THE ROLE OF ISOMETRIES IN COSMOLOGICAL MODELS

by

AIDAN JOSEPH KEANE B.Sc.

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University of Glasgow
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Department of Physics
and Astronomy,
University of Glasgow,
Glasgow G12 8QQ.

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Summary.

This work is concerned with the role of symmetries in cosmological models in the context of the general theory of relativity.

We review the observational evidence for the homogeneity and isotropy of our universe and discuss briefly the Friedmann-Robertson-Walker (FRW) models which describe homogeneous and isotropic cosmological models.

The rest of the thesis provides an overview of those concepts of group theory and differential geometry relevant to the study of symmetry properties of homogeneous spacetimes. The techniques of differential geometry provide a method for describing the structures which can exist on a manifold. Once the manifold structure has been established it is possible to explore the symmetry properties of this manifold. The aspects of group theory relevant to the symmetry properties of a spacetime are then expounded. In particular we study connected Lie groups and their corresponding Lie algebras.

The symmetry transformations that leave the metric invariant are called isometries and the set of isometries form a group which can be split into a continuous component and a discrete component. The continuous isometries have associated with them infinitesimal isometries and these can be described by Killing vectors. These Killing vectors form the Lie algebra of the underlying symmetry group. The Killing vector fields therefore characterise the symmetry properties of the spacetime. The properties of isometries are discussed and some examples are given. In particular, to each Killing vector there corresponds a conserved quantity. The consequences of Lie group structure and the classification scheme for spatially homogeneous cosmologies (Bianchi classification) are outlined.

We compute the Killing vector fields for the FRW models, discuss their algebraic properties and the conservation laws derivable from them. These can be used to derive simply and directly some of the familiar results of the Friedmann cosmologies.

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Contents

Chapter 1

Homogeneous and Isotropic Cosmology.

1.1 Introduction.

This work is concerned with the role of symmetries in cosmological models. Our approach to cosmology is based on the general theory of relativity since it is so far the most successful theory of gravitation to have emerged.

A cosmological model is a spacetime with a specific matter/energy content. The spacetime is a 4-dimensional manifold M endowed with a metric tensor **g**. Once M is chosen then the global topology has been specified. The line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

describes the local geometry of the spacetime manifold.

The energy-momentum tensor **T** describes the energy content (ie matter, radiation, fields) of the model. The metric components $g_{\mu\nu}$ and energy-momentum tensor components $T_{\mu\nu}$ can be regarded as satisfying Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.2}$$

where we have put the gravitational constant G and the speed of light c both equal to 1. Of course, the energy-momentum tensor associated with a cosmological model must have reasonable physical properties.

Why consider symmetries in cosmological models? Because of the complexity of the field equations (a set of coupled, non-linear, partial differential equations), one cannot find exact solutions except for cases of rather high symmetry. Exact solutions give an idea of the qualitative features that can arise in general relativity, and so of possible properties of realistic solutions of the field equations. In cosmology we simplify the Einstein field equations by imposing symmetries on the solutions i.e. on the metric tensor. The solutions which possess symmetries are the most useful and best known.

As in classical mechanics there is often associated with a symmetry a conserved quantity (conserved quantities arise in general relativity when we begin to describe the symmetries associated with our solutions). The conservation of angular momentum for the central force problem arises from the fact that the Hamiltonian is independent of θ. This is an expression of the rotational symmetry of the system ie there is no preferred orientation in the plane. $\frac{\partial H}{\partial \theta} = 0$ means that the energy of the system is unchanged if we rotate it through $\delta\theta$ without changing r, p_r, p_θ. So there is a transformation which leaves H invariant and a corresponding conserved quantity. In an analagous manner certain symmetry transformations on a manifold can have associated conserved quantities. A symmetry transformation on a manifold manifests itself in the invariance of the metric under this particular transformation and these motions are described by *Killing vectors.*

The symmetry transformations that leave the metric invariant are called isometries. The set of isometries form a group which can be split into a continuous component and a discrete component. The continuous isometries can be described by Killing vectors.

The Killing vector fields therefore characterise the symmetry properties of the spacetime. For example, if a spacetime is stationary i.e invariant under time translations, there exists a timelike Killing vector field. Similarly, spherical symmetry implies the existence of Killing vector fields which correspond to rotations.

The Friedmann-Robertson-Walker cosmological models are homogeneous and isotropic or more accurately have spatial sections which are homogeneous and isotropic.

Isotropy at all points in space demands spatial homogeneity but not vice versa. Homogeneity and isotropy are best described in terms of the isometries of the manifold. A spacetime doesn't necessarily have to be invariant under translations in order for it to be spatially homogeneous. A space is homogeneous if there exists a group of isometries that takes every point into any other point on the manifold, thus leaving no point fixed. Isotropy at a point in the space requires the 3-dimensional spatial sections to be invariant under rotations at that point.

1.1.1 Outline of Thesis.

The purpose of this thesis is to deal with the matters mentioned above in a precise mathematical form and discuss briefly the Friedmann-Robertson-Walker (FRW) model.

In chapter 1 we review the observational evidence for the homogeneity and isotropy of our universe.

Chapter 2 deals with differential geometry, which we rely upon for a concise description of the structure on a manifold.

Chapter 3 explores those aspects of group theory relevant to the symmetry properties of a spacetime. In particular we study connected Lie groups: these are continuous groups which can be described completely by their local subgroup which is in one to one correspondence with the Lie algebra of the Lie group.

In chapter 4 the properties of isometries are discussed and some examples are given. The consequences of Lie group structure and the classification scheme for spatially homogeneous cosmologies (Bianchi classification) are outlined.

In chapter 5 we compute the Killing vector fields for the FRW models and discuss their algebraic properties. These are used to derive an expression for the redshift using conserved quantities. As far as we are aware, the work in this chapter has not been presented in any of the literature to date.

1.2 The Friedmann Models.

The universe appears to be spatially homogeneous and isotropic on the largest scales. The standard big bang model, based on the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) spacetimes is a remarkably successful operating hypothesis describing the structure and evolution of the universe. The FRW model describes an expanding universe - the volume of the spatial sections change with time. The expansion leads to a singularity at a finite time in the past when the volume of the spatial section becomes zero and matter becomes infinitely dense and infinitely hot. The FRW model provides a theoretical framework for such observations as

(1) the Hubble law of recession of galaxies ie the expansion of the universe, (2) the abundances of the light elements, in accordance with predictions from primordial nucleosynthesis and (3) the Cosmic Microwave Background Radiation (CMBR) which is believed to result from the initial hot dense phase of the universe.

The assumptions of homogeneity and isotropy on large scales have strong observational support. The evidence for angular isotropy on large scales comes from (1) the smallness of the CMBR large-angle anisotropy detected by COBE, (2) from the isotropy of radiation backgrounds at other wavelengths (3) the isotropy of deep galaxy and radio source counts. The galaxy surveys give 2-dimensional information about the distribution of *luminous* matter in the universe. On the other hand, the large-angle CMBR measurements directly probe the gravitational potential and are thus sensitive to the mass distribution itself ie both luminous and dark matter. We should be wary of associating the distribution of light with that of mass. Dark matter is certain to exist since luminous matter alone cannot account for the structure which is observed in the universe today.

Evidence for large-scale homogeneity comes in part from galaxy redshift surveys which give 3-dimensional information about the galaxy distribution. As mentioned before, large structures do exist e.g. superclusters, voids etc. but the net fluctuations in galaxy density become small on the largest scales.

When considering the large scale structure of the universe, the basic constituents can be taken to be the galaxies. Galaxies tend to occur in groups or clusters, each containing a few to a few thousand galaxies, which can aslo cluster to form superclusters. There is no evidence for clustering on large scales. Large-scale means on scales which are large compared to the distance between typical nearest galaxies (of the order of a million light years). From the CMBR results and galaxy ditribution surveys the universe appears to be homogeneous and isotropic *around us.* It is important to note that in order to step from this to a cosmological model we must add a further assumption (since we observe the universe from a single vantage point). This assumption is called the *Copernican Principle* and says that we do not occupy a special position in the universe. By itself the Copernican principle would simply imply homogeneity but when combined with observations of isotropy it implies that the universe should appear isotropic around *every* point. This is called the *Cosmological principle.*

The Einstein equations applied to a homogeneous and isptropic universe yield the Friedmann equations (the energy momentum tensor is assumed to be that of a perfect fluid). The metric incorporates the geometrical properties (e.g. symmetries) of the spacetime. The Einstein equations describe the dynamics of the model i.e. the manner in which the matter behaves and the system evolves. It should be noted that the model says nothing about *why* the universe should be homogeneous and isotropic.

1.2.1 The Metric.

The assumptions of spatial homogeneity and isotropy severely restrict the form of the spacetime geometry.

Each FRW spacetime consists of a 3-dimensional spatial section parameterised by the time t. These spatial sections have the global topology of either flat Euclidean space, the 3-sphere or the 3-hyperboloid.

The constant-time 3-surfaces have uniform spatial curvature and the metric on these surfaces can be described in polar coordinates by

$$ds^2 = S^2(t)\Big[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big].$$

where S(t) is the scale factor which describes the overall expansion or contraction. r, θ, ϕ are comoving coordinates and k is the sign of the spatial curvature. The case k=0 corresponds to flat Euclidean space (E^3), k=1 correponds to the geometry of the 3-sphere (S^3) and k=-1 the 3-hyperboloid (H^3), the three dimensional analogue of a hyperbolic saddle. Thus models with $k \leq 0$ are spatially infinite (open), while those with k=+1 are spatially finite (closed). The full FRW spacetime metric can then be written in the form:

$$ds^2 = dt^2 - S^2(t)\Big[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]. \tag{1.3}$$

1.2.2 The Friedmann Equations and Dynamics.

It is possible to make predictions from our model about the evolution of the universe: we are particularly interested in the early universe and in the ultimate fate of the universe. It would be a difficult task to formulate the dynamics of the FRW model in terms of a general solution valid for all time because of the complexity of the resulting field equations. Instead we consider the early universe as being dominated by radiation and relativistic matter and later as being matter dominated. The Friedmann models take the cosmological constant Λ to be zero. The energy-momentum tensor is assumed be that of a perfect fluid, viz.

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} \tag{1.4}$$

where p is the pressure and ϵ is the energy density measured in the rest frame of the fluid element and u^{μ} is the four-velocity of the fluid.

Einstein's equations for the FRW model are

$$3(\dot{S}^2 + k) = 8\pi G\epsilon S^2 \tag{1.5}$$

$$2S\ddot{S} + \dot{S}^2 + k = -8\pi G p S^2 \tag{1.6}$$

The first corresponds to the 00-component of Einstein's field equations and the second is the 11-component (the 11, 22 and 33-components are identical). The Freidmann equations are obtained from (1.5) and (1.6):

$$3(\dot{S}^2 + k) = 8\pi G\epsilon S^2 \tag{1.7}$$

$$\ddot{S} = -\frac{4\pi G}{3}(\epsilon + 3p)S \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\dot{\epsilon} + 3(p + \epsilon)\frac{\dot{S}}{S} = 0 \tag{1.9}$$

which states conservation of energy (of course, equation(1.9) is a direct consequence of $T^{\mu\nu}_{;\nu} = 0$).

Now, we have two equations i.e. (1.7) and (1.8), for three unknown functions $S(t), \epsilon$ and p. In order to solve this system of differential equations we need a third equation. The third equation is provided by an equation of state $p = p(\epsilon)$. Some

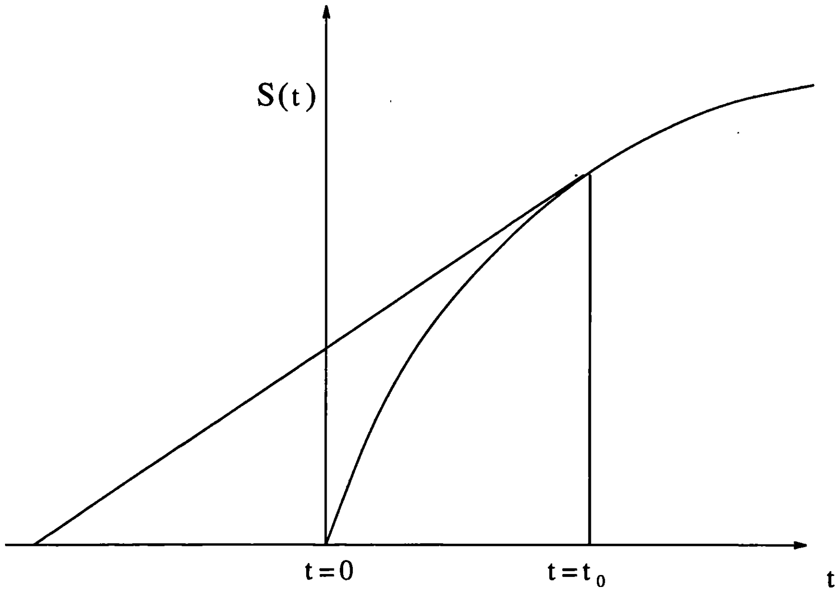

Figure 1.1: The variation (qualitative) of the scale factor

information can be obtained, however, without solving the Friedmann equations explicitly if we make a few reasonable assumptions about pressure and density.

On physical grounds, it seems reasonable to assume that $(\epsilon + 3p)$ will always be positive. It follows from (1.8) that $\ddot{S}$ will always be negative. S is positive by definition and $\dot{S}_0$ is positive since we see redshifted photons - not blue shifted (we shall denote values evaluated for the present epoch by a subscript 0 e.g. $S_0 = S(t_0)$). This tells us that the function S(t) should be concave downwards and will go to zero at, say, t=0. The form of the function S(t) can be seen in figure 2.

From (1.9) we find that

$$\frac{d(\epsilon S^3)}{dS} = -3pS^2 \tag{1.10}$$

For p=0, ϵ decreases as S^{-3} and for $p > 0$, ϵ decreases faster than S^{-3}. Therefore, ϵS^2 decreases faster than S^{-1}. Equation (1.5) can be written

$$\dot{S}^2 = \frac{8\pi G\epsilon S^2}{3} - k.$$

We recall that $\dot{S}_0$ is positive. For the case k=-1, $\dot{S}^2$ decreases but never goes to zero, in fact, S tends to t as $t \mapsto \infty$. k=0, $\dot{S}^2$ decreases and goes to zero as $t \mapsto \infty$. However, for the case k=+1, $\dot{S}^2$ will go to zero but since $\ddot{S}$ is always negative, $\dot{S}^2$ will

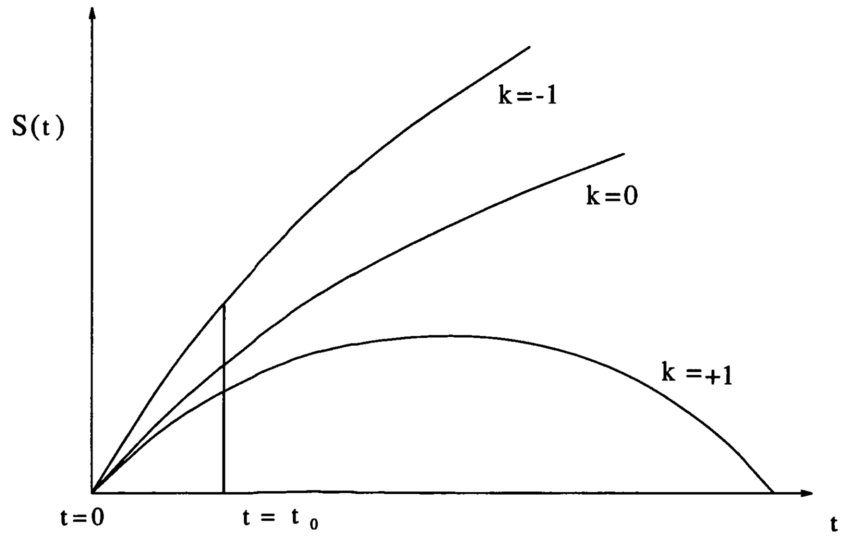

Figure 1.2: The scale factor.

become negative and S will decrease to zero. This sets out three distinct scenarios for the evolution of the universe. We are confronted here with the possibility of the universe's expansion halting and giving way to contraction ie a universe with a finite lifetime. The actual solution will dictate the present state of the universe eg its density and we shall consider this next.

The first of the Freidmann equations (1.5) can be written

$$\frac{\dot{S}^2}{S^2} = \frac{8\pi G\epsilon}{3} - \frac{k}{S^2}$$

The definition of the Hubble parameter H(t) is $H(t) = \frac{\dot{S}(t)}{S(t)}$ so it follows that

$$\frac{3H^2}{8\pi G} = \epsilon - \frac{3k}{8\pi G S^2}$$

and evaluating this for the present epoch

$$\begin{aligned} \frac{3{H_0}^2}{8\pi G} &= \epsilon_0 - \frac{3k}{8\pi G {S_0}^2} \\ \text{or} \qquad \epsilon_c &= \epsilon_0 - \frac{3k}{8\pi G {S_0}^2} \end{aligned}$$

where $\epsilon_c = \frac{3{H_0}^2}{8\pi G}$ is the critical density. It is then obvious that when k=+1 $\epsilon_c < \epsilon_0$, k=0 has $\epsilon_c = \epsilon_0$ and k=-1 has $\epsilon_c > \epsilon_0$. The deceleration parameter is defined as

$$q(t) = -\frac{\ddot{S}(t)S(t)}{\dot{S}(t)^2}$$

and (1.8) gives for the present epoch

$$(\epsilon_0 + 3p_0) = \left(\frac{3}{4\pi G}\right) q_0 {H_0}^2.$$

If we now assume that p_0 can be considered negligible in comparison to ϵ_0 then this equation can be expressed as

$$\frac{k}{{S_0}^2} = (2q_0 - 1){H_0}^2$$

There is a simple relationship between the ratio of the present day energy density and the critical density and the deceleration parameter q_0 ie

$$\frac{\epsilon_0}{\epsilon_c} = 2q_0 = \Omega_0.$$

The universe is open if $q_0 \leq \frac{1}{2}(\Omega_0 \leq 1)$ and closed if $q_0 > \frac{1}{2}(\Omega_0 > 1)$. The value of q_0 is not known exactly but we do have some idea of its possible values.

We can find exact solutions for the case of zero pressure but they will not, of course, give an accurate description of the early universe. The energy density varies in the following manner

$$\frac{\epsilon}{\epsilon_0} = (\frac{S_0}{S})^3$$

k=0. The Einstein-deSitter model.

$$\frac{S(t)}{S_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

k=+1.

$$S(t) = \frac{1}{2}\alpha(1 - \cos\theta) \qquad \text{where} \qquad \alpha = \frac{2q_0}{H_0(2q_0 - 1)^{\frac{3}{2}}} \qquad \text{and} \qquad \cos\theta = \frac{1-q}{q}.$$

k=-1.

$$S(t) = \frac{1}{2}\beta(\cosh\psi - 1) \qquad \text{where} \qquad \beta = \frac{2q_0}{H_0(1 - 2q_0)^{\frac{3}{2}}} \qquad \text{and} \qquad \cosh\psi = \frac{1-q}{q}.$$

For a radiation dominated era $p = \frac{1}{3}\epsilon$ and an exact solution can be obtained as before. However, we might use a different energy-momentum tensor when considering this early stage in the evolution of the universe. An exact solution does exist which applies to both radiation and matter dominated eras (i.e. when $T^{\mu\nu}$ has contributions from both matter and radiation) - see Islam (1991) [11] for details.

The general acceptance of the FRW model has obscured some of its difficulties however the main problems do not directly contradict the observational data. We shall consider these briefly.

There is an initial singularity at t=0: a finite energy is concentrated within zero volume and hence the energy density ϵ becomes infinite. Such an occurrence contradicts common sense and voilates accepted physical laws. An initial singularity might be removed by adding a cosmological constant Λ to the field equations (eg the deSitter model has a non zero Λ and the scale factor is never zero valued - hence there is no singularity) but it is not certain whether the actual Λ has a non zero value. Even the assumption of isotropy was loosened in an attempt to restore a physically acceptable solution near t=0 but still the singularity persists. The singularity seems inherent in the FRW model.

Let t be the age of the universe. Two arbitrary regions can be causally related only if they are within ct of each other and this distance ct is called the horizon. The horizon of events is currently about 10^{23}km which coincides with the size of the universe but the size of the universe changes more slowly than the size of the horizon and so the horizon was once smaller. How is it possible that when one part of the universe is not causally related to another, they can have the same density etc and the universe can appear isotropic? This is the horizon problem.

Given the large scale homogeneity, the distribution of matter at around 10^{-45} seconds must have been very smooth (but not absolutely smooth) - otherwise we would observe very large inhomogeneities. This is the smoothness problem. Currently Ω lies between 0.1 and 2. Ω must have been equal to 1 within 1 in 10^{15} at the beginning of the universe which seems unlikely and this is called the flatness problem. These are problems of initial conditions and attempts have been made to solve the latter three using the inflationary scenario - see the paper by J.A.Frieman (1994) [7].

What are the values of the cosmological parameters H_0 and q_0? The redshift z can be expressed in terms of the distance l from us

$$z = H_0 l + \frac{1}{2}(1 + q_0){H_0}^2 l^2 + O({H_0}^3 l^3)$$

Thus from the observed redshifts it is possible to determine the parameters H_0 and q_0 if an independent estimate can be obtained for the distance. Cepheid variables provide the answer: they possess a period-luminosity relationship and so can be used as "standard candles". Supernovae can also be used.

Whether a cosmological constant is necessary or not, is unknown and in order that cosmological models can be considered in all generality it is reasonable to consider H_0, q_0 and Λ as the three unknown parameters.

Chapter 2

Differential Geometry.

In this chapter we set out the preliminary mathematical and geometric concepts relevant to our study of general relativity. This is essentially the mathematics of curved spaces - or differential geometry. The description of the symmetriesof such spaces is provided by the theory of Lie groups and Lie algebras which we shall discuss in chapter 3.

2.1 Manifolds.

Our mathematical treatment of curved spaces begins with the idea of a topological space and manifold.

Let S and T be sets. In order to define a continuous function $f : S \mapsto T$ we need to give the sets S and T a topological structure. A topological structure (or a topology) on a set S is a collection of subsets of S called *open sets*, satisfying the axioms:

(1) The union of any number of open sets is open.

(2) The intersection of any finite number of open sets is open.

(3) Both S itself and the empty set ϕ are open.

When such a structure is given on a set S we call it a *topological space* and we refer to its elements as points. For a detailed discussion see Wald (1984) [23]. R^n is a topological space: we denote by R^n the set of all n-tuples of real numbers $(x^1, x^2, ...x^n)$, that is, every point of R^n can be labelled by a set of these numbers.

A topological space is said to be *Hausdorff* if for every pair of distinct points p, q in the space one can find open sets O_p, O_q such that $p \in O_p$, $q \in O_q$ and $O_p \cap O_q = 0$ ie that the open sets containing the distinct points do not intersect. Therefore, in a

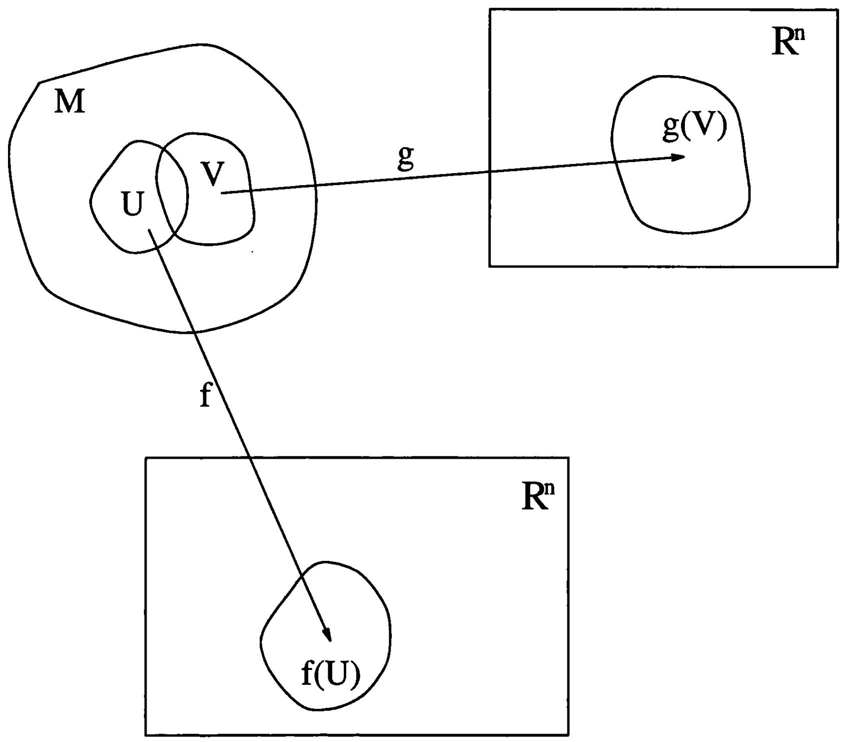

Figure 2.1: Mapping from M to R^n.

Hausdorff space, any line joining two distinct points can be infinitely subdivided.

We are now in a position to define a manifold.

Definition. *(manifold) An n-dimensional manifold M is a set of points which constitute a topological space, together with an open covering U_i of M and mappings $f_i : U_i \mapsto R^n$ where f_i is a continuous one to one mapping of U_i onto the open subset $f_i(U_i)$ of R^n. The pair (U_i, f_i) is called a coordinate patch and if $p \in U_i$ then $x^i = f_i(p)$ are called the coordinates of p in the coordinate system (U_i, f_i).*

We will only be considering manifolds which are Hausdorff spaces.

Each point of the manifold M lies in an open set (also called an open neighbourhood) with a continuous one to one map onto an open set of R^n. This means that M is locally like R^n which is an important property as it effectively insures that every region of M can be assigned at least one set of coordinates, that is every point in that neighbourhood can be uniquely identified by a set of numbers $(x^1, x^2, ...x^n)$.

