GENERATORS OF THE SECOND HOMOTOPY MODULE OF GROUP PRESENTATIONS WITH APPLICATIONS

by

YOUNG - GHEEL BAIK

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To my wife YOUNG - WOO ,

my son MUN - IHL, my daughter HYE - IHN

CONTENTS

STATEMENT

Chapter 1 covers basic material and motivation for the thesis. Most of this is taken from [35].

Chapter 2, §5.2.2, §5.3 and chapter 6 are my own work. The results in Chapter 2 may be "well-known", but except for the material on relation modules I cannot find it in the literature.

§3.1 will appear in a joint paper with Howie and Pride [1]. §3.2, §3.3,§4.1 will appear in the more general form in a joint paper with Pride [2].

Almost all material in §5.1 is taken from [5], and the results in §5.2.1 are known [5] [18].

ACKNOWLEDGEMENT

I would like to express my deepest gratitute to my supervisor Dr. S. J. Pride, FRSE for suggesting the topics studied in this thesis, and for his assistance and encouragement throughout the research period.

I would also like to thank Prof. A. C. Kim in Pusan National University for first introducing me to the mathematics world.

Finally, I would like to thank the British Council for financial support for my first research year.

Abstract

The main work of this thesis starts with Chapter 2.

The main problems of the thesis are the following (*) and its applications.

Let $\mathscr{P}'$ be a subpresentation of a group presentation $\mathscr{P}$ and X be a set of generators of the second homotopy module $\pi_2(\mathscr{P}')$ of $\mathscr{P}'$.

(*) *What extra elements are needed to generate* $\pi_2(\mathscr{P})$*?*

In Chapter 2, we study the answer about (*) when $\mathscr{P}$ and $\mathscr{P}'$ define the same group, i.e., $\mathscr{P}$ is a presentation obtained from $\mathscr{P}'$ by a sequence of Tietze transformations. And we also study the relationship between the relation modules of $\mathscr{P}$ and $\mathscr{P}'$.

In Chapter 3 and 4, we study the answers about (*) when $\mathscr{P}$ is

(1) a graph product in §3.1

(2) a graph of groups (HNN extension, amalgamated free product) in §3.2

(3) a split extension in §4.1,

and we apply the results to get some properties - computing the second integral (co)homology, the efficiency and minimality, and the Cockroft property of $\mathscr{P}$. In particular, in §3.3.3, we get some exact sequences concerning graphs of groups.

In Chapter 5, we study situations where *no* extra elements are needed to generate $\pi_2(\mathscr{P})$ ("relative asphericity").

In Chapter 6, we compute the second integral (co)homology of aspherical Coxeter groups and consider the efficiency and minimality of Coxeter presentations and Coxeter groups. This work is related to the theme of the rest of the thesis, since it makes use of generators of π_2 of aspherical Coxeter presentations.

Notations

Let G, H, K be groups. page

$G \oplus H$	the direct sum	
$G * H$	the free product	
H^G	the normal closure of H in G	
$G \cong H$	G is isomorphic to H	
G/H	the quotient of G by H	
AutG	the automorphism group of G	
$H \times_\rho K$	the split extension of H by K where $\rho: K \longrightarrow \mathrm{Aut}(H)$ is a homomorphism	
sgpA	the subgroup generated by a subset A	
rk(G)	the rank of the torsion-free part (when G is abelian)	
d(G)	the least number of generators	
[a,b]	the commutator of a and b ($= aba^{-1}b^{-1}$)	
$\mathbb{Z}G$	the integral group ring	
IG	the augmentation ideal	
$- \otimes_G -$	the tensor product of $\mathbb{Z}G$-modules	
$\upsilon(G) = 1 - rk(H_1(G)) + d(H_2(G))$		16
ρ_1, ρ_2	the standard surjections	12
μ_1, μ_2	the standard injections	12
$H_k(G,A)$	the k-th homology group of G with coefficients in A	
$H^k(G,B)$	the k-th cohomology group of G with coefficients in B	
$H_2(G)$	the second integral homology of G	
$H_2(G)$	the second integral homology of G	

We adopt the usual notation in set theory.

$A \cup B$	the union of the sets A and B
$A \setminus B$	the set difference

$A \subseteq B$	A is a subset of B	
$a \in A$	a belongs to A	
$\lvert A \rvert$	the cardinality of A	
$A \times B$	the Cartesian product	
$A \xrightarrow{\text{incl}} B$	the inclusion of A into B	
$\mathbb{Z}$	the integer	
(m,n)	the greatest common divisor	
$\mathrm{Ker}\alpha$	the kernel of α	
$\mathrm{Im}\alpha$	the image of α	

Notation concerning presentations.

Let $\mathscr{P}$ be a group presentation.

$\pi_2(\mathscr{P})$	the second homotopy module	2
$M(\mathscr{P})$	the relation module	12
$I_2(\mathscr{P})$	the second Fox ideal	17
$\mathscr{P}^{st}$	the star-complex	81
$\chi(\mathscr{P})$	the Euler characteristic	16

Notation concerning pictures.

Let $\mathbb{P}$ be a picture.

$W(\mathbb{P})$	the label of $\mathbb{P}$	4
$\partial(\mathbb{P})$	the boundary of $\mathbb{P}$	3
$-\mathbb{P}$	the mirror image of $\mathbb{P}$	6
A^*	the symmetrized closure of a set A of pictures	10
$\mathbb{P}^W$	the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label W	8
$< \mathbb{P} >$	the equivalence class containing $\mathbb{P}$	8
$W(\gamma)$	the label of a path	5
$W(c)$	the label of a corner	77

$\theta(c)$	the angle of a corner	85
$\exp_R(\mathbb{P})$	the exponent sum of R in $\mathbb{P}$	13
$\exp_x(W)$	the exponent sum of x in W	13

Notation concerning graphs.

Let Γ be a graph.

$v=v(\Gamma)$	the vertex set	18
$e=e(\Gamma)$	the edge set	18
e^+	an orientation of e	18
$\iota(e)$	the initial vertex of e	18
$\tau(e)$	the terminal vertex of e	18
$\phi(e)$	the weight of e	105
$Adj(v)=\{e;\ e\in e^+$, one of endpoints of e is v$\}$		105

Notation concerning graphs of groups.

G_v	vertex group	36
$\mathcal{P}_v$	a presentation of G_v	37
T	a maximal forest	36
G_e, $\bar{G}_e$	edge groups	36
$a_{i,e}$, $\bar{a}_{i,e}$		37
y_e, $\bar{y}_e$		38
w_e	the set of all words on y_e	38
γ_e	the isomorphism of G_e into $\bar{G}_e$	36
θ_e	the natural epimorphism $F_e \longrightarrow G_e$	38
N_e	the kernel of θ_e	38

Miscellaneous notation.

Let σ be a sequence of words.

$\Pi\sigma$	the product of terms of σ	1
$< \sigma >$	the equivalence class containing σ	2
$\sigma(\underline{\gamma})$	the sequence associated with $\underline{\gamma}$	5

Let R be a relator.

Chapter 1 Preliminaries

In this chapter we will introduce the basic concepts. Our main reference is [35].

1.1 Second homotopy modules

Let $\mathcal{P} = \langle \mathbf{x} ; \mathbf{r} \rangle$ be a group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of cyclically reduced words on $\mathbf{x} \cup \mathbf{x}^{-1}$.

Let N be the normal closure of $\mathbf{r}$ in F, where F is the free group on $\mathbf{x}$. Then the quotient G of F by N is called the *group defined by* $\mathcal{P}$.

We denote by $\mathbf{w}$ the set of all words on $\mathbf{x} \cup \mathbf{x}^{-1}$. If $\mathbf{s}$ is a subset of $\mathbf{r}$ then $\mathbf{s}^{\mathbf{W}}$ is the set of all words of the form

$$WS^{\varepsilon}W^{-1} \ (W \in \mathbf{w},\ S \in \mathbf{s},\ \varepsilon = \pm 1).$$

Let σ be a finite sequence of elements of $\mathbf{r}^{\mathbf{W}}$, say $\sigma = (c_1, \dots, c_n)$, where $c_i \in \mathbf{r}^{\mathbf{W}}$ $(i = 1, \dots, n)$. Then we define $\Pi\sigma$ to be the product $c_1 c_2 \dots c_n$. If $\Pi\sigma$ is freely equal to 1, then σ is called an *identity sequence*. We define the *inverse* σ^{-1} of σ to be $(c_n^{-1}, \dots, c_1^{-1})$, and for $W \in \mathbf{w}$ we define the *conjugate* $W\sigma W^{-1}$ of σ by W to be $(Wc_1W^{-1}, \dots, Wc_nW^{-1})$.

We define operations on sequences as follows. Let $c_i = W_i R_i^{\varepsilon_i} W_i^{-1}$ $(W_i \in \mathbf{w}$, $R_i \in \mathbf{r}$, $\varepsilon_i = \pm 1$, $i = 1, \dots, n)$.

($\#_1$) Replace each W_i by a word freely equal to it.

($\#_2$) Delete two consecutive terms if one is identically equal to the inverse of the other.

($\#_3$) The opposite of ($\#_2$).

($\#_4$) Replace two consecutive terms c_i, c_{i+1} by either

$$c_{i+1},\ c_{i+1}^{-1} c_i c_{i+1} \quad \text{or} \quad c_i c_{i+1} c_i^{-1},\ c_i.$$

Two sequences σ, σ' will be said to be *(Peiffer) equivalent* if one can be obtained from the other by a finite number of applications of the operations

$(\#_1)$, $(\#_2)$, $(\#_3)$, $(\#_4)$. The equivalence class containing σ will be denoted by $<\sigma>$.

The set of all equivalence classes forms a group under the following binary operation

$$<\sigma_1> + <\sigma_2> = <\sigma_1\sigma_2>$$

where $\sigma_1\sigma_2$ is the juxtaposition of the two sequences σ_1, σ_2.

We let $\pi_2(\mathcal{P})$ denote the subgroup consisting of all elements $<\sigma>$ where σ is an identity sequence. We can think of an identity sequence as *a relation (an identity) among relators*. So $\pi_2(\mathcal{P})$ gives us a description of all relations among relators.

We note that the group of all equivalence classes is not abelian under the operation + but the subgroup $\pi_2(\mathcal{P})$ is abelian [35,p687].

We can also consider $\pi_2(\mathcal{P})$ as a left $\mathbb{Z}G$-module by the G-action given by

$$WN.<\sigma> = <W\sigma W^{-1}> \qquad (W\in F).$$

We call $\pi_2(\mathcal{P})$ the *second homotopy module of* $\mathcal{P}$.

In the next section an element of $\pi_2(\mathcal{P})$ will be represented by a geometric configuration.

1.2 Pictures

A *picture* $\mathbb{P}$ is a geometric configuration consisting of the following:

(a) A disc D^2 with basepoint O on ∂D^2.

(b) Disjoint discs $\Delta_1,\ldots,\Delta_n$ in the interior of D^2. Each disc Δ_λ $(\lambda=1,\ldots,n)$ has a basepoint O_λ on $\partial\Delta_\lambda$.

(c) A finite number of disjoint arcs $\alpha_1,\ldots,\alpha_m$. Each arc lies in the closure of $D^2\setminus\bigcup_{\lambda=1}^{n}\Delta_\lambda$ and is either a simple closed curve having trivial intersection with $\partial D^2\cup\partial\Delta_1\cup\ldots\cup\partial\Delta_n$, or is a simple non-closed curve which joins two points of $\partial D^2 \cup \partial \Delta_1 \cup \ldots \cup \partial \Delta_n$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting the arc transversely.

A picture $\mathbb{P}$ is called to be *connected* if $\cup\{\Delta_1,\ldots,\Delta_n\} \cup \cup\{\alpha_1,\ldots,\alpha_n\}$ is connected.

For each disc Δ, the *corners* of Δ are the closures of the connected components of $\partial\Delta \backslash \cup\{\alpha_1,\ldots,\alpha_m\}$, where $\alpha_1,\ldots,\alpha_m$ are arcs of Δ. The *regions* of $\mathbb{P}$ are the closures of connected components of $D^2 \backslash (\cup\{\text{discs}\} \cup \cup\{\text{arcs}\})$. An *inner* region of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet ∂D^2.

We remark that when we refer to the discs of $\mathbb{P}$ we mean the discs $\Delta_1,\ldots,\Delta_n$, but not the ambient disc D^2. We define $\partial\mathbb{P}$ to be ∂D^2.

We say that $\mathbb{P}$ is *spherical* if no arcs meet $\partial\mathbb{P}$. If $\mathbb{P}$ is spherical then we often omit $\partial\mathbb{P}$.

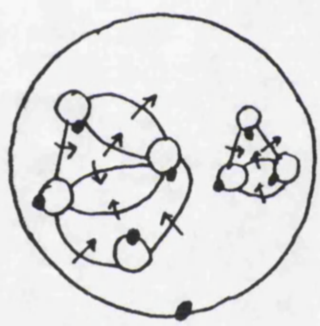


Figure 1.1

Definition A picture $\mathbb{P}$ is *over* $\mathcal{P}$ if the following conditions hold:

(1) Each arc is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$.

(2) If we travel around $\partial\Delta_\lambda$ once in the clockwise direction starting at O_λ and read off the labels on the arcs encounted then we obtain a word which belongs to $\mathbf{r} \cup \mathbf{r}^{-1}$ and we call this word the *label* of Δ_λ.

Example 1.1. Let $\mathcal{P} = \langle x,y,z; x^3, yzy^{-1}z^{-1}, xyx^{-2}y^{-1}, xzx^{-2}z^{-1} \rangle$. Then $\mathbb{P}$ is a spherical picture over $\mathcal{P}$.

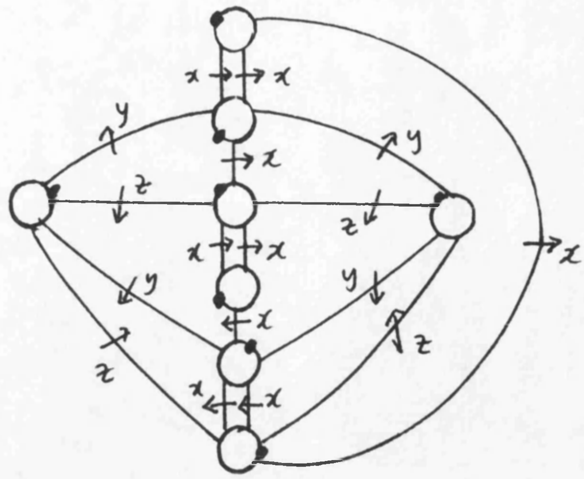
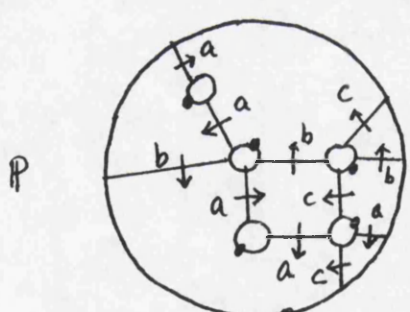


Figure 1.2

Let y and s be subsets of x and r respectively. An arc labelled by an element of $y \cup y^{-1}$ is called a y-*arc* and a disc labelled by an element of $s \cup s^{-1}$ is called an s-*disc*.

The *label* on $\mathbb{P}$ (denoted $W(\mathbb{P})$) is the word read off by travelling around ∂D^2 once in the clockwise starting at O.

Example 1.2. Let $\mathscr{P} = < a,b,c; a^2,(ab)^2,[b,c],[a,c] >$.



Then $W(\mathbb{P}) = b^{-1}ac^{-1}b^{-1}ac$.

Figure 1.3

A *circle* in a picture over $\mathscr{P}$ consists of a collection of distinct arcs $\alpha_1,\ldots,\alpha_n$ and distinct discs $\Delta_1,\ldots,\Delta_n$ such that α_i joins Δ_i to Δ_{i+1} ($i=1,\ldots,n$ subscripts mod n), and $\alpha_1,\ldots,\alpha_n$ all have the same label and the same orientation. We call a circle C *minimal* if there is no circle contained in the region enclosed by C. A disc Δ in the region enclosed by C will be said to be *adjacent* to C if Δ is joined to one of the discs $\Delta_1,\ldots,\Delta_n$ by an arc.

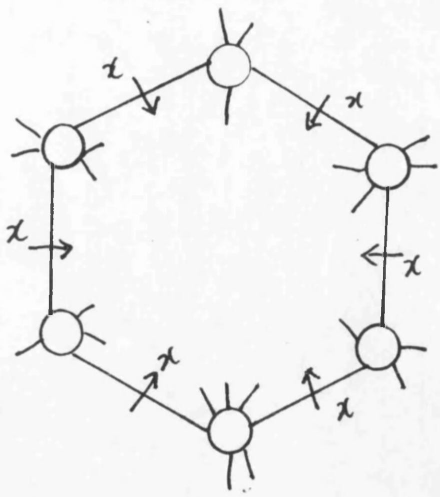


Figure 1.4

A (*transverse*) *path* in $\mathbb{P}$ is a path in the closure of $D^2 \setminus \bigcup_{\lambda=1}^{n} \Delta_\lambda$ which intersects the arcs of $\mathbb{P}$ only finitely many times.

If we travel along a path γ from its initial point to its terminal point we will cross various arcs, and we can read off the labels on these arcs, giving a word $W(\gamma)$, the *label on* γ.

A *spray* for $\mathbb{P}$ is a sequence $\underline{\gamma} = (\gamma_1,\ldots,\gamma_n)$ of simple paths satisfying the following; for $\lambda=1,\ldots,n$, γ_λ starts at O and ends at the basepoint $O_{\theta(\lambda)}$ of $\Delta_{\theta(\lambda)}$, where θ is a permutation of $\{1,\ldots,n\}$ (depending on $\underline{\gamma}$); for $1 \le \lambda < \mu \le n$, γ_λ and γ_μ intersect only at O; travelling around O clockwise in $\mathbb{P}$ we encounter the paths in the order $\gamma_1,\ldots,\gamma_n$. The *sequence* $\sigma(\underline{\gamma})$ *associated with* $\underline{\gamma}$ is

$$(W(\gamma_1)W(\Delta_{\theta(1)})W(\gamma_1)^{-1},\ldots,\ W(\gamma_n)W(\Delta_{\theta(n)})W(\gamma_n)^{-1}).$$

A picture will be said to *represent* a sequence σ if there is a spray for the picture whose associated sequence is σ.

Example 1.3. Let $\sigma=(b^{-1}a^2b,\ b^{-1}a^{-1}(ab)^{-2}ab,\ a^2, c^{-1}a^{-1}[a,c]ac, c^{-1}a^{-1}[b,c]^{-1}ac)$. Then the following picture represents σ.

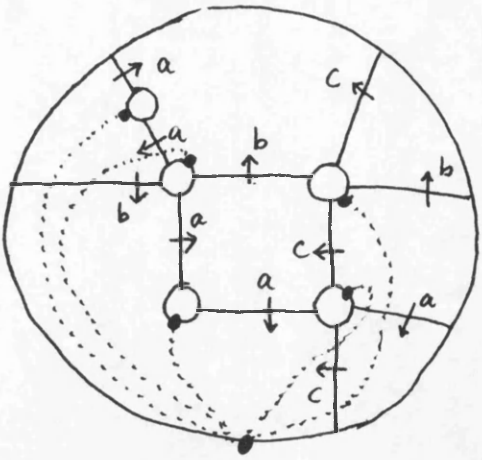


Figure 1.5

Remark 1.1. (1) If $\mathbb{P}$ represents σ then the mirror image $-\mathbb{P}$ of $\mathbb{P}$ represents σ^{-1}.

(2) If $\mathbb{P}_1, \mathbb{P}_2$ represent σ_1, σ_2 respectively then $\mathbb{P}_1 + \mathbb{P}_2$ represents $\sigma_1\sigma_2$, where $\mathbb{P}_1 + \mathbb{P}_2$ is the picture like Fig.1.6.

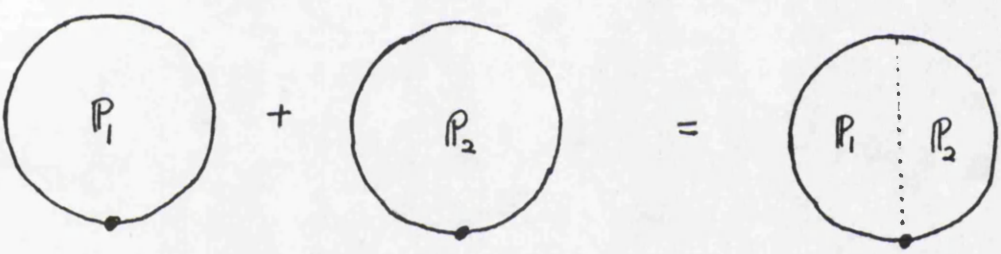


Figure 1.6

Theorem 1.2.1.[35,Theorem 2.1(ii)] *Every identity sequence can be represented by a spherical picture.*

Now we introduce the basic operations on pictures.

(A) Deletion of a closed arc which encircles no discs or arcs of $\mathbb{P}$ (such a closed arc is called a *floating circle*).

$(A)^{-1}$ Insertion of a floating circle.

A *cancelling pair* is a spherical picture with exactly two discs, and when their basepoints lie in the same region.

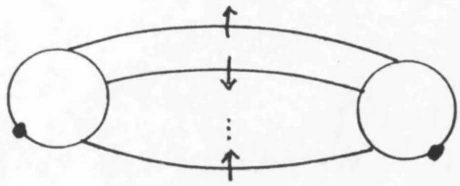


Figure 1.7

(B) If there is a simple closed path β in $\mathbb{P}$ such that the part of $\mathbb{P}$ encircled by β is a cancelling pair, then remove that part of $\mathbb{P}$ encircled by β.

$(B)^{-1}$ The opposite of (B).

(C) Bridge move.

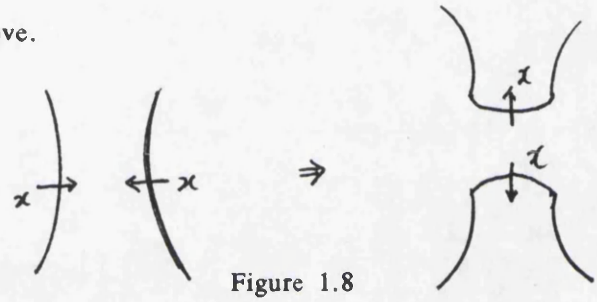


Figure 1.8

Remark 1.2. Since we allow only one basepoint on each disc, when a relator is a proper power, we need more caution. That is to say,

Figure 1.9

is a cancelling pair, whereas

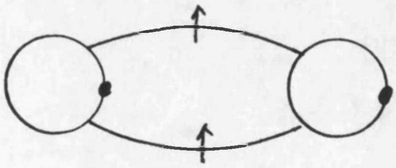


Figure 1.10

is not. So we will only insert basepoints for discs whose labels are proper powers.

Two spherical pictures will be said to be *equivalent* if one can be transformed to the other by a finite number of operations $(A)^{\pm 1}$, $(B)^{\pm 1}$, (C).

We let $< \mathbb{P} >$ denote the equivalence class containing $\mathbb{P}$.

The set $\sum$ of all equivalence classes of all spherical pictures over $\mathcal{P}$ forms a group under the following binary operation

$$< \mathbb{P}_1 > + < \mathbb{P}_2 > = < \mathbb{P}_1 + \mathbb{P}_2 >$$

where the inverse of $< \mathbb{P} >$ is $< -\mathbb{P} >$ and the identity is the equivalence class containing the empty picture.

Let $\mathbb{P}^W$ be the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label W like Fig.1.11.

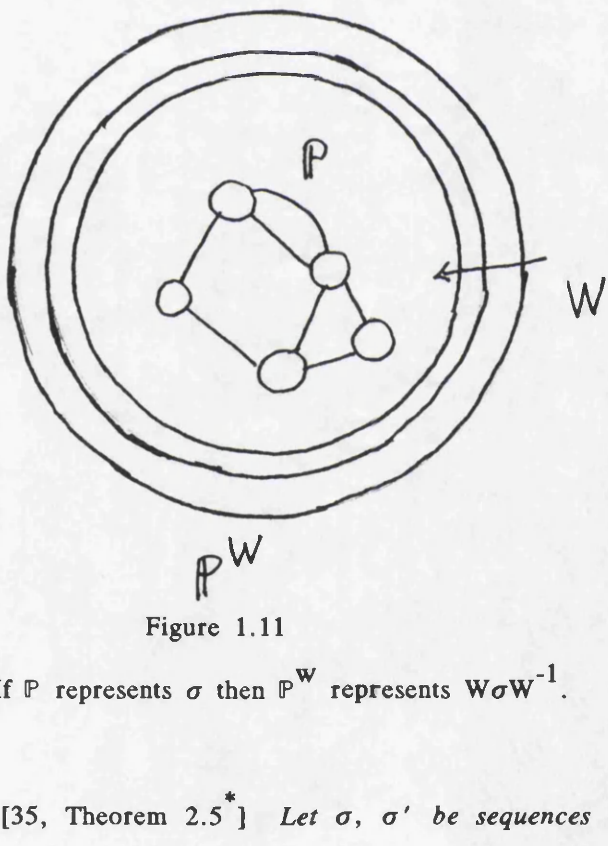


Figure 1.11

Remark 1.3. If $\mathbb{P}$ represents σ then $\mathbb{P}^W$ represents $W\sigma W^{-1}$.

Theorem 1.2.2.[35, Theorem 2.5*] *Let σ, σ' be sequences represented by $\mathbb{P}$, $\mathbb{P}'$ respectively. Then σ and σ' are equivalent if and only if $\mathbb{P}$ and $\mathbb{P}'$ are equivalent.*

Remark 1.4. The group $\sum$ is abelian under the operation $+$. Consider the following Fig.1.12. Then the sequences $\sigma(\underline{\gamma})$, $\sigma(\underline{\gamma}')$ are equivalent since they are sprays for $\mathbb{P}_1+\mathbb{P}_2$ [35,Theorem 2.4*]. And since $\sigma(\underline{\gamma}')=\sigma(\underline{\gamma}'')$, by Theorem 1.2.2 $\mathbb{P}_1+\mathbb{P}_2$ and $\mathbb{P}_2+\mathbb{P}_1$ are equivalent.

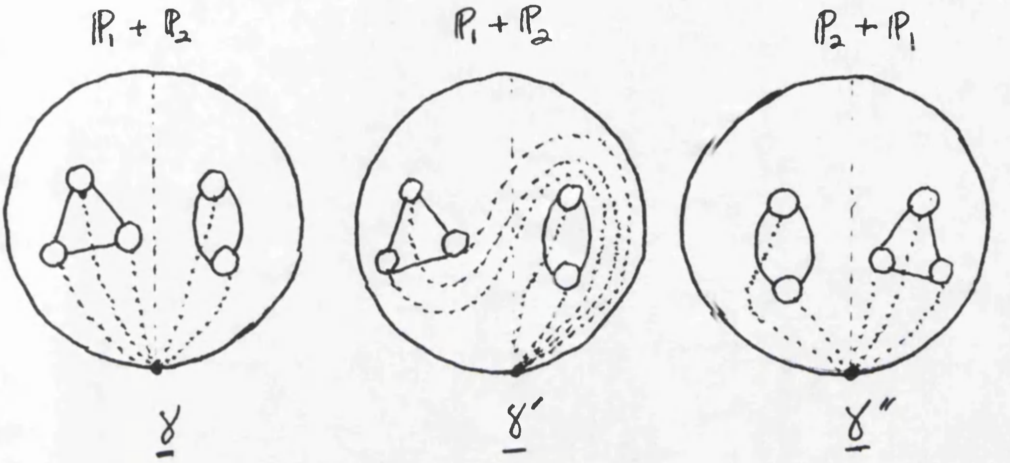


Figure 1.12

We can consider $\sum$ as a left $\mathbb{Z}G$-module by the G-actiom given by

$$WN. < \mathbb{P} > \; = \; < \mathbb{P}^W > \qquad (W \in F).$$

Now we define a map

$$\psi\colon \pi_2(\mathcal{P}) \longrightarrow \textstyle\sum, \qquad < \sigma > \longmapsto < \mathbb{P} >$$

where $\mathbb{P}$ is a spherical picture representing σ. By Theorem 1.2.2, ψ is well-defined and injective. By Theorem 1.2.1, ψ is surjective. And by the above Remarks 1.1 and 1.3, ψ is a module homomorphism.

From now, we will identify $\pi_2(\mathcal{P})$ with $\sum$.

Consider a collection X of spherical pictures over $\mathcal{P}$. We introduce two further operations on $\pi_2(\mathcal{P})$ as follows.

(D) If there is a simple closed path β in a picture such that the part of the picture enclosed by β is a copy of $\mathbb{P}$ or $-\mathbb{P}$ ($\mathbb{P} \in X$), then delete that part of the picture enclosed by β.

$(D)^{-1}$ The opposite of (D).

Two spherical picture will be said to be *equivalemt* (*rel*X) if one can be transformed to the other by a finite number of operations $(A)^{\pm 1}$, $(B)^{\pm 1}$, (C), $(D)^{\pm 1}$.

Theorem 1.2.3.[35, Theorem 2.6 Corollary 1] *The elements* $<\mathbb{P}>$ $(\mathbb{P}\in X)$ *generate* $\pi_2(\mathcal{P})$ *if and only if every spherical picture is equivalent* (*rel*X) *to the empty picture.*

Remark 1.5. Theorem 2.6 in [35] actually refers to the situation where several basepoints are allowed. But it is easily modified to our situation.

If the elements $<\mathbb{P}>$ $(\mathbb{P}\in X)$ generate $\pi_2(\mathcal{P})$ then we will say that X *generates* $\pi_2(\mathcal{P})$.

For technical reasons it is convenient to work with the *symmetrized closure* of a collection A of (connected) pictures. Think of the connected spherical picture $\mathbb{P}$ as being drawn not in the plane, but on the surface of a sphere. Then there are different representations of $\mathbb{P}$ in the plane according to which region we choose to contain the point from which we perform stereographic projection. There are also the mirror images of these pictures. We call this collection of pictures the *symmetrized closure* of $\mathbb{P}$, and we define the *symmetrized closure* A^* of A to be the union of the symmetrized closures of all the $\mathbb{P}\in A$.

Example 1.4.

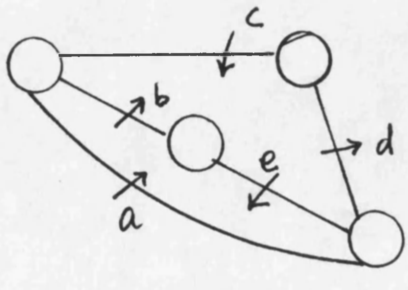


Figure 1.13

Then the symmetrized closure of $\mathbb{P}$ consists of $\mathbb{P}, -\mathbb{P}$ together with the following pictures and their mirror images.

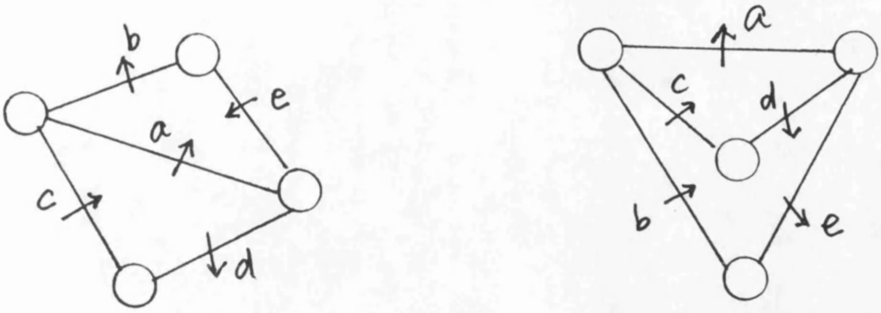


Figure 1.14

Proposition 1.2.4. *The relations "equivalence (relA)" and "equivalence rel(A^*)" are the same.*

Proof. We think of a picture $\mathbb{A}^* \in A^*$ at the situation before performing stereographic projection. Thus some inner region contains its basepoint O. Consider a transverse path γ from a point on $\partial\mathbb{A}^*$ to O with label $x_1x_2 \ldots x_n$. We insert a collection of conscentric circles with total label $x_1x_2 \ldots x_n$ left side of $\mathbb{A}^*$. Performing bridge moves we get a picture $\mathbb{A}_1^*$ like Fig.1.15.

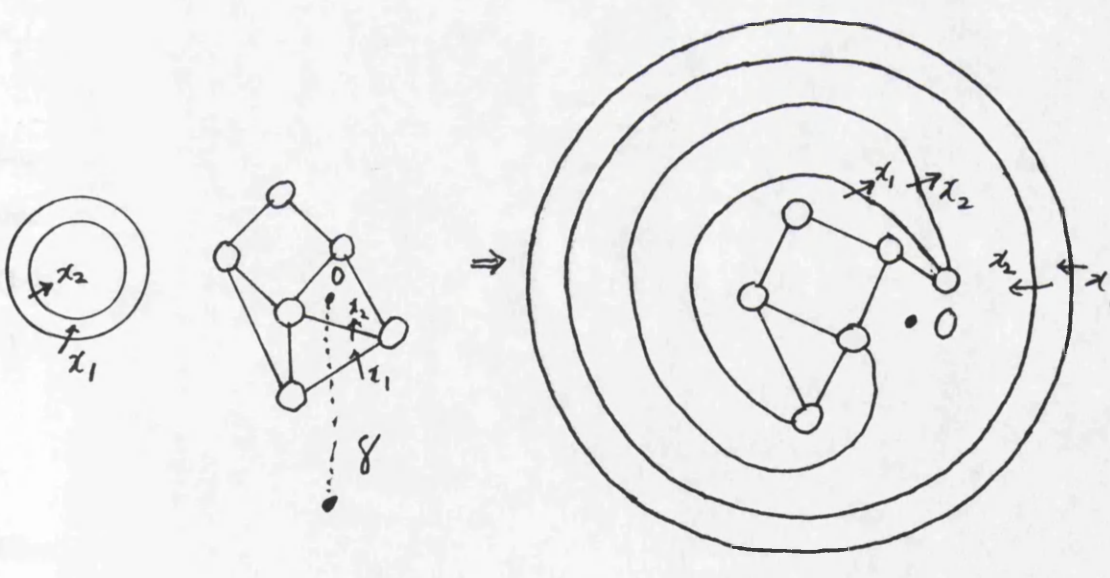


Figure 1.15

The subpicture of $\mathbb{A}_1^*$ which is the part inside the consecentric circles is a picture contained in A. Therefore $\mathbb{A}_1^*$ is equivalent (relA) to the empty picture. The reverse is trivial.

1.3 Some exact sequences concerning $\pi_2(\mathcal{P})$

Let $\mathcal{P} = < \mathbf{x} ; \mathbf{r} >$ and X a generating set for $\pi_2(\mathcal{P})$. Let G be the group defined by $\mathcal{P}$, that is, G=F/N, where F is the free group on $\mathbf{x}$ and N is the normal closure of $\mathbf{r}$ in F.

The *relation module* $M(\mathcal{P})$ of $\mathcal{P}$ is the abelianization N/N′ of N regarded as a left $\mathbb{Z}G$-module, with G-action given by

$$WN.UN' = WUW^{-1}N' \qquad (W \in F,\ U \in N).$$

Let $P_3 = \bigoplus_{\mathbb{P} \in X} \mathbb{Z}Gt_{\mathbb{P}}$, $P_2 = \bigoplus_{R \in \mathbf{r}} \mathbb{Z}Gt_R$, $P_1 = \bigoplus_{x \in \mathbf{x}} \mathbb{Z}Gt_x$, $P_0 = \mathbb{Z}G$. Then we have the following exact sequences [35].

(1-1) $$0 \longrightarrow \pi_2(\mathcal{P}) \xrightarrow{\mu_2} P_2 \xrightarrow{\rho_2} M(\mathcal{P}) \longrightarrow 0$$

$$<\mathbb{P}> \longmapsto \sum_{i=1}^{n} \varepsilon_i W_i N t_{R_i} \qquad (\mathbb{P} \in X)$$

$$t_R \longmapsto RN' \qquad (R \in \mathbf{r})$$

where $\mathbb{P}$ represents $\sigma = (W_1 R_1^{\varepsilon_1} W_1^{-1}, \ldots, W_n R_n^{\varepsilon_n} W_n^{-1})$. We often write $\mu_2(\mathbb{P})$ instead of $\mu_2(<\mathbb{P}>)$.

We regard $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with trivial G-action. There is the *augmentation map* ε: $P_0 \longrightarrow \mathbb{Z}$ which takes each element of G to 1. The kernel of this map is called the *augmentation ideal* denoted by IG.

(1-2) $$0 \longrightarrow IG \xrightarrow{\text{incl}} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(1-3) $$0 \longrightarrow M(\mathcal{P}) \xrightarrow{\mu_1} P_1 \xrightarrow{\rho_1} IG \longrightarrow 0$$

$$WN' \longmapsto \sum_{x \in \mathbf{x}} \rho\left(\frac{\partial W}{\partial x}\right) t_x \qquad (W \in N)$$

$$t_x \longmapsto xN - 1 \qquad (x \in \mathbf{x}).$$

(Here $\frac{\partial}{\partial x}$: $\mathbb{Z}F \longrightarrow \mathbb{Z}F$ is Fox derivation[32,§II.3], and ρ: $\mathbb{Z}F \longrightarrow \mathbb{Z}G$ is induced by the natural epimorphism $F \longrightarrow G$.)

We call μ_1, μ_2 the *standard injections* and ρ_1, ρ_2 the *standard surjections.*

If we put the three sequences (1-1),(1-2),(1-3) together we get the exact sequence

(1-4) $$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$\partial_3 : t_{\mathbb{P}} \longmapsto \mu_2(\mathbb{P})$$
$$\partial_2 = \mu_1 \rho_2$$
$$\partial_1 = \rho_1 .$$

1.4 Reasons for computing generators of $\pi_2(\mathscr{P})$

In this section we will survey some reasons why it is of interest to compute a set X of generators of $\pi_2(\mathscr{P})$.

For any picture $\mathbb{P}$ over $\mathscr{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $\exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled R minus the number of discs labelled R^{-1}.

Example 1.5. Let $\mathscr{P} = \langle x,y \ ; \ S, \ R \rangle$, where $S = x^3$, $R = [x,y]$.

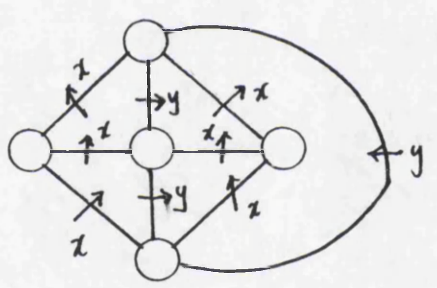


Figure 1.16

Then $\exp_S(\mathbb{P}) = 0$, $\exp_R(\mathbb{P}) = 3$.

For any word W on **x** and any $x \in \mathbf{x}$, the *exponent sum* of x in W, denoted by $\exp_x(W)$ is the number of occurences of x in W minus the number of occurences of x^{-1}.

(a) Identities among relators

As we described in §1.2, $\pi_2(\mathscr{P})$ consists of the relations among relators. Computing generators of $\pi_2(\mathscr{P})$ thus amounts to determining a collection of identities among the relators of $\mathcal{P}$ from which all other identities are derivable.

(b) Relation modules and higher (co)homology

The short exact sequence (1-1) gives us a presentation

$$< t_R \ (R \in r); \ \mu_2(\mathbb{P})=0 \ (\mathbb{P} \in X) >$$

for M($\mathcal{P}$) from X. So we can sometimes know the structure of M($\mathcal{P}$).

From (1-2) and (1-3), we get

$$0 \longrightarrow M(\mathcal{P}) \xrightarrow{\mu_1} P_1 \xrightarrow{\rho_1} P_0 \xrightarrow{\varepsilon} 0.$$

Then by dimension shifting we get

$$H^{n+2}(G, -) \cong \mathrm{Ext}^n_G(M(\mathcal{P}), -)$$

$$H_{n+2}(G, -) \cong \mathrm{Tor}^G_n(-, M(\mathcal{P})), \qquad n \geq 1.$$

See [26, p189]. So if we know the structure of M($\mathcal{P}$) then we can compute the the higher (co)homology groups of G.

