Nontrivial Solutions of Nonlinear Functional and Differential Equations

by

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Almost all of the results of this thesis are the original work of the author with the exception of several results specifically mentioned in the text and attributed there to the authors concerned.

Chapter one covers basic definitions, notions and some known results which will be used in this thesis.

Much of chapters three and four is a joint work with Prof. J.R.L. Webb and has been submitted for publication in [21] and [22].

Chapter two and five are the original work of the author alone. Some parts of these chapters have appeared in [18], [20] and have been submitted for publication in [19].

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Summary

This thesis is concerned with the problem of finding nontrivial solutions of nonlinear equations.

Chapter one is an introduction to the concepts used through the thesis, including the notion of topological degree, measure of noncompactness, fixed point index and so on.

The work of chapter two builds a new definition of spectrum for nonlinear, finitely continuous maps using the class of A-proper mappings. In this chapter we also investigate the properties of the new spectrum and we discuss advantages and disadvantages of such a finite-dimensional approach.

In chapter three, by using fixed point index theory, we establish new results for some three point boundary value problems (BVPs) that have been previously studied by various authors, for example by Gupta *et al.* in [13, 15]. For certain values of a parameter α these particular BVPs can generate a continuous kernel that changes sign, so that positive solutions may not exist. We obtain existence of at least one or of multiple nonzero solutions.

In chapter four we extend the results of chapter three, allowing more general functions f and discontinuous kernels. We focus on a particular BVP that leads precisely to this situation, obtaining again, under suitable conditions, existence of nonzero solutions.

Finally, in chapter five, we turn our attention to the problem of eigenvalues of some three point BVPs. By using some results of chapter three and four together with a well known theorem on eigenvalues, we prove the existence of positive (and negative) eigenvalues.

Introduction

This thesis is divided in two parts, and reflects the variety of interests and problems that I came across in my journey through the kingdom of Nonlinear Analysis while I was studying for my Ph.D.

The first part is related to the problem of finding a new definition of spectrum of Nonlinear Operators. Due to the importance of spectral theory for linear operators it is not surprising that several attempts have been made to define and study a spectrum for nonlinear operators. One of the first attempts is due to Kachurovskij in 1969 [25]. Kachurovskij gave a definition of spectrum for continuous maps. His idea of regularity involves the bijectivity of the function and imposes constraints on the properties of the inverse as well. Later, in 1978 [11], Furi, Martelli and Vignoli introduced a spectrum with interesting applications. This spectrum is defined for continuous operators by using a number attached to the measure of noncompactness and the concept of *stably solvable* maps. Other authors gave [3, 5, 8, 10, 36] different definitions of spectrum of nonlinear operators and each of them focused on a particular class of maps.

In chapter two we introduce a new definition of spectrum for finitely continuous operators, which we call the A-spectrum. To do this we use the notion of approximate solvability (hence the concept of A-proper maps), a modification of the Furi-Martelli-Vignoli spectrum for finite dimensional maps and the theory of *topological degree.* We investigate the topological properties of the A-spectrum, its closedness, boundedness, nonemptiness and the relation with the linear spectrum. In section 2.5 attention is given to positively homogeneous operators, extending some known results valid for eigenvalues for linear operators to the new spectrum. In particular we show that if $\lambda \in \sigma_A(f)$ and $|\lambda| > \gamma(f)$, then there exists $\bar{t} \in (0, 1]$ such that $\lambda/(\bar{t})$ is an eigenvalue of f. This is used in section 2.9 to prove a result similar to the Birkhoff-Kellogg Theorem for finitely continuous maps. Finally in section 2.12 we show that in general eigenvalues are not contained in the A-spectrum and we compare it with other spectra.

The second part of the thesis is concerned with the study of nonzero solutions and eigenvalues of certain nonlocal boundary value problems (BVPs). In order to solve these problems we do not use the topological degree directly but we use its restriction to cones, the *fixed point index*, extensively.

In chapter three we investigate two second order differential equations of the form

$$u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1) \tag{0.0.1}$$

under one of the boundary conditions (BCs)

$$u'(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1, \tag{0.0.2a}$$

$$u(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1. \tag{0.0.2b}$$

These so-called three-point BVPs, and more general m-point BVPs, have been well studied in recent years, see for example Gupta *et al.* [13, 15] and the references therein.

The idea we use to find nontrivial solutions is to write the BVP as an equivalent Hammerstein Integral Equation

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds := Tu(t) \tag{0.0.3}$$

and look for fixed points of the operator T on a cone of functions that are positive on an interval, namely

$$K = \{u \in C[0,1] : \min\{u(t) : a \leq t \leq b\} \geq c\|u\|\}.$$

The reason we use this particular cone rather than the cone of positive functions commonly used in the literature is due to the fact that positive solutions may not exist. For example for (0.0.2a) when $\alpha < 0$, the kernel $k(t,s)$ is not positive for all values of t, s, indeed $k(1,s) < 0$ for all s. Therefore, when g and f are positive, a fixed point of the operator T cannot be positive on $[0,1]$.

The improvement with respect to the classical theory is that this new cone works for a wider class of kernels, allowing kernels that change sign.

In section 3.2 we show that the operator T sends the cone into itself. By means of a well known result of fixed point index theory, we prove that, under suitable conditions, the operator T has one or more fixed points. In sections 3.3 and 3.4 we show how to apply the abstract theorems to our particular BVPs. In practice this means finding upper and lower bounds for the kernel on a suitable interval $[a, b]$.

In chapter four we extend the results of chapter three by allowing more general functions f and *discontinuous* kernels that change sign. The BVP

$$u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{0.0.4}$$

with boundary condition

$$u(1) = \alpha u'(\eta), \; u(0) = 0, \; 0 < \eta < 1. \tag{0.0.5}$$

leads precisely to this situation. The kernel of the associated integral equation has a discontinuity on the line $s = \eta$.
Again, using a technique similar to the one of chapter three we are able to prove first existence of multiple fixed points of the associated integral equation and then give results for our particular BVP.

Chapter five deals with the problem of finding eigenvalues of the Hammerstein Integral Equation of the form

$$\lambda u(t) = \int_G k(t, s) f(s, u(s))\, ds := Tu(t),$$

where we allow k and f to be discontinuous and k to change sign. In this case we use the results obtained in chapter three and four together with a well known theorem on eigenvalues to show the existence of positive (and negative) eigenvalues. We apply our results to the BVP

$$\lambda u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{0.0.6}$$

subject to the boundary condition:

$$u'(0) = 0, \; \alpha u'(\eta) = u(1), \; 0 < \eta < 1, \tag{0.0.7}$$

and also to the other BCs seen above.

Contents

Chapter 1

Preliminaries

Throughout this thesis we will be interested in the solvability of nonlinear functional and differential equations. Depending on the nature of the problem we will use different tools of Nonlinear Functional Analysis, for example *topological degree* and *fixed point index theory*, and use notions such as *approximation solvability*.
In this chapter we give an introduction to these concepts.

1.1 Banach spaces, linear operators

In the sequel X and Y will denote Banach spaces. We say that a Banach space is *real* if the scalar field (which we indicate by $\mathbb{K}$) is $\mathbb{R}$ and *complex* if the field is $\mathbb{C}$. A Banach space X is said to be *separable* if it has a countable dense subset. A sequence $\{e_i\} \subset X$ is a *Schauder basis* if every $x \in X$ has a unique convergent expansion

$$x = \sum_{i \geq 1} x_i e_i,$$

where $x_i \in \mathbb{K}$. Given a map $L : X \to Y$ we say that L is a *linear operator* if

$$L(\alpha x + \beta y) = \alpha Lx + \beta Ly \text{ for every } x, y \in \operatorname{dom} L \text{ and } \alpha, \beta \in \mathbb{K}.$$

The *norm* of a linear operator is given by the formula

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|}.$$

If $\|L\|$ is finite we say that L is *bounded.*

The following Theorem links continuity and boundedness of a linear operator:

Theorem 1.1.1. *[26] Let L be a linear operator. Then*

i) L is continuous if and only if L is bounded,

ii) if L is continuous at a single point, it is continuous.

Note that, in the case of a finite dimensional space, we have the stronger result:

Theorem 1.1.2. *[26] If a normed space is finite dimensional, then every linear operator is bounded.*

We say that a linear operator $L : X \to Y$ is *compact* if L maps bounded sets into *relatively compact* sets in Y, that is, for every given bounded set M in X, the closure $\overline{T(M)}$ is compact. A *linear functional* f is a linear operator with domain X and range $\mathbb{K}$. A *bounded linear functional* is a linear functional with finite norm. The *dual space* X^* of a Banach space X is the vector space of all bounded linear functionals on X.

Example 1.1.3. An important example of a Banach space that will be used in the thesis is the Banach space $C[a, b]$ of continuous functions from $[a, b]$ to $\mathbb{K}$ endowed with the norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|.$$

Example 1.1.4. By definition the space l^2 is the set of all sequences of numbers $(x_1, x_2, \ldots)$ such that

$$\sum_{j=1}^{\infty} |x_j|^2 < \infty,$$

endowed with the norm

$$\|x\| = \Big(\sum_{j=1}^{\infty} |x_j|^2\Big)^{\frac{1}{2}}.$$

A simple basis for l^2 is $\{e_n\} = \{\delta_{in}\}_{i\in N}$ where

$$\delta_{in} = \begin{cases} 1 \text{ if } i = n, \\ 0 \text{ if } i \neq n. \end{cases}$$

The space l^2 is very interesting (see for example [26]) and it is actually the *prototype* of all infinite dimensional Hilbert spaces. In fact it can be shown that every infinite dimensional separable Hilbert space is isomorphic to l^2, a good reference is Theorem 16.19 in [16].

1.2 Brouwer degree

In this section we define the *Brouwer degree* for continuous mapping and state some results that we will use later, when dealing with finite-dimensional problems.

Let Ω be a bounded open set in $\mathbb{R}^n$. For each continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ and $y \notin f(\partial\Omega)$ we can define an integer $\deg(f, \Omega, y)$ which, roughly speaking, corresponds to the number of solutions $x \in \Omega$ of the equation $f(x) = y$. If f is a smooth function and y is not a critical value for f, the degree is given by the simple formula

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} J_f(x),$$

where $J_f(x) = \det f'(x)$. When y is a critical value we can define the degree by approximation (see for example [33] for details). In general for continuous functions the Brouwer degree is constructed via approximation with a smooth function g. Let $g \in C^1(\overline{\Omega})$ be such that

$$\|f(x) - g(x)\| < \operatorname{dist}(y, f(\partial\Omega)).$$

We define the degree of f by setting

$$\deg(f, \Omega, y) = \deg(g, \Omega, y).$$

It can be shown that this definition does not depend on the choice of the function g, again [33] is a good reference.

We are now able to state some properties of the Brouwer degree.

Theorem 1.2.1. *[6] Let Ω be a open bounded set in $\mathbb{R}^n$, $f \in C(\overline{\Omega})$ and $y \notin f(\partial\Omega)$. Then the Brouwer degree has the following properties:*

(d1) *(Normalization)* $\deg(I, \Omega, y) = 1$ *for* $y \in \Omega$

(d2) *(Additivity)* $\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y)$ *whenever* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$

(d3) *(Homotopy)* $\deg(h(t,\cdot),\Omega,y(t))$ *is independent of* t *whenever* $h : [0,1]\times\overline{\Omega} \to \mathbb{R}^n$ *and* $y : [0,1] \to \mathbb{R}^n$ *are continuous and* $y(t) \notin h(t,\partial\Omega)$ *for every* $t \in [0,1]$.

(d4) *(Existence)* $\deg(f,\Omega,y) \neq 0$ *implies* $f^{-1}(y) \neq \emptyset$.

(d5) $\deg(\cdot,\Omega,y)$ *is constant on* $\{g \in C(\overline{\Omega}) : \|g-f\| < r\}$, *where* $r = \operatorname{dist}(y, f(\partial\Omega))$.

(d6) $\deg(f,\Omega,\cdot)$ *is constant on* $B_r(y) \subset \mathbb{R}^n$. *Furthermore,* $\deg(f,\Omega,\cdot)$ *is constant on every component of* $\mathbb{R}^n \setminus f(\partial\Omega)$.

(d7) *(Boundary dependence)* $\deg(f,\Omega,y) = \deg(g,\Omega,y)$ *whenever* $f|_{\partial\Omega} = g|_{\partial\Omega}$.

(d8) *(Excision property)* $\deg(f,\Omega,y) = \deg(f,\Omega_1,y)$ *for every open* $\Omega_1 \subset \Omega$ *such that* $y \notin f(\overline{\Omega}\setminus\Omega_1)$.

The following is the well-known *Brouwer fixed point theorem* (see for example Theorem 3.2 of [6]):

Theorem 1.2.2. *Let* $D = \overline{B_1(0)}$ *be the unit ball in* $\mathbb{R}^n$ *and* $f : D \to D$ *continuous. Then* f *has a fixed point.*

We can prove this result by means of degree theory.

Proof. We assume that $f(x) \neq x$ for $x \in \partial D$, otherwise there is nothing further to prove. Consider the homotopy

$$h(t,x) = x - tf(x),\ t \in [0,1] \text{ and } x \in D.$$

It is clear that, for $t \in [0,1)$, $tf(x) \notin \partial D$ and hence

$$h(t,x) \neq 0 \text{ for } x \in \partial D \text{ and } t \in [0,1].$$

By the homotopy property of the degree we have

$$\deg(I-f, B_1(0), 0) = \deg(I, B_1(0), 0).$$

Since $\deg(I, B_1(0), 0) = 1$, we obtain $\deg(I-f, B_1(0), 0) = 1$. Hence there exists $x \in B_1(0)$ such that $f(x) = x$. $\square$

Remark 1.2.3. Although we said that the degree is roughly a "count" of the number of solutions of an equation $f(x) = y$, there are cases of maps with zero degree for which the equation $f(x) = y$ admit solutions, as we can see in the following simple example:

Example 1.2.4. Take $\Omega = (-1, 1)$, $y = \frac{1}{4}$ and $f(x) = x^2$. Obviously $-\frac{1}{2}$ and $\frac{1}{2}$ are solutions of the equation $x^2 = \frac{1}{4}$ and they lie in Ω. Nevertheless, let $g \equiv 1$ and obtain $\deg(g, (-1, 1), \frac{1}{4}) = 0$ by $(d4)$ and $\deg(g, (-1, 1), \frac{1}{4}) = \deg(f, (-1, 1), \frac{1}{4})$ by $(d7)$. Hence $\deg(f, (-1, 1), \frac{1}{4}) = 0$.

In the case of odd maps this does not happen. We recall that Ω is said to be *symmetric* with respect to the origin if $\Omega = -\Omega$. A map $f : \Omega \subset X \to Y$ is *odd* on Ω if and only if for every $x \in \Omega$ we have $f(-x) = -f(x)$. We can state Borsuk's Theorem (see Theorem 4.1 of [6] for a quick proof).

Theorem 1.2.5. *Let $\Omega \subset \mathbb{R}^n$ be open bounded symmetric with $0 \in \Omega$. Let $f \in C(\overline{\Omega})$ be odd and $0 \notin f(\partial\Omega)$. Then $\deg(f, \Omega, 0)$ is odd.*

1.3 Leray-Schauder degree

We recall that a nonlinear map $f : X \to Y$ is said to be *compact* if f maps bounded sets into relatively compact sets in Y.

The *Leray-Schauder degree* is an extension of the Brouwer degree to the case of infinite dimensional spaces, in the particular case of maps of the form $T = I - C$, where I is the identity and C is a compact map. The key theorem used in order to define the Leray-Schauder degree is the following:

Theorem 1.3.1. *[33] Let $\Omega \subset X$ be a bounded open set and $C : \bar{\Omega} \to Y$ compact. Given $\varepsilon > 0$, there exists a continuous map $C_\varepsilon : \bar{\Omega} \to Y$, whose range $C_\varepsilon(\bar{\Omega})$ is finite dimensional such that, for every $x \in \bar{\Omega}$*

$$\|C(x) - C_\varepsilon(x)\| < \varepsilon.$$

By virtue of Theorem 1.3.1 we can define the Leray-Schauder degree for a map of the type $T = I - C$ by using Brouwer degree. Indeed let $\tilde{T} = I - C_\varepsilon$, where C_ε

is a continuous map on $\overline{\Omega}$ with finite dimensional range such that

$$\sup_{\overline{\Omega}} \|C_\varepsilon x - Cx\| < \operatorname{dist}(y, \partial T(\Omega)) = \varepsilon$$

and $\tilde{\Omega}$ be the finite dimensional subspace of X which contains y and $C_\varepsilon(\overline{\Omega})$. Then we can set

$$\deg_{LS}(T, \Omega, y) = \deg(\tilde{T}, \tilde{\Omega}, y).$$

In [33] it is shown that $\deg_{LS}(T, \Omega, y)$ does not depend on the particular C_ε chosen to approximate C.

We are ready to state the main properties of the Leray-Schauder degree (details can be found in [6, 33]).

Theorem 1.3.2. *Let Ω be a open bounded set in X. Let $T = I - C : \overline{\Omega} \to X$ be such that $C : \overline{\Omega} \to X$ is compact and $y \notin T(\partial\Omega)$. Then the Leray-Schauder* $\deg_{LS}(T, \Omega, y)$ *is well defined and inherits the properties* $(d1) - (d8)$ *of the Brouwer degree.*

In this thesis we will not use the Leray-Schauder degree directly, but we will consider its restriction to cones, the fixed point index.

1.4 Fixed point index

The *fixed point index* is, loosely speaking, an algebraic count of the number of fixed points of a map in a closed convex set (usually a cone).

Definition 1.4.1. [6] We say that a set K is *convex* if $tx + (1-t)y \in K$ for every $x, y \in K$ and $t \in [0, 1]$. We define the *convex hull* of a set D to be the set

$$\operatorname{co} D = \Big\{ \sum_{i=1}^{n} t_i x_i : x_i \in D,\ t_i \in [0,1] \text{ and } \sum_{i=1}^{n} t_i = 1 \Big\}.$$

The definition of the fixed point index for compact maps in infinite dimensional spaces involves the Leray-Schauder degree and is given by the following:

Definition 1.4.2. [6] Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K = D \cap K \neq \emptyset$. Let $T : \overline{D}_K \to K$ be compact.

Suppose that $x \neq T(x)$ for all $x \in \partial_K D$, the boundary of D relative to K.

We define the fixed point index by the equation

$$i_K(T, D_K) = \deg_{LS}(I - Tr, r^{-1}(D_K) \cap B_\rho, 0),$$

where r is a retraction from X onto K, $B_\rho \supset \overline{D}_K$.

Remark 1.4.3. Tr has the same fixed points as T for if $Trx = x$ then $x \in K$ because $T : \overline{D}_K \to K$, hence $rx = x$. So $x \neq Tx$ for $x \in \partial_K D$, implies $x \neq Trx$ for $x \in \partial(r^{-1}(D_K) \cap B_\rho)$ hence the Leray-Schauder degree $\deg(I - Tr, r^{-1}(D_K) \cap B_\rho, 0)$ is defined. It can be shown that the degree is independent of the choice of the retraction r and the radius ρ. Hence, the index $i_K(T, D_K)$ is well defined.

Hence we can state the basic properties of the fixed point index.

Theorem 1.4.4. *[6] Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K := D \cap K \neq \emptyset$. Let $T : \overline{D}_K \to K$ be a compact map. Suppose that $x \neq T(x)$ for all $x \in \partial_K D$. The fixed point index has the following properties:*

(P_1) *(Existence) If $i_K(T, D_K) \neq 0$, then T has a fixed point in D_K.*

(P_2) *(Normalisation) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.*

(P_3) *(Additivity) If V^1, V^2 are disjoint relatively open subsets of D_K such that $x \neq T(x)$ for $x \in \overline{D}_K \setminus (V^1 \cup V^2)$, then*

$$i_K(T, D_K) = i_K(T, V^1) + i_K(T, V^2).$$

(P_4) *(Homotopy) Let $h : [0,1] \times \overline{D}_K \to K$ be compact such that $x \neq h(t, x)$ for $x \in \partial_K D$ and $t \in [0,1]$. Then*

$$i_K(h(0, .), D_K) = i_K(h(1, .), D_K).$$

As we said at the beginning of this section, the best candidate for a closed convex set to work with, for example when dealing with integral equations, is a cone.

Definition 1.4.5. A *cone* $K \subset X$ is a closed convex set such that $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

A key result in this area is *Krasnosel'skiĭ's fixed point theorem* on cones. It is a standard technique to use this theorem to prove the existence of solutions of some integral equations (the proof is illustrated in Figure 1.1).

Theorem 1.4.6. *[28] Let $K \subset X$ be a cone, Ω_1 and Ω_2 be open subsets in X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be compact. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ if either*

i) $\|Tx\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_2$

or

ii) $\|Tx\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_2$.

Proof. If T has a fixed point on $\partial(K \cap \Omega_1)$ or $\partial(K \cap \Omega_2)$ we are done. Otherwise, assume that *i)* is satisfied, obtaining $i_K(T, \Omega_1) = 1$, see proof of (2) of Lemma 1.4.7. The hypothesis $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$ implies $i_K(T, \Omega_2) = 0$, see for example [12]. Using (3) of Lemma 1.4.7 we have that T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. The case *ii)* is treated in a similar manner. □

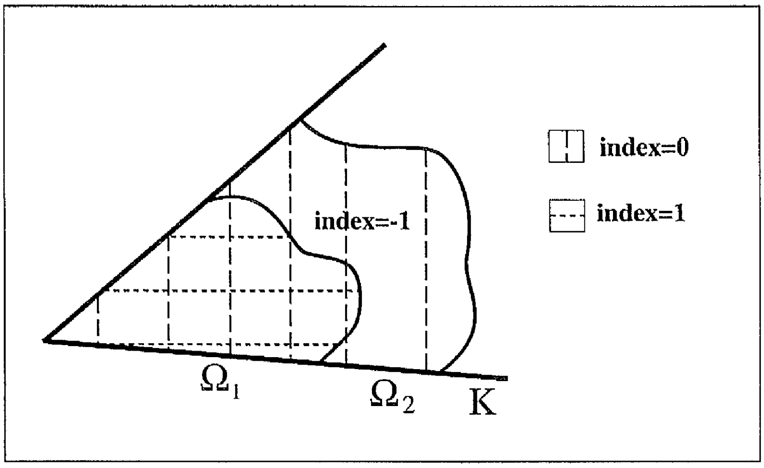

Figure 1.1: Krasnosel'skiĭ's fixed point theorem

The following lemma (see for example Theorem 12.3 in [1]) will be useful later in the thesis since, replacing the common "cone expansion" assumption $\|Tx\| \geq \|x\|$ with $x \neq Tx + \lambda e$, enables us to obtain different results:

Lemma 1.4.7. *Let D be an open bounded set with $D_K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $T : \overline{D}_K \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_K$. Then the fixed point index $i_K(T, D_K)$ has the following properties.*

(1) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.

(2) If $\|Tx\| \leq \|x\|$ for $x \in \partial D_K$, then $i_K(T, D_K) = 1$.

(3) Let D^1 be open in X with $\overline{D^1} \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(T, D^1_K) = 0$, then T has a fixed point in $D_K \setminus \overline{D^1_K}$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, D^1_K) = 1$.

Proof. For (1), suppose $i_K(T, D_K) \neq 0$ where $D = B_\rho$ and for arbitrary $m > 0$ consider $h(t, x) = Tx + tme$. By hypothesis $x \neq h(t, x)$ so by the homotopy property we obtain $i_K(T + me, D_K) \neq 0$. Hence, by the existence property, for each $m \in \mathbb{N}$ there is $x_m \in K$ with $\|x_m\| < \rho$ such that $x_m = Tx_m + me$. As T maps bounded sets to bounded sets this is impossible.
For (2), consider the homotopy $h(t, x) := tTx$. Then $x = h(t, x)$ would imply $r = \|x\| = t\|Tx\| \leq t\|x\|$ and $t = 1$ is excluded by assumption.
(3) is just the Additivity property. □

1.5 Measure of noncompactness

As we have seen compactness does play a central role in functional analysis, therefore is not surprising that tools there have been developed to measure, roughly speaking, how far a map is from being compact. To be more precise:

Definition 1.5.1. Let $B \subset X$ be bounded, we call the number

$$\alpha(B) = \inf\{d > 0 \text{ such that } B \text{ admits a finite cover by sets of diameter} \leq d\}$$

the *Kuratowski* (set) measure of noncompactness and

$$\beta(B) = \inf\{r > 0 \text{ such that } B \text{ can be covered by finitely many balls of radius } r\}$$

the *Hausdorff* (ball) measure of noncompactness.
The properties of α and β can be found for example in [6].

Given a measure of noncompactness attached to sets it is natural to define an analogue of this number for functions in the following manner:

$$\alpha(f) = \inf\{k \geq 0 : \alpha(f(B)) \leq k\alpha(B) \text{ for every bounded } B \subset X\},$$
$$\omega(f) = \sup\{k \geq 0 : \alpha(f(B)) \geq k\alpha(B) \text{ for every bounded } B \subset X\}$$
$$\beta(f) = \inf\{k \geq 0 : \beta(f(B)) \leq k\beta(B) \text{ for every bounded } B \subset X\},$$
$$\tilde{\omega}(f) = \sup\{k \geq 0 : \beta(f(B)) \geq k\beta(B) \text{ for every bounded } B \subset X\}$$

The main properties of $\alpha(f)$ and $\omega(f)$ can be found in [11].

Remark 1.5.2. Recall that for every bounded set $B \subset X$ we have the useful relation between the two measures (see for example [40]):

$$\beta(B) \leq \alpha(B) \leq 2\beta(B).$$

One can also show that

$$\frac{1}{2}\alpha(f) \leq \beta(f) \leq 2\alpha(f)$$

and that

$$\frac{1}{2}\tilde{\omega}(f) \leq \omega(f) \leq 2\tilde{\omega}(f).$$

Furthermore note that a function $f : X \to X$ is compact, i.e. f maps bounded sets into sets with compact closure, if and only if $\alpha(f) = \beta(f) = 0$.

Definition 1.5.3. We say that a map $f : D \subset X \to Y$ is a *β-contraction* if $\beta(f(G)) \leq k\beta(G)$, for every bounded set $G \subset D$ for some $k > 0$. f is said to be *ball condensing* if $\beta(f(G)) < \beta(G)$, for every bounded set $G \subset D$ with $\beta(G) \neq 0$.

We have the simple result:

Lemma 1.5.4. *Let f be a β-contraction with $\beta(f) < k$. Then there exists $\varepsilon > 0$ such that $\mu I + f$ is a β-contraction with $\beta(\mu I + f) < k$ for all $\mu \in \mathbb{K}$ with $|\mu| < \varepsilon$.*

Proof. We have that $\beta(\mu I + f) \leq \beta(\mu I) + \beta(f) = |\mu| + \beta(f)$. Therefore $\mu I + f$ is a β-contraction. If we choose $\varepsilon = k - \beta(f)$ we have $\beta(\mu I + f) < k$. $\square$

1.6 A-proper maps

The theory of A-proper maps was introduced by Petryshyn in [38], motivated by the need of constructing a solution of an infinite dimensional equation $f(x) = y$ by a limit of finite dimensional approximations $f_n(x) = y_n$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "Q_n"] \\ X_n \arrow[u, "V_n"] \arrow[r, "f_n"] & Y_n \end{tikzcd}$$

Definition 1.6.1. Let X and Y be Banach spaces, $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$, be sequences of oriented finite dimensional subspaces, V_n an injective map of X_n into X and Q_n a continuous linear map from Y to Y_n. We say that the projection scheme $\Gamma = \{X_n, V_n; Y_n, Q_n\}$ is *admissible* if $\text{dist}(x, X_n) \to 0$ for every $x \in X$ and $\sup_n \|Q_n\| \leq \eta$. If furthermore $\text{dist}(y, Y_n) \to 0$ for every $y \in Y$ we say that the projection scheme is *complete* (for further information on projection schemes see [40]).

Definition 1.6.2. [40] Given a Banach space X and a map $f : X \to Y$, we say that the equation $f(x) = y$ is approximation solvable (or simply *A-solvable*) with respect to $\Gamma = \{X_n, V_n; Y_n, Q_n\}$ if there exists $n_0 \in \mathbb{N}$ such that $Q_n f(x_n) = Q_n y$ has a solution $x_n \in X_n$ for every $n \geq n_0$ and $x_{n_k} \to x_0$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $f(x_0) = y$.

Remark 1.6.3. Although we give the general definitions of A-solvability and A-properness we will work, unless otherwise stated, with the special case of a particular *projection scheme* for (X, X). In this case $Y_n = X_n$ and $Q_n = P_n : X \to X_n$ is a linear projection with $\|P_n\| = 1$ for every n. We indicate this particular scheme by

$$\Gamma_1 = \{X_n, P_n\}.$$

Remark 1.6.4. A-solvability means that not only can we find a solution of the infinite-dimensional equation, but we can construct a solution via a limit of finite-dimensional approximations. Note that the A-solvability of an equation implies its solvability, but the converse need not be true as the following example shows:

Example 1.6.5. Take $l^2(\mathbb{C})$ with $\Gamma_1 = \{X_n, P_n\}$, where $X_n = \text{span}\{e_1, \ldots, e_n\}$, $\{e_i\}$ is the standard basis of l^2, P_n is the projection over the basis and $f : l^2 \to l^2$ is defined by

$$f(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_1, x_4, x_3, \ldots).$$

The equation $f(x) = y$ is clearly solvable and $f_n(x) = P_n(y)$ is solvable if n is even but not if n is odd (take for example $y = \{\frac{1}{n}\}$). Therefore $f(x) = y$ is not A-solvable (in fact it is feebly A-solvable as defined in [40]).

Definition 1.6.6. [40] We recall that a map $f : X \to Y$ is *demicontinuous* at x if $\{x_j\} \in X$ and $x_j \to x$ imply $f(x_j) \rightharpoonup f(x)$, where the symbol "$\rightharpoonup$" denotes weak convergence. f is *weakly continuous* if $\{x_j\} \in X$ and $x_j \rightharpoonup x$ imply $f(x_j) \rightharpoonup f(x)$; f is *finitely continuous* if for every finite dimensional subspace V of X and every sequence $\{x_j\} \in V$, $x_j \to x \in V$ implies $f(x_j) \rightharpoonup f(x)$.

Definition 1.6.7. [40] Given a map $f : X \to Y$, we say that f is *A-proper* relative to $\Gamma = \{X_n, V_n; Y_n, Q_n\}$ if

$$f_n := Q_n f : X_n \to Y_n$$

is continuous for each $n \in \mathbb{N}$ and if $\{x_{n_j} | x_{n_j} \in X_{n_j}\}$ is a bounded sequence such that

$$\|Q_n f(x_{n_j}) - Q_n y\| \to 0 \text{ as } j \to \infty, \;\; y \in Y,$$

there exists a subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$ and $x \in X$ such that $x_{n_{j(k)}} \to x$ and $f(x) = y$.

Theorem 1.6.8. *[40] Let $f : X \to Y$ be A-proper and $C : X \to Y$ be compact, then $f + C$ is A-proper.*

A similar property is valid for ball condensing maps (see Corollary 2.2 of [40])

Proposition 1.6.9. *Let $f : X \to X$ be ball condensing. Then*

$$T_t := I + tf : X \to X$$

is A-proper for every $t \in [-1, 1]$.

