A dissertation submitted for the degree of Master of Science in the University of Glasgow

WELL-BOUNDED OPERATORS

by

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1974

SUMMARY OF THESIS

WELL-BOUNDED OPERATORS

by B.H. LIM

This thesis is primarily concerned with the structure theory of well-bounded operators and the relationships between various classes of well-bounded operators and prespectral operators.

In chapter I, we follow Ringrose (11) to discuss well-bounded operators in a non-reflexive Banach space X. It turns out that well-boundedness of $T \in L(X)$ is equivalent to the existence of a family of projections $\{E(t) : t \in R\}$ on X^*, called the decomposition of the identity for T, satisfying certain natural properties and such that

$$\langle Tx, x^* \rangle = b \langle x, x^* \rangle - \int_a^b \langle x, E(t)x^* \rangle \, dt \qquad (x \in X,\ x^* \in X^*).$$

In this case, the family $\{E(t) : t \in R\}$ is not necessarily unique, and a necessary and sufficient condition for its uniqueness is given.

In chapter II and III, we discuss three subclasses of well-bounded operators. These are well-bounded operators decomposable in X and well-bounded operators of type (A) and type (B). The main results are that if T is a well-bounded operator decomposable in X then it is uniquely decomposable and that if T is a well-bounded operator of type (A), the algebra homomorphism from AC(J) into L(X) can be extended to an algebra homomorphism from NBV(J) into L(X). We also give some examples in the last section. In chapter III, we follow Spain (14) to use an elementary integration theory to establish directly the characterisation of the type (B) operators. (Theorem III.4.3). We also show that if T is a well-bounded operator of type (B) and $\{F^*(t) : t \in R\}$ is the unique decomposition of the identity for T, then for $f \in AC(J)$, we have

$$f(T) = \int_{a-}^{b} f(t) \, dF(t)$$

where the integral exists as a strong limit of Riemann sums. Moreover, $F(s) - F(s-)$ is a projection on X onto $\{x : Tx = sx\}$ and the residual spectrum of T is empty.

In the fourth and final chapter, we prove some results concerning relationships between various classes of well-bounded operators and prespectral operators. The main results are that an adjoint of an operator $T \in L(X)$ with $\sigma(T) \subseteq R$ is a scalar-type operator of class X if and only if T is well-bounded with a decomposition of the identity of bounded variation and that a well-bounded spectral operator is a scalar-type spectral operator and of type (B). Moreover, a well-bounded prespectral operator which is decomposable in X is a scalar-type operator. Finally, we give a counter-example showing that there is a well-bounded operator of type (B) which is not a scalar-type spectral operator.

PREFACE

This dissertation is submitted in accordance with the regulations for the degree of Master of Science in the University of Glasgow.

The work presented here has been done under the supervision of Dr. H. R. Dowson. I express my deep gratitude to him for his guidance, constant interest and encouragement throughout the period of research. I also thank Dr. P. G. Spain who also have helped in many aspects of my work here.

CONTENTS

INTRODUCTION

The theme of this dissertation is "well-bounded operators on Banach spaces".

Well-bounded operators were first introduced by Smart (13). He proved that if T is a well-bounded operator on a reflexive Banach space X, then for any real number t, there exists a unique projection $F(t)$ such that

(i) $F(t)$ commutes with any bounded operator commuting with T.

(ii) $\|F(t)\| \leqslant 2K$, for all $t \in R$.

(iii) $F(t) = 0$ for $t < a$ and $F(t) = I$ for $t \geqslant b$ where $[a,b]$ is a bounded interval containing $\sigma(T)$.

(iv) $F(s) = F(s)F(t) = F(t)F(s)$ for $s \leqslant t$.

(v) $\lim_{t \to s+} F(t)x = F(s)x$ for all x in X.

(vi) $\lim_{t \to s-} F(t) = F(s-)$ exists in the strong operator topology.

(vii) $\sigma(T/F(t)X) \subset (-\infty, t] \cap \sigma(T)$ and

$\sigma(T/(I - F(t))X) \subset [t, \infty) \cap \sigma(T)$ for all real t.

He also proved the existence of the "scalar operator"

$$S = \int_{a-}^{b} t \; dF(t)$$

where the integral exists as a strong limit of Riemann sums. Ringrose (10) improved this result and showed that

$$T = \int_{a-}^{b} t \; dF(t) \quad .$$

The approach used by Smart and Ringrose was based in part on the fact that a well-bounded operator admits a functional calculus for absolutely continuous functions.

Sills (12) presented a different method for obtaining the spectral theorem for this operator. The method consists in introducing Arens multiplication in $AC_0^{**}([0,1])$ the second dual space of $AC([0,1])$ and in identifying a collection of idempotents in $AC_0^{**}([0,1])$ corresponding to the non-zero multiplicative linear functionals on $L^{\infty}([0,1])$ which is isometrically isomorphic to $AC_0^{*}([0,1])$; these can be associated with the points of $[0,1]$. If T is well-bounded on X, there is an algebra homomorphism from $AC_0([0,1])$ into L(X) and if X is reflexive, this homomorphism can be extended to a homomorphism of the algebra $AC_0^{**}([0,1])$ into L(X). The extended homomorphism maps the idempotents of $AC_0^{**}([0,1])$ into projection operators from which the integral representation of T can be derived. Moreover, the extended homomorphism is defined on a quotient algebra of $AC_0^{**}([0,1])$ which turns out to be a copy of $BV_0([0,1])$.

In chapter I, we follow Ringrose (11) to discuss well-bounded operators in a non-reflexive Banach space X. It turns out that well-boundedness of $T \in L(X)$ is equivalent to the existence of a family of projections $\{E(t) : t \in R\}$ on X^*, called the decomposition of the identity for T, satisfying certain natural properties and such that

$$\langle Tx,x^*\rangle = b\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle \, dt \qquad (x \in X,\ x^* \in X^*).$$

In this case, the family $\{E(t) : t \in R\}$ is not necessarily unique, and a necessary and sufficient condition for its uniqueness is given.

In chapter II and III, we discuss three subclasses of well-bounded operators. These are well-bounded operators decomposable in X and well-bounded operators of type (A) and type (B). The main results are that if T is a well-bounded operator decomposable in X then it is uniquely decomposable and that if T is a well-bounded operator of type (A), the algebra homomorphism from AC(J) into L(X) can be extended to an algebra homomorphism from NBV(J) into L(X). We also give some examples in the last section. In chapter III, we follow Spain(14) to use an elementary integration theory to establish directly the characterisation of the type (B) operators.

(Theorem III.4.3.). We also show that if T is well-bounded operator of type (B) and $\{F^*(t) : t \in R\}$ is the unique decomposition of the identity for T, then for $f \in AC(J)$, we have

$$f(T) = \int_{a-}^{b} f(t)\, dF(t)$$

where the integral exists as a strong limit of Riemann sums. Moreover, $F(s) - F(s-)$ is a projection on X onto $\{x : Tx = sx\}$ and the residual spectrum of T is empty.

In the fourth and final chapter, we prove some results concerning relationships between various classes of well-bounded operators and prespectral operators. The main results are that an adjoint of an operator $T \in L(X)$ with $\sigma(T) \subset R$ is scalar-type operator of class X if and only if T is well-bounded with a decomposition of the identity of bounded variation and that a well-bounded spectral operator is scalar-type spectral operator and of type (B). Moreover, a well-bounded prespectral operator which is decomposable in X is a scalar-type operator. Finally, we give a counterexample showing that there is a well-bounded operator of type (B) which is not a scalar-type spectral operator.

NOTATION

$L(X)$	Banach algebra of bounded linear operators on X.
$\langle x,x^*\rangle$	denotes the value of the functional x^* in X^* at x in X.
R	Real line.
C	Complex plane.
$\underset{\sim}{N}$	The set of all natural numbers.
$\overline{sp}\{M\}$	denotes the closed subspace generated by M.
$sp\{M\}$	denotes the linear subspace generated by M.
$\\|f\\|_\infty$	essential supremum of $\|f\|$.
χ_τ	denotes the characteristic function of the set τ.
$\sigma(T)$	denotes the spectrum of the linear operator T.
$\rho(T)$	denotes the resolvent set of the linear operator T.
T/Y	denotes the restriction of T to Y.
st lim	denotes the limit in the strong operator topology.
M^w	denotes the weak closure of M.
$\mathcal{J}^s$	denotes the strong closure of $\mathcal{J}$.
$\subset, \supset$	These symbols denote inclusion.
$\equiv$	This symbol means " is identically equal to ".

CHAPTER I:

THE STRUCTURE OF GENERAL WELL-BOUNDED OPERATORS:-

1. Preliminaries:-

Let $J = [a,b]$ be a compact interval in the real line R. Let BV(J) be the Banach algebra of complex-valued functions of bounded variation on J with norm $|||\cdot|||$

$$|||f||| = |f(b)| + \operatorname{var}(f,J)$$

where var(f,J) is the total variation of f over J.

Let AC(J) be the Banach subalgebra of absolutely continuous functions on J. For f in AC(J),

$$|||f||| = |f(b)| + \int_a^b |f'(t)|\, dt.$$

Let $AC_0(J)$ and $BV_0(J)$ be the Banach subalgebras of AC(J) and BV(J) consisting of the functions in AC(J) and BV(J) respectively that vanish at b.

Let NBV(J) be the Banach subalgebra of BV(J) consisting of those functions f in BV(J) which are normalized by the requirement that f is continuous on the left on $(a,b]$.

Let $\mathcal{P}(J)$ be the subalgebra of AC(J) consisting of the polynomials on J. $\mathcal{P}(J)$ is dense in AC(J).

Let T be a bounded operator on X. We define p(T) in the natural way for each polynomial p by setting $p(T) = \sum_{n=0}^{K} a_n T^n$ where $p(s) = \sum_{n=0}^{K} a_n s^n$. The map $p \longmapsto p(T)$ is an algebra homomorphism.

We say that T is well-bounded if there is a compact interval J and a constant K such that

$$||p(T)|| \leq K|||p||| \qquad p \in \mathcal{P}(J). \tag{1}$$

If T is well-bounded then so is T* (with the same J and K).

Smart (13) introduced this definition and proved the following fundamental result.

1.1. Lemma:- Let T in L(X) be a well—bounded operator with natural algebra homomorphism $\phi : p \longmapsto p(T)$ from $\mathcal{P}(J)$ into L(X). Let K and J be chosen such that (1) is satisfied. Then ϕ has a unique extension to an algebra homomorphism (also denoted by) $\phi : f \longmapsto f(T)$ from AC(J) into L(X) such that

(i) $||f(T)|| \leqslant K|||f|||$ $\qquad (f \in AC(J))$,

(ii) if S in L(X) satisfies TS = ST, then

$$Sf(T) = f(T)S \qquad (f \in AC(J)),$$

(iii) $f(T^*) = f(T)^*$ $\qquad (f \in AC(J))$.

Proof: If f is absolutely continuous, then the derivative f' of f is in $L^1[a,b]$. There exists a sequence q_n of polynomials such that

$$\int_J |q_n - f'| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Let $\qquad p_n(t) = -\int_t^b q_n(u)\,du + f(b)$.

Obviously, p_n is a polynomial and

$$|||p_n - f||| = \int_J |q_n - f'| \longrightarrow 0 \quad \text{as } n \to \infty .$$

Then $\qquad ||p_n(T) - p_m(T)|| \leqslant K\, |||p_n - p_m||| \longrightarrow 0 \quad$ as $m,n \longrightarrow \infty$,

so that $p_n(T)$ converges in the uniform operator topology to an operator (independent the choice of p_n) which will be called f(T). Thus the extension is unique. Moreover, it is obvious that

$$||f(T)|| \leqslant K\, |||f||| \qquad (\, f \in AC(J)\,).$$

Since ϕ is an algebra homomorphism from $\mathcal{P}(J)$ into L(X), by continuity of ϕ, it is also an algebra homomorphism from AC(J) into L(X). Since (ii) and (iii) are true for polynomials, by continuity of ϕ, they must also be true for absolutely continuous functions. This completes the proof.

Thr notion of a decomposition of the identity was introduced by Ringrose in (11).

1.2. Definition:- A decomposition of the identity for X (on J) is a family $\{E(s) : s \in R\}$ of projections on X^* such that

(i) $E(s) = 0 \quad s < a, \quad E(s) = I \quad s \geqslant b,$

(ii) $E(s)E(t) = E(t)E(s) = E(s) \quad s \leqslant t,$

(iii) there is a constant K such that

$$||E(s)|| \leqslant K \qquad (s \in R),$$

(iv) the function $s \longmapsto \langle x, E(s)x^* \rangle$ is Lebesgue measurable for $x \in X$ and $x^* \in X^*$,

(v) if $x \in X$, $x^* \in X^*, s \in [a,b)$ and if the function $t \longmapsto \int_a^t \langle x, E(u)x^* \rangle \, du$ is right differentiable at s, then the right derivative at s is $\langle x, E(s)x^* \rangle$,

(vi) for each $x \in X$, the map $x^* \longmapsto \langle x, E(\cdot)x^* \rangle$ from X^* into $L^{\infty}(a,b)$ is continuous when X^* and $L^{\infty}(a,b)$ are given their weak*-topologies (as duals of X and $L^1(a,b)$).

1.3. Definition:- An operator T in $L(X)$ is said to be decomposable (on J) if there is a decomposition of the identity for X (on J) such that

$$\langle Tx, x^* \rangle = b \langle x, x^* \rangle - \int_a^b \langle x, E(t)x^* \rangle \, dt \quad (x \in X, \; x^* \in X^*). \quad (2)$$

In this case, we say that the family $\{E(s) : s \in R\}$ is a decomposition of the identity for T.

2. The structure of well-bounded operators (General):-

In this section, we shall show that T is well-bounded on J if and only if T is decomposable on J, and two constants K coincide. Also, we can choose the family $\{E(s) : s \in R\}$ so that

$$S^*E(s) = E(s)S^* \qquad (s \in R)$$

for all $S \in L(X)$ satisfying $ST = TS$. Furthermore the algebra homomorphism of Lemma 1.1 is given by

$$\langle f(T)x, x^* \rangle = f(b) \langle x, x^* \rangle - \int_a^b \langle x, E(t)x^* \rangle f'(t) \, dt \quad (x \in X, x^* \in X^*, f \in AC(J)). (3)$$

We need the following Lemma to prove our first theorem.

2.1. Lemma:- If g, $f_1, f_2, \ldots, f_n$ be any $n+1$ linear functionals on a Banach space X, and if $f_i(x) = 0$ for $i=1,2,\ldots n$, implies $g(x) = 0$, then g is a linear combination of the f_i.

Proof:- Consider the linear mapping $U: X \longrightarrow C^n$, defined by

$$U(x) = [f_1(x), \ldots\ldots, f_n(x)] \qquad (x \in X)$$

On the subspace $U(X)$ of C^n, define the mapping ϕ by

$$\phi(Ux) = \phi[f_1(x), \ldots, f_n(x)] = g(x)$$

The map ϕ is well-defined, since $U(x) = U(y)$ implies that $U(x-y) = 0$, so that, by hypothesis, $g(x) = g(y)$. It is obvious that ϕ is a linear functional on the subspace $U(X)$ of C^n. By Hahn-Banach theorem, it can be extended to a linear functional ϕ' on C^n. Hence ϕ' has the form

$$\phi'[y_1 \ldots, y_n] = \sum_{i=1}^{n} \beta_i y_i \qquad (\beta_i \in C,\ i=1,\ldots,n)$$

Thus
$$g(x) = \sum_{i=1}^{n} \beta_i f_i(x) \qquad (\beta_i \in C, i=1,\ldots,n) .$$

2.2. Theorem:- Let $\{E(t) : t \in R\}$ be a decomposition of the identity for X. Then there is a unique operator T in L(X) which satisfies (2).

Proof: The uniqueness of such an operator is trivial, and it is there-fore sufficient to construct one.

Let $L(x,x^*)$ be the bilinear form on $X \times X^*$ defined by

$$L(x,x^*) = b\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle \, dt \quad (x \in X,\ x^* \in X^*) \quad (4)$$

We may deduce from 1.2.(iii) that

$$| L(x,x^*) | \leqslant \{ |b| + K(b-a)\} \, ||x^*|| \; ||x|| . \qquad (5)$$

We now choose and fix an element x in X and consider $L(x,x^*)$ as a linear functional on X^*. By virtue of condition (vi) of §.1.2.

$$\int_a^b \langle x,E(t)x^*\rangle \, dt$$

is a continuous function of x^* ($\in X^*$) if we consider X^* with its weak*-topology. This is true also of $\langle x,x^*\rangle$. Hence $L(x,x^*)$ is a weak*-continuous linear functional on X^*. There exists a weak*-neighbourhood

$$N(0,x_1,\ldots,x_n,\varepsilon) = \{x^* \in X^* : |\langle x_k,x^*\rangle| < \varepsilon \text{ for } k = 1,\ldots,n\}$$

which is mapped by $L(x,\cdot)$ into the unit sphere of $\mathbb{C}$. For $x^{**} \in X^{**}$, let

$$H_{x^{**}} = \{x^* \in X^* : \langle x^*,x^{**}\rangle = 0\}$$

and suppose that $x_0^* \in \bigcap_{i=1}^n H_{\hat{x}_i}$ where $\hat{x}_i$ is defined by $\langle x^*,\hat{x}_i\rangle = \langle x_i,x^*\rangle$ $(i = 1,\ldots,n)$. Then $x_0^* \in N(0,x_1,\ldots,x_n,\varepsilon)$ and hence $|L(x,x_0^*)| < 1$. Since $\bigcap_{i=1}^n H_{\hat{x}_i}$ is a linear space, it contains mx_0^* for every integer m. Hence $m\,|L(x,x_0^*)| = |L(x,mx_0^*)| < 1$, from which we conclude that $L(x,x_0^*) = 0$. That is, $L(x,x_0^*) = 0$ whenever $\langle x_i,x_0^*\rangle = 0$ for $i = 1,\ldots,n$. It follows from Lemma 1 that

$$L(x,\cdot) = \sum_{i=1}^n \alpha_i\hat{x}_i \qquad \text{for some } \alpha_i \in C \ (i = 1,\ldots,n).$$

Let $y = \sum_{i=1}^n \alpha_i x_i$. Then

$$L(x,x^*) = \langle x^*,\hat{y}\rangle = \langle y,x^*\rangle .$$

Hence for each $x \in X$, there is an element $y = y(x)$ in X such that

$$L(x,x^*) = \langle y,x^*\rangle \qquad (x^* \in X^*). \tag{6}$$

It is obvious that y depends linearly on x and from (5) we deduce that

$$||y|| \leq \{|b| + K(b-a)\}\,||x||.$$

Hence there is an operator T in $L(X)$ such that $Tx = y$ $(x \in X)$. The required results now follows from (4) and (6).

2.3. <u>Theorem</u>:- Let $\{E(t) : t \in R\}$ be a decomposition of the identity for X, and let T be the associated decomposable operator defined by (2). Then

(i) T is well-bounded and satisfies (1),

(ii) if $f \longmapsto f(T)$ is the algebra homomorphism of Lemma 1.1 then

$$\langle f(T)x,x^*\rangle = f(b)\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle f'(t)\,dt \qquad (x \in X,\ x^* \in X^*).$$

Proof: We shall show by induction that

$$\langle T^n x,x^*\rangle = b^n\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\, nt^{n-1}\,dt \qquad (x \in X,\ x^* \in X^*) \tag{7}$$

for all positive integers n.

When $n = 1$, (7) is valid by (2) and if we assume the validity of (7) for a particular integer n then

$$\begin{aligned}\langle T^{n+1}x,x^*\rangle &= \langle T(T^n x),x^*\rangle \\ &= b\langle T^n x,x^*\rangle - \int_a^b \langle T^n x,E(t)x^*\rangle\, dt \\ &= b\Big\{ b^n\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\, nt^{n-1}\, dt\Big\} \\ &\quad - \int_a^b \Big\{ b^n\langle x,E(t)x^*\rangle - \int_a^b \langle x,E(s)E(t)x^*\rangle\, ns^{n-1}\, ds\Big\} dt. \quad (8)\end{aligned}$$

Now,

$$\begin{aligned}&\int_a^b\int_a^b \langle x,E(s)E(t)x^*\rangle\, ns^{n-1}\, ds\, dt \\ &= \int_a^b \Big\{\int_a^t \langle x,E(s)x^*\rangle\, ns^{n-1}\, ds + \int_t^b \langle x,E(t)x^*\rangle\, ns^{n-1}\, ds\Big\}\, dt \\ &= \int_a^b \langle x,E(s)x^*\rangle\, ns^{n-1}(b-s)\,ds + \int_a^b \langle x,E(t)x^*\rangle\,(b^n - t^n)\,dt .\end{aligned}$$

By substituting this value for the last integral in (8), we obtain

$$\langle T^{n+1}x,x^*\rangle = b^{n+1}\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\,(n+1)t^n\, dt.$$

This completes the inductive proof of (7). It follows that for any polynomial p,

$$\langle p(T)x,x^*\rangle = p(b)\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\, p'(t)\, dt . \quad (9)$$

Thus
$$\begin{aligned}|\langle p(T)x,x^*\rangle| &\le ||x^*||\ ||x||\ \Big\{|p(b)| + K\int_a^b |p'(t)|\,dt\Big\} \\ &\le K||x^*||\ ||x||\ \Big\{|p(b)| + \mathrm{var}(p,J)\Big\}\end{aligned}$$

and therefore $||p(T)|| \le K\{|p(b)| + \mathrm{var}(p,J)\}$.

This proves the first part of the theorem.

Let $x \in X$, $x^* \in X^*$ be fixed and define

$$L_1(f) = \langle f(T)x,x^*\rangle \qquad (f \in AC(J)),$$
$$L_2(f) = f(b)\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle f'(t)\, dt \qquad (f \in AC(J)),$$

and by virtue of (9),

$$L_1(p) = L_2(p)$$

for every polynomial p. From the argument in the proof of Lemma 1.1, we know that $\mathcal{P}(J)$ forms a dense subset of $AC(J)$. Hence $L_1(f) = L_2(f)$ $(f \in AC(J))$ as required.

2.4. Theorem:- Under the hypotheses of Theorem 2.3, define

$$M_s = E(s)X^*, \quad N_s = \bigcap_{t<s} (I - E(t))X^*$$

and let L_s (respectively R_s) be the class of all functions in $AC(J)$ such that $f(t) = 0$ when $t \leqslant s$ (respectively $t \geqslant s$). Then, if $f \longmapsto f(T^*)$ is the algebra homomorphism of Lemma 1.1,

(i) $M_s = \{ x^* \in X^* : f(T^*)x^* = 0,\ f \in L_s \}$,

(ii) $N_s = \{ x^* \in X^* : f(T^*)x^* = 0,\ f \in R_s \}$,

(iii) $M_s \cap N_s = \{ x^* \in X^* : T^*x^* = sx^* \}$,

(iv) $M_s = \bigcap_{t>s} M_t$,

(v) the subspaces M_s, N_s are invariant under T^* and $\sigma(T^*/M_s) \subset [a,s]$, $\sigma(T^*/N_s) \subset [s,b]$ $\quad (a \leqslant s \leqslant b)$.

Proof: (i) The result is obvious if $s \notin [a,b)$, since

$$L_s = \begin{cases} (0) & (s \geqslant b) \\ AC(J) & (s < a) \end{cases} .$$

We may therefore suppose that $s \in [a,b)$. Let $x^* \in X^*$, then

$$\begin{aligned}
& x^* \in M_s \\
\Longleftrightarrow \quad & E(t)x^* = x^* \qquad (s \leqslant t \leqslant b) \\
\Longleftrightarrow \quad & \langle x, E(t)x^* \rangle = \langle x, x^* \rangle \qquad (s \leqslant t \leqslant b,\ x \in X) \\
\Longleftrightarrow \quad & \langle x, E(t)x^* \rangle = \langle x, x^* \rangle \text{ for almost all } t \in [s,b] \text{ and } x \in X. \text{ (By §1.2(v))} \\
\Longleftrightarrow \quad & \langle x, x^* \rangle \int_a^b f'(t)\,dt - \int_a^b \langle x, E(t)x^* \rangle f'(t)\,dt = 0 \qquad (f \in L_s,\ x \in X) \\
\Longleftrightarrow \quad & f(b)\langle x, x^* \rangle - \int_a^b \langle x, E(t)x^* \rangle f'(t)\,dt = 0 \qquad (f \in L_s,\ x \in X) \\
\Longleftrightarrow \quad & \langle x, f(T^*)x^* \rangle = 0 \qquad (f \in L_s,\ x \in X) \\
\Longleftrightarrow \quad & f(T^*)x^* = 0 \qquad (f \in L_s).
\end{aligned}$$

This proves (i). The proof of (ii) is similar, and is omitted.

(iv) If $s \in [a,b)$, the result follows from the fact that $x^* \in M_s$ is equivalent to: for $x \in X$, $\langle x, E(t)x^* \rangle = \langle x, x^* \rangle$, for almost all t in $[s,b]$. Since M_s is constant on each of the complementary intervals of $[a,b)$, the result is trivial, when $s \notin [a,b)$.

(v): Since L_s and R_s are ideals in the Banach algebra $AC(J)$, it follows easily from (i) and (ii) that M_s and N_s are invariant under $f(T^*)$ $(f \in AC(J))$ and in particular, under T^*.

Suppose that $r \notin [a,s]$ and that g in $AC(J)$ is such that

$$(t-r)g(t) = 1 \qquad (a \leqslant t \leqslant s) .$$

Then the function $(t-r)g(t)-1 = g(t)(t-r) - 1$ is in L_s and by (i)

$$(T^*- rI)g(T^*)x^* - x^* = g(T^*)(T^* - rI)x^* - x^* = 0 \qquad (x^* \in M_s).$$

Thus T^*-rI, as an operator acting on M_s, has the inverse $g(T^*)$. It follows that $r \notin \sigma(T^*/ M_s)$. Hence $\sigma(T^*/M_s) \subset [a,s]$. The proof of the second inclusion is similar.

