**SKEW GROUP RINGS AND MAXIMAL ORDERS**

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This work is dedicated to the memory of Chris Martin,

who will always be an inspiration.

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# CONTENTS.

## SUMMARY.

In this thesis, we are concerned with the question of when a skew group ring of a finite group over a prime Noetherian ring is a maximal order. Contrary to what one might first expect, it is not necessary for the coefficient ring itself to be a maximal order; however, we do require that it be a $G$-maximal order (where $G$ is a group). Such objects are defined and discussed in Chapter 2.

Chapter 3 is devoted to the proof of the main result of this work, which is as follows.

**THEOREM 3.2.2 (Main Theorem).** *Let $S$ be a prime Noetherian ring and $G$ a finite group acting on $S$. Suppose further that the action of $G$ is X-outer. Let $T$ denote the skew group ring $S*G$, and consider the following hypotheses:*

*(a) $S$ is a $G$-maximal order, and*

*(b) $p_0T$ is a prime ideal of $T$ for all height-1 reflexive $G$-prime ideals $p_0$ of $S$.*

*(i) If (a) and (b) both hold, then $T$ is a prime maximal order.*

*(ii) Suppose that the order of $G$ is a unit in $S$. If $T$ is a prime maximal order, then (a) and (b) both hold.*

Note that it is necessary to have the order of $G$ being a unit of $S$ in part (ii) of the Theorem, as we show by means of Example 6.2.11. However, as we shall see, this hypothesis can be dropped when we allow the coefficient ring to be commutative.

A major tool in the proof of the Main Theorem is a result which we refer to as the Test Theorem. In the following, $R_0$ denotes the set of elements $q$ of the quotient ring of $R$ such that $qI \subseteq R$ for some non-zero ideal $I$ of $R$; $R_0'$ is the analogous subring where the ideals multiply on the left.

**THEOREM 3.1.4 (Test Theorem).** *Let $R$ be a prime Noetherian ring and let $\Omega$ denote the set of height-1 reflexive prime ideals of $R$. Then $R$ is a maximal order if and only if*

*(i) Each $P \in \Omega$ is localisable;*

*(ii) $R_P$ is a maximal order for all $P \in \Omega$;*

*(iii) $R_0 = R_0'$;*

*(iv) $R = R_0 \cap (\cap\{R_P : P \in \Omega\})$.*

In Chapter 4 we state and prove an important result which gives necessary and sufficient conditions for a crossed product to be local. Here, for an ideal $p$ of a ring $S$, $K(p)$ is the stabiliser of $p$ in $G$. The precise result is as follows.

**THEOREM 4.2.8** *Let $S$ be a ring and $G$ a finite group acting on $S$ with twisting $\tau$. Let $T$ be the crossed product $S*G$. Then $T$ is local if and only if*

*(a) $S$ is $G$-local;*

*(b) $(S/p)*K$ is local, where $p$ is a maximal ideal of $S$ and $K := K(p)$.*

We then turn to Chapter 5, where we specialise Theorem 3.2.2 to give necessary and sufficient conditions for a skew group ring $T$ of a finite group $G$ over a commutative Noetherian domain $S$ to be a maximal order. As well as being able to relax the hypothesis on the order of $G$ being a unit of $S$ (as we pointed out earlier), there is also a nice test for when the ideal $(P \cap S)T$ of $T$ is prime, where $P$ is a height-1 prime ideal of $T$. So condition (b) of Theorem 3.2.2 can be replaced by a condition which is relatively easy to check. Moreover, every commutative $G$-maximal order is a maximal order (see Lemma 2.1.12). The result is:

**THEOREM 5.2.7** *Let $S$ be a commutative Noetherian domain and $G$ a finite group acting on $S$; let $T$ denote the skew group ring $S*G$. Then $T$ is a prime maximal order if and only if*

*(a) $S$ is integrally closed;*

*(b) there exists no non-identity element $g$ of $G$ such that $I(g) \subseteq p$ for some height-1 prime ideal $p$ of $S$.*

Here, for all non-identity elements $g$ of $G$, $I(g)$ denotes the ideal $\{s - s^g : s \in S\}S$ of $S$.

We continue our discussion, in Chapter 6, with a further result. This gives sufficient conditions for a skew group ring of a finite group over a prime Noetherian PI ring integral over its centre to be a tame order. If we allow the order of $G$ to be a unit in the coefficient ring, then one of these conditions becomes redundant. Moreover, if the coefficient ring is commutative, then the given conditions are also necessary.

**THEOREM 6.2.9** *Let $S$ be a prime Noetherian PI ring integral over its centre and let $G$ be a finite group acting on $S$. Put $T = S*G$, the skew* *group ring, and suppose that $T$ is prime. Consider the following conditions:*

*(a) $S$ is a tame order;*

*(b) for all height-1 $G$-prime ideals $Q$ of $S$, $QT$ is a semiprime ideal of $T$.*

*(i) If (a) and (b) both hold, $T$ is a tame order.*

*(ii) Suppose that $S$ is commutative. If $T$ is a tame order, then (a) and (b) both hold.*

*(iii) Suppose that the order of $G$ is a unit in $S$. If (a) holds, then $T$ is a tame order.*

The aforementioned Example 6.2.11 shows also that (ii) is false for non-commutative coefficient rings.

Finally, in Chapter 7, we give sufficient conditions for a skew group ring of a finitely generated nilpotent group over a prime Noetherian ring to be a maximal order.

**COROLLARY 7.1.5** *Let $R$ be a prime Noetherian ring and $G$ a finitely generated nilpotent group acting on $R$. Denote by $H$ the torsion subgroup of $G$ and suppose that the elements of $H$ are X-outer on $R$. Suppose that the skew group ring $R*H$ is a maximal order. Then the skew group ring $R*G$ is a maximal order.*

# INTRODUCTION.

Throughout this thesis, all rings are associative with a multiplicative identity, and all modules are unitary. Any conditions (such as Noetherian, hereditary, reflexive) will be assumed to hold both on the left and the right unless explicitly stated otherwise. A similar convention applies to bimodules. In each section, a paragraph detailing the notation used in a particular situation is given, and this notation is then referred to (for example, in the statement of a result) wherever it is subsequently used.

