaut and generalized Clairaut equations will be treated as reductions of these simple systems.

4.1 Definition and Properties of Lie Derivative

The Lie derivative of an object σ describes the infinitesimal change along a path, determined by a vector field V. It is denoted as $\boldsymbol{L}_V(\sigma)$.

Lie derivative of vector fields: If V and W are vector fields on M then the Lie derivative of W with respect to V is in coordinate representation

$$L_V(w^i \frac{\partial}{\partial x^i}) = (v^j w,_j^i - w^j v,_j^i) \frac{\partial}{\partial x^i}$$

where $(\cdot),_j \equiv \frac{\partial}{\partial x^j}(\cdot)$. Hence the Lie derivative of a vector is again a vector and is denoted by the Lie bracket of V and W

$$L_V(W) = [V, W].$$

Lie derivative of functions: The Lie derivative of a function or 0-form f is then the directional derivative with respect to V

$$L_V f = V \rfloor df = v^i f,_i \, .$$

Lie derivative of differential forms: Let σ be a differential form and V be a vector field on M, then

$$L_V(\sigma) = V \rfloor (d\sigma) + d(V \rfloor \sigma).$$

The Lie derivative of a p-form σ is again a p-form.

Further properties:

$$L_V(a\sigma + b\omega) \quad = \quad aL_V(\sigma) + bL_V(\omega)$$

$$\begin{aligned} L_V(d\sigma) &= dL_V(\sigma) \\ L_V(\sigma \wedge \omega) &= (L_V\sigma) \wedge \omega + \sigma \wedge (L_V\omega) \\ L_V(W\rfloor\sigma) &= [V,W]\rfloor\sigma + W\rfloor(L_V\sigma) \end{aligned}$$

where a and b are constants.

4.2 Using differential forms to find symmetries

Consider an ordinary differential equation $F(x, y, y', ..., y^{(n)}) = 0$. It can be formulated using the language of differential forms. Then the equation is represented by a set of forms,

$$\begin{aligned} \theta_1 &= dy - pdx \\ \theta_2 &= dp - \omega dx \\ \dots & \\ \theta_{n+1} &= dF. \end{aligned}$$

To obtain the symmetries of the given set of forms a vector field V has to be found that determines a flow under which the ideal of forms θ_i is invariant. This is the case when

- the Lie derivative of θ_i vanishes, $L_V\theta_i = 0$, or
- the Lie derivative is linear in the forms θ_i, $i = 1, ..., n$, that is

$$L_V\theta_i = \sum_j \lambda_i^j \theta_j$$

  where the λ_i^j are general functions.

Then the vector field V represents the direction of an infinitesimal symmetry transformation.

4.3 Symmetries of First Order Differential Equations

4.3.1 Symmetries of $F(x, y, p) = 0$

A general first order differential equation is given by $F(x, y, p)$ where $p = y'$. It is equivalent to the system

$$\begin{aligned} \theta_1 &= dy - pdx \\ \theta_2 &= dF. \end{aligned}$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p}$$

where X, Y and P are functions of x, y and p, if V satisfies the symmetry conditions

$$\begin{aligned} L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 \\ L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2. \end{aligned}$$

4.3.2 *Symmetries of* $p = f(x, y)$

A first order differential equation where the highest derivative can be isolated is given by $F(x, y, p) = p - f(x, y) = 0$ with $p = y'$. It is equivalent to the system

$$\theta_1 = dy - f dx.$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}$$

where X and Y are functions of x and y, if V satisfies the symmetry condition

$$L_V\theta_1 = \lambda_1\theta_1$$

from which we get

$$\begin{aligned} L_V\theta_1 = V\rfloor(dx \wedge df) + d(Y - fX) &= \lambda_1\theta_1 \\ (-f_x X - f_y Y - fX_x + Y_x)dx + (Y_y - fX_y)dy &= -f\lambda_1 dx + \lambda_1 \end{aligned}$$

which determines $\lambda_1 = Y_y - fX_y$. Therefore we get

$$Xf_x + Yf_y - Y_x - fY_y + f(X_x + fX_y) = 0.$$

4.3.3 *Symmetries of* $y' = 0$

The first order differential equation given by $y' = 0$ is equivalent to the system

$$\theta_1 = dy.$$

The symmetry condition above is then

$$0 = Y_x.$$

The general solution of $y' = 0$ is known to be $y = I$, where I is a constant. A general symmetry is given by

$$V = \alpha(x, I)\frac{\partial}{\partial x} + \sigma_1(I)\frac{\partial}{\partial I}.$$

Note that $\alpha\frac{\partial}{\partial x}\rfloor\theta_1 = 0$ is true for all α so $\alpha(x,y)\frac{\partial}{\partial x}$ is the characteristic symmetry which can be neglected. The symmetry condition is then

$$L_V\theta_1 = \lambda_1\theta_1$$

which gives

$$d\sigma_1 = \lambda_1 dI$$

leading to the conditions

$$\begin{aligned}\sigma_{1,I} &= \lambda_1\\ \sigma_{1,x} &= 0\end{aligned}$$

which are satisfied for any function $\sigma_1(I)$. This determines the general symmetry as

$$V = \sigma_1\frac{\partial}{\partial y}$$

where σ_1 is an arbitrary function of y.

4.4 Symmetries of Second Order Differential Equations

4.4.1 Symmetries of $F(x,y,p,\omega) = 0$

A general second order differential equation is given by $F(x,y,p,\omega)$ where $p = y'$, $\omega = y''$. It is equivalent to the system

$$\begin{aligned}\theta_1 &= dy - pdx\\ \theta_2 &= dp - \omega dx\\ \theta_3 &= dF.\end{aligned}$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p} + \Omega\frac{\partial}{\partial \omega}$$

where X, Y, P and Ω are functions of x, y, p and ω, if V satisfies the symmetry conditions

$$\begin{aligned}L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 + \lambda_{13}\theta_3\\ L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2 + \lambda_{23}\theta_3\\ L_V\theta_3 &= \lambda_{31}\theta_1 + \lambda_{32}\theta_2 + \lambda_{33}\theta_3.\end{aligned}$$

4.4.2 Symmetries of $\omega = f(x,y,p)$

A second order differential equation where the highest derivative can be isolated is given by $F(x,y,p,\omega) = \omega - f(x,y,p) = 0$ with $p = y'$, $\omega = y''$. It is equivalent to the system

$$\begin{aligned} \theta_1 &= dy - p dx \\ \theta_2 &= dp - f dx. \end{aligned}$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p}$$

where X, Y and P are functions of x, y and p, if V satisfies the symmetry conditions

$$\begin{aligned} L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 \\ L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2 \end{aligned}$$

from which we get

$$\begin{aligned} L_V\theta_1 &= V\rfloor(dx \wedge dp) + d(Y - pX) = \lambda_{11}\theta_1 + \lambda_{12}\theta_2 \\ &= (-P + Y_x - pX_x)dx + (Y_y - pX_y)dy + (Y_p - pX_p)dp + (Y_\omega - pX_\omega)d\omega \\ &= (-p\lambda_{11} - f\lambda_{12})dx + (\lambda_{11})dy + (\lambda_{12})dp \end{aligned}$$

which determines P as

$$P = Y_x + p(Y_y - pX_y - X_x) + f(Y_p - pX_p).$$

We also get

$$\begin{aligned} L_V\theta_2 &= V\rfloor(dx \wedge df) + d(P - fX) = \lambda_{21}\theta_1 + \lambda_{22}\theta_2 \\ &= (-f_xX - f_yY - f_pP - fX_x + P_x)dx + (P_y - fX_y)dy + (P_p - fX_p)dp \\ &= (-p\lambda_{21} - f\lambda_{22})dx + (\lambda_{21})dy + (\lambda_{22})dp \end{aligned}$$

which determines $\lambda_{21} = P_y - fX_y$ and $\lambda_{22} = P_p - fX_p$. Therefore we get

$$Xf_x + Yf_y + Pf_p - P_x - pP_y - fP_p + f(X_x + pX_y + fX_p) = 0.$$

4.4.3 Symmetries of $y'' = 0$

The second order differential equation given by $y'' = 0$ is equivalent to the system of

$$\begin{aligned} \theta_1 &= dy - p\,dx \\ \theta_2 &= dp. \end{aligned}$$

The symmetry conditions above are then

$$P = Y_x + p(Y_y - pX_y - X_x)$$

$$0 = P_x + pP_y.$$

Suppose that X and Y depend only on x and y, then the system breaks into components

$$\begin{aligned} 0 &= Y_{xx} \\ 0 &= -X_{xx} + 2Y_{xy} \\ 0 &= -2X_{yx} + Y_{yy} \\ 0 &= X_{yy} \end{aligned}$$

with the solution

$$\begin{aligned} X &= a_1xy + a_2y + a_3x^2 + a_4x + a_5 \\ Y &= a_3xy + a_6x + a_1y^2 + a_7y + a_8 \end{aligned}$$

where a_1, ..., a_8 are constants. This is the eight-parameter group of point symmetries of $y'' = 0$. If we want to get not only point symmetries, a different approach is needed. A general solution of $y'' = 0$ is known to be $y = I_1x + I_2$, where I_1 and I_2 are constants, and the space of solutions is therefore given in coordinates

$$\begin{aligned} y &= I_1x + I_2 \\ p &= I_1. \end{aligned}$$

Changing coordinates from (x, y, p) to $(x, I_1 = p, I_2 = y - px)$ transforms the above system into

$$\begin{aligned} \theta_1 &= xdI_1 + dI_2 \\ \theta_2 &= dI_1. \end{aligned}$$

Symmetry condition for a general symmetry

$$V = \alpha(x, I_1, I_2)\frac{\partial}{\partial x} + \sigma_1(I_1, I_2)\frac{\partial}{\partial I_1} + \sigma_2(I_1, I_2)\frac{\partial}{\partial I_2}$$

with $\alpha(x, I_1, I_2)\frac{\partial}{\partial x}$ as the characteristic part is then

$$\begin{aligned} L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 \\ L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2. \end{aligned}$$

For θ_1 this gives

$$xd\sigma_1 + d\sigma_2 = (\lambda_{11}x + \lambda_{12})dI_1 + (\lambda_{11})dI_2$$

leading to the conditions

$$x\,\sigma_{1,I_1} + \sigma_{2,I_1} = \lambda_{11}x + \lambda_{12}$$

$$\begin{aligned} x\,\sigma_{1,I_2} + \sigma_{2,I_2} &= \lambda_{11} \\ x\,\sigma_{1,x} + \sigma_{2,x} &= 0. \end{aligned}$$

For θ_2 this gives

$$d\sigma_1 = (\lambda_{21}x + \lambda_{22})dI_1 + (\lambda_{21})dI_2$$

leading to the conditions

$$\begin{aligned} \sigma_{1,I_1} &= \lambda_{11}x + \lambda_{12} \\ \sigma_{1,I_2} &= \lambda_{11} \\ \sigma_{1,x} &= 0 \end{aligned}$$

which are satisfied if σ_1 and σ_2 are arbitrary functions of I_1 and I_2. Transforming back to coordinates (x, y, p) determines the symmetry $V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p}$ as the general symmetry

$$V = (\sigma_1 x + \sigma_2)\frac{\partial}{\partial y} + \sigma_1\frac{\partial}{\partial p}$$

where σ_1 and σ_2 are arbitrary functions of p and $y - px$ and $X = 0$.
Note that a symmetry

$$V = X(x,y)\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y}$$

has the same action on solutions of a given differential equation as

$$V = [Y(x,y) - X(x,y)p]\frac{\partial}{\partial y}.$$

Since σ_1 and σ_2 are arbitrary, the symmetry generator above includes cases in which $Y(x,y,p) = \sigma_1 x + \sigma_2$ is linear in p and can be written as $Y(x,y) - X(x,y)p$. Then the admitted symmetry

$$V = Y(x,y,p)\frac{\partial}{\partial y}$$

is a point symmetry.