(iii) We first note that, by taking $s=b$ in (v) we have $\sigma(T^*) \subset [a,b]$. From this, it is obvious that

$$M_s \cap N_s = \{ 0 \} = \{ x^* \in X^* : T^*x^* = sx^* \}$$

when $s \notin [a,b]$. We may therefore suppose that $s \in [a,b]$. We claimed that

$$M_s \cap N_s = \{x^* \in X^* : f(T^*)x^* = f(s)x^* \ (f \in AC(J))\} . \qquad (10)$$

In fact, if $f(T^*)x^* = f(s)x^* \quad (f \in AC(J))$, then $f(T^*)x^* = 0$ when $f \in L_s \cup R_s$ so $x^* \in M_s \cap N_s$.

Conversely, if $x^* \in M_s \cap N_s$ and $f \in AC(J)$, then the function g defined by

$$g(t) = f(t) - f(s)$$

can be expressed as the sum of some g_1 in L_s and g_2 in R_s and hence

$$\begin{aligned} f(T^*)x^* - f(s)x^* &= g(T^*)x^* \\ &= g_1(T^*)x^* - g_2(T^*)x^* \\ &= 0 \end{aligned}$$

This proves (10).

By taking $f(t) \equiv t$, we deduce that $T^*x^* = sx^*$ whenever $x^* \in M_s \cap N_s$.

On the other hand, if $T^*x^* = sx^*$, then

$$f(T^*)x^* = f(s)x^*$$

whenever f is a polynomial, and hence (by the continuity of the homomorphism $f \longmapsto f(T^*)$ in Lemma 1.1) whenever f is in $AC(J)$.

Corollary 1. $\sigma(T) \subset [a,b]$.

Proof: We have already seen that in proving (iii) that $\sigma(T^*) \subset [a,b]$. Hence the result follows immediately.

Corollary 2: If T is a decomposable operator then the subspaces M_s, N_s depend only on T and not on the choice of the decomposition of the identity $\{E(t) : t \in R\}$.

Proof: The result follows immediately from Theorem 4(i) and (ii).

Now, let T be a well-bounded operator on X. Following Ringrose (11) we shall construct a decomposition of the identity $\{E(t) : t \in R\}$ for X such that (2) is satisfied. First, we need the following lemmas.

2.5. Lemma: Let L be a bounded linear functional on $AC(J)$. Then there exist a constant m_L and a function w_L in $L^\infty(a,b)$ such that

$$L(f) = m_L f(b) - \int_a^b w_L(t)\, df(t) \qquad (f \in AC(J)) .$$

Proof: Since $AC(J)$ is a closed subspace of $BV(J)$, by the Hahn-Banach theorem, L can be extended to a linear functional (also denoted by) L on $BV(J)$. Let

$$w_L(t) = L(\chi_{(a,t]}) \qquad a < t \leq b$$
$$= 0 \qquad t=a$$
$$m_L = w_L(b).$$

For every f in $AC(J)$ and $\varepsilon > 0$, there is a step function

$$f_\varepsilon = \sum_{i=1}^{n} a_i \chi_{(u_{i-1},u_i]}$$

where $a = u_0 < u_1 < \dots < u_n = b$, such that

$$|||f - f_\varepsilon||| < \varepsilon \quad \text{and}$$

$$L(f_\varepsilon) = \sum_{i=1}^{n} \alpha_i \, L(\chi_{(u_{i-1}, u_i]})$$
$$= \sum_{i=1}^{n} \alpha_i \, (w(u_i) - w(u_{i-1}))$$
$$= w_L(b)\alpha_n - \sum_{i=1}^{n} w(u_i)\, (\alpha_i - \alpha_{i-1})$$
$$= w_L(b) f_\varepsilon(b) - \int_a^b w_L(t)\, df(t) \qquad (f \in AC(J)).$$

Moreover, $\displaystyle\int_a^b w(t)^n dt \leq \int_a^b |L(\chi_{(a,t]})|^n dt$

$$\leq |L|^n (b-a).$$

If we let E_m be the set where $|w_L(t)| \geq m$ and μ Lebesgue measure on R, this shows that

$$m^n \mu(E_m) \leq |L|^n (b-a)$$

i.e. $\quad \mu(E_m) / (b-a) \leq (|L| / m)^n .$

Letting $n \to \infty$, we find that $\mu(E_m) = 0$ if $m > |L|$.

Thus $\quad ||w_L||_\infty \leq |L| \quad$ i.e. $\quad w_L \in L^\infty(a,b)$.

Obviously, $\quad m_L \leq |L|$.

Hence $\quad \max \{ m_L, ||w_L||_\infty \} \leq |L|$.

On the other hand, let g in $AC_0(J)$, then

$$|L(g)| = | \int_a^b w_L(t) g'(t) dt|$$
$$\leq ||w_L||_\infty |||g|||.$$

For every $f \in AC(J)$, $f(x) = f(b) + (f(x) - f(b))$

Let $g(x) = f(x) - f(b)$, so $\quad f(x) = f(b) + g(x)$.

$$|L(f)| = |L(g)| \leq ||w_L||_\infty |||g||| \leq ||w_L||_\infty |||f|||$$
$$|L| \leq ||w_L||_\infty .$$

Hence $\quad |L| = \max \{ m_L, ||w_L||_\infty \}$.

2.6. <u>Lemma</u>:- Given any $x \in X$, $x^* \in X^*$, there exists a function w_{x,x^*} in $L^\infty(a,b)$, uniquely determined to within a null function, such that

$$\langle x, f(T^*)x^* \rangle = f(b) \langle x, x^* \rangle - \int_a^b w_{x,x^*}(t)\, df(t) \qquad (11)$$

where $f \longmapsto f(T^*)$ is the algebra homomorphism in Lemma 1.1 . The function w_{x,x^*} satisfies

$$||w_{x,x^*}||_\infty \leq K\ ||x^*||\ ||x||$$

and its equivalence class depends linearly on both x and x^*.

Proof: For fixed $x \in X$ and $x^* \in X^*$, define

$$L_{x,x^*}(f) = \langle x, f(T^*)x^* \rangle \qquad (f \in AC(J)).$$

It is obvious from Lemma 1.1 that L_{x,x^*} is a continuous linear functional on AC(J) and that

$$||\ L_{x,x^*}|| \leq K\ ||x^*||\ ||x|| \ .$$

Hence there exist a constant m_{x,x^*} and a function w_{x,x^*} in $L^\infty(a,b)$ such that

$$L_{x,x^*}(f) = m_{x,x^*}\ f(b) - \int_a^b w_{x,x^*}(t)\ dt \qquad (\ f \in AC(J)\ )$$

$$\max\ (\ |m_{x,x^*}|,\ ||w_{x,x^*}||_\infty) \leq K\ ||x^*||\ ||x|| \ .$$

By considering the function $f(t) \equiv 1$ we obtain, $m_{x,x^*} = \langle x, x^* \rangle$. Hence

$$\langle x, f(T^*)x^* \rangle = f(b)\ \langle x, x^* \rangle - \int_a^b w_{x,x^*}(t)\ df(t) \qquad (f \in AC(J)\ ).$$

Furthermore, the integral in this equation is uniquely determined and depends linearly on both x and x^*, for any f in AC(J). Hence the equivalence class w_{x,x^*} is likewise uniquely determined and a linear function of both x and x^*.

Now, let $NBV_0(J)$ be the subalgebra of NBV(J) consisting of those functions f in NBV(J) whose continuous singular parts vanish identically. i.e. f can be written in form $f = f_{ac} + f_b$ where $f_{ac} \in AC(J)$ and f_b is a pure break function(a uniformly convergent sum of step functions). In fact,

$$f_b(t) = \sum_{a \leq s < t} (\ f(s+) - f(s)\ ).$$

We shall attempt to use (11) as a method of defining an operator $f(T^*)$ for a general f in $NBV_0(J)$. If we choose s with $a \leq s \leq b$, then $\chi_{(-\infty,s]} \in NBV_0(J)$. Formal substitution in (11) yeilds

$$\langle x, \chi_{(-\infty,s]}(T^*)x^* \rangle = w_{x,x^*}(s).$$

Hence in order that a linear operator $\chi_{(-\infty,s]}(T^*)$ can be defined in this way for each $s \in R$, it is necessary that the functions w_{x,x^*} themselves (not merely their equivalence classes) shall depend linearly on both $x \in X$ and

$x^* \in X^*$. Therefore w_{x,x^*} has to be restricted. We are thus led to the problem of selecting representatives from the equivalence classes of L^∞ functions in such a way that a linear relation between equivalence classes implies the corresponding relation between the functions representing these classes.

This problem was solved by J. Von Neumann with perfectly general algebraic relations. However, Ringrose gives below a less sophisticated construction of a set of representatives which has the properties that we require. In the following lemmas, we use the symbol ' $\overset{\circ}{=}$ ' to denote equality almost everywhere on $[a,b)$ and ' $\equiv$ ' for equality everywhere on $[a,b)$.

Let $\mathcal{F}$ be the filter on $(0,\infty)$ generated by $\{(0,\frac{1}{n}] : n=1,2,\ldots\ldots\}$ and let $\mathcal{G}$ be $\mathcal{F}$ as filter base on $\beta(0,\infty)$ where $\beta(0,\infty)$ is the Čech compactification of $(0,\infty)$.

2.7. Lemma:- Let $\mathcal{U}$ be ultrafilter on $\beta(0,\infty)$ containing $\mathcal{G}$. Let w be any function which is essentially bounded and Lebesgue measurable on the interval $[a,b)$. Then for every s in $[a,b)$

$$w_{\mathcal{U}}(s) = \lim_{h \to 0, \mathcal{U}} \int_0^1 w(s+ht)dt$$

exists. Furthermore, if $v,w,z \in L^\infty(a,b)$, then

(i): $w_{\mathcal{U}} \overset{\circ}{=} w$ and if $v \overset{\circ}{=} w$, then $v_{\mathcal{U}} \equiv w_{\mathcal{U}}$.

(ii): $w_{\mathcal{U}}$ is bounded and Lebesgue measurable on $[a,b)$ and

$$\sup_{s \in [a,b)} |w_{\mathcal{U}}(s)| = ||w||_\infty .$$

(iii): if w is continuous on the right throughout $[a,b)$, then $w \equiv w_{\mathcal{U}}$.

(iv): if c, d are constants and $cw + dv \overset{\circ}{=} z$, then $cw_{\mathcal{U}} + dv_{\mathcal{U}} \equiv z_{\mathcal{U}}$.

(v): $z \overset{\circ}{=} wv$ and $v_{\mathcal{U}}$ is continuous on the right throughout $[a,b)$, then $z_{\mathcal{U}} \equiv w_{\mathcal{U}} v_{\mathcal{U}}$.

Proof: Since $\beta(0,\infty)$ is compact, the ultrafilter $\mathcal{U}$ converges to some point α of $\beta(0,\infty)$.

We may define $w(s) = 0$ when $s \notin [a,b)$. For fixed w and s, the function

$$r(h) = \int_0^1 w(s+ht)dt$$

is continuous on $(0,\infty)$ and

$$\begin{aligned} |\, r(h) \,| &= \left| \int_0^1 w(s+ht)\, dt \right| \\ &\leq \int_0^1 |\, w(s+ht) |\, dt \\ &\leq ||\, w \,||_\infty . \end{aligned} \tag{12}$$

Since r is bounded, there is a unique continuous function r_o on $\beta(0,\infty)$ whose restriction to $(0,\infty)$ is r. Thus

$$w_{\mathcal{U}}(s) = \lim_{\substack{h\to 0\\ \mathcal{U}}} r(h) = \lim_{h\to\alpha} r_o(h) = r_o(\alpha).$$

This completes the proof of the first part of the lemma.

(i) The result is trivial.

(ii) From (12) we have

$$\sup{}_{s\in[a,b)} w_{\mathcal{U}}(s) \leq ||w||_\infty .$$

so $w_{\mathcal{U}}$ is a bounded and Lebesgue measurable on $[a,b)$.

Let $G = \{ t \in [a,b) : w_{\mathcal{U}}(t) = w(t) \}$. By (i), $w_{\mathcal{U}} \overset{o}{=} w$ and $[a,b) \setminus G$ is of Lebesgue measure zero. Hence

$$||\, w \,||_\infty \leq \sup{}_{t\in G} |\, w(t) \,| = \sup{}_{t\in G} |w_{\mathcal{U}}(t)| \leq \sup{}_{t\in[a,b)} |w_{\mathcal{U}}(t)|.$$

Thus $\qquad \sup_{t\in[a,b)} |\, w_{\mathcal{U}}(t) \,| = ||w||_\infty$.

(iii) Suppose w is continuous on the right throughout $[a,b)$. For every $s \in [a,b)$ $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\, w(s+ht) - w(s) \,| < \varepsilon \qquad \text{with} \quad |\, h \,| < \delta \;\; , \; t \in [0,1] .$$

Hence $w_{\mathcal{U}}(s) = w(s)$, for all $s \in [a,b)$.

(iv) Since

$$\begin{aligned} (cw + dv\,)_{\mathcal{U}} &= \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 (cw + dv\,)(s+ht)\, dt \\ &= \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 cw(s+ht)\, dt + \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 dv(s+ht)\, dt \\ &= cw_{\mathcal{U}} + dv_{\mathcal{U}} . \end{aligned}$$

and by (i), $\qquad (cw + dv\,)_{\mathcal{U}} \equiv z_{\mathcal{U}}$.

Hence $\qquad cw_{\mathcal{U}} + dv_{\mathcal{U}} \equiv z_{\mathcal{U}}$.

(v) The proof is similar to (iv) and is omitted.

We shall refer to $w_{\mathcal{U}}$ as the $\mathcal{U}$-representative of the equivalence class containing w.

Now, we shall extend the algebra homomorphism $f \mapsto f(T)$ of Lemma 1.1 to a homomorphism of the algebra $NBV_o(J)$ into $L(X^*)$ such that for all x in X and x* in X*,

$$\langle x, f_{\mathcal{U}}(T^*)x^* \rangle = f(b) \langle x, x^* \rangle - \int_{[a,b)} w_{x,x^*}(t)\, df(t) \quad (f \in NBV_o(J)) \qquad (13)$$

where w_{x,x^*} are the functions of Lemma 6, which are selected by taking in each case the $\mathcal{U}$-representative of the relevant equivalence class.

2.8. Lemma:- (i) Suppose that w_{x,x^*} are the functions mentioned in the last paragraph. Then, given any $f \in NBV_o(J)$, there is a unique operator $f_{\mathcal{U}}(T^*)$ in $L(X^*)$ such that (13) is satisfied.

(ii) For any f in AC(J), $f_{\mathcal{U}}(T^*) = f(T^*)$.

Proof: (i) Since w_{x,x^*} is the $\mathcal{U}$-representative of an equivalence class which depends linearly on both x and x*, we may deduce from Lemma 7(iv) that the function w_{x,x^*} itself has the same property. Furthermore,

$$\sup_{t \in [a,b)} |w_{x,x^*}(t)| = \| w_{x,x^*} \|_\infty \leq K \|x^*\| \, \|x\|.$$

For every f in $NBV_o(J)$, f can be written in the form $f = f_{ac} + f_b$ where f_{ac} is in AC(J) and $f_b(t) = \sum_{a \leq s < t} (f(s+) - f(s))$. Hence the integral

$$\int_{[a,b)} w_{x,x^*}(t)\, df(t)$$

is well-defined and

$$\int_{[a,b)} w_{x,x^*}(t)\, df(t) = \int_{[a,b)} w_{x,x^*}(t)\, df_{ac}(t) + \int_{[a,b)} w_{x,x^*}(t)\, df_b(t)$$
$$= \int_{[a,b)} w_{x,x^*}(t) f'_{ac}(t)\, dt + \sum_{a \leq t < b} w_{x,x^*}(t) (f(t+) - f(t)).$$

Thus the equation

$$L(x,x^*) = f(b) \langle x, x^* \rangle - \int_{[a,b)} w_{x,x^*}(t)\, df(t)$$

defines a bilinear form L on $X \times X^*$. Also

$$| L(x,x^*) | \leq \|x^*\| \, \|x\| \left\{ |f(b)| + K \operatorname{var}(f,J) \right\}$$
$$\leq K \|x^*\| \, \|x\| \, |||f|||.$$

It is easily follows from this fact that, given any x^* in X^*, the equation

$$\langle x, f_{\mathcal{U}}(T^*)x^* \rangle = L(x,x^*)$$

defines uniquely a linear functional $f_{\mathcal{U}}(T^*)x^*$ on X and that the operator $f_{\mathcal{U}}(T^*)$ has all the required properties.

(ii): When $f \in AC(J)$, we may deduce from (11) and (13) that $f_{\mathcal{U}}(T^*) = f(T^*)$.

2.9. Lemma: Suppose that $S \in L(X)$ and $ST = TS$. Then

$$S^*f_{\mathcal{U}}(T^*) = f_{\mathcal{U}}(T^*)S^* \qquad (f \in NBV_0(J)).$$

Proof: For any $x \in X$ and $x^* \in X^*$,

$$\begin{aligned}\langle x, S^*f_{\mathcal{U}}(T^*)x^* \rangle &= \langle Sx, f_{\mathcal{U}}(T^*)x^* \rangle \\ &= f(b)\langle Sx, x^* \rangle - \int_{[a,b)} w_{Sx,x^*}(t)\, df(t)\end{aligned}$$

and

$$\langle x, f_{\mathcal{U}}(T^*)S^*x^* \rangle = f(b)\langle x, S^*x^* \rangle - \int_{[a,b)} w_{x,S^*x^*}(t)\, df(t).$$

It follows that $f_{\mathcal{U}}(T^*)S^* = S^*f_{\mathcal{U}}(T^*)$ if and only if

$$\int_{[a,b)} \left\{ w_{Sx,x^*}(t) - w_{x,S^*x^*}(t) \right\} df(t) = 0 \qquad (x \in X,\ x^* \in X^*). \quad (14).$$

By virtue of Lemma 1.1(i) and Lemma 7(ii), (14) is satisfied whenever $f \in AC(J)$. It follows that

$$w_{Sx,x^*} \overset{\circ}{=} w_{x,S^*x^*}.$$

Thus these two functions are $\mathcal{U}$ - representatives of the same equivalence class and are therefore identically equal. Hence (14) is satisfied for every $f \in NBV_0(J)$, as required.

2.10. Theorem. The mapping $f \longmapsto f_{\mathcal{U}}(T^*)$ is an algebra homomorphism from $NBV_0(J)$ into $L(X^*)$.

Proof: It is immediate from (13) that this mapping is linear. It remains to establish the multiplicative property.

Let $f, g \in NBV_0(J)$, $x \in X$, and $x^* \in X^*$. In the computations that follow, we shall omit the suffix $\mathcal{U}$. By virtue of Lemma (ii), this causes no inconsistency. We have

$$\langle x, f(T^*)g(T^*)x^*\rangle = f(b)\langle x, g(T^*)x^*\rangle - \int_{[a,b)} w_{x,g(T^*)x^*}(t)\,df(t)$$
$$= f(b)g(b)\langle x,x^*\rangle - f(b)\int_{[a,b)} w_{x,x^*}(t)\,dg(t)$$
$$- \int_{[a,b)} w_{x,g(T^*)x^*}(t)\,df(t)$$

while

$$\langle x, fg(T^*)x^*\rangle = fg(b)\langle x,x^*\rangle - \int_{[a,b)} w_{x,x^*}(t)\,dfg(t)$$
$$= f(b)g(b)\langle x,x^*\rangle - \int_{[a,b)} w_{x,x^*}(t)f(t+)\,dg(t)$$
$$- \int_{[a,b)} w_{x,x^*}(t)g(t+)\,df(t).$$

So
$$\langle x, f(T^*)g(T^*)x^*\rangle - \langle x, fg(T^*)x^*\rangle$$
$$= \int_{[a,b)} \left\{ g(t+)w_{x,x^*}(t) - w_{x,g(T^*)x^*}(t) \right\} df(t) - f(b)\int_{[a,b)} w_{x,x^*}(t)\,dg(t)$$
$$+ \int_{[a,b)} f(t)\,d\Big(\int_{[a,t]} w_{x,x^*}(s)\,dg(s) \Big)$$
$$= \int_{[a,b)} \left\{ g(t+)w_{x,x^*}(t) - w_{x,g(T^*)x^*}(t) \right\} df(t)$$
$$- \int_{[a,b)} \int_{[a,t]} w_{x,x^*}(s)\,dg(s)\,df(t)$$
$$= \int_{[a,b)} \left\{ g(t+)w_{x,x^*}(t) - w_{x,g(T^*)x^*}(t) - \int_{[a,t]} w_{x,x^*}(s)\,dg(s) \right\} df(t). \quad (15)$$

By virtue of Lemma 11.1 the left-hand side of (115) is zero whenever $f,\ g \in AC(J)$. By fixing g and varying f, we deduce that

$$g(t+)w_{x,x^*}(t) - w_{x,g(T^*)x^*}(t) - \int_{[a,t)} w_{x,x^*}(s)\,dg(s) \doteq 0 \quad (x \in X,\ x^* \in X^*) \quad (116)$$

whenever $g \in AC(J)$. Since the functions

$$g(t+), \quad \int_{[a,t]} w_{x,x^*}(s)\,dg(s)$$

are both continuous from the right on $[a,b)$, we may deduce from Lemma 7 (iii) (v) that each of the three terms in (16) is the $\mathcal{U}$-representative of its equivalence class. Hence, by Lemma 7(iv), the left-hand side of (16) vanishes identically whenever $g \in AC(J)$. We may therefore deduce from (15) that

$$f(T^*)g(T^*) = fg(T^*) \qquad (f \in NBV_0(J),\ g \in AC(J)\ ). \qquad (117)$$

However, when $g \in AC(J)$, $g(T)$ commutes with T and so (Lemma 9) $g(T^*)$ commutes with $f(T^*)$ for every $f \in NBV_0(J)$. Since, further, $fg(t) = gf(t)$, we may rewrite (117) in the form

$$g(T^*)f(T^*) = gf(T^*) \qquad (f \in NBV_0(J),\ g \in AC(J))$$

which is equivalent to

$$f(T^*)g(T^*) = fg(T^*) \qquad (f \in AC(J),\ g \in NBV_0(J)\ ).$$

We may now deduce from (15) by varying f in $AC(J)$ that (16) holds whenever $g \in NBV_0(J)$. The same argument as before shows that the left-hand side of (16) vanishes identically $(g \in NBV_0(J))$ and hence that

$$f(T^*)g(T^*) = fg(T^*) \qquad (\ f,g \in NBV_0(J)\ )\ .$$

2.11. Lemma:- Let $E(s) = \chi_{(-\infty,s]}(T^*)$ and for each s in R and $h>0$, let $k_{s,h}(t)$ be an absolutely continuous function defined as follows:

$$k_{s,h}(t) = \begin{cases} 1 & t \leqslant s \\ 1 + (s-t)/h & s \leqslant t \leqslant s+h \\ 0 & s+h \leqslant t \end{cases}$$

Then in the weak operator topology on $L(X^*)$,

$$E(s) = \lim_{\substack{h\to 0 \\ \mathcal{U}}} k_{s,h}(T^*) \tag{18}$$

Proof: If $x \in X$ and $x^* \in X^*$, we have

$$\begin{aligned}\langle x, E(s)x^*\rangle &= \chi_{(-\infty,s]}(b)\,\langle x,x^*\rangle - \int_{[a,b)} w_{x,x^*}(t)\, d\,\chi_{(-\infty,s]}(t) \\ &= \begin{cases} 0 & s < a \\ w_{x,x^*}(s) & a \leqslant s < b \\ \langle x,x^*\rangle & s \geqslant b\ . \end{cases}\end{aligned}$$

Since w_{x,x^*} is the $\mathcal{U}$-representative of its equivalence class, it follows that, when $a \leqslant s < b$,

$$\begin{aligned}\langle x,E(s)x^*\rangle &= w_{x,x^*}(s) \\ &= \lim_{\substack{h\to 0 \\ \mathcal{U}}} \int_0^1 w_{x,x^*}(s+ht)\, dt \\ &= \lim_{\substack{h\to 0 \\ \mathcal{U}}} \Big\{ k_{s,h}(b)\,\langle x,x^*\rangle - \int_{[a,s)} w_{x,x^*}(t)\, d(k_{s,h}(t)) \\ &\qquad - \int_{[s,s+h)} w_{x,x^*}(t)\, d(k_{s,h}(t)) - \int_{[s+h,b)} w_{x,x^*}(t)\, d(k_{s,h}(t))\Big\} \\ &= \lim_{\substack{h\to 0 \\ \mathcal{U}}} \Big\{ k_{s,h}(b)\,\langle x,x^*\rangle - \int_{[a,b)} w_{x,x^*}(t)\, d(k_{s,h}(t))\Big\}.\end{aligned}$$

Thus

$$\langle x,E(s)x^*\rangle = \lim_{\substack{h\to 0 \\ \mathcal{U}}} \langle x, k_{s,h}(T^*)x^*\rangle \qquad (\ x \in X,\ x^* \in X^*\ ).$$

2.12. Lemma:- The operators $\{E(s) : s \in R\}$ introduced in Lemma 11 form a decomposition of the identity for X. The associated decomposable operator is T.

Proof: Since $\chi^2_{(-\infty,s]}(t) = \chi_{(-\infty,s]}(t)$ it follows that $E(s)^2 = E(s)$ and so E(s) is a projection in X*. We now have to verify that the conditions (i)....., (vi) in § 1.2 are satisfied. Now, (i) and (iv) follow from (18) and (ii) is an immediate consequence of the corresponding relations for the functions $\chi_{(-\infty,t]}$ and $\chi_{(-\infty,s]}$. Since

$$\|E(t)\| \leq K \,|||\chi_{(-\infty,t]}||| \leq K,$$

(iii) is satisfied. Since w_{x,x^*} is the $\mathcal{U}$-representative of its equivalence class we may deduce from (18) that

$$\langle x,E(s)x^*\rangle = \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 \langle x,E(s+ht)x^*\rangle\, dt \qquad (x\in X,\ x^*\in X^*)$$

from which property (v) follows at once.

It remains to prove (vi). From (10),(18) and Lemma 1.1 (iii), we deduce that

$$\langle f(T)x,x^*\rangle = f(b)\,\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle f'(t)\, dt \quad (f\in AC(J), x\in X, x^*\in X^*). \quad (19)$$

Given u in $L^1(a,b)$, set

$$f_u(s) = \int_s^b u(t)\, dt.$$

Then $f_u \in AC(J)$. For any fixed $x\in X$, the mapping A of $L^1(a,b)$ into X defined by

$$Au = f_u(T)x$$

is clearly a continuous linear operator, and by using (19), we obtain

$$\langle u,A^*x^*\rangle = \langle Au,x^*\rangle = \langle f_u(T)x,x^*\rangle = \int_a^b \langle x,E(t)x^*\rangle\, u(t)\, dt .$$

Thus the mapping considered in condition (vi) of § 1.2 is A*, and is the adjoint of a continuous linear operator A and so it has the required property.