References used in the text refer to the source from which the result was taken at the time (not necessarily the *original* source, however), and so consequently many results are credited to books such as those by D.S. Passman, or J.C. McConnell and J.C. Robson. At the end of each chapter, there is a section entitled Additional Remarks, in which every effort has been made to give credit where credit is due to unoriginal results used in the thesis. Any aberrations or ommissions in this respect are unintentional, and entirely the fault of the author.

The theory of maximal orders is essentially the study of "non-commutative arithmetic" and has its origins in the work of R. Dedekind (1831-1916); in particular his investigations into the factorisation properties of ideals of a ring of algebraic integers in an algebraic number field.

We can generalise the idea of a commutative Noetherian integral domain being a maximal order if and only if it is integrally closed - if $R$ is a (non-commutative) prime Noetherian ring then its integral closure in its quotient ring is a maximal order containing $R$ and equivalent to $R$. Many questions involving, for example, integral representations of groups, or those concerning matrices with integer entries, reduce to the study of non-maximal orders. Then these rings can be embedded in maximal orders to which they are equivalent and known facts about maximal orders can be used; information can then be pulled down to the original ring in question.

Maximal orders do occur naturally; for example, principal ideal domains, unique factorisation domains and Dedekind domains, all being integrally closed commutative domains are all maximal orders. Also, simple rings are maximal orders, so in particular Weyl algebras over a field of characteristic zero provide another source of examples. Consider the associated graded ring of the universal enveloping algebra $U$ of a finite-dimensional Lie algebra over a field $k$; this ring is just a commutative polynomial ring over $k$, which is a Noetherian unique factorisation domain, and hence a maximal order. It is not hard to deduce that $U$ itself is a maximal order. We also have that group algebras of poly-(infinite cyclic) groups are all maximal orders.

We begin this thesis with some introductory material, including a discussion on maximal orders, and some more general background work on projective modules and crossed products of both finite and infinite groups. We also give a brief summary of work done to date concerning the question of when skew group rings (and group rings) are maximal orders. We quote results due to P.F. Smith in 1984 ([S, Theorem 1.4]) and K.A. Brown in 1985 ([B, Theorem F]), which give necessary and sufficient conditions for a group ring to be a maximal order. In Smith's theorem, the group is a nilpotent-by-finite group which is also polycyclic-by-finite, and the coefficient ring a Noetherian maximal order in a simple Artinian ring. Brown's result deals with the case of a group ring of a polycyclic-by-finite group over a commutative Noetherian domain; it was shown by Brown and Smith, together with M. Marubayashi in 1991 (see [B-M-S]) how Brown's result can be generalised to the case of an arbitrary Noetherian coefficient ring.

This contrasts greatly with what we find in later chapters when dealing with skew group rings, namely that the generalisation from a coefficient ring which is a commutative Noetherian domain to one which is a (non-commutative) prime Noetherian ring is often not possible without imposing extra hypotheses on the group and the ring concerned. But more of this later.

There is a result due to M. Auslander, O. Goldman and D.S. Rim, which is hinted at in the paper [A-R] and appears in full in [R, Theorem 40.14], which says that a skew group ring of a finite group over a Dedekind domain $S$ is a maximal order if and only if $S/R$ is unramified, where $S$ is the integral closure of $R$ in the quotient field of $S$. Without going into too much detail about what this actually means, we "translate" this result and restate it (as Theorem 1.5.16) using language compatible with work done later in the thesis, and find that it is a special case of Theorem 5.2.7, a result which gives necessary and sufficient conditions for a skew group ring of a finite group over a commutative Noetherian domain (not necessarily Dedekind) to be a maximal order.

In Chapter 2 we introduce the concept of a $G$-maximal order (where $G$ is a group), which as its name suggests, can be thought of as a $G$-equivariant version of a maximal order. The definition is analogous to that of a maximal order: we impose an equivalence relation on the $G$-invariant right orders contained in some (Artinian) quotient ring $Q$ of a prime Noetherian ring on which a finite group $G$ acts. Then a right order $R$ in $Q$ is a $G$-maximal right order precisely when it is maximal within its equivalence class (see 2.1.3 and 2.1.4).

Of course, all maximal orders are $G$-maximal orders, and a commutative Noetherian domain is a $G$-maximal order precisely when it is integrally closed (see Lemma 2.1.12 and compare this with Example 1.1.2), so that our definition yields nothing new in the commutative case. However, there do exist genuine non-commutative $G$-maximal orders which are not maximal orders, and an example of such is given in 2.1.13. As in the ordinary case, the verification that a particular (prime Noetherian) ring is a $G$-maximal order is most easily effected by considering the right and left orders of the $G$-invariant ideals of the ring (see Theorem 2.1.8).

As we will see in later chapters, in order for a skew group ring of a finite group over a prime Noetherian ring to be a prime maximal order, it is not necessarily the case that the coefficient ring itself is a maximal order; however we do require it to be a $G$-maximal order, and it is for this reason that we investigate such objects.

As is to be expected, there are many results concerning maximal orders whose proofs can be adapted to give results about $G$-maximal orders (where $G$ is a finite group acting on the ring). Several such results are offered in §2.2, including the fact that any non-zero maximal (right and left) reflexive $G$-invariant ideal of a prime Noetherian $G$-maximal order is a height-1 $G$-prime ideal (Lemma 2.2.4); this result is a $G$-equivariant analogue of [C-S2, Theorem 1.6], and our proof is modelled on theirs.

Proposition 2.2.5 then gives a description of such ideals in a $G$-maximal order, namely in terms of the intersection of the $G$-orbit of a height-1 (left or right or 2-sided) reflexive prime ideal. This result also tells us when a ring actually contains a proper $G$-invariant left and right reflexive ideal. We then go on to define the notion of a $G$-local ring (that is, a prime Noetherian semilocal ring with $G$-prime Jacobson radical); when the Jacobson radical of such a ring $R$ is reflexive, we have that $R$ is a $G$-maximal order precisely when it is hereditary, and this happens if and only if the Jacobson radical is invertible (see Corollary 2.2.11). This result is a $G$-equivariant version of Proposition 1.2.8, largely due to C.R. Hajarnavis and T.H. Lenegan in [H-L, Proposition 1.3].

