SECOND ORDER DEHN FUNCTIONS OF FINITELY PRESENTED GROUPS AND MONOIDS

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Contents

STATEMENT

Chapter 1 covers basic materials such as two-complexes, pictures over a two-complexes, group presentations, monoid presentations, monoid pictures, (first order) Dehn functions of groups and monoids. Most of these are standard and can be found, for example, in [Al1, BoPr, ECHLPT, Jo, Ki, Mo, NaPr, Pr1, Pr2, Pr3] as indicated.

Chapter 2 is my own work which had been done before the joint paper [ABBPW1] with Alonso, Bogley, Burton and Pride and the results in this chapter were then extended in [ABBPW1].

§3.1 is joint work with S. J. Pride, and §3.2, §3.3 are my own work.

Chapters 4, 6, 7 are my own work.

Chapter 5 is joint work with Pride which will appear in a joint paper with Alonso, Bogley, Burton and Pride [ABBPW2] as Sections 5, 6.

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ABSTRACT

The main work of this thesis starts with Chapter 2.

Chapter 2 concerns the second homotopy module of a finitely presented group of type F_3. We define the second higher order Dehn functions by considering the comparison between the "volume" and the "surface area" of nullhomotopies of spherical maps into CW complexes. We show that the second order Dehn function of groups of type F_3 is an invariant of quasi-isometry type.

In Chapter 3, we translate the concept of the second order Dehn function of finite group presentations to FDT monoid presentations by introducing a *well-placed* retraction relation between any two two-complexes and showing some invariance results. We show that the second order Dehn function of an FDT monoid at a fixed element is an invariant of isomorphism type.

In Chapters 4, we give upper bounds for asynchronously combable groups with departure function.

In Chapter 5, we first give the general upper bounds for direct products. Then we concentrate on the calculations for the optimal bounds of second order Dehn functions of direct products of the form $G_0 \times F$ where the second order Dehn function of G_0 is bounded by a linear function and F is a free group of finite rank. Some interesting examples are given.

In Chapters 6, we carry out calculations for the upper bounds of second order Dehn functions of HNN-extensions, amalgamated free products, and split extensions, and finally in Chapter 7, some nice upper bounds as well as lower bounds for the second order Dehn functions of groups of the form $\mathbb{Z}^2 \rtimes_\phi F$ are established where F is a free group of finite rank.

NOTATIONS

Let G and H be groups, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}^+$ be the sets of all integers, all natural numbers, and all non-negative real numbers respectively

$G \times H$	the direct product
$G * H$	the free product of G and H
$G \rtimes_\phi H$	split extension of G by H with H-action ϕ
$G \cong H$	G is isomorphic to H
G/K	the quotient group of G by a normal subgroup K
$\mathbb{Z}G$	the integral group ring
$\delta_G^{(1)}$	the first order Dehn function of G
$\delta_G^{(2)}$	the second order Dehn function of G
G'	derived group (commutator subgroup) of G
$[a,b]$	the commutator of a and b ($= aba^{-1}b^{-1}$, $a,\ b \in G$)
a^b	the conjugate of a by b in G ($= b^{-1}ab$, $a,\ b \in G$)
$\mathbb{Z}^n$	the free abelian group of rank n

If $\mathcal{P} = < \boldsymbol{x}; \boldsymbol{r} >$ is a presentation, and if W is a word on $\boldsymbol{x}$

$F(\boldsymbol{x})$	the free group generated by $\boldsymbol{x}$
$G(\mathcal{P})$	the group defined by $\mathcal{P}$
$\pi_1(\mathcal{P})$	the first homotopy group
$\pi_2(\mathcal{P})$	the second homotopy module
$[W]$	the free equivalence class containing W
$\overline{W}$	the element of $G(\mathcal{P})$ represented by W
$L(W)$	the word length of W

Let $\mathbb{P}$ be a picture over $\mathcal{P}$, $\boldsymbol{X}$ be a set of generating set of $\pi_2(\mathcal{P})$

$\partial\mathbb{P}$	the boundary of $\mathbb{P}$
$A(\mathbb{P})$	the area (the disc number) of $\mathbb{P}$
$< \mathbb{P} >$	the equivalence class containing $\mathbb{P}$
$V_{\mathcal{P},\boldsymbol{X}}(\mathbb{P})$	the volume of $\langle \mathbb{P} \rangle$
$\delta_{\mathcal{P}}^{(1)}$	the first order Dehn function of $\mathcal{P}$
$\delta_{\mathcal{P},\boldsymbol{X}}^{(2)}$	the second order Dehn function of $\mathcal{P}$ with respect to $\boldsymbol{X}$

If $\Gamma = (\boldsymbol{v},\ \boldsymbol{e})$ is a graph and $\boldsymbol{\mathcal{D}} = \langle \Gamma;\ \boldsymbol{Z} \rangle$ is a two-complex,

and $\boldsymbol{X}$ is a set of closed paths in Γ

ι	the initial function with domain $\boldsymbol{e}$ and range $\boldsymbol{v}$
τ	the terminal function with domain $\boldsymbol{e}$ and range $\boldsymbol{v}$
$^{-1}$	the inverse function with both domain and range $\boldsymbol{e}$
$\boldsymbol{\mathcal{D}}^{(1)}$	the 1-skeleton of $\boldsymbol{\mathcal{D}}$
$\pi_n(\boldsymbol{\mathcal{D}}, v)$	the n-th homotopy group of $\boldsymbol{\mathcal{D}}$ based on v, $n = 1,\ 2$
$[\gamma]$	the free equivalence class containing the path γ

Also if γ is a path in $\boldsymbol{\mathcal{D}}$, and if S is a monoid

$\overline{\gamma}$	the homotopy equivalence class containing γ
$Area_{\boldsymbol{\mathcal{D}}}(\gamma)$	the area of γ if γ is contractible in $\boldsymbol{\mathcal{D}}$
$Area_{\boldsymbol{\mathcal{D}},\boldsymbol{X}}(\gamma)$	the area of γ relative to $\boldsymbol{X}$ if γ is contractible in $\boldsymbol{\mathcal{D}}^{\boldsymbol{X}}$
$(\boldsymbol{\mathcal{D}},\ S)$	the object with S-action on $\boldsymbol{\mathcal{D}}$ on both sides compatibly
$\mathfrak{C}$	the collection of all objects

Let $\hat{\boldsymbol{\mathcal{P}}} = [\boldsymbol{x}\ ;\ \boldsymbol{r}]$ be a finite monoid presentation

$\hat{F}(\boldsymbol{x})$	the free monoid generated by $\boldsymbol{x}$
$S(\hat{\boldsymbol{\mathcal{P}}})$	the monoid defined by $\hat{\boldsymbol{\mathcal{P}}}$
$\boldsymbol{\mathcal{D}}(\hat{\boldsymbol{\mathcal{P}}})$	the two-complex arising from $\hat{\boldsymbol{\mathcal{P}}}$

Let $\mathbb{P}$ be a path in $\boldsymbol{\mathcal{D}}(\hat{\boldsymbol{\mathcal{P}}})$ (or a picture over $\hat{\boldsymbol{\mathcal{P}}}$),

$\boldsymbol{X}$ be a trivialiser of $(\boldsymbol{\mathcal{D}}(\hat{\boldsymbol{\mathcal{P}}}), \hat{F}(\boldsymbol{x}))$, and W be a word on $\boldsymbol{x}$

$L(\mathbb{P})$	the path length (or the disc number) of $\mathbb{P}$
$< \mathbb{P} >$	the equivalence class containing $\mathbb{P}$
$\overline{W}$	the element of $S(\hat{\boldsymbol{\mathcal{P}}})$ represented by W
$L(W)$	the word length of W
$\hat{\delta}^{(1)}_{\hat{\boldsymbol{\mathcal{P}}}}$	the first order Dehn function of $\hat{\boldsymbol{\mathcal{P}}}$
$\hat{\delta}^{(2)}_{\hat{\boldsymbol{\mathcal{P}}},\boldsymbol{X},W}$	the second order Dehn function of $\hat{\boldsymbol{\mathcal{P}}}$ with respect to $\boldsymbol{X}$ at W

We adopt the usual notations in set theory

$A \cup B$	the union of the sets A
$A - B$	the set difference
$A \subseteq B$	A is a subset of B
$a \in A$	a belongs to A
$\|A\|$	the cardinality of A

Also, if $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a function

$\overline{f}$	the subnegative closure of f

Chapter 0

Introduction

Due to the introduction of *word hyperbolic* [Gr] and *automatic* [ECHLPT] groups, the (first order) isoperimetric properties of finitely presented groups have become a central issue in geometric group theory. In general, to measure the algorithmic complexity of the word problem of finitely presented groups one can consider the first order Dehn function of these groups, which arranges comparison between "circumference" and "area" of van Kampen diagrams into an integer-valued function that is an invariant of quasi-isometry type. This also raises another very interesting issue, namely to classify finitely presented groups in terms of quasi-isometries.

In [ABBPW1], we defined the Dehn functions of groups in all dimensions in terms of topology by considering the comparison between the "volume" and the "surface area" of nullhomotopies of spherical maps into CW-complexes. There, we proved that if $\mathcal{D}$ is the finite $(n+1)$-skeleton of a $K(G,1)$-complex, then the higher order Dehn functions of $\mathcal{D}$ through dimension n are invariants of the quasi-isometry type of the group G, and moreover, satisfy quasi-retract inequalities.

However, to apply the definition and the quasi-retract inequality as well as the quasi-isometry invariance to particular groups in low dimensions, it is natural to ask for a combinatorial version of the theory, namely in terms of a combination of geometry and algebra. In fact, the (first order) Dehn function or more generally, the (first order) isoperimetric function is originally defined combinatorially by Gromov [Gr]. See also [Al1, Brc, Gel].

Aiming at this, we give a combinatorial definition of the second order Dehn function $\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}$ of any finite group presentation $\boldsymbol{\mathcal{P}}$ with respect to a given generating set $\boldsymbol{X}$ (finite or not) of the associated second homotopy module $\pi_2(\boldsymbol{\mathcal{P}})$ in §2.1, namely, the second order Dehn function of a finite presentation will be defined by considering the comparison between the "volume" and the "surface area" of spherical pictures over this presentation.

We will prove that the definition is unambiguous.

That the first order Dehn function of finitely presented groups is an invariant of quasi-isometry type was first discovered by Alonso [Al1], where he proved that if the Cayley graphs of two finitely presented groups are quasi-isometric (see Definition 2.2.8) then the first order Dehn functions of these two groups with respect to the generating sets are equivalent. We will show a further statement (Theorem 2.2.13) in §2.3, that if groups G and H are finitely presented by $\mathcal{P}$ and $\mathcal{Q}$ respectively, and if G is a quasi-retract of H with H being of type F_3, then G is also of type F_3 and the second order Dehn function of $\mathcal{P}$ with respect to any finite generating set of $\pi_2(\mathcal{P})$ is bounded by the second order Dehn function of $\mathcal{Q}$ with respect to any finite generating set of $\pi_2(\mathcal{Q})$. The quasi-isometry invariance property then is a corollary of this result. Thus, up to equivalence, $\delta^{(2)}$ is independent of the choice of different finite presentations of G. We then use $\delta_G^{(2)}$ to denote a particular representative of the equivalence class.

We point out here that this work had been done before the paper [ABBPW1] and this idea then was extended to all dimensions in [ABBPW1].

Besides, in order to obtain the quasi-retract inequalities, in §2.2 we establish some general relationships between pictures over different two-complexes and mappings from one to the other.

Pride [Pr3] introduced the concept of (first order) Dehn function of finitely presented monoids. See also [MaOt1] where it is proved that the word problem of a given finitely presented monoid is solvable if and only if its *first order Dehn function* is bounded by a recursive function. In his paper [Pr4] (also see [Sq2, GuSa]), Pride developed a geometric technique in the low-dimensional homotopy theory of monoids by associating a two-complex $\mathcal{D}(\hat{\mathcal{P}})$ with a monoid presentation $\hat{\mathcal{P}} = [\boldsymbol{x}\ ;\ \boldsymbol{r}]$ (see §1.3.3). The free monoid $\hat{F}(\boldsymbol{x})$ on $\boldsymbol{x}$ acts on $\mathcal{D}(\hat{\mathcal{P}})$ on both the left and right. The two-complex $\mathcal{D}(\hat{\mathcal{P}})$ is also a geometric (in terms of pictures) interpretation of the idea of a two-complex associated with a *string rewriting system* introduced by Squier in [Sq2]. Moreover, Squier [Sq2] also introduced the notion of a monoid of *finite derivation type* (FDT). That means, one can find a finite presentation $\hat{\mathcal{P}}$ for the monoid, say S, and a finite set $\boldsymbol{X}$ of closed paths in $\mathcal{D} = \mathcal{D}(\hat{\mathcal{P}})$ such that the two-complex obtained from $\mathcal{D}(\hat{\mathcal{P}})$ by attaching additional

2-cells with boundaries from $\hat{F}(\boldsymbol{x}) \cdot \boldsymbol{X} \cdot \hat{F}(\boldsymbol{x})$ has trivial fundamental group (i.e. $\boldsymbol{X}$ is a *trivialiser* of $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$). Squier then proved two interesting results. First, the notion is independent of the choice of finite presentation of a given monoid; second, a monoid with a finite complete presentation is FDT. By another paper [Sq1], a monoid with a *finite complete presentation* is of type left and right FP_3. (The popularity of a monoid with a finite complete presentation is due to the fact that there is a syntactically simple algorithm for solving the word problem, that is, given two words on the generators, reduce them to irreducible words respectively, and then compare these two irreducible words literally. See [MaOt2] for a reference.) Squier also asked if an FDT monoid is left and right FP_3. Cremanns and Otto [CrOt], Lafont [La] and Pride [Pr3, §3.3; Pr4, Theorems 3.3, 4.2] independently have given this question a positive solution.

In §3.1 we first consider the definition of the *delta function* of a pair $(\mathcal{D}, S)$ where $\mathcal{D}$ is a locally finite two-complex, S is a monoid acting on $\mathcal{D}$ on both sides compatibly, and then show some invariance results. The concept of a *well-placed retraction* between two pairs $(\mathcal{D}, S)$, $(\mathcal{D}_0, S_0)$ plays the key role.

We then in §3.2 apply these invariance results to the two-complexes associated to finite FDT monoid presentations. We show (see Theorem 3.2.2) that our definition of the second order Dehn function $\hat{\delta}^{(2)}$ of a given FDT monoid S for a fixed element $s \in S$ of the monoid is independent of the choice of finite monoid presentation. We then use $\hat{\delta}^{(2)}_{S,s}$ to denote a particular representative of the equivalence class. Establishing this theory also gives an alternative proof of Squier's result mentioned above, that being FDT is independent of the choice of finite presentation.

But the weakness of this theory is that it does not give an upper bound for all Dehn functions of a given FDT monoid for different elements of the monoid. To overcome this, in §3.3 we focus on the shortest words of each word congruence class. We then are able to define the second order Dehn function of a finite monoid presentation (Definition 3.3.6) by which we can give an upper bound of the second order Dehn functions of the presentation of all words (see (3.4)). But I am unable to show this definition is independent of the choice of presentation of a given monoid.

Since groups are monoids, it is natural to ask what are the relationships among the lower order Dehn functions: $\delta^{(1)}_G$, $\hat{\delta}^{(1)}_G$, $\delta^{(2)}_G$, and $\hat{\delta}^{(2)}_{G,1}$ when G is of type F_3. (Note that, from [CrOt], a group is FDT if and only if it is of type F_3.) I have established some inequalities among these functions as follows. (These results are not proved in the thesis, but appear in [Wa].)

(i) Let G be a finite presented group. Then

$$\delta_G^{(1)} \preceq \hat{\delta}_G^{(1)} \preceq (\delta_G^{(1)})^2 + n\delta_G^{(1)}.$$

(ii) Let G be a group of type F_3. Then

$$\hat{\delta}_{G,1}^{(2)}(n) \preceq \delta_G^{(2)}(n) \preceq \overline{\hat{\delta}}_{G,1}^{(2)}(n^2)$$

for all $n \in \mathbb{N}$.

(Here $\preceq$ is a relation on number functions and $\overline{f}$ is the *subnegative closure* of a number function both of which will be defined in §1.2.3.)

We mention that one can use second order Dehn functions (of groups and monoids) to discuss certain decision problems concerning second homotopy problems.

From Chapter 4, we start the calculations of second order Dehn functions of groups. We remark that J. Wang [Waj] has done some calculations about second order Dehn functions of monoids.

In Chapter 4, we obtain upper bounds for second order Dehn functions of combable groups and asynchronously combable groups with departure function. This is an account of the analysis of Gersten's proof of any asynchronously combable group with a departure function being of type F_3 [Ge3]. The result that any word hyperbolic group has a linear second order Dehn function was proved by Bogley and Burton and appears in [ABBPW2]. This result is taken as a theorem (Theorem 4.2.1) of this chapter without giving a proof.

In Chapter 5, we first give general bounds for the second order Dehn function of a direct product of groups of type F_3. The results (though not the proofs) are similar to the first order case. We show (Theorem 5.1.3) that if G_0, G_1 are of type F_3, then

$$\max\{\delta_{G_0}^{(2)}, \delta_{G_0}^{(2)}\} \preceq \delta_{G_0 \times G_1}^{(2)} \preceq \max\{\overline{\delta}_{G_0}^{(2)}, \overline{\delta}_{G_0}^{(2)}\} + n^2.$$

We then concentrate on direct products of the form $G_0 \times F$, where F is free of finite rank and G_0 is a group of type F_3 with $\delta_{G_0}^{(2)}$ is bounded by a linear function. Let $h : [1, \infty) \longrightarrow \mathbb{R}^+$ be a strictly increasing continuous function such that

(i) *the restriction of* h *on* $\mathbb{N}$ *is subnegative;*

(ii) $h(x) \geq x$ *for all* $x \in [1, \infty)$*; and*

(iii) *the function* $x \longmapsto \frac{x}{h^{-1}(x)}$ *is increasing for* $x > n_0$ *for some natural number* $n_0 \in \mathrm{Im}h$.

and assume $\delta_{G_0}^{(1)} \preceq h$. Then (Theorem 5.2.1)

$$\delta_{G_0 \times F}^{(2)} \preceq \frac{n^2}{h^{-1}(n)}.$$

Furthermore, if G_0 has an aspherical presentation $\boldsymbol{\mathcal{P}}_0$ and there is a sequence $\mathbb{D}_i$ ($i = 1, 2, \cdots$) of *stable* pictures over $\boldsymbol{\mathcal{P}}_0$ such that

$$L(\partial \mathbb{D}_{[h^{-1}(n)]}) \leq b_1 h^{-1}(n); \quad \text{and}$$
$$c_1 n \leq A(\mathbb{D}_{[h^{-1}(n)]}) \leq c_2 n$$

for all integers $n \geq h(n_0)$, then (Theorem 5.2.6)

$$\delta_{\boldsymbol{\mathcal{P}},X}^{(2)} \succeq \frac{n^2}{h^{-1}(n)}.$$

By some interesting examples we show that there are infinitely many real numbers $\frac{3}{2} \leq \alpha \leq 2$ such that there exist groups of the form $G_0 \times F$ whose second order Dehn functions are exactly n^α, where G_0 has an aspherical presentation.

The calculation for an upper bound of second order Dehn functions of HNN-extensions, amalgamated free products, and split extensions are given in Chapter 6. For example, if G is an HNN-extension of a group G_0 of type F_3 with two isomorphic associated subgroups (finitely presented) $H, \check{H}$, then (Theorem 6.1.3)

$$\delta_G^{(2)} \preceq \overline{\delta}_{G_0}^{(1)}(\overline{\delta}_H^{(1)}),$$

and if $G = H \rtimes_\phi K$ is a split extension where H and K are of type F_3 then (Theorem 6.2.2) *either* there exists a constant $a > 1$ such that

$$\delta_G^{(2)} \preceq \overline{\delta}_H^{(2)}(\overline{\delta}_H^{(1)}(a^n)) + \overline{\delta}_K^{(2)}(n) + a^n,$$

or

$$\delta_G^{(2)} \preceq \overline{\delta}_H^{(2)}(\overline{\delta}_H^{(1)}(n^2)) + \overline{\delta}_K^{(2)}(n) + n^2.$$

People may be interested in making comparison of this chapter with Brick's work on the first order Dehn functions of extensions in [Brc].

Finally, in Chapter 7 we focus on estimating the upper bounds and lower bounds of the particular split extensions of the form $\mathbb{Z}^2 \rtimes_\phi F$, where F is a free group of finite rank. We show that the second order Dehn function of such a group is bounded over by $n^{\frac{3}{2}}$ and bounded below by $n \log n$. Moreover, if for some generator t of F, ϕ_t has eigenvalues ± 1, then the second order Dehn function of this group is bounded below by $n^{\frac{4}{3}}$; and if ϕ_t has finite order for some generator t of F, then the second order Dehn function of this group is exactly $n^{\frac{3}{2}}$.

Chapter 1

Preliminaries

1.1 Two-complexes

Most concepts and results in this section can be found in [Mo, NaPr, Pr1].

1.1.1 Graphs

A *graph* (in the sense of Serre, [Se]) $\Gamma = (\boldsymbol{v}, \boldsymbol{e})$ consists of two disjoint sets $\boldsymbol{v} = \boldsymbol{v}(\Gamma)$ (of *vertices*) and $\boldsymbol{e} = \boldsymbol{e}(\Gamma)$ (of *edges*) together with three functions:

$$\iota : \boldsymbol{e} \longrightarrow \boldsymbol{v}, \quad \tau : \boldsymbol{e} \longrightarrow \boldsymbol{v}, \quad {}^{-1} : \boldsymbol{e} \longrightarrow \boldsymbol{e}$$

called *initial, terminal* and *inverse* respectively with the properties that $\iota(e) = \tau(e^{-1})$, $(e^{-1})^{-1} = e$, and $e^{-1} \neq e$ for all $e \in \boldsymbol{e}$. (A graph is also called a *one complex.* See [Sta] and [Sti].)

A non-empty *path* α of Γ is a non-empty finite sequence of edges written in the form $\alpha = e_1 e_2 \cdots e_m$ such that $\tau(e_i) = \iota(e_{i+1})$, $1 \leq i \leq m-1$. The *initial vertex* $\iota(\alpha)$ of α is defined to be $\iota(e_1)$, the *terminal vertex* $\tau(\alpha)$ of α is defined to be $\tau(e_m)$, and the *inverse* α^{-1} of α is defined to be $e_m^{-1} e_{m-1}^{-1} \cdots e_1^{-1}$ which is also a path in Γ. The *length* of α then is m, denoted $L(\alpha) = m$. If $e_i^{-1} \neq e_{i+1}$ for all $i = 1, \cdots, m-1$ then we say that α is *freely reduced.* When $\tau(\alpha) = \iota(\alpha)$ we then say that α is a *closed path.* A closed path is *cyclically reduced* if all its cyclic permutations are freely reduced. For each $v \in \boldsymbol{v}$ we introduce the *empty path* 1_v at v which has no edges. We have $L(1_v) = 0$, and $1_v^{-1} = 1_v$.

If α and β are two paths of Γ with $\tau(\alpha) = \iota(\beta)$ then the *product* $\alpha\beta$ of α and β is defined to be the path starting with α followed by β.

Let $\boldsymbol{s}$ be any set of closed paths in Γ. By *symmetrical closure* of $\boldsymbol{s}$ we mean the set of all cyclic permutations (also paths in Γ) of each element of $\boldsymbol{s}$ and their inverses.

Let $\boldsymbol{v}_0 \subseteq \boldsymbol{v}$, $\boldsymbol{e}_0 \subseteq \boldsymbol{e}$. We say $\Gamma_0 = (\boldsymbol{v}_0, \boldsymbol{e}_0)$ is a *subgraph* of Γ if Γ_0 is closed under ι, τ and $^{-1}$.

A graph Γ is *connected* if given any two vertices of Γ there is a path in Γ joining them. A maximal connected subgraph of Γ is called a *component* of Γ.

The *star* of a vertex v of a graph Γ is the set $Star(v) = \{e : e \in \boldsymbol{e}, \iota(e) = v\}$. A graph Γ is *locally finite* if $Star(v)$ is finite for all $v \in \boldsymbol{v}$.

Let Γ, Γ' be any two graphs. A *mapping of graphs* $\phi : \Gamma \longrightarrow \Gamma'$ is a function sending $\boldsymbol{v}(\Gamma)$ to $\boldsymbol{v}(\Gamma')$ and edges in Γ to paths in Γ' so that $\phi(\iota(e)) = \iota(\phi(e))$, $\phi(\tau(e)) = \tau(\phi(e))$ and $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in \boldsymbol{e}(\Gamma)$. By this, ϕ extends to all paths of Γ, i.e., if $\alpha = e_1 \cdots e_m$ is a non-empty path of Γ then $\phi(\alpha) = \phi(e_1) \cdots \phi(e_m)$, and for any empty path 1_v we require that $\phi(1_v) = 1_{\phi(v)}$.

A mapping (of graphs) $\phi : \Gamma \longrightarrow \Gamma'$ is *rigid* if it maps edges to edges. Suppose that ϕ is rigid, and let $v \in \boldsymbol{v}(\Gamma)$. If $e \in Star(v)$ then $\phi(e) \in Star(\phi(v))$, thus $\phi(Star(v)) \subseteq Star(\phi(v))$. We say that a rigid mapping ϕ is *locally bijective* if for all $v \in \boldsymbol{v}(\Gamma)$, $\phi : Star(v) \longrightarrow Star(\phi(v))$ is bijective.

Let α' be a path in Γ'. A path α in Γ is a *lift* of α' at $\iota(\alpha)$ if $\phi(\iota(\alpha)) = \iota(\alpha')$ and $\phi(\alpha) = \alpha'$. We have the following lemma (see [Mo, Lemmas 1.1A.1, 1.1A.2]).

Lemma 1.1.1 *Let $\phi : \Gamma \longrightarrow \Gamma'$ be a locally bijective mapping of graphs. Then for any path $\alpha' \in \Gamma'$ and any vertex v of Γ with $\phi(v) = \iota(\alpha')$ there exists a unique lift of α' at v.*

1.1.2 Monoids acting on graphs

Let $\Gamma = (\boldsymbol{v}, \boldsymbol{e})$ be a graph, and let S be a monoid. We say that S *acts* on Γ on the *left* if S acts on the set $\boldsymbol{v} \cup \boldsymbol{e}$ in such a way that for any $v \in \boldsymbol{v}$, $e \in \boldsymbol{e}$ and any $s \in S$

(i) $s \cdot v \in \boldsymbol{v}, \quad s \cdot e \in \boldsymbol{e}$;

(ii) $\iota(s \cdot e) = s \cdot \iota(e),\ \ \tau(s \cdot e) = s \cdot \tau(e),\ \ (s \cdot e)^{-1} = s \cdot e^{-1}$.

This *left action* then extends to paths: if $\alpha = e_1 e_2 \cdots e_m$ is a non-empty path with $e_i \in \boldsymbol{e}$ $(1 \leq i \leq m)$ then

(iii) $s \cdot \alpha = (s \cdot e_1)(s \cdot e_2) \cdots (s \cdot e_m)$.

By (ii), $s \cdot \alpha$ is a path of Γ. For each $v \in \boldsymbol{v}$ we require $s \cdot 1_v = 1_{s \cdot v}$. Note that for any path α of Γ and any $s \in S$, we have $s \cdot \alpha^{-1} = (s \cdot \alpha)^{-1}$ by (ii) and (iii).

Similarly, we can define a *right action* of S on Γ.

We say that a left action and a right action of S on Γ are *compatible* if for any s, $s' \in S$ $(s \cdot \gamma) \cdot s' = s \cdot (\gamma \cdot s')$ for any $\gamma \in \Gamma$.

1.1.3 Two-complexes

A *two-complex* $\mathcal{D}$ is a pair

$$\mathcal{D} = \langle \Gamma :\ Z \rangle$$

where Γ is a graph (the 1-*skeleton* $\mathcal{D}^{(1)}$ of $\mathcal{D}$) and Z is a set of closed paths (called *defining paths*) of $\mathcal{D}$. We say $\mathcal{D}$ is *finite* if $\boldsymbol{v}(\Gamma)$, $\boldsymbol{e}(\Gamma)$ and Z all are finite. We say $\mathcal{D}$ is *locally finite* if Γ is, and we say $\mathcal{D}$ is *connected* if Γ is.

Let Γ' be a subgraph of Γ and let Z' be a subset of Z such that Z' is a set of closed paths of Γ'. Then the two-complex $\mathcal{D}' = \langle \Gamma';\ Z' \rangle$ is called a *subcomplex* of $\mathcal{D}$.

For any two-complex $\mathcal{D} = \langle \Gamma;\ Z \rangle$ there are four elementary *operations* on the paths of $\mathcal{D}$ introduced as follows (where we suppose that $(\mathrm{I})^{-1}$ and $(\mathrm{II})^{-1}$ are applicable).

- (I) Deletion of an inverse pair ee^{-1} of two successive edges.
- $(\mathrm{I})^{-1}$ The inverse of (I).
- (II) Deletion of a subpath β of a path γ with $\gamma = \gamma_1 \beta \gamma_2$ and $\beta \in Z \cup Z^{-1}$.
- $(\mathrm{II})^{-1}$ The inverse of (II).

We point out that in the presence of the operations (I) and $(\mathrm{I})^{-1}$, the operations (II) and $(\mathrm{II})^{-1}$ are equivalent to the following operation.

(II′) Replace a subpath β of a path $\gamma = \gamma_1\beta\gamma_2$ with $(\beta')^{-1}$ where either $\beta\beta'$ or $\beta'\beta$ is in $\boldsymbol{Z} \cup \boldsymbol{Z}^{-1}$.

Let $\boldsymbol{X}$ be another set of closed paths of Γ. Then $\mathcal{D}$ is a subcomplex of the two-complex $\mathcal{D}^{\boldsymbol{X}} = \langle\Gamma;\ \boldsymbol{Z},\ \boldsymbol{X}\rangle$. The operation (II′) above on the paths of $\mathcal{D}^{\boldsymbol{X}}$ then is divided into the following two operations.

(II'_1) Replace a subpath β of a path $\gamma = \gamma_1\beta\gamma_2$ with $(\beta')^{-1}$ where either $\beta\beta'$ or $\beta'\beta$ is in $\boldsymbol{Z} \cup \boldsymbol{Z}^{-1}$.

(II'_2) Replace a subpath α of a path $\gamma = \gamma_1\alpha\gamma_2$ with $(\alpha')^{-1}$ where either $\alpha\alpha'$ or $\alpha'\alpha$ is in $\boldsymbol{X} \cup \boldsymbol{X}^{-1}$.

As usual, any two paths γ, ρ in $\mathcal{D}$ are said to be *freely equal*, denoted by $\gamma \sim^{(1)} \rho$, if one can be obtained from the other by a finite sequence of applications of operations (I) and $(\mathrm{I})^{-1}$; they are said to be *homotopic* (or *equivalent*), denoted by $\gamma \sim \rho$, if one can be obtained from the other by a finite sequence of applications of operations (I), $(\mathrm{I})^{-1}$ and (II'_1); and are said to be *equivalent relative to* $\boldsymbol{X}$, denoted by $\gamma \sim \rho\,(\mathrm{rel}\ \boldsymbol{X})$, if one can be obtained from the other by a finite sequence of applications of operations (I), $(\mathrm{I})^{-1}$, (II'_1), and (II'_2). Note that if two paths of $\mathcal{D}$ are equivalent relative to $\boldsymbol{X}$ then they are homotopic in $\mathcal{D}^{\boldsymbol{X}}$. A closed path γ of $\mathcal{D}$ which is homotopic to an empty path is said to be *contractible* in $\mathcal{D}$.

For each path γ in $\mathcal{D}$, we write $[\gamma]$ for the free equivalence class consisting of all paths freely equal to γ, and write $\overline{\gamma}$ for the homotopy equivalence class consisting of all paths homotopic to γ.

Let $\mathcal{D} = \langle\Gamma;\ \boldsymbol{Z}\rangle$ and $\mathcal{D}' = \langle\Gamma';\ \boldsymbol{Z}'\rangle$ be any two two-complexes. A *mapping of two-complexes* $\phi : \mathcal{D} \longrightarrow \mathcal{D}'$ is a mapping of graphs from Γ to Γ' with the property that the image of each element of $\boldsymbol{Z}$ is a contractible path in $\mathcal{D}'$. Furthermore, we say that ϕ is *locally bijective* if

(i) ϕ is a locally bijective mapping of graphs; and

(ii) $\boldsymbol{Z}$ consists of all lifts of elements of $\boldsymbol{Z}'$.

The following two lemmas are standard and can be easily proved.

Lemma 1.1.2 *Let $\phi : \mathcal{D} \longrightarrow \mathcal{D}'$ be a mapping of two-complexes. Let γ, ρ be two paths in $\mathcal{D}$ with $\iota(\gamma) = \iota(\rho)$. If $\gamma \sim \rho$ in $\mathcal{D}$ then $\phi(\gamma) \sim \phi(\rho)$ in $\mathcal{D}'$.*

Lemma 1.1.3 *Let $\phi : \mathcal{D} \longrightarrow \mathcal{D}'$ be a locally bijective mapping of two-complexes. Let γ', ρ' be two paths in $\mathcal{D}'$ with $\iota(\gamma') = \iota(\rho')$, and let γ, ρ respectively be lifts of γ' and ρ' in $\mathcal{D}$ at some vertex v. If $\gamma' \sim \rho'$ in $\mathcal{D}'$ then $\gamma \sim \rho$ in $\mathcal{D}$.*

In the remainder of this section we suppose $\mathcal{D} = \langle \Gamma;\ \boldsymbol{Z} \rangle$ is a two-complex with $\Gamma = (\boldsymbol{v},\ \boldsymbol{e})$, and $\boldsymbol{X}$ is a set of closed paths of $\mathcal{D}$.

We now define the *area* function of a closed path γ as follows.

Definition 1.1.4 (i) *Let γ be a path contractible in $\mathcal{D}$. The* area *of γ, denoted by $Area_{\mathcal{D}}(\gamma)$ is the smallest number of operations of type* (II)$'_1$ *used in any transformation of γ to an empty path.*

(i) *Let γ be a path in $\mathcal{D}$ contractible in $\mathcal{D}^{\boldsymbol{X}}$. The* area *of γ* with respect to $\boldsymbol{X}$, *denoted by $Area_{\mathcal{D},\boldsymbol{X}}(\gamma)$, is the smallest number of operations of type* (II$'_2$) *used in any transformation of γ to an empty path.*

The following lemma will be used several times.

Lemma 1.1.5 *Let γ be any arbitrary closed path in $\mathcal{D}$ at some vertex v of $\mathcal{D}$. If γ is contractible in $\mathcal{D}^{\boldsymbol{X}}$ with $Area_{\mathcal{D},\boldsymbol{X}}(\gamma) = r$, then γ is homotopic in $\mathcal{D}$ to a product of conjugates of the form*

$$\gamma \sim \prod_{i=1}^{r} \rho_i \beta_i \rho_i^{-1}, \tag{1.1}$$

where $\beta_i \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$, ρ_i is a path in $\mathcal{D}$, $1 \le i \le r$.

Proof. If $r = 0$, then γ is contractible in $\mathcal{D}$ and hence $\gamma \sim 1_v$. Now let $r > 0$. By definition, there is a finite sequence of paths $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_n = 1_v$ such that for each i $(0 \le i \le n-1)$ one of γ_i and γ_{i+1} is obtained from the other by an application of operation (I), or (II$'_1$), or (II$'_2$), and there are precisely r applications of operation (II$'_2$).

Let m be the first number for which an operation of type (II$'_2$) is applied. Thus, $Area_{\mathcal{D},\boldsymbol{X}}(\gamma_{m+1}) = r - 1$. By induction hypothesis we then have $\gamma_{m+1} \sim \prod_{j=2}^{r} \rho_j \beta_j \rho_j^{-1}$ in $\mathcal{D}$, where each $\beta_j \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$. Let $\gamma_m = \rho_1 \alpha \rho_2$ and $\gamma_{m+1} = \rho_1 \alpha'^{-1} \rho_2$ where $\alpha\alpha'$ or

$\alpha'\alpha \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$. We can assume $\alpha\alpha' \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$ (the other case can be proved in the same way). Note that $\gamma_m \sim \gamma$. We then have

$$\begin{aligned}\gamma \sim \gamma_m &= \rho_1\alpha\rho_2 \sim^{(1)} \rho_1\alpha\alpha'\rho_1^{-1}\rho_1\alpha'^{-1}\rho_2 \\ &= \rho_1\beta_1\rho_1^{-1}\gamma_{m+1} \sim \rho_1\beta_1\rho_1^{-1}\prod_{j=2}^{r}\rho_j\beta_j\rho_j^{-1} \\ &= \prod_{i=1}^{r}\rho_i\beta_i\rho_i^{-1}\end{aligned}$$

where $\beta_1 = \alpha\alpha' \in \boldsymbol{X} \cup \boldsymbol{X}^{-1}$. This completes the proof. □

We will call the product of conjugates in (1.1) a *defining product* of γ relative to $\boldsymbol{X}$.

Remark 1.1.6 *In particular, if we let $\mathcal{D}_0 = \langle \Gamma; \emptyset \rangle$ be a two-complex where $\emptyset$ is the empty set, then $\mathcal{D} = \mathcal{D}_0^{\boldsymbol{Z}}$. Thus, if γ is contractible in $\mathcal{D}$, then by the above lemma we have*

$$\gamma \sim^{(1)} \prod_{i=1}^{r}\rho_i\beta_i\rho_i^{-1} \tag{1.2}$$

with $r = Area_{\mathcal{D}}(\gamma)$, certain paths ρ_i of $\mathcal{D}$, and some $\beta_i \in \boldsymbol{Z} \cup \boldsymbol{Z}^{-1}$, $1 \le i \le r$. We then call this product of conjugates a defining product *of γ.*

For the calculation of the area function we have the following lemma by (1.1), (1.2) and Definition 1.1.4.

Lemma 1.1.7 (i) *If two closed paths γ, γ' contractible in $\mathcal{D}^{\boldsymbol{X}}$ are homotopic in $\mathcal{D}$, then $Area_{\mathcal{D},\boldsymbol{X}}(\gamma) = Area_{\mathcal{D},\boldsymbol{X}}(\gamma')$.*

(ii) *If γ is contractible in $\mathcal{D}^{\boldsymbol{X}}$ and β is any path of $\mathcal{D}$ with $\tau(\beta) = \iota(\gamma)$ then $Area_{\mathcal{D},\boldsymbol{X}}(\gamma) = Area_{\mathcal{D},\boldsymbol{X}}(\beta\gamma\beta^{-1})$; if γ is contractible in $\mathcal{D}$ and β is any path of $\mathcal{D}$ with $\tau(\beta) = \iota(\gamma)$ then $Area_{\mathcal{D}}(\gamma) = Area_{\mathcal{D}}(\beta\gamma\beta^{-1})$.*

1.1.4 Fundamental groups of two-complexes

For any $v \in \boldsymbol{v}$ we let $\pi_1(\mathcal{D}, v)$ be the set of all homotopy equivalence classes each of which is of the form $\overline{\gamma}$ with γ a closed path in $\mathcal{D}$ at v. A multiplication can be defined on $\pi_1(\mathcal{D}, v)$ by $\overline{\gamma}_1\overline{\gamma}_2 = \overline{\gamma_1\gamma_2}$, and this multiplication can be easily checked to be well-defined.

By this multiplication, $\pi_1(\mathcal{D}, v)$ is then a group, the *fundamental group* of $\mathcal{D}$ at v. If $u \in \boldsymbol{v}$ is in the same component of $\mathcal{D}$ as v then $\pi_1(\mathcal{D}, u) \cong \pi_1(\mathcal{D}, v)$. Hence, if Γ is connected, then all fundamental groups of $\mathcal{D}$ are isomorphic.

Suppose that $\boldsymbol{X}$ is a set of closed paths in $\mathcal{D}$ such that every closed path in $\mathcal{D}$ is equivalent (rel $\boldsymbol{X}$) to the empty path. Thus, all fundamental groups of $\mathcal{D}^{\boldsymbol{X}}$ are trivial. We then say that $\boldsymbol{X}$ is a *trivialiser* of $\mathcal{D}$.

1.1.5 Monoids acting on two-complexes

Let $\mathcal{D} = \langle \Gamma;\ \boldsymbol{Z} \rangle$ be a two-complex. By a *left action* of S on $\mathcal{D}$ we mean that S acts on Γ on the left and $S \cdot \boldsymbol{Z} = \boldsymbol{Z}$. Similarly, we have the definition of a *right action* of S on $\mathcal{D}$. If this pair of actions of S are compatible on Γ then we say they are *compatible* on $\mathcal{D}$.

By $(\mathcal{D},\ S)$ we mean that the monoid S acts on $\mathcal{D}$ on both sides compatibly. Let $(\mathcal{D},\ S)$, $(\mathcal{D}',\ S')$ be two such pairs. By a *mapping* $\phi : (\mathcal{D},\ S) \longrightarrow (\mathcal{D}',\ S')$ we mean that

(i) $\phi : \mathcal{D} \longrightarrow \mathcal{D}'$ is a mapping of two-complexes;

(ii) $\phi : S \longrightarrow S'$ is a monoid homomorphism;

(iii) $\phi(s \cdot \gamma \cdot s') = \phi(s) \cdot \phi(\gamma) \cdot \phi(s')$ for any $s, s' \in S$ and any $\gamma \in \mathcal{D}$.

It can be checked that we then have a *category* $\mathfrak{C}$ with *objects* all pairs of the form $(\mathcal{D},\ S)$ and *morphisms* all mappings of these objects. We will say that an object $(\mathcal{D},\ S)$ of $\mathfrak{C}$ is *locally finite* if $\mathcal{D}$ is a locally finite two-complex.

Let $(\mathcal{D},\ S)$ be an object of $\mathfrak{C}$. Let $\mathcal{D}_0$ be a subcomplex of $\mathcal{D}$ and let S_0 be submonoid of S. If $\mathcal{D}_0$ is *S_0-invariant*, i.e., for any $\gamma \in \mathcal{D}_0$ we have $S_0 \cdot \gamma \cdot S_0 \subseteq \mathcal{D}_0$, then (with S_0 acting by restriction) $(\mathcal{D}_0,\ S_0)$ is also an object of $\mathfrak{C}$ which we call a *subobject* of $(\mathcal{D},\ S)$.

Let $(\mathcal{D},\ S)$ be an object of $\mathfrak{C}$ where $\mathcal{D} = \langle \Gamma;\ \boldsymbol{Z} \rangle$. Let $\boldsymbol{X}$ be a set of closed paths of $\mathcal{D}$. We write $\boldsymbol{X}^S$ for $S \cdot \boldsymbol{X} \cdot S$. Since S also acts on the two-complex $\mathcal{D}^{\boldsymbol{X}^S} = \langle \Gamma;\ \boldsymbol{Z}, \boldsymbol{X}^S \rangle$, thus, $(\mathcal{D}^{\boldsymbol{X}^S},\ S)$ is an object of $\mathfrak{C}$. We say that $\boldsymbol{X}$ *trivialises* $(\mathcal{D},\ S)$ (and then $\boldsymbol{X}$ is a *trivialiser* of $(\mathcal{D},\ S)$) if $\boldsymbol{X}^S$ is a trivialiser of $\mathcal{D}$.

Definition 1.1.8 *Let $(\mathcal{D},\ S)$ be an object of $\mathfrak{C}$ and let $(\mathcal{D}_0,\ S_0)$ be a subobject of $(\mathcal{D},\ S)$. A mapping $\phi : (\mathcal{D},\ S) \longrightarrow (\mathcal{D}_0,\ S_0)$ is a* retraction *if $\phi|_{\mathcal{D}_0} = id$ and $\phi|_{S_0} = id$, where id* *denotes the identity mapping. (We then say that $(\mathcal{D}_0, S_0)$ is a* retract *of $(\mathcal{D}, S)$ if there is a retraction from $(\mathcal{D}, S)$ to $(\mathcal{D}_0, S_0)$).*

1.2 Group presentations and monoid presentations

1.2.1 Words

Let $\boldsymbol{x}$ be non-empty set and let $\boldsymbol{x}^{-1}$ to be a set in 1:1 correspondence with $\boldsymbol{x}$ ($x \longmapsto x^{-1}$, $x \in \boldsymbol{x}$). The elements of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ are *letters*, and a *word* W on $\boldsymbol{x}$ is an expression

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$$

where $n \geq 0$, $x_i \in \boldsymbol{x}$, $\varepsilon_i = \pm 1$, and $1 \leq i \leq n$. When $n = 0$ we have the *empty word* denoted 1. We say W is a *positive* word on $\boldsymbol{x}$ if either $n = 0$ or $n > 0$ and $\varepsilon_i = +1$, $1 \leq i \leq n$. The *inverse* of W, denoted W^{-1}, is the word $x_1^{-\varepsilon_1} x_2^{-\varepsilon_2} \cdots x_n^{-\varepsilon_n}$. The *length* of W, denoted by $L(W)$, is the number n of the letters involved in W. (Note that we also use $L(\gamma)$ for the length of a path γ in a two-complex.)

Let W, U be two words on $\boldsymbol{x}$. The product of W and U, denoted WU, is the *juxtaposition* of W followed by U. By this binary operation, the set $\hat{F}(\boldsymbol{x})$ of all positive words on $\boldsymbol{x}$ then is a monoid called the *free monoid* on $\boldsymbol{x}$. We now introduce the following operation on the set of all words on $\boldsymbol{x}$.

(†) Deletion/insertion of a pair of inverse letters $x^{\varepsilon} x^{-\varepsilon}$, $\varepsilon = \pm 1$.

Two words W, W' on $\boldsymbol{x}$ are *freely equal*, denoted $W \sim^{(1)} W'$, if one can be obtained from the other by a number of applications of operation (†). We again denote the free equivalence class containing W by $[W]$. Let $F(\boldsymbol{x})$ be the set of all free equivalence classes of words on $\boldsymbol{x}$. A multiplication can be defined on $F(\boldsymbol{x})$ by $[W][U] = [WU]$, and one can check that this multiplication is well-defined. By this multiplication, $F(\boldsymbol{x})$ is then a group, the *free group* on $\boldsymbol{x}$. See [Jo, §1.2] for detail. We remark that sometimes we may simply write W for the free equivalence class $[W]$ for any word W on $\boldsymbol{x}$, if it does not cause any confusion.

1.2.2 Group presentations

A *group presentation* $\mathcal{P}$ is a pair $\langle \boldsymbol{x};\ \boldsymbol{r}\rangle$ where $\boldsymbol{x}$ is a set (the *generating symbols*) and $\boldsymbol{r}$ is a set of non-empty, cyclically reduced words on $\boldsymbol{x}$ (the *defining relators*). We say that $\mathcal{P}$ is *finite*, if both $\boldsymbol{x}$ and $\boldsymbol{r}$ are finite.

Alternatively, we regard $\mathcal{P}$ as a two-complex where the underlying graph $\mathcal{P}^{(1)}$ consists of a *single vertex* o, the set $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ of *edges* and the set $\boldsymbol{r}$ of *defining paths.* We remark that, in the sequel, without further comment we will often regard $\mathcal{P}$ as a two-complex. Moreover, each word W then is a (closed) path in $\mathcal{P}$, and $\overline{W}$ is the homotopy equivalence class containing W.

The (unique) fundamental group (at o) of $\mathcal{P}$ is denoted by $G(\mathcal{P})$ (or formally by $\pi_1(\mathcal{P}) = \pi_1(\mathcal{P}, o)$). A group G is said to be *presented* (or *defined*) by $\mathcal{P}$ if $G \cong G(\mathcal{P})$. Let N be the *normal closure* of $\{[R] : R \in \boldsymbol{r}\}$ in $F(\boldsymbol{x})$. Then by the definition of $G(\mathcal{P})$ we have (see [Ki, Proposition 1.5.1]) a one-to-one map sending $\overline{W}$ to $[W]N$ for each word W on $\boldsymbol{x}$ so that $G(\mathcal{P}) \cong F(\boldsymbol{x})/N$.

A *van Kampen diagram* over a presentation $\mathcal{P} = \langle \boldsymbol{x}\ ;\ \boldsymbol{r}\rangle$ is a finite connected planar graph $\Lambda \subset \mathbb{R}^2$ (here $\mathbb{R}^2$ is the real plane and the term *graph* is in the sense of basic graph theory) whose edges are directed and labelled by elements of $\boldsymbol{x}$ in such a way that every face of Λ (bounded component of $\mathbb{R}^2 \backslash \boldsymbol{x}$) is a *disc* whose boundary label (for some starting point and orientation) belongs to $\boldsymbol{r}$. The van Kampen Lemma (for example, see [LySc, Proposition 9.2]) says that a word W on $\boldsymbol{x}$ represents the identity of $G(\mathcal{P})$ if and only if there is a van Kampen diagram over $\mathcal{P}$ with boundary label W (for certain starting point and orientation).

1.2.3 Equivalence and subnegativity of number functions

Given two increasing functions $f_1,\ f_2 : \mathbb{N} \longrightarrow \mathbb{R}^+$ we write $f_1 \preceq f_2$ if there are constants $A,\ B > 0$ and $C \geq 0$, where B is an integer, such that

$$f_1(n) \leq Af_2(Bn) + Cn \qquad (n \in \mathbb{N}),$$

and we say that f_1 is *equivalent* to f_2, denoted $f_1 \sim f_2$, if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

Following Brick [Brc] we will say that a function $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ is *subnegative* if $f(n_1) + f(n_2) \leq f(n_1 + n_2)$ for all $n_1,\ n_2 \in \mathbb{N}$. Given any function $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ there is a least subnegative function $\overline{f}$ greater than or equal to f called the *subnegative closure* of f defined by

$$\bar{f}(n) = \max\{\sum_{i=1}^{r} f(n_i) :\ n_1 + n_2 + \cdots n_r = n,\ n_i \in \mathbb{N}\,(1 \leq i \leq r)\}.$$

Then if $f_1 \preceq f_2$ we have $\bar{f}_1 \preceq \bar{f}_2$ and so if $f_1 \sim f_2$ then $\bar{f}_1 \sim \bar{f}_2$.

1.2.4 First order Dehn functions of groups

Consider a finite group presentation $\mathcal{P} = \langle \boldsymbol{x}\ ;\ \boldsymbol{r}\rangle$. Let W be a word on $\boldsymbol{x}$. If $\overline{W} = 1$ in $G(\mathcal{P})$ then by Remark 1.1.6 there exist $r \in \mathbb{N}$ such that in $\mathcal{P}$

$$W \sim^{(1)} \prod_{i=1}^{r} U_i R_i^{\varepsilon_i} U_i^{-1} \tag{1.3}$$

for certain words U_i on $\boldsymbol{x}$, $R_i \in \boldsymbol{r}$, and $\varepsilon_i = \pm 1$ $(1 \leq i \leq r)$.

Definition 1.2.1 *The* first order Dehn function *of $\mathcal{P}$ is the integer valued function*

$$\delta^{(1)}_{\mathcal{P}}(n) = \max\{Area_{\mathcal{P}}(W) :\ L(W) \leq n,\ \overline{W} = 1\}, \quad n \in \mathbb{N}.$$

By the main result of [All], up to equivalence $\delta^{(1)}$ is independent of the choice of different finite presentations. Thus, if a group G is finitely presented, then we can write $\delta^{(1)}_G$ for a typical representative of the equivalence class.

1.2.5 Cayley graphs and universal coverings

Let G be a group finitely presented by $\mathcal{P} = \langle \boldsymbol{x}\ ;\ \boldsymbol{r}\rangle$. We identify G with $G(\mathcal{P})$. Then G is generated by $\{\bar{x} :\ x \in \boldsymbol{x}\}$.

The *Cayley graph* $\Gamma_{\boldsymbol{x}}(G)$ of G with respect to $\boldsymbol{x}$ consists of the vertex set $\boldsymbol{v} = \boldsymbol{v}(\Gamma_{\boldsymbol{x}}(G)) = G$, and the edge set

$$\boldsymbol{e} = \boldsymbol{e}(\Gamma_{\boldsymbol{x}}(G)) = \{(g,\ x^{\varepsilon}) :\ g \in G,\ x \in \boldsymbol{x},\ \varepsilon = \pm 1\}$$

satisfying $\iota(g,\, x^\varepsilon) = g$, $\tau(g,\, x^\varepsilon) = g\bar{x}^\varepsilon$ and $(g,\, x^\varepsilon)^{-1} = (g\bar{x}^\varepsilon,\, x^{-\varepsilon})$. Thus, for each vertex g of $\Gamma_{\boldsymbol{x}}(G)$, $|Star(g)| = 2|\boldsymbol{x}|$. Let γ be a path in $\Gamma_{\boldsymbol{x}}(G)$ from g to another element g' of G, say $\gamma = (g,\, x_1^{\varepsilon_1})(g\bar{x}_1^{\varepsilon_1},\, x_2^{\varepsilon_2})\cdots(g\overline{x_1^{\varepsilon_1}\cdots x_{n-1}^{\varepsilon_{n-1}}},\, x_n^{\varepsilon_n})$. Reading off the second coordinates of the edges of γ gives a unique word $W_\gamma = x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$ on $\boldsymbol{x}$. We then can define the *projection* map $p_o : \Gamma_{\boldsymbol{x}}(G) \longrightarrow \mathcal{P}^{(1)}$ which is a locally bijective mapping of graphs given by

$$p_o(\gamma) = W_\gamma, \quad p_o(g) = o, \qquad \text{for any path } \gamma \text{ and any } g \in G.$$

Thus, for any word $W = x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$ on $\boldsymbol{x}$ and for any $g \in G$ we have a unique lift of W in $\mathcal{P}^{(1)}$ at g

$$t_g(W) = (g,\, x_1^{\varepsilon_1})(g\bar{x}_1^{\varepsilon_1},\, x_2^{\varepsilon_2})\cdots(g\overline{x_1^{\varepsilon_1}\cdots x_{n-1}^{\varepsilon_{n-1}}},\, x_n^{\varepsilon_n}).$$

We then call t_g the *lift map* from the set of all words on $\boldsymbol{x}$ to the set of all paths at g of $\Gamma_{\boldsymbol{x}}(G)$.

The group G acts on $\Gamma_{\boldsymbol{x}}(G)$ by multiplication on the left: the element $g \in G$ defines a map ϕ_g, which maps a vertex g' to gg', an edge $(g',\, x^\varepsilon)$ to $(gg',\, x^\varepsilon)$ ($x \in \boldsymbol{x}$, $\varepsilon = \pm 1$).

A *(word) metric* $d_{\boldsymbol{x}} = d$ is defined by assigning a *unit length* to each edge of $\Gamma_{\boldsymbol{x}}(G)$ and defining the *distance* between two vertices $g_1,\, g_2 \in G$ to be the minimum length of paths in $\Gamma_{\boldsymbol{x}}(G)$ joining them denoted $d(g_1,\, g_2)$. The paths joining g_1 and g_2 with the minimum length are called *geodesics.* It can be show that we then have defined a *metric space* denoted by $(G,\, d)$.

We also define the *length* of an element $g \in G$ with respect to $\boldsymbol{x}$, written $|g|_{\boldsymbol{x}}$ or simply $|g|$, to be the length of a geodesic in $\Gamma_{\boldsymbol{x}}(G)$ from the identity to g (i.e. the length of a shortest word on $\boldsymbol{x}$ representing g). Thus, for any two elements $g_1,\, g_2 \in G$, $d(g_1,\, g_2) = |g_1^{-1}g_2|$. The left action of G on itself then is by *isometries* since $d(gg_1,\, gg_2) = |(gg_1)^{-1}gg_2| = |g_1^{-1}g_2|$.

Let $\tilde{\boldsymbol{r}} = \{t_g(R) : R \in \boldsymbol{r},\, g \in G\}$. We then obtain a two-complex, the *universal covering* $\tilde{\mathcal{P}} = \langle \Gamma_{\boldsymbol{x}}(G);\, \tilde{\boldsymbol{r}}\rangle$ of $\mathcal{P}$, with vertex set G, edge set

$$\{(g,\, x^\varepsilon) : g \in G,\, x \in \boldsymbol{x},\, \varepsilon = \pm 1\},$$

and set of defining paths $\tilde{\boldsymbol{r}}$. Let $g,\, g'$ be any two elements of G. Then there exists $g'' \in G$ such that $g = g'g''$. Let $W = x_1^{\varepsilon_1}\cdots x_m^{\varepsilon_m}$ be a word on $\boldsymbol{x}$ such that $\overline{W} = g''$ in G. This then gives a path in $\tilde{\mathcal{P}}$ from g' to g:

$$(g',\, x_1^{\epsilon_1})(g'\overline{x}_1^{\epsilon_1},\, x_2^{\epsilon_2})\cdots(g'\overline{x_1^{\epsilon_1}\cdots x_{m-1}^{\epsilon_{m-1}}},\, x_m^{\epsilon_m}).$$

Thus, $\tilde{\mathcal{P}}$ is connected, and has trivial fundamental groups [LySc, Proposition 4.1].

Since $p_o(t_g(R)) = R$ for any $t_g(R) \in \tilde{\boldsymbol{r}}$, $g \in G$, p_o is a mapping of two-complexes. We call p_o the *projection map* from $\tilde{\mathcal{P}}$ to $\mathcal{P}$ and call t_g the *lift map* from $\mathcal{P}$ to $\tilde{\mathcal{P}}$ with respect to g. Note that t_g is not a mapping of two-complexes. Since p_o is locally bijective, by Lemmas 1.1.2 and 1.1.3 we have

Lemma 1.2.2 *Let $\tilde{W}$, $\tilde{W}'$ be any two paths in $\tilde{\mathcal{P}}$.*

(i) *If $\tilde{W} \sim \tilde{W}'$ in $\tilde{\mathcal{P}}$ then $p_o(\tilde{W}) \sim p_o(\tilde{W}')$ in $\mathcal{P}$, and*

(ii) *if $W \sim W'$ in $\mathcal{P}$ then $t_g(W) \sim t_g(W')$ in $\tilde{\mathcal{P}}$.*

1.2.6 Monoid presentations

A *monoid presentation* $\hat{\mathcal{P}}$ is a pair $[\boldsymbol{x}; \boldsymbol{r}]$ where $\boldsymbol{x}$ is a set (the *generating symbols*) and each $R \in \boldsymbol{r}$ (a *defining relation*) is an ordered pair $(R_{+1},\, R_{-1})$, where R_{+1} and R_{-1} are distinct positive words on $\boldsymbol{x}$. We write $R_{+1} = R_{-1}$ instead of $(R_{+1},\, R_{-1})$. Sometimes, we need to list the elements of $\boldsymbol{r}$. We write each element as $R_{+1,1} = R_{-1,1},\, \cdots,\, R_{+1,k} = R_{-1,k},\, \cdots$. We say that $\hat{\mathcal{P}}$ is *finite*, if both $\boldsymbol{x}$ and $\boldsymbol{r}$ are finite.

We now introduce the following operation on positive words on $\boldsymbol{x}$.

(‡) If positive word W contains a subword R_ε, where $\varepsilon = \pm 1$ and $R_{+1} = R_{-1} \in \boldsymbol{r}$, then replace it by $R_{-\varepsilon}$.

Two positive words W_1, W_2 on $\boldsymbol{x}$ are *equivalent*, denoted $W_1 \sim W_2$, if one of them can be obtained from the other by a number of applications of (‡). Clearly, if U, V are any two words on $\boldsymbol{x}$ and if $W_1 \sim W_2$ then $UW_1V \sim UW_2V$. We then say that this equivalence relation $\sim$ is a *congruence* relation. Let W be a positive word on $\boldsymbol{x}$. We still denote the congruence class containing W by $\overline{W}$. Let $S(\hat{\mathcal{P}})$ be the the set of all congruence classes. We now have a well-defined multiplication on $S(\hat{\mathcal{P}})$ given by $\overline{W}_1\overline{W}_2 = \overline{W_1W_2}$ and then $S(\hat{\mathcal{P}})$ is a monoid (for example, see [Ki, Lemma 1.2.1] for detail). Let S be any monoid. If $S \cong S(\hat{\mathcal{P}})$ then we say that S is *presented* (or *defined*) by $\hat{\mathcal{P}}$.

1.3 Pictures over two-complexes

1.3.1 Pictures over presentations

We refer the readers to [BoPr] and [Pr2] for the reference in this subsection.

In this subsection and the following section we let $\mathcal{P} = \langle \boldsymbol{x} ; \boldsymbol{r} \rangle$ be a group presentation.

A *picture* $\mathbb{P}$ over $\mathcal{P}$ is a geometric configuration consisting of the following.

(1) A *disc* D^2 with *basepoint* o on the *boundary* ∂D^2 of D^2.

(2) Disjoint *discs* $\Omega_1, \Omega_2, \cdots, \Omega_n$ in the interior of D^2. Each disc Ω_i has a *basepoint* o_i on the *boundary* $\partial\Omega_i$ of Ω_i.

(3) A finite number of disjoint *arcs* $\alpha_1, \alpha_2, \cdots, \alpha_m$. Each arc lies in the closure of $D^2 - \bigcup_{i=1}^n \Omega_i$ and is *either* a simple closed curve having trivial intersection with $\partial D^2 \cup (\bigcup_{i=1}^n \Omega_i)$, *or* is a simple non-closed curve which joins two points of $\partial D^2 \cup (\bigcup_{i=1}^n \partial\Omega_i)$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting the arc transversely, and is labelled by an element $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ which is called the *label* of the arc and this arc is said to be an $\boldsymbol{x}$*-arc*.

(4) Reading off the labels on the arcs encountered while travelling around $\partial\Omega_i$ $(1 \leq i \leq n)$ in the clockwise direction from o_i to o_i gives a word which belongs to $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$. We call this word the *label* of Ω_i and say that Ω_i is a $\boldsymbol{r}$*-disc.*

We define the *boundary* $\partial\mathbb{P}$ of $\mathbb{P}$ to be ∂D^2. The *label* on $\mathbb{P}$, denoted by $W(\mathbb{P})$, is the word read off by travelling around $\partial\mathbb{P}$ once in the clockwise direction starting from o. When we refer to the discs we mean the discs in the interior of the ambient disc D^2 not D^2 itself. A *region* of $\mathbb{P}$ is the closure of a component of $D^2 - ((\bigcup_{i=1}^n \Omega) \cup (\bigcup_{j=1}^m \alpha_j))$. An *inner region* of $\mathbb{P}$ is a region that does not meet $\partial\mathbb{P}$ and all other regions of $\mathbb{P}$ are *outer regions.* We write $A(\mathbb{P})$ for the *disc number* in $\mathbb{P}$ (also called the *area* of $\mathbb{P}$). We say that $\mathbb{P}$ is *spherical* if no arc meets $\partial\mathbb{P}$. Thus, a spherical picture only has one outer region labelled by its basepoint. We remark that sometimes we would drop off the ambient disc D^2 of a spherical picture.

Example 1.3.1 Let $\mathcal{Q} = \langle x,\ y,\ z;\ x^{-1}y^{-1}z,\ y^{-1}zx^{-1},\ y^2\rangle$. If $\mathbb{P}$ is as illustrated in Fig. 1.1, then $\mathbb{P}$ is a picture over $\mathcal{Q}$ (forgetting the three broken arcs together with their labels γ_1, γ_2, γ_3 which will be introduced later) with $W(\mathbb{P}) = x^{-1}xz^{-1}yyyy^{-1}xx^{-1}z$.

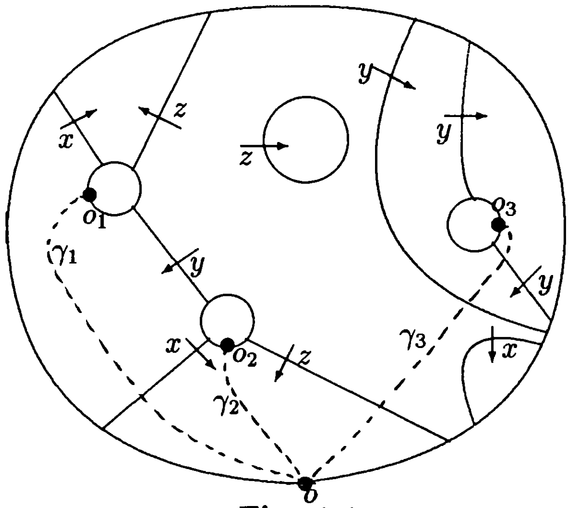
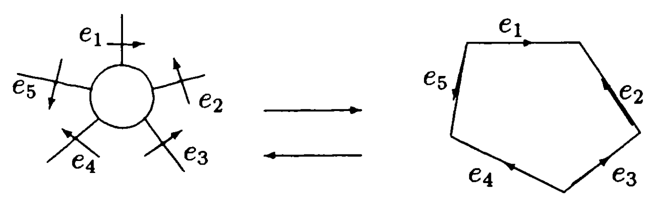


Fig. 1.1

Obviously, every picture $\mathbb{P}$ over $\mathcal{P}$ uniquely corresponds to a van Kampen diagram Λ over $\mathcal{P}$ by replacing a disc together with its incident arcs by a 2-cell with boundary label the disc label:



We call $\mathbb{P}$ the *dual* of Λ and vice versa. Thus there is a pictorial version of the van Kampen Lemma:

Lemma 1.3.2 *There exists a picture $\mathbb{P}$ over $\mathcal{P}$ with label W if and only if $\overline{W} = 1$ in $G(\mathcal{P})$.*

We will say that a van Kampen diagram over $\mathcal{P}$ is a *spherical diagram* if its dualization is an spherical picture over $\mathcal{P}$.