(c) Computing of $H_2(G)$ (Schur multiplier) and $H^2(G)$

If A, B are any right and left G-modules respectively, then from (1-4) we have

$$H_2(G,A) = \frac{\mathrm{Ker}\, 1 \otimes \partial_2}{\mathrm{Im}\, 1 \otimes \partial_3}$$

$$H^2(G,B) = \frac{\mathrm{Ker}\,\mathrm{Hom}_{\mathbb{Z}G}(\partial_3, 1)}{\mathrm{Im}\,\mathrm{Hom}_{\mathbb{Z}G}(\partial_2, 1)}.$$

In particular, taking A=$\mathbb{Z}$ and B=$\mathbb{Z}$ (with trivial G-action) we have

$$H_2(G) = \mathrm{Ker}\delta_2/\mathrm{Im}\delta_3$$

$$H^2(G) = \mathrm{Ker}\delta_3^*/\mathrm{Im}\delta_2^*$$

where

(1-5) $$\delta_2: \bigoplus_{R \in r} \mathbb{Z}t_R \longrightarrow \bigoplus_{x \in x} \mathbb{Z}t_x, \qquad t_R \longmapsto \sum_{x \in x} \exp_x(R) t_x$$

(1-6) $$\delta_3: \bigoplus_{\mathbb{P} \in X} \mathbb{Z}t_{\mathbb{P}} \longrightarrow \bigoplus_{R \in r} \mathbb{Z}t_R, \qquad t_{\mathbb{P}} \longmapsto \sum_{R \in r} \exp_R(\mathbb{P}) t_R$$

(1-7) $\delta_2^*: \bigoplus_{x\in x} \mathbb{Z}t_x^* \longrightarrow \bigoplus_{R\in r} \mathbb{Z}t_R^*, \quad t_x^* \longmapsto \sum_{R\in r} \exp_x(R)t_R^*$

(1-8) $\delta_3^*: \bigoplus_{R\in r} \mathbb{Z}t_R^* \longrightarrow \bigoplus_{\mathbb{P}\in X} \mathbb{Z}t_{\mathbb{P}}^*, \quad t_R^* \longmapsto \sum_{\mathbb{P}\in X} \exp_R(\mathbb{P})t_{\mathbb{P}}^*.$

So we can compute them easily.

(d) Other (co)homology properties

(1) We also have another exact sequence

$$0 \longrightarrow H_3(G) \longrightarrow \mathbb{Z}\otimes_G \pi_2(\mathcal{P}) \xrightarrow{\delta} \mathrm{Ker}\delta_2 \longrightarrow H_2(G) \longrightarrow 0.$$

See [35, Therem 1.2]. Since $H_3(G) \cong \mathrm{Ker}\delta$, if we know the structure of $\pi_2(\mathcal{P})$ then we can get some knowledge about $H_3(G)$. We also have a similar conclusion about $H^3(G)$.

(2) We consider a finiteness condition of a group. We say that a group K is of *type* FP_n $(0\le n\le \infty)$ if there is a partial projective resolution of the trivial K-module $\mathbb{Z}$

$$Q_n \longrightarrow \dots \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where Q_i $(i=0,1,\dots,n)$ is finitely generated [7,§VIII.4].

Proposition 1.4.1. *Let* K *be the group defined by a finite presentation* $\mathcal{Q}=\langle y;s\rangle$. *Then* K *is of type* FP_3 *if and only if* $\pi_2=\pi_2(\mathcal{Q})$ *is finitely generated.*

Proof. Suppose that π_2 is finitely generated. From (1-4) we get the exact sequence

$$0 \longrightarrow \pi_2 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $P_0=\mathbb{Z}K$, $P_1=\bigoplus_{|y|}\mathbb{Z}K$, $P_2=\bigoplus_{|s|}\mathbb{Z}K$. Let P_3 be a finitely generated free $\mathbb{Z}K$-module mapping onto π_2. Then we have the partial projective (in fact free) resolution

$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Now suppose that K is of type FP_3, so there is a partial projective resolution

$$Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where Q_i $(i=0,1,2,3)$ is finitely generated. Let $A=\mathrm{Im}(Q_3 \to Q_2)$. Then we have the exact sequence

$$0 \longrightarrow A \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

So by [7, Lemma 8.4.4], we have

$$\pi_2 \oplus Q_2 \oplus P_1 \oplus Q_0 \cong A \oplus P_2 \oplus Q_1 \oplus P_0.$$

Since the RHS is finitely generated, so is the LHS. Hence π_2 is finitely generated.

(3) A group K has *cohomological dimension* k (we write $\mathrm{cd}K=k$) if $H^q(K,A)=0$ for all $q>k$ and all left $\mathbb{Z}K$-modules A, but there exists a left $\mathbb{Z}K$-module B such that $H^k(K,B)\neq 0$ [23,p119].

Proposition 1.4.2. (i) $\mathrm{cd}G\leq 2$ *if* $\pi_2(\mathcal{P})=0$.

(ii) $\mathrm{cd}\leq 3$ *if and only if* $\pi_2(\mathcal{P})$ *is projective.*

Proof. By (1-4) and [23,Proposition 8.1].

(e) Efficiency, minimality and Cockroft property

We can regard a finite presentation $\mathcal{P} = \langle x;r \rangle$ as a 2-CW complex with one vertex.

Figure 1.17

And the Euler characteristic $\chi(\mathcal{P})$, say

$$\chi(\mathcal{P}) = 1 - |\mathbf{x}| + |\mathbf{r}|$$

is bounded below by

$$\upsilon(G) = 1 - \mathrm{rk}(H_1(G)) + d(H_2(G))$$

where G is the group defined by $\mathcal{P}$, rk( ) means the rank of the torsion-free part and d( ) means the least number of generators. See [4].

Definitions Consider the collection **G** of all finite presentations which define a group G.

(a) $\mathcal{P}_0 \in \mathbf{G}$ is called *minimal* if $\chi(\mathcal{P}_0) \leq \chi(\mathcal{P})$ for all $\mathcal{P} \in \mathbf{G}$.

(b) $\mathcal{P}_0 \in \mathbf{G}$ is called *efficient* if $\chi(\mathcal{P}_0) = \upsilon(G)$.

(c) G is called *efficient* if there is an efficient presentation for G.

(d) A spherical picture $\mathbb{P}$ is called *Cockroft* if for all $R \in \mathbf{r}$, $\exp_R(\mathbb{P}) = 0$.

(e) $\mathcal{P}$ is called *Cockroft* if all $\mathbb{P} \in \pi_2(\mathcal{P})$ are Cockroft.

(f) $\mathcal{P}$ is called *Cockroft (mod p)* where $p > 1$ is an integer if for each $\mathbb{P} \in \pi_2(\mathcal{P})$ and for all $R \in \mathbf{r}$, $\exp_R(\mathbb{P}) \equiv 0 \pmod p$.

Remark 1.7. (i) Classes of efficient groups are given in [4],[8].

(ii) Examples of non-efficient groups were given by Swan [42], and their minimal presentations were given by Wamsley [44].

(iii) We will introduce new examples of non-efficient groups in §3.3.2.

Proposition 1.4.3.(Brandenberg and Dyer [6]) *If $\mathcal{P}$ is Cockroft then $\mathcal{P}$ is efficient.*

Proof. If $\mathcal{P}$ is Cockroft then $H_2(G) = H_2\mathcal{P}$. Since

$$\begin{aligned}\chi(\mathcal{P}) &= \mathrm{rk}(H_0\mathcal{P}) - \mathrm{rk}(H_1\mathcal{P}) + \mathrm{rk}(H_2\mathcal{P}) \\ &= 1 - \mathrm{rk}G/G' + \mathrm{rk}(H_2\mathcal{P}),\end{aligned}$$

we get the result.

Let $I_2(\mathcal{P})$ be the 2-sided ideal in $\mathbb{Z}G$ generated by the non-zero coefficients of elements of $\mathrm{Im}\mu_2$. This ideal is called the *second Fox ideal.*

Theorem 1.4.4. (Lustig [30]) *If there is a ring homomorphism*

$$\rho \colon \mathbb{Z}G \longrightarrow M_k(L)$$

with $\rho(1)=1$ *into the* $(k\times k)$*-matrix ring* $(k\geq 1)$ *over a commutative ring* L *with* 1 *such that* $I_2(\mathcal{P})$ *is contained in* $\mathrm{Ker}\rho$*, then* $\mathcal{P}$ *is minimal.*

In §3.3.2, we will use this Theorem to get non-efficient but minimal presentations.

Corollary 1.4.5. *If* $\mathcal{P}$ *is Cockroft (mod* p*) then* $\mathcal{P}$ *is minimal.*

Proof. We take $L=\mathbb{Z}_p$, $k=1$ and ρ the composition

$$\mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow \mathbb{Z}_p.$$

If $\sum_{i=1}^{n} \varepsilon_i g_i$ is any generator of $I_2(\mathcal{P})$ then $\sum_{i=1}^{n} \varepsilon_i$ is a multiple of p. So $I_2(\mathcal{P})$ is contained in $\mathrm{Ker}\rho$. Thus $\mathcal{P}$ is minimal by Theorem 1.4.4.

1.5 Graphs.

A *graph* Γ consists of two disjoint sets $\mathbf{v}=\mathbf{v}(\Gamma)$ (vertices) and $\mathbf{e}=\mathbf{e}(\Gamma)$ (edges) and three functions

$$\iota\colon \mathbf{e} \longrightarrow \mathbf{v},\quad \tau\colon \mathbf{e} \longrightarrow \mathbf{v},\quad {}^{-1}\colon \mathbf{e} \longrightarrow \mathbf{e}$$

satisfying: $\iota(e)=\tau(e^{-1})$, $(e^{-1})^{-1}=e$, $e^{-1}\neq e$ for all $e\in\mathbf{e}$. We call $\iota(e)$ and $\tau(e)$ the *initial* and the *terminal* point of $e\in\mathbf{e}$ respectively. And an orientation $\mathbf{e}^{+}$ of Γ consists of a choice of exactly one edge from each *edge pair* e, e^{-1} ($e\in\mathbf{e}$). We will refer to the pair $(\mathbf{v},\mathbf{e}^{+})$ with the functions ι, τ as an *oriented* graph with oriented edge set $\mathbf{e}^{+}$. We call $e\in\mathbf{e}$ a *loop* if $\iota(e)=\tau(e)$.

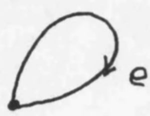


Figure 1.18

A graph Γ is *simple* if whenever $\iota(e_1)=\iota(e_2)$ and $\tau(e_1)=\tau(e_2)$, then $e_1=e_2$.

We call Γ *finite* if e and v both are finite. A simple graph Γ is called *complete* if for any two distinct vertices u and v, there is an edge e with $\iota(e)=u$, $\tau(e)=v$.

A *path* is a finite sequence $e_1,\ldots,e_n$ of edges such that $\tau(e_i)=\iota(e_{i+1})$ for all $i<n$. A *closed* path is a path $e_1,\ldots,e_n$ with $\iota(e_1)=\tau(e_n)$. A path is *simple* if $\iota(e_i)$ $(i=1,\ldots,n)$ are all distinct. We write $e_1e_2\ldots e_n$ instead of $e_1,\ldots,e_n$.

A *subgraph* Γ' of Γ consists of subsets v′ of v and e′ of e such that for any $e\in e'$, $e^{-1}\in e'$ and $\iota(e)\in v'$.

We say that Γ is *connected* if given any two vertices of Γ there is a path joining them. And we call a connected subgraph of Γ a *component* of Γ if no connected subgraph of Γ properly contains it.

A *forest* is a graph in which there are no non-empty simple closed paths, and a *tree* is a connected forest. A *maximal forest* T in a graph is a forest with the property that no subgraph of Γ which properly contains T is a forest.

Proposition 1.5.1.[13] *Any graph has a maximal forest.*

1.6 Detailed description of the thesis.

In Chapter 2, we study the question of how the second homotopy modules and the relation modules of group presentations transform by Tietze transformations.

Suppose that two presentations

$$\mathcal{P}_1 = < a_1,\ldots, a_n ; R_1,\ldots, R_m >$$

$$\mathcal{P}_2 = < b_1,\ldots, b_k ; S_1,\ldots, S_p >$$

define the same group G. Then we show that

$$\pi_2(\mathcal{P}_1) \oplus (\underset{p+n}{\oplus} \mathbb{Z}G) \cong \pi_2(\mathcal{P}_2) \oplus (\underset{m+k}{\oplus} \mathbb{Z}G)$$

in Theorem 2.4 and

$$M(\mathcal{P}_1) \oplus (\underset{k}{\oplus} \mathbb{Z}G) \cong M(\mathcal{P}_2) \oplus (\underset{n}{\oplus} \mathbb{Z}G)$$

in Corollary 2.6. There are already alternative proofs of Corollary 2.6, for example [31], but our result gives us the rank of the free module explicitly.

In Chapter 3, graph products and fundamental groups of graphs are considered.

We calculate generating sets of π_2's (Theorems 3.1.4, 3.2.1, 3.2.4, 3.2.6). And we describe the second integral (co)homology of the fundamental group of a graph of groups and consider necessary and sufficient conditions for its presentation to be Cockroft (Proposition 3.3.1). We also introduce new class of minimal presentations $< x, y\ ;\ x^n, xyx^{-q}y^{-1} >$ ($1 \leq q < n$, $(q,n)=1$) which are not efficient (Theorem 3.3.3) by using the theorem due to Lustig mentioned above (Theorem 1.4.4). In § 3.3.3, we get a short exact sequence concerning a graph of groups which involves the second homotopy module of a presentation $\mathcal{P}$ of the whole group, the second homotopy modules of presentations $\mathcal{P}_v$ ($v \in \mathbf{v}$) of the vertex groups and the relation modules of presentations of edge groups (Theorem 3.3.4). From this we can derive a short exact sequence (due to Hannerbaur[25]) involving the relation module of $\mathcal{P}$, the relation modules of $\mathcal{P}_v$ ($v \in \mathbf{v}$) and the augmentation ideals of edge groups.

In Chapter 4, split extensions are considered.

We calculate a generating set of π_2 (Theorem 4.1.3). And we describe the second integral (co)homology. In [43], Tahara proved that $H_2(K)$ is a direct summand in $H_2(G)$ and described the complement of $H_2(K)$ in $H_2(G)$ theoretically, where G is a split extension of H by K. But we can describe the complements of $H_2(K)$ and $H^2(K)$ in $H_2(G)$ and $H^2(G)$ more practically (Proposition 4.2.1).

Bogley and Pride [5] introduced the concept of a relative presentation

< H, x ; r >, and introduced the notion of the asphericity of such presentation. They considered the asphericity of relative presentations with one single defining relator of the form $xaxbx^{\varepsilon}c$, where $a,b,c \in H$ and $\varepsilon = \pm 1$.

In §§ 5.1, 5.2.1, we survey the basic concepts, the important theorems for relative presentations and the tests for asphericity.

In § 5.2.2, we give new results about relative presentations with one defining relator of length 4 and 5.

In the final chapter, Chapter 6, we compute the second integral (co)homology of aspherical Coxeter groups and consider the efficiency of Coxeter presentations and Coxeter groups. This work is related to the theme of the rest of the thesis, since it makes use of generators of π_2 of aspherical Coxeter presentations (already computed in [38]).

Howlett [27] described the Schur multiplier of Coxeter groups. Pride and Stöhr [38] introduced the concept of an aspherical Coxeter group and calculated a generating set of π_2 of an aspherical Coxeter presentation. And they also described the Schur multiplier of an aspherical Coxeter group.

We give a third description of the Schur multiplier of an aspherical Coxeter group by using a generating set of π_2 which was calculated by Pride and Stöhr [38], and describe the second integral cohomology of an aspherical Coxeter group by a similar calculation (Corollary 6.1.3 and Theorem 6.1.5). And we prove that a Coxeter presentation is efficient if and only if the graph used to define the Coxeter group has no odd edges (Theorem 6.1.6). We also give a sufficient condition for a Coxeter group defined by a graph with some odd edges to have an efficient presentation (Theorem 6.1.7).

Chapter 2. Tietze transformations

In this chapter, we will study the question of how the second homotopy modules and the relation modules of group presentations transform by Tietze transformations.

Given a presentation $\mathscr{P}_1 = < \mathbf{x} ; \mathbf{r} >$, each Tietze transformation T_1, T_2 transforms it into a presentation $\mathscr{P}_2$ in accordance with the following definition.

(T_1) If each $S \in \mathbf{s}$ is a consequence of $\mathbf{r}$ then let

$$\mathscr{P}_2 = < \mathbf{x} ; \mathbf{r}, \mathbf{s} > .$$

(T_2) If each $y \in \mathbf{y}$ is a symbol not in $\mathbf{x}$ and each U_y $(y \in \mathbf{y})$ is a word on $\mathbf{x}$, then let

$$\mathscr{P}_2 = < \mathbf{x}, \mathbf{y} ; \mathbf{r}, y^{-1}U_y \ (y \in \mathbf{y}) > .$$

We can prove easily that if a presentation is obtained from another presentation by any of T_1, T_1^{-1}, T_2, T_2^{-1} then they define isomorphic groups. In the case T_1 or T_1^{-1} we know that $\mathscr{P}_1$ and $\mathscr{P}_2$ define the same group because the normal closures of $\mathbf{r}$ and $\mathbf{r} \cup \mathbf{s}$ in the free group on $\mathbf{x}$ are the same. In the case T_2 or T_2^{-1} we can define the following homomorphisms θ and ψ. Let G_1 and G_2 be the groups defined by $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively, say $G_1 = F_1/N_1$ and F_2/N_2.

$$\theta: G_1 \longrightarrow G_2, \qquad xN_1 \longmapsto xN_2 \ (x \in \mathbf{x}).$$

$$\psi: G_2 \longrightarrow G_1, \qquad xN_2 \longmapsto xN_1 \ (x \in \mathbf{x})$$

$$yN_2 \longmapsto U_yN_1 \ (y \in \mathbf{y}).$$

Then we know that they are mutually inverse isomorphisms.

Proposition 2.1. *Suppose that* $\mathscr{P}_2 = < \mathbf{x} ; \mathbf{r}, \mathbf{s} >$ *is obtained from* $\mathscr{P}_1 = < \mathbf{x} ; \mathbf{r} >$

by an operation T_1 *where each* $S \in \mathbf{s}$ *is a consequence of* $\mathbf{r}$. *Then*

$$\pi_2(\mathcal{P}_2) \cong \pi_2(\mathcal{P}_1) \oplus \left(\bigoplus_{|\mathbf{s}|} \mathbb{Z}G \right),$$

where G *is the group defined by* $\mathcal{P}_1$.

Proof. Let X be a generating set for $\pi_2(\mathcal{P}_1)$. Since S ($S \in \mathbf{s}$) is a consequence of $\mathbf{r}$, S is freely equal to a product

$$\prod_{i=1}^{n} W_i R_i^{\varepsilon_i} W_i^{-1} \quad (R_i \in \mathbf{r},\ \varepsilon_i = \pm 1,\ W_i \text{ a word on } \mathbf{x},\ i=1,\ldots,n).$$

Then there is a picture $\mathbb{D}_S$ over $\mathcal{P}_1$ which consists of R_i-discs and $\mathbf{x}$-arcs, and $\partial\mathbb{D}_S = S$. Now we can construct a spherical picture $\mathbb{P}_S$ over $\mathcal{P}_2$ of the form depicted in Fig 2.1,

$\mathbb{P}_S$

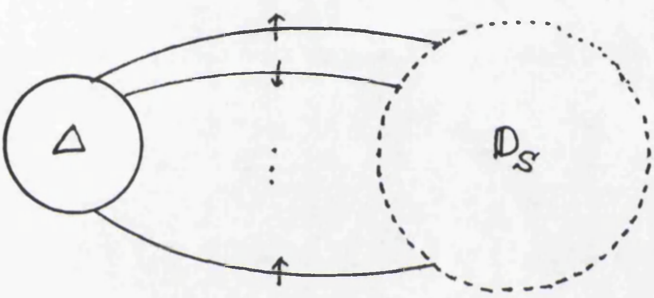


Figure 2.1

where Δ is an S-disc.

Suppose a reduced spherical picture $\mathbb{P}$ over $\mathcal{P}_2$ has some S-discs. We draw a simple closed curve β such that β encloses only one S-disc. Next we insert an element $\bar{\mathbb{P}}_S$ of $\{\mathbb{P}_S\}^*$ inside β. By bridge moves, the S-disc inside β and the S-disc of $\bar{\mathbb{P}}_S$ make a cancelling pair which is removed. The subpicture of $\mathbb{P}$ which is outside β and $\mathbb{D}_S$ of $\bar{\mathbb{P}}_S$ make another spherical picture $\mathbb{P}'$ over $\mathcal{P}_2$ with one fewer S-discs. We can repeat the above argument with $\mathbb{P}'$ in place of $\mathbb{P}$ and so on. We continue the above procedure until we get a picture $\hat{\mathbb{P}}$ without S-discs. Since we can consider $\hat{\mathbb{P}}$ as a spherical picture over $\mathcal{P}_1$, $\hat{\mathbb{P}}$ is equivalent (relX) to the empty picture. Consequently, $\mathbb{P}$ is equivalent (rel $X \cup \{\mathbb{P}_S; S \in \mathbf{s}\}$) to the empty picture. So $\pi_2(\mathcal{P}_2)$ is generated by $X \cup \{\mathbb{P}_S; S \in \mathbf{s}\}$.

Let $<X>$ be the submodule of $\pi_2(\mathcal{P}_2)$ generated by X. Consider the following diagram

$$\begin{array}{ccc} \pi_2(\mathcal{P}_1) & \xrightarrow{\mu_2^{(1)}} & \bigoplus_{R\in r} \mathbb{Z}Gt_R \\ & & \downarrow \iota \\ \pi_2(\mathcal{P}_2) & \xrightarrow{\mu_2^{(2)}} & \bigoplus_{R\in r} \mathbb{Z}Gt_R \oplus (\bigoplus_{S\in s} \mathbb{Z}Gt_S) \end{array}$$

where $\mu_2^{(1)}$, $\mu_2^{(2)}$ are the standard injections and ι is an embedding. Since the image of $<X>$ under $\mu_2^{(2)}$ lies in $\bigoplus_{R\in r} \mathbb{Z}Gt_R$ and the image of $<\mathbb{P}_S>$ under $\mu_2^{(2)}$ has the form $\zeta_S - t_S$ where $\zeta_S \in \bigoplus_{R\in r} \mathbb{Z}Gt_R$, the images of $<X>$ and $\{<\mathbb{P}_S>;S\in s\}$ of $\mu_2^{(2)}$ are mutually disjoint. So $<X>$ and $\{<\mathbb{P}_S>;S\in s\}$ mutually are disjoint in $\pi_2(\mathcal{P}_2)$ because $\mu_2^{(2)}$ is injective. And the equivalence classes $<\mathbb{P}_S>$'s are independent. Thus

$$\begin{aligned} \pi_2(\mathcal{P}_2) &\cong <X> \oplus (\bigoplus_{S\in s} <\mathbb{P}_S>) \\ &\cong \pi_2(\mathcal{P}_1) \oplus (\bigoplus_{|s|} \mathbb{Z}G). \end{aligned}$$

In the case T_2, we will consider a more general situation.

Let $\mathcal{P} = <y;t>$ be a group presentation defining a group $G=F/N$. Let $\mathcal{P}_0=<y_0;t_0>$ be a *full* subpresentation of $\mathcal{P}$ (i.e., y_0 is a subset of y and t_0 consists of all relators involving y_0). Let $G_0=F_0/N_0$ be the group defined by $\mathcal{P}_0$. We say that $\mathcal{P}_0$ is an *injective* subpresentation of $\mathcal{P}$ if the natural map $G_0 \longrightarrow G$ is injective.

Let X_0 be the set of all spherical pictures over $\mathcal{P}_0$. If $\mathbb{P}$ is a spherical picture over $\mathcal{P}_0$ then the element of $\pi_2(\mathcal{P}_0)$ represented by $\mathbb{P}$ will be denoted by $<\mathbb{P}>_0$. Of course, $\mathbb{P}$ also represents an element of $\pi_2(\mathcal{P})$, which will be denoted by $<\mathbb{P}>$.

Theorem 2.2. *If* $\mathcal{P}_0$ *is an injective subpresentation of* $\mathcal{P}$ *then the submodule of* $\pi_2(\mathcal{P})$ *generated by* X_0 *is isomorphic to* $\mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0)$ *under the map*

$$<\mathbb{P}> \longmapsto 1 \otimes <\mathbb{P}>_0 \qquad (\mathbb{P} \in X_0).$$

Proof. From (1-1) in §1.3, we get the standard injections

$$\mu_2 : \pi_2(\mathcal{P}) \longrightarrow (\bigoplus_{T \in t_0} \mathbb{Z}Gt_T) \oplus (\bigoplus_{S \in t \setminus t_0} \mathbb{Z}Gt_S)$$

$$\mu_2^0 : \pi_2(\mathcal{P}_0) \longrightarrow \bigoplus_{T \in t_0} \mathbb{Z}G_0\bar{t}_T.$$

If we apply $\mathbb{Z}G \otimes_{G_0} -$, then we get an embedding

$$1 \otimes \mu_2^0 : \mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0) \longrightarrow \bigoplus_{T \in t_0} \mathbb{Z}Gt_T$$

where t_T is identified with $1 \otimes \bar{t}_T$.

Let $<X_0>$ be the submodule of $\pi_2(\mathcal{P})$ generated by X_0. Then

$$1 \otimes \mu_2^0(\mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0)) = \mu_2(<X_0>).$$

Since $1 \otimes \mu_2^0$ and μ_2 are injective, we get

$$<X_0> \cong \mathbb{Z}G \otimes_{G_0} \pi_2(\mathcal{P}_0)$$

where the isomorphism is the composition of μ_2 and $(1 \otimes \mu_2^0)^{-1}$.

Corollary 2.3. *Suppose that* $\mathcal{P}_2 = <\mathbf{x}, \mathbf{y}\ ;\ \mathbf{r}, \{y^{-1}U_y ; y \in \mathbf{y}\}>$ *is obtained from* $\mathcal{P}_1 = <\mathbf{x};\mathbf{r}>$ *by an operation* T_2, *where each* y ($y \in \mathbf{y}$) *is a symbol not in* $\mathbf{x}$ *and each* U_y ($y \in \mathbf{y}$) *is a word on* $\mathbf{x}$. *Then*

$$\pi_2(\mathcal{P}_2) \cong \pi_2(\mathcal{P}_1).$$

Proof. Since every reduced spherical picture over $\mathcal{P}_2$ has no $y^{-1}U_y$-discs ($y \in \mathbf{y}$), $<X_1> = \pi_2(\mathcal{P}_2)$ where X_1 is the set of all spherical pictures over $\mathcal{P}_1$. By Theorem 2.2, $\pi_2(\mathcal{P}_2) \cong \mathbb{Z}G_2 \otimes_{G_1} \pi_2(\mathcal{P}_1)$. But $\mathbb{Z}G_2 \otimes_{G_1} \pi_2(\mathcal{P}_1) \cong \pi_2(\mathcal{P}_1)$ because $G_1 \cong G_2$. So we get the result.

Theorem 2.4. *Let a group* G *be defined by the following two finite presentations*

$$\mathcal{P}_1 = \langle a_1,\ldots,a_n; R_1,\ldots,R_m \rangle$$

$$\mathcal{P}_2 = \langle b_1,\ldots,b_k; S_1,\ldots,S_p \rangle$$

where $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are disjoint. Then*

$$\pi_2(\mathcal{P}_1) \oplus (\oplus_{p+n} \mathbb{Z}G) \cong \pi_2(\mathcal{P}_2) \oplus (\oplus_{m+k} \mathbb{Z}G).$$

Proof. The first part of our proof is taken from [33, Theorem 1.5]. Let $\mathbf{a}=\{a_1,\ldots,a_n\}$, $\mathbf{r}=\{R_1,\ldots,R_m\}$, $\mathbf{b}=\{b_1,\ldots,b_k\}$, $\mathbf{s}=\{S_1,\ldots,S_p\}$. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are presentations under the functions $a_i \longmapsto g_i (i=1,..,n)$ and $b_j \mapsto h_j (j=1,\ldots,k)$ respectively. Since $h_j \in G$, we can express h_j in terms of $g_1,\ldots,g_n$. So we get

$$h_1 = B_1(g_i),\ldots, h_k = B_k(g_i).$$

By applying T_2, we get the presentation

$$\mathcal{P}_3 = \langle \mathbf{a}, \mathbf{b} ; \mathbf{r}, b_1=B_1(a_i),\ldots, b_k=B_k(a_i) \rangle.$$

We note that each S_r $(r=1,\ldots,p)$ is a consequence of relators of $\mathcal{P}_3$. Thus by applying T_1 we get

$$\mathcal{P}_4 = \langle \mathbf{a}, \mathbf{b} ; \mathbf{r}, \mathbf{s}, b_1=B_1(a_i),\ldots,b_k=B_k(a_i) \rangle.$$

Expressing $g_1,\ldots,g_n$ in terms of $h_1,\ldots,h_k$, we get $g_1=A_1(h_j),\ldots, g_n=A_n(h_j)$. So we get $a_1=A_1(b_j),\ldots,a_n=A_n(b_j)$. By applying T_1, we get the presentation

$$\mathcal{P}^* = \langle \mathbf{a}, \mathbf{b} ; \mathbf{r}, \mathbf{s}, b_1=B_1(a_i),\ldots,b_k=B_k(a_i), a_1=A_1(b_j),\ldots,a_n=A_n(b_j) \rangle.$$

Similarly, we can get $\mathcal{P}^*$ from $\mathcal{P}'$ by applying T_2 and then T_1 twice. Therefore by Proposition 2.1 and Collorary 2.3, we get our result.

Now we consider the relation module case. There are already alternative proofs, for example [31], but our result gives us the rank of the free module explicitly.

Theorem 2.5. (i) *If* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the same as in Proposition* 2.1, *then*

$$M(\mathcal{P}_1) = M(\mathcal{P}_2).$$

(ii) *If* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the same as in Collorary* 2.3, *then*

$$M(\mathcal{P}_2) \cong M(\mathcal{P}_1) \oplus \big(\bigoplus_{|\mathbf{y}|} \mathbb{Z}G_2 \big).$$

Proof. (i) It is clear because the normal closures of $\mathbf{r}$ and $\mathbf{r} \cup \mathbf{s}$ in the free group on $\mathbf{x}$ are the same.

(ii) Consider the following diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(\mathcal{P}_1) & \xrightarrow{\mu_2^{(1)}} & \bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_1\bar{t}_R & \longrightarrow & M(\mathcal{P}_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\phi} & & & & \\
0 & \longrightarrow & \pi_2(\mathcal{P}_2) & \xrightarrow{\mu_2^{(2)}} & \big(\bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_2t_R\big) \oplus \big(\bigoplus_{y\in\mathbf{y}} \mathbb{Z}G_2t_y\big) & \longrightarrow & M(\mathcal{P}_2) & \longrightarrow & 0
\end{array}$$

where ϕ is an embedding given by $\bar{t}_R \longmapsto t_R$ (since $G_1 \cong G_2$) and α is the isomorphism in Corollary 2.3. Then we have

$$M(\mathcal{P}_1) \cong \operatorname{coker} \mu_2^{(1)}$$

$$M(\mathcal{P}_2) \cong \operatorname{coker} \mu_2^{(2)}$$

$$\cong \big(\bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_2t_R/\mathrm{Im}\mu_2^{(2)} \big) \oplus \big(\bigoplus_{y\in\mathbf{y}} \mathbb{Z}G_2t_y \big).$$

Since $\mathrm{Im}\ \mu_2^{(1)} \cong \mathrm{Im}\ \mu_2^{(2)}$ and $G_1 \cong G_2$, we have an induced isomorphism

$$\bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_1\bar{t}_R/\mathrm{Im}\mu_2^{(1)} \longrightarrow \bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_2t_R/\mathrm{Im}\mu_2^{(2)}.$$

So,

$$M(\mathcal{P}_2) \cong \big(\bigoplus_{R\in\mathbf{r}} \mathbb{Z}G_1\bar{t}_R/\mathrm{Im}\mu_2^{(1)} \big) \oplus \big(\bigoplus_{y\in\mathbf{y}} \mathbb{Z}G_2t_y \big)$$

$$\cong M(\mathcal{P}_1) \oplus \big(\bigoplus_{y\in\mathbf{y}} \mathbb{Z}G_2t_y \big).$$

Corollary 2.6. *If* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the same as in Theroem 2.4, then*

$$M(\mathcal{P}_1) \oplus \big(\bigoplus_{k} \mathbb{Z}G \big) \cong M(\mathcal{P}_2) \oplus \big(\bigoplus_{n} \mathbb{Z}G \big).$$

Proof. By Thereom 2.5 and the proof of Theorem 2.4.

Example 2.7. We consider the cyclic group of order 6. The application of a sequence of Tietze transformations and their results are indicated by the following scheme.

$$\mathcal{P} = \langle\, t \,;\, t^6 \,\rangle$$

T_2, $a=t^3$; $\quad \langle\, t, a \,;\, t^6,\ a=t^3 \,\rangle$

T_2, $b=t^2$; $\quad \langle\, t, a, b \,;\, t^6,\ a=t^3,\ b=t^2 \,\rangle$

T_1, a^2 ; $\quad \langle\, t, a, b \,;\, t^6,\ a=t^3,\ b=t^2,\ a^2 \,\rangle$

T_1, b^3 ; $\quad \langle\, t, a, b \,;\, t^6,\ a=t^3,\ b=t^2,\ a^2,\ b^3 \,\rangle$

T_1, $ab=ba$; $\quad \langle\, t, a, b \,;\, t^6,\ a=t^3,\ b=t^2,\ a^2,\ b^3,\ ab=ba \,\rangle$

T_1, $t=ab^{-1}$; $\quad \langle\, t, a, b \,;\, t^6,\ a=t^3,\ b=t^2,\ a^2,\ b^3,\ ab=ba,\ t=ab^{-1} \,\rangle$

Now we start with $\mathcal{P}' = \langle\, a, b \,;\, a^2,\ b^3,\ ab=ba \,\rangle$.

T_2, $t=ab^{-1}$; $\quad \langle\, t, a, b \,;\, a^2,\ b^3,\ ab=ba,\ t=ab^{-1} \,\rangle$

T_1, $b=t^2$; $\langle\, t, a, b \,;\, a^2,\ b^3,\ ab=ba,\ t=ab^{-1},\ b=t^2 \,\rangle$

$$t^2=(ab^{-1})^2=a^2b^{-2}=b^{-2}=b$$

T_1, $a=t^3$; $\langle\, t, a, b \,;\, a^2,\ b^3,\ ab=ba,\ t=ab^{-1},\ b=t^2,\ a=t^3 \,\rangle$

$$t^3=(ab^{-2})^3=a^3b^6=a$$

T_1, t^6 ; $\mathcal{P}^*$

$$t^6=(ab^{-1})^6=a^6b^{-6}=1$$

Thus we get $\pi_2(\mathcal{P}) \oplus (\mathbb{Z}G)^4 \cong \pi_2(\mathcal{P}') \oplus (\mathbb{Z}G)^3$, $M(\mathcal{P}) \oplus (\mathbb{Z}G)^2 \cong M(\mathcal{P}') \oplus \mathbb{Z}G$.

If the result of Theorem 2.4 was cancellative, then we would have

$$\pi_2(\mathcal{P}) \oplus \mathbb{Z}G \cong \pi_2(\mathcal{P}').$$

In particular, $\pi_2(\mathcal{P}')$ would be generated by two elements, because $\pi_2(\mathcal{P})$ is generated by only one picture. By Theorem 3.1.4 we can get a generating set of

$\pi_2(\mathscr{P}')$ which consists of four elements. At present I propose that $\pi_2(\mathscr{P}')$ can not be generated by less than 4 generators. However, I am unable to prove this.

Chapter 3 Calculation of generators of the second homotopy module I: graph products, fundamental groups of graphs of groups.

In this chapter, we study the calculation of generators of the second homotopy modules of graph products and fundamental groups of graphs of groups, and we give some applications.

3.1 Graph products

Let Γ be a simple oriented graph with vertex set $\mathbf{v}$ and oriented edge set $\mathbf{e}$. For each $v\in\mathbf{v}$ let a *vertex group* G_v be the group given by a presentation $\mathcal{P}_v=<\mathbf{x}_v;\mathbf{s}_v>$, where the elements of $\mathbf{s}_v$ are cyclically reduced words on $\mathbf{x}_v$, and for each $e\in\mathbf{e}$ with $\iota(e)=u$ and $\tau(e)=v$, let $\mathbf{r}_e$ consist of some cyclically reduced words on $\mathbf{x}_u \cup \mathbf{x}_v$ each involving at least one $\mathbf{x}_u$-symbol and at least one $\mathbf{x}_v$-symbol. Let $\mathcal{P}_e=<\mathbf{x}_u,\mathbf{x}_v;\mathbf{s}_u,\mathbf{s}_v,\mathbf{r}_e>$ and $\mathbf{x}=\bigcup_{v\in\mathbf{v}}\mathbf{x}_v$, $\mathbf{s}=\bigcup_{v\in\mathbf{v}}\mathbf{s}_v$, $\mathbf{r}=\bigcup_{e\in\mathbf{e}}\mathbf{r}_e$. Let $\mathcal{P}$ be the group presentation $<\mathbf{x};\mathbf{s},\mathbf{r}>$. The group G defined by $\mathcal{P}$ is called a *general graph product of the groups* $G_v(v\in\mathbf{v})$. Such groups have been studied in [34],[36],[37] (they did not use this terminology). Especially Pride [36] considered the following question and gave a partial answer.

Let X_e be a set of generating pictures for $\pi_2(\mathcal{P}_e)$.

When is $\pi_2(\mathcal{P})$ generated by $\bigcup_{e\in\mathbf{e}} X_e$?

Let e be an edge with $\iota(e)=u$ and $\tau(e)=v$ of Γ, we will say $\mathcal{P}_e$ has *property*-W_k if no non-trivial element of $G_u * G_v$ of free product length less than or equal to 2k lies in the kernel of the natural epimorphism

$$G_u * G_v \longrightarrow G_e.$$

Theorem 3.1.1.[36] *$\pi_2(\mathcal{P})$ is generated by $\bigcup_{e\in\mathbf{e}} X_e$ if one of the following* *conditions is satisfied*:

(1) Each $\mathcal{P}_e$ *has property-*W_2

(2) Γ *is triangle-free and each* $\mathcal{P}_e$ *has property-*W_1.

An example of this Theorem was mentioned by Pride [36] but was explicitly given in [38]. We will introduce this example in §6.1.

If each $\mathbf{r}_e$ ($e\in\mathbf{e}$) consists of all words $[a,b]=aba^{-1}b^{-1}$ ($a\in\mathbf{x}_{\iota(e)}$, $b\in\mathbf{x}_{\tau(e)}$) then G is called a *graph product of the groups* G_v ($v\in\mathbf{v}$) [10],[11],[20],[21],[22]. If all G_v ($v\in\mathbf{v}$) are infinite cyclic groups then G is called a *graph group* [14],[15],[16],[17],[40],[41]. A graph product has two extreme cases. If the edge set $\mathbf{e}$ is empty then G is the free product of the groups G_v ($v\in\mathbf{v}$). If Γ is complete then G is the direct product of the groups G_v ($v\in\mathbf{v}$). Therefore a graph product shares many of the interesting properties of these extreme cases [20].

For each triangle {u,v,w} in Γ we have a collection of spherical pictures of the form depicted in Fig.3.1 with $a\in\mathbf{x}_u$, $b\in\mathbf{x}_v$, $c\in\mathbf{x}_w$. Let Z be the union of all these collections over all triangles of Γ.