In §2.3 we define the (right or left) symbolic powers of a height-1 $G$-prime ideal of a prime Noetherian ring. Our definition is based on that given by J.H. Cozzens and F.L. Sandomierski in [C-S1], and agrees with that of A.W. Goldie's symbolic powers, introduced in 1967 in [G]. These are used to show, in Proposition 2.3.12, that a reflexive height-1 $G$-prime ideal of a prime Noetherian $G$-maximal order ($G$ being a finite group) is localisable, a fact based on a similar result due to Hajarnavis and S. Williams in [H-W].

In Chapter 3 we prove the main result of the thesis, which appears as Theorem 3.2.2. First of all, §3.1 is concerned with the statement and proof of a result referred to as the Test Theorem (Theorem 3.1.4), which we use to test when a prime Noetherian ring is a maximal order. The result was partially proved by M. Chamarie, in [C, Proposition 1.10(b)], but is new in the form presented here. We also offer, as Theorem 3.1.13, a $G$-equivariant analogue of the Test Theorem, which in a similar way allows us to test when a prime Noetherian ring is a $G$-maximal order. These results are instrumental in the proof of the Main Theorem.

§3.2 is concerned with the proof of the Main Theorem: let $S$ be a prime Noetherian ring and $G$ a finite group acting on $S$ such that $G$ is X-outer on $S$. Then sufficient conditions for the skew group ring $T = S*G$ to be a prime maximal order are that $S$ is a $G$-maximal order, and that $p_0T$ is a prime ideal of $T$ for all height-1 $G$-prime ideals $p_0$ of $S$. It turns out that these conditions are not necessary in general, however, unless we insist that the order of $G$ is a unit of $S$. An example is given in 6.2.11 which shows (amongst other things) that this extra hypothesis really is needed. Where the theory breaks down in this respect is illustrated by Lemma 3.2.8 and Example 3.2.9; if $S$ is a semilocal $G$-maximal order with reflexive Jacobson radical (where $G$ is a finite group acting on $S$, of course), then we need the order of $G$ to be a unit in $S$ for the skew group ring $T = S*G$ to be semilocal and hereditary, which in turn is essential for the ideals $p_0T$ of $T$ described above to be prime.

In Chapter 4 we discuss when group rings and crossed products are local. Our main result is Theorem 4.2.8, which is new. The (well-known) group ring case is dealt with in §4.1, culminating in Theorem 4.1.4, which gives necessary and sufficient conditions for a group ring of a finite group over a ring to be local. We prove the result in full in the absence of a suitable reference, but when the coefficient ring is a field, the result can be found in [P2, Theorem 10.1.16].

This theorem (4.1.4) is then used in the proof of Theorem 4.2.8 which gives necessary and sufficient conditions for a crossed product of a finite group over a ring to be local (one of these being that the coefficient ring is $G$-local, as defined in Chapter 2). Also, we have Lemma 4.2.5, which only applies to skew group rings, and is a key ingredient in the proof of the main result of Chapter 5. We give an example in Remark 4.2.5 to show that the stumbling block, so to speak, in the case of crossed products is genuine.

Chapter 5 deals with the question of when a skew group ring of a finite group over a commutative Noetherian domain is a prime maximal order; the answer to this is the result of which Auslander, Goldman and Rim's theorem is a special case, as we mentioned earlier.

In §5.1 some background material is given, in particular we quote a result due to B.J. Müller (see [M]) which says that if $R$ is a ring finitely generated as a module over its centre $Z$, with $Z$ Noetherian, then the clique of a prime ideal $P$ of $R$ is just the set of those primes of $R$ which have the same intersection in $Z$ as $P$. We then use this theorem to prove that a prime ideal $P$ of a prime Noetherian ring finitely generated as a module over its centre $Z$ is localisable if and only if $P$ is the one and only prime of $R$ lying over $P \cap Z$. This is an important result, and is used in §5.2.

The second section of this chapter is concerned with the proof of the main result, Theorem 5.2.7, which gives necessary and sufficient conditions for a skew group ring $T$ of a finite group $G$ over a commutative Noetherian domain $S$ to be a prime maximal order. Note that, in contrast to the part of Theorem 3.2.2 which deals with necessity, we no longer require the hypothesis that the order of the group be invertible in the coefficient ring. We define, for all non-identity elements $g$ of $G$, the ideal

$$\mathrm{I}(g) = \{s - s^g : s \in S\}S$$

of $S$. These ideals play a crucial rôle in the answer to the question of when the ideal $p_0T$ of $T$ (where $p_0$ is a height-1 $G$-prime ideal of $S$) is prime, and hence localisable (see Proposition 5.2.8 and Lemma 5.2.9). To be precise, Theorem 5.2.7 asserts that the skew group ring $T$ described above is a prime maximal order if and only if the coefficient ring $S$ is integrally closed, and the ideal $\mathrm{I}(g)$ is not contained in any height-1 prime ideal of $S$ for all (non-identity) elements $g$ of $G$. The sufficiency of these conditions is a direct consequence of the Main Theorem (Theorem 3.2.2), and so we concentrate here on proving their necessity. Observe that the condition involving the $\mathrm{I}(g)$s is much easier to verify than the corresponding hypothesis in Theorem 3.2.2. A key tool in the proof of Theorem 5.2.7 is one of the results of Chapter 4, Lemma 4.2.5.

In 5.2.16 we show by way of an example how Theorem 5.2.7 can be compared to Brown's theorem ([B, Theorem F]); indeed, either result can be used to test for a maximal order in the case of group rings of Abelian-by-finite semidirect products considered here. Also, an example is given in 6.2.11 which demonstrates how the $\mathrm{I}(g)$ condition is not a reliable test for an ideal $p_0T$ of a skew group ring $T$ to be prime (where $p_0$ is a height-1 prime ideal of the coefficient ring) when the coefficient ring is non-commutative.

