Ito, Luke John (2019) *Almost finiteness of groupoid actions and Z-stability of C*-algebras associated to tilings.* PhD thesis.

http://theses.gla.ac.uk/74404/

Copyright and moral rights for this work are retained by the author

A copy can be downloaded for personal non-commercial research or study, without prior permission or charge

This work cannot be reproduced or quoted extensively from without first obtaining permission in writing from the author

The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the author

When referring to this work, full bibliographic details including the author, title, awarding institution and date of the thesis must be given

Enlighten: Theses
https://theses.gla.ac.uk/
research-enlighten@glasgow.ac.uk

Almost finiteness of groupoid actions and $\mathcal{Z}$-stability of C^*-algebras associated to tilings

by

Luke John Ito

A thesis submitted to the
College of Science and Engineering
at the University of Glasgow
for the degree of
Doctor of Philosophy

August, 2019

Abstract

The property of almost finiteness was first introduced by Matui for locally compact Hausdorff totally disconnected étale groupoids with compact unit spaces, and has since been extended by Suzuki to drop the assumption of being totally disconnected. A property of the same name has been defined by Kerr for free actions of discrete groups on compact metric spaces. In this setting, Kerr shows that almost finiteness has direct relevance to the classification programme for simple, separable, unital, nuclear, infinite dimensional C^*-algebras, as it implies that the associated crossed product is $\mathcal{Z}$-stable.

The motivating example for this thesis comes from the theory of aperiodic tilings. A tiling is a covering of Euclidean space by a collection of sets (called *tiles*) which overlap only on their boundaries. A tiling is called *aperiodic* if it does not contain arbitrarily large periodic patterns. Such tilings find physical applications, acting as models for quasicrystals. One may associate a groupoid to certain aperiodic tilings, and the C^*-algebras of such groupoids encode information about physical observables in quasicrystalline molecules.

In this thesis, we generalise Kerr's notion of almost finiteness of group actions to allow for actions of groupoids. We show that the canonical action of the groupoid associated to any aperiodic, repetitive tiling with finite local complexity on its unit space is almost finite, and we use this to show that the C^*-algebra of the tiling is $\mathcal{Z}$-stable. We develop a groupoid version of the Ornstein-Weiss quasitiling machinery, which we use to prove our $\mathcal{Z}$-stability result. Finally, we give a direct proof that tiling C^*-algebras are quasidiagonal, which eases the route to classification in the case that the algebra has unique trace.

Contents

List of Figures

Acknowledgements

Many thanks to EPSRC for providing the funding necessary to complete this project.

This thesis would not have been possible without the support of a great many people. In particular, I would like to thank:

- All of my friends, for sharing in the best days, and brightening the worst ones.
- Everyone in the Analysis Group in Glasgow, for cultivating such an amazing research environment. It has been a pleasure to be a part of it.
- Andy, for all your help organising the Young Functional Analysts' Workshop in 2017, for all the cheeky pre-seminar pastries, and for always being a huge source of encouragement.
- Nanny Alma and Grandad John, for giving me the confidence to attempt so much, and the determination I needed to succeed.
- Jake, for always making time when I needed to decompress. Thanks for being the best brother anyone could ask for.
- Mum and Dad, for all your love and support, and for always pushing me to shoot for the stars. I love you to the moon and back.
- Jeanette, for much more than I could possibly list here. Thank you for believing in me through all the self-doubt and struggle, and for patiently listening whenever I needed to talk an idea through.

Finally, many thanks to my supervisors Joachim and Mike, for all of your guidance throughout this project. You always had an explanation when I was confused, a suggestion when I was stuck, and constant enthusiasm for my efforts. I couldn't have completed this thesis without you.

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

A *tiling* is a covering of Euclidean space by a countable collection of sets which are homeomorphic to the closed unit ball, and which only intersect on their boundaries. We call a tiling *periodic* if it is identical to some non-zero translate of itself. The modern study of tilings has its roots in [77], in which Wang attempts to determine the decidability of the Domino Problem. This problem asks whether there exists an algorithm which, given any finite set of labelled unit squares and "matching rules" determining allowable placements, would determine, in finite time, whether or not these squares admit a tiling of the plane. Wang proved that the Domino Problem is decidable if and only if every such tile set admits a periodic tiling, and conjectured that this was the case. Shortly afterwards, Wang's student Berger found a set of 20,426 tiles which admits tilings of the plane, but only non-periodic ones, and used this tile set to demonstrate the undecidability of the domino problem in [7]. A tile set which can tile the plane, but only non-periodically, is called *aperiodic*. Nowadays, many more examples of aperiodic tile sets are known, perhaps the most famous being the Penrose tiles originally constructed in [47], [48]. The tilings constructed from such sets have physical applications as mathematical models for *quasicrystals* – molecular structures which are highly ordered, but not periodic [39], [70].

Given a tiling of $\mathbb{R}^d$ formed from an aperiodic tile set, we can allow $\mathbb{R}^d$ to act on the tiling by translation, and thereby obtain a dynamical system. We make three standard assumptions on our tilings: aperiodicity, repetitivity, and finite local complexity. These correspond to freeness, minimality, and compactness of the associated dynamical system, respectively. The construction of C^*-algebras from such a system is initially discussed in [10, Section 2.3] for Penrose tilings, and for general aperiodic tilings in [33] (see also [5] for a version which encodes the continuous dynamics). The algebra introduced in [33] arises from a groupoid which is constructed from the tiling, and will be of particular interest to us. C^*-algebras have historically found physical applications in quantum mechanics, by modelling algebras of observables, and hence the C^*-algebras associated to tilings have strong links to the physics of quasicrystals.

One of the main topics of interest in the theory of C^*-algebras is the study of a certain classification programme. The goal of a classification programme is the construction of a subclass of C^*-algebras along with an invariant which classifies the algebras in the subclass in the sense that any two algebras within the class with isomorphic invariants are isomorphic themselves. The origins of the programme of most current interest were laid by Elliott in [18], who later conjectured that simple separable unital nuclear C^*-algebras were classified by an invariant consisting of ordered topological K-theory along with the trace simplex. Alongside Elliott's original results in [18], positive evidence for the conjecture was provided by Kirchberg and Phillips [38]. However, a string of counterexamples were found by Rørdam [60] and Toms [74], [75]. Rørdam's counterexample was able to be classified using the additional information of the real rank of the algebras, but Toms' counterexamples were more severe, demonstrating the necessity of a new regularity assumption on the subclass of algebras to be classified, rather than an extension of the invariant.

In the 1990's, Jiang and Su constructed a simple separable unital nuclear C^*-algebra $\mathcal{Z}$ which is KK-equivalent to $\mathbb{C}$. It follows that under some mild assumptions, the Elliott invariant of a C^*-algebra A is isomorphic to the invariant of its tensor product with the Jiang-Su algebra $A \otimes \mathcal{Z}$. With Toms' counterexamples in mind, the existence of an isomorphism $A \otimes \mathcal{Z} \cong A$ became recognised as a necessary regularity assumption for the classification, and became known as *$\mathcal{Z}$-stability.*

Another regularity condition came in 2010, when Winter and Zacharias introduced a noncommutative analogue of topological covering dimension, known as *nuclear dimension* [84]. As a result of decades of work by many researchers [82], [24], [19], [73], finiteness of nuclear dimension has now been shown to be part of a sufficient regularity condition for Elliott's classification programme. It was shown by Winter in [81] that finite nuclear dimension implies $\mathcal{Z}$-stability for simple separable unital infinite dimensional C^*-algebras, and the converse implication (with the additional necessary hypothesis of nuclearity) comprises part of a conjecture by Toms and Winter (which appears as [84, Conjecture 9.3]). After several partial results ([43], [65] in the unique trace case and [8] when the set of traces forms a Bauer simplex) the recent breakthrough in [9] establishes the equivalence of finite nuclear dimension and $\mathcal{Z}$ stability for such C^*-algebras in full generality. Therefore, to establish the classifiability of a given algebra one is free to show either of these properties.

The main result of this thesis establishes $\mathcal{Z}$-stability for the C^*-algebras associated to tilings, and hence shows that these algebras are classifiable by the Elliott invariant. This result forms joint work with the author's supervisors Whittaker and Zacharias [31].

Theorem 1.0.1 (Theorem 6.4.3)**.** *The C^*-algebra associated to an aperiodic and reptitive tiling with finite local complexity is $\mathcal{Z}$-stable.*

Corollary 1.0.2. *The C^*-algebra associated to an aperiodic and reptitive tiling with finite local complexity is classifiable by its Elliott invariant.*

We now briefly describe the previously known results in this direction (see page 152 for a more precise treatment). In [62], Sadun and Williams consider tilings by polygons whose edges are described by vectors with rational coordinates. They show that from each such tiling, one can obtain a tiling by unit squares whose associated dynamical system is topologically conjugate to that of the original tiling. This provides a description of the associated C^*-algebra as a crossed product by $\mathbb{Z}^d$, which is shown to be classifiable by Winter [83], making use of the nuclear dimension estimates of Szabó [72].

Another result can be obtained as a consequence of work by Deeley and Strung [12], which classifies the C^*-algebras associated to Smale spaces by bounding the dynamic asymptotic dimension [27] of the associated groupoids. A well-known method of generating tilings makes use of a process of inflation and subdivision on a finite set of tiles. The tilings that are generated in this manner are known as *substitution tilings.* Anderson and Putnam [5] have shown that the C^*-algebras associated to substitution tilings can be thought of as arising from a Smale space, and hence are classifiable by the results of [12].

Our method provides a unified framework which encapsulates both of these classification results. We place no restriction on the shapes of the tiles, and we do not require a substitution structure. Thus, our result newly classifies any aperiodic, repetitive, non-rational, non-substitution tiling with finite local complexity. Some concrete examples of such tilings can be found in [69].

The methodology behind our result makes use of a property known as almost finiteness, which manifests in many related ways. The notion of almost finiteness for second countable étale groupoids with compact and totally disconnected unit spaces was introduced by Matui in [41]. The definition was designed as a weakening of the AF (approximately finite) property of groupoids, motivated by the fact that AF étale groupoids with totally disconnected unit spaces were already known to be completely classified up to isomorphism [53]. In [71], the definition was extended to more general étale groupoids whose unit spaces are not necessarily totally disconnected. It is also shown that under the assumption of minimality, the reduced C^*-algebra associated to an almost finite étale groupoid has stable rank one (the lowest possible value). Stable rank is a sort of non-commutative covering dimension which was introduced in [57], and stable rank one implies nice behaviour of the K-groups of the C^*-algebra.

In [35], Kerr introduced the notion of almost finiteness for free actions of groups on compact metric spaces. The idea encapsulated by the definition is that the space can be almost partitioned into finitely many pairwise disjoint subsets, which can be arranged into collections called *towers*, such that the sets within each collection (the *levels* of the tower) can be permuted by the action. This is an extension of Matui's original definition in the sense that groupoids which arise from almost finite actions on totally disconnected spaces are almost finite in the sense of [41]. In this setting, the motivation for the definition is to act as a dynamical substitute for the properties of $\mathcal{Z}$-stability from the topological setting, and of hyperfiniteness from the measure-theoretic setting. It is shown in [35] that the definition accomplishes this goal; almost finite, free, minimal actions of infinite groups on compact metrisable spaces of finite covering dimension have $\mathcal{Z}$-stable crossed product C^*-algebras.

One may instead allow a groupoid G to act on a space X by making use of a fibration of X over the unit space $G^{(0)}$ of G. This induces an action of G on $C(X)$ in a similar manner to the group case, where $C(X)$ makes use of the fibration on X to fibre over $G^{(0)}$ in a natural way. From this setup, one can form a *groupoid crossed product* C^*-algebra $C(X) \rtimes G$ as in [54] (see also [44], [23]), which is isomorphic to the usual crossed product in the case that G is a group being viewed as a groupoid with one unit. It is then natural to ask in what capacity the results in [35] can be generalised to this new setting. To this end, we extend the definition of almost finiteness from [35] to allow for actions of groupoids.

Motivated by the results in [35], we produce our main result via the following scheme. The groupoid associated to a tiling can be thought of as the transformation groupoid arising from the canonical action of the tiling groupoid on its own unit space. We use [41, Remark 6.4] to show that this transformation groupoid is almost finite. We prove (Theorem 4.3.15) an analogue of [71, Lemma 5.2] which then shows that the action generating the tiling groupoid is almost finite in the sense we define (Definition 4.3.13). By [23, Proposition 4.38], the C^*-algebra associated to any transformation groupoid can be thought of as a groupoid crossed product by the induced action. Thus, the tiling groupoid can be thought of as a groupoid crossed product arising from an almost finite action. We then follow the techniques used in the proof of [35, Theorem 12.4] and use this to prove that the C^*-algebras associated to tilings are $\mathcal{Z}$-stable. The method of proof appearing in [35] makes use of the Ornstein-Weiss quasitiling machinery for groups [46], and we develop an analogue for use in the setting of étale groupoids with compact unit space (Theorem 6.2.8). Although the proof of our $\mathcal{Z}$-stability result presently makes heavy use of the geometry of the tiling groupoid, it should be thought of as a proof-of-concept towards extending the existing theory to more general classes of groupoids. In particular, we make the following conjecture.

Conjecture 1.0.3. *Let $\alpha : G \curvearrowright X$ be an almost finite, continuous, free, minimal action* *of a second countable, topologically amenable, locally compact, Hausdorff, étale groupoid* G *such that* $G_u = s^{-1}(u)$ *is infinite for each* $u \in G^{(0)}$ *on a compact, metrisable space* X*. Then the associated reduced groupoid crossed product* $C(X) \rtimes_r G$ *is* $\mathcal{Z}$*-stable.*

We comment that in the group case, almost finiteness of the action automatically ensures amenability of the acting group. An example in [6] involving expander graphs shows that almost finiteness of a groupoid (in the sense of [41]) does not imply amenability. Nevertheless, using a characterisation of amenability from [55], we suspect that if a locally compact Hausdorff étale groupoid admits an almost finite action, then it is topologically amenable. Therefore, our assumption of topological amenability may be superfluous. This assumption is supposed to cover an amenability-type condition which is required by our method of proof. More precisely, it is meant to guarantee the existence of arbitrarily long sequences of suitable tiles for our quasitiling machinery (see Theorem 6.2.8). However, we have no proof that topological amenability actually implies the existence of such sequences, so it would be interesting to confirm a sufficient condition for this purpose.

With these sequences in hand, the only other complication to the proof of Theorem 6.4.3 compared to the group case comes from the fibred nature of the action. Because each groupoid element only acts on part of the space, it is possible for one of the tiles to only act on part of one of the tower levels, or to act on a whole tower level, but only send a portion of it into any other tower level. We need to arrange that, for each pair of tower levels, each of these tiles either admits a subset which moves the entirety of the first tower level into the second, or that the image of the first tower level under the action of the tile does not intersect the second tower level at all. It is in this part of our proof that the geometry of the tiling groupoid plays a large role, though we believe a similar argument should be possible in more generality.

The thesis will be structured as follows. Chapter 2 contains an introduction to groupoids, with a focus on the étale groupoids to which our results are applicable. In Chapter 3, we provide a detailed overview of the theory of tilings, including full proofs of many "folklore" results which are often stated without proof in the literature. We explore the substitution method of constructing tilings, which produces examples with a particularly nice structure. We also include the construction of a groupoid from the tiling, and the associated C^*-algebra. We also explore the infinite rotation case, which was introduced in [78]. In Chapter 4, we detail the varied notions of almost finiteness which exist in the literature, and introduce our new notion of almost finiteness of actions of groupoids, showing that it extends the existing theory. It is in this chapter that we prove that tiling groupoids are almost finite. This chapter also contains our construction and exploration of Cantor graphs from certain classes of tilings. Chapter 5 contains a presentation of the essentials of the theory of groupoid crossed products. Finally, Chapter 6 contains the majority of our new material, including the groupoid version of the tiling technology used in [35] and our classification results.

We note that the new material found in Sections 4.3, 4.4 and 6.2 to 6.5, including the main results of this thesis, appears in joint work with the author's supervisors Whittaker and Zacharias [31].

Chapter 2

Groupoids

2.1 Basics

The purpose of this chapter is to acquaint the reader with groupoids, which generalise the notion of groups. The material presented here will loosely follow the development in [67, Chapter 2]. The main difference between the two notions is that a groupoid consists of a set G equipped with a *partial* binary operation $G \times G \to G$ (that is, the operation need only be defined on a *subset* of $G \times G$, in contrast to the group case, in which it would be required to be defined everywhere). The following (algebraic) set of axioms is not the most efficient for the purpose of checking whether an object is a groupoid, but is among the more instructive presentations.

Definition 2.1.1. A *groupoid* is a set G equipped with the following data:

- a subset $G^{(0)} \subset G$ of elements known as *units*. We refer to $G^{(0)}$ as the *unit space* of G;
- maps $r, s : G \to G^{(0)}$, known as the *range* and *source* maps;
- a composition map $(g, h) \mapsto gh$ from the set of *composable pairs*, $G^{(2)} := \{(g, h) \in G \times G \mid r(h) = s(g)\}$, to G; and
- an inverse map $g \mapsto g^{-1}$ from G to G.

The above data is also required to satisfy the following assumptions. For every $u \in G^{(0)}$ and $f, g, h \in G$, we have:

(G1) $r(u) = u = s(u)$;

(G2) $r(g)g = g = gs(g)$;

(G3) $r(g^{-1}) = s(g)$ and $s(g^{-1}) = r(g)$ (in other words, (g, g^{-1}) and (g^{-1}, g) are elements of $G^{(2)}$);

(G4) $g^{-1}g = s(g)$ and $gg^{-1} = r(g)$;

(G5) if $(g, h) \in G^{(2)}$, then $r(gh) = r(g)$ and $s(gh) = s(h)$; and

(G6) if $(f, g) \in G^{(2)}$ and $(g, h) \in G^{(2)}$, then $(fg)h = f(gh)$.

A *subgroupoid* of G is a subset $K \subset G$ which is closed under the composition and inverse maps of G. By (G4), this implies that K is also closed under the source and range maps.

We picture a groupoid element $g \in G$ as a directed arrow between two points, $s(g)$ and $r(g)$, of $G^{(0)}$. The element g^{-1} is obtained by flipping the direction of the arrow corresponding to g. It is sometimes more convenient to visualise elements of the unit space $G^{(0)}$ as distinguished loops attached to the endpoints of these arrows, instead of points. We also allow for more loops than this – that is, there may exist $g \in G \setminus G^{(0)}$ such that $s(g) = r(g)$. Composition of groupoid elements becomes the usual composition of paths in this viewpoint, and is read right-to-left.

Given a subset $C \subset G$, we denote

$$C^{(2)} := (C \times C) \cap G^{(2)}.$$

Observe that, in the case that C is a subgroupoid, this notation is already implicitly defined by Definition 2.1.1.

We also allow for the composition of subsets $A, B \subset G$, defined by

$$AB := \{ab \mid a \in A,\ b \in B,\ \text{and } (a, b) \in G^{(2)}\},$$

and, given $g \in G$, we also define

$$Ag := \{ag \mid a \in A,\ (a, g) \in G^{(2)}\},$$

and

$$gA := \{ga \mid a \in A,\ (g, a) \in G^{(2)}\}.$$

It is important to note that these products are a lot less well-behaved than in the group case. For instance, it is possible that AB is empty when A and B are not! The notation will be used particularly often with an element $u \in G^{(0)}$ to produce the shorthands

$$Au = A \cap s^{-1}(u) \quad \text{and} \quad uA = A \cap r^{-1}(u),$$

which we refer to as the *source bundle* and *range bundle* of u in A, respectively. The respective notations A_u and A^u are less suggestive, but more common in the literature. They are particularly compatible with the notation for Haar systems, indicating the support of the associated measure in the system. We will not need to work with Haar systems often in this thesis, but we will make use of this latter notation when we do.

We prove that groupoids have cancellation.

Lemma 2.1.2. *Let G be a groupoid and $f, g, h \in G$. If $(f, g), (h, g) \in G^{(2)}$ and $fg = hg$, then $f = h$, and if $(g, f), (g, h) \in G^{(2)}$ and $gf = gh$, then $f = h$.*

Proof. We only prove right-cancellation, as the proof of left-cancellation is similar. Suppose that $fg = hg$. Using (G5) we see that $s(fg) = s(g) = s(hg)$. Therefore, by (G3), $(fg, g^{-1}) \in G^{(2)}$ and $(hg, g^{-1}) \in G^{(2)}$, and we may compose $fg = hg$ on the right by g^{-1} to obtain $(fg)g^{-1} = (hg)g^{-1}$. Using associativity and (G4), this yields $fr(g) = f(gg^{-1}) = h(gg^{-1}) = hr(g)$. Now, since (f, g) and (h, g) were in $G^{(2)}$, we have $s(f) = r(g) = s(h)$ so we have shown that $fs(f) = hs(h)$. By applying (G2), we see that $f = h$. □

We prove some basic properties which will help us to recognise units and inverses in our groupoids.

Lemma 2.1.3. *Let G be a groupoid.*

(i) We have $u \in G^{(0)}$ if and only if $u^2 = u$.

(ii) Property (G4) uniquely characterises the inverse of each element of G, in the sense that if $a, g \in G$ are such that either $ag = s(g)$ or $ga = r(g)$, then $a = g^{-1}$, and both equalities automatically hold.

Proof.

(i) First, if $u \in G^{(0)}$, observe that $u^2 = r(u)u = u$ by properties (G1) and (G2). Next, if $u^2 = u$, we observe that $(u, u) \in G^{(2)}$ and so $r(u) = s(u)$. Using this together with the equality $u^2 = u$ and properties (G2) and (G4) yields $u = r(u)u = s(u)u = (u^{-1}u)u = u^{-1}u^2 = u^{-1}u = s(u) \in G^{(0)}$, so that u is a unit.

(ii) Let $g \in G$ and suppose that $a \in G$ satisfies $ag = s(g) = g^{-1}g$. By right-cancellation, this shows that $a = g^{-1}$, and by (G4) we then have $ga = r(g)$. The proof is similar when we assume $ga = r(g)$. □

Let us consider some examples.

Examples 2.1.4.

(i) Any group Γ can be thought of as a groupoid whose unit space has a single element $\Gamma^{(0)} = \{e\}$. The multiplication and inverse maps are inherited directly from the group, and every $\gamma \in \Gamma$ has $r(\gamma) = s(\gamma) = e$.

(ii) Let $(X, \sim)$ be a set equipped with an equivalence relation and consider the set of equivalent pairs $R = \{(x, y) \subset X \times X \mid x \sim y\}$. This becomes a groupoid with $R^{(0)} = \{(x, x) \mid x \in X\}$ (which we identify with X), and operations $s((x, y)) = y$, $r((x, y)) = x$, $(x, y)^{-1} = (y, x)$ and $(x, y)(y, z) = (x, z)$ for $x, y, z \in X$. From now on we will drop the double brackets to denote $s((x, y))$ and $r((x, y))$ more simply by $s(x, y)$ and $r(x, y)$.

(iii) Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X. This gives rise to the *transformation groupoid* $\Gamma \ltimes X$ as follows:

- as a set, $\Gamma \ltimes X = \Gamma \times X$;
- $(\Gamma \ltimes X)^{(0)} := \{(e, x) \mid x \in X\}$ is identified with X;
- for $(\gamma, x) \in \Gamma \ltimes X$, we define $s(\gamma, x) = x$ and $r(\gamma, x) = \gamma \cdot x$;
- $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma \cdot x)$; and
- when $\gamma \cdot x = y$, the following multiplication is defined for all $\sigma \in \Gamma$: $(\sigma, y)(\gamma, x) = (\sigma\gamma, x)$.

Alternatively, we can picture the transformation groupoid as a graph whose vertex set is X, and so that, for each $\gamma \in \Gamma$ and $x \in X$, there is an arrow labelled by γ pointing from x to $\gamma \cdot x$.

We now make precise what is required for two groupoids to be "the same".

Definition 2.1.5. Let G and H be groupoids. We say that a map $\phi : G \to H$ is a *groupoid homomorphism* if, whenever $(g_1, g_2) \in G^{(2)}$, we have that $(\phi(g_1), \phi(g_2)) \in H^{(2)}$ and $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.

A bijective groupoid homomorphism is called a *groupoid isomorphism*. We say that the groupoids G and H are *isomorphic* if there exists a groupoid isomorphism between them.

As one would expect, groupoid homomorphisms preserve not only the multiplication, but all structure maps of the groupoids. We will prove this for the range and source map in the following proposition. For completeness, we detail the proof for the inverse map here. Given that ϕ preserves the source map, we have $s(\phi(g)) = \phi(s(g)) = \phi(g^{-1}g) = \phi(g^{-1})\phi(g)$. Combining this with part (ii) of Lemma 2.1.3, we see that $\phi(g^{-1}) = \phi(g)^{-1}$, as required.

Proposition 2.1.6. *The inverse of a bijective groupoid homomorphism is a groupoid homomorphim.*

Proof. Suppose that $\phi : G \to H$ is a bijective groupoid homomorphism, and let $(h_1, h_2) \in H^{(2)}$. Since ϕ is surjective, there exist $g_1, g_2 \in G$ such that $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$.

We claim that $(g_1, g_2) \in G^{(2)}$. To prove this, we show that $r(h_2) = \phi(r(g_2))$ and that $s(h_1) = \phi(s(g_1))$. Since $(h_1, h_2) \in H^{(2)}$ implies $r(h_2) = s(h_1)$, this will yield $\phi(r(g_2)) = r(h_2) = s(h_1) = \phi(s(g_1))$, whence we obtain $r(g_2) = s(g_1)$ using the injectivity of ϕ. We comment that this also proves that ϕ preserves the range and source maps of the groupoids.

Observe that $r(\phi(g_2))\phi(g_2) = \phi(g_2) = \phi(r(g_2)g_2) = \phi(r(g_2))\phi(g_2)$. By cancelling $\phi(g_2)$, this implies that $r(h_2) = r(\phi(g_2)) = \phi(r(g_2))$, as we claimed. The proof of the second equality is similar, and can be obtained by considering $\phi(g_1)s(\phi(g_1))$.

To finish the proof, we need to show that $\phi^{-1}(h_1h_2) = \phi^{-1}(h_1)\phi^{-1}(h_2) = g_1g_2$. Observe that $\phi(g_1g_2) = \phi(g_1)\phi(g_2) = h_1h_2$, since ϕ is a groupoid homomorphism and $(g_1, g_2) \in G^{(2)}$. Applying ϕ^{-1} to both sides gives $g_1g_2 = \phi^{-1}(h_1h_2)$, as required. □

Groupoids from Example 2.1.4(ii) can be characterised by the following property.

Definition 2.1.7. A groupoid is called *principal* if there exists at most one arrow between each pair of objects. That is, if the assigment $g \mapsto (r(g), s(g))$ for $g \in G$ is injective.

Proposition 2.1.8. *Let G be a groupoid. Then G is principal if and only if G is isomorphic to an equivalence relation groupoid (as in Example 2.1.4(ii)).*

Proof. Let G be the groupoid associated to the equivalence relation $(X, \sim)$ and let (x_1, y_1), $(x_2, y_2) \in G$. In this setting, the expression $(r(x_1, y_1), s(x_1, y_1)) = (r(x_2, y_2), s(x_2, y_2))$ becomes $(x_1, y_1) = (x_2, y_2)$, so G is principal.

Now, let G be any principal groupoid, and consider the set $R = \{(r(g), s(g)) \mid g \in G\}$. We claim that R defines an equivalence relation $\sim$ on $G^{(0)}$ via $x \sim y$ if and only if $(y, x) \in R$, and that G is isomorphic to R when we view R as a groupoid as in Example 2.1.4(ii).

Let $x, y, z \in G^{(0)}$. We have $(x, x) = (r(x), s(x)) \in R$, so $x \sim x$. If $x \sim y$ then there exists $g \in G$ such that $s(g) = x$ and $r(g) = y$. Then we have $(x, y) = (s(g), r(g)) = (r(g^{-1}), s(g^{-1}))$, so $y \sim x$. Finally, if $x \sim y$ and $y \sim z$ then there exist $g, h \in G$ such that $s(h) = x$, $r(h) = y$, $s(g) = y$, and $r(g) = z$. Since $r(h) = s(g) = y$, we see that $(g, h) \in G^{(2)}$. We then have $(r(gh), s(gh)) = (r(g), s(h)) = (z, x)$ so that $x \sim z$, so $\sim$ is an equivalence relation.

Define a map $\phi : G \to R$ via $\phi(g) = (r(g), s(g))$. We show that ϕ is a bijective groupoid homomorphism. Suppose that $(g, h) \in G^{(2)}$, so that $s(g) = r(h)$. Then we have $s(\phi(g)) =$

$s(r(g), s(g)) = s(g) = r(h) = r(r(h), s(h)) = r(\phi(h))$, so $(\phi(g), \phi(h)) \in R^{(2)}$. Furthermore, we have $\phi(g)\phi(h) = (r(g), s(g))(r(h), s(h)) = (r(g), s(h)) = (r(gh), s(gh)) = \phi(gh)$, so ϕ is a groupoid homomorphism. Surjectivity of ϕ is clear from the definition of R. Finally, ϕ is precisely the map which appears in the definition of principality, so ϕ is injective since G is principal. □

2.2 Topological groupoids

We now wish to equip our groupoids with topologies. The following definition generalises the notion of a topological group.

Definition 2.2.1. A *topological groupoid* is a groupoid G equipped with a topology such that the relative topology on $G^{(0)} \subset G$ is Hausdorff, r and s are continuous with respect to the relative topology on $G^{(0)}$, the map $g \mapsto g^{-1}$ is continuous, and the multiplication $(g, h) \mapsto gh$ is continuous with respect to the relative topology on $G^{(2)}$ induced from the product topology on $G \times G$.

The proof of the following result is taken from [67, Lemma 2.3.2].

Proposition 2.2.2. *Let G be a topological groupoid. Then $G^{(0)}$ is closed in G if and only if G is Hausdorff.*

Proof. First, suppose G is Hausdorff, so that limits of nets in G are unique. Consider a net $\{x_i\}_{i\in I} \subset G^{(0)}$ which converges to $g \in G$. Since $x_i \in G^{(0)}$, we have $r(x_i) = x_i$ for each $i \in I$, and since the range map is continuous, we see that $x_i = r(x_i) \to r(g)$, so that the net $\{x_i\}_{i\in I}$ converges to both g and $r(g)$. Since the limit is unique, $\lim x_i = r(g) \in G^{(0)}$, which shows that $G^{(0)}$ is closed.

Next, suppose that $G^{(0)}$ is closed in G. In order to show that G is Hausdorff, we will prove that limits of nets in G are unique. To this end, let $\{g_i\}_{i\in I} \subset G$ be a net, and suppose that $\{g_i\}_{i\in I}$ converges to $a \in G$ and $b \in G$. Since the range map is continuous and $G^{(0)}$ is Hausdorff, $r(a) = r(b)$, so that $(a, b^{-1}) \in G^{(2)}$. Since the inverse and multiplication maps are continuous, we see that $g_i g_i^{-1} \to ab^{-1}$. On the other hand, $g_i g_i^{-1} = r(g_i) \in G^{(0)}$ for each $i \in I$, so closure of $G^{(0)}$ implies that $ab^{-1} \in G^{(0)}$, say $ab^{-1} = x \in G^{(0)}$. Observe that $x = s(x) = s(ab^{-1}) = r(b)$. Thus $ab^{-1} = r(b)$ and, multiplying both sides on the right by b, we obtain $as(b) = r(b)b = b$. Finally, since the source map is continuous and $G^{(0)}$ is Hausdorff, $s(a) = s(b)$, so that $b = as(b) = as(a) = a$. Thus, the limit of the net was unique. □

In fact, we will usually be interested in a more restricted type of topological groupoid, which are analogous to discrete groups. To describe these groupoids, we will need a stronger requirement than continuity of the structure maps. The correct notion is given by the following definition.

Definition 2.2.3. Let X and Y be topological spaces. A *local homeomorphism* is a function $f : X \to Y$ such that, for every $x \in X$, there exists an open set $x \in U_x \subset X$ such that the restriction $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism.

Definition 2.2.4. A topological groupoid is called *étale* if the source map is a local homeomorphism *as a map from G to G*.

Remark 2.2.5. The étale property is an analogue for discreteness of groups (we will justify this in a little more detail shortly – see Proposition 2.2.10). It is important to note that $s : G \to G$ is required to be a local homeomorphism. In particular, it is *not* sufficient that $s : G \to G^{(0)}$ be a local homeomorphism, where the unit space is equipped with the subspace topology. For example, in Proposition 2.2.7, the latter condition is not sufficient to ensure that $G^{(0)}$ is open in G.

Lemma 2.2.6. *Any local homeomorphism $f : X \to Y$ is a continuous and open map. Therefore, a bijective local homeomorphism is a homeomorphism.*

Additionally, if $f : X \to Y$ and $g : Y \to Z$ are local homeomorphisms, then the composition $g \circ f$ is again a local homeomorphism.

Proof. First, we show that f is continuous. Let $V \subset Y$ be open, and assume $f^{-1}(V) \neq \varnothing$. Given $x \in f^{-1}(V)$, there exists an open set $x \in U_x \subset X$ such that $f|_{U_x}$ is a homeomorphism. Consider the open subset $W = f(U_x) \cap V$ of Y, observing that $f(x) \in W$. Then $f|_{U_x}^{-1}(W)$ is open in X since $f|_{U_x}^{-1}$ is a homeomorphism on W, and we have $x \in f|_{U_x}^{-1}(W) \subset f^{-1}(V)$.

Next, we show that f is open, so let $U \subset X$ be open. Let $y \in f(U)$, so that there exists $x \in U$ such that $f(x) = y$. By definition, there exists an open set $x \in U_x \subset X$ such that $f|_{U_x}$ is a homeomorphism. Then $U \cap U_x$ is open in X and $f|_{U_x}$ is a homeomorphism on $U \cap U_x$, so $f|_{U_x}(U \cap U_x)$ is open in Y. Observe that $y \in f|_{U_x}(U \cap U_x) \subset f(U)$, which shows that f is an open map.

To prove the final statement, let $f : X \to Y$ and $g : Y \to Z$ be local homeomorphisms. Take any point $x \in X$ and find an open set $x \in U_x \subset X$ such that $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism. In particular, notice that $f(U_x)$ is open in Y. Since g is also a local homeomorphism, for any point $y \in Y$, there exists an open set $y \in V_y \subset Y$ such that $g|_{V_y}$ is a homeomorphism onto its image. Let $y = f(x)$ and consider the restriction of $g|_{V_y}$ to the nonempty open set $W := V_y \cap f(U_x)$. This restriction is once again a homeomorphism. Since $W \subset f(U_x)$, we can consider the preimage $f|_{U_x}^{-1}(W) \subset U_x$. Notice that, by construction, the restriction $(g \circ f)|_{f|_{U_x}^{-1}(W)}$ is a homeomorphism onto its image, which proves that $g \circ f$ is a local homeomorphism. □

Proposition 2.2.7. *Let G be an étale topological groupoid. Then the range, multiplication, and inverse maps of G are local homeomorphisms, and $G^{(0)}$ is open in G.*

Proof. First, notice that $G^{(0)} = s(G)$. Since $s : G \to G$ is an open map by Lemma 2.2.6, this shows that $G^{(0)}$ is open in G.

Secondly, notice that the inverse map of any topological groupoid is a homeomorphism (because it is continuous and self-inverse), so it is a local homeomorphism.

Now, denoting the inverse map of a groupoid by i, we see by assumption (G3) of groupoids that $r = s \circ i$ is the composition of two local homeomorphisms, which is a local homeomorphism by Lemma 2.2.6.

It remains to show that the multiplication map is a local homeomorphism. It is proven in [67, Lemma 2.4.11] that multiplication on an étale groupoid is an open map. Arguing similarly to [53, Proposition 2.8], since r and s are local homeomorphisms, given a point $(g, h) \in G^{(2)}$, we can find open neighbourhoods $g \in U \subset G$ and $h \in V \subset G$ such that the restrictions $r|_U : U \to r(U)$ and $s|_V : V \to s(V)$ are homeomorphisms. Then $(g, h) \in (U \times V) \cap G^{(2)}$, which is open in $G^{(2)}$. We claim that the restriction of the multiplication map to $(U \times V) \cap G^{(2)}$ is injective. Indeed, suppose $(g_1, h_1), (g_2, h_2) \in (U \times V) \cap G^{(2)}$ are such that $g_1h_1 = g_2h_2$. Then we have $g_1, g_2 \in U$ and $h_1, h_2 \in V$ and also that

$$s(h_1) = s(g_1h_1) = s(g_2h_2) = s(h_2).$$

Since s was a homeomorphism on V, this implies that $h_1 = h_2$. Similarly, we have

$$r(g_1) = r(g_1h_1) = r(g_2h_2) = r(g_2),$$

so that $g_1 = g_2$, since r was a homeomorphism on U.

Since the restriction of the multiplication to $(U \times V) \cap G^{(2)}$ is continuous, open, and bijective onto its image, it is a homeomorphism. □

Example 2.2.8. Let X be a second countable locally compact Hausdorff space, and let Γ be a locally compact group acting on X by homeomorphisms. Then the associated transformation groupoid becomes a topological groupoid when equipped with the product topology.

Notation 2.2.9. Given a group Γ and an element $\gamma \in \Gamma$, we denote by $L_\gamma : \Gamma \to \Gamma$ the left-multiplication map by γ. That is for each $\sigma \in \Gamma$ we have

$$L_\gamma(\sigma) = \gamma\sigma.$$

Proposition 2.2.10. *A transformation groupoid is étale if and only if the acting group is discrete.*

Proof. First, suppose that the transformation groupoid $\Gamma \ltimes X$ is étale. By Proposition 2.2.7, this means that $(\Gamma \ltimes X)^{(0)} = \{e\} \times X$ is open. Since the projection $\pi_\Gamma : \Gamma \times X \to \Gamma$ is an open map, we see that $\pi_\Gamma(\{e\} \times X) = \{e\}$ is open in Γ. Since the group multiplication is continuous on Γ, we see that for each $\gamma \in \Gamma$, the preimage $(L_{\gamma^{-1}})^{-1}(\{e\}) = L_\gamma(\{e\}) = \{\gamma\}$ is also open, so that Γ is discrete.

Now, suppose that Γ is discrete. We wish to prove that $s : \Gamma \ltimes X \to \Gamma \ltimes X$ is a local homeomorphism. So, choose $(\gamma, x) \in \Gamma \ltimes X$. Then (γ, x) is in $\{\gamma\} \times X$, which is open in $\Gamma \ltimes X$. We have $s(\{\gamma\} \times X) = \{e\} \times X$, which is open in $\Gamma \ltimes X$. All that remains is to observe that $s|_{\{\gamma\}\times X}$ is injective onto its image, because it is just projection to the X coordinate. Thus $s|_{\{\gamma\}\times X}$ is a continuous open map which is bijective onto its image, and hence it is a homeomorphism. □

The following sets form the building blocks of the topology of an étale groupoid.

Definition 2.2.11. A subset B of an étale groupoid G is a *bisection* if there is an open set U containing B such that $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms.

The following result appears in [53]. The proof we present is taken from [67].

Proposition 2.2.12 ([53, Proposition 2.8])**.** *Any étale groupoid G has a base of open bisections. Furthermore, if G is second-countable, then it has a countable base of open bisections.*

Proof. We give the proof under the assumption of second countability, but it is easy to modify to obtain a proof of the former statement. Since G is second-countable, it has a countable dense subset $\{g_n\}_{n\in\mathbb{N}}$. Since G is also first-countable, each point of G has a countable neighbourhood basis, so, for each $n \in \mathbb{N}$, we can find a countable neighbourhood basis $\{U_{n,i}\}_{i\in\mathbb{N}}$ of g_n. Without loss of generality, we may assume that each set $U_{n,i}$ is open. Since r is a local homeomorphism, we can assume, without loss of generality, that, for every $n \in \mathbb{N}$ and $i \in \mathbb{N}$, r is a homeomorphism on $U_{n,i}$. Similarly, for each $n \in \mathbb{N}$, we can construct a neighbourhood basis $\{V_{n,i}\}_{i\in\mathbb{N}}$ of g_n consisting of open sets such that s is a homeomorphism on each $V_{n,i}$. Then, for each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, the set $U_{n,i} \cap V_{n,i}$ is an open bisection, and the collection $\{U_{n,i} \cap V_{n,i} \mid n, i \in \mathbb{N}\}$ is a countable base for the topology of G. □

The following consequence of the above result will become critical to the development in Chapter 4.

Proposition 2.2.13. *Let G be an étale groupoid, and let $C \subset G$ be a compact subset. Then there exists $n \in \mathbb{N}$ such that $|Cu| \leq n$ and $|uC| \leq n$ for every $u \in G^{(0)}$.*

Proof. By Propostion 2.2.12, G has a base $\{B_i\}_{i \in I}$ of open bisections. Since C is compact and is covered by the collection $\{B_i\}_{i \in I}$, there exists a finite subcover $\{B_1, \ldots, B_n\}$. Let $j \in \{1, \ldots, n\}$. Since s and r are injective on B_j, we have $|B_j u| = |B_j \cap s^{-1}(u)| \leq 1$ and $|uB_j| = |B_j \cap r^{-1}(u)| \leq 1$ for each $u \in G^{(0)}$. It follows that $|Cu| \leq n$, and $|uC| \leq n$. □

The following special class of étale groupoids have a particularly nice topological structure.

Definition 2.2.14. A groupoid G is called *ample* if it is étale and $G^{(0)}$ is zero-dimensional (that is, the topology of $G^{(0)}$ admits a base of clopen subsets).

Proposition 2.2.15 ([20, Proposition 4.1])**.** *Any (second-countable) locally compact Hausdorff ample groupoid G has a (countable) base of compact open bisections.*

Proof. As before, we establish the proof under the assumption of second-countability. Let $g \in U \subset G$, with U open. We seek a compact open bisection C such that $g \in C \subset U$. By Proposition 2.2.12, find an open bisection B such that $g \in B \subset U$. We have that $s(g) \in s(B) \subset G^{(0)}$, and we know that $s(g)$ has a countable neighbourhood basis of clopen subsets $\{U_n\}_{n \in \mathbb{N}}$. In fact, since $G^{(0)}$ is locally compact and Hausdorff, $s(g)$ admits a neighbourhood basis of compact sets, so we may choose each set U_n to be compact and open. Find an $n \in \mathbb{N}$ such that $s(g) \in U_n \subset s(B)$. Since $s|_B : B \to s(B)$ was a homeomorphism, $C \coloneqq (s|_B)^{-1}(U_n)$ is the compact open bisection that we seek. □

2.3 A category-theoretic viewpoint

In this section, we present an alternative viewpoint of groupoids which allows for a particularly slick definition. First, we acquaint the reader with the basic concepts which we will use.

2.3.1 Some category-theoretic concepts

Definition 2.3.1. A *small category* C is a triplet consisting of:

- a set $\mathrm{ob}(C)$, elements of which are called *objects*;

- a set $\hom(C)$, elements of which are called *arrows*, such that each arrow is associated to two objects, called the *source* and *range* of the arrow. For $a, b \in \text{ob}(C)$, we denote by $\hom(a, b)$ the subset of arrows whose source is a and whose range is b; and

- a partial binary operation $\circ$ on arrows, called *composition*, such that, for any $a, b, c \in \text{ob}(C)$, we have $\circ : \hom(b, c) \times \hom(a, b) \to \hom(a, c)$. The operation is further required to be associative, so that, for any $f \in \hom(a, b)$, $g \in \hom(b, c)$, and $h \in \hom(c, d)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

It is also required that, for each $x \in \text{ob}(C)$, there exists an *identity arrow* $1_x \in \hom(x, x)$. Given $f \in \hom(a, b)$, we require that these identity arrows satisfy $1_b \circ f = f = f \circ 1_a$. We say that an arrow $f \in \hom(a, b)$ is *invertible* if there exists an arrow $g \in \hom(b, a)$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$.

Definition 2.3.2. A *covariant functor* F between two categories C and D consists of two assignments. One on objects, $F : \text{ob}(C) \to \text{ob}(D)$, and one on arrows, $F : \hom(a, b) \to \hom(F(a), F(b))$, which satisfy:

(i) for every $x \in \text{ob}(C)$, we have $F(1_x) = 1_{F(x)}$; and

(ii) for every $f \in \hom(a, b)$ and $g \in \hom(b, c)$, we have $F(g \circ f) = F(g) \circ F(f)$.

One may also define *contravariant functors* which invert the direction of arrows, so that if F is contravariant then, for every $a, b \in \text{ob}(C)$, $F : \hom(a, b) \to \hom(F(b), F(a))$. The *identity functor* $1_C : C \to C$ satisfies $1_C(x) = x$ for every $x \in \text{ob}(C)$, and $1_C(f) = f$ for every $a, b \in \text{ob}(C)$ and $f \in \hom(a, b)$. It is easily seen that this is a covariant functor.

Definition 2.3.3. Two categories C and D are said to be *isomorphic* if there exist functors $F : C \to D$ and $G : D \to C$ which are mutually inverse in the sense that $GF = 1_C$ and $FG = 1_D$ are the identity functors on C and D respectively. In this case, we say that the functors F and G are *isomorphisms* of the categories C and D.

2.3.2 Groupoids as categories

We can neatly encapsulate Definition 2.1.1 in the following category-theoretic package.

Definition 2.3.4. A *groupoid* is a small category in which every arrow is invertible.

Although not as explicit as Definition 2.1.1, this definition does come with some advantages. For instance, by phrasing the definition in category-theoretic language, the above definition comes pre-equipped with notions of isomorphism and equivalence between groupoids. We devote some time to checking that the category-theoretic notions agree with their counterparts defined above. First, we explain how to move between the two definitions.

A correspondence between groupoids and small categories with inverses is obtained by setting $\mathrm{ob}(G) = G^{(0)}$ and $\mathrm{hom}(x, y) = \{g \in G \mid s(g) = x, r(g) = y\}$ (so that the terminologies "source" and "range" are the same in the groupoid as in the category). The composition is then given by groupoid multiplication $g \circ h = gh$, and is defined when $r(h) = s(g)$ – that is, when $(g, h) \in \mathrm{hom}(s(g), r(g)) \times \mathrm{hom}(s(h), s(g))$. Associativity of the composition is given by assumption (G6) of groupoids. For $x \in \mathrm{ob}(G) = G^{(0)}$, the identity morphism $1_x = x$ satisfies $r(x) = s(x) = x$ so that $x \in \mathrm{hom}(x, x)$. By assumption (G2) of groupoids, for any $g \in G$, we have $1_{r(g)} \circ g = r(g)g = g = gs(g) = g \circ 1_{s(g)}$.

Proposition 2.3.5 ([67, Lemma 2.1.12])**.** *Let G and H be groupoids, and $\phi : G \to H$. Then ϕ is a groupoid homomorphism if and only if it is a covariant functor between the categories G and H.*

Proof. First, suppose that ϕ is a groupoid homomorphism. Given $x \in G^{(0)}$, we have $\phi(x)^2 = \phi(x^2) = \phi(x)$, so applying part (i) of Lemma 2.1.3 shows that $\phi(x) \in H^{(0)}$. Thus, $\phi : \mathrm{ob}(G) \to \mathrm{ob}(H)$.

In the proof of Proposition 2.1.6 we saw that, for any $g \in G$, $\phi(r(g)) = r(\phi(g))$ and $\phi(s(g)) = s(\phi(g))$. In the language of categories, this shows that, for any $x, y \in G^{(0)} = \mathrm{ob}(G)$, $\phi : \mathrm{hom}(x, y) \to \mathrm{hom}(\phi(x), \phi(y))$.

For each $x \in G^{(0)}$, the identity arrow at x is just x, so we have $\phi(1_x) = \phi(x) = 1_{\phi(x)}$, and we see that property (i) of Definition 2.3.2 is satisfied.

Finally, when phrased in the language of groupoids, property (ii) of Definition 2.3.2 is satisfied precisely when ϕ is a groupoid homomorphism. This also shows that the reverse implication holds. □

Combining this result with Proposition 2.1.6 implies that groupoid isomorphisms are precisely isomorphisms of the corresponding categories.

Chapter 3

Tilings

3.1 Tiling dynamical systems

In this section, we introduce the basic theory of tilings. The constructions presented here are standard in the literature, and largely follow the relatively modern development presented in [34]. See also [26] for a comprehensive introduction to the topic, and [64] for another modern reference.

A tiling of the Euclidean space $\mathbb{R}^d$ is a covering of the space by subsets of prescribed shapes in such a way that distinct subsets only intersect on their boundary. We begin by describing the "prescribed shapes" that we wish to use.

Definition 3.1.1. A *prototile* p is a labelled subset of $\mathbb{R}^d$ that is homeomorphic to the closed unit ball.

We usually think of the labelling of prototiles in terms of colours. In particular, this labelling allows for the inclusion of prototiles of the same shape which follow different placement or substitution rules.

Definition 3.1.2. Let $\mathcal{P}$ be a finite set of prototiles. A *tiling* T with prototile set $\mathcal{P}$ is a set $\{t_1, t_2, \ldots\}$ of subsets of $\mathbb{R}^d$, henceforth referred to as *tiles*, which satisfy

(T1) for each $i \in \{1, 2, \ldots\}$, there exists $x_i \in \mathbb{R}^d$ and $p_i \in \mathcal{P}$ such that $t_i = p_i + x_i$. Two tiles which are translates of the same prototile are said to have the same *tile type*;

(T2) $\bigcup_{i=1}^{\infty} t_i = \mathbb{R}^d$; and

(T3) $\operatorname{int}(t_i) \cap \operatorname{int}(t_j) = \varnothing$ for $i \neq j$, where int denotes the interior of a subset of $\mathbb{R}^d$.

The restriction to a finite set of prototiles is not strictly necessary, but it allows us to avoid some technicality later which might result from having prototiles of arbitrary sizes.

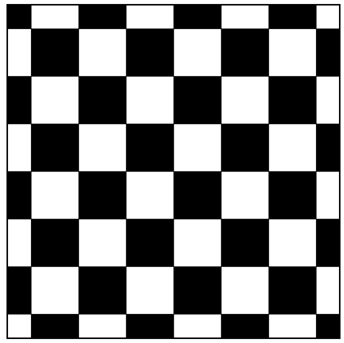

Figure 3.1: The chequerboard tiling

Of course, this could also be avoided with a different restriction, such as enforcing bounds on the diameter of prototiles (see Definition 3.1.14). In any case, the tilings of interest to us will necessarily have finite prototile sets as a consequence of finite local complexity (see Definition 3.1.13), so we prefer to make this restriction part of the definition.

Example 3.1.3. The chequerboard tiling (see Figure 3.1) is a tiling of $\mathbb{R}^2$. The prototile set consists of two elements. Both are unit squares in $\mathbb{R}^2$, but one is labelled by the colour "white" and the other by the colour "black".

Definition 3.1.4. A *patch* P of a tiling T of $\mathbb{R}^d$ is a finite subset of tiles of T whose union is connected as a subset of $\mathbb{R}^d$.

We introduce a convenient piece of notation that can be used to extract patches from a tiling. The idea can be found in [26, p. 19], and a similar notation in [78, p. 3].

Notation 3.1.5. If $X \subset \mathbb{R}^d$ and T is a tiling of $\mathbb{R}^d$, denote by $T \sqcap X$ the subset of T consisting of tiles which intersect X:

$$T \sqcap X \coloneqq \{t \in T \mid t \cap X \neq \varnothing\}.$$

When X is bounded and connected, $T \sqcap X$ is a patch of the tiling T. Patches of this form will shortly be useful in defining a metric between tilings.

Notation 3.1.6. Given a tiling T of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we denote by $T+x$ the tiling obtained by translating each tile in T by x:

$$T + x \coloneqq \{t + x \mid t \in T\}.$$

We refer to $T + x$ as the *translate of* T *by* x. We use this to define the *set of translates* of T:

$$T + \mathbb{R}^d \coloneqq \{T + x \mid x \in \mathbb{R}^d\}.$$

This notation leads to the following easy result, which is often stated without proof in the literature. We provide a proof for completeness.

Proposition 3.1.7. *The elements of $T + \mathbb{R}^d$ are tilings with the same prototile set as T.*

Proof. Choose an element $T' \in T + \mathbb{R}^d$. By definition, we can write $T' = T + x$ for some $x \in \mathbb{R}^d$. Denote by $\mathcal{P}$ the prototile set of T. We show that the three properties of a tiling hold for T'.

(T1): For every tile $t' \in T'$, there exists a tile $t \in T$ such that $t' = t + x$. By property (T1) of the tiling T, we see that $t = p + y$ for some prototile $p \in \mathcal{P}$ and $y \in \mathbb{R}^d$. Therefore, $t' = p + (y + x)$ is a translate of an element of $\mathcal{P}$.

(T2): Let $y \in \mathbb{R}^d$ be arbitrary. By property (T2) of the tiling T, we know that there exists a tile $t \in T$ such that $y - x \in t$. By definition, $t + x$ is a tile in $T + x$, and we have $y = (y - x) + x \in t + x \in T'$. This shows that the tiles in T' cover $\mathbb{R}^d$.

(T3): Consider tiles $t'_1, t'_2 \in T'$ such that $t'_1 \neq t'_2$. Then $t'_1 - x$ and $t'_2 - x$ are tiles in T with $t'_1 - x \neq t'_2 - x$, and we have by property (T3) of the tiling T that $\operatorname{int}(t'_1 - x) \cap \operatorname{int}(t'_2 - x) = \varnothing$. On the other hand, $y \in \operatorname{int}(t'_1 - x) \cap \operatorname{int}(t'_2 - x)$ if and only if $y + x \in \operatorname{int}(t'_1) \cap \operatorname{int}(t'_2)$, so we see that $\operatorname{int}(t'_1) \cap \operatorname{int}(t'_2) = \varnothing$. □

We are now ready to define a metric on a given set of tilings. For now, the set of tilings that one should have in mind is the set of translates of a particular tiling. The intuitive notion captured by the definition is that two tilings are close if we can translate each one slightly to get them to agree on a large ball around the origin.

Definition 3.1.8. We define the *tiling metric*, d, on a given set of tilings of $\mathbb{R}^d$ as follows. Let T_1 and T_2 be tilings in the set. Given $\epsilon \in (0, 1)$, we say that T_1 and T_2 are *ϵ-close* if there exist vectors $x_1, x_2 \in \mathbb{R}^d$ such that $|x_1|, |x_2| \leq \epsilon$ and which satisfy

$$(T_1 + x_1) \sqcap B_{\epsilon^{-1}}(0) = (T_2 + x_2) \sqcap B_{\epsilon^{-1}}(0),$$

where $B_r(x)$ denotes the open ball of radius r and centre x. The distance between the tilings is defined to be the infimum of the set of $\epsilon \in (0, 1)$ for which T_1 and T_2 are ϵ-close. If no such ϵ exists, we say that the distance between the tilings is 1.

We remark that, in the context of the above definition, the notion of ϵ-closeness cannot be expressed as a single inequality involving d, because knowing that $d(T_1, T_2) = \epsilon$ tells us nothing about whether T_1 and T_2 are ϵ-close. However, knowing that $d(T_1, T_2) < \epsilon$ is enough to ensure that T_1 and T_2 are ϵ-close, which implies that $d(T_1, T_2) \leq \epsilon$.

The proof of the following result appears to often be omitted from the literature.

Proposition 3.1.9. *The tiling metric is a metric on any given set of tilings.*

Proof. Symmetricity and non-negativity are clear. It is also easy to see that if $T = T'$ then $d(T,T') = 0$. The philosophy of the remainder of the proof of positive definiteness is also fairly straightforward, but implementing the technical details is a little tricky. The philosophy is to choose an appropriate point x of any tile $t \in T$, and see that the tilings agree on a large enough ball, up to a small enough translation, that it forces the tile $t' \in T'$ which contains x to agree with t.

More precisely, suppose that $d(T,T') = 0$, so that, for any $\epsilon > 0$, T and T' are ϵ-close. Choose an arbitrary tile $t \in T$, and choose a point $x \in \text{int}(t)$. Since T' covers $\mathbb{R}^d$, there exists $t' \in T$ such that $x \in t'$. We may further assume that $x \in \text{int}(t')$. Indeed, since $x \in \text{int}(t)$, there exists $R > 0$ such that $B_R(x) \subset \text{int}(t)$, and, since $x \in t'$, this ball also intersects $\text{int}(t')$. We may then replace x by a point in the intersection of this ball and $\text{int}(t')$ to fulfil our additional assumption.

Thus, let $x \in \text{int}(t) \cap \text{int}(t')$, and find $r > 0$ such that $B_r(x) \subset \text{int}(t) \cap \text{int}(t')$. Observe that this implies that, for any $y, z \in \mathbb{R}^d$ such that $|y|, |z| < r$, the interiors of $t + y$ and $t' + z$ will still intersect at x. Choose $0 < \epsilon < \min\{1, r, r^{-1}\}$ to be sufficiently small that $t \in T \sqcap B_{\epsilon^{-1}-r}(0)$ and $t' \in T' \sqcap B_{\epsilon^{-1}-r}(0)$. Since T and T' are ϵ-close, there exist $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| \leq \epsilon < r$ such that $(T + x_1) \sqcap B_{\epsilon^{-1}}(0) = (T' + x_2) \sqcap B_{\epsilon^{-1}}(0)$. Observe that $(T+x_1) \sqcap B_{\epsilon^{-1}}(0) = (T \sqcap B_{\epsilon^{-1}}(-x_1)) + x_1 \supset (T \sqcap B_{\epsilon^{-1}-r}(0)) + x_1 \ni t + x_1$. Similarly, we see that $t' + x_2 \in (T' + x_2) \sqcap B_{\epsilon^{-1}}(0)$. Since these patches were equal, and since the interiors of $t + x_1$ and $t' + x_2$ intersect, we must have that $t + x_1 = t' + x_2$, so that $t' = t + (x_1 - x_2)$.

Suppose that $x_1 \neq x_2$, so that $|x_1 - x_2| > 0$. If we now repeat the argument of the paragraph above, instead utilising the fact that T and T' are $(|x_1 - x_2|/4)$-close, we see that there also exist $y_1, y_2 \in \mathbb{R}^d$ with $|y_1|$, $|y_2| \leq |x_1 - x_2|/4$ such that $t + (x_1 - x_2) = t' = t + (y_1 - y_2)$, which requires that $x_1 - x_2 = y_1 - y_2$, because t is bounded. Thus, we have $|x_1 - x_2| = |y_1 - y_2| \leq |y_1| + |y_2| \leq |x_1 - x_2|/2$, which requires that $|x_1 - x_2| = 0$, a contradiction. Therefore, we must have $x_1 = x_2$, which shows that $t = t' \in T'$. Since $t \in T$ was arbitrary, we have shown that $T \subset T'$. By switching the roles of T and T' in the proof, we see that $T = T'$.

It only remains to prove the triangle inequality. To do so, we prove that, for any $\epsilon > 0$, we have $d(T_1, T_3) \leq d(T_1, T_2) + d(T_2, T_3) + \epsilon$. To save notation, let $a' = d(T_1, T_2)$ and $b' = d(T_2, T_3)$, and fix $\epsilon > 0$ small enough that $a' + b' + \epsilon < 1$, otherwise the proof is trivial. Also, let $a = a' + \epsilon/2$ and $b = b' + \epsilon/2$, to ensure that T_1 and T_2 are a-close, and that T_2 and T_3 are b-close. Thus, we can find x_1, $x_2 \in \mathbb{R}^d$ with $|x_1|$, $|x_2| \leq a$ such that $T_1 + x_1 = T_2 + x_2$ on tiles which intersect $B_{a^{-1}}(0)$. Similarly, there exist y_1, $y_2 \in \mathbb{R}^d$ with $|y_1|$, $|y_2| \leq b$ such that $T_2 + y_1 = T_3 + y_2$ on $B_{b^{-1}}(0)$. Using the first of these equalities, we have that

$T_1 + x_1 - x_2 = T_2$ on $B_{a^{-1}}(-x_2)$, so that $(T_1 + x_1 - x_2) + y_1 = T_2 + y_1$ on $B_{a^{-1}}(y_1 - x_2)$. Applying the second equality to this last statement yields $(T_1 + x_1 - x_2) + y_1 = T_3 + y_2$ on tiles which intersect $B_{a^{-1}}(y_1 - x_2) \cap B_{b^{-1}}(0)$. Translating all this by x_2 finally gives $T_1 + x_1 + y_1 = T_3 + x_2 + y_2$ on $B_{a^{-1}}(y_1) \cap B_{b^{-1}}(x_2)$. Notice that $|x_1 + y_1|$ and $|x_2 + y_2|$ are no larger than $a + b$, as required for T_1 and T_3 to be $(a + b)$-close, so to finish the proof it suffices to show that $B_{(a+b)^{-1}}(0) \subset B_{a^{-1}}(y_1) \cap B_{b^{-1}}(x_2)$.