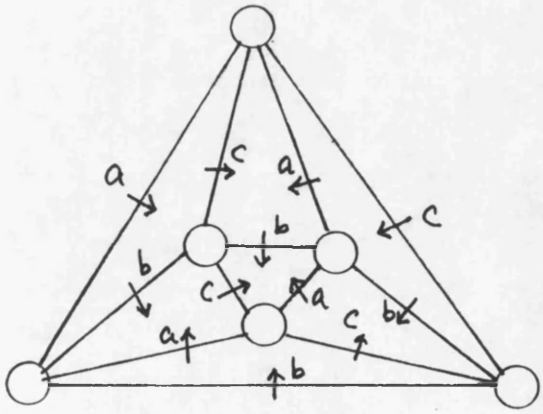


Figure 3.1

For each $e\in\mathbf{e}$, with $\iota(e)=u$, $\tau(e)=v$ say, let $S=x_1x_2...x_n\in\mathbf{s}_u$. Then for each $y\in\mathbf{x}_v$, we have a spherical picture $\mathbb{P}_{S,b}$ over $\mathcal{P}$ of the form depicted in Fig.3.2. Similarly we get $\mathbb{P}_{T,x}$ ($T\in\mathbf{s}_v$, $x\in\mathbf{x}_u$).

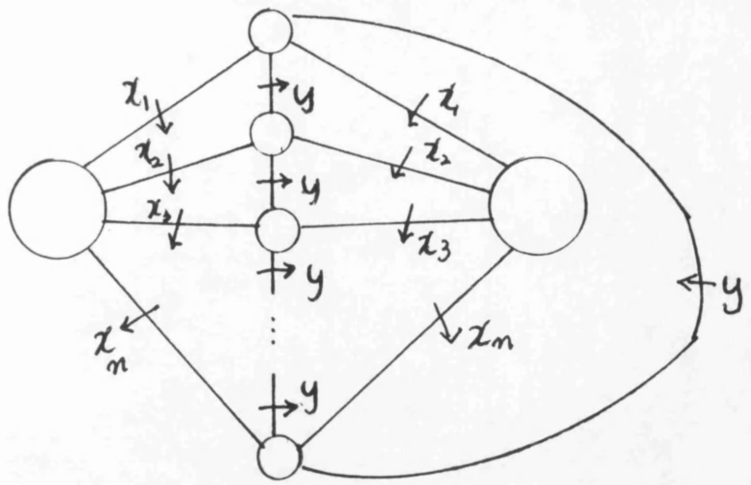


Figure 3.2

Let

$$Y_{e,u} = \{ \mathbb{P}_{S,y};\ S \in s_u,\ y \in x_v \}$$

$$Y_{e,v} = \{ \mathbb{P}_{T,x};\ T \in s_v,\ x \in x_u \}.$$

And let $Y_e = Y_{e,u} \cup Y_{e,v}$ and $Y = \bigcup_{e \in \mathbf{e}} Y_e$.

Let X_v be a collection of spherical pictures over $\mathcal{P}_v$ such that $\pi_2(\mathcal{P}_v)$ is generated by X_v and let $X = \bigcup_{v \in \mathbf{v}} X_v$. Let $\mathcal{P}'$ be the subpresentation $< \mathbf{x} ; \mathbf{r} >$ of $\mathcal{P}$.

Lemma 3.1.2. *Every spherical picture over $\mathcal{P}$ is equivalent (rel($X \cup Y)^*$) to a spherical picture over $\mathcal{P}'$.*

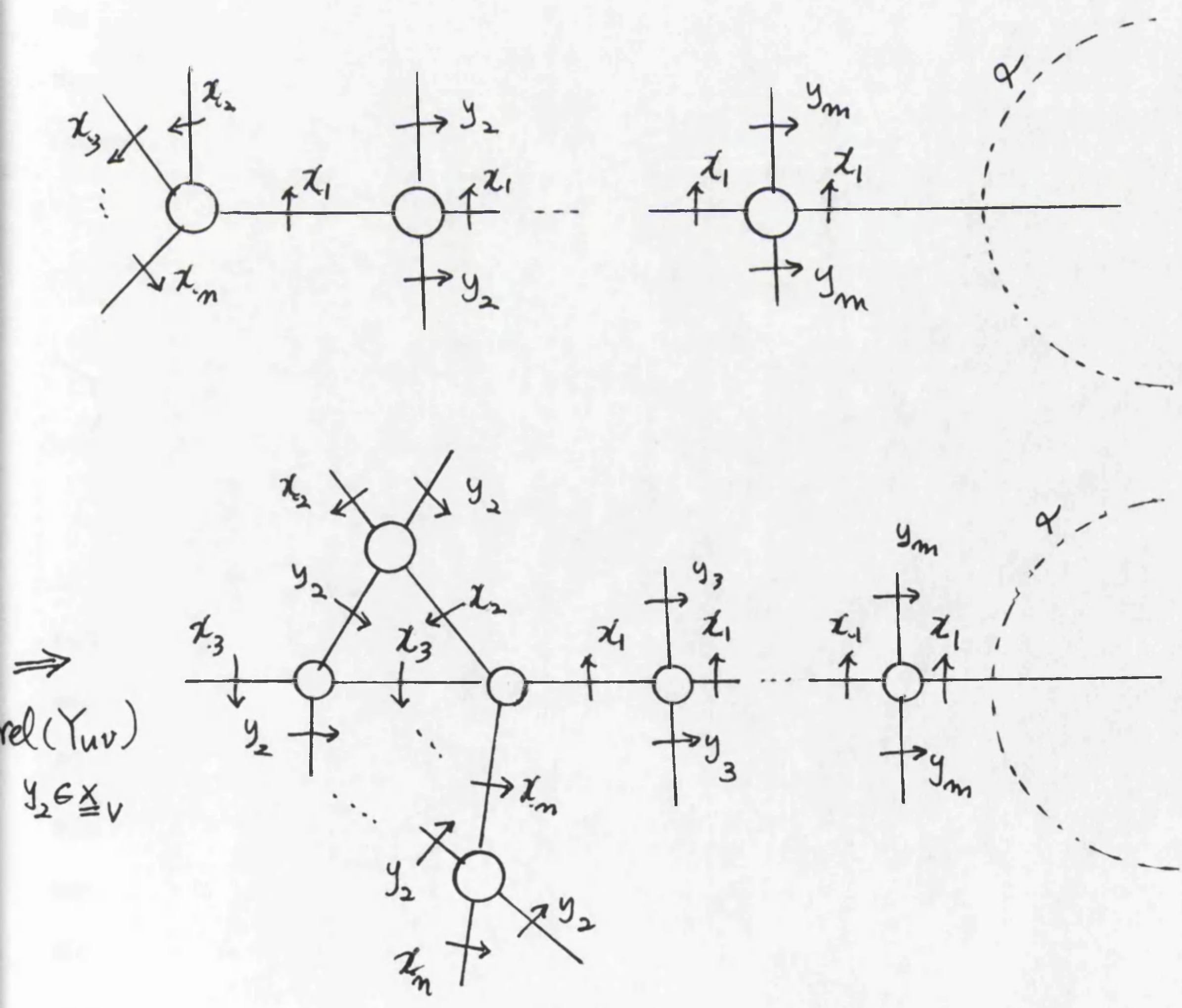
Proof. Let $v \in \mathbf{v}$ and suppose that a given spherical picture $\mathbb{P}$ over $\mathcal{P}$ contains at least one s_v-disc. We claim that $\mathbb{P}$ is equivalent (rel$(X \cup Y)^*$) to a picture $\mathbb{P}'$ with fewer s_v-discs than $\mathbb{P}$, and the same number of s_u-discs for each $u \in \mathbf{v} \setminus \{v\}$. Repeating this process, we eventually obtain a picture equivalent (rel$(X \cup Y)^*$) to $\mathbb{P}$, with no s-discs at all, which establishes our result.

Let α be a simple closed transverse path in $\mathbb{P}$ such that all discs lying inside α are s_v-discs, and all arcs inside α are $\mathbf{x}_v$-arcs. We also require that at least one disc lies inside α. (Such a path α certainly exists: for example the boundary of a small neighbourhood of any s_v-disc.)

Now any arc β meeting α is labelled by an element $a \in \mathbf{x}_v$, and is the beginning of a path of arcs labelled a and non s_v-discs $\Delta_1, \ldots, \Delta_{m-1}$ $(m \geq 1)$ in the exterior of α, ending *either* (i) with an s_v-disc Δ_m in the exterior of α, *or* (ii) with an arc β', labelled a, incident at Δ_{m-1} and also meeting α.

In case (ii) we insist that $\beta' \neq \beta$ if $m \geq 2$. If $m=1$ in case (ii) then $\beta = \beta'$ is an arc crossing α twice.

The first step in our argument is to eliminate possibility (i) above. If this possibility occurs, then we can modify $\mathbb{P}$ and α as in Fig.3.3 to bring Δ_m inside α. This modification does not chage the number of s_u-discs for any $u \in \mathbf{v}$, but increases the number of s_v-discs inside α. After a finite number of such modifications, we may assume that possibility (i) does not occur.



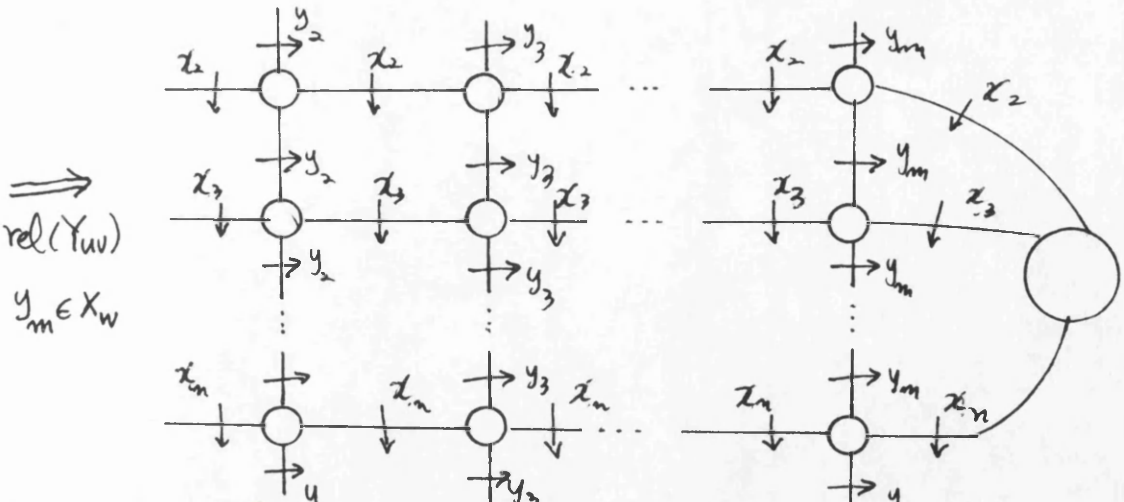


Figure 3.3

Thus every arc β crossing α is the start of a path of non s_v-discs and $\mathbf{x}_v$-arcs (all with the same label as β) in the exterior of α, that eventually recrosses α. Note also that no two such paths can cross in the exterior of α, since they contain no s_v-discs. (See Fig.3.4.) It follows that the label of the transverse path α is a word in $\mathbf{x}_v$ freely equal to 1. A sequence of bridge moves near α creates a spherical picture over $\mathscr{P}_v$ inside α, which can be removed to obtain the desired picture $\mathbb{P}'$.

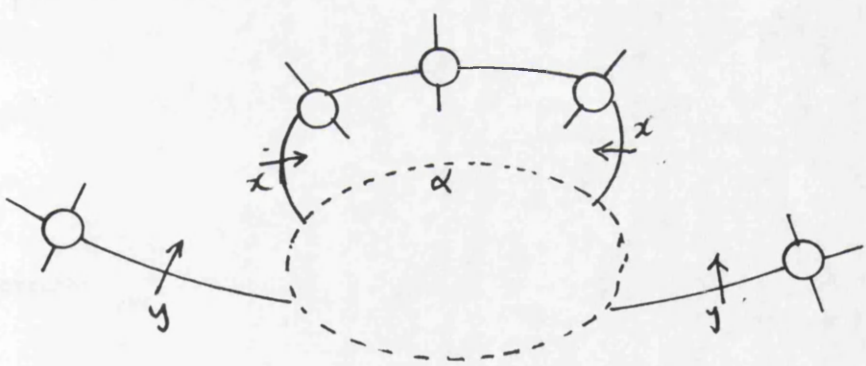


Figure 3.4

Lemma 3.1.3. *Every spherical picture over $\mathscr{P}'$ is equivalent ($relZ^*$) to the empty picture.*

Proof. Let $\mathbb{P}$ be a non-empty spherical picture over $\mathscr{P}'$ and let c($\mathbb{P}$) denote the number of circles in $\mathbb{P}$. (For the definition of circle see p4.) Let C be a minimal circle in $\mathbb{P}$ and let δ(C) denote the number of discs of $\mathbb{P}$ lying inside the region enclosed by C. Our aim is to modify $\mathbb{P}$ to obtain a new picture $\mathbb{P}'$ such that:

(i) C is changed to a new minimal circle C′ in $\mathbb{P}'$,

(ii) $\delta(C') = 0$,

(iii) $\mathbb{P}'$ is equivalent (rel Z^*) to $\mathbb{P}$,

(iv) $c(\mathbb{P}') = c(\mathbb{P})$.

Suppose $\delta(C) > 0$ and let Δ be an adjacent disc to C. Then we can insert a suitable picture from Z^* near Δ, and perform a succession of bridge moves and deletions of cancelling pairs to obtain a picture $\mathbb{P}_1$ with a minimal circle C_1, where $\delta(C_1) < \delta(C)$ and $c(\mathbb{P}_1) = c(\mathbb{P})$. See Fig 3.5.

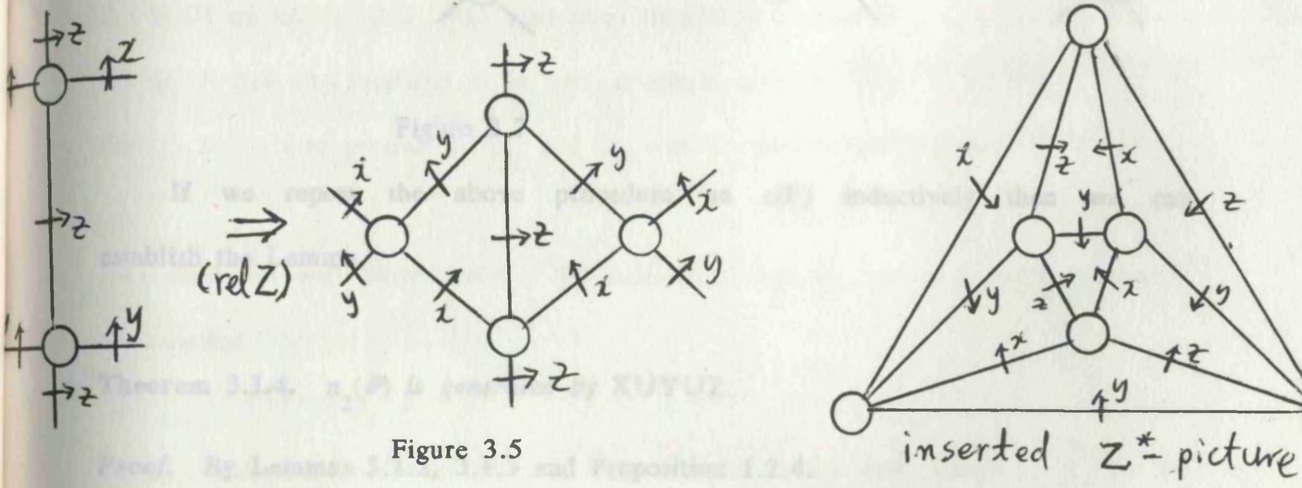


Figure 3.5

We can then repeat the above procedure with $\mathbb{P}_1$ in place of $\mathbb{P}$, and so on, eventually arriving at the required picture $\mathbb{P}'$.

Now in $\mathbb{P}'$ we meet the sitation illustrated in Fig.3.6.

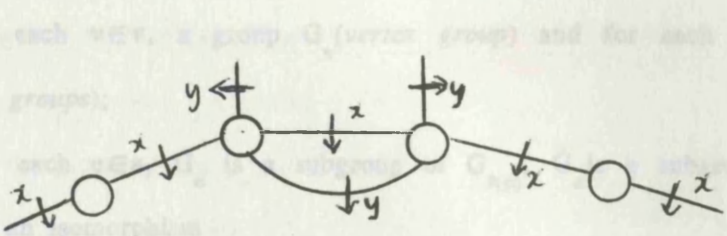


Figure 3.6

Then we do bridge moves in $\mathbb{P}'$ and deletion of a cancelling pair to get a circle C^* as in Fig.3.7. If C^* has no discs then we are finished. Otherwise, we can repeat the above argument with C^* in place of C'. Eventually we can get a floating circle which will be removed. That is to say we get a spherical picture $\mathbb{P}^*$ over $\mathcal{P}'$ which is equivalent (relZ^*) to $\mathbb{P}$ and $c(\mathbb{P}^*) < c(\mathbb{P}') = c(\mathbb{P})$.

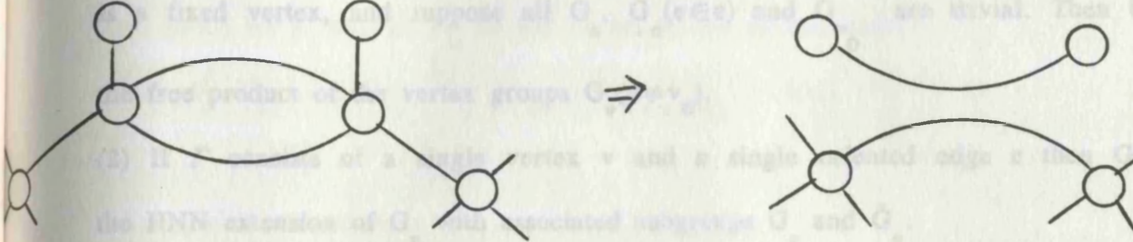


Figure 3.7

If we repeat the above procedure on $c(\mathbb{P})$ inductively then we can establish the Lemma.

Theorem 3.1.4. *$\pi_2(\mathcal{P})$ is generated by* $X \cup Y \cup Z$.

Proof. By Lemmas 3.1.2, 3.1.3 and Proposition 1.2.4.

3.2 Graphs of groups

A *graph of groups* $\mathcal{G}$ consists of

(i) an oriented graph Γ (with vertex set **v** and oriented edge set **e**);

(ii) for each $v \in \mathbf{v}$, a group G_v (*vertex group*) and for each $e \in \mathbf{e}$, groups G_e, $\bar{G}_e$ (*edge groups*);

(iii) for each $e \in \mathbf{e}$, G_e is a subgroup of $G_{\iota(e)}$, $\bar{G}_e$ is a subgroup of $G_{\tau(e)}$, and there is an isomorphism

(3-1) $\gamma_e : G_e \longrightarrow \bar{G}_e$.

Let T be a maximal forest in Γ and let F(**e**) be the free group with basis

e. Let G be the quotient of $F(e)*(\ast_{v\in \mathbf{v}} G_v)$ by the normal closure of the set $\{g_e e(\gamma_e(g_e))^{-1}e^{-1};\ e\in \mathbf{e},\ g_e\in G_e\} \cup \{e; e\in T\}$. Then G is called the *fundamental group of the graph of groups* $\mathcal{G}$ [13].

Remarks (1) Let Γ be a simple graph such that for each $e\in \mathbf{e}$, $\iota(e)=v_0$ where v_0 is a fixed vertex, and suppose all G_e, $\bar{G}_e (e\in \mathbf{e})$ and G_{v_0} are trivial. Then G is the free product of the vertex groups $G_v (v\neq v_0)$.

(2) If Γ consists of a single vertex v and a single oriented edge e then G is the HNN extension of G_v with associated subgroups G_e and $\bar{G}_e$.

(3) If Γ has two vertices u, v and a single oriented edge e joining u and v, then G is the free product of G_u and G_v with G_e and $\bar{G}_e$ amalgamated.

Now we will write down a presentation $\mathcal{P}$ for G. For each $v\in \mathbf{v}$, choose a presentation

$$\mathcal{P}_v = < \mathbf{x}_v;\ \mathbf{s}_v >$$

for G_v. Thus G_v is (isomorphic to) the quotient of the free group on $\mathbf{x}_v$ by the normal closure N_v of $\mathbf{s}_v$. For each $e\in \mathbf{e}$ let

$$a_{i,e}\ ,\ \bar{a}_{i,e} (i\in I(e))$$

be non-empty freely reduced words on $\mathbf{x}_{\iota(e)}$, $\mathbf{x}_{\tau(e)}$ respectively such that

$$G_e = \mathrm{sgp}\{\ a_{i,e}N_{\iota(e)};\ i\in I(e)\ \}$$

$$\bar{G}_e = \mathrm{sgp}\{\ \bar{a}_{i,e}N_{\tau(e)};\ i\in I(e)\ \},$$

and such that the correspondence

$$a_{i,e} \longrightarrow \bar{a}_{i,e} \quad (i\in I(e))$$

induces the isomorphism γ_e in (3-1).

For $e\in \mathbf{e}$, $i\in I(e)$, let

$$R_{i,e} = a_{i,e}\hat{e}\,\bar{a}_{i,e}^{-1}\hat{e}^{-1}$$

where $\hat{e}=e$ if $e\notin T$, and $\hat{e}$ is empty if $e\in T$. Let $\mathbf{r}$ denote the collection of all the $R_{i,e}$'s and let $s = \bigcup_{v\in\mathbf{v}} s_v$, $x = \bigcup_{v\in\mathbf{v}} x_v$ and $I = \bigcup_{e\in\mathbf{e}} I(e)$. Then

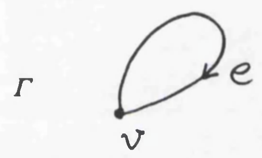
$$\mathcal{P} = \langle\ x,\ e\ (e\in\mathbf{e}\backslash T)\ ;\ s,\ \mathbf{r}\ \rangle$$

is a presentation for G.

For $e\in\mathbf{e}$, choose (disjoint) sets $\mathbf{y}_e = \{\ y_{i,e};\ i\in I(e)\}$ and $\bar{\mathbf{y}}_e = \{\ \bar{y}_{i,e};\ i\in I(e)\}$. Let $\mathbf{w}_e$ be the set of all words (reduced or not) on $\mathbf{y}_e$. Let F_e be the free group on $\mathbf{y}_e$ and let N_e be the kernel of the epimorphism

(3-2) $\theta_e : F_e \longrightarrow G_e$, $\qquad y_{i,e} \longmapsto a_{i,e}N_{\iota(e)}$ $(i\in I(e))$.

3.2.1 HNN extensions



For convenience, let $\mathcal{P}_v = \langle x;s \rangle$ be a presentation for G_v and let G_e and $\bar{G}_e$ be the subgroups generated by $\{\ a_iN_v \mid i\in I\ \}$ and $\{\ b_iN_v \mid i\in I\ \}$ respectively, where the correspondence $a_i \longrightarrow b_i$ induces the isomorphism γ_e. Then the HNN extension G of G_v with associated subgroups G_e and $\bar{G}_e$ has a presentation

$$\mathcal{P} = \langle\ x,\ e;\ s,\ \mathbf{r}\ \rangle$$

where $\mathbf{r} = \{\ a_ieb_i^{-1}e^{-1} \mid i\in I\}$.

Let X be the collection of all spherical pictures over $\mathcal{P}_v$. If an element $W(y_i)\in\mathbf{w}_e$ defines an element of N_e then $W(a_i)$ defines the identity in G_v. So there is a picture $\mathbb{A}_W$ over $\mathcal{P}_v$ with the boundary label $W(a_i)$. We note that though $\mathbb{A}_W$ is not unique, it is unique up to equivalence (rel X), because if the pictures $\mathbb{A}_W$ and $-\mathbb{A}'_W$ can be combined to make a spherical picture over $\mathcal{P}_v$.

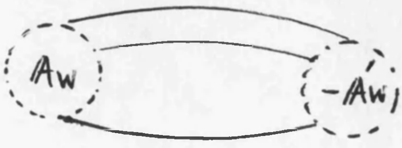


Figure 3.8

Thus we can make a collection A by choosing one picture $\mathbb{A}_W$ over $\mathcal{P}_v$ with boundary label $W(a_i)$ for each element $W(y_i) \in w_e$ which defines an element of N_e.

Since γ_e is an isomorphism, if $W(a_iN_v)$ is 1 in G_v then also $W(b_iN_v)$ is 1 in G_v. Thus for each element $W(y_i) \in w_e$ which defines an element of N_e, we get another picture $\mathbb{B}_W$ over $\mathcal{P}_v$ unique up to equivalence (relX) with boundary label $W(b_i)$. Therefore we can get another collection B consisting of pictures $\mathbb{B}_W$ over $\mathcal{P}_v$ with boundary label $W(b_i)$ for each element $W(y_i) \in w_e$ which defines an element of N_e.

Let $W = W(y_i) = y_{i_1}^{\varepsilon_1} y_{i_2}^{\varepsilon_2} \ldots y_{i_n}^{\varepsilon_n}$ $(y_{i_j} \in y, \ \ \varepsilon_j \pm 1, \ \ j=1,2,\ldots,n)$. Then we can construct a spherical picture $\mathbb{P}_W$ over $\mathcal{P}$ of the form depicted in Fig.3.9.

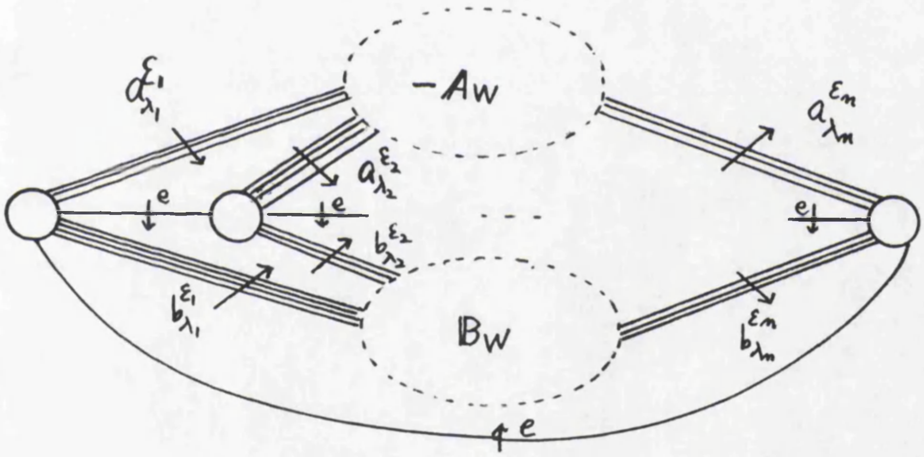


Figure 3.9

Let Y be the collection of all spherical pictures $\mathbb{P}_W$ (W defines an element of N_e).

Theorem 3.2.1. X ∪ Y *generates* $\pi_2(\mathcal{P})$.

Proof. Let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$. If $\mathbb{P}$ has no **r**-discs then $\mathbb{P}$ is contained in X. So we can assume that $\mathbb{P}$ has at least one **r**-disc. Each **r**-disc has just two arcs labelled by e and these two e-arcs have the same direction.

So we can consider a circle C consisting of **r**-discs and e-arcs with the property that there are no **r**-discs contained in the region enclosed by C. Then the part enclosed by C constitute a picture $\mathbb{P}_1$ over $<\mathbf{x},e;s>$ of which the boundary label is *either* (i) a word $W(a_i)$ *or* (ii) a word $W(b_i)$, according to whether the orientation of e-arcs is inwards to, or outwards from, the region enclosed by C. Now we can draw a simple closed curve β in $\mathbb{P}$ "just outside" C, so that the discs of the subpicture $\mathbb{P}_2$ of $\mathbb{P}$ enclosed by β are precisely the discs of $\mathbb{P}_1$ and the **r**-discs of C. The label on β will be *either* $W(b_i)$ in case (i) *or* $W(a_i)$ in case (ii).

Figure 3.10

Since the correspondence $a_i \longrightarrow b_i (i \in I)$ induces γ_e (3-1) and $W(\mathbb{P}_1)$ defines the identity of G_v, the label on β defines the identity of G_v, so there is a $\mathbb{B}_W \in \mathbf{B}$ in case (i) or an $\mathbb{A}_W \in \mathbf{A}$ in case (ii) whose boundary label is equal to the label on β. We assume the case (i). Now we insert the spherical picture

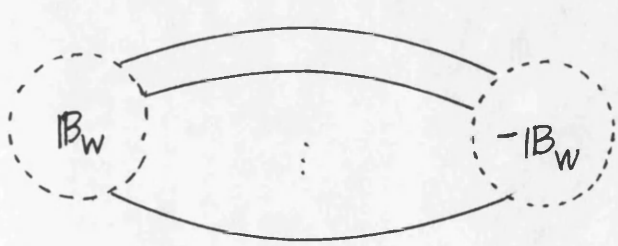


Figure 3.11

(we can consider it as a collection of cancelling pairs) in $\mathbb{P}$ near β. By bridge moves, $\mathbb{P}_2$ and $-\mathbb{B}_W$ make a spherical picture which is equivalent (rel X) to a spherical picture in Y^*. We remove it. Then $\mathbb{B}_W$ and the subpicture of $\mathbb{P}$

outside β makes a spherical picture $\mathbb{P}'$ with n fewer r-discs which is equivalent (rel $X \cup Y$) to $\mathbb{P}$. We can repeat the above argument with $\mathbb{P}'$ in place of $\mathbb{P}$, and so on, eventually arriving at a picture $\hat{\mathbb{P}}$ with no r-discs. Then $\hat{\mathbb{P}}$ is equivalent (rel X) to the empty picture. So we get our result.

Example 3.2.2. Suppose that we have the followings:

$$\mathcal{P}_v = \langle a,b,c,d; a^2b^2a^4b^4a^6b^6,\ c^2d^2c^4d^4c^6d^6 \rangle$$

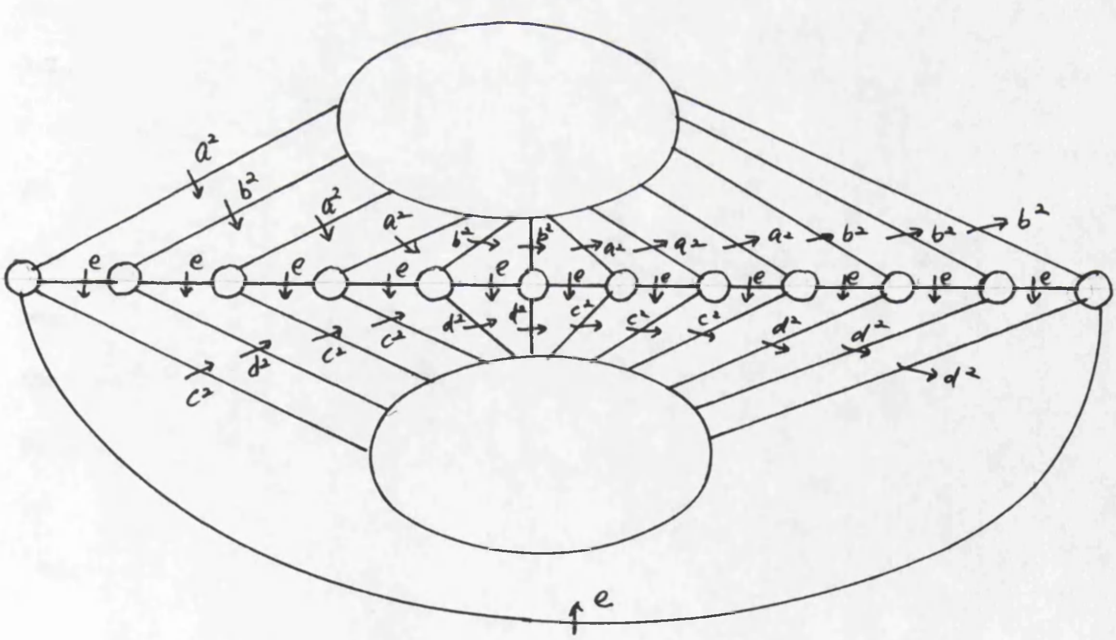
$$G_e = \text{sgp}\{a^2N_v,\ b^2N_v\}, \qquad \bar{G}_e = \text{sgp}\{c^2N_v, d^2N_v\}$$

$$\gamma_e : G_e \longrightarrow \bar{G}_e\ ,\quad a^2N_v \longmapsto c^2N_v,\quad b^2N_v \longmapsto d^2N_v.$$

Then

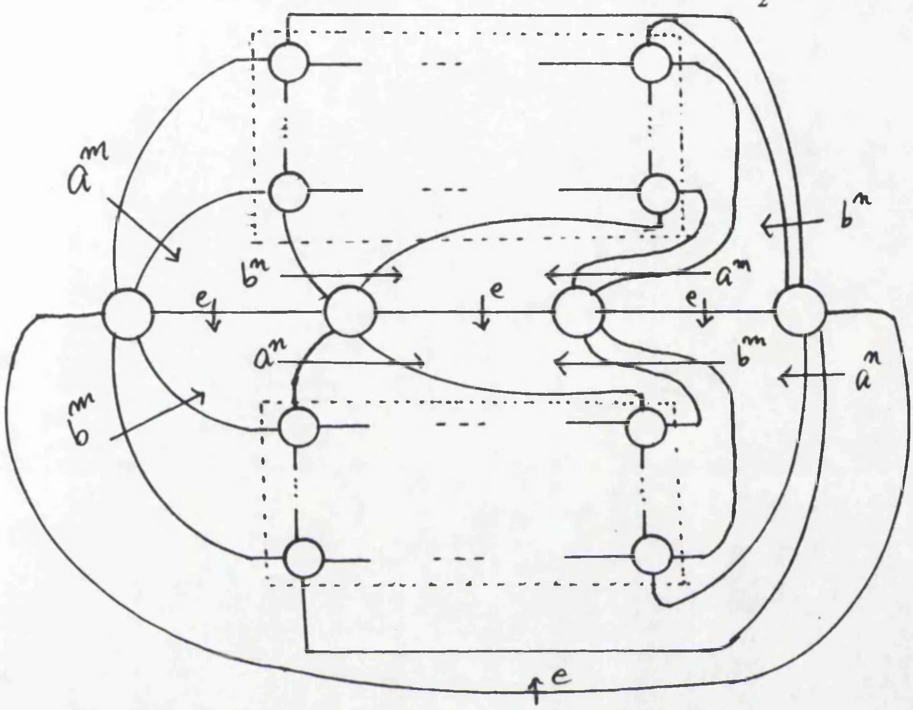
$$\mathcal{P} = \langle\ a,b,c,d,e\ ;\ a^2b^2a^4b^4a^6b^6,\ c^2d^2c^4d^4c^6d^6,\ a^2ec^{-2}e^{-1},\ b^2ed^{-2}e^{-1} \rangle$$

Let θ: F $\longrightarrow G_e$, $x \longmapsto a^2N_v$, $y \longmapsto b^2N_v$, where F is the free group on {x,y}. Then using small cancellation theory N_e is the normal closure of $\{xyx^2y^2x^3y^3\}$. So $\pi_2(\mathcal{P})$ is generated by $\mathbb{P}$.

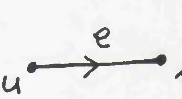


Example 3.2.3. Let $\mathcal{P}_v = \langle a,b;[a,b] \rangle$, $G_e = sgp\{a^mN_v, b^nN_v\}$, $\bar{G}_e = \{b^mN_v, a^nN_v\}$ $\gamma_e: G_e \longrightarrow \bar{G}_e$, $a^mN_v \longmapsto b^mN_v$, $b^nN_v \longmapsto a^nN_v$ and $\theta: F \longrightarrow G_e$, $x \longmapsto a^mN_v$, $y \longmapsto b^nN_v$. Then N_e is the normal closure of $[x,y]$. So

$\mathcal{P} = \langle a,b,e;\ [a,b],\ a^meb^{-m}e^{-1},\ b^nea^{-n}e^{-1} \rangle$ and $\pi_2(\mathcal{P})$ is generated by



3.2.2 Amalgamated free products

Γ 

Let G_e and $\bar{G}_e$ be generated by $\{a_iN_u \mid i \in I\}$ and $\{b_iN_v \mid i \in I\}$ respectively, where the correspondence $a_i \longrightarrow b_i$ induces the isomorphism γ_e in (3-1). Then the free product of G_u and G_v with amalgamated subgroup G_e and $\bar{G}_e$ has a presentation

$$\mathcal{P} = \langle\ x_u,\ x_v;\ s_u,\ s_v,\ r\ \rangle,$$

where $r = \{a_ib_i^{-1} \mid i \in I\}$.

Consider the presentation

$$\hat{\mathcal{P}} = \langle\ x_u,\ x_v,\ e\ ;\ s_u,\ s_v,\ \hat{r}\ \rangle,$$

where $\hat{r} = \{a_ieb_i^{-1}e^{-1} \mid i \in I\}$. Then we can consider $\hat{\mathcal{P}}$ as a presentation of the HNN extension of $G_u * G_v$ with associated subgroups G_e and $\bar{G}_e$.

Let X_u and X_v be the collections of all spherical pictures over $\mathscr{P}_u$ and $\mathscr{P}_v$ respectively and $X=X_u \cup X_v$.

For each element $W(y_i)$ which defines an element of N_e, we get a spherical picture $\mathbb{P}_W$ over $\hat{\mathscr{P}}$ like Fig.3.9. Here $\mathbb{A}_W$ and $\mathbb{B}_W$ are pictures over $\mathscr{P}_u$ and $\mathscr{P}_v$ respectively. Let $\mathbb{Q}_W$ be the spherical picture over $\mathscr{P}$ obtained from $\mathbb{P}_W$ by eliminating all the e-arcs like Fig.3.12 and let Y be the collection of all the $\mathbb{Q}_W$'s.

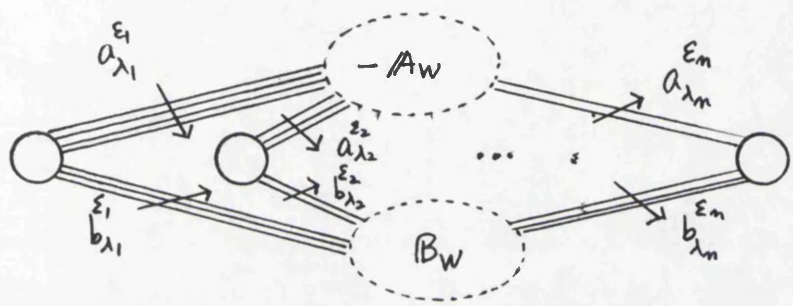


Figure 3.12

Theorem 3.2.4. X $\cup$ Y *generates* $\pi_2(\mathscr{P})$.

Proof. Let $\mathbb{P}$ be a spherical picture over $\mathscr{P}$. Convert it to a spherical picture $\hat{\mathbb{P}}$ over $\hat{\mathscr{P}}$ as follows.

(a) For each arc labelled by an element $x \in x_u^{\pm 1}$ replace it by three parallel arcs.

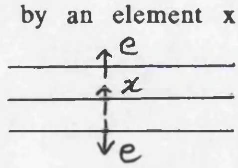


Figure 3.13

(b) If, when reading around a disc we encounter two successive arcs labelled by e and e^{-1}, then perform a bridge move to delete them.

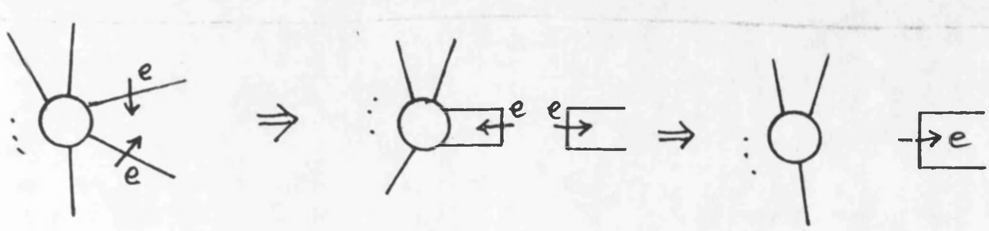


Figure 3.14

(c) Removing all floating e-arcs.

The procedure in the proof of Theorem 3.2.1 will give a reduction of $\hat{\mathbb{P}}$ to the empty picture. Rubbing out all the e-arcs at each stage of the reduction will establish that $\mathbb{P}$ is equivalent (rel $X \cup Y$) to the empty picture.