This completes the proof that $\{E(s) : s\in R\}$ is a decomposition of the identity for X. By taking $f(t) \equiv t$ in (19), it follows that the associated decomposable operator is T.

Now, we can summarize the main results of this section in the following theorem. Part(iii) is a consequence of Lemma 9.

2. 13. Theorem:- Let T be a well-bounded operator on X. Then,

(i) T is decomposable ,

(ii) If $\mathcal{U}$ is the ultrafilter mentioned in Lemma 7, then the operators $\{E(t) : t \in R\}$ defined by (18) are projections and form a decomposition of the identity for X, whose associated decomposable operator is T.

(iii) The decomposition of the identity $\{E(t) : t \in R\}$ constructed as in (ii) has the following property: if $S \in L(X)$ and $ST = TS$, then

$$S^*E(t) = E(t)S^* \qquad (t \in R).$$

Combining the results in Theorem 3 and 13, we get the following result.

2.14. Theorem: An operator T is well-bounded if and only if T is decomposable.

3. The uniqueness problem:-

Given a decomposable operator T, we may ask whether or not the associated decomposition of the identity is unique. When this the case, we shall say that T is uniquely decomposable. We shall give an example to show that ,in general, T is not uniquely decomposable.

We now introduce a number of definitions and notation which will be of fundamental importance throughout this section .

3.1. Definition:- Let $u \in L^1(a,b)$. We shall say that u is C-limitable on the right at a point s of $[a,b)$ if the indefinite integral of u is differentiable on the right at s. This property is not affected if the values of u are altered on a null set, and depends only on the equivalence class of u. It is therefore meaningful to refer to this definition when u is known only to within a null function.

3.2. Definition:- A function $u \in L^1(a,b)$ is C-continuous on the right at a point s of $[a,b)$ if it is C-limitable on the right at s and the derivative of the indefinite integral of u at the point s is equal to u(s).

3.3. Theorem:- Let $T \in L(X)$ be well-bounded and satisfy $(\mathbb{1})$. Then T is uniquely decomposable if and only if for every $x \in X$ and $x^* \in X^*$, the function w_{x,x^*} of Lemma 2.6 is C-limitable on the right throughout $[a,b)$.

Proof: The decomposition of the identity $\{E(t) : t \in R\}$ as constructed in § 2 satisfies

$$E(t) = \begin{cases} 0 & (t<a) \\ \mathbb{1} & (t \geq b). \end{cases} \tag{20}$$

It follows from Theorem 2.4 corollary 2, that all such decompositions satisfy (20).

Suppose now that $\{E(t) : t \in R\}$ is one of these decompositions. From Theorem 2.3 (ii) and Lemmas 1.1 (iii), 2.6, we may deduce that

$$\int_a^b \langle x, E(t)x^* \rangle f'(t)\, dt = \int_a^b w_{x,x^*}(t) f'(t)\, dt \qquad (f \in AC(J))$$

and hence that

$$x, E(t)x^* = w_{x,x^*}(t)$$

for almost all t in $[a,b]$. If we assume that each of the functions w_{x,x^*} is C-limitable on the right throughout $[a,b)$, then the same is true of the functions $\langle x, E(t)x^* \rangle$. Property (v) in § 1.2 now implies that

$$\begin{aligned} \langle x, E(s)x^* \rangle &= \lim_{h \to 0+} \int_0^1 \langle x, E(s+ht)x^* \rangle\, dt \\ &= \lim_{h \to 0+} \int_0^1 w_{x,x^*}(s+ht)\, dt \qquad (x \in X,\ x^* \in X^*,\ a \leq s < b). \end{aligned}$$

We deduce that $E(s)$ is uniquely determined when $s \in [a,b)$ and we have already proved this to be the case when $s \notin [a,b)$.

Conversely, suppose on the contrary that there exist $x \in X$, $x^* \in X^*$ and $s \in [a,b)$ such that w_{x,x^*} is not C-limitable on the right at s, then we may choose ultrafilters $\mathcal{U}_1$, $\mathcal{U}_2$ on $(0,\infty)$ such that

$$\lim_{\substack{h \to 0 \\ \mathcal{U}_1}} \int_0^1 w_{x,x^*}(s+ht)\, dt \neq \lim_{\substack{h \to 0 \\ \mathcal{U}_2}} \int_0^1 w_{x,x^*}(s+ht)\, dt.$$

If $\{E(t)^{(i)} : t \in R\}$ is the decomposition of the identity obtained by taking $\mathcal{U} = \mathcal{U}_i$ ($i=1,2.$) throughout § 2, then

$$\langle x, E(s)^{(i)} x^* \rangle = \lim_{\substack{h\to 0 \\ \mathcal{U}_i}} \int_0^1 w_{x,x^*}(s+ht)\, dt \quad (i=1,2).$$

Hence $E(s)^{(1)} \neq E(s)^{(2)}$ and there exist two distinct decompositions of the identity which give rise to the operator T.

Corollary: Suppose that $T \in L(X)$ is decomposable but not uniquely. Then there exist two distinct associated decomposition of the identity, both having property (iii) of Theorem 2.13 .

Proof: The two decompositions constructed in the proof of the last theorem have this property.

3.4. Example: Let X be the complex Banach space $L^\infty[0,1] \oplus L^1[0,1]$ with the norm defined as follows:

$$\| \{x,y\} \| = \|x\|_\infty + \int_0^1 |y(t)|\, dt \qquad (x \in L^\infty[0,1] ,\ y \in L^1[0,1]).$$

Define operators S,N and T on X by

$$S : \{x(t),y(t)\} \longrightarrow \{tx(t),ty(t)\} \qquad (t \in [0,1]),$$

$$N : \{x(t),y(t)\} \longrightarrow \{0,\ x(t)\} \qquad (t \in [0,1]),$$

$$T = S + N .$$

We claim that T is well-bounded. In fact, if p is a complex polynomial, a routine calculation shows that

$$p(T): \{x(t),y(t)\} \longrightarrow \{p(t)x(t),\ p'(t)x(t)+p(t)y(t)\} \qquad (t \in [0,1])$$

and

$$\begin{aligned} \| \{px,\ p'x+py\} \| &= \|px\|_\infty + \int_0^1 |p'x + py| \\ &\leq \Big[\sup_{t\in[0,1]} |p(t)|\Big] \Big[\|x\|_\infty\Big] + \int_0^1 |p'x| + \int_0^1 |py| \\ &\leq \Big[\sup_{t\in[0,1]} |p(t)|\Big] \Big[\|x\|_\infty\Big] + \Big[\|x\|_\infty\Big]\Big[\int_0^1 |p'|\Big] \\ &\quad + \Big[\sup_{t\in[0,1]} |p(t)|\Big]\Big[\int_0^1 |g|\Big] \\ &\leq \Big(\sup_{t\in[0,1]} |p(t)| + \mathrm{var}(p, [0,1])\Big) \Big(\|\{x,y\}\| \Big) \\ &\leq (|p(1)| + 2\,\mathrm{var} (p, [0,1])) (\|\{x,y\}\|) \end{aligned}$$

Hence

$$\begin{aligned} \|p(T)\| &\leq 2 (|p(1)| + \mathrm{var} (p, [0,1])) \\ &= 2\, |||p|||. \end{aligned}$$

Thus T is a well-bounded operator. It is easily seen that (by its uniqueness) that the homomorphism of Lemma 1.1 is determined by the equation

$$f(T):\ \{x(t),y(t)\} \longmapsto \{f(t)x(t), f'(t)x(t) + f(t)y(t)\} \quad (t\in[0,1], f\in AC([0,1]))$$

For every $x^*\oplus y^* \in L^1[0,1] \oplus L^\infty[0,1] \subseteq X^*$ and $f\in AC([0,1])$, we have

$$\begin{aligned}
&\langle f(T)\{x,y\},\ x^*\oplus y^*\rangle \\
&= \langle \{fx, f'x + fy\},\ x^*\oplus y^*\rangle \\
&= \langle fx, x^*\rangle + \langle f'x + fy, y^*\rangle \\
&= \int_0^1 fxx^* + \int_0^1 f'xy^* + \int_0^1 fyy^* \\
&= f(1)\int_0^1 x(t)x^*(t)\,dt - \int_0^1 \Big(\int_0^t x(u)x^*(u)\,du\Big)\,df(t) + \int_0^1 x(t)y(t)f'(t)\,dt \\
&\qquad + f(1)\int_0^1 y(t)y^*(t)\,dt - \int_0^1 \Big(\int_0^t y(u)y^*(u)\,du\Big)\,df(t) \\
&= f(1)\langle \{x,y\},\ x^*\oplus y^*\rangle - \int_0^1 \Big\{\int_0^t (x(u)x^*(u)+y(u)y^*(u))\,du - x(t)y^*(t)\Big\} f'(t)\,dt
\end{aligned}$$

Hence the function $w = w_{\{x,y\},x^*\oplus y^*}$ of Lemma 2.6 are given by

$$w(t) = \int_0^t (\ x(u)x^*(u) + y(u)y^*(u)\ )\,du - x(t)y^*(t)\ .$$

If we let $x(t) \equiv 1$, $y(t) \equiv 1$, $x^*(t) \equiv t$, and

$$y^*(t) = \begin{cases} 2 + \sin(\log|t-\tfrac{1}{2}|) + \cos(\log|t-\tfrac{1}{2}|) & t \neq \tfrac{1}{2} \\ 0 & t = \tfrac{1}{2} \end{cases}$$

then, it is easily to verify that $y^*(t)$ is not C-limitable on the right at $t = \frac{1}{2}$. It follows that, for such $\{x,y\}$ and $x^*\oplus y^*$, $w_{\{x,y\},\ x^*\oplus y^*}$ is not C-limitable on the right throughout $[0,1)$. We deduce from Theorem 3 that T is not uniquely decomposable.

To end this chapter, we note that the homomorphism of Lemma 1.1 can be extended to a homomorphism from NBV(J) into L(X*), whenever T is uniquely decomposable. Since, in this case, by Theorem 3, the $\mathcal{U}$-representative of w_{x,x^*} is the limit of the sequence of continuous functions whose nth member is

$$w_n(s) = n\int_0^{\frac{1}{n}} w_{x,x^*}(s+t)\,dt$$

and hence is a Borel measurable function.

CHAPTER II:

SOME SPECIAL CLASSES OF WELL-BOUNDED OPERATORS:-

In this chapter and the following chapter, we shall deal with three subclasses of well-bounded operators. These are well-bounded operators decomposable in X and well-bounded operators of type (A) and type (B).

Let T be a well-bounded operator on X. T is called decomposable in X if and only if there is a family $\{F(t) : t \in R\}$ of projections on X such that $\{F^*(t) : t \in R\}$ forms a decomposition of the identity for T. We shall prove in §.1 that if T is decomposable in X then it is uniquely decomposable. T is said to be of type (A) if and only if it is decomposable in X and its unique decomposition of the identity $\{F^*(t) : t \in R\}$ satisfies $\lim_{t \to s+} F(t)x = F(s)x$ for all $s \in R$ and x in X. T is said to be of type (B) if and only if it is of type (A) and its unique decomposition of identity $\{F(t)^* : t \in R\}$ satisfies the condition:

For each s in R, as $t \to s-$, F(t) converges in the strong operator topology to an operator, henceforth denoted by F(s-).

The first notion was introduced by Ringrose (11) and the other two were introduced by Berkson and Dowson (4).

1. Properties of well-bounded operators decomposable in X:-

The following results are due to Ringrose (11).

1.1. Theorem:- Suppose that T in L(X) is decomposable in X. Let $\{F(t) : t \in R\}$ be a family of projections in X whose adjoints $\{F(t)^* : t \in R\}$ form a decomposition of the identity for T and satisfy (2). Let $s \in [a,b]$. Then

(i) $\quad F(s)f(T) = f(T)F(s) \qquad (f \in AC(J))$

where $f \mapsto f(T)$ is the algebra homomorphism of Lemma I.1.1.

(ii): $\sigma(\,T/\,F(s)(X)\,) \subseteq [a,s]$,

$\sigma(\,T/(I-F(s))(X)\,) \subseteq [s,b]$.

(iii) T is uniquely decomposable.

(iv) if $S \in L(X)$ and $ST = TS$, then

$$SF(s) = F(s)S.$$

(v) Given any x in X and x^* in X^*, the function $\langle F(t)x,x^*\rangle$ is everywhere C-continuous on the right.

Proof: (i). Since $\langle x,F^*(s)x^*\rangle = \langle F(s)x,x^*\rangle$ the equation in Theorem I.3.2 may be rewritten in the form

$$\langle f(T)x,x^*\rangle = f(b)\langle x,x^*\rangle - \int_a^b \langle F(t)x,x^*\rangle\, df(t) \quad (x\in X,\ x^*\in X^*). \quad (21)$$

By using this and the relation $F(t)F(s) = F(s)F(t)$, we obtain

$$\begin{aligned}\langle F(s)f(T)x,x^*\rangle &= \langle f(T)x,F^*(s)x^*\rangle \\ &= f(b)\langle x,F^*(s)x^*\rangle - \int_a^b \langle F(t)x,F^*(s)x^*\rangle\, df(t) \\ &= f(b)\langle F(s)x,x^*\rangle - \int_a^b \langle F(t)F(s)x,x^*\rangle\, df(t) \\ &= \langle f(T)F(s)x,x^*\rangle \qquad (\,x\in X,\ x^*\in X^*\,).\end{aligned}$$

Hence $\quad F(s)f(T) = f(T)F(s) \qquad (\,f\in AC(J)\,).$

(ii): Suppose that $r \notin [s,b]$. Then, there exists g in AC(J) such that

$$(t-r)g(t) = 1 \qquad (\,s \leq t \leq b\,).$$

Then the function

$$(t-r)g(t) - 1 = g(t)\,(t-r) - 1 .$$

is in L_s (as defined in Theorem I.2.4) and Theorem I.2.4 (i) implies that

$$(T^*-rI)g(T^*)x^* - x^* = 0 = g(T^*)(T^*-rI)x^* - x^*$$

for all $x^* \in (\,I-F^*(s)\,)(X^*)$.

i.e. $\quad (T^*-rI)g(T^*)x^* = x^* = g(T^*)(T^*-rI)x^* \qquad (x^*\in(I-F^*(s))(X^*)).$

From this and by use of (i) and Lemma I.1.1 we obtain

$$(T-rI)g(T)(I-F(s)) = I-F(s) = g(T)(T-rI)(I-F(s)).$$

Hence, if all the operators are resticted to the subspace $(I-F(s))(X)$, g(T) is the inverse of T-rI. Thus $r \notin \sigma(\,T/(I-F(s))(X)\,)$ and

$$\sigma(\,T/(I-F(s))(X)\,) \subseteq [s,b]\ .$$

The proof that $\sigma(\,T/F(s)(X)\,) \subseteq [a,s]$ is similar.

(iii): Suppose that $\{E(t) : t \in R\}$ is any decomposition of the identity which gives rise to the operator T and has the property (iii) of Theorem I.2.13. Since (by(i)) F(t) commutes with T, it follows that $F^*(t)$ commutes with E(t). Hence E(t) and $F^*(t)$ are commuting projections with the same range space (Theorem I.2.4, Corollary 2) and are therefore equal. Thus $\{F^*(t) : t \in R\}$ is the only decomposition of the identity of this type and from the corollary to Theorem I.3.3 we deduce that T is uniquely decomposable.

(iv): By Theorem I.2.13(ii) there exists a decomposition of the identity $\{E(t) : t \in R\}$ such that the property (iii) of Theorem I.2.13 is satisfied. From (iii) we have $E(t) = F^*(t)$; hence it follows that $SF(t) = F(t)S$ whenever TS = ST.

(v) Since T is uniquely decomposable, the functions w_{x,x^*} are C-limitable on the right throughout $[a,b)$. Hence the function $\langle F(t)x,x^*\rangle$, which is the $\mathcal{U}$-representative of the equivalence class containing w_{x,x^*}, is C-continuous on the right. It is constant, and hence (trivially) C-continuous on the right on each of the complement intervals of $[a,b)$.

1.2. <u>Theorem</u>:- Suppose that T is a well-bounded operator on a weakly complete Banach space X. Then T is uniquely decomposable if and only if T is decomposable in X.

Proof: The implication in one direction has already been established in Theorem 1(iii).

Suppose that T is uniquely decomposable. Then the functions w_{x,x^*} are C-limitable on the right throughout $[a,b)$. Hence by Lemma I.2.11,

$$\begin{aligned}\langle x,E(s)x^*\rangle &= \lim_{h\to 0+} \langle x,\ k_{s,h}(T^*)x^*\rangle \\ &= \lim_{h\to 0+} \langle k_{s,h}(T)x,x^*\rangle \qquad (\ x\in X,\ x^*\in X^*\ ).\end{aligned}$$

Since X is weakly complete, we deduce that as $h \to 0+$ through any sequence, $k_{s,h}(T)x$ conveges in weak topology of X. For each x in X, define $F(s)x = \lim_{h \to 0+} k_{s,h}(T)x$. It is clear that F(s) is a projection on X and $F^*(s) = E(s)$. Hence T is decomposable in X.

I.3. Definition:- If X is a Banach space and $V \subset X$, the set

$$V^{\perp} = \{ x^* : x^* \in X^*, \langle y, x^* \rangle = 0 \text{ for all } y \in V \}$$

is called the annihilator ~~or othrogonal complement~~ of V.

I.4. Theorem:- Let T be a decomposable operator on a Banach space X and suppose that $\sigma_p(T^*) = \phi$. Then T is decomposable in X.

Proof: Let

$$V_s = \overline{sp} \{ f(T)x : f \in R_s, x \in X \}$$
$$W_s = \overline{sp} \{ f(T)x : f \in L_s, x \in X \}$$

where R_s and L_s are defined as in Theorem I.2.4. Then by Theorem I.2.4, (i), (ii) and Lemma I.2.1 (ii), we have $V_s^{\perp} = N_s$ and $W_s^{\perp} = M_s$.

Now,

$$E(s)X^* = M_s,$$
$$(I-E(s))(X^*) \subseteq N_s,$$
$$E(s)(X^*) + (I-E(s))(X^*) = X^*,$$
$$M_s \cap N_s = (0);$$

the last equation holds because $\sigma_p(T^*) = \phi$; we deduce that

$$N_s = (I-E(s))(X^*) \text{ and that}$$
$$M_s + N_s = X^*.$$

Hence E(s) is the projection from X^* onto M_s parallel to N_s.

It is clear that

$$(V_s + W_s)^{\perp} = V_s^{\perp} \cap W_s^{\perp} = M_s \cap N_s = (0).$$

On the other hand, for every $x^* \in V_s^{\perp} + W_s^{\perp}$, there exists $y^* \in V_s^{\perp}$ and $z^* \in W_s^{\perp}$ such that $x^* = y^* + z^*$ and then

$$\langle x, x^* \rangle = \langle x, y^* \rangle + \langle x, z^* \rangle = 0 \qquad (x \in V_s \cap W_s)$$

Hence $x^* \in (V_s \cap W_s)^{\perp}$.

i.e. $(V_s \cap W_s)^{\perp} \supseteq V_s^{\perp} + W_s^{\perp} = M_s + N_s = X^*$

Thus $(V_s \cap W_s)^{\perp} = X^*$.

It follows that $V_s \cap W_s = (0)$ and $V_s + W_s = X$.

Let $F(s)$ be the projection from X onto V_s parallel to W_s. It is obvious that $F^*(s) = E(s)$. Since the above construction of $F(s)$ can be carried out for any $s \in [a,b)$ and we let $F(s) = 0$ for all $s<a$ and $F(s) = I$ for all $s \geq b$, we deduce that T is decomposable in X.

We shall give an example in the last section to show that there is a uniquely decomposable operator which is not decomposable in X.

2. <u>The structure of well-bounded operators of type (A):-</u>

In this section, we shall show that when T is well-bounded of type (A), the algebra homomorphism from $AC(J)$ into $L(X)$ of Lemma I.1.1 can be extended to a homomorphism of the Banach algebra $NBV(J)$ into $L(X)$ such that it can be expressed in terms of a Riemann-Stieltjes integral. First, we shall give a sufficient condition for the existence of Riemann-Stieltjes integral.

We say that a sequence $u = (u_k : 0 \leq k \leq m)$ is a partition of J, if $a = u_0 < u_1 < \ldots\ldots < u_m = b$. We note that u can mean $(I_1, \ldots, I_m)$ where $I_k = [u_{k-1}, u_k]$ $(1 \leq k \leq m)$. We write $u \geq u'$ (u is a refinement of u') if and only if each closed interval I_k $(1 \leq k \leq m)$ is contained in some $I'_j = [u'_{j-1}, u'_j]$ $(1 \leq j \leq n)$. The family U_J of all partitions of J is directed by the relation $\geq$. We shall denote by $u + u'$ the totality of dividing points in both u and u' arranged in linear order.

Let f be a complex-valued function on J and g be a function on J taking values in a Banach space X. When $u \in U_J$, we define

$$\sum_u g\Delta f = \sum_1^m g(v_k)(f(u_k) - f(u_{k-1})) \quad \text{with} \quad u_{k-1} \leq v_k \leq u_k .$$

Now, let I be any subinterval of J, i.e. $I = [c,d]$ with $a \leq c < d \leq b$.

We define

$$w(g,I) = \text{l.u.b}\left\{ ||g(t_1) - g(t_2)|| : t_1, t_2 \in I \right\}$$

which is called the oscillation of g on I and

$$w(Sg\Delta f,I) = \text{l.u.b}\left\{ ||\sum_u g\Delta f - \sum_{u'} g\Delta f|| : u, u' \in U_J \right\}.$$

We say that g is Riemann-Stieltjes integrable with respect to f, if $\lim_{U_J} \sum_u g\Delta f$ exists as a net limit in the strong topology of X and define

$$R - S\int_a^b g\, df = \lim_{U_J} \sum_u g\Delta f.$$

2.1. Lemma:- If g is a bounded function on J and f is of bounded variation on J, for $I = [c,d]$, let $f(I) = f(d) - f(c)$, then

$$w(Sg\Delta f,I) \leq w(g,I)\,\text{var}(f,I).$$

Proof:- Take any partitions $u' = (I_1', \ldots, I_m')$ and $u'' = (I_1'', \ldots, I_n'')$ of I and set $u = u' + u''$; then,

$$\begin{aligned} &|\textstyle\sum_{i=1}^{m} g(v_i)f(I_i') - \sum_{j=1}^{n} g(v_j'')f(I_j'')| \\ = \; &|\textstyle\sum_{i,j} g(v_j')f(I_{ij}) - \sum_{i,j} g(v_j'')f(I_{ij})| \\ \leq \; &w(g,I)\textstyle\sum_{i,j}|f(I_{ij})| \\ \leq \; &w(g,I)\,\text{var}(f,I). \end{aligned}$$

Hence $w(Sg\Delta f,I) \leq w(g,I)\,\text{var}(f,I)$.

2.2. Theorem:- If f is a continuous function of bounded variation on J and g is a bounded function, then sufficient condition that $\int_a^b g\, df$ exist is that the set D of all discontinuities of g be countable.

Proof:- Since D is a countable set, we can let $D = \{t_n : n = 1,2,\ldots\}$. For each n and $\varepsilon > 0$ there exists an open interval J_n with $t_n \in J_n^{\circ}$ (interior of J_n) such that $\text{var}(f,J_n) < \frac{\varepsilon}{2^n}$ and $J_i \cap J_j = \phi$ whenever $i \neq j$. So there exists a sequence $\{J_n\}$ of disjoint open intervals such that

$$\sum_n \text{var}(f,J_n) < \varepsilon .$$

Since J_n are open intervals, $G = \bigcup_{n=1}^{\infty} J_n$ is an open set. Let F be the closed set complementary to G relative to J. i.e. $F = J \setminus G$. Then g is continuous for each points of F, and consequently uniformly continuous, so that there exists

$\delta > 0$ such that

$$\| g(t) - g(s) \| < \varepsilon \qquad \text{with} \qquad | t-s | < \delta$$

and $\quad w(g, [s-\delta , s+ \delta]) < 2\varepsilon$.

Consider now any partition u such that $\|u\| < \delta$, where $\|u\| = \max \left\{ |u_k - u_{k-1}| : k = 1,\ldots,m \right\}$. Let $I_1',\ldots,I_r'$ be the closed interval of u containing at least one point F and $I_1'',\ldots,I_s''$ be the complementary intervals none of which contains a point of F. Then

$$\sum_{i=1}^{s} \operatorname{var}(f,I_i'') \leqslant \sum_{n=1}^{\infty} \operatorname{var}(f,I_n) < \varepsilon$$

and so

$$\sum_{i=1}^{s} w(g,I_i'')\operatorname{var}(f,I_i'') < 2M\varepsilon$$

where $M = \sup_J \|g(t)\|$.

On the other hand,

$$\begin{aligned} &\sum_{j=1}^{r} w(g,I_j')\operatorname{var}(f,I_j') \\ &\leqslant 2\varepsilon \sum_{j=1}^{r} \operatorname{var}(f,I_j') \\ &\leqslant 2\varepsilon \operatorname{var}(f,J) . \end{aligned}$$

Consequently, $\qquad \sum_u w(g,I)\operatorname{var}(f,I) < 2\varepsilon(M + \operatorname{var}(f,J))$

for any subdivision such that $\|u\| < \delta$.

