HOMOLOGICAL PROPERTIES OF NOETHERIAN RINGS AND NOETHERIAN RING EXTENSIONS

By

ZHONG YI

A Thesis Submitted for the Degree of Doctor of Philosophy

in the University of Glasgow

Department of Mathematics

University of Glasgow

July, 1993

TO

JING AND SAN

ACKNOWLEDGEMENTS

I am most grateful to Professor K. A. Brown, my supervisor, for his direction and encouragement for all this work. It is he who stimulated my interest for this topic. Throughout the preparation of this thesis he gave me a lot of useful ideas and suggestions. It is also my pleasure to thank him for his consideration and kindness to my life in Glasgow.

I would like to thank the University of Glasgow and the ORS Committee for their financial support for my Ph.D. project from October 1990 to September 1993, and this Mathematics Department for its support to let me attend many conferences and seminars throughout the United Kingdom and in Europe.

My thanks are also offered to Guangxi Teachers' University and its Mathematics Department for their support during my stay in the United Kingdom.

HOMOLOGICAL PROPERTIES OF NOETHERIAN RINGS AND NOETHERIAN RING EXTENSIONS

TABLE OF CONTENTS

SUMMARY

This thesis is devoted to the study of the homological dimension, homological homogeneity and injective homogeneity of the skew group rings, crossed products, group graded rings and the Ore extensions; and to the study of the Auslander-Gorenstein, the Auslander-regular and the Macaulay properties of the injectively homogeneous rings and the homologically homogeneous rings.

In chapter 2, we study the global dimension of skew group rings, crossed products and group graded rings. About the global dimension of a crossed product of a finite group over a right *FBN* and left coherent ring, we obtain

THEOREM A. *Suppose that* R *is a right* *FBN* *left coherent ring and that* $r.gl.dim.(R) < \infty$. *Let* G *be a finite group and let* $S = R*G$ *be a crossed product. Suppose that for each maximal ideal* M *of* R *with* $char(R/M) = p > 0$, $(R/M)*G_M$ *is semisimple Artinian, where*

$$G_M = \{g \in G \mid M^g = M\}.$$

Then $r.gl.dim.(R*G) = r.gl.dim.(R) < \infty$.

Some necessary conditions and sufficient conditions for $R*G$, a skew group ring of a finite group over a local or semilocal right Noetherian

Summary

ring, to have finite right global dimension are also given. In particular if R is commutative Noetherian and G is finite, we obtain some necessary and sufficient conditions for $R*G$, a skew group ring, to have finite global dimension. Then we further extend these results to larger classes of groups and prove, for example, the following

THEOREM B. *Let R be a commutative Noetherian ring, let G be a polycyclic-by-finite group acting on R and let $R*G$ be the skew group ring. Then the following statements are equivalent:*

(i) $gl.dim.(R*G) < \infty$;

(ii) (a) $gl.dim.(R) < \infty$;

(b) for every maximal ideal M of R with $char(R/M) = p > 0$ and for every finite subgroup T of G, $(R/M)*T_M$ is semisimple Artinian, where

$$T_M = \{g \in T \mid M^g = M\};$$

(iii) (a) $gl.dim.(R) < \infty$;

(b) for every maximal ideal M of R with $char(R/M) = p > 0$, $G(M)$ contains no element of order p, where

$$G(M) = \{g \in G \mid r^g - r \in M, \text{ for all } r \in R\}.$$

In chapter 3 we first study the injective homogeneity of crossed products, then use the smash products machinery to extend our results to strongly group graded rings and obtain

THEOREM C. *Let G be a finite group and let $S = R(G)$ be a strongly*

Summary

G-graded ring with coefficient ring R. Then S is right injectively homogeneous (respectively right injectively smooth) FBN if and only if so is R.

Then we study the injectively homogeneous Noetherian P. I. rings. It is proved that such rings are always Auslander-Gorenstein. We also give some necessary and sufficient conditions to injective homogeneity and homological homogeneity for a Noetherian P. I. ring all of whose cliques of maximal ideals are localizable.

In chapter 4, We come to study the Auslander-Gorenstein, the Auslander-regular and the Macaulay properties of injectively homogeneous and homologically homogeneous Noetherian rings which are integral over their centres. The main result in this aspect is

THEOREM D. *Let R be a Noetherian ring integral over its centre.*

(i) R is inj. hom. if and only if R is Auslander-Gorenstein and locally Macaulay.

(ii) R is hom. hom. if and only if R is Auslander-regular and locally Macaulay.

The final chapter, chapter 5, is devoted to the study of the homological homogeneity and the injective homogeneity of the Ore extensions. We show that in many cases the homological homogeneity and the injective homogeneity of the coefficient rings can be passed to the Ore extensions. The results can be summarized as

Summary

THEOREM E. *Let* R *be a ring, let* σ *be an automorphism of* R *and let* δ *be a* σ*-derivation of* R*. Let* S *be the ring* $R[x; \sigma, \delta]$ *or* $R[x, x^{-1}; \sigma]$*. Suppose that* R *is a finitely generated module over a central subring* K*, which is* (σ, δ)*-trivial, and suppose that* S *is also a finitely generated module over its own centre.*

(i) *If* R *is inj. hom., then* S *is also inj. hom. and*

$$inj.dim.(S_S) = inj.dim.(R_R) + 1.$$

(ii) *If* R *is hom. hom., then* S *is also hom. hom. and*

$$gl.dim.(S) = gl.dim.(R) + 1.$$

INTRODUCTION

In ring theory, the investigation of the variations of some ring properties under different extensions and restrictions and the study of the homological properties of rings are two classical and active research directions. This thesis is mainly aimed at these two topics. Specifically, here in our research, the ring extensions are skew group rings, crossed products, group graded rings and the Ore extensions, and the restrictions are to the coefficient rings of these rings. Those ring properties which we are interested in are the global dimensions, the homological homogeneity, the injective homogeneity, the Auslander-Gorenstein property, the Auslander-regular property and the Macaulay property.

From the very beginning of the development of homological algebra, global dimension played a prominent role. It is a very useful invariant attached to a ring. Especially, as demonstrated by many classical results, the finiteness of global dimension is a very important property for a ring. For example, a commutative Noetherian local ring of finite global dimension is a unique factorization domain, as a result of celebrated work of M. Auslander and D. Buchsbaum [ABu2], and some difficult problems about non-commutative Noetherian rings can be solved under the condition of finite global dimension; see [GW, p.287] for example. Moreover, the study of (Noetherian) rings of finite global dimension is a popular and active current area of research; see [AS], [ATV1], [ATV2] and [SZ] for example.

About the study of the global dimension of group rings, we really have to trace back to Maschke's Theorem, as it characterizes the group rings of global dimension 0. Then we have Serre's Theorem (see [Pa1, Theorem 10.3.12]), which relates the global dimension of a group ring to that of a subgroup ring. E. Aljadeff and S. Rosset studied the global dimension of crossed products over commutative rings. They show that the hierarchy of extensions, first group rings, then skew group rings and then crossed products, corresponds to an increasing likelihood that the global dimension will be finite; see [AR, 3.3 and 3.4]. So the moral is that the more structure on an extension ring the more likely it will have finite global dimension. In [Al2] Aljadeff proved a version of Serre's Theorem for crossed products over commutative rings. About the global dimension of crossed products there are also a few results in [MR] and [Al1].

In the study of global dimension (and some other properties) of group rings, skew group rings, crossed products and group graded rings, the case that the group is finite plays an essential role since the elements of finite order (especially when the order divides the characteristic of the coefficient ring) affect the structure quite a lot. Serre's Theorem mentioned above is a useful tool to transfer some of the results from finite groups to larger classes of groups.

Introduction

Comparing with the profound study of the primeness, semiprimeness and various quotient rings of skew group rings, crossed products and the group graded rings, we know much less about the global dimensions of these ring extensions. After briefly recalling the definitions and fixing our notations in chapter 1, we start our own research on this topic in chapter 2. Chapter 2 is mainly devoted to the study of the necessary conditions and sufficient conditions under which the global dimensions of the skew group rings, crossed products and group graded rings are finite. We first give two general lemmas about the global dimensions of strongly group graded rings. Then, as the first step of the study of the global dimension of crossed products, we study their simplicity and semisimplicity properties. It is shown that the semisimplicity of crossed products can be reduced to the simplicity of crossed products over some simple Artinian rings. (See the introduction of § 2.2 for the background on this topic.)

In § 2.3 we prove that for a right *FBN* and left coherent ring R, the global dimension of a crossed product over R is controlled by the semisimplicity of some crossed products over some Artinian factor rings of R. If the coefficient ring is local or semilocal, in § 2.4, some necessary conditions and sufficient conditions about the finiteness of the global dimension of crossed products or skew group rings are given. When the coefficient ring is commutative Noetherian, in § 2.5, we obtain some necessary and sufficient conditions for a skew group ring to have finite global dimension. This result can be regarded as a considerable generalization of the result of Maschke mentioned on the previous page.

Then using the generalized Serre's Theorem [Al1, Theorem 0.3], we can extend our results to some larger classes of groups. Some examples of rings are given in § 2.6. These examples are used to illustrate that the conditions and results appearing in our main theorems are best possible. In § 2.7 we describe some relationships between the trace maps and the global dimension of skew group rings and strongly group graded rings. Having investigated the finiteness of global dimensions of skew group rings and crossed products to some extent, we then indicate that, in § 2.8, unlike global dimension, the finitistic dimensions of strongly group graded rings are always stable.

In chapter 3, we study the injective homogeneity and the Auslander-Gorenstein property. This topic has two sources. The first one is homologically homogeneous rings and injectively homogeneous rings, which are introduced for Noetherian rings which are integral over their centres by K. A. Brown and C. R. Hajarnavis in [BH1] and [BH2]. Recently Stafford and Zhang [SZ] generalized the definition to all *FBN* rings. The homologically homogeneous rings and the injectively homogeneous rings are appropriate generalizations of commutative regular rings and the commutative Gorenstein rings respectively. In [BH1], [BH2] and [SZ] the authors have shown that the homologically homogeneous rings and the injectively homogeneous rings share many nice properties of the commutative regular rings and the commutative Gorenstein rings respectively. The second resource is the Auslander-Gorenstein, the Auslander-regular, and the Macaulay properties; cf. [Ek], [Bj], [Lev] and [SZ]. The Auslander condition is a useful property to let one use homological methods effectively on noncommutative Noetherian rings. In the introduction of chapter 3 we will give more details about the history and background of the Auslander condition.

The recent research by Stafford and Zhang [SZ] shows that injective homogeneity, homological homogeneity, the Auslander-Gorenstein property, the Auslander-regular property and the Macaulay property are closely related to one another. In [SZ] it is shown that the injectively smooth Noetherian P. I. rings are Auslander-Gorenstein and Macaulay. We prove that the converse is also true. We also generalize some results in [SZ] to give an equivalent condition to injective homogeneity for a Noetherian P. I. ring all of whose cliques of maximal ideals are localizable. Using the tool of smash product, in this chapter, we prove that a strongly group graded ring by a finite group is *FBN* and injectively homogeneous if and only if so is its coefficient ring.

In chapter 4, we prove that for a Noetherian ring integral over its centre, it is injectively homogeneous if and only if it is Auslander-Gorenstein and locally Macaulay. This result is parallel to that about the injectively homogeneous Noetherian P. I. rings given in chapter 3; but the proof is different. We use localization at central elements and the C-grade of ideals, which are developed in [BHM2].

In chapter 5, we first investigate the Artinian quotient rings and the injective dimension of the Ore extensions. We prove that the Ore extension has an Artinian quotient ring if its coefficient ring has. With some natural hypothesis we prove that an Ore extension is injectively homogeneous or homologically homogeneous if so is its coefficient ring.

Introduction

Particularly, it is shown that a large class of Weyl algebras are injectively homogeneous or homologically homogeneous.

In the text most of our references are quoted from the original authors. Only for simplicity and brevity reasons refer we sometimes to the books [GW], [MR] and [Pa2] for some well-known results. The internal references are given in such a way that, for example, 5.2 Theorem means 5.2 Theorem in the current chapter, whereas 2.5.2 Theorem indicates 5.2 Theorem in chapter 2.

Finally, we would like to remark that if not otherwise stated all the results presented in this thesis are the author's original work under the direction of Professor K. A. Brown. As a convenient reference, a short section named notes is placed at the end of each chapter to indicate whether a result appearing in that chapter is well-known or is a new one. Most parts of chapter 2 will soon appear in the *Journal of Algebra* with the title *Homological dimension of skew group rings and crossed products*; cf. [Yi1]; and chapter 3 has been organized into a paper, (submitted for publication), with the title *Injective homogeneity and the Auslander-Gorenstein property*; see [Yi2].

CHAPTER 1

PRELIMINARIES

We would like to explain our terminology, fix our notation and state a few well-known results in this preliminary chapter. These are the ones most frequently used in this thesis. Some other terminology and notation will be explained when they appear the first time in the text. Our main references for the definitions, notation and well-known results are [GW], [MR], [Rot], [Pa1] and [Pa2].

§ 1.1 NOTATION AND CONVENTIONS

In this thesis all rings are associative rings with identity elements (the identity element is usually denoted by 1). Usually, we use the letter R to denote a ring. A subring of a ring R always contains the identity element of R. A ring homomorphism from a ring R_1 to a ring R_2 always maps the identity element of R_1 to the identity element of R_2. Modules (either right or left) are unitary modules. Module homomorphisms will be written on the side opposite to that of the ring action on the module. Suppose that R and S are rings. We use M_R (respectively ${}_RM$ and ${}_SM_R$) to denote that M is a right R-module (respectively left R-module and (S, R)-bimodule). If we state a concept, which has both a left hand side and a right hand side version, without a prefix, we always mean it holds on both sides. For example, a Noetherian ring means a left and right Noetherian ring, and an ideal means a left and right ideal, etc. Suppose that R is a ring and n is a positive integer. We use $M_n(R)$ to denote the ring of all $n \times n$ matrices with entries in R.

For a ring R and a (right or left) R-module M, the projective dimension and injective dimension of M are denoted by $pr.dim._R(M)$ and $inj.dim._R(M)$ respectively. If there is no ambiguity we may omit the ring R and simply denote them by $pr.dim.(M)$ and $inj.dim.(M)$ respectively. The right (respectively left) global dimension of R is denoted by $r.gl.dim.(R)$ (respectively $l.gl.dim.(R)$). If the right global dimension and the left global dimension are equal, we simply denote the common value by $gl.dim.(R)$. For the definitions of these concepts, see [Rot, p.233 and p.235]. The injective dimension of the ring R as a right (respectively left) R-module is denoted by $r.inj.dim.(R)$ (respectively $l.inj.dim.(R)$), and simply by $inj.dim.(R)$ if the two values are the same.

§ 1.2 SKEW GROUP RINGS, CROSSED PRODUCTS AND GROUP GRADED RINGS

Let G be a group and let R be a ring. It is well-known that we can form the group ring of G over R, denoted by $R[G]$; see [Pa1]. Suppose that N is a normal subgroup of G, then the group ring $R[N]$ of N over R is a subring of $R[G]$. It turns out that $R[N]$ and $R[G]$ have some special relationships as described in the following more general concept.

2.1 DEFINITION. *Let R be a ring with identity and let G be a multiplicative group. A crossed product of G over R, denoted by $R*G$, is an associative ring which is a free right R-module with basis the set $\bar{G}$, a copy of G. Thus $R*G = \oplus_{g\in G}\bar{g}R$. Addition in $R*G$ is the same as in the module structure and multiplication is defined distributively by:*

$$\bar{g}\bar{h} = \overline{gh}\alpha(g, h), \text{ for all } g, h \in G,$$

where $\alpha: G\times G \longrightarrow U(R)$, the group of units of R; and

$$r\bar{g} = \bar{g}r^{t(g)}, \text{ for all } r \in R \text{ and all } g \in G,$$

where $t: G \longrightarrow Aut(R)$.

In order that the maps α and t define a crossed product, that is, the multiplication defined above is associative, α and t need to satisfy some special relationships. For example, we have the following

2.2 LEMMA. [Pa2, Lemma 1.1] *The associativity of* $R*G$ *is equivalent to the assertions that for all* $x, y, z \in G$

(i) $\alpha(xy, z)\alpha(x, y)^{t(z)} = \alpha(x, yz)\alpha(y, z)$;

(ii) $t(y)t(z) = t(yz)\eta(y, z)$, *where* $\eta(y, z)$ *denotes the automorphism of* R *induced by the unit* $\alpha(y, z)$. ∎

2.3 REMARKS. (i) Let R be a ring and let G be a group with a normal subgroup N. It is easy to see that the group ring $R[G]$ is a crossed product of the quotient group G/N over the group ring $R[N]$. (In most cases both maps t and α are not trivial.)

(ii) In the definition of crossed product the map t is in general not a group homomorphism. By 2.2 Lemma (ii) t is a group homomorphism if and only if for all $y, z \in G$, $\alpha(y, z)$ is in the centre of R.

In the definition of crossed product, if α is trivial, that is $\alpha(g, h) = 1$ for all $g, h \in G$, the crossed product $R*G$ is called a *skew group ring*. If t is trivial, that is $t(g)$ is the identity map for every g in G, $R*G$ is called a *twisted group ring*. If both α and t are trivial, then the crossed product becomes the ordinary group ring. Usually we write $r^{t(g)}$ simply as r^g. We may write $R*G$ as $R_t^{\alpha}G$ in case we need to point out the maps α and t. For more details and the basic properties of skew group rings and crossed products, see [Mo1] and [Pa2].

Let R be a ring and let t be an automorphism of R. We call t an *inner* automorphism if there exists a unit $u \in R$ such that

$t(r) = u^{-1}ru$ for all $r \in R$. An automorphism is called *outer* if it is not inner. (See [Mo1, p.1] for details.) Let R be a ring, let G be a group and let $R*G = R_t^{\alpha}G$ be a crossed product. As pointed out in 2.3 Remarks (ii) the map t is in general not a group homomorphism; but it is easy to see that t yields a group homomorphism $G \longrightarrow Aut(R)/Inn(R)$, where $Inn(R)$ is the group of inner automorphisms of R. This gives G an action, in the proper sense, on the set of ideals of R.

Let $R*G$ be a crossed product and let I be an ideal of R. For every element $g \in G$, we define

$$I^g = \{a^g \mid a \in I\}$$

and

$$G_I = \{g \in G \mid I^g = I\}.$$

It is obvious that I^g is an ideal of R and G_I is a subgroup of G. We call G_I the *invariant group* of I. If $I^g = I$ for all $g \in G$, that is $G_I = G$, then I is said to be *G-invariant*. It is easy to see that G_I is the unique largest subgroup H of G such that I is H-invariant. The ring R is said to be *G-prime* if the product of any two non-zero G-invariant ideals is also non-zero.

In order to explain some other concepts, we give the following

2.4 PROPOSITION. [Pa2, Proposition 10.4] *Let R be a prime ring. Then there exists a ring $Q_s(R)$ uniquely determined by the properties*

(i) $Q_s(R) \supseteq R$ with the same identity element;

(ii) if $q \in Q_s(R)$ then there exist non-zero ideals A, B of R such that Aq, $qB \subseteq R$;

(iii) if $q \in Q_S(R)$ and I is a non-zero ideal of R, then either $Iq = 0$ or $qI = 0$ implies $q = 0$;

(iv) let $f\colon {}_RA \longrightarrow {}_RR$ and $g\colon B_R \longrightarrow R_R$ be given, where A, B are non-zero ideals of R. Suppose that for all $a \in A$ and all $b \in B$ we have $(af)b = a(gb)$. Then there exists $q \in Q_S(R)$ with $af = aq$ and $gb = qb$ for all $a \in A$, $b \in B$. ■

Let R be a prime ring. The ring $Q_S(R)$ determined in 2.4 Proposition is called the *symmetric Martindale ring of quotients*; see [Pa2, section 10] for details.

Let R be a prime ring and let σ be an automorphism of R. Then σ is said to be *X-inner* if there exists a unit $q \in Q_S(R)$ with $r^{\sigma} = q^{-1}rq$ for all $r \in R$, and σ is called *X-outer* if it is not X-inner; see [Pa2, p.107]. Analogously, we can also define X-inner and X-outer automorphisms for semiprime rings; see [Mo1, Chapter3] for details. Suppose that R is a prime ring and $R*G$ is a crossed product. Define

$$G_{inn} = \{g \in G \mid \bar{g} \text{ is X-inner on } R\}.$$

Then G_{inn} is a normal subgroup of G; cf. [Pa2, Lemma 12.3 (iii)]. We say G is *X-inner* on R if $G_{inn} = G$. Similarly, G is called *X-outer* on R if $G_{inn} = \langle 1 \rangle$. As shown in many works (see [Pa2, section 12] for example) G_{inn} plays an important role in the determination of the structure of $R*G$.

In our study of crossed products two classes of groups will play key roles. First, of course, there are the finite groups; next there are the polycyclic-by-finite groups as defined in the following

2.5 DEFINITION. *A group* G *is called polycyclic-by-finite if* G *has a series of subgroups*

$$G = G_n \geq G_{n-1} \geq \dots \geq G_0 = 1$$

with $G_{i+1} \triangleright G_i$ *and each* G_{i+1}/G_i *either infinite cyclic or finite,* $0 \leq i < n.$

As the foundation of our research, there exists a version of Hilbert's Basis Theorem for crossed products.

2.6 PROPOSITION. [Pa2, Proposition 1.6] *If* R *is a right Noetherian ring and* G *is a polycyclic-by-finite group, then* $R*G$, *a crossed product, is also right Noetherian.* ■

2.7 REMARKS. (i) It is still an open question whether the converse of the above proposition is also true for group rings, that is whether $R[G]$, the group ring of G over R, Noetherian requires G polycyclic-by-finite; cf. [MR, p.25].

(ii) The converse of the above proposition is not true for twisted group rings, as shown by the following

2.8 EXAMPLE. Let k be a field, let $G = \langle\mathbb{Q}, +\rangle$ be the additive group of all the rational numbers and let $H = \langle\mathbb{Z}, +\rangle$ be the additive group of all the integers. Thus H is a subgroup of G. It is obvious that G/H

is an infinite locally finite group, so G/H is not polycyclic-by-finite. It is clear that $k[G]$ is a commutative domain. Let $\mathscr{C} = k[H] \setminus \{0\}$. Then $\mathscr{C}$ is a multiplicatively closed subset of $k[G]$, and it is easy to see that

$$k[G]\mathscr{C}^{-1} = k[H]\mathscr{C}^{-1}*(G/H),$$

which is the quotient field of $k[G]$. Clearly, $k[H]\mathscr{C}^{-1}*(G/H)$ is a twisted group ring.

Many properties of crossed products are shared by a more general class of ring extensions, the group-graded rings, which were first introduced by E. C. Dade in [Da].

2.9 DEFINITION. *Let G be a multiplicative group with identity element 1 and let R be a ring with identity. We call R a G-graded ring if there is a family of additive subgroups $\{R_g \mid g \in G\}$ of R, such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If further $R_g R_h = R_{gh}$ for all $g, h \in G$, then R is called strongly G-graded.*

It is clear that for a G-graded ring R, R_1, the component corresponding to the identity element of G, always contains the identity element of R; see [Da, Proposition 1.4]; and that R_1 is a subring of R; see [Da, 1.3 (a)]. We call R_1 the *coefficient ring of* R. For simplicity we may denote R as $R = R_1(G)$.

Let G be a group and let R be a G-graded ring. Suppose S is a subset of G. We denote $R_S = \sum_{g\in S} R_g$. If S is a subgroup of G, then, obviously, R_S is a subring of R and R_S is an S-graded ring. (For more details and the classical results about group graded rings, see [NsV].)

We can easily to check that R is a strongly G-graded ring if and only if $1_R \in R_g R_y$ for all $g \in G$, where $y = g^{-1}$ for typographical reasons. It is obvious that every crossed product is a strongly group graded ring; but the converse is not true in general as the following example demonstrates.

2.10 EXAMPLE. [Pa2, Exercise 3 on p.18] Let k be a field, let $S = M_3(k)$ and let $G = \{1, x\}$ be a group of order 2. Then S is a strongly G-graded ring by the decomposition $S = S_1 \oplus S_x$, where

$$S_1 = \begin{pmatrix} k & k & 0 \\ k & k & 0 \\ 0 & 0 & k \end{pmatrix} \text{ and } S_x = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ k & k & 0 \end{pmatrix}.$$

But this decomposition is not a crossed product since $dim._k(S_1) = 5$ while $dim._k(S_x) = 4$.

In fact, about the relationships between group graded rings and crossed products, we have the following

2.11 LEMMA. [Pa2, Exercise 2 on p.18] *Let G be a group and let R be a G-graded ring with decomposition $R = \oplus_{g\in G} R_g$. Then R is a crossed product of G over R_1 with the same decomposition if and only if R_g contains a unit of R for each $g \in G$.* ■

§ 1.3 ORE EXTENSIONS

Let R be a ring and let σ be an endomorphism of R. A (left) *σ-derivation* of R is an additive map $\delta: R \longrightarrow R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$.

3.1 PROPOSITION. [GW, Proposition 1.10] *Let R be a ring, let σ be an endomorphism of R and let δ be a σ-derivation of R. Then there exists a ring S, containing R as a subring, such that S is a free left R-module with a basis of the form $1, x, x^2, \dots$ and $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.* ∎

The ring S determined in the above proposition is denoted as $R[x; \sigma, \delta]$ and is called an *Ore extension* of R. Analogously, for a ring R and an endomorphism σ of R, we can also define a *right σ-derivation*, which is an additive map δ of R satisfying the rule: $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$ for all $r, s \in R$. By using a right σ-derivation δ, we can also construct an Ore extension, which is a free right R-module. Suppose that σ is an automorphism of R and δ is an additive map of R. Then it is clear that δ is a left σ-derivation if and only if $-\delta\sigma^{-1}$ is a right σ^{-1}-derivation and the Ore extensions $R[x; \sigma, \delta]$ and $R[x; \sigma^{-1}, -\delta\sigma^{-1}]$ coincide.

We would like to point out that (as discussed, for example, in

[MR, § 1.2]) if a ring S is generated by a subring ring R and an element x, and if

(i) ${}_R S$ is a free left R-module with a basis $\{x^i \mid i \geq 0\}$, and

(ii) $xR \subseteq Rx + R$,

then there exist an endomorphism σ of R and a σ-derivation δ such that $S = R[x; \sigma, \delta]$. (Hence justifying the concept.)

We also have a version of Hilbert's Basis Theorem for Ore extensions.

3.2 THEOREM. [GW, Theorem 1.12] *Let R be a ring, let σ be an automorphism of R and let δ be a σ-derivation of R. If R is right (resp. left) Noetherian, then the Ore extension $S = R[x; \sigma, \delta]$ is also right (resp. left) Noetherian.* ■

3.3 REMARK. In order that the above theorem is true, σ must be an automorphism. The following example shows that assuming σ is a monomorphism is not enough to ensure the truth of the theorem.

3.4 EXAMPLE. [GW, Exercise 1N] Let $R = k[x]$ be a polynomial ring over a field k, and let σ be the k-algebra endomorphism of R given by the rule $\sigma(f) = f(x^2)$. Then $R[y; \sigma]$ is neither right nor left Noetherian.

§ 1.4 NOTES

4.1 All the concepts and results appearing in this chapter are well-known.

4.2 [Rot] and [CE] are the main references for homological algebra results and concepts.

4.3 [Pa2] and [Mol] are our main references for crossed products and skew group rings. All the concepts and results appearing in section 1.2 can be found in [Pa2].

4.4 For Ore extensions, we use [GW] and [MR] as main references. All the results and concepts appearing in section 1.3 can be found in [GW] or [MR].

CHAPTER 2

FINITENESS OF GLOBAL DIMENSIONS OF SKEW GROUP RINGS AND CROSSED PRODUCTS

Let R be a ring with finite global dimension, let G be a group and let $R*G$ be a crossed product. It is well-known that the global dimension of $R*G$ may be infinite. For example the group ring $K[G_p]$, where K is a field of characteristic $p > 0$ and G_p is a cyclic group of order p, has infinite global dimension (cf. 1.5 Lemma (i) below). In this chapter we would like to study the necessary conditions and sufficient conditions for a skew group ring and a crossed product to have finite global dimension. The main purpose is to demonstrate that the global dimension of certain skew group rings and crossed products are controlled by the global dimension of certain skew group rings and crossed products of finite groups over simple Artinian rings.

This chapter contains many of our main results of this thesis. In § 2.1 we give several general results about the global dimension of strongly group graded rings. These results are useful in later sections and chapters. In § 2.2, as the first step to study the finiteness of global dimension of crossed products, we study the conditions under which the global dimension of a crossed product is zero, that is, the conditions under which a crossed product is semisimple Artinian. Our 2.3 Theorem shows that the semisimplicity of crossed products can be reduced to the simplicity of some crossed products over simple Artinian rings. We also give a result to describe the semisimplicity of crossed products over some factor rings of the coefficient rings; see 2.6 Proposition. In § 2.3 we show that for a crossed product of a finite group over a right *FBN* and left coherent ring (see § 2.3 below for the definitions of *FBN* rings and coherent rings), its global dimension is controlled by the semisimplicity of some crossed products over the simple Artinian factors of the coefficient ring; see 3.3 Theorem for details. In § 2.4 we study the finiteness of global dimension of crossed products and skew group rings when the coefficient rings are Noetherian with all their maximal ideals right localizable or the coefficient rings are local or semilocal. In these cases some necessary conditions and sufficient conditions for a crossed product or a skew group ring to have finite global dimension are obtained; see 4.6 Theorem, 4.10 Proposition and 4.12 Corollary for details. In § 2.5 we apply our previous results to the case of commutative coefficient rings and obtain some necessary and sufficient conditions for a skew group ring of a finite group over a commutative Noetherian ring to have finite global dimension; see 5.2 Theorem. Then we use a version of Serre's Theorem for crossed products, which was given by Aljadeff in [Al2] (see 5.5 Theorem), to extend our results to larger classes of groups. For example, for polycyclic-by-finite groups we obtain 5.7 Corollary. Then we give some examples in § 2.6 to show that the conditions in our main results are essential and the results are the best possible. In § 2.7 we describe some relationships between the trace maps (see § 2.7 for the definition) and the global dimension of strongly group graded rings and skew group rings; see 7.2 Proposition for example. In § 2.8 we briefly describe the finitistic dimension of strongly group graded rings; see 8.4 Theorem.