Example 3.2.5. Let

$$\mathcal{P}_u = \langle a,b;[a,b],\ a^{p\alpha},\ b^{q\beta} \rangle \ , \quad \mathcal{P}_v = \langle c,d;[c,d],\ c^{p\lambda},\ d^{q\delta} \rangle$$

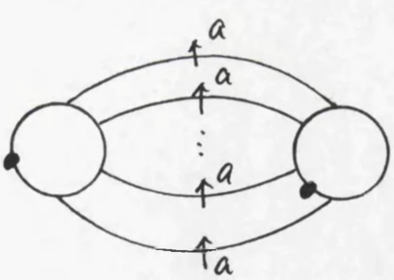
$$G_e = \mathrm{sgp}\{a^{\alpha}N_v,\ b^{\beta}N_v\}, \quad \bar{G}_e = \mathrm{sgp}\{c^{\lambda}N_v,\ d^{\delta}N_v\}$$

$$\gamma\colon G_e \longrightarrow \bar{G}_e\ , \quad a^{\alpha}N_v \longmapsto c^{\lambda}N_v,\quad b^{\beta}N_v \longmapsto d^{\delta}N_v .$$

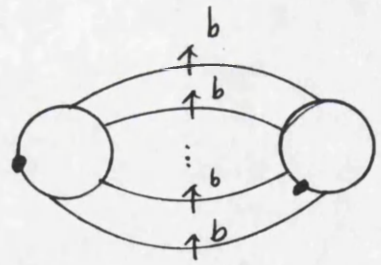
Then

$$\mathcal{P} = \langle a,b,c,d;[a,b],[c,d],\ a^{p\alpha},\ b^{q\beta},\ c^{p\lambda},\ d^{q\delta}, a^{\alpha}c^{-\lambda},\ b^{\beta}d^{-\delta} \rangle .$$

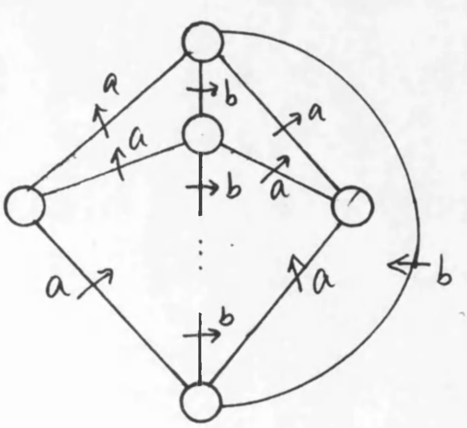
Let $\theta\colon F \longrightarrow G_e$, $x \longmapsto a^{\alpha}N_v$, $y \longmapsto b^{\beta}N_v$ where F is the free group on $\{x,y\}$. Then N_e is the normal closure of $[x,y]$, x^p, y^q. So $\pi_2(\mathcal{P})$ is generated the following pictures.



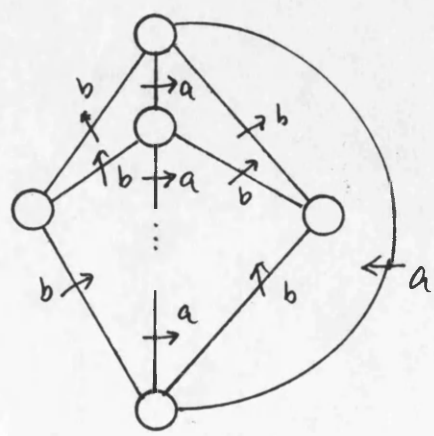
$\mathbb{P}_1$



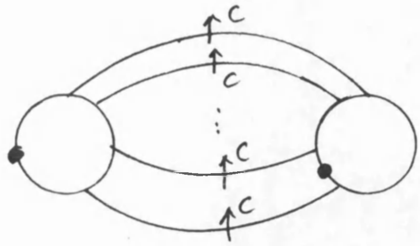
$\mathbb{P}_2$



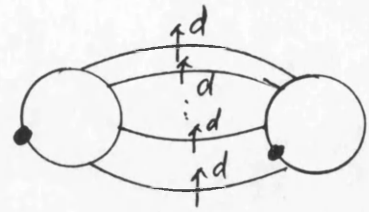
$\mathbb{P}_3$



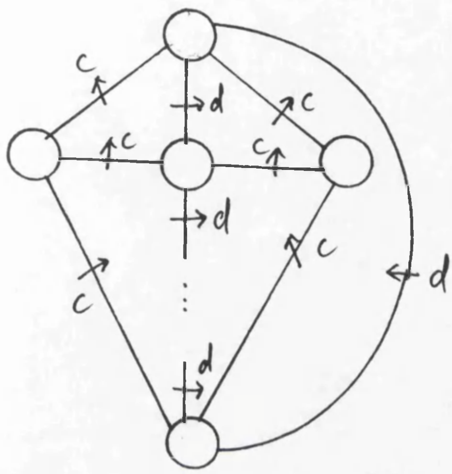
$\mathbb{P}_4$



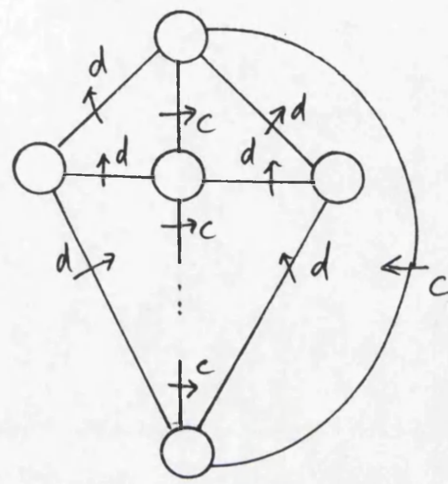
$\mathbb{P}_5$



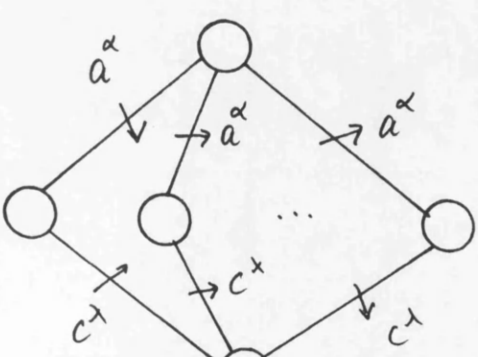
$\mathbb{P}_6$



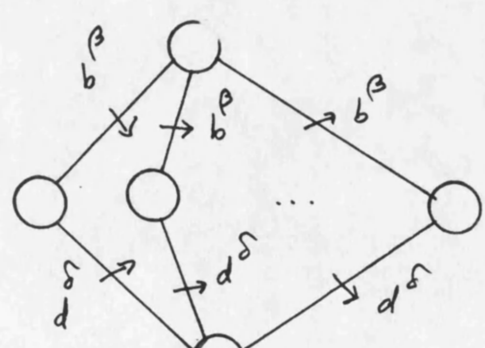
$\mathbb{P}_7$



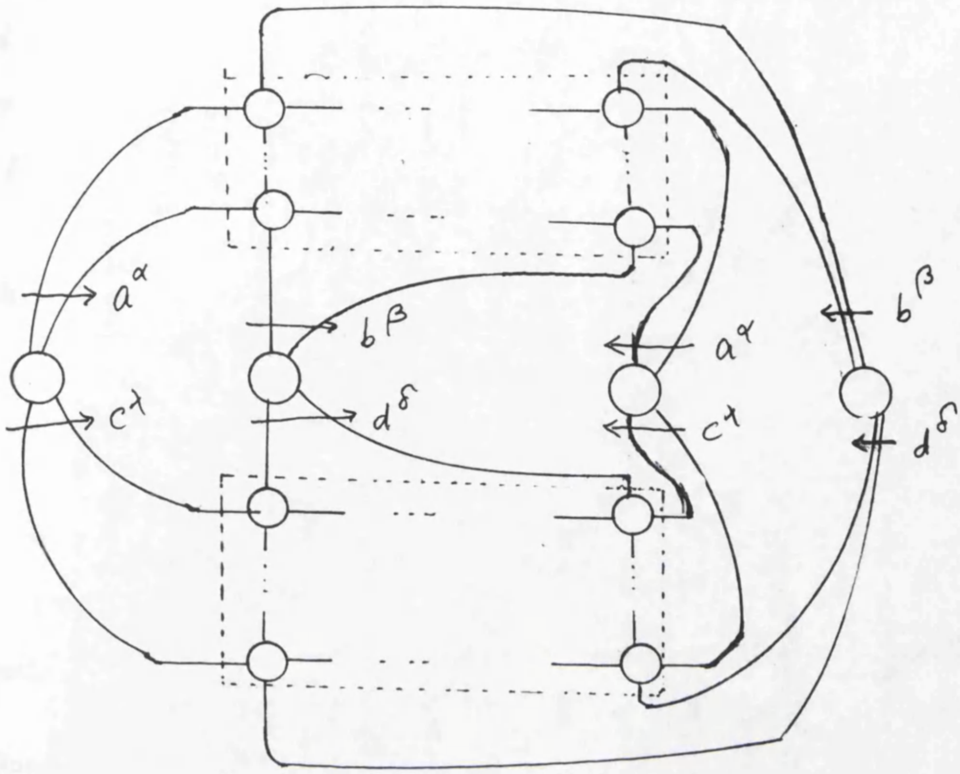
$\mathbb{P}_8$



$\mathbb{P}_9$



$\mathbb{P}_{10}$



3.2.3 Fundamental groups of graphs of groups

We will consider the general case of a graph Γ. If $\mathbb{P}$ is a spherical picture over $\mathcal{P}$ then $\mathbb{P}$ is a picture over a finite subpresentation of $\mathcal{P}$, so we can assume that Γ is finite.

Let X_v ($v \in \mathbf{v}$) be the collection of all spherical pictures over $\mathcal{P}_v$ ($v \in \mathbf{v}$) and let $X = \bigcup_{v \in \mathbf{v}} X_v$. If $e \notin T$ then Y_e denotes the collection of all spherical pictures $\mathbb{P}_W$ as in Fig.3.9 and if $e \in T$ then Y_e denotes the collection of all spherical pictures $\mathbb{Q}_W$ as in Fig.3.12 ($W \in \mathbf{w}_e$, W defines an element of N_e). And let $Y = \bigcup_{e \in \mathbf{e}} Y_e$.

Theorem 3.2.6. $X \cup Y$ *generates* $\pi_2(\mathcal{P})$.

Proof. We will prove it by induction on the number of edges of Γ. The case that Γ has only one edge is treated in §§ 3.2.1 and 3.2.2. For the induction step we consider the following two cases separately.

(i) $\Gamma = T$.

Choose an extremal vertex w of T and let f be the unique edge incident with w. Then we can assume that Γ is like Fig 3.15.

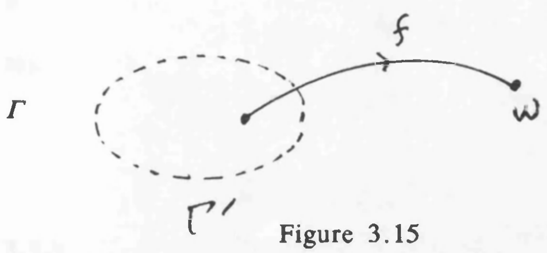
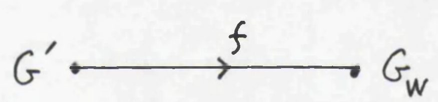


Figure 3.15

Let $\mathcal{P}'$ be the subpresentation of $\mathcal{P}$ arising from Γ'. Then by induction hypothesis $\pi_2(\mathcal{P}')$ is generated by $(\bigcup_{\substack{v \in V \\ v \neq w}} X_v) \cup (\bigcup_{\substack{e \in E \\ e \neq f}} Y_e)$. Let G′ be the groups defined by $\mathcal{P}'$. Now we consider the following graph of groups.

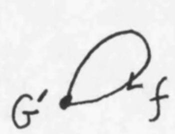


Since $\mathcal{P}$ is the presentation arising from this graph of groups, by Theorem 3.2.4, we get our conclusion.

(ii) $\Gamma \neq T$.

Choose an edge $f \in \Gamma \backslash T$ and let $\mathcal{P}'$ be the subpresentation of $\mathcal{P}$ arising from $\Gamma \backslash \{f\}$ and G′ the group defined by $\mathcal{P}'$. Then $\pi_2(\mathcal{P}')$ is generated by $X \cup (Y \backslash Y_f)$.

Now we consider the following graph of groups.



Since $\mathcal{P}$ is the presentation arising from this graph of groups by Theorem 3.2.1 we get our conclusion.

3.3 Applications

In this section, we describe the second integral (co)homology of the fundamental groups of graphs of groups and consider necessary and sufficient conditions for their presentations to be Cockroft, and we get some short exact sequences. We also introduce some new classes of presentations which are minimal but not efficient.

Terminology will be as in §3.2.

3.3.1 Second (co)homology and Cockroft property

For $S \in \mathbf{s}$, let

$$n_W(S) = \exp_S(\mathbb{A}_W) - \exp_S(\mathbb{B}_W).$$

Then δ_2, δ_3, δ_2^* and δ_3^* in §1.4 are given as follows.

$$\delta_3 : (\bigoplus_{\mathbb{P} \in X} \mathbb{Z}t_{\mathbb{P}}) \oplus (\bigoplus_{\mathbb{P}_W \in Y} \mathbb{Z}t_{\mathbb{P}_W}) \longrightarrow (\bigoplus_{S \in \mathbf{s}} \mathbb{Z}t_S) \oplus (\bigoplus_{i \in I} \mathbb{Z}t_i)$$

$$t_{\mathbb{P}} \longmapsto \sum_{S \in \mathbf{s}} \exp_S(\mathbb{P}) t_S$$

$$t_{\mathbb{P}_W} \longmapsto \sum_{S \in \mathbf{s}} n_W(S) t_S + \sum_{i \in I} \exp_{y_i}(W) t_i$$

$$\delta_2 : (\bigoplus_{S \in \mathbf{s}} \mathbb{Z}t_S) \oplus (\bigoplus_{i \in I} \mathbb{Z}t_i) \longrightarrow (\bigoplus_{x \in \mathbf{x}} \mathbb{Z}t_x) \oplus (\bigoplus_{e \in \mathbf{e}\Gamma} \mathbb{Z}t_e)$$

$$t_S \longmapsto \sum_{x \in \mathbf{x}} \exp_x(S) t_x$$

$$t_i \longmapsto \sum_{x \in \mathbf{x}} \exp_x(a_i b_i^{-1}) t_x$$

$$\delta_3^* : (\bigoplus_{\mathbb{P} \in X} \mathbb{Z}t_{\mathbb{P}}^*) \oplus (\bigoplus_{\mathbb{P}_W \in Y} \mathbb{Z}t_{\mathbb{P}_W}^*) \longleftarrow (\bigoplus_{S \in \mathbf{s}} \mathbb{Z}t_S^*) \oplus (\bigoplus_{i \in I} \mathbb{Z}t_i^*)$$

$$t_S^* \longmapsto \sum_{\mathbb{P} \in X} \exp_S(\mathbb{P}) t_{\mathbb{P}}^* + \sum_{\mathbb{P}_W \in Y} n_W(S) t_{\mathbb{P}_W}^*$$

$$t_i^* \longmapsto \sum_{\mathbb{P}_W \in Y} \exp_{y_i}(W) t_{\mathbb{P}_W}^*$$

$$\delta_2^* : (\bigoplus_{S \in \mathbf{s}} \mathbb{Z}t_S^*) \oplus (\bigoplus_{i \in I} \mathbb{Z}t_i^*) \longleftarrow (\bigoplus_{x \in \mathbf{x}} \mathbb{Z}t_x^*) \oplus (\bigoplus_{e \in \mathbf{e}\Gamma} \mathbb{Z}t_e^*)$$

$$t_x^* \longmapsto \sum_{S\in \mathbf{s}} \exp_x(S)t_S^* + \sum_{x\in \mathbf{x}} \exp_x(a_ib_i^{-1})t_i^*$$

$$t_e^* \longmapsto 0.$$

Then we get

(i) $H_2(G) \cong \mathrm{Ker}\delta_2/\mathrm{Im}\delta_3$

(ii) $H^2(G) \cong \mathrm{Ker}\delta_3^*/\mathrm{Im}\delta_2^*$.

Remark There are the following long exact (co)homology sequences [9 ,Theorem 2]

$$\dots \longrightarrow \bigoplus_{e\in \mathbf{e}} H_n(G_e) \longrightarrow \bigoplus_{v\in \mathbf{v}} H_n(G_v) \longrightarrow H_n(G) \longrightarrow \bigoplus_{e\in \mathbf{e}} H_{n-1}(G_e) \longrightarrow \dots$$

and

$$\dots\longleftarrow H^{n+1}(G) \longleftarrow \prod_{e\in \mathbf{e}} H^n(G_e) \longleftarrow \prod_{v\in \mathbf{v}} H^n(G_v) \longleftarrow H^n(G) \longleftarrow \dots$$

So we can in theory compute H^2 and H_2 by using the above long exact sequences. But our method is practical.

Proposition 3.3.1. *$\mathscr{P}$ is Cockroft if and only if the following conditions hold:*

(i) $\mathscr{P}_v$ ($v\in\mathbf{v}$) is Cockroft;

(ii) $N_e \subseteq F'_e$ ($e\in\mathbf{e}$);

(iii) If $e\in\mathbf{e}$ and $\iota(e)=\tau(e)$ then for all $W\in\mathbf{w}_e$, $n_W(S)=0$ ($S\in\mathbf{s}_{\iota(e)}$);

(iv) If $e\in\mathbf{e}$ and $\iota(e)\neq\tau(e)$ then for all $W\in\mathbf{w}_e$

$$exp_S(\mathbb{B}_W) = 0 \quad (S\in\mathbf{s}_{\iota(e)})$$

$$exp_S(\mathbb{A}_W) = 0 \quad (S\in\mathbf{s}_{\tau(e)}).$$

Proof. ($\Rightarrow$) Suppose $\mathscr{P}$ is Cockroft. Then for all $\mathbb{P}\in X$, for all $S\in\mathbf{s}$, $\exp_S(\mathbb{P})=0$. Thus $\mathscr{P}_v$ is Cockroft. We note that if W defines an element of N_e then $\exp_{R_i}(\mathbb{P}_W)=\exp_{y_i}(W)$ for all i. Thus (ii) holds. Since $\exp_S(\mathbb{P}_W)=0$, we get (iii) and (iv).

($\Leftarrow$) By (i), all $\mathbb{P}\in X$ are Cockroft. By (ii), (iii) and (iv), all $\mathbb{P}_w\in Y$ are Cockroft. So $\mathcal{P}$ is Cockroft by Theorem 3.2.6.

Example 3.3.2. We consider the same presentation as in Example 3.2.5.

$$\mathcal{P} = < a, b, c, d \; ; S_1, S_2, S_3, T_1, T_2, T_3, R_1, R_2 >$$

where $S_1=[a,b]$, $S_2=a^{p\alpha}$, $S_3=b^{q\beta}$, $T_1=[c,d]$, $T_2=c^{p\lambda}$, $T_3=d^{q\delta}$, $R_1=a^{\alpha}c^{-\lambda}$, $R_2=b^{\beta}d^{-\delta}$.

$\mathbb{P}_1,\ldots,\mathbb{P}_{11}$ are the same as in Example 3.2.5.

δ_3:

$$t_{\mathbb{P}_i} \longmapsto 0 \ (i=1,2,5,6)$$
$$t_{\mathbb{P}_3} \longmapsto p\alpha t_{S_1}$$
$$t_{\mathbb{P}_4} \longmapsto -q\beta t_{S_1}$$
$$t_{\mathbb{P}_7} \longmapsto p\lambda t_{T_1}$$
$$t_{\mathbb{P}_8} \longmapsto -q\delta t_{T_1}$$
$$t_{\mathbb{P}_9} \longmapsto t_{T_2} - t_{S_2} + pt_{R_1}$$
$$t_{\mathbb{P}_{10}} \longmapsto t_{T_3} - t_{S_3} + qt_{R_2}$$
$$t_{\mathbb{P}_{11}} \longmapsto \lambda\delta t_{T_1} - \alpha\beta t_{S_1}$$

δ_3^*:

$$t^*_{S_1} \longmapsto p\alpha t^*_{\mathbb{P}_3} - q\beta t^*_{\mathbb{P}_4} - \alpha\beta t^*_{\mathbb{P}_{11}}$$
$$t^*_{S_2} \longmapsto -t^*_{\mathbb{P}_9}$$
$$t^*_{S_3} \longmapsto -t^*_{\mathbb{P}_{10}}$$
$$t^*_{T_1} \longmapsto p\lambda t^*_{\mathbb{P}_7} - q\delta t^*_{\mathbb{P}_8} + \lambda\delta t^*_{\mathbb{P}_{11}}$$
$$t^*_{T_2} \longmapsto t^*_{\mathbb{P}_9}$$
$$t^*_{T_3} \longmapsto t^*_{\mathbb{P}_{10}}$$
$$t^*_{R_1} \longmapsto t^*_{\mathbb{P}_9}$$
$$t^*_{R_2} \longmapsto qt^*_{\mathbb{P}_{10}}$$

δ_2:

$$t_{S_1}, t_{T_1} \longmapsto 0$$
$$t_{S_2} \longmapsto p\alpha t_a$$
$$t_{S_3} \longmapsto q\beta t_b$$
$$t_{T_2} \longmapsto p\lambda t_c$$
$$t_{T_3} \longmapsto q\delta t_d$$
$$t_{R_1} \longmapsto \alpha t_a - \lambda t_c$$
$$t_{R_2} \longmapsto \beta t_b - \delta t_d.$$

δ_2^*:

$$t^*_a \longmapsto p\alpha t^*_{S_2} + \alpha t^*_{R_1}$$
$$t^*_b \longmapsto q\beta t^*_{S_3} + \beta t^*_{R_2}$$
$$t^*_c \longmapsto p\lambda t^*_{T_2} - \lambda t^*_{R_1}$$
$$t^*_d \longmapsto q\delta t^*_{T_3} - \delta t^*_{R_2}$$

Suppose that

$$k_1(p\alpha t_a)+k_2(q\beta t_b)+k_3(p\lambda t_c)+k_4(q\delta t_d)+k_5(\alpha t_a-\lambda t_c)+k(\beta t_b-\delta t_d)=0.$$

Then

$$k_1p + k_5 = 0$$

$$k_2q + k_6 = 0$$

$$k_3p - k_5 = 0$$

$$k_4q - k_6 = 0.$$

Thus we have $k_1= -k_3$, $k_5= pk_3$, $k_2= -k_4$, $k_6= qk_4$. Therefore,

$\mathrm{Ker}\delta_2$ is generated by t_{S_1}, t_{T_1}, $pt_{R_1} + t_{T_2} - t_{S_2}$ and $qt_{R_2} + t_{T_3} - t_{S_3}$.

$\mathrm{Im}\delta_3$ is generated by $pt_{R_1} + t_{T_2} - t_{S_2}$, $qt_{R_2} + t_{T_3} - t_{S_3}$, $p\alpha t_{S_1}$, $q\beta t_{S_1}$,

$p\lambda t_{T_1}$, $q\delta t_{T_1}$, $\lambda\delta t_{T_1} - \alpha\beta t_{S_1}$.

So we get

$$H_2(G) \cong < x, y ; [x,y], x^n, y^m, x^{\lambda\delta}y^{-\alpha\beta} >$$

where $n=(p\alpha,q\beta)$ and $m=(p\lambda,q\delta)$.

Now we calculate $H^2(G)$. Suppose that

$$k_1(p\alpha t^*_{\mathbb{P}_3} - q\beta t^*_{\mathbb{P}_4} - \alpha\beta t^*_{\mathbb{P}_{11}})+k_2(-t^*_{\mathbb{P}_9})+k_3(-t^*_{\mathbb{P}_{10}})+k_4(p\lambda t^*_{\mathbb{P}_7} - q\delta t^*_{\mathbb{P}_8} + \lambda t^*_{\mathbb{P}_{11}})$$

$$+k_5t_{\mathbb{P}_9} + k_6t_{\mathbb{P}_{10}} + k_7t_{\mathbb{P}_9} + k_8t_{\mathbb{P}_{10}} =0.$$

so we have

$$k_1 = k_4 = 0$$

$$k_2 - k_5 - pk_7 = 0$$

$$k_3 - k_6 - qk_8 = 0.$$

Then we have solutions;

$$(k_2,k_5,k_7) = (p, 0, 1) \text{ or } (1, 1, 0)$$

$$(k_3,k_6,k_8) = (q, 0, 1) \text{ or } (1, 1, 0).$$

Therefore

$\mathrm{Ker}\delta^*_3$ is generated by $\omega_1=t^*_{S_2} + t^*_{T_2}$, $\omega_2=pt^*_{S_2} + t^*_{R_1}$, $\omega_3=t^*_{S_3} + t^*_{T_3}$,

$\omega_4=qt^*_{S_3} + t^*_{R_2}$.

$\mathrm{Im}\delta^*_2$ is generated by $\alpha\omega_2$, $\beta\omega_4$, $\lambda(p\omega_1- \omega_2)$, $\delta(q\omega_3- \omega_4)$. So we get

$$H^2(G) \cong < \omega_1, \omega_2, \omega_3, \omega_4 ; [\omega_i,\omega_j](1 \le i < j \le 4), \omega_2^{\alpha}, \omega_4^{\beta}, (\omega_1^p\omega_2^{-1})^{\lambda}, (\omega_3^q\omega_4^{-1})^{\delta} >.$$

$(n,\lambda\delta) \neq 1$, $(m,\alpha\beta) \neq 1$ and $(n,m) \neq 1$ then $d(H_2(G))=2$. So $\upsilon(G)=1-0+2=3$. Therefore

$$\mathcal{P}' = < a, b, c, d ; [a,b], [c,d], a^{p\alpha}, b^{q\beta}, a^{\alpha}c^{-\lambda}, b^{\beta}d^{-\delta} >$$

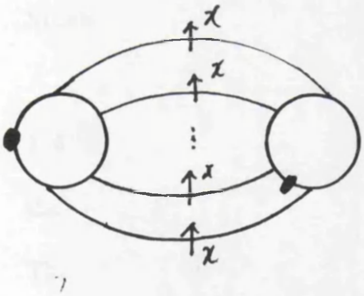
is an efficient presentation for G.

3.3.2 Some minimal presentations which are not efficient

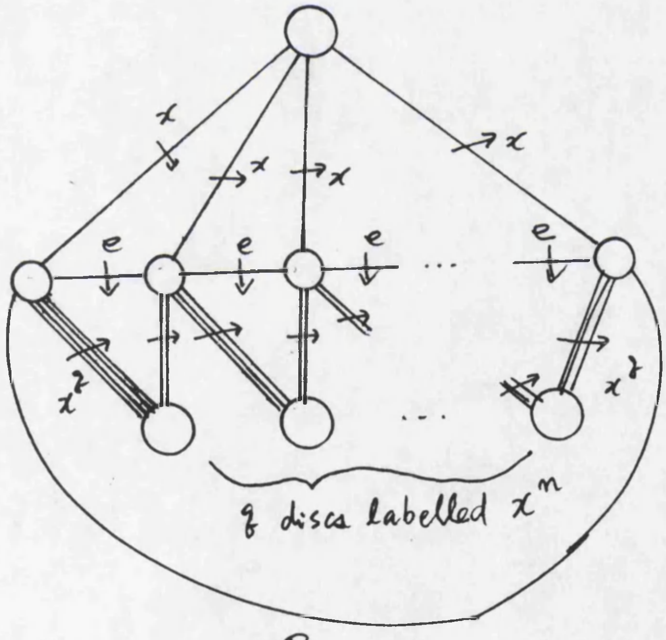
Let $\mathcal{P}_v = <x;x^n>$ and G_e and $\bar{G}_e$ the subgroups generated by xN_v and x^qN_v, where $(n,q)=1$. Then we get $\gamma_e : G_e \longrightarrow \bar{G}_e$, $xN_v \longmapsto x^qN_v$. Then

$$\mathcal{P} = <x,e;x^n, xex^{-q}e^{-1}>.$$

By Theorem 3.2.1, $\pi_2(\mathcal{P})$ is generated by $\mathbb{P}_1$ and $\mathbb{P}_2$.



$\mathbb{P}_1$



$\mathbb{P}_2$

Theorem 3.3.3. *The presentation $\mathcal{P}$ above is always minimal, but it is efficient if and only if $(q-1,n) \neq 1$.*

Let $S = x^n$, $R = xex^{-q}e^{-1}$. Then

$$\delta_3: \quad t_{\mathbb{P}_1} \longmapsto 0 \qquad\qquad \delta_3^*: \quad t_S^* \longmapsto (q-1)t_{\mathbb{P}_2}^*$$

$$t_{\mathbb{P}_2} \longmapsto (q-1)t_S + nt_R \qquad\qquad t_R^* \longmapsto nt_{\mathbb{P}_2}^*$$

$$\delta_2: \quad t_S \longmapsto nt_x \qquad\qquad \delta_2^*: \quad t_x^* \longmapsto nt_S^* + (q-1)t_R^*$$

$$t_R \longmapsto (q-1)t_x \qquad\qquad t_e^* \longmapsto 0$$

Thus

$\text{Im}\delta_3$ is generated by $\xi = (q-1)t_S + nt_R$

$\text{Ker}\delta_2$ is generated by $1/(q-1,n)\xi$

$\text{Im}\delta_2^*$ is generated by $\xi^* = (q-1)t_R^* + nt^*_S$

$\text{Ker}\delta_3^*$ is generated by $1/(q-1,n)\xi^*$.

So

$$H_2(G) \cong H^2(G) \cong \mathbb{Z}_{(q-1,n)}.$$

Hence we have that

$$\upsilon(G) = \begin{cases} 0, & (q-1,n) = 1 \\ 1, & (q-1,n) \neq 1. \end{cases}$$

Since $\chi(\mathscr{P}) = 1-2+2 = 1$, $\mathscr{P}$ is efficient if and only if $(q-1,n) \neq 1$.

Now we will prove that $\mathscr{P}$ is minimal when $(q-1,n) = 1$ by using Theorem 1.4.4(Lustig). Let C be the infinite cyclic group generated by t and consider the homomorphism $G \longrightarrow C$ induced by the mapping: $x \longmapsto 1$, $e \longmapsto t$. This gives rise to a ring homomorphism

$$\eta: \mathbb{Z}G \longrightarrow \mathbb{Z}_nC.$$

Let A be the quotient of $\mathbb{Z}_nC$ by the ideal generated by $\bar{q}t-1$, and let ψ be the composition

$$\psi: \mathbb{Z}G \xrightarrow{\ \eta\ } \mathbb{Z}_nC \xrightarrow{\ \varphi\ } A.$$

Now $\pi_2(\mathscr{P})$ is generated by the two pictures $\mathbb{P}_1$, $\mathbb{P}_2$ as described above. The image of these generators under the standard injection of $\pi_2(\mathscr{P})$ into $\mathbb{Z}Gt_S \oplus \mathbb{Z}Gt_R$ are $(xN-1)t_S$, $(qeN-1)t_S + (1+xN+\ldots+x^{n-1}N)t_R$. Thus $I_2(\mathscr{P})$ is generated by

$$xN-1,\ qeN-1,\ 1+xN+\ldots+x^{n-1}N.$$

Since

$$\psi(xN-1) = \varphi\eta(xN-1) = 0$$

$$\psi(qeN-1) = \varphi\eta(qeN-1)$$

$$= \varphi(\bar{q}t-1) = 0$$

$$\psi(1+xN+\ldots+x^{n-1}N) = \varphi(\bar{n}1)$$

$$= \varphi(0) = 0,$$

by Theorem 1.4.4 (Lustig) $\mathcal{P}$ is minimal.

3.3.3 Some exact sequences

If $\mathbb{P}$ is a spherical picture over $\mathcal{P}_v$ then the element of $\pi_2(\mathcal{P}_v)$ represented by $\mathbb{P}$ will be denoted by $<\mathbb{P}>_v$. Of course, $\mathbb{P}$ also represents an element of $\pi_2(\mathcal{P})$, which will be denoted (as usual) by $<\mathbb{P}>$. We will write $-\otimes_v-$, $-\otimes_e-$ instead of $-\otimes_{G_v}-$, $-\otimes_{G_e}-$ respectively.

Theorem 3.3.4. *There is a short exact sequence*

$$0 \longrightarrow \bigoplus_{v\in \mathbf{v}} (\mathbb{Z}G\otimes_v \pi_2(\mathcal{P}_v)) \longrightarrow \pi_2(\mathcal{P}) \longrightarrow \bigoplus_{e\in \mathbf{e}} (\mathbb{Z}G\otimes_e N_e/N'_e) \longrightarrow 0.$$

$$1\otimes <\mathbb{P}>_v \longmapsto <\mathbb{P}> \qquad (\mathbb{P}\in X_v,\ v\in \mathbf{v})$$

$$<\mathbb{P}> \longmapsto 0 \qquad (\mathbb{P}\in X_v,\ v\in \mathbf{v})$$

$$<\mathbb{P}_W> \longmapsto 1\otimes WN'_e \qquad (\mathbb{P}_W\in Y_e,\ e\in \mathbf{e}).$$

Corollary 3.3.5.[12] *$\mathcal{P}$ is aspherical if and only if all $\mathcal{P}_v$ ($v\in\mathbf{v}$) are aspherical and all G_e ($e\in\mathbf{e}$) are free on the given basis $a_{i,e}$ ($i\in I(e)$).*

Proof. ($\Rightarrow$) By Theorem 3.3.4, $\mathbb{Z}G\otimes_v\pi_2(\mathcal{P}_v) = 0$ and $\mathbb{Z}G\otimes_e N_e/N'_e = 0$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}G_e$- and $\mathbb{Z}G_v$-module, $\pi_2(\mathcal{P}_v) = 0$ and $N_e/N'_e = 0$. Therefore $\mathcal{P}_v$ is aspherical and G_e is free on the given basis $a_{i,e}$ $(i \in I(e))$.

$(\Leftarrow)$ clear.

Remark As we would know it in Example 3.2.2, the freeness of G_e is crucial.

Corollary 3.3.6. [25] *There is a short exact sequence*

$$0 \longrightarrow \bigoplus_{v\in \mathbf{v}} (\mathbb{Z}G \otimes_v M(\mathcal{P}_v)) \longrightarrow M(\mathcal{P}) \longrightarrow \bigoplus_{e\in \mathbf{e}} (\mathbb{Z}G \otimes_e IG_e) \longrightarrow 0.$$

$$\begin{array}{lll} 1\otimes WN'_v \longmapsto WN' & & (W\in N_v) \\ SN' \longmapsto 0 & & (S \in \mathbf{s}) \\ R_iN' \longmapsto 1\otimes(a_iN_e - 1) & & (R_i\in \mathbf{r},\ i\in I). \end{array}$$

Proof. Consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{v\in\mathbf{v}} (\mathbb{Z}G_v \otimes_v \pi_2(\mathcal{P}_v)) & \longrightarrow & \pi_2(\mathcal{P}) & \longrightarrow & \bigoplus_{e\in\mathbf{e}} (\mathbb{Z}G\otimes_e N_e/N'_e) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \oplus(1\otimes\mu_2^v)} & (1) & \downarrow{\scriptstyle \mu_2} & (2) & \downarrow{\scriptstyle \oplus(1\otimes\mu_1^e)} & & \\
0 & \longrightarrow & \bigoplus_{v\in\mathbf{v}} (\mathbb{Z}G\otimes_v P_2^v) & \xrightarrow{\alpha} & P_2 & \xrightarrow{\beta} & \bigoplus_{e\in\mathbf{e}} (\mathbb{Z}G\otimes_e P_2^e) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \oplus(1\otimes\rho_2^v)} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \oplus(1\otimes\rho_1^e)} & & \\
 & & \bigoplus_{v\in\mathbf{v}} (\mathbb{Z}G\otimes_v M(\mathcal{P}_v)) & & M(\mathcal{P}) & & \bigoplus_{e\in\mathbf{e}} (\mathbb{Z}G\otimes_e IG_e) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $P_2^v = \bigoplus_{S\in \mathbf{s}_v} \mathbb{Z}G_v\bar{t}_S$, $P_2^e = \bigoplus_{i\in I(e)} \mathbb{Z}G_e\bar{t}_{i,e}$, $P_2 = (\bigoplus_{S\in\mathbf{s}} \mathbb{Z}Gt_S) \oplus (\bigoplus_{e\in\mathbf{e}} (\bigoplus_{i\in I} \mathbb{Z}Gt_{i,e}))$ and

$\alpha: 1\otimes\bar{t}_S \longmapsto t_S$,

$\beta: t_S \longmapsto 0,\ \ t_{i,e} \longmapsto 1\otimes\bar{t}_{i,e}$.

The middle column is given in (1-1). The first and third columns are given from (1-1), (1-3) and by tensoring by $\mathbb{Z}G\otimes_v -$ and $\mathbb{Z}G\otimes_e -$ respectively. Since $\mathbb{Z}G\otimes_v P_2^v = \mathbb{Z}G\otimes_v (\bigoplus_{S\in\mathbf{s}_v} \mathbb{Z}G_v\bar{t}_S) \cong \bigoplus_{S\in\mathbf{s}_v} \mathbb{Z}Gt_S$ and $\mathbb{Z}G\otimes_e P_2^e = \mathbb{Z}G\otimes_e (\bigoplus_{i\in I(e)} \mathbb{Z}G_e\bar{t}_{i,e})$

$\cong \bigoplus_{i \in I(e)} \mathbb{Z}Gt_{i,e}$, we get the middle row. The top row is given from Theorem 3.3.5.

Now we consider commutativity:

$$(1)\ 1 \otimes \mathbb{P}_v \longmapsto \mathbb{P} \longmapsto \mu_2(\mathbb{P}), \qquad 1 \otimes \mathbb{P}_v \longmapsto 1 \otimes \mu_2^v(\mathbb{P}_v) \longmapsto \mu_2(\mathbb{P})$$

$$(2)\ \mathbb{P} \longmapsto 0\ (\mathbb{P} \in X_v,\ v \in \mathbf{v}),\quad \mathbb{P}_w \longmapsto 1 \otimes WN'_e \longmapsto 1 \otimes \sum_{i \in I(e)} \theta \frac{\partial W}{\partial y_i} \bar{t}_{i,e} \quad (\mathbb{P}_w \in Y_e,\ e \in \mathbf{e})$$

$$\mathbb{P} \longmapsto \sum_{S \in s_v} g_S t_S \longmapsto 0\ (\mathbb{P} \in X_v,\ v \in \mathbf{v}),$$

$$\mathbb{P}_w \longmapsto \sum_{S \in s_v} g_S t_S + \sum_{i \in I(e)} \theta \frac{\partial W}{\partial y_i} t_{i,e} \longmapsto 1 \otimes \sum_{i \in I(e)} \theta \frac{\partial W}{\partial y_i} \bar{t}_{i,e} \quad (\mathbb{P}_w \in Y_e,\ e \in \mathbf{e}).$$

Then by snake lemma and the exactness of three columns we get the following.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\oplus(1 \otimes \mu_2^v)) & \xrightarrow{\alpha^*} & \mathrm{Coker}\mu_2 & \xrightarrow{\beta^*} & \mathrm{Coker}(\oplus(1 \otimes \mu_2)) & \longrightarrow & 0 \\
 & & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\
 & & \bigoplus_{v \in \mathbf{v}} (\mathbb{Z}G \otimes_v M(\mathscr{P}_v)) & & M(\mathscr{P}) & & \bigoplus_{e \in \mathbf{e}} (\mathbb{Z}G \otimes_e IG_e) & &
\end{array}$$

where α^*, β^* are the induced homomorphisms of α, β respectively and ϕ_1, ϕ_2, ϕ_3 are isomorphisms.

$$\begin{aligned}
\phi_3\beta^*\phi_2^{-1}(SN') &= \phi_3\beta^*(t_S + \mathrm{Im}\mu_2) \\
&= \phi_3(\beta(t_S) + \mathrm{Im} \oplus (1 \otimes \mu_1^e)) \\
&= \phi_3(0) = 0
\end{aligned}$$

$$\begin{aligned}
\phi_3\beta^*\phi_2^{-1}(R_{i,e}N') &= \phi_3\beta^*(t_{i,e} + \mathrm{Im}\mu_2) \\
&= \phi_3(\beta(t_{i,e}) + \mathrm{Im} \oplus (1 \otimes \mu_i^e)) \\
&= \phi_3(1 \otimes \bar{t}_{i,e} + \mathrm{Im} \oplus (1 \otimes \mu_i^e)) = 1 \otimes (a_{i,e} N_e - 1)
\end{aligned}$$

$$\begin{aligned}
\phi_2\alpha^*\phi_1^{-1}(1 \otimes WN'_v) &= \phi_2\alpha^*\Big(\sum_{j=1}^{n} (\varepsilon_j U_j N \otimes \bar{t}_{S_j}) + \mathrm{Im} \oplus (1 \otimes \mu_2^v)\Big) \\
&= \phi_2\Big(\alpha\Big(\sum_{j=1}^{n} \varepsilon_j U_j N \otimes \bar{t}_{S_j}\Big) + \mathrm{Im}\mu_2\Big) \\
&= \phi_2\Big(\sum_{j=1}^{n} \varepsilon_j U_j N t_{S_j} + \mathrm{Im}\mu_2\Big) = WN',
\end{aligned}$$

where W is freely equal to a product $\prod_{j=1}^{n} U_j S_j^{\varepsilon_j} U_j^{-1} \in N_v$. Therefore we get the result.