4.5 Symmetries of Third and Higher Order Differential Equations

4.5.1 Symmetries of $F(x, y, p, \omega, s) = 0$

A general third order differential equation is given by $F(x, y, p, \omega, s) = 0$ where $p = y'$, $\omega = y''$ and $s = y'''$. It is equivalent to the system

$$\begin{aligned} \theta_1 &= dy - pdx \\ \theta_2 &= dp - \omega dx \end{aligned}$$

$$\begin{aligned}
\theta_3 &= d\omega - s dx \\
\theta_4 &= dF.
\end{aligned}$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p} + \Omega\frac{\partial}{\partial \omega} + S\frac{\partial}{\partial s}$$

where X, Y, P, Ω and S are functions of x, y, p, ω and s, if V satisfies the symmetry conditions

$$\begin{aligned}
L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 + \lambda_{13}\theta_3 + \lambda_{14}\theta_4 \\
L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2 + \lambda_{23}\theta_3 + \lambda_{24}\theta_4 \\
L_V\theta_3 &= \lambda_{31}\theta_1 + \lambda_{32}\theta_2 + \lambda_{33}\theta_3 + \lambda_{34}\theta_4 \\
L_V\theta_4 &= \lambda_{41}\theta_1 + \lambda_{42}\theta_2 + \lambda_{43}\theta_3 + \lambda_{44}\theta_4.
\end{aligned}$$

4.5.2 Symmetries of $s = f(x, y, p, \omega)$

A third order differential equation where the highest derivative can be isolated is given by $F(x, y, p, \omega, s) = s - f(x, y, p, \omega) = 0$ where $p = y'$, $\omega = y''$ and $s = y'''$. It is equivalent to the system

$$\begin{aligned}
\theta_1 &= dy - p dx \\
\theta_2 &= dp - \omega dx \\
\theta_3 &= d\omega - f dx.
\end{aligned}$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p} + \Omega\frac{\partial}{\partial \omega}$$

where X, Y, P and Ω are functions of x, y, p and ω, if V satisfies the symmetry conditions

$$\begin{aligned}
L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 + \lambda_{13}\theta_3 \\
L_V\theta_2 &= \lambda_{21}\theta_1 + \lambda_{22}\theta_2 + \lambda_{23}\theta_3 \\
L_V\theta_3 &= \lambda_{31}\theta_1 + \lambda_{32}\theta_2 + \lambda_{33}\theta_3
\end{aligned}$$

from which we get

$$\begin{aligned}
L_V\theta_1 = V\rfloor(dx \wedge dp) + d(Y - pX) &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 + \lambda_{13}\theta_3 \\
(-P + Y_x - pX_x)dx &= (-p\lambda_{11} - \omega\lambda_{12} - f\lambda_{13})dx \\
+(Y_y - pX_y)dy + (Y_p - pX_p)dp & \quad +(\lambda_{11})dy + (\lambda_{12})dp \\
+(Y_\omega - pX_\omega)d\omega & \quad +(\lambda_{13})d\omega
\end{aligned}$$

which determines P as

$$P = Y_x + p(Y_y - pX_y - X_x) + \omega(Y_p - pX_p) + f(Y_\omega - pX_\omega).$$

We also get

$$\begin{array}{rcl} L_V\theta_2 = V\rfloor(dx \wedge d\omega) + d(P - \omega X) & = & \lambda_{21}\theta_1 + \lambda_{22}\theta_2 + \lambda_{23}\theta_3 \\ (-\Omega + P_x - \omega X_x)dx & = & (-p\lambda_{21} - \omega\lambda_{22} - f\lambda_{23})dx \\ +(P_y - \omega X_y)dy + (P_p - \omega X_p)dp & & +(\lambda_{21})dy + (\lambda_{22})dp \\ +(P_\omega - \omega X_\omega)d\omega & & +(\lambda_{23})d\omega \end{array}$$

which determines Ω as

$$\Omega = P_x + p(P_y - \omega X_x) + \omega(P_p - \omega X_p - X_x) + f(P_\omega - pX_\omega).$$

Finally, we get

$$\begin{array}{rcl} L_V\theta_3 = V\rfloor(dx \wedge df) + d(\Omega - fX) & = & \lambda_{31}\theta_1 + \lambda_{32}\theta_2 + \lambda_{33}\theta_3 \\ (-f_xX - f_yY - f_pP - f_\omega - fX_x + \Omega_x)dx & = & (-p\lambda_{31} - \omega\lambda_{32} - f\lambda_{33})dx \\ +(\Omega_y - fX_y)dy + (\Omega_p - fX_p)dp & & +(\lambda_{31})dy + (\lambda_{32})dp \\ +(\Omega_\omega - fX_\omega)d\omega & & +(\lambda_{33})d\omega \end{array}$$

which determines $\lambda_{31} = \Omega_y - fX_y$, $\lambda_{32} = \Omega_p - fX_p$ and $\lambda_{33} = \Omega_\omega - fX_\omega$. Therefore we obtain

$$Xf_x + Yf_y + Pf_p + \Omega f_\omega - \Omega_x - p\Omega_y - \omega\Omega_p - f\Omega_\omega + f(X_x + pX_y + \omega X_p + fX_\omega) = 0.$$

4.5.3 Symmetries of $y''' = 0$

The third order differential equation given by $y''' = 0$ is equivalent to the system

$$\begin{array}{rcl} \theta_1 & = & dy - p\,dx \\ \theta_2 & = & dp - \omega\,dx \\ \theta_3 & = & d\omega. \end{array}$$

The symmetry conditions above are then

$$\begin{array}{rcl} P & = & Y_x + p(Y_y - pX_y - X_x) + \omega(Y_p - pX_p) \\ \Omega & = & P_x + p(P_y - \omega X_x) + \omega(P_p - \omega X_p - X_x) \\ 0 & = & \Omega_x + p\Omega_y + \omega\Omega_p \end{array}$$

which leads to a system of PDEs.
A general solution of $y''' = 0$ is known to be $y = \frac{1}{2}I_1x^2 + I_2x + I_3$, where I_1, I_2

and I_3 are constants, and the space of solutions is therefore given in coordinates

$$\begin{aligned} y &= \frac{1}{2}I_1x^2 + I_2x + I_3 \\ p &= I_1x + I_2 \\ \omega &= I_1. \end{aligned}$$

Changing coordinates from (x, y, p, ω) to $(x, I_1 = \omega, I_2 = p - \omega x, I_3 = y + \frac{1}{2}\omega x^2 - px)$ transforms the above system into

$$\begin{aligned} \theta_1 &= \frac{1}{2}x^2 dI_1 + x dI_2 + dI_3 \\ \theta_2 &= x dI_1 + dI_2 \\ \theta_3 &= dI_1. \end{aligned}$$

Symmetry condition for a general symmetry

$$V = \sigma_1(I_1, I_2, I_3)\frac{\partial}{\partial I_1} + \sigma_2(I_1, I_2, I_3)\frac{\partial}{\partial I_2} + \sigma_3(I_1, I_2, I_3)\frac{\partial}{\partial I_3}$$

neglecting the characteristic part, is then

$$L_\sigma \theta_i = \sigma \rfloor d\theta_i + d(V \rfloor \theta_i) = \lambda_{i1}\theta_1 + \lambda_{i2}\theta_3 + \lambda_{i3}\theta_3$$

for $i = 1, 2, 3$.
For θ_1 this gives

$$\frac{1}{2}x^2 d\sigma_1 + x d\sigma_2 + d\sigma_3 = (\lambda_{11}\frac{1}{2}x^2 + \lambda_{12}x + \lambda_{13})dI_1 + (\lambda_{11}x + \lambda_{12})dI_2 + (\lambda_{11})dI_3$$

leading to the conditions

$$\begin{aligned} \frac{1}{2}x^2\,\sigma_{1,I_1} + x\sigma_{2,I_1} + \sigma_{3,I_1} &= \lambda_{11}\frac{1}{2}x^2 + \lambda_{12}x + \lambda_{13} \\ \frac{1}{2}x^2\,\sigma_{1,I_2} + x\sigma_{2,I_2} + \sigma_{3,I_2} &= \lambda_{11}x + \lambda_{12} \\ \frac{1}{2}x^2\,\sigma_{1,I_3} + x\sigma_{2,I_3} + \sigma_{3,I_3} &= \lambda_{11} \\ \frac{1}{2}x^2\,\sigma_{1,x} + x\sigma_{2,x} + \sigma_{3,x} &= 0. \end{aligned}$$

For θ_2 this gives

$$x d\sigma_1 + d\sigma_2 = (\lambda_{21}\frac{1}{2}x^2 + \lambda_{22}x + \lambda_{23})dI_1 + (\lambda_{21}x + \lambda_{22})dI_2 + (\lambda_{21})dI_3$$

leading to the conditions

$$x\,\sigma_{1,I_1} + \sigma_{2,I_1} = \lambda_{21}\frac{1}{2}x^2 + \lambda_{22}x + \lambda_{23}$$

$$
\begin{aligned}
x\,\sigma_{1,I_2} + \sigma_{2,I_2} &= \lambda_{21}x + \lambda_{22} \\
x\,\sigma_{1,I_3} + \sigma_{2,I_3} &= \lambda_{21} \\
x\,\sigma_{1,x} + \sigma_{2,x} &= 0.
\end{aligned}
$$

For θ_3 this gives

$$d\sigma_1 = (\lambda_{31}\frac{1}{2}x^2 + \lambda_{32}x + \lambda_{33})dI_1 + (\lambda_{31}x + \lambda_{32})dI_2 + (\lambda_{31})dI_3$$

leading to the conditions

$$
\begin{aligned}
\sigma_{1,I_1} &= \lambda_{31}\frac{1}{2}x^2 + \lambda_{32}x + \lambda_{33} \\
\sigma_{1,I_2} &= \lambda_{31}x + \lambda_{32} \\
\sigma_{1,I_3} &= \lambda_{31} \\
\sigma_{1,x} &= 0
\end{aligned}
$$

which are satisfied for σ_1, σ_2 and σ_3 as arbitrary functions of I_1, I_2 and I_3. Transforming back to coordinates (x, y, p, ω) determines the symmetry $V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + P\frac{\partial}{\partial p} + \Omega\frac{\partial}{\partial \omega}$ as the general symmetry

$$V = (\frac{1}{2}\sigma_1 x^2 + \sigma_2 x + \sigma_3)\frac{\partial}{\partial y} + (\sigma_1 x + \sigma_2)\frac{\partial}{\partial p} + \sigma_1\frac{\partial}{\partial \omega}$$

where σ_1, σ_2 and σ_3 are arbitrary functions of ω, $p - \omega x$ and $y + \frac{1}{2}\omega x^2 - px$ and $X = 0$.
Point symmetries are given if $Y(x, y, p) = \frac{1}{2}\sigma_1 x^2 + \sigma_2 x + \sigma_3$ is linear in p and can be written as $Y(x, y) - X(x, y)p$.

