Using Charge Quantisation Rules to Extend the Standard Model

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To my family and Sharon.

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Declaration

With the exception of chapter 1, the work of this thesis has been performed by myself with the assistance and guidance of my supervisor, Dr. C.D. Froggatt, and Prof. H.B. Nielsen.

Abstract

We examine extensions of the Standard Model (SM), trying to base our assumptions on what has already been observed. We consider our models to be the most obvious extensions of the SM in the sense that we don't consider anything fundamentally different such as grand unification or supersymmetry which are not directly suggested by the SM itself.

We use features of the SM to guide our extensions. This method has the advantage that all our models will be based (at least in part) on experimental observations. The disadvantage is that we cannot expect such models to give us any fundamentally new explanations.

The main features we use from the SM are small representations and charge quantisation. By small representations we mean fundamental or singlet representations of each non-abelian group and weak hypercharges close to zero. We use generalisations of the weak hypercharge quantisation rule observed in the Standard Model to specify the weak hypercharge modulo 2 for any given representation of the non-Abelian part of the gauge group. When we combine these principles with the requirement, for a theoretically consistent model, that there are no anomalies, we are left with a very restricted choice of models.

For most of this thesis we concentrate on the possibility of additional low mass fermions (relative to the Planck mass) and search for combinations of allowed representations which don't produce any gauge anomalies. We put strong experimental constraints on these models by using the renormalisation group equations to estimate fixed point masses for the new fermions in our models, and also to

check that there is no $U(1)$ Landau pole below the Planck scale. This is required since we are assuming a desert up to the Planck scale.

In our most promising model we show that a fourth generation of quarks without leptons is possible and can soon be tested experimentally. In this model we replace the fourth generation of leptons (required in the SM to cancel anomalies) with a generation of $SU(5)$-"quarks" which are a generalisation of the SM quarks but coupling to a new $SU(5)$ group instead of $SU(3)$. We discuss how well this model agrees with experiment and give estimates for the masses of the new fermions.

In the final chapter we examine a different model where we don't introduce any new low mass fermions. Instead we try to explain the mass structure of the SM in a natural way. The problem with the SM is that the masses require different fermions to have different Yukawa couplings to the SM Higgs boson. The smallest and largest couplings differ by a factor of about 10^5. In this model all fundamental Yukawa couplings are of order 1 (which we assume to be more natural). The range of masses we observe are due to the different symmetry breaking scales associated with this model breaking down to the SM. The results are compared to results for a very similar model.

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Chapter 1

Introduction

The so-called Standard Model (SM) has been widely accepted as an accurate description of all observed physics other than the gravitational interaction. However, not many people believe that the SM is a complete description of physics. The main reason is the large number ($\sim$ 20) of free parameters in the model. There is also the fact that current accelerators have not yet found a Higgs boson and so many features of the SM have not even been observed. This means that it is certainly possible that the SM is only the low energy part of another model. This other model could be observed at scales higher than energies probed by todays accelerators. In fact such a model could be at a scale as high as the Planck scale or as low as the electroweak scale. In this thesis we wish to investigate some of these models and check to ensure that they are self-consistent and agree with current experimental data.

In order for any model to be valid it must be self-consistent. This is not trivial for chiral gauge theories. When quantised, anomalies can arise which would spoil the gauge symmetry and make the theory useless for calculations. The absence of such anomalies provides many constraints on our models. These are discussed in section 1.1.

Another requirement for self-consistency is that the gauge coupling constants (the $U(1)$ coupling constant in particular) remain finite over the range of energy scales in which the model is supposed to be valid. This must be checked carefully since the coupling 'constants' vary with the energy scale. We shall describe the dependence of the coupling constants on the energy scale in section 1.2. In particular, in section 1.2.1, we will discuss the constraint that the finiteness of the gauge coupling constants provides on the weak hypercharges of fermions in our models.

The success of the SM means that it must almost certainly have great significance to any attempt to produce any other model to describe nature. Therefore we have chosen to base our models on the SM, extending and generalising features of the SM to produce models as similar to the SM as possible. The basic ideas and alternatives are outlined in section 1.4.

1.1 Anomalies

When calculating using quantum field theory, it is found that diagrams involving loops introduce infinities and so would give infinite cross-sections. This is obviously not physically possible and to get round this the theory must be regularised. In simple terms this means that some sort of momentum cut-off is introduced so that the infinite terms (which arise from integrating over infinite momenta) are made finite. This is equivalent to using a set of running parameters (running because they depend on some energy or momentum scale) instead of the bare parameters which appear in the Lagrangian. In a sense the infinities are put into the bare parameters. It does not matter that they are then infinite since they are not physical observables. This procedure is called renormalisation. However, this is only possible for certain theories. One of the requirements is the absence of some kinds of anomalies.

Anomalies are purely quantum effects. They correspond to some quantity which is conserved classically not being conserved in the quantised theory. Some anomalies are harmless such as the axial-vector current Ward identity anomaly in current algebra which explains the high rate of the neutral pion decay:

$$\pi^0 \to 2\gamma$$

However, some anomalies are harmful in the sense that they spoil the renormalisation procedure, making it impossible to calculate anything meaningful from the theory. In this section we shall discuss these types of anomalies and the constraints imposed on our models by the requirement that such anomalies vanish.

1.1.1 Gauge Anomalies

In any chiral gauge theory, gauge anomalies can arise. These anomalies lead to an inconsistent theory and so they must not be present in a good theory. Each fermion representation makes its contribution to each type of anomaly. We say that there is an anomaly present if the total contribution to an anomaly from all the fermion representations is non-zero.

As we shall discuss in section 2.2.1 the models considered in this thesis have gauge groups of the general form

$$U(1) \otimes \prod_i SU(N_i)/D \tag{1.1.1}$$

The discrete group D leads to charge quantisation but has no direct relevance to the anomalies. We assume all fermions to be in $\mathbf{N}$, $\overline{\mathbf{N}}$ or singlet $(\mathbf{1})$ representations of each $SU(N)$, as will be discussed in section 2.2.2. We define n to be the N-ality of a representation ($n = 1$ (-1) for representation $\mathbf{N}$ ($\overline{\mathbf{N}}$) and $n = 0$ for singlet representation). We can also define the size, S, of each representation as the dimension of the representation (e.g. in the SM, $S = 6$ for the $(\mathbf{2},\mathbf{3})$ representation of $SU(2) \otimes SU(3)$ which is equivalent to the fact that there are 6 left-handed quarks in each generation).

For gauge anomalies we sum the contribution for all left-handed fermions and subtract the sum over all right-handed fermions. This is equivalent to summing

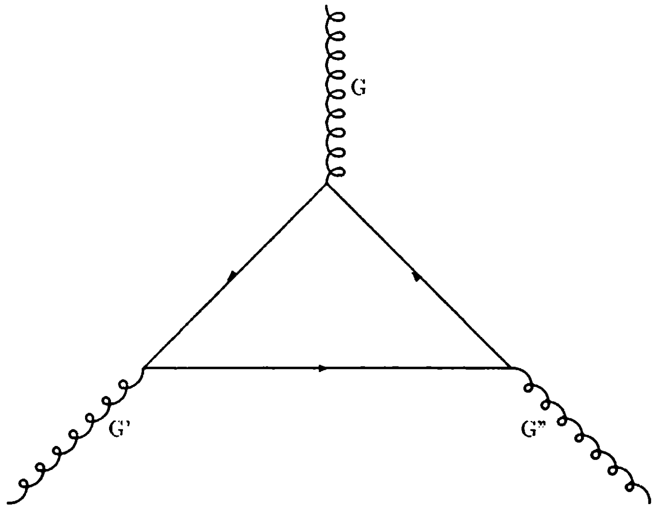

Figure 1.1: For the theory to be anomaly-free, the amplitude of this Feynman diagram must be zero for all choices of external gauge bosons after summing over all possible fermions in the internal loop (triangle).

over left-handed fermions and left-handed anti-fermions. We have now introduced all the necessary notation to write down general equations for all types of gauge anomalies.

The requirement that there are no anomalies present in a theory is analogous to the triangle Feynman diagram in fig. 1.1 with a fermion loop and three external gauge bosons (labelled by G, G' and G'') having zero amplitude for all possible choices of gauge bosons G, G' and G''. The contribution from each fermion representation is calculated by making particular choices for the fermions in the internal loop. These contributions must then sum to give zero amplitude if there is to be no anomaly. When this is true we say that the anomaly has been cancelled. The general condition for anomaly cancellation is,

$$Tr[\{T_L^a, T_L^b\}T_L^c] = Tr[\{T_R^a, T_R^b\}T_R^c] \tag{1.1.2}$$

where the 'T's are the transformation matrices for the fermions at the three vertices. When we consider left-handed anti-fermions instead of right-handed fermions, the condition becomes simply,

$$Tr[\{T^a, T^b\}T^c] = 0 \tag{1.1.3}$$

The trace corresponds to summing over all individual fermion representations and we can split the condition into different conditions for each choice of external gauge bosons. In our models we have the following type of anomalies which must be cancelled by an appropriate choice of fermion representations.

If each of G, G' and G'' is an $SU(N)$ gauge boson where $N \geq 3$ then each representation gives a relative contribution of $Sn^3 = Sn$ (since $n = -1$, 0 or 1 in our models). The total contribution is therefore $\sum_i S_i n_i$ where i labels each left-handed fermion (and anti-fermion) representation. We label this type of anomaly $[SU(N)]^3$ and require

$$\sum_i S_i n_i = 0 \tag{1.1.4}$$

Another type of anomaly corresponds to the diagram with one $U(1)$ gauge boson and two $SU(N)$ gauge bosons where $N \geq 2$, labelled as $[SU(N)]^2 U(1)$. Each representation gives a relative contribution $Sn^2 y$ where y is the conventional weak hypercharge [1] . Therefore we require

$$\sum_i S_i (n_i)^2 y_i = 0 \tag{1.1.5}$$

The final type of gauge anomaly corresponds to the diagram with all the gauge bosons G, G' and G'' being $U(1)$ gauge bosons. This is labelled as $[U(1)]^3$ and each representation gives a relative contribution Sy^3. Therefore we require

$$\sum_i S_i y_i^3 = 0 \tag{1.1.6}$$

[1]Throughout this thesis we take the normalisation for the weak hypercharge that the right handed electron has $y = 2$.

1.1.2 Other Anomalies

There is also a mixed gravitational and gauge anomaly [1] which corresponds to one $U(1)$ gauge boson and two gravitons in figure 1.1. We will label this as $G^2U(1)$. Each representation gives a relative contribution Sy and so this leads to the constraint

$$\sum_i S_i y_i = 0 \tag{1.1.7}$$

This anomaly comes from theories involving quantum gravity. At first this may not appear important for our models since we are not considering quantum gravity. But, since all such theories require this constraint in the low energy limit, we must make sure this anomaly doesn't exist in our models if we want them to be low energy effective theories of a complete theory which includes gravity.

Another possible anomaly is the Witten discrete $SU(2)$ anomaly [2]. This states that if the number of left-handed $SU(2)$ doublets is odd then the theory is inconsistent. This is different from the other anomalies considered in the sense that this is a global anomaly whereas the other are all local. The anomaly corresponds to the requirement that the theory should be consistent with a global $SU(2)$ gauge transformation. However, if there are an odd number of Weyl doublets, it is possible to perform such a transformation and introduce a change of sign in the Lagrangian. This means that the theory cannot be used to calculate in a general gauge. As we shall see later in section 2.4 we will always have an even number of Weyl $SU(2)$ doublets and so this anomaly does not give us any problems.

1.2 Renormalisation Group Equations

The renormalisation procedure introduces an arbitrary scale μ and the value of all the renormalised couplings depend on this scale. The renormalisation group equations (RGEs) describe how to relate parameters at different scales. We use these equations to calculate the running gauge coupling constants from the electroweak scale up to the Planck scale. If one of these becomes infinite then there is a Landau pole. If this happens then the model cannot be self-consistent up to the Planck scale. Since we calculate the RGEs perturbatively this would mean that the model was not perturbatively consistent up to the Planck scale. It is possible that the theory may still make sense in this case if non-perturbative methods were used but it is not clear whether or not this would happen. Therefore we will take the view that a consistent theory must be perturbatively consistent.

We also use the RGEs to calculate the Yukawa couplings at different scales. Here we go in the opposite direction to the calculation of the gauge coupling constants by calculating the Yukawa couplings at the electroweak scale from values chosen at the Planck scale. The reason we do this is that if the value of a Yukawa coupling at the Planck scale is increased, normally the resultant value at the electroweak scale is increased. However, there is a quasi-fixed point limit on the Yukawa couplings at the electroweak scale. This occurs when the Yukawa coupling at the Planck scale reaches a certain value (typically of order 1), and when increased further the value at the electroweak scale becomes fairly insensitive to the precise value at the Planck scale. This can be taken to be an upper limit on the Yukawa coupling at the electroweak scale. However, there is not a precise limit for each fermion and when there are several heavy fermions the limit on any one depends on the relative values of the other Yukawa couplings.

We use first order (1-loop) RGEs. More accurate results are available in the literature but further corrections only change predictions by a few percent. This will not affect the existence or otherwise of a Landau pole in the models we consider. Also, since the quasi-fixed point limits for the Yukawa couplings are only used to give approximate upper limits for the electroweak Yukawa couplings (and corresponding masses) of fermions, the error introduced by neglecting 2-loop corrections is not significant.

We will now describe the 1-loop RGEs for the gauge couplings and the Yukawa couplings. We use these to check that there are no Landau poles and to give estimates of the Yukawa couplings at the electroweak scale. Then we will describe the RGE for the Higgs quartic coupling which is related to the Higgs mass. Finally we discuss the definitions of mass for fermions; relating the Yukawa coupling to the running mass and the pole mass.

1.2.1 Gauge Couplings

The Lagrangian density of the gauge fields in a quantum field theory with a simple gauge group G is given by,

$$\mathcal{L} = -\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu}) \tag{1.2.8}$$

where the anti-symmetric field tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{1.2.9}$$

g is the bare gauge coupling and A_μ is defined in terms of the gauge fields, A_μ^a, and the adjoint representation of generators of G, T^a, by:

$$A_\mu = T^a A_\mu^a \tag{1.2.10}$$

The commutation relations of the generators are given in terms of the group structure constants, f^{abc}:

$$[T^a, T^b] = i f^{abc} T^c \tag{1.2.11}$$

and are normalised so that:

$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{1.2.12}$$

For semi-simple groups the Lagrangian density is simply a sum of the Lagrangian densities for each of the simple factors. For a $U(1)$ factor the gauge field tensor is simply defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.2.13}$$

When quantum field theories defined by these Lagrangian densities are renormalised, the physical gauge couplings depend on an arbitrary scale. To 1-loop order the equation describing the running gauge coupling of the gauge group $SU(N)$, g_N, at scale μ is:

$$\frac{dg_N}{dt} = \beta_N(g_N) \tag{1.2.14}$$

where $t = \ln(\mu)$ and the first order beta function, $\beta_N(g_N)$, is given by:

$$\beta_N(g_N) = g_N^3 K_N \tag{1.2.15}$$

The constant K_N is defined as [3]:

$$K_N = \frac{1}{16\pi^2}\left[-\frac{11}{3}C_2(G) + \frac{4}{3}\kappa S_2(F) + \frac{1}{6}S_2(S)\right] \tag{1.2.16}$$

The group factors are; the eigenvalue of the quadratic Casimir operator acting on the adjoint representation (the representation of the gauge bosons), $C_2(G)$ and the Dynkin indices for fermion and scalar representations, $S_2(F)$ and $S_2(S)$ respectively. $\kappa = \frac{1}{2}$ for 2-component fermions (Weyl fermions) and $\kappa = 1$ for 4-component fermions. In all our models the fermions are 2-component and will be in fundamental or singlet representations of each $SU(N)$ gauge group. The only scalar will be the SM Higgs boson. So we have:

$$C_2(G) = N \tag{1.2.17}$$

$$S_2(F) = n_F \tag{1.2.18}$$

$$S_2(S) = n_S \tag{1.2.19}$$

$$\kappa = \frac{1}{2} \tag{1.2.20}$$

where n_F (n_S) is the number of fermions (scalars) in fundamental representations of $SU(N)$. Since the only scalar we have is the SM Higgs boson, $n_S = 1$ for the $SU(2)$ subgroup of the SMG and $n_S = 0$ otherwise.

We can integrate eqs. (1.2.14) and (1.2.15) analytically by writing them as,

$$\int_{g_N(\mu_0)}^{g_N(\mu)} \frac{dg_N}{g_N^3} = K_N \int_{t_0}^{t} dt \tag{1.2.21}$$

This leads to the result

$$\frac{1}{2g_N^2(\mu_0)} - \frac{1}{2g_N^2(\mu)} = K_N \ln\left(\frac{\mu}{\mu_0}\right) \tag{1.2.22}$$

We now use the definition of the fine structure constants,

$$\alpha_N(\mu) = \frac{g_N^2(\mu)}{4\pi} \tag{1.2.23}$$

along with the definition of K_N to write:

$$\frac{1}{\alpha_N(\mu)} = \frac{1}{\alpha_N(\mu_0)} + \frac{1}{12\pi}(22N - 4n_F - n_S)\ln\left(\frac{\mu}{\mu_0}\right) \tag{1.2.24}$$

This equation allows us to calculate the fine structure constant at any scale μ provided we know it's value at some scale μ_0.

The $U(1)$ fine structure constant at scale μ, $\alpha_1(\mu)$, can be calculated in a similar way and to 1-loop order we have:

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - \frac{1}{12\pi}(Y^2 + Y_S^2)\ln\left(\frac{\mu}{\mu_0}\right) \tag{1.2.25}$$

where Y^2 is the sum of weak hypercharges squared over all fermions and $Y_S^2 = 1$ for the SM Higgs boson.

However, a more usual convention is to use the normalisation for the $U(1)$ gauge coupling constant used in grand unified theories (GUTs). This is because the coupling constant would be normalised differently if the $U(1)$ group was the subgroup of a simple group. In particular, for $U(1) \subset SU(N)$ we have the following normalisation:

$$(g_1^2)_{\mathrm{GUT}} \equiv \frac{5}{3}(g_1^2)_{\mathrm{SM}} \tag{1.2.26}$$

$$(\alpha_1^{-1})_{\mathrm{GUT}} \equiv \frac{3}{5}(\alpha_1^{-1})_{\mathrm{SM}} \tag{1.2.27}$$

This then leads to a modification of eq.(1.2.25):

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - \frac{1}{20\pi}(Y^2 + Y_S^2)\ln\left(\frac{\mu}{\mu_0}\right) \tag{1.2.28}$$

Landau Poles

A Landau pole is a point where the value of a gauge coupling constant becomes infinite. This corresponds to α^{-1} becoming zero. It is only for the $U(1)$ gauge group that the value of α^{-1} necessarily decreases when run from low to high scales. Therefore it is the $U(1)$ sector that is likely to have a Landau pole. In fact there must be a Landau pole at some scale; what we are interested in is whether this occurs above or below the Planck scale. Above the Planck scale we do not expect our model to be a good description of physics so we do not worry about effects above the Planck scale. However, a Landau pole below the Planck scale would not be acceptable since the model would not then be (perturbatively) consistent.

We can use eq. (1.2.28) to calculate an upper limit on Y^2. This is because the requirement of no $U(1)$ Landau pole below the Planck scale means that $\frac{1}{\alpha(M_{Planck})} > 0$. This then provides a limit on Y^2 for any choice of thresholds where new particles are introduced. We will use this in section 4.2.2 to rule out several models.

Thresholds

The equations given in this section for the running of the gauge coupling constants involve numbers of fermions or Higgs bosons coupling to the groups or sums over all fermions or Higgs bosons of weak hypercharges squared. These numbers depend on all fermions or Higgs bosons in the theory whatever their masses. However, we should really include threshold effects which mean that particles do not contribute significantly unless the scale, μ, is of the same order of magnitude as, or greater than, their mass. A full calculation of such effects is complicated so we will use a very simple approximation. We will include all particles with pole masses below the scale in the RGEs. When the scale is the same as a pole mass, that particle will be included in (removed from) the RGEs if we are running from low to high (high to low) scales.

This method is accurate enough for our purposes since we are really only interested in finding out if there is a $U(1)$ Landau pole below the Planck scale. In all the models we consider the existence or otherwise of a Landau pole below the Planck scale is not dependent on small changes in the low energy value of α_1 caused by incomplete threshold analysis.

1.2.2 Yukawa Couplings

Let us consider the simplest case of two fermions, p and m, getting a mass via the SM Higgs mechanism. In order for this mechanism to work we must have a left-handed $SU(2)$ doublet, $F_L \equiv \begin{pmatrix} p \\ m \end{pmatrix}_L$, and two right-handed $SU(2)$ singlets, p_R and m_R. p_R must have a value of weak hypercharge, $y_{p_R} = y_{p_L} + 1$, and m_R must have a value, $y_{m_R} = y_{m_L} - 1$ (p_L and m_L must obviously have the same value). Otherwise the fermions must couple to the gauge group in the same way. The Lagrangian density is given by:

$$\mathcal{L} = -\overline{F}_L y_p i\tau_2 \Phi^* p_R - \overline{F}_L y_m \Phi m_R + \text{h.c.} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \tag{1.2.29}$$

where y_p (y_m) is the Yukawa coupling of fermion p (m) and the Higgs potential is defined by:

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \tag{1.2.30}$$

If we parameterise the complex scalar doublet by 4 real fields:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1^+ + i\phi_2^+ \\ \phi_1^0 + i\phi_2^0 \end{pmatrix} \tag{1.2.31}$$

then after the electroweak symmetry breaking one of the fields gets a vacuum expectation value (VEV):

$$< \phi_1^0 > = v \tag{1.2.32}$$

where

$$v = \sqrt{\frac{\mu^2}{\lambda}} \tag{1.2.33}$$

We label this VEV, $< \phi_{WS} >$.

The fermion interactions with the Higgs then lead to fermion masses. After the symmetry breaking the running mass of fermion f, m_f, is related to its Yukawa coupling, y_f, by:

$$m_f = \frac{y_f}{\sqrt{2}} < \phi_{WS} > \tag{1.2.34}$$

Here we present the RGEs for the Yukawa couplings to order 1-loop for fermion f. We label its left-handed $SU(2)$ doublet partner by f'.

$$\frac{dy_f}{dt} = \frac{1}{16\pi^2}\beta_f(t) \tag{1.2.35}$$

where, to 1-loop, $\beta_f(t)$ is defined by [4]:

$$\begin{aligned}
\beta_f(t) &= y_f(t)\left[\frac{3}{2}\left(y_f^2(t) - y_{f'}^2(t)\right) + Y_2(S)(t) - G_f(t)\right] \\
Y_2(S)(t) &= Tr(Y^\dagger(t)Y(t)) && (1.2.36) \\
G_f(t) &= 6\sum_N g_N^2(t)C_2^N(f) && (1.2.37)
\end{aligned}$$

$g_N(t)$ is the $SU(N)$ ($U(1)$ for $N = 1$) gauge coupling constant, $Y(t)$ is the Yukawa coupling matrix and $C_2^N(f)$ is the quadratic Casimir operator for the fermion representations of gauge group N. For $SU(N)$, we have the definition:

$$C_2^N(f) = \frac{N^2 - 1}{2N} \tag{1.2.38}$$

and for $U(1)$ we replace $g_1^2 C_2^1(f)$ with:

$$\frac{1}{8}\left((Q_f^L)^2 + (Q_f^R)^2\right) g_1^2 \tag{1.2.39}$$

where we have used Q^L (Q^R) for the weak hypercharge of the left- (right-) handed fermions. Using the conventional GUT normalisation this becomes:

$$\frac{3}{40}\left((Q_f^L)^2 + (Q_f^R)^2\right) g_1^2 \tag{1.2.40}$$

Y is the Yukawa coupling matrix and in this case is a diagonal matrix containing the Yukawa couplings of the fermions. There is (an identical) Yukawa coupling for each of the N fermions in each $\mathbf{N}$ representation of $SU(N)$ $(N \geq 3)$. For example, if the p and m fermions were the up and down quarks we would get $Y_2(S) = 3y_p^2 + 3y_m^2$.

If we have several fermions getting a mass in this way then we can simply generalise the above Lagrangian density by adding more terms. If several fermions have the same quantum numbers then they can mix together and so the gauge eigenstates are different from the mass eigenstates. In the SM this happens for the three generations of quarks. However, the mixing is a small effect and in this thesis we shall ignore it for simplicity. Therefore the Yukawa coupling matrix, Y, will remain diagonal and the above equations will remain unchanged.

1.2.3 The SM Higgs Self-Coupling

We will present the RGEs to 1-loop order for the SM Higgs boson self-coupling, λ. To first order this parameter is related to the Higgs mass by the equation,

$$M_H^2 = \frac{\lambda}{\sqrt{2}G_\mu} = \lambda < \phi_{WS} >^2 \tag{1.2.41}$$

where G_μ is the coefficient of the effective four fermion interaction in an effective low energy model of the weak interactions:

$$\frac{G_\mu}{\sqrt{2}}[\overline{\nu}_e \gamma^\beta (1-\gamma_5)e][\overline{\mu}\gamma_\beta(1-\gamma_5)\nu_\mu] \tag{1.2.42}$$

From the measured value of the muon lifetime [5],

$$\tau_\mu = 2.19703 \pm 0.00004 \times 10^{-6} \text{ s} \tag{1.2.43}$$

we can calculate,

$$G_\mu = 1.16637 \pm 0.00002 \times 10^{-5} \text{ GeV}^{-2} \tag{1.2.44}$$

There is no experimental value for λ at any scale since the Higgs has not yet been observed. However, as for the Yukawa couplings of new fermions, we can observe a quasi-fixed point for $\lambda(\mu = M_Z)$. We can use this to put upper limits on the Higgs mass in our models.

The equation for the running of $\lambda(\mu)$ is given by,

$$\frac{d\lambda}{dt} = \beta_\lambda(t) \tag{1.2.45}$$

where, to 1-loop [4]:

$$\begin{aligned}\beta_\lambda(t) &= \frac{1}{16\pi^2}\left[12\lambda^2 - \left(\frac{9}{5}g_1^2 + 9g_2^2\right)\lambda + \right.\\ &\left.\frac{9}{4}\left(\frac{3}{25}g_1^4 + \frac{2}{5}g_1^2g_2^2 + g_2^4\right) + 4Y_2(S)\lambda - 4H(S)\right]\end{aligned} \tag{1.2.46}$$

where the conventional (GUT type) normalisation is used for g_1. $Y_2(S)$ and $H(S)$ are defined in terms of the Yukawa coupling matrix Y by,

$$Y_2(S) = Tr(Y^\dagger Y) \tag{1.2.47}$$

$$H(S) = Tr(Y^\dagger Y Y^\dagger Y) \tag{1.2.48}$$

1.3 Definitions of Mass

For heavy quarks the experimentally measured mass is the pole mass. The mass calculated from the Yukawa coupling by,

$$m_f = \frac{y_f}{\sqrt{2}} < \phi_{WS} > \tag{1.3.49}$$

is known as the running mass. We wish to relate these two definitions of mass so that we can compare running Yukawa couplings to experimentally measured masses.

To first order the pole mass, M, is related to the running mass at scale μ, $m(\mu)$, by simply:

$$M = m(M) \tag{1.3.50}$$

However there are further corrections arising from loop diagrams. We shall calculate the leading correction to this formula for fermions in fundamental representations of the gauge group $SU(N)$. For $N=3$ this is a good approximation for quarks in the SM since the corrections due to the strong force are much larger than those due to QED. This is simply because $\alpha_S \gg \alpha_{QED}$.

We use dimensional regularisation where the number of dimensions is,

$$D = 4 - 2\varepsilon \tag{1.3.51}$$

We define m_0 to be the bare mass and g_0 to be the bare $SU(N)$ gauge coupling. The renormalised $SU(N)$ gauge coupling at scale μ is $g(\mu)$ and we define the running fine structure constant,

$$\alpha(\mu) \equiv \frac{g^2(\mu)}{4\pi} \tag{1.3.52}$$

Working to 1-loop order will allow us to calculate the corrections to eq. (1.3.50) to order $\alpha(\mu)$. To this order the following equations hold:

$$M = m_0 \left[1 + \frac{g_0^2}{(4\pi)^{\frac{D}{2}} M^{2\varepsilon}} C_2^N(f) \frac{D-1}{D-3} \Gamma(\varepsilon)\right] \tag{1.3.53}$$

$$m_0 = m(\mu) \left[1 - \frac{3\alpha(\mu)}{4\pi\varepsilon} C_2^N(f)\right] \tag{1.3.54}$$

$$\frac{g_0^2}{4\pi} = \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^\varepsilon \alpha(\mu) \tag{1.3.55}$$

For fermions in the fundamental representation of $SU(N)$,

$$C_2^N(f) = \frac{N^2-1}{2N} \tag{1.3.56}$$

The constant $\gamma \approx 0.5772$ is Euler's constant. We use the approximation for the gamma function for small ε,

$$\Gamma(\varepsilon) \approx \frac{1}{\varepsilon} - \gamma \tag{1.3.57}$$

It is now straightforward to eliminate all the bare couplings and relate the pole mass to the running mass.

$$\frac{M}{m(\mu)} = \left[1 - \frac{3\alpha(\mu)}{4\pi\varepsilon} C_2^N(f)\right] \left[1 + \frac{g_0^2}{(4\pi)^{\frac{D}{2}} M^{2\varepsilon}} C_2^N(f) \frac{D-1}{D-3} \Gamma(\varepsilon)\right] \tag{1.3.58}$$

Expressing D in terms of ε and neglecting terms of order $\alpha^2(\mu)$ gives the following equation.

$$\frac{M}{m(\mu)} = 1 + \frac{\alpha(\mu)}{4\pi} C_2^N(f) \left[-\frac{3}{\varepsilon} + \left(\frac{\mu^2 e^\gamma}{M^2} \right)^\varepsilon \frac{3-2\varepsilon}{1-2\varepsilon} \Gamma(\varepsilon) \right] \tag{1.3.59}$$

Now we take the limit $\varepsilon \to 0$. For small ε,

$$\frac{3-2\varepsilon}{1-2\varepsilon} \approx (3-2\varepsilon)(1+2\varepsilon) \approx 3 + 4\varepsilon \tag{1.3.60}$$

and

$$\left(\frac{\mu^2 e^\gamma}{M^2} \right)^\varepsilon \approx 1 + \varepsilon \ln \left(\frac{\mu^2 e^\gamma}{M^2} \right) \tag{1.3.61}$$

Combining these we get,

$$\begin{aligned} \left(\frac{\mu^2 e^\gamma}{M^2} \right)^\varepsilon \frac{3-2\varepsilon}{1-2\varepsilon} \Gamma(\varepsilon) &\approx \left[1 + \varepsilon \ln \left(\frac{\mu^2 e^\gamma}{M^2} \right) \right] [3 + 4\varepsilon] \left[\frac{1}{\varepsilon} - \gamma \right] \\ &\approx -3\gamma + \frac{1}{\varepsilon} \left[3 + 4\varepsilon + 3\varepsilon \ln \left(\frac{\mu^2 e^\gamma}{M^2} \right) \right] \\ &= -3\gamma + \frac{3}{\varepsilon} + 4 + 3 \ln \left(\frac{\mu^2}{M^2} \right) + 3\gamma \\ &= \frac{3}{\varepsilon} + 4 + 3 \ln \left(\frac{\mu^2}{M^2} \right) \end{aligned} \tag{1.3.62}$$

Therefore, in the limit $\varepsilon \to 0$, to order $\alpha(\mu)$, we can relate the pole mass to the running mass of fermion f by:

$$M_f = m_f(\mu) \left[1 + C_2^N(f) \frac{\alpha(\mu)}{\pi} \left(1 + \frac{3}{4} \ln \left(\frac{\mu^2}{M^2} \right) \right) \right] \tag{1.3.63}$$

and in the special case where $\mu = M$:

$$M_f = m_f(M) \left[1 + C_2^N(f) \frac{\alpha(M)}{\pi} \right] \tag{1.3.64}$$

In particular:

$$M_f = m_f(M) \left[1 + \frac{4\alpha_3(M)}{3\pi} \right] \tag{1.3.65}$$

$$M_f = m_f(M) \left[1 + \frac{12\alpha_5(M)}{5\pi} \right] \tag{1.3.66}$$

for fermions in fundamental representations of the gauge groups $SU(3)$ and $SU(5)$ respectively. Eq. (1.3.65) agrees with [6].