To do this, we prove the two inequalities $|y_1| + (a + b)^{-1} < a^{-1}$ and $|x_2| + (a + b)^{-1} < b^{-1}$. Since $|y_1| < b$ and $|x_2| < a$, we instead show that $b+(a+b)^{-1} < a^{-1}$ and $a+(a+b)^{-1} < b^{-1}$. We only show the first of these, as the second can be obtained by swapping the roles of a and b. By multiplying out the denominators (recalling that $0 < a,\, b,\, a + b < 1$, so there is no problem with sign changes), rearranging, and cancelling, we see that it is equivalent to show that $b(1-a^2-ab) > 0$. Since $b > 0$ it is equivalent to show that $1 > a^2+ab = a(a+b)$, and since both multiplicands on the right hand side are positive and smaller than 1, we are done. □

Proposition 3.1.10. *Given a finite set $\mathcal{P}$ of prototiles, the space of all tilings with prototile set $\mathcal{P}$ is complete in the tiling metric.*

Proof. Our proof follows the same method as the proof of [2, Lemma 3.2.3].

Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of tilings with prototile set $\mathcal{P}$, and for $k \in \mathbb{N}$ write $s_k = d(T_k, T_{k+1})$. Pass to a subsequence to assume without loss of generality that the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ is decreasing and $\sum_{n=1}^{\infty} s_n < \infty$. Assume further that for each $k \in \mathbb{N}$ we have $s_{k+1}^{-1} > s_{k+1} + s_k + s_k^{-1}$.

Fix $k \in \mathbb{N}$. Since we have $d(T_k, T_{k+1}) = s_k$, there exist $x_k, y_k \in \mathbb{R}^d$ such that $|x_k|, |y_k| \le s_k$ and so that $(T_k + x_k) \sqcap B_{s_k^{-1}}(0) = (T_{k+1} + y_k) \sqcap B_{s_k^{-1}}(0)$. In this way, define vectors x_n and y_n for every $n \in \mathbb{N}$. Write $\delta_k = \sum_{n=k}^{\infty}(x_n - y_n)$, noting that the sum converges because $|x_n - y_n| \le s_n$ and $\sum_{n=1}^{\infty} s_n < \infty$ by assumption.

Consider the sequence $\Big(\big(T_n \sqcap B_{s_n^{-1}}(-x_n)\big) + \delta_n\Big)_{n\in\mathbb{N}}$. We claim that this defines an increasing sequence of patches which agree where they intersect. Indeed, for each fixed $k \in \mathbb{N}$ we have

$$\begin{aligned}\big(T_k \sqcap B_{s_k^{-1}}(-x_k)\big) + x_k - y_k &= \big((T_k + x_k) \sqcap B_{s_k^{-1}}(0)\big) - y_k \\ &= \big((T_{k+1} + y_k) \sqcap B_{s_k^{-1}}(0)\big) - y_k = T_{k+1} \sqcap B_{s_k^{-1}}(-y_k).\end{aligned}$$

Now observe that since $|x_{k+1}| \le s_{k+1}$, $|y_k| \le s_k$, and $s_{k+1}^{-1} > s_{k+1} + s_k + s_k^{-1}$, we have $B_{s_{k+1}^{-1}}(-x_{k+1}) \supset B_{s_k+s_k^{-1}}(0) \supset B_{s_k^{-1}}(-y_k)$. Using this along with the calculation above, it follows that

$$\Big(T_k \sqcap B_{s_k^{-1}}(-x_k)\Big) + x_k - y_k = T_{k+1} \sqcap B_{s_k^{-1}}(-y_k) \subset T_{k+1} \sqcap B_{s_{k+1}^{-1}}(-x_{k+1}).$$

By adding δ_{k+1} to both sides, and using the equality $\delta_k = (x_k - y_k) + \delta_{k+1}$, we obtain

$$\Big(T_k \sqcap B_{s_k^{-1}}(-x_k)\Big) + \delta_k \subset \Big(T_{k+1} \sqcap B_{s_{k+1}^{-1}}(-x_{k+1})\Big) + \delta_{k+1}$$

as we claimed.

Let $T' = \bigcup_{n=1}^{\infty} \Big(\Big(T_n \sqcap B_{s_n^{-1}}(-x_n)\Big) + \delta_n\Big)$. We show that T' satisfies the three properties of a tiling with prototile set $\mathcal{P}$.

(T1): Let $t' \in T'$ be a tile, so that for some $k \in \mathbb{N}$ we have $t' \in \Big(T_k \sqcap B_{s_k^{-1}}(-x_k)\Big) + \delta_k = (T_k + \delta_k) \sqcap B_{s_k^{-1}}(\delta_k - x_k)$. It follows that there exists $t \in T_k$ such that $t' = t + \delta_k$. By property (T1) of the tiling T_k, we have $t = p + v$ for some element $p \in \mathcal{P}$ and some $v \in \mathbb{R}^d$, and it follows that $t' = p + (v + \delta_k)$, so that t' is a translate of $p \in \mathcal{P}$.

(T2): Since the procedure at the start of the proof defines arbitrarily large patches of T' as patches of a tiling with prototile set $\mathcal{P}$, it is clear that T' covers $\mathbb{R}^d$.

(T3): Again, the procedure at the start of the proof allows us to view any patch of T' as a subpatch of a patch in some tiling with prototile set $\mathcal{P}$, where this condition is satisfied.

To finish the proof, we must show that $d(T_n, T') \to 0$ as $n \to \infty$, so fix $\epsilon > 0$. Observe that for each $k \in \mathbb{N}$ we have

$$(T_k + \delta_k) \sqcap B_{s_k^{-1}}(\delta_k - x_k) \subset T'.$$

By adding $x_k - \delta_k$ to both sides, it follows that

$$(T_k + x_k) \sqcap B_{s_k^{-1}}(0) = (T' + x_k - \delta_k) \sqcap B_{s_k^{-1}}(0).$$

Now observe that $|x_k| \leq s_k \to 0$ and $|x_k - \delta_k| \to 0$ as $k \to \infty$. Therefore, when K is large enough that $|x_k|, |x_k - \delta_{k+1}| < \epsilon$ and $s_k^{-1} > \epsilon$ whenever $k > K$, it follows that $d(T_k, T') < \epsilon$ whenever $k > K$. □

We can now complete the set of translates of a given tiling to obtain a more interesting space, which is where we will work from now on.

Definition 3.1.11. Suppose T is a tiling of $\mathbb{R}^d$ with prototile set $\mathcal{P}$. The *continuous hull* Ω_T of T is the closure of $T + \mathbb{R}^d$ with respect to the tiling metric in the space consisting of all tilings with prototile set $\mathcal{P}$.

By Proposition 3.1.10, the elements of Ω_T are tilings with the same prototile set as T.

Examples 3.1.12.

(i) If T is a tiling of $\mathbb{R}$ by unit intervals all of which have the same label (see Figure 3.2), then $\Omega_T = T + \mathbb{R}$, and can be identified with the circle $\mathbb{R}/\mathbb{Z}$.

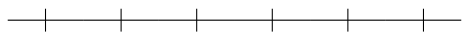

Figure 3.2: A tiling of $\mathbb{R}$ by unit intervals

(ii) The hull of the chequerboard tiling is a torus. In this case, the hull is the same as the set of translates. One can see a repeating square in Figure 3.3.

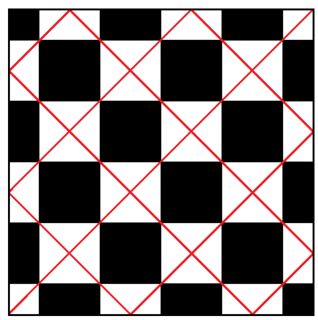

Figure 3.3: Toroidal structure in the chequerboard tiling

(iii) Consider the tiling T' of $\mathbb{R}$ pictured in Figure 3.4. This is a tiling by unit intervals with integer endpoints, all of which are labelled "w" (for "white") except the one whose left endpoint lies on zero which we label "b" (for "black"). This tiling is not

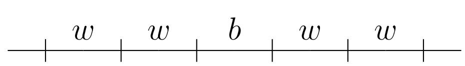

Figure 3.4: A tiling of $\mathbb{R}$ with one distinguished tile

periodic – when $x \neq 0$ we have $T' + x \neq T'$. Therefore, the set of translates of T' can be identified (as a set) with $\mathbb{R}$. One can check that the sequence $\{T' + n \mid n \in \mathbb{N}\}$ is Cauchy in the tiling metric, and converges to the tiling T of Figure 3.2. Therefore, $\Omega_{T'}$ also contains a copy of the hull of the tiling T, which is identified with a circle. Putting this together, the hull $\Omega_{T'}$ looks like a copy of $\mathbb{R}$ "wrapping around" the circle, pictured in Figure 3.5. Tilings of the form $T' + z$ for $z \in \mathbb{Z}$ occur on the dashed line between the marked points inside and outside the circle, which are identified and represent the tiling T'. The top of the circle represents the tiling T of the first example, and the bottom of the circle is the tiling $T + 1/2$.

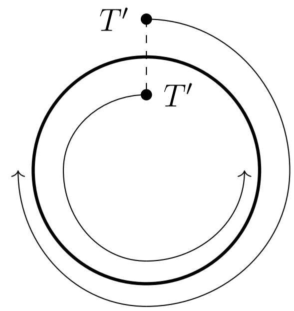

Figure 3.5: The hull of the tiling from Figure 3.4

When we defined the set of translates of a tiling, $T + \mathbb{R}^d$, there was an implicit action of $\mathbb{R}^d$ on this set by translation. We extend our consideration of this action to the entire hull Ω_T, and in this way we obtain a dynamical stystem $(\Omega_T, \mathbb{R}^d)$. We now describe several important properties of tilings and also see how these properties translate into the language of the dynamical system.

Definition 3.1.13. Suppose T is a tiling of $\mathbb{R}^d$.

(i) T is said to have *finite local complexity* (FLC) if, for each $R > 0$, the collection of patches $\{T \sqcap B_R(x) \mid x \in \mathbb{R}^d\}$ is finite up to translation.

(ii) T is said to be *nonperiodic* if, for every $x \in \mathbb{R}^d$, $T + x = T$ implies $x = 0$. We say T is *aperiodic* if every tiling in the continuous hull Ω_T is nonperiodic.

(iii) T is said to be *repetitive* if, for every patch $P \subset T$, there exists $R > 0$ such that, for every $x \in \mathbb{R}^d$, P appears in $T \sqcap B_R(x)$. The idea is that copies of any patch appearing somewhere in the tiling can be found spread uniformly throughout the tiling. A different viewpoint is that knowledge of your local surroundings in a repetitive tiling gives no information as to your position within the tiling.

Note that the terms "nonperiodic" and "aperiodic" do not have a completely standardised meaning in the literature. In particular, "aperiodic" has been used to describe a prototile set which is unable to tile periodically, or as a replacement for "nonperiodic" in the above definition. In this case, our usage of the term "aperiodic" would usually be replaced with "strong nonperiodicity" or "strong aperiodicity", respectively.

Definition 3.1.14. For $U \subset \mathbb{R}^d$ we denote the *diameter* of U by

$$\operatorname{diam}(U) = \sup\{d(x, y) \mid x, y \in U\}.$$

Theorem 3.1.15. *Let T be a tiling of $\mathbb{R}^d$.*

(i) T has FLC if and only if Ω_T is compact.

(ii) T is nonperiodic if and only if the translational action of $\mathbb{R}^d$ on $T + \mathbb{R}^d$ is free, and T is aperiodic if and only if the action of $\mathbb{R}^d$ on Ω_T is free.

(iii) If T is repetitive, then the action of $\mathbb{R}^d$ on Ω_T is minimal. If T has FLC, then the converse is also true. Furthermore, if T is reptitive, then the continuous hull Ω_T is independent of T in the sense that the continuous hull of any other tiling in Ω_T will also be Ω_T.

Proof. ((i), $\Rightarrow$): We follow the proof of [52, Lemma 2], and show that Ω_T is sequentially compact. Consider a sequence $(T_i)_{i\in\mathbb{N}}$ in Ω_T. Since T has FLC, the collection of patches $\{T_i \sqcap B_1(0)\}_{i\in\mathbb{N}}$ is finite up to translation, so there exists a patch P_1 of some tiling in Ω_T such that infinitely many patches in the collection are translates of P_1. Without loss of generality, by considering a translate of the tiling containing P_1 if necessary, we may assume that $\text{int}(P_1)$ contains the origin. Construct a subsequence $(T_i^{P_1})$ of (T_i) consisting of the tilings such that $T_i^{P_1} \sqcap B_1(0)$ is a translate of P_1. For each i, say that $T_i^{P_1} \sqcap B_1(0) = P_1 + x_i$. Equivalently, $P_1 = (T_i^{P_1} \sqcap B_1(0)) - x_i = (T_i^{P_1} - x_i) \sqcap B_1(-x_i)$, which shows that $B_1(-x_i) \subset P_1$, and that this ball intersects every tile in P_1. Since $0 \in P_1$ and $-x_i \in P_1$, it follows that $x_i \in \overline{B_{\text{diam}(P_1)}(0)}$.

Observe that, since $T_i^{P_1} \sqcap B_1(0) = P_1 + x_i$, we have $P_1 + x_j = P_1 + x_i + (x_j - x_i) = (T_i^{P_1} \sqcap B_1(0)) + (x_j - x_i) = (T_i^{P_1} + x_j - x_i) \sqcap B_1(x_j - x_i)$. Now, observe that the ball $B_1(x_j - x_i)$ is in the same relative position in $P_1 + x_j$ as the ball $B_1(-x_i)$ would be in P_1. We know that $P_1 \sqcap B_1(-x_i) = P_1 = P_1 \sqcap B_1(-x_j)$ so, in the equality at the start of the paragraph, we get the same result by replacing $B_1(x_j - x_i)$ with $B_1(0)$, since the latter ball is in the same relative position in $P_1 + x_j$ as $B_1(-x_j)$ is in P_1. Therefore, $P_1 + x_j = (T_i^{P_1} + x_j - x_i) \sqcap B_1(x_j - x_i) = (T_i^{P_1} + x_j - x_i) \sqcap B_1(0)$. On the other hand, $P_1 + x_j = T_j^{P_1} \sqcap B_1(0)$, so we have shown that $(T_i^{P_1} + x_j - x_i) \sqcap B_1(0) = T_j^{P_1} \sqcap B_1(0)$ for every $i, j \in \mathbb{N}$.

Since $(x_i)_{i\in\mathbb{N}} \subset \overline{B_{\text{diam}(P_1)}(0)}$, it admits a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$, converging to x, say. Consider the subsequence $(T_{n_i}^{P_1})_{i\in\mathbb{N}} \subset (T_i)_{i\in\mathbb{N}}$. Choose $\epsilon > 0$ and suppose $N \in \mathbb{N}$ is large enough that, whenever $i, j \geq N$, we have $|x_{n_i} - x| < \epsilon/2$ and $|x_{n_j} - x| < \epsilon/2$, so that $|x_{n_j} - x_{n_i}| < \epsilon$. Then, as long as $i, j \geq N$, the previous paragraph shows that there exist vectors $v_1, v_2 \in \mathbb{R}^d$ with $|v_1|, |v_2| < \epsilon$ such that $(T_{n_i}^{P_1} + v_1) \sqcap B_1(0) = (T_{n_j}^{P_1} + v_2) \sqcap B_1(0)$, by choosing $v_1 = x_{n_j} - x_{n_i}$, and $v_2 = 0$. For short, we say that $(T_{n_i}^{P_1})_{i\in\mathbb{N}}$ *converges on* $B_1(0)$.

Now, we repeat this procedure, beginning with the finite collection of patches $\{T_{n_i}^{P_1} \sqcap B_2(0)\}_{i\in\mathbb{N}}$, to find a subsequence of $(T_{n_i}^{P_1})_{i\in\mathbb{N}}$ which converges on $B_2(0)$. We continue to repeat this argument, obtaining, for each $n \in \mathbb{N}$ a subsequence of $(T_i)_{i\in\mathbb{N}}$ which converges on $B_n(0)$. Apply the Cantor diagonalization argument, forming a subsequence of $(T_i)_{i\in\mathbb{N}}$ by taking the first element of the subsequence which converges on $B_1(0)$, the second element of the subsequence which converges on $B_2(0)$, and so on. Then, for any $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \epsilon^{-1}$, and observe that any two terms of this subsequence beyond the N-th can be made to agree on $B_{\epsilon^{-1}}(0)$ up to a translation of each by a vector of magnitude smaller than ϵ, showing that any such terms are ϵ-close. This shows that the sequence is Cauchy in the tiling metric, and hence it must be convergent, since Ω_T is complete.

((i), $\Leftarrow$): We will show that if T does not have FLC, then Ω_T is not sequentially compact. Suppose that there exists $R > 0$ such that the collection $\{T \sqcap B_R(x) \mid x \in \mathbb{R}^d\}$ contains infinitely many patches which are not translations of one another. Since, for any $R' > R$, this implies that the collection $\{T \sqcap B_{R'}(x) \mid x \in \mathbb{R}^d\}$ also contains infinitely many patches which are not pairwise translationally equivalent, we may increase R if necessary to assume that, for some fixed $\delta > 0$, the collection $\{T \sqcap B_R(x) \mid \text{there exists } t \in T \text{ such that } B_\delta(x) \subset t\}$ contains infinitely many patches which are not pairwise translationally equivalent. In fact, by further increasing R by the maximal diameter of a prototile (which is finite because the prototile set is finite), we may only consider a single point x from the interior of each tile of t when forming the set above, and we may choose these points so that whenever two tiles have the same tile type, the associated x are in the same relative position in each tile. In other words, for each prototile p, we distinguish a point $x(p) \in \text{int}(p)$ such that $B_\delta(x(p)) \subset p$, and so that if $P = T \sqcap B_R(x)$ is one of the patches we are considering, and x is in the interior of a tile $t \in T$ such that $t = p + y$, then $x = x(p) + y$. Take a subcollection $\{P_i\}_{i \in \mathbb{N}}$ such that no two patches in this collection are translates of each other. For each i, say that $P_i = T \sqcap B_R(x_i)$, so that $P_i - x_i = (T \sqcap B_R(x_i)) - x_i = (T - x_i) \sqcap B_R(0)$. Consider the sequence of tilings $(T - x_i)_{i \in \mathbb{N}}$. Then, in the tiling $T - x_i$, the ball $B_\delta(0)$ is contained in the interior of a tile of the patch $P_i - x_i$.

Choose $0 < \epsilon < \delta/2$ small enough that $R < \epsilon^{-1} - \epsilon$, and suppose that $T - x_i$ and $T - x_j$ are ϵ-close. Then there exist $v_1, v_2 \in \mathbb{R}^d$ with $|v_1|, |v_2| \leq \epsilon$ such that $(T - x_i + v_1) \sqcap B_{\epsilon^{-1}}(0) = (T - x_j + v_2) \sqcap B_{\epsilon^{-1}}(0)$. Observe that in the left-hand patch, the tile on the origin is $t_i - x_i + v_1$, where $t_i \in P_i$ is the tile which contains x_i, and that $t_i - x_i + v_1$ contains the origin in its interior, because $|V_1| \leq \epsilon < \delta/2$. Similarly, in the right-hand patch, the tile $t_j - x_j + v_2$ contains the origin in its interior. Since the interiors of these tiles overlap, and the two patches are the same, we must have $t_i - x_i + v_1 = t_j - x_j + v_2$. This shows that the tiles t_i and t_j have the same tile type, both being translates of the prototile p, say. Observe that $t_i - x_i = t_j - x_j + (v_2 - v_1)$, and that the origin sits in the same relative position in $t_i - x_i$ as $x(p)$ sits in p. On the other hand, since the origin sits in this same position in $t_j - x_j$, we see that the origin sits in $t_i - x_i = t_j - x_j + (v_2 - v_1)$ in the same relative position that $x(p) + v_1 - v_2$ does in p. This shows that $v_1 - v_2 = 0$, so that $v_1 = v_2 = v$, say.

Now, we have shown that there exists $|v| < \epsilon$ such that $(T - x_i + v) \sqcap B_{\epsilon^{-1}}(0) = (T - x_j +$

$v) \sqcap B_{\epsilon^{-1}}(0)$. Equivalently, $((T - x_i) \sqcap B_{\epsilon^{-1}}(-v)) + v = ((T - x_j) \sqcap B_{\epsilon^{-1}}(-v)) + v$, so that $(T - x_i) \sqcap B_{\epsilon^{-1}}(-v) = (T - x_j) \sqcap B_{\epsilon^{-1}}(-v)$, and hence that $((T - x_i) \sqcap B_{\epsilon^{-1}}(-v)) \sqcap B_R(0) = ((T - x_j) \sqcap B_{\epsilon^{-1}}(-v)) \sqcap B_R(0)$. Now, $B_R(0) \subset B_{\epsilon^{-1}-\epsilon}(0) \subset B_{\epsilon^{-1}}(-v)$, so we see that $((T - x_i) \sqcap B_{\epsilon^{-1}}(-v)) \sqcap B_R(0) = (T - x_i) \sqcap B_R(0) = P_i - x_i$, and similarly for j. This shows that $P_i - x_i = ((T - x_i) \sqcap B_{\epsilon^{-1}}(-v)) \sqcap B_R(0) = ((T - x_j) \sqcap B_{\epsilon^{-1}}(-v)) \sqcap B_R(0) = P_j - x_j$, so that $P_i = P_j + (x_i - x_j)$. Since no two distinct patches P_i were translates of one another, this implies that $i = j$, showing that $d(T - x_i, T - x_j) \geq \epsilon$ whenever $i \neq j$.

(ii): First, suppose that T is nonperiodic, and consider a tiling $T + y \in T + \mathbb{R}^d$ and $x \in \mathbb{R}^d$ such that $(T + y) + x = T + y$. Then we see that $T + x = (T + y + x) - y = (T + y) - y = T$, so that $x = 0$ by the definition of nonperiodicity. Thus, the translational action on $T + \mathbb{R}^d$ is free. The converse is clear: T is an element of $T + \mathbb{R}^d$, and nonperiodicity is just the restriction of the definition of freeness to this element.

Next, T is aperiodic if and only if every $T' \in \Omega_T$ is nonperiodic. That is, T is aperiodic if and only if, for every $T' \in \Omega_T$, $T' + x = T'$ implies $x = 0$, which is precisely the definition of freeness of the translational action on Ω_T.

((iii), $\Rightarrow$): Observe that, for any tiling T, the collection of patches which appear in any element $T' \in \Omega_T$ is, up to a possible translation of each patch, a subset of the collection of patches appearing in T. We claim that if T is repetitive, then these collections are equal up to translation.

This is certainly true if T' is a translate of T, so we prove it if T' is a limit of a Cauchy sequence of these translates. Indeed, choose any patch P of T, and find the associated R from the definition of repetitivity. In a Cauchy sequence of translates, $(T - x_i)_{i \in \mathbb{N}}$, we can always see a translation of P inside $(T - x_i) \sqcap B_R(0)$, because there exists $y \in \mathbb{R}^d$ such that $P + y \subset T \sqcap B_R(x_i)$ by reptitivity, and then $P + y - x_i \subset (T \sqcap B_R(x_i)) - x_i = (T - x_i) \sqcap B_R(0)$. All tilings far enough along the sequence (once the distance between any two is smaller than some $\epsilon > 0$ with $\epsilon^{-1} - \epsilon > R$) can be made to agree on $B_{R+\epsilon}(0)$ by applying a translation of size smaller than ϵ to each. This requires that, for large enough i, all of the patches $(T - x_i) \sqcap B_R(0)$ are translates of the same patch, and hence the limit tiling also contains a translate of this patch. Since every element of the sequence contains a translate of P in $B_R(0)$, this shows that the limit tiling also contains a translate of P, which proves our claim.

Now, let T' be an arbitrary tiling in Ω_T, and fix a second tiling $T'' \in \Omega_T$. We have shown that, up to translation, the collection of patches appearing in T' and T'' are equal. Choose a patch in T'' which (as a subset of $\mathbb{R}^d$) contains a ball around 0 of some arbitrarily large radius. Since a translate of this patch also appears in T', we can find a translate of T' which agrees with T'' on this patch. Since we can choose the radius to be as large as we like, this agreement means that the translates of T' get arbitrarily close to T'', and the system is seen to be minimal. Note that this also proves the final claim – if the action by translation on Ω_T is minimal, then for any $T' \in \Omega_T$, the closure of the orbit of T' in the tiling metric is Ω_T. On the other hand, this is precisely how $\Omega_{T'}$ is defined.

((iii),$\Leftarrow$): Suppose that T is a tiling with FLC that is not repetitive, so that there exists a patch $P \subset T$ such that, for any $R > 0$, we can find an $x \in \mathbb{R}^d$ such that P does not appear in $T \sqcap B_R(x)$. Associated to the sequence $(n)_{n\in\mathbb{N}}$ of choices of R, we find a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ that witness this condition, so that, for each $n \in \mathbb{N}$, no translate of P appears in the patch $T \sqcap B_n(x_n)$. Consider the sequence of tilings $(T_n)_{n\in\mathbb{N}}$ defined by $T_n = T - x_n$, so that, for each $n \in \mathbb{N}$, $T_n \sqcap B_n(0)$ doesn't contain a translate of P. Since T has FLC, (i) tells us that Ω_T is compact, so there exists a convergent subsequence of $(T_n)_{n\in\mathbb{N}}$ with limit T'. Notice that no translate of P appears anywhere in the tiling T', and so the orbit of T' cannot be dense. Indeed, if $y \in P \subset T$, choose $\epsilon > 0$ small enough that, for any $x \in \mathbb{R}^d$ such that $|x| < \epsilon$, we have $P - y + x \subset (T - y + x) \sqcap B_{\epsilon^{-1}}(0)$. Then, since T' contains no translate of P, it follows that $d(T' + z, T - y) > \epsilon$ for every $z \in \mathbb{R}^d$. □

Examples 3.1.16.

(i) If T has a finite prototile set consisting of polytopes, and tiles in T meet full-edge-to-full-edge, then T has FLC. Tilings satisfying these criteria are sometimes known as *simple tilings* (for example, see [63]).

(ii) Consider a tiling of $\mathbb{R}^2$ by (unlabelled) unit squares, arranged as in Figure 3.6. On the lower half plane, the squares are arranged periodically, with corners on the integer lattice $\mathbb{Z}^2$. On the upper half plane, the n^{th} row of tiles are placed so that there is a horizontal offset of $1/(n+1)$ with the row below. Thus, for example, the tiles in the row which covers $\mathbb{R} \times [0, 1]$ have corners on $(\mathbb{Z} + 1/2) \times \{0, 1\}$, and the tiles in the row which covers $\mathbb{R} \times [1, 2]$ have corners on $(\mathbb{Z} + 1/2 + 1/3) \times \{1, 2\}$. This tiling does not have FLC, because the collection of two-tile patches consisting of a tile from one row and an adjacent tile from the row above is infinite.

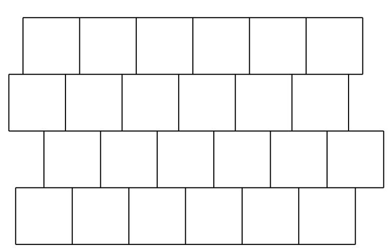

Figure 3.6: A patch of a tiling without finite local complextiy

(iii) If T is a tiling of $\mathbb{R}^d$ which is periodic (that is, not nonperiodic), and such that there are d linearly independent vectors $x_1, \ldots, x_d$ such that $T + x_i = T$ for each

$i \in \{1, \dots, d\}$, then T is repetitive. In particular, any periodic tiling of $\mathbb{R}$ is repetitive. Sometimes, periodic tilings which do not satisfy this condition are instead called *subperiodic*.

(iv) The tiling T' seen in Figure 3.4 is not repetitive – take the one-tile patch P consisting of the tile labelled "b", for instance. We saw that the hull of this tiling contains the periodic tiling of Figure 3.2, so T' is nonperiodic but not aperiodic.

3.2 Substitution tilings

The tilings of most interest to us will be those with FLC which are both aperiodic and repetitive. The repetitivity property can be thought of as enforcing some weaker sort of periodicity, or "long-range order" in a tiling, which leads to some push-and-pull between the properties. In this section, we briefly study a method of generating tilings which is particularly well suited to enable us to acquire tilings with all three of the properties we seek. The constructions presented here are well-known (see, for example, [5], [64]).

We begin with a finite set of prototiles, $\mathcal{P}$. Associated to the set of prototiles is a *substitution map*, ω, which assigns to each prototile $p \in \mathcal{P}$ a patch $\omega(p)$ of a particular form consisting of translates of prototiles in $\mathcal{P}$. More specifically, the patch $\omega(p)$ is generated by a process of expansion and subdivision. First, the prototile is inflated by a scaling factor $\lambda_\omega > 1$ (which depends only on the map ω, and not on the particular prototile), and then this inflated tile is subdivided into translates of elements of the prototile set $\mathcal{P}$ to form $\omega(p)$. We also define ω on translates of prototiles using the rule $\omega(p + x) = \omega(p) + \lambda_\omega x$. This allows us to iterate the map ω, and by doing so one obtains larger and larger patches $\omega^n(p)$ for $n \in \mathbb{N}$. The patch of the form $\omega^n(p)$ will be referred to as a *level n supertile associated to p*. Under certain assumptions, we can use these patches to describe a tiling.

Example 3.2.1. Let $\mathcal{P}$ be a set of prototiles of the plane consisting of two elements: a unit square, and a right-angled triangle whose non-hypotenuse sides have unit length. A substitution map for this system with scaling factor $\lambda = 2$ is shown in Figure 3.7.

Substitution systems may be used to generate tilings as follows. Since the prototile set is finite, for each prototile p, there exists $n \in \mathbb{N}$ large enough that the level n supertile $\omega^n(p)$ contains tiles which do not intersect the boundary $\partial\omega^n(p) \subset \mathbb{R}^d$ (here we are thinking of $\omega^n(p)$ as a subset of $\mathbb{R}^d$, rather than a patch). Furthermore, we claim that there exists some prototile p and some $n \in \mathbb{N}$ such that $\omega^n(p)$ contains a translate t of p which does not intersect $\partial\omega^n(p)$. Indeed, consider the following procedure. Begin with any $p_1 \in \mathcal{P}$, and find $n_1 \in \mathbb{N}$ such that $\omega^{n_1}(p_1)$ contains a tile which does not intersect its boundary. This tile is a translate of some prototile, which we call p_2. Repeat this procedure, at each step finding an n_k such that $\omega^{n_k}(p_k)$ contains a tile which does not intersect its boundary,

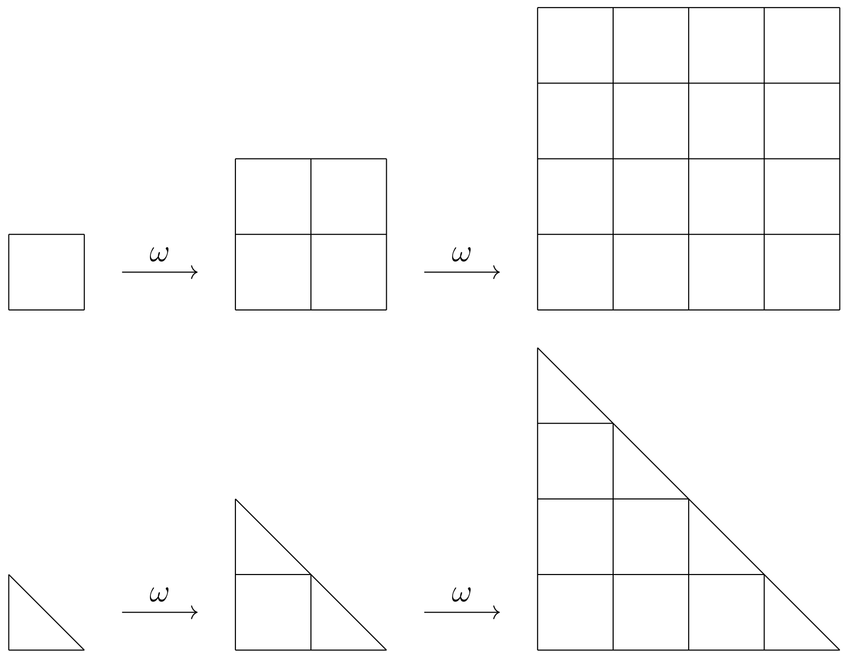

Figure 3.7: A substitution system

and is a translate of p_{k+1}. Since the prototile set is finite, there exists $k \in \mathbb{N}$ such that $p_{k+1} = p_{k-m}$ for some $0 \leq m \leq k-1$. Then, observe that $\omega^{n_k+\cdots+n_{k-m}}(p_k)$ contains a translate of p_k which does not intersect its boundary.

Let $p \in \mathcal{P}$ and $n \in \mathbb{N}$ be such that $\omega^n(p)$ contains a translate of p which does not intersect its boundary. We imagine subdividing p according to the substitution rule n times, but without inflating it, and then posititioning p so that the origin is contained in the interior of the miniaturised copy of t that we see in the subdivision. Since t is itself a copy of p, we may further subdivide t according to ω^n, and then newly position the origin in the interior of the miniaturised tile in this subdivision of t which is in the same relative position in t as t is in the subdivision of p. Iterating this procedure yields a Cauchy sequence of placements of the origin. These placements converge to a position $x(p)$ in the interior of p because they were all chosen to lie in the miniaturised copy of t in the subdivision of p, which is a closed set which doesn't intersect the boundary of p. If we now assume that p is positioned so that $x(p)$ lies on the origin, then $\omega^n(p)$ is a level n supertile which contains a translate of p such that $x(p)$ still lies on the origin. Thus, the sequence of patches $\{\omega^{nj}(p)\}_{j\in\mathbb{N}}$ agree where they intersect, and, due to the expansion factor of ω and the fact that the origin lies in the interior of p, these patches grow to cover $\mathbb{R}^d$, so that $\bigcup_{j\in\mathbb{N}} \omega^{nj}(p)$ defines a tiling. Of course, the structure of the tiling generated by the procedure outlined above will depend on the choice of prototile, and the procedure is not guaranteed to work for every prototile in general. The following assumption on ω solves both of these problems.

Definition 3.2.2. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of prototiles, and ω a substitution rule on

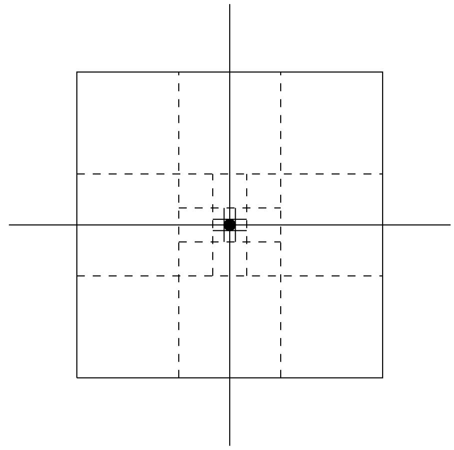

Figure 3.8: Positioning the origin in a substitution system

$\mathcal{P}$. The substitution system $(\mathcal{P}, \omega)$ is said to be *primitive* if there exists $k \in \mathbb{N}$ such that, for each $i \in \{1, \ldots, n\}$, the patch $\omega^k(p_i)$ contains a translate of every prototile $p_j \in \mathcal{P}$.

Observe that, whenever $j \in \mathbb{N}$, every supertile $\omega^{k+j}(p)$ will also contain a translate of every prototile, because, for each $t \in \omega^j(p)$, it contains a translate of the patch $\omega^k(t)$.

Under this condition, we may use the procedure outlined above to generate a tiling starting from *any* $p \in \mathcal{P}$. More precisely, there exists $n \in \mathbb{N}$ such that, for every $p \in \mathcal{P}$, the supertile $\omega^n(p)$ contains tiles which do not intersect its boundary. Let $t \in \omega^n(p)$ be such a tile. Then, if $k \in \mathbb{N}$ is the primitivity constant of the substitution, we see that, for every $p \in \mathcal{P}$, the supertile $\omega^k(t) \subset \omega^{n+k}(p)$ contains a translate of p which does not intersect the boundary of $\omega^{n+k}(p)$, whence the process from above can be used to generate a tiling.

A more important consequence of primitivity is that the resulting continuous hull does not depend on the prototile we begin with.

Theorem 3.2.3. *If $(\mathcal{P}, \omega)$ is a primitive substitution system, and T, T' are tilings generated by this substitution system as in the procedure above, then $\Omega_T = \Omega_{T'}$.*

Proof. Let p_T and $p_{T'}$ denote the prototiles from which we obtained T and T' respectively. With no loss of generality, we may assume that p_T and $p_{T'}$ contain the origin at the locations $x(p_T)$ and $x(p_{T'})$ constructed above. Let k be the primitivity constant for ω, and let n be as in the paragraph above. Denote the tile in T containing the origin by t, which is a copy of the prototile p_T such that $x(p_T)$ lies on 0. First, notice that we have $\omega^{n+k}(T) = T$, since T is obtained as a limit of applications of ω^{n+k} to a prototile. This means that $\omega^{n+k}(t)$ is a patch in T. Observe that $t \in \omega^{n+k}(t)$ is in the same position as before we applied the substitution, and that $\omega^{n+k}(t)$ contains a translate of every prototile by primitivity. In particular, $\omega^{n+k}(t)$ contains a translate of $p_{T'}$, which we denote by $t' = p_{T'} + y$. Since $\omega^{n+k}(t)$ is a patch in T, we see that $\omega^{n+k}(t) - y$ is a patch in $T - y$, and this patch contains a copy of $p_{T'}$ with $x(p_{T'})$ at the origin. Then the sequence $\{\omega^{j(n+k)}(T-y)\}_{j\in\mathbb{N}}$ converges to T' by the definition of T'. On the other hand, we have that $\omega^{j(n+k)}(T-y) = T - \lambda_\omega^{j(n+k)}y$ for each $j \in \mathbb{N}$, so this is a convergent sequence of translates of T, which means that the limit is in Ω_T. Thus we have shown that $T' \in \Omega_T$. It follows easily that $\Omega_{T'} \subset \Omega_T$, and we obtain the reverse inclusion by switching the roles of T and T' in the proof. □

In the light of this theorem, when talking about substitution tilings, we will drop the T from Ω_T and instead refer simply to Ω (or Ω_ω to emphasise the substitution rule).

The substitution map ω extends to tilings T with prototile set $\mathcal{P}$ by applying the substitution to each tile:

$$\omega(T) = \{t' \in \omega(t) \mid t \in T\}.$$

Some authors (see [5]) define Ω as the collection of tilings such that every patch is a subset of a patch of the form $\omega^n(p+x)$ for some $n \in \mathbb{N}$, $p \in \mathcal{P}$ and $x \in \mathbb{R}^d$. This definition is equivalent to ours, as is recorded in [5, Proposition 3.4]. Using this new definition, and making use of our newly defined extension of ω to tilings, it becomes clear that $\omega(\Omega) \subset \Omega$, so that we can consider the map $\omega : \Omega \to \Omega$.

To finish the section, we introduce the second assumption required for our substitution rule to produce the kind of tilings that we are looking for.

Definition 3.2.4. Let $(\mathcal{P}, \omega)$ be a primitive substitution system. We say that ω is *recognisable* if it is injective as a map $\omega : \Omega \to \Omega$.

One can show that every substitution map $\omega : \Omega \to \Omega$ is surjective (for example, see [5, Proposition 2.2]), so recognisable substitutions are bijective as maps $\Omega \to \Omega$. In particular, the recognisability property implies that, for every $n \in \mathbb{N}$, every tiling in Ω admits a unique decomposition into level n supertiles. Existence follows because every patch in every tiling in Ω is a subpatch of a translate of a supertile, and each such patch admits decompositions into supertiles of each lower level. To see uniqueness, suppose that a tiling admits two distinct decompositions into level n supertiles for some $n \geq 1$. For each of these decompositions in turn, define a new tiling as follows. Obtain a collection of subsets of $\mathbb{R}^d$ by taking the union of all the tiles appearing in each supertile in the decomposition, and then perform an enlargement of each of these subsets with scale factor λ_ω^{-n} and centre 0. Observe that the resulting collection of subsets is a tiling in Ω, and that the two tilings we obtain are distinct because the supertile decompositions we started with were. Furthermore, applying ω^n to either of these tilings produces the tiling we began with, showing that ω cannot be injective.

Example 3.2.5. Consider the substitution associated to the square prototile seen in Figures 3.7 and 3.8. This generates a periodic tiling by unit squares, so the hull Ω contains the

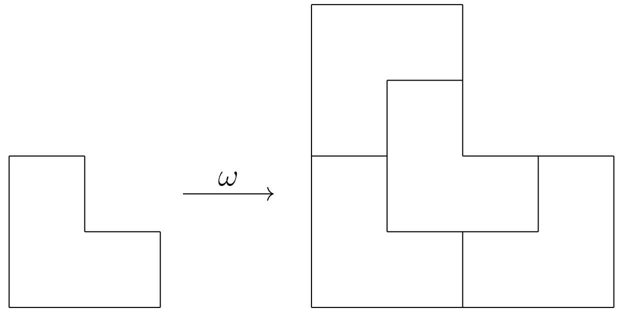

Figure 3.9: The chair substitution

tiling T by unit squares whose vertices lie on the integers. Notice that $\omega(T) = T$, and $\omega(T + (1/2, 1/2)) = \omega(T) + (2 \times (1/2, 1/2)) = T + (1, 1) = T$, so that this substitution is not recognisable. One may also observe that the tiling by unit squares generated by this substitution does not admit a unique decomposition into level 1 supertiles.

Example 3.2.6. The substitution system defined by the rule depicted in Figure 3.9 along with its $\pi/2$ rotations is easily seen to be primitive (with $k = 2$). It is recognisable because the level 1 supertiles are unable to non-trivially intersect. The tilings which this system generates are known as chair tilings.

The proof of the following theorem shows how (in the presence of FLC) primitivity implies repetitivity, and recognisability implies aperiodicity.

Theorem 3.2.7. *Let T be a tiling with FLC generated from a primitive, recognisable substitution system. Then T is aperiodic and repetitive.*

Proof. Suppose that $T' \in \Omega$ is such that $T' + x = T'$ for some $x \in \mathbb{R}^d$. By recognisability, for each n, T' has a unique decomposition into level n supertiles. Observe that we can obtain the (unique) decomposition of $T' + x$ into level n supertiles by translating each supertile in the decomposition of T' by x. Choose n large enough that, for each $p \in \mathcal{P}$, each supertile $\omega^n(p)$ overlaps its translate $\omega^n(p) + x$ (in the sense that these patches have some tiles in common). Since $T' = T' + x$, we see that, for any level n supertile P in the decomposition of T', T' contains both patches P and $P + x$. Furthermore, by our choice of n, the supertiles P and $P + x$ overlap in T'. Thus, if $x \neq 0$, the decomposition of T' into the level n supertiles $P + x$ is distinct from the decomposition into level n supertiles P, which contradicts recognisability.

Next, we show that T is repetitive, so fix a patch $P \subset T$. Since T was generated by applying ω to a prototile, all patches appearing in T arise from the substitution rule in the sense that there exists some $p \in \mathcal{P}$ and $j \in \mathbb{N}$ such that the supertile $\omega^{j(n+k)}(p)$ contains a copy of P, where k is the primitivity constant of ω, and n is chosen as in the construction of tilings from a primitive substitution. By primitivity, a copy of this supertile (and hence also a copy of the patch P) appears in every level $(j+1)(k+n)$ supertile. By FLC, the diameters of these supertiles are uniformly bounded by some constant D. Set $R = 2D$ and fix $x \in \mathbb{R}^d$. We claim that P appears in $T \sqcap B_R(x)$. Since T was generated by ω, we have that $\omega^{n+k}(T) = T$, so that also $\omega^{(j+1)(n+k)}(T) = T$. This means that we can partition T into level $(j+1)(n+k)$ supertiles (with one supertile corresponding to each tile in T). Since tiles in T cover $\mathbb{R}^d$, this means that x is contained in some level $(j+1)(k+n)$ supertile. By our choice of R, $B_R(x)$ then contains this supertile. On the other hand, as we showed above, the supertile itself contains a copy of P. Thus we have $P \subset T \sqcap B_R(x)$, as required. □

3.3 Tiling groupoids

The material contained in this section first appears in [33].

Let T be a tiling with prototile set $\mathcal{P}$. We first provide a simplification of the space Ω_T, making it more tractable for our analysis. Distinguish a point $x(p)$ in the interior of each prototile $p \in \mathcal{P}$. We refer to $x(p)$ as the *puncture* of the prototile p. Now, since each tile $t \in T$ satisfies $t = p + y$ for some $p \in \mathcal{P}$ and $y \in \mathbb{R}^d$, we assign punctures to tiles by $x(t) = x(p) + y$. Thus, every tile in every tiling in Ω_T is punctured. We now use these punctures to restrict the set of tilings we consider.

Definition 3.3.1. Suppose Ω is the continuous hull of an FLC tiling. The *discrete* (or *punctured*) *hull* Ω_{punc} of the tiling consists of all tilings $T \in \Omega$ for which the origin is a puncture of some tile $t \in T$.

Since these punctures always lie in the interiors of tiles, for each $T \in \Omega_{\text{punc}}$ there is a unique tile in T which contains the origin, which we denote by $T(0)$.

The set of punctures in a tiling has the following special structure.

Definition 3.3.2. Let (M, d) be a metric space and $X \subset M$.

- We say that X is *uniformly discrete* if $\inf\{d(x, y) \mid x, y \in X \text{ such that } x \neq y\} > 0$.
- We say that X is *relatively dense* if there exists $R > 0$ such that the collection $\{B_x(R) \mid x \in X\}$ covers M.
- We say that X is a *Delone set* if it is both uniformly discrete and relatively dense.

Lemma 3.3.3. *Let Ω_{punc} be the discrete hull arising from a punctured tiling with FLC. Then the set of punctures of tiles in any tiling $T \in \Omega_{\text{punc}}$ is a Delone set in $\mathbb{R}^d$.*

Proof. Since T has FLC, it has a finite punctured prototile set $\mathcal{P} = \{p_1, \ldots, p_n\}$. For each $j \in \{1, \ldots, n\}$, there exists $r_j > 0$ such that $B_{r_j}(x(p_j)) \subset p_j$, since $x(p_j) \in \text{int}(p_j)$.

Let $r = \min\{r_j \mid j = 1, \dots, n\}$ so that, for each $p \in \mathcal{P}$, we have $B_r(x(p)) \subset p$. Then, for each $t \in T$, there exists $p_t \in \mathcal{P}$ and $x_t \in \mathbb{R}^d$ such that $t = p_t + x_t$, and so we have $B_r(x(t)) = B_r(x(p_t) + x_t) \subset p_t + x_t = t$. Since each tile $t \in T$ contains exactly one puncture, this shows that the infimum of the distances between any two punctures in tiles of T is bounded below by r, so the set of punctures is uniformly discrete.

Since prototiles are homeomorphic to the closed unit ball of $\mathbb{R}^d$, they have finite diameter. Let $R = \max\{\text{diam}(p) \mid p \in \mathcal{P}\}$. We claim that the collection of open balls of radius $2R$ around punctures of T is a cover of $\mathbb{R}^d$, so let $y \in \mathbb{R}^d$. Since T is a cover of $\mathbb{R}^d$, there exists $t \in T$ such that $y \in t$. Since t is a translate of some prototile, $\text{diam}(t) \leq R$. Since $y \in t$ and $x(t) \in t$, we have $d(y, x(t)) \leq R$, so $y \in B_{2R}(x(t))$, which proves our claim. □

Remark 3.3.4. Let Ω_{punc} be the punctured hull arising from an FLC tiling T with prototile set $\mathcal{P}$. Observe that in the proof of Lemma 3.3.3, the constant of uniform discreteness was obtained only by referencing the prototile set. By Proposition 3.1.10, each $T' \in \Omega_{\text{punc}}$ is composed of prototiles from $\mathcal{P}$ (possibly a proper subset of $\mathcal{P}$), so the constant obtained by running T' through the proof will be no smaller than the constant associated to T. Therefore, the collection of uniform discreteness constants for tilings in the same hull is uniformly bounded below by some $r > 0$, and hence we can use the same constant for all these tilings. We record this observation in the following lemma, which will simplify the topological arguments to come.

Lemma 3.3.5. *Let Ω_{punc} be the discrete hull arising from a punctured tiling with FLC. Then there exists $r > 0$ such that, for every $T \in \Omega_{\text{punc}}$ and $x \in \mathbb{R}^d$ which satisfies $0 < |x| < r$, we have $T + x \notin \Omega_{\text{punc}}$.*

For the remainder of this section, we assume that Ω_{punc} is the discrete hull arising from a tiling with FLC. We equip Ω_{punc} with the subspace topology inherited from the topology induced by the tiling metric on the associated continuous hull Ω. We now present a base for this topology, which will be of more use to us than the usual base given by open balls.

Notation 3.3.6. Given a patch P of any tiling in Ω_{punc}, and a tile $t \in P$, define the following subset of Ω_{punc}:

$$U(P, t) := \{T \in \Omega_{\text{punc}} \mid P - x(t) \subset T\}.$$

In other words, tilings in $U(P, t)$ look like the patch P around the origin, with this patch aligned so that t is the (unique) tile containing the origin. These sets are reminiscent of cylinder sets in sequence spaces.

Lemma 3.3.7. *For any patch P of any tiling in Ω_{punc}, and any $t \in P$, the subset $U(P, t)$ is clopen in the metric topology on Ω_{punc}.*

Proof. Given $T \in U(P,t)$, let $0 < \epsilon < r/2$, where $r > 0$ is as in Lemma 3.3.5. Furthermore, choose ϵ to be small enough that $\epsilon^{-1} - \epsilon > \operatorname{diam}(P)$. Then, for any tiling $T' \in \Omega_{\text{punc}}$ with $d(T,T') < \epsilon$, there exist $x, y \in \mathbb{R}^d$ with $|x|, |y| \leq \epsilon$ such that $(T+x) \sqcap B_{\epsilon^{-1}}(0) = (T'+y) \sqcap B_{\epsilon^{-1}}(0)$. This implies that $(T'+y-x) \sqcap B_{\epsilon^{-1}}(-x) = T \sqcap B_{\epsilon^{-1}}(-x)$. Observe that $B_{\epsilon^{-1}}(-x) \supset B_{\epsilon^{-1}-\epsilon}(0) \supset B_{\operatorname{diam}(P)}(0)$. Since $P - x(t) \subset T \sqcap B_{\operatorname{diam}(P)}(0) \subset T \sqcap B_{\epsilon^{-1}}(-x) = (T'+y-x) \sqcap B_{\epsilon^{-1}}(-x)$, we see that $P - x(t) \subset T' + y - x$, and hence $T' + y - x \in U(P,t) \subset \Omega_{\text{punc}}$. Suppose that $|x - y| > 0$. Since $T' + y - x \in \Omega_{\text{punc}}$, and since $|x-y| \leq 2\epsilon < r$, it follows from Lemma 3.3.5 that $T' = (T' + y - x) + (x - y) \notin \Omega_{\text{punc}}$, a contradiction. This shows that $x = y$, so that $T' = T' + y - x \in U(P,t)$. Since $T' \in B_\epsilon(T) \subset \Omega_{\text{punc}}$ was arbitrary, we see that $B_\epsilon(T) \subset U(P,t)$, so that $U(P,t)$ is open in Ω_{punc}.

Next, let $(T_n)_{n\in\mathbb{N}}$ be a convergent sequence in $U(P,t)$, with limit $T \in \Omega_T$. Choose $N \in \mathbb{N}$ so that when $n > N$, $d(T_n, T) < \epsilon$, where $0 < \epsilon < r/2$ and $\epsilon^{-1} - \epsilon > \operatorname{diam}(P)$. Arguing as in the first part of the proof, when $n > N$ we obtain $P - x(t) \subset T_n \sqcap B_{\operatorname{diam}(P)}(0) = T \sqcap B_{\operatorname{diam}(P)}(0)$, which shows that $T \in U(P,t)$. □

Proposition 3.3.8. *The collection $\{U(P,t) \mid P \subset T' \in \Omega_{\text{punc}}, t \in P\}$ generates the metric topology on Ω_{punc}.*

Proof. We check that each topology is finer than the other. By Lemma 3.3.7, we know that each $U(P,t)$ is open in the metric topology, and thus the metric topology is finer than the new topology. To finish the proof, it suffices to show that open balls in the metric topology are also open in the new topology. In other words, for each $T \in \Omega_{\text{punc}}$ and $\epsilon > 0$, we must exhibit a set $U(P,t)$ such that $T' \in U(P,t)$ implies $d(T,T') < \epsilon$. To do so, choose $P = T \sqcap B_{2\epsilon^{-1}}(0)$, and $t = T(0)$ to be the tile in T which contains the origin. Then, when $T' \in U(P,t)$, T and T' agree on $B_{2\epsilon^{-1}}(0)$, which means that $d(T,T') \leq \epsilon/2 < \epsilon$, as required. □

Theorem 3.3.9. *The discrete hull of an aperiodic, repetitive tiling with FLC is a Cantor space. That is, it is compact, Hausdorff, has no isolated points, and has a countable base of clopen sets.*

Proof. We first prove that Ω_{punc} is a closed subset of Ω. Combining this with Theorem 3.1.15(i) shows that Ω_{punc} is compact. In fact, we prove that the complement of Ω_{punc} is open. So, take $T \in \Omega \setminus \Omega_{\text{punc}}$. Since the set of punctures in T is uniformly discrete, we have that $R \coloneqq \inf\{|x| \mid x \in \mathbb{R}^d,\ T + x \in \Omega_{\text{punc}}\} > 0$. Then $B_{R/2}(T) \subset \Omega \setminus \Omega_{\text{punc}}$, so the complement of Ω_{punc} is open.

The punctured hull Ω_{punc} is Hausdorff, because it inherits the metric topology.

We claim that, up to translation, there are countably many patches in any given tiling. Indeed, the collection of patches in a tiling up to translation can be expressed as the union of the collections of n-tile patches up to translation over $n \in \mathbb{N}$, and, by FLC, each of the sets in this union is finite. By combining this with Lemma 3.3.7 and Proposition 3.3.8, we see that the collection $\{U(P,t) \mid P \subset T, t \in P\}$ is a countable base of clopen sets.

It remains to show that Ω_{punc} contains no isolated points. Consider any $T \in \Omega_{\text{punc}}$ and take any neighbourhood $T \in U \subset \Omega_{\text{punc}}$. Since the collection of open balls is a base for the topology on Ω_{punc}, we can find $\epsilon > 0$ small enough that $B_\epsilon(T) \subset U$, so to prove that T is not an isolated point we just have to exhibit $T' \in \Omega_{\text{punc}}$ such that $0 < d(T,T') < \epsilon$. Consider the patch $P = T \sqcap B_{2\epsilon^{-1}}(0)$. By repetitivity, translates of this patch appear infinitely often in T, so we can find $x \neq 0$ such that $T \sqcap B_{2\epsilon^{-1}}(x)$ is a translate of P, and so that x sits in the same relative position in this translate of P as 0 does in P (so that, in fact, $T \sqcap B_{2\epsilon^{-1}}(x) = P + x$). Consider the tiling $T' = T - x$. Notice that by construction $T' \sqcap B_{2\epsilon^{-1}}(0) = (T-x) \sqcap B_{2\epsilon^{-1}}(0) = (T \sqcap B_{2\epsilon^{-1}}(x)) - x = P + x - x = P = T \sqcap B_{2\epsilon^{-1}}(0)$, so that $d(T,T') \leq \epsilon/2$. To conclude, notice that, since $x \neq 0$, we have $T \neq T - x = T'$ by aperiodicity. Therefore, $0 < d(T,T') < \epsilon$, as required. □

Definition 3.3.10. We associate a groupoid R_{punc} to a tiling using translational equivalence on Ω_{punc}. That is, we define

$$R_{\text{punc}} := \{(T - x(t), T) \mid T \in \Omega_{\text{punc}} \text{ and } t \in T\}.$$

This has the structure of an equivalence relation groupoid (see Example 2.1.4(ii)). We define a metric on R_{punc} by

$$d((T-x,T),(T'-y,T')) = d(T,T') + |x-y|.$$

Notice that, due to the presence of the $|x-y|$ term, the topology on R_{punc} is **not** inherited from the product topology on $\Omega_{\text{punc}} \times \Omega_{\text{punc}}$. Instead, it arises from $\Omega_{\text{punc}} \times \mathbb{R}^d$. Indeed, under the assumption of repetitivity, for any tiling $T \in \Omega_{\text{punc}}$ and any $0 < \epsilon < 1$, translates of the patch $P = T \sqcap B_{\epsilon^{-1}}(0)$ appear infinitely often in the tiling. Thus, we can find a sequence of tilings $\{T_i\}_{i\in\mathbb{N}} \subset U(P, P(0))$ such that $T_i = T_1 - x_i$ where $|x_i| \to \infty$ as $i \to \infty$. Since all of these tilings contain the patch P, we have $d(T_i, T_j) \leq \epsilon$ for every $i, j \in \mathbb{N}$. Consider the sequence of arrows $\{(T_1, T_i)\}_{i\in\mathbb{N}} \subset R_{\text{punc}}$. In the product metric arising from $\Omega_{\text{punc}} \times \Omega_{\text{punc}}$, the distance between any two of these arrows is no larger than ϵ, whereas in the topology induced by the metric above, the distances are not bounded, which shows that the two metrics are not equivalent.

Notation 3.3.11. Let P be a patch, and consider tiles $t, t' \in P$. Define a subset of R_{punc}

by

$$V(P,t,t') \coloneqq \{(T',T) \mid T \in U(P,t),\ T' \in U(P,t') \text{ and } T' = T - (x(t') - x(t))\}.$$

Note that arrows in $V(P,t,t')$ are required to have their source and range in the *same* copy of the patch P (as opposed to distinct translates of the patch, which appear in different locations in the tiling). This is enforced by the presence of the vector $x(t') - x(t)$ in the definition.

Proposition 3.3.12. *The topology generated by the collection $\{V(P,t,t')\}$, taken over all patches P of tilings of Ω_{punc} and tiles $t,t' \in P$, coincides with the metric topology on R_{punc}.*

Proof. We prove that each topology is finer than the other. First, let $(T - (x(t') - x(t)), T) \in V(P,t,t')$. We need to find $\epsilon > 0$ such that $B_\epsilon((T - (x(t') - x(t)), T)) \subset V(P,t,t')$. As in the proof of Lemma 3.3.7, choose $0 < \epsilon < r/2$ to be small enough that $d(T,T') < \epsilon$ implies $T' \in U(P,t)$, where r is as in Lemma 3.3.5. Let $(T' - y, T') \in R_{\text{punc}}$ be such that $d((T'-y,T'),(T-(x(t')-x(t)),T)) < \epsilon$, so that $d(T,T') < \epsilon$ and $|x(t') - x(t) - y| < \epsilon < r$. By our choice of ϵ, this implies that $T' \in U(P,t)$, and hence that $T' - (x(t') - x(t)) \in \Omega_{\text{punc}}$. Since $T' - y \in \Omega_{\text{punc}}$ and $(T' - y) + (y - x(t') + x(t)) = T' - (x(t') - x(t)) \in \Omega_{\text{punc}}$, Lemma 3.3.5 implies that $y = x(t') - x(t)$. Therefore, $(T' - y, T') = (T' - (x(t') - x(t)), T') \in V(P,t,t')$, which shows that $V(P,t,t')$ is open in the metric topology.

Now, consider an arbitrary open ball $B_\epsilon((T - x, T))$ around a point $(T - x, T) \in R_{\text{punc}}$, and let $(T' - y, T') \in B_\epsilon((T - x, T))$. We need to find a patch P, and $t,t' \in P$, such that $(T' - y, T') \in V(P,t,t') \subset B_\epsilon((T - x, T))$. In fact, choose $\delta > 0$ such that $B_\delta((T' - y, T')) \subset B_\epsilon((T - x, T))$. We will find P, t, and t' such that $(T' - y, T') \in V(P,t,t') \subset B_\delta((T' - y, T'))$. Let $R > \max(|y|, \delta^{-1})$ and choose $P = T' \sqcap B_R(0)$, $t = T'(0)$, and $t' = T'(y)$, observing that $(T' - y, T') \in V(P,t,t')$. Let $(T'' - z, T'') \in V(P,t,t')$. First, notice that $z = x(t') - x(t) = y$, so $|y - z| = 0$. Second, we have $P = P - x(t) \subset T''$, so that $T'' \sqcap B_R(0) = P = T' \sqcap B_R(0)$, implying that $d(T', T'') \leq R^{-1} < \delta$. Putting these together, we obtain $d((T'' - z, T''), (T' - y, T')) = d(T', T'') + |y - z| < \delta$, as required. $\square$

Lemma 3.3.13. *Let P be a patch of a tiling of Ω_{punc}, and let $t,t' \in P$. Then, when we identify $R^{(0)}_{\text{punc}}$ with Ω_{punc}, s is a homeomorphism from $V(P,t,t')$ to $U(P,t)$, and r is a homeomorphism from $V(P,t,t')$ to $U(P,t')$.*

Proof. We only prove the result for r, as the proof for s is similar.

We first prove that r is injective. Any element of $V(P,t,t')$ has the form $(T - (x(t') - x(t)), T)$, for some $T \in U(P,t)$. Take $(T - (x(t') - x(t)), T)$ and $(T' - (x(t') - x(t)), T')$

in $V(P,t,t')$, and suppose that $r(T-(x(t')-x(t)),T) = r(T'-(x(t')-x(t)),T')$, so that $T-(x(t')-x(t)) = T'-(x(t')-x(t))$. This implies that $T = T'$, and the arrows that we started with are seen to be equal.