The diagram shows two of the open neighbourhoods, U and V, of M but there may be others. A particular manifold may have a neighbourhood which includes all of M, but we shall consider the general case where there are many coordinate patches needed to cover M. All the U_i constitute the open covering of M. The neighbourhood U has its own map onto the region f(U) of R^n and similarly for V.

The neighbourhood U_i together with its map f_i is called a *chart* or *coordinate patch* and is denoted (U_i, f_i).Put another way, a coordinate patch is an open set in which coordinates x^i uniquely describe points. These patches cover the whole of the manifold M and the complete system of coordinate patches is known as an *atlas.*

We now consider the further characterisation of a manifold: the C^k manifold. Consider a point p lying in both open sets U and V, with their corresponding charts f and g. The map $(g \circ f^{-1})$ from R^n to R^n expresses the coordinates of any point p $\in V \cap U$ in the (V,g) coordinate system in terms of the (U,f) system. Using somewhat abbreviated notation, if p has coordinates $x^i(p)$ in (U,f) then it has coordinates $y^i(p) = y^i(x^i(p))$ in (V,g). This is simply a coordinate transformation and can be carried out for any overlapping coordinate patches on M. If the map is C^k i.e. k times continuously differentiable, for all such overlapping coordinate patches U_i and U_j in M, then M is said to be a C^k manifold or a C^k differentiable manifold.

It should be noted that we have not introduced the concept of distance in defining the manifold. The concept of a manifold has numerous uses many of which do not require any distance function to be defined between points but we have ensured the *local* topology is that of R^n. The differentiability property of a C^k manifold allows us to define, for example, tensors, differential forms and Lie derivatives without the need for any distance function.

The surface of a sphere S^2 is an example of a manifold for which one needs more than one coordinate patch to completely cover it. Consider polar coordinates θ, ϕ on S^2. In order for this system to completely cover all of S^2 there would have to exist a corresponding mapping $\Phi : M \mapsto R^2$ but this is not the case since at the poles Φ is not even a map. At the poles on S^2, ϕ can have any value and so the north pole corresponds to the whole line $(0, \phi)$ in R^2 and the south pole to the whole line (π, ϕ). Similarly the line $\phi = 0$ corresponds to the line $\phi = 2\pi$ and each point on this line corresponds to two separate points on R^2.

At least two coordinate patches are needed to completely cover the sphere: one could be the coordinate patch above but with the poles and the line $\phi = 2\pi$ removed and the other an identical system but with its line along the equator of the first and

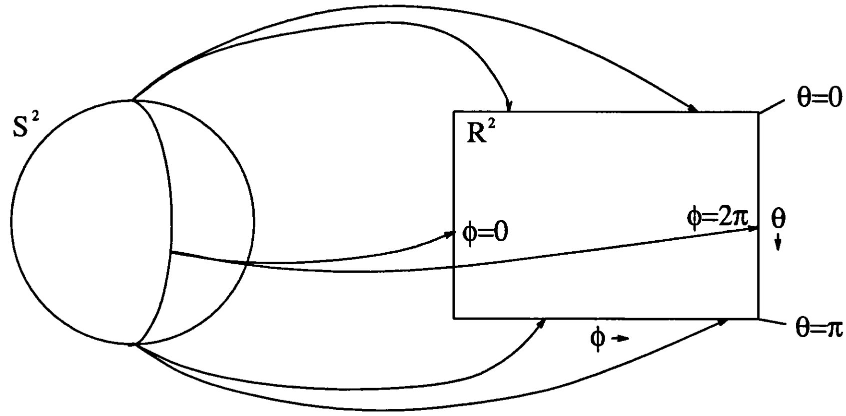

Figure 2.2: The mapping of S^2 onto R^2.

the poles at $\phi = \frac{1}{2}\pi$ and $\phi = \frac{3}{2}\pi$.

2.2 Geometrical Objects.

We are now in a position to place "geometrical objects" on the manifold, such as functions and vector fields.

Let us begin by considering a *curve* on M. A curve is not just a continuous set of points on M. It is defined as a mapping from the set of real numbers R^1 into M. Thus a curve is a parameterised path.

Two curves which pass through the same points but with different parameterisation are different curves. A congruence of curves is a set of curves filling the manifold, so that there is a unique curve passing through each point in M. Hence, the paths of different curves in the congruence will never cross.

A *function* is a scalar field: a rule which assigns a real number to every point in M. This is shown in the diagram below

The function f is a map from M to R^1 and there is the usual map of the region U of M onto R^n given by g. The function then becomes a function on R^n since

$$f(p) \equiv f(x^i(p)).$$

If the function is differentiable on R^n then it is said also to be differentiable on M.

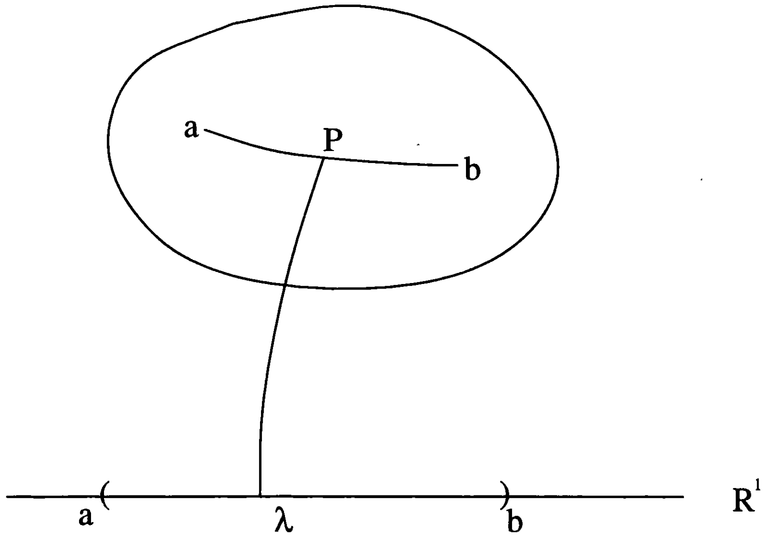

Figure 2.3: Curve parameterised by λ.

2.2.1 The Tangent Space and Vector Fields.

A *linear vector space* V of dimension n over a field f is a set of n elements v_i together with a binary operation + called addition and scalar multiplication $\cdot$ such that for all $u, v \in V$ and $a, b \in f$

$a \cdot u \in V$

$a \cdot (u + v) = a \cdot u + b \cdot v.$

$(a + b) \cdot u = a \cdot u + b \cdot u.$

$(ab) \cdot u = a \cdot (b \cdot u).$

$1 \cdot u = u.$

We are free to choose any n linearly independent vectors to form a basis for the vector space.

Let us now define tangent vectors and show how they from a vector space. Consider an arbitrary function defined on the manifold and a curve $x^\mu = x^\mu(\lambda)$. The derivative of f along the curve may be written

$$\frac{df}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda} \frac{\partial f}{\partial x^i} \tag{2.1}$$

We thus define the *tangent vector* as a mapping of functions into R.

$$\frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda} \frac{\partial}{\partial x^i} \tag{2.2}$$

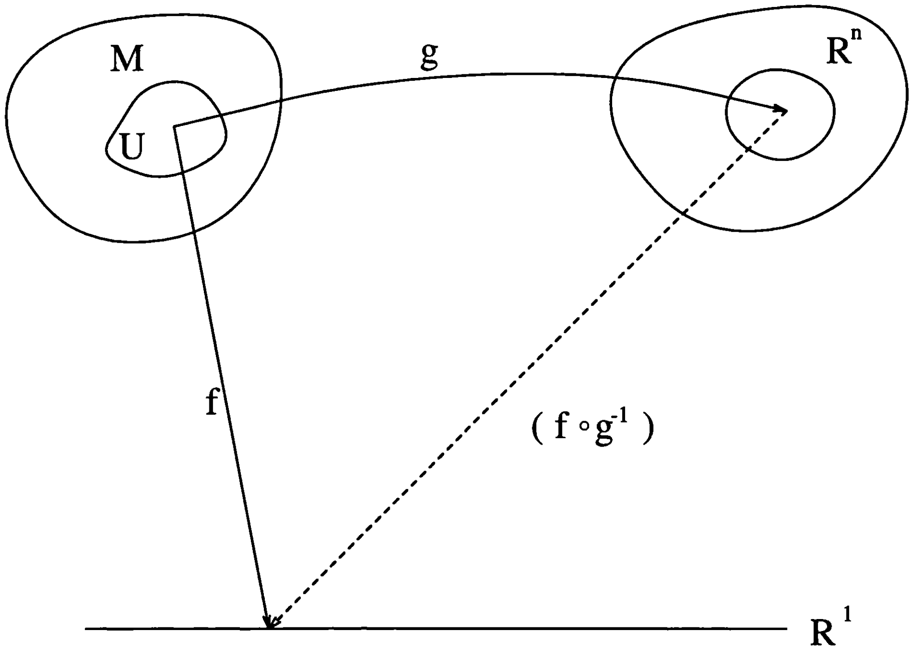

Figure 2.4: function.

We can consider the set $\{\frac{dx^i}{d\lambda}\}$ to be the components of a tangent vector with respect to the basis $\{\frac{\partial}{\partial x^i}\}$ in the tangent space at the point p of M. We can show that the tangent vectors form a vector space and the $\frac{\partial}{\partial x^i}$ are the basis for this vector space. At every point of M a vector is defined and this constitutes a vector field. It is obvious from the above that to each congruence defined on the manifold there will be a corresponding (unique) vector field.

Thus, at any point p, the tangent space T_p is a vector space with the same dimension n as the manifold M. By choosing a basis in each T_p we arrive at a basis for vector fields.

2.2.2 Coordinated and Non-coordinated Bases.

The n operators $\{\frac{\partial}{\partial x^i}\}$ are basis vector fields. We call a set of basis vectors derived from a set of coordinates a *coordinated basis.* Not every basis is a coordinated basis - any n vector fields linearly independent in an open set (neighbourhood) of M may be used as a basis in that open set. n vector fields are linearly independent on M if at every point p of M the corresponding n vectors are linearly independent. To arrive at an example of a non-coordinated basis consider ordinary spherical coordinates.

In the coordinated basis $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$ the velocity of a particle is

$$V = \frac{dr}{dt}\frac{\partial}{\partial r} + \frac{d\theta}{dt}\frac{\partial}{\partial \theta} + \frac{d\phi}{dt}\frac{\partial}{\partial \phi}$$

This can be rewritten in terms of the non-coordinated basis $X_r = \frac{\partial}{\partial r}, X_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}, X_\phi = \frac{1}{rsin\theta}\frac{\partial}{\partial \phi}$ as

$$V = \frac{dr}{dt}X_r + r\frac{d\theta}{dt}X_\theta + rsin\theta\frac{d\phi}{dt}X_\phi$$

The $\{X_\mu\}$ form a non-coordinated basis. For any basis $\{X_\mu\}$ we can write the commutator

$$[X_\mu, X_\nu] = D^\lambda_{\mu\nu}X_\lambda. \tag{2.3}$$

The functions $D^\lambda_{\mu\nu}$ are the *structure coefficients* corresponding to the basis $\{X_\mu\}$. In a general basis the $D^\lambda_{\mu\nu}$ do not vanish. In fact it can be shown that if and only if $D^\lambda_{\mu\nu} = 0$ is the basis a coordinated basis. If the manifold admits a group of isometries (see chapter 3) then the most convenient basis is one with the $D^\lambda_{\mu\nu}$ determined by the group structure. We shall have more to say on this when we consider the invariant basis.

2.2.3 Tensor Fields.

The coordinate independent concept of a vector field is that of a differential operator V on M, which carries differentiable functions on M into other differentiable functions. The concept of a vector field can be generalised to that of a tensor field.

We begin by defining a *one-form* field as a real-valued linear mapping of vector fields.

Consider the set V* of linear maps ω from the tangent vector space V into the set of real numbers

$$\omega : V \mapsto R.$$

We can define addition and scalar multiplication of such linear maps in such a way that we get a natural vector space structure on V*. We call V* the *dual vector space* to V and the elements of V* are called *dual vectors* or *one-forms.* It should be noted that taking the double dual V** gives nothing new: we can naturally identify

V** with V. A rule which gives a one-form at every point of the manifold (ie a rule which defines a one-form in every dual tangent space) is called a one-form field.

If $\{X_\mu\}$ is a general basis we define a corresponding dual set of one-forms $\{\omega^\mu\}$ by

$$\omega^\mu(X_\nu) = \delta^\mu_\nu \tag{2.4}$$

The most general one-form field can be written $\omega = a_\sigma \omega^\sigma$. If $V = b^\gamma X_\gamma$ then

$$\omega(V) = a_\sigma b^\sigma$$

We have now made sufficient progress to define a tensor. As before, let V be a finite dimensional vector space and let V* be its dual vector space. Then a tensor field T of type $\binom{k}{l}$ is a multilinear map (ie linear on every arguement)

$$T : \underbrace{V^* \times V^* \times .... \times V^*}_{k} \times \underbrace{V \times V \times .... \times V}_{l} \mapsto R$$

that is, when the tensor T operates on k one-forms and l vectors it produces a real number. The tensor field T is an operator on k one-form fields and l vector fields which produces a function.

The tensor product (cartesian product) $\omega \otimes \sigma$ of two one-forms is a bilinear operator acting on pairs of vector fields, U and V say

$$\omega \otimes \sigma(U, V) = \omega(U)\sigma(V). \tag{2.5}$$

The tensor product on forms and vectors is used to build up tensors of arbitrary rank. The product of k one-forms and l vectors is a tensor of covariant rank k and contravariant rank l. The typical tensor can be written as

$$T = T^{\alpha....\beta}_{\mu....\nu} X_\alpha \otimes ....X_\beta \otimes \omega^\mu \otimes ....\otimes \omega^\nu. \tag{2.6}$$

The functions $T^{\alpha....\beta}_{\mu....\nu}$ are said to be the components of the tensor with respect to the basis $X_\alpha \otimes ....\otimes X_\beta \otimes \omega^\mu \otimes ....\otimes \omega^\nu$.

Traditionally we look upon the metric as being a structure which determines the distance between two nearby points on a manifold. The square of the distance between two such points in the same coordinate patch is then given by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

the dx^μ being infinitesimal displacements.

From the modern viewpoint of differential geometry, the metric is a tensor that operates on any two tangent vectors A and B to give a real number, denoted by

$$g(A, B) = A \cdot B \tag{2.7}$$

In an arbitrary basis we can write

$$g = g_{\mu\nu} \omega^\mu \otimes \omega^\nu$$

So far, we have considered the manifold and the various geometrical objects defined upon it. As soon as we define a metric on our manifold then we introduce the notion of distance and curvature on M (although a metric tensor need not be defined in order to define curvature - but we shall discuss this later). We may decide not to introduce a metric and it is still possible to describe some of a manifold's properties and the geometrical objects which exist upon it without reference to a metric (for example differential forms and the Lie derivative).

Next we consider the importance of an affine connection and begin to describe curvature in tensorial terms. These arise in Cartan's equations: in fact, both the connection coefficients and the curvature components corresponding to a particular metric can be "read off" from Cartan's first and second equations respectively.

2.3 Differentiation on Manifolds.

We now consider the subject of differentiation on a manifold. The concept of the derivative of a function on a manifold is familiar and straightforward: The derivative of the function f along a curve parameterised by λ is $\frac{df}{d\lambda}$ and we can choose any curve on the manifold with which to form such a derivative. However, it will be useful to have some way of describing the variation of a vector field (or indeed, any tensor field) on the manifold.

All the structures we wish to place on the manifold M can be defined in terms of differential operators, as we have seen for the tensors above. In order to accommodate the notion of the variation of a tensor field we introduce three derivative operators with distinct properties: they are the covariant derivative, the Lie derivative and the exterior derivative. The latter two are defined purely by the manifold structure whereas the covariant derivative is defined by placing extra structure on the manifold and we refer to this as adding a connection. We need not define a metric tensor on the manifold in order to define a connection. Let us first consider the problems which arise when we try to compare vectors defined at different points on a curved manifold.

Consider, to begin with, a manifold with no curvature. We can define a basis vector field on this manifold. For example, in E^3 we could use the orthogonal basis i,j,k. In such a flat space, parallel lines remain parallel and so we can use the notion of parallel when talking about vectors at different points. Therefore, if we have a vector field V defined on this manifold we can describe its variation with respect to this basis vector field. However, in general our manifold will have curvature and we loose the notion of parallel vectors at different points. That is, the question now arises: how do we "translate" our vector to compare it with a vector at another point in order to determine the "variation" of a vector field? It turns out we can still determine the variation of a vector field along a curve by considering the notion of *parallel transport.* Locally the manifold can be considered flat (which allows us to use the notion of parallel) and so we can compare two vectors which are infinitesimally close to each other.

If the tangent vectors corresponding to a curve prameterised by λ at infinitesimally close points are parallel and of equal length then the tangent vector is said to be parallel transported along the curve. This idea can be extended to the whole congruence parameterised by λ so that we then have the notion of a vector field whose tangent vectors are parallel transported along a certain congruence.

Any deviation from this manifests itself in the *covariant derivative* of the vector field V along the curve. If the vector field V has tangent vectors which are parallel transported along the curves parameterised by λ then

$$\frac{dV}{d\lambda} = 0$$

The covariant derivative operator ∇_U is defined formally as one which acts on a vector field V to give a vector field $\nabla_U(V)$ such that

(1) for any C^1 functions f and g and C^1 vector fields V,W,

$$\nabla_{fU+gV}W = f\nabla_U W + g\nabla_V W.$$

(2) $\nabla_U V$ is linear in V, i.e. for C^1 vector fields V,W and $\alpha, \beta \in R^1$,

$$\nabla_U(\alpha V + \beta W) = \alpha\nabla_U V + \beta\nabla_U W.$$

(3) for any C^1 function f and C^1 vector field V,

$$\nabla_U(fV) = (Uf)V + f\nabla_U V.$$

The *connection coefficients* $\Gamma^{\mu}_{\beta\alpha}$ are defined in any basis by

$$\nabla_{X_\alpha} X_\beta = \Gamma^{\mu}_{\beta\alpha} X_\mu \tag{2.8}$$

It follows from the properties of ∇_U that the covariant derivative of the vector field V with respect to the vector field U can then be written

$$\nabla_U V = [v^{\mu}_{,\sigma} + \Gamma^{\mu}_{\tau\sigma} v^{\tau}] u^{\sigma} X_\mu \tag{2.9}$$

or equivalently

$$\nabla_U V = v^{\mu}_{;\sigma} u^{\sigma} X_\mu \tag{2.10}$$

where $v^{\mu}{}_{;\sigma}$ is defined to be $v^{\mu}_{,\sigma} + \Gamma^{\mu}_{\tau\sigma} v^{\tau}$. The covariant derivative of V along a curve takes account of both the change in the components and the variation of the basis vector fields. On a flat manifold the components of the covariant derivative reduce to the partial derivatives of the components of the vector field V.

In the above analysis we have added more structure to the manifold: an affine connection. That is, a connection has been added in order to define an absolute parallelism. Primarily, equation (2.8) enables us to compute the change in the basis vector fields.

Consider a manifold that has defined upon it a congruence of curves parameterised by λ. Consider a mapping of *each* point on *each* curve to another point on the same curve a distance $\Delta\lambda$ along, as shown

The congruence provides a natural mapping of M onto itself. See fig 2.5

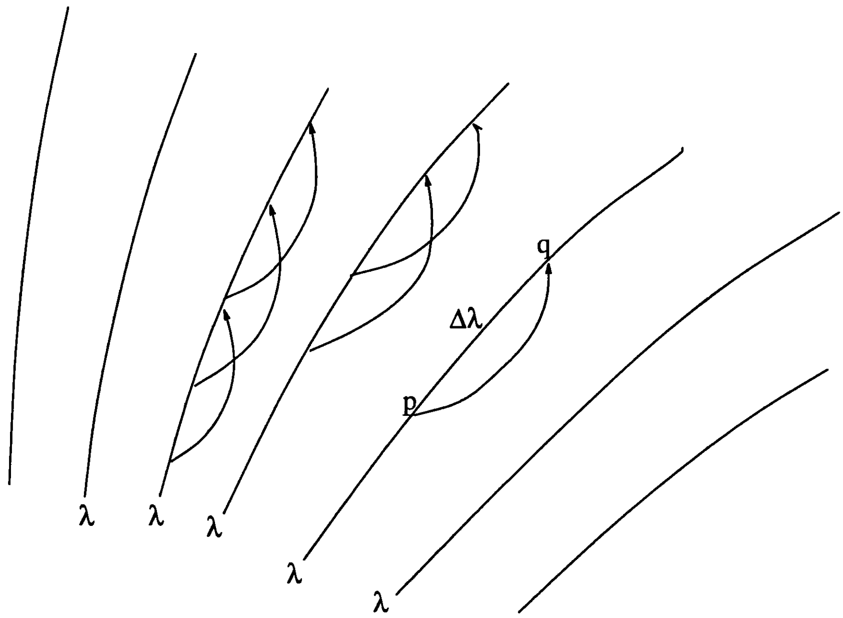

Figure 2.5: Mapping of a congruence onto itself.

2.3.1 Lie Dragging a Function and the Lie Derivative of a Function.

Now consider a function f defined on the manifold. The mapping or Lie dragging defines a new function f' by demanding that it has the value at q that f has at p. ie $f'(q) = f(p)$ for all p and q on the same curve separated by $\Delta\lambda$. The function f is said to be Lie dragged if $f'(q) = f(q)$ for *all* $\Delta\lambda$. A function that is Lie dragged must be constant along any curve of the congruence ie

$$\frac{df}{d\lambda} = 0.$$

This is the first step towards describing a derivative using Lie dragging. It is now obvious that we can do this by comparing the function to the Lie dragged one. Taking the derivative of f involves defining the new function f' at all points on the manifold by $f'(\lambda + \Delta\lambda) = f(\lambda)$ for one particular $\Delta\lambda$ and forming the difference between this and the original function f at p ie $f'(p) - f(p)$. We then divide by $\Delta\lambda$ and take the limit as $\Delta\lambda \mapsto 0$ ie

$$\lim_{\Delta\lambda \to 0} \left[\frac{f'(p) - f(p)}{\Delta\lambda} \right]$$

This is the Lie derivative of the function f. If the function f is Lie dragged then $f'(p) = f(p)$ and the Lie derivative of f along the congruence is zero.

If V is the vector field defined by the λ congruence then the Lie derivative of f with respect to V is denoted

$$\pounds_V f = \frac{df}{d\lambda} = V(f). \tag{2.11}$$

2.3.2 Lie Dragging a Vector Field and the Lie Derivative of a Vector Field.

We have the λ congruence corresponding to the vector field $V = \frac{d}{d\lambda}$ and the μ congruence corresponding to $U = \frac{d}{d\mu}$. λ defines a map of the manifold as before, and the μ congruence is acted on by this map. The new congruence is given by parameter $\mu*$ and the corresponding vector field is $\frac{d}{d\mu*}$. If the $\mu*$ congruence equals the μ congruence ie $\frac{d}{d\mu*} = \frac{d}{d\mu}$ for *all* $\Delta\lambda$ then the vector field is said to be Lie dragged. We define the Lie derivative of U with respect to V as the vector field which operates on an arbitrary function f to give

$$[\pounds_V U](f) = \lim_{\Delta\lambda \to 0} \left[\frac{U'(p) - U(p)}{\Delta\lambda} \right](f)$$

It turns out that

$$\pounds_V U = [V, U] \tag{2.12}$$

2.3.3 The Lie Derivative of a Tensor Field.

The Lie derivatives of one-forms and tensors of higher rank are defined in terms of the Lie derivatives of vector fields and functions. For example, consider the one-form ω: we define $\pounds_V \omega$ by the required property of Lie derivatives,

$$\pounds_V(\omega(W)) = (\pounds_V \omega)(W) + \omega(\pounds_V(W))$$

$\omega(W)$ is just a function, $(\pounds_V(W))$ is a vector and so $\omega(\pounds_V(W))$ is a function. Hence we can deduce the Lie derivative of the one-form $(\pounds_V \omega)$. We define the Lie derivative of a tensor field T through

$$\pounds_V(T(\omega, ...; U, ...)) \quad = \quad (\pounds_V T)(\omega, ...; U, ...)$$

$$
\begin{aligned}
&+ \quad T(\pounds_V\omega, ...; U, ...) \\
&+ \quad T(\omega, ...; \pounds_V U, ...) \qquad (2.13)
\end{aligned}
$$

One of the principal uses of Lie derivatives in physics is to express the notion that a tensor field is invariant under some transformation.

$$\pounds_V T = 0$$

2.4 Curvature in the Language of Differential Forms.

A p-form is defined to be a *completely antisymmetric* $\binom{0}{p}$ tensor. As before a one-form is a $\binom{0}{1}$ tensor and a scalar is a zero-form. Just as $\binom{0}{2}$ tensors could be made from $\binom{0}{1}$ tensors using the tensor product, we define an operation $\wedge$ called the wedge product for constructing two-forms from one-forms. If p and q are two one-forms then

$$p \wedge q = p \otimes q - q \otimes p \qquad (2.14)$$

This can be extended to

$$
\begin{aligned}
p \wedge (q \wedge r) = (p \wedge q) \wedge r \;&=\; p \wedge q \wedge r \\
&=\; p \otimes q \otimes r + q \otimes r \otimes p + ... \\
&-\; q \otimes p \otimes r - p \otimes r \otimes q - ....
\end{aligned}
$$

In this way we can construct a general p-form from one-forms. We can also construct new p-forms using the exterior derivative which operates on p-forms to produce (p+1)-forms. The definition of the operator d is that d is a linear operator carrying a function f into the unique one-form df defined by df(U)=U(f) where U is any vector field. d is extended to forms of higher rank by requiring that

(1). d converts a p-form into a (p+1)-form. (2). $d(d\omega) = 0$ for any p-form ω. (3). $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma$ if ω is a p-form.