Moreover, if u' be any partition such that $u' \geqslant u$, then, by rearrangement of terms so as to bring together the terms in each subinterval of u, we find that

$$\begin{aligned} &\Big\| \sum_{i=1}^{m} g(v_i)(f(u_i) - f(u_{i-1})) - \sum_{j=1}^{n} g(v_j')(f(u_j') - f(u_{j-1}')) \Big\| \\ \leqslant \quad & \sum_{i=1}^{m} w(Sg\Delta f, I_i) \\ \leqslant \quad & \sum_{i=1}^{m} w(g,I_i)\operatorname{var}(f,I_i) \qquad \text{(by Lemma 11)} \\ < \quad & 2\varepsilon (M + \operatorname{var}(f,J)) \end{aligned}$$

Finally, for any two partitions u',u" such that $u' \geqslant u$, $u'' \geqslant u$,

$$\begin{aligned} &\Big\| \sum_{u'} g\Delta f - \sum_{u''} g\Delta f \Big\| \\ &\leqslant \Big\| \sum_{u'} g\Delta f - \sum_{u} g\Delta f \Big\| + \Big\| \sum_{u} g\Delta f - \sum_{u''} g\Delta f \Big\| \\ &< 4\varepsilon (M + \operatorname{var}(f,J)) \end{aligned}$$

Hence the integral $\int_a^b g\, df$ exists.

2.3. Theorem:- (Integration by parts) If $\int_a^b g\, df$ exists , then $\int_a^b f\, dg$ exists and $\int_a^b f\, dg = g(b)f(b) - g(a)f(a) - \int_a^b g\, df$.

Proof: If $\int_a^b g\, df$ exists then given $\mathcal{E} > 0$, there exists a $u_\mathcal{E} \in U_J$ such that

$$|| \int_a^b g\, df - \sum_u g\Delta f || < \mathcal{E} \quad \text{for all } u \geq u_\mathcal{E} .$$

Select a $u \geq u_\mathcal{E}$. Then

$$|| \sum_{i=1}^n f(v_i)(g(u_i) - g(u_{i-1})) - g(b)f(b) - g(a)f(a) + \int_a^b g\, df ||$$

$$= || \sum_{i=0}^n g(u_i)(f(v_{i+1}) - f(v_i) - \int_a^b g\, df ||$$

$$= || \sum_{i=0}^n \{ g(u_i)(f(v_{i+1}) - f(v_i)) + g(u_i)(f(u_i) - f(v_i)) \} - \int_a^b g\, df || < \mathcal{E} .$$

Since $v = (a = u_0 \leq v_1 \leq u_1 \leq v_2 \leq \ldots\ldots \leq v_{n+1} = u_n = b)$ includes u and so $v \geq u_\mathcal{E}$. Consequently $\int_a^b f\, dg$ exists and is equal to

$$f(b)g(b) - f(a)g(a) - \int_a^b g\, df.$$

2.4. Lemma:- Let h be a right continuous function on R with values in a metric space (M, ρ). Then h has only a countable number of discontinuities.

Proof:- For each discontinuity t of h, define

$$d(t) = \overline{\lim}_{t_1,t_2 \to t} \rho(h(t_1),h(t_2)).$$

Let $S_n = \{ t \in R : d(t) > \frac{1}{n} \}$ and let s be any point in S_n. Since h is continuous on the right, there exist $\mathcal{E}_n > 0$ such that

$$d(t) \leq \frac{1}{n} \qquad (t \in (s+ \mathcal{E}_n , s+2\mathcal{E}_n))$$

i.e. $\quad S_n \cap (s+ \mathcal{E}_n , s+2 \mathcal{E}_n) = \phi$.

Choose a rational number in ($s+ \mathcal{E}_n , s+2 \mathcal{E}_n$). This maps S_n one-to-one into the set of all rational numbers. Hence S_n is countable. Thus the set of all discontinuities of h, being $\bigcup_{n=1}^{\infty} S_n$, is countable.

We note that the discontinuities of a function of bounded variation are all of the first kind, i.e. f(t+) and f(t-) exist for each t, and are at most countable in number.

Let f be in NBV(J) and let $\{ t_n : n = 1,2,\ldots. \}$ be the set of all discontinuities of f. Then,

$$\sum_{n=1}^{\infty} | f(t_n+) - f(t_n) | \leq var(f,J)$$

and it follows that the series

$$\sum_{n=1}^{\infty} (f(t_n+) - f(t_n)) \chi_{(t_n,b]}(t)$$

is absolutely and uniformly convergent on $[a,b]$.

So let $f_b(t) = \sum_{n=1}^{\infty}(f(t_n+) - f(t_n)) \chi_{(t_n,b]}(t)$ and

$$f_c(t) = f(t) - f_b(t).$$

Then f_c is a continuous function of boundedvvariation on $[a,b]$. Hence f can be rewritten in the form $f = f_c + f_b$, where f_c is a continuous function on $[a,b]$ and f_b is a uniformly convergent sum of step functions. This notion will be need in the proof of our main result.

Now, we can prove our main result in this section.

2.5. Theorem:- Let T be a well-bounded operator of type (A) on X and let K be chosen so that

$$|| p(T) || \leq K |||p|||$$

Then, the homomorphism of Lemma I.11.1 can be extended to a homomorphism Ψ of the Banach algebra NBV(J) into L(X) such that

$$\langle \Psi(f)x,x^* \rangle = f(b) \langle x,x^* \rangle - \int_a^b \langle F(t)x,x^* \rangle \; df(t) \quad (x \in X, x^* \in X^*, f \in NBV(J))$$

where $\{ F^*(t) : t \in R \}$ is the unique decomposition of the identity for T. Furthurmore,

$$||\Psi(f) || \leq K |||f||| \qquad (f \in NBV(J))$$

If S in L(X) and ST = TS, then $S\Psi(f) = \Psi(f)S$ for all f in NBV(J).

Proof: Since T is well-bounded of type (A) for each $x \in X$, the function $t \longmapsto F(t)x$ is continuous on the right in the strong topology and by Lemma 4 its set of discontinuities is countable ; moreover

$$|| E(t)x || \leq K ||x|| .$$

It follows from Theorem 2 that

$$R\text{-}S \int_a^b F(t)x \;\; df_c(t) \qquad \text{exists.}$$

For each n, the function $f_{t_n}(t) = (f(t_n+) - f(t_n))\chi_{(t_n,b]}(t)$ has a discontinuity at t_n. If $u \in U_J$ included t_n (i.e. $\exists$ k s.t. $u_k = t_n$), then

$$\sum_u F(\cdot)x\,\Delta f_{t_n} = (f(t_n+) - f(t_n))F(t')x \quad \text{with } u_k = t_n \leq t' \leq u_{k+1}.$$

Hence

$$\begin{aligned}\text{R-S}\int_a^b F(t)x\,df_{t_n} &= \lim_{U_J}\sum_u F(\cdot)x\,\Delta f_{t_n}\\ &= \lim_{t'\to t_n+}(f(t_n+) - f(t_n))F(t')x\\ &= (f(t_n+) - f(t_n))F(t_n)x.\end{aligned}$$

It follows that

$$\text{R-S}\int_a^b F(t)x\,df_b(t) = \sum_{n=1}^{\infty}(f(t_n+) - f(t_n))F(t_n)x.$$

Thus $\quad \text{R-S}\int_a^b F(t)x\,df(t) \quad$ exists.

Now we define $\Psi: NBV(J) \longrightarrow L(X)$ by

$$\Psi(f)x = f(b)x - \text{R-S}\int_a^b F(t)x\,df(t) \qquad (f \in NBV(J),\ x \in X). \tag{22}$$

Then for each $x^* \in X^*, x \in X$ and $f \in NBV(J)$

$$\sum_u \langle F(\cdot)x, x^*\rangle\,\Delta f = \Big\langle \sum_u F(\cdot)x\,\Delta f, x^*\Big\rangle$$

$$\begin{aligned}\lim_{U_J}\sum_u \langle F(\cdot)x, x^*\rangle\,\Delta f &= \Big\langle \lim_{U_J}\sum_u F(\cdot)x\,\Delta f, x^*\Big\rangle\\ &= \Big\langle \text{R-S}\int_a^b F(t)x\,df(t), x^*\Big\rangle.\end{aligned}$$

Hence $\quad \text{R-S}\int_a^b \langle F(t)x, x^*\rangle\,df(t)$ exists and

$$\text{R-S}\int_a^b \langle F(t)x, x^*\rangle\,df(t) = \Big\langle \text{R-S}\int_a^b F(t)x\,df(t), x^*\Big\rangle.$$

It follows that

$$\begin{aligned}\langle \Psi(f)x, x^*\rangle &= \langle f(b)x, x^*\rangle - \Big\langle \text{R-S}\int_a^b F(t)x\,df(t), x^*\Big\rangle\\ &= f(b)\langle x, x^*\rangle - \text{R-S}\int_a^b \langle F(t)x, x^*\rangle\,df(t).\end{aligned}$$

Moreover, from (22)

$$\begin{aligned}\|\Psi(f)x\| &\leq |f(b)|\,\|x\| + \Big\|\text{R-S}\int_a^b F(t)x\,df(t)\Big\|\\ &\leq (|f(b)| + K\,\mathrm{var}(f,J))\|x\|.\end{aligned}$$

It follows that

$$\|\Psi(f)\| \leq K\,|||f||| \qquad (f \in NBV(J)).$$

Hence the map Ψ is linear and bounded; also

$$\Psi(\, t \longmapsto 1\,)\, x = x \qquad (\, x \in X\,)$$

i.e. $$\Psi(\, t \longmapsto 1\,) = I$$

and $$\langle \Psi(\, t \longmapsto t\,)x, x^* \rangle = b\,\langle x, x^* \rangle - \int_a^b \langle F(t)x, x^* \rangle\, dt$$
$$= \langle Tx, x^* \rangle \qquad (\, x \in X,\ x^* \in X^*\,).$$

Hence $\Psi(\, t \mapsto t\,) = T$.

Moreover, from (22) and Theorem 3,

$$\Psi(f)x = f(a)F(a) + \text{R-S}\int_a^b f(t)\, dF(t)x. \tag{23}$$

Since $F(s)F(t) = F(t)F(s) = F(s)$ when $s \leq t$, we have

$$\Big\{f(a)F(a) + \textstyle\sum_{k=1}^{m} f(v_k)(F(u_k) - F(u_{k-1}))\Big\}\Big\{g(a)F(a) + \textstyle\sum_{k=1}^{m} g(v_k)(F(u_k) - F(u_{k-1}))\Big\}$$
$$= fg(a)F(a) - \textstyle\sum_{k=1}^{m} f(v_k)g(v_k)(F(u_k) - F(u_{k-1}))\Big\} \quad (\, f,g \in NBV(J),\ u \in U_J).$$

Hence $\Psi(f)\,\Psi(g) = \Psi(fg)$.

Thus Ψ is an algebra homomorphism. By Lemma I.1.1

$$\Psi(f) = f(T) \qquad (f \in AC(J))$$

where $f \longmapsto f(T)$ is the algebra homomorphism mentioned in Lemma I.1.1 .

Finally, by Theorem 1.1 , $SF(t) = F(t)S$ $(t \in R)$ and so by the definition of $\Psi(f)$, we have

$$S\Psi(f) = \Psi(f)S \qquad (\, f \in NBV(J)\,).$$

3. <u>Examples:-</u>

3.1. Example:- Let $X = C[0,1]$ and define T in $L(X)$ by

$$(Tx)(t) = t\, x(t)\,, \qquad (\, 0 \leq t \leq 1\,).$$

Obviously, $$(T^n x)(t) = t^n\, x(t) \qquad (\, n = 0,1,2,\ldots\ldots).$$

Hence, for any complex polynomial p,

$$(p(T)x)(t) = p(t)x(t)\ .$$

It follows easily that

$$\|p(T)\| \leq \sup_{t \in [0,1]} |p(t)| \leq |||p|||.$$

Thus T is a well-bounded operator. It is clear that the homomorphism of

Lemma I.1.1 is determined by the equation

$$(f(T)x)(t) = f(t)x(t) \qquad (x \in X,\ f \in AC([0,1])).$$

We shall now make the usual identification of X^* with the space of Radon measures on $[0,1]$. When $y \in X^*$, $x \in X$, $f \in AC(0,1)$, we have

$$\begin{aligned}\langle f(T)x,y \rangle &= \int_{[0,1]} f(t)x(t)\ dy(t) \\ &= \int_{[0,1]} f(t)\ \ d\int_{[0,t]} x(t)\ dy(t) \\ &= f(1)\int_{[0,1]} x(t)\ dy(t) - \int_{[0,1]}\int_{[0,t]} x(u)\ dy(u)\ f'(t)dt.\end{aligned}$$

Hence the functions $w_{x,y}$ in Lemma I.2.6 are given by

$$w_{x,y}(t) = \int_{[0,t]} x(u)\ dy(u). \qquad (24)$$

It is clear that $w_{x,y}$ is continuous (hence C-limitable) on the right throughout $[0,1)$. Hence T is uniquely decomposable. From (24) we deduce that

$$\int_{[0,1]} x(u)\ d(E(t)y)(u) = \langle x,E(t)y\rangle = w_{x,y}(t) = \int_{[0,t]} x(u)\ dy(u)\ .$$

Thus $E(t)y$ is the restriction of the measure y to the interval $[0,t]$.

Now, suppose that on the contrary that there is an operator $F(t)$ on X whose adjoint is $E(t)$. Then

$$\begin{aligned}\int_{[0,1]} (F(t)x)(u)\ dy(u) &= \langle F(t)x,y\rangle = \langle x,E(t)y\rangle \\ &= \int_{[0,t]} x(u)\ dy(u) \\ &= \int_{[0,t]} \chi_{[0,t]}(u)x(u)\ dy(u)\ .\end{aligned}$$

Hence $\quad (F(t)x)(u) = \chi_{[0,t]}(u)x(u).$

However, if we take $t=\frac{1}{2}$ and $x(u) = 1$, then

$$\chi_{[0,t]}x \notin X\ .$$

This is a contradiction. Thus, there is no operator on X whose adjoint is $E(t)$. Hence T is uniquely decomposable, but not decomposable in X.

3.2. <u>Example:</u>- Let $X = L^{\infty}[0,1]$. Define T in $L(X)$ by

$$(Tx)(t) = t\ x(t) \qquad (x \in L^{\infty}[0,1],\ t \in [0,1])$$

Then, if p is any complex polynomial,

$$(p(T)x)(t) = p(t)x(t).$$

It follows that

$$||p(T)x|| = \int_0^1 |p(t)x(t)|\, dt \leqslant \sup_{t \in [0,1]} |p(t)| \int_0^1 |x(t)|\, dt$$

$$||p(T)|| \leqslant \sup_{t \in [0,1]} |p(t)| \leqslant |||p|||.$$

Hence T is well-bounded operator. Moreover, the homomorphism in Lemma I.1.1 is determined by

$$(f(T)x)(t) = f(t)x(t) \qquad (f \in AC([0,1]),\ x \in X).$$

Now, X^* is isometrically isomorphic with the Banach space ba$\{[0,1], \mathcal{L}, \mu\}$ consisting of all finitely additive complex-valued set functions λ, defined on $\mathcal{L}$, the σ-algebra of Lebesgue measurable sets, which vanish on sets of Lebesgue measure zero, and which have finite total variation on $[0,1]$ with respect to $\mathcal{L}$, the norm of λ being given by

$$||\lambda|| = \mathrm{var}_{\mathcal{L}}(\lambda, [0,1]).$$

The correspondence is given by

$$\langle x, x^* \rangle = \int_{[0,1]} x(t)\, \lambda(dt) \qquad (x^* \in X^*, \quad \lambda \in \mathrm{ba}\{[0,1], \mathcal{L}, \mu\}).$$

(See, for example, Theorem IV.8.15 of (6))

Let $\{E(t) : t \in R\}$ be the decomposition of the identity for T constructed in Chapter I § 2. Then, by Lemma I.2.11,

$$\begin{aligned}
\langle x, E(s)\lambda \rangle &= \lim_{\substack{h \to 0 \\ \mathcal{U}}} \langle x,\ k_{s,h}(T^*)\lambda \rangle \\
&= \lim_{\substack{h \to 0 \\ \mathcal{U}}} \langle k_{s,h}(T)x,\ \lambda \rangle \\
&= \lim_{\substack{h \to 0 \\ \mathcal{U}}} \int_{[0,1]} k_{s,h}(t)x(t)\ \lambda(dt) \\
&= \int_{[0,s]} x(t)\lambda(dt) + \lim_{\substack{h \to 0 \\ \mathcal{U}}} \int_{[s,s+h)} (1 + (s-t)/h)x(t)\, \lambda(dt).
\end{aligned}$$

It is easily seen that

$$\lim_{\substack{h \to 0 \\ \mathcal{U}}} \int_{[s,s+h)} (1 + (s-t)/h)x(t)\lambda(dt) = 0.$$

Hence

$$\langle x, E(s)\lambda \rangle = \int_{[0,s]} x(t)\, \lambda(dt).$$

Thus $\qquad (E(s)\lambda)(\delta) = \lambda(\delta \cap [0,s]) \qquad (\lambda \in \mathrm{ba}\{[0,1], \mathcal{L}, \mu\}, \delta \in \mathcal{L}).$

For each real s define a projection $F(s)$ on X by

$$(F(s)x)(t) = \chi_{(-\infty,s]}(t)x(t) \qquad (x \in X,\ t \in [0,1]).$$

It is obvious that $F^*(s) = E(s)$. Hence the well-bounded operator T is decomposable in X. However, let $x(t) \equiv 1$,

$$\begin{aligned} \| F(s)x - F(s')x \|_\infty &= \| \chi_{[0,s]}(t) - \chi_{[0,s']}(t) \|_\infty \\ &= \| \chi_{(s',s]}(t) \|_\infty \\ &= 1 \qquad (s,s' \in [0,1],\ s > s'). \end{aligned}$$

Hence the strong operator limits $\lim_{s \to u+} F(s)$ and $\lim_{s \to u-} F(s)$ fail to exist at any point u of $(0,1)$. Hence T is not a well-bounded operator of type (A).

3.3. Example:- Let X be a Banach space of all convergent sequences $x = \{x_n\}$ of complex numbers under the norm

$$\| x \| = \sup_n |x_n| .$$

The pairing of X* with ℓ^1 given by

$$\langle x,y \rangle = y_1 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} x_n y_{n+1}$$

where $y = \{y_n\} \in \ell^1$ induces an isometric isomorphism of ℓ^1 onto X*. Define T, in L(X) by

$$T \{x_n\} = \{ -\tfrac{1}{n} x_n \} .$$

Obviously, if p is any polynomial ,

$$p(T) \{x_n\} = \{ p(-\tfrac{1}{n})x_n \} .$$

It follows that

$$\begin{aligned} \| p(T)x \| &\leq \sup_n | p(-\tfrac{1}{n})x_n | \\ &\leq \sup_{t \in [-1,0]} | p(t) | \sup_n |x_n| . \end{aligned}$$

Hence

$$\| p(T) \| \leq \sup_{t \in [-1,0]} | p(t) | .$$

Thus T is well-bounded.

It is easily seen that the homomorphism of Lemma I.2.1 is given by

$$f(T) \{x_n\} = \{ f(-\tfrac{1}{n})x_n \} .$$

Let $\{E(t) : t \in R\}$ be the decomposition of the identity for T constructed in Chapter 1. § 2; then by Lemma I.2.11,

$$\langle x, E(s)y\rangle = \lim_{\substack{h\to 0\\ \mathcal{U}}} \langle x, k_{s,h}(T^*)y\rangle$$
$$= \lim_{\substack{h\to 0\\ \mathcal{U}}} \langle k_{s,h}(T)x, y\rangle \qquad (x \in X,\ y \in X^*)$$

where $\mathcal{U}$ is the ultrafilter mentioned in Lemma I.2.7 and $k_{s,h}(t)$ is the function defined in Lemma I.2.11. For each $s \in [-1,0)$, there exists a positive integer m such that $s \in [-\frac{1}{m}, -\frac{1}{m+1})$. We assume that $h > 0$ and $h < |s + \frac{1}{m+1}|$, then

$$\langle k_{s,h}(T)x, y\rangle = \langle \{k_{s,h}(-\tfrac{1}{n})x_n\}, y\rangle$$
$$= y_1 \lim_{n\to\infty} \{k_{s,h}(-\tfrac{1}{n})x_n\} + \sum_{n=1}^{\infty} k_{s,h}(-\tfrac{1}{n})x_n y_{n+1}$$
$$= \sum_{n=1}^{m} x_n y_{n+1} \qquad (x \in X,\ y \in X^*).$$

Hence

$$\langle x, E(s)y\rangle = \sum_{n=1}^{m} x_n y_{n+1} \qquad (x \in X,\ y \in X^*).$$

Thus

$$E(s)\{y_n\} = \{0, y_1, \ldots\ldots\ldots, y_m, 0, \ldots\ldots\} \quad (s \in [-\tfrac{1}{m}, -\tfrac{1}{m+1})).$$

Moreover, if $s \in [-1,0)$, there is a $\delta_s > 0$ such that the function $\langle x, E(\cdot)y\rangle$ is constant in $[s, s+\delta_s)$. Hence $\langle x, E(\cdot)y\rangle$ is continuous on the right throughout $[-1,0)$. It follows that $\langle x, E(\cdot)y\rangle$ is C-limitable on the right throughout $[-1,0)$. Hence $\{E(s) : s \in R\}$ is the unique decomposition of the identity for T. Define the function $F(\cdot)$ by

$$F(s)\{x_n\} = \begin{cases} 0 & s < -1 \\ \{x_1, \ldots, x_m, 0, \ldots\} & s \in [-\frac{1}{m}, -\frac{1}{m+1}) \\ \{x_n\} & s \geq 0 \end{cases}$$

It is easy to verify that

$$F^*(s) = E(s) \qquad \text{for each } s \in R.$$

It follows that T is decomposable in X. Furthermore, if $s \in R$, there is $\epsilon_s > 0$ such that the function $F(\cdot)$ is constant in $[s, s+\epsilon_s)$. Hence $F(\cdot)$ is continuous on the right in the strong operator topology. However, let $\{x_n\}$ be the sequence whose terms are π, then

$$\| F(-\tfrac{1}{m})x - F(-\tfrac{1}{n})x \| = \pi$$

for all positive integers m and n with $m \neq n$. Hence $\lim_{s \to 0-} F(s)$ fails to exist. Thus the well-bounded operator T is of type (A) but not of type (B).

We note that the first example is due to Ringrose (11), and the second and third example are due to Berkson and Dowson (4).

CHAPTER III:-

THE STRUCTURE OF WELL-BOUNDED OPERATORS OF TYPE (B):

1. Integration theory:-

The integrals described here are based on the modified Stieltjes integrals of Krabbe (9). We shall use them to describe the extension of the algebra homomorphism of well-bounded operators of type (B).

Let U_J be the family of all partitions of J. We recall that $u = (u_k: 0 \leq k \leq m) \geq v = (v_j: 0 \leq j \leq n)$ if and only if each $[u_{k-1}, u_k]$ $(1 \leq k \leq m)$ is contained in some $[v_{j-1}, v_j]$ $(1 \leq j \leq n)$.

Let M(u) be the family of sequences $u^* = (u_k^* : 1 \leq k \leq m)$ such that

$$u_{k-1} \leq u_k^* \leq u_k \qquad (1 \leq k \leq m)$$

for each u in U_J.

A pair $\bar{u} = (u,u^*)$ with $u \in U_J$ and $u^* \in M(u)$ is called a marked partition of J. We write π_J for the family of marked partitions of J and define the pre-order $\geq$ on π_J by setting $(u,u^*) \geq (v,v^*)$ if and only if $u \geq v$.

Let $\pi_J^i = \{\bar{u} = (u,u^*) \in \pi_J : u_{k-1} < u_k^* < u_k \, , \, 1 \leq k \leq m\}$

and let $\pi_J^r = \{\bar{u} = (u,u^*) \in \pi_J : u_k^* = u_k, \, 1 \leq k \leq m\}$.

The sets U_J, π_J, π_J^i and π_J^r are directed by $\geq$. Also, π_J^i and π_J^r are cofinal in π_J.

Let Φ and Ψ be functions on J, one taking values in C, the other in L(X) or C. When $\bar{u} \in \pi_J$, we define

$$\sum \Phi(\Psi\Delta\bar{u}) = \sum_1^m \Phi(u_k^*)(\Psi(u_k) - \Psi(u_{k-1})).$$

The following integrals are defined as net limits in the strong operator topology (when they exist):

$$\int_J \Phi \, d\Psi = \text{st} \lim_{\pi_J} \sum \Phi(\Psi\Delta\bar{u}).$$

This is the ordinary Stieltjes refinement integral and has been discussed in Chapter 2.

$$\int_J^r \Phi \, d\Psi = \operatorname{st} \lim_{\pi_J^r} \sum \Phi(\Psi \Delta \bar{u})$$

This integral is called a right Cauchy integral.

$$\int_J^i \Phi \, d\Psi = \operatorname{st} \lim_{\pi_J^i} \sum \Phi(\Psi \Delta \bar{u})$$

This integral is called a modified Stieltjes integral.

Let $\mathcal{E}(J)$ be the family of functions $E(\cdot)$ from R into L(X) satisfying

(i) $E(s) = E(s+) = \operatorname{st} \lim_{t \to s+} E(t)$, $\quad s \in R$,

(ii) $E(s-) = \operatorname{st} \lim_{t \to s-} E(t)$ exists, $\quad s \in R$,

(iii) $E(s) = 0$, $\quad s < a$,

(iv) $E(s) = E(b)$, $\quad s \geqslant b$.

11.11. Lemma:- Let $E \in \mathcal{E}(J)$. Then $\sup_R ||E(s)|| = \sup_J ||E(s)|| < \infty$.

Proof: Let $x \in X$. Since $E(s+)$ and $E(s-)$ exist ($s \in J$), $||E(t)x||$ is bounded for all t in some neighborhood of s. Since J is compact, $\sup_J ||E(s)x||$ is finite. By the uniform boundedness principle, $\sup_J ||E(s)||$ is finite, and clearly, $\sup_R ||E(s)|| = \sup_J ||E(s)|| < \infty$.

For T in L(X) and $-\infty < c < d \leqslant \infty$, we define

$$T\chi_{[c,d)}(t) = \begin{cases} T & t \in [c,d) \\ 0 & t \notin [c,d). \end{cases}$$

We note that if $a \leqslant c < d \leqslant b$, then

$$T\chi_{[c,d)} \in \mathcal{E}(J) \quad \text{and}$$
$$T\chi_{[b,\infty)} \in \mathcal{E}(J).$$

Let $E_u = \sum_1^m E(u_{k-1})\chi_{[u_{k-1},u_k)} + E(b)\chi_{[b,\infty)}$ when $E \in \mathcal{E}(J)$ and $u = (u_k : 0 \leqslant k \leqslant m) \in U_J$. Then, $E_u \in \mathcal{E}(J)$.