§ 2.1 GLOBAL DIMENSION OF STRONGLY GROUP GRADED RINGS

We would like to give two elementary lemmas concerning the global dimension of strongly group graded rings in this section. They will be frequently used later.

1.1 LEMMA. *Let* G *be an arbitrary group and let* $R = \oplus_{g\in G}R_g$ *be a strongly* G*-graded ring. Suppose* H *is a subgroup of* G.

(*i*) *For any* $g \in G$, $R_{gH} = \oplus_{h\in H}R_{gh}$ *is a finitely generated projective right* R_H*-module. In particular* R *is a projective right* R_H*-module.*

(*ii*) $r.gl.dim.(R_1) \leq r.gl.dim.(R_H) \leq r.gl.dim\ (R)$.

PROOF. Suppose that R, G and H are as stated.

(i) Let $g \in G$ and write y for g^{-1} for typographical reasons.

Obviously R_{gH} is a right R_H-module. Since $R_gR_y = R_1 \ni 1$, there exist finitely many $r_i \in R_g$ and $s_i \in R_y$, such that $\sum_{i=1}^{n} r_is_i = 1$. Then

$$f_i : R_{gH} \longrightarrow R_H : r \longrightarrow s_ir, \text{ for all } r \in R_{gH},$$

is an R_H-homomorphism. Therefore $\sum_{i=1}^{n} r_if_i(r) = r$ for all $r \in R_{gH}$, and by the dual basis lemma (see [MR, Lemma 3.5.2]) R_{gH} is a finitely generated projective R_H-module.

(ii) This follows directly from (i) and [MR, Theorem 7.2.8]. ∎

1.2 REMARK. In the above lemma (ii) is a generalization of [MR, Corollary 7.5.6 (i)].

Let G be a finite group and let $R = \oplus_{g\in G}R_g$ be a strongly G-graded ring. Then $R_yR_g = R_1$ for all $g \in G$, where we write y for g^{-1} as before. Thus there exist $a_i^g \in R_y$ and $b_i^g \in R_g$ such that $\sum_{I_g} a_i^gb_i^g = 1$, where I_g is a finite set. Suppose M and N are right R-modules. If $f \in Hom_{R_1}(M, N)$, we define a map from M to N by

$$\tilde{f}(m) = \sum_{g\in G}\sum_{I_g} (f(ma_i^g))b_i^g \text{ , for all } m \in M. \tag{*}$$

By direct calculation, we have the following

1.3 LEMMA. [Na, Lemma 2.1] *Let G be a finite group, let R be a strongly G-graded ring and let M and N be right R-modules. Suppose that $f \in Hom_{R_1}(M, N)$. Then $\tilde{f}$, which is defined as in (*), is an R-homomorphism.* ∎

Chapter 2

Suppose that G is a finite group and R is a strongly G-graded ring. Let M be a right R-module. If there exists an $f \in Hom_{R_1}(M, M)$ such that $\tilde{f} = 1_M$, then M is called *R-regular* (cf. [Na, Section 2]). Suppose that $|G|$ is invertible in R_1. Let $n = |G|$. It is easy to see that $(\widetilde{1/n}) = 1_M$ for any right R-module M. Thus all the right R-modules are R-regular if $|G|$ is invertible in R_1.

Before giving our next result, we state the following well-known theorem. It is useful in later sections and chapters.

1.4 THEOREM. [Na, Theorem 2.1] *Let G be a finite group, let R be a strongly G-graded ring with coefficient ring R_1, let M be a right R-module and let $N = \oplus_{g \in G} N_g$ be a graded right R-module. Then for each non-negative integer n and each $g \in G$, we have*

(i) $Ext^n_R(M, N) \cong Ext^n_{R_1}(M, N_g)$;

(ii) $Ext^n_R(N, M) \cong Ext^n_{R_1}(N_g, M)$;

(iii) $pr.dim._{R_1}(M) \le pr.dim._R(M)$, *and the equality holds if* $pr.dim._R(M) < \infty$;

(iv) $inj.dim._{R_1}(M) \le inj.dim._R(M)$, *and the equality holds if* $inj.dim._R(M) < \infty$. ∎

Now we can prove the following

1.5 LEMMA. *Let G be a finite group, let $R = \oplus_{g \in G} R_g$ be a strongly G-graded ring and let H be a subgroup of G.*

(i) If $r.gl.dim.(R) < \infty$, *then*

$$r.gl.dim.(R_1) = r.gl.dim.(R_H) = r.gl.dim.(R).$$

(ii) Suppose that $|G|$ *is invertible in* R_1 . *Then for every right* R*-module* P, P_{R_1} *is projective (respectively injective) if and only if* P_R *is projective (respectively injective). Moreover, for every right* R*-module* M,

$$pr.dim.(M_R) = pr.dim.(M_{R_1}); \quad inj.dim.(M_R) = inj.dim.(M_{R_1}).$$

(iii) If $|G|$ *is invertible in* R_1 , *then*

$$r.gl.dim.(R_1) = r.gl.dim.(R_H) = r.gl.dim.(R).$$

PROOF. (i) Suppose that $r.gl.dim.(R) < \infty$. Let M be a right R-module such that $pr.dim.(M_R) = r.gl.dim.(R)$. By 1.4 Theorem (iii) we have

$$pr.dim.(M_{R_1}) = pr.dim.(M_R) = r.gl.dim.(R).$$

Therefore by 1.1 Lemma (ii), we have $r.gl.dim.(R_1) = r.gl.dim.(R)$. Similarly, $r.gl.dim.(R_1) = r.gl.dim.(R_H)$.

(ii) Let $n = |G|$ and suppose that n is invertible in R_1 . Let P be a right R-module. If P_R is projective, then P_{R_1} is projective since R_{R_1} is projective by 1.1 Lemma (i). Suppose that P_{R_1} is projective. Let N be a right R-module and let

$$N \xrightarrow{f} P \longrightarrow 0 \qquad (1)$$

be an exact sequence of R-modules and R-homomorphisms. Since P_{R_1} is projective, (1) splits as a sequence of R_1-modules and R_1-homomorphisms. Thus there exists an R_1-homomorphism g, say, from P to N such that $fg = 1_P$. By simple calculation, we have

$f(n^{-1}\tilde{g}) = 1_P$, where $\tilde{g}$ is defined as in (*). By 1.3 Lemma $n^{-1}\tilde{g}$ is an R-homomorphism. Thus (1) splits as a sequence of R-modules and R-homomorphisms and so P_R is projective.

Let M be a right R-module. By 1.4 Theorem (iii), $pr.dim._{R_1}(M) \leq pr.dim._R(M)$. Now we prove that

$$pr.dim._R(M) \leq pr.dim._{R_1}(M).$$

We may suppose that $pr.dim._{R_1}(M) = t < \infty$. Let

$$\cdots \longrightarrow P_{t-1} \xrightarrow{\delta_{t-1}} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{2}$$

be a projective resolution of M_R . Since R_{R_1} is projective by 1.1 Lemma (i), (2) is also a projective resolution of M_{R_1}. Because $pr.dim._{R_1}(M) = t$, $ker(\delta_{t-1})$ is a projective right R_1-module by [Rot, Theorem 9.5]. Since $ker(\delta_{t-1})$ is a right R-module, by our above discussion, $ker(\delta_{t-1})$ is a projective right R-module. Then $pr.dim._R(M) \leq t$ by [Rot, Theorem 9.5] again. Thus

$$pr.dim._{R_1}(M) = pr.dim._R(M).$$

Analogously, we can prove the statements about the injective properties.

(iii) This follows directly from (ii) and 1.1 Lemma (ii). ■

1.6 REMARKS. (i) In the above lemma (ii) and (iii) are generalizations of [MR, Theorem 7.5.6 (ii)] and [MR, Theorem 7.5.6 (iii)] respectively.

(ii) In the setting of 1.5 Lemma, suppose that $|G|$ is invertible in R_1 . Then every right R-module is R-regular by our discussion above 1.4 Theorem. Thus 1.5 Lemma (ii) and (iii) can be obtained from 1.1 Lemma

(ii) and the following well-known result. Here we have presented a detailed and self-contained proof.

1.7 PROPOSITION. [Na, Corollary, 2.13] *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. Suppose that M is a right R-module such that M is R-regular. Then*

$$pr.dim._R(M) = pr.dim._{R_1}(M) \quad \text{and} \quad inj.dim._R(M) = inj.dim._{R_1}(M). \qquad \blacksquare$$

As shown by the above proposition, for a finite group G and a strongly G-graded ring R, the behaviour of projective dimension and injective dimension of R-regular modules is quite simple. We list the following lemma, which indicates that there are plenty of R-regular modules.

1.8 LEMMA. [Na, Lemma 2.4] *Let G be a finite group and let R be a strongly G-graded ring. Then every graded right R-module is R-regular.* ■

Using [Da, Theorem 2.8], 1.7 Proposition and 1.8 Lemma, we obtain

1.9 COROLLARY. [Na, Corollary 2.7] *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. Suppose that $N = \oplus_{g \in G} N_g$ is a graded right R-module. Then for each $g \in G$*

$$inj.dim._R(N) = inj.dim._{R_1}(N_g). \qquad \blacksquare$$

Chapter 2

§ 2.2 SIMPLICITY AND SEMISIMPLICITY OF CROSSED PRODUCTS

It is well-known that the right global dimension of a ring is zero if and only if the ring is semisimple Artinian. Let G be a finite group and let R be a strongly G-graded ring. Suppose that R_1, the coefficient ring, is semisimple Artinian. Then by 1.5 Lemma (i) R has finite global dimension if and only if R is semisimple Artinian. Thus as the first step in the study of finiteness of global dimension of crossed products, we would like to study the semisimplicity of crossed products in this section.

About the simplicity and semisimplicity of crossed products and skew group rings, there are many well-known results. If R is a semisimple Artinian ring and G is a finite group with $|G|$ invertible in R, Maschke's Theorem says that $R*G$, a crossed product, is also semisimple Artinian; see [MR, Theorem 7.5.6]. Suppose that R is a simple ring, G is a finite group and $R*G$ is a crossed product. If G is outer, (note that here outer and X-outer are the same since R is simple), then $R*G$ is also simple. (This result is a special case of [Pa2, Corollary 12.6]; also see [Mo1, Theorem 2.3]; but it is essentially due to Azumaya [Az].) If R is semisimple Artinian and if G is finite and X-outer then $R*G$, a skew group ring, is also semisimple Artinian. (This is a special case of a result of Montgomery; see [Mo2, Theorem 3.1] or cf. [Mo1, Corollary 3.18 (3)].) In [HLS] the simplicity of skew group rings is also studied and many conditions for $R*G$, a skew group ring, to be simple are given there. In 2.3 Theorem below, we prove that the semisimplicity of crossed products can be reduced to the simplicity of crossed products over simple Artinian rings.

At first let us give a lemma.

2.1 LEMMA. *Let R be a ring, let G be a group and let $R_t^{\alpha}G$ be a crossed product. Suppose that $R = T\oplus S$ is a direct sum of two ideals S and T such that for each $g \in G$, $T^g \subseteq T$ and $S^g \subseteq S$. Then*

$$R_t^{\alpha}G \cong T_{t_1}^{\beta}G \oplus S_{t_2}^{\gamma}G,$$

where t_1 and t_2 are the restrictions of t to T and S respectively and $\alpha(g_1, g_2) = \beta(g_1, g_2) + \gamma(g_1, g_2)$ for all $g_1, g_2 \in G$.

PROOF. Suppose that all the stated conditions are satisfied. Clearly, there exist crossed products $T_{t_1}^{\beta}G$ and $S_{t_2}^{\gamma}G$. It is obvious that the map

$$\Phi:\ R_t^{\alpha}G \longrightarrow T_{t_1}^{\beta}G\oplus S_{t_2}^{\gamma}G;\ \ \sum\bar{g}_i(t_i+s_i) \longrightarrow \sum\bar{g}_it_i + \sum\bar{g}_is_i\ ,$$

where $g_i \in G$, $t_i \in T$ and $s_i \in S$, is a ring isomorphism. ∎

Now we can prove

2.2 PROPOSITION. *Let R be a ring, let G be a finite group and let $R*G$ be a crossed product.*

(*i*) *Suppose that R has a ring decomposition $R = S_1\oplus\ldots\oplus S_n$, as* *finite number of rings and suppose that* G *permutes the set* $\{S_1, \ldots, S_n\}$. *Let*

$$\Omega_j = \{S_k \mid S_k^g = S_j \text{ for some } g \in G\},\ j = 1, \ldots, m,$$

be all the orbits of G *on* $\{S_1, \ldots, S_n\}$, *and let* $m_i = [G : G_{S_i}]$, *where* G_{S_i} *is defined as in § 1.2. Then*

$$R*G \cong \oplus_{i=1}^m M_{m_i}(S_i*G_{S_i}),$$

where $M_q(S)$ *denotes the* $q \times q$ *matrix ring over the ring* S. *Therefore*

$$r.gl.dim.(R*G) = \max_{i=1}^n\{r.gl.dim.(S_i*G_{S_i})\}$$

(ii) Suppose that R *is semisimple Artinian. Let* $R = S_1 \oplus \ldots \oplus S_n$, *where each* S_i *is simple Artinian. Then* $R*G$ *is semisimple Artinian if and only if* $S_i*G_{S_i}$ *is semisimple Artinian for each* $i = 1, \ldots, n$.

PROOF. Suppose that R, G and $R*G$ are as stated.

(i) Suppose that S_i, m_i, $i = 1, \ldots, n$, and Ω_j, $j = 1, \ldots, m$, are as stated. Then by 2.1 Lemma $R*G$ is the direct sum of the crossed products

$$T_i = (\oplus_{S \in \Omega_i} S)*G.$$

Let $e_i \in S_i$ be the identity element of S_i and write $G = \bigcup_{j=1}^{m_i} G_{S_i} g_j$ (disjoint union). Then $1 = \sum_{j=1}^{m_i} e_i^{g_j}$ is a decomposition of $1 \in T_i$ into orthogonal idempotents which are permuted transitively by the subgroup $U(\oplus_{S \in \Omega_i} S)\bar{G}$ of $U(T_i)$, where $U(\ )$ denotes the group of units, and $e_i T_i e_i = S_i*G_{S_i}$. Therefore it follows from [Pal, Lemma 6.1.6] that

$$R*G \cong \oplus_{i=1}^m M_{m_i}(S_i*G_{S_i}).$$

Thus we have

$$r.gl.dim.(R*G) = max_{i=1}^{n}\{r.gl.dim.(S_i*G_{S_i})\}.$$

(ii) Suppose that R is semisimple Artinian, so $R = S_1 \oplus ... \oplus S_n$, where each S_i is simple Artinian. Then each S_i is a minimal ideal of R, so G permutes the set $\{S_1, ..., S_n\}$. Then the result follows from (i). ∎

Using 2.2 Proposition we can prove the following theorem, which is one of our main results in this section.

2.3 THEOREM. *Let $R = S_1 \oplus ... \oplus S_n$, where S_i is simple Artinian for $i = 1, ..., n$. Let G be a finite group and let $R*G$ be a crossed product. For each i, define*

$$H_i = \{g \in G \mid S_i^g = S_i \text{ and } g \text{ is inner on } S_i\},$$

and let N_i be a Sylow p_i-subgroup of H_i where $p_i = char(S_i)$ if this is positive, and $N_i = 1$ if $char(S_i) = 0$.

(i) $R*G$ is semisimple Artinian if and only if S_i*N_i is simple Artinian for $i = 1, ..., n$.

(ii) $R*G$ is semisimple Artinian if and only if S_i*P_i is simple Artinian for every elementary Abelian subgroup P_i of N_i, for $i = 1, ..., n$.

PROOF. Suppose that $R, G, R*G, N_i$ and H_i are as stated.

(i) ($\Longrightarrow$) Suppose that $R*G$ is semisimple Artinian. From 1.1 Lemma (ii) and 2.1 Lemma we know that S_i*N_i is semisimple Artinian.

Then by [LP1, Corollary 3.10] S_i*N_i must be simple Artinian.

($\Leftarrow$) Suppose that S_i*N_i is simple Artinian for $i = 1,\dots, n$. By [Pa2, Corollary 18.11 and Corollary 12.6] $S_i*G_{S_i}$ is semisimple Artinian. So by 2.2 Proposition (ii) $R*G$ is semisimple Artinian.

(ii) By 1.1 Lemma (ii), [Pa2, Theorem 18.10] and [LP1, Corollary 3.10] S_i*N_i is simple Artinian if and only if S_i*P_i is simple Artinian for every elementary abelian subgroup P_i of N_i. Then the result follows from (i). ∎

From 2.3 Theorem, we see that the semisimplicity of $R*G$ has been reduced to the simplicity of $S*P$, for a simple Artinian ring S with $char(S) = p > 0$ and an elementary Abelian p-subgroup P. Since P is elementary Abelian, P has the form $P = P_1\oplus\dots\oplus P_n$, where each P_i has order p. Then $S*P \cong ((S*P_1)*\dots*P_{n-1})*P_n$. Therefore $S*P$ is semisimple Artinian if and only if all $((S*P_1)*\dots*P_{i-1})*P_i$ are simple Artinian for $i = 0,\dots, n$, where $P_0 = \langle 1\rangle$.

Therefore the problem one is left to consider is the simplicity of $S*P$ with S simple Artinian of characteristic $p > 0$ and P a cyclic group of order p. If P is outer on S, then $S*P$ is a simple Artinian ring by the above theorem or by [Pa2, Corollary 12.6]. (As pointed out before, it is due to Azumaya [Az].) If P is inner on S, then $S*P$ may fail to be simple (e.g if $S*P = S[P]$ is the ordinary group ring), and, on the other hand, may still be simple sometimes (see 2.5 Example below). In the case of skew group rings, the following proposition throws some light on this problem.

2.4 PROPOSITION. *Let S be a simple ring (not necessarily Artinian) with $char(S) = p > 0$. Let $P = \langle g \rangle$ be a cyclic group of order p, with P inner on S. Let $S*P$ be the skew group ring. Then the following are equivalent:*

(i) $S*P$ is a simple ring;

(ii) if $v \in S$ is such that $s^g = vsv^{-1}$, for all $s \in S$, then $v^p \neq 1$;

(iii) $S*P$ is not isomorphic to an ordinary group ring of S over a cyclic group of order p.

PROOF. Let S, P and $S*P$ be as stated.

(i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) are obvious.

(ii) $\Longrightarrow$ (i) Suppose that (ii) holds. Since P is inner on S. We can choose $v \in S$ with v invertible such that $s^g = vsv^{-1}$, for all $s \in S$. By [HLS, Proposition 1.1 (b)] $S*P$ is simple if and only if $c(S*P)$, the centre of $S*P$, is a field. Let K be the centre of S. By [HLS, Proposition 1.6 (ii)] we have

$$c(S*P) = K[vg] \cong K[z]/(z^p - v^p),$$

where z is an indeterminate over K. Therefore $c(S*P)$ is a field if and only if $z^p - v^p$ is a maximal ideal of $K[z]$, if and only if v^p is not a pth power of an element of K [Jac, Lemma on p.225]. If v^p is a pth power of an element $k \in K$, so $v^p = k^p$, then $k \neq 0$ and $(k^{-1}v)^p = k^{-p}v^p = 1$. So $s^g = vsv^{-1} = (k^{-1}v)s(k^{-1}v)^{-1}$, contradicting (ii). ∎

2.5 EXAMPLE. A cyclic group of order p acts on a division ring S with $char(S) = p > 0$. The action is inner, but $S*P$, the skew group ring, is simple Artinian.

Let $R = A_1(K) = K[x][y, d/dx]$ be the first Weyl algebra over a field K, where $char(K) = p > 0$. (See [GW, p.15] for example.) Let S be the classical quotient ring of $A_1(K)$. So S is a division ring. Let g be the automorphism of S defined by the conjugation by x, that is $s^g = xsx^{-1}$ for all $s \in S$. Since x^p is contained in the centre of S, $\langle g \rangle$ is cyclic of order p. Let $P = \langle g \rangle$ and let $S*P$ be the skew group ring. By some direct calculations we know that $C(R)$, the centre of R, is $K[x^p, y^p]$. Obviously, R is a finitely generated module over its centre, so R is a P. I. ring. Let $\mathcal{C} = C(R) \setminus \{0\}$. By [MR, Theorem 13.6.5] we know that $R\mathcal{C}^{-1}$ is a central simple algebra with centre $Z = C(R)\mathcal{C}^{-1}$. Since $R\mathcal{C}^{-1}$ is a finitely generated module over its centre, we have $S = R\mathcal{C}^{-1}$ and so the centre of S is $K(x^p, y^p)$, the quotient field of $K[x^p, y^p]$. Suppose that $v \in S$ is invertible and $v^p = 1$, such that $s^g = xsx^{-1} = vsv^{-1}$. Then $v^{-1}x \in K(x^p, y^p)$. Let $v^{-1}x = f(x^p, y^p)/h(x^p, y^p)$, where $f(x^p, y^p)$ and $h(x^p, y^p)$ are elements of $K[x^p, y^p]$. So $x = v[f(x^p, y^p)/h(x^p, y^p)]$, and therefore

$$x^p = v^p[f(x^p, y^p)/h(x^p, y^p)]^p = [f(x^p, y^p)/h(x^p, y^p)]^p.$$

It is easy to see that this is impossible. By 2.4 Proposition, $S*P$ is a simple Artinian ring.

Now we give an application of 2.2 Proposition to the semisimplicity of certain factor rings of crossed products. This result is needed for the proof of our main theorems in later sections.

Suppose that R is a ring, G is a group and $R*G$ is a crossed product. For any ideal M of R, denote $\bigcap_{g\in G}M^g$ by M^0, so that M^0 is the unique largest G-invariant ideal contained in M.

2.6 PROPOSITION. *Suppose that R is an arbitrary ring, G is a finite group and $R*G$ is a crossed product. Let M be a maximal ideal of R. Then*

$$(R/M^0)*G \cong M_n((R/M)*G_M),$$

where $n = [G : G_M]$. In particular $(R/M^0)*G$ is semisimple Artinian if and only if $(R/M)*G_M$ is semisimple Artinian.

PROOF. Suppose that R, G, M and n are as stated. Let $G = \bigcup_{i=1}^{n} G_M g_i$ (disjoint union) with $g_1 = 1$. Then

$$M^0 = \bigcap_{g\in G}M^g = M^{g_1} \cap \dots \cap M^{g_n},$$

where $M^{g_1}, \dots, M^{g_n}$ are different. By the Chinese remainder theorem, we have

$$R/M^0 \cong R/M^{g_1} \oplus \dots \oplus R/M^{g_n}.$$

Consider R/M^{g_i} as ideals of R/M^0. Thus G permutes the set $\{R/M, R/M^{g_2}, \dots, R/M^{g_n}\}$ transitively and $(R/M)^g = R/M^g$, for all $g \in G$. By the proof of 2.2 Proposition (i), we have

$$(R/M^0)*G \cong M_n((R/M)*G_M).$$

In particular $(R/M^0)*G$ is semisimple Artinian if and only if $(R/M)*G_M$ is semisimple Artinian. ∎

Chapter 2

§ 2.3 RIGHT FBN LEFT COHERENT COEFFICIENT RINGS

A ring R is called *right bounded* if every essential right ideal of R contains an ideal which is essential as a right ideal. A ring R is *right fully bounded* if every prime factor ring of R is right bounded. A right *FBN* ring is any right fully bounded right Noetherian ring. For details about these rings refer to [GW, Chapter 8]. Obviously, a commutative Noetherian ring is *FBN*. By [MR, 13.6.6 (iii)] we know that Noetherian P. I. rings (and thus rings which are finitely generated modules over commutative Noetherian rings) are all *FBN*. From [BHM2, 3.5 Lemma (i)] or by an argument similar to the proof of [GW, Proposition 8.1 (b)] we know that Noetherian rings which are integral over their centres are also *FBN*.

A ring R is called *left coherent* if every direct product of flat right R-modules is flat, cf. [AF, 19.20 Theorem]. It is obvious that every left Noetherian ring is left coherent, see [Rot, p.113], but the converse is not true. Moreover there exist rings which are right *FBN* and left coherent but not left Noetherian; see 3.5 Example below. In this section, we discuss when a crossed product has finite right global dimension if the coefficient ring is right *FBN* and left coherent. By 1.1 Lemma (ii) we may assume that the coefficient ring has finite right global dimension.

The following well-known proposition gives a very useful property of right *FBN* rings.

3.1 PROPOSITION. [GW, Proposition 8.4] *Let R be a right FBN ring. If P is a right primitive ideal of R (in particular, if P is a maximal ideal), then R/P is a simple Artinian ring.* ■

3.2 LEMMA. *Let R be a right FBN ring, let G be a finite group and let $S = R{*}G$ be a crossed product. Then S is also a right FBN ring. Let M^* be a maximal ideal of S. Then $M^* \cap R = \cap_G M^g$ for a maximal ideal M of R, so in particular $R/(M^* \cap R)$ is a semisimple Artinian ring.*

PROOF. Suppose that R, G and S are as stated. Since $S = R{*}G$ is a finitely generated right R-module, by [Let, Proposition 4.9] or [So, Theorem 21], we know that $S = R{*}G$ is right *FBN*. Let M^* be a maximal ideal of S. By [Pa2, Lemma 14.2 (i)] we have $M^* \cap R = \cap_G M^g$, where M is a minimal prime over $M^* \cap R$. Using [Pa2, Theorem 16.6], it follows that M is a maximal ideal of R. Since primitive factor rings of right *FBN* rings are simple Artinian as shown in 3.1 Proposition, it is easy to see that $R/(M^* \cap R)$ is a semisimple Artinian ring. ■

We can now prove one of our main results in this thesis.

3.3 THEOREM. *Suppose that R is a right FBN left coherent ring and that $r.gl.dim.(R) < \infty$. Let G be a finite group and let $S = R{*}G$ be a* *crossed product. Suppose that for each maximal ideal* M *of* R *with* $char(R/M) = p > 0$, $(R/M)*G_M$ *is semisimple Artinian, where*

$$G_M = \{g \in G \mid M^g = M\}.$$

Then $r.gl.dim.(R*G) = r.gl.dim.(R) < \infty$.

PROOF. Suppose that S and G are as stated. By 3.2 Lemma we know that $S = R*G$ is also right *FBN*. Since $R*G$ is a free left R-module of finite rank, by [Row, p.266 Exercises 8 and 9′] for any set A we have

$$R^A \otimes_R (R*G) \cong (R \otimes_R (R*G))^A \cong (R*G)^A$$

as right $R*G$-modules. Since R is left coherent, by [AF, 19.20 Theorem], R^A is a flat right R-module. Therefore $(R*G)^A \cong R^A \otimes_R (R*G)$ is a flat right $R*G$-module. By [AF, 19.20 Theorem] again, $R*G$ is left coherent.

Suppose that I is a right ideal of R. Since $r.gl.dim.(R) < \infty$, there exists a finite projective resolution of $(R/I)_R$. Let

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow R/I \longrightarrow 0$$

be such a resolution. Since ${}_RS$ is free,

$$0 \longrightarrow P_n \otimes_R S \longrightarrow \dots \longrightarrow P_0 \otimes_R S \longrightarrow (R/I) \otimes_R S \longrightarrow 0$$

is a projective resolution of the right S-module $(R/I) \otimes_R S \cong S/IS$. Therefore

$$pr.dim._S(S/IS) \le pr.dim._R(R/I) \le r.gl.dim.(R). \tag{1}$$

Now we prove that for any simple right S-module V_S , we always have $pr.dim.(V_S) \le r.gl.dim.(R)$. Let $M^* = ann_S(V)$. Since S is right *FBN* by 3.2 Lemma, M^* is a maximal ideal of S and S/M^* is a simple Artinian ring by 3.1 Proposition. Thus there exists a positive integer n

such that $V^{(n)} \cong S/M^*$ as right S-modules. Therefore, we have

$$pr.dim._S(V) = pr.dim._S(V^{(n)}) = pr.dim._S(S/M^*). \qquad (2)$$

By 3.2 Lemma $R/(M^* \cap R)$ is a semisimple Artinian ring and

$$M^* \cap R = \cap_G M^g = M^0,$$

where M is a maximal ideal of R. If $char(R/M) = 0$, by 1.5 Lemma (iii) we have

$$r.gl.dim.((R/M^0)*G) = r.gl.dim.(R/M^0) = 0.$$

Therefore $(R/M^0)*G$ is semisimple Artinian. If $char(R/M) = p > 0$, by hypothesis $(R/M)*G_M$ is semisimple Artinian, and so by 2.6 Proposition $(R/M^0)*G$ is semisimple Artinian. Therefore we always have $(R/M^0)*G$ semisimple Artinian. Since

$$S/((M^* \cap R)S) = (R*G)/((M^* \cap R)*G) \cong (R/(M^* \cap R))*G = (R/M^0)*G,$$

it follows that $S/(M^* \cap R)S$ is a semisimple right S-module. Because

$$0 \longrightarrow M^*/((M^* \cap R)S) \longrightarrow S/((M^* \cap R)S) \longrightarrow S/M^* \longrightarrow 0 \qquad (3)$$

is an exact sequence of right S-modules and $S/((M^* \cap R)S)$ is a semisimple S-module, the sequence (3) splits. Therefore

$$S/((M^* \cap R)S) \cong (S/M^*) \oplus (M^*/((M^* \cap R)S)) \qquad (4)$$

as right S-modules. Thus by (2), (4) and (1) we have

$$\begin{aligned} pr.dim._S(V) = pr.dim._S(S/M^*) &\le pr.dim._S(S/((M^* \cap R)S)) \\ &\le pr.dim._R(R/(M^* \cap R)) \le r.gl.dim.(R). \end{aligned}$$

Then from [Ra, Theorem 8] we obtain

$$r.gl.dim.(S) = sup\ \{pr.dim._S(V) \mid V_S \text{ is simple}\} \le r.gl.dim.(R) < \infty.$$

By 1.5 Lemma (i) we have $r.gl.dim.(R*G) = r.gl.dim.(R) < \infty.$ ∎

3.4 REMARKS. (i) The converse of 3.3 Theorem is not true even in the case of R being commutative; see 6.4 Example. The converse of 3.3 Theorem is still not true even in the case of skew group rings and R being a finite module over its centre; see 6.2 Example.