2.4.1 Cartan's Equations and Curvature.

Let $\{\omega^\mu\}$ be a basis for one-forms dual to a basis $\{X_\mu\}$. The exterior derivative of any one-form ω^μ is a two-form $d\omega^\mu$ and hence a linear combination of the basis two-forms $\{\omega^\mu \wedge \omega^\nu\}$. It can be shown that

$$d\omega^\mu = -\frac{1}{2} D^\mu_{\alpha\beta}\omega^\alpha \wedge \omega^\beta \tag{2.15}$$

where the $D^\mu_{\alpha\beta}$ are the structure coefficients of the basis X_μ. In the basis X_μ the corresponding structure coefficients are given by

$$D^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha} - \Gamma^\mu_{\alpha\beta}$$

which implies

$$d\omega^\mu = -\frac{1}{2}\Gamma^\mu_{\beta\alpha}\omega^\alpha \wedge \omega^\beta$$

The *connection forms* $\omega^\mu_\nu = \Gamma^\mu_{\nu\sigma}\omega^\sigma$ can be replaced in the last equation. Then

$$d\omega^\mu = -\omega^\mu_\sigma \wedge \omega^\sigma \tag{2.16}$$

This is *Cartan's first equation.*

In general, for a Riemannian space we have that

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\alpha},_\beta + g_{\sigma\beta},_\alpha - g_{\alpha\beta},_\sigma) + \frac{1}{2}(-D^\mu_{\alpha\beta} + g_{\alpha\sigma}g_{\mu\tau}D^\sigma_{\tau\beta} + g_{\sigma\beta}g_{\mu\tau}D^\sigma_{\tau\alpha}).$$

When $D^\mu_{\alpha\beta} = 0$ (a coordinated basis) the connection coefficients $\Gamma^\mu_{\alpha\beta}$ become the familiar Christoffel symbols and are symmetric ie $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$.

The curvature of a manifold is described in terms of a curvature tensor. The Riemann curvature tensor $\mathbf{R}$ is a $\binom{1}{3}$ tensor field. Covariant derivatives do not generally commute and this failure of successive operations of differentiation to commute when applied to a vector field manifests itself in the Riemann curvature tensor which is defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

The tensor $\mathbf{R}$ also expresses the failure of a vector to return to its original value when parallel transported around a loop. In terms of a basis, the components $R^\sigma_{\alpha\mu\nu}$ of the Riemann curvature tensor are given by

$$R(X_\mu, X_\nu)X_\alpha = R^\sigma_{\alpha\mu\nu}X_\sigma$$

The *curvature forms*

$$\theta^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\mu}_{\alpha}\wedge\omega^{\alpha}_{\nu} \tag{2.17}$$

and

$$\theta^{\mu}_{\nu} = \frac{1}{2}R^{\mu}_{\nu\sigma\tau}\omega^{\sigma}\wedge\omega^{\tau} \tag{2.18}$$

This is Cartan's second equation. In a general basis $\{X_{\mu}\}$

$$R^{\sigma}_{\alpha\mu\nu} = \Gamma^{\sigma}_{\alpha\nu},_{\mu} - \Gamma^{\sigma}_{\alpha\mu},_{\nu} + \Gamma^{\tau}_{\alpha\nu}\Gamma^{\sigma}_{\tau\mu} - \Gamma^{\tau}_{\alpha\mu}\Gamma^{\sigma}_{\tau\nu} - D^{\tau}_{\mu\nu}\Gamma^{\sigma}_{\alpha\tau} \tag{2.19}$$

Once again, if $\{X_{\mu}\}$ is a coordinated basis we obtain the familiar expression for the components of the curvature tensor $R^{\sigma}_{\alpha\mu\nu}$.

Cartan's first and second equations are all we need to compute the connection coefficients and the components of the Riemann curvature tensor **R** for a metric in a particular basis. We solve for ω^{μ}_{ν} and then use these to compute θ^{μ}_{ν} and we can then "read off" the components of **R** directly. We can choose the basis which makes the metric components constant or one which is adapted to the symmetries of the spacetime ie an invariant basis (see chapter 4): this may reduce the number of non-zero components in the curvature tensor **R** and will certainly be less tedious than working out the components in a coordinated basis.

2.5 Maps of Manifolds.

We have discussed the notion of a manifold and some of the structures which can be placed upon it. There is also the possibility of defining various maps between manifolds and mappings of a manifold onto itself, to each mapping there may be a corresponding map induced on some tensor field. This is our next consideration.

If we have a map ϕ between the manifolds M and N (not necessarily being of the same dimension) $\phi : M \mapsto N$ then ϕ can map upper index $\binom{r}{0}$ tensor fields from M onto N and $\binom{0}{r}$tensors fields from N onto M.

More precisely, when ϕ maps points from M to N and f is a function on N, ϕ_* defines the function $\phi_* f$ on M by

$$\phi_* f(p) = f(\phi(p)).$$

Similarly, if $\lambda(t)$ is a curve through p in M, then the image curve $\phi(\lambda(t))$ in N passes through the point $\phi(p)$. The correponding tangent vector at p in M can be mapped into that at $\phi(p)$ and so we can define the map ϕ_* as: for each C^1 function f at $\phi(p)$ and vector X at p

$$X(\phi^* f)|_p = \phi_* X(f)|_{\phi(p)}.$$

These can be extended to tensor fields (See Hawking and Ellis (1973) [10]) so that ϕ^* is a map which takes all upper index tensor fields from M to N and the map ϕ_* takes all lower index tensor fields from N to M.

The ϕ^* and ϕ_* cannot be extended to mixed tensors. A C^∞ map $\phi : M \mapsto N$ is said to be a diffeomorphism if it is one to one, onto, and has a C^∞ inverse. A map is said to be from M *into* N if the map is defined for all points in M - if in addition every point of N has an inverse image (not necessarily a unique one) then we say it is a mapping from M *onto* N. If ϕ is a diffeomorphism then we can use ϕ^{-1} to extend the definition of ϕ^* to tensors of all types since $(\phi^{-1})^*$ goes from $V_{\phi(p)}$ to V_p (with similar reasoning for ϕ^*) and it can be shown that $\phi_* = (\phi^{-1})^*$ so we need only consider ϕ^* and $(\phi^{-1})^*$.

If $\phi : M \mapsto N$ is a diffeomorphism then M and N are necessarily of the same dimension and have identical manifold structure.

Let us now consider a special case where $\phi : M \mapsto M$ is a diffeomorphism. There exists a tensor field $\mathbf{T}$ on M. ϕ induces a map ϕ^* of $\mathbf{T}$ and we can compare $\mathbf{T}$ and $\phi^*\mathbf{T}$. If $\phi^*\mathbf{T} = \mathbf{T}$ then ϕ is a *symmetry transformation* for the tensor field $\mathbf{T}$. Associated with a one-parameter family of diffeomorphisms ϕ_t is a vector field v: v is said to generate ϕ_t. If ${\phi_t}^*\mathbf{T} = \mathbf{T}$ then the Lie derivative of $\mathbf{T}$ with respect to v is zero

$$\pounds_v \mathbf{T} = 0$$

The set of all vector fields V under which the tensor field $\mathbf{T}$ is invariant forms a *Lie algebra* (see chapter 3). In order for a set to form an algebra the elements must form a vector space over the real numbers R and the commutator of any two elements must also be an element in the set. The first is satisfied since the set of fields is a vector space over R.

Now, we know $\pounds_v \mathbf{T} = 0$ and $\pounds_w \mathbf{T} = 0$ for $v, w \in V$. This means that $[\pounds_v, \pounds_w]\mathbf{T} = 0$ and for the second condition to hold we have to prove that this implies $\pounds_{[v,w]}\mathbf{T} = 0$.

It is straightforward to prove $[\pounds_v, \pounds_w]\mathbf{T} = \pounds_{[v,w]}\mathbf{T}$ for any arbitrary tensor field $\mathbf{T}$. This can be seen by first considering a function: $[\pounds_v, \pounds_w]f = \pounds_{[v,w]}f$ for an arbitrary function f since

$$\begin{aligned}[\pounds_v, \pounds_w]f &= \pounds_v\pounds_w(f) - \pounds_w\pounds_v(f) \\ &= v[w(f)] - w[v(f)] \\ &= [v, w]f \\ &= \pounds_{[v,w]}f\end{aligned}$$

and for vector fields

$$\begin{aligned}[\pounds_v, \pounds_w]u &= \pounds_v\pounds_w(u) - \pounds_w\pounds_v(u) \\ &= v[w(u)] - w[v(u)] \\ &= [[v, w], u] \qquad \text{by the Jacobi identity for vector fields} \\ &= \pounds_{[v,w]}u\end{aligned}$$

This does in fact apply to all tensor fields $\mathbf{T}$ by (2.13) stated earlier.

Thus the vector fields that Lie drag any fixed tensor must form a Lie algebra (see chapter 3).

A diffeomorphism $\phi : M \mapsto M$ for which $\phi^*\mathbf{g} = \mathbf{g}$ is called an isometry of $\mathbf{g}$ on M. There will in this case be a set of vector fields ξ for which

$$\pounds_\xi \mathbf{g} = 0$$

and the ξ are referred to as *Killing vector fields* on M. The Killing vector fields for a particular manifold will, therefore, form the Lie algebra of a Lie group and we refer to this group as a group of isometries. This equation can also be expressed in component form:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{2.20}$$

We shall investigate the properties of Killing vectors in chapter 4.

The theory of groups is a self contained abstract theory which has been widely studied and has produced many far reaching results applicable to many physical problems. So any set endowed with a group structure has in addition, many properties which can be studied and interpreted solely in terms of its group structure. We devote the next chapter to those parts of group theory relevant to our study of symmetries of the metric tensor.

Chapter 3

The Theory of Groups and Lie Groups.

We shall consider the theory of groups in detail now, in particular that of Lie groups, in order that we can justify our previous conclusions on the properties of isometry groups. A Lie group need not be associated with the invariance of a tensor field **T**. That is, some Lie groups describe the symmetries of systems and some Lie groups describe non-symmetries of other systems. However, we are interested primarily in isometry groups. It is obvious that we can talk of both discrete isometries (eg reflections) and continuous isometries (eg translations, rotations). In fact, all continuous groups can be considered as having a discrete component (the discrete groups cannot be described by an infinitesimal generator) and a continuous connected component. When we say isometry group we shall be referring only to the latter and we will look, in particular, at continuous groups. The Lie groups have additional properties (they can be considered as having the structure of a manifold as well as group properties) and so are even more specific than this but we shall progress to these in due course. Associated with every Lie group is a Lie algebra which indicates the local properties of the Lie group. We consider their relationship and the classification of Lie groups. This has relevance to the classification of cosmologies according to their isometry groups. This is important since we would like to determine the properties of the symmetry group of a spacetime from its infinitesimal generators ie isometries (Killing vector fields) or conformal Killing vector fields and, of course, it is the infinitesimal generators which form the Lie algebra.

3.1 Group Axioms.

A group G is a set of elements $g_1, g_2, ...g_n \in G$ together with an operation o called group multiplication (or composition), such that

1. $g_i \in G, g_j \in G \quad g_i \circ g_j \in G$: closure
2. $g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$: associativity
3. $g_1 \circ g_i = g_i = g_i \circ g_1$ for all g_i : existence of identity
4. $g_i \circ g_i^{-1} = g_1 = g_i^{-1} \circ g_i$: unique inverse.

Groups can be discrete or continuous. The *order* of a discrete group is the number of discrete elements which constitute the group. This may be finite or infinite. We say that a continuous group has a *dimension* equal to the number of parameters required to specify a particular element of the group. It may be finite or infinite dimensional.

The following are examples of groups:

(1).The set of all positive rational numbers form a group with respect to multiplication. The product of two positive rational numbers is another positive rational number, as is the inverse of a positive rational number and the unit element is the number 1. Associativity is assured by the laws of arithmetic. This is an infinite Abelian group.

(2).The set of integers Z forms an Abelian group with respect to addition.

(3).The set of nonsingular matrices over a field F form a group under matrix multiplication.

The set of negative rational numbers do not form a group (under multiplication) and neither does the set of positive integers (under multiplication) since each element lacks an inverse.

Since a group G is a collection of elements, the usual definitions and notations of set theory can be applied to G. A subgroup is a subset of a group G which obeys the group postulates.

3.2 Cosets, Invariant Subgroups and Factor Groups.

Cosets: A group G can be partitioned into disjoint classes of mutually equivalent elements-called right cosets. We choose a subgroup H of G and define the equivalence of a and b. All elements of G which satisfy

$$ab^{-1}{\in}H \qquad \text{for a,b}{\in}G.$$

are said to be equivalent and constitute a right coset of the subgroup H (similarly, elements wich satisfy $a^{-1}b{\in}H$ constitute a left coset of H). Hence every subset of the form Ha is a right coset of H, for a $\in$ G.

Generally, when an equivalence relation has been defined on a set, this set can be expressed as the union of all the distinct equivalent classes. In the present case, the equivalence classes are the right cosets so that G is the union of all the distinct cosets. In order to express this result more formally, we select one representative from each coset. If a_i is one of the representatives, the corresponding coset may be denoted by Ha_i and G may be expressed as

$$G = \sum_i^n Ha_i$$

where n is the number of distinct right cosets.

If A is simultaneously a right and left coset of the subgroup H then A=Ha=aH where a is any element of A. If *every* right coset is simultaneously a left coset then we must have Ha=aH ie $a^{-1}Ha = H$ for all $a{\in}G$.

Definition. *An invariant subgroup (or normal subgroup) N satisfies for all $n{\in}N$ and $a{\in}G$*

$$a^{-1}Na = N.$$

In order that the partition of the group G into the left end right cosets of the subgroup N should coincide, it is necessary and sufficient that N should be an invariant subgroup.

Definition. *Let N be an invariant subgroup of the group G and let A and B be cosets of N ie A=Na, B=Nb. For the product AB we have AB=NaNb=NNab=Nab so that AB is again a coset of N. This multiplication operation satisfies the group* *axioms. The group of cosets thus obtained is called the factor group of the group G by the normal subgroup N and is denoted G/N.*

3.3 The Classical Groups.

Groups can be continuous or discrete and can be infinite or finite (in dimension). Our starting point for the study of continuous groups is the enumeration of all classical matrix groups. We introduce the concept of change of basis in a vector space V_N and associate with each change of basis an $N \times N$ nonsingular matrix. Such matrices belong to the General Linear Groups Gl(N,-), which we look at first. Secondly, the subgroups of the general linear groups which are volume preserving are defined: the Special Linear Groups Sl(N,-). The concept of the metric is then presented and the corresponding metric preserving groups are defined. The classical groups are then defined as having these or having combinations of the above properties. It can be misleading to think of the classical groups just in terms of their matrix representations - it is worthwhile remembering they do actually have important physical relevance, relating to symmetry. Therefore the exact definitions of the above classical groups are set out and not just the resulting conditions on the matrices constituting the group representation.

Every set of basis vectors in V_N can be related to every other by an $N \times N$ nonsingular matrix (the matrix must be nonsingular to possess an inverse). The $N \times N$ matrix groups involved in changing bases in vector spaces over the fields R and C are Gl(N,r) and Gl(N,c) respectively.

Volume preserving groups:

The completely antisymmetric subspace of $(V_N)^r$ is spanned by $\binom{N}{r}$ basis vectors (see chapter 2). In particular, for r=N there is only one basis vector

$$e_1 \wedge e_2 \wedge e_3 ....e_N.$$

This basis is called the volume element associated with the basis $\{e_1, e_2, ...e_N\}$. Under a change of basis $e_1{}' = A_i^j e_j$, the new basis $e_1{}' \wedge e_2{}' \wedge e_3{}' ....e_N{}'$ is a multiple of the basis $e_1 \wedge e_2 \wedge e_3 ....e_N$.

$$e_1{}' \wedge e_2{}' \wedge e_3{}' ....e_N{}' = (det A) e_1 \wedge e_2 \wedge e_3 ....e_N$$

The subset of Gl(N,-) which preserves volume forms a group defined by detA=+1. These volume preserving groups of transformations in V_N over R,C are called special linear groups Sl(N,r) and Sl(N,c) respectively.

Metric preserving groups:

A metric function on a vector space V_N is a mapping of a pair of vectors into a number in the field F associated with the vector space V_N.

$$g(v_1, v_2) = f \qquad v_1, v_2 \in V, \qquad f \in F.$$

This mapping obeys

$$
\begin{aligned}
& g(v_1, \alpha v_2 + \beta v_3) &=& \alpha g(v_1, v_2) + \beta g(v_1, v_3) \quad (3.1)\\
\text{and} \quad & g(\alpha v_1 + \beta v_2, v_3) &=& g(v_1, v_3)\alpha + g(v_2, v_3)\beta \quad (3.2)\\
\text{or} \quad & g(\alpha v_1 + \beta v_2, v_3) &=& g(v_1, v_3)\bar{\alpha} + g(v_2, v_3)\bar{\beta} \quad (3.3)
\end{aligned}
$$

Metrics obeying (3.1) and (3.2) are called *bilinear* metrics and those obeying (3.1) and (3.3) are *sesqulinear* metrics. Further possibilities are that the metric may be symmetric and/or positive-definite but the latter is not considered when defining the classical groups since it is too restrictive. Writing the metric in terms of the basis vectors gives

$$g_{ij} = g(e_i, e_j).$$

The transformation properties of the metric function g_{ij} under the change of basis $e_1{}' = A_i^j e_j$ are given by

$$g_{ij} = A_j^k g_{lk} A_i^l.$$

Groups preserving bilinear symmetric metrics are called *Orthogonal.* The Orthogonal groups preserving metrics with signature (N_+, N_-) in vector spaces over R^N,C^N,Q^N where $N = (N_+ + N_-)$, are $O(N_+, N_-; r)$, $O(N_+, N_-; c)$, $O(N_+, N_-; q)$ respectively.

Groups preserving bilinear antisymmetric metrics are called *Symplectic.* Similarly, the Symplectic groups preserving metrics with signature (N_+, N_-) in vector spaces over R^N,C^N,Q^N are $Sp(N_+, N_-; r)$, $Sp(N_+, N_-; c)$, $Sp(N_+, N_-; q)$.

Groups preserving sesquilinear symmetric metrics are called *Unitary.* The Unitary groups preserving metrics with signature (N_+, N_-) in vector spaces over R^N,C^N,Q^N are $U(N_+, N_-; r)$, $U(N_+, N_-; c)$, $U(N_+, N_-; q)$.

The classical groups can have combinations of the above properties, see Gilmore (1974) [8]. We denote the intersection of two groups in the following manner eg. the intersection of $U(N,c)$ and $Sp(2N,c)$ is $USp(2N,c)$. The metric-preserving groups which are also volume-preserving are called the *special* metric-preserving groups and denoted by an additional S, eg.

$$Sl(N,c) \cap U(N,c) = SU(N,c)$$

Not all of the classical groups are distinct - there exist isomorphisms and homomorphisms between some groups of the same dimension. The matrix representation of a classical group does, of course, form a group itself. The dimension of a particular group can be determined by considering the conditions on the corresponding matrices.

3.4 Continuous Groups.

A *continuous* or *topological* group has two distinct kinds of structures on it. It has an algebraic structure and it also has a topological structure. Algebraically it is a group and it therefore obeys the group axioms. Topologically, it is a manifold. In order that a group be a continuous group it has to obey the two additional axioms:

$$\begin{array}{lll} \text{The mapping} & \alpha\times\beta\mapsto\alpha\beta & \text{is continuous.} \\ \text{The mapping} & \alpha\mapsto\alpha^{-1} & \text{is continuous.} \end{array} \tag{3.4}$$

This ensures that the product of any group element near α with a group element near β is a group element near $\alpha\beta$ and that if α is a group element near β then α^{-1} is a group element near β^{-1}. These are the only axioms required to connect the algebraic with the topological properties of continuous groups. The theory of Lie groups is the result of imposing these axioms and extra conditions of differentiability on the group manifold.

Our concern now is with the *global* properties of continuous groups. In light of the above, it is possible to *define* both the continuous group and the continuous group of transformations. We are already familiar with the continuous function and the differentiable manifold.

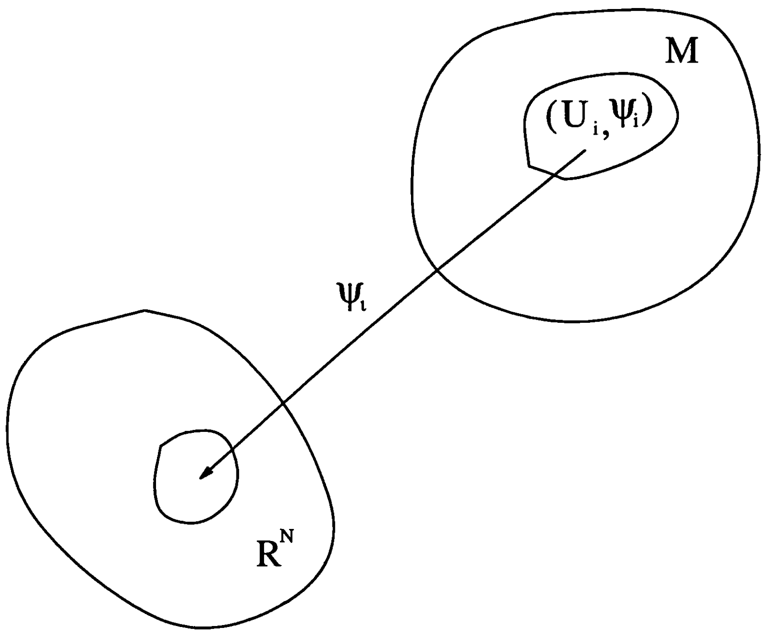

Figure 3.1: The group manifold.

Definition. *A continuous (or topological) group consists of*

1. An underlying n-dimensional manifold M.

2. An operation Φ, with corresponding group composition functions ϕ^μ $(\mu = 1, ...n)$ which map a pair of elements in the group into another element of the group ie

$$\Phi \ : M \times M \mapsto M$$

$$\text{that is} \qquad \Phi \ (\alpha, \beta) \mapsto \gamma \qquad \alpha, \beta, \gamma \in M.$$

The corresponding map which relates the parameters of the group elements is

$$\phi_i \ : R^N \times R^N \mapsto R^N$$

$$\text{equivalently} \qquad \phi_i^\mu \ (\alpha^\mu, \beta^\mu) = \gamma^\mu \qquad \alpha, \beta, \gamma \in M.$$

Each coordinate patch on M has the corresponding chart (U_i, ψ_i). To each chart there will be a corresponding map ϕ_i. Therefore there may be a number of different ϕ_i corresponding to a particular continuous group. Each arises from a different choice of ψ_i on M. By α we mean the group element $(\alpha^1, \alpha^2, ...\alpha^n)$.

3. The mappings

$$\Phi \ : \ \alpha o \beta \mapsto \alpha\beta$$

$$\text{and} \qquad \chi \ : \ \alpha \mapsto \alpha^{-1} \qquad \text{are continuous.}$$

The group multiplication properties may be translated into conditions on the functions ϕ^{μ}, since the group axioms must be obeyed. There can be an arbitrary number of different ϕ corresponding to the particular group, but they will all fulfil these conditions.

Definition. *A continuous group of transformations consists of*

A:

1. An underlying topological space τ, which is an n-dimensional manifold.
2. An operation Φ, the group composition functions which map a pair of elements in the group into another element of the group.

$$\Phi(\alpha, \beta) \mapsto \gamma \qquad \alpha, \beta, \gamma \in M.$$

3. The mappings

$$\Phi \ : \ \alpha \circ \beta \mapsto \alpha\beta$$
$$\text{and} \qquad \chi \ : \ \alpha \mapsto \alpha^{-1} \qquad \text{are continuous.}$$

and B:

1. A geometrical space H, which is an N-dimensional manifold.
2. A mapping f which maps an element in the product space $\tau \times H$ into H.

$$f : \tau \times H \mapsto H$$

$f \equiv f^i \equiv f^i(\alpha, x)$ where x=1,...N and of course by α we mean the group element $(\alpha^1, \alpha^2, ...\alpha^n)$ and by x we mean the point in the geometrical manifold $(x^1, x^2, ...x^N)$.

3. This mapping is continuous.

Similarly, there will be conditions set on ϕ and f by the group multiplication properties.

It is important to make a clear distinction between the underlying topological space of the group and the geometrical space on which the group acts. Every continuous group may be considered as a continuous group of transformations if we allow it to act on itself ie

$$G \equiv \tau, \qquad f \equiv \phi$$

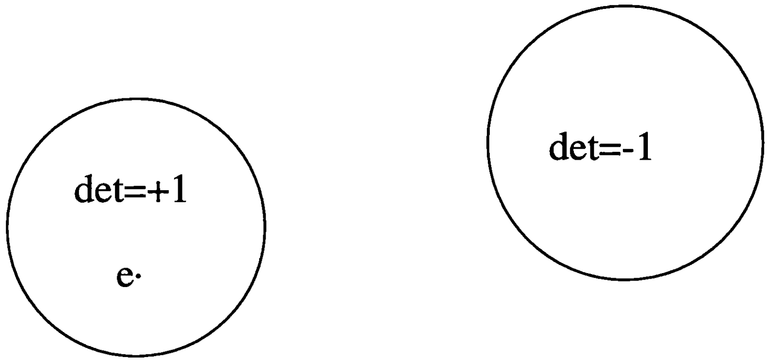

Figure 3.2: The continuous group O(3) is composed of two disconnected parts, each having its own determinant. e is the identity element.

We consider a few ideas relevant to our discussion of continuous groups

A space is said to be *connected* if any two points in the space can be joined by a path which lies entirely within that space. A continuous group is connected if its underlying manifold is connected. For example, the group O(3) is disconnected. The group O(3) consists of all transformations which leave the quantity $x^2 + y^2 + z^2$ invariant and has a representation in terms of orthogonal matrices: the determinant of every orthogonal matrix is either +1 (corresponding to pure rotations and denoted SO(3)) or -1 (rotation-reflection).