A concept closely related to A-properness is A-stability

Definition 1.6.10. [40] Given a function $f : X \to X$, we say that f is *A-stable* if there exists a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|P_n f(x) - P_n f(y)\| \geq h(\|x - y\|)$$

for all $x, y \in X_n$ with $x \neq y$ and $n \geq n_0$.

We have the following theorem (see Theorem 1.5 of [40])

Theorem 1.6.11. *Let $f : X \to X$ be continuous and A-stable. Then the following are equivalent:*

(A1) f is A-proper,

(A2) $f(x) = y$ is uniquely A-solvable for every $y \in X$,

(A3) f is surjective,

(A4) f is pseudo-A-proper, that is if $\{x_{n_j} | x_{n_j} \in X_{n_j}\}$ is a bounded sequence such that

$$\|f_{n_j}(x_{n_j}) - P_{n_j} y\| \to 0 \text{ as } j \to \infty, \ y \in X,$$

there exists and $x \in X$ such that $f(x) = y$.

Chapter 2

A new definition of Spectrum

In this chapter we discuss a new definition of spectrum for nonlinear, finitely continuous maps, which we call the A-spectrum. The newly defined spectrum has, in some cases, nice topological properties and reduces to the usual spectrum in the case of linear compact maps. In particular positively homogeneous operators have special spectral properties. We discuss the advantages and disadvantages of this finite-dimensional approach to spectral theory. We also stress that, whereas the spectra [8, 11] are defined only for continuous maps, the A-spectrum is defined for a wider class of maps. Furthermore we obtain approximate-solvability results rather than mere solvability. In section 2.9 we illustrate how to use A-spectral properties to solve infinite-dimensional problems. Some parts of this chapter appear in [18, 20].

We begin with a quick look over the classical linear spectral theory.

2.1 Linear spectral theory

In the case of linear operators the spectrum is a well studied concept. Let $L : X \to X$ be a bounded linear operator (i.e. continuous) and I be the identity from X to X.

Definition 2.1.1. We define the *resolvent* of L to be the set

$$\rho(L) = \{\lambda \in \mathbb{K} : (\lambda I - L) \text{ is an isomorphism}\}$$

and the *spectrum* of L is defined as

$$\sigma(L) = \mathbb{K} \setminus \rho(L).$$

Remark 2.1.2. We can split the spectrum into the following disjoint subsets:

1. A lack of injectivity: The point spectrum (*eigenvalues*)

$$\sigma_p(L) = \{\lambda \in \mathbb{K} : \lambda I - L \text{ is not injective}\}$$

2. A lack of surjectivity:

   (a) The continuous spectrum

$$\sigma_c(L) = \{\lambda \in \mathbb{K} : \lambda I - L \text{ is injective}, \overline{R(\lambda I - L)} = X, R(\lambda I - L) \neq X\}$$

   (b) The residual spectrum

$$\sigma_r(L) = \{\lambda \in \mathbb{K} : \lambda I - L \text{ is injective}, \overline{R(\lambda I - L)} \neq X\}$$

Remark 2.1.3. If $\dim X < \infty$ we have $\sigma(L) = \sigma_p(L)$.

Topologically speaking, the spectrum of a linear operator is a nice (compact) subset of the Complex plane. In fact we have the Theorem:

Theorem 2.1.4. *[26] The spectrum of a bounded linear operator has the following properties:*

i) The spectrum is non empty,

ii) The spectrum is closed,

iii) The spectrum is bounded.

Definition 2.1.5. We call the extended real number

$$r_\sigma(L) = \sup\{|\lambda| : \lambda \in \sigma(L)\}$$

the *spectral radius* of f.

We have an interesting result concerning this number:

Theorem 2.1.6. *[26]. If L is a bounded linear operator then*

$$r_\sigma(L) = \lim_n \sqrt[n]{\|L^n\|} \text{ (Gel'fand formula).}$$

2.2 Nonlinear spectra

The spectrum of linear operators plays a central role in functional analysis, in view of the applications to the study of differential equations. Therefore it is not surprising that several attempts have been made to define and study spectra for nonlinear operators. Various definitions of spectra for nonlinear operators have been given, for example [3, 5, 8, 10, 11, 25, 36], and each focused on a particular class of map, for example continuous maps, Fréchet differentiable maps, linearly bounded maps, k-epi maps.

For our purposes we will focus on some these spectra (for a recent survey on nearly all the nonlinear spectra mentioned [4] is a good reference).

One of the first attempts is due to Kachurovskij in 1969:

Definition 2.2.1. [25] Let $f : X \to X$ be continuous. f is said to be *Lip-regular* if f is bijective and f^{-1} is *Lipschitz*, that is there exists a constant $k \in \mathbb{R}$ such that

$$\|f^{-1}(x) - f^{-1}y\| \leq k\|x - y\| \text{ for every } x, y \in X.$$

The *Lip-resolvent* is defined by

$$\rho_{Lip}(f) = \{\lambda \in \mathbb{K} : \lambda I - f \text{ is Lip-regular}\}$$

and the *Lip-spectrum* by

$$\sigma_{Lip}(f) = \mathbb{K} \setminus \rho_{lip}(f).$$

Later, in 1978, Furi, Martelli and Vignoli introduced a spectrum with interesting applications:

Definition 2.2.2. [11] Let $f : X \to X$ be a map. We define the numbers

$$d(f) = \liminf_{\|x\| \to +\infty} \frac{\|f(x)\|}{\|x\|} \text{ and } q(f) = \limsup_{\|x\| \to +\infty} \frac{\|f(x)\|}{\|x\|}.$$

The main properties of $d(f)$, $q(f)$ can be found in [11]. A continuous function $f : X \to X$ is said to be *fmv-regular* if $\tilde{\omega}(f)$ and $d(f)$ and are both positive and f is *stably solvable*, i.e. if the equation $f(x) = h(x)$ has a solution for every continuous and compact map $h : X \to X$ with $q(h) = 0$. This leads to the *fmv-resolvent*

$$\rho_{fmv}(f) = \{\lambda \in \mathbb{K} : \lambda I - f \text{ is } fmv\text{-regular}\}$$

and the *fmv-spectrum*

$$\sigma_{fmv}(f) = \mathbb{K} \setminus \rho_{fmv}(f).$$

Remark 2.2.3. Note that, roughly speaking, the stably solvable condition measures the lack of surjectivity and $d(f)$ measures the lack of injectivity. Furthermore note that $d(f)$ and $q(f)$ depend on *asymptotic* properties of f.

We shall need the following finite dimensional version of Proposition 6.1.3 of [11].

Proposition 2.2.4. *Let* $\dim(X) < \infty$ *and* $f : X \to X$ *be fmv-regular and let* $g : X \to X$ *be such that* $q(g) < d(f)$ *then* $f + g$ *is fmv-regular.*

More recently Feng (1997) gave a definition of nonlinear spectrum that involves a different concept of solvability.

Definition 2.2.5. [8] Let $f : X \to X$ be a map. We define the number

$$m(f) = \inf_{x \neq 0} \frac{\|f(x)\|}{\|x\|} \text{ and } M(f) = \sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|}.$$

Now, let f be continuous, for $r > 0$ we denote by

$$\nu_r(f, 0) = \inf\{k \geq 0, \text{ there exists } g : B_r \to X, \text{ with } \alpha(g) \leq k,$$
$$g \equiv 0 \text{ on } \partial B_r : \ f(x) = g(x) \text{ has no solutions in } B_r\},$$

where $B_r = \{x \in X : \|x\| \leq r\}$ and ∂B_r denotes the boundary of B_r.
We call the number

$$\nu(f) = \inf\{\nu_r(f, 0), r > 0\}$$

the *measure of solvability* of f at 0. We say that f is *Feng regular* if

$$\tilde{\omega}(f) > 0, \ m(f) > 0 \text{ and } \nu(f) > 0.$$

The *Feng resolvent* is defined by

$$\rho_f(f) = \{\lambda \in \mathbb{K} : \lambda I - f \text{ is Feng-regular}\}$$

and the *Feng spectrum* by

$$\sigma_f(f) = \mathbb{K} \setminus \rho_f(f).$$

Remark 2.2.6. Note that (unlike $d(f)$ and $q(f)$), $m(f)$ and $M(f)$ depend on *global* conditions.

Also in 1997 another nonlinear spectrum appeared:

Definition 2.2.7. [3] Given a continuous function $f : X \to X$ we define the *Dörfner resolvent* by

$$\rho_D(f) = \{\lambda \in \mathbb{K} : \lambda I - f \text{ is bijective and } M([\lambda I - f]^{-1}) < \infty\}$$

and the *Dörfner spectrum* by

$$\sigma_D(f) = \mathbb{K} \setminus \rho_D(f).$$

In some cases this is a larger than the fmv-spectrum:

Proposition 2.2.8. *[4] Let $f : X \to X$ be a continuous map with $M(f) < \infty$. Then*

$$\sigma_{fmv}(f) \subseteq \sigma_D(f).$$

Appell *et al.* in the paper [5] (2001) studied a nonlinear spectrum that uses a modification of the concept of stably solvable maps.

Definition 2.2.9. [5] A continuous map $f : X \to Y$ is said to be *(a,p)-stably solvable* if the equation $f(x) = g(x)$ has a solution for every continuous map $g : X \to Y$ with $\alpha(g) \leq a$ and $q(g) \leq p$. If the constants a, p are positive, then f is said to be *strictly stably solvable.*

The next proposition links the notions of stably solvable and (a,p)-stably solvable.

Proposition 2.2.10. *[5] Let $f : X \to Y$ be stably solvable with $d(f) > 0$. Then f is $(0,p)$-stably solvable for every $p \leq d(f)$. If also $\omega(f) > 0$, then f is (a,p)-stably solvable for every $a < \omega(f)$ and $p < d(f)$.*

Definition 2.2.11. [5] If $f : X \to X$ is strictly stably solvable and $d(f) > 0$ we say that f is *Appell-Giorgieri-Väth regular.* This leads to the *Appell-Giorgieri-Väth resolvent*

$$\rho_{agv}(f) = \{\lambda \in \mathbb{K} : \lambda I - f \text{ is Appell-Giorgieri-Väth regular}\}$$

and the *Appell-Giorgieri-Väth spectrum*

$$\sigma_{agv}(f) = \mathbb{K} \setminus \rho_{agv}(f).$$

In lieu of stably solvability, Santucci and Väth [43] use stably 0-epi maps.

Definition 2.2.12. [43] Let $\Omega \subset X$ be a open, bounded set. A continuous map $f : X \to X$ is said to be *stably 0-epi* on Ω if $\text{dist}(0, f(\partial\Omega)) > 0$ and there exists $k > 0$ such that, for every continuous map $g : X \to X$ with $g|_{\partial\Omega} = 0$ and $\omega_\Omega(g) \leq k$, where

$$\omega_\Omega(f) = \sup\{k \geq 0 : \alpha(f(B)) \geq k\alpha(B) \text{ for every bounded } B \subset \Omega\},$$

the equation $f(x) = g(x)$ has a solution $x \in \Omega$.
Given a closed and bounded set $K \subset X$, with $0 \in K$, following [43] we define the *Phantom* of f to be the set

$$\phi(f) = \{\lambda \in \mathbb{K} : \text{ for every open } \Omega \subset K,\ \lambda I - f \text{ fails to be 0-epi on } \Omega\}.$$

We have the useful spectral inclusion:

Proposition 2.2.13. *[44] Let $f : X \to X$ be continuous. Then*

$$\phi(f) \subseteq \sigma_{agv}(f) \subseteq \sigma_{fmv}(f).$$

In the case of a *linear* operator all these spectra coincide with the usual linear spectrum.

Theorem 2.2.14. *Let $L : X \to X$ be a bounded liner operator. Then*

$$\sigma_{Lip}(L) = \sigma_{fmv}(L) = \sigma_{fw}(L) = \sigma_D(L) = \sigma_{agv}(L) = \phi(f) = \sigma(L).$$

Proof. The proof of this fact is a mere collection of results in [3, 8, 44]. □

2.3 The A-spectrum

The linear resolvent is stable under small perturbations of the identity. The A-properness property instead is invariant under *compact* perturbations but it is not in general invariant under small perturbations of the identity, as the following example shows:

Example 2.3.1. Consider the Hilbert space l^2 and take $f : l^2 \to l^2$ defined by $f(x) = \rho(x)x$ where $\rho(x) = e^{-\|x\|}$. First of all note that f is a β-condensing map. To see this we use a method analogous to the one in [37]. Let A be any set in l^2. Then $f(A) \subseteq \overline{\text{co}}\{A \cup \{0\}\}$ and $\beta(f(A)) \leq \beta(A \cup \{0\}) = \beta(A)$. Suppose now that $A \subset l^2$ and $\beta(A) = d > 0$. We can choose $r < \frac{d}{2}$, and define $A_1 = A \cap \overline{B_r(0)}$ and $A_2 = A \cap (\overline{B_r(0)})^c$, and consider $f(A) = f(A_1) \cup f(A_2)$. We have

$$\beta(f(A_1)) \leq 2r < d = \beta(A).$$

As ρ is a strictly decreasing function and $\|x\| \geq r$ for $x \in A_2$,

$$f(A_2) \subseteq \{tx \mid 0 \leq t \leq \rho(r), x \in A_2\} \subseteq \overline{\text{co}}\{\rho(r)A \cup \{0\}\}$$

and $\beta(f(A_2)) \leq \rho(r)\beta(A) < \beta(A)$. Thus

$$\beta(f(A)) = \max\{\beta(f(A_1)), \beta(f(A_2))\} < \beta(A)$$

and f is a β-condensing map. Hence $I - f$ is A-proper by Proposition 1.6.9.

But $I - f$ does not remain A-proper under small perturbations of the identity. In fact for every fixed positive ε we can give a bounded sequence $\{x_n\}$ such that $P_n[(1-\varepsilon)I - f](x_n) \to 0$ but $\{x_n\}$ has no convergent subsequence. Considering the sequence $\{x_n\} = \{-\ln(1-\varepsilon)e_n\}$, where $\{e_n\} = \{\delta_{in}\}_{i \in N}$, we have

$$\begin{aligned} P_n[(1-\varepsilon)I - f](x_n) &= (1-\varepsilon)P_n(x_n) - e^{-\|x_n\|}P_n(x_n) \\ &= [(1-\varepsilon) - e^{-\|x_n\|}]P_n(x_n) \\ &= 0 \qquad \text{for every } n, \end{aligned}$$

but $\|x_n - x_m\| = -\sqrt{2}\ln(1-\varepsilon)$ for every $n \neq m$. Hence $(1-\varepsilon)I - f$ is not A-proper.

Therefore it is convenient to introduce the following definition:

Definition 2.3.2. We say that $f : X \to X$ is *A-proper stable* if there exists $\varepsilon > 0$ such that $f + \mu I$ is A-proper for all $\mu \in \mathbb{K}$ with $|\mu| < \varepsilon$.

We will also need the numbers

$$\tau(f) = \sup\{\tau \in \mathbb{R}^+ : \mu I + f \text{ is } A\text{-proper for every } \mu \in \mathbb{K} \text{ with } |\mu| < \tau\},$$

$$\gamma(f) = \inf\{\gamma \in \mathbb{R}^+ : \mu I + f \text{ is } A\text{-proper for every } \mu \in \mathbb{K} \text{ with } |\mu| > \gamma\}.$$

Proposition 2.3.3. *Let f be A-proper stable. Then, for $\mu \neq 0$, μf is A-proper stable with $\tau(\mu f) = |\mu|\tau(f)$.*

Proof. It is known from [40] that if f is A-proper then μf is A-proper. It follows that $\varepsilon I + \mu f = \mu(\frac{\varepsilon}{\mu} I + f)$ is A-proper whenever $|\frac{\varepsilon}{\mu}| < \tau(f)$. Then μf is A-proper stable with $\tau(\mu f) = |\mu|\tau(f)$. □

The following theorem gives conditions for β-contractions to be A-proper stable.

Theorem 2.3.4. *Suppose that $D \subseteq X$ is closed and $T : D \to X$ is a continuous map such that there exists $\mu_0 > 0$ such that*

$$\beta\left(\{P_n T x_n\}\right) \geq \mu_0 \beta(\{x_n\}) \tag{2.3.1}$$

for each bounded sequence $\{x_n \mid x_n \in D_n = D \cap X_n\}$. If $f : D \to X$ is a β- contraction such that $\beta(f) < \mu_0$, then $T_t \equiv T + tf : D \to X$ is A-proper stable for each $t \in \mathbb{K}$ with $|t| \leq 1$.

Proof. First we note that $T_{1n} : D_n \subset X_n \to X_n$ is continuous for each $n \in \mathbb{N}$. Now let $\{x_{n_j} \mid x_{n_j} \in D_{n_j}\}$ be a bounded sequence such that $P_{n_j} T_1(x_{n_j}) - P_{n_j}(g) \to 0$ as $j \to \infty$ for some $g \in X$. Since $P_{n_j}(g) \to g$ in X, we see that

$$g_{n_j} \equiv P_{n_j} T(x_{n_j}) + P_{n_j} f(x_{n_j}) \to g \text{ in } X.$$

Since $\{g_{n_j}\}$ is precompact and

$$P_{n_j} T(x_{n_j}) = g_{n_j} - P_{n_j} f(x_{n_j}) \text{ for all } j \in \mathbb{N}, \tag{2.3.2}$$

it follows from (2.3.1), (2.3.2), Lemma 1 of [45], and the condition on f that

$$\mu_0 \beta(\{x_{n_j}\}) \leq \beta(\{P_{n_j} T(x_{n_j})\}) = \beta(\{P_{n_j} f(x_{n_j})\}) \leq \beta(f)\beta(\{x_{n_j}\}).$$

Thus $\beta(\{x_{n_j}\}) = 0$, so $\{x_{n_j}\}$ has a convergent subsequence $\{x_{n_{j(k)}}\}$ with $x_{n_{j(k)}} \to x$ for some $x \in D$. Hence, by continuity of T and f, we see that $Tx + f(x) = g$, that is T_1 is A-proper.

We note that, for each $t \in \mathbb{K}$ with $|t| \leq 1$, we have that $T_t \equiv T + tf : D \to X$ is also a map of the same kind with $\beta(tf) = |t|\,\beta(f) < |t|\,\mu_0 \leq \mu_0$, so that T_t is A-proper for every fixed t with $|t| \leq 1$.

To prove the A-proper stability note that $\lambda I + T_t \equiv \lambda I + T + tf = T + (\lambda I + tf)$ and $\lambda I + tf$ is a β-contraction with $\beta(\lambda I + tf) \leq |\lambda| + |t|\,\beta(f)$. Therefore $\lambda I + T_t$ is A-proper for every λ such that $|\lambda| < \mu_0 - |t|\beta(f)$, that is T_t is A-proper stable. □

Corollary 2.3.5. *When f is a β-contraction with $\beta(f) < 1$, then*

$$T_t = I - tf : D \subset X \to X$$

is A-proper stable for each $t \in \mathbb{K}$ with $|t| \leq 1$.

Proof. Note that the condition (2.3.1) is satisfied with $\mu_0 = 1$, so the corollary follows directly from Theorem 2.3.4. □

Remark 2.3.6. It follows from Remark 1.5.2 that the same result holds when f is a k-set contraction with $\alpha(f) < \frac{1}{2}$. From Theorem 2.3.4 it also follows that $\gamma(f) \leq \beta(f)$.

Definition 2.3.7. We say that $f : X \to X$ is A-*stably solvable* if there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $f_n(x)$ is stably solvable.

Definition 2.3.8. Let $f : X \to X$ be a map. We define the numbers:

$$d_R(f) = \inf_{\|x\| \geq R} \frac{\|f(x)\|}{\|x\|}, \quad d'_R(f) = \liminf_{n \to \infty} d_R(f_n),$$
$$d'(f) = \liminf_{n \to \infty} d(f_n), \quad m'(f) = \liminf_{n \to \infty} m(f_n)$$

The following Lemma clarifies the usefulness of the condition $d'_R(f) > 0$.

Lemma 2.3.9. *Let $f : X \to X$ and suppose that there is R such that $d'_R(f) > 0$. Fix $y \in X$. If x_n is a solution of the finite dimensional equation $P_n f(x_n) = P_n y$, then $\{x_n\}$ is bounded.*

Proof. Let $\liminf_{n\to\infty} d_R(f_n) = l > 0$, then there exists $\bar{n}$ such that for all $n \geq \bar{n}$,

$$d_R(f_n) = \inf_{\|x_n\| \geq R} \frac{\|P_n f(x_n)\|}{\|x_n\|} > \frac{l}{2}.$$

If $\|x_n\| \geq R$ we have that $\|y\| \geq \|P_n y\| = \|P_n f(x_n)\| \geq \frac{l}{2}\|x_n\|$. Therefore if we set

$$K = \max\left\{R, \frac{2\|y\|}{l}, \|x_1\|, \ldots, \|x_{\tilde{n}}\|\right\}$$

then

$$\|x_n\| \leq K < \infty \text{ for all } n.$$

□

We can now give a definition of regularity as follows:

Definition 2.3.10. Let $f : X \to X$ be a finitely continuous map. f is said to be *A-regular* if the following conditions hold:

i) f is A-stably solvable,

ii) f is A-proper stable,

iii) there exists R such that $d'_R(f) > 0$.

We define the *A-resolvent* by

$$\rho_A(f) = \{\lambda \in \mathbb{K} \text{ such that } \lambda I - f \text{ is } A\text{-regular}\}$$

and the *A-spectrum* by

$$\sigma_A(f) = \mathbb{K} \setminus \rho_A(f).$$

As in the linear spectral theory we define the *A-spectral radius* by

$$r_{\sigma_A}(f) = \sup\{|\lambda| : \ \lambda \in \sigma_A(f)\}.$$

From the definition of A-regularity and Lemma 2.3.9 we obtain the following:

Corollary 2.3.11. *Let $f : X \to X$ be A-regular. Then for all $y \in X$ the equation $f(x) = y$ is A-solvable.*

Proof. Fix $y \in X$ and consider the equation $P_n f(x_n) = P_n y$. Since f_n is stably solvable for $n \geq n_0$, we can find a sequence of solutions $\{x_n\}$, which is bounded by Lemma 2.3.9. For this sequence we have that $\|P_n f(x_n) - P_n y\| = 0$ for all $n \geq n_0$. By A-properness there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in X$ and $f(x) = y$. □

2.4 Some properties

In some cases the newly defined spectrum shares some of the familiar properties with the usual spectrum of linear operators.

Lemma 2.4.1. *Let*

$$q_R(g) = \sup_{\|x\| \geq R} \frac{\|g(x)\|}{\|x\|}.$$

Then $d'_R(f-g) \geq d'_R(f) - q_R(g)$.

Proof. For each n we have

$$d_R(P_nf - P_ng) = \inf_{\substack{\|x\| \geq R \\ x \in X_n}} \frac{\|P_nf(x) - P_ng(x)\|}{\|x\|} \geq \inf_{\substack{\|x\| \geq R \\ x \in X_n}} \frac{\|P_nf(x)\|}{\|x\|} - \sup_{\substack{\|x\| \geq R \\ x \in X_n}} \frac{\|P_ng(x)\|}{\|x\|}$$

$$= d_R(P_nf) - q_R(P_ng).$$

Now

$$q_R(P_ng) = \sup_{\substack{\|x\| \geq R \\ x \in X_n}} \frac{\|P_ng(x)\|}{\|x\|} \leq \sup_{\substack{\|x\| \geq R \\ x \in X_n}} \frac{\|g(x)\|}{\|x\|} \leq \sup_{\substack{\|x\| \geq R \\ x \in X}} \frac{\|g(x)\|}{\|x\|} = q_R(g).$$

Therefore

$$d_R(P_nf - P_ng) \geq d_R(P_nf) - q_R(g)$$

and

$$d'_R(f-g) \geq d'_R(f) - q_R(g).$$

□

Theorem 2.4.2. *The A-spectrum is a closed set.*

Proof. We show that the resolvent is open. Let $\lambda \in \rho_A(f)$ and take μ such that $|\lambda - \mu| < \min\{d'_R(\lambda I - f), \tau(\lambda I - f)\}$ and let $\theta = \mu - \lambda$. By Lemma 2.4.1 we have

$$d'_R(\mu I - f) = d'_R(\lambda I - f + \theta I) \geq d'_R(\lambda I - f) - |\theta| > 0$$

and with a simple sketch in the Complex plane one can see also that

$$\tau(\mu I - f) \geq \tau(\lambda I - f) - |\mu - \lambda| > 0.$$

It remains to show that $\mu I - f$ is A-stably solvable, i.e. for $n \geq n_0$ and for every continuous map $h : X_n \to X_n$ with $q(h) = 0$ we can solve the finite dimensional equation

$$P_n[(\mu I - f)(x)] = h(x). \tag{2.4.1}$$

But now note that we can apply Proposition 2.2.4, because we can rewrite (2.4.1) as

$$P_n[(\lambda I - f)(x)] + P_n[(\theta I)(x)] = h(x),$$

where $P_n[(\lambda I - f)(x)]$ is stably solvable and $P_n[(\theta I)(x)]$ plays the role of g. Therefore we can find a solution x of (2.4.1). $\square$

In general it is not clear if this spectrum is bounded or not. For some classes of operators we can achieve boundedness of the spectrum. We use the following Proposition due to Schäfer (the proof can be found in Corollary 8.1 of [6]).

Proposition 2.4.3. *Let $F : X \to X$ be completely continuous. Then the following alternative holds: Either $x - tF(x) = 0$ has a solution for every $t \in [0, 1]$ or $S = \{x$ such that $x = tF(x)$ for some $t \in (0, 1)\}$ is unbounded.*

Theorem 2.4.4. *We have the estimate:*

$$r_{\sigma_A}(f) \leq \max\{M(f), \gamma(f)\}.$$

Proof. Let $|\lambda| > \max\{M(f), \gamma(f)\}$, then by Lemma 2.4.1 we have

$$d_R'(\lambda I - f) \geq |\lambda| - q_R(f) \geq |\lambda| - M(f) > 0.$$

If we choose $|\lambda| > \gamma(f)$, $\lambda I - f$ is A-proper stable. To complete the proof, we have to show that the equation

$$\lambda x_n - P_n f(x_n) = h(x_n) \tag{2.4.2}$$

has a solution $x_n \in X_n$, for every continuous map $h : X_n \to X_n$ with $q(h) = 0$. In order to use Schäfer's result we will consider the equation

$$x_n - \frac{t}{\lambda} P_n f(x_n) - \frac{t}{\lambda} h(x_n) = 0.$$

Let $\varepsilon < |\lambda| - M(f)$. Since $q(h) = 0$ there exists $\bar{r} \in \mathbb{R}$ such that $\dfrac{\|h(x_n)\|}{\|x_n\|} < \varepsilon$ whenever $\|x_n\| \geq \bar{r}$. Then we have, for $\|x_n\| = R \geq \bar{r}$,

$$R^{-1}\|x_n - t(\lambda^{-1}(P_n f(x_n) - h(x_n)))\| \geq 1 - t(|\lambda|^{-1}M(f) + |\lambda|^{-1}\varepsilon) > 0$$

for all $t \in (0, 1)$. Thus we can apply Proposition 2.4.3 to find a solution x_n of (2.4.2).

□

Corollary 2.4.5. *If f is a β-contraction, we have the estimate:*

$$r_{\sigma_A}(f) \leq \max\{M(f), \beta(f)\}.$$

Proof. To show the A-proper stability, by Proposition 2.3.3, it is sufficient to study $I - \frac{1}{\lambda}f$. If f is a β-contraction then $I - \frac{1}{\lambda}f$ is A-proper stable by Corollary 2.3.5 when $\beta(\frac{1}{\lambda}f) < 1$, where $\beta(\frac{1}{\lambda}f) = \frac{1}{|\lambda|}\beta(f)$. So if we choose $|\lambda| > \beta(f)$, $\lambda I - f$ is A-proper stable. □

One may ask what is the relation between the A-spectrum and the fmv-spectra of the finite-dimensional projections. If we denote

$$\mathfrak{S}(f) = \{\lambda \in \mathbb{K} : \text{there exists a sequence } \{\lambda_{n_j}\} \subset \sigma_{fmv}(f_{n_j}) \text{ and } \lambda_{n_j} \to \lambda\}$$

we have the following inclusion:

Proposition 2.4.6. *Let $f : X \to X$ be finitely continuous. Then*

$$\mathfrak{S}(f) \subset \sigma_A(f).$$

Proof. For $\lambda \in \mathfrak{S}(f)$, we can find a sequence $\{\lambda_{n_j}\}$ such that $\lambda_{n_j} \to \lambda$. If $\lambda \in \sigma_A(f)$ we are done. Otherwise we have $\lambda \in \rho_A(f)$ and there are two cases.

Case 1. $\lambda_{n_j} \in \sigma_A(f)$ for a subsequence. Since $\sigma_A(f)$ is a closed set, we have that λ also lies in $\sigma_A(f)$, impossible.

Case 2. $\lambda_{n_j} \in \rho_A(f)$ for all but finitely many j and $\lambda \in \rho_A(f)$. Then $\lambda I - f$ is A-regular therefore $\lambda I - f_n$ is stably solvable for $n \geq n_0$ and $d'_R(\lambda I - f) = l > 0$. Then $d(\lambda I - f_n) \geq d_R(\lambda I - f_n) > \frac{l}{2}$ whenever $n \geq \bar{n}$. Notice that since $\lambda_{n_j} \to \lambda$ there exists j_0 such that $|\lambda - \lambda_{n_j}| < \frac{l}{2}$ for all $j \geq j_0$. Then, using Proposition 2.2.4, we have that $\lambda_{n_j}I - f_{n_j}$ is fmv-regular for j sufficiently large, contradicting the hypothesis that $\lambda_{n_j} \in \sigma_{fmv}(f_{n_j})$. $\square$

Remark 2.4.7. Are the $\sigma(f_n)$ nested, that is $\sigma(f_n) \subseteq \sigma(f_{n+1}) \subseteq \ldots \subseteq \mathfrak{S}(f)$? Example 2.6.4 will give an answer to this question.

We now show that the class of continuous A-regular maps is smaller than the stably solvable one.

Proposition 2.4.8. *Let f be continuous map. If f is A-regular then f is stably solvable.*

Proof. To see this we have to show that the equation $f(x)+h(x)=0$ has a solution $x \in X$ for every continuous and compact map h with $q(h)=0$. First notice that $f+h$ is A-proper by Theorem 1.6.8. We can now use a similar argument to that of Theorem 2.4.2 to solve, for $n \geq n_0$, the finite dimensional equation

$$P_n[(f+h)(x)] = 0,$$

using Proposition 2.2.4 to find a solution x_n. It suffices to show $\{x_n\}$ is bounded, for then, by A-properness, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$, $x_{n_k} \to x$ and x is a solution of the equation $f(x)+h(x)=0$. Suppose that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k}\| \to \infty$ and $f_n(x_{n_k}) = h_n(x_{n_k})$ for every n. Since $d'_R(f) > 0$, there exist $n_1 \in \mathbb{N}$ and $\delta > 0$ such that $d_R(f_n) \geq \delta$ for every $n \geq n_1$. Since $\|x_{n_k}\| \to \infty$, there exists $n_2 \in \mathbb{N}$ such that $\|x_{n_k}\| \geq R$ for every $n \geq n_2$. Let $\bar{n} \geq \max\{n_0, n_1, n_2\}$. Then for every $n \geq \bar{n}$ we have

$$\delta \leq \frac{\|f_n(x_{n_k})\|}{\|x_{n_k}\|} = \frac{\|h_n(x_{n_k})\|}{\|x_{n_k}\|} \leq \frac{\|h(x_{n_k})\|}{\|x_n\|} \to 0$$

a contradiction. Therefore $\{x_n\}$ is bounded and f is stably solvable. $\square$

2.5 Positively homogeneous maps

Recall that a map $f : X \to X$ is said to be *positively homogeneous* if $f(tx) = tf(x)$ for every $x \in X$ and $t \in \mathbb{R}_+$. For positively homogeneous operators more can be said about the A-spectrum. In particular, in the continuous case, it contains eigenvalues. By *eigenvalue* of a function $f : X \to X$, we mean a scalar $\lambda \in \mathbb{K}$ such that there exists $x \in X$, $x \neq 0$ such that $f(x) = \lambda x$. Note that this is not the only possible definition for the term eigenvalue, see for example [42].