4.5.4 Symmetries of $F(x, y, y', y'', ..., y^{(n)}) = 0$

A general nth order differential equation is given by $F(x, y, y'', y''', ..., y^{(n)}) = 0$ and is equivalent to the system

$$
\begin{aligned}
\theta_1 &= dy - y'dx \\
\theta_2 &= dy' - y''dx \\
\dots & \\
\theta_{(n+1)} &= dF.
\end{aligned}
$$

The system admits a symmetry

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Y'\frac{\partial}{\partial y'} + Y''\frac{\partial}{\partial y''} + \dots + Y^{(n)}\frac{\partial}{\partial y^{(n)}}$$

where X, Y, Y',..., $Y^{(n)}$ are functions of x, y, y',..., $y^{(n)}$, if V satisfies the symmetry conditions

$$\begin{aligned}
L_V\theta_1 &= \lambda_{11}\theta_1 + \lambda_{12}\theta_2 + \ldots + \lambda_{1(n+1)}\theta_{(n+1)} \\
&\ldots \\
L_V\theta_{(n+1)} &= \lambda_{(n+1)1}\theta_1 + \lambda_{(n+1)2} + \ldots + \lambda_{(n+1)(n+1)}\theta_{(n+1)}.
\end{aligned}$$

4.5.5 *Symmetries of* $y^{(n)} = 0$

The nth order differential equation given by $y^{(n)} = 0$ is equivalent to the system

$$\begin{aligned}
\theta_1 &= dy - y'\,dx \\
\theta_2 &= dy' - y''\,dx \\
&\ldots \\
\theta_{(n+1)} &= dy^{(n-1)}.
\end{aligned}$$

The method used for second and third order equations can be continued for higher order equations. The general solution of $y^{(n)} = 0$ is known to be

$$y = \frac{1}{(n-1)!} I_1\, x^{n-1} + \frac{1}{(n-2)!} I_2\, x^{n-2} + \ldots + \frac{1}{2!} I_{n-2}\, x^2 + I_{n-1}\, x + I_n$$

where I_1, I_2, ... I_n are constants, and the space of solutions is therefore given in coordinates

$$\begin{aligned}
y &= \frac{1}{(n-1)!} I_1\, x^{n-1} + \frac{1}{(n-2)!} I_2\, x^{n-2} + \ldots + \frac{1}{2!} I_{n-2}\, x^2 + I_{n-1}\, x + I_n \\
y' &= \frac{1}{(n-2)!} I_1\, x^{n-2} + \frac{1}{(n-3)!} I_2\, x^{n-3} + \ldots + I_{n-2}\, x + I_{n-1} \\
&\ldots \\
y^{(n-2)} &= I_1\, x + I_2 \\
y^{(n-1)} &= I_1.
\end{aligned}$$

Following the method used for lower values of n we then find a general symmetry

$$\begin{aligned}
V &= \left(\frac{1}{(n-1)!}\sigma_1 x^{n-1} + \frac{1}{(n-2)!}\sigma_2 x^{n-1} + \ldots + \sigma_{(n)}\right)\frac{\partial}{\partial y} \\
&\quad + \ldots \\
&\quad + (\sigma_1 x + \sigma_2)\frac{\partial}{\partial y^{(n-2)}} + \sigma_1 \frac{\partial}{\partial y^{(n-1)}}
\end{aligned}$$

where σ_1, σ_2, ..., $\sigma_{(n)}$ are arbitrary functions of

$$\begin{aligned}
I_1 &= y^{(n-1)} \\
I_2 &= y^{(n-2)} - I_1 x
\end{aligned}$$

$$
\begin{aligned}
I_3 &= y^{(n-3)} - \frac{1}{2!} I_1 x^2 - I_2 x \\
&\dots \\
I_n &= y + \frac{1}{(n-1)!} I_1 x^{n-1} - \frac{1}{(n-2)!} I_2 x^{n-2} - \dots - \frac{1}{2} I_{(n-2)} x^2 - I_{(n-1)} x.
\end{aligned}
$$

4.6 Summary

This chapter introduced to a method to find symmetries using differential forms. The method was illustrated for simple examples of the form $y^{(n)} = 0$. Their symmetries could be determined and these results are useful for further investigations on differential equations with singular solutions.

5. CLAIRAUT AND GENERALIZED CLAIRAUT EQUATIONS

This chapter gives an introduction to differential equations with singular solutions. Special emphasis is on the known example of a Clairaut equation and its generalizations, as we aim to consider generalized Clairaut equations as reductions of equations $y^{(n)} = 0$.

5.1 Clairaut equation

A Clairaut equation is a first order differential equation given by

$$F(x, y, p) = y - xp - f(p) = 0$$

with $p = y'$. The method to find solutions for this kind of equation is described for example in [12] and [16]. Differentiating leads to

$$0 = \frac{dp}{dx}(x + f'(p))$$

which is satisfied if either

- $\frac{dp}{dx} = 0$ or
- $x + f'(p) = 0$.

Consider the first case, if $\frac{dp}{dx} = 0$ is true, then a general integral is given by

$$y = cx + f(c)$$

where c is a constant. This solution describes a family of straight lines. If $x + f'(p) = 0$ is true, then another solution can be given by the two equations

$$\begin{aligned} 0 &= x + f'(p) \\ y &= px + f(p). \end{aligned}$$

Eliminating p leads to the envelope of the straight lines represented by the general solution [6], [7]. This solution cannot be obtained by choosing a particular value for the constant c and is therefore called a singular solution.

Example: An equation is given by

$$F(x, y, p) = y - px - (1/2)p^2 = 0$$

. Differentiating gives

$$0 = \omega(x + p)$$

where $\omega = \frac{dp}{dx}$. If $\omega = 0$, then $y = c_1 x + c_2$ where c_1 and c_2 are constants satisfying $F(x, c_1 x + c_2, c_1) = 0$. Therefore we get the solution

$$y = cx + (1/2)c^2$$

where c is a constant. This is the general solution. Another solution is obtained by $x+p = 0$ which inserted into the original equation gives $y + x^2 - (1/2)(-x)^2 = 0$ which gives another solution as

$$y = -(1/2)x^2.$$

This solution is a singular solution as it is not part of the general solution. Figure 5.1 shows some general solution curves and the singular solution of the example equation.

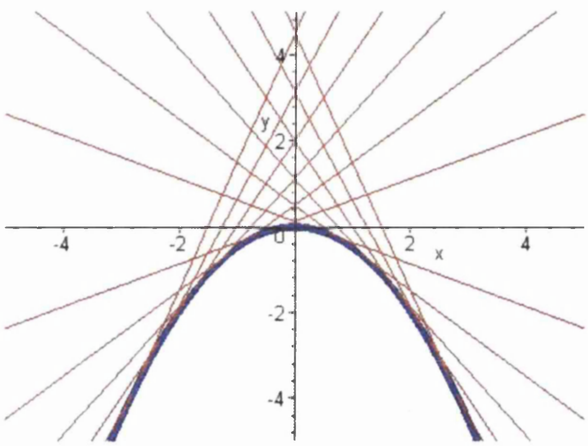

Fig. 5.1: General and singular solution of $y - px - (1/2)p^2 = 0$.

5.2 First order generalized Clairaut equation

5.2.1 Generalized Clairaut equation $F(p, y - px) = 0$

A generalized Clairaut equation is given by $F(p, y - px) = 0$. Using the same method as described above and differentiating the equation leads to

$$\begin{aligned} F(p, y - px) &= 0 \\ \frac{\partial F}{\partial p}(p, y - px) &= 0. \end{aligned}$$

Example: An equation is given by

$$F(x,y,p) = (y-px)^2 - p^2 = 1.$$

Differentiating gives

$$0 = \omega(x(y-px)+p)$$

where $\omega = \frac{dp}{dx}$. If $\omega = 0$, then $y = c_1x + c_2$ where c_1 and c_2 are constants satisfying $F(x, c_1x + c_2, c_1) = 0$. Therefore we get the solution

$$(y-cx)^2 - c^2 = 1$$

where c is a constant, which is the general solution. Another solution is obtained by $x(y-px)+p$ which leads to a singular solution

$$y^2 + x^2 = 1$$

which is the unit circle as illustrated in figure 5.2 along with some general solution curves.

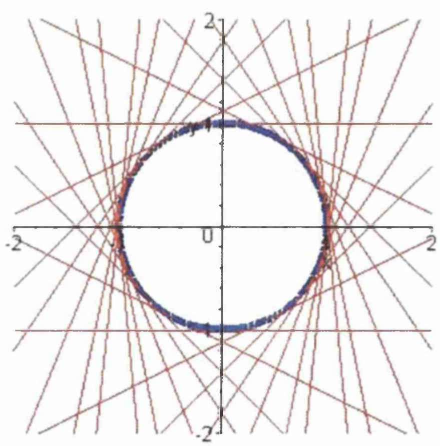

Fig. 5.2: General and singular solution of $(y-px)^2 - p^2 = 1$.

5.2.2 Goursat's generalized Clairaut equation

Generalizing the preceding example, a generalized Clairaut equation is described by Goursat [12]. Consider a curve for which the product of the distances from two points F and F' to its tangent is equal to a constant b^2. The distance between the points F and F' shall be $2c$, and the points shall be on the x-axis as shown in the sketch in figure 5.3. An equation

$$y - px = k = \text{const}$$

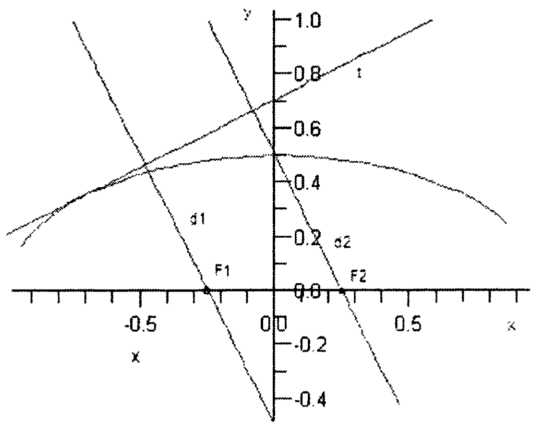

Fig. 5.3: Sketch of Goursat's generalization.

represents the tangent. The distances d_1 and d_2 from the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ to the tangent are

$$d_1 = \frac{|cp - k|}{\sqrt{p^2+1}} \qquad d_2 = \frac{|-cp - k|}{\sqrt{p^2+1}}.$$

Setting the product of the distances to a constant gives

$$\begin{aligned} d_1 \, d_2 &= b^2 \\ &= \frac{|k^2 - c^2p^2|}{p^2+1} \end{aligned}$$

Suppose that F_1 and F_2 are on the same side of the tangent we get $b^2 = \frac{k^2-c^2p^2}{p^2+1}$ and because $k^2 = (y - px)^2$ we get the equation

$$(y - px)^2 - c^2p^2 = b^2(1 + p^2).$$

The equation reduces to a Clairaut type equation of the form

$$y = px \pm \sqrt{b^2 + a^2p^2}$$

where $a^2 = b^2 + c^2$. The general solution is given by the family of straight lines

$$y = Cx \pm \sqrt{b^2 + a^2C^2}.$$

The singular solution is determined by

$$\begin{aligned} y - px - \sqrt{b^2 + a^2p^2} &= 0 \\ x + \frac{a^2p}{\sqrt{b^2 + a^2p^2}} &= 0 \end{aligned}$$

from which we obtain a solution as the equation of a general ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as illustrated in figure 5.4.

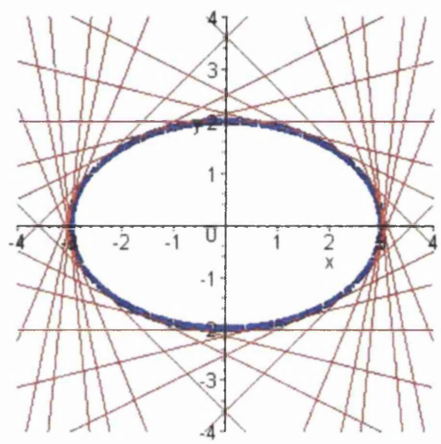

Fig. 5.4: Ellipse for $a = 3$ and $b = 2$.