1.4 Choosing the Type of Model

We have now discussed the basic theory behind model building. In section 1.1 we have described the constraints imposed on a chiral gauge theory by the requirement of anomaly cancellation. In section 1.2.1 we have discussed the requirement in our models of no $U(1)$ Landau pole below the Planck scale. We have also presented the RGEs necessary to examine quasi-fixed point Yukawa couplings, in section 1.2.2. We can use these as upper limits on the Yukawa couplings of the fermions in our models. Then in section 1.3 we showed how to convert these Yukawa couplings to running masses and pole masses. This will be used to provide upper limits on the mass of new fermions in our models. Now we must decide what sort of models we wish to examine. In this section we will outline our requirements for extending the SM.

First we shall briefly describe some other methods of extending the SM and then describe our method and compare it to some of these other methods. Over the years there have been numerous attempts at extending the SM. Some of these models have been proposed with the purpose of explaining some particular feature of the SM. For example, GUTs 'explain' the convergence of coupling constants at some energy (typically of order 10^{15}GeV) as a manifestation of a single fundamental unified interaction. Other models such as supersymmetry (SUSY) have been proposed for mainly aesthetic reasons; SUSY introduces a symmetry between bosons and fermions. But so far none of these attempts has been entirely successful, although SUSY GUTs are phenomenologically consistent with the unification of the SM gauge coupling constants and do not suffer from the gauge hierarchy problem (why the electroweak scale is so small compared to expected radiative corrections from the more fundamental theory which should be of the order of the GUT or Planck scale).

Another approach to extending the SM is to look at the SM itself and look for distinctive features which could be generalised or assumed to hold in an extended theory. The SM has been so successful that, within our experimental and calculational accuracy, it has proved to be a perfect description of nature (except for the gravitational interaction). So we have good reason to say that taking guidance from the SM is akin to "listening to God".

Having accepted this point of view we must now try and interpret the message of the SM. By this we mean that we must look for fundamental features in the SM which could distinguish it from similar and, without experimental evidence, equally plausible models. We propose that one such feature is charge quantisation. This can be expressed as

$$\frac{y}{2} + \frac{1}{2}\text{“duality”} + \frac{1}{3}\text{“triality”} \equiv 0 \pmod 1 \tag{1.4.67}$$

where y is the conventional weak hypercharge. The duality has value 1 if the representation is an $SU(2)$ doublet (**2**) and 0 if it is an $SU(2)$ singlet (**1**). The triality has value 1 if the representation is an $SU(3)$ triplet (**3**), 0 if it is an $SU(3)$ singlet (**1**), and -1 if it is an $SU(3)$ anti-triplet ($\overline{\mathbf{3}}$). In general we can define the N-ality of a representation of $SU(N)$ to be the minimum number of N-plet representations of $SU(N)$ which must be combined to construct the representation. In particular N-ality has value 1 if a representation is an $SU(N)$ N-plet (**N**), 0 if it is an $SU(N)$ singlet (**1**), and -1 if it is an $SU(N)$ anti-N-plet ($\overline{\mathbf{N}}$). Note that in $SU(2)$ the $\overline{\mathbf{2}}$ representation is equivalent to the **2** representation. We expect that in an extension of the SM this charge quantisation relation or some generalisation of it will hold.

An obvious way of extending the SM is to extend the gauge group. The Standard Model Group (SMG) is [7, 8]:

$$SMG \equiv S(U(2) \otimes U(3)) = U(1) \otimes SU(2) \otimes SU(3)/\hat{D}_3 \tag{1.4.68}$$

where the discrete group

$$\hat{D}_3 \equiv \{(e^{i2\pi/6}, -I_2, e^{i2\pi/3}I_3)^n : n \in \mathcal{Z}_6\} \tag{1.4.69}$$

ensures the quantisation rule eq. (1.4.67) (I_N is the identity of $SU(N)$). We argue that the most obvious extension is to add more groups to the sequence $U(1) \otimes SU(2) \otimes SU(3)$ and to use a different discrete group so that the quantisation rule is generalised to involve all the group components. One of the groups we consider is

$$G_5 \equiv U(1) \otimes SU(2) \otimes SU(3) \otimes SU(5)/\hat{D}_5 \tag{1.4.70}$$

where the discrete group $\hat{D}_5$ is defined as

$$\hat{D}_5 \equiv \{(e^{i2\pi/N_5}, -I_2, e^{i2\pi/3} I_3, e^{i2\pi m_5/5} I_5)^n : n \in \mathcal{Z}_{N_5}\} \tag{1.4.71}$$

where $N_5 = 30$ $(= 2 \times 3 \times 5)$ and m_5 is an integer which is not a multiple of 5. This group gives a generalised quantisation rule,

$$\frac{y}{2} + \frac{1}{2}\text{“duality”} + \frac{1}{3}\text{“triality”} + \frac{\mathrm{m}_5}{5}\text{“quintality”} \equiv 0 \pmod 1 \tag{1.4.72}$$

which is the most obvious generalisation of the SM charge quantisation rule. Further generalisations are obtained by extending the sequence $U(1) \otimes SU(2) \otimes SU(3)$ with a set of $SU(N)$ factors, where the 'N's are greater than 3 and mutually prime [7].

We will consider the fundamental scale to be the Planck mass (M_{Planck}) and our models will be a full description of physics without gravity below this scale. The assumptions we make about our models essentially lead to the conclusion that all new fermions with a mass significantly below M_{Planck} must have a mass below the TeV scale as explained in section 3.2. Therefore our models all describe low energy physics (below the TeV scale) and have a desert up to the Planck scale where new physics will occur. We don't specify any details about the Planck scale physics since it is largely irrelevant to low energy physics.

We shall describe the gauge groups considered in this thesis and the motivation for choosing such groups in more detail in section 2.2.1. We shall consider general types of gauge groups and also give specific examples, concentrating on the group G_5 defined by eqs. (1.4.70) and (1.4.71). When we also impose the condition that all fermions are in fundamental representations, as in the SM, we are limited to the models which we shall consider in this thesis. After choosing the gauge group we want to examine which low mass fermions (low relative to the Planck scale) can exist in the model. We must check that the model is then consistent, both theoretically and experimentally.

The main theoretical constraint is that there are no anomalies as described in section 1.1. This greatly limits the choice of fermions and their weak hypercharges in our models.

There is one important fact to keep in mind when proposing any extended model which has extra non-Abelian gauge groups such as $SU(N)$. As we already know from the SM, the $SU(3)$ group acts as a technicolour group [9] and gives a contribution to the $W^{\pm}$ and Z^0 masses. In the SM this contribution is very small but when confining groups with $N > 3$ are considered we must carefully consider the effect this will have. Since we are not wanting the complications of extended technicolour in order to generate quark and lepton masses, we assume that there is a Higgs doublet and that the masses of the weak gauge bosons are generated by a combination of the Higgs sector of the theory and the technicolour effects of the gauge groups. This happens in exactly the same way as in the SM where QCD gives a small contribution to the $W^{\pm}$ and Z^0 masses.

For our models to be perturbatively valid, all Yukawa couplings at the electroweak scale must be not much greater than 1 and consequently none of the fermions can have a mass much greater than the electroweak scale. This means that we would expect the thresholds for including the new fermions into the RGEs to be approximately at the electroweak scale. However, we will sometimes take a somewhat higher threshold scale for all the new fermions when checking to see if a model could be perturbatively valid up to the Planck scale. For example, we can calculate the running gauge coupling constants, assuming that all the new fermions can be included in the RGEs at the TeV scale. Thus we can check to see if any gauge coupling constant becomes infinite below the Planck scale (i.e.

if there are any Landau poles, especially for the $U(1)$ coupling). If the threshold was lower then the new fermions would affect the coupling constants even more but this would only be a small effect. Obviously we do not want the coupling constants to become infinite or the theory will be inconsistent. When we do this we find that there are few self-consistent models allowed by our assumptions, in the sense that for any particular gauge group only a few combinations of fermions which cancel the anomalies do not cause the $U(1)$ gauge coupling to diverge.

We will show that in the model with gauge group G_5 we can add new fermions with masses accessible to present or planned future accelerators, in particular a fourth generation of quarks without any new leptons. Although the model is consistent and can be tested experimentally in the near future, it is not called for theoretically and does not resolve any of the outstanding problems of the SM. Nevertheless it is the simplest alternative to the SM which has the same characteristic properties as the SM itself.

1.5 Outline of thesis

In chapter 2 we will discuss our method of constructing models and describe in detail the types of models our method leads us to consider. We shall compare these models to some alternatives and try to justify our approach in comparison to these others. We will also compare our method of choosing the fermion content of our models to previous methods of deriving the SM generation of quarks and leptons. We shall see that our methods provide a consistent derivation of the SM generation without introducing any phenomenological arguments.

In chapter 3 we shall discuss the experimental constraints which arise from the consistency of the SM with experiments. This includes the experimental limits on the mass of the top quark and the masses of new, undetected fermions. We will show that current experimental limits provide lower limits for new quark masses. We will also show that no more massless fermions are allowed in our models and so we are justified in using the simplification of the anomaly constraints when all fermions get a mass via the SM Higgs mechanism, derived in section 2.4. We will also consider the technicolour-like effects of fermion condensates in new non-abelian groups and the resultant reduction of the SM Higgs VEV relative to its SM value. This will lead to a corresponding reduction in fermion masses. Finally we will discuss the constraints imposed by the effects of loop corrections in the electroweak theory. The deviations from tree-level relations are measured by precision electroweak measurements and can be parameterised in such a way that we can derive some simple constraints on the number of new fermion $SU(2)$ doublets in our models.

In chapter 4 we shall show the difficulty of constructing a model where all the new fermions are in 5-plet or anti-5-plet representations of $SU(5)$. We will begin by considering the general problem of producing an anomaly-free set of fermions when we make no simplifying assumption about the fermions getting a mass via the SM Higgs mechanism. Then we shall consider the simpler case where we find anomaly-free sets of massive fermions but cannot satisfy the condition that there is no $U(1)$ Landau pole below the Planck scale.

In chapter 5 we will see how the difficulties of chapter 4 can be overcome by also adding fermions which are $SU(5)$ singlets; in particular a fourth generation of quarks but no fourth generation of leptons. We will also show how such a solution can be formulated in a more general gauge group. Once we have produced an acceptable model we then investigate in detail how well it agrees with the precision electroweak data and experimental limits on quark masses. We show that the model can be chosen in such a way that it is acceptable using current experimental data but that it is very close to current limits and will soon be confirmed or rejected by new data.

In chapter 6 we will discuss a different type of model. In some ways this model is similar to the others discussed in this thesis. However, the type of group is quite different and cannot really be claimed to be suggested by the SM. We will still have a charge quantisation rule (in fact 4 rules) but we will not introduce any more low mass fermions. The gauge group is an extension of the SMG but in this model the SMG is a diagonal subgroup of the full group. The SM fermion Yukawa couplings to the SM Higgs boson are viewed as effective couplings in a low energy effective theory. The fundamental Yukawa couplings to the Higgs bosons responsible for the symmetry breaking down to the SMG are assumed to be of order 1. The details of this symmetry breaking are parameterised and we vary the parameters to obtain the best order of magnitude fit to the SM fermion masses and mixing angles. We compare our results to a very similar model.

In chapter 7 we shall sum up the results of this thesis and discuss the overall merits of such models.

Chapter 2

Building a Consistent Extension of the Standard Model

2.1 Types of Extensions

There are many ways to extend the SM so the first step is to decide what type of extensions to consider. To do this we have decided to use the SM itself as a guide. By this we mean that we shall only consider extensions with features similar to the SM. This does, of course, rule out many popular models. Some of the models ruled out by our approach are; GUTs, SUSY models and any model which includes quantum gravity. GUTs are ruled out because the idea of coupling constant unification is not directly suggested by the SM. It is true that the gauge coupling constants almost converge at an energy of approximately 10^{15} GeV but it is now known that they do not meet exactly. It can be argued that this is a sign that there is unification and the reason it is not apparent is that the SM and simple GUTs are not correct and so do not give the correct RGEs. Indeed it is now known that SUSY GUTs can allow unification consistent with the current experimental measurements of the gauge coupling constants.

However, we do not consider SUSY theories because there is no evidence for such theories, either from experiment or the structure of the SM itself. Introducing SUSY to explain the gauge hierarchy problem (why the electroweak scale is so much smaller than expected radiative corrections from a fundamental theory at the GUT or Planck scale) or to allow coupling constant unification is not justified in our approach of using known features of the SM which are apparent at low energies (energies accessible to current accelerators).

Models involving quantum gravity must be considered at some stage since it is widely accepted that the existence of classical relativity requires the existence of a fundamental quantum theory of gravity. Superstring theory is the current candidate for a theory that combines quantum field theory and quantum gravity in a consistent way. However, so far no-one has managed to solve the theory to predict physics below the Planck scale. So even if this theory is accepted, there are still many possibilities for models below the Planck scale. So our approach is to start from the SM and try to extend this accepted model to other possible models below the Planck scale. We will assume that our models are valid up to the Planck scale and that some fundamental theory such as string theory will then unify the model with quantum gravity.

In this chapter we will discuss in detail the type of extensions we do consider and try to justify our method. Then we shall discuss how our extensions fit in with the theoretical constraints of anomaly cancellation. Finally we shall discuss how our methods can be used to reproduce the generation of quarks and leptons within the SM itself. This can be seen as a check that our methods are consistent with the idea of using only fundamental features of the SM.

2.2 Extrapolations From the SM

In this section we discuss aesthetic extrapolations from the SM. These are features of the SM which have no obvious explanation but in some way can be used to specify the model almost uniquely. We try to pick out these features and carry them over to or generalise them in our extended models. This is a method of selecting a particular type of model and our view is that this is the most logical method although the features chosen may of course be subject to personal prejudice.

2.2.1 Extending the Gauge Group and Charge Quantisation

As stated in section 1.4, an obvious way of extending the SM is to extend the gauge group. The SMG is:

$$SMG \equiv U(1) \otimes SU(2) \otimes SU(3)/\hat{D}_3 \tag{2.2.1}$$

where the discrete group

$$\hat{D}_3 \equiv \{(e^{i2\pi/6}, -I_2, e^{i2\pi/3}I_3)^n : n \in \mathcal{Z}_6\} \tag{2.2.2}$$

ensures the quantisation rule, eq. (1.4.67). We believe that the most obvious extension is to add more special unitary groups to the sequence $U(1) \otimes SU(2) \otimes SU(3)$ and to use a different discrete group so that the quantisation rule above is generalised. In [7] it is argued that the group should be of the form

$$G_p \equiv U(1) \otimes SU(2) \otimes SU(3) \otimes SU(5) \otimes \cdots \otimes SU(p)/\hat{D}_p \tag{2.2.3}$$

where the product is over all $SU(q)$ where q is a prime number less than or equal to the prime number p. The discrete group $\hat{D}_p$ is defined as

$$\hat{D}_p \equiv \{(e^{i2\pi/N_p}, -I_2, e^{i2\pi/3}I_3, e^{i2\pi m_5/5}I_5, \ldots, e^{i2\pi m_p/p}I_p)^n : n \in \mathcal{Z}_{N_p}\} \tag{2.2.4}$$

where $N_p = 2 \times 3 \times 5 \times \cdots \times p$ and m_N is an integer which is not a multiple of N. In fact we can obviously choose $0 \leq m_N \leq N-1$ since m_N is really only defined modulo N. We also have the freedom to choose that there are, for example, at least as many $SU(2)$ doublets which are $\mathbf{N}$ representations of $SU(N)$ as $\overline{\mathbf{N}}$

representations since we can conjugate $SU(N)$ and set $m_N \to -m_N \pmod N$. We will use this fact later to eliminate duplicate solutions where all N-plets and anti-N-plets have been interchanged. This also allows us to fix $m_3 = 1$ rather than 2 as in the SM.

The group $\hat{D}_p$ gives a generalised quantisation rule:

$$\begin{aligned} &\frac{y}{2} + \frac{1}{2}\text{"duality"} + \frac{1}{3}\text{"triality"} + \\ \frac{m_5}{5}\text{"quintality"} + \cdots + &\frac{\mathrm{m_p}}{\mathrm{p}}\text{"p} - \text{ality"} \equiv 0 \pmod 1 \end{aligned} \tag{2.2.5}$$

We will also consider the more general groups defined as:

$$SMG_{2N_1N_2\ldots N_k} \equiv U(1) \otimes SU(2) \otimes SU(N_1) \otimes \cdots \otimes SU(N_k)/D_{2N_1\ldots N_k} \tag{2.2.6}$$

where

$$D_{2N_1\ldots N_k} \equiv \{(e^{i2\pi/\hat{N}}, -I_2, e^{i2\pi m_{N_1}/N_1} I_{N_1}, \ldots, e^{i2\pi m_{N_k}/N_k} I_{N_k})^m : m \in \mathcal{Z}_{\hat{N}}\} \tag{2.2.7}$$

Here $\hat{N} = 2 \times N_1 \times \cdots \times N_k$ and the N_i are odd and mutually prime (we can obviously assume they are arranged in ascending order). So the charge quantisation rule is:

$$\frac{y}{2} + \frac{1}{2}d + \frac{m_{N_1}}{N_1}n_1 + \cdots + \frac{m_{N_k}}{N_k}n_k \equiv 0 \pmod 1 \tag{2.2.8}$$

where we have defined d to be the duality and n_i to be the N_i-ality of a representation. The groups SMG_{23N} are the minimal extensions of the SMG ($\equiv SMG_{23}$) which are inspired by the SMG, in the sense that each is also a cross product of $U(1)$ and a set of distinct special unitary groups, with a charge quantisation rule involving all the direct factors, and contains the SMG as a subgroup.

It has been suggested that a defining property of the SMG is that it has few outer automorphisms relative to the rank of the group [10]. This can be described by saying that it is very skew. If we accept this principle, which is suggested by random dynamics [7], then the groups $SMG_{2N_1N_2\ldots N_k}$ are naturally suggested as alternatives to the SMG. In particular, the requirement that all the N_i be mutually prime and the definition of the discrete group $D_{2N_1N_2...N_k}$ follow from this principle. In fact, the stronger requirement that all N_i should be prime is suggested [11]. Alternatively we can derive eq. (2.2.8) directly as a natural generalisation of the SM charge quantisation rule, eq. (1.4.67).

We can also consider the charge quantisation rules and the condition that the N_i should be mutually prime to be suggested by the SM charge quantisation rule. This is because the generalised quantisation rule shares the property with the SM rule, eq. (1.4.67), that a given allowed value of $\frac{y}{2}$ implies a unique combination of N-alities: (duality, triality, ..., N_i-ality, ...) [1]. This is true provided we also make the assumption about small representations which we discuss in the next section.

Of course it is possible that the apparent charge quantisation rule in the SM is simply due to chance; i.e. the fermions in the SM just happen to obey that particular rule. However we believe that the quantisation rule is a fundamental feature of the SM; so we argue that it is very difficult to see how there cannot be a generalisation of this rule in an extended model, while still retaining the general features of the SM. In fact the form of the generalised charge quantisation rule is suggested from the SM and there seems to be little choice in selecting the rule since the SM rule appears to be the one which involves all the direct factors equivalently. It could even be argued that the choice of the most complicated charge quantisation rule in some way defines the SMG. This is why we have divided out the discrete groups $\hat{D}_p$ and $D_{2N_1...N_k}$.

2.2.2 Small Representations

In the SM, for each $SU(N)$ group, the fermion representations are either N-plet $(\mathbf{N})$, anti-N-plet $(\overline{\mathbf{N}})$ or singlet $(\mathbf{1})$. This can be described by saying that all the

[1]This corresponds to the global group, associated with the generalised charge quantisation rule, having a connected centre [7].

fermions lie in fundamental representations of each $SU(N)$ group to which they couple. We pick this as a feature of the SM which we shall extend to our models. We note here that this is in contrast to some other attempts to extend the SM. For example in SUSY there are fermions in other representations (e.g. gauginos in adjoint representations). Fundamental representations are also suggested in [12] since these make the Weyl equation most stable when considering random dynamics [2].

Another feature is that the weak hypercharge is in some way minimised in the SM, subject of course to the constraints of anomaly cancellation and charge quantisation, as we shall show in section 2.5. So in our extended model we will choose hypercharge values close to zero whenever possible. More precisely, we choose to minimise the sum of weak hypercharges squared over all fermions. This will also minimise the running of the $U(1)$ gauge coupling constant and so give each model the best chance of being consistent up to the Planck scale, which we require as stated in section 2.2.3.

2.2.3 Higher Energies - Desert Hypothesis

The SM has been tested at energies up to a few hundred GeV. There have been many theories proposed which would be valid at energy scales ranging from 1 TeV up to the Planck scale around 10^{16} TeV. Many of these theories have a large range of energy where no new physics occurs. One example is GUTs where there is typically no new physics from the SM energy scale up to the grand unification scale around 10^{12} TeV. An alternative is that there is no new physics until the Planck scale where we can be almost certain that quantum gravity will have a

[2]In fact, from this point of view, each representation of the full gauge group should only be non-singlet with respect to one non-Abelian factor. This is not true for the left-handed quarks but is true for all other fermions in the SM. However the left-handed quarks are required in order that there are no gauge anomalies. So we can consider that the Weyl equation is as stable as possible if we only have small representations.

significant effect. We shall adopt this view for our extended models. This means that once we have set the mass scale for the fermions in the extended model, we can calculate the running coupling constants and check to see if there is a Landau pole below the Planck scale, i.e. whether the $U(1)$ gauge coupling becomes infinite below the Planck scale. If there is a Landau pole then we will conclude that such a model is not consistent.

2.3 Alternative Groups

In this section we shall describe some alternative extensions of the SM. We will consider groups similar to those we are examining in the main part of this thesis in the sense that they contain the SMG and additional special unitary group factors. This obviously does not include models which unify the individual components of the SMG or models which involve SUSY. There have been many such models and the additional symmetries are usually used to explain; coupling constant unification, the number of families in the SM, or the fermion mass hierarchy, in a fairly natural way.

In the models described in section 2.2.1 the SM fermions cannot couple to any new gauge fields because of the charge quantisation rule. This is due to the fact that all values of $\frac{y}{2}$ in the SM are multiples of $\frac{1}{6}$ and so the charge quantisation rule, eq. (2.2.8), forces the SM fermions to be singlets of all $SU(N)$ groups where $N > 3$ due to our assumption about small representations.

However the situation is more complicated if we allow more than one $SU(N)$ gauge group for any particular N. Where we have $N = 2$ or 3 there are two distinct cases. In the first case the SM group $SU(N)$ is an invariant subgroup of the extended group. We then call the extra $SU(N)$ groups a horizontal symmetry. In the other case the $SU(N)$ group in the SMG is not an invariant subgroup and is generally a diagonal subgroup of the extended group.

2.3.1 Invariant Subgroup Case: Horizontal Symmetries

If we have one more $SU(2)$ or $SU(3)$ group then we can have a horizontal symmetry (a non-abelian symmetry which places fermions from different generations in the same multiplet). The idea of a gauged horizontal symmetry is not new and has been used to try and explain the mass hierarchy of the SM fermions [13]. However, an $SU(N)$ group with $N > 3$ is not a possible horizontal symmetry without introducing many more fermions because there are only 3 generations of SM fermions and the smallest non-trivial representation of $SU(N)$ is the N-plet. For example if $N = 5$ we would have an $SU(5)$ horizontal symmetry and so we would need at least 5 generations of SM fermions. Even with 5 generations this would not fit into our type of models since the SM fermions could not then obey the charge quantisation rule,

$$\frac{y}{2} + \frac{d}{2} + \frac{t}{3} + m_5 \frac{q}{5} \equiv 0 \pmod 1 \tag{2.3.9}$$

where $q \equiv$ quintality $\equiv$ 5-ality. Therefore we will only consider $SU(3)$ and $SU(2)$ groups as candidates for a horizontal symmetry.

If the horizontal symmetry gauge group is $SU(3)_H$ then we must place fermions from different generations in the same triplet (or anti-triplet). It turns out that the only way to do this, avoiding anomalies (see section 1.1) and not introducing any new fermions, is to put all fermions in the same (or conjugate) representation of $SU(3)_H$ as they are in the colour group, $SU(3)_C$, of the SM; so that all three generations of left-handed quarks are put in a triplet (or anti-triplet) of $SU(3)_H$ etc. However, the SM fermions would not then obey the charge quantisation rule which might be expected, similar to eq. (2.2.8):

$$\frac{y}{2} + \frac{1}{2}d + \frac{1}{3}t_C + \frac{1}{3}t_H \equiv 0 \pmod 1 \tag{2.3.10}$$

If the horizontal symmetry group is $SU(2)_H$ then we can make some or all SM fermions triplets of $SU(2)_H$. This would allow the fermions to satisfy the charge quantisation rule but triplets are not the smallest representations of $SU(2)$

and so we do not favour this as explained in section 2.2.2. We could place some fermions in doublets of $SU(2)_H$. This could be done, without introducing any anomalies, by placing two generations of quarks in the same doublet or taking two generations and placing the fermions in the same representation of $SU(2)_H$ as they are in the electroweak group $SU(2)_L$. Different doublets could connect fermions from a different pair of generations. For example left-handed quarks from the first and second generations could be in the same doublet, right-handed 'up' quarks from the first and third generations could be in the same doublet and right-handed 'down' quarks from the second and third generations could be in the same doublet. This would not give any anomalies though it is difficult to see how this could be used to explain the fermion masses.

The main problem with these types of models is that fermions in different generations with very different masses are put in the same multiplet. This means that the fermions would naturally get the same mass. It is difficult to break the symmetry in such a way that the masses of all the different fermions are split by realistic amounts [13].

To sum up, we do not consider these possibilities in this thesis because triplets of $SU(2)$ are not fundamental representations and the other possibilities, with fermions in fundamental representations of the gauge groups $SU(2)_H$ or $SU(3)_H$, mean that the fermions could not obey the extended charge quantisation rule. Of course models involving horizontal symmetries do not enforce such charge quantisation rules or require small representations of $SU(2)_H$.

2.3.2 Non-invariant Subgroup Case: SMG as Diagonal Subgroup

In the case where, for example, the $SU(3)_C$ subgroup of the SMG is not an invariant subgroup of the full gauge group, the only possibility is that it is a diagonal (or anti-diagonal) subgroup of $SU(3)^n$ for some integer n. In this type of model different generations can couple to different $SU(2)$ and $SU(3)$ gauge groups in the full gauge group. There would then be symmetry breaking to produce the SMG in such a way that $SU(3)_C$ could be said to be a diagonal subgroup of all the $SU(3)$ groups in the full group which exists at energies higher than the symmetry breaking scale. In other words, $SU(3)_C$ is then the subgroup in which all the $SU(3)$ groups undergo the same transformations. In this way it is trivial to cancel all the anomalies since each generation of quarks and leptons cancel all anomalies separately and couple to a $U(1) \otimes SU(2) \otimes SU(3)$ subgroup of the full group in the same way as they couple to the SMG. This is in contrast to the invariant subgroup case where the SM fermions had to couple to the SMG and also to other subgroups of the full gauge group. Also, in the diagonal case, the dimension of each representation is the same as in the SM, whereas, in the invariant subgroup case, the dimensions were larger since different SM representations were combined under the horizontal symmetry.

This type of model has been proposed [14] as an alternative to horizontal symmetries or grand unification. Examples include topcolour models [15] and the anti-grand unification model [16], where the group $SMG^3 \equiv SMG \otimes SMG \otimes SMG$ has been used to successfully predict the values of the gauge coupling constants. The anti-grand unification model has also been analysed as a model to explain the hierarchy of SM fermion masses [17]. Here the extended model with gauge group $SMG^3 \otimes U(1)_f$ has been fairly successful at reproducing the observed fermion masses in an order of magnitude approximation (reproducing all SM fermion masses within a factor of 2 or 3). The extra $U(1)_f$ gauge symmetry is called a flavour symmetry and is required to produce the observed mass differences within the second and third generations, e.g. $m_b \ll m_t$.