(ii) If R is commutative and $R*G$ is a skew group ring, then the converse of 3.3 Theorem is true and we have more equivalent conditions; see 5.2 Theorem.

(iii) In the proof of 3.3 Theorem, the fully bounded hypothesis is used crucially in the following two ways:

(a) For a right *FBN* ring R every simple right R-module has co-Artinian annihilator; see 3.1 Proposition.

(b) For a right *FbN* and left coherent ring, the right global dimension equals the supremum of the projective dimensions of its simple right modules; see [Ra, Theorem 8].

It is still an open question whether (b) is valid for an arbitrary Noetherian ring, that is whether the global dimension of a (two-sided) Noetherian ring equals the supremum of the projective dimensions of its simple right modules; see [GW, p.287]. (A counter example in the one-sided Noetherian case was given by Fields [Fi, p.348]) If the answer for this question is positive, then 3.3 Theorem can be extended to all the (two-sided) Noetherian rings such that the simple right modules have co-Artinian annihilators.

3.5 EXAMPLE. We give a ring which is right *FBN* and left coherent; but it is not left Noetherian. Let $R = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{pmatrix}$, where $\mathbb{Q}$ and $\mathbb{R}$ are the field of rational numbers and the field of real numbers respectively. Then R is a right Noetherian P. I. ring. Thus R is right *FBN*. Obviously, R is not left Noetherian. By [MR, 7.5.1 Proposition] the left global dimension of R is 1. Thus R is left hereditary and so every finitely generated left ideal of R is projective. Therefore R is left coherent. (See [Rot, p.128].)

§ 2.4 LOCAL COEFFICIENT RINGS

A semiprime ideal N in a right Noetherian ring R is called *right localizable* provided $\mathscr{C}(N)$, the set of elements which are regular modulo N, is a right Ore set. A ring R is said to be *semilocal* if the factor ring $R/J(R)$ of R by its Jacobson radical $J(R)$ is semisimple Artinian. If $R/J(R)$ is simple Artinian, then R is called *local*. If $R/J(R)$ is a division ring, then R is called *scalar local*. In this section, we study the homological dimension of a crossed product or a skew group ring when the coefficient ring is Noetherian with all the maximal ideals right localizable or the coefficient ring is Noetherian local or semilocal.

4.1 LEMMA. *Let R be a local ring such that $char(R/J(R)) = p > 0$, let G be a cyclic group of order p acting on R and let $R*G$ be the skew group ring. Then $R*G$ is a local ring, and either $J(R*G) = J(R)*G$, or*

$(R/J(R))*G$ is isomorphic to the ordinary group ring $(R/J(R))[G]$, so that $(R*G)/J(R*G) \cong R/J(R)$.

PROOF. Suppose that R and G are as stated. Then $R/J(R)$ is simple Artinian. By [Pa2, Theorem 4.2], we always have $J(R)*G \subseteq J(R*G)$. If $(R/J(R))*G$ is simple Artinian, then $J(R*G) = J(R)*G$ and $R*G$ is local. Suppose that $(R/J(R))*G$ is not simple Artinian. Then by 2.4 Proposition $(R/J(R))*G$ is isomorphic to the group ring of G over $R/J(R)$. Let $\omega(\ )$ denote the augmentation ideal of group rings. Then by [Pa1, Lemma 8.1.17] the Jacobson radical of $(R/J(R))[G]$ is just its augmentation ideal. Therefore we have

$$(R/J(R))*G\big/J((R/J(R))*G) \cong (R/J(R)[G])\big/\omega((R/J(R))[G]) \cong R/J(R).$$

Since $J(R)*G \subseteq J(R*G)$ by [Pa2, Theorem 4.2] and

$$J(R*G)/(J(R)*G) = J((R*G)/(J(R)*G)) = J((R/J(R))*G),$$

the result follows. ∎

Recall that in a ring a *right denominator set* is a right Ore right reversible set; see [GW, Chapter 9]. The quotient rings of group rings have been studied be P. F. Smith in [Sm]. We give the following lemma, which is about the quotient rings of crossed products. Its part (i) is a generalization of [Sm, 2.6 Lemma] and part (ii) is well-known, see [LP2, Lemma 1.5 (i)].

4.2 LEMMA. *Let R be a ring, let G be a group and let $R*G$ be a crossed product.*

(i) Suppose that $\mathcal{C}$ is a right denominator set of R such that $\mathcal{C}$ is G-invariant, that is $c^g \in \mathcal{C}$ for each $c \in \mathcal{C}$ and each $g \in G$. Then $\mathcal{C}$ is a right denominator set of $R*G$ and

$$(R*G)\mathcal{C}^{-1} \cong R\mathcal{C}^{-1}*G,$$

where G can be an arbitrary group.

(ii) Suppose that R is a semiprime right Goldie ring and G is finite. Then $R*G$ is a right order in an Artinian ring and $Q(R*G) = Q(R)*G$, where $Q(\ )$ denotes the classical right quotient ring.

PROOF. (i) Suppose that $R, G, R*G$ and $\mathcal{C}$ are as stated. At first we check that $\mathcal{C}$ is a right Ore set of $R*G$. Let $c \in \mathcal{C}$ and let $\sum_{i=1}^{n} \bar{g}_i r_i$ be an element of $R*G$. Then $c^g \in \mathcal{C}$ for each $g \in G$ because $\mathcal{C}$ is G-invariant. Since $\mathcal{C}$ is a right Ore set of R, for each g_i, $i = 1,\dots, n$, there exists $c_i \in \mathcal{C}$ and $r'_i \in R$ such that $r_i c_i = c^{g_i} r'_i$. By [GW, Lemma 9.2 (a)] $\mathcal{C} \cap (\bigcap_{i=1}^{n} c_i R)$ is non-empty. Choose $c' \in \mathcal{C} \cap (\bigcap_{i=1}^{n} c_i R)$ and suppose that $c' = c_i a_i$, where each $a_i \in R$. Then it is clear that

$$(\textstyle\sum_{i=1}^{n} \bar{g}_i r_i)c' = c(\sum_{i=1}^{n} \bar{g}_i r'_i a_i).$$

Therefore $\mathcal{C}$ is a right Ore set of $R*G$. By some direct calculations we can see that $\mathcal{C}$ is also a right reversible set of $R*G$. (We omit the details.) Therefore $\mathcal{C}$ is a right denominator set of $R*G$. By 1.2.2 Lemma ([Pa2, Lemma 1.1]) it is easy to see that there exists a crossed product $R\mathcal{C}^{-1}*G$ induced from the crossed product $R*G$. Clearly,

$$(R*G)\mathcal{C}^{-1} \cong R\mathcal{C}^{-1}*G.$$

(ii) This is [LP2, Lemma 1.5 (i)]. ∎

Chapter 2

Let R be a ring. A nonzero right R-module U is called *uniform* if the intersection of any two nonzero submodules of U is nonzero; see [GW, p.71]. Let M_R be a right R-module. Suppose that $E(M)$, the injective hull of M_R, is a direct sum of n uniform submodules. Then we say that the *uniform rank*, or just the *rank* of M_R is n, and denote it as $rank(M) = n$. If no such n exists, we say the rank of M is infinite; cf. [GW, p.74].

From [Pa2, Theorem 31.6] and its proof we obtain the following theorem; but the original result is due to Nakayama and Azumaya; see [NA, Theorem 15].

4.3 THEOREM. [Pa2, Theorem 31.6] *Let R be a simple Artinian ring let G be a finite group acting on R and let $R{*}G$ be the skew group ring. Assume that $R{*}G$ is simple. For example, this occurs if G is outer on R. Then there exists a division ring D and an integer k with $R^G \cong M_k(D)$, $R{*}G \cong M_{k|G|}(D)$ and $k \mid rank(R_R)$.* ■

The above theorem plays a key role in the proof of the following lemma.

4.4 LEMMA. *Let R be a semiprime local right Noetherian ring with $char(R/J(R)) = p > 0$, $rank(R/J(R)) = m$ and $p \nmid m$. Let G be a finite elementary Abelian p-group acting on R. Suppose that the skew group ring $R{*}G$ has finite right global dimension. Then $(R/J(R)){*}G$ is simple Artinian.*

PROOF. Suppose that R, G and $R*G$ are as stated and that the stated conditions are satisfied. Since G is a finite elementary Abelian p-group, $G = P_1\times\ldots\times P_n$, where each P_i is a cyclic group of order p. Clearly, $R*G \cong ((R*P_1)*\ldots*P_{n-1})*P_n$, where each $*$ denotes a skew group ring. For $0 \le i \le n$, let $R_i = ((R*P_1)*\ldots*P_{i-1})*P_i$ and $\bar{R}_i = (((R/J(R))*P_1)*\ldots*P_{i-1})*P_i$, where $R_0 = R$ and $\bar{R}_0 = R/J(R)$. Suppose that $(R/J(R))*G$ is not simple Artinian. Then there exists a q, $1 \le q \le n$, such that $\bar{R}_i$ is simple Artinian for $i < q$, but $\bar{R}_q$ is not simple Artinian.

Let $\bar{R}_{q-1} = M_t(L)$, where L is a division ring. By 4.1 Lemma and our choice of q, R_{q-1} is local with $J(R_{q-1}) = J(R)*(P_1\times\ldots\times P_{q-1})$, and R_q is local with $R_q/J(R_q) \cong R_{q-1}/J(R_{q-1}) = \bar{R}_{q-1} = M_t(L)$. Since $R_q = R*(P_1\times\ldots\times P_q)$, by 4.2 Lemma (ii) R_q has an Artinian right quotient ring. By 1.1 Lemma (ii) R_q has finite right global dimension. Then by [BHM1, Corollary 3.3 and 3.4] we may suppose that

$$R_q \cong M_s(D), \text{ where } D \text{ is a local domain and } s\,|\,t. \tag{1}$$

Therefore R_q is a prime ring.

Since $\bar{R}_{q-1} = (R/J(R))*(P_1\times\ldots\times P_{q-1})$ is simple Artinian, by 4.3 Theorem, we have

$$t = p^{q-1}k, \text{ where } k\,|\,rank((R/J(R))_{R/J(R)}) = m. \tag{2}$$

By 4.2 Lemma $Q(R)*(P_1\times\ldots\times P_q) = Q(R*(P_1\times\ldots\times P_q)) = Q(R_q)$, which is simple Artinian because R_q is prime right Noetherian. Using 4.3 Theorem again, we have

$$rank((R_q)_{R_q}) = rank((Q(R_q))_{Q(R_q)}) = p^q r, \tag{3}$$

where $r\,|\,rank(Q(R)_{Q(R)})$. But by (1) $R_q \cong M_s(D)$, therefore from (3) we have

$$s = rank((R_q)_{R_q}) = p^q r. \tag{4}$$

Since $s|t$ by (1), from (4) and (2) we have $p^q r|p^{q-1}k$. Therefore $p|k$. But $k|m$ by (2), thus $p|m$. This is a contradiction. So $(R/J(R))*G$ is simple Artinian. ∎

An ideal I in a ring R is said to have the *right AR-property* if for every right ideal K of R, there is a positive integer n such that $K \cap I^n \subseteq KI$; see [GW, p.190].

4.5 LEMMA. *Let R be a right Noetherian ring, let G be a finite group and let $R*G$ be a crossed product. If I is a G-invariant ideal of R with the right AR-property, then $I*G$ is an ideal of $R*G$ and has the right AR-property.*

PROOF. Suppose that R, G and I are as stated. Let K be a right ideal of $R*G$. Since R is right Noetherian and G is finite, K is a finitely generated right R-module. Because I has the right AR-property, by [GW, Lemma 11.11] there exists a positive integer n such that

$$K \cap (R*G)I^n \subseteq KI. \tag{1}$$

Because $(R*G)I^n = I^n*G = (I*G)^n$ and $KI \subseteq K(I*G)$, from (1) we have $K \cap (I*G)^n \subseteq K(I*G)$. Therefore $I*G$ has the right AR-property. ∎

4.6 THEOREM. *Let R be a right Noetherian ring and let G be a finite* *group.*

(i) Let $R*G$ be a crossed product. Suppose that R has finite right global dimension and $J(R)$ has the right AR-property. If $(R/J(R))*G$ is semisimple Artinian, then $r.gl.dim.(R*G) < \infty$.

(ii) Let $R*G$ be a skew group ring. Suppose that R is semiprime local and $char(R/J(R)) \nmid rank(R/J(R))$. If $r.gl.dim.(R*G) < \infty$, then $(R/J(R))*G$ is semisimple Artinian.

PROOF. Suppose that R and G are as stated.

(i) Suppose $J(R)$ has the right AR-property and $(R/J(R))*G$ is semisimple Artinian. Since $J(R)*G \subseteq J(R*G)$ by [Pa2, Theorem 4.2], we have $J(R*G) = J(R)*G$. So $R*G$ is a semilocal ring. By 4.5 Lemma $J(R*G)$ has the right AR-property. Therefore by [Bo, Corollary], we have

$$r.gl.dim.(R*G) = \sup \{pr.dim.(V_{R*G}) \mid V_{R*G} \text{ is simple}\}. \qquad (1)$$

Because the primitive images of semilocal rings are Artinian, we can deduce (i) from (1) by an argument similar to that used in proving 3.3 Theorem. We omit the details.

(ii) If $char(R/J(R) = 0$, then by 1.5 Lemma (iii) $(R/J(R))*G$ is semisimple Artinian. Now we suppose that $char(R/J(R)) = p > 0$. Suppose $(R/J(R))*G$ is not semisimple Artinian. Then by [Pa2, Theorem 18.10] there exists an elementary Abelian p-subgroup P of G such that $(R/J(R))*P$ is not semisimple Artinian. Since $R*G$ has finite right global dimension, from 1.1 Lemma (ii) it follows that $R*P$ has finite right global dimension. Therefore by 4.4 Lemma $(R/J(R))*P$ must be simple Artinian. This is a contradiction. ∎

4.7 REMARKS. (i) It has long been an open question whether the Jacobson radical of every semilocal Noetherian ring has the AR-property; see [GW, Exercise 120 (c)]; but there exist one-sided Noetherian counter examples; see [GW, p.285] for details.

(ii) It is an open question whether every local Noetherian ring R of finite global dimension is a matrix ring over a domain (and so in particular is prime). By [BHM1, Corollary 3.3] this is equivalent to asking whether R must have an Artinian classical quotient ring. The answer is positive if R is a finite module over its centre [BHM2, Theorem 6.7], or if $gl.dim.(R) \leq 3$ [Sn, Theorem]. By recent work of Stafford and Zhang, see [SZ, Abstract], it is also true if R satisfies a polynomial identity.

(iii) The hypothesis that $p \nmid m$ in 4.6 Theorem (ii) cannot simply be omitted; see 6.3 Example.

(iv) 4.6 Theorem (ii) is not true in the case of crossed products; see 6.4 Example.

From 4.6 Theorem we obviously have

4.8 COROLLARY. *Let R be a semiprime scalar local right Noetherian ring and let G be a finite group acting on R. Let $R*G$ be the skew group ring. If $R*G$ has finite right global dimension, then $(R/J(R))*G$ is semisimple Artinian.* ■

For want of a convenient reference, we include the following well-known result.

4.9 LEMMA. *Let P be a right localizable prime ideal of a semiprime right Noetherian ring R and let $\mathcal{C}(P)$ be the set of elements which are regular modulo P. Then*

$$\{r \in R \mid rx = 0, \text{ for some } x \in \mathcal{C}(P)\}$$

$$= \bigcap\{Q \mid Q \text{ is a minimal prime ideal of } R \text{ and } Q \subseteq P\}.$$

In particular, R_P is semiprime.

PROOF. Let $I = \{r \in R \mid rx = 0$ for some $x \in \mathcal{C}(P)\}$. Since $\mathcal{C}(P)$ is a right Ore set of R, it is easy to see that I is an ideal of R. Let $K = \cap\{Q \mid Q$ is a minimal prime ideal of R and $Q \subseteq P\}$. Suppose that Q is a minimal prime ideal of R such that $Q \subseteq P$. If I is not contained in Q, then $(I + Q)/Q$ is a non-zero ideal of R/Q, so $(I + Q)/Q$ is essential as a right ideal of R/Q. By [GW, Proposition 5.9], there exists $d \in I \cap \mathcal{C}(Q)$. Then $dx = 0 \in Q$ for some $x \in \mathcal{C}(P)$. Since $x \notin Q$, this is a contradiction, so $I \subseteq K$.

Conversely, let $J = \cap\{Q \mid Q$ is a minimal prime ideal of R and Q is not contained in $P\}$. By the same analysis which we used in the above paragraph there exists $b \in J \cap \mathcal{C}(P)$. Since R is semiprime, $Kb \subseteq KJ = 0$, so $K \subseteq I$. Therefore $K = I$. In particular, by [GW, Proposition 9.19 (b)] R_P is semiprime. ■

We globalise 4.6 Theorem (ii) as the following

4.10 PROPOSITION. *Let R be a semiprime right Noetherian ring and let G be a finite group acting on R. Suppose that the skew group ring $R*G$ has finite right global dimension. Let M be a maximal ideal of R with R/M Artinian and $char(R/M) = p > 0$. Suppose that M is right localizable and that $p \nmid rank(R/M)$. Then $(R/M)*G_M$ is semisimple Artinian.*

PROOF. Suppose that R, G, M and $R*G$ are as stated. By 1.1 Lemma (ii) $R*G_M$ also has finite right global dimension. So we may suppose that M is G-invariant. By 4.2 Lemma (i) we know that

$$\mathscr{C}(M) = \{x \in R \mid x + M \text{ is regular in } R/M\}$$

is also a right Ore set in $R*G$ and $(R*G)_{\mathscr{C}(M)} = R_{\mathscr{C}(M)}*G = R_M*G$. By hypothesis R/M is simple Artinian. By [GW, Lemma 12.18] we know that R_M is a local ring, $J(R_M) = MR_M$ and $R_M/MR_M \cong R/M$. Since R is semiprime, R_M is semiprime by 4.9 Lemma. Then by [MR, Corollary 7.4.3] we have

$$r.gl.dim.(R_M*G) = r.gl.dim.((R*G)_{\mathscr{C}(M)}) \le r.gl.dim.(R*G),$$

which is finite by hypothesis. Since

$$char(R_M/J(R_M)) = char(R/M) \nmid rank(R/M) = rank(R_M/J(R_M)),$$

by 4.6 Theorem (ii) $(R_M/J(R_M))*G$ is semisimple Artinian, so $(R/M)*G$ is semisimple Artinian. ■

4.11 LEMMA. *Let R be a Noetherian ring such that every maximal ideal of R is right localizable.*

(i) R is semiprime if and only if R_M is semiprime for every

maximal ideal M *of* R.

(ii) If R *is a finite module over its centre and the global dimension of* R *is finite, then* R *is semiprime.*

PROOF. Suppose that R is as stated.

(i) ($\Longrightarrow$) follows from 4.9 Lemma.

($\Longleftarrow$) Suppose that for every maximal ideal M of R, R_M is semiprime. Let N be the nilpotent radical of R and let $I = r.ann_R(N)$. Suppose $N \neq 0$. Then $I \neq R$. So there exists a maximal ideal M of R such that $I \subseteq M$. If N is $\mathscr{C}(M)$-torsion, where $\mathscr{C}(M)$ is the set of elements which are regular modulo M, then there exists a $c \in \mathscr{C}(M)$ such that $Nc = 0$. This contradicts to $I \subseteq M$. Thus $N_M \neq 0$. Suppose $N^t = 0$ for some positive integer t. Since N_M is an ideal of R_M by [GW, Theorem 9.20], it is easy to see that $(N_M)^t = 0$. This contradicts the semiprimeness of R_M. So $N = 0$ and then R is semiprime.

(ii) Suppose that R is a finite module over its centre. Let M be an arbitrary maximal ideal of R. From [Mü, Section 3.1] it is easy to deduce that R_M is a finite module over its own centre. Because R_M is local and $gl.dim.(R_M) \leq gl.dim.(R) < \infty$, by [BHM2, Theorem 6.7] R_M is prime. So R is semiprime by (i). ∎

4.12 COROLLARY. *Let* R *be a Noetherian ring which is a finite module over its centre and let the global dimension of* R *be finite. Suppose that every maximal ideal of* R *is right localizable and that for every* *maximal ideal M of R with $char(R/M) = p > 0$, $p \nmid rank(R/M)$. Let G be a finite group and let $R*G$ be a skew group ring. Then $gl.dim.(R*G)$ is finite if and only if $(R/M)*G_M$ is semisimple Artinian for every maximal ideal M of R such that $char(R/M) = p > 0$.*

PROOF. Let R, G and $R*G$ be as stated. Then R is an FBN ring by [GW, Proposition 8.1 (b)]. So ($\Longleftarrow$) follows directly from 3.3 Theorem.

($\Longrightarrow$) By 4.11 Lemma (ii) R is semiprime. The result therefore follows from 4.10 Proposition. ∎

§ 2.5 COMMUTATIVE NOETHERIAN COEFFICIENT RINGS

Specializing the results of § 2.4 to commutative Noetherian coefficient rings, we can obtain some equivalent conditions for $R*G$, a skew group ring, to have finite global dimension. For convenience of statement, we give the following definition. Suppose that R is a ring and G is a group acting on R. Let M be an ideal of R. We define

$$G(M) = \{g \in G \mid r^g - r \in M, \text{ for all } r \in R\}.$$

$G(M)$ is usually called the *inertia group* of M. By some direct calculations we immediately obtain the following

5.1 LEMMA. *Suppose that R is a ring, G is a group acting on R and*

M is an ideal of R.

(i) $G(M)$ is the unique largest subgroup H of G such that M is H-invariant and H acts trivially on R/M.

(ii) If N and M are ideals of R, such that $N \subseteq M$, then $G(N) \subseteq G(M)$.

(iii) If $N \subseteq M$ are ideals of R such that N is G-invariant, then $G(M) = G(M/N)$. ∎

The following theorem is one of our main results in this chapter.

5.2 THEOREM. *Let R be a commutative Noetherian ring, let G be a finite group acting on R and let $R*G$ be the skew group ring. Then the following are equivalent:*

(i) $gl.dim.(R*G) < \infty$;

(ii) a) $gl.dim.(R) < \infty$;

b) for every maximal ideal M of R with $char(R/M) = p > 0$, $(R/M)*G_M$ is semisimple Artinian;

(iii) a) $gl.dim.(R) < \infty$;

b) for every maximal ideal M of R with $char(R/M) = p > 0$, $G(M)$ contains no elements of order p.

PROOF. Suppose that R, G and $R*G$ are as stated. Since R is commutative, it is easy to see that all the conditions in 4.12 Corollary are satisfied. So (i) $\Longleftrightarrow$ (ii) follows directly from 4.12 Corollary and 1.1 Lemma (ii).

(ii) $\Longrightarrow$ (iii) It is obvious that $G(M) \subseteq G_M$ for every maximal ideal M of R. By 5.1 Lemma (i), $G(M)$ acts trivially on R/M. Suppose that $G(M)$ contains an element of order p. Then from [Pa1, Corollary 10.3.7 (i)] we know that $R*G(M)$ is not semisimple Artinian. Thus $R*G_M$ is not semisimple Artinian by 1.1 Lemma (ii). This contradicts the hypothesis (ii).

(iii) $\Longrightarrow$ (ii) Suppose that (iii) holds. Let M be a maximal ideal of R such that $char(R/M) = p > 0$. Suppose that $(R/M)*G_M$ is not semisimple Artinian. By [Pa2, Corollary 18.11], there exists a Sylow p-subgroup P of G_M such that $(R/M)*P$ is not semisimple Artinian. By [Mo1, Theorem 2.3] P is not outer on R/M, and so there exists $g \in P - \{1\}$ with g inner on R/M. We may suppose the order of g is p. Since R/M is commutative, it follows that g acts trivially on R/M. By 5.1 Lemma (i), $g \in G(M)$, contradicting the hypothesis of (iii).

■

5.3 COROLLARY. *Let R be a commutative Noetherian ring, let G be a finite group acting on R and let $R*G$ be the skew group ring. Then*

$$gl.dim.(R*G) = \sup_M \{gl.dim.(R_M*G_M)\},$$

where M ranges over all the maximal ideals of R.

PROOF. Suppose that R and G are as stated. By [MR, Corollary 7.4.3], 4.2 Lemma (i) and 1.1 Lemma (ii) we always have

$$gl.dim.(R*G) \geq \sup_M \{gl.dim.(R_M*G_M)\},$$

where M ranges over all the maximal ideals of R. Suppose that

$$\sup_{M} \{gl.dim.(R_M * G_M)\} < \infty,$$

where M ranges over all the maximal ideals of R. For each maximal ideal M of R, by 5.2 Theorem $(R/M)*G_M = (R_M/J(R_M))*G_M$ is semisimple Artinian. By 5.2 Theorem again $gl.dim.(R*G) < \infty$. Then by 1.5 Lemma (i) and [No, Theorem 9.2.10], we have

$$gl.dim.(R*G) = gl.dim.(R) = \sup_{M} \{gl.dim.(R_M) = \sup_{M} \{gl.dim.(R_M * G_M)\},$$

where M ranges over all the maximal ideals of R. ∎

5.4 COROLLARY. *Let R be a commutative Noetherian ring of finite global dimension and let G be a finite group acting on R. Let $S = R*G$ be the skew group ring.*

(i) $gl.dim.(R*G) < \infty$ if and only if, for all primes p which are not units in R, $gl.dim.(R*G_p) < \infty$, where G_p is any Sylow p-subgroup of G.

(ii) $gl.dim.(R*G) < \infty$ if and only if, for all primes p which are not units of R, $gl.dim.(R*P) < \infty$, where P is any elementary Abelian p-subgroup of G.

PROOF. Suppose that R, G and $R*G$ are as stated.

(i) $(\Longrightarrow)$ This follows directly from 1.5 Lemma (i).

$(\Longleftarrow)$ Suppose that the condition is satisfied. Let M be a maximal ideal of R. Suppose that $char(R/M) = p > 0$ and there exists $g \in G(M)$ with order p. Then there exists a Sylow p-subgroup G_p of G such that $g \in G_p$. By hypothesis $gl.dim.(R*G_p) < \infty$, so according to 5.2 Theorem $G_p(M) = 1$. But $g \in G_p(M)$, a contradiction. Therefore $G(M)$ has no elements of order p. By 5.2 Theorem $gl.dim.(R*G) < \infty$.

(ii) The proof is similar to the proof of (i). ∎

We can extend 5.2 Theorem to larger classes of groups by using a result of Aljadeff [Al2, Theorem 0.3], which is a version of Serre's Theorem for crossed products.

5.5 THEOREM. [Al2, Theorem 0.3] *Let R be a commutative ring, let G be a group and let $R*G$ be a crossed product. Suppose that H is a subgroup of G of finite index. Then the following conditions are equivalent:*

(i) $r.gl.dim.(R*G) < \infty$;

(ii) $r.gl.dim.(R*H) < \infty$, and for each finite subgroup T of G, the crossed product $R*T$ is R-semisimple; that is , for each right $R*T$-module M, if M is projective as a right R-module then M is projective as a right $R*T$-module;

(iii) $r.gl.dim.(R*H) < \infty$ and for each finite subgroup T of G, $r.gl.dim.(R*T) < \infty$.

Moreover in this situation $r.gl.dim.(R*G) = r.gl.dim.(R*H)$. ∎

Using 5.5 Theorem and 5.2 Theorem, we immediately obtain the following

5.6 PROPOSITION. *Let R be a commutative Noetherian ring, let G be an arbitrary group and let $R*G$ be a skew group ring. Suppose that H is a* *subgroup of* G *of finite index. Then the following statements are equivalent:*

(i) $gl.dim.(R*G) < \infty$;

(ii) (a) $gl.dim.(R*H) < \infty$;

(b) *for every maximal ideal* M *of* R *with* $char(R/M) = p > 0$ *and for every finite subgroup* T *of* G, $(R/M)*T_M$ *is semisimple Artinian;*

(iii) (a) $gl.dim.(R*H) < \infty$;

(b) *for every maximal ideal* M *of* R *with* $char(R/M) = p > 0$, $G(M)$ *contains no element of order* p. ∎

PROOF. Suppose that R, G, H and $R*G$ are as stated.

(i) ($\Longrightarrow$) (ii) Suppose that $gl.dim.(R*G) < \infty$. Let M be a maximal ideal of R with $char(R/M) = p > 0$ and let T be a finite subgroup of G. Then by 1.1 Lemma (ii) we have $gl.dim.(R*H) < \infty$ and $gl.dim.(R*T) < \infty$. Using 5.2 Theorem, we obtain (ii).

(ii) ($\Longrightarrow$) (i) Suppose that (ii) is true. Using (ii) and 5.2 Theorem, we know that for each finite subgroup T of G, $gl.dim.(R*T) < \infty$. Then (i) follows from 5.5 Theorem.

The proof of the equivalence of (i) and (iii) is analogous to the above argument. ∎

Specializing 5.6 Proposition to polycyclic-by-finite groups, we obtain

5.7 COROLLARY. *Let R be a commutative Noetherian ring, let G be a polycyclic–by–finite group acting on R and let $R*G$ be the skew group ring. Then the following statements are equivalent:*

(i) $gl.dim.(R*G) < \infty$;

(ii) *(a)* $gl.dim.(R) < \infty$;

(b) for every maximal ideal M of R with $char(R/M) = p > 0$ and for every finite subgroup T of G, $(R/M)*T_M$ is semisimple Artinian;

(iii) *(a)* $gl.dim.(R) < \infty$;

(b) for every maximal ideal M of R with $char(R/M) = p > 0$, $G(M)$ contains no element of order p.

PROOF. Since G is polycyclic–by–finite, there exists a normal subgroup H of G with finite index such that H is poly–infinite cyclic [Pa1, Lemma 10.2.5]. If R has finite global dimension, then $R*H$ has finite global dimension by [MR, Theorem 7.5.3 (ii)]. Thus the corollary follows directly from 5.6 Proposition. ∎

5.8 REMARKS. (i) In 5.2 Theorem, 5.6 Proposition and 5.7 Corollary, if R is only a finite module over its centre rather than a commutative ring, then their conditions (ii) and (iii) are no longer necessary conditions for $R*G$ to have finite global dimension; see 6.1 Example and 6.2 Example.