A *transverse path* γ in a picture $\mathbb{P}$ is a path in the closure of $D^2 - \bigcup_{i=1}^{n} \Omega_i$ which intersects the arcs of $\mathbb{P}$ only finitely many times. Reading off the labels on the arcs encountered while travelling along a transverse path from its initial point to its terminal point gives a word on $\boldsymbol{x}$ denoted $W(\gamma)$. Let γ be a simple closed transverse path in $\mathbb{P}$. The picture enclosed by γ is called a *subpicture* of $\mathbb{P}$.

A *spray* for $\mathbb{P}$ is a sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ of simple transverse paths satisfying the following: for $i = 1, 2, \cdots, n$, γ_i starts at o and ends at a basepoint of some $\Omega_{\chi(i)}$, where χ is a permutation of $\{1, 2, \cdots, n\}$ (depending on $\underline{\gamma}$); for $1 \leq i < j \leq n$, γ_i and γ_j intersect only at o; travelling around o clockwise in $\mathbb{P}$ we encounter these transverse paths in order $\gamma_1, \gamma_2, \cdots, \gamma_n$. Let $R^{\varepsilon_i}_{\chi(i)}$ be the label of $\Omega_{\chi(i)}$ and let W_i be the label of γ_i. Then associated with $\underline{\gamma}$ we have a *sequence* over $\mathcal{P}$:

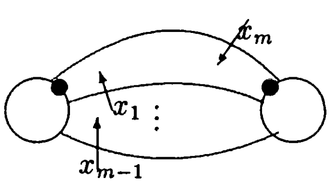
$$(W_1 R^{\varepsilon_1}_{\chi(1)} W_1^{-1}, W_2 R^{\varepsilon_2}_{\chi(2)} W_2^{-1}, \cdots, W_n R^{\varepsilon_n}_{\chi(n)} W_n^{-1}).$$

We call this the *sequence associated with* $\mathbb{P}$ (relative to the given spray $\underline{\gamma}$).

Example 1.3.1 (continued) The sequence associated with the spray $\underline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ illustrated in Fig. 1.1 is

$$(x^{-1}(y^{-1}zx^{-1})^{-1}x, x^{-1}y^{-1}z, z^{-1}yy^{-1}(y^2)yy^{-1}z).$$

Let $\mathbb{P}$ be a picture over $\mathcal{P}$. A *floating circle* of $\mathbb{P}$ is a closed arc which encloses no discs or arcs of $\mathbb{P}$. In the example above, the circle labelled by z in Fig. 1.1 is a floating circle. A *semifloating circle* of $\mathbb{P}$ is an arc which starts and ends on $\partial\mathbb{P}$ and which is such that all other arcs and discs of $\mathbb{P}$ lie on the same side of this arc as the basepoint o of $\mathbb{P}$. In Fig. 1.1, we see there is a floating semicircle labelled by x. A *cancelling pair* of $\mathbb{P}$ is a spherical picture with exactly two discs whose basepoints lie in the same region.



We now introduce some elementary operations on spherical pictures as follows. Let $\mathbb{P}$ be a picture over $\mathcal{P}$.

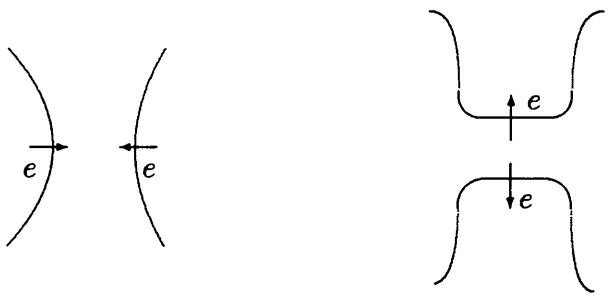
(A) Deletion of a floating circle.

(A)$^{-1}$ Insertion of a floating circle.

(B) Deletions of a cancelling pair.

$(B)^{-1}$ Insertion of a cancelling pair.

(C) *Bridge move*:

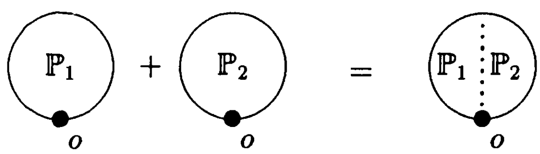


Bridge move

Two spherical pictures are *equivalent* if one can be obtained from the other by a finite number of operations (A), $(A)^{-1}$, (C), $(C)^{-1}$, (D).

Let $\mathbb{P}$ be any spherical picture over $\mathcal{P}$. We denote $\langle\mathbb{P}\rangle$ the equivalence class of spherical picturers over $\mathcal{P}$ containing $\mathbb{P}$. We say that $\mathbb{P}$ is *minimal* if $A(\mathbb{P}) = \min\{A(\mathbb{Q}) : \mathbb{Q} \in \langle\mathbb{P}\rangle\}$.

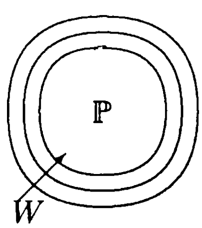
The *mirror image* of a picture $\mathbb{P}$ over $\mathcal{P}$, denoted $-\mathbb{P}$, is also a picture over $\mathcal{P}$. We form the *sum* of any two pictures $\mathbb{P}_1$, $\mathbb{P}_2$ over $\mathcal{P}$ in the obvious way:



and we will write $\mathbb{P}_1 - \mathbb{P}_2$ for $\mathbb{P}_1 + (-\mathbb{P}_2)$. Clearly, for any picture $\mathbb{P}$ over $\mathcal{P}$, $\mathbb{P} - \mathbb{P}$ is equivalent to the empty picture, and if $\mathbb{P}_1$, $\mathbb{P}_2$ are both spherical then $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_2 + \mathbb{P}_1$. The set of all equivalence classes of spherical pictures over $\mathcal{P}$ forms a abelian group, denoted $\pi_2(\mathcal{P})$, under the following binary operation:

$$\langle\mathbb{P}_1\rangle + \langle\mathbb{P}_2\rangle = \langle\mathbb{P}_1 + \mathbb{P}_2\rangle.$$

Let W be a word on $\boldsymbol{x}$, and let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$. We then can form a new spherical picture over $\mathcal{P}$ denoted $\mathbb{P}^W$ of the form



We now consider the $G(\mathcal{P})$-action on $\pi_2(\mathcal{P})$.

Lemma 1.3.3 *The $G(\mathcal{P})$-action on $\pi_2(\mathcal{P})$ given by*

$$\overline{W} \cdot \langle \mathbb{P} \rangle = \langle \mathbb{P}^W \rangle, \qquad W \text{ a word on } \boldsymbol{x}, \ \langle \mathbb{P} \rangle \in \pi_2(\mathcal{P})$$

is well-defined and so $\pi_2(\mathcal{P})$ is a left $\mathbb{Z}G(\mathcal{P})$-module.

Proof. Let $\mathbb{P}$ be any spherical picture over $\mathcal{P}$ and let W_1, W_2 be any two words on $\boldsymbol{x}$ with $\overline{W}_1 = \overline{W}_2$ in $G(\mathcal{P})$. By Lemma 1.3.2, there exists a picture $\mathbb{Q}$ over $\mathcal{P}$ with $W(\mathbb{Q}) = W_1 W_2^{-1}$. Consider the picture $\mathbb{P}^{W_1} + (\mathbb{Q} - \mathbb{Q}) - \mathbb{P}^{W_2}$. By applying bridge moves and removing of cancelling pairs, we see that this picture is equivalent to the empty picture as shown in Fig. 1.2.

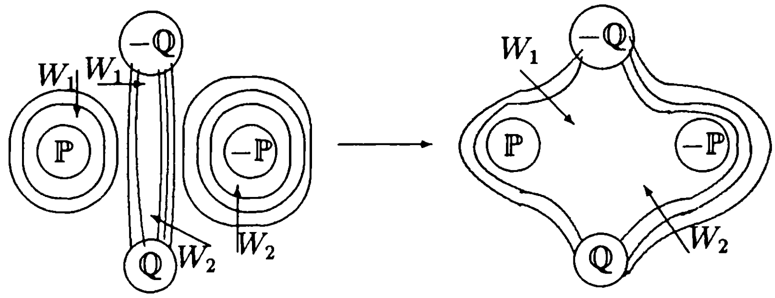


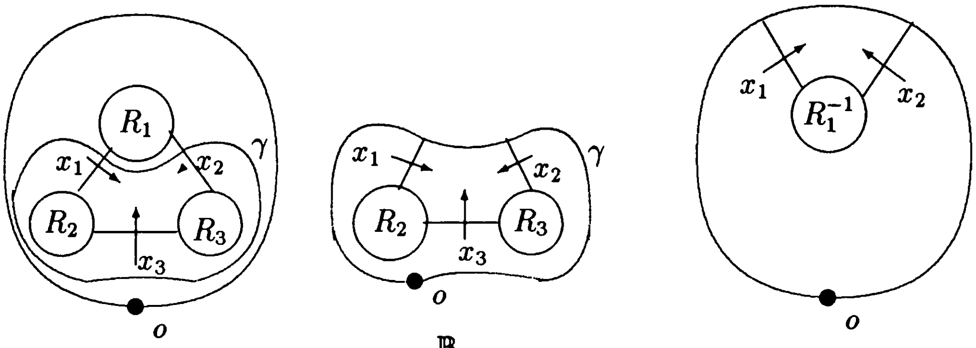
Fig. 1.2

Thus,

$$\begin{aligned} \langle \mathbb{P}^{W_1} \rangle - \langle \mathbb{P}^{W_2} \rangle &= \langle \mathbb{P}^{W_1} \rangle + \langle \mathbb{Q} - \mathbb{Q} \rangle - \langle \mathbb{P}^{W_2} \rangle \\ &= \langle \mathbb{P}^{W_1} + (\mathbb{Q} - \mathbb{Q}) - \mathbb{P}^{W_2} \rangle \\ &= 0 \end{aligned}$$

and so $\langle \mathbb{P}^{W_1} \rangle = \langle \mathbb{P}^{W_2} \rangle$ as required. □

Let γ be a simple closed path in a spherical picture $\mathbb{A}$ over $\mathcal{P}$, and let $\mathbb{B}$ be the subpicture of $\mathbb{A}$ enclosed by γ. The *complement* of $\mathbb{B}$ in $\mathbb{A}$ is defined as follows. Delete the interior of $\mathbb{B}$ to form an oriented annulus. Identification of $\partial\mathbb{A}$ to the point o produces an oriented disc that has boundary γ, and which supports a new picture over $\mathcal{P}$. The complement of $\mathbb{B}$ in $\mathbb{A}$ is obtained from this new picture by a planar reflection. The complement has the same boundary label as $\mathbb{B}$ and its discs are those of $\mathbb{A}$ which are not in $\mathbb{B}$, taken with the inverse labels. See Fig. 1.3.



$\mathbb{A}$ $\mathbb{B}$ complement of $\mathbb{B}$ in $\mathbb{A}$

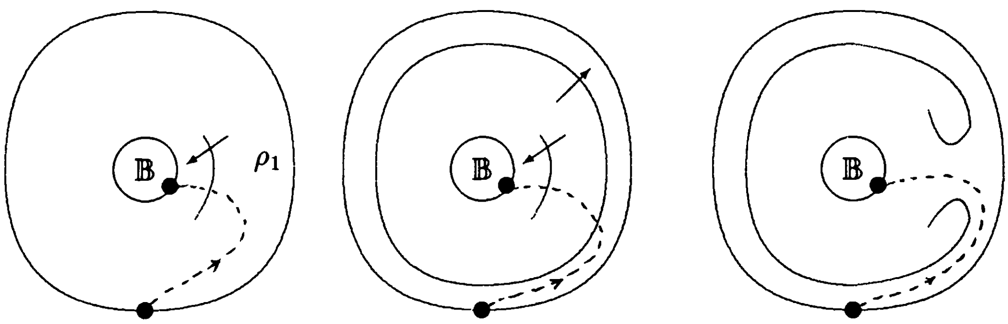
Fig. 1.3

Lemma 1.3.4 (see [BoPr, Lemma 1.5]) *Let $\mathbb{A}$ be a spherical picture over $\mathcal{P}$. Let $\mathbb{B}$ be a subpicture of $\mathbb{A}$, and let $\mathbb{B}'$ be the complement of $\mathbb{B}$ in $\mathbb{A}$. Suppose $\mathbb{P}$ is a spherical picture over $\mathcal{P}$ having $\mathbb{B}$ as a subpicture, and suppose $\mathbb{P}'$ is obtained from $\mathbb{P}$ by replacing $\mathbb{B}$ by $\mathbb{B}'$. Then*

$$\langle \mathbb{P} \rangle - \langle \mathbb{P}' \rangle = \langle \mathbb{A}^W \rangle$$

for some word W on $\boldsymbol{x}$.

Proof. Let ρ_1 be a transverse path from the basepoint of $\mathbb{A}$ to the basepoint of $\mathbb{B}$ with label W_1. A sequence of bridge moves applied to the spherical picture $\mathbb{A}^{W_1^{-1}}$ yields a picture $\mathbb{A}_1$ containing $\mathbb{B}$, and where the basepoint of $\mathbb{B}$ "exposed", lying in the boundary outer region of $\mathbb{A}_1$. See Fig. 1.4.

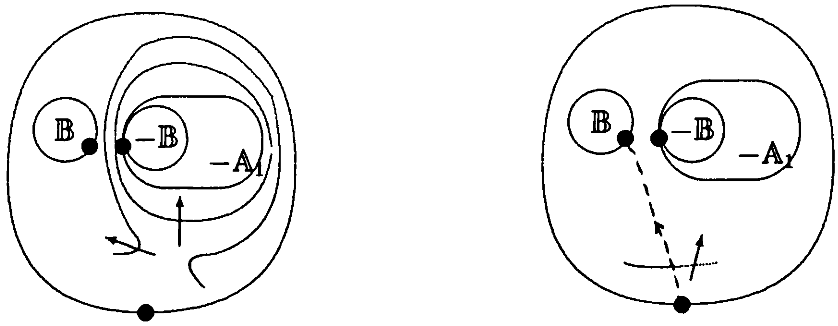


$\mathbb{A}$ $\mathbb{A}^{W_1^{-1}}$ $\mathbb{A}_1$

Fig. 1.4

Let ρ_2 be a transverse path from the basepoint of $\mathbb{P}$ to the basepoint of $\mathbb{B}$ (as a subpicture in $\mathbb{P}$) with label W_2. To insert $-\mathbb{A}_1$ into the region of $\mathbb{P}$ where the basepoint of the subpicture $\mathbb{B}$ lies we apply bridge moves on $\mathbb{P} + (-\mathbb{A}_1)^{W_2}$. Denote this new picture (spherical) by $\mathbb{P}_1$ we then see that (see Fig. 1.5)

$$\langle \mathbb{P}_1 \rangle = \langle \mathbb{P} \rangle - \langle \mathbb{A}^{W_2 W_1^{-1}} \rangle.$$



$\mathbb{P} + (-\mathbb{A}_1)^{W_2}$ $\mathbb{P}_1$

Fig. 1.5

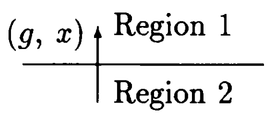
On the other hand, the oppositely oriented and adjacent copies of $\mathbb{B}$ in $\mathbb{P}_1$ can be removed by a sequence of bridge moves and deletions of cancelling pairs. The resulting picture is then exactly $\mathbb{P}'$, and so the lemma follows. □

In the situation of the above lemma, we will say that $\mathbb{P}'$ is obtained from $\mathbb{P}$ *modulo* $\mathbb{A}$.

1.3.2 Pictures over universal coverings

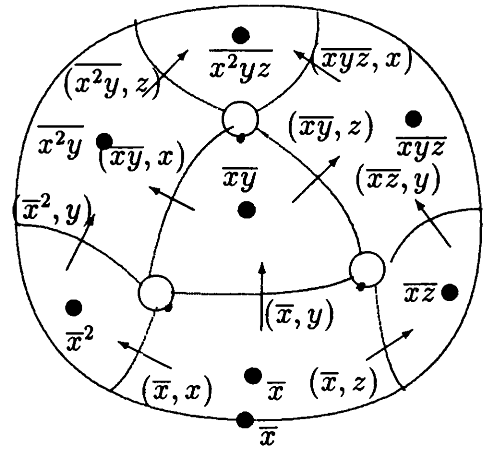
There is a general notion of pictures over two-complexes (for example, see [BoPr, BrHu, CoHu] for reference). Here, for our use we will concentrate on the notion of pictures over the universal covering $\tilde{\mathcal{P}}$.

The basepoint of the picture is labelled by an element of $G(\mathcal{P})$. The arcs of the picture are labelled by edges of $\tilde{\mathcal{P}}$. Each disc has a basepoint and the label on the disc is either a defining path or the inverse of a defining path. Each region of the picture is assigned an element of $G(\mathcal{P})$. It is required that if we have an arc in the picture, with label (g, x) ($g \in G(\mathcal{P})$, $x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$) say, separating two regions:

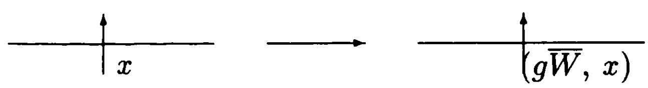


and if g_1 and g_2 are the elements of $G(\mathcal{P})$ assigned to Region 1 and Region 2 respectively, then $g_1 = \iota(g,\ x) = g$ and $g_2 = \tau(g,\ x) = g\overline{x}$.

Example 1.3.5 Let $\mathcal{P} = \langle x, y, z;\ [x,y], [y,z], [z,x]\rangle$. The following then is a picture over $\tilde{\mathcal{P}}$.



Given a picture $\mathbb{P}$ over $\mathcal{P}$, then for any $g \in G(\mathcal{P})$ we have a unique picture $t_g(\mathbb{P}) = \tilde{\mathbb{P}}_g$ over $\tilde{\mathcal{P}}$ of $\mathbb{P}$ at g, defined as follows. Label the basepoint of $\mathbb{P}$ by g and assign g to the outer region (containing the basepoint of $\mathbb{P}$). For each arc of $\mathbb{P}$ choose a transverse path in $\mathbb{P}$ from the basepoint of $\mathbb{P}$ to the start of the arrow on the arc. Then relabel the arc by $(g\overline{W},\ x)$, where W is the label of the transverse path and $x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$ is the label of the arc, i.e.



Note that if W' is the label of another transverse path from the basepoint of $\mathbb{P}$ to the start of the arrow of the given arc, then from [Pr2, Theorem 2.4] we know that $\overline{W}' = \overline{W}$ and so $(g\overline{W}', x) = (g\overline{W}, x)$. In addition, we assign the element $g\overline{W}$ of $G(\mathcal{P})$ to the region where the start of the arrow on the given arc lies, and assign the element $g\overline{Wx}$ to the region where the end of the arrow on the given arc lies. Again, from [Pr2] we see that these assignments are independent of the choice of transverse paths. For each disc of $\mathbb{P}$ labelled by R^ε $(R \in \boldsymbol{r})$, choose a transverse path in $\mathbb{P}$ from the basepoint of $\mathbb{P}$ to the basepoint of this disc. Then relabel the disc by $t_{g\overline{W}}(R^\varepsilon) = \tilde{R}^\varepsilon_{g\overline{W}}$, where W is the label of the transverse path (this label is once again independent of the choice of transverse paths).

Applying p_o to the labels of the arcs of $\tilde{\mathbb{P}}_g$, one recovers $\mathbb{P}$. We write $p_o(\tilde{\mathbb{P}}_g) = \mathbb{P}$ and so $p_o t_g(\mathbb{P}) = \mathbb{P}$.

Conversely, let $\tilde{\mathbb{P}}$ be any picture over $\tilde{\mathcal{P}}$ with $\iota(\tilde{\mathbb{P}}) = g$ for some $g \in G$. Here we write $\iota(\tilde{\mathbb{P}})$ for the label of the basepoint of $\tilde{\mathbb{P}}$. Then the basepoint and the region containing this basepoint are labelled by g. Consider any arc in $\tilde{\mathbb{P}}$. Suppose that the arrow riding on this arc is labelled by an edge (g', x) of $\tilde{\mathcal{P}}$ with $g' \in G$, $x \in \boldsymbol{x}$. Then the region containing the start of this arrow has the label g' and the region containing the end of this arrow has the label $g'\overline{x}$. Thus, each transverse path of $\tilde{\mathbb{P}}$ from the basepoint of $\tilde{\mathbb{P}}$ to any region of $\tilde{\mathbb{P}}$ labelled by g'' say, is also a path in $\tilde{\mathcal{P}}$ from g to g''. Moreover, let Ω be a disc of $\tilde{\mathbb{P}}$ and suppose that the region containing the basepoint of Ω has the label g_1. Reading off the labels on the arcs meeting Ω clockwise gives a defining path $\tilde{R}_{g_1}$ of $\tilde{\mathcal{P}}$ or its inverse. So, applying p_o to the arcs of $\tilde{\mathbb{P}}$ gives a picture $\mathbb{P}$ over $\mathcal{P}$ denoted $p_o(\tilde{\mathbb{P}}) = \mathbb{P}$. Now, by the definition of t_g we have $t_g(\mathbb{P}) = \tilde{\mathbb{P}}$, namely $\tilde{\mathbb{P}}$ is the unique lift $\tilde{\mathbb{P}}_g$ of $\mathbb{P}$ at g and

$$t_g p_o(\tilde{\mathbb{P}}) = \tilde{\mathbb{P}}.$$

Thus, the map t_g on the set of all pictures over $\mathcal{P}$ and the restriction of p_o on the set of all pictures over $\tilde{\mathcal{P}}$ at g are mutually inverse. Hence, we have proved the following lemma.

Lemma 1.3.6 *For any $g \in G$ the restriction of t_g to the set of all pictures over $\mathcal{P}$ and the restriction of p_o to the set of all pictures at g over $\tilde{\mathcal{P}}$ defined in the above are mutually inverse.*

The elementary operations on pictures over $\mathcal{P}$ are translated to elementary operations on pictures over $\tilde{\mathcal{P}}$. (One can check that all these operations have no affect on the basepoint of a given picture over $\tilde{\mathcal{P}}$.) Thus, for each $g \in G(\mathcal{P})$, we have the notion of equivalent spherical pictures over $\tilde{\mathcal{P}}$ at g. Let $\langle\tilde{\mathbb{P}}\rangle_g$ (or simply $\langle\tilde{\mathbb{P}}\rangle$ without causing any confusion) denote the equivalence class containing the spherical picture $\tilde{\mathbb{P}}$ over $\tilde{\mathcal{P}}$ at g, and let $\pi_2(\tilde{\mathcal{P}}, g)$ denote the set of all equivalence classes of spherical picture over $\tilde{\mathcal{P}}$ at g. Then, as for the situation for $\pi_2(\mathcal{P})$, $\pi_2(\tilde{\mathcal{P}}, g)$ forms a abelian group.

We remark that Lemma 1.3.4 also can be extended to the situation of pictures over $\tilde{\mathcal{P}}$.

1.3.3 Fox derivations

Let $F = F(\boldsymbol{x})$, and let $x \in \boldsymbol{x}$. The *Fox derivation* [CrFo]

$$\frac{\partial}{\partial x} : \mathbb{Z}F \longrightarrow \mathbb{Z}F$$

satisfies (here for simplicity we drop off the square brackets as we remarked in §1.2.1)

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial x^{-1}}{\partial x} = -x^{-1}, \quad \frac{\partial UV}{\partial x} = \frac{\partial U}{\partial x} + U\frac{\partial V}{\partial x}, \quad \frac{\partial y}{\partial x} = 0 \quad (y \in \boldsymbol{x}, y \neq x).$$

Let $\theta : \mathbb{Z}F \longrightarrow \mathbb{Z}G$ be induced by the natural epimorphism $F \longrightarrow G$. From now on, whenever we have this composition of $\frac{\partial}{\partial x} : \mathbb{Z}F \longrightarrow \mathbb{Z}F$ and $\theta : \mathbb{Z}F \longrightarrow \mathbb{Z}G$ we will use the notation $\frac{\partial^G}{\partial x}$ for $\theta\frac{\partial}{\partial x}$, i.e.

$$\frac{\partial^G W}{\partial x} = \theta(\frac{\partial W}{\partial x}), \quad \text{for any } W \in F.$$

1.3.4 Some exact sequences

Let $\mathcal{P} = \langle \boldsymbol{x} \,;\, \boldsymbol{r} \rangle$ be a finite presentation and let $G = G(\mathcal{P}) \cong F(\boldsymbol{x})/N$ where N is the normal closure of $\{[R] : R \in \boldsymbol{r}\}$ in $F(\boldsymbol{x})$. The *relation module* is the abelianization N/N' of N regarded as a left $\mathbb{Z}G$-module with G-action given by

$$\overline{W} \cdot [U]N' = [WUW^{-1}]N' \quad (\overline{W} \in G(\mathcal{P}),\ [U] \in N).$$

Let $\boldsymbol{X} = \{\mathbb{P}_1, \mathbb{P}_2, \cdots\}$ be a set of spherical pictures over $\mathcal{P}$. We say that $\boldsymbol{X}$ *generates* $\pi_2(\mathcal{P})$ or is a set of *generating pictures* for $\pi_2(\mathcal{P})$ if $\{\langle\mathbb{P}_1\rangle, \langle\mathbb{P}_2\rangle, \cdots\}$ generates $\pi_2(\mathcal{P})$ as a $\mathbb{Z}G$-module.

Note that a set $\boldsymbol{X}$ of spherical pictures is a generating set if and only if every spherical picture can be transformed to the empty picture by a sequence of bridge moves, insertions/deletions of floating circles, insertions/deletions of cancelling pairs, insertions/deletions of elements of $\boldsymbol{X}$ and their inverses (see [Pr2, Theorem 2.5*] or [BoPr, §1.2]).

Suppose $\boldsymbol{X}$ is a generating set for $\pi_2(\mathcal{P})$. Let

$$P_3 = \bigoplus_{\mathbb{P}\in \boldsymbol{X}} \mathbb{Z}Gt_{\mathbb{P}}, \quad P_2 = \bigoplus_{R\in \boldsymbol{r}} \mathbb{Z}Gt_R, \quad P_1 = \bigoplus_{x\in \boldsymbol{x}} \mathbb{Z}Gt_x, \quad P_0 = \mathbb{Z}G$$

be free $\mathbb{Z}G$-modules. We then have the following short exact sequence [Pr2]:

$$0 \longrightarrow \pi_2(\mathcal{P}) \xrightarrow{\mu_2} P_2 \xrightarrow{\theta_2} N/N' \longrightarrow 0 \tag{1.4}$$

$$\mu_2 : \langle \mathbb{P} \rangle \longmapsto \sum_{i=1}^{n} \varepsilon_i \overline{W}_i t_{R_i}, \qquad \theta_2 : t_R \longmapsto [R]N' \quad (\mathbb{P} \in \boldsymbol{X},\ R \in \boldsymbol{r}),$$

where $(W_1 R_1^{\varepsilon_1} W_1^{-1}, \cdots, W_n R_n^{\varepsilon_n} W_n^{-1})$ is a sequence associated with $\mathbb{P}$. The embedding μ_2 is called the *standard embedding* from $\pi_2(\mathcal{P})$ to P_2.

Regard $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with trivial G-action. There is the *augmentation map* $\epsilon : P_0 \longrightarrow \mathbb{Z}$ which sends each element of G to 1. Let $\ker \epsilon = IG$, the *augmentation ideal*, and let *incl.* $: IG \longrightarrow P_0$ be the *inclusion* map. Then we have a short exact sequence:

$$0 \longrightarrow IG \xrightarrow{incl.} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0. \tag{1.5}$$

From [BrHu, p196], we also have the following short exact sequence

$$0 \longrightarrow N/N' \xrightarrow{\mu_1} P_1 \xrightarrow{\theta_1} IG \longrightarrow 0 \tag{1.6}$$

where μ_1 and θ_1 are respectively defined by

$$\mu_1 : [W]N' \longmapsto \sum_{x\in \boldsymbol{x}} \frac{\partial^G W}{\partial x} t_x, \qquad \theta_1 : t_x \longmapsto \overline{x} - 1 \quad (x \in \boldsymbol{x})$$

for all $[W] \in N$ and all $x \in \boldsymbol{x}$.

Now, by (1.4), (1.5) and (1.6) we get an exact sequence

$$0 \longrightarrow \pi_2(\mathcal{P}) \xrightarrow{\mu_2} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \tag{1.7}$$

where $\partial_1 = incl.\theta_1$, $\partial_2 = \mu_1\theta_2$.

We say that a group G is of type FP_n $(0 \leq n \leq \infty)$ if there is a *partial projective resolution* of the trivial G-module $\mathbb{Z}$ (see [Bro] for reference):

$$Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where Q_i $(0 \leq i \leq n)$ is a finitely generated projective $\mathbb{Z}G$-module.

Lemma 1.3.7 *Let G be a group finitely presented by $\mathcal{P} = \langle \boldsymbol{x} \,;\, \boldsymbol{r} \rangle$. Then G is of type FP_3 if and only if $\pi_2(\mathcal{P})$ is finitely generated as a $\mathbb{Z}G$-module.*

Proof. Consider the exact sequence (1.7). Suppose $\boldsymbol{X}$ is a finite set of generating pictures for $\pi_2(\mathcal{P})$. Then $P_3 = \bigoplus_{\mathbb{P} \in \boldsymbol{X}} \mathbb{Z}Gt_{\mathbb{P}}$ is a finitely generated free $\mathbb{Z}G$-module, and the mapping $t_{\mathbb{P}} \longrightarrow \langle \mathbb{P} \rangle$ ($\mathbb{P} \in \boldsymbol{X}$) induces a epimorphism from P_3 to $\pi_2(\mathcal{P})$. Hence, we have a partial projective (in fact, free) resolution of $\mathbb{Z}$

$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Conversely, if G is of type FP_3, then there exists a partial resolution of $\mathbb{Z}$

$$Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where Q_i $(1 \leq i \leq 3)$ is finitely generated. Let $A = \mathrm{Im}(Q_3 \longrightarrow Q_2)$. Then we have the exact sequence

$$0 \longrightarrow A \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Hence, by Lemma 8.4.4 of [Bro] we have

$$\pi_2(\mathcal{P}) \oplus Q_2 \oplus P_1 \oplus Q_0 \cong A \oplus P_2 \oplus Q_1 \oplus P_0.$$

Since the right hand side is finitely generated, the left hand side also is finitely generated. So $\pi_2(\mathcal{P})$ is finitely generated. □

A group G is said to be of *type* F_3 if it is finitely presented and is of type FP_3. (In fact, a group G is of *type* F_n $(n \geq 1)$ if there exists an *Eilenberg-Maclane complex* $K(G,1)$ with *finite* n-skeleton.) Conditions F_1, F_2 are equivalent, respectively, to G being finitely generated, finitely presented. For $n \geq 3$, G is of type F_n if and only if G is finitely presented and of type FP_n. For example, see [Al3] for reference.)

1.4 Pictures over monoid presentations

1.4.1 Pictures over monoid presentations

Consider a monoid presentation

$$\hat{\mathcal{P}} = [\boldsymbol{x}\,;\,\boldsymbol{r}].$$

As in [Pr3], for each $R \in \boldsymbol{r}$ and any words U, V on $\boldsymbol{x}$ we can define two geometric objects called *atomic pictures* $\mathbb{E} = (U,\ R,\ +1,\ V)$ and $\mathbb{E}^{-1} = (U,\ R,\ -1,\ V)$ over $\hat{\mathcal{P}}$ as depicted in Fig. 1.6:

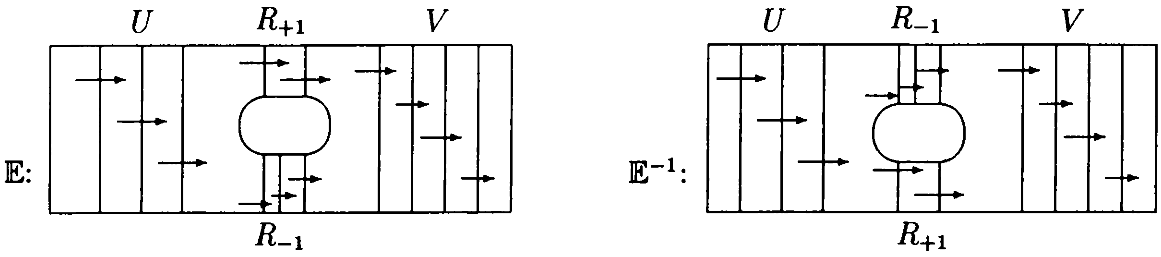


Fig. 1.6

where each arc in the rectangles is transversely orientated from left to right and labelled by an element of $\boldsymbol{x}$; each disc represents the relator $R : R_{+1} = R_{-1}$, with *upper* (*half*) *label* R_ε and *lower* (*half*) *label* $R_{-\varepsilon}$ for the edge $\mathbb{E}^\varepsilon$ ($\varepsilon = \pm 1$). The word $UR_\varepsilon V$ we read off by travelling along the top of the picture $\mathbb{E}^\varepsilon$ from left to right is called the *upper* (*half*) *boundary label* of $\mathbb{E}^\varepsilon$ and the word $UR_{-\varepsilon}V$ we read off along the bottom is called the *lower* (*half*) *boundary label* of $\mathbb{E}^\varepsilon$.

1.4.2 Associated two-complexes

Regarding those atomic pictures defined in the previous subsection as edges with *initial* $\iota(\mathbb{E}^\varepsilon) = UR_\varepsilon V$ and *terminal* $\tau(\mathbb{E}^\varepsilon) = UR_{-\varepsilon}V$ as well as *inverse* $(\mathbb{E}^\varepsilon)^{-1} = (\mathbb{E})^{-\varepsilon}$, $\varepsilon = \pm 1$, we can associate with $\hat{\mathcal{P}}$ the graph $\Gamma(\hat{\mathcal{P}}) = (\hat{F}(\boldsymbol{x}),\ \boldsymbol{e})$ where $\boldsymbol{e}$ is the set of all the atomic pictures. A path $\mathbb{P}$ in $\Gamma(\hat{\mathcal{P}})$ will also be called a *monoid picture* over $\hat{\mathcal{P}}$ with *upper* (half) boundary label $\iota(\mathbb{P})$ and *lower* (half) boundary label $\tau(\mathbb{P})$, and a closed path $\mathbb{P}$ in $\Gamma(\hat{\mathcal{P}})$ will also be called a *spherical* (*monoid*) *picture* over $\hat{\mathcal{P}}$. An *arc* of a path $\mathbb{P}$ consists of a number of edge arcs which are labelled by the same element of $\boldsymbol{x}$ and geometrically connected one by one. We see that the length $L(\mathbb{P})$ of a path $\mathbb{P}$ is the number of discs in the geometric representation of $\mathbb{P}$.

Note that if $\hat{\mathcal{P}}$ is finite then $\Gamma(\hat{\mathcal{P}})$ is locally finite.

We also have left and right actions of $\hat{F}(\boldsymbol{x})$ on $\Gamma(\hat{\mathcal{P}})$, that is, for any word W on $\boldsymbol{x}$, any vertex V (also a word on $\boldsymbol{x}$) and any edge $\mathbb{E} = (U,\ R,\ \varepsilon,\ U')$ ($\varepsilon = \pm 1$) in $\Gamma(\hat{\mathcal{P}})$,

$$W \cdot V = WV, \quad V \cdot W = VW \quad \text{(product in } \hat{F}(\boldsymbol{x}))$$

$$W \cdot \mathbb{E} = (WU,\ R,\ \varepsilon,\ U'), \quad \mathbb{E} \cdot W = (U,\ R,\ \varepsilon,\ U'W).$$

Obviously, these actions are compatible.

Example 1.4.1 Let $\hat{\mathcal{P}} = [x,\ y:\ x^3 = 1,\ xy = yx^2]$, and let

$$\mathbb{E}_1 = (\,1,\ x^3\ = 1,\ -1,\ y\,), \quad \mathbb{E}_2 = (x^2,\ xy = yx^2,\ +1,\ 1),$$

$$\mathbb{E}_3 = (x,\ xy = yx^2,\ +1,\ x^2), \quad \mathbb{E}_4 = (1,\ xy = yx^2,\ +1,\ x^4),$$

$$\mathbb{E}_5 = (yx^3,\ x^3 = 1,\ +1,\ 1), \quad \mathbb{E}_6 = (\,y,\ x^3 = 1,\ +1,\ 1\,).$$

Then $\tau(\mathbb{E}_i) = \iota(\mathbb{E}_{i+1})$, $i = 1,\ 2,\ \cdots,\ 5$ and $\iota(\mathbb{E}_1) = \tau(\mathbb{E}_6) = y$. Thus, $\mathbb{P} = \mathbb{E}_1 \cdots \mathbb{E}_6$ is a closed path at y in $\Gamma(\hat{\mathcal{P}})$, where, for example, the curve starting at the lower boundary of the disc in $\mathbb{E}_1$ and ending at the upper boundary of the disc in $\mathbb{E}_4$ labelled by x is an arc of $\mathbb{P}$. Now by a left action of y and a right action by x, we obtain another closed path $y \cdot \mathbb{P} \cdot x$ at y^2x as shown in Fig. 1.7.

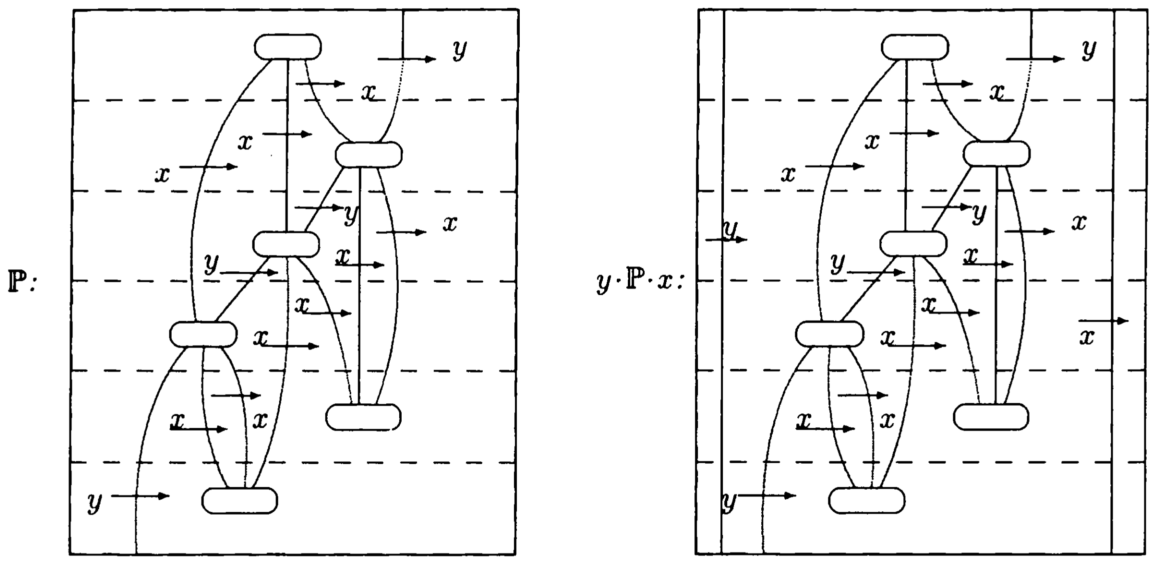


Fig. 1.7

By introducing a set $\hat{\boldsymbol{Z}}$ of the following defining paths we then form a locally finite two-complex $\langle \Gamma(\hat{\mathcal{P}});\ \hat{\boldsymbol{Z}} \rangle$ denoted $\mathcal{D}(\hat{\mathcal{P}})$. For any two edges $\mathbb{A}$, $\mathbb{B}$ in $\Gamma(\hat{\mathcal{P}})$, the defining path (closed) of $\mathcal{D}(\hat{\mathcal{P}})$ defined by $\mathbb{A}$ and $\mathbb{B}$ is

$$[\mathbb{A},\ \mathbb{B}] = (\mathbb{A} \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B})(\mathbb{A}^{-1} \cdot \tau(\mathbb{B})(\iota(\mathbb{A}) \cdot \mathbb{B}^{-1}) \tag{1.8}$$

as shown in Fig. 1.8.

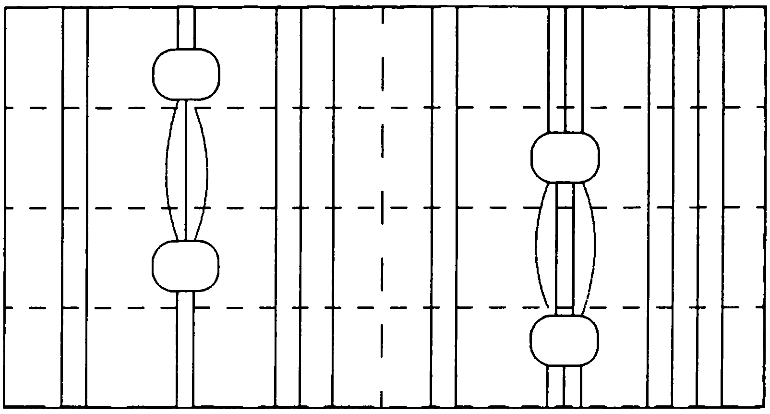


Fig. 1.8

Obviously we have $\hat{F}(\boldsymbol{x}) \cdot \hat{\boldsymbol{Z}} \cdot \hat{F}(\boldsymbol{x}) = \hat{\boldsymbol{Z}}$. Thus, $\hat{F}(\boldsymbol{x})$ acts on $\mathcal{D}(\hat{\mathcal{P}})$ on both sides compatibly and so if $\hat{\mathcal{P}}$ is finite then $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ is a locally finite object of $\mathfrak{C}$.

Definition 1.4.2 *A finite monoid presentation $\hat{\mathcal{P}} = [\boldsymbol{x}\,;\, \boldsymbol{r}]$ is FDT if $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ has a finite trivialiser; and a monoid S is FDT if it has a finite presentation which is FDT.*

We now extend (1.8) to get the following lemma.

Lemma 1.4.3 (Pull-down and push-up) *Let $\mathbb{A}$, $\mathbb{B}$ be any two paths in $\mathcal{D}(\hat{\mathcal{P}})$. Then $(\mathbb{A} \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B})$ and $(\iota(\mathbb{A}) \cdot \mathbb{B})(\mathbb{A} \cdot \tau(\mathbb{B})$ are equivalent, namely*

$$(\mathbb{A} \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B})(\mathbb{A}^{-1} \cdot \tau(\mathbb{B})(\iota(\mathbb{A}) \cdot \mathbb{B}^{-1}) \sim 1_{\iota(\mathbb{A})\iota(\mathbb{B})}.$$

Proof. Let $\mathbb{A} = \mathbb{A}_1\mathbb{A}_2 \cdots \mathbb{A}_m$ and let $\mathbb{B} = \mathbb{B}_1\mathbb{B}_2 \cdots \mathbb{B}_n$ with each $\mathbb{A}_i$ and each $\mathbb{B}_j$ are edges. Thus,

$$\begin{aligned}(\mathbb{A} \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B}) &= (\mathbb{A}_1 \cdot \iota(\mathbb{B})) \cdots (\mathbb{A}_m \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B}_1) \cdots (\tau(\mathbb{A}) \cdot \mathbb{B}_n) \\ &= (\mathbb{A}_1 \cdot \iota(\mathbb{B}_1)) \cdots (\mathbb{A}_m \cdot \iota(\mathbb{B}_1))(\tau(\mathbb{A}_m) \cdot \mathbb{B}_1) \cdots (\tau(\mathbb{A}_m) \cdot \mathbb{B}_n).\end{aligned}$$

But

$$(\mathbb{A}_m \cdot \iota(\mathbb{B}_1))(\tau(\mathbb{A}_m) \cdot \mathbb{B}_1) \sim (\iota(\mathbb{A}_m) \cdot \mathbb{B}_1)(\mathbb{A}_m \cdot \tau(\mathbb{B}_1)).$$

So,

$$\begin{aligned}&(\mathbb{A}_1 \cdot \iota(\mathbb{B}_1)) \cdots (\mathbb{A}_m \cdot \iota(\mathbb{B}_1))(\tau(\mathbb{A}_m) \cdot \mathbb{B}_1) \cdots (\tau(\mathbb{A}_m) \cdot \mathbb{B}_n) \\ \sim\ &(\mathbb{A}_1 \cdot \iota(\mathbb{B}_1)) \cdots (\mathbb{A}_{m-1} \cdot \iota(\mathbb{B}_1))(\iota(\mathbb{A}_m) \cdot \mathbb{B}_1)(\mathbb{A}_m \cdot \tau(\mathbb{B}_1))(\tau(\mathbb{A}_m) \cdot \mathbb{B}_2) \cdots (\tau(\mathbb{A}_m) \cdot \mathbb{B}_n).\end{aligned}$$

Repeating this procedure we eventually get

$$\begin{aligned}
&(\mathbb{A}_1 \cdot \iota(\mathbb{B}_1)) \cdots (\mathbb{A}_m \cdot \iota(\mathbb{B}_1))(\tau(\mathbb{A}_m) \cdot \mathbb{B}_1) \cdots (\tau(\mathbb{A}_m) \cdot \mathbb{B}_n) \\
\sim\ &(\iota(\mathbb{A}_1) \cdot \mathbb{B}_1) \cdots (\iota(\mathbb{A}_1) \cdot \mathbb{B}_n)(\mathbb{A}_1 \cdot \tau(\mathbb{B}_n)) \cdots (\mathbb{A}_m \cdot \tau(\mathbb{B}_n)) \\
\sim\ &(\iota(\mathbb{A}) \cdot \mathbb{B})(\mathbb{A} \cdot \tau(\mathbb{B})).
\end{aligned}$$

This completes our proof. □

This lemma means that in the geometric configuration of $(\mathbb{A} \cdot \iota(\mathbb{B}))(\tau(\mathbb{A}) \cdot \mathbb{B})$ one can simply pull down the part representing $\mathbb{A}$ and push up the part representing $\mathbb{B}$ without changing the homotopy type.

1.4.3 First order Dehn functions of monoids

We still consider the monoid presentation $\hat{\mathcal{P}}$ defined at the beginning of this section. We can see that each path $\mathbb{P} = \mathbb{E}_1\mathbb{E}_2 \cdots \mathbb{E}_m$ of $\mathcal{D}(\hat{\mathcal{P}})$ represents a *derivation* from $\iota(\mathbb{P})$ to $\tau(\mathbb{P})$ by means of the relators. For example, suppose $\mathbb{E}_i = (U_i,\ R_i,\ \varepsilon_i,\ V_i)$, $\varepsilon_i = \pm 1$, $i = 1,\ 2,\ \cdots,\ m$. Then we have a derivation of length m from the word $U_1 R_{\varepsilon_1,1} V_1$ to the word $U_m R_{-\varepsilon_m,m} V_m$ of the form:

$$\begin{aligned}
&U_1 R_{\varepsilon_1,1} V_1 \xrightarrow{R_{\varepsilon_1,1}=R_{-\varepsilon_1,1}} U_2\, R_{\varepsilon_2,2} V_2 \xrightarrow{R_{\varepsilon_2,2}=R_{-\varepsilon_2,2}} \cdots \\
&\qquad \cdots \xrightarrow{R_{\varepsilon_{m-1},m-1}=R_{-\varepsilon_{m-1},m-1}} U_m R_{\varepsilon_m,m} V_m \xrightarrow{R_{\varepsilon_m,m}=R_{-\varepsilon_m,m}} U_m R_{-\varepsilon_m,m} V_m
\end{aligned}$$

where $U_{i+1} R_{\varepsilon_{i+1},i+1} V_{i+1} = U_i R_{-\varepsilon_i,i} V_i$ as words on $\boldsymbol{x}$, $i = 1,\ 2,\ \cdots,\ m-1$. Thus, $\mathcal{D}(\hat{\mathcal{P}})$ consists of components such that any two vertices lie in a component of $\mathcal{D}(\hat{\mathcal{P}})$ if and only if these two vertices represent the same element of $S(\hat{\mathcal{P}})$. Thus, if we let $\Delta(W)$ denote the component of $\mathcal{D}(\hat{\mathcal{P}})$ containing the vertex W then the map: $\overline{W} \longmapsto \Delta(W)$ is a one-to-one map from $S(\hat{\mathcal{P}})$ to the set of all components of $\mathcal{D}(\hat{\mathcal{P}})$.

The *first order Dehn function* of a finite monoid presentation (see [Pr3]) is defined as follows.

Let $\hat{\mathcal{P}} = [\boldsymbol{x}\ ;\ \boldsymbol{r}]$ be a finite monoid presentation for a monoid S and let U, V be two words on $\boldsymbol{x}$ with $L(U) + L(V) = n$ for some integer n. If $\overline{U} = \overline{V}$ then U and V lie in the same component of $\mathcal{D}(\hat{\mathcal{P}})$. So there are paths in $\mathcal{D}(\hat{\mathcal{P}})$ from U to V. Following

Pride [Pr3] we define the *derivation length* $Der_{\hat{\mathcal{P}}}(U, V)$ (in [Pr3], Pride uses the notation $Area_{\hat{\mathcal{P}}}(U, V)$) of U and V to be the length of a shortest path in $\mathcal{D}(\hat{\mathcal{P}})$ from U to V. Then the *first order Dehn function* of the monoid presentation $\hat{\mathcal{P}}$ is the function

$$\hat{\delta}^{(1)}_{\hat{\mathcal{P}}}(n) = \max\{Der_{\hat{\mathcal{P}}}(U, V) : \overline{U} = \overline{V},\ U,\ V \in \hat{F}(\boldsymbol{x}) \text{ with } L(U) + L(V) \leq n\}.$$

By using Tietze transformations Pride [Pr3, Theorem] proved that up to equivalence $\hat{\delta}^{(1)}$ is independent of the choice of finite monoid presentations of S. We then use $\hat{\delta}^{(1)}_S$ to denote a particular representative of the equivalence class.

Chapter 2

The definition, the quasi-retract inequalities and the quasi-isometry invariance

2.1 The definition of second order Dehn function of groups

2.1.1 The definition

Let

$$\mathcal{P} = \langle \boldsymbol{x} \,;\, \boldsymbol{r} \rangle$$

be a finite group presentation for a given group G, and let $F(\boldsymbol{x})$ be the free group on $\boldsymbol{x}$. Let $\boldsymbol{X}$ be a (not necessarily finite) set of generators of $\pi_2(\mathcal{P})$ as a $\mathbb{Z}G$-module. Then any $\xi = \langle \mathbb{P} \rangle \in \pi_2(\mathcal{P})$ can be written as a sum

$$\sum_{i=1}^{m} \varepsilon_i g_i \xi_i, \tag{2.1}$$

where $\varepsilon_i = \pm 1$, $g_i \in G$, $\xi_i = \langle \mathbb{P}_i \rangle \in \boldsymbol{X}$ *for* $i = 1, \cdots, m$. To give a description for ξ of the form (2.1), we hope that the value of m is as small as possible.

Definition 2.1.1 *For each $\xi \in \pi_2(\mathcal{P})$ the volume $V_{\mathcal{P},\boldsymbol{X}}(\xi)$ (or simply $V_{\boldsymbol{X}}(\xi)$) of ξ with respect to $\boldsymbol{X}$ is the minimal value of m over all sums of the form (2.1) equal to ξ. If*

$\langle \mathbb{P} \rangle = \xi$, *we sometimes write* $V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\mathbb{P})$ *for* $V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi)$.

We then define the *second order Dehn function* as follows.

Definition 2.1.2 *The second order Dehn function of* $\boldsymbol{\mathcal{P}}$ *with respect to a generating set* $\boldsymbol{X}$ *of the second homotopy module* $\pi_2(\boldsymbol{\mathcal{P}})$ *is the function*

$$\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}} : \mathbb{N} \longrightarrow \mathbb{R}^+$$

given by

$$\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(n) = \max\{V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi) : \ A(\xi) \leq n\}.$$

We must prove this definition is valid. To do so it suffices to show

Lemma 2.1.3 *The set*

$$\chi_n = \{V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi) : \ A(\xi) \leq n\}$$

is finite for all $n \in \mathbb{N}$.

Proof. We call an element κ of $\pi_2(\boldsymbol{\mathcal{P}})$ *connected* if there is a minimal connected picture representing κ. Since $\boldsymbol{\mathcal{P}}$ is finite there are only finitely many connected elements of $\pi_2(\boldsymbol{\mathcal{P}})$ with a fixed area. Let

$$\chi^*_n = \max\{V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\kappa) : \ A(\kappa) \leq n, \kappa \text{ connected}\}.$$

Then clearly χ^*_n is finite. Now let ξ be any element of $\pi_2(\boldsymbol{\mathcal{P}})$ with $A(\xi) \leq n$, and let $\mathbb{P}$ be a minimal picture representing ξ. Then $\mathbb{P}$ will have a non-empty connected spherical subpicture $\mathbb{D}$. Let $\mathbb{P}_1$ be obtained from $\mathbb{P}$ by removing $\mathbb{D}$. Then if ξ_1, κ_1 are the elements of $\pi_2(\boldsymbol{\mathcal{P}})$ represented by $\mathbb{P}_1$,$\mathbb{D}$ respectively, we have, by Lemma 1.3.4

$$\xi = \xi_1 + g_1\kappa_1$$

for some $g_1 \in G$. Since $A(\xi_1) < A(\xi)$ we may repeat the argument with ξ_1 in place of ξ, and so on. Eventually we get

$$\xi = g_1\kappa_1 + g_2\kappa_2 + \cdots + g_l\kappa_l$$

where $g_1, g_2, \cdots, g_l \in G$, and $\kappa_1, \kappa_2, \cdots, \kappa_l$ are connected. Noting that $l \leq n$ and $A(\kappa_i) \leq n$ for $i = 1, 2, \cdots, l$, we deduce that

$$V_{\boldsymbol{X}}(\xi) \leq n\chi_n^*$$

which implies that χ_n is finite. □

The following remark will be used, usually without further comment, in the sequel.

Remark 2.1.4 Let $\mathcal{Q} = \langle \boldsymbol{x}; \boldsymbol{r} \rangle$ be a subpresentation of $\mathcal{P} = \langle \boldsymbol{x}, \boldsymbol{y}; \boldsymbol{r}, \ \boldsymbol{s} \rangle$ where $\boldsymbol{s} \subseteq F(\boldsymbol{y})$. The inclusion map $\phi: \ \mathcal{Q} \longrightarrow \mathcal{P}$ induces an embedding ϕ_* from the group $H = G(\mathcal{Q})$ to the group $G = G(\mathcal{P})$. Also ϕ induces an abelian group homomorphism

$$\phi_*: \ \pi_2(\mathcal{Q}) \longrightarrow \pi_2(\mathcal{P}) \qquad \langle \mathbb{P} \rangle_{\mathcal{Q}} \longmapsto \langle \mathbb{P} \rangle_{\mathcal{P}},$$

and we have

$$\phi_*(h \cdot \xi) = \phi_*(h) \cdot \phi_*(\xi) \qquad (h \in H, \ \xi \in \pi_2(\mathcal{Q})).$$

Here we write $\langle \mathbb{P} \rangle_{\mathcal{Q}}$ if we regard it as an element of $\pi_2(\mathcal{Q})$ and write $\langle \mathbb{P} \rangle_{\mathcal{P}}$ if we regard it as an element of $\pi_2(\mathcal{P})$.

Now suppose we have a finite set $\boldsymbol{X}$ of generating pictures of $\pi_2(\mathcal{P})$ containing a set $\boldsymbol{Y}$ of generating pictures of $\pi_2(\mathcal{Q})$. Since every expression for $\langle \mathbb{P} \rangle$ for any spherical picture $\mathbb{P}$ over $\mathcal{Q}$ of form (2.1) in $\pi_2(\mathcal{Q})$ is also an expression for $\langle \mathbb{P} \rangle$ in $\pi_2(\mathcal{P})$, it follows from the previous paragraph that

$$V_{\boldsymbol{X}}(\langle \mathbb{P} \rangle_{\mathcal{P}}) \leq V_{\boldsymbol{Y}}(\langle \mathbb{P} \rangle_{\mathcal{Q}}). \qquad (2.2)$$

2.2 Quasi-retract inequalities and quasi-isometry invariance

2.2.1 Universal covers

Let $\mathcal{P} = \langle \boldsymbol{x} \, ; \, \boldsymbol{r} \rangle$ be a finite presentation and let $G = G(\mathcal{P})$. We regard $\mathcal{P}$ as a 2-complex with a single vertex as explained in §1.2.2, and let $\tilde{\mathcal{P}}$ be the universal covering of $\mathcal{P}$. Let

$p_0 : \tilde{\mathcal{P}} \longrightarrow \mathcal{P}$ be the projection map, and for each $g \in G$ let $t_g : \mathcal{P} \longrightarrow \tilde{\mathcal{P}}$ be the lift map with respect to g defined in §1.2.5. We further have the following.

Lemma 2.2.1 (Homotopy lifting) *Let $\tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}'$ be any two spherical pictures over $\tilde{\mathcal{P}}$. We have that*

(i) *if $\tilde{\mathbb{P}} \sim \tilde{\mathbb{P}}'$ in $\tilde{\mathcal{P}}$ then $p_o(\tilde{\mathbb{P}}) \sim p_o(\tilde{\mathbb{P}}')$ in $\mathcal{P}$,*

(ii) if $\mathbb{P} \sim \mathbb{P}'$ in $\mathcal{P}$ then $t_g(\mathbb{P}) \sim t_g(\mathbb{P}')$ in $\tilde{\mathcal{P}}$.

Proof. Suppose that $\tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}'$ are any two spherical pictures over $\tilde{\mathcal{P}}$ and $\tilde{\mathbb{P}} \sim \tilde{\mathbb{P}}'$ in $\tilde{\mathcal{P}}$. In order to show $p_o(\tilde{\mathbb{P}}) \sim p_o(\tilde{\mathbb{P}}')$ in $\mathcal{P}$, by symmetry and induction it suffices to show that if $\tilde{\mathbb{P}}'$ is obtained from $\tilde{\mathbb{P}}$ by a single application of one of the operations: deletion of a floating circle, deletion of a cancelling pair, bridge move, then so is $p_o(\tilde{\mathbb{P}}')$ obtained from $p_o(\tilde{\mathbb{P}})$.

Now the geometric configurations of $p_o(\tilde{\mathbb{P}})$ and $p_o(\tilde{\mathbb{P}}')$ are the same as those of $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}}'$ respectively. Therefore, the proof for the first two cases are trivial. For the third case, suppose that there is a neighbourhood in $\tilde{\mathbb{P}}$ containing exactly two arc segments and the arrows riding on these arcs have the same label, say (g^*, x), with opposite directions (see Fig. 2.1).

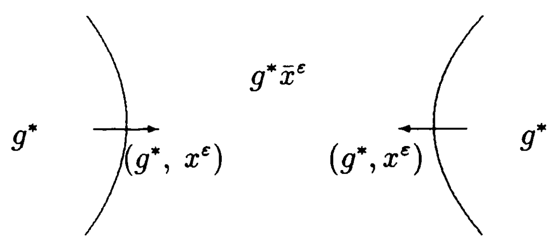


Fig. 2.1

Passing from $\tilde{\mathbb{P}}$ to $\tilde{\mathbb{P}}'$ by a bridge move, this neighbourhood becomes that as shown in Fig. 2.2.

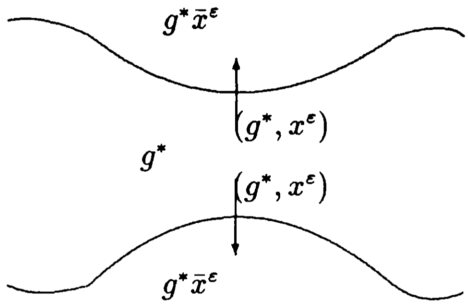


Fig. 2.2

We point out that $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}}'$ differ only by these neighbourhoods.

Moving from $\tilde{\mathbb{P}}$ to $p_o(\tilde{\mathbb{P}})$ by p_o, the labels of the arrows riding on the two arcs in Fig. 2.1 become x in the corresponding neighbourhood of $p_o(\tilde{\mathbb{P}})$, and moving from $\tilde{\mathbb{P}}'$ to $p_o(\tilde{\mathbb{P}}')$ by p_o, the labels of the arrows riding on the two arcs in Fig. 2.2 become x as well in the corresponding neighbourhood of $p_o(\tilde{\mathbb{P}}')$. Also, $p_o(\mathbb{P})$ and $p_o(\mathbb{P}')$ differ only by these neighbourhoods. Thus, picture $p_o(\tilde{\mathbb{P}}')$ can be obtained from picture $p_o(\tilde{\mathbb{P}})$ by a bridge move. So, we have $p_o(\tilde{\mathbb{P}}') \sim p_o(\tilde{\mathbb{P}})$ in $\mathcal{P}$.

Let $\mathbb{P}$ and $\mathbb{P}'$ be any two pictures over $\mathcal{P}$ with $\mathbb{P} \sim \mathbb{P}'$ in $\mathcal{P}$. In order to show (ii), by symmetry and induction again it suffices to show that if $\mathbb{P}'$ is obtained from $\mathbb{P}$ by a single application of one of the operations: deletion of a floating circle, deletion of a cancelling pair, bridge move, then so is $t_g(\mathbb{P}')$ obtained from $t_g(\mathbb{P})$. The proof for the first two cases is also trivial since the geometry configurations of $t_g(\mathbb{P})$ and $t_g(\mathbb{P}')$ again are precisely those of $\mathbb{P}$ and $\mathbb{P}'$ respectively. For the third case, suppose that there is a neighbourhood in $\mathbb{P}$ containing exactly two arc segments and the arrows riding on these arcs have the same label, say x, with opposite directions as shown in Fig. 2.3.

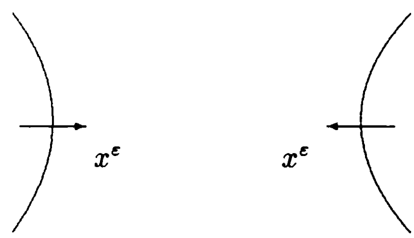


Fig. 2.3

Passing from $\mathbb{P}$ to $\mathbb{P}'$ by performing a bridge move, this neighbourhood becomes as shown in Fig. 2.4.

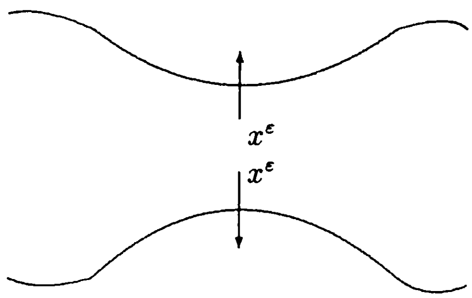


Fig. 2.4

Choose a transverse path in $\mathbb{P}$ from the basepoint to the start of the left arrow labelled by x in the above neighbourhood of $\mathbb{P}$ and suppose that this path represents an element $g' \in G$. By extending this path to the end of the same arrow and to the start of the other arrow labelled also by x by simply crossing the left arc first and then the right arc in the neighbourhood we obtain a transverse path from the basepoint of $\mathbb{P}$ to the end of the left arrow and a transverse path from the basepoint of $\mathbb{P}$ to the start of the right arrow labelled by x in this neighbourhood and these two paths represent the elements $g'\overline{x}$ and $g'\overline{xx^{-1}} = g'$ in G respectively. Let $g^* = gg'$. Then when we lift $\mathbb{P}$ to $t_g(\mathbb{P})$, this neighbourhood becomes exactly that as shown in Fig. 2.1. Similarly, when we lift $\mathbb{P}'$ to $t_g(\mathbb{P}')$ at g, the neighbourhood in $\mathbb{P}'$ we mentioned in the above also becomes exactly that as shown in Fig. 2.2.

We point out that $t_g(\mathbb{P})$ and $t_g(\mathbb{P}')$ differ by these two neighbourhoods. Thus, picture $t_g(\mathbb{P}')$ can be obtained from picture $t_g(\mathbb{P})$ by performing a bridge move. So, we also have $t_g(\mathbb{P}') \sim t_g(\mathbb{P})$. □

Corollary 2.2.2 *The function*

$$p_*\langle\tilde{\mathbb{P}}\rangle = \langle p_o(\tilde{\mathbb{P}})\rangle$$

is well-defined on equivalence classes, and for any $g \in G$ the restriction

$$p_* : \pi_2(\tilde{\mathcal{P}}, g) \longrightarrow \pi_2(\mathcal{P})$$

is an isomorphism of abelian groups.