A connected space is *simply connected* if a curve connecting any two points in the space can be continuously deformed into every other curve connecting the same two points.

A Lie group is a continuous group with specific properties:

Definition. *A Lie group is a C^∞ manifold endowed with a group structure in which multiplication and taking of inverses are C^∞ operations.*

This is the definition given by Hausner and Schwartz (1968) [9] but there are others which define a Lie group as being the connected component of a continuous group. We have chosen to use the above definition so we can emphasise any properties which depend upon the connectedness of a group.

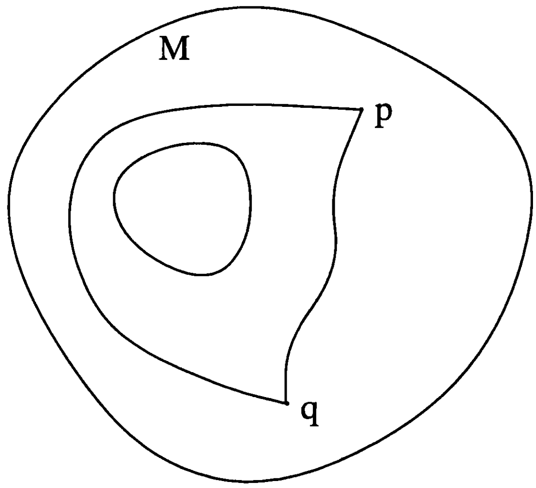

Figure 3.3: This manifold M is not simply connected.

3.5 Local Concepts

It is true that we will shortly be able to describe the (global) properties of a connected Lie group by knowing its properties only in the vicinity of the origin. To make this possible, we define an object that behaves essentially like the *neighbourhood of the identity element* in a Lie group. We find that it is simpler to deal with such a local Lie group.

In the connected component of a Lie group it is always possible to choose a finite number of points on the path joining the identity e to β ie

$$\alpha_0 = e, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n = \beta$$

with the properties:

1. α_i and α_{i+1} lie within a common neighbourhood.
2. $\alpha_{i+1} \circ {\alpha_i}^{-1}$ lie inside some neighbourhood of the identity e, for all i.

The group operation β can then be written as:

$$\beta = (\alpha_n \circ {\alpha_{n-1}}^{-1})....(\alpha_3 \circ {\alpha_2}^{-1}) \circ (\alpha_2 \circ {\alpha_1}^{-1}) \circ (\alpha_1 \circ {\alpha_0}^{-1}) \circ I$$

that is, β is the product of group operations that all lie close to the identity. This is the basis of the exponential mapping procedure which allows us to obtain Lie group elements from the local properties of a connected Lie group, which amounts to repeated application of the infinitesimal operators. We shall discuss this in more detail shortly.

Definition. *A local Lie group is that subgroup of a Lie group which consists of elements corresponding to infinitesimal group elements.*

3.6 Lie Groups and Lie Algebras.

As stated earlier, finite elements of a connected Lie group can be obtained by repeated application of the infinitesimal elements. This means the properties of the connected Lie group can be completely determined by its local properties (that is, the properties of the group near the identity element). The infinitesimal generators (from which the infinitesimal elements are formed) describe these local properties and form what we call a Lie algebra: the Lie algebra characterises the Lie group. It was shown in chapter 2 that the infinitesimal generators of an isometry group formed an algebra. Now we are considering Lie groups in general and show that a corresponding Lie algebra exists. Lie's three theorems provide a mechanism for constructing the Lie algebra for any Lie group (see later).

3.6.1 Infinitesimal Generators of a Lie Group. The Lie Algebra and Structure Constants.

The infinitesimal generators of a Lie group of transformations:

We consider the effect an infinitesimal transformation has on the structure of a function F(p) defined on the geometrical space H. It will be useful to have an expression for the change induced by an infinitesimal element of a Lie group.

First of all, consider a Lie group with composition function ϕ which induces the transformation $f(\alpha, x)$ on the coordinates of the geometrical space H (α denotes a group element). We define a coordinate sytem S on H and label these coordinates $x^i(p)$. The function F(p) is defined for all points $p \in H$. Then

$$F(p) = F^s[x^1(p), x^2(p), ...x^N(p)].$$

Under the coordinate transformation $f(\alpha, x)$, the coordinates x are transformed into coordinates x' and we shall refer to this coordinate system as S'. We can write

$$F^{s'}[x'^1(p), x'^2(p), ...x'^N(p)] = F^{s'}[x'(p)].$$

The existence of the field F(p) is independent of any coordinate system and so

$$F(p) = F^s[x(p)] = F^{s'}[x'(p)].$$

which defines our new function $F^{s'}$ of coordinates x'. Our aim is to express $F^{s'}$ in terms of F^s.

The coordinates x' are related to the coordinates x by the transformation $f(\alpha, x)$ ie

$$\begin{aligned} x' &= f(\alpha, x) \\ \text{therefore} \quad x &= f(\alpha^{-1}, x') \end{aligned}$$

but since we are dealing with infinitesimal transformations only then

$$\begin{aligned} x' &= f(\delta\alpha, x) \\ \text{and} \quad x &= f(-\delta\alpha, x') \end{aligned}$$

A Taylor expansion around $\delta\alpha = 0$ allows us to write our new coordinates (to first order) as

$$x^j(p) = x'^j(p) - \delta\alpha^\mu \frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\Big|_{\beta=0}$$

We recall

$$F^{s'}[x'(p)] = F^s[x(p)].$$

Expanding the function about the identity element

$$F^{s'}[x'(p)] = F^s[x'(p)] - \delta\alpha^\mu \frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\Big|_{\beta=0} \frac{\partial}{\partial x'^j} F^s[x'(p)].$$

The change in the structural form induced in F(p) is then given by

$$F^{s'}[x'(p)] - F^s[x'(p)] = -\delta\alpha^\mu \frac{\partial f^j[\beta, x'(p)]}{\partial \beta^\mu}\Big|_{\beta=0} \frac{\partial}{\partial x'^j} F^s[x'(p)],$$

which can be written

$$F^{s'}[x'] - F^s[x'] = -\delta\alpha^\mu X_\mu(x') F^s[x']$$

where

$$X_\mu(x') = -\frac{\partial f^j[\beta, x']}{\partial \beta^\mu}\Big|_{\beta=0} \frac{\partial}{\partial x'^j}.$$

The X_μ are called the *infinitesimal generators* of the Lie group and they can be considered as being tangent vectors in the tangent space to the group manifold at the identity e. These infinitesimal generators form a vector space or more precisely they from an algebra, as we shall see shortly. If the elements of an algebra are the infinitesimal generators of a Lie group then the algebra is called a *Lie algebra.*

The operator that effects the infinitesimal transformation is (to first order)

$$1 + \delta\alpha^\mu X_\mu \tag{3.5}$$

The properties of the infinitesimal generators characterise the Lie group and the Lie algebra corresponding to the Lie group. We can see this via Lie's theorems. From (3.5) we can see that the elements of the Lie algebra exist in one to one correspondence with the elements of a local Lie group ie X_μ is an element of the Lie algebra and $1 + \delta\alpha^\mu X_\mu$ is an element of the Local Lie group.

In chapter 2 we saw that the set of Killing vector fields corresponding to a group of isometries form an algebra (the multiplication operator being the commutator), that is, the commutator of any two Killing vector fields u and v, denoted $[u, v]$, is also a Killing vector field. In fact, the infinitesimal generators of every Lie group form a Lie algebra. We shall show this by considering the commutator of infinitesimal Lie group elements.

If α and β are elements in an abelian Lie group then $\alpha\beta\alpha^{-1} = \beta$ but for a non-abelian group, γ is a measure of the non commutivity of α and β:

$$\alpha\beta\alpha^{-1} = \gamma\beta$$
$$\alpha\beta\alpha^{-1}\beta^{-1} = \gamma$$
$$(\alpha\beta)(\beta\alpha)^{-1} = \gamma$$

Therefore γ must be an element of the group. $(\alpha\beta)(\beta\alpha)^{-1}$ is called the commutator of the elements α and β.

If two elements are close to the identity of the group then we can write them as (to second order) as

$$\alpha = I + \delta\alpha^\mu X_\mu \delta\alpha^\nu X_\nu$$

$$\beta = I + \delta\beta^\mu X_\mu \delta\beta^\nu X_\nu$$

The X_μ form a basis for the vector space of infinitesimal generators. Evaluating the commutator $(\alpha\beta)(\beta\alpha)^{-1}$ we find that (to second order)

$$\begin{aligned}
(\alpha\beta)(\beta\alpha)^{-1} &= I + \delta\alpha^\mu X_\mu(\delta\beta^\nu X_\nu) - \delta\beta^\mu X_\mu(\delta\alpha^\nu X_\nu) \\
&= I + \delta\alpha^\mu\delta\beta^\nu[X_\mu X_\nu - X_\nu X_\mu] \\
&= I + \delta\alpha^\mu\delta\beta^\nu[X_\mu, X_\nu]
\end{aligned}$$

Since $(\alpha\beta)(\beta\alpha)^{-1}$ is a group element then so is $I + \delta\alpha^\mu\delta\beta^\nu[X_\mu, X_\nu]$ and $[X_\mu, X_\nu]$ *must* exist within the vector space of the infinitesimal group generators X_μ. Hence $[X_\mu, X_\nu]$ can be expanded in terms of the basis X_μ i.e.

$$[X_\mu, X_\nu] = C^\lambda_{\mu\nu} X_\lambda$$

The $C^\lambda_{\mu\nu}$ are called the *structure constants* of the Lie group and transform by a change of basis of the Lie algebra as the components of a third rank tensor. The structure constants are characteristic of the Lie group and are very important since Lie algebras with the same structure constants are isomorphic. That is, Lie algebras (of the same dimension) with different elements but the same structure constants are isomorphic. Conversely, given a set of structure constants $C^\lambda_{\mu\nu}$ there is a Lie group G (not unique) which has a Lie algebra with those $C^\lambda_{\mu\nu}$.

Lie's 3rd theorem states that the structure constants obey:

$$C^\lambda_{\mu\nu} = -C^\lambda_{\nu\mu}$$

and the Jacobi identity,

$$C^\lambda_{\mu\nu} C^\tau_{\lambda\eta} + C^\lambda_{\nu\eta} C^\tau_{\lambda\mu} + C^\lambda_{\eta\mu} C^\tau_{\lambda\nu} = 0.$$

The converse to Lie's 3rd theorem shall be considered shortly.

We have just seen how the infinitesimal generators of a Lie group arise and how they form a Lie algebra. The Lie algebra can also be specified solely in terms of the group manifold structure. Let us consider the following transformations of a Lie group onto itself. Consider the group element g. Let G be the group manifold. The mapping

$L_g \; : G \mapsto G$	defined by	$L_g h = gh \forall h \in G$	is called left translation by g
$R_g \; : G \mapsto G$	defined by	$R_g h = hg \forall h \in G$	is called right translation by g

That is, left (right) translation maps all elements h of G into the elements gh (hg) of G. There are two groups of transformations of G into G, one defined by left translations and the other by right translations.

We assume that we can define vector fields on the group manifold G. Then L'_g maps $v(h)$ into $L'_g v(h)$ for all $h \in G$. The set of vector fields invariant under the left (right) translations are called left (right) invariant vector fields on G. A vector field v on G is *left invariant* if

$$L'_g v(h) = v(L_g h) \qquad \forall g, h \in G$$

i.e. the newly generated vector field $L'_g v(h)$ is identical to $v(gh)$ for all $g \in G$. It follows that for a left invariant vector field v

$$L'_g v(e) = v(g) \qquad \forall g \in G.$$

We conclude that there is a one to one correspondence between the set of left invariant vector fields on G and the set of tangent vectors to G at the identity element e. Further, it can be shown that the set of left invariant vector fields is closed under the Lie bracket operation. So the set of left invariant vector fields on G form an algebra and are in one to one correspondence with the tangent vectors in the tangent space of G at e. Earlier, we saw how the infinitesimal generators spanned the tangent space at the identity and formed a Lie algebra: we must conclude that the left invariant vector fields on the group manifold similarly specify the Lie algebra of the Lie group.

From the definition of a Lie group (or a Lie group of transformations) we can see that it can have a number of different functions ϕ, each corresponding to a different parameterisation (different realization) of the Lie group. ϕ is the group multiplication/composition law since it relates any two points α, β to another γ

$$\phi(\alpha,\beta)\mapsto\gamma \qquad \alpha,\beta,\gamma\in M.$$

However, all the different ϕ's will have the same *local* properties. This suggests that a canonical mapping ϕ can be constructed using exponential mapping and it turns out this is the case. Also, there are many different Lie groups which have the same Lie algebra, that is, the elements of each of the Lie algebras have the same commutation relations (or structure constants). Lie groups which have isomorphic Lie algebras are said to be locally isomorphic.

Taylors Theorem for connected Lie groups. *The Lie group operation corresponding to the Lie algebra element $\alpha^\mu X_\mu$ is $exp(\alpha^\mu X_\mu)$. This is a faithfull representation of the group and ϕ can be determined.*

From the Lie algebra, we take the X_μ and exponentiate to get a ϕ. It is possible to determine the *global* properties of a connected Lie group from exponentiation of the elements of the Lie algebra. This is the same as saying that if G is a connected Lie group and U is a neighbourhood of the identity e (ie the local subgroup of G) then G is generated by U ie any $g \in G$ is a finite product $g = g_1 \circ g_2 .....g_n$ with $g_i \in U$ and n depending on U and g. This is what we set out to achieve and should become clearer with an example, given later.

When we solve Killing's equation for the infinitesimal isometries on a manifold we then have knowledge of the Lie algebra. We can exponentiate to get all the group elements and the actual isometry group.

If two Lie algebras have the same structure constants ie are isomorphic, then the corresponding groups are said to have the same Lie algebra and are locally isomorphic. If, in addition to to the structure constants, we also know the infinitesimal generators corresponding to a particular Lie algebra then it is straightforward to exponentiate and find the corresponding (unique) Lie group. The following allows us to find all the (global) Lie groups corresponding to a particular Lie algebra.

3.6.2 The Exact Relationship between Lie Groups and Lie Algebras.

As we have mentioned before, many Lie groups may have the same Lie algebra and so there is obviously not a one-to-one relationship between the two, ie there is *not* in general a unique Lie group corresponding to a particular Lie algebra. How much can we say, though, about the lie groups corresponding to a particular Lie algebra?

It is the converse to Lie's third theorem which leads us to the answer, for a Lie algebra over the field of real numbers.

Converse to Lie's third theorem. *Let A be an n-dimensional Lie algebra over R. Then there is a* unique *simply connected n-dimensional Lie group SG whose Lie algebra is isomorphic with A.*

Many Lie groups may have the same Lie algebra, but amongst these *only one* of them is simply connected. This simply connected Lie group is called the *Universal covering group* SG. Further, all locally isomorphic (ie having the same Lie algebra) Lie groups can be obtained from the universal covering group in a straightforward manner: if SG is the simply connected Lie group and D_i is one of it's invariant subgroups then the factor group

$$G_i = \frac{SG}{D_i}$$

is a Lie group whose Lie algebra is isomorphic with the Lie algebra of SG.

This means that the problem of determining all Lie groups with the same Lie algebra reduces to computing all possible invariant subgroups of a given simply connected Lie group.

Summarising the above we can state the following theorem:

Theorem. *Two Lie groups with isomorphic Lie algebras are locally isomorphic and either:*

1. Globally isomorphic, or
2. Homomorphic images of the universal covering group.

So we see that knowledge of the structure constants alone is not enough to specify a unique Lie group. If we have in addition the elements of the Lie algebra of a connected Lie group then we can determine this group completely ie we are in

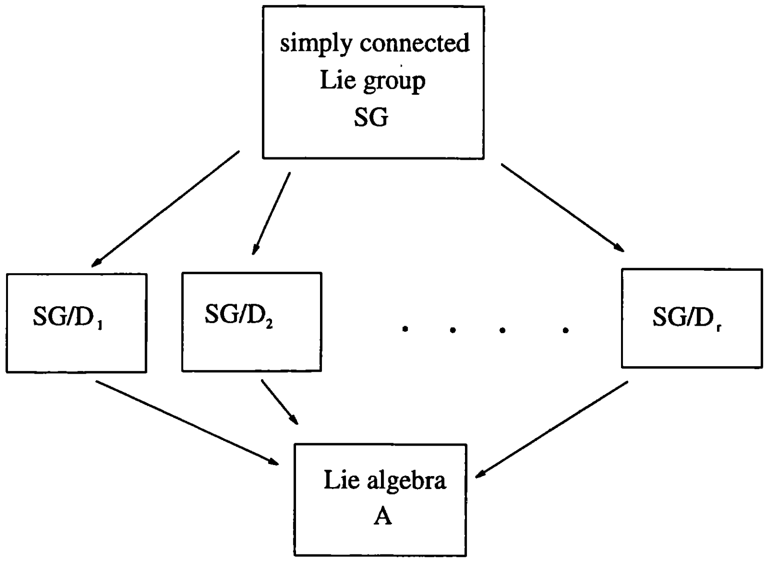

Figure 3.4: The SG/D_i form the complete set of Lie groups with Lie algebra A.

a position to determine its global properties. Otherwise, we can only narrow down the possibilities to a set of locally isomorphic connected Lie groups.

3.7 The Classification of Lie Groups.

The above theorems reduce the problem of classifying all Lie groups to those of
(1) Classifying all possible Lie algebras.
(2) Constructing all possible invariant subgroups associated with a given simply connected Lie group.

We shall assume that the latter presents no problem and consider the first. Classifying Lie algebras amounts to being able to find some kind of canonical commutation relations for an arbitry Lie algebra. However, it turns out that this is, as yet, not possible - we cannot do this for an arbitrary Lie algebra.

There are a few "tools" which allow us in some cases, to determine a canonical form for a particular type of Lie algebra. These are discussed below.
(a) The study of the *regular representation* of a Lie algebra leads to a semi-canonical form for an arbitrary Lie algebra. This is the analysis of the transformation properties of the structure constants (which transform as a tensor). The regular representation is not faithful in general - it may not distinguish between two distinct

elements of one Lie algebra but it does distinguish between different Lie algebras.
(b) To every Lie algebra there is a corresponding *secular equation* and we can find its roots and from it determine the *rank* of the Lie algebra. The analysis is valid in all representations. We can classify all root spaces for semisimple Lie algebras - root diagrams give a straightforward portrayal of the root vectors. An even simpler diagramatic technique for the description of a semisimple Lie algebra is the *Dynkin diagram.* Hence it is possible to classify all semisimple Lie algebras. There is not yet a canonical form for the commutation relations of solvable and nonsemisimple Lie algebras, nor is there a complete classification scheme for these Lie algebras. Such a canonical form does exist for the semisimple Lie algebras.

The analysis gives us the first criterion of solvability.

(c) The *Cartan-Killing metric* for an arbitrary Lie algebra is introduced and the information gained is utilised in the second criterion of solvability.

The two criteria mentioned above can be recast into *Cartan's Criterion* which then provides us with a procedure for decomposing an arbitrary Lie algebra into a semisimple part and a solvable part.

3.8 Some Examples.

3.8.1 The group SO(2).

Consider a system symmetric under rotations in a plane about a fixed point O. We adopt a cartesian coordinate frame on the plane with e_1 and e_2 as the orthonormal basis vectors. $R(\theta)$ denotes a rotation through an angle θ.

$$\begin{aligned} R(\theta)e_1 &= e_1\cos\theta + e_2\sin\theta \\ R(\theta)e_2 &= -e_1\sin\theta + e_2\cos\theta \end{aligned}$$

Which can be written

$$R(\theta)e_i = e_j R(\theta)^j_i$$

The corresponding coordinate transformation is

$$x'^i = \{R(\theta)^{-1}\}^i_j x^j.$$

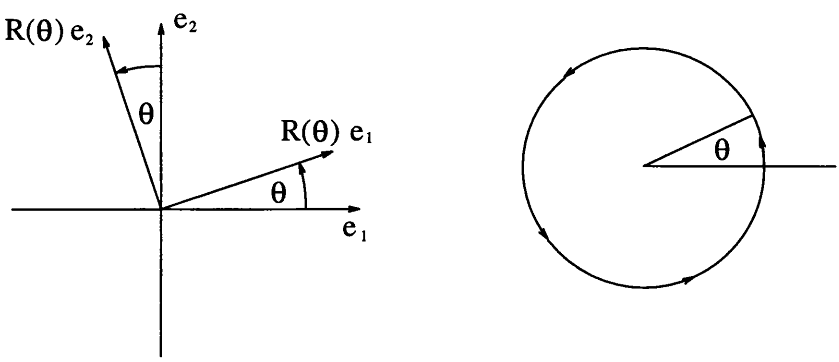

Figure 3.5: Rotation through θ and the manifold of SO(2).

where

$$R(\theta)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It is obvious that the composition law can be expressed as

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta_2)R(\theta_1) \tag{3.6}$$

The law of multiplication (composition), the existence of an identity given by R(θ=0) and the existence of a unique inverse ie $R(\theta)^{-1} = R(-\theta)$ means that the rotations in 2 dimensions form a one-parameter group, which we call SO(2). The composition law tells us that this is an abelian group. The group consists of group elements which are labelled by one continuous variable and therefore is a continuous group of transformations. It is also connected and is therefore a Lie group.

For rotations through a small angle $d\theta$ the corresponding $R(d\theta)$ is

$$\begin{aligned} R(d\theta) \approx \begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & d\theta \\ -d\theta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\theta \\ &= I + J d\theta \end{aligned}$$

The J is the infinitesimal (matrix) generator of the Lie group and the $R(d\theta)$ corresponds to a group operation near the identity. Repeated application of the infinitesimal operation should give us a finite element. Consider

$$\lim_{N\to\infty} \left(I + \frac{\theta}{N} J\right)^N = exp(\theta J) = R(\theta).$$

We can see that the local behaviour of the group determines the important properties of a connected Lie group ie the global properties.

3.8.2 The groups SU(2) and SO(3).

A commonly quoted example of two locally isomorphic Lie groups is that of SO(3) and SU(2). We shall show here how they are homomorphic and locally isomorphic and that the group SU(2) is simply connected.

A linear transformation R of the variables x^1, x^2, x^3 which leaves the form $(x^1)^2 + (x^2)^2 + (x^3)^2$ invariant, is called a three-dimensional rotation. The set of all such transformations forms a continuous group called the 3-dimensional rotation group. These transformations can be represented by the set of all real orthogonal 3-dimensional matrices. As mentioned before the determinant of every orthogonal matrix is either +1 corresponding to a pure rotation or -1 corresponding to a rotation-reflection: the set of all pure rotations forms a continuous group called SO(3). Rotations can also be specified by 2-dimensional unitary matrices with determinant +1 and this group is denoted SU(2). The Lie groups SO(3) and SU(2) are distinct groups.

Let us consider the Lie group SO(3). A rotation R is a transformation **x=Rx**. A general rotation $R(\psi_1, \psi_2, \psi_3)$ is given by the product of the 3 matrices $r_1(\psi_1)$, $r_2(\psi_2)$ and $r_3(\psi_3)$ corresponding to rotations around the Ox^1, Ox^2 and Ox^3 axes respectively. These matrices are

$$r_1(\psi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & \sin\psi_1 & \cos\psi_1 \end{pmatrix}, r_2(\psi_2) = \begin{pmatrix} \cos\psi_2 & 0 & \sin\psi_2 \\ 0 & 1 & 0 \\ -\sin\psi_2 & 0 & \cos\psi_2 \end{pmatrix},$$

$$r_3(\psi_3) = \begin{pmatrix} \cos\psi_3 & -\sin\psi_3 & 0 \\ \sin\psi_3 & \cos\psi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.7}$$

Any rotation can be completely specified by the above matrices. The infinitesimal matrices g_i corresponding to these rotations about the axes Ox^i are given by

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

Their Lie algebra is given by

$$[g_i, g_j] = \epsilon^k_{ij} g_k \tag{3.9}$$

It is straightforward to exponentiate back to regain the group elements.

Let us now consider the group SU(2). Associated with the coordinate system x^i there is a hermitian matrix $P = x^i \sigma_i$ where σ_i are the Pauli spin matrices. If this frame is rotated into another x^k with corresponding P' then the two matrices are related by $P' = UPU^\dagger$ where U is a member of the group SU(2).

The matrices that represent SU(2) are of the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \text{where} \qquad \alpha\bar{\alpha} + \beta\bar{\beta} = 1. \tag{3.10}$$

The complex numbers can be written $\alpha = a + ib$ and $\beta = c + id$ with a, b, c, d real. The matrices of SU(2) corresponding to the matrices $r_i(\psi_i)$ are given below

$$\begin{aligned}
r_1(\psi_1) &\quad\mapsto\quad \pm U_1(\psi_1) = \pm \begin{pmatrix} \cos\frac{1}{2}\psi_1 & i\sin\frac{1}{2}\psi_1 \\ i\sin\frac{1}{2}\psi_1 & \cos\frac{1}{2}\psi_1 \end{pmatrix}, \\
r_2(\psi_2) &\quad\mapsto\quad \pm U_2(\psi_2) = \pm \begin{pmatrix} \cos\frac{1}{2}\psi_2 & -\sin\frac{1}{2}\psi_2 \\ \sin\frac{1}{2}\psi_2 & \cos\frac{1}{2}\psi_2 \end{pmatrix}, \\
r_3(\psi_3) &\quad\mapsto\quad \pm U_3(\psi_3) = \pm \begin{pmatrix} e^{\frac{i}{2}\psi_3} & 0 \\ 0 & e^{-\frac{i}{2}\psi_3} \end{pmatrix}
\end{aligned} \tag{3.11}$$

There is obviously a $2 \mapsto 1$ correspondence between the group elements of SU(2) and SO(3) and we say there is a $2 \mapsto 1$ homomorphism between them.