Lemma 2.5.1. *Let $f : X \to X$ be a positively homogeneous map. Then*

a) $d_R(f) = d(f) = m(f) = \inf_{\|y\|=1} \|f(y)\|$.

b) $q_R(f) = q(f) = M(f) = \sup_{\|y\|=1} \|f(y)\|$.

c) $d'_R(f) = d'(f) = m'(f)$.

Proof. *a)* Since f is positively homogeneous we have

$$m(f) = \inf_{x\neq 0} \frac{\|f(x)\|}{\|x\|} = \inf_{x\neq 0}\left\| f\left(\frac{x}{\|x\|}\right)\right\| = \inf_{\|y\|=1} \|f(y)\|.$$

Similarly we have $d_R(f) = \inf_{\|y\|=1} \|f(y)\|$ and

$$d(f) = \lim_{R\to\infty} \inf_{\|x\|\geq R} \frac{\|f(x)\|}{\|x\|} = \lim_{R\to\infty} \inf_{\|y\|=1} \|f(y)\| = \inf_{\|y\|=1} \|f(y)\|.$$

b) Similar to *a)*.

c) $d(f_n) = d_R(f_n) = m(f_n)$ for every n by a). □

Proposition 2.5.2. *Let $f : X \to X$ be a positively homogeneous, finitely continuous map. If $\lambda I - f$ is A-proper and $d'(\lambda I - f) = 0$, then λ is an eigenvalue of f that lies in the spectrum.*

Proof. If $d'(\lambda I - f) = 0$ we have that

$$\liminf_{n\to\infty} \left\{ \inf_{\|x_n\|=1} \|P_n[\lambda x_n - f(x_n)]\| \right\} = 0, x_n \in X_n.$$

Since X_n is finite dimensional and f is finitely continuous, there exists a sequence $\{y_n\}, y_n \in X_n$, $\|y_n\| = 1$, such that

$$\|P_n\left[\lambda y_n - f\left(y_n\right)\right]\| = \inf_{\|x_n\|=1} \|P_n[\lambda x_n - f(x_n)]\|.$$

Therefore there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\left\|P_{n_j}\left[\lambda y_{n_j} - f\left(y_{n_j}\right)\right]\right\| \to 0 \text{ as } j \to \infty.$$

We can now use the A-properness to find a subsequence $\{y_{n_{j(k)}}\}$ of $\{y_{n_j}\}$ such that $y_{n_{j(k)}} \to y \in X$, $\|y\| = 1$ and $\lambda y - f(y) = 0$, that is λ is an eigenvalue. □

One nice property of the new spectrum is that, in some cases, we can compute the eigenvalues by an approximation process.

Corollary 2.5.3. *Let f be a positively homogeneous, finitely continuous map, $\{\lambda_n\}$ be a sequence of eigenvalues for f_n, that is there exist $n_0 \in \mathbb{N}$ and a nonzero sequence $\{\bar{x}_n\}$ with $\bar{x}_n \in X_n$ such that $f_n(\bar{x}_n) = \lambda_n \bar{x}_n$ for $n > n_0$. Suppose there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to \lambda$. Then $\lambda \in \sigma_A(f)$ and either λ is an eigenvalue of f or $\lambda I - f$ is not A-proper.*

Proof. All we need to show is that $m'(\lambda I - f) = 0$ and apply Proposition 2.5.2. This is true since

$$\begin{aligned} \liminf_{n\to\infty} \left\{ \inf_{x_n \neq 0} \frac{\|P_n[\lambda x_n - f(x_n)]\|}{\|x_n\|} \right\} &\leq \liminf_{n\to\infty} \left\{ \frac{\|\lambda \bar{x}_n - P_n f(\bar{x}_n)\|}{\|\bar{x}_n\|} \right\} \\ = \liminf_{n\to\infty} \left\{ \frac{\|\lambda \bar{x}_n - \lambda_n \bar{x}_n\|}{\|\bar{x}_n\|} \right\} &= \liminf_{n\to\infty} |\lambda - \lambda_n| \leq \lim_{j\to\infty} |\lambda - \lambda_{n_j}| = 0. \end{aligned}$$

□

Example 2.5.4. As an application of the previous Corollary we can take the operator $f : l^2 \to l^2$ defined by

$$f(z_1, z_2, \ldots, z_j, \ldots) = \left(|z_1|, \frac{|z_2|}{2}, \ldots, \frac{|z_j|}{j}, \ldots \right)$$

f is a compact, positively homogeneous map. For $\lambda \neq 0$ the map $\lambda I - f$ is A-proper by Theorem 1.2 of [40] and we can compute some of the eigenvalues of f by approximation. In fact if we take $\{y_n\}$ defined by $y_n = e_j$ for all $n \in \mathbb{N}$, and j is fixed, we have that

$$f_n(y_n) = \frac{1}{j}(y_n) \text{ for all } n \geq j.$$

We can apply the Corollary to prove that j^{-1} is an eigenvalue for every $j \in \mathbb{N}$.

Note that f is also a continuous map with $d(f_n) > 0$ for all n but $d(f) = 0$. In fact we have

$$\|f_n(z_1, z_2, \ldots, z_n)\| = \sqrt{\sum_{j=1}^{n} \frac{|z_j|^2}{j^2}} \geq \sqrt{\sum_{j=1}^{n} \frac{|z_j|^2}{n^2}} = \frac{1}{n}\|(z_1, z_2, \ldots, z_n)\|$$

and

$$\|f_n(0,\ldots,0,1)\| = \frac{1}{n}\|(0,\ldots,0,1)\|.$$

Therefore, since f_n is positively homogeneous map, we have $d(f_n) = \frac{1}{n}$ but $d(f) = 0$. Moreover note that f is not A-proper since, given $\{y_n\}$ as above, $\|P_n f(y_n)\| = \frac{1}{n} \to 0$ as $n \to \infty$ and $\{y_n\}$ is a bounded sequence with no convergent subsequence. This example suggests that a compact map may not be A-proper. We will prove this later in Proposition 2.6.1.

Remark 2.5.5. When can we a priori bound the set $\{\lambda_n\}$? For example when $M(f)$ is finite. In fact if λ_n is an eigenvalue of f_n we have

$$\|\lambda x_n\| = \|f_n(x_n)\| \leq \|P_n\|\|f(x_n)\| \leq \|f(x_n)\|$$

for some $x_n \in X_n$. Therefore, dividing by $\|x_n\|$ we have

$$|\lambda_n| = \frac{\|\lambda_n x_n\|}{\|x_n\|} \leq \frac{\|f(x_n)\|}{\|x_n\|} \leq M(f).$$

Can we achieve *all* the eigenvalues by an approximation process? In the continuous case we have the following:

Proposition 2.5.6. *Let $f : X \to X$ be a positively homogeneous, continuous map. If λ is an eigenvalue then $d'(\lambda I - f) = 0$. In particular the eigenvalues lie in the spectrum.*

Proof. Let λ be an eigenvalue, let y be a corresponding eigenvector and since f is positively homogeneous there is no restriction in letting $\|y\| = 1$. Set $y_n = \frac{P_n y}{\|P_n y\|}$. Note that $\{y_n\}$ is well defined since $P_n y \to y \neq 0$ implies $P_n y \neq 0$ for $n \geq n_0$. Note also that $\|y_n\| = 1$, $y_n \to y$ and $P_n f(y_n) \to f(y)$, by continuity. Thus

$$\liminf_{n\to\infty} \inf_{\|x_n\|=1} \|P_n[\lambda x_n - f(x_n)]\| \leq \liminf_{n\to\infty} \|\lambda y_n - f_n(y_n)\| = 0.$$

Therefore $d'(\lambda I - f) = 0$. $\square$

It is known that for a linear operator L, if $\lambda \in \sigma(L)$ and $|\lambda| > \alpha(L)$, then λ is an eigenvalue of L [35]. If f is positively homogeneous we have the following results.

Theorem 2.5.7. *Let $f : X \to X$ be a positively homogeneous, odd finitely continuous map and let $\lambda \in \sigma_A(f)$ and $|\lambda| > \gamma(f)$. Then λ is an eigenvalue of f.*

Proof. Since $|\lambda| > \gamma(f)$ then $\lambda I - f$ is A-proper stable. If $d'(\lambda I - f) = 0$, then, by Proposition 2.5.2, λ is an eigenvalue of f and we are done. If $d'(\lambda I - f) > 0$, we will show that $\lambda I - f$ is A-stably solvable and therefore $|\lambda| \notin \sigma_A(f)$. Since $d'(\lambda I - f) > 0$ and f is odd, there exists $\bar{n} \in \mathbb{N}$ such that $\deg(\lambda I - f_n, B_r(0), 0) \neq 0$ for all $n \geq \bar{n}$ and $r \in \mathbb{R}^+$. Using degree theory we show that, given a continuous map $h : X_n \to X_n$ with $q(h) = 0$ the equation $\lambda x_n - P_n f(x_n) = h(x_n)$ has a solution x_n. Let $\varepsilon < d_R(f_n)$. Then there exists $\tilde{r} \in \mathbb{R}_+$ such that $q_{\tilde{r}}(h) \leq \varepsilon$. Consider the homotopy

$$H(t, x_n) = \lambda x_n - P_n f(x_n) - th(x_n).$$

Then for $\|x_n\| = \tilde{r}$ we have

$$\tilde{r}^{-1} \|\lambda x_n - P_n f(x_n) - th(x_n)\| \geq d_{\tilde{r}}(\lambda I - f_n) - t\varepsilon > 0 \text{ for all } t \in [0, 1].$$

Therefore, by degree theory, $\deg(\lambda I - f_n - h, B_{\tilde{r}}(0), 0) \neq 0$. □

When f is not odd a weaker conclusion can be drawn.

Theorem 2.5.8. *Let $\lambda \neq 0$ and let $f : X \to X$ be a positively homogeneous, finitely continuous map such that $\lambda I - tf$ is A-proper for all $t \in (0, 1]$. Then either $\lambda I - f$ is A-stably solvable or there exists $\bar{t} \in (0, 1]$ such that $\lambda / \bar{t}$ is an eigenvalue of f.*

Proof. If $d'(\lambda I - \bar{t} f) = 0$ for some $\bar{t} \in (0, 1]$ then $\lambda / \bar{t}$ is an eigenvalue of f by Proposition 2.5.2 and we are done. Therefore suppose $d'(\lambda I - tf) > 0$ for all $t \in (0, 1]$. Consider the set

$$V_n = \{x_n \in X_n \text{ such that } \|x_n\| = 1 \text{ and } \lambda x_n - t_n f(x_n) = 0 \text{ for some } t_n \in (0, 1]\}.$$

If $V_n \neq \emptyset$ then there exist $\bar{t}_n \in (0, 1]$ and $\bar{x}_n \in X_n$ with $\|\bar{x}_n\| = 1$ such that $\lambda \bar{x}_n - \bar{t}_n f_n(\bar{x}_n) = 0$. If $V_n \neq \emptyset$ for infinitely many n then, since $\{\bar{t}_n\}$ is bounded, there exists $\bar{t} \in (0, 1]$ and a subsequence $\{\bar{x}_{n_j}\}$ of $\{\bar{x}_n\}$ such that $\|\bar{t} f_{n_j}(\bar{x}_{n_j}) - \lambda \bar{x}_{n_j}\| \to 0$ as $j \to \infty$, contradicting the hypothesis that $d'(\lambda I - tf) > 0$ for all $t \in (0, 1]$. Note that $\bar{t} \neq 0$, otherwise we would have $\|\lambda \bar{x}_{n_j}\| \to 0$ as $j \to \infty$, contradicting the fact that $\|\lambda \bar{x}_{n_j}\| = |\lambda|$ for all j.

If $V_n = \emptyset$, consider the homotopy

$$H(t, x_n) = \lambda x_n - tP_n f(x_n).$$

Then $\deg(\lambda I, B_1(0), 0) = \deg(\lambda I - f_n, B_1(0), 0) \neq 0$ since X_n is finite dimensional. Note that, since f is positively homogeneous, this is true for all $B_r(0)$ with $r \in \mathbb{R}_+$. Therefore $\lambda I - f_n$ is stably solvable. If $V_n = \emptyset$ for all but finitely many n then $\lambda I - f$ is A-stably solvable. ☐

Corollary 2.5.9. *Let $f : X \to X$ be a positively homogeneous, finitely continuous map and let $\lambda \in \sigma_A(f)$ and $|\lambda| > \gamma(f)$. Then there exists $\bar{t} \in (0, 1]$ such that $\lambda/\bar{t}$ is an eigenvalue of f.*

Proof. Follows directly from Theorem 2.5.8. ☐

As we have seen, for a continuous linear map L, the radius of the spectrum is given by $r_\sigma(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}$, where $\|L\| = \sup_{\|x\|=1} \|L(x)\|$. We also use this notation when f is a positively homogeneous, nonlinear map, rather than $M(f)$. The following theorem gives an estimate for the radius of the A-spectrum in the case of positively homogeneous operators, which is more precise than the result of Theorem 2.4.4.

Theorem 2.5.10. *Let X be a Banach space and $f : X \to X$ be a positively homogeneous, finitely continuous map with $\gamma(f) < \infty$ and $\liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}} < \infty$. If $\lambda \in \mathbb{R}_+$ with*

$$\lambda > \max\{\gamma(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\},$$

then $\lambda \in \rho_A(f)$. If also $\|x_1\| = \|x_2\|$ implies $\|f(x_1)\| = \|f(x_2)\|$, then

$$r_{\sigma_A}(f) \leq \max\{\gamma(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\}.$$

Proof. Take $|\lambda| > \max\{\gamma(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\}$. Suppose that $\lambda \in \sigma_A(f)$. Since f is positively homogeneous, we can apply Corollary 2.5.9 to show that there exist

$\bar{t} \in (0,1]$ and $\bar{x} \in X$, with $\bar{x} \neq 0$, such that $\bar{t}f(\bar{x}) = \lambda\bar{x}$. Without loss of generality assume also that $\|\bar{x}\| = 1$. If $\lambda \in \mathbb{R}_+$, then

$$\|f\| = \sup_{\|x\|=1} \|f(x)\| \geq \|f(\bar{x})\| = \frac{|\lambda|}{\bar{t}} \geq |\lambda|.$$

Also

$$\left\|f^2(\bar{x})\right\| = \left\|f\Big(\frac{\lambda}{\bar{t}}\bar{x}\Big)\right\| = \left\|\frac{\lambda}{\bar{t}}f(\bar{x})\right\| \geq \frac{|\lambda|^2}{\bar{t}^2} \geq |\lambda|^2.$$

By induction we obtain $\|f^n\|^{\frac{1}{n}} \geq |\lambda|$. This contradicts the hypothesis

$$|\lambda| > \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}.$$

Therefore $\lambda \in \rho_A(f)$. If also $\|x_1\| = \|x_2\|$ implies $\|f(x_1)\| = \|f(x_2)\|$, then for $\lambda = \rho e^{i\theta}$, $|\lambda| = \rho > 0$ we have

$$\|f(\bar{x})\| = \left\|\frac{\lambda}{\bar{t}}\bar{x}\right\| = \frac{|\lambda|}{\bar{t}} = \frac{\rho}{\bar{t}}$$

and

$$\|f^2(\bar{x})\| = \left\|f\Big(\frac{\lambda}{\bar{t}}\bar{x}\Big)\right\| = \left\|f\Big(\frac{\rho}{\bar{t}}\bar{x}\Big)\right\| = \left\|\frac{\rho}{\bar{t}}f(\bar{x})\right\| = \frac{\rho^2}{\bar{t}^2} \geq \rho^2 = |\lambda|^2.$$

Therefore if

$$|\lambda| > \max\{\gamma(f), \liminf_{n\to\infty} \|f^n\|^{\frac{1}{n}}\},$$

then $\lambda \in \rho_A(f)$. □

2.6 Nonemptiness of the A-spectrum

It is known that in the case of linear operators the spectrum is always non-empty, here we prove a nonlinear analogue for compact maps.

Proposition 2.6.1. *Let* $\dim(X) = \infty$ *and* $f : X \to X$ *be a compact map. Then* $\sigma_A(f) \neq \emptyset$.

Proof. It is sufficient to show that f is not A-proper and therefore $0 \in \sigma_A(f)$. Since $\dim(X) = \infty$, there exist a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that (see for example Proposition 7.1 of [6])

$$\|x_n - x_m\| \geq 1 \text{ for } n \neq m.$$

Since f is compact, $\{f(x_n)\}$ is precompact. Then there exists $y \in X$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $f(x_{n_j}) \to y$. Therefore

$$\|P_n f(x_{n_j}) - P_n y\| \leq \|P_n\| \|f(x_{n_j}) - y\| \leq \|f(x_{n_j}) - y\| \to 0 \text{ as } n_j \to \infty$$

and $\{x_{n_j}\}$ has no convergent subsequence. □

Open question: Is the A-spectrum *always* nonempty? The usual example of nonlinear theories fails here, as we will see in example 2.6.4.

Definition 2.6.2. Following [11] we denote by

$$\sigma_\pi(f) = \{\lambda \in \sigma_{fmv}(f) \text{ such that } d(\lambda I - f) = 0 \text{ or } \omega(\lambda I - f) = 0\}.$$

The following Proposition has been shown in ([11], Theorem 8.1.2).

Proposition 2.6.3. *Let $f : X \to X$ be continuous. Then*

$$\partial\sigma_{fmv}(f) \subset \sigma_\pi(f).$$

Example 2.6.4. Consider the space $l^2(\mathbb{C})$ with the standard basis and standard projection and the map from $l^2(\mathbb{C}) \to l^2(\mathbb{C})$ defined by

$$f(z_1, z_2, z_3, z_4, \ldots) = (\overline{z}_2, i\overline{z}_1, \overline{z}_4, i\overline{z}_3, \ldots).$$

From the definition of f it follows that $f(x+y) = f(x) + f(y)$ and $f(\lambda x) = \bar{\lambda} f(x)$ for all $x, y \in l^2(\mathbb{C})$ and $\lambda \in \mathbb{C}$; in particular f is positively homogeneous. Since f is positively homogeneous, $\lambda I - f_n$ is positively homogeneous and by Lemma 2.5.1 we have

$$d(\lambda I - f_n) = \inf_{\|x_n\|=1} \|[\lambda I - f_n](x_n)\|.$$

It is convenient to split into two cases:

Case 1. In the "even" case we have

$$(\lambda I - f_{2n})(z_1, z_2, \ldots, z_{2n-1}, z_{2n})$$
$$= (\lambda z_1 - \overline{z}_2, \lambda z_2 - i\overline{z}_1, \ldots, \lambda z_{2n-1} - \overline{z}_{2n}, \lambda z_{2n} - i\overline{z}_{2n-1}).$$

In [11] Furi *et al.* studied the function $f_2(z_1, z_2) = (\overline{z}_2, i\overline{z}_1)$ and proved that $\sigma_{fmv}(f_2) = \emptyset$. We can extend their result to show that $\sigma_{fmv}(f_{2n}) = \emptyset$ for all n. To verify this first notice that $d(\lambda I - f_{2n}) > 0$ for all λ. Otherwise suppose that $d(\lambda I - f_{2n}) = 0$ for some λ in $\mathbb{C}$. Since $\mathbb{C}^{2n}$ is finite dimensional there exists $y \in \mathbb{C}^{2n}$ with $\|y\| = 1$ such that

$$\|[\lambda I - f_{2n}](y)\| = 0. \tag{2.6.1}$$

Rewriting (2.6.1) by components we have

$$\begin{cases} \lambda y_1 - \overline{y}_2 = 0 \\ \lambda y_2 - i\overline{y}_1 = 0 \\ \quad\cdots \\ \lambda y_{2n-1} - \overline{y}_{2n} = 0 \\ \lambda y_{2n} - i\overline{y}_{2n-1} = 0 \end{cases}$$

Notice that $\lambda \neq 0$ otherwise we would have $y = 0$. Since y is a solution of (2.6.1) with $\|y\| = 1$ it has at least one nonzero component. Without loss of generality take $y_{2k} \neq 0$. Then we would have

$$\begin{cases} \lambda y_{2k-1} = \overline{y}_{2k} \\ \lambda y_{2k} = i\overline{y}_{2k-1} \end{cases}$$

and obtain

$$i\overline{y}_{2k-1}y_{2k-1} = \overline{y}_{2k}y_{2k}$$

a contradiction. Therefore $d(\lambda I - f_{2n}) > 0$ for all λ. Since $\partial\sigma_{fmv}(f_{2n}) \subset \sigma_\pi(f_{2n})$ by Proposition 2.6.3, from the fact that $\mathbb{C}^{2n}$ is finite dimensional and f_{2n} continuous implies that

$$\sigma_\pi(f_{2n}) \equiv \{\lambda \in \mathbb{C} : d(\lambda I - f_{2n}) = 0\},$$

we have that $\partial\sigma_{fmv}(f_{2n}) = \emptyset$. This implies that $\sigma_{fmv}(f_{2n})$ is either $\mathbb{C}$ or empty. Since

$$q(f_{2n}) = \limsup_{\|z\|\to\infty} \sqrt{\frac{|\overline{z}_2|^2 + |i\overline{z}_1|^2 + \ldots + |\overline{z}_{2n}|^2 + |i\overline{z}_{2n-1}|^2}{|z_1|^2 + |z_2|^2 + \ldots + |z_{2n-1}|^2 + |z_{2n}|^2}} = 1,$$

we can apply Proposition 8.1.2 of [11] to show that $\lambda I - f_{2n}$ is fmv-regular when $|\lambda| > 1$. Therefore $\sigma_{fmv}(f_{2n}) = \emptyset$.

Case 2. In the "odd" case we have

$$(\lambda I - f_{2n+1})(z_1, z_2, \ldots, z_{2n}, z_{2n+1})$$
$$= (\lambda z_1 - \overline{z}_2, \lambda z_2 - i\overline{z}_1, \ldots, \lambda z_{2n} - i\overline{z}_{2n-1}, \lambda z_{2n+1}).$$

Take $\zeta = (0, 0, \ldots, 0, 1)$ and note that

$$d(f_{2n+1}) = \inf_{\|x_n\|=1} \|f_{2n+1}(x_n)\| \leq \|f_{2n+1}(\zeta)\| = 0.$$

Thus $0 \in \sigma_{fmv}(f_{2n+1})$ (incidentally note also that f_{2n+1} is not surjective and therefore not stably solvable).

Since $0 \in \sigma_{fmv}(f_{2n+1})$ for all n we have that $0 \in \mathfrak{S}(f)$ and we can apply Proposition 2.4.6 to show that $0 \in \sigma_A(f)$.

Therefore $\sigma_A(f)$ is not empty.

Remark 2.6.5. Example 2.6.4 shows that the finite dimensional spectra $\sigma_{fmv}(f_n)$ need not be nested since

$$\sigma_{fmv}(f_{2n+1}) \not\subset \sigma_{fmv}(f_{2n+2}).$$

Furthermore it shows that Proposition 2.5.2 fails to be true when we drop the A-properness hypothesis. In fact, since $d(f_{2n+1}) = 0$ for all n, we have $d'(f) = 0$, but clearly 0 is not an eigenvalue. To see that f is not A-proper take $\{e_i\}$ the standard basis of $l^2(\mathbb{C})$. We have $\|e_i\| = 1$ for all i,

$$\|f_{2n+1}(e_{2n+1})\| = 0 \text{ for all } n$$

but $\{e_i\}$ does not have a convergent subsequence.

2.7 The linear case

One requirement one would expect for the new spectrum is that when the operator is linear it reduces to the usual spectrum for linear operators. As finite dimensional projections are involved in this theory, in general this fails to occur. First note that in the linear case the definition of A-stability [40] reduces to the following:

Definition 2.7.1. Let L be a bounded linear map. We say that $L : X \to X$ is *A-stable* if there exists a constant $c > 0$ and $n_0 \in \mathbb{N}$ such that $\|L_n(x)\| \geq c\|x\|$ for all $x \in X_n$ and $n \geq n_0$.

Recall that if a linear map $L : X \to X$ is a homeomorphism there exists $h > 0$ such that $\|L(x)\| \geq h\|x\|$. So the requirement $\|L_n(x)\| \geq c\|x\|$ is a natural one. The following theorem (see Theorem 1.3 of [40] for example) is a characterization of linear A-proper maps.

Theorem 2.7.2. *Let $L : X \to Y$ be a linear bounded map and Γ be a projection scheme for L. Then the following assertions are equivalent:*

1. *L is injective and A-proper.*
2. *L is surjective and A-stable.*
3. *The equation $L(x) = y$ is uniquely A-solvable.*

Lemma 2.7.3. *Let $L : X \to X$ be a linear A-proper isomorphism. Then L is A-proper stable.*

Proof. Since L is a linear isomorphism there exists $\theta \in \mathbb{R}^+$ such that $L + \varepsilon I$ is a linear isomorphism for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \theta$. Note that L is A-stable by Theorem 2.7.2, that is, $\|L_n(x)\| \geq c\|x\|$ for all $x \in X_n$ and $n \geq n_0$. Let $|\varepsilon| < \min\{\theta, c\}$. Then

$$\|L_n(x) + \varepsilon x\| \geq \|L_n(x)\| - |\varepsilon|\,\|x\| \geq (c - |\varepsilon|)\|x\| \text{ for all } x \in X_n \text{ and } n \geq n_0.$$

This proves that $L + \varepsilon I$ is an A-stable isomorphism and, by Theorem 2.7.2, $L + \varepsilon I$ is A-proper. $\square$

We have the following characterization:

Theorem 2.7.4. *Let $L : X \to X$ be a bounded linear map. Then*

$$\sigma_A(L) = \sigma(L) \cup \{\lambda \in \mathbb{C} \text{ such that } \lambda I - L \text{ is not } A\text{-proper}\}.$$

Proof. Take λ in $\rho_A(L)$. Then, by Corollary 2.3.11, $\lambda I - L$ is surjective. Since $d'_R(\lambda I - L) > 0$ then $\lambda I - L$ is A-stable. We can apply Theorem 2.7.2 to show the injectivity. Therefore $\lambda I - f$ is a A-proper isomorphism. Now take λ such that $\lambda I - L$ is a A-proper isomorphism then, by Lemma 2.7.3, $\lambda I - L$ is A-proper stable. By Theorem 2.7.2, $\lambda I - L$ is A-stable and therefore $d'_R(\lambda I - L) > 0$. Theorem 2.7.2 shows that $\lambda I - f$ is A-solvable and therefore $P_n(L)$ is fmv-regular for $n \geq \bar{n}$. Thus $\lambda I - L$ is A-regular. □

Intuitively this means that the spectrum is made up by points where either the operator is not a bijection or somehow the approximation process fails to occur. The following example shows that $\sigma(L) \subsetneq \sigma_A(L)$.

Example 2.7.5. Take f defined as in Example 1.6.5. Since f is a linear isomorphism, $0 \notin \sigma(f)$. But f is not A-proper, therefore $0 \in \sigma_A(f)$. To check that f is not A-proper we proceed, mutatis mutandis, as in Example 2.6.4.

In the case of compact linear map we have the following:

Corollary 2.7.6. *Let* $\dim(X) = \infty$ *and* $L : X \to X$ *be a compact linear map. Then*

$$\sigma_A(L) = \sigma(L).$$

Proof. Since L is compact, $\lambda I - L$ is A-proper when $\lambda \neq 0$. Since L is compact we have that $0 \in \sigma(L)$. By Theorem 2.7.4 we have $\sigma_A(L) = \sigma(L)$. □

Open question: In Theorem 2.7.4 we have shown that, given a linear map L and a projection scheme Γ, $\sigma(L) \subseteq \sigma_A(L)$. Does there exists a suitable projection scheme such that $\sigma_A(L) = \sigma(L)$?
Note that if $L : X \to X$ is a linear isomorphism, by Lemma 2.2 of [41], one can show that there exists an approximation scheme Γ_L such that L is A-proper with respect to Γ_L. But in general this fails to be a Γ_1 situation.

2.8 An interesting example

In the next example, rather than giving an estimate, we shall compute the A-spectrum in detail. This will show that the A-resolvent in general is not a connected set.

Example 2.8.1. Consider the map from $l^2(\mathbb{C}) \to l^2(\mathbb{C})$ defined by

$$f(z_1, z_2, z_3, z_4, \ldots) = (\overline{z}_2, \overline{z}_1, \overline{z}_4, \overline{z}_3, \ldots).$$

First of all notice that, for $|\lambda| \neq 1$, $\lambda I - f$ is a surjective operator with inverse

$$[\lambda I - f]^{-1}(w_1, w_2, w_3, w_4, \ldots) = \left(\frac{\overline{\lambda} w_1 + \overline{w}_2}{|\lambda|^2 - 1}, \frac{\overline{\lambda} w_2 + \overline{w}_1}{|\lambda|^2 - 1}, \frac{\overline{\lambda} w_3 + \overline{w}_4}{|\lambda|^2 - 1}, \frac{\overline{\lambda} w_4 + \overline{w}_3}{|\lambda|^2 - 1}, \ldots \right).$$

Let us now compute $\sigma(f)$. From the definition of f it follows that $f(x+y) = f(x) + f(y)$ and $f(\lambda x) = \bar{\lambda} f(x)$ for all $x, y \in l^2(\mathbb{C})$ and $\lambda \in \mathbb{C}$, in particular f is positively homogeneous. Since f is positively homogeneous, $\lambda I - f_n$ is positively homogeneous.