We note that the fermions in some of these models obey the extended charge quantisation rules which we would expect. For example the fermions in the SMG^3

model obey the charge quantisation rules:

$$\frac{y_i}{2} + \frac{1}{2}d_i + \frac{1}{3}t_i \equiv 0 \pmod 1 \tag{2.3.11}$$

where the three copies of the SMG are labelled by $i = 1$, 2 and 3. With three separate charge quantisation rules, this is not truly a straightforward extrapolation of the SM charge quantisation rule. However it is similar in the sense that these rules are required to produce the group SMG^3 which has as large a value of χ [3] as the SMG itself. The quantity χ measures how strongly intermingled the $U(1)$ subgroups are with the semi-simple part via dividing discrete groups (i.e equivalently via the quantisation rule(s)). It happens that groups of the form SMG^n have the largest possible value of this measure; $\chi = log(6)/4$. The charge quantisation rules:

$$\begin{aligned} \frac{y_i}{2} + \frac{1}{2}d_i + \frac{1}{3}t_i &\equiv 0 \pmod 1 \quad (2.3.12) \\ y_f &\equiv 0 \pmod 1 \quad (2.3.13) \end{aligned}$$

are chosen to maximise χ for the group $SMG^3 \otimes U(1)_f$ among all those with the same algebra although this group does not have as large a value of χ as the SMG. In fact $\chi = \ln(6^3)/13 = \frac{12}{13}\ln(6)/4$ for the group $SMG^3 \otimes U(1)_f$.

However, the symmetry breaking scale of the group SMG^3 is taken to be just below the Planck scale in the anti-grand unification model and in most of this thesis we wish to study the possibilities of new physics at much lower energies; energies of the same order of magnitude as the electroweak scale rather than the Planck scale. This is still possible in such a model but it then loses its ability to predict the gauge coupling constants. Topcolour models do introduce new dynamics at the TeV scale but in this thesis we shall not consider such models.

[3]The quantity χ is defined in [11] for any group G as $\chi(G) = \ln(q(G))/r(G)$ where $r(G)$ is the rank of the group G. Further, $q(G)$ is defined as the order of the factor group, obtained by dividing the group of all abelian charge combinations $(y_1, y_2, \ldots, y_r)$ allowed for any representations of the group G, by the group of those abelian charge combinations allowed for representations trivial under the semi-simple part of the group G.

We will discuss the model with gauge group $SMG^3 \otimes U(1)_f$ in chapter 6 where we will use it to try to explain the fermion masses and mixing angles in the SM. However, this is only a small part of this thesis so we will continue with our main approach where we look for models with new low mass fermions.

2.4 Fermion Mass and Anomaly Cancellation

In the SM fermions get a mass via the SM Higgs mechanism. To do this in a general gauge group of the form

$$U(1) \otimes SU(2) \otimes G/D$$

where G is any Lie group and D is a discrete group, a left-handed fermion representation $(y, \mathbf{2}, \mathbf{R})$ should occur together with the left-handed anti-fermion representations $(-[y+1], \mathbf{1}, \overline{\mathbf{R}})$ and $(-[y-1], \mathbf{1}, \overline{\mathbf{R}})$. We shall refer to this as the mass grouping $\{y, \mathbf{R}\}$ where $\mathbf{R}$ is a representation of G (irreducible in our models). As explained in section 3.1.1 we assume that all fermions in our models, other than the leptons which have already been observed, get a mass by this mechanism. We shall now describe what consequences this has for anomaly cancellation in our models, where G is a product of $SU(N_i)$ groups with $N_i \geq 3$.

We consider the grouping $\{y, \mathbf{R}\}$ for the gauge group

$$U(1) \otimes SU(2) \otimes \prod_{i=1}^{k} SU(N_i)/D_{2N_1 \ldots N_k}$$

where the irreducible representation $\mathbf{R}$ is made up of fundamental ($\mathbf{N}_i$ or $\overline{\mathbf{N}}_i$) or singlet representations of each factor $SU(N_i)$. The contribution to each type of anomaly from this grouping, $\{y, \mathbf{R}\}$, is easily calculated, using the results of section 1.1, to be as follows.

$$
\begin{array}{llcl}
[SU(N_i)]^3 & \rightarrow \; 2S_R n + S_R(-n) + S_R(-n) & = & 0 \\
[SU(N_i)]^2\, U(1) & \rightarrow \; 2S_R n^2 y - S_R n^2 (y+1) - S_R n^2 (y-1) & = & 0 \\
[Grav]^2\, U(1) & \rightarrow \; 2S_R y + S_R(-y-1) + S_R(-y+1) & = & 0 \\
[U(1)]^3 & \rightarrow \; 2S_R y^3 + S_R(-y-1)^3 + S_R(-y+1)^3 & = & -6S_R y \\
[SU(2)]^2\, U(1) & \rightarrow & & 2S_R y
\end{array}
$$

Here n_i is the N_i-ality of the representation $\mathbf{R}$ and S_R is its dimension (size).

So we can see that the above grouping which is necessary to give a mass to the fermions also simplifies the anomaly constraints. In particular, if we take all fermions to be grouped in this way then we are only left with the single constraint for the absence of the mixed gauge-gravitational and gauge anomalies

$$\sum_j S_j y_j = 0 \tag{2.4.14}$$

where j labels each grouping $\{y_j, \mathbf{R}_j\}$.

There will also be no Witten anomaly, since we must have an even number of $SU(2)$ doublets to satisfy eq. (2.4.14). This follows from the charge quantisation rule (2.2.8), the fact that N_i are all odd and the assumption of fundamental or singlet representations for each $SU(N_i)$ subgroup. Using the charge quantisation rule and defining

$$\frac{e_j}{d_j} = \sum_{i=1}^{k} \frac{m_{N_i}}{N_i}(n_i)_j \tag{2.4.15}$$

we can write

$$\frac{y_j}{2} = c_j + \frac{1}{2} + \frac{e_j}{d_j} \tag{2.4.16}$$

where c_j, d_j and e_j are integers and d_j are odd. Therefore, since eq. (2.4.14) can be written as $\sum_j S_j \frac{y_j}{2} = 0$, we must have $\sum_j S_j \frac{1}{2} \equiv 0 \pmod 1$. In other words $\sum_j S_j \equiv 0 \pmod 2$, which means that there are an even number of $SU(2)$ doublets and so no Witten anomaly.

2.5 Deriving the SM Generation

In this section we shall first give a short description of the SM quarks and leptons which form all the known elementary fermions. We shall then show how the application of anomaly cancellation and some other assumptions can be used to derive a SM generation of quarks and leptons without making any specific assumptions about the fermion representations in the SM gauge group. We can derive not only the non-abelian representations but also the abelian representations (weak hypercharge) by using a charge quantisation rule. This is one of the reasons we consider the charge quantisation rule to be a fundamental feature of the SM and so justify generalising it in our extended models. Also, the assumptions required to derive the SM generation are used to derive the properties of fermions in our extended models.

Finally we compare our derivation to alternative methods of deriving the SM generation. The main difference is our use of the charge quantisation rule as a fundamental property of the gauge group rather than simply a consequence of the SM fermion representations. We consider this to greatly simplify the other assumptions needed to derive the SM generation.

2.5.1 The SM Generation

In the SM there are 3 generations of fermions which are identical except for their masses. Each generation consists of 15 Weyl fermions and can be divided into a lepton generation and a quark generation. The quarks couple to the $SU(3)$ gauge group whereas the leptons are $SU(3)$ singlets and so do not 'feel' the strong force. The properties of these fermions are shown in table 2.1. The fermions are labelled as in the first (lightest) generation.

Table 2.1: The lightest SM generation.

Generation	Fermion Label	Representation of $SU(2) \otimes SU(3)$	Representation of U(1), $\frac{Y}{2}$	Electric Charge Q
Quark	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\mathbf{2}, \mathbf{3}$	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
	$\overline{u}_L$	$\mathbf{1}, \overline{\mathbf{3}}$	$-\frac{2}{3}$	$-\frac{2}{3}$
	$\overline{d}_L$	$\mathbf{1}, \overline{\mathbf{3}}$	$\frac{1}{3}$	$\frac{1}{3}$
Lepton	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\mathbf{2}, \mathbf{1}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	$\overline{e}_L$	$\mathbf{1}, \mathbf{1}$	1	1

The quark generation is formed by the representations $(\frac{1}{3}, \mathbf{2}, \mathbf{3})_L$, $(-\frac{4}{3}, \mathbf{1}, \overline{\mathbf{3}})_L$ and $(\frac{2}{3}, \mathbf{1}, \overline{\mathbf{3}})_L$ of the gauge group $U(1) \otimes SU(2) \otimes SU(3)$. This is precisely the mass grouping $\{\frac{1}{3}, \mathbf{3}\}$ (where the representation $\mathbf{3}$ is of the gauge group $SU(3)$) described in section 2.4. All the quarks get a mass by the Higgs mechanism. The lepton generation is formed by the representations $(-1, \mathbf{2}, \mathbf{1})_L$ and $(2, \mathbf{1}, \mathbf{1})_L$ of the same gauge group. However, this is not the same as the mass grouping $\{-1, \mathbf{1}\}$ because there is no right-handed neutrino (no anti-neutrino representation $(0, \mathbf{1}, \mathbf{1})_L$) in the SM. This means that the neutrino is massless in the SM but the electron can still get a mass by the Higgs mechanism. However, the lepton generation gives the same contribution to all anomalies as the mass grouping $\{-1, \mathbf{1}\}$ would, since the right-handed neutrino would be totally neutral (i.e. would not interact with any gauge fields).

2.5.2 Derivation of the SM Generation

In fact, we can derive the SM generation using the following assumptions:
(i) *The SM gauge group*: $SMG \equiv S(U(2) \otimes U(3))$. This includes the charge quantisation rule eq. (1.4.67).
(ii) *Mass protection*: This means that we cannot have left- and right-handed fermions with the same representation of the SMG. Also we cannot have a right-handed neutrino since it can get a Majorana mass.
(iii) *Anomaly cancellation*: In addition to the cancellation of gauge anomalies, the Witten global $SU(2)$ anomaly and the mixed gauge and gravitational anomaly must also be absent.
(iv) *Small representations*: This means (c.f. section 2.2.2) that all fermions are in either fundamental or singlet representations of the $SU(2)$ and $SU(3)$ subgroups and the sum of weak hypercharge squared for all fermions is as small as possible.

So our aim is to minimise the value of $\sum_i S_i \left(\frac{y_i}{2}\right)^2$ (where S_i is the dimension of representation i with weak hypercharge y_i) for all possible choices of mass protected fermions in fundamental or singlet representations of $SU(2)$ and $SU(3)$, assuming the charge quantisation rule, eq. (1.4.67), and cancelling all relevant anomalies. We note that for one SM generation (which satisfies assumptions (i) to (iii))

$$\sum_i S_i \left(\frac{y_i}{2}\right)^2 = \frac{10}{3} \tag{2.5.17}$$

and we show that there is no other mass protected solution of the anomaly constraints with

$$\sum_i S_i \left(\frac{y_i}{2}\right)^2 \leq \frac{10}{3} \tag{2.5.18}$$

So we shall prove that one SM generation also satisfies assumption (iv) and thus we will show that assumptions (i) to (iv) define the SM generation. Note that

Table 2.2: Contributions of $S\left(\frac{y}{2}\right)^2$ for all fundamental and singlet representations of $SU(2)$ and $SU(3)$ for any value of weak hypercharge which satisfies eq. (1.4.67). All 'N's are integers so that the charge quantisation rule, eq. (1.4.67), is satisfied. S is the dimension of the non-abelian representation.

Type	Representation of $SU(2) \otimes SU(3)$	$\frac{y}{2}$	$S\left(\frac{y}{2}\right)^2$
a	$\mathbf{2},\mathbf{3}$	$N_a + \frac{1}{6}$	$6N_a^2 + 2N_a + \frac{1}{6}$
b	$\mathbf{2},\overline{\mathbf{3}}$	$N_b - \frac{1}{6}$	$6N_b^2 - 2N_b + \frac{1}{6}$
c	$\mathbf{1},\mathbf{3}$	$N_c - \frac{1}{3}$	$3N_c^2 - 2N_c + \frac{1}{3}$
d	$\mathbf{1},\overline{\mathbf{3}}$	$N_d + \frac{1}{3}$	$3N_d^2 + 2N_d + \frac{1}{3}$
e	$\mathbf{2},\mathbf{1}$	$N_e - \frac{1}{2}$	$2N_e^2 - 2N_e + \frac{1}{2}$
f	$\mathbf{1},\mathbf{1}$	N_f	N_f^2

in order to satisfy assumption (iv) we must satisfy eq. (2.5.18) [4]. So in the following analysis we will implicitly assume eq. (2.5.18). Table 2.2 shows all allowed representations and their contribution of $S\left(\frac{y}{2}\right)^2$.

[4]This requirement of small values of weak hypercharge is different from other approaches where the aim is usually to find the minimum number of fermions. The charge quantisation rule means that all fermions will have non-zero weak hypercharge (we don't consider right-handed neutrinos) and so the solution will not have a large number of fermions, but it will not necessarily be the minimum number. When considering groups with $SU(N)$ where $N > 5$ we will look for the minimum number of fermions since this is simpler and will usually produce the solution with minimum sum of weak hypercharges squared.

In order to satisfy eq. (2.5.18) we must choose $N_a = N_b = 0$, $N_c \in \{0,1\}$, $N_d \in \{-1,0\}$, $N_e \in \{0,1\}$ and $N_f \in \{-1,1\}$. (We don't consider $N_f = 0$ because this would be a right-handed neutrino which would not contribute to any anomalies and would be expected to get a Majorana mass of the order of the Planck mass). This means that we cannot have mass protected fermions of both types a and b. So we can choose, without loss of generality, that there are no fermions of type b [5]. So we get table 2.3 which shows all allowed fermions and contributions to some anomalies.

For mass protection we cannot have any of the following combinations; types c_1 and d_1, types c_2 and d_2, types e_1 and e_2, or types f_1 and f_2 (all defined in table 2.3). Also note that all the types of representations in table 2.3 contribute to the mixed anomaly, $\sum_i S_i y_i$. This means that we cannot use only type f fermions to produce an anomaly-free set of mass protected fermions. Therefore, if no fermions couple to the $SU(3)$ group, we would require some fermions of either type e_1 or e_2. But then there would be no way to cancel the $[SU(2)]^2U(1)$ anomaly. So we can conclude that some fermions must couple to $SU(3)$.

Suppose there are no fermions of type a. Then the above arguments mean that, to cancel the $[SU(3)]^3$ anomaly, we must have equal numbers of either types c_1 and d_2 or types c_2 and d_1. But then there is no way to cancel the $[SU(3)]^2U(1)$ anomaly. So we have a contradiction which, means that there must be at least one type a.

The $[SU(2)]^2U(1)$ anomaly must be cancelled by having as many type e_1 as type a. So there are no type e_2 due to the principle of mass protection. Again using the principle of mass protection, the only way to cancel the $[SU(3)]^3$ and $[SU(3)]^2U(1)$ anomalies is by having the number of types a, d_1 and d_2 the same. So we can now cancel the $[U(1)]^3$ and mixed anomalies using table 2.4.

So we can see that the anomaly-free set of mass protected fermions which

[5] Choosing no fermions of type a would lead to an equivalent solution with opposite chirality.

Table 2.3: All allowed representations of fermions which could be used to satisfy eq. (2.5.18) and their contributions to some anomalies.

Type	Representation of $SU(2) \otimes SU(3)$	$\frac{y}{2}$	$S\left(\frac{y}{2}\right)^2$	$[SU(3)]^3$	$[SU(3)]^2U(1)$	$[SU(2)]^2U(1)$
a	$\mathbf{2},\mathbf{3}$	$\frac{1}{6}$	$\frac{1}{6}$	2	$\frac{1}{3}$	$\frac{1}{2}$
c_1	$\mathbf{1},\mathbf{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$	0
c_2	$\mathbf{1},\mathbf{3}$	$\frac{2}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	0
d_1	$\mathbf{1},\overline{\mathbf{3}}$	$\frac{1}{3}$	$\frac{1}{3}$	-1	$\frac{1}{3}$	0
d_2	$\mathbf{1},\overline{\mathbf{3}}$	$-\frac{2}{3}$	$\frac{4}{3}$	-1	$-\frac{2}{3}$	0
e_1	$\mathbf{2},\mathbf{1}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
e_2	$\mathbf{2},\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
f_1	$\mathbf{1},\mathbf{1}$	-1	1	0	0	0
f_2	$\mathbf{1},\mathbf{1}$	1	1	0	0	0

Table 2.4: Allowed combinations of fermions and their contribution to the remaining anomalies.

Types	$G^2U(1)$	$[U(1)]^3$	$S\left(\frac{y}{2}\right)^2$
$a + d_1 + d_2 + e_1$	$1 + 1 - 2 - 1 = -1$	$\frac{1}{36} + \frac{1}{9} - \frac{8}{9} - \frac{1}{4} = -1$	$\frac{7}{3}$
f_1	-1	-1	1
f_2	1	1	1

minimises the sum of the weak hypercharges squared is one of type a, d_1, d_2, e_1 and f_2. This is one SM quark-lepton generation.

2.5.3 Alternative Derivations of the SM Generation

There have been other attempts to derive the SM generation using various assumptions. Most notably Geng and Marshak [18] have tried to derive the SM generation using the constraints due to cancellation of anomalies. They also assume mass protection but not the charge quantisation rule eq. (1.4.67). Instead of minimising the sum of weak hypercharges squared they try to find the minimum number of fermions required to satisfy these assumptions.

The smallest number of Weyl fermions found by Marshak is 14. This solution consists of the following representations of the gauge group $U(1) \otimes SU(2) \otimes SU(3)$: $(0, \mathbf{2}, \mathbf{3})_L$, $(y, \mathbf{1}, \overline{\mathbf{3}})_L$, $(-y, \mathbf{1}, \overline{\mathbf{3}})_L$ and $(0, \mathbf{2}, \mathbf{1})_L$. Geng and Marshak rule out this solution because the $SU(2)$ doublet cannot acquire a Dirac or Majorana mass, even with the spontaneous symmetry breaking of the gauge group. However, we know from the SM that the neutrino is massless and so there doesn't appear to be any reason why massless fermions should be excluded from such an analysis. (We could obviously use phenomenological arguments but that would defeat the purpose of trying to derive the SM generation). They also object to this solution because they feel it trivialises the cancellation of the mixed gravitational and gauge anomaly. In what sense the anomaly condition is trivial is not entirely clear since not all fermions have zero weak hypercharge; but also why should it matter if a constraint is trivially satisfied? In our derivation this solution does not occur because of the charge quantisation rule. So by enforcing the charge quantisation rule, which we have taken as one of the defining properties of the SMG in section 2.2.1, we can avoid this solution without introducing dubious arguments about fermion masses or not allowing 'trivial' cancellation of anomalies.

So, if we add the assumption of the charge quantisation rule, we would expect to find that the SM generation is the smallest possible number of Weyl fermions. However, there are smaller solutions which have not been considered by Geng and Marshak. These solutions have 12 Weyl fermions and do not couple to the $SU(3)$ subgroup. The fermions belong to 6 $SU(2)$ doublets with values of weak hypercharge given by, for example, $-9, -9, 1, 1, 5, 11$. This set of 12 Weyl fermions has a huge sum of weak hypercharges squared but this could obviously be changed by scaling all the weak hypercharges to smaller values. Without assuming the charge quantisation rule this would be possible. This solution appears to have been ignored because Geng and Marshak implicitly assumed that at least one fermion must couple to each part of the gauge group. However, we wish to find a consistent method of deriving the SM generation and so this must be considered as an additional assumption.

So if we then also add the assumption that all subgroups must have some fermion coupling to them, we can almost derive the SM generation. The problem is that we can scale all values of weak hypercharge for the SM fermions by a factor of $(6n + 1)$ where n is any integer [6]. The SM generation is obviously the

[6]Without the charge quantisation rule we could scale the weak hypercharges by an arbitrary amount. Then we couldn't use the procedure of minimising the sum of hypercharges squared since this would obviously force all values to zero. There is then no way to fix the scale other

solution with the values of hypercharge closest to zero. We can express this by choosing to minimise the sum of hypercharges squared for this solution. But since we must introduce such an assumption why not use it from the start?! This then allows us to drop two of the above assumptions; that all subgroups must have a fermion coupling to them and that we should look for the smallest number of Weyl fermions. We are then left with the four assumptions used in section 2.5.2 which we have already tried to justify in this thesis. This seems more reasonable than introducing more assumptions with no justification.

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than by assuming the fermions get a mass by the Higgs mechanism and fixing the scale to the weak hypercharge of the Higgs boson. So the charge quantisation rule effectively introduces a scale for the weak hypercharge independent of any Higgs bosons.

Chapter 3

Experimental Constraints on New Fermions

In this section we shall discuss the constraints on our models which are due to experimental evidence. In particular we are concerned with the possibilities for the existence of more fermions and what restrictions can be imposed both directly and indirectly on their mass. Some difficulty arises since fermions may be confined and so not directly observable. This means that direct experimental restrictions will refer to the mass of particles which are combinations of these fermions, like hadrons in the case of quarks.

3.1 Experimental Limits on Fermion Masses

First we shall discuss the constraints on fermion masses due to the fact that so far no non-SM fermions have been observed. We shall show that this rules out any extra massless fermions and then give current limits on the masses of different type of new fermions.

3.1.1 Massless Fermions

Only three massless fermions have been observed and they are the three massless neutrinos described in the SM (even if the neutrinos do have a small mass we know that there are only 3 with a mass less than $\frac{1}{2}M_Z$). Any other massless fermions, which had any significant coupling to the SM fermions or gauge bosons, would have been observed if they were not confined. When we assume fermions belong only to fundamental and singlet representations (as postulated in section 2.2.2), the charge quantisation rule in our models ensures that the only possible fermions which would not be electrically charged would be neutrinos. A left-handed neutrino without a right-handed neutrino would be massless as in the SM. We already know that there are only three such neutrinos and so we cannot consider this as a possibility for new fermions. A right-handed neutrino would be completely decoupled from the gauge group and so it could get a gauge invariant Majorana mass. So we would expect that it would have a mass $\sim M_{Planck}$ and so it is excluded as a low mass fermion in our models. Therefore any new massless fermions in our models must be electrically charged and so must also be confined by a new interaction well above the QCD scale, on phenomenological grounds.

If there is a confined gauge group then we assume that fermion condensates will be formed as in QCD. If a fermion doesn't have a chiral partner with respect to some confined group H, the condensates formed will break the group H. So if we assume that there is no spontaneous gauge symmetry breaking, other than that of the electroweak symmetry group, no fermions can be chiral w.r.t. G where the full gauge group is $U(1) \otimes SU(2) \otimes G/D$ (where D is some discrete group). In our models the extra $SU(N)$ gauge groups are all confining (with negative beta functions), so that $G \equiv H$. This leads to the phenomenological requirement that all new fermions with a mass much lower than the fundamental scale (Planck scale) should get a mass via the SM Higgs mechanism.

If the left- and right-handed fermions occur with the same representations of the full gauge group $U(1)\otimes SU(2)\otimes G/D$, then the fermions can form a Dirac mass term in the Lagrangian. So they would be expected to get a mass comparable to the fundamental scale, which we take to be the Planck mass in our models. Such fermions would not contribute to any anomalies and would not be observable because of their high mass. We shall therefore ignore them in our models. If a fermion cannot form such a fundamental Dirac (or Majorana) mass term then we say it is mass protected, since it would be fundamentally massless and could only get a mass indirectly through some interaction such as the Higgs mechanism. All the fermions considered in our models are mass protected by the electroweak interactions.

We conclude that all new fermions in our models must get their mass from the Higgs mechanism. Furthermore, they must couple to the usual SM Higgs particle in the same way as the SM fermions. In other words, the fermion condensates must have the same quantum numbers as the SM Higgs boson; otherwise their contributions to the $W^{\pm}$ and Z^0 masses, via the usual technicolour [9] mechanism, would be analogous to those from the vacuum expectation values of Higgs particles with non-standard weak isospin and hypercharges. This would lead to a significant deviation of the ρ parameter ($\rho \equiv \frac{M_W^2}{M_Z^2 cos^2\theta_W}$) from unity [19] in contradiction with precision electroweak data.

3.1.2 Massive Fermions

In the SM there are two different types of fermions, quarks and leptons, which differ by the fact that quarks couple to the $SU(3)$ gauge fields and so are confined, whereas leptons have no direct coupling to the $SU(3)$ gauge fields and are not confined. There are experimental limits on the masses of any quarks and leptons which have not yet been observed. If there are any more leptons then they must have a mass greater than 45 GeV [5]. We shall assume that there are no more leptons, since even the neutrino would have to get a mass larger than this and it is difficult to see how a neutrino could naturally be given a mass greater than 45 GeV but still much lower than the fundamental scale (which is the Planck scale in our models). This is because a right-handed neutrino, as already discussed in section 3.1.1, would naturally get a Majorana mass and so the see-saw mechanism [20] would leave the left-handed neutrino with a very small mass. For this reason we cannot allow any more generations of SM leptons. However the limits on the quark masses are dependent on the type of quark and its decay modes.

The top quark has recently been observed by the CDF [21] and D0 [22] collaborations. The mass is in the range 150-220 GeV. We will assume that $M_t \approx 170$ GeV, with a lower limit of 160 GeV which agrees well with the CDF analysis which is statistically better than the D0 analysis. For the purpose of this thesis we take the limit on possible fourth generation quarks, t' and b', to be

$$M_{t'}, M_{b'} > 130 \text{ GeV}$$

from the dilepton analyses of the CDF [23] and D0 [24] groups (less restrictive limits apply if other decay modes are dominant). Note that experimental limits are taken to apply to the pole masses for heavy quarks.

The above experimental limits do not apply to new fermions which are not singlets of the additional $SU(N)$ gauge groups. These fermions would be more difficult to detect experimentally and would anyway be confined inside 'hadrons' with a confinement scale (generically at the electroweak scale) much higher than the QCD scale. For this reason we will concentrate on the masses of new SM fermions and not make any precise assumptions about experimental lower limits for fermions coupling to $SU(N)$ subgroups with $N > 3$.

We require our models to remain perturbative in the desert from the TeV scale to the Planck scale. So we can use the RGEs to examine how the Yukawa couplings evolve from the Planck scale down to the electroweak scale. In particular we study the infra-red quasi-fixed-point structure of the RGEs. In the SM the fixed point values provide upper limits on the mass of the top quark, M_t, and the Higgs scalar, M_H. Similarly in extended models we get upper limits on the masses of the heaviest fermions, though the precise values depend on the relative masses of these fermions and also the unknown gauge coupling strength, g_N, of the $SU(N)$ groups to which the fermions couple. Also we must be careful to point out that the RGEs describe the running of the Yukawa couplings and, as we discuss in section 3.2, the actual masses will be less than naively expected, due to the technicolour-like contribution from $SU(N)$ to the electroweak VEV, $v = 246$ GeV. As we shall see, this will enable us to quite accurately predict the masses of some of the fermions we introduce in our model in chapter 5, since we have theoretical upper limits and experimental lower limits.

3.2 Technicolour Contributions

Technicolour theories [9] have been proposed as an alternative to the Higgs mechanism to provide a mass for the weak gauge bosons. This is based on the fact that QCD would provide a (very small) mass for these bosons without any Higgs scalars. Similarly any other confining $SU(N)$ gauge groups, with fermions which are in non-trivial representations of $U(1) \otimes SU(2)$, are expected to form fermion condensates which would contribute to the $W^{\pm}$ and Z^0 masses. In our models the charge quantisation rule ensures that all fermions (except a right-handed neutrino) would be non-trivial under $U(1)$. Thus all $SU(N)$ groups in our models, which are coupled to fermions, will contribute to the weak boson masses.

We stress that we are not proposing a technicolour model as such, but simply taking into account the unavoidable effect that adding an $SU(N)$ group has. We are assuming that the Higgs sector of our models is the same as in the SM, i.e. one Higgs doublet and that the fermion condensates have the same quantum numbers as the Higgs doublet. Then the VEV due to the Higgs field, $< \phi_{WS} >$, is related to the total VEV, v, and the contribution from $SU(N)$ due to fermion condensates,

F_{π_N}, by the relation

$$< \phi_{WS} >^2 + F_{\pi_N}^2 = v^2 = (246.22 \text{ GeV})^2 \tag{3.2.1}$$

which is exactly the same as in technicolour models with a scalar [25].

The fermion running masses, m_f, are related to the Higgs field VEV in the usual way:

$$m_f = \frac{y_f}{\sqrt{2}} < \phi_{WS} > \tag{3.2.2}$$

where y_f is the Yukawa coupling constant for the fermion f (y is used for both Yukawa coupling and weak hypercharge but it should be obvious from the context which is being referred to). The running masses of the SM quarks and general $SU(N)$-"quarks" are related to the pole masses by eqs. (1.3.65) and (1.3.64) respectively. For SM quarks with a mass of order M_Z,

$$M_f \approx 1.05 m_f(M_Z) \tag{3.2.3}$$

where M_f is the pole mass and $m_f(M_Z)$ is the running mass at $\mu = M_Z$.

In order to avoid any significant suppression of the top quark and other fermion masses, due to the reduction of $< \phi_{WS} >$ below its SM value, we usually imagine taking

$$F_{\pi_N} \quad \leq 75 \text{ GeV} \tag{3.2.4}$$

and thus

$$< \phi_{WS} > \quad \geq \quad 234 \text{ GeV} \tag{3.2.5}$$

In fact we shall quote limits on fermion pole masses based on taking,

$$F_{\pi_N} \quad = \quad 75 \text{ GeV} \tag{3.2.6}$$

$$< \phi_{WS} > \quad = \quad 234 \text{ GeV} \tag{3.2.7}$$

This means that we expect the $SU(5)$ gauge group to have a confinement scale above the electroweak scale. By confinement scale we mean the mass of the lightest 'hadrons' other than Goldstone bosons. We can estimate this scale either by scaling QCD or using the estimates of [26]. We would expect the confinement scale to be approximately $5F_{\pi_N} \approx 400$ GeV. In fact [26] suggests that this estimate should be inversely proportional to $\sqrt{N}$ so we have given the estimate for $N = 5$.