The infinitesimal matrices corresponding to SU(2) are

$$\pm u_i = \pm(\frac{i}{2})\sigma_i \tag{3.12}$$

where again, the σ_i are the Pauli spin matrices. The u_i have the Lie algebra

$$[u_i, u_j] = \epsilon^k_{ij} u_k \tag{3.13}$$

We have established that SO(3) and SU(2) have the same Lie algebra.

Let us now consider the group manifold of SU(2). We do this by considering the parameter space of the group. If A is any matrix in SU(2) then we can express it in terms of the identity e and the Pauli spin matrices i.e.

$$A = ae + b\sigma_1 + c\sigma_2 + d\sigma_3 \qquad \text{where} \qquad a^2 + b^2 + c^2 + d^2 = 1 \tag{3.14}$$

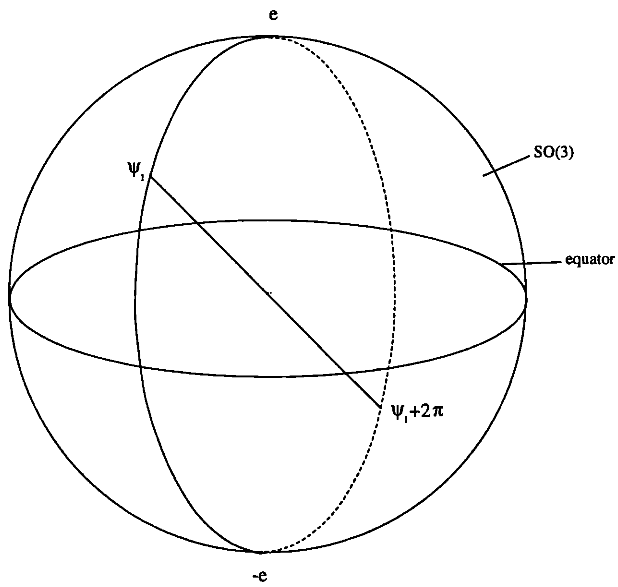

Figure 3.6: The manifold of SU(2).

This is the equation for the 3-sphere i.e. the region S^3 of E^4 and there is a one to one correspondence between the point (a, b, c, d) on S^3 and the matrix A. Hence the group manifold of SU(2) is S^3.

We have just shown that there is a $2 \mapsto 1$ homomorphism between SU(2) and SO(3) and that they have the same Lie algebras. This enables us to specify the group manifold of SO(3). The group manifold of SU(2) is a double covering of that of SO(3): see figure.

One can demonstrate this homomorphism more clearly by considering the one-parameter subgroups $r_1(\psi_1)$ of SO(3) and $U_1(\psi_1)$ of SU(2):

$$r_1(\psi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & \sin\psi_1 & \cos\psi_1 \end{pmatrix}, \qquad U_1(\psi_1) = \begin{pmatrix} \cos\frac{1}{2}\psi_1 & i\sin\frac{1}{2}\psi_1 \\ i\sin\frac{1}{2}\psi_1 & \cos\frac{1}{2}\psi_1 \end{pmatrix}$$

$U_1(\psi_1)$ begins at $\psi_1 = 0$ and ends at $\psi_1 = 4\pi$ as shown in the figure. It is also clear that the points ψ_1 and $\psi_1 + 2\pi$ are diametrically opposite on S^3. On the same manifold, $r_1(\psi_1)$ begins at $\psi_1 = 0$ and ends at $\psi_1 = 2\pi$ i.e. at diametrically opposite points on S^3. Every point on S^3 corresponds to a distinct element of the group SU(2) but every two diametrically opposite points correspond to just one element of SO(3). So we associate the group SO(3) with the top half of S^3 which is equivalent to identifying diametrically opposite points on S^3.

Since the manifold S^3 is simply connected then so too is the Lie group SU(2). However SO(3) is not simply connected. For a manifold to be simply connected we must be able to shrink any closed curve to a point. A closed curve which passes through diametrically opposite points on the equator cannot be brought to a point since they must always remain diametrically opposite.

There is a one to one correspondence between the tangent space at the identity e of SO(3) and that of SU(2) and this is why they have the same Lie algebra. Since SU(2) and SO(3) have the same Lie algebra and SU(2) is simply connected then it follows that SU(2) is the universal covering group of SO(3).

Chapter 4

Isometries of Space and Spacetime.

A spacetime consists of a 4-dimensional manifold M and a metric tensor **g**, denoted (M,**g**). The metric tensor **g** describes only the local geometry of a spacetime (M,**g**). We cannot infer the global properties of a manifold from **g** alone, however we can say that only certain global topologies are compatible with a particular **g**. In this work we only consider the transformations (isometries) of points on a manifold separated by small distances ie locally seperated: no mention is made of any transformations preserving distances on a large scale. These isometries describe exactly the local geometry ie the symmetry of the metric **g**. Infinitesimal isometries are described by Killing vector fields and correspondimg to every infinitesimal isometry is a finite isometry obtained by exponentiation which again preserves the distance between locally separated points. There may exist discrete isometries which include such things as reflections. The complete symmetry group of the manifold includes its isometries and any global symmetries ie those arising from transforming points with a non local separation.

We discuss the possible symmetry properties of a general manifold (general in the sense that it is of dimension n and has no particular signature) and will later apply this to the usual 4-dimensional spacetimes encountered in General Relativity.

4.1 Isometries of space and spacetime

Any transformation which leaves the distance between any two (locally seperated) points of the manifold invariant is called an *isometry* of that metric. This is the case if $g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta$ is invariant under a transformation x^μ to x'^μ ie

$$g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta = g_{\alpha\beta}(x'^\mu)dx'^\alpha dx'^\beta \tag{4.1}$$

Thus, these transformations are length preserving transformations on the manifold.

As an example we examine flat Euclidean space E^2 and its isometries. E^2 has the line element:

$$ds^2 = dx^2 + dy^2. \tag{4.2}$$

Under the following transformation

$$x' = x + a; \qquad y' = y + b \qquad \text{for } a,b \text{ constant} \tag{4.3}$$

then $dx' = dx$ and $dy' = dy$ and so the new line element is

$$dx'^2 + dy'^2 = dx^2 + dy^2. \tag{4.4}$$

The new line element is identical to the original (4.2), and so the metric of E^2 is invariant under translations. Secondly, consider the transformation from coordinates x, y to:

$$x' = \cos\theta x - \sin\theta y; \qquad y' = \cos\theta x + \sin\theta y \tag{4.5}$$

then $dx' = \cos\theta dx - \sin\theta dy$ and $dy' = \sin\theta dx + \cos\theta dy$ and so, in this case, the metric (4.2) is transformed to

$$dx'^2 + dy'^2 = (dx^2 + dy^2). \tag{4.6}$$

This line element has the same functional form as (4.2) and so the transformation (4.5) is an isometry of the space E^2. The transformation (4.5) corresponds to a rotation around the origin of the coordinate system.

Reflections are also isometries ie $x \mapsto -x$ and $y \mapsto -y$. It is obvious that the metric is unchanged under reflections.

Hence the metric of the space E^2 is invariant under translations, rotations and reflections: every point of E^2 is equivalent to every other point.

4.2 Killing Vector Fields

We will now give these isometries a more precise mathematical description in terms of transformation properties of the metric tensor.

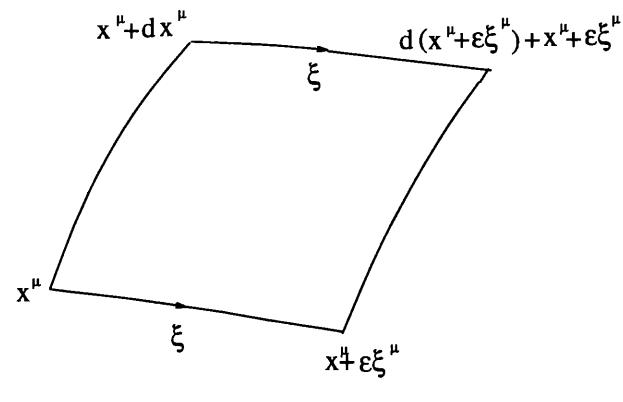

Figure 4.1: An infinitesimal transformation on the manifold.

An infinitesimal isometry is described by a vector field, a *Killing vector field* ξ in the following manner: An infinitesimal isometry from x^μ to x'^μ is described by the transformation

$$x'^\mu = x^\mu + \epsilon\xi^\mu \tag{4.7}$$

with ϵ constant and $|\epsilon| << 1$. The Killing vector ξ is the *infinitesimal generator* of the infinitesimal isometry. The requirement that $g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta = g_{\alpha\beta}(x'^\mu)dx'^\alpha dx'^\beta$ demands that the vector field ξ's covariant components ξ_α satisfy

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{4.8}$$

Such a vector field is called a *Killing vector field* of the metric. Equations (5.8) are called *Killing's equations.* If there exists a solution of Killing's equations for a given $g_{\alpha\beta}$, then the corresponding ξ^μ represents an infinitesimal isometry of the metric $g_{\alpha\beta}$ and implies that the metric has a certain symmetry. Since equation (5.8) is covariantly expressed, that is, it is a tensor equation, if the metric has an isometry in one coordinate system the transformed metric will also have a corresponding isometry (but this is obvious from the fact that the line element is invariant).

Killing's equations can be derived by considering length preserving transformations. Consider two neighbouring points on a curve $x^\mu(\lambda)$ transformed to a new curve $x^\mu(\lambda)+\epsilon\xi^\mu$ by an infinitesimal isometry, the ξ^μ being of course the components of the infinitesimal generator or Killing Vector field ξ.

In order that the transformation be length preserving it is necessary that the infinitesimal distance between any two arbitrary points remains invariant ie

$$g_{\alpha\beta}(x^{\mu})dx^{\alpha}dx^{\beta} = g_{\alpha\beta}(x'^{\mu})dx'^{\alpha}dx'^{\beta}$$

Writing this out explicitly gives

$$g_{\alpha\beta}(x^{\mu})dx^{\alpha}dx^{\beta} = g_{\alpha\beta}(x^{\mu} + \epsilon\xi^{\mu})d(x^{\alpha} + \epsilon\xi^{\alpha})d(x^{\beta} + \epsilon\xi^{\beta})$$
$$= (g_{\alpha\beta}(x^{\mu}) + \epsilon\xi^{\kappa}g_{\alpha\beta,\kappa}(x^{\mu}))(dx^{\alpha} + \epsilon dx^{\sigma}\xi^{\alpha}{}_{,\sigma})(dx^{\beta} + \epsilon dx^{\tau}\xi^{\beta}{}_{,\tau})$$

Where we have expanded $g_{\alpha\beta}(x^{\mu} + \epsilon\xi^{\mu})$ to first order in ϵ using a Taylor series for brevity now put $g_{\alpha\beta}(x^{\mu})$=$g_{\alpha\beta}$ and retaining only terms of first order in ϵ we get

$$g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{\alpha\beta}dx^{\alpha}dx^{\beta} + g_{\alpha\beta}dx^{\alpha}\epsilon\xi^{\beta}{}_{,\tau}\,dx^{\tau} + g_{\alpha\beta}dx^{\sigma}\epsilon\xi^{\alpha}{}_{,\sigma}\,dx^{\beta} + \epsilon\xi^{\kappa}g_{\alpha\beta,\kappa}dx^{\alpha}dx^{\beta}$$

That is

$$g_{\alpha\beta}dx^{\alpha}\xi^{\beta}{}_{,\tau}\,dx^{\tau} + g_{\alpha\beta}dx^{\sigma}\xi^{\alpha}{}_{,\sigma}\,dx^{\beta} + \xi^{\kappa}g_{\alpha\beta,\kappa}dx^{\alpha}dx^{\beta} = 0.$$

Now, factorising out the $dx^{\alpha}dx^{\beta}$ since the dx^{α} are arbitrary gives

$$g_{\alpha\tau}\xi^{\tau}{}_{,\beta} + g_{\sigma\beta}\xi^{\sigma}{}_{,\alpha} + \xi^{\kappa}g_{\alpha\beta,\kappa} = 0.$$

In geodesic coordinates this becomes

$$g_{\alpha\tau}\xi^{\tau}_{;\beta} + g_{\sigma\beta}\xi^{\sigma}_{;\alpha} + \xi^{\kappa}g_{\alpha\beta;\kappa} = 0$$

Since $\Gamma^{\alpha}_{\beta\gamma} = 0$ for all α, β, γ. Because of the covariant nature of this equation, it holds generally. Also since $g_{\alpha\beta;\kappa} = 0$ always then

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

which is the required result.

It is important to notice that if ξ_1 and ξ_2 are two Killing vector fields, then the linear combination $a_1\xi_1 + a_2\xi_2$ is also a Killing vector if a_1 and a_2 are constants. This is obvious from the form of the Killing equations (5.8). However, if a_1 and a_2 are functions of the coordinates on M then $a_1\xi_1 + a_2\xi_2$ is a vector field but not necessarily a Killing vector field. This suggests that the set of Killing vector fields form an algebra. This is indeed the case: the commutator $[\xi_i, \xi_j]$ is also a Killing vector field. This is obvious from the properties of the Lie derivative which we discussed at the end of chapter 2. In fact the set of isometies of a manifold form a Lie group and the Killing vectors form the corresponding Lie algebra. To which element of the Lie algebra does this commutator correspond? From section (3.6.1) we see that the commutator is equivalent to the element $C^k_{ij}\xi_k$.

4.3 Isometry Groups.

The set of isometries on a manifold M form a group: An associative product is defined, an inverse exists for each element, and an identity exists.

The group of isometries (sometimes called a *group of motions*) is the symmetry group of the manifold M and will be denoted G or G_r where r is the dimension of the group. The dimension of the group G_r is not necessarily equal to, and can be greater or less than, the dimension n of the manifold M. More precisely, the isometries form a Lie group, a Lie group of transformations. The infinitesimal isometries or Killing vectors considered above form the Lie algebra of the Lie group of transformations. The isometries of M can be obtained from the infinitesimal isometries by exponentiation in the same way that Lie group elements are obtained from the infinitesimal generators which form the Lie algebra.

In order to determine the infinitesimal isometries on a manifold M we have to solve the Killing equations for the particular metric. We have to find all linearly independent solutions of these equations and the number of such solutions is the dimension r of the group G_r. It will be useful to know the order r of the group of motions beforehand. It is not a trivial procedure to find r in general - but we can do so, and it turns out that spaces with constant curvature admit the maximum number possible for a manifold M_n and this number is n(n+1)/2. We will shortly consider (and prove) a theorem which allow us to determine the order r of a group of motions G_r. Firstly, we shall state a theorem which gives us a feeling for the notion of an isometry and the associated Killing field.

Theorem. *A necessary and sufficient condition that a manifold M admits a one-dimensional group G_1 of motions is that there exists a coordinate system for which all of the metric components do not involve one coordinate, say x^1; then the curves of parameter x^1 are the trajectories of the motion.*

Consider, for example, a manifold M whose metric components $g_{\alpha\beta}$ relative to some coordinated basis dx^α are independent of one coordinate x^k. Then:

$$\frac{\partial g_{\alpha\beta}}{\partial x^k} = 0 \tag{4.9}$$

Any curve $x^\alpha = x^\alpha(\lambda)$ in the space (M,g) can be translated in the x^k direction by the infinitesimal coordinate shift $\Delta x^k = \epsilon$ (ie x^α is transformed to $x^\alpha + \epsilon\xi^\alpha$)

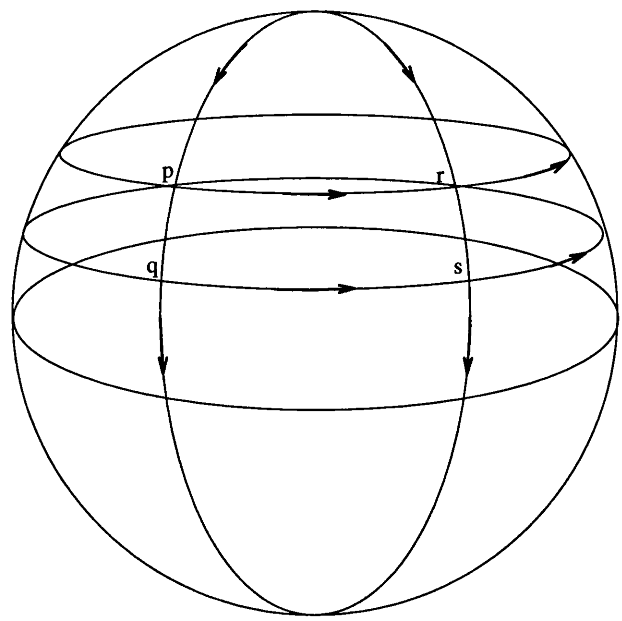

Figure 4.2: $\frac{\partial}{\partial\phi}$ is a Killing vector field on S^2.

to form a congruent curve with identical length to the original curve - because the line element is invariant under this transformation. The vector $\xi = \frac{\partial}{\partial x^k}$ is the infinitesimal generator of these length preserving translations. ξ is a Killing vector, or more precisely a Killing vector field on the manifold M.

The isometries of the two-sphere S^2 demonstrate this idea clearly. We consider S^2 embedded in flat Euclidean space E^3. E^3 has the metric $ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$. S^2 is the region r=constant and so we can choose r=1 and the metric of S^2 is $ds^2 = d\theta^2 + sin^2\theta d\phi^2$. We have coordinates θ,ϕ on the sphere and in the coordinated basis $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial\phi}$ the metric components are $g_{11} = 1$, $g_{22} = sin^2\theta$ and $g_{21} = g_{12} = 0$. It is obvious that the metric is independent of ϕ and so $\frac{\partial}{\partial\phi}$ is a Killing vector field. However, the metric depends upon θ and so $\frac{\partial}{\partial\theta}$ is not a Killing vector field. These fields are illustrated in figure 4.2.

The field $\frac{\partial}{\partial\phi}$ transports the curve pq into the curve rs and it can be seen that the infinitesimal transformation generated by $\frac{\partial}{\partial\phi}$ is length preserving and so this is indeed an isometry. The field $\frac{\partial}{\partial\theta}$ transports the curve pr into qs and this transformation is obviously not length preserving.

4.4 Minkowski Spacetime.

Let us consider the isometries of Minkowski spacetime. The line element is given by

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)$$

The Minkowski spacetime manifold is maximally symmetric i.e. admits the maximum possible number of independent Killing vectors allowed on a 4-dimensional manifold (see section 4.6), which is 10.

The 10 Killing vectors are

$$\begin{aligned}
\xi_1 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, & \eta_1 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\
\xi_2 &= t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}, & \eta_2 &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\
\xi_3 &= t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}, & \eta_3 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}
\end{aligned}$$
$$\zeta_1 = \frac{\partial}{\partial x}, \qquad \zeta_2 = \frac{\partial}{\partial y}, \qquad \zeta_3 = \frac{\partial}{\partial z}, \qquad \zeta_4 = \frac{\partial}{\partial t} \tag{4.10}$$

The Killing vectors reflect the symmetry properties of the manifold. The ζ_i correspond to translations which are obviously isometries because the spacetime is flat. The η_i have the commutation relations $[\eta_i, \eta_j] = -\epsilon_{ij}^k \eta_k$ which suggests that the corresponding subgroup is locally isomorphic to SO(3). However, it is obvious from the form of the metric that it is invariant under rotations. The ξ_i correspond to boosts of spacetime. In fact, the isometry group of this manifold is the 10-parameter Poincaré group. This group consists of all transformations of the form

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + c^{\mu} \qquad \text{which satisfy} \qquad \eta_{\mu\nu}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

where of course, $\eta_{\mu\nu}$ are the components of the Minkowski metric.

4.5 Constants of Motion

In any geometry endowed with a symmetry described by a Killing vector field ξ, motion *along any geodesic* leaves constant the scalar product of the tangent vector with the Killing vector ie

$$\mathbf{p}\cdot\xi = constant \tag{4.11}$$

It is straight forward to verify this result. We take the derivative of $p^\alpha \xi_\alpha$ along the curve and show this to be zero. For a geodesic with parameter λ, the derivative is given by

$$\frac{d}{d\lambda}(p^\alpha \xi_\alpha)$$

We evaluate this as

$$\frac{d}{d\lambda}(p^\alpha \xi_\alpha) = \frac{dp^\alpha}{d\lambda}\xi_\alpha + p^\alpha \frac{d\xi_\alpha}{d\lambda}$$

But from the geodesic equation we have

$$\frac{dp^\alpha}{d\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma$$

Hence

$$\frac{d}{d\lambda}(p^\alpha \xi_\alpha) = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \xi_\alpha + p^\alpha \frac{d\xi_\alpha}{d\lambda}$$

and

$$\frac{d\xi_\alpha}{d\lambda} = \frac{dx^\sigma}{d\lambda}\frac{\partial \xi_\alpha}{\partial x^\sigma} = p^\sigma(\xi_{\alpha,\sigma})$$

The covariant derivative of ξ_α is given by

$$\xi_{\alpha;\sigma} = \xi_{\alpha,\sigma} - \Gamma^\tau_{\alpha\sigma}\xi_\tau$$

Therefore

$$\begin{aligned}
\frac{d}{d\lambda}(p^\alpha \xi_\alpha) &= -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \xi_\alpha + p^\beta p^\gamma(\xi_{\beta;\gamma} + \Gamma^\alpha_{\beta\gamma}\xi_\alpha) \\
&= p^\beta p^\gamma(\xi_{\beta;\gamma}) \\
&= \frac{1}{2} p^\beta p^\gamma(\xi_{\beta;\gamma} + \xi_{\gamma;\beta}) \\
&= 0
\end{aligned}$$

The last step coming from Killing's equation. Hence the result

$$\mathbf{p}\cdot\xi = constant$$

4.6 The Dimension of a Group of Motions

A main property of a manifold M with a metric g is the number of isometries it admits. We now consider the conditions under which the Killing equations admit solutions. The first step is to show that if a solution exists and we are given both the values of ξ_α and its first derivative $\xi_{\alpha,\beta}$ at a point P on M, then the Killing vector field is uniquely determined at every point on the manifold M. We derive an equation which relates the second derivative of a Killing field to the Riemann tensor.

Now by the definition of the Riemann tensor, for an arbitrary vector field V^α

$$R_{\alpha\beta\gamma}{}^{\tau}V_\tau = V_{\gamma;\beta\alpha} - V_{\gamma;\alpha\beta}. \tag{4.12}$$

Taking the first covariant derivative of Killing's equations gives

$$\xi_{\gamma;\beta\alpha} = -\xi_{\beta;\gamma\alpha}. \tag{4.13}$$

Hence, for a Killing vector field

$$R_{\alpha\beta\gamma}{}^{\tau}\xi_\tau = -\xi_{\beta;\gamma\alpha} - \xi_{\gamma;\alpha\beta}.$$

Taking this equation and the equations obtained by permuting the indices α, β, γ gives

$$R_{\alpha\beta\gamma}{}^{\tau}\xi_\tau = -\xi_{\beta;\gamma\alpha} - \xi_{\gamma;\alpha\beta}. \tag{4.14}$$

$$R_{\gamma\alpha\beta}{}^{\tau}\xi_\tau = -\xi_{\alpha;\beta\gamma} - \xi_{\beta;\gamma\alpha}. \tag{4.15}$$

$$R_{\beta\gamma\alpha}{}^{\tau}\xi_\tau = -\xi_{\gamma;\alpha\beta} - \xi_{\alpha;\beta\gamma}. \tag{4.16}$$

Adding equations (4.14) and (4.16) and subtracting (4.15) gives

$$(R_{\alpha\beta\gamma}{}^{\tau} + R_{\beta\gamma\alpha}{}^{\tau} - R_{\gamma\alpha\beta}{}^{\tau})\xi_\tau = -2\xi_{\gamma;\alpha\beta}. \tag{4.17}$$

The identity

$$R_{[\alpha\beta\gamma]}{}^{\tau} = 0$$

can be written

$$R_{\alpha\beta\gamma}{}^{\tau} + R_{\beta\gamma\alpha}{}^{\tau} = -R_{\gamma\alpha\beta}{}^{\tau}$$

Accordingly

$$\begin{aligned} -2R_{\gamma\alpha\beta}{}^{\tau}\xi_\tau &= -2\xi_{\gamma;\alpha\beta} \\ \text{thus} \qquad \xi_{\gamma;\alpha\beta} &= -R_{\alpha\gamma\beta}{}^{\tau}\xi_\tau. \end{aligned} \tag{4.18}$$

From this final equation it is clear that a Killing field ξ^α is completely determined on the manifold M by the values of ξ^α and $L_{\alpha\beta} \equiv \xi_{\alpha;\beta}$ at any point p in the manifold M; namely, if we are given as our initial set of data $(\xi^\alpha, L_{\alpha\beta})$ at a point p, then $(\xi^\alpha, L_{\alpha\beta})$ at any other point q is determined by integration of the system of partial differential equations

$$L_{\alpha\beta} = \xi_{\alpha;\beta} \quad ,$$
$$\xi_{\gamma;\alpha\beta} = -R_{\alpha\gamma\beta}{}^{\tau}\xi_\tau \quad , \tag{4.19}$$

along any curve connecting p and q. It is obvious that : (1) If a Killing vector field and its derivative vanish at a point, then the Killing vector field vanishes everywhere. (2) Consider the "initial data space" $(\xi^\alpha, L_{\alpha\beta})$. The manifold M has dimension n. In order to specify a Killing field completely at every point on M, we require $(\xi^\alpha, L_{\alpha\beta})$ at a point p. There are n of ξ^α and n^2 of $L_{\alpha\beta}$ but from the definition of the latter the $L_{\alpha\beta}$ cannot be completely independently specifiable, that is, because of the antisymmetry of $L_{\alpha\beta}$ ie $L_{\alpha\beta}$=-$L_{\beta\alpha}$, there are intrinsically only $\frac{n}{2}(n-1)$ possible independent $L_{\alpha\beta}$. There are no such restrictions on the ξ^α. As stated before, the data $(\xi^\alpha, L_{\alpha\beta})$ specifies one Killing field. There may be further restrictions on the data space $(\xi^\alpha, L_{\alpha\beta})$ imposed by the integrability conditions of the set of equations (4.19) but we can consider the case where the n ξ^α and $\frac{n}{2}(n-1)$ $L_{\alpha\beta}$ can be prescribed independently of each other.