In Section 6.1 we give some background material on integral extensions, polynomial identity (PI) rings, Krull domains and prime Noetherian rings integral over their centres. To end §6.1, we quote four results proved by Yi Zhong in [Y], which will be used in the proof of the main result of this chapter.

In Section 6.2 we define a tame order, and observe (in Remark 6.2.2) that a prime Noetherian maximal order integral over its centre is a tame order. This generalisation of the concept of a maximal order was introduced by R.M. Fossum in 1968 (see [F1, Page 325]). The crucial property distinguishing a tame order from a maximal order is that the localisations of height-1 primes of the centre are required to be hereditary, but not necessarily *local*, as would be the case in a maximal order. In particular, we show in Lemma 6.2.3 that a commutative Noetherian domain is a tame order precisely when it is a maximal order (and this happens if and only if it is integrally closed). It is not true, however, that every tame order is maximal, and an example is provided in 6.2.12.

Our main result, Theorem 6.2.9, deals with the question of when a skew group ring is a tame order. Let $S$ be a prime Noetherian PI ring integral over its centre $C$, and assume that $C$ is integrally closed. Let $G$ be a finite group acting on $S$; as usual, $T$ denotes the skew group ring $S*G$. Suppose also that $T$ is prime. Then what Theorem 6.2.9 says is, that if $S$ is a tame order and $QT$ is a semiprime ideal of $T$ for all height-1 $G$-prime ideals $Q$ of $S$, then $T$ is a tame order (compare this with Theorem 3.2.2). However, these conditions are not necessary unless $S$ is commutative. On the other hand, if the order of $G$ is a unit in $S$ and $S$ is tame, then $T$ is also tame. This last sentence is a special case of a result proved by E. Nauwelaerts and F.

van Oystaeyen in 1986 ([N-O, Theorem 3.1]), which gives a similar result for a ring $R$ strongly graded by a finite group $G$, with the order of $G$ being a unit of $R$.

It can be seen, by comparing Theorem 6.2.9 with Theorem 3.2.2 that, restricting ourselves to PI rings integral over their centres as coefficient rings, the key property which distinguishes skew group rings $T$ which are maximal orders from those which are tame orders is the *primeness* (as opposed to the *semiprimeness*) of the ideals $QT$ of $T$, where $Q$ is a height-1 $G$-prime ideal of the coefficient ring. Example 6.2.12 (as we mentioned above) shows that this is a genuine distinction.

In view of Theorem 3.2.2, one may be inclined to define the notion of a "$G$-tame order" ($G$ being a group, of course), whereby one would expect the analogue of Theorem 3.2.2 for tame orders to require that the coefficient ring $S$ be a "$G$-tame order". However, it is easy to check that, replacing the conditions of Theorem 6.2.9 with appropriate hypotheses involving height-1 $G$-prime ideals of the centre of $S$ yields nothing new; we are merely in the same situation as before. So it seems unlikely that any refinement in this direction is possible.

Finally, in our last (and shortest) chapter, we consider a prime Noetherian ring $R$ and a poly-(infinite cyclic) group $G$. It is shown in Theorem 7.1.3 that if $R$ is a maximal order, then the crossed product $R*G$ is a maximal order. This is not a new result, but a complete proof is given in the absence of a suitable reference. The converse to this theorem is not true however, and an example illustrating this is given in 7.1.6. This should be compared to the fact that, as in Theorem 3.2.2, the skew group ring $S*G$ being a maximal order does not imply that the coefficient ring $S$ is a maximal order.

What is new, though, is a corollary to Theorem 7.1.3 (Corollary 7.1.5), where we let $R$ be a prime Noetherian ring and $G$ a finitely generated nilpotent group with torsion subgroup $H$. Suppose that the elements of $H$ are X-outer on $R$. Then if the crossed product $R*H$ is a maximal order, it follows that the crossed product $R*G$ is also a maximal order. The proof of this hinges on the fact that, since $G/H$ is a finitely generated torsionfree nilpotent group, it is therefore poly-(infinite cyclic) (as we point out in Remark 1.4.10). Then by induction, there is no loss in assuming $G/H$ to be infinite cyclic. So we have that $R*G = (R*H) * (G/H)$ is a crossed product of an infinite cyclic group over a maximal order, and so Theorem 7.1.3 applies.

# CHAPTER 1. PRELIMINARIES.

## §1.1 MAXIMAL ORDERS.

1.1.1 We begin by giving some definitions: Let $R$ be a prime Noetherian ring with simple Artinian quotient ring $Q$. Then $R$ is an *order* in $Q$ and an order $S$ in $Q$ is said to be equivalent to $R$ if there exist units $a$, $b$, $c$, $d$ in $Q$ such that $aRb \subseteq S$ and $cSd \subseteq R$. We call $R$ a *maximal order* if it is maximal within its equivalence class, that is if $S$ is an order in $Q$ equivalent to $R$ and containing $R$, then $S$ must be equal to $R$.

Let $I$ be a non-zero ideal of $R$. Define

$$O_l(I) = \{q \in Q : qI \subseteq I\}$$

and

$$O_r(I) = \{q \in Q : Iq \subseteq I\}.$$

$O_l(I)$ and $O_r(I)$ are orders in $Q$ equivalent to $R$.

**1.1.2 THEOREM. [M-R, Proposition I.3.1]** *A prime Noetherian ring $R$ is a maximal order if and only if $O_l(I) = O_r(I) = R$ for all non-zero ideals $I$ of $R$.*

**EXAMPLE.** A commutative Noetherian domain is a maximal order if and only if it is integrally closed.

**1.1.3 LEMMA.** *Let $R$ be a prime Noetherian ring with the property that $O_l(P) = O_r(P) = R$ for all (non-zero) prime ideals $P$ of $R$. Then $R$ is a maximal order.*

**PROOF.** Suppose that

$$O_l(P) = O_r(P) = R$$

for all non-zero prime ideals $P$ of $R$, but that $R$ is not a maximal order. Therefore, by Theorem 1.1.2 we may assume that there exists a (non-zero) ideal $I$ of $R$ such that $R \subset O_l(I)$, since if $R \subset O_r(I)$ the argument is symmetric. Since $R$ is Noetherian, we can choose $I$ to be maximal with respect to this property. We claim that $I$ is in fact prime.