Eq. (1.3.65) gives the following relation for the pole mass of quark f:

$$M_f = \left(1 + \frac{4\alpha_S(M_f)}{3\pi}\right) \frac{y_f(M_f)}{\sqrt{2}} < \phi_{WS} > \tag{3.2.8}$$

In the approximation $M_f \approx M_Z$ we get,

$$M_f \approx 174 y_f(M_Z) \text{ GeV} \tag{3.2.9}$$

This gives a quick guide to the value of the pole mass of a quark with mass of order M_Z but we will, of course, use eq. (3.2.8) when quoting the actual values of the pole mass for given values of Yukawa coupling.

For $SU(N)$-"quarks" (with $N > 3$) we will quote the pole masses based on the reduced value of $< \phi_{WS} >$ but this will not make much difference compared to the difference between two choices of $SU(N)$ gauge couplings. Also, we will not be too concerned about the masses of $SU(N)$-"quarks" since they will be confined and the $SU(N)$ confinement scale may be much higher than the electroweak scale. In this case the pole masses may not even be relevant.

Upper limits for fermion masses are obtained by using quasi-fixed-point values for the Yukawa coupling constants, y_f, as determined from the RGEs in viable models with a desert above the TeV scale. These infra-red fixed point Yukawa couplings are of order unity which would lead to thresholds for including these fermions in the RGEs at the electroweak scale. However for the purposes of investigating the behaviour of the gauge coupling constants, and especially to demonstrate that the $U(1)$ coupling constant develops a Landau pole in our model without new SM fermions (chapter 4), we take a more generous single threshold of ten times the electroweak scale ~ 1.7 TeV for all new fermions in that model. For our discussion in chapter 5 of the model with a fourth generation of quarks we take the more stringent lower threshold value of M_Z, to demonstrate the absence of Landau poles in this case.

3.3 Precision Electroweak Data

Measurements of electroweak interactions are now accurate enough to be sensitive to loop corrections to propagators and vertex corrections. These effects are model dependent and can be sensitive to the values of some parameters such as fermion and Higgs masses. So far the SM seems to be consistent with the precision electroweak measurements and there is no experimental evidence that the SM is not correct. Obviously any other viable model should also agree with the data and in this section we discuss the experimental measurements of radiative corrections and the theoretical methods of calculating them.

There are many ways of parameterising the precision electroweak data. We choose the parameters S, T and U [27] which fully parameterise the precision data in the limit that all new fermions have infinite masses. However, these parameters are detailed enough provided none of the new fermions have masses less than the electroweak scale. These three parameters correspond to different types of radiative corrections.

The three parameters can be calculated perturbatively. If we consider an $SU(2)$ doublet $\binom{U}{D}$ with fermion masses m_U and m_D, in the limit:

$$\delta m \equiv |m_U - m_D| \ll m_U \ , \ m_D \tag{3.3.10}$$

and

$$m \equiv m_U \approx m_D \gg m_Z \tag{3.3.11}$$

we obtain the following relations:

$$S \approx \frac{1}{6\pi} \tag{3.3.12}$$

$$T \approx \frac{1}{12\pi s^2 c^2}\left[\frac{(\delta m)^2}{m_Z^2}\right] \tag{3.3.13}$$

$$U \approx \frac{2}{15\pi}\left[\frac{(\delta m)^2}{m^2}\right] \tag{3.3.14}$$

where $s \equiv \sin\theta_W$ and $c \equiv \cos\theta_W$.

These equations are really only valid when $m \gg m_Z$ but are a good approximation when $m > m_Z$. The main uncertainty comes from the fact that they are perturbative calculations and there is evidence from scaling known QCD effects that the contributions from new fermions may be larger than these estimates. However, it is by no means clear how to calculate these parameters non-perturbatively and so we shall assume that the perturbative calculation will be accurate enough. If we assume that the perturbative calculations are a lower limit then we can at least be sure that any model which appears to contradict experimental data is in fact ruled out.

The T parameter is a measure of the loop corrections to the ρ parameter. Contributions from each massive fermion are proportional to the difference of the masses squared between the fermions in the $SU(2)$ doublet. Since the SM (including the top quark) is consistent with the measured value of T, we want the contribution from the new fermions, T_{new} to be small. We can arrange $T_{new} \approx 0$ by choosing the masses of the new fermions to be degenerate within each $SU(2)$ doublet.

The SM is also consistent with the experimental value of U. So we want $U_{new} \approx 0$. Usually the U parameter is unimportant and can be assumed to be close to zero provided the model does not introduce anomalous W interactions [28] since it is suppressed relative to the T parameter by a factor of $\frac{m_Z^2}{m^2}$. Our models do not introduce such interactions but we will not always be considering $m \gg m_Z$. However, as is the case for T_{new}, a non-zero value of U_{new} requires a mass splitting in the $SU(2)$ doublets. We are already making this small so that $T_{new} \approx 0$. Therefore we can safely consider the U parameter to be consistent with experiment for all our models provided the T parameter is, and so neglect it in our analysis.

So we are left to consider the S parameter. This parameter does not vanish in the limit of $\delta m = 0$ and $m \gg m_Z$. In fact, in this limit, the S parameter gets a contribution of $\frac{1}{6\pi}$ from each $SU(2)$ doublet. So if we introduce $N_{Doublets}$ new $SU(2)$ doublets, the contribution to S is given by,

$$S_{new} = \frac{N_{Doublets}}{6\pi} \tag{3.3.15}$$

An analysis of the precision electroweak data gives [29]:

$$S_{new} = -0.21 \pm 0.24^{-0.08}_{+0.17} \tag{3.3.16}$$

where the second error is from the Higgs mass M_H. The central value is for $M_H = 300$ GeV, the upper second error for $M_H = 1000$ GeV and the lower one for $M_H = 60$ GeV. So if we take the lower limit $M_H = 60$ GeV,

$$S_{new} \approx 0 \pm 0.24 \tag{3.3.17}$$

and so $N_{Doublets}$ new $SU(2)$ doublets would differ from the mean value by approximately $0.22 N_{Doublets}$ standard deviations. Note here that the perturbative calculation of S_{new} predicts a positive contribution from the new fermions and so there is no way to cancel these contributions with other fermions. Therefore we can limit $N_{Doublets}$ by choosing how many standard deviations we are prepared to allow the model to differ from precision electroweak data. For example, if we wish our model to agree with the data for the S parameter to within 2 standard deviations, we must ensure that $N_{Doublets} \leq 9$.

Chapter 4

The SMG_{235} Model Without New SM Fermions

Here we will examine the model based on the gauge group $SMG_{235} \equiv G_5$ defined by eqs. (1.4.70) and (1.4.71), since it is the absolute minimal extension to the SM among all the possible groups we have proposed in section 2.2.1. In chapter 5 we will consider models based on the more general groups SMG_{23N} of eqs. (2.2.6) and (2.2.7), including new SM fermions to highlight the general features of all such extensions to the SM. However we will only analyse the consequences in detail for the group SMG_{235}.

In this section we will discuss the two possibilities: (i) that there are no new fermions beyond those of the SM and (ii) that there are new fermions which all couple to the $SU(5)$ gauge group. This latter possibility may seem to be tantamount to adding a completely separate sector to the SM rather than extending the SM, since the new fermions will be confined under a new gauge group. However, it is really no more a separate sector than the SM is three separate sectors (one for each generation), since these extra fermions will still couple to the electro-weak group due to the charge quantisation rule. We will discuss the other possibility, that there are new fermions, some coupling to the $SU(5)$ gauge group and others not, in chapter 5

4.1 No New Fermions

There is of course the possibility that there are no extra fermions associated with this enlarged group. If this is so then the only possible observations would be the detection of $SU(5)$-"glueballs". In this case the $SU(5)$ gauge group would be decoupled from the SMG and so the only way to observe these "glueballs" would be through their gravitational interactions. They could have been produced in the very early universe and the lightest state would be essentially stable since they could only decay via the gravitational interaction. Therefore they would only be observable as dark matter.

So this case is essentially uninteresting (at least from the point of view of extending the SM) and will not be considered further. Instead we turn to the possibility that there exist more types of fermions than have been currently observed and consider whether or not they can be incorporated into a consistent model.

4.2 New Fermions Coupling to $SU(5)$

Of course fermions all contribute to anomalies which must be cancelled. The fermions in the SM cancel all anomalies on their own; so the extra fermions must cancel all anomalies amongst themselves. There are two cases examined in this section. Firstly the general case where massless fermions are allowed. In section 3.1.1 we have already argued that there should be no more massless fermions. However, we shall examine this general case for completeness since the argument against massless fermions was phenomenological. We will proceed with the analysis as far as possible but it will become clear how difficult it is to find the minimum solution in this case.

As we have discussed in section 2.4, the anomaly conditions are much simpler in the case where all fermions must get a mass by the SM Higgs mechanism. This is also expected to be the only phenomenologically viable type of model as discussed in section 3.1.1. We will find the smallest number of fermions which cancel all anomalies and then show what happens to the running gauge coupling constants. We will see that this model contains a Landau pole and will also prove that all other models of this type also contain a Landau pole below the Planck scale.

4.2.1 General Case Including Massless Fermions

We have studied the case where we do not make the requirement that all fermions can get a mass via the SM Higgs mechanism. We used the general equations for anomaly cancellation and, by making assumptions of maximum numbers of fermions, we used some simple techniques to derive constraints on the number and types of fermions allowed. We also derived constraints on the allowed values of weak hypercharge. The detailed analysis is shown in appendix A.

The general constraint for the absence of the $[U(1)]^3$ anomaly is complicated. All the other constraints are linear and can be easily manipulated and simplified. So what we ended up with was some simple linear constraints and a complicated $[U(1)]^3$ anomaly constraint. This meant that we could not find any solutions of the anomaly equations. However, we did put several restrictions on the possible types of solution. We could solve the equations using a computer but since we don't think that solutions with massless fermions are likely to be phenomenologically acceptable we didn't proceed any further. Instead we will now examine the phenomenologically acceptable case where all fermions get a mass via the SM Higgs mechanism.

4.2.2 Only Massive Fermions

As explained in section 2.4 the anomaly equations in our models are greatly simplified when all the fermions are massive due to the SM Higgs mechanism. In fact they are reduced to just one equation, $\sum_i S_i y_i = 0$. If we label each mass-grouping of fermion representations by the label $\{y, \mathbf{R}\}$ where $\mathbf{R}$ is the representation of the group $SU(3) \otimes SU(5)$, then table 4.1 shows all six possible groupings, a to f, and their relative contributions, $S_i y_i$, to the anomaly equation. We use eq. (1.4.72), with the definition $m \equiv m_5$ to simplify the notation, giving us the charge quantisation rule,

$$\frac{y}{2} + \frac{1}{2}\text{“duality”} + \frac{1}{3}\text{“triality”} + \frac{\mathrm{m}}{5}\text{“quintality”} \equiv 0 \pmod 1 \qquad (4.2.1)$$

where the integer m is fixed in any given model [1]. So we can determine $\frac{y}{2} \pmod 1$ for any given representation $\mathbf{R}$.

For a solution to the anomaly equation $\sum_i S_i y_i = 0$, we must obviously combine the fractions $\frac{m}{5}$ so that the 5 is cancelled in the denominator since all Ns are integers. We must also have an even number of groupings so that the '$\frac{1}{2}$'s combine to give an integer. This automatically ensures that there can be no Witten anomaly as explained in section 2.4. This can be done by using equal numbers of type a and type b groupings. The two smallest solutions are in fact: (i) one type a grouping and one type b grouping and (ii) two groupings of type a and two of type b. The smallest solution, (i), is not possible without giving the fermions a fundamental Dirac mass, since the anomaly constraints require that $N_a + N_b = 0$ giving pairs of representations, $(y, \mathbf{2}, \mathbf{1}, \mathbf{5})$ and $(-y, \mathbf{2}, \mathbf{1}, \overline{\mathbf{5}})$ etc., which are not mass protected.

The smallest allowed solution with mass protected fermions is therefore solution (ii) with two groupings of type a and two of type b. This solution is shown

[1] In fact we can limit m to be 1 or 2 since it is only defined modulo 5 and, by replacing m with $-m \pmod 5$ and all representations of $SU(5)$ with their conjugates, we are left with an equivalent model.

Table 4.1: Allowed mass groupings $\{y, \mathbf{R}\}$ of new fermions in the SMG_{235} model, using the charge quantisation rule, eq. (4.2.1), and fundamental representations of $SU(5)$. Their relative contributions to the anomaly equation, eq. (2.4.14), are given in the final column. A particular mass grouping of type t is given by choosing a particular value of weak hypercharge, i.e. by choosing a particular value of the integer N_t.

Type	$\mathbf{R}$	$\frac{y}{2}$	$\frac{1}{10}Sy$
a	$\mathbf{1}, \mathbf{5}$	$N_a - \frac{m}{5} - \frac{1}{2}$	$N_a - \frac{m}{5} - \frac{1}{2}$
b	$\mathbf{1}, \overline{\mathbf{5}}$	$N_b + \frac{m}{5} + \frac{1}{2}$	$N_b + \frac{m}{5} + \frac{1}{2}$
c	$\mathbf{3}, \mathbf{5}$	$N_c - \frac{m}{5} + \frac{1}{6}$	$3N_c - \frac{3m}{5} + \frac{1}{2}$
d	$\mathbf{3}, \overline{\mathbf{5}}$	$N_d + \frac{m}{5} + \frac{1}{6}$	$3N_d + \frac{3m}{5} + \frac{1}{2}$
e	$\overline{\mathbf{3}}, \mathbf{5}$	$N_e - \frac{m}{5} - \frac{1}{6}$	$3N_e - \frac{3m}{5} - \frac{1}{2}$
f	$\overline{\mathbf{3}}, \overline{\mathbf{5}}$	$N_f + \frac{m}{5} - \frac{1}{6}$	$3N_f + \frac{3m}{5} - \frac{1}{2}$

in detail in table 4.2. All anomalies cancel provided $\sum_{i=1}^{4} N_i = 0$. We can now choose values of the N_i.

The fermion contribution to the (first order)beta function for the $U(1)$ running gauge coupling constant is proportional to $\sum y^2$. We therefore want to choose values of N_i so as to minimise $\sum y^2$, in order that any $U(1)$ Landau pole is at as high an energy as possible. This gives us the best chance that the solution of table 4.2 will be perturbatively valid up to the Planck scale and hence that our model will be self-consistent. However, this condition of minimising $\sum y^2$ is also suggested by the small representation structure of the SM, as explained in

Table 4.2: Smallest anomaly-free (subject to the constraint $N_1+N_2+N_3+N_4=0$) set of mass protected fermions which all couple to $SU(5)$.

Representation under $SU(2)\otimes SU(3)\otimes SU(5)$	$U(1)$ Representation $\frac{y}{2}$
$\mathbf{2,1,5}$	$N_1-\frac{m}{5}-\frac{1}{2}$
$\mathbf{1,1,\overline{5}}$	$-N_1+\frac{m}{5}$
$\mathbf{1,1,\overline{5}}$	$-N_1+\frac{m}{5}+1$
$\mathbf{2,1,5}$	$N_2-\frac{m}{5}-\frac{1}{2}$
$\mathbf{1,1,\overline{5}}$	$-N_2+\frac{m}{5}$
$\mathbf{1,1,\overline{5}}$	$-N_2+\frac{m}{5}+1$
$\mathbf{2,1,\overline{5}}$	$N_3+\frac{m}{5}+\frac{1}{2}$
$\mathbf{1,1,5}$	$-N_3-\frac{m}{5}-1$
$\mathbf{1,1,5}$	$-N_3-\frac{m}{5}$
$\mathbf{2,1,\overline{5}}$	$N_4+\frac{m}{5}+\frac{1}{2}$
$\mathbf{1,1,5}$	$-N_4-\frac{m}{5}-1$
$\mathbf{1,1,5}$	$-N_4-\frac{m}{5}$

section 2.2.2. Keeping in mind that the N_i are integers, $\sum_{i=1}^{4} N_i = 0$, and that the particles must be mass protected, we find that the minimum value of $\sum y^2$ is given by

$$N_1 = N_2 = 1 \quad N_3 = 0 \quad N_4 = -2$$

or

$$N_3 = N_4 = -1 \quad N_1 = 0 \quad N_2 = 2$$

where $m = 2$. These values of N_i give $\sum y^2 = 203.2$, for the solution of table 4.2, which is much larger than the $\frac{40}{3}$ per generation of the SM particles.

In section 3.2 we explained that it was reasonable to consider that all new fermions could be included at a threshold no higher than 1.7 TeV. This should provide an accurate enough upper limit for the threshold for our purposes. Therefore, since the fermions will have the least effect on the running coupling constants if they are included at the highest possible threshold, we will assume that all these extra fermions can be included with a simple threshold at 1.7 TeV. We can now check whether or not this model has a Landau pole below the Planck scale.

There are four fine structure constants which we shall label by α_1, α_2, α_3 and α_5 corresponding to the four gauge groups $U(1)$, $SU(2)$, $SU(3)$ and $SU(5)$ respectively. The fine structure constants, α_i, are related to the gauge coupling constants, g_i, by the relation $\alpha_i = \frac{g_i^2}{4\pi}$. The equations governing the running coupling constants to first order in perturbation theory (a good discussion of RGEs in the SM is given in [30]) can be integrated analytically (see section 1.2.1) to give

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - \frac{1}{12\pi}\left(Y^2 + n_H\right) ln\left(\frac{\mu}{\mu_0}\right) \tag{4.2.2}$$

$$\frac{1}{\alpha_2(\mu)} = \frac{1}{\alpha_2(\mu_0)} + \frac{1}{12\pi}\left(44 - 2n_{2f} - n_H\right) ln\left(\frac{\mu}{\mu_0}\right) \tag{4.2.3}$$

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(\mu_0)} + \frac{1}{12\pi}\left(66 - 2n_{3f}\right) ln\left(\frac{\mu}{\mu_0}\right) \tag{4.2.4}$$

$$\frac{1}{\alpha_5(\mu)} = \frac{1}{\alpha_5(\mu_0)} + \frac{1}{12\pi}(110 - 2n_{5f})\, ln\left(\frac{\mu}{\mu_0}\right) \tag{4.2.5}$$

where we calculate $\alpha_i(\mu)$ (the running coupling constants at the energy scale $\mu > \mu_0$) in terms of $\alpha_i(\mu_0)$. $Y^2 \equiv \sum y^2$ is the sum of the weak hypercharges squared for all fermions included at a threshold below μ_0 and n_{mf} are the number of fermion $\mathbf{m}$ and $\overline{\mathbf{m}}$ representations of $SU(m)$ included at a threshold below μ_0. n_H is the number of Higgs doublets included at a threshold below μ_0. These equations assume that there are no fermions or Higgs scalars included at a threshold between μ_0 and μ. In order to calculate the value of $\alpha_i(\mu)$ when there are fermions or Higgs bosons included at a threshold between μ_0 and μ we must do the calculation in steps, calculating the value of α_i up to the included at a threshold each particle. So we use the experimental values of the fine structure constants at M_Z (including the top quark and Higgs boson in the beta functions at this scale) to calculate the coupling constants at 1.7 TeV, where we include the new fermions, and then run the coupling constants up to the Planck scale. This is a crude method since there would really be complicated threshold effects as each fermion was included. However these effects can reasonably be assumed to be small, relative to the changes in the coupling constants caused by the running from the electroweak scale to the Planck scale, and so we will use this much simpler method. Second order RGEs [31] could be used but the improvement over the first order RGEs would not be significant when compared to the error introduced by the naive assumptions made about threshold effects.

From [5] we find

$$\alpha_1^{-1}(M_Z) = 98.08 \pm 0.16 \tag{4.2.6}$$

$$\alpha_2^{-1}(M_Z) = 29.794 \pm 0.048 \tag{4.2.7}$$

$$\alpha_3^{-1}(M_Z) = 8.55 \pm 0.37 \tag{4.2.8}$$

We can now use the above equations to examine how the coupling constants behave up to the Planck scale. Since there is no experimental value for α_5 at any energy scale we shall assume that $\alpha_5^{-1}(M_Z) = 2$, so that the $SU(5)$ interaction is stronger than QCD at M_Z and confines above the electroweak scale. Fig. 4.1 shows what happens for each group. For the graphs, we normalise the $U(1)$ gauge coupling as if the $U(1)$ group were embedded in a simple group. This essentially corresponds to redefinition of g_1.

$$(g_1^2)_{\mathrm{GUT}} \equiv \frac{5}{3}(g_1^2)_{\mathrm{SM}} \tag{4.2.9}$$

$$(\alpha_1^{-1})_{\mathrm{GUT}} \equiv \frac{3}{5}(\alpha_1^{-1})_{\mathrm{SM}} \tag{4.2.10}$$

as explained in section 1.2.1. So henceforth we use the standard GUT normalisation. Eqs. (4.2.2) and (4.2.6) now become,

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - \frac{1}{20\pi}\left(Y^2 + n_H\right) ln\left(\frac{\mu}{\mu_0}\right) \tag{4.2.11}$$

$$\alpha_1^{-1}(M_Z) = 58.85 \pm 0.10 \tag{4.2.12}$$

As we can see from fig. 4.1, $\frac{1}{\alpha_1}$ becomes negative at about 10^7 GeV which means that there is a $U(1)$ Landau pole. So we can conclude that this theory would be inconsistent, at least as far as perturbation theory is concerned, without new interactions below 10^7 GeV.

In fact we can show that there is no anomaly-free model, having all new fermions getting a mass via the SM Higgs mechanism and belonging to fundamental representations of $SU(5)$, with a desert above the TeV scale, which does not have a Landau pole below the Planck scale. The condition for no Landau pole below the Planck scale is $\frac{1}{\alpha_1(M_{Pl})} > 0$. Therefore eq. (4.2.11) can be rearranged to give

$$Y^2 + n_H < \frac{20\pi}{\alpha_1(\mu_0) ln\left(\frac{M_{Pl}}{\mu_0}\right)} \tag{4.2.13}$$

Since, for the SM, $Y_{SM}^2 = 40$ and $n_H = 1$,

$$Y^2 + n_H \geq 41 \tag{4.2.14}$$

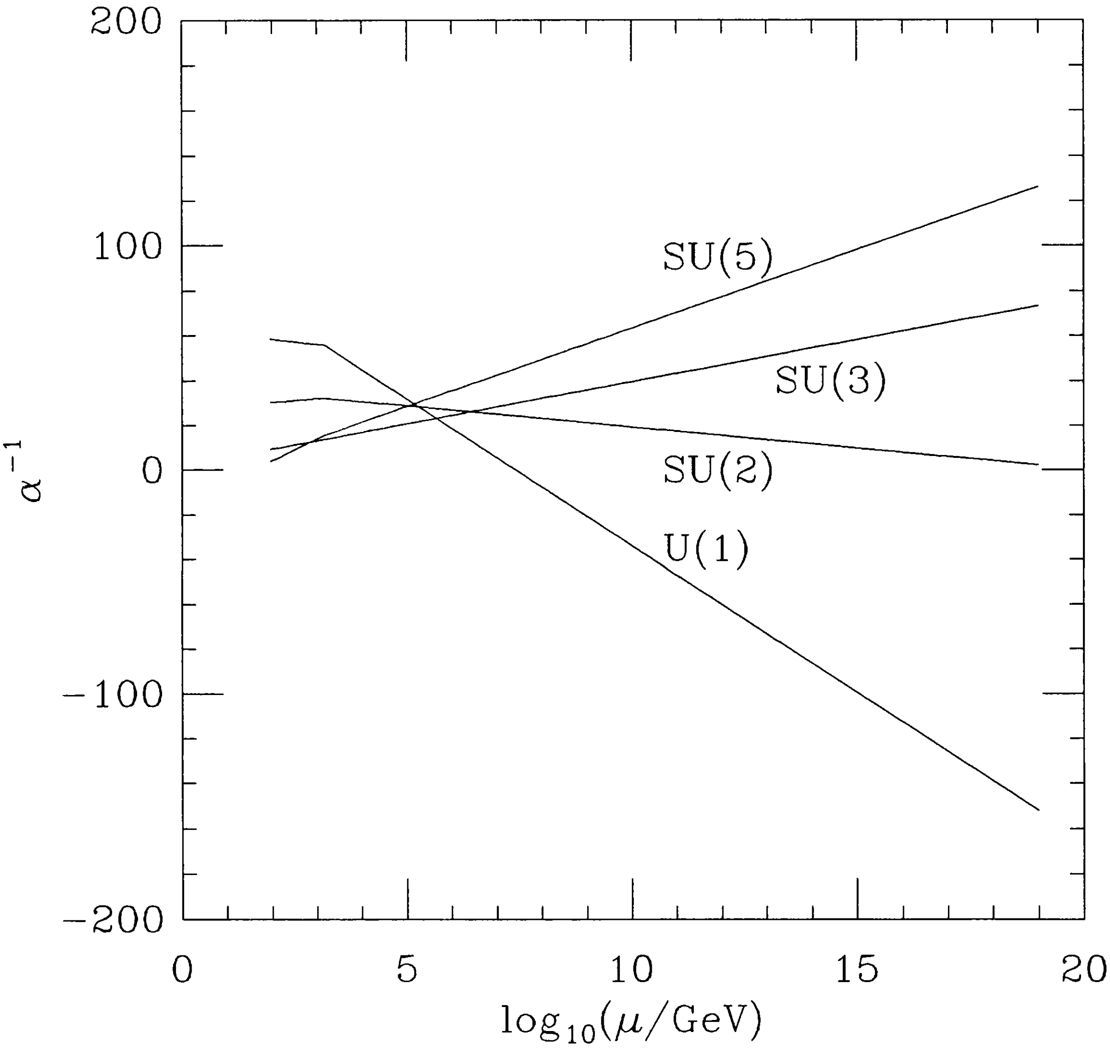

Figure 4.1: α^{-1} from M_Z to the Planck scale for each component group in the SMG_{235} model without new SM fermions. There is clearly a $U(1)$ Landau pole at $\mu \sim 10^7$ GeV and $SU(2)$ also loses asymptotic freedom. $\alpha_5^{-1}(M_Z) = 2$ has been chosen as a specific example.

above the electroweak scale and so we can use eqs. (4.2.11) and (4.2.12) to calculate an upper limit for $\frac{1}{\alpha_1(1.7TeV)}$:

$$\frac{1}{\alpha_1(1.7TeV)} \leq 57 \tag{4.2.15}$$

We then use $Y^2 = Y^2_{SM} + Y^2_{new}$ in eq. (4.2.13) with $\mu_0 = 1.7$ TeV and conclude that

$$Y^2_{new} < 57.5 \tag{4.2.16}$$

assuming the new fermions can be included naively at a threshold no higher than 1.7 TeV.

For each mass grouping $\{y, \mathbf{R}\}$, with $P_R = 4S_R$ fermions, we can calculate the value of Y^2:

$$Y^2 = S_R[2y^2 + (y+1)^2 + (y-1)^2] = S_R(4y^2 + 2) \tag{4.2.17}$$

Therefore we have,

$$Y^2 \quad \geq \quad 2S_R \quad = \quad \frac{1}{2}P_R \tag{4.2.18}$$

If there are several mass groupings, $Y^2 \geq \frac{1}{2}\sum P_R \equiv \frac{1}{2}P$ where P is the total number of fermions. So if we define P_{new} to be the number of non-SM fermions, we can conclude,

$$P_{new} \quad \leq \quad 2Y^2_{new} \quad < \quad 115 \tag{4.2.19}$$

So now we have shown that there must be less than 115 extra fermions. However the smallest solutions, subject to the constraints in this section, larger than two type a and two type b representations are three type a and one type c representations etc. which contain 120 fermions and so must cause a Landau pole below the Planck scale [2]. Therefore there are no possible anomaly-free models without

[2]Using second order RGEs or a more complete analysis of thresholds would obviously change the precise limit in eq. (4.2.16). However, the charge quantisation rule in our model means that y cannot be zero and so it is not possible to attain the limit of eq. (4.2.18). So in fact, the value of Y^2_{new} will generally be much greater than this limit. For example, three type a and one type c lead to $Y^2_{new} \geq \frac{1708}{15} \approx 114$ which is much greater than the required maximum given by eq. (4.2.16).

a Landau pole, where all the new fermions couple to the $SU(5)$ gauge group.

We will now examine the case where we allow some new $SU(5)$ singlet fermions, as well as some fermions which couple to $SU(5)$, in order to cancel the anomalies. We shall show that it is possible to have more SM fermions in such a model.

Chapter 5

The SMG_{235} Model With New SM Fermions

In this chapter we shall first examine sets of fermions (which are generalisations of the SM quarks) in groups, SMG_{2M} and SMG_{2MN}, defined by eqs. (2.2.6) and (2.2.7), similar to the SMG. We shall then examine the particular case of a 4th generation of SM quarks along with a generation of $SU(5)$-"quarks" in a model with gauge group SMG_{235}. After showing this model to be self-consistent (with no Landau poles below the Planck scale), we will discuss the possibility of experimental evidence for and against the model. Finally, we will discuss the more general models with gauge group SMG_{23MN} with a generation of $SU(M)$- and $SU(N)$-"quarks".

5.1 Fermions in the groups SMG_{2M} and SMG_{2MN}

In section 5.1.1 we shall examine the group SMG_{2M}. The SMG is an example of this type of group, with the particular choice of $M = 3$. We shall show that this general group allows anomaly-free sets of fermions which consist of a generation of SM leptons and a generation of $SU(M)$-"quarks" which are a simple generalisation of the $SU(3)$ quarks in the SM.

We shall then show in section 5.1.2 that we can have anomaly-free sets of fermions in the group SMG_{2MN} without any leptons. We shall then examine the particular case of the group SMG_{235} which we shall discuss in detail since it contains the SMG and with $N = 5$ it is the smallest extension to the SMG allowed by our method.

5.1.1 Fermions in the Group SMG_{2M}

In the SM, each generation is formed by taking the two mass groupings $\{\frac{1}{3}, \mathbf{3}\}$ and $\{-1, \mathbf{1}\}$ (where the representations $\mathbf{3}$ and $\mathbf{1}$ are of the group $SU(3)$) as explained in sections 2.4 and 2.5. We will now consider a more general situation where we have the gauge group SMG_{2M} defined in section 2.2.1 (where $M \geq 3$ is an odd integer) and the fermions are in the groupings $\{y_1, \mathbf{M}\}$ and $\{y_2, \mathbf{1}\}$ (where the representations $\mathbf{M}$ and $\mathbf{1}$ are of the group $SU(M)$).