Now we will prove Theorem 3.3.4. We will use the notation of §3.2.3.

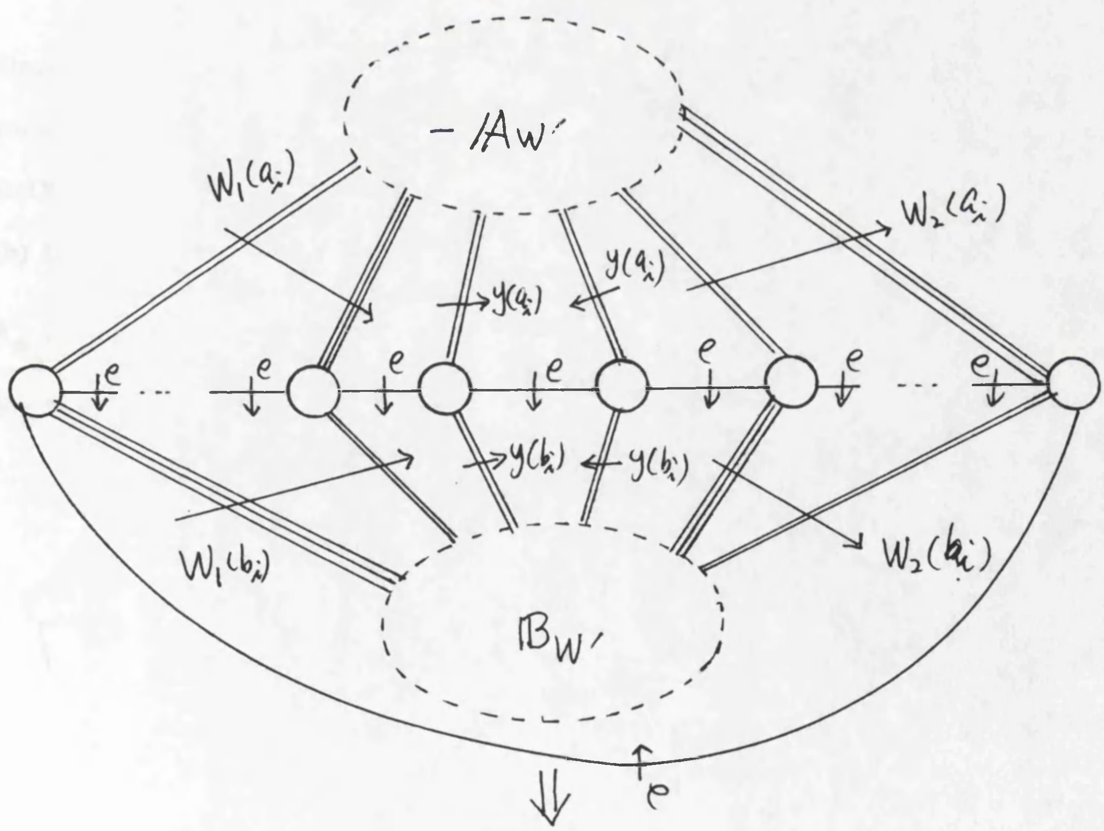
We will need the following Lemma.

Lemma 3.3.7. (a) *If* $\mathbb{P}_W \in Y_e$ *and* W′ *is a word on* $\mathbf{y}_e$ *freely equal to* W *then* $\mathbb{P}_{W'}$ *is equivalent* (*rel*$X_{\iota(e)} \cup X_{\tau(e)}$) *to* $\mathbb{P}_W$.

(b) *If* $\mathbb{P}_{W_1}$ *and* $\mathbb{P}_{W_2}$ *are in* Y_e *then* $\mathbb{P}_{W_1W_2}$ *is equivalent* (*rel*$X_{\iota(e)} \cup X_{\tau(e)}$) *to* $\mathbb{P}_{W_1} + \mathbb{P}_{W_2}$.

(c) *If* $\mathbb{P}_W \in Y$ *and* $U \in \mathbf{w}_e$, *then* $\mathbb{P}_{UWU^{-1}}$ *is equivalent* (*rel*$X_{\iota(e)} \cup X_{\tau(e)}$) *to the picture* $\hat{\mathbb{P}}$ *obtained by surrounding* $\mathbb{P}_W$ *by a collection of concentric closed arcs with total label* $U(a_i)$.

Proof. (a) We may assume that $W = W_1W_2$ and $W' = W_1yy^{-1}W_2$. Then we perform a sequence of bridge moves as follows.



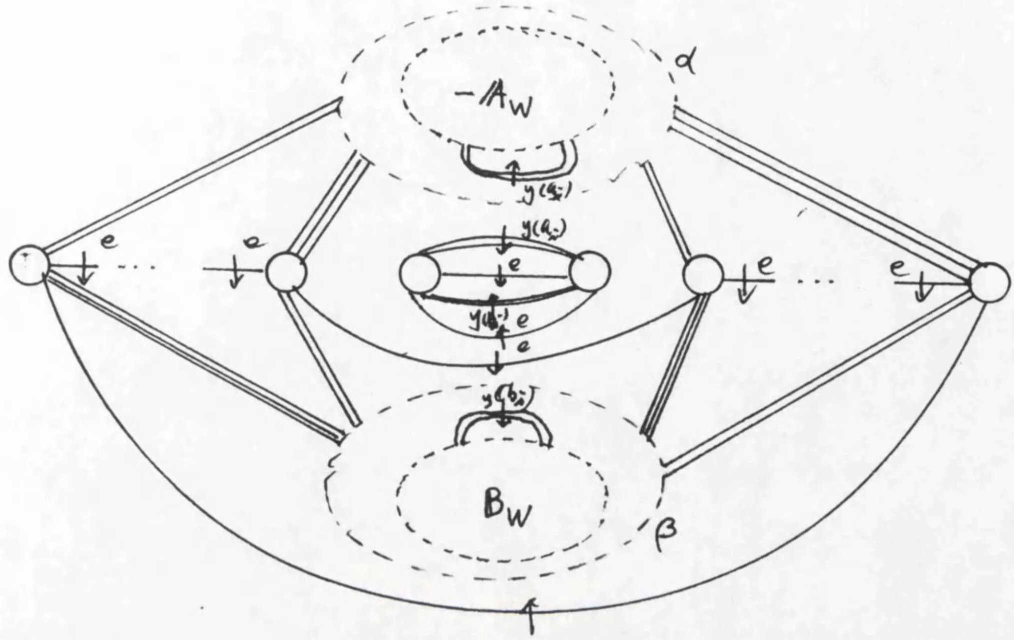


Figure 3.16

Since the labels of α and β are $W(a_i)$ and $W(b_i)$, we can replace subpictures enclosed by α and β with $\mathbb{A}_W$ and $\mathbb{B}_W$ respectively. So $\mathbb{P}_{W'}$ is equivalent (rel$X_{\iota(e)} \cup X_{\tau(e)}$) to $\mathbb{P}_W$.

(b) $\mathbb{B}_{W_1 W_2}$ (resp. $\mathbb{A}_{W_1 W_2}$) is equivalent (rel$X_{\iota(e)} \cup X_{\tau(e)}$) to $\mathbb{B}_{W_1} + \mathbb{B}_{W_2}$ (resp. $\mathbb{A}_{W_1} + \mathbb{A}_{W_2}$). So $\mathbb{P}_{W_1 W_2}$ is equivalent (rel$X_{\iota(e)} \cup X_{\tau(e)}$) to the spherical picture like Fig.3.17.

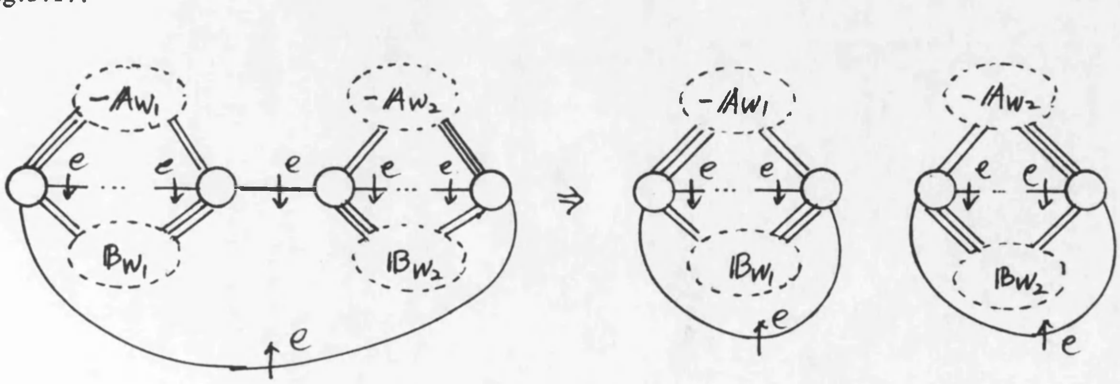


Figure 3.17

By bridge move this is equivalent to $\mathbb{P}_{W_1} + \mathbb{P}_{W_2}$.

(c) By bridge moves, we get the spherical picture like Fig.3.18 from $\mathbb{P}_{UWU^{-1}}$.

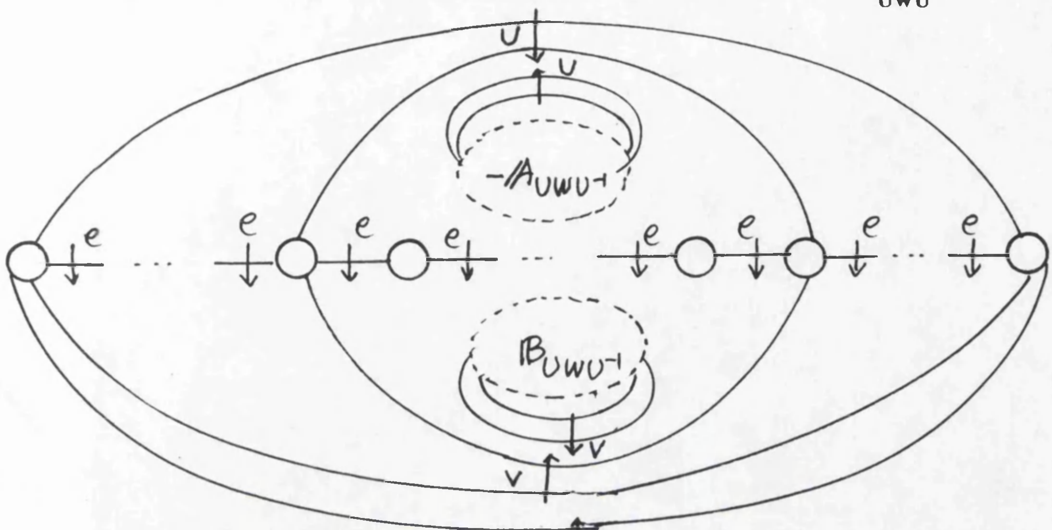
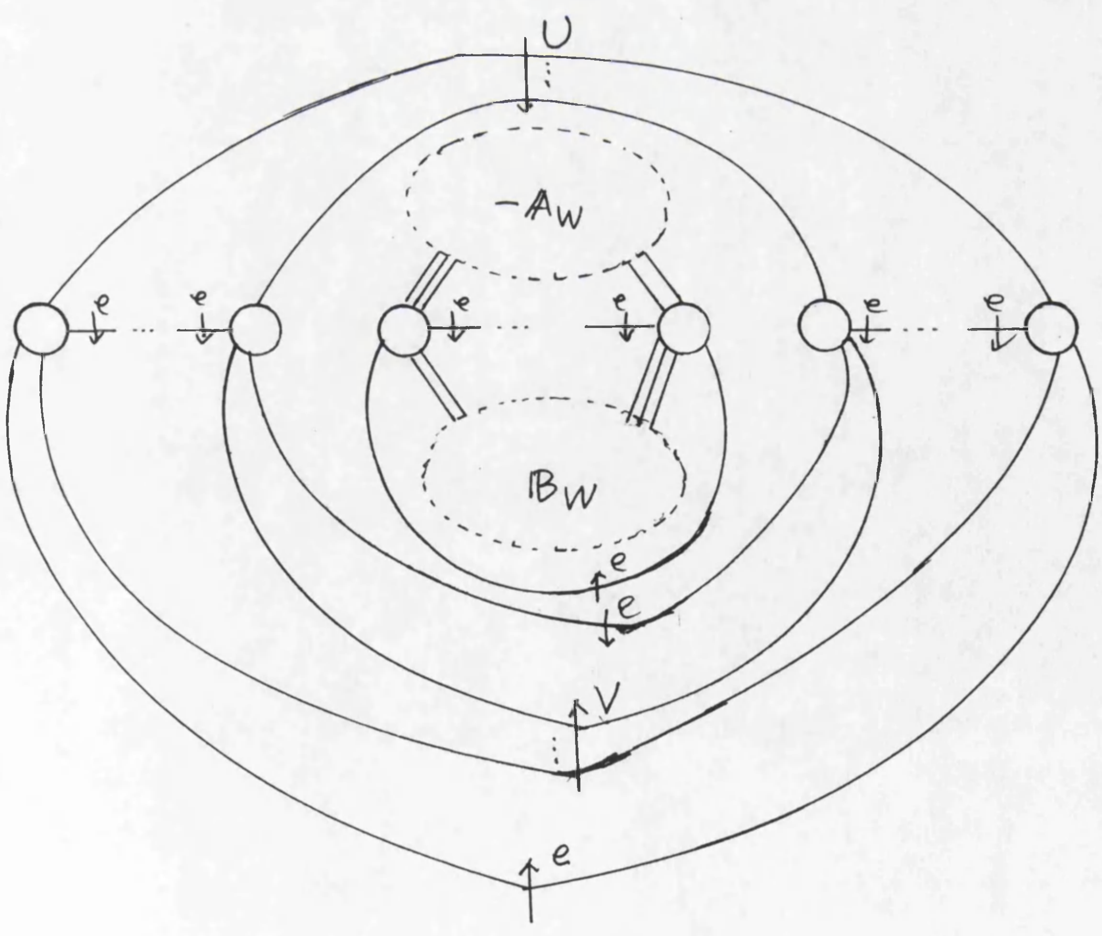


Figure 3.18

Then by replacement of $\mathbb{B}_{UWU^{-1}}$ (resp. $\mathbb{A}_{UWU^{-1}}$) by $\mathbb{B}_W$ (resp. $\mathbb{A}_W$) and by bridge move, we get the spherical picture like Fig.3.19.



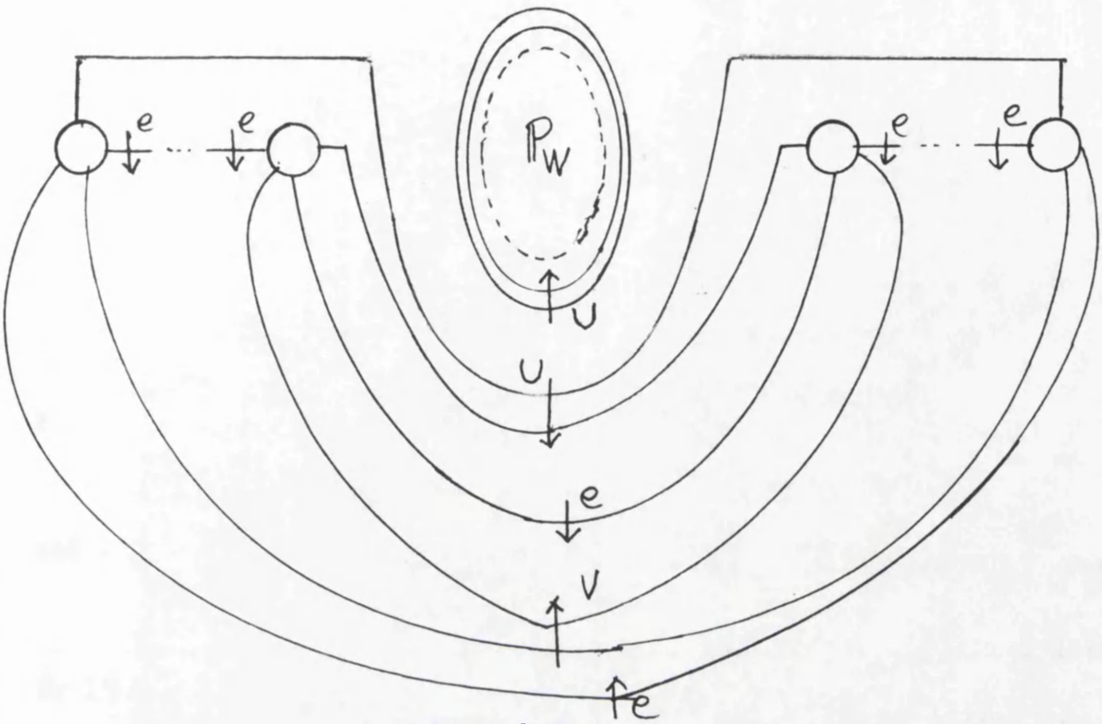


Figure 3.19

Let $< X >$, $< X_v >$ $(v \in \mathbf{v})$ be the submodules of $\pi_2(\mathcal{P})$ generated by X, X_v $(v \in \mathbf{v})$ respectively. Then from Theorem 2.2 we get

$$\mathbb{Z}G \otimes_v \pi_2(\mathcal{P}_v) \cong < X_v >$$

since $\mathcal{P}_v$ is an injective subpresentation of $\mathcal{P}$. So

$$\bigoplus_{v \in \mathbf{v}} (\mathbb{Z}G \otimes_v \pi_2(\mathcal{P}_v)) \cong \bigoplus_{v \in \mathbf{v}} < X_v > .$$

Since the images of the $< X_v >$'s under the standard injection

(3-3) $$\mu_2 : \pi_2(\mathcal{P}) \longrightarrow (\bigoplus_{S \in \mathbf{s}} \mathbb{Z}Gt_S) \oplus (\bigoplus_{e \in \mathbf{e}} (\bigoplus_{i \in I(e)} \mathbb{Z}Gt_{i,e}))$$

have pairwise trivial intersection, we get

$$\bigoplus_{v \in \mathbf{v}} < X_v > \cong < X > .$$

Thus the proof amounts to showing that

$$\bigoplus_{e \in \mathbf{e}} (\mathbb{Z}G \otimes_e N_e / N'_e) \cong \pi_2(\mathcal{P}) / < X > .$$

We define a map

$$N_e \longrightarrow \pi_2(\mathcal{P})/< X >, \quad W \longmapsto < \mathbb{P}_W > + < X >.$$

This is well-defined by Lemma 3.3.7 (a), and by (b) it is a group homomorphism. Since $\pi_2(\mathcal{P})/< X >$ is abelian we get an induced group homomorphism

$$N_e/N'_e \longrightarrow \pi_2(\mathcal{P})/< X >, \quad WN'_e \longmapsto < \mathbb{P}_W > + < X >.$$

By (c), this is a $\mathbb{Z}G_e$-homomorphism. So by tensoring by $\mathbb{Z}G \otimes_e -$ we get a $\mathbb{Z}G$-homomorphism

$$\phi_e: \mathbb{Z}G \otimes_e N_e/N'_e \longrightarrow \pi_2(\mathcal{P})/< X >, \quad 1 \otimes WN'_e \longmapsto < \mathbb{P}_W > + < X >.$$

and adding over all $e \in \mathbf{e}$, we get a $\mathbb{Z}G$-homomorphism

$$\phi : \bigoplus_{e \in \mathbf{e}} (\mathbb{Z}G \otimes_e N_e/N'_e) \longrightarrow \pi_2(\mathcal{P})/< X >.$$

By Theorem 3.2.6, ϕ is surjective.

From (1-3) in §1.3, we get an embedding

$$\mu_1^e : N_e/N'_e \longrightarrow \bigoplus_{i \in I(e)} \mathbb{Z}G_e \bar{t}_{i,e}, \quad WN'_e \longmapsto \sum_{i \in I(e)} \theta_e \frac{\partial W}{\partial y_i} \bar{t}_{i,e}.$$

Applying $\mathbb{Z}G \otimes_e -$ gives an embedding

$$1 \otimes \mu_1^e: \mathbb{Z}G \otimes_e N_e/N'_e \longrightarrow \bigoplus_{i \in I(e)} \mathbb{Z}Gt_{i,e}, \quad 1 \otimes WN'_e \longmapsto \sum_{i \in I(e)} \bar{\theta}_e \frac{\partial W}{\partial y_i} t_{i,e},$$

where $t_{i,e}$ is identified with $1 \otimes_e \bar{t}_{i,e}$ and $\bar{\theta}_e$ is the ring homomorphism induced by the composition

$$F_e \xrightarrow{\theta_e} G_e \xrightarrow{\text{incl}} G.$$

And adding over all $e \in \mathbf{e}$ gives us a $\mathbb{Z}G$-homomorphism

$$1 \otimes \mu: \bigoplus_{e \in \mathbf{e}} (\mathbb{Z}G \otimes_e N_e/N'_e) \longrightarrow \bigoplus_{e \in \mathbf{e}} (\bigoplus_{i \in I(e)} \mathbb{Z}Gt_{i,e}).$$

Since the image of $< X >$ under the standard injection μ_2 in (3-3) lies in $\bigoplus_{S \in \mathbf{s}} \mathbb{Z}Gt_S$, there is an induced homomorphism

$$\bar{\mu}: \pi_2(\mathcal{P})/< X > \longrightarrow \bigoplus_{e \in \mathbf{e}} (\bigoplus_{i \in I(e)} \mathbb{Z}Gt_{i,e}), \quad < \mathbb{P}_W > \longmapsto \sum_{e \in \mathbf{e}} \sum_{i \in I(e)} \bar{\theta}_e \frac{\partial W}{\partial y_i} t_{i,e}.$$

Since $1 \otimes \mu = \bar{\mu}\phi$, and $1 \otimes \mu$ is injective, ϕ is injective. Therefore we get the result.

Chapter 4. Calculation of generators of the second homotopy module II: Split extensions

In this chapter we study the calculation of generators of the second homotopy modules of split extentions, and we give some applications.

4.1 Split extensions

Let K and H be groups, and let

$$\rho: K \longrightarrow \mathrm{Aut}(H)$$

be a homomorphism. We write ρ_k instead of $\rho(k)$. The set of all ordered pairs (h,k) ($h\in H, k\in K$) forms a group $G=H\times_\rho K$ under the binary operation defined by

$$(h,k)\,(h',k') = (h\rho_k(h'),kk') \qquad (h,h'\in H,\ k,k'\in K).$$

We call this group the *split extension* (or *semi-direct product*) *of* H *by* K [39].

Let $H_o = \{(h,1)\ ;\ h\in H\}$. Then H_o is a subgroup of G and is the isomorphic image of H under

$$\mu: h \longmapsto (h,1) \qquad (h\in H).$$

We also have an epimorphism

$$\psi: G \longrightarrow K, \quad (h,k) \longmapsto k \quad (h\in H,\ k\in K)$$

with $\ker\psi=H_o$, so we get

$$K \cong G/H_o.$$

Thus we have a short exact sequence

$$1 \longrightarrow H \xrightarrow{\ \mu\ } G \xrightarrow{\ \psi\ } K \longrightarrow 1\ .$$

Suppose that $\mathscr{H} = <\mathbf{x}\ ;\ \mathbf{s}>$ and $\mathscr{K} = <\mathbf{y}\ ;\ \mathbf{t}>$ are presentations for H and K respectively under the maps

$$x \longmapsto h_x\ (x\in\mathbf{x}), \qquad y \longmapsto k_y\ (y\in\mathbf{y}).$$

Then we have a presentation for G

(4-1) $\mathscr{P} = < \mathbf{x}, \mathbf{y} ; \mathbf{s}, \mathbf{t}, \mathbf{r} >$

where $\mathbf{r} = \{xy\lambda_{xy}^{-1}y^{-1} \mid x \in \mathbf{x}, y \in \mathbf{y}\}$ and λ_{xy} is a word on $\mathbf{x}$ representing the element $\rho_{k_y}^{-1}(h_x)$ of H [28,Proposition 10.1].

Now we consider the group G^* defined by the subpresentation

(4-2) $\mathscr{P}^* = < \mathbf{x}, \mathbf{y} ; \mathbf{s}, \mathbf{r} >$

of $\mathscr{P}$. Then G^* is the fundamental group of a graph of groups, where the underlying graph has a single vertex v and a loop y for each $y \in \mathbf{y}$, and

$$G_v = H, \; G_y = \bar{G}_y = H \; (y \in \mathbf{y}), \quad \gamma_y = \rho_{k_y}^{-1}.$$

Let X_H be the collection of all spherical pictures over $\mathscr{H}$. For $y \in \mathbf{y}$, $S \in \mathbf{s}$, we get the spherical picture over $\mathscr{P}^*$ of the form depicted in Fig.4.1

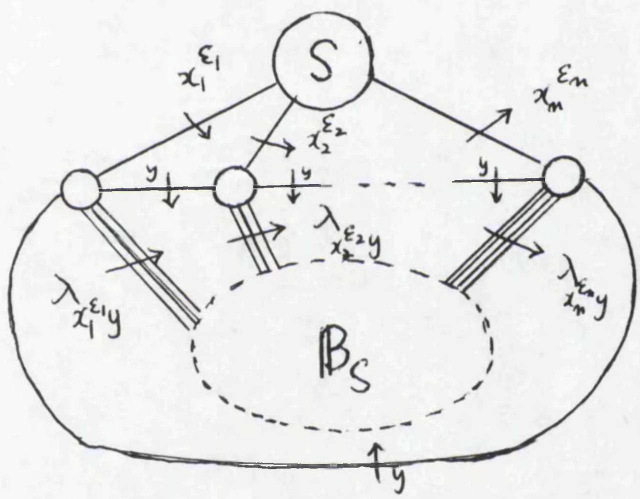


Figure 4.1

where $\mathbb{B}_S$ is a picture over $\mathscr{H}$. Let Y_y be the collection of all spherical pictures $\mathbb{P}_{S,y}$ $(S \in \mathbf{s})$.

Lemma 4.1.1. *$\pi_2(\mathscr{P}^*)$ is generated by* $X_H \cup Y$, *where* $Y = \bigcup_{y \in \mathbf{y}} Y_y$.

Proof. By Theorem 3.2.6.

Let $\mathbf{t}^{\#}$ be the set of all cyclic permutations of elements of $\mathbf{t} \cup \mathbf{t}^{-1}$ ending with an element of $\mathbf{y}^{-1}$ (rather than with an element of $\mathbf{y}$). For each $T \in \mathbf{t}^{\#}$, say

$$T = y_1^{\varepsilon_1} \ldots y_n^{\varepsilon_n} y^{-1}$$

$(y,\ y_i \in \mathbf{y},\ \varepsilon_i = \pm 1,\ i = 1,\ldots,n)$, then

$$\rho_{k_{y_1}}^{\varepsilon_1} \ldots \rho_{k_{y_n}}^{\varepsilon_n} \rho_{k_y}^{-1}$$

is the identity of Aut(H). Thus for any word V on **x**,

$$Uy^{-1}VyU^{-1}V^{-1}$$

represents the identity of G^*, where $U = y_1^{\varepsilon_1}\ldots y_n^{\varepsilon_n}$. Hence taking V to be $x \in \mathbf{x}$, and noting that $y^{-1}xy = \lambda_{xy}$ in G^*, we get that $U\lambda_{xy}U^{-1}x^{-1}$ represents 1 in G^*. So we get a picture $\mathbb{D}_{T,x}$ over $\mathcal{P}^*$ with the boundary label $U\lambda_{xy}U^{-1}x^{-1}$. Such a picture is unique up to equivalence (rel $X_H \cup Y$) by an similar argument to that in §3.2.1. We can then get a spherical picture $\mathbb{P}_{T,x}$ over $\mathcal{P}$ of the form depicted in Fig.4.2.

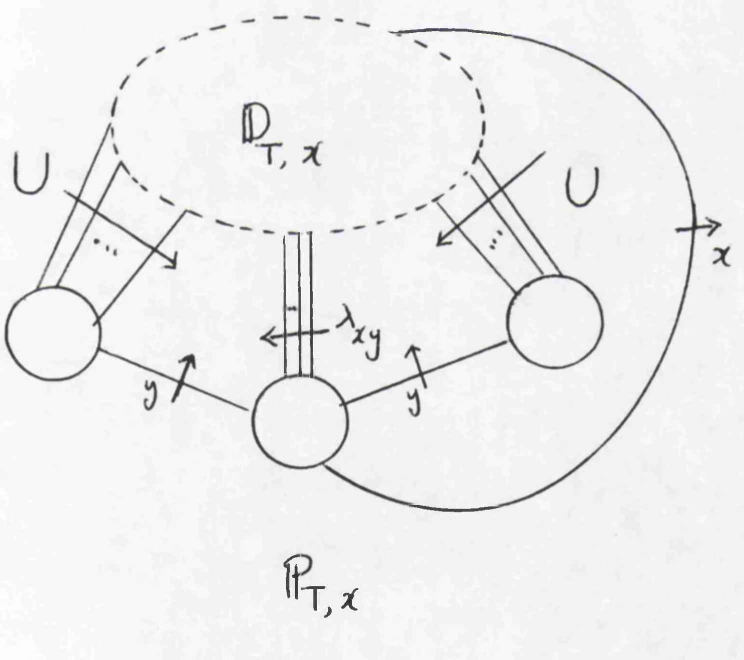


Figure 4.2

Let Z be the collection of all spherical pictures $\mathbb{P}_{T,x}$ $(x \in \mathbf{x}, T \in \mathbf{t}^{\#})$ and X_K a collection of spherical pictures over $\mathcal{K}$ which generate $\pi_2(\mathcal{K})$.

Lemma 4.1.2. *Every spherical picture over* $\mathcal{P}$ *is equivalent (rel* $Z \cup X_K$*) to a picture over* $\mathcal{P}^*$.

Proof. We will prove it by a similar procedure as in the proof of Lemma 3.1.2.

Let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$ and assume that $\mathbb{P}$ has at least one t-disc. Let α be a simple closed transverse path in $\mathbb{P}$ such that all discs lying inside α are t-discs, and all arcs inside α are y-arcs. We also require that at least one disc lies inside α.

Any arc β meeting α is labelled by an element $y \in \mathbf{y}$, and is the beginning of a path of arcs labelled by y and non t-discs $\Delta_1, \ldots, \Delta_{n-1} (n \geq 1)$ in the exterior of α, ending *either* (i) with a t-disc Δ in the exterior of α, *or* (ii) with an arc β', labelled y, incident at Δ_{n-1} and also meeting α.

We assume that possibility (i) occurs and we will show how to move Δ into the interior of α. Suppose that $n > 1$ and the labels of Δ and Δ_{n-1} are $(y_1^{\varepsilon_1} \ldots y_m^{\varepsilon_m} y^{-1})^{-1}$ and $x^{-1} y \lambda_{xy} y^{-1}$ respectively. Then our situation is like Fig.4.3.

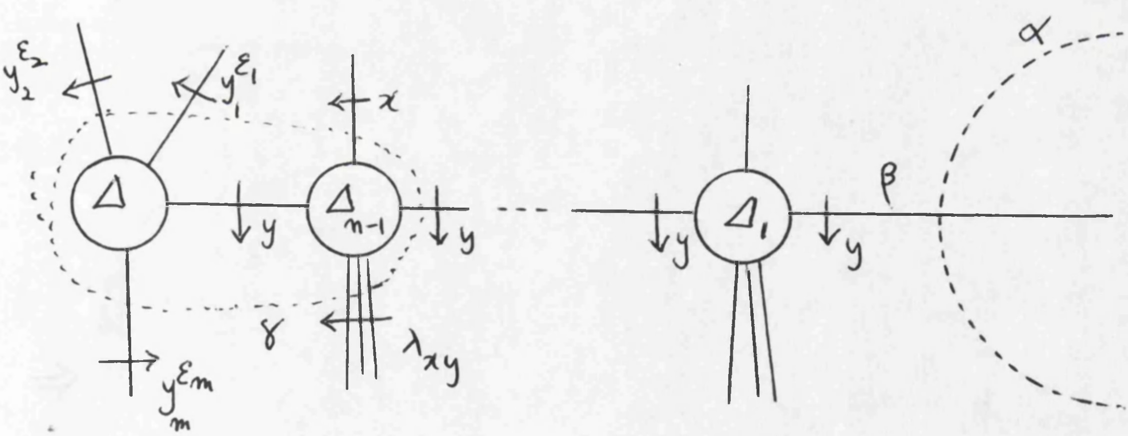


Figure 4.3

Now we draw a simple closed curve γ such that γ encloses only Δ and Δ_{n-1}. Next insert an element of $\{\mathbb{P}_{T,x}\}^*$ inside γ, where $T = y_1^{\varepsilon_1} \ldots y_m^{\varepsilon_m} y^{-1}$. Then by a sequence of bridge moves, we have two cancelling pairs which are removed. And the subpicture of $\mathbb{P}$ outside γ and the subpicture of $\mathbb{P}_{T,x}$ consisting of $\mathbb{D}_{T,x}$ and one t-disc make the subpicture of another spherical picture $\mathbb{P}'$ over $\mathscr{P}$ like Fig.4.4.

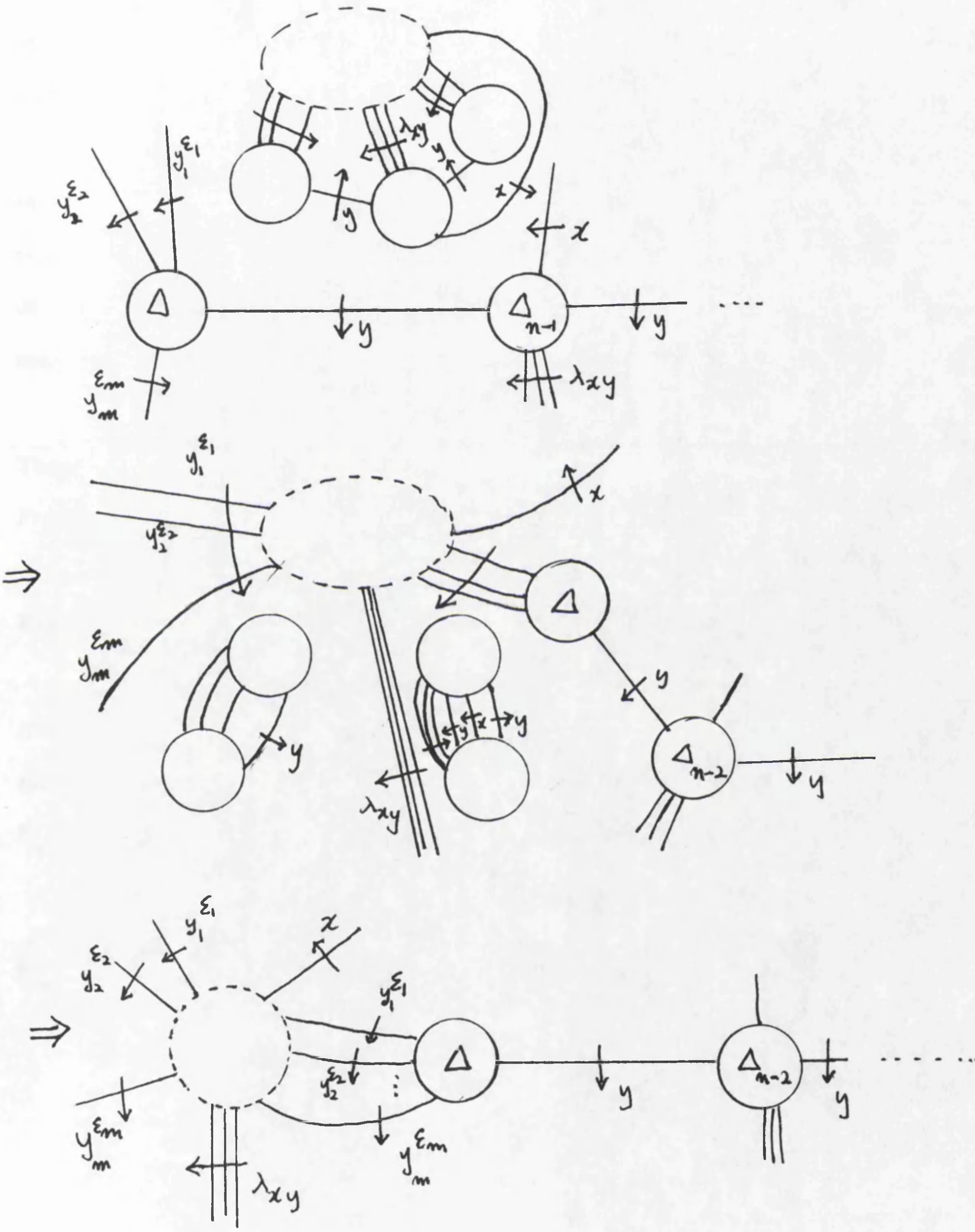


Figure 4.4

We note that the number of t-discs is not changed and that Δ moves closer to α by the above operation. Also we can perform the same operation to the case that the labels of Δ, Δ_{n-1} are $(y_1^{\varepsilon_1}\dots y_m^{\varepsilon_m} y^{-1})^{-1}$ and $(x^{-1}y\lambda_{xy}y^{-1})^{-1}$ by an element of $\{\mathbb{P}_{T,x}\}^*$. After a finite number of the above operations, we may eventually move Δ inside α.

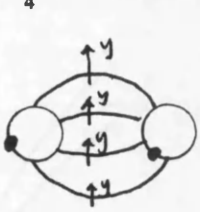
In this way, we can assume that possibility (i) does not occur. By the same argument as in Lemma 3.1.2, the label of α is then a word in y freely equal to 1. A sequence of bridge moves near α creates a spherical picture over $\mathcal{K}$ which can be removed. Repeating the above performance, we get the conclusion.

Theorem 4.1.3. *$\pi_2(\mathcal{P})$ is generated by* $Z \cup Y \cup X_H \cup X_K$.

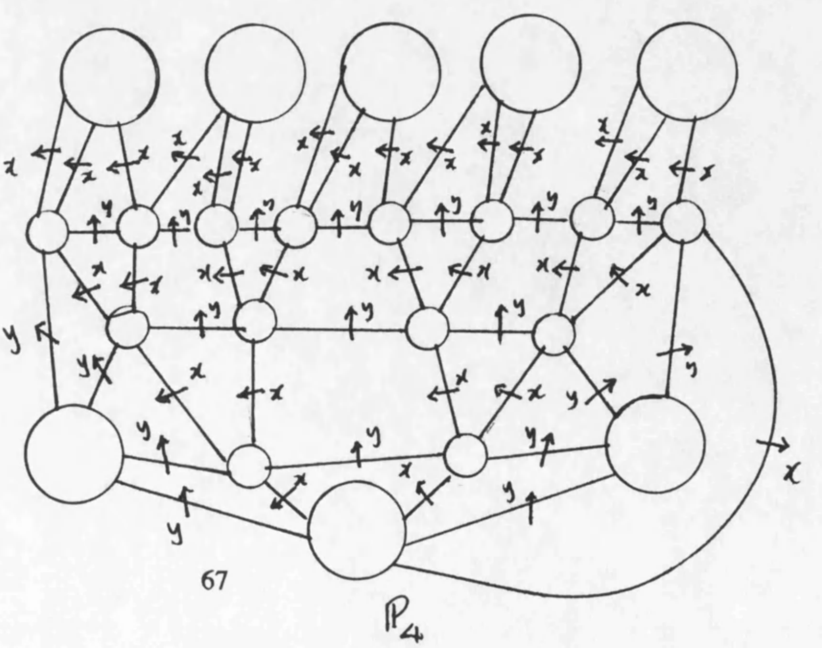
Proof. By Lemmas 4.1.1 and 4.1.2.