Let g be any function in BV(J), we define

$$\pi_J^g = \begin{cases} \pi_J & g \in NBV(J) \\ \pi_J^i & g \in BV(J) \setminus NBV(J). \end{cases}$$

The following integral is also defined as a net limit in the strong operator topology (when it exists):

$$\oint_J E\, dg = \operatorname{st}\lim_{\pi_J^g} \sum E(g\Delta \bar{u}) \qquad (g\in BV(J),\ E\in \mathcal{E}(J)).$$

It is easy to verify that if $\oint_J E_1\, dg$ and $\oint_J E_2\, dg$ exist, then

$$\oint_J (E_1 + E_2)\, dg = \oint_J E_1\, dg + \oint_J E_2\, dg .$$

Now, we shall prove the existence of above integral. First, we need the following elementary results.

1.2. Lemma:- $\lim_{U_J} \sup_J \| E(s)x - E_u(s)x \| = 0 \qquad (x\in X,\ E\in \mathcal{E}(J)).$

Proof: Let $E\in\mathcal{E}(J)$, $x\in X$. Let $\varepsilon > 0$.

For each s in $[a,b)$, there exists r_s ($s< r_s < b$) such that $\| E(t)x - E(t')x \| \leq \varepsilon$ when $t,t'\in [s,r_s)$, since $E(s) = E(s+)$.

For each s in $(a,b]$, there exists l_s ($a < l_s < s$) such that $\| E(t)x - E(t')x \| \leq \varepsilon$ when $t,t'\in [l_s,s)$, since $E(s-)$ exists.

The sets $[a,r_a)$, $(l_b,b]$, (l_s,r_s) ($a < s < b$) form an open cover of J. Let $[a,r_a)$, $(l_b,b]$, (l_{s_j},r_{s_j}) (j in some finite set) be a finite subcover.

Let v be the partition with points a, b, r_a, l_b, s_j, l_{s_j}, r_{s_j} (j in some finite set). Then for any $u = (u_k : 0 \leq k \leq m) \geq v$, since any $[u_{k-1},u_k)$ is a subset of $[a,r_a)$, $[l_b,b)$, $[l_{s_j}, s_j)$ or $[s_j,r_{s_j})$ (some j), we have

$$\sup_J \| E(s)x - E_u(s)x \| < \varepsilon .$$

Hence $\qquad \lim_{U_J} \sup_J \| E(s)x - E_u(s)x \| = 0 \qquad x\in X,\ E\in\mathcal{E}(J).$

1.3 Lemma:-

(i) $\oint_J T \chi_{[b,\infty)}\, dg = 0 , \qquad g\in BV(J),\ T\in L(X);$

(ii) $\oint_J T\chi_{[s,t)}\, dg = (g(t) - g(s))T, \quad g\in BV(J),\ T\in L(X);\ a \leq s < t \leq b;$

(iii) $\oint_J E_u\, dg = \sum_1^m E(u_{k-1})(g(u_k) - g(u_{k-1}))$

$\qquad = \operatorname{st}\lim_{\pi_J^g} \sum E_u(g\Delta \bar{v}) , \qquad g\in BV(J), E\in\mathcal{E}(J), u\in U_J.$

Proof: (i): Let $\bar{u}\in\pi_J^g$. Then

$$\sum T\chi_{[b,\infty)}(g\Delta\bar{u}) = \sum_1^m T\chi_{[b,\infty)}(u_k^*)(g(u_k) - g(u_{k-1}))$$
$$= \begin{cases} (g(b) - g(u_{m-1}))T & u_m^* = b, \\ 0 & u_m^* < b. \end{cases}$$

Hence

$$\operatorname{st}\lim_{\pi_J^g} \sum T\chi_{[b,\infty)}(g\Delta\bar{u}) = 0 .$$

(ii) Let $a < s \leq b$, $\bar{u} \in \pi_J^g$, $u \geqslant (a, s, b)$ (no condition if $s = b$). Then $s = u_n$ for some n with $1 \leq n \leq m$, and

$$\sum T\chi_{[a,s)}(g\Delta\bar{u}) = \sum_1^{n-1}(g(u_k) - g(u_{k-1}))T + T\chi_{[a,s)}(u_n^*)(g(s) - g(u_{n-1}))$$
$$= \begin{cases} (g(u_{n-1}) - g(a))T, & u_n^* = u_n = s, \\ (g(s) - g(a))T, & u_n^* < u_n = s. \end{cases}$$

Hence $\operatorname{st}\lim_{\pi_J^g} \sum T\chi_{[a,s)}(g\Delta\bar{u}) = (g(s) - g(a))T$. Since

$$\chi_{[s,t)} = \chi_{[a,t)} - \chi_{[a,s)} \qquad (a \leq s < t \leq b),$$

Hence

$$\oint_J T\chi_{[s,t)}\,dg = \oint_J T\chi_{[a,t)}\,dg - \oint_J T\chi_{[a,s)}\,dg$$
$$= (g(t) - g(a))T - (g(s) - g(a))T$$
$$= (g(t) - g(s))T .$$

(iii) From the definition of E_u, it is obvious that the result follows directly from (i) and (ii).

1.4. <u>Theorem</u>:- Let g be in $BV(J)$ and E in $\mathcal{E}(J)$. Then $\oint_J E\,dg$ exists, and

$$\oint_J E\,dg = \operatorname{st}\lim_{U_J} \oint_J E_u\,dg .$$

Also,
$$\|\oint_J E\,dg\| \leq \operatorname{var}(g,J)\sup_J\|E(s)\|, \tag{25}$$
$$\|\oint_J E\,dg\,x\| \leq \operatorname{var}(g,J)\sup_J\|E(s)x\|, \quad x \in X. \tag{26}$$

Proof: It is easy to verify that

$$\|\sum E(g\Delta\bar{u}) - \sum F(g\Delta\bar{u})\| \leq \operatorname{var}(g,J)\sup_J\|E(s) - F(s)\|, \tag{27}$$
$$(g \in BV(J),\ E,\ F \in \mathcal{E}(J))$$

and

$$\|\sum E(g\Delta\bar{u})x - \sum F(g\Delta\bar{u})x\| \leq \operatorname{var}(g,J)\sup_J\|E(s)x - F(s)x\| \tag{28}$$
$$(g \in BV(J),\ E, F \in \mathcal{E}(J),\ x \in X).$$

Setting F = 0 in (27) and (28), we have

$$|| \sum E(g\Delta \bar{u})|| \leqslant \mathrm{var}(g,J)\sup_J||E(s)||$$

and $$|| \sum E(g\Delta \bar{u})x|| \leqslant \mathrm{var}(g,J)\sup_J||E(s)x||.$$

Hence (25) and (26) follow immediately.

Now, let $u \in U_J$, let $\bar{v}$, $\bar{w} \in \pi_J^g$ and let $x \in X$. Then

$$|| \sum E(g\Delta \bar{v})x - \sum E(g\Delta \bar{w})x||$$

$$\leqslant \quad || \sum E(g\Delta \bar{v})x - \sum E_u(g\Delta \bar{v})x|| + || \sum E(g\Delta \bar{w})x - \sum E_u(g\Delta \bar{w})x||$$

$$+ \quad || \sum E_u(g\Delta \bar{v})x - \sum E_u(g\Delta \bar{w})x||$$

$$\leqslant \ 2\,\mathrm{var}(g,J)\sup_J||E(s)x - E_u(s)x|| + || \sum E_u(g\Delta \bar{v})x - \sum E_u(g\Delta \bar{w})||.$$

Then, Lemma 2 and Lemma 3 show that $|| \sum E(g\Delta \bar{v})x - \sum E(g\Delta \bar{w})x|| \longrightarrow 0$ as $\bar{v}$ and $\bar{w}$ increase in π_J^g. Therefore, $\left\{ \sum E(g\Delta \bar{v}) : \bar{v} \in \pi_J^g \right\}$ is a uniformly bounded strongly Cauchy net in L(X) and so converges to its unique strong limit. Hence $\oint_J E\, dg$ esists. Moreover,

$$|| \oint_J E_u\, dg\, x - \oint_J E\, dg\, x ||$$

$$= \quad || \oint_J (E_u - E)\, dg\, x ||$$

$$\leqslant \quad \mathrm{var}(g, J)\sup_J|| E_u(s)x - E(s)x ||.$$

It follows from Lemma 2 that

$$\oint_J E\, dg = \operatorname*{st\,lim}_{U_J} \oint_J E_u\, dg \ .$$

1.5. Theorem:- Let $E \in \mathcal{E}(J)$, $g \in BV(J)$ and let $\left\{ g_\alpha : \alpha \in \sigma \right\}$ be a net in BV(J) with $\sup_\sigma \mathrm{var}(g_\alpha, J) < \infty$ and $g(s) = \lim g_\alpha(s)$ $\quad (s \in J)$.
Then $\oint_J E\, dg = \operatorname*{st\,lim}_{\sigma} \oint_J E\, dg_\alpha$.

Proof: Let $u \in U_J$. Then

$$\oint_J E\, dg - \oint_J E\, dg_\alpha = \oint_J (E-E_u)dg - \oint_J (E-E_u)dg_\alpha + \oint_J E_u d(g-g_\alpha).$$

Let x in X. It follows from Theorem 4 and Lemma 3(iii) that

$$|| \oint_J E\, dg\, x - \oint_J E\, dg_\alpha\, x ||$$

$$\leqslant \quad || \oint_J (E-E_u)\, dg\, x|| + || \oint_J (E-E_u)\, dg_\alpha\, x|| + || \oint_J E_u\, d(g-g_\alpha)\, x||$$

$$\leq \mathrm{var}(g,J)\sup_J||E(s)x - E_u(s)x|| + \sup_{\sigma} \mathrm{var}(g_\alpha,J)||E(s)x - E_u(s)x||$$
$$+ \sup_J||E(s)x|| \sum_1^m |(g-g_\alpha)(u_k) - (g-g_\alpha)(u_{k-1})|$$
$$\leq (\mathrm{var}(g,J) + \sup_{\sigma} \mathrm{var}(g_\alpha,J))\sup_J||E(s)x - E_u(s)x||$$
$$+ \sup_J||E(s)x|| \sum_1^m |(g-g_\alpha)(u_k) - (g-g_\alpha)(u_{k-1})|$$

and this expression can be made arbitrarily small by choosing u fine enough (Lemma 2) and then α large enough. Hence

$$\oint_J E\, dg = \mathrm{st} \lim_{\sigma} \oint_J E\, dg_\alpha.$$

Let $S(g,E) = g(b)E(b) - \oint_J E\, dg$ when $g \in BV(J)$, $E \in \mathcal{E}(J)$.

T.6. Lemma:

(i) $S(g, \chi_{[s,\infty)}T) = g(s)T$ $\quad g \in BV(J)$, $T \in L(X)$, $a \leq s \leq b$.

(ii) $||S(g,E)|| \leq |||g|||\sup_J||E(s)||$ $\quad g \in BV(J)$, $E \in \mathcal{E}(J)$.

(iii) $||S(g,E)x|| \leq |||g|||\sup_J||E(s)x||$ $\quad g \in BV(J)$, $E \in \mathcal{E}(J)$.

(iv) $S(\chi_{[a,s]},E) = E(s)$ $\quad E \in \mathcal{E}(J)$, $s \in J$.

Proof: (i), (ii) and (iii) follow directly from Lemma 2 and Theorem 3.

(iv) If $s = b$, then

$$S(\chi_J, E) = E(b) - \oint_J E\, d\chi_J$$
$$= E(b).$$

If $s < b$, then

$$S(\chi_{[a,s]},E) = - \oint_J E\, d\chi_{[a,s]}$$
$$= - \mathrm{st} \lim_{U_J} \oint_J E_u\, d\chi_{[a,s]} \qquad \text{(Theorem 4)}$$
$$= - \mathrm{st} \lim_{U_J} \sum_1^m E(u_{k-1})(\chi_{[a,s]}(u_k) - \chi_{[a,s]}(u_{k-1}))$$
$$= - \mathrm{st} \lim_{U_J} (-E(u_{n-1})) \text{ where } s \in [u_{n-1},u_n)$$
$$= \mathrm{st} \lim_{U_J} E_u(s)$$
$$= E(s) \qquad \text{(Lemma 2)}$$

1.7. Lemma:- Let $g \in BV(J)$ and $\bar{u} \in \pi_J(g,r)$ where

$$\pi_J(g,r) = \begin{cases} \pi_J & g \in NBV(J), \\ \pi_J^r & g \in BV(J)\setminus NBV(J). \end{cases}$$

Define $$g_{\bar{u}} = g(a)\chi_{\{a\}} + \sum_1^m g(u_k^*)\chi_{(u_{k-1},u_k]} .$$

Then $g_{\bar{u}} \in BV(J)$ and

$$g(s) = \lim_{\pi_J(g,r)} g_{\bar{u}}(s) \qquad s \in J.$$

Also, $var(g_{\bar{u}},J) \leq 2\sup_J|g(s)|$ if g is real monotonic increasing.

Proof: It is obvious that $g_{\bar{u}} \in BV(J)$ and $g_{\bar{u}}(a) = g(a)$. If $a < s \leq b$ and $u \geqslant (a,s,b)$ (no condition if $s = b$) then $s = u_n$ for some n ($1 \leq n \leq m$) and

$$g_{\bar{u}}(s) = g(u_n^*) = \begin{cases} g(u_n^*) & g \in NBV(J), \\ g(u_n) & g \in BV(J)\setminus NBV(J). \end{cases}$$

Therefore $$\lim_{\pi_J(g,r)} g_{\bar{u}}(s) = g(s) \qquad (s \in J) .$$

If g is real monotonic increasing, then $var(g,J) \leq 2\sup_J|g(s)|$ and $g_{\bar{u}}$ is also monotonic increasing. Hence

$$var(g_{\bar{u}},J) \leq 2\sup_J|g_{\bar{u}}(s)| \leq 2\sup_J|g(s)|.$$

1.8 Theorem:- Let $E \in \mathcal{E}(J)$. Then

$$S(g,E) = \begin{cases} g(a)E(a) + \oint_J g\, dE & g \in NBV(J) \\ g(a)E(a) + \int_J^r g\, dE & g \in BV(J)\setminus NBV(J). \end{cases}$$

Proof: Let $\bar{u} \in \pi_J(g,r)$. Then

$$S(g_{\bar{u}},E) = g(a)S(\chi_{\{a\}},E) + \sum_1^m g(u_k^*)(S(\chi_{[a,u_k]},E) - S(\chi_{[a,u_{k-1}]},E))$$
$$= g(a)E(a) + \sum g(E\Delta\bar{u}) \qquad \text{(Lemma 6).}$$

Since every g in $BV(J)$ can be expressed in the form

$$g = t_1g_1 + t_2g_2 + t_3g_3 + t_4g_4$$

where $t_i \in C$ ($i = 1,2,3,4$), and g_i is real monotonic-increasing. It suffices to prove the case where g is real monotonic increasing. Then, the result follows from Lemma 8 and Theorem 5.

We shall write $\int_J^{\oplus} g\, dE$ instead of $S(g,E)$ when $g \in BV(J)$ and $E \in \mathcal{E}(J)$.

2. A convergence theorem in BV(J) :-

Let Σ_J be the algebra of subsets of $[a,b)$ generated by sets of the form $[s,t)$ $(a \leq s < t \leq b)$. Let $\mathfrak{D}$ be the set of all linear combinations of characteristic functions of sets in Σ_J. Let $\mathcal{Q}_J$ be the closure of $\mathfrak{D}$ in the supremum norm. Then it is easy to verify that $\mathcal{Q}_J$ is a Banach space under the supremum norm. It follows from Lemma 1.1 (or (8), Theorem 4.5) that $\mathcal{Q}_J$ consists of all functions, vanishing on $(-\infty,a)$ and on $[b,\infty)$, and right continuous and left limitable on R.

From the scalar version of Theorem 1.4 the integral $\int_J^{i} w\, dg$ exists for g in BV(J) and w in $\mathcal{Q}_J$. Moreover, if $w = \lim_n w_n$ in the supremum norm where $\{w_n\} \subset \mathfrak{D}$, then

$$\int_J^{i} w\, dg = \lim_n \int_J^{i} w_n\, dg. \tag{29}$$

and from the scalar version of Lemma 3,

$$\int_J^{i} \chi_{[s,t)}\, dg = g(t) - g(s). \tag{30}$$

We use this notion to prove the following well-known result.

2.1. Theorem:- There is an isometric isomorphism between $\mathcal{Q}_J^*$ and $BV_o(J)$, determined by the identity

$$\langle w,w^*\rangle = \int_J^{i} w\, dg \qquad (w \in \mathcal{Q}_J,\ w^* \in \mathcal{Q}_J^*,\ g \in BV_o(J)). \tag{31}$$

Proof: It follows from Theorem 1.4 that

$$\left| \int_J^{i} w\, dg \right| \leq \sup |w(s)|\ |||g||| \qquad (w \in \mathcal{Q}_J,\ g \in BV_o(J)).$$

Hence, for each g in $BV_o(J)$, (31) defines a point w^* in $\mathcal{Q}_J^*$ with

$$||w^*|| < |||g|||. \tag{32}$$

To show that every $w^* \in \mathcal{Q}_J^*$ is given by some $g \in BV_o(J)$, we have only to define

$$g(s) = \begin{cases} \langle -\chi_{[s,b)}, w^*\rangle & (a \leq s < b) \\ 0 & s = b\ . \end{cases}$$

We shall show that g is of bounded variation. Consider $u = (u_k : 0 \leq k \leq m) \in U_J$ and let

$$\lambda_k = \overline{\operatorname{sgn}}\,(g(u_k) - g(u_{k-1})) \qquad (k = 1, 2, \ldots, m)$$

where

$$\overline{\operatorname{sgn}}\, c = \begin{cases} 0 & \text{if } c = 0 \\ \bar{c}/|c| & \text{if } c \neq 0. \end{cases}$$

Obviously,

$$\lambda_k(g(u_k) - g(u_{k-1})) = |g(u_k) - g(u_{k-1})| \qquad (k = 1, 2, \ldots, m).$$

Then,

$$\begin{aligned} &\sum_1^m |g(u_k) - g(u_{k-1})| \\ &= \sum_1^m \lambda_k(g(u_k) - g(u_{k-1})) \\ &= \sum_1^{m-1} \lambda_k(\langle -\chi_{[u_k,b)}, w^*\rangle - \langle -\chi_{[u_{k-1},b)}, w^*\rangle) \\ &\qquad - \lambda_m\langle -\chi_{[u_{m-1},b)}, w^*\rangle \\ &= \sum_1^{m-1} \lambda_k(\langle \chi_{[u_{k-1},b)} - \chi_{[u_k,b)}, w^*\rangle + \langle \lambda_m [u_{m-1},b), w^*\rangle \\ &= \langle \sum_1^m \lambda_k \chi_{[u_{k-1},u_k)}, w^*\rangle \\ &\leq ||w^*|| \max\{|\lambda_1|, \ldots, |\lambda_m|\} \\ &\leq ||w^*|| . \end{aligned}$$

Hence g is of bounded variation and

$$\operatorname{var}(g,J) \leq ||w^*|| . \tag{33}$$

From (30), we have

$$\int_J \chi_{[s,b)}\, dg = -g(s) = \langle \chi_{[s,b)}, w^*\rangle .$$

Hence (31) holds for every function w in $\mathfrak{D}$. Since $\mathfrak{D}$ is dense in $\mathcal{Q}_J$ and therefore, since both sides of (31) are continuous in w, it follows that (31') holds for all $w \in \mathcal{Q}_J$. Moreover, it follows from (32), (33) that

$$\operatorname{var}(g,J) = ||w^*||.$$

Since the linearity of correspondence between w^* and g is clear, the theorem is proved.

2.2. Lemma:- Let $\{ g_\alpha : \alpha \in \mathfrak{S} \}$ be a uniformly bounded net in $BV_o(J)$ and let $g \in BV_o(J)$. Then $g = \lim g_\alpha$ in the $\mathcal{Q}_J$-topology of $BV_o(J)$ if and only if $g(s) = \lim g_\alpha(s) \quad (a \leq s < b)$.

Proof: From (30), we have

$$\langle -\chi_{[s,b)}, g \rangle = \int_J^{i} -\chi_{[s,b)}\, dg = g(s) \qquad (a \leq s < b).$$

If $g = \lim_{\mathfrak{S}} g_\alpha$ in the $\mathcal{Q}_J$-topology of $BV_o(J)$, then

$$g(s) = \langle -\chi_{[s,b)}, g \rangle = \lim_{\mathfrak{S}} \langle -\chi_{[s,b)}, g_\alpha \rangle = \lim_{\mathfrak{S}} g_\alpha(s) \qquad (a \leq s < b).$$

Conversely, if $g(s) = \lim_{\mathfrak{S}} g_\alpha(s) \quad (a \leq s < b)$, then it follows from Theorem 1.5 that

$$\langle w, g \rangle = \int_J^{i} w\, dg = \lim_{\mathfrak{S}} \int_J^{i} w\, dg_\alpha = \lim_{\mathfrak{S}} \langle w, g_\alpha \rangle \qquad (w \in \mathcal{Q}_J).$$

Hence $g = \lim g_\alpha$ in the $\mathcal{Q}_J$-topology of $BV_o(J)$.

2.3. Theorem:- Let $g \in BV(J)$. Then there is a net $\{ g_\alpha : \alpha \in \mathfrak{S} \}$ in $AC(J)$ such that $g = \lim_{\mathfrak{S}} g_\alpha$ pointwise on J and $\sup_{\mathfrak{S}} |||g_\alpha||| \leq |||g|||$.

Proof: Since g can be written as $(g - g(b)\chi_J) + g(b)\chi_J$ we see that it suffices to show that if $g \in BV_o(J)$ then there is a net $\{ g_\alpha : \alpha \in \mathfrak{S} \}$ in $AC_o(J)$ such that $g = \lim_{\mathfrak{S}} g_\alpha$ pointwise on $[a,b)$ and $\sup_{\mathfrak{S}} \operatorname{var}(g_\alpha, J) \leq \operatorname{var}(g,J)$.

Now, for each w in $\mathcal{Q}_J$, we define

$$P_w(f) = \int_J w\, df \qquad (f \in AC_o(J)).$$

It is easy to verify that P_w is a bounded linear functional on $AC_o(J)$ and

$$||P_w|| = \sup_J |w(s)|.$$

Therefore $\mathcal{Q}_J$ can be identified with a subspace of $AC_o{}^*(J)$.

From Theorem 1, each function g in $BV_o(J)$ defines a bounded linear functional L_g on $\mathcal{Q}_J$ by

$$L_g(w) = \int_J^{i} w\, dg \qquad (w \in \mathcal{Q}_J)$$

and $\qquad || L_g || = \operatorname{var}(g,J).$

By the Hahn-Banach theorem, we can extend L_g to a linear functional (also denoted by) L_g on $AC_o{}^*(J)$ without increasing its norm. So $L_g \in AC_o^{**}(J)$.

By Goldstine's Theorem (6,V.4.5.) there is a net $\{g_\alpha : \alpha \in \mathfrak{S}\}$ in $AC_o(J)$ converging to L_g in the $AC_o^*(J)$-topology of $AC_o^{**}(J)$ and satisfying $var(g_\alpha, J) \leq ||L_g|| = var(g,J)$. Then, $g = \lim_{\mathfrak{S}} g_\alpha$ in the $\mathcal{Q}_J$-topology of $BV_o(J)$, so it follows from Lemma 2 that $g(s) = \lim g_\alpha(s) \quad (a \leq s < b)$.

3. Naturally ordered nets of operators:-

Suppose $\{E_\alpha : \alpha \in \mathfrak{S}\}$ is a set of commuting projections on X. A projection E such that

$$E(X) = \overline{sp}\left\{ \bigcup_{\mathfrak{S}} E_\alpha(X) \right\} \qquad \text{and}$$

$$(I - E)(X) = \bigcap_{\mathfrak{S}} (I - E_\alpha)(X)$$

is the supremum of $\{E_\alpha : \alpha \in \mathfrak{S}\}$ and is denoted by $\bigvee_{\mathfrak{S}} E_\alpha$.

A net $\{E_\alpha : \alpha \in \mathfrak{S}\}$ of projections is said to be naturally ordered if $E_\alpha \leq E_\beta$ whenever $\alpha \leq \beta$.

Spain in (14), extended this teminology and said that the net $\{E_\alpha : \alpha \in \mathfrak{S}\}$ is a naturally ordered net of operators if $E_\alpha = E_\alpha E_\beta = E_\beta E_\alpha$, whenever $\alpha < \beta$.

We need the following Lemma which is due to Banach (1,Theorem 6, P.58).

3.1. Lemma:- Let G be a subset of X and y be any point of X. Then, there esists a sequence $\{g_n\}$ which is a linear combination of elements in G with $\lim_{n\to\infty} g_n = y$ if and only if for each $x^* \in X^*$,

$$\langle x, x^* \rangle = 0 \qquad (x \in G)$$

implies that $\langle y, x^* \rangle = 0$.

Proof:- Suppose there exists a sequence $\{g_n\}$ which is a linear combination of elements of G with $\lim_{n\to\infty} g_n = y$. Then, for each $x^* \in X^*$,

$$\langle x, x^* \rangle = 0 \qquad (x \in G)$$

implies that $\langle g_n, x^* \rangle = 0 \qquad (n \in \underset{\sim}{N})$

and hence $\langle y, x^* \rangle = \langle \lim_{n\to\infty} g_n, x^* \rangle$

$$= 0 .$$

Conversely, suppose that for each $x^* \in X^*$,

$$\langle x, x^* \rangle = 0 \qquad (x \in G)$$

implies that $\langle y, x^* \rangle = 0$.