Let $A$ and $B$ be non-zero ideals of $R$ both strictly containing $I$ such that $AB \subseteq I$. Let $x \in O_l(I)\backslash R$. Then $xI \subseteq I$ so that

$$xAB \subseteq xI \subseteq I.$$

Without loss of generality,

$$A = \{r \in R : rB \subseteq I\}.$$

Due to the maximality of $I$, $O_l(B) = R$. Hence

$$xA \subseteq \{q \in Q(R) : qB \subseteq I\} \subseteq O_l(B) = R$$

so that $xA \subseteq A$. We have $x$ belonging to $O_l(A)$ but not $R$, and $A \supset I$; a contradiction to our initial choice of $I$. Hence $I$ is prime as claimed.

We now have a (non-zero) prime ideal $I$ of $R$ with $O_l(I) \supset R$, which is impossible. Therefore such an ideal $I$ does not exist and $R$ is a maximal order, as required.

**1.1.4 DEFINITIONS.** Let $R$ be a right (or left) order in a quotient ring $Q$. A *fractional right R-ideal* is an $R$-submodule $I$ of $Q_R$ such that $aI \subseteq R$ and $bR \subseteq I$ for some units $a$, $b$ of $Q$. Fractional left $R$-ideals and fractional (2-sided) $R$-ideals are defined similarly. A *right R-ideal* is a fractional right $R$-ideal which is contained in $R$; left $R$-ideals and $R$-ideals are defined in a similar way.

Note that any essential right ideal of a prime Noetherian ring $R$ is a right $R$-ideal.

For a fractional right $R$-ideal $I$ define

$$I_l^* := \{q \in Q : qI \subseteq R\},$$

and for a fractional left $R$-ideal $I$ put

$$I_r^* := \{q \in Q : Iq \subseteq R\}.$$

Then $I_l^*$ is a left $R$-submodule of $Q$ and $I_r^*$ is a right $R$-submodule of $Q$. We say that a fractional $R$-ideal $I$ is *invertible* if

$$(I_l^*)I = I(I_r^*) = R.$$

**1.1.5 PROPOSITION. [Mc-R, Proposition 3.1.15].** *Let $R$ be a right order in a quotient ring $Q$ and $I$ a fractional right (resp. left) R-ideal. Then $I_l^* \cong Hom_R(I_R, R_R)$ as left R-modules (resp. $I_r^* \cong Hom_R({}_RI, {}_RR)$ as right R-modules).*

**1.1.6 REMARK.** Note that, when $R$ is a prime Noetherian maximal order and $I$ a non-zero ideal of $R$, $I_l^* = I_r^*$. This is because, by Theorem 1.1.2 $O_l(I) = O_r(I) = R$, so that

$$\begin{aligned} I_l^* &= \{q \in Q : qI \subseteq R\} \\ &= \{q \in Q : qI \subseteq O_r(I)\} \\ &= \{q \in Q : IqI \subseteq I\} \\ &= \{q \in Q : Iq \subseteq O_l(I)\} \\ &= \{q \in Q : Iq \subseteq R\} \\ &= I_r^*. \end{aligned}$$

If this is the case we write $I_l^* = I_r^* = I^*$.

**1.1.7 DEFINITION.** Let $R$ be as in 1.1.4 and $I$ a non-zero ideal of $R$. Then $I$ is *right reflexive* if

$$I = (I_l^*)_r^*$$

and $I$ is *left reflexive* if

$$I = (I_r^*)_l^*.$$

Furthermore, $I$ is *reflexive* if $I$ is both right and left reflexive. It is clear for a non-zero ideal $I$ of $R$ that $I \subseteq (I_l^*)_r^*$ and $I \subseteq (I_r^*)_l^*$.

**REMARK.** It follows from Remark 1.1.6 that if $R$ is a maximal order then

$$(I_l^*)_r^* = (I_r^*)_l^*;$$

in this case we denote this set by $I^{**}$, and $I$ is reflexive precisely when

$$I = I^{**}.$$

It is always the case that $I^*$ and $I^{**}$ are right and left reflexive.

**1.1.8 PROPOSITION.** *Let $R$ be a prime Noetherian ring with quotient ring $Q$, and let $I$ be a non-zero proper ideal of $R$. Then $I$ is right (resp. left) reflexive if and only if $I$ is the annihilator of a non-zero $R$-submodule of $(Q/R)_R$ (resp. ${}_R(Q/R)$).*

**PROOF.** Suppose first that $I$ is right reflexive. Then

$$I = (I_l{}^*)_r{}^*$$

so that

$$I = \{q \in Q : (I_l{}^*)q \subseteq R\}.$$

But $1 \in I_l{}^*$ and $q \in (I_l{}^*)_r{}^* = I$ imply that $q$ belongs to $R$. Therefore

$$I = \{r \in R : (I_l{}^*)r \subseteq R\}$$

which is precisely the annihilator in $R$ of the $R$-submodule $((I_l{}^*)/R)_R$ of $(Q/R)_R$.

On the other hand, let $Y_R$ be an $R$-submodule of $Q_R$ strictly containing $R$, and suppose that $I$ is the annihilator in $R$ of the $R$-module $(Y/R)_R$. We have that $YI \subseteq R$; that is $Y \subseteq I_l{}^*$. Now, $(I_l{}^*)_r{}^*$ is by definition the annihilator in $Q$ of the $R$-module $((I_l{}^*)/R)_R$, which as above we see to be equal to the annihilator in $R$ of $((I_l{}^*)/R)_R$, and since $Y \subseteq I_l{}^*$ it is contained in the annihilator in $R$ of the $R$-module $(Y/R)_R$. We have shown that

$$(I_l{}^*)_r{}^* \subseteq \mathrm{Ann}_R((Y/R)_R) = I.$$

The reverse inclusion is clear, and so I is right reflexive as required. The left hand case is proved analogously.