The Killing vector fields corresponding to the data sets

$$(\xi^1, 0, 0...0; 0, 0, ...0), (0, \xi^2, 0, ...0; 0, 0, ...0).....etc$$

$$(0, 0, ...0; L_{12}, 0, 0, ...0), (0, 0, ...0; 0, L_{13}, 0, ...0), ...etc$$

of which there are $n + \frac{n}{2}(n-1)$, are all linearly independent and since the ξ^α and $L_{\alpha\beta}$ are *all* independently specifiable then the above represents the maximum freedom in choice of data. This must, therefore, correspond to the maximum possible number of linearly independent Killing vector fields allowed on a manifold of dimension n. The maximum number of independent Killing fields is the dimension of the initial data space which is $n + \frac{n}{2}(n-1) = \frac{n}{2}(n+1)$. A manifold M which admits the maximum possible number of Killing vector fields ie $\frac{n}{2}(n+1)$ is refered to as *maximally symmetric* which we shall discuss shortly.

However, in general a manifold does not admit the maximum number of Killing fields, that is, ξ^α and $L_{\alpha\beta}$ cannot be prescribed independently and are connected by the integrability conditions of the set of differential equations (4.19). The number of Killing fields admitted by a particular manifold (M,g) can be determined by considering the integrability conditions of (4.19). We aim to establish a theorem concerning certain systems of partial differential equations.

Consider a system of partial differential equations:

$$\begin{aligned} \frac{\partial\theta^\alpha}{\partial x^i} &= \psi_i^\alpha(\theta^1,...\theta^m;x^1,...x^n) \\ &\equiv \psi_i^\alpha(\theta;x) \qquad \text{where} \qquad \alpha=1,...,m \qquad i=1,...,n \end{aligned} \tag{4.20}$$

These are equivalent to the system of ordinary differential equations

$$d\theta^\alpha = \psi_i^\alpha dx^i \qquad \text{where} \qquad \alpha=1,...,m \qquad i=1,...,n \tag{4.21}$$

The conditions of integrability of (4.20) are

$$\frac{\partial\psi_i^\alpha}{\partial x^j} + \frac{\partial\psi_i^\alpha}{\partial\theta^\beta}\psi_j^\beta = \frac{\partial\psi_j^\alpha}{\partial x^i} + \frac{\partial\psi_j^\alpha}{\partial\theta^\gamma}\psi_i^\gamma \tag{4.22}$$

If these equations are satisfied identically, the system (4.20) is said to be *completely integrable.* In this case there are m independent solutions expressible in the form

$$\theta^\alpha = c^\alpha + \left(\frac{\partial\theta^\alpha}{\partial x^i}\right)_0 (x^i - x_0^i) + \frac{1}{2}\left(\frac{\partial^2\theta^\alpha}{\partial x^i\partial x^j}\right)_0 (x^i - x_0^i)(x^j - x_0^j) + ... \tag{4.23}$$

If equations (4.22) are not satisfied identically, we have a set F_1 of algebraic equations which establish conditions upon the θ's as functions of the x's. If we differentiate each of these equations with respect to the x's and substitute for $\frac{\partial\theta^\alpha}{\partial x^i}$ from (4.20), we get a further set of algebraic equations and these resulting algebraic equations are either a consequence of F_1 or we get a new set F_2. Proceeding in this way we get a sequence of sets F_1, F_2,.... of algebraic equations, which *must* be compatible if (4.20) are to have a solution. If one of these sets is not a consequence of the preceeding sets, it introduces at least one additional condition. As a result, if equations (4.20) are to admit a solution, there must be a positive integer N such that the equations of the (N+1)th set are satisfied because of the equations of the preceeding N sets; otherwise we should obtain more than m independent equations which would imply a relation between the x's. It follows that $N \leq m$.

Now suppose that there is a number N such that the equations of the sets

$$F_1, ..., F_N \tag{4.24}$$

are compatible and each set introduces one or more conditions independent of the conditions imposed by the equations of the other sets, and that all the equations of the set

$$F_{N+1} \tag{4.25}$$

are satisfied identically because of the equations of the sets (4.24). Assume there are p(<m) independent conditions imposed by (4.24), say $G_\gamma(\theta, x) = 0$. Since the jacobian matrix is of rank p, these equations may be solved for p of the θ's in terms of the remaining (m-p) θ's and the x's. The equations of the set (4.24) are then of the form (by suitable numbering)

$$\theta^\sigma = \phi^\sigma(\theta^{p+1}, ..., \theta^m, x) \qquad \text{where} \qquad \sigma = 1, ..., p \tag{4.26}$$

We can now reconsider the original set of equations (4.20). Since equations (4.26) were derived from the p independent conditions then replacing them back into (4.20) will render the corresponding integrability conditions (4.22) - now in terms of the $\theta^{p+1}, ..., \theta^m$ and the x's - completely integrable. This is equivalent to saying the equations

$$\frac{\partial \theta^\nu}{\partial x^i} = \bar{\psi}_i^\nu(\theta^{p+1}, ...\theta^m; x^1, ...x^n) \qquad \text{where} \qquad \nu = p+1, ..., m \tag{4.27}$$

are completely integrable, the $\bar{\psi}_i^\nu$ being obtained from the ψ_i^ν by replacing θ^σ by their expressions (4.26). Hence we have a solution involving (m-p) constants, ie we have (m-p) independent solutions.

When p=m, we have in place of (4.26), $\theta^\alpha = \psi^\alpha(x)$. In this case there are no constants of integration.

Theorem. *In order that a system of equations*

$$\frac{\partial \theta^\alpha}{\partial x^i} = \psi_i^\alpha(\theta^1, ...\theta^m; x^1, ...x^n) \qquad \textit{where} \qquad \alpha = 1, ..., m \qquad i = 1, ..., n$$

admit a solution, it is necessary and sufficient that there exists a positive integer $N \leq m$ such that the equations of the sets F_1,...,F_N are compatible for all values of *the x's in a domain, and that the equations of the set F_{N+1} are satisfied because of the former sets; if p is the number of independent equations in the first N sets, there are m-p independent solutions.*

The above theorem also applies to the case where there are a certain number of functional relations F_0 between the θ's and the x's which must be satisfied in addition to the differential equations (4.20). We include in the set F_1 also the conditions that arise from F_0 by differentiation and substitution from (4.20).

We now return to the number of Killing fields admitted by a manifold with metric, (M,**g**), and consider the integrability conditions of the equations (4.19).

The integrability conditions of the equations

$$\xi_{\gamma;\alpha\beta} = -R_{\alpha\gamma\beta}{}^{\tau}\xi_{\tau}$$

are (see Eisenhart (1933) [5])

$$\xi_{\tau}(R^{\tau}_{\gamma\alpha\beta;\sigma} - R^{\tau}_{\sigma\alpha\beta;\gamma}) + \xi_{\tau;\sigma}R^{\tau}_{\gamma\alpha\beta} - \xi_{\tau;\gamma}R^{\tau}_{\sigma\alpha\beta} + \xi_{\tau;\beta}R^{\tau}_{\alpha\gamma\sigma} + \xi_{\alpha;\tau}R^{\tau}_{\beta\gamma\sigma} = 0 \qquad (4.28)$$

Equations (4.28) constitute the set F_1 and the Killing equations

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \qquad (4.29)$$

are the set F_0 using the above notation.

If (4.28) are a consequence of (4.29) then there are no conditions p in F_1 and the number of independent solutions is m-p = m, and is equal to the number of equations in (4.29) which is $\frac{n}{2}(n+1)$. This corresponds to the case of a manifold with maximal symmetry. It is obviuos that $m = \frac{n}{2}(n+1)$.

If (4.28) are not a consequence of (4.29) and there are p conditions in F_1 then the number of independent solutions is $r = \frac{n}{2}(n+1) - p$. This, of course, only holds if all the equations are consistent with each other. This gives us a criterion for determining the order (dimension) r of an isometry group G_r.

Let us consider the simple example of flat Euclidean space E^2. The line element can be written $ds^2 = dx^2 + dy^2$ and we already know that the manifold admits three independent Killing vectors:

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

We will investigate the integrability conditions of the corresponding Killing equations to illustrate the theory. The Killing equations are

$$\xi_{x,x} = 0, \qquad \xi_{x,y} + \xi_{y,x} = 0, \qquad \xi_{y,y} = 0.$$

We adopt the following notation:

$$\begin{aligned} \eta_1 &= \xi_x & \eta_4 &= \xi_{y,y} \\ \eta_2 &= \xi_y & \eta_5 &= \xi_{x,y} \\ \eta_3 &= \xi_{x,x} & \eta_6 &= \xi_{y,x} \end{aligned}$$

With these definitions we have

$$\eta_{1,1} = \eta_3, \qquad \eta_{2,2} = \eta_4, \qquad \eta_{1,2} = \eta_5, \qquad \eta_{2,1} = \eta_6 \tag{4.30}$$

and the Killing equations become

$$\eta_3 = 0, \qquad \eta_5 + \eta_6 = 0, \qquad \eta_4 = 0 \tag{4.31}$$

The set F_0 are the Killing equations which are

$$\eta_{1,1} = 0, \qquad \eta_{1,2} = \eta_6, \qquad \eta_{2,2} = 0 \tag{4.32}$$

Taking the partial derivatives of the Killing equations gives

$$\begin{aligned} \eta_{3,1} &= 0, & \eta_{4,2} &= 0, \\ \eta_{3,2} &= 0, & \eta_{5,1} &= 0 \\ \eta_{4,1} &= 0, & \eta_{5,2} &= 0 \\ \eta_{6,1} &= 0, & \eta_{6,2} &= 0 \end{aligned} \tag{4.33}$$

The integrability conditions of the Killing equation can be written

$$\eta_{i,jl} = \eta_{i,lj}$$

And these are the set F_1. These are automatically satisfied because of the equations (4.33) and we now only have to consider the case

$$\eta_{1,21} = \eta_{11,2}.$$

Since $\eta_{11,2} = 0$ then we have

$$\eta_{6,1} = 0,$$

which is already in the set F_1. Hence, this sets no new conditions on the η_i and p=0. The manifold thus has the maximum possible number of Killing vector fields which is 3: it has maximal symmetry.

We consider further the case of maximal symmetry. In this case the quantities ξ_α and $\xi_{\alpha;\beta}$ are independently specifiable ie both parts of equation (4.28) equal zero seperately

$$\begin{aligned} \xi_\tau(R^\tau_{\gamma\alpha\beta;\sigma} &- R^\tau_{\sigma\alpha\beta;\gamma}) = 0 \\ \text{and} \quad \xi_{\tau;\sigma}R^\tau_{\gamma\alpha\beta} &- \xi_{\tau;\gamma}R^\tau_{\sigma\alpha\beta} + \xi_{\tau;\beta}R^\tau_{\alpha\gamma\sigma} + \xi_{\alpha;\tau}R^\tau_{\beta\gamma\sigma} = 0 \end{aligned}$$

This sets the condition on the Riemann curvature tensor that

$$R_{\lambda\mu\nu\sigma} = \frac{R}{n(n-1)}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu})$$

where, of course, n is the dimension of the manifold M (see De felice and Clarke (1992) for details).

The equation characterising a space of constant curvature is

$$R_{\lambda\mu\nu\sigma} = k(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) \qquad \text{where k=constant}$$

which is exactly equivalent to saying the Ricci scalar is constant ie $R_{,\mu} = 0$. Hence M is a space of constant curvature. The next theorem follows from this.

Theorem. *When and only when M is a space of constant curvature, the Killing equations admit $\frac{n}{2}(n+1)$ independent solutions; in all other cases there are fewer solutions.*

A maximally symmetric manifold is of constant curvature and so it is possible to state immediately that the dimension of the corresponding isometry group is $\frac{n}{2}(n+1)$.

4.7 4-dimensional Spacetimes and Groups of Motions.

The manifold (space or spacetime) admits a number r of independent Killing fields. The manifold is said to be invariant under an isometry group G_r of dimension r.

So far we have talked only about the number of independent Killing fields admitted by a manifold and have briefly considered some examples and their derivation. We have not discussed their role, in detail, with regards the symmetry properties of the manifold.

It is important to realise that there are different types of isometry group. Even in the case where there are two manifolds with the same number of independent Killing fields r, the two G_r can have entirely different properties with regards symmetry of the manifold. The precise properties of the G_r (or equivalently the Killing vector fields) reflect the symmetry properties of the manifold. We are already familiar with the terms homogeneuos, isotropic, spherically symmetric and static etc. as applied to a space or spacetime but we can now define these concepts in terms of isometries. We say that a manifold must admit a certain isometry in order to have a particular symmmetry eg homogeneity, isotropy,...etc.

An isometry group may have subgroups, each of which may reflect a different symmetry property of the manifold, or the whole isometry group itself may correspond to a particular symmetry.

First, we define the transitive group (or subgroup) and the isotropy group (or subgroup).

An isometry group G_r is said to be *transitive* on a manifold M_n if for any two distinct points $p, q \in M$ there exists an isometry $g \in G$ such that g(p)=q.

It is possible that only part of a transitive isometry group is responsible for its transitivity and this component of G we call the transitive subgroup of G. One must note that the identity must be included in order for the component to form a subgroup, although, strictly speaking the identity has no transitive action. As an example the closed FRW model has an isometry group isomorphic to SO(4) with a transitive subgroup locally isomorphic to SO(3).

Consider a transitive isometry group (or subgroup). Its transitivity is reflected in the nature of the corresponding Killing vector fields: It is always possible to construct a Killing vector $a^i\xi_i$ (i.e. by linear combination of the basis Killing vector fields) which has components in all n coordinate directions. This ensures that the corresponding group element $I + a^i\xi_i$ takes every point into any other point on the manifold (see discussion of the isometries of the manifold S^2 in section 4.7.4 and chapter 5 for the Killing vectors of the FRW models).

A *simply transitive* group (or subgroup) is a transitive group which has the same dimension as the manifold M_n on which it acts ie n=r. In this case one can use the r linearly independent Killing vector fields as a basis for vector fields. A *multiply transitive* group (or subgroup) is a transitive group which has greater dimension than the manifold M_n on which it acts ie $n > r$.

Isotropic means "the same in all directions". We note that if a space or spacetime is isotropic about one point only then we normally refer to this as a spherically symmetric space or spacetime. When we mean isotropy at every point we shall say so explicitly. The isotropy group I_p of a point p is the set of all isometries which leave p fixed. It is a subgroup of the complete isometry group G_r. If G_r is simply or multiply transitive ie $n \geq r$ then all I_p are isomorphic.

4.7.1 Homogeneity.

Loosely speaking, a manifold is homogeneous if it is the same at all points. More precisely, a manifold is homogeneous if there exists a transitive group of isometries on the manifold. This is, of course, equivalent to saying there exists an isometry which maps every point on the manifold into any other point on the manifold.

The description of a spatially homogeneous 4-dimensional spacetime is as follows: we imagine our 4-dimensional spacetime as being foliated by 3-dimensional spacelike hypersurfaces Σ_t (parameterised by the time coordinate t) which fulfill the above criterion for homogeneity eg the FRW models.

This provides us with the definition:

Definition. *A spacetime is said to be spatially homogeneous if there exists a one-parameter family of spacelike hypersurfaces Σ_t foliating the spacetime such that the metric* g *has a group of isometries transitive on the Σ_t.*

There are two types: we can obviously identify two distinct classes of spatially homogeneous spacetimes, those with simply transitive isometry groups and those with multiply transitive isometry groups.

(1). The former are further classified according to the Lie algebra of the corresponding simply transitive component of the isometry group. This classification is due to Bianchi and there are nine distinct types of Lie algebra.

(2). The second type are called Kantowski-Sachs models. There are only two possible distinct cases. We shall mention these later.

Cosmological models in which the metric is the same at all points of space *and* time, homogeneous in space and time or ST homogeneous. Such a model is a 4-dimensional manifold. M_4 on which a transitive (simply or multiply) group of isometries G_r acts, ie with $r \geq 4$. This means that it is possible to define on the 4-dimensional manifold an isometry which will map every point into any other point on the manifold.

4.7.2 Other Spacetime Symmetries.

How can one describe other symmetries such as stationary, static, and spherically symmetry in terms of Killing vector fields?

A spacetime is said to be stationary if it admits a timelike Killing vector field. The metric components would then be independent of the time coordinate t and the spacetime has "time translational symmmetry". A spacetime is said to be static if it is stationary and the metric is invariant under the transformation $t \mapsto -t$. Such a metric has no $dtdx^i$ terms, ie all the components g_{0i} are zero. The static nature of such a spacetime requires the existence of a spacelike hypersurface orthogonal to the orbits of the "time translational" isometry. Time reflection isometries do not form a continuous group but from a discrete group and there are of course no corresponding infinitesimal isometries.

Spherical symmetry (isotropy) of a spacetime is granted through the existence of an isotropy group isomorphic to SO(3), the group of rotations in 3-dimensions. If a spacetime admits such an isotropy group at every point then we say it is isotropic at all points and this demands that the spacetime be spatially homogeneous. Also, if we have a spacetime which is spatially homogeneous and there exists an isotropy group at one point then it follows that the spacetime is isotropic about every point - we make use of this in chapter 5 when we consider the Killing vectors of the FRW models.

4.7.3 Classification and Generation of Solutions.

Assuming knowledge of the metric tensor we can determine all the independent Killing vector fields on a manifold. The commutators of these infinitesimal group elements define the structure constants of the isometry group. Thus the ξ are the basis elements of the Lie algebra (we can of course choose a different basis). Any set of C^k_{ij} which satisfy the equations

$$C^k_{ij} = -C^k_{ji}$$

and the Jacobi identity

$$C^l_{ij}C^m_{lk} + C^l_{jk}C^m_{li} + C^l_{ki}C^m_{lj} = 0.$$

are the structure constants for some Lie algebra. In order to find all Lie algebras of dimension n it is necessary to find all possible independent solutions of the above two equations. Each of the independent solutions can be expressed in a canonical form and we can always change the C^k_{ij} to a new set $C^{k\prime}_{ij}$ which matches one of the canonical forms.

It seems obvious to classify our solutions (metric tensors) according to their isometry groups. This does indeed make sense since it turns out that it is possible in some cases to generate solutions by assuming that they have a particular isometry group.

Solutions of Killings equations for a particular metric gives us the infinitesimal isometries, Lie algebra and Lie group corresponding to that metric. Conversely, it should be possible to take a set of Killing vectors and generate a metric - the reverse process of solving for the Killing vectors. We shall only consider the case of spatially homogeneous and ST homogeneous spacetimes in this work. We shall deal with classification and generation of spatially homogeneous and ST homogeneous solutions together. We have already established that every Lie group has an associated Lie algebra and we are interested in three cases:

(1). ST homogeneous models. There are eight 4-dimensional Lie algebras which are tabulated in Petrov (1969) [16]. An example of such a model is the Einstein static universe.

(2). Spatially homogeneous models. There are nine 3-dimensional Lie algebras. A list is given in Taub (1951) (and is included here as an appendix) and includes, for each type, an invariant basis (see later). The numbering system is due to Bianchi: if a space is spatially homogeneous and the Lie algebra of the simply transitive isometry subgroup is Bianchi type N (N=I, II,....,IX) then the model is said to be of Bianchi type N. The association of a spacetime with a Bianchi type is, therefore, with reference to the transitive subgroup only. It does not give any information about any other symmetry properties eg isotropy.

The FRW models are spatially homogeneous *and* isotropic:
For the closed (k=+1) case the complete Lie algebra is isomorphic to that of SO(4). The isometry group is SO(4) which is transitive on the 3-dimensional hypersurfaces (the transitive subgroup is SO(3) - corresponding to Bianchi type IX) and this isometry group has an isotropy subgroup SO(3).
The isometry group for the flat (k=0) model is the product of translations in 3 dimensions (the transitive subgroup corresponding to Bianchi type I) and rotations in 3 dimensions (the isotropy group SO(3)).
For the open case (k=-1) the isometry group is that of the Lorentz group SO(3,1). The transitive subgroup corresponds to Bianchi type V. Again, the isotropy subgroup is SO(3).
The Kasner solution is another spatially homogeneous cosmological model which is Bianchi type I. It is anisotropic since the spatial sections have different expansion rates in different directions. For this model

$$ds^2 = -dt^2 + t^{2p_1}(dx^1)^2 + t^{2p_2}(dx^2)^2 + t^{2p_3}(dx^3)^2$$

where $p_1 + p_2 + p_3 = {p_1}^2 + {p_2}^2 + {p_3}^2 = 1$. The conditions on the p_i make the model necessarily anisotropic, however axial symmetry is possible with $p_1 = -\frac{1}{3}$ and $p_2 = p_3 = \frac{2}{3}$.
(3). There are only two possible distinct Kantowski-Sachs models. One possible manifold structure is $S^2 \times R$ and the corresponding isometry group is $SO(3) \times R$ which is 4-dimensional and so has a 4-dimensional Lie algebra: the isometry group is thus multiply transitive. See Kantowski and Sachs (1966) [12] for the Lie algebras.

Our aim now, is to construct homogeneous (ST and spatially homogeneous) cosmological models ie generate their solutions. We can do this for the cases (1) and (2) outlined above (for case (3) we require spatial techniques which we will not discuss). We can treat these two cases in a straightforward manner by constructing homogeneous spaces from knowledge of a Lie algebra: it relies on the fact that the Killing vectors are transitive on the spaces.

Consider a manifold of dimension n with a transitive isometry group G (of dimension n). Let us begin by introducing the *invariant basis*. An invariant basis $\{X_\mu\}$ is one for which

$$\pounds_{\xi_i} X_\mu = 0. \tag{4.34}$$

for $\mu = 1, ..n$ and $i = 1, ..n$. ξ_i are of course the basis Killing vector fields on M.

An invariant basis is useful because the metric components are constant when it is written in terms of this basis. The metric components can be written as $\mathbf{g}(X_\mu, X_\nu)$. Consider the Killing vector ξ, then

$$\pounds_\xi(\mathbf{g}(X_\mu, X_\nu)) = (\pounds_\xi \mathbf{g})(X_\mu, X_\nu) + \mathbf{g}((\pounds_\xi X_\mu), X_\nu) + \mathbf{g}(X_\mu, (\pounds_\xi X_\nu)).$$

All the terms on the right-hand side vanish. Hence, the $g_{\mu\nu}$ are constant. Also, the structure coefficients $D^\lambda_{\mu\nu}$, defined by $[X_\mu, X_\nu] = D^\lambda_{\mu\nu} X_\lambda$ are constant on the homogeneous manifold. An invariant basis can always be found for a manifold M when we have a set of ξ_i simply transitive on M. Each X_μ is a Lie dragged vector field and if we have ξ_i then we can solve equation (4.34) for the basis $\{X_\mu\}$. Once we have this basis it is straightforward to write down the metric.

It is worthwhile noting that $D^\lambda_{\mu\nu} = -C^\lambda_{\mu\nu}$ on M and that $\pounds_{\xi_i} X_\mu = 0$ implies $\pounds_{\xi_i} \omega^\mu = 0$. Cartan's first equation then becomes

$$d\omega^\mu = \frac{1}{2} C^\mu_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \tag{4.35}$$

It is equally valid to solve this equation for an invariant dual basis $\{\omega^\mu\}$ and construct the corresonding metric.

Case (1). For ST homogeneous models the metric takes the form

$$\mathbf{g} = a_{\mu\nu} \omega^\mu \otimes \omega^\nu$$

where $a_{\mu\nu}$ is any symmetric matrix of constants and $\{\omega^\mu\}$ is the invariant dual basis.

Case (2). For spatially homogeneous models the spatial part of the metric takes the form

$$\mathbf{g}_{(s)} = a_{ij} \omega^i \otimes \omega^j$$

where a_{ij} is any symmetric matrix of constants and $\{\omega^i\}$ is the invariant dual basis. Then the complete spacetime metric is

$$\mathbf{g} = -dt^2 + a_{ij}\omega^i \otimes \omega^j$$

For the ST homogeneous models we could choose the $C^{\lambda}_{\mu\nu}$ from a list of canonical structure constants as listed in Petrov (1969) [16]. Similarly for spatially homogeneous models the C^k_{ij} can be chosen from a list ie one of the Bianchi types. It is possible to add further constraints on the metric eg that of isotropy.

Note that the Lie derivative of the metric with respect to the corresonding Killing vectors will indeed be zero since the metric components are *constant* with respect to an *invariant* basis.

4.7.4 The 2-Sphere: Homogeneous and Isotropic.

Let us now consider the two-sphere S^2 and its isometries. We shall demonstrate how these isometries make the sphere homogeneous and isotropic. Now, the two-sphere S^2 can be embedded in R^3, S^2 being the region $x^2 + y^2 + z^2 = r^2$ where r is a constant. The rotations in 3-dimensions generate the 2-dimensional surface of S^2. Since the two-sphere S^2 is of maximal symmetry and of dimension 2 then the number of independent Killing fields on S^2 is $2(2+1)/2 = 3$. These Killing fields are

$$\eta_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$
$$\eta_2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$
$$\eta_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

The homogeneity of this manifold is granted through its invariance under SO(3), which acts transitively on S^2: At every point on S^2 it is possible to construct a Killing vector which takes that point into any other point. Each single element η_i cannot be termed transitive on its own, but the existence of the others ensures that together there is no point which doesn't have an isometry defined which can take one point into any other.

The isotropy subgroup of a point p on M consists of all elements of the isometry group which leave the point p invariant. The isotropy is granted through invariance under the one-parameter subgroup of SO(3), SO(2). That is, at any point we can rotate the sphere about its axis, and this obviously corresponds to an isometry. Consider the points in R^3 on the z-axis $(0,0,\pm r)$. Isotropy about these points is granted by the Killing field

$$\eta_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

This field has value zero at the point $(0,0,\pm r)$ and rotates the rest of the manifold into itself. Similarly for $(\pm r,0,0)$: η_2 and $(0,\pm r,0)$: η_3. In fact it is obvious that we can do this for any point on S^2 simply by defining an appropriate coordinate system. The isotropy subgroups so obtained will of course just be isomorphic to SO(2).