Proof. By the above lemma, p_* is well-defined on equivalence classes. Let $\tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}'$ be two spherical pictures with $\iota(\tilde{\mathbb{P}}) = \iota(\tilde{\mathbb{P}}') = g$. Clearly, by the definition of p_o we have that

$$p_o(\tilde{\mathbb{P}} + \tilde{\mathbb{P}}') = p_o(\tilde{\mathbb{P}}) + p_o(\tilde{\mathbb{P}}').$$

Thus

$$\begin{aligned} p_*(\langle\tilde{\mathbb{P}}\rangle + \langle\tilde{\mathbb{P}}'\rangle) &= \langle p_o(\tilde{\mathbb{P}} + \tilde{\mathbb{P}}')\rangle = \langle p_o(\tilde{\mathbb{P}}) + p_o(\tilde{\mathbb{P}}')\rangle = \langle p_o(\tilde{\mathbb{P}})\rangle + \langle p_o(\tilde{\mathbb{P}}')\rangle \\ &= p_*\langle\tilde{\mathbb{P}}\rangle + p_*\langle\tilde{\mathbb{P}}'\rangle. \end{aligned}$$

So p_* is a homomorphism of abelian groups. Now by the above lemma the function

$$t_{g*} : \pi_2(\mathcal{P}) \longrightarrow \pi_2(\tilde{\mathcal{P}}, g), \quad \langle\mathbb{P}\rangle \longmapsto \langle t_g(\mathbb{P})\rangle \tag{2.3}$$

is also well-defined and again it is easily shown to be a homomorphism of abelian groups which by Lemma 1.3.6 is the inverse of the above restriction of p_*. □

Given the universal cover $\tilde{\mathcal{P}}$ of $\mathcal{P}$ it will be assumed that for any pair of elements $g_1,\, g_2 \in G$, a fixed geodesic from g_1 to g_2 is chosen denoted γ_{g_1,g_2}. We can suppose that $\gamma_{g_2,g_1} = \gamma_{g_1,g_2}^{-1}$ for any pair $g_1,\, g_2 \in G$.

2.2.2 Mappings of groups

Now let $\mathcal{Q} = \langle \boldsymbol{y}\ ;\ \boldsymbol{s}\rangle$ be another finite group presentation and let $H = G(\mathcal{Q})$. Let $\tilde{\mathcal{Q}}$ be the universal cover of $\mathcal{Q}$ with projection map q_o.

Let $\phi :\ G \longrightarrow H$ be any function (not necessary a homomorphism). Then ϕ can be regarded as a function sending vertices of $\tilde{\mathcal{P}}$ to vertices of $\tilde{\mathcal{Q}}$. We extend ϕ as follows. For e an edge of $\tilde{\mathcal{P}}$ we define

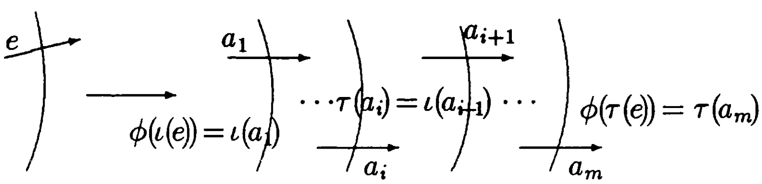
$$\phi(e) = \gamma_{\phi(\iota(e)),\phi(\tau(e))}.$$

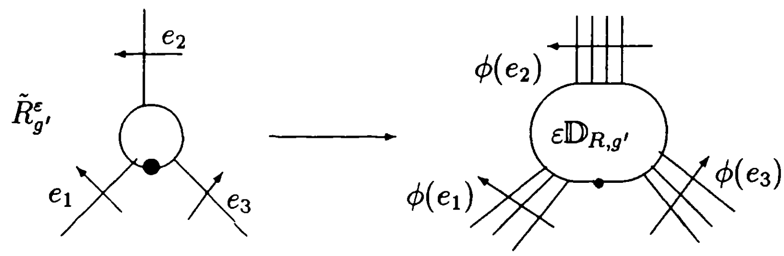
If $\alpha = e_1 e_2 \cdots e_n$ is a path in $\tilde{\mathcal{P}}$ we define

$$\phi(\alpha) = \phi(e_1)\phi(e_2)\cdots\phi(e_n).$$

Consider a defining path $\tilde{R}_g$ of $\tilde{\mathcal{P}}$. Then $\phi(\tilde{R}_g)$ is a closed path in $\tilde{\mathcal{Q}}$, and so $q_o(\phi(\tilde{R}_g))$ defines the identity of H (and so ϕ is a mapping of two-complexes). Thus there is picture over $\mathcal{Q}$ with boundary label $q_o(\phi(\tilde{R}_g))$. We choose such a picture and let $\mathbb{D}_{R,g}$ denote the lift over $\tilde{\mathcal{Q}}$ at $\phi(g)$ of this chosen picture.

Now let $\tilde{\mathbb{P}}$ be any picture over $\tilde{\mathcal{P}}$ with the basepoint labelled by some element ς of G. We convert it to a picture denoted $\phi(\tilde{\mathbb{P}})$ over $\tilde{\mathcal{Q}}$ as follows: given a region Ψ labelled by an element g^* of G, replace the label by $\phi(g^*)$; given an arc, labelled e say, replace it by a sequence of parallel arcs with total label $\phi(e)$, and if $\phi(e) = a_1a_2\cdots a_m$ for some edges a_1, a_2, $\cdots$, a_m of $\tilde{\mathcal{Q}}$, then the region of $\phi(\tilde{\mathbb{P}})$ shared by the arcs labelled by a_i and a_{i+1} $(i = 1, \cdots, a_m)$ is labelled by $\tau(a_i) = \iota(a_{i+1})$, and in particular, the regions at the start of the arrow labelled by a_1 and at the end of the arrow labelled by a_m had been labelled by $\phi(\iota(e)) = \iota(\phi(e)) = \iota(a_1)$ and $\phi(\tau(e)) = \tau(\phi(e)) = \tau(a_m)$ respectively; given a disc labelled by $\tilde{R}^{\varepsilon}_{g'}$ $(R \in \boldsymbol{r}, g' \in G, \varepsilon = \pm 1)$ replace it by the picture $\varepsilon\mathbb{D}_{R,g'}$. Thus, $\phi(\tilde{\mathbb{P}})$ is a picture over $\tilde{\mathcal{Q}}$ at $\phi(g)$.





We then have

Lemma 2.2.3 *The function* ϕ_* *given by*

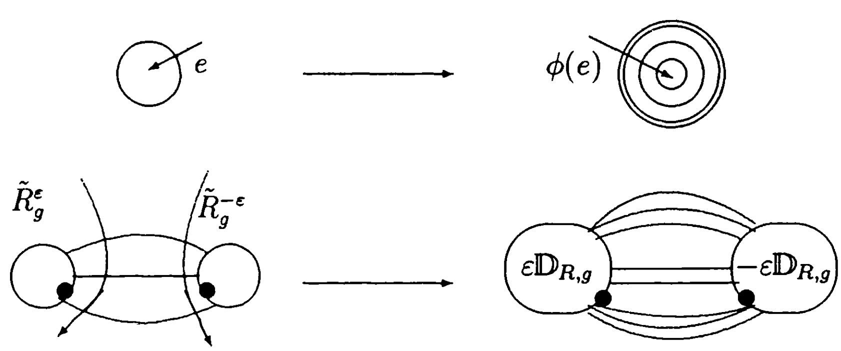
$$\phi_*\langle\tilde{\mathbb{P}}\rangle = \langle\phi(\tilde{\mathbb{P}})\rangle$$

is well-defined on equivalence classes; in particular, for any $g \in G$ *the restriction*

$$\phi_* : \ \pi_2(\tilde{\mathcal{P}}, g) \longrightarrow \pi_2(\tilde{\mathcal{Q}}, \phi(g))$$

is a homomorphism of abelian groups.

Proof. To show that ϕ_* is well-defined we must show that if $\tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}'$ are any pictures over $\tilde{\mathcal{P}}$ with $\tilde{\mathbb{P}} \sim \tilde{\mathbb{P}}'$ in $\tilde{\mathcal{P}}$ then $\phi(\tilde{\mathbb{P}}) \sim \phi(\tilde{\mathbb{P}}')$ in $\tilde{\mathcal{Q}}$. In order to do this, by the symmetry and the induction, it suffices to show that if $\tilde{\mathbb{P}}'$ is obtained from $\tilde{\mathbb{P}}$ by one application of each operation of deletions of floating circles, deletions of cancelling pairs, and bridge moves then so is $p_o(\tilde{\mathbb{P}}')$ obtained from $p_o(\tilde{\mathbb{P}})$. Thus, in the first case, the floating circle in $\tilde{\mathbb{P}}$ is converted to a sequence of concentric closed arcs in $\phi(\tilde{\mathbb{P}})$. By a sequence of applications of deletions of floating circles these closed arcs are removed and then $\phi(\tilde{\mathbb{P}})$ is transformed to $\phi(\tilde{\mathbb{P}}')$. In the second case, the cancelling pair in $\tilde{\mathbb{P}}$ is converted to a trivial spherical subpicture of $\phi(\tilde{\mathbb{P}})$ which can be modified to a sequence of cancelling pairs by a succession of bridge moves, and so by removing these cancelling pairs we obtain the picture $\phi(\tilde{\mathbb{P}}')$. In the third case, the bridge move used on $\tilde{\mathbb{P}}$ to obtain $\tilde{\mathbb{P}}'$ is converted to a sequence of bridge moves on $\phi(\tilde{\mathbb{P}})$ to get $\phi(\tilde{\mathbb{P}}')$. The proof for the restriction of ϕ_* on $\pi_2(\tilde{\mathcal{P}}, g)$ being a homomorphism of abelian groups is similar with that in the proof of Corollary 2.2.2 for the restriction of p_* on $\pi_2(\tilde{\mathcal{P}}, g)$ being a homomorphism of abelian groups. □



We will denote the composition

$$\pi_2(\mathcal{P}) \xrightarrow{p_*^{-1}} \pi_2(\tilde{\mathcal{P}}, g) \xrightarrow{\phi_*} \pi_2(\tilde{\mathcal{Q}}, \phi(g)) \xrightarrow{q_*} \pi_2(\mathcal{Q})$$

by ϕ_g.

We now prove the following result.

Lemma 2.2.4 *For any $\xi \in \pi_2(\mathcal{P})$ and $h,\ g \in G$*

$$\phi_g(h \cdot \xi) = \phi(g)^{-1}\phi(gh) \cdot \phi_{gh}(\xi).$$

Proof. Let ξ be any element of $\pi_2(\mathcal{P})$ and let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$ representing ξ. Then,

$$\begin{aligned} \phi_g(h \cdot \xi) &= \phi_g(h\langle\mathbb{P}\rangle) \\ &= \phi_g\langle\mathbb{P}^W\rangle \quad \text{(for some word } W \text{ on } \boldsymbol{x} \text{ representing } h) \\ &= q_*\phi_* p_*^{-1}\langle\mathbb{P}^W\rangle \end{aligned}$$

by the definition of ϕ_g. Let $\tilde{W}_g$ be the unique lift of W at g and let $\tilde{\mathbb{P}}_{gh}$ be the lift of $\mathbb{P}$ at gh. Thus, the lift of $\mathbb{P}^W$ at g is $\tilde{\mathbb{P}}_{gh}^{\tilde{W}_g}$. Then by the definition of p_*^{-1} (i.e. the function t_g defined by (2.3)) we have

$$\phi_g(h \cdot \xi) = \phi_g(h\langle \mathbb{P} \rangle) = q_* \phi_* \langle \tilde{\mathbb{P}}_{gh}^{\tilde{W}_g} \rangle.$$

Let $\tilde{W}_{\phi(g)}$ be $\phi(\tilde{W}_g)$ which is a path in $\tilde{\mathcal{Q}}$ from $\phi(g)$ to $\phi(gh)$. Then

$$\phi_* \langle \tilde{\mathbb{P}}_{gh}^{\tilde{W}_g} \rangle = \langle \phi(\tilde{\mathbb{P}}_{gh})^{\tilde{W}_{\phi(g)}} \rangle.$$

Thus,

$$\phi_g(h \cdot \xi) = \phi_g(h\langle \mathbb{P} \rangle) = q_* \langle \phi(\tilde{\mathbb{P}}_{gh})^{\tilde{W}_{\phi(g)}} \rangle.$$

Moreover, by the definition of q_o, $q_o(\tilde{W}_{\phi(g)})$ is a word on $\boldsymbol{y}$ representing the element $\phi(g)^{-1}\phi(gh)$ of H. Therefore,

$$\begin{aligned}
\phi_g(h \cdot \xi) &= \phi_g(h\langle \mathbb{P} \rangle) \\
&= \langle q_o(\phi(\tilde{\mathbb{P}}_{gh})^{\tilde{W}_{\phi(g)}}) \rangle \\
&= \langle q_o(\phi(\tilde{\mathbb{P}}_{gh}))^{q_o(\tilde{W}_{\phi(g)})} \rangle \\
&= \phi(g)^{-1}\phi(gh) \langle q_o(\phi(\tilde{\mathbb{P}}_{gh})) \rangle \\
&= \phi(g)^{-1}\phi(gh) \cdot q_* \phi_* \langle \tilde{\mathbb{P}}_{gh} \rangle \\
&= \phi(g)^{-1}\phi(gh) \cdot q_* \phi_* p_*^{-1} \langle \mathbb{P} \rangle \\
&= \phi(g)^{-1}\phi(gh) \cdot \phi_{gh}(\xi)
\end{aligned}$$

as required. □

Suppose now that we have another function $\psi : H \longrightarrow G$. For any edge e in $\tilde{\mathcal{P}}$ we have the closed path

$$\mu_e = e\gamma_{\tau(e),\psi\phi(\tau(e))} \psi\phi(e)^{-1} \gamma^{-1}_{\iota(e),\psi\phi(\iota(e))}$$

in $\tilde{\mathcal{P}}$ as shown in Fig. 2.5.

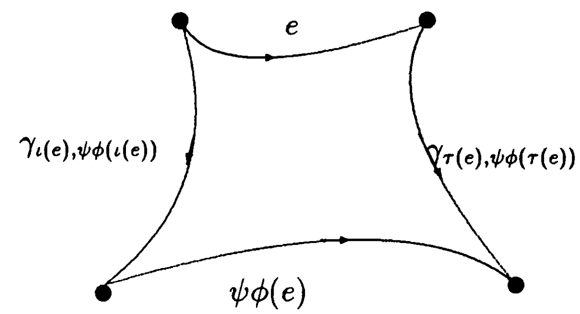


Fig. 2.5

Then $p_o(\mu_e)$ defines the identity of G. So there is a picture over $\mathcal{P}$ with boundary label $p_o(\mu_e)$. We choose such a picture, and let Δ_e denote the lift of this picture at $\iota(e)$. For any defining path $\tilde{R}_g$ of $\tilde{\mathcal{P}}$ we then have a spherical picture $\mathbb{A}_{R,g}$ over $\tilde{\mathcal{P}}$ as depicted in Fig. 2.6:

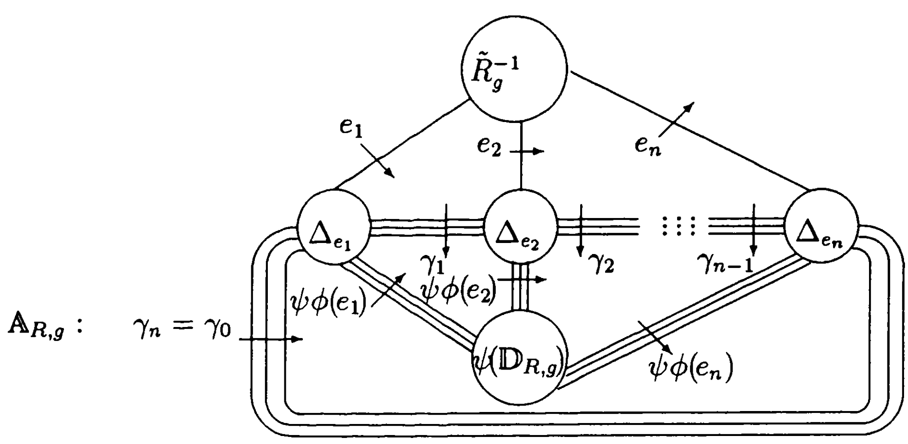


Fig. 2.6

where $\gamma_i = \gamma_{\tau(e_i),\psi\phi(\tau(e_i))}$, $i = 1, 2, \cdots, n$, and respectively, circles labelled Δ_{e_i}, $i = 1, 2, \cdots, n$ and circle labelled $\psi(\mathbb{D}_{R,g})$ represent subpictures over $\tilde{\mathcal{P}}$.

We let $\mathbb{B}_{R,g}$ denote the mirror image of the complement in $\mathbb{A}_{R,g}$ of the subpicture consisting of the disc labelled $\tilde{R}_g^{-1}$ and the adjacent arcs.

Let $\boldsymbol{Z}$ denote the set of elements of $\pi_2(\mathcal{P})$ represented by the pictures $p_o(\varepsilon\mathbb{A}_{R,g})$ ($R \in \boldsymbol{r}$, $g \in G$, $\varepsilon = \pm 1$). We prove the following key proposition which tells the difference between any $\xi \in \pi_2(\mathcal{P})$ and the double image $\psi_{\phi(g)}\phi_g(\xi)$ for all $g \in G$.

Proposition 2.2.5 (Mapping difference) *Let* $\xi \in \pi_2(\mathcal{P})$ *and* $g \in G$. *Then there exist* $h_0, \cdots, h_{A(\xi)} \in G$, $\zeta_1, \zeta_2, \cdots, \zeta_{A(\xi)} \in \boldsymbol{Z}$ *such that*

$$\xi = h_0\psi_{\phi(g)}\phi_g(\xi) + \sum_{i=1}^{A(\xi)} h_i\zeta_i. \tag{2.4}$$

Proof. Let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$ representing ξ, where $\mathbb{P}$ has $n = A(\xi)$ discs. Let $\tilde{\mathbb{P}}_g$ be the lift of $\mathbb{P}$ at g for some fixed $g \in G$ and suppose that the discs $\Omega_1, \Omega_2, \cdots, \Omega_n$ of $\tilde{\mathbb{P}}_g$ are labelled $\tilde{R}_{1,g_1}^{\varepsilon_1}$, $\tilde{R}_{2,g_2}^{\varepsilon_2}$, $\cdots$, $\tilde{R}_{n,g_n}^{\varepsilon_n}$, respectively. Let $\tilde{\mathbb{P}}_g^*$ be the picture obtained from $\tilde{\mathbb{P}}_g$ by replacing the disc Ω_i by the picture $\varepsilon_i\mathbb{B}_{R_i,g_i}$ for $i = 1, , 2, \cdots, n$. We will show below that $\gamma_{\psi\phi(g),g} \cdot \tilde{\mathbb{P}}_g^*$ is equivalent to $\psi\phi(\tilde{\mathbb{P}}_g)$, and from this we obtain the proposition as follows. We have

$$\begin{aligned}
\psi_{\phi(g)}\phi_g(\xi) &= p_*\psi_*q_*^{-1}q_*\phi_*p_*^{-1}\langle\mathbb{P}\rangle \\
&= p_*\psi_*\phi_*\langle\tilde{\mathbb{P}}_g\rangle \\
&= \langle p_o\psi\phi(\tilde{\mathbb{P}}_g)\rangle \\
&= \langle p_o(\gamma_{\psi\phi(g),g} \cdot \tilde{\mathbb{P}}_g^*)\rangle.
\end{aligned}$$

Since $\gamma_{\psi\phi(g),g}$ is a path in $\tilde{\mathcal{P}}$ from $\psi\phi(g)$ to g, by the definition of p_o, $p_o(\gamma_{\psi\phi(g),g})$ is a word on $\boldsymbol{x}$ representing the element $(\psi\phi(g))^{-1}g$. Therefore,

$$\begin{aligned}
\psi_{\phi(g)}\phi_g(\xi) &= \langle p_o(\gamma_{\psi\phi(g),g} \cdot \tilde{\mathbb{P}}_g^*)\rangle \\
&= \langle p_o(\tilde{\mathbb{P}}_g^*)^{p_o(\gamma_{\psi\phi(g),g})}\rangle \\
&= (\psi\phi(g))^{-1}g \cdot \langle p_o(\tilde{\mathbb{P}}_g^*)\rangle.
\end{aligned}$$

But $p_o(\tilde{\mathbb{P}}_g^*)$ is obtained from $\mathbb{P}$ by n replacements of subpictures of elements of the set

$$\{p_o(\varepsilon\mathbb{A}_{R,g'}) : \ R \in \boldsymbol{r},\ g' \in G,\ \varepsilon = \pm 1\},$$

so by Lemma 1.3.4,

$$\langle\mathbb{P}\rangle - \langle p_o(\tilde{\mathbb{P}}_g^*)\rangle = h_1\zeta_1 + h_2\zeta_2 + \cdots + h_n\zeta_n$$

for certain elements $\zeta_i \in \boldsymbol{Z}$, $h_i \in G\,(i = 1, 2, \cdots, n)$ and then the proposition follows by taking $h_0 = g^{-1}\psi\phi(g)$.

To see that $\gamma_{\psi\phi(g),g} \cdot \tilde{\mathbb{P}}_g^*$ is equivalent to $\psi\phi(\tilde{\mathbb{P}}_g)$, observe that if we have an arc in $\tilde{\mathbb{P}}_g$ labelled e say, then when we pass to $\psi\phi(\tilde{\mathbb{P}}_g)$ this arc is replaced by a sequence of parallel arcs with total label $\psi\phi(e)$. We can modify this sequence of parallel arcs as shown in Fig. 2.7.

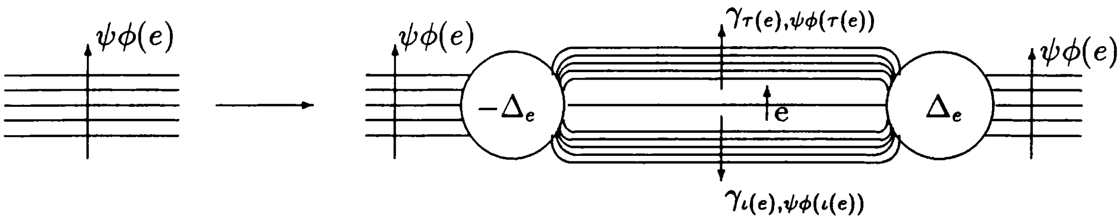


Fig. 2.7

For simplicity we will depict the configuration on the right by the following:

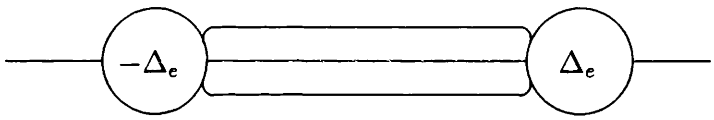


Fig. 2.8

We carry out the above procedure for each arc of $\tilde{\mathbb{P}}_g$, obtaining a picture $\tilde{\mathbb{P}}'$ equivalent to $\psi\phi(\tilde{\mathbb{P}}_g)$.

Now consider a typical disc Ω_i of $\tilde{\mathbb{P}}_g$ labelled by $\tilde{R}^{\varepsilon_i}_{i,g_i} = e_1 e_2 \cdots e_m$. When we pass from $\tilde{\mathbb{P}}_g$ to $\tilde{\mathbb{P}}'$, a neighbourhood of Ω_i becomes modified as shown in Fig. 2.9.

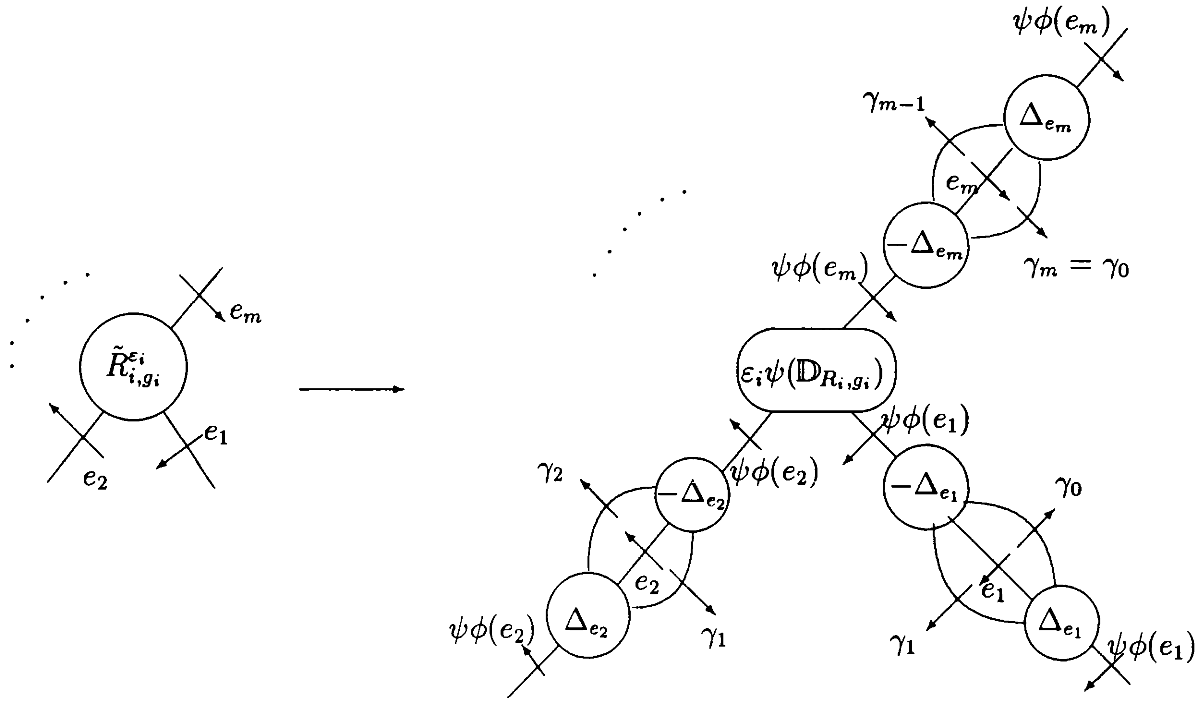


Fig. 2.9

By performing bridge moves we can further modify this neighbourhood to that as depicted in Fig. 2.10.

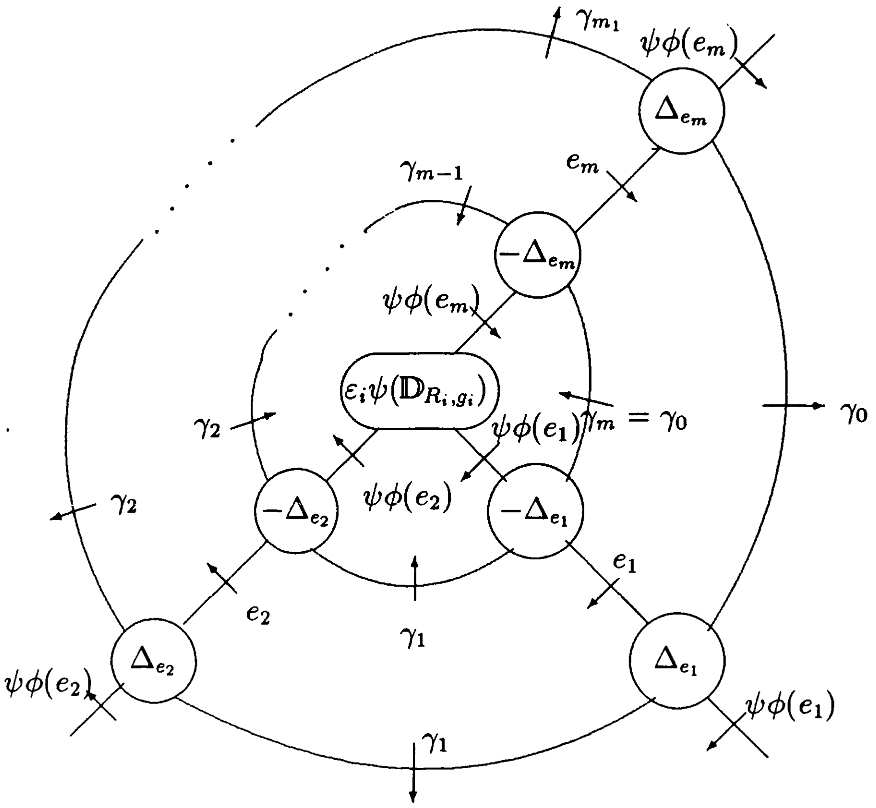


Fig. 2.10

Note that the picture made up of $\varepsilon_i\psi(\mathbb{D}_{R_i,g_i})$ and $-\Delta_{e_1}$, $-\Delta_{e_2}$, $\cdots$, $-\Delta_{e_m}$ is $\varepsilon_i\mathbb{B}_{R_i,g_i}$. We still call this modified picture $\tilde{\mathbb{P}}'$.

Consider a neighbourhood of a typical region Ψ of $\tilde{\mathbb{P}}_g$ as shown in Fig. 2.11.

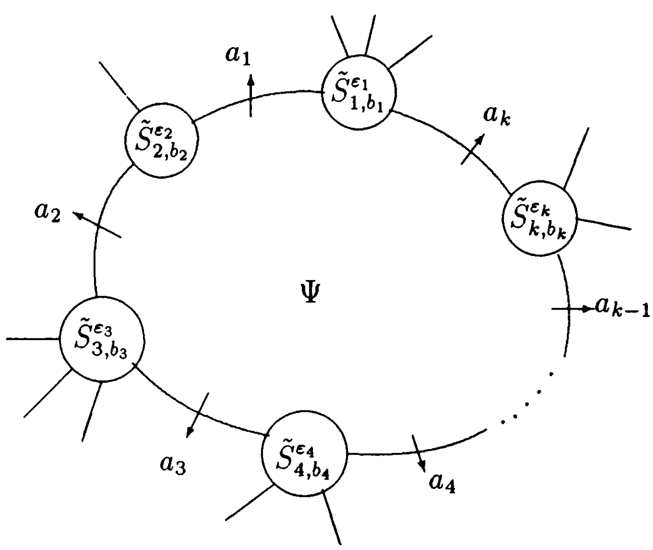


Fig. 2.11

Here $a_1, a_2, \cdots, a_k$ are edges in $\tilde{\mathcal{P}}$, $S_1, S_2, \cdots, S_k \in \boldsymbol{r}$, $b_1, b_2, \cdots, b_k \in G$, and $\epsilon_1, \epsilon_2, \cdots, \epsilon_k = \pm 1$. Let $\alpha_j = \gamma_{\tau(a_j),\psi\phi(\tau(a_j))}$, $j = 1, 2, \cdots, k$. Then when we pass from $\tilde{\mathbb{P}}_g$ to $\tilde{\mathbb{P}}'$, this neighbourhood becomes that as shown in Fig. 2.12.

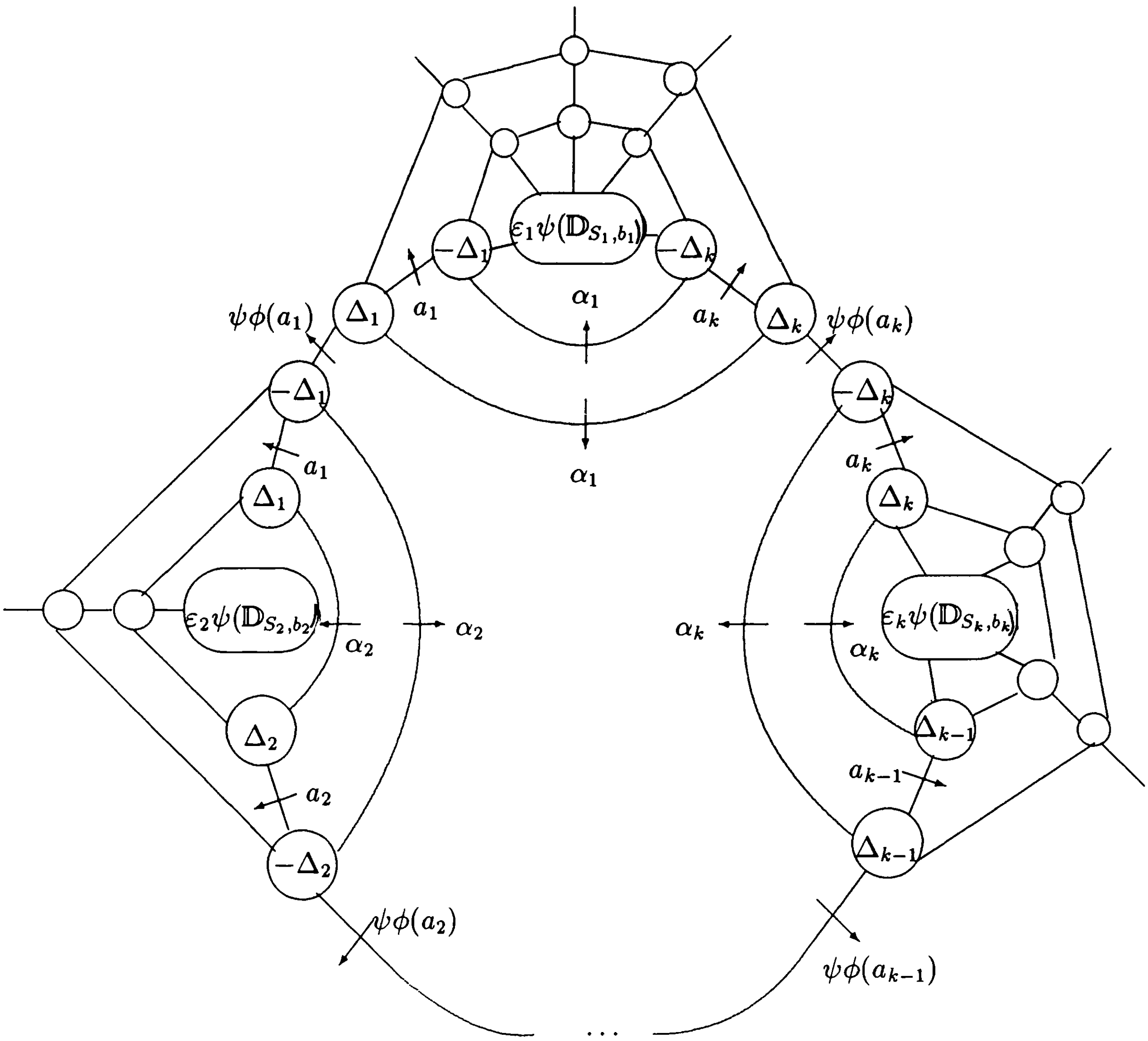


Fig. 2.12

Since $\iota(a_1), \iota(a_2), \cdots, \iota(a_k)$ all are the label of region Ψ, $\alpha_1, \alpha_2, \cdots, \alpha_k$ are all the same. Note that this situation also happen to the neighbourhood of any other region (including the outer region) of $\tilde{\mathbb{P}}_g$ and their corresponding neighbourhoods in $\tilde{\mathbb{P}}'$. Thus, from Fig. 2.10 we can see that the pairs of discs Δ_j and $-\Delta_j$ joined by arcs labelled by $\psi\phi(a_j)$, $j = 1, 2, \cdots, k$ are cancelling pairs. By bridge moves around these cancelling pairs we can remove these cancelling pairs and if Ψ is not the outer region of $\tilde{\mathbb{P}}_g$ we can remove the sequence of concentric closed arcs with the total label α_1. After we have finished this modification on every region of $\tilde{\mathbb{P}}_g$, the neighbourhood of $\tilde{\mathbb{P}}'$ corresponding to the typical region Ψ is then modified as shown in Fig. 2.13.

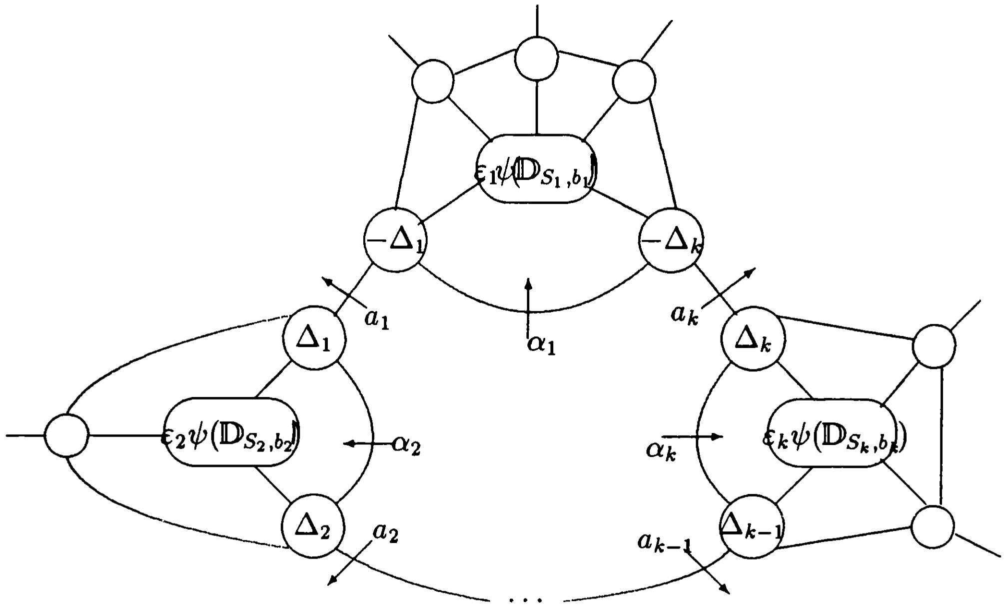


Fig. 2.13

This gives that this modified picture from $\tilde{\mathbb{P}}'$ is precisely the picture $\tilde{\mathbb{P}}_g^*$ surrounded by a sequence of concentric closed arcs obtained from the above modification on the neighbourhood corresponding to the outer region of $\tilde{\mathbb{P}}_g$. Since the basepoint of $\tilde{\mathbb{P}}_g^*$ is g and the basepoint of $\tilde{\mathbb{P}}'$ is $\psi\phi(g)$, the total label of this sequence of concentric closed arcs is a path from $\psi\phi(g)$ to g. So, we have shown that $\gamma_{\psi\phi(g),g} \cdot \tilde{\mathbb{P}}_g^*$ is equivalent to $\psi\phi(\tilde{\mathbb{P}}_g)$. □

2.2.3 Quasi-retractions and Quasi-isometries

We state the definitions of quasi-retraction and quasi-isometry between metric spaces as follows, referring to [Al3].

Definition 2.2.6 *Let $(X,\, d)$, $(X',\, d')$ be two metric spaces where d, d' are the metrics in the corresponding spaces X and X' respectively. A pair of maps*

$$f : X \longrightarrow X', \qquad f' : X' \longrightarrow X$$

is called a quasi-retraction of X' to X if there exist constants $c_1 > 0$, and $c_2 \geq 0$ such that for all $x,\, y \in X,\; x',\, y' \in X'$

$$d'(f(x),\, f(y)) \leq c_1 d(x,\, y) + c_2,$$

$$d(f'(x'),\, f'(y')) \leq c_1 d'(x',\, y') + c_2,$$

$$d(f'f(x),\, x) \leq c_2.$$

The metric space $(X,\, d)$ is called a quasi-retract of $(X',\, d')$ if there is a quasi-retraction of X' to X.

Definition 2.2.7 *Let $(X,\, d)$, $(X',\, d')$ be two metric spaces where d, d' are the metrics in the corresponding spaces X and X' respectively. A pair of maps*

$$f : X \longrightarrow X', \qquad f' : X' \longrightarrow X$$

is called a quasi-isometry if there exist constants $c_1 > 0$, and $c_2 \geq 0$ such that for all $x,\, y \in X$, $x',\, y' \in Y$

$$d'(f(x),\, f(y)) \leq c_1 d(x,\, y) + c_2,$$

$$d(f'(x'),\, f'(y')) \leq c_1 d'(x',\, y') + c_2,$$

$$d(f'f(x),\, x) \leq c_2,$$

$$d(ff'(x'),\, x') \leq c_2.$$

The metric spaces (X, d) and (X', d') are called quasi-isometric if there is a quasi-isometry between them.

By the above definitions, a metric space $(X,\, d)$ is quasi-isometric to another metric space $(X',\, d')$ if and only if $(X,\, d)$ is a quasi-retract of $(X',\, d')$, and vice versa.

Consider a Cayley graph of a finitely generated group as a word metric space as described in §1.2.4. We then have the following definition.

Definition 2.2.8 *Let G, H be two groups finitely generated by $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively. Let $\Gamma_{\boldsymbol{x}}(G)$ and $\Gamma_{\boldsymbol{y}}(H)$ be their corresponding Cayley graphs with word metrics d, d' respectively. We say that G is a quasi-retract of H if the metric space $(G,\, d)$ with respect to $\boldsymbol{x}$ is a quasi-retract of the metric space $(H,\, d')$ with respect to $\boldsymbol{y}$ and we say that G and H are quasi-isometric if $(G,\, d)$ and $(H,\, d')$ are quasi-isometric.*

It is trivial that quasi-retraction of groups is a geometric property, that is, if a group G is a quasi-retract of a group H, and G, H are quasi-isometric to groups G_1, H_1 respectively, then G_1 is also a quasi-retract of H_1. By [Al3, Lemma6], quasi-retraction of groups is independent of the choice of presentations. Furthermore, by [Al3, Theorem 8], a quasi-retract of a group H of type F_n (resp. of type FP_n) again is of type F_n (resp. of type FP_n) for any integer $n \geq 2$.

Given a finitely generated group H, we can give some examples of quasi-retracts of H.

Example 2.2.9 *By Definition 2.2.7, if a group G is quasi-isometric to H then G is a quasi-retract of H. Thus, (see* [GhHa]*) the following groups are quasi-retracts of H.*

(i) All subgroups of H of finite index (in fact, all such subgroups are quasi-isometric to H).

(ii) All groups commensurable with H. Here, we say a group G is *commensurable* with H if there are subgroups $G_1 \leq G$, $H_1 \leq H$ of finite indices such that $G_1 \cong H_1$.

Example 2.2.10 *Every finite subgroup of H is a quasi-retract of H.*

To see this is true, simply note that H and G are quasi-isometric to H and 1 respectively.

Example 2.2.11 *A (homomorphic) retract of H.*

Here, we say a group G is a *retract* of H if there are homomorphisms

$$G \xrightarrow{\phi} H \xrightarrow{\psi} G$$

such that $\psi\phi$ is the identity map of G. Following [Al3, Example (3)], G then is a quasi-retract of H.

Let K_0, H_0 be two groups together with a homomorphism $\phi : K_0 \longrightarrow Aut(H_0)$, where $Aut(H_0)$ is the automorphism group of H_0. For each $k \in K_0$ we write ϕ_k for $\phi(k) \in Aut(H_0)$. In particular, if K_0 is a cyclic group generated by k, then we identify ϕ with ϕ_k. The map ϕ determines an *action* of K_0 on H_0 given by

$$h^k := \phi_k(h), \quad h \in H_0, \ k \in K_0.$$

The set of all ordered pairs (h, k), $h \in H_0$, $k \in K_0$ forms a *split extension* denoted $H = H_0 \rtimes_\phi K_0$ of H_0 by K_0 under the binary operation defined by

$$(h, k)(h', k') = (h\phi_k(h'), kk'), \quad h, h' \in H_0, k, k' \in K_0.$$

Furthermore, if $\mathcal{H}_0 = \langle \boldsymbol{x}; \boldsymbol{r} \rangle$ and $\mathcal{K}_0 = \langle \boldsymbol{t}; \boldsymbol{s} \rangle$ are presentations for H_0 and K_0 respectively, then H has a presentation

$$\mathcal{H} = \langle \boldsymbol{x}, \boldsymbol{t}; \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{\alpha} \rangle,$$

where $\boldsymbol{\alpha} = \{t^{-1}xt\lambda_{xt}^{-1} : x \in \boldsymbol{x}, t \in \boldsymbol{t}\}$ with λ_{xt} a word on $\boldsymbol{x}$ representing the element $\phi_{\bar{t}}^{-1}(\bar{x})$ of $G(\mathcal{H}_0)$ for each pair $x \in \boldsymbol{x}$, $t \in \boldsymbol{t}$. We have two natural embeddings $H_0 \longrightarrow H$ and $K_0 \longrightarrow H$ respectively defined by

$$h \longmapsto (h, 1), \text{ and } k \longmapsto (1, k), \quad h \in H_0, k \in K_0.$$

Identifying H_0 and K_0 with their images respectively gives that H_0 is normal subgroup of H with complement K_0. Note that K_0 is a retract of H as the maps

$$K_0 \overset{k \mapsto (1,k)}{\longrightarrow} H \overset{(h,k) \mapsto k}{\longrightarrow} K_0$$

show.

The following proposition is a standard fact.

Proposition 2.2.12 *A group K is a retract of a group H if and only if H is a split extension of a normal subgroup H_0 by another subgroup K_0 with $K_0 \cong K$.*

Proof. Suppose that K is a retract of H. Then there are homomorphisms

$$K \overset{\phi}{\longrightarrow} H \overset{\psi}{\longrightarrow} K$$

such that $\psi\phi$ is the identity map of K. Let $H_0 = \ker\psi$, $K_0 = \mathrm{im}\phi$. Then for any $h \in H$, we have

$$h = h(\phi\psi(h))^{-1}\phi\psi(h).$$

Since

$$\begin{aligned}
\psi(h(\phi\psi(h))^{-1}) &= \psi(h)\psi((\phi\psi(h))^{-1}) \\
&= \psi(h)(\psi\phi\psi(h))^{-1} \\
&= \psi(h)(\psi(h))^{-1} \quad \text{(since } \psi\phi \text{ is the identity map of } K\text{)} \\
&= 1,
\end{aligned}$$

we have $h(\phi\psi(h))^{-1} \in H_0$. Thus, $H = H_0K_0$ and $K \cong K_0$ since ϕ is injective. Moreover, if $h_0 \in K_0 \cap H_0$, then $\psi(h_0) = 1$ and there is an element $h_1 \in H$ such that $h_0 = \phi(h_1)$. Thus,

$$1 = \psi(h_0) = \psi\phi(h_1) = h_1$$

and hence $h_0 = \phi(h_1) = 1$, that is $H_0 \cap K_0 = \{1\}$.

The converse follows from the previous discussion. □

From the above proposition we can see that if G is a finite subgroup of a group H and if G has no normal complement in H, then G is a quasi-retract but not a retract H. It may be very interesting to investigate the behaviour of quasi-retractivity between groups.

2.2.4 Inequalities and invariance theorems

Let G and H be two groups finitely presented by $\mathcal{P} = \langle \boldsymbol{x}\,;\boldsymbol{r}\rangle$ and $\mathcal{Q} = \langle \boldsymbol{y}\,;\boldsymbol{s}\rangle$, respectively. Let $\Gamma_{\boldsymbol{x}}$ and $\Gamma_{\boldsymbol{y}}$ be their corresponding Cayley graphs with word metrics d, d' respectively. We suppose that G is a quasi-retract of H with a quasi-retraction

$$\phi:\ (G,\ d) \longrightarrow (H,\ d'), \qquad \psi:\ (H,\ d') \longrightarrow (G,\ d)$$

for a pair of constants $c_1 > 0$, $c_2 \geq 0$ as defined in Definition 2.2.6, where $(G,\ d)$ and $(H,\ d')$ are the word metric spaces with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively as defined in §1.2.5. Further supposing that H is of type F_3, we do some calculations as follows.

For convenience, we replace $c_1 + c_2$ by c. Let r_0 be the maximum of the lengths of the relators of $\mathcal{P}$ and $\mathcal{Q}$. Also, let $\boldsymbol{Y}$ be a finite set of generators of $\pi_2(\mathcal{Q})$, and $\boldsymbol{X}$ be any set of generators of $\pi_2(\mathcal{P})$. Let a be the maximum of the areas of elements of $\boldsymbol{Y}$, and let

$$b = \max\{\delta^{(1)}_{\mathcal{P}}(cr_0),\ \delta^{(1)}_{\mathcal{Q}}(cr_0)\}.$$

Thus, each picture $\mathbb{D}_{R,g}$ defined in §2.2.2 can be chosen to have at most b discs. We will assume that such pictures have been chosen. We will also assume that analogous pictures $\mathbb{D}_{S,h}$ $(S \in \boldsymbol{s},\ h \in H)$ over $\mathcal{Q}$ with at most b discs have been chosen. We then have the following:

(2.2.1) Each picture $\psi(\mathbb{D}_{R,g})$ or each picture $\phi(\mathbb{D}_{S,h})$ has at most b^2 discs;

(2.2.2) $A(\phi_g(\xi)) \leq bA(\xi)$, $A(\psi_h(\eta)) \leq bA(\eta)$ for all $\xi \in \pi_2(\mathcal{P})$, $\eta \in \pi_2(\mathcal{Q})$, $g \in G$, $h \in H$;

(2.2.3) For any $\eta \in \boldsymbol{Y}$ and $h \in H$, $V(\psi_h(\eta)) \leq \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(ab)$.

Let η be any element of $\pi_2(\mathcal{Q})$. Then

$$\eta = \sum_{i=1}^{V(\eta)} \varepsilon_i h_i \eta_i$$

for certain elements $h_i \in H$, $\eta_i \in \boldsymbol{Y}$, $\varepsilon_i = \pm 1$ ($i = 1, ,2, \cdots, V(\eta)$). Then by Lemma 2.2.4

$$\psi_h(\eta) = \sum_{i=1}^{V(\eta)} \varepsilon_i g_i \psi_{hh_i}(\eta_i)$$

for certain elements g_i ($i = 1, 2, \cdots, V(\eta)$) of G. So using (2.2.3) we get:

(2.2.4) For any $\eta \in \pi_2(\mathcal{Q})$, and $h \in H$, $V(\psi_h(\eta)) \leq \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(ab)V(\eta)$.

Now each path $\mu_e = e\gamma_{\tau(e),\psi\phi(\tau(e))}\psi\phi(e)^{-1}\gamma^{-1}_{\iota(e),\psi\phi(\iota(e))}$ (e an edge of $\tilde{\mathcal{P}}$) has length of at most $(1+c)^2$, so the picture Δ_e in §2.2.2 can be chosen to have at most $\delta^{(1)}_{\mathcal{P}}((c+1)^2)$ discs. We assume that such pictures have been chosen. Then (using (2.2.1)) we have:

(2.2.5) The area of each picture $\mathbb{A}_{R,g}$ depicted in Fig. 2.6 is at most α where

$$\alpha = 1 + r_0\delta^{(1)}_{\mathcal{P}}((c+1)^2) + b^2.$$

Theorem 2.2.13 *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations for groups G and H respectively. Let $\boldsymbol{X}$ be a set of module generators for $\pi_2(\mathcal{P})$ and suppose that $\boldsymbol{Y}$ is a finite set of module generators for $\pi_2(\mathcal{Q})$. Suppose that G is a quasi-retract of H as defined in Definition 2.2.8. Then*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}},$$

and G is of type F_3.

Proof. Let $\xi \in \pi_2(\mathcal{P})$ with $A(\xi) \le n$. Using Proposition 2.2.5 we get

$$\xi = h_0 \psi_{\phi(1)} \phi_1(\xi) + \sum_{i=1}^{A(\xi)} h_i \zeta_i \tag{2.5}$$

for certain elements $h_0, h_1, \cdots, h_{A(\xi)} \in G$, $\zeta_1, \zeta_2, \cdots, \zeta_{A(\xi)} \in \boldsymbol{Z}$, where $\boldsymbol{Z}$ is the same as defined in §2.2.2. Since $\mathcal{P}$ is finite, $\boldsymbol{Z}$ is also finite. Thus, we have the following inequalities.

$$\begin{aligned} V(\xi) &\le V(\psi_{\phi(1)}\phi_1(\xi)) + \sum_{i=1}^{A(\xi)} V(\zeta_i) \\ &\le \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(ab) V(\phi_1(\xi)) + n\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(\alpha) \qquad \text{(using (2.2.4) and (2.2.5))} \\ &\le \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(ab)\delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}(bn) + n\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(\alpha) \qquad \text{(using (2.2.2)).} \end{aligned}$$

Hence

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(n) \le \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(ab)\delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}(bn) + n\delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}(\alpha),$$

so

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}$$

as required. From (2.5) we also see that ξ has a description in terms of the elements of $\boldsymbol{Z}$ and $\psi(\boldsymbol{Y})$. Thus, the finite set $\boldsymbol{Z} \cup \psi(\boldsymbol{Y})$ is a generating set of $\pi_2(\mathcal{P})$. So G is of type F_3 and hence the rest of the theorem follows. □

From §2.2.3 we have the following corollaries.

Corollary 2.2.14 *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations for groups G and H respectively. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two finite sets of module generators for $\pi_2(\mathcal{P})$ and $\pi_2(\mathcal{Q})$, respectively. Suppose that G and H are quasi-isometric as defined in Definition 2.2.8. Then $\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \sim \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}$.*

Corollary 2.2.15 *Let G be a retract of H, and let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations for G and H respectively. If $\boldsymbol{Y}$ is a finite generating set for $\pi_2(\mathcal{Q})$, then for any generating set $\boldsymbol{X}$ of $\pi_2(\mathcal{P})$, one has $\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}$.*

Corollary 2.2.16 *Suppose that G is a group of type F_3 and H is a subgroup of finite index in G. Let $\mathcal{P}$ and $\mathcal{Q}$ be finite presentations for G and H, and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two finite generating sets for $\pi_2(\mathcal{Q})$ and $\pi_2(\mathcal{P})$, respectively. Then $\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \sim \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}$; in particular, if $\mathcal{P}$ and $\mathcal{Q}$ are finite presentations of isomorphic groups, then $\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \sim \delta^{(2)}_{\mathcal{Q},\boldsymbol{Y}}$.*

It follows from this corollary that for any group G of type F_3 we can define $\delta_G^{(2)}$ to be the equivalence class of functions containing $\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}$ for any given pair $\boldsymbol{\mathcal{P}}$, $\boldsymbol{X}$ ($\boldsymbol{\mathcal{P}}$ a finite presentation of G, and $\boldsymbol{X}$ any finite generating set of $\pi_2(\boldsymbol{\mathcal{P}})$). Then we have (by Corollary 2.2.14)

Theorem 2.2.17 *The equivalence class $\delta_G^{(2)}$ is a quasi-isometry invariant on the class of groups of type F_3.*

Chapter 3

Second order Dehn functions of monoids

3.1 Delta functions of two-complexes

3.1.1 Delta functions

For a locally finite object $(\mathcal{D}, S) \in \mathfrak{C}$ with trivialiser $\boldsymbol{X}$, let n be any positive integer and let v be any vertex. We consider the set of paths in $\mathcal{D}$

$$\Upsilon_n(v) = \{\gamma : L(\gamma) \leq n, \ \iota(\gamma) = \tau(\gamma) = v\}.$$

Since $(\mathcal{D}, S)$ is locally finite, $\Upsilon_n(v)$ is finite. Thus, the following concept is well-defined.

Definition 3.1.1 *Let $(\mathcal{D}, S)$ be a locally finite object of $\mathfrak{C}$ with trivialiser $\boldsymbol{X}$, and let v be any vertex of $(\mathcal{D}, S)$. The following function is said to be the* delta function *of $(\mathcal{D}, S)$ with respect to $\boldsymbol{X}$ at vertex v:*

$$\delta_{\mathcal{D},S,\boldsymbol{X},v}(n) = \max\{Area_{\mathcal{D},\boldsymbol{X}^S}(\gamma) : \gamma \in \Upsilon_n(v)\}.$$

If there is not any confusion, we will simply use the notation $\delta_{\mathcal{D},\boldsymbol{X},v}$ for $\delta_{\mathcal{D},S,\boldsymbol{X},v}$ in the sequel.

3.1.2 Invariance over vertices in a single component and invariance over trivialisers

Throughout this subsection we suppose that $(\mathcal{D}, S)$ is a locally finite object of $\mathfrak{C}$ with finite trivialiser $\boldsymbol{X}$.

Lemma 3.1.2 *Suppose that u, v are any two vertices in the same component of $\mathcal{D}$. Then*

$$\delta_{\mathcal{D},\boldsymbol{X},v} \sim \delta_{\mathcal{D},\boldsymbol{X},u}.$$

Proof. Let α be a path from u to v of length b, say. Let n be any positive integer and let $\gamma \in \Upsilon_n(v)$ be any closed path at v of length at most n. Then $\alpha\gamma\alpha^{-1} \in \Upsilon_{n+2b}(u)$. By Lemma 1.1.7, $Area_{\mathcal{D},\boldsymbol{X}^S}(\alpha\gamma\alpha^{-1}) = Area_{\mathcal{D},\boldsymbol{X}^S}(\gamma)$. Thus

$$\{Area_{\mathcal{D},\boldsymbol{X}^S}(\gamma) : \gamma \in \Upsilon_n(v)\} \subseteq \{Area_{\mathcal{D},\boldsymbol{X}^S}(\beta) : \beta \in \Upsilon_{n+2b}(u)\}$$

and so

$$\delta_{\mathcal{D},\boldsymbol{X},v}(n) \leq \delta_{\mathcal{D},\boldsymbol{X},u}(n+2b).$$

By symmetry we also have

$$\delta_{\mathcal{D},\boldsymbol{X},u}(n) \leq \delta_{\mathcal{D},\boldsymbol{X},v}(n+2b),$$

and this completes the proof. □

Lemma 3.1.3 *Suppose $\boldsymbol{X}'$ is another finite trivialiser of $(\mathcal{D}, S)$. Then for any vertex v in $\mathcal{D}$ we have*

$$\delta_{\mathcal{D},\boldsymbol{X},v} \sim \delta_{\mathcal{D},\boldsymbol{X}',v}.$$

Proof. Let

$$a = \max\{Area_{\mathcal{D},\boldsymbol{X}^S}(\alpha'),\ Area_{\mathcal{D},\boldsymbol{X}'^S}(\alpha) : \alpha \in \boldsymbol{X},\ \alpha' \in \boldsymbol{X}'\}.$$

Then, by Lemma 1.1.5, for any closed path γ of $\mathcal{D}$ at vertex v there is a defining product $\prod_{i=1}^{r} \beta_i(s_i \cdot \alpha_i^{\varepsilon_i} \cdot s_i')\beta_i^{-1}$ (relative to $\boldsymbol{X}^S$) for γ with $r = Area_{\mathcal{D},\boldsymbol{X}^S}(\gamma)$, $\alpha_i \in \boldsymbol{X}$, s_i, $s_i' \in S$, certain $\beta_i \in \mathcal{D}$, $\varepsilon_i = \pm 1$, $1 \leq i \leq r$. Also, for each α_i $(1 \leq i \leq r)$ there is a defining product $\prod_{j=1}^{r_i} \rho_{ij}(s_{ij} \cdot \alpha'^{\epsilon_{ij}}_{ij} \cdot s'_{ij})\rho_{ij}^{-1}$ (relative to $\boldsymbol{X}'^S$) for α_i with $r_i = Area_{\boldsymbol{\mathcal{D}},\boldsymbol{X}'^S}(\alpha_i) \leq a$, $\alpha'_{ij} \in \boldsymbol{X}'$, s_{ij}, $s'_{ij} \in S$, certain $\rho_{ij} \in \boldsymbol{\mathcal{D}}$, $\epsilon_{ij} = \pm 1$, $1 \leq j \leq r_i$. Thus,

$$\begin{aligned}
\gamma &\sim \prod_{i=1}^{r} \beta_i(s_i \cdot \alpha_i^{\varepsilon_i} \cdot s'_i)\beta_i^{-1} \\
&\sim \prod_{i=1}^{r}\prod_{j=1}^{r_i} \beta_i(s_i \cdot (\rho_{ij}(s_{ij} \cdot \alpha'^{\epsilon_{ij}}_{ij} \cdot s'_{ij})^{\varepsilon_i}\rho_{ij}^{-1}) \cdot s'_i)\beta_i^{-1} \\
&= \prod_{i=1}^{r}\prod_{j=1}^{r_i} \beta_i(s_i \cdot \rho_{ij} \cdot s'_i)(s_i s_{ij} \cdot \alpha'^{\varepsilon_i\epsilon_{ij}}_{ij} \cdot s'_{ij}s'_i)(s_i \cdot \rho_{ij}^{-1} \cdot s'_i)\beta_i^{-1} \\
&= \prod_{i=1}^{r}\prod_{j=1}^{r_i} (\beta_i(s_i \cdot \rho_{ij} \cdot s'_i)) \left(s_i s_{ij} \cdot \alpha'^{\varepsilon_i\epsilon_{ij}}_{ij} \cdot s'_{ij}s'_i\right) \left((s_i \cdot \rho_{ij} \cdot s'_i)^{-1}\beta_i^{-1}\right),
\end{aligned}$$

since $s_i \cdot \rho_{ij}^{-1} \cdot s'_i = (s_i \cdot \rho_{ij} \cdot s'_i)^{-1}$. Hence,

$$Area_{\boldsymbol{\mathcal{D}},\boldsymbol{X}'^S}(\gamma) \leq a Area_{\boldsymbol{\mathcal{D}},\boldsymbol{X}^S}(\gamma)$$

and so

$$\delta_{\boldsymbol{\mathcal{D}},\boldsymbol{X}',v}(n) \leq a\delta_{\boldsymbol{\mathcal{D}},\boldsymbol{X},v}(n).$$

Similarly, we also have

$$\delta_{\boldsymbol{\mathcal{D}},\boldsymbol{X},v}(n) \leq a\delta_{\boldsymbol{\mathcal{D}},\boldsymbol{X}',v}(n)$$

and then the lemma follows. □

Let Δ be a component of $\boldsymbol{\mathcal{D}}$ and let v be any vertex in Δ. The above two lemmas allow us to write $\delta_{\boldsymbol{\mathcal{D}},\Delta}$ as a typical representative of the equivalence class containing $\delta_{\boldsymbol{\mathcal{D}},\boldsymbol{X},v}$.

3.1.3 Invariance over well-placed retractions

In this subsection we suppose that $(\boldsymbol{\mathcal{D}}, S)$ is a locally finite object of $\mathfrak{C}$ and $(\boldsymbol{\mathcal{D}}_0, S_0)$ is a subobject of $(\boldsymbol{\mathcal{D}}, S)$.

Proposition 3.1.4 *Let $\phi : (\boldsymbol{\mathcal{D}}, S) \longrightarrow (\boldsymbol{\mathcal{D}}_0, S_0)$ be a retraction. Suppose that $\boldsymbol{X}$ is a trivialiser of $(\boldsymbol{\mathcal{D}}, S)$. Then $\phi(\boldsymbol{X})$ is a trivialiser of $(\boldsymbol{\mathcal{D}}_0, S_0)$. Moreover, if v_0 is any vertex of $\boldsymbol{\mathcal{D}}_0$ then*

$$\delta_{\boldsymbol{\mathcal{D}}_0, S_0,\phi(\boldsymbol{X}),v_0}(n) \leq \delta_{\boldsymbol{\mathcal{D}},S,\boldsymbol{X},v_0}(n).$$

Proof. Let n be any positive integer and let γ be any closed path of length n in $\mathcal{D}_0$ with $\iota(\gamma) = v_0$. By Lemma 1.1.5, there is a defining product (relative to $\boldsymbol{X}^S$) in $\mathcal{D}$ of the form

$$\gamma \sim \prod_{i=1}^{r} \beta_i (s_i \cdot \alpha_i^{\varepsilon_i} \cdot s_i') \beta_i^{-1},$$

with $r \leq \delta_{\mathcal{D},S,\boldsymbol{X},v_0}(n)$, $\alpha_i \in \boldsymbol{X}$, s_i, $s_i' \in S$, $\beta_i \in \mathcal{D}$ are certain paths, $\varepsilon_i = \pm 1$, $1 \leq i \leq r$.

Hence, in $\mathcal{D}_0$,

$$\gamma = \phi(\gamma) \sim \prod_{i=1}^{r} \phi(\beta_i)(\phi(s_i) \cdot \phi(\alpha_i)^{\varepsilon_i} \cdot \phi(s_i'))\phi(\beta_i)^{-1}.$$

This means that $\phi(\boldsymbol{X})$ is a trivialiser of $(\mathcal{D}_0,\ S_0)$ and

$$Area_{\mathcal{D}_0, \phi(\boldsymbol{X})^{S_0}}(\gamma) \leq r$$

as required. □

Let $\boldsymbol{Y}^* = \boldsymbol{Y}^*(\mathcal{D},\ \mathcal{D}_0)$ denote the set of paths $\eta = e_1 e_2 \cdots e_n$ in $\mathcal{D}$ with the following properties:

(1) η is a shortest path in the homotopy equivalence class containing η in $\mathcal{D}$;

(2) $\iota(\eta), \tau(\eta) \in \mathcal{D}_0$;

(3) $\iota(e_2), \tau(e_2), \cdots, \iota(e_n) \in \mathcal{D} - \mathcal{D}_0$;

(4) if $n = 1$ then $e_1 \notin \mathcal{D}_0$ (if $n > 1$ then all $e_i \notin \mathcal{D}_0$ by (3)).

Definition 3.1.5 *If*

(i) *every component of $\mathcal{D}$ contains a vertex of $\mathcal{D}_0$;*

(ii) *for every path $\eta \in \boldsymbol{Y}^*$ there is a path $\hat{\eta}$ in $\mathcal{D}_0$ such that $\iota(\hat{\eta}) = \iota(\eta)$, $\tau(\hat{\eta}) = \tau(\eta)$ (we then fix such a $\hat{\eta}$ and write $\boldsymbol{Y}' = \{\eta\hat{\eta}^{-1} : \eta \in \boldsymbol{Y}^*\}$);*

(iii) *there is a finite subset $\boldsymbol{Y} \subseteq \boldsymbol{Y}'$ such that $\boldsymbol{Y}' = S \cdot (\boldsymbol{Y} \cup \boldsymbol{Y}^{-1}) \cdot S$,*

*we then say that $(\mathcal{D}_0,\ S_0)$ is $\boldsymbol{Y}$-*well-placed *in $(\mathcal{D},\ S)$.*

We say that $(\mathcal{D}_0,\ S_0)$ is well-placed *in $(\mathcal{D},\ S)$ if it is $\boldsymbol{Y}$-well-placed for some $\boldsymbol{Y}$.*

Let $\boldsymbol{Y}^*$ and $\boldsymbol{Y}$ be defined as above.