**1.1.9 REMARK.** Note that if a proper ideal $I$ of a ring $R$ is right reflexive then $R \subset I_l{}^*$. Similarly, if $I$ is left reflexive then $R \subset I_r{}^*$.

**1.1.10 LEMMA.** *Let $R$ be a prime Noetherian maximal order and $P$ a maximal ideal of $R$ which is reflexive. Then $P$ is invertible.*

**PROOF.** Since $R$ is a maximal order, $P_l{}^* = P_r{}^* = P^*$, by Remark 1.1.6. It is clear that $P^*P$ is an ideal of $R$ and that $P \subseteq P^*P$. If $P = P^*P$ then

$$P^* \subseteq O_1(P) = R,$$

a contradiction to the reflexivity of $P$ by Remark 1.1.9. Therefore $P \subset P^*P$. But $P$ is a maximal ideal of $R$ so that

$$P^*P = R.$$

Similarly, $PP^* = R$ and $P$ is invertible, as required.

## §1.2 PROJECTIVE MODULES AND GENERATORS.

**1.2.1 DEFINITION.** A module is *projective* if it is a direct summand of a free module.

**REMARK.** If $R$ is a semisimple Artinian ring, then all $R$-modules are projective.

**1.2.2 DEFINITION.** Let $R$ be any ring and $M$ a right $R$-module. We call $M_R$ a *generator* if

$$\sum\{f(M) \ : \ f \in \mathrm{Hom}_R(M, R)\} = R.$$

**1.2.3 LEMMA.** [Mc-R, Lemma 3.5.3] *Let $R$ be any ring and $M$ a right $R$-module. Then*

(i) *$M_R$ is finitely generated projective if and only if $M$ is isomorphic to a direct summand of $R^{(n)}$ for some $n \in \mathbb{N}$, and*

(ii) *$M_R$ is a generator if and only if $R$ is isomorphic to a direct summand of $M^{(n)}$ for some $n \in \mathbb{N}$.*

**1.2.4 LEMMA.** [Mc-R, 5.1.7] *Let $R$ be any ring and $M$ a right $R$-module. If $M_R$ is projective then $M$ is right reflexive.*

**1.2.5** The following result can be generalised, but is stated here in as much generality as suits our present purposes.

**LEMMA** **[Mc-R, Lemma 5.2.5]** *Let $R$ be a prime Noetherian ring and $I$ a non-zero ideal of $R$. Then the following are equivalent:*

*(i)* *$I$ is invertible;*

*(ii)* *${}_RI$ and $I_R$ are generators;*

*(iii)* *$O_l(I) = O_r(I) = R$ and ${}_RI$, $I_R$ are finitely generated projective;*

*(iv)* *$O_l(I) = O_r(I) = R$ and ${}_RI$, $I_R$ are projective.*

**1.2.6 DEFINITIONS.** Let $R$ be any ring and $M$ a right $R$-module. $M$ has *projective dimension* n (written pr.dim($M$) = n) if $M$ has a projective resolution of length n, with n minimal such. Note that if $M$ is projective then

$$\text{pr.dim.}(M) = 0,$$

and if $M$ is a projective right ideal of $R$ then

$$\text{pr.dim.}(R/M) \leqslant 1.$$

The *right global dimension* of $R$, written r.gl.dim.($R$), is then defined as follows:

$$\text{r.gl.dim.}(R) = \sup\{\text{pr.dim.}(M) : M \text{ a right } R\text{-module}\}.$$

The left global dimension of $R$, l.gl.dim.($R$), is defined similarly. When $R$ is Noetherian,

$$\text{r.gl.dim.}(R) = \text{l.gl.dim.}(R)$$

so in this case we talk of the *global dimension* of $R$, denoted gl.dim.($R$).

We say that a Noetherian ring $R$ is *hereditary* if and only if

$$\text{gl.dim.}(R) \leqslant 1.$$

It is clear from the definition that $R$ is semisimple Artinian precisely when gl.dim.($R$) = 0.

**REMARK.** If $R$ is a hereditary ring, then all right and left ideals of $R$ are projective.

**1.2.7 DEFINITION.** Let $R$ be a ring with Jacobson radical $J$. Then $R$ is said to be *semilocal* if $R/J$ is semisimple Artinian, and *local* if $R/J$ is simple Artinian.

**1.2.8 PROPOSITION.** *Let $R$ be a prime Noetherian local ring with the Jacobson radical of $R$, denoted $J$, reflexive. Then the following are equivalent:*

*(i) $R$ is a maximal order;*

*(ii) $R$ is hereditary;*

*(iii) $J$ is invertible.*

**PROOF.** (ii) $\Longleftrightarrow$ (iii). [H-L, Proposition 1.3]

(i) $\Longrightarrow$ (iii). This is immediate from Lemma 1.1.10.

(iii) $\Longrightarrow$ (i). Suppose that $J$ is invertible, but that $R$ is not a maximal order. Let $0 \neq I$ be an ideal of $R$ such that $R \subset O_l(I)$, and choose $I$ to be maximal with respect to this property. Now, $O_l(J)J \subseteq J$ implies that

$$O_l(J) = O_l(J)J(J_r{}^*) \subseteq J(J_r{}^*) = R,$$

since $J$ is invertible. Hence

$$O_l(J) = R.$$

A similar argument gives $O_r(J) = R$. Therefore by choice of $I$, $I \subset J$ and by the argument of Remark 1.1.6,

$$J_r{}^* = J_l{}^* = J^*.$$

Put $K = IJ^*$, an ideal of $R$. Then

$$KJ = IJ^*J = I,$$

since $J$ is invertible. It is clear that $I \subseteq K$; suppose that $I = K$. Then $KJ = K$, contradicting Nakayama's Lemma. Therefore $I \subset K$.