Next, we prove surjectivity. Given $T \in U(P,t')$, observe that $T-(x(t)-x(t')) \in U(P,t)$. Then, write $T' = T-(x(t)-x(t'))$ to see that $(T, T-(x(t)-x(t'))) = (T'+(x(t)-x(t')), T') = (T'-(x(t')-x(t)), T') \in V(P,t,t')$. By definition, $r(T, T-(x(t)-x(t'))) = T$, so $r : V(P,t,t') \to U(P,t')$ is surjective.

Observe that r preserves distance on $V(P,t,t')$, since all elements of $V(P,t,t')$ implement translations by the same vector $x(t')-x(t) \in \mathbb{R}^d$. It follows that r is continuous, and that $r|_{V(P,t,t')}^{-1}$ also preserves distances on $U(P,t')$, so is continuous. □

The previous lemma shows that each $V(P,t,t')$ is compact and open in R_{punc}. Since these sets formed a basis for the topology on R_{punc}, and r is a homeomorphism on each, it follows that R_{punc} is étale. Since the unit space of R_{punc} is the Cantor set Ω_{punc}, we see that R_{punc} is ample.

3.4 The infinite rotation case

In this section, we only consider tilings of $\mathbb{R}^2$. Up to now, the only Euclidean motions we have explicitly been considering on our tilings have been translations. Of course, by increasing the size of the prototile set to include copies of prototiles in different orientations, it is possible for the hull of a tiling to also include rotations of each tiling. However, our restriction to a finite prototile set means that we will only see finitely many orientations of each tiling in the hull. We now follow the ideas of [78] to discuss how to expand the situation to allow for infinitely many orientations of prototiles. We note that our main classification result (Theorem 6.4.3) does not apply to the tilings introduced in this section, though the quasidiagonality result at the end of the thesis (Theorem 6.5.2) does. For the avoidance of doubt, the only places that we consider tilings whose prototiles appear in infinitely many orientations outside of this section are Sections 3.5 and 6.5.

Unfortunately, it will be necessary to tweak the definition of a tiling slightly to account for this added flexibility. We attempt to do so in a way which minimises the disruption to the exposition, and which easily restricts back to the finite rotation case.

Consider the group of orientation preserving isometries of the plane, $\mathbb{R}^2 \rtimes S^1$, where we have identified the group of translations of $\mathbb{R}^2$ with $\mathbb{R}^2$ itself, and the group of rotations about the origin, $SO(2,\mathbb{R})$, is identified with S^1. Denote by $R_\theta \in S^1 \subset \mathbb{R}^2 \rtimes S^1$ the anticlockwise rotation about the origin by angle θ. That is,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

There is a metric on $\mathbb{R}^2 \rtimes S^1$ given by

$$d((x, R_\theta), (y, R_\phi)) = |x - y| + \left(\sum_{i,j=1}^{2} (a_{ij} - b_{ij})^2 \right)^{1/2},$$

where $R_\theta = (a_{ij})$ and $R_\phi = (b_{ij})$ are the associated matrices.

We allow prototiles to appear in infinitely many orientations by requiring each tile to be the image of a prototile under an element of a subgroup $\mathbb{R}^2 \subset \Gamma \subset \mathbb{R}^2 \rtimes S^1$ which contains the subgroup of translations, and which is closed in the topology induced by the metric above. This philosophy allows us to maintain our assumption that tilings have a finite prototile set.

Definition 3.4.1. Let $\mathcal{P}$ be a finite set of prototiles, and let Γ be a closed subgroup of $\mathbb{R}^2 \rtimes S^1$. A *tiling* T with prototile set $\mathcal{P}$ is a set $\{t_1, t_2, \ldots\}$ of subsets of $\mathbb{R}^2$, henceforth referred to as *tiles*, which satisfy

(T1) for each $i \in \{1, 2, \ldots\}$ there exists $\gamma_i \in \Gamma$ and $p_i \in \mathcal{P}$ such that $t_i = \gamma_i(p_i)$. Two tiles which are images of the same prototile are said to have the same *tile type*;

(T2) $\bigcup_{i=1}^{\infty} t_i = \mathbb{R}^d$; and

(T3) $\mathrm{int}(t_i) \cap \mathrm{int}(t_j) = \varnothing$ for $i \neq j$.

Observe that, since Γ is a proper closed subgroup of $\mathbb{R}^2 \rtimes S^1$ containing $\mathbb{R}^2$, either Γ contains only finitely many elements $R_\theta \in S^1$, or the collection $\{R_\theta \in \Gamma\}$ is dense in S^1, which (by closure of Γ) would imply that $\Gamma = \mathbb{R}^2 \rtimes S^1$. In the former case, each prototile appears in only finitely many orientations, so by enlarging the prototile set to include all of these rotated prototiles, we may choose $\gamma_i \in \mathbb{R}^2$ for each $i \in \mathbb{N}$. This presents a dichotomy between tilings whose prototile set may be chosen to ensure that the choice $\Gamma = \mathbb{R}^2$ works, and those for which we must choose $\Gamma = \mathbb{R}^2 \rtimes S^1$. We say that tilings of the former type have *finite rotational symmetry*, and tilings of the latter type have *infinite rotational symmetry.* For the rest of this chapter, all tilings will be assumed to have infinite rotational symmetry. After this chapter, however, we make as standard the assumption that tilings have *finite* rotational symmetry, and any results which extend to the more general setting will be highlighted.

Example 3.4.2. The pinwheel tilings were introduced by Radin in [51]. They are substitution tilings based on an unpublished construction of Conway (see Figure 3.10), which can be thought of as a substitution rule with inflation factor $\sqrt{5}$. Observe that the substitution causes the small triangle which does not share a face with the large triangle to appear at an angle of $\tan^{-1}(1/2)$ relative to the large triangle. It follows that the pinwheel tiling generated by this substitution has infinite rotational symmetry.

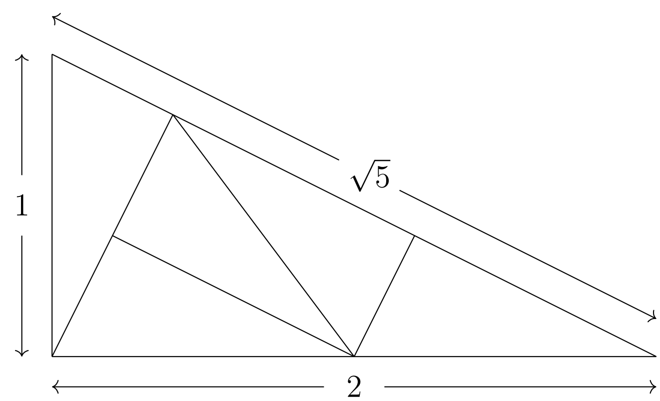

Figure 3.10: Conway's triangle substitution

We now wish to redefine the tiling metric to make use of the rotation. We allow elements of Γ to act on tilings by $(x, R_\theta) \cdot T = R_\theta(T + x)$. As the metric was previously used to complete the set of translates, and since only finitely many orientations of prototiles were possible, we didn't need to keep track of the difference in the amount of rotation between two tilings. Indeed, when tilings were sufficiently close, they had to have the same tile type on the origin, which required the tile on the origin to have the same shape, same label, and same orientation. In the infinite rotation setting, this will no longer be the case. In particular, we may have a sequence of tilings, $\{R_{\theta_n}(T)\}_{n\in\mathbb{N}}$, which are all rotations of some initial tiling T, such that the sequence of the angles of the associated rotations $(\theta_n)_{n\in\mathbb{N}}$ is converging to $\theta \in S^1$, say. It is reasonable to ask that this sequence of tilings should converge in the tiling metric to $R_\theta(T)$. Under our previous tiling metric, this sequence will not necessarily be Cauchy, so a new metric is required. However, the updated definition of the tiling metric is identical in spirit to the previous definition; tilings will be close in the new metric if they agree on a large ball up to a transformation which is close to the identity of $\mathbb{R}^2 \rtimes S^1$.

Definition 3.4.3 ([78, Definition 1.3])**.** Let $0 < \epsilon < 1$. In order that tilings T and T' with infinite rotational symmetry be ϵ-close, we ask that there are elements $(x, R_\theta), (y, R_\phi) \in \Gamma$ which are both a distance no larger than ϵ from the identity $(0, R_0) \in \Gamma$, and are such that $R_\theta(T + x) \sqcap B_{\epsilon^{-1}}(0) = R_\phi(T' + y) \sqcap B_{\epsilon^{-1}}(0)$. We define the *tiling metric* d_Γ by defining $d_\Gamma(T, T')$ to be the infimum of the set of $0 < \epsilon < 1$ such that T and T' are ϵ-close, or 1 if no such ϵ exists.

Although the next definition looks very similar to our previous one, the new tiling metric means that the resulting hull is quite different.

Definition 3.4.4 ([78, Definition 1.4])**.** The *continuous hull* of a tiling T with infinite rotational symmetry, denoted Ω_T, is the completion of $T + \mathbb{R}^2$ in the tiling metric d_Γ.

Of course, the correct notion of finite local complexity for this setting must allow for rotations as well as translations.

Definition 3.4.5 ([78, Definition 1.5])**.** A tiling with infinite rotational symmetry has *finite local complexity* (FLC) if, for all $R > 0$, the set $\{T \sqcap B_R(x) \mid x \in \mathbb{R}^2\}/\Gamma$ is finite.

The notions of repetitivity and aperiodicity do not change from the finite rotation case (see Definition 3.1.13).

We will now reintroduce the blanket assumption that our tilings are aperiodic, repetitive, and have FLC. As before, we equip our prototiles with punctures and consider the restriction Ω_{punc} of the continuous hull to the subset of tilings which have a puncture on the origin. This time, we wish to choose the punctures to maximally break the symmetry of each prototile, in the sense that the symmetry group of each pointed prototile $(p, x(p))$ is trivial. This will ensure that any distinct rotations of a prototile (using the puncture as the centre of rotation) are also distinct as subsets of $\mathbb{R}^2$. This will be important when we wish to begin keeping track of the deviation of a tiling from its "canonical orientation". As before, every tile in every tiling is punctured at the image of the puncture of the corresponding prototile under the action of the associated element of Γ. In this setting, there is a natural subset, $\Omega^0_{\text{punc}} \subset \Omega_{\text{punc}}$, consisting of the tilings for which the tile on the origin is a translate of a prototile, with no rotation involved:

$$\Omega^0_{\text{punc}} \coloneqq \{T \in \Omega_{\text{punc}} \mid \text{there exists } y \in \mathbb{R}^2 \text{ such that } T(0) + y \in \mathcal{P}\}.$$

Clearly, each tiling $T \in \Omega_{\text{punc}}$ can be expressed as $R_\theta(T_0)$ for some tiling $T_0 \in \Omega^0_{\text{punc}}$. Furthermore, by [78, Lemma 2.4], when $\Gamma = \mathbb{R}^2 \rtimes S^1$, the map $\psi : \Omega_{\text{punc}} \to \Omega^0_{\text{punc}} \times S^1$ defined by $\psi(T) = (T_0, R_\theta)$ is a homeomorphism.

The identification of images of the same patch under elements of Γ leads us to define the following basis sets for the metric topology on Ω_{punc}. In order to treat the finite and infinite rotation cases of $\Gamma = \mathbb{R}^2$ and $\Gamma = \mathbb{R}^2 \rtimes S^1$ simultaneously, we denote by $\pi_{S^1} : \mathbb{R}^2 \rtimes S^1 \to S^1$ the projection $(x, R_\theta) \mapsto \theta$. We abuse notation slightly to take the convention that $\mathbb{R}^2 \subset \mathbb{R}^2 \rtimes S^1$, by identifying $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$, and thus we will write $\pi_{S^1}(\mathbb{R}^2) = \pi_{S^1}(\mathbb{R}^2 \times \{0\}) = \{0\}$. Given a patch P of some tiling in Ω^0_{punc}, a tile $t \in P$, and a subset $W \subset S^1$, define

$$U(P, t, W) \coloneqq \{T \in \Omega_{\text{punc}} \mid R_\theta(P - x(t)) \subset T \text{ for some } \theta \in W \cap \pi_{S^1}(\Gamma)\}.$$

When $\Gamma = \mathbb{R}^2$, any choice of $0 \in W \subset S^1$ provides the subset $U(P, t)$ constructed earlier. We also note that when $\Gamma = \mathbb{R}^2 \rtimes S^1$, $U(P, t, \{0\})$ can be identified with the open set $U(P, t)$ from the finite rotation case. We can obtain a basis for the topology on Ω_{punc} by only considering open subsets $W \subset S^1$. Using the homeomorphism ψ defined above, these sets can be identified with $U(P, t, \{0\}) \times W$, and thus generate the metric topology on $\Omega_{\text{punc}} \cong \Omega^0_{\text{punc}} \times S^1$.

Just as in the finite rotation case, we form the following groupoid of translational equivalence on Ω_{punc}:

$$R_{\text{punc}} := \{(T - x(t), T) \mid T \in \Omega_{\text{punc}}, t \in T\},$$

and equip it with the following metric

$$d((T - x(t), T), (T' - x(t'), T')) = d_\Gamma(T, T') + |x(t) - x(t')|.$$

The following sets can be used to form a basis for the metric topology on R_{punc}. Given a patch P of a tiling in Ω^0_{punc}, tiles $t, t' \in P$, and a subset $W \subset S^1$, define

$$V(P, t, t', W) := \{(T - R_\theta(x(t') - x(t)), T) \mid R_\theta(P - x(t)) \subset T \text{ for some } \theta \in W \cap \pi_{S^1}(\Gamma)\}.$$

As above, in the finite rotation case, the set $V(P, t, t', W)$ is equal to $V(P, t, t')$ as long as $0 \in W$. In the infinite rotation case we identify $V(P, t, t', \{0\})$ with the set $V(P, t, t')$ from the finite rotation case. If we take the collection of sets $V(P, t, t', W)$ such that $W \subset S^1$ is open, we obtain a basis for the metric topology on R_{punc}. As is recorded in [78, Lemma 2.6], $s(V(P, t, t', W)) = U(P, t, W)$, and $r(V(P, t, t', W)) = U(P, t', W)$.

When $\Gamma = \mathbb{R}^2 \rtimes S^1$, we distinguish in particular the following clopen sets of Ω_{punc} and R_{punc}:

$$U(P, t) := U(P, t, S^1) \quad \text{and} \quad V(P, t, t') := V(P, t, t', S^1).$$

Notice that these subsets are *not* the same as those which use this notation in the finite rotation case, so it is important to keep track of which group Γ is in the background when we use this notation. This is unfortunate, but would be necessary sooner or later, as we will wish to group all copies of a patch (with the same distinguished tile) together, irrespective of rotation. This notation can be thought of in both the finite and infinite rotation case as encapsulating all images of a given patch with a given distinguished tile under our chosen group of isometries (recalling that in the finite rotation case we allow the prototile set to handle the rotational aspect, rather than implementing the rotations as isometries). In other words, to encapsulate both definitions of $U(P, t)$ and $V(P, t, t')$

simultaneously, we have

$$U(P,t) \coloneqq U(P,t,\pi_{S^1}(\Gamma)) = \{T \in \Omega_{\text{punc}} \mid R_\theta(P - x(t)) \subset T \text{ for some } \theta \in \pi_{S^1}(\Gamma)\},$$

and

$$\begin{aligned} V(P,t,t') &\coloneqq V(P,t,t',\pi_{S^1}(\Gamma)) \\ &= \{(T - R_\theta(x(t') - x(t)), T) \mid R_\theta(P - x(t)) \subset T \text{ for some } \theta \in \pi_{S^1}(\Gamma)\}. \end{aligned}$$

3.5 Tiling C^*-algebras

In the previous sections, we have associated an étale groupoid to any aperiodic and repetitive tiling with finite local complexity. The groupoid is principal, and is associated to the the relation of translational equivalence on Ω_{punc}: $T \sim T'$ if and only if there exists $x \in \mathbb{R}^d$ such that $T = T' - x$. The construction of a C^*-algebra from this groupoid now proceeds following [53] (in which the full details of the construction are contained). Start with the space $C_c(R_{\text{punc}})$ of continuous complex-valued functions on R_{punc} with compact support. Introduce a product and involution given by

$$f * g(T_1, T_2) = \sum_{T_3 \sim T_1} f(T_1, T_3) g(T_3, T_2) \tag{3.5.1}$$

and

$$f^*(T_1, T_2) = \overline{f(T_2, T_1)}. \tag{3.5.2}$$

We also introduce a norm on $C_c(R_{\text{punc}})$ by considering a family of representations indexed by tilings. First, we give the Hilbert space onto which $C_c(R_{\text{punc}})$ will be represented.

Notation 3.5.1. For short, given a tiling $T \in \mathbb{R}^d$, denote by $[T] \coloneqq T + \mathbb{R}^d$ the translational equivalence class of T. Also denote the Hilbert space

$$\ell^2([T]) \coloneqq \left\{ (x_{T'})_{T' \in [T]} \;\middle|\; x_{T'} \in \mathbb{C} \text{ for every } T' \in [T] \text{ and } \sum_{T' \in [T]} |x_{T'}|^2 < \infty \right\}.$$

We can now introduce the family of representations. Given $T_1 \in \Omega_{\text{punc}}$, define $\pi_{T_1} : C_c(R_{\text{punc}}) \to B(\ell^2([T_1]))$ by

$$(\pi_{T_1}(f)\psi)(T_2) = \sum_{T_3 \in [T_1]} f(T_2, T_3)\psi(T_3),$$

where $f \in C_c(R_{\text{punc}})$, $\psi \in \ell^2([T_1])$, and $T_2 \in [T_1]$. Let us unpack this definition a little. The idea of the representation is that it implements translation using indicator functions in the following sense. If f is an indicator function for a translation by a vector x:

$$f(T_2, T_3) = \begin{cases} 1 & \text{if } T_2 = T_3 + x \\ 0 & \text{otherwise,} \end{cases}$$

then $\pi_{T_1} f$ shifts the indices in the sequence ψ by the same translate, so that the complex number corresponding to the tiling T_3 in ψ now corresponds to the tiling $T_3 + x$ in $\pi_{T_1} f(\psi)$. We comment that such functions f are in $C_c(R_{\text{punc}})$, because, under the FLC assumption, their support is precisely a union of finitely many compact open sets of the form $V(P, P(0), t)$, where $P = T \sqcap B_R(0)$ for some $T \in \Omega_{\text{punc}}$ and $R > |x|$, and $t \in P$.

We use these representations to define the *reduced norm* of $f \in C_c(R_{\text{punc}})$,

$$\|f\|_r := \sup_{T' \in \Omega_{\text{punc}}} \{\|\pi_{T'} f\|\},$$

where $\|\pi_{T'} f\|$ denotes the operator norm. We complete $C_c(R_{\text{punc}})$ in $\|\cdot\|_r$ to obtain a C^*-algebra $A_{\text{punc}} = C_r^*(R_{\text{punc}})$ – the C^*-algebra associated to our original tiling. That the representations we used are nondegenerate, and that the above norm is indeed a C^*-norm, is verified in [53].

Definition 3.5.2. Let T be an aperiodic and repetitive tiling with FLC, and let R_{punc} be its tiling groupoid. When equipped with the product and involution defined by (3.5.1) and (3.5.2), the completion of $C_c(R_{\text{punc}})$ in the reduced norm is a C^*-algebra, denoted by any of $C_r^*(R_{\text{punc}})$, A_T, or A_{punc}, and referred to as the C^*-algebra of the tiling T.

This is not the only sensible C^*-algebra that one could construct from a tiling. For example, in [34, Section 4], the crossed product C^*-algebra arising from the action $\mathbb{R}^d \curvearrowright \Omega$ on the continuous hull is considered (for an excellent reference on crossed product C^*-algebras, see [80]). In fact, since our tiling C^*-algebra is constructed from a transversal to the action considered here, this C^*-algebra is strongly Morita equivalent to the one constructed above [45].

We now comment on some properties of the tiling C^*-algebra $C_r^*(R_{\text{punc}})$. It is unital, with unit given by

$$1_\Delta(T, T') = \begin{cases} 1 & \text{if } T = T' \\ 0 & \text{otherwise,} \end{cases}$$

where Δ is the diagonal of R_{punc}:

$$\Delta = \{(T, T) \mid T \in \Omega_{\text{punc}}\}.$$

Since R_{punc} is a locally compact and σ-compact metric space (because it has a countable basis of compact open sets $V(P,t,t')$), $C_c(R_{\text{punc}})$ is separable, which implies that $C_r^*(R_{\text{punc}})$ is also separable. Since the action of $\mathbb{R}^d$ on Ω is free and minimal, and the primitive ideal space of $C(\Omega)$ is just Ω, [25, Corollary 3.3] shows that the crossed product C^*-algebra is simple. Since $C(\Omega)$ is commutative, it is nuclear. Therefore, [80, Theorem 7.18] shows that the crossed product C^*-algebra is nuclear. Both of these properties are preserved by the strong Morita equivalence (see, for example, [56, Theorem 3.1] for simplicity, and [86] or [30] for nuclearity), showing that $C_r^*(R_{\text{punc}})$ is also simple and nuclear. By [4, Corollary 6.2.14 and Proposition 6.1.8], it follows that R_{punc} is measurewise amenable, and hence that $C_r^*(R_{\text{punc}}) \cong C^*(R_{\text{punc}})$ (here $C^*(R_{\text{punc}})$ denotes the *full* C^*-algebra obtained by completing in the norm arising from considering *all* representations of $C_c(R_{\text{punc}})$, not just those defined above). Since R_{punc} is étale and measurewise amenable, it is topologically amenable by [4, Theorem 3.3.7], and, by [76], this implies that $C_r^*(R_{\text{punc}})$ satisfies the Universal Coefficient Theorem of [61].

We record all of these properties in the following result.

Theorem 3.5.3. *The C^*-algebra associated to an aperiodic and repetitive tiling with FLC is simple, separable, unital, nuclear, infinite dimensional, and satisfies the UCT.*

3.5.1 A densely spanning set

Now that we have our algebra in hand, we wish to present a countable set of elements that spans a dense subalgebra of $C_r^*(R_{\text{punc}})$. It will be convenient to work with these elements later on, in Section 6.5. The following construction will work in both the finite and infinite rotation cases, and can be found in [78, Section 3].

Given a patch P in the tiling, and tiles $t, t' \in P$, denote the indicator function of the compact open set $V(P,t,t')$ by $e(P,t,t') \in C_c(R_{\text{punc}})$. Given any tile of any tiling $t \in T \in \Omega_{\text{punc}}$, so that $t = \gamma(p)$ is the image of some prototile $p \in \mathcal{P}$ by some element $\gamma = (x, R_\theta) \in \Gamma$, we introduce the notation $\angle t := \theta$ to denote the angle of rotation which was applied to the prototile p to obtain t. Observe that $\angle$ is well defined, because the punctures were chosen to maximally break the symmetry of the prototiles. We use this to define a unitary element $z \in C_c(R_{\text{punc}})$, which is supported on the diagonal of R_{punc}, by

$$z(T,T') = \begin{cases} e^{i\angle T(0)} & \text{if } T = T' \\ 0 & \text{otherwise.} \end{cases}$$

We note that in the finite rotation case $\angle T(0) = 0$ for all tilings $T \in \Omega_{\text{punc}}$, and so in this case $z = 1_\Delta$ is just the unit of the C^*-algebra. By [78, Lemma 3.2], these functions have the following properties:

(i) $e(P,t,t') \cdot e(P',t',t'') = e(P \cup P', t, t'')$ if P and P' both contain t', and agree where they intersect (otherwise, the product is zero);

(ii) $e(P,t,t')^* = e(P,t',t)$;

(iii) $z \cdot e(P,t,t') = e^{i(\angle t' - \angle t)}(e(P,t,t') \cdot z)$; and

(iv) the linear span of $\mathcal{E} = \{z^k \cdot e(P,t,t')\}$ over $k \in \mathbb{Z}$, all patches P of tilings in Ω_{punc}, and all tiles $t, t' \in P$, is dense in A_{punc}.

To conclude the section, we describe the action of the images of elements of $\mathcal{E}$ under the induced representation $\pi_T : C_c(R_{\text{punc}}) \to B(\ell^2([T_1]))$. We now think of elements of $\ell^2([T])$ as being indexed by tiles in T, rather than tilings in $[T]$ (though of course we can recover the whole tiling just by knowing which tile of T lies at the origin). Consider the orthonormal basis $\{\delta_t \mid t \in T\}$ of $\ell^2([T])$. We compute that

$$\begin{aligned}
\pi_T(z^k \cdot e(P,t,t'))\delta_{t''}(t''') &= \sum_{q \in T} \left(z^k \cdot e(P,t,t')\right)(T - x(t'''), T - x(q))\, \delta_{t''}(q) \\
&= \left(z^k \cdot e(P,t,t')\right)(T - x(t'''), T - x(t'')) \\
&= \sum_{q \in T} z^k(T - x(t'''), T - x(q)) e(P,t,t')(T - x(q), T - x(t'')) \\
&= z^k(T - x(t'''), T - x(t''')) e(P,t,t')(T - x(t'''), T - x(t'')) \\
&= \begin{cases} e^{ik\angle t'''} & \text{if } (T - x(t'''), T - x(t'')) \in V(P,t,t') \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In other words,

$$\pi_T(z^k \cdot e(P,t,t'))\delta_{t''} = \begin{cases} e^{ik(\angle t' + \theta)}\delta_{t'''} & \text{if } R_\theta(P - x(t)) \subset T - x(t'') \\ & \quad \text{and } x(t''') = x(t'') + R_\theta(x(t') - x(t)) \\ 0 & \text{otherwise.} \end{cases}$$

Morally, one expects an index shift along the vector $x(t') - x(t)$ in the patch P, since $e(P,t,t')$ is the indicator for such a vector. Since the copy of P may be rotated, we have to rotate our vector to ensure that this index shift is valid in the tiling. In order for the index t'' which we are shifting to return a nonzero element of the sequence, it must act as a copy of the tile $t \in P$. Finally, by the definition of the multiplication on $C_c(R_{\text{punc}})$, the unitary z will pick up the rotation at the origin of the tiling $(T - x(t'')) - R_\theta(x(t') - x(t))$, whose origin tile t''' appears as a copy of the tile t' in the same rotated translate of P in which t'' is acting as t. Since this copy of P is sitting at an angle of θ, we thus pick up both the rotation of the tile t' sitting in P, and this rotation by θ.

Chapter 4

Almost finiteness

In this chapter, we will explore the property of *almost finiteness* introduced by Matui in [41]. We consider several translations of this property into different settings. We will also spend some time linking each new setting back to the original definition to make the relationships between the concepts clear. Each notion of almost finiteness will boil down to requiring our object to be arbitrarily well approximated in some sense by chopping it into finitely many "blocks", which are further subdivided into finitely many pieces on which our object can be thought of as acting by permutations. In each case, we introduce a notion of invariance which is used to decrease the tolerance of the approximation, forcing the blocks to be subdivided more finely, and thus increasing the number of pieces which are permuted.

In Section 4.1, we detail the notion of almost finiteness for groupoids. The original definition for étale groupoids with totally disconnected unit spaces was introduced by Matui in [41], but we present a generalisation due to Suzuki [71] to general étale groupoids.

In Section 4.2, we consider the definition of almost finiteness of actions of groups introduced by Kerr in [35]. It is shown in [35] that this definition generalises Matui's in the sense that an action on a zero-dimensional compact metrisable space is almost finite if and only if the associated transformation groupoid is. Furthermore, the main result of [35] shows that the crossed product C^*-algebras associated to almost finite actions are $\mathcal{Z}$-stable, and thus exhibits the relevance of almost finiteness to Elliott's classification programme.

Our contribution begins in Section 4.3, in which we explore the topic of groupoid actions and generalise Kerr's definition to allow for actions of étale groupoids (Definition 4.3.13), linking it back to the existing notions (Theorem 4.3.15). In Section 4.4, we show that tiling groupoids are almost finite in the sense of Matui and translate this into the viewpoint provided by Definition 4.3.13. This latter viewpoint will allow us to consider tiling C^*-algebras as crossed products by almost finite groupoid actions, which will allow us to follow the methods used by Kerr [35] to prove our main result (Theorem 6.4.3).

Sections 4.5 and 4.6 aim to explore a sufficient criterion to obtain almost finiteness for group actions on the Cantor set. In Section 4.5, we present a criterion used by Elek in [17] which implies almost finiteness of such actions. Elek's criterion manifests as a growth condition on the size of balls in the path metric of a graph associated to the action. In Section 4.6, we associate a graph to the action of a tiling groupoid on its unit space, and prove that it satisfies Elek's criterion. We suggest that generalisations of Elek's work to allow for the actions of groupoids could be explored in the future.

4.1 Almost finiteness for groupoids

We begin with some concepts required to define almost finite groupoids, following the development in [71]. Throughout this section, we assume that G is a locally compact Hausdorff étale groupoid whose unit space is compact. First, we introduce the notion of invariance which will be used to define the tolerance of the approximation.

Definition 4.1.1 ([71, Definition 3.1]; c.f. [41, Definition 6.2])**.** Let $A, C \subset G$ be compact and let $\epsilon > 0$. We say that A is (C, ϵ)-*invariant* if the following inequality holds for every $u \in s(A)$:

$$\frac{|CAu \setminus Au|}{|Au|} < \epsilon.$$

Remark 4.1.2. Both cardinalities in the inequality are finite by Proposition 2.2.13 (CA is compact by [27, Lemma 5.2]). We will see a related formulation later (see Lemma 4.3.11), which will be more useful when we come to define almost finiteness for groupoid actions.

Next, we move towards defining the blocks (and subdivisions thereof) which will make up the approximation of an almost finite groupoid. We first need a supplementary definition, which helps groupoids to more closely mimic group actions.

Definition 4.1.3. A subset $V \subset G$ is said to be a G-set if both the range and the source map are injective on V.

Notice that this is slightly different than the notion of a principal subgroupoid given by Definition 2.1.7. A principal groupoid may contain multiple arrows with the same source or range, as long as a single arrow exists between any *pair* of units. In a G-set, each unit is attached to at most two arrows: one which "enters" and one which "leaves" that unit. Notice in particular that subgroupoids do not make good G-sets – any subgroupoid which is also a G-set is necessarily a subset of $G^{(0)}$.

Such subsets are useful when we wish to consider the natural action of a groupoid on its unit space (see Example 4.3.2(ii)). In this viewpoint, the action of a G-set on $G^{(0)}$

behaves somewhat similarly to the action of a single group element on a space. The largest differences are the fact that a G-set might not act on some elements, and that the image of $G^{(0)}$ under the action might not be all of $G^{(0)}$.

Definition 4.1.4. Let K be a compact groupoid. We say that a clopen subset F of $K^{(0)}$ is a *fundamental domain* of K if there exist a natural number $n \in \mathbb{N}$, natural numbers $N_i \in \mathbb{N}$ for each $i \in \{1, \ldots, n\}$, and a pairwise disjoint collection of clopen subsets $\{F_j^{(i)}\}_{\substack{i\in\{1,\ldots,n\}\\ j\in\{1,\ldots,N_i\}}}$ of $K^{(0)}$ such that

(F1) $F = \bigsqcup_{i=1}^{n} F_1^{(i)}$;

(F2) $K^{(0)} = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{N_i} F_j^{(i)}$;

(F3) for each $i \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, N_i\}$ there is a unique compact open K-set $V_{j,k}^{(i)}$ satisfying $r(V_{j,k}^{(i)}) = F_j^{(i)}$ and $s(V_{j,k}^{(i)}) = F_k^{(i)}$; and

(F4) $K = \bigsqcup_{i=1}^{n} \bigsqcup_{1\leq j,k\leq N_i} V_{j,k}^{(i)}$.

Definition 4.1.5. A (compact) subgroupoid K of G is called *elementary* if $G^{(0)} \subset K$ and if K admits a fundamental domain.

Definition 4.1.6 ([71, Definition 3.6])**.** We say that G is *almost finite* if it satisfies

(i) the union of all compact open G-sets covers G; and

(ii) for any compact subset $C \subset G$ and $\epsilon > 0$, there exists a (C, ϵ)-invariant elementary subgroupoid of G.

Remark 4.1.7. The first condition ensures that the elementary subgroupoids provided by the second condition contain enough information to reconstruct G. In the case that $G^{(0)}$ is totally disconnected, this is automatic (see for example [20]), so this definition is easily seen to generalise [41, Definition 6.2].

Matui [41] introduced the class of almost finite groupoids as a generalisation of the class of AF groupoids (AF stands for *approximately finite*), which were shown to be characterised up to isomorphism in [21]. A groupoid is AF if it can be written as an increasing union of elementary groupoids, and so AF groupoids are easily seen to be almost finite.

The intuition behind the definition is that the unit space decomposes into collections of subsets $\{F_j^{(1)}\}_{j=1,\ldots,N_1}$, $\{F_j^{(2)}\}_{j=1,\ldots,N_2}$ and so on, such that we can think of the sets within the collection $\{F_j^{(i)}\}_{j=1,\ldots,N_i}$ as being permuted by the groupoid via the sets $V_{j,k}^{(i)}$. In some sense, these decompositions are required to provide arbitrarily good approximations for the groupoid as a whole. Tightening the invariance condition (by increasing C and reducing ϵ) forces the unit space to be subdivided more finely in order that more arrows are included in the elementary subgroupoid at each unit, thereby forcing the elementary subgroupoid to better approximate G.

Examples 4.1.8.

(i) As remarked above, AF groupoids are almost finite.

(ii) The transformation groupoids associated to free actions of $\mathbb{Z}^d$ on the Cantor set are almost finite [41, Lemma 6.3].

(iii) If Γ is a countably infinite amenable group such that every finitely generated subgroup of Γ has subexponential growth, then the transformation groupoid of every free action of Γ on a zero-dimensional compact metrisable space is almost finite [14]. In fact, the same is true for every free action of Γ on a compact metrisable space of finite covering dimension [37].

(iv) An example of a non-transformation groupoid which is almost finite can be found in [6]. It is a coarse groupoid associated to the space obtained by attaching a copy of $\mathbb{N}$ to each vertex of an expander graph.

4.2 Almost finiteness for actions of groups

In this section, we detail the notion of almost finiteness for group actions introduced by Kerr in [35] (see Definition 4.2.8). We also link Definition 4.2.8 back to Definition 4.1.6 (see Theorem 4.2.10). The section will conclude with the statement of [35, Theorem 12.4], which highlights the applications of this property to the classification programme for C^*-algebras (see Section 6.1 for more details on this programme).

Throughout the section, Γ will denote a countable discrete group, which will act freely and continuously on a compact Hausdorff space X. For $\gamma \in \Gamma$ and $x \in X$, we denote the image of x under the action of γ by γx. This notation passes to subsets so that, for $A \subset X$, $\gamma \in \Gamma$, and $\Sigma \subset \Gamma$, we write $\gamma A := \{\gamma x \mid x \in A\}$ and $\Sigma A := \{\gamma x \mid \gamma \in \Sigma, x \in A\}$.

Roughly, for an action to be almost finite, it must partition "most of" the space X into finitely many finite collections (or *towers*) of subsets (the *levels* of the towers), such that the levels of each tower may be permuted by the action. The partition is allowed to miss a small part of the space, which we express by comparing it to a union of small portions of each tower. We begin by defining the relation that we use for this comparison.

Definition 4.2.1 ([35, Definition 3.1])**.** Given $A, B \subset X$, we write $A \prec B$ if, for every closed set $C \subset A$, there exist

- a finite collection $\mathcal{U}$ of open subsets of X which cover C; and

- an element $s_U \in \Gamma$ for each $U \in \mathcal{U}$ such that $\{s_U U \mid U \in \mathcal{U}\}$ is a pairwise disjoint collection of subsets of B.

Examples 4.2.2.

(i) Let $X = \Gamma = \mathbb{Z}_N$ with action given by left translation $\gamma x := \gamma + x \pmod N$, and equip X with the discrete topology. Then $A \prec B$ if and only if $|A| \leq |B|$.

Proof. Suppose $|A| \leq |B|$. Write $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$. Since any subset of A is closed, let $C = \{a_{i_1}, \dots, a_{i_k}\} \subset A$ be an arbitrary subset. Define $\mathcal{U} = \{\{a_{i_1}\}, \dots, \{a_{i_k}\}\}$, which is a finite open cover of C. Since $|A| \leq |B|$, we have $k \leq n$, so we can set $s_{\{a_{i_j}\}} = b_j - a_{i_j}$ (modulo N) for each $j \in \{1, \dots, k\}$. Then the images $s_{\{a_{i_j}\}}\{a_{i_j}\} = \{b_j\}$ are pairwise disjoint subsets of B for $j \in \{1, \dots, k\}$, so $A \prec B$.

On the other hand, suppose $A \prec B$ and choose $C = A$. Notice that, for every $\gamma \in \Gamma$ and $U \subset X$, we have $|U| = |\gamma U|$. Find $\mathcal{U}$ as in the definition of $\prec$. Since $\mathcal{U}$ covers A, we have $\sum_{U \in \mathcal{U}} |U| \geq |A|$. On the other hand, $\sum_{U \in \mathcal{U}} |U| = \sum_{U \in \mathcal{U}} |s_U U| = |\bigsqcup_{U \in \mathcal{U}} s_U U| \leq |B|$, since the sets $s_U U \subset B$ are pairwise disjoint. Thus, we see that $|A| \leq |B|$. □

(ii) More generally, let Γ be any countable, discrete group acting on a finite set X equipped with the discrete topology. Then, if $A, B \subset X$, we have $A \prec B$ if and only if $|A| \leq |B \cap \mathrm{Orb}(A)|$, where $\mathrm{Orb}(A) := \{x \in X \mid \exists a \in A \text{ and } \gamma \in \Gamma \text{ such that } x = \gamma a\}$. The first part of the proof of the previous example generalises easily, as does the second part if we assume (without loss of generality) that $\mathcal{U}$ consists of subsets of A.

(iii) Let $X = \{0,1\}^{\mathbb{N}}$ be the Cantor set, equipped with the topology arising from the following metric. Given two sequences $(x_n), (y_n) \in \{0,1\}^{\mathbb{N}}$, the distance between them is defined to be 2^{-k}, where k is the smallest index such that $x_k \neq y_k$, and the distance is 0 if no such k exists. Let $\Gamma = \mathbb{Z}$ act on X by the odometer action generated by

$$1 \cdot (1, 1, \dots, 1, 0, a_{k+1}, \dots) = (0, 0, \dots, 0, 1, a_{k+1}, \dots) \text{ and}$$
$$1 \cdot (1, 1, \dots) = (0, 0, \dots).$$

For $a_1, \dots, a_m \in \{0,1\}$, we denote the associated cylinder set by

$$[a_1 \dots a_m] := \{(a_1, \dots, a_m, x_{m+1}, x_{m+2}, \dots) \mid x_j \in \{0,1\} \ \forall j \geq m+1\}.$$

These are clopen sets which generate the metric topology on X. For two such cylinder sets, we have $[a_1 \dots a_m] \prec [b_1 \dots b_n]$ if and only if $m \geq n$.

Definition 4.2.3 ([35, Definition 4.1])**.** A *tower* is a pair (W, S) consisting of a subset W of X, and a finite subset S of Γ such that the collection $\{sW\}_{s \in S}$ is pairwise disjoint. We refer to the set W as the *base* of the tower, the set S as the *shape* of the tower, and the sets sW as the *levels* of the tower. We say that the tower is *open* (respectively *closed*, *clopen*) if W is open (respectively closed, clopen), and we say that a collection $\{(W_i, S_i)\}_{i \in I}$ of towers *covers* X if $\bigcup_{i \in I} S_i W_i = X$.

Definition 4.2.4 ([35, Definition 8.1])**.** A *castle* is a finite collection of towers $\{(W_i, S_i) \mid i = 1, \dots, n\}$ such that the sets $S_i W_i$ for $i \in \{1, \dots, n\}$ are pairwise disjoint. The sets sW_i for $i \in \{1, \dots, n\}$ and $s \in S_i$ are called the *levels* of the castle. We say the castle is *open* (respectively *closed*, *clopen*) if each tower is open (respectively closed, clopen).

Such castles will take on the role of partitioning a "large portion" of the space in the definition of almost finiteness. In fact, we wish to be able to find castles whose shapes encapsulate the Følner requirement from the measure-theoretic setting of hyperfiniteness. To do so, we introduce the required notion of invariance.

Definition 4.2.5. Let C be a finite subset of Γ and let $\epsilon > 0$. A finite subset A of Γ is said to be (C, ϵ)-*invariant* if

$$|CA \setminus A| < \epsilon |A|.$$

We say that a sequence of finite subsets $\{A_n\}_{n \in \mathbb{N}}$ of Γ *becomes arbitrarily approximately invariant* if, for any finite subset $C \subset \Gamma$ and any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that A_n is (C, ϵ)-invariant whenever $n \geq N$.

There are many different formulations of this property, which are equivalent for the purposes of specifying arbitrary levels of approximate invariance. Some of these formulations are more natural to use for certain purposes than others, so we present several here. Before we do so, we need one more piece of notation.

Notation 4.2.6. Let Γ be a group and let $C, A \subset \Gamma$ be finite. The C-*boundary* of A is the set

$$|\partial_C A| := |\{s \in \Gamma \mid Cs \cap A \neq \varnothing \text{ and } Cs \cap A^c \neq \varnothing\}|.$$

Lemma 4.2.7. *Let $C, A \subset \Gamma$ be finite and $\epsilon > 0$. Consider the following properties.*

(i) A is (C, ϵ)-invariant;

(ii) $|\{s \in A \mid Cs \subset A\}| > (1 - \epsilon)|A|$;

(iii) $|\partial_C A| < \epsilon |A|$; and

(iv) $|A \Delta CA| < \epsilon |A|$, where Δ denotes the symmetric difference.

If a sequence of finite subsets $A_n \subset \Gamma$ becomes arbitrarily approximately invariant according to any one of the conditions (in the sense that, for any finite $C \subset \Gamma$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that A_n satisfies the condition for C and ϵ whenever $n \geq N$), then the sequence also becomes arbitrarily approximately invariant according to any of the other conditions.

The set $\partial_C A$ appearing in property (iii) is known as the *C-boundary* of A.

Proof (see [36, pgs. 92–93]). (ii) $\implies$ (iv): First, we claim that whenever $C, A \subset \Gamma$ are finite, we have

$$\{s \in A \mid Cs \subset A\} = \bigcap_{c \in C} (A \cap c^{-1}A). \tag{4.2.1}$$

Indeed, $s \in A$ is contained in the left hand side if and only if $cs \in A$ for every $c \in C$, if and only if $s \in c^{-1}A$ for every $c \in C$.

Secondly, note that for any two sets $A, B \subset \Gamma$ we have

$$A \cap B = A \setminus (A \Delta B). \tag{4.2.2}$$

This is clear to picture geometrically. Alternatively, notice that the right-hand side expands to $A \setminus ((A \setminus B) \cup (B \setminus A))$. Elements of this set must clearly lie in A, and due to the removal of $A \setminus B$, they must lie in B as well.

Now, let $C \subset \Gamma$ be finite and $\epsilon > 0$, and assume that A is a finite subset of Γ which witnesses property (ii) for a tolerance of $(C^{-1}, \epsilon/(2|C|))$. Then we have by equations (4.2.1) and (4.2.2) that

$$\begin{aligned}\left(1 - \frac{\epsilon}{2|C|}\right)|A| < |\{s \in A \mid C^{-1}s \subset A\}| &= \left|\bigcap_{r \in C^{-1}} (A \cap r^{-1}A)\right| \\ &= \left|\bigcap_{c \in C} (A \cap cA)\right| \\ &= \left|\bigcap_{c \in C} (A \setminus (A \Delta cA))\right|.\end{aligned} \tag{4.2.3}$$

Therefore, for any fixed $t \in C$, we have that

$$\begin{aligned}\left(1 - \frac{\epsilon}{2|C|}\right)|A| &< \left|\bigcap_{c \in C} A \setminus (A \Delta cA)\right| \\ &\leq |A \setminus (A \Delta tA)| = |A| - |A \cap (A \Delta tA)|.\end{aligned}$$

Rearranging this yields

$$|A \cap (A \Delta tA)| < \frac{\epsilon}{2|C|}|A|. \tag{4.2.4}$$

Next, we note that the map $A \to tA$ given by $g \mapsto tg$ is a bijection, so that $|A| = |tA|$. We use this together with an application of inequality (4.2.3) and equation (4.2.2) (this time swapping the roles of A and cA) to compute that

$$\begin{aligned}\left(1 - \frac{\epsilon}{2|C|}\right)|A| &< \left|\bigcap_{c\in C}(cA \cap A)\right| \\ &= \left|\bigcap_{c\in C}(cA \setminus (cA\Delta A))\right| \\ &\le |tA \setminus (A\Delta tA)| \\ &= |tA| - |tA \cap (A\Delta tA)| \\ &= |A| - |tA \cap (A\Delta tA)|.\end{aligned}$$

Rearranging this yields

$$|tA \cap (A\Delta tA)| < \frac{\epsilon}{2|C|}|A|. \tag{4.2.5}$$

Now, for any fixed $t \in C$, summing the inequalities (4.2.4) and (4.2.5) shows that

$$|A\Delta tA| = |A \cap (A\Delta tA)| + |tA \cap (A\Delta tA)| < \frac{\epsilon}{|C|}|A|. \tag{4.2.6}$$

To conclude, we note that

$$|A\Delta CA| \le \sum_{t\in C}|A\Delta tA| < \epsilon|A|,$$

so that A satisfies property (iv) for C and ϵ.

(iii) $\implies$ (ii): We will prove that if a sequence $\{A_n\}$ does not become arbitrarily invariant for property (ii), then it does not become arbitrarily invariant for property (iii). In fact, we will show that if a finite $A \subset \Gamma$ does not satisfy property (ii) for a given choice of C and ϵ, then it does not satisfy property (iii) for $C \cup \{e\}$ and ϵ. To this end, suppose that $C \subset \Gamma$ is finite, and $\epsilon > 0$. Take a finite subset $A \subset \Gamma$ which does not satisfy property (ii) for C and ϵ. We claim that A also does not satisfy property (ii) for $C \cup \{e\}$ and ϵ. Indeed, we have

$$|\{s \in A \mid (C\cup\{e\})s \subset A\}| = |\{s \in A \mid Cs\cup\{s\} \subset A\}| = |\{s \in A \mid Cs \subset A\}| \le (1-\epsilon)|A|.$$

Thus, we may replace C by $C \cup \{e\}$ to assume that $e \in C$. Combining this with equality (4.2.1), we see that

$$\left|\bigcap_{t\in C} t^{-1}A\right| = |\{s \in A \mid Cs \subset A\}| \le (1-\epsilon)|A|. \tag{4.2.7}$$

We claim that

$$\partial_C A = \bigcup_{t\in C} t^{-1}A \setminus \bigcap_{t\in C} t^{-1}A. \tag{4.2.8}$$

Indeed, both sides describe elements $s \in \Gamma$ for which $Cs \cap A \neq \varnothing$ (this is clear in the definition of $\partial_C A$, and arises via the union in the right-hand description: there exists some $t \in C$ such that $s \in t^{-1}A$, whence $ts \in A$). The second condition that $Cs \cap A^c \neq \varnothing$ from the left-hand side is expressed on the right-hand side by the removal of those $s \in \Gamma$ for which every element of Cs lies within A.

Notice also that

$$\bigcap_{t\in C} t^{-1}A \subset \bigcup_{t\in C} t^{-1}A,$$

so we obtain from equation (4.2.8) that

$$|\partial_C A| = \left|\bigcup_{t\in C} t^{-1}A\right| - \left|\bigcap_{t\in C} t^{-1}A\right|.$$

Combining this with equation (4.2.7) yields

$$|\partial_C A| = \left|\bigcup_{t\in C} t^{-1}A\right| - \left|\bigcap_{t\in C} t^{-1}A\right| \geq |A| - (1-\epsilon)|A| = \epsilon|A|,$$

so we see that A does not satisfy property (iii) for C and ϵ either.

(ii) $\implies$ (iii): Given a finite subset $C \subset \Gamma$, and $\epsilon > 0$, choose a finite subset $A \subset \Gamma$ which satisfies property (ii) up to a tolerance of $(CC^{-1}, \epsilon/(2|C|^2))$. Arguing as in the proof of (ii) $\implies$ (iv), but replacing C^{-1} by CC^{-1}, and $\epsilon/(2|C|)$ by $\epsilon/(2|C|^2)$, we derive for every $r \in CC^{-1}$ an inequality similar to (4.2.6):

$$|rA\Delta A| < \frac{\epsilon}{|C|^2}|A|. \tag{4.2.9}$$

We claim that

$$\partial_C A = \bigcup_{s,t\in C} (s^{-1}A\Delta t^{-1}A). \tag{4.2.10}$$

Indeed, each set in the union on the right-hand side describes elements $g \in \Gamma$ for which precisely one of $sg \in A$ or $tg \in A$ is true, so that we have both $Cg \cap A \neq \varnothing$ (via the statement which was true), and $Cg \cap A^c \neq \varnothing$ (via the statement which was false), so each set in the union is contained in $\partial_C A$. On the other hand, for each element $g \in \partial_C A$, there is at least one element $s \in C$ for which $sg \in A$, and at least one element $t \in C$ for which $tg \notin A$, so that $g \in (s^{-1}A \setminus t^{-1}A) \subset (s^{-1}A\Delta t^{-1}A)$ and so is contained in the union on the right-hand side.

Finally, notice that since $h(D\Delta E) = hD\Delta hE$ for any $h \in \Gamma$ and any $D, E \subset \Gamma$, and since multiplication by any element $h \in \Gamma$ is a bijection, we have for any elements $g, h \in \Gamma$ that

$$|gA\Delta hA| = |h(h^{-1}gA\Delta A)| = |h^{-1}gA\Delta A|.$$

Combining this with equation (4.2.10) and inequality (4.2.9) yields

$$|\partial_C A| = \left|\bigcup_{s,t\in C}(s^{-1}A\Delta t^{-1}A)\right| \leq \sum_{s,t\in C}|ts^{-1}A\Delta A| < |C|^2\left(\frac{\epsilon|A|}{|C|^2}\right) = \epsilon|A|,$$

so that A satisfies property (iii) for C and ϵ.

(iv) $\implies$ (i): Let $C \subset \Gamma$ be finite and $\epsilon > 0$, and suppose that A satisfies property (iv) for this choice of C and ϵ. Then we have

$$|CA \setminus A| \leq |CA \setminus A| + |A \setminus CA| = |CA\Delta A| < \epsilon|A|,$$

so that A is (C, ϵ)-invariant.

(i) $\implies$ (ii): Let $C \subset \Gamma$ be finite, and $\epsilon > 0$, and suppose that A is $(C^{-1}, \epsilon/|C|)$-invariant. In particular, we have for each $t \in C$ that

$$|t^{-1}A \setminus A| \leq |C^{-1}A \setminus A| < \frac{\epsilon}{|C|}|A|. \tag{4.2.11}$$

Observe that we can obtain from equation (4.2.1) that

$$\begin{aligned} |\{s \in A \mid Cs \subset A\}| &= \left|\bigcap_{t\in C}(A \cap t^{-1}A)\right| \\ &\geq |A| - \sum_{t\in C}|A \setminus t^{-1}A| \\ &= |A| - \sum_{t\in C}|t^{-1}A \setminus A|. \end{aligned} \tag{4.2.12}$$

To obtain the inequality on the second line, we begin with A and think of the intersection $A \cap t^{-1}A$ as removing the set $A \setminus t^{-1}A$ from A. Of course, we obtain a lower bound on the cardinality of the intersection by subtracting each $|A \setminus t^{-1}A|$ from $|A|$, because the sets $A \setminus t^{-1}A$ may intersect for different choices of $t \in C$. For the final equality, we used that

$$|A \setminus t^{-1}A| = |A| - |A \cap t^{-1}A| = |t^{-1}A| - |A \cap t^{-1}A| = |t^{-1}A \setminus A|.$$

Thus, we combine inequality (4.2.12) and inequality (4.2.11) to obtain

$$|\{s \in A \mid Cs \subset A\}| \geq |A| - \sum_{t \in C} |t^{-1}A \setminus A| > |A| - |C| \left(\frac{\epsilon |A|}{|C|} \right) = (1-\epsilon)|A|,$$

so that A satisfies condition (ii) for C and ϵ. □

Definition 4.2.8 ([35, Definition 8.2])**.** A free action $\Gamma \curvearrowright X$ on a compact metric space is said to be *almost finite* if, for every $m \in \mathbb{N}$, finite set $C \subset \Gamma$, and $\epsilon > 0$,

(i) there is an open castle $\{(W_i, S_i)\}_{i \in \{1,\dots,n\}}$ whose shapes are (C, ϵ)-invariant and whose levels have diameter less than ϵ; and

(ii) for each $i \in \{1, \dots, n\}$, there are sets $S_i' \subset S_i$ such that $|S_i'| < |S_i|/m$ and

$$X \setminus \bigsqcup_{i=1}^{n} S_i W_i \prec \bigsqcup_{i=1}^{n} S_i' W_i.$$

The second condition enforces the "smallness" of the complement of the castle in X compared to the number of levels in the towers, while the first condition encodes the approximate invariance requirement from hyperfiniteness.

Examples 4.2.9.

(i) The action of $\mathbb{Z}_N$ on itself by translation is almost finite, using the single-tower castle $(\{1\}, \mathbb{Z}_N)$. In fact, any free action of a finite group Γ on a finite set X with the discrete topology is almost finite using a castle of the form $\{(\{x_i\}, \Gamma)\}_{i \in I}$ with each $x_i \in X$ chosen from a different orbit of the action.

(ii) The odometer action on the Cantor set $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ is almost finite. Given a finite $C \subset \mathbb{Z}$ and $\epsilon > 0$, we can find $N > 0$ so that $C \subset \{-N, \dots, N\}$. Then the set $\{0, \dots, k-1\} \subset \mathbb{Z}$ will be (C, ϵ)-invariant when

$$\frac{|\{-N, \dots, N+k-1\} \Delta \{0, \dots, k-1\}|}{k} = \frac{2N}{k} < \epsilon$$

or, in other words, when $k > 2N/\epsilon$. Choose $l \in \mathbb{N}$ such that $l > \log_2(2N/\epsilon)$, so that $k = 2^l$ has $k > 2N/\epsilon$. A cylinder set of length l has diameter $1/2^{l+1}$, so for such a cylinder set to have diameter smaller than ϵ we require $1/2^{l+1} < \epsilon$, so that $l > \log_2(1/\epsilon) - 1$. So, choose $l > \max(\log_2(2N/\epsilon), \log_2(1/\epsilon - 1))$, and consider the cylinder set $[x_1 \dots x_l]$ of length l such that $x_1 = x_2 = \cdots = x_l = 0$. Then the single-tower castle $([x_1 \dots x_l], \{0, \dots, 2^l - 1\})$ covers $\{0,1\}^{\mathbb{N}}$, and thus witnesses almost finiteness for C and ϵ.

(iii) Generalising the previous example, every free $\mathbb{Z}^m$-action on a zero-dimensional compact metrisable space is almost finite (see [41, Lemma 6.3] for the statement in the language of groupoids, and [35, Section 10] for the translation into the language of Definition 4.2.8).

(iv) For a simple negative example, let $\theta = M/N$ be a rational number in lowest terms and consider the action of rotation $\alpha_\theta : \mathbb{Z}_N \curvearrowright \mathbb{T}$ on the unit circle. This action is *not* almost finite. Since $\mathbb{T}$ has covering dimension 1, there exists no cover of $\mathbb{T}$ by disjoint open sets. Therefore the complement of any castle $X \setminus \bigsqcup S_i W_i$ will be nonempty. For $n > N$, the condition $|S_i'| < |S_i|/n$ requires that $S_i' = \varnothing$, because the largest shape we can choose is $S_i = \mathbb{Z}_N$. Putting this together, for the action to be almost finite we would require the existence of a castle $\{(W_i, S_i)\}$ such that $X \setminus \bigsqcup S_i W_i \prec \varnothing$, which cannot happen as the left-hand side must be nonempty.

(v) Now let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and consider the irrational rotation $\alpha_\theta : \mathbb{Z} \curvearrowright \mathbb{T}$. This action is almost finite by [35, Theorem 9.3] since it has comparison (for the definition of comparison, see for example [35, Definition 3.2]).

We have the following link back to Definition 4.1.6.

Theorem 4.2.10 ([71, Lemma 5.2])**.** *Let $\alpha : \Gamma \curvearrowright X$ be a free, continuous action of a countable discrete group on a compact metric space. Suppose that the corresponding transformation groupoid $\Gamma \ltimes_\alpha X$ is almost finite in the sense of Definition 4.1.6, with the additional condition that the clopen subsets $F_j^{(i)}$ appearing in Definition 4.1.4 can be chosen to become arbitrarily small in diameter as the invariance of the elementary subgroupoid increases. Then α is almost finite.*

Conversely, suppose that α is almost finite, with the additional property that the castles in the definition can be chosen to be clopen and to partition X. Then $\Gamma \ltimes_\alpha X$ is almost finite.

We briefly elaborate on the proof (but see also Theorem 4.3.15). A correspondence is created between $(C \times X, \epsilon)$-invariant elementary subgroupoids of $\Gamma \ltimes X$ and (C, ϵ)-invariant clopen castles whose levels partition X. Given an elementary subgroupoid, the levels of the corresponding castle are precisely the subsets $F_j^{(i)}$ appearing in Definition 4.1.4, and vice-versa.

In the case that X is zero-dimensional, [35, Lemma 10.1 and Theorem 10.2] show that we can arrange, with no loss of generality, for the castles to be clopen, to partition X, and for their levels to have arbitrarily small diameters. Thus, we obtain the following simplified result.

Theorem 4.2.11. *Let $\alpha : \Gamma \curvearrowright X$ be a free, continuous action of a countable discrete* *group on a zero-dimensional compact metric space. Then α is almost finite if and only if the corresponding transformation groupoid $\Gamma \ltimes_\alpha X$ is almost finite in the sense of Definition 4.1.6.*

The following result highlights the applications of almost finiteness to the classification programme.

Theorem 4.2.12 ([35, Theorem 12.4]). *Suppose that Γ is infinite. Let $\alpha : \Gamma \curvearrowright X$ be a free minimal action which is almost finite. Then $C(X) \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable.*

In particular, this result provides classification of the crossed products arising from almost finite group actions by their Elliott invariant (see Section 6.1 for more precise details).

4.3 Almost finiteness for actions of groupoids

In this section, we introduce the notion of almost finiteness for actions of groupoids. We first make precise what it means for a groupoid to act on a set. The definition generalises that for group actions, with the only complication being the need to fibre the set receiving the action to ensure compatibility with the natural fibration of the groupoid over its unit space. To the best of the author's knowledge, the following notion first appears in [15]. The exact formulation presented here appears as [23, Definition 1.55].

Definition 4.3.1. Let G be a groupoid and X a set. We say that G *acts (on the left) of* X if there is a surjection $r_X : X \to G^{(0)}$ and a map $(g, x) \mapsto g \cdot x$ from $G * X := \{(g, x) \in G \times X \mid s(g) = r_X(x)\}$ to X such that

(i) if $(h, x) \in G * X$ and $(g, h) \in G^{(2)}$, then $(g, h \cdot x) \in G * X$ and
$$g \cdot (h \cdot x) = gh \cdot x; \text{ and}$$

(ii) $r_X(x) \cdot x = x$ for all $x \in X$.

It is also possible to define right-actions of groupoids analogously, in which case we denote the surjection by $s_X : X \to G^{(0)}$. In this thesis, all groupoids will act on the left.

We will usually drop the central dot from the notation, referring to the image of $x \in X$ under $g \in G$ by gx. We will preserve the dot in the case that $X = G^{(0)}$, to distinguish between groupoid multiplication gx, and the action $g \cdot x$. For $W \subset X$ and $S \subset G$, we denote
$$S \cdot W := \{g \cdot x \mid g \in S,\ x \in W \text{ and } s(g) = r_X(x)\}.$$

Again, we will be lax with notation and often denote this set by SW, but we will always attempt to make the meaning clear from context. Just as with products of subsets of G, in contrast to the group case, $S \cdot W$ is not nicely behaved. For instance, it is possible that $S \cdot W$ is empty when both S and W are not!

Examples 4.3.2.

(i) If we view a group Γ as a groupoid, then the definition of a groupoid action $\Gamma \curvearrowright X$ is equivalent to the usual definition of a group action. In this case, since $\Gamma^{(0)} = \{e\}$, the surjection becomes $r_X(x) = e$ for every $x \in X$.

(ii) There is a natural action $G \curvearrowright G^{(0)}$ of a groupoid on its unit space. The surjection $r_{G^{(0)}}$ is just the identity map and the action becomes $g \cdot s(g) = r(g)$.

We make the following simple observation.

Lemma 4.3.3. *Let $G \curvearrowright X$ be any groupoid action. If $(g, x) \in G * X$, then $r_X(gx) = r(g)$.*

Proof. Since $(g, x) \in G * X$ and $(g^{-1}, g) \in G^{(2)}$, we have $(g^{-1}, gx) \in G * X$ by condition (i) in Definition 4.3.1, so that $r(g) = s(g^{-1}) = r_X(gx)$. □

Definition 4.3.4. Let G be a topological groupoid which acts on a topological space X.