Proposition 3.1.6 *Suppose that* $(\mathcal{D}_0,\ S_0)$ *is* $\boldsymbol{Y}$*-well-placed in* $(\mathcal{D},\ S)$. *Let* $\boldsymbol{X}_0$ *be a trivialiser of* $(\mathcal{D}_0,\ S_0)$. *Then* $\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{Y}$ *is a trivialiser of* $(\mathcal{D},\ S)$. *Moreover, if* v *is a vertex in* $\mathcal{D}$ *and if* $v_0 \in \mathcal{D}_0$ *lies in the same component of* $\mathcal{D}$ *as* v, *then*

$$\delta_{\mathcal{D},S,\boldsymbol{X},v} \preceq \delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}.$$

Proof. Let

$$\lambda = \max\{L(\eta\hat{\eta}^{-1}) :\ \eta\hat{\eta}^{-1} \in \boldsymbol{Y}\}.$$

Since $\boldsymbol{Y}' = S \cdot (\boldsymbol{Y} \cup \boldsymbol{Y}^{-1}) \cdot S$, $L(\eta\hat{\eta}^{-1}) \leq \lambda$ for all $\eta\hat{\eta}^{-1} \in \boldsymbol{Y}'$.

Let n be any positive integer and let ρ be any closed path in $\mathcal{D}$ of length n starting at v. Let β be a path in $\mathcal{D}$ from v to a vertex v_0 of $\mathcal{D}_0$ (by the condition (i) of Definition 3.1.5 such a v_0 exists). Let $b = L(\beta)$ and let γ be a shortest path in the homotopy equivalence class in $\mathcal{D}$ containing $\beta\rho\beta^{-1}$. Then $L(\gamma) \leq 2b + n$ and $\iota(\gamma) = v_0 = \tau(\gamma)$. We factorize γ into a product of the form

$$\gamma = \beta_1\eta_1\beta_2\eta_2 \cdots \beta_q\eta_q\beta_{q+1}$$

where $\eta_i \in \boldsymbol{Y}^*$, $1 \leq i \leq q$, $q \leq 2b + n$, and $\beta_1,\ \beta_2,\ \cdots,\ \beta_{q+1}$ are certain paths in $\mathcal{D}_0$. We then have a path in $\mathcal{D}_0$ (by (ii) of Definition 3.1.5):

$$\gamma' = \beta_1\hat{\eta}_1\beta_2\hat{\eta}_2 \cdots \beta_q\hat{\eta}_q\beta_{q+1}.$$

Note that we have $L(\gamma') \leq \lambda L(\gamma) \leq \lambda(2b + n)$, and in $\mathcal{D}$

$$\gamma \sim^{(1)} \gamma' \prod_{i=1}^{q} \alpha_i(\eta_i\hat{\eta}_i^{-1})\alpha_i^{-1}$$

for certain paths α_i in $\mathcal{D}$, $1 \leq i \leq q$.

Since γ' lies in $\mathcal{D}_0$, it is equivalent in $\mathcal{D}_0$ to a product of at most $\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}(\lambda(2b+n))$ conjugates of elements of $S_0 \cdot (\boldsymbol{X}_0 \cup \boldsymbol{X}_0^{-1}) \cdot S_0$. Thus, we see that γ is equivalent in $\mathcal{D}$ to at most

$$\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}(\lambda(2b + n)) + 2b + n$$

conjugates of elements of $S \cdot ((\boldsymbol{X}_0 \cup \boldsymbol{Y}) \cup (\boldsymbol{X}_0 \cup \boldsymbol{Y})^{-1}) \cdot S$. Since γ is equivalent to $\beta\rho\beta^{-1}$, we thus have, by Lemma 1.1.7, that ρ is also equivalent in $(\mathcal{D},\ S)$ to at most

$$\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}(\lambda(2b + n)) + 2b + n$$

conjugates of elements of $S \cdot ((\boldsymbol{X}_0 \cup \boldsymbol{Y}) \cup (\boldsymbol{X}_0 \cup \boldsymbol{Y})^{-1}) \cdot S$. Hence, $\boldsymbol{X}_0 \cup \boldsymbol{Y}$ is a trivialiser of $(\mathcal{D}, S)$ and

$$\delta_{\mathcal{D},S,\boldsymbol{X},v}(n) \leq \delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}(\lambda(2b+n)) + 2b + n$$

as required. □

To combine Propositions 3.1.4 and 3.1.6, we suppose, for some set $\boldsymbol{Y}$ of closed paths of $\mathcal{D}$, that $(\mathcal{D}_0, S_0)$ is a $\boldsymbol{Y}$-well-placed retract of $(\mathcal{D}, S)$ with $\phi : (\mathcal{D}, S) \longrightarrow (\mathcal{D}_0, S_0)$ a retraction. Since each component of $\mathcal{D}$ contains a vertex of $\mathcal{D}_0$, it suffices (by Lemma 3.1.2) to restrict attention to vertices of $\mathcal{D}_0$.

Suppose $\boldsymbol{X}_0$ is a finite trivialiser of $(\mathcal{D}_0, S_0)$. Then $\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{Y}$ is a finite trivialiser of $(\mathcal{D}, S)$ by Proposition 3.1.6, and so, by Proposition 3.1.4, $\phi(\boldsymbol{X}) = \boldsymbol{X}_0 \cup \phi(\boldsymbol{Y})$ is again a finite trivialiser of $(\mathcal{D}_0, S_0)$. By Proposition 3.1.4 we have, for any vertex v_0 of $\mathcal{D}_0$, that

$$\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0 \cup \phi(\boldsymbol{Y}),v_0} \preceq \delta_{\mathcal{D},S,\boldsymbol{X}_0 \cup \boldsymbol{Y},v_0}.$$

By Lemma 3.1.2,

$$\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0 \cup \phi(\boldsymbol{Y}),v_0} \sim \delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}.$$

Thus,

$$\delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0} \preceq \delta_{\mathcal{D},S,\boldsymbol{X}_0 \cup \boldsymbol{Y},v_0}.$$

Also, by Proposition 3.1.6,

$$\delta_{\mathcal{D},S,\boldsymbol{X}_0 \cup \boldsymbol{Y},v_0} \preceq \delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}.$$

We then have

Theorem 3.1.7 *Let v_0 be any vertex in $\mathcal{D}_0$ and $\boldsymbol{Y}$ be a finite set of closed paths in $\mathcal{D}$. If $(\mathcal{D}_0, S_0)$ is a $\boldsymbol{Y}$-well-placed retract of $(\mathcal{D}, S)$, and if $\boldsymbol{X}_0$ is a finite trivialiser of $(\mathcal{D}_0, S_0)$, then $\boldsymbol{X}_0 \cup \boldsymbol{Y}$ is a finite trivialiser of $(\mathcal{D}, S)$ and*

$$\delta_{\mathcal{D},S,\boldsymbol{X}_0 \cup \boldsymbol{Y},v_0} \sim \delta_{\mathcal{D}_0,S_0,\boldsymbol{X}_0,v_0}.$$

If $(\mathcal{D}_0, S_0)$ is well-placed in $(\mathcal{D}, S)$ then any component Δ of $\mathcal{D}$ contains a component Δ_0 of $\mathcal{D}_0$, and we see from the above theorem that (assuming $(\mathcal{D}_0, S_0)$ is a retract and $(\mathcal{D}, S)$ and $(\mathcal{D}_0, S_0)$ are two locally finite objects of $\mathfrak{C}$)

$$\delta_{\mathcal{D}_0,\Delta_0} \sim \delta_{\mathcal{D},\Delta}.$$

3.2 Application to monoids

3.2.1 Associated two-complexes

Consider a finite monoid presentation

$$\hat{\mathcal{P}} = [\boldsymbol{x}\,;\,\boldsymbol{r}].$$

Associated with $\hat{\mathcal{P}}$ we have a locally finite graph $\Gamma(\hat{\mathcal{P}})$ and a two-complex $\mathcal{D}(\hat{\mathcal{P}}) = \langle \Gamma(\hat{\mathcal{P}});\ \hat{\boldsymbol{Z}}\rangle$ as defined in §1.4.2.

Now, the left and right actions of $\hat{F}(\boldsymbol{x})$ on $\Gamma(\hat{\mathcal{P}})$ satisfy $\hat{F}(\boldsymbol{x}) \cdot \hat{\boldsymbol{Z}} \cdot \hat{F}(\boldsymbol{x}) = \hat{\boldsymbol{Z}}$. Thus, $\hat{F}(\boldsymbol{x})$ acts on $\mathcal{D}(\hat{\mathcal{P}})$ on both sides compatibly and so $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ is a locally finite object of $\mathfrak{C}$.

Definition 3.2.1 *Let $\hat{\mathcal{P}} = [\boldsymbol{x}; \boldsymbol{r}]$ be a finite monoid presentation which is FDT. Suppose that $\hat{\boldsymbol{X}}$ is a finite trivialiser of $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$. For any $W \in \hat{F}(\boldsymbol{x})$ the* second order Dehn function $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{F}(\boldsymbol{x}),\hat{\boldsymbol{X}},W}$ *of $\hat{\mathcal{P}}$ with respect to $\hat{\boldsymbol{X}}$ at W then is the delta function of $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ with respect to $\hat{\boldsymbol{X}}$ at vertex W, i.e.*

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{F}(\boldsymbol{x}),\hat{\boldsymbol{X}},W} = \delta_{\mathcal{D}(\hat{\mathcal{P}}),\hat{F}(\boldsymbol{x}),\hat{\boldsymbol{X}},W},$$

or simply (if there is not any confusion)

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W} = \delta_{\mathcal{D}(\hat{\mathcal{P}}),\hat{\boldsymbol{X}},W}.$$

In particular, if $\Delta(W)$ is the component of $\mathcal{D}(\hat{\mathcal{P}})$ containing W and $\hat{\mathcal{P}}$ is FDT then we write

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\overline{W}} = \delta_{\mathcal{D}(\hat{\mathcal{P}}),\Delta(W)}$$

for a typical representative of the equivalence class containing $\delta_{\mathcal{D}(\hat{\mathcal{P}}),\hat{F}(\boldsymbol{x}),\hat{\boldsymbol{X}},W}$.

Recall that the components of $\mathcal{D}(\hat{\mathcal{P}})$ are one to one correspondence with the elements of the monoid $S(\hat{\mathcal{P}})$ defined by $\hat{\mathcal{P}}$. Thus we see that, up to equivalence we can define, for each element $\overline{W}$ of $S(\hat{\mathcal{P}})$, a second order Dehn function $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\overline{W}}$.

In this section, we aim to prove the following theorem.

Theorem 3.2.2 *Let $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ be two finite monoid presentations and suppose $\varphi : S(\hat{\mathcal{P}}_1) \longrightarrow S(\hat{\mathcal{P}}_2)$ is an isomorphism.*

(i) *If* $\hat{\mathcal{P}}_1$ *is FDT then so is* $\hat{\mathcal{P}}_2$.

(ii) *For any* $\overline{W} \in S(\hat{\mathcal{P}}_1)$ *we have* $\hat{\delta}^{(2)}_{\hat{\mathcal{P}}_1,\overline{W}} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{P}}_2,\varphi(\overline{W})}$.

We remark that part (i) of this theorem was originally proved by Squier [Sq2].

Definition 3.2.3 *Let* S *be an FDT monoid, and let* $s \in S$. *We define* $\hat{\delta}^{(2)}_{S,s}$ *(up to equivalence) to be* $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\overline{W}}$, *where* $\hat{\mathcal{P}}$ *is some finite monoid presentation such that* $S(\hat{\mathcal{P}})$ *is isomorphic to* S *under an isomorphism* $\varphi : S \longrightarrow S(\hat{\mathcal{P}})$ *say, and* $\varphi(s) = \overline{W}$.

Theorem 3.2.2 shows that this definition is valid. For suppose that S is any monoid and $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ are any two finite monoid presentations for S. Then there are two monoid isomorphisms $\varphi_1 : S \longrightarrow S(\hat{\mathcal{P}}_1)$ and $\varphi_2 : S \longrightarrow S(\hat{\mathcal{P}}_2)$, and hence the composition $\varphi = \varphi_2\varphi_1^{-1} : S(\hat{\mathcal{P}}_1) \longrightarrow S(\hat{\mathcal{P}}_2)$ is also a monoid isomorphism.

3.2.2 Tietze transformations

We now define *elementary Tietze transformations* on any monoid presentation $\hat{\mathcal{P}} = [\boldsymbol{x}; \boldsymbol{r}]$ as follows.

Type I: Add to $\hat{\mathcal{P}}$ a new relation $W_0 = V_0$, where $\overline{W}_0 = \overline{V}_0$ in $S(\hat{\mathcal{P}})$, to get the presentation $\hat{\mathcal{Q}}_1 = [\boldsymbol{x};\, \boldsymbol{r},\, W_0 = V_0]$.

Type II: Add to $\hat{\mathcal{P}}$ a new generator y, and a new relation $y = U_0$ where U_0 is a word on $\boldsymbol{x}$, to get the presentation $\hat{\mathcal{Q}}_2 = [\boldsymbol{x},\, y;\, \boldsymbol{r},\, y = U_0]$.

In both cases, the inclusion mappings $\hat{\mathcal{P}} \longrightarrow \hat{\mathcal{Q}}_i$ ($i = 1, 2$) of monoid presentations induce monoid isomorphisms

$$S(\hat{\mathcal{P}}) \longrightarrow S(\hat{\mathcal{Q}}_i), \quad i = 1,\, 2$$

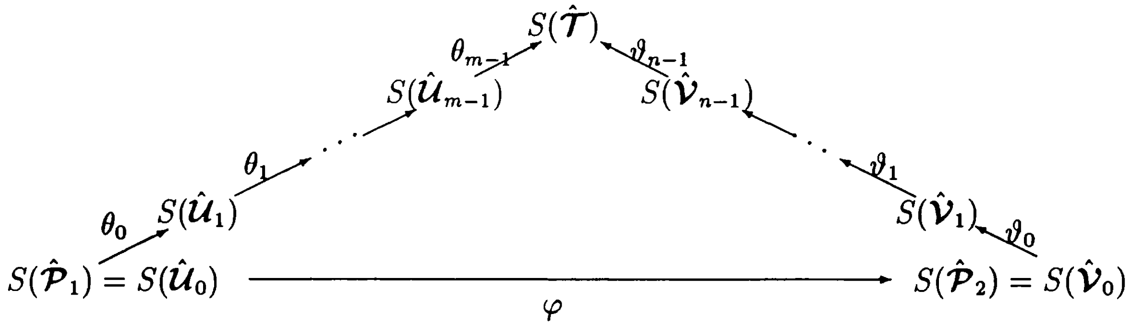
(called the *elementary isomorphisms*).

Proposition 3.2.4 *Let* $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ *be two finite monoid presentations such that* $\varphi : S(\hat{\mathcal{P}}_1) \longrightarrow S(\hat{\mathcal{P}}_2)$ *is an isomorphism. Then there is a finite monoid presentation* $\hat{\mathcal{T}}$ *and two sequences of finite monoid presentations*

$$\hat{\mathcal{P}}_1 = \hat{\mathcal{U}}_0,\, \hat{\mathcal{U}}_1,\, \cdots,\, \hat{\mathcal{U}}_m = \hat{\mathcal{T}}, \tag{3.1}$$

$$\hat{\mathcal{P}}_2 = \hat{\mathcal{V}}_0,\ \hat{\mathcal{V}}_1,\ \cdots,\ \hat{\mathcal{V}}_n = \hat{\mathcal{T}} \tag{3.2}$$

such that each $\hat{\mathcal{U}}_{i+1}$ (resp. each $\hat{\mathcal{V}}_{j+1}$) is obtained from $\hat{\mathcal{U}}_i$ (resp. $\hat{\mathcal{V}}_j$) by an elementary Tietze transformation ($0 \le i \le m-1$, $0 \le j \le n-1$). Moreover, the following diagram commutes:



where $\theta_i : S(\hat{\mathcal{U}}_i) \longrightarrow S(\hat{\mathcal{U}}_{i+1})$ and $\vartheta_j : S(\hat{\mathcal{V}}_j) \longrightarrow S(\hat{\mathcal{V}}_{j+1})$ are the corresponding elementary isomorphisms, $0 \le i \le m-1$, $0 \le j \le n-1$.

Proof. Let

$$\hat{\mathcal{P}}_1 = \hat{\mathcal{U}}_0 = [x_1,\ x_2,\ \cdots,\ x_k;\ R_1,\ R_2,\ \cdots,\ R_l],$$

$$\hat{\mathcal{P}}_2 = \hat{\mathcal{V}}_0 = [y_1,\ y_2,\ \cdots,\ y_p;\ T_1,\ T_2,\ \cdots,\ T_q].$$

Suppose that $V_i = V_i(y_1, \cdots, y_p)$, $U_j = U_j(x_1, \cdots, x_k)$ are words on $\{y_1,\ \cdots,\ y_p\}$ and $\{x_1,\ \cdots,\ x_k\}$ respectively such that

$$\varphi(\overline{x}_i) = \overline{V}_i \quad \text{and} \quad \varphi^{-1}(\overline{y}_j) = \overline{U}_j,$$

$1 \le i \le k$, $1 \le j \le p$. First, for each $0 \le j \le p-1$, by adding a new generator y_{j+1} (Type II) to $\hat{\mathcal{U}}_j$ we obtain the presentation

$$\hat{\mathcal{U}}_{j+1} = [x_1,\ \cdots,\ x_k, y_1,\ \cdots,\ y_{j+1}; R_1,\ \cdots,\ R_l,\ y_1 = U_1,\ \cdots,\ y_{j+1} = U_{j+1}]$$

and we have the corresponding elementary isomorphism $\theta_j : S(\hat{\mathcal{U}}_j) \longrightarrow S(\hat{\mathcal{U}}_{j+1})$.

Consider a relation ($1 \le j \le q$)

$$T_j :\ T_{+1,j} = T_{-1,j}$$

of $\hat{\mathcal{P}}_2$. Here $T_{+1,j}, T_{-1,j}$ are positive words $T_{+1,j}(y_1, \cdots, y_p)$, $T_{-1,j}(y_1, \cdots, y_p)$ on $\{y_1, \cdots, y_p\}$. Since φ^{-1} is an isomorphism, we have

$$\overline{T}_{+1,j}(\overline{U}_1, \cdots, \overline{U}_p) = \overline{T}_{-1,j}(\overline{U}_1, \cdots, \overline{U}_p)$$

in $S(\hat{\mathcal{P}}_1)$ (and hence in $S(\hat{\mathcal{U}}_p)$), and so (since $\overline{y}_l = \overline{U}_l$, $1 \le l \le p$),

$$\overline{T}_{+1,j}(\overline{y}_1, \cdots, \overline{y}_p) = \overline{T}_{-1,j}(\overline{y}_1, \cdots, \overline{y}_p)$$

in $S(\hat{\mathcal{U}}_p)$. Thus, for each $0 \le j \le q-1$, by adding a new relation T_{j+1} (Type I) to $\hat{\mathcal{U}}_{p+j}$ we obtain the presentation

$$\hat{\mathcal{U}}_{p+j+1} = [x_1, \cdots, x_k, y_1, \cdots, y_p;\ R_1, \cdots, R_l, y_1 = U_1, \cdots, y_p = U_p, T_1, \cdots, T_{j+1}]$$

and the corresponding elementary isomorphism θ_{p+j+1}.

Now in $S(\mathcal{P}_1)$,

$$\overline{x}_i = \varphi^{-1}(\overline{V}_i) = \varphi^{-1}(\overline{V}_i(\bar{y}_1, \cdots, \bar{y}_p)) = \overline{V}_i(\overline{U}_1, \cdots, \overline{U}_p)$$

and so in $S(\hat{\mathcal{U}}_{p+q})$

$$\overline{x}_i = \overline{V}_i(\overline{U}_1, \cdots, \overline{U}_p) = \overline{V}_i$$

for all $1 \le i \le k$. Thus, for each $0 \le i \le k-1$, by adding a new relation $x_{i+1} = V_{i+1}$ (Type I) to $\hat{\mathcal{U}}_{p+q+i}$ we obtain the presentation

$$\begin{aligned}\hat{\mathcal{U}}_{p+q+i+1} = [\ & x_1, \cdots, x_k, y_1, \cdots, y_p; \\ & R_1, \cdots, R_l, y_1 = U_1, \cdots, y_p = U_p, T_1, \cdots, T_q, x_1 = V_1, \cdots, x_{i+1} = V_{i+1}]\end{aligned}$$

and the corresponding isomorphism $\theta_{p+q+i+1}$. We then let $\hat{\mathcal{T}} = \hat{\mathcal{U}}_{p+q+k}$.

By symmetry we also have a sequence of presentations

$$\hat{\mathcal{P}}_2 = \hat{\mathcal{V}}_0, \cdots, \hat{\mathcal{V}}_{k+l+p} = \hat{\mathcal{T}}$$

such that $\hat{\mathcal{V}}_{j+1}$ is obtained from $\hat{\mathcal{V}}_j$ by an application of Type I or Type II, $0 \le j \le k+l+p-1$. For each j, we let ϑ_j be the corresponding isomorphism. Then we have

$$\begin{aligned}\theta_{p+q+k-1} \cdots \theta_0(\overline{x}_i) &= \bar{x}_i = \overline{V}_i \quad (\text{in } S(\hat{\mathcal{T}})) \\ &= \vartheta_{k+l+p-1} \cdots \vartheta_0(\overline{V}_i) \\ &= \vartheta_{k+l+p-1} \cdots \vartheta_0 \varphi(\overline{x}_i), \qquad (1 \le i \le k)\end{aligned}$$

and so the diagram commutes as required. This completes our proof. □

On the other hand, associated with the two types of Tietze transformations we have two monoid homomorphisms: ϕ_1 the identity monoid homomorphism of $\hat{F}(\boldsymbol{x})$ and ϕ_2 the monoid homomorphism from $\hat{F}(\boldsymbol{x}, y)$ to $\hat{F}(\boldsymbol{x})$ defined by

$$\phi_2 : x \longmapsto x \ \ (x \in \boldsymbol{x}), \quad y \longmapsto U_0. \tag{3.3}$$

(In fact, each ϕ_i induces the isomorphism from $S(\hat{\mathcal{Q}}_i)$ to $S(\hat{\mathcal{P}})$ which is the inverse of the corresponding elementary isomorphism from $S(\hat{\mathcal{P}})$ to $S(\hat{\mathcal{Q}}_i)$, $i = 1,\ 2$.)

Consider the Type I operation. Since $\overline{W}_0 = \overline{V}_0$ in $S(\hat{\mathcal{P}})$, there is path in $\mathcal{D}(\hat{\mathcal{P}})$ from V_0 to W_0, say $\mathbb{B}_0$. Then $\mathbb{B}_0$ together with the edge $\mathbb{E}_0 = (1, W_0 = V_0, +1, 1)$ form a closed path $\mathbb{A}_0 = \mathbb{B}_0\mathbb{E}_0$ at v_0.

We now extend ϕ_1 to be a mapping: $(\mathcal{D}(\hat{\mathcal{Q}}_1),\ \hat{F}(\boldsymbol{x})) \longrightarrow (\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ as follows.

(1) For any vertex $V \in \hat{F}(\boldsymbol{x})$ of $\mathcal{D}(\hat{\mathcal{Q}}_1)$, $\phi_1(V) = V$;

(2) For any edge of the form $\mathbb{E} = (W,\ R,\ \varepsilon,\ W')$, $R \in \boldsymbol{r}$ of $\mathcal{D}(\hat{\mathcal{Q}}_1)$, $\phi_1(\mathbb{E}) = \mathbb{E}$, and for any edge of the form $W \cdot \mathbb{E}_0^{\pm 1} \cdot W' = (W,\ W_0 = V_0,\ \pm 1,\ W')$ of $\mathcal{D}(\hat{\mathcal{Q}}_1)$, $\phi_1(W \cdot \mathbb{E}_0^{\pm 1} \cdot W') = W \cdot \mathbb{B}_0^{\mp 1} \cdot W'$.

First, it is easy to check, this extension is a mapping of graphs from $\Gamma(\hat{\mathcal{Q}}_1)$ to $\Gamma(\hat{\mathcal{P}})$. Thus, by Lemma 1.4.3, it is a mapping of two-complexes and by (1) and (2) it is a retraction.

Now consider the Type II operation. We extend ϕ_2 to be a mapping: $(\mathcal{D}(\hat{\mathcal{Q}}_2), \hat{F}(\boldsymbol{x}, y)) \longrightarrow (\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ as follows.

(3) For any vertex $V \in \hat{F}(\boldsymbol{x}, y)$ of $\mathcal{D}(\hat{\mathcal{Q}}_2)$, $\phi_2(V)$ is as defined in (3.3);

(4) For any edge $\mathbb{E} = (W, R, \varepsilon, W')$ of $\mathcal{D}(\hat{\mathcal{Q}}_2)$ with $R \in \boldsymbol{r}$, $\phi_2(\mathbb{E}) = (\phi_2(W), R, \varepsilon, \phi_2(W'))$, and for any edge of the form $\mathbb{E} = (W,\ y = U_0,\ \varepsilon,\ W')$ of $\mathcal{D}(\hat{\mathcal{Q}}_2)$, $\phi_2(\mathbb{E}) = 1_{\phi_2(W)U_0\phi_2(W')}$.

Again, it is easy to check using Lemma 1.4.3, that ϕ_2 is a mapping of two-complexes. Moreover, by (3) and (4), ϕ_2 is a retraction.

We will further prove in the following subsection (§3.2.3) that $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ is well-placed in $(\mathcal{D}(\hat{\mathcal{Q}}_1),\ \hat{F}(\boldsymbol{x}))$ and $(\mathcal{D}(\hat{\mathcal{Q}}_2),\ \hat{F}(\boldsymbol{x},\ y))$. We will then use this in §3.2.4 to prove Theorem 3.2.2.

The above notations will remain unchanged throughout the rest of this section.

3.2.3 Invariance over presentations

Lemma 3.2.5 *The object* $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ *is a well-placed retract of* $(\mathcal{D}(\hat{\mathcal{Q}}_1),\ \hat{F}(\boldsymbol{x}))$ *and a well-placed retract of* $(\mathcal{D}(\hat{\mathcal{Q}}_2),\ \hat{F}(\boldsymbol{x},y))$.

Proof. In the previous subsection we have seen that both ϕ_1, ϕ_2 are retractions.

Since all vertices of $\mathcal{D}(\hat{\mathcal{Q}}_1)$ are vertices of the subcomplex $\mathcal{D}(\hat{\mathcal{P}})$, we can see that the set $\boldsymbol{Y}_1^* = \boldsymbol{Y}^*(\mathcal{D}(\hat{\mathcal{Q}}_1),\ \mathcal{D}(\hat{\mathcal{P}}))$ as defined in Definition 3.1.5 only contains single edges of the form $(W',\ W_0 = V_0,\ \pm 1,\ V')$. Thus, if we let $\boldsymbol{Y}_1$ be $\{\mathbb{E}_0\mathbb{B}_0\}$ then $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ is $\boldsymbol{Y}_1$-well-placed in $(\mathcal{D}(\hat{\mathcal{Q}}_1),\ \hat{F}(\boldsymbol{x}))$.

Consider the set $\boldsymbol{Y}_2^* = \boldsymbol{Y}^*(\mathcal{D}(\hat{\mathcal{Q}}_2), \mathcal{D}(\hat{\mathcal{P}}))$ defined as in Definition 3.1.5. If $\mathbb{B} \in \boldsymbol{Y}_2^*$ is a non-trivial path, say $\mathbb{B} = \mathbb{E}_1\mathbb{E}_2\cdots\mathbb{E}_m$, then $\iota(\mathbb{E}_1), \tau(\mathbb{E}_m) \in \hat{F}(\boldsymbol{x})$, and $\tau(\mathbb{E}_1), \cdots, \tau(\mathbb{E}_{m-1}) \notin \hat{F}(\boldsymbol{x})$. Since $\iota(\mathbb{E}_1) \in \hat{F}(\boldsymbol{x})$, $\tau(\mathbb{E}_1) = \iota(\mathbb{E}_2) \notin \hat{F}(\boldsymbol{x})$, $\mathbb{E}_1$ must be of the form $(W,\ y = U_0,\ -1,\ V)$. Furthermore, since $\tau(\mathbb{E}_m) \in \hat{F}(\boldsymbol{x})$, the y-arc in $\mathbb{E}_1$ must terminate at some $\mathbb{E}_j$, $2 \le j \le m$, namely $\mathbb{E}_j$ must be of the form $(W',\ y = U_0,\ +1,\ V')$ and hence $\mathbb{B}$ must be of the form as depicted in Fig. 3.1.

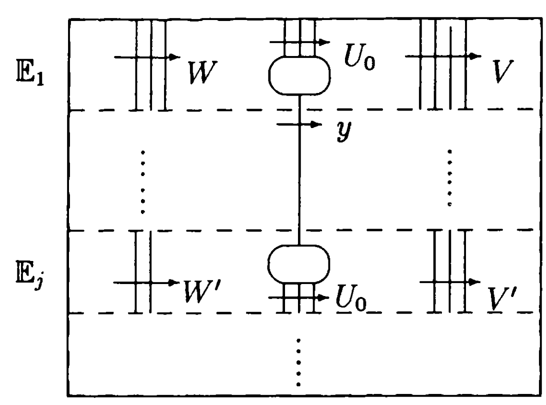


Fig. 3.1

Thus, $\mathbb{B}$ is not a shortest path in the homotopy equivalence classes containing $\mathbb{B}$. Hence, $\boldsymbol{Y}_2^* = \emptyset$, and so $(\mathcal{D}(\hat{\mathcal{P}}),\ \hat{F}(\boldsymbol{x}))$ is $\emptyset$-well-placed in $(\mathcal{D}(\hat{\mathcal{Q}}_2),\ \hat{F}(\boldsymbol{x},\ y))$, and this completes the proof. □

3.2.4 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2: Consider (3.1) and (3.2). If $\hat{\mathcal{U}}_i$ is FDT, by Lemma 3.2.5 and Proposition 3.1.6, $\hat{\mathcal{U}}_{i+1}$ is also FDT $(0 \leq i \leq m-1)$. Thus, if $\hat{\mathcal{U}}_0 = \hat{\mathcal{P}}_1$ is FDT, then $\hat{\mathcal{T}}$ is FDT. Moreover, by Lemma 3.2.5 and Proposition 3.1.4 if $\hat{\mathcal{V}}_j$ is FDT then $\hat{\mathcal{V}}_{j-1}$ is also FDT $(1 \leq j \leq n-1)$. Hence, we have proved (i) that if $\hat{\mathcal{P}}_1$ is FDT then so is $\hat{\mathcal{P}}_2$.

Let $\hat{\boldsymbol{X}}$, $\hat{\boldsymbol{Y}}$ be any finite trivialisers of $(\mathcal{D}(\hat{\mathcal{P}}_1), \hat{F}(\boldsymbol{x}))$ and $(\mathcal{D}(\hat{\mathcal{P}}_2), \hat{F}(\boldsymbol{y}))$ respectively. Let W be any element of $\hat{F}(\boldsymbol{x})$ and let V be any element of $\hat{F}(\boldsymbol{y})$ such that $\overline{V} = \varphi(\overline{W})$ in $S(\hat{\mathcal{P}}_2)$. By the commutative diagram in the Proposition 3.2.4, we have $\theta_{m-1}\cdots\theta_0(\overline{W}) = \vartheta_{n-1}\cdots\vartheta_0(\overline{V})$ and so W and V (as elements of $\hat{F}(\boldsymbol{x}, \boldsymbol{y})$) lie in the same component of $\mathcal{D}(\hat{\mathcal{T}})$.

Now by successively using Lemma 3.2.5 and Theorem 3.1.7 we have

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}}_1, \hat{F}(\boldsymbol{x}), \hat{\boldsymbol{X}}, W} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}), \hat{\boldsymbol{X}}', W}$$

for some finite trivialiser $\hat{\boldsymbol{X}}'$ of $(\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}))$ and

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}}_2, \hat{F}(\boldsymbol{y}), \hat{\boldsymbol{Y}}, V} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}), \hat{\boldsymbol{Y}}', V}$$

for some finite trivialiser $\hat{\boldsymbol{Y}}'$ of $(\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}))$. Since W and V lie in the same component of $\mathcal{D}(\hat{\mathcal{T}})$, and $\hat{\boldsymbol{X}}'$ and $\hat{\boldsymbol{Y}}'$ are finite, by Lemmas 3.1.2 and 3.1.3 we have

$$\hat{\delta}^{(2)}_{\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}), \hat{\boldsymbol{X}}', W} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{T}}, \hat{F}(\boldsymbol{x}, \boldsymbol{y}), \hat{\boldsymbol{Y}}', V}$$

and so

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}}_1, \hat{F}(\boldsymbol{x}), \hat{\boldsymbol{X}}, W} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{P}}_2, \hat{F}(\boldsymbol{y}), \hat{\boldsymbol{Y}}, V},$$

namely,

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}}_1, \overline{W}} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{P}}_2, \varphi(\overline{W})}$$

for any $\overline{W} \in S(\hat{\mathcal{P}}_1)$ and this completes the proof. □

3.3 The definition concerning the shortest words

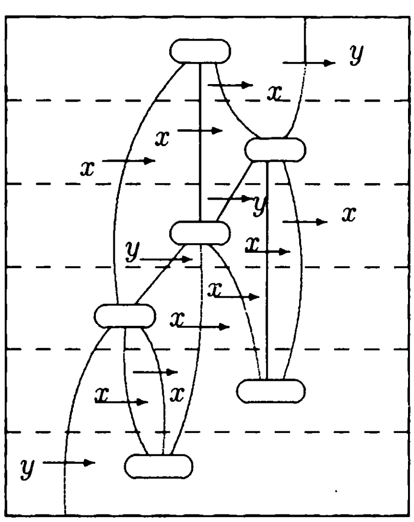
In this section, we will define a global second order Dehn function of a finite monoid presentation. This function gives an overall upper bound for all second order Dehn functions at shortest words in every component. However, I am unable to prove this definition is independent of the choice of finite presentations.

Let $\hat{\mathcal{P}} = [\boldsymbol{x} \,;\, \boldsymbol{r}]$ be a finite monoid presentation. Suppose that $\mathbb{P}$ is a path in $\mathcal{D}(\hat{\mathcal{P}})$. We will say that $\mathbb{P}$ is a *connected picture*, if the corresponding geometrical configuration of $\mathbb{P}$, regarded as a graph in the plane where the discs are vertices and the arcs are edges, is connected. If $\mathbb{P}$ is not connected, then it consists of a number of (connected) *components*. A component is *non-trivial* if it contains at least one disc.

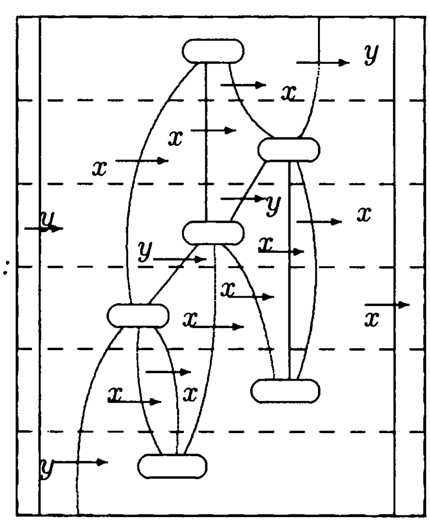
Furthermore, suppose that $\mathbb{P}$ is a closed path in $\mathcal{D}(\hat{\mathcal{P}})$. Since $\iota(\mathbb{P}) = \tau(\mathbb{P})$ we are able to glue the upper boundary of the geometrical configuration of $\mathbb{P}$ with its lower boundary to obtain a spherical geometrical configuration. We will say that $\mathbb{P}$ is a *connected spherical picture* over $\hat{\mathcal{P}}$ if this spherical geometrical configuration is connected. If the spherical picture $\mathbb{P}$ is not connected, then it consists of a number of *components* (connected spherical pictures). A component of a closed path (a spherical picture) is *non-trivial* if it contains at least one disc.

Example 3.3.1 *The picture $\mathbb{P}$ in the Example 1.4.1 is connected. But the picture $y{\cdot}\mathbb{P}{\cdot}x$ is not connected; it has one non-trivial component (the picture $\mathbb{P}$) and two trivial components the two arcs labelled y and x respectively.*

$\mathbb{P}$:



$y{\cdot}\mathbb{P}{\cdot}x$:



Example 3.3.2 *Let*

$$\hat{\mathcal{P}}_0 = [x, y;\ xy = 1,\ yx = 1,\ x^2 = y^2].$$

Then the closed path

$$\mathbb{P}_1 = (1,\ xy = 1,\ +1,\ xyxyxyxy)(xyxyxyxy,\ xy = 1,\ -1,\ 1)$$

with the following geometrical configuration

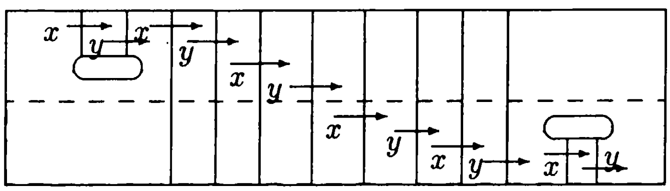


Fig. 3.2

is not a connected picture but a connected spherical picture. The closed path

$$\begin{aligned}\mathbb{P}_2 \ =\ & (x,\ xy = 1,\ +1,\ (xy)^2yx^2)(x(xy)^2,\ xy = 1,\ -1,\ yx^2)\\ & (x(xy)^3yx,\ xy = 1,\ -1,\ x)(x(xy)^3y,\ x^2 = y^2,\ +1,\ yx)\\ & (x(xy)^3y^2,\ x^2 = y^2,\ -1,\ x)(x(xy)^3y,\ yx = 1,\ +1,\ x^2)\end{aligned}$$

with the following geometrical configuration

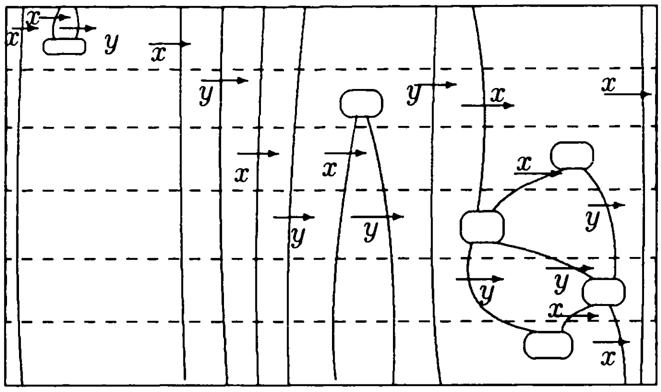


Fig. 3.3

is not a connected spherical picture; it consists of three trivial components and two non-trivial components.

Example 3.3.3 *Let*

$$\hat{\mathcal{P}}_1 = [x, y, z;\ xy = x,\ yz = z].$$

Then the closed path

$$\mathbb{P}_3 = (1,\ xy = x,\ +1,\ y^m z)(xy^m,\ yz = z,\ -1,\ 1)$$

with the following geometric configuration

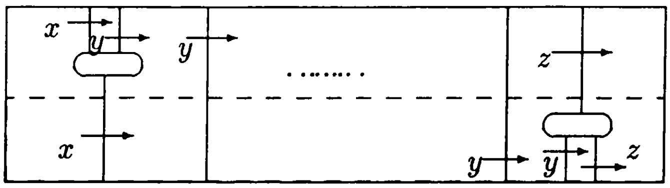


Fig. 3.4

is a connected spherical picture (but not a connected picture).

Let $\hat{\mathcal{P}} = [\boldsymbol{x}\ ;\ \boldsymbol{r}]$ be a finite presentation, and let

$$\boldsymbol{w} = \{W :\ W \in \hat{F}(\boldsymbol{x}),\ L(W) \le L(U) \text{ for any } U \in \hat{F}(\boldsymbol{x}) \text{ with } \overline{U} = \overline{W}\},$$

that is, $\boldsymbol{w}$ *is the set of minimal representatives of elements of* $S(\hat{\mathcal{P}})$.

Lemma 3.3.4 *For any* $n \in \mathbb{N}$ *the set*

$$\Upsilon_n^* = \{\mathbb{P} :\ L(\mathbb{P}) \le n, \mathbb{P} \text{ a connected spherical picture over } \hat{\mathcal{P}} \text{ at } W \text{ for some } W \in \boldsymbol{w}\}$$

is finite.

Proof. Let $\mathbb{P}$ be any connected spherical picture at W over $\hat{\mathcal{P}}$ for some $W \in \boldsymbol{w}$ with $L(\mathbb{P}) \le n$. We first show that $\mathbb{P}$ is a connected picture. Suppose $\mathbb{P}$ is not a connected picture. Let $\mathbb{P}_1$ be the first non-trivial component of $\mathbb{P}$ to the left. Since $\mathbb{P}$ is a connected spherical picture, there are not trivial components on the left of $\mathbb{P}_1$ (otherwise these trivial components would be trivial connected spherical components of $\mathbb{P}$). Let $\mathbb{P}_2$ be the picture obtained from $\mathbb{P}$ by removing $\mathbb{P}_1$. Then $\mathbb{P}_2$ is not trivial by the same reason. Thus, either $\iota(\mathbb{P}_1) = W_1$, $\tau(\mathbb{P}_1) = W_1W_2$, or $\iota(\mathbb{P}_1) = W_1W_2$, $\tau(\mathbb{P}_1) = W_1$ for some words W_1, W_2 on $\boldsymbol{x}$ with W_2 not a empty word. Without any loss we suppose it is the first case. Note that if we glue the lower boundary of $\mathbb{P}$ with its upper boundary then those arcs of $\mathbb{P}_1$ labelled W_2 must be connected to the upper boundary of $\mathbb{P}_2$. Thus, $\iota(\mathbb{P}_2) = W_2W_3$, $\tau(\mathbb{P}_2) = W_3$ for some word W_3 on $\boldsymbol{x}$ as shown in Fig. 3.5 since $\mathbb{P}$ is a connected spherical picture.

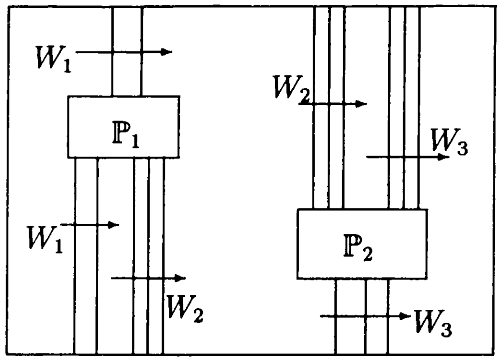


Fig. 3.5

From $\mathbb{P}_1$ we have $\overline{W_1} = \overline{W_1 W_2}$. Thus $\overline{W} = \overline{W_1 W_2 W_3} = \overline{W_1 W_3}$. This is a contradiction since $L(W_1 W_3) < L(W)$.

We now show that for any non-trivial connected picture $\mathbb{P}$ with $L(\mathbb{P}) \leq n$, $L(\iota(\mathbb{P})) \leq an$ where

$$a = \max\{L(R_{+1}),\ L(R_{-1}) :\ R \in \boldsymbol{r}\}.$$

If $n = 1$, then $\iota(\mathbb{P}) = R_{+1}$ or R_{-1} for some $R \in \boldsymbol{r}$, and so $L(\iota(\mathbb{P})) \leq a$.

Suppose that the result is true for $n - 1 \geq 1$. Let $\mathbb{P}$ be a connected picture with $L(\mathbb{P}) = n$, say

$$\mathbb{P} = \mathbb{E}_1 \mathbb{E}_2 \cdots \mathbb{E}_n,$$

where $\mathbb{E}_1, \cdots, \mathbb{E}_n$ are edges in $\mathcal{D}(\hat{\mathcal{P}})$. Let $\mathbb{P}_{n-1} = \mathbb{E}_1, \cdots, \mathbb{E}_{n-1}$. If $\mathbb{P}_{n-1}$ is connected, then by the induction hypothesis we have

$$L(\iota(\mathbb{P}_n)) = L(\iota(\mathbb{P}_{n-1})) \leq a(n-1) < an.$$

If $\mathbb{P}_{n-1}$ is not connected, then $\mathbb{P}_{n-1}$ consists of a number of components. Suppose that $\mathbb{P}_{n-1}$ has n_0 trivial components which are arcs starting at the upper half boundary of $\mathbb{P}$ and going all the way down to the upper half boundary of $\mathbb{E}_n$. Since $\mathbb{P}$ is connected, all these arcs must join the upper half of the unique disc of $\mathbb{E}_n$. Thus, the number of these arcs is bounded by a, namely $n_0 \leq a$. By dropping off these arcs from $\mathbb{P}_{n-1}$, we obtain a picture $\mathbb{P}'_{n-1}$ with $L(\mathbb{P}'_{n-1}) = n - 1$ consisting of a number of non-trivial components, say $\mathbb{Q}_1, \mathbb{Q}_2, \cdots, \mathbb{Q}_m$ (see Fig. 3.6) with

$$\sum_{i=1}^{m} L(\mathbb{Q}_i) = n - 1.$$

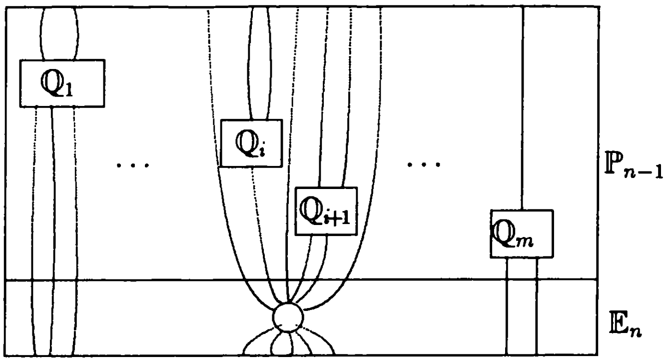


Fig. 3.6

By induction hypothesis, we have $L(\iota(\mathbb{Q}_i)) \leq aL(\mathbb{Q}_i)$ for all i and so

$$\begin{aligned} L(\iota(\mathbb{P})) &= L(\iota(\mathbb{P}_{n-1})) \leq a + \sum_{i=1}^{m} L(\iota(\mathbb{Q}_i)) \leq a + \sum_{i=1}^{m} a(L(\mathbb{Q}_i)) \\ &= a(1 + n - 1) = an \end{aligned}$$

as required.

Since $\boldsymbol{x}$ is finite, there are only finitely many words on $\boldsymbol{x}$ with length bounded by an for any fixed integer $n \in \mathbb{N}$. Thus, since $\boldsymbol{r}$ is finite, the number of paths of length bounded by n each of which joins two words of length bounded by an then also is finite; in particular, Υ_n^* is finite. □

Lemma 3.3.5 *Let $\hat{\boldsymbol{X}}$ be any trivialiser of the two-complex $(\mathcal{D}(\hat{\mathcal{P}}), \hat{F}(\boldsymbol{x}))$. For any $n \in \mathbb{N}$ the set*

$$B_n = \{Area_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}^{\hat{F}(\boldsymbol{x})}}(\mathbb{P}) : \mathbb{P} \textit{ closed at } W \textit{ in } \mathcal{D}(\hat{\mathcal{P}}) \textit{ for some } W \in \boldsymbol{w} \textit{ with } L(\mathbb{P}) \leq n\}$$

is finite.

Proof. Let

$$B_n^{\Upsilon} = \{Area_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}^{\hat{F}(\boldsymbol{x})}}(\mathbb{Q}) : \mathbb{Q} \in \Upsilon_n^*\}.$$

Then B_n^{Υ} is a finite set by the previous lemma.

Consider any closed path $\mathbb{P}$ at W in $\mathcal{D}(\hat{\mathcal{P}})$ for some $W \in \boldsymbol{w}$ with $L(\mathbb{P}) \leq n$. Suppose that $\mathbb{P}$ has h non-trivial components (as connected spherical pictures), say $\mathbb{P}_1, \mathbb{P}_2, \cdots, \mathbb{P}_h$.

Let $L(\mathbb{P}_i) = n_i$, $i = 1, 2, \cdots, h$, where $\sum_{i=1}^{h} n_i = n$. Now $\mathbb{P}$ is homotopic to a closed path $\mathbb{P}'$ of the form

$$\mathbb{P}' = \prod_{i=1}^{h} U_i \cdot \mathbb{P}_i \cdot V_i$$

for some words U_i, V_i on $\boldsymbol{x}$, $i = 1, \cdots, h$ (see Fig. 3.7).

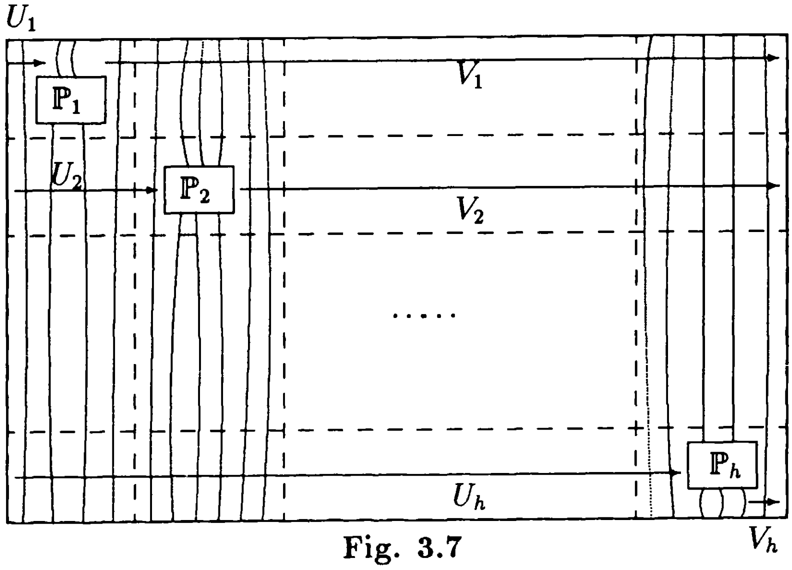


Fig. 3.7

Since for each i, $\iota(\mathbb{P}_i)$ is a subword of W, $\iota(\mathbb{P}_i) \in \boldsymbol{w}$. Thus, for each i, $\mathbb{P}_i \in B_{n_i}$. Let $r_i = Area_{\hat{\mathcal{P}},\hat{X}^{\mathcal{F}(\boldsymbol{x})}}(\mathbb{P}_i)$ for each i, then by the above lemma $r_i \leq |B_{n_i}^{\Upsilon}| \leq |B_n^{\Upsilon}|$ which is finite. Thus, by (1.1), each $U_i \cdot \mathbb{P}_i \cdot V_i$ is homotopic to a closed path $\mathbb{P}_i'$ of the form

$$\mathbb{P}_i' = \prod_{j=1}^{r_i} \mathbb{D}_{i_j} \mathbb{A}_{i_j} \mathbb{D}_{i_j}^{-1}.$$

Therefore, $\mathbb{P}$ is homotopic to a closed path $\mathbb{P}''$ of the form

$$\mathbb{P}'' = \prod_{i=1}^{h} \prod_{j=1}^{r_i} U_i \cdot \mathbb{D}_{i_j} \mathbb{A}_{i_j} \mathbb{D}_{i_j}^{-1} \cdot V_i.$$

Hence

$$Area_{\hat{\mathcal{P}},\hat{X}^{\mathcal{F}(\boldsymbol{x})}}(\mathbb{P}) \leq h \max\{r_i : 1 \leq i \leq h\} \leq h|B_n^{\Upsilon}| \leq n|B_n^{\Upsilon}|$$

and our lemma follows. □

From this lemma, we are able to define the second order Dehn function of a monoid presentation as follows.

Definition 3.3.6 *The second order Dehn function of a finite monoid presentation $\hat{\mathcal{P}}$ with respect to a finite trivialiser $\hat{\boldsymbol{X}}$ is the function*

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}}(n) = \max\{B_n\},\ n \in \mathbb{N}$$

where $\hat{\boldsymbol{X}}$ is a trivialiser of $\mathcal{D}(\hat{\mathcal{P}})$ and B_n is defined in the above lemma.

In fact,

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}}(n) = \max\{\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{F}(\boldsymbol{x}),\hat{\boldsymbol{X}},W}(n) :\ W \in \boldsymbol{w}\}.$$

The advantage of this definition is that for any $U \in \hat{F}(\boldsymbol{x})$, there is a $W \in \boldsymbol{w}$ such that $\overline{W} = \overline{U}$. Then there is a path in $\mathcal{D}(\hat{\mathcal{P}})$ of length $d = Der_{\hat{\mathcal{P}}}(W,\ U)$ from W to U. By Lemma 3.1.2, we then have

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},U}(n) \leq \hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W}(n + 2d) \leq \hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}}(n + 2d). \tag{3.4}$$

Thus, we can give an upper bound for $\delta^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},U}$ by means of $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}}$.

However, suppose that $\hat{\mathcal{P}}$ is FDT, $\hat{\boldsymbol{X}}$ is finite, and $\hat{\mathcal{Q}} = [\boldsymbol{y};\ \boldsymbol{s}]$ is another finite monoid presentation such that there is an isomorphism $\varphi : S(\hat{\mathcal{Q}}) \longrightarrow S(\hat{\mathcal{P}})$. Let $\hat{\boldsymbol{Y}}$ be any finite trivialiser of $(\mathcal{D}(\hat{\mathcal{Q}}),\ \hat{F}(\boldsymbol{y}))$. In order to show $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}} \preceq \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}$ we must find constants $A,\ B > 0$ and $C \geq 0$ such that

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}}(n) \leq A\hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}(Bn) + Cn,$$

namely, for each $W \in \boldsymbol{w}$ and all $n \in \mathbb{N}$,

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W}(n) \leq A\hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}(Bn) + Cn.$$

By Theorem 3.2.2, for a fixed $W \in \boldsymbol{w}$ there is a $U'_W \in \hat{F}(\boldsymbol{y})$ such that $\varphi(\overline{W}) = \overline{U}'_W$ in $S(\hat{\mathcal{Q}})$ and

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W} \preceq \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},U'_W}.$$

So, there are constants $A_W,\ B_W > 0,\ C_W \geq 0$, such that for all $n \in \mathbb{N}$

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W}(n) \leq A_W\hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},U'_W}(B_W n) + C_W n.$$

Since $\hat{\mathcal{P}}$ is finite, for each $W \in \boldsymbol{w}$ the set of all shortest words in $\overline{W}$ is finite. Therefore, if we let $A_{\overline{W}} = \max\{A_V : V \in \boldsymbol{w} \cap \overline{W}\}$, $B_{\overline{W}} = \max\{B_V : V \in \boldsymbol{w} \cap \overline{W}\}$, and $C_{\overline{W}} = \max\{C_V : V \in \boldsymbol{w} \cap \overline{W}\}$ then for all $n \in \mathbb{N}$ and all $V \in \boldsymbol{w} \cap \overline{W}$ we have

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},V}(n) \leq A_{\overline{W}} \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},U'_W}(B_{\overline{W}} n) + C_{\overline{W}} n.$$

For each U'_W we now let $W' \in \hat{F}(\boldsymbol{y})$ be a shortest word representing $\overline{U}'_W$. Then by (3.4),

$$\hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},U'_W}(n) \leq \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},W'}(n + 2D_W) \leq \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}(n + 2D_W)$$

with $D_W = Der_{\hat{\mathcal{Q}}}(W', U'_W)$. Since $\boldsymbol{w} \cup \overline{W}$ is finite, the set $\{U'_W : W \in \boldsymbol{w} \cup \overline{W}\}$ is finite. We then have a constant $D_{\overline{W}} \geq D_W$, $W \in \boldsymbol{w} \cap \overline{W}$ such that

$$\begin{aligned} \hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},V}(n) &\leq A_{\overline{W}} \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}},W'}(B_{\overline{W}} n + 2D_{\overline{W}} B_{\overline{W}}) + C_{\overline{W}} n \\ &\leq A_{\overline{W}} \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}(B_{\overline{W}} n + 2D_{\overline{W}} B_{\overline{W}}) + C_{\overline{W}} n \end{aligned}$$

for all $V \in \boldsymbol{w} \cap \overline{W}$.

Thus, these constants $A_{\overline{W}}$, $B_{\overline{W}}$, $C_{\overline{W}}$ and $D_{\overline{W}}$ are fixed for the component of $\mathcal{D}(\hat{\mathcal{P}})$ corresponding to $\overline{W}$. But, if $\mathcal{D}(\hat{\mathcal{P}})$ has infinitely many components then we are unable to find constants A, $B > 0$, $C \geq 0$ such that

$$\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}},W}(n) \leq A \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}(Bn) + Cn$$

for all $W \in \boldsymbol{w}$. Hence, we are unable to show $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}} \preceq \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}$. Nevertheless, we have seen that if the monoid $S(\hat{\mathcal{P}})$ is finite then $\mathcal{D}(\hat{\mathcal{P}})$ has only finitely many components and so we can find constants A, B, C so that the above inequality holds. Hence, by symmetry we have

Theorem 3.3.7 *If $S(\hat{\mathcal{P}})$ is finite then $\hat{\delta}^{(2)}_{\hat{\mathcal{P}},\hat{\boldsymbol{X}}} \sim \hat{\delta}^{(2)}_{\hat{\mathcal{Q}},\hat{\boldsymbol{Y}}}$.*

Chapter 4

Calculations of second order Dehn functions of groups I: (a)synchronously combable groups

4.1 Definitions and notations

Throughout this chapter, we let G be a group finitely presented by $\mathcal{P} = \langle \boldsymbol{x} \,;\, \boldsymbol{r} \rangle$ and we identify G with $G(\mathcal{P})$. We let $\mu : \hat{F}(\boldsymbol{x}, \boldsymbol{x}^{-1}) \longrightarrow G$ be the monoid homomorphism given by $x^\varepsilon \longmapsto \bar{x}^\varepsilon$ ($x \in \boldsymbol{x}$, $\varepsilon = \pm 1$).

Consider the Cayley graph $\Gamma_{\boldsymbol{x}}(G)$. A *combing* $\sigma : G \longrightarrow \hat{F}(\boldsymbol{x}, \boldsymbol{x}^{-1})$ is a section of μ, or equivalently, σ is a set of chosen paths in $\Gamma_{\boldsymbol{x}}(G)$ such that for any $g \in G$, $\sigma(g)$ is a choice of a path from the identity to g. We write σ_g for this chosen path and call this path a *combing line.* If $\sigma_g = e_1 e_2 \cdots e_m$, then for any non-negative integer time t if $t \leq m$ we write $\sigma_g(t) = \tau(e_t)$ and if $t > m$ we write $\sigma_g(t) = \tau(e_m) = g$.

A combing σ of $\Gamma_{\boldsymbol{x}}(G)$ is said to have the *(synchronous) K-fellow traveller property* if there exists a non-negative constant K such that the combing paths to any vertices g_1, g_2 with $d(g_1, g_2) \leq 1$ in $\Gamma_{\boldsymbol{x}}(G)$ are within a distance K of each other at any integer time $t \geq 0$, i.e.

$$d(\sigma_{g_1}(t),\, \sigma_{g_2}(t)) \leq K$$

and we will say that σ_{g_1} and σ_{g_2} are *(synchronous) K-fellow travellers* in $\Gamma_{\boldsymbol{x}}(G)$.

We say that G is (*synchronously*) *combable* if it admits a combing having the K-fellow traveller property. Any *synchronously automatic group* (for definition see [ECHLPT, §2.3) is a synchronously combable group [ECHLPT, p84], and any combable group is of type F_3 [Al2, Theorem 2].

Let

$$\Omega = \{\psi : \mathbb{N} \longrightarrow \mathbb{N};\ \psi(0) = 0,\ \psi(n+1) = \psi(n) \text{ or } \psi(n) + 1,\ n \in \mathbb{N}\}$$

where all ψ are unbounded. Given a combing σ of G and for any $g,\ h \in G$, we set

$$E_\sigma(g,h) = \min_{\psi,\, \psi' \in \Omega} \{\max_{t \in \mathbb{N}} \{d(\sigma_g(\psi(t)),\ \sigma_h(\psi'(t)))\}\}.$$

Then the *asynchronous width* of σ is defined to be

$$\Phi(n) = \max\{E_\sigma(g,h) :\ d(1,g),\ d(1,h) \leq n,\ d(g,h) = 1,\ g,\ h \in G\}$$

for all $n \in \mathbb{N}$. If Φ is bounded by a constant K then we say that σ has the *asynchronous K-fellow traveller property.*

We say that G is *asynchronously combable* if it admits a combing having the asynchronous K-fellow traveller property. Any *asynchronously automatic group* (for definition see [ECHLPT, §7.2) is asynchronously combable [ECHLPT, Theorem 7.3.6].

The *length* $L_\sigma(n)$ of a given combing σ is defined by:

$$L_\sigma(n) = \max_{g \in G}\{|L(\sigma_g)|\ :\ d(1,g) \leq n\}.$$

If there is a increasing function $f\colon\ \mathbb{N} \longrightarrow \mathbb{N}$ so that $L_\sigma(n) \leq f(n)$ for all n, then we say that the length of σ is bounded by f.

Let σ be a combing of G. Let $D\colon\ \mathbb{N} \longrightarrow \mathbb{N}$ be a function. If for any integer $r > 0$, $g \in G$ and for all integers $s,\ t$ with $0 \leq s,t \leq L(\sigma_g)$ one has that $|s - t| > D(r)$ implies $d(\sigma_g(s),\ \sigma_g(t)) > r$, we then say that D is a *departure function* for the combing σ. Any asynchronously combable group with a departure function is of type F_3 [Ge3, Theorem 1.1].

All asynchronously automatic groups are asynchronously combable with departure function [ECHLPT, Theorems 7.2.4, 7.2.8]. Bridson [Brd1, Theorem A′] showed that the group $\mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ is asynchronously combable with departure function for all $\phi \in GL_n(\mathbb{Z})$.

4.2 Word hyperbolic groups

In his influential paper [Gr], M. Gromov introduced the class of *word hyperbolic* groups (also see [ABC]). These are the finitely presented groups with linear first order Dehn function. It is well-known that word hyperbolic groups are all automatic groups [BGSS, ECHLPT, Ol]. For our use in the following chapters, we state a result of Bogley and Burton [ABBPW2, Theorem 4.1] as follows.

Theorem 4.2.1 *Each word hyperbolic group has linear second order Dehn function.*

Thus it happens that a finitely presented group with a linear first order Dehn function also has a linear second order Dehn function.

We point out that the converse to Theorem 4.2.1 is false. For example, any group with a finite aspherical presentation has linear second order Dehn function. Such groups need not be word hyperbolic, as demonstrated by the free abelian group of rank two. Also (see Corollary 5.2.2) if $G = G_0 \times \mathbb{Z}$ where G_0 is word hyperbolic then $\delta_G^{(2)}$ is linear. Such a group G is word hyperbolic if and only if G_0 is finite. (Note that the proof of Corollary 5.2.2 depends on Theorem 4.2.1.)

4.3 Asynchronously combable groups with departure functions

For the first order Dehn functions of asynchronously combable groups we have the following:

(a) if G is asynchronously combable then $\delta_G^{(1)} \preceq 2^n$ [Brd2, Theorem 6.1];

(b) if G admits an asynchronously bounded combing σ, and if $L_\sigma(n) \leq f(n)$ by a function f for all positive integer n, then $\delta_G^{(1)} \preceq nf(n)$ [BrPi, Lemma 4.1; Br3, Proposition 5.1].

Example 4.3.1 *Every automatic group [ECHLPT, Lemma 2.3.9] or every* semihyperbolic group *(see [AlBr] for definition) admits a synchronously bounded combing σ with*

$L_\sigma(n)$ linear in n and so by (b) the first order Dehn function of each such group is bounded above by a quadratic function.

Example 4.3.2 *Let $G_0 = B_{p,q}$ be the Baumslag-Solita group defined by the (aspherical) presentation $\mathcal{P}_0 = \langle y,\ z\ ;\ zy^pz^{-1}y^{-q}\rangle$ $(1 \le p < q)$. By Theorem E1 of [BGSS] this is an asynchronously automatic group and so by (a) $\delta^{(1)}_{G_0} \preceq 2^n$. (In fact, by a result of Gersten [Ge3, the proof of Theorem B] we know that $\delta^{(1)}_{G_0} \sim 2^n$.)*

Example 4.3.3 *Consider groups of the form $G_0 = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$, $\phi \in GL_n(\mathbb{Z})$. By the proof of the Main Lemma of [Brd1], Proposition 5.2 and Theorem A′ of [Brd1] we know that these groups are asynchronously combable with departure functions. Moreover, the combings admitted by these groups have lengths bounded by polynomial functions or exponential functions according to the absolute values of the eigenvalues of ϕ being 1 or not. Thus, $\delta^{(1)}_{G_0}$ is bounded by a polynomial function or an exponential function [BrGe, Main Theorem].*

The estimate of $\delta^{(2)}$ for asynchronously combable groups below derives from the work of Gersten [Ge3]. Gersten showed that any asynchronously combable group with a departure function is of type F_3. It was pointed out to S. Pride by S. Rosebrock that an analysis of Gersten's proof enables an estimate for $\delta^{(2)}$ to be given. Pride then by giving an account of the analysis proved that if G admits a synchronously bounded combing σ with the length L_σ bounded by function f then (b) holds for $\delta^{(2)}_G$. Here we show by developing the same technique that (b) holds for $\delta^{(2)}_G$ if G admits an asynchronous combing with departure function. We also show (a) is true for (synchronously) combable groups.