(ii) In 5.2 Theorem, 5.6 Proposition and 5.7 Corollary, suppose that $R*G$ is a crossed product rather than a skew group ring. Then their conditions (ii) and (iii) are no longer necessary for $R*G$ to have finite global dimension; see 6.4 Example; but they are still sufficient conditions for $R*G$ to have finite global dimension. For example, we give the following corollary.

5.9 COROLLARY. *Let R be a commutative Noetherian ring with finite global dimension. Let G be a polycyclic-by-finite group and let $S = R*G$ be a crossed product. If one of the following conditions is satisfied, then $gl.dim.(R*G) < \infty$.*

(i) For every maximal ideal M of R with $char(R/M) = p > 0$ and every finite subgroup T of G, $(R/M)*T_M$ is semisimple Artinian.

(ii) For every maximal ideal M of R with $char(R/M) = p > 0$, $G(M)$ contains no element of order p.

PROOF. Let R, G and $R*G$ be as stated. At first, suppose that (i) is satisfied. Using 5.5 Theorem and 3.3 Theorem, by an argument used in the proof of 5.7 Corollary we obtain $gl.dim.(R*G) < \infty$. Next, suppose that (ii) is satisfied. Suppose that $R*G$ is determined by the maps α and t, that is $R*G = R_t^{\alpha}G$. By [AR, Corollary 3.3] and [A12, Remark 0.4] we have

$$gl.dim.(R_t^{\alpha}G) \leq gl.dim.(R_tG).$$

So the result follows from 5.7 Corollary (iii). ∎

Notice that in the proof of our 5.2 Theorem (i) $\Leftrightarrow$ (ii) we used the results of Section 4; that is, the localization method. We would like to remark that we could bypass Section 4 to give another proof for 5.2

Theorem; but we need to use the following proposition. We include it here for interest.

5.10 PROPOSITION. *Let R be a ring, let G be a finite group and let $S = R*G$ be a skew group ring. If there exists a proper ideal M of R such that $char(R/M) = m > 0$ and $G(M)$ has an element $g \neq 1$ of order dividing m, then $r.gl.dim.(R*G) = \infty$.*

PROOF. Let R, G and M be as stated. Suppose that $g \in G(M)$, $g \neq 1$, with order dividing m. We may suppose that $|g| = p$ is a prime, and $m = pn$. Because $char(R/M) = m = pn$, p is not a unit of R/M. So $pR + M$ is a proper ideal of R, and $char(R/(pR+M)) = p$. By 5.1 Lemma (ii) $g \in G(pR + M)$. Therefore we may suppose that $char(R/M) = p$ is a prime and $G(M)$ has an element g of order p. By 1.1 Lemma (ii) $r.gl.dim.(R*\langle g\rangle) \leq r.gl.dim.(R*G)$. Therefore we may suppose that G is cyclic of order p. By 5.1 Lemma (i) M is G-invariant and G acts trivially on R/M. So $(R/M)*G$ is the ordinary group ring.

Let $\hat{g} = 1 + g + ... + g^{p-1}$. it is easy to see that

$$0 \longrightarrow \hat{g}S \xrightarrow{i} S \xrightarrow{\Phi} (g-1)S \longrightarrow 0$$

and

$$0 \longrightarrow (g-1)S \xrightarrow{j} S \xrightarrow{\Psi} \hat{g}S \longrightarrow 0$$

are exact sequences of right S-modules, where i and j are imbedding, and Φ and Ψ are left multiplication by $(g-1)$ and $\hat{g}$ respectively.

Suppose that one of $\hat{g}S$ and $(g-1)S$ is projective. Then $S \cong (g-1)S \oplus \hat{g}S$ as right S-modules. Suppose in this isomorphism s_1 and

s_2 correspond to $g - 1$ and $\hat{g}$ respectively. Then $S = s_1S \oplus s_2S$, and $s_1\hat{g}$ corresponds to $(g-1)\hat{g} = 0$. Therefore $s_1\hat{g} = 0$. Similarly $s_2\hat{g} = 0$. Let $\bar{\ }$ denote images in $\bar{S} = S/MS = (R/M)[G]$, the group ring. Then $\bar{\hat{g}}$ is in the centre of $\bar{S}$. So

$$(S/MS)\hat{g} = ((s_1S + s_2S)/MS)\hat{g} = \bar{s}_1\bar{S}\bar{\hat{g}} + \bar{s}_2\bar{S}\bar{\hat{g}} = \bar{s}_1\bar{\hat{g}}\bar{S} + \bar{s}_2\bar{\hat{g}}\bar{S} = 0.$$

Therefore $\hat{g} \in MS = M{*}G$, and so $1 \in M$. Since M is a proper ideal, this is a contradiction. Therefore $\hat{g}S$ and $(g-1)S$ are not projective S-modules. By [Pal, Lemma 10.3.3] we have $r.gl.dim.(R{*}G) = \infty$. ∎

5.11 REMARK. Another proof of 5.2 Theorem: (i) $\Longrightarrow$ (iii) follows directly from 1.1 Lemma (ii) and 5.10 Proposition. (iii) $\Longrightarrow$ (ii) is the same as in the proof of 5.2 Theorem. (ii) $\Longrightarrow$ (i) follows from 3.3 Theorem.

§ 2.6 EXAMPLES

We give several examples to show that the conditions in our main results are essential. The following two examples show that the converse of 3.3 Theorem is not true and the conditions in 5.2 Theorem are not equivalent if R is only a finite module over its centre.

6.1 EXAMPLE. Suppose that K is a field of characteristic $p > 0$ and

$R = M_2(K)$ is the $2{\times}2$ matrix ring over K. Let $G = \langle u \rangle$ be the multiplicative group generated by u, where $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let G act on R by conjugation by u, and let $R*G$ be the skew group ring. Then $R*G$ is not semisimple Artinian since $gu - 1$ is a central nilpotent element. Therefore $gl.dim.(R*G) = \infty$ by 1.5 Lemma (i). But $G(0) = 1$, this example satisfies the condition (iii) but not (i) of 5.2 Theorem.

6.2 EXAMPLE. Let $R = K[x, x^{-1}][y, d/dx]$ be the differential operator ring over $K[x, x^{-1}]$, where $K = \mathbb{Z}/2\mathbb{Z}$. Let $G = \langle g \rangle$ be a cyclic group of order 2 acting on R by $r^g = xrx^{-1}$, for all $r \in R$. Because R is a finitely generated module over its centre, R is FBN. Let $M = (x^2-1)R + (y^2-1)R$. Obviously M is G-invariant. It is easy to check that M is a maximal ideal of R and $\bar{R} = R/M \cong M_2(\mathbb{Z}/2\mathbb{Z})$. Note that $\bar{R}*G$ is not semisimple Artinian since $(\bar{x}g - \bar{1})^2 = \bar{x}^2g^2 - \bar{1} = 0$ and $\bar{x}g - \bar{1}$ is a non-zero central element of $\bar{R}*G$.

Next we check that $gl.dim.(R*G) = 2$. Let $S = R*G$. Consider the right S-module R_S. Since

$$\textstyle\sum_G (yx)^g = yx + x(yx)x^{-1} = yx + xy = xy - yx = 1,$$

by [ARS, Proposition 1.7] R_S is projective. Since

$$y(1+g)x + x(1+g)y = yx + ygx + xy + xgy = yx + xy + g(xy+xy)$$

$$= yx + xy = xy - yx = 1,$$

by [ARS, Corollary 1.5], R_S is a generator. But R_S is cyclic, so R_S is a finitely generated projective generator. Therefore S is Morita equivalent to $End_S(R_S)$ and the latter is R^G. We can directly check that

$$R^G = K[x,\ x^{-1},\ y^2] \cong K[x,\ x^{-1}][z],$$

where z is an indeterminate over $K[x,\ x^{-1}]$. Therefore

$$gl.dim.(R{*}G) = gl.dim.(R^G) = gl.dim.(K[x,\ x^{-1}][z]),$$

which is 2 by [MR, Theorem 7.5.3].

This example shows that the converse of 3.3 Theorem is not true even in the case of skew group rings and R being a finite module over its centre. Moreover the conditions (i) and (ii) in 5.2 Theorem are not equivalent if R is only a finite module over its centre.

6.3 EXAMPLE. Localize the ring $R = K[x,\ x^{-1}][y,\ d/dx]$ of 6.2 Example at its maximal ideal M, which we can do since M is generated by central elements [Mü, Theorem 7]. Set $R' = R_M$ and extend the action of G to R'; cf. 4.2 Lemma (i). It is easy to see that $gl.dim.(R'{*}G) \le 2$. By 4.1 Lemma we know that $R'{*}G$ is a local ring. Then by [Sn, Theorem] $R'{*}G$ is prime. But $(R'/J(R')){*}G = (R/M){*}G$ is not semisimple Artinian by 6.2 Example. This shows that the condition $p \nmid m$ in 4.6 Theorem (ii) can not simply be omitted.

When we consider crossed products rather than just skew group rings, the sufficient conditions in 4.6 Theorem (i) and 5.2 Theorem (ii) and (iii) for finite global dimension are all easily seen to fail to be necessary.

6.4 EXAMPLE. Let p be a prime, K a field of characteristic p, and $S = K[x,\ x^{-1}]$. Let $R = K[x^p,\ x^{-p}] \subseteq S$, so that $S = R{*}G$, a crossed product, where G is the group of order p. Thus S is hereditary, but $gl.dim.((R/M)*G) = \infty$ and $G(M) = G$ when $M = (x^p - 1)R$. Localizing at M gives an example with a local coefficient ring, with the same properties.

§ 2.7 TRACES AND GLOBAL DIMENSION

In this section, we would like to study the relationships between the global dimension and the trace maps (definitions are given below) for strongly group graded rings and skew group rings.

Let G be a finite group, let R be a ring and let $R*G$ be a skew group ring. Then we can define a map tr from R to R by

$$tr(r) = \sum_{g\in G} r^g \quad \text{for all } r \in R.$$

The map tr is called the *trace* of G on R; see [Mol, p.1].

Let G be a finite group and let $R = \oplus_{g\in G} R_g$ be a strongly G-graded ring. For each $g \in G$, since $1 \in R_y R_g = R_1$, where $y = g^{-1}$, we may fix a decomposition of the identity

$$1 = \sum_{i\in I_g} v_y^{(i)} u_g^{(i)},$$

where I_g is a finite set, $u_g^{(i)} \in R_g$ and $v_y^{(i)} \in R_y$, $y = g^{-1}$. Thus for each $g \in G$, we can define a map $(\ )^g$ from R_1 to R_1 by

$$r^g = \sum_{i\in I_g} v_y^{(i)} r u_g^{(i)}, \text{ for all } r \in R_1 . \tag{*}$$

Let $R*G = R_t^{\alpha}G = \oplus_{g\in G}\bar{g}R$ be a crossed product or a skew group ring.

Then for each $g \in G$, since $\bar{g}^{-1} \in \overline{g^{-1}}R$ and $\bar{g} \in \bar{g}R$, we can choose $1 = \bar{g}^{-1}\bar{g}$ as the decomposition of the identity. Thus in this situation the map defined in (*) sends each element r of R to $\bar{g}^{-1}r\bar{g}$, which is $r^{t(g)}$. Therefore $(\ )^g$ coincide with the automorphism $t(g)$ in this case.

Usually, $(\ )^g$, defined in (*), may not be an endomorphism of the coefficient ring R_1 for general strongly group graded rings (1.2.10 Example can demonstrate this fact). Fortunately, by some direct calculations, we have the following

7.1 LEMMA. [CVV, Section 1] *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. For each $g \in G$, let $(\ )^g$ be as defined in (*). Then the restriction of $(\ )^g$ to $Z(R_1)$, the centre of R, is an automorphism of $Z(R_1)$.* ∎

Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. We call the following map

$$tr:\ Z(R_1) \longrightarrow Z(R_1);\ \ r \longrightarrow \textstyle\sum_{g\in G} r^g \quad \text{for all } r \in Z(R_1),$$

where $(\ )^g$ is defined as in (*), the *trace* of G on $Z(R_1)$; see [CVV, Section 1] for details.

As illustrated in [CVV] and [NuV], if there exists a $c \in Z(R_1)$ such that $tr(c) = 1$, then R has some strong properties. For example, in this case there exists a version of Maschke's Theorem; see [CVV, 1.1.2 Proposition]. We will give some more results to demonstrate this fact.

7.2 PROPOSITION. *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1. Suppose that there exists an element $c \in Z(R_1)$ such that $tr(c) = 1$. Then every right R-module is R-regular. In particular, we have*

(i) for each right R-module M,

$$pr.dim._R(M) = pr.dim._{R_1}(M); \quad inj.dim._R(M) = inj._{R_1}dim.(M);$$

(ii) $r.gl.dim.(R) = r.gl.dim.(R_1)$.

PROOF. Suppose that the stated conditions are satisfied. For each $g \in G$, let $1 = \sum_{i\in I_g} v_y^{(i)}u_g^{(i)}$, where I_g is a finite set, $u_g^{(i)} \in R_g$ and $v_y^{(i)} \in R_y$, $y = g^{-1}$, be the fixed decomposition of the unit such that $tr(c) = 1$ for some $c \in Z(R_1)$. For each right R-module M, let ϕ be the map of right multiplication by c. Since c is in the centre of R_1, ϕ is an R_1-homomorphism of M. For each element m of M

$$\tilde{\phi}(m) = \sum_{g\in G}\sum_{i\in I_g} \phi(mv_y^{(i)})u_g^{(i)} = \sum_{g\in G}\sum_{i\in I_g} (mv_y^{(i)}c)u_g^{(i)} = m.tr(c) = m,$$

where $\tilde{\phi}$ is defined as in § 2.1. Thus $\tilde{\phi}$ is the identity map and so M is R-regular by definition (see § 2.1).

From the above fact and 1.7 Proposition, we obtain (i).

(ii) follows from (i) and 1.1 Lemma (ii). ∎

7.3 REMARK. It is clear that 1.5 Lemma (ii) and (iii) can be deduced from 7.2 Proposition.

Using 7.2 Proposition, we can deduce some results of [NuV].

7.4 PROPOSITION. *Let G be a finite group and let R be a strongly G-graded ring with coefficient ring R_1.*

(i) If R is right hereditary, then so is R_1. (This part is valid for arbitrary groups.)

(ii) If R is right semihereditary, then so is R_1.

Suppose that there exists an element $c \in Z(R_1)$ such that $tr(c) = 1$. We have

(i)′ if R_1 is right hereditary, then so is R;

(ii)′ if R_1 is right semihereditary, then so is R.

PROOF. Suppose that R, G and R_1 are as stated.

(i) follows directly from 1.1 Lemma (ii). Since R is a finitely generated right R_1-module by 1.1 Lemma (i), for each finitely generated right ideal I of R_1, $I \otimes_{R_1} R$ is isomorphic to a finitely generated right ideal of R, so (ii) follows quickly.

Suppose that there exists an element $c \in Z(R_1)$ such that $tr(c) = 1$. (i)′ follows from 7.2 Proposition (ii). (ii)′ can be deduced easily by using 1.1 Lemma (i) and 7.2 Proposition (i). ∎

7.5 REMARKS. (i) 7.4 Proposition (i) and (i)′ are well-known; see [NuV, 2.1 Proposition (1) and 2.3 Proposition].

(ii) In the special case of skew group rings. 7.4 Proposition (i)′ and (ii)′ appeared in a recent preprint [GD]; see [GD, Lemma 2.6].

(iii) In 7.2 and 7.4 Propositions the condition $c \in Z(R_1)$ is essential. Without this condition 7.2 and 7.4 Propositions are easily seen not to be true; see 7.8 Example.

Recall that if $S = R*G$ is a skew group ring, then R is a right S-module, where the elements of R act by right multiplication, and if $r \in R$ and $g \in G$, then $r.g = r^g$. This module is called the *principal* *S-module.*

In the special case of a skew group ring over a commutative coefficient ring. We obtain

7.6 COROLLARY. *Let G be a finite group acting on a commutative ring R and let $S = R*G$ be the skew group ring. Then the following are equivalent:*

(i) $r.gl.dim.(R*G) < \infty$;

(ii) *(a)* $r.gl.dim.(R) < \infty$;

(b) *R is projective as a principal right $R*G$-module;*

(iii) *(a)* $r.gl.dim.(R) < \infty$;

(b) *there exists an element $c \in R$ such that $tr(c) = 1$.*

PROOF. Let R, G, and $R*G$ be as stated.

(i) ($\Longrightarrow$) (ii) Suppose that $r.gl.dim.(R*G) < \infty$. from 1.1 Lemma (ii) we obtain $r.gl.dim.(R) < \infty$. From 1.4 Theorem (iii) we know that the principal right $R*G$-module R is projective.

(ii) ($\Longrightarrow$) (iii) This is obvious since R is a projective right S-module if and only if the trace map is surjective; see [ARS, Proposition 1.7]

(iii) ($\Longrightarrow$) (i) This follows from 7.2 Proposition (ii). ∎

7.7 REMARKS. (i) The equivalence of (i) and (ii) in 7.6 Corollary also appears in a recent preprint of Aljadeff; see [Al3, Proposition 3.4]. Related results (about hereditary and semihereditary) also appear in [GD]; see [GD, Corollary 3.7].

(ii) Note that (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) in the above theorem is valid without any commutativity hypothesis on R. However, when R is non-commutative, (iii) and (ii) do not imply (i), as the following example (due to [GD]) demonstrates.

7.8 EXAMPLE. (6.1 Example, [GD, Section 3]) Let k be a field of characteristic $p > 0$, let R be the 2x2 matrix ring over k and let G be the multiplicative group generated by the unit $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let G act on R by conjugation by u. Then $gl.dim.(R*G) = \infty$ (see 6.1 Example); but by some direct calculations, we have $tr(u) = 1$. This example supports the statements of 7.5 remarks (iii) and 7.7 Remarks (ii).

§ 2.8 FINITISTIC DIMENSIONS OF STRONGLY GROUP GRADED RINGS

Finitistic dimensions (see 8.1 Definition below) are useful tools to study rings of infinite global dimension as shown in [ABu1], [Ba1] and [Se]. Recently the finitistic dimensions of Noetherian rings and fixed subrings are also studied in [KKS] and [KK]. Having studied the finiteness of global dimensions of skew group rings and crossed products in the previous sections, in this section we would like to remark that the behaviour of the finitistic dimensions of strongly group graded rings is much simpler than that of their global dimensions. We refer to [Ba1] for the basic properties and definitions of finitistic dimensions.

8.1 DEFINITION. [Ba1, Section 5] *Let* R *be a ring. The right finitistic dimensions of* R *are defined as follows:*

$rFPD(R) = \sup\{pr.dim.(A) \mid A$ *is a right* R*-module with* $pr.dim.(A) < \infty\}$.

$rFWD(R) = \sup\{w.dim.(A) \mid A$ *is a right* R*-module with* $w.dim.(A) < \infty\}$, *where* $w.dim.(A)$ *denotes the flat dimension of* A; *see [Rot, p.238] for details.*

$rFID(R) = \sup\{inj.dim.(A) \mid A$ *is a right* R*-module with* $inj.dim.(A) < \infty\}$.

$rfPD(R) = \sup\{pr.dim.(A) \mid A$ *is a finitely generated right* R*-module and* $pr.dim.(A) < \infty\}$.

The following lemma and proposition, which are analogues of [Na, Lemma 2.3] and 1.4 Theorem ([Na, Theorem 2.1]) respectively, can be proved by a similar argument as the proof of [Na Lemma 2.3] and 1.4 Theorem. So we omit their proofs.

8.2 LEMMA. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Let M be a right S-module and let $N = \oplus_{g\in G}N_g$ be a graded left S-module. Then as Abelian groups*

$$M\otimes_S N \cong M\otimes_R N_g ,$$

for each $g \in G$. ∎

8.3 PROPOSITION. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Let M be a right S-module and let $N = \oplus_{g\in G}N_g$ be a graded left S-module. Then*

(i) for each $g \in G$ and each non-negative integer n,

$$Tor_n^S(M, N) \cong Tor_n^R(M, N_g),$$

as Abelian groups;

(ii) $w.dim._R(M) \leq w.dim._S(M)$, and the equality holds if $w.dim._S(M)$ is finite. ∎

Using 8.3 Proposition and 1.4 Theorem, we can obtain the following Theorem, which describes the relationships between the finitistic dimensions of a strongly group graded ring and those of its coefficient ring.

8.4 THEOREM. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then*

(i) $rFPD(R) = rFPD(S)$;

(ii) $rFWD(R) = rFWD(S)$;

(iii) $rFID(R) = rFID(S)$;

(iv) $rfPD(R) = rfPD(S)$.

PROOF. Let R, G and S be as stated.

(i) Let M be a right S-module with $pr.dim._S(M) < \infty$. By 1.4 Theorem (iii) we have $pr.dim._R(M) = pr.dim._S(M) < \infty$. Thus $rFPD(S) \le rFPD(R)$. Let N be a right R-module with $pr.dim._R(N) < \infty$. Since ${}_RS$ is projective by 1.1 Lemma (i), when we apply the functor $-\otimes_R S$ to a projective resolution of N_R, it becomes a projective resolution of the right S-module $N\otimes_R S$, so

$$pr.dim._S(N\otimes_R S) \le pr.dim._R(N) < \infty. \tag{1}$$

When we consider R and S as R-bimodules, R is a direct summand of S. Thus as right R-modules N_R is a direct summand of $N\otimes_R S$. Using this factor and 1.4 Theorem (iii) we have

$$pr.dim._R(N) \le pr.dim._R(N\otimes_R S) \le pr.dim._S(N\otimes_R S). \tag{2}$$

Combine (1) and (2), we have $pr.dim._S(N\otimes S) = pr.dim._R(N) < \infty$. Therefore $rFPD(R) \le rFPD(S)$ and so we obtain (i).

Using 8.3 Proposition (ii) and 1.4 Theorem (iv), by a similar argument of the proof of (i) we obtain (ii) and (iii).

Since S is finitely generated projective both as a left and as a right R-module by 1.1 Lemma (i), every finitely generated right

S-module is finitely generated as a right R-module, and for each finitely generated right R-module N, $N \otimes_R S$ is a finitely generated right S-module. Then by a similar argument of the proof of (i), we can obtain (iv). ■

8.5 REMARKS. (i) Let G be a finite group and let $R(G)$ be a strongly G-graded ring. 8.4 Theorem indicates that it is the finiteness, or not, of the global dimension of $R(G)$ which is the key property of $R(G)$, rather than the values of the projective dimensions of its modules of finite projective dimension. The latter tend to follow the same numbers over R.

(ii) Let G be a finite group and let $R(G)$ be a strongly G-graded ring. Unlike the global dimension, the injective dimension of $R(G)$ is quite stable. In fact, from 1.9 Corollary, we have

$$r.inj.dim._{R(G)}(R(G)) = r.inj.dim._R(R).$$

§ 2.9 NOTES

9.1 Most parts of this chapter (§ 2.1 to § 2.6) have been put into the paper [Yi1], which will soon appear in the *Journal of Algebra.*

9.2 1.1 Lemma and 1.5 Lemma are our generalizations of the corresponding crossed products results. 1.3, 1.4, 1.7 and 1.8 are from [Na].

9.3 2.1 is simple and must be well known. It seems that 2.2, 2.3, 2.4 and 2.6 are new results. We consider 2.3 and 2.6 as two of our main results.

9.4 3.1 is well-known. 3.2 is easy. 3.3 is one of our main results.

9.5 4.2 (ii) is from [LP2]. 4.2 (i) is the generalization of the corresponding group ring result [Sm, 2.6 Corollary]. 4.3 is due to [NA]. The rest of the results appearing in § 2.4 are new results. We consider 4.6 Theorem, 4.10 Proposition and 4.12 Corollary as some of our main results.

9.6 5.5 is from [Al2]. All the other results in § 2.5 are new ones. We consider 5.2 Theorem and 5.7 Corollary as two of our main results.

9.7 Parts (i) and (i)′ of 7.4 appeared in [NuV]. In the special case of skew group rings, Parts (i)′ and (ii)′ of 7.4 appeared in [GD]. Parts (i) and (ii) of 7.6 was obtained with a different proof in [Al3]. 7.2 is a new result.

9.8 8.2 and 8.3 are the analogues of some results in [Na]. 8.4 is a new result.

CHAPTER 3

INJECTIVE HOMOGENEITY AND THE AUSLANDER-GORENSTEIN PROPERTY

In the late sixties M. Auslander introduced various homological conditions on Noetherian, or more generally, coherent rings. These conditions were expressed in terms of the vanishing of some *Ext*-groups, and were studied both for commutative and non-commutative rings; cf. [ABr] for example. Imposing some of these conditions on Noetherian rings with finite injective dimension (respectively global dimension), people defined and studied the so called Auslander-Gorenstein (respectively regular) rings; cf. [FGR]. (See § 3.1 for the definitions and [Bj] for a survey.) It turns out that the Auslander-Gorenstein condition is very useful as it permits one to make effective use of homological techniques in non-commutative Noetherian rings; but it is very difficult to determine whether a non-commutative Noetherian ring satisfies the Auslander-Gorenstein condition or not. In [Bj] and [Ek] the Auslander-Gorenstein condition on graded rings and filtered rings are studied. For example, it is proved that if R is an Auslander-Gorenstein (respectively regular) ring, then so is any Ore extension over R; see [Ek, 4.2 Theorem]. In [Lev] and [SZ], the authors have proved that a large class of rings studied by M. Artin, W. Schelter, J. Tate and M. Van Den Bergh in [AS], [ATV1] and [ATV2] are also Auslander-Gorenstein.

Injectively homogeneous rings and homologically homogeneous rings (see § 3.1 below for the definitions) are some other classes of rings which are related to the Auslander-Gorenstein conditions. These rings are first introduced and studied for Noetherian rings which are integral over their centres by K. A. Brown and C. R. Hajarnavis in [BH1] and [BH2], and then are generalized and studied for all *FBN* rings by T. Stafford and J. J. Zhang in [SZ]. For Auslander-Gorenstein rings, another condition, the Macaulay condition, is also introduced in [SZ]. In particular, it is shown, in [SZ], that every injectively smooth Noetherian P. I. ring is Auslander-Gorenstein and Macaulay; see [SZ, Theorem 3.10]; and that homologically homogeneous Noetherian P. I. rings are Auslander-regular and integral over their centres.

We devote this chapter to the study of injectively homogeneous rings and homologically homogeneous rings, and to the study of the Auslander-Gorenstein, the Auslander-regular and the Macaulay properties of these rings. In § 3.1 all the concepts mentioned above are defined. In § 3.2 we prove that a ring strongly group graded by a finite group is right injectively homogeneous *FBN* if and only if so is its coefficient ring; see 2.13 Theorem. In § 3.3 we prove that an injectively homogeneous Noetherian P. I. ring is Auslander-Gorenstein; see 3.12 Theorem, and then in 3.14 Corollary, we give some equivalent characterizations of inj. hom. and hom. hom. Noetherian P. I. rings which have all their cliques of maximal ideals localizable.

The Auslander-regular and Auslander-Gorenstein conditions are very powerful, but they are in practice often very difficult conditions to check for a given ring. One of the main aims of this chapter is thus to give alternative, but more easily checked conditions, which are equivalent to the Auslander conditions in the presence of a rather natural additional condition, namely the locally Macaulay condition, which are introduced in 3.8 Definition. It is 3.14 Corollary which affords the desired equivalent conditions.

§ 3.1 DEFINITIONS

Let us recall several terminologies and define our concepts in this section. Suppose that R is a Noetherian ring, and that Q and P are prime ideals of R. If there exists an ideal A of R such that $QP \subseteq A \subseteq Q \cap P$ and $(Q \cap P)/A$ is nonzero and torsionfree as a right (R/P)-module and as a left (R/Q)-module, then we say there is a (second layer) *link* from Q to P; see [GW, p.178]. The *graph of links* of R is the directed graph whose vertices are the elements of $Spec(R)$ with an arrow from Q to P whenever there exists a link from Q to P. The connected components of this graph are called *cliques*, and if $P \in Spec(R)$ then we denote by $Cl(P)$ the unique clique containing P; see [GW, p.178].

Chapter 3

1.1 Let R be a ring and let M be a finitely generated right R-module. We define the *upper grade* and *grade* of M, denoted by $u.gr._R(M)$ and $j_R(M)$ respectively, or simply by $u.gr.(M)$ and $j(M)$, as

$$u.gr.(M) = sup\{n \mid Ext^n_R(M, R) \neq 0\},$$

and

$$j(M) = inf\{n \mid Ext^n_R(M, R) \neq 0\};$$

cf. [BH2], [BH1], [Bj] and [Lev].

1.2 Let R be a Noetherian ring and let M be a finitely generated right or left R-module. We say that M satisfies the *Auslander condition* if the following holds: for every integer i and every submodule N of $Ext^i_R(M, R)$, $j(N) \geq i$. (Note that $Ext^i_R(M, R)$ is an R-module, on the opposite side to M, in view of the fact that R is an R-bimodule.) If every finitely generated right and left R-module satisfies the Auslander condition, then we say R satisfies the Auslander condition. A Noetherian ring is called *Auslander-Gorenstein* (respectively *regular*), if it satisfies the Auslander condition and has finite (right and left) injective dimension (respectively global dimension); cf. [Bj] for details. (By [Za] if a Noetherian ring has finite right and left injective dimension, then the two numbers are equal.) It is clear that there exist (commutative) Noetherian rings which are Auslander-Gorenstein but not Auslander-regular. By [Ba2] we know that for commutative Noetherian rings, the Auslander condition is a direct consequence of the finiteness of injective dimension. (Thus the concept of an Auslander-Gorenstein ring is a generalization of the concept of a commutative Gorenstein ring.) Unlike the commutative case, for non-commutative Noetherian rings, the Auslander condition is not a direct consequence of the finiteness of injective dimension, and not even of the finiteness of global dimension. The following is such an example given by I. Reiten.

1.3 EXAMPLE. (see [Re, Example 2.4.6] or [Bj]) Let K be a field and let $R = \begin{pmatrix} K & 0 \\ V & K \end{pmatrix}$, where V is a 2-dimensional vector space over K. Then $gl.dim.(R) = 1$ by [MR, 7.5.1 Proposition]; but R is not Auslander-regular. In fact, if we let $J = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$ and consider it as a left R-module, then we can check that $Ext^1_R(R/J, R)$ contains a submodule, N say, which is isomorphic to the right ideal $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$, so $Ext^0_R(N, R) \neq 0$.