(i) We say that the action is *continuous* if the maps $r_X : X \to G^{(0)}$ and $(g, x) \mapsto gx$ from $G * X \to X$ are continuous.

(ii) We say that the action is *free* if, for every $x \in X$, $gx = x$ implies $g = r_X(x)$.

(iii) We say that the action is *minimal* if, for every $x \in X$, the subset $\{gx \mid g \in G\}$ is dense in X.

The following notion of dynamical comparison is a generalisation of Definition 4.2.1 to the groupoid setting.

Definition 4.3.5. Let $G \curvearrowright X$ be a groupoid action, and let $A, B \subset X$. We write $A \prec B$ if, for every closed $C \subset A$, there exist

- a finite collection $\mathcal{U}$ of open subsets of X which cover C; and
- a subset $S_U \subset G$ for each $U \in \mathcal{U}$ such that $r_X(U) \subset s(S_U)$ and so that the collection $\{tU \mid U \in \mathcal{U}, t \in S_U\}$ consists of pairwise disjoint subsets of B.

Example 4.3.6. Consider the canonical action $R_{\text{punc}} \curvearrowright \Omega_{\text{punc}}$ of a tiling groupoid R_{punc} (see Definition 3.3.10) on its unit space. Consider patches P, P' such that $P' \subset P$, and tiles $t \in P$ and $t' \in P'$. Then (recalling Notation 3.3.6) we claim that $U(P, t) \prec U(P', t')$. First, observe that $U(P, t) \subset U(P', t')$ if and only if $t = t'$. Given any closed subset $C \subset U(P, t)$, consider the open cover $\mathcal{U} = \{U(P, t)\}$ of C. Put $S_{U(P,t)} = V(P, t, t')$, and observe that $S_{U(P,t)}U(P, t) = U(P, t') \subset U(P', t')$, as required.

In Example 2.1.4(iii), we saw how to construct a groupoid from a group action. We now generalise this construction to allow for groupoid actions. Such groupoids will be vitally important in the sequel.

Definition 4.3.7. Let G be a groupoid acting on a set X. The associated *transformation groupoid* $G \ltimes X$ is the set $G * X$ with the following structure. The set of composable pairs is

$$(G \ltimes X)^{(2)} := \{((g,x),(h,y)) \in (G \ltimes X) \times (G \ltimes X) \mid h \cdot y = x\}.$$

The product of such a pair is given by $(g,x)(h,y) = (gh,y)$. The inverse operation is $(g,x)^{-1} := (g^{-1}, gx)$.

In the case that G is a topological groupoid acting on a topological space X, $G \ltimes X = G * X$ inherits the relative topology from the product topology on $G \times X$.

We can compute the missing structure maps as follows. The range and source of $(g,x) \in G \ltimes X$ are given by

$$s(g,x) = (g,x)^{-1}(g,x) = (g^{-1}, gx)(g,x) = (g^{-1}g, x) = (s(g), x) = (r_X(x), x),$$

and

$$r(g,x) = (g,x)(g,x)^{-1} = (gg^{-1}, gx) = (r(g), gx) = (r_X(gx), gx).$$

Hence, we see that $(G \ltimes X)^{(0)}$ can be naturally identified with X via $(r_X(x), x) \mapsto x$. Under this identification, $s(g,x) = x$ and $r(g,x) = gx$. From all this, it is easy to see that when G is a group, we recover the same groupoid as Example 2.1.4(iii). In the case that G is a groupoid, all that changes from our previous picture of a transformation groupoid is that not every element of G is "attached" to each element of X.

The following observations about the topology of groupoid actions will be useful.

Lemma 4.3.8. *Let G be a locally compact Hausdorff groupoid which acts continuously on a locally compact Hausdorff space X. Then*

(i) if $W \subset X$ and $S \subset G$ are compact, then $S \cdot W$ is compact in X;

(ii) [23, Proposition 1.72] if G is étale, then $G \ltimes X$ is étale; and

(iii) if G is étale and $W \subset X$ and $S \subset G$ are open, then $S \cdot W$ is open in X.

Proof.

(i) Observe that $S \cdot W$ is the image of the compact subset $(S \times W) \cap (G * X) \subset G * X$ under the continuous action map $(g,x) \mapsto gx$, and is therefore compact in X.

(ii) As is shown in [53], G is étale if and only if it admits a Haar system and the range and source maps are open. In this case, by [23, Proposition 1.72], the range and source of $G \ltimes X$ are also open and $G \ltimes X$ admits a Haar system, so that $G \ltimes X$ is étale.

(iii) As above, when G is étale, the range map of $G \ltimes X$ is open. Then $S \cdot W = r((S \times W) \cap (G \ltimes X))$ is open in $(G \ltimes X)^{(0)} \cong X$. □

Lemma 4.3.9. *Let G be a locally compact Hausdorff étale groupoid acting continuously on a locally compact Hausdorff space X. Then the range map $r_X : X \to G^{(0)}$ is open.*

Proof. By Lemma 4.3.8, $G \ltimes X$ is étale. Suppose that $U \subset X$ is open. We identify $U \subset X$ with $V = \{(r_X(u), u) \mid u \in U\} \subset (G \ltimes X)^{(0)}$. Since U is open in X and $X \cong (G \ltimes X)^{(0)}$ is open in $G \ltimes X$, it follows that $V \cong U$ is open in $G \ltimes X$. Observe that $r_X(U) = \pi_G(V)$, where π_G is projection to the G-coordinate on $G \times X$. Since π_G is an open map, this shows that $r_X(U)$ is open in G. □

Continuous actions $G \curvearrowright X$ such that r_X is open are sometimes called *strongly continuous*. By Lemma 4.3.9, these concepts coincide for étale groupoids.

Motivated by Lemma 4.2.7, we introduce a property similar to the notion of (C, ϵ)-invariance for subsets of étale groupoids which will turn out to be more useful for our purposes than Definition 4.1.1. First, we recall our existing notion of (C, ϵ)-invariance for groupoids (Definition 4.1.1. Also c.f. Definition 4.2.5).

Definition 4.3.10. Let G be an étale groupoid. Let $A, C \subset G$ be compact, and let $\epsilon > 0$. We say that A is (C, ϵ)-*invariant* if the following inequality holds for every $u \in s(A)$:

$$\frac{|CAu \setminus Au|}{|Au|} < \epsilon. \tag{4.3.1}$$

We say that a sequence of compact subsets $\{A_n\}_{n \in \mathbb{N}}$ of G *becomes arbitrarily approximately invariant* if, for any compact subset $C \subset G$ and any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that A_n is (C, ϵ)-invariant whenever $n \geq N$.

Lemma 4.3.11. *Let G be an étale groupoid. A sequence of nonempty compact subsets $\{A_n\}_{n \in \mathbb{N}}$ of G becomes arbitrarily approximately invariant if and only if, for any compact $C \subset G$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, whenever $n \geq N$, A_n satisfies*

$$|\{a \in A_n u \mid Ca \subset A_n u\}| > (1 - \epsilon)|A_n u| \tag{4.3.2}$$

for every $u \in s(A_n)$.

Remark 4.3.12. Motivated by this lemma, we will say that A is (C,ϵ)-*invariant* if it satisfies either condition (4.3.1) or (4.3.2). It is important to note that this dual usage of the terminology is rather sloppy, since (C,ϵ)-invariance of a *single* set A in the sense of (4.3.1) does *not* imply that A satisfies (4.3.2) for the same choice of C and ϵ, nor does the reverse implication hold. Our abuse of the terminology is justified by the fact that we will only care about sequences of sets which become *arbitrarily* approximately invariant. Even so, we will endeavour to make clear which definition we are working with at any given time.

Proof of Lemma 4.3.11. First, notice that

$$|\{a \in Au \mid Ca \subset Au\}| = |Au| - |\{a \in Au \mid Ca \not\subset Au\}|$$

so that (4.3.2) is equivalent to requiring that

$$|\{a \in Au \mid Ca \not\subset Au\}| < \epsilon|Au|$$

for each $u \in s(A)$. Therefore, given a compact set $C \subset G$, it will suffice to find constants $c_1, c_2 > 0$ such that for every compact set $A \subset G$ and every $u \in s(A)$

$$\frac{1}{c_1}|CAu \setminus Au| \leq |\{a \in Au \mid Ca \not\subset Au\}| \leq c_2|CAu \setminus Au|.$$

($\Rightarrow$): Fix $A, C \subset G$ compact and nonempty, and $\epsilon > 0$, and suppose that A is (C,ϵ)-invariant in the sense of (4.3.1), so that for every $u \in s(A)$

$$|CAu \setminus Au| < \epsilon|Au|.$$

Set $c_2 = \sup_{v \in G^{(0)}} |vC| > 0$, which is finite by Proposition 2.2.13. Notice that it is possible for distinct $a_1, a_2 \in \{a \in Au \mid Ca \not\subset Au\}$ to admit $t_{a_1}, t_{a_2} \in C$ such that $t_{a_1}a_1 = t_{a_2}a_2 \in CAu \setminus Au$, and that this requires by cancellation that t_{a_1} and t_{a_2} are also distinct, but share a range. Therefore this situation can occur for at most c_2 distinct elements $a_1, \ldots, a_{c_2}$. In other words, any map from $\{a \in Au \mid Ca \not\subset Au\}$ to $CAu \setminus Au$ defined by $a \mapsto t_a a$, where $t_a \in C$ is any element as above, is at most c_2-to-one. It follows that

$$|\{a \in Au \mid Ca \not\subset Au\}| \leq c_2|CAu \setminus Au| < c_2\epsilon|Au|,$$

so that A is $(C, c_2\epsilon)$-invariant in the sense of (4.3.2).

($\Leftarrow$): This time, suppose that A is (C,ϵ)-invariant in the sense of (4.3.2), so that

$$|\{a \in Au \mid Ca \not\subset Au\}| < \epsilon|Au|,$$

and let $c_1 = \sup_{w \in G^{(0)}} |Cw| > 0$, which is again finite by Proposition 2.2.13. Observe that

$$\begin{aligned} |CAu \setminus Au| &= \left| \bigcup_{w \in r(Au)} \bigcup_{c \in Cw} \{ca \mid a \in wAu, ca \notin Au\} \right| \\ &\leq \sum_{w \in r(Au)} \sum_{c \in Cw} |\{cau \mid a \in wAu, ca \notin Au\}|. \end{aligned}$$

Now, for each fixed $w \in r(Au)$, the second sum contributes $|Cw| \leq c_1$ terms. Each of these terms corresponds to some $c \in Cw$, and counts the $a \in wAu$ for which $ca \notin Au$. Therefore, if $a \in wAu$ is such that $Ca \subset Au$, it will never be counted in this sum, whereas if $a \in wAu$ has $Ca \not\subset Au$, it can be counted at most $|Cw| \leq c_1$ times. Thus, we obtain

$$\begin{aligned} |CAu \setminus Au| &\leq \sum_{w \in r(Au)} \sum_{c \in Cw} |\{cau \mid a \in wAu, ca \notin Au\}| \\ &\leq \sum_{w \in r(Au)} |Cw| |\{a \in wAu \mid Ca \not\subset Au\}| \\ &\leq c_1 \sum_{w \in r(Au)} |\{a \in wAu \mid Ca \not\subset Au\}| \\ &= c_1 \left| \bigsqcup_{w \in r(Au)} \{a \in wAu \mid Ca \not\subset Au\} \right| \\ &= c_1 |\{a \in Au \mid Ca \not\subset Au\}| \\ &< c_1 \epsilon |Au|. \end{aligned}$$

showing that A is $(C, c_1\epsilon)$-invariant in the sense of (4.3.1). □

For the remainder of the section, we assume that G is a locally compact Hausdorff étale groupoid whose unit space is compact which acts continuously on a compact metric space X.

Definition 4.3.13. Let $\alpha : G \curvearrowright X$ be an action. Let $C \subset G$ be compact, and let $\epsilon > 0$.

- A *tower* of α is a pair (W, S) consisting of a nonempty subset $W \subset X$ and a nonempty compact open subset $S \subset G$ such that $S = \bigsqcup_{j=1}^{N} S_j$ decomposes into compact open S_j with $s(S_j) = r_X(W)$ for each j, so that the range and source maps are injective on each S_j, and such that the sets S_jW are pairwise disjoint for $j \in \{1, \dots, N\}$. We refer to the set W as the *base*, S as the *shape*, and the sets S_jW as the *levels* of the tower.

- Consider a finite collection $\{(W_1, S_1), \dots, (W_n, S_n)\}$ of towers such that for each $i \in \{1, \dots, n\}$, the shape of the i-th tower has decomposition $S_i = \bigsqcup_{j=1}^{N_i} S_{i,j}$. Such a sequence is called a *castle* if the collection of all tower levels $\{S_{i,j}W_i \mid i \in \{1, \dots, n\}, j \in \{1, \dots, N_i\}\}$ is pairwise disjoint. The sets W_i are called the *bases*,

S_i the *shapes*, and $S_{i,j}W_i$ the *levels* of the castle $\{(W_i, S_i)\}_{i=1}^n$. A castle is called a *tower decomposition of* α if the levels of the castle partition X.

- A castle $\{(W_i, S_i)\}_{i=1}^n$ is said to be (C, ϵ)-*invariant* if all of the shapes $S_1, \ldots, S_n$ are (C, ϵ)-invariant subsets of G, in the sense defined by condition (4.3.2) from Lemma 4.3.11.
- A castle $\{(W_i, S_i)\}_{i=1}^n$ is said to be *open* (respectively *closed*, *clopen*) if each W_i is open (respectively closed, clopen) in X.
- A groupoid action is said to be *almost finite* if, for every $m \in \mathbb{N}$, every compact $C \subset G$, and every $\epsilon > 0$, there exist
  (i) an open castle $\{(W_i, S_i)\}_{i=1}^n$ whose shapes are (C, ϵ)-invariant, and whose levels have diameter less than ϵ; and
  (ii) sets $S_i' \subset S_i$ for each $i \in \{1, \ldots, n\}$ such that, for each $u \in G^{(0)}$, $|S_i'u| \leq |S_iu|/m$, so that (W_i, S_i') is a *subtower* of (W_i, S_i), in the sense that if $S_i = \bigsqcup_{j=1}^{N_i} S_{i,j}$, then there exist $L_i \in \mathbb{N}$ and a subcollection $\{j_{i,1}, \ldots, j_{i,L_i}\} \subset \{1, \ldots, N_i\}$ such that $S_i' = \bigsqcup_{l=1}^{L_i} S_{i,j_{i,l}}$, and such that
$$X \setminus \bigsqcup_{i=1}^{n} S_iW_i \prec \bigsqcup_{i=1}^{n} S_i'W_i.$$

In a tower (W, S), where $S = \bigsqcup_{j=1}^N S_j$, each set S_j should be thought of as corresponding to a single group element from Definition 4.2.8. The injectivity condition of the source and range together with the condition that $s(S_j) = r_X(W)$ ensure that each element of W has exactly one image under the action of S_j. Aside from this modification to ensure compatibility with the fibred structure of the groupoid, the definition is identical in spirit to Definition 4.2.8.

We show that the diameter condition appearing in the statement is no obstacle in our case of interest. In the group setting, closedness of the towers comes with no loss of generality by [35, Lemma 10.1]. We conjecture that this should also be true in the groupoid setting, but we do not have a proof, so it appears as an assumption in the following result. Another lingering question to consider is whether the castles witnessing almost finiteness can always be chosen to partition X when X is totally disconnected (c.f. [35, Theorem 10.2]).

Lemma 4.3.14 (c.f. [35, Theorem 10.2])**.** *Let G be a locally compact Hausdorff étale groupoid, X a totally disconnected compact metric space, and $\alpha : G \curvearrowright X$ a free and continuous action which admits arbitrarily invariant* ***clopen*** *castles as in the definition of almost finiteness, but which do not satisfy the diameter condition. Then α is almost finite. That is, we can choose the invariant clopen castles to satisfy the diameter condition, with* *no loss of generality.*

Proof. Let $C \subset G$ be compact, let $\epsilon > 0$, and let (W, S) be a (C, ϵ)-invariant clopen tower as in the statement of the lemma, so that $S = \bigsqcup_{i=1}^{m} S_i$, where S_i is compact and open in G. We first show that we can refine (W, S) into a clopen castle whose levels have diameter smaller than ϵ. The bases of the towers in the castle will partition W, and the shape of each new tower will simply be the subset of S consisting of arrows which act on its base. Since S is (C, ϵ)-invariant, it will follow that the shape of each new tower is also (C, ϵ)-invariant.

Consider the first tower level $S_1 \cdot W$. Since X is totally disconnected, it has a basis of clopen sets. Therefore, since $S_1 \cdot W$ is compact and open, there exists a finite collection of clopen subsets $\{U_{n_1}\}_{n_1=1,\ldots,N}$ of X with diameter smaller than ϵ such that $S_1 \cdot W = \bigcup_{n_1=1}^{N} U_{n_1}$. By replacing U_{n_1} by $U_{n_1} \setminus \bigcup_{j=1}^{n_1-1} U_j$ for each $n_1 = 1, 2, \ldots, N$ in turn, we may assume without loss of generality that the collection $\{U_{n_1}\}_{n_1=1,\ldots,N}$ is pairwise disjoint.

For each $n_1 \in \{1, \ldots, N\}$, let $V_{n_1} = S_1^{-1} \cdot U_{n_1}$, which is a compact open subset of X by Lemma 4.3.8. Observe that $S_1 \cdot V_{n_1} = U_{n_1}$ since $r_X(U_{n_1}) \subset r(S_1) = s(S_1^{-1})$. Furthermore, since the collection $\{U_{n_1}\}_{n_1=1,\ldots,N}$ is pairwise disjoint and the source map is injective on S_1, the collection $\{V_{n_1}\}_{n_1=1,\ldots,N}$ is pairwise disjoint, and we have $W = S_1^{-1} \cdot \bigsqcup_{n_1=1}^{N} U_{n_1} = \bigsqcup_{n_1=1}^{N} S_1^{-1} \cdot U_{n_1} = \bigsqcup_{n_1=1}^{N} V_{n_1}$ so that the collection $\{V_{n_1}\}_{n_1=1,\ldots,N}$ partitions W. Associate to V_{n_1} the subset H_{n_1} of S_1 which is able to act upon it. That is, put $H_{n_1} = S_1 \cap s^{-1}(r_X(V_{n_1}))$, so that $H_{n_1} \cdot V_{n_1} = S_1 \cdot V_{n_1} = U_{n_1}$. Observe that H_{n_1} is compact and open because r_X is continuous and open by Lemma 4.3.9 and s is a local homeomorphism.

In this manner, we have created N "single-level" towers (V_{n_1}, H_{n_1}) for $n_1 \in \{1, \ldots, N\}$ whose levels have diameter smaller than ϵ and partition $S_1 \cdot W$, and such that for each $n_1 \in \{1, \ldots, N\}$ and each $x \in V_{n_1}$, we have $H_{n_1} r_X(x) = S_1 r_X(x)$.

Now, for each $n_1 = 1, \ldots, N$ in turn, consider the subset $S_2 \cdot V_{n_1}$ of $S_2 \cdot W$. Arguing similarly as above, construct a cover of $S_2 \cdot V_{n_1}$ by pairwise disjoint clopen subsets $\{U_{n_1,n_2}\}_{n_2=1,\ldots,N_{n_1}}$ of X with diameter smaller than ϵ. For each $n_2 \in \{1, \ldots, N_{n_1}\}$ put $V_{n_1,n_2} = S_2^{-1} \cdot U_{n_1,n_2}$ and $H_{(n_1,n_2)} = (H_{n_1} \cup S_2) \cap s^{-1}(r_X(V_{n_1,n_2}))$ so that V_{n_1,n_2} and $H_{(n_1,n_2)}$ are both compact and open, and so that $\{V_{n_1,n_2}\}_{n_2=1,\ldots,N_{n_1}}$ partitions V_{n_1}. Define $H_{(n_1,n_2),1} = S_1 \cap H_{(n_1,n_2)}$ and $H_{(n_1,n_2),2} = S_2 \cap H_{(n_1,n_2)}$ so that $H_{(n_1,n_2)} = \bigsqcup_{i=1}^{2} H_{(n_1,n_2),i}$. Observe that $H_{(n_1,n_2),1} \cdot V_{n_1,n_2} \subset U_{n_1}$ and $H_{(n_1,n_2),2} \cdot V_{n_1,n_2} \subset U_{n_1,n_2}$ both have diameter smaller than ϵ. In this way, for each $n_1 \in \{1, \ldots, N\}$ we construct N_{n_1} "two-level" towers $(V_{n_1,n_2}, H_{(n_1,n_2)})$ for $n_2 \in \{1, \ldots, N_{n_1}\}$. The collection of all of the levels of these towers is pairwise disjoint and partitions $(S_1 \cup S_2) \cdot W$, each level has diameter smaller than ϵ, and for each $x \in V_{n_1,n_2}$ we have $H_{(n_1,n_2)} r_X(x) = (S_1 \cup S_2) r_X(x)$.

Continue to iterate this procedure, at the k-th step beginning with a collection of towers $(V_{n_1,n_2,\dots,n_{k-1}}, H_{(n_1,n_2,\dots,n_{k-1})})$ for $n_1 \in \{1,\dots,N\}$, $n_2 \in \{1,\dots,N_{n_1}\}$, $\dots$, $n_{k-1} \in \{1,\dots,N_{n_1,\dots,n_{k-2}}\}$, and covering $S_k \cdot V_{n_1,n_2,\dots,n_{k-1}}$ by finitely many pairwise disjoint clopen subsets $\{U_{n_1,\dots,n_k}\}_{n_k=1,\dots,N_{n_1,\dots,n_{k-1}}}$ of X with diameter smaller than ϵ. Set $V_{n_1,\dots,n_k} = S_k^{-1} \cdot U_{n_1,\dots,n_k}$ so that the collection $\{V_{n_1,\dots,n_k}\}_{n_k=1,\dots,N_{n_1,\dots,n_{k-1}}}$ partitions $V_{n_1,\dots,n_{k-1}}$. Also set $H_{(n_1,\dots,n_k)} = (H_{(n_1,\dots,n_{k-1})} \cup S_k) \cap s^{-1}(r_X(V_{n_1,\dots,n_k}))$ (with decomposition $H_{(n_1,\dots,n_k),i} = S_i \cap H_{(n_1,\dots,n_k)}$ for each $i \in \{1,\dots,k\}$) to obtain a collection of "k-level" towers $\{(V_{n_1,\dots,n_k}, H_{(n_1,\dots,n_k)})\}$ indexed by n_1 up to n_k such that the collection of all tower levels is pairwise disjoint and partitions $(S_1 \cup \dots \cup S_k) \cdot W$, each level of each tower has diameter smaller than ϵ, and so that for each $x \in V_{n_1,\dots,n_k}$ we have $H_{(n_1,\dots,n_k)} r_X(x) = (S_1 \cup \dots \cup S_k) r_X(x)$.

The procedure terminates after m steps (where m is the number of levels in the tower (W,S)) to produce a clopen castle $\{(V_{n_1,\dots,n_m}, H_{(n_1,\dots,n_m)})\}$, where $n_1 \in \{1,\dots,N\}$, $n_2 \in \{1,\dots,N_{n_1}\}$, $\dots$, $n_m \in \{1,\dots,N_{n_1,\dots,n_{m-1}}\}$, such that every level of every tower has diameter smaller than ϵ. The levels of this castle will partition $(S_1 \cup \dots \cup S_m) \cdot W = S \cdot W$. For each $x \in V_{n_1,\dots,n_m} = r_X^{-1}(s(H_{(n_1,\dots,n_m)}))$ we have $H_{(n_1,\dots,n_m)} r_X(x) = (S_1 \cup \dots \cup S_m) r_X(x) = S r_X(x)$, so that for each $u \in s(H_{(n_1,\dots,n_m)})$ we can use (C,ϵ)-invariance of S to obtain $|CH_{(n_1,\dots,n_m)}u \setminus H_{(n_1,\dots,n_m)}u| = |CSu \setminus Su| < \epsilon |Su| = \epsilon |H_{(n_1,\dots,n_m)}u|$, which shows that $H_{(n_1,\dots,n_m)}$ is (C,ϵ)-invariant.

It remains to show that given a clopen castle $\{(W_j,S_j)\}_{j=1,\dots,J}$ witnessing condition (ii) of almost finiteness, the clopen castle $\{(V^{(j)}_{n_1,\dots,n_{m_j}}, H^{(j)}_{(n_1,\dots,n_{m_j})})\}_{j,n_1,\dots,n_{m_j}}$ obtained by applying the procedure above on each tower in the castle satisfies condition (ii) of almost finiteness. For each $j \in \{1,\dots,J\}$ find sets $S'_j \subset S_j$ such that

$$X \setminus \bigsqcup_{j=1}^{J} S_j W_j \prec \bigsqcup_{j=1}^{J} S'_j W_j.$$

Observe that for each j we have

$$\bigsqcup_{n_1,\dots,n_{m_j}} (S'_j \cap H^{(j)}_{(n_1,\dots,n_{m_j})}) V^{(j)}_{n_1,\dots,n_{m_j}} = S'_j W_j$$

and that the sets appearing in the union on the left-hand side are levels of the tower $(V^{(j)}_{n_1,\dots,n_{m_j}}, H^{(j)}_{(n_1,\dots,n_{m_j})})$. Observe also that for each $u \in G^{(0)}$, $|(S'_j \cap H^{(j)}_{(n_1,\dots,n_{m_j})})u| \le |S'_j u|$. It follows that

$$\begin{aligned} X \setminus \bigsqcup_{j=1}^{J} \bigsqcup_{n_1,\dots,n_{m_j}} H^{(j)}_{(n_1,\dots,n_{m_j})} V^{(j)}_{n_1,\dots,n_{m_j}} &= X \setminus \bigsqcup_{j=1}^{J} S_j W_j \\ &\prec \bigsqcup_{j=1}^{J} S'_j W_j = \bigsqcup_{j=1}^{J} \bigsqcup_{n_1,\dots,n_{m_j}} (S'_j \cap H^{(j)}_{(n_1,\dots,n_{m_j})}) V^{(j)}_{n_1,\dots,n_{m_j}}. \end{aligned}$$

Therefore, the choices $(H^{(j)}_{(n_1,\ldots,n_{m_j})})' = (S_j' \cap H^{(j)}_{(n_1,\ldots,n_{m_j})})$ witness property (ii) of almost finiteness for the new castle. □

Theorem 4.3.15 (c.f. Theorems 4.2.10 and 4.2.11)**.** *Let G be a locally compact Hausdorff étale groupoid with compact unit space, X a compact metric space, and $\alpha : G \curvearrowright X$ a continuous action. Let $C \subset G$ be a compact subset, and fix $\epsilon > 0$.*

Suppose that α admits a (C, ϵ)-invariant clopen tower decomposition. Then the transformation groupoid $G \ltimes X$ admits a $(C \times X, \epsilon)$-invariant elementary subgroupoid. The converse holds in the case that G is ample.

Proof. $(\Rightarrow)$: Take a clopen tower decomposition $\{(W_i, S_i)\}_{i=1}^n$ of α as in the theorem, so that $S_i = \bigsqcup_{j=1}^{N_i} S_{i,j}$ for each $i \in \{1, \ldots, n\}$, where the range map of G is injective on each $S_{i,j}$. Define an elementary subgroupoid

$$K = \bigsqcup_{i=1}^{n} \bigsqcup_{1\leq j,k\leq N_i} V^{(i)}_{j,k}$$

as follows. We will let $V^{(i)}_{j,k}$ be a collection of arrows from the k-th to the j-th level of the i-th tower (W_i, S_i), which we will obtain by moving from the k-th level $S_{i,k}W_i$ down to the base along $S_{i,k}^{-1}$, and then from the base to the j-th level along $S_{i,j}$. More precisely, for each $j \in \{1, \ldots, N_i\}$ consider the collection

$$V^{(i)}_{j,\text{base}} = (S_{i,j} \times W_i) \cap (G \ltimes X) = \{(g, x) \mid g \in S_{i,j},\ x \in W_i,\ s(g) = r_X(x)\}.$$

Observe that each element of $V^{(i)}_{j,\text{base}}$ has source in the base of the i-th tower (W_i, S_i), and range in the j-th level $S_{i,j}W_i$. Notice that, for each $x \in W_i$, there is exactly one associated element $(g, x) \in V^{(i)}_{j,\text{base}}$, because $s(S_{i,j}) = r_X(W_i)$ (so there is at least one) and the source map is injective on $S_{i,j}$. Thus, the element $(g, x) \in V^{(i)}_{j,\text{base}}$ has $s(g, x) = x$ and is the unique element in $V^{(i)}_{j,\text{base}}$ with source equal to x. Notice that

$$(V^{(i)}_{j,\text{base}})^{-1} = \{(g^{-1}, gx) \mid g \in S_{i,j},\ x \in W_i,\ s(g) = r_X(x)\}$$

also contains exactly one element associated to each $x \in W_i$. This time, $x = r(g^{-1}, gx)$. Define

$$V^{(i)}_{j,k} = V^{(i)}_{j,\text{base}}(V^{(i)}_{k,\text{base}})^{-1}.$$

An element of $V^{(i)}_{j,k}$ has the form $(g, x)(h, y)^{-1} = (g, x)(h^{-1}, hy)$, where $x, y \in W_i$, $g \in S_{i,j}$, $h \in S_{i,k}$, $s(g) = r_X(x)$, and $s(h) = r_X(y)$. For these arrows to be composable we require that $y = r(h^{-1}, hy) = s(g, x) = x$, and then the product looks like $(g, x)(h^{-1}, hx) = (gh^{-1}, hx)$, and has source $hx \in S_{i,k} \cdot W_i$, and range $gh^{-1}hx = gx \in S_{i,j}W_i$. Since the arrow (h^{-1}, hx) (respectively (g, x)) was unique in $(V^{(i)}_{k,\text{base}})^{-1}$ (respectively $V^{(i)}_{j,\text{base}}$) with range (respectively source) equal to x, this construction shows that $V^{(i)}_{j,k}$ consists of a choice of exactly one arrow for each $x \in W_i$, with source $hx \in S_{i,k}W_i$ and range $gx \in S_{i,j}W_i$. Note in particular that the specification of an element $x \in W_i$ is enough to recover the associated arrow in $V^{(i)}_{j,k}$.

From this computation, we see that a candidate fundamental domain for the elementary subgroupoid is $F = \bigsqcup_{i=1}^{n} F^{(i)}_1$, where $F^{(i)}_1 = S_{i,1}W_i$, and this leads us to set $F^{(i)}_j = S_{i,j}W_i$.

We check that K is indeed an elementary subgroupoid of $G \ltimes X$.

- The $F^{(i)}_j$ are clopen in X. Since W_i is closed in the compact X, it is compact. Since $S_{i,j}$ is also compact, Lemma 4.3.8 implies that $F^{(i)}_j = S_{i,j}W_i$ is compact. Thus, $F^{(i)}_j$ is a compact subset of the Hausdorff X, so it is closed. Since both $S_{i,j}$ and W_i are open, and G is étale, Lemma 4.3.8 also impies that $F^{(i)}_j$ is open.
- $r(V^{(i)}_{j,k}) = F^{(i)}_j$, and $s(V^{(i)}_{j,k}) = F^{(i)}_k$. Since $s(S_{i,j}) = r_X(W_i)$, we see that, for each j,

$$s(V^{(i)}_{j,\text{base}}) = \{x \in W_i \mid \exists g \in S_{i,j} \text{ such that } s(g) = r_X(x)\} = W_i.$$

  It is also clear from the definitions involved that $r(V^{(i)}_{j,\text{base}}) = S_{i,j}W_i$. Therefore, $r((V^{(i)}_{k,\text{base}})^{-1}) = s(V^{(i)}_{k,\text{base}}) = W_i = s(V^{(i)}_{j,\text{base}})$. This shows that the multiplication $V^{(i)}_{j,\text{base}}(V^{(i)}_{k,\text{base}})^{-1}$ associates an arrow to every element of $r(V^{(i)}_{j,\text{base}})$, and hence that $r(V^{(i)}_{j,k}) = r(V^{(i)}_{j,\text{base}}) = S_{i,j}W_i = F^{(i)}_j$. Similarly, we see that $s(V^{(i)}_{j,k}) = s((V^{(i)}_{k,\text{base}})^{-1}) = r(V^{(i)}_{k,\text{base}}) = S_{i,k}W_i = F^{(i)}_k$, as required.
- $X = (G \ltimes X)^{(0)} \subset K$. Choose $x \in X$. Since $\{(W_i, S_i)\}_{i=1}^{n}$ was a clopen tower decomposition, there exist unique $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, N_i\}$ such that $x \in F^{(i)}_j$. Observe that $V^{(i)}_{j,j} \subset K$ is defined by

$$V^{(i)}_{j,j} = V^{(i)}_{j,\text{base}}(V^{(i)}_{j,\text{base}})^{-1} \supset r(V^{(i)}_{j,\text{base}}) = F^{(i)}_j,$$

  where the last equality follows from the previous bullet point. Therefore, $x \in V^{(i)}_{j,j} \subset K$.
- The collection $\left\{F^{(i)}_j \;\middle|\; i \in \{1, \ldots, n\}, j \in \{1, \ldots, N_i\}\right\}$ is a partition of $K^{(0)}$. By the previous bullet point we see that $K^{(0)} = X$. Since we assumed that $\{(W_i, S_i)\}_{i=1}^{n}$ was a clopen tower decomposition, we obtain

$$K^{(0)} = X = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{N_i} (S_{i,j} \cdot W_i) = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{N_i} F^{(i)}_j.$$

- $V^{(i)}_{j,k}$ is a $(G \ltimes X)$-set; that is, we claim that the range and source maps are injective on $V_{j,k}^{(i)}$. To see this, we use the injectivity of the range and source on $S_{i,j}$. To be precise, if $(g_1h_1^{-1}, h_1x_1), (g_2h_2^{-1}, h_2x_2) \in V_{j,k}^{(i)}$ have equal ranges, then $g_1x_1 = g_2x_2$, where $g_1, g_2 \in S_{i,j}$. This requires that $r(g_1) = r_X(g_1x_1) = r_X(g_2x_2) = r(g_2)$. Since the range map was assumed to be injective on $S_{i,j}$, this necessitates that $g_1 = g_2 = g$, say, so that $gx_1 = gx_2$. Acting on the left by g^{-1}, we see that $x_1 = x_2 = x$, say, and so we are considering arrows (gh_1^{-1}, h_1x) and (gh_2^{-1}, h_2x) in $V_{j,k}^{(i)}$. Now, since h_1x and h_2x are defined, we have $s(h_1) = r_X(x) = s(h_2)$ for $h_1, h_2 \in S_{i,k}$. Since the source map was injective on $S_{i,k}$, this forces $h_1 = h_2$, and therefore the elements of $V_{j,k}^{(i)}$ that we started with are equal.

Similarly, if the sources of the above elements of $V_{j,k}^{(i)}$ are equal, so that $h_1x_1 = s(g_1h_1^{-1}, h_1x_1) = s(g_2h_2^{-1}, h_2x_2) = h_2x_2$, injectivity of the range map on $S_{i,k}$ shows that $h_1 = h_2$ and thus also that $x_1 = x_2$. Then, we obtain two arrows (g_1h^{-1}, hx) and (g_2h^{-1}, hx) of $V_{j,k}^{(i)}$, and observe that this requires $g_1h^{-1}hx = g_1x$ and $g_2h^{-1}hx = g_2x$ to be defined. This necessitates that $s(g_1) = r_X(x) = s(g_2)$, and so, by injectivity of the source map on $S_{i,j}$, we must have $g_1 = g_2$.

- $V_{j,k}^{(i)}$ is compact and open. Recall that $V_{j,\text{base}}^{(i)} = (S_{i,j} \times W_i) \cap (G \ltimes X)$. By assumption, $S_{i,j}$ was compact and open in G, and W_i was clopen in X. Furthermore, since W_i is closed in the compact space X, it is compact. Therefore, $S_{i,j} \times W_i$ is compact and open in $G \times X$, and thus we see that $V_{j,\text{base}}^{(i)}$ is compact and open in $G \ltimes X$ (whose topology is inherited from the product topology on $G \times X$) for each j. Since the inverse map of $G \ltimes X$ is a homeomorphism, we see also that $(V_{k,\text{base}}^{(i)})^{-1}$ is compact and open. Since groupoid multiplication is continuous, this shows that $V_{j,k}^{(i)} = V_{j,\text{base}}^{(i)}(V_{k,\text{base}}^{(i)})^{-1}$ is compact. As is shown in [27, Lemma 5.2], the product of two open sets in an étale groupoid is again open, which also proves that $V_{j,k}^{(i)}$ is open since $G \ltimes X$ is étale by Lemma 4.3.8.

- $V_{j,k}^{(i)}$ is the unique compact open $(G \ltimes X)$-subset of K with the above properties; that is, we claim that it is the only compact open subset $V \subset K$ such that the range and source are injective on V and satisfy $r(V) = F_j^{(i)}$ and $s(V) = F_k^{(i)}$. Indeed, let $V \subset K$ be any such set. Observe that the only arrows of K whose range intersects $F_j^{(i)}$ are elements of $\bigsqcup_{k=1}^{N_i} V_{j,k}^{(i)}$, so we must have $V \subset \bigsqcup_{k=1}^{N_i} V_{j,k}^{(i)}$ since $V \subset K$ and $r(V) = F_j^{(i)}$. In fact, since $s(V) = F_k^{(i)}$, and the collection $\left\{F_l^{(i)} \;\middle|\; l \in \{1, \dots, N_i\}\right\}$ is pairwise disjoint, this shows that $V \subset V_{j,k}^{(i)}$. Fix an arbitrary element $(gh^{-1}, hx) \in V_{j,k}^{(i)}$. Observe that by injectivity of the source map on $S_{i,k}$, and the range map on $S_{i,j}$, the elements $h \in S_{i,k}$ and $g \in S_{i,j}$ are unique in their respective sets such that hx and gx are defined, and therefore $(gh^{-1}, hx) \in V_{j,k}^{(i)}$ is the unique element of $V_{j,k}^{(i)}$ with range gx. Since $r(V) = F_j^{(i)}$, we must therefore have $(gh^{-1}, hx) \in V$, which shows that $V = V_{j,k}^{(i)}$.

Now, fix a compact subset $C \subset G$ and $\epsilon > 0$ and suppose that the clopen tower decomposition $\{(W_1, S_1), \ldots, (W_n, S_n)\}$ is (C, ϵ)-invariant, so that for each $i \in \{1, \ldots, n\}$ and $u \in s(S_i)$, we have

$$\frac{|CS_i u \setminus S_i u|}{|S_i u|} < \epsilon.$$

Construct the associated elementary subgroupoid

$$K = \bigsqcup_{i=1}^{n} \bigsqcup_{1 \leq j,k \leq N_i} V_{j,k}^{(i)}.$$

We want to show that K is $(C \times X, \epsilon)$-invariant, that is, that for every $x \in K^{(0)} = X$,

$$\frac{|(C \times X)Kx \setminus Kx|}{|Kx|} < \epsilon.$$

Fix $x \in X$. By construction, there exists a unique $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, N_i\}$ such that $x \in F_j^{(i)}$. Then the only elements $k \in K$ for which $s(k) = x$ lie in the sets $V_{l,j}^{(i)}$ for $l \in \{1, \ldots, N_i\}$. From each such set, there exists a unique $k_l \in K$ with $s(k_l) = x$, so $|Kx| = N_i$. On the other hand, we assumed that we can partition S_i into subsets $S_i = \bigsqcup_{j=1}^{N_i} S_{i,j}$ such that each $S_{i,j}$ has the same source, and such that the source map is injective on each $S_{i,j}$. For any $u \in s(S_i)$, it follows that $|S_i u| = N_i$, which shows that $|Kx| = |S_i u|$.

For the choice of $x \in F_j^{(i)} = S_{i,j} W_i$ from above, write $x = hy$ for $h \in S_{i,j}$ and $y \in W_i$. We can compute

$$\begin{aligned} Kx &= \{(gh^{-1}, x) \mid g \in S_{i,l}, s(g) = s(h), l = 1, \ldots, N_i\} \\ &= \{(gh^{-1}, hy) \mid g \in S_{i,l}, s(g) = s(h), l = 1, \ldots, N_i\}, \end{aligned}$$

which gives

$$(C \times X)Kx = \{(c, z)(gh^{-1}, hy) \mid g \in S_{i,l}, s(g) = s(h), l = 1, \ldots, N_i, c \in C, z \in X\}.$$

These elements are composable when $s(c) = r(g)$ and $gy = z$, producing the product (cgh^{-1}, hy). Thus, with h and y fixed as above, we see that

$$\begin{aligned} (C \times X)Kx \setminus Kx = \{(cgh^{-1}, hy) \mid & g \in S_{i,l}, s(g) = s(h), l = 1, \ldots, N_i, \\ & c \in C, s(c) = r(g), cg \notin S_{i,m} \text{ for } m = 1, \ldots, N_i\} \\ = \{(cgh^{-1}, hy) \mid & g \in S_i, s(g) = s(h), c \in C, s(c) = r(g), cg \notin S_i\}. \end{aligned}$$

On the other hand, for $u = r_X(y) \in S_i^{(0)}$, we have

$$\begin{aligned} CS_iu \setminus S_iu &= \{cgu \mid g \in S_i, s(g) = u = r_X(y), c \in C, s(c) = r(g), cg \notin S_i\} \\ &= \{cgu \mid g \in S_i, s(g) = s(h), c \in C, s(c) = r(g), cg \notin S_i\}, \end{aligned}$$

so we see that $|CS_iu \setminus S_iu| = |(C \times X)Kx \setminus Kx|$ via the bijection $(cgh^{-1}, hy) \mapsto cgu$. Thus, combining this with our previous observation that $|Kx| = |S_iu| = N_i$ for each $u \in S_i^{(0)}$, we have obtained

$$\frac{|(C \times X)Kx \setminus Kx|}{|Kx|} = \frac{|CS_iu \setminus S_iu|}{|S_iu|} < \epsilon,$$

where the last inequality uses (C, ϵ)-invariance of S_i.

$(\Leftarrow)$: Let G be ample, and let $K = \bigsqcup_{i=1}^{n} \bigsqcup_{1 \le j,k \le N_i} V_{j,k}^{(i)}$ be a $(C \times X, \epsilon)$-invariant elementary subgroupoid of $G \ltimes X$. We wish to form a clopen tower decomposition of α via the following scheme.

- Set the base of the i-th tower to be $W_i = F_1^{(i)} = s(V_{j,1}^{(i)})$ (for any choice of $j \in \{1, \ldots, N_i\}$).
- Set the shape S_i of this tower to be the projection of $\bigsqcup_{j=1}^{N_i} V_{j,1}^{(i)}$ to the G-coordinate, which decomposes into subsets $S_{i,j}$ obtained by projecting $V_{j,1}^{(i)}$ to the G-coordinate, so that the j-th level of the tower is $V_{j,1}^{(i)} F_1^{(i)} = F_j^{(i)}$.

Before this will work, however, there is a small issue to resolve. It is possible that there exists $x, y \in F_1^{(i)}$ such that $r_X(x) = r_X(y)$, and so that $V_{j,1}^{(i)}$ contains arrows (g, x) and (h, y), where $g \neq h$. Then, when we pass to the G-coordinate to obtain the shape of the tower, the subset $S_{i,j}$ will contain both g and h. Since $s(g) = r_X(x) = r_X(y) = s(h)$, this violates the injectivity of the source map on $S_{i,j}$. In order to fix this, we modify the sets $V_{j,k}^{(i)}$ to ensure that each set associates the same element of G to all elements of $F_k^{(i)}$ with the same image under r_X.

Since each $V_{j,k}^{(i)}$ is a compact open subset of $G \ltimes X$, the image $\pi_G(V_{j,k}^{(i)})$ under the projection to the G coordinate is also compact and open. By Proposition 2.2.15, since G is ample, the topology on G has a base of clopen bisections $\{B_m\}_{m \in I}$. Therefore, using compactness and taking relative complements if necessary, we can write $\pi_G(V_{j,k}^{(i)}) = \bigsqcup_{q=1}^{Q} B_q$ as a union of finitely many pairwise disjoint clopen bisections, each of which will be compact in G because they are closed in the compact subset $\pi_G(V_{j,k}^{(i)})$. We replace $V_{j,k}^{(i)}$ by the collection $\{V_{j,k}^{(i)} \cap (B_q \times F_k^{(i)})\}_{q=1}^{Q}$ of smaller compact open $(G \ltimes X)$-sets to assume, without loss of generality, that the range and source maps are injective on $\pi_G(V_{j,k}^{(i)})$. We note that from the newly defined subsets $V_{j,k}^{(i)}$, we also obtain redefined clopen sets $F_k^{(i)} = s(V_{j,k}^{(i)}) \subset X$ and $F_j^{(i)} = r(V_{j,k}^{(i)}) \subset X$, and, since the original $V_{j,k}^{(i)}$ was a $(G \ltimes X)$-set, the new collection $\{F_j^{(i)}\}_{i,j}$ is still pairwise disjoint.

We then form a clopen tower decomposition $\{(W_i, S_i)\}_{i=1,\dots,n}$ as follows. Set $W_i = F_1^{(i)}$ and

$$S_i = \pi_G(s^{-1}(F_1^{(i)}) \cap K) = \pi_G\left(\bigsqcup_{l=1}^{N_i} V_{l,1}^{(i)}\right),$$

with associated decomposition $S_i = \bigsqcup_{j=1}^{N_i} S_{i,j}$, where $S_{i,j} = \pi_G(V_{j,1}^{(i)})$. The levels of the i-th tower are then precisely the clopen sets $F_j^{(i)}$, for $j \in \{1, \dots, N_i\}$. Using the properties of the elementary subgroupoid, we see that the base of each tower is clopen, each $S_{i,j}$ is compact and open, $s(S_{i,j}) = r_X(W_i)$, and that the collection of tower levels is pairwise disjoint and partitions X. Due to the way we modified $V_{j,k}^{(i)}$, each $S_{i,j}$ is a subset of a bisection, and it follows that the source and range maps are injective on $S_{i,j}$. This shows that $\{(W_i, S_i)\}_{i=1,\dots,n}$ is a clopen tower decomposition, as we claimed.

Next, we show that this clopen tower decomposition is (C, ϵ)-invariant. Notice that for each $x \in W_i$ we have

$$\begin{aligned}
|(C \times X)Kx \setminus Kx| &= |\{(cg, x) \mid c \in C, (g, x) \in Kx, (cg, x) \notin Kx\}| \\
&= |\{cg \mid c \in C, g \in \pi_G(Kx), cg \notin \pi_G(Kx)\}| \\
&= |\{cg \mid c \in C, g \in S_i r_X(x), cg \notin S_i r_X(x)\}| \\
&= |CS_i r_X(x) \setminus S_i r_X(x)|.
\end{aligned}$$

Next, notice that since $\pi_G|_{Kx}$ is a bijection, this also gives $|Kx| = |\pi_G((s^{-1}(W_i) \cap K)x)| = |S_i r_X(x)|$. Putting this all together gives, for each $x \in F_1^{(i)} \subset K^{(0)}$, that

$$\frac{|CS_i r_X(x) \setminus S_i r_X(x)|}{|S_i r_X(x)|} = \frac{|(C \times X)Kx \setminus Kx|}{|Kx|} < \epsilon.$$

To conclude, observe that every $u \in s(S_i)$ has the form $u = r_X(x)$ for some $x \in F_1^{(i)}$, so what we have proven is enough to show that S_i is (C, ϵ)-invariant, as required. □

Corollary 4.3.16. *Let G be a locally compact Hausdorff étale groupoid with compact unit space, X a totally disconnected compact metric space, and $\alpha : G \curvearrowright X$ a continuous action.*

If α is almost finite, with the additional requirement that the open castles in Definition 4.3.13 can be chosen to be clopen tower decompositions of X, then the transformation groupoid $G \ltimes X$ is almost finite. Conversely, if G is ample and $G \ltimes X$ is almost finite, then α is almost finite and, furthermore, the castles appearing in the definition can be chosen to be clopen tower decompositions.

Proof. Fix $\epsilon > 0$ to be used throughout the proof.

First assume that α is almost finite, and that the open castles in the definition can be chosen to be clopen tower decompositions. Fix a compact subset $A \subset G \ltimes X$. We aim to produce an (A, ϵ)-invariant elementary subgroupoid of $G \ltimes X$. Denote by $\pi_G : G \ltimes X \to G$ the projection map to the G-coordinate. Since π_G is continuous, $\pi_G(A)$ is a compact subset of G. By our assumption on α, we can find a $(\pi_G(A), \epsilon)$-invariant clopen tower decomposition. Applying Theorem 4.3.15 produces a $(\pi_G(A) \times X, \epsilon)$-invariant elementary subgroupoid of $G \ltimes X$. Since $A \subset \pi_G(A) \ltimes X$, this elementary subgroupoid is also (A, ϵ)-invariant, so we see that $G \ltimes X$ satisfies condition (ii) of Definition 4.1.6. By Remark 4.1.7, since $(G \ltimes X)^{(0)} \cong X$ is totally disconnected, $G \ltimes X$ also satisfies condition (i) of Definition 4.1.6, and is seen to be almost finite.

Now assume that G is ample and $G \ltimes X$ is almost finite. Fix a compact subset $C \subset G$. We aim to construct a (C, ϵ)-invariant open castle which satisfies conditions (i) and (ii) from the final part of Definition 4.3.13. By almost finiteness of $G \ltimes X$, since $C \times X$ is a compact subset of $G \ltimes X$, there exists a $(C \times X, \epsilon)$ elementary subgroupoid of $G \ltimes X$. Since G is ample, we can apply Theorem 4.3.15 to obtain a (C, ϵ)-invariant clopen tower decomposition of α. Thus, we have obtained a (C, ϵ)-invariant open castle which covers X, so that condition (ii) from Definition 4.3.13 is automatically satisfied. It only remains to show that we can arrange for the levels of this castle to have diameter smaller than ϵ, which follows from Lemma 4.3.14 since X is totally disconnected. □

4.4 Almost finiteness of tiling groupoids

We now present a viewpoint of tiling groupoids as transformation groupoids associated to a special sort of groupoid action, and prove that they are almost finite in the sense of Definition 4.1.6.

Definition 4.4.1 ([22, Definition 2.1]). Let $d \in \mathbb{N}$. Let φ be a free action of $\mathbb{R}^d$ on a compact, metrisable space Ω. We call a closed subset $X \subset \Omega$ a *flat Cantor transversal* if it satisfies

(i) X is homeomorphic to a Cantor space;

(ii) for any $\omega \in \Omega$, there exists $p \in \mathbb{R}^d$ such that $\varphi^p(\omega) \in X$;

(iii) there exists a positive real number $M > 0$ such that

$$C_M = \{\varphi^p(x) \mid x \in X, p \in B_M(0)\}$$

is open in Ω, and

$$X \times B_M(0) \ni (x, p) \mapsto \varphi^p(x) \in C_M$$

is a homeomorphism; and

(iv) for any $x \in X$ and $r > 0$, there exists an open neighbourhood $U \subset X$ of x such that, for every $y \in U$, $\{p \in B_r(0) \mid \varphi^p(x) \in X\} = \{p \in B_r(0) \mid \varphi^p(y) \in X\}$.

The following appears as [41, Remark 6.4] (see also the proof of [41, Lemma 6.3]).

Lemma 4.4.2 ([41, Remark 6.4]). *Suppose that $\varphi : \mathbb{R}^d \curvearrowright \Omega$ is a free action on a compact metrisable space, and let $X \subset \Omega$ be a flat Cantor transversal. Construct an étale principal groupoid G as in [22] as follows:*

$$G \coloneqq \{(x, \varphi^p(x)) \mid p \in \mathbb{R}^d \text{ and } x, \varphi^p(x) \in X\}.$$

Then G is almost finite.

Notice that when we consider the free action of $\mathbb{R}^d$ on the continuous hull of an aperiodic and repetitive tiling with finite local complexity, and set $X = \Omega_{\text{punc}}$, the groupoid G constructed in the lemma above is isomorphic to R_{punc}. So, in order to prove that R_{punc} is almost finite, it suffices to prove the following lemma.

Lemma 4.4.3. *Let Ω be the continuous hull of an aperiodic and repetitive tiling with FLC. Then $\Omega_{\text{punc}} \subset \Omega$ is a flat Cantor transversal.*

Proof. By Theorem 3.3.9, Ω_{punc} is homeomorphic to a Cantor set. Given $T \in \Omega$, and denoting by $T(0)$ any choice of tile in T which contains the origin, and by $x(t)$ the puncture location of a tile $t \in T$, observe that $\varphi^{-x(T(0))}(T) = T - x(T(0)) \in \Omega_{\text{punc}}$, so (ii) is satisfied.

By Lemma 3.3.3, the set of punctures in any tiling in Ω forms a Delone set in $\mathbb{R}^d$, so it is uniformly discrete. Furthermore, all the patches in any tiling in Ω_{punc} already appear in the initial tiling that we chose, so we can choose the same constant which witnesses uniform discreteness in all of these Delone sets. Take $M = r/2$, where r is the chosen constant witnessing uniform discreteness, so that the distance between any two distinct punctures in any tiling is greater than r. Let $C_M = \{T' + p \mid T' \in \Omega_{\text{punc}}, p \in B_M(0)\}$ as in Definition 4.4.1.

We check that C_M is open. Take a tiling $T \in C_M$. Say that $T = T' + p$ for some $T' \in \Omega_{\text{punc}}$ and $p \in B_M(0)$. Take $0 < \epsilon < (M - |p|)/2$ to be small enough that $\epsilon^{-1} - \epsilon > M$. Then, when $d(T, T'') < \epsilon$, we can find vectors $x, y \in \mathbb{R}^d$ with $|x|, |y| \leq \epsilon$ such that $(T - x) \sqcap B_{\epsilon^{-1}}(0) = (T'' - y) \sqcap B_{\epsilon^{-1}}(0)$. Since $T - p \in \Omega_{\text{punc}}$, T has a puncture at $p \in B_M(0)$, so $T - x$ has a puncture at $p - x \in B_{M+\epsilon}(0) \subset B_{\epsilon^{-1}}(0)$. Since $T - x$ and $T'' - y$ agree on this latter ball, we see that $T'' - y$ also has a puncture at $p - x$, so that T'' has a puncture at $p - x + y$. Observe that

$$|p - x + y| \leq |p| + |y| + |x| \leq |p| + 2\epsilon < |p| + 2(M - |p|)/2 = M.$$

Thus, if $v = p - x + y$, then $T'' - v \in \Omega_{\text{punc}}$ and $v \in B_M(0)$, which shows that $T'' \in C_M$, so that $B_\epsilon(T) \subset C_M$. This proves that C_M is open.

The map defined in (iii) is just a restriction of the (continuous) right-action map $\Omega \times \mathbb{R}^d \to \Omega$, so it is continuous onto its image. It is surjective by the definition of C_M.

Suppose that $T, T' \in \Omega_{\text{punc}}$ and $p, p' \in B_M(0)$ have $\varphi^p(T) = \varphi^{p'}(T')$, so that $T+p = T'+p'$. This implies that $T' = T + p - p' = T - (p' - p)$. Since $T' \in \Omega_{\text{punc}}$, there is a puncture on the origin of $T - (p' - p)$, and hence there is a puncture on $p' - p$ in T. On the other hand, $T \in \Omega_{\text{punc}}$, so there is a puncture on the origin of T. Observe that $|p' - p| < 2M = r$, so that we have obtained two punctures (on 0 and on $p' - p$ in T) with a distance smaller than r between them. Therefore, these punctures must be equal, so that $p' - p = 0$, and we see that $T' = T - (p' - p) = T$. This proves that the map is injective.

To conclude the proof that this continuous bijective map is a homeomorphism, we prove that it is open. It suffices to show that the images of subsets of the form $U(P,t) \times B_r(x)$ are open in Ω. So, choose some $T' \in U(P,t) \subset \Omega_{\text{punc}}$ and $p \in B_r(x) \subset B_M(0)$, take $T = \varphi^p(T')$, and choose $0 < \epsilon < (r - |p - x|)/2$ to be small enough that $P \subset B_{\epsilon^{-1}-\epsilon-|p|}(x(t))$. Choose $T'' \in \Omega$ such that $d(T, T'') < \epsilon$. Then there exist vectors $v_1, v_2 \in \mathbb{R}^d$ with $|v_1|, |v_2| \leq \epsilon$ such that $(T'' + v_2) \sqcap B_{\epsilon^{-1}}(0) = (T + v_1) \sqcap B_{\epsilon^{-1}}(0) = (T' + p + v_1) \sqcap B_{\epsilon^{-1}}(0) = (T' \sqcap B_{\epsilon^{-1}}(-p - v_1)) + p + v_1$. Observe that $T' \sqcap B_{\epsilon^{-1}}(-p - v_1) \supset T' \sqcap B_{\epsilon^{-1}-\epsilon-|p|}(0) \supset P - x(t)$, which implies that $T'' + v_2$ contains a copy of $P - x(t) + p + v_1$. This implies that $T'' + v_2 - v_1 - p$ contains a copy of $P - x(t)$, so that $T'' + v_2 - v_1 - p \in U(P,t)$. Observe that $|-(v_2 - v_1 - p) - x| \leq 2\epsilon + |p - x| < r$, so that $-(v_2 - v_1 - p) \in B_r(x)$. This shows that T'' is in the image of $U(P,t) \times B_r(x)$. We conclude that $B_\epsilon(T)$ is a subset of the image of $U(P,t) \times B_r(x)$, so this latter set is open in Ω. Thus, the map from (iii) is a homeomorphism.

Finally, for property (iv), choose $T \in \Omega_{\text{punc}}$ and $r > 0$. Let $P \subset T$ be a patch which contains $B_r(0) \subset T$, and set $U = U(P,t)$. Then whenever $T' \in U$, we see that T and T' agree on $B_r(0)$, and thus the set of punctures in T which lie within $B_r(0)$ is the same subset of $\mathbb{R}^d$ as the set of punctures in T' which lie within $B_r(0)$. Since $\varphi^p(T) = T + p \in \Omega_{\text{punc}}$ if and only if there is a puncture at $-p$ in T, this shows that

$$\{p \in B_r(0) \mid \varphi^p(T) \in \Omega_{\text{punc}}\} = \{p \in B_r(0) \mid \varphi^p(T') \in \Omega_{\text{punc}}\}. \qquad \square$$

We can alternatively think of the groupoid $G \cong R_{\text{punc}}$ from above as the transformation groupoid $R_{\text{punc}} \ltimes \Omega_{\text{punc}}$ associated to the canonical action $R_{\text{punc}} \curvearrowright \Omega_{\text{punc}}$. Lemmas 4.4.2 and 4.4.3 then show that the transformation groupoid $R_{\text{punc}} \cong G \cong R_{\text{punc}} \ltimes \Omega_{\text{punc}}$ is almost finite. Although we could now directly apply Theorem 4.3.15 to this viewpoint of tiling groupoids to show that the action $R_{\text{punc}} \curvearrowright \Omega_{\text{punc}}$ is almost finite, we instead wish to argue directly, and specialise the proof of Lemma 4.3.14 to the tiling situation. The reasons to do this are threefold. It allows us to present a rather concrete, geometrical realisation of the ideas contained in the proof, it highlights the methods used when working with our main examples, and, most importantly, it will allow us to show that we can choose the towers to be formed of sets from the bases of the topologies on Ω_{punc} and R_{punc}.

Lemma 4.4.4. *Let T be an aperiodic and repetitive tiling with FLC. Then the tiling groupoid $R_{\text{punc}} \cong R_{\text{punc}} \ltimes \Omega_{\text{punc}}$ is almost finite (see Definition 4.1.6). Furthermore, if $C \subset R_{\text{punc}}$ is compact and $\epsilon > 0$, then the subsets $F_j^{(i)}$ appearing in the definition of the fundamental domain (Definition 4.1.4) of any $(C \times \Omega_{\text{punc}}, \epsilon)$-invariant elementary subgroupoid of $R_{\text{punc}} \ltimes \Omega_{\text{punc}}$ can be chosen to have diameter smaller than ϵ.*

Proof. We have already shown that R_{punc} is almost finite in the sense of Definition 4.1.6, so all that remains is to show that we can arrange for the small diameter condition to be satisfied. We make use of the proof of Theorem 4.3.15 to phrase the almost finiteness of this groupoid in terms of clopen tower decompositions of Ω_{punc} (see Definition 4.3.13). We will use a similar argument as in the proof of Lemma 4.3.14 to show that we can iteratively modify any given tower decomposition to obtain the diameter condition.

Let $C \subset R_{\text{punc}}$ be compact and open, and let $\epsilon > 0$. Obtain a (C, ϵ)-invariant clopen tower decomposition of Ω_{punc}. Choose any tower (W, S) in this decomposition. We aim to modify the tower by splitting it into finitely many towers which are still (C, ϵ)-invariant, and such that the collection of the levels of the new towers is a partition of the union of the levels of (W, S), but so that the levels of the new towers all have diameter smaller than ϵ. Applying this procedure to all of the towers simultaneously will produce a clopen tower decomposition with the properties we seek.