The following lemma is known, for example, see [Br3, 6.1].

Lemma 4.3.4 *Suppose that G admits a combing σ having the asynchronous K-fellow traveller property. Let $\mathcal{P}_1 = \langle \boldsymbol{x};\ \boldsymbol{r}_1\rangle$ where $\boldsymbol{r}_1$ is the set of all words $W = x_1 \cdots x_n$ on $\boldsymbol{x}$ such that $n \le 2(K+1)$ and $\overline{x_1 \cdots x_n} = 1$ in G. Then $\mathcal{P}_1$ is a (finite) presentation for G.*

Proof. Consider an edge $e = (g, x^\varepsilon)$ of $\Gamma_{\boldsymbol{x}}(G)$ $(x \in \boldsymbol{x},\ g \in G,\ \varepsilon = \pm 1)$. We have a pair of monotone unbounded functions ψ_g and $\psi_{g\bar{x}^\varepsilon}$ such that $d(\sigma_g(\psi_g(t)), \sigma_{g\bar{x}^\varepsilon}(\psi_{g\bar{x}^\varepsilon}(t))) \le K$ for all $t \in \mathbb{N}$. Thus, if we choose a geodesic (called *space-like* segments) in $\Gamma_{\boldsymbol{x}}(G)$ from $\sigma_g(\psi_g(t))$ to $\sigma_{g\overline{x}^\varepsilon}(\psi_{g\bar{x}^\varepsilon}(t))$ then we have a subgraph of $\Gamma_{\boldsymbol{x}}(G)$ of the form as depicted in Fig. 4.1 which consists of some triangles and/or trapezoids.

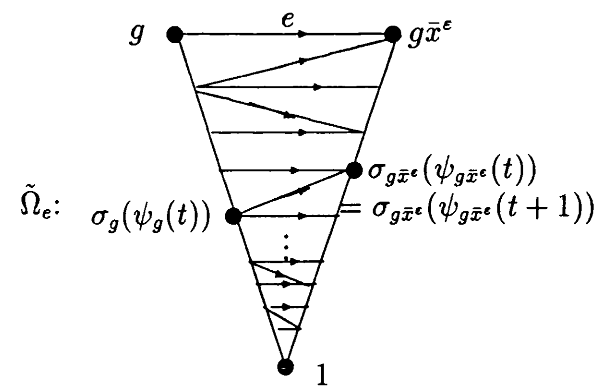


Fig. 4.1.

We will fix such a subgraph for e denoted $\tilde{\Omega}_e$. Let ρ be the boundary path of a triangle or trapezoid in $\tilde{\Omega}_e$. Then ρ has a length at most $2K+2$. Thus, if the projection of this path to $\mathcal{P}_1$ is U, i.e. $p_o(\rho) = U$ then $L(U) \leq 2(K+1)$, where p_o is the projection map as defined in §1.2.4.

Now, for any given word $W = x_1 \cdots x_n$ on $\boldsymbol{x}$ with $\overline{x_1 x_2 \cdots x_n} = 1$ in G and for any $g \in G$, we lift W to a closed path $t_g(W)$ in $\Gamma_{\boldsymbol{x}}(G)$. We can fill $t_g(W)$ in with some $\tilde{\Omega}_e$ for each edge e of $t_g(W)$ to obtain a planar subgraph of $\Gamma_{\boldsymbol{x}}(G)$ denoted $\tilde{\Lambda}_{W,g}$ as demonstrated in Fig. 4.2. By projecting $\tilde{\Lambda}_{W,g}$ to $\mathcal{P}_1$ we obtain a van Kampen diagram Λ_W (the images of space-like segments in $\tilde{\Lambda}_{W,g}$ under p_o will also be called the *space-like words* of Λ_W) for $p_o(t_g(W)) = W$ over $\mathcal{P}_1$ and this completes our proof. □

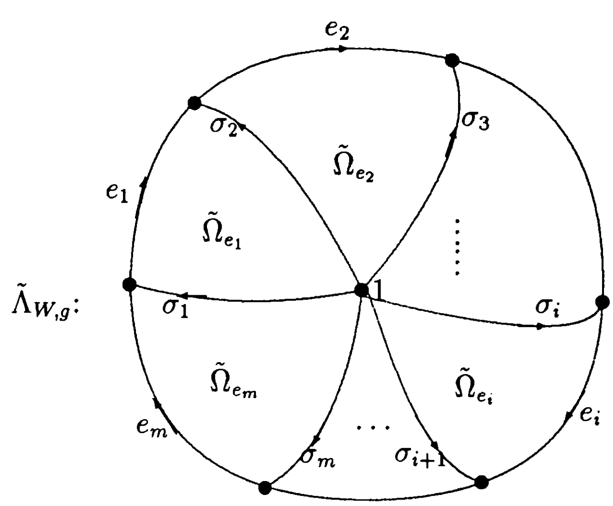


Fig. 4.2

We now further suppose that L_σ is bounded by an increasing function f and σ admits a departure function D.

For each $R \in \boldsymbol{r}_1$ and each $g \in G$, by lifting R at g in $\Gamma_{\boldsymbol{x}}(G)$ we obtain the universal covering $\tilde{\boldsymbol{\mathcal{P}}}_1$ of $\boldsymbol{\mathcal{P}}_1$. Let $\tilde{R}_g = e_1 e_2 \cdots e_m$ be a defining path of $\tilde{\boldsymbol{\mathcal{P}}}_1$, $R \in \boldsymbol{r}_1$, $g \in G$. Since $\iota(e_1) = g$, and since $m \le 2(K+2)$, it follows that $L(\sigma_{\iota(e_i)}) \le f(|g| + K + 1)$. By padding terms which are trivial paths to the ends of $\sigma_{\iota(e_i)}$'s if necessary, we may extend them to paths $\sigma_i = e_{i1} e_{i2} \cdots e_{ih}$ $(1 \le i \le m)$ with e_{ij}'s are empty paths for $j > L(\sigma_{\iota(e_i)})$ where $h = \max\{L(\sigma_{\iota(e_i)}) : 1 \le i \le m\} \le f(|g| + K + 1)$.

Consider $\tilde{\Lambda}_{R,g}$ (defined in the proof of Lemma 4.3.4). For each $1 \le j < h$, starting at $\tau(e_{1j})$ on σ_1 we trave along m space-like segments back to σ_1 at some $\tau(e_{1,T_j})$ as shown in Fig. 4.3. Denote this path by γ_j'. Then the length of γ_j' is at most mK. Obviously we have $T_{j-1} \le T_j$ by the monotonicity of each element of $\tilde{\Omega}$. We note that the diagrams $\Lambda_{R,g}$ $(R \in \boldsymbol{r}_1,\ g \in G)$ are spherical.

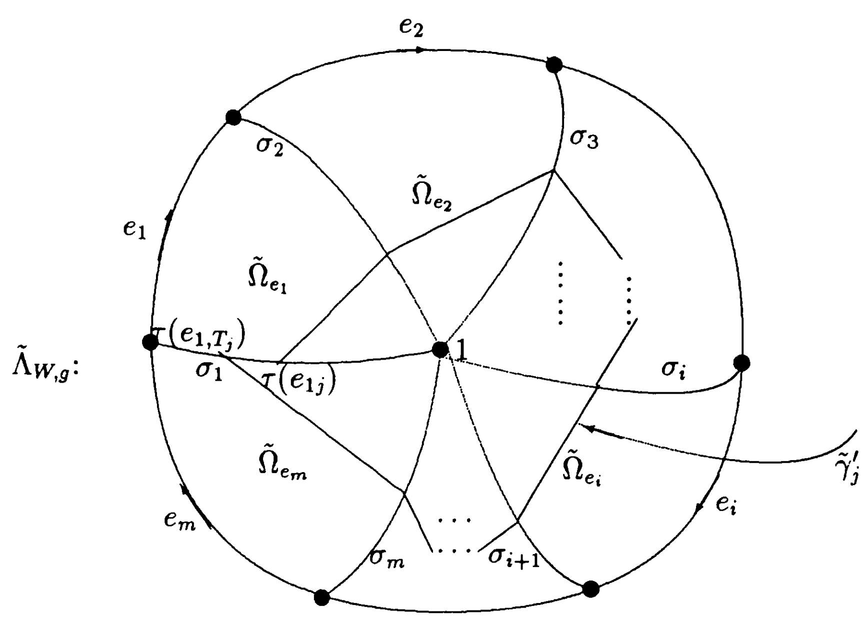


Fig. 4.3

We have the following properties.

(i) $|T_j - j| \le D(Km)$, $0 \le j \le h$;

(ii) If γ_j' and γ_{j-1}' meet σ_i at $\tau(e_{i,t_1})$ and $\tau(e_{i,t_2})$ respectively (clearly $t_1 \ge t_2$), then $t_1 - t_2 \le D(2Km + 1)$;

(iii) $|T_j - T_{j-1}| \leq D(2Km+1)$, $0 \leq j \leq h$.

Let γ_j be the path by adding the segment denoted ζ_j of σ_1 from $\tau(e_{1,T_j})$ to $\tau(e_{1,j})$ to γ'_j. Then $L(\gamma_j) \leq mK + D(mK)$ by (i). In addition, we require that γ_0 is the empty path at 1 and γ_h is $\tilde{R}_g$. Between γ_{j-1} and γ_j we have a *drum* of $\tilde{\Lambda}_{R,g}$ looking like (by cutting it along e_{1j})

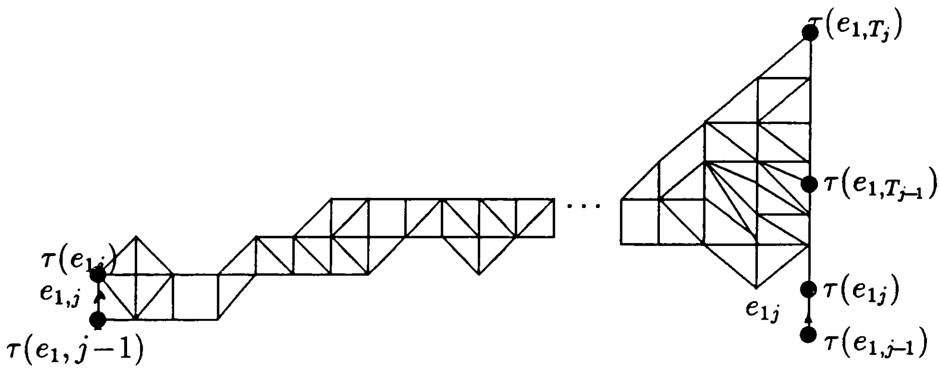


Fig. 4.4

consisting of at most $2mK \cdot D(2mK+1)$ those basic triangles and trapezoids. We then have a picture (the dualization of this drum) $\tilde{\mathbb{C}}^{(j)}_{R,g}$ over $\tilde{\mathcal{P}}$ containing at most $2mK \cdot D(2mK+1)$ discs and at most $4mK(K+1) \cdot D(2mK+1)$ arcs.

Since $L(\gamma_j) \leq mK + D(mK)$, we can choose a picture $\mathbb{A}^{(j)}_R$ over $\mathcal{P}_1$ with boundary label $p_0(\gamma_j)$ and $A(\mathbb{A}^{(j)}_R) \leq \delta^{(1)}_{\mathcal{P}_1}(mK + D(mK))$. We also can assume that the total number of arcs in $\mathbb{A}^{(j)}_R$ is at most $2(K+1)\delta^{(1)}_{\mathcal{P}_1}(mK + D(mK))$.

Let $\tilde{\mathbb{A}}^{(j)}_{R,g}$ be the lift of $\mathbb{A}^{(j)}_R$ at $\tau(e_{1j})$. Then $\tilde{\mathbb{A}}^{(j)}_{R,g}$ is a picture over $\tilde{\mathcal{P}}_1$ with boundary label γ_j. We then obtain a spherical picture $\tilde{\mathbb{B}}^{(j)}_{R,g}$ over $\tilde{\mathcal{P}}$ at $\iota(e_{1j})$ of the form as shown in Fig. 4.5 with

$$A(\tilde{\mathbb{B}}^{(j)}_{R,g}) \leq 2\delta^{(1)}_{\mathcal{P}_1}(mK + D(mK)) + 2mK \cdot D(2mK+1),$$

$$\# \text{ arcs of } \tilde{\mathbb{B}}^{(j)}_{R,g} \leq 4mK(K+1) \cdot D(2mK+1) + 4(K+1)\delta^{(1)}_{\mathcal{P}_1}(mK + D(mK)).$$

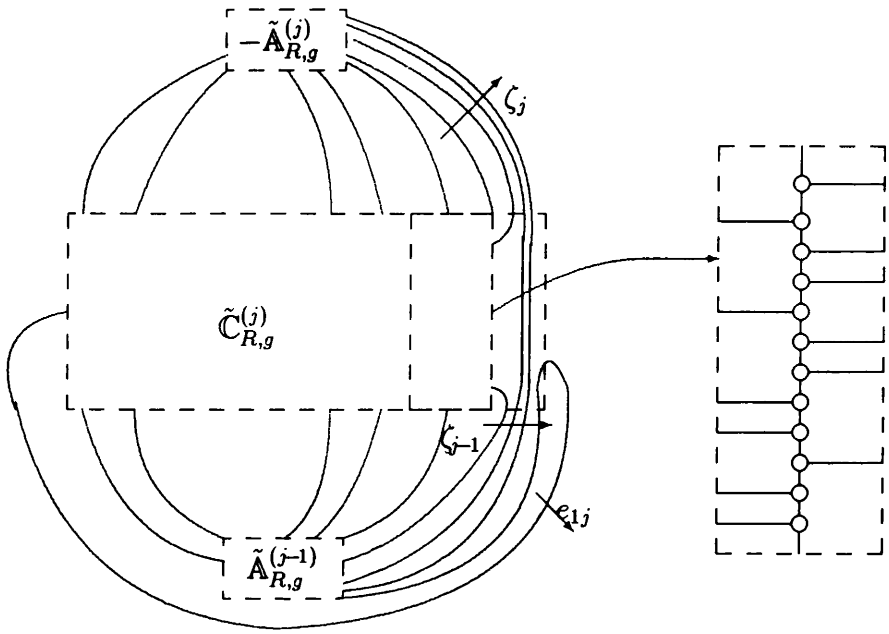


Fig. 4.5

Corresponding to $\tilde{\Lambda}_{R,g}$ we also have a spherical picture $\tilde{\mathbb{P}}_{R,g}$ which is of the form as demonstrated in Fig. 4.6.

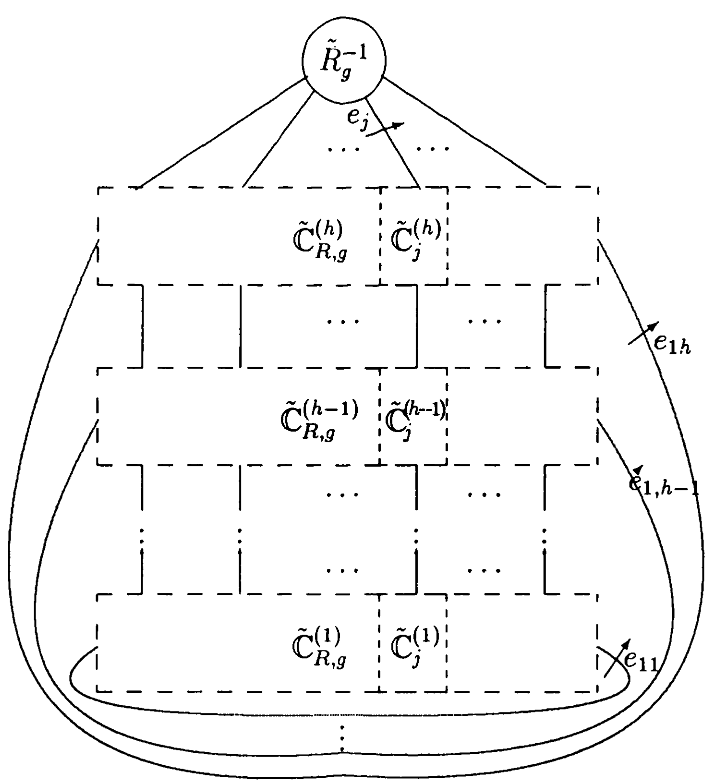


Fig. 4.6

The following lemma now is true.

Lemma 4.3.5 *We have*

$$\tilde{\mathbb{P}}_{R,g} \sim \sum_{j=1}^{h} \left(\tilde{\mathbb{B}}^{(j)}_{R,g}\right)^{e_{1,j+1}\cdots e_{1h}}.$$

Since the disc numbers of all spherical pictures $\tilde{\mathbb{B}}^{(j)}_{R,g}$ are bounded by $2\delta^{(1)}_{\mathcal{P}_1}(mK + D(mK)) + 2mK \cdot D(2mK+1)$ and the arc numbers of these pictures are bounded by $4mK(K+1) \cdot D(2mK+1) + 4(K+1)\delta^{(1)}_{\mathcal{P}_1}(mK + D(mK))$, if we let $\boldsymbol{X}$ be the set of all images of these pictures under the projection p_o then $\boldsymbol{X}$ is finite.

Theorem 4.3.6 *We have that*

(i) $\boldsymbol{X}$ *generates* $\pi_2(\mathcal{P}_1)$;

(ii) $\delta^{(2)}_G \preceq nf(n)$.

Proof. Let n be any positive integer, and let $\mathbb{P}$ be a minimal connected spherical picture over $\mathcal{P}_1$ with n discs $\Delta_1, \Delta_2, \cdots, \Delta_n$, labelled $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \cdots, R_n^{\varepsilon_n}$. Let the word $x_{i1}\cdots x_{i,j_i}$ $(1 \le i \le n)$ on $\boldsymbol{x}$ be the label of a minimal transverse path in $\mathbb{P}$ from the basepoint of $\mathbb{P}$ to the basepoint of Δ_i. Then $j_i \le 2(K+1)n$ by elementary graph theory. Let $\tilde{\mathbb{P}}_1$ be the lift of $\mathbb{P}$ at 1 in $\tilde{\mathcal{P}}$. Then the discs $\tilde{\Delta}_1, \cdots, \tilde{\Delta}_n$ of $\tilde{\mathbb{P}}_1$ are labelled by $\tilde{R}^{\varepsilon_1}_{1,g_1}, \cdots, \tilde{R}^{\varepsilon_n}_{n,g_n}$ where $g_i = \overline{x_{i1}\cdots x_{i,j_i}}$, $1 \le i \le n$. We convert $\tilde{\mathbb{P}}_1$ to a picture $\tilde{\mathbb{P}}_1'$ by replacing each $\tilde{\Delta}_i$ by the complement of the disc labelled $\tilde{R}^{-\varepsilon_i}_{i,g_i}$ in $\varepsilon_i \tilde{\mathbb{P}}_{R_i,g_i}$. Suppose that there is an arc labelled e_j connecting $\tilde{\Delta}_i$ and $\tilde{\Delta}_{i+1}$. Then in $\tilde{\mathbb{P}}_1'$ we see that all subpictures $\tilde{\mathbb{C}}^{(q)}_j$ $(1 \le q \le h)$ as shown in Fig. 4.6 will be cancelled as in $\tilde{\mathbb{P}}^{\varepsilon_{i+1}}_{R_{i+1},g_{i+1}}$ we have the same subpictures with opposite symbols since $\tilde{\Omega}_{e_j}$ is fixed. Thus, $\tilde{\mathbb{P}}'$ can be transformed to the empty picture by bridge moves and eliminations of cancelling pairs. Thus, by Lemma 1.3.4, for certain paths λ_i $(i = 1, \cdots, n)$

$$\langle \tilde{\mathbb{P}}_1 \rangle = \sum_{i=1}^{n} \varepsilon_i \langle \tilde{\mathbb{P}}^{\lambda_i}_{R_i,g_i} \rangle.$$

By Lemma 4.3.5, applying the projection p_o then gives an expression for $\langle \mathbb{P} \rangle$ involving at most $nf(2(K+1)n + K + 1)$ terms of $\boldsymbol{X}$.

Now let $\mathbb{P}$ be an arbitrary spherical picture $\mathbb{P}$ over $\mathcal{P}$ with n discs having nontrivial components $\mathbb{P}_1, \mathbb{P}_2, \cdots, \mathbb{P}_q$ with $n_1, n_2, \cdots, n_q$ discs respectively, where $n_1 + n_2 + \cdots + n_q = n$. Then there are words $U_1, U_2, \cdots, U_q$ on $\boldsymbol{x}$ such that

$$\langle \mathbb{P} \rangle = \sum_{i=1}^{q} \overline{U}_i \cdot \langle \mathbb{P}_i \rangle.$$

Thus, using the previous paragraph we get

$$\begin{aligned} V_{\mathcal{P}_1,\boldsymbol{X}}(\mathbb{P}) &\leq \sum_{i=1}^{q} n_i f(2(K+1)n_i + K + 1) \\ &\leq nf(2(K+1)n + K + 1) \end{aligned}$$

as required. □

If G is asynchronously automatic, then by the proof of Theorem 7.3.4 of [ECHLPT] the function f in the above theorem then can be taken as a simple exponential one. Furthermore, if G is automatic or semihyperbolic, then f can be taken as a linear one. Thus, we have

Corollary 4.3.7 *If G is an asynchronously automatic then $\delta_G^{(2)} \preceq 2^n$; and if G is synchronously automatic or semihyperbolic then $\delta_G^{(2)} \preceq n^2$.*

Example 4.3.3 (continued) We see that if $G_0 = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ ($\phi \in GL_n(\mathbb{Z})$ then $\delta_{G_0}^{(2)}$ is bounded by a polynomial function or an exponential function according to the absolute values of the eigenvalues of ϕ being 1 or not. We will discuss the situation when $n = 2$ in Chapter 7.

4.4 General combable groups

We now consider any arbitrary combable groups (without restriction on length of combing lines).

Theorem 4.4.1 *If G is (synchronously) combable then $\delta_G^{(2)} \preceq 2^n$.*

Proof. We follow the proofs of Lemmas 4.3.4, 4.3.5 and Theorem 4.3.6.

Suppose that σ is a combing of $\Gamma_{\boldsymbol{x}}(G)$ with (synchronously) K-fellow traveller property. Let $\mathcal{P}_1$ be a finite presentation as defined in Lemma 4.3.4 which obviously is a presentation for G.

Let n be any positive integer and let $\mathbb{P}$ be any connected minimal spherical picture over $\mathcal{P}_1$ with n discs $\Delta_1, \cdots, \Delta_n$ labelled by $R_1^{\varepsilon_1}, \cdots, R_n^{\varepsilon_n}$ respectively. Let the word

$x_{i1} \cdots x_{i,j_i}$ $(1 \leq i \leq n)$ on $\boldsymbol{x}$ be the label of a minimal transverse path in $\mathbb{P}$ from the basepoint of $\mathbb{P}$ to the basepoint of Δ_i. Then $j_i \leq 2(K+1)n$ by elementary graph theory. Now the discs $\tilde{\Delta}_1, \cdots, \tilde{\Delta}_n$ of $\tilde{\mathbb{P}}$ are labelled by $\tilde{R}^{\epsilon_1}_{1,g_1}, \cdots, \tilde{R}^{\epsilon_n}_{n,g_n}$ where $g_i = \overline{x_{i1} \cdots x_{i,j_i}}$, $1 \leq i \leq n$.

The point which we have to take extra care is about the number of drums in the spherical diagrams $\Lambda_{R_i^{\epsilon_i}}$, $i = 1, 2, \cdots, n$. By the proofs of Lemmas 4.3.4, 4.3.5 and Theorem 4.3.6, it suffices to show that we can uniformly reduce these diagrams to new ones such that the number of drums in each of the new diagrams is bounded by an exponential function with n. Here the term *uniformly* means that these reductions have no affect on the cancellations in the proof of Theorem 4.3.6.

Let P be the number of all possibilities of the event of dividing a path of integer length not bigger than $2K(K+1)$ into at most $2(K+1)$ segments of integer length not bigger than K. Then P is also a constant. Following the proof of Theorem 4.3.6, we consider the defining paths $\tilde{R}_{i,g_i}$ of $\tilde{\mathcal{P}}_1$, say

$$\tilde{R}_{i,g_i} = e_{i1}e_{i2}\cdots e_{i,m_i}, \qquad m_i \leq 2(K+1),\ 1 \leq i \leq n.$$

Let

$$h = \max\{L(\sigma_{\iota(e_{iq})}) : 1 \leq q \leq m_i,\ 1 \leq i \leq n,\}$$

and let σ_{iq} be the padded path obtained from $\sigma_{\iota(e_{iq})}$ written in h edges.

Consider the van Kampen diagrams $\Lambda_i = \Lambda_{R_i^{\epsilon_i}}$. If $h > (2P|\boldsymbol{x}|^{2K(K+1)})^n$, then there are at least $P^n((2|\boldsymbol{x}|)^{2K(K+1)})^{n-1}$ space-like words W_{1j}'s ($W_{ij} = p_o(\gamma_{ij})$) in Λ_1 are the same words on $\boldsymbol{x}$, and hence there are at least $P^{n-1}((2|\boldsymbol{x}|)^{2K(K+1)})^{n-1}$ of these words having the same division by the images of the combing lines in $\tilde{\Lambda}_1$ under p_o (note that all γ_{ij}'s are synchronously paths of lengths $\leq K(2K+1)$), say $W_{1,1_1}, \cdots, W_{1,1_{k_1}}$ for some positive integer $k_1 \geq P^{n-1}((2|\boldsymbol{x}|)^{2K(K+1)})^{n-1}$. Similarly, in Λ_2, among the space-like words $W_{2,1_1}, \cdots, W_{2,1_{k_1}}$ there are at least $P^{n-2}((2|\boldsymbol{x}|)^{2K(K+1)})^{n-2}$ of them are the same words on $\boldsymbol{x}$ and the same division by the images of the combing lines in $\tilde{\Lambda}_2$, say $W_{2,2_1}, \cdots, W_{2,2_{k_2}}$ for some positive integer $k_2 \geq P^{n-2}((2|\boldsymbol{x}|)^{2K(K+1)})^{n-2}$, and so on. Finally, there is a positive integer $k_n \geq P(2|\boldsymbol{x}|)^{2K(K+1)}$ such that for each $1 \leq i \leq n$

$$W_{i,i_1}, W_{i,i_2}, \cdots, W_{i,i_{k_n}}$$

are the the same words on $\boldsymbol{x}$ and have the same division by the images of the combing lines in $\tilde{\Lambda}_i$. We then can apply a surgery to cut off the drums of all Λ_i between each pair of these two paths and obtain n new diagrams Λ_i', $1 \le i \le n$. We repeat this procedure if necessary so that we can assume that $h \le (2P|\boldsymbol{x}|^{2K(K+1)})^n$. This is the desired result. $\square$

It seems difficult to use the above technique for asynchronously combable groups since in the proof of Theorem 4.3.6, the jth space-like segments of $\gamma_{i,q}$ and $\gamma_{i+1,q}$ may be not the same and therefore, this will have affect on the cancellations after the above uniform reduction.

Chapter 5

Calculations of second order Dehn functions of groups II: direct and free products

5.1 General bounds for direct and free products

In this section we let G_0, G_1 be groups of type F_3 finitely presented by $\boldsymbol{\mathcal{P}}_0 = \langle \boldsymbol{x}; \boldsymbol{r}_0 \rangle$, $\boldsymbol{\mathcal{P}}_1 = \langle \boldsymbol{t};\, \boldsymbol{r}_1 \rangle$ respectively, and let $\boldsymbol{X}_i$ be a finite set of generating pictures for $\pi_2(\boldsymbol{\mathcal{P}}_i)$, $(i = 0,\, 1)$.

A presentation for the direct product $G = G_0 \times G_1$ is given by

$$\boldsymbol{\mathcal{P}} = \langle \boldsymbol{x},\, \boldsymbol{t}\,;\, \boldsymbol{r}_0,\, \boldsymbol{r}_1,\, \boldsymbol{s} \rangle$$

where $\boldsymbol{s} = \{[x,t] : x \in \boldsymbol{x},\ t \in \boldsymbol{t}\}$. Since both G_0 and G_1 are retracts of G, by Corollary 2.2.15 we have

$$\delta_G^{(2)} \succeq \max\{\delta_{G_0}^{(2)},\, \delta_{G_1}^{(2)}\}. \tag{5.1}$$

If $\mathbb{D}$ is a picture over $\boldsymbol{\mathcal{P}}_0$ then for each $t \in \boldsymbol{t}$ we have a corresponding spherical picture $\mathbb{P}_{\mathbb{D},t}$ over $\boldsymbol{\mathcal{P}}$ as in Fig. 5.1, where the top oval labelled $-\mathbb{D}$ is the subpicture $-\mathbb{D}$, the mirror image of $\mathbb{D}$, the bottom oval is the subpicture $\mathbb{D}$, and the middle discs corresponds to the commutators. We let $\xi_{\mathbb{D},t}$ denote the element of $\pi_2(\boldsymbol{\mathcal{P}}_0)$ represented by $\mathbb{P}_{\mathbb{D},t}$. When $\mathbb{D}$ consists of a single disc labelled by some $R \in \boldsymbol{r}_0$ then we write $\mathbb{P}_{R,t}$ (resp. $\xi_{R,t}$) instead of $\mathbb{P}_{\mathbb{D},t}$ (resp. $\xi_{\mathbb{D},t}$). In similar way, for any $S \in \boldsymbol{r}_1$ and $x \in \boldsymbol{x}$ we have a spherical picture $\mathbb{P}_{S,x}$ over $\boldsymbol{\mathcal{P}}$ and a corresponding element $\xi_{S,x}$ of $\pi_2(\boldsymbol{\mathcal{P}})$.

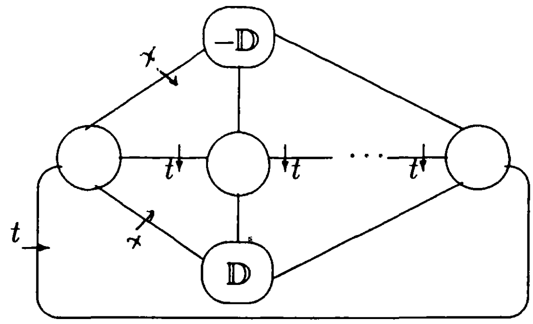


Fig. 5.1

Let W be the label on $\mathbb{D}$ where the discs of $\mathbb{D}$ are labelled $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \cdots, R_m^{\varepsilon_m}$. Choose a spray $(\gamma_1, \gamma_2, \cdots, \gamma_m)$ for $\mathbb{D}$ and let U_i be the label on γ_i, $(i = 1, 2, \cdots, m)$. By Remark 1.1.6, W is freely equal to

$$\prod_{i=1}^{m} U_i R_i^{\varepsilon_i} U_i^{-1}.$$

Moreover, let μ_2 be the standard embedding (see (1.7)). Then we have

$$\mu_2(\xi_{\mathbb{D},t}) = (\bar{t}-1)\sum_{i=1}^{m} \varepsilon_i \overline{U}_i e_{R_i} + \sum_{x\in\boldsymbol{x}} \frac{\partial^G W}{\partial x} e_{[x,t]}. \tag{5.2}$$

Lemma 5.1.1 *We have that*

$$\xi_{\mathbb{D},t} = \sum_{i=1}^{m} \varepsilon_i \overline{U}_i \xi_{R_i,t}.$$

Proof. Consider the image of $\xi_{\mathbb{D},t}$ under the embedding μ_2. Note that by the definition of Fox derivation (§1.3.1) we have

$$\begin{aligned}
&\frac{\partial^G \prod_{i=1}^m U_i R_i^{\varepsilon_i} U_i^{-1}}{\partial x} \\
&= \frac{\partial^G U_1 R_1^{\varepsilon_1} U_1^{-1}}{\partial x} + \overline{U_1 R_1^{\varepsilon_1} U_1^{-1}} \frac{\partial^G U_2 R_2^{\varepsilon_2} U_2^{-1}}{\partial x} + \cdots + \prod_{j=1}^{m-1} \overline{U_j R_j^{\varepsilon_j} U_j^{-1}} \frac{\partial^G U_m R_m^{\varepsilon_m} U_m^{-1}}{\partial x} \\
&= \sum_{i=1}^{m} \frac{\partial^G U_i R_i^{\varepsilon_i} U_i^{-1}}{\partial x} \qquad (\text{since } \overline{U_j R_j^{\varepsilon_j} U_j^{-1}} = 1,\ 1 \le j \le m-1) \\
&= \sum_{i=1}^{m} \left(\frac{\partial^G U_i}{\partial x} + \overline{U_i} \frac{\partial^G R_i^{\varepsilon_i}}{\partial x} - \overline{U_i R_i^{\varepsilon_i} U_i^{-1}} \frac{\partial^G U_i}{\partial x} \right) \\
&= \sum_{i=1}^{m} \varepsilon_i \overline{U_i} \frac{\partial^G R_i}{\partial x} \qquad (\text{since } \overline{U_i R_i^{\varepsilon_i} U_i^{-1}} = 1,\ 1 \le i \le m).
\end{aligned}$$

Hence,

$$\mu_2(\xi_{\mathbb{D},t}) = (\bar{t}-1)\sum_{i=1}^{m} \varepsilon_i \overline{U}_i e_{R_i} + \sum_{x\in\boldsymbol{x}} \frac{\partial^G W}{\partial x} e_{[x,t]}$$

$$= (\bar{t}-1)\sum_{i=1}^{m}\varepsilon_i\overline{U}_i e_{R_i} + \sum_{x\in\boldsymbol{x}}\frac{\partial^G \prod_{i=1}^m U_i R_i^{\varepsilon_i} U_i^{-1}}{\partial x} e_{[x,t]}$$

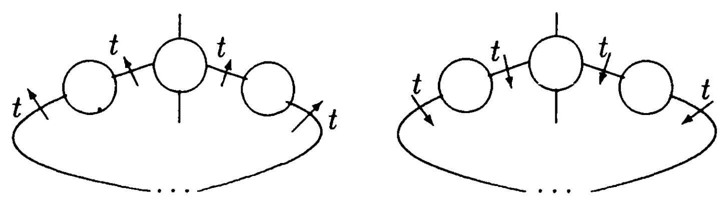
$$= (\bar{t}-1)\sum_{i=1}^{m}\varepsilon_i\overline{U}_i e_{R_i} + \sum_{x\in\boldsymbol{x}}\sum_{i=1}^{m}\varepsilon_i\overline{U}_i\frac{\partial^G R_i}{\partial x} e_{[x,t]}$$

$$= \sum_{i=1}^{m}\varepsilon_i\overline{U}_i\left((\bar{t}-1)e_{R_i} + \sum_{x\in\boldsymbol{x}}\frac{\partial^G R_i}{\partial x} e_{[x,t]}\right)$$

$$= \mu_2\left(\sum_{i=1}^{m}\varepsilon_i\overline{U}_i\xi_{R_i,t}\right)$$

and the lemma follows since μ_2 is injective. □

Let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$ and let $t \in \boldsymbol{t}$. A non-trivial *t-circle* (*outward directed or inward directed*) in $\mathbb{P}$ consists of a collection of $\boldsymbol{s}$-discs $\Delta_1, \cdots, \Delta_k$ and a collection of t-arcs $\alpha_1, \cdots, \alpha_k$ where α_i joins Δ_i and Δ_{i+1} (subscripts mod k).



outward directed t-circle   inward directed t-circle

Fig. 5.2

We also allow a *trivial* t-circle consisting of a single closed arc labelled t. A t-circle C is *minimal* if there is no t-circle contained in the region enclosed by C.

Let

$$\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{X}_1 \cup \boldsymbol{Y}_0 \cup \boldsymbol{Y}_1$$

where $\boldsymbol{Y}_0 = \{\mathbb{P}_{R,t} : R \in \boldsymbol{r}_0,\ t \in \boldsymbol{t}\}$, $\boldsymbol{Y}_1 = \{\mathbb{P}_{S,x} : S \in \boldsymbol{r}_1,\ x \in \boldsymbol{x}\}$.

Proposition 5.1.2 *We have*

(i) $\boldsymbol{X}$ *generates* $\pi_2(\mathcal{P})$;

(ii) *suppose that* $\delta^{(2)}_{\mathcal{P}_i,\boldsymbol{X}_i}(n) \le f(n)$ *for all* n $(i = 0,\ 1)$, *where* f *is subnegative. Let* ξ *be an element of* $\pi_2(\mathcal{P})$ *with* $A(\xi) = n$ *and let* $\mathbb{P}$ *be a minimal picture representing* ξ. *Then*

$$V_{\boldsymbol{X}}(\xi) \le f(n) + (a+1)n^2,$$

where $a = \max\{L(R);\ R \in \boldsymbol{r}_1\}$ *(take* $a = 0$ *if* $\boldsymbol{r}_1$ *is empty).*

From this and (5.1) we immediately deduce the following.

Theorem 5.1.3 *We have*

$$\max\{\delta^{(2)}_{G_0},\ \delta^{(2)}_{G_1}\} \preceq \delta^{(2)}_{G_0\times G_1} \preceq \max\{\overline{\delta}^{(2)}_{G_0},\ \overline{\delta}^{(2)}_{G_1}\} + n^2.$$
$$\{\delta^{(2)}_{G_0},\ \delta^{(2)}_{G_1}\} \preceq \delta^{(2)}_{G_0\times G_1} \preceq \max\{\overline{\delta}^{(2)}_{G_0},\ \overline{\delta}^{(2)}_{G_1}\} + n^2$$

Proof of Proposition 5.1.2. We will concentrate on proving (ii), as (i) has already been proved in [BHP]. In fact, our discussion below amounts to an analysis of the proof in [BHP] to get the required estimate for $\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}$.

We let $\boldsymbol{\mathcal{P}}^* = \langle \boldsymbol{x},\ \boldsymbol{t};\ \boldsymbol{r}_0,\ \boldsymbol{s}\rangle$.

Let n be any positive integer and let ξ be an element of $\pi_2(\boldsymbol{\mathcal{P}})$ with $A(\xi) \leq n$. Let $\mathbb{P}$ be a minimal picture representing ξ, and let n_0, n_1, m be the numbers of $\boldsymbol{r}_0-$, $\boldsymbol{r}_1-$, $\boldsymbol{s}$-discs in $\mathbb{P}$ respectively.

Let $\mathbb{P}^{(1)}$ be the configuration obtained from $\mathbb{P}$ by removing all $\boldsymbol{x}$-arcs. Two $\boldsymbol{r}_1$-discs of $\mathbb{P}$ will be said to be in the same *1-component* of $\mathbb{P}$ if they lie in the same component of $\mathbb{P}^{(1)}$. If Δ, Δ' are two $\boldsymbol{r}_1$-discs lying in the same 1-component then they can be connected by a path ρ of $\boldsymbol{t}$-arcs and $(\boldsymbol{r}_1 \cup \boldsymbol{s})$-discs. In fact, if we regard $\mathbb{P}^{(1)}$ as a graph, where the edges are the arcs and the vertices are the discs, then ρ is just a path in this graph. It will be assumed that a maximal forest Φ in $\mathbb{P}^{(1)}$ has been chosen, and that the paths connecting $\boldsymbol{r}_1$-discs are geodesics in Φ.

Consider a 1-component Ω of $\mathbb{P}$ containing $\boldsymbol{r}_1$-discs. Let $\Delta_0,\ \Delta_1,\ \cdots,\ \Delta_k$ be the $\boldsymbol{r}_1$-discs in this 1-component and let ρ_λ $(\lambda = 0,\ 1,\ \cdots,\ k)$ be the (geodesic) path in Φ from Δ_0 to Δ_λ. Let d_λ be the number of $\boldsymbol{s}$-discs in ρ_λ. We may assume that

$$0 = d_0 = d_1 = \cdots = d_j \leq d_{j+1} \leq \cdots \leq d_k.$$

We will show that we can modify $\mathbb{P}$ modulo $\boldsymbol{Y}_1$-pictures so that all the d_λ's are 0.

Suppose $j < k$ (otherwise no modifications are necessary) and consider Δ_{j+1}. Then the discs of ρ_{j+1} together with their incident arcs give a subpicture $\mathbb{Q}$ of $\mathbb{P}$, which has the form as shown in Fig. 5.3 where the disc Θ is an $\boldsymbol{s}$-disc.

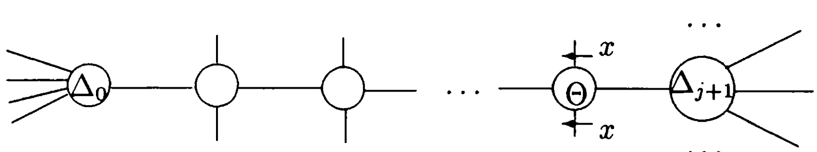


Fig. 5.3

We then have a $\boldsymbol{Y}_1$-picture $\mathbb{P}_{S,x}$ such that $\mathbb{P}_{S,x}$ (or $\mathbb{P}_{S,x}^{-1}$) is of the form as depicted in Fig. 5.4.

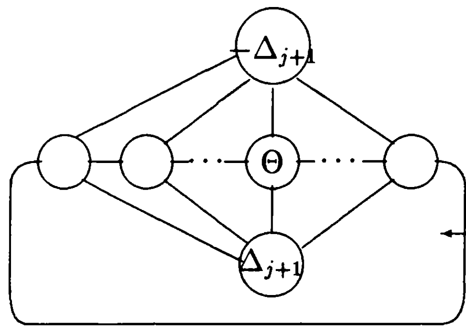


Fig. 5.4

Modulo the $\boldsymbol{Y}_1$-picture $\mathbb{P}_{S,x}$ we may move Δ_{j+1} nearer Δ_0 as indicated in Fig. 5.5. This gives a new picture $\mathbb{P}'$. A maximal forest Φ' for $\mathbb{P}'$ arises from the maximal forest Φ of $\mathbb{P}^{(1)}$ as follows. Remove all $\boldsymbol{x}$-arcs of $\mathbb{P}'$ to obtain $\mathbb{P}'^{(1)}$. Since the above operation has affect only on the 1-component Ω of $\mathbb{P}$, $\mathbb{P}'^{(1)}$ consists of all 1-components of $\mathbb{P}^{(1)}$ which are not Ω and a new 1-component Ω' obtained from Ω by the above operation. If T is the chosen maximal tree of Ω, then by the above operation, T is transformed to a maximal tree T' of Ω'. Then $\Phi' = (\Phi - \{T\}) \cup \{T'\}$. Note that this operation also does not affect the distances from Δ_0 to Δ_λ $(j+1 \le \lambda \le k)$. We then get new geodesics ρ'_λ $(\lambda = 0,\ 1,\ ,\cdots,\ k)$ with

$$d'_\lambda = 0\ (0 \le \lambda \le j),\ d'_{j+1} = d_{j+1} - 1,\ d'_\lambda = d_\lambda\ (j+1 < \lambda \le k).$$

We point out that this operation adds less than a new $\boldsymbol{s}$-discs to $\mathbb{P}$.

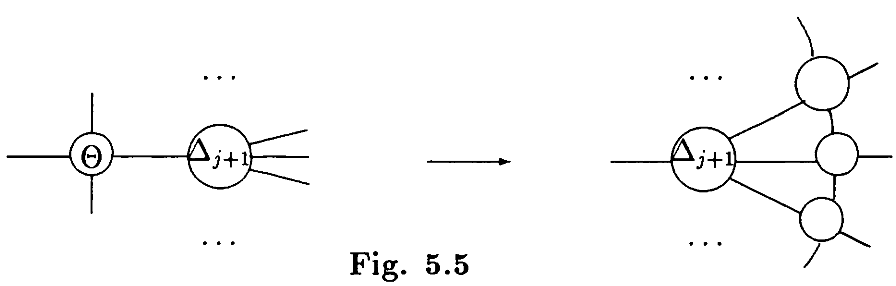


Fig. 5.5

We repeat the above procedure as often as is necessary to decrease d_{j+1} to 0. Note that this requires at most m operations. We then repeat the process successively for $\Delta_{j+2},\ \cdots,\ \Delta_k$, finally arriving (after at most mk operations) at a picture $\mathbb{P}_1$. Now in $\mathbb{P}_1$ there will be a simple closed transverse path α such that the subpicture of $\mathbb{P}_1$ enclosed by

α consists precisely of the discs Δ_0, Δ_1, $\cdots$, Δ_k and their incident arcs. Thus, every arc say β, crossing α is the start of a path consisting of non-$\boldsymbol{r}_1$-discs and $\boldsymbol{t}$-arcs (having the same label as β) in the exterior of α, and therefore, eventually recrossing α as illustrated in Fig. 5.6. Note that no two such paths can cross in the exterior of α since they contain no $\boldsymbol{r}_1$-discs. This establishes that the label on α is freely equivalent to the empty word, and so by bridge moves we can create a spherical picture $\mathbb{Q}_1$ over $\boldsymbol{\mathcal{P}}_1$ inside α with discs Δ_0, Δ_1, $\cdots$, Δ_k. Note that passing from $\mathbb{P}$ to $\mathbb{P}_1$ we create no new $\boldsymbol{t}$-circles. Since α encloses k $\boldsymbol{r}_1$-discs, the number of arcs which can intersect α is at most ak. Thus when we perform bridge moves to create the spherical subpicture $\mathbb{Q}_1$ inside α we can create at most ak new $\boldsymbol{t}$-circles.

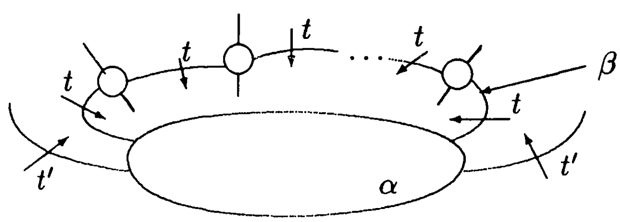


Fig. 5.6

We may carry out the above procedure for all the 1-components of $\mathbb{P}$ arriving (after at most mn_1 operations) at a picture $\mathbb{P}^*$ with the following properties:

(i) *$\mathbb{P}^*$ has spherical subpictures $\mathbb{B}_1$, $\mathbb{B}_2$, $\cdots$, $\mathbb{B}_q$ each of which is a picture over $\boldsymbol{\mathcal{P}}_1$, and where the total number of discs in $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \cdots \cup \mathbb{B}_q$ is n_1;*

(ii) *The picture $\mathbb{P}^{**}$ obtained from $\mathbb{P}^*$ by removing all $\mathbb{B}_1$, $\cdots$, $\mathbb{B}_q$ is a picture over $\boldsymbol{\mathcal{P}}^*$ having n_0 $\boldsymbol{r}_0$-discs, at most $amn_1 + m$ $\boldsymbol{s}$-discs, and at most $m + an_1$ non-trivial $\boldsymbol{t}$-circles (at most m $\boldsymbol{t}$-circles coming from the original $\boldsymbol{s}$-discs plus at most an_1 new $\boldsymbol{t}$-circles).*

Let $\xi^{**} = \langle \mathbb{P}^{**} \rangle$. We deduce from Lemma 1.3.4 that

$$V_{\boldsymbol{X}}(\xi) \leq V_{\boldsymbol{X}}(\xi^{**}) + f(n_1) + mn_1.$$

Now from the lemma below we get $V_{\boldsymbol{X}}(\xi^{**}) \leq f(n_0) + (m + an_1)n_0$. Thus

$$\begin{aligned} V_{\boldsymbol{X}}(\xi) &\leq f(n_0) + (m + an_1)n_0 + f(n_1) + mn_1 \\ &\leq (f(n_0) + f(n_1)) + an_0n_1 + m(n_0 + n_1) \\ &\leq f(n) + (a+1)n^2 \end{aligned}$$

as required. □

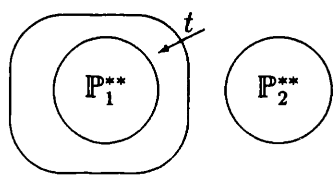
Lemma 5.1.4 *Let* $\mathbb{P}^{**}$ *be a picture over* $\mathcal{P}^*$ *with* n_0 $\boldsymbol{r}_0$*-discs and* q *non-trivial* $\boldsymbol{t}$*-circles. Then*

$$V_{\boldsymbol{X}}(\mathbb{P}^{**}) \leq f(n_0) + qn_0.$$

Proof. By induction on the total number of $\boldsymbol{t}$-circles in $\mathbb{P}^{**}$.

If $\mathbb{P}^{**}$ has no $\boldsymbol{t}$-circles then $\mathbb{P}^{**}$ is a picture over $\mathcal{P}_0$ and so $V_{\boldsymbol{X}}(\mathbb{P}^{**}) \leq f(n_0)$.

Suppose $\mathbb{P}^{**}$ has a trivial $\boldsymbol{t}$-circle. Then $\mathbb{P}^{**}$ has the form



where $\mathbb{P}_1^{**}$ and $\mathbb{P}_2^{**}$ are spherical pictures over $\mathcal{P}^*$.

Let $n_0^{(1)}$, $n_0^{(2)}$ be the numbers of $\boldsymbol{r}_0$-discs and q_1, q_2 be the numbers of non-trivial $\boldsymbol{t}$-circles in $\mathbb{P}_1^{**}$ and $\mathbb{P}_2^{**}$, respectively. Then (using induction and Lemma 1.3.4)

$$\begin{aligned} V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}^{**}) &\leq V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}_1^{**}) + V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}_2^{**}) \\ &\leq f(n_0^{(1)}) + f(n_0^{(2)}) + q_1 n_0^{(1)} + q_2 n_0^{(2)} \\ &\leq f(n_0) + qn_0. \end{aligned}$$

Now, suppose $\mathbb{P}^{**}$ has no trivial $\boldsymbol{t}$-circles but has non-trivial $\boldsymbol{t}$-circles. Choose a minimal $\boldsymbol{t}$-circle, say C, as indicated in Fig. 5.7. We then can modify $\mathbb{P}^{**}$ into two new spherical pictures $\mathbb{P}_0$ and $\mathbb{P}_{\mathbb{D},t}^{W}$ for some word W on $\boldsymbol{x} \cup \boldsymbol{t}$ also as shown in Fig. 5.7.

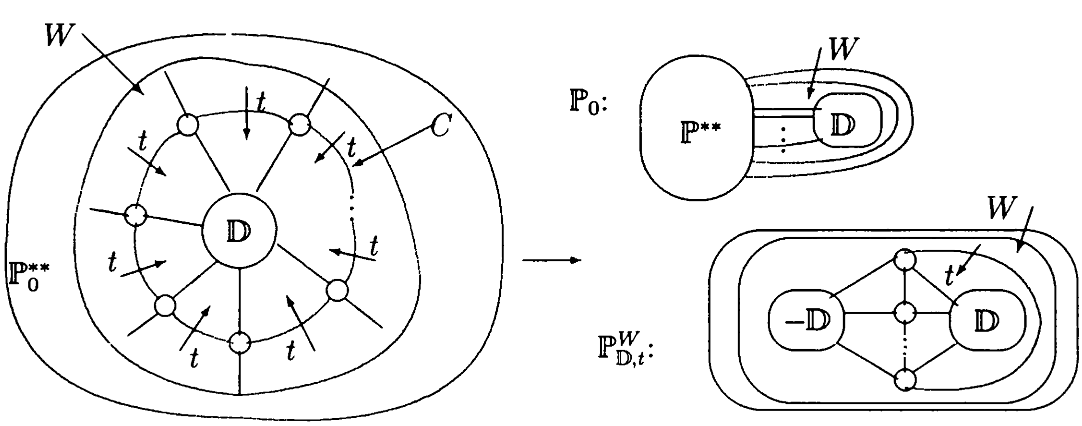


Fig. 5.7

By Lemma 1.3.4 we then have $V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}^{**}) \leq V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}_0) + V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}_{\mathbb{D},t})$. By Lemma 5.1.1, $V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}_{\mathbb{D},t}) \leq A(\mathbb{D}) \leq n_0$. Note that $\mathbb{P}_0$ contains n_0 $\boldsymbol{r}_0$-discs and $q-1$ t-circles. Thus, by induction hypothesis we have

$$\begin{aligned} V_{\boldsymbol{X}_0 \cup \boldsymbol{Y}_0}(\mathbb{P}^{**}) &\leq (f(n_0) + (q-1)n_0) + n_0 \\ &= f(n_0) + qn_0 \end{aligned}$$

as required. □

We now consider the free product $G = G_0 * G_1$. A presentation for G is given by $\mathcal{P} = \langle \boldsymbol{x},\ \boldsymbol{t};\ \boldsymbol{r}_0,\ \boldsymbol{r}_1 \rangle$.

Note that G_0 and G_1 are retracts of G. Thus, by simplifying the proof of Proposition 5.1.2 (there are no $\boldsymbol{s}$-discs) or a direct proof we can obtain the following theorem.

Theorem 5.1.5 *Let G_0, G_1 are two groups of type F_3. Then*

$$\max\{\delta^{(2)}_{G_0},\ \delta^{(2)}_{G_1}\} \preceq \delta^{(2)}_{G_0 * G_1} \preceq \max\{\overline{\delta}^{(2)}_{G_0},\ \overline{\delta}^{(2)}_{G_1}\}.$$

5.2 Some exact calculations for direct products

Our aim in this section is to give some exact calculations for $\delta^{(2)}_G$ with $G = G_0 \times F$ where F is *free* (of finite rank) and G_0 is a group of type F_3 with $\delta^{(2)}_{G_0}$ linear.

A presentation for G is given by

$$\mathcal{P} = \langle \boldsymbol{x},\ \boldsymbol{t};\ \boldsymbol{r}_0,\ \boldsymbol{s} \rangle$$

where $\mathcal{P}_0 = \langle \boldsymbol{x};\ \boldsymbol{r}_0 \rangle$ is a finite presentation for G_0, $\boldsymbol{t}$ is a set with $|\boldsymbol{t}|$ =rank(F), $\boldsymbol{s} = \{[x,t]\ :\ x \in \boldsymbol{x},\ t \in \boldsymbol{t}\}$. We let $\boldsymbol{X}_0$ be a finite set of spherical pictures which generates $\pi_2(\mathcal{P}_0)$ and let

$$\boldsymbol{X} = \boldsymbol{X}_0 \cup \{\mathbb{P}_{R,t}\ :\ R \in \boldsymbol{r}_0,\ t \in \boldsymbol{t}\}.$$

Then $\boldsymbol{X}$ generates $\pi_2(\mathcal{P})$.

We also require the following notation. Let

$$h : [1, \infty) \longrightarrow \mathbb{R}^+$$

be a strictly increasing continuous function such that

(i) *the restriction of* h *on* $\mathbb{N}$ *is subnegative;*

(ii) $h(x) \geq x$ *for all* $x \in [1, \infty)$*; and*

(iii) *the function* $x \longmapsto \frac{x}{h^{-1}(x)}$ *is increasing for* $x > n_0$ *for some natural number* $n_0 \in \mathrm{Im} h$.

Note that by (iii) we have $\frac{x}{h^{-1}(x)} \geq 1$ for $x > n_0$. This fact will be used in the following subsection without a further comment.

Throughout the reminder of this section the above notations will remain fixed.

5.2.1 Upper bounds for $\delta^{(2)}_{G_0 \times F}$ when $\delta^{(2)}_{G_0}$ is linear

Theorem 5.2.1 *If* $\delta^{(2)}_{G_0}$ *is linear and* $\delta^{(1)}_{\mathcal{P}_0}(n) \leq bh(n)$ *for all natural number* n *and some integer constant* $b > 1$*, then*

$$\delta^{(2)}_{\mathcal{P}, \boldsymbol{X}}(n) \leq \frac{3cn^2}{h^{-1}(n)},$$

for all natural number $n > n_0$ *and some constant* c.

Proof. Since $\delta^{(2)}_{G_0}$ is linear we can assume that $\delta^{(2)}_{\mathcal{P}_0, \boldsymbol{X}_0}(n) \leq bn$ with the same constant b as above. Let c_0 be the maximum volume of all elements ξ of $\pi_2(\mathcal{P})$ with $A(\xi) \leq n_0$, and let

$$c = \max\{c_0,\ b,\ h(n_0)\}.$$

Let $n > n_0$ be any natural number and let ξ be an element of $\pi_2(\mathcal{P})$ with $A(\xi) = n$. Let $\mathbb{P}$ be a minimal spherical picture over $\mathcal{P}$ representing ξ. By induction on n we argue that

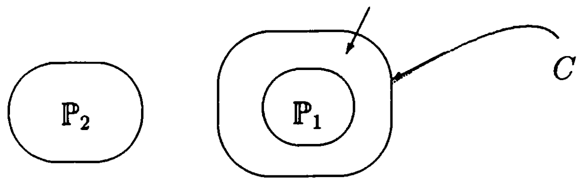
$$V_{\boldsymbol{X}}(\xi) \leq \frac{3cn^2}{h^{-1}(n)}.$$

Case 1. $\mathbb{P} = W \cdot \mathbb{P}_0$ *where* $\mathbb{P}_0$ *is a picture over* $\mathcal{P}_0$, W *is a word on* $\boldsymbol{x} \cup \boldsymbol{t}$.

We then have $V_{\boldsymbol{X}}(\xi) \leq bn \leq cn \leq \frac{3cn^2}{h^{-1}(n)}$.

Case 2. $\mathbb{P}$ *is not as in Case 1;* $\mathbb{P}$ *contains a trivial* t*-circle* C *for some* $t \in \boldsymbol{t}$.

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the subpictures of $\mathbb{P}$ lying *just* inside and outside C, respectively. Since C is trivial, both $\mathbb{P}_1$ and $\mathbb{P}_2$ are spherical. Hence, $\mathbb{P}$ has the form:



Thus, if we let $\xi_i = \langle \mathbb{P}_i \rangle$ $(i = 1,\ 2)$, then by Lemma 1.3.4 we have

$$V_{\boldsymbol{X}}(\xi) \leq V_{\boldsymbol{X}}(\xi_1) + V_{\boldsymbol{X}}(\xi_2).$$

Let $n_i = A(\xi_i)$, $i = 1,\ 2$. We distinguish three subcases.

Subcase 2.1. $A(\xi_i) = n_i \leq n_0,\ i = 1,\ 2.$

Since $\frac{n^2}{h^{-1}(n)} \geq 1$, we have

$$V_{\boldsymbol{X}}(\xi) \leq V_{\boldsymbol{X}}(\xi_1) + V_{\boldsymbol{X}}(\xi_2) \leq 2c \leq \frac{3cn^2}{h^{-1}(n)}.$$

Subcase 2.2. $A(\xi_1) > n_0,\ A(\xi_2) \leq n_0$ (similarly $A(\xi_2) > n_0,\ A(\xi_1) \leq n_0$).

Then

$$\begin{aligned}
V(\xi) &\leq V(\xi_1) + V(\xi_2) \\
&\leq \frac{3cn_1^2}{h^{-1}(n_1)} + c \qquad \text{(by induction hypothesis)} \\
&\leq 3c\left(\frac{n_1^2}{h^{-1}(n_1)} + 1\right) \\
&\leq 3c\left(\frac{nn_1}{h^{-1}(n)} + 1\right) \qquad \left(\text{since } \frac{n}{h^{-1}(n)} \geq \frac{n_1}{h^{-1}(n_1)}\right) \\
&= \frac{3cn^2}{h^{-1}(n)}\left(\frac{n_1}{n} + \frac{h^{-1}(n)}{n^2}\right) \\
&\leq \frac{3cn^2}{h^{-1}(n)} \qquad \left(\text{since } n_1 + \frac{h^{-1}(n)}{n} \leq n_1 + 1 \leq n\right).
\end{aligned}$$

Subcase 2.3. $A(\xi_i) > n_0,\ i = 1,\ 2.$

Then

$$\begin{aligned}
V_{\boldsymbol{X}}(\xi) &\leq V_{\boldsymbol{X}}(\xi_1) + V_{\boldsymbol{X}}(\xi_2) \\
&\leq \frac{3cn_1^2}{h^{-1}(n_1)} + \frac{3cn_2^2}{h^{-1}(n_2)} \qquad \text{(by induction hypothesis)} \\
&\leq 3c\left(\frac{n_1(n_1+n_2)}{h^{-1}(n_1+n_2)} + \frac{n_2(n_1+n_2)}{h^{-1}(n_1+n_2)}\right)
\end{aligned}$$

$$(\text{since } \frac{n_1+n_2}{h^{-1}(n_1+n_2)} \geq \frac{n_i}{h^{-1}(n_i)},\ i=1,2)$$

$$= \frac{3cn^2}{h^{-1}(n)}.$$

Case 3. $\mathbb{P}$ is not as in Case 1 and Case 2.

Then $\mathbb{P}$ contains nontrivial $\boldsymbol{t}$-circles. we pick a minimal one, say C. Suppose C contains q discs. Let $\mathbb{D}$ and $\mathbb{Q}$ be the subpictures of $\mathbb{P}$ lying *just* inside and outside C. Then $\mathbb{D}$ is a picture over $\boldsymbol{\mathcal{P}}_0$. We then can modify $\mathbb{P}$ into two spherical subpictures $\mathbb{P}'$ and $\mathbb{P}^W_{\mathbb{D},t}$ for some word W on $\boldsymbol{x} \cup \boldsymbol{t}$ as shown in Fig. 5.8, where the picture $\mathbb{P}_{\mathbb{D},t}$ is as defined in Fig. 5.1. We have $A(\mathbb{P}') = n - q$.

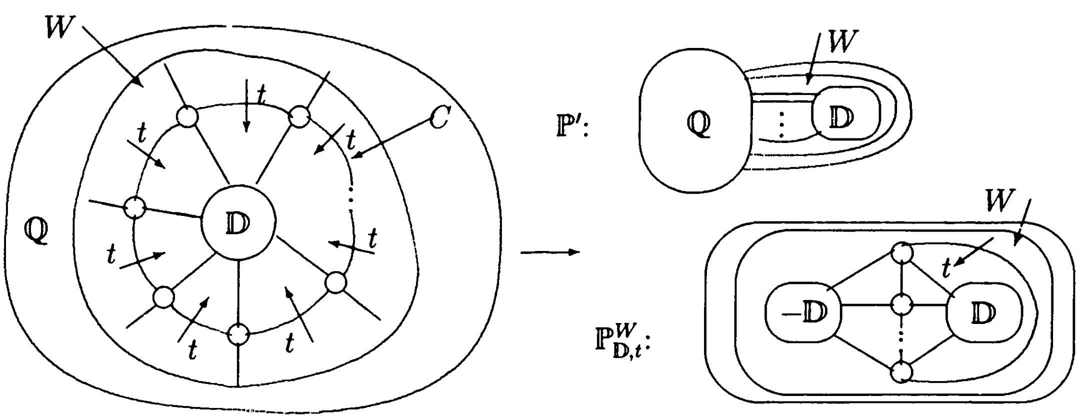


Fig. 5.8

Let $\xi' = \langle \mathbb{P}' \rangle$. Then by by Lemma 1.3.4,

$$\xi = \xi' + \varepsilon g \xi_{\mathbb{D},t} \tag{5.3}$$

where g is the element of G represented by W, ε is $+1$ or -1 according as to whether C is inward directed or outward directed.

We let m denote the number of discs in $\mathbb{D}$, and we distinguish three subcases.

Subcase 3.1. $(m \leq) n - q \leq n_0$.

Then

$$\begin{aligned} V_{\boldsymbol{X}}(\xi) &\leq c + (n-q) \quad \text{(using (5.3) and Lemma 5.1.1)} \\ &\leq c + n_0 \\ &< c(1+n_0) \quad (\text{since } c > 1) \\ &\leq cn \\ &\leq \frac{3cn^2}{h^{-1}(n)}. \end{aligned}$$

Subcase 3.2. $n - q > n_0$, and $2bh(q) > m$.

Then

$$\begin{aligned}
V_{\mathbf{X}}(\xi) &\leq \frac{3c(n-q)^2}{h^{-1}(n-q)} + m \\
&\quad \text{(using (5.3), the induction hypotheses and Lemma 5.1.1)} \\
&\leq \frac{3cn(n-q)}{h^{-1}(n)} + m \quad (\text{since } \tfrac{n-q}{h^{-1}(n-q)} \leq \tfrac{n}{h^{-1}(n)}) \\
&= \frac{3cn^2}{h^{-1}(n)}\left(1 - \frac{q}{n} + \frac{mh^{-1}(n)}{3cn^2}\right).
\end{aligned}$$

Now, if $m \leq n_0$, then $m \leq n_0 \leq h(n_0) \leq c$. Hence,

$$\begin{aligned}
1 - \frac{q}{n} + \frac{mh^{-1}(n)}{3cn^2} &\leq 1 - \frac{q}{n} + \frac{h^{-1}(n)}{3n^2} \\
&\leq 1 - \frac{q}{n} + \frac{1}{3n} \quad (\text{since } \tfrac{h^{-1}(n)}{n} \leq 1) \\
&\leq 1,
\end{aligned}$$

and so

$$V_{\mathbf{X}}(\xi) \leq \frac{3cn^2}{h^{-1}(n)}.$$

Suppose $m > n_0$. By the subnegativity of h, $m < 2bh(q) < h(3bq) \leq h(3cq)$, so $h^{-1}(m) < 3cq$. Hence,

$$\begin{aligned}
1 - \frac{q}{n} + \frac{mh^{-1}(n)}{3cn^2} &\leq 1 - \frac{q}{n} + \frac{mh^{-1}(m)}{3cnm} \quad (\text{since } \tfrac{h^{-1}(n)}{n} \leq \tfrac{h^{-1}(m)}{m}) \\
&\leq 1 - \frac{q}{n} + \frac{3cq}{3cn} \\
&= 1.
\end{aligned}$$

Thus we also have

$$V_{\mathbf{X}}(\xi) \leq \frac{3cn^2}{h^{-1}(n)}.$$

Subcase 3.3. $n - q > n_0$, and $2bh(q) \leq m$.