4.8 Conformal Killing Vectors and Killing Tensors.

An isometry ϕ is a mapping (diffeomorphism) $\phi : M \mapsto M$ and $\phi^*\mathbf{g} \mapsto \mathbf{g}$ and is described by a Killing vector ξ (the infinitesimal generators) such that

$$\pounds_\xi \mathbf{g} = 0.$$

The motion preserves lengths and hence angles between vectors defined on M.

The above can be considered as a special case when considering *conformal isometries.* A conformal isometry ϕ is a mapping (diffeomorphism) $\phi : M \mapsto M$ and

$$\phi^*\mathbf{g} \mapsto \Omega^2\mathbf{g}$$

The infinitesimal generators of the corresponding transformations are conformal Killing vectors (CKV) Θ such that

$$\pounds_\Theta \mathbf{g} = \psi(x^k)\mathbf{g}.$$

$\psi(x^k)$ is called the conformal factor. The conformal motion preserves angles between vectors defined on M. There are in fact three special cases:

(1). Killing vector. $\psi = 0$.

(2). homothetic Killing vector. $\psi \neq 0$, $\psi_{,\alpha} = 0$.

(3). special conformal Killing vector. $\psi_{;\alpha\beta} = 0$.

The set of all conformal isometries on M have the structure of a Lie group, in precisely the same manner as for isometries. Hence, the set of all CKV for a particular manifold forms a Lie algebra (It should be noted that the more general conformal transformation is not necessarily associated with a diffeomorphism of M). The conformal Killing equation can be written in component form as

$$\Theta_{\alpha;\beta} + \Theta_{\beta;\alpha} = \frac{2}{n}\Theta^{\gamma}{}_{;\gamma}g_{\alpha\beta} \tag{4.36}$$

where n is the dimension of the manifold M. The scalar product of the conformal Killing vector Θ and the tangent vector field p can be written in component form as

$$p^{\beta}(\Theta_{\alpha}p^{\alpha})_{;\beta} = \frac{1}{n}(\Theta^{\beta}{}_{;\beta})p^{\alpha}p_{\alpha} \tag{4.37}$$

Conformal Killing vectors give rise to constants of motion for null geodesics (for which $p^{\alpha}p_{\alpha} = 0$).

A Killing Tensor is a totally symmetric tensor K which satifies

$$K_{(\alpha_1.....\alpha_m;\alpha_{m+1})} = 0 \tag{4.38}$$

where parentheses indicate total symmetrization. Killing tensor fields do not arise in any natural way from groups of diffeomorphisms on M. The generators are not spacetime vector fields but rather depend on the geodesic tangent vector and lie in the "jet space" of geodesic equations. It is found that the metric is not invariant under Killing tensor symmetries.

However, like Killing vectors, Killing tensors give rise to constants of motion:

$$K_{\alpha_1.....\alpha_m}p^{\alpha_1}.....p^{\alpha_m} = constant. \tag{4.39}$$

These Killing tensor constants of motion correspond to symmetries of the geodesic equations.

The conserved quantities associated with geodesics and each of the Killing vectors, Killing tensors and Conformal Killing vectors do in some cases allow expressions to be obtained for the geodesics, somewhat reducing for example,the problem of solving the geodesic equation. With all but the simplest metrics it would involve a fair amount of labour to directly solve the geodesic equation. As an example, with the Schwarzchild solution the constants of motion associated with the Killing vectors reduce the problem of finding the geodesics to the problem of one-dimensional motion of a particle in a potential. The gravitational redshift can also be obtained by comparing the frequency of emitted and observed photons using conserved quantities.

The Kerr metric admits a 2nd rank Killing tensor $K_{\alpha\beta}$ and the associated constants of motion allows us to obtain *all* the geodesics explicitly.

In the next chapter an expression for the redshift in a FRW model is obtained using Killing vectors explicitly.

Chapter 5

Killing vectors for the FRW models.

In this chapter we find the Killing vectors for each type of FRW geometry and show that there are 3 distinct Lie algebras (each corresponding to k=+1, 0 or -1). The closed model is Bianchi type IX and the complete isometry group is locally isomorphic to SO(4), the flat model is Bianchi type I and has a complete isometry group given by the product of translations T_3 and rotations SO(3) and the open model is Bianchi type V with isometry group locally isomorphic to the (proper,orthochronus) Lorentz group SO(3,1). We then show how each corresponding distinct isometry group is transitive and has an isotropy subgroup. Finally, a redshift calculation is carried out using the conserved quantities arising from Killing vectors.

The assumed geometries of the FRW models allow us to assign a metric to each type. The FRW metric can be written in the form:

$$ds^2 = dt^2 - S^2(t)\Big[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]. \tag{5.1}$$

We shall refer to this as being the coordinate system (t, r, θ, ϕ). It is worthwhile noting that since the FRW models have expanding spatial sections, the vector field $\frac{\partial}{\partial t}$ is not a Killing vector field. This is obvious from the metric and the presence of the scale factor S(t).

Through a transformation to pseudo-cartesian coordinates we have the isotropic form of the metric

$$ds^2 = dt^2 - \frac{S^2(t)}{\left(1+\dfrac{k\bar{r}^2}{4}\right)^2}(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) \tag{5.2}$$

The transformation is

$$\begin{aligned}
\bar{x} &= \bar{r}\sin\theta\cos\phi \\
\bar{y} &= \bar{r}\sin\theta\sin\phi \\
\bar{z} &= \bar{r}\cos\theta \\
\bar{r} &= \frac{2r}{1+(1-kr^2)^{\frac{1}{2}}}
\end{aligned} \tag{5.3}$$

and we shall refer to this coordinate system as $(t, \bar{x}, \bar{y}, \bar{z})$.

Before proceeding with our analysis we note that in this form it is obvious that the spatial part of the metric is invariant under rotations i.e. interchanging $\bar{x}$, $\bar{y}$, and $\bar{z}$ leaves the metric invariant. It follows that the following vector fields are Killing vector fields for all three values of k.

$$\bar{x}\frac{\partial}{\partial \bar{y}} - \bar{y}\frac{\partial}{\partial \bar{x}} \tag{5.4}$$

$$\bar{y}\frac{\partial}{\partial \bar{z}} - \bar{z}\frac{\partial}{\partial \bar{y}} \tag{5.5}$$

$$\bar{z}\frac{\partial}{\partial \bar{x}} - \bar{x}\frac{\partial}{\partial \bar{z}} \tag{5.6}$$

As we already know, these are the infinitesimal generators of SO(3), the group of rotations in 3 dimensions. These Killing vector fields reflect the isotropy of the 3-dimensional spatial sections and will form the isotropy subgroup of the whole isometry group for each of the three FRW models. In fact these Killing fields are the generators of the isotropy group of the point $\bar{x} = \bar{y} = \bar{z} = 0$ on the 3-dimensional spatial sections. Since the FRW models are also homogeneous there is a similar group at every point of the spatial sections which is isomorphic to SO(3). Hence it is possible to say that the isotropy subgroup of the isometry group for each FRW model is isomorphic to SO(3).

Also, except in the case of the flat model, the Killing vector fields corresponding to translations

$$\frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{z}} \tag{5.7}$$

are not admitted. As we will discover, the open and closed models have transitive Killing fields acting on the spatial sections which are not just simple translations. We shall make further use of the isotropic nature of this coordinate system later.

This is all we can say about the Killing fields of the FRW models without actually solving the Killing equations explicitly. Our objective now is to find all independent solutions to Killing's equation.

5.1 Solving Killing's Equation.

In order to determine the Killing vectors of a particular metric for a spacetime (M,**g**), it is necessary to find all linearly independent solutions of the Killing equations:

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{5.8}$$

We can state beforehand the number of isometries the metric **g** will admit by considering the integrability conditions of (5.8), as discussed earlier. Equation (5.8) can be written as:

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} - 2\Gamma^{\gamma}_{\alpha\beta}\xi_{\gamma} = 0. \tag{5.9}$$

Obviously we choose the coordinate system most sympathetic to the symmetries of the space, which will still allow us to solve for all the Killing vectors.

For the moment we choose the form (5.1) because of its polar form and its generality in curvature (k= +1,0,-1). Consider a general vector field on the manifold with contravariant components (A,B,C,D) with respect to the coordinated basis $\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$, where A,B,C and D are functions of t,r,θ, ϕ. It is our objective to find the conditions on these components of this vector for it to be a Killing vector. The components of the metric are:

$$g_{00} = 1; \qquad g_{11} = -\frac{S^2}{(1-kr^2)}; \qquad g_{22} = -S^2r^2; \qquad g_{33} = -S^2r^2\sin^2\theta \tag{5.10}$$

The covariant components of the vector field are thus given by

$$\xi_0 = A; \qquad \xi_1 = -B\frac{S^2}{(1-kr^2)}; \qquad \xi_2 = -CS^2r^2; \qquad \xi_3 = -DS^2r^2\sin^2\theta \tag{5.11}$$

The Christoffel symbols are computed from the metric components (5.10). We recall that

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}g^{\gamma\sigma}(g_{\sigma\alpha;\beta} + g_{\sigma\beta;\alpha} - g_{\alpha\beta;\sigma})$$

The ten partial differential equations are as follows:

$$\begin{aligned}
\frac{\partial A}{\partial t} &= 0\\
-\frac{\partial A}{\partial r}+\frac{S^2}{(1-kr^2)}\frac{\partial B}{\partial t} &= 0\\
\frac{\partial A}{\partial \theta}+\frac{\partial C}{\partial t}S^2r^2 &= 0\\
-\frac{\partial A}{\partial \phi}+\frac{\partial D}{\partial t}S^2r^2\sin^2\theta &= 0\\
\frac{Bkr}{(1-kr^2)}+\frac{\partial B}{\partial r}+\frac{A\dot{S}}{S} &= 0\\
\frac{\partial C}{\partial r}r^2+\frac{\partial B}{\partial \theta}\frac{1}{(1-kr^2)} &= 0\\
\frac{\partial D}{\partial r}r^2\sin^2\theta+\frac{\partial B}{\partial \phi}\frac{1}{(1-kr^2)} &= 0\\
\frac{\partial C}{\partial \theta}+\frac{A\dot{S}}{S}+\frac{B}{r} &= 0\\
\frac{\partial C}{\partial \phi}+\frac{\partial D}{\partial \theta}\sin^2\theta &= 0\\
\frac{\partial D}{\partial \phi}\sin^2\theta+\frac{A\dot{S}}{S}\sin^2\theta+\frac{B}{r}\sin^2\theta+C\sin\theta\cos\theta &= 0
\end{aligned} \tag{5.12}$$

where $\dot{S}$ denotes differentiation with respect to t, ie $\frac{dS}{dt}$. It would be impossible to find a completely general solution for these partial differential equations, however it was possible to find some particular solutions by initially assuming a particular form for the solution. Solutions of the form (A,B,0,0) could not be found. There were solutions of the form (0,0,C,D), which were the following:

$$(0,0,-\sin\phi,-\cot\theta\cos\phi) \tag{5.13}$$

$$(0,0,\cos\phi,-\cot\theta\sin\phi) \tag{5.14}$$

$$(0,0,0,1) \tag{5.15}$$

Note that these are all independent of k, and so are common to all the spaces, whether k equals +1, 0 or -1 . Also, we found a solution of the form (0,A,B,0) whose particular form is dependent on k, ie

$$(0,\cos\theta(1-kr^2)^{\frac{1}{2}},-\frac{\sin\theta(1-kr^2)^{\frac{1}{2}}}{r},0) \tag{5.16}$$

The Killing vector (5.15) is the vector field $\frac{\partial}{\partial\phi}$ and corresponds to a rotation around the z-axis on the manifold, by definition of the coordinate system. The metric (5.1) is independent of ϕ and so this particular solution was obvious from the beginning anyway. We also note that if we consider the flat case, k=0, then $\bar{r} = r$ and (5.16) becomes

$$(0, \cos\theta, \frac{-\sin\theta}{r}, 0)$$

which when transformed into cartesians is equivalent to $\frac{\partial}{\partial z}$. This suggests that (5.16) might be a generalisation of translations in the flat case, ie that it may form part of the transitive subgroup of the isometry group for each FRW model.

At this stage we are unable to say much more about the other solutions (5.13), (5.14) and (5.16) given above. However we can analyse these Killing vectors in the pseudo-cartesian coordinate system $(t, \bar{x}, \bar{y}, \bar{z})$. T We racall that the new coordinates are related to the old by:

$$\begin{aligned} \bar{x} &= \bar{r}\sin\theta\cos\phi \\ \bar{y} &= \bar{r}\sin\theta\sin\phi \\ \bar{z} &= \bar{r}\cos\theta \\ \text{where} \qquad \bar{r} &= \frac{2r}{1+(1-kr^2)^{\frac{1}{2}}} \end{aligned} \tag{5.17}$$

First, make the transformation from (t, r, θ, ϕ) to $(t, \bar{r}, \theta, \phi)$ and then from $(t, \bar{r}, \theta, \phi)$ to $(t, \bar{x}, \bar{y}, \bar{z})$. The components of the vectors are transformed in the usual way

$$\bar{\xi}^\alpha = \frac{\partial\bar{x}^\alpha}{\partial x^\beta}\xi^\beta.$$

The three Killing vectors (5.13),(5.14) and (5.15) become, with respect to the coordinated basis $\frac{\partial}{\partial t}, \frac{\partial}{\partial\bar{x}}, \frac{\partial}{\partial\bar{y}}, \frac{\partial}{\partial\bar{z}}$, respectively:

$$\begin{gathered} (0, 0, -\bar{z}, \bar{y}) \\ (0, \bar{z}, 0, -\bar{x}) \\ (0, -\bar{y}, \bar{x}, 0) \end{gathered} \tag{5.18}$$

Writing this explicitly in terms of the basis vectors as

$$-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi} \qquad \equiv \qquad \bar{y}\frac{\partial}{\partial\bar{z}} - \bar{z}\frac{\partial}{\partial\bar{y}}$$

$$\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \equiv \bar{z}\frac{\partial}{\partial\bar{x}} - \bar{x}\frac{\partial}{\partial\bar{z}} \tag{5.19}$$

$$\frac{\partial}{\partial\phi} \equiv \bar{x}\frac{\partial}{\partial\bar{y}} - \bar{y}\frac{\partial}{\partial\bar{x}}$$

we see that these are simply the rotations around the $\bar{x}$,$\bar{y}$ and $\bar{z}$ axes respectively, which we encountered earlier. These are common to the open, closed and flat FRW models. As we already know, they will form the isotropy subgroup of the whole isometry group for each of the three FRW models. We label these three Killing vectors as

$$\eta_1 = \bar{y}\frac{\partial}{\partial\bar{z}} - \bar{z}\frac{\partial}{\partial\bar{y}} \tag{5.20}$$

$$\eta_2 = \bar{z}\frac{\partial}{\partial\bar{x}} - \bar{x}\frac{\partial}{\partial\bar{z}} \tag{5.21}$$

$$\eta_3 = \bar{x}\frac{\partial}{\partial\bar{y}} - \bar{y}\frac{\partial}{\partial\bar{x}} \tag{5.22}$$

Next we come to the Killing vector (5.16)

$$\left(\cos\theta(1-kr^2)^{\frac{1}{2}}\right)\frac{\partial}{\partial r} - \left(\frac{\sin\theta(1-kr^2)^{\frac{1}{2}}}{r}\right)\frac{\partial}{\partial\theta} \tag{5.23}$$

which we already suspect to be one of the transitive Killing fields. Again make the transformation from (t, r, θ, ϕ) to $(t, \bar{r}, \theta, \phi)$ and then from $(t, \bar{r}, \theta, \phi)$ to $(t, \bar{x}, \bar{y}, \bar{z})$, all the time using a coordinated basis. We find that in the pseudo-cartesian coordinate system (5.23) becomes:

$$\left(\frac{k\bar{x}\bar{z}}{2}\right)\frac{\partial}{\partial\bar{x}} + \left(\frac{k\bar{y}\bar{z}}{2}\right)\frac{\partial}{\partial\bar{y}} + \left(\frac{k\bar{z}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial\bar{z}}. \tag{5.24}$$

Because of the isotropic nature of this coordinate system we can simply permute $\bar{x}$, $\bar{y}$, and $\bar{z}$ to get another two Killing vectors. They are

$$\left(\frac{k\bar{x}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial\bar{x}} + \left(\frac{k\bar{y}\bar{x}}{2}\right)\frac{\partial}{\partial\bar{y}} + \left(\frac{k\bar{z}\bar{x}}{2}\right)\frac{\partial}{\partial\bar{z}}. \tag{5.25}$$

and

$$\left(\frac{k\bar{x}\bar{y}}{2}\right)\frac{\partial}{\partial\bar{x}} + \left(\frac{k\bar{y}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial\bar{y}} + \left(\frac{k\bar{z}\bar{y}}{2}\right)\frac{\partial}{\partial\bar{z}}. \tag{5.26}$$

It would be invalid to ask at this point if these three Killing vector fields are the infinitesimal generators of the transitive subgroup of the isometry group since we do not know whether they form a subalgebra at all.

We have already derived the infinitesimal generators of the isotropy subgroup for each of the FRW models. It was sufficient to find the commutation relations between these Killing vectors (and determine the structure constants for this subgroup) to decide that it was locally isomorphic to SO(3) (it was only the fact that the generators are exactly equal to those of SO(3) that meant the actual corresponding subgroup was SO(3)). We similarly determine the commutation relations (lie algebra) of the set of Killing vectors (5.24), (5.25) and (5.26). We label them as follows

$$\xi_1 = \left(\frac{k\bar{x}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{k\bar{y}\bar{x}}{2}\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{k\bar{z}\bar{x}}{2}\right)\frac{\partial}{\partial \bar{z}}. \tag{5.27}$$

$$\xi_2 = \left(\frac{k\bar{x}\bar{y}}{2}\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{k\bar{y}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{k\bar{z}\bar{y}}{2}\right)\frac{\partial}{\partial \bar{z}}. \tag{5.28}$$

$$\xi_3 = \left(\frac{k\bar{x}\bar{z}}{2}\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{k\bar{y}\bar{z}}{2}\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{k\bar{z}^2}{2} - \frac{k\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{z}}. \tag{5.29}$$

We recall that the Lie derivative of a vector field B with respect to the vector field A, $L_A B$ is equal to their commutator $[A, B]$. In component form then

$$[A, B]^i = (L_A B)^i = A^j \frac{\partial B^i}{\partial x^j} - B^j \frac{\partial A^i}{\partial x^j}.$$

This enables us to calculate the commutation relations between all our derived Killing vectors:

$$[\eta_i, \eta_j] = -\epsilon^k_{ij}\eta_k \tag{5.30}$$

$$[\xi_i, \xi_j] = -k\epsilon^k_{ij}\eta_k \tag{5.31}$$

$$[\xi_i, \eta_j] = -\epsilon^k_{ij}\xi_k \tag{5.32}$$

where ϵ^k_{ij} is the alternating symbol in 3 dimensions. It is now obvious that our three "suspected" transitive Killing vectors ξ_i do not even form a subalgebra of the Lie algebra of the isometry group. In order for them to do so their commutation relations would have to have the form $[\xi_i, \xi_j] = C^k_{ij}\xi_k$ for some constants C^k_{ij}. The η_i do, as noted before, form a subalgebra of the complete algebra and hence are the infinitesimal generators of a subgroup of the complete isometry group. It is possible to manipulate the Lie algebra obtained above to put it in a form where we do have a transitive subalgebra (and still retain the isotropy subgroup). We would like to find linear combinations of the ξ_i and η_i so that these do form a subalgebra of the complete algebra. It is important to note that the Lie algebra obtained in this way will still have the same canonical form as the original ie they are isomorphic to each other. We will have simply changed the basis to another equally admissible basis.

The desired form of the Lie algebra is obtained as follows. The linear combination

$$\xi'_i = \xi_i + a_i^m \eta_m \tag{5.33}$$

is required to satisfy the subalgebra commutation relations

$$[\xi'_i, \xi'_j] = C_{ij}^k \xi'_k. \tag{5.34}$$

Now we have

$$[(\xi_i + a_i^m \eta_m), (\xi_j + a_j^n \eta_n)] = C_{ij}^k (\xi_k + a_k^q \eta_q). \tag{5.35}$$

It is now clear that we can solve for the matrices a_i^m using the established relations (5.30), (5.31) and (5.32), which will give us the desired Lie subalgebra for each value of k, whilst still retaining the subalgebra corresponding to the isotropy subgroup. Expanding the left-hand side gives

$$\begin{aligned}
[(\xi_i + a_i^m \eta_m) \quad &, \quad (\xi_j + a_j^n \eta_n)] \\
&= [\xi_i, \xi_j] + a_j^n [\xi_i, \eta_n] + a_i^m [\eta_m, \xi_j] + a_i^m a_j^n [\eta_m, \eta_n] \\
&= -k\epsilon_{ij}^m \eta_m + a_j^n (-\epsilon_{in}^m \xi_m) + a_i^m (-\epsilon_{mj}^\tau \xi_\tau) + a_i^m a_j^n (-\epsilon_{mn}^\sigma \eta_\sigma) \\
&= -k\epsilon_{ij}^m \eta_m - a_j^n \epsilon_{in}^m (\xi'_m - a_m^\sigma \eta_\sigma) - a_i^m \epsilon_{mj}^\tau (\xi'_\tau - a_\tau^\sigma \eta_\sigma) - a_i^m a_j^n \epsilon_{mn}^\sigma \eta_\sigma \\
&= (-k\epsilon_{ij}^k + a_j^n a_\sigma^k \epsilon_{in}^\sigma + a_i^\sigma a_\tau^k \epsilon_{\sigma j}^\tau - a_i^\sigma a_j^n \epsilon_{\sigma n}^k) \eta_k - (a_j^n \epsilon_{in}^k + a_i^\tau \epsilon_{\tau j}^k) \xi'_k
\end{aligned}$$

This must equal $C_{ij}^k \xi'_k$. We require therefore

$$\begin{aligned}
(-k\epsilon_{ij}^k + a_j^n a_\sigma^k \epsilon_{in}^\sigma + a_i^\sigma a_\tau^k \epsilon_{\sigma j}^\tau - a_i^\sigma a_j^n \epsilon_{\sigma n}^k) = 0 \\
\text{and} \qquad C_{ij}^k = -(a_j^n \epsilon_{in}^k + a_i^\tau \epsilon_{\tau j}^k)
\end{aligned} \tag{5.36}$$

The set of equations (5.36) must be examined separately according to each value of k. For each k, we have found suitable matrices a_j^i and the corresponding structure constants C_{ij}^k.

k=+1.

For the case k =+1, we find that $a^i_j = -\delta^i_j$ and $C^k_{ij} = 2\epsilon^k_{ij}$. The structure constants can be reduced further by taking $\xi''_i = \frac{1}{2}\xi'_i$ so that

$$[\xi''_i, \xi''_j] = \epsilon^k_{ij}\xi''_k. \tag{5.37}$$

The newly computed elements of the desired Lie algebra are thus given by

$$\begin{aligned} \xi'_i &= \xi_i - \delta^j_i \eta_j \\ &= \xi_i - \eta_i \\ \text{Therefore} \qquad \xi''_i &= \frac{1}{2}(\xi_i - \eta_i) \end{aligned}$$

We can write these out explicitly as

$$\xi''_1 = \frac{1}{2}\left[\left(\frac{\bar{x}^2}{2} - \frac{\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{\bar{y}\bar{x}}{2} + \bar{z}\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{\bar{z}\bar{x}}{2} - \bar{y}\right)\frac{\partial}{\partial \bar{z}}.\right] \tag{5.38}$$

$$\xi''_2 = \frac{1}{2}\left[\left(\frac{\bar{x}\bar{y}}{2} - \bar{z}\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{\bar{y}^2}{2} - \frac{\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{\bar{z}\bar{y}}{2} + \bar{x}\right)\frac{\partial}{\partial \bar{z}}.\right] \tag{5.39}$$

$$\xi''_3 = \frac{1}{2}\left[\left(\frac{\bar{x}\bar{z}}{2} + \bar{y}\right)\frac{\partial}{\partial \bar{x}} + \left(\frac{\bar{y}\bar{z}}{2} - \bar{x}\right)\frac{\partial}{\partial \bar{y}} + \left(\frac{\bar{z}^2}{2} - \frac{\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial \bar{z}}.\right] \tag{5.40}$$

We can simply commute these to check the commutation relations (5.37). Are these transitive on the 3 dimensional spacelike hypersurfaces?

For a group of isometries to be transitive on a manifold M_n there must exist an isometry which will transform any point on the manifold into any other point on the manifold. In other words, there must be no point at which it is impossible to construct an infinitesimal generator (through linear combination) which has components along all n coordinate directions.

In the case of our four dimensional spacetime (M,g) we are considering 3 dimensional hypersurfaces parameterised by the "time" coordinate t. We usually refer to these as H(t) and they foliate our spacetime.

It can be seen clearly that at every point on H(t) for the case k=+1 we can construct a Killing vector which has components in all three coordinate directions. Even when $\bar{x} = \bar{y} = \bar{z} = 0$ we can still construct the linear combination

$$A\frac{\partial}{\partial \bar{x}} + B\frac{\partial}{\partial \bar{y}} + C\frac{\partial}{\partial \bar{z}}$$

where, of course, A,B and C are constants. Therefore, the Killing vectors ${\xi''}_1$,${\xi''}_2$ and ${\xi''}_3$ above are the infinitesimal generators of the transitive subgroup of the isometry group for the k=+1 FRW model.