Recall that S decomposes into finitely many compact open G-sets $S = \bigsqcup_{k=1}^{K} S_k$, and first consider the G-set S_1, which moves the base of the tower to the first level. Consider the collection of patches $\{T \sqcap B_{R_1}(0) \mid T \in \Omega_{\text{punc}}\}$ for a sufficiently large R_1, to be defined. By FLC, there are finitely many such patches $\{P_1^1, \dots, P_{N_1}^1\}$. Notice that each patch contains the origin, and that the collection $\{U(P_n^1, P_n^1(0))\}_{n=1}^{N_1}$ is pairwise disjoint and covers Ω_{punc}, where $P_n^1(0)$ denotes the (unique) tile in P_n^1 which contains the origin.

Since the G-set S_1 is compact and open in R_{punc}, and the collection of sets $V(P, t, t') \subset R_{\text{punc}}$ is a basis for the topology on R_{punc}, we can choose R_1 large enough so that there exists a subcollection $\{P_{n_1}^1, \dots, P_{n_{M_1}}^1\}$ of $\{P_1^1, \dots, P_{N_1}^1\}$, and a collection of tiles $\{t_{m,l}^1\}_{l=1,\dots,L_m} \subset P_{n_m}^1$ for each $m \in \{1, \dots, M_1\}$ such that $S_1 = \bigsqcup_{m=1}^{M_1} \bigsqcup_{l=1}^{L_m} V(P_{n_m}^1, P_{n_m}^1(0), t_{m,l}^1)$. Since $W = s(S_1)$, this implies that the base of the tower is $W = \bigsqcup_{m=1}^{M_1} U(P_{n_m}^1, P_{n_m}^1(0))$, and that the first level is $S_1 \cdot W = \bigcup_{m=1}^{M_1} \bigcup_{l=1}^{L_m} U(P_{n_m}^1, t_{m,l}^1)$.

Since S_1 is a G-set, and since, for any fixed $m \in \{1, \ldots, M_1\}$ and $l \neq l'$, we have

$$V(P^1_{n_m}, P^1_{n_m}(0), t^1_{m,l}) \cap V(P^1_{n_m}, P^1_{n_m}(0), t^1_{m,l'}) = \varnothing$$

and

$$s(V(P^1_{n_m}, P^1_{n_m}(0), t^1_{m,l})) = U(P^1_{n_m}, P^1_{n_m}(0)) = s(V(P^1_{n_m}, P^1_{n_m}(0), t^1_{m,l'})),$$

the injectivity of the source map of S_1 implies that all the tiles $t^1_{m,l}$ for $l \in \{1, \ldots, L_m\}$ must be equal, to $t^1_m \in P^1_{n_m}$, say. Then, in fact, we have $S_1 = \bigsqcup_{m=1}^{M_1} V(P^1_{n_m}, P^1_{n_m}(0), t^1_m)$.

Since, for $m \neq m'$, we have

$$V(P^1_{n_m}, P^1_{n_m}(0), t^1_m) \cap V(P^1_{n_{m'}}, P^1_{n_{m'}}(0), t^1_{m'}) = \varnothing,$$

the injectivity of the range map on S_1 tells us that the collection

$$\{r(V(P^1_{n_m}, P^1_{n_m}(0), t^1_m))\}_{m=1,\ldots,M_1} = \{U(P^1_{n_m}, t^1_m)\}_{m=1,\ldots,M_1}$$

must be pairwise disjoint.

By choosing R_1 to be large enough, we may assume that we have $B_{\epsilon^{-1}}(x(t^1_m)) \subset P^1_{n_m}$ for each $m \in \{1, \ldots, M_1\}$, so that the diameter of $U(P^1_{n_m}, t^1_m)$ is smaller than ϵ. Indeed, since the vectors of translation associated to elements of S_1 are already prescribed above, and since enlarging R_1 produces a new finite collection of patches, each of which contains some patch P_{n_m} from above, increasing R_1 corresponds to finding a partition of each $V(P^1_{n_m}, P^1_{n_m}(0), t^1_m)$ into subsets $\{V(P^1_{n_m,q}, P^1_{n_m}(0), t^1_m)\}_{q=1,\ldots,Q_m}$, where $P^1_{n_m} \subset P^1_{n_m,q}$. Since t^1_m does not depend on q, we simply ensure that $B_{\epsilon^{-1}}(x(t^1_m)) \subset P^1_{n_m,q}$.

Split the clopen tower (W, S) into the collection of "one-level" clopen towers $\{(W^1_m, H^1_m) \mid m = 1, \ldots, M_1\}$ by taking $W^1_m := W \cap U(P^1_{n_m}, P^1_{n_m}(0)) = U(P^1_{n_m}, P^1_{n_m}(0))$ to be the base of the m-th tower, and $H^1_m := S_1 \cap s^{-1}(W^1_m) = V(P^1_{n_m}, P^1_{n_m}(0), t^1_m)$ to be the shape of the m-th tower. The single level of this tower is $U(P^1_{n_m}, t^1_m)$, and, by our choice of R_1, it has diameter smaller than ϵ. In addition, as we saw above, the collection of the tower levels $\{H^1_m \cdot W^1_m\}_{m=1,\ldots,M_1} = \{U(P^1_m, t^1_m)\}_{m=1,\ldots,M_1}$ is pairwise disjoint, and so partitions the first level $S_1 \cdot W = \bigsqcup_{m=1}^{M_1} U(P^1_{n_m}, t^1_m)$ of the tower (W, S). So, from the "one-level" tower (W, S_1), we have obtained a collection of towers whose levels have diameter less than ϵ, and still partition $S_1 \cdot W$.

Now, we iterate this procedure. At the k-th step we replace W by the base of each of the towers (W^{k-1}_m, H^{k-1}_m), for $m = 1, \ldots, M_{k-1}$, which were constructed in the $(k-1)$-th step, in turn, and replace S_1 by S_k. We find $R_k \geq R_{k-1}$ large enough that there exists a subcollection $\{P^k_{n_1}, \ldots, P^k_{n_{M_k}}\}$ of the finite set of patches $\{P^k_1, \ldots, P^k_{N_k}\}$ of the form $T \sqcap B_{R_k}(0)$

such that $S_k = \bigsqcup_{m=1}^{M_k} V(P^k_{n_m}, P^k_{n_m}(0), t^k_m)$ (the fact that there is just one tile associated to each patch uses the fact that S_k is a G-set, and an argument as above). We further choose R_k to ensure that $B_{\epsilon^{-1}}(x(t^k_m)) \subset P^k_{n_m}$ for each $m \in \{1, \ldots, M_k\}$. Then, to form the towers for the k-th step, we take bases of the form $W^k_m = U(P^k_{n_m}, P^k_{n_m}(0))$. Observe that, for each $m \in \{1, \ldots, M_k\}$, we have $W^k_m \subset W^{k-1}_{\tilde{m}}$ for some $\tilde{m} \in \{1, \ldots, M_{k-1}\}$. We define the shape $H^k_m = (H^{k-1}_{\tilde{m}} \cup S_k) \cap s^{-1}(W^k_m)$. That is, we "pick up" all of the arrows that our construction had previously associated to this subset, and also include the new arrows contained in the subset $S_k \subset S$ of the shape of the tower (W, S) whose sources lie in this subset.

After finitely many iterations, the process terminates. At the end of the process, we have obtained an $R = R_K$ large enough to work for all of the sets S_k in the construction above simultaneously. In particular, the levels of each tower (W^K_m, H^K_m) for $m \in \{1, \ldots, M_K\}$ will have diameter smaller than ϵ, and will partition $\bigsqcup_{k=1}^{K}(S_k \cdot W)$. We claim that the shapes of the towers we constructed are still (C, ϵ)-invariant. Let $m \in \{1, \ldots, M_K\}$ and let $T \in s(H^K_m)$ be arbitrary. By construction, the collection of arrows from S with source T is equal to the set of arrows from H^K_m with source T. This, combined with the fact that S was (C, ϵ)-invariant, shows that H^K_m is (C, ϵ)-invariant too. □

Remark 4.4.5. By following the procedure in the proof above, we may assume, without loss of generality, that each tower (W, S) in a clopen tower decomposition of the action $R_{\text{punc}} \curvearrowright \Omega_{\text{punc}}$ has $W = U(P, t)$ and $S = \bigsqcup_{t' \in Q} V(P, t, t')$ for some patch P, some $t \in P$, and some subset $Q \subset P$. Given any $\epsilon > 0$, we may enlarge P to further assume that $B_{\epsilon^{-1}}(x(t')) \subset P$ for each $t' \in Q$.

4.5 Almost finiteness for countable graphs

Recently, Elek introduced an alternative notion of almost finiteness for group actions on the Cantor set [17, Definition 5.1]. The most attractive feature of this setting is the presence of a simple criterion for almost finiteness of the action. A groupoid is associated to the action, which is isomorphic to the usual transformation groupoid in the case that the action is free. When the group is finitely generated, this groupoid may be equipped with the structure of a graph with bounded vertex degrees. A notion of almost finiteness for these graphs is introduced [17, Definition 1.3], and a relatively simple sufficient criterion to imply this property is obtained [17, Theorem 10]. Elek shows that the almost finiteness of these graphs implies the almost finiteness of the action in the sense he defines [17, Proposition 5.2]. In addition, when the action is free, almost finiteness (of the action) in the sense defined by Elek [17, Definition 5.1] is equivalent to almost finiteness in the sense defined by Kerr [35, Definition 8.2] (Definition 4.2.8). Therefore, studying the graph gives us a way to test for almost finiteness of the action.

Definition 4.5.1. Let X be a set, and let $E \subset X \times X$ be a graph. We say that E is *countable* if X is countable. We say that E has *bounded vertex degrees* with vertex degree bound $d \in \mathbb{N}$ if, for every $x \in X$,

$$\deg(x) \coloneqq |\{y \in X \mid (x,y) \in E\}| \leq d.$$

For $d \in \mathbb{N}$, we denote by Gr_d the collection of countable graphs with vertex degree bound d. We will denote the set of vertices of a graph E by $V(E)$.

Definition 4.5.2. Let $E \in \mathrm{Gr}_d$ for some $d \in \mathbb{N}$, and let $H \subset V(E)$ be finite.

(i) The *boundary of H in E* is the set

$$\partial_E(H) \coloneqq \{x \in H \mid \text{there exists } y \in V(E) \setminus H \text{ such that } (x,y) \in E\}.$$

When the choice of graph is clear we will drop the E to denote $\partial_E(H)$ by $\partial(H)$.

(ii) The *isoperimetric constant of H in E* is given by

$$i_E(H) \coloneqq \frac{|\partial_E(H)|}{|H|}.$$

We now introduce the appropriate notion of almost finiteness for this setting.

Definition 4.5.3 ([17, Definition 1.3])**.** A collection $\mathcal{G} \subset \mathrm{Gr}_d$ is *almost finite* if, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for each $G \in \mathcal{G}$ we can find a partition $V(G) = \bigsqcup_{j=1}^{\infty} H_j$ such that, for any $j \geq 1$,

(i) $\mathrm{diam}_{d_G}(H_j) \leq C_\epsilon$; and

(ii) $i_G(H_j) \leq \epsilon$.

We call such a partition of $V(G)$ an (ϵ, C_ϵ)-partition. We say that a graph $G \in \mathrm{Gr}_d$ is almost finite if the class $\{G\} \subset \mathrm{Gr}_d$ is almost finite.

In fact, we will be interested in an a priori stronger property (but see [17, Conjecture 1.1]).

Definition 4.5.4. A class $\mathcal{G} \subset \mathrm{Gr}_d$ is called *distributed almost finite* if it is almost finite and if, for any $\epsilon > 0$, there exists a constant-time distributed algorithm (see [17, Section 7]) that computes an (ϵ, C_ϵ)-partition for any $G \in \mathcal{G}$.

The following example gives a link back to Definition 4.2.8.

Example 4.5.5. Let Γ be a finitely generated group with symmetric finite generating set Σ, and let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a continuous action on the Cantor set. Construct a graph $E_{\alpha,\Sigma}$ from the action α and generating set Σ as follows. Set $V(E_{\alpha,\Sigma}) = \mathcal{C}$. If $x, y \in \mathcal{C}$ are distinct, let $(x, y) \in E_{\alpha,\Sigma}$ if there exists $\sigma \in \Sigma$ such that $\sigma x = y$. Also, let $(x, x) \in E_{\alpha,\Sigma}$ if $\gamma x = x$ for every $\gamma \in \Gamma$. Observe that for each $x \in \mathcal{C}$, the component of $E_{\alpha,\Sigma}$ containing x is an element of $\mathrm{Gr}_{|\Sigma|}$. Then distributed almost finiteness of the collection of components of $E_{\alpha,\Sigma}$ implies almost finiteness of the action $\alpha : \Gamma \curvearrowright \mathcal{C}$ in the sense of [17, Definition 5.1] (see [17, Proposition 5.2] for the details). We record this observation in the following proposition.

Proposition 4.5.6 ([16, Proposition 5.2])**.** *Let Γ be a finitely generated group, let Σ be a finite generating set, and let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a continuous action on the Cantor set. Suppose that the collection of components of the graph $E_{\alpha,\Sigma}$ is a distributed almost finite class. Then α is almost finite in the sense of [17, Definition 5.1].*

The following definition is crucial for the main result of [17].

Definition 4.5.7. Let E be a graph with bounded vertex degrees, and let $D \in \mathbb{N}$. We say that E is *D-doubling* if for any $x \in V(E)$ and $s \geq 1$ we have

$$|B_{2s}(E, x)| \leq D|B_s(E, x)|,$$

where the balls are defined with respect to the metric d_E on $V(E)$.

The following provides a sufficient criterion for almost finiteness in this setting.

Theorem 4.5.8 ([17, Theorem 10])**.** *For any $D > 1$, the class of D-doubling countable graphs is distributed almost finite.*

As in [17, Section 5], we now construct a locally compact Hausdorff principal ample groupoid G_α from the action α as follows. We set $G_\alpha^{(0)} = \mathcal{C}$. The elements of G_α are given by the pairs $(x, y) \in \mathcal{C} \times \mathcal{C}$ such that $y = \gamma x$ for some $\gamma \in \Gamma$. The source and range maps are $s(x, y) = x$ and $r(x, y) = y$, and the multiplication is $(x, y)(y, z) = (x, z)$. In general, G_α will be different from the transformation groupoid $\Gamma \ltimes \mathcal{C}$ of example 2.1.4(iii), but they are isomorphic if the action is free. The difference between G_α and $\Gamma \ltimes \mathcal{C}$ is that G_α "forgets" the group element implementing the action, whereas $\Gamma \ltimes \mathcal{C}$ remembers it. For this reason, for any $x, y \in \mathcal{C}$, the collection of arrows in $\Gamma \ltimes \mathcal{C}$ with source x and range y, $(\Gamma \ltimes \mathcal{C})_x^y$, will be no smaller than $(G_\alpha)_x^y$.

Proposition 4.5.9 (c.f. [17, Proposition 5.1])**.** *Let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a free and continuous action of a countable group. Then the following are equivalent.*

(i) α is almost finite in the sense of [17, Definition 5.1].

(ii) G_α is almost finite in the sense of [41, Definition 6.2] (see Definition 4.1.6).

(iii) α is almost finite in the sense of [35, Definition 8.2] (see Definition 4.2.8).

Proof. The equivalence of (i) and (ii) is given by [17, Proposition 5.1], while the equivalence of (ii) and (iii) is Theorem 4.2.11. □

Corollary 4.5.10. *Let Γ be a finitely generated group, let Σ be a finite generating set, and let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a free and continuous action on the Cantor set. Suppose that there exists $D \in \mathbb{N}$ such that each component of the graph $E_{\alpha,\Sigma}$ is D-doubling. Then α is almost finite.*

Proof. Combine Theorem 4.5.8 with Propositions 4.5.6 and 4.5.9. □

4.6 Almost finiteness of graphs associated to tilings

We now discuss some applications of the theory of Section 4.5 to our main examples. The construction of the groupoid G_α from page 85 can easily be extended to allow for groupoid actions on the Cantor set, and we will again retrieve the transformation groupoid in the case that the action is free. We discuss the situation for tiling groupoids, which can be thought of as being "finitely generated" in a certain sense, and admit natural actions on the Cantor set. Following [17], we obtain a graph from the tiling groupoids and show that it is distributed almost finite using the D-doubling criterion (Theorem 4.5.8). Future work could explore the question of associating graphs to more general groupoid actions, and whether Elek's D-doubling criterion can be used to imply almost finiteness of the action in the groupoid setting.

We aim to follow a construction similar to Example 4.5.5 to obtain a graph $E_{\alpha,\Sigma}$ from an aperiodic, repetitive tiling of $\mathbb{R}^d$ with finite local complexity. In our case, the action $\alpha : R_{\text{punc}} \curvearrowright \Omega_{\text{punc}}$ is the canonical groupoid action of R_{punc} on its unit space (see Example 4.3.2(ii)). We think of R_{punc} as being generated by the subset $\Sigma \subset R_{\text{punc}}$ consisting of translates between distinct tiles which share a $(d-1)$-dimensional subset in the tiling. Observe that Σ is symmetric and, by FLC, ΣT is finite for each $T \in \Omega_{\text{punc}}$. This allows us to obtain a bounded vertex degree graph $E_{\alpha,\Sigma}$ in exactly the same way as in Example 4.5.5. Given a tiling $T \in \Omega_{\text{punc}}$, we denote the component of this graph which contains T by E_T. We think of the set of vertices of E_T as the subset of $\mathbb{R}^d$ consisting of punctures of tiles of T, with an edge between the punctures of distinct tiles if the tiles share a $(d-1)$-dimensional face in T.

In the remainder of the section, we use volume arguments to show that E_T is D-doubling, for some $D > 1$ which depends only on the hull Ω_{punc}. Since we already know that R_{punc} is almost finite, this provides a promising hint towards the extension of Corollary 4.5.10 to the setting of groupoid actions. The existence of such a result would provide a convenient sufficient condition for almost finiteness of free groupoid actions on the Cantor set which are "finitely generated" (in the sense that the generating set is finite at each unit).

Our volume arguments will work smoothly to obtain an upper bound on $|B_{2s}(G_\alpha, x)|$, but a priori it is not clear that a similar argument will work to establish a lower bound for $|B_s(G_\alpha, x)|$. The reason is that we may observe a vertex in our tiling which is contained in multiple triangular prototiles, which share edges and "wrap around" the vertex. If we form a patch from such tiles, this behaviour could prevent the minimal spanning distance across the patch from increasing quickly enough as we append tiles to the patch. More precisely, perhaps we could double the number of tiles included in the patch, while not increasing the maximal radius of a ball in $\mathbb{R}^d$ which is contained in the patch. It turns out that the FLC property prevents this from happening in the long-term. To see this, we will utilise a weaker version of [1, Lemma 2.2] (Lemma 4.6.4). First, we introduce some terminology we will need.

Definition 4.6.1. Let T be a tiling. An *adjacency* is a reflexive, symmetric binary relation $\sim$ on T which satisfies:

(i) for any tiles t, $t' \in T$, there exist $t_0, \dots, t_n \in T$ such that $t = t_0$, $t' = t_n$ and $t_i \sim t_{i+1}$ for every $i \in \{0, \dots, n-1\}$;

(ii) there exists some positive integer M_1 such that if $t \cap t' \neq \varnothing$, then the number n from condition (i) can be chosen to be no larger than M_1; and

(iii) there exists $R > 0$ such that if $t \sim t'$, then the Hausdorff distance $d_H(t, t') \leq R$.

Note that the relation between tilings that we used to define our tiling graph for a tiling T of $\mathbb{R}^d$ can be expressed in terms of tiles as follows:

$$t \sim t' \Leftrightarrow \dim(t \cap t') \geq d - 1. \tag{4.6.1}$$

For the remainder of this section we will denote the ball with centre $x \in \mathbb{R}^d$ and radius $R > 0$ by $B(x, R)$.

Lemma 4.6.2. *Let T be a tiling of $\mathbb{R}^d$ with FLC. Then the relation (4.6.1) defines an adjacency on T.*

Proof.

(i) Given $t, t' \in T$ consider the line segment L between the punctures $x(t)$ and $x(t')$. Notice that L only passes through finitely many tiles by FLC (because any ball around $x(t)$ whose radius is larger than the length of L contains L and also intersects only finitely many tiles). If L doesn't pass through any points which lie in three or more tiles, then all the points of L lie in the interiors or on $(d-1)$-dimensional faces of tiles. In this case, we can number the tiles in the order that L passes through them as it moves from t to t' to obtain the tiles from the condition.

If L does pass through a point x that lies in three or more tiles, consider the ball of radius ϵ around x for some $\epsilon > 0$, to be specified. By FLC, we can choose ϵ small enough that this ball only intersects the tiles that intersect at x. Then, replacing the segment of L that lies in this ball with a path contained in the ball through the interiors and $(d-1)$-dimensional faces of tiles allows the method described above to work. Repeat this procedure for each point at which this behaviour occurs (there will be only finitely many due to FLC) to finish.

(ii) If $\dim(t \cap t') \geq d-1$, then we can choose $n = 1$. So, suppose $\dim(t \cap t') < d-1$. We may choose a point $x \in t \cap t'$ and $R_{t,t'} > 0$ large enough that $t, t' \subset B(x, R_{t,t'})$. Since this ball is convex, we can use a method similar to part (i) to find a path of adjacent tiles between t and t' inside this ball. By FLC, there is a finite number $M_{t,t'}$ of tiles intersecting this ball, so the length of the path is bounded by $M_{t,t'}$. We now repeat this procedure at each location in the tiling where two tiles intersect. Since the radii of the balls can be chosen to depend only on the prototiles in the configuration (e.g. choosing $R_{t,t'}$ to be twice the diameter of the larger prototile), and since we have only finitely many prototiles, all of these radii are bounded by some $R > 0$. By FLC, there is a uniform bound on the number of tiles intersecting any ball $B(x, R)$ for $x \in \mathbb{R}^d$. Set M_1 to be the value of this uniform bound.

(iii) By FLC there exists a prototile of maximal diameter $D_{\max}$, so we may choose $R = 3D_{\max}$. □

Definition 4.6.3. Let T be a tiling equipped with an adjacency $\sim$. For $t \in T$ and $n \in \mathbb{N}$, the n^{th} *corona* $t^{(n)}$ of t is defined recursively by:

$$t^{(0)} := t, \qquad t^{(n)} := \{t'' \in T \mid \text{there exists } t' \in t^{(n-1)} \text{ such that } t'' \sim t'\}.$$

Lemma 4.6.4 ([1, Lemma 2.2]). *Suppose T is a tiling of $\mathbb{R}^d$ with FLC, and let $\sim$ be an adjacency on T. Then there exists a constant $c > 0$ (which depends only on T and $\sim$) such that, for every $t \in T$ and every $x \in t$, we have, for sufficiently large s, that*

$$B\left(x, \frac{s}{c}\right) \subset t^{(s)} \subset B(x, cs).$$

Remark 4.6.5. The original statement replaces the FLC condition with the requirement for T to be a Delone Tiling, that is, that T satisfies the following axiom of Grünbaum-

Shephard [26]:

(N) There exist two constants a, $A > 0$ such that for every $t \in T$ there exists a ball of radius a inside t and a ball of radius A containing t.

Since FLC tilings have finitely many prototiles, it is easy to see that all FLC tilings satisfy this axiom.

Proof of Lemma 4.6.4. We present the same proof appearing in [1], but elaborate on the details.

Choose an arbitrary $t \in T$ and $x \in t$. By FLC, there exists a prototile of maximal diameter $A > 0$. Combining this with the constant R from condition (iii) of adjacencies, we have that

$$t^{(0)} = t \subset B(x, 2A), \qquad t^{(1)} \subset B(x, R + 2A).$$

Indeed, condition (iii) of adjacencies implies that if $t^{(n)} \subset B(x, nR + 2A)$ then $t^{(n+1)} \subset B(x, (n+1)R + 2A)$. By induction, we have for $s \geq 2$ that

$$t^{(s)} \subset B(x, sR + 2A) \subset B(x, s(R + A))$$

so setting $c \geq R + A$ makes the right-hand inclusion hold. Note that this constant depends only on T and $\sim$, taking into account the maximal prototile diameter and the constant from condition (iii) of the adjacency.

On the other hand, for each $s \in \mathbb{N}$, we can use compactness to find a covering of the Euclidean boundary $\partial(t^{(s)})$ by finitely many balls $\{B(x_i, 1)\}_{i=1,\dots,k}$ such that $x_i \in t^{(s)}$. Furthermore, we may choose the balls to cover the set

$$\partial(t^{(s)})\left[\frac{1}{2}\right] := \left\{x \in \mathbb{R}^d \mid \text{there exists } y \in \partial(t^{(s)}) \text{ such that } \|x - y\| \leq \frac{1}{2}\right\},$$

because $\partial(t^{(s)})\left[\frac{1}{2}\right]$ is compact, and is covered by the collection $\{B(x, 1) \mid x \in t^{(s)}\}$.

By FLC, there is a uniform bound M_0 on the number of tiles in T intersecting each ball of radius 1, so each of $B(x_i, 1)$ intersects no more than M_0 tiles. For each $i \in \{1, \dots, k\}$ let $t_i \in t^{(s)}$ be a tile such that $x_i \in t_i$. By applying condition (ii) of adjacencies M_0 times starting from t_i, we see that

$$B(x_i, 1) \subset t_i^{(M_0 M_1)} \subset t^{(s + M_0 M_1)}.$$

Therefore, for each $t \in T$ and $s \in \mathbb{N}$, we have that

$$t^{(s)}\left[\frac{1}{2}\right] := \left\{x \in \mathbb{R}^d \mid \text{there exists } y \in t^{(s)} \text{ such that } \|x - y\| \leq \frac{1}{2}\right\} \subset t^{(s + M_0 M_1)}.$$

In other words, taking M_0M_1 coronas increases the diameter of a patch by at least 1, where M_0 is the FLC constant associated to balls of radius 1. In order to apply this to obtain the left-hand inclusion, given $s \in \mathbb{N}$, write

$$s = b_0 M_0 M_1 + b_1,$$

where $b_0, b_1 \in \mathbb{N}$, and $0 \leq b_1 < M_0M_1$. Then, by iterating the above inclusion, we obtain

$$t^{(s)} \supset t^{(s-M_0M_1)}\left[\frac{1}{2}\right] \supset t^{(s-2M_0M_1)}[1] \supset \cdots \supset t^{(b_1)}\left[\frac{b_0}{2}\right] \supset B\left(x, \frac{b_0}{2}\right).$$

Choose $c \geq 4M_0M_1$, so that

$$\frac{s}{c} \leq \frac{s}{4M_0M_1} = \frac{bM_0M_1 + b_1}{4M_0M_1} = \frac{b_0}{4} + \frac{b_1}{4M_0M_1}.$$

Since $b_1 < M_0M_1$, we have $\frac{b_1}{4M_0M_1} < 1/4$, so that

$$\frac{s}{c} \leq \frac{b_0}{4} + \frac{1}{4}.$$

When $s = b_0M_0M_1 + b_1$ is sufficiently large, we have $\frac{b_0}{4} + \frac{1}{4} < \frac{b_0}{2}$. Indeed, this is true whenever $b_0 > 1$. So, when $s \geq 2M_0M_1$ and $c \geq 4M_0M_1$, the following inclusion holds:

$$B\left(x, \frac{s}{c}\right) \subset B\left(x, \frac{b_0}{2}\right) \subset t^{(s)}.$$

Once again, notice that this constant c only depends on the tiling T (via the FLC constant M_0) and the adjacency $\sim$ (via M_1 which arises from the second condition of adjacencies). To conclude the proof, set $c \geq \max(R + A, 4M_0M_1)$. □

Theorem 4.6.6. *Let T be a repetitive and aperiodic tiling with FLC. Then there exists some constant $D > 0$ such that the tiling graph E_T is D-doubling. Hence, E_T is almost finite.*

Proof. We wish to find $D > 1$ so that, when s is sufficiently large and $T' \in \Omega_T$, we have

$$\frac{|B_{2s}(E_T, T')|}{|B_s(E_T, T')|} \leq D.$$

Thus, we seek an upper bound on $|B_{2s}(E_T, T')|$ and a lower bound on $|B_s(E_T, T')|$.

To obtain the upper bound, we compute the maximum volume of a patch associated to a ball $B_{2s}(E_T, T')$. By this, we mean the volume of the patch made up of the tiles in T' which sit at the origin of some tiling in the ball. This patch can alternatively be described as the $2s^{\text{th}}$ corona of the tile sitting at the origin in T'. By FLC, there exists a prototile of maximal diameter $D_{\max}$, and the volume of the patch we just described is bounded by the case that each tile is a ball of this diameter. Since $B_{2s}(E_T, T')$ contains strings of tiles which are $2s+1$ tiles long starting from the origin tile of T', we have a bound on the diameter of the patch of $2(2s+1)D_{\max}$. If x is any point contained within the origin tile of T', then the patch is contained in $\overline{B_{(2s+1)D_{\max}}(x)}$. Denote by V_d the volume of the unit ball in $\mathbb{R}^d$. Then the volume of our patch is bounded by $V_d((2s+1)D_{\max})^d$. To link back to $|B_{2s}(E_T, T')|$, we must consider what the largest number of tiles that could fit into this patch of maximal volume could be. By FLC, there exists a prototile of minimal volume $V_{\min} > 0$, and thus the bound becomes

$$|B_{2s}(E_T, T')| \leq \frac{((2s+1)D_{\max})^d V_d}{V_{\min}}. \tag{4.6.2}$$

The lower bound is obtained similarly. By Lemma 4.6.4, there exists $c > 0$ depending only on T (since our adjacency is fixed) such that, for any $t \in T'$ and large enough s,

$$B\left(x, \frac{s}{c}\right) \subset t^{(s)}.$$

Note that the patch prescribed by $B_s(E_T, T')$ is precisely the s^{th} corona of the tile $T'(0)$ in T' which contains the origin. The volume of $B\left(x, \frac{s}{c}\right)$ is given by $V_d\left(\frac{s}{c}\right)^d$, so we have

$$\mathrm{Vol}(T'(0)^{(s)}) \geq V_d \frac{s^d}{c^d}.$$

On the other hand, if $T'(0)^{(s)} = \bigcup_{i=1}^{|B_s(E_T,T')|} t_i$, and $V_{\max} > 0$ is the maximal volume of a prototile, then

$$\mathrm{Vol}(T'(0)^{(s)}) = \sum_{i=1}^{|B_s(E_T,T')|} \mathrm{Vol}(t_i) \leq |B_s(E_T, T')| V_{\max}.$$

Putting this together yields

$$V_d \frac{s^d}{c^d} \leq |B_s(E_T, T')| V_{\max}$$

so that

$$|B_s(E_T, T')| \geq \frac{s^d V_d}{V_{\max} c^d}.$$

Combining this with (4.6.2), we obtain

$$\frac{|B_{2s}(E_T, T')|}{|B_s(E_T, T')|} \leq \left(2 + \frac{1}{s}\right)^d \frac{D_{\max}^d V_{\max} c^d}{V_{\min}} \leq \frac{(3D_{\max}c)^d V_{\max}}{V_{\min}},$$

and the right-hand side is a constant depending only on the tiling T. $\square$

Chapter 5

Groupoid crossed products

In this chapter, we present the construction of a C^*-algebra from data contained in the action of a groupoid on a space, as introduced in [54]. The resulting algebra is known as a *crossed product*. The construction works in significantly more generality than we present here, but, for brevity, we stick to our case of interest. In particular, throughout the chapter, we make the blanket assumption that all groupoids are locally compact, Hausdorff, étale, and have compact unit spaces, unless otherwise specified. As such, we are sweeping many technical details present in the general case under the rug, preferring to give a grounding of the key ideas of the theory. For more complete general treatments, see [44], [23].

Familiarity with group crossed products is not strictly necessary to understand the material in this chapter, but will certainly be helpful. There are many excellent references available – see, for example, [80], [50]. In the case of group crossed products, one begins by constructing a C^**-dynamical system*, which consists of a (locally compact) group Γ, a C^*-algebra A, and a continuous homomorphism $\alpha : \Gamma \to \mathrm{Aut}(A)$. We think of the group as acting on the algebra via $\gamma \cdot a = \alpha_\gamma(a)$. Roughly, for some Hilbert space H, one obtains a unitary representation of the group on H, and a representation of the algebra on H, and requires that these representations should interact in a manner that implements the dynamics of the system. These representations are combined to form a representation of the set $C_c(\Gamma, A)$ of continuous compactly-supported functions $\Gamma \to A$ on H. Completing $C_c(\Gamma, A)$ in the universal norm arising from these combined representations yields a C^*-algebra, $A \rtimes_\alpha \Gamma$, known as the *(full) crossed product*. One can also complete in the norm arising from the regular representation of Γ, yielding the *reduced crossed product*, $A \rtimes_{\alpha,r} \Gamma$. In the case that Γ is amenable, these constructions coincide [80, Theorem 7.13], and the converse also holds by results in the paragraph at the end of this introduction. If it is clear from context what the action, α, is, then we often drop this symbol from the notation to denote the full crossed product by $A \rtimes \Gamma$, and reduced crossed product by $A \rtimes_r \Gamma$.

The groupoid case follows a similar setup, with a wrinkle (which by now the reader should anticipate). As has been a recurring theme with groupoids, their fibred structure means that they must act on fibred spaces. This limits the C^*-algebras A on which a particular groupoid G can act. In particular, we require there to be some fibration of A over the unit space of G into a bundle $\mathcal{A} = \bigsqcup_{u \in G^{(0)}} A(u)$. The natural way to represent a groupoid is not on a single Hilbert space, but rather to consider a bundle of Hilbert spaces over the unit space of G. Then an element $g \in G$ can be implemented as a unitary operator from the Hilbert space fibre at $s(g)$ to that at $r(g)$. The representation of the algebra must be compatible with this structure, decomposing into representations of each fibre $A(u)$ onto the corresponding Hilbert space fibre. The replacement for $C_c(\Gamma, A)$ in this setting should consist of a collection of sections of a bundle over G. Since the only bundles in sight are over $G^{(0)}$, we take the pull-back of $\mathcal{A}$ by the range map of G. This allows us to consider the set of (compactly supported) sections of the pull-back bundle, $\Gamma_c(G, r^*\mathcal{A})$. Taking cues from the group case, we can combine the representations of G and A to yield a representation of $\Gamma_c(G, r^*\mathcal{A})$ and complete to get a groupoid crossed product. As before, there is a choice of norm depending on which representations we choose to consider, which again leads to a *full crossed product*, and a *reduced crossed product*.

The concept of amenability for locally compact groupoids is more complex than the group case (see [4] for a thorough reference), featuring at least three notions of differing strength. From strongest to weakest, these are topological amenability (sometimes simply referred to as "amenability" in the literature), measurewise amenability, and metric amenability (see [68] for this last notion, which states that the full and reduced groupoid C^*-algebras are isomorphic, and is also referred to as the *weak containment property*). We have the following relationships between these properties.

- Topological amenability implies measurewise amenability. (This is apparent from the definitions. See also [4, Proposition 3.3.5]).
- Measurewise amenability implies metric amenability [4, Proposition 6.1.8].
- In the étale case, measurewise amenability implies topological amenability (in fact, this is true in even more generality – see [4, Theorem 3.3.7]).
- Metric amenability does *not* imply measurewise amenability in general, even for étale groupoids [79]. However, under the assumption of *inner exactness*, this implication does hold [3, Theorem 2.10].

When G is locally compact, the full and reduced crossed products coincide if and only if G is measurewise amenable [3, Corollary 2.11]. If G is a locally compact group, [4, Theorem 3.3.7 and Example 3.3.10(2)] show that G is measurewise amenable if and only if it is topologically amenable, which coincides with the usual notion of amenability for groups. Since the groupoid crossed products of a group are isomorphic to the usual crossed products [23, Proposition 4.4], this establishes the converse to [80, Theorem 7.13] mentioned in the previous paragraph.

5.1 $C_0(Y)$-algebras

The following restricted class of C^*-algebras are those able to be acted upon by a given groupoid.

Definition 5.1.1 ([23, Definition 3.12])**.** Let Y be a locally compact Hausdorff space. We say that a C^*-algebra A is a $C_0(Y)$-*algebra* if there exists a $*$-homomorphism Φ_A from $C_0(Y)$ into the centre of the multiplier algebra $ZM(A)$ which is nondegenerate in the sense that the set

$$\Phi_A(C_0(Y)) \cdot A \coloneqq \operatorname{span}\{\Phi_A(f)a \mid f \in C_0(Y), a \in A\}$$

is dense in A.

We will ignore much of the generality offered by the theory and instead stick to developing the following motivating example.

Proposition 5.1.2. *Suppose $G \curvearrowright X$ is a continuous action of a groupoid on a compact Hausdorff space. Then $A = C(X)$ is a $C_0(G^{(0)})$-algebra.*

Proof. Since the action is continuous, the associated surjection $r_X : X \to G^{(0)}$ is continuous. Since A is unital and commutative, $ZM(A) = A$, so we are looking for a $*$-homomorphism $\Phi_A : C(G^{(0)}) \to A$. Set $\Phi_A(f) = f \circ r_X$. One easily checks that this is a $*$-homomorphism. Furthermore, Since $C(G^{(0)})$ is unital, the unit of A is in $\Phi_A(C_0(G^{(0)}))$. It follows that Φ_A is nondegenerate in the required sense. □

Remark 5.1.3. Since $G^{(0)} = r_X(X)$ is the continuous image of a compact space, we are implicitly assuming that $G^{(0)}$ is compact. In this situation, the terminology "$C_0(G^{(0)})$-algebra" is sometimes replaced by "$C(G^{(0)})$-algebra".

We intend to form bundles over such algebras, so we introduce the precise sort of bundle we will be defining.

Definition 5.1.4 ([23, Definition 3.1, 3.4])**.** An *upper-semicontinuous Banach bundle* over a locally compact Hausdorff space X is a topological space $\mathcal{A}$ with a continuous, open surjection $p : \mathcal{A} \to X$ and Banach space structures on each fibre $\mathcal{A}_x \coloneqq p^{-1}(x)$ such that

(i) the map $a \mapsto \|a\|_{\mathcal{A}_{p(a)}}$ is upper-semicontinuous from $\mathcal{A}$ to $\mathbb{R}^+$;

(ii) for each $\lambda \in \mathbb{C}$, the map $a \mapsto \lambda a$ is continuous from $\mathcal{A}$ to $\mathcal{A}$;

(iii) the map $(a, b) \mapsto a + b$ is continuous from $\mathcal{A} * \mathcal{A} := \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid p(a) = p(b)\}$ to $\mathcal{A}$; and

(iv) if $\{a_i\}$ is a net in $\mathcal{A}$ such that $p(a_i) \to x$ and $\|a_i\| \to 0$, then $a_i \to 0_x$, where 0_x is the zero element in the fibre $\mathcal{A}_x$.

If we make the stronger assumption that each fibre is equipped with the structure of a C^*-algebra, and enforce the additional conditions

(v) the map $a \mapsto a^*$ is continuous from $\mathcal{A}$ to $\mathcal{A}$; and

(vi) the map $(a, b) \mapsto ab$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$;

then we say that $\mathcal{A}$ is an *upper-semicontinuous C^*-bundle.*

Definition 5.1.5 ([23, Definition 3.7])**.**

(i) Let $\mathcal{A}$ be an upper-semicontinuous Banach bundle over X. A *section* of $\mathcal{A}$ is a continuous map $f : X \to \mathcal{A}$ such that $p(f(x)) = x$ for every $x \in X$. We denote the space of sections by $\Gamma(X, \mathcal{A})$.

(ii) We say that $f \in \Gamma(X, \mathcal{A})$ *vanishes at infinity* if the set $\{x \in X \mid \|f(x)\| \geq \epsilon\}$ is compact for all $\epsilon \geq 0$. We denote the set of such sections by $\Gamma_0(X, \mathcal{A})$.

(iii) We say that $f \in \Gamma(X, \mathcal{A})$ has *compact support* if $\{x \in X \mid \|f(x)\| > 0\}$ is compact. We denote the set of sections with compact support by $\Gamma_c(X, \mathcal{A})$.

We equip $\Gamma(X, \mathcal{A})$ with the operations of pointwise addition and pointwise scalar multiplication. In addition, we equip $\Gamma_0(X, \mathcal{A})$ with the uniform norm $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$. If $\mathcal{A}$ is a C^*-bundle, then we also equip $\Gamma(X, \mathcal{A})$ with the operations of pointwise multiplication and involution.

We now detail the formation of such a bundle from the action of G on X. A similar construction will work for a general $C_0(Y)$-algebra by replacing $G^{(0)}$ with Y and ignoring the specifics. Consider the ideal $J_u \subset C_0(G^{(0)})$ of functions which vanish at $u \in G^{(0)}$. Denote $I_u := \overline{\Phi_A(J_u) \cdot A}$, so in the specific case of $A = C(X)$ with $C_0(G^{(0)})$-algebra structure arising from an action $G \curvearrowright X$, I_u will be the set of functions which vanish on $r_X^{-1}(u)$. Denote the quotient by $A(u) = A/I_u$ and the image of $a \in A$ under the quotient map by $a(u)$. The quotients $A(u)$ will form the fibres of the bundle which we will associate to our algebra, and in our case $A(u) = C(r_X^{-1}(u))$. We think of elements of $A(u)$ as functions $f : X \to \mathbb{C}$ such that $f(x) = 0$ if $x \notin r_X^{-1}(u)$. Denote $\mathcal{A} := \bigsqcup_{u \in G^{(0)}} A(u)$. We would like to think of the set of continuous sections $\Gamma_0(G^{(0)}, \mathcal{A})$ of the bundle which vanish at infinity as isomorphic to A via the map $a \mapsto (u \mapsto a(u))$. Of course, the definition of

$\Gamma_0(G^{(0)}, \mathcal{A})$ relies on the notion of continuity, and hence depends on the topology of $\mathcal{A}$. Fortunately, [23, Corollary 3.26] ensures that there is a (unique) topology on $\mathcal{A}$ for which this identification makes sense.

With the bundle structure in hand, we can now define what it means for a groupoid to act on an algebra.

Definition 5.1.6. Let G be a locally compact Hausdorff étale groupoid. Let A be a $C_0(G^{(0)})$-algebra, and let $\mathcal{A}$ be its associated upper-semicontinuous bundle. An *action* of G on A is a family of functions $\alpha = \{\alpha_g\}_{g\in G}$ such that

(i) for each $g \in G$, the map $\alpha_g : A(s(g)) \to A(r(g))$ is an isomorphism;

(ii) for all $(g, h) \in G^{(2)}$, $\alpha_{gh} = \alpha_g \circ \alpha_h$; and

(iii) the map $g \cdot a := \alpha_g(a)$ defines a continuous action of G on $\mathcal{A}$ (in the sense of Definition 4.3.1).

We refer to the triple (A, G, α) as a *groupoid dynamical system*. We say that (A, G, α) is *separable* if A is separable and G is second countable.

Example 5.1.7. Suppose that $\alpha : G \curvearrowright X$ is a continuous groupoid action on a compact Hausdorff space. We define the *induced action* of G on $C(X)$, which is also denoted by α, as follows. Given $f \in C(X)$ and $g \in G$, define

$$\alpha_g(f)(x) = \begin{cases} f(g^{-1}x) & \text{if } x \in r_X^{-1}(r(g)) \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\alpha_g(f)$ is *not* necessarily continuous on X. Rather, it is an element of $C(r_X^{-1}(r(g)))$. More properly, one can restrict each α_g to a map

$$\alpha_g : C(r_X^{-1}(s(g))) \to C(r_X^{-1}(r(g))),$$

whence we obtain a groupoid action of G on $C(X)$ in the sense of Definition 5.1.6.

5.2 Covariant representations

In this section, we will define the *covariant representations* which will be used to encode the dynamics of our system. These come in two halves – a representation of the groupoid, and a representation of the $C_0(Y)$-algebra. First, we introduce the objects onto which we will be representing (Definition 5.2.2). Roughly, the philosophy is that we will represent our bundle $\mathcal{A}$ of C^*-algebras onto a bundle of Hilbert spaces. For convenience, we recall some preliminary definitions.

Definition 5.2.1 ([80, Appendix D.2])**.**

(i) A *Polish space* is a topological space which is separable and completely metrisable. Given a Polish space P, we denote by $\mathscr{B}(P)$ the σ-algebra of Borel subsets of P. A subset A of a Polish space P is called *analytic* if there exists a continuous map from another Polish space Q into P such that $f(Q) = A$. Given an analytic subset A of P, the *relative Borel structure* on A arising from P is the σ-algebra $\mathscr{M} = \{A \cap B \mid B \in \mathscr{B}(P)\}$.

(ii) We say that measurable spaces $(Y, \mathscr{Y})$ and $(Z, \mathscr{Z})$ are *isomorphic* if there exists a bijective measurable map $f : Y \to Z$ such that f^{-1} is also measurable.

(iii) A measurable space $(Y, \mathscr{Y})$ is called an *analytic Borel space* if it is isomorphic to $(A, \mathscr{M})$, where A is an analytic subset of a Polish space and $\mathscr{M}$ is the relative Borel structure on A.

Definition 5.2.2 ([23, Definition 3.61])**.** Suppose $\mathfrak{H} = \{H(y)\}_{y \in Y}$ is a collection of separable nonzero complex Hilbert spaces indexed by an analytic Borel space Y. The *total space* is defined to be

$$Y * \mathfrak{H} := \{(y, h) \mid y \in Y, h \in H(y)\}$$

and is equipped with the obvious projection map $\pi : Y * \mathfrak{H} \to Y$. We say that $Y * \mathfrak{H}$ is an *analytic Borel Hilbert bundle* if it is equipped with a σ-algebra which makes it an analytic Borel space such that

a) π is measurable; and

b) there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of sections (called a *fundamental sequence* for $Y * \mathfrak{H}$) such that

   (i) the maps $\bar{f}_n : Y * \mathfrak{H} \to \mathbb{C}$ given by

   $$\bar{f}_n(y, h) = \langle f_n(y), h \rangle_{H(y)}$$

   are measurable (with respect to the Borel σ-algebra on $\mathbb{C}$) for each n;

   (ii) for each n and m, the map $Y \to \mathbb{C}$ given by

   $$y \mapsto \langle f_n(y), f_m(y) \rangle_{H(y)}$$

   is measurable; and

   (iii) the collection of functions $\{\pi\} \cup \{\bar{f}_n\}_{n \in \mathbb{N}}$ separates points of $Y * \mathfrak{H}$.

Remark 5.2.3. We are being deliberately sloppy with our notation for sections. Given a section f of $Y * \mathfrak{H}$, the image $f(y)$ of a point of Y is an element of $Y * \mathfrak{H}$ and thus really has the form $f(y) = (y, g_f(y))$ for some $g_f : Y \to \bigsqcup H(y)$. Of course, f is determined entirely by g_f, so we will continue to use the two interchangeably.

Notation 5.2.4. Given an analytic Borel Hilbert bundle $Y * \mathfrak{H}$, we denote the set of measurable sections $Y \to Y * \mathfrak{H}$ by $B(Y * \mathfrak{H})$.

5.2.1 Groupoid representations

For us, $G^{(0)}$ will form the base of an analytic Borel Hilbert bundle. An element $g \in G$ will be represented as a unitary transformation $U_g : H(s(g)) \to H(r(g))$ between the fibres at its range and source. Let us make this precise.

Notation 5.2.5. Given Hilbert spaces H_1 and H_2, denote the the collection of unitary transformations $U : H_1 \to H_2$ by $U(H_1, H_2)$.

Definition 5.2.6. If $Y * \mathfrak{H}$ is an analytic Borel Hilbert bundle, then its *isomorphism groupoid* is defined as

$$\mathrm{Iso}(Y * \mathfrak{H}) := \{(x, V, y) \mid x, y \in Y,\ V \in U(H(y), H(x))\}$$

equipped with the smallest σ-algebra such that the map $\mathrm{Iso}(Y * \mathfrak{H}) \to \mathbb{C}$ defined by $(x, V, y) \mapsto \langle V f(y), g(x)\rangle_{H(x)}$ is measurable (with respect to the Borel σ-algebra on $\mathbb{C}$) for all measurable sections $f, g \in B(Y * \mathfrak{H})$.

We define the set of composable pairs as

$$\mathrm{Iso}(Y * \mathfrak{H})^{(2)} := \{((x, V, y), (w, U, z)) \in \mathrm{Iso}(Y * \mathfrak{H}) \times \mathrm{Iso}(Y * \mathfrak{H}) \mid y = w\}$$

and the operations are given by

$$(x, V, y)(y, U, z) = (x, VU, z) \quad \text{and} \quad (x, V, y)^{-1} = (y, V^*, x).$$

Definition 5.2.7. Let G be a locally compact étale groupoid. A *groupoid representation* of G is a triple $(\mu, G^{(0)} * \mathfrak{H}, U)$ where μ is a (finite) Radon measure on $G^{(0)}$, $G^{(0)} * \mathfrak{H}$ is an analytic Borel Hilbert bundle, and $U : G \to \mathrm{Iso}(G^{(0)} * \mathfrak{H})$ is a measurable groupoid homomorphism (with respect to the Borel σ-algebra on G) such that, for each $g \in G$, there exists a unitary $U_g : H(s(g)) \to H(r(g))$ such that $U(g) = (r(g), U_g, s(g)) \in \mathrm{Iso}(G^{(0)} * \mathfrak{H})$.

Observe that, in order that U be a homomorphism, for each $u \in G^{(0)}$, U_u must be the identity operator on $H(u)$.

Remark 5.2.8. The measure μ in the above definition is usually assumed to be *quasi-invariant* (see Definition 5.2.17). This has nice theoretical consequences. For example, when G is Hausdorff, enforcing quasi-invariance of μ allows us to use a groupoid representation to obtain a representation of $C_c(G)$ on the direct integral of $G^{(0)} * \mathfrak{H}$ (see Definition 5.2.9). However, quasi-invariance isn't necessary for the given definition to make sense. Thus, we have opted to skip this technicality for now, and introduce it later when it becomes vital to our development. The interested reader can refer to [44, Section 7] for an exploration of some of the results we are forgoing by taking this approach.

5.2.2 $C_0(Y)$-linear representations

We now detail the representation of a $C_0(Y)$-algebra onto a Hilbert space arising from an analytic Borel Hilbert bundle. Due to the fibred structure of everything, the representation will decompose into representations on the level of each fibre. The precise details are fairly technical, so we refer the reader to [23, Section 3.3.2] for a treatment of all the relevant proofs, and to [80, Appendix F.3] for a more complete treatment of the theory as a whole.

To obtain a representation of the groupoid, we were able to implement $g \in G$ as a unitary between the relevant Hilbert spaces. We wish to follow a similar philosophy when representing the algebra. We have already seen that elements of A correspond to sections of $\mathcal{A} = \bigsqcup_{u \in G^{(0)}} A(u)$, so we must represent A as sections of the Hilbert bundle $G^{(0)} * \mathfrak{H}$. It will turn out that our representations will decompose fibrewise to yield representations of each $A(u)$ onto $H(u)$, and that we can also combine such fibrewise representations into a representation of A as sections of the Hilbert bundle (see Proposition 5.2.13 and Proposition 5.2.16).

We begin by pinning down the sections of $Y * \mathfrak{H}$ onto which A will be represented.

Definition 5.2.9 ([23, Definition 3.80])**.** Suppose $Y * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a measure on Y. Consider

$$\mathcal{L}^2(Y * \mathfrak{H}, \mu) := \left\{ f \in B(Y * \mathfrak{H}) \;\middle|\; \int_Y \|f(y)\|^2_{H(y)} d\mu(y) < \infty \right\}.$$

Equip $\mathcal{L}^2(Y * \mathfrak{H}, \mu)$ with the operations of pointwise addition and scalar multiplication and let $L^2(Y * \mathfrak{H}, \mu)$ be the quotient of $\mathcal{L}^2(Y * \mathfrak{H}, \mu)$ such that sections which agree μ-almost everywhere are identified. When equipped with the operations inherited from $\mathcal{L}^2(Y * \mathfrak{H}, \mu)$ and the inner product

$$\langle f, g \rangle = \int_Y \langle f(y), g(y) \rangle_{H(y)} d\mu(y),$$

$L^2(Y * \mathfrak{H}, \mu)$ becomes a Hilbert space, which we call the *direct integral* of $\mathfrak{H}$ with respect to μ.

Remark 5.2.10. Definition 5.2.9 contains a number of assertions, which we do not justify. Most are routine, besides the fact that the inner product we put on $L^2(Y * \mathfrak{H}, \mu)$ makes sense. The interested reader is directed to [80, Appendix F.2] for a full justification. In addition, we remark that $L^2(Y * \mathfrak{H}, \mu)$ will always be separable in our setting (see, for example, [80, Lemma F.16]).

Now we move on to the representations themselves. First, we make precise our comment about combining representations on each fibre into a representation of A. The following notion will be used to ensure that a collection of representations is in some sense compatible with the Borel structure.

Definition 5.2.11. Suppose $Y * \mathfrak{H}$ is an analytic Borel Hilbert bundle with a fundamental sequence $\{f_n\}$. A family of bounded linear operators $T(y) : H(y) \to H(y)$ is a *Borel field of operators* if the map

$$y \mapsto \langle T(y)(f_n(y)), f_m(y)\rangle_{H(y)}$$

is measurable with respect to the Borel σ-algebra on $\mathbb{C}$ for all m and n.

We can combine such a family of operators into a single operator, as follows.

Proposition 5.2.12 ([23, Proposition 3.91])**.** *Suppose $Y * \mathfrak{H}$ is an analytic Borel Hilbert bundle, and that μ is a σ-finite measure on Y. Let $\{T(y)\}$ be a Borel field of operators which is essentially bounded in the sense that*

$$\Lambda := \operatorname{ess\,sup}_{y \in Y} \|T(y)\| < \infty,$$

where ess sup *denotes the essential supremum. Then there exists a bounded linear operator on $L^2(Y * \mathfrak{H}, \mu)$ of norm Λ, called the direct integral of the $T(y)$ and denoted $\int_Y^{\oplus} T(y)d\mu(y)$, defined by*

$$\left(\int_Y^{\oplus} T(y)d\mu(y)\right)(f)(x) = T(x)f(x)$$

for all $f \in \mathcal{L}^2(Y * \mathfrak{H}, \mu)$ and $x \in Y$.

This machinery can be applied to form a single representation from an appropriate collection of representations on the fibres.

Proposition 5.2.13 ([23, Proposition 3.93])**.** *Suppose Y is a second countable locally compact Hausdorff space, A a separable $C_0(Y)$-algebra, $Y * \mathfrak{H}$ an analytic Borel Hilbert bundle, and μ a σ-finite Borel measure on Y. Given a collection of representations $\{\pi_y : A(y) \to B(H(y)) \mid y \in Y\}$ such that for each $a \in A$ the set $\{\pi_y(a(y)) \mid y \in Y\}$ is a Borel field of operators, we can form a representation*

$$\pi = \int_Y^{\oplus} \pi_y d\mu(y)$$

of A on $L^2(Y * \mathfrak{H}, \mu)$ called the direct integral *of the collection $\{\pi_y\}$, and defined for $a \in A$*

by

$$\pi(a) = \int_Y^{\oplus} \pi_y(a(y))d\mu(y).$$

We now consider the other direction, and turn to the construction of representations of A on $L^2(Y * \mathfrak{H}, \mu)$ which decompose into representations of each fibre. Since we can view $C_0(Y)$ as an ideal in any $C_0(Y)$-algebra A, we don't just want these representations of A to be multiplicative, but also to somehow respect $C_0(Y)$. We will need the following operators to make this $C_0(Y)$-multiplicativity precise.

Definition 5.2.14 ([23, Definition 3.85])**.** Let $Y * \mathfrak{H}$ be an analytic Borel Hilbert bundle and μ a finite measure on Y. An operator T on $L^2(Y * \mathfrak{H}, \mu)$ is called *diagonal* if there exists a bounded measurable function $\phi : Y \to \mathbb{C}$ such that

$$T(h)(y) = \phi(y)h(y)$$

for μ-almost every $y \in Y$. We denote the collection of such operators by $\Delta(Y * \mathfrak{H}, \mu)$. Given such a bounded measurable map ϕ, the associated diagonal operator is denoted by T_ϕ.

Now, as one might expect, we want to pull a function $\phi \in C_0(Y)$ out of the representation by applying T_ϕ.

Definition 5.2.15 ([23, Definition 3.98])**.** Let Y be a second countable locally compact Hausdorff space, A a $C_0(Y)$-algebra, $Y * \mathfrak{H}$ an analytic Borel Hilbert bundle, and μ a finite Borel measure on Y. We say that a representation $\pi : A \to B(L^2(Y * \mathfrak{H}, \mu))$ is $C_0(Y)$*-linear* if

$$\pi(\phi \cdot a) = T_\phi \pi(a)$$

for every $a \in A$ and $\phi \in C_0(Y)$.

As we mentioned at the start of the section, we wanted to get our hands on representations which decompose to give representations on the level of each fibre. Indeed, these $C_0(Y)$-linear representations are those that we seek.

Proposition 5.2.16 ([23, Proposition 3.99])**.** *Suppose Y is a second countable locally compact Hausdorff space, A a separable $C_0(Y)$-algebra, $Y * \mathfrak{H}$ an analytic Borel Hilbert bundle, and μ a finite Borel measure on Y. Given a $C_0(Y)$-linear representation $\pi : A \to L^2(Y * \mathfrak{H}, \mu)$, there exists, for each $y \in Y$, a (possibly degenerate) representation $\pi_y : A(y) \to B(H(y))$ such that, for each $a \in A$, the set $\{\pi_y(a(y))\}_{y \in Y}$ is an essentially bounded Borel field of operators, and*

$$\pi = \int_Y^{\oplus} \pi_y d\mu(y).$$

With this useful fact about $C_0(Y)$-linear representations in hand, it is particularly useful to note that every representation of a $C_0(Y)$-algebra is equivalent to a $C_0(Y)$-linear representation. See [23, Proposition 3.101] or [13, Theorem 8.3.2] for the details.

5.2.3 Covariant representations

Now that we have the two halves of a covariant representation, we will see how they interact to encode the dynamics of the system. It is in this section that quasi-invariance of the measure μ from Definition 5.2.7 will become crucial, so we finally make good on our promise to define it.

Definition 5.2.17 ([23, Definition 3.70])**.** Suppose G is a locally compact Hausdorff groupoid with Haar system $\lambda = \{\lambda^u\}$, where λ^u is supported on $G^u = r^{-1}(u)$. Denote the images of these measures under the inverse map of the groupoid by $\lambda_u = (\lambda^u)^{-1}$, so that λ_u is supported on $G_u = s^{-1}(u)$. Given a Radon measure μ on $G^{(0)}$, we define the *induced measures* ν and ν^{-1} to be the Radon measures on G defined by the equations

$$\nu(f) := \int_{G^{(0)}} \int_G f(g) d\lambda^u(g) d\mu(u) \quad \text{and}$$

$$\nu^{-1}(f) := \int_{G^{(0)}} \int_G f(g) d\lambda_u(g) d\mu(u)$$

for all $f \in C_c(G)$. That is, if $A \subset G$ is compact, we set $\nu(A) = \nu(1_A)$. We say that the measure μ is *quasi-invariant* if ν and ν^{-1} are mutually absolutely continuous. In this case, we write Δ for the Radon-Nikodym derivative $d\nu/d\nu^{-1}$, and call it the *modular function* of μ.

We remark that Δ can (and will) be chosen to be a groupoid homomorphism from G to $(\mathbb{R}_+, \times)$ [28, Corollary 3.14].

Example 5.2.18. When G is étale, the system of counting measures becomes a Haar system, under which the modular function $\Delta(g)$ can be thought of as giving the ratio of the weights that the measure μ assigns to $r(g)$ and $s(g)$. In particular, when $u \in G^{(0)}$, we obtain $\Delta(u) = 1$.

From now on, we wish to enforce the quasi-invariance of the measures μ in our groupoid representations. We will endeavour to indicate this everywhere we need it, but for the avoidance of doubt, from this point forward, one should consider this quasi-invariance as an assumption in Definition 5.2.7.

Definition 5.2.19. Suppose (A, G, α) is a separable groupoid dynamical system, and that μ is a quasi-invariant measure on $G^{(0)}$. A *covariant representation* $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ of (A, G, α) consists of a unitary representation $(\mu, G^{(0)} * \mathfrak{H}, U)$ of G, and a $C_0(G^{(0)})$-linear representation $\pi : A \to B(L^2(G^{(0)} * \mathfrak{H}, \mu))$. If $\{\pi_u\}$ is a decomposition of π and ν is the measure induced by μ, we require that there exists a ν-null set $N \subset G$ such that, for all $g \notin N$, we have

$$U_g \pi_{s(g)}(a) = \pi_{r(g)}(\alpha_g(a))U_g$$

for every $a \in A(s(g))$.

Remark 5.2.20. We will usually simply denote the covariant representation by (π, U) and take for granted that there is a quasi-invariant measure μ and an analytic Borel Hilbert bundle $G^{(0)} * \mathfrak{H}$ in the background.