Goursat's 2nd generalization: Goursat [12] also mentions another example of generalizing a Clairaut equation. It is given as the system

$$\begin{aligned} F(y - px, z - qx, p, q) &= 0 \\ G(y - px, z - qx, p, q) &= 0 \end{aligned}$$

where $p = y'$ and $q = z'$. Differentiating these equations w.r.t. x gives the relations

$$\begin{aligned} \frac{\partial p}{\partial x}(F_p - xF_{y-px}) + \frac{\partial q}{\partial x}(F_q - xF_{z-qx}) &= 0 \\ \frac{\partial p}{\partial x}(G_p - xG_{y-px}) + \frac{\partial q}{\partial x}(G_q - xG_{z-qx}) &= 0. \end{aligned}$$

5.3 Second order generalizations of Clairaut equation

5.3.1 Generalized Clairaut equation $F(y + \frac{1}{2}x^2\omega - px, p - \omega x, \omega) = 0$

A generalized Clairaut equation is given by

$$F(y + \frac{1}{2}x^2\omega - px, p - \omega x, \omega) = 0$$

where $p = y'$ and $\omega = y''$. Similarly to treating first order generalizations, we differentiate the equation which leads to

$$\begin{aligned} F(y + \frac{1}{2}x^2\omega - px, p - \omega x, \omega) &= 0 \\ \frac{\partial F}{\partial \omega}(y + \frac{1}{2}x^2\omega - px, p - \omega x, \omega) &= 0. \end{aligned}$$

Example: An equation is given by

$$F = y + \frac{1}{2}x^2\omega - px + p - \omega x + f(\omega).$$

Differentiating leads to

$$0 = s(\frac{1}{2}x^2 - x + \omega)$$

where $s = \frac{d\omega}{dx}$. If $s = 0$, then $y = \frac{1}{2}c_1x^2 + c_2x + c_3$ and c_1, c_2 and c_3 are constants satisfying $F(x, \frac{1}{2}c_1x^2 + c_2x + c_3, c_1x + c_2, c_1) = 0$, i.e.

$$\frac{1}{2}c_1^2 + c_2 + c_3 = 0$$

must hold. The family of parabolas given by $y = \frac{1}{2}c_1x^2 + c_2x + c_3$ is the general solution. The general solutions in the (x, y, p)-space are given by the two-parameter family

$$\begin{aligned} y &= c_2(x-1) + \frac{1}{2}c_1(x^2 - c_1) \\ p &= c_2 + c_1x. \end{aligned}$$

If $\frac{1}{2}x^2 - x + \omega = 0$ then we get a first order differential equation

$$y + p - px - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 = 0$$

which can be integrated and the singular solution is then

$$y = (1-x)(\frac{1}{24}x^3 - \frac{1}{8}x^2 - \frac{1}{8}x - \frac{1}{8}\frac{1}{(x-1)}) + \frac{\text{const}}{1-x}.$$

5.4 Higher order generalizations of Clairaut equation

Higher order generalized Clairaut equations are given by

$$F(I_1, I_2, ..., I_n) = 0$$

where the I_1, I_2, ..., I_n are given by

$$\begin{aligned}
I_1 &= y^{(n-1)} \\
I_2 &= y^{(n-2)} - I_1 x \\
I_3 &= y^{(n-3)} - \frac{1}{2} I_1 x^2 - I_2 x \\
I_4 &= y^{(n-4)} - \frac{1}{3!} I_1 x^3 - \frac{1}{2} I_2 x^2 - I_3 x \\
&\dots \\
I_n &= y + \frac{1}{(n-1)!} I_1 x^{n-1} - \frac{1}{(n-2)!} I_2 x^{n-2} - \dots - \frac{1}{2} I_{(n-2)} x^2 - I_{(n-1)} x.
\end{aligned}$$

A Clairaut equation is the special case where $n = 2$. The general solution can be given as

$$y = \frac{c_1}{(n-1)!} x^n + \frac{c_2}{(n-2)!} x^{n-1} + \dots + \frac{c_{(n-2)}}{2!} x^2 + c_{(n-1)} x + c_n.$$

Note that the I_1, I_2, ..., I_n are integrals of $y^{(n)} = 0$. Singular solutions appear in higher order generalizations and can be investigated similarly to the case of a Clairaut equation [17].

5.5 Summary

In this chapter we investigated the Clairaut equation and explained how the general and singular solutions can be obtained. We introduced generalizations of the Clairaut equation and determined general and singular solutions for example equations with low order. Finally we outlined higher order generalized Clairaut equations.

6. SYMMETRIES OF CLAIRAUT AND GENERALIZED CLAIRAUT EQUATION

In this chapter Clairaut and generalized Clairaut equations are investigated from the symmetry point of view. Their symmetries are determined by using findings of the previous chapter. Symmetries of the general and symmetries of the singular solution of a Clairaut-type equation are distinguished.

6.1 Symmetries of a Clairaut equation

6.1.1 General Symmetries of a Clairaut equation

For determining the symmetries of a Clairaut equation we look at the equation $y'' = 0$ and its symmetries. As mentioned in a previous chapter it has the general integral $y = I_1 x + I_2$ so that $I_1 = p$ and $I_2 = y - px$. The characteristic vector field is

$$X_c = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}$$

which satisfies $L_{X_c}\theta = 0$ for $\theta = dy - p\,dx$. We know that symmetries of $y'' = 0$ are all vector fields of the form

$$\alpha(I_1, I_2)\frac{\partial}{\partial I_1} + \beta(I_1, I_2)\frac{\partial}{\partial I_2} + \gamma(x, I_1, I_2)\frac{\partial}{\partial x}$$

where $\gamma(x, I_1, I_2)\partial_x$ is the characteristic part. We look at the symmetries generated by $\frac{\partial}{\partial I_1}$ and $\frac{\partial}{\partial I_2}$.

$\frac{\partial}{\partial I_1}$ generates the symmetry:

$$I_1 \rightarrow I_1 + \varepsilon\,,\; I_2 \rightarrow I_2$$

i.e.

$$\begin{aligned}
p' = p + \varepsilon \quad &\Rightarrow \quad p' - p = \varepsilon \\
y' - p'x' = y - px \quad &\Rightarrow \quad y' - y = p(x' - x) + \varepsilon x' \\
\delta p = \varepsilon, \quad & \qquad \delta y = p\delta x + \varepsilon x' \\
\frac{\partial}{\partial I_1} \quad &= \quad \dot{x}\frac{\partial}{\partial x} + (p\dot{x} + x)\frac{\partial}{\partial y} + \frac{\partial}{\partial p} = x\frac{\partial}{\partial y} + \frac{\partial}{\partial p} + \dot{x}X_c.
\end{aligned}$$

$\frac{\partial}{\partial I_2}$ generates the symmetry:

$$I_1 \to I_1,\, I_2 \to I_2 + \varepsilon$$

i.e.

$$\begin{aligned} p' = p \quad &\Rightarrow \quad p' - p = 0 \\ y' - p'x' = y - px + \varepsilon \quad &\Rightarrow \quad y' - y = p(x' - x) + \varepsilon \\ \delta p = 0,\, \delta y \quad &= \quad p\delta x + \varepsilon \\ \frac{\partial}{\partial I_2} \quad &= \quad \dot{x}\frac{\partial}{\partial x} + (p\dot{x} + 1)\frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \dot{x}X_c. \end{aligned}$$

Any λX_c is a symmetry and $\{\lambda X_c | \lambda \text{ a function}\}$ is a Lie algebra ideal inside $\{\sigma | L_\sigma \theta = 0 \pmod{\theta}\}$. So the characteristic symmetries can be factored out to leave the noncharacteristic symmetry algebra [2], [3].

$$\alpha(I_1, I_2)\frac{\partial}{\partial I_1} + \beta(I_1, I_2)\frac{\partial}{\partial I_2}$$

where $I_1 = p$, $I_2 = y - px$, $\frac{\partial}{\partial I_1} \sim x\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$, $\frac{\partial}{\partial I_2} \sim \frac{\partial}{\partial y}$. These symmetries permute elements of the general solution and the characteristic symmetries map each particular solution to itself.
Solutions to $y'' = 0$ are all the straight lines in the (x, y)-space. They lift to straight lines in the (x, y, p)-space where p is the slope of the lines in the (x, y)-space.
A Clairaut equation is given by $y - px = f(p)$. In coordinates I_1, I_2 this is $f(I_1) = I_2$. It is a choice of a one-parameter family of lines from the two-parameter general solution of $y'' = 0$. Symmetries of a Clairaut equation are therefore either the characteristic symmetry X or symmetries which permute members of the one-parameter family. Such a symmetry has the form $\sigma = \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2}$ and must satisfy $\sigma(f(I_1) - I_2) = 0$.
This leads to $\alpha f' = \beta$, hence we get $\sigma = \alpha(\frac{\partial}{\partial I_1} + f'(I_1)\frac{\partial}{\partial I_2})$. Inserting the symmetries generated by $\frac{\partial}{\partial I_1}$ and $\frac{\partial}{\partial I_2}$ we get

$$\sigma = \alpha((x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p})$$

which generates the flow

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= x + f'(p) \\ \dot{p} &= 1. \end{aligned}$$

Hence we have

$$x = x_0$$

$$\begin{aligned} y &= x_0 t + f(t + p_0) + a, \quad a = y_0 - f(p_0) \\ p &= t + p_0 \end{aligned}$$

which represents the map

$$\begin{pmatrix} x \\ y \\ p \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + xt + f(p) - f(p-t) \\ t + p \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{p} \end{pmatrix}.$$

Then the general solution $y = cx + f(c)$ with $p = c$ will be mapped into

$$y + xt + f(c) - f(c-t) = cx + f(c)$$

which leads to

$$y = (c-t)x + f(c-t).$$

Hence the symmetry permutes the members of the general solution.
Considering also the characteristic part, then symmetries of a Clairaut equation are given as

$$\lambda X_c + \alpha(\frac{\partial}{\partial I_1} + f'(I_1)\frac{\partial}{\partial I_2})$$

which equals

$$\lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}) + \alpha((x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}).$$

6.1.2 Symmetries of the singular solution of a Clairaut equation

In forms the Clairaut equation $F = y - px - f(p) = 0$ is equivalent to the system Θ given by

$$\begin{aligned} \theta_1 &= dy - pdx \\ \theta_2 &= -(x + f'(p))dp. \end{aligned}$$

Θ has rank less than 2 when $x + f' = 0$. An ideal is given by the reduced system $\Theta' = \langle -(x + f'(p))dp \rangle$.
A quotient ring is the quotient of a ring and one of its ideals [23]. Considering Θ and the ideal Θ' then the quotient ring is denoted as Θ/Θ'. It has elements of Θ where $\Theta' = 0$. This is the case if

- $x + f' \neq 0$ and $dp = 0$ or
- $x + f' = 0$ and $dp \neq 0$.