We can now write the new Lie algebra for the case k=+1 explicitly in terms of the ${\xi''}_i$ and η_i. The complete Lie algebra for this model is now given by

$$[{\xi''}_i, {\xi''}_j] = \epsilon_{ij}^k {\xi''}_k \tag{5.41}$$

$$[\eta_i, \eta_j] = -\epsilon_{ij}^k \eta_k \tag{5.42}$$

$$[{\xi''}_i, \eta_j] = -\epsilon_{ij}^k {\xi''}_k \tag{5.43}$$

Where the last relation comes from replacing the expression ${\xi''}_i = \frac{1}{2}(\xi_i - \eta_i)$ in equation (5.32).

Now make the replacement ${\eta'}_i = -\eta_i$ to get

$$[{\xi''}_i, {\xi''}_j] = \epsilon_{ij}^k {\xi''}_k \tag{5.44}$$

$$[{\eta'}_i, {\eta'}_j] = \epsilon_{ij}^k {\eta'}_k \tag{5.45}$$

$$[{\xi''}_i, {\eta'}_j] = \epsilon_{ij}^k {\xi''}_k \tag{5.46}$$

It is now obvious that the complete Lie algebra above is isomorphic to that of SO(4). See Wybourne (1974) [24] for a list of the generators and Lie algebra of SO(4). The subalgebra given by (5.44) is isomorphic to the Lie algebra of SO(3).This subalgebra is the Lie algebra of the transitive subgroup and corresponds to Bianchi type IX. The subalgebra given by (5.45) is also isomorphic to the Lie algebra of SO(3) and the associated Killing vectors ${\eta'}_i$ are the generators of the isotropy subgroup - this was discussed at the beginning of the chapter.

Since the spacetime is spatially homogeneous and there is an isotropy subgroup at at least one point then the spatial sections are isotropic at all points.

k=-1.

For the case k=-1 we have that a_1^2=-1, a_2^1=1 and the rest of a_j^i=0. The structure constants are

$$C_{13}^1 = -C_{31}^1 = 1 \tag{5.47}$$

$$C_{23}^2 = -C_{32}^2 = 1 \tag{5.48}$$

$$\text{and the rest of} \qquad C_{ij}^k = 0 \tag{5.49}$$

The newly computed elements of the new Lie algebra are now given by

$$\begin{aligned}\xi'_1 &= \xi_1 - \eta_2\\ \xi'_2 &= \xi_2 + \eta_1\\ \xi'_3 &= \xi_3\end{aligned}$$

These can be written explicitly as

$$\xi'_1 = \left[\left(-\frac{\bar{x}^2}{2} + \frac{\bar{r}^2}{4} + 1 - \bar{z}\right)\frac{\partial}{\partial\bar{x}} + \left(-\frac{\bar{y}\bar{x}}{2}\right)\frac{\partial}{\partial\bar{y}} + \left(-\frac{\bar{z}\bar{x}}{2} + \bar{x}\right)\frac{\partial}{\partial\bar{z}}.\right] \tag{5.50}$$

$$\xi'_2 = \left[\left(-\frac{\bar{x}\bar{y}}{2}\right)\frac{\partial}{\partial\bar{x}} + \left(-\frac{\bar{y}^2}{2} + \frac{\bar{r}^2}{4} + 1 - \bar{z}\right)\frac{\partial}{\partial\bar{y}} + \left(-\frac{\bar{z}\bar{y}}{2} + \bar{y}\right)\frac{\partial}{\partial\bar{z}}.\right] \tag{5.51}$$

$$\xi'_3 = \left[\left(-\frac{\bar{x}\bar{z}}{2}\right)\frac{\partial}{\partial\bar{x}} + \left(-\frac{\bar{y}\bar{z}}{2}\right)\frac{\partial}{\partial\bar{y}} + \left(-\frac{\bar{z}^2}{2} + \frac{\bar{r}^2}{4} + 1\right)\frac{\partial}{\partial\bar{z}}.\right] \tag{5.52}$$

Once again, there is no point on the hypersurface H(t) where it is impossible to construct a Killing vector with components in all three coordinate directions, ie even when $\bar{x} = \bar{y} = \bar{z} = 0$ we can still construct the linear combination

$$A\frac{\partial}{\partial\bar{x}} + B\frac{\partial}{\partial\bar{y}} + C\frac{\partial}{\partial\bar{z}}$$

where A,B and C are constants. Hence the ξ'_i are the infinitesimal generators of the transitive subgroup of the isometry group of the open FRW model. The complete Lie algebra for this model is now given by

$$[\eta_i, \eta_j] = -\epsilon^k_{ij}\eta_k \tag{5.53}$$

$$[\xi'_1, \xi'_3] = \xi'_1 \tag{5.54}$$

$$[\xi'_2, \xi'_3] = \xi'_2 \tag{5.55}$$

and

$$\begin{aligned}[\xi'_1, \eta_1] &= -\eta_3 \qquad [\xi'_2, \eta_1] = \xi'_3\\ [\xi'_1, \eta_2] &= -\xi'_3 \qquad [\xi'_2, \eta_2] = -\xi'_3\\ [\xi'_1, \eta_3] &= \xi'_2 \qquad [\xi'_2, \eta_3] = -\xi'_1\end{aligned}$$

The subalgebra (5.53) is isomorphic to SO(3) and corresponds to the Lie algebra of the isotropy subgroup of the isometry group.

The subalgebra given by (5.54) and (5.55) is the Lie algebra of the transitive subgroup and is isomorphic to Bianchi type V as can be seen from the structure constants given above.

We turn now to the original form of the Lie algebra given by (5.30), (5.31) and (5.32). The value of k is inserted and the substitution ${\eta'}_i = -\eta_i$ is made to give

$$\begin{aligned} [{\eta'}_i, {\eta'}_j] &= \epsilon^k_{ij}{\eta'}_k \\ [\xi_i, \xi_j] &= -\epsilon^k_{ij}{\eta'}_k \\ [{\eta'}_i, \xi_j] &= \epsilon^k_{ij}\xi_k \end{aligned}$$

This is the Lie algebra of the (proper,orthochronus)Lorentz group SO(3,1) - See Carmeli (1977) [1] for details of the Lorentz group. Therefore, the complete isometry group for the open FRW model is SO(3,1) and the model is Bianchi type V. Again this spacetime is isotropic at all points on the spatial sections.

k=0.

For the case k=0 we have that all a^i_j=0. The structure constants are all zero also C^k_{ij} =0. The elements of the new Lie algebra are now simply

$$\xi'_i = \xi_i \tag{5.56}$$

Replacing k=0 in the expressions for ξ_i ie (5.27), (5.28) and (5.29), produces

$$\begin{aligned} {\xi'}_1 &= \frac{\partial}{\partial \bar{x}} \\ {\xi'}_2 &= \frac{\partial}{\partial \bar{y}} \\ {\xi'}_3 &= \frac{\partial}{\partial \bar{z}} \end{aligned} \tag{5.57}$$

It is obvious that the ${\xi'}_i$ above are the infinitesimal generators of the transitive subgroup of the isometry group of the flat FRW model. Again we put ${\eta'}_i = -\eta_i$ and the complete Lie algebra for the flat model is given by

$$[\xi'_i,\xi'_j] = 0 \quad (5.58)$$

$$[\eta',\eta'] = \epsilon^k_{ij}\eta'_k \quad (5.59)$$

$$[\xi'_i,\eta'_j] = \epsilon^k_{ij}\xi'_k \quad (5.60)$$

For the flat model the isometry group is the product of translations in 3 dimensions T_3 and rotations SO(3).

5.2 The Redshift for the FRW Models.

In the FRW model it is straight forward to derive the redshift, owing to the form of the metric:

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Radial null geodesics are defined so $d\theta = d\phi = 0$ for an observer at the origin of the coordinate system and hence we can derive the redshift from the remaining expression:

$$dt = \pm\frac{Sdr}{(1-kr^2)^{\frac{1}{2}}}$$

We choose the minus sign since the radial coordinate r decreases as time increases. Integrating this expression gives

$$\int_{t_o}^{t_e}\frac{dt}{S(t)} = \int_{r_o}^{r_e}\frac{dr}{(1-kr^2)^{\frac{1}{2}}} \quad (5.61)$$

Suppose wave crests are emitted at t_e and $t_e + \Delta t_e$ and observed at t_e and $t_o + \Delta t_o$ respectively, then in a similar manner to (5.61) we have

$$\int_{t_o+\Delta t_o}^{t_e+\Delta t_e}\frac{dt}{S(t)} = \int_{r_e}^{r_o}\frac{dr}{(1-kr^2)^{\frac{1}{2}}} \quad (5.62)$$

S(t) is slowly varying with time and so remains unchanged over small time intervals δt and so by subtracting (5.61) from (5.62)

$$\frac{\Delta t_e}{S(t_e)} - \frac{\Delta t_o}{S(t_o)} = 0 \qquad \text{that is,} \qquad \frac{\Delta t_e}{\Delta t_o} = \frac{S(t_e)}{S(t_o)} \quad (5.63)$$

The redshift z of a photon emitted at t_e and reaching the observer at t_o is defined in terms of the respective wavelengths as

$$z + 1 = \frac{\lambda_o}{\lambda_e}$$

The time interval between successive wave crests is proportional to their wavelength λ. This gives the standard result that the redshift z is given by

$$1 + z = S(t_o)/S(t_e) \tag{5.64}$$

where t_e and t_o refer to time of emission and observation respectively. However, with any other metric the neat result might not be as easily obtained. It is, therefore, desirable to find another, more general procedure for determining the redshift. It turns out that we can make use of some of the conserved quantities associated with the FRW Killing vector fields and derive an equivalent expression for the redshift.

We recall that: In any geometry endowed with a symmetry described by a Killing vector field ξ, motion *along any geodesic* leaves constant the scalar product of the tangent vector p with the Killing vector ie

$$p{\cdot}\xi = constant \tag{5.65}$$

The FRW Killing vector field (5.16) corresponds to a transitive element of the isometry group and so the conservation law () will relate one point on the manifold to another. The Killing vector fields corresponding to the isotropy subgroup only decscribe the symmetry about a point and so, do not contribute any useful information regarding the redshift.

We now procede to evaluate this expression for the Killing vector (5.16) which in the coordinated basis $\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$, was

$$(0, \cos\theta(1 - kr^2)^{\frac{1}{2}}, -\frac{\sin\theta(1 - kr^2)^{\frac{1}{2}}}{r}, 0) \tag{5.66}$$

We denote the tangent vector to the geodesic by p^λ ie

$$p^\lambda \equiv (E, p^r, p^\theta, p^\phi)$$

Take the observer as being at the origin and consider the path of a radial photon (which is a geodesic), then $p^\theta = p^\phi = 0$ and we are free to choose $\theta = 0$. Therefore, the sum $p^\lambda \xi_\lambda = p^r \xi_r$ since all other terms are zero.

$$\xi_r = g_{rr}\xi^r = -\Big(\frac{S^2}{(1-kr^2)}\Big)\cos\theta(1-kr^2)^{\frac{1}{2}}$$

Then

$$p^\lambda \xi_\lambda = -\frac{p^r S^2}{(1-kr^2)^{\frac{1}{2}}} = constant = A.$$

Hence

$$p^r = \frac{A(1-kr^2)^{\frac{1}{2}}}{S^2}$$

For a photon

$$p^\lambda p_\lambda = 0$$

This implies that

$$(p^0)^2 - \frac{(p^r)^2 S^2}{(1-kr^2)} = 0.$$

replacing in the value of p^r we get the result that

$$(p^0)^2 = \frac{A}{S^2}$$

We recall that S=S(t). Then

$$\frac{(p^0)_o}{(p^0)_e} = \frac{S(t_e)}{S(t_o)}$$

but $p^0 = E$, the energy of the photon and is inversely proportional to its wavelength. From (5.64), therefore

$$1 + z = \frac{(p^0)_e}{(p^0)_o} = \frac{S(t_o)}{S(t_e)}$$

which is the result we would expect.

Conformal Killing vectors for the FRW models.

The FRW models admit in general 15 independent conformal Killing vectors (CKV), 6 of which are the ordinary Killing vectors. A list of the CKV for each value of k is given in a paper by Maartens and Maharaj (1985). They all have in common the CKV $S(t)\frac{\partial}{\partial t}$. The FRW model with k=0 has the same CKV as minkowski spacetime and hence the same Lie algebra. In order to find the CKV of the $k = \pm 1$ cases Maartens and Maharaj generalise a k=0 CKV and then commute this with the CKV $S(t)\frac{\partial}{\partial t}$ and the 6 Killing vectors to generate the complete Lie agebra.

The Killing vectors for k=0,+1,-1 are independent of the scale factor S(t) but the CKV do involve S(t). Certain choices of S(t) will make certain CKV become Killing vectors. A complete list is given in Maartens and Maharaj (1985). That is, Minkowski, de Sitter, Anti-de Sitter and Einstein static spaces can all be considered as special cases of the general FRW spacetime.

Consider the conformal Killing vector $S(t)\frac{\partial}{\partial t}$. We recall that conformal Killing vectors give rise to constants of motion for null geodesics, for which $p^\alpha p_\alpha = 0$, i.e.

$$p^\beta(\Theta_\alpha p^\alpha)_{;\beta} = \frac{1}{n}(\Theta^\beta{}_{;\beta})p^\alpha p_\alpha = 0 \tag{5.67}$$

where $\mathbf{p}$ is the tangent vector to the null geodesic. Therefore,

$$\mathbf{p}\cdot\Theta = constant.$$

We can carry out a similar calculation of the redshift in the FRW models using the above and the CKV $\Theta = S(t)\frac{\partial}{\partial t}$. The CKV can be written in (contravariant) component form as $(S(t), 0, 0, 0)$ and the above scalar product can then be written

$$p^\alpha\Theta_\alpha = p^0\Theta_0 = p^0 S(t) = A = constant.$$

Hence,

$$p^0 = \frac{A}{S(t)}$$

and we immediately get the result that

$$\frac{(p^0)_e}{(p^0)_o} = \frac{S(t_o)}{S(t_e)} = 1 + z$$

5.3 Conclusion.

We have illustrated the relevance of our work on group theory and differential geometry by considering the symmetry properties of the FRW models. The highly symmetric and idealised form of each model allows a clear demonstration of the role of each type of isometry (transitive or isotropic) within the complete isometry group. The neat results obtained for the redshift calculation illustrates the usefulness of the local isometry group.

The Bianchi Types.

This is a list of all possible 3-dimensional Lie algebras, as first given by Bianchi. The Lie algebras are characterised by their structure constants C^i_{jk}.

Also given is a set of Killing vectors ξ_i which exhibit the commutation relations given by the structure constants. The X_i are a corresponding invariant basis i.e. $\pounds_{\xi_i} X_i = 0$ for all i,j=1,2,3. The ω^i are the duals to the X_i. This information allows us to construct a metric for a 3-dimensional space

$$\mathbf{g} = g_{ij}\omega^i \otimes \omega^j$$

(the g_{ij} are constants) such that $\pounds_{\xi_i}\mathbf{g} = 0$ i.e. the ξ_i are Killing vectors corresponding to the isometries of the metric $\mathbf{g}$. See section 4.7.3 for details.

Type I

$$\begin{array}{lllll} C^i_{jk} = 0. & \xi_1 = \dfrac{\partial}{\partial x^1} & X_1 = \dfrac{\partial}{\partial x^1} & \omega^1 = dx^1 & d\omega^1 = 0 \\ & \xi_2 = \dfrac{\partial}{\partial x^2} & X_2 = \dfrac{\partial}{\partial x^2} & \omega^1 = dx^2 & d\omega^2 = 0 \\ & \xi_3 = \dfrac{\partial}{\partial x^3} & X_3 = \dfrac{\partial}{\partial x^3} & \omega^3 = dx^3 & d\omega^3 = 0 \end{array}$$

Type II

$$\begin{array}{lll} C^1_{23} = -C^1_{32} = 1. & \xi_1 = \dfrac{\partial}{\partial x^2} & X_1 = \dfrac{\partial}{\partial x^2} \\ \text{rest of } C^i_{jk} = 0. & \xi_2 = \dfrac{\partial}{\partial x^3} & X_2 = x^1\dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial x^3} \\ & \xi_3 = \dfrac{\partial}{\partial x^1} + x^3\dfrac{\partial}{\partial x^2} & X_3 = \dfrac{\partial}{\partial x^1} \end{array}$$

$$\begin{array}{ll} \omega^1 = dx^2 - x^1 dx^3 & d\omega^1 = \omega^2 \wedge \omega^3 \\ \omega^2 = dx^3 & d\omega^2 = 0 \\ \omega^1 = dx^1 & d\omega^3 = 0 \end{array}$$

Type III

$$C^1_{13} = -C^1_{31} = 1. \qquad \xi_1 = \frac{\partial}{\partial x^2} \qquad X_1 = e^{x^1}\frac{\partial}{\partial x^2}$$

$$\text{rest of } C^i_{jk} = 0. \qquad \xi_2 = \frac{\partial}{\partial x^3} \qquad X_2 = \frac{\partial}{\partial x^3}$$

$$\xi_3 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} \qquad X_3 = \frac{\partial}{\partial x^1}$$

$$\begin{aligned} \omega^1 &= e^{-x^1}dx^2 & d\omega^1 &= \omega^1 \wedge \omega^2 \\ \omega^2 &= dx^3 & d\omega^2 &= 0 \\ \omega^3 &= dx^1 & d\omega^3 &= 0 \end{aligned}$$

Type IV

$$C^1_{13} = -C^1_{31} = 1 \qquad \xi_1 = \frac{\partial}{\partial x^2} \qquad X_1 = e^{x^1}\frac{\partial}{\partial x^2}$$

$$C^1_{23} = -C^1_{32} = 1 \qquad \xi_2 = \frac{\partial}{\partial x^3} \qquad X_2 = x^1e^{x^1}\frac{\partial}{\partial x^2} + e^{x^1}\frac{\partial}{\partial x^3}$$

$$C^2_{23} = -C^2_{32} = 1 \qquad \xi_3 = \frac{\partial}{\partial x^1} + (x^2+x^3)\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} \qquad X_3 = \frac{\partial}{\partial x^1}$$

rest of $C^i_{jk} = 0$.

$$\begin{aligned} \omega^1 &= e^{-x^1}dx^2 - x^1e^{-x^1}dx^3 & d\omega^1 &= \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \\ \omega^2 &= e^{-x^1}dx^3 & d\omega^2 &= \omega^2 \wedge \omega^3 \\ \omega^3 &= dx^1 & d\omega^3 &= 0 \end{aligned}$$

Type V

$$C^1_{13} = -C^1_{31} = 1 \qquad \xi_1 = \frac{\partial}{\partial x^2} \qquad X_1 = e^{x^1}\frac{\partial}{\partial x^2}$$

$$C^1_{23} = -C^1_{32} = 1 \qquad \xi_2 = \frac{\partial}{\partial x^3} \qquad X_2 = e^{x^1}\frac{\partial}{\partial x^3}$$

$$\text{rest of } C^i_{jk} = 0. \qquad \xi_3 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} \qquad X_3 = \frac{\partial}{\partial x^1}$$

$$\begin{aligned} \omega^1 &= e^{-x^1}dx^2 & d\omega^1 &= \omega^1 \wedge \omega^3 \\ \omega^2 &= e^{-x^1}dx^3 & d\omega^2 &= \omega^2 \wedge \omega^3 \\ \omega^3 &= dx^1 & d\omega^3 &= 0 \end{aligned}$$

Type VI

$$
\begin{array}{lll}
C^1_{13} = -C^1_{31} = 1 & \xi_1 = \dfrac{\partial}{\partial x^2} & X_1 = e^{x^1}\dfrac{\partial}{\partial x^2} \\
C^2_{23} = -C^2_{32} = h & \xi_2 = \dfrac{\partial}{\partial x^3} & X_2 = e^{hx^1}\dfrac{\partial}{\partial x^3} \\
(h \neq 0, 1) & \xi_3 = \dfrac{\partial}{\partial x^1} + x^2\dfrac{\partial}{\partial x^2} + hx^3\dfrac{\partial}{\partial x^3} & X_3 = \dfrac{\partial}{\partial x^1}
\end{array}
$$

rest of $C^i_{jk} = 0$.

$$
\begin{aligned}
\omega^1 &= e^{-x^1}dx^2 & d\omega^1 &= \omega^1 \wedge \omega^3 \\
\omega^2 &= e^{-hx^1}dx^3 & d\omega^2 &= h\omega^2 \wedge \omega^3 \\
\omega^3 &= dx^1 & d\omega^3 &= 0
\end{aligned}
$$

Type VII

$$
\begin{array}{lll}
C^2_{13} = -C^2_{31} = 1 & \xi_1 = \dfrac{\partial}{\partial x^2} & X_1 = (A + kB)\dfrac{\partial}{\partial x^2} - B\dfrac{\partial}{\partial x^3} \\
C^1_{23} = -C^1_{32} = -1 & \xi_2 = \dfrac{\partial}{\partial x^3} & X_2 = B\dfrac{\partial}{\partial x^2} + (A - kB)\dfrac{\partial}{\partial x^3} \\
C^2_{23} = -C^2_{32} = h & \xi_3 = \dfrac{\partial}{\partial x^1} - x^3\dfrac{\partial}{\partial x^2} + (x^2 + hx^3)\dfrac{\partial}{\partial x^3} & X_3 = \dfrac{\partial}{\partial x^1} \\
(h^2 < 4) & &
\end{array}
$$

rest of $C^i_{jk} = 0$.

$$
\begin{aligned}
\omega^1 &= (C - kD)dx^2 - Ddx^3 & d\omega^1 &= -\omega^2 \wedge \omega^3 \\
\omega^2 &= Ddx^2 + (C + kD)dx^3 & d\omega^2 &= \omega^1 \wedge \omega^3 + h\omega^2 \wedge \omega^3 \\
\omega^3 &= dx^1 & d\omega^3 &= 0
\end{aligned}
$$

where $A = e^{kx^1} cosax^1, \quad B = -\dfrac{1}{a}e^{kx^1} sinax^1,$

$C = e^{-kx^1} cosax^1, \quad D = -\dfrac{1}{a}e^{-kx^1} sinax^1$

and $k = \dfrac{h}{2}, \quad a = (1 - k^2)^{\frac{1}{2}}.$

Type VIII

$$\begin{aligned}
C^1_{23} &= -C^1_{32} = -1 & \xi_1 &= \frac{1}{2}e^{-x^3}\frac{\partial}{\partial x^1} + \frac{1}{2}[e^{x^3} - (x^2)^2e^{-x^3}]\frac{\partial}{\partial x^2} - x^2e^{-x^3}\frac{\partial}{\partial x^3} \\
C^2_{31} &= -C^2_{13} = 1 & \xi_2 &= \frac{\partial}{\partial x^3} \\
C^3_{12} &= -C^3_{21} = 1 & \xi_3 &= \frac{1}{2}e^{-x^3}\frac{\partial}{\partial x^1} - \frac{1}{2}[e^{x^3} + (x^2)^2e^{-x^3}]\frac{\partial}{\partial x^2} - x^2e^{-x^3}\frac{\partial}{\partial x^3}
\end{aligned}$$

rest of $C^i_{jk} = 0.$

$$\begin{aligned}
X_1 &= \frac{1}{2}[1 + (x^1)^2]\frac{\partial}{\partial x^1} + \frac{1}{2}[1 - 2x^1x^2]\frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial x^3} \\
X_2 &= -x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \\
X_3 &= \frac{1}{2}[1 - (x^1)^2]\frac{\partial}{\partial x^1} + \frac{1}{2}[-1 + 2x^1x^2]\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3}
\end{aligned}$$

$$\begin{aligned}
\omega^1 &= dx^1 + [1 + (x^1)^2]dx^2 + [x^1 - x^2 - (x^1)^2x^2]dx^3 & d\omega^1 &= -\omega^2 \wedge \omega^3 \\
\omega^2 &= 2x^1dx^2 + (1 - 2x^1x^2)dx^3 & d\omega^2 &= \omega^3 \wedge \omega^1 \\
\omega^3 &= dx^1 + [-1 + (x^1)^2]dx^2 + [x^1 + x^2 - (x^1)^2x^2]dx^3 & d\omega^3 &= \omega^1 \wedge \omega^2
\end{aligned}$$

Type IX

$$\begin{aligned}
C^1_{23} &= -C^1_{32} = 1 & \xi_1 &= \frac{\partial}{\partial x^2} \\
C^2_{31} &= -C^2_{13} = 1 & \xi_2 &= \cos x^2\frac{\partial}{\partial x^1} - \cot x^1 \sin x^2\frac{\partial}{\partial x^2} + \frac{\sin x^2}{\sin x^1}\frac{\partial}{\partial x^3} \\
C^3_{12} &= -C^3_{21} = 1 & \xi_3 &= -\sin x^2\frac{\partial}{\partial x^1} - \cot x^1 \cos x^2\frac{\partial}{\partial x^2} + \frac{\cos x^2}{\sin x^1}\frac{\partial}{\partial x^3}
\end{aligned}$$

rest of $C^i_{jk} = 0.$

$$\begin{aligned}
X_1 &= -\sin x^3\frac{\partial}{\partial x^1} + \frac{\cos x^3}{\sin x^1}\frac{\partial}{\partial x^2} - \cot x^1\cos x^3\frac{\partial}{\partial x^3} \\
X_2 &= \cos x^3\frac{\partial}{\partial x^1} + \frac{\sin x^3}{\sin x^1}\frac{\partial}{\partial x^2} - \cot x^1\sin x^3\frac{\partial}{\partial x^3} \\
X_3 &= \frac{\partial}{\partial x^3}
\end{aligned}$$

$$
\begin{aligned}
\omega^1 &= -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2 \qquad & d\omega^1 &= \omega^2 \wedge \omega^3 \\
\omega^2 &= \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2 \qquad & d\omega^2 &= \omega^3 \wedge \omega^1 \\
\omega^3 &= \cos x^1 dx^2 + dx^3 \qquad & d\omega^3 &= \omega^1 \wedge \omega^2
\end{aligned}
$$

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