We have a picture $\mathbb{D}'$ over $\mathcal{P}_0$ with at most $h(q)$ discs and having the same boundary label as $\mathbb{D}$. Then the spherical picture $\mathbb{B}$ obtained by putting together a copy of $\mathbb{D}$ and a copy of $-\mathbb{D}'$ represents an element of $\pi_2(\mathcal{P}_0)$ containing at most $m + bh(q)$ discs. Thus, $\mathbb{P}$ can be modified into two spherical pictures $\mathbb{P}''$ and $\mathbb{B}^W$ for some word W on $\boldsymbol{x} \cup \boldsymbol{t}$ as shown in Fig. 5.9.

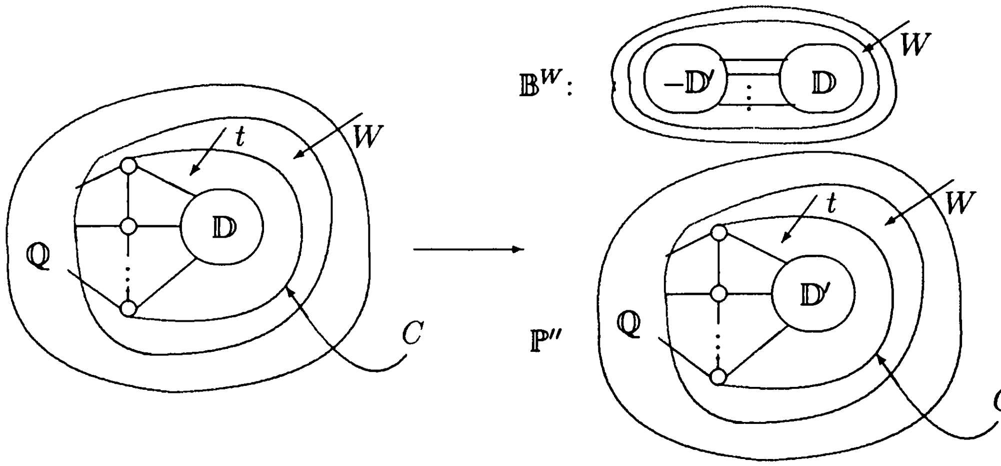


Fig. 5.9

Let $\xi'' = \langle \mathbb{P}'' \rangle$, $\xi^* = \langle \mathbb{B} \rangle$. We have

$$\begin{aligned} V_{\mathcal{P},\boldsymbol{X}}(\xi) &\leq V_{\mathcal{P},\boldsymbol{X}}(\xi'') + V_{\mathcal{P}_0,\boldsymbol{X}_0}(\xi^*) \quad \text{(by Lemma 1.3.4)} \\ &\leq V_{\mathcal{P},\boldsymbol{X}}(\xi'') + b(m + bh(q)). \end{aligned}$$

Since $2bh(q) \leq m$ and $b \leq c$, $b(m + bh(q)) \leq 2cm$. Thus, if $n - m + bh(q) \leq n_0$, then $V_{\mathcal{P},\boldsymbol{X}}(\xi'') \leq c_0 \leq c$ and so

$$V_{\boldsymbol{X}}(\xi) \leq c_0 + b(m + bh(q)) \leq 3cm \leq \frac{3cn^2}{h^{-1}(n)}.$$

If $n - m + bh(q) > n_0$, then by induction hypothesis $V_{\mathcal{P},\boldsymbol{X}}(\xi'') \leq \frac{3c(n-m+bh(q))^2}{h^{-1}(n-m+bh(q))}$ and so

$$\begin{aligned} V_{\boldsymbol{X}}(\xi) &\leq \frac{3c(n - m + bh(q))^2}{h^{-1}(n - m + bh(q))} + b(m + bh(q)) \\ &= \frac{3cn^2}{h^{-1}(n)} \cdot \frac{(n - m + bh(q))/(h^{-1}(n - m + bh(q))}{n/h^{-1}(n)} \left(1 - \frac{m - bh(q)}{n} + \right. \\ &\qquad \left. \frac{bh^{-1}(n - m + bh(q))(m + bh(q))}{3cn(n - m + bh(q))}\right). \end{aligned}$$

Since $\frac{x}{h^{-1}(x)}$ is increasing for $x > n_0$, we then have

$$\frac{(n - m + bh(q))/h^{-1}(n - m + bh(q))}{n/h^{-1}(n)} \leq 1.$$

Now $2bh(q) \leq m$, thus $(m + bh(q))/(m - bh(q)) \leq 3$. Moreover, we also have $h^{-1}(n - m + bh(q))/(n - m + bh(q)) \leq 1$. We thus have

$$1 - \frac{m - bh(q)}{n} + \frac{bh^{-1}(n - m + bh(q))(m + bh(q))}{3cn(n - m + bh(q))}$$

$$
\begin{aligned}
&= 1 - \frac{1}{n}\left(m - bh(q) - \frac{bh^{-1}(n-m+bh(q))(m+bh(q))}{3c(n-m+bh(q))}\right) \\
&\leq 1 - \frac{1}{n}\left(m - bh(q) - \frac{b(m+bh(q))}{3c}\right) \\
&\leq 1 - \frac{1}{n}\left(m - bh(q) - \frac{b(m-bh(q))}{c}\right) \\
&\leq 1 - \frac{1}{n}(m - bh(q))\left(1 - \frac{b}{c}\right) \\
&\leq 1 \qquad \text{(since } b \leq c \text{ and } m - bh(q) \leq n).
\end{aligned}
$$

Hence, we also have

$$V_{\mathbf{X}}(\xi) \leq \frac{3cn^2}{h^{-1}(n)}$$

and this completes our proof. □

From this theorem and Theorem 4.2.1, we have

Corollary 5.2.2 *If G_0 is word hyperbolic then $\delta^{(2)}_{G_0 \times F}$ is linear.*

Now let H be word hyperbolic and let $G_0 = H \times F^*$ where F^* is a finitely generated free group. Then G_0 is automatic and so $\delta^{(1)}_{G_0} \preceq n^2$. By Corollary 5.2.2, $\delta^{(2)}_{G_0}$ is linear. Thus we have

Corollary 5.2.3 *If H is word hyperbolic and F^*, F are finitely generated free groups, then $\delta^{(2)}_{H \times F^* \times F} \preceq n^{\frac{3}{2}}$.*

We will see below (Example 5.2.8), the upper bound in Corollary 5.2.3 is often exact.

Now let $\mathcal{P}_0$ be an aspherical presentation. Then $\delta^{(2)}_{\mathcal{P}_0,\emptyset} = 0$, and so $\delta^{(2)}_{G_0}$ is linear. By Theorem 5.2.1 we have

Corollary 5.2.4 *Suppose that G_0 has an aspherical presentation $\mathcal{P}_0$ such that $\delta^{(1)}_{\mathcal{P}_0}(n) \leq bh(n)$ for all natural number n and some constant $b > 1$. Then $\delta^{(2)}_{G_0 \times F} \preceq \frac{n^2}{h^{-1}(n)}$.*

5.2.2 Lower bounds for $\delta^{(2)}_{G_0 \times F}$ where G_0 has an aspherical presentation

Lemma 5.2.5 *Suppose that $\mathcal{P}_0$ is aspherical. Then the second homotopy module $\pi_2(\mathcal{P})$ is a free $\mathbb{Z}G$-module with basis $\xi_{R,t}(R \in \boldsymbol{r}_0,\ t \in \boldsymbol{t})$.*

Proof. Since $\mathcal{P}_0$ is aspherical we may take the set $\boldsymbol{X}_0$ of generating pictures of $\pi_2(\mathcal{P}_0)$ to be empty. Thus the elements $\xi_{R,t}$ ($R \in \boldsymbol{r}_0$, $t \in \boldsymbol{t}$) generates $\pi_2(\mathcal{P})$. To show they are a free basis consider the standard exact sequence (1.7) for $\mathcal{P}_0$. Since $\pi_2(\mathcal{P}_0) = 0$,

$$\partial_2 : \bigoplus_{R \in \boldsymbol{r}_0} \mathbb{Z}G_0 \hat{e}_R^{(0)} \longrightarrow \bigoplus_{x \in \boldsymbol{x}} \mathbb{Z}G_0 \hat{e}_x^{(0)} \qquad \hat{e}_R^{(0)} \longmapsto \sum_{x \in \boldsymbol{x}} \frac{\partial^{G_0} R}{\partial x} \hat{e}_x^{(0)}$$

is injective. Applying the exact functor $\mathbb{Z}G \otimes_{\mathbb{Z}G_0}$- we then obtain (with $e_R = 1 \otimes \hat{e}_R^{(0)}$ ($R \in \boldsymbol{r}_0$), $e_x = 1 \otimes \hat{e}_x^{(0)}$ ($x \in \boldsymbol{x}$)) an injection

$$\bigoplus_{R \in \boldsymbol{r}_0} \mathbb{Z}Ge_R \xrightarrow{1 \otimes \partial_2} \bigoplus_{x \in \boldsymbol{x}} \mathbb{Z}Ge_x \qquad e_R \longmapsto \sum_{x \in \boldsymbol{x}} \frac{\partial^{G} R}{\partial x} e_x.$$

Taking a copy of this injection for each $t \in \boldsymbol{t}$ we obtain an injection

$$\varphi : \bigoplus_{R \in \boldsymbol{r}_0, t \in \boldsymbol{t}} \mathbb{Z}Ge_R^t \longrightarrow \bigoplus_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \mathbb{Z}Ge_x^t \qquad e_R^t \longmapsto \sum_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \frac{\partial^{G} R}{\partial x} e_x^t.$$

Since $\bigoplus_{R \in \boldsymbol{r}_0, t \in \boldsymbol{t}} \mathbb{Z}Ge_R^t$ is a free $\mathbb{Z}G$-module, and the set $\{\xi_{R,t} : R \in \boldsymbol{r}_0, t \in \boldsymbol{t}\}$ generates $\pi_2(\mathcal{P})$, we have an epimorphism of $\mathbb{Z}G$-modules

$$\phi : \bigoplus_{R \in \boldsymbol{r}_0, t \in \boldsymbol{t}} \mathbb{Z}Ge_R^t \longrightarrow \pi_2(\mathcal{P}) \qquad e_R^t \longmapsto \xi_{R,t}.$$

Let ψ be the mapping given by the composition

$$\pi_2(\mathcal{P}) \xrightarrow{\mu_2} \left(\bigoplus_{R \in \boldsymbol{r}_0} \mathbb{Z}Ge_R \right) \bigoplus \left(\bigoplus_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \mathbb{Z}Ge_{[x,t]} \right) \xrightarrow{\sigma} \bigoplus_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \mathbb{Z}Ge_x^t,$$

where μ_2 is the embedding as in (1.7) and the second mapping σ is given by $e_R \mapsto 0$, $e_{[x,t]} \mapsto e_x^t$, $x \in \boldsymbol{x}$, $t \in \boldsymbol{t}$. Since $\bigoplus_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \mathbb{Z}Ge_{[x,t]} \cong \bigoplus_{x \in \boldsymbol{x}, t \in \boldsymbol{t}} \mathbb{Z}Ge_x^t$, σ is an epimorphism of $\mathbb{Z}G$-modules. Thus, ψ is also a homomorphism of $\mathbb{Z}G$-modules. By the definition of μ_2 we have

$$\mu_2(\xi_{R,t}) = (\bar{t} - 1)e_R + \sum_{x \in \boldsymbol{x}} \frac{\partial^G R}{\partial x} e_{[x,t]}.$$

Thus,

$$\psi\phi(e_R^t) = \sigma\mu_2(\xi_{R,t}) = \sigma((\bar{t} - 1)e_R + \sum_{x \in \boldsymbol{x}} \frac{\partial^G R}{\partial x} e_{[x,t]}) = \sum_{x \in \boldsymbol{x}} \frac{\partial^G R}{\partial x} e_x^t = \varphi(e_R^t)$$

and so $\psi\phi = \varphi$. Since φ is injective, ϕ is also injective and so ϕ is a bijection as required.

□

If Σ is a free abelian group with basis $\boldsymbol{Y}$ then every element $\sigma \in \Sigma$ has a unique representation in the form

$$n_1 y_1 + n_2 y_2 + \cdots + n_q y_q$$

where $q \geq 0$, $y_1, y_2, \cdots, y_q$ are distinct elements of $\boldsymbol{Y}$, and $n_1, n_2, \cdots, n_q$ are non-zero integers. We write $|\sigma|$ for $|n_1| + |n_2| + \cdots |n_q|$.

In particular, by Lemma 5.2.5 (together with the fact that $\mathbb{Z}G$ is free abelian on G) we have that $\pi_2(\boldsymbol{\mathcal{P}})$ is free abelian on the set $\{g \cdot \xi_{R,t} : g \in G, R \in \boldsymbol{r}_0, t \in \boldsymbol{t}\}$, and so we may consider $|\xi|$ for any $\xi \in \pi_2(\boldsymbol{\mathcal{P}})$. Note then that $|\xi| = V_{\boldsymbol{X}}(\xi)$.

Suppose $\mathbb{D}$ is a picture over $\boldsymbol{\mathcal{P}}_0$ and $t \in \boldsymbol{t}$. We have seen in Lemma 5.1.1 that

$$\xi_{\mathbb{D},t} = \sum_{i=1}^{A(\mathbb{D})} \varepsilon_i g_i \cdot \xi_{R_i,t}$$

for certain $g_i \in G_0$, $R_i \in \boldsymbol{r}_0$, $\varepsilon_i \in \{1, -1\}$ $(i = 1, \cdots, A(\mathbb{D}))$. However, it is conceivable that there could be cancellations in the sum on the right hand side of this expression. We will say that $\mathbb{D}$ is *stable* if no such cancellations occur. Thus $\mathbb{D}$ is stable if and only if

$$V_{\boldsymbol{X}}(\xi_{\mathbb{D},t}) = |\xi_{\mathbb{D},t}| = A(\mathbb{D}).$$

More generally, for any positive integer q, let

$$\xi_{\mathbb{D},t}^{(q)} = (1 + \bar{t} + \bar{t}^2 + \cdots + \bar{t}^{q-1})\xi_{\mathbb{D},t}.$$

Then if $\mathbb{D}$ is stable we have

$$V(\xi_{\mathbb{D},t}^{(q)}) = qA(\mathbb{D}). \tag{5.4}$$

Note that $\xi_{\mathbb{D},t}^{(q)}$ can be represented by the spherical picture $\mathbb{P}_{\mathbb{D},t}^{(q)}$ depicted in Fig. 5.10 below. So

$$A(\xi_{\mathbb{D},t}^{(q)}) \leq 2A(\mathbb{D}) + qL(\partial\mathbb{D}). \tag{5.5}$$

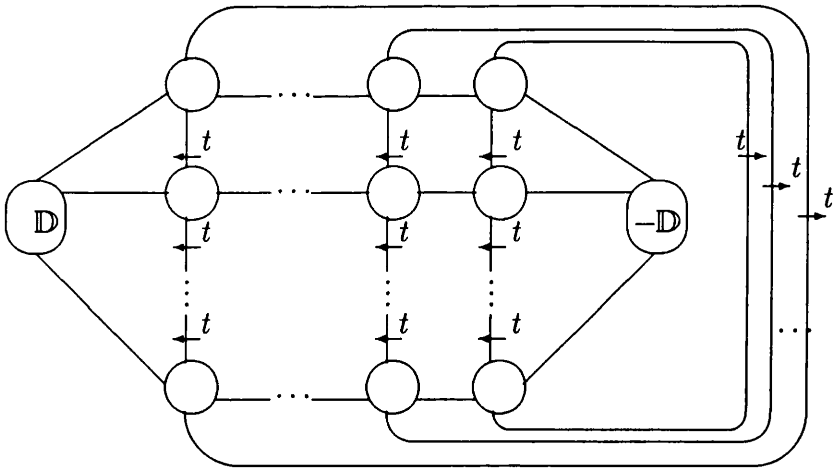


Fig. 5.10

Theorem 5.2.6 *Assume $\mathcal{P}_0$ is aspherical. Suppose there are positive constants b_1, c_1, c_2 (with b_1, c_2 positive integers), and a sequence $\mathbb{D}_i$ $(i = 1, 2, \cdots)$ of stable pictures over $\mathcal{P}_0$ such that*

$$L(\partial\mathbb{D}_{[h^{-1}(n)]}) \leq b_1 h^{-1}(n); \quad \textit{and} \tag{5.6}$$

$$c_1 n \leq A(\mathbb{D}_{[h^{-1}(n)]}) \leq c_2 n \tag{5.7}$$

for all integers $n \geq h(n_0)$ (where $[\,]$ denotes the integral part). Then

$$\delta^{(2)}_{\mathcal{P},X} \succeq \frac{n^2}{h^{-1}(n)}.$$

Proof. Let n be an integer with $n \geq h(n_0)$, and let $m = [h^{-1}(n)]$. Write

$$\frac{A(\mathbb{D}_m)}{L(\partial\mathbb{D}_m)} = q - \lambda$$

where q is a positive integer and $0 \leq \lambda < 1$. Then for any $t \in \boldsymbol{t}$ we have

$$\begin{aligned} A(\xi^{(q)}_{\mathbb{D}_m,t}) &\leq 3A(\mathbb{D}_m) + \lambda L(\partial\mathbb{D}_m) \quad \text{(using (5.5))} \\ &\leq 3c_2 n + \lambda b_1 h^{-1}(n) \quad \text{(by (5.6), (5.7))} \\ &\leq (3c_2 + b_1)n. \end{aligned}$$

Also

$$\begin{aligned} V(\xi^{(q)}_{\mathbb{D}_m,t}) &= qA(\mathbb{D}_m) \quad \text{(using (5.4))} \\ &= \frac{A(\mathbb{D}_m)^2}{L(\partial\mathbb{D}_m)} + \lambda A(\mathbb{D}_m) \\ &\geq \frac{c_1^2 n^2}{b_1 h^{-1}(n)} \quad \text{(by (5.6), (5.7)).} \end{aligned}$$

Thus

$$\frac{n^2}{h^{-1}(n)} \le \frac{b_1}{c_1^2}\delta^{(2)}_{\mathcal{P},X}((3c_2+b_1)n)$$

which proves the theorem. □

5.2.3 Examples

In the following examples, F always denotes a free group of finite rank. The result of Lyndon [Ly], that a presentation with a single relator which is not a proper power is aspherical, will be used without being further mentioned.

Example 5.2.7 *Let $G_0 = \mathbb{Z}^2$. Then*

$$\delta^{(2)}_{G_0\times F} \sim n^{\frac{3}{2}}.$$

An aspherical presentation for G_0 is $\mathcal{P}_0 = \langle y,\, z\,;\, [y,z]\rangle$. Let $h(x) = x^2$, $n_0 = 1$. Then all of the conditions we imposed on h in the previous two subsections hold. It is well-known (see [ECHLPT]) that there is a constant $c_1 > 1$ such that $\delta^{(1)}_{\mathcal{P}_0}(n) \le c_1 n^2 = c_1 h(n)$. Moreover, let $W_i = y^i z^i y^{-i} z^{-i}$ $(i = 0,\, 1,\, \cdots,\,)$, and let $\mathbb{D}_i$ to the picture with boundary label W_i as depicted in Fig. 5.11 where $L(\partial\mathbb{D}_i) = 4i$, $A(\mathbb{D}_i) = i^2$. We show that $\mathbb{D}_i$ is stable. Let $R = [y,\, z]$. Since $W_i \sim^{(1)} \prod_{j=0}^{i-1}\prod_{k=0}^{i-1} y^j z^k R z^{-k} y^{-j}$, we have

$$\xi_{\mathbb{D}_i,t} = \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} \bar{y}^j \bar{z}^k \cdot \xi_{R,t}.$$

Clearly, there are not any cancellations in the above sum and so $\mathbb{D}_i$ is stable.

Now, for any $n \ge 1$

$$L(\partial\mathbb{D}_{[h^{-1}(n)]}) \le 4h^{-1}(n)$$

and $A(\mathbb{D}_{[h^{-1}(n)]}) = [h^{-1}(n)]^2 = [n^{\frac{1}{2}}]^2 \le n$. Since $n^{\frac{1}{2}} \ge 1$, $[n^{\frac{1}{2}}] \ge \frac{1}{2}n^{\frac{1}{2}}$. Thus,

$$[n^{\frac{1}{2}}]^2 \ge (\frac{1}{2}n^{\frac{1}{2}})^2 = \frac{1}{4}n,$$

i.e.

$$\frac{1}{4}n \le A(\mathbb{D}_{[h^{-1}(n)]}) \le n.$$

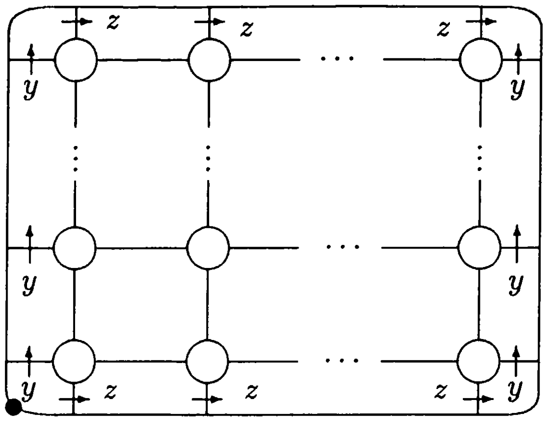


Fig. 5.11

We deduce from Corollary 5.2.4 and Theorem 5.2.6 that $\delta^{(2)}_{\mathbb{Z}^2\times F} \sim n^{\frac{3}{2}}$; in particular, $\delta^{(2)}_{\mathbb{Z}^3} \sim n^{\frac{3}{2}}$.

Example 5.2.8 *Let $G_0 = H \times F^*$ where H is word hyperbolic and F^* is any free group of finite rank. If the group $\mathbb{Z}$ is a (quasi) retract of H then*

$$\delta^{(2)}_{G_0\times F} \sim n^{\frac{3}{2}}.$$

By Corollary 5.2.3, $\delta^{(2)}_{H\times F^*\times F} \preceq n^{\frac{3}{2}}$. A direct generalization of the result of the above example gives that if F^{**} is again a free group of finite rank then

$$\delta^{(2)}_{F^{**}\times F^*\times F} \sim n^{\frac{3}{2}}.$$

If $\mathbb{Z}$ is a (quasi) retract of H, then $\mathbb{Z}\times F^*\times F$ will be a (quasi) retract of $H\times F^*\times F$, so $\delta^{(2)}_{H\times F^*\times F} \succeq \delta^{(2)}_{\mathbb{Z}^3} \sim n^{\frac{3}{2}}$. Thus, $\delta^{(2)}_{H\times F^*\times F} \sim n^{\frac{3}{2}}$.

Example 5.2.9 *Let $G_0 = F_m \rtimes F_2$ be the split extension of the free group F_m $(m \geq 2)$ by another free group F_2 on y_1, y_2 defined by the presentation*

$$\mathcal{P}_0 = \langle x_1, \cdots, x_m,\ y_1,\ y_2\ ;\ x_m^{y_\lambda} = x_m,\ x_k^{y_\lambda} = x_k x_{k+1}\ \ (1 \leq k \leq m-1,\ \lambda = 1,\ 2)\rangle.$$

Then

$$\delta^{(2)}_{G_0\times F} \sim n^{2-\frac{1}{m+1}}.$$

By Theorem 3.1 of [CCH] or Theorem 6.1.4 of next chapter, $\mathcal{P}_0$ is aspherical. Let $h(x) = x^{m+1}$, $n_0 = 1$. Then again all conditions imposed on h in the previous two subsections hold. Bridson [Brd3] has shown that $\delta^{(1)}_{\mathcal{P}_0} \sim h$. Let $R_{m,\lambda} = x_m^{y_\lambda} x_m^{-1}$, $R_{k,\lambda} = x_k^{y_\lambda} x_{k+1}^{-1} x_k^{-1}$, $1 \le k \le m-1$, $\lambda = 1,\, 2$, and consider the picture $\mathbb{D}_i$ for word

$$W_i = y_2^{-(i+m-2)} x_1^{i+m-2} y_2^{i+m-2} y_1^{-(i+m-2)} x_1^{-(i+m-2)} y_1^{i+m-2} \quad (i \in \mathbb{N}).$$

Fig. 5.12 shows the top half of $\mathbb{D}_3$ for $m = 3$. (The bottom half is obtained by reflecting the top half through a horizontal line and replacing all y_2-arcs with y_1-arcs.)

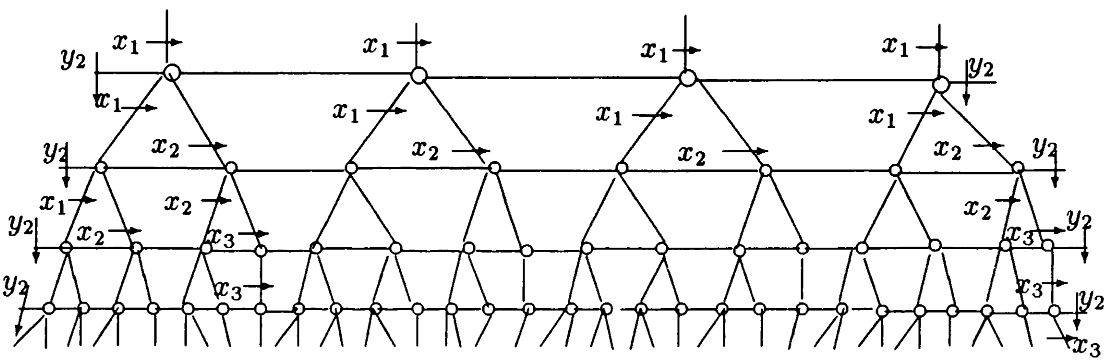


Fig. 5.12

Since each disc in the top half has positive orientation and each disc in the bottom half has negative orientation, we have

$$\xi_{\mathbb{D}_i,t} = \sum_{j=1}^{\frac{1}{2}A(\mathbb{D}_i)} g_j \cdot \xi_{R_{q_j,1},t} - \sum_{j=1}^{\frac{1}{2}A(\mathbb{D}_i)} g'_j \cdot \xi_{R_{q_j,2},t}$$

for certain $g_j,\, g'_j \in G_0$, $1 \le q_j \le m$, $1 \le j \le \frac{1}{2}A(\mathbb{D}_i)$. Clearly, within each sum there are not cancellations since all terms of this sum have the same symbol, and also there are not cancellations between two sums due to the different second subsubscrips. Thus, $\mathbb{D}_i$ is stable. Clearly $L(\partial\mathbb{D}_i) = 6(i+m) - 12$.

Let $B_k^{(j)}$ denote the number of the $R_{k,2}$-discs of the jth row of $\mathbb{D}_i$ from the top, $1 \le k \le m$, $1 \le j \le m+i-2$. For convenience we let $l = m+i-2$. Then we have

$$B_1^{(j)} = l, \; 1 \le j \le l; \tag{5.8}$$

$$B_k^{(1)} = B_k^{(2)} = \cdots = B_k^{(k-1)} = 0, \quad 1 < k \le m; \tag{5.9}$$

and

$$B_k^{(p)} = B_{k-1}^{(p-1)} + B_k^{(p-1)}, \quad p \ge k,\; 1 < k \le m. \tag{5.10}$$

Thus, for $p \geq k$, $1 < k \leq m$, we have

$$\begin{aligned}
B_k^{(p)} &= B_{p-1}^{(p-1)} + B_k^{(p-1)} \qquad \text{(by (5.10))} \\
&= B_{k-1}^{(p-1)} + B_{k-1}^{(p-2)} + \cdots + B_{k-1}^{(k-1)} + B_k^{(k-1)} \\
&\qquad \text{(by successively using (5.10) } p-k \text{ times)} \\
&= B_{k-1}^{(p-1)} + B_{k-1}^{(p-2)} + \cdots + B_{k-1}^{(k-1)} \qquad (\text{since } B_k^{(k-1)} = 0 \text{ by (5.9)}) \\
&= \sum_{j=1}^{p-k+1} B_{k-1}^{(p-j)},
\end{aligned}$$

namely, we have

$$B_k^{(p)} = \sum_{j=1}^{p-k+1} B_{k-1}^{(p-j)}, \quad p \geq k,\ 1 < k \leq m. \tag{5.11}$$

By (5.8), in total we have

$$\sum_{p=1}^{l} B_1^{(p)} = l^2$$

$R_{1,2}$-discs in $\mathbb{D}_i$. By (5.11) and (5.8), for $p > 1$ we have

$$B_2^{(p)} = \sum_{j=1}^{p-1} B_1^{(p-j)} = l(p-1).$$

We now show by induction on $1 < k \leq m$ that if $p \geq k$ then

$$B_k^{(p)} = \frac{l}{(k-1)!}(p-1)(p-2)\cdots(p-k+1). \tag{5.12}$$

In order to do this we need the following elementary fact (which can be simply proved by induction on n):

$$\sum_{j'=1}^{n} (j'+m-1)(j'+m-2)\cdots(j'+1)j' = \frac{1}{m+1}(n+m)(n+m-1)\cdots(n+1)n \tag{5.13}$$

for any pair of integers $n > 0$, $m \geq 0$.

We have

$$\begin{aligned}
B_k^{(p)} &= \sum_{j=1}^{p-k+1} B_{k-1}^{(p-j)} \qquad \text{(using (5.11))} \\
&= \frac{l}{(k-2)!} \sum_{j=1}^{p-k+1} (p-j-1)(p-j-2)\cdots(p-j-(k-1)+1)
\end{aligned}$$

by using induction hypothesis. Let $j' = p - k - j + 2$. Then $j' = 1$ when $j = p - k + 1$ and $j' = p - k + 1$ when $j = 1$. Thus,

$$\begin{aligned}
&\sum_{j=1}^{p-k+1} (p-j-1)(p-j-2)\cdots(p-j-(k-1)+1) \\
&= \sum_{j'=1}^{p-k+1} (j'+(k-2)-1)(j'+(k-2)-2)\cdots(j'+(k-2)-(k-2)) \\
&= \frac{1}{k-1}(p-1)(p-2)\cdots(p-(k-1)) \quad \text{(by (5.13))}
\end{aligned}$$

and this completes the proof of (5.12).

Now, by using (5.12) and (5.13) we have

$$\begin{aligned}
\sum_{p=k}^{l} B_k^{(p)} &= \frac{l}{(k-1)!}\sum_{p=k}^{l}(p-1)(p-2)\cdots(p-k+1) \\
&= \frac{l}{(k-1)!}\sum_{j=1}^{l-k+1} (j+(k-1)-1)(j+(k-1)-2)\cdots(j+1)j \\
&= \frac{l}{(k-1)!}\frac{1}{k}l(l-1)\cdots(l-k+1) \\
&= \frac{l}{k!}l(l-1)\cdots(l-k+1) \\
&\le l^{k+1} \\
&\le (m+i)^{k+1}.
\end{aligned}$$

On the other hand, Since

$$l^2(l-k+1) = (l-k+2)^2(l-k+1) + (k-2)^2(l-k+1) + 2(k-2)(l-k+2)(l-k+1)$$

and

$$(l-k+2)^3 = (l-k+2)^2(l-k+1) + (l-k+2)(l-k+1) + (l-k+2),$$

we have

$$l^2(l-k+1) \ge (l-k+2)^3, \quad \text{if } k > 2,\ l > k-1.$$

So, for $k > 2$ and $i > 1$ we have

$$\sum_{p=k}^{l} B_k^{(p)} \ge \frac{(m+i-k)^{k+1}}{k!}.$$

Also,

$$\sum_{p=2}^{l} B_2^{(p)} = \frac{l^2(l-1)}{2} \geq \frac{l^3}{4}$$

for $i > 1$. Thus, we have for all $1 < k \leq m$

$$\sum_{p=k}^{l} B_k^{(p)} \geq \frac{(m+i-k)^{k+1}}{2 \cdot k!}.$$

Hence, by symmetry,

$$\begin{aligned} A(\mathbb{D}_i) &= 2\sum_{k=1}^{m}\sum_{p=k}^{l} B_k^{(p)} \leq 2\sum_{k=2}^{m+1}(m+i)^k \\ &= 2(m+i)^2\frac{(m+i)^m - 1}{m+i-1} \leq 3(m+i)^{m+1} \leq 3m^{m+1}i^{m+1}, \end{aligned}$$

and for all $i > 1$,

$$\begin{aligned} A(\mathbb{D}_i) &= 2\left(\sum_{p=1}^{l} B_1^{(p)} + \sum_{k=2}^{m}\sum_{p=k}^{l} B_k^{(p)}\right) \\ &\geq 2\sum_{p=m}^{l} B_m^{(p)} \\ &\geq \frac{(m+i-m)^{m+1}}{m!} \\ &= \frac{i^{m+1}}{m!}. \end{aligned}$$

But $\frac{1}{m} \leq A(\mathbb{D}_1) = 2(1+m-2) = 2(m-1)$ since $m \geq 2$. Hence, if let $\alpha = \frac{1}{m!}$, $\beta = 3m^{m+1}$, then $\alpha i^{m+1} \leq A(\mathbb{D}_i) \leq \beta i^{m+1}$ for all i. Thus, $A(\mathbb{D}_{[h^{-1}(n)]}) \leq \beta[h^{-1}(n)]^{m+1} \leq \beta(h^{-1}(n))^{m+1} = \beta n$. Moreover, since $h^{-1}(n) \geq 1$ for $n > n_0 = 1$, $[h^{-1}(n)] \geq \frac{1}{2}h^{-1}(n)$. We then have

$$A(\mathbb{D}_{[h^{-1}(n)]}) \geq \alpha(\frac{h^{-1}(n)}{2})^{m+1} \geq \frac{\alpha}{2^{m+1}}n.$$

Thus, for all $n > h(n_0)$,

$$L(\partial\mathbb{D}_{[h^{-1}(n)]}) = L(W_{[h^{-1}(n))]}) = 6([h^{-1}(n)] + m - 2) \leq 6(h^{-1}(n) + m) \leq 6mh^{-1}(n),$$

and

$$\frac{\alpha}{2^{m+1}}n \leq A(\mathbb{D}_{[h^{-1}(n)]}) \leq \beta n.$$

We then deduce from Corollary 5.2.4 and Theorem 5.2.6 that $\delta^{(2)}_{(F_m \rtimes F_2)\times F} \sim n^{2-\frac{1}{m+1}}$.

Example 5.2.10 *Let $G_0 = B_{p,q}$ be the Baumslag-Solitar group defined by the (aspherical) presentation $\mathcal{P}_0 = \langle y,\ z\ ;\ zy^pz^{-1}y^{-q}\rangle$ $(1 \le p < q)$. Then*

$$\delta^{(2)}_{G_0\times F} \sim \frac{n^2}{\log_\alpha n},$$

where $\alpha = \frac{q}{p}$.

Let $h(x) = \alpha^x$, and let $n_0 = 3$ be the smallest natural number bigger than the number e (base of the natural logarithm). Let $f(x) = \frac{x}{h^{-1}(x)} = \frac{x}{\log_\alpha x}$. Since for $x > n_0$, the first order derivative $f'(x) = \frac{\log_\alpha x - 1/\ln\alpha}{\log^2_\alpha x}$ of $f(x)$ is positive, $f(x)$ is increasing for $x > n_0$, and all conditions we imposed on $h(x)$ in the previous two subsections hold.

By Theorem E1 of [BGSS] $B_{p,q}$ is asynchronously automatic and hence $\delta^{(1)}_{B_{p,q}} \preceq h$ [Brd2, Theorem 6.1]. Following Gersten [Ge2, the proof of Theorem B], define non-negative integers a_j, b_j $(j = 1, 2, \cdots,)$ inductively as follows: $a_1 = p$, $b_1 = 0$; b_{j+1} is the least natural number with $\frac{q}{p}a_j + b_{j+1}$ divisible by p, $a_{j+1} = \frac{q}{p}a_j + b_{j+1}$. Let

$$W_i = [z^iy^pz^{-1}y^{b_2}z^{-1}y^{b_3}\cdots z^{-1}y^{b_i}z^{-1}, y]$$

then $L(W_i) \le 2(p+2)i+2$. Let $\mathbb{D}_i$ be the picture over $\mathcal{P}_0$ with boundary label W_i obtained by dualizing the van Kampen diagram in Figure 2 of [Ge2]. (This picture is illustrated in Fig. 5.13 for $p = 1$, $q = 2$, and $i = 4$.) We then have $A(\mathbb{D}_i) = \frac{2}{p}(a_1 + a_2 + \cdots + a_i)$.

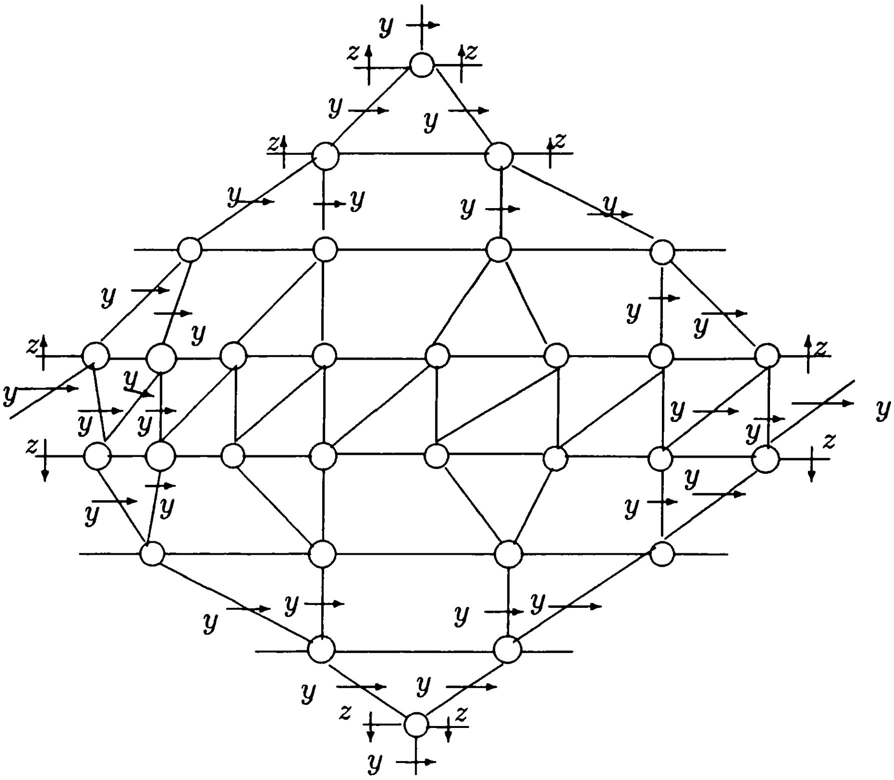


Fig. 5.13

Let $R = zy^pz^{-1}y^{-q}$. Since each disc in the top half has positive orientation and each disc in the bottom half has negative orientation, we have

$$\xi_{\mathbb{D}_i,t} = \sum_{j=1}^{\frac{1}{2}A(\mathbb{D}_i)} g_j \cdot \xi_{R,t} - \sum_{j=1}^{\frac{1}{2}A(\mathbb{D}_i)} h_j \cdot \xi_{R,t}.$$

Clearly, within each sum there are not cancellations since all terms of each sum have the same symbol. Also, by Lemma 4.3 of [Ge2] there are not cancellations between the two sums. Thus, $\mathbb{D}_i$ is stable.

Now inductively we have

$$\begin{aligned}
a_j &= \alpha a_{j-1} + b_j = \alpha\left(\alpha a_{j-2} + b_{j-1}\right) + b_j \\
&= \alpha^2\left(\alpha a_{j-3} + b_{j-2}\right) + \alpha b_{j-1} + b_j \\
&\vdots \\
&= \alpha^{j-1} a_1 + \alpha^{j-2} b_2 + \cdots + \alpha b_{j-1} + b_j \\
&\leq p\alpha^{j-1} + p\alpha^{j-2} + \cdots + p\alpha + p \qquad (\text{since } a_1 = p,\ b_1,\ \cdots, b_j \leq p) \\
&= \frac{\alpha^j - 1}{\alpha - 1} p
\end{aligned}$$

$$\leq \alpha^j \frac{p^2}{q-p} \leq \alpha^j p^2.$$

So

$$A(\mathbb{D}_i) \leq \frac{2}{p}(p + \alpha p^2 + \cdots \alpha^i p^2) = \frac{2}{p}\left(p + \alpha p^2 \frac{\alpha^i - 1}{\alpha - 1}\right) \leq 2 + \frac{2}{q-p}\alpha^{i+1}p^3 < 4\alpha^{i+1}p^3.$$

On the other hand, since $a_1 \geq 1$, $a_j \geq \alpha^{j-1}$ $(j \geq 1)$,

$$A(\mathbb{D}_i) \geq \frac{2}{p}\left(1 + \alpha + \cdots + \alpha^{i-1}\right) = \frac{2(\alpha^i - 1)}{p(\alpha - 1)} = \frac{2(\alpha^i - 1)}{q-p}.$$

Thus, if $i \geq \log_\alpha 2$, then $2(\alpha^i - 1) \geq \alpha^i$, and so

$$A(\mathbb{D}_i) \geq \frac{1}{q-p}\alpha^i.$$

We thus have that for any $n \geq \max\{h(n_0), \log_\alpha 2\}$,

$$L(\partial\mathbb{D}_{[h^{-1}(n)]}) = L(W_{[h^{-1}(n)]}) \leq 2(h^{-1}(n) + p + h^{-1}(n)p + h^{-1}(n) + 1) \leq 2(3+2p)h^{-1}(n),$$

and

$$A(\mathbb{D}_{[h^{-1}(n)]}) < 4\alpha^{h^{-1}(n)+1}p^3 = 4p^3\alpha n.$$

Let $[h^{-1}(n)] = h^{-1}(n) - \lambda$ with $0 \leq \lambda < 1$. Then

$$A(\mathbb{D}_{[h^{-1}(n)]}) \geq \frac{\alpha^{[h^{-1}(n)]}}{q-p} = \frac{\alpha^{h^{-1}(n)-\lambda}}{q-p} = \frac{n}{(q-p)\alpha^\lambda} > \frac{n}{\alpha(q-p)}.$$

We then deduce from Corollary 5.2.4 and Theorem 5.2.6 that

$$\delta^{(2)}_{B_{p,q}\times F} \sim \frac{n^2}{\log_\alpha n}.$$

Example 5.2.11 *Let A be the split extension $F^* \rtimes_\phi \mathbb{Z}$ where F^* is a free group of rank 3 on z_1, z_2, z_3, and let $G_0 = A *_{F^*} A \cong F^* \rtimes_\phi B$ where B is a free group of rank 2 on s_1, s_2, and both s_1 and s_2 act on A by the automorphism ϕ. If ϕ is induced by the mapping*

$$z_1 \longmapsto z_3, \quad z_2 \longmapsto z_1 z_3, \quad z_3 \longmapsto z_2 z_3,$$

then

$$\delta^{(2)}_{G_0\times F} \sim \frac{n^2}{\log_2 n}.$$

It was shown in [BBMS] that the automorphism ϕ satisfies the condition that for any g, g' in F^*, $g \neq 1$ in F^* and any positive integer m, $\phi^m(g) \neq g'^{-1}gg'$, and so A is word hyperbolic and G_0 is asynchronously automatic with $\delta^{(1)}_{G_0} \sim 2^n$.

By [CCH, Theorem 3.1] or Theorem 6.1.4 of the next chapter, both A and G_0 are aspherical. Now, the group G_0 has a presentation

$$\mathcal{P}_0 = \langle z_1,\ z_2,\ z_3,\ s_1\ s_2\ ;\ s_l^{-1}z_1s_l = z_3,\ s_l^{-1}z_2s_l = z_1z_3,\ s_l^{-1}z_3s_l = z_2z_3,\ (l = 1,\ 2)\rangle.$$

Let $R_{1,l} = s_l^{-1}z_1s_lz_3^{-1}$, $R_{2,l} = s_l^{-1}z_2s_lz_3^{-1}z_1^{-1}$, and $R_{3,l} = s_l^{-1}z_3s_lz_3^{-1}z_2^{-1}$, $l = 1, 2$. Consider the word $W_i = s_1^{-i}z_3s_1^is_2^{-i}z_3^{-1}s_2^i$. Then $\overline{W}_i = 1$ in G_0 $(i = 1, 2, \cdots)$. We have a sequence of pictures $\mathbb{D}_i$ of the form shown as in Fig. 5.14 for $i = 4$.

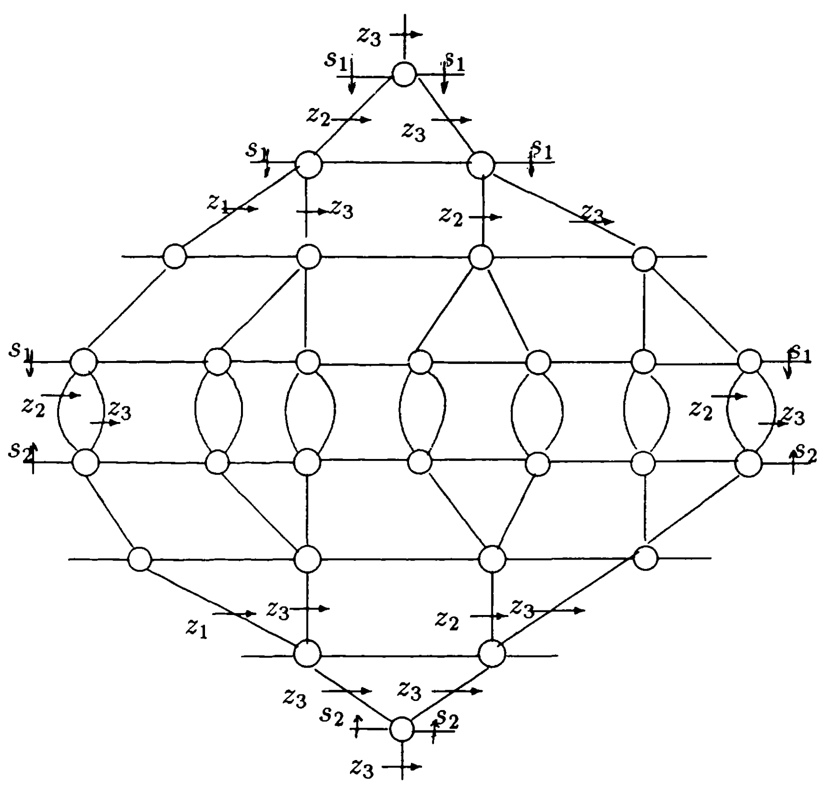


Fig. 5.14

Similarly as we did in Example 5.2.9, we can show that $\mathbb{D}_i$ is stable.

We have that $L(\partial\mathbb{D}_i) = 4i + 2$. Let $C_l^{(j)}$ be the number of $R_{l,1}$-discs in the jth row of $\mathbb{D}_i$ from the top. Then we have

$$C_1^{(1)} = 0,\ C_2^{(1)} = 0,\ C_3^{(1)} = 1,$$
$$C_1^{(j)} = C_2^{(j-1)},\ C_2^{(j)} = C_3^{(j-1)},$$
$$C_3^{(j)} = C_1^{(j-1)} + C_2^{(j-1)} + C_1^{(j-1)}.$$

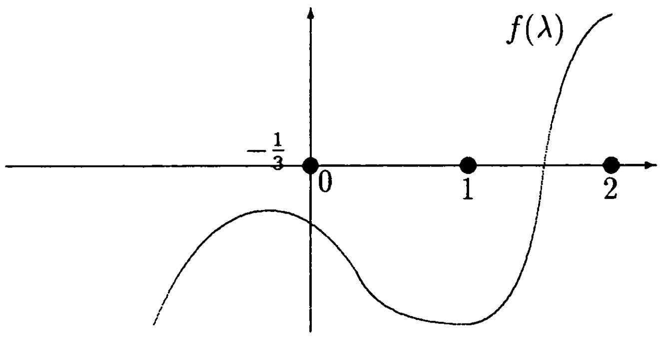
Let a_j denote the number of discs in the jth row of the *upper half* of $\mathbb{D}_i$. Then we have

$$a_j = C_1^{(j)} + C_2^{(j)} + C_3^{(j)}, \quad 2 \le j \le i$$

and so, for $j > 3$,

$$\begin{aligned} a_j &= C_1^{(j)} + C_2^{(j)} + C_3^{(j)} \\ &= C_2^{(j-1)} + C_3^{(j-1)} + C_1^{(j-1)} + C_2^{(j-1)} + C_3^{(j-1)} \\ &= a_{j-1} + C_2^{(j-1)} + C_3^{(j-1)} \\ &= a_{j-1} + C_3^{(j-2)} + C_1^{(j-2)} + C_2^{(j-2)} + C_3^{(j-2)} \\ &= a_{j-1} + a_{j-2} + C_3^{(j-2)} \\ &= a_{j-1} + a_{j-2} + C_1^{(j-3)} + C_2^{(j-3)} + C_3^{(j-3)} \\ &= a_{j-1} + a_{j-2} + a_{j-3}. \end{aligned}$$

We further show that the polynomial $f(\lambda) = \lambda^3 - \lambda^2 - \lambda - 1$ has only one real root, which lies in the interval between 1 and 2. First, $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = 1$ are the roots of $f'(\lambda) = 3\lambda^2 - 2\lambda - 1$, where f' is the first order derivative of f. Since $f'(\lambda)$ is positive for $\lambda < -\frac{1}{3}$ or $\lambda > 1$, we see that f has its local maximum value at $-\frac{1}{3}$ and has its local minimum value at 1. So, f is strictly increasing on $(-\infty, -\frac{1}{3}] \cup [1, \infty)$ and strictly decreasing on $[-\frac{1}{3}, 1]$. But $f(-\frac{1}{3}) < 0$, so f has no root in $(-\infty, 1)$. Since $f(2) > 0$, $f(1) < 0$, by the monotonicity of f on $(1, \infty)$, f has only one real root, say b, with $1 < b < 2$. We sketch the curve of f on the real plane $\mathbb{R}^2$ as below.



We now show that

$$\frac{1}{2}b^j \le a_j \le 4b^j$$

for all $j \in \mathbb{N}$ by induction.

First, we have

$$\frac{1}{2}b < a_1 = 1 < 4b, \quad \frac{1}{2}b^2 < a_2 = 2 < 4b^2, \quad \frac{1}{2}b^3 < a_3 = 4 < 4b^3.$$

For $j \geq 3$, since $a_j = a_{j-1} + a_{j-2} + a_{j-3}$, by induction hypothesis we have

$$a_j \leq 4b^{j-1} + 4b^{j-2} + 4b^{j-3} = 4b^{j-3}(1 + b + b^2) = 4b^{j-3}b^3 = 4b^j$$

and

$$a_j \geq \frac{1}{2}b^{j-1} + \frac{1}{2}b^{j-2} + \frac{1}{2}b^{j-3} = \frac{1}{2}b^{j-3}(1 + b + b^2) = \frac{1}{2}b^{j-3}b^3 = \frac{1}{2}b^j,$$

as required. Thus, for all $i \in \mathbb{N}$ we have $A(\mathbb{D}_i) = 2(a_1 + a_2 + \cdots + a_i) \geq b^i$, and

$$A(\mathbb{D}_i) = 2(a_1 + a_2 + \cdots + a_i) \leq 8b\frac{b^i - 1}{b - 1} \leq 8b\frac{b^i}{b - 1}.$$

But $b^3 - b^2 - b - 1 = 0$, so we have $\frac{1}{b-1} = \frac{b^2}{b+1} \leq b$. Hence,

$$A(\mathbb{D}_i) \leq 8b^{i+2}.$$

Let $h(x) = b^x$ and let $n_0 = 3$ be the smallest natural number bigger than e (base of the natural logarithm). Similar to Example 5.2.10 we have that, for all $x \geq n_0$, $\frac{x}{h^{-1}(x)}$ is increasing and all conditions we imposed on $h(x)$ in the previous two subsections hold.

Therefore, for all $n \geq h(n_0)$ we have

$$L(\partial\mathbb{D}_{[h^{-1}(n)]}) \leq 6h^{-1}(n) \quad \text{and} \quad A(\mathbb{D}_{[h^{-1}(n)]}) \leq 8b^2 n.$$

On the other hand, let $[h^{-1}(n)] = h^{-1}(n) - \varepsilon$ with $0 \leq \varepsilon < 1$. Then

$$A(\mathbb{D}_{[h^{-1}(n)]}) \geq b^{[h^{-1}(n)]} = \frac{n}{b^\varepsilon} \geq \frac{1}{b}n.$$

Now, by Corollary 5.2.4 and Theorem 5.2.6 we then have $\delta^{(2)}_{G_0 \times F} \sim \frac{n^2}{\log_2 n}$ since $\frac{n^2}{\log_b n} \sim \frac{n^2}{\log_2 n}$.

Furthermore, let D be the direct product $A \times B$. Then by Theorem 2 of [BBMS], G_0 can be embedded into D. Thus, $G_0 \times F$ can be embedded into $D \times F$, and (see Example 5.2.8) $\delta^{(2)}_{D \times F} \sim n^{\frac{3}{2}}$.

This example gives the following theorem which shows that there are pairs of groups G, H of type F_3 with H isomorphic to a proper subgroup of G, but $\delta^{(2)}_H \succ \delta^{(2)}_G$.

Theorem 5.2.12 *There are groups H, G of types F_3 with $H \leq G$ such that*

$$\delta_H^{(2)} \sim \frac{n^2}{\log_2 n} \quad \text{and} \quad \delta_G^{(2)} \sim n^{\frac{3}{2}}.$$

Example 5.2.13 *Let G_0 be defined by the (aspherical) presentation*

$$\mathcal{P}_0 = \langle \boldsymbol{x},\, \boldsymbol{y}\,;\, U^p V^{-q} \rangle,$$

where U, V are non-empty reduced words on $\boldsymbol{x}$, $\boldsymbol{y}$ respectively, and p, q are integers > 1. Then

$$\delta_{G_0 \times F}^{(2)} \sim n^{\frac{3}{2}}.$$

Obviously, if let $h(x) = x^2$, $n_0 = 1$, then for all $x > n_0$, the function $\frac{x}{x^{1/2}}$ is increasing and all conditions we imposed on $h(x)$ in the previous two subsections hold.

It is shown in [BMS] that $\delta_{\mathcal{P}_0}^{(1)} \sim h$. For any positive integer i, let W_i denote the word $V(UV)^i U^{ip}(UV)^{-i}V^{-1}U^{-ip}$. Then we can construct a picture $\mathbb{D}_i$ with boundary label W_i. (The picture for $i = 2$ is shown in Fig. 5.15.) Note that, in $\mathbb{D}_i$, we have $2i+2$ rows and i column, and so $A(\mathbb{D}_i) = 2(i+1)i$. Since all discs in $\mathbb{D}_i$ have positive orientation, similarly as we did in Example 5.2.9 we can show that $\mathbb{D}_i$ is stable.

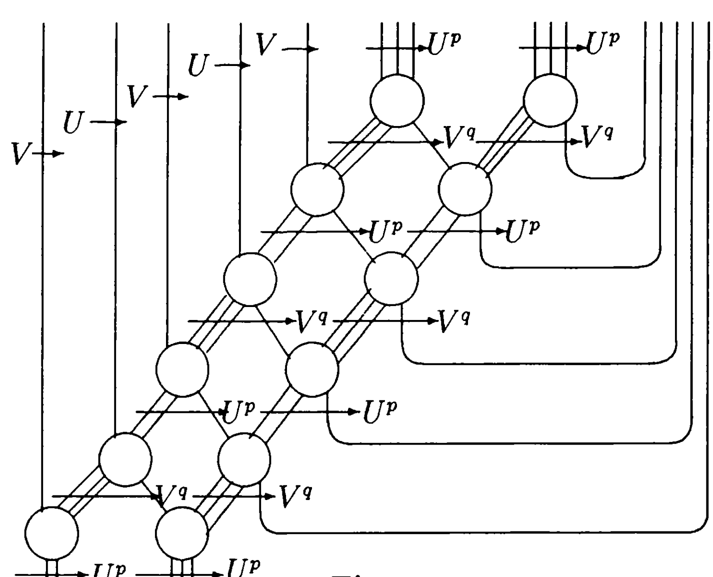


Fig. 5.15

Letting $\alpha = \max\{L(U),\, L(V)\}$ we then have that for any $n \geq h(n_0) = 1$,

$$L(\partial \mathbb{D}_{[h^{-1}(n)]}) \leq (2p+6)\alpha h^{-1}(n).$$

Also

$$A(\mathbb{D}_{[h^{-1}(n)]}) = 2([h^{-1}(n)]+1)[h^{-1}(n)] \le 2(h^{-1}(n))^2 + 2h^{-1}(n) \le 4n.$$

On the other hand, since $n \ge 1$, $h^{-1}(n) \ge 1$. Thus, $[h^{-1}(n)] \ge \frac{1}{2}h^{-1}(n)$. Hence,

$$A(\mathbb{D}_{[h^{-1}(n)]}) \ge 2(\frac{1}{2}h^{-1}(n)+1)(\frac{1}{2}h^{-1}(n)) \ge \frac{1}{2}n.$$

Consequently, by Corollary 5.2.4 and Theorem 5.2.6 we have $\delta^{(2)}_{G_0 \times F} \sim n^{\frac{3}{2}}$.

Chapter 6

Calculations of second order Dehn functions of groups III: HNN-extensions, amalgamated free products and split extensions

6.1 HNN-extensions and amalgamated free products

6.1.1 Generators

Let G_0 be a group of type F_3 finitely presented by $\mathcal{P}_0 = \langle \boldsymbol{x}_0 \,;\, \boldsymbol{r}_0 \rangle$. Let H and $\check{H}$ be two finitely presented subgroups of G_0 together with an isomorphism $\gamma : H \longrightarrow \check{H}$. Choose a finite set $\boldsymbol{a} = \{a_y \,:\, y \in \boldsymbol{y}\}$ of words on $\boldsymbol{x}_0$ which represents a generating set for H. Let $\check{\boldsymbol{a}} = \{\check{a}_y \,:\, y \in \boldsymbol{y}\}$ be such that for each $y \in \boldsymbol{y}$, $\check{a}_y$ represents $\gamma(\bar{a}_y)$. Let $F(\boldsymbol{y})$ be the free group on $\boldsymbol{y}$, and let $\boldsymbol{s}$ be a finite set of words on $\boldsymbol{y}$ whose normal closure in $F(\boldsymbol{y})$ is the kernel of the epimorphism

$$F(\boldsymbol{y}) \longrightarrow H, \qquad [y] \longmapsto \bar{a}_y,\ y \in \boldsymbol{y}.$$

Thus, $\mathcal{H} = \langle \boldsymbol{y};\ \boldsymbol{s} \rangle$ is a finite presentation for H. The *HNN-extension* G of G_0 with associated subgroups H and $\check{H}$ is then finitely presented by (see [Co] for reference)

$$\mathcal{P} = \langle \boldsymbol{x}_0,\, t \,;\, \boldsymbol{r}_0,\ t^{-1} a_y t \check{a}_y^{-1}\ (y \in \boldsymbol{y}) \rangle.$$

If $W = W(\boldsymbol{y})$ is a word on $\boldsymbol{y}$, we then write $W(\boldsymbol{a})$ and $W(\check{\boldsymbol{a}})$ for the words on $\boldsymbol{x}$ obtained from $W(\boldsymbol{y})$ by replacing each $y \in \boldsymbol{y}$ with a_y and $\check{a}_y$ respectively. Since for each $S = S(\boldsymbol{y}) \in \boldsymbol{s}$, $\overline{S}(\boldsymbol{a}) = 1$ in $G(\mathcal{P}_0)$, we then can choose two pictures over $\mathcal{P}_0$, say $\mathbb{S}$ and $\check{\mathbb{S}}$, with boundary labels $S(\boldsymbol{a})$ and $S(\check{\boldsymbol{a}})$ respectively.

For each $y \in \boldsymbol{y}$, let $a_{y^{-1}} = a_y^{-1}$ and $\check{a}_{y^{-1}} = \check{a}_y^{-1}$. Then we have in $G(\mathcal{P})$

$$t^{-1}a_{y^{-1}}t = t^{-1}a_y^{-1}t = \check{a}_y^{-1} = \check{a}_{y^{-1}}.$$

If $\mathbb{D}$ is any picture over $\mathcal{H}$ with discs $\Delta_1, \cdots, \Delta_n$ labelled $S_1^{\varepsilon_1}, \cdots, S_n^{\varepsilon_n}$ ($S_i \in \boldsymbol{s}$, $\varepsilon_i = \pm 1$, $1 \le i \le n$) say, then let $\mathbb{D}(\boldsymbol{a})$ (resp. $\mathbb{D}(\check{\boldsymbol{a}})$) be the picture over $\mathcal{P}_0$ obtained by replacing each arc labelled by $y \in \boldsymbol{y}$ by a sequence of parallel arcs labelled by a_y (resp. $\check{a}_y$), and replacing each disc Δ_i by the picture $\varepsilon_i\mathbb{S}_i$ (resp. $\varepsilon_i\check{\mathbb{S}}_i$). Then if $W(\boldsymbol{y}) = y_1 \cdots y_n$ ($y_i \in \boldsymbol{y} \cup \boldsymbol{y}^{-1}$, $1 \le i \le n$) is the boundary label of $\mathbb{D}$, $W(\boldsymbol{a}) = a_{y_1} \cdots a_{y_n}$ and $W(\check{\boldsymbol{a}}) = \check{a}_{y_1} \cdots \check{a}_{y_n}$ are the boundary labels of $\mathbb{D}(\boldsymbol{a})$ and $\mathbb{D}(\check{\boldsymbol{a}})$ respectively. Therefore, we can construct a spherical picture $\mathbb{P}_{\mathbb{D},t}$ over $\mathcal{P}$ as depicted in Fig. 6.1, and we let $\langle \mathbb{P}_{\mathbb{D},t} \rangle = \xi_{\mathbb{D},t}$.

$\mathbb{P}_{\mathbb{D},t}$:

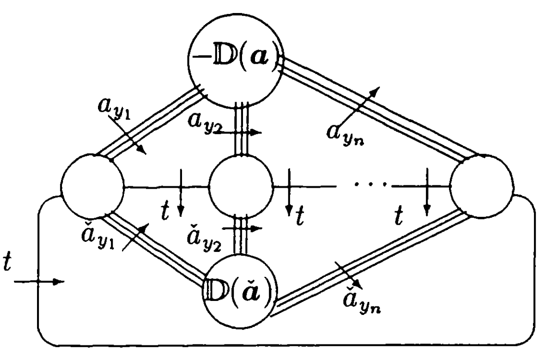


Fig. 6.1

In particular, if $\mathbb{D}$ consists of a single disc labelled $S \in \boldsymbol{s}$, then we write $\mathbb{P}_{S,t}$ instead of $\mathbb{P}_{\mathbb{D},t}$, and let $\xi_{S,t} = \langle \mathbb{P}_{S,t} \rangle$.

As in Chapter 5 we have the following property.

Lemma 6.1.1 *If $\mathbb{D}$ has discs labelled $S_1^{\varepsilon_1}, S_2^{\varepsilon_2}, \cdots, S_n^{\varepsilon_n}$, and a spray with labels $U_1(\boldsymbol{y})$, $U_2(\boldsymbol{y}), \cdots, U_n(\boldsymbol{y})$, then*

$$\xi_{\mathbb{D},t} = \sum_{i=1}^{n} \varepsilon_i \overline{U}_i(\boldsymbol{a}) \xi_{S_i,t}.$$

Proof. Let $R_{y,t} = t^{-1}a_y t \check{a}_y^{-1}$ ($y \in \boldsymbol{y}$). Suppose that for each $1 \le i \le n$, $\mathbb{S}_i$ has discs labelled $S_{i,1}^{\varepsilon_{i,1}}, \cdots, S_{i,j_i}^{\varepsilon_{i,j_i}}$, and has a spray labelled $V_{i,1}, \cdots, V_{i,j_i}$, words on $\boldsymbol{x}$. Thus, we have a spray for $\mathbb{D}(\boldsymbol{a})$ labelled

$$U_1(\boldsymbol{a})V_{1,1},\ \cdots,\ U_1(\boldsymbol{a})V_{1,j_1},\ \cdots,\ U_n(\boldsymbol{a})V_{n,1},\ \cdots,\ U_n(\boldsymbol{a})V_{n,j_n},$$

and a spray for $\mathbb{D}(\check{\boldsymbol{a}})$ labelled

$$U_1(\check{\boldsymbol{a}})V_{1,1},\ \cdots,\ U_1(\check{\boldsymbol{a}})V_{1,j_1},\ \cdots,\ U_n(\check{\boldsymbol{a}})V_{n,1},\ \cdots,\ U_n(\check{\boldsymbol{a}})V_{n,j_n}.$$

Then as in the proof of Lemma 5.1.1 we have

$$\begin{aligned}\mu_2(\xi_{\mathbb{D},t}) &= \sum_{i=1}^{n}\sum_{l=1}^{j_i}\Big(\varepsilon_i\epsilon_{i,l}\bar{t}\,\overline{U}_i(\check{\boldsymbol{a}})\overline{V}_{i,l}e_{S_{i,l}} - \varepsilon_i\epsilon_{i,l}\overline{U}_i(\boldsymbol{a})\overline{V}_{i,l}e_{S_{i,l}}\Big) + \sum_{i=1}^{n}\sum_{y\in\boldsymbol{y}}\varepsilon_i\overline{U}_i(\boldsymbol{a})\frac{\partial^H S_i}{\partial y}(\boldsymbol{a})e_{R_{y,t}}\\ &= \sum_{i=1}^{n}\sum_{l=1}^{j_i}\Big(\varepsilon_i\epsilon_{i,l}\overline{U}_i(\boldsymbol{a})\bar{t}\overline{V}_{i,l}e_{S_{i,l}} - \varepsilon_i\epsilon_{i,l}\overline{U}_i(\boldsymbol{a})\overline{V}_{i,l}e_{S_{i,l}}\Big) + \sum_{i=1}^{n}\sum_{y\in\boldsymbol{y}}\varepsilon_i\overline{U}_i(\boldsymbol{a})\frac{\partial^H S_i}{\partial y}(\boldsymbol{a})e_{R_{y,t}}.\end{aligned}$$

But

$$\mu_2(\xi_{S_i,t}) = \sum_{l=1}^{j_i}\Big(\epsilon_{i,l}\bar{t}\overline{V}_{i,l}e_{S_{i,l}} - \epsilon_{i,l}\overline{V}_{i,l}e_{S_{i,l}}\Big) + \sum_{y\in\boldsymbol{y}}\frac{\partial^H S_i}{\partial y}(\boldsymbol{a})e_{R_{y,t}}.$$

So, we have

$$\mu_2(\xi_{\mathbb{D},t}) = \sum_{i=1}^{n}\varepsilon_i\overline{U}_i\xi_{S_i,t}$$

as required. □

Let $\mathbb{P}$ be any spherical picture over $\boldsymbol{\mathcal{P}}$. We introduce the *t-circles* (*outward or inward directed*), *minimal t-circles* and *trivial t-circles* in an obvious way as we did in §5.1.