Example 4.1.4. Let

$\mathcal{P} = \langle x, y \,;\, x^n, y^m, xyx^{-q}y^{-1} \rangle$. where $q^m - 1 = \alpha n$ (α is a positive integer). By Theorem 4.1.3, a set of generators of $\pi_2(\mathcal{P})$ consists of the two pictures given at the start of § 3.3.2 together with two further pictures $\mathbb{P}_3$, $\mathbb{P}_4$. The pictures $\mathbb{P}_3$, $\mathbb{P}_4$ are illustrated below for the case n=3, m=4, q=2.



$\mathbb{P}_3$



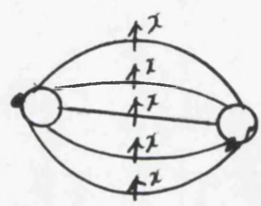
$\mathbb{P}_4$

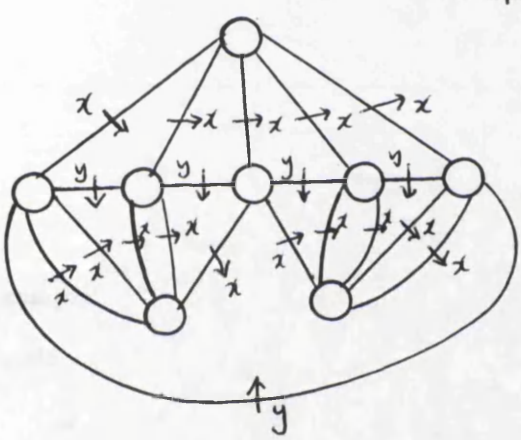
Example 4.1.5. Let

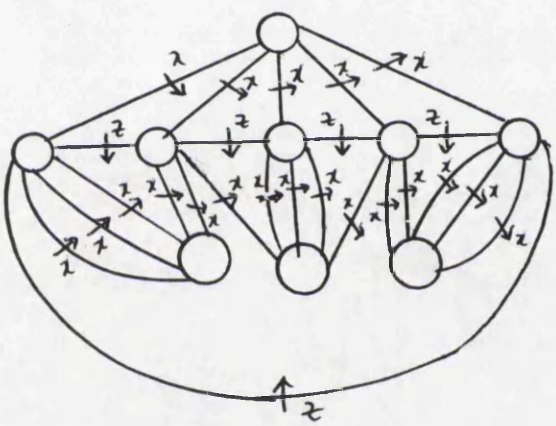
$$\mathscr{H} = \langle\, x \,;\, x^{m} \,\rangle, \quad \mathscr{K} = \langle\, y, z \,;\, yzy^{-1}z^{-1} \,\rangle$$

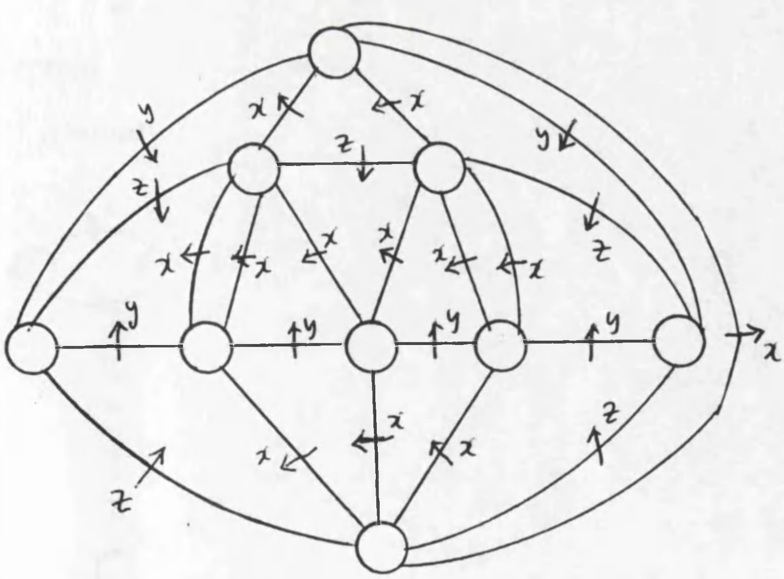
$$\mathscr{P} = \langle\, x, y, z \,;\, x^{m},\ yzy^{-1}z^{-1},\ xyx^{-p}y^{-1},\ xzx^{-q}z^{-1} \,\rangle,\ (p,m)=(q,m)=1.$$

Then by Theorem 4.1.3 we get a set of generators for $\pi_2(\mathscr{P})$ consisting of the following seven spherical pictures. We illustrate it for the case m=5,p=2,q=3.

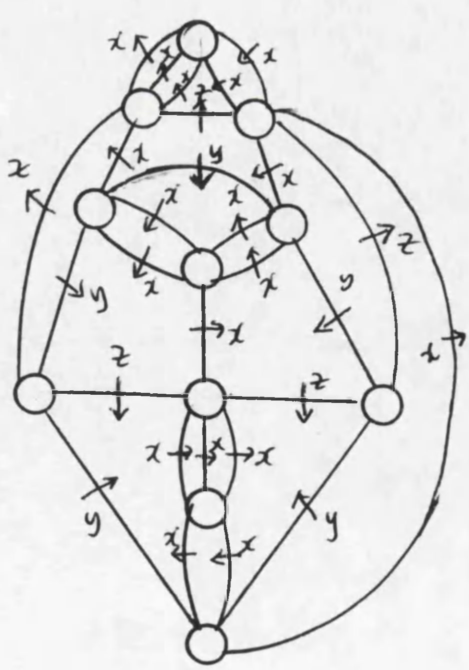




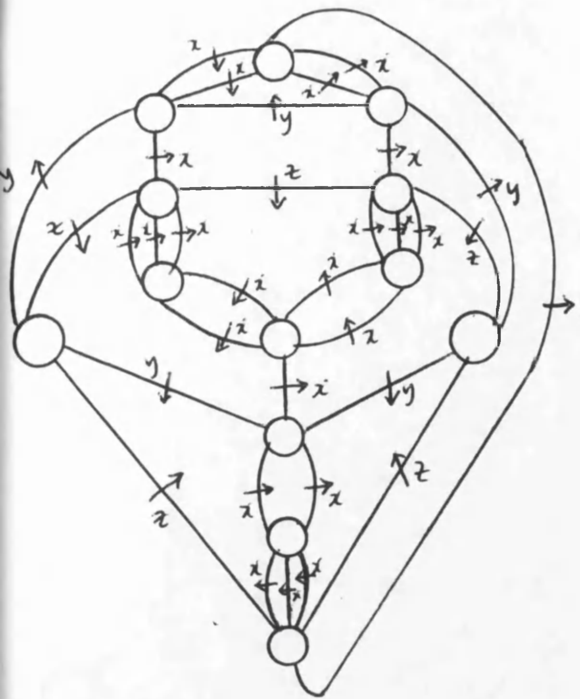




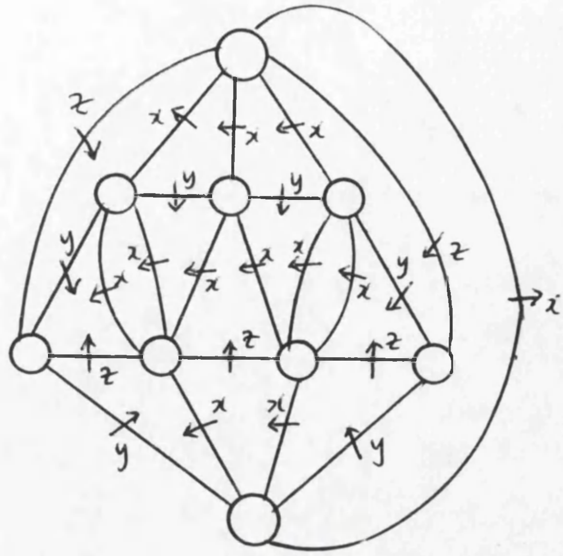
$\mathbb{P}_{yzy^{-1}z^{-1},\, x}$



$\mathbb{P}_{z^{-1}yzy^{-1},\, x}$



$\mathbb{P}_{y^{-1}zyz^{-1},\, x}$



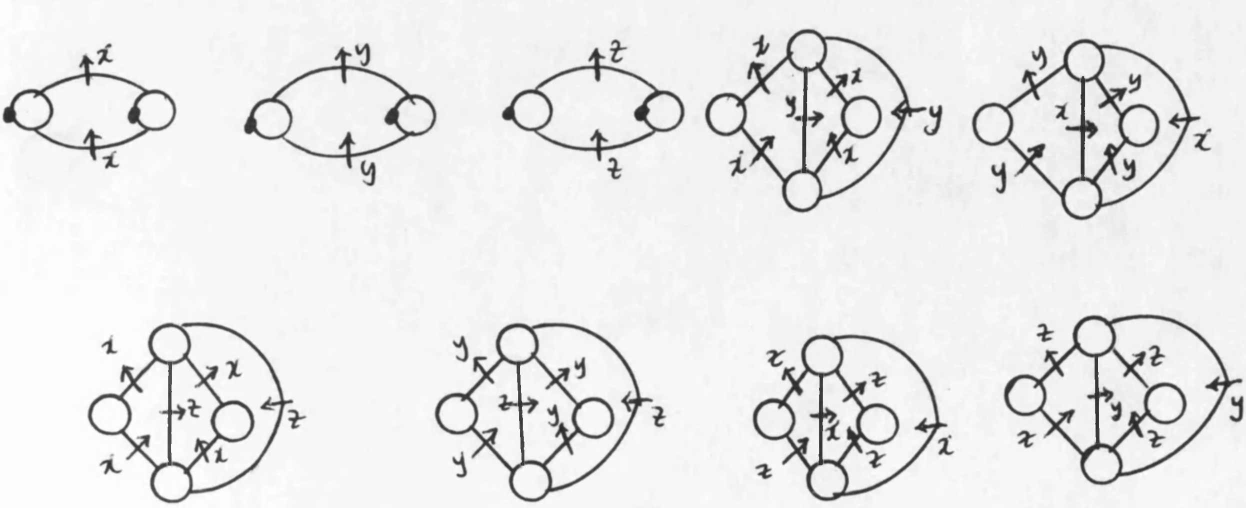
$\mathbb{P}_{zyz^{-1}y^{-1},\, x}$

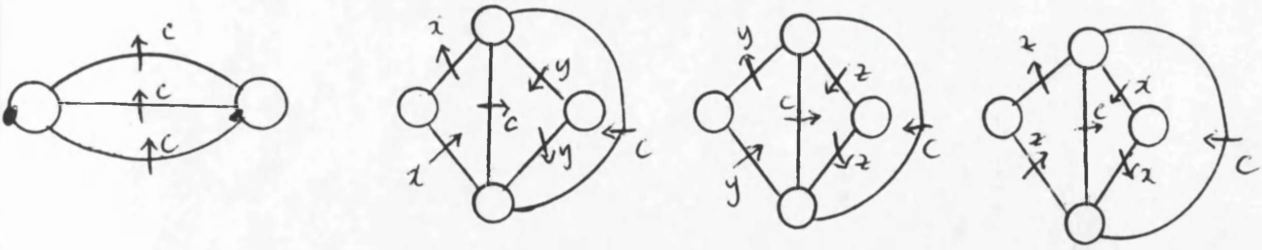
Example 4.1.6. Theorem 4.1.3, of course, applies to wreath products. As a simple example let

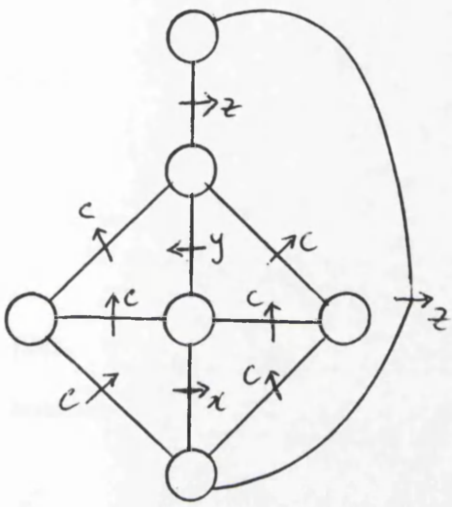
$$\mathcal{H} = < x, y\ z\ ;\ x^2,\ y^2,\ z^2,\ [x,y],\ [y,z],\ [z,x] >, \quad \mathcal{K} = < c\ ;\ c^3 >$$

$$\mathcal{P} = < x,y,z,c;x^2,y^2,z^2,[x,y],[y,z],[z,x],c^3,xcyc^{-1},yczc^{-1},zcxc^{-1} >$$

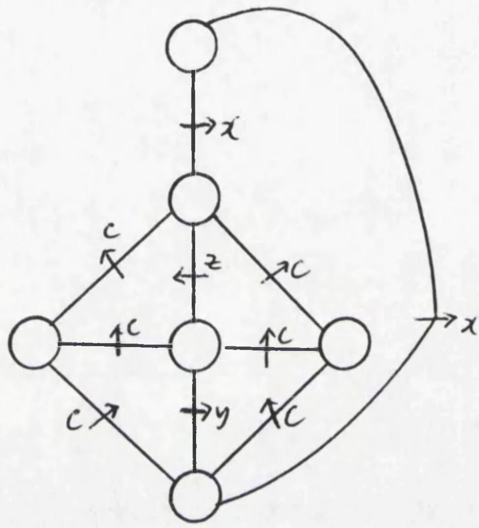
Then we get a set of generators for $\pi_2(\mathcal{P})$ consisting of the following spherical pictures.



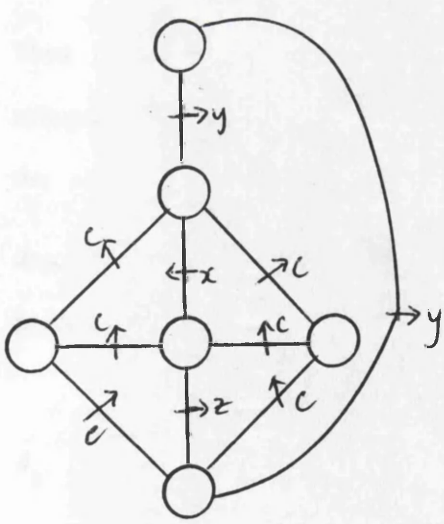




$\mathbb{P}_{c^{-3},x}$



$\mathbb{P}_{c^{-3},y}$



$\mathbb{P}_{c^{-3},z}$

4.2. Applications

In this section we describe the second integral (co)homology of a split extension, and consider necessary and sufficient conditions for its presentation to be Cockroft.

4.2.1 Calculation of H_2 and H^2

Let $G = H\times_\rho K$. Then there are homomorphisms

$$G \underset{\varphi}{\overset{\psi}{\rightleftarrows}} K \qquad \psi\varphi=1,$$

[28]. Since $H_n(\ )$ is a covariant functor [26,p188], we get the induced homomorphisms

$$H_n(G) \underset{\varphi_*}{\overset{\psi_*}{\rightleftarrows}} H_n(K) \qquad \psi_*\varphi_*=1.$$

Since $H_n(G)$, $H_n(K)$ are abelian and φ_* is injective, we get

$$\begin{aligned} H_n(G) &\cong \mathrm{Im}\varphi_* \oplus \mathrm{Ker}\psi_* \\ &\cong H_n(K) \oplus \mathrm{Ker}\psi_*. \end{aligned}$$

Thus $H_n(K)$ is a direct summand in $H_n(G)$. In a similar way, $H^n(K)$ is a direct summand in H^n. In the case of second homology, a theoretical description of the complement C of $H_2(K)$ in $H_2(G)$ has been in [43] (see also [24]). We can describe the complements of $H_2(K)$ and $H^2(K)$ in $H_2(G)$ and $H^2(G)$ more practically.

From Theorem 4.1.3, (1-5),(1-6),(1-7),(1-8) in §1.4 are given as follows.

$$\delta_3: \Big(\bigoplus_{\mathbb{P}\in Y\cup Z} \mathbb{Z}t_{\mathbb{P}}\Big) \oplus \Big(\bigoplus_{\mathbb{P}\in X_H} \mathbb{Z}t_{\mathbb{P}}\Big) \oplus \Big(\bigoplus_{\mathbb{P}\in X_K} \mathbb{Z}t_{\mathbb{P}}\Big) \longrightarrow \Big(\bigoplus_{S\in s} \mathbb{Z}t_S\Big) \oplus \Big(\bigoplus_{R\in r} \mathbb{Z}t_R\Big) \oplus \Big(\bigoplus_{T\in t} \mathbb{Z}t_T\Big)$$

$$t_{\mathbb{P}} \longmapsto \sum_{S\in s} \exp_S(\mathbb{P})t_S \qquad (\mathbb{P}\in X_H)$$

$$t_{\mathbb{P}} \longmapsto \sum_{T\in t} \exp_T(\mathbb{P})t_T \qquad (\mathbb{P}\in X_K)$$

$$t_{\mathbb{P}} \longmapsto \sum_{R\in r} \exp_R(\mathbb{P}) + \sum_{S\in s} \exp_S(\mathbb{P}) \qquad (\mathbb{P}\in Y\cup Z)$$

$$\delta_2 : (\bigoplus_{S\in s} \mathbb{Z}t_S) \oplus (\bigoplus_{R\in r} \mathbb{Z}t_R) \oplus (\bigoplus_{T\in t} \mathbb{Z}t_T) \longrightarrow (\bigoplus_{x\in x} \mathbb{Z}t_x) \oplus (\bigoplus_{y\in y} \mathbb{Z}t_y)$$

$$t_S \longmapsto \sum_{x\in x} \exp_x(S)t_x$$

$$t_R \longmapsto \sum_{x\in x} (1-\exp_x(\lambda_{xy}))t_x$$

$$t_T \longmapsto \sum_{y\in y} \exp_y(T)t_y$$

$$\delta_3^* : (\bigoplus_{S\in s} \mathbb{Z}t_S^*) \oplus (\bigoplus_{R\in r} \mathbb{Z}t_R^*) \oplus (\bigoplus_{T\in t} \mathbb{Z}t_T^*) \longrightarrow (\bigoplus_{\mathbb{P}\in Y\cup Z} \mathbb{Z}t_{\mathbb{P}}^*) \oplus (\bigoplus_{\mathbb{P}\in X_H} \mathbb{Z}t_{\mathbb{P}}^*) \oplus (\bigoplus_{\mathbb{P}\in X_K} \mathbb{Z}t_{\mathbb{P}}^*)$$

$$t_S^* \longmapsto \sum_{\mathbb{P}\in Y\cup Z\cup X_H} \exp_S(\mathbb{P})t_{\mathbb{P}}^*$$

$$t_R^* \longmapsto \sum_{\mathbb{P}\in Y\cup Z} \exp_R(\mathbb{P})t_{\mathbb{P}}^*$$

$$t_T^* \longmapsto \sum_{\mathbb{P}\in X_K} \exp_T(\mathbb{P})t_{\mathbb{P}}^*$$

$$\delta_2^* : (\bigoplus_{x\in x} \mathbb{Z}t_x^*) \oplus (\bigoplus_{y\in y} \mathbb{Z}t_y^*) \longrightarrow (\bigoplus_{S\in s} \mathbb{Z}t_S^*) \oplus (\bigoplus_{R\in r} \mathbb{Z}t_R^*) \oplus (\bigoplus_{T\in t} \mathbb{Z}t_T^*)$$

$$t_x^* \longmapsto \sum_{S\in s} \exp_x(S)t_S^* + \sum_{R\in r} \exp_x(R)t_R^*$$

$$t_y^* \longmapsto \sum_{T\in t} \exp_y(T)t_T^*.$$

So, we get

$$H_2(G) = \mathrm{Ker}\delta_2/\mathrm{Im}\delta_3 \qquad H_2(G) = \mathrm{Ker}\delta_3^*/\mathrm{Im}\delta_2^*.$$

Proposition 4.2.1. (i) $H_2(G) \cong H_2(K) \oplus \mathrm{Ker}\hat{\delta}_2/\mathrm{Im}\hat{\delta}_3$.

(ii) $H^2(G) \cong H^2(K) \oplus \mathrm{Ker}\hat{\delta}_3^*/\mathrm{Im}\hat{\delta}_2^*$.

where $\hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_2^*, \hat{\delta}_3^*$ *are the restrictions of* $\delta_2, \delta_3, \delta_2^*, \delta_3^*$ *on* $(\bigoplus_{R\in r} \mathbb{Z}t_R) \oplus (\bigoplus_{S\in s} \mathbb{Z}t_S)$, $(\bigoplus_{\mathbb{P}\in Y\cup Z} \mathbb{Z}t_{\mathbb{P}}) \oplus (\bigoplus_{\mathbb{P}\in X_H} \mathbb{Z}t_{\mathbb{P}})$, $(\bigoplus_{S\in s} \mathbb{Z}t_S^*) \oplus (\bigoplus_{R\in r} \mathbb{Z}t_R^*)$, $\bigoplus_{x\in x} \mathbb{Z}t_x^*$ *respectively.*

Example 4.2.2. Consider the same presentation as in Example 4.1.4.

$$\mathcal{P} = \langle x,\ y\ ;\ S,\ T,\ R \rangle$$

where $S=x^n$, $T=y^m$, $R=xyx^{-q}y^{-1}$. Then we get

$$\delta_3:\quad t_{\mathbb{P}_1} \longmapsto 0 \qquad\qquad \delta_3^*:\quad t_S^* \longmapsto (q-1)t_{\mathbb{P}_2}^* + \alpha t_{\mathbb{P}_4}^*$$

$$t_{\mathbb{P}_2} \longmapsto (q-1)t_S + nt_R \qquad\qquad t_T^* \longmapsto 0$$

$$t_{\mathbb{P}_3} \longmapsto 0 \qquad\qquad t_R^* \longmapsto nt_{\mathbb{P}_2}^* + \frac{q^m-1}{q-1} t_{\mathbb{P}_4}^*$$

$$t_{\mathbb{P}_4} \longmapsto \frac{q^m-1}{q-1} t_R + \alpha t_S \qquad\qquad \delta_2^*:\quad t_x^* \longmapsto nt_S^* + (1-q)t_R^*$$

$$\delta_2:\quad t_S \longmapsto nt_x \qquad\qquad t_x^* \longmapsto nt_S^* + (1-q)t_R^*$$

$$t_T \longmapsto mt_y \qquad\qquad t_y^* \longmapsto mt_T^*$$

$$t_R \longmapsto (1-q)t_x$$

Suppose that $k_1nt_x + k_2mt_y + k_3(1-q)t_x = 0$

$$l_1\left(nt_{\mathbb{P}_2}^* + \frac{q^m-1}{q-1} t_{\mathbb{P}_4}^*\right) + l_2\left((q-1)t_{\mathbb{P}_2}^* + \alpha t_{\mathbb{P}_4}^*\right) = 0.$$

Then

$$k_2 = 0$$

$$nk_1 + (1-q)k_3 = 0$$

and $\quad l_1n + l_2(q-1) = 0.$

Let $\kappa_1 = (q-1, n, \frac{\alpha n}{q-1}, \alpha)$ and $\kappa_2 = (\frac{q-1}{\kappa_1}, \frac{n}{\kappa_1})$. Then

$\mathrm{Ker}\delta_2$ is generated by $\xi = \frac{q-1}{\kappa_1\kappa_2} t_S + \frac{n}{\kappa_1\kappa_2} t_T$.

$\mathrm{Im}\delta_3$ is generated by $\kappa_1\kappa_2\xi$ and $\kappa_1\kappa_3\xi = \alpha t_S + \frac{q^m-1}{q-1} t_R$, where $\kappa_3 = \alpha\frac{\kappa_2}{q-1}$.

$\mathrm{Ker}\delta_3^*$ is generated by t_T^*, $\frac{n}{\kappa_1\kappa_2} t_S^* + \frac{1-q}{\kappa_1\kappa_2} t_R^*$.

$\mathrm{Im}\delta_2^*$ is generated by $nt_S^* + (1-q)t_R^*$, mt_T^*.

Therefore

$$H_2(G) \cong \mathbb{Z}_{\kappa_1} \quad \text{and} \quad H^2(G) \cong \mathbb{Z}_m \oplus \mathbb{Z}_{\kappa_1\kappa_2}.$$

Thus $\mathcal{P}$ is efficient if and only if $\kappa_1 \neq 1$. But when $\kappa_1 = 1$, an efficient presentation was given by Wamsley[45] and Beyl[3].

4.2.2 Cockroft property

Theorem 4.2.3. *$\mathcal{P}$ is Cockroft if and only if the following conditions hold:*

(i) *$\mathcal{H}$ and $\mathcal{K}$ are Cockroft.*

(ii) $\mathbf{s}\subseteq F'$, *where* F *is the free group on* $\mathbf{x}$.

(iii) *For all* $S\in\mathbf{s}$, *for each* $W, \exp_S(\mathbb{P}_W)=0$, *where* W *defines* 1 *in* H.

(iv) *For each* $\mathbb{D}_{T,x}$

$$\exp_R(\mathbb{D}_{T,x}) = -1 \qquad (R=xy\lambda_{xy}^{-1}y^{-1})$$

$$\exp_{R'}(\mathbb{D}_{T,x})=\exp_S(\mathbb{D}_{T,x})=0 \qquad (R'\neq R\ (R'\in\mathbf{r}), S\in\mathbf{s}).$$

Proof. ($\Rightarrow$) By Theorem 4.1.3, for each $\mathbb{P}\in Z\cup X_K\cup X_H\cup Y$, $\exp_S(\mathbb{P})=\exp_T(\mathbb{P})=\exp_R(\mathbb{P})$ $=0$. So we get the following:

(a) $\mathbb{P}\in X_H\cup X_K$ then (i) must hold,

(b) $\mathbb{P}\in Y$ then (ii) and (iii) must hold,

(c) $\mathbb{P}\in Z$ then (iv) must hold.

($\Leftarrow$) By Theorem 4.1.3 and the reverse argument of the above.

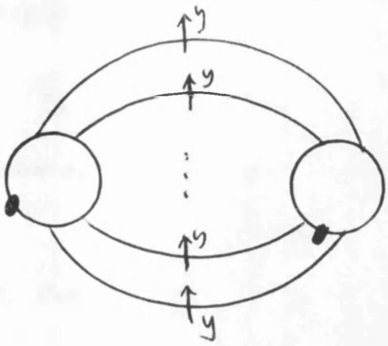
Example 4.2.4. Let

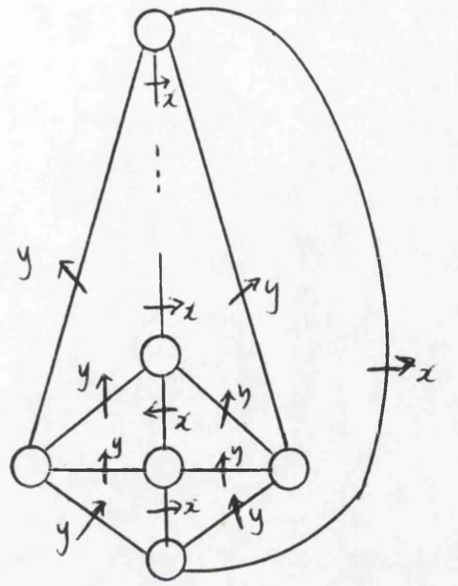
$$\mathcal{H}=\langle x,z; xzx^{-1}z^{-1}\rangle$$

$$\mathcal{K}=\langle y; y^{2m}\rangle$$

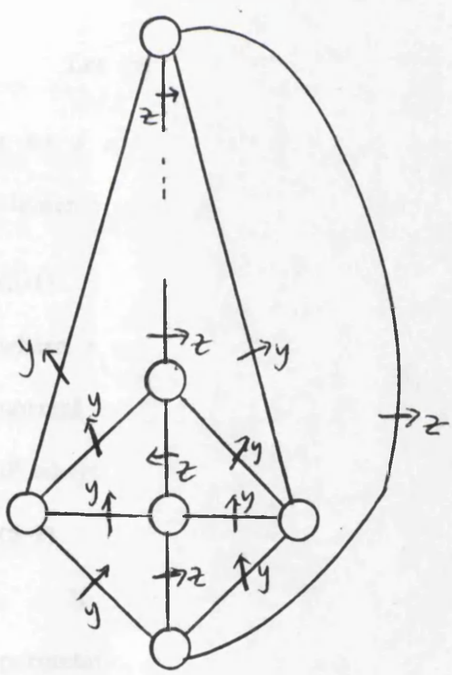
$$\mathcal{P}=\langle x,y,z;\ xzx^{-1}z^{-1}, y^{2m}, xyxy^{-1}, zyzy^{-1}\rangle.$$

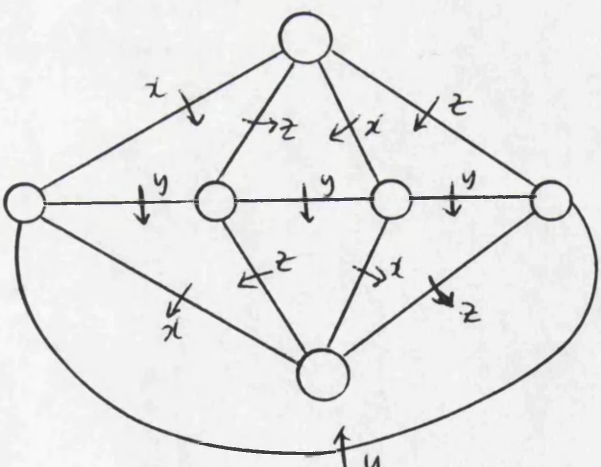
Then $\mathcal{P}$ is Cockroft because a generating set of $\pi_2(\mathcal{P})$ is as follows:





$P_{y^{-2m}, x}$





$P_{y^{-2m}, z}$

Chapter 5 Relative presentation

In this chapter we study when relative presentations with one defining relator are aspherical.

5.1 Preliminaries

In this section we introduce the notation, the definitions and the techniques required for § 5.2 and § 5.3 and the important theorems for relative presentations.

5.1.1 Relative presentations

Let H be a group and $< \mathbf{x} >$ the free group on a set $\mathbf{x}$ of symbols, and let $\mathbf{r}$ be a subset of the free product $H * < \mathbf{x} >$ consisting of cyclically reduced elements of the form

(5-1) $\quad x_1^{\varepsilon_1} h_1 x_2^{\varepsilon_2} h_2 \ldots x_n^{\varepsilon_n} h_n$

where $x_i \in \mathbf{x}$, $\varepsilon_i = \pm 1$, $h_i \in H$ $(i=1,\ldots,n)$. The quotient G of $H * < \mathbf{x} >$ by the normal closure N of $\mathbf{r}$ is called *the group defined by the relative presentation* $\mathcal{P}$ where $\mathcal{P}$ is the triple

(5-2) $\quad < H, \mathbf{x} ; \mathbf{r} >.$

If $\mathbf{s}$ is a subset of $\mathbf{r}$ then we denote by $\mathbf{s}^*$ the set of all cyclic permutations of elements of $\mathbf{s} \cup \mathbf{s}^{-1}$ of the form (5-1), that is, all cyclic permutations which begin with an $\mathbf{x}$ -symbol.

For $R \in \mathbf{r}^*$ write $R = Sh$ where $h \in H$ and S begins and ends with $\mathbf{x}$ -symbols. We let

(5-3) $\quad \bar{R} = S^{-1}h^{-1}.$

If R is an element of $\mathbf{r}^*$ then R can be written in the form $\hat{R}^{p(R)}$ where $\hat{R}$

is not proper power, and p(R) is a positive integer. We call $\hat{R}$ the *root* of R, and p(R) the *period.*

We will say that $\mathcal{P}$ is *orientable* if for each $R \in \mathbf{r}$, $\{R\}^* \cap \mathbf{r} = \{R\}$ and no element of **r** is a cyclic permutation of its inverse. From now, we will assume that presentations in this chapter are orientable because we treat only orientable presentations.

5.1.2 Relative pictures over relative presentations

We prepare the concept of *a relative picture over a relative presentation.* Fix a relative presentation $\mathcal{P} = < H, \mathbf{x} ; \mathbf{r} >$. A relative picture $\mathbb{P}$ has the same geometric shape as an ordinary picture as in §1.2 ,but the labelling is different and additional conditions are needed.

A (relative) picture $\mathbb{P}$ is labelled as follows.

Each arc is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$ and each corner of $\mathbb{P}$ is to be oriented clockwise (with respect to the ambient disc of $\mathbb{P}$) and labelled by an element of H. If c is a corner of Δ then we denote by W(c) the word obtained by reading in clockwise order the labels on the arcs and corners meeting $\partial\Delta$ beginning with the label on the arc at the head of the clockwise oriented corner c. And the following two conditions are satisfied:

(i) For each corner c of $\mathbb{P}$, $W(c) \in \mathbf{r}^*$.

(ii) If $h_1,\ldots,h_m$ is the sequence of corner labels encountered in a clockwise traversal of the boundary of an inner region of $\mathbb{P}$, then $h_1\ldots h_m = 1$ in H.

Example 5.1.1. Let $\mathcal{P} = <H, x ; x^3axa >$ and $a^4=1$ in H.

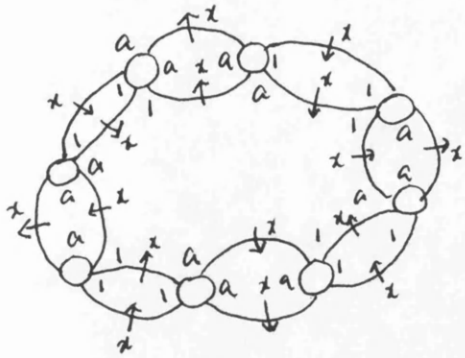


Figure 5.1

Remark. An ordinary presentation can be regarded as a relative presentation with H={1}, and an ordinary picture also can be regarded as a relative picture where every corner is labelled by 1.

5.1.3 Asphericity

Let $\mathscr{P}$ be a relative presentation. A *dipole* in a picture over $\mathscr{P}$ consists of a pair of corners c, c′ of the picture together with an arc α joining the head of one corner with the tail of the other such that the following conditions hold:

(i) c and c′ lie in the same region of the picture;

(ii) W(c) = W(c′).

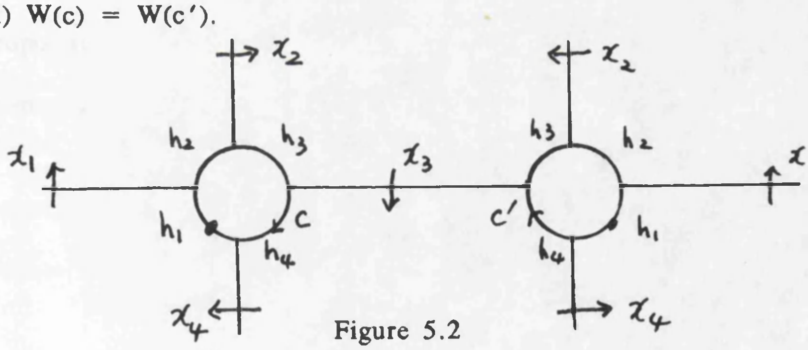


Figure 5.2

A picture over $\mathscr{P}$ is *reduced* if it does not contain a dipole.

Definition A relative presentation $\mathscr{P}$ is *aspherical* if every connected spherical picture over $\mathscr{P}$ contains a dipole. In fact we can assume connectedness for by considering a suitable connected spherical subpicture, we see that if $\mathscr{P}$ is aspherical then *no* spherical picture over $\mathscr{P}$ is reduced.

Now we obtain an ordinary group presentation $\mathcal{P}^*$ defining the same group G as follows.

Let $\mathcal{H}$ = < **a** ; **s** > be an ordinary presentation of H. Then there is a homomorphism φ from the free group on **a** onto H with kernel the normal closure of **s**. For each $h \in H$ we choose an element of $\varphi^{-1}(h)$, represented by a freely reduced word on **a**. Now φ extends to a homomorphism from the free group on $\mathbf{a} \cup \mathbf{x}$ to $H * \langle \mathbf{x} \rangle$

$$a \longmapsto \varphi(a) \qquad (a \in \mathbf{a})$$

$$x \longmapsto x \qquad (x \in \mathbf{x})$$

and the lifting of elements of H described above induces a lifting of elements of $H * \langle \mathbf{x} \rangle$. In particular, for each $R \in \mathbf{r}$ we have its lift $\tilde{R}$ (a cyclically reduced word on $\mathbf{a} \cup \mathbf{x}$). We let

$$\mathcal{P}^* = \langle \mathbf{a}, \mathbf{x} \; ; \; \mathbf{s}, \; \tilde{\mathbf{r}} \rangle$$

where $\tilde{\mathbf{r}} = \{\tilde{R} \; ; \; R \in \mathbf{r}\}$.

Let ϕ: H $\longrightarrow$ G be the composition:

$$H \xrightarrow[\text{inclusion}]{} H * \langle \mathbf{x} \rangle \xrightarrow[\text{natural surj}]{} G.$$

Let X be a generating set of $\pi_2(\mathcal{H})$. If $\mathcal{P}$ is aspherical then by Proposition 5.1.3 below we get a generating set of $\pi_2(\mathcal{P}^*)$ consisting of all elements of X together with the pictures depicted in Fig.5.3.

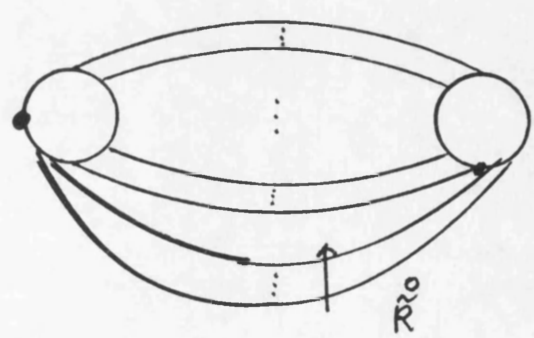


Figure 5.3

where $\tilde{R} = (\overset{0}{\tilde{R}})^n$ $(\tilde{R} \in \tilde{\mathbf{r}})$.

Lemma 5.1.2. *If $\mathcal{P}$ is aspherical, then every picture over $\mathcal{P}^*$ having at least one $\tilde{\mathbf{r}}$-disc and having no* x*-arcs meeting the boundary of the picture, contains an $\tilde{\mathbf{r}}$-dipole.*

Proof. See [5,Lemma 1.5].

Proposition 5.1.3. *If $\mathcal{P}$ is aspherical then $\pi_2(\mathcal{P}^*)$ is generated by* X ∪ Y, *where* Y *is the collection of all pictures as in Fig.*5.3.

Proof. Let $\mathbb{P}^*$ be a spherical picture over $\mathcal{P}^*$ without $\tilde{\mathbf{r}}$-discs. If $\mathbb{P}^*$ has x-arcs then these x-arcs are floating circles or simple closed arcs surrounding a subpicture consisting of s-discs. Floating circles can be removed and in the other case $\mathbb{P}^*$ is equivalent (relX) to the empty picture. If $\mathbb{P}^*$ has no x-arcs then we can consider $\mathbb{P}^*$ as a picture over $\mathcal{H}$. So we assume that $\mathbb{P}^*$ has at least one $\tilde{\mathbf{r}}$-disc. Then by Lemma 5.1.2, $\mathbb{P}^*$ has an $\tilde{\mathbf{r}}$-dipole. By bridge move and deletion of $\tilde{\mathbf{r}}$-dipole we get another picture over $\mathcal{P}^*$ with less r-discs than $\mathbb{P}^*$. By induction on the number of r-discs, we can get a picture over $\mathcal{P}^*$ without $\tilde{\mathbf{r}}$-discs. So we can get the conclusion.

A connected spherical picture $\mathbb{P}$ over $\mathcal{P}$ is defined to be *strictly spherical* if the product of the corner labels in the outer annular region (taken in anticlockwise order) defines the identity in H. The relative presentation $\mathcal{P}$ is *weakly aspherical* if each strictly spherical picture over $\mathcal{P}$ contains a dipole.