It is convenient to split into two cases:

Case 1. In the "even" case we have

$$(\lambda I - f_{2n})(z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = (\lambda z_1 - \overline{z}_2, \lambda z_2 - \overline{z}_1, \ldots, \lambda z_{2n-1} - \overline{z}_{2n}, \lambda z_{2n} - \overline{z}_{2n-1})$$

with inverse

$$[\lambda I - f_{2n}]^{-1}(w_1, w_2, \ldots, w_{2n}) = \left(\frac{\overline{\lambda} w_1 + \overline{w}_2}{|\lambda|^2 - 1}, \frac{\overline{\lambda} w_2 + \overline{w}_1}{|\lambda|^2 - 1}, \ldots, \frac{\overline{\lambda} w_{2n} + \overline{w}_{2n-1}}{|\lambda|^2 - 1} \right).$$

We will show that $\sigma_{fmv}(f_{2n}) = S^1$, where $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Note that if $|\lambda| = 1$ then λ is an eigenvalue of f_{2n}. In fact

$$f_{2n}\left(\frac{1}{2}, \frac{\overline{\lambda}}{2}, \ldots, \frac{1}{2n}, \frac{\overline{\lambda}}{2n}\right) = \left(\frac{\overline{\overline{\lambda}}}{2}, \frac{1}{2}, \ldots, \frac{\overline{\overline{\lambda}}}{2n}, \frac{1}{2n}\right) = \lambda \left(\frac{1}{2}, \frac{\overline{\lambda}}{2}, \ldots, \frac{1}{2n}, \frac{\overline{\lambda}}{2n}\right).$$

Since f_{2n} is positively homogeneous the eigenvalues of f_{2n} lie in $\sigma_{fmv}(f_{2n})$. Since

$$M(f_{2n}) = \sup_{z \neq 0} \frac{\|f_{2n}(z)\|}{\|z\|} = \sup_{z \neq 0} \sqrt{\frac{|\overline{z}_2|^2 + |\overline{z}_1|^2 + \ldots + |\overline{z}_{2n}|^2 + |\overline{z}_{2n-1}|^2}{|z_1|^2 + |z_2|^2 + \ldots + |z_{2n-1}|^2 + |z_{2n}|^2}} = 1$$

we can use Proposition 2.2.8 to show that $\sigma_{fmv}(f_{2n}) \subset \sigma_D(f_{2n})$. Since $\sigma_D(f_{2n}) = S^1$ we have $S^1 \subset \sigma_{fmv}(f_{2n}) \subset \sigma_D(f_{2n}) = S^1$ and therefore

$$\sigma_{fmv}(f_{2n}) = S^1.$$

Case 2. In the "odd" case we have

$$(\lambda I - f_{2n+1})(z_1, z_2, \dots, z_{2n}, z_{2n+1}) = (\lambda z_1 - \overline{z}_2, \lambda z_2 - \overline{z}_1, \dots, \lambda z_{2n} - \overline{z}_{2n-1}, \lambda z_{2n+1}).$$

with inverse

$$[\lambda I - f_{2n}]^{-1}(w_1, w_2, \dots, w_{2n+1}) = \left(\frac{\overline{\lambda} w_1 + \overline{w}_2}{|\lambda|^2 - 1}, \frac{\overline{\lambda} w_2 + \overline{w}_1}{|\lambda|^2 - 1}, \dots, \frac{\overline{\lambda} w_{2n} + \overline{w}_{2n-1}}{|\lambda|^2 - 1}, \frac{w_{2n+1}}{\lambda}\right).$$

We will show that $\sigma_{fmv}(f_{2n+1}) = \{0\} \cup S^1$. Note that if $|\lambda| = 1$ then λ is an eigenvalue of f_{2n+1}. In fact

$$f_{2n+1}\left(\frac{1}{2}, \frac{\overline{\lambda}}{2}, \dots, \frac{1}{2n}, \frac{\overline{\lambda}}{2n}, 0\right) = \lambda\left(\frac{1}{2}, \frac{\overline{\lambda}}{2}, \dots, \frac{1}{2n}, \frac{\overline{\lambda}}{2n}, 0\right).$$

Also 0 is an eigenvalue of f_{2n+1} since

$$f_{2n+1}(0, \dots, 0, 1) = (0, \dots, 0).$$

Since f_{2n+1} is positively homogeneous the eigenvalues of f_{2n+1} lie in $\sigma_{fmv}(f_{2n+1})$, therefore $\{\{0\} \cup S^1\} \subset \sigma_{fmv}(f_{2n+1})$ and $d(\lambda I - f_{2n+1}) > 0$ for all n when $\lambda \notin \{\{0\} \cup S^1\}$. Since

$$\begin{aligned} M(f_{2n+1}) = \sup_{z \neq 0} \frac{\|f_{2n+1}(z)\|}{\|z\|} &= \sup_{z \neq 0} \sqrt{\frac{|\overline{z}_2|^2 + |\overline{z}_1|^2 + \dots + |\overline{z}_{2n}|^2 + |\overline{z}_{2n-1}|^2}{|z_1|^2 + |z_2|^2 + \dots + |z_{2n-1}|^2 + |z_{2n}|^2 + |z_{2n+1}|^2}} \\ &\leq \sup_{z \neq 0} \sqrt{\frac{|\overline{z}_2|^2 + |\overline{z}_1|^2 + \dots + |\overline{z}_{2n}|^2 + |\overline{z}_{2n-1}|^2}{|z_1|^2 + |z_2|^2 + \dots + |z_{2n-1}|^2 + |z_{2n}|^2}} = 1, \end{aligned}$$

we can use Proposition 2.2.8 to show that $\sigma_{fmv}(f_{2n+1}) \subset \sigma_D(f_{2n+1})$. Since

$$\sigma_D(f_{2n+1}) = \{\{0\} \cup S^1\}$$

we have

$$\sigma_{fmv}(f_{2n+1}) = \{\{0\} \cup S^1\}.$$

Therefore

$$\sigma_{fmv}(f_{2n+1}) \not\subset \sigma_{fmv}(f_{2n+2}).$$

Is $\lambda I - f$ A-proper? First note that $\lambda I - f$ is surjective for all $\lambda \notin S^1$. In fact, given $w \in l^2(\mathbb{C})$ where $w = (w_1, w_2, w_3, w_4, \ldots)$, we have

$$[\lambda I - f]\left(-\frac{\overline{\lambda} w_1 + \overline{w}_2}{1-|\lambda|^2}, -\frac{\overline{\lambda} w_2 + \overline{w}_1}{1-|\lambda|^2}, -\frac{\overline{\lambda} w_3 + \overline{w}_4}{1-|\lambda|^2}, -\frac{\overline{\lambda} w_4 + \overline{w}_3}{1-|\lambda|^2}, \ldots\right)$$
$$= (w_1, w_2, w_3, w_4, \ldots).$$

When $|\lambda| \neq 1$ and $\lambda \neq 0$, we shall see that $\lambda I - f$ is a surjective A-stable map and therefore, by Theorem 1.6.11, A-proper. To show the A-stability it is enough to show that there exists $\alpha \in \mathbb{R}_+$ such that

$$\|P_k[\lambda I - f](x-y)\| \geq \alpha \|x - y\| \text{ for all } x, y \in \mathbb{C}^k, x \neq y, k \in \mathbb{N}.$$

If $|\lambda| < 1$ and $\lambda \neq 0$ set $\alpha = \min\{(1-|\lambda|), |\lambda|\}$.

When $k = 2n$ we have

$$\begin{aligned}\|[\lambda I - f_{2n}](x-y)\| \geq& \|f_{2n}(x-y)\| - |\lambda|\|x-y\| = \|x-y\| - |\lambda|\|x-y\| \\ =&(1-|\lambda|)\|x-y\| \geq \alpha\|x-y\|.\end{aligned}$$

When $k = 2n+1$, we have

$$\begin{aligned}&\|[\lambda I - f_{2n+1}](x-y)\|^2 \\ &\quad= |\lambda(x_1 - y_1) - \overline{x}_2 + \overline{y}_2|^2 + \ldots + |\lambda(x_{2n} - y_{2n}) - \overline{x}_{2n-1} + \overline{y}_{2n-1}|^2 \\ &\qquad + |\lambda(x_{2n+1} - y_{2n+1})|^2 \\ &\qquad\quad = \|[\lambda I - f_{2n}](P_{2n}(x-y))\|^2 + |\lambda|^2 |x_{2n+1} - y_{2n+1}|^2 \\ &\qquad\quad \geq (1-|\lambda|)^2 \|P_{2n}(x-y)\|^2 + |\lambda|^2 |x_{2n+1} - y_{2n+1}|^2 \\ &\qquad\qquad \geq \alpha^2 \|P_{2n}(x-y)\|^2 + \alpha^2 |x_{2n+1} - y_{2n+1}|^2 = \alpha^2 \|x-y\|^2.\end{aligned}$$

Then, for all k, $\|[\lambda I - f_k](x-y)\| \geq \alpha \|x-y\|$.

If $|\lambda| > 1$ set $\alpha = |\lambda| - 1$.

When $k = 2n$ we have

$$\begin{aligned}\|[\lambda I - f_{2n}](x-y)\| \geq& |\lambda|\|x-y\| - \|f_{2n}(x-y)\| = |\lambda|\|x-y\| - \|x-y\| \\ =&(|\lambda| - 1)\|x-y\| \geq \alpha\|x-y\|.\end{aligned}$$

When $k = 2n+1$, we have

$$\begin{aligned}
&\|[\lambda I - f_{2n+1}](x-y)\|^2 \\
&\quad = |\lambda(x_1 - y_1) - \overline{x}_2 + \overline{y}_2|^2 + \ldots + |\lambda(x_{2n} - y_{2n}) - \overline{x}_{2n-1} + \overline{y}_{2n-1}|^2 \\
&\qquad + |\lambda(x_{2n+1} - y_{2n+1})|^2 \\
&\qquad = \|[\lambda I - f_{2n}](P_{2n}(x-y))\|^2 + |\lambda|^2 |x_{2n+1} - y_{2n+1}|^2 \\
&\qquad \geq (|\lambda| - 1)^2 \|P_{2n}(x-y)\|^2 + |\lambda|^2 |x_{2n+1} - y_{2n+1}|^2 \\
&\qquad\qquad \geq \alpha^2 \|P_{2n}(x-y)\|^2 + \alpha^2 |x_{2n+1} - y_{2n+1}|^2 = \alpha^2 \|x-y\|^2.
\end{aligned}$$

Therefore, for all k, $\|[\lambda I - f_k](x-y)\| \geq \alpha\|x-y\|$.

This shows that, when $\lambda \notin \{0\} \cup S^1$, $\lambda I - f$ is A-proper stable. It remains to show that $d'_R(\lambda I - f) > 0$. Notice that, since $\|[\lambda I - f_k](x-y)\| \geq \alpha\|x-y\|$ when $x \neq y$, putting $y = 0$ and dividing by $\|x\|$ we have,

$$d_R(\lambda I - f_k) \geq \frac{\|[\lambda I - f_k](x)\|}{\|x\|} \geq \alpha \text{ for all } k \in \mathbb{N}.$$

Therefore $d'_R(\lambda I - f) \geq \alpha > 0$ when $\lambda \notin \{0\} \cup S^1$.

Note that since $\{\{0\} \cup S^1\} \subset \sigma_{fmv}(f_{2n+1})$ for all n we have that

$$\{\{0\} \cup S^1\} \subset \mathfrak{S}(f)$$

and we can apply Proposition 2.4.6 to show that $\{\{0\} \cup S^1\} \subset \sigma_A(f)$. Therefore (see Figure 2.1)

$$\sigma_A(f) = \{0\} \cup S^1.$$

Note that, since $d(f_{2n+1}) = 0$ for all n, we have $d'(f) = 0$. In view of Proposition 2.5.2 we can say that f is not A-proper because otherwise 0 would be an eigenvalue of f, and clearly it is not. We can also show that f is not A-proper directly. In fact given $\{e_i\}$ the standard basis of $l^2(\mathbb{C})$ we have $\|e_i\| = 1$ for all i,

$$\|f_{2n+1}(e_{2n+1})\| = 0 \text{ for all } n$$

but $\{e_i\}$ does not have a convergent subsequence.

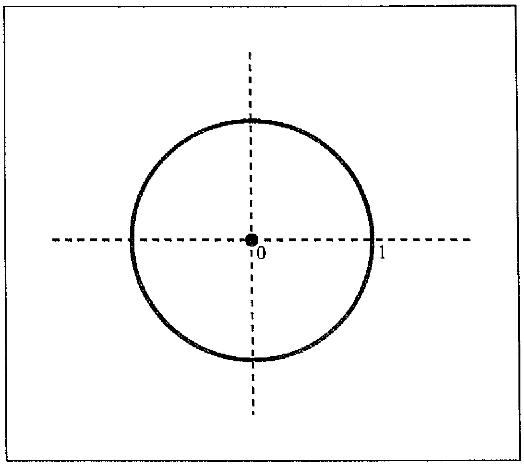

Figure 2.1: A non-connected resolvent

2.9 An application

As an application of the theory for example we can prove, using Proposition 2.4.6 and Corollary 2.5.9, a result similar to the Birkhoff-Kellogg Theorem for finitely continuous maps. First we state the Birkoff-Kellogg Theorem (see for example Theorem 10.1.5 of [11]):

Theorem 2.9.1. *Let $S = \{x \in X : \|x\| = 1\}$, X be an infinite dimensional Banach space and $f : S \to X$ be compact such that*

$$\inf_{x \in S} \|f(x)\| > 0.$$

Then f has a positive eigenvalue.

Lemma 2.9.2. *Let $S = \{x \in X : \|x\| = 1\}$, X be an infinite dimensional Banach space and $f : S \to X$ be a finitely continuous, bounded map. Let $\tilde{f} : X \to X$ be the positively homogeneous operator defined as follows:*

$$\tilde{f}(x) = \begin{cases} \|x\| f\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $\gamma(f) = \gamma(\tilde{f})$.

Proof. It is sufficient to show that f is A-proper if and only if $\tilde{f}$ is A-proper. Suppose that f is A-proper and let $\{x_n\}$ be a bounded sequence such that

$$\|P_n \tilde{f}(x_n) - P_n y\| \to 0 \text{ as } n \to \infty.$$

By boundedness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j}\| \to \alpha$. If $\alpha \neq 0$ we have

$$\left\|\alpha P_{n_j} f\Big(\frac{x_{n_j}}{\|x_{n_j}\|}\Big) - P_{n_j} y\right\| \to 0.$$

Since f is A-proper there exists a subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$ and $x \in S$ such that $\frac{x_{n_{j(k)}}}{\|x_{n_{j(k)}}\|} \to x$ and $f(x) = \frac{y}{\alpha}$. Therefore $\tilde{f}(x) = y$. If $\alpha = 0$ we have $x_{n_j} \to 0$, $P_{n_j} y \to 0$ and 0 is a solution of the equation $\tilde{f}(x) = 0$. To show that f is A-proper if $\tilde{f}$ is A-proper, simply note that $f = \tilde{f}|_S$. □

Definition 2.9.3. Let $f : X \to X$ be a continuous map. A point $\lambda \in \mathbb{K}$ is said to be a *asymptotic bifurcation point* (see [27]) if there exists a sequence $\{(\lambda_n, x_n)\}$ in $\mathbb{K} \times X$ such that $\|x_n\| \to \infty$, $\lambda_n \to \lambda$ and $\lambda_n x_n = f(x_n)$ for every n.

Theorem 2.9.4. *Let $S = \{x \in X : \|x\| = 1\}$, X be an infinite dimensional real Banach space and $f : S \to X$ be a finitely continuous, bounded map such that*

$$\liminf_{n\to\infty} \inf_{\|x_n\|=1} \|P_n f(x_n)\| > \gamma(f).$$

Then f has an eigenvalue.

Proof. Let $\tilde{f} : X \to X$ be the positively homogeneous operator defined as in Lemma 2.9.2, then

$$d(\tilde{f}) = \inf_{\|x\|=1} \|f(x)\|, \; M(\tilde{f}) = \sup_{\|x\|=1} \|f(x)\|$$

and $\gamma(\tilde{f}) = \gamma(f)$. Let $B(\tilde{f}_{2n+1})$ be the set of all asymptotic bifurcation points of $\tilde{f}_{2n+1}$. By Theorem 11.1.3 of [11], there exists $\mu_{2n+1} \in \mathbb{R}$ such that $\mu_{2n+1} \in B(\tilde{f}_{2n+1})$. Since μ_{2n+1} is an asymptotic bifurcation point we have $d(\mu_{2n+1} I - \tilde{f}_{2n+1}) = 0$. So $\mu_{2n+1} \in \sigma_{fmv}(\tilde{f}_{2n+1})$. Since, by our assumption, $d'(\tilde{f}) > \gamma(f)$, there exists $\varepsilon_0 \in \mathbb{R}_+$ such that $d'(\tilde{f}) \geq \gamma(f) + \varepsilon_0$ and there exists $\bar{n}$ such that $d(\tilde{f}_{2n+1}) \geq \gamma(f) + \frac{\varepsilon_0}{2}$ whenever $n \geq \bar{n}$. Assume $|\mu_{2n+1}| \leq \gamma(f) + \frac{\varepsilon_0}{3}$. Then $|\mu_{2n+1}| < d(\tilde{f}_{2n+1})$ for every $n \geq \bar{n}$. Therefore

$$d(\mu_{2n+1} I - \tilde{f}_{2n+1}) \geq d(\tilde{f}_{2n+1}) - |\mu_{2n+1}| > 0.$$

This contradicts $d(\mu_{2n+1}I - \tilde{f}_{2n+1}) = 0$. Therefore $|\mu_{2n+1}| > \gamma(f) + \frac{\varepsilon_0}{3}$. The sequence $\{\mu_{2n+1}\}$ is bounded since we have

$$r_{\sigma_{fmv}}(\tilde{f}_{2n+1}) \leq q(\tilde{f}_{2n+1}) \leq M(\tilde{f}).$$

Therefore there exists a subsequence $\{\mu_j\}$ of $\{\mu_{2n+1}\}$ such that $\mu_j \to \mu$ with $|\mu| > \gamma(f)$. Since $\mu_j \in \sigma_{fmv}(\tilde{f}_j)$ and $\mu_j \to \mu$, by Proposition 2.4.6, we have $\mu \in \sigma_A(\tilde{f})$. By Corollary 2.5.9, it follows that there exists $\bar{t} \in (0, 1]$ such that $\mu/\bar{t}$ is an eigenvalue of $\tilde{f}$. Let $\bar{x} \in X$ with $\|\bar{x}\| = 1$ be such that $\tilde{f}(\bar{x}) = \mu/\bar{t}(\bar{x})$. Then $f(\bar{x}) = r\bar{x}$ where $r = \mu/\bar{t}$. □

The following example shows that there exists a map f to which Theorem 2.9.1 does not apply (since f is not compact) but Theorem 2.9.4 can be used.

Example 2.9.5. Consider the space $l^2(\mathbb{R})$ and let g be the radial retraction of $l^2(\mathbb{R})$ onto the unit ball given by

$$g(x) = \begin{cases} \dfrac{x}{\|x\|} & \text{if } \|x\| > 1, \\ x & \text{if } \|x\| \leq 1. \end{cases}$$

Note that $\beta(g) = 1$ (see [6]). Fix now $y \in l^2(\mathbb{R})$ with

$$y = \sum_{i=1}^{n_0} a_i e_i,$$

$\{e_i\}$ being the standard basis of $l^2(\mathbb{R})$, $a_i \in \mathbb{R}$ and $\|y\| > 2$. Let $f : S \to l^2(\mathbb{R})$ be defined by

$$f(x) = y + g(x).$$

Then

$$\begin{aligned} \inf_{\|x_n\|=1} \|P_n f(x_n)\| &= \inf_{\|x_n\|=1} \|P_n y + P_n g(x_n)\| \\ &\geq \|P_n y\| - \sup_{\|x_n\|=1} \|P_n g(x_n)\| = \|y\| - 1 > 1, \end{aligned}$$

whenever $n \geq n_0$. Now

$$\beta(f) = \beta(y + g) = \beta(g) = 1.$$

Therefore

$$\liminf_{n\to\infty} \inf_{\|x_n\|=1} \|P_n f(x_n)\| > \beta(f).$$

Furthermore we have

$$\sup_{\|x|=1} \|f(x)\| = \sup_{\|x|=1} \|y + g(x)\| \leq \|y\| + 1.$$

Hence f satisfies the conditions of Theorem 2.9.4. Therefore f has an eigenvalue.

2.10 On the growth properties

In this section we investigate the relation between the various growth conditions so far encountered, we will also study

$$\tilde{d}(f) = \liminf_{\|x\|\to+\infty} \frac{|(f(x), J(x))|}{\|x\|^2}$$

and show why it is not suitable as a "lack of injectivity" type of condition for the spectrum.

Remark 2.10.1. Note that for all $x_n \in X_n$ we have that

$$\liminf_{\|x_n\|\to+\infty} \frac{\|P_n f(x_n)\|}{\|x_n\|} \leq \|P_n\| \liminf_{\|x_n\|\to+\infty} \frac{\|f(x_n)\|}{\|x_n\|} \leq \liminf_{\|x_n\|\to+\infty} \frac{\|f(x_n)\|}{\|x_n\|}.$$

Let $n \to \infty$ and take the lim inf.

$$d'(f) = \liminf_{n\to+\infty} \left\{ \liminf_{\|x_n\|\to+\infty} \frac{\|P_n f(x_n)\|}{\|x_n\|} \right\} \leq \liminf_{n\to+\infty} \liminf_{\|x_n\|\to+\infty} \frac{\|f(x_n)\|}{\|x_n\|}$$

If f is continuous, do we have $d'(f) \leq d(f)$? Does $d'(f) > 0$ imply $d(f) > 0$? We do not know the answer to this questions, but Lemma 2.10.4 and 2.10.5 give partial answers.

Lemma 2.10.2. *Let $f : X \to X$ with $m'(f) > 0$. Fix $y \in X$ and let x_n be a solution of the finite dimensional equation $P_n f(x_n) = P_n y$. Then $\{x_n\}$ is bounded.*

Proof. If x_n is a solution of $P_n f(x_n) = P_n y$ we have that

$$\|y\| \geq \|P_n y\| = \|P_n f(x_n)\| \geq m(f_n)\|x_n\|$$

Since $m'(f) = \liminf_{n\to+\infty} m(f_n) = l > 0$ there exists $\bar{n}$ such that $m(f_n) > \frac{l}{2}$ for all $n \geq \bar{n}$. Setting $K = \max\left\{\frac{2\|y\|}{l}, \|x_1\|, \ldots, \|x_{\bar{n}}\|\right\}$ then

$$\|x_n\| \leq K \text{ for all } n.$$

$\square$

Remark 2.10.3. Can we have a similar result by replacing the condition $m'(f) > 0$ with $d'(f) > 0$? Probably not. It is true that $d'(f) = l > 0$ implies that there exists $\bar{n}$ such that

$$d(f_n) = \liminf_{\|x_n\|\to+\infty} \frac{\|P_n f(x_n)\|}{\|x_n\|} > \frac{l}{2} \text{ for all } n > \bar{n}.$$

But the condition $d(f_n) > \frac{l}{2}$ implies that there exists R such that

$$\frac{\|P_n f(x_n)\|}{\|x_n\|} > \frac{l}{3} \text{ for } \|x_n\| \geq R.$$

The problem is that R can depend on n. This is the reason why we introduce the condition $d'_R(f)$, similar to $d'(f)$, but slightly stronger.

When do $d'_R(f)$ and $d'(f)$ coincide? For example when f is positively homogeneous (in particular when f is linear).

Lemma 2.10.4. *Let f be a positively homogeneous, Lipschitz continuous map, with Lipschitz constant k and $d(f) = 0$. Then $d'(f) = 0$.*

Proof. By Lemma 2.5.1 we have that $d(f) = \inf_{\|x\|=1} \|f(x)\|$. Since $d(f) = 0$ there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ and $\|f(x_n)\| < \varepsilon_n$, where $\varepsilon_n \to 0$. As $\overline{\bigcup X_n} = X$, there exists $P_{m(n)}$ such that $y_m = P_{m(n)}x_n$ satisfies $\|y_m - x_n\| < \varepsilon_n$. Then $\|f(y_m) - f(x_n)\| \leq k\varepsilon_n$ so

$$\|f(y_m)\| \leq k\varepsilon_n + \varepsilon_n$$

and

$$\|P_m f(y_m)\| \leq \|f(y_m)\| \leq (k+1)\varepsilon_n \text{ with } 1 - \varepsilon_n \leq \|y_m\| \leq 1.$$

Therefore

$$\frac{\|P_m f(y_m)\|}{\|y_m\|} \leq (k+1)\frac{\varepsilon_n}{1-\varepsilon_n}.$$

By Lemma 2.5.1 we have

$$d(P_m f) \leq (k+1)\frac{\varepsilon_n}{1-\varepsilon_n}.$$

This implies $d'(f) = 0$. □

Lemma 2.10.5. *Let f be a positively homogeneous, finitely continuous A-proper map with $d'(f) = 0$. Then $d(f) = 0$.*

Proof. By Lemma 2.5.1 we have that

$$\inf_{\|x_{n_j}\|=1} \|P_{n_j} f(x_{n_j})\| \to 0 \text{ as } j \to \infty, x_{n_j} \in X_{n_j}.$$

Since X_{n_j} is finite dimensional we can build a sequence $\{y_{n_j}\}$ with $\|y_{n_j}\| = 1$ such that $P_{n_j} f y_{n_j} \to 0$ as $j \to \infty$. By A-properness there exists a subsequence $\{y_{n_{j(k)}}\}$ of $\{y_{n_j}\}$ such that $y_{n_{j(k)}} \to y$ and $f(y) = 0$. Therefore $d(f) = 0$. □

Remark 2.10.6. Lemma 2.10.5 does not hold when f is not A-proper. In fact take f as in Example 1.6.5, by Lemma 2.5.1 we have

$$d(f_{2n+1}) = \inf_{\|y_{2n+1}\|=1} \|f_{2n+1}(y)\| \leq \|f_{2n+1}(e_{2n+1})\| = 0 \text{ for every } n.$$

Therefore $d'(f) = 0$, whereas $d(f) = 1$.

Lemma 2.10.7. *Let $f : X \to X$ be a continuous map. Then*

$$m'(f) \leq m(f).$$

Proof. Fix $\varepsilon > 0$. Then there exists $x_\varepsilon \in X$, $x_\varepsilon \neq 0$, such that

$$m(f) \leq \frac{\|f(x_\varepsilon)\|}{\|x_\varepsilon\|} < m(f) + \varepsilon.$$

Take now $x_{\varepsilon,n} = P_n x_\varepsilon$. Since $x_{\varepsilon,n} \to x_\varepsilon$ and $f(x_{\varepsilon,n}) \to f(x_\varepsilon)$ we have

$$\inf_{\substack{x_n \neq 0 \\ x_n \in X_n}} \frac{\|f_n(x_n)\|}{\|x_n\|} \leq \frac{\|f_n(x_{\varepsilon,n})\|}{\|x_{\varepsilon,n}\|} \leq \frac{\|f(x_{\varepsilon,n})\|}{\|x_{\varepsilon,n}\|} \leq \frac{\|f(x_\varepsilon)\|}{\|x_\varepsilon\|} + \varepsilon < m(f) + 2\varepsilon$$

whenever $n \geq \bar{n}$. Since ε is arbitrary we have

$$\liminf_{n\to\infty} \inf_{x_n \neq 0} \frac{\|f_n(x_n)\|}{\|x_n\|} \leq \inf_{x \neq 0} \frac{\|f(x)\|}{\|x\|}.$$

□

A similar inequality holds also for $d(f)$.

Lemma 2.10.8. *Let $f : X \to X$ be a continuous map. Then $d'_R(f) \leq d(f)$.*

Proof. Let R be fixed and take $\varepsilon > 0$. Then there exists $x_\varepsilon \in X$, $\|x_\varepsilon\| > 2R$, such that

$$d(f) - \varepsilon < \frac{\|f(x_\varepsilon)\|}{\|x_\varepsilon\|} < d(f) + \varepsilon.$$

Take now $x_{\varepsilon,n} = P_n x_\varepsilon$. Since $x_{\varepsilon,n} \to x_\varepsilon$ and $f(x_{\varepsilon,n}) \to f(x_\varepsilon)$ we have

$$d_R(f_n) = \inf_{\substack{\|x_n\| \geq R \\ x_n \in X_n}} \frac{\|f_n(x_n)\|}{\|x_n\|} \leq \frac{\|f_n(x_{\varepsilon,n})\|}{\|x_{\varepsilon,n}\|} \leq \frac{\|f(x_{\varepsilon,n})\|}{\|x_{\varepsilon,n}\|} \leq \frac{\|f(x_\varepsilon)\|}{\|x_\varepsilon\|} + \varepsilon < d(f) + 2\varepsilon$$

whenever $n \geq \bar{n}$. Therefore

$$\liminf_{n \to \infty} d_R(f_n) \leq d(f) + 2\varepsilon.$$

As ε is arbitrary this shows $d'_R(f) \leq d(f)$. □

What is the relation between the condition $\tilde{d}(f) > 0$ and the stably solvable requirement? If X is a finite dimensional complex Banach space of dimension greater than 1 with X^* strictly convex and $f : X \to X$ with $\tilde{d}(f) > 0$ one can show that f is stably solvable.

Lemma 2.10.9. *Let S^1 be the unit circle in the complex plane $\mathbb{C}$. Let X be a finite dimensional complex Banach space of dimension n greater than 1. Let $S_R = \{x \in X : \|x\| = R\}$ and let $g : S_R \to S^1$ be continuous. Then g is homotopic to the constant map 1, that is, there is a continuous map $h : [0,1] \times S_R \to S^1$ such that $h(0,x) = 1$ and $h(1,x) = g(x)$ for all $x \in S_R$.*

This is a consequence of the fact that S_R is homeomorphic to the real sphere S^{2m-1} and the homotopy group $\pi_j(S^1) = 0$ for $j > 1$.

Theorem 2.10.10. *Let X be a finite dimensional complex Banach space of dimension greater than 1 with X^* strictly convex and let $T : X \to X$ be continuous. Suppose there exists $R > 0$ such that $(Tx, Jx) \neq 0$ for all $x \in S_R$. Then there is $x_0 \in B_R$ such that $Tx_0 = 0$.*

Proof. For $\|x\| = R$ let $g(x) = \dfrac{\overline{(Tx, Jx)}}{|(Tx, Jx)|}$, where $\overline{z}$ denotes the complex conjugate of z. As T and J are continuous and $(Tx, Jx) \neq 0$ for all $x \in S_R$, $g : S_R \to S^1$ is continuous. By Lemma 2.10.9, there is a continuous map $h : [0,1] \times S_R \to S^1$ such that $h(0,x) = 1$ and $h(1,x) = g(x)$ for all $x \in S_R$. Now define $H(t,x) = h(t,x)Tx$ for $0 \leq t \leq 1$ and $x \in S_R$. Then $H(t,x) \neq 0$ so by homotopy invariance the Brouwer degrees $d(T, B_R, 0)$ and $d(H_1, B_R, 0)$ are equal. [We take any continuous extension of H_1 to $\overline{B}_R$.] Consider the homotopy $M(t,x) = tH_1(x) + (1-t)x$ for $x \in \overline{B}_R$, $0 \leq t \leq 1$. We claim that $M(t,x) \neq 0$ for $x \in S_R$ and $t \in [0,1]$. Indeed

$$(M(t,x), Jx) = t|(Tx, Jx)| + (1-t)(x, Jx) > 0.$$

Therefore $d(H_1, B_R, 0) = d(I, B_r, 0) = 1$ and hence $d(T, B_R, 0) = 1$ so there exists $x_0 \in B_r$ with $Tx_0 = 0$. □

Theorem 2.10.11. *Let X be a finite dimensional complex Banach space of dimension greater than 1 with X^* strictly convex and let $f : X \to X$ be continuous with $\tilde{d}(f) > 0$. Then f is stably solvable, that is, for every continuous map $h : X \to X$ with $q(h) = 0$, there is a solution of the equation $f(x) = h(x)$.*

Proof. Let $T(x) = f(x) - h(x)$. Then

$$\frac{|(Tx, Jx)|}{\|x\|^2} \geq \frac{|(fx, Jx)|}{\|x\|^2} - \frac{\|hx\|}{\|x\|}$$

so

$$\liminf_{\|x\| \to \infty} \frac{|(Tx, Jx)|}{\|x\|^2} \geq \tilde{d}(f)$$

and there is $R > 0$ such that $(Tx, Jx) \neq 0$ for $\|x\| = R$. Hence by Theorem 2.10.10 there exists $x_0 \in B_R$ with $Tx_0 = 0$. □

The converse is not true as the following example shows:

Example 2.10.12. Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by $f(z_1, z_2) = (\overline{z}_2, -\overline{z}_1)$. First of all note that since

$$\frac{\|f(z)\|}{\|z\|} = \sqrt{\frac{z_2\overline{z}_2 + z_1\overline{z}_1}{z_1\overline{z}_1 + z_2\overline{z}_2}} = 1 \text{ for all } z \neq 0$$

we have that $d(f) = 1$. We will now show that $[f] = [I]$, where $[f]$ is the homotopy class associated with f (for further information on homotopy classes over the spheres see for example [17]). Using proposition 6.2.2 of [11], this will make f fmv-regular and therefore stably solvable.