In §6.1.2, 6.1.3 below the above notations will remain fixed.

6.1.2 Upper bounds

Suppose $\boldsymbol{X}_0$ is a finite set of generating pictures of $\pi_2(\boldsymbol{\mathcal{P}}_0)$, and let $\boldsymbol{X}_t$ be the set of all spherical pictures $\mathbb{P}_{S,t}$, $S \in \boldsymbol{s}$. Let

$$b = \max\{A(\mathbb{S}),\ A(\check{\mathbb{S}});\ S \in \boldsymbol{s}\},$$

and let

$$\boldsymbol{\alpha} = \{t^{-1}a_y t\check{a}_y^{-1}:\ y \in \boldsymbol{y}\}.$$

Theorem 6.1.2 *The following statements hold.*

(i) *The set $\boldsymbol{X}_0 \cup \boldsymbol{X}_t$ generates $\pi_2(\boldsymbol{\mathcal{P}})$.*

(ii) *Let $n > 0$, $m \geq 0$ be any two integers. If $\mathbb{P}$ is a spherical picture over $\mathcal{P}$ with $A(\mathbb{P}) = n$, and the number of $\boldsymbol{\alpha}$-discs of $\mathbb{P}$ is m. Then*

$$V_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t}(\mathbb{P}) \leq \overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(2b\overline{\delta}^{(1)}_{\mathcal{H}}(m) + n - m) + \overline{\delta}^{(1)}_{\mathcal{H}}(m).$$

From this theorem we immediately have the following.

Theorem 6.1.3 *We have*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t} \preceq \overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\overline{\delta}^{(1)}_{\mathcal{H}}).$$

In particular, if $\mathcal{P}_0$ is aspherical, then

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t} \preceq \overline{\delta}^{(1)}_{\mathcal{H}};$$

and if H is word hyperbolic, then

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t} \preceq \overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}.$$

Proof of Theorem 6.1.2:

We will concentrate on proving (ii) as Theorem 2′ of [BaPr] implies (i). However, the following argument also implicitly gives a proof of (i).

Suppose $\mathbb{P}$ contains q t-circles, say C_1, C_2, $\cdots$, C_q, and each C_i contains m_i $\boldsymbol{\alpha}$-discs. For each $1 \leq i \leq q$, if C_i is not minimal, then there are a number of t-circles, say C_{j_1}, C_{j_2}, $\cdots$, $C_{j_{\alpha_i}}$ ($1 \leq j_l \leq q$, $1 \leq l \leq \alpha_i$) for some natural number $\alpha_i > 0$, which lie inside C_i such that there are no other t-circles in the region bounded by C_i and all C_{j_l} ($1 \leq l \leq \alpha_i$) as illustrated in Fig. 6.2. Note that these t-circles C_{j_l} may be not minimal. We denote by $\mathbb{T}_i$ the subpicture (over $\mathcal{P}_0$) between C_i and these C_{j_l}'s. If C_i is a minimal t-circle, then we also denote by $\mathbb{T}_i$ the subpicture (over $\mathcal{P}_0$) enclosed by C_i, and we let $\alpha_i = 0$. We denote the subpicture (over $\mathcal{P}_0$) outside all t-circles by $\mathbb{T}_{q+1}$. Let $n_i = A(\mathbb{T}_i)$ ($1 \leq i \leq q+1$), then $n = \sum_{i=1}^{q}(m_i + n_i) + n_{q+1}$.

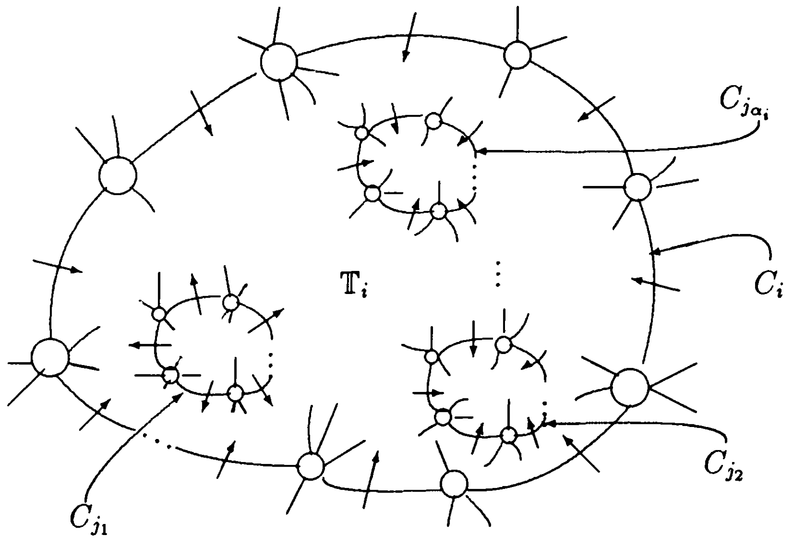


Fig. 6.2

Consider a neighbourhood of a typical t-circle C_j of $\mathbb{P}$ as illustrated in Fig. 6.3. Note that if C_j is minimal then the subpicture $\mathbb{T}'_j = \mathbb{T}_j$. Let $\beta_j^{(1)}$ and $\beta_j^{(2)}$ be the closed transverse paths in $\mathbb{P}$ lying *just* outside and inside C_j respectively. Then there is a word $W(\boldsymbol{y})$ such that the labels on $\beta_j^{(1)}$, $\beta_j^{(2)}$ are $W_j(\boldsymbol{a})$, $W_j(\check{\boldsymbol{a}})$ if C_j is inward directed (resp. $W_j(\check{\boldsymbol{a}})$, $W_j(\boldsymbol{a})$ if C_j is outward directed). Suppose, for definiteness that C_j is outward directed, so that the label on $\beta_j^{(2)}$ is $W_j(\boldsymbol{a})$. Note that if C_j is trivial, then both $W_j(\boldsymbol{a})$ and $W_j(\check{\boldsymbol{a}})$ are the empty words. Since $L(W_j(\boldsymbol{y})) = m_j$, we can choose a picture $\mathbb{D}_j^*$ over $\boldsymbol{\mathcal{H}}$ with boundary label $W_j(\boldsymbol{y})$ such that $A(\mathbb{D}_j^*) \leq \delta_{\boldsymbol{\mathcal{H}}}^{(1)}(m_j)$. Thus, if we let $\mathbb{D}_j = \mathbb{D}_j^*(\boldsymbol{a})$ and $\check{\mathbb{D}}_j = \mathbb{D}_j^*(\check{\boldsymbol{a}})$ over $\boldsymbol{\mathcal{P}}_0$, then

$$A(\mathbb{D}_j), A(\check{\mathbb{D}}_j) \leq b\delta_{\boldsymbol{\mathcal{H}}}^{(1)}(m_j).$$

Note that $\mathbb{D}_j$ and $\check{\mathbb{D}}_j$ have boundary labels $W_j(\boldsymbol{a})$ and $W_j(\check{\boldsymbol{a}})$ respectively.

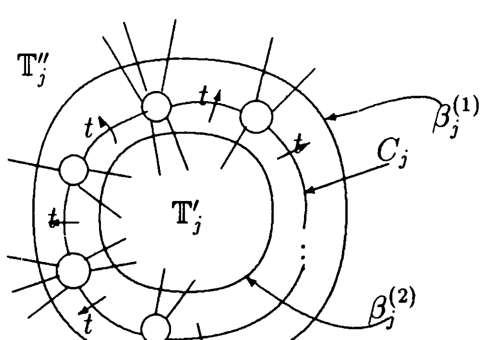


Fig. 6.3

We now carry out modifications on each *minimal* t-circle C_j as shown in Fig. 6.4.

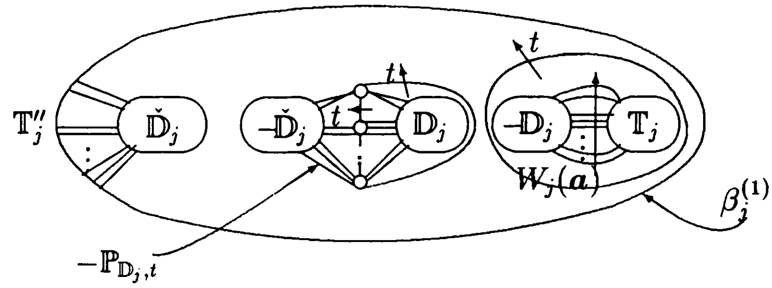


Fig. 6.4

We let $\mathbb{Q}_j$ be the spherical picture consisting of $-\mathbb{D}_j$ and $\mathbb{T}_j$. Note that if C_j is inward directed then $\mathbb{Q}_j$ consists of $-\check{\mathbb{D}}_j$ and $\mathbb{T}_j$, and the spherical picture in the center is $\mathbb{P}_{\mathbb{D}_j,t}$. Removing all these $\mathbb{Q}_j$'s and $-\mathbb{P}_{\mathbb{D}_j,t}$'s or $\mathbb{P}_{\mathbb{D}_j,t}$'s from $\mathbb{P}$ gives a new spherical picture $\mathbb{P}'$ over $\mathcal{P}$. Repeat the above modifications on all minimal t-circles of $\mathbb{P}'$ (if there are any). Now, if C_i is a minimal t-circle of $\mathbb{P}'$, then C_i is also a t-circle of $\mathbb{P}$. Since C_i is not minimal, we can suppose that C_i is of the form as illustrated in Fig. 6.2 (and hence, the t-circles C_{j_1}, C_{j_2}, $\cdots$, $C_{j_{\alpha_i}}$ are minimal in $\mathbb{P}$). Thus, in $\mathbb{P}'$, C_i together with its interior subpicture has the form as illustrated in Fig. 6.5.

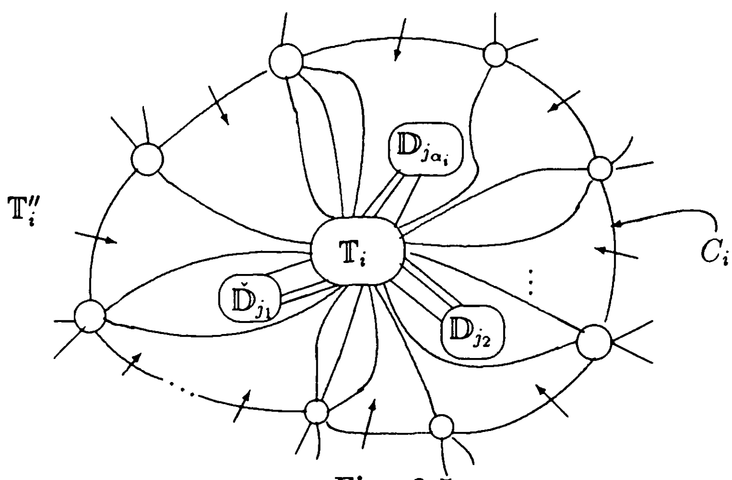


Fig. 6.5

Thus, after the modification on C_i we obtain a geometric configuration similar to Fig. 6.4 except that the spherical subpicture $\mathbb{Q}_j$ now is the spherical subpicture $\mathbb{Q}_i$ of the form as illustrated in Fig. 6.6.

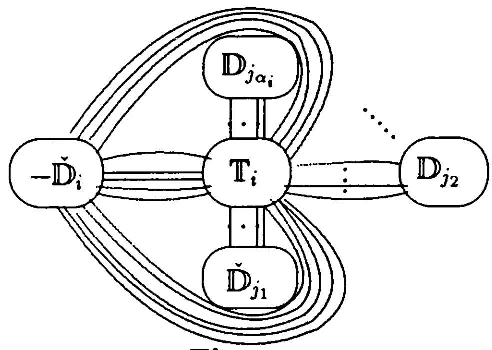


Fig. 6.6

We may repeat the modifications and finally we arrive at a spherical picture $\mathbb{Q}_{q+1}$ of the form as illustrated in Fig. 6.7, where we suppose that $C_{l_1}, \cdots, C_{l_h}$ are all the t-circles bounding $\mathbb{T}_{q+1}$, and each $\mathbb{D}'_{lk}$ $(1 \le k \le h)$ is $\check{\mathbb{D}}_{l_k}$ or $\mathbb{D}_{l_k}$ according to whether C_{l_k} is outward directed or inward directed.

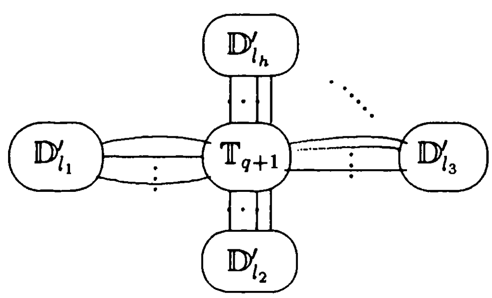


Fig. 6.7

Thus, we see that $\mathbb{P}$ is equivalent to a spherical picture $\mathbb{P}^*$ which is a collection of

(i) q spherical subpictures of the form $\mathbb{C}_i = \epsilon_i \mathbb{P}^{U_i}_{\mathbb{D}_i,t}$ $(1 \le i \le q)$ where $\mathbb{P}_{\mathbb{D}_i,t}$ is as defined in Fig. 6.1, U_i are certain words on $\boldsymbol{x}_0 \cup \{t\}$, and $\epsilon_i = \pm 1$;

(ii) q spherical subpictures of the form $\mathbb{Q}_i^{V_i}$ $(1 \le i \le q)$ where V_i are certain words on $\boldsymbol{x}_0 \cup \{t\}$; and

(iii) a spherical picture $\mathbb{Q}_{q+1}$ consisting of $\mathbb{T}_{q+1}$ and some $\mathbb{D}_i$'s or $\check{\mathbb{D}}_i$'s (only one of each this pair) if C_i bounds $\mathbb{T}_{q+1}$, $1 \le i \le q$.

Thus, by Lemma 1.3.4 we have

$$V_{\mathcal{P},\boldsymbol{X}_0 \cup \boldsymbol{X}_t}(\mathbb{P}) \le \sum_{i=1}^{q} V_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{Q}_i) + V_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{Q}_{q+1}) + \sum_{i=1}^{q} V_{\mathcal{P},\boldsymbol{X}_0 \cup \boldsymbol{X}_t}(\mathbb{C}_i).$$

Let $\varepsilon_i = 0$ or 1 according to whether or not C_i bounds $\mathbb{T}_{q+1}$. Then $A(\mathbb{Q}_{q+1}) \leq n_{q+1} + \sum_{i=1}^{q}(1-\varepsilon_i)b\delta^{(1)}_{\mathcal{H}}(m_i)$. Also, for each $1 \leq i \leq q$, if C_i is minimal, then $A(\mathbb{Q}_i) \leq n_i + b\delta^{(1)}_{\mathcal{H}}(m_i)$; if C_i is not minimal, say as depicted in Fig. 6.2, then $\varepsilon_{j_l} = 1$ $(1 \leq l \leq \alpha_i)$, and

$$A(\mathbb{Q}_i) \leq n_i + b\delta^{(1)}_{\mathcal{H}}(m_i) + \sum_{l=1}^{\alpha_i} \varepsilon_{j_l} b\delta^{(1)}_{\mathcal{H}}(m_{j_l}).$$

We observe that, for all $1 \leq l \leq \alpha_i$, $\delta^{(1)}_{\mathcal{H}}(m_{j_l})$ could not be counted elsewhere for any another $A(\mathbb{Q}_k)$ with $1 \leq k \leq q$ and $k \neq i, j_l$. Moreover, by Lemma 6.1.1, $V_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t}(\mathbb{C}_i) = V^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{P}_{\mathbb{D}_i,t}) \leq \delta^{(1)}_{\mathcal{H}}(m_i)$ $(1 \leq i \leq q)$. Now,

$$\begin{aligned}
\sum_{i=1}^{q} V_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{Q}_i) \leq & \sum_{\substack{1\leq i\leq q\\ \alpha_i=0}} \delta^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(n_i + b\delta^{(1)}_{\mathcal{H}}(m_i)) \\
& + \sum_{\substack{1\leq i\leq q\\ \alpha_i>0}} \delta^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(n_i + b\delta^{(1)}_{\mathcal{H}}(m_i) + \sum_{l=1}^{\alpha_i} \varepsilon_{j_l} b\delta^{(1)}_{\mathcal{H}}(m_{j_l})) \\
\leq & \ \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\sum_{\substack{1\leq i\leq q\\ \alpha_i=0}} (n_i + b\delta^{(1)}_{\mathcal{H}}(m_i))) \\
& + \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\sum_{\substack{1\leq i\leq q\\ \alpha_i>0}} (n_i + b\delta^{(1)}_{\mathcal{H}}(m_i) + \sum_{l=1}^{\alpha_i} \varepsilon_{j_l} b\delta^{(1)}_{\mathcal{H}}(m_{j_l}))).
\end{aligned}$$

By the above observation, we have

$$\sum_{\substack{1\leq i\leq q\\ \alpha_i>0}} \sum_{l=1}^{\alpha_i} \varepsilon_{j_l} b\delta^{(1)}_{\mathcal{H}}(m_{j_l}) \leq \sum_{i=1}^{q} \varepsilon_i b\delta^{(1)}_{\mathcal{H}}(m_i).$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{q} V_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{Q}_i) \leq & \ \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\sum_{\substack{1\leq i\leq q\\ \alpha_i=0}} (n_i + b\delta^{(1)}_{\mathcal{H}}(m_i))) \\
& + \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\sum_{\substack{1\leq i\leq q\\ \alpha_i>0}} (n_i + b\delta^{(1)}_{\mathcal{H}}(m_i)) + \sum_{i=1}^{q} \varepsilon_i b\delta^{(1)}_{\mathcal{H}}(m_i)).
\end{aligned}$$

Moreover,

$$V^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\mathbb{Q}_{q+1}) \leq \delta^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(n_{q+1} + \sum_{i=1}^{q}(1-\varepsilon_i)b\delta^{(1)}_{\mathcal{H}}(m_i)).$$

Thus,

$$V_{\mathcal{P},\boldsymbol{X}_0\cup\boldsymbol{X}_t}(\mathbb{P}) \leq \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\sum_{\substack{1\leq i\leq q\\ \alpha_i=0}} (n_i + b\delta^{(1)}_{\mathcal{H}}(m_i)))$$

$$+\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}\Big(\sum_{\substack{1\le i\le q\\ \alpha_i>0}}(n_i+b\delta^{(1)}_{\mathcal{H}}(m_i))+\sum_{i=1}^{q}\varepsilon_i b\delta^{(1)}_{\mathcal{H}}(m_i)\Big)$$

$$+\delta^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}\Big(n_{q+1}+\sum_{i=1}^{q}(1-\varepsilon_i)b\delta^{(1)}_{\mathcal{H}}(m_i)\Big)+\sum_{i=1}^{q}\delta^{(1)}_{\mathcal{H}}(m_i)$$

$$\le \overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}\Big(\sum_{i=1}^{q+1}n_i+2b\overline{\delta}^{(1)}_{\mathcal{H}}\big(\sum_{i=1}^{q}m_i\big)\Big)+\overline{\delta}^{(1)}_{\mathcal{H}}\big(\sum_{i=1}^{q}m_i\big)$$

$$\le \overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(n-m+2b\overline{\delta}^{(1)}_{\mathcal{H}}(m))+\overline{\delta}^{(1)}_{\mathcal{H}}(m)$$

as required. □

Theorem 6.1.3 can be generalized as follows. Let $\boldsymbol{t}$ be a finite set such that, for each $t\in\boldsymbol{t}$, there are a pair of finitely presented subgroups H_t and $\check{H}_t$ of G_0 together with an isomorphism $\gamma_t:\ H_t\longrightarrow\check{H}_t$. For each $t\in\boldsymbol{t}$, choose a finite set $\boldsymbol{a}_{y,t}=\{a_{y,t}:\ y\in\boldsymbol{y}_t\}$ of words on $\boldsymbol{x}_0$ representing a generating set of H_t. Let $\check{a}_{y,t}=\{\check{a}_{y,t}: y\in\boldsymbol{y}_t\}$ be such that for each $y\in\boldsymbol{y}_t$, $\check{a}_{y,t}$ represents the image under γ_t of $\overline{a}_{y,t}$ for each $y\in\boldsymbol{y}_t$. Let $\mathcal{H}_t=\langle\boldsymbol{y}_t;\ \boldsymbol{s}_t\rangle$ be a finite presentation for H_t under the map $y_t\longmapsto a_{y,t}$ $(y\in\boldsymbol{y}_t,\ t\in\boldsymbol{t})$. The *HNN-extension* of the *base group* G_0 with *stable letters* $t\in\boldsymbol{t}$ and associated subgroups H_t, $\check{H}_t$ $(t\in\boldsymbol{t})$ has the following presentation (see [Co] for reference)

$$\mathcal{Q}=\langle\boldsymbol{x}_0,\ \boldsymbol{t};\ \boldsymbol{r}_0,\ t^{-1}a_{y,t}t\check{a}^{-1}_{y,t}\ (y\in\boldsymbol{y}_t,\ t\in\boldsymbol{t})\rangle.$$

For each $t\in\boldsymbol{t}$, we let $\boldsymbol{X}_t=\{\mathbb{P}_{S_t,t}:\ S_t\in\boldsymbol{s}_t\}$ to be the set of spherical pictures over $\mathcal{Q}$ as we defined in Fig.6.1. Since in any spherical picture $\mathbb{P}$ over $\mathcal{Q}$, each pair of $\boldsymbol{t}$-circles can not meet, thus, by an analysis of the proof of Theorem 6.1.2 (by taking account of $\boldsymbol{t}$-circles instead of single t-circles) we have

Theorem 6.1.4 (i) *The set* $\boldsymbol{X}=\boldsymbol{X}_0\cup(\cup_{t\in\boldsymbol{t}}\boldsymbol{X}_t)$ *generates* $\pi_2(\mathcal{Q})$.

(ii) *Suppose* $\phi:\mathbb{N}\longrightarrow\mathbb{R}^+$ *is a subnegative function satisfying*

$$\delta^{(1)}_{\mathcal{H}_t}(n)\le\phi(n),\quad n\in\mathbb{N},\ t\in\boldsymbol{t}.$$

Then

$$\delta^{(2)}_{\mathcal{Q},\boldsymbol{X}}\preceq\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\phi).$$

Corollary 6.1.5 *If* $\delta^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}\preceq n^{\alpha}$ *and* $\delta^{(1)}_{\mathcal{H}_t}\preceq n^{\beta}$, $t\in\boldsymbol{t}$, *for some real numbers* $\alpha\ge 0$ *and* $\beta\ge 1$, *then*

$$\delta^{(2)}_{\mathcal{Q},\boldsymbol{X}}\preceq n^{\alpha\beta}.$$

We can use Theorem 6.1.4 for split extensions of the form $G_0 \rtimes_\phi F$, where F is a free group of finite rank, i.e. $G_0 \rtimes_\phi F$ is also the HNN-extension of the base group G_0 with stable letters a free generating set $\boldsymbol{t}$ of F and associated subgroup G_0. Thus, we have the following corollary.

Corollary 6.1.6 *Suppose*

$$\mathcal{Q} = \langle \boldsymbol{x}_0,\ \boldsymbol{t};\ \boldsymbol{r}_0,\ t^{-1}xt^{-1}\lambda_{xt}^{-1}\ (t \in \boldsymbol{t},\ x \in \boldsymbol{x}_0)\rangle$$

is a presentation for $G_0 \rtimes_\phi F$, where λ_{xt} are words on $\boldsymbol{x}_0$ representing $\phi_{\bar{t}}(\bar{x})$ ($t \in \boldsymbol{t}$, $x \in \boldsymbol{x}_0$). If $\boldsymbol{X}$ is a finite set of generating pictures for $\pi_2(\mathcal{Q})$, then

$$\delta^{(2)}_{\mathcal{Q},\boldsymbol{X}} \preceq \bar{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\bar{\delta}^{(1)}_{\mathcal{P}_0}).$$

In particular, by using the facts that $\delta^{(2)}_{\mathbb{Z}^3} \preceq n^{\frac{3}{2}}$, $\delta^{(2)}_{\mathbb{Z}^m} \preceq n^2$ ($m > 3$), and $\delta^{(1)}_{\mathbb{Z}^q} \preceq n^2$ ($q \geq 1$) we have the following (for the case of the groups $\mathbb{Z}^2 \rtimes_\phi F$ we will have particular discussion in Chapter 7).

Corollary 6.1.7 *The following inequalities are true.*

(i) $\delta^{(2)}_{\mathbb{Z}^3 \rtimes_\phi F} \preceq n^3$;

(ii) $\delta^{(2)}_{\mathbb{Z}^m \rtimes_\phi F} \preceq n^4$ $(m \geq 4)$.

6.1.3 Amalgamated free products

Let G_1 be another group of type F_3 finitely presented by $\mathcal{P}_1 = \langle \boldsymbol{x}_1;\ \boldsymbol{r}_1\rangle$. Suppose that H and $\check{H}$ are finitely presented subgroups of G_0 and G_1 respectively such that there is an isomorphism $\gamma : H \longrightarrow \check{H}$. We still use the notations $\boldsymbol{y}$, $\boldsymbol{a}$, a_y, $\check{\boldsymbol{a}}$, $\check{a}_y$, and $\mathcal{H}$ as defined in the previous subsection except that, here $\check{\boldsymbol{a}}$ is a finite set of words on $\boldsymbol{x}_1$.

The *amalgamated free product* $G = G_0 *_H G_1$ of G_0 and G_1 with associated subgroups H and $\check{H}$ is then finitely presented by

$$\mathcal{P} = \langle \boldsymbol{x}_0,\ \boldsymbol{x}_1;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1,\ \boldsymbol{\alpha}_0\rangle,$$

where $\boldsymbol{\alpha}_0 = \{a_y\check{a}_y^{-1} :\ y \in \boldsymbol{y}\}$.

Consider the presentation

$$\mathcal{P}^* = \langle \boldsymbol{x}_0,\ \boldsymbol{x}_1,\ t;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1,\ \boldsymbol{\alpha} \rangle$$

where $\boldsymbol{\alpha} = \{t^{-1}a_y t \check{a}_y^{-1};\ y \in \boldsymbol{y}\}$. Then $\mathcal{P}^*$ is a presentation for the HNN-extension G^* of the free product $G_0 * G_1$ (presented by $\mathcal{P}_0 * \mathcal{P}_1 = \langle \boldsymbol{x}_0,\ \boldsymbol{x}_1;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1 \rangle$) with associated subgroups H and $\check{H}$. Let $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ be two finite sets of generating pictures for $\pi_2(\mathcal{P}_0)$ and $\pi_2(\mathcal{P}_1)$ respectively, and let $\boldsymbol{X}_t = \{\mathbb{P}_{A,t} :\ A \in \boldsymbol{\alpha}\}$ be the set of spherical pictures over $\mathcal{P}^*$ obtained as in Fig. 6.1. Let $\boldsymbol{X}^* = \boldsymbol{X}_0 \cup \boldsymbol{X}_1 \cup \boldsymbol{X}_t$. Then $\boldsymbol{X}^*$ generates $\pi_2(\mathcal{P}^*)$. By Theorem 6.1.3 and Theorem 5.1.5 we have

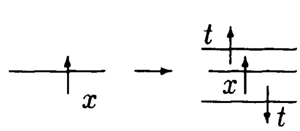
$$\delta^{(2)}_{\mathcal{P}^*,\boldsymbol{X}^*} \preceq \overline{\delta}^{(2)}_{\mathcal{P}_1 * \mathcal{P}_0, \boldsymbol{X}_0 \cup \boldsymbol{X}_1}(\overline{\delta}^{(1)}_{\mathcal{P}_t}) \preceq \max\left\{\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\overline{\delta}^{(1)}_{\mathcal{P}_t}),\ \overline{\delta}^{(2)}_{\mathcal{P}_1,\boldsymbol{X}_1}(\overline{\delta}^{(1)}_{\mathcal{P}_t})\right\}.$$

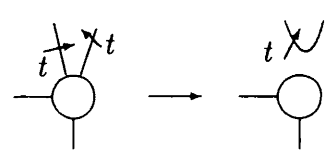
As remarked in [BaPr], each spherical picture $\mathbb{P}$ over $\mathcal{P}$ can be converted to a spherical picture $\mathbb{P}^*$ over $\mathcal{P}^*$ as follows:

(a) for each arc labelled by an element $x \in \boldsymbol{x}_1 \cup \boldsymbol{x}_1^{-1}$ replace it by three parallel arcs labelled by t^{-1}, x, t respectively;

(b) if, while reading around a disc we encounter two successive arcs labelled by t and t^{-1}, then perform a bridge move to delete them;

(c) remove all floating circles.

(a): 

(b): 

Example 6.1.8 Let $G_0 = \mathbb{Z}_6 \oplus \mathbb{Z}_2$, $G_1 = \mathbb{Z}_9 \oplus \mathbb{Z}_4$ with presentations

$$\mathcal{P}_0 = \langle a,\ b;\ [a,\ b],\ a^6,\ b^2 \rangle, \quad \text{and} \quad \mathcal{P}_1 = \langle c,\ d;\ [c,\ d],\ c^9,\ d^4 \rangle$$

respectively. If H and $\check{H}$ are the subgroups of G_0 and G_1 generated by $\{\overline{a}^2, \overline{b}\}$ and $\{\overline{c}^3, \overline{d}^2\}$ respectively, then the mapping

$$\gamma : a^2 \longmapsto c^3, \quad b \longmapsto d^2$$

induces an isomorphism from H to $\check{H}$, and

$$\mathcal{P} = \langle a,\ b,\ c,\ d;\ [a,\ b],\ [c,\ d],\ a^6,\ b^2,\ c^9,\ d^4,\ a^2c^{-3},\ bd^{-2} \rangle$$

is a presentation for $G_0 *_H G_1$.

Let

$$\mathcal{P}^* = \langle a,\ b,\ c,\ d,\ t;\ [a,\ b],\ [c,\ d],\ a^6,\ b^2,\ c^9,\ d^4,\ t^{-1}a^2tc^{-3},\ t^{-1}btd^{-2}\rangle,$$

and consider the following conversion (as shown in Fig. 6.8) of a spherical picture $\mathbb{P}$ over $\mathcal{P}$ to a spherical picture $\mathbb{P}^*$ over $\mathcal{P}^*$. (Since the labels in the bottom half are in $\{c,\ d\}$, the conversion is only applied on the top half.)

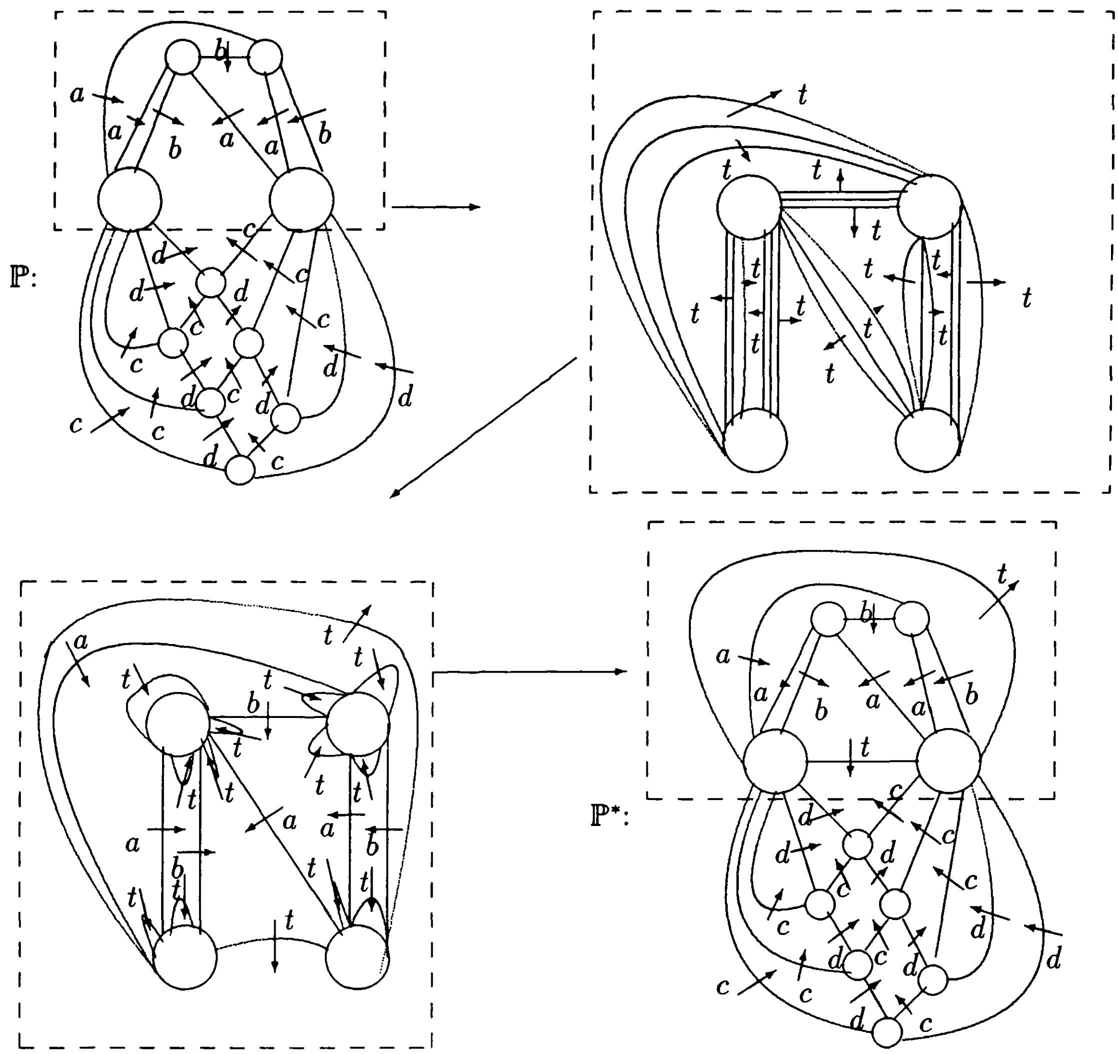


Fig. 6.8

Now we let $\boldsymbol{Y}_t$ be the set obtained from $\boldsymbol{X}_t$ by rubbing of all t-arcs. and consider a spherical picture $\mathbb{P}$ over $\mathcal{P}$. Let $\mathbb{P}^*$ be the picture over $\mathcal{P}^*$ converted from $\mathbb{P}$ by the above operations (a), (b) and (c). Suppose $\mathbb{P}^*$ has q t-circles $C_1, \cdots, C_q$ say, for some natural number q. Then as we did in the proof of Theorem 6.1.2 $\mathbb{P}^*$ is divided into $q+1$

subpictures, say $\mathbb{T}_1$, $\mathbb{T}_2$, $\cdots$, , $\mathbb{T}_{q+1}$, over $\mathcal{P}_0 * \mathcal{P}_1$ plus these q t-circles. Rubbing off all t-arcs of $\mathbb{P}^*$ recovers the picture $\mathbb{P}$ and this has no affect on the $q+1$ subpictures $\mathbb{T}_1$, $\mathbb{T}_2$, $\cdots$, , $\mathbb{T}_{q+1}$. Using broken arcs for t-arcs of $\mathbb{P}^*$ in $\mathbb{P}$ gives q circles $C_1^{(0)}$, $\cdots$, $C_q^{(0)}$ in $\mathbb{P}$ consisting of these broken arcs and $a_y\check{a}_y$-discs ($y \in \boldsymbol{y}$) in $1:1$ correspondence with the t-circles C_1, $\cdots$, C_q of $\mathbb{P}^*$. We call these circles the $t^{(0)}$-circles. Also, if a t-circle C_i say, of $\mathbb{P}^*$ is inward (resp. outward) directed then we say that the corresponding $t^{(0)}$-circle $C_i^{(0)}$ of $\mathbb{P}$ is also inward (resp. outward) directed as illustrated in Fig. 6.9. In particular, all elements of $\boldsymbol{Y}_t$ have a single (inward directed) $t^{(0)}$-circle.

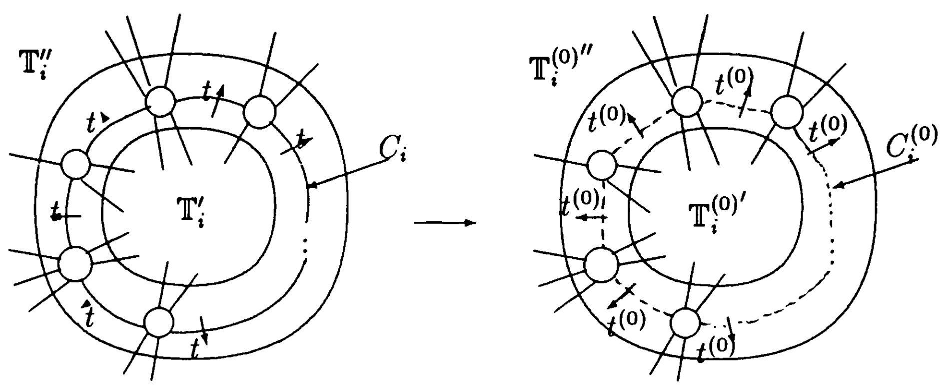


Fig. 6.9

We now can proceed with modifications for each $t^{(0)}$-circle $C_i^{(0)}$ as shown in Figs. 6.4- 6.7, where the t-circles are replaced by the corresponding $t^{(0)}$-circles. Thus, by taking account of $t^{(0)}$-circles and the subpictures $\mathbb{T}_1$, $\mathbb{T}_2$, $\cdots$, , $\mathbb{T}_{q+1}$ and following the proof of Theorem 6.1.2 we obtain the following theorem.

Theorem 6.1.9 (i) *The set* $\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{X}_1 \cup \boldsymbol{Y}_t$ *generates* $\pi_2(\mathcal{P})$.

(ii) *We have*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \max\left\{\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\overline{\delta}^{(1)}_{\mathcal{H}}),\ \overline{\delta}^{(2)}_{\mathcal{P}_1,\boldsymbol{X}_1}(\overline{\delta}^{(1)}_{\mathcal{H}})\right\}.$$

In particular, if $\mathcal{P}_0$ *and* $\mathcal{P}_1$ *are both aspherical, then*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \overline{\delta}^{(1)}_{\mathcal{H}};$$

and if $\mathcal{P}_0$ *and* $\mathcal{P}_1$ *are both word hyperbolic, then*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}} \preceq \max\left\{\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0},\ \overline{\delta}^{(2)}_{\mathcal{P}_1,\boldsymbol{X}_1}\right\}.$$

Remarks 6.1.10 (i) *People could not obtain results similar to our Theorems 6.1.3 and 6.1.9 for the first order Dehn functions. For example, Miller [Mi] proved that there is a finitely presented group G_0 which has unsolvable word problem and can be obtained from a finitely generated free group by applying three successive HNN-extensions, where the associated subgroups are finitely generated free groups. Thus, $\delta^{(1)}_{G_0}$ is faster than any recursive function since G_0 has unsolvable word problem.*

(ii) *As with Theorem 6.1.4, Theorem 6.1.9 can be extended to the case where there is a finite set $\boldsymbol{t}$ such that for each $t \in \boldsymbol{t}$ there are a pair of isomorphic subgroups H_t (finitely presented by $\mathcal{H}_t = \langle \boldsymbol{y}_t;\ \boldsymbol{s}_t\rangle$ say,) and $\check{H}_t$ of G_0 and G_1 respectively. The* extended amalgamated free product *of G_0 and G_1 with associated subgroups $\{H_t,\ \check{H}_t;\ t \in \boldsymbol{t}\}$ has the presentation*

$$\mathcal{Q} = \langle \boldsymbol{x}_0,\ \boldsymbol{x}_1;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1,\ a_{y,t}\check{a}_{y,t}^{-1}\ (y \in \boldsymbol{y}_t,\ t \in \boldsymbol{t})\rangle.$$

One then can obtain

(1) *The set $\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{X}_1 \cup (\cup_{t\in\boldsymbol{t}} \boldsymbol{Y}_t)$ generates $\pi_2(\mathcal{Q})$.*

(2) *Suppose $\phi : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a subnegative function satisfying*

$$\delta^{(1)}_{\mathcal{H}_t}(n) \leq \phi(n), \quad n \in \mathbb{N},\ t \in \boldsymbol{t}.$$

Then

$$\delta^{(2)}_{\mathcal{Q},\boldsymbol{X}} \preceq \max\{\overline{\delta}^{(2)}_{\mathcal{P}_0,\boldsymbol{X}_0}(\phi),\ \overline{\delta}^{(2)}_{\mathcal{P}_1,\boldsymbol{X}_1}(\phi)\}.$$

(iii) *The approaches developed in this section could be further extended to arbitrary* graphs of groups.

6.2 Split extensions

6.2.1 Upper bounds in general

Let K and H be groups of type F_3 finitely presented by presentations $\mathcal{H} = \langle \boldsymbol{x}\,;\boldsymbol{r}\rangle$ and $\mathcal{K} = \langle \boldsymbol{t}\,;\boldsymbol{s}\rangle$ respectively. Let

$$\phi : K \longrightarrow Aut(H)$$

be a homomorphism. For each $k \in K$, we write ϕ_k instead of $\phi(k)$. Consider the split extension $G = H \rtimes_\phi K$ defined by the finite presentation

$$\mathcal{P} = \langle \boldsymbol{x}, \boldsymbol{t}\,; \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{\alpha} \rangle,$$

where for simplicity we require $\boldsymbol{\alpha} = \left\{ t^{-1}xt\lambda_{xt}^{-1};\ x \in \boldsymbol{x},\ t \in \boldsymbol{t} \cup \boldsymbol{t}^{-1} \right\}$ with λ_{xt} a word on $\boldsymbol{x}$ representing the element $\phi_{\bar{t}}(\bar{x})$ of $G(\mathcal{H})$ for each pair $x \in \boldsymbol{x}$, $t \in \boldsymbol{t} \cup \boldsymbol{t}^{-1}$. We point out that the presentation $\mathcal{P}$ chosen for $H \rtimes_\phi K$ is not standard. The standard one is of the form

$$\mathcal{P}' = \langle \boldsymbol{x}, \boldsymbol{t};\ \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{\alpha}' \rangle,$$

where $\boldsymbol{\alpha}' = \{t^{-1}xt\lambda_{xt}^{-1} : t \in \boldsymbol{t},\ x \in \boldsymbol{x}\}$. The reason we choose $\mathcal{P}$ here is for the simplicity of constructing the pictures $\mathbb{D}_{U,x}$ over $\mathcal{P}$ as illustrated in Fig. 6.10 below.

When K is a free group of finite rank, we already have obtained an upper bound for $H \rtimes_\phi K$ in Theorem 6.1.6. Here, we will consider the general case.

Since $t^{-1}x^{-1}t = \lambda_{xt}^{-1}$ in G for each pair $x \in \boldsymbol{x}$ and $t \in \boldsymbol{t} \cup \boldsymbol{t}^{-1}$, we let $\lambda_{x^{-1}t} = \lambda_{xt}^{-1}$. For any word $W = x_1 x_2 \cdots x_k$ on $\boldsymbol{x}$ ($x_i \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$, $1 \leq i \leq k$) and each $t \in \boldsymbol{t} \cup \boldsymbol{t}^{-1}$, we have in G that

$$t^{-1}Wt = \lambda_{x_1 t}\lambda_{x_2 t} \cdots \lambda_{x_k t}.$$

Moreover, for any word $U = t_1 t_2 \cdots t_m$ on $\boldsymbol{t}$ ($t_j \in \boldsymbol{t} \cup \boldsymbol{t}^{-1}$, $1 \leq j \leq m$) and each $x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$, we inductively define a word denoted λ_{xU} on $\boldsymbol{x}$ such that $\lambda_{xU} = U^{-1}xU$ in G as follows. When $m = 1$, then $\lambda_{xU} = \lambda_{xt_1}$. Suppose that we have defined word $\lambda_{xt_1 \cdots t_{m-1}} = x_1 \cdots x_k$ on $\boldsymbol{x}$ ($x_i \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$, $1 \leq i \leq k$) which is equal to $t_{m-1}^{-1}\lambda_{xt_1 \cdots t_{m-2}}t_{m-1}$ in G. Then we define

$$\lambda_{xU} = \lambda_{xt_1 \cdots t_m} = \lambda_{x_1 t_m}\lambda_{x_2 t_m} \cdots \lambda_{x_k t_m}$$

which is equal to $t_m^{-1}\lambda_{xt_1 \cdots t_{m-1}}t_m$ in G. This corresponds to a picture denoted $\mathbb{D}_{U,x}$ over $\mathcal{P}$ as illustrated in Fig. 6.10. We let $a_{U,x}$ be the maximum disc number of the columns of $\mathbb{D}_{U,x}$.

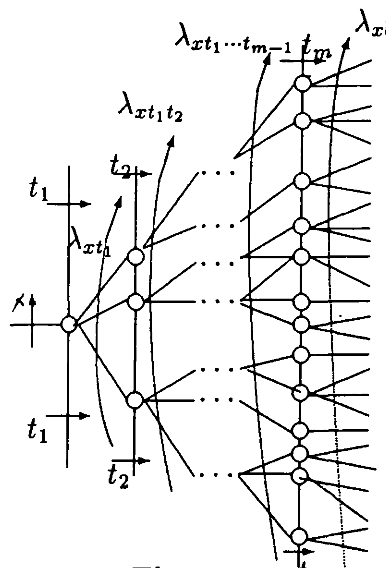


Fig. 6.10

Let G^* be the group defined by the subpresentation

$$\mathcal{P}^* = \langle \boldsymbol{x}, \boldsymbol{t};\ \boldsymbol{r},\ \boldsymbol{\alpha} \rangle$$

of $\mathcal{P}$. Let $G_0 = H_t = \check{H}_t = H$ $(t \in \boldsymbol{t})$. We also have the isomorphism $\gamma_t : H_t \longrightarrow \check{H}_t$ defined by $\phi_{\bar{t}}$. Then G^* is the HNN-extension of the base group H with stable letters $t \in \boldsymbol{t}$ and associated subgroup H.

Let $\boldsymbol{s}^\sharp$ be the set of all cyclic permutations of $\boldsymbol{s} \cup \boldsymbol{s}^{-1}$. For each $S \in \boldsymbol{s}^\sharp$, say

$$S = t_1 t_2 \cdots t_m t^{-1}, \quad t^{-1},\ t_i \in \boldsymbol{t} \cup \boldsymbol{t}^{-1},\ \varepsilon_i = \pm 1,\ i = 1,\ 2,\ \cdots,\ m.$$

Then $\phi_{\bar{t}_1}\phi_{\bar{t}_2} \cdots \phi_{\bar{t}_m}\phi_{\bar{t}}^{-1} = \phi_1$, the identity of $Aut(H)$. Thus, for each $x \in \boldsymbol{x}$, the word $Ut^{-1}xtU^{-1}x^{-1}$ represents the identity of G^*, where $U = t_1 t_2 \cdots t_m$. But $t^{-1}xt = \lambda_{xt}$ in G^*, so $\lambda_{xt} = U^{-1}xU = \lambda_{xU}$ in G^*. Hence $\lambda_{xt} = \lambda_{xU}$ in H (since λ_{xt} and λ_{xU} are words on $\boldsymbol{x}$ and G^* is an HNN-extension of H). So we may choose a picture $\mathbb{H}_{U,x}$ over $\mathcal{H}$ with boundary label $\lambda_{xt}\lambda_{xU}^{-1}$. We then have a spherical picture $\mathbb{Q}_{S,x}$ over $\mathcal{P}$ of the form depicted as in the Fig. 6.11. We then let $\boldsymbol{Z} = \left\{\mathbb{Q}_{S,x};\ x \in \boldsymbol{x},\ S \in \boldsymbol{s}^\sharp\right\}$. Clearly, $\boldsymbol{Z}$ is finite.

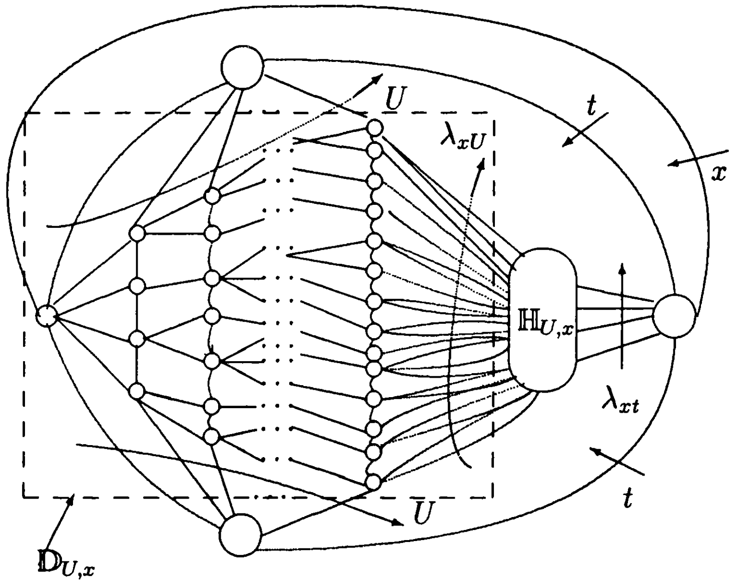


Fig. 6.11

Referring to §6.1, for each $R = x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k} \in \boldsymbol{r}$ and each $t \in \boldsymbol{t}$, we have a picture $\mathbb{P}_{R,t}$ over $\boldsymbol{\mathcal{H}}$ as shown in Fig. 6.12. We then let $\boldsymbol{Y} = \{\mathbb{P}_{R,t} : R \in \boldsymbol{r}, t \in \boldsymbol{t}\}$.

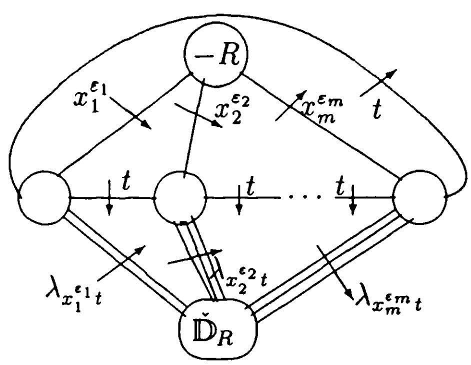


Fig. 6.12

Let $\boldsymbol{X}_H$ be a set of generating pictures of the $\mathbb{Z}H$-module $\pi_2(\boldsymbol{\mathcal{H}})$, and let $\boldsymbol{X}_K$ be a set of generating pictures of the $\mathbb{Z}K$-module $\pi_2(\boldsymbol{\mathcal{K}})$. Let

$$\boldsymbol{X} = \boldsymbol{X}_H \cup \boldsymbol{X}_K \cup \boldsymbol{Y} \cup \boldsymbol{Z}, \quad \boldsymbol{X}^* = \boldsymbol{X}_H \cup \boldsymbol{Y},$$

and let

$$\boldsymbol{s}^* = \{U : U \text{ a word on } \boldsymbol{t} \text{ with } Ut^{-1} \in \boldsymbol{s}^\sharp \text{ for some } t \in \boldsymbol{t} \cup \boldsymbol{t}^{-1}\}.$$

We then let

$$b = \max\{A(\mathbb{P}_{R,t}) : R \in \boldsymbol{r}, t \in \boldsymbol{t}\},$$

and let

$$a_0 = \max\{A(\mathbb{H}_{U,x}) : U \in \boldsymbol{s}^*,\ x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}\}, \quad a_1 = \max\{A(\mathbb{D}_{U,x}) : U \in \boldsymbol{s}^*,\ x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}\},$$
$$a_2 = \max\{a_{U,x} : U \in \boldsymbol{s}^*,\ x \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}\}.$$

(Recall that $a_{U,x}$ is the maximum disc number of the columns of $\mathbb{D}_{U,x}$ in Fig. 6.10.)

Proposition 6.2.1 (i) *The set $\boldsymbol{X}^*$ generates the $\mathbb{Z}G^*$-module $\pi_2(\mathcal{P}^*)$ and the set $\boldsymbol{X}$ generates the $\mathbb{Z}G$-module $\pi_2(\mathcal{P})$.*

(ii) *Let ξ be an element of $\pi_2(\mathcal{P})$ with $A(\xi) = n$, and let $\mathbb{P}$ be a minimal picture over $\mathcal{P}$ representing ξ. Then if $a_2 > 1$,*

$$V_{\mathcal{P},\boldsymbol{X}}(\xi) \leq \overline{\delta}^{(2)}_{\mathcal{H},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\mathcal{H}}(n + (a_1 - 1)na_2^n) + n + a_0 n a_2^n) + \overline{\delta}^{(2)}_{\mathcal{K},\boldsymbol{X}_K}(n) + na_2^n,$$

and if $a_2 = 1$, then

$$V_{\mathcal{P},\boldsymbol{X}}(\xi) \leq \overline{\delta}^{(2)}_{\mathcal{H},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\mathcal{H}}(n + (a_1 - 1)n^2) + a_0 n^2 + n) + \overline{\delta}^{(2)}_{\mathcal{K},\boldsymbol{X}_K}(n) + n^2.$$

From this proposition we then have the following theorem.

Theorem 6.2.2 *If $a_2 > 1$ then*

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(n) \preceq \overline{\delta}^{(2)}_{\mathcal{H},\boldsymbol{X}_H}(\overline{\delta}^{(1)}_{\mathcal{H}}(a_2^n)) + \overline{\delta}^{(2)}_{\mathcal{K},\boldsymbol{X}_K}(n) + a_2^n,$$

and if $a_2 = 1$ then

$$\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(n) \preceq \overline{\delta}^{(2)}_{\mathcal{H},\boldsymbol{X}_H}(\overline{\delta}^{(1)}_{\mathcal{H}}(n^2)) + \overline{\delta}^{(2)}_{\mathcal{K},\boldsymbol{X}_K}(n) + n^2,$$

for all $n \in \mathbb{N}$.

We remark that (i) of Proposition 6.2.1 has also been proved in [BoPr].

Proof of Proposition 6.2.1. Let n be any positive integer and let $\mathbb{P}$ be a minimal spherical picture over $\mathcal{P}$ with $A(\mathbb{P}) = n$. Let n_0, n_1, m be the numbers of $\boldsymbol{r}$-, $\boldsymbol{s}$-, $\boldsymbol{\alpha}$-discs in $\mathbb{P}$ respectively. We will follow the proof of Proposition 5.1.2.

Let $\mathbb{P}^{(1)}$ be the configuration obtained from $\mathbb{P}$ by removing all $\boldsymbol{x}$-arcs. Two $\boldsymbol{s}$-discs of $\mathbb{P}$ will be said to be in the same *1-component* of $\mathbb{P}$ if they lie in the same component of $\mathbb{P}^{(1)}$. If Δ, Δ' are two $\boldsymbol{s}$-discs lying in the same 1-component then they can be connected by a path ρ of $\boldsymbol{t}$-arcs and $(\boldsymbol{s} \cup \boldsymbol{\alpha})$-discs. Regard $\mathbb{P}^{(1)}$ as a graph, then ρ is just a path in this graph. It will be assumed that a maximal forest Φ in $\mathbb{P}^{(1)}$ has been chosen, and that the paths connecting $\boldsymbol{s}$-discs are geodesics in Φ.

Consider a 1-component Ω of $\mathbb{P}$ containing n_Ω $\boldsymbol{s}$-discs and m_Ω $\boldsymbol{\alpha}$-discs. Let $\Delta_0, \Delta_1, \cdots, \Delta_k$ be the $\boldsymbol{s}$-discs in this 1-component and let ρ_λ $(\lambda = 0, 1, \cdots, k)$ be the (geodesic) path in Φ from Δ_0 to Δ_λ. Let d_λ be the number of $\boldsymbol{\alpha}$-discs in ρ_λ. We may assume that

$$0 = d_0 = d_1 = \cdots = d_j \leq d_{j+1} \leq \cdots \leq d_k.$$

We will show that we can modify $\mathbb{P}$ modulo $\boldsymbol{Z}$-pictures so that all the d_λ's are 0.

Suppose $j < k$ (otherwise no modifications are necessary) and consider Δ_{j+1}. Then the discs of ρ_{j+1} together with their incident arcs give a subpicture $\mathbb{Q}$ of $\mathbb{P}$, which has the form as shown in Fig. 6.13 where the disc Θ is an $\boldsymbol{\alpha}$-disc.

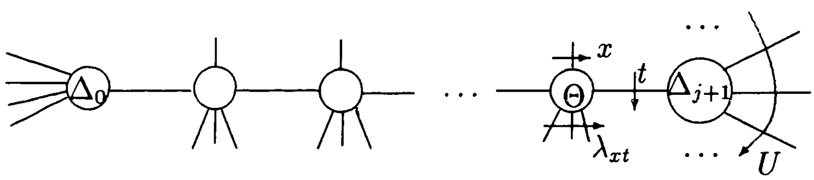


Fig. 6.13

Modulo the $\boldsymbol{Z}$-picture $\mathbb{Q}_{S,x}$ (where $S = Ut^{-1} \in \boldsymbol{s}^\sharp$) we may move Δ_{j+1} nearer Δ_0 as indicated in Fig. 6.14. This gives a new picture $\mathbb{P}'$. Let Ω' is the geometric configuration obtained from Ω by the above operation. A maximal forest Φ' for $\mathbb{P}'^{(1)}$ arises from the maximal forest Φ of $\mathbb{P}^{(1)}$ as follows. Remove all $\boldsymbol{x}$-arcs of $\mathbb{P}'$ to obtain $\mathbb{P}'^{(1)}$. Since the above operation has affect only on the 1-component Ω of $\mathbb{P}$, thus, $\mathbb{P}'^{(1)} = (\mathbb{P}^{(1)} - \Omega) \cup \Omega'$. Now, if $T \subseteq \Phi$ is the maximal tree of Ω, then ρ_{j+1} is a path in T. Let T' be the tree obtained from T by replacing ρ_{j+1} with ρ'_{j+1} as illustrated in Fig. 6.15. Obviously, T' is a maximal tree for Ω'. We then let $\Phi' = (\Phi - T) \cup T'$. (Note that this operation may affect the distances from Δ_0 to Δ_λ $(j + 1 < \lambda \leq k)$.) This operation adds at most a_1 new $\boldsymbol{\alpha}$-discs and at most a_0 $\boldsymbol{r}$-discs to $\mathbb{P}$, and eliminates one $\boldsymbol{\alpha}$-disc (the disc Θ). So, the number of $\boldsymbol{\alpha}$-discs of Ω' is at most $m_\Omega + a_1 - 1$. Moreover, there still are n_1 $\boldsymbol{s}$-discs in $\mathbb{P}'$. We then get new geodesics ρ'_λ $(\lambda = 0, 1, ,\cdots, k)$ with

$$d'_\lambda = 0\,(0 \leq \lambda \leq j),\ d'_{j+1} = d_{j+1} - 1,\ d'_\lambda \leq d_\lambda + a_2 - 1\,(j + 1 < \lambda \leq k).$$

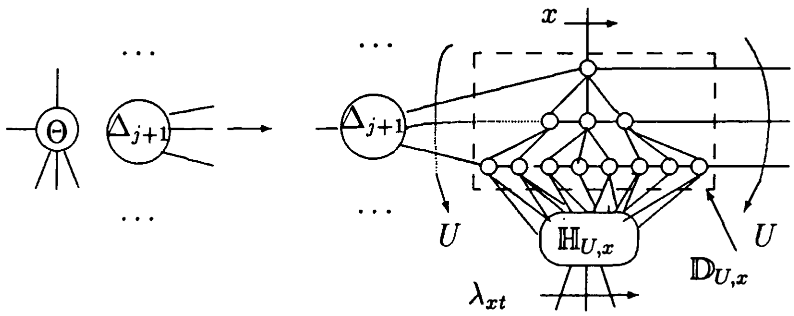


Fig. 6.14

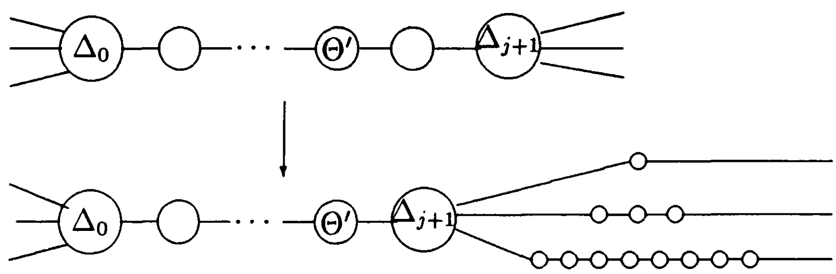


Fig. 6.15

We repeat the above procedure as often as is necessary to decrease d_{j+1} to 0. Note that this requires at most m_Ω operations. At this stage, the new picture has at most $(a_1 - 1)m_\Omega + m_\Omega$ $\boldsymbol{\alpha}$-discs and $a_0 m_\Omega + n_0$ $\boldsymbol{r}$-discs, and in this new picture the geodesic from Δ_0 to Δ_{j+2} will have length at most $(a_2 - 1)m_\Omega + m_\Omega = a_2 m_\Omega$. Inductively, if this process requires at most $a_2^l m_\Omega$ operations for Δ_{j+l+1}, then since this increases the distance from Δ_0 to Δ_{j+l+2} by at most $(a_2 - 1)a_2^l m_\Omega$, the process for Δ_{j+l+2} requires at most $(a_2 - 1)a_2^l m_\Omega + a_2^l m_\Omega = a_2^{l+1} m_\Omega$ operations. We repeat the process successively for $\Delta_{j+2}, \cdots, \Delta_k$, finally arriving (after at most

$$m_\Omega + a_2 m_\Omega + \cdots + a_2^{n_\Omega - 1} m_\Omega = m_\Omega(1 + a_2 + \cdots + a_2^{n_\Omega - 1})$$

operations) at a picture $\mathbb{P}_1$. Now in $\mathbb{P}_1$ there will be a simple closed transverse path α such that the subpicture of $\mathbb{P}_1$ enclosed by α consists precisely of the discs $\Delta_0, \Delta_1, \cdots, \Delta_k$ and their incident arcs. By the same argument as in the proof of Proposition 5.1.2, the label on α is then freely equivalent to the empty word, and so by bridge moves we can create a spherical picture $\mathbb{Q}_1$ over $\mathcal{K}$ inside α with discs $\Delta_0, \Delta_1, \cdots, \Delta_k$.