Lemma 5.1.4. *If $\mathcal{P}$ is weakly aspherical and if the natural map of H into G is an embedding, then $\mathcal{P}$ is aspherical.*

Proof. See [5, Lemma 1.7].

If $\mathcal{P}$ is an aspherical relative presentation then we get the following theorems [5].

Theorem 5.1.5. *The natural homomorphism* $H \longrightarrow G$ *is injective.*

Theorem 5.1.6. *If* $R \in \mathbf{r}$ *then* $\mathring{R}$ *defines an element of order precisely* $p(R)$ *in* G.

Theorem 5.1.7. *For any left* ZG*-module* A*, and any right* ZG*-module* B*, we have*

$$H^n(G,A) = H^n(H,A) \oplus \Big(\prod_{R \in \mathbf{r}} H^n(sgp\{\mathring{R}N\},A)\Big)$$

$$H_n(G,B) = H_n(H,B) \oplus \Big(\bigoplus_{R \in \mathbf{r}^n} H_n(sgp\{\mathring{R}N\},B)\Big)$$

for all $n \geq 3$.

Theorem 5.1.8. *Any finite subgroup of* G *is contained in a conjugate* H *or in a conjugate of one of the cyclic subgroups* $sgp\{\mathring{R}N\}$ $(R \in \mathbf{r})$.

For the proofs of these results see Theorem 1.1, Corollary 1, Theorem 1.1 Corollary 4, Theorem 1.3, Theorem 1.4 of [5], respectively.

5.1.4 Tests for asphericity

(1) Weight test

The *star-complex* $\mathcal{P}^{st}$ of $\mathcal{P}$ is a certain graph whose edges are labelled by elements of the group H. The definition is as follows.

The vertex and edges sets are $\mathbf{x} \cup \mathbf{x}^{-1}$, $\mathbf{r}^*$ respectively. For $R \in \mathbf{r}^*$, write $R = Sh$ where $h \in H$ and S begins and ends with $\mathbf{x}$-symbols. The initial and terminal functions are given by: $\iota(R)$ is the first symbol of S, $\tau(R)$ is the inverse of the last symbol of S. The inverse function on edges is given by the operaror $\bar{}$ defined in (5-3). The labelling function is defined by $\lambda(R)=h^{-1}$, and is extended to paths in the obvious way. Note that $\lambda(\bar{R})=\lambda(R)^{-1}$. A non-empty cyclically reduced closed path in $\mathscr{P}^{st}$ will be called *admissible* if it has trivial label in H.

Example 5.1.9. Let $\mathscr{P}=\langle H,x;(x^2h)^2\rangle$. Then $\mathscr{P}^{st}$ is as follows:

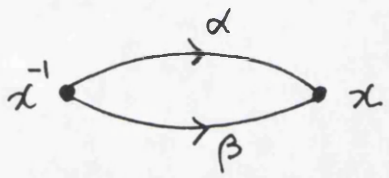


Figure 5.4

where $\lambda(\alpha)=h$ and $\lambda(\beta)=1$. If h has infinite order then no admissible paths exist. And if h has finite order $r\geq 2$ then the powers of $(\alpha\beta^{-1})^r$ (or $(\alpha^{-1}\beta)^r$) are admissible.

A *weight function* θ on $\mathscr{P}^{st}$ is a real valued function on the set of edges of $\mathscr{P}^{st}$ which satifies $\theta(\bar{R}) = \theta(R)$ for all $R \in \mathbf{r}^*$. The *weight* of a path is the sum of the weights of the constituent edges.

A weight function θ on $\mathscr{P}^{st}$ is *weakly aspherical* if the following two conditions are satisfied:

Let $R \in \mathbf{r}$, say $R = x_1^{\varepsilon_1}h_1\ldots x_n^{\varepsilon_n}h_n$ as in (4-2). Then

(i) $\sum_{i=1}^{n}(1-\theta(x_i^{\varepsilon_i}h_i\ldots x_n^{\varepsilon_n}h_n x_1^{\varepsilon_1}h_1\ldots x_{i-1}^{\varepsilon_{i-1}}h_{i-1}))\geq 2$

(ii) Each admissible path in $\mathscr{P}^{st}$ has weight at least 2.

A weakly aspherical weight function on $\mathscr{P}^{st}$ is *aspherical* if each edge of

$\mathcal{P}^{st}$ has non-negative weight.

Theorem 5.1.10. *(i) If $\mathcal{P}^{st}$ has a weakly aspherical weight function then $\mathcal{P}$ is weakly aspherical.*

(ii) If $\mathcal{P}^{st}$ has an aspherical weight function then $\mathcal{P}$ is aspherical.

Proof. See [5, Theorem 2.1].

Example 5.1.11. Let $\mathcal{P}$ be the same presentation as in Example 5.1.9. If h has infinite order, then no admissible paths exist. So if we take $\theta(\alpha) = \theta(\beta) = 0$ then θ is an aspherical weight function. If h has order $r \geq 2$ then the admissible paths are powers of $(\alpha\beta^{-1})^r$ (or $(\alpha^{-1}\beta)^r$). So if we take $\theta(\alpha) = \theta(\beta) = 1/r$ then θ is also aspherical. Therefore $\mathcal{P}$ is aspherical.

(2) Small cancellation conditions

Let k be a positive integer. A k-*wheel over* $\mathcal{P}$ is a (non-trivial) connected picture $\mathbb{W}$ over $\mathcal{P}$ which has discs $\{\Delta_0, \Delta_1, \ldots, \Delta_k\}$ and which satisfies:

(i) each arc of $\mathbb{W}$ meets a disc Δ_j for some $j \in \{1,\ldots,k\}$;

(ii) each arc of $\mathbb{W}$ either meets Δ_0 or $\partial\mathbb{W}$;

(iii) each disc of $\mathbb{W}$ has a corner which lies in a region of $\mathbb{W}$ that meets $\partial\mathbb{W}$.

A typical k-wheel is depicted in Fig.5.5.

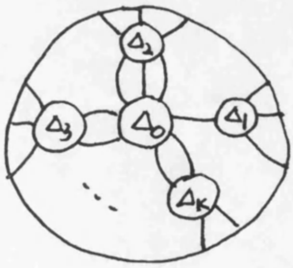


Figure 5.5

Definition Let p be a positive integer. Then $\mathcal{P}$ satisfies *C(p)* if there are no reduced k-wheels over $\mathcal{P}$ for $k<p$.

Definition Let q be a positive integer. Then $\mathcal{P}$ satisfies *T(q)* if there are no admissible paths in $\mathcal{P}^{st}$ of length m for $3 \leq m < q$.

Therem 5.1.12. *If $\mathcal{P}$ satisfies* C(p), T(q) *where* $1/p + 1/q = 1/2$ *then $\mathcal{P}$ is aspherical.*

Proof. See [5,Theorem 2.2].

Example 5.1.13. Let $\mathcal{P} = <$ H,x ; xaxbxcxd $>$, where a,b,c,d are distinct. Then $\mathcal{P}^{st}$ is as follows:

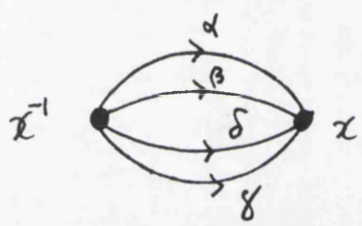


Figure 5.6

where $\lambda(\alpha) = a$, $\lambda(\beta) = b$, $\lambda(\delta) = c$, $\lambda(\gamma) = d$. So there are no admissible cycles of length ≤ 3. Thus $\mathcal{P}$ satisfies T(4). If two discs Δ_1 and Δ_2 of a picture over $\mathcal{P}$ share at least two consecutive x-arcs then they constitute a dipole. So there are no reduced k-wheels over $\mathcal{P}$ for $k<4$. Thus $\mathcal{P}$ satisfies C(4). Therefore $\mathcal{P}$ is aspherical.

(3) Change of variables

A general change of variable result is given in [5]. We will just state the result for the situation we will be interested in.

Let $\mathcal{P} = <$ H, x ; R $>$, where $R = xh_1xh_2...xh_n$. Let t be another symbol. Rewrite each element of H $*$ $<$ x $>$ using $x = th_1^{-1}$, $x^{-1} = h_1t^{-1}$. Let $R' = t^2h_1^{-1}h_2...th_1^{-1}h_n$. Then we let

$$\mathcal{P}' = < H,t\ ;\ R' >.$$

Lemma 5.1.14. *$\mathcal{P}$ is apherical if and only if $\mathcal{P}'$ is apherical.*

Proof. See [5,Theorem 2.4].

(4) Curvature arguments

We make use of the following result, which is the dual of [14,Proposition 4.4].

Lemma 5.1.15. *Let $\mathbb{P}$ be a connected spherical picture. Suppose that to each corner c of $\mathbb{P}$ we assign a real number $\theta(c)$, which we call the angle of c. Then it is impossible for the following two conditions to both hold:*

(5-4) *For each disc of $\mathbb{P}$, if $c_1,\ldots,c_n$ are the corners of the disc then*

$$\sum \theta(c_i) \geq 2,$$

(5-5) *For each region of $\mathbb{P}$ including the outer annular region, if $c_1',\ldots,c_m'$ are the corners of the region then*

$$\sum \theta(c_i') \leq m-2.$$

Proof. See [14,Proposition 4.4].

If we can assign angles such that (5-5) holds then there must exist a disc Δ such that

$$\sum \theta(c_i) < 2$$

where $c_1,\ldots,c_n$ are the corners of Δ. We will call such a disc *exceptional.*

Alternatively, if we can assign angles such that (5-4) holds then there must exist a region Θ which could be the outer annular region such that

$$\sum \theta(c_i') > m-2$$

where $c_1',\ldots,c_m'$ are the corners of Θ. We will call such a region *exceptional.*

5.2 Relative presentations with one defining relator

5.2.1 Known results

We collect some known results.

Theorem 5.2.1. [5] *If* $\mathcal{P} = \langle H, x ; xh_1xh_2\ldots xh_n \rangle$ *then the natural homomorphism* $H \longrightarrow G$ *is injective.*

Lemma 5.2.2. *Suppose* $\langle H, x ; xh_1xh_2\ldots xh_n \rangle$ *is aspherical. Then an element of the form* $x^m h$ $(m < n)$ *cannot have finite order.*

Proof. Suppose $x^m h$ has finite order. Then by Theorem 5.1.8, $x^m h$ belongs to a conjugate of H. However $x^m h$ does not lie in the normal closure H^G of H in G, since we have

$$G/H^G = \langle x ; x^n \rangle.$$

Theorem 5.2.3. *Let* $\mathcal{P} = \langle H, x ; xaxb \rangle$ *where* $a \neq b$. *Then* $\mathcal{P}$ *is apherical if and only if* $a^{-1}b$ *has infinite order.*

Proof. ($\Leftarrow$) $\mathcal{P}^{st}$ is as follows:

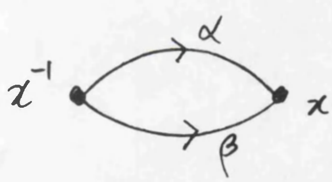


Figure 5.7

where $\lambda(\alpha) = a$, $\lambda(\beta) = b$. Since no admissible paths exist in $\mathcal{P}^{st}$, if $\theta(\alpha) = \theta(\beta) = 0$, then θ is aspherical.

($\Rightarrow$) Suppose that $a^{-1}b$ has a finite order. Since $(xa)^2 = b^{-1}a$ in G, we deduce that xa has a finite order. It is impossible by Lemma 5.2.2.

Theorem 5.2.4. *Let* $\mathcal{P} = \langle H, x ; xax^{-1}b \rangle$ $(a \neq 1 \neq b)$. *Then* $\mathcal{P}$ *is aspherical if and only if* a *and* b *both have infinite order.*

Proof. ($\Rightarrow$) Suppose a has finite order. Then we have a reduced spherical picture as follows:

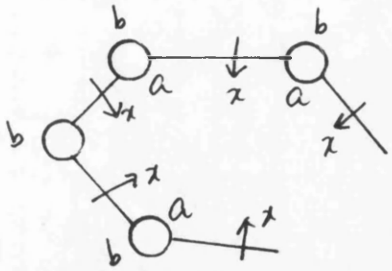


Figure 5.8

Therefore $\mathcal{P}$ is not aspherical. Similary if b has finite order.

($\Leftarrow$) Consider $\mathcal{P}^{st}$

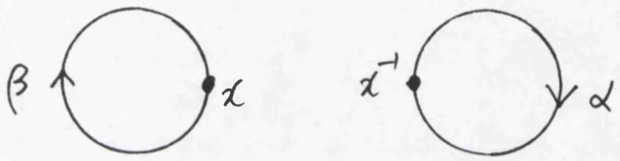


Figure 5.9

where $\lambda(\beta) = b$, $\lambda(\alpha) = a$. If a and b both have infinite order then $\mathcal{P}^{st}$ has no admissible paths. So we let $\theta(\alpha) = 0 = \theta(\beta)$ then θ is aspherical.

Theorem 5.2.5. [5,Theorem 3.1] *Let* $\mathcal{P} = < H, x \; ; \; xh_1xh_2xh_3 >$, *where* $\{h_1,h_2,h_3\}$ *contains at least two distinct elements of* H. *Then* $\mathcal{P}$ *is aspherical if and only if neither of the following conditions hold:*

(i) *For* $i = 1,2,3$, $h_ih_{i+1}^{-1}$ *has finite order* $p_i > 0$ (*subscripts mod* 3) *and* $1/p_1 + 1/p_2 + 1/p_3 > 1$.

(ii) *There* are $j \in \{1,2,3\}$, $p > 2$ *and* $0 \le k < p$ *such that* $sgp\{h_ih_{i+1}^{-1}; i=1,2,3\}$ *is finite cyclic with generator* $h_jh_{j+1}^{-1}$ *of order* p, *and* $h_{j+1}h_{j+2}^{-1} = (h_jh_{j+1}^{-1})^k$ *where either:*

(a) $k = 1$;

(b) $p = k+2$ or $2k+1$;

(c) $p = 6$ and $k = 2$ or 3.

Theorem 5.2.6. [5,Theorem 3.5] *Let* $\mathcal{P} = < H, x ; xaxbx^{-1}c >$ *where* a,b,c $\in$ H *and* b $\neq$ 1 $\neq$ c. $\mathcal{P}$ *is aspherical except possibly if* b *and* c *have finite orders* p, q *and either* $1/p + 1/q > 1/2$, *or* $a^{-1}ba = c^k$ *for some* k, *or* $aca^{-1} = b^k$ *for some* k.

Theorem 5.2.7. [18] *Let* $\mathcal{P} = < H, x ; xaxbx^{-1}c >$ *where* p *and* q $(1 < q \leq p < \infty)$ *are the orders of* b *and* c *respectively.*

(I) *If* $1/p + 1/q > 1/2$ *then* $\mathcal{P}$ *is aspherical if and only if none of the following conditions hold:*

(i) $q = p = 2$ *and* $(a^{-1}bac)^n$ *for some* $n \geq 1$;

(ii) $q = 2$, $p = 3$ *and at least one of* $a^{-1}baca^{-1}b^{-1}ac$, $(a^{-1}bac)^2(a^{-1}b^{-1}ac)^2$, $(a^{-1}bac)^r$ $(2 \leq r \leq 5)$ *is trivial*;

(iii) $q = 2$, $p = 4$ *and at least one of* $a^{-1}baca^{-1}b^{-1}ac$, $a^{-1}b^2ac$, $(a^{-1}bac)^2$, $(a^{-1}bac)^3$ *is trivial*;

(iv) $q = 2$, $p = 5$ *and at least one of* $a^{-1}baca^{-1}b^{-1}ac$, $(a^{-1}bac)^2$, $(a^{-1}bac)^3$ *is trivial*;

(v) $q = 2$,$p = 6$ *and* $a^{-1}b^3ac = (a^{-1}bac)^m = 1$ *for some* $m \geq 3$;

(vi) $q = 2$, $p \geq 6$ *and* $(a^{-1}bac)^2 = 1$;

(vii) $q = 2$, $p \geq 6$ *and* $a^{-1}baca^{-1}b^{-1}ac = 1$;

(viii) $q = 3$, $p = 3$ *and at least one of* $a^{-1}bac^{-1}$, $a^{-1}baca^{-1}b^{-1}ac^{-1}$, $(a^{-1}bac^{-1})^2$ *is trivial*;

(iv) $q = 3$, $4 \leq p \leq 5$ *and* $(a^{-1}bac^{-1})^2 = 1$;

(II) *If* $1/p + 1/q \leq 1/2$ *with* $(p,q) \neq (8,4)$ *or* $(9,3)$ *then* $\mathcal{P}$ *is aspherical if and only if none of the following conditions hold*:

(x) $q \geq 3$ *and* $a^{-1}bac^{\varepsilon} = 1$ $(\varepsilon = \pm 1)$;

(xi) $q \geq 3$ *and* $a^{-1}b^2ac = 1$;

(xii) $q = 3$, $p = 6$ *and* $a^{-1}b^2ac^{-1}$;

(xiii) $q = p = 5$, *and* $a^{-1}b^2ac^{-1} = 1$;

(xiv) $q = p = 7$ *and* $a^{-1}b^2ac^{-1} = 1$;

(xv) $q = p = 9$ *and* $a^{-1}b^2ac^{-1} = 1$.

5.2.2 New results

Let $\mathcal{P} = < H, x ; xh_1xh_2,\ldots,xh_n >$ where $h_i \in H$ $(i=1,\ldots,n)$. In this section we will study the conditions for $\mathcal{P}$ to be aspherical. If h_i are all distinct then $\mathcal{P}$ is aspherical by a similar argument as in Example 5.1.13. Therefore we need to consider what happens when the h_i are not all distinct. We will consider this question for $n=4,5$. The following Theorems deal with the essentially different cases.

(1) Length 4

Theorem 5.2.8. *Let* $\mathcal{P} = < H, x ; xaxbxaxd >$. *Then* $\mathcal{P}$ *is aspherical if and only if* $b^{-1}d$ *has infinite order.*

(Here we assume a,b,d are all distinct. A similar convention applies to the statements of the other theorems.)

Theorem 5.2.9. *Let* $\mathcal{P} = < H, x ; xaxaxcxd >$. *Then* $\mathcal{P}$ *is aspherical except possibly if one of the following hold*:

(i) $a^{-1}c = d^{-1}a$, $(d^{-1}a)^{2n} = 1$;

(ii) $a^{-1}d = (a^{-1}c)^2$, $(a^{-1}c)^n = 1$ *or* ($a^{-1}c = (a^{-1}d)^2$, $(a^{-1}d)^n = 1$);

(iii) $(a^{-1}c)^2 = 1$, $(c^{-1}d)^n = 1$ *or* ($(a^{-1}d)^2 = 1$, $(c^{-1}d)^n = 1$);

(iv) $(a^{-1}c)^3 = 1$, $(c^{-1}d)^n = 1$ *or* ($(a^{-1}d)^3 = 1$, $(c^{-1}d)^n = 1$).

Theorem 5.2.10. *Let* $\mathcal{P} = < H, x ; xaxbxaxb >$. *Then* $\mathcal{P}$ *is aspherical.*

Theorem 5.2.11. *Let* $\mathcal{P} = \langle H, x\ ;\ xaxaxaxd \rangle$. *Then* $\mathcal{P}$ *is aspherical if and only if* $a^{-1}d$ *has infinite order.*

Theorem 5.2.12. *Let* $\mathcal{P} = \langle H, x\ ;\ xaxaxcxc \rangle$. *Then* $\mathcal{P}$ *is aspherical if and only if* $a^{-1}c$ *has infinite order.*

Theorem 5.2.10 is easy. The most difficult is Theorem 5.2.9 which needs the curvature arguments.

(2) Length 5

Theorem 5.2.13. *Let* $\mathcal{P} = \langle H, x\ ;\ xaxaxcxdxe \rangle$. *Then* $\mathcal{P}$ *is aspherical.*

Theorem 5.2.14. *Let* $\mathcal{P} = \langle H, x\ ;\ xaxbxaxdxe \rangle$. *Then* $\mathcal{P}$ *is aspherical except possibly if one of the following holds:*

(i) $a^{-1}b = (a^{-1}e)^2$, $a^{-1}e = (a^{-1}b)^2$, $a^{-1}d = (a^{-1}b)^2$, $a^{-1}b = (a^{-1}d)^2$;

(ii) $b = da^{-1}e$, $e = da^{-1}b$, $d = ba^{-1}e$;

(iii) $be^{-1}de^{-1} = 1$, $(be^{-1})^2 = 1$, $(bd^{-1})^2 = 1$, $bd^{-1}ed^{-1} = 1$.

Theorem 5.2.15. *Let* $\mathcal{P}$ *be one of the following presentations:*

(a) $\langle H, x\ ;\ xaxaxaxbxc \rangle$;

(b) $\langle H, x\ ;\ xaxaxbxaxc \rangle$.

Then $\mathcal{P}$ *is aspherical except possibly if one of the following holds:*

(i) $a^{-1}ba^{-1}c = 1$, $(b^{-1}c)^n = 1$;

(ii) $(a^{-1}b)^2 = a^{-1}c$, $(b^{-1}c)^n = 1$ *or* ($(a^{-1}c)^2 = a^{-1}b$, $(b^{-1}c)^n = 1$);

(iii) $(a^{-1}b)^2 = 1$, $(b^{-1}c)^n = 1$ *or* ($(a^{-1}c)^2 = 1$, $(b^{-1}c)^n = 1$);

(iv) $(b^{-1}c)^2 = 1$.

Theorem 5.2.16. *Let* $\mathcal{P}$ = < H, x ; xaxaxbxbxc >. *Then* $\mathcal{P}$ is *aspherical except possibly if one of the following holds:*

(i) $a^{-1}ba^{-1}c = 1$;

(ii) $(a^{-1}b)^2 = a^{-1}c$ *or* ($(a^{-1}c)^2 = a^{-1}b$);

(iii) $(a^{-1}b)^2 = 1$ *or* ($(a^{-1}c)^2 = 1$);

(iv) $(b^{-1}c)^2 = 1$.

Theorem 5.2.17. *Let* $\mathcal{P}$ = < H, x ; xaxaxbxcxb >. *Then* $\mathcal{P}$ *is aspherical except possibly if one of the following holds:*

(i) $a^{-1}ba^{-1}c = 1$;

(ii) $(a^{-1}b)^2 = a^{-1}c$ *or* ($(a^{-1}c)^2 = a^{-1}b$);

(iii) $(a^{-1}b)^2 = 1$ *or* ($(a^{-1}c)^2 = 1$).

Theorem 5.2.18. *Let* $\mathcal{P}$ *be one of the following presentations:*

(a) < H, x ; xaxaxaxaxb >*;*

(b) < H, x ; xaxaxbxaxb >*;*

(c) < H, x ; xaxaxaxbxb >.

Then $\mathcal{P}$ *is aspherical if and only if* $a^{-1}b$ *has infinite order.*

The only case not covered by the above theorems is

$$\mathcal{P} = < H, x ; xaxbxaxbxc >.$$

Making the change of variables $x=ta^{-1}$ we get

$$\mathcal{P}' = < H, t ; t^2hth^2tg >.$$

If we make the substitution $u=t^2h$ we get (using Tietze transformations)

$$\mathcal{P}'' = < H, u ; u^3h^{-1}gu^2g >$$

which (after a change of variables) is covered by Theorem 5.2.15(b). However, it is not clear how to connect asphericity of $\mathcal{P}'$ and $\mathcal{P}''$.

5.3 Proofs

We will use "change of variable" as in § 5.1(3) by

$$x = ta^{-1}.$$

We will also make use of Lemma 5.1.4 and Theorem 5.2.1 without comment. So in order to verify asphericity we only need to check weak asphericity.

Lemma 5.3.1. *Let* $\mathcal{P} = \langle H, x; xh_1xh_2\ldots xh_n\rangle$. *If* $h_1,\ldots,h_n$ *generate an infinite cyclic subgroup of H then* $\mathcal{P}$ *is aspherical.*

Proof. Let c be a generator of the infinite cyclic group generated by $h_1,\ldots,h_n$. Suppose that $\mathbb{P}$ is a reduced strictly spherical picture over $\mathcal{P}$. We change each h_i (i=1,...,n) appearing in the label of the corners of $\mathbb{P}$ to the appropriate power of c. And we draw short arcs of the same number as the absolute value of the power of c which labelled c or c^{-1} at each corner of $\mathbb{P}$. Since the number of short arcs labelled c must be equal to the number of short arcs labelled c^{-1}, we can connect two short arcs of which labels are c and c^{-1} by an arc labelled by c with an appropriate orientation in a way that c-arcs do not cross each other.

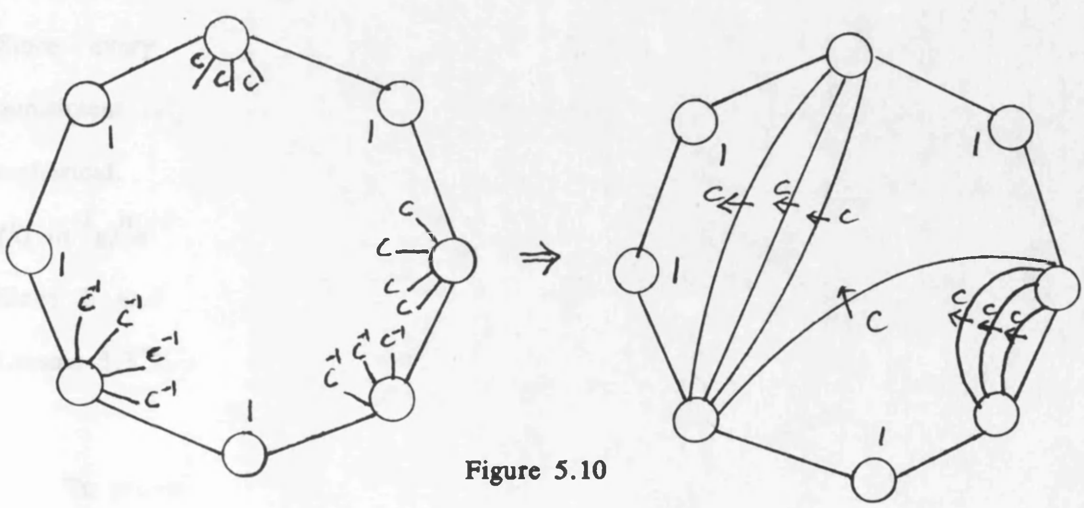


Figure 5.10

Then we get a reduced spherical picture over $\mathcal{Q} = \langle c,x; xc^{m_1}xc^{m_2}\ldots xc^{m_n} \rangle$ from $\mathbb{P}$ where $h_i = c^{m_i}$ $(i=1,\ldots,n)$. But it is impossible because $\mathcal{Q}$ is a one-relator ordinary presentation and so every spherical picture over $\mathcal{Q}$ has a dipole [12].

5.3.1 Length 4 case

Proof of Theorem 5.2.8. We consider $\mathcal{P} = \langle H, t ; t^2ht^2g \rangle$ $(h = a^{-1}b,\ g = a^{-1}d)$. Then we will prove that $\mathcal{P}$ is aspherical if and only if $h^{-1}g$ has infinite order. Suppose that $h^{-1}g$ has finite order. Then $(t^2h)^2 = (h^{-1}g)^{-1}$ so t^2h has finite order. But by Lemma 5.2.2, it is impossible. Suppose that $h^{-1}g$ has infinite order. We consider the following:

$\mathcal{P}^{st}$

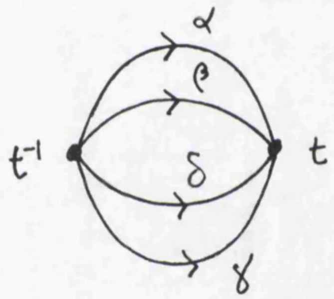


Figure 5.11

where $\lambda(\alpha)=h$, $\lambda(\beta)=g$, $\lambda(\delta)=1$, $\lambda(\gamma)=1$.

(i) No relation of the form $(h^{-1}g)^nh^{-1}$ or $g(h^{-1}g)^n$ holds in H.

Since every admissible paths have at least two of $\delta^{\pm 1}$ or $\gamma^{\pm 1}$ as the constituent edges, if we let $\theta(\alpha) = 0 = \theta(\beta)$ and $\theta(\delta) = 1 = \theta(\gamma)$ then θ is aspherical.

(ii) $(h^{-1}g)^nh^{-1}$ or $g(h^{-1}g)^n$ holds in H.

Since h and g generate an infinite cyclic subgroup of H, $\mathcal{P}$ is aspherical by Lemma 5.3.1.

To prove Theorem 5.2.9, we need the folllowing Lemmas. Let

$\mathcal{P}' = < H, t \; ; \; t^3htg >$, where $h = a^{-1}c$, $g = a^{-1}d$.

Lemma 5.3.2. *If* $h^{-1}g$ *has infinite order then* $\mathcal{P}'$ *is aspherical.*

Proof. By the same argument as the "if" part of the proof of Theorem 5.2.8.

Lemma 5.3.3. $\mathcal{P}'$ *is aspherical except possibly if one of the following hold*:

(i) $hg = 1$;

(ii) $h^2g^{-1} = 1$ *or* ($g^2h^{-1} = 1$);

(iii) $(h^{-1}g)^2 = 1$;

(iv) $h^2 = 1$ *or* ($g^2 = 1$).

Proof. We will use curvature argument as in §5.1(4). Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P}'$. For each m-gon of $\mathbb{P}$ (including the outer annular region) we give $\theta(c_i) = (m-2)/m$ for $i = 1,\ldots,m$. Then (5-5) holds. So there exists an exceptional disc Δ such that

$$\theta(c_1) + \theta(c_2) + \theta(c_3) + \theta(c_4) < 2$$

because all discs of $\mathbb{P}$ have four corners. Since the smallest non-zero value of θ is 1/2 and the next is 2/3, one of $\theta(c_i)$ is 0 and one of $\theta(c_i)$ is 1/2. Thus Δ is as follows.

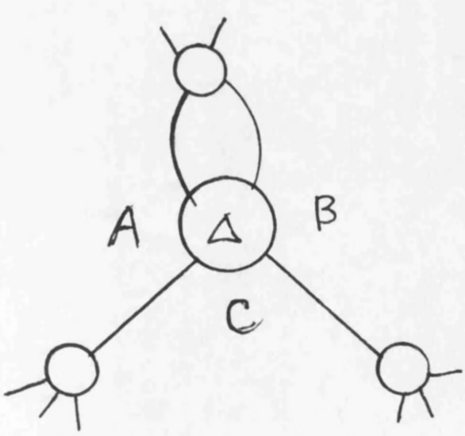


Figure 5.12

where one of A,B and C is a 4-gon. (Note that A,B or C could be the outer annular region.) Now we study the possible labellings of A,B and C. We note that the following labellings constitute dipoles:

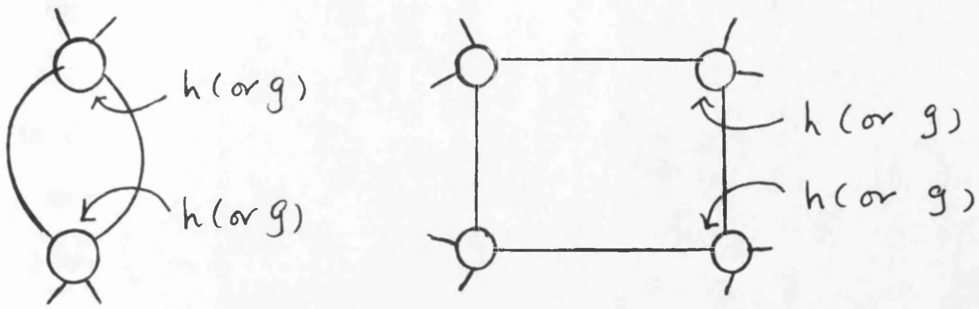


Figure 5.13

(1). A (or B) is a 4-gon

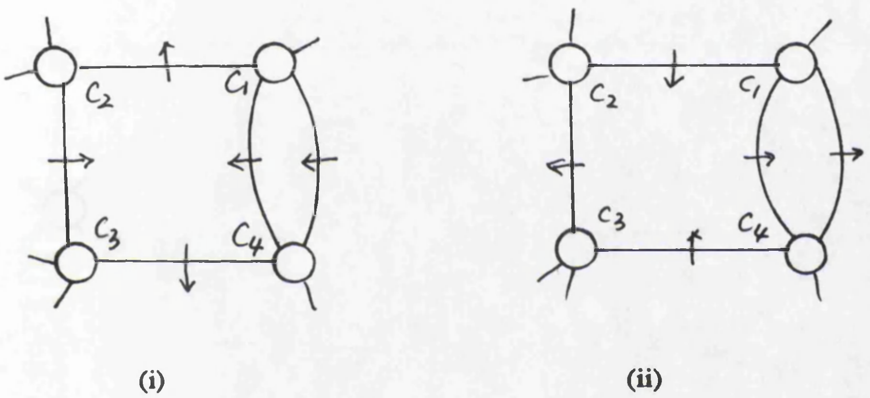


Figure 5.14

First we consider the situation (i).

(1-1) $\lambda(c_1)=1$ then $\lambda(c_4)$ must be h.

(a) $\lambda(c_2)=1$ (impossible)

(b) $\lambda(c_2)=h$

$\lambda(c_3)=1$ then we get $h^2=1$

$\lambda(c_3)=g$ then we get $h^2g^{-1}=1$

(c) $\lambda(c_2)=g$

$\lambda(c_3)=1$ so we get $hg=1$

(1-2) $\lambda(c_3)=h$ then $\lambda(c_4)$ must be 1.

(a) $\lambda(c_2)=1$

$\lambda(c_3)=h$ then we get $h^2=1$

$\lambda(c_3)=g$ then we get $hg=1$

(b) $\lambda(c_2)=g$

$\lambda(c_3)=h$ so we get $h^2g^{-1}=1$

In the situation (ii), if $\lambda(c_1)=1$ then $\lambda(c_4)$ must be g and if $\lambda(c_1)=g$ then $\lambda(c_4)$ must be 1. So we get the results for the situation (ii) from the results for (i) by exchanging g and h with each other.

(2). C is a 4-gon

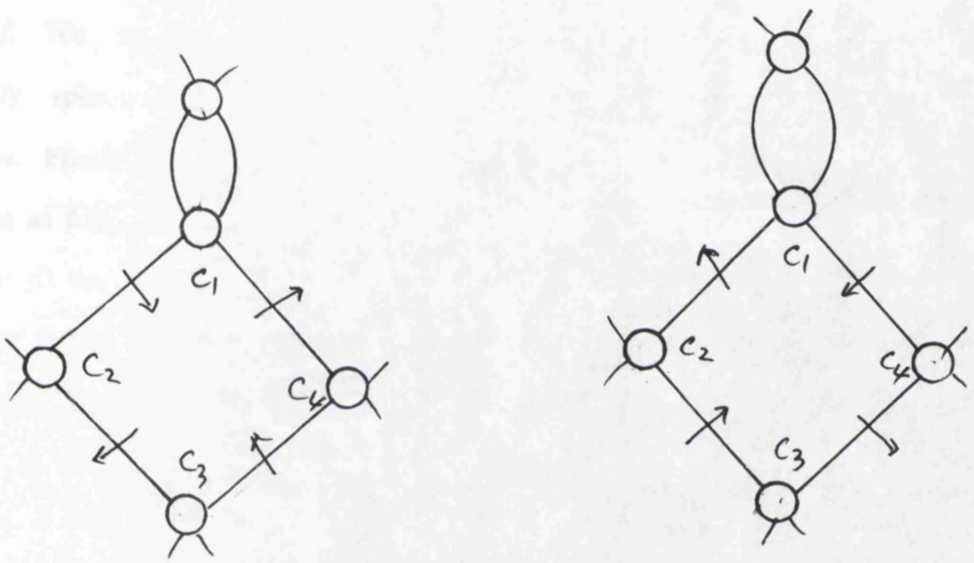


Figure 5.15

We note that $\lambda(c_1)\neq 1$.

(2-1) $\lambda(c_1)=h$

(a) $\lambda(c_2)=1$

$\lambda(c_4)=1$ then $\lambda(c_3)$ is h or g. So we get $h^2=1$ or $hg=1$.

$\lambda(c_4)=g$ then $\lambda(c_3)=h$. So we get $h^2g^{-1}=1$.

(b) $\lambda(c_2)=g$

$\lambda(c_4)=1$ then $\lambda(c_3)=h$. So we get $h^2g^{-1}=1$.

$\lambda(c_4)=g$ then $\lambda(c_3)$ is 1 or h. So we get $(hg^{-1})^2=1$ or $g^2h^{-1}=1$.

(2-2) $\lambda(c_2)=g$ and (ii).

As mentioned in the second part of (1) we get the results from (2-1) by exchanging g and h with each other. So we can get the conclusion.

Lemma 5.3.4. *$\mathscr{P}'$ is aspherical except possibly if one of the following hold*:

(i) $hg=1$;

(ii) $h^2g^{-1}=1$ *or* $(g^2h^{-1}=1)$;

(iii) $h^3=1$ *or* $(g^3=1)$;

(iv) $h^2=1$ *or* $(g^2=1)$.

Proof. We will use once again curvature arguments. Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathscr{P}'$. In this case we assign two types of angles. Firstly, for each disc of $\mathbb{P}$ which is adjacent to a 2-gon, we give the angles as follows:

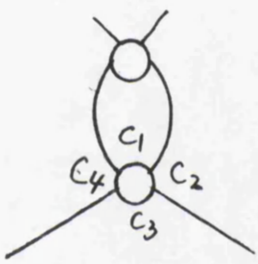


Figure 5.16

where $\theta(c_1)=0$, $\theta(c_2)= \theta(c_4)= 3/4$ and $\theta(c_3)= 1/2$.

Otherwise $\theta(c_i)= 1/2$ $(i=1,2,3,4)$. Then (5-4) holds. Thus there exists an exceptional region Θ (which could be the outer annular region) such that

$$\sum \theta(c_i') > m-2.$$

Since the largest value of θ is 3/4, it is impossible for $m\geq 8$. And $\mathbb{P}$ has no subpictures as follows:

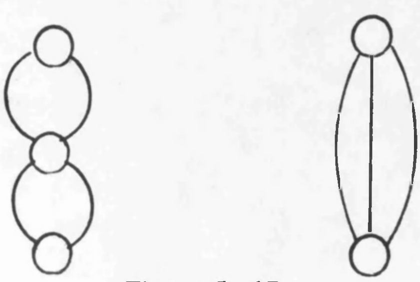
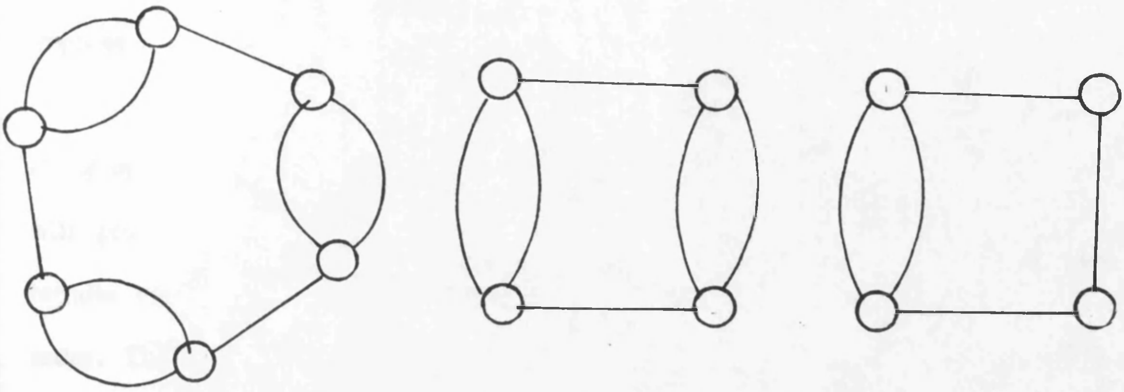


Figure 5. 17

So we have only three possibilities for Θ.



(i) (ii) (iii)

Figure 5. 18

From (i) we get $g^3=1$ or $h^3=1$. From (ii), we get $g^2=1$ or $h^2=1$. The case (iii) is the same as in the first part of proof the of Lemma 5.3.3. So we get the conclusions.