(Note that there are two errors in [Bj] about Reiten's Example, that is $gl.dim.(R) = 2$ and $Ext^0_R(Ext^1_R(R/J, R), R) \neq 0$ are not true.)

An Auslander-Gorenstein ring R is called *Macaulay* if $j(M) + K.dim.(M) = K.dim.(R)$ holds for every finitely generated right or left R-module M, where $K.dim.(\ )$ denotes the (Gabriel-Rentschler) Krull dimension; see [SZ, Section 1] for details. (This definition is motivated by the concept of a commutative Macaulay local ring, which satisfies an equidimensionality identity; cf. [Ma, Theorem 17.4] for example.) We list the following examples to show that Auslander-regular rings may not be Macaulay. This contrasts with the situation for commutative local rings. Recall that a commutative Noetherian regular local ring, (and even, in fact, a commutative Noetherian local Gorenstein ring) is Macaulay; see, for example, [Ka, Theorem 215] or 3.10 Lemma below. (This fact leads us to define the concept of a locally Macaulay ring in 3.8 Definition.)

1.4 EXAMPLE [SZ] Let $R = K[[x]][y]$, where K is a field and $K[[x]]$ denote the power series ring. Then it is clear that R is commutative Noetherian with global dimension 2, so it is Auslander-regular. However, R is not Macaulay, since the module $R/(1-xy)R$ is simple of projective dimension 1; cf. [SZ, Section 2] for details.

1.5 EXAMPLE. Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is any field. Then R is an Artinian ring with global dimension 1. By some direct calculations (or see also [SZ, Section 5]), R is Auslander-regular; but it is not Macaulay since $j_R(R/M) = 1$, where M is the right ideal $\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$.

In fact it is rather easy to see, and is a consequence of our later results (see 3.1 Lemma and 2.1 Lemma), that an Artinian ring is Auslander-regular and Macaulay if and only if it is semisimple; and also that an Artinian ring is Auslander-Gorenstein and Macaulay if and only if it is quasi-Frobenius.

The research of Stafford and Zhang [SZ] shows that the Auslander-Gorenstein, the Auslander-regular and the Macaulay properties are closely related to some other homological properties, which are explained in the following

1.6 DEFINITION. *Let R be an FBN ring.*

(i) If R has finite injective dimension and for each pair of maximal ideals P and Q in the same clique

$$u.gr.(R/P) = u.gr.(R/Q),$$

then R is called a right injectively homogeneous ring, right inj. hom. ring for short. If for every maximal ideal P

$$u.gr.(R/P) = inj.dim.(R),$$

then R is called right injectively smooth, right inj. smooth for short.

(ii) If R has finite global dimension and for each pair of maximal ideals P and Q in the same clique

$$pr.dim.(R/P) = pr.dim.(R/Q),$$

then R is called a right homologically homogeneous ring, right hom. hom. ring for short. If further for every maximal ideal P of R

$$pr.dim.(R/P) = gl.dim.(R),$$

then R is called a right homologically smooth ring, a right hom. smooth ring for short.

In the above definition R/P and R/Q are considered as right R-modules. We also have the left hand side versions of these concepts, which are defined analogously.

1.7 REMARK. Hom. hom. rings and inj. hom. rings are first introduced and studied by K. A. Brown and C. R. Hajarnavis in [BH1] and [BH2]. There the definitions are slightly different and the rings are supposed to be integral over their centres. We will study these kinds of rings in the later chapters. Our above definition is adopted from Stafford and Zhang [SZ]. In this chapter when we mention hom. hom. rings and inj. hom. rings we always mean in the sense of our 1.6 Definition.

§ 3.2 INJECTIVE HOMOGENEITY OF CROSSED PRODUCTS AND STRONGLY GROUP GRADED RINGS

Let G be a group and let R be a ring. Recall that we use $R*G$ to denote a skew group ring or crossed product of G over R. Suppose S is a G-graded ring, as defined in § 1.2, and suppose the component of S corresponding to the identity element of G is R. Recall that we call R the coefficient ring of S and denote $S = R(G)$. In this section, we first study the injective homogeneity of crossed products, and then use the duality machinery of smash products (defined later) to transfer our results to strongly group graded rings.

For the proof of our main results, we first give some lemmas.

2.1 LEMMA. *Let R be a right Noetherian ring and let M be a finitely generated right R-module. Then*

$$u.gr.(M) \leq \inf\{pr.dim.(M),\ r.inj.dim.(R)\}.$$

If $pr.dim.(M)$ *is finite, then* $u.gr.(M) = pr.dim.(M)$*. In particular, if* $r.gl.dim.(R) < \infty$*, then* $r.inj.dim.(R) = r.gl.dim.(R)$*.*

PROOF. The inequality is obvious.

Suppose that $pr.dim.(M) = n < \infty$. Then $Ext_R^{n+1}(M, -) = 0$ and there exists a finitely generated right R-module N such that $Ext_R^n(M, N) \neq 0$ [No, Proposition 9, p.147]. Let

$$0 \longrightarrow K \longrightarrow R^{(m)} \longrightarrow N \longrightarrow 0$$

be an exact sequence. Then we have an exact sequence

$$Ext_R^n(M, R^{(m)}) \longrightarrow Ext_R^n(M, N) \longrightarrow Ext_R^{n+1}(M, K) = 0.$$

Since $Ext_R^n(M, N) \neq 0$, We deduce that $Ext_R^n(M, R^{(m)}) \neq 0$. By [Rot, Theorem 7.14] we have

$$Ext_R^n(M, R^{(m)}) \cong \oplus\textstyle\sum_{(m\ copies)} Ext_R^n(M, R).$$

Thus $Ext_R^n(M, R) \neq 0$. Therefore $u.gr.(M) = pr.dim.(M)$. The final statement follows easily. ■

2.2 REMARK. It is possible that $pr.dim.(M) = \infty$, but $u.gr.(M)$ is finite. For example, let k be a field of characteristic $p > 0$, let G be the cyclic group of order p and let $R = k[G]$ be the group ring. Let k be the principal $k[G]$-module. Then $pr.dim.(k) = \infty$, but $u.gr.(k) = 0$ because $k[G]$ is self injective by [Ste, p.279].

From 2.1 Lemma we have

2.3 LEMMA. *Let R be an FBN ring. Then R is right hom. hom. (respectively right hom. smooth) if and only if R is right inj. hom. (respectively right inj. smooth) and has finite global dimension.* ∎

Recall that two rings R and S are said to be *Morita equivalent* if there exists a progenerator (finitely generated projective generator) M_R such that $S \cong End(M_R)$; see [MR, 3.5.5] for details. Let us quote the following well-known results from [MR] and use the same notation as being used there.

2.4 THEOREM. [MR, 3.5.7 Proposition and 3.5.9 Theorem] *Let R and S be Morita equivalent rings with progenerator ${}_SM_R$.*

(i) The functor $N_R \longrightarrow (N \otimes_R M^*)_S$ provides a category equivalence between right R-modules and right S-Modules.

(ii) The functor ${}_RN_R \longrightarrow {}_S(M \otimes_R N \otimes_R M^*)_S$ provides a category equivalence between R-bimodules and S-bimodules.

(iii) The map $A \longrightarrow MAM^*$ gives a semigroup isomorphism between the ideals of R and those of S. In particular there is a (1, 1)-correspondence between their prime ideals which preserves primitivity.

(iv) If A is an ideal of R, then R/A and S/MAM^* are Morita equivalent rings. ∎

It is well-known that *FBN* property is Morita invariant. Now we show that inj. homogeneity, inj. smoothness, hom. homogeneity and hom.

smoothness are all Morita invariant properties. These facts are needed for the proof of our later results.

2.5 LEMMA. *Let R and S be Morita equivalent rings with progenerator ${}_SM_R$. Then*

(i) R is FBN if and only if so is S;

(ii) R is FBN and right inj. hom. (resp. right inj. smooth) if and only if so is S;

(iii) R is FBN and right hom. hom. (resp. right hom. smooth) if and only if so is S.

PROOF. We only give the proof of part (iii) as an illustration. By some similar arguments we can prove the other parts.

Let R and S be as stated. Suppose that S is right hom. hom. *FBN*. By (i) we know that R is *FBN*. Let P and Q be two maximal ideals of R such that there is a link from Q to P. Then by 2.4 Theorem we can see that there is a link from MQM^* to MPM^*. Suppose that V_1 and V_2 are two simple right R-modules with right annihilators Q and P respectively. Then by 2.4 Theorem (i) $V_1 \otimes_R M^*$ and $V_2 \otimes_R M^*$ are simple right S-modules and it is clear that their right annihilators are MQM^* and MPM^* respectively. By 2.4 Theorem (i) we have

$$pr.dim._R(V_i) = pr.dim._S(V \otimes_R M^*),\ i = 1,\ 2. \tag{1}$$

Since S is right hom. hom. we have

$$pr.dim._S(V_1 \otimes_R M^*) = pr.dim._S(V_2 \otimes_R M^*) \tag{2}$$

From (1) and (2) we have $pr.dim._R(V_1) = pr.dim._R(V_2)$, so R is right hom. hom.. Symmetrically, if R is right hom. hom. FBN, then so is S. ∎

2.6 Let R be a ring with an automorphism σ and let M be a right R-module. We can construct a new R-module, denoted by M^σ, as follows. The underlying Abelian group of M^σ is that of M, but with the elements labelled by m^σ rather than m; and multiplication is defined by $m^\sigma r = (m\sigma^{-1}(r))^\sigma$. (See [MR, 7.3.4] for details.) It is easy to see that $R \cong R^\sigma$ as right R-modules and for any two right R-modules M and N we have an isomorphism of Abelian groups

$$Ext^n_R(M, N) \cong Ext^n_R(M^\sigma, N^\sigma).$$

Therefore for any finitely generated right R-module M

$$u.gr.(M) = u.gr.(M^\sigma), \tag{1}$$

and

$$j(M) = j(M^\sigma). \tag{2}$$

Suppose that I is a right ideal of R. Then we can see that $(R/I)^\sigma \cong R/\sigma(I)$ as right R-modules. Therefore

$$u.gr.(R/I) = u.gr.(R/\sigma(I)). \tag{3}$$

Since the following fact will be used quite often, we list it as

2.7 LEMMA. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Suppose that M is a finitely generated right S-module. Then $u.gr._S(M) = u.gr._R(M)$, and*

$j_S(M) = j_R(M)$.

PROOF. Let G, R, S and M be as stated. By 2.1.4 Theorem (i), $Ext_R^i(M, R) \cong Ext_S^i(M, S)$ for each non-negative integer i, so our lemma follows directly. ∎

Let S be a ring with a subring R, let P be a prime ideal of S and let p be a prime ideal of R. We recall that, see [GW] or [Let, Section 4], the prime ideal P of S is said to *lie over* p if p is a minimal prime over $P \cap R$.

2.8 LEMMA. *Let G be a finite group, let R be an FBN ring and let $S = R*G$ be a crossed product. Suppose P is a maximal ideal of S lying over a prime ideal p of R. Then*

$$u.gr._S(S/P) = u.gr._R(R/p).$$

PROOF. Let G, R, S and P be as stated. By [Pa2, Theorem 16.6], p is a maximal ideal of R and $P \cap R = \bigcap_{g \in G} p^g$, where p^g is the image of p under the automorphism of R induced by g. (The context should prevent any ambiguity with the notation of 2.6.) Since R is FBN, S is also FBN by [Let, Proposition 4.9]. By 2.3.1 Proposition S/P and R/p are simple Artinian rings. Suppose

$$S/P \cong V^{(n)}, \text{ where } V \text{ is a simple right } S\text{-module.} \tag{1}$$

Being a finitely generated module over the semisimple Artinian ring $R/(P \cap R)$, V as a right R-module is semisimple Artinian, and we may suppose

$$V \cong V_1 \oplus V_2 \oplus \ldots \oplus V_m \tag{2}$$

as right R-modules, where V_i are simple right R-modules. Then $P \cap R = ann_R(V) = \bigcap_{i=1}^{m} ann_R(V_i)$. Thus for each i, $ann(V_i) = p^{g_i}$, for some $g_i \in G$. As right R-modules

$$R/p^{g_i} \cong V_i^{(k_i)}, \text{ for some positive integer } k_i . \tag{3}$$

Therefore by (1), (2), (3) and 2.6 (3) we have

$$u.gr._R(S/P) = u.gr._R(V) = max\{u.gr._R(V_i)\}$$
$$= max\{u.gr._R(R/p^{g_i})\} = u.gr._R(R/p). \tag{4}$$

By 2.7 Lemma,

$$u.gr._S(S/P) = u.gr._R(S/P). \tag{5}$$

Combine (4) and (5) we have $u.gr._S(S/P) = u.gr._R(R/p)$. ■

We need the following lemma. It is a generalization of the corresponding result for group rings; see [Bu, 2.5] for details. Using [AF, 5.19 Lemma] or by a similar argument as the proof of [Bu, 2.5], we can easily prove it.

2.9 LEMMA. *Let G be an arbitrary group let R be a ring and let $S = R{*}G$ be a crossed product. Suppose that M is a right R-module and N is a submodule of M_R. Then N_R is essential in M_R if and only if $N \otimes_R S$ is essential in $M \otimes_R S$ as right S-modules.* ■

Let R be a ring and let M be a nonzero right R-module. Recall that an *affiliated submodule* of M is any submodule of the form

$ann_M(P)$ where P is an ideal of R maximal among the annihilators of nonzero submodules of M. An *affiliated series* for M is a series of submodules of the form

$$0 = M_0 < M_1 <\ldots < M_n = M,$$

where for each $i = 1,\ldots, n$ the module M_i/M_{i-1} is an affiliated submodule of M/M_{i-1}. If $P_i = ann_R(M_i/M_{i-1})$ then the series $P_1 ,\ldots, P_n$ is called the series of *affiliated primes* of M corresponding to the giving affiliated series. (See [GW, p.33] for details.)

We refer to [GW] for the basic properties of links between two prime ideals of a Noetherian ring, but recall the definitions given at the beginning of § 3.1. The following proposition is motivated by [Br1, Lemma 2.2].

2.10 PROPOSITION. *Let R be an FBN ring, let G be a finite group and let $S = R{*}G$ be a crossed product. Suppose that P and Q are two maximal ideals of S such that there is a link from Q to P. Let $Q \cap R = \bigcap_{g\in G} q^g$ and $P \cap R = \bigcap_{g\in G} p^g$. Then there exists an $h \in G$ such that either $q = p^h$, (in which case $P\cap R = Q\cap R$), or there is a link from q to p^h.*

PROOF. Suppose that R, G, S, P and Q are as stated. By [Pa2, Theorem 16.6] there exist maximal ideals p and q of R such that $Q\cap R = \bigcap_{g\in G} q^g$ and $P\cap R = \bigcap_{g\in G} p^g$. Since there is a link from Q to P, by [GW, Theorem 11.2] there exists a finitely generated uniform right

S-module M with an affiliated series $0 < U < M$ such that U is isomorphic to a uniform right ideal of S/P and M/U is isomorphic to a uniform right ideal of S/Q, $ann_S(U) = P$ and $ann_S(M/U) = Q$.

Let $V = ann_M(P\cap R)$. Then V is an S-module. Hence if $U \subsetneq V$, then $P\cap R \subseteq Q$. Therefore $P\cap R \subseteq Q\cap R$. Thus there exists an $h \in G$ such that $p^h \subseteq q$. Because p and q are maximal ideals of R, $p^h = q$.

Thus we may suppose that

$$U = V. \tag{1}$$

Let $H = \{g \in G \mid p^g = p\}$ and let T be a right transversal set of H in G. For $g \in T$, let E_g be the R-injective hull of the right R-module R/p^g. By 2.9 Lemma $(R/p^g)\otimes_R S$ is essential in $E_g\otimes_R S$ both as right R-modules and hence as right S-modules. By [Na, Corollary 2.6], $E_g\otimes_R S$ is an injective right S-module. Therefore $E_g\otimes_R S$ is the injective hull of $(R/p^g)\otimes_R S$. Let $E = \oplus_{g\in T}(E_g\otimes_R S)$ and let $A = \oplus_{g\in T}((R/p^g)\otimes_R S)$. Then E is the S-injective hull of A.

For all $g \in T$, p^g is a maximal ideal of R, so

$$R/(\textstyle\bigcap_{g\in T}p^g) \cong \oplus_{g\in T}(R/p^g)$$

as right R-modules. Thus as right R-modules

$$\begin{aligned} A = \oplus_{g\in T}((R/p^g)\otimes_R S) &\cong (\oplus_{g\in T}(R/p^g))\otimes_R S \\ &\cong (R/(\textstyle\bigcap_{g\in T}p^g))\otimes_R S \cong (R/(\textstyle\bigcap_{g\in T}p^g))*G. \end{aligned}$$

Let $\bar{S} = (R/(\bigcap_{g\in T}p^g))*G$ and $\bar{P} = P/(\cap p^g)S$. Because $R/(\cap_{g\in T}p^g)$ is a semisimple Artinian ring and G is finite, $\bar{S}$ is a quasi-Frobenius ring by [Na, Corollary 2.10]. By [Ste, p.276 Definition and Proposition 3.1], $\bar{P}$ is a right annihilator ideal of $\bar{S}$; that is, there exists a non-zero right ideal $\bar{I}$ of $\bar{S}$ such that $\bar{I}\bar{P} = 0$. Therefore $\bar{I}$ is a right

$\bar{S}/\bar{P}$-module, so it is a right S/P-module. Because S/P is a simple Artinian ring and U is isomorphic to a uniform right ideal of S/P, U is a simple right S/P-module, so we may suppose that $U \subseteq \bar{I}$ as right S-modules. Since U is essential in M, we may suppose that $M \subseteq E$. It is obvious that $A(P \cap R) = 0$. By (1) $A \cap M = U$. Therefore

$$M/U = M/(A \cap M) \cong (M+A)/A \subseteq E/A.$$

Choose a uniform R-submodule C/A of E/A such that $C/A \subseteq M/U$ and $ann_R(C/A) = L$, say, is a prime ideal of R. Since $Q \cap R = ann_R(M/U) \subseteq L$, $q^f \subseteq L$, for some $f \in G$, so $L = q^f$ by the maximality of q^f. As R-modules

$$C/A \subseteq E/A \cong \oplus_{\substack{g \in T \\ h \in G}} (E(R/p^g) \otimes_R h)/((R/p^g) \otimes_R h).$$

Since C/A is uniform, among all the projection maps from the above direct sum to its direct summands, there exists one which is an injection when restricted to C/A. Thus there exist $g \in T$ and $h \in G$ such that C/A is isomorphic to a submodule of $(E(R/p^g) \otimes_R h)/((R/p^g) \otimes_R h)$. Let D be a submodule of $E(R/p^g) \otimes_R h$ containing $(R/p^g) \otimes_R h$ such that

$$C/A \cong D/((R/p^g) \otimes_R h).$$

Let $F = ann_D(p^{gh})$. If $F = D$ then $p^{gh} \subseteq L$ and so $p^{gh} = L = q^f$. Therefore $q = p^{ghf^{-1}}$. Thus we may suppose that $F \neq D$. Choose $D' \subseteq D$ such that $0 \subset F \subset D'$ is an affiliated series. It is easy to see that its affiliated primes are p^{gh} and L. By Jategaonkar's main Lemma [GW, Theorem 11.1] there is a link from L to p^{gh}; that is, from q^f to p^{gh}. Therefore there is a link from q to $p^{ghf^{-1}}$. ■

2.11 PROPOSITION. *Let R be an FBN ring, let G be a finite group* *and let $S = R*G$ be a crossed product. Then R is right inj. hom. (resp. right inj. smooth) if and only if so is S.*

PROOF. Suppose that R, G and S are as stated. Since R is FBN, by [Let, Proposition 4.9] S is also FBN. By 2.1.9 Corollary, we have

$$inj.dim._S(S) = inj.dim._R(R),$$

so R has finite injective dimension if and only if S has finite injective dimension.

($\Longrightarrow$) Suppose that R is right inj. hom.. Let P and Q be two maximal ideals of S in the same clique. By [Pa2, Theorem 16.6] we may suppose that

$$P \cap R = \bigcap_{g \in G} p^g, \text{ and } \quad Q \cap R = \bigcap_{g \in G} q^g,$$

where p and q are two maximal ideals of R. By 2.10 Proposition $q \in \bigcup_{g \in G} cl(p^g)$, so there exists an $h \in G$ such that $q \in cl(p^h)$. Since R is right inj. hom. we have $u.gr._R(R/q) = u.gr._R(R/p^h)$. By 2.6 (3) we have

$$u.gr._R(R/p) = u.gr._R(R/q). \tag{1}$$

By 2.8 Lemma and (1), we obtain $u.gr._S(S/P) = u.gr._S(S/Q)$. Therefore S is right inj. hom..

($\Longleftarrow$) Suppose that S is right inj. hom.. Let p and q be two maximal ideals of R such that there is a link from q to p. By [Let, Theorem 5.3], there exist prime ideals Q and P of S with Q lying over q and P lying over p such that Q and P are in the same clique. By [Pa2, Theorem 16.6] P and Q are maximal ideals of S. By 2.8 Lemma and the right injective homogeneity of S we have

$$u.gr._R(R/p) = u.gr._S(S/P) = u.gr._S(S/Q) = u.gr._R(R/q).$$

Therefore R is right inj. hom..

The same argument will give the proof of the right inj. smooth version. ∎

2.12 For the use in the proof of our main results, let us briefly recall the definition of smash product. Suppose G is a finite group and suppose S is a G-graded ring. The *smash product* of G over S, denoted by $S\#G^*$, is an associative ring with identity and having S as a subring. As an Abelian group $S\#G^*$ is a free right S-module with a basis $\{p_x\}_{x\in G}$. So $S\#G^* = \oplus\sum_{x\in G} p_x S$. Addition in $S\#G^*$ is the same as in the module structure and the multiplication is defined distributively by

$$(p_x s)p_y = p_x s_{zy} = s_{zy} p_y$$

for all $x, y \in G$ and $s \in S$, where $s = \sum_{x\in G} s_x$, and $z = x^{-1}$ for typographical reasons. We remind the reader that there exists an action of G on $S\#G^*$ defined by $(p_x s)^g = p_{(g^{-1}x)} s$, so we can form a skew group ring $(S\#G^*)*G$; see [CM], [Qu] and [Pa2, Section 2] for details.

It is well-known that the smash product is a useful tool to translate skew group ring results to the context of group graded rings. We now demonstrate this fact in generalizing 2.11 Proposition to strongly group graded rings.

2.13 THEOREM. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then* *(i) R is FBN if and only if S is FBN;*

(ii) R is a right inj. hom. (resp. right inj. smooth) FBN ring if and only if so is S.

PROOF. Suppose that G, R and S are as stated. We use $\approx$ to denote an equivalence of categories. By [CM, Theorem 2.2] and [Da, Theorem 2.8] we have

$$Mod(S\#G^{*}) \approx GrMod(S) \approx Mod(R), \tag{1}$$

where $Mod(\ )$ and $GrMod(\ )$ denote the categories of right modules and graded right modules respectively. As stated before there is a skew group ring $(S\#G^{*})*G$. Using [Pa2, Theorem 2.5] we have

$$(S\#G^{*})*G \cong M_{|G|}(S). \tag{2}$$

(i) Since S is a finitely generated right R-module by 2.1.1 Lemma (i), if R is FBN, then S is FBN by [Let, Proposition 4.9].

Since $S\#G^{*}$ is a finitely generated S-module, if S is FBN then $S\#G^{*}$ is FBN, again using [Let, Proposition 4.9]. By (1) and 2.5 Lemma (i) R is FBN.

(ii) ($\Longrightarrow$) Suppose that R is right inj. hom. and FBN. By 2.5 Lemma and (1) $S\#G^{*}$ is right inj. hom. and FBN. Then by 2.11 Proposition $(S\#G^{*})*G$, a skew group ring, is also right inj. hom. and FBN, and so is S by (2) and 2.5 Lemma.

($\Longleftarrow$) Suppose that S is right inj. hom. and FBN. By (i) R is FBN. By (2), 2.5 Lemma and 2.11 Proposition $S\#G^{*}$ is right inj. hom.. Then by (1) R is right inj. hom..

The same argument will give the proof of the right inj. smooth version. ∎

We give the following example to show that 2.13 Theorem (ii) fails if we assume only S is G-graded, instead of strongly G-graded.

2.14 EXAMPLE. Let K be an arbitrary field and let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Let $G = \langle x \rangle$ be a group of order 2. Then R is G-graded by the decomposition $R = R_1 \oplus R_x$, where

$$R_1 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \text{ and } R_x = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}.$$

Obviously, R is not strongly G-graded. By [MR, 7.5.1 Proposition] it is clear that $gl.dim.(R) = 1$. It is well-known that R has only one clique and the simple right R-module $\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ is obviously projective. Therefore R is not hom. hom.; but R_1 is clearly hom. smooth.

From the proof of 2.13 Theorem, we obviously have the following

2.15 COROLLARY. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then*

(i) S is FBN if and only if $S\#G^*$ is FBN;

(ii) S is right inj. hom. (resp. right inj. smooth) if and only if $S\#G^*$ is right inj. hom. (resp. right inj. smooth). ∎

As we saw in chapter 2, for a finite group G and a strongly G-graded Noetherian ring S with coefficient ring R, when R has finite global dimension, the global dimension of S may be infinite (even in the group ring case). Thus we have the following modified version of 2.13 Theorem for right hom. hom. properties, which follows easily from 2.13 Theorem and 2.3 Lemma, noting also 2.1.1 Lemma (ii).

2.16 COROLLARY. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R.*

(i) If S is right hom. hom. (resp. right hom. smooth), then so is R.

(ii) If R is right hom. hom. (resp. right hom. smooth) and S has finite global dimension, then S is also right hom. hom. (resp. right hom. smooth). ■

§ 3.3 INJ. HOM. NOETHERIAN P. I. RINGS ARE AUSLANDER-GORENSTEIN

Stafford and Zhang have proved that right inj. smooth Noetherian P. I. rings are also left inj. smooth, and these rings are Auslander-Gorenstein and Macaulay [SZ, Theorem 1.3]. We prove that the converse of this result is also true; see 3.2 Proposition below, and that right inj. hom. Noetherian P. I. rings are also left inj. hom., so in this case we may simply call them inj. hom. rings. [SZ, Theorem 5.6] shows that hom. hom. Noetherian P. I. rings are Auslander-regular. In this section, as one of our main purpose, we generalize this result to prove that inj. hom. Noetherian P. I. rings are Auslander-Gorenstein. The other main purpose in this section is to give equivalent characterizations of inj. hom. and hom. hom. Noetherian P. I. rings which have all their cliques of maximal ideals localizable. At the end of this section we point out that a strongly graded ring by a finite group is Auslander-Gorenstein (or Macaulay) *FBN* if and only if so is its coefficient ring.

3.1 LEMMA. *Let R be an Auslander-Gorenstein, Macaulay, FBN ring. Then for each maximal ideal Q of R,*

$$j(R/Q) = u.gr.(R/Q) = inj.dim.(R) = K.dim.(R),$$

where R/Q and R may both be considered as right or left R-modules. In particular, R is right and left inj. smooth.

PROOF. We only state our proof for left modules since for right modules the argument is similar. Suppose that $inj.dim.(R) = n$. Then there exists a finitely generated left R-module M such that $Ext^n_R(M, R) \neq 0$. Denote $Ext^n_R(M, R)$ by $E^n(M)$. By [Br2, Lemma 2.1 (i)] we know that $E^n(M)$ is a finitely generated right R-module. By the Auslander-Gorenstein condition, $j(E^n(M)) \geq n$. By the Macaulay condition, we have

$$n \leq j(E^n(M)) \leq j(E^n(M)) + K.dim.(E^n(M)) = K.dim.(R). \tag{1}$$

Suppose that Q is a maximal ideal of R, so R/Q is a simple Artinian ring by 2.3.1 Proposition, and then $K.dim.(R/Q) = 0$. Thus by the Macaulay condition and [Lev, 2.2 Remarks (1)]

$$K.dim.(R) = K.dim.(R/Q) + j(R/Q) = j(R/Q) \leq n. \tag{2}$$

From (1) and (2), we obtain

$$j(R/Q) = u.gr.(R/Q) = inj.dim.(R) = K.dim.(R). \qquad \blacksquare$$

Since Noetherian P. I. rings are *FBN* (see [MR, 13.6.6 (iii)]), from 3.1 Lemma and [SZ, Theorem 3.10] we have

3.2 PROPOSITION. *Let R be a Noetherian P. I. ring. Then R is inj. smooth if and only if R is Auslander-Gorenstein and Macaulay.* ■

Suppose that R is a ring. Recall that the *Laurent series ring* of R, denoted by $R((x))$, is $R((x)) = \{\sum_{j=n}^{\infty} r_j x^j \mid r_j \in R,\ n \in \mathbb{Z}\}$; cf. [GS]. It is the localization of the power series ring $R[[x]]$ with respect to its Ore set $\{x^i \mid i = 0, 1, \ldots\}$. For a finitely generated right R-module M, we denote $M \otimes_R R((x))$ by $M((x))$. Since it is not known at present whether the cliques of a Noetherian P. I. ring are always localizable (see 3.5 below), we are forced to pass from R to $R((x))$ in proving 3.12 Theorem.

Let R be a ring. Recall that a module ${}_RM$ is *faithfully flat* if it is flat and also for any N_R, $N \otimes_R M = 0$ implies $N = 0$; see [MR, 7.2.3]. By using [AF, 19.20 Theorem] and [MR, 7.2.3 Proposition], we have

3.3 LEMMA. *Let R be a Noetherian ring. Then $R((x))$ is a faithfully flat R-module.* ■

Using the above lemma, we easily obtain the following

3.4 LEMMA. *Let R be a Noetherian ring and let M be a finitely generated left R-module. Then*

(i) $Ext^i_R(M, R)\otimes_R R((x)) \cong Ext^i_{R((x))}(M((x)), R((x)))$ *as right $R((x))$-modules;*

(ii) $Ext^i_R(M, R) = 0$ *if and only if* $Ext^i_{R((x))}(M((x)), R((x))) = 0$;

(iii) $j_R(M) = j_{R((x))}(M((x)))$; $u.gr._R(M) = u.gr._{R((x))}(M((x)))$.