If $dp = 0$, then $p = c$, where c is constant and $\Theta/(p = c)$ has then elements of $\langle dy - c\,dx \rangle$. This system is closed. Since $d(y - cx) = 0$, a general solution is given by $y - cx - f(c) = 0$.
If $x + f' = 0$, then $\Theta/(x + f' = 0)$ has elements of $\langle dy + pf''dp \rangle$. This system

is also closed. Because of $d(y + pf' - f) = 0$, a singular solution is given by

$$\begin{aligned} y + pf' - f &= \text{const.} \\ x + f' &= 0. \end{aligned}$$

As shown in the previous section symmetries of a Clairaut equation $F(x, y, p) = y - px - f(p) = 0$ are given by

$$\sigma = \alpha((x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}).$$

Singular solutions of Clairaut are given by the two conditions:

$$\begin{aligned} F(x, y, p) &= y - px - f(p) = 0 \\ \frac{\partial F}{\partial p}(x, y, p) &= -x - f'(p) = 0. \end{aligned}$$

However, whilst

$$\sigma(F) = \{(x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}\}F = (x + f') + (-x - f') = 0$$

the action of σ on $\partial_p F$ is

$$\sigma(\frac{\partial F}{\partial p}) = \{(x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}\}F_p = -f''(p) \neq 0.$$

But σ is an equivalence class of vector fields in the factor algebra, so some λX_c can be added to it. Therefore we get

$$(\sigma + \lambda X_c)(F) = \sigma(F) + \lambda X_c(F) = 0$$

and

$$\begin{aligned} (\sigma + \lambda X_c)(\frac{\partial F}{\partial p}) &= \{(x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p} + \lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y})\}(-x - f'(p)) \\ &= -f''(p) - \lambda. \end{aligned}$$

This is equal to zero for $\lambda = -f''(p)$. Hence we can specify the symmetries as

$$\sigma + \lambda X_c = (x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p} - f''(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y})$$

from which we obtain

$$-f''\frac{\partial}{\partial x} + (x + f' - pf'')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$$

as the symmetry of the singular solution.

Example: Let F be given as the Clairaut equation $F(x,y,p) = y - px - \frac{1}{2}p^2$. Figure 6.1 shows the general solution lines in the (x,y,p) space. The projection

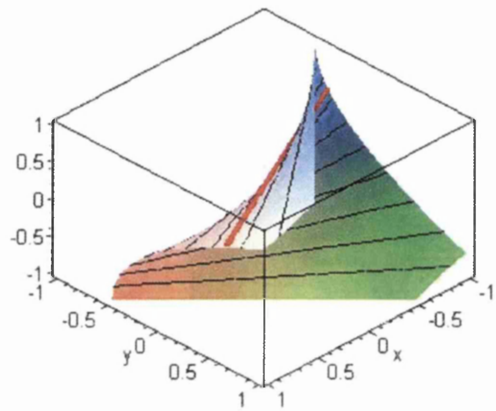

Fig. 6.1: General solution in (x,y,p) space.

onto the (x,y) plane shows the singular solution as illustrated in figure 6.2. The singular solution is given if $F = 0$ and $F_p = -x - p = 0$ is satisfied. Symmetry of the singular solution is

$$V = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial p}.$$

The symmetry conditions are satisfied because $V(F) = 0$ and $V(F_p) = 0$.

6.2 Symmetries of $F(p, y - px) = 0$

6.2.1 General Symmetries of $F(p, y - px) = 0$

Given the first order differential equation $F(p, y - px) = 0$ or $F(I_1, I_2) = 0$ in coordinates I_1, I_2. It is equivalent to the system of 1-forms

$$\begin{aligned} \theta_1 &= dy - p\,dx \\ \theta_2 &= d(F). \end{aligned}$$

A symmetry has the form $\sigma = \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2}$ and must satisfy $\sigma(F(I_1, I_2)) = 0$. Therefore we get $\beta = -\alpha\frac{F_{I_1}}{F_{I_2}}$ leading to $\sigma = \alpha(\frac{\partial}{\partial I_1} - \frac{F_{I_1}}{F_{I_2}}\frac{\partial}{\partial I_2})$. The symmetry σ is then given as

$$\sigma = \alpha((x - \frac{F_{I_1}}{F_{I_2}})\frac{\partial}{\partial y} + \frac{\partial}{\partial p})$$

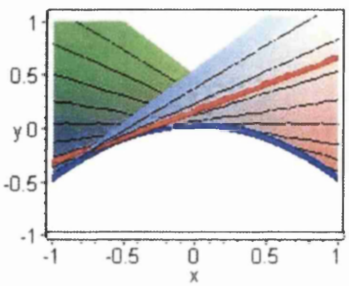

Fig. 6.2: Singular solution in (x, y, p) space.

which generates the flow

$$\begin{aligned}\dot{x} &= 0\\ \dot{y} &= x - \frac{F_{I_1}}{F_{I_2}}\\ \dot{p} &= 1.\end{aligned}$$

Considering the characteristic part, then the symmetries of $F(p, y - px) = 0$ are given by

$$\lambda X_c + \alpha(I_1, I_2)(\frac{\partial}{\partial I_1} - \frac{F_{I_1}}{F_{I_2}}\frac{\partial}{\partial I_2})$$

which is equal to

$$\lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}) + \alpha(p, y - px)((x - \frac{F_{I_1}}{F_{I_2}})\frac{\partial}{\partial y} + \frac{\partial}{\partial p}).$$

6.2.2 Symmetries of the Singular Solution of $F(p, y - px) = 0$

Singular solutions of $F(p, y - px) = 0$ are given by the two conditions:

$$\begin{aligned}F(p, y - px) &= 0\\ \frac{\partial F}{\partial p}(p, y - px) &= 0.\end{aligned}$$

Let F_{I_1} be denoted as F_1 and F_{I_2} as F_2. A symmetry

$$\sigma = \alpha((x - \frac{F_1}{F_2})\frac{\partial}{\partial y} + \frac{\partial}{\partial p})$$

applied to the first condition gives

$$\sigma(F) = \{(x - \frac{F_1}{F_2})\frac{\partial}{\partial y} + \frac{\partial}{\partial p})\}F = F_1 - \frac{F_1}{F_2}F_2 = 0.$$

But for the second condition we get

$$\sigma(\frac{\partial F}{\partial p}) = \{(x - \frac{F_1}{F_2})\frac{\partial}{\partial y} + \frac{\partial}{\partial p})\}F_p = F_{1,1} - \frac{F_1}{F_2}F_{1,2} \neq 0$$

Because σ is an equivalence class of vector fields in the factor algebra, some λX_c can be added to it. This leads to

$$(\sigma + \lambda X_c)(F) = \sigma(F) + \lambda X_c(F) = 0$$

and

$$\begin{aligned}(\sigma + \lambda X_c)(\frac{\partial F}{\partial p}) &= \{(x - \frac{F_1}{F_2})\frac{\partial}{\partial y} + \frac{\partial}{\partial p} + \lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y})\}(F_p) \\ &= (x - \frac{F_1}{F_2})F_{py} + F_{pp} + \lambda(F_{px} + pF_{py}).\end{aligned}$$

This is equal to zero for

$$\lambda = -\frac{(x - \frac{F_1}{F_2})F_{py} + F_{pp}}{F_{px} + pF_{py}}.$$

Therefore the symmetry of the singular solution can be given as

$$\sigma + \lambda X_c = \lambda\frac{\partial}{\partial x} + (x - \frac{F_1}{F_2} + \lambda p)\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$$

for the above λ.

6.2.3 Symmetries of Goursat's generalization

Goursat's generalization of the Clairaut equation is given by $y = px \pm \sqrt{b^2 + a^2p^2}$. This is equivalent to the system Θ given by

$$\begin{aligned}\theta_1 &= dy - pdx \\ \theta_2 &= -(x \pm \frac{a^2p}{\sqrt{b^2 + a^2p^2}})dp.\end{aligned}$$

As shown in a previous section symmetries of a general Clairaut equation $y = px + f(p)$ are given by $\sigma = (x + f')\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$. Therefore symmetries of Goursat's generalization should be determined by

$$\sigma = (x \pm \frac{a^2p}{\sqrt{b^2 + a^2p^2}})\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$$

where any characteristic symmetries $\lambda(\frac{\partial}{\partial x}+p\frac{\partial}{\partial y})$ can be added. It must satisfy the symmetry conditions

$$\begin{aligned} L_\sigma\theta_1 &= \lambda_{11}\theta_1+\lambda_{12}\theta_2 && (6.1)\\ L_\sigma\theta_2 &= \lambda_{21}\theta_1+\lambda_{22}\theta_2. && (6.2)\end{aligned}$$

Regarding 6.1 we get

$$\sigma\rfloor(-dp\wedge dx)+d(x+\frac{a^2p}{\sqrt{b^2+a^2p^2}})=\frac{a^2b^2+a^4p^2-a^4p^4}{(b^2+a^2p^2)^{(3/2)}}dp=\lambda_{11}\theta_1+\lambda_{12}\theta_2$$

which is satisfied for

$$\begin{aligned}\lambda_{11} &= 0\\ \lambda_{12} &= \frac{a^2b^2+a^4p^2-a^4p^4}{x(b^2+a^2p^2)^{(3/2)}+a^2p(b^2+a^2p^2)}.\end{aligned}$$

Considering 6.2 we get

$$\sigma\rfloor(dp\wedge dx)+d(-(x+\frac{a^2p}{\sqrt{b^2+a^2p^2}}))=-\frac{a^2b^2+a^4p^2-a^4p^4}{(b^2+a^2p^2)^{(3/2)}}dp=\lambda_{21}\theta_1+\lambda_{22}\theta_2$$

which is satisfied for

$$\begin{aligned}\lambda_{21} &= 0\\ \lambda_{22} &= -\frac{a^2b^2+a^4p^2-a^4p^4}{x(b^2+a^2p^2)^{(3/2)}+a^2p(b^2+a^2p^2)}.\end{aligned}$$

Symmetries of the singular solution of an equation $y=px+f(p)$ are given by $\sigma_s=-f''\frac{\partial}{\partial x}+(x+f'-pf'')\frac{\partial}{\partial y}+\frac{\partial}{\partial p}$ as mentioned in a previous section. In the case of Goursat's generalization we then get

$$\sigma_s=-(\frac{a^2b^2+a^4p^2-a^4p^4}{(b^2+a^2p^2)^{(3/2)}})\frac{\partial}{\partial x}+(x+\frac{a^2p}{\sqrt{b^2+a^2p^2}}-p\frac{a^2b^2+a^4p^2-a^4p^4}{(b^2+a^2p^2)^{(3/2)}})\frac{\partial}{\partial y}+\frac{\partial}{\partial p}$$

as the symmetry for the singular solution.

6.3 Symmetries of $F(\omega,p-\omega x,y+\frac{1}{2}\omega x^2-px)=0$

6.3.1 General Symmetries of $F(\omega,p-\omega x,y+\frac{1}{2}\omega x^2-px)=0$

For determining the symmetries of the Clairaut equation we started by considering the symmetries of $y''=0$. Similarly we now look at $y'''=0$ to determine the symmetries of the generalization

$$F(\omega,p-\omega x,y+\frac{1}{2}\omega x^2-px)=0.$$

$y''' = 0$ has the general integral $y = \frac{1}{2}I_1x^2 + I_2x + I_3$ so that $I_1 = \omega$, $I_2 = p - \omega x$ and $I_3 = y + \frac{1}{2}\omega x^2 - px$. The characteristic vector field is

$$X_c = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}.$$

Symmetries of $y''' = 0$ are all vector fields of the form

$$\alpha(I_1, I_2, I_3)\frac{\partial}{\partial I_1} + \beta(I_1, I_2, I_3)\frac{\partial}{\partial I_2} + \gamma(I_1, I_2, I_3)\frac{\partial}{\partial I_3} + \delta(x, I_1, I_2, I_3)\frac{\partial}{\partial I_x}$$

where $\delta(x, I_1, I_2, I_3)\partial_x$ is the characteristic part. We look at the symmetries generated by $\frac{\partial}{\partial I_1}$, $\frac{\partial}{\partial I_2}$ and $\frac{\partial}{\partial I_3}$.