To see this, we define the homotopy $H : S_1 \times [0,1] \to S_1$ by

$$H(z,t) = \frac{tz + (1-t)f(z)}{\|tz + (1-t)f(z)\|}.$$

We claim that, for all $t \in [0,1]$ and z with $\|z\| = 1$, H is a continuous map with $H(z,0) = f(z)$ and $H(z,1) = z$. To check the continuity of H all we need to do is to show that $\|tz + (1-t)f(z)\| \neq 0$ when $\|z\| = 1$ and $t \in [0,1]$. Therefore we can study the system of equations

$$\begin{cases} tz_1 + (1-t)\overline{z}_2 = 0 \\ tz_2 + (t-1)\overline{z}_1 = 0 \end{cases}$$

Setting $z_1 = a + ib$ and $z_2 = c + id$ where $a, b, c, d \in \mathbb{R}$ and splitting the real and imaginary parts this is equivalent to solve the system

$$A = \begin{cases} ta + c - tc = 0 \\ tb - d + td = 0 \\ tc + ta - a = 0 \\ td - tb + b = 0 \end{cases}, \quad \text{say } A(t) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

Notice that the equation $\det A(t) = 0$ has no real solution and therefore A has only the trivial solution $0 \notin S^1$.

Then f is a stably solvable map with $d(f) = 1$, but in this case $\tilde{d}(f) = 0$ since $\langle f(z), z \rangle = 0$ for all $z \in \mathbb{C}^2$.

Remark 2.10.13. In the previous example we provided an antilinear map with $\tilde{d}(f) = 0$ and $d(f) > 0$. One could ask what happens when f is a linear map from a finite dimensional complex Banach space to itself. Are the two conditions equivalent? The answer is no as the following shows:

Example 2.10.14. Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by $f(z_1, z_2) = (-z_2, z_1)$. Note that $d(f) = 1$. If we let $\{x_n\} = \{(n, 0)\}$ we have that $f(x_k) = (0, k)$ and therefore $\langle f(x_k), x_k\rangle = 0$ for all $k \in \mathbb{N}$ and $\tilde{d}(f) = 0$.

2.11 Some continuation principles

Proposition 2.2.4 is a continuation principle that plays an important role in the theory of fmv-regular maps. We can prove similar results for A-regular maps:

Proposition 2.11.1. *Let $f : X \to X$ be a continuous, A-regular map and suppose there exists $\rho \in \mathbb{R}_+$ such that*

$$\beta(\{P_n f x_n\}) \geq \rho\beta(\{x_n\}) \tag{2.11.1}$$

is satisfied for each bounded sequence $\{x_n \mid x_n \in X_n\}$. If $g : X \to X$ is a β-contraction such that $\beta(g) < \mu_0$ and $q_r(g) < d'_R(f)$, then $f + g$ is A-regular.

Proof. By Theorem 2.3.4, $f + g$ is A-proper stable. By Lemma 2.4.1

$$d'_R(f+g) \geq d'_R(f) - q_R(g) > 0.$$

Since $q_R(g) < d'_R(f)$, there exists $\varepsilon_1 > 0$ such that $q_R(g) < d'_R(f) - \varepsilon_1 < d'_R(f)$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$d(f_n) \geq d_R(f_n) \geq d'_R(f) - \varepsilon_1 > q_R(g) \geq q_R(g_n)$$

for every $n \geq n_0$. Since $d(f_n) > q(g_n)$, we can apply Proposition 2.2.4 to show that $f_n + g_n$ is stably solvable. □

Unfortunately, condition (2.11.1) is not easy to check. A stronger, but easier to verify, requirement is given by the following.

Lemma 2.11.2. *Let $f : X \to X$ be a continuous, surjective map with*

$$\|P_n f(x_n) - P_n f(y_n)\| \geq \rho\|x_n - y_n\| \tag{2.11.2}$$

for every $x_n, y_n \in X_n$ *and* $n \geq n_0$. *Then*

$$\beta(\{f_n x_n\}) \geq \rho\beta(\{x_n\})$$

for each bounded sequence $\{x_n | x_n \in X_n\}$ *and* $n \geq n_0$.

Proof. For $n \geq n_0$, let $\{x_n | x_n \in X_n\}$ be any bounded sequence and suppose that $\{f_n(x_n)\}$ is covered by finitely many balls $B(y_j, r)$ with $\{y_1, \ldots, y_k\} \subset X$. Since f is surjective there exist $\{u_1, \ldots, u_k\} \subset X$ such that $f(u_j) = y_j$ for all j. Set $u_{j,n} = P_n u_j$. Obviously $u_{j,n} \to u_j$ and, by continuity of f and P_n,

$$f_n(u_{j,n}) \to f(u_j) \text{ for every } j$$

In particular, if we fix $\varepsilon > 0$, there exists $n_1 \in N$, $n_1 \geq n_0$, such that, for $n \geq n_1$, we have

$$\|u_j - u_{j,n}\| < \varepsilon \text{ and } \|f_n(u_{j,n}) - f(u_j)\| < \varepsilon \text{ for every } j$$

and therefore, for those n such that $\{f_n(x_n)\}$ belongs to $B(y_j, r)$ and exceeds n_1, we have

$$\begin{aligned}
\rho\|x_n - u_j\| \leq& \rho\|x_n - u_{j,n}\| + \rho\|u_j - u_{j,n}\| \\
\leq& \|f_n(x_n) - f_n(u_{j,n})\| + \rho\varepsilon \\
\leq& \|f_n(x_n) - f_n(u_{j,n}) + y_j - y_j\| + \rho\varepsilon \\
\leq& \|f_n(x_n) - y_j\| + \|f_n(u_{j,n}) - f(u_j)\| + \rho\varepsilon \leq r + \varepsilon(1 + \rho).
\end{aligned}$$

It follows that the set $\{x_n | n \geq n_1\}$ is covered by the balls $B(u_j, (r + \varepsilon(1 + \rho))\rho^{-1})$. As ε is arbitrary, we have shown that

$$\beta(\{f_n x_n\}) \geq \rho\beta(\{x_n\})$$

for $n \geq n_0$ as required. □

Note that Lemma 2.11.2, unlike Lemma 2.4 of [40], does not require the finite dimensional subspaces $\{X_n\}$ to be nested.

Corollary 2.11.3. *Let $f : X \to X$ be a continuous, A-regular map such that* (2.11.2) *holds, and let $g : X \to X$ be such that $q_R(g) < d'_R(f)$ and $\beta(g) < \rho$. Then $f + g$ is A-regular.*

Proof. Note that f satisfies the hypothesis of Proposition 2.11.1 since, by Lemma 2.11.2, (2.11.1) holds. □

Corollary 2.11.4. *If f satisfies the hypothesis of Proposition 2.11.1, then f is strictly stably solvable.*

Proof. Since $f + g$ is A-regular it is surjective and in particular we can solve the equation $f(x) + g(x) = 0$. In particular f is (a, p)-stably solvable for every $a < \dfrac{\rho}{2}$ and $p < d'_R(f)$ by Proposition 2.11.1. □

Remark 2.11.5. Note that condition (2.11.2) depends on λ, i.e.

$$(f \text{ satisfies } (2.11.2) \not\Rightarrow f + \lambda I \text{ satisfies } (2.11.2)).$$

In the case of a *demicontinuous* map we have the following:

Theorem 2.11.6. *Let X be a reflexive Banach space with X^* strictly convex and suppose that $D \subseteq X$ is closed and $T : D \to X$ is a demicontinuous map such that there exists $\mu_0 > 0$ with*

$$\beta(\{P_n f x_n\}) \geq \mu_0 \beta(\{x_n\})$$

for each bounded sequence $\{x_n \mid x_n \in D_n\}$. If $f : D \to X$ is a β-contraction such that $\beta(f) < \mu_0$, then $T_t \equiv T + tf : D \to X$ is A-proper stable for each $t \in \mathbb{K}$ with $|t| \leq 1$.

Proof. First we note that $T_{1n} : D_n \subset X_n \to X_n$ is continuous for each $n \in \mathbb{N}$. Now let $\{x_{n_j} \mid x_{n_j} \in D_{n_j}\}$ be a bounded sequence such that $P_{n_j}T(x_{n_j}) + P_{n_j}f(x_{n_j}) - g \to 0$ as $j \to \infty$ for some $g \in Y$. Now

$$\mu_0 \beta\{x_{n_j}\} \leq \beta\{P_{n_j}T(x_{n_j})\} = \beta\{g - P_{n_j}f(x_{n_j})\} = \beta\{P_{n_j}f(x_{n_j})\} \leq \beta(f)\beta\{x_{n_j}\},$$

and therefore $\beta\{x_{n_j}\} = 0$. Then there exists a subsequence, $\{x_{n_{j(k)}}\}$ such that $x_{n_{j(k)}} \to x$. Since f is continuous we have $P_{n_{j(k)}} f(x_{n_{j(k)}}) \to f(x)$ and

$$P_{n_{j(k)}} T(x_{n_{j(k)}}) \to g - f(x)$$

By demicontinuity of T we have $T(x_{n_{j(k)}}) \rightharpoonup T(x)$. Let $J : X \to X^*$ be a duality map. Then for every $y \in \bigcup X_j$ we have

$$(P_{n_{j(k)}} T(x_{n_{j(k)}}), Jy) = (T(x_{n_{j(k)}}), Jy) \text{ for } n \geq n_1.$$

Now

$$(P_{n_{j(k)}} T(x_{n_{j(k)}}), Jy) \to (g - f(x), Jy) \text{ and } (T(x_{n_{j(k)}}), Jy) \to (T(x), Jy).$$

Therefore

$$(T(x) + f(x) - g, Jy) = 0 \text{ for every } y \in \bigcup X_j.$$

Under our assumptions J is demicontinuous (see [6]) and therefore

$$(T(x) + f(x) - g, Jy) = 0 \text{ for every } y \in \overline{\bigcup X_j} = X.$$

Since $J(X)$ is dense in X^* this is true for every $x^* \in X^*$. Therefore $T(x) + f(x) = g$, i.e. T_1 is A-proper.

We note that, for each $t \in \mathbb{K}$ with $|t| \leq 1$, we have that $T_t \equiv T + tf : D \to Y$ is also a map of the same kind with $\beta(tf) = |t|\,\beta(f) < |t|\,\mu_0\eta^{-1} \leq \mu_0\eta^{-1}$, so that T_t is A-proper for every fixed t with $|t| \leq 1$.

To prove the A-proper stability note that $\lambda I + T_t \equiv \lambda I + T + tf = T + (\lambda I + tf)$ and $\lambda I + tf$ is a β-contraction with $\beta(\lambda I + tf) \leq |\lambda| + |t|\,\beta(f)$, by Lemma 1.5.4. Therefore $\lambda I + T_t$ is A-proper for every λ such that $|\lambda| < \mu_0\eta^{-1} - |t|\,\beta(f)$ (i.e. T_t is A-proper stable). □

Proposition 2.11.7. *Let X be a reflexive Banach space with X^* strictly convex and $f : X \to X$ be a demicontinuous, A-regular map and suppose there exists $\rho \in \mathbb{R}_+$ such that* (2.11.1) *is satisfied for each bounded sequence $\{x_n \mid x_n \in X_n\}$. If $g : X \to X$ is a β-contraction such that $\beta(g) < \mu_0$ and $q_r(g) < d'_R(f)$, then $f + g$ is A-regular.*

Proof. The result follows from Theorem 2.11.6 and a similar argument as in Proposition 2.11.1. □

In the case of compact perturbations weaker assumptions are needed.

Proposition 2.11.8. *Let $f : X \to X$ be a demicontinuous, A-regular map and let $g : X \to X$ be compact such that $q_R(g) < d'_R(f)$. Then $f + g$ is A-regular.*

Proof. Since A-proper maps are invariant under compact perturbations, $f + g$ is A-proper stable. Note that, by Lemma 2.4.1,

$$d'_R(f+g) \geq d'_R(f) - q_R(g) > 0.$$

Since $q_R(g) < d'_R(f)$, there exists $\varepsilon > 0$ such that $q_R(g) < d'_R(f) - \varepsilon < d'_R(f)$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$d_R(f_n) \geq d'_R(f) - \varepsilon > q_R(g) \geq q_R(g_n)$$

for every $n \geq n_0$. Since $d(f_n) > q(g_n)$, we can apply Proposition 2.2.4 to show that $f_n + g_n$ is stably solvable. □

2.12 A comparison

Theorem 2.12.1. *Let $f : X \to X$ be a continuous, A-regular map. Then f is agv-regular.*

Proof. Since f is A-proper, f is not compact by the proof of Lemma 2.6.1 and therefore $\alpha(f) > 0$. Since $d'_R(f) > 0$, by Lemma 2.10.8, we have that $d(f) > 0$. Furthermore we can use Theorem 2.4.8 to show that f is stably solvable. By Proposition 2.2.10 we have that f is agv-regular. □

Theorem 2.12.2. *Let $f : X \to X$ be a continuous map. Then*

$$\phi(f) \subseteq \sigma_{agv}(f) \subseteq \sigma_A(f).$$

Proof. From Proposition 2.2.13 $\phi(f) \subseteq \sigma_{agv}(f)$. Theorem 2.12.1 gives the inclusion $\sigma_{agv}(f) \subseteq \sigma_A(f)$. □

The following example shows that the inclusion $\sigma_{agv}(f) \subseteq \sigma_A(f)$ may be strict.

Example 2.12.3. Take f defined as in Example 1.6.5. Since f is a linear isomorphism, $\sigma_{agv}(f) = \sigma(f)$ by Proposition 11 of [5] and therefore $0 \notin \sigma_{agv}(f)$. But f is not A-proper, so that $0 \in \sigma_A(f)$.

Proposition 2.12.4. *Let* $\dim X = \infty$ *and* $f : X \to X$ *be a compact map. Then*

$$\sigma_{fmv}(f) \subseteq \sigma_A(f).$$

Proof. Take $\lambda \in \rho_A(f)$. By Lemma 2.6.1, $\lambda \neq 0$ and therefore $\omega(\lambda I - f) = |\lambda| > 0$. Furthermore f is stably solvable by Theorem 2.4.8 and $d(\lambda I - f) > 0$, by Lemma 2.10.8. Then $\lambda \in \rho_{fmv}(f)$. $\square$

Example 2.12.5. A map that is A-regular but is not Feng-regular.

Take $f_1 : \mathbb{R} \to \mathbb{R}$ (see Figure 2.2) defined by

$$f_1(x) = \begin{cases} x & \text{if } x \in (-\infty, \frac{1}{2}] \cup [2, +\infty), \\ -x+1 & \text{if } x \in (\frac{1}{2}, 1) \\ 2x-2, & \text{if } x \in [1, 2). \end{cases}$$

Let $f : l^2(\mathbb{R}) \to l^2(\mathbb{R})$ be defined by

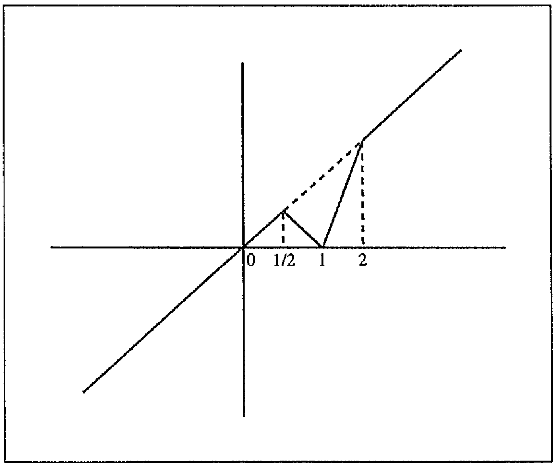

Figure 2.2: $f_1(x)$

$$f(x_1, x_2, x_3, \ldots) = (f_1(x_1), x_2, x_3, \ldots).$$

To show that f is A-regular, set

$$g(x_1, x_2, x_3, \ldots) = (g_1(x_1), 0, 0, \ldots),$$

where

$$g_1(x) = \begin{cases} 0 & \text{if } x \in (-\infty, \frac{1}{2}] \cup [2, +\infty), \\ -2x+1 & \text{if } x \in (\frac{1}{2}, 1), \\ x-2 & \text{if } x \in [1, 2). \end{cases}$$

Obviously $f = I + g$ and g is compact. Furthermore

$$q_R(g) = \sup_{\|x\| \geq R} \frac{\|g(x)\|}{\|x\|} \leq \frac{1}{R} \leq \frac{1}{2} < d'_R(I) = 1$$

whenever $R \geq 2$. Therefore, by Proposition 2.11.8, $f+g$ is A-regular and $0 \in \rho_A(f)$. Note that, since 0 is an eigenvalue (with corresponding eigenvector $(1, 0, 0, \ldots)$), $m(f) = 0$ and therefore f is not Feng-regular and $0 \in \sigma_{fw}(f)$.

Remark 2.12.6. Note that the function f defined in Example 2.12.5 is an A-proper map which is not A-stable. To check this, assume that if f is A-stable. Then there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha(0) = 0$ and $\alpha(r) > 0$ when $r > 0$, such that

$$\|f_n(x) - f_n(y)\| \geq \alpha(\|x - y\|).$$

for every $x, y \in X_n$ and $n \geq n_0$. Set $\tilde{x} = (0, 0, \ldots, 0)$ and $\tilde{y} = (1, 0, \ldots, 0)$ then $0 = \|f_n(\tilde{x}) - f_n(\tilde{y})\| \geq \alpha(\|\tilde{x} - \tilde{y}\|) = \alpha(1) > 0$, a contradiction.

Example 2.12.5 also sheds light on the fact that in the *linear* case the A-regularity of a map L implies unique A-solvability of the equation $L(x) = y$. In the *nonlinear* case this is no longer true. In fact the equation $f(x) = (0, \ldots, 0)$ has two solutions $x_1 = (0, \ldots, 0)$ and $x_2 = (1, 0, \ldots, 0)$.

2.13 Conclusions

In this chapter we discussed an approach to nonlinear spectral theory via finite dimensional approximations. We have studied some properties of the A-spectrum, partially investigating the relations between the A-spectrum, eigenvalues and some other nonlinear spectra. As a further development of the theory, it could be interesting, for example, to study the applications of the A-spectrum to differential equations. In particular this means checking whether it is possible to use these techniques to obtain results not achievable by other methods. Furthermore we used stable solvability as our concept of solvability for the finite dimensional approximations, but, as can be seen in section 2.2, this is not the only possible approach.

Chapter 3

Nonzero solutions of some boundary value problems with continuous kernels

In the second part of this thesis, chapters three to five, we study some nonlocal boundary value problems (BVPs) for second order ordinary differential equations (ODEs). Such type of problems have been studied by Il'in and Moiseev [23]. Gupta *et al.* in [13, 15] widely studied these BVPs, proving existence of solutions. Since 0 is often a possible solution, an existence theorem alone may be of little use, also in applications *positive* solutions are often of importance. Ma in [34] studied the existence of positive solutions of such problems under superlinear and sublinear growth of the nonlinear term. We study problems where positive solutions need not exist. We do not impose global growth assumptions on the nonlinearity and use the theory of fixed point index to prove existence of one or more nonzero solutions under conditions which strictly include the sublinear and superlinear cases.

The BVPs in chapter three generate a continuous kernel that changes sign. In order to tackle these problems we introduce a cone of functions positive on an interval $[a, b]$ that enables us to prove the existence of nontrivial solutions.

The BVP in chapter four is different, since it generates a discontinuous kernel and the theory of chapter three no longer applies. Thus we generalise the theory of chapter three, in order to deal with such a discontinuity. We also allow a more general nonlinear term.

In chapter five we use the results of chapter three and four to prove existence of positive (and negative) eigenvalues of a variety of nonlocal BVPs studied in the previous chapters and a BVP of a new type.

3.1 Introduction

In this chapter we study the existence of nonzero solutions of second order differential equations of the form

$$u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1) \tag{3.1.1}$$

under one of the boundary conditions (BCs)

$$u'(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1, \tag{3.1.2a}$$

$$u(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1. \tag{3.1.2b}$$

These are the three-point boundary value problems for which existence has been extensively studied by Gupta *et al.*, often assuming f grows sublinearly.

One approach to finding positive solutions is to write the BVP as an equivalent Hammerstein integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds := Tu(t) \tag{3.1.3}$$

and find a solution as a fixed point of the operator T by using the classical theory of fixed point index in cones to establish the existence of one or multiple positive solutions.

Let $P = \{x \in C[0,1] : x(t) \geq 0 \text{ for } t \in [0,1]\}$ be the cone of non-negative continuous functions. In general, it can be hard to use the cone P to obtain existence of nonzero fixed points. Some recent progress was made by Lan and Webb [32] who used the cone

$$\tilde{K} = \{u \in C[0,1] : u \geq 0, \ \min\{u(t) : a \leq t \leq b\} \geq c\|u\|\}$$

(which is of a type due to D.Guo, see for example [12]) to prove that at least one positive solution existed for some boundary conditions of separated type. These results strictly included f being either sublinear or superlinear. These results have been improved by Lan [29] to yield existence of multiple positive solutions under suitable conditions on f for the separated BCs.

Webb [46] used Lan's results for the Hammerstein integral equation to establish the existence of multiple positive solutions for the three point BCs above, when $0 < \alpha\eta < 1$ for (3.1.2a) and $0 < \alpha < 1$ for (3.1.2b). Webb's results improved some of Ma's [34] who dealt with the sublinear and superlinear case only for (3.1.2b).

In this chapter we shall consider the other possible ranges for the parameter α. For (3.1.2a) when $\alpha < 0$, the kernel $k(t,s)$ is not positive for all values of t, s, indeed $k(1,s) < 0$ for all s. Therefore, when g and f are positive, a fixed point of the operator T cannot be positive on $[0,1]$.

Nevertheless, as we intend to show in this chapter, it is possible to prove that nonzero solutions exist which have the property that they are positive (or negative) on some subinterval $[a,b]$ of $[0,1]$.

We shall show that one or more nonzero solutions exists under conditions on f exactly similar to those of Lan for each of the other possible range of parameter α in each of the BCs above.

The methods we use are rather similar to those of Lan but we seek solutions of a different type, hence we employ a larger cone.

The conditions we impose on g are quite weak, for example we can allow g to be a non-negative L^1 function which is positive on a set of positive measure.

We suppose f is positive; some of our other hypotheses involve

$$\lim_{x\to 0+} f(x)/x \quad \text{and} \quad \lim_{x\to\infty} f(x)/x.$$

Our conditions strictly include the sublinear and superlinear cases.

The results of this chapter are based on [21].

3.2 Existence of nonzero solutions of Hammerstein integral equations

We begin by giving some results for the following Hammerstein integral equation.

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds \equiv Tu(t). \tag{3.2.1}$$

We shall make the following assumptions on f, g and the kernel k, throughout the chapter, even if not mentioned explicitly.

(F) $f : \mathbb{R} \to [0, \infty)$ is continuous.

(C) $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and there exist a measurable function $\Phi : [0,1] \to [0,\infty)$ and a number $c \in (0,1]$ such that

$$|k(t,s)| \leq \Phi(s) \text{ for } t, s \in [0,1], \text{ and}$$
$$c\Phi(s) \leq k(t,s) \text{ for } t \in [a,b] \text{ and } s \in [0,1]$$

(G) $g : [0,1] \to [0,\infty)$ is measurable and $\int_0^1 \Phi(s)g(s)\,ds < \infty$.

The hypothesis (C) means finding upper bounds for $|k(t,\cdot)|$ when $t \in [0,1]$ and lower bounds of the same form for $k(t,\cdot)$ with $t \in [a,b]$. In applications we have some freedom of choice in determining a, b but we are constrained by needing $k(t,s)$ to be positive for all $t \in [a,b]$ and $s \in [0,1]$.

These hypotheses will allow us to work in the cone

$$K = \{u \in C[0,1] : \min\{u(t) : a \leq t \leq b\} \geq c\|u\|\}.$$

This is a larger cone than the one used by Lan [29]. Note that functions in K are positive on the subinterval $[a,b]$ but may change sign on $[0,1]$.

Remark 3.2.1. We check that K is a cone. Let $u, v \in K$ and $a_1, a_2 \in [0,+\infty)$. We have

$$\begin{aligned}\min_{t\in[a,b]}\{a_1u(t) + a_2v(t)\} &\geq \min_{t\in[a,b]} a_1u(t) + \min_{t\in[a,b]} a_2v(t)\\ &\geq c\|a_1u\| + c\|a_2v\| \geq c\|a_1u + a_2v\|.\end{aligned}$$

Furthermore it is obvious that if $u \in K$ and $-u \in K$ then $u = 0$. Therefore K is a cone.

In order to use the well-known fixed point index for compact maps, we need to prove that $T : K \to K$ is compact, that is, T is continuous and $\overline{T(Q)}$ is compact for each bounded subset $Q \subset K$.

Theorem 3.2.2. *Assume that (F), (G) and (C) hold. Then T maps K into K and is compact.*

Proof. Let $\mathcal{T} : C[0,1] \to C[0,1]$ be defined by $(\mathcal{T}x)(t) = \int_0^1 k(t,s)g(s)x(s)\,ds$. Then the kernel has the properties:

(i) $\int_0^1 |k(t,s)|g(s)\,ds \leq \int_0^1 \Phi(s)g(s)\,ds$ for all $t \in [0,1]$.

(ii) For each $\tau \in [0,1]$, $\lim_{t\to\tau} \int_0^1 |k(t,s)g(s) - k(\tau,s)g(s)|\,ds = 0$.

To see (ii), note that if $t_n \to \tau$, then $|k(t_n,s)g(s) - k(\tau,s)g(s)| \to 0$ and

$$|k(t_n,s)g(s) - k(\tau,s)g(s)| \leq 2\Phi(s)g(s) \text{ for every } n.$$

Therefore, by the dominated convergence theorem, (ii) holds. Since $[0,1]$ is compact, the limit in (ii) is uniform in τ. Hence Proposition 3.4 (p.167) of [35] shows that $\mathcal{T} : C[0,1] \to C[0,1]$ is compact. As f is continuous, it follows that the operator $T : C[0,1] \to C[0,1]$ is compact.

Furthermore we see that $T : K \to K$. Indeed, we have

$$|Tu(t)| \leq \int_0^1 |k(t,s)|g(s)f(u(s))\,ds \text{ so that}$$
$$\|Tu\| \leq \int_0^1 \Phi(s)g(s)f(u(s))\,ds.$$

Also

$$\min_{a\leq t\leq b}\{Tu(t)\} \geq c\int_0^1 \Phi(s)g(s)f(u(s))\,ds.$$

Hence $Tu \in K$ for every $u \in K$. □

We require some knowledge of the classical fixed point index for compact maps, see for example [1] or [12] for further information.

Let K be a cone in a Banach space X. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K.

Notation: Let $q : C[0,1] \to \mathbb{R}$ denote the function

$$q(u) = \min\{u(t) : a \leq t \leq b\}.$$

Proposition 3.2.3. *Let $X = C[0,1]$ and $[a,b] \subset [0,1]$. Then q is continuous on X.*

Proof. Let $u_n \to u$ in X. Then $u_n(t) \to u(t)$ uniformly on $[a,b]$. There exists $t_0 \in [a,b]$ such that $q(u) = u(t_0)$, and $t_n \in [a,b]$ such that $q(u_n) = u_n(t_n)$ for every n. Since $u_n(t_0) \to u(t_0)$ and $u_n(t_0) \geq q(u_n)$ for every n we have $u(t_0) \geq \limsup q(u_n)$. Also $u(t_0) \leq u(t_n) \leq u_n(t_n) + |u_n(t_n) - u(t_n)|$, hence $u(t_0) \leq q(u_n) + |u_n(t_n) - u(t_n)|$ and so $u(t_0) \leq \liminf q(u_n)$. This proves $u(t_0) = \lim q(u_n)$. □

Following Lan [29], for $\rho > 0$, we shall use the set $\Omega_\rho = \{u \in K : q(u) < c\rho\}$. We write $K_r = \{u \in K : \|u\| < r\}$ and $\overline{K}_r = \{u \in K : \|u\| \leq r\}$.

Lemma 3.2.4. *Ω_ρ defined above has the following properties.*

(*a*) *Ω_ρ is open relative to K.*

(*b*) *$K_{c\rho} \subset \Omega_\rho \subset K_\rho$.*

(*c*) *$u \in \partial\Omega_\rho$ if and only if $q(u) = c\rho$.*

(*d*) *If $u \in \partial\Omega_\rho$, then $c\rho \leq u(t) \leq \rho$ for $t \in [a,b]$.*

The proof is exactly similar to Lan's [29], but we give the proof for completeness.