Let $x \in O_1(I)$. Then $xI \subseteq I$ so that $xKJ \subseteq KJ$. We have

$$xK = xKJJ^* \subseteq KJJ^* = K$$

since $J$ is invertible, and so $x \in O_1(K)$. Therefore

$$R \subset O_1(I) \subseteq O_1(K)$$

with $I \subset K$, contradicting the initial choice of $I$. So there exists no such ideal $I$. A similar argument works on the right, so $R$ is a maximal order. This completes the proof.

**1.2.9** The following result should be compared with [F-S, Theorem 1], in which it is proved that if all non-zero projective right modules over a semilocal ring are generators, then they are direct sums of a fixed idempotent-generated principal right ideal. The proof of the following theorem is an adaptation of this.

**THEOREM.** *Let $R$ be a semilocal Noetherian ring in which each finitely generated projective right $R$-module is a generator. Then there exists a primitive idempotent $0 \neq e \in R$ such that each finitely generated projective right $R$-module is isomorphic to a direct sum of copies of $eR$.*

**PROOF.** Since $R$ is Noetherian, it is clear that there exists a (non-zero) element $e = e^2 \in R$ with $eR$ indecomposable. Let $R$ have Jacobson radical $J$, and let

$$\{T_i : i = 1,\dots,n\}$$

be the representatives of isomorphism classes of simple right $R$-modules. Then

$$\frac{(eR)}{(eJ)} = \frac{(eR)}{(eR)J} \cong T_1^{(k_1)} \oplus \ldots \oplus T_n^{(k_n)} \qquad \ldots (1)$$

where $k_1,\ldots,k_n \in \mathbb{N}$.

By hypothesis, $eR$ and $(eR)/(eJ)$ are generators. Therefore $(eR)/(eJ)$ maps onto each $T_i$, and so $k_i \neq 0$ for all $i = 1,\ldots,n$. Let $P$ be a non-zero finitely generated projective right $R$-module. By Nakayama's Lemma, $P/PJ$ is non-zero. Now $R$ is semilocal, so $R/J$ is semisimple Artinian so that the $R/J$-module $P/PJ$ is completely reducible. Therefore

$$\frac{P}{PJ} \cong T_1^{(c_1)} \oplus \ldots \oplus T_n^{(c_n)} \qquad \ldots (2)$$

with $c_1,\ldots,c_n \in \mathbb{N}$.

Reindex the $T_i$ so that

$$c_1/k_1 = \min\{c_i/k_i : i = 1,\ldots,n\}.$$

Note that this gives

$$c_i k_1 \geqslant k_i c_1$$

for each $i = 1,\ldots,n$. Then

$$\frac{P^{(k_1)}}{(PJ)^{(k_1)}} \cong T_1^{(c_1 k_1)} \oplus \ldots \oplus T_n^{(c_n k_1)} \qquad \ldots (3)$$

and also

$$\frac{(eR)^{(c_1)}}{(eJ)^{(c_1)}} \cong T_1^{(k_1 c_1)} \oplus \ldots \oplus T_n^{(k_n c_1)}. \qquad \ldots (4)$$

So there exists an epimorphism $f$ from $P^{(k_1)}$ onto $\dfrac{(eR)^{(c_1)}}{(eJ)^{(c_1)}}$. Of course we also have the canonical epimorphism $g$ from $(eR)^{(c_1)}$ onto $\dfrac{(eR)^{(c_1)}}{(eJ)^{(c_1)}}$.

Consider the following diagram:

$$\begin{tikzcd}
 & P^{(k_1)} \arrow[dl, dashed, "\hat{f}"'] \arrow[d, "f"] & \\
(eR)^{(c_1)} \arrow[r, "g"] & \frac{(eR)^{(c_1)}}{(eJ)^{(c_1)}} \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

There exists the homomorphism $\hat{f}$ as shown; we claim that $\hat{f}$ is surjective. If not, then

$$\mathrm{Im}(\hat{f}) + (eJ)^{(c_1)} \subset (eR)^{(c_1)}.$$

This means that

$$(g \circ \hat{f})(P^{(k_1)}) \subset (eR)^{(c_1)}/(eJ)^{(c_1)},$$

so that

$$f(P^{(k_1)}) \subset (eR)^{(c_1)}/(eJ)^{(c_1)}$$

since the diagram commutes. This contradicts the surjectivity of $f$, so that $\hat{f}$ is an epimorphism as claimed.

Now, since $(eR)^{(c_1)}$ is projective, the sequence

$$P^{(k_1)} \xrightarrow{\hat{f}} (eR)^{(c_1)} \longrightarrow 0$$

splits, and so there exists a finitely generated projective right $R$-module $Q$ with

$$P^{(k_1)} \cong (eR)^{(c_1)} \oplus Q.$$

Notice that the homogeneous components in $T_1$ in (3) and (4) are of the same length. Therefore $Q$ does not map onto $T_1$. Assume $Q \neq 0$. Since $Q$ is finitely generated projective it is a generator by hypothesis, so that $Q$ must map onto $T_1$. This contradiction means that $Q = 0$. Therefore

$$P^{(k_1)} \cong (eR)^{(c_1)} \qquad \ldots (5)$$

and $P$ is a direct sum of indecomposable finitely generated right $R$-modules.