$\frac{\partial}{\partial I_1}$ generates the symmetry:

$$I_1 \to I_1 + \varepsilon\,,\ I_2 \to I_2\,,\ I_3 \to I_3$$

i.e.

$$\begin{aligned} \omega' = \omega + \varepsilon \quad &\Rightarrow \quad \omega' - \omega = \varepsilon \\ p' - \omega' x' = p - \omega x \quad &\Rightarrow \quad p' - p = \omega(x' - x) + \varepsilon x' \\ y' + \frac{1}{2}\omega' x'^2 - p'x' = y + \frac{1}{2}\omega x^2 - px \quad &\Rightarrow \quad y' - y = \frac{1}{2}\omega(x' - x)^2 + p(x' - x) + \frac{1}{2}\varepsilon x'^2. \end{aligned}$$

Hence

$$\begin{aligned} \delta\omega &= \varepsilon \\ \delta p &= \omega\delta x + \varepsilon x' \\ \delta y &= p\delta x + \frac{1}{2}\varepsilon x'^2. \end{aligned}$$

Then we get

$$\begin{aligned} \frac{\partial}{\partial I_1} &= \dot{x}\frac{\partial}{\partial x} + (p\dot{x} + \frac{1}{2}x^2)\frac{\partial}{\partial y} + (\omega\dot{x} + x)\frac{\partial}{\partial p} + \frac{\partial}{\partial \omega} \\ &= \frac{1}{2}x^2\frac{\partial}{\partial y} + x\frac{\partial}{\partial p} + \frac{\partial}{\partial \omega} + \dot{x}X_c. \end{aligned}$$

$\frac{\partial}{\partial I_2}$ generates the symmetry:

$$I_1 \to I_1,\ I_2 \to I_2 + \varepsilon,\ I_3 \to I_3$$

i.e.

$$\begin{aligned} \omega' = \omega \quad &\Rightarrow \quad \omega' - \omega = 0 \\ p' - \omega' x' = p - \omega x + \varepsilon \quad &\Rightarrow \quad p' - p = \omega(x' - x) + \varepsilon \end{aligned}$$

$$y' + \frac{1}{2}\omega' x'^2 - p'x' = y + \frac{1}{2}\omega x^2 - px \quad \Rightarrow \quad y' - y = \frac{1}{2}\omega(x'-x)^2 + p(x'-x) + \varepsilon x'.$$

Hence

$$\begin{aligned}\delta\omega &= 0\\ \delta p &= \omega\delta x + \varepsilon\\ \delta y &= p\delta x + \varepsilon x'.\end{aligned}$$

Then we get

$$\begin{aligned}\frac{\partial}{\partial I_2} &= \dot{x}\frac{\partial}{\partial x} + (p\dot{x} + x)\frac{\partial}{\partial y} + (\omega\dot{x} + 1)\frac{\partial}{\partial p}\\ &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial p} + \dot{x}X_c.\end{aligned}$$

$\frac{\partial}{\partial I_3}$ generates the symmetry:

$$I_1 \to I_1,\, I_2 \to I_2,\, I_3 \to I_3 + \varepsilon$$

i.e.

$$\begin{aligned}\omega' = \omega \quad &\Rightarrow \quad \omega' - \omega = 0\\ p' - \omega' x' = p - \omega x \quad &\Rightarrow \quad p' - p = \omega(x' - x)\\ y' + \frac{1}{2}\omega' x'^2 - p'x' = y + \frac{1}{2}\omega x^2 - px + \varepsilon \quad &\Rightarrow \quad y' - y = \frac{1}{2}\omega(x'-x)^2 + p(x'-x) + \varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}\delta\omega &= 0\\ \delta p &= \omega\delta x\\ \delta y &= p\delta x + \varepsilon.\end{aligned}$$

Finally we get

$$\frac{\partial}{\partial I_3} = \dot{x}\frac{\partial}{\partial x} + (p\dot{x} + 1)\frac{\partial}{\partial y} + (\omega\dot{x})\frac{\partial}{\partial p} = \frac{\partial}{\partial y} + \dot{x}X_c.$$

The characteristic symmetries can be factored out to leave the non characteristic symmetry algebra.

$$\alpha(I_1, I_2, I_3)\frac{\partial}{\partial I_1} + \beta(I_1, I_2, I_3)\frac{\partial}{\partial I_2} + \gamma(I_1, I_2, I_3)\frac{\partial}{\partial I_3}$$

where $I_1 = \omega$, $I_2 = p - \omega x$, $I_3 = y + \frac{1}{2}\omega x^2 - px$ and $\frac{\partial}{\partial I_1} \sim \frac{1}{2}x^2\frac{\partial}{\partial y} + x\frac{\partial}{\partial p} + \frac{\partial}{\partial \omega}$, $\frac{\partial}{\partial I_2} \sim x\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$, $\frac{\partial}{\partial I_3} \sim \frac{\partial}{\partial y}$.
Given an equation $F(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) = 0$ or $F(I_1, I_2, I_3) = 0$ in

coordinates I_1, I_2 and I_3. It is a choice of a two-parameter family of parabolas from the three-parameter general solution of $y''' = 0$. Symmetries are either the characteristic symmetry X_c or symmetries of the form $\sigma = \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} + \gamma\frac{\partial}{\partial I_3}$ and must satisfy $\sigma(F(I_1, I_2, I_3)) = 0$.
This leads to $\gamma = -\frac{\alpha F_1+\beta F_2}{F_3}$ and we get

$$\sigma = \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} - \frac{\alpha F_1 + \beta F_2}{F_3}\frac{\partial}{\partial I_3}$$

equal to

$$\sigma = (\alpha\frac{1}{2}x^2 + \beta x - \frac{\alpha F_1 + \beta F_2}{F_3})\frac{\partial}{\partial y} + (\alpha x + \beta)\frac{\partial}{\partial p} + \alpha\frac{\partial}{\partial \omega}.$$

Considering the characteristic part, then symmetries of $F(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) = 0$ are

$$\lambda X_c + \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} - \frac{\alpha F_1 + \beta F_2}{F_3}\frac{\partial}{\partial I_3}$$

equal to

$$\lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}) + (\alpha\frac{1}{2}x^2 + \beta x - \frac{\alpha F_1 + \beta F_2}{F_3})\frac{\partial}{\partial y} + (\alpha x + \beta)\frac{\partial}{\partial p} + \alpha\frac{\partial}{\partial \omega}.$$

6.3.2 Symmetries of the Singular Solution of $F(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) = 0$

Singular solutions of $F(\omega, p-\omega x, y+\frac{1}{2}\omega x^2 - px) = 0$ are given by the conditions:

$$\begin{aligned} F(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) &= 0 \\ \frac{\partial F}{\partial \omega}(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) &= 0. \end{aligned}$$

Applying symmetry

$$\sigma = \alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} - \frac{\alpha F_1 + \beta F_2}{F_3}\frac{\partial}{\partial I_3}$$

to the first condition leads to

$$\sigma(F) = \{\alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} - \frac{\alpha F_1 + \beta F_2}{F_3}\frac{\partial}{\partial I_3}\}F = \alpha F_1 + \beta F_2 - \frac{\alpha F_1 + \beta F_2}{F_3}F_3 = 0$$

and is therefore satisfied. However

$$\sigma(\frac{\partial F}{\partial p}) = \{\alpha\frac{\partial}{\partial I_1} + \beta\frac{\partial}{\partial I_2} - \frac{\alpha F_1 + \beta F_2}{F_3}\frac{\partial}{\partial I_3}\}F_\omega = \alpha F_{1,1} + \beta F_{1,2} - \frac{\alpha F_1 + \beta F_2}{F_3}F_{1,3} \neq 0.$$

Because σ is an equivalence class of vector fields in the factor algebra, some λX_c can be added to it. The first condition is then still satisfied

$$(\sigma + \lambda X_c)(F) = \sigma(F) + \lambda X_c(F) = 0.$$

For the second condition we get

$$\begin{aligned}(\sigma + \lambda X_c)(\frac{\partial F}{\partial \omega}) &= \{(\alpha\frac{1}{2}x^2 + \beta x - \frac{\alpha F_1 + \beta F_2}{F_3})\frac{\partial}{\partial y} + (\alpha x + \beta)\frac{\partial}{\partial p} + \alpha\frac{\partial}{\partial \omega} \\ &\quad + \lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p})\}(F_\omega)\end{aligned}$$

which is satisfied for

$$\lambda = \frac{(\alpha\frac{1}{2}x^2 + \beta x - \frac{\alpha F_1 + \beta F_2}{F_3})F_{\omega y} + (\alpha x + \beta)F_{\omega p} + \alpha F_{\omega\omega}}{F_{\omega x} + pF_{\omega y} + \omega F_{\omega p}}.$$

Symmetry of the singular solution is then

$$(\alpha\frac{1}{2}x^2 + \beta x - \frac{\alpha F_1 + \beta F_2}{F_3})\frac{\partial}{\partial y} + (\alpha x + \beta)\frac{\partial}{\partial p} + \alpha\frac{\partial}{\partial \omega} + \lambda(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p})$$

for the above λ.

6.4 Symmetries of Higher Order Generalized Clairaut Equations

The method introduced to determine symmetries of $F(y - px, p) = 0$ and $F(\omega, p - \omega x, y + \frac{1}{2}\omega x^2 - px) = 0$ can be applied to higher order generalized Clairaut equations. Symmetries of a generalized Clairaut equation

$$F(I_1, I_2, ..., I_n) = 0$$

where the I_1, I_2, ..., I_n are

$$\begin{aligned}I_1 &= y^{(n-1)} \\ I_2 &= y^{(n-2)} - I_1\,x \\ &\dots \\ I_n &= y + \frac{1}{(n-1)!}I_1\,x^{n-1} - \frac{1}{(n-2)!}I_2\,x^{n-2} - \dots - \frac{1}{2}I_{n-2}\,x^2 - I_{n-1}\,x\end{aligned}$$

are then obtained as

$$\sigma = \alpha_1\frac{\partial}{\partial I_1} + \alpha_2\frac{\partial}{\partial I_2} + \dots + \alpha_{(n-1)}\frac{\partial}{\partial I_{n-1}} - \frac{\alpha_1 F_1 + \alpha_2 F_2 + \dots + \alpha_{(n-1)}F_{(n-1)}}{F_{(n)}}\frac{\partial}{\partial I_3}$$

where α_1, α_2, ... $\alpha_{(n)}$ are arbitrary functions of I_1, I_2, ... I_n.

6.5 Summary

In this chapter we have investigated the Clairaut equation and its generalizations. Their symmetries have been determined for the Clairaut equation and first and second order generalizations. This was generalized for higher order generalized Clairaut equations by using characteristic symmetries of $y^{(n)} = 0$.

7. SINGULAR SOLUTIONS AND REDUCTION USING SYMMETRIES

Symmetries can be useful in achieving a reduction of order of a given differential equation. In this chapter we investigate the method of reducing the order and take results of the previous chapters into account, in particular the knowledge we gained concerning symmetries of equations with singular solutions.

7.1 *Symmetries and Reduction*

We first look at the reduction of order in general to gain more understanding on the reduction procedure regarding systems of differential forms. This will be illustrated with examples of known cases.