If $y \in G$, then the result is obvious. We may assume that $y \notin G$. Let $M = \mathrm{sp}\{G\}$. We shall show that $y \in \bar{M}$. Suppose on the contrary that $y \notin \bar{M}$. Then, by the Hahn Banach theorem, there is a functional $z^* \in X^*$ with

$$\langle y, z^* \rangle = 1 \tag{34}$$

and

$$\langle x, z^* \rangle = 0 \qquad (x \in M). \tag{35}$$

Hence, (35) implies that

$$\langle y, z^* \rangle = 0,$$

which contradicts (34). Hence $y \in \bar{M}$. Thus there exists a sequence $\{g_n\}$ which is a linear combination of elements in G such that

$$\lim_{n\to\infty} g_n = y.$$

3.2. <u>Definition</u>: Let $\{T_\alpha : \alpha \in \sigma\}$ be a net in $L(X)$. y_x is a weak x-cluster point of $\{T_\alpha\}$ if y_x is a weak cluster point of the net $\{T_\alpha x : \alpha \in \sigma\}$.

The definition is due to Barry (2) who proved the following result.

3.3. <u>Theorem</u>:- Let $\{E_\alpha : \alpha \in \sigma\}$ be a naturally ordered uniformly bounded net of projections on X. Then st $\lim\limits_{\sigma} E_\alpha = \bigvee_\sigma E_\alpha$ if and only if $\{E_\alpha : \alpha \in \sigma\}$ has a weak x-cluster point for each x in X.

Proof:- Suppose $\{E_\alpha : \alpha \in \sigma\}$ has a weak x-cluster point for each x in X. For any x in X and $\alpha_o \in \sigma$, let y_x be a weak x-cluster point of $\{E_\alpha : \alpha \in \sigma\}$. For any $\varepsilon > 0$ and $x^* \in X^*$, let

$$N_\varepsilon(y_x) = \{ z : |\langle z - y_x, E^*_{\alpha_o} x^* \rangle| < \varepsilon \}.$$

Then $N_\varepsilon(y_x)$ is a neighbourhood of y_x in the weak topology of X. Since y_x is the weak cluster point of $\{E_\alpha x : \alpha \in \sigma\}$, then there is a $\beta \geq \alpha_o$ such that

$$|\langle E_\beta(x) - y_x, E^*_{\alpha_o} x^* \rangle| < \varepsilon.$$

Hence
$$|\langle E_{\alpha_o} x - E_{\alpha_o} y_x, x^* \rangle|$$
$$= |\langle E_{\alpha_o} E_\beta x - E_{\alpha_o} y_x, x^* \rangle|$$
$$= |\langle E_\beta x - y_x, E^*_{\alpha_o} x^* \rangle| < \varepsilon \qquad (x^* \in X^*).$$

Thus
$$E_{\alpha_o} x = E_{\alpha_o} y_x \qquad (\alpha_o \in \sigma). \tag{36}$$

Moreover, since $y_x \in \{E_\alpha x : \alpha \in \sigma\}^w$, it follows from Lemma 11 that
$$y_x = \lim_{n\to\infty} T_n x \tag{37}$$

where $T_n = \sum_{k=1}^{m} c_k^n E_{\alpha_k}$.

Clearly, for each $n \in N$, there is a $\beta_o \in \sigma$ such that $E_\alpha T_n = T_n$ whenever $\alpha \geq \beta_o$.

Hence $\quad \text{st} \lim_{\sigma} E_\alpha T_n = T_n$.

Furthermore, it follows from (36) that
$$\| E_\alpha x - E_\alpha T_n x \| = \| E_\alpha y_x - E_\alpha T_n x \|$$
$$\leq \sup_\sigma \|E_\alpha\| \, \|y_x - T_n x\|.$$

Thus $\lim_{n\to\infty} E_\alpha T_n x = E_\alpha x$ uniformly in α. By the E.H. Moore theorem on the interchange of limits (6, I.7.6)
$$y_x = \lim_{n\to\infty} \lim_{\sigma} E_\alpha T_n x$$
$$= \lim_{\sigma} \lim_{n\to\infty} E_\alpha T_n x$$
$$= \lim_{\sigma} E_\alpha x \quad .$$

Set $Ex = y_x$ $(x \in X)$. It is now easy to verify that E is a projection in X. Finally, if $x \in E(X)$ i.e. $Ex = x$, then $x = y_x = \lim_{n\to\infty} T_n x$. Thus $x \in \overline{sp}\{\bigcup_\sigma E_\alpha(X)\}$. i.e. $E(X) \subset \overline{sp}\{\bigcup_\sigma E_\alpha(X)\}$. If $x \in \bigcap_\sigma (I - E_\alpha)(X)$, then $x \in \lim_\sigma (I - E_\alpha)(X) = (I - E)(X)$. Thus $(I - E)(X) \supset \bigcap_\sigma (I - E_\alpha)(X)$. Since $EE_\alpha = E_\alpha E = E_\alpha$ and $(I - E_\alpha)(I - E) = I - E$, it is clear that $E(X) \supset \overline{sp}\{\bigcup_\sigma E_\alpha(X)\}$ and $(I - E)(X) \subset \bigcap_\sigma (I - E_\alpha)(X)$. Therefore $E = \bigvee_\sigma E_\alpha$.

We shall also need the following result, which has a similar proof to Theorem 3.

3.4. Theorem:- Let $\{E_\alpha : \alpha \in \sigma\}$ be a naturally ordered uniformly bounded net of operators on X. Then, st $\lim_{\sigma} E_\alpha$ exists if and only if $\{E_\alpha : \alpha \in \sigma\}$ has a weak x-cluster point for each $x \in X$.

Proof: The proof is similar to Theorem 3 and is therefore omitted.

4. The structure of well-bounded operators of type (B):-

In this section, we follow Spain (14), using an elementary integration theory to establish directly the characterisation of well-bounded operators of type (B) and then show that the algebra homomorphism mentioned in Lemma I.1.1 can be extended to an algebra homomorphism from BV(J) into L(X).

We need the following notion.

4.1. Definition:- Let M be a subset of X. The absolutely convex hull (denoted by aco(M)) of M is the set of all linear combinations $\sum_{i=1}^{n} \alpha_i x_i$ of elements x_i in M in which $\alpha_i \in C, |\alpha_i| \leq 1$ and $\sum_{i=1}^{n} |\alpha_i| = 1$. The closed absolutely convex hull is the closure of aco(M) and is denoted by $\overline{\text{aco}}(M)$.

4.2. Lemma:- Let $M \subset X$ be totally bounded. Then $\overline{\text{aco}}(M)$ is compact.

Proof: The set $\overline{\text{aco}}(M)$, being a closed subset of a complete space X is complete. Hence it suffices to show that $\overline{\text{aco}}(M)$ is totally bounded.

Let $\varepsilon > 0$. Since M is totally bounded, there is a finite subset $\{z_1, \ldots\ldots, z_n\} \subset M$ such that

$$M \subset \bigcup_{i=1}^{n} S(z_i, \tfrac{\varepsilon}{4})$$

where $S(z_i, \frac{\varepsilon}{4})$ is a sphere with center z_i and radius $\frac{\varepsilon}{4}$.

Let $N = \text{aco}(\{z_1, z_2, \ldots\ldots, z_n\})$. Now, $\overline{\text{aco}} \subset \bigcup_{x \in \text{aco}(M)} S(x, \frac{\varepsilon}{4})$. But, if $y \in \text{aco}(M)$, then $y = \sum_{i=1}^{m} \alpha_i y_i$, where $y_i \in M$, $\alpha_i \in C$, $|\alpha_i| \leq 1$ and $\sum_{i=1}^{m} |\alpha_i| = 1$. Let v be a function on M into $\{1, 2, \ldots\ldots, n\}$ such that if $x \in M$, $|x - z_{v(x)}| < \frac{\varepsilon}{4}$. Then

$$\left| y - \sum_{i=1}^{m} \alpha_i z_{v(y_i)} \right| = \left| \sum_{i=1}^{m} \alpha_i (y_i - z_{v(y_i)}) \right| \leq \sum_{i=1}^{m} |\alpha_i| \, |y_i - z_{v(y_i)}| < \frac{\varepsilon}{4},$$

and thus $\overline{aco}(M) \subset \bigcup_{x \in N} S(x, \frac{\varepsilon}{2})$.

Now, $N = \left\{ \sum_{i=1}^{n} \alpha_i z_i : \alpha_i \in C, \ |\alpha_i| \leq 1 \text{ and } \sum_{i=1}^{n} |\alpha_i| = 1 \right\}$. Let $Y = \left\{ (a_1, \ldots, a_n) \in C^n : |\alpha_i| \leq 1, \ \sum_{i=1}^{n} |\alpha_i| = 1 \right\}$. Then, the mapping ϕ defined by

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \alpha_i z_i$$

is a continuous mapping of the compact set Y onto N. Thus N is compact and hence totally bounded. Then, there is a finite subset $\{w_1, \ldots, w_m\}$ of N such that $N \subset \bigcup_{i=1}^{m} S(w_i, \frac{\varepsilon}{2})$. But then $\overline{aco}(M) \subset \bigcup_{i=1}^{m} S(w_i, \varepsilon)$. Hence $\overline{aco}(M)$ is totally bounded. This completes the proof.

Now, we give three conditions on a well-bounded operator equivalent to the definition of type (B).

4.3. Theorem:- Let T be a well-bounded operator on X and let $J = [a, b]$, K be chosen so that (1) is satisfied. Let $f \longmapsto f(T)$ be the algebra homomorphism from AC(J) into L(X) mentioned in Lemma I.1.1. Then the following conditions are equivalent:

(i) T is of type (B).

(ii) For every x in X, $f \longmapsto f(T)x$ is a compact linear map of AC(J) into X.

(iii) For every x in X, $f \longmapsto f(T)x$ is weakly compact linear map of AC(J) into X.

(iv) There exists a family $\{E(s) : s \in R\}$ of projections on X such that

$$E \in \mathcal{E}(J), \quad E(b) = I,$$

$$E(t)E(s) = E(s)E(t) = E(s) \qquad (s < t),$$

$$\| E(s) \| < K \qquad (s \in R),$$

and

$$T = \int_J^{\oplus} s \, dE(s).$$

Proof: We show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). Let $\{E^*(s) : s \in R\}$ be the unique decomposition of the identity for T. By the definition of type (B) operator, we have $E \in \mathcal{E}(J)$. We define a map ψ from AC(J) into L(X) by

$$\psi(f) = \int_J^{\oplus} f\, dE \qquad f \in AC(J).$$

The map ψ is linear and bounded; also

$$\psi(s \longmapsto 1) = \int_J^{\oplus} dE = I.$$

Moreover,

$$\begin{aligned} \langle \psi(s \longmapsto s)x, x^* \rangle &= \int_J^{\oplus} s\, d\langle E(s)x, x^* \rangle \\ &= b\langle x, x^* \rangle - \int_J \langle E(s)x, x^* \rangle\, ds \\ &= \langle Tx, x^* \rangle . \end{aligned}$$

Hence $\qquad \psi(s \longmapsto s) = T.$

Since $E(t)E(s) = E(t)E(s) = E(s) \qquad (s < t)$, it follows that

$$\begin{aligned} &\left\{f(a)E(a) + \sum f(E\Delta\bar{u})\right\}\left\{g(a)E(a) + \sum g(E\Delta\bar{u})\right\} \\ &= fg(a)E(a) + \sum fg(E\Delta\bar{u})) \qquad f,g \in AC(J),\ \bar{u} \in \pi_J . \end{aligned}$$

Hence $\qquad \psi(f)\psi(g) = \psi(fg).$

Thus ψ is an algebra homomorphism. By Lemma I.1.1,

$$\psi(f) = f(T) \qquad (f \in AC(J)).$$

For each x in X, we let $\mathcal{E}_X = \{E(s)x : s \in R\}$. Since $E(s) = 0 \quad (s < a)$ and $E(s) = I \quad (s \geq b)$. Then, for any $\delta > 0$, $\mathcal{E}_X = \{E(s)x : s \in [a-\delta, b]\}$. Let $\varepsilon > 0$. By the same argument as the proof of Lemma 1.2, there exists $s_1(= a-\delta)$, $s_2(= a)$, ... $s_n(=b)$ such that

$$\mathcal{E}_X \subseteq \bigcup_{i=1}^{n} S(E(s_i)x, \varepsilon).$$

Hence $\mathcal{E}_X$ is totally bounded. By Lemma 1, $\overline{aco}(\mathcal{E}_X)$ is compact. Moreover, let $f \in AC(J)$, $|||f||| \leq 1$ and $\bar{u} \in \pi_J$. Then,

$$\begin{aligned} &f(b)E(b)x - \sum E(f\Delta\bar{u})x \\ &= f(b)x - \sum_1^n (f(u_k) - f(u_{k-1}))E(u_k^*)x \in \overline{aco}(\mathcal{E}_X) . \end{aligned}$$

Therefore $f(T)x \in \overline{aco}(\mathcal{E}_x)$. Hence, for each x in X, $f \longmapsto f(T)x$ is a compact linear map from AC(J) into X.

(ii) $\Longrightarrow$ (iii). Trivial.

(iii) $\Longrightarrow$ (iv). Let $\mathcal{U}$ be the ultrafilter on $\beta(0,\infty)$ mentioned in Lemma I.2.7. For each $x \in X$ and $x^* \in X^*$, we define

$$L_{x,x^*}(f) = \langle f(T)x, x^* \rangle \qquad (f \in AC(J)).$$

It is obvious that L_{x,x^*} is a bounded linear functional on AC(J). By Lemmas I.2.5 and I.2.7, we have

$$L_{x,x^*}(f) = m_{x,x^*} f(b) - \int_a^b w_{x,x^*}(s) f'(s)\, ds \quad (x \in X, x^* \in X^*, f \in AC(J))$$

where $m_{x,x^*} \in C$, $w_{x,x^*} \in L^\infty(J)$, and

$$w_{x,x^*}(s) = \lim_{\substack{h \to 0 \\ \mathcal{U}}} \int_0^1 w_{x,x^*}(s+ht)\, dt .$$

Then, $\langle x, x^* \rangle = L_{x,x^*}(s \longmapsto 1) = m_{x,x^*}$.

Let $k_{s,h}$ be the function defined in Lemma I.2.11. Then

$$\langle k_{s,h}(T)x, x^* \rangle = L_{x,x^*}(k_{s,h}) = \int_0^1 w_{x,x^*}(s+ht)\, dt \quad (a \leq s < s+h < b).$$

Now, for each $x \in X$, let

$$\mathcal{K}_x = \left\{ k_{s,h}(T)x : a \leq s < s+h < b \right\}^w .$$

By hypothesis, $\mathcal{K}_x$ is weakly compact. Hence $\mathcal{K}_x$ may therefore be considered as a compact space. For fixed s in $[a,b)$, we define a vector-valued function τ_s from $(0,\infty)$ into the compact space $\mathcal{K}_x$ as follows:

$$\tau_s(h) = k_{s,h}(T)x \qquad (0 < h < \infty).$$

Since $\| \tau_s(h) \| = \| k_{s,h}(T)x \| \leq K \||k_{s,h}\|| \, \|x\| \leq K \|x\|$, then τ_s is bounded. Moreover, for every $h, h' \in (0,\infty)$,

$$\begin{aligned} \| \tau_s(h) - \tau_s(h') \| &= \| k_{s,h}(T)x - k_{s,h'}(T)x \| \\ &= \|(k_{s,h} - k_{s,h'})(T)x\| \\ &\leq K \||k_{s,h} - k_{s,h'}\|| \, \|x\| \\ &\leq K \|x\| \frac{|h-h'|}{\max\{h,h'\}} . \end{aligned}$$

Hence, τ_s is continuous on $(0,\infty)$. It follows from Stone's Theorem (7,6.5) that there is a unique continuous function $\overline{\tau}_s$ on $\beta(0,\infty)$ whose restriction to $(0,\infty)$ is τ_s. Let α be the limit of $\mathcal{U}$; then

$$\lim_{\substack{h\to 0\\ \mathcal{U}}} \langle \tau_s(h),x^*\rangle = \lim_{h\to\alpha} \langle \overline{\tau}_s(h),x^*\rangle = \langle \overline{\tau}_s(\alpha),x^*\rangle \quad (x^*\in X^*).$$

Hence, for each $s\in[a,b)$, we define

$$E(s)x = \overline{\tau}_s(\alpha).$$

Obviously, $E(s)x \in \mathcal{K}_x$ and since

$$\begin{aligned}\lim_{\substack{h\to 0\\ \mathcal{U}}} \langle \tau_s(h),x^*\rangle &= \lim_{\substack{h\to 0\\ \mathcal{U}}} \langle k_{s,h}(T)x,x^*\rangle \\ &= \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 w_{x,x^*}(s+ht)\,dt \\ &= w_{x,x^*}(s) \qquad (x^*\in X^*).\end{aligned}$$

Hence $\quad \langle E(s)x,x^*\rangle = w_{x,x^*}(s) \qquad (x\in X,\ x^*\in X^*). \qquad (38)$

Let $E(s) = 0 \quad (s < a)$,and $E(s) = I \quad (s \geq b)$.

Since $k_{s,h}k_{t,k} = k_{t,k}k_{s,h} = k_{s,h}$ for $0 < h < t-s$, $0 < k$, we have

$$E(s)E(t) = E(t)E(s) = E(s)$$

when $s < t$. By Theorem 3.4, and the weak compactness of $\mathcal{K}_x$, the strong limits $E(s+)$ and $E(s-)$ exist for all s in R.

From (38), we have

$$\begin{aligned}\langle E(s)x,x^*\rangle &= \lim_{\substack{h\to 0\\ \mathcal{U}}} \int_0^1 w_{x,x^*}(s+ht)\,dt \\ &= w_{x,x^*}(s+) \\ &= \langle E(s+)x,x^*\rangle \qquad (a \leq s < b.).\end{aligned}$$

Therefore $E(s) = E(s+) \quad (a \leq s < b)$, hence each $E(s)$ is a projection.

Thus $E\in\mathcal{E}(J)$.

Moreover, since

$$\begin{aligned}\langle Tx,x^*\rangle &= L_{x,x^*}(s \longmapsto s) \\ &= b\langle x,x^*\rangle - \int_a^b w_{x,x^*}(s)\,ds \\ &= b\langle x,x^*\rangle - \int_a^b \langle E(s)x,x^*\rangle\,ds \\ &= \int_J^{\oplus} s\, d\langle E(s)x,x^*\rangle.\end{aligned}$$

Hence $\quad T = \int_J^{\oplus} s \, dE.$

Finally, since $\quad \langle E(s)x,x^*\rangle = \lim_{h \to 0} \langle k_{s,h}(T)x,x^*\rangle \quad (x^* \in X^*)$ and

$$| \langle k_{s,h}(T)x,x^*\rangle | \leqslant || k_{s,h}(T) || \; ||x|| \; ||x^*|| \leqslant K||x|| \; ||x^*||.$$

Hence, $\quad | \langle E(s)x,x^*\rangle | \leqslant K \; ||x|| \; ||x^*|| \qquad (x \in X, \; x^* \in X^*).$

Thus $\quad || E(s) || \leqslant K \qquad (s \in R).$

(iv) $\Longrightarrow$ (i). It suffices to show that $\{E^*(s) : s \in R\}$ forms a decomposition of the identity for T. The conditions (i),(ii),(iii) of Definition I.1.2 follow immediately. Since $E \in \mathcal{E}(J)$, it follows that $\langle x,E^*(s)x^*\rangle = \langle E(s)x,x^*\rangle$ is everywhere right-continuous on R, for each $x \in X$ and $x^* \in X^*$. Hence conditions (iv) and (v) are satisfied. Moreover, since $T = \int_J^{\oplus} s \, dE$, then

$$\begin{aligned}\langle Tx,x^*\rangle &= \int_J^{\oplus} s \; d\langle E(s)x,x^*\rangle \\ &= b \langle x,x^*\rangle - \int_a^b \langle x,E^*(s)x^*\rangle \, ds \, .\end{aligned}$$

From the proof of Theorem I.2.3, we deduce that

$$\langle f(T)x,x^*\rangle = f(b) \langle x,x^*\rangle - \int_a^b \langle x,E^*(s)x^*\rangle f'(s) \, ds \qquad (f \in AC(J)).$$

It remain to proof (vi). Fix x in X and consider the map A from $L^1(J)$ into X defined by

$$Au = f_u(T)x$$

where

$$f_u(s) = \int_s^b u(t) \, dt \qquad (u \in L^1(J)).$$

A is clearly bounded and linear. For u in $L^1(J)$ and x^* in X^*, we have

$$\begin{aligned}\langle u,A^*x^*\rangle = \langle Au,x^*\rangle &= \langle f_u(T)x,x^*\rangle \\ &= \int_a^b \langle x,E^*(s)x^*\rangle \, u(s) \, ds \quad .\end{aligned}$$

It follows that the map A^* from X^* into $L^{\infty}(J)$ is given by

$$A^*x^* = \langle x,E^*(\cdot)x^*\rangle \; .$$

Since A is continuous, A^* is continuous when X^* and $L^{\infty}(J)$ are endowed with their weak*-topologies. This completes the proof that (iv) implies (i).

In our next theorem, we give some properties of well-bounded operators of type (B).

4.4. <u>Theorem</u>:- Let T be a well-bounded operator of type (B) on X and let $J = [a,b]$ be chosen so that (1) is satisfied. Let $\{F^*(t) : t \in R\}$ be the unique decomposition of the identity for T.

(i) $$f(T) = \int_{a-}^{b} f(t)\, dF(t) \qquad (f \in AC(J))$$

where the integral exists as a strong limit of Riemann sums. (This is valid for well-bounded operators of type (A).)

(ii) If for some s in R and x in X we have $(T-sI)^2x = 0$, then also $(T-sI)x = 0$. (This is valid for an arbitrary well-bounded operator.)

(iii) For each s in R, $F(s) - F(s-)$ is a projection on X whose range is given by

$$\{F(s) - F(s-)\}(X) = \{x \in X : Tx = sx\} \quad .$$

(iv) The residual spectrum of T is empty.

Proof: (i) For each $x \in X$, it follows from the proof of Theorem II.2.5 that

$$f(T)x = f(b)x - \text{R-S}\int_{a}^{b} F(t)x\, df(t) \qquad (f \in AC(J)).$$

For any $\varepsilon > 0$, we may assume that f is absolutely continuous on $[a-\varepsilon, b]$. It is easy to verify that

$$\text{R-S}\int_{a-\varepsilon}^{a} F(t)x\, df(t) = 0.$$

Hence

$$\begin{aligned} f(T)x &= f(b)x - \text{R-S}\int_{a-\varepsilon}^{a} F(t)x\, df(t) + \text{R-S}\int_{a}^{b} F(t)x\, df(t) \\ &= f(b)x - \text{R-S}\int_{a-\varepsilon}^{b} F(t)x\, df(t). \end{aligned}$$

Then, from Theorem II.2.3, we deduce that

$$\text{R-S}\int_{a-\varepsilon}^{b} f(t)\, dF(t)x \quad \text{exists and}$$

$$\begin{aligned} f(T)x &= f(b)x - f(b)x + f(a-\varepsilon)F(a-\varepsilon)x + \text{R-S}\int_{a-\varepsilon}^{b} f(t)\, dF(t)x \\ &= \text{R-S}\int_{a-\varepsilon}^{b} f(t)\, dF(t)x. \end{aligned}$$

Let $\varepsilon \to 0$. Then,

$$f(T)x = \text{R-S}\int_{a-}^{b} f(t)\, dF(t)x.$$

This completes the proof of (i).

(ii) If $(T-sI)^2x = 0$, then for any $M > 0$,

$$(I+M(T-sI)^2)x = x$$

so that $$(I + M(T-sI)^2)^{-1}x = x.$$

Thus

$$\begin{aligned}||(T-sI)x|| &= ||(T-sI)(I + M(T-sI)^2)^{-1}x|| \\ &\leqslant K||x||\ |||(t-s)(1 + M(t-s)^2)^{-1}||| \\ &\leqslant K||x||\ \tfrac{5}{2} M^{-\frac{1}{2}}.\end{aligned}$$

Hence $(T-sI)x = 0$.

(iii) Since $F(t)F(s) = F(s)F(t) = F(s)$ for $s < t$, then,

$$F(s)F(s-) = F(s-)F(s) = F(s-) \qquad (s \in R).$$

Hence $F(s) - F(s-)$ is a projection.

Now, let $x \in \{F(s) - F(s-)\}(X)$. For any $\varepsilon > 0$, let $u = (u_k : 0 \leqslant k \leqslant m)$ be a partition of $[a-\varepsilon, b]$. Then

$$\begin{aligned}&\textstyle\sum_{k=1}^{m} u_k^*(F(u_k)x - F(u_{k-1})x) \\ = \quad & u_n^* \{F(s)x - F(s-)x\} \qquad\qquad (s \in (u_{n-1}, u_n]) \\ = \quad & u_n^* x .\end{aligned}$$

This is valid for any partition of $[a-\varepsilon, b]$.

Hence, it follows from (i) that

$$Tx = sx .$$

Conversely, let $Tx = sx$ and $\theta > 0$, then

$$TF(s-\theta)x = F(s-\theta)Tx = sF(s-\theta)x$$

$$T\{I - F(s+\theta)\}x = \{I - F(s+\theta)\}Tx = s\{I - F(s+\theta)\}x.$$

Since by Theorem II.1.1 ,

$$\sigma(T/F(s-\theta)(X)) \subset (-\infty, s-\theta]$$

$$\sigma(T/(I - F(s+\theta))(X) \subset [s+\theta, \infty).$$

Hence $$F(s-\theta)x = \{I - F(s+\theta)\}x = 0.$$

Thus $x = \{F(s+\theta) - F(s-\theta)\}x$ for any $\theta > 0$. Letting $\theta \longrightarrow 0$ yields

$$x = \{F(s) - F(s-)\}x$$

and the proof of (iii) is complete.

(iv) It follows from I.2.4 that

$$\left\{ x^* \in X^* : T^*x^* = sx^* \right\} = F^*(s)(X^*) \cap \left\{ I - F^*(s-) \right\}(X^*)$$
$$= \left\{ F^*(s) - F^*(s-) \right\} (X^*).$$

Hence, by (iii), s is an eigenvalue for T if and only if it is an eigenvalue for T^*, and so by Exercise VII.5.9 of (6) (p.581) the residual spectrum of T is empty.

We end this section with the following result.

4.5. Theorem:- Let T be a well-bounded operator of type (B) on X and let $J = [a,b]$, K be chosen so that (1) is satisfied. Then the algebra homomorphism $f \mapsto f(T)$ can be extended to an algebra homomorphism Ψ from $BV(J)$ into $L(X)$ such that

$$\|\Psi(f)\| < K|||f||| \qquad (f \in BV(J)).$$

If $S \in L(X)$ and $ST = TS$, then $S\Psi(f) = \Psi(f)S \qquad (f \in BV(J))$.