5.3 Groupoid crossed products

Now that we have obtained a pair of representations which interact to implement the dynamics of our system, the next step in defining a crossed product is to combine the representations U and π into one. In analogue to the group case, one might anticipate that this combined representation will be an integrated form, and might expect it to represent sections of the bundle $\mathcal{A}$. However, we will first need to deal with the detail that U is defined on G, while sections of $\mathcal{A}$ are defined on $G^{(0)}$. To fix this incompatibility, we will need to be able to apply π to sections of a bundle over G. This leads us to the following.

Definition 5.3.1. Suppose X and Y are locally compact Hausdorff spaces, $\mathcal{A}$ is an upper-semicontinuous Banach bundle over X with associated projection map $p : \mathcal{A} \to X$, and $\tau : Y \to X$ is continuous. The *pull-back of* $\mathcal{A}$ *by* τ is defined to be the set

$$\tau^*\mathcal{A} \coloneqq \{(y, a) \in Y \times \mathcal{A} \mid \tau(y) = p(a)\}$$

equipped with the relative topology and projection map $q : \tau^*\mathcal{A} \to Y$ defined by $q(y, a) = y$.

It is a fact (see for example [23, Proposition 3.34]) that $\tau^*\mathcal{A}$ is an upper-semicontinuous Banach bundle over Y, and that if $\mathcal{A}$ is a C^*-bundle, then $\tau^*\mathcal{A}$ is too. We will make use of this by taking τ to be the range map of G.

In contrast to $\Gamma_c(G^{(0)}, \mathcal{A})$, we do not equip $\Gamma_c(G, r^*\mathcal{A})$ with the operations of pointwise multiplication and involution. Instead, we follow [44, Proposition 4.4] or [23, Proposition 3.54] and utilise the following convolution and involution which incorporate the action into the $*$-algebra structure:

$$f * g(\gamma) = \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \quad \text{and} \quad f^*(\gamma) = \alpha_\gamma(f(\gamma^{-1})^*).$$

Definition 5.3.2 ([23, Proposition 3.118])**.** Suppose (A, G, α) is a separable dynamical system, and that $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ is a covariant representation. Let $\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$ be a decomposition of π. Then there is a I-norm decreasing nondegenerate $*$-representation $\pi \rtimes U$ of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(G^{(0)} * \mathfrak{H}, \mu)$ called the *integrated form* of (π, U), and given by

$$\pi \rtimes U(f)(h)(u) = \int_G \pi_u(f(g))U_g h(s(g))\Delta(g)^{-\frac{1}{2}} d\lambda^u(g)$$

for $f \in \Gamma_c(G, r^*\mathcal{A})$, $h \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$ and $u \in G^{(0)}$.

Let us explain this definition a little. In the integral, the only g which contribute are those in $\mathrm{supp}(\lambda^u) = G^u$, so that $r(g) = u$. For such g, $h(s(g)) \in H(s(g))$ is sent into $H(r(g)) = H(u)$ by U_g. Then $f(g) \in A(r(g)) = A(u)$, so $\pi_u(f(g)) \in B(H(u))$ operates on $U_g h(s(g)) \in H(u)$. Thus, for each $g \in G^u$, we obtain an element of $H(u)$, which the integral combines to give a single element of $H(u)$.

We are finally in a position to define the (full) groupoid crossed product.

Definition 5.3.3 ([23, Definition 3.126])**.** Suppose (A, G, α) is a separable groupoid dynamical system. We define the *universal norm* on $\Gamma_c(G, r^*\mathcal{A})$ by

$$\|f\| := \sup\{\|\pi \rtimes U(f)\| \mid (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

The completion of $\Gamma_c(G, r^*\mathcal{A})$ in this norm is a C^*-algebra called the *groupoid crossed product* of A by G, and denoted by $A \rtimes_\alpha G$, or just $A \rtimes G$ if the choice of action is clear.

Mimicking the group case, we can also define a norm by taking the supremum over a special type of covariant representation to obtain the reduced crossed product. It is this construction which we now detail. See [23, Examples 3.105 and 3.122] for a more detailed exposition.

As above, begin with a separable groupoid dynamical system (A, G, α), and a nondegenerate $C_0(G^{(0)})$-linear representation $\pi : A \to B(L^2(G^{(0)} * \mathfrak{H}, \mu))$ with decomposition

$$\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u),$$

so that $\pi_u : A(u) \to B(H(u))$. As in [23, Example 3.84], form the pull-back bundle by the source map of G:

$$s^*(G^{(0)} * \mathfrak{H}) = \{(g, h) \mid h \in H(s(g))\}.$$

Let ν be the induced measure, as in Definition 5.2.17. By [80, Theorem I.5], we can use the source map of G to decompose ν into finite measures ν_u on G_u. Denote by ν^u the image of ν_u under inversion in G, which is a finite measure supported on G^u. Thus, for each $u \in G^{(0)}$, we obtain a separable Hilbert space $K(u) := L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^u}, \nu^u)$. We use the Hilbert spaces $K(u)$ as fibres to form an analytic Borel Hilbert bundle $G^{(0)} * \mathfrak{K}$. Denote by Δ the modular function which arises from μ. As in [23, Example 3.105], for each $g \in G$, one can obtain a unitary

$$\lambda_g : L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^{s(g)}}, \nu^{s(g)}) \to L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^{r(g)}}, \nu^{r(g)})$$

by defining $\lambda_g f(h) = \Delta(g)^{1/2} f(g^{-1}h)$ for each $h \in G^{r(g)}$.

Also, define $\tilde{\pi} : A \to B(L^2(G^{(0)} * \mathfrak{K}, \mu))$ by

$$((\tilde{\pi}(a)f)(u))(g) = \pi_{s(g)}(\alpha_g^{-1}(a(u)))\,(f(u)(g))$$

where $a \in A$, $f \in L^2(G^{(0)} * \mathfrak{K}, \mu)$, $u \in G^{(0)}$, and $g \in G^u$. To describe what is happening here, first working through the left-hand side, we see that $(\tilde{\pi}(a)f)(u)$ should be an element of $K(u)$, and hence assign to any $g \in G^u$ an element of $H(s(g))$. On the right-hand side, $f(u)$ is an element of $K(u)$, so $f(u)(g) \in H(s(g))$. Now, since $u = r(g)$, we obtain $\alpha_g^{-1}(a(u)) \in A(s(g))$, and so $\pi_{s(g)}(\alpha_g^{-1}(a(u))) \in B(H(s(g)))$, and this operator is being applied to the element $f(u)(g) \in H(s(g))$.

It is not obvious, but one can show that $(\tilde{\pi}, \lambda)$ defines a covariant representation of (A, G, α). Covariant representations of this special form are known as *left regular representations*. We now discuss the integrated form $\tilde{\pi} \rtimes \lambda$ of such representations, which will be representations of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(G^{(0)} * \mathfrak{K}, \mu)$. In fact, instead of working directly with the integrated form $\tilde{\pi} \rtimes \lambda$, it is easier to work with the following representation of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$, which is unitarily equivalent to the integrated form [23, Example 3.122 and Remark 3.123]:

$$L_\pi(f)h(g) = \int_{G^{r(g)}} \pi_{s(g)}(\alpha_g^{-1}(f(t)))h(t^{-1}g)d\lambda^{r(g)}(t),$$

where $f \in \Gamma_c(G, r^*\mathcal{A})$, $h \in L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$, and $g \in G$. Similarly to the definition of $\tilde{\pi}$ above, $h(t^{-1}g)$ is an element of $H(s(t^{-1}g)) = H(s(g))$, which is being transformed by the operator $\pi_{s(g)}(\alpha_g^{-1}(f(t))) \in B(H(s(g)))$. This makes sense because $f(t) \in A(r(t)) = A(r(g))$, so that $\alpha_g^{-1}(f(t)) \in A(s(g))$.

Due to the equivalence of L_π and $\tilde{\pi} \rtimes \lambda$, we refer to L_π as the *integrated left regular representation* associated to π.

Definition 5.3.4. Suppose (A, G, α) is a separable groupoid dynamical system. We define the *reduced norm* on $\Gamma_c(G, r^*\mathcal{A})$ by

$$\|f\|_r := \sup\{\|L(f)\| \mid L \text{ is an integrated left regular representation of } (A, G, \alpha)\}.$$

The completion of $\Gamma_c(G, r^*\mathcal{A})$ in this norm is a C^*-algebra called the *reduced groupoid crossed product* of A by G, and is denoted $A \rtimes_{\alpha,r} G$, or just $A \rtimes_r G$ if the choice of action is clear.

Chapter 6

Classification of tiling C^*-algebras

In this chapter, we prove our main result: that the C^*-algebras associated to aperiodic and repetitive tilings with finite local complexity tensorially absorb the Jiang-Su algebra $\mathcal{Z}$ defined in [32] (Theorem 6.4.3). The proof follows that of [35, Theorem 12.4], making use of a groupoid version of the quasitiling technology of Ornstein-Weiss [46] (see also [36, Theorem 4.36] for the formulation which inspires our result), which we present as Theorem 6.2.8. The ideas behind the proofs are identical to the group case, with the proviso that we will spend much of our time adjusting for the fact that groupoids have multiple identities.

6.1 Cuntz subequivalence, order zero maps, and classification

In this section, we give a basic introduction to the classification machinery (Theorem 6.1.6) which we will use. The following notion was introduced in [11] (see also [49, Remark 1.2] for this formulation).

Definition 6.1.1. Let A be a C^*-algebra. Given positive elements $a, b \in A$, we say that a is *Cuntz subequivalent to b over A*, and write $a \precsim_A b$, if there exists a sequence $(r_n)_{n\in\mathbb{N}}$ in A with $a = \lim_{n\to\infty} r_n b r_n^*$.

Definition 6.1.2. Let A be a C^*-algebra. We say that $a, b \in A$ are *orthogonal*, and write $a \perp b$, if $ab = ba = a^*b = ab^* = 0$.

Definition 6.1.3 ([84, Definition 1.3])**.** Let A and B be C^*-algebras, and let $\phi : A \to B$ be a completely positive map. We say that ϕ has *order zero* if it preserves orthogonality. That is, if, for every $a, b \in A$ such that $a \perp b$, we have $\phi(a) \perp \phi(b)$.

The following structure theorem classifies completely positive order zero maps, and is often useful to check whether a particular linear map is order zero.

Theorem 6.1.4 ([84, Theorem 2.3])**.** *Let A and B be C^*-algebras, and let $\phi : A \to B$ be a completely positive order zero map. Set $C = C^*(\phi(A)) \subset B$. Denote by $\mathcal{M}(C)$ the multiplier algebra of C, and denote by C' the commutant of C. Then there exist a positive element $h \in \mathcal{M}(C) \cap C'$ and a $*$-homomorphism $\psi : A \to \mathcal{M}(C) \cap \{h\}'$ such that*

$$\phi(a) = h\psi(a)$$

for each $a \in A$. Moreover, if A is unital, then $h = \phi(1_A)$.

Definition 6.1.5. Let A be a $C*$-algebra, and $a, b \in A$. The *commutator* of a and b, denoted $[a, b]$, is defined by

$$[a, b] := ab - ba.$$

As in [35], we will make use of the following criterion to check for $\mathcal{Z}$-stability, which was obtained following the methods of [42].

Theorem 6.1.6 ([29, Theorem 4.1])**.** *Let A be a simple separable unital nuclear C^*-algebra which is not isomorphic to $\mathbb{C}$. Write $\precsim$ for the relation of Cuntz subequivalence over A. Suppose that, for every $n \in \mathbb{N}$, finite set $F \subset A$, $\epsilon > 0$, and nonzero positive element $a \in A$, there exists an order zero completely positive and contractive map $\phi : M_n \to A$ such that*

(i) $1 - \phi(1) \precsim a$; and

(ii) $\|[w, \phi(B)]\| < \epsilon$ for all $w \in F$ and $B \in M_n$ with $\|B\| = 1$.

Then A is $\mathcal{Z}$-stable.

We now turn to the application of $\mathcal{Z}$-stability in the classification programme. Recall that this programme is concerned with the construction of an invariant which encapsulates enough information about a certain class of simple, separable, unital, nuclear, infinite dimensional C^*-algebras to ensure that an isomorphism of the invariants of any two C^*-algebras in this class lifts to an isomorphism of the algebras themselves. The *Elliott invariant* consists of ordered topological K-theory along with the trace simplex. With this invariant in hand, the question of the precise class of algebras which it classifies comes to the fore. The property of $\mathcal{Z}$-stability arises to prominence as part of a sufficient condition for an algebra to be classifiable using this invariant.

The following is a partial statement of the Toms-Winter conjecture.

Theorem 6.1.7. *Let A be a simple, separable, unital, nuclear, infinite dimensional C^*-*

algebra. Then the following are equivalent.

(i) A is $\mathcal{Z}$-stable.

(ii) A has finite nuclear dimension (see [85]).

The implication (ii)⇒(i) was shown to hold in [81], and the implication (i)⇒(ii) is shown in [9]. Consider the following capstone result, which brings together decades of work by many authors (see for example [82], [24], [19]).

Theorem 6.1.8 ([73, Corollary D]). *Let A and B be simple, separable, unital, and infinite dimensional C^*-algebras with finite nuclear dimension which satisfy the Universal Coefficient Theorem (UCT) of [61]. Then A is isomorphic to B if and only if A and B have isomorphic Elliott invariants.*

It was shown in [61] that the UCT holds for all separable nuclear C^*-algebras which are KK-equivalent to abelian ones. It remains an open problem whether this class in fact contains *all* separable nuclear C^*-algebras. Regardless, combining the above result with Theorem 6.1.7 guarantees the classifiability of all simple, separable, unital, nuclear, infinite dimensional, $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT.

6.2 Ornstein-Weiss quasitiling machinery for groupoids

A fundamental result in ergodic theory is the Rokhlin lemma, which states that any aperiodic invertible measure-preserving transformation T on a probability space (X, μ) admits for any $n \in \mathbb{N}$ and $\epsilon > 0$ a measurable subset $E \subset X$ such that the sets $E, TE, \ldots, T^{n-1}E$ are pairwise disjoint and such that $\mu(E \cup TE \cup \cdots \cup T^{n-1}E) > 1 - \epsilon$. This theorem can be thought of as a statement about actions of $\mathbb{Z}$, whence it becomes natural to ask about generalisations of the result for actions of different groups.

It turns out that this problem is closely related to the question of tileability of the acting group. Given a group G, this question asks whether, for each $\epsilon > 0$, there exists a finite set of finite subsets of G such that for each finite subset $F \subset G$ there exists a collection of placements of each subset so that the collection of all of the "placed subsets" is a pairwise disjoint collection of subsets of G which covers all but $\epsilon|F|$ of the elements of F. We wish to immediately point out that this notion of tileability is very different from the tilings of $\mathbb{R}^d$ discussed in Chapter 3! The Ornstein-Weiss quasitiling theorem [46, Section I.2, Theorem 6] shows that when G is amenable such collections of subsets do exist, as long as we allow an ϵ overlap between the placements of each subset (but the placements of different subsets need not overlap). Such a collection of subsets is said to ϵ-quasitile G.

It is also shown [46, Section II.2, Theorem 5] that the existence of ϵ-quasitilings implies a Rokhlin lemma-type result for free actions of the group.

In order to follow the proof of $\mathcal{Z}$-stability in [35], we require a generalisation of the Ornstein-Weiss quasitiling theorem to the groupoid setting. In this section, we work towards a result (Theorem 6.2.8) which fills this role by mimicking [36, Theorem 4.36].

Throughout, we will assume that G is a locally compact Hausdorff étale groupoid with compact unit space unless otherwise noted. We will need the following concepts.

Definition 6.2.1 ([36, Definition 4.29])**.** Let (X, μ) be a finite measure space. Let $\beta, \epsilon \geq 0$. We say that a collection $\{A_i\}_{i \in I}$ of measurable subsets of X

(i) is a β-*even* covering of X if there exists a positive integer M such that $\sum_{i \in I} \mathbf{1}_{A_i}(x) \leq M$ for every $x \in X$ and $\sum_{i \in I} \mu(A_i) \geq \beta M \mu(X)$, in which case M is called a *multiplicity* of the β-even covering.

(ii) β-*covers* X if $\mu(\bigcup_{i \in I} A_i) \geq \beta\mu(X)$.

(iii) is ϵ-*disjoint* if, for each $i \in I$, there exists a subset $\widehat{A_i} \subset A_i$ such that $\mu(\widehat{A_i}) \geq (1-\epsilon)\mu(A_i)$, and such that the collection $\{\widehat{A_i}\}_{i \in I}$ is pairwise disjoint.

We modify the above definition to apply to locally compact Hausdorff groupoids, as follows. Let G be a locally compact Hausdorff groupoid admitting a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ of Radon measures, where λ^u is supported on $G^u = r^{-1}(u)$. Consider the image of λ^u under the inverse map of G. This is a Radon measure on $G_u = s^{-1}(u)$, which we denote by λ_u. Given a compact subset $A \subset G$, the measure space (Au, λ_u) is finite for each $u \in G^{(0)}$. Thus, we can rephrase the properties from the above definition by requiring that they hold at each source bundle.

Definition 6.2.2. Let G be a locally compact Hausdorff groupoid admitting a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$, and let $\beta \geq 0$. For each $u \in G^{(0)}$, equip the source bundle G_u with the measure $\lambda_u = (\lambda^u)^{-1}$ constructed above. Let $A \subset G$ be compact, so that (Au, λ_u) is a finite measure space for each $u \in G^{(0)}$. We say that a collection $\{A_i\}_{i \in I}$ of compact subsets of A

(i) is a β-*even covering* of A with multiplicity M if, for every $u \in s(A)$, the collection $\{A_i u\}_{i \in I}$ is a β-even covering of (Au, λ_u) with multiplicity M.

(ii) β-*covers* A if $\{A_i u\}_{i \in I}$ β-covers (Au, λ_u) for each $u \in s(A)$.

(iii) is ϵ-*disjoint* if $\{A_i u\}_{i \in I}$ is ϵ-disjoint in (Au, λ_u) for each $u \in s(A)$.

Notice that the constant M appearing in (i) does not depend on u – the same multiplicity must work at every unit. In the case that G is étale, each measure in the Haar system (and thus also each measure λ_u) can be chosen to be the counting measure, and we will work in this situation for the remainder of the section.

The following pair of lemmas will be applied repeatedly in the proof of our quasitiling result.

Lemma 6.2.3 (c.f. [36, Lemma 4.31])**.** *Let A be a compact subset of a locally compact Hausdorff étale groupoid (whose unit space is not necessarily compact). Let $0 \leq \epsilon \leq \frac{1}{2}$ and $0 < \beta \leq 1$, and let $\{A_i\}_{i\in I}$ be a β-even covering of A. Then, for each $u \in s(A)$, there is an ϵ-disjoint subcollection $\{A_j u\}_{j\in J_u}$ of $\{A_i u\}_{i\in I}$, indexed by $J_u \subset I$, which $\epsilon\beta$-covers Au.*

Proof. By definition, for each $u \in s(A)$, the collection $\{A_i u\}_{i\in I}$ is a β-even covering of the finite measure space Au (equipped with the counting measure) in the sense of Definition 6.2.1. Thus, for each $u \in s(A)$, applying [36, Lemma 4.31] yields the ϵ-disjoint subcollection of $\{A_i u\}_{i\in I}$ which $\epsilon\beta$-covers Au that we seek. □

In this section, we will use as standard the notion of approximate invariance defined in Lemma 4.3.11.

Definition 6.2.4. Let G be an étale groupoid. Let $C, A \subset G$ be nonempty and compact, and let $\epsilon > 0$. We denote

$$I_C(A) \coloneqq \{a \in A \mid Ca \subset A\},$$

and we say that A is (C,ϵ)-*invariant* if, for each $u \in s(A)$, we have $|I_C(Au)| \geq (1-\epsilon)|Au|$.

Lemma 6.2.5 (c.f. [36, Lemma 4.33])**.** *Let G be a locally compact Hausdorff étale groupoid with compact unit space. Let $C \subset G$ be a nonempty, compact subset containing $G^{(0)}$. Choose $\epsilon > 0$, and let*

$$\delta \geq \max\left(0, 1 - \frac{\inf_{w\in G^{(0)}} |Cw|}{\sup_{w\in G^{(0)}} |wC|}(1-\epsilon)\right).$$

Suppose that $A \subset G$ is a nonempty, compact subset which is (C,ϵ)-invariant. Then the collection $\{Ca \mid a \in I_C(A)\}$ is a $(1-\delta)$-even covering of A with multiplicity $\sup_{w\in G^{(0)}} |wC|$.

Proof. First, we show that for every $u \in s(A)$,

$$\sum_{a\in I_C(A)} |Cau| \geq (1-\delta) \sup_{w\in G^{(0)}} |wC||Au|.$$

To this end, fix $u \in s(A)$. Since A is (C,ϵ)-invariant, we have $|I_C(Au)| \geq (1-\epsilon)|Au|$.

Therefore, we see that

$$\begin{aligned}\sum_{a\in I_C(A)} |Cau| &= \sum_{a\in I_C(Au)} |Cau| \\ &\geq |I_C(Au)| \inf_{a\in I_C(Au)} |Cr(a)| \\ &\geq (1-\epsilon)|Au| \inf_{w\in G^{(0)}} |Cw| \\ &= (1-\epsilon)|Au| \sup_{w\in G^{(0)}} |wC| \frac{\inf_{w\in G^{(0)}} |Cw|}{\sup_{w\in G^{(0)}} |wC|} \\ &\geq (1-\delta) \sup_{w\in G^{(0)}} |wC||Au|.\end{aligned}$$

Next, we show that for every $g \in Au$,

$$\sum_{a\in I_C(A)} \mathbf{1}_{Cau}(g) \leq \sup_{w\in G^{(0)}} |wC|.$$

Fix $g \in Au$. It is equivalent to show that

$$\sum_{a\in I_C(Au)} \mathbf{1}_{Cau}(g) \leq \sup_{w\in G^{(0)}} |wC|,$$

because $Cau = \varnothing$ whenever $s(a) \neq u$. Suppose that $g \in Ca_1u \cap Ca_2u$, where $a_1, a_2 \in I_C(Au)$ are distinct. Then there exist $c_1, c_2 \in C$ such that $c_1a_1 = g = c_2a_2$, which requires that $r(c_1) = r(g) = r(c_2)$. Since $a_1 \neq a_2$, the fact that $c_1a_1 = c_2a_2$ implies by cancellation that $c_1 \neq c_2$. Extending this argument, we see that the intersection of $n \in \mathbb{N}$ sets from the collection $\{Cau \mid a \in I_C(Au)\}$ at a common element g requires the existence of n distinct elements of $r(g)C$. This shows that, for each $u \in s(A)$ and $g \in Au$,

$$\sum_{a\in I_C(Au)} \mathbf{1}_{Cau}(g) \leq |r(g)C| \leq \sup_{w\in G^{(0)}} |wC|. \qquad \square$$

In the group case, one may alternatively characterise approximate invariance by using a "small boundary" condition. We introduce an analogue of the notion for groupoids.

Definition 6.2.6 (c.f. [36, Definition 4.34])**.** Let G be an étale groupoid, and let $C, A \subset G$ be compact. The C-*boundary* of A is

$$\partial_C(A) := \{g \in G \mid Cg \cap A \neq \varnothing \text{ and } Cg \cap A^c \neq \varnothing\}.$$

Notice that $\partial_C(A)$ may be infinite. Therefore, to obtain the aforementioned "small boundary" condition, we need to consider the C-boundary of the source bundle at each unit, and ask that, for every $u \in s(A)$, we have

$$|\partial_C(Au)| \leq \epsilon |Au|.$$

In particular, notice that $\partial_C(Au) \subset C^{-1}Au$ and so it is always finite since $C^{-1}A$ is compact.

We now introduce our analogue of quasitilings. When reading the following definition, notice that cutting down to the finite source bundles Au is still necessary for the "ϵ" of "ϵ-quasitiling" to be meaningful, but is built into our terminology via Definition 6.2.2.

Definition 6.2.7 (c.f. [36, Definition 4.35])**.** Let G be a locally compact Hausdorff étale groupoid with compact unit space. Let A be a compact subset of G, and fix $\epsilon > 0$. A finite collection $\{S_1, \ldots, S_n\}$ of compact subsets of G is said to *ϵ-quasitile* A if there exist $C_1, \ldots, C_n \subset G$ such that $\bigcup_{i=1}^n S_i C_i \subset A$ and so that the collection of right translates $\{S_i c \mid i \in \{1, \ldots, n\}, c \in C_i\}$ is ϵ-disjoint and $(1-\epsilon)$-covers A.

The subsets $S_1, \ldots, S_n$ are referred to as the *tiles*, and $C_1, \ldots, C_n$ as the *centres* of the quasitiling.

The following is our analogue of the Ornstein-Weiss quasitiling theorem. We briefly describe the ideas behind the proof. Given a compact subset C of a groupoid G, and $\epsilon > 0$, the general philosophy will be to start with a (C, ϵ)-invariant compact subset A of G, apply Lemma 6.2.5 to construct an even covering of A whose tolerance is limited by C and ϵ, and then use Lemma 6.2.3 to turn our even covering into an ϵ-disjoint collection which still covers some fixed portion of each source bundle in A. The assumptions of Theorem 6.2.8 are designed to ensure that we can iterate this procedure $n \in \mathbb{N}$ times. In particular, at each stage, the relative complement in A of everything covered up to that point will be sufficiently invariant to allow another application of this pair of lemmas. In addition, the assumptions ensure that when this procedure terminates, the union of all the ϵ-disjoint subcollections that have been constructed will $(1-\epsilon)$-cover the invariant compact set A that we started with.

Theorem 6.2.8 (c.f. [36, Theorem 4.36])**.** *Let G be a locally compact Hausdorff étale groupoid with compact unit space. Let $0 < \epsilon < 1/2$, and choose $m \in \mathbb{N}$ such that $m \geq 2$. Let n be a positive integer such that $(1-\epsilon/m)^n < \epsilon$. Suppose that $S_1 \subset S_2 \subset \cdots \subset S_n$ are compact subsets of G which contain $G^{(0)}$ and are such that, for each $i \in \{1, \ldots, n\}$,*

(i) $1/m < \dfrac{\inf_{w \in G^{(0)}} |S_i w|}{\sup_{w \in G^{(0)}} |w S_i|} \leq 1$*;*

(ii) $\dfrac{\inf_{w \in G^{(0)}} |S_i w|}{\sup_{w \in G^{(0)}} |w S_i|}(1-\epsilon) \geq 1/m$*; and*

(iii) $|\partial_{S_{i-1}}(S_i w)| \leq (\epsilon^2/8)|S_i w|$ *for each* $w \in G^{(0)}$*.*

Then every $(S_n, \epsilon^2/4)$-invariant compact subset of G is ϵ-quasitiled by $\{S_1, \ldots, S_n\}$.

Proof. Let A be a nonempty $(S_n, \epsilon^2/4)$-invariant compact subset of G. We will recursively construct $C_n, \dots, C_1 \subset G$ (in that order) such that, for each $k \in \{1, \dots, n\}$, $\bigcup_{i=k}^n S_i C_i \subset A$, and so that the collection of translates $\{S_i c \mid i \in \{k, \dots, n\}, c \in C_i\}$ is ϵ-disjoint and λ_k-covers A, where

$$\lambda_k = \min(1 - \epsilon,\ 1 - (1 - \epsilon/m)^{n-k+1}).$$

By our assumption on n it will then follow that $\{S_1, \dots, S_n\}$ ϵ-quasitiles A. Indeed, we have that $(1 - \epsilon/m)^n < \epsilon$, so $\lambda_1 = 1 - \epsilon$, as required.

For the base step (when $k = n$), we apply Lemma 6.2.5 to the $(S_n, \epsilon^2/4)$-invariant compact subset A to notice that the collection $\{S_n a \mid a \in I_{S_n}(A)\}$, where $I_{S_n}(A) = \{a \in A \mid S_n a \subset A\}$ is as in Definition 6.2.4, is a β-even covering of A, where

$$\beta = \frac{\inf_{w \in G^{(0)}} |S_n w|}{\sup_{w \in G^{(0)}} |w S_n|}(1 - \epsilon^2/4).$$

Observe that $\beta \geq \frac{\inf |S_n w|}{\sup |w S_n|}(1 - \epsilon)$, and so, by condition (ii), $\beta \geq 1/m$. This shows that the collection $\{S_n a \mid a \in I_{S_n}(A)\}$ is a $(1/m)$-even covering of A. Then we can apply Lemma 6.2.3 to find, for each $u \in G^{(0)}$, a subset $C_{n,u} \subset I_{S_n}(Au)$ such that the subcollection $\{S_n c \mid c \in C_{n,u}\}$ is ϵ-disjoint and (ϵ/m)-covers Au. This allows us to construct $C_n = \bigsqcup_{u \in G^{(0)}} C_{n,u}$, which satisfies the properties we seek.

Suppose that, for some $k \in \{1, \dots, n-1\}$, we have constructed $C_n, C_{n-1}, \dots, C_{k+1} \subset G$ with the properties we desire. Set $A'_k = A \setminus \bigcup_{i=k+1}^n S_i C_i$. If $|A'_k u| < \epsilon |Au|$ for every $u \in s(A)$, then we can finish our construction by taking each of $C_k, \dots, C_1$ to be the empty set, so suppose that the set $Z_k := \{u \in s(A) \mid |A'_k u| \geq \epsilon |Au|\}$ is nonempty. Then, for each $u \in G^{(0)} \setminus Z_k$, we will take $C_k u = \varnothing$, and we define $A_k = \bigcup_{u \in Z_k} A'_k u$. We now wish to show that A_k is $(S_k, \frac{\epsilon}{2})$-invariant.

First, observe that, for each $u \in G^{(0)}$, $i \in \{k+1, \dots, n\}$, and $c \in C_i u$, we have $|\partial_{S_k}(S_i c u)| \leq |\partial_{S_{i-1}}(S_i c u)|$, since $S_k \subset S_{i-1}$. Notice that the map $a \mapsto ac^{-1}$ is an injection which maps $\partial_{S_{i-1}}(S_i c s(c))$ into $\partial_{S_{i-1}}(S_i r(c))$. Using this along with the boundary condition (iii) on the S_i, we see that

$$|\partial_{S_k}(S_i c s(c))| \leq |\partial_{S_{i-1}}(S_i c s(c))| \leq |\partial_{S_{i-1}}(S_i r(c))| \leq (\epsilon^2/8)|S_i r(c)|.$$

Since the collection $\{S_i c s(c) \mid i \in \{k+1, \dots, n\},\ c \in C_i\}$ is $\frac{1}{2}$-disjoint (since $\epsilon < \frac{1}{2}$), and since $|\bigcup_{i=k+1}^n S_i C_i u| \leq |Au| \leq \epsilon^{-1}|A_k u|$ for every $u \in Z_k$, we obtain, for each $u \in Z_k$, that

$$\left| \bigcup_{i=k+1}^{n} \bigcup_{c \in C_i u} \partial_{S_k}(S_i c u) \right| \leq \frac{\epsilon^2}{8} \sum_{i=k+1}^{n} \sum_{c \in C_i u} |S_i r(c)|$$

$$
\begin{aligned}
&= \frac{\epsilon^2}{8} \sum_{i=k+1}^{n} \sum_{c \in C_i u} |S_i c| \\
&\leq \frac{\epsilon^2}{4} \left| \bigcup_{i=k+1}^{n} S_i C_i u \right| \\
&\leq \frac{\epsilon}{4} |A_k u|, \qquad (6.2.1)
\end{aligned}
$$

where we used $\frac{1}{2}$-disjointness to obtain the penultimate inequality, as follows. Use $\frac{1}{2}$-disjointness to find, for each $i \in \{k+1, \ldots, n\}$ and $c \in C_i u$, a subset $\widehat{S_{i,c}} \subset S_i$ such that $|\widehat{S_{i,c}} c| \geq \frac{1}{2} |S_i c|$, and such that the collection $\{\widehat{S_{i,c}} c \mid i \in \{k+1, \ldots, n\}, c \in C_i u\}$ is pairwise disjoint. Then we have

$$
\begin{aligned}
\left| \bigcup_{i=k+1}^{n} S_i C_i u \right| &= \left| \bigcup_{i=k+1}^{n} \bigcup_{c \in C_i u} S_i c \right| \\
&\geq \left| \bigsqcup_{i=k+1}^{n} \bigsqcup_{c \in C_i u} \widehat{S_{i,c}} c \right| \\
&= \sum_{i=k+1}^{n} \sum_{c \in C_i u} |\widehat{S_{i,c}} c| \\
&\geq \frac{1}{2} \sum_{i=k+1}^{n} \sum_{c \in C_i u} |S_i c|.
\end{aligned}
$$

Notice that all the sums in the computation of (6.2.1) are finite. In particular, one can see that $|C_i u| < \infty$ for each i and each $u \in G^{(0)}$, because $S_i C_i u \subset Au$ which is finite, and so $\infty > |S_i C_i u| = \sum_{c \in C_i u} |S_i r(c)| \geq |C_i u| \inf_{w \in G^{(0)}} |S_i w| \geq |C_i u|$ because $G^{(0)} \subset S_i$.

Since A is $(S_n, \epsilon^2/4)$-invariant, and $S_k \subset S_n$, A is also $(S_k, \epsilon^2/4)$-invariant. Thus, for each $u \in s(A)$, $|I_{S_k}(Au)| = |\{a \in Au \mid S_k a \subset A\}| \geq (1 - \epsilon^2/4)|Au|$, and so, for each $u \in s(A_k) \subset Z_k$, we have

$$
\begin{aligned}
|I_{S_k}(A_k u)| &= |\{a \in A_k u \mid S_k a \subset A_k u\}| \\
&= \left| I_{S_k}(Au) \setminus \bigcup_{i=k+1}^{n} \left(S_i C_i u \cup \bigcup_{c \in C_i u} \partial_{S_k}(S_i c u) \right) \right| \\
&\geq |I_{S_k}(Au)| - \left| \bigcup_{i=k+1}^{n} S_i C_i u \right| - \left| \bigcup_{i=k+1}^{n} \bigcup_{c \in C_i u} \partial_{S_k}(S_i c u) \right| \\
&\geq \left(1 - \frac{\epsilon^2}{4}\right) |Au| - (|Au| - |A_k u|) - \frac{\epsilon}{4} |A_k u| \\
&= |A_k u| - \frac{\epsilon}{4} |A_k u| - \frac{\epsilon^2}{4} |Au| \\
&\geq \left(1 - \frac{\epsilon}{2}\right) |A_k u|,
\end{aligned}
$$

which shows that A_k is $(S_k, \frac{\epsilon}{2})$-invariant. In the first equality, we think of the complement as enforcing additional constraints on the elements of $I_{S_k}(Au)$ to get elements of the left-hand side. Clearly, such elements cannot lie in $\bigcup_{i=k+1}^n S_iC_iu$, as this set is the complement of A_ku in Au. Furthermore, if an element $a \in I_{S_k}(Au)$ lies in $\partial_{S_k}(S_icu)$ for some $i \in \{k+1, \ldots, n\}$ and $c \in C_i$, then there exists $t \in S_k$ such that $ta \in S_icu \subset \bigcup_{i=k+1}^n S_iC_iu = Au \setminus A_ku$, so such elements cannot lie in the left-hand set. To obtain the third line, we have used inequality (6.2.1) and the fact that $\bigcup_{i=k+1}^n S_iC_iu = Au \setminus A_ku$. The final line uses the fact that $|Au| \leq \epsilon^{-1}|A_ku|$, since $u \in Z_k$.

Thus, we can apply Lemma 6.2.5 to see that the collection of right translates of S_k which lie in A_k form a β_k-even covering of A_k, where

$$\beta_k = \frac{\inf_{w\in G^{(0)}} |S_kw|}{\sup_{w\in G^{(0)}} |wS_k|}\left(1 - \frac{\epsilon}{2}\right).$$

Observe that $\beta_k \geq \frac{\inf |S_kw|}{\sup |wS_k|}(1-\epsilon)$ and so, by condition (ii), we see that $\beta_k > 1/m$. Therefore, the collection $\{S_ka \mid a \in A_k,\ S_ka \subset A_k\}$ is a $(1/m)$-even covering of A_k. Use Lemma 6.2.3 to obtain, for each $u \in s(A_k)$, an ϵ-disjoint subcollection $\{S_kcu \mid c \in C_{k,u}\}$ of these translates which (ϵ/m)-covers A_k, and set $C_k = \bigsqcup_{u\in s(A_k)} C_{k,u}$.

Note that $\bigcup_{i=k}^n\{S_ic \mid c \in C_i\}$ is ϵ-disjoint, because it is the union of two ϵ-disjoint collections, $\{S_kc \mid c \in C_k\}$ and $\bigcup_{i=k+1}^n\{S_ic \mid c \in C_i\}$, which are such that the members of each collection are disjoint from all the members of the other. All that remains to show is that the new collection λ_k-covers A. Indeed, it is enough to show that it λ_k-covers Au for each $u \in Z_k$, since we already know the collection $(1-\epsilon)$-covers Au for each $u \notin Z_k$.

We assumed that $\bigcup_{i=k+1}^n S_iC_i$ was a λ_{k+1}-cover of A, so, at each unit $u \in s(A)$, the remainder has cardinality at most $(1-\lambda_{k+1})|Au|$. For each $u \in Z_k$, we constructed an (ϵ/m)-cover of this remainder. Thus, for each $u \in Z_k$, the collection $\bigcup_{i=k}^n S_iC_iu$ is an α_k-cover of Au where

$$\alpha_k = \lambda_{k+1} + \frac{\epsilon}{m}(1-\lambda_{k+1}).$$

If $\lambda_{k+1} = 1-\epsilon$, we obtain $\alpha_k = 1-\epsilon+\epsilon^2/m > 1-\epsilon \geq \lambda_k$.

If $\lambda_{k+1} = 1-(1-\epsilon/m)^{n-k}$, we obtain

$$\begin{aligned}\alpha_k &= 1-\left(1-\frac{\epsilon}{m}\right)^{n-k} + \frac{\epsilon}{m}\left(1-\frac{\epsilon}{m}\right)^{n-k}\\ &= 1-\left(1-\frac{\epsilon}{m}\right)^{n-k}\left(1-\frac{\epsilon}{m}\right)\\ &= 1-\left(1-\frac{\epsilon}{m}\right)^{n-k+1}\\ &\geq \lambda_k.\end{aligned}$$

In both cases, $\alpha_k \geq \lambda_k$, so $\bigcup_{i=k}^{n} S_i C_i$ is a λ_k-cover. □

Example 6.2.9. We show that groupoids arising from aperiodic, repetitive tilings of $\mathbb{R}^d$ with finite local complexity admit sequences of subsets S_l as in Theorem 6.2.8 of arbitrary length. Let Ω_{punc} be the punctured hull of such a tiling, and let R_{punc} be the associated groupoid. By FLC, there is a prototile of minimal volume $V_{\min}$, and one of maximal volume $V_{\max}$. Find $m \in \mathbb{N}$ such that $m > \max(V_{\max}/V_{\min}, 2)$. For $n \in \mathbb{N}$, define $S_n \subset R_{\text{punc}}$ as follows. For each $w \in \Omega_{\text{punc}}$, consider the set of punctures in w which lie in $B_n(0)$. For each such puncture, $x(t)$, include the arrow $(w - x(t), w)$ in S_n. In other words, S_n is the collection of allowable translates by vectors of magnitude smaller than n:

$$S_n = \{(w + x, w) \in R_{\text{punc}} \mid w \in \Omega_{\text{punc}}, |x| < n\}.$$

We claim that S_n is closed in the metric topology on R_{punc}. Indeed, if $\{(w_j + x_j, w_j)\}_{j\in\mathbb{N}}$ is a sequence in S_n converging to $(w + x, w)$, then both $d(w, w_j) \to 0$ and $|x - x_j| \to 0$. As in the proof of Lemma 3.3.7, we can choose j large enough to ensure $w_j \sqcap B_n(0) = w \sqcap B_n(0)$. Furthermore, since $|x_j| < n$ for each j, using the uniform discreteness of the puncture set together with the last sentence and the convergence of x_j to x, it follows that eventually $x_j = x$ for all j. Thus, we see that $|x| < n$, and so $(w + x, w) \in S_n$, showing that S_n is closed. In addition, we have that

$$S_n \subset \bigcup_{w\in\Omega_{\text{punc}}} \bigcup_{t\in w\sqcap B_n(0)} V(w \sqcap B_n(0), w(0), t).$$

By FLC, there are only finitely many patches $w \sqcap B_n(0)$ involved in the union on the right, each of which contain finitely many tiles. Thus the right-hand side is a finite union of compact sets, and so is itself compact. Thus we see that S_n is a closed subset of a compact set in a metric space, so it is compact. Note that S_n is closed under taking inverses, and therefore $|wS_n| = |S_n w|$ for each $w \in \Omega_{\text{punc}}$. It follows that

$$\frac{\inf_{w\in\Omega_{\text{punc}}} |S_n w|}{\sup_{w\in\Omega_{\text{punc}}} |wS_n|} \leq 1.$$

We look to bound $\inf_{w\in\Omega_{\text{punc}}} |S_n w|$ and $\sup_{w\in\Omega_{\text{punc}}} |wS_n| = \sup_{w\in\Omega_{\text{punc}}} |S_n w|$. We will use volume estimates to do this.

First, for $w \in \Omega_{\text{punc}}$ to maximise $|S_n w|$, we wish there to be as many tiles of w intersecting $B_n(0)$ as possible, so that we can assume the punctures of all these tiles lie in this ball. By FLC, there exists a prototile of largest diameter $D_{\max}$, one of minimal volume $V_{\min}$, and one of maximal volume $V_{\max}$. Consider the ball $B_{n+D_{\max}}(0)$. If a tile t intersects the complement of this ball then, since its diameter is at most $D_{\max}$, it cannot intersect $B_n(0)$.

Therefore, since t intersects $B_n(0)$ only if it is contained within $B_{n+D_{\max}}(0)$, we ask how many tiles can fit within the latter ball. An obvious upper bound is obtained by assuming every tile in the ball has minimal volume, and is given by

$$\sup_{w\in\Omega_{\text{punc}}}|S_n w| \leq \frac{\text{Vol}(B_{n+D_{\max}}(0))}{V_{\min}} = \frac{V_d(n+D_{\max})^d}{V_{\min}},$$

where V_d denotes the volume of the unit ball in $\mathbb{R}^d$.

We argue similarly to establish our second bound. From this point onwards we will assume that $n > D_{\max}$. If a tile intersects $B_{n-D_{\max}}(0)$, then its puncture must lie in $B_n(0)$, so we seek to minimise the number of tiles intersecting this ball. To do so, we may assume any such tile has volume $V_{\max}$ and is completely contained within the ball. Then we see that

$$\inf_{w\in\Omega_{\text{punc}}}|S_n w| \geq \frac{V_d(n-D_{\max})^d}{V_{\max}}. \tag{6.2.2}$$

Combining our bounds yields

$$\frac{\inf_{w\in\Omega_{\text{punc}}}|S_n w|}{\sup_{w\in\Omega_{\text{punc}}}|wS_n|} \geq \frac{V_d V_{\min}(n-D_{\max})^d}{V_d V_{\max}(n+D_{\max})^d} \underset{n\to\infty}{\to} \frac{V_{\min}}{V_{\max}} > \frac{1}{m}.$$

Therefore, for large enough n, and small enough $0<\epsilon<1/2$, we have that

$$\frac{\inf_{w\in\Omega_{\text{punc}}}|S_n w|}{\sup_{w\in\Omega_{\text{punc}}}|wS_n|}(1-\epsilon) \geq \frac{1}{m}.$$

Now, fix $k \in \mathbb{N}$. Since S_n is the set of translates in R_{punc} with magnitude smaller than n, we see that any translate in $\partial_{S_n}(S_{n+k})$ must have magnitude larger than $(n+k)-n=k$, and smaller than $(n+k)+n=2n+k$. Thus, to obtain an upper bound for $|\partial_{S_n}(S_{n+k}w)|$, we wish to maximise the number of punctures which appear in the annulus with centre 0, inner radius k, and outer radius $2n+k$. A tile which intersects the complement of the annulus with centre 0, inner radius $k-D_{\max}$, and outer radius $2n+k+D_{\max}$ cannot have its puncture in the region of interest, so we estimate the number of tiles contained in this latter annulus to obtain our upper bound, as follows:

$$|\partial_{S_n}(S_{n+k}w)| \leq \frac{V_d((2n+k+D_{\max})^d-(k-D_{\max})^d)}{V_{\min}},$$

where the top of the fraction is just the volume of the second annulus. On the other hand, we know by (6.2.2) that

$$|S_{n+k}w| \geq \frac{V_d(n+k-D_{\max})^d}{V_{\max}}.$$

Let

$$C_{n,k} := \frac{V_{\max}((2n+k+D_{\max})^d - (k-D_{\max})^d)}{V_{\min}(n+k-D_{\max})^d}$$

so that

$$\begin{aligned}|\partial_{S_n}(S_{n+k}w)| &\leq \frac{V_d((2n+k+D_{\max})^d - (k-D_{\max})^d)}{V_{\min}} \\ &= C_{n,k}\frac{V_d(n+k-D_{\max})^d}{V_{\max}} \\ &\leq C_{n,k}|S_{n+k}w|.\end{aligned}$$

Notice that, for each fixed n, we have $C_{n,k} \to 0$ as $k \to \infty$. Therefore, for each n, we can find $k(n) \geq 1$ such that $C_{n,k} \leq \epsilon^2/8$ whenever $k \geq k(n)$. Then the inequality above becomes

$$|\partial_{S_n}(S_{n+k(n)}w)| \leq C_{n,k(n)}|S_{n+k(n)}w| \leq \epsilon^2/8|S_{n+k(n)}w|.$$

We now relabel the sequence (S_n) by removing the first terms and shifting index, to assume that every set in the sequence satisfies properties (i) and (ii) from the statement of Theorem 6.2.8. We construct a subsequence by choosing indices n_i recursively as follows. Let $n_1 = 1$ and put $n_{i+1} = n_i + k(n_i)$. By construction, the sequence $(S_{n_i})_{i\in\mathbb{N}}$ will satisfy all of the requirements of Theorem 6.2.8.

6.3 Conditional expectations on groupoid crossed products

Throughout this section, G will be a locally compact Hausdorff étale groupoid acting continuously on a compact Hausdorff space X (since the range surjection $r_X : X \to G^{(0)}$ is continuous, this implies that the unit space of G is compact). We will denote by α the induced action of G on $C(X)$ from Example 5.1.7. Consider the crossed product $C(X) \rtimes G$, and the associated bundle $\mathcal{A} = \bigsqcup_{u\in G^{(0)}} C(r_X^{-1}(u))$. We will wish to choose the positive element appearing in Theorem 6.1.6 to be contained within $C(X) \cong \Gamma_0(G^{(0)}, \mathcal{A}) \subset \Gamma_c(G, r^*\mathcal{A}) \subset C(X) \rtimes G$. Arguing similary to [49], the results of this section will allow us to do so.

We make a remark on notation. In the sequel, it will be important to distinguish between a section of the pull-back bundle, $f \in \Gamma_c(G, r^*\mathcal{A})$, and the images of elements of G under this section, $f_g \in C(r_X^{-1}(r(g)))$. To do this, we will denote the section $g \mapsto f_g$ for each $g \in G$ by $\sum_{g\in G} f_g U_g$. Observe that this notation is highly suggestive of the image of the section under the integrated form of some covariant representation (π, U), and, in fact, we have every intention of multiplying sections as follows:

$$\left(\sum_{g\in G} a_g U_g\right)\left(\sum_{h\in G} b_h U_h\right) = \sum_{(g,h)\in G^{(2)}} a_g U_g b_h U_h = \sum_{(g,h)\in G^{(2)}} a_g \alpha_g(b_h) U_{gh}.$$

In order to formally justify this calculation, we should check that the images of the sections under the integrated form L of the left-regular representation from Section 5.3 interact in this way. Equation (6.3.2) (see page 126) gets us most of the way there, and does not make use of the formula above, so we forward-reference it to compute, for $(g,h) \in G^{(2)}$, that

$$\begin{aligned} L(a_gU_g)L(b_hU_h) &= (\Delta(g)^{-1/2}\tilde{\pi}_{r(g)}(a_g)\lambda_g)(\Delta(h)^{-1/2}\tilde{\pi}_{r(h)}(b_h)\lambda_h) \\ &= \Delta(gh)^{-1/2}\tilde{\pi}_{r(g)}(a_g)\lambda_g\tilde{\pi}_{s(g)}(b_h)\lambda_h \\ &= \Delta(gh)^{-1/2}\tilde{\pi}_{r(g)}(a_g)\tilde{\pi}_{r(g)}(\alpha_g(b_h))\lambda_{gh} \\ &= \Delta(gh)^{-1/2}\tilde{\pi}_{r(g)}(a_g\alpha_g(b_h))\lambda_{gh} \\ &= L(a_g\alpha_g(b_h)U_{gh}), \end{aligned}$$

where we made use of the fact that Δ is a groupoid homomorphism, $(\tilde{\pi}, \lambda)$ is a covariant representation, and $\tilde{\pi}(a_g) = \tilde{\pi}_{r(g)}(a_g)$ since $a_g \in A(r(g))$, using the decomposition of $\tilde{\pi}$ from Theorem 5.2.16 (and, similarly, $\tilde{\pi}(b_h) = \tilde{\pi}_{r(h)}(b_h)$).

The following notation will come in handy in the remainder of the section.

Notation 6.3.1. Given any subset $C \subset G$ and any element $f = \sum_{g\in G} f_g U_g \in \Gamma_c(G, r^*\mathcal{A})$, we define the following map $f|_C : G \mapsto r^*\mathcal{A}$:

$$f|_C(g) := \begin{cases} f_g & \text{if } g \in C \\ 0_{A(r(g))} & \text{otherwise.} \end{cases}$$

We remark that this notation is different from the usual notion of restriction, because $f|_C$ is still defined on all of G. This is a little unfortunate, as we will occasionally need to use the two together. However, the notion given here is the only one of the two which we will apply to sections of a bundle, and in all other cases the usual meaning of the notation is assumed. When C is clopen in G, the pasting lemma implies that $f|_C$ is continuous as a map from G to $r^*\mathcal{A}$ (the map $X \setminus C \to r^*\mathcal{A}$ given by $g \mapsto 0_{A(r(g))}$ is continuous by [23, Proposition 3.40]), so that $f|_C \in \Gamma_c(G, r^*\mathcal{A})$.

Consider the following map, defined for $\sum_{g\in G} f_g U_g \in \Gamma_c(G, r^*\mathcal{A})$:

$$E\left(\sum_{g\in G} f_g U_g\right) := \sum_{u\in G^{(0)}} f_u,$$

where $f_g \in C(r_X^{-1}(r(g)))$ for each $g \in G$, so that the right-hand side is just a complex-valued function on X (recall that we think of elements of $C(r_X^{-1}(r(g)))$ as functions which are defined on all of X, but supported inside $r_X^{-1}(r(g))$). For each $f \in \Gamma_c(G, r^*\mathcal{A})$, we claim that $E(f) \in C(X)$.

Write $f = \sum_{g\in G} f_g U_g$. By [23, Proposition 4.38], there is an isomorphism $\iota : C_c(G \ltimes X) \to \Gamma_c(G, r^*\mathcal{A})$, with inverse j given by $j(f)(g, x) = f_g(gx)$. Since G is étale and Hausdorff, we see, by combining Proposition 2.2.2 and Proposition 2.2.7, that $G^{(0)}$ is clopen in G. This implies that $f|_{G^{(0)}} \in \Gamma_c(G, r^*\mathcal{A})$, so that $j(f|_{G^{(0)}}) \in C_c(G \ltimes X)$. In fact, because $(f|_{G^{(0)}})_g = 0_{r_X^{-1}(r(g))}$ unless $g \in G^{(0)}$, observe that $j(f|_{G^{(0)}})$ is supported on $G^{(0)} * X$, which is homeomorphic to X via $\Phi(x) = (r_X(x), x)$. Thus, under the identification of $G^{(0)} * X$ with X, we see that $j(f|_{G^{(0)}})|_{G^{(0)}*X} \in C(X)$. On the other hand, for each $x \in X$,

$$E(f)(x) = f_{r_X(x)}(x) = j(f|_{G^{(0)}})(r_X(x), x) = j(f|_{G^{(0)}})(\Phi(x)),$$

so $E(f) = j(f|_{G^{(0)}}) \circ \Phi$, which shows that $E(f) \in C(X)$ too. In fact, we remark for later that an almost identical argument works to show that for any clopen bisection $B \subset G$, the sum $\sum_{g\in B} f_g$ defines an element of $C(X)$. The only modification that needs to be made is in the final equality, where we must precompose $j(f|_B)$ with the homeomorphism $(r|_{B*X})^{-1}$ arising from $r : G \ltimes X \to X$, to obtain

$$\sum_{g\in B} f_g = j(f|_B) \circ (r|_{B*X})^{-1} \circ \Phi \in C(X).$$

We wish to extend E to a map defined on the whole crossed product $C(X) \rtimes_r G$. It is easy to check that $E : \Gamma_c(G, r^*\mathcal{A}) \to C(X)$ is linear, and we claim that it is bounded. Given $f \in \Gamma_c(G, r^*\mathcal{A})$

$$\begin{aligned}\|f\| = \sup_{g\in G} \|f_g\|_{C(X)} \geq \sup_{u\in G^{(0)}} \sup_{x\in X} |f_u(x)| &= \sup_{u\in G^{(0)}} \sup_{x\in r_X^{-1}(u)} |f_u(x)| \\ &= \sup_{x\in X} |f_{r_X(x)}(x)| \\ &= \|E(f)\|.\end{aligned}$$

This shows that $\|E\| \leq 1$, so $E : \Gamma_c(G, r^*\mathcal{A}) \to C(X)$ is a bounded linear map defined on a dense subalgebra of $C(X) \rtimes_r G$. Thus, by the bounded linear transformation theorem, it can be uniquely extended to a bounded linear map $C(X) \rtimes_r G \to C(X)$, which we also denote by E.

Lemma 6.3.2. *The map $E : C(X) \rtimes_r G \to C(X)$ is a faithful conditional expectation. That is, E is a positive linear map which satisfies*

(i) $E(1_{C(X)\rtimes G}) = 1_{C(X)}$;

(ii) $E(f_1 a f_2) = f_1 E(a) f_2$ *for every* $a \in C(X) \rtimes G$ *and* $f_1, f_2 \in C(X)$; *and*

(iii) (Faithfulness) if $a \in C(X) \rtimes_r G$ *is positive and nonzero, then* $E(a)$ *is nonzero.*

Proof. We will check positivity and property (ii) on elements of $\Gamma_c(G, r^*\mathcal{A})$. By continuity, both of these properties, as well as linearity, then extend to $C(X) \rtimes G$.

To check positivity, let $f = \sum_{g\in G} f_g U_g \in \Gamma_c(G, r^*\mathcal{A})$, so that $f^* = \sum_{g\in G} U_{g^{-1}} f_g^* = \sum_{g\in G} \alpha_{g^{-1}}(f_g^*) U_{g^{-1}}$. Then we have

$$ff^* = \sum_{g,h\in G} f_g U_g \alpha_{h^{-1}}(f_h^*) U_{h^{-1}} = \sum_{(g,h^{-1})\in G^{(2)}} f_g \alpha_{gh^{-1}}(f_h^*) U_{gh^{-1}},$$

so that

$$E(ff^*) = \sum_{gh^{-1}\in G^{(0)}} f_g \alpha_{gh^{-1}}(f_h^*) = \sum_{g\in G} f_g \alpha_{r(g)}(f_g^*) = \sum_{g\in G} f_g f_g^* \geq 0.$$

Next, we check property (i). We have

$$E(1_{C(X)\rtimes G}) = E\left(\sum_{u\in G^{(0)}} 1_{r_X^{-1}(u)} U_u\right) = \sum_{u\in G^{(0)}} 1_{r_X^{-1}(u)} = 1_{C(X)}.$$

To show property (ii), choose any $a = \sum_{g\in G} a_g U_g \in \Gamma_c(G, r^*\mathcal{A})$, and $f_1, f_2 \in C(X)$. We think of f_1 and f_2 as elements of $\Gamma_c(G, r^*\mathcal{A})$ by identifying f_i with $\sum_{u\in G^{(0)}} (f_i)|_{r_X^{-1}(u)} U_u$ for $i = 1, 2$. Then

$$\begin{aligned}
E(f_1 a f_2) &= E\left(\left(\sum_{u\in G^{(0)}} (f_1)|_{r_X^{-1}(u)} U_u\right)\left(\sum_{g\in G} a_g U_g\right)\left(\sum_{v\in G^{(0)}} (f_2)|_{r_X^{-1}(v)} U_v\right)\right) \\
&= E\left(\sum_{g\in G} (f_1)|_{r_X^{-1}(r(g))} a_g \alpha_g\left((f_2)|_{r_X^{-1}(s(g))}\right) U_g\right) \\
&= \sum_{u\in G^{(0)}} (f_1)|_{r_X^{-1}(u)} a_u \alpha_u((f_2)|_{r_X^{-1}(u)}).
\end{aligned}$$

Now, observe that $\alpha_u((f_2)|_{r_X^{-1}(u)}) = f_2|_{r_X^{-1}(u)}$. In addition, since the support of a_u is already contained within $r_X^{-1}(u)$, we see that

$$f_1 a_u f_2 = (f_1)|_{r_X^{-1}(u)} a_u (f_2)|_{r_X^{-1}(u)} = (f_1)|_{r_X^{-1}(u)} a_u \alpha_u((f_2)|_{r_X^{-1}(u)}),$$

so that

$$E(f_1 a f_2) = \sum_{u \in G^{(0)}} f_1 a_u f_2 = f_1 E(a) f_2.$$

It remains to show faithfulness, for which we follow the proof of [66, Lemma 1.2.1]. First, we show that E is *equivariant* – that is, that $\alpha_g(E(b)) = E(U_g b U_g^*)$ for every $g \in G$ and $b \in C(X) \rtimes_r G$. By continuity, it will be enough to show this for $b \in \Gamma_c(G, r^*\mathcal{A})$. In this case, $\alpha_g(E(b)) \in C(r_X^{-1}(r(g)))$. If $b = \sum_{h \in G} b_h U_h$ then, for each $x \in r_X^{-1}(r(g))$, we have

$$\alpha_g(E(b))(x) = E(b)(g^{-1}x) = \left(\sum_{u \in G^{(0)}} b_u \right) (g^{-1}x) = b_{s(g)}(g^{-1}x).$$

On the other hand,

$$\begin{aligned}
E(U_g b U_g^*)(x) &= E\left(U_g \left(\sum_{h \in G} b_h U_h \right) U_{g^{-1}} \right)(x) \\
&= E\left(\sum_{h \in G_{s(g)}^{s(g)}} \alpha_g(b_h) U_{ghg^{-1}} \right)(x) \\
&= \left(\sum_{u \in G^{(0)} \cap G_{s(g)}^{s(g)}} \alpha_g(b_u) U_{gug^{-1}} \right)(x) \\
&= \alpha_g(b_{s(g)})(x) \\
&= b_{s(g)}(g^{-1}x),
\end{aligned}$$

so that $\alpha_g(E(b)) = E(U_g b U_g^*)$, showing that E is equivariant.

Consider a faithful $C_0(G^{(0)})$-linear representation π of $C(X)$ onto an analytic Borel Hilbert bundle $G^{(0)} * \mathfrak{H}$, with integrated form $\int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$, where $\pi_u : C(r_X^{-1}(u)) \to B(H(u))$ are the representations of the fibres (see Theorem 5.2.16). Let $\mathcal{H} = L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$ and, for $g \in G$ and $\xi \in H(s(g))$, consider $\delta_{g,\xi} \in \mathcal{H}$ defined by

$$\delta_{g,\xi}(h) = \delta_{g,h}\xi$$

for $h \in G$, where $\delta_{g,h}$ is the Kronecker delta. That is, $\delta_{g,\xi}$ is the section of $s^*(G^{(0)} * \mathfrak{H})$ which is zero everywhere except at g, where it takes the value ξ.

Form a Hilbert bundle $G^{(0)} * \mathfrak{K} = \bigsqcup_{u \in G^{(0)}} \mathcal{H}(u)$ over $G^{(0)}$ from $\mathcal{H}$ by setting $\mathcal{H}(u)$ to be the collection of elements in $\mathcal{H}$ whose support is contained in $G^u = r^{-1}(u)$ (that is, if $F \in \mathcal{H}(u)$, then $g \notin G^u$ implies $F(g) = 0_{H(s(g))}$). For $g \in G$, define $V_g : H(s(g)) \to \mathcal{H}(r(g))$ by $\xi \mapsto \Delta(g)^{1/2}\delta_{g,\xi}$, so that $V_g^* : \mathcal{H}(r(g)) \to H(s(g))$ is given by $V_g^*(F) = \Delta(g)^{-1/2}F(g)$, for each $F \in \mathcal{H}(r(g))$.