7.1.1 Reduction of a system $\Theta = \langle\theta_1, \theta_2\rangle$

Let Θ be a system of forms and $\Theta = \langle\theta_1, \theta_2\rangle$ be closed under d. Suppose it has symmetries τ and σ. Choosing $\theta_2 = \tilde{\theta}$ such that $\tau\rfloor\tilde{\theta} = 0$ and assuming $\sigma\rfloor\tilde{\theta} \neq 0$ then we obtain the reduced system $\Theta' = \langle\tilde{\theta}\rangle$ which we shall show to be an ideal of Θ.
The two symmetries τ and σ form a Lie algebra, hence their commutator is a linear combination

$$[\sigma, \tau] = c_1\sigma + c_2\tau$$

where c_1 and c_2 are structure constants. Either both structure constants are zero or one or both are non-zero. The symmetry generators can be transformed so that only two distinct cases have to be considered:

$$[\sigma, \tau] = \tau$$

$$[\sigma, \tau] = 0.$$

We can choose $\tilde{\theta}$ so that $\tau\rfloor\tilde{\theta} = 0$. The definition of $d\theta$ of a 1-form θ is

$$d\theta(\sigma, \tau) = \sigma(\theta\rfloor\tau) - \tau(\theta\rfloor\sigma) - \theta\rfloor[\sigma, \tau].$$

Here we have $d\tilde{\theta}(\sigma, \tau) = -\tau(\tilde{\theta}\rfloor\sigma)$. Therefore it can be shown that σ is also symmetry of the reduced system $\langle\tilde{\theta}\rangle$.

$$\sigma\rfloor d\tilde{\theta} + d(\sigma\rfloor\tilde{\theta}) \quad = \quad \lambda_{11}\theta_1 + \lambda_{12}\tilde{\theta}$$

$$\begin{aligned}
\tau\rfloor(\sigma\rfloor d\tilde{\theta}) + \tau\rfloor d(\sigma\rfloor\tilde{\theta}) &= \lambda_{11}(\tau\rfloor\theta_1) \\
d\tilde{\theta}(\sigma,\tau) + \tau(\sigma\rfloor\tilde{\theta}) &= \lambda_{11}(\tau\rfloor\theta_1) \\
0 &= \lambda_{11}(\tau\rfloor\theta_1).
\end{aligned}$$

$d\tilde{\theta} = \omega \wedge \tilde{\theta}$ holds. $\Theta' =< \tilde{\theta} >$ is a differential ideal of Θ with symmetry σ. The form $\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}}$ is then closed, since

$$\begin{aligned}
d(\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}}) &= \frac{1}{\sigma\rfloor\tilde{\theta}}d\tilde{\theta} - \frac{1}{(\sigma\rfloor\tilde{\theta})^2}d(\sigma\rfloor\tilde{\theta}) \wedge \tilde{\theta} \\
&= \frac{1}{(\sigma\rfloor\tilde{\theta})^2}\{(\sigma\rfloor\tilde{\theta})d\tilde{\theta} + (\sigma\rfloor d\tilde{\theta}) \wedge \tilde{\theta}\} \\
&= \frac{1}{(\sigma\rfloor\tilde{\theta})^2}\{\sigma\rfloor(\tilde{\theta} \wedge d\tilde{\theta})\} = 0.
\end{aligned}$$

7.1.2 Known cases for the reduction of $\Theta = \langle\theta_1, \theta_2\rangle$

Let a system be given as $\Theta = \langle dy - pdx, dp\rangle$.

Example I: Suppose the system has the symmetries

$$\tau = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p}$$

and

$$\sigma = \frac{\partial}{\partial x}$$

where $[\sigma, \tau] = 0$. To satisfy $\tilde{\theta}\rfloor\tau = 0$, $\tilde{\theta}$ can be chosen as

$$\tilde{\theta} = p\theta_1 - y\theta_2 = pdy - ydp - p^2dx$$

Then $\sigma = \partial_x$ is a symmetry for $\tilde{\theta}$ since $L_\sigma\tilde{\theta} = 2pdp + d(-p^2) = 0$. A form

$$\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}} = -\frac{1}{p}dy + \frac{y}{p^2}dp + dx$$

is then closed, because of

$$d(\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}}) = \frac{1}{p^2}dp \wedge dy + \frac{1}{p^2}dy \wedge dp = 0.$$

An ideal of Θ is then the reduced system

$$\Theta' = \langle\tilde{\theta} = pdy - ydp - p^2dx\rangle.$$

Example II: Let the system have the symmetries

$$\tau = (x + f'(p))\frac{\partial}{\partial y} + \frac{\partial}{\partial p}$$

and

$$\sigma = \frac{\partial}{\partial y}$$

where $[\sigma, \tau] = 0$. To satisfy $\tilde{\theta} \rfloor \tau = 0$, $\tilde{\theta}$ can here be chosen as

$$\tilde{\theta} = \theta_1 - (x + f'(p))\theta_2 = dy - pdx - (x + f'(p))dp.$$

Then $\sigma = \partial_y$ is a symmetry for $\tilde{\theta}$ since $L_\sigma \tilde{\theta} = 0$. The form

$$\frac{\tilde{\theta}}{\sigma \rfloor \tilde{\theta}} = (x + f'(p))dy + dp$$

is then closed, because $d(\frac{\tilde{\theta}}{\sigma \rfloor \tilde{\theta}}) = 0$. An ideal of Θ is here the reduced system

$$\Theta' = \langle \tilde{\theta} = dy - pdx - (x + f'(p))dp \rangle.$$

7.1.3 Reduction of the system $\Theta = \langle \theta_1, \theta_2 \rangle$ in invariant coordinates

In invariant coordinates $I_1 = y - px$ and $I_2 = p$ the above system is given by $\Theta = \langle dI_1, dI_2 \rangle$ where the symmetries are

$$\tau = \tau_1(I_1, I_2)\frac{\partial}{\partial I_1} + \tau_2(I_1, I_2)\frac{\partial}{\partial I_2}$$

and

$$\sigma = \sigma_1(I_1, I_2)\frac{\partial}{\partial I_1} + \sigma_2(I_1, I_2)\frac{\partial}{\partial I_2}.$$

We now reconsider the examples above in invariant coordinates.

Example I: The symmetries τ and σ are then

$$\tau = I_1 \frac{\partial}{\partial I_1} + I_2 \frac{\partial}{\partial I_2}$$

and

$$\sigma = -I_2 \frac{\partial}{\partial I_1}.$$

$\tilde{\theta} \rfloor \tau = 0$ is satisfied for

$$\tilde{\theta} = dI_1 - \frac{I_1}{I_2} dI_2.$$

Then $\sigma = -I_2\partial_{I_1}$ is a symmetry for $\tilde{\theta}$ since $L_\sigma\tilde{\theta} = dI_2 + d(-I_2) = 0$. A form

$$\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}} = -\frac{1}{I_2}dI_1 + \frac{I_1}{{I_2}^2}dI_2$$

is then closed, because

$$d(\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}}) = \frac{1}{{I_2}^2}dI_2 \wedge dI_1 + \frac{1}{{I_2}^2}dI_1 \wedge dI_2 = 0.$$

The reduced system Θ' is given as

$$\Theta' = \langle\tilde{\theta} = dI_1 - \frac{I_1}{I_2}dI_2\rangle.$$

Example II: Let the symmetries be

$$\tau = f'(I_2)\frac{\partial}{\partial I_1} + \frac{\partial}{\partial I_2}$$

and

$$\sigma = \frac{\partial}{\partial I_1}.$$

To satisfy $\tilde{\theta}\rfloor\tau = 0$, $\tilde{\theta}$ can here be chosen as

$$\tilde{\theta} = dI_1 - f'(I_2)dI_2.$$

$\sigma = \partial_{I_1}$ is a symmetry for $\tilde{\theta}$ because of $L_\sigma\tilde{\theta} = 0$. The form $\tilde{\theta} = dI_1 - f'(I_2)dI_2$ is then closed, because

$$d(\frac{\tilde{\theta}}{\sigma\rfloor\tilde{\theta}}) = 0.$$

The reduced system is here

$$\Theta' = \langle\tilde{\theta} = dI_1 - f'(I_2)dI_2\rangle = \langle d(I_1 - f(I_2))\rangle$$

corresponding to a Clairaut equation.

7.1.4 *Reduction of the system* $\Theta = \langle\theta_1, \theta_2, \theta_3\rangle$

In the case of a third order system

$$\Theta = \langle\theta_1, \theta_2, \theta_3\rangle$$

let Θ be closed and admit symmetries σ and τ. Then we can choose $\tilde{\theta}_1$ and $\tilde{\theta}_2$ such that $\tau\rfloor\tilde{\theta}_1 = 0$ and $\tau\rfloor\tilde{\theta}_2 = 0$. We assume that $\sigma\rfloor\tilde{\theta}_1 \neq 0$ and $\sigma\rfloor\tilde{\theta}_2 \neq 0$. Then the reduced system $\Theta' = \langle\tilde{\theta}_1, \tilde{\theta}_2\rangle$ is closed. It is an ideal of Θ and admits the symmetry σ. As in the second order case this can be shown by considering the Lie derivative of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ with respect to σ.

$$\begin{aligned}
\sigma\rfloor d\tilde{\theta}_1 + d(\sigma\rfloor\tilde{\theta}_1) &= \lambda_1\tilde{\theta}_1 + \lambda_2\tilde{\theta}_2 + \lambda_3\theta_3 \\
\tau\rfloor(\sigma\rfloor d\tilde{\theta}_1) + \tau\rfloor d(\sigma\rfloor\tilde{\theta}_1) &= \lambda_3(\tau\rfloor\theta_3) \\
d\tilde{\theta}_1(\sigma,\tau) + \tau(\sigma\rfloor\tilde{\theta}_1) &= \lambda_3(\tau\rfloor\theta_3) \\
0 &= \lambda_3(\tau\rfloor\theta_3)
\end{aligned}$$

Hence $L_\sigma\tilde{\theta}_1 = \lambda_{11}\tilde{\theta}_1 + \lambda_{12}\tilde{\theta}_2$ and analogically $L_\sigma\tilde{\theta}_2 = \lambda_{21}\tilde{\theta}_1 + \lambda_{22}\tilde{\theta}_2$. Then we have the reduced system $\Theta' = \langle\tilde{\theta}_1, \tilde{\theta}_2\rangle$. It is closed under d.

$$\begin{aligned}
d\tilde{\theta}_1 &= \omega_{11}\wedge\tilde{\theta}_1 + \omega_{12}\wedge\tilde{\theta}_2 \\
d\tilde{\theta}_2 &= \omega_{21}\wedge\tilde{\theta}_1 + \omega_{22}\wedge\tilde{\theta}_2
\end{aligned}$$

7.1.5 *Known cases for the reduction of* $\Theta = \langle\theta_1, \theta_2, \theta_3\rangle$

Considering the case where the system Θ is given by

$$\begin{aligned}
\theta_1 &= dy - pdx \\
\theta_2 &= dp - \omega dx \\
\theta_3 &= d\omega.
\end{aligned}$$

Example I: Let the system admit symmetries σ and τ given by

$$\begin{aligned}
\sigma &= \frac{\partial}{\partial x} \\
\tau &= y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} + \omega\frac{\partial}{\partial\omega}.
\end{aligned}$$

$\tilde{\theta}_1$ and $\tilde{\theta}_2$ can be chosen as

$$\begin{aligned}
\tilde{\theta}_1 &= p\theta_1 - y\theta_2 - 0\theta_3 = pdy - p^2dx - ydp - y\omega dx \\
\tilde{\theta}_2 &= \omega\theta_1 + 0\theta_2 - y\theta_3 = \omega dy - \omega pdx - yd\omega
\end{aligned}$$

so that $\tau\rfloor\tilde{\theta}_1 = 0$ and $\tau\rfloor\tilde{\theta}_2 = 0$. Then $\tilde{\theta}_1$ and $\tilde{\theta}_2$ admit σ and

$$\begin{aligned}
L_\sigma\tilde{\theta}_1 &= 0 \\
L_\sigma\tilde{\theta}_2 &= 0.
\end{aligned}$$

There is a reduced system $\Theta' = \langle\tilde{\theta}_1, \tilde{\theta}_2\rangle$ with

$$\begin{aligned}
\tilde{\theta}_1 &= pdy - p^2dx - ydp - y\omega dx \\
\tilde{\theta}_2 &= \omega dy - \omega pdx - yd\omega
\end{aligned}$$

admitting σ as a symmetry. Symmetries σ and τ are given by

$$\begin{aligned}
\sigma &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial p} \\
\tau &= (\frac{1}{2}x^2 + x + f'(\omega))\frac{\partial}{\partial y} + (x+1)\frac{\partial}{\partial p} + \frac{\partial}{\partial \omega}.
\end{aligned}$$

$\tilde{\theta}_1$ and $\tilde{\theta}_2$ can be chosen as

$$\begin{aligned}
\tilde{\theta}_1 &= \theta_1 - (\frac{1}{2}x^2 + x + f'(\omega))\theta_3 = dy - pdx - (\frac{1}{2}x^2 + x + f'(\omega))d\omega \\
\tilde{\theta}_2 &= \theta_2 - (x+1)\theta_3 = dp - \omega dx - (x+1)d\omega
\end{aligned}$$

so that $\tau \rfloor \tilde{\theta}_1 = 0$ and $\tau \rfloor \tilde{\theta}_2 = 0$. Then $\tilde{\theta}_1$ and $\tilde{\theta}_2$ admit σ as a symmetry and

$$\begin{aligned}
L_\sigma \tilde{\theta}_1 &= 0 \\
L_\sigma \tilde{\theta}_2 &= 0.
\end{aligned}$$

There is a reduced system $\Theta' = \langle \tilde{\theta}_1, \tilde{\theta}_2 \rangle$ with

$$\begin{aligned}
\tilde{\theta}_1 &= dy - pdx - (\frac{1}{2}x^2 + x + f'(\omega))d\omega \\
\tilde{\theta}_2 &= dp - \omega dx - (x+1)d\omega
\end{aligned}$$

admitting σ as a symmetry.