Furthermore let $\{g_\alpha : \alpha \in \mathfrak{S}\}$ be a uniformly bounded net in $AC(J)$ converging pointwise to a function g in $BV(J)$. Then $\Psi(g) = \text{st}\lim_{\mathfrak{S}} \Psi(g_\alpha)$. Also,

$\{\Psi(g) : g \in BV(J)\} \subset \{f(T) : f \in AC(J)\}^s$.

Proof: Let $\{E^*(s) : s \in R\}$ be the unique decomposition of the identity for T. By the definition of a type (B) operator, $E \in \mathcal{E}(J)$. We define Ψ from $BV(J)$ into $L(X)$ by

$$\Psi(f) = \int_J^{\oplus} f \, dE \qquad (f \in BV(J)).$$

The argument in the proof of Theorem 3 ((i)$\Rightarrow$(ii)) shows that Ψ is an algebra homomorphism from $BV(J)$ into $L(X)$ and if $f \in AC(J)$ then $\Psi(f) = f(T)$.

Moreover, it follows from Lemma 1.6 that

$$\|\Psi(f)\| = \left\| \int_J^{\oplus} f \, dE \right\| \leq \sup_R \|E(s)\| \; |||f||| < K|||f|||.$$

If $S \in L(X)$ and $ST=TS$, then by Theorem II.1.1(iv), we have

$$SE(s) = E(s)S \qquad (s \in R).$$

Hence it follows from the definition of $\Psi(f)$ that

$$\Psi(f)S = S\Psi(f).$$

Now, let $\{g_\alpha : \alpha \in \mathfrak{S}\}$ be a uniformly bounded net in AC(J) converging pointwise to a function g in BV(J). From Theorem 1.5, we deduce that

$$\text{st} \lim_{\mathfrak{S}} \Psi(g_\alpha) = \text{st} \lim_{\mathfrak{S}} \int_J^{\oplus} g_\alpha \, dE = \int_J^{\oplus} g \, dE = \Psi(g) . \quad (39)$$

Moreover, given g in BV(J), it follows from Theorem 2.3 that there is a uniformly bounded net $\{g_\alpha : \alpha \in \mathfrak{S}\}$ in AC(J) converging pointwise to a function g in BV(J). Then, by (39), we have

$$\Psi(g) = \text{st} \lim_{\mathfrak{S}} g_\alpha(T).$$

Hence $\quad \{\Psi(g) : g \in BV(J)\} \subset \{f(T) : f \in AC(J)\}^s$.

5. <u>Well-bounded operators on a reflexive Banach space:-</u>

Throughout this section, X will denote a reflexive Banach space. Our main result is that every well-bounded operator on X is of type (B). First, we need the following result.

5.1. <u>Theorem</u>:- Let Y be a Banach space and U be a bounded operator from Y into X. Then U is weakly compact.

Proof: Since X is reflexive, we have $X = X^{**}$. Hence U^{**} is a map from Y^{**} into X. It is easily seen that U^{**} is continuous when Y^{**} is endowed with the weak*-topology and X with the weak topology.

Now, let S and S^{**} be the closed unit spheres in Y and Y^{**} respectively. By the theorem of Alaoglu (6, V.4.2), S^{**} is compact in the weak*-topology. It follows that the continuous image $U^{**}(S^{**})$ is weakly compact. But $U(S) \subset U^{**}(\hat{S}) \subset U^{**}(S^{**})$ where $\hat{S} = \{\hat{y} : y \in S\}$ and $y \mapsto \hat{y}$ is the natural embedding of Y into Y^{**}. Hence $\overline{U(S)}^{w}$ is weakly compact. Thus U is weakly compact.

Now, we are in position to prove our main result.

5.2. Theorem:- Every well-bounded operator T on X is of type (B).

Proof: Let $f \longmapsto f(T)$ be the algebra homomorphism from $AC(J)$ into $L(X)$ mentioned in Lemma I.1.1. Then, for each x in X, $f \longmapsto f(T)x$ is a linear map from $AC(J)$ into X. Since X is reflexive, it follows from Theorem 1 that $f \longmapsto f(T)x$ is weakly compact and hence, by Theorem 4.3, T is of type (B).

We restate our main result in the following form.

5.3. Theorem:- Let T be a bounded operator on X. Then T is well-bounded if and only if there exist a compact interval $J = [a,b]$, a constant K and a family $\{E(s) : s \in R\}$ of projections such that

(i) $||E(s)|| \leqslant K \qquad (s \in R)$

(ii) $E(s) = 0 \quad (s < a), \quad E(s) = I \quad (s \geqslant b)$

(iii) $E(s)E(t) = E(t)E(s) = E(s) \qquad (s < t)$

(iv) $\lim_{t \to s+} E(t)x = E(s)x$

(v) $\lim_{t \to s-} E(t)$ exists in the strong operator topology.

(vi) $T = \int_J^{\oplus} s \, dE(s)$.

Proof: Suppose T is well-bounded. Then the result follows from Theorem 2 and Theorem 4.3. Conversely, let $\{E(s) : s \in R\}$ be a family of projections satisfying conditions (i) $\longrightarrow$ (vi). Then $E \in \mathcal{E}(J)$. For any f in $AC(J)$, we define

$$\Psi(f) = \int_J^{\oplus} f \, dE.$$

Then, the argument in the proof of Theorem 4.3 ((i) $\Longrightarrow$ (ii)) shows that Ψ is an algebra homomorphism. Moreover, by (vi),

$$\Psi(s \longmapsto s) = T \qquad \text{and}$$

$$\Psi(s \longmapsto 1) = \int_J^{\oplus} 1 \, dE(s) = I.$$

Then, we conclude that

$$\Psi(p) = \int_J^{\oplus} p(s) \, dE(s) = p(T)$$

for any complex polynomial p.

Hence, it follows from Lemma 1.6 that

$$\begin{aligned} ||p(T)|| &= ||\int_J^{\oplus} p(s)\ dE(s)|| \\ &\leq \sup_R ||E(s)||\ |||p||| \\ &\leq K|||p|||. \end{aligned}$$

Thus T is well-bounded.

CHAPTER IV:-

ON WELL-BOUNDED AND SCALAR-TYPE OPERATORS:-

1. Some relationships between well-bounded and scalar-type operators:-

In this section, we shall use various properties of scalar-type operators, prespectral operators and spectral operators. For the definitions and properties of these classes of operators the reader is referred to (3) and (6).

1.1. Definition:- Let T be a well-bounded operator on X. A decomposition of the identity $\{E(s) : s \in R\}$ for T is said to be of bounded variation if and only if the function $\langle x, E(\cdot)x^* \rangle$ is of bounded variation on R for every x in X and x* in X*.

1.2. Theorem:- Let T be a bounded operator on X and $\sigma(T) \subseteq R$. Then the following conditions are equivalent:

(i) T is a well-bounded operator with a decomposition of the identity of bounded variation.

(ii) There are a compact interval J and a constant M such that

$$||p(T)|| \leq 4M \sup_J |p(t)|$$

for every complex polynomial p.

(iii) T* is a scalar-type operator of class X. If (i) holds, then T is a uniquely decomposable well-bounded operator.

Proof: We prove the second statement of the theorem first. Suppose that $\{E(t) : t \in R\}$ is a decomposition of the identity of bounded variation for T. Then from Theorem I.3.3, we deduce that

$$\langle f(T)x, x^* \rangle = f(b) \langle x, x^* \rangle - \int_a^b \langle x, E(t)x^* \rangle f'(t)\, dt \quad (f \in AC(J))$$

for each x in X and x* in X*. It follows that if w_{x,x^*} is the function constructed in Lemma I.2.6, then

$$w_{x,x^*}(t) = \langle x, E(t)x^* \rangle \qquad (\text{a.e. on } [a,b]).$$

Now, since $\langle x, E(\cdot)x^* \rangle$ is in BV(J), its discontinuities are at most countable.

Hence it is equal a.e. to a right continuous function u_{x,x^*} in $BV(J)$ and so

$$u_{x,x^*}(t) = w_{x,x^*}(t) \qquad \text{(a.e. on } [a,b] \text{) .}$$

It follows that the indefinite integral of w_{x,x^*} is differentiable on the right, with right-hand derivative u_{x,x^*} at every point of $[a,b)$. Hence, by Theorem I.4.3, T is uniquely decomposable. Moreover, by condition (v) of Definition I.1.2,

$$u_{x,x^*}(t) = \langle x, E(t)x^* \rangle \qquad (t \in [a,b)).$$

Since the function $\langle x, E(t)x^* \rangle$ is constant on the intervals $(-\infty, a)$ and $[b, \infty)$, this function is continuous on the right at every point of R.

We now prove the first statement of the theorem. Suppose that (i) holds. Then we can find a decomposition of the identity $\{E(s) : s \in R\}$ for T, a compact interval $J = [a,b]$ and a constant K such that

$$E(s) = 0 \ (s \leq a), \quad E(s) = I \ (s \geq b), \quad \|E(s)\| \leq K \ (s \in R)$$

and for every x in X, x^* in X^*, the function $\langle x, E(\cdot)x^* \rangle$ is a right continuous function of bounded variation on R. Let Σ be the algebra of subsets of $(a,b]$ expressible as a finite disjoint union of intervals of the form $(a_i, b_i]$. We define μ on Σ as follows

$$\mu\left(\bigcup_{i=1}^{n} (a_i, b_i] \right) = \sum_{i=1}^{n} \{ E(b_i) - E(a_i) \} \quad .$$

It is easy to verify that μ is well-defined and finitely additive on Σ . For x in X and x^* in X^*

$$\left| \langle x, \mu\left(\bigcup_{i=1}^{n} (a_i, b_i] \right) x^* \rangle \right| \leq \sum_{i=1}^{n} | \langle x, (E(b_i) - E(a_i)) x^* \rangle |$$
$$\leq \operatorname{var} \langle x, E(\cdot)x^* \rangle < \infty \quad .$$

Hence by the uniform boundedness principle there is a constant M such that

$$\|\mu(\sigma)\| \leq M \qquad (\sigma \in \Sigma).$$

Now,

$$| \langle x, \mu(\sigma)x^* \rangle | \leq M \, \|x\| \, \|x^*\| \qquad (\sigma \in \Sigma),$$

and so ~~it~~ from Lemma III.1.5 of (6) (p.97) ~~that~~

$$\operatorname{var} \langle x, E(\cdot)x^* \rangle \leq 4 \, M \, \|x\| \, \|x^*\|.$$

Let p be any complex polynomial. Then

$$\langle p(T)x,x^*\rangle = p(b)\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\, p'(t)\, dt$$
$$= p(b)\langle x,x^*\rangle - \int_a^b \langle x,E(t)x^*\rangle\, dp(t) \qquad (x\in X,\ x^*\in X^*).$$

On integrating by parts, we obtain

$$\langle p(T)x,x^*\rangle = \int_a^b p(t)\, d\langle x,E(t)x^*\rangle \qquad (x\in X,\ x^*\in X^*).$$

$$\therefore\ |\langle p(T)x,x^*\rangle| \leq \sup_J |p(t)|\ \mathrm{var}\langle x,E(\cdot)x^*\rangle$$
$$\leq 4\, M\, ||x||\ ||x^*||\ \sup_J |p(t)|;$$
$$||p(T)|| \leq 4\, M \sup_J |p(t)|$$

for every complex polynomial p. Thus (i) implies (ii). Now suppose that (ii) holds. By the Weierstrass polynomial theorem we may extend the map $p \longmapsto p(T)$ in the obvious way to get a continuous algebra homomorphism of $C(J)$ into $L(X)$ such that

$$||f(T)|| \leq 4\, M \sup_J |f(t)|,$$

and so, by Theorem XVII.2.4 of (6) (p.2184), (ii) implies (iii). Now suppose that (iii) holds. Then by Theorem 5.2 of (3) (p.306) T^* has a unique resolution of the identity, $\mathcal{F}(\cdot)$ say, of class X. Let $J = [a,b]$ be a compact interval containing $\sigma(T)$ such that $a \notin \sigma(T)$ and let p be any complex polynomial. Then if $||\mathcal{F}(\cdot)|| \leq M$, we have

$$||p(T)|| = ||p(T^*)|| \leq 4\, M \sup_J |p(t)|$$
$$= 4\, M \sup_J \left|p(b) - \int_t^b p'(s)\, ds\right|$$
$$\leq 4\, M\ |||p|||. \qquad (40)$$

Hence T and T^* are well-bounded. Now, let $f \in AC(J)$. Then by the Weierstrass polynomial theorem there is a sequence $\{p_n\}$ of polynomials converging uniformly to f on J. The inequality (40) shows that $\{p_n(T)\}$ converges in the norm of $L(X)$ to an operator $f(T)$ and, moreover, $f(T)$ is independent of the approximating sequence chosen. Also the map $f \longmapsto f(T)$ is multiplicative and

$$||f(T)|| \leq 4\, M \sup_J |f(t)| \leq 4\, M\ |||f||| \qquad (f \in AC(J)).$$

Hence by uniqueness clause in Lemma I.1.1, the map $f \longmapsto f(T)$ is the homomorphism of AC(J) into L(X) described in the statement of that Lemma.

Define

$$E(t) = \mathcal{G}((-\infty,t]) \qquad (t \in R). \tag{41}$$

We shall show that $\{E(t) : t \in R\}$ is a decomposition of the identity of bounded variation for T. Let $x \in X$ and $x^* \in X^*$. Since $\mathcal{G}(\cdot)$ is of class X, it follows that

$$\begin{aligned}\lim_{n\to\infty} \langle x, E(t+\tfrac{1}{n})x^*\rangle &= \lim_{n\to\infty} \langle x, \mathcal{G}((-\infty, t+\tfrac{1}{n}])\rangle \\ &= \langle x, \mathcal{G}((-\infty,t])\rangle \\ &= \langle x, E(t)x^*\rangle \qquad (t \in R).\end{aligned}$$

Hence $\langle x, E(\cdot)x^*\rangle$ is everywhere right-continuous. Moreover, since $\langle x, \mathcal{G}(\cdot)x^*\rangle$ is a bounded countably additive set function, by Lemma III.1.5 of (6), we have

$$\operatorname{var}\langle x, \mathcal{G}(\cdot)x^*\rangle \leq 4\,M\,||x||\;||x^*||.$$

Hence $\quad \operatorname{var}\langle x, E(\cdot)x^*\rangle \leq 4\,M\,||x||\;||x^*||.$

Thus the function $\langle x, E(\cdot)x^*\rangle$ is of bounded variation on R. The conditions (i), (ii), (iii) of Definition I.1.2 follow immediately from (41). Conditions (iv) and (v) follow from the right-continuity of $\langle x, E(\cdot)x^*\rangle$.

If $f \in AC(J)$,

$$\begin{aligned}\langle f(T)x, x^*\rangle = \langle x, f(T^*)x^*\rangle &= \int_a^b f(t)\, d\langle x, E(t)x^*\rangle \\ &= f(b)\langle x, x^*\rangle - \int_a^b \langle x, E(t)x^*\rangle\, df(t),\end{aligned} \tag{42}$$

on integrating by parts. In particular, let $f(t) \equiv t$; we have

$$\langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(t)x^*\rangle\, dt.$$

It remains to prove (vi). Fix x in X and consider

$$Au = f_u(T)x,$$

where

$$f_u(s) = \int_s^b u(t)\, dt \qquad (u \in L^1(J)).$$

A is clearly bounded and linear. For u in $L^1(J)$ and x* in X*, we have

$$\begin{aligned}\langle u, A^*x^* \rangle &= \langle Au, x^* \rangle = \langle f_u(T)x, x^* \rangle \\ &= \langle x, f_u(T^*)x^* \rangle \\ &= \int_a^b f_u(t)\, d\langle x, E(t)x^* \rangle \\ &= \int_a^b \langle x, E(t)x^* \rangle\, u(t)\, dt\end{aligned}$$

using (42) and then integrating by parts. It follows that the map A* from X* into $L^{\infty}(J)$ is given by

$$A^*x^* = \langle x, E(\cdot)x^* \rangle \quad .$$

Since A is continuous, A* is continuous when X* and $L^{\infty}(J)$ are endowed with their weak*-topologies. This completes the proof that (iii) implies (i).

1.3. Theorem:- Let X be weakly complete , and T in L(X) satisfy $\sigma(T) \subseteq R$. Then the following three conditions are equivalent.

(i) T is a well-bounded operator with a decomposition of the identity of bounded variation.

(ii) There are a compact interval J and a constant M such that

$$||p(T)|| \leq 4\, M\, \sup_J |p(t)|$$

for every complex polynomial p.

(iii) T is a scalar-type spectral operator.

If this is the case, then T is decomposable in X.

Proof: The equivalence of (i) and (ii) was shown in the previous theorem. Now, assume (iii) holds. Then there is a resolution of identity $\mathcal{E}(\cdot)$ of class X* such that

$$T = \int_{\sigma(T)} t\, \mathcal{E}(dt) \quad .$$

Let J = [a,b] be a compact interval containing $\sigma(T)$ and p be any complex polynomial. Then

$$p(T) = \int_a^b p(t)\, \mathcal{E}(dt)$$

and

$$|\langle p(T)x,x^*\rangle| = |\langle \int_a^b p(t)\,\mathcal{G}(dt)x,x^*\rangle|$$
$$\leq 4\,M\,\sup_J|p(t)|\ ||x||\ ||x^*||$$

where M is the constant such that $||\mathcal{G}(\cdot)|| \leq M$. Hence

$$||p(T)|| \leq 4\,M\,\sup_J|p(t)|.$$

This proves that (iii) $\Longrightarrow$ (ii). Now assume (ii) holds. By the Weierstrass polynomial theorem we may extend the map $p \mapsto p(T)$ in the obvious way to get a continuous algebra homomorphism of $C(J)$ into $L(X)$ such that

$$||f(T)|| \leq 4\,M\,\sup_J|f(t)|.$$

Then,since X is weakly complete, by Theorem XVII.2.5 of (6)(p.2186) (ii) implies (iii).

If (i) holds, then by Theorem 2, T is uniquely decomposable and hence by Theorem II.1.2, T is decomposable in X.

1.4. Theorem:- A scalar-type spectral operator T with $\sigma(T) \subseteq R$ is a well-bounded operator of type (B).

Proof: Let $\mathcal{G}(\cdot)$ be the resolution of the identity for T. Then T^* is a scalar-type operator with unique resolution of the identity $\mathcal{G}^*(\cdot)$ of class X, by Theorem 3.11 of (3) (p.299). Hence T is well-bounded with unique decomposition of the identity $\{E(s) : s \in R\}$ given by

$$E(s) = \mathcal{G}^*((-\infty,s]) \qquad (s \in R).$$

Let $F(s) = \mathcal{G}((-\infty,s])$, then $F^*(s) = E(s)$ $(s \in R)$. Hence T is decomposable in X. Let $x \in X$. For any strictly decreasing sequence $\{t_n\}$ (respectively, strictly increasing sequence $\{s_n\}$) of real numbers converging to t (respectly, to s), then by using the countable additivity of $\mathcal{G}(\cdot)$, we have

$$\lim_{n\to\infty}(F(t_n) - F(t))x = \lim_{n\to\infty}\mathcal{G}((t,t_n])x$$
$$= \lim_{n\to\infty}\sum_{j=n}^{\infty}\mathcal{G}((t_{j+1},t_j])x = 0.$$

Thus $$\lim_{n\to\infty}F(t_n)x = F(t)x \qquad (x \in X).$$

Since

$$\bigcup_{n=1}^{\infty} (-\infty, s_n] = (-\infty, s) ,$$

$$\begin{aligned} \mathcal{G}((-\infty,s))x &= \sum_{k=1}^{\infty} \mathcal{G}_n((-\infty,s_n])x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mathcal{G}((-\infty,s_k])x \\ &= \lim_{n\to\infty} \mathcal{G}((-\infty,s_n])x \\ &= \lim_{n\to\infty} F(s_n)x \qquad (x \in X) . \end{aligned}$$

Hence

$$\lim_{n\to\infty} F(s_n)x = \mathcal{G}\{(-\infty,s]\}x \qquad (x \in X).$$

It follows that, in the strong operator topology, $\lim_{t\to s+} F(t) = F(s)$, $\lim_{t\to s-} F(t) = \mathcal{G}\{(-\infty,s]\}$, and so T is well-bounded of type (B).

In order to prove further relationships between the classes of scalar-type operators and well-bounded operators we require the concept of the single-valued extension property. Let $T \in L(X)$ and $x \in X$. An X-valued function f is called an analytic extension of $(\xi I-T)^{-1}x$ if and only if it is defined and analytic on an open set D(f) containing $\rho(T)$ and such that

$$(\xi I - T)f(\xi) = x \qquad \xi \in D(f).$$

It is clear that, for such an extension

$$f(\xi) = (\xi I - T)^{-1}x \qquad \xi \in \rho(T).$$

1.5. <u>Definition</u>:- The function $(\xi I - T)^{-1}x$ is said to have the single-valued extension property if for every pair f,g of such extensions we have $f(\xi) = g(\xi)$ for all ξ in $D(f) \cap D(g)$. The union of the sets D(f) as f varies over all analytic extensions of $(\xi I - T)^{-1}x$ is called the resolvent set of x and is denoted by $\rho(x)$. The spectrum $\sigma(x)$ of x is defined to be the complement of $\rho(x)$ in the complex plane.

It is clear that if $(\xi I - T)^{-1}x$ has the single-valued extension property, then there is a maximal analytic extension $x(\xi)$ of $(\xi I - T)^{-1}x$ whose domain is $\rho(x)$. If $(\xi I - T)^{-1}x$ has the single-valued extension property for every x in X, then T is said to have the single-valued extension property. It follows readily from the definitions that if $T \in L(X)$ and $\sigma(T)$ is nowhere dense, then T has the single-valued extension property.

We need the following lemmas.

1.6. Lemma:- Let $T \in L(X)$ and let Y be a closed subspace of X with $TY \subset Y$ such that $\sigma(T/Y)$ is nowhere dense. Then $\rho(T) \subseteq \rho(T/Y)$.

Proof: If $\lambda \in \rho(T)$ then $(\lambda I - T)^{-1}$ exists as a bounded linear operator on X. Suppose on the contrary that $\lambda \notin \rho(T/Y)$ i.e. $\lambda \in \sigma(T/Y)$, then $(\lambda I - T)^{-1}Y \not\subset Y$. Since otherwise $(\lambda I - T)^{-1}/Y$ is a bounded linear operator on Y inverse to $(\lambda I - T)/Y$. Hence there is y in Y and y^* in $Y^{\perp}$ such that

$$\langle (\lambda I - T)^{-1} y, y^* \rangle \neq 0 . \tag{43}$$

It is well-known that the resolvent function $R(\xi, T) = (\xi I - T)^{-1}$ defined on $\rho(T)$ is analytic in $\rho(T)$. Hence the function defined by

$$\phi(\xi) = \langle (\xi I - T)^{-1} y, y^* \rangle$$

is continuous on $\rho(T)$. It follows from (43) that there is a neighbourhood $N_\varepsilon(\lambda)$ such that

$$\phi(\xi) = \langle (\xi I - T)^{-1} y, y^* \rangle \neq 0 \qquad (\xi \in N_\varepsilon(\lambda)).$$

Therefore, for all $\xi \in N_\varepsilon(\lambda)$

$$(\xi I - T)^{-1} Y \not\subseteq Y,$$

and hence $\xi \in \sigma(T/Y)$.

Thus $N_\varepsilon(\lambda) \subseteq \sigma(T/Y)$ which contradicts the fact that $\sigma(T/Y)$ is nowhere dense. Therefore $\lambda \in \rho(T/Y)$ and hence $\rho(T) \subseteq \rho(T/Y)$.

1.7. Lemma:- Let T, in $L(X)$, have nowhere dense spectrum. Let Y be closed subspace of X with $TY \subset Y$ such that $\sigma(T/Y)$ is nowhere dense. Then if $y \in Y$ and $\sigma_X(y)$, $\sigma_Y(y)$ denote the spectra of y with respect to T and T/Y, respectively, we have $\sigma_X(y) = \sigma_Y(y)$.

Proof: Let $y_X(\xi)$ and $y_Y(\xi)$ be the maximal X-valued and Y-valued analytic functions satisfying

$$(\xi I - T) y(\xi) = y.$$

Let $R_X(y)$ and $R_Y(y)$ be the closed subspaces defined by

$$R_X(y) = \overline{sp}\left\{(\zeta I - T)^{-1}y : \zeta \in \rho(T)\right\},$$

$$R_Y(y) = \overline{sp}\left\{(\zeta I/Y - T/Y)^{-1}y : \zeta \in \rho(T/Y)\right\}.$$

Since by Lemma 6, $\rho(T/Y) \supseteq \rho(T)$, we have

$$R_Y(y) \supseteq R_X(y). \tag{44}$$

Let $\rho_X(y)$ and $\rho_Y(y)$ be the domains of definition of $y_X(\zeta)$ and $y_Y(\zeta)$ respectively. Since a Y-valued function is also X-valued, the maximality of $\rho_X(y)$ implies that $\rho_Y(y) \subseteq \rho_X(y)$. Now, by hypothesis, $\rho(T)$ is dense in the plane, and there is therefore a sequence of points of $\rho(T)$ converging to each point in $\rho_X(y)$. Since $y_X(\zeta)$ is analytic we have $y_X(\zeta) \in R_X(y)$ for every ζ in $\rho_X(y)$. Hence

$$R_X(y) = \overline{sp}\left\{y_X(\zeta) : \zeta \in \rho_X(y)\right\}.$$

Similarly, $$R_Y(y) = \overline{sp}\left\{y_Y(\zeta) : \zeta \in \rho_Y(y)\right\}.$$

Since $\rho_Y(y) \subseteq \rho_X(y)$, we have

$$R_Y(y) \subseteq R_X(y). \tag{45}$$

From (44) and (45) $R_Y(y) = R_X(y)$, and so $y_X(\zeta)$ is a Y-valued function. By the maximality of $\rho_Y(y)$ it now follows that $\rho_X(y) \subseteq \rho_Y(y)$. Hence $\rho_X(y) = \rho_Y(y)$, $\sigma_X(y) = \sigma_Y(y)$, and $y_Y(\zeta) = y_Y(\zeta)$ $\quad (\zeta \in \rho_X(y))$.