Let $\tilde{\pi}$ and λ be the maps from the left regular representation (see page 106). Note that

$$\lambda(g)\delta_{h,\xi}(\eta) = \Delta(g)^{1/2}\delta_{h,\xi}(g^{-1}\eta) = \begin{cases} \Delta(g)^{1/2}\xi & \text{if } \eta = gh \\ 0 & \text{otherwise,} \end{cases}$$

so that $\lambda(g)\delta_{h,\xi} = \Delta(g)^{1/2}\delta_{gh,\xi}$ when $(g,h) \in G^{(2)}$. Since $\Delta(u) = 1$ for each $u \in G^{(0)}$, observe that, for each $g \in G$ and $\xi \in H(s(g))$,

$$\lambda(g)V_{s(g)}\xi = \lambda(g)\Delta(s(g))^{1/2}\delta_{s(g),\xi} = \lambda(g)\delta_{s(g),\xi} = \Delta(g)^{1/2}\delta_{g,\xi} = V_g\xi,$$

so that

$$\lambda(g)V_{s(g)} = V_g. \tag{6.3.1}$$

Denote by a_gU_g the map $G \to r^*\mathcal{A}$ which sends g to a_g and all other elements $\eta \in G$ to $0_{A(r(\eta))}$. Then, for $h \in L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^{s(g)}}, \nu^{s(g)})$ and $\gamma \in G^{r(g)}$, and denoting the integrated left regular representation by L, we also have

$$\begin{aligned} L(a_gU_g)(h)(\gamma) &= \sum_{r(\eta)=r(\gamma)=r(g)} \pi_{s(\gamma)}\left(\alpha_{\gamma^{-1}}(a_gU_g(\eta))\right) h(\eta^{-1}\gamma) \\ &= \pi_{s(\gamma)}\left(\alpha_{\gamma^{-1}}(a_g)\right) h(g^{-1}\gamma), \end{aligned}$$

and

$$\begin{aligned} \left(\tilde{\pi}(a_g)\lambda(g)(h)\right)(\gamma) &= \tilde{\pi}(a_g)\left(\Delta(g)^{1/2}\alpha_g(h)\right)(\gamma) \\ &= \pi_{s(\gamma)}\left(\alpha_{\gamma^{-1}}(a_g)\right)\Delta(g)^{1/2}\alpha_g(h)(\gamma) \\ &= \Delta(g)^{1/2}\pi_{s(\gamma)}\left(\alpha_{\gamma^{-1}}(a_g)\right) h(g^{-1}\gamma), \end{aligned}$$

so that

$$L(a_gU_g) = \Delta(g)^{-1/2}\tilde{\pi}(a_g)\lambda(g). \tag{6.3.2}$$

Now, we identify $\delta_{g,\xi}$ with the element of $L^2(G^{(0)} * \mathfrak{K}, \mu)$ which sends $r(g)$ to $\delta_{g,\xi}$, and all other $u \in G^{(0)}$ to $0 \in \mathcal{H}(u)$. We compute that

$$\begin{aligned} (\tilde{\pi}(a_g)(\delta_{g,\xi}))(u)(h) &= \pi_{s(h)}(\alpha_h^{-1}((a_g)|_{r_X^{-1}(u)}))(\delta_{g,\xi}(u)(h)) \\ &= \begin{cases} \pi_{s(h)}(\alpha_h^{-1}((a_g)|_{r_X^{-1}(r(g))}))(\delta_{g,\xi}(h)) & \text{if } u = r(g) \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \pi_{s(g)}(\alpha_g^{-1}(a_g))\xi & \text{if } u = r(g) \text{ and } h = g \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This shows that

$$\tilde{\pi}(a_g)\delta_{g,\xi} = \delta_{g,\pi_{s(g)}(\alpha_g^{-1}(a_g))\xi}, \tag{6.3.3}$$

where the right-hand side is being identified with an element of $L^2(G^{(0)} * \mathfrak{K}, \mu)$ in the same way that $\delta_{g,\xi}$ was.

Thus, using the equalities (6.3.2), (6.3.1), and (6.3.3), notice that, for $u, w \in G^{(0)}$ and $g \in G_u$,

$$\begin{aligned} V_w^* L(a_g U_g) V_u \xi &= \Delta(g)^{-1/2} V_w^* \tilde{\pi}(a_g)(\lambda(g) V_u \xi) \\ &= \Delta(g)^{-1/2} V_w^* \tilde{\pi}(a_g) V_g \xi \\ &= V_w^* \tilde{\pi}(a_g)\delta_{g,\xi} \\ &= V_w^* \delta_{g,\pi_u(\alpha_{g^{-1}}(a_g))\xi} \\ &= \Delta(w)^{-1/2}\delta_{g,\pi_u(\alpha_{g^{-1}}(a_g))\xi}(w) \\ &= \delta_g(w)\pi_u(\alpha_{g^{-1}}(a_g))\xi, \end{aligned}$$

which is zero unless $g = w = u$. Therefore,

$$\begin{aligned} V_u^* L\left(\sum_{g\in G} a_g U_g\right) V_u \xi &= \pi_u(\alpha_{u^{-1}}(a_u))\xi \\ &= \pi_u(a_u)\xi \\ &= \pi_u\left(\sum_{w\in G^{(0)}} a_w\right)\xi \\ &= \pi_u\left(E\left(\sum_{g\in G} a_g U_g\right)\right)\xi, \end{aligned}$$

so that we can use continuity of E to see that, for any $b \in C(X) \rtimes_r G$, we have

$$V_u^* L(b) V_u = \pi_u\left(E(b)\right). \tag{6.3.4}$$

Now, suppose that $b \in \Gamma_c(G, r^*\mathcal{A})$ satisfies $b^*b = \sum_{h\in G} c_h U_h$, and observe that, using the equalities (6.3.1), (6.3.2), and (6.3.4), we have

$$\begin{aligned} V_g^* L(b^*b) V_g &= V_{s(g)}^* \lambda(g)^* L\left(\sum_{h\in G} c_h U_h\right)\lambda(g) V_{s(g)} \\ &= V_{s(g)}^* \lambda(g^{-1}) \sum_{h\in G} L(c_h U_h)\lambda(g) V_{s(g)} \\ &= V_{s(g)}^* \sum_{h\in Gr(g)} \lambda(g^{-1})\Delta(h)^{-1/2}\tilde{\pi}(c_h)\lambda(hg) V_{s(g)} \end{aligned}$$

$$
\begin{aligned}
&= V_{s(g)}^* \sum_{h \in Gr(g)} \lambda(g^{-1})\Delta(h)^{-1/2}\tilde{\pi}_{r(h)}(c_h)\lambda(hg)V_{s(g)} \\
&= V_{s(g)}^* \sum_{h \in r(g)Gr(g)} \Delta(h)^{-1/2}\tilde{\pi}_{s(g)}(\alpha_{g^{-1}}(c_h))\lambda(g^{-1}hg)V_{s(g)} \\
&= V_{s(g)}^* \sum_{h \in r(g)Gr(g)} \Delta(h)^{-1/2}\tilde{\pi}(\alpha_{g^{-1}}(c_h))\lambda(g^{-1}hg)V_{s(g)} \\
&= V_{s(g)}^* \sum_{h \in r(g)Gr(g)} L(\alpha_{g^{-1}}(c_h)U_{g^{-1}hg})V_{s(g)} \\
&= V_{s(g)}^* L\left(\sum_{h \in r(g)Gr(g)} \alpha_{g^{-1}}(c_h)U_{g^{-1}hg}\right)V_{s(g)} \\
&= V_{s(g)}^* L\left(\sum_{h \in G} U_{g^{-1}}c_hU_hU_g\right)V_{s(g)} \\
&= V_{s(g)}^* L\left(U_g^*\left(\sum_{h \in G} c_hU_h\right)U_g\right)V_{s(g)} \\
&= V_{s(g)}^* L\left(U_g^*b^*bU_g\right)V_{s(g)} \\
&= \pi_{s(g)}(E(U_g^*b^*bU_g)).
\end{aligned} \tag{6.3.5}
$$

Now, suppose $b \in C(X) \rtimes_r G$ is such that $E(b^*b) = 0$. By equivariance, we have for every $g \in G$ that $E(U_g^*b^*bU_g) = \alpha_{g^{-1}}(E(b^*b)) = 0$. By equality (6.3.5), this yields

$$
0 = \pi_{s(g)}(E(U_g^*b^*bU_g)) = V_g^*L(b^*b)V_g. \tag{6.3.6}
$$

Now, observe that, for every $g \in G$ and $\xi \in H(s(g))$, we can use equality (6.3.6) to obtain

$$
\begin{aligned}
\|L(b)\delta_{g,\xi}\|_{\mathcal{H}}^2 &= \langle L(b^*b)\delta_{g,\xi}, \delta_{g,\xi}\rangle_{\mathcal{H}(r(g))} \\
&= \Delta(g)^{-1}\langle L(b^*b)V_g\xi, V_g\xi\rangle_{\mathcal{H}(r(g))} \\
&= \Delta(g)^{-1}\left\langle \left(V_g^*L(b^*b)V_g\right)\xi, \xi\right\rangle_{H(s(g))} \\
&= 0,
\end{aligned}
$$

from which it follows that $b = 0$ (L is faithful by [23, Remark 3.125]). □

Lemma 6.3.3 (c.f. [49] Lemma 7.8)**.** *Let G be a locally compact Hausdorff étale groupoid acting freely and continuously on a compact Hausdorff space. Let $a \in \Gamma_c(G, r^*\mathcal{A}) \subset C(X) \rtimes G$, and let $\epsilon > 0$. Then there exists $f \in C(X)$ such that $0 \leq f(x) \leq 1$ for every $x \in X$, $fa^*af \in C(X)$, and $\|fa^*af\| \geq \|E(a^*a)\| - \epsilon$.*

Proof. Write $b = a^*a$. If $\|E(b)\| \leq \epsilon$, then we can simply take $f = 0$, so suppose there exists $x \in X$ such that $|E(b)(x)| > \epsilon$. Since a is a compactly supported section, and the product of compact subsets of G is always compact, there exists a compact subset $K \subset G$, and $b_g \in C(r_X^{-1}(r(g)))$ for each $g \in K$, such that $b = \sum_{g \in K} b_g U_g$. Since b is positive, $E(b)$ is too. Let

$$U = \{x \in X \mid E(b)(x) > \|E(b)\| - \epsilon\},$$

which is a nonempty open subset of X. We wish to find a nonempty open set $W \subset U$ such that the sets in the collection $\{gW \mid g \in K\}$ are pairwise disjoint.

Notice that for $g, h \in K$, $gW \cap hW \neq \varnothing$ if and only if there exist $w_1, w_2 \in W$ such that $gw_1 = hw_2$ if and only if $w_1 = g^{-1}hw_2$ if and only if $W \cap K^{-1}KW \neq \varnothing$, so it is equivalent to ask that whenever $g, h \in K$ are such that $g^{-1}h \notin G^{(0)}$, we have $W \cap g^{-1}hW = \varnothing$.

Since G is étale, the topology on G has a base of open bisections $\{B_i\}_{i \in I}$. Therefore, $\{B_i\}_{i \in I}$ is an open cover of $K^{-1}K$, which is the product of compact subsets in an étale groupoid, and so is compact itself by [27, Lemma 5.2]. So, there exists a finite subcover $\{B_1, \ldots, B_N\}$ of $K^{-1}K$. In fact, since G is étale and Hausdorff, $G^{(0)}$ is a clopen bisection. Therefore, by putting $B_0 = G^{(0)}$ and $B_k = B_k \setminus G^{(0)}$ for each $k \in \{1, \ldots, N\}$, then redefining N and relabelling, we may assume that $B_1 = G^{(0)}$ and that $B_i \cap G^{(0)} = \varnothing$ for each $i \in \{2, \ldots, N\}$. We will construct an open $W \subset U$ such that $W \cap \bigcup_{k=2}^N B_k W = \varnothing$.

Now, we wish to find a subset $V \subset U$ such that, for each $k \in \{1, \ldots, N\}$, either $r_X(V) \subset s(B_k)$, or $r_X(V) \cap s(B_k) = \varnothing$. In other words, for each k, either every element of V will be acted upon by B_k, or none of them will be. We will construct a nested sequence of nonempty open sets $U \supset V_1 \supset V_2 \supset \cdots \supset V_N = V$ so that, for each $k \in \{1, \ldots, N\}$, and each $1 \leq j \leq k$, either $r_X(V_k) \subset s(B_j)$, or $r_X(V_k) \cap s(B_j) = \varnothing$. Observe that since $B_1 = G^{(0)}$, the choice $V_1 = U$ works.

Now, suppose we have constructed open sets $U = V_1 \supset V_2 \supset \cdots \supset V_{k-1}$ as above for some $2 \leq k \leq N$, and proceed in cases as follows.

Case 1: If $r_X(V_{k-1}) \cap s(B_k) = \varnothing$, then set $V_k = V_{k-1}$.

Case 2: If $r_X(V_{k-1}) \cap s(B_k) \neq \varnothing$, then set $V_k = V_{k-1} \cap r_X^{-1}(s(B_k))$. Observe that V_k is open because $r_X^{-1}(s(B_k))$ is open, since B_k is open, s is an open map, and r_X is continuous.

Choose any $x \in V = V_N$. For each $k \in \{1, \ldots, N\}$, since B_k is a bisection, there is at most one element $g \in B_k$ such that $s(g) = r_X(x)$, so the set $\bigcup_{k=1}^N B_k x \subset X$ is finite with cardinality at most N. Since the action is free, and since $B_k \cap G^{(0)} = \varnothing$ whenever $2 \leq k \leq N$, for each $g \in \bigcup_{k=2}^N B_k r_X(x)$ we have $gx \neq x$. Since X is Hausdorff, for each such $gx = y$, we can find disjoint open subsets $x \in V_y$ and $y \in Z_y$. Let $V' = V \cap \left(\bigcap_{y \in \bigcup_{k=2}^N B_k x} V_y\right)$, which is nonempty since $x \in V'$, and is open in X because it is the intersection of finitely many open sets. Observe also that $V' \cap Z_y = \varnothing$ for each $y \in \bigcup_{k=2}^N B_k x$.

We now wish to find a second finite sequence of nested open subsets $V' \supset W_1 \supset W_2 \supset \cdots \supset W_N \ni x$, such that, for each $k \in \{2, \ldots, N\}$, we have $W_k \cap gW_k = \varnothing$ for every

$g \in \bigcup_{j=2}^{k} B_j$. Then $W = W_N$ will be such that $gW \cap hW = \varnothing$ for all $g \neq h \in K$, as we wanted. Start with $W_1 = V'$, and proceed inductively as follows.

Suppose that we have constructed $V' = W_1 \supset \cdots \supset W_{k-1} \ni x$ as above for some $2 \leq k \leq N$, and construct W_k by proceeding in cases as follows. First, observe that since $W_{k-1} \subset V$, either $r_X(W_{k-1}) \subset s(B_k)$, or $r_X(W_{k-1}) \cap s(B_k) = \varnothing$.

Case 1: Suppose $r_X(W_{k-1}) \cap s(B_k) = \varnothing$. This means that $B_k W_{k-1} = \varnothing$, so we have $gW_{k-1} = \varnothing$ for each $g \in B_k$. We assumed that for each $g \in \bigcup_{j=2}^{k-1} B_j$ we have $W_{k-1} \cap gW_{k-1} = \varnothing$, and so for each $g \in \bigcup_{j=2}^{k} B_j$ we still have $W_{k-1} \cap gW_{k-1} = \varnothing$. Thus, we see that the choice $W_k = W_{k-1}$ is suitable.

Case 2: Suppose $r_X(W_{k-1}) \subset s(B_k)$. Then, since $x \in W_{k-1}$, there exists $g \in B_k$ such that $s(g) = r_X(x)$. Write $y = gx$. Since B_k is open in G, and since G is étale, we see by Lemma 4.3.8 that $B_k W_{k-1}$ is an open subset of X. Thus, Z_y and $B_k W_{k-1}$ are open subsets of X which both contain y, so that $Z_y \cap B_k W_{k-1}$ is a nonempty open set in X. Put $W_k = B_k^{-1}(Z_y \cap B_k W_{k-1}) \subset W_{k-1}$. Observe that $x \in W_k$, and W_k is open by Lemma 4.3.8, because B_k^{-1} is open in X. All we have to check is that for each $g \in B_k$ we have $W_k \cap gW_k = \varnothing$ (note that this automatically holds for $g \in \bigcup_{j=2}^{k-1} B_j$, because $W_k \subset W_{k-1}$). Indeed, $gW_k \subset B_k W_k \subset Z_y$, which is disjoint from $V' \supset W_k$, as required.

So, construct an open set $W \subset U$ such that the collection $\{gW \mid g \in K\}$ is pairwise disjoint. Note in particular that, by construction of K, $r(K) \subset K$, so that $r(g)W$ and gW are disjoint for each $g \in K$. Choose $x_0 \in W$, and let $f \in C(X)$ satisfy $0 \leq f(x) \leq 1$ for every $x \in X$, $\operatorname{supp}(f) \subset W$, and $f(x_0) = 1$. Then

$$fbf = \sum_{g \in K} f b_g U_g f = \sum_{g \in K} f b_g \alpha_g(f) U_g.$$

Since $\operatorname{supp}(f) \subset W$ and $\operatorname{supp}(\alpha_g(f)) \subset gW$, observe that $f b_g \alpha_g(f) = b_g f \alpha_g(f)$ is supported inside $W \cap gW$. Since $gW \subset r_X^{-1}(r(g))$, we have $W \cap gW = r(g)(W \cap gW) = r(g)W \cap gW$, and we thus see that $f b_g \alpha_g(f)$ is supported within $r(g)W \cap gW$. So, for $g \in K \setminus G^{(0)}$, our construction of W provides $f b_g \alpha_g(f) = 0$. Thus, we have

$$fbf = \sum_{u \in K \cap G^{(0)}} f b_u \alpha_u(f) U_u = \sum_{u \in K \cap G^{(0)}} f b_u f|_{r_X^{-1}(u)},$$

where we have identified the central expression with a complex-valued function on X. Since $\operatorname{supp}(b_u) \subset r_X^{-1}(u)$, we see that $f b_u f|_{r_X^{-1}(u)} = f b_u f$, so that

$$fbf = f \left(\sum_{u \in G^{(0)}} b_u \right) f = f E(b) f \in C(X).$$

In addition, we have

$$\|fbf\| \geq |fbf(x_0)| = |f(x_0)b_{r_X(x_0)}(x_0)f(x_0)| = |b_{r_X(x_0)}(x_0)| = |E(b)(x_0)| > \|E(b)\| - \epsilon,$$

where the final inequality follows from the fact that $x_0 \in U$. □

The following is another straightforward generalisation of a result from [49] to the groupoid setting, and the proof is almost unchanged.

Lemma 6.3.4 (c.f. [49, Lemma 7.9])**.** *Let $G \curvearrowright X$ be a free continuous action of a locally compact Hausdorff étale groupoid on a compact Hausdorff space. Let $B \subset C(X) \rtimes_r G$ be a unital subalgebra such that*

(i) $C(X) \subset B$; and

(ii) $B \cap \Gamma_c(G, r^*\mathcal{A})$ is dense in B.

Let $a \in B_+ \setminus \{0\}$. Then there exists $b \in C(X)_+ \setminus \{0\}$ such that $b \precsim_B a$, where $\precsim_B$ denotes the relation of Cuntz subequivalence over B (see Definition 6.1.1).

Proof. Without loss of generality, we may assume $\|a\| \leq 1$. Since the conditional expectation $E : C(X) \rtimes_r G \to C(X)$ is faithful, $E(a) \in C(X)$ is a nonzero positive element. Let $\epsilon = \frac{1}{8}\|E(a)\|$ and note that $\epsilon \leq 1$, since $\|E(a)\| \leq \|a\|$. Since $B \cap \Gamma_c(G, r^*\mathcal{A})$ is dense in B, choose $c \in B \cap \Gamma_c(G, r^*\mathcal{A})$ such that $\|c - a^{1/2}\| < \epsilon$. Without loss of generality, we may choose c such that $\|c\| \leq 1$. Notice that

$$\begin{aligned}
\epsilon^2 > \|c - a^{1/2}\|^2 &= \|(c - a^{1/2})(c - a^{1/2})^*\| \\
&= \|cc^* + a - ca^{1/2} - c^*a^{1/2}\| \\
&\geq \|cc^* - a\| - \|2a - ca^{1/2} - c^*a^{1/2}\| \\
&\geq \|cc^* - a\| - \|a^{1/2}\|\|2a^{1/2} - c - c^*\|,
\end{aligned}$$

so that

$$\|cc^* - a\| < \epsilon^2 + \|a^{1/2}\|\|a^{1/2} - c\| + \|a^{1/2}\|\|(a^{1/2} - c)^*\| < 3\epsilon.$$

Similarly, $\|c^*c - a\| < 3\epsilon$. Since E is norm decreasing, we see that $\|E(c^*c - a)\| < 3\epsilon$, which implies that $\|E(a)\| < \|E(c^*c)\| + 3\epsilon$. Applying Lemma 6.3.3 with c in place of a, and with the ϵ defined above, yields $f \in C(X)$ with $0 \leq f \leq 1$, which satisfies

$$\|fc^*cf\| \geq \|E(c^*c)\| - \epsilon > \|E(a)\| - 4\epsilon = 8\epsilon - 4\epsilon = 4\epsilon.$$

Therefore, $(fc^*cf - 3\epsilon)_+$ is a nonzero positive element of $C(X)$. To conclude the proof, we apply [49, Lemma 1.4(6)] at the first step, [49, Lemma 1.7] and the fact that $cf^2c^* \leq cc^*$ because $0 \leq f \leq 1$ at the second step, and [49, Lemma 1.4(10)] and $\|cc^* - a\| < 3\epsilon$ at the final step, to see that

$$(fc^*cf - 3\epsilon)_+ \sim_B (cf^2c^* - 3\epsilon)_+ \precsim_B (cc^* - 3\epsilon)_+ \precsim_B a. \qquad \square$$

6.4 $\mathcal{Z}$-stability of tiling C^*-algebras

This section contains our main result (Theorem 6.4.3). Before we get to it, we need one last lemma, which will allow us to witness the Cuntz subequivalence in Theorem 6.1.6. For convenience, we first recall our notion of dynamical comparison for groupoids.

Definition 6.4.1. Let $G \curvearrowright X$ be a groupoid action, and let $A, B \subset X$. We write $A \prec B$ if, for every closed $C \subset A$, there exist a finite collection $U_1, \ldots, U_M$ of open subsets of X which cover C, and a subset $S_m \subset G$ for each $m \in \{1, \ldots, M\}$, which are such that $r_X(U_m) \subset s(S_m)$ and so that the collection $\{tU_m \mid m \in \{1, \ldots, M\}, t \in S_m\}$ consists of pairwise disjoint subsets of B.

Lemma 6.4.2 (c.f. [35, Lemma 12.3])**.** *Let $G \curvearrowright X$ be a continuous action of a groupoid on a compact metrisable space. Let A be a closed subset of X and B an open subset of X such that $A \prec B$. Let $f, g : X \to [0,1]$ be continuous functions such that $f(x) = 0$ for any $x \in X \setminus A$, and $g(b) = 1$ for any $b \in B$. Then there exists $v \in C(X) \rtimes_r G$ such that $v^*gv = f$.*

Proof. Since $A \prec B$, and A is a closed subset of A, find a finite collection $U_1, \ldots, U_M$ of open subsets of X which cover A, and subsets $S_1, \ldots, S_M$ of G as in the definition. Without loss of generality, by removing any element $t \in S_m$ such that $s(t) \notin r_X(U_m)$, we may assume that $r_X(U_m) = s(S_m)$. Furthermore, by preserving one element from each source bundle and removing all other elements from S_m if necessary, we may assume that $s : S_m \to r_X(U_m)$ is injective for each $m \in \{1, \ldots, M\}$.

Using a partition of unity argument, produce, for each $m \in \{1, \ldots, M\}$, a continuous function $h_m : X \to [0,1]$ such that $h_m(x) = 0$ for any $x \in X \setminus U_m$, and which satisfy $0 \leq \sum_{m=1}^M h_m(x) \leq 1$ for any $x \in X$, and $\sum_{m=1}^M h_m(a) = 1$ for any $a \in A$. Put $v = \sum_{m=1}^M \sum_{t \in S_m} U_t(fh_m)^{1/2}$.

Denote by α the induced action of G on $C(X)$, given by

$$\alpha_g(f)(x) = \begin{cases} f(g^{-1}x) & \text{if } x \in r_X^{-1}(r(g)) \\ 0 & \text{otherwise.} \end{cases}$$

We compute that

$$\begin{aligned} v^*gv &= \left(\sum_{m=1}^{M}\sum_{t\in S_m}(fh_m)^{1/2}U_{t^{-1}}\right) g \left(\sum_{\tilde{m}=1}^{M}\sum_{\tilde{t}\in S_{\tilde{m}}}U_{\tilde{t}}(fh_{\tilde{m}})^{1/2}\right) \\ &= \sum_{m,\tilde{m}=1}^{M}\sum_{t\in S_m}\sum_{\tilde{t}\in S_{\tilde{m}}} U_{t^{-1}}g\alpha_t(f)^{1/2}\alpha_t(h_m)^{1/2}\alpha_{\tilde{t}}(h_{\tilde{m}})^{1/2}\alpha_{\tilde{t}}(f)^{1/2}U_{\tilde{t}}. \end{aligned}$$

Note that $\alpha_t(h_m)$ is supported on tU_m, and $\alpha_{\tilde{t}}(h_{\tilde{m}})$ is supported on $\tilde{t}U_{\tilde{m}}$, which are disjoint unless $m=\tilde{m}$ and $t=\tilde{t}$. Thus, the only nonzero contributions to the above sum require $t=\tilde{t}$ and $m=\tilde{m}$, so we obtain

$$\begin{aligned} v^*gv &= \sum_{m=1}^{M}\sum_{t\in S_m} U_{t^{-1}}g\alpha_t(h_m)\alpha_t(f)U_t \\ &= \sum_{m=1}^{M}\sum_{t\in S_m} \alpha_{t^{-1}}(g)h_m f. \end{aligned}$$

The term corresponding to a particular $m\in\{1,\dots,M\}$ and $t\in S_m$ in the above sum is supported in $U_m\cap r_X^{-1}(s(t))$. For $x\in U_m\cap r_X^{-1}(s(t))$, we have $\alpha_{t^{-1}}(g)(x)=g(tx)$, where $tx\in tU_m\subset B$, so we obtain $\alpha_{t^{-1}}(g)=1$ wherever this term is supported, because $g(b)=1$ for any $b\in B$ by assumption. This gives

$$v^*gv = \sum_{m=1}^{M}\sum_{t\in S_m}(fh_m)|_{U_m\cap r_X^{-1}(s(t))}$$

Since $s(S_m)=r_X(U_m)$, since $s:S_m\to r_X(U_m)$ was injective for each m, and using the facts that h_m is supported inside U_m and f is supported inside A, together with our assumption that $\sum_{m=1}^{M}h_m(a)=1$ for each $a\in A$, we see that

$$v^*gv = \sum_{m=1}^{M}(fh_m)|_{U_m} = \sum_{m=1}^{M} fh_m = \sum_{m=1}^{M} f(h_m)|_A = f\sum_{m=1}^{M}(h_m)|_A = f. \qquad \square$$

Observe that for any groupoid G, the transformation groupoid associated to the canonical action $\alpha : G \curvearrowright G^{(0)}$ is isomorphic to G itself. This allows us to cast any groupoid with compact unit space as a groupoid crossed product as follows. Denote by $\tilde{\alpha}$ the induced action of G on $C(G^{(0)})$. By [23, Proposition 4.38], the groupoid crossed product $C(G^{(0)})\rtimes_{\tilde{\alpha}}G$ is isomorphic to the groupoid C^*-algebra $C^*(G\ltimes_\alpha G^{(0)})$ of the transformation groupoid associated to α. Since $G\ltimes_\alpha G^{(0)}\cong G$, we have also that $C^*(G\ltimes_\alpha G^{(0)})\cong C^*(G)$, and so we obtain the chain of isomorphisms $C^*(G)\cong C^*(G\ltimes G^{(0)})\cong C(G^{(0)})\rtimes_{\tilde{\alpha}}G$. In the following result, we will use this to think of tiling groupoids (see Definition 3.3.10) as groupoid crossed products.

We now present our main result. Our proof is heavily inspired by that of [35, Theorem 12.4], with the ideas translated into the groupoid setting. As written, the proof depends quite heavily on the geometric structure of the tiling groupoid, though it should be possible to generalise the result significantly.

Theorem 6.4.3 (c.f. [35, Theorem 12.4]). *Let $G = R_{\text{punc}}$ be the groupoid associated to an aperiodic and repetitive tiling with finite local complexity acting on its unit space $X = \Omega_{\text{punc}}$ in the canonical way. Then $C_r^*(R_{\text{punc}}) \cong C(X) \rtimes_r G$ is $\mathcal{Z}$-stable.*

Proof. As per usual, we let α denote the induced action of G on $C(X)$, so that $\alpha_g : C(X) \to C(r_X^{-1}(r(g)))$ is given by $\alpha_g(f)(x) = f(g^{-1}x)$ for $x \in r_X^{-1}(r(g))$, and we define $\alpha_g(f)(x) = 0$ if $r_X(x) \neq r(g)$.

Let $n \in \mathbb{N}$, $\epsilon > 0$, and let $a \in C(X) \rtimes_r G$ be a nonzero positive element. Let $F \subset C(X) \rtimes_r G$ be a finite set. We will show that there exists a map $\phi : M_n \to C(X) \rtimes_r G$ as in Theorem 6.1.6, from which we will be able to conclude that $C(X) \rtimes_r G$ is $\mathcal{Z}$-stable.

Notice that, since $\Gamma_c(G, r^*\mathcal{A})$ is dense in $A \rtimes_r G$, it will be enough to consider finite subsets $F \subset \Gamma_c(G, r^*\mathcal{A})$. Each such subset is generated by a finite subset Υ of the unit ball of $C(X)$ and a compact subset $K \subset G$ which, without loss of generality, we may assume contains $G^{(0)}$ and is closed under taking inverses. To see that Υ can be chosen to be finite, fix any $f = \sum_{g \in C} f_g U_g \in F$ and observe that, since C is compact and G is ample, C admits a finite cover of pairwise disjoint compact open bisections $\{B_1, \ldots, B_N\}$. Then f is generated by $\{U_g \mid g \in C\}$ together with the finite collection of functions $\{\sum_{g \in B_i} f_g\}_{i=1,\ldots,N}$, each of which is in $C(X)$ by the argument at the start of Section 6.3. Since $C(X) \rtimes_r R_{\text{punc}}$ is generated by sets of this form, it will be enough to check condition (ii) in Theorem 6.1.6 for each element $w \in \Upsilon$ and for $w = \sum_{g \in K} U_g$.

Since K is a compact subset of R_{punc}, there exists a real number $R_K > 0$ such that if $(T - x, T) \in K$ then $|x| < R_K$. By enlarging K we may further assume, without loss of generality, that $R_K^{-1} < r/2$, where r is as in Lemma 3.3.5. This ensures that whenever $d(T, T') < R_K^{-1}$ in the tiling metric, the subsets KT and $KT' \subset R_{\text{punc}}$ consist of translations by the same set of vectors in $\mathbb{R}^d$. We enlarge K one last time to assume that $K = \{(T - x, T) \in R_{\text{punc}} \mid |x| < R_K\}$. Reasoning as for the sets constructed in Example 6.2.9, this choice of K is seen to be compact.

By Lemma 6.3.4, we may assume that $a \in C(X)$. Then, since a is a nonzero positive element, there exists $x_0 \in X$ and $\theta > 0$ such that a is strictly positive on the closed ball of radius 2θ centred at x_0. Thus, we may assume that a is a $[0, 1]$-valued function which takes the value 1 on all points within distance 2θ from x_0, and the value 0 at all points at distance at least 3θ from x_0. Denote by O the open ball of radius θ centred at x_0.

Because the action is minimal, there exists a subset $D \subset G$ such that $D^{-1} \cdot O = X$ and so that there exists $m \in \mathbb{N}$ such that, for each $u \in G^{(0)}$, $|D \cap r^{-1}(u)| < m$ and $|D \cap s^{-1}(u)| < m$. To see this, since G is étale it has a base of open bisections $\{B_i\}_{i\in I}$. For each $i \in I$, the set B_iO is open in X by Lemma 4.3.8, because B_i is open in G and O is open in X. Because B_i is a bisection, for each point $x \in O$, there exists at most one $g \in B_i$ such that $s(g) = r_X^{-1}(x)$, and therefore at most one image $gx \in B_iO$. Since the action is minimal, the collection $\{B_iO\}_{i\in I}$ is an open cover of X and, since X is compact, there exists a finite subcover $\{B_1O, \dots, B_mO\}$. Let $D = \bigcup_{j=1}^m B_j^{-1}$, which has the properties we seek.

Let $0 < \kappa < 1$, to be determined on page 151. Choose an integer $Q > (n^2 \sup_{u\in G^{(0)}} |uK|)/\epsilon$.

Denote by K^Q the product of K with itself Q times in G:

$$K^Q := \{k_Q \cdots k_1 \mid (k_{i+1}, k_i) \in K^{(2)} \text{ for every } i = 1, \dots, Q-1\}.$$

Take $\beta > 0$ to be small enough so that if T is a nonempty compact subset of G which is sufficiently invariant under left-translation by K^Q, then, for every $T' \subset T$ with $|T'u| \geq (1 - n\beta)|Tu|$ at every $u \in G^{(0)}$, we have, for every $u \in G^{(0)}$, that

$$\left| \bigcap_{g\in K^Q} \left(gT' \sqcup \bigsqcup_{v\in G^{(0)}\setminus r(g)} vT' \right) u \right| \geq (1-\kappa)|Tu|. \tag{6.4.1}$$

To ease thinking, note that since $G^{(0)} \subset K$ and K is closed under taking inverses, the left-hand set is the subset consisting of $t \in T'u$ such that $K^Qt \subset T'u$. Choose $L \in \mathbb{N}$ large enough so that $(1 - \beta/2)^L < \beta$. By Example 6.2.9, there exist compact subsets $G^{(0)} \subset T_1 \subset \cdots \subset T_L$ of G such that $|\partial_{T_{l-1}}(T_lu)| \leq (\beta^2/8)|T_lu|$ for each $l = 2, \dots, L$ and $u \in G^{(0)}$. We may also enlarge each T_l in turn to assume that it is sufficiently invariant under left translation by K^Q so that we can use the previous paragraph to ensure that for every subset $T' \subset T_l$ satisfying $|T'u| \geq (1 - n\beta)|T_lu|$ for each $u \in G^{(0)}$, we have, for each $u \in G^{(0)}$, that (6.4.1) holds for this choice of T' and with T replaced with T_l.

Since T_L was constructed as in Example 6.2.9, there exists $R_L > 0$ such that $T_L = \{(T - x, T) \in R_{\text{punc}} \mid |x| < R_L\}$. By the uniform continuity of the functions in $\Upsilon \cup \Upsilon^2$, and of the action map $T_L \times X \to X$, there exists $0 < \eta < (QR_K + R_L)^{-1}$ so that if $x, y \in X$, and $t, t' \in T_L$ are such that $s(t) = x$ and $s(t') = y$, and satisfy $d(x, y) < \eta$ and $d(t, t') < \eta$, then $|f(tx) - f(t'y)| < \epsilon^2/(4n^4)$. Let $\mathcal{U} = \{U_1, \dots, U_M\}$ be an open cover of X whose members all have diameter less than η. Let $\eta > \eta' > 0$ be a Lebesgue number for $\mathcal{U}$ which is no larger than θ.

Let E be a compact subset of G containing T_L, and let $\delta > 0$ be such that $\delta < \beta^2/4$. Since G_u is infinite for each $u \in G^{(0)}$, we may enlarge E and shrink δ as necessary to ensure that, for each nonempty (E, δ)-invariant compact set $S \subset G$ and each $u \in G^{(0)}$, we have that

$$\beta|Su| \geq Mn \sum_{l=1}^{L} \max_{v \in G^{(0)}} |T_l v|. \tag{6.4.2}$$

By combining Lemma 4.4.4 and Theorem 4.3.15, we see that the action $G \curvearrowright X$ admits clopen tower decompositions of arbitrary invariance. Therefore, for some $N \in \mathbb{N}$, we can find nonempty clopen sets $W_1, \ldots, W_N \subset X$ and nonempty (E, δ)-invariant compact open sets $S_1, \ldots, S_N \subset G$ such that the family $\{(W_k, S_k)\}_{k=1}^{N}$ is a clopen tower decomposition of X with levels of diameter less than η'. Write $S_k = \bigsqcup_{j=1}^{n_k} S_{k,j}$ for the decomposition of each shape. By Remark 4.4.5, we may assume, without loss of generality, that, for each $k \in \{1, \ldots, N\}$, there exists a patch P_k and a tile $t_k \in P_k$ such that $W_k = U(P_k, t_k)$, and so that, for each $j \in \{1, \ldots, n_k\}$, there exists $t_{k,j} \in P_k$ such that $S_{k,j} = V(P_k, t_k, t_{k,j})$. Note in particular that, for each $l \in \{1, \ldots, L\}$, since $T_l = \{(T - x, T) \in R_{\text{punc}} \mid |x| < R_l\}$ for some $0 < R_l \leq R_L$, our choice of η' ensures that if x and y are in the same tower level, then $T_l x$ and $T_l y$ consist of arrows which implement exactly the same set of vectors of translation.

Let $k \in \{1, \ldots, N\}$. Since S_k is $(T_L, \beta^2/4)$-invariant, and since the sets $T_1, \ldots, T_L$ satisfy the assumptions of Theorem 6.2.8, we can find $C_{k,1}, \ldots, C_{k,L} \subset S_k$ such that $\bigcup_{l=1}^{L} T_l C_{k,l} \subset S_k$, and so that the collection $\{T_l c \mid l \in \{1, \ldots, L\}, c \in C_{k,l}\}$ is β-disjoint and $(1-\beta)$-covers S_k. We claim that we may assume that, for each $l \in \{1, \ldots, L\}$ and each $p \in \{1, \ldots, n_k\}$, we have either $S_{k,p} \subset T_l C_{k,l}$ or $S_{k,p} \cap T_l C_{k,l} = \varnothing$. That is, $T_l C_{k,l}$ either contains all the groupoid elements corresponding to a given level of the tower, or none of them. We prove this assertion over a number of steps.

To elaborate, recall that $C_{k,L}$ was constructed from the set $I = \{c \in S_k \mid T_L c \subset S_k\}$. Suppose $c \in S_{k,j} = V(P_k, t_k, t_{k,j})$ is an element of I. Then, for each fixed $t \in T_L r(c)$, $tc \in S_k$, and so, in particular, $tc \in S_{k,i} = V(P_k, t_k, t_{k,i})$ for some i. Recall that T_L was the collection of allowable vectors of translation of magnitude smaller than R_L. Thus, if $tc \in V(P_k, t_k, t_{k,i}) \subset S_k$ for some $c \in S_{k,j}$, then the vector $x(t_{k,i}) - x(t_{k,j})$ was associated to t, and was allowable in $r(c) \in U(P_k, t_{k,j})$. Let c' be any other element of $S_{k,j}$. By the diameter condition on the tower levels, $T_L r(c')$ consists of arrows which implement the same set of vectors as the arrows in $T_L r(c)$. By the argument at the start of the paragraph, we see that any such arrow $t' \in T_L r(c')$ implements the vector $x(t_{k,i}) - x(t_{k,j})$ for some i. Since $c' \in S_{k,j} = V(P_k, t_k, t_{k,j})$, we see that $t'c' \in V(P_k, t_k, t_{k,i}) = S_{k,i} \subset S_k$. This shows that $c' \in I$, so that $S_{k,j} \subset I$. Hence, we have shown that, for each j, either $S_{k,j} \cap I = \varnothing$ or $S_{k,j} \subset I$.

To obtain $C_{k,L}$, we applied Lemma 6.2.3 to get a subset $C_{k,L} \subset I$ such that the collection $\{T_L c \mid c \in C_{k,L}\}$ was β-disjoint. In doing so, we claim that we can choose to either remove or keep each tower level as a whole. Indeed, if we fix any $x \in W_k$ and apply Lemma 6.2.3 to the collection $\{T_L cx \mid c \in I\}$, then we obtain $C_{k,L}x \subset Ix$ such that $\{T_L cx \mid c \in C_{k,L}x\}$ is β-disjoint. Now, choose any other $y \in W_k = U(P_k, t_k)$. By the previous paragraph, I contains all or none of the elements of each $S_{k,p}$, so, by the structure of the tower (W_k, S_k), Ix and Iy consist of arrows which implement the same vectors of translation. This allows us to choose $C_{k,L}y$ to consist of arrows which implement the same vectors as those in $C_{k,L}x$. Since $I \subset S_k$, if $c, c' \in I$ implement the same vector of translation and have $s(c) = x$ and $s(c') = y$, then $r(c)$ and $r(c')$ are in the same tower level, and thus (as tilings) they agree on $B_{\eta^{-1}}(0)$. Therefore, $T_L r(c)$ and $T_L r(c')$ consist of elements which implement the same set of vectors. This shows that $\{T_L cx \mid c \in C_{k,L}x\}$ and $\{T_L c'y \mid c' \in C_{k,L}y\}$ implement the same set of vectors. Combining this with the fact that the collection $\{T_L cx \mid c \in C_{k,L}x\}$ was β-disjoint, we see that the collection $\{T_L c'y \mid c' \in C_{k,L}y\}$ is also β-disjoint. Repeat this construction for each $y \in W_k$, and set $C_{k,L} = \bigsqcup_{y \in W_k} C_{k,L}y$, so that the collection $\{T_L c \mid c \in C_{k,L}\}$ is β-disjoint. Observe that by construction, for each j, we have either $S_{k,j} \cap C_{k,l} = \varnothing$ or $S_{k,j} \subset C_{k,l}$.

By construction, since $C_{k,L} \subset I$, we have $T_L C_{k,L} \subset S_k$. Now, suppose that $g \in S_{k,p} = V(P_k, t_k, t_{k,p})$ is such that $g \in T_L C_{k,L}$. Then there exists $c \in C_{k,L}$ such that $c \in S_{k,j} = V(P_k, t_k, t_{k,j})$ for some j, and $t \in T_L$ implementing the vector $x(t_{k,p}) - x(t_{k,j})$, such that $tc = g$. Choose any other $g' \in S_{k,p}$, and consider the groupoid element c' with source $s(g')$ and range $s(g') + x(t_{k,j}) - x(t_k)$. Observe that c' is in $S_{k,j} = V(P_k, t_k, t_{k,j})$, which is a subset of $C_{k,L}$ by the previous paragraph, since $c \in S_{k,j} \cap C_{k,L}$. Since $r(c')$ is in the same tower level as $r(c) = s(t)$, the arrows in $T_L r(c')$ implement exactly the same vectors as the arrows in $T_L s(t)$, so, in particular, there is an element $t' \in T_L r(c')$ which implements the same vector $x(t_{k,p}) - x(t_{k,j})$ that t does. Then $t'c'$ implements the vector $x(t_{k,p}) - x(t_k)$, to $s(t'c') = s(g')$, so we see that $t'c' = g'$, and hence that $g' \in T_L C_{k,L}$. This shows that, for each p, either $S_{k,p} \cap T_L C_{k,L} = \varnothing$, or $S_{k,p} \subset T_L C_{k,L}$.

We now simply repeat the procedure of the previous three paragraphs for each $l \in \{L-1, \ldots, 1\}$ in turn. As in the proof of Lemma 6.2.8, at each step we consider $I = \{c \in S_k \mid T_l c \subset S_k \setminus \bigsqcup_{j=l+1}^{L} T_j C_{k,j}\}$. Notice that when forming $S_k \setminus \bigsqcup_{j=l+1}^{L} T_j C_{k,j}$, we have removed entire tower levels from consideration, so at each step we are working with a subtower of (W_k, S_k).

Since the levels of the tower (W_k, S_k) have diameter less than η', which is a Lebesgue number for $\mathcal{U}$, for each $l \in \{1, \ldots, L\}$ there is a partition

$$C_{k,l} = C_{k,l,1} \sqcup C_{k,l,2} \sqcup \cdots \sqcup C_{k,l,M}$$

such that $cW_k \subset U_m$ whenever $m \in \{1, \ldots, M\}$ and $c \in C_{k,l,m}$. In fact, by the diameter condition on the tower levels, for each j such that $S_{k,j} \subset C_{k,l}$, there exists m such that $S_{k,j}W_k \subset U_m$, whence, for each $c \in S_{k,j}$, we may put $c \in C_{k,l,m}$. In this way, we may assume that, for each j and m, either $S_{k,j} \cap C_{k,l,m} = \varnothing$ or $S_{k,j} \subset C_{k,l,m}$. In addition, since $T_l C_{k,l,m} \subset S_k$, and making use of the structure of T_l and $S_{k,j}$, it is then automatic that, for each j and m, either $S_{k,j} \cap T_l C_{k,l,m} = \varnothing$ or $S_{k,j} \subset T_l C_{k,l,m}$ by following a similar argument as that for $T_L C_{k,L}$ above.

For each l and m, choose pairwise disjoint subsets $C^{(1)}_{k,l,m}, \ldots, C^{(n)}_{k,l,m}$ of $C_{k,l,m}$ such that, for each $u \in G^{(0)}$ and $i \in \{1, \ldots, n\}$, we have $|C^{(i)}_{k,l,m}u| = \lfloor |C_{k,l,m}u|/n \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. Further enforce that, if $S_{k,j} \subset C_{k,l,m}$ and $c \in S_{k,j}$ is included in $C^{(i)}_{k,l,m}$, then $S_{k,j} \subset C^{(i)}_{k,l,m}$. Since the source map is injective on $S_{k,j}$, this amounts to adding a single element to $C^{(i)}_{k,l,m}u$ for each $u \in s(S_{k,j}) = W_k$. In this way, each $C^{(i)}_{k,l,m}$ corresponds to a choice of a $(1/n)$-th of the tower levels associated to $C_{k,l,m}$ (ignoring any remainder), so that, for each j, either $S_{k,j} \cap C^{(i)}_{k,l,m} = \varnothing$, or $S_{k,j} \subset C^{(i)}_{k,l,m}$. As in the last paragraph, it is then automatic that either $S_{k,j} \cap T_l C^{(i)}_{k,l,m} = \varnothing$, or $S_{k,j} \subset T_l C^{(i)}_{k,l,m}$ for each j.

For each $i \in \{2, \ldots, n\}$, choose a bijection

$$\Lambda_{k,i} : \bigsqcup_{l,m} C^{(1)}_{k,l,m} \to \bigsqcup_{l,m} C^{(i)}_{k,l,m}$$

which sends $C^{(1)}_{k,l,m}u$ to $C^{(i)}_{k,l,m}u$ for every $l \in \{1, \ldots, L\}$, $m \in \{1, \ldots, M\}$, and every $u \in G^{(0)}$. Further enforce that, whenever $S_{k,j} \subset C^{(1)}_{k,l,m}$, we have, for each i, that $\Lambda_{k,i}(S_{k,j}) = S_{k,p}$ for some p such that $S_{k,p} \subset C^{(i)}_{k,l,m}$, so that $\Lambda_{k,i}$ sends the set of groupoid elements associated to one particular tower level to the set of groupoid elements associated to some other tower level. Also, define $\Lambda_{k,1}$ to be the identity map on $\bigsqcup_{l,m} C^{(1)}_{k,l,m}$, and denote

$$\Lambda_{k,i,j} \coloneqq \Lambda_{k,i} \circ \Lambda^{-1}_{k,j} : \bigsqcup_{l,m} C^{(j)}_{k,l,m} \to \bigsqcup_{l,m} C^{(i)}_{k,l,m}.$$

Since the collection $\{T_l c \mid l \in \{1, \ldots, L\}, c \in C_{k,l}\}$ is β-disjoint, for every $l \in \{1, \ldots, L\}$ and $c \in C_{k,l}$, we can find a $T_{k,l,c} \subset T_l$ satisfying $|T_{k,l,c}u| \geq (1-\beta)|T_l u|$ for each $u \in G^{(0)}$, which is such that the collection $\{T_{k,l,c}c \mid l \in \{1, \ldots, L\}, c \in C_{k,l}\}$ is pairwise disjoint. Recall that, for each $c \in C_{k,l}$, we have $T_l c \subset S_k \setminus \bigsqcup_{j=l+1}^{L} T_j C_{k,j}$. Therefore, given $c \in C_{k,l}$ and $c' \in C_{k',l'}$, observe that $T_l c$ and $T_{l'}c'$ can only intersect when $k = k'$ (otherwise the levels of the k-th and k'-th tower would intersect), and when $l = l'$. Therefore, by following a similar argument as in the construction of $C_{k,L}$, we may perform this construction in such a way that, whenever $S_{k,j} \subset C_{k,l}$ and $c, c' \in S_{k,j}$, we choose $T_{k,l,c}r(c)$ and $T_{k,l,c'}r(c')$ to consist of arrows which implement the same set of vectors. This shows that, for each j, either $S_{k,j} \cap \bigsqcup_{c \in C_{k,l}} T_{k,l,c}c = \varnothing$ or $S_{k,j} \subset \bigsqcup_{c \in C_{k,l}} T_{k,l,c}c$. We are also free to choose

$T_{k,l,c}u = T_l u$ for every $u \in G^{(0)} \setminus \{r(c)\}$.

In addition, by construction, for every $c \in C_{k,l,m}^{(j)}$ and every $i \in \{1,\dots,n\}$, we have $r(c) \in U_m$, and $r(\Lambda_{k,i,j}(c)) \in U_m$. In particular, this means that $d(r(c), r(\Lambda_{k,i,j}(c))) < \eta$ in the tiling metric, so the tilings $r(c)$ and $r(\Lambda_{k,i,j}(c))$ agree on $B_{\eta^{-1}}(0)$. Therefore, for each i, $T_l r(c)$ and $T_l r(\Lambda_{k,i,j}(c))$ consist of groupoid elements which implement exactly the same vectors to $r(c)$ and $r(\Lambda_{k,i,j}(c))$, respectively. Consider the set $\widetilde{T_{k,l,c}}$, which we obtain from $T_{k,l,c}$ by removing the arrows which correspond to the same vectors as the arrows which are removed from any set of the form $T_l r(\Lambda_{k,i,j}(c))$ to construct $T_{k,l,\Lambda_{k,i,j}(c)}$. In other words, if an arrow corresponding to translating $r(\Lambda_{k,i,j}(c))$ by some vector is removed from $T_l r(\Lambda_{k,i,j}(c))$ when constructing $T_{k,l,\Lambda_{k,i,j}(c)}$ for any i, then the arrow corresponding to translating $r(c)$ by the same vector does not appear in $\widetilde{T_{k,l,c}}$. Since we had $|T_{k,l,\Lambda_{k,i,j}(c)} r(\Lambda_{k,i,j}(c))| \geq (1-\beta)|T_l r(\Lambda_{k,i,j}(c))|$ for each $i \in \{1,\dots,n\}$, we see that $|\widetilde{T_{k,l,c}} r(c)| \geq (1-n\beta)|T_l r(c)|$. In addition, the sets of vectors of translation associated to the elements of any two of the sets $\widetilde{T_{k,l,\Lambda_{k,i,j}(c)}} r(\Lambda_{k,i,j}(c))$, for $i \in \{1,\dots,n\}$, are the same. Also, when $u \neq r(c)$, we have $\widetilde{T_{k,l,c}} u = T_l u$. Observe that, whenever $c, c' \in S_{k,j}$, the sets $\widetilde{T_{k,l,c}} r(c)$ and $\widetilde{T_{k,l,c'}} r(c')$ will consist of arrows which implement the same vectors of translation, so that, for each j, either $S_{k,j} \cap \bigsqcup_{c \in C_{k,l}} \widetilde{T_{k,l,c}} c = \varnothing$ or $S_{k,j} \subset \bigsqcup_{c \in C_{k,l}} \widetilde{T_{k,l,c}} c$.

Now, for each $j \in \{1,\dots,n\}$ and each $c \in C_{k,l,m}^{(j)}$, consider the set $T'_{k,l,c} \coloneqq \bigcap_{i=1}^n \widetilde{T_{k,l,\Lambda_{k,i,j}(c)}}$. Observe that, for each $u \in G^{(0)}$,

$$|T'_{k,l,c}u| \geq (1-n\beta)|T_l u|.$$

In addition, we still have, for each j, that whenever $c, c' \in S_{k,j}$, the sets $T'_{k,l,c} r(c)$ and $T_{k,l,c'} r(c')$ implement the same vectors of translation, so that, for each j, either $S_{k,j} \cap \bigsqcup_{c \in C_{k,j}} T'_{k,l,c} c = \varnothing$ or $S_{k,j} \subset \bigsqcup_{c \in C_{k,j}} T'_{k,l,c} c$. Since $T'_{k,l,c} r(c)$ and $T'_{k,l,c} r(\Lambda_{k,i,j}(c))$ consist of arrows which implement the same vectors for each i, given $t \in T'_{k,l,c} r(c)$, we will denote by $t^{(i)}$ the element of $T'_{k,l,c} r(\Lambda_{k,i,j}(c))$ which implements the same vector as t.

Set

$$B_{k,l,c,Q}u = \bigcap_{g \in K^Q} \left(gT'_{k,l,c} \sqcup \bigsqcup_{v \in G^{(0)} \setminus r(g)} vT'_{k,l,c} \right),$$

noting that, by our assumption on the invariance of the sets T_l, and using (6.4.1), we have, for each $u \in G^{(0)}$, that $|B_{k,l,c,Q}u| \geq (1-\kappa)|T_l u|$. We can alternatively describe $B_{k,l,c,Q}$ as the collection of elements $t \in T'_{k,l,c}$ such that $K^Q t \subset T'_{k,l,c}$.

Suppose $g \in S_{k,p}$ is such that $g \in \bigcup_{\tilde{c} \in C_{k,l}} B_{k,l,\tilde{c},Q}\tilde{c}$, so that $g = tc$, with $t \in B_{k,l,c,Q}$ for $c \in S_{k,j} \subset C_{k,l}$, say. Given $c' \in S_{k,j}$, define $t' \in T'_{k,l,c'}$ to be the element of $Gr(c')$ which implements the same vector of translation that t does. Observe that, since $r(t)$ and $r(t')$ are in the same tower level, they agree on $B_{\eta^{-1}}(0)$, and so $K^Q r(t)$ and $K^Q r(t')$ consist of arrows implementing the same vectors of translation. Since $t \in B_{k,l,c,Q}$, it follows that $t' \in B_{k,l,c',Q}$, and therefore $t'c' \in \bigsqcup_{\tilde{c} \in C_{k,l}} B_{k,l,\tilde{c},Q}\tilde{c}$. Since $t'c'$ was an arbitrary element of $S_{k,p}$, this shows that either $S_{k,p} \cap \bigsqcup_{\tilde{c} \in C_{k,l}} B_{k,l,\tilde{c},Q}\tilde{c} = \varnothing$, or $S_{k,p} \subset \bigsqcup_{\tilde{c} \in C_{k,l}} B_{k,l,\tilde{c},Q}\tilde{c}$.

For each $q \in \{0, \dots, Q-1\}$, set

$$B_{k,l,c,q} = K^{Q-q}B_{k,l,c,Q} \setminus K^{Q-q-1}B_{k,l,c,Q}.$$

Since $G^{(0)} \subset K$, we have, for each $q \in \{0, \dots, Q\}$, that $B_{k,l,c,q} \subset (G^{(0)})^q(K^{Q-q}B_{k,l,c,Q} \setminus K^{Q-q-1}B_{k,l,c,Q}) \subset K^Q B_{k,l,c,Q}$, so these sets partition $K^Q B_{k,l,c,Q}$. In addition, we have $B_{k,l,c,Q} \subset (G^{(0)})^Q B_{k,l,c,Q} \subset K^Q B_{k,l,c,Q}$, so that, for each $u \in G^{(0)}$, $|K^Q B_{k,l,c,Q}u| \geq |B_{k,l,c,Q}u| \geq (1-\kappa)|T_l u|$.

Since $K^Q B_{k,l,c,Q}$ is a subset of $T'_{k,l,c}$, we have $K^Q B_{k,l,c,Q}c \subset S_k$. Also, since $K^q x$ and $K^q y$ consist of groupoid elements which implement the same vectors of translation whenever $q \in \{1, \dots, Q\}$ and $d(x, y) < \eta$, and since we have either $S_{k,p} \subset \bigsqcup_{c \in C_{k,l}} B_{k,l,c,Q}c$ or $S_{k,p} \cap \bigsqcup_{c \in C_{k,l}} B_{k,l,c,Q}c = \varnothing$, we see that, for each k, l, q and j, we have either $S_{k,j} \subset \bigsqcup_{c \in C_{k,l}} B_{k,l,c,q}c$ or $S_{k,j} \cap \bigsqcup_{c \in C_{k,l}} B_{k,l,c,q}c = \varnothing$. In other words, we may ensure that all the elements of S_k which correspond to any one clopen tower level are all associated to the same q.

We claim that, for each $q \in \{1, \dots, Q\}$ and $i \in \{1, \dots, n\}$, whenever $t \in B_{k,l,c,q}$ we have $t^{(i)} \in B_{k,l,c,q}$ as well. Suppose that $s(t) = r(\Lambda_{k,r,j}(c))$, and observe that $s(t^{(i)}) = r(\Lambda_{k,i,j}(c))$. By construction, $s(t)$ and $s(t^{(i)})$ are in the same U_m, and so the distance between them is smaller than η, so these tilings agree on $B_{QR_K+R_L}$. By construction, this means that the groupoid elements in $K^Q T_L$ implement the same set of translations at both of these tilings. We assumed that $t \in B_{k,l,c,q}$, so we can find $b \in B_{k,l,c,Q}$ and $g \in K^{Q-q}$ such that $gb \notin K^{Q-q-1}B_{k,l,c,Q}$, and such that $t = gb$. By construction, since $gb \in K^Q T_L$, the vector which implements b is also allowable at $s(t^{(i)})$, and is implemented by $\tilde{b} \in T_L$, say, and then the vector which implements g at $r(b)$ is also allowable at $r(\tilde{b})$, and is implemented by $\tilde{g} \in R_{\text{punc}}$, say. Since $r(b)$ and $r(\tilde{b})$ agree on $B_{QR_K}(0)$, and since $g \in K^{Q-q}$, it follows from our choice of K that $\tilde{g} \in K^{Q-q}$. Next, we show that $\tilde{b} \in B_{k,l,c,Q}$. To do so, we prove that $K^Q\tilde{b} \subset T'_{k,l,c}$. Indeed, we assumed that $b \in B_{k,l,c,Q}$, so we know that $K^Q b \subset T'_{k,l,c}$. Since $r(b)$ and $r(\tilde{b})$ agree on $B_{QR_K}(0)$, we know that $K^Q r(b)$ and $K^Q r(\tilde{b})$ consist of the same vectors of translation. By construction of $T'_{k,l,c}$, since $s(b)$ and $s(\tilde{b})$ agree on $B_{QR_K+R_L}(0)$, the sets $T'_{k,l,c}s(b)$ and $T'_{k,l,c}s(\tilde{b})$ also implement the same set of vectors. Thus, since b and $\tilde{b}$ implemented the same vector, we see that, whenever $k \in K^Q r(b)$ and $\tilde{k} \in K^Q r(\tilde{b})$ implement the same translation, we have $kb \in T'_{k,l,c}$ if and only if $\tilde{k}\tilde{b} \in T'_{k,l,c}$. Since $K^Q b \subset T'_{k,l,c}$ by assumption, this shows that $K^Q\tilde{b} \subset T'_{k,l,c}$, so $\tilde{b} \in B_{k,l,c,Q}$, as required. Finally, we must show that $\tilde{g}\tilde{b} \notin K^{Q-q-1}B_{k,l,c,Q}$. This follows because $\tilde{g}\tilde{b}$ implements the same vector as gb, which was not an element of $K^{Q-q-1}B_{k,l,c,Q}$, and the elements of the sets $K^{Q-q-1}B_{k,l,c,Q}s(b)$ and $K^{Q-q-1}B_{k,l,c,Q}s(\tilde{b})$ are implemented by precisely the same set of vectors.

Given $g \in K$, it is clear that

$$gB_{k,l,c,Q} \subset KB_{k,l,c,Q} \subset B_{k,l,c,Q-1} \cup B_{k,l,c,Q}. \tag{6.4.3}$$

For $q \in \{1, \ldots, Q-1\}$, we have

$$gB_{k,l,c,q} \subset B_{k,l,c,q-1} \cup B_{k,l,c,q} \cup B_{k,l,c,q+1}, \tag{6.4.4}$$

because it is clear that $gB_{k,l,c,q} \subset K^{Q-q+1}B_{k,l,c,Q} = \bigsqcup_{j=q-1}^{Q} B_{k,l,c,j}$, while, given $h \in B_{k,l,c,q}$, if we had $gh \in K^{Q-q-2}B_{k,l,c,Q} = \bigsqcup_{j=q+2}^{Q} B_{k,l,c,j}$ then the closure of K under inverses would provide $h \in g^{-1}K^{Q-q-2}B_{k,l,c,Q} \subset KK^{Q-q-2}B_{k,l,c,Q} = K^{Q-q-1}B_{k,l,c,Q}$, which contradicts the membership of h in $B_{k,l,c,q}$.