From section 2.4 all the gauge anomalies will cancel if

$$My_1 + y_2 = 0 \tag{5.1.1}$$

Since we also have the charge quantisation rule

$$\frac{y}{2} + \frac{1}{2}\text{“duality”} + \frac{\mathrm{m_M}}{\mathrm{M}}\text{“M} - \text{ality”} \equiv 0 \pmod 1 \tag{5.1.2}$$

we can write

$$\frac{y_1}{2} = -\frac{1}{2} - \frac{m_M}{M} + c_1 \tag{5.1.3}$$
$$\frac{y_2}{2} = -\frac{1}{2} + c_2 \tag{5.1.4}$$

where c_1 and c_2 are integers. We now have the condition that for no anomalies to be present

$$-\frac{M+1}{2} - m_M + Mc_1 + c_2 = 0 \tag{5.1.5}$$

In the SM a lepton generation is formed (with the addition of a right-handed neutrino which can be removed without affecting any anomalies) when we have $c_2 = 0$ as explained in section 2.5. If we insert this value into eq. (5.1.5) then we find,

$$c_1 = \frac{1}{M}\left(\frac{M+1}{2} + m_M\right) \tag{5.1.6}$$

This can always be solved by setting $m_M = \frac{M-1}{2}$ and $c_1 = 1$. In fact if $M = 3$ then this is simply one of the anomaly-free SM quark-lepton generations.

However, this is not a good solution for an extension of the SM (which would be obtained by considering $SMG_{2M} \subset SMG_{23M}$) since it contains an extra massless neutrino which has already been ruled out by experiment. It is difficult to produce a neutrino with a mass so large that it wouldn't already have been detected, as explained in section 3.1.2. We could choose not to set $c_2 = 0$ or 1 above, which would force all the extra leptons to be massive (by leptons we mean any fermions which are only coupled to the electroweak subgroup, $SU(2) \otimes U(1)$). This is because there would then be two $SU(2)$ singlets which were charged (and at least one would have an electric charge of magnitude two or more, which is against our principle of small representations) and so both would be required to cancel anomalies unlike the case of a hypothetical right-handed neutrino. They would both then get a mass by the usual SM Higgs mechanism since neither could get a Majorana mass. But even if we assumed that these leptons had masses higher than experimental limits this solution is not really favoured by our postulate of small values of weak hypercharge discussed in section 2.2.2. So in order to find a satisfactory solution we shall look at a similar, more general, case.

5.1.2 Fermions in the Group SMG_{2MN}

Suppose we have the gauge group SMG_{2MN}, where both M and $N > M \geq 3$ are mutually prime odd integers, which has the charge quantisation rule

$$\frac{y}{2} + \frac{1}{2}\text{“duality”} + \frac{\mathrm{m_M}}{\mathrm{M}}\text{“M} - \text{ality”} + \frac{\mathrm{m_N}}{\mathrm{N}}\text{“N} - \text{ality”} \equiv 0 \pmod 1 \tag{5.1.7}$$

Then with fermions in mass groupings $\{y_1, (\mathbf{M}, \mathbf{1})\}$ and $\{y_2, (\mathbf{1}, \mathbf{N})\}$ (where the representations $(\mathbf{M}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{N})$ are of the group $SU(M) \otimes SU(N)$) the condition for no anomalies is

$$My_1 + Ny_2 = 0 \tag{5.1.8}$$

The charge quantisation rule means that we can write

$$\frac{y_1}{2} = -\frac{1}{2} - \frac{m_M}{M} + c_1 \tag{5.1.9}$$

$$\frac{y_2}{2} = -\frac{1}{2} - \frac{m_N}{N} + c_2 \tag{5.1.10}$$

where c_1 and c_2 are integers. We then find that the condition for no anomalies becomes

$$2Nc_2 = N + [2(m_M + m_N) + (1 - 2c_1)M] \tag{5.1.11}$$

Since N and M are both odd there will always be a solution since we can choose m_M and m_N so that $(m_M + m_N) = M$ and c_1 so that $(3 - 2c_1)$ is an odd multiple of N. In general there will also be other solutions.

In particular, for the gauge group $G_5 \equiv SMG_{235}$ we can have a fourth generation of quarks without any extra leptons by choosing $M = 3$, $N = 5$, $m_3 = 1$ and $c_1 = 1$ above. Then

$$10c_2 = 5 + [2(1 + m_5) - 3] \tag{5.1.12}$$

or equivalently

$$5c_2 = 2 + m_5 \tag{5.1.13}$$

So we have a solution with $c_2 = 1$ and $m_5 = 3$.

The representations of the left-handed fermions which couple to the $SU(5)$ subgroup are shown in table 5.1. This is a generalisation of the quarks in the SM, coupling to $SU(5)$ rather than $SU(3)$.

In fact we have a solution with a fourth generation of quarks for the general case, where N is any odd integer greater than 3 by choosing $c_2 = 1$ and $m_N = \frac{1}{2}(N + 1)$. This means that if a fourth generation of quarks without leptons

Table 5.1: Left-handed fermions coupling to $SU(5)$ in the mass grouping $\{-\frac{1}{5},(\mathbf{1},\mathbf{5})\}$. The electric charges are in units of $\frac{1}{5}$ due to the charge quantisation rule. These $SU(5)$-"quarks" form an anomaly-free set of fermions together with a fourth generation of SM quarks.

Representation under $SU(2)\otimes SU(3)\otimes SU(5)$	Type	$U(1)$ Representation $\frac{y}{2}$	Electric Charge Q
$2,1,5$	$\begin{pmatrix} 5u \\ 5d \end{pmatrix}_L$	$-\frac{1}{10}$	$\begin{pmatrix} \frac{2}{5} \\ -\frac{3}{5} \end{pmatrix}$
$1,1,\overline{5}$	$\overline{5u}_L$	$-\frac{4}{10}$	$-\frac{2}{5}$
$1,1,\overline{5}$	$\overline{5d}_L$	$\frac{6}{10}$	$\frac{3}{5}$

Table 5.2: Fermions coupling to $SU(N)$ which would form an anomaly-free set of fermions together with a fourth generation of quarks. These fermions form the mass-grouping $\{-\frac{1}{N}, (\mathbf{1}, \mathbf{N})\}$.

Representation under $SU(2) \otimes SU(3) \otimes SU(N)$	Type	$U(1)$ Representation $\frac{y}{2}$	Electric Charge Q
$\mathbf{2}, \mathbf{1}, \mathbf{N}$	$\begin{pmatrix} Nu \\ Nd \end{pmatrix}_L$	$-\frac{1}{2N}$	$\begin{pmatrix} \frac{N-1}{2N} \\ -\frac{N+1}{2N} \end{pmatrix}$
$\mathbf{1}, \mathbf{1}, \overline{\mathbf{N}}$	$\overline{Nu}_L$	$-\frac{N-1}{2N}$	$-\frac{N-1}{2N}$
$\mathbf{1}, \mathbf{1}, \overline{\mathbf{N}}$	$\overline{Nd}_L$	$\frac{N+1}{2N}$	$\frac{N+1}{2N}$

was detected, there would be no immediate way of deducing the value of N. Table 5.2 shows the properties of the left-handed fermions which couple to the $SU(N)$ subgroup. Note that this is a generalisation of the SM quarks, coupling to $SU(N)$ with the specific choice of $m_N = \frac{1}{2}(N+1)$. If we set $N = 3$ we would in fact get a generation of quarks with the opposite chirality to those in the SM. This is to be expected since we are using these fermions to cancel the anomaly contribution of a 4th generation of SM quarks (with the usual chirality).

This solution, with a fourth generation of quarks and the fermions of table 5.1, for the gauge group SMG_{235} is analogous to one SM quark-lepton generation in the gauge group SMG, in the sense that it is the smallest anomaly-free set of mass protected fermions which couple non-trivially to all the gauge fields. The SM quark-lepton generation has been shown to be the smallest such set of fermions for the SMG in section 2.5. Note that although a generation of SM leptons and the fermions conjugate to those in table 5.1 is a smaller anomaly-free set of fermions in the gauge group SMG_{235}, none of these fermions couples to the $SU(3)$ subgroup. However, as in section 2.5, we do not feel that requiring some fermions to couple to all parts of the gauge group is necessary. So this is not a valid reason for choosing this solution in comparison to the one with leptons and so the choice is really made on a phenomenological basis.

As stated in section 3.1.2, we take the limits for the masses of a fourth generation of quarks to be $M_{b'} > 130$ GeV, $M_{t'} > 130$ GeV and for the top quark mass to be $M_t \sim 170$ GeV. We can now use the RGEs, first to show that these additional fermions do not cause any inconsistencies such as gauge coupling constants becoming infinite below the Planck scale, and then to estimate upper limits on the values of the Yukawa couplings to the SM Higgs field of these fermions. This will lead to upper limits on the masses indicating that the t' and b' quarks would be almost within reach of present experiments.

5.2 No Landau Poles

As in chapter 4 we can investigate how the gauge coupling constants vary with energy up to the Planck scale. Here we set the thresholds for all the unknown fermions (4th generation quarks and fermions coupling to $SU(5)$), as well as for the top quark and Higgs boson, to M_Z. The absence of Landau poles in this case will guarantee their absence if some of the thresholds are set higher than M_Z. From experimental limits we would expect that all these thresholds should be greater than M_Z.

We use eqs. (4.2.3)-(4.2.5) and (4.2.11) with $Y^2 = \frac{872}{15}$ to run the gauge coupling constants up to the Planck scale as shown in fig. 5.1. Now we see that with a fourth generation of quarks and the fermions in table 5.1, with far fewer fermions than the model in section 4.2.2 where all the new fermions coupled to $SU(5)$, there are no problems with Landau poles below the Planck scale. So our SMG_{235} model with new SM fermions appears to be consistent.

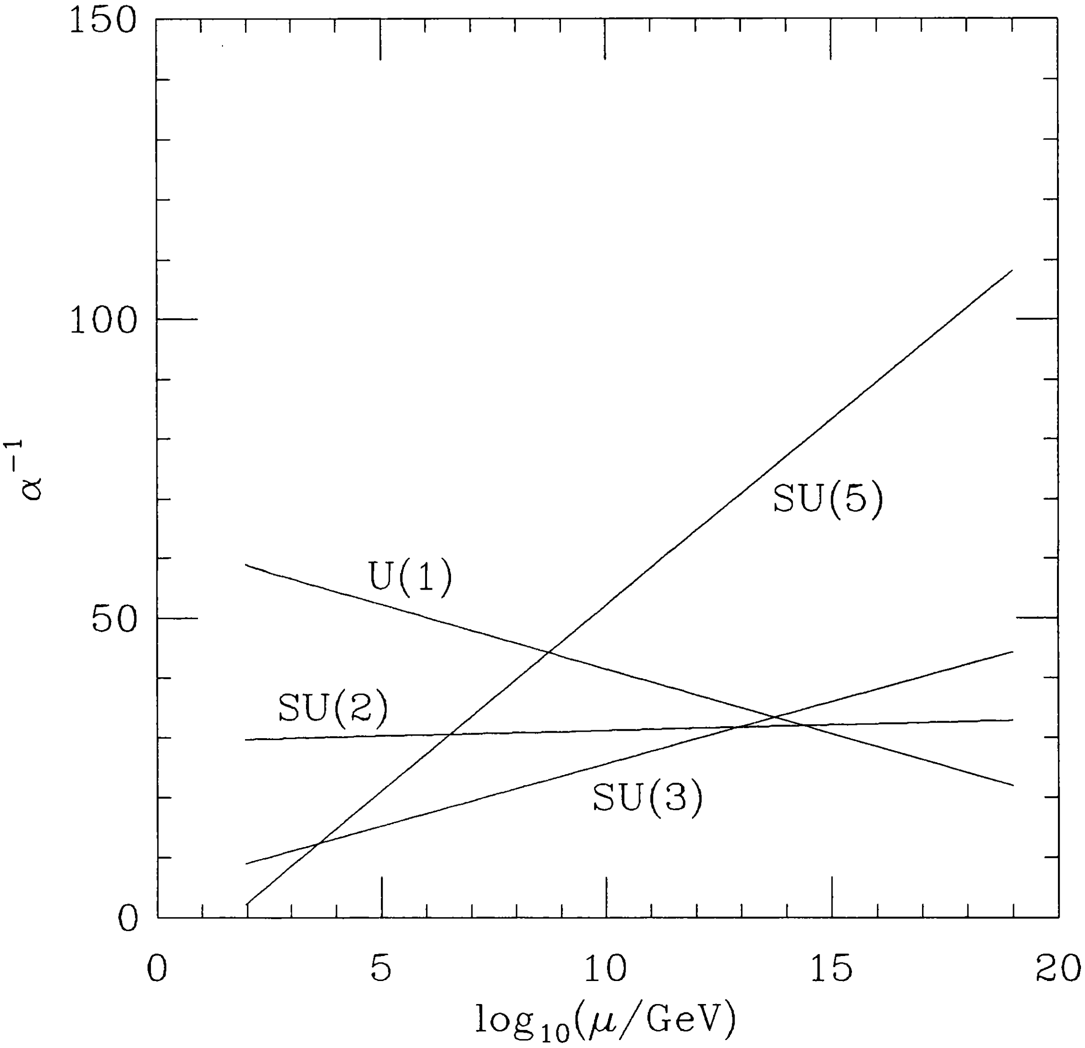

Figure 5.1: α^{-1} from M_Z to the Planck scale for each component group in the SMG_{235} model with a fourth generation of quarks and the fermions of table 5.1 which couple to $SU(5)$. The initial value of $\alpha_5^{-1}(M_Z) = 2$ was chosen so that the $SU(5)$ group would confine above the electroweak scale. There are obviously no Landau poles between m_Z and the Planck scale so this model is self-consistent.

5.3 Upper Limits for Yukawa Couplings and Higgs Mass

Now we can choose initial values for the Yukawa couplings at the Planck scale and use the RGEs to see how they evolve as they are run down to the electro-weak scale. Assuming no mixing for the quarks and neglecting the masses of all SM fermions except the top quark (a good approximation), the RGEs are, to one loop order in perturbation theory as described in section 1.2.2:

$$\frac{dy_t}{dt} = y_t \frac{1}{16\pi^2}\left(\frac{3}{2}y_t^2 + Y_2(S) - G_{3u}\right) \tag{5.3.14}$$

$$\frac{dy_{t'}}{dt} = y_{t'} \frac{1}{16\pi^2}\left(\frac{3}{2}(y_{t'}^2 - y_{b'}^2) + Y_2(S) - G_{3u}\right) \tag{5.3.15}$$

$$\frac{dy_{b'}}{dt} = y_{b'} \frac{1}{16\pi^2}\left(\frac{3}{2}(y_{b'}^2 - y_{t'}^2) + Y_2(S) - G_{3d}\right) \tag{5.3.16}$$

$$\frac{dy_{5u}}{dt} = y_{5u} \frac{1}{16\pi^2}\left(\frac{3}{2}(y_{5u}^2 - y_{5d}^2) + Y_2(S) - G_{5u}\right) \tag{5.3.17}$$

$$\frac{dy_{5d}}{dt} = y_{5d} \frac{1}{16\pi^2}\left(\frac{3}{2}(y_{5d}^2 - y_{5u}^2) + Y_2(S) - G_{5d}\right) \tag{5.3.18}$$

where the $SU(5)$ fermions have been labelled 5u and 5d as generalisations of the naming of $SU(3)$ quarks, as shown in table 5.1. The other variables are defined as

$$Y_2(S) = 5y_{5u}^2 + 5y_{5d}^2 + 3y_{t'}^2 + 3y_{b'}^2 + 3y_t^2 \tag{5.3.19}$$

$$G_{3u} = \frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2 \tag{5.3.20}$$

$$G_{3d} = \frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2 \tag{5.3.21}$$

$$G_{5u} = \frac{153}{500}g_1^2 + \frac{9}{4}g_2^2 + \frac{72}{5}g_5^2 \tag{5.3.22}$$

$$G_{5d} = \frac{333}{500}g_1^2 + \frac{9}{4}g_2^2 + \frac{72}{5}g_5^2 \tag{5.3.23}$$

Here $Y_2(S)$ is really $Tr(Y^\dagger Y)$ where Y is the Yukawa matrix for all the fermions. We have used the above approximation since the Yukawa couplings of the other fermions (SM fermions other than the top quark) are much less than the Yukawa couplings of the new fermions.

We can also examine the quasi-fixed point limit of the Higgs self-coupling, $\lambda(\mu)$, using eq. (1.2.46) with the above definition of $Y_2(S)$ and,

$$H(S) = 5y_{5u}^4 + 5y_{5d}^4 + 3y_{t'}^4 + 3y_{b'}^4 + 3y_t^4 \tag{5.3.24}$$

We can choose values for the Yukawa couplings at the Planck scale and then use the RGEs to see what values the Yukawa couplings will have at any other scale. We have chosen the low energy scale to be M_Z. We observe quasi-fixed points similar to the case for the top quark in the SM [32] and these will provide upper limits on the fermion masses. However, the resulting Yukawa coupling for any fermion at M_Z depends on the Yukawa couplings of the other fermions. But there is an approximate infrared fixed point limit on $Y_2(S)$ and so one Yukawa coupling can be increased at the expense of the others. This limit on $Y_2(S)$ is quite precise if there is only one strong interaction at low energies, such as QCD in the SM [1]. We observe numerically that $Y_2(S) \approx 7.7 \pm 0.3$, provided the Yukawa couplings of the three heavy quarks are greater than 1 at the Planck scale and that the Yukawa couplings of the fermions coupling to the $SU(5)$ gauge group are less than the Yukawa couplings of the heavy quarks at the Planck scale. See, for example, figure 5.6.

First we shall discuss the quasi-fixed points in detail for this model. We shall examine the limits on the Yukawa couplings when we consider only the quarks and when we consider only the $SU(5)$-"quarks" as well as the more general case when all the new fermions have significant Yukawa couplings.

Then we shall go on to discuss upper limits on the Yukawa couplings from quasi-fixed point values when all fermions get a mass consistent with experimental limits. This will provide strong constraints on the allowed masses of these new fermions.

[1] Detailed results for a general number of heavy SM generations are derived in [33].

5.3.1 Examples Of Quasi-Fixed Points

In this section we shall give various examples of quasi-fixed points in this particular model. The simplest examples are to set either, all SM quark Yukawa couplings to zero at the Planck scale, or all $SU(5)$-"quark" Yukawa couplings to zero at the Planck scale. Since we wish our models to be consistent with precision electroweak data we will set the Yukawa couplings of fermions in the same $SU(2)$ doublet to the same value at the Planck scale. This will ensure that to a good approximation $T_{new} = U_{new} = 0$, as required by experimental results described in section 3.3.

Only the Top Quark: The SM Case

First we shall examine the familiar case of fixed points in the SM. The top quark is the only fermion with a significant Yukawa coupling and all the fermion mixing angles are small so we are justified in neglecting the effects due to the other fermions. So we can examine the Yukawa coupling of the top quark, y_t, and the Higgs boson self-coupling, λ, alone. Since there are only 2 parameters we are interested in, the simplest way to examine the fixed point behaviour is to plot y_t v λ as the couplings are run from chosen initial values at the Planck scale down to the electroweak scale. Figure 5.2 shows how several different initial values of y_t and λ all converge to the same value at the electroweak scale. We have used 2-loop RGEs for this case since they are well known for the SM (see [30] for example). This should make this section consistent with previous calculations although for the other sections we shall simply use the 1-loop RGEs for convenience and also because the effect of varying the unknown value of the $SU(5)$ gauge coupling constant will cause a much greater difference than that between 1-loop and 2-loop calculations.

We can clearly see that the final values of y_t and λ at the electroweak scale are not sensitive to the values at the Planck scale provided $y_t(M_{Planck}) > 1$. For $y_t(M_{Planck}) = 1$ we can see that there is a small difference at the electroweak

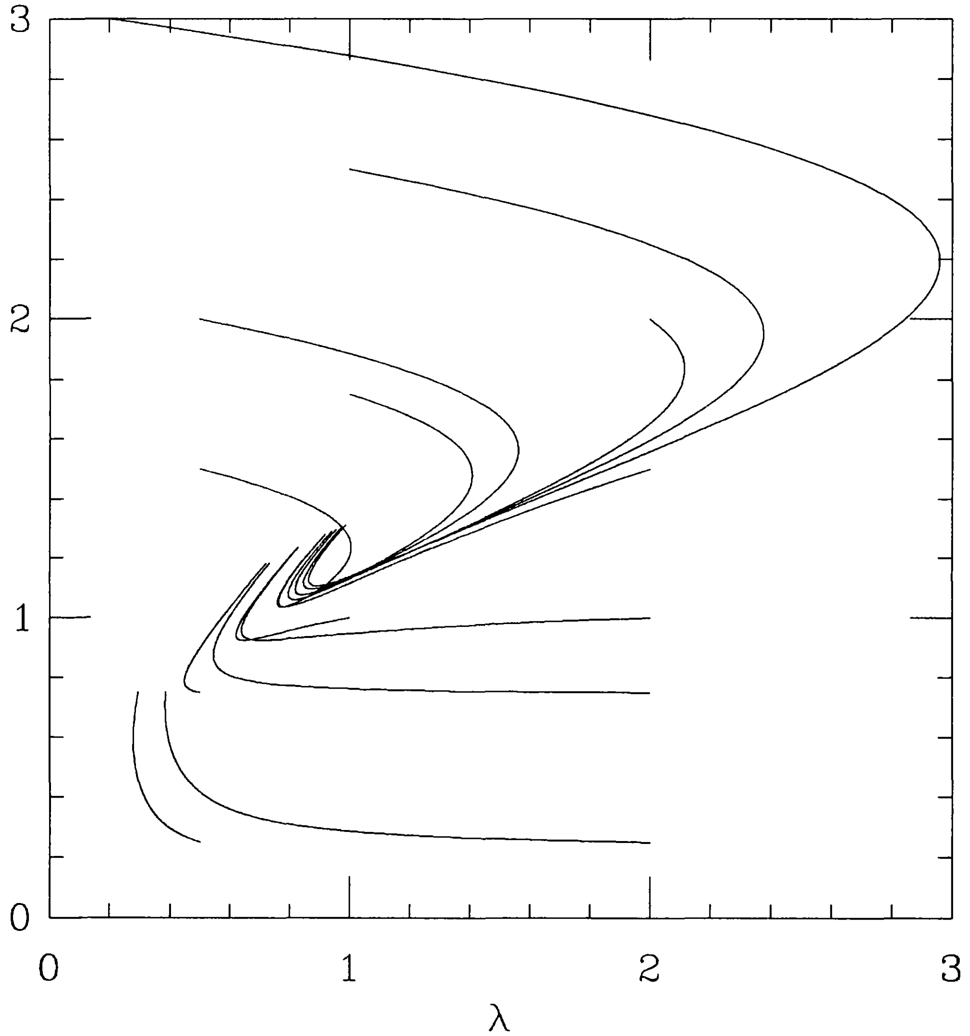

Figure 5.2: Fixed point of y_t and λ in the SM. The lines show the running of different initial values at the Planck scale down to the fixed point values at the electroweak scale.

scale. For $y_t(M_{Planck}) < 1$ the values at the electroweak scale are clearly sensitive to the initial values at the Planck scale. This is an important point. If the Yukawa coupling is not large enough at the Planck scale then the fixed point will not be reached at the electroweak scale. In this example the cases with $y_t(M_{Planck}) \leq 1$ would eventually reach a fixed point below the electroweak scale if we continued to run the equations. However, the exact value of y_t and λ at the fixed point depends on the scale of the fixed point. This is because of the dependence of the RGEs for y_t and λ on the gauge coupling constants. Therefore, if a set of initial conditions does not converge on the fixed point at the electroweak scale, it will reach a different fixed point at a lower scale. But in this case that will mean that the resultant pole masses of the top quark and Higgs boson will be different since they depend on the values of y_t and λ at the electroweak scale. So here we can say that fixed points below the electroweak scale have no physical relevance. This will also be true for our model with a generation of $SU(5)$-"quarks" since they can attain fixed point pole masses of the same order as the electroweak scale.

We can see that the lines with $y_t(M_{Planck}) > 1$ converge before reaching the fixed point at the electroweak scale. This is because the fixed point values of y_t and λ depend on the gauge coupling constants, g_i. These in turn depend on the scale since they are running coupling constants. So what we are observing is the dependence of the fixed point on the scale. The change in direction of the lines just before the fixed point at the electroweak scale is due to the rapid increase in g_3 near the electroweak scale which increases the top quark Yukawa coupling. We will see clearer examples of this behaviour when analysing the $SU(5)$-"quark" Yukawa couplings since we choose $g_5(M_Z) > g_3(M_Z)$.

The fixed point values of y_t and λ can be read off figure 5.2. We get,

$$y_t(M_Z) \approx 1.30 \qquad (5.3.25)$$

$$\lambda(M_Z) \approx 0.96 \qquad (5.3.26)$$

When we use the SM VEV,

$$< \phi_{WS} > = \ v = 246.22 \text{ GeV} \tag{5.3.27}$$

the corresponding pole masses are, using eqs. (1.3.65) and (1.2.41),

$$M_t \approx 232 \text{ GeV} \tag{5.3.28}$$

$$M_H \approx 236 \text{ GeV} \tag{5.3.29}$$

For comparison with the next sections, when we use the value of $< \phi_{WS} >$ reduced by the technicolour-like contribution to the VEV from the condensates of $SU(5)$-"quarks", $< \phi_{WS} >= 234$ GeV, we get,

$$M_t \approx 221 \text{ GeV} \tag{5.3.30}$$

$$M_H \approx 225 \text{ GeV} \tag{5.3.31}$$

Only the Top and 4th Generation Quarks

Now we shall include the 4th generation quarks and examine the quasi-fixed point behaviour for various values of Yukawa couplings at the Planck scale. The choice of Yukawa couplings is simplified by our requirement that the masses of the fourth generation quarks should be equal so that the model can be consistent with the precision electroweak data; i.e. $T_{new} \approx 0$ and $U_{new} \approx 0$. For simplicity we choose $y_{t'}(M_{Planck}) = y_{b'}(M_{Planck})$. This will produce a splitting of the pole masses of about 3%. This will give a negligible contribution to the T and U parameters. We will now show some graphs of the running Yukawa couplings to illustrate the quasi-fixed point behaviour and to show the resultant masses without the $SU(5)$-"quarks".

First we show the fourth generation quarks alone without the top quark in figure 5.3. This will give an estimate of the maximum possible mass of the fourth generation quarks. The inclusion of the top quark will reduce this estimate. From

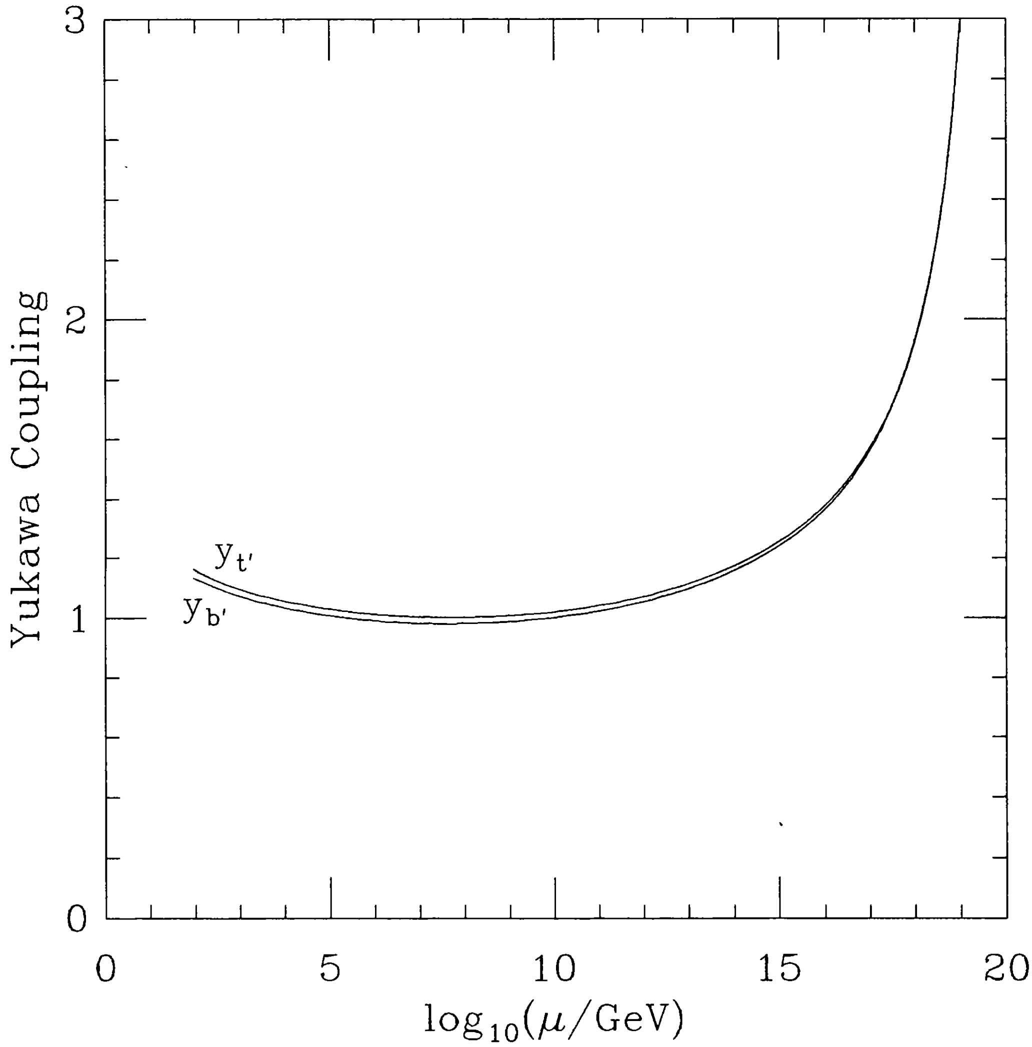

Figure 5.3: Fixed point of $y_{t'}$ and $y_{b'}$ without the top quark or any $SU(5)$-"quarks".

this graph we can calculate the pole masses of the fourth generation quarks. They are,

$$M_{t'} = 200 \text{ GeV} \tag{5.3.32}$$

$$M_{b'} = 196 \text{ GeV} \tag{5.3.33}$$

We can certainly assume that these masses are upper limits since the effect of including more fermions is to reduce the average Yukawa coupling. This effect can be observed in the graphs where the top quark alone (figures 5.4 and 5.5) or the top quark with the $SU(5)$-"quarks" (figures 5.10 and 5.11) are included. We can see that the t' and b' end up with almost the same Yukawa couplings. This is because the RGEs are the same apart from a small difference due to the different coupling of the quarks to the $U(1)$ gauge group.