Proof. (*a*) holds since q is continuous. (*c*) see 3.2.5-3.2.7. Let $u \in K_{c\rho}$. Then

$$c\|u\| \leq q(u) \leq \|u\| < c\rho$$

and $u \in \Omega_\rho$. If $u \in \Omega_\rho$, then $c\|u\| \leq q(u) < c\rho$. This implies $\|u\| < \rho$ and $u \in K_\rho$. Hence, (*b*) holds. If $u \in \partial\Omega_\rho$, by (*c*) we have $c\|u\| \leq q(u) = c\rho \leq u(t)$ for all $t \in [a,b]$, so (*d*) holds. □

Remark 3.2.5. In general, given a Banach space X and a continuous function $q : X \to \mathbb{R}$, we have that for $\rho > 0$, the set $\Omega_\rho = \{x \in X : q(x) < \rho\}$ is open but in general $\partial\Omega_\rho \neq \{x \in X : q(x) = \rho\}$, as the following example shows:

Example 3.2.6. Let X be a Banach space and

$$q(x) = \begin{cases} \|x\| & \text{if} \quad 0 \leq \|x\| < 1, \\ 1 & \text{if} \quad 1 \leq \|x\| \leq 2, \\ \|x\| - 1 & \text{if} \quad \|x\| > 2. \end{cases}$$

Note that q is continuous in X. For $\|x\| < 1$, $\|x\| > 2$ and $1 < \|x\| < 2$ this is clear. When $\|x_0\| = 1$ we have

$$q(x) = \begin{cases} \|x\| & \text{for } \|x - x_0\| < 1 \\ 1 & \end{cases}$$

and

$$q(x) - q(x_0) = \begin{cases} \|x\| - 1 & \text{if} \quad 0 \leq \|x\| < 1, \\ 0 & \text{if} \quad \|x\| > 1. \end{cases}$$

Hence $\|q(x) - q(x_0)\| \leq \big|\|x\| - \|x_0\|\big|\|x - x_0\|$. When $\|x_0\| = 2$

$$q(x) = \begin{cases} 1 & \text{if} \quad \|x\| < 2, \ \|x - x_0\| < 1, \\ 0 & \text{if} \quad \|x\| > 2. \end{cases}$$

Thus

$$|q(x) - q(x_0)| = \begin{cases} |1 - 1| = 0 \\ \big|\|x\| - 1 - 1\big| \end{cases}$$

and $|q(x) - q(x_0)| \leq \big|\|x\| - 2\big| \leq \big|\|x\| - \|x_0\|\big| \leq \|x - x_0\|$. Therefore q is continuous. Note also that $\Omega_1 = \{x : q(x) < 1\} = B(0,1)$, $\partial\Omega_1 = \{x : \|x\| = 1\}$ but

$$\{x : q(x) = 1\} = \{x : 1 \leq \|x\| \leq 2\} \supsetneq \partial\Omega_1.$$

If $q(x)$ satisfies some extra property we have the stronger result:

Lemma 3.2.7. *Let $q : X \to \mathbb{R}$ be continuous and $q(tx)$ be strictly increasing in t for every x. Then*

$$\partial\Omega_\rho = \{x \in X : q(x) = \rho\}.$$

Proof. Since the set $\{x \in X : q(x) < \rho\}$ is open we have

$$\partial\Omega_\rho \subseteq \{x \in X : q(x) = \rho\}.$$

If $q(x_0) = \rho$ we have, for $t < 1$, $q(tx_0) < \rho$ and, for $\tau > 1$, $q(\tau x_0) > \rho$. Hence a neighborhood of x_0 contains points of $\{x \in X : q(x) < \rho\}$ and $\{x \in X : q(x) > \rho\}$, that is, x_0 is a boundary point. $\square$

Notation: Let

$$f_{c\rho,\rho} = \min\{f(u)/\rho : u \in [c\rho, \rho]\}, \qquad f^{-\rho,\rho} = \max\{f(u)/\rho : u \in [-\rho, \rho]\},$$
$$M = \Big(\min_{a\le t\le b} \int_a^b k(t,s)g(s)\,ds\Big)^{-1} \text{ and } m = \Big(\max_{0\le t\le 1} \int_0^1 |k(t,s)|g(s)\,ds\Big)^{-1}.$$

We now prove two lemmas which give conditions when the fixed point index is either 0 or 1.

Lemma 3.2.8. *Suppose* $\int_a^b \Phi(s)g(s)\,ds > 0$ *and that*

$(*)$ $f_{c\rho,\rho} \ge Mc$ *and* $x \ne Tx$ *for* $x \in \partial\Omega_\rho$.

Then $i_K(T, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in K$. We prove that

$$x \ne Tx + \lambda e \quad \text{for } x \in \partial\Omega_\rho \quad \text{and } \lambda > 0.$$

In fact, if not, there exist $x \in \partial\Omega_\rho$ and $\lambda > 0$ such that $x = Tx + \lambda e$. By condition $(*)$ and (d) of Lemma 3.2.4, we have for $t \in [a,b]$,

$$\begin{aligned} x(t) = &\int_0^1 k(t,s)g(s)f(x(s))\,ds + \lambda \ge \int_a^b k(t,s)g(s)f(x(s))\,ds + \lambda \\ \ge &cM\rho \int_a^b k(t,s)g(s)\,ds + \lambda \ge c\rho + \lambda. \end{aligned}$$

This implies that $q(x) \ge c\rho + \lambda > c\rho$ contradicting (c) of Lemma 3.2.4. Hence (1) of Lemma 1.4.7 gives $i_K(T, \Omega_\rho) = 0$. □

Later, in Remark 4.2.5, we compare the assumption (1) of Lemma 1.4.7 with the commonly used $\|Tu\| \ge \|u\|$ for $\|u\| = \rho$.

Lemma 3.2.9. *Suppose* $\max_{0\le t\le 1} \int_0^1 |k(t,s)|g(s)\,ds > 0$ *and that* f *satisfies*

$(**)$ $f^{-\rho,\rho} \le m$ *and* $x \ne Tx$ *for* $x \in \partial K_\rho$.

Then $i_K(T, K_\rho) = 1$.

Proof. By $(**)$, for $u \in \partial K_\rho$ and $t \in [0,1]$, we have

$$\begin{aligned} |Tu(t)| = \left|\int_0^1 k(t,s)g(s)f(u(s))\,ds\right| &\le \int_0^1 |k(t,s)|\,g(s)f(u(s))\,ds \\ &\le m\rho \int_0^1 |k(t,s)|\,g(s)\,ds \le \rho = \|u\|. \end{aligned}$$

Therefore $\|Tu\| \le \|u\|$ for $u \in \partial K_\rho$. By means of (2) of Lemma 1.4.7, we have $i_K(T, K_\rho) = 1$. □

We now give our new result which asserts that Eq. (3.2.1) has at least two nonzero solutions which are positive on the subinterval $[a,b]$ (the proof is illustrated in Figures 3.1, 3.2).

Theorem 3.2.10. *Assume that* $\int_a^b \Phi(s)g(s)\,ds > 0$ *and one of the following conditions holds:*

(S_1) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < c\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$$f^{-\rho_1,\rho_1} \le m, \quad f_{c\rho_2,\rho_2} \ge Mc, \quad x \neq Tx \quad \textit{for } x \in \partial\Omega_{\rho_2}, \quad \textit{and} \quad f^{-\rho_3,\rho_3} \le m.$$

(S_2) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < \rho_2 < c\rho_3$ *such that*

$$f_{c\rho_1,\rho_1} \ge Mc, \quad f^{-\rho_2,\rho_2} \le m, \quad x \neq Tx \quad \textit{for } x \in \partial K_{\rho_2}, \quad \textit{and } f_{c\rho_3,\rho_3} \ge Mc.$$

Then Eq. (3.2.1) *has two solutions in* K *each of which is positive on* $[a,b]$. *Moreover, if in* (S_1), $f^{-\rho_1,\rho_1} \le m$ *is replaced by* $f^{-\rho_1,\rho_1} < m$, *then Eq.* (3.2.1) *has a third solution* $x_0 \in K_{\rho_1}$.

Proof. Assume that (S_1) holds. We show that either T has a fixed point x_1 in ∂K_{ρ_1} or in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. If $x \neq Tx$ for $x \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$, by Lemmas 3.2.8 and 3.2.9, we have $i_K(T, K_{\rho_1}) = 1$, $i_K(T, \Omega_{\rho_2}) = 0$ and $i_K(T, K_{\rho_3}) = 1$. By (b) of Lemma 3.2.4, we have $\overline{K}_{\rho_1} \subset K_{c\rho_2} \subset \Omega_{\rho_2}$ since $\rho_1 < c\rho_2$. It follows from (3) of Lemma 1.4.7 that T has a fixed point x_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Similarly, T has a fixed point x_2 in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. The proof is similar when (S_2) holds. □

Remark 3.2.11. Note that the third solution $x_0 \in K_{\rho_1}$ might be zero. The other solutions are not because their norms are bounded away from zero. Although the statement and proof is almost identical to the similar result in [29] which deals with positive solutions, our new result allows solutions that are only positive on a subinterval and may change sign, and indeed this happens in the differential equations we consider below.

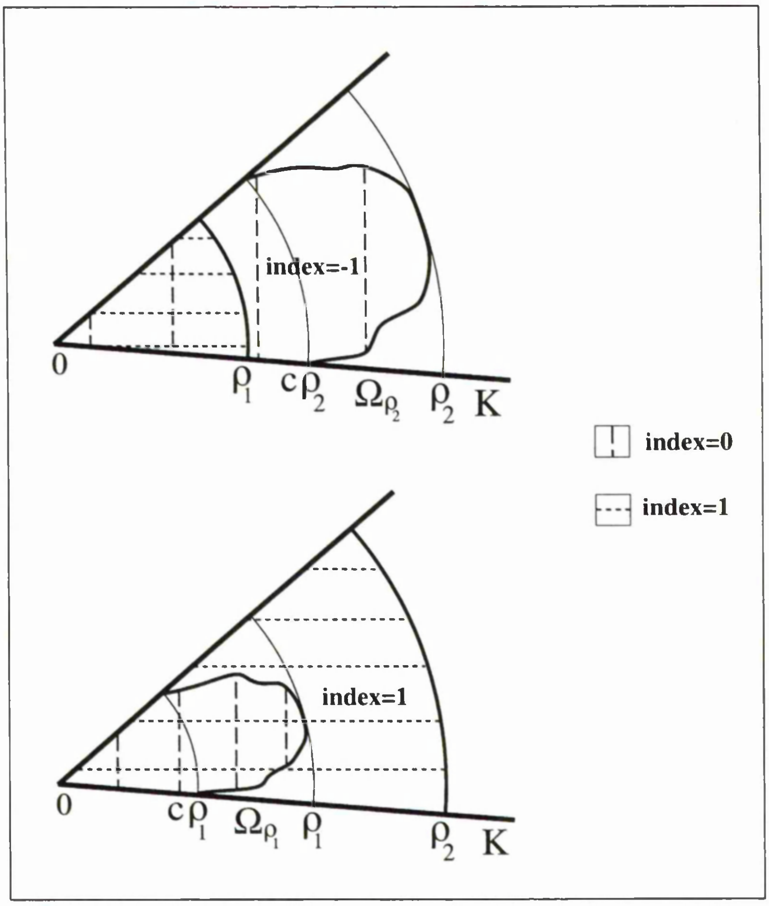

Figure 3.1: One nonzero solution

Remark 3.2.12. It is possible to give results for more than two solutions by merely adding more conditions of the same type to the list in (S_1) or (S_2). We do not state such results leaving them to the reader who may refer to [29] for the type of result that may be stated.

Notation: Let

$$f^0 = \limsup_{u\to 0} \frac{f(u)}{|u|}, \quad f_0 = \liminf_{u\to 0} \frac{f(u)}{|u|}, \quad f^\infty = \limsup_{u\to\infty} \frac{f(u)}{u} \text{ and } f_\infty = \liminf_{u\to\infty} \frac{f(u)}{u}.$$

As a special case of Theorem 3.2.10 we have the following result.

Corollary 3.2.13. *Assume that $\int_a^b \Phi(s)g(s)\,ds > 0$ and there exists $\rho > 0$ such that one of the following conditions holds.*

(E_1) $0 \le f^0 < m$, $f_{c\rho,\rho} \ge Mc$, $x \ne Tx$ *for* $x \in \partial\Omega_\rho$, *and* $0 \le f^\infty < m$.

(E_2) $M < f_0 \le \infty$, $f^{-\rho,\rho} \le m$, $x \ne Tx$ *for* $x \in \partial K_\rho$, *and* $M < f_\infty \le \infty$.

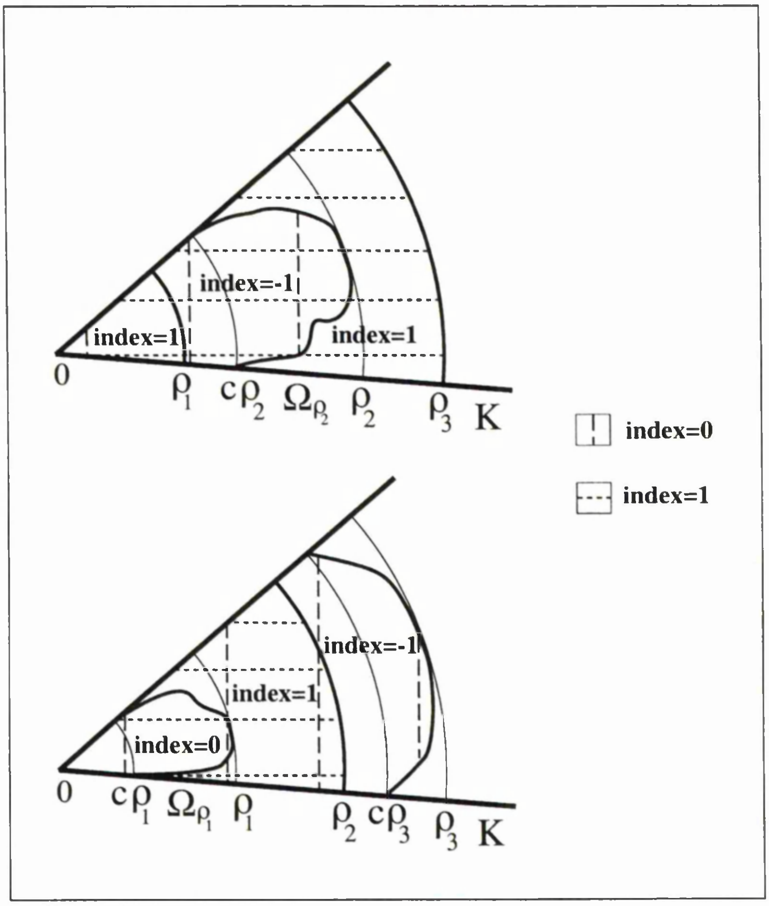

Figure 3.2: Two nonzero solutions

Then Eq. (3.2.1) *has two nonzero solutions in* K.

Proof. We show that (E_1) implies (S_1). In fact, $0 \le f^0 < m$ implies that there exists $\rho_1 \in (0, c\rho)$ such that $f^{-\rho_1,\rho_1} < m$. Let $\tau \in (f^\infty, m)$. Then there exists $r > \rho$ such that $f(u) \le \tau u$ for $u \in [r, \infty)$ since $0 \le f^\infty < m$. Let $\beta = \max\{f(u) : 0 \le u \le r\}$ and $\rho_3 > \beta/(m - \tau)$. Then we have

$$f(u) \le \tau u + \beta \le \tau\rho_3 + \beta < m\rho_3 \quad \text{for } u \in [0, \rho_3].$$

This implies $f^{-\rho_3,\rho_3} < m$, hence (S_1) holds. Similarly, (E_2) implies (S_2). □

By a similar argument to that of Theorem 3.2.10, we obtain the following new results on existence of at least one nonzero solution of Eq. (3.2.1).

Theorem 3.2.14. *Assume that* $\int_a^b \Phi(s)g(s)\,ds > 0$ *and one of the following conditions holds.*

(H_1) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < c\rho_2$ *such that*

$$f^{-\rho_1,\rho_1} \le m \quad \text{and} \quad f_{c\rho_2,\rho_2} \ge Mc.$$

(H_2) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that*

$$f_{c\rho_1,\rho_1} \ge Mc \quad \text{and} \quad f^{-\rho_2,\rho_2} \le m.$$

Then Eq. (3.2.1) *has a nonzero solution in* K.

Theorem 3.2.14 generalises Theorem 2.2 in [32] by allowing solutions that change sign.

Remark 3.2.15. We shall see below that, for certain values of the parameter α, the kernel $k(t, s)$ is negative for t in some interval $[a, b]$, for all s. In this case, assuming g and f are positive, we can show that nonzero solutions exist that are *negative* on $[a, b]$. Indeed, u is a solution of

$$u(t) = \int_0^1 k(t, s)g(s)f(u(s))\, ds$$

if and only if $v := -u$ is a solution of

$$v(t) = \int_0^1 \tilde{k}(t, s)g(s)\tilde{f}(v(s))\, ds \equiv \tilde{T}v(t)$$

where $\tilde{k} = -k$ and $\tilde{f}(v) = f(-v)$. Moreover v is positive on $[a, b]$ if and only if u is negative on $[a, b]$. Hence we can obtain results, exactly similar to ones above, for the existence of solutions that are negative on $[a, b]$. We do not state the obvious theorems thus obtained.

3.3 Multiple nonzero solutions of problem (3.1.2a)

We investigate the BVP

$$u'' + g(t)f(u) = 0, \qquad \text{a.e on } [0, 1], \tag{3.3.1}$$

with boundary conditions

$$u'(0) = 0, \; \alpha u(\eta) = u(1), \; 0 < \eta < 1. \tag{3.3.2}$$

By a solution of this BVP we will mean a solution of the corresponding Hammerstein Integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds. \tag{3.3.3}$$

The solution of $u'' + y = 0$ with the BCs (3.3.2) is (by routine integration)

$$u(t) = \frac{1}{1-\alpha}\int_0^1 (1-s)y(s)\,ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)y(s)\,ds - \int_0^t (t-s)y(s)\,ds.$$

Thus the kernel [Green's function] of (3.3.3) is

$$k(t,s) = \frac{1}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha}{1-\alpha}(\eta - s), & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t. \end{cases}$$

We shall study separately the cases $\alpha < 0$ and $\alpha > 1$. The existence of positive solutions when $\alpha = 0$ has been studied in [32], and when $0 < \alpha < 1$ in [46]. $\alpha = 1$ is the resonance case and can not be dealt with by the methods here but existence in this case was studied in [9].

The case $\alpha < 0$.

To simplify the calculations we write $-\beta$ in place of α, so that $\beta > 0$.

We have to exhibit $\Phi(s)$, a subinterval $[a,b] \subset [0,1]$ and a constant $c < 1$ such that

$$\begin{aligned} |k(t,s)| &\le \Phi(s) \quad \text{for every } t, s \in [0,1], \\ k(t,s) &\ge c\Phi(s) \quad \text{for every } s \in [0,1],\ t \in [a,b]. \end{aligned}$$

We show that for these BCs we can take $\Phi(s) = (1-s)$.

Upper bounds

<u>Case 1.</u> $s \le \eta$. If $s > t$ then $k(t,s) \ge 0$ and

$$k(t,s) = \frac{1-s}{1+\beta} + \frac{\beta}{1+\beta}(\eta - s) \le \frac{1 - s + \beta(1-s)}{1+\beta} = (1-s).$$

If $s \le t$ then

$$k(t,s) = \frac{1-s}{1+\beta} + \frac{\beta}{1+\beta}(\eta - s) - (t-s) = \frac{1 + \beta\eta - t(1+\beta)}{1+\beta}.$$

If $t \le \frac{1+\beta\eta}{1+\beta}$ then $k(t,s) \ge 0$ and

$$k(t,s) = \frac{1+\beta\eta - t(1+\beta)}{1+\beta} \le \frac{1+\beta\eta - s(1+\beta)}{1+\beta} \le (1-s).$$

If $t > \frac{1+\beta\eta}{1+\beta}$ then $k(t,s) \leq 0$ and

$$-k(t,s) = \frac{-1-\beta\eta+t(1+\beta)}{1+\beta} \leq \frac{-1-\beta\eta+(1+\beta)}{1+\beta} = \frac{\beta(1-\eta)}{1+\beta} \leq \frac{\beta(1-s)}{1+\beta}.$$

Case 2. $s > \eta$. If $s > t$ then

$$k(t,s) = \frac{(1-s)}{1+\beta} \geq 0$$

and we are done. If $s \leq t$ then

$$k(t,s) = \frac{1+\beta s - t(1+\beta)}{1+\beta}.$$

If $t \leq \frac{1+\beta s}{1+\beta}$ then $k(t,s) \geq 0$ and

$$k(t,s) \leq \frac{1+\beta s - s(1+\beta)}{1+\beta} = \frac{(1-s)}{1+\beta}.$$

If $t > \frac{1+\beta s}{1+\beta}$ then $k(t,s) \leq 0$ and

$$-k(t,s) = \frac{-1-\beta s+t(1+\beta)}{1+\beta} \leq \frac{-1-\beta s+(1+\beta)}{1+\beta} = \frac{\beta(1-s)}{1+\beta}.$$

Lower bounds

We show that we may take arbitrary $[a,b] \subset [0,\eta]$

Case 1. $s \leq \eta$. If $s > t$ then

$$k(t,s) = \frac{1-s}{1+\beta} + \frac{\beta}{1+\beta}(\eta - s) \geq \frac{(1-s)}{1+\beta}.$$

If $s \leq t$, since $t \leq b \leq \eta$ we have

$$k(t,s) \geq \frac{1-s}{1+\beta} + \frac{\beta}{1+\beta}(\eta - s) - (\eta - s) = \frac{1-\eta}{1+\beta} \geq \frac{1-\eta}{1+\beta}(1-s).$$

Case 2. $s > \eta$. If $s > t$ then $k(t,s) = (1-s)/(1+\beta)$ and we are done. Since we take $b \leq \eta$ the case $s \leq t$ does not occur.

The conclusion is that we may take $c = (1-\eta)/(1+\beta)$.

Theorem 3.3.1. *Let $a, b \in [0,\eta]$ and $c = (1-\eta)/(1+\beta)$. Let m, M be as defined previously and suppose that $\int_a^b g(s)\,ds > 0$. Then for $\alpha < 0$ the BVP* (3.3.1), (3.3.2) *has at least one nonzero solution, positive on $[a,b]$, if either*

(h_1) *$0 \leq f^0 < m$ and $M < f_\infty \leq \infty$*

or

(h_2) *$0 \le f^\infty < m$ and $M < f_0 \le \infty$,*

and has two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either

(E_1) *$0 \le f^0 < m$, $f_{c\rho,\rho} \ge cM$, $x \ne Tx$ for $x \in \partial\Omega_\rho$, and $0 \le f^\infty < m$,*

or

(E_2) *$M < f_0 \le \infty$, $f^{-\rho,\rho} \le m$, $x \ne Tx$ for $x \in \partial K_\rho$, and $M < f_\infty \le \infty$.*

We give a simple example to illustrate the theorem.

Example 3.3.2. Set $g \equiv 1$ and $f \equiv 2$, in this case $f^\infty = 0$, $f_0 = \infty$. The solution of (3.3.1) with (3.3.2) is

$$u(t) = -t^2 + \frac{(1+\beta\eta^2)}{(1+\beta)}.$$

This is a solution that is positive on an interval containing $(0,\eta]$ but negative at $t = 1$ (in Figure 3.3 we illustrate the special case $\eta = \frac{1}{2}$ and $\beta = 1$, obtaining $u(t) = -t^2 + \frac{5}{8}$).

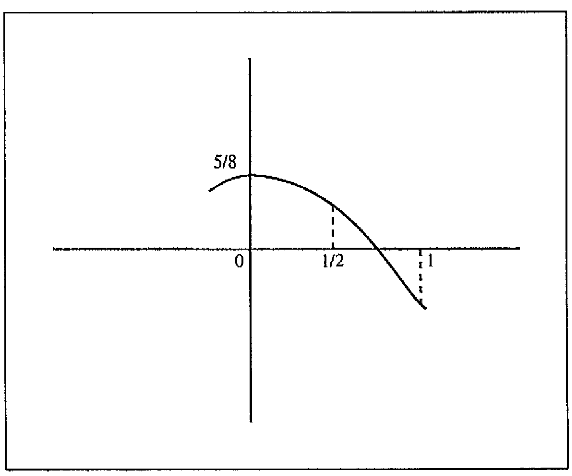

Figure 3.3: A solution positive on an interval

As an application of Theorem 3.3.1 we consider the following eigenvalue problem:

$$\lambda u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1) \tag{3.3.4}$$

subject to BCs

$$u'(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1. \tag{3.3.5}$$

Theorem 3.3.3. *Let $[a,b] \subset [0,\eta]$, $c = (1-\eta)/(1+\beta)$ and $\alpha < 0$. Suppose that $\int_a^b g(s)\,ds > 0$. Let m, M be as defined previously. Then λ is an eigenvalue of the boundary value problem (3.3.4)-(3.3.5), with a corresponding eigenvector that is positive on $[a,b]$, if either*

(P_1) $f^0/m < \lambda < f_\infty/M$

or

(P_2) $f^\infty/m < \lambda < f_0/M$.

Proof. Take $\lambda \in (f^0/m, f_\infty/M)$ and consider the equation

$$u''(t) + g(t)\tilde{f}(u(t)) = 0 \tag{3.3.6}$$

with BC (3.3.5), where $\tilde{f}(u) = \lambda^{-1}f(u)$. From (P_1) it follows that $\tilde{f}^0 < m$ and $M < \tilde{f}_\infty$. We can apply Theorem 3.3.1 to the BVP (3.3.6)-(3.3.5), hence obtaining a nontrivial solution of the BVP (3.3.4)-(3.3.5). The case (P_2) is treated in a similar manner. □

Similar results are valid for the other BCs we consider below. We leave the statements to the reader.

The case $\alpha > 1$.

For these BCs the kernel k is negative on an interval so we apply remark 3.2.15 and consider $-k$ in place of k. Thus we have to find Φ such that $|k(t,s)| \leq \Phi(s)$ for every $t, s \in [0,1]$ and show that there exists $[a,b] \subset [0,1]$ and a constant c such that $-k(t,s) \geq c\Phi(s)$ for every $s \in [0,1]$ and $t \in [a,b]$. In fact we show that we can take

$$\Phi(s) = \frac{\alpha}{\alpha-1}(1-s).$$

Upper bounds

Case 1. $s \leq \eta$. If $s > t$ then

$$-k(t,s) = \frac{1-s}{\alpha-1} - \frac{\alpha}{\alpha-1}(\eta - s) = \frac{1-s-\alpha\eta+\alpha s}{\alpha-1}.$$

If $s \geq \frac{-1+\alpha\eta}{\alpha-1}$ (this occurs in particular when $\alpha\eta < 1$) $-k(t,s) \geq 0$ and

$$-k(t,s) \leq \frac{1-s-\alpha\eta+\alpha\eta}{\alpha-1} = \frac{(1-s)}{\alpha-1}.$$

If $s < \frac{-1+\alpha\eta}{\alpha-1}$ then $k(t,s) > 0$ and

$$k(t,s) = \frac{-1+s+\alpha\eta-\alpha s}{\alpha-1} \leq \frac{-1+s+\alpha-\alpha s}{\alpha-1} = (1-s).$$

If $s \leq t$ then

$$-k(t,s) = \frac{1-s}{\alpha-1} - \frac{\alpha}{\alpha-1}(\eta-s) + (t-s) = \frac{1-\alpha\eta+\alpha t-t}{\alpha-1}.$$

If $t \geq \frac{-1+\alpha\eta}{\alpha-1}$ then $-k(t,s) \geq 0$ and

$$-k(t,s) \leq \frac{1-\alpha\eta+\alpha-1}{\alpha-1} \leq \frac{\alpha(1-s)}{\alpha-1}.$$

If $t < \frac{-1+\alpha\eta}{\alpha-1}$ then $k(t,s) > 0$ and

$$k(t,s) = \frac{-1+\alpha\eta-\alpha t+t}{\alpha-1} \leq \frac{-1+\alpha-\alpha s+s}{\alpha-1} = (1-s)$$

<u>Case 2.</u> $s > \eta$. If $s > t$ then

$$0 \leq -k(t,s) = \frac{(1-s)}{\alpha-1}$$

and we are done.

If $s \leq t$ then

$$0 \leq -k(t,s) = \frac{(1-s)}{\alpha-1} + (t-s) \leq \frac{\alpha(1-s)}{\alpha-1}.$$

Lower bounds

We will show that we may take $a = \eta$ and $b \in (\eta, 1]$ which will yield a solution that is negative on $[\eta, b]$. But, if also $\alpha\eta < 1$, we may take an arbitrary $[a,b] \subset [0,1]$. In particular this means that there exists a solution which is negative on the whole interval $[0,1]$ when $\alpha > 1$ and $\alpha\eta < 1$.

<u>Case 1.</u> $s \leq \eta$. If $s > t$ then

$$-k(t,s) = \frac{1-s-\alpha\eta+\alpha s}{\alpha-1} \geq \frac{1-s-\alpha\eta+\alpha\eta s}{\alpha-1} = (1-\alpha\eta)\frac{(1-s)}{\alpha-1}.$$

If $\alpha\eta < 1$ we may take an arbitrary a but if $\alpha\eta > 1$ we would have a problem. However, if $\alpha\eta > 1$ we choose $a \geq \eta$ so that this case does not occur.

If $s \leq t$ and $\alpha\eta < 1$ then

$$-k(t,s) \geq \frac{1-\alpha\eta+\alpha s-s}{\alpha-1} \geq (1-\alpha\eta)\frac{(1-s)}{\alpha-1}.$$

If $s \le t$ and $\alpha\eta > 1$, as we choose $a \ge \eta$ we have

$$-k(t,s) = \frac{1-\alpha\eta+\alpha t - t}{\alpha - 1} \ge \frac{1-\alpha\eta+\alpha\eta-\eta}{\alpha-1} \ge (1-\eta)\frac{(1-s)}{\alpha-1}.$$

Case 2. $s > \eta$. If $s > t$ then

$$-k(t,s) = \frac{(1-s)}{\alpha-1}$$

and we are done. If $s \le t$ then

$$-k(t,s) = \frac{1+\alpha t - t - \alpha s}{\alpha-1} \ge \frac{1+\alpha s - s - \alpha s}{\alpha-1} = \frac{(1-s)}{\alpha-1}.$$

The conclusion is that we may take either $a = \eta$, $b \in (\eta, 1]$ and $c = (1-\eta)/\alpha$ or, when $\alpha\eta < 1$, we may take a, b arbitrary and $c = (1-\alpha\eta)/\alpha$. Thus we can state the following results:

Theorem 3.3.4. *Let $\alpha > 1$, let $a = \eta$, $b \in (\eta, 1]$ (or a, b arbitrarily chosen in $[0,1]$ if $\alpha\eta < 1$). Suppose that $\int_a^b g(s)\,ds > 0$. Let c be as given above and let m, M be as defined previously. Then the BVP* (3.3.1), (3.3.2) *has at least one nonzero solution, negative on $[a,b]$, if either*

(h_1) $0 \le f^0 < m$ *and* $M < f_\infty \le \infty$

or

(h_2) $0 \le f^\infty < m$ *and* $M < f_0 \le \infty$,

and has two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either

(E_1) $0 \le f^0 < m$, $f_{c\rho,\rho} \ge cM$, $x \ne Tx$ *for* $x \in \partial\Omega_\rho$, *and* $0 \le f^\infty < m$,

or

(E_2) $M < f_0 \le \infty$, $f^{-\rho,\rho} \le m$, $x \ne Tx$ *for* $x \in \partial K_\rho$, *and* $M < f_\infty \le \infty$.

We illustrate the theorem with the following simple example.

Example 3.3.5. Let $\alpha > 1$, and take $g \equiv 1$, $f \equiv 2$. The solution of the BVP (3.3.1), (3.3.2) is

$$u(t) = \frac{(\alpha\eta^2 - 1)}{(\alpha - 1)} - t^2.$$

If $\alpha\eta < 1$ this solution is negative on all of $[0,1]$. When $\alpha\eta^2 > 1$ the solution is negative on an interval, for example, when $\eta = 1/2$ and $\alpha = 5$ the solution is $u(t) = 1/16 - t^2$ (we illustrate this in Figure 3.4). By taking α very large, the interval on which the solution is negative approaches $(\eta, 1]$, hence our choice of $[a, b]$ is optimal in giving the largest interval on which the solution is negative.

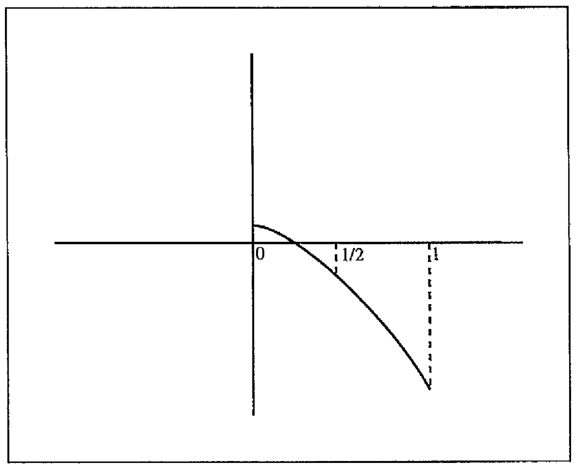

Figure 3.4: A solution negative on an interval

3.4 Multiple nonzero solutions of problem (3.1.2b)

We now investigate the second BVP

$$u'' + g(t)f(u) = 0, \text{ a.e on } [0,1], \tag{3.4.1}$$

with boundary conditions

$$u(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1. \tag{3.4.2}$$

The kernel in this case is

$$k(t,s) = \frac{1}{1-\alpha\eta}t(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha\eta}(\eta - s), & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t. \end{cases}$$

The case $\alpha < 0$.