Example II: As another example we consider now the case of the two symmetries σ and τ chosen as

$$\begin{aligned}
\sigma &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial p} \\
\tau &= F_\omega \frac{\partial}{\partial y} + F_p \frac{\partial}{\partial p} + F_y \frac{\partial}{\partial \omega}
\end{aligned}$$

where $F = F(x, y, p, \omega)$. To satisfy symmetry conditions F must be of the form $F = c_1(xp + x^2\omega) + c_2(p + \omega x) + c_3 y + f(\omega)$ where c_1, c_2 and c_3 are constants and f an arbitrary function of ω.
$\tilde{\theta}_1$ and $\tilde{\theta}_2$ can be chosen as

$$\begin{aligned}
\tilde{\theta}_1 &= \theta_1 - F_\omega \theta_3 = dy - pdx - (c_1 x^2 + c_2 x + f'(\omega))d\omega \\
\tilde{\theta}_2 &= \theta_2 - F_p \theta_3 = dp - \omega dx - (c_1 x + c_2)d\omega
\end{aligned}$$

so that $\tau \rfloor \tilde{\theta}_1 = 0$ and $\tau \rfloor \tilde{\theta}_2 = 0$. Then $\tilde{\theta}_1$ and $\tilde{\theta}_2$ admit σ as a symmetry and

$$\begin{aligned}
L_\sigma \tilde{\theta}_1 &= 0 \\
L_\sigma \tilde{\theta}_2 &= 0.
\end{aligned}$$

The reduced system $\Theta' = \langle\tilde{\theta}_1, \tilde{\theta}_2\rangle$ is then

$$\begin{aligned}\tilde{\theta}_1 &= dy - pdx - (c_1x^2 + c_2x + f'(\omega))d\omega \\ \tilde{\theta}_2 &= dp - \omega dx - (c_1x + c_2)d\omega\end{aligned}$$

admitting σ as a symmetry.

7.2 Singular solution and Reduction

This section investigates the role singular solutions play in reductions of a system. It is explained how differential equations of the form $y^{(n)} = 0$ can be reduced to Clairaut-type equations.

7.2.1 Singular solution and reduction of a 2nd order system

Given $y'' = 0$, equivalent to $\Theta = \langle dy - pdx, dp\rangle$. We know that Θ has the symmetry

$$\sigma = F_2\frac{\partial}{\partial y} + (F_1 - xF_2)\frac{\partial}{\partial p}$$

where $F = F(p, y - px)$ and F_1 and F_2 denote $F_1 = \frac{\partial F}{\partial p}$ and $F_2 = \frac{\partial F}{\partial(y-px)}$. This symmetry can be used in a reduction process. Since

$$F_2(dy - pdx) + (F_1 - xF_2)dp = d(F(p, y - px))$$

the given equation $y'' = 0$ is reduced to a first order differential equation

$$F(p, y - px) = F(I_1, I_2) = 0$$

where F is a generalized Clairaut equation. The reduced system representing F is then

$$\Theta = \langle dy - pdx, (F_1 - xF_2)dp\rangle.$$

Generally, this reduced system has rank 2. However, if $(F_1 - xF_2)dp$ equals zero, the rank is less than 2. Two cases can then be distinguished:

- Let $dp = 0$. This leads to a general solution $y = c_1x + c_2$, where c_1 and c_2 are constants, satisfying $F(c_1, c_2) = 0$.
- Let $(F_1 - xF_2)$ be equal to zero. Then we get $x = \frac{F_1}{F_2}$. This leads to the singular solution of $F(y - px, p) = 0$ which is given by two conditions

  $$F(y - px, p) = F(I_1, I_2) = 0$$

  and

  $$x = \frac{F_1}{F_2}.$$

This leads to a system

$$\langle dy - p \quad d(\frac{F_1}{F_2})\rangle.$$

Symmetries of $F(p, y - px) = 0$ have the form

$$\sigma = \alpha((F_2 x - F_1)\frac{\partial}{\partial y} - F_2\frac{\partial}{\partial p}).$$

Hence the equation $y'' = 0$ is reduced to an equation of Clairaut-type with general and singular solution.

Example : The symmetry

$$\sigma = \frac{\partial}{\partial y} - (f'(p) + x)\frac{\partial}{\partial p}$$

reduces $y'' = 0$ to the Clairaut equation $F = y - px - f(p)$ with the general solution $y = cx + f(c)$ and the singular solution

$$\begin{aligned} y + pf' - f &= \text{const} \\ x &= -f'. \end{aligned}$$

7.2.2 Singular solution and reduction of a 3rd order system

Given $y''' = 0$, equivalent to $\Theta = \langle dy - pdx, dp - \omega dx, d\omega\rangle$. We know that Θ has the symmetry

$$\sigma = F_3\frac{\partial}{\partial y} + (F_2 - xF_3)\frac{\partial}{\partial p} + (F_1 - xF_2 + \frac{1}{2}x^2F_3)\frac{\partial}{\partial \omega}.$$

Since

$$F_3(dy-pdx)+(F_2-xF_3)(dp-\omega dx)+(F_1-xF_2+\frac{1}{2}x^2F_3)d\omega = d(F(\omega, p-\omega x, y+\frac{1}{2}x^2\omega-px))$$

Θ can be reduced to a function

$$F(\omega, p - \omega x, y + \frac{1}{2}x^2\omega - px) = F(I_1, I_2, I_3) = 0$$

which is of second order. The reduced system representing F is then

$$\Theta = \langle dy - pdx, dp - \omega dx, (F_1 - xF_2 + \frac{1}{2}x^2F_3)d\omega\rangle.$$

This system has in general rank 3. If $(F_1 - xF_2 + \frac{1}{2}x^2F_3)d\omega$ equals zero, the rank is less than 3. Again two cases can be distinguished here.

- Let $d\omega = 0$, which leads to the general solution $y = \frac{1}{2}c_1x^2 + c_2x + c_3$, where c_1, c_2 and c_3 are constants, satisfying $F(c_1, c_2, c_3) = 0$.

- Let $(F_1 - xF_2 + \frac{1}{2}x^2F_3)$ be equal to zero. Then $x = \frac{F_2 \pm \sqrt{F_2^2 - 2F_3F_1}}{F_3}$ and we get the system

$$\langle dy - pd(\frac{F_2 \pm \sqrt{F_2^2 - 2F_3F_1}}{F_3}), dp - \omega d(\frac{F_2 \pm \sqrt{F_2^2 - 2F_3F_1}}{F_3})\rangle$$

which describes the singular solution.

The equation $y''' = 0$ is reduced to an equation of Clairaut-type with general and singular solution.

Example : Let F be given as the generalized Clairaut equation

$$F = y + \frac{1}{2}x^2\omega - px - f(\omega, p - \omega x).$$

Then the general solution is $y = \frac{1}{2}c_1x^2 + c_2x + f(c_1, c_2)$ and a singular solution is given by the system

$$\langle dy - pd(-f_2 \pm \sqrt{f_2^2 + 2f_1}), dp - \omega d(-f_2 \pm \sqrt{f_2^2 + 2f_1})\rangle.$$

Let f be given by $f = p - \omega x - g(\omega)$ then the above system is equal to

$$\begin{aligned} \theta_1 &= dy + p(\frac{g''(\omega)}{\sqrt{1 - 2g'(\omega)}})d\omega \\ \theta_2 &= dp + \omega(\frac{g''(\omega)}{\sqrt{1 - 2g'(\omega)}})d\omega. \end{aligned}$$

Choosing $\theta_1 = \theta_1 - \frac{p}{\omega}\theta_2$ and $\theta_2 = \frac{\sqrt{1-2g'(\omega)}}{\omega g''(\omega)}\theta_2$ transforms the system into

$$\begin{aligned} \theta_1 &= dy - \frac{p}{\omega}dp \\ \theta_2 &= d\omega + \frac{\sqrt{1 - 2g'(\omega)}}{\omega g''(\omega)}dp \end{aligned}$$

as a reduced system.

7.3 Summary

We illustrated that generalized Clairaut equations can be obtained by a symmetry reduction of equations $y^{(n)} = 0$. We transformed an nth order system where all the solutions are known into a system of order $(n-1)$ with general and singular solution. As the reduced system is a generalization of a Clairaut equation, further symmetries can be determined with the methods used in the previous chapters.

8. CONCLUSIONS

We have studied how symmetries relate to singularities of differential equations by investigating symmetries and reduction of differential equations to equations with singular solutions, in particular Clairaut-type equations.
For this purpose we have given an introduction into the practice of finding and using symmetries for integration of differential equations.
Furthermore, differential forms and the method to use them in finding symmetries of differential equations have been introduced. This method has been applied to examples of simple structure, that are of the form $y^{(n)} = 0$. Their symmetries could be determined using functions of their first integrals. Putting these first integrals in relation leads to differential equations with singularities: Clairaut and generalized Clairaut equations.
Symmetries of Clairaut-type equations could be derived from the previously determined symmetries of equations $y^{(n)} = 0$. The general solution of $y^{(n)} = 0$ includes the general solution of Clairaut and generalized Clairaut equations, therefore symmetries of both types of equations are the same and are in general a combination of characteristic and non-characteristic symmetries of $y^{(n)} = 0$. These symmetries permute the elements of the general solution.
Symmetries of the singular solution have been considered separately, since singularity conditions have to be satisfied. These symmetries have been used in the reduction of the chosen differential equations. A general description how symmetries can be used to reduce the order of differential equations has been given. Special cases where symmetries are retained after reduction have been described in general as well as by regarding known examples.
Applying the reduction method for the previously chosen examples reveals, that the Clairaut equation and its generalizations can be obtained by reduction of differential equations of the form $y^{(n)} = 0$.
It remains to investigate whether any generalized Clairaut equation can be obtained as a reduction of an equation $y^{(n)} = 0$. Instead of reducing we could then increase the order of an equation with singular solutions to obtain a simple equation of the form $y^{(n)} = 0$ for which all solutions are known.
Further investigations could also be done by considering other classes of differential equations with singular solutions. Instead of starting with equations of the form $y^{(n)} = 0$, more general equations could be chosen. By considering for example a differential equation $y'' = g(x)$ and putting its first integrals in a relation we obtain $F(p - G', y - px - G + G'x) = 0$, where $G''(x) = g(x)$ for which the Clairaut equation is a special case.

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