Now we show the fixed point behaviour when we consider all three heavy quarks, still without the $SU(5)$-"quarks", in figure 5.4. It is obvious that, although they start with equal Yukawa couplings at the Planck scale ($y_t(M_{Planck}) = y_{t'}(M_{Planck}) = y_{b'}(M_{Planck}) = 3.0$), the top quark Yukawa coupling is reduced at the electroweak scale because its doublet partner, the bottom quark, has a much smaller Yukawa coupling (approximated to 0 here). This effect is obviously much larger than the small change between the t' and b' Yukawa couplings due to their different weak hypercharges. The resultant pole masses are,

$$M_t = 130 \text{ GeV} \tag{5.3.34}$$

$$M_{t'} = 177 \text{ GeV} \tag{5.3.35}$$

$$M_{b'} = 172 \text{ GeV} \tag{5.3.36}$$

Because the top tends to get a smaller Yukawa coupling than the fourth generation quarks, it is necessary to give it a much larger Yukawa coupling at the Planck scale so that it can attain a pole mass within the current experimental limits.

In figure 5.5 we can see that the three quarks will get a similar mass at the

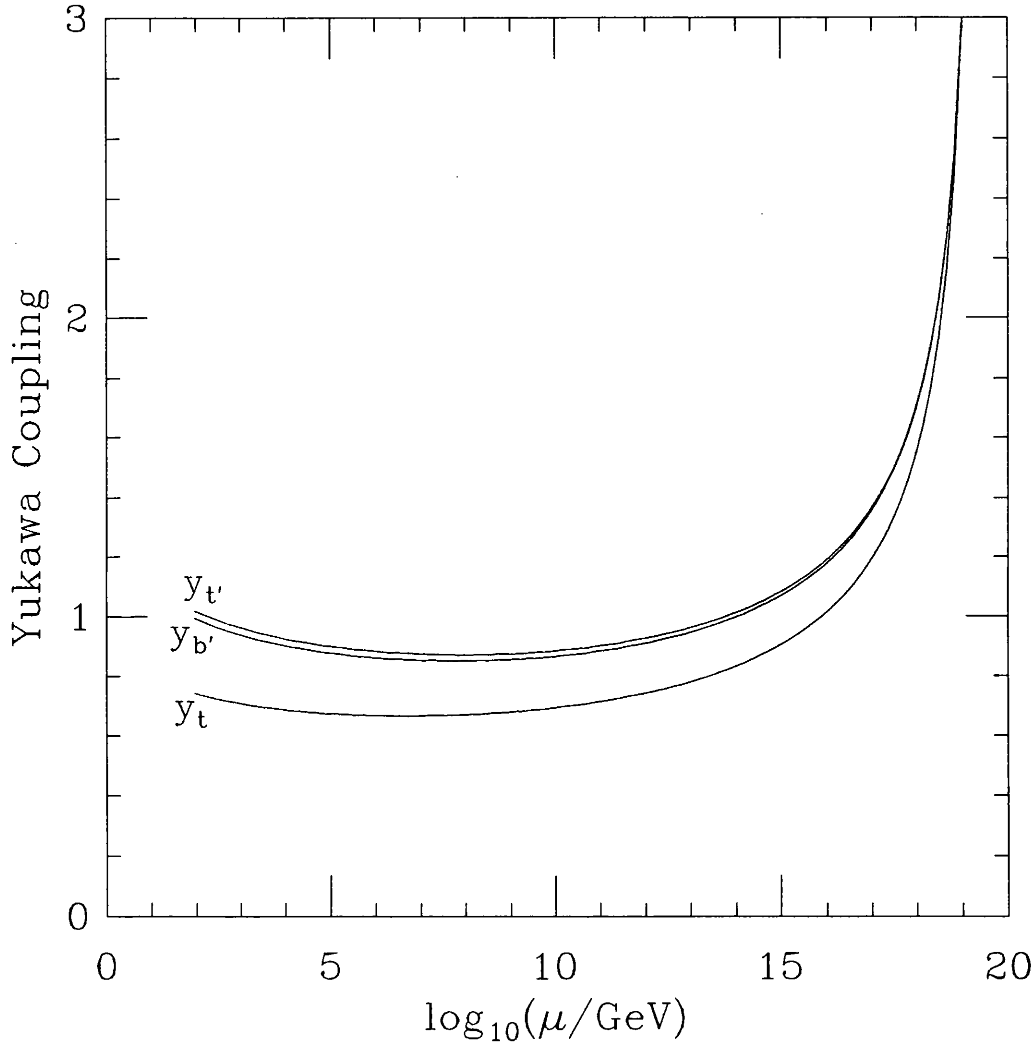

Figure 5.4: Running Yukawa couplings for the top and fourth generation quarks. All Yukawa couplings are chosen to be 3.0 at the Planck scale.

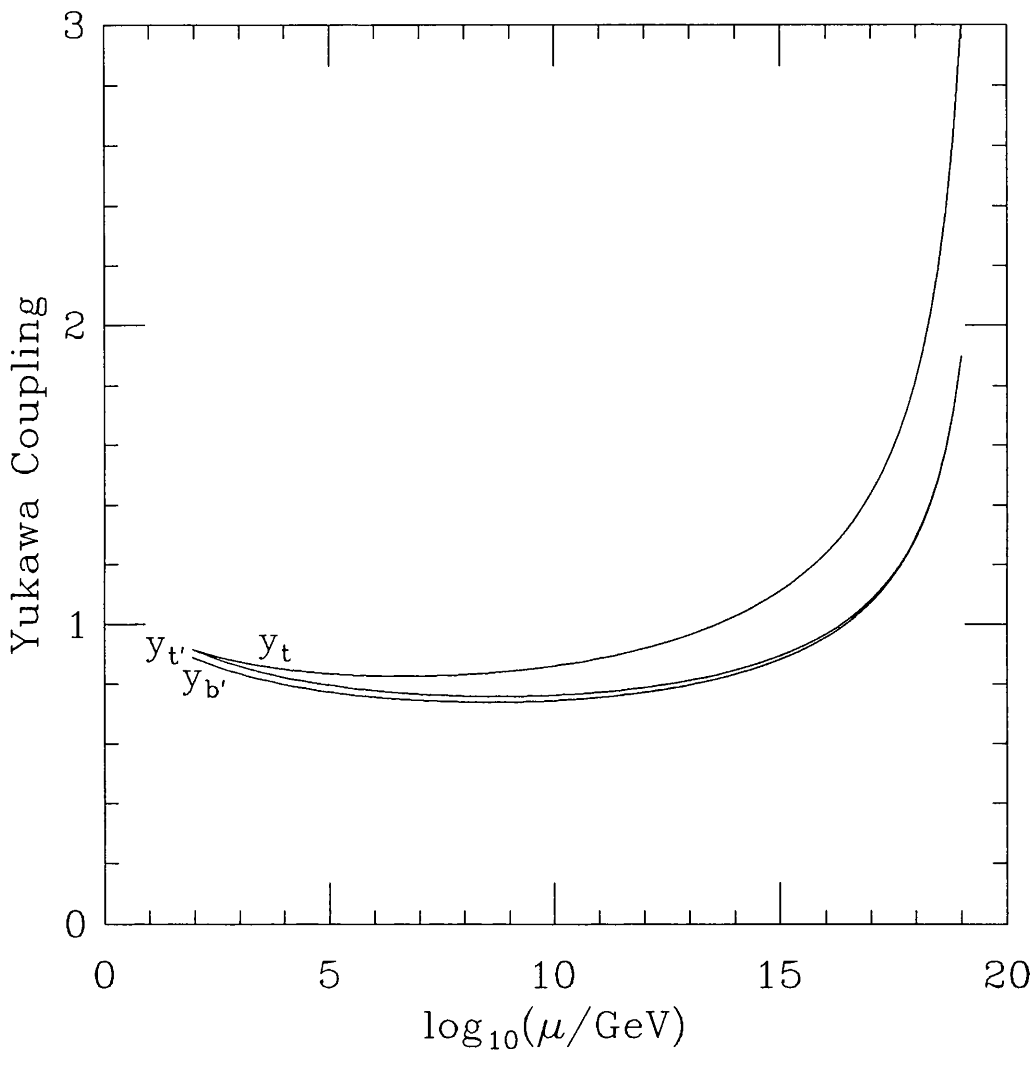

Figure 5.5: Running Yukawa couplings for the top and fourth generation quarks. The top quark gets a pole mass of 160 GeV, near the bottom end of the experimental range. The fourth generation quarks end up with a similar mass.

electroweak scale if we choose,

$$y_t(M_{Planck}) = 3.0 \tag{5.3.37}$$

$$y_{t'}(M_{Planck}) = 1.9 \tag{5.3.38}$$

$$y_{b'}(M_{Planck}) = 1.9 \tag{5.3.39}$$

These values lead to the following pole masses:

$$M_t = 160 \text{ GeV} \tag{5.3.40}$$

$$M_{t'} = 159 \text{ GeV} \tag{5.3.41}$$

$$M_{b'} = 155 \text{ GeV} \tag{5.3.42}$$

The initial values of the Yukawa couplings at the Planck scale were chosen so that the top quark would just lie within the experimental range for its measured pole mass. Since the fourth generation quarks then got roughly the same mass, we can assume that they cannot be heavier than the top quark. This is certainly the case when we consider that the inclusion of the $SU(5)$-"quarks" will further reduce the average Yukawa coupling at the electroweak scale.

In figure 5.6 we can see the value of $Y_2(S)$ for the three cases considered in this section. We see that although the different cases produced different quark masses, the value of $Y_2(S)$ was practically the same in all three cases. This demonstrates that for a fixed number of fermions we can increase the mass of some at the expense of others since the total sum of Yukawa couplings squared must remain approximately the same.

Only $SU(5)$-"Quarks"

We shall now examine the $SU(5)$ quasi-fixed point without any heavy SM fermions. This will show the similarity between the quasi-fixed point for the $SU(5)$-"quarks" and a generation of SM quarks. We can also see how the value of the quasi-fixed point depends on the strength of the $SU(5)$ gauge coupling constant at the electroweak scale. Since there is no experimental limit on the coupling constant we

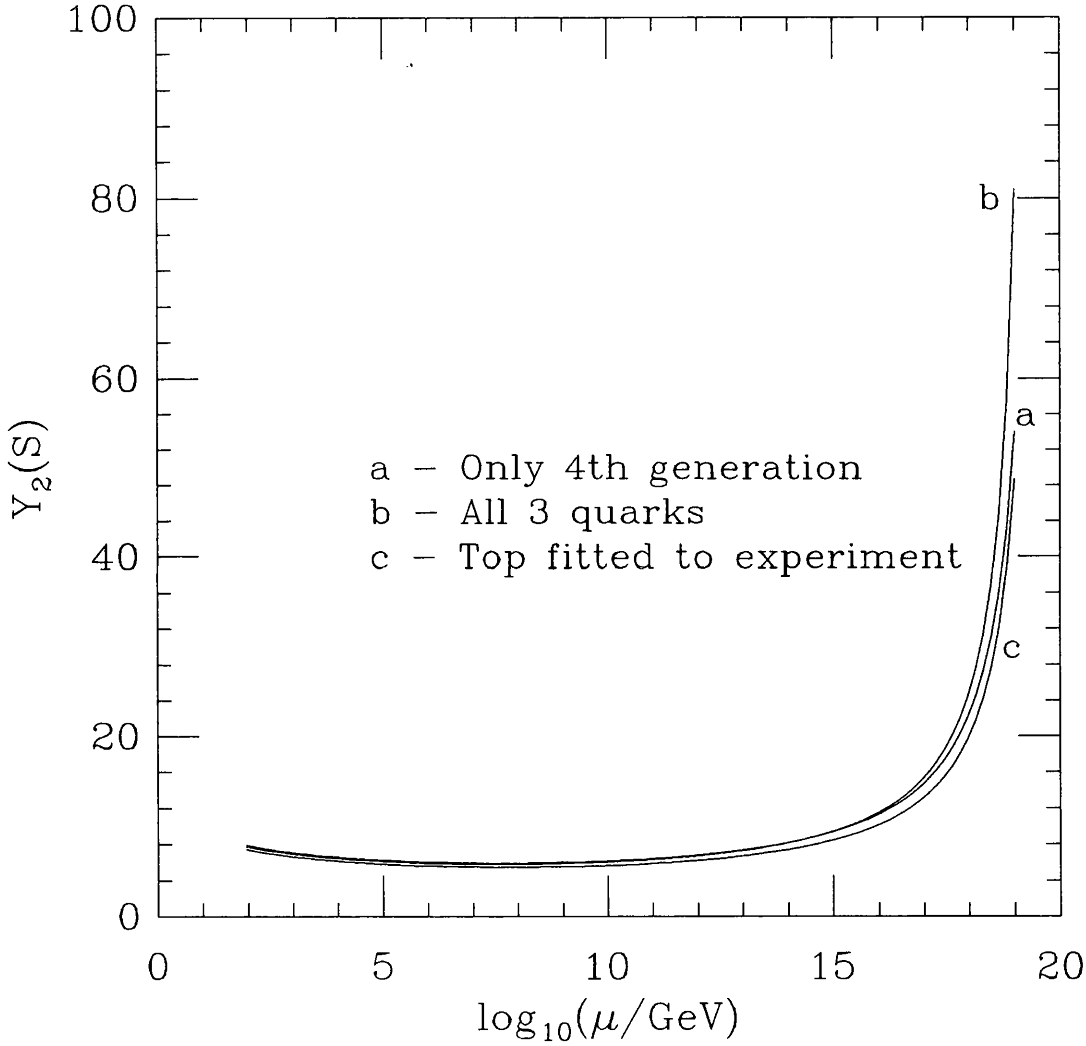

Figure 5.6: Fixed point value of $Y_2(S)$ for figures 5.3, 5.4 and 5.5.

are free to choose any value we wish. However, since no $SU(5)$-"quarks" have been observed we must ensure that either they have masses large enough to avoid detection or that the $SU(5)$ confinement scale is so high that the lightest $SU(5)$ bound states would have masses beyond the limits of current accelerators. Again we will choose the fermions within the $SU(2)$ doublet to be almost degenerate so that $T_{new} \approx 0$ and $U_{new} \approx 0$. To do this simply we choose $y_{5u}(M_{Planck}) = y_{5d}(M_{Planck})$ as we did for the fourth generation quarks in the last section.

In figure 5.7 we start with the Yukawa couplings equal to 3.0 at the Planck scale and choose $\alpha^{-1}(M_Z) = 2.0$. We can see that between the Planck scale and the electroweak scale the Yukawa couplings are about 0.8 but at the electroweak scale they are about 1.5. This is because the value of the quasi-fixed point Yukawa couplings depends on the strength of the gauge coupling constants. The $SU(5)$ gauge coupling constant is relatively small at energies considerably higher than the electroweak scale and so the fixed point values at these scale are not very large. It is only when the $SU(5)$ gauge coupling dramatically increases in strength near its confinement scale (of the same order of magnitude as the electroweak scale) that the quasi-fixed point values increase [2]. Since in this example we have chosen $\alpha_5(M_Z) \gg \alpha_3(M_Z)$ the $SU(5)$-"quarks" attain larger Yukawa couplings than the quarks in the corresponding example with only a fourth generation of SM quarks (More precisely, this difference is due to the fact that $\frac{72}{5}g_5^2(M_Z) > 8g_3^2(M_Z)$ which is the main difference between the RGEs for the $SU(5)$-"quarks" and the fourth generation of SM quarks in eqs. (5.3.15)-(5.3.23)). In this example the pole masses of the $SU(5)$-"quarks" are, using eq. (1.3.66),

$$M_{5u} = 233 \text{ GeV} \tag{5.3.43}$$

[2]The $U(1)$ and $SU(2)$ gauge coupling constants are much smaller than the $SU(5)$ gauge coupling constant and so do not have much effect. The $SU(3)$ gauge coupling constant has no effect since the $SU(5)$-"quarks" do not couple to the $SU(3)$ group, though in the general case it would affect the $SU(5)$-"quark" Yukawa couplings indirectly through its effect on the quark Yukawa couplings

$$M_{5d} \quad = \quad 236 \text{ GeV} \tag{5.3.44}$$

In figure 5.8 we can see the difference caused by the choice of a weaker $SU(5)$ interaction. With $\alpha_5^{-1}(M_Z) = 10.0$ we have $\alpha_5(M_Z) \approx \alpha_3(M_Z)$. We observe the same behaviour for the Yukawa couplings above the electroweak scale as in figure 5.7 but now the weaker $SU(5)$ coupling at the electroweak scale means that there is not such a sudden increase in the quasi-fixed point values. This leads to the following $SU(5)$-"quark" pole masses,

$$M_{5u} \quad = \quad 160 \text{ GeV} \tag{5.3.45}$$

$$M_{5d} \quad = \quad 162 \text{ GeV} \tag{5.3.46}$$

Figure 5.9 shows the value of $Y_2(S)$ for the $SU(5)$-"quark" fixed points with the different choices of $\alpha_5(M_Z)$. We can see that the value of $Y_2(S)$ is almost the same in both cases until close to the electroweak scale where it increases greatly for the case where $\alpha_5^{-1}(M_Z) = 2.0$ and only increases a little for the case where $\alpha_5^{-1}(M_Z) = 10.0$. This is simply due to the fact that $\alpha_5 \approx 0$ in both cases at scales above the electroweak scale but the difference between the two cases is much more significant at lower scales. Therefore we cannot give accurate predictions of the masses of the $SU(5)$-"quarks" because the choice of $\alpha_5(M_Z)$ is arbitrary and the masses are so sensitive to this value.

All New Fermions and Top Quark with Equal Yukawa Couplings

We will now set the Yukawa couplings of all new fermions and the top quark to be equal and examine their relative values at the electroweak scale. We choose the value for the $SU(5)$ fine structure constant to be

$$\alpha_5^{-1}(M_Z) = 2.0 \tag{5.3.47}$$

In figure 5.10 we can see that when we choose all the Yukawa couplings of the $SU(5)$-"quarks", fourth generation quarks and the top quark to be 2.0 at

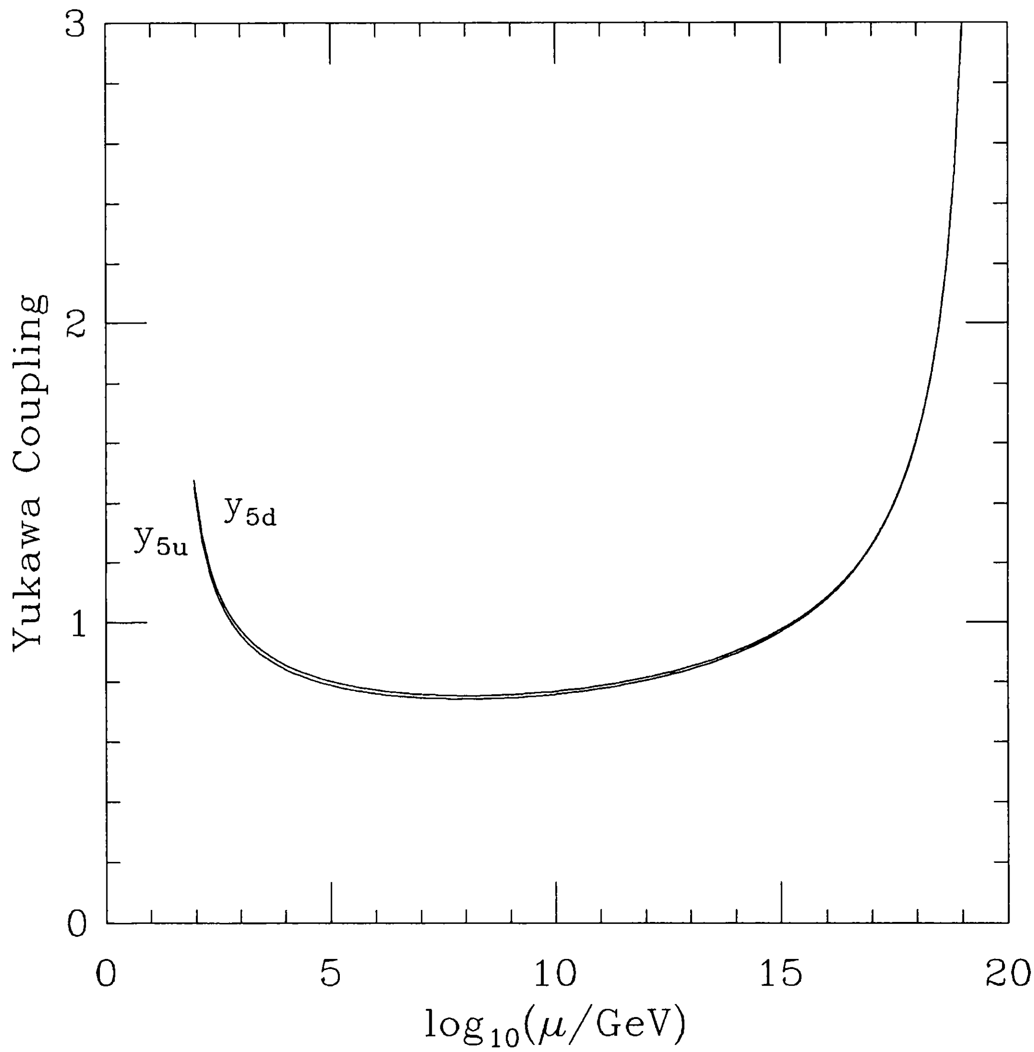

Figure 5.7: Fixed point value of the Yukawa couplings of the $SU(5)$-“quarks” when they are set to 3.0 at the Planck scale. The $SU(5)$ fine structure constant is chosen to be $\alpha_5^{-1}(M_Z) = 2.0$.

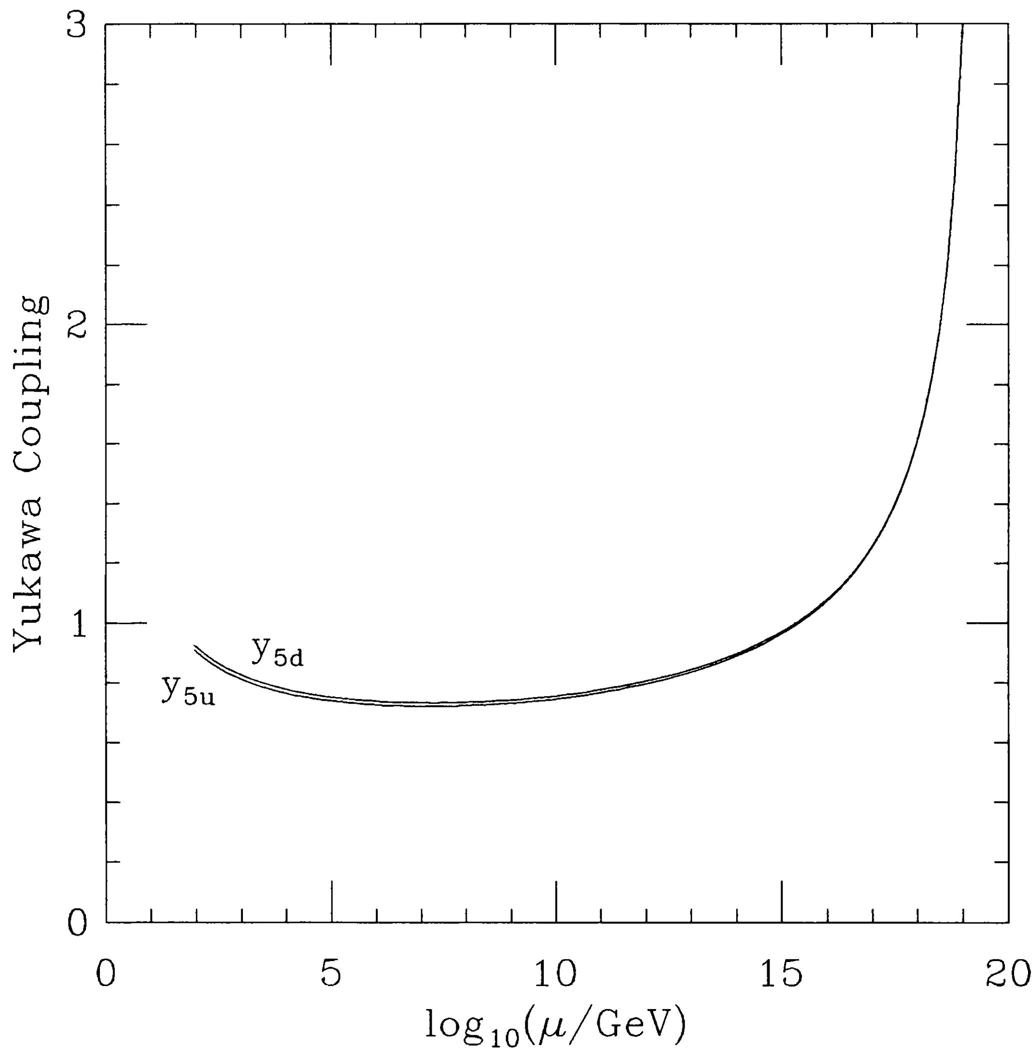

Figure 5.8: Fixed point value of the Yukawa couplings of the $SU(5)$-"quarks" when they are set to 3.0 at the Planck scale. The $SU(5)$ fine structure constant is chosen to be $\alpha_5^{-1}(M_Z) = 10.0$.

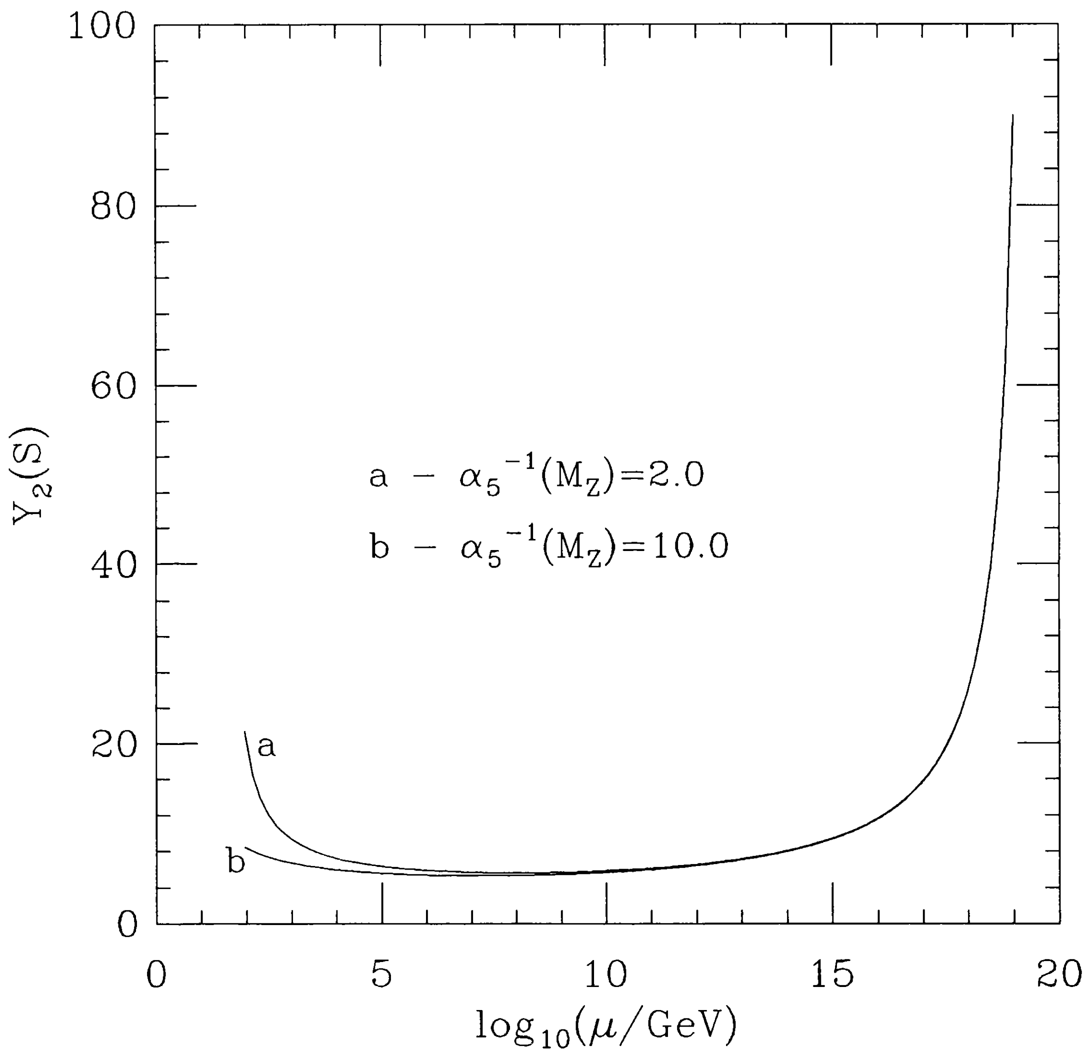

Figure 5.9: Fixed point values of $Y_2(S)$ for the $SU(5)$-"quarks" with different values of $\alpha_5(M_Z)$ corresponding to figures 5.7 and 5.8.