We may carry out the above procedure for all the 1-components of $\mathbb{P}$ arriving (after at most

$$\sum_\Omega m_\Omega(1 + a_2 + \cdots + a_2^{n_\Omega - 1}) \leq m(1 + a_2 + \cdots + a_2^{n_1 - 1})$$

operations) at a picture $\mathbb{P}^*$ with the following properties:

(i) $\mathbb{P}^*$ *has spherical subpictures* $\mathbb{B}_1, \mathbb{B}_2, \cdots, \mathbb{B}_q$ *each of which is a picture over* $\boldsymbol{\mathcal{K}}$, *and where the total number of discs in* $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \cdots \cup \mathbb{B}_q$ *is* n_1;

(ii) *The picture* $\mathbb{P}^{**}$ *obtained from* $\mathbb{P}^*$ *by removing all* $\mathbb{B}_1, \cdots, \mathbb{B}_q$ *is a picture over* $\boldsymbol{\mathcal{P}}^*$ *having at most* $m+(a_1-1)m(1+a_2+\cdots+a_2^{n_1-1})$ $\boldsymbol{\alpha}$*-discs and at most* $n_0+a_0m(1+a_2+\cdots+a_2^{n_1-1})$ $\boldsymbol{r}$*-discs, i.e.*

$$A(\mathbb{P}^{**}) \leq m + n_0 + (a_0 + a_1 - 1)m(1 + a_2 + \cdots + a_2^{n_1-1}).$$

Now, if $a_2 > 1$, then $1 + a_2 + \cdots + a_2^{n_1-1} \leq a_2^{n_1}$, and so

$$m + n_0 + (a_0 + a_1 - 1)m(1+a_2+\cdots+a_2^{n_1-1}) \leq m+n_0+(a_0+a_1-1)ma_2^{n_1}.$$

Thus, we deduce from Lemma 1.3.4 that, if $a_2 > 1$, then

$$\begin{aligned}
V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi) &\leq V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\mathbb{P}^{**}) + \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1) + ma_2^{n_1} \\
&\leq V_{\boldsymbol{\mathcal{P}}^*,\boldsymbol{X}^*}(\mathbb{P}^{**}) + \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1) + ma_2^{n_1} \\
&\leq \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(m+(a_1-1)ma_2^{n_1})+m+n_0+(a_0+a_1-1)ma_2^{n_1}-m-(a_1-1)ma_2^{n_1}) \\
&\quad +\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1)+ma_2^{n_1} \\
&\qquad \text{(by Corollary 6.1.6 and Theorem 6.1.2)} \\
&= \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(m + (a_1 - 1)ma_2^{n_1})+n_0 + a_0ma_2^{n_1})+\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1)+ma_2^{n_1} \\
&\leq \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(n + (a_1 - 1)na_2^{n}) + n + a_0na_2^{n})+\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n)+na_2^{n}.
\end{aligned}$$

If $a_2 = 1$, then $1 + a_2 + \cdots + a_2^{n_1-1} = n_1$, and so

$$m + n_0 + (a_0 + a_1 - 1)m(1 + a_2 + \cdots + a_2^{n_1-1}) = m + n_0 + (a_0 + a_1 - 1)mn_1.$$

Thus, we deduce from Lemma 1.3.4 that, if $a_2 = 1$, then

$$\begin{aligned}
V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi) &\leq V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\mathbb{P}^{**}) + \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1) + mn_1 \\
&\leq V_{\boldsymbol{\mathcal{P}}^*,\boldsymbol{X}^*}(\mathbb{P}^{**}) + \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1) + mn_1 \\
&\leq \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(m+(a_1-1)mn_1)+m+n_0+(a_0+a_1-1)mn_1-m-(a_1-1)mn_1) \\
&\quad +\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1)+mn_1 \\
&\qquad \text{(by Corollary 6.1.6 and Theorem 6.1.2)} \\
&= \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(m + (a_1 - 1)mn_1)+n_0 + a_0mn_1)+\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n_1)+mn_1 \\
&\leq \overline{\delta}^{(2)}_{\boldsymbol{\mathcal{H}},\boldsymbol{X}_H}(2b\overline{\delta}^{(1)}_{\boldsymbol{\mathcal{H}}}(n + (a_1 - 1)n^2)+n + a_0n^2)+\overline{\delta}^{(2)}_{\boldsymbol{\mathcal{K}},\boldsymbol{X}_K}(n)+n^2.
\end{aligned}$$

This completes our proof. □

We point out that, as shown in the following example, sometimes the subpicture $\mathbb{D}_{U,x}$ illustrated in Fig. 6.11 could be chosen more easily and a better upper bound can be obtained.

Example 6.2.3 *Let*

$$\mathcal{P}' = \langle x_1,\ x_2,\ t_1,\ t_2;\ [x_1,x_2],\ [t_1,t_2],\ x_1^{t_i}x_2^{-2}x_1^{-1},\ x_2^{t_i}x_2^{-1}x_1^{-1}\ (i=1,\ 2)\rangle$$

be a (standard) presentation for the split extension $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}^2$*, where bath* $\phi_{\bar{t}_1}$ *and* $\phi_{\bar{t}_2}$ *are given by the mapping:*

$$x_1 \longmapsto x_1x_2^2, \quad x_2 \longmapsto x_1x_2.$$

Since we have in $G(\mathcal{P}')$ that $x_1^{t_i^{-1}} = x_1^{-1}x_2^2$ and $x_2^{t_i^{-1}} = x_1x_2^{-1}$ $((i=1,\ 2))$, we can change $\mathcal{P}'$ to the presentation

$$\mathcal{P} = \langle x_1, x_2, t_1, t_2;\ [x_1,x_2], [t_1,t_2], x_1^{t_i}x_2^{-2}x_1^{-1}, x_2^{t_i}x_2^{-1}x_1^{-1}, x_1^{t_i^{-1}}x_2^{-2}x_1, x_2^{t_i^{-1}}x_2x_1^{-1},\ (i=1,2)\rangle$$

for the group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}^2$. Now,

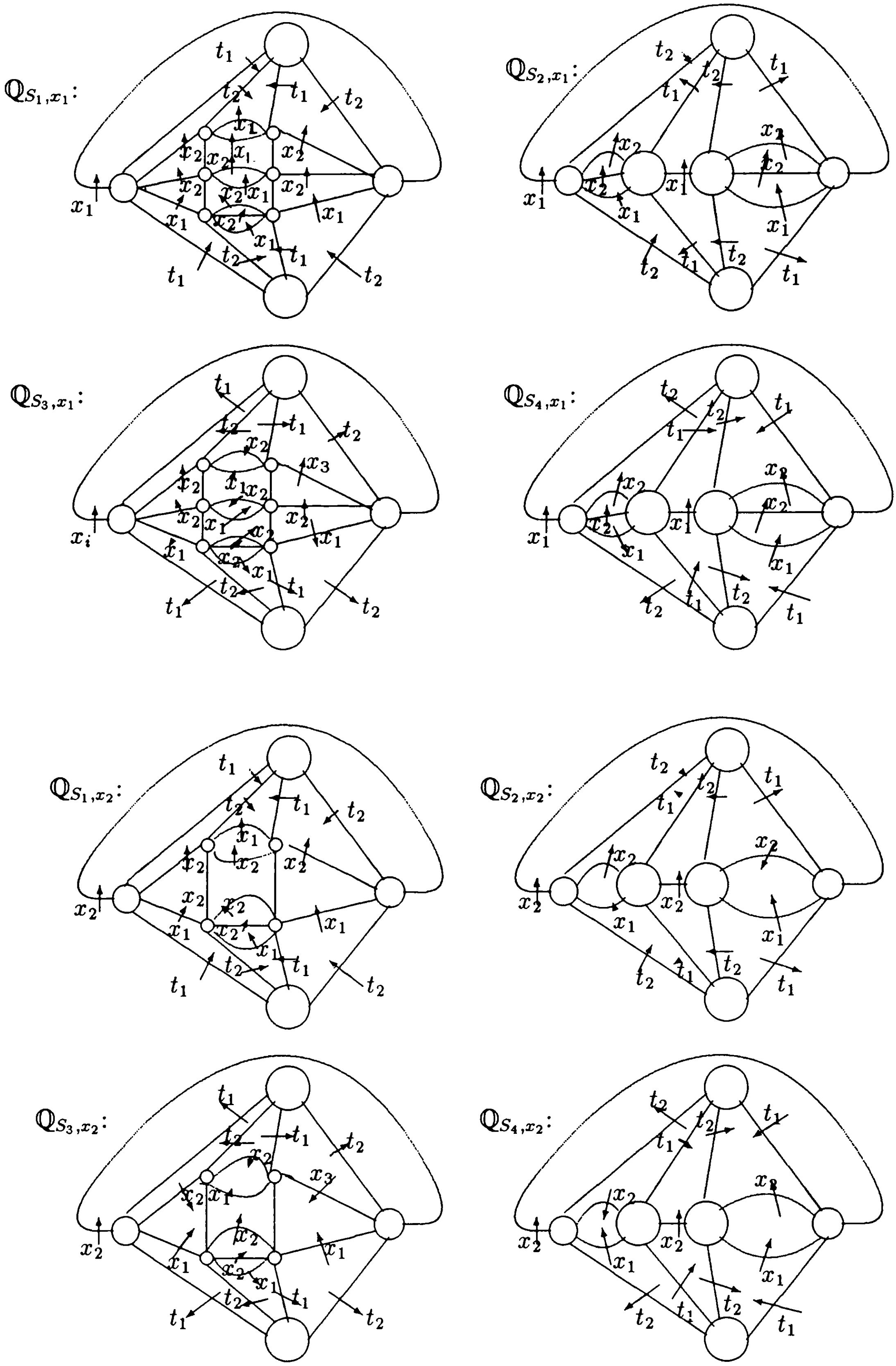
$$\begin{aligned} \boldsymbol{s}^\sharp \ = \ \{ & S_1 = t_1t_2t_1^{-1}t_2^{-1},\ S_2 = t_2t_1^{-1}t_2^{-1}t_1,\ S_3 = t_1^{-1}t_2^{-1}t_1t_2,\ S_4 = t_2^{-1}t_1t_2t_1^{-1} \\ & S_5 = t_2t_1t_2^{-1}t_1^{-1},\ S_6 = t_1t_2^{-1}t_2^{-1}t_2,\ S_7 = t_2^{-1}t_1^{-1}t_2t_1,\ S_8 = t_1^{-1}t_2t_1t_2^{-1}\}, \end{aligned}$$

and so

$$\begin{aligned} \boldsymbol{s}^* \ = \ \{ & U_1 = t_1t_2t_1^{-1},\ U_2 = t_2t_1^{-1}t_2^{-1},\ U_3 = t_1^{-1}t_2^{-1}t_1,\ U_4 = t_2^{-1}t_1t_2 \\ & U_5 = t_2t_1 2t_2^{-1},\ U_6 = t_1t_2^{-1}t_2^{-1},\ U_7 = t_2^{-1}t_1^{-1}t_2,\ U_8 = t_1^{-1}t_2t_1\}. \end{aligned}$$

Let $\mathcal{P}_0 = \langle x_1,\ x_2;\ [x_1,x_2]\rangle$, and $\mathcal{P}_1 = \langle t_1,\ t_2;\ [t_1,t_2]\rangle$. We now can construct $\mathbb{Q}_{S_j,x_i}$ $(i=1,\ 2,\ 1 \le j \le 4)$ as illustrated in Fig. 6.16. (By symmetry, the pictures $\mathbb{Q}_{S_j,x_i}$, $i=1,\ 2,\ 5 \le j \le 8$ can be obtained from $\mathbb{Q}_{S_{j-4},x_i}$ by replacing each arc labelled t_1 with an arc labelled t_2 and vice versa.) Applying the proof of Proposition 6.2.1 to this example we see that $a_0 = 0$, $a_1 = 7$ and $a_2 = 3$. Thus, by the asphericities of $\mathcal{P}_0$ and $\mathcal{P}_1$ we have

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}^2} \preceq 0 + 0 + n3^n \preceq 3^n.$$



$\mathbb{Q}_{S_1,x_1}$:
$\mathbb{Q}_{S_2,x_1}$:
$\mathbb{Q}_{S_3,x_1}$:
$\mathbb{Q}_{S_4,x_1}$:
$\mathbb{Q}_{S_1,x_2}$:
$\mathbb{Q}_{S_2,x_2}$:
$\mathbb{Q}_{S_3,x_2}$:
$\mathbb{Q}_{S_4,x_2}$:

Fig. 6.16

Chapter 7

Upper and lower bounds for $\mathbb{Z}^2 \rtimes_\phi F$

7.1 Preface of the chapter

In this chapter we focus on estimating the upper bounds and lower bounds of the particular split extensions of the form $\mathbb{Z}^2 \rtimes_\phi F$, where F is a free group of finite rank freely generated by $\boldsymbol{t}$.

Since for each $t \in \boldsymbol{t}$, $\phi_{\bar{t}} \in Aut(\mathbb{Z}^2)$, $\phi_{\bar{t}}$ can be identified to be a matrix $M_t \in GL_2(\mathbb{Z})$, the *general linear group* of dimension 2 over $\mathbb{Z}$. We will say that $\phi_{\bar{t}}$ has *eigenvalues* α, β if M_t has. In particular, if $|\boldsymbol{t}| = 1$ then ϕ is identified with a 2×2 matrix over $\mathbb{Z}$. From Theorems 6.1.3 and 6.1.9 we see that the group $\mathbb{Z}^2 \rtimes_\phi F$ satisfies a quadratic second order isoperimetric inequality. We will improve this upper bound, and will also show that the second order Dehn function of such a group is not linear.

Theorem 7.1.1 *Let $\phi : F \longrightarrow GL_2(\mathbb{Z})$. Then*

$$n \ln n \preceq \delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \preceq n^{\frac{3}{2}}.$$

Moreover, if for some $t \in \boldsymbol{t}$, $\phi_{\bar{t}}$ has eigenvalues ± 1, then

$$n^{\frac{4}{3}} \preceq \delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \preceq n^{\frac{3}{2}},$$

and if for some $t \in \boldsymbol{t}$, $\phi_{\bar{t}}$ has finite order, then

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \sim n^{\frac{3}{2}}.$$

This Theorem, in fact, is the combination of Propositions 7.2.5 and 7.3.2 below.

In their paper [BrGe], Bridson and Gersten classified first order Dehn functions of groups $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, in terms of the automorphism $\phi \in GL_2(\mathbb{Z})$ by using Theorem 5.5 of [Sc], as follows.

Lemma 7.1.2 (i) *If ϕ has finite order, then $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is quasi-isometric to $\mathbb{Z}^3$ and so $\delta^{(1)}_{\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}} \sim n^2$.*

(ii) *If the eigenvalues of ϕ are ± 1 and ϕ has infinite order, then $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is quasi-isometric to the* three dimensional integral Heisenberg group *presented by*

$$\mathcal{P}_1 = \langle x, y, t\,;\; xy = yx,\; t^{-1}xt = xy,\; t^{-1}yt = y \rangle$$

and so $\delta^{(1)}_{\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}} \sim n^3$.

(iii) *If the eigenvalues of ϕ are not ± 1, then $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ is quasi-isometric to the group presented by*

$$\mathcal{P}_2 = \langle x, y, t\,;\; xy = yx,\; t^{-1}xt = x^2y,\; t^{-1}yt = xy \rangle$$

and so $\delta^{(1)}_{\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}} \sim 2^n$.

By (i) of this lemma and Corollary 2.2.15 we have the following result.

Proposition 7.1.3 *Suppose that $\phi \in GL_2(\mathbb{Z})$ has finite order. Then*

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}} \sim n^{\frac{3}{2}}.$$

7.2 Lower bounds

7.2.1 Some geometric techniques

Let $\mathcal{P}_0 = \langle \boldsymbol{x};\ \boldsymbol{s} \rangle$ be a finite presentation for a group G_0 of type F_3. Suppose we have a split extension $G_0 \rtimes_\phi F$ presented by

$$\mathcal{P} = \langle \boldsymbol{x},\ \boldsymbol{t};\ \boldsymbol{s},\ t^{-1}xt = \check{a}_{x,t}\ (x \in \boldsymbol{x},\ t \in \boldsymbol{t}) \rangle,$$

where $\check{a}_{x,t}$ $(x \in \boldsymbol{x},\ t \in \boldsymbol{t})$ are words on $\boldsymbol{x}$. Let $\check{\boldsymbol{a}}_t = \{\check{a}_{x,t} :\ x \in \boldsymbol{x}\}$, $t \in \boldsymbol{t}$. For each word $W = W(\boldsymbol{x})$ on $\boldsymbol{x}$ and each $t \in \boldsymbol{t}$, we will write $\phi_t(W)$ for $W(\check{\boldsymbol{a}}_t)$ obtained from $W(\boldsymbol{x})$ by replacing each $x \in \boldsymbol{x}$ with $\check{a}_{x,t}$. As in §6.1, for each $S = S(\boldsymbol{x}) \in \boldsymbol{s}$ and each $t \in \boldsymbol{t}$, choose a picture $\check{\mathbb{S}}_t$ over $\mathcal{P}_0$ with boundary $\phi_t(S)$. Moreover, for each $t \in \boldsymbol{t}$ and each picture $\mathbb{D}$ over $\mathcal{P}_0$ with discs $\Delta_1, \cdots, \Delta_n$ labelled $S_1^{\varepsilon_1}, \cdots, S_n^{\varepsilon_n}$ ($S_i \in \boldsymbol{s}$, $\varepsilon_i = \pm 1$, $1 \le i \le n$) say, let $\mathbb{D}(\check{\boldsymbol{a}}_t)$ be the picture over $\mathcal{P}_0$ obtained by replacing each arc labelled by $x \in \boldsymbol{x}$ by a sequence of parallel arcs labelled by $\check{a}_{x,t}$, and replacing each disc Δ_i by the picture $\varepsilon_i \check{\mathbb{S}}_{i,t}$ ($1 \le i \le n$). Then if $W(\boldsymbol{x}) = x_1 \cdots x_n$ ($x_i \in \boldsymbol{x} \cup \boldsymbol{x}^{-1}$, $1 \le i \le n$) is the boundary label of $\mathbb{D}$, $W(\check{\boldsymbol{a}}_t) = \check{a}_{x_1,t} \cdots \check{a}_{x_n,t}$ is the boundary label of $\mathbb{D}(\check{\boldsymbol{a}}_t)$ respectively. We will write $\phi_t(\mathbb{D})$ for $\mathbb{D}(\check{\boldsymbol{a}}_t)$. We now construct a spherical picture $\mathbb{P}_{\mathbb{D},t}$ over $\mathcal{P}$ as depicted in Fig. 7.1, and we denote $\langle \mathbb{P}_{\mathbb{D},t} \rangle$ by $\xi_{\mathbb{D},t}$. In particular, if $\mathbb{D}$ consists of a single disc labelled $S \in \boldsymbol{s}$, as before we then write $\mathbb{P}_{S,t}$ instead of $\mathbb{P}_{\mathbb{D},t}$, and let $\xi_{S,t} = \langle \mathbb{P}_{S,t} \rangle$.

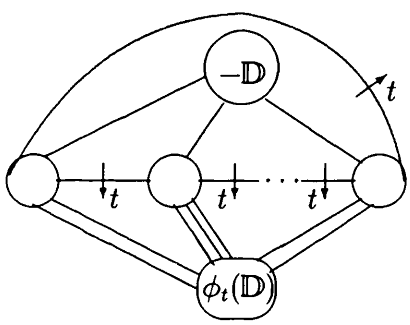


Fig. 7.1

Now, for any positive integer m and each $t \in \boldsymbol{t}$, we further construct a spherical picture $\mathbb{P}_{\mathbb{D},t}^{(m)}$ over $\mathcal{P}$ as depicted in Fig. 7.2. We let $\langle \mathbb{P}_{\mathbb{D},t}^{(m)} \rangle = \xi_{\mathbb{D},t}^{(m)}$.

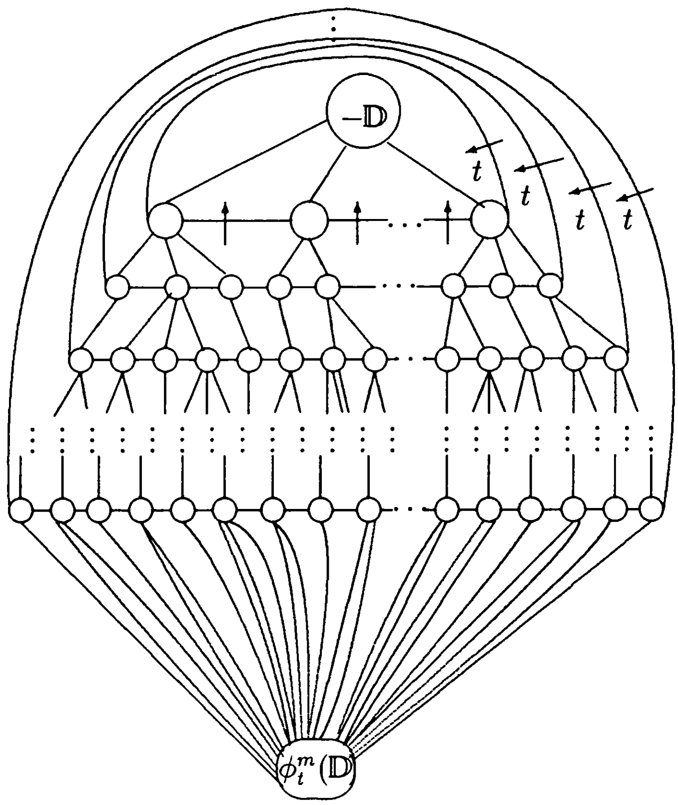


Fig. 7.2

Let $\boldsymbol{X}_0$ be a finite set of generating pictures of $\pi_2(\boldsymbol{\mathcal{P}}_0)$, and let

$$\boldsymbol{X}_{\boldsymbol{t}} = \{\mathbb{P}_{S,t} : \ S \in \boldsymbol{s},\ t \in \boldsymbol{t}\}.$$

Then by Theorem 6.1.4 $\boldsymbol{X} = \boldsymbol{X}_0 \cup \boldsymbol{X}_{\boldsymbol{t}}$ is a finite set of generating pictures of $\pi_2(\boldsymbol{\mathcal{P}})$. We have the following properties.

Lemma 7.2.1 (i) *If* $\mathbb{D}$ *has discs labelled* $S_1^{\varepsilon_1}$, $S_2^{\varepsilon_2}$, $\cdots$, $S_n^{\varepsilon_n}$, *and a spray with labels* $U_1(\boldsymbol{x})$, $U_2(\boldsymbol{x})$, $\cdots$, $U_n(\boldsymbol{x})$, *then*

$$\xi_{\mathbb{D},t} = \sum_{i=1}^{n} \varepsilon_i \overline{U}_i \xi_{S_i,t},\ t \in \boldsymbol{t}.$$

(ii) *If* $\boldsymbol{\mathcal{P}}_0$ *is aspherical, then* $\pi_2(\boldsymbol{\mathcal{P}})$ *is free on the elements* $\xi_{S,t}$, $S \in \boldsymbol{s}$, $t \in \boldsymbol{t}$.

(iii) *If* $\boldsymbol{\mathcal{P}}_0$ *is aspherical and if* $\phi_t^i(\mathbb{D})$ *is stable for each* $1 \le i \le m$ *and some* $t \in \boldsymbol{t}$, *then*

$$V_{\boldsymbol{\mathcal{P}},\boldsymbol{X}}(\xi_{\mathbb{D},t}^{(m)}) = \sum_{i=0}^{m-1} A(\phi_t^i(\mathbb{D})) = \sum_{i=0}^{m-1} Area(\phi_t^i(W)),$$

where W *is the boundary label of* $\mathbb{D}$.

Proof. The argument (i) is a special case of Lemma 6.1.1.

The proof for (ii) is same as the proof of Lemma 5.2.5 by using Theorem 6.1.4.

To prove (iii) we note that

$$\xi^{(m)}_{\mathbb{D},t} = \xi_{\mathbb{D},t} + \bar{t}\xi_{\phi_t(\mathbb{D}),t} + \cdots + \bar{t}^{m-1}\xi_{\phi^{m-1}(\mathbb{D}),t}.$$

By (i), for each $0 \le i \le m-1$,

$$\xi_{\phi^i_t(\mathbb{D}),t} = \sum_{j=1}^{A(\phi^i_t(\mathbb{D}))} \varepsilon_{ij}\overline{U}_{ij}\xi_{S_{ij},t}.$$

There are no cancellations in this expression since $\phi^i_t(\mathbb{D})$ is stable. Moreover, because of the different powers of $\bar{t}$ we see that there also are not cancellations between $\varepsilon_{ij}\bar{t}^i\overline{U}_{ij}\xi_{S_{ij},t}$ and $\varepsilon_{lk}\bar{t}^l\overline{U}_{lk}\xi_{S_{lk},t}$ with $0 \le i < l \le m-1$, $1 \le j \le A(\phi^i_t(\mathbb{D}))$, and $1 \le k \le A(\phi^l_t(\mathbb{D}))$. This proves (iii) by (ii). □

Note that $A(\mathbb{P}^{(m)}_{\mathbb{D},t}) = A(\mathbb{D}) + A(\phi^m_t(\mathbb{D})) + \sum_{i=0}^{m-1} L(\phi^i_t(W))$. Thus, we can use the idea in Chapter 5 to obtain a lower bound for $\delta^{(2)}_{\mathcal{P},\boldsymbol{X}}$ as demonstrated in the following subsection.

7.2.2 Lower bounds for $\mathbb{Z}^2 \rtimes \mathbb{Z}$

From now on, we let $G_0 \cong \mathbb{Z}^2$ be presented by $\mathcal{P}_0 = \langle x,\, y;\, [x,\, y]\rangle$. Note that $\mathcal{P}_0$ is aspherical (for example, by Theorem 6.1.2).

Consider the split extension $G_1 = \mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$ (the Heisenberg group) presented by

$$\mathcal{P}_1 = \langle x,\, y,\, t;\, xy = yx,\, t^{-1}xt = xy,\, t^{-1}yt = y\rangle,$$

where the automorphism $\phi_1 = \phi_{\bar{t}}$ is given by the mapping

$$x \longmapsto xy, \quad y \longmapsto y.$$

By Theorem 6.1.4 and Lemma 7.2.1 we know that if $\boldsymbol{X}_1 = \{\mathbb{P}_1\}$ where $\mathbb{P}_1$ is the spherical picture illustrated in the Fig. 7.3, then $\boldsymbol{X}_1$ freely generates $\pi_2(\mathcal{P}_1)$.

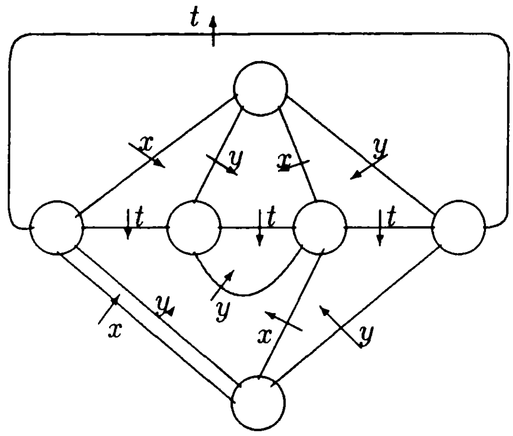
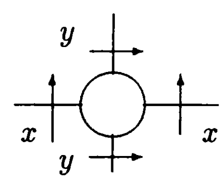
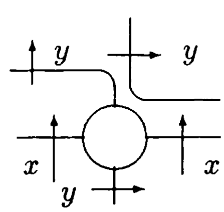


Fig. 7.3

If $\mathbb{D}$ is a picture over $\mathcal{P}_0$, the picture $\phi_1(\mathbb{D})$ can be obtained as follows. For each x-arc of $\mathbb{D}$, replace it by a pair of parallel arcs with total label xy, and replace each disc



by the following picture.



Then the picture $\phi_1(\mathbb{D})$ has boundary label $\phi_1(W)$ where W is the boundary label of $\mathbb{D}$. Moreover, we have

$$A(\mathbb{D}) = A(\phi_1(\mathbb{D})) = Area_{\mathcal{P}_0}(\phi_1(W)). \tag{7.1}$$

Let $W = x^{\alpha}y^{\beta}x^{-\alpha}y^{-\beta}$ for any pair of positive integers α and β. Then $\overline{W} = 1$ in G_0. Let $\mathbb{D}$ be the picture illustrated in Fig. 7.4.

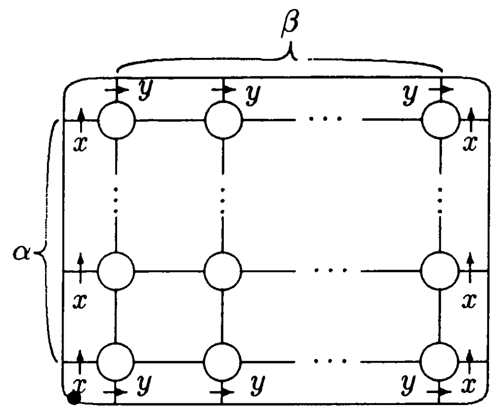


Fig. 7.4

Since all discs in $\mathbb{D}$ are positively oriented, by (i) of Lemma 7.2.1 we see that $\mathbb{D}$ is stable and $A(\mathbb{D}) = \alpha\beta$. By the construction of $\phi_1(\mathbb{B})$ for any picture $\mathbb{B}$ over $\mathcal{P}_0$, we see that if all discs in $\mathbb{B}$ are positively oriented, then all discs in $\phi_1(\mathbb{B})$ are also positively oriented. Thus, for all $i \in \mathbb{N}$, all discs of $\phi_1^i(\mathbb{D})$ are positively oriented, and hence, $\phi_1^i(\mathbb{D})$ is stable. By (iii) of Lemma 7.2.1 and (7.1) we then have

$$V_{\mathcal{P}_1,\boldsymbol{X}_1}(\xi_{\mathbb{D},t}^{(m)}) = \sum_{i=1}^{m} A(\phi_1^{i-1}(\mathbb{D})) = \sum_{i=1}^{m} A(\mathbb{D}) = m\alpha\beta. \tag{7.2}$$

Since $\phi_1^{i-1}(W) = (xy^{i-1})^{\alpha}y^{\beta}(xy^{i-1})^{-\alpha}y^{-\beta}$, we have

$$L(\phi_1^{i-1}(W)) = 2\alpha + 2\beta + 2\alpha(i-1) = 2\alpha i + 2\beta,$$

and so

$$A(\mathbb{P}_{\mathbb{D},t}^{(m)}) = 2A(\mathbb{D}) + \sum_{i=1}^{m} L(\phi_1^{i-1}(W)) = 2\alpha\beta + \sum_{i=1}^{m}(2\alpha i + 2\beta) = 2\alpha\beta + 2\beta m + \alpha m(m+1). \tag{7.3}$$

Proposition 7.2.2 *For any positive integer n we have*

$$\delta_{\mathcal{P}_1,\boldsymbol{X}_1}^{(2)}(6n) \geq \frac{1}{8}n^{\frac{4}{3}}.$$

Proof. Let n be any positive integer and let $\alpha = [n^{\frac{1}{3}}]$, $\beta = [n^{\frac{2}{3}}]$, $m = [n^{\frac{1}{3}}]$. Then by (7.3) we have

$$A(\mathbb{P}_{\mathbb{D},t}^{(m)}) \leq 2n^{\frac{1}{3}}n^{\frac{2}{3}} + 2n^{\frac{2}{3}}n^{\frac{1}{3}} + n^{\frac{1}{3}}n^{\frac{1}{3}}(n^{\frac{1}{3}}+1) \leq 6n.$$

Since if $\sigma \geq 1$ then $[\sigma] \geq \frac{1}{2}\sigma$, we have

$$\alpha \geq \frac{1}{2}n^{\frac{1}{3}}, \ \beta \geq \frac{1}{2}n^{\frac{2}{3}}, \ m \geq \frac{1}{2}n^{\frac{1}{3}}.$$

Thus, by (7.2),

$$V_{\mathcal{P}_1,\boldsymbol{X}_1}(\mathbb{P}_{\mathbb{D},t}^{(m)}) \geq \frac{1}{8}n^{\frac{4}{3}}$$

and so

$$\delta^{(2)}_{\mathcal{P}_1, \boldsymbol{X}_1}(6n) \geq \frac{1}{8} n^{\frac{4}{3}}$$

as required. □

We now consider the split extension $G_2 = \mathbb{Z}^2 \rtimes_{\phi_2} \mathbb{Z}$ presented by

$$\mathcal{P}_2 = \langle x,\, y,\, t;\; xy = yx,\; t^{-1}xt = x^2y,\; t^{-1}yt = xy \rangle,$$

where the automorphism $\phi_2 = \phi_{\bar{t}}$ is given by the mapping

$$x \longmapsto x^2y, \quad y \longmapsto xy.$$

By Lemmas 6.1.6 and 7.2.1 we know that if $\boldsymbol{X}_2 = \{\mathbb{P}_2\}$ where $\mathbb{P}_2$ is the spherical picture defined in the Fig. 7.5, then $\boldsymbol{X}_2$ freely generates $\pi_2(\mathcal{P}_2)$.

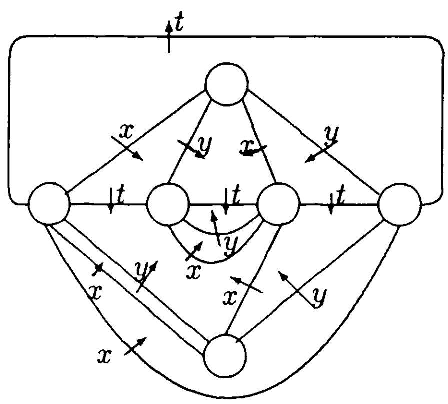
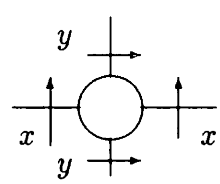
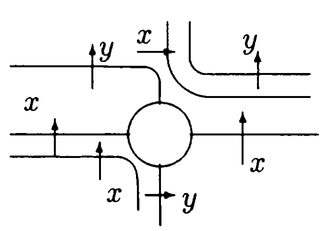


Fig. 7.5

If $\mathbb{D}$ is a picture over $\mathcal{P}_0$, the picture $\phi_2(\mathbb{D})$ can be obtained as follows. For each x-arc of $\mathbb{D}$, replace it by three parallel arcs with total label x^2y, for each y-arc of $\mathbb{D}$, replace it by a pair of parallel arcs with total label xy, and replace each disc



by the following picture.



Then the picture $\phi_2(\mathbb{D})$ has boundary label $\phi_2(W)$ where W is the boundary label of $\mathbb{D}$. Moreover, we have

$$A(\mathbb{D}) = A(\phi_2(\mathbb{D})) = Area_{\mathcal{P}_0}(\phi_2(W)). \tag{7.4}$$

Let $W_\alpha = x^\alpha y^\alpha x^{-\alpha} y^{-\alpha}$. Then $\overline{W}_\alpha = 1$ in G_0. We still let $\mathbb{D}$ be the picture as in Fig. 7.4 (here we let $\beta = \alpha$). Since all discs in $\mathbb{D}$ are positively oriented, as in the previous situation we see that for all $0 \leq i$, $\phi_2^i(\mathbb{D})$ is stable. Thus, by (iii) of Lemma 7.2.1 and (7.4) we have

$$V_{\mathcal{P}_2,\boldsymbol{X}_2}(\xi_{\mathbb{D},t}^{(m)}) = \sum_{i=1}^{m} A(\phi_2^{i-1}(\mathbb{D})) = \sum_{i=1}^{m} A(\mathbb{D}) = m\alpha^2. \tag{7.5}$$

Let U be any word on $\{x, y\}$, and let $L_x(U)$, $L_y(U)$ denote the numbers of occurrences of $x^{\pm 1}$ and $y^{\pm 1}$ in U respectively. Then $L(U) = L_x(U) + L_y(U)$. By the ϕ_2-action, we have $L_x(\phi_2(U)) = 2L_x(U) + L_y(U)$, and $L_y(\phi_2(U)) = L_x(U) + L_y(U)$. So,

$$L(\phi_2(U)) = L_x(\phi_2(U)) + L_y(\phi_2(U)) = 3L_x(U) + 2L_y(U) \leq 3L(U).$$

Inductively, we thus have $L(\phi_2^i(U)) \leq 3^i L(U)$. In particular, we have

$$L(\phi_2^i(W_\alpha)) \leq 4 \times 3^i \alpha \quad \text{for all } i \in \mathbb{N},$$

and hence

$$A(\mathbb{P}_{\mathbb{D},t}^{(m)}) = 2A(\mathbb{D}) + \sum_{i=1}^{m} L(\phi_2^{i-1}(W)) \leq 2\alpha^2 + \sum_{i=1}^{m} (4 \cdot 3^{i-1}\alpha) = 2\alpha^2 + 2\alpha(3^m - 1). \tag{7.6}$$

Proposition 7.2.3 *For any integer $n \geq 3^2$ we have*

$$\delta_{\mathcal{P}_2,\boldsymbol{X}_2}^{(2)}(3n) \geq \frac{1}{16} n \log_3 n.$$

Proof. Let n be any positive integer and let $\alpha = [n^{\frac{1}{2}}]$, $m = [\log_3 n^{\frac{1}{2}}]$. Then by (7.6) we have

$$A(\mathbb{P}_{\mathbb{D},t}^{(m)}) \leq 2(n^{\frac{1}{2}})^2 + 2n^{\frac{1}{2}}\left(3^{\log_3 n^{\frac{1}{2}}}\right) = 2n + 2n = 4n.$$

Also, if $\log_3 n^{\frac{1}{2}} \geq 1$, i.e. if $n \geq 3^2$, then

$$\alpha \geq \frac{1}{2} n^{\frac{1}{2}}, \ m \geq \frac{1}{2} \log_3 n^{\frac{1}{2}}.$$

Thus, by (7.5),

$$V_{\mathcal{P}_2,\boldsymbol{X}_2}(\mathbb{P}_{\mathbb{D},t}^{(m)}) \geq \frac{1}{8} n \log_3 n^{\frac{1}{2}} = \frac{1}{16} n \log_3 n$$

and so

$$\delta_{\mathcal{P}_2,\boldsymbol{X}_2}^{(2)}(3n) \geq \frac{1}{16} n \log_3 n$$

as required. □

7.2.3 Lower bounds for $\mathbb{Z}^2 \rtimes_\phi F$

We now consider the presentation

$$\boldsymbol{\mathcal{P}} = \langle x,\ y,\ \boldsymbol{t};\ [x,\ y],\ t^{-1}xt = \check{a}_{x,t},\ t^{-1}yt = \check{a}_{y,t}\ (t \in \boldsymbol{t})\rangle$$

for the group $\mathbb{Z}^2 \rtimes_\phi F$. Let $\boldsymbol{X} = \{\mathbb{P}_{[x,y],t} : t \in \boldsymbol{t}\}$. Then $\boldsymbol{X}$ generates $\pi_2(\boldsymbol{\mathcal{P}})$ by Theorem 6.2.1.

Proposition 7.2.4 *Let t be any element of $\boldsymbol{t}$, and let*

$$\boldsymbol{\mathcal{P}}_t = \langle x,\ y,\ t;\ [x,\ y],\ t^{-1}xt = \check{a}_{x,t},\ t^{-1}yt = \check{a}_{y,t}\rangle$$

be a presentation for $\mathbb{Z}^2 \rtimes_{\phi_{\bar{t}}} \mathbb{Z}$. Then $\boldsymbol{X}_t = \{\mathbb{P}_{[x,y],t}\}$ generates $\pi_2(\boldsymbol{\mathcal{P}}_t)$ and

$$\delta^{(2)}_{\boldsymbol{\mathcal{P}},\boldsymbol{X}} \succeq \delta^{(2)}_{\boldsymbol{\mathcal{P}}_t,\boldsymbol{X}_t}.$$

Proof. By Theorem 6.2.1 it is clear that $\boldsymbol{X}_t = \{\mathbb{P}_{[x,y],t}\}$ generates $\pi_2(\boldsymbol{\mathcal{P}}_t)$.

Note that $G(\boldsymbol{\mathcal{P}})$ is the HNN-extension of the base group $G(\boldsymbol{\mathcal{P}}_t)$ with associated subgroup $G(\boldsymbol{\mathcal{P}}_0)$. Therefore, there is a natural embedding $\psi : G(\boldsymbol{\mathcal{P}}_t) \longrightarrow G(\boldsymbol{\mathcal{P}})$ given by the mapping

$$\psi : x \longmapsto x, \quad y \longmapsto y, \quad t \longmapsto t.$$

Since ψ is injective, Theorem 1.3 of [Pr1] implies that the induced homomorphism $\psi_* : \pi_2(\boldsymbol{\mathcal{P}}_t) \longrightarrow \pi_2(\boldsymbol{\mathcal{P}})$ given by the mapping

$$\langle\mathbb{P}\rangle_{\boldsymbol{\mathcal{P}}_t} \longmapsto \langle\mathbb{P}\rangle_{\boldsymbol{\mathcal{P}}} \quad (\mathbb{P} \text{ is a spherical picture over } \boldsymbol{\mathcal{P}}_t)$$

is also an embedding. In our situation, the injectiveness of ψ_* can be proved directly as follows. First, ψ induces an embedding (also denoted ψ) $\psi : \mathbb{Z}G(\boldsymbol{\mathcal{P}}_t) \longrightarrow \mathbb{Z}G(\boldsymbol{\mathcal{P}})$ of group rings. Let $\mathbb{P}$ be any spherical picture over $\boldsymbol{\mathcal{P}}_t$ such that $\langle\mathbb{P}\rangle \in ker\psi_*$. Suppose in $\pi_2(\boldsymbol{\mathcal{P}}_t)$ we have

$$\langle\mathbb{P}\rangle_{\boldsymbol{\mathcal{P}}_t} = \sum_{i=1}^{m} \varepsilon_i \overline{W}_i \langle\mathbb{P}_{[x,y],t}\rangle_{\boldsymbol{\mathcal{P}}_t},$$

where $\varepsilon_i = \pm 1$, $\overline{W}_i \in G(\boldsymbol{\mathcal{P}}_t)$, $1 \leq i \leq r$. Then

$$\psi_*\langle\mathbb{P}\rangle_{\boldsymbol{\mathcal{P}}_t} = \sum_{i=1}^{m} \varepsilon_i \psi(\overline{W}_i)\psi_*(\langle\mathbb{P}_{[x,y],t}\rangle_{\boldsymbol{\mathcal{P}}_t}) = \sum_{i=1}^{r} \varepsilon_i \psi(\overline{W}_i)\langle\mathbb{P}_{[x,y],t}\rangle_{\boldsymbol{\mathcal{P}}} = 0.$$

Since $\pi_2(\boldsymbol{\mathcal{P}})$ is free on basis $\{\langle\mathbb{P}_{[x,y],t}\rangle : t \in \boldsymbol{t}\}$ by Lemma 7.2.1, we have

$$\sum_{i=1}^{m} \varepsilon_i \psi(\overline{W}_i) = \psi(\sum_{i=1}^{m} \varepsilon_i \overline{W}_i) = 0.$$

But ψ is injective, so $\sum_{i=1}^{m} \varepsilon_i \overline{W}_i = 0$, and hence, $\langle \mathbb{P} \rangle = 0$. Thus, ψ_* is injective.

Let n be any positive integer, and let $\mathbb{Q}$ be a spherical picture over $\mathcal{P}_t$ with $A(\mathbb{Q}) = n$. Then $\mathbb{Q}$ is also a picture over $\mathcal{P}$. Suppose we have

$$\langle \mathbb{Q} \rangle_{\mathcal{P}_t} = \sum_{i=1}^{r} \epsilon_i \overline{U}_i \langle \mathbb{P}_{[x,y],t} \rangle_{\mathcal{P}_t}$$

in $\pi_2(\mathcal{P}_t)$, where $\overline{U}_i \in G(\mathcal{P}_t)$, $\epsilon_i = \pm 1$, $1 \leq i \leq r$, and $r = V_{\mathcal{P}_t, \boldsymbol{X}_t}(\mathbb{Q})$. Then in $\pi_2(\mathcal{P})$ we have

$$\langle \mathbb{Q} \rangle_{\mathcal{P}} = \psi_*(\langle \mathbb{Q} \rangle_{\mathcal{P}_t}) = \sum_{i=1}^{r} \varepsilon_i \psi(\overline{W}_i) \langle \mathbb{P}_{[x,y],t} \rangle_{\mathcal{P}}.$$

By Lemma 7.2.1, $\pi_2(\mathcal{P})$ is a free module with basis $\boldsymbol{X}$. So, this expression is the unique one for $\langle \mathbb{Q} \rangle$ in $\pi_2(\mathcal{P})$. Hence, we have

$$V_{\mathcal{P},\boldsymbol{X}}(\mathbb{Q}) = V_{\mathcal{P}_t, \boldsymbol{X}_t}(\mathbb{Q}),$$

and so

$$\begin{aligned} \delta^{(2)}_{\mathcal{P},\boldsymbol{X}}(n) &\geq \max\{V_{\mathcal{P},\boldsymbol{X}}(\mathbb{Q}) : \mathbb{Q} \text{ a spherical picture over } \mathcal{P}_t \text{ with } A(\mathbb{Q}) \leq n\} \\ &= \max\{V_{\mathcal{P}_t,\boldsymbol{X}_t}(\mathbb{Q}) : \mathbb{Q} \text{ a spherical picture over } \mathcal{P}_t \text{ with } A(\mathbb{Q}) \leq n\} \\ &= \delta^{(2)}_{\mathcal{P}_t,\boldsymbol{X}_t}(n) \end{aligned}$$

as required. □

The following Proposition now is true by Lemma 7.1.2 and the above proposition.

Proposition 7.2.5 *We have*

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \succeq n \ln n.$$

Moreover, if for some $t \in \boldsymbol{t}$, $\phi_{\bar{t}}$ has eigenvalues ± 1, then

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \succeq n^{\frac{4}{3}},$$

and if for some $t \in \boldsymbol{t}$, $\phi_{\bar{t}}$ has finite order then

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \succeq n^{\frac{3}{2}}.$$

7.3 Upper bounds

7.3.1 Upper bounds for groups $\mathbb{Z}^2 \rtimes_\phi F$

By Lemma 7.1.2, the upper bounds of the second order Dehn functions of groups $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ can be obtained by establishing the upper bounds of the second order Dehn functions of groups $\mathbb{Z}^2 \rtimes_{\phi_i} \mathbb{Z}$ (i = 1, 2) defined in §7.2.2. But, we need to consider the general situation for the groups $G = \mathbb{Z}^2 \rtimes_\phi F$.

For each $t \in \boldsymbol{t}$, we let $M_t \in GL_2(\mathbb{Z})$ be the matrix corresponding to $\phi_{\bar{t}}$ of the form

$$M_t = \begin{pmatrix} i_t & j_t \\ k_t & h_t \end{pmatrix}, \quad i_t, j_t, k_t, h_t \in \mathbb{Z}$$

such that $i_t h_t - j_t k_t = \pm 1$. Then the split extension $\mathbb{Z}^2 \rtimes_\phi F$ has the standard presentation

$$\mathcal{P} = \langle x, y, \boldsymbol{t}\,;\, xy = yx,\ t^{-1}xt = x^{i_t}y^{k_t},\ t^{-1}yt = x^{j_t}y^{h_t}\ (t \in \boldsymbol{t})\rangle.$$

However, in general this presentation is not adequate for our purposes.

For any word W on $\{x,\ y\}$, we use $exp_x(W)$ and $exp_y(W)$ to denote the exponent sum of x and y in W respectively. By Corollary N4 of [MKS, Theorem 3.9], there is an epimorphism from the automorphism group of the free group of rank 2 on $\{x,\ y\}$ to $GL_2(\mathbb{Z})$, i.e., for each $M_t \in GL_2(\mathbb{Z})$, there is an automorphism of the free group on $\{x,\ y\}$ given by

$$x \longmapsto U_t, \qquad y \longmapsto V_t$$

for some word U_t, V_t on $\{x,\ y\}$ with

$$\begin{pmatrix} exp_x(U_t) & exp_x(V_t) \\ exp_y(U_t) & exp_y(V_t) \end{pmatrix} = M_t.$$

Thus, U_t, V_t freely generate the free group on $\{x,\ y\}$ and

$$\mathcal{P}' = \langle x, y, \boldsymbol{t}\,;\, xy = yx,\ t^{-1}xt = U_t,\ t^{-1}yt = V_t\ (t \in \boldsymbol{t})\rangle$$

is also a presentation for G. Now, by a theorem of Nielsen [MKS, Theorem 3.9], $[U_t,\ V_t]$ is freely conjugate to $[x,\ y]$ or $[x,\ y]^{-1}$ (this is called the *commutator-generator property* in [HiPr]). Thus, for each $t \in \boldsymbol{t}$, there is a picture $\check{\mathbb{S}}_t$ over $\mathcal{P}_0$ with *one* $[x,\ y]$-disc and boundary label $[U_t,\ V_t]$. This is the key point for the approach below to establish our upper bound.

Let ϕ'_t be the mapping : $x \longmapsto U_t,\ y \longmapsto V_t,\ t \in \boldsymbol{t}$. For any picture $\mathbb{D}$ over $\mathcal{P}_0$, as in §7.2.1, we have a picture $\phi'_t(\mathbb{D})$ over $\mathcal{P}_0$ obtained from $\mathbb{D}$ by replacing each $[x,\ y]^{\pm 1}$-disc with $\pm\check{\mathbb{S}}_t$ and replacing each arc labelled x (resp. y) with a sequence of parallel arcs labelled U_t (resp. V_t). Thus, we can construct a spherical picture $\mathbb{P}_{\mathbb{D},t}$ over $\mathcal{P}$ as in Fig. 7.1 with $A(\mathbb{P}_{\mathbb{D},t}) = 2A(\mathbb{D})$. In particular, we have the spherical picture $\mathbb{P}_{[x,y],t}$ over $\mathcal{P}'$. By Theorem 6.1.4, the set $\boldsymbol{X} = \{\mathbb{P}_{[x,y],t} : t \in \boldsymbol{t}\}$ generates $\pi_2(\mathcal{P}')$.

Example 7.3.1 Let $G = \mathbb{Z}^2 \rtimes_\phi F$ where F is a free group of rank 2 on t_1, t_2, and where

$$M_{t_1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad M_{t_2} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

The standard presentation for G is

$$\mathcal{P} = \langle x, y, t_1, t_2;\ xy = yx,\ x^{t_1} = xy^2,\ y^{t_1} = xy,\ x^{t_2} = x^2y,\ y^{t_2} = x^3y^2\rangle.$$

Note that xy^2, xy freely generate the free group on $\{x,\ y\}$. So we may take $U_{t_1} = xy^2$ and $V_{t_1} = xy$. However, since x^2y, x^3y^2 do not satisfy the commutator-generator property (see Fig. 7.7 below) they do not freely generate the free group on $\{x,\ y\}$. Consider the following procedure:

$$(x,\ y) \longrightarrow (y,\ x)$$
$$(y,\ x) \longrightarrow (yx,\ x)$$
$$(yx,\ x) \longrightarrow (x,\ yx)$$
$$(x,\ yx) \longrightarrow (xyx,\ yx)$$
$$(xyx,\ yx) \longrightarrow (yx,\ xyx)$$
$$(yx,\ xyx) \longrightarrow (yx^2yx,\ xyx).$$

Let $U_{t_2} = xyx$, $V_{t_2} = yx^2yx$. Then U_{t_2}, V_{t_2} freely generate the free group on $\{x,\ y\}$. Since

$$\begin{pmatrix} exp_x(U_{t_2}) & exp_x(V_{t_2}) \\ exp_y(U_{t_2}) & exp_y(V_{t_2}) \end{pmatrix} = M_{t_2},$$

the modified presentation for G is

$$\mathcal{P}' = \langle x, y, t_1, t_2;\ xy = yx, x^{t_1} = xy^2, y^{t_1} = xy, x^{t_2} = xyx, y^{t_2} = yx^2yx\rangle.$$

The generating pictures for $\pi_2(\mathcal{P}')$ are

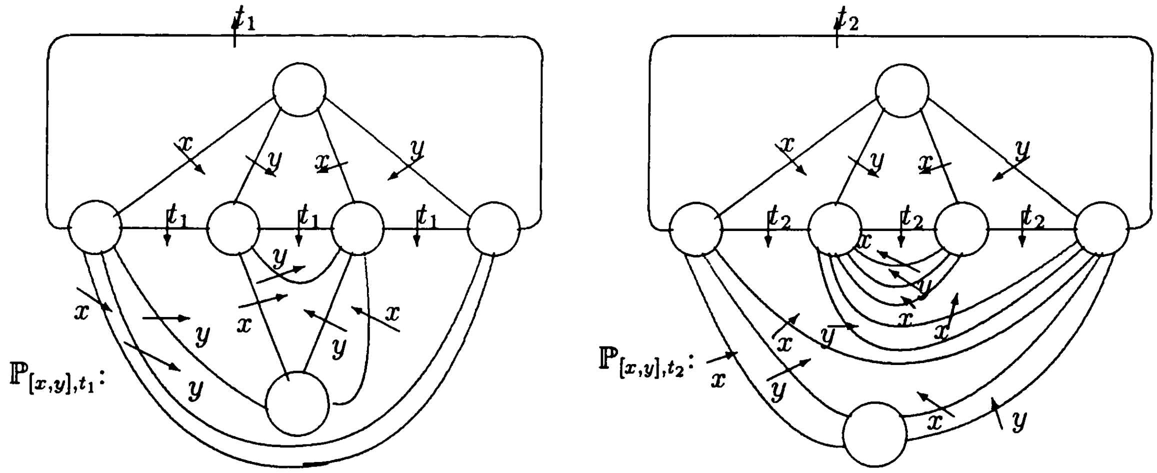


Fig. 7.6

Note that the generating pictures for $\pi_2(\mathcal{P})$ would be

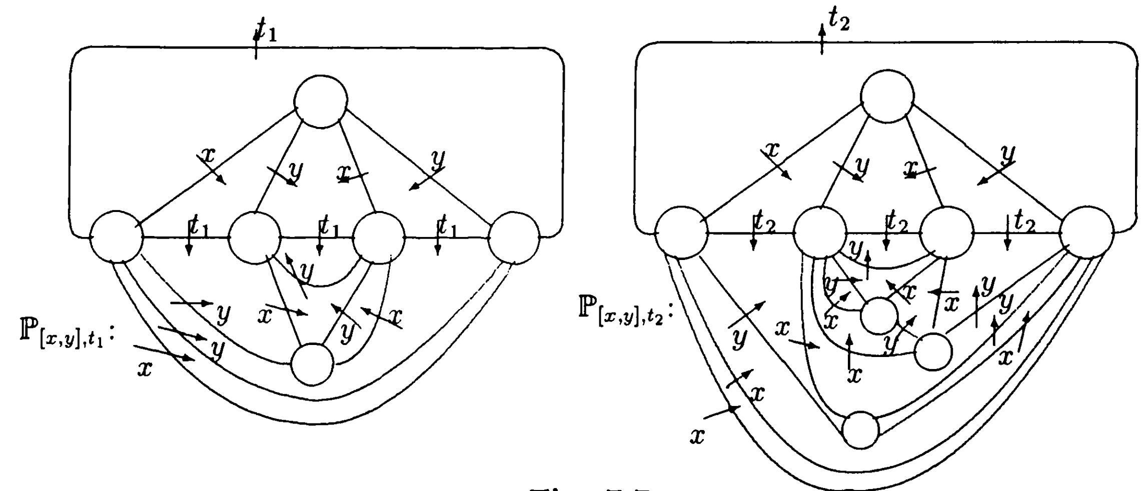


Fig. 7.7

In the second of these the number of $[x,\ y]$-discs is too big for our purposes.

Proposition 7.3.2 *We have*

$$\delta^{(2)}_{\mathbb{Z}^2 \rtimes_\phi F} \preceq n^{\frac{3}{2}}.$$

Proof. Let n be any positive integer and let $\mathbb{P}$ be any arbitrary spherical picture over $\mathcal{P}'$ containing n discs. Since $\delta^{(1)}_{\mathbb{Z}^2}(q) \preceq q^2$, we may suppose that $\delta^{(1)}_{\mathcal{P}_0}(q) \leq \alpha q^2$ for some constant $\alpha \geq 1$ and all $q \in \mathbb{N}$.

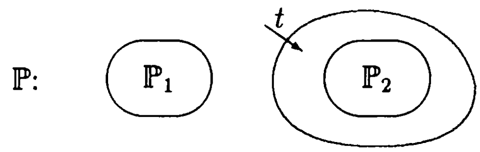
We argue that

$$V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) \leq \alpha n^{\frac{3}{2}}$$

by induction on the number m of $\boldsymbol{t}$-circles of $\mathbb{P}$.

If $m = 0$ then $\mathbb{P}$ is a spherical picture over $\mathcal{P}_0$ and hence equivalent to the empty picture as $\mathcal{P}_0$ is aspherical. Thus $V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) = 0 \leq \alpha n^{\frac{3}{2}}$.

Let $m \geq 1$. Suppose $\mathbb{P}$ contains at least one trivial $\boldsymbol{t}$-circle, say a t-circle C. Then $\mathbb{P}$ consists of two spherical subpictures $\mathbb{P}_1$, $\mathbb{P}_2$ and C:



where the numbers of $\boldsymbol{t}$-circles of $\mathbb{P}_1$ and $\mathbb{P}_2$ are less that m. Let n_1, n_2 be the disc numbers of $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively. Thus, by induction hypothesis and Lemma 1.3.4 we have

$$\begin{aligned} V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) &\leq V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}_1) + V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}_2) \\ &\leq \alpha n_1^{\frac{3}{2}} + \alpha n_2^{\frac{3}{2}} \\ &\leq \alpha(n_1 + n_2)^{\frac{3}{2}} = \alpha n^{\frac{3}{2}}. \end{aligned}$$

Suppose $\mathbb{P}$ contains only non-trivial $\boldsymbol{t}$-circles. We take a minimal one, say C, a t-circle in $\mathbb{P}$ for some $t \in \boldsymbol{t}$ consisting of q t-arcs and q discs for some positive integer q. Let $\mathbb{D}_1$, $\mathbb{D}_2$ be the subpictures lying *just* inside and outside C with boundary label W_1 and W_2, words on $\{x, y\}$, respectively.

Suppose C is outward directed as illustrated in Fig. 7.8. Then $L(W_1) = q$. We can assume that $\mathbb{D}_1$ contains $Area_{\mathcal{P}_0}(W_1) \leq \alpha q^2$ discs. Otherwise, replace $\mathbb{D}_1$ by a picture $\mathbb{D}_1'$ over $\mathcal{P}_0$ containing $Area_{\mathcal{P}_0}(W_1)$ discs and having the same boundary label W_1. Then the consequent picture is equivalent to $\mathbb{P}$ by the asphericity of $\mathcal{P}_0$. Now, $\mathbb{P}$ is equivalent to two spherical pictures $\mathbb{P}_1$ and $\mathbb{P}^U_{\mathbb{D}_1,t}$ for some word U on $\{x, y, t\}$ also as shown in Fig. 7.8.

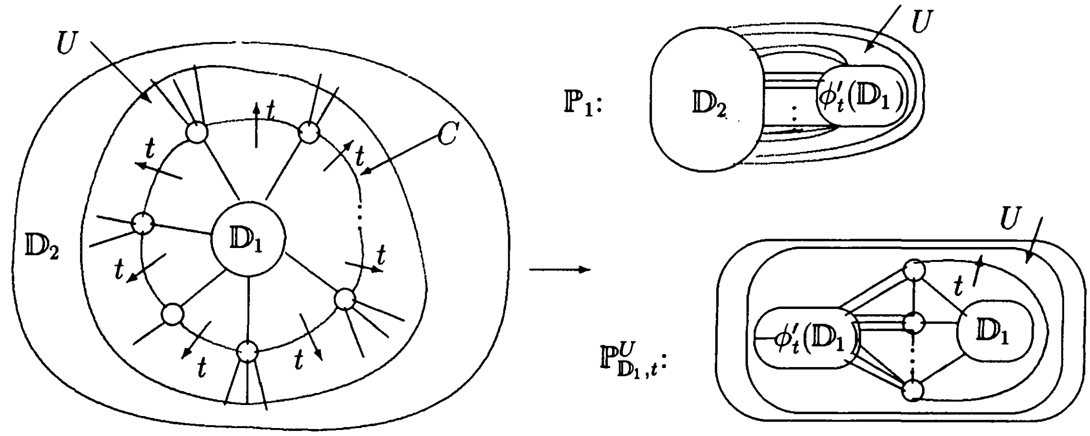


Fig. 7.8

By Lemma 1.3.4 we then have $V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) \le V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}_1) + V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}_{\mathbb{D}_1,t})$. By the construction of $\mathbb{P}_{\mathbb{D}_1,t}$ and Lemma 7.2.1, $V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}_{\mathbb{D}_1,t}) \le A(\mathbb{D}_1) \le \min\{\alpha q^2,\ n-q\}$. Thus, $A(\mathbb{P}_1) = n - q$ and $\mathbb{P}_1$ contains $m-1$ $\boldsymbol{t}$-circles. Hence, by induction hypothesis we have

$$V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) \le \begin{cases} \alpha q^2 + \alpha(n-q)^{\frac{3}{2}} & \text{if } \alpha q^2 \le n-q \\ n-q+\alpha(n-q)^{\frac{3}{2}} & \text{if } \alpha q^2 > n-q \text{ (but } Area_{\mathcal{P}_0}(W_1) \le n-q). \end{cases}$$

If $\alpha q^2 \le n-q$, then

$$\begin{aligned} V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) &\le \alpha(n-q)^{\frac{3}{2}} + \alpha q^2 \\ &= \alpha n^{\frac{3}{2}}\left(\left(1-\frac{q}{n}\right)^{\frac{3}{2}} + \frac{q^2}{n^{3/2}}\right) \\ &= \alpha n^{\frac{3}{2}}\left(1-\frac{q}{n}\right)^{\frac{1}{2}}\left(1-\frac{q\left(1-\frac{q}{(n-q)^{1/2}}\right)}{n}\right) \\ &\le \alpha n^{\frac{3}{2}}\left(1-\frac{q}{n}\right)^{\frac{1}{2}}\left(1-\frac{q\left(1-\frac{1}{\alpha^{1/2}}\right)}{n}\right) \\ &\quad \text{(since } (n-q)^{\frac{1}{2}} \ge \alpha^{\frac{1}{2}} q) \\ &\le \alpha n^{\frac{3}{2}}. \end{aligned}$$

If $\alpha q^2 > n-q$, then

$$\begin{aligned} V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) &\le \alpha(n-q)^{\frac{3}{2}} + n - q \\ &= \alpha n^{\frac{3}{2}}\left(\left(1-\frac{q}{n}\right)^{\frac{3}{2}} + \frac{n-q}{\alpha n^{\frac{3}{2}}}\right) \\ &= \alpha n^{\frac{3}{2}}\left(1-\frac{q}{n}\right)^{\frac{1}{2}}\left(1-\frac{\alpha q-(n-q)^{\frac{1}{2}}}{\alpha n}\right) \end{aligned}$$

$$
\begin{aligned}
&< \alpha n^{\frac{3}{2}}\left(1-\frac{q}{n}\right)^{\frac{1}{2}}\left(1-\frac{q\left(\alpha-\alpha^{\frac{1}{2}}\right)}{\alpha n}\right) \\
&\quad (\text{since } (n-q)^{\frac{1}{2}} < \alpha^{\frac{1}{2}} q) \\
&\leq \alpha n^{\frac{3}{2}}.
\end{aligned}
$$

We then have

$$V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) \leq \alpha n^{\frac{3}{2}}.$$

It remains to consider the situation that C is inward directed as illustrated in Fig. 7.9.

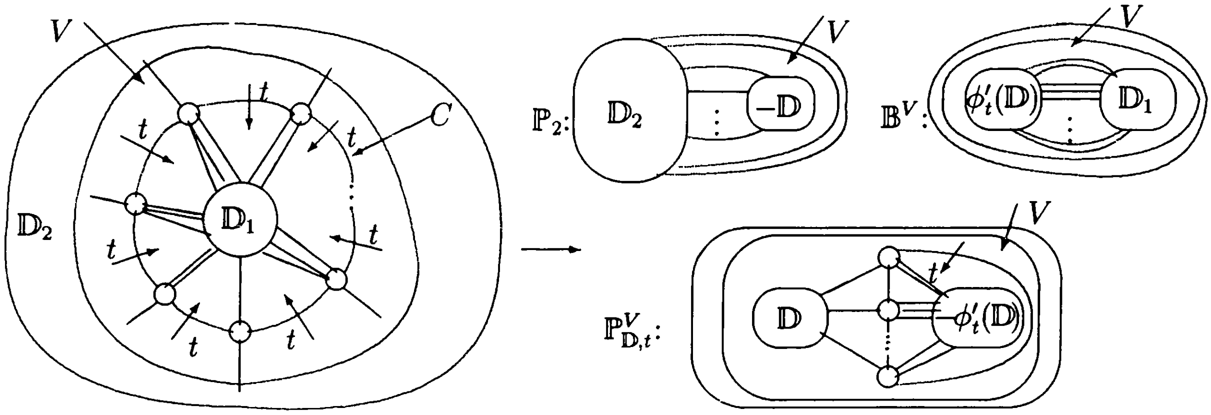


Fig. 7.9

Thus, we have $L(W_2) = q$. Let $\mathbb{D}$ be a picture over $\mathcal{P}_0$ with boundary label W_2 and $A(\mathbb{D}) = Area(W_2) \leq \alpha q^2$. Thus, $\mathbb{P}$ is equivalent to the sum of three spherical pictures $\mathbb{P}_2$, $\mathbb{P}^V_{\mathbb{D},t}$, and $\mathbb{B}^V$ for some word V on $\{x, y, t\}$. (This is also illustrated in Fig.7.9.) Since $\mathbb{B}$ is a spherical picture over $\mathcal{P}_0$, $\mathbb{B}$ is equivalent to the empty picture by the asphericity of $\mathcal{P}_0$. By the t-action we have

$$Area(W_1) = Area(W_2) = A(\mathbb{D}) = A(\phi'_t(\mathbb{D})) \leq A(\mathbb{D}_1).$$

Hence, $A(\mathbb{P}_2) \leq n-q$ and $\mathbb{P}_2$ contains $m-1$ t-circles. Thus, as we did in the first situation we also have that $V_{\mathcal{P}',\boldsymbol{X}}(\mathbb{P}) \leq \alpha n^{\frac{3}{2}}$ as required. □

We remark that there are two key points in the above approach to the estimation of the upper bound, i.e. the asphericity of $\mathcal{P}_0$ and the commutator-generator property of the free group of rank 2. We then have difficulty to extend this approach to the groups of the form $\mathbb{Z}^m \rtimes_\phi F$ for $m > 2$ without these two properties.

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