1.8. Lemma:- Let T, in L(X), have the single-valued extension property. Then the spectrum $\sigma(x)$ of x is void if and only if $x = 0$.

Proof: If $\sigma(x)$ is void, then the hypotheses imply that $x(\zeta)$ is everywhere defined single-valued, and hence entire. By Liouville's theorem, $x(\zeta)$ is constant. Moreover, the series

$$(\zeta I - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\zeta^{n+1}}$$

converges in the uniform operator topology provided that $|\zeta| \geqslant \sup\left\{|\lambda| : \lambda \in \sigma(T)\right\}$.
Hence, for $x^* \in X^*$

$$\lim_{\zeta\to\infty} \langle x(\zeta), x^*\rangle = \lim_{\zeta\to\infty} \langle(\zeta I - T)^{-1}x, x^*\rangle$$

$$= \lim_{\xi\to\infty} \left(\frac{\langle x,x^*\rangle}{\xi} + \frac{\langle Tx,x^*\rangle}{\xi^2} + \ldots\ldots\ldots\ldots\ldots\ldots \right)$$
$$= 0 \ .$$

Thus $\langle x(\xi),x^*\rangle = 0$ for all ξ and all $x^* \in X^*$. It follows that $x(\xi) = 0$ and thus $x = (\xi I - T)x(\xi) = 0$. The converse is obvious.

Before proving our next result, we observe that if $T \in L(X)$ and $\sigma(T) \subseteq R$, then $\sigma(T^*) \subseteq R$ and so T^* has the single-valued extension property.

II.9. Theorem:- Let T be a well-bounded operator on X, and let $\{E(s) : s \in R\}$ be a decomposition of the identity for T. Then

$$E(s)X^* = \{ y \in X^* : \sigma(y) \subseteq (-\infty, s] \} \qquad (s \in R).$$

Proof: It was shown in Theorem I.2.4 that for every s in R, $E(s)X^*$ is a closed subspace of X^* invariant under T^* and this subspace does not depend on the particular decomposition of the identity chosen. Since by Theorem I.2.13 there is a decomposition of the identity for T which commutes with T^*, there is no loss of generality in assuming that $T^*E(s) = E(s)T^*$ ($s \in R$). It was shown in Theorem I.2.4 that $\sigma(T^*/E(s)X^*) \subseteq (-\infty, s]$ and in the course of proving Theorem II.1.1 that $\sigma(T^*/(I^* - E(s))X^*) \subseteq [s, \infty)$. Let $y \in E(s)X^*$ and $Y = E(s)X^*$. Then by Lemma 7

$$\sigma(y) = \sigma_Y(y) \subseteq \sigma(T^*/E(s)X^*) \subseteq (-\infty, s]$$

where $\sigma_Y(y)$ denotes the spectrum of y with respect to $T^*/E(s)X^*$, so that

$$E(s)X^* \subseteq \{ y \in X^* : \sigma(y) \subseteq (-\infty, s] \} .$$

Conversely, let $\sigma(y) \subseteq (-\infty, s]$ and $\theta > 0$. Then since

$$E(s+\theta)T^* = T^*E(s+\theta)$$

we have

$$(\xi I^* - T^*)(I^* - E(s+\theta))y(\xi)$$
$$= (I^* - E(s+\theta))(\xi I^* - T^*)y(\xi)$$
$$= (I^* - E(s+\theta))y$$

so $(I^* - E(s+\theta))y(\xi)$ is an analytic extension of $(\xi I^* - T^*)^{-1}(I^* - E(s+\theta))y$

to $\rho(y)$. Hence $\rho((I^* - E(s+\theta))y) \supseteq \rho(y)$ so that

$$\sigma((I^* - E(s+\theta))y) \subseteq \sigma(y) \subseteq (-\infty, s].$$

Let $Y = \{I^* - E(s+\theta)\} X^*$. Then, by Lemma 7

$$\sigma((I^* - E(s+\theta))y) = \sigma_Y((I^* - E(s+\theta))y)$$
$$\subseteq \sigma(T^*/\{I^* - E(s+\theta)\}X^*) \subseteq [s+\theta, \infty).$$

Hence $\sigma((I^* - E(s+\theta))y)$ is void and so by Lemma 8, $E(s+\theta)y = y$ for all $\theta > 0$.

Therefore, by Theorem I.2.4, $x^* \in E(s)X^*$. Thus

$$E(s)X^* = \{y \in X^* : \sigma(y) \subset (-\infty, s]\}$$

and the proof is complete.

1.10. <u>Theorem</u>:- Let T, in L(X), be a well-bounded operator which is decomposable in X. Then if $\{F(s) : s \in R\}$ is the family of projections on X whose adjoints form the unique decomposition of the identity for T, we have

$$F(s)X = \{x \in X : \sigma(x) \subseteq (-\infty, s]\} \qquad (s \in R).$$

Proof: By Theorem II.1.1, we have $\sigma(T/F(s)X) \subseteq (-\infty, s]$ and $\sigma(T/(I - F(s))X) \subseteq [s, \infty)$, for all s in R. Moreover, since $F(t)F(s) = F(s)$ for all $s < t$, we have $F(s)X \subseteq \bigcap_{t>s} F(t)X$. On the other hand, if $F(t)x = x$ for all $t > s$, then for any x^* in X^*, the funct-ion $\langle F(t)x, x^*\rangle = \langle x, x^*\rangle$ is constant on (s, ∞). It follows from Theorem II.1.1(v) that

$$\langle F(s)x, x^*\rangle = \langle x, x^*\rangle$$

for all $x^* \in X^*$. Hence $F(s)x = x$. Thus

$$F(s)X = \bigcap_{t>s} F(t)X.$$

The argument of Theorem 9, then suffices to establish the present theorem.

1.11. <u>Theorem</u>:- Suppose that T, in L(X), is both a spectral operator and a well-bounded operator. Then T is a scalar-type spectral operator.

Proof: Let $\mathcal{E}(\cdot)$ be the resolution of the identity for T, and let

$$S = \int_{\sigma(T)} s\,\mathcal{E}(ds).$$

Since T is well-bounded, by Theorem I.2.13, it possesses a decomposition of the identity $\{E(s) : s \in R\}$ with the following property: if $A \in L(X)$ and $AT = TA$ then

$$A^*E(s) = E(s)A^* \qquad (s \in R).$$

Now by Theorem 9,

$$E(s)X^* = \{y \in X^* : \sigma(y) \subseteq (-\infty, s]\} \qquad (s \in R).$$

Also by Theorem 3.11 of (3), T^* is a prespectral operator on X^* with unique resolution of the identity $\mathcal{E}^*(\cdot)$ of class X. Hence by Theorem XV.3.4 of (6),

$$\mathcal{E}^*((-\infty, s])X^* = \{y \in X^* : \sigma(y) \subseteq (-\infty, s]\} .$$

Since $\mathcal{E}((-\infty, s])T = T\mathcal{E}((-\infty, s])$, we have

$$\mathcal{E}^*((-\infty, s])E(s) = E(s)\mathcal{E}^*((-\infty, s]).$$

Therefore $\mathcal{E}^*((-\infty, s])$ and $E(s)$ are commuting projections with the same range and so are equal. It follows that for all x in X and y in X^*, the function $\langle x, E(\cdot)y\rangle$ is everywhere continuous on the right and of bounded variation on R. Since $\{E(s) : s \in R\}$ is a decomposition of the identity for T , we have

$$\langle Tx, y\rangle = b\langle x, y\rangle - \int_a^b \langle x, E(s)y\rangle \, ds \quad (x \in X, y \in X^*) \qquad (46)$$

where $J = [a, b]$ is the compact interval such that

$$E(s) = 0 \quad (s < a) \quad \text{and} \quad E(s) = I \quad (s \geq b).$$

Then $\langle x, E(s)x^*\rangle = 0$ whenever $s < a$. Hence (46) can be rewritten as follows:

$$\langle Tx, y\rangle = b\langle x, y\rangle - \int_{a-\theta}^b \langle x, E(s)y\rangle \, ds$$

for any $\theta > 0$. On integrating by parts, we obtain

$$\langle Tx, y\rangle = \int_{a-\theta}^b s \, d\langle x, E(s)y\rangle \qquad (x \in X, \ y \in X^*).$$

However, by Corollary 1 of Theorem I.2.4, $\sigma(T) \subseteq [a, b] \subseteq [a-\theta, b]$, and so

$$\langle Sx, y\rangle = \int_{a-\theta}^b s\mu(ds) = \int_{a-\theta}^b s \, d\langle x, E(s)y\rangle \qquad (x \in X, \ y \in X^*)$$

where $\mu(\cdot) = \langle \mathcal{G}(\cdot)x,y \rangle$. It follows that T = S and this completes the proof.

1.12. Theorem:- Suppose that T, in L(X), is both a prespectral operator of class Γ and a well-bounded operator decomposable in X. Then T is a scalar-type operator of class Γ.

Proof: Let $\{F(s) : s \in R\}$ be the family of projections on X whose adjoints form the unique decomposition of the identity for T. Let

$$E(s) = F^*(s) \qquad (s \in R).$$

Let $\mathcal{G}(\cdot)$ be the resolution of the identity of class Γ for T, and let

$$S = \int_{\sigma(T)} s \,\mathcal{G}(ds) \;.$$

By Theorem 10,

$$F(s)X = \{x \in X : \sigma(x) \subseteq (-\infty,s]\} \qquad (s \in R).$$

Also by Theorem XV.3.4 of (6)

$$\mathcal{G}((-\infty,s])X = \{x \in X : \sigma(x) \subseteq (-\infty,s]\} \qquad (s \in R).$$

Now, $\mathcal{G}((-\infty,s])$ commutes with T and so by Theorem II.1.1(iv),

$$\mathcal{G}((-\infty,s])F(s) = F(s)\mathcal{G}((-\infty,s]) \qquad (s \in R).$$

Hence $\mathcal{G}((-\infty,s])$ and F(s) are commuting projections with the same range and so are equal. It follows that for all x in X and y in Γ the function $\langle F(\cdot)x,y \rangle$ is everywhere continuous on the right and of bounded variation on R. Since $\{E(s) : s \in R\}$ is a decomposition of the identity for T, we have

$$\begin{aligned}\langle Tx,y \rangle &= b\langle x,y \rangle - \int_a^b \langle x,E(s)y \rangle \, ds \\ &= b\langle x,y \rangle - \int_a^b \langle F(s)x,y \rangle \, ds \;.\end{aligned}$$

Without loss of generality we may assume that E(a) = 0, otherwise we may replace a by a-θ (any θ > 0) as shown in the proof of Theorem 11. On integrating by parts, we obtain

$$\langle Tx,y \rangle = \int_a^b s \; d\langle F(s)x,y \rangle \qquad (x \in X,\ y \in \Gamma).$$

However by Corollary 1 of Theorem I.2.4, $\sigma(T) \subseteq [a,b]$, and so

$$\langle Sx,y\rangle = \int_a^b s\,\mu(ds) = \int_a^b s\, d\langle F(s)x,y\rangle \qquad (x\in X,\ y\in \Gamma)$$

where $\mu(\cdot) = \langle \mathcal{G}(\cdot)x,y\rangle$. Since Γ is total, S = T, and the proof is complete.

We conclude this section by giving an example of an operator which is both prespectral and well-bounded, but which is not a scalar-type operator.

1.13. Example:- Let X be the complex Banach space $L^{\infty}[0,1] \oplus L^{1}[0,1]$ with the norm defined as follows: if $f\in L^{\infty}[0,1]$ and $g\in L^{1}[0,1]$

$$|||\{f,g\}||| = ||f||_{\infty} + \int_0^1 |g(t)|\,dt.$$

Define operators S,N and T on X by

$$S : \{f(t),g(t)\} \longrightarrow \{tf(t),tg(t)\} \qquad (t\in[0,1]),$$

$$N : \{f(t),g(t)\} \longrightarrow \{0,f(t)\} \qquad (t\in[0,1]),$$

$$T = S + N.$$

We have shown in the Example I.3.4 that T is well-bounded operator. Moreover, it is easy to verify that S is a scalar-type operator on X with a resolution of the identity E(·) of class $L^{1}[0,1] \oplus L^{\infty}[0,1]$ given by

$$E(\tau) : \{f(t),g(t)\} \longrightarrow \{\chi_{\tau}(t)f(t),\ \chi_{\tau}(t)g(t)\} \quad (\tau\in\Sigma_p)$$

where Σ_p denotes the σ-algebra of Borel subsets of ~~the~~ complex plane. Observe that $E(\tau)N = NE(\tau)$ $(\tau\in\Sigma_p)$ and so by Theorem 3.5(ii) of (3) (p.296) T is prespectral on X with a resolution of the identity E(·) of class $L^{1}[0,1] \oplus L^{\infty}[0,1]$. T = S + N is the corresponding Jordan decomposition of T. Also T is not a scalar-type operator, for, if this were the case, then since $\sigma(T) = [0,1]$, T would have unique Jordan decomposition T + 0 by Theorem 5.2 of (3) (p.306).

2. <u>A counterexample</u>:-

In the last section, we have shown that a scalar-type spectral operator with real spectrum is also a well-bounded operator of type (B). We shall show that the converse is not true. The following counterexample is due to Dowson and Spain (5).

Let $1 < p \leqslant \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Let Z be the locally compact abelian group of integers with counting measure. Let $Z^{\wedge}$ be its dual group, the circle, which is isomorphic to the interval $[0,2\pi]$ with its endpoints identified. Haar measure on $[0,2\pi]$ is Lebesgue measure divided by 2π. We denote the norm on $L^p(Z)$ by $||\cdot||_p$.

Let B be the Hilbert form defined on $L^p(Z) \times L^q(Z)$ by

$$B(\xi,\lambda) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{\xi(m)\lambda(n)}{m-n}$$

where the dash implies the omission of the terms in which m = n.

It is well-known that there is constant B_p, depending on p such that

$$\left| \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{\xi(m)\lambda(n)}{m-n} \right| \leqslant B_p||\xi||_p||\lambda||_q. \qquad (47)$$

We need the following result which is due to Steckin (15).

2.1. <u>Theorem</u>:- Let f be of bounded variation on $[0,2\pi]$ with $f(0) = f(2\pi)$.
Define

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-int}\, dt .$$

Then the map $F: \xi \longrightarrow c * \xi$ is a bounded linear operator on $L^p(Z)$ where $c = \{c_k\}$ and $c * \xi\,(n) = \sum_{m=-\infty}^{\infty} c_{n-m}\,\xi(m)$.

Proof: Let $\lambda \in L^q(Z)$. Consider

$$\begin{aligned} T_N &= \sum_{m=-N}^{N} \sum_{n=-N}^{N} c_{n-m}\,\xi(m)\lambda(n) \\ &= c_0\sum_{m=-N}^{N}\xi(m)\lambda(m) + \sum_{m=-N}^{N} \sum_{n=-N}^{N}{}' c_{n-m}\,\xi(m)\lambda(n) \end{aligned}$$

where the dash implies the omission of the terms in which m = n.

Since

$$| T_N' | = | \sum_{m=-N}^{N} \sum_{n=-N}^{N}{}' c_{n-m} \xi(m)\lambda(n) |$$

$$= | \sum_{m=-N}^{N} \sum_{n=-N}^{N}{}' \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-i(n-m)t}\, dt\, \xi(m)\lambda(n) |$$

$$= | \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{m=-N}^{N} \sum_{n=-N}^{N}{}' \xi(m)\lambda(n)\, e^{-i(n-m)t} \right) dt |.$$

On integrating by parts, we have

$$| T_N' | = \left| \frac{1}{2\pi i} \int_0^{2\pi} \left(\sum_{m=-N}^{N} \sum_{n=-N}^{N}{}' \frac{\xi(m)\lambda(n)\, e^{-i(n-m)t}}{n-m} \right) df(t) \right|$$

$$\leqslant \frac{1}{2\pi} \sup_{t\in[0,2\pi]} \left| \sum_{n=-N}^{N} \sum_{n=-N}^{N}{}' \frac{\xi(m)\, e^{imt}\lambda(n)\, e^{-int}}{n-m} \right| \mathrm{var}(f,[0,2\pi])$$

$$\leqslant \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\, ||\xi||_p\, ||\lambda||_q \qquad \text{(From (47)).}$$

Hence

$$| T_N | \leqslant |c_0|\, ||\xi||_p\, ||\lambda||_q + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\, ||\xi||_p\, ||\lambda||_q$$

$$= \left(|c_0| + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\right) ||\xi||_p\, ||\lambda||_q.$$

Since the right-hand side of the above inequality is independent of N, we let $N \to \infty$, and obtain

$$\left| \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{n-m} \xi(m)\lambda(n) \right| \leqslant \left(|c_0| + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\right) ||\xi||_p ||\lambda||_q.$$

i.e.

$$\left| \sum_{n=-\infty}^{\infty} (c * \xi)(n)\lambda(n) \right|$$

$$\leqslant |c_0| + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\, ||\xi||_p\, ||\lambda||_q$$

for all $\lambda \in L^q(Z)$, which implies that $F(\xi) = c * \xi \in L^p(Z)$ and

$$||F(\xi)|| \leqslant |c_0| + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi])\, ||\xi||_p .$$

Hence F is a bounded linear operator on $L^p(Z)$ with

$$||F|| \leqslant |c_0| + \frac{B_p}{2\pi} \mathrm{var}(f,[0,2\pi]).$$

Now, let $\tau : L^2(Z^\wedge) \longrightarrow L^2(Z)$ be the Plancherel extension of the Fourier transform and let $\wedge$ be the inverse of τ . Then,

$$(\tau x)(n) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, e^{-int}\, dt \qquad (n \in Z, x \in L(Z^\wedge)). \qquad (48)$$

If $x, y \in L^2(Z^\wedge)$, then by (48) and Parseval's formula, we have

$$\begin{aligned}\mathcal{T}(xy)(n) &= \frac{1}{2\pi}\int_0^{2\pi} x(t)y(t)\, e^{-int}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} x(t)\overline{\overline{y(t)}\, e^{int}}\, dt \\ &= \sum_{m=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-imt}\right)\overline{\left(\frac{1}{2\pi}\int_0^{2\pi}\overline{y(t)}\, e^{int}\, e^{-imt}\, dt\right)} \\ &= \sum_{m=-\infty}^{\infty}\mathcal{T}(x)(m)\mathcal{T}(y)(m-n).\end{aligned}$$

Hence $\mathcal{T}(xy) = \mathcal{T}(x) * \mathcal{T}(y)$ where $*$ denotes convolution.

Let $e_\theta = \mathcal{T}\chi_{[0,\theta]}$ $(0 \leq \theta \leq 2\pi)$. Then $e_\theta(0) = \theta/2\pi$ and $e_\theta(n) = (1 - e^{-in\theta})/2\pi n i$ $(n \neq 0)$. Thus $e_\theta \in L^p(Z)$ $(p > 1)$. Moreover, it is easy to verify that the set $\{e_\theta : 0 \leq \theta \leq 2\pi\}$ is dominated in $L^p(Z)$ $(p > 1)$.

Let j be the identity function from $[0,2\pi]$ onto $[0,2\pi]$, i.e. $j(t) = t$ $(t \in [0,2\pi])$ and let $\mathfrak{z} = \mathcal{T}j$. Then $\mathfrak{z}(0) = \pi$ and $\mathfrak{z}(n) = \frac{1}{n}$ $(n \neq 0)$. The Hilbert transform H is defined on $L^p(Z)$ $(p > 1)$ by

$$H\xi = \xi * (\ldots,-1/n,\ldots,-1/2,-1,0,1,1/2,\ldots,1/n,\ldots). \tag{49}$$

It follows from Theorem 1 that $E(\theta) : \xi \longmapsto e_\theta * \xi$ $(0 \leq \theta \leq 2\pi)$ defines a bounded linear operator on $L^p(Z)$ $(p > 1)$ and

$$||E(\theta)|| \leq \frac{\theta}{2\pi} + B_p/2\pi \qquad (0 \leq \theta \leq 2\pi).$$

Hence $\sup\{||E(\theta)|| : \theta \in [0,2\pi]\} = K < 1 + B_p/2\pi$.

Now, let p be fixed in the range (1,2). Let $0 \leq \theta \leq \varphi \leq 2\pi$ and let $\xi \in L^p(Z) \subseteq L^2(Z)$. Then

$$\begin{aligned}E(\varphi)E(\theta) &= e_\varphi * (e_\theta * \xi) \\ &= \mathcal{T}\chi_{[0,\varphi]} * \mathcal{T}(\chi_{[0,\theta]}\hat{\xi}) \\ &= \mathcal{T}(\chi_{[0,\varphi]}\chi_{[0,\theta]}\hat{\xi}) \\ &= (\chi_{[0,\theta]}\hat{\xi}) \\ &= e_\theta * \xi = E(\theta)\xi.\end{aligned}$$

Similarly, $E(\theta)E(\varphi)\mathfrak{f} = E(\theta)\mathfrak{f}$. In particular, $E^2(\theta) = E(\theta)$. Hence $\{E(\theta) : 0 \leq \theta \leq 2\pi\}$ is a uniformly bounded family of projections on $L^p(Z)$.

Since

$$\| e_\theta - e_\varphi \|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} | \chi_{[0,\theta]}(t) - \chi_{[0,\varphi]}(t) |^2 \, dt\right)^{\frac{1}{2}}$$
$$= (|\theta-\varphi|/2\pi)^{\frac{1}{2}},$$

we have $\lim_{\varphi \to \theta} e_\varphi(n) = e_\theta(n)$ $\qquad (n \in Z)$.

By Lebesgue's dominated convergence theorem,

$$\lim_{\varphi \to \theta} \| e_\varphi - e_\theta \|_p = 0 .$$

For $\mathfrak{f} \in L^p(Z)$ and $n \geq 1$ define

$$\mathfrak{f}_n = (\ldots\ldots\ldots,0,\mathfrak{f}(-n),\mathfrak{f}(-n+1),\ldots\ldots,\mathfrak{f}(n-1),\mathfrak{f}(n),0,\ldots)$$

Then

$$\|\mathfrak{f} * (e_\theta - e_\varphi)\|_p \leq \|(\mathfrak{f} - \mathfrak{f}_n) * (e_\theta - e_\varphi)\|_p + \|\mathfrak{f}_n * (e_\theta - e_\varphi)\|_p$$
$$\leq \|(\mathfrak{f}-\mathfrak{f}_n) * e_\theta\|_p + \|(\mathfrak{f}-\mathfrak{f}_n) * e_\varphi\|_p$$
$$+ \|\mathfrak{f}_n * (e_\theta - e_\varphi)\|_p$$
$$\leq 2K \|\mathfrak{f} - \mathfrak{f}_n\|_p + \|\mathfrak{f}_n\|_1 \|e_\theta - e_\varphi\|_p.$$

Hence, if $0 < \theta < 2\pi$

$$E(\theta+) = \lim_{\varphi \to \theta+} E(\varphi) = E(\theta),$$
$$E(\theta-) = \lim_{\varphi \to \theta-} E(\varphi) = E(\theta),$$
$$E(0+) = E(0) = 0, \qquad E(2\pi-) = E(2\pi) = I,$$

the limits being defined in the strong operator topology. Therefore, by Theorem III.5.3

$$T = \int_{[0,2\pi]} t \; dE(t)$$

is a well-bounded operator of type (B).

Let $(t_k : k = 0,1,\ldots,m)$ be a partition of $[0,2\pi]$ and let p be any complex polynomial. Then for every $\mathfrak{f}$ in $L^p(Z)$,

$$\sum_{k=1}^{m} p(t_k)[E(t_k) - E(t_{k-1})]\xi$$
$$= \tau\left\{\sum_{k=1}^{m} p(t_k)[\chi_{[0,t_k]} - \chi_{[0,t_{k-1}]}]\hat{\xi}\right\}.$$

Therefore $$p(T)\xi = \tau(f\hat{\xi}) \quad (50)$$

In particular, $T\xi = \tau(j\hat{\xi}) = \varsigma * \xi$; thus $T = \pi I + iH$.

Now, suppose that T is a scalar-type spectral operator. Then by Theorem 1.3, there are a compact interval J containing $\sigma(T)$ and a constant M such that

$$||p(T)|| \leq 4\, M\, \sup_J |p(t)| \quad (51)$$

for every complex polynomial p. From (48) and (50) we obtain

$$(p(T)e_{2\pi})(n) = \frac{1}{2\pi}\int_0^{2\pi} p(t)\, e^{-int}\, dt \quad (52)$$

for every complex polynomial p. Let f be any complex function continuous on $[0,2\pi]$; then

$$f(T) = \int_{[0,2\pi]} f(t)\, dE(t)$$

is a bounded linear operator on $L^p(Z)$ and from (51),(52) and the Weierstrass polynomial approximation theorem,

$$(f(T)e_{2\pi})(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-int}\, dt .$$

This gives a contradiction, since not every complex function continuous on $[0,2\pi]$ has $L^p(Z)$-summable Fourier coefficients (16, V.4.11). We have therefore shown that if $1 < p < 2$ then $T = \pi I + iH$ is well-bounded but not scalar-type spectral. Now consider the case $2 < p < \infty$ and let $1/p + 1/q = 1$. Then $1 < q < 2$ and from (49) $-H^*$ is the Hilbert transform on $L^q(Z)$. Therefore $T^* = \pi I^* + i(-H^*)$ is well-bounded but not scalar-type spectral. Since $||p(T)|| = ||p(T^*)||$, it follows from Theorem 1.3 that T is well-bounded but not scalar-type spectral. We observe at this point that T is not even spectral, if $1 < p < 2$ or $2 < p < \infty$, by Theorem 1.4. Finally, if $p = 2$, then

$$\begin{aligned}\langle E(\theta)\xi, \lambda\rangle &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e_\theta(n-m)\xi(m)\overline{\lambda(n)} \\ &= \sum_{m=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \xi(m)\, \overline{e_\theta(m-n)}\, \overline{\lambda(n)} \\ &= \langle \xi, E(\theta)\lambda\rangle \ .\end{aligned}$$

Hence the projections $\{E(\theta) : \theta \in [0,2\pi]\}$ are self-adjoint and so T is self-adjoint. Hence in this case T is both scalar-type spectral and well-bounded.

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