If we combine Lemmas 5.3.2, 5.3.3 and 5.3.4, then we get the following result:

Lemma 5.3.5. *$\mathcal{P}'$ is aspherical except possibly one of the following hold*:

(i) $hg=1$, $(h^{-1}g)^n=1$;

(ii) $h^2g^{-1}=1$, $(h^{-1}g)^n=1$ *or* ($g^2h^{-1}=1$, $(h^{-1}g)^n=1$);

(iii) $h^2=1$, $(h^{-1}g)^n=1$ *or* ($g^2=1$, $(h^{-1}g)^n=1$);

(iv) $h^3=1$, $(h^{-1}g)^n=1$ *or* ($g^3=1$, $(h^{-1}g)^n=1$).

So we get the conclusion of Theorem 5.2.9.

Proof of Theorem 5.2.10. We consider $\mathcal{P}' = < H,t\ ;\ (t^2h)^2 >$, where $h = a^{-1}b$. By Example 5.1.11, $\mathcal{P}'$ is aspherical.

Proof of Theorem 5.2.11. We consider $\mathcal{P}' = < H, t ; t^4h >$, where $h = a^{-1}d$. We will prove that $\mathcal{P}'$ is aspherical if and only if h has infinite order. If h has finite order then t also has finite order. It is impossible by Lemma 5.2.2. Suppose h has infinite order. Then by Lemma 5.3.1, $\mathcal{P}'$ is aspherical.

Proof of Theorem 5.2.12. We consider $\mathcal{P}' = < H, t ; t^3hth >$, where $h = a^{-1}c$. We will prove that $\mathcal{P}'$ is aspherical if and only if h has infinite order. If h has infinite order then $\mathcal{P}'$ is aspherical by Lemma 5.3.1. Suppose that h has finite order. Then we can get the following reduced picture over $\mathcal{P}'$, for example ,the order of h is 5. So $\mathcal{P}'$ is not aspherical.

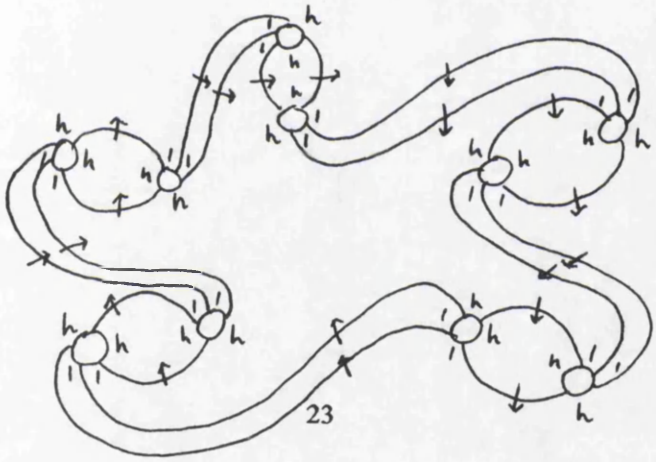


Figure 5. 19

5.3.2 Length 5 case

Since the methods are similar to the length 4 case, we will just give outlines of proofs.

Proof of Theorem 5.2.13. We consider $\mathcal{P}' = <H,t;t^3htgtk>$ $(h=a^{-1}c,\ g=a^{-1}d,\ k=a^{-1}e)$. Since $\mathcal{P}'$ satisfies C(4) and T(4), we get the result.

Definition We call the following subpicture *a double 2-gon.*

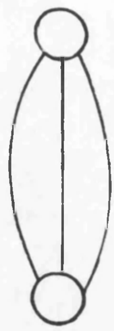


Figure 5.20

Lemma 5.3.6. *Let* $\mathbb{P}$ *be a reduced strictly spherical picture over* $\mathcal{P} = <H,x;xaxbxcxdxe>$ *where* a,b,c,d,e *are elements of* H. *If* $\mathbb{P}$ *has no double 2-gons, then* $\mathbb{P}$ *has an exceptional disc* Δ *as follows:*

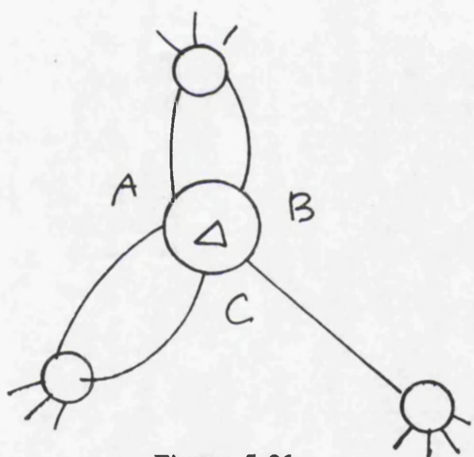


Figure 5.21

where one of A, B, C is a 4-gon. (Note that A,B, *or* C *could be the outer annular region.)*

Proof. We will use the curvature arguments in §5.1.(4). For each m-gon of $\mathbb{P}$ (including the outer annular region), we give $\theta(c_i)=(m-2)/m$ $(i=1,\ldots,m)$. Then (5-5) holds. So there exists an exceptional disc Δ such that $\theta(c_1)+\ldots+\theta(c_5)<2$. Since the smallest non-zero value of θ is 1/2 and the next is 2/3, two of $\theta(c_i)$ is 0 and one of $\theta(c_i)$ is 1/2. Thus we can get the result.

Proof of Theorem 5.2.14. We consider $\mathcal{P}' = \langle H,t;t^2ht^2gtk \rangle$ $(h=a^{-1}b, g=a^{-1}d, k=a^{-1}e)$. Since a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}'$ has no double 2-gons, $\mathbb{P}$ has an exceptional disc like Fig.5.21 by Lemma 5.3.6. Then we can get the result from the possible labellings of A,B,C.

Proof of Theorem 5.2.15.

(a) $\mathcal{P} = \langle H, x ; xaxaxaxbxc \rangle$.

We consider $\mathcal{P}' = \langle H,t;t^4htg \rangle$ $(h=a^{-1}b, g=a^{-1}c)$. By a similar argument as the "if" part of the proof of Theorem 5.2.8, if $h^{-1}g$ has infinite order then $\mathcal{P}'$ is aspherical. Now we will use the curvature arguments. Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P}'$ then $\mathbb{P}$ may have discs of 4 different types. We give angles as follows:

(1)

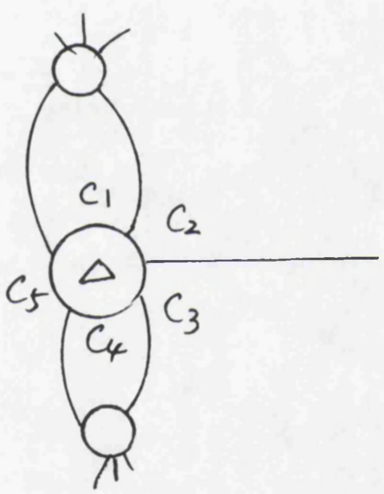


Figure 5.22

we give $\theta(c_1)=\theta(c_4)=0$, $\theta(c_2)=\theta(c_3)=3/4$, $\theta(c_5)=1/2$;

(2)

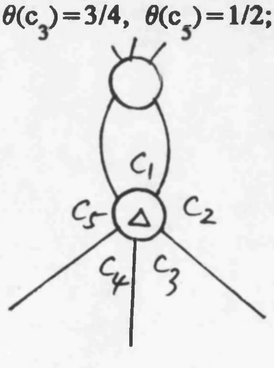


Figure 5.23

we give $\theta(c_1)=0$, $\theta(c_i)=1/2$ (i=2,3,4,5);

(3)

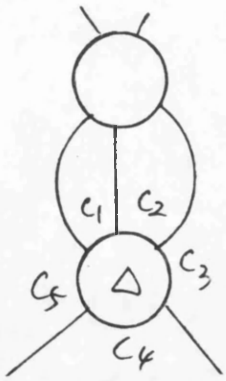


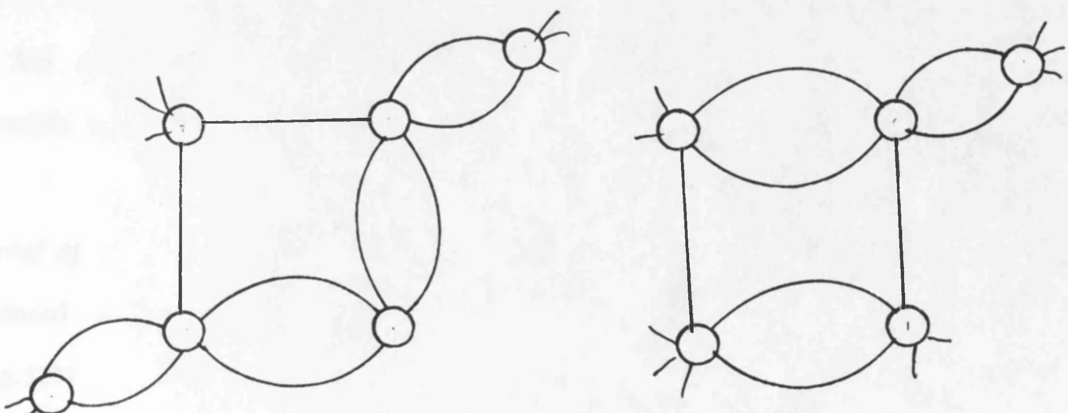
Figure 5.24

we give $\theta(c_1)=\theta(c_2)=0$, $\theta(c_3)=\theta(c_4)=\theta(c_5)=2/3$;

(4) otherwise, we $\theta(c_i)=2/5$ (i=1,...,5). Then (5-4) holds. Therefore there is an exceptional region Θ (which could be the outer annular region) such that

$$\sum_{i=1}^{m} \theta(c_i') > m-2.$$

Since the largest value of θ is 3/4, it is impossible for $m \geq 8$. So it is enough to consider 4- or 6-gons. In 6-gon case, Θ has at most three 3/4-corners because 3/4-corners cannot appear consecutively (if so, they form a dipole). If Θ has a 3/4-corner then it must have at least one 1/2-corner. Therefore only the case that Θ has three 3/4-corners, two 2/3-corners and one 1/2-corner can be considered. But this case also is impossible because if Θ has three 3/4-corners then it must have at least two 1/2-corners. So we have the following possibilities for Θ.



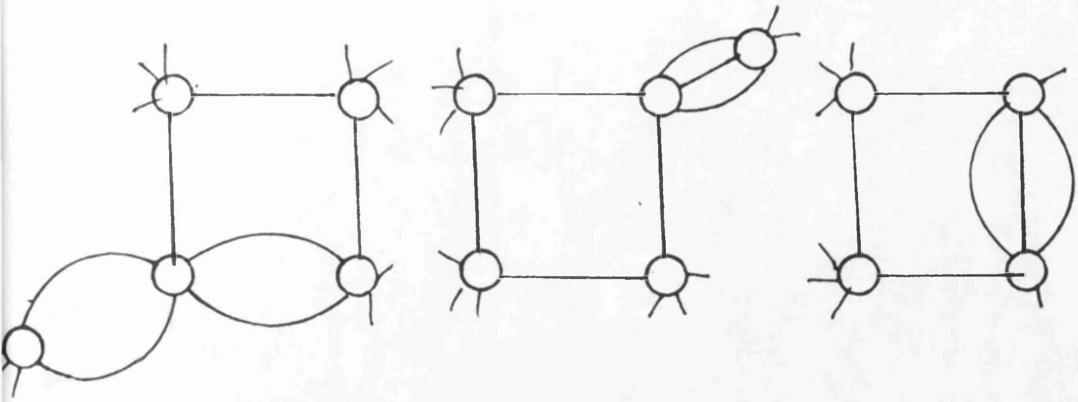


Figure 5.25

At any case, all labels of corners of Θ cannot be 1. So we can get the result from the possible labelling of each 4-gon.

(b) $< H, x ; xaxaxbxaxc >$.

We consider $\mathcal{P} = <H,t;t^3ht^2g>$ $(h=a^{-1}b, g=a^{-1}c)$. By a similar argument as the "if" part of the proof of Theorem 5.2.8, if $h^{-1}g$ has infinite order then $\mathcal{P}$ is aspherical. Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P}$. Then $\mathbb{P}$ has no double 2-gons, so it has an exceptional disc like Fig.5.21. We can get the result from the possible labellings of A,B,C.

Proof of Theorem 5.2.16. We consider $\mathcal{P} = <H,t;t^2(th)^2tg>$ $(h=a^{-1}b, g=a^{-1}c)$. Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P}$. Since $\mathbb{P}$ has no double 2-gons, $\mathbb{P}$ has an exceptional disc like Fig.5.21. So we can get the result from the possible labellings of A,B,C.

Proof of Theorem 5.2.17. We consider $\mathcal{P} = <H,t;t^3htgth>$ $(h=a^{-1}b, g=a^{-1}c)$. Since a reduced strictly spherical picture over $\mathcal{P}$, we get an exceptional disc like Fig.5.21. But in this case at least one label of corners of A,B,C is 1. So we get the result.

Proof of Theorem 5.2.18.

(a) < H, x ; xaxaxaxaxb >. We consider $\mathscr{P}' = <H,t;t^5h>$. By a similar argument of the proof of Theorem 5.2.11, we get the conclusion.

(b) < H, x ; xaxaxbxaxb >. We get $\mathscr{P}' = <H,t;t^3ht^2h>$ $(h=a^{-1}b)$ by changing variable. Suppose that h has finite order. By Tietze transformation $u=t^2h$, we get $<H,u;u^5h^{-1}>$. So u has finite order. Then t^2h has finite order.By Lemma 5.2.2, $\mathscr{P}$ is not aspherical. Now we suppose that h has infinite order. Then $\mathscr{P}'$ is aspherical by Lemma 5.3.1.

(c) < H, x ; xaxaxaxbxb >. We consider $\mathscr{P}' = <H,t;t^4hth>$ $(h=a^{-1}b)$. By a similar argument of the proof of Theorem 5.2.12, we get the conclusion.

Chapter 6. Homology, cohomology and efficiency of Coxeter groups

In this chapter we will compute the second integral (co)homology of aspherical Coxeter groups and consider the efficiency of Coxeter presentations and Coxeter groups. This work is related to the theme of the rest of the thesis, since it makes use of generators of π_2 of aspherical Coxeter presentations (already computed in [38]).

6.1. Notation and statement of results

Let $\Gamma = (\mathbf{v},\mathbf{e})$ be a finite simple graph, let $\mathbf{e}^+$ be an orientation of $\mathbf{e}$ and let

$$\phi\colon \mathbf{e} \longrightarrow \{2,3,\ldots\}$$

be a function with $\phi(e^{-1})=\phi(e)$ $(e\in\mathbf{e})$. Let $\mathscr{C} = \mathscr{C}(\Gamma,\phi)$ be the presentation

(6-1) $\quad < \mathbf{v} \;;\; v^2 \;(v\in\mathbf{v}),\; (\iota(e)\tau(e))^{\phi(e)} \;(e\in\mathbf{e}^+) >.$

We call $\mathscr{C}$ a *Coxeter presentation* and the group C defined by $\mathscr{C}$ a *Coxeter group*. For each $e\in\mathbf{e}$, we call $\phi(e)$ the *weight* of e.

Definition [38] $\mathscr{C}$ is called *aspherical* if it satisfied the condition: If e_1, e_2, e_3 are three distinct edges of Γ which form a triangle then

$$1/\phi(e_1) + 1/\phi(e_2) + 1/\phi(e_3) \leq 1.$$

Let $\mathrm{Adj}(v) = \{e \;;\; e\in\mathbf{e}^+$, one of endpoints of e is v$\}$. We call an edge of Γ *even* (*odd*) according to whether its weight is even (odd). We call a vertex *even* if all edges in Adj(v) are even. A vertex which is not even will be called *odd*.

Notation

$m = |\mathbf{v}|$

$n = |\mathbf{e}^+|$

n_e; the number of even edges in $\mathbf{e}^+$

n_0; the number of odd edges in e^+

Γ_i; i-th component of Γ after removing all even edges (i=1,...,d)

T_i; a maximal tree of Γ_i (i=1,...,d)

v_i; the set of all vertices of Γ_i (i=1,...,d)

l ; the number of edges in e^+ lying in $\bigcup_{i=1}^{d} (\Gamma_i \backslash T_i)$

$\hat{n}$; the number of edges in e^+ of weight≥ 3

t ; the number of $\sim$-equivalence classes on the set of edges in e^+ of weight 2, where $\sim$ is the equivalence relation defined as follows.

Let $A=\{ e\in e^+ ;\ \phi(e)=2\}$. Let us write $e \underset{\sim}{\sim} f$ for $e,f\in A$ if e and f form two edges of a triangle whose third edge is odd. $\sim$ is the transitive closure of $\underset{\sim}{\sim}$.

We first discuss the second homology and cohomology.

Howlett proved the following Theorem.

Theorem 6.1.1. [26] *The Schur multiplier of C is an elementary abelian 2-group of rank* $\hat{n}+t+d-m$.

Corollary 6.1.2. *Suppose the following holds.*

(6-2) *Γ has no triangle $e_1e_2e_3$ with $\phi(e_1)=\phi(e_2)=2$, and $\phi(e_3)$ odd.*

Then $H_2(C)$ *is an elementary abelian 2-group of rank* $n+d-m$.

Corollary 6.1.3. *If $\mathscr{C}$ is aspherical then* $H_2(C)$ *is an elementary abelian 2-group of rank* n_e+l.

An alternative proof of Corollary 6.1.3 has been given by Pride and Stöhr [38]. We will give a third proof of this result by using the following Theorem 6.1.4. Similar calculation using Theorem 6.1.4 will enable us to compute cohomology.

Theorem 6.1.4.[38] *If $\mathscr{C}$ is aspherical then $\pi_2(\mathscr{C})$ is generated by* $\mathbb{P}_{\mathbf{v}}$ $(v \in \mathbf{v})$, $\mathbb{P}_e$ $(e \in \mathbf{e}^+)$, $\mathbb{Q}_e$ $(e \in \mathbf{e}^+)$.

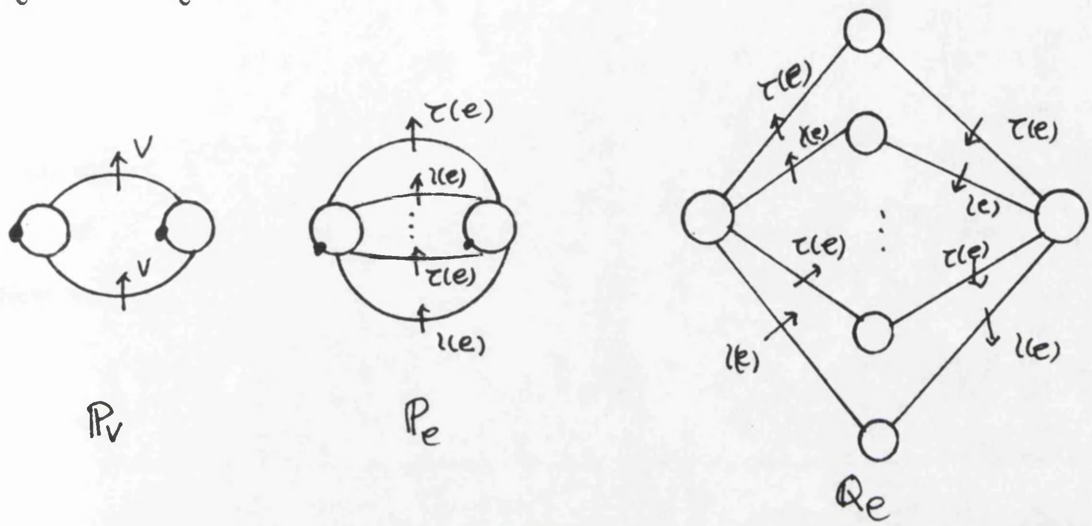


Figure 6.1

Theorem 6.1.5. *If C is the Coxeter group defined by an aspherical Coxeter presentation then* $H^2(C)$ *is an elementary abelian 2-group of rank* d.

We now consider the efficiency of Coxeter groups.

Theorem 6.1.6. *The Coxeter presentation $\mathscr{C}$ is efficient if and only if Γ has no odd edges.*

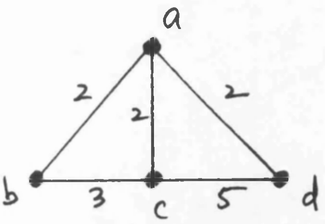
Theorem 6.1.7. *The Coxeter group C is efficient if (6-2) holds.*

Corollary 6.1.8. *The Coxeter group defined by an aspherical Coxeter presentation is efficient.*

We have not been able to decide about the necessity of (6-2) in Theorem 6.1.7. In this connection, consider the following example.

Example 6.1.9. Let Γ be a graph as follows:

Γ



Then we get the presentation

$$\mathscr{C} = < a, b, c, d \; ; \; a^2, b^2, c^2, d^2, (ab)^2, (ac)^2, (ad)^2, (bc)^3, (cd)^5 > .$$

Now we perform a sequence of Tietze transformations as follows:

let $bc=t$, then $b=tc$

$$< a, t, c, d \; ; \; a^2, (tc)^2, c^2, (atc)^2, (ac)^2, t^3, d^2, (ad)^2, (cd)^5 >$$

$1=atcatc=atcact^{-1}=[a,t]$

$$< a, t, c, d \; ; \; a^2, (tc)^2, c^2, [a,t], (ac)^2, t^3, d^2, (ad)^2, (cd)^5 >$$

let $x=at$, then $x^3=a^3t^3=a$ and $t=ax=x^4$

$$< x, c, d \; ; \; x^6, (x^4c)^2, c^2, [x^3,x^4], (x^3c)^2, x^{12}, d^2, (x^3d)^2, (cd)^5 >$$

$1=x^4cx^4c=xcx^7c=xcxc$

$$< x, c, d \; ; \; x^6, (xc)^2, c^2, (x^3c)^2, d^2, (x^3d)^2, (cd)^5 >$$

$(x^3c)^2$ is a consequence of $(xc)^2$ and c^2

$$< x, c, d \; ; \; x^6, (xc)^2, c^2, d^2, (x^3d)^2, (cd)^5 >$$

let $y=x^3cd$, then $y^5=x^{15}(cd)^5=x^3$

$$< x, c, d, y \; ; \; x^6, (xc)^2, c^2, (x^3d)^2, d^2, (cd)^5, y=x^3cd, y^5=x^3 >$$

$cd=x^3y=y^6$ and $(cd)^5=y^{30}$ and $d=cy^6$

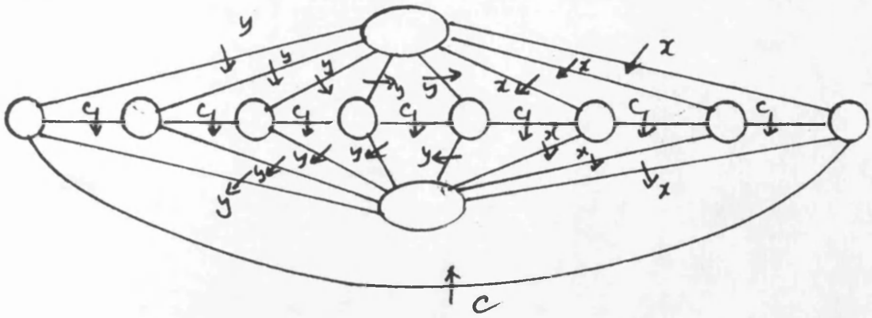
$$< x, c, y \; ; \; y^{10}, (xc)^2, c^2, (cy^6)^2, (y^5cy^6)^2, y^{30}, y=y^5ccy^6, y^5=x^3 >$$

$(y^5cy^6)^2=(cy)^2$ and $(cy^6)^2$ is a consequence of $(cy)^2$ and c^2

$$\mathscr{C}' = < x, c, y \; ; \; y^{10}, xcxc^{-1}, c^2, ycyc^{-1}, y^5x^{-3} > .$$

By Theorem 6.1.1, $d(C)=2+1+2-4=1$ but $\chi(\mathscr{C}')=3$. So $\mathscr{C}'$ is not efficient. But we cannot apply Theorem 1.4.3 to prove that $\mathscr{C}'$ is minimal. Because the following picture belongs to a generating set for $\pi_2(\mathscr{C}')$ and so $y^5x^{-3}N+y^5x^{-2}N+y^5x^{-1}N$,

$1+yN+y^2N+y^3N+y^4N$, $1+cx^3N$ are contained in $I_2(\mathscr{C}')$, where N is the normal closure of the relators of $\mathscr{C}'$ in the free group on $\{x,c,y\}$. And the numbers of terms of them are relative prime.



6.1.2 Calculation of $H_2(C)$ and $H^2(C)$ of aspherical Coxeter groups

Let $\mathscr{C}$ be an aspherical Coxeter presentation as in (6-1). Then Theorem 6.1.4 enables us to work out the maps ∂_2, ∂_3 in (1-4).

For computational reasons it is useful, in part, to work with $\mathbb{Z}[1/2]$-coefficients, rather than $\mathbb{Z}$-coefficients. Thus we let

$$D_2 = (\bigoplus_{v\in\mathbf{v}} \mathbb{Z}[1/2]a_v) \oplus (\bigoplus_{e\in\mathbf{e}^+} \mathbb{Z}[1/2]a_e)$$

$$D_1 = \bigoplus_{v\in\mathbf{v}} \mathbb{Z}[1/2]t_v$$

$$D_2^* = (\bigoplus_{v\in\mathbf{v}} \mathbb{Z}[1/2]a_v^*) \oplus (\bigoplus_{e\in\mathbf{e}^+} \mathbb{Z}[1/2]a_e^*)$$

$$D_3^* = \bigoplus_{e\in\mathbf{e}^+} \mathbb{Z}[1/2]b_e^*$$

And we let P_2 (resp. P_1, P_2^*, P_3^*) consist of the elements of D_2 (resp. D_1, D_2^*, D_3^*) whose coefficients lie in $\mathbb{Z}$.

For $v\in\mathbf{v}$, let

$$\lambda_v = a_v^* + \sum_{e\in Adj(v)} \phi(e)/2\ a_e^* \in D_2^*$$

and for $e\in\mathbf{e}^+$, let

$$\mu_e^{\varepsilon} = \varepsilon(\ a_e - \phi(e)/2\ (a_{\iota(e)} + a_{\tau(e)})) \in D_2 \qquad (\varepsilon=\pm1).$$

Define maps

$$\delta : D_2 \longrightarrow D_1,\quad a_u \longmapsto 2t_u,\quad a_e \longmapsto \phi(e)(t_{\iota(e)}+t_{\tau(e)})$$

$$\delta^* : D_2^* \longrightarrow D_3^*,\quad a_u^* \longmapsto -\sum_{e\in \mathrm{Adj}(u)} \phi(e)b_e^*,\quad a_e^* \longmapsto 2b_e^*.$$

Then the maps δ_2 and δ_3^* in (1-5), (1-8) are the restrictions

$$\delta_2 = \delta : P_2 \longrightarrow P_1$$

$$\delta_3^* = \delta^* : P_2^* \longrightarrow P_3^*$$

The maps (1-6), (1-7) are given by

$$\delta_3 : \bigoplus_{e\in \mathbf{e}^+} \mathbb{Z}b_e \longrightarrow P_2,\qquad b_e \longmapsto 2\mu_e$$

$$\delta_2^* : \bigoplus_{v\in \mathbf{v}} \mathbb{Z}t_v^* \longrightarrow P_2^*,\qquad t_v^* \longmapsto 2\lambda_v$$

Now we will prove Corollary 6.1.3 and Theorem 6.1.5. To do this, we need the following Lemmas.

Lemma 6.1.10. (i) $\{a_v\ (v\in\mathbf{v}),\ \mu_e\ (e\in\mathbf{e}^+)\}$ *is a basis for* D_2.

(ii) $\mathrm{Ker}\delta$ *is generated by the elements* μ_e $(e\in\mathbf{e}^+)$.

(iii) $\mathrm{Ker}\delta_2 = \mathrm{Ker}\delta \cap P_2$ *is generated by*

μ_e ($e\in\mathbf{e}^+$;*even*)

$2\mu_e$ ($e\in\mathbf{e}^+$;*odd*)

μ_c (c;*odd closed path*), *where an closed path is a path consisting of odd edges, and if* $e_1,\ldots,e_n$ *are the edges making up* c *then*

$$\mu_c = \sum_{i=1}^{k} \mu_{e_i}.$$

Proof. (i) and (ii) are clear. For (iii), let

$$\mu = \mu_{e_1} + \ldots + \mu_{e_k}$$

be an element of $\mathrm{Ker}\delta_2$, where $e_1,\ldots,e_k$ are distinct odd edges. It suffices to show that μ is a sum of μ_c's (c;odd closed path), since μ_e (e;even) and $2\mu_e$ are elements of $\mathrm{Ker}\delta_2$. Let $\iota(e_1)=v_0$ and $\tau(e_1)=v_1$. Since $\mu\in P_2$, one of the other e_j's, which we can assume to be e_2 without loss of generality, must have v_1 as one of its endpoints. Now replacing e_2 by e_2^{-1} if necessary, we can assume that $\iota(e_2)=v_1$. (This replacement is permissible, since

$$\mu - 2\mu_{e_2} = \mu_{e_1} - \mu_{e_2} + \mu_{e_3} + \ldots + \mu_{e_k}$$
$$= \mu_{e_1} + \mu_{e_2^{-1}} + \mu_{e_3} + \ldots + \mu_{e_k}$$

and $2\mu_{e_2} \in \mathrm{Ker}\delta_2$.) Let $\tau(e_2)=v_2$. Since Γ is simple $v_2 \neq v_0$. So there must exist another edge, say e_3, with one endpoint v_2. By a similar argument to the above, we can suppose that $\iota(e_3)=v_2$. Let $\tau(e_3)=v_3$. If v_3 is v_0 then $e_1e_2e_3$ is a closed odd path. Otherwise, there exists another edge, say e_4, with one endpoint v_3. And so on. Eventually we must get a closed odd path c_1. By induction $\mu - \mu_{c_1}$ is a sum of μ_c's (c a closed odd path).

Lemma 6.1.11. (i) $\{\lambda_v\ (v\in \mathbf{v}),\ a_e^*\ (e\in e^+)\}$ *is a basis for* D_2^*.

(ii) $\mathrm{Ker}\delta^*$ *is generated by the elements* λ_v $(v\in \mathbf{v})$.

(iii) $\mathrm{Ker}\delta_3^* = \mathrm{Ker}\delta^* \cap P_2^*$ *is generated by*

$$2\lambda_v\ (v\in \mathbf{v},\ v;\mathit{odd}),\quad \sum_{v\in \mathbf{v}_i} \lambda_v\ (i=1,\ldots,d).$$

Proof. (i) and (ii) are clear. For (iii), let

$$\lambda = \lambda_{v_1} + \ldots + \lambda_{v_k}$$

be an element of $\mathrm{Ker}\delta_3^*$, where $v_1,\ldots,\ v_k$ are distinct odd vertices. It suffices to show that λ is a sum of $\sum_{v\in \mathbf{v}_i} \lambda_v$, since $2\lambda_v$ $(v\in \mathbf{v},\ v;\mathrm{odd})$ are elements of $\mathrm{Ker}\delta_3^*$ and each even vertex constitutes a $\mathbf{v}_j$ for some $j=1,\ldots,d$. Let $V=\{v_1,\ \ldots\ ,\ v_k\}$. We claim that if $v_i\in V$ then all vertices of Γ joined to v_i by an odd edge are also in V. For suppose that v is joined to v_i by an odd edge e. If $v\notin V$ then the coefficient of a_e^* in λ would be $\phi(e)/2\ \notin \mathbb{Z}$.

It now follows that V is the union of some subsets $\mathbf{v}_{j_1},\ \ldots\ ,\mathbf{v}_{j_p}$ of the $\mathbf{v}_j$'s.

Now a $\mathbb{Z}$-basis for $\mathrm{Ker}\delta_2$ is obtained as follows. Fix a base vertex of T_i and for each $e \in \Gamma_i$, let the closed path $c_e = c_1 e c_2^{-1}$, where c_1 and c_2 are geodesic paths from the base vertex to the initial and terminal vertices of e in T_i like Fig.6.2.

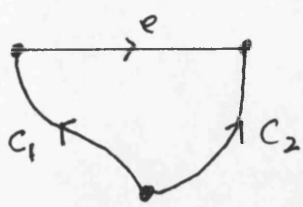


Figure 6.2.

If $c = e_1 \ldots e_k$ is any odd closed path we have

$$\mu_c = \mu_{c_{e_1}} + \ldots + \mu_{c_{e_k}}$$

Also $\mu_{c_e} = 0$ if $e \in T_i$. Thus we get the following Lemma.

Lemma 6.1.12. A *$\mathbb{Z}$-basis for* $\mathrm{Ker}\delta_2$ *consists of*

$$\mu_e \ (e \in e^+ ;\textit{even})$$

$$2\mu_e \ (e \in e^+ ;\textit{odd edge in } \bigcup_{i=1}^{d} T_i)$$

$$\mu_{c_e} \ (e \in e^+ ;\text{odd edge outside } \bigcup_{i=1}^{d} T_i).$$

Now we take a $\mathbb{Z}$-basis for $\mathrm{Ker}\delta_3^*$. For each i, let $\lambda_i = \sum_{v \in v_i} \lambda_v$ and let $\hat{v}_i$ be the set obtained from v_i by removing one vertex. Then we get the following Lemma.

Lemma 6.1.13. $\{\lambda_1, \ldots, \lambda_d, 2\lambda_v \ (v \in \hat{v}_i, i = 1, \ldots, d)\}$ *is a $\mathbb{Z}$-basis for* $\mathrm{Ker}\delta_3^*$.

And we get the following Lemmas from Theorem 6.1.4.

Lemma 6.1.14. (i) $\mathrm{Im}\delta_3$ *is generated by*

$2\mu_e$ ($e \in \mathbf{e}^+$;*even*)

$2\mu_e$ ($e \in \mathbf{e}^+$;*odd edge in* $\bigcup_{i=1}^{d} T_i$)

$2\mu_{c_e}$ ($e \in \mathbf{e}^+$;*odd edge outside* $\bigcup_{i=1}^{d} T_i$)

(ii) $\mathrm{Im}\delta_2^*$ *is generated by*

$2\lambda_i$ ($i=1,\ldots,d$)

$2\lambda_v$ (v;*odd*).

By Lemmas 6.1.12 - 6.1.14, we get Corollary 6.1.3 and Theorem 6.1.5.

6.1.3 Efficiency

Consider a presentation $\mathcal{Q} = \langle \mathbf{x}, y ; x^2, y^2, (xy)^{2\alpha+1}, \mathbf{r} \rangle$, where $x \in \mathbf{x}$ and α is a positive integer. Let $z \notin \mathbf{x} \cup \{y\}$ and $\mathbf{s}$ be the set obtained from $\mathbf{r}$ by replacing all occurences of y in each relator $R \in \mathbf{r}$ by $xz^{-\alpha}$. Then we get another presentation $\mathcal{Q}_y = \langle \mathbf{x}, z ; x^2, xz^{-\alpha}xz^{\alpha+1}, \mathbf{s} \rangle$. We call this the *deformaton of* $\mathcal{Q}$ *at y.*

Lemma 6.1.15. *$\mathcal{Q}$ and $\mathcal{Q}_y$ define the same group but* $\chi(\mathcal{Q}_y) = \chi(\mathcal{Q}) - 1$.

Proof. We will use a sequence of Tietze transformations.

$\mathcal{Q} = \langle \mathbf{x}, y ; x^2, y^2, (xy)^{2\alpha+1}, \mathbf{r} \rangle$

replace y with xy

$\langle \mathbf{x}, y ; x^2, (xy)^2, y^{2\alpha+1}, \mathbf{r}' \rangle$, where $\mathbf{r}'$ is the set obtained from $\mathbf{r}$ by replacing all occurences of y in each relator $R \in \mathbf{r}$ by xy

let $z = y^2$

$\langle \mathbf{x}, y, z ; x^2, (xy)^2, y^{2\alpha+1}, z = y^2, \mathbf{r} \rangle$

$y = z^{-\alpha}$, $z^{2\alpha+1} = 1$

$\langle \mathbf{x}, z ; x^2, (xz^{-\alpha})^2, z^{2\alpha+1}, \mathbf{s} \rangle$

$xz^{-\alpha}x^{-1}=z^{\alpha}$ then $xz^{-\alpha}x^{-1}=z^{-(\alpha+1)}$

$< x, z ; x, xz^{-\alpha}xz^{\alpha+1}, z^{2\alpha+1}, s >$

$$z^{\alpha^2} = x^2z^{\alpha^2}x^{-2} = x(xz^{\alpha}x^{-1})^{\alpha}x^{-1} = x(z^{\alpha+1})^{\alpha}x^{-1} = (xz^{\alpha}x^{-1})^{\alpha+1} = z^{(\alpha+1)^2}$$

$\therefore z^{2\alpha+1}$ is a consequece of x^2 and $xz^{-\alpha}xz^{\alpha+1}$

$< x, z ; x^2, xz^{-\alpha}xz^{\alpha+1}, s > = \mathcal{Q}_y$.

Proof of Theorem 6.1.6. If Γ has an odd edge then $\mathscr{C}$ is not efficient by Lemma 6.1.15. Suppose that all edges of $\mathscr{C}$ are even. Then by Corollary 6.1.3,

$$\upsilon(C) = 1-0+n.$$

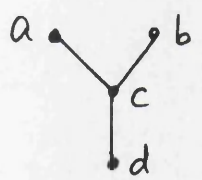
And

$$\chi(\mathscr{C}) = 1-m+(n+m)=1+n.$$

Thus $\mathscr{C}$ is efficient.

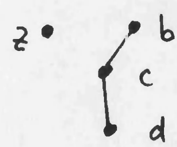
Proof of Theorem 6.1.7. Let $\Phi = \bigcup_{i=1}^{d} T_i$. Then we will perform deformations at extremal vertices iteratively. For example, let

$$\mathscr{C} = < a, b, c, d ; a^2, b^2, c^2, d^2, (ab)^3, (ac)^5, (bc)^3, (ad)^2, (cd)^3 >.$$

Φ 

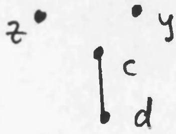
Firstly, we perform deformation at a, then we get

$$\mathscr{C}_a = < b,c,d,z;b^2,c^2,d^2,(cz^{-2}b)^3,cz^{-2}cz^3,(bc)^3,(cz^{-2}d)^2,(cd)^3 >.$$

Φ_a 

Performing deformation at b gives us

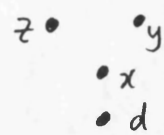
$$\mathscr{C}_{a,b} = < c,d,z,y;c^2,d^2,(cz^{-2}cy^{-1})^3,cz^{-2}cz^3,cy^{-1}cy^2,(cz^{-2}d)^2, (cd)^3 >.$$

$\Phi_{a,b}$ 

Once again, we perform deformation at c, then we get

$$\mathscr{C}_{a,b,c} = \langle d,z,y,x; d^2, (dx^{-1}z^{-2}dx^{-1}y^{-1})^3, dx^{-1}z^{-2}dx^{-1}z^3, dx^{-1}y^{-1}dx^{-1}y^2,$$
$$(dx^{-1}z^{-2}d)^2, dx^{-1}dx^2 \rangle$$

$\Phi_{a,b,c}$



Let $\mathscr{C}'$ be the presentation obtained from $\mathscr{C}$ by successively deforming all edges of Φ in the above way. Then $\mathscr{C}$ and $\mathscr{C}'$ define the same group and by Lemma 6.1.15,

$$\chi(\mathscr{C}') = \chi(\mathscr{C}) - (\text{ the number of edges of } \Phi \cap e^+).$$
$$= 1-m+(m+n_e+n_o)-(n_o-l)$$
$$= 1+n_e+l$$

because the number of edges of $\Phi \cap e^+$ is equal to n_o-l. Since $l=n_o-m+d$, by Corollary 6.1.2, we get

$$\upsilon(C) = 1+(n+d-m)$$
$$= 1+(n+l-n_o)$$
$$= 1+n_e+l.$$

Therefore $\mathscr{C}'$ is an efficient presentation for C.

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