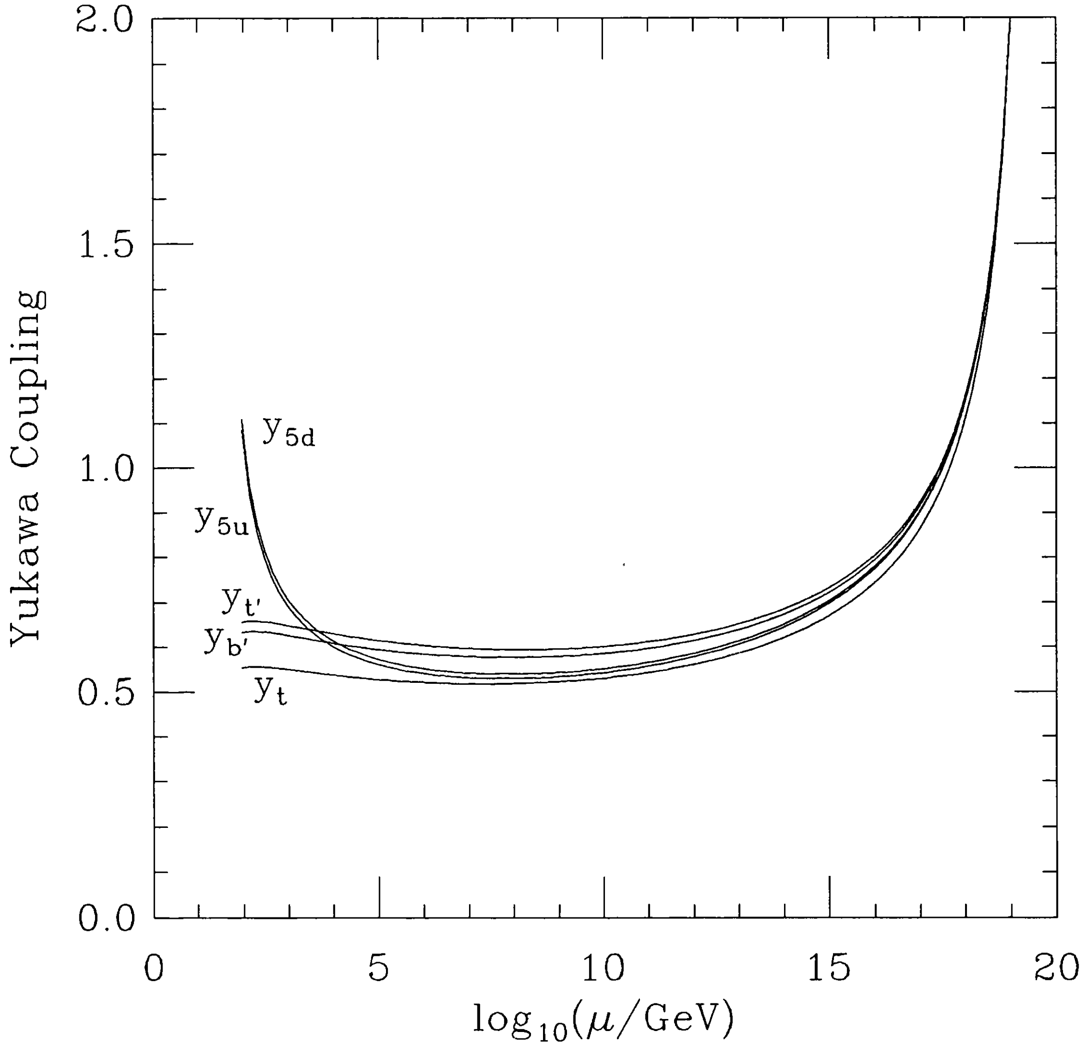

Figure 5.10: Quasi-fixed Yukawa couplings at the electroweak scale when all Yukawa couplings are chosen to be 2.0 at the Planck scale and $\alpha_5^{-1}(M_Z) = 2.0$.

the Planck scale, the $SU(5)$-"quarks" get much larger masses at the electroweak scale due to the fact that $g_5(M_Z) > g_3(M_Z)$. This is to be expected from the comparison figures 5.4 and 5.7 where the $SU(5)$-"quarks" got a much larger mass than the SM quarks. The actual pole masses in this case are,

$$M_{5u} = 184 \text{ GeV} \tag{5.3.48}$$
$$M_{5d} = 187 \text{ GeV} \tag{5.3.49}$$
$$M_{t'} = 116 \text{ GeV} \tag{5.3.50}$$
$$M_{b'} = 112 \text{ GeV} \tag{5.3.51}$$
$$M_t = 98 \text{ GeV} \tag{5.3.52}$$

Obviously the top quark gets a mass much lower than its experimental limit and so we must choose the Yukawa coupling of the top quark to be larger than the Yukawa couplings of the $SU(5)$-"quarks" at the Planck scale. The top quark also ends up with a Yukawa coupling less that those of the fourth generation quarks as in figure 5.4. In some sense the small mass of the top quark makes the model a bit unnatural, especially when we consider the fact that we end up choosing the Yukawa coupling of the top quark to be much larger than the Yukawa couplings of all the other fermions to fit in with experimental limits.

5.3.2 Limits on Masses Consistent with Experimental Limits

In this section we shall try to choose Yukawa couplings at the Planck scale for all the heavy fermions in order to get an experimentally acceptable model. We will use the examples in section 5.3.1 to guide us. We also comment on the consistency of the model with precision electroweak data.

The values of the Yukawa couplings at the Planck scale in fig. 5.11 have been chosen so that the top quark pole mass is consistent with current experimental limits ($M_t \approx 170$ GeV) and the fourth generation quark pole masses are above the current experimental lower limit of 130 GeV. Also $M_{b'} \sim M_{t'}$ and $M_{5u} \sim M_{5d}$ have been chosen so that there is only a small contribution to the ρ parameter described in section 3.3.

We have discussed the radiative corrections in section 3.3 and this model appears to be consistent with current experimental data since we have arranged T_{new} and U_{new} to be approximately zero. The only non-zero contribution to the three parameters used to describe the precision electroweak data is the contribution to the S parameter from the 8 new $SU(2)$ doublets. This causes the S parameter to deviate from the experimental mean value by slightly less than 2 standard deviations. However, as noted in section 3.3, it is difficult to calculate the theoretical contributions and the perturbative estimates used may not be very accurate. Nevertheless we would consider this model to be consistent with the current experimental precision electroweak data.

Table 5.3 gives the values of the Yukawa couplings at M_Z and the corresponding pole masses using eq. (1.3.65) for the quarks and eq. (1.3.66) for the $SU(5)$-"quarks". These masses should be considered upper limits on the masses of the fermions only for this particular choice of Yukawa couplings at the Planck scale. For other choices of Yukawa couplings at the Planck scale we could, for example, increase the mass of the fourth generation of quarks but this would have to be compensated for by a reduction in the mass of some of the other fermions.

These values for the masses are consistent with current experimental limits but are not so high that the new fermions could remain undetected for long. In fact the quark masses may even be within the limits of current accelerators. It is not clear whether the fermions coupling to $SU(5)$ could be observed, since they would obviously be confined by the $SU(5)$ gauge interaction which we take to confine above the electroweak scale. So even if they have masses of about 100 GeV, they would be much more difficult to detect than quarks with greater masses. For this reason we consider the clearest evidence for this model would come from the detection of a fourth generation quark. The masses of some of the new fermions

Table 5.3: Infrared fixed point Yukawa couplings and corresponding running masses (for $F_\pi = 75$ GeV) for a particular choice of Yukawa couplings at the Planck scale.

Fermion	Yukawa Coupling	Pole Mass (GeV)
y_t	1.00	175
$y_{t'}$	0.77	135
$y_{b'}$	0.75	131
y_{5u}	0.38	94
y_{5d}	0.40	97

could be increased, but not by much, since this would mean a reduction in the mass of other fermions. This means that this model is consistent and relatively easy to test.

For completeness we present the fixed point value of λ which determines the mass of the Higgs boson. Figure 5.12 shows how the value of λ is large near the Planck scale when we choose $\lambda(M_{Planck}) = 3.0$ but decreases at lower energies. The final value of $\lambda(M_Z)$ is only about 0.5. This is a fixed point value since we would obtain this low energy value of λ for a wide range of initial choices of λ at the Planck scale. This leads to a Higgs running mass of,

$$M_H \approx 172 \text{ GeV} \tag{5.3.53}$$

We would consider this to be a maximum limit on the Higgs mass if this model is to be perturbatively valid up to the Planck scale. Notice that the effect of the extra fermions is to reduce the limit on the Higgs mass relative to its SM limit, eq.(5.3.29). Part of this effect is due to the reduction of the Higgs VEV but the limit of eq. (5.3.53) is still much less than the value given by eq.(5.3.31). Since we are at the fixed point for the fermion masses, we can assume that the Higgs must have a mass very close to the mass given in eq. (5.3.53). This is because a lower mass would mean that when running λ from the electroweak scale up to the Planck scale it would become negative and so the vacuum would be unstable. If we are exactly at the fixed point for the Yukawa couplings the Higgs must get this mass. But if the Yukawa couplings are slightly lower then there will be an upper bound on $\lambda(M_Z)$ above which λ will become infinite when run up to the Planck scale and a lower bound below which it will become negative when run up to the Planck scale.

5.4 Conclusions

We have shown that we can have a self-1consistent model with a fourth generation of quarks and a generation of $SU(N)$-"quarks" where $N \geq 5$ is odd. By examining precision electroweak data we have limited the models to the one with $N = 5$. We can then produce a model which also appears to be consistent with experiment. However, there are several difficulties with this model.

We can produce masses of new fermions so that they are heavier than current experimental limits but it is not easy to do so. We must carefully choose the Yukawa couplings of all the heavy fermions at the Planck scale. One of the main problems is that it is not easy to produce a large enough top mass. To do this we must choose the Yukawa coupling of the top quark at the Planck scale to be greater than all the other Yukawa couplings. The variables are constrained so much by experiment that it could reasonably be argued that there isn't much room left for the model. However, even with the choice of parameters made in the last section of this chapter, the model would not be very difficult to test experimentally and so we would claim that it does at least have the advantage of being easily testable.

Another example of the limited range of parameters in our model is the fact that we have required the masses of the t' and b' as well as the masses of the $SU(5)$-"quarks" to be similar. This is required so that our model doesn't contribute to the T and U parameters and also so that all these fermions are more massive than the experimental limits. In some ways it could be argued that it is natural for fermions in the same $SU(2)$ doublet to have similar masses but this is not what is observed in the SM.

Perhaps the weakest point in our argument that this model could be consistent with experiment is that the S parameter differs from its measured value by 2 standard deviations. On its own this wouldn't be too bad but if we consider that most non-perturbative estimates of contributions to the S parameter are greater than the perturbative estimates then we can conclude that the S parameter predicted by our model probably differs by more than 2 standard deviations from the measured value.

So overall we could say that our model is consistent with experiment but only just. The only reason we would not consider investigating the agreement with precision electroweak data in more detail is that the masses for the fourth generation of quarks are so close to the current experimental limits that direct evidence for or against the model should soon be available.

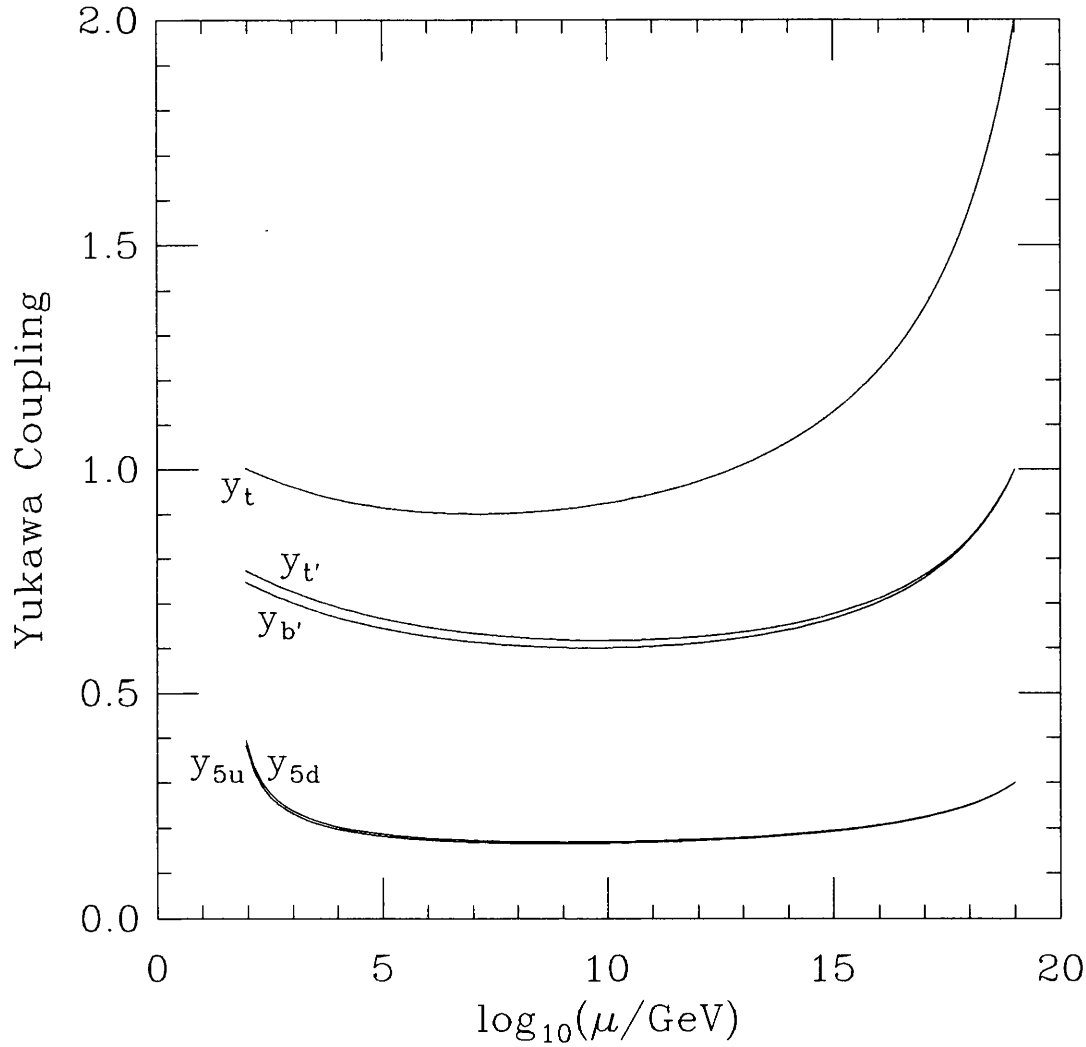

Figure 5.11: An example of running Yukawa couplings for all fermions with a mass the same order of magnitude as the electroweak scale. The values were chosen at the Planck scale and run down to M_Z so that all the fermions would have a mass allowed by current experimental limits.

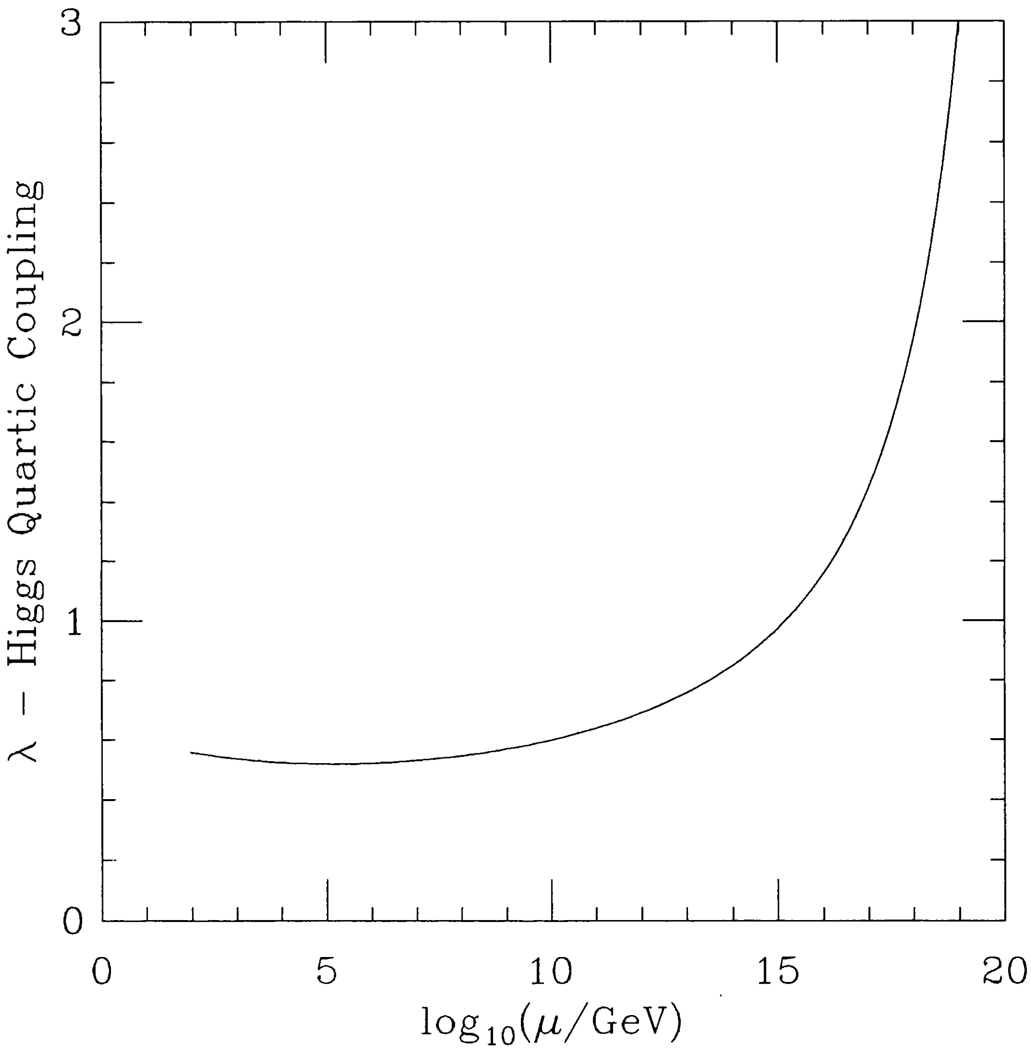

Figure 5.12: Fixed point value of λ, the Higgs quartic coupling. This graph, along with the estimated value of $< \phi_{WS} >= 234$ GeV, leads to the approximate upper bound on the Higgs mass of 172 GeV.

Chapter 6

Fermion Masses from Diagonal Symmetry

6.1 Introduction

This chapter is about a slightly different type of model. In the previous chapters we have been examining models with new physics close to the electroweak scale. But now we turn our attention towards a model where we do not introduce any new low mass fermions and where all new physics will occur near the Planck scale. One model examined in [17] has gauge group,

$$SMG_1 \otimes SMG_2 \otimes SMG_3 \otimes U(1)_f \tag{6.1.1}$$

The SMG will emerge as a diagonal subgroup. In the full group the ith SM generation was considered to transform only under SMG_i. The $U(1)_f$ charges are not determined for the SM fermions and can be chosen freely, provided all anomalies are cancelled.

As in the other models in this thesis, we have a charge quantisation rule. For this group there are, in fact, 4 charge quantisation rules. As we have already discussed in section 2.3.2, the charge quantisation rules are chosen to maximise the χ parameter. The four rules are; a SM charge quantisation rule for each SMG

factor and a quantisation of the $U(1)_f$ charges:

$$\frac{y_i}{2} + \frac{1}{2}d_i + \frac{1}{3}t_i \equiv 0 \pmod 1 \tag{6.1.2}$$

$$y_f \equiv 0 \pmod 1 \tag{6.1.3}$$

To simplify the notation we will normalise all charges to be integer by defining:

$$Q_i \equiv 3y_i \tag{6.1.4}$$

$$Q_f \equiv y_f \tag{6.1.5}$$

This model was used to provide an explanation of the vast range of fermion masses in the SM without requiring such a range of fundamental Yukawa couplings. This was done by assuming that the fermion masses (apart from the top quark) were suppressed by different amounts due to the details of the symmetry breaking down to the *SMG*. This suppression is due to the fermion transformation properties under the full gauge symmetries which are assumed to be partially conserved at low energies. These symmetries are called partially conserved chiral symmetries (PCCSs). The details depend on the complete symmetry breaking mechanism but an order of magnitude estimate of the amount of suppression of each Yukawa coupling can be obtained from the ratios of each symmetry breaking scale to the fundamental scale. We assume that the fundamental scale is the Planck scale. Essentially, the Yukawa couplings of the fermions to the SM Higgs are viewed as effective couplings in a low energy effective theory. The fundamental Yukawa couplings are couplings to the heavy Higgs bosons responsible for the diagonal symmetry breaking and these couplings are assumed to be of order 1.

We can write the part of the low energy Lagrangian responsible for fermion mass generation as,

$$\mathcal{L} = \overline{U_R} M_U U_L + \overline{D_R} M_D D_L + \overline{l_R} M_l l_L + \text{h.c.} \tag{6.1.6}$$

where U, D and l are the three generations of up quarks, down quarks and electron-like leptons. In other words:

$$U = \begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad D = \begin{pmatrix} d \\ s \\ b \end{pmatrix} \qquad l = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \tag{6.1.7}$$

The mass matrices are related to the Yukawa matrices by,

$$M_t = Y_t \frac{< \phi_{WS} >}{\sqrt{2}} \tag{6.1.8}$$

where $t = U$, D or l.

Using the notation of [17] we define δ_i to be the ratio of the symmetry breaking scale of $SU(3)_i$ to the fundamental scale. Similarly we use ϵ_i for $SU(2)_i$. We then estimate the values of the Yukawa matrices by assuming that all elements are of order 1 unless suppressed. We make the assumption that each entry is suppressed by a factor of δ_i if it connects a triplet of $SU(3)_i$ to a singlet of $SU(3)_i$, and similarly by ϵ_i for elements which connect doublets of $SU(2)_i$ to singlets of $SU(2)_i$. We use a general metric to parameterise the suppression due to the abelian symmetries. This suppresses each element $Y_{\alpha\beta}$ by a factor, $exp(-\sqrt{(Q^L_{i\alpha} - Q^R_{i\beta})g_{ij}(Q^L_{j\alpha} - Q^R_{j\beta})}$, where g_{ij} is a metric and a sum over i and j which run over 1, 2, 3 and f is implicit. $Q^L_{i\alpha}$ ($Q^R_{i\alpha}$) is the $U(1)_i$ charge of the left- (right-) handed fermion in generation α (type U, D or l depending on which Yukawa matrix is being considered).

The Yukawa matrices are diagonalised algebraically to give an order of magnitude estimate for the masses and mixing angles of all the SM fermions. Numerical methods for diagonalising the matrices cannot be used because we are only estimating the order of magnitude of each entry in the matrices and we have no information about the relative phases of the entries which would be complex numbers.

The masses obtained from the Yukawa matrices are assumed to be the running masses at the Planck scale. We compare the order of magnitude estimates for the masses and mixing angles to the experimental masses run up to the Planck scale. However, in order to make the results more meaningful we will present the approximate values at 1 GeV except for the c and b quarks where we will present the pole masses. For simplicity, since everything is calculated only order of magnitude, we don't use the full RGEs to calculate our estimates at 1 GeV from our estimates at the Planck scale. Instead we simply multiply each fermion mass by a factor which approximates its observed ratio of running masses at 1 GeV to the Planck scale for the values of the experimental masses. The mixing angles don't vary much with scale as shown in [30] and so we assume that the estimates we obtain from the mass matrices are directly comparable to experimentally measured values. We use all SM fermions (except the top quark) and the mixing angles; V_{us}, V_{cb} and V_{ub}, in the fit. We use a computer program to find the best choice of the parameters g_{ij}, ϵ_k and δ_k (where i and j run over 1, 2, 3 and f and k runs over 1, 2 and 3) so that we minimise,

$$\chi^2 \equiv \sum_f [\ln(m_f^{est}) - \ln(m_f^{exp})]^2 + \sum [\ln(V_{f'}^{est}) - \ln(V_{f'}^{exp})]^2 \tag{6.1.9}$$

where est refers to the estimated values from the mass matrices and exp refers to the experimentally measured values. Also, f labels all 8 massive SM fermions other than the top quark and f' labels us, cb and ub corresponding to the three mixing angles stated above. For the top quark, we use the value of the pole mass:

$$M_t = 174 \text{ GeV} \tag{6.1.10}$$

The top mass affects the fit in two ways. Firstly, the RGEs for the Yukawa couplings of all fermions depend on the large Yukawa coupling of the top quark. But, since the same methods and a modified version of the computer program used for the analysis of [17] was used, another effect is present. This is due to the fact that the Yukawa matrices were normalised so that all the entries were given relative to the top Yukawa coupling. The top Yukawa coupling has been chosen not to be suppressed but that still means that it is of order 1, not exactly

1. Therefore, by defining it to be exactly 1, the precise value has to be absorbed into all the other entries. This will bias the fit depending on whether the top quark Yukawa coupling happens to be greater than or less than 1. Therefore, we can only really compare different models where we have chosen the same top mass. The top mass used means that the Yukawa coupling is approximately 1 at the electroweak scale and so the fit should be roughly the same as a fit where this normalisation is not used. We will give all the results for the top mass chosen to be, $M_t = 174$ GeV.

6.2 Model

One of the models considered in [17] was based on the gauge group of eq. (6.1.1). The $U(1)_f$ group had been added to the simpler group, $SMG_1 \otimes SMG_2 \otimes SMG_3$, also examined in [17], in order to produce a splitting of masses within each generation. Since this was not necessary for the first (lightest) generation, it was assumed that all fermions in the first generation did not couple to the $U(1)_f$ group, i.e. $Q_f = 0$ for all first generation fermions. It was found in [17] that the only way to cancel all anomalies was to give the second and third generation fermions values of Q_f that were proportional to their values of conventional weak hypercharge or the values shown in table 6.1. Since the values of weak hypercharge couldn't produce the correct mass structure, the latter alternative obviously had to be used.

The number of free parameters was reduced by the requirement that the top mass should not be suppressed. Since it was assumed that the top mass would come from the entry $(M_U)_{3\,2}$, the following seven conditions were required:

$$\epsilon_3 = 1 \tag{6.2.11}$$

$$\delta_2 = 1 \tag{6.2.12}$$

$$\delta_3 = 1 \tag{6.2.13}$$

$$g_{ij}(Q^L_{j\,t} - Q^R_{j\,c}) = 0 \tag{6.2.14}$$

Table 6.1: Values of Q_f for the second and third generation fermions in the model of [17]. The first generation fermions have $Q_f = 0$.

Fermion	Q_f	Fermion	Q_f
c_L	0	t_L	0
c_R	1	t_R	-1
s_R	-1	b_R	1
μ_L	0	τ_L	0
μ_R	-1	τ_R	1

The last condition allows us to eliminate 4 of the 10 parameters from the metric. We choose to eliminate g_{13}, g_{23}, g_{2f} and g_{3f}. In fact it was also found that the best fit was always for $\epsilon_2 = 1$.

However, the conditions on the parameters; ϵ_2, ϵ_3, δ_2 and δ_3, mean that essentially the PCCSs were considered to be:

$$SU(3)_1 \otimes SU(2)_1 \otimes U(1)_1 \otimes U(1)_2 \otimes U(1)_3 \otimes U(1)_f \tag{6.2.15}$$

rather than the full group given in eq. (6.1.1). There are two ways to look at this. Firstly, if we imagine that the group of eq. (6.1.1) is the full group in this model then it is not surprising that the symmetry breaking scales for different parts of the group are different. By chance some parts will break at higher energies than others. This means that they would not contribute to suppression of elements in the mass matrices since these symmetries would not be partially conserved. The alternative view is that the fact that the top quark mass is not suppressed and $\epsilon_2 = 1$ is due to the fact that the full group of PCCSs is not in fact the group

Table 6.2: Values of Q_f for the second and third generation fermions in the model with group given by eq. (6.2.15). The first generation fermions have $Q_f = 0$ as in the model of [17].

Fermion	Q_f	Fermion	Q_f
c_L	0	t_L	0
c_R	3	t_R	-3
s_R	-3	b_R	3
μ_L	-4	τ_L	4
μ_R	-5	τ_R	5

of eq. (6.1.1) but only the subgroup given by eq. (6.2.15). If we accept the latter point of view, we can then choose different values of the $U(1)_f$ charges since there are not as many anomaly constraints for the subgroup as for the larger group. In particular, we can choose the $U(1)_f$ charges given in table 6.2.

We will now examine both models. If the full gauge group is not $SMG^3 \otimes U(1)_f$ then we would not expect the $U(1)_f$ charges which are required to cancel the anomalies in this larger group to give a better fit than another set of charges which also cancels the anomalies in the smaller group. So, by fitting both models to the data, we would expect the best fit in both cases to be reasonably good. If it turns out that the original set of $U(1)_f$ charges gives a much better fit then we would suspect that the full gauge group was in fact $SMG^3 \otimes U(1)_f$. Obviously if the second choice of $U(1)_f$ charges gives a better fit then we would prefer this model and conclude that the full group was not in fact $SMG^3 \otimes U(1)_f$.

6.3 Results

In the model with $U(1)_f$ charges given in table 6.1, we obtained a good fit with the experimental measurements of the fermion masses and mixing angles. The value of χ^2 was:

$$\chi^2 = 5.2 \tag{6.3.16}$$

and the estimated masses and mixing angles are given in table 6.3. As in [17], it was found that some of the parameters were not required for the best fit. In fact we could choose; either δ_1 or ϵ_1 to be equal to 1 and either g_{12} or g_{1f} to be zero as well as $\epsilon_2 = 1$. This means that the fit to the 11 parameters was actually made using only 6 parameters.

In our new model with $U(1)_f$ charges given in table 6.2, we obtained the following fit with the larger value of χ^2:

$$\chi^2 = 6.9 \tag{6.3.17}$$

and the estimated masses and mixing angles are given in table 6.3.

In the original model, the important elements in the mass matrices were:

$$M_U = \begin{bmatrix} m_u & 0 & 0 \\ 0 & 0 & m_t \\ 0 & m_c & 0 \end{bmatrix} \tag{6.3.18}$$

$$M_D = \begin{bmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{bmatrix} \tag{6.3.19}$$

Table 6.3: Masses and mixing angles fitted using the $U(1)_f$ charges given in table 6.1.