Again we write $\beta = -\alpha > 0$. We show that we may take

$$\Phi(s) = (1+\beta)\frac{s(1-s)}{1+\beta\eta}.$$

Upper bounds

Case 1. $s \le \eta$. If $s > t$ then $k(t,s) \ge 0$ and

$$k(t,s) = \frac{t(1-s) + \beta t(\eta - s)}{1+\beta\eta} \le \frac{s(1-s) + \beta s(1-s)}{1+\beta\eta} = (1+\beta)\frac{s(1-s)}{1+\beta\eta}.$$

If $s \le t$,

$$k(t,s) = \frac{t(1-s)}{1+\beta\eta} + \frac{\beta t}{1+\beta\eta}(\eta - s) - (t-s) = \frac{-st - \beta ts + s + \beta\eta s}{1+\beta\eta},$$

and $k(s,t)$ is negative for $t > \frac{1+\beta\eta}{1+\beta}$ and $s \ne 0$; note also that $\frac{1+\beta\eta}{1+\beta} > \eta$.

For $k(t,s) \ge 0$ we have

$$k(t,s) = \frac{-st - \beta ts + s + \beta\eta s}{1+\beta\eta} \le \frac{s(1-s) + \beta s(\eta - s)}{1+\beta\eta} \le (1+\beta)\frac{s(1-s)}{1+\beta\eta},$$

and for $k(t,s) < 0$ we have

$$-k(t,s) = \frac{st + \beta ts - s - \beta\eta s}{1+\beta\eta} \le \frac{\beta s(1-\eta)}{1+\beta\eta} \le \beta\frac{s(1-s)}{1+\beta\eta}.$$

Case 2. $s > \eta$. If $s > t$ then $k(t,s) \ge 0$

$$k(t,s) = \frac{t(1-s)}{1+\beta\eta} \le \frac{s(1-s)}{1+\beta\eta}.$$

If $s \le t$ then

$$k(t,s) = \frac{-ts - \beta\eta t + s + \beta\eta s}{1+\beta\eta}$$

and $k(s,t)$ is negative for $t > \frac{s(1+\beta\eta)}{s+\beta\eta}$. When $k(t,s) \ge 0$ we have

$$k(t,s) = \frac{-ts - \beta\eta t + s + \beta\eta s}{1+\beta\eta} \le \frac{-s^2 - \beta\eta s + s + \beta\eta s}{1+\beta\eta} = \frac{s(1-s)}{1+\beta\eta},$$

and when $k(t,s) < 0$ we have

$$-k(t,s) = \frac{ts + \beta\eta t - s - \beta\eta s}{1+\beta\eta} \le \frac{\beta\eta(1-s)}{1+\beta\eta} \le \beta\frac{s(1-s)}{1+\beta\eta}.$$

Lower bounds

We show that we may take arbitrary $[a,b] \subset (0,\eta]$.

Case 1. $s \le \eta$. If $s > t$ then

$$k(t,s) = \frac{t - st + \beta t\eta - \beta ts}{1+\beta\eta} \ge \frac{t(1-s)}{1+\beta\eta} \ge \frac{a(1-s)}{1+\beta\eta} \ge a\frac{s(1-s)}{1+\beta\eta}.$$

If $s \le t$ and $t \le b \le \eta$,

$$k(t,s) \ge \frac{-s\eta - \beta\eta s + s + \beta\eta s}{1+\beta\eta} = \frac{s(1-\eta)}{1+\beta\eta} \ge (1-\eta)\frac{s(1-s)}{1+\beta\eta}.$$

Case 2. $s > \eta$. If $s > t$ then

$$k(t,s) = \frac{t(1-s)}{1+\beta\eta} \geq \frac{a(1-s)}{1+\beta\eta} \geq a\frac{s(1-s)}{1+\beta\eta}.$$

The case $s \leq t$ does not occur since we take $b \leq \eta$. Therefore we may take

$$c = \min\{a, 1-\eta\}/(1+\beta).$$

Theorem 3.4.1. *Let $a, b \in (0, \eta]$ and suppose that $\int_a^b g(s)\,ds > 0$. Let c be as given above. Let m, M be as defined previously. Then for $\alpha\eta < 0$ the BVP* (3.4.1), (3.4.2) *has at least one nonzero solution, positive on $[a,b]$, if either*

(h_1) $0 \leq f^0 < m$ *and* $M < f_\infty \leq \infty$

or

(h_2) $0 \leq f^\infty < m$ *and* $M < f_0 \leq \infty$,

and has two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either

(E_1) $0 \leq f^0 < m$, $f_{c\rho,\rho} \geq cM$, $x \neq Tx$ *for* $x \in \partial\Omega_\rho$, *and* $0 \leq f^\infty < m$,

or

(E_2) $M < f_0 \leq \infty$, $f^{-\rho,\rho} \leq m$, $x \neq Tx$ *for* $x \in \partial K_\rho$, *and* $M < f_\infty \leq \infty$.

The following simple example illustrates result.

Example 3.4.2. Set $g \equiv 1$ and $f \equiv 2$. The solution of (3.4.1) with (3.4.2) is

$$u(t) = \frac{1+\beta\eta^2}{1+\beta\eta}\,t - t^2.$$

Thus $u(t)$ is positive on $[0,\eta]$ but $u(1) < 0$. (in Figure 3.4.2 we illustrate the special case $\eta = \frac{1}{2}$ and $\beta = 1$ obtaining $u(t) = -t^2 + \frac{5}{6}t$).

The case $\alpha\eta > 1$.

For these BCs the kernel k is negative on an interval so we apply remark 3.2.15. We show that for this BCs we may take

$$\Phi(s) = \alpha\frac{s(1-s)}{\alpha\eta - 1}.$$

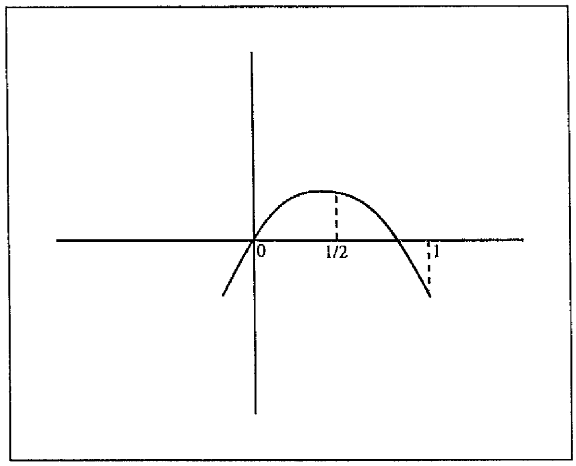

Figure 3.5: A nontrivial solution

Upper bounds

Case 1. $s \leq \eta$. If $s > t$ then

$$-k(t,s) = \frac{t(1-s-\alpha\eta+\alpha s)}{\alpha\eta-1} \leq \frac{t(1-s)}{\alpha\eta-1} \leq \frac{s(1-s)}{\alpha\eta-1}.$$

Also

$$k(t,s) = \frac{t(-1+s+\alpha\eta-\alpha s)}{\alpha\eta-1} \leq \frac{t(-1+s+\alpha-\alpha s)}{\alpha\eta-1} \leq (\alpha-1)\frac{s(1-s)}{\alpha\eta-1}.$$

If $s \leq t$ then

$$-k(t,s) = \frac{-st+\alpha ts-\alpha\eta s+s}{\alpha\eta-1} = \frac{s(-t+\alpha t-\alpha\eta+1)}{\alpha\eta-1}.$$

When $t \geq \frac{(\alpha\eta-1)}{\alpha-1}$ then $-k(s,t) \geq 0$ and

$$-k(t,s) \leq \frac{s(\alpha-\alpha\eta)}{\alpha\eta-1} \leq \alpha\frac{s(1-s)}{\alpha\eta-1}.$$

If $t < \frac{(\alpha\eta-1)}{\alpha-1}$ then $k(s,t) \geq 0$ and

$$-k(t,s) \leq \frac{s(s-1+\alpha(\eta-s))}{\alpha\eta-1} \leq \frac{s(s-1+\alpha(1-s))}{\alpha\eta-1} \leq (\alpha-1)\frac{s(1-s)}{\alpha\eta-1}.$$

Case 2. $s > \eta$. If $s > t$ then $-k(t,s) \geq 0$ and

$$-k(t,s) = \frac{t(1-s)}{\alpha\eta-1} \leq \frac{s(1-s)}{\alpha\eta-1}.$$

If $s \leq t$ then $-k(t,s) \geq 0$ and

$$\begin{aligned}-k(t,s) &= \frac{-ts+\alpha\eta t-\alpha\eta s+s}{\alpha\eta-1} \leq \frac{-s+\alpha\eta-\alpha\eta s+s}{\alpha\eta-1}\\ &= \frac{\alpha\eta(1-s)}{\alpha\eta-1} < \alpha\frac{s(1-s)}{\alpha\eta-1}.\end{aligned}$$

Lower bounds

We show that we may take an arbitrary $[a,b] \subset [\eta,1]$.

Case 1. $s \le \eta$. Since we take $a \ge \eta$ we only have the case $s \le t$ and then

$$-k(t,s) = \frac{s(-t+\alpha t-\alpha\eta+1)}{\alpha\eta-1}.$$

Since $\eta > \frac{\alpha\eta-1}{\alpha-1}$ we have $-k(t,s) \ge 0$ and

$$-k(t,s) \ge \frac{s(-\eta+\alpha\eta-\alpha\eta+1)}{\alpha\eta-1} \ge (1-\eta)\frac{s(1-s)}{\alpha\eta-1}.$$

Case 2. $s > \eta$. If $s > t$ then

$$-k(t,s) = \frac{t(1-s)}{\alpha\eta-1} \ge \frac{a(1-s)}{\alpha\eta-1} \ge a\frac{s(1-s)}{\alpha\eta-1}.$$

If $s \le t$ then

$$-k(t,s) = \frac{-ts+\alpha\eta t-\alpha\eta s+s}{\alpha\eta-1} \ge \frac{-s^2+\alpha\eta s-\alpha\eta s+s}{\alpha\eta-1} = \frac{s(1-s)}{\alpha\eta-1}.$$

Thus we may take $c = \min\{a, 1-\eta\}/\alpha$.

Theorem 3.4.3. *Let $a,b \in [\eta,1]$ and suppose that $\int_a^b g(s)\,ds > 0$. Let m, M be as defined previously and let $c = \min\{a,1-\eta\}/\alpha$. Then for $\alpha > 1$ the BVP (3.4.1), (3.4.2) has at least one nonzero solution, negative on $[a,b]$, if either*

(h_1) *$0 \le f^0 < m$ and $M < f_\infty \le \infty$*

or

(h_2) *$0 \le f^\infty < m$ and $M < f_0 \le \infty$,*

and has two nonzero solutions, negative on $[a,b]$, if there is $\rho > 0$ such that either

(E_1) *$0 \le f^0 < m$, $f_{c\rho,\rho} \ge cM$, $x \ne Tx$ for $x \in \partial\Omega_\rho$, and $0 \le f^\infty < m$,*

or

(E_2) *$M < f_0 \le \infty$, $f^{-\rho,\rho} \le m$, $x \ne Tx$ for $x \in \partial K_\rho$, and $M < f_\infty \le \infty$.*

3.5 Radial solutions of elliptic PDEs

Consider the problem of existence of positive radial solutions in an annulus in $\mathbb{R}^n$, $n \geq 2$, for the equation

$$\triangle u + h(|x|)f(u) = 0, \quad \text{for a.e. } |x| \in [R_1, R_0]. \tag{3.5.1}$$

with boundary condition:

$$\frac{\partial u}{\partial r} = 0 \quad \text{for} \quad |x| = R_0 \quad \text{and} \quad u(R_1) = \alpha u(R_\eta). \tag{3.5.2}$$

We assume

(1) $0 < R_1 < R_\eta < R_0 < \infty$.

(2) $f : \mathbb{R} \to [0, \infty)$ is continuous.

(3) $h \in L^1(R_0, R_1)$ and $h(r) \geq 0$ a.e..

For radial solutions $u = u(r)$, $r = |x|$ we can write (3.5.1) in the form

$$u''(r) + \frac{n-1}{r}u'(r) + h(r)f(u(r)) = 0 \quad \text{a.e. on } [R_1, R_0]. \tag{3.5.3}$$

Eq.(3.5.3) can be transformed into the ODE

$$u'' + g(t)f(u) = 0$$

by means of the following variables. Put $u(t) = u(r(t))$ where

$$r(t) = (\gamma + (\beta - \gamma)t)^{-1/(n-2)} \text{ for } t \in [0, 1],$$

where $\gamma = R_0^{-(n-2)}$ and $\beta = R_1^{-(n-2)}$, and let

$$\phi(t) = ((\beta - \gamma)/(n-2))^2(\gamma + (\beta - \gamma)t)^{-2(n-1)/(n-2)}.$$

Then Eq. (3.5.3) becomes

$$u''(t) + \phi(t)h(r(t))f(u(t)) = 0, \quad \text{a.e. on } [0, 1]. \tag{3.5.4}$$

subject to the boundary conditions

$$u'(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1, \tag{3.5.5}$$

[In 2-dimensions we use $r(t) = R_0^{1-t}R_1^t$ and $\phi(t) = (R_0(1-t)\log(R_0/R_1))^2$.]

Hence we can apply, for example, Theorem 3.3.1 to obtain at least one nonzero radial solution of the BVP (3.5.1)-(3.5.2).

Remark 3.5.1. Similar results are valid the problem

$$\triangle u + h(|x|)f(u) = 0, \quad \text{for a.e. } |x| \in [R_1, R_0], \tag{3.5.6}$$

with boundary conditions

$$u(R_0) = 0 \quad \text{and} \quad u(R_1) = \alpha u(R_\eta). \tag{3.5.7}$$

Radial solutions can be studied by transforming the BVP (3.5.6)-(3.5.7) into the ODE

$$u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1) \tag{3.5.8}$$

with boundary conditions

$$u(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1. \tag{3.5.9}$$

Chapter 4

Nonzero solutions of some boundary value problems with discontinuous kernels

4.1 Introduction

In this chapter we extend the results of chapter three to allow for discontinuities in the kernel and more general functions f. One motivation is that certain nonlocal boundary value problems lead to precisely this situation. We shall study in detail the problem

$$u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{4.1.1}$$

with boundary condition

$$u(1) = \alpha u'(\eta), \ u(0) = 0, \ 0 < \eta < 1. \tag{4.1.2}$$

In this case the kernel of the corresponding integral equation has a discontinuity. We shall use our theory to show that multiple nonzero (but not necessarily positive) solutions exist, under suitable conditions on f, when either $0 \leq \alpha < 1 - \eta$ or $\alpha < 0$. These results are completely new and have been submitted for publication in [22].

4.2 Existence of nontrivial solutions of Hammerstein integral equations

We begin by giving some new results for the following Hammerstein integral equation.

$$u(t) = \int_G k(t,s)f(s,u(s))\,ds := Tu(t), \tag{4.2.1}$$

where G is a compact set in $\mathbb{R}^n$ of positive measure. We will work in the space $C(G)$ of continuous functions endowed with the usual supremum norm. We shall make the following assumptions on f, g and the kernel k. Recall that f is said to satisfy the Carathéodory conditions if for each u, $s \mapsto f(s,u)$ is measurable and for almost every s, $u \mapsto f(s,u)$ is continuous.

(C_1) Suppose that for every $r > 0$, $f : G \times [-r,r] \to [0,\infty)$ satisfies Carathéodory conditions on $G\times[-r,r]$ and there exists a measurable function $g_r : G \to [0,\infty)$ such that

$$f(s,u) \leq g_r(s) \quad \text{for almost all } s \in G \text{ and all } u \in [-r,r].$$

(C_2) $k : G \times G \to \mathbb{R}$ is measurable, and for every $\tau \in G$ we have

$$\lim_{t\to\tau} \int_G |k(t,s) - k(\tau,s)|g_r(s)\,ds = 0.$$

(C_3) There exist a closed subset $G_0 \subset G$ with $\operatorname{meas}(G_0) > 0$, a measurable function $\Phi : G \to [0,\infty)$ and a constant $c \in (0,1]$ such that

$$|k(t,s)| \leq \Phi(s) \text{ for } t \in G \text{ and almost every } s \in G$$
$$c\Phi(s) \leq k(t,s) \text{ for } t \in G_0 \text{ and almost every } s \in G.$$

(C_4) For each r there is $M_r < \infty$ such that $\int_G \Phi(s)g_r(s)\,ds \leq M_r$.

These hypotheses allow us to work in the cone

$$K = \{u \in C(G) : \min\{u(t) : t \in G_0\} \geq c\|u\|\}.$$

This is similar to but larger than the cone used by Lan [30]. In order to use the well-known fixed point index for compact maps, we need to prove that $T : K \to K$ is compact.

Theorem 4.2.1. *Assume that (C_1)-(C_4) hold for some $r > 0$. Then T maps $\overline{K}_r$ into K and is compact.*

Proof. The compactness of T follows from Proposition 3.1, p.164, of [35] since, as G is compact, the limit in (C_2) is readily shown to be uniform in $\tau \in G$. To see that $T : \overline{K}_r \to K$, for $u \in \overline{C}_r$ and $t \in G$, we have,

$$|Tu(t)| \leq \int_G |k(t,s)| f(s, u(s))\, ds$$

so that

$$\|Tu\| \leq \int_G \Phi(s) f(s, u(s))\, ds.$$

Also

$$\min_{t \in G_0}\{Tu(t)\} \geq c \int_G \Phi(s) f(s, u(s))\, ds.$$

Hence $Tu \in K$ for every $u \in \overline{K}_r$. □

Remark 4.2.2. In Theorem 4.2.1, if the hypotheses hold for each $r > 0$, then T maps K into K and is compact. We shall only consider this case.

Let $q : C(G) \to \mathbb{R}$ denote the function $q(u) = \min\{u(t) : t \in G_0\}$. The proof of Proposition 3.2.3 shows that q is continuous. As in chapter three, for $\rho > 0$, we shall use the set $\Omega_\rho = \{u \in K : q(u) < c\rho\}$.

Lemma 4.2.3. *Ω_ρ defined above has the following properties.*

(a) Ω_ρ is open relative to K.

(b) $K_{c\rho} \subset \Omega_\rho \subset K_\rho$.

(c) $u \in \partial\Omega_\rho$ if and only if $q(u) = c\rho$.

(d) If $u \in \partial\Omega_\rho$, then $c\rho \leq u(t) \leq \rho$ for $t \in G_0$.

We omit the simple proof as it is exactly similar to the one in chapter three. We now prove a lemma which implies the index is zero, this is a more general version of Lemma 3.2.8

Lemma 4.2.4. *Assume that there exists $\rho > 0$ such that $u \neq Tu$ for $u \in \partial\Omega_\rho$ and the following conditions hold.*

$(H_\rho^\geq)$ *There exists a measurable function* $\psi_\rho : G_0 \to \mathbb{R}_+$ *such that*

$$f(s,u) \geq c\rho\,\psi_\rho(s) \quad \textit{for all } u \in [c\rho, \rho] \quad \textit{and almost all } s \in G_0,$$

and $\inf_{t\in G_0} \int_{G_0} k(t,s)\psi_\rho(s)\,ds \geq 1$.

Then $i_K(T, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in G$. Then $e \in K$. We prove that

$$u \neq Tu + \lambda e \quad \text{for } u \in \partial\Omega_\rho \quad \text{and } \lambda > 0.$$

In fact, if not, there exist $u \in \partial\Omega_\rho$ and $\lambda > 0$ such that $u = Tu + \lambda e$. By $(H_\rho^\geq)$, we have for $t \in G_0$,

$$\begin{aligned} u(t) = &\int_G k(t,s) f(s,u(s))\,ds + \lambda \geq \int_{G_0} k(t,s) f(s,u(s))\,ds + \lambda \\ \geq &c\rho \int_{G_0} k(t,s)\psi_\rho(s)\,ds + \lambda \geq c\rho + \lambda. \end{aligned}$$

This implies $q(u) \geq c\rho + \lambda > c\rho$, contradicting (c) of Lemma 4.2.3. Hence (1) of Lemma 1.4.7 implies $i_K(T, \Omega_\rho) = 0$. □

Note that if strict inequality holds in $(H_\rho^\geq)$, taking $\lambda = 0$ we see that $u \neq Tu$ for $u \in \partial\Omega_\rho$.

Remark 4.2.5. A commonly used assumption in place of (1) of Lemma 1.4.7 is $\|Tu\| \geq \|u\|$ for $\|u\| = \rho$. We observe that this follows from a stronger version of $(H_\rho^\geq)$ namely $f(s,u) \geq \rho\psi_\rho(s)$ for $c\rho \leq u \leq \rho$ where $\inf_{t\in G_0} \int_{G_0} k(t,s)\psi_\rho(s)\,ds \geq 1$. Indeed, for $t \in G_0$ and $u \in K$ with $\|u\| = \rho$ we have

$$|Tu(t)| = \int_G k(t,s) f(s,u(s)\,ds \geq \int_{G_0} k(t,s)\rho\psi_\rho(s)\,ds \geq \rho = \|u\|.$$

This remark shows that using the open set Ω_ρ and (1) of Lemma 1.4.7 gives a stronger result.

We now give a more general version of Lemma 3.2.9, which implies the index is 1.

Lemma 4.2.6. *Assume that there exists $\rho > 0$ such that $u \neq Tu$ for $u \in \partial K_\rho$ and f satisfies the following condition.*

$(H_\rho^\leq)$ *There exists a measurable function $\phi_\rho : G \to \mathbb{R}_+$ such that*

$$f(s,u) \leq \rho\, \phi_\rho(s) \quad \text{for all } u \in [-\rho,\rho] \quad \text{and almost all } s \in G$$

and $\sup_{t\in G} \int_G |k(t,s)|\phi_\rho(s)\, ds \leq 1$.

Then $i_K(T, K_\rho) = 1$.

Proof. By $(H_\rho^\leq)$ we have for $u \in \partial K_\rho$ and $t \in G$,

$$\begin{aligned} |Tu(t)| = \left|\int_G k(t,s) f(s,u(s))\, ds\right| &\leq \int_G |k(t,s)|\, f(s,u(s))\, ds \\ &\leq \rho \int_G |k(t,s)|\phi_\rho(s)\, ds \leq \rho = \|u\|. \end{aligned}$$

This implies $\|Tu\| \leq \|u\|$ for $u \in \partial K_\rho$. By means of (2) of Lemma 1.4.7, we obtain $i_K(T, K_\rho) = 1$. □

Note that if strict inequality holds in $(H_\rho^\leq)$, then $u \neq Tu$ for $u \in \partial K_\rho$.

We now give our new result which asserts that Eq. (4.2.1) has at least one or at least two nonzero solutions which are positive on the subset G_0 of G.

Theorem 4.2.7. *The integral equation Eq.* (4.2.1) *has a nonzero solution in K if either of the following conditions hold.*

(H_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < c\rho_2$ such that*
$(H_{\rho_1}^\leq)$, $\quad (H_{\rho_2}^\geq), u \neq Tu$ *for $u \in \partial\Omega_{\rho_2}$.*

(H_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that*
$(H_{\rho_1}^\geq)$, $\quad (H_{\rho_2}^\leq)$, *$u \neq Tu$ for $u \in \partial K_{\rho_2}$.*

Eq. (4.2.1) *has two nonzero solutions in K if one of the following conditions hold.*

(S_1) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < c\rho_2$ and $\rho_2 < \rho_3$ such that*
$(H_{\rho_1}^\leq)$, $\quad (H_{\rho_2}^\geq)$, *$u \neq Tu$ for $u \in \partial\Omega_{\rho_2}$ and $(H_{\rho_3}^\leq)$ hold.*

(S_2) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2 < c\rho_3$ such that*

$$(H^{\geq}_{\rho_1}), \quad (H^{\leq}_{\rho_2}),\ u \neq Tu \text{ for } u \in \partial K_{\rho_2} \quad \text{and } (H^{\geq}_{\rho_3}) \text{ hold.}$$

Moreover, if in (S_1), strict inequality holds in $(H^{\leq}_{\rho_1})$, then Eq. (4.2.1) has a third solution $u_0 \in K_{\rho_1}$.

Proof. Assume that (S_1) holds. We show that either T has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$ or on its boundary. If $u \neq Tu$ for u in the boundary, by Lemmas 4.2.4 and 4.2.6, we have $i_K(T, K_{\rho_1}) = 1$, $i_K(T, \Omega_{\rho_2}) = 0$. By (b) of Lemma 4.2.3, we have $\overline{K}_{\rho_1} \subset K_{c\rho_2} \subset \Omega_{\rho_2}$ since $\rho_1 < c\rho_2$. It follows from (3) of Lemma 1.4.7 that T has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Similarly, T has a fixed point u_2 in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$ or on its boundary. When strict inequality holds then $u \neq Tu$ for $u \in \partial K_{\rho_1}$ so $i_K(T, K_{\rho_1}) = 1$ and T has a fixed point u_0 in K_{ρ_1}. The other assertions are proved similarly. □

Remark 4.2.8. Comments similar to Remarks 3.2.11- 3.2.11 also apply in this case.

In the particular case when $f(t,u) = g(t)h(u)$ where $\Phi g \in L^1$ and h is continuous it is possible to give conditions that are more easily verified.

Definition 4.2.9. We define the following numbers:

$$m = \left(\max_{t \in G} \int_G |k(t,s)| g(s)\, ds\right)^{-1} \text{ and } \quad M = \left(\min_{t \in G_0} \int_{G_0} k(t,s) g(s)\, ds\right)^{-1}$$
$$h^{-\rho,\rho} = \sup_{u \in [-\rho,\rho]} \frac{h(u)}{\rho}, \quad h^0 = \limsup_{u \to 0} \frac{h(u)}{|u|}, \quad h^\infty = \limsup_{|u| \to \infty} \frac{h(u)}{|u|}$$
$$h_{c\rho,\rho} = \inf_{u \in [c\rho,\rho]} \frac{h(u)}{\rho}, \quad h_0 = \liminf_{u \to 0+} \frac{h(u)}{u}, \quad h_\infty = \liminf_{u \to \infty} \frac{h(u)}{u}.$$

Lemma 4.2.10. *We have the following implications.*

1. *$h^0 < m$ implies $h^{-\rho,\rho} < m$ for some ρ (small) and $h^{-\rho,\rho} \leq m$ implies $(H^{\leq}_{\rho})$.*

2. *$h^\infty < m$ implies $h^{-\rho,\rho} < m$ holds for some ρ (large).*

3. *$h_0 > M$ implies $h_{c\rho,\rho} > cM$ for some ρ and $h_{c\rho,\rho} \geq cM$ implies $(H^{\geq}_{\rho})$.*

4. *$h_\infty > M$ implies $h_{c\rho,\rho} > cM$ holds for some ρ.*

Proof. (1) For $\varepsilon > 0$ there is $\rho_\varepsilon > 0$ such that $h(u)/|u| < h^0 + \varepsilon$ for $|u| \le \rho_\varepsilon$ which implies there is $\rho > 0$ such that $h^{-\rho,\rho} < m$ when $h^0 < m$. Also $h^{-\rho,\rho} \le m$ implies $h(u)g(s) \le m\rho g(s)$ so that $(H_\rho^{\le})$ holds with $\phi_\rho(s) = mg(s)$. (2) Let $\beta > m$. There is r such that $h(u)/|u| < \beta$ for $|u| \ge r$. As h is continuous there exists γ such that $h(u) < \beta|u| + \gamma$ for all u. Let $\rho = \frac{\gamma}{m-\beta}$, then $h(u) < m\rho$ for $|u| \le \rho$. The proofs of (3) and (4) are straightforward. □

We now give a more easily checked version of Theorem 4.2.7.

Theorem 4.2.11. *Let $f(t,u) = g(t)h(u)$ be as above and assume that $\int_{G_0} \Phi(s)g(s)\,ds > 0$. Then Eq. (4.2.1) has a nonzero solution in K if one of the following conditions hold:*

(H_1') *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < c\rho_2$ such that*

$$h^{-\rho_1,\rho_1} \le m \quad \textit{and} \quad h_{c\rho_2,\rho_2} \ge cM;$$

(H_2') *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that*

$$h_{c\rho_1,\rho_1} \ge cM \quad \textit{and} \quad h^{-\rho_2,\rho_2} \le m.$$

Eq. (4.2.1) has two nonzero solutions in K if there is $\rho > 0$ such that either of the following conditions hold:

(S_1') $0 \le h^0 < m,\quad h_{c\rho,\rho} \ge cM,\ u \ne Tu$ *for* $u \in \partial\Omega_\rho$ *and* $0 \le h^\infty < m,$

(S_2') $M < h_0 \le \infty,\quad h^{-\rho,\rho} \le m,\ u \ne Tu$ *for* $u \in \partial K_\rho$ *and* $M < h_\infty \le \infty.$

Theorem 4.2.11 generalises Theorem 2.9 of [21] by allowing discontinuous kernels and generalises Theorem 2.2 of [30] by allowing kernels that are not positive everywhere hence giving existence of solutions that change sign.

4.3 Multiple nonzero solutions of equation (4.1.1).

We now investigate the BVP

$$u'' + f(t,u(t)) = 0, \text{ a.e. on } [0,1] \tag{4.3.1}$$

with boundary conditions

$$u(1) = \alpha u'(\eta),\ u(0) = 0,\ 0 < \eta < 1,\ \alpha < 1 - \eta. \tag{4.3.2}$$

By a solution of this BVP we will mean a solution of the corresponding Hammerstein Integral equation

$$u(t) = \int_0^1 k(t,s) f(s, u(s))\, ds. \tag{4.3.3}$$

The kernel [Green's function] in (4.3.3) is

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha}, & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t \end{cases}.$$

Note that the kernel is discontinuous on the line $s = \eta$ but does satisfy (C_2). We shall study separately the cases $\alpha > 0$ and $\alpha < 0$. In the special case $\alpha = 0$, existence of one positive solution is covered by the results of [32]. The results we obtain are new.

The case $\alpha > 0$.