PROOF. (i) is a direct consequence of [BL, 1.6 Proposition]. (ii) follows from (i) since $R((x))$ is a faithfully flat R-module by 3.3 Lemma, and (iii) follows directly from (ii). ∎

Suppose that R is a Noetherian ring and P is a prime ideal of R. We use $\mathcal{C}(P)$ to denote the set of all the elements of R which are regular modulo P. Let X be a set of prime ideals of R, we say X is *localizable* if: (i) $\bigcap\{\mathcal{C}(P) \mid P \in X\}$, denoted by $\mathcal{C}(X)$, is an Ore set; (ii) R_X/PR_X, for each $P \in X$, is simple Artinian and PR_X, $P \in X$, are all the primitive ideals of R_X. We call X *classically localizable* if X is localizable and has the following property: (iii) for each $P \in X$ the injective hull of the (both right and left) R_X-module R_X/PR_X is the union of its socle series; see [GW, p.219] and [Be, p.37] for details.

3.5 REMARK. It is an open question whether the cliques of a Noetherian P. I. ring are always localizable. It is also an open question whether localizable cliques of Noetherian rings are necessary classically localizable; see [GW, p.289]. But we have the following positive results:

(i) if R is a Noetherian P. I. ring which is a finitely generated algebra over its centre, then every clique of R is classically localizable;

(ii) if R is an FBN ring which contains an uncountable set F of central units such that the difference of two distinct elements of F is a unit, then every clique of R is classically localizable ([Wa, Theorem 8]);

(iii) if R is an FBN ring, then every finite clique of R is classically localizable.

See [Be, Proposition 6.1 and Theorem 6.11] for further details.

We quote the following lemma from [Sta]. It will be used later. As the paper [Sta] is not widely available, we give a proof for this lemma.

3.6 LEMMA. *Let R be a Noetherian P. I. ring. If Ω is a clique of prime ideals of R, then*

$$\Omega[[x]] = \{Q[[x]] \mid Q \in \Omega\}$$

is a clique of prime ideals of the Noetherian P. I. ring $R[[x]]$, and

$$\Omega((x)) = \{Q((x)) \mid Q \in \Omega\}$$

is a clique of prime ideals of the Noetherian P. I. ring $R((x))$.

PROOF. Suppose that R, Ω, $\Omega[[x]]$ and $\Omega((x))$ are as stated. Since R is a Noetherian P. I. ring, both $R[[x]]$ and $R((x))$ are Noetherian P. I. rings [GS]. Thus R, $R[[x]]$ and $R((x))$ all satisfy the second layer condition. (See [GW] for the definition and basic properties of the second layer condition.) In the following we denote $S = R[[x]]$.

Suppose that $Q, P \in \Omega$ and there is a link from Q to P. Then there exists an ideal A of R such that $QP \subseteq A \subseteq Q \cap P$, $(Q \cap P)/A$ is non-zero and torsionfree both as a left R/Q-module and as a right R/P-module. Clearly, we have

$$Q[[x]]P[[x]] \subseteq (QP)[[x]] \subseteq A[[x]] \subseteq (Q \cap P)[[x]] \subseteq Q[[x]] \cap P[[x]]$$

and $(Q[[x]] \cap P[[x]])/A[[x]]$ is non-zero and torsionfree both as a left $R[[x]]/Q[[x]]$-module and as a right $R[[x]]/P[[x]]$-module. So there is a link from $Q[[x]]$ to $P[[x]]$. Therefore $\Omega[[x]]$ is link connected.

To prove that $\Omega[[x]]$ is link closed. We first prove the following assertion:

suppose that Q' and P' are prime ideals of $R[[x]]$ such that there is a link from Q' to P', then either $Q' \cap R = P' \cap R$ or there is a link from $Q' \cap R$ to $P' \cap R$.

Denote $P = P' \cap R$ and $Q = Q' \cap R$. Clearly both P and Q are prime ideals of R. Suppose that there is no link from Q to P. By [GW, Exer. 12G, p.205] the bimodule ${}_{R/Q}((Q \cap P)/QP)_{R/P}$ is unfaithful at least on one side, say on the right hand side. Let $I = r.ann._R((Q \cap P)/QP)$. Therefore $P \subsetneq I$, so I is not contained in P'. Since there is a link from Q' to P', there exists an ideal A of S such that $Q'P' \subseteq A \subseteq Q' \cap P'$ and $(Q' \cap P')/A$ is non-zero and torsionfree both as a left S/Q'-module and as a right S/P'-module. Since $(((Q \cap P)S + A)/A)IS = 0$ and I is not contained in P', we have $(Q \cap P)S \subseteq A$. By factoring out $Q \cap P$, we may suppose that $Q \cap P = 0$. Then R

is a semiprime ring, and Q and P are minimal primes of R. Thus Q and P are disjoint from $\mathscr{C}_R(0)$. Localizing at $\mathscr{C}_R(0)$, we may suppose that R is semisimple Artinian. Then $Q = Re$ for some central idempotent e of R. Since $e \in Q \subseteq Q'$ and there is a link from Q' to P', by [GW, Lemma 11.7] we have $e \in P'$ and so $e \in P$. Thus we obtain $Q \subseteq P$. Symmetrically we have $P \subseteq Q$. Therefore we have $Q = P$ and the assertion is proved.

Now let $Q \in \Omega$ and let P' be a prime ideal of $R[[x]]$ such that there exists a link from $Q[[x]]$ to P'. Let $P = P' \cap R$. Obviously, we have $P[[x]] \subseteq P'$. By the assertion proved above we have either $Q = P$ or there exists a link from Q to P. If the former happens, by [GW, Corollary 12.6] we have $P' = Q[[x]]$. Suppose the latter happens. Then by our proof above there exists a link from $Q[[x]]$ to $P[[x]]$. By [GW, Corollary 12.6] again we have $P' = P[[x]]$. Thus $\Omega[[x]]$ is left link closed. Symmetrically, $\Omega[[x]]$ is also right link closed and so $\Omega[[x]]$ is a clique in $R[[x]]$.

Since $\Omega[[x]]$ is disjoint from the set $X = \{x^i \mid i = 0, 1, \ldots\}$, when we localize $R[[x]]$ with respect to X, we know that $\Omega((x))$ is a clique of $R((x))$. ∎

3.7 LEMMA. *Let R be a Noetherian P. I. ring and let N be a finitely generated right R-module. If for each clique Ω of maximal ideals of R, $N((x))_{\Omega((x))} = 0$, then $N = 0$.*

PROOF. If Ω is a clique of R, by 3.6 Lemma $\Omega((x)) = \{Q((x)) \mid Q \in \Omega\}$

is a clique of the Noetherian P. I. ring $R((x))$. By [Wa, Theorem 8], $\Omega((x))$ is classically localizable.

Without loss of generality, we may suppose that N_R is a simple module. Let $P = ann_R(N)$, a maximal ideal of R. Suppose Ω is the clique of R which contains P. If $N((x))_{\Omega((x))} = 0$, then $N((x))$ is $\mathcal{C}(\Omega((x)))$- torsion, where $\mathcal{C}(\Omega((x)))$ denotes the set of elements of R which are regular modulo each prime ideal in $\Omega((x))$. Since $R/P \cong N^{(n)}$ for some positive integer n, $R((x))/P((x)) \cong N((x))^{(n)}$. Therefore $R((x))/P((x))$ is $\mathcal{C}(\Omega((x)))$-torsion. In particular there exists a $c \in \mathcal{C}(\Omega((x)))$ such that $(1 + P((x))).c = 0$, that is $c \in P((x))$. This is a contradiction. ∎

Being motivated by 3.2 Proposition and the concept of a commutative Noetherian Cohen-Macaulay ring (see [Ma, p.136] and cf. our discussion above 1.4 Example), we would like to give the following

3.8 DEFINITION. *Let R be an Auslander-Gorenstein Noetherian P. I. ring with all its cliques of maximal ideals localizable. If for each clique Ω of maximal ideals of R, R_Ω is Auslander-Gorenstein and Macaulay, then we call R a locally Macaulay ring.*

We give the following proposition to show that the Auslander-Gorenstein and Auslander-regular properties are invariant under localizations.

3.9 PROPOSITION. *Let R be a Noetherian ring and let $\mathcal{C}$ be an Ore set of R. If R is Auslander-Gorenstein (respectively regular), then so is $R\mathcal{C}^{-1}$.*

PROOF. We only give the proof for the Auslander-Gorenstein property as the proof for the Auslander-regular property is similar.

Let R and $\mathcal{C}$ be as stated. Since $\mathcal{C}$ is a (left and right) Ore set, we have $R\mathcal{C}^{-1} = \mathcal{C}^{-1}R$ and this is a flat R-module on both sides. Let A be a right R-module. By [BL, 1.6 Proposition] we have

$$\mathcal{C}^{-1}(Ext_R^*(A,\ R)) \cong Ext^*_{R\mathcal{C}^{-1}}(A\mathcal{C}^{-1},\ R\mathcal{C}^{-1}). \tag{1}$$

Assume that A is a right $R\mathcal{C}^{-1}$-module. Then $A \cong A\mathcal{C}^{-1}$ as right $R\mathcal{C}^{-1}$-modules by [GW, Exer. 9k (a)]. So from (1) we have

$$\mathcal{C}^{-1}(Ext_R^*(A,\ R)) \cong Ext^*_{R\mathcal{C}^{-1}}(A,\ R\mathcal{C}^{-1}).$$

Therefore

$$r.inj.dim._{R\mathcal{C}^{-1}}(R\mathcal{C}^{-1}) \le r.inj.dim._R(R).$$

(As pointed out in [GJ], we do not have such an inequality for arbitrary R-modules.)

Suppose that M' is a finitely generated right $R\mathcal{C}^{-1}$-module. Then it is clear that we can find a finitely generated right R-module M such that $M' = M\mathcal{C}^{-1}$. Let i be a non-negative integer and let N' be a finitely generated $R\mathcal{C}^{-1}$-submodule of $Ext^i_{R\mathcal{C}^{-1}}(M',\ R\mathcal{C}^{-1})$. As discussed above we have

$$\mathcal{C}^{-1}(Ext_R^i(M,\ R)) \cong Ext^i_{R\mathcal{C}^{-1}}(M\mathcal{C}^{-1},\ R\mathcal{C}^{-1}) \cong Ext^i_{R\mathcal{C}^{-1}}(M',\ R\mathcal{C}^{-1}).$$

By [GW, Theorem 9.17 (a)] we can find a finitely generated submodule N of $Ext_R^i(M,\ R))$ such that $N' = \mathcal{C}^{-1}N$. Since R is Auslander-Gorenstein, by (1) we obtain $j(N') = j(\mathscr{C}^{-1}N) \geq j(N) \geq i$. The above argument also applies to left modules. So $R\mathscr{C}^{-1}$ is Auslander–Gorenstein. ∎

Note that in view of the preceding proposition, the key point of 3.8 Definition is the requirement that the ring R_Ω should be Macaulay.

Recall that a commutative Noetherian ring A is called *Cohen–Macaulay* if A_m is a Cohen–Macaulay local ring for every maximal ideal m of A; see [Ma, p.136]. Thus our locally Macaulay ring defined in 3.8 (for non-commutative rings) is the natural generalization of the concept of a commutative Cohen–Macaulay ring. For example, with our terminology of a locally Macaulay ring, the well-known result that a commutative Noetherian ring of finite injective dimension is Cohen Macaulay can be restated as the following

3.10 LEMMA. *Every commutative Noetherian ring of finite injective dimension is locally Macaulay.*

PROOF. Suppose that R is a commutative Noetherian ring of finite injective dimension. Of course the cliques of R are singletons and are localizable. Let P be a prime ideal of R. Then R_P is a commutative Noetherian local ring of finite injective dimension, so it is obviously inj. smooth. By [SZ, Theorem 3.10] R_P is Macaulay. Therefore R is locally Macaulay. ∎

3.11 REMARK. Let $R = k[[x]][y]$, where k is an arbitrary field and

$k[[x]]$ is the power series ring. Obviously, R is commutative Noetherian of finite global dimension, so it is locally Macaulay by the above Lemma; but it is not Macaulay as pointed out in 1.4 Example.

3.12 THEOREM. *Suppose that R is a right inj. hom. Noetherian P. I. ring. Then R is Auslander-Gorenstein, and R is also left inj. hom.. If each clique of maximal ideals of R is localizable, then R is locally Macaulay.*

PROOF. Let Ω be a clique of maximal ideals of R. By 3.6 Lemma $\Omega((x)) = \{Q((x)) \mid Q \in \Omega\}$ is a clique of $R((x))$ and by [Wa, Theorem 8] $\Omega((x))$ is classically localizable. We first prove that $R((x))_{\Omega((x))}$ is inj. smooth. By [GJ, Theorem 1.1]

$$inj.dim.(R((x))_{\Omega((x))}) \leq inj.dim.(R((x))) \leq inj.dim.(R[[x]]).$$

But

$$inj.dim.(R[[x]]) \leq inj.dim.(R) + 1 < \infty$$

by [Rot, Corollary 11.68]. Therefore $inj.dim.(R((x))_{\Omega((x))}) < \infty$. Since $\Omega((x))$ is classically localizable, $\{P((x))_{\Omega((x))} \mid P \in \Omega\}$ are all the maximal ideals of $R((x))_{\Omega((x))}$. Suppose that $P \in \Omega$. Then

$$Ext^{i}_{R((x))_{\Omega((x))}}(R((x))_{\Omega((x))}\big/P((x))_{\Omega((x))}, R((x))_{\Omega((x))})$$

equals 0 if and only if $Ext^{i}_{R((x))}(R((x))/P((x)), R((x))) = 0$ by 3.7 Lemma, and if and only if $Ext^{i}_{R}(R/P, R) = 0$ by 3.4 Lemma. Thus

$$u.gr._{R((x))_{\Omega((x))}}(R((x))_{\Omega((x))}\big/P((x))_{\Omega((x))}) = u.gr._{R}(R/P). \quad (1)$$

For any other element $Q \in \Omega$, since R is inj. hom. we have

$$u.gr.(R/P) = u.gr.(R/Q).$$

Then by (1) we know that

$u.gr.(R((x))_{\Omega((x))}/Q((x))_{\Omega((x))}) = u.gr.(R((x))_{\Omega((x))}/P((x))_{\Omega((x))})$,

and this number must be $inj.dim.(R((x))_{\Omega((x))})$ by [SZ, Lemma 3.12]. Therefore $R((x))_{\Omega((x))}$ is inj. smooth. By [SZ, Theorem 3.10] $R((x))_{\Omega((x))}$ is Auslander-Gorenstein.

Let M be a finitely generated right or left R-module, let n be a non-negative integer. Suppose that N is a submodule of $Ext^n_R(M, R)$. Then

$$N((x)) \subseteq Ext^n_{R((x))}(M((x)), R((x)))$$

by 3.4 Lemma and 3.3 Lemma. For every clique Ω of maximal ideals of R, we have

$$N((x))_{\Omega((x))} \subseteq Ext^n_{R((x))_{\Omega((x))}}(M((x))_{\Omega((x))}, R((x))_{\Omega((x))}).$$

Since $R((x))_{\Omega((x))}$ is Auslander-Gorenstein

$$Ext^m_{R((x))_{\Omega((x))}}(N((x))_{\Omega((x))}, R((x))_{\Omega((x))}) = 0, \text{ for all } m < n.$$

By 3.4 Lemma and 3.7 Lemma we have $Ext^m_R(N, R) = 0$, for all $m < n$. Therefore R is Auslander-Gorenstein.

Let us use $l.u.gr.(M)$ to denote the upper grade of a left module M. Suppose that R is a right inj. hom. Noetherian P. I. ring and that P is a maximal ideal of R. Let Ω be the clique of R which contains P. Since we have proved that $R((x))_{\Omega((x))}$ is (right and left) inj. smooth. By (1) and its left hand side version, we have

$$l.u.gr.(R/P) = u.gr.(R/P).$$

Therefore R is also left inj. hom..

For the final part, suppose that all the cliques of maximal ideals of R are localizable. Let Ω be a clique of maximal ideals of R. By a simpler version of the above argument, R_Ω is inj. smooth. Thus R_Ω is Auslander–Gorenstein and Macaulay by [SZ, Theorem 3.10], so R is locally Macaulay. ■

3.13 REMARK. When R is commutative Noetherian, then as indicated in 3.10 Lemma, R Auslander–Gorenstein implies R locally Macaulay. But we do not have this implication for noncommutative Noetherian P. I. rings. Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is any field. As indicated in 1.5 Example, R is Auslander–regular; but it is not Macaulay. Since R has only one clique Ω, say, and $R_\Omega = R$, it is not locally Macaulay. It is also easy to see that R is not inj. hom., so this example also shows that the Auslander–Gorenstein property does not imply injective homogeneity.

Fortunately, for Noetherian P. I. rings with all their cliques of maximal ideals localizable, we have the following more precise result.

3.14 COROLLARY. *Let R be a Noetherian P. I. ring with each clique of maximal ideals of R localizable. Then*

(i) R is inj. hom. if and only if R is Auslander–Gorenstein and locally Macaulay;

(ii) R is hom. hom. if and only if it is Auslander–regular and locally Macaulay.

PROOF. (i) ($\Longrightarrow$) This is a consequence of 3.12 Theorem.

($\Longleftarrow$) Suppose that R is Auslander–Gorenstein and locally Macaulay.

Let P and Q be two maximal ideals of R in the same clique Ω, say. Then R_Ω is Auslander-Gorenstein and Macaulay by hypothesis. By 3.2 Proposition R_Ω is inj. smooth. As shown in the proof of 3.12 Theorem we have

$$u.gr.(R/P) = u.gr.(R_\Omega/P_\Omega) = u.gr.(R_\Omega/Q_\Omega) = u.gr.(R/Q),$$

so R is inj. hom..

The proof of (ii) follows from (i) and 2.3 Lemma. ∎

We studied the injective homogeneity and homological homogeneity of strongly group-graded rings in section 2. As illustrated in the present section and [SZ], these properties are closely related to the Auslander-Gorenstein, Auslander-regular and Macaulay properties, so we would like to finish this section with a proposition about the Auslander-Gorenstein and Macaulay properties of the strongly group graded rings. To prove the proposition we need the following lemma.

3.15 LEMMA. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Suppose that R is Noetherian. Let $N = \oplus_{g\in G}N_g$ be a finitely generated graded S-module. Then for each $g \in G$, we have*

$$K.dim._R(N_g) = K.dim._R(N) = K.dim._S(N).$$

PROOF. Suppose that R, G, S and N are as stated. It is obvious that all the stated Krull dimensions exist. For each $g \in G$, we define a map from the lattice of R-submodules of N_g to the lattice of R-submodules of N_1 by sending L $(\subseteq N_g)$ to $LS_{g^{-1}}$. Since S is strongly G-graded, this map preserves strict inclusion. By [GW, Exer. 13Q, p.235], we have $K.dim._R(N_g) \le K.dim._R(N_1)$. Symmetrically, we also have the reversed inequality. Therefore by [GW, Corollary 13.2] and [Na, Theorem 1.2] we have

$$K.dim._R(N_g) = K.dim._R(N) = K.dim._S(N). \qquad \blacksquare$$

3.16 PROPOSITION. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Suppose that R is Noetherian, (but not necessarily fully bounded) Then*

(i) R is Auslander-Gorenstein if and only if S is Auslander-Gorenstein;

(ii) R is Auslander-Gorenstein and Macaulay if and only if so is S.

PROOF. Suppose that G, R and S are as stated. By 2.1.9 Corollary we have

$$inj.dim._R(R) = inj.dim._S(S),$$

so R has finite injective dimension if and only if S has. In the following we only state our proof for right modules since the same argument applies to left modules.

(i) $(\Longrightarrow)$ Suppose that R is Auslander-Gorenstein. Let M be a finitely generated right S-module and let i be a non-negative integer. Suppose that N is an S-submodule of $Ext^i_S(M, S)$. Since $Ext^i_S(M, S)$ is isomorphic to $Ext^i_R(M, R)$ by 2.1.4 Theorem, by 2.7 Lemma and the

Auslander-Gorenstein property of R, we have

$$j_S(N) = j_R(N) \geq i.$$

Thus S is Auslander-Gorenstein.

($\Longleftarrow$) Suppose that S is Auslander-Gorenstein. For each finitely generated right R-module M, every non-negative integer i and every submodule N of $Ext_R^i(M, R)$, $S\otimes_R N$ is a submodule of $S\otimes_R Ext_R^i(M, R)$; but

$$S\otimes_R Ext_R^i(M, R) \cong Ext_S^i(M\otimes_R S, S)$$

by [BL, 1.6 Proposition]. Since S is Auslander-Gorenstein,

$$j_S(S\otimes_R N) \geq i. \tag{1}$$

Because as R-modules N is a direct summand of $S\otimes_R N$, it follows that $j_R(N) \geq j_R(S\otimes_R N)$. Thus by 2.7 Lemma and (1) we have

$$j_R(N) \geq j_R(S\otimes_R N) = j_S(S\otimes_R N) \geq i.$$

Therefore R is Auslander-Gorenstein.

(ii) For every finitely generated right S-module M, by [Na, Theorem 1.2] we have

$$K.dim._R(M) = K.dim._S(M). \tag{2}$$

For every finitely generated right R-module N, by 3.15 Lemma we have

$$K.dim._R(N) = K.dim._R(N\otimes S) = K.dim._S(N\otimes S). \tag{3}$$

Then (ii) follows easily from (2), (3), 2.7 Lemma and the definition of the Macaulay condition. ■

§ 3.4 NOTES

4.1 § 3.1 consists of definitions and classical examples.

4.2 2.1 is well-known. 2.3 is an analogue of [BH2, Theorem 6.5]. 2.4 is from [MR]. 2.9 is a generalization of [Bu, 2.5]. 2.10 is motivated by [Br2, Lemma 2.2]. All the other results are new. 2.10 Proposition, 2.13 Theorem and 2.16 Corollary are our main results in § 3.2.

4.3 One part of 3.2 is from [SZ]. 3.3, 3.4 and 3.10 are well-known. 3.6 is from [Sta]. One part of 3.15 is from [Na]. The concept of a locally Macaulay ring (3.8 Definition) is new. 3.2, 3.12 and 3.14 are our main results in § 3.3.

CHAPTER 4

INJECTIVELY HOMOGENEOUS NOETHERIAN RINGS INTEGRAL OVER THEIR CENTRES

In this chapter, we study injectively homogeneous and homologically homogeneous Noetherian rings in the sense of K. A. Brown and C. R. Hajarnavis's original definitions; see § 4.1 for details. So the rings which we study in this chapter are always supposed to be integral over their centres. The main purpose is to prove that a Noetherian ring integral over its centre is injectively smooth if and only if it is Auslander-Gorenstein and Macaulay, see 2.11 Theorem; and that a Noetherian ring integral over its centre is injectively homogeneous if and only if it is Auslander-Gorenstein and locally Macaulay, see 2.12 Theorem. These results are parallel to those about Noetherian P. I. rings given in 3.3.2 Proposition (also cf. [SZ, Theorem 3.10]) and 3.3.14 Corollary (also cf. [SZ, Theorem 5.6]).

chapter 4

§ 4.1 DEFINITIONS AND RELATIONSHIPS WITH 3.1.6

As pointed out in the previous chapter, injectively homogeneous rings and homologically homogeneous are first introduced and studied by K. A. Brown and C. R. Hajarnavis in [BH2] and [BH1]. There these rings are supposed to be integral over their centres. In this section, we would like to explain their definitions and study the relationships between these definitions and the ones defined by Stafford and Zhang as given in 3.1.6.

1.1 DEFINITION. *Let R be a Noetherian ring integral over a central subring C. Suppose that R has finite right injective dimension. We say that R is right injectively homogeneous (inj. hom. for short) over C if*

$$u.gr._R(R/P) = u.gr._R(R/Q)$$

for all maximal ideals P and Q of R with $P \cap C = Q \cap C$. (See [BH2] for details.) As before, cf. 3.1.6, we say that R is injectively smooth (inj. smooth for short) if $u.gr._R(R/P) = r.inj.dim._R(R)$ for every maximal ideal P of R.

Let R be a Noetherian ring with centre Z and let C be a subring of Z over which R is integral. By [BH2, Corollary 3.6], R is right inj. hom. over C if and only if R is right inj. hom. over Z.

By [BH2, Corollary 4.4] R is right inj. hom. if and only if it is left inj. hom.. So from now on we may simply call a right inj. hom. ring over a central subring a inj. hom. ring.

1.2 DEFINITION. *Let R be a right Noetherian ring with finite right global dimension and integral over a central subring C. We call R a right homologically homogeneous (hom. hom.) ring over C if*

$$pr.dim._R(R/P) = pr.dim._R(R/Q)$$

for all maximal ideals P and Q of R with $P \cap C = Q \cap C$. (See [BH1] for details.) As defined in 3.1.6, we say that R is homologically smooth (hom. smooth for short) if $pr.dim._R(R/P) = r.gl.dim.(R)$ for each maximal ideal P of R.

Let R be a right Noetherian ring integral over a central subring C. Suppose that Z is the centre of R. By [BH1, 2.7 Proposition] R is right hom. hom. over C if and only if it is right hom. hom. over Z. So we simply call R a right hom. hom. ring. Suppose that R is a Noetherian ring integral over its centre. Then by [BH2, Theorem 6.5] or our 3.2.1 Lemma, we know that R is (right or left) hom. hom. (resp. hom. smooth) if and only if it is inj. hom. (resp. inj. smooth) and has finite global dimension.

Now we give two examples to show that, for the class of inj. hom. (smooth) Noetherian P. I. rings as defined in 3.1.6 and the class of inj. hom. (smooth) Noetherian rings integral over their centres as defined in 1.1, neither one is a special case of the other.

1.3 EXAMPLE. [SZ, Example 5.10] There exists a Noetherian P. I. ring R which is inj. smooth (in the sense of 3.1.6); but it is not integral over its centre.

PROOF. Let k be a field which contains an element λ transcendental over the prime subfield of k. Let $S = k\{x, y\}/(xy-\lambda yx)$ and let $R = S/x^2S$. Then R is a Noetherian P. I. ring and R is also Auslander–Gorenstein and Macaulay; see [SZ, Example 5.10] for full details. Thus R is inj. smooth by 3.3.2 Proposition. Since λ is transcendental, it is clear that the centre of R is k and so R is not integral over its centre. ∎

1.4 EXAMPLE. There exists a division ring D which is integral over its centre, so is a hom. smooth ring in the sense of 1.2 Definition; but D is not a P. I. ring.

PROOF. Let k be a field of characteristic $p > 0$. Let

$$R = k[x_1 , x_2 ,\dots][\theta_1 , \theta_2 ,\dots; \partial/\partial x_1 , \partial/\partial x_2 ,\dots].$$

Then R is an Ore domain since each element of R is contained in an n^{th} Weyl Algebra over k for some positive integer n and the n^{th} Weyl Algebra is an Ore domain. Let D be the quotient division ring of R. Then

$$D = k(x_i , \theta_i , \mid i = 1, 2,\dots),$$

subject to the relationships: $x_i\theta_j-\theta_jx_i = \delta_{ij}$, $x_ix_j - x_jx_i = 0$, $\theta_i\theta_j - \theta_j\theta_i = 0$.

It is easy to see that the centre of D is

$$C = k(x_i^p\ ,\ \theta_i^p \mid i = 1, 2, \ldots).$$

It is obvious that D is not finite dimensional over C. Therefore D is not a P. I. ring.

For each element $f \in D$, there exists a positive integer n such that

$$f \in D_n = k(x_1 ,\ldots, x_n\ ;\ \theta_1 ,\ldots, \theta_n \mid \ldots\ ).$$

It is clear that D_n is finite dimensional over its centre, which is

$$C_n = k(x_1 ,\ldots, x_n\ ,\ \theta_1 ,\ldots, \theta_n\ ),$$

so D_n is integral over C_n. Therefore D is integral over its centre C. ∎

Note that we use exactly the same conditions to define inj. smoothness and hom. smoothness in 1.1, 1.2 and 3.1.6 Definitions. Although the conditions used to define inj. hom. and hom. hom. in 1.1, 1.2 and 3.1.6 are slightly different, there are still close relationships between these definitions, especially for Noetherian P. I. rings.

Let R be a Noetherian ring integral over its centre Z, say. (Then R is *FBN* by [BHM2, 3.5 Lemma (i)]). By [GW, Lemma 11.7] for every two prime ideals Q and P of R in the same clique, we have $Q \cap Z = P \cap Z$. Therefore if R is inj. (resp. hom.) hom. in the sense of 1.1 (resp. 1.2), then R is also inj. (resp. hom.) hom. in the sense of 3.1.6.

If R is a Noetherian P. I. ring which is hom. hom. in the sense of 3.1.6 Definition, then R is integral over its centre by [SZ, Theorem 6.5] and R is also hom. hom. in the sense of 1.2 definition by [SZ, Remarks 5.7 (ii)]. But the converse is not true in general as demonstrated by 1.4 Example, and we do not have such a statement for inj. hom. Noetherian P. I. rings as shown by 1.3 Example.

We know that the cliques of a Noetherian ring which is a finite module over its centre can be determined precisely by the following

1.5 PROPOSITION. (B. Müller, [GW, Theorem 11.20]) *Let R be a Noetherian ring which is a finite module over its centre Z. Then for each prime ideal P of R, the clique of R which contains P is determined by*

$$cl(P) = \{Q \in Spec(R) \mid Q \cap Z = P \cap Z\}. \quad \blacksquare$$

From the above proposition, it is obvious that the definition of inj. hom. rings and hom. hom. rings given above are equivalent to those given in 3.1.6 for Noetherian rings which are finite modules over their centres. We would naturally conjecture that for all Noetherian rings integral over their centres the above definitions for inj. hom. and hom. hom. rings are always equivalent to those of 3.1.6. This leads to the following more general (well-known) conjecture.

1.6 CONJECTURE. *Let R be a Noetherian ring with centre Z. Suppose that R is integral over Z. Then for each prime ideal P of R,*

$$cl(P) = \{Q \in Spec(R) \mid Q \cap Z = P \cap Z\}.$$

By [GW, Lemma 11.7] we know that the set defined on the right of the above identity is a union of cliques ($cl(P)$ and probably some others). So the problem in the conjecture is to show that the set on the right hand side is "link connected".