We obtain a linear map $\psi : M_n \to C(X) \rtimes_r G$ by defining it on the standard matrix units $\{e_{ij}\}_{i,j=1}^{n}$ of M_n by

$$\psi(e_{ij}) = \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}$$

where $t^{(i)} \in T_{k,l,\Lambda_{k,i,j}(c)}r(\Lambda_{k,i,j}(c))$ is the element constructed earlier, and extending linearly. In fact, we claim that ψ is a $*$-homomorphism, so we check that $\psi(e_{ij})^* = \psi(e_{ji})$ and that $\psi(e_{ij})\psi(e_{i'j'}) = \psi(e_{ij'})$ when $j = i'$ and 0 otherwise. Indeed, we have

$$\begin{aligned}\psi(e_{ij})^* &= \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)} 1_{tcW_k}U_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}\\ &= \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)} U_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}1_{t^{(i)}\Lambda_{k,i,j}(c)W_k}.\end{aligned}$$

On the other hand,

$$\psi(e_{ji}) = \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(i)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)} U_{t^{(j)}\Lambda_{k,j,i}(c)c^{-1}t^{-1}}1_{tcW_k}.$$

Since $\Lambda_{k,j,i} : C_{k,l,m}^{(i)} \to C_{k,l,m}^{(j)}$ is a bijection with inverse $\Lambda_{k,i,j}$, for each $c \in C_{k,l,m}^{(i)}$, by putting $\tilde{c} = \Lambda_{k,j,i}(c)$ and $\tilde{t} = t^{(j)}$ (so that $\tilde{t}^{(i)} = t$), we can change the index in the above sum to obtain

$$\begin{aligned}\psi(e_{ji}) &= \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{\tilde{c}\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{\tilde{t}\in B_{k,l,\tilde{c},q}r(\tilde{c})} U_{\tilde{t}\tilde{c}\Lambda_{k,i,j}(\tilde{c})^{-1}(\tilde{t}^{(i)})^{-1}}1_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})W_k}\\ &= \psi(e_{ij})^*.\end{aligned}$$

Working similarly,

$$\begin{aligned}\psi(e_{ij})\psi(e_{i'j'}) = &\left(\sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}\right)\\ &\left(\sum_{\tilde{k}=1}^{N}\sum_{\tilde{l}=1}^{L}\sum_{\tilde{m}=1}^{M}\sum_{\tilde{c}\in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j')}}\sum_{\tilde{q}=1}^{Q}\sum_{\tilde{t}\in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}}r(\tilde{c})} U_{\tilde{t}^{(i')}\Lambda_{\tilde{k},i',j'}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}1_{\tilde{t}\tilde{c}W_{\tilde{k}}}\right)\\ = \sum_{k,\tilde{k}=1}^{N}\sum_{l,\tilde{l}=1}^{L}\sum_{m,\tilde{m}=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}} &\sum_{\tilde{c}\in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j')}}\sum_{q,\tilde{q}=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)}\sum_{\tilde{t}\in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}}r(\tilde{c})} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}U_{\tilde{t}^{(i')}\Lambda_{\tilde{k},i',j'}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}1_{\tilde{t}\tilde{c}W_{\tilde{k}}}\\ = \sum_{k,\tilde{k}=1}^{N}\sum_{l,\tilde{l}=1}^{L}\sum_{m,\tilde{m}=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{\tilde{c}\in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j')}}\sum_{q,\tilde{q}=1}^{Q} &\sum_{t\in B_{k,l,c,q}r(c)}\sum_{\tilde{t}\in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}}r(\tilde{c})} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}1_{\tilde{t}^{(i')}\Lambda_{\tilde{k},i',j'}(\tilde{c})W_{\tilde{k}}}U_{\tilde{t}^{(i')}\Lambda_{\tilde{k},i',j'}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}\end{aligned}$$

The supports of the indicator functions in this final summand are disjoint (and hence the corresponding term is zero) unless $k = \tilde{k}$ and $tc = \tilde{t}^{(i')}\Lambda_{k,i',j'}(\tilde{c})$. Since $tc \in T_{k,l,c}c$ and $\tilde{t}^{(i')}\Lambda_{k,i',j'}(\tilde{c}) \in T_{k,\tilde{l},\Lambda_{k,i',j'}(\tilde{c})}\Lambda_{k,i',j'}(\tilde{c})$, the disjointness conditions on these sets then imply that $l = \tilde{l}$ and $c = \Lambda_{k,i',j'}(\tilde{c})$, which in turn implies that $m = \tilde{m}$ and $i' = j$. Thus, we see that the whole sum returns zero if $i' \neq j$. Now the equality $tc = \tilde{t}^{(i')}\Lambda_{k,i',j'}(\tilde{c}) = \tilde{t}^{(j)}c$ shows that $t = \tilde{t}^{(j)}$, and hence that $q = \tilde{q}$. Eliminating all the duplicated variables from the sum above gives

$$\begin{aligned}\psi(e_{ij})\psi(e_{jj'}) = \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M} &\sum_{\tilde{c}\in C_{k,l,m}^{(j')}}\sum_{q=1}^{Q}\sum_{\tilde{t}\in B_{k,l,\tilde{c},q}r(\tilde{c})} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\Lambda_{k,j,j'}(\tilde{c}))\Lambda_{k,j,j'}(\tilde{c})^{-1}(\tilde{t}^{(j)})^{-1}}1_{\tilde{t}^{(j)}\Lambda_{k,j,j'}(\tilde{c})W_k}U_{\tilde{t}^{(j)}\Lambda_{k,j,j'}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}\\ &= \sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{\tilde{c}\in C_{k,l,m}^{(j')}}\sum_{q=1}^{Q}\sum_{\tilde{t}\in B_{k,l,\tilde{c},q}r(\tilde{c})} U_{\tilde{t}^{(i)}\Lambda_{k,i,j'}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}1_{\tilde{t}\tilde{c}W_k}\end{aligned}$$

$$= \psi(e_{ij'}).$$

For each $k \in \{1, \dots, N\}$, let $h_k : X \to [0,1]$ denote the indicator function for the clopen set W_k, and note that, since W_k is clopen, $h_k \in C(X)$. For each $k \in \{1, \dots, N\}$, $l \in \{1, \dots, L\}$, $m \in \{1, \dots, M\}$, $i, j \in \{1, \dots, n\}$ and $c \in C^{(j)}_{k,l,m}$, we set

$$\begin{aligned} h_{k,l,c,i,j} &= \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(i)} \Lambda_{k,i,j}(c) c^{-1} t^{-1}} \alpha_{tc}(h_k) \\ &= \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(i)} \Lambda_{k,i,j}(c) c^{-1} t^{-1}} 1_{tcW_k}. \end{aligned}$$

Define a linear map $\phi : M_n \to C(X) \rtimes_r G$ by setting

$$\phi(e_{ij}) = \sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{c \in C^{(j)}_{k,l,m}} h_{k,l,c,i,j}$$

and extending linearly. Set

$$h = \sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C^{(i)}_{k,l,m}} h_{k,l,c,i,i}$$

so that $h : X \to [0,1]$ is a continuous function. We check that h commutes with the image of ψ, and that

$$\phi(b) = h\psi(b)$$

for every $b \in M_n$, which will show that ϕ is an order-zero c.p.c map.

We have

$$\begin{aligned} h\psi(e_{ij}) = &\left(\sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{r=1}^{n} \sum_{c \in C^{(r)}_{k,l,m}} \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(r)} \Lambda_{k,r,r}(c) c^{-1} t^{-1}} 1_{tcW_k} \right) \\ &\left(\sum_{\tilde{k}=1}^{N} \sum_{\tilde{l}=1}^{L} \sum_{\tilde{m}=1}^{M} \sum_{\tilde{c} \in C^{(j)}_{\tilde{k},\tilde{l},\tilde{m}}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}} r(\tilde{c})} U_{\tilde{t}^{(i)} \Lambda_{\tilde{k},i,j}(\tilde{c}) \tilde{c}^{-1} \tilde{t}^{-1}} 1_{\tilde{t}\tilde{c}W_{\tilde{k}}} \right) \\ = &\sum_{k,\tilde{k}=1}^{N} \sum_{l,\tilde{l}=1}^{L} \sum_{m,\tilde{m}=1}^{M} \sum_{r=1}^{n} \sum_{c \in C^{(r)}_{k,l,m}} \sum_{\tilde{c} \in C^{(j)}_{\tilde{k},\tilde{l},\tilde{m}}} \\ &\sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}} r(\tilde{c})} \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(r)} t^{-1}} 1_{tcW_k} U_{\tilde{t}^{(i)} \Lambda_{\tilde{k},i,j}(\tilde{c}) \tilde{c}^{-1} \tilde{t}^{-1}} 1_{\tilde{t}\tilde{c}W_{\tilde{k}}} \end{aligned}$$

$$= \sum_{k,\tilde{k}=1}^{N} \sum_{l,\tilde{l}=1}^{L} \sum_{m,\tilde{m}=1}^{M} \sum_{r=1}^{n} \sum_{c \in C_{k,l,m}^{(r)}} \sum_{\tilde{c} \in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j)}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}} r(\tilde{c})} \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(r)}t^{-1}} 1_{tcW_k} 1_{\tilde{t}^{(i)}\Lambda_{\tilde{k},i,j}(\tilde{c})W_{\tilde{k}}} U_{\tilde{t}^{(i)}\Lambda_{\tilde{k},i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}$$

Notice that, since t is associated to $c \in C_{k,l,m}^{(r)}$, we have $t^{(r)} = t$. Indeed, let $t = (r(c) - x, r(c))$, so that $t^{(r)} = (r(\Lambda_{k,r,r}(c)) - x, r(\Lambda_{k,r,r}(c))) = (r(c) - x, r(c)) = t$. Now, we have $c \in C_{k,l,m}^{(r)} \subset C_{k,l}$ and $t \in T'_{k,l,c} \subset T_{k,l,c} \subset T_l$, and similarly $\Lambda_{\tilde{k},i,j}(\tilde{c}) \in C_{\tilde{k},\tilde{l}}$ and $\tilde{t}^{(i)} \in T_{\tilde{l}}$, so that $tc \in S_k$ and $\tilde{t}^{(i)}\Lambda_{\tilde{k},i,j}(\tilde{c}) \in S_{\tilde{k}}$, so for the supports of the indicator functions in the above sum to intersect, and hence the corresponding term to be nonzero, we must have $k = \tilde{k}$ (because levels of different towers are disjoint). Since tc and $\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})$ are both elements of S_k, for the indicators accociated to them to intersect, we must have $tc = \tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})$. But $tc \in T_{k,l,c}c$, and $\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c}) \in T_{k,\tilde{l},\Lambda_{k,i,j}(\tilde{c})}\Lambda_{k,i,j}(\tilde{c})$, and these sets are disjoint unless $l = \tilde{l}$ and $c = \Lambda_{k,i,j}(\tilde{c})$ (which implies also that $m = \tilde{m}$ and $r = i$). Combining this with the fact that $tc = \tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c}) = \tilde{t}^{(i)}c$ shows that $t = \tilde{t}^{(i)}$, and hence that $q = \tilde{q}$. Therefore, we only need to sum over k, l, m, $\tilde{c} \in C_{k,l,m}^{(j)}$, and $\tilde{q}$ and $\tilde{t} \in B_{k,l,\tilde{c},\tilde{q}} r(\tilde{c})$ (because knowing $\tilde{c}$ and $\tilde{t}$ allows us to determine c and t). Putting this all together yields

$$\begin{aligned}
h\psi(e_{ij}) &= \sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{\tilde{c} \in C_{k,l,m}^{(j)}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{k,l,\tilde{c},\tilde{q}} r(\tilde{c})} \frac{\tilde{q}}{Q} U_{\tilde{t}^{(i)}(\tilde{t}^{(i)})^{-1}} 1_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})W_k} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{\tilde{c} \in C_{k,l,m}^{(j)}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{k,l,\tilde{c},\tilde{q}} r(\tilde{c})} \frac{\tilde{q}}{Q} U_{r(\tilde{t}^{(i)})} 1_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})W_k} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{\tilde{c} \in C_{k,l,m}^{(j)}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{k,l,\tilde{c},\tilde{q}} r(\tilde{c})} \frac{\tilde{q}}{Q} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}} 1_{\tilde{t}\tilde{c}W_k} \\
&= \phi(e_{ij}).
\end{aligned}$$

Similarly, we compute that

$$\begin{aligned}
\psi(e_{ij})h &= \left(\sum_{\tilde{k}=1}^{N} \sum_{\tilde{l}=1}^{L} \sum_{\tilde{m}=1}^{M} \sum_{\tilde{c} \in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j)}} \sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}} r(\tilde{c})} U_{\tilde{t}^{(i)}\Lambda_{\tilde{k},i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}} 1_{\tilde{t}\tilde{c}W_{\tilde{k}}} \right) \\
&\quad \left(\sum_{k=1}^{N} \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{r=1}^{n} \sum_{c \in C_{k,l,m}^{(r)}} \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(r)}\Lambda_{k,r,r}(c)c^{-1}t^{-1}} 1_{tcW_k} \right) \\
&= \sum_{k,\tilde{k}=1}^{N} \sum_{l,\tilde{l}=1}^{L} \sum_{m,\tilde{m}=1}^{M} \sum_{r=1}^{n} \sum_{c \in C_{k,l,m}^{(r)}} \sum_{\tilde{c} \in C_{\tilde{k},\tilde{l},\tilde{m}}^{(j)}}
\end{aligned}$$

$$\sum_{\tilde{q}=1}^{Q}\sum_{\tilde{t}\in B_{\tilde{k},\tilde{l},\tilde{c},\tilde{q}}r(\tilde{c})}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)}\frac{q}{Q}U_{\tilde{t}^{(i)}\Lambda_{\tilde{k},i,j}(\tilde{c})\tilde{c}^{-1}\tilde{t}^{-1}}1_{\tilde{t}\tilde{c}W_{\tilde{k}}}U_{r(t)}1_{tcW_k}.$$

This time, the only nonzero contribution occurs when $k=\tilde{k}$ and when $\tilde{t}\tilde{c}=tc$. Using similar arguments as above, this forces $c=\tilde{c}$ and $t=\tilde{t}$, and hence $q=\tilde{q}$, $l=\tilde{l}$, $m=\tilde{m}$, and $r=j$. Thus, the sum above becomes

$$\begin{aligned}\psi(e_{ij})h&=\sum_{k=1}^{N}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{c\in C_{k,l,m}^{(j)}}\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)}\frac{q}{Q}U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}\\&=\phi(e_{ij}).\end{aligned}$$

Next, we verify condition (ii) in Theorem 6.1.6 for the element $w=\sum_{g\in K}U_g$. Let $1\leq i,j\leq n$. Using the fact that K is closed under taking inverses in the sum over K in the second term, we have

$$\begin{aligned}wh_{k,l,c,i,j}-h_{k,l,c,i,j}w&=\sum_{q=1}^{Q}\sum_{t\in B_{k,l,c,q}r(c)}\sum_{g\in Kr(t^{(i)})}\frac{q}{Q}U_{gt^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}\\&\quad-\sum_{\tilde{q}=1}^{Q}\sum_{\tilde{t}\in B_{k,l,c,\tilde{q}}r(c)}\sum_{\tilde{g}\in Kr(\tilde{t})}\frac{\tilde{q}}{Q}U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(c)c^{-1}\tilde{t}^{-1}}1_{\tilde{t}cW_k}U_{\tilde{g}^{-1}}\\&=\sum_{q,\tilde{q}=1}^{Q}\sum_{\substack{t\in B_{k,l,c,q}r(c)\\ \tilde{t}\in B_{k,l,c,\tilde{q}}r(c)}}\sum_{\substack{g\in Kr(t^{(i)})\\ \tilde{g}\in Kr(\tilde{t})}}\frac{q}{Q}U_{gt^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}-\frac{\tilde{q}}{Q}U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(c)c^{-1}(\tilde{g}\tilde{t})^{-1}}1_{\tilde{g}\tilde{t}cW_k}.\end{aligned}$$

In view of (6.4.3) and (6.4.4), we may pair up the elements in the first and second terms as follows. Because $gt^{(i)}\in B_{k,l,c,q+a}$ for $a\in\{-1,0,1\}$, we want to associate the element $\frac{q}{Q}U_{gt^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}}1_{tcW_k}$ to an element corresponding to $\tilde{q}=q+a$. To find an appropriate $\tilde{g}$ and $\tilde{t}$, observe that by construction there is an element h of K with source equal to $r(t)$ which implements the same vector of translation as g. Set $\tilde{t}=ht$, noting that by our construction, $\tilde{t}\in B_{k,l,c,q+a}$, and set $\tilde{g}=h^{-1}$ so that $\tilde{g}\tilde{t}=t$. Observe that by construction, we obtain $\tilde{t}^{(i)}=gt^{(i)}$, so the term of the sum which corresponds to these choices of $g,\tilde{g}$, $q,\tilde{q}$ and $t,\tilde{t}$ becomes

$$\frac{-a}{Q}U_{gt^{(i)}\Lambda_{k,i.j}(c)c^{-1}t^{-1}}1_{tcW_k}=\frac{-a}{Q}1_{gt^{(i)}\Lambda_{k,i,j}(c)W_k}U_{gt^{(i)}\Lambda_{k,i.j}(c)c^{-1}t^{-1}},$$

and the norm of this element is no larger than $1/Q<\epsilon/(n^2\sup_{u\in G^{(0)}}|uK|)$. Now, all of the associated indicator functions are supported on tower levels $gt^{(i)}\Lambda_{k,i,j}(c)W_k$, and the only time two of these levels can intersect is if they are associated to two different $g\in K$

with the same range. Thus, we obtain

$$\|wh_{k,l,c,i,j} - h_{k,l,c,i,j}w\| \leq \frac{\sup_{u\in G^{(0)}} |uK|}{Q} < \frac{\epsilon}{n^2}.$$

All of the associated indicator functions in sight here are supported within $K^Q B_{k,l,c,Q} cW_k$. Since these sets are pairwise disjoint for distinct k, l, and c, we obtain

$$\|w\phi(e_{ij}) - \phi(e_{ij})w\| = \sup_{k,l,c} \|wh_{k,l,c,i,j} - h_{k,l,c,i,j}w\| \leq \frac{\epsilon}{n^2}.$$

Hence, for any norm-one $b = (b_{ij}) \in M_n$, we have

$$\begin{aligned}\|[w, \phi(b)]\| &= \|w\phi(b) - \phi(b)w\| \\ &\leq \sum_{i,j=1}^{n} \|w\phi(b_{ij}) - \phi(b_{ij})w\| \leq n^2 \cdot \frac{\epsilon}{n^2} = \epsilon.\end{aligned}$$

Next, we check condition (ii) in Theorem 6.1.6 for the functions in Υ. Let $i, j \in \{1, \ldots, n\}$ and $f \in \Upsilon \cup \Upsilon^2$. Let $k \in \{1, \ldots, N\}$ and $l \in \{1, \ldots, L\}$. Let $c \in C^{(j)}_{k,l,m}$. Since k, i, j are fixed in the calculation below, we write $\Lambda = \Lambda_{k,i,j}$ for short.

$$\begin{aligned}h^*_{k,l,c,i,j} f h_{k,l,c,i,j} &= \left(\sum_{q=1}^{Q} \sum_{t\in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}} \alpha_{tc}(1_{W_k})\right)^* f \\ &\quad \left(\sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t}\in B_{k,l,c,\tilde{q}} r(c)} \frac{\tilde{q}}{Q} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(c)c^{-1}\tilde{t}^{-1}} \alpha_{\tilde{t}c}(1_{W_k})\right) \\ &= \sum_{q,\tilde{q}=1}^{Q} \sum_{t\in B_{k,l,c,q} r(c)} \sum_{\tilde{t}\in B_{k,l,c,\tilde{q}} r(c)} \frac{q\tilde{q}}{Q^2} 1_{tcW_k} U_{tc\Lambda(c)^{-1}(t^{(i)})^{-1}} f U_{\tilde{t}^{(i)}\Lambda(c)c^{-1}\tilde{t}^{-1}} 1_{\tilde{t}cW_k}.\end{aligned}$$

To simplify the notation, let $g_t = tc$ and $h_t = t^{(i)}\Lambda(c)$. Then we have

$$\begin{aligned}h^*_{k,l,c,i,j} f h_{k,l,c,i,j} &= \sum_{q,\tilde{q}=1}^{Q} \sum_{t\in B_{k,l,c,q} r(c)} \sum_{\tilde{t}\in B_{k,l,c,\tilde{q}} r(c)} \frac{q\tilde{q}}{Q^2} 1_{g_tW_k} \alpha_{g_t h_t^{-1}}(f) U_{g_t h_t^{-1} h_{\tilde{t}} g_{\tilde{t}}^{-1}} 1_{g_{\tilde{t}}W_k} \\ &= \sum_{q,\tilde{q}=1}^{Q} \sum_{t\in B_{k,l,c,q} r(c)} \sum_{\tilde{t}\in B_{k,l,c,\tilde{q}} r(c)} \frac{q\tilde{q}}{Q^2} \alpha_{g_t h_t^{-1}}(f) 1_{g_tW_k} 1_{g_t h_t^{-1} h_{\tilde{t}} W_k} U_{g_t h_t^{-1} h_{\tilde{t}} g_{\tilde{t}}^{-1}}.\end{aligned}$$

Extracting the important part, this term is associated to the product of indicator functions

$$1_{tcW_k} 1_{tc\Lambda(c)^{-1}(t^{(i)})^{-1}\tilde{t}^{(i)}\Lambda(c)W_k}.$$

The groupoid element $(t^{(i)})^{-1}\tilde{t}^{(i)}$ is only defined when $r(t^{(i)}) = r(\tilde{t}^{(i)})$. We already know that $s(t^{(i)}) = s(\tilde{t}^{(i)}) = r(\Lambda_{k,i,j}(c))$, so, by principality of R_{punc}, this element is only defined when $t^{(i)} = \tilde{t}^{(i)}$. Since $t^{(i)}$ and $\tilde{t}^{(i)}$ are associated to the same vectors in $r(\Lambda_{k,i,j}(c))$ as t and $\tilde{t}$ are in c, this shows that the only contributions to the sum occur when $t = \tilde{t}$, and hence when $q = \tilde{q}$. Thus, we obtain

$$h^*_{k,l,c,i,j} f h_{k,l,c,i,j} = \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q^2}{Q^2} \alpha_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}(f) 1_{tcW_k}. \tag{6.4.5}$$

Similarly, we obtain

$$\begin{aligned} f h^*_{k,l,c,i,j} h_{k,l,c,i,j} &= f \left(\sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q}{Q} U_{t^{(i)}\Lambda_{k,i,j}(c)c^{-1}t^{-1}} \alpha_{tc}(1_{W_k}) \right)^* \\ &\quad \left(\sum_{\tilde{q}=1}^{Q} \sum_{\tilde{t} \in B_{k,l,c,\tilde{q}} r(c)} \frac{\tilde{q}}{Q} U_{\tilde{t}^{(i)}\Lambda_{k,i,j}(c)c^{-1}\tilde{t}^{-1}} \alpha_{\tilde{t}c}(1_{W_k}) \right) \\ &= \sum_{q,\tilde{q}=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \sum_{\tilde{t} \in B_{k,l,c,\tilde{q}} r(c)} \frac{q\tilde{q}}{Q^2} f 1_{tcW_k} U_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}\tilde{t}^{(i)}\Lambda_{k,i,j}(c)c^{-1}\tilde{t}^{-1}} 1_{\tilde{t}cW_k}. \end{aligned}$$

As before, this term is only defined when $((t^{(i)})^{-1}, \tilde{t}^{(i)}) \in G^{(2)}$, which forces $t = \tilde{t}$ and $q = \tilde{q}$, whence we obtain

$$f h^*_{k,l,c,i,j} h_{k,l,c,i,j} = \sum_{q=1}^{Q} \sum_{t \in B_{k,l,c,q} r(c)} \frac{q^2}{Q^2} f 1_{tcW_k}. \tag{6.4.6}$$

Now, let $x \in W_k$. By definition of $C_{k,l,m}$ and $\Lambda_{k,i,j}$, both $\Lambda_{k,i,j}(c)x$ and cx belong to U_m, which had diameter less than η. In addition, we have $d(t, t^{(i)}) < \eta$ for all t and i. Therefore, by the definition of η, we obtain that

$$|f(t^{(i)}\Lambda_{k,i,j}(c)x) - f(tcx)| < \frac{\epsilon^2}{4n^4}$$

and hence

$$\begin{aligned} \sup_{y \in tcW_k} \left| \left(\alpha_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}(f) - f \right)(y) \right| &= \sup_{x \in W_k} \left| \left(\alpha_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}(f) - f \right)(tcx) \right| \\ &= \sup_{x \in W_k} \left| \alpha_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}(f)(tcx) - f(tcx) \right| \\ &= \sup_{x \in W_k} |f(t^{(i)}\Lambda_{k,i,j}(c)x) - f(tcx)| \\ &\leq \frac{\epsilon^2}{4n^4}. \end{aligned}$$

Using this along with (6.4.5) and (6.4.6), and the fact that the tower levels tcW_k are disjoint for distinct t, we obtain

$$\begin{aligned}\|h^*_{k,l,c,i,j}fh_{k,l,c,i,j} - fh^*_{k,l,c,i,j}h_{k,l,c,i,j}\| &= \max_{q\in\{1,\dots,Q\}}\sup_{t\in B_{k,l,c,q}}\frac{q^2}{Q^2}\|(\alpha_{tc\Lambda_{k,i,j}(c)^{-1}(t^{(i)})^{-1}}(f) - f)1_{tcW_k}\| \\ &< \frac{\epsilon^2}{3n^4} \qquad (6.4.7)\end{aligned}$$

Now, set $w = \phi(e_{ij})$ and fix $f \in \Upsilon$. Since the indicator functions associated to $h_{k,l,c,i,j}$ have pairwise disjoint supports for distinct $k \in \{1,\dots,N\}$, $l \in \{1,\dots,L\}$, $m \in \{1,\dots,M\}$ and $c \in C^{(j)}_{k,l,m}$, it follows from (6.4.7) that

$$\|w^*gw - gw^*w\| < \frac{\epsilon^2}{3n^4}$$

where g is equal to f or f^2 (because in (6.4.7) we were working with $f \in \Upsilon \cup \Upsilon^2$). From this, it follows that

$$\begin{aligned}\|w^*f^2w - fw^*fw\| &\leq \|w^*f^2w - f^2w^*w\| + \|f^2w^*w - fw^*fw\| \\ &< \frac{\epsilon^2}{3n^4} + \|f(fw^*w - w^*fw)\| \\ &\leq \frac{\epsilon^2}{3n^4} + \|f\|\|fw^*w - w^*fw\| \\ &< \frac{\epsilon^2}{3n^4} + \frac{\epsilon^2}{3n^4} \\ &= \frac{2\epsilon^2}{3n^4}.\end{aligned}$$

Hence

$$\begin{aligned}\|fw - wf\|^2 &= \|(fw - wf)^*(fw - wf)\| \\ &= \|w^*f^2w - fw^*fw + fw^*wf - w^*fwf\| \\ &\leq \|w^*f^2w - fw^*fw\| + \|(fw^*w - w^*fw)f)\| \\ &< \frac{2\epsilon^2}{3n^4} + \frac{\epsilon^2}{3n^4} \\ &= \frac{\epsilon^2}{n^4}\end{aligned}$$

so that

$$\|fw - wf\| < \frac{\epsilon}{n^2}.$$

Therefore, for every norm-one $b = (b_{ij}) \in M_n$, we have

$$\|[f,\phi(b)]\| \leq \sum_{i,j=1}^{n}\|[f,\phi(b_{ij})]\| < n^2 \cdot \frac{\epsilon}{n^2} = \epsilon.$$

To complete the proof, we now show that $1 - \phi(I) \precsim a$. By enlarging E to include D, and shrinking δ if necessary, we can ensure that the sets $S_1, \ldots, S_N$ are sufficiently left-invariant under D so that, for each $k \in \{1, \ldots, N\}$ and $u \in G^{(0)}$, the set $R_k u \coloneqq \{s \in S_k u \mid Ds \subset S_k\}$ has cardinality at least $|S_k u|/2$. Note that, since $D^{-1} \cdot O = X$, we have $s(D) = X = G^{(0)}$. Thus, for any $t \in G$, we have $t \in D^{-1}Dt$, because $Dt \neq \varnothing$.

Fix $k \in \{1, \ldots, N\}$. Let $R_k = \bigsqcup_{u \in G^{(0)}} R_k u \subset S_k$. Let R_k' be a maximal subset of R_k with the property that the collection $\{Ds \mid s \in R_k'\}$ is pairwise disjoint. Observe that if $s, t \in R_k$ have $Ds \cap Dt \neq \varnothing$, then $s \in D^{-1}Dt$. Therefore, if R_k' is chosen in such a way that there exists $t \in R_k$ such that $R_k' \cap D^{-1}Dt = \varnothing$, then, by including any element of $D^{-1}Dt \cap R_k$ in R_k' (for instance, the choice $t \in D^{-1}Dt \cap R_k$ is always valid), we obtain a larger set which satisfies the disjointness condition on R_k', contradicting the maximality of R_k'. So, for each $u \in G^{(0)}$, $R_k'u$ contains at least one element from each subset of the form $D^{-1}Dt$ for $t \in R_k u$. Now, the smallest that we can make $R_k'u$ is to assume that we choose exactly one element from each such set, and to assume that all of these sets are subsets of $R_k u$. In this case, we get

$$|R_k'u| \geq \frac{|R_k u|}{\max_{t \in R_k u} |D^{-1}Dt|} \geq \frac{|S_k u|}{2 \max_{u \in G^{(0)}} |D^{-1}Du|}.$$

Put $P = \max_{u \in G^{(0)}} |D^{-1}Du|$, and note that $P < \infty$ since we assumed that $|Du|$ and $|D^{-1}u| = |uD|$ were uniformly bounded for $u \in G^{(0)}$.

By constructing D using the basis of clopen bisections $\{V(T \sqcap B_r(0), t, t')\}$, where we only consider sufficiently large values of r, we may ensure that the sets Dx and Dy consist of groupoid elements which implement the same vectors whenever x and y are sufficiently close. This will allows us to choose R_k' to contain either all of the groupoid elements (of some $S_{k,j}$) which make up a particular level of the tower (W_k, S_k), or none of them. Combining this with the inequality above, we may assume that (W_k, R_k') is a subtower of (W_k, S_k) which contains at least one $(2P)$-th of the levels of the tower (W_k, S_k).

Since $D^{-1}O = X$, for each $s \in R_k'$ there exists $t \in D$ such that tsW_k intersects O. Indeed, $s \in S_k$, so $sW_k \neq \varnothing$, and thus we can find $t \in D$ and $w \in O$ such that $t^{-1}w = r(s) \in sW_k$, which implies that $w \in tsW_k$ and hence that $tsW_k \cap O \neq \varnothing$. By construction of R_k, we have $ts \in S_k$, so tsW_k is a subset of a level of our tower, which was assumed to have diameter less than $\eta' \leq \theta$. Thus, since O was the ball of radius θ centred at x_0, and since tsW_k intersects O and has diameter smaller than θ, the tower level containing tsW_k is contained in the ball of radius 2θ centred at x_0, and thus a takes value 1 on this entire tower level. Furthermore, since the sets Ds for $s \in R_k'$ are disjoint, all the the elements $ts \in S_k$ constructed here are distinct. Thus, we have constructed an injective association $s \mapsto \tilde{s}$ of elements $s \in R_k'$ to elements $\tilde{s} = ts \in S_k$, such that a takes the constant value

1 on the tower level which contains $r(\tilde{s})$. This shows that, for each $u \in G^{(0)}$, the set $S_k^\sharp u$ of elements $t \in S_k u$ such that a takes the constant value 1 on the tower level containing $r(t)$ has cardinality $|S_k^\sharp u| \geq |R_k' u| \geq |S_k u|/(2P)$. Note that, for any j, if $t \in S_{k,j}$ is an element of $S_k^\sharp$, then all elements $s \in S_{k,j}$ are in $S_k^\sharp$ as well, since $t \in S_k^\sharp$ implies that a takes the constant value 1 on the tower level $S_{k,j}W_k$. In other words, $(W_k, S_k^\sharp)$ is a subtower of (W_k, S_k), and contains at least one $(2P)$-th of the levels of this tower.

Set

$$S_k'' = \bigsqcup_{l=1}^{L} \bigsqcup_{m=1}^{M} \bigsqcup_{i=1}^{n} \bigsqcup_{c \in C_{k,l,m}^{(i)}} B_{k,l,c,Q}c,$$

noting that (W_k, S_k'') is a subtower of (W_k, S_k), and fix $u \in G^{(0)}$. We have, for each $v \in G^{(0)}$, that $|B_{k,l,c,Q}v| \geq (1-\kappa)|T_l v|$, so we obtain

$$\begin{aligned}
|S_k'' u| &\geq \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C_{k,l,m}^{(i)} u} |B_{k,l,c,Q} r(c)| \\
&\geq (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C_{k,l,m}^{(i)} u} |T_l r(c)| \\
&= (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \sum_{c \in C_{k,l,m}^{(i)} u} |T_l c| \\
&\geq (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} \left| \bigcup_{c \in C_{k,l,m}^{(i)} u} T_l c \right| \\
&= (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} |T_l C_{k,l,m}^{(i)} u|.
\end{aligned}$$

Recall that $\bigcup_{i=1}^{n} C_{k,l,m}^{(i)} \subset C_{k,l,m}$ contains all except at most n elements of $C_{k,l,m}$. We use this to continue the computation above, obtaining

$$\begin{aligned}
|S_k'' u| &\geq (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{n} |T_l C_{k,l,m}^{(i)} u| \\
&\geq (1-\kappa) \sum_{l=1}^{L} \sum_{m=1}^{M} (|T_l C_{k,l,m} u| - n \max_{v \in G^{(0)}} |T_l v|) \\
&\geq (1-\kappa) \sum_{l=1}^{L} \left(\left| \bigcup_{m=1}^{M} T_l C_{k,l,m} u \right| - Mn \max_{v \in G^{(0)}} |T_l v| \right) \\
&\geq (1-\kappa) \sum_{l=1}^{L} (|T_l C_{k,l} u| - Mn \max_{v \in G^{(0)}} |T_l v|) \\
&\geq (1-\kappa) \left(\left| \bigcup_{l=1}^{L} T_l C_{k,l} u \right| - Mn \sum_{l=1}^{L} \max_{v \in G^{(0)}} |T_l v| \right)
\end{aligned}$$

$$\begin{aligned}&\geq (1-\kappa)\left((1-\beta)|S_k u| - \beta|S_k u|\right)\\ &\geq (1-\kappa)(1-2\beta)|S_k u|,\end{aligned}$$

where we used the fact that $\{T_l c \mid l \in \{1,\dots,L\}, c \in C_{k,l}\}$ is a $(1-\beta)$-cover of S_k, and the inequality (6.4.2) together with (E,δ)-invariance of S_k to obtain the second-to-last line.

So, if κ and β are small enough, we can ensure that, for each $u \in G^{(0)}$, we have

$$|(S_k \setminus S_k'')u| \leq \frac{|S_k u|}{4P} \leq \frac{|S_k^\sharp u|}{2}.$$

In this case, choose an injection $f_k : S_k \setminus S_k'' \to S_k^\sharp$ which sends $(S_k \setminus S_k'')u$ to $S_k^\sharp u$ for each $u \in G^{(0)}$. Further enforce that if $t, t' \in S_{k,j}$ for some j, and $f_k(t) \in S_{k,p}$ for some p, then $f_k(t') \in S_{k,p}$ as well. In this way, f_k will map the collection of elements associated to any tower level to a collection of elements associated to some other tower level. Since $\{(W_k, S_k)\}_{k=1}^N$ is a clopen tower decomposition of X, we have $X = \bigsqcup_{k=1}^N S_k W_k$, and hence

$$X \setminus \bigsqcup_{k=1}^N S_k'' W_k = \bigsqcup_{k=1}^N (S_k \setminus S_k'')W_k.$$

Since $(W_k, S_k \setminus S_k'')$ and $(W_k, S_k^\sharp)$ are subtowers of (W_k, S_k), write $S_k \setminus S_k'' = \bigsqcup_{j \in J} S_{k,j}$, and $S_k^\sharp = \bigsqcup_{j^\sharp \in J^\sharp} S_{k,j^\sharp}$, where J and $J^\sharp$ are finite index sets. Observe that the collection of subsets $S_{k,j}W_k$ for $k \in \{1,\dots,N\}$ and $j \in J$ covers $X \setminus \bigsqcup_{k=1}^N S_k'' W_k$. For each $j \in J$, and each $t \in S_{k,j}$, consider the element $f_k(t)t^{-1} \in G$, noting that $s(f_k(t)) = s(t) = r(t^{-1})$ by construction of f_k, so that the multiplication is defined. Then $\bigsqcup_{t \in S_{k,j}} (f_k(t)t^{-1})tW_k = \bigsqcup_{t \in S_{k,j}} f_k(t)W_k$ is a level $S_{k,j^\sharp}$ of the tower $(W_k, S_k^\sharp)$, so, since f_k was injective and all the levels $S_k W_k$ of the tower (W_k, S_k) were disjoint for $k \in \{1,\dots,N\}$, the collection $\{\bigsqcup_{t \in S_{k,j}} f_k(t)W_k \mid k \in \{1,\dots,N\}, j \in J\}$ of open subsets of $\bigsqcup_{k=1}^N S_k^\sharp W_K$ is pairwise disjoint, and consists of images (under the action map) of sets which covered $X \setminus \bigsqcup_{k=1}^N S_k'' W_k$. Thus, we have shown that

$$X \setminus \bigsqcup_{k=1}^N S_k'' W_k \prec \bigsqcup_{k=1}^N S_k^\sharp W_k.$$

Since the function $1 - \phi(I)$ is supported on $X \setminus \bigsqcup_{k=1}^N S_k'' W_k$, and a takes the constant value 1 on $\bigsqcup_{k=1}^N S_k^\sharp W_k$, it follows by Lemma 6.4.2 that there exists a $v \in C(X) \rtimes_r G$ such that $v^* a v = 1 - \phi(I)$, which shows that $1 - \phi(I) \precsim a$. □

Corollary 6.4.4. *The C^*-algebra associated to any aperiodic, repetitive tiling with finite local complexity is classified by its Elliott invariant.*

Proof. Combine Theorems 3.5.3, 6.4.3, 6.1.7, and 6.1.8. □

We now discuss the previously known classification results for tiling C^*-algebras, and give some examples which are newly classifiable via our approach.

The first result we discuss is concerned with *rational* tilings. A tiling is called rational if the edge of every tile is given by a vector with rational coordinates. In [62, Lemma 5], Sadun and Williams prove that the dynamical system arising from any punctured rational tiling is topologically conjugate to the dynamical system $(\Omega_{\mathrm{sq}}, \mathbb{Z}^d)$ associated to some tiling by unit squares punctured at their centres, which is naturally equipped with the $\mathbb{Z}^d$ action of translation between punctures. The associated crossed product $C(\Omega_{\mathrm{sq}}) \rtimes \mathbb{Z}^d$ is shown to be classifiable by Winter in [83, Corollary 3.2] via the nuclear dimension estimates of Szabó [72, Theorem 5.3].

The second result concerns the tilings generated from a primitive and recognisable substitution (see Section 3.2). Anderson and Putnam [5] prove that the extension of the substitution map to the continuous hull $\omega : \Omega \to \Omega$ gives Ω the structure of a Smale space, and that the unstable equivalence relation on this Smale space is precisely the relation of translational equivalence used to define the tiling groupoid R_{punc} (see Definition 3.3.10). Therefore, the unstable C^*-algebra of this Smale space is isomorphic to the tiling C^*-algebra $C^*(R_{\mathrm{punc}})$. Deeley and Strung [12] are able to show that these algebras are classifiable via the dynamic asymptotic dimension of Guentner, Willett and Yu [27].

We remark that the work in this thesis was originally inspired by Deeley and Strung's results. We originally hoped to classify general tiling algebras by proving that the dynamic asymptotic dimension of the associated groupoids is finite. This relies on assignments of finitely many colours to elements of the unit space of a groupoid such that the set of elements of each colour is open, and so that the groupoids generated by arrows whose source and range are assigned the same colour are all relatively compact. In the tiling groupoid, open sets are collections of patterns with distinguished tiles, so we hoped to assign a colour to each pattern of some fixed size in a systematic way to witness finite dynamic asymptotic dimension. We quickly ran into a complication: it is possible for two copies of the same pattern to overlap in a tiling. This behaviour can create "fault lines" in which infinitely many copies of the same pattern sit along a ray in the tiling. This creates an infinite chain of composable arrows between elements of one colour, and the subgroupoid generated by these arrows is not relatively compact. In order to solve this problem, one might hope to increase the size of the pattern so as to make this self-intersection rare enough to eliminate fault lines. In doing so, one must also increase the number of colours (and hence the dimension estimate), and it was not clear that this process would terminate in finite time. In contrast, [41, Remark 6.4] guarantees the existence of an almost finite tower structure within the tiling groupoid, allowing us to circumvent the issues with this construction.

Our result requires no substitution structure, and places no restriction on the shapes of the tiles. Another advantage of our approach over the methods of [62] is that we avoid the need to deform the tiling in any way. Concrete examples of tilings which are newly classified by our result can be found in work of Socolar [69], in which aperiodicity is enforced by an "alternation condition" on rhombic tiles whose edges are given by irrational vectors. More abstractly, the undecidability of the Domino Problem implies that there exist infinitely many distinct methods to enforce aperiodicity in a tiling, although only a few constructions are currently known. Any aperiodic, repetitive, non-rational, non-substitution tiling with finite local complexity is newly captured by our classification methods.

6.5 Quasidiagonality

In this section, we give a direct proof that the C^*-algebra A_T associated to any aperiodic, repetitive tiling T with finite local complexity is *quasidiagonal*. Our result holds even in the case that the tiling has infinite rotational symmetry. We stress that these algebras were already known to be quasidiagonal as a consequence of [73, Corollary B], and our result is simply a more direct proof of this fact.

Quasidiagonality does not give us any new information regarding the classifiability of these algebras, but it does simplify the machinery required for their classification in certain cases. In particular, Matui and Sato [43, Corollary 6.2] classify simple separable unital nuclear quasidiagonal C^*-algebras in the UCT class which have strict comparison and a unique trace. Their result exploits Winter's localisation technique [82] by combining [43, Theorem 6.1] and the localised classification result of Lin and Niu [40, Theorem 5.4] to show that these algebras are classifiable up to $\mathcal{Z}$-stability. Earlier work by Matui and Sato [42, Theorem 1.1] shows that strict comparison and $\mathcal{Z}$-stability are equivalent for these algebras, so in fact they obtain the non-localised classification result in [43]. In this way, C^*-algebras which are quasidiagonal, $\mathcal{Z}$-stable, and monotracial are seen to be classifiable without reference to some of the complicated constructions culminating in [73].

In the light of the above, for the purposes of this section, we are interested in tilings whose C^*-algebras have unique trace. It is shown in [33] that the C^*-algebras associated to aperiodic, repetitive *substitution* tilings with finite local complexity are monotracial. As was noted on page 152, these C^*-algebras were already known to be classified by results of Deeley and Strung [12, Corollary 3.8 and Theorem 4.7], who prove that the C^*-algebras associated to substitution tilings (i.e. the unstable C^*-algebras of the associated Smale space) have finite nuclear dimension. Their main classification result is only stated for the homoclinic algebra of the Smale space, which they prove is quasidiagonal. Our quasidiagonality result therefore provides the "missing piece" necessary to access the simplified route to classification detailed in the previous paragraph for the unstable C^*-algebras of the Smale spaces associated to substitution tilings.

In the non-substitution case, the class of *quasiperiodic* tilings is detailed in [59, Section 8.2], in which it is stated [59, Corollary 8.5] (see also [58, Section 12]) that the dynamical systems associated to these tilings are uniquely ergodic, and hence their C^*-algebras have unique trace.

Definition 6.5.1. Let H be a separable Hilbert space. A subset $A \subset B(H)$ is called a *quasidiagonal set of operators* if there exists an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \leq \cdots$, which converge strongly to the identity operator (that is, for any $x \in H$, $\|P_n(x) - x\| \to 0$ as $n \to \infty$), and are such that, for every $a \in A$,

$$\|[a, P_n]\| \coloneqq \|aP_n - P_n a\| \to 0 \text{ as } n \to \infty.$$

A C^*-algebra A is called *quasidiagonal* if there exists a faithful representation $\pi : A \to B(H)$ such that $\pi(A)$ is a quasidiagonal set of operators.

Theorem 6.5.2. *The C^*-algebra associated to any aperiodic and repetitive tiling with finite local complexity (with either finite or infinite rotational symmetry) is quasidiagonal.*

Proof. Let Ω_{punc} be the punctured hull of such a tiling, and fix a tiling $T \in \Omega_{\text{punc}}$. In [53] it is shown that the induced representation from the unit space $\pi \coloneqq \pi_T$ described on the generating set $\mathcal{E}$ defined in Section 3.5.1 extends to a faithful nondegenerate representation of A_T on $\ell^2(T) \coloneqq \ell^2([T])$. We will prove that $\pi(A_T)$ is a quasidiagonal set of operators. Define a sequence of projections $Q_n \in B(\ell^2(T))$ by

$$Q_n(\delta_t) \coloneqq \begin{cases} \delta_t & \text{if } t \in T \sqcap B_n(0) \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(Q_n)_{n\in\mathbb{N}}$ is an increasing sequence of projections. Since T has FLC, Q_n has finite rank for each $n \in \mathbb{N}$, and $(Q_n)_{n\in\mathbb{N}}$ converges strongly to the identity by square-summability of the elements of $\ell^2(T)$.

We now show that, for each element $z^k \cdot e(P, t, t') \in \mathcal{E}$, we have

$$\lim_{n\to\infty} \|Q_n \pi(z^k \cdot e(P, t, t')) - \pi(z^k \cdot e(P, t, t'))Q_n\|_{\text{op}} = 0.$$

This will prove that A_T is quasidiagonal, since $\overline{\text{span}\{\mathcal{E}\}} = A_T$. We first observe that

$$\lim_{n\to\infty} \|Q_n \pi(z^k \cdot e(P, t, t')) - \pi(z^k \cdot e(P, t, t'))Q_n\|_{\text{op}}$$

$$= \lim_{n\to\infty} \sup_{t''\in T} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2. \tag{6.5.1}$$

Set $\epsilon > 0$. Then there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} &\sup_{t''\in T} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2 \\ &\qquad \le \epsilon + \sup_{t''\in T\sqcap B_m(0)} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2. \end{aligned} \tag{6.5.2}$$

Since the supremum in (6.5.2) is over a finite set, we can use this bound in (6.5.1) and then interchange the limit and supremum on the right-hand side to obtain

$$\begin{aligned} &\lim_{n\to\infty} \|Q_n\pi(z^k \cdot e(P,t,t')) - \pi(z^k \cdot e(P,t,t'))Q_n\|_{\mathrm{op}} \\ &\qquad \le \epsilon + \sup_{t''\in T\sqcap B_m(0)} \lim_{n\to\infty} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2. \end{aligned} \tag{6.5.3}$$

Choose $n \in \mathbb{N}$ large enough that $T \sqcap B_n(0)$ includes all tiles that intersect the ball $B_{m+|x(t')-x(t)|}(0)$. Then we compute that

$$\begin{aligned} &\sup_{t''\in T\sqcap B(0,m)} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2 \\ &= \begin{cases} \sup_{t''\in T\sqcap B(0,m)} \|e^{k(\angle t+\theta)i}Q_n\delta_{t'''} - \pi(z^k \cdot e(P,t,t'))\delta_{t''}\|_2 \\ \qquad \text{if } \exists (0,R_\theta) \in \mathbb{R}^2 \rtimes S^1 \text{ such that } R_\theta(P-x(t)) \subset T - x(t'') \\ \qquad \text{and } x(t''') = x(t'') + R_\theta(x(t')-x(t)) \\ 0 \qquad \text{otherwise} \end{cases} \\ &= \begin{cases} \sup_{t''\in T\sqcap B(0,m)} \|e^{k(\angle t+\theta)i}\delta_{t'''} - e^{k(\angle t+\theta)i}\delta_{t'''}\|_2 \\ \qquad \text{if } \exists (0,R_\theta) \in \mathbb{R}^2 \rtimes S^1 \text{ such that } R_\theta(P-x(t)) \subset T - x(t'') \\ \qquad \text{and } x(t''') = x(t'') + R_\theta(x(t')-x(t)) \\ 0 \qquad \text{otherwise} \end{cases} \\ &= 0. \end{aligned}$$

Since this holds for all sufficiently large $n \in \mathbb{N}$, we see that

$$\sup_{t''\in T\sqcap B_m(0)} \lim_{n\to\infty} \|Q_n\pi(z^k \cdot e(P,t,t'))\delta_{t''} - \pi(z^k \cdot e(P,t,t'))Q_n\delta_{t''}\|_2 = 0. \tag{6.5.4}$$

Using (6.5.4) in (6.5.3), we obtain

$$\lim_{n\to\infty} \|Q_n\pi(z^k \cdot e[P,t,t']) - \pi(z^k \cdot e[P,t,t'])Q_n\|_{\mathrm{op}} \le \epsilon,$$

which gives the desired result. □

References

[1] S. Akiyama, J. Caalim, K. Imai, and H. Kaneko. Corona limits of tilings: Periodic case. *Discrete Comput. Geom.*, 61(3):626–652, 2019.

[2] H. Alzahrani. *Cut-and-project tilings constructed from crystallographic tilings*. PhD thesis, University of Liverpool, 2016.

[3] C. Anantharaman-Delaroche. Some remarks about the weak containment property for groupoids and semigroups. arXiv:1604.01724v3 [math.OA].

[4] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique*. Enseignement mathématique, Geneva, 2000.

[5] J. E. Anderson and I. F. Putnam. Topological invariants for substitution tilings and their associated C^*-algebras. *Ergod. Theory Dyn. Syst.*, 18(3):509–537, 1998.

[6] P. Ara, C. Bönicke, J. Bosa, and K. Li. Almost finite ample groupoids and their C^*-algebras. 2019. In preparation.

[7] R. Berger. The undecidability of the domino problem. *Mem. Amer. Math. Soc.*, 66, 1966.

[8] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter. Covering dimension of C^*-algebras and 2-coloured classification. *Mem. Am. Math. Soc.*, 257(1233), 2019.

[9] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter. Nuclear dimension of simple C^*-algebras. arXiv:1901.05853v2 [math.OA].

[10] A. Connes. *Noncommutative Geometry*. Academic Press, 1994.

[11] J. Cuntz. Dimension Functions on Simple C^*-Algebras. *Math. Ann.*, 233(2):145–153, 1978.

[12] R. J. Deeley and K. R. Strung. Nuclear dimension and classification of C^*-algebras associated to Smale spaces. *Trans. Amer. Math. Soc.*, 370(5):3467–3485, 2018.

[13] J. Dixmier. *C^*-algebras*, volume 15 of *North-Holland Mathematical Library*. North-Holland Publishing Co., 1977.

[14] T. Downarowicz and G. Zhang. The comparison property of amenable groups. arXiv:1712.05129v1 [math.DS], 2017.

[15] C. Ehresmann. Gattungen von lokalen Strukturen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 60:49–77, 1958.

[16] G. Elek. Cantor combinatorics and almost finiteness. arXiv:1802.01908v1 [math.DS].

[17] G. Elek. Qualitative graph limit theory. Cantor dynamical systems and constant-time distributed algorithms. arXiv:1812.07511v2 [math.DS].

[18] G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.

[19] G. A. Elliott, G. Gong, H. Lin, and Z. Niu. On the classification of simple amenable C^*-algebras with finite decomposition rank, II. 2016. arXiv:1507.03437v3 [math.OA].

[20] R. Exel. Reconstructing a totally disconnected groupoid from its ample semigroup. *Proc. Am. Math. Soc.*, 138(8):2991–3001, 2010.

[21] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau. Affable equivalence relations and orbit structure of Cantor dynamical systems. *Ergod. Theory Dynam. Syst.*, 24(2):441–475, 2004.

[22] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau. Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.*, 179(1):119–158, 2010.

[23] G. Goehle. *Groupoid Crossed Products*. PhD thesis, Dartmouth College, 2009. arXiv:0905.4681v1 [math.OA].

[24] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras. 2015. arXiv:1501.00135v6 [math.OA].

[25] E. C. Gootman and J. Rosenberg. The structure of crossed product C^*-algebras: A proof of the generalized Effros-Hahn conjecture. *Invent. Math.*, 52(3):283–298, 1979.

[26] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. Freeman, New York, 1987.

[27] E. Guentner, R. Willett, and G. Yu. Dynamic Asymptotic Dimension: relation to dynamics, topology, coarse geometry, and C^*-algebras. *Math. Ann.*, 367(1–2):785–829, 2017.

[28] P. Hahn. Haar measure for measure groupoids. *Trans. Amer. Math. Soc.*, 242:1–33, 1978.

[29] I. Hirshberg and J. Orovitz. Tracially $\mathcal{Z}$-absorbing C^*-algebras. *J. Funct. Anal.*, 265(5):765–785, 2013.

[30] A. an Huef, I. Raeburn, and D. P. Williams. Properties preserved under Morita equivalence of C^*-algebras. *Proc. Amer. Math. Soc.*, 135:1495–1503, 2007.

[31] L. J. Ito, M. F. Whittaker, and J. Zacharias. Classification of tiling C^*-algebras. arXiv:1908.00770v1 [math.OA].

[32] X. Jiang and H. Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.

[33] J. Kellendonk. Noncommutative Geometry of Tilings and Gap Labelling. *Rev. Math. Phys.*, 7(7):1133–1180, 1995.

[34] J. Kellendonk and I. F. Putnam. Tilings, C^*-algebras, and K-theory. In *Directions in Mathematical Quasicrystals*, volume 13 of *CRM Monogr. Ser.*, pages 177–206, Providence, 2000. American Mathematical Society.

[35] D. Kerr. Dimension, comparison, and almost finiteness. *J. Eur. Math. Soc. (to appear).* arXiv:1710.00393v1 [math.DS].

[36] D. Kerr and H. Li. *Ergodic Theory: Independence and Dichotomies.* Springer, 2017.

[37] D. Kerr and G. Szabó. Almost finiteness and the small boundary property. *Comm. Math. Phys. (to appear).* arXiv:1807.04326v1 [math.OA].

[38] E. Kirchberg and N. C. Phillips. Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$. *J. Reine Angew. Math.*, 525:17–54, 2000.

[39] D. Levine and P. J. Steinhardt. Quasicrystals: A new class of ordered structures. *Phys. Rev. Lett.*, 53(26):2477–2480, 1984.

[40] H. Lin and Z. Niu. Lifting KK-elements, asymptotical unitary equivalence and classification of simple C^*-algebras. *Adv. Math.*, 219(5):1729–1769, 2008.

[41] H. Matui. Homology and topological full groups of étale groupoids on totally disconnected spaces. *Proc. Lond. Math. Soc.*, 104(1):27–56, 2012.

[42] H. Matui and Y. Sato. Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras. *Acta Math.*, 209(1):179–196, 2012.

[43] H. Matui and Y. Sato. Decomposition rank of UHF-absorbing C^*-algebras. *Duke Math. J.*, 163(14):2687–2708, 2014.

[44] P. Muhly and D. P. Williams. Renault's Equivalence Theorem for Groupoid Crossed Products. *New York Journal of Mathematics Monographs*, 3:1–83, 2008.

[45] P. S. Muhly, J. N. Renaut, and D. P. Williams. Equivalence and isomorphism for groupoid C^*-algebras. *J. Operator Theory*, 17(1):3–22, 1987.

[46] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[47] R. Penrose. The role of aesthetics in pure and applied mathematical research. *Bulletin of the Institute of Mathematics and Its Applications*, 10(2):266–271, 1974.

[48] R. Penrose. Pentaplexity: A Class of Non-Periodic Tilings of the Plane. *The Mathematical Intelligencer*, 2:32–37, 1979.

[49] N. C. Phillips. Large Subalgebras. arXiv:1408.5546v1 [math.OA], 2014.

[50] N. C. Phillips. *An introduction to crossed product C^*-algebras and minimal dynamics*. 2017. `https://pages.uoregon.edu/ncp/Courses/CRMCrPrdMinDyn/Notes_20170205.pdf`.

[51] C. Radin. The pinwheel tilings of the plane. *Ann. Math.*, 139(3):661–702, 1994.

[52] C. Radin and M. Wolff. Space tilings and local isomorphism. *Geom. Dedicata*, 42(3):355–360, 1992.

[53] J. Renault. *A Groupoid Approach to C^*-Algebras*, volume 793 of *Lect. Notes in Math.* 1980.

[54] J. Renault. Représentation des produits croisés d'algèbres de groupoïdes. *J. Operator Theory*, 18(1):67–97, 1987.

[55] J. Renault. Topological amenability is a Borel property. *Math. Scand.*, 117(1):5–30, 2015.

[56] M. A. Rieffel. *Unitary Representations of Group Extensions; an Algebraic Approach to the Theory of Mackey and Blattner*, volume 4 of *Adv. Math. supplementary studies.* Academic Press, 1979.

[57] M. A. Rieffel. Dimension and stable rank in the K-theory of C^*-algebras. *Proc. Lond. Math. Soc.*, 46(2):301–333, 1983.

[58] E. A. Robinson Jr. The Dynamical Theory of Tilings and Quasicrystallography. In *Ergodic Theory of $\mathbb{Z}^d$-Actions*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 451–474. Cambridge Univ. Press, 1996.

[59] E. A. Robinson Jr. Symbolic Dynamics and Tilings of $\mathbb{R}^d$. In *Symbolic Dynamics and its Applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 81–120, Providence, 2004. Amer. Math. Soc.

[60] M. Rørdam. A simple C^*-algebra with a finite and an infinite projection. *Acta Math.*, 191(1):109–142, 2003.

[61] J. Rosenberg and C. Schochet. The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. *Duke Math. J.*, 55(2):431–474, 1987.

[62] L. Sadun and R. F. Williams. Tiling spaces are Cantor set fiber bundles. *Ergod. Theory Dyn. Syst.*, 23(1):307–316, 2003.

[63] L. A. Sadun. Tilings, tiling spaces and topology. *Philos. Mag.*, 86(6–8):875–881, 2006.

[64] L. A. Sadun. *Topology of Tiling Spaces*. University lecture series. American Mathematical Society, 2008.

[65] Y. Sato, S. White, and W. Winter. Nuclear dimension and $\mathcal{Z}$-stability. *Invent. Math.*, 202(2):893–921, 2015.

[66] A. Sierakowski. *Discrete Crossed product C^*-algebras*. PhD thesis, University of Copenhagen, 2009.

[67] A. Sims. Hausdorff étale groupoids and their C^*-algebras. Lecture notes, arXiv:1710.10897v1 [math.OA], 2017.

[68] A. Sims and D. P. Williams. Amenability for Fell bundles over groupoids. *Illinois J. Math.*, 57(2):429–444, 2013.

[69] J. E. S. Socolar. Weak matching rules for quasicrystals. *Commun. Math. Phys.*, 129(3):599–619, 1990.

[70] P. J. Steinhardt and S. Ostlund. *The physics of quasicrystals*. World Scientific Publishing Co., 1989.

[71] Y. Suzuki. Almost finiteness for general étale groupoids and its applications to stable rank of crossed products. *Int. Math. Res. Notices (to appear)*. arXiv:1702.04875v3 [math.OA].

[72] G. Szabó. The Rokhlin dimension of topological $\mathbb{Z}^m$-actions. *Proc. Lond. Math. Soc.*, 110:673–694, 2015.

[73] A. Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C^*-algebras. *Ann. Math.*, 185(1):229–284, 2017.

[74] A. S. Toms. On the independence of K-theory and stable rank for simple C^*-algebras. *J. Reine Angew. Math.*, 578:185–199, 2005.

[75] A. S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. Math.*, 167(3):1029–1044, 2008.

[76] J. L. Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.

[77] H. Wang. Proving theorems by pattern recognition – II. *Bell Syst. Tech. J.*, 40(1):1–41, 1961.

[78] M. F. Whittaker. C^*-algebras of tilings with infinite rotational symmetry. *J. Operator Theory*, 64(2):299–319, 2010.

[79] R. Willett. A non-amenable groupoid whose maximal and reduced C^*-algebras are the same. *Münster J. Math.*, 8:241–252, 2015.

[80] D. P. Williams. *Crossed products of C^*-algebras*, volume 134 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2007.

[81] W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. *Invent. Math.*, 187(2):259–342, 2012.

[82] W. Winter. Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras. *J. Reine Angew. Math.*, 692:193–231, 2014.

[83] W. Winter. Classifying crossed product C^*-algebras. *Amer. J. Math.*, 138(3):793–820, 2016.

[84] W. Winter and J. Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2:311–324, 2009.

[85] W. Winter and J. Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.

[86] H. H. Zettl. Strong Morita equivalence of C^*-algebras preserves nuclearity. *Arch. Math. (Basel)*, 38:448–452, 1982.