In this case we shall suppose that $0 < \alpha < 1 - \eta$. This is necessary for our method in order to obtain appropriate lower bounds. We have to exhibit $\Phi(s)$, a subinterval $[a, b] \subset [0, 1]$ and a constant $c < 1$ such that

$$\begin{aligned} |k(t,s)| \le& \Phi(s) \text{ for every } t \in [0,1] \text{ and almost every } s \in [0,1], \\ k(t,s) \ge& c\Phi(s) \text{ for every } t \in [a,b] \text{ and almost every } s \in [0,1]. \end{aligned}$$

We show that we may take

$$\Phi(s) = \max\left\{1, \frac{\alpha}{\eta}\right\} \frac{s(1-s)}{1-\alpha}.$$

<u>Case 1.</u> $s > \eta$. If $t < s$ then $k(t,s) \ge 0$ and

$$k(t,s) = \frac{t}{1-\alpha}(1-s) \le \frac{s(1-s)}{1-\alpha}.$$

If $t \ge s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - (t-s) = \frac{s(1-\alpha) + t(\alpha - s)}{1-\alpha}.$$

The minimum/maximum occur when $t = 1$ or $t = s$. Thus $k \geq 0$. If $s > \alpha$ then

$$k(t,s) = \frac{s(1-\alpha)+t(\alpha-s)}{1-\alpha} \leq \frac{s(1-\alpha)+s(\alpha-s)}{1-\alpha} = \frac{s(1-s)}{1-\alpha}.$$

If $s \leq \alpha$ then

$$k(t,s) = \frac{s(1-\alpha)+t(\alpha-s)}{1-\alpha} \leq \frac{s(1-\alpha)+\alpha-s}{1-\alpha} = \frac{\alpha(1-s)}{1-\alpha} < \frac{\alpha}{\eta}\frac{s(1-s)}{(1-\alpha)}.$$

Case 2. $s \leq \eta$. If $t < s$

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} = \frac{t(1-s-\alpha)}{1-\alpha}.$$

When $s \leq 1-\alpha$ we have $k(t,s) \geq 0$ and

$$k(t,s) \leq \frac{s(1-s-\alpha)}{1-\alpha} \leq \frac{s(1-s)}{1-\alpha}.$$

The case $\eta \geq s > 1-\alpha$ cannot occur since we have $0 < \alpha < 1-\eta$.

If $t \geq s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} - (t-s) = \frac{s(1-t-\alpha)}{1-\alpha}.$$

If $t \leq 1-\alpha$ then $k(t,s) \geq 0$ and

$$k(t,s) \leq \frac{s(1-t)}{1-\alpha} \leq \frac{s(1-s)}{1-\alpha}.$$

If $t > 1-\alpha$ then $k(t,s) \leq 0$ and

$$-k(t,s) = \frac{s(-1+t+\alpha)}{1-\alpha} \leq \frac{\alpha s}{1-\alpha} < \frac{s(1-\eta)}{1-\alpha} \leq \frac{s(1-s)}{1-\alpha}.$$

Lower bounds

We show that we may take an arbitrary $[a,b] \subset (0, 1-\alpha)$

Case 1. $s > \eta$. If $t < s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) \geq a\frac{(1-s)}{1-\alpha} \geq a\frac{s(1-s)}{1-\alpha}.$$

If $t \geq s$

$$k(t,s) = \frac{s+\alpha t-\alpha s-st}{1-\alpha} \geq s\frac{(1-b)}{1-\alpha} \geq (1-b)\frac{s(1-s)}{1-\alpha}.$$

Case 2. $s \leq \eta$. If $t < s$ then

$$k(t,s) = t\frac{1-s-\alpha}{1-\alpha} \geq a\frac{(1-\eta-\alpha)}{1-\alpha} \geq a(1-\eta-\alpha)4\frac{s(1-s)}{1-\alpha}.$$

If $t \geq s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} - (t-s) = \frac{s - st - \alpha s}{1-\alpha}$$
$$\geq \frac{s(1-b-\alpha)}{1-\alpha} \geq (1-b-\alpha)\frac{s(1-s)}{1-\alpha}.$$

The conclusion is that we may take

$$c = \frac{\min\{4a(1-\eta-\alpha),(1-b-\alpha)\}}{\max\{1,\frac{\alpha}{\eta}\}}.$$

We state a result when $f(t,u) = g(t)h(u)$, of course there is a more general result analogous to Theorem 4.2.7.

Theorem 4.3.1. *Let* $[a,b] \subset (0,1-\alpha)$ *and suppose that* $\int_a^b \Phi(s)g(s)\,ds > 0$. *Let* c *be as given above. Let* m, M *be as defined previously. Then for* $0 < \alpha < 1-\eta$ *the* BVP (4.3.1), (4.3.2) *has at least one nonzero solution, positive on* $[a,b]$, *if either*

(h_1') $0 \leq h^0 < m$ *and* $M < h_\infty \leq \infty$, *or*

(h_2') $0 \leq h^\infty < m$ *and* $M < h_0 \leq \infty$,

and has two nonzero solutions, positive on $[a,b]$, *if there is* $\rho > 0$ *such that either*

(S_1') $0 \leq h^0 < m$, $h_{c\rho,\rho} \geq cM$, $u \neq Tu$ *for* $u \in \partial\Omega_\rho$, *and* $0 \leq h^\infty < m$, *or*

(S_2') $M < h_0 \leq \infty$, $h^{-\rho,\rho} \leq m$, $u \neq Tu$ *for* $u \in \partial K_\rho$, *and* $M < h_\infty \leq \infty$.

We give a simple example to illustrate the theorem.

Example 4.3.2. Set $f(t,u) \equiv 2$. In this case the solution is

$$u(s) = -s\Big(s - \frac{1-2\alpha\eta}{1-\alpha}\Big).$$

For $\eta \leq 1/2$ and $\eta + \alpha < 1$, the solution is actually positive on all of $[0,1]$ For $\eta > 1/2$ the solution is negative for $t > t_0 = \frac{1-2\alpha\eta}{1-\alpha}$, but is positive on $(0, 1-\alpha)$.

The case $\alpha < 0$.

To simplify the calculations we write $-\beta$ in place of α, so that $\beta > 0$.

We show that for these BCs we can take

$$\Phi(s) = \max\Big\{\frac{(1-\eta+\beta)}{1-\eta}, \frac{\beta}{\eta}\Big\}\frac{s(1-s)}{1+\beta}.$$

Upper bounds

<u>Case 1.</u> $s > \eta$. If $t < s$ then $k(t,s) \geq 0$ and

$$k(t,s) = \frac{t}{1+\beta}(1-s) \leq \frac{s(1-s)}{1+\beta}.$$

If $t \geq s$ then

$$k(t,s) = \frac{s-\beta t+\beta s-st}{1+\beta} \leq \frac{s-\beta s+\beta s-st}{1+\beta} = \frac{s(1-t)}{1+\beta} \leq \frac{s(1-s)}{1+\beta}.$$

If $t \leq \frac{s(1+\beta)}{\beta+s}$, $k(t,s) \geq 0$ and we are done.
If $t > \frac{s(1+\beta)}{\beta+s}$ we have

$$-k(t,s) = \frac{ts+\beta t-s-s\beta}{1+\beta} \leq \frac{s+\beta-s-s\beta}{1+\beta} \leq \frac{\beta}{\eta}\frac{s(1-s)}{1+\beta}.$$

<u>Case 2.</u> $s \leq \eta$. Note that in this case $\frac{(1-s)}{(1-\eta)} \geq 1$. If $t < s$ then $k(t,s) \geq 0$ and

$$\begin{aligned} k(t,s) =& \frac{t}{1+\beta}(1-s) + \frac{\beta t}{1+\beta} = t\frac{1-s+\beta}{1+\beta} \\ \leq& \frac{s(1-s+\beta)}{1+\beta} \leq \frac{s(1-s+\beta\frac{(1-s)}{(1-\eta)})}{1+\beta} \leq \frac{(1-\eta+\beta)}{1-\eta}\frac{s(1-s)}{1+\beta}. \end{aligned}$$

If $t \geq s$ then $k(t,s) \geq 0$ and

$$\begin{aligned} k(t,s) = \frac{t}{1+\beta}(1-s) + \frac{\beta t}{1+\beta} - (t-s) &= \frac{s-st+\beta s}{1+\beta} \\ &\leq \frac{s(1-s+\beta)}{1+\beta} \leq \frac{(1-\eta+\beta)}{1-\eta}\frac{s(1-s)}{1+\beta}. \end{aligned}$$

Lower bounds

We show that we may take an arbitrary $[a,b] \subset (0,\eta]$.

<u>Case 1.</u> $s > \eta$. If $t < s$ then

$$k(t,s) = \frac{t}{1+\beta}(1-s) \geq a\frac{(1-s)}{1+\beta} \geq a\frac{s(1-s)}{1+\beta}.$$

Since we take $b \leq \eta$ the (awkward) case $t \geq s$ does not occur.

<u>Case 2.</u> $s \leq \eta$. If $t < s$ then

$$k(t,s) = \frac{t-st+\beta t}{1+\beta} = t\frac{1-s+\beta}{1+\beta} \geq a\frac{s(1-s)}{1+\beta}$$

If $t \geq s$ then

$$k(t,s) = \frac{s - st + \beta s}{1+\beta} \geq \beta \frac{s}{1+\beta} \geq \beta \frac{s(1-s)}{1+\beta}.$$

The conclusion is that we may take

$$c = \frac{\min\{a, \beta\}}{\max\left\{(1-\eta+\beta), \frac{\beta}{\eta}\right\}}.$$

Remark 4.3.3. In this case it is possible to take a somewhat larger b namely any $b < b_0$, where

$$b_0 := \frac{\eta(1+\beta)}{\eta+\beta},$$

but the corresponding c is more complicated.

For the case when $f(t,u) = g(t)h(u)$ we have the following result.

Theorem 4.3.4. *Let $[a,b] \subset (0,\eta]$ and suppose that $\int_a^b \Phi(s)g(s)\,ds > 0$. Let c be as given above. Let m, M be as defined previously. Then for $\alpha < 0$ the BVP* (4.3.1), (4.3.2) *has at least one nonzero solution, positive on $[a,b]$, if either (h_1') or (h_2') of Theorem 4.3.1 is satisfied. There are two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either (S_1') or (S_2') of Theorem 4.3.1 holds.*

The following example illustrates the result.

Example 4.3.5. Let $g(t) = 1$ and

$$h(u) = \begin{cases} 2 & \text{if } |u| \leq 3/m, \\ u^p & \text{for } u \text{ very large,} \end{cases}$$

where $p > 1$. Then $h_0 = \infty$ and $h_\infty = \infty$ and choosing ρ with $2/m < \rho < 3/m$ we have $h^{-\rho,\rho} < m$. Hence (S_2') holds and the BVP has two nonzero solutions which are positive on $(0,\eta]$, the 'small' solution being as written in Example 4.3.2.

Remark 4.3.6. As in section 3.5 it is possible to state results for the existence of radial solutions of PDEs in an annulus. For example radial solutions of the BVP

$$\triangle u + h(|x|)f(u) = 0, \quad \text{for a.e. } |x| \in [R_1, R_0]. \tag{4.3.4}$$

with BC:

$$u(R_0) = 0 \quad \text{and, for} \quad |x| = R_\eta, \quad u(R_1) = \alpha \frac{\partial u}{\partial r}(x). \tag{4.3.5}$$

can be studied by means of the ODE

$$u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{4.3.6}$$

with BC

$$u(1) = \alpha u'(\eta), \ u(0) = 0, \ 0 < \eta < 1. \tag{4.3.7}$$

Chapter 5

Eigenvalues of some nonlocal boundary value problems

5.1 Introduction

In this chapter we study the existence of eigenvalues for a Hammerstein Integral Equation of the form

$$\lambda u(t) = \int_G k(t,s) f(s,u(s))\, ds := Tu(t), \tag{5.1.1}$$

where G is a compact set in $\mathbb{R}^n$ with $\text{meas}(G) > 0$ and k and f are allowed to be discontinuous. The tool we use is a well known result for compact maps in order to establish existence of eigenvalues, working on the cone

$$K = \{u \in C(G) : \min\{u(t) : t \in G_0\} \geq c\|u\|\},$$

where G_0 is a closed subset of G. This type of cone was introduced in chapter three and is a larger cone than the one used by Lan [31].

Our results apply to second order differential equations of the form

$$\lambda u''(t) + f(t,u(t)) = 0, \quad (0 < t < 1), \tag{5.1.2}$$

subject to suitable boundary conditions (BCs). In this chapter we concentrate on the following nonlocal boundary value problems:

$$u'(0) = 0,\ \alpha u'(\eta) = u(1),\ 0 < \eta < 1, \tag{5.1.3a}$$

$$u(0) = 0, \ \alpha u'(\eta) = u(1), \ 0 < \eta < 1, \tag{5.1.3b}$$

$$u'(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1, \tag{5.1.3c}$$

$$u(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1. \tag{5.1.3d}$$

The boundary condition (5.1.3a) is studied or the first time as far as we know. Condition (5.1.3b) has been studied in chapter four. The two conditions (5.1.3c),(5.1.3d) have been widely studied by Gupta & co-authors, see for example [13], [14] and the reference therein, and also by Webb [46]. The results are new and have been submitted for publication in [19].

5.2 Existence of eigenvalues of Hammerstein integral equations

We begin by giving some results for the following Hammerstein integral equation.

$$\lambda u(t) = \int_G k(t,s) f(s, u(s))\, ds := Tu(t), \tag{5.2.1}$$

where G is a compact set in $\mathbb{R}^n$ of positive measure. Throughout the chapter, even if not mentioned explicitly, we shall make the following assumptions on f, g and the kernel k for a fixed $r > 0$ (the assumptions $(C_1) - (C_4)$ are the same as chapter four, but we repeat them for convenience):

(C_1) $f : G \times [-r, r] \to [0, \infty)$ satisfies Carathéodory conditions on $G \times [-r, r]$ and there exists a measurable function $g_r : G \to [0, \infty)$ such that

$$f(t, u) \le g_r(t) \quad \text{for almost all } t \in G \quad \text{and all } u \in [-r, r].$$

(C_2) $k : G \times G \to \mathbb{R}$ is measurable, and for every $\tau \in G$ we have

$$\lim_{t \to \tau} \int_G |k(t,s) - k(\tau, s)| g_r(s)\, ds = 0.$$

(C_3) There exist a closed subset $G_0 \subset G$ with $\text{meas}(G_0) > 0$, a measurable function $\Phi : G \to [0, \infty)$ and a constant $c \in (0, 1]$ such that

$$|k(t,s)| \le \Phi(s) \text{ for } t \in G \text{ and almost every } s \in G$$

$$c\Phi(s) \le k(t,s) \text{ for } t \in G_0 \text{ and almost every } s \in G.$$

(C_4) For each r there is $M_r < \infty$ such that $\int_G \Phi(s) g_r(s)\, ds \leq M_r$.

We use the following well known result (see for example Lemma 1.1, Chapter 5 of [27]).

Lemma 5.2.1. *Let $T : \overline{K}_r \to K$ be compact and suppose that*

$$\inf_{x \in \partial K_r} \|Tx\| > 0.$$

Then there exist $\lambda_0 > 0$ and $x_0 \in \partial K_r$ such that $\lambda_0 x_0 = Tx_0$.

The following theorem generalises Lan's results, allowing operators with kernels that may have both signs:

Theorem 5.2.2. *Assume that there exists $\rho \in (0, r]$ such that:*

(i) There exists a measurable function $m_\rho : G_0 \to \mathbb{R}_+$ such that

$$f(s,u) \geq m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \quad \text{and almost all } s \in G_0,$$

(ii) $\tau := \sup_{t \in G_0} \int_{G_0} k(t,s) m_\rho(s) ds > 0$.

Then there exist λ_0 and $u_0 \in \partial K_\rho$ such that $\lambda_0 u_0 = Tu_0$.

Proof. Since T satisfies the hypotheses of Theorem 4.2.1, $T : \overline{K}_r \to K$ and is compact. Let $u \in \partial K_\rho$, then we have, for every $s \in G_0$, $c\rho \leq u(s) \leq \rho$. For $t \in G_0$ we have $k(t,s) \geq 0$ and

$$|Tu(t)| \geq \int_{G_0} k(t,s) f(s, u(s)) ds \geq \int_{G_0} k(t,s) m_\rho(s) ds$$

Thus $\|Tu\| = \sup_{t \in G} |Tu(t)| \geq \sup_{t \in G_0} |Tu(t)| \geq \tau$ and $\inf_{u \in \partial K_\rho} \|Tu\| > 0$. By Lemma 5.2.1 we obtain the existence of an eigenvalue $\lambda_0 > 0$. □

Remark 5.2.3. In the paper [31], due to the positive nature of the kernel, Lan is able to take a larger τ, namely $\tau = \sup_{t \in G} \int_{G_0} k(t,s) m_\rho(s) ds > 0$.

Remark 5.2.4. We shall see below that, for certain values of the parameter α, the kernel $k(t,s)$ is negative for t in some interval G_0, for all s. In this case, assuming f

is positive, we can show that a negative eigenvalue exists by studying the operator $-T$. Indeed, λ is an eigenvalue for

$$\lambda u(t) = \int_G k(t,s) f(s,u(s))\, ds$$

if and only if $\tilde{\lambda}$ is an eigenvalue of

$$\tilde{\lambda} u(t) = \int_G \tilde{k}(t,s) f(s,u(s))\, ds \equiv \tilde{T} u(t)$$

where $\tilde{k} = -k$ and $\tilde{\lambda} = -\lambda$. Hence we can obtain a result, exactly similar to one above, for the existence of negative eigenvalues. We do not state the obvious theorem thus obtained.

5.3 Eigenvalues of problem (5.1.3a).

As an application of the theory, we investigate in this section the existence of eigenvalues for equations of the form

$$\lambda u''(t) + f(t,u(t)) = 0, \quad \text{a.e on } [0,1], \tag{5.3.1}$$

with boundary conditions

$$u'(0) = 0,\ \alpha u'(\eta) = u(1),\ 0 < \eta < 1. \tag{5.3.2}$$

By an eigenvalue of this problem we mean an eigenvalue of the related Hammerstein integral equation

$$\lambda u(t) = \int_G k(t,s) f(s,u(s))\, ds. \tag{5.3.3}$$

The solution of $u'' + y = 0$ with these BCs is

$$u(t) = \int_0^1 (1-s) y(s) ds - \alpha \int_0^\eta y(s) ds - \int_0^t (t-s) y(s) ds$$

with Green's function

$$k(t,s) = (1-s) + \begin{cases} -\alpha, & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t. \end{cases}$$

Note that, for $\alpha \neq 0$, the kernel is discontinuous on the line $s = \eta$. We shall study separately the cases $\alpha < 0$ and $\alpha > 1$. The case $\alpha = 0$ is included in the results of Lan [31], who studied separated BCs.

The case $\alpha < 0$.

To simplify the calculations we write $-\beta$ in place of α, so that $\beta > 0$.

We have to exhibit $\Phi(s)$, a subinterval $[a,b] \subset [0,1]$ and a constant $c < 1$ such that

$$\begin{aligned} |k(t,s)| \leq& \Phi(s) \quad \text{for every } t, s \in [0,1], \\ k(t,s) \geq& c\Phi(s) \quad \text{for every } s \in [0,1],\ t \in [a,b]. \end{aligned}$$

We show that for these BCs we can take

$$\Phi(s) = (1-s)\Big(1 + \frac{\beta}{1-\eta}\Big).$$

Upper bounds

Indeed

$$k(t,s) \leq (1-s)\Big(1 + \frac{\beta}{1-\eta}\Big),$$

since $\frac{1-s}{1-\eta} \geq 1$ for $s \leq \eta$.

Lower bounds

We show that we may take arbitrary $[a,b] \subset [0,1)$.

Case 1. $s \leq \eta$. If $s > t$ then

$$k(t,s) = (1-s) + \beta \geq (1-s).$$

If $s \leq t$ then

$$k(t,s) = (1-s) + \beta - (t-s)$$

which is a function decreasing in t and therefore the minimum is achieved when $t = 1$. So

$$k(t,s) \geq \beta(1-s).$$

Case 2. $s > \eta$. If $s > t$ then

$$k(t,s) = (1-s).$$

If $s \leq t$ then

$$k(t,s) = (1-s) - (t-s) = 1 - t \geq (1-b)(1-s)$$

Thus we can take

$$c = \frac{\min\{\beta, 1-b\}}{(1 + \frac{\beta}{1-\eta})}. \tag{5.3.4}$$

We can now state the following result on the existence of eigenvalues of Equation (5.3.1) with BC (5.3.2):

Theorem 5.3.1. *Let $\alpha < 0$, $[a,b] \subset [0,1)$, c be as in (5.3.4) and assume that there exists $\rho \in (0,r]$ such that:*

(i) There exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that

$$f(s,u) \geq m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \quad \text{and almost all } s \in [a,b],$$

(ii) $\sup_{t\in[a,b]} \int_a^b k(t,s)m_\rho(s)ds > 0$.

Then the boundary value problem (5.3.1)-(5.3.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a,b]$.

The case $0 < \alpha < 1 - \eta$.

When $\alpha > 0$ note that $k(1,s) = -\alpha < 0$ for every $s \in [0,\eta]$. We have to find Φ such that $|k(t,s)| \leq \Phi(s)$ for every $t,s \in [0,1]$ and show that there exists $[a,b] \subset [0,1]$ and a constant c such that $k(t,s) \geq c\Phi(s)$ for every $s \in [0,1]$ and $t \in [a,b]$. In fact we show that we can take

$$\Phi(s) = (1-s).$$

Upper bounds

Clearly $k(t,s) \leq (1-s)$ in all cases. $k(t,s)$ is negative when $s \leq \eta$ and $t \geq s$ and $1 - t - \alpha < 0$. In this case we have then

$$-k(t,s) = -1 + t + \alpha \leq \alpha < 1 - \eta \leq (1-s)$$

and we are done.

Lower bounds

We will show that we may take $[a,b] \subset [0,\eta]$.

Case 1. $s \leq \eta$. If $s > t$ then

$$k(t,s) = 1 - s - \alpha \geq (1 - \eta - \alpha)(1-s).$$

If $s \leq t$, since we chose $\alpha < 1 - \eta$, we obtain

$$k(t,s) = 1 - t - \alpha \geq 1 - \eta - \alpha \geq (1 - \eta - \alpha)(1-s).$$

Case 2. $s > \eta$. If $s > t$ then

$$k(t,s) = (1-s)$$

and we are done. Since we take $b \leq \eta$, the case $s \leq t$ does not occur.

Therefore we may set $c = (1 - \eta - \alpha)$.

Theorem 5.3.2. *Let $0 < \alpha < 1 - \eta$, $[a, b] \subset [0, \eta]$, $c = (1 - \eta - \alpha)$ and assume that there exists $\rho \in (0, r]$ such that:*

(i) There exists a measurable function $m_\rho : [a, b] \to \mathbb{R}_+$ such that

$$f(s, u) \geq m_\rho(s) \quad \textit{for all } u \in [c\rho, \rho] \quad \textit{and almost all } s \in [a, b],$$

(ii) $\sup_{t \in [a,b]} \int_a^b k(t, s) m_\rho(s) ds > 0.$

Then the boundary value problem (5.3.1)-(5.3.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a, b]$.

We illustrate the theorem with two simple examples.

Example 5.3.3. Let $[a, b] = [0, \eta]$ and $f(s, u(s))$ be defined as

$$f(s, u) = \begin{cases} |u(s)|(\eta - s), & 0 \leq s \leq \eta, \\ 0, & \eta < s \leq 1. \end{cases}$$

Take $0 < \rho \leq r < +\infty$ and $g_r = r\eta$. In this case we have $f(s, u) \leq g_r$ for every $u \in [-\rho, \rho]$ and $f(s, u) \geq c\rho(\eta - s)$ for $u \in [c\rho, \rho]$ and $s \in [0, \eta]$. Also

$$\int_0^\eta k(t, s) c\rho(\eta - s) ds \geq c^2 \rho \int_0^\eta (1 - s)(\eta - s) ds > 0.$$

By Theorem 5.3.2 we obtain the existence of a positive eigenvalue for the BVP (5.3.1)-(5.3.2).

Example 5.3.4. Let $f(s, u) \equiv 2$. For every fixed $\rho > 0$, $\lambda = (1 - 2\alpha\eta)/\rho$ is a positive eigenvalue of the boundary value problem (5.3.1)-(5.3.2) with corresponding eigenfunction

$$u(t) = \frac{(1 - 2\alpha\eta) - t^2}{\lambda}.$$

$u(t)$ is positive on $[0, \eta]$ since $\alpha < 1 - \eta$ and u changes sign ($u(1) < 0$).

5.4 Eigenvalues of problem (5.1.3b).

We now investigate the second BVP

$$\lambda u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{5.4.1}$$

with boundary conditions

$$u(0) = 0,\ \alpha u'(\eta) = u(1),\ 0 < \eta < 1,\ \alpha < 1 - \eta \tag{5.4.2}$$

The kernel in this case is

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha}, & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t \end{cases}$$

We study separately the cases when $\alpha < 0$ and $\alpha \le 1 - \eta$. The existence of positive eigenvalues when $\alpha = 0$ is covered by the results of Lan [31].

The case $\alpha < 0$.

In chapter four it has been shown that we can take

$$\Phi(s) = \max\left\{\frac{(1-\eta-\alpha)}{1-\eta}, -\frac{\alpha}{\eta}\right\}\frac{s(1-s)}{1-\alpha},$$

$[a, b] \subset (0, \eta]$ and $c = \min\{a, -\alpha\}/\max\{(1-\eta-\alpha), -\alpha/\eta\}$. Now it is clear that a theorem exactly similar to Theorem 5.3.1 holds, we leave the statement to the reader.

The case $0 < \alpha < 1 - \eta$.

In chapter four it has been shown that we may take

$$\Phi(s) = \max\left\{1, \frac{\alpha}{\eta}\right\}\frac{s(1-s)}{1-\alpha}, \ c = \frac{\min\{a(1-\eta-\alpha), (1-b-\alpha)\}}{\max\{1, \frac{\alpha}{\eta}\}}$$

and $[a, b] \subset (0, 1-\alpha)$. A result similar to Theorem 5.3.2 holds. We omit the obvious statement.

5.5 Eigenvalues of problem (5.1.3c).

We now investigate the BVP

$$\lambda u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{5.5.1}$$

with boundary conditions

$$u'(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1, \tag{5.5.2}$$

The kernel in this case is

$$k(t,s) = \frac{1}{1-\alpha}(1-s) - \begin{cases} \frac{\alpha}{1-\alpha}(\eta - s), & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \le t \\ 0, & s > t. \end{cases}$$

We shall study separately the cases $\alpha < 0$, $0 < \alpha < 1$ and $\alpha > 1$. The case $\alpha = 0$ has given by Lan in [31].

The case $\alpha < 0$.

In chapter three it has been shown that the kernel satisfies $|k(t,s)| \le (1-s)$ for every $s, t \in [0,1]$ and $k(t,s) > c(1-s)$ for every $t \in [a,b]$ and $s \in [0,1]$, where $[a,b] \subset [0,\eta]$ and $c = (1-\eta)/(1-\alpha)$. Therefore we can state the following theorem:

Theorem 5.5.1. *Let $\alpha < 0$, $[a,b] \subset [0,\eta]$, $c = (1-\eta)/(1-\alpha)$ and assume that there exists $\rho \in (0, r]$ such that:*

(i) There exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that

$$f(s,u) \ge m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \quad \text{and almost all } s \in [a,b],$$

(ii) $\sup_{t\in[a,b]} \int_a^b k(t,s) m_\rho(s) ds > 0.$

Then the boundary value problem (5.5.1)-(5.5.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a,b]$.

The case $0 < \alpha < 1$.

In [46] Webb proved that we can take

$$\Phi(s) = \frac{1-s}{1-\alpha},$$

$[a,b] \subset [0,1]$ and $c = \alpha(1-\eta)$. Thus we can state a similar result to Theorem 5.5.1. We omit the obvious statement.

The case $\alpha > 1$.

For these BCs the kernel k is negative on an interval so we apply Remark 5.2.4 and consider $-k$ in place of k. In chapter three it has been shown that we may take

$$\Phi(s) = \frac{\alpha}{\alpha - 1}(1-s)$$

and then $-k(t,s) > c\Phi(s)$ for $t \in [a,b]$ and $s \in [0,1]$, where $a = \eta$, $b \in (\eta, 1]$ and $c = (1-\eta)/\alpha$. Therefore we have the following result related to the existence of negative eigenvalues:

Theorem 5.5.2. *Let $\alpha > 1$, $[a,b]$ and c be as above and assume that there exists $\rho \in (0,r]$ such that:*

(i) There exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that

$$f(s,u) \geq m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \quad \text{and almost all } s \in [a,b],$$

(ii) $\sup_{t\in[a,b]} \int_a^b -k(t,s) m_\rho(s) ds > 0$.

Then the boundary value problem (5.5.1)-(5.5.2) has a negative eigenvalue and a corresponding eigenfunction that is negative on $[a,b]$.

We illustrate the theorem with the following example.

Example 5.5.3. Take $[a,b] = [\eta, 1]$, $c = (1-\eta)/\alpha$ and $f(s,u(s))$ be defined as

$$f(s,u) = \begin{cases} |u(s)|(s-\eta), & \eta \leq s \leq 1, \\ 1, & 0 \leq s < \eta. \end{cases}$$

The function f is positive and discontinuous, but satisfies Carathéodory conditions, $f(s,u) \leq g_r$, where $g_r = \max\{1, r\}$. Also $f(s,u) \geq c\rho(s-\eta)$ for $u \in [c\rho, \rho]$ and $s \in [\eta, 1]$. Clearly $\int_\eta^1 -k(t,s)(s-\eta)ds > 0$. By Theorem 5.5.2 the BVP (5.5.1)-(5.5.2) has a negative eigenvalue.

5.6 Eigenvalues of problem (5.1.3d).

We now investigate the BVP

$$\lambda u''(t) + f(t, u(t)) = 0, \quad (0 < t < 1), \tag{5.6.1}$$

with boundary conditions (BCs)

$$u(0) = 0, \ \alpha u(\eta) = u(1), \ 0 < \eta < 1. \tag{5.6.2}$$

The kernel in this case is

$$k(t,s) = \frac{1}{1-\alpha\eta}t(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha\eta}(\eta - s), & s \leq \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \leq t \\ 0, & s > t. \end{cases}$$

We shall study separately the cases $\alpha\eta < 0$, $0 < \alpha\eta < 1$ and $\alpha\eta > 1$. The case $\alpha = 0$ is covered by results of Lan [31].

The case $\alpha\eta < 0$.

In chapter three it has been shown that we can take

$$\Phi(s) = (1-\alpha)\frac{s(1-s)}{1-\alpha\eta},$$

$[a, b] \subset (0, \eta]$ and $c = \min\{a, 1-\eta\}/(1-\alpha)$. Now it is clear that a theorem exactly similar to Theorem 5.5.1 holds, we leave the statement to the reader.

The case $0 < \alpha\eta < 1$.

In [46] Webb proved that we can take

$$\Phi(s) = \max\{1, \alpha\}\frac{1-s}{1-\alpha\eta},$$

$[a, b] \subset (0, 1]$ and that for $\alpha < 1$ we may take $c = \min\{a, \alpha\eta, 4a(1-\eta), \alpha(1-\eta)\}$ and for $\alpha \geq 1$ we may take $c = \min\{a\eta, 4a(1-\alpha\eta)\eta, \eta(1-\alpha\eta)\}$. A result similar to Theorem 5.5.1 holds. We omit the obvious statement.

The case $\alpha\eta > 1$.

For these BCs the kernel k is negative on an interval so we apply Remark 5.2.4 and consider $-k$ in place of k. In chapter three it has been shown that we may take

$$\Phi(s) = \alpha\frac{s(1-s)}{\alpha\eta - 1}.$$

Indeed $-k(t,s) > c\Phi(s)$ for $t \in [a,b]$ and $s \in [0,1]$, where $[a,b] \subset [\eta,1]$ and $c = \min\{a, 1-\eta\}/\alpha$. A theorem exactly similar to Theorem 5.5.2 holds, we leave the statement to the reader.

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