§ 4.2 THE AUSLANDER-GORENSTEIN AND THE MACAULAY PROPERTIES

In this section our inj. hom. and hom. hom. rings are always in the sense of 1.1 and 1.2 Definitions, and we always suppose that R is a Noetherian ring integral over a central subring C.

Let A be a finitely generated right R-module and let I be an ideal of C. We define a *C-sequence* in I on A to be an ordered sequence of elements $x_1, x_2, \ldots, x_n$ of I such that

(i) $A(\sum_{i=1}^{n} x_i R) \neq A$;

(ii) x_1 is not a zero divisor of A (that is $ax_1 = 0$, $a \in A$, implies $a = 0$), and for $i = 1, \ldots, n-1$ x_{i+1} is not a zero divisor of $A/(A(\sum_{j=1}^{i} x_j R))$.

A C-sequence $x_1, x_2, \ldots, x_n$ in I on A is called *maximal* if I consists of zero divisors on $A/(A(\sum_{i=1}^{n} x_i R))$. If I is an ideal of R, then we define a C-sequence in I on A as the above with each x_i in $I \cap C$. By [BHM2, 4.4 Proposition] we know that for each ideal I of C and each finitely generated right R-module A, any two maximal

C-sequences in I on A have the same length. We define the common length of maximal C-sequences in I on A as the *C-grade of I on A* and denote it by $G_C(I, A)$. When $A = R$, we may denote it by $G_C(I)$, or simply by $G(I)$ and call it the *C-grade of I*; see [BHM2] for full details. (Note the differences of the C-grade of an ideal with the grade and the upper grade of a module given in 3.1.1.)

We state the following well-known result since we need it in the proof of the next lemma.

2.1 THEOREM. [Ek, 4.3 Theorem] *Let R be a Noetherian ring which contains an element t such that t is normal; that is, $tR = Rt$. Suppose that t is contained in the Jacobson radical of R and that t is not a zero divisor. If $R/(t)$ is Auslander-Gorenstein (respectively regular), then so is R.* ∎

2.2 LEMMA. *Let R be a Noetherian ring integral over its centre Z. Suppose Z is local and R is inj. hom.. Then R is Auslander-Gorenstein.*

PROOF. By [BH2, Corollary 3.5] we have $K.dim.(R) = inj.dim.(R) = n$, say. We induct on n.

When $n = 0$, R is quasi-Frobenius. Thus for each (right or left) R-module N, we have $Ext_R^i(N, R) = 0$ for all $i > 0$. Therefore R obviously satisfies the Auslander condition.

Now suppose that $K.dim.(R) = n > 0$ and that the lemma is true for rings with $K.dim.(R) < n$. Let m be the unique maximal ideal of Z. By [BH2, Corollary 3.5], for any maximal ideal M of R,

$$G(M) = rank(m) = K.dim.(Z) = K.dim.(R) = n > 0.$$

According to the definition of $G(M)$, there exists an element $c \in m$ such that c is regular in R. By [BH2, Theorem 4.2], R/cR is inj. hom. and

$$K.dim.(R/cR) = inj.dim.(R/cR) = n - 1.$$

Since $c \in m$, Z/cZ is also local, by the induction hypothesis R/cR is Auslander-Gorenstein. Now $c \in m = J(Z)$, which is contained in $J(R)$ by [BHM2, 3.6 Lemma], where $J(\ )$ denotes the Jacobson radical. Therefore by 2.1 Theorem R is Auslander-Gorenstein. ■

The following lemma is an improved version of [No, Lemma 2, p.187]. We refer the reader to [No, Lemma 2, p.187] for its proof.

2.3 LEMMA. *Let R be a ring with a central subring Z. Suppose that A is a right (or left) R-module. If $A_m = 0$ for each maximal ideal m of Z, then $A = 0$.* ■

2.4 LEMMA. *Let R be a Noetherian ring and let Z be a central subring of R. If for each maximal ideal m of Z, R_m satisfies the Auslander condition, then R satisfies the Auslander condition.*

PROOF. Let M be a finitely generated non-zero right (or left) R-module. Let i be a non-negative integer and let N be an R-submodule of $Ext_R^i(M, R)$. Because localization is an exact functor, using [BL, Proposition 1.6] we have

$$N_m \leq (Ext_R^i(M,\ R))_m = Ext_{R_m}^i(M_m,\ R_m). \tag{1}$$

By [BL, Proposition 1.6] and the Auslander condition on R_m, for all $j < i$ we have, in view of (1),

$$(Ext_R^j(N,\ R))_m \cong Ext_{R_m}^j(N_m,\ R_m) = 0.$$

By 2.3 Lemma, $Ext_R^j(N,\ R) = 0$. Therefore R satisfies the Auslander condition. ∎

2.5 THEOREM. *Let R be a Noetherian ring integral over its centre Z. If R is inj. hom., then R is Auslander-Gorenstein.*

PROOF. Suppose that R is inj. hom. with centre Z and suppose that m is a maximal ideal of Z. Then Z_m is a local ring and R_m is inj. hom. over Z_m [BH2, Lemma 3.3]. By 2.2 Lemma R_m is Auslander-Gorenstein. By 2.4 Lemma R is Auslander-Gorenstein. ∎

2.6 PROPOSITION. [Bj, 1.8 Proposition] *Let R be an Auslander-Gorenstein ring. If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of finitely generated right R-modules, then*

$$j(M) = inf\{j(M'),\ j(M'')\}.$$ ∎

The following result is an analogue of [Lev, 5.10 Sublemma].

2.7 LEMMA. *Let R be an Auslander-Gorenstein ring and let*

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

be an exact sequence of finitely generated right (or left) R-modules. If

$$j_R(M_i) + K.dim.(M_i) = K.dim.(R),$$

for $i = 1, 2$, *then*

$$j_R(M) + K.dim.(M) = K.dim.(R).$$

PROOF. By 2.6 Proposition, we have

$$j(M) = min\{j(M_i) \mid i = 1, 2.\}. \tag{1}$$

By [GW, Lemma 13.1]

$$K.dim.(M) = max\{K.dim.(M_i) \mid i = 1, 2.\}. \tag{2}$$

Then by (1), (2) and hypothesis, we have

$$\begin{aligned} K.dim.(M) &= max\{K.dim.(M_i) \mid i = 1, 2.\} \\ &= max\{K.dim.(R) - j(M_i) \mid i = 1, 2.\} \\ &= K.dim.(R) - min\{j(M_i) \mid i = 1, 2.\} \\ &= K.dim.(R) - j(M). \end{aligned}$$

Therefore we have

$$K.dim.(M) + j(M) = K.dim.(R). \qquad \blacksquare$$

2.8 REMARK. Even for an Auslander-Gorenstein ring R, it is not true in general that the validity of the "Macaulay identity" passes to submodules and factor modules. For example, let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is an arbitrary field. As pointed out in 3.1.5 Example, R is Auslander-regular. Let $M_1 = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$, let $M = R$ and let $M_2 = M/M_1$. Then $K.dim.(M) + j(M) = K.dim.(R)$, but

$$K.dim.(M_2) + j(M_2) = 1 \neq 0 = K.dim.(R).$$

The first part of the following lemma is well-known.

2.9 LEMMA. *Let R be a Noetherian ring. Then for each prime ideal P of R, for each finitely generated torsionfree right (or left) (R/P)-module N,*

$$K.dim._R(R/P) = K.dim._R(N).$$

If R is Auslander-Gorenstein, then

$$j_R(R/P) = j_R(N).$$

PROOF. By [GW, Proposition 6.19] there is an imbedding map of R-modules

$$N \longrightarrow (R/P)^{(n)} \tag{1}$$

for some positive integer n. By [GW, Corollary 6.26 (a)], there is an imbedding map of R-modules

$$R/P \longrightarrow N^{(k)} \tag{2}$$

for some positive integer k. By [GW, Lemma 13.1] and using (1) and (2), we have

$$K.dim._R(N) = K.dim._R(R/P).$$

Suppose that R is Auslander-Gorenstein. Using 2.6 Proposition, (1) and (2), we have

$$j_R(N) = j_R(R/P). \qquad \blacksquare$$

Suppose that R is a Noetherian ring integral over its centre. [BH2, Corollary 3.5 (i)] shows that if R is inj. hom., then for each maximal ideal M of R $j_R(R/M) = rank(M)$. We generalize this result to show that, in fact, this identity is true for all prime ideals of R.

2.10 LEMMA. *Let R be a Noetherian ring integral over its centre. Suppose that R is inj. hom.. Then for each prime ideal P of R,*

$$j_R(R/P) = rank(P),$$

and

$$K.dim._R(R/P) = max\{rank(M) \mid P \subseteq M \in maxspec\{R\}\} - rank(P).$$

PROOF. Let R be a Noetherian ring integral over its centre and let P be a prime ideal of R. Suppose that Z is the centre of R. Let $p = P \cap Z$. Then p is a prime ideal of Z. By [BL, Proposition 1.6]

$$(Ext^i_R(R/P\ ,\ R))_p \cong Ext^i_{R_p}(R_p/P_p\ ,\ R_p),$$

so

$$j_R(R/P) \leq j_{R_p}(R_p/P_p). \tag{1}$$

By [BH2, Theorem 4.1] R_p is inj. hom. and by [BH2, Corollary 3.5]

$$j_{R_p}(R_p/P_p) = rank(P_p) = rank(P). \tag{2}$$

Since R is inj. hom., by [BH2, Theorem 3.4 (ii)] R is Z-Macaulay in the sense of [BHM2, 4.10 Definition]. (See [BHM2] for details.) Therefore by [BHM2, 4.11 Theorem (i)] we have

$$rank(P) = G(P). \tag{3}$$

By [BH1, 3.1] we know that

$$G(P) \leq j_R(R/P). \tag{4}$$

Combine (1), (2), (3) and (4) we obtain

$$j_R(R/P) = rank(P).$$

Since R is integral over its centre, R is FBN. By [GW, Theorem 13.13] we have

$$K.dim._R(R/P) = K.dim._{R/P}(R/P) = cl.k.dim.(R/P). \tag{5}$$

But

$$cl.k.dim.(R/P) = max\{rank(M/P) \mid P \subseteq M \in maxspec\{R\}\}. \tag{6}$$

Since R is inj. hom., by [BH2, Theorem 3.4 (i) and (ii)] and [BHM2, 5.2 Theorem], R satisfies the saturated chain condition, so

$$rank(M/P) = rank(M) - rank(P). \tag{7}$$

Combine (5), (6) and (7), we have

$$K.dim._R(R/P) = max\{rank(M) \mid P \subseteq M \in maxspec\{R\}\} - rank(P). \qquad \blacksquare$$

The following Theorem is a parallel result to 3.3.2 Proposition.

2.11 THEOREM. *Let R be a Noetherian ring integral over its centre.*

(i) R is inj. smooth if and only if R is Auslander-Gorenstein and Macaulay.

(ii) R is hom. smooth if and only if R is Auslander-regular and Macaulay.

PROOF. Suppose that R is a Noetherian ring integral over its centre.

(i) ($\Longleftarrow$) This is a consequence of 3.3.1 Lemma.

($\Longrightarrow$) Suppose that R is inj. smooth. By 2.5 Theorem R is Auslander-Gorenstein. We need only to prove that R is Macaulay. Let M be a finitely generated right (or left) R-module. Since R is FBN, by [GW, Theorem 8.6], there exist a submodule series

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n = M$$

of M such that for each $i = 1, 2, \dots, n$, $P_i = ann_R(M_i/M_{i-1})$ is a prime ideal of R and M_i/M_{i-1} is isomorphic to a uniform right ideal of R/P_i, so as a R/P_i-module, M_i/M_{i-1} is torsionfree. By 2.9 Lemma and 2.10 Lemma, we have

$$j_R(M_i/M_{i-1}) = j_R(R/P_i) = rank(P_i) \tag{1}$$

and

$$K.dim._R(M_i/M_{i-1}) = K.dim._R(R/P_i)$$
$$= max\{rank(M) \mid P_i \subseteq M \in maxspec\{R\}\} - rank(P_i). \tag{2}$$

Since R is inj. smooth, for each maximal ideal M of R, by [BH2, Corollary 3.5] we have

$$rank(M) = u.gr.(R/M) = inj.dim.(R) = K.dim.(R).$$

From (2) and the above identity we have

$$K.dim._R(M_i/M_{i-1}) = K.dim.(R) - rank(P_i). \tag{3}$$

Thus from (1) and (3) we have

$$j_R(M_i/M_{i-1}) + K.dim.(M_i/M_{i-1}) = rank(P_i) + K.dim.(R) - rank(P_i)$$
$$= K.dim.(R). \tag{4}$$

By 2.7 Lemma and (4), using induction, we have

$$j_R(M_i) + K.dim.(M_i) = K.dim.(R)$$

for $i = 1, ..., n$. In particular, we have

$$j_R(M) + K.dim.(M) = K.dim.(R).$$

Therefore R is Macaulay.

(ii) follows directly from (i) and [BH2, Theorem 6.5] or 3.2.1 Lemma. ■

Let R be a Noetherian ring integral over its centre. We call R *locally Macaulay* if R_m is Auslander-Gorenstein and Macaulay for every maximal ideal m of the centre of R. Note that such a ring is Auslander–Gorenstein, by 2.4 Lemma, if it has finite injective dimension. Obviously, the ring given in 3.1.4 Example, that is, $R = F[[x]][y]$, where F is an arbitrary field, is locally Macaulay but not Macaulay.

Now we can prove the following theorem. It is one of our main results in this chapter and is a parallel result to 3.3.14.

2.12 THEOREM. *Let R be a Noetherian ring integral over its centre.*

(i) R is inj. hom. if and only if R is Auslander–Gorenstein and locally Macaulay.

(ii) R is hom. hom. if and only if R is Auslander–regular and locally Macaulay.

PROOF. Suppose that R is a Noetherian ring integral over its centre. Let Z be the centre of R.

(i) ($\Longrightarrow$) Suppose that R is inj. hom.. By 2.5 Theorem R is Auslander–Gorenstein. By [BH2, Lemma 3.3 and Corollary 3.5], for each maximal ideal m of Z R_m is inj. smooth. Therefore R_m is Auslander–Gorenstein and Macaulay by 2.11 Theorem (i), so R is locally Macaulay by definition.

($\Longleftarrow$) Suppose that R is Auslander–Gorenstein and locally Macaulay. Let P and Q be two maximal ideals of R such that $P \cap Z = Q \cap Z = m$, say. Then R_m is Auslander–Gorenstein and Macaulay. By 2.11 Theorem (i), R_m is inj. smooth. So

$$u.gr._{R_m}(R_m/P_m) = u.gr._{R_m}(R_m/Q_m). \tag{1}$$

Let I be a right ideal of R which contains m. Considering the localization of $Ext^*_R(R/I, R)$ with respect to m we know that

$$u.gr._{R_m}(R_m/I_m) \le u.gr._R(R/I).$$

Using 2.3 Lemma we obtain

$$u.gr._{R_m}(R_m/I_m) \ge u.gr._R(R/I).$$

Therefore we have

$$u.gr._{R_m}(R_m/I_m) = u.gr._R(R/I) \tag{2}$$

From (1) and (2) we obtain

$$u.gr._R(R/P) = u.gr._R(R/Q).$$

Therefore R is inj. hom..

(ii) follows from (i) and [BH2, Theorem 6.5] or 3.2.1 Lemma. ∎

In view of the results of this chapter and chapter 3, it is natural to conjecture that all our main results obtained in these two chapters are also true for inj. hom. and hom. hom. FBN rings. Specifically, we have

2.13 CONJECTURE. *Let R be an FBN ring.*

(i) R is inj. smooth if and only if it is Auslander-Gorenstein and Macaulay.

(ii) R is inj. hom. (as defined in 3.1.6) if and only if it is Auslander-Gorenstein and locally Macaulay.

In 2.13 Conjecture (ii), it is an implicit part of the conjecture that the concept of a locally Macaulay ring can be successfully defined for an arbitrary *FBN* ring; for example, it is possible that the cliques of all such rings are classically localizable.

§ 4.3 NOTES

3.1 The main results of this chapter are parallel to those given in chapter 3. The C-grade developed in [BHM2] is an essential tool used in this chapter.

3.2 § 4.1 consists of definitions and discussions.

3.3 2.1, 2.3, 2.6 and the first part of 2.9 are well-known. 2.10 is motivated by [BH2, Corollary 3.5 (i)]. 2.5, 2.11 and 2.12 are our main results of this chapter.

CHAPTER 5

INJECTIVE HOMOGENEITY AND HOMOLOGICAL HOMOGENEITY OF THE ORE EXTENSIONS

Let R be a ring, let σ be an endomorphism of R and let δ be a σ-derivation of R. The Ore extension of R determined by σ and δ is denoted by $R[x; \sigma, \delta]$. The Ore extension is one of the most important ring extensions. It is a useful tool to provide many examples and counter examples in ring theory. Its *FBN* and P. I. properties, its global dimension and Krull dimension have all been studied extensively; see [MR], [Ca] and [Ha] etc. The localization of the Ore extensions have also been studied recently; see [Po] and [Go] etc. As pointed out in chapter 3, the Auslander-Gorenstein and the Auslander-regular properties can be passed to the Ore extensions from their coefficient rings; see [Ek, 4.2 Theorem]. In this chapter, we would like to study the injective homogeneity and homological homogeneity of these rings. The main purpose is to prove that under some natural conditions, if the coefficient ring is an injectively homogeneous or homologically homogeneous ring, then so is the Ore extension; see 2.2 Theorem and 2.3 Theorem. We also give a result about the hom. smoothness of the Ore extensions; see 2.6

Proposition. In this chapter our injectively homogeneous and homologically homogeneous rings are always in the sense of K. A. Brown and C. R. Hajarnavis's original definitions; that is as given in 4.1.1 and 4.1.2 Definitions.

§ 5.1 ARTINIAN QUOTIENT RINGS AND INJECTIVE DIMENSION OF THE ORE EXTENSIONS

Let R be a ring and let σ be a monomorphism of R. It has been proved by Jategaonkar, [Jat, Theorem 3.1], that $R[x; \sigma]$ has a right Artinian right quotient ring if R has. We generalize this result to prove that, in fact, for an automorphism σ of R and a σ-derivation δ, $R[x; \sigma, \delta]$ and $R[x, x^{-1}; \sigma]$ always have right Artinian right quotient rings if R has. We need this result for our arguments in the later sections; although no doubt well-known, the easy direct proof does not appear to be in the literature; but see 1.5 Remarks of this section.

The proof for the result about skew Laurent polynomial rings is quite simple.

1.1 LEMMA. *Let R be a ring which has a right Artinian right quotient ring. Suppose that σ is an automorphism of R. Then $R[x, x^{-1}; \sigma]$ also has a right Artinian right quotient ring.*

PROOF. By [Jat, Theorem 3.1] $R[x;\ \sigma]$ has a right Artinian right quotient ring. Because

$$R[x,\ x^{-1};\ \sigma] = R[x;\ \sigma]X^{-1}, \text{ where } X = \{x^i \mid i = 0, 1, \ldots\},$$

by [GW, Exer. 9W, p.161] $R[x,\ x^{-1};\ \sigma]$ has a right Artinian right quotient ring. ∎

To prove the result for Ore extensions, we need the following lemma, which is an analogue of [Jat, Theorem 1.3].

1.2 LEMMA. *Let R be a right Artinian ring, let σ be an automorphism of R and let δ be a σ-derivation. Then $R[x;\ \sigma,\ \delta]$ has a right Artinian right quotient ring.*

PROOF. Let R, σ and δ be as stated and let $S = R[x;\ \sigma,\ \delta]$. Let $\mathcal{C}$ be the set of monic polynomials in S. Then $\mathcal{C}$ is a multiplicatively closed set and $\mathcal{C} \subseteq \mathcal{C}_S(0)$ by [MR, 7.9.2 Lemma (i)]. By [MR, 7.9.3 Proposition] $\mathcal{C}$ is a right Ore set of S, and $K.\dim.(S_{\mathcal{C}}) = K.\dim.(R)$ by [MR, 7.9.4 Theorem], and this dimension is 0 because R is right Artinian. Therefore $S_{\mathcal{C}}$ is right Artinian. By [GW, Exer. 9W] S has a right Artinian right quotient ring. ∎

To prove our main theorem, we need the following lemma. Its parts (i) and (ii) appeared in [Ca, Proposition 7.1.2], and part (iii) is a natural analogue of part (ii).

1.3 LEMMA. *Let R be a ring, let σ be an endomorphism of R and let δ be a (right) σ-derivation. Suppose that $\mathcal{C}$ is a right Ore set of regular elements of R such that $\sigma(\mathcal{C}) \subseteq \mathcal{C}$. Let $Q = R\mathcal{C}^{-1}$.*

(i) We can extend σ uniquely to an endomorphism σ' of Q and extend δ uniquely to a (right) σ'-derivation δ' of Q.

(ii) The set $\mathcal{C}$ is a right Ore set of regular elements of $S = R[x; \sigma, \delta]$ and $S\mathcal{C}^{-1} \cong R\mathcal{C}^{-1}[x; \sigma', \delta']$.

(iii) Suppose that σ is an automorphism of R. Then $\mathcal{C}$ is a right Ore set of regular elements of $S = R[x, x^{-1}; \sigma]$, σ can be extended uniquely to an automorphism σ' of Q and $S\mathcal{C}^{-1} \cong R\mathcal{C}^{-1}[x, x^{-1}; \sigma']$.

PROOF. Let R, σ, δ, $\mathcal{C}$ and Q be as stated.

(i) Suppose that τ is an endomorphism of Q which is an extension of σ. Then for each $c \in \mathcal{C}$, we have $\tau(c^{-1}c) = 1 = \tau(cc^{-1})$, so $\tau(c^{-1}) = \tau(c)^{-1} = \sigma(c)^{-1}$. Thus by some direct calculations we can see that the following map

$$\sigma': Q \longrightarrow Q; \quad \sigma'(rc^{-1}) = \sigma(r)\sigma(c)^{-1}, \text{ for all } r \in R \text{ and } c \in \mathcal{C},$$

is an endomorphism of Q and σ' is the unique extension of σ.

Suppose that β is a σ'-derivation of Q which is an extension of δ. Then for each $c \in \mathcal{C}$, we have

$$0 = \beta(1) = \beta(c^{-1}c) = \beta(c^{-1})\sigma'(c) + c^{-1}\beta(c),$$

so

$$\beta(c^{-1}) = -c^{-1}\delta(c)\sigma(c)^{-1}. \tag{1}$$

Thus we can directly check that the following map

$$\delta': Q \longrightarrow Q; \quad \delta'(rc^{-1}) = \delta(c)\sigma(c)^{-1} + r\delta'(c^{-1}),$$

for all $r \in R$ and $c \in \mathscr{C}$, where $\delta'(c^{-1})$ is defined as in (1), is the unique σ'-derivation of Q which is the extension of δ.

(ii) Let $S = R[x, \sigma; \delta]$. By considering the leading coefficients of the elements of S, we can see that each element of $\mathscr{C}$ is regular in S. In order to prove that $\mathscr{C}$ is a right Ore set of S, we first prove the following assertion by induction:

For each positive integer n, for each element $c \in \mathscr{C}$ there exist $c' \in \mathscr{C}$ and $s' \in S$ such that $x^n c' = cs'$.

Using the above assertion and the fact that $\mathscr{C}$ is a right Ore set of R, we can easily see that $\mathscr{C}$ is a right Ore set of S, cf. [Ca] for full details. Clearly, $S\mathscr{C}^{-1}$ and $R\mathscr{C}^{-1}[x; \sigma', \delta']$ are isomorphic rings as their elements have the same expressions if we consider them as right $R\mathscr{C}^{-1}$-modules.

The proof of (iii) is analogous to that of (ii). ∎

Now we can prove our

1.4 THEOREM. *Let R be a right Noetherian ring, let σ be an automorphism of R and let δ be a σ-derivation of R. Let S be $R[x, x^{-1}; \sigma]$ or $R[x; \sigma, \delta]$. If R has a right Artinian right quotient ring Q, then S also has a right Artinian right quotient ring, W say. Moreover, W is simple or semisimple Artinian if Q is simple or semisimple Artinian respectively.*

PROOF. Suppose that R, σ, and δ are as stated and that R has a right Artinian right quotient ring. If $S = R[x, x^{-1}; \sigma]$, Then by 1.1 Lemma S has a right Artinian right quotient ring. Now suppose that $S = R[x; \sigma, \delta]$. Let $\mathcal{C}$ be the set of regular elements of R. Then $\mathcal{C}$ is a right Ore set of R and $Q = R\mathcal{C}^{-1}$ is a right Artinian ring. By 1.3 Lemma σ can be extended to an automorphism σ' of Q, δ can be extended to a σ'-derivation δ' of Q, $\mathcal{C}$ is a right Ore set of S and $S\mathcal{C}^{-1} = Q[x; \sigma', \delta']$. By 1.2 Lemma $S\mathcal{C}^{-1}$ has a right Artinian right quotient ring. Therefore S also has a right Artinian right quotient ring by [GW, Exer. 9W].

Let W be the right Artinian quotient ring of $R[x, x^{-1}; \sigma]$ or $R[x; \sigma, \delta]$. By [GW, Theorems 5.10 and 5.12] Q (or W) is simple or semisimple Artinian if and only if R (or S) is prime or semiprime respectively. But S is prime (semiprime) if R is prime (semiprime) by [MR, 1.2.9 Theorem (iii) and 7.9.14 Proposition (i)]. So the final result follows. ∎

1.5 REMARKS. (i) At the end of [JJ] (also see [JJV, Corollary 2.3]), the authors announced that for a positively filtered ring, if the corresponding graded ring has a right Artinian right quotient ring, then so has the ring itself. Since $R[x; \sigma, \delta]$ is a positively filtered ring with $R[x; \sigma]$ as its graded ring, so our result about $R[x; \sigma, \delta]$ in 1.4 Theorem can be a consequence of this result and [Jat, Theorem 3.1]; but here we have given an easy and direct proof.

(ii) The example $R = \mathbb{C}\oplus\mathbb{C}$, $\sigma: (a, b) \longrightarrow (b, a)$, $a, b \in \mathbb{C}$; $S = R[x; \sigma]$ or $R[x, x^{-1}; \sigma]$ shows that in 1.4 Theorem W can be simple even if Q is not simple.

(iii) Let $R = k[y]/\langle y^p \rangle$, where p is a prime number and k is a field of characteristic p. Let $S = R[x; d/dy]$. This example shows that in 1.4 Theorem W can be simple even when Q is not semisimple.

Using 1.4 Theorem and 2.4.2 Lemma, we immediately obtain

1.6 COROLLARY. *Let R be a right Noetherian ring, let G be a polycyclic-by-finite group and let $R{*}G$ be a crossed product. If R has a right Artinian right quotient ring, then $R{*}G$ also has a right Artinian right quotient ring.* ∎

A ring R is called a *Jacobson ring* if every prime factor ring of R has zero Jacobson radical; see [MR, 9.1.2].

1.7 LEMMA. *Let K be a commutative Noetherian Jacobson ring, let R be an affine K-algebra. Suppose that σ is an automorphism of R and that δ is a σ-derivation such that K is (σ, δ)-trivial. Let $S = R[x; \sigma, \delta]$ or $R[x, x^{-1}; \sigma]$. If S is a P. I. ring, then every maximal ideal of S contains a regular element. (In fact, we prove that every maximal ideal contains a monic polynomial, which is obviously regular).*

PROOF. Suppose that K, R and S are as stated. It is obvious that S is an affine K-algebra. Let P be a maximal ideal of S. Then S/P is a simple Artinian ring by 2.3.1 Proposition. So there exists a simple right S-module V and a positive integer n such that $S/P \cong V^{(n)}$ as right S-modules. By [MR, 13.10.4 Theorem (i)], S/P is a finite dimensional vector space over the field $K/ann_K(S/P)$. Thus S/P is a finitely generated right R-module. Suppose that $\bar{1}, \bar{x}, \ldots, \overline{x^n} \in S/P$ generate S/P as a right R-module. Then

$$\overline{x^{n+1}} = \bar{1}r_0 + \ldots + \overline{x^n}r_n .$$

Thus P contains $x^{n+1} - r_0 - \ldots - x^n r_n$, which is a monic polynomial.

■

1.8 REMARKS. (i) In 1.7 Lemma suppose that R has a right Artinian right quotient ring. Then S also has a right Artinian right quotient ring by 1.4 Theorem. In this case, using [GW, Theorems 9.22 and 10.9], we can see that the result of 1.7 Lemma is equivalent to the statement that no maximal ideal of S is minimal.

(ii) In 1.7 Lemma, suppose that R is an Artinian ring which is a finite module over its centre K, say, and suppose that K is (σ, δ)-trivial. Then K is Artinian by [Ei, Theorem 1 (b)], so it is a Jacobson ring. If S is also a finite module over its own centre, then it is a P. I. ring. So in this setting all the conditions in 1.7 Lemma are satisfied. We will meet this situation later.

The relationships between the projective dimension and flat dimension of modules over an Ore extension or a skew Laurent polynomial ring and that over their coefficient rings are given by the following result.

1.9 PROPOSITION. [MR, 7.5.2 Proposition] *Let* $S = R[x; \sigma, \delta]$ *or* $S = R[x, x^{-1}; \sigma]$, *where* σ *is an automorphism of* R, *and let* M *be a right* S*-module.*

(i) There is an exact sequence of S*-modules*

$$0 \longrightarrow M^{\sigma} \otimes_R S \xrightarrow{\beta} M \otimes_R S \xrightarrow{\alpha} M \longrightarrow 0$$

with $\alpha(m \otimes s) = ms$ *and* $\beta(m^{\sigma} \otimes s) = mx \otimes s - m \otimes xs$.

(ii) $pr.dim._S(M) \le pr.dim._R(M) + 1$; $w.dim._S(M) \le w.dim._R(M) + 1$. ■

We also have a similar result for injective dimension.