Quantity	FittedValue	ExperimentalValue
$m_e(1\ \mathrm{GeV})$	1.0 MeV	0.5 MeV
$m_\mu(1\ \mathrm{GeV})$	156 MeV	105 MeV
$m_\tau(1\ \mathrm{GeV})$	1.51 GeV	1.78 GeV
$m_d(1\ \mathrm{GeV})$	4.9 MeV	9.2 MeV
$m_u(1\ \mathrm{GeV})$	4.9 MeV	5.2 MeV
$m_s(1\ \mathrm{GeV})$	758 MeV	194 MeV
M_c	0.76 GeV	1.5 GeV
M_b	5.7 GeV	4.9 GeV
V_{us}	0.21	0.22
V_{cb}	0.013	0.042
V_{ub}	0.0027	0.0027

Table 6.4: Masses and mixing angles fitted using the $U(1)_f$ charges given in table 6.2.

Quantity	FittedValue	ExperimentalValue
$m_e(1\text{ GeV})$	0.82 MeV	0.50 MeV
$m_\mu(1\text{ GeV})$	580 MeV	105 MeV
$m_\tau(1\text{ GeV})$	4.6 GeV	1.78 GeV
$m_d(1\text{ GeV})$	4.0 MeV	9.2 MeV
$m_u(1\text{ GeV})$	4.0 MeV	5.2 MeV
$m_s(1\text{ GeV})$	1060 MeV	194 MeV
M_c	1.3 GeV	1.5 GeV
M_b	3.4 GeV	4.9 GeV
V_{us}	0.22	0.22
V_{cb}	0.012	0.042
V_{ub}	0.0028	0.0027

$$M_l \; = \begin{bmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{bmatrix} \tag{6.3.20}$$

In this model the matrices has the same form except that the element in M_l associated with m_μ was generally larger than the one associated with m_τ. Therefore, we assumed the lepton matrix to take the form:

$$M_l = \begin{bmatrix} m_e & 0 & 0 \\ 0 & m_\tau & 0 \\ 0 & 0 & m_\mu \end{bmatrix} \tag{6.3.21}$$

The elements labelled 0 are not exactly zero but they are not significant for producing order of magnitude masses. The structure of these matrices leads to the following order of magnitude predictions for the masses (relative to the top mass) and mixing angles in terms of the free parameters used in the fit: for the original model:

$$m_u \approx \epsilon_1 e^{-\sqrt{9g_{11}}} \tag{6.3.22}$$

$$m_c \approx e^{-\sqrt{45g_{22}+\frac{45}{4}g_{33}-\frac{9}{4}g_{ff}}} \tag{6.3.23}$$

$$m_t = 1 \tag{6.3.24}$$

$$m_d \approx m_u \tag{6.3.25}$$

$$m_s \approx e^{-\sqrt{-3g_{22}+\frac{3}{4}g_{33}+\frac{9}{4}g_{ff}}} \tag{6.3.26}$$

$$m_b \approx e^{-\sqrt{16g_{22}+8g_{33}}} \tag{6.3.27}$$

$$m_e \approx m_u \tag{6.3.28}$$

$$m_\mu \approx m_s \tag{6.3.29}$$

$$m_\tau \approx m_b \tag{6.3.30}$$

$$V_{us} \approx \frac{\epsilon_1 \delta_1}{m_c} e^{-\sqrt{g_{11}+64g_{22}+12g_{33}-3g_{ff}-32g_{12}-6g_{1f}}} \tag{6.3.31}$$

$$V_{cb} \approx \frac{1}{m_b} e^{-\sqrt{45g_{22}+\frac{9}{4}g_{33}-\frac{5}{4}g_{ff}}} \tag{6.3.32}$$

$$V_{ub} \approx V_{us} V_{cb} \tag{6.3.33}$$

and for the new model:

$$m_u \approx \epsilon_1 e^{-\sqrt{9g_{11}}} \tag{6.3.34}$$

$$m_c \approx e^{-\sqrt{45g_{22}+\frac{45}{4}g_{33}-\frac{81}{4}g_{ff}}} \tag{6.3.35}$$

$$m_t = 1 \tag{6.3.36}$$

$$m_d \approx m_u \tag{6.3.37}$$

$$m_s \approx e^{-\sqrt{-3g_{22}+\frac{3}{4}g_{33}+\frac{9}{4}g_{ff}}} \tag{6.3.38}$$

$$m_b \approx e^{-\sqrt{48g_{22}+6g_{33}-18g_{ff}}} \tag{6.3.39}$$

$$m_e \approx m_u \tag{6.3.40}$$

$$m_\mu \approx e^{-\sqrt{16g_{22}+8g_{33}-8g_{ff}}} \tag{6.3.41}$$

$$m_\tau \approx e^{-\sqrt{5g_{22}+\frac{1}{4}g_{33}-\frac{5}{4}g_{ff}}} \tag{6.3.42}$$

$$V_{us} \approx \frac{\epsilon_1 \delta_1}{m_c} e^{-\sqrt{g_{11}+64g_{22}+12g_{33}-27g_{ff}-32g_{12}-18g_{1f}}} \tag{6.3.43}$$

$$V_{cb} \approx \frac{1}{m_b} e^{-\sqrt{45g_{22}+\frac{9}{4}g_{33}-\frac{45}{4}g_{ff}}} \tag{6.3.44}$$

$$V_{ub} \approx V_{us} V_{cb} \tag{6.3.45}$$

These estimates give some explanation why this model does not provide such a good fit to the experimental data. In comparison to the original model, we do not immediately get the good Planck scale prediction:

$$m_b \approx m_\tau \tag{6.3.46}$$

However, we find that we do, as in the original model, automatically get the fairly good relation between the mixing angles:

$$V_{ub} \approx V_{us} V_{cb} \tag{6.3.47}$$

and the reasonable relation at the Planck scale that:

$$m_u \approx m_d \approx m_e \tag{6.3.48}$$

Overall, the fit is reasonably good but there are some problems. As in the original model, we find that the mass of the s quark is too large; even larger in this model. Also we now find that the μ and τ leptons get a mass larger than expected. The fit to the mixing angles is good apart from V_{cb} which is a bit small.

However, another problem with this model is that it requires one more free parameter than the original model. In the original model we could remove the parameters, ϵ_1, ϵ_2 and g_{1f} for example. Here we have already removed ϵ_2 by the definition of the gauge group. We can also set $\epsilon_1 = 1$ but we require all the remaining parameters for the fit. This is because we don't automatically have the relations between the masses which were present in the first model algebraically:

$$m_\mu \approx m_s \tag{6.3.49}$$

$$m_\tau \approx m_b \tag{6.3.50}$$

If we really want to reduce the number of parameters, it is possible to set $g_{12} = 0$ and get a slightly worse fit but then the original model is even more obviously better. So this model really requires 7 parameters to fit the data and even so it doesn't fit as well as the original model with only 6 parameters.

6.4 Conclusions

We have shown that the masses and mixing angles of the SM fermions can be fitted using the idea of the SMG being the diagonal subgroup of a larger group. The first model had already been analysed and was based on the largest group, $SMG^3 \otimes U(1)_f$. The second model had not been analysed before and was based on the smaller group, with PCCSs, $SU(3)_1 \otimes SU(2)_1 \otimes U(1)_1 \otimes U(1)_2 \otimes U(1)_3 \otimes U(1)_f$.

It was obvious that the first model gave a better fit than the second model. Both in terms of χ^2 being lower and the number of free parameters being fewer, the original model is better. However, the fit was not so much better that we could conclude that the full group must be $SMG^3 \otimes U(1)_f$. The particular model examined (defined by the particular choice of $U(1)_f$ charges to cancel the anomalies in the particular subgroup of $SMG^3 \otimes U(1)_f$) was not as good but there are other possible choices of subgroup and (more relevant) more choices of $U(1)_f$ charges to cancel the anomalies. Without studying these to see how they compare it isn't really possible to draw any strong conclusions. But we can perhaps conclude that this does at least point towards the full group, $SMG^3 \otimes U(1)_f$, being correct.

Chapter 7

Summary and Conclusions

We have discussed extensions of the SM having a similar gauge group structure to the SM itself. In particular we have been guided by the requirement of an anomaly-free theory, with additional mass protected fermions satisfying a generalised charge quantisation rule. We were thereby led to extend the SM cross product group, $U(1) \otimes SU(2) \otimes SU(3)$, by adding extra $SU(N)$ direct factors, with the 'N's greater than 3 and mutually prime. A generalised charge quantisation rule, involving each direct factor was then obtained by dividing out an appropriate discrete group. Extending the SM in this fairly obvious way produces the groups SMG_{23N}, SMG_{23MN} etc. Another feature we take over from the SM is the principle of using only small (fundamental or singlet) fermion non-abelian representations. For the abelian representations we take the condition that weak hypercharges should be chosen to be close to zero. More precisely, we minimise the sum of weak hypercharges squared over all the fermions.

The extra $SU(N)$ groups introduced confine and form fermion condensates having the same quantum numbers as the SM Higgs doublet. It follows that the extra $SU(N)$ groups act as partial technicolour groups and must confine near the electroweak scale. However, the SM Higgs field is still responsible for all the fermion masses, albeit with a somewhat reduced VEV.

We have studied in detail the conditions for anomaly cancellation in our minimal extension of the SM gauge group, SMG_{235}. It is not possible to construct an anomaly-free model using new mass protected fermions which are all non-singlet under $SU(5)$, without encountering a Landau pole in the $U(1)$ fine structure constant well below the Planck scale. However it is possible to construct a consistent model with a fourth generation of quarks but, instead of an extra generation of leptons, with a generation of the fermions coupling to $SU(5)$; the $SU(5)$-"quarks" given in table 5.1.

A similar solution with a fourth generation of quarks and a generation of $SU(N)$-"quarks", as given in table 5.2, is possible for the gauge group SMG_{23N}. However the number of $SU(2)$ doublets in the model increases with N and hence their contribution to the electroweak radiative corrections becomes more important. The SMG_{235} model is just consistent with the precision electroweak data but SMG_{23N} models with $N > 5$ are probably ruled out depending on how many standard deviations we allow the S parameter in our model to differ from experiment (see section 3.3). Similarly the SMG_{23MN} models, with both M and N greater than 3, would be inconsistent with the precision electroweak data.

The SMG_{235} model with a fourth generation of quarks and a generation of $SU(5)$-"quarks" seems to be phenomenologically consistent. However agreement with precision electroweak data is not certain. We have used the perturbative estimate of the contribution to the S parameter and our model differs from the experimental mean value by 2 standard deviations. If we accepted the non-perturbative estimate obtained by scaling QCD we would then differ by 4 standard deviations and conclude that none of our models could be consistent. However, we argue that there is no reason to prefer one method to the other. The only certain method is to calculate the S parameter non-perturbatively for this model which unfortunately, as with many strong interaction phenomena, cannot be reliably done.

Definite experimental evidence for or against this model will soon be available since it requires the existence of t' and b' quarks at or below the top quark mass scale. This is consistent with current experimental limits but they could not remain undetected for long. However it is unlikely that the $SU(5)$-"quarks" could be observed with current accelerators; they would be confined inside $SU(5)$-"hadrons" which would have a mass of the order of the $SU(5)$ confinement scale. We take this to be greater than the electroweak scale. Also, they would have a small production cross section at present hadron colliders. Even if this model doesn't turn out to be correct we hope that the derivation might at least highlight some of the important features of the SM and some of the unique qualities of the SM which appears (admittedly almost by definition) as the smallest case of our more general models.

In the final chapter we examined a different type of model. The original motivation was that the group SMG^3 could be used to predict the values of the gauge coupling constants. It was then used to explain the structure of the SM masses and mixing angles in a natural way. The group $U(1)_f$ was then added to greatly improve the fit. However, not all parameters and PCCSs were needed for the fit. So the natural step was to examine the subgroup which was needed for the fit and try a different set of $U(1)_f$ charges. The charges could be chosen differently since there are not as many anomaly constraints in the subgroup as in the full group. However, it was found that the choice of charges used in this thesis did not improve on the choice for the full group. In fact it gave a worse fit. This could be said to provide evidence for the full $SMG^3 \otimes U(1)_f$ group. However, this is an area where more work could be done since there are other choices of $U(1)_f$ charges which cancel the anomalies. Only when all the models have been analysed will it be possible to provide a definite conclusion.

Appendix A

Massless Fermions All Coupling to $SU(5)$

In this appendix we shall show what can be done in the general case where we do not assume that the fermions must get a mass via the SM Higgs mechanism. We examine the case where all the fermions are assumed to couple to the $SU(5)$ gauge group. Our principle of small representations then forces all the fermions to be in **5** or $\overline{\mathbf{5}}$ representations of $SU(5)$. The condition for the absence of the $[U(1)]^3$ anomaly is non-linear but all the other constraints are linear. So we ignore the $[U(1)]^3$ anomaly to start with and manipulate the equations for the cancellation of the other anomalies. Using the charge quantisation rule, eq. (1.4.72), we can express all the anomaly constraints in terms of integers. We then simplify the linear equations using simple techniques such as showing that certain combinations of variables must be divisible by 5. Then we can often constrain that combination of variables to be equal to zero provided we assume some limit on the total number of fermions. We finally end up with several simple constraints on the allowed types of fermions.

We shall start by writing all representations of the gauge group SMG_{235} which are **5** or $\overline{\mathbf{5}}$ representations of the $SU(5)$ subgroup, are fundamental or singlet representations of the $SU(2)$ and $SU(3)$ subgroups and have weak hypercharge values obeying the charge quantisation rule, eq. (1.4.72). Tables A.1 and A.2 show the relative contributions of each of these particle representations to each type of gauge anomaly. In this case the mixed gauge and gravitational anomaly is cancelled whenever the $[SU(5)]^2U(1)$ anomaly is since all fermions are in fundamental representations of $SU(5)$. In tables A.1 and A.2, each integer N is different and there can be any number of each representation with the same or different values of N. The notation N_A is used to mean the sum of the A values of N for the A $(1,1,5)$ representations. Similar notation is used for the other representations of $SU(2)\otimes SU(3)\otimes SU(5)$. Keeping this in mind it is now easy to sum the columns except for the $[U(1)]^3$ anomaly. The resulting constraints are, apart from the $[U(1)]^3$ anomaly,

$$\begin{aligned}
A - B + 2C - 2D + 3E - 3F + 3G - 3H + & \\
6I - 6J + 6K - 6L &= 0 \qquad \text{(A.1)}\\
E + F - G - H + 2I + 2J - 2K - 2L &= 0 \qquad \text{(A.2)}\\
10(N_C + N_D + 3N_I + 3N_J + 3N_K + 3N_L) & \\
+2m(-C + D - 3I + 3J - 3K + 3L) & \\
+5(-C + D + I + J - K - L) &= 0 \qquad \text{(A.3)}\\
10(N_A + N_B + 2N_C + 2N_D) & \\
-2m(A - B + 2C - 2D) + 10(-C + D) &= 0 \qquad \text{(A.4)}\\
10(N_A + N_B + 2N_C + 2N_D + 3N_E + 3N_F + 3N_G + 3N_H & \\
+6N_I + 6N_J + 6N_K + 6N_L) & \\
-2m(A - B + 2C - 2D + 3E - 3F & \\
+3G - 3H + 6I - 6J + 6K - 6L) & \\
-10(C - D + E + F - G - H - I - J + K + L) &= 0 \qquad \text{(A.5)}
\end{aligned}$$

Table A.1: All fundamental and singlet fermion representations of $SU(2)$ and $SU(3)$ which are fundamental representations of $SU(5)$. The weak hypercharges obey the charge quantisation rule, eq. (1.4.72) and the 'N's are integers which need not be the same, even for two identical non-abelian representations. The relative contributions to the $[U(1)]^3$ and $[SU(2)]^2U(1)$ gauge anomalies are shown for each type of representation.

Rep.	No.	y	$[U(1)]^3$	$[SU(2)]^2[U(1)]$
$1,1,5$	A	$2N - \frac{2m}{5}$	$\frac{8}{25}(5N-m)^3$	0
$1,1,\overline{5}$	B	$2N + \frac{2m}{5}$	$\frac{8}{25}(5N+m)^3$	0
$2,1,5$	C	$2N - \frac{2m}{5} - 1$	$\frac{2}{25}(10N-2m-5)^3$	$10N-2m-5$
$2,1,\overline{5}$	D	$2N + \frac{2m}{5} + 1$	$\frac{2}{25}(10N+2m+5)^3$	$10N+2m+5$
$1,3,5$	E	$2N - \frac{2m}{5} - \frac{2}{3}$	$\frac{8}{225}(15N-3m-5)^3$	0
$1,3,\overline{5}$	F	$2N + \frac{2m}{5} - \frac{2}{3}$	$\frac{8}{225}(15N+3m-5)^3$	0
$1,\overline{3},5$	G	$2N - \frac{2m}{5} + \frac{2}{3}$	$\frac{8}{225}(15N-3m+5)^3$	0
$1,\overline{3},\overline{5}$	H	$2N + \frac{2m}{5} + \frac{2}{3}$	$\frac{8}{225}(15N+3m+5)^3$	0
$2,3,5$	I	$2N - \frac{2m}{5} + \frac{1}{3}$	$\frac{2}{225}(30N-6m+5)^3$	$30N-6m+5$
$2,3,\overline{5}$	J	$2N + \frac{2m}{5} + \frac{1}{3}$	$\frac{2}{225}(30N+6m+5)^3$	$30N+6m+5$
$2,\overline{3},5$	K	$2N - \frac{2m}{5} - \frac{1}{3}$	$\frac{2}{225}(30N-6m-5)^3$	$30N-6m-5$
$2,\overline{3},\overline{5}$	L	$2N + \frac{2m}{5} - \frac{1}{3}$	$\frac{2}{225}(30N+6m-5)^3$	$30N+6m-5$

Table A.2: Types of representations A to L shown in table A.1 with their relative contributions to the $[SU(3)]^2U(1)$, $[SU(5)]^2U(1)$, $[SU(3)]^3$ and $[SU(5)]^3$ gauge anomalies. Since all the fermions are in fundamental representations of $SU(5)$ the condition for the absence of the mixed gauge and gravitational anomaly, $G^2U(1)$, is the same as for the absence of the $[SU(5)]^2U(1)$ gauge anomaly.

Rep.	No.	$[SU(3)]^2[U(1)]$	$[SU(5)]^2[U(1)]$	$[SU(3)]^3$	$[SU(5)]^3$
$1,1,5$	A	0	$10N-2m$	0	1
$1,1,\overline{5}$	B	0	$10N+2m$	0	-1
$2,1,5$	C	0	$20N-4m-10$	0	2
$2,1,\overline{5}$	D	0	$20N+4m+10$	0	-2
$1,3,5$	E	$30N-6m-10$	$30N-6m-10$	1	3
$1,3,\overline{5}$	F	$30N+6m-10$	$30N+6m-10$	1	-3
$1,\overline{3},5$	G	$30N-6m+10$	$30N-6m+10$	-1	3
$1,\overline{3},\overline{5}$	H	$30N+6m+10$	$30N+6m+10$	-1	-3
$2,3,5$	I	$60N-12m+10$	$60N-12m+10$	2	6
$2,3,\overline{5}$	J	$60N+12m+10$	$60N+12m+10$	2	-6
$2,\overline{3},5$	K	$60N-12m-10$	$60N-12m-10$	-2	6
$2,\overline{3},\overline{5}$	L	$60N+12m-10$	$60N+12m-10$	-2	-6

Also the total number of additional fermions is,

$$P = 5(A + B + 2C + 2D + 3E + 3F + 3G + 3H + 6I + 6J + 6K + 6L) \quad \text{(A.6)}$$

Using the above equations, keeping in mind that all the variables are integers, it is fairly straightforward to find the smallest possible solutions using some simple numerical analysis.

First of all, using eqs. (A.1) and (A.5) we obtain,

$$\begin{aligned} N_A + N_B + 2N_C + 2N_D + 3N_E + 3N_F + 3N_G + 3N_H + \\ 6N_I + 6N_J + 6N_K + 6N_L &= \\ C - D + E + F - G - H - I - J + K + L & \end{aligned} \quad \text{(A.7)}$$

From eq. (A.1), $(A - B + 2C - 2D)$ must be a multiple of 3, i.e.

$$3|(A - B + 2C - 2D) \quad \text{(A.8)}$$

Similarly, we can get the following equations; from eq. (A.4),

$$5|(A - B + 2C - 2D) \quad \text{(A.9)}$$

from eq. (A.1),

$$2|(A - B + 3E - 3F + 3G - 3H) \quad \text{(A.10)}$$

from eq. (A.2),

$$2|(E + F - G - H) \quad \text{(A.11)}$$

and from eq. (A.3),

$$5|(-C + D - 3I + 3J - 3K + 3L) \quad \text{(A.12)}$$

and

$$2|(-C + D + I + J - K - L) \quad \text{(A.13)}$$

Now, using eqs. (A.8) and (A.9) we get,

$$15|(A - B + 2C - 2D) \quad \text{(A.14)}$$

and using eqs. (A.10) and (A.11) we obtain,

$$2|(A - B - 6F + 6G) \tag{A.15}$$

which implies,

$$2|(A - B) \tag{A.16}$$

Eqs. (A.14) and (A.16) then give,

$$30|(A - B + 2C - 2D) \tag{A.17}$$

Using eq. (A.17), if $A - B + 2C - 2D \neq 0$ then $|A - B + 2C - 2D| \geq 30$ and so from eq. (A.1) $|E - F + G - H + 2I - 2J + 2K - 2L| \geq 10$. Eq. (A.6) then tells us that the total number of additional fermions must be at least 300. Therefore, for any solution with $P < 300$,

$$A - B + 2C - 2D = 0 \tag{A.18}$$

Eqs. (A.1), (A.2) and (A.4) now become,

$$\begin{aligned} E - H + 2I - 2L &= 0 && \text{(A.19)} \\ F - G + 2J - 2K &= 0 && \text{(A.20)} \\ N_A + N_B + 2N_C + 2N_D &= C - D && \text{(A.21)} \end{aligned}$$

while eqs. (A.3) and (A.7) can be rearranged to give,

$$\begin{aligned} N_E + N_F + N_G + N_H + {} & \\ 2N_I + 2N_J + 2N_K + 2N_L &= -I - J + K + L && \text{(A.22)} \\ 5(N_A + N_B + N_E + N_F + N_G + {} & \\ N_H - 4N_I - 4N_J - 4N_K - 4N_L) &= m(A - B - 6I + {} \\ & \quad 6J - 6K + 6L) && \text{(A.23)} \end{aligned}$$

In order that there is no Witten discrete $SU(2)$ anomaly,

$$2|(C + D + I + J + K + L) \tag{A.24}$$

This means that

$$2|(-C + D - 3I + 3J - 3K + 3L) \tag{A.25}$$

So

$$4|[(-2C + 2D) - 6I + 6J - 6K + 6L] \tag{A.26}$$

and using eq. (A.18),

$$4|(A - B - 6I + 6J - 6K + 6L) \tag{A.27}$$

But, from eq. (A.23),

$$5|(A - B - 6I + 6J - 6K + 6L) \tag{A.28}$$

Therefore

$$20|(A - B - 6I + 6J - 6K + 6L) \tag{A.29}$$

Again using eq. (A.6), for $P < 200$ we must have,

$$A - B - 6I + 6J - 6K + 6L = 0 \tag{A.30}$$

or equivalently, using eq. (A.18),

$$C - D + 3I - 3J + 3K - 3L = 0 \tag{A.31}$$

eq. (A.23) now becomes,

$$N_A + N_B + N_E + N_F + N_G + N_H - 4N_I - 4N_J - 4N_K - 4N_L = 0 \tag{A.32}$$

From eq. (A.31), $3|(C - D)$. Now if $|C - D| \geq 6$ then eqs. (A.31), (A.18), (A.19), and (A.20) along with eq. (A.6) show that $P \not< 200$. So we are left with $|C - D| = 0$ or 3.

If $C - D = 3$ then, using eq. (A.18), $A - B = -6$, and from eq. (A.31) $I - J + K - L = -1$. Table A.3 gives the possible combinations which allow $P < 200$ in this case. Here $P_1 = 120 + 10(\alpha + 2\beta + 3\gamma + 3\delta)$,and α, β, γ, and δ

Table A.3: Allowed combinations of fermion representations which could cancel all anomalies with less than 200 fermions for $C - D = 3$.

I	J	K	L	A	B	C	D	E	F	G	H	P
1	0	0	2	α	$\alpha+6$	3	0	2	0	0	0	$180+10\alpha$
1	1	0	1	α	$\alpha+6$	3	0	0	0	2	0	$180+10\alpha$
0	1	1	1	α	$\alpha+6$	3	0	2	0	0	0	$180+10\alpha$
0	2	1	0	α	$\alpha+6$	3	0	0	0	2	0	$180+10\alpha$
0	1	0	0	α	$\alpha+6$	$\beta+3$	β	γ	δ	$\delta+2$	γ	P_1
0	0	0	1	α	$\alpha+6$	$\beta+3$	β	$\delta+2$	γ	γ	δ	P_1

Table A.4: Allowed combinations of fermions representations which cancel all anomalies except the $[U(1)]^3$ gauge anomaly, with less than 200 fermions.

	p	q	r	s	t	u	Minimum P
I	0	0	0	0	0	0	0
II	-6	3	2	0	-1	0	120
III	-6	3	0	-2	0	1	120
IV	6	-3	-2	0	1	0	120
V	6	-3	0	2	0	-1	120

are all whole numbers [1].

If $C - D = -3$ then we have the same solutions as above with A and B interchanged ($A \leftrightarrow B$), $C \leftrightarrow D$, $E \leftrightarrow H$, $F \leftrightarrow G$, $I \leftrightarrow L$, and $J \leftrightarrow K$.

If $C = D$ then, $A = B$ from eq. (A.18) and from eq. (A.31), $I + K = J + L$. However, from eq. (A.21), $2|(N_A + N_B)$. Using this in eqs. (A.22) and (A.32) gives, $2|(I + J - K - L)$. But eqs. (A.18), (A.19), (A.20) and (A.31) along with eq. (A.6) show that if $|I+J-K-L| \geq 2$ then $P \not< 200$. Therefore $I+J = K+L$ and so $I = L$ and $J = K$.

Writing $p \equiv A - B$, $q \equiv C - D$, $r \equiv E - H$, $s \equiv F - G$, $t \equiv I - L$, and $u \equiv J - K$ the above results can be summarised in table A.4. Types II and IV are equivalent, as are types III and V by interchanging A and B etc. which is equivalent to relabelling representations 3, $\overline{3}$ and 5, $\overline{5}$.

We can also define $N_P \equiv N_A + N_B$ and $N_Q \equiv N_C + N_D$ etc. Eqs. (A.19)-(A.22)

[1] We use the convention that whole numbers are non-negative integers.

and (A.30)-(A.32) can now be rewritten as

$$
\begin{aligned}
p &= 6t - 6u && \text{(A.33)} \\
q &= -3t + 3u && \text{(A.34)} \\
r &= -2t && \text{(A.35)} \\
s &= -2u && \text{(A.36)} \\
N_Q + 3N_T + 3N_U &= -2t + u && \text{(A.37)} \\
N_P + 2N_Q &= -3t + 3u && \text{(A.38)} \\
N_R + N_S + 2N_T + 2N_U &= -t - u && \text{(A.39)}
\end{aligned}
$$

Now we must consider the $[U(1)^3]$ anomaly. It can be shown that when eqs. (A.33)-(A.39) hold the $[U(1)]^3$ anomaly leads to the following constraint:

$$
\begin{aligned}
5[N_{P3} + 2N_{Q3} + 3N_{R3} + 3N_{S3} + 6N_{T3} + 6N_{U3} & \\
-3(N_{Q2} + N_{R2} + N_{S2} - N_{T2} - N_{U2}) & \\
-3(t + 2N_T + 2N_U)] & \\
-3m[N_{P2} + 2N_{Q2} + 3N_{R2} - 3N_{S2} + 6N_{T2} - 6N_{U2} & \\
+2(2t + u + 2N_S + 6N_T + 4N_U)] &= 0 \qquad \text{(A.40)}
\end{aligned}
$$

where the following definitions have been used:

$N_{Ai} \equiv \sum N_a^i$ etc. for $i = 2$ or 3.

$N_{P3} \equiv N_{A3} + N_{B3}$, $N_{Q3} \equiv N_{C3} + N_{D3}$ etc.

$N_{P2} \equiv N_{A2} - N_{B2}$, $N_{Q2} \equiv N_{C2} - N_{D2}$ etc.

There is no obvious way of simplifying the above equations (especially eq. (A.40)) any further so we shall now assume that the fermions are all massive as discussed in section 2.4 and continue with the massive case in section 4.2.2. The general case could be solved by trial and error using a computer but we have not yet done that. If we did this we could look for the solution of the anomaly constraints which minimised the sum of weak hypercharges squared. This would allow us to show whether any model could be perturbatively valid up to the Planck scale.

However, this method would obviously be far too complicated for any progress to be made analytically.

We can show that the smallest solution has 30 fermions but a very large sum of weak hypercharge squared. To cancel the anomalies we must have the same number of **5** and $\overline{\mathbf{5}}$ representations of $SU(5)$. So the smallest solutions are: one **5** and one $\overline{\mathbf{5}}$, 10 fermions; two **5**s and two $\overline{\mathbf{5}}$s, 20 fermions. However, it is simple to show that in order to to cancel the anomalies these solutions cannot have mass protected fermions. The smallest set of fermions which can cancel the anomalies with mass protected fermions is three **5**s and three $\overline{\mathbf{5}}$s, 30 fermions. The solution with the smallest sum of weak hypercharges squared is:

$$(-\tfrac{22}{5},\mathbf{5}) \quad (\tfrac{18}{5},\mathbf{5}) \quad (\tfrac{18}{5},\mathbf{5})$$

$$(-\tfrac{8}{5},\overline{\mathbf{5}}) \quad (-\tfrac{8}{5},\overline{\mathbf{5}}) \quad (\tfrac{2}{5},\overline{\mathbf{5}})$$

where only the representations of $U(1)\otimes SU(5)$ are shown since they are all singlets of $SU(2) \otimes SU(3)$. The sum of weak hypercharge squared is

$$\sum y^2 = 252.8 \tag{A.41}$$

As we showed in section 4.2.2 this would cause a $U(1)$ Landau pole below the Planck scale. So we really must search for the solution with the smallest sum of weak hypercharge squared if we want a solution which would be consistent. However, we don't believe that massless fermions would be phenomenologically acceptable anyway and so we have not pursued this case any further.

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