1.10 PROPOSITION. *Let* $S = R[x; \sigma, \delta]$ *or* $R[x, x^{-1}; \sigma]$, *where* σ *is an automorphism of* R *and* δ *is a* σ*-derivation. Let* M_S *be a right* S*-module. Then*

$$inj.dim.M_S \le inj.dim.M_R + 1.$$

In particular, when R *is right Noetherian, then*

$$inj.dim.S_S \le inj.dim.R_R + 1.$$

PROOF. Let R and S be as stated and let M be a right S-module. For any right S-module N_S, by 1.9 Proposition there is an exact sequence of right S-modules

$$0 \longrightarrow N^{\sigma} \otimes_R S \longrightarrow N \otimes_R S \longrightarrow N \longrightarrow 0,$$

where N^{σ} is defined as in 3.2.6. Applying $Ext_S^*(-, M)$ to this, we have an exact sequence

$$Ext_S^i(N^\sigma \otimes_R S,\ M) \longrightarrow Ext_S^{i+1}(N,\ M) \longrightarrow Ext_S^{i+1}(N \otimes_R S,\ M). \tag{1}$$

Because ${}_RS$ is free, by [Rot, Theorem 11.65] we have

$$Ext_S^i(N \otimes_R S,\ M) \cong Ext_R^i(N,\ M). \tag{2}$$

If $inj.dim.(M_R) = \infty$, the result is obviously true.

Suppose that $inj.dim.(M_R) = n < \infty$. Then $Ext_R^i(V,\ M) = 0$, for all $i > n$ and for all right R-modules V. From (1) and (2) we have $Ext_S^{i+1}(N,\ M) = 0$, for all $i > n$ and for all right S-modules N. Therefore

$$inj.dim.(M_S) \leq n + 1 = inj.dim.(M_R) + 1.$$

In particular, we have $inj.dim.(S_S) \leq inj.dim.(S_R) + 1$. When R is right Noetherian, the direct sum of injective right R-modules is also injective [AF, 18.13 Proposition]. Because S_R is free we have $inj.dim.(S_R) = inj.dim.(R_R)$. Thus $inj.\ dim.(S_S) \leq inj.dim.(R_R) + 1$. ■

1.11 REMARK. 1.10 Proposition is a generalization of [BH2, Lemma 6.1] since it is proved in [BH2, Lemma 6.1] (also cf. [BH2, Theorem 6.2]) that for a Noetherian ring R with finite right injective dimension $r.inj.dim.(R[x]) = r.inj.dim.(R[x,\ x^{-1}]) = r.inj.dim.(R) + 1$.

§ 5.2 HOMOLOGICAL HOMOGENEITY AND INJECTIVE HOMOGENEITY OF THE ORE EXTENSIONS

In this section, our purpose is to prove 2.2 Theorem, 2.3 Theorem and 2.6 Proposition, which are the main results of this chapter.

2.1 LEMMA. *Let R be an Artinian selfinjective ring with centre K over which R is a finitely generated module. Let σ be an automorphism of R and let δ be a σ-derivation such that K is (σ, δ)-trivial. Let $S = R[x; \sigma, \delta]$ or $R[x, x^{-1}; \sigma]$. If S is a finitely generated module over its centre C, then for each maximal ideal P of S, $u.gr._S(S/P) = 1$. In particular S is inj. smooth.*

PROOF. Let R, K and S be as stated. Suppose that P is a maximal ideal of S. Then by 1.10 Proposition we have $inj.dim.(S_S) \leq 1$. Since selfinjective Noetherian rings are Artinian; see [Ste, p.276], it follows that S is not selfinjective, so $r.inj.dim.(S) = 1$. Therefore $Ext^i_S(S/P, S) = 0$, for all $i > 1$. According to the definition of upper grade, we need only to prove that $Ext^1_S(S/P, S) \neq 0$. Because P is a maximal ideal of S and S is a finite module over its centre, thus is FBN, so S/P is a simple Artinian ring by 2.3.1 Proposition. Let $S/P \cong V^{(n)}$, where V_S is a simple right S-module. So $Ext^1_S(S/P, S) \neq 0$ if and only if $Ext^1_S(V, S) \neq 0$. By [Rot, Corollary 7.20] $Ext^1_S(V, S) \neq 0$

if and only if there exists a non-split extension of S by V,

$$0 \longrightarrow S \longrightarrow V' \longrightarrow V \longrightarrow 0.$$

Because V is simple, this is equivalent to S having an essential extension V' such that $V'/S \cong V$. By 1.4 Theorem S has an Artinian quotient ring Q, say. Since S is essential in Q and S/P is simple Artinian, we need only to prove that Q/S has a non-zero submodule annihilated by P.

Since Q is Artinian, by [GW, Theorem 10.9] $\mathcal{C}_S(0) = \mathcal{C}_S(N)$, where N is the prime radical of S. Because S is a finite module over its centre C, $N \cap C$ is the prime radical of C. Using the same argument as [Mü, 3.1] we have

$$S\mathcal{C}_S^{-1}(N) = S\mathcal{C}_C^{-1}(N \cap C).$$

Therefore every element in $\mathcal{C}_C(N \cap C)$ is regular in C, that is

$$\mathcal{C}_C(N \cap C) \subseteq \mathcal{C}_C(0).$$

But $\mathcal{C}_C(0) \subseteq \mathcal{C}_C(N \cap C)$ [GW, Lemma 10.8], therefore $\mathcal{C}_C(0) = \mathcal{C}_C(N \cap C)$ and

$$Q = S\mathcal{C}_S^{-1}(0) = S\mathcal{C}_S^{-1}(N) = S\mathcal{C}_C^{-1}(N \cap C) = S\mathcal{C}_C^{-1}(0).$$

Since P is a maximal ideal of S, P is not a minimal prime ideal of S by 1.8 Remarks (i). Because Q is Artinian, by [GW, Theorem 9.22] $P \cap \mathcal{C}_C(0) \neq \phi$. Let $c \in P \cap \mathcal{C}_C(0)$. Since c is regular in S, by [GW, Lemma 13.6]

$$k.dim._{S/cS}(S/cS) \leq k.dim._S(S) - 1,$$

which is zero since $k.dim._S(S) = 1$ by [MR, 6.5.4 Proposition]. Therefore S/cS is Artinian. By [Rot, Corollary 11.68]

$$inj.dim._{S/cS}(S/cS) \leq inj.dim._S(S) - 1 = 0.$$

Therefore S/cS is selfinjective. Thus S/cS is a quasi-Frobenius ring.

Since P/cS is a proper ideal of S/cS, by [Ste, p.276 Definition and p.277 Proposition 3.4], we have

$$P/cS = r.l.(P/cS),$$

where $l.(\ )$ and $r.(\ )$ denote the left annihilator and right annihilator respectively. Let $l.(P/cS) = Y/cS$, it is non-zero and $YP \subseteq cS$. Therefore Y/cS is a submodule of S/cS and right annihilated by P. It is obvious that

$$\Phi:\ S \longrightarrow c^{-1}S/S;\ \ s \longrightarrow \overline{c^{-1}s}$$

is an S-epimorphism with $Ker(\Phi) = cS$. Therefore

$$S/cS \cong c^{-1}S/S \subseteq Q/S.$$

Then Q/S has a submodule isomorphic to Y/cS, and hence annihilated by P. By our above argument $Ext^1_S(S/P,\ S) \neq 0$ and $u.gr._S(S/P) = 1$. By definition S is inj. smooth. ■

We can now prove the following Theorem. It is one of the main results of this chapter. In the special case where σ is the identity and δ is trivial, it was obtained as [BH2, Theorem 6.2].

2.2 THEOREM. *Let $S = R[x;\ \sigma,\ \delta]$ or $S = R[x,\ x^{-1};\ \sigma]$, where σ is an automorphism of R and δ is a σ-derivation of R. Suppose that R is inj. hom. and it is a finitely generated module over a central subring K, which is $(\sigma,\ \delta)$-trivial. If S is a finitely generated module over its own centre C, then S is inj. hom. and*

$$inj.dim.(S_S) = inj.dim.(R_R) + 1.$$

PROOF. Suppose that R and S are as stated. By Hilbert's Basis Theorem [GW, Theorem 1.12] S is Noetherian. By 1.10 Proposition

$$inj.dim.(S_S) \leq inj.dim.(R_R) + 1 < \infty.$$

Since K is (σ, δ)-trivial, it is obvious that $K \subseteq C$. Suppose that P and Q are two maximal ideals of S such that $P \cap C = Q \cap C$. Then $P \cap K = Q \cap K = p$, say, is a prime ideal of K. Suppose that for some non-negative integer n

$$Ext^n_S(S/P, S) \neq 0.$$

Let $\mathcal{C} = K \setminus p$. Since we suppose that S is a finitely generated module over its centre, S is FBN, so S/P is a simple Artinian ring by 2.3.1 Proposition. Then each non-zero element of $(K + P)/P$ is regular in S/P, so is invertible in S/P. Thus each element of $\mathcal{C}$ induces an automorphism of S/P as right S-module. Therefore

$$(Ext^n_S(S/P, S))_{\mathcal{C}} \neq 0.$$

By [BL, 1.6 Proposition]

$$(Ext^n_S(S/P, S))_{\mathcal{C}} = Ext^n_{S_{\mathcal{C}}}(S_{\mathcal{C}}/PS_{\mathcal{C}}, S_{\mathcal{C}}) \neq 0.$$

Therefore $u.gr._{S_{\mathcal{C}}}(S_{\mathcal{C}}/P_{\mathcal{C}}) \geq u.gr._S(S/P)$. The reverse inequality is always true, so we have

$$u.gr._{S_{\mathcal{C}}}(S_{\mathcal{C}}/P_{\mathcal{C}}) = u.gr._S(S/P).$$

By the same argument we have

$$u.gr._{S_{\mathcal{C}}}(S_{\mathcal{C}}/Q_{\mathcal{C}}) = u.gr._S(S/Q).$$

Therefore we need only to prove that

$$u.gr._{S_{\mathcal{C}}}(S_{\mathcal{C}}/P_{\mathcal{C}}) = u.gr._{S_{\mathcal{C}}}(S_{\mathcal{C}}/Q_{\mathcal{C}}).$$

After localizing S with respect to $\mathcal{C}$, from now on we may suppose that

K is local with a unique maximal ideal p. Since R is inj. hom., by [BH2, Corollary 3.5] we have

$$inj.dim.(R) = K.dim.(K) = rank(p) = G(P) = t, \text{ say.}$$

So there exists a K-sequence $\{x_1, x_2, \ldots, x_t\}$ in p by the definition of $G(p)$. (See chapter 4 for the definition.) Since S_R is a free module, it is obvious that $\{x_1, x_2, \ldots, x_t\}$ is also a C-sequence in p. Let

$$\bar{S} = S/\langle x_1, x_2, \ldots, x_t \rangle S.$$

Then

$$\bar{S} \cong (R/\langle x_1, x_2, \ldots, x_t \rangle R)[x; \sigma, \delta]$$

(respectively $(R/\langle x_1, x_2, \ldots, x_t \rangle R)[x, x^{-1}; \sigma]$). By [Rot, Corollary 11.68] we have

$$u.gr._S(S/P) = u.gr._{\bar{S}}(\bar{S}/\bar{P}) + t, \tag{1}$$

and

$$u.gr._S(S/Q) = u.gr._{\bar{S}}(\bar{S}/\bar{Q}) + t. \tag{2}$$

Let $\bar{R} = R/\langle x_1, x_2, \ldots, x_t \rangle R$. Since R is inj. hom. and $inj.dim.(R) = rank(p) = t$, by [BH2, Theorem 4.2] and [BH2, Corollary 3.5] we know that $\bar{R}$ is also inj. hom. and $inj.dim.(\bar{R}) = t - t = 0$. So $\bar{R}$ is a quasi-Frobenius ring. Thus $\bar{R}$ is an Artinian selfinjective ring. It is easy to see that $\bar{R}$ is a finitely generated module over the central subring $\bar{K} = K/\langle x_1, x_2, \ldots, x_t \rangle$. By 2.1 Lemma, $\bar{S}$ is inj. smooth and

$$u.gr._{\bar{S}}(\bar{S}/\bar{P}) = u.gr._{\bar{S}}(\bar{S}/\bar{Q}) = 1. \tag{3}$$

Then by (1), (2) and (3) we obtain

$$u.gr._S(S/P) = 1 + t = u.gr._S(S/Q).$$

Therefore S is inj. hom..

To prove the final statement, by [BH2, Corollary 3.5] we can choose a maximal ideal p of K such that $rank(p) = inj.dim.(R)$. Choose a maximal ideal P of S such that $pS \subseteq P$. Then by the above argument, we can easily see that $u.gr._S(S/P) \geq 1 + inj.dim.(R)$. Then by 1.10 Proposition we obtain $inj.dim.(S) = inj.dim.(R) + 1$. ∎

2.3 THEOREM. *Let* $S = R[x; \sigma, \delta]$ *or* $S = R[x, x^{-1}; \sigma]$, *where* σ *is an automorphism of* R *and* δ *is a* σ*-derivation of* R. *Suppose that* R *is hom. hom. and it is a finitely generated module over a central subring* K, *which is* (σ, δ)*-trivial. If* S *is a finitely generated module over its own centre* C, *then* S *is hom. hom. and*

$$gl.dim.(S) = gl.dim.(R) + 1.$$

PROOF. Suppose that R, K and S are as stated. When R has finite global dimension, S also has finite global dimension by [MR, 7.5.3 Theorem (i) and (ii)]. Thus S is hom. hom. by 2.2 Theorem and [BH2, Theorem 6.5]. Since both S and R are hom. hom, by 3.2.1 Lemma and 2.2 Theorem we have

$$gl.dim.(S) = inj.dim.(S) = inj.dim.(R) + 1 = gl.dim.(R) + 1.$$

Therefore $gl.dim.(S) = gl.dim.(R) + 1$. ∎

The Weyl algebra (cf. [GW, p.15]) is one of the most commonly used Ore extensions. The following corollary shows that many of the Weyl algebras are injectively or homologically homogeneous.

2.4 COROLLARY. *Let R be a commutative Noetherian ring of positive characteristic. Let*

$$A_n(R) = R[x_1,\dots, x_n][\theta_1,\dots, \theta_n; \partial/\partial x_1,\dots, \partial/\partial x_n]$$

be the n^{th} Weyl algebra over R. If R has finite injective dimension (resp. global dimension), then $A_n(R)$ is injectively homogeneous (resp. homologically homogeneous).

PROOF. Let R and $A_n(R)$ be as stated. Suppose that the characteristic of R is p (p may not be a prime number). It is easy to see that $R[x_1^p ,\dots, x_n^p , \theta_1^p ,\dots, \theta_n^p]$ is contained in the centre of $A_n(R)$, and over which $A_n(R)$ is a finitely generated module. Then the corollary follows directly from 2.2 and 2.3 Theorems. ∎

The following result, announced in [BH1, 7.4] is a consequence of our theorem above.

2.5 COROLLARY. *Let R be a ring which is a finitely generated module over its centre K, let σ be an automorphism of R of finite order (that is $\sigma^n = 1$ for some positive integer n). Suppose that σ is the identity on K. If R is inj. hom. (resp. hom. hom.), then both $R[x; \sigma]$ and $R[x, x^{-1}; \sigma]$ are inj. hom. (resp. hom. hom.).*

PROOF. Suppose that R and σ are as stated. Let n be the order of σ. Let $S = R[x; \sigma]$ (resp. $R[x, x^{-1}; \sigma]$). It is easy to see that $K[x^n]$ (resp. $K[x^n, x^{-n}]$) is contained in the centre of S and S is a finitely generated module over $K[x^n]$ (resp. $K[x^n, x^{-n}]$). Then the corollary follows directly from 2.2 and 2.3 Theorems. ■

Having studied the injective homogeneity and homologically homogeneity of the Ore extensions, we would like to remark that, as indicated before, the Auslander-Gorenstein and Auslander-regular properties can always be passed to the Ore extensions from the coefficient rings. However, in general cases the injective smoothness, the homological smoothness and the Macaulay properties can not be passed to the Ore extensions, even not to the polynomial extensions from their coefficient rings. The ring given in 3.1.4 Example, that is $S = F[[x]][y]$, where F is an arbitrary field and $F[[x]]$ is the power series ring, can demonstrate these facts. It is well-known that $F[[x]]$ is a local ring with $gl.dim.(F[[x]]) = 1$. Thus $F[[x]]$ is hom. smooth and Macaulay; but $pr.dim._S(S/\langle 1-xy\rangle S) = 1$ and $gl.dim.(S) = 2$. Therefore S is neither hom. smooth nor Macaulay.

Note that in the example given above, $F[[x]]$ is not an affine algebra over F. For an Ore extension with a coefficient ring which is an affine algebra over a commutative Noetherian Jacobson, the following proposition throws some light on this problem.

2.6 PROPOSITION. *Let K be a commutative Noetherian Jacobson ring, let R be an affine K-algebra with finite global dimension. Suppose that σ is an automorphism of R and δ is a σ-derivation of R such that K is (σ, δ)-trivial. Let $S = R[x; \sigma, \delta]$ or $R[x, x^{-1}; \sigma]$. Suppose that S*

is a P. I. ring. If for every maximal ideal m *of* R $pr.dim._R(R/m) = gl.dim.(R)$, *then for every maximal ideal* P *of* S

$$pr.dim._S(S/P) = gl.dim.(S) = gl.dim.(R) + 1.$$

PROOF. Suppose that K, R and S are as stated. Then it is obvious that S is an affine algebra over K. Let P be a maximal ideal of S. Then as shown in the proof of 1.7 Lemma, S/P is a finite dimensional vector space over the field $K/ann_K(S/P)$, so S/P is a finitely generated right R-module. Clearly, $R/(P\cap R)$ is a subspace of S/P. So $R/(P\cap R)$ is a finite dimensional vector space over $K/ann_K(S/P)$. Therefore $R/(P\cap R)$ is an Artinian R-module. Choose a simple submodule V of $R/(P\cap R)$. By hypothesis $pr.dim._R(V) = gl.dim.(R)$. Then by [Ka, Theorem B (3) p.124], $pr.dim._R(S/P) = gl.dim.(R)$. By [MR, 7.9.16 Theorem and 7.9.17 Corollary] we have

$$pr.dim._S(S/P) = pr.dim._R(S/P) + 1 = gl.dim.(R) + 1.$$

But $gl.dim.(S) \leq gl.dim.(R) + 1$ by [MR, 7.5.3 (i)]. Therefore

$$pr.dim._S(S/P) = gl.dim.(S) = gl.dim.(R) + 1. \quad \blacksquare$$

In this chapter our injectively homogeneous rings and homologically homogeneous rings are in the sense of K. A. Brown and C. R. Hajarnavis's original definitions, that is, they are as defined in 4.1.1 and 4.1.2 Definitions; so these rings have to be integral over their centres. However, if these terms are given the meanings assigned by Stafford and Zhang, that is, they are defined as in 3.1.6 Definition, then it is an open question whether they are preserved under Ore extensions. Thus we would like to make the following conjecture.

2.7 CONJECTURE. *Let R be an inj. (resp. hom.) hom. FBN ring as defined in 3.1.6 Definition. Suppose that σ is an automorphism of R and δ is a σ-derivation of R. Let $S = R[x; \sigma, \delta]$ or $R[x, x^{-1}; \sigma]$. If S is FBN, then it is also inj. (resp. hom.) hom. in the sense of 3.1.6 Definition.*

§ 5.3 NOTES

3.1 1.4 must be well-known and it has also been announced in [JJ]. 1.9 is from [MR]. 1.10 is a parallel result of [MR, 7.5.2 Proposition], and it is also a generalization of [BH2, Lemma 6.1].

3.2 2.2 Theorem, 2.3 Theorem and 2.6 Proposition are our main results of this chapter. 2.2 Theorem is a generalization of [BH2, Theorem 6.2].

REFERENCES

[Al1] E. Aljadeff, Homological duality for crossed products, J. Algebra 134 (1990), 157-181.

[Al2] E. Aljadeff, Serre's extension theorem for crossed products, J. London Math. Soc. 44 (1991), 47-54.

[Al3] E. Aljadeff, On the surjectivity of some trace maps, preprint, Technion-Israel Institute of Technology, 1992.

[AR] E. Aljadeff and S. Rosset, Global dimensions of crossed products, J. Pure Appl. Algebra 40 (1986), 104-113.

[AF] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-verlag, (New York), 1974.

[AS] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987), 171-216.

[ATV1] M. Artin, J. Tate and M. Van Den Berg, Some algebras associated to automorphisms of curves, in *The Grothendieck Festschrift* (Eds. P. Cartier, et al), Birkhauser, Basel, 1990.

[ATV2] M. Artin, J. Tate and M. Van Den Berg, Modules over regular algebras of dimension 3, Inventions Math. 106 (1991), 335-389.

[ABr] M. Auslander and M. Bridger, Stable module theory, Memoirs of Amer. Math. Society 94, 1969.

[ABu1] M. Auslander and D. Buchsbaum, Homological dimension in local rings, Trans. Amer. Math. Soc. vol. 85 (1957), 390-405.

References

[ABu2] M. Auslander and D. Buchsbaum, Unique facterization in regular local rings, Proc. Natl. Acad. Sc. USA 45 (1959), 733-734.

[ARS] M. Auslander, I. Reiten and S. O. Smalo, Galois actions on rings and finite Galois coverings, Math. Scand. 65 (1989), 5-32.

[Az] G. Azumaya, New foundation of the theory of simple rings, Proc. Japan Acad. 22 (1946), 325-332.

[Ba1] H. Bass, finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 95 (1960), 466-488.

[Ba2] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963), 8-28.

[Be] A. D. Bell, Notes on Localization in Noncommutative Noetherian Rings, Cuadernos De Algebra 9 (1988), Universidad de Granada.

[Bj] J. E. Bjork, The Auslander condition on Noetherian rings, Lecture Notes in Math. 1404, Springer Verlag (1989), 137-173.

[Bo] M. Boratynski, A change of rings theorem and the Artin-Rees property, Proc. Amer. Math. Soc. 53 (1975), 307-310.

[Br1] K. A. Brown, Localization at cliques in group rings, J. Pure and Applied Algebra, 41 (1986), 9-16.

[Br2] K. A. Brown, Fully bounded Noetherian rings of finite injective dimension, Quart. J. Math. Oxford, 41 (1990), 1-13.

[BH1] K. A. Brown and C. R. Hajarnavis, Homologically homogeneous rings, Trans. Amer. Math. Soc. 281 (1984), 197-208.

[BH2] K. A. Brown and C. R. Hajarnavis, Injectively homogeneous rings, J. Pure Appl. Algebra 51 (1988), 65-77.

[BHM1] K. A. Brown, C. R. Hajarnavis and A. B. MacEacharn, Noetherian rings of finite global dimension, Proc. London Math. Soc. 44 (1982), 349-371.

[BHM2] K. A. Brown, C. R. Hajarnavis and A. B. MacEacharn, Rings of finite global dimension integral over their centre, Comm. Algebra 11 (1983), 67-93.

[BL] K. A. Brown and T. Levasseur, Cohomology of bimodules over enveloping algebras, Math. Z. 189 (1985), 393-413.

[Bu] W. D. Burgess, Rings of quotients of group rings, Can. J. Math. 21 (1969), 865-975.

[CVV] S. Caenepeel, M. Van Den Bergh and F. Van Oystaeyen, Generalized crossed products applied to maximal orders, Brauer groups and related exact sequences, J. Pure Applied Algebra 33 (1984), 123-149.

[CE] H. Cartan and S. Eilenberg, Homological Algebras, Princenton University Press, Princeton, 1956.

[Ca] G. Cauchon, Les T-anneaux et les anneaux à identités polynomiales Noethériens, Thèse, Orsay (1977).

[CM] M. Cohen and S. Montgomery, Group-graded rings, smash products, and group actions, Trans. Amer. Math. Soc. 282 (1984), 237-258.

[Da] E. C. Dade, Group-graded rings and modules, Math. Z. 174 (1980), 241-262.

[Ei] D. Eisenbud, Subrings of Artinian and Noetherian rings, Math. Ann, 185 (1970), 247-249.

[Ek] E. K. Ekström, The Auslander condition on graded and filtered Noetherian rings, Séminairen Dubreil-Malliavin, 1987-8 (Lecture

Notes in Math., No. 1404), Springer-Verlag, Berlin, (1989).

[Fi] K. L. Field, On the global dimension of residue rings, Pacific J. Math. 32 (1970), 345-349.

[FGR] R. Fossum, P. Griffith and I. Reiten, Trivial extensions of Abelian categories, Lecture Notes in Math. 456 (1975), Springer Verlag.

[GD] J. J. García and A. Del Río, Groups acting on rings, projectivity and global dimension, preprint, Universidad de Murcia, Spain, 1993.

[Go] K. R. Goodearl, Prime links in differential operator rings, Quart. J. Math. Oxford Ser. (2) 42 (1991), No. 168, 457-487.

[GJ] K. R. Goodearl and D. A. Jordan, Localizations of injective modules, Proc. Edinburgh Math. Soc. 28 (1985), 289-299.

[GS] K. R. Goodearl and L. W. Small, Krull versus global dimension in Noetherian P. I. rings, Proc. Amer. Math. Soc. 92(2) (1984), 175-178.

[GW] K. R. Goodearl and R. B. Warfield , Jr., An Introduction to Noncommutative Noetherian Rings, Cambridge University Press, (Cambridge), 1989.

[HLS] D. Handelman, J. Lawrence and W. Schelter, Skew group rings, Houston J. Math. 4 (1978), 175-198.

[Ha] A. M. Hassanein, Sur les anneaux semi-primiers differentiels a identite polynomiale, Comm. Algebra, 18(8) (1990), 2735-2746.

[Jac] N. Jacobson, Basic Algebra I, W. H. Freeman and Company, (San Francisco), 1974.

References

[Jat] A. V. Jategaonkar, Skew polynomial rings over orders in Artinian rings, J. Algebra, 21 (1972), 51-59.

[JJ] A. Jensen and S. Jøndrup, Classical quotient rings of group graded rings, Comm. Algebra, 20(10) (1992), 2923-2936.

[JJV] A. Jensen, S. Jøndrup and M. Van Den Berg, Artinian quotient rings of filtered rings, to appear.

[Ka] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.

[KK] E. Kirkman and J. Kuzmanovich, On the finitistic dimension of fixed subrings, preprint, Wake Forest University (Winston-Salem).

[KKS] E. Kirkman, J. Kuzmanovich and L. Small, Finitistic dimensions of Noetherian rings, J. Algebra, vol.147 (2) (1992), 350-364.

[Let] E. S. Letzter, Prime ideals in finite extension of Noetherian rings, J. Algebra 135 (1990), 412-439.

[Lev] T. Levasseur, Some properties of non-commutative regular graded rings, Glasgow Math. J. 34(3) (1992), 277-300.

[LP1] M. Lorenz and D. S. Passman, Prime ideals in crossed products of finite groups, Israel J. Math. 33 (1979), 89-132.

[LP2] M. Lorenz and D. S. Passman, Prime ideals in group algebras of polycyclic-by-finite groups, Proc. London Math. Soc. 43 (1981), 520-543.

[Ma] H. Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematice 8, Cambridge University Press, (Cambridge), 1992.

[Mo1] S. Montgomery, Fixed Rings of Finite Automorphism Groups of Associative Rings, Lecture Notes in Math. 818, Springer, (Berlin), 1980.

[Mo2] S. Montgomery, Outer automorphisms of semiprime rings, J. London Math. Soc. 18 (1978), 209-221.

[MR] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Wiley-Interscience, (New York), 1987.

[Mü] B. J. Müller, Localization in non-commutative Noetherian rings, Can. J. Math. 28 (1976), 600-610.

[NA] T. Nakayama and G. Azumaya, On irreducible rings, Ann. Math. 48 (1947), 949-965.

[Na] C. Nastasescu, Strongly graded rings of finite groups, Comm. Algebra 11(10) (1983), 1033-1071.

[NsV] C. Nastasescu and F. Van Oystaeyen, Graded Ring Theory, North Holland Math. Library 28 (1982).

[NuV] E. Nauwelaerts and F. Van Oystaeyen, Finite generalized crossed products over tame and maximal orders, J. Algebra 101 (1986), 61-68.

[No] D. G. Northcott, An Introduction to Homological Algebra, Cambridge University Press, 1960.

[Pa1] D. S. Passman, The Algebraic Structure of Group Rings, Wiley-Interscience, (New York), 1977.

[Pa2] D. S. Passman, Infinite Crossed Products, Academic Press, (San Diego), 1989.

[Po] D. G. Poole, Localization in Ore extensions of commutative Noetherian rings, J. Algebra, 128 (1990), 434-445.

[Qu] D. Quinn, Group-graded rings and duality, Trans. Amer. Math. Soc.

292 (1985), 155-167.

[Ra] J. Rainwater, Global dimension of fully bounded Noetherian rings, Comm. Algebra 15(10) (1987), 2143-2156.

[Re] I. Reiten, Trivial extensions and Gorenstein rings, Thesis, Chicago University (1971).

[Rot] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, (New York), 1979.

[Row] L. H. Rowen, Ring Theory (Second Edition), Academic Press, (San Diego), 1991.

[Se] J. P. Serre, Sur la dimension homologique des anneaux et des modules Noethériens, Proceedings International Symposium on Algebraic Number Theory, Tokyo and Nikko, (1955), 175-189.

[Sm] P. F. Smith, Quotient rings of group rings, J. London Math Soc. (2), 3 (1971), 645-660.

[Sn] R. L. Snider, Noncommutative regular local rings of dimension 3 II, Comm. Algebra 19(9) (1991), 2495-2496.

[So] L. Soueif, Normalizing extensions and injective modules, essentially bounded normalizing extensions, Comm. Algebra 15(8) (1987), 1607-1619.

[Sta] J. T. Stafford, Infinite cliques in Noetherian rings, preprint, University of Michigan.

[SZ] J. T. Stafford and J. J. Zhang, Homological properties of (graded) Noetherian P. I. rings, preprint, University of Michigan. (Ann Arbor), 1992.

[Ste] B. Stenstrom, Rings of Quotients, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[Wa] R. B. Warfield, Jr., Non-commutative localized rings, in *Seminaire Malliavin* (Lecture notes in Math. 1220), Springer Verlag, Berlin (1986), 178-200.

[Yi1] Z. Yi, Homological dimension of skew group rings and crossed products, preprint, University of Glasgow, 1992, (to appear in Journal of Algebra).

[Yi2] Z. Yi, Injective homogeneity and the Auslander-Gorenstein property, preprint, University of Glasgow, 1993.

[Za] A. Zaks, Injective dimension of semi-primary rings, J. Algebra, 13 (1969), 73-86.