Without loss of generality, suppose that $P$ is indecomposable. From the isomorphism (5) we have that

$$\frac{P^{(k_1)}}{(P^{(k_1)})J} \cong \frac{(eR)^{(c_1)}}{(eR)^{(c_1)}J}$$

and so considering homogeneous components in the $T_i$, via the isomorphisms (1) and (2), we must have that

$$c_i k_1 = k_i c_1$$

for all $i = 1,\ldots,n$. Now suppose that $k_1 < c_1$. Then $k_i < c_i$ for $i = 2,\ldots,n$. From the definition of the $c_i$, $k_i$ it follows that there exists an epimorphism $\varphi$ mapping $P$ onto $(eR)/(eJ)$. We have

$$\begin{tikzcd}
 & P \arrow[d, "\varphi"] \arrow[dl, dashed, "\hat{\varphi}"'] & \\
eR \arrow[r] & \frac{eR}{eJ} \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

Since $P$ is projective, $\hat{\varphi}$ exists as shown such that the diagram commutes. As we saw with $\hat{f}$, the map $\hat{\varphi}$ is an epimorphism. Therefore, since $P$ is projective, the sequence

$$P \xrightarrow{\hat{\varphi}} eR \longrightarrow 0$$

splits; a contradiction to the fact that $P$ is indecomposable. Hence $k_1 \not< c_1$. Similarly, if $c_1 < k_1$ then $k_i > c_i$ for all i and so $eR$ maps, via $\theta$ say, onto $P/PJ$. We see, on completing the following diagram

$$
\begin{tikzcd}
 & eR \arrow[dl, dashed, "\hat{\theta}"'] \arrow[d, "\theta"] & \\
P \arrow[r] & \dfrac{P}{PJ} \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}
$$

that there exists an epimorphism $\hat{\theta}$ from $eR$ onto $P$. Again this is impossible, so we must have $c_1 = k_1$. It follows that

$$c_i = k_i$$

for all $i = 1,\dots,n$. Hence $P \cong eR$, and the result follows.

**1.2.10 DEFINITION.** Let $R$ be a ring and $A$ a right $R$-module. The *trace ideal* of $A$, denoted $\mathrm{Tr}(A)$ is defined to be

$$\mathrm{Tr}(A) = \sum\{f(A) \ : \ f \in \mathrm{Hom}_R(A,R)\}.$$

**REMARK.** It is clear that $\mathrm{Tr}(A)$ is a right ideal of $R$. Let $r \in R$, $f \in \mathrm{Hom}_R(A,R)$; then

$$rf \in \mathrm{Hom}_R(A,R).$$

Define $r(f(a)) = rf(a)$ for all $a \in A$. Then $\mathrm{Tr}(A)$ is also a left ideal of $R$.

**1.2.11 LEMMA.** *Let $R$ be a ring and $P$ a projective $R$-module. Then the trace ideal $Tr(P)$ of $P$ is idempotent.*

**PROOF.** Put $I := \mathrm{Tr}(P)$. First we claim that $P = PI$. Clearly

$$PI \subseteq P;$$

suppose that $PI \subset P$. Then

$$P/PI \neq 0.$$

Choose a non-zero cyclic factor $X$ of $P/PI$. Then $X$ is an $(R/I)$-module, and

$$X \cong R/K$$

for some right ideal $K$ of $R$. We have that $I \subseteq K$. Let $\varphi$ be the map from $P$ onto $R/K$ obtained from composing the map from $P$ onto $X$ with the above isomorphism. Let $\pi$ be the canonical epimorphism from $R$ onto $R/K$. Consider the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\hat{\varphi}"'] \arrow[d, "\varphi"] & \\ R \arrow[r, "\pi"] & \frac{R}{K} \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

Since $P$ is projective, the map $\hat{\varphi}$ exists as shown. Now,

$$\hat{\varphi} \in \mathrm{Hom}_R(P,R)$$

and so by definition of $I$, $\hat{\varphi}(P) \subseteq I$. Therefore

$$\pi\hat{\varphi}(P) \subseteq \pi(I) = 0,$$

since $I \subseteq K$. But the above diagram commutes, so $\pi\hat{\varphi}(P) = \varphi(P)$. Therefore

$$\varphi(P) \subseteq \pi(I) = 0$$

which means that $P \subseteq \mathrm{Ker}(\varphi)$, a contradiction. Hence

$$P = PI,$$

as claimed.

We now have that

$$f(P) = f(PI) = f(P)I$$

for all $f \in \mathrm{Hom}_R(P,R)$. Let $u \in I$ and write

$$u = \sum\{f_i(p_i) : i = 1,\ldots,m\}$$

where $f_i \in \mathrm{Hom}_R(P,R)$ and $p_i \in P$ for all i. But since $P = PI$, there exist elements $h_j \in I$ such that

$$p_i = \sum\{(p_{ij})(h_j) \ : \ j = 1,\ldots,n(i)\}$$

where $p_{ij} \in P$ for all $i = 1,\ldots,m$. Therefore

$$u = \sum_{i=1}^{m} f_i\Big(\sum_{j=1}^{n(i)} (p_{ij})(h_j)\Big) = \sum_{i,j} f_i(p_{ij})h_j \in I$$

since $f_i(p_i) \in I$ and $h_j \in I$. Therefore $I = I^2$ and $I$ is idempotent, as required.

**1.2.12 DEFINITION.** An ideal $I$ of a ring $R$ is said to have the *right AR-property* if for each right ideal $A$ of $R$, there exists $n \in \mathbb{N}$ such that $A \cap I^n \subseteq AI$.

The left $AR$-property is defined similarly, and $I$ has the $AR$-property when $I$ has both the left and right $AR$-property.

**1.2.13 LEMMA.** *Let $R$ be a prime Noetherian ring in which all maximal (2-sided) ideals have the right AR-property. Then $R$ has no non-zero idempotent ideals.*

**PROOF.** Suppose that $I$ is a non-zero ideal of $R$ such that $I = I^2$. If $I \subset R$ then $I$ is contained in some maximal ideal $M$ (say) of $R$. It follows that $I \subseteq M^i$ for all $i \in \mathbb{N}$. Let

$$0 \neq x \in \cap\{M^i \ : \ i = 1,\ldots,\infty\},$$

and since $R$ is prime we can assume that $x$ is regular. Now $xR$ is a right ideal of $R$ and $M$ has the right $AR$-property, so there exists $n \in \mathbb{N}$ such that

$$xR \subseteq (xR \cap M^n) \subseteq xRM = xM.$$

Therefore $M = R$, a contradiction. It follows that

$$\cap\{M^i \ : \ i = 1,\ldots,\infty\} = 0$$

so that $I = 0$, again not possible. Hence $I = R$, and the result is proved.

**1.2.14 LEMMA.** *Let $R$ be a prime Noetherian ring in which all maximal (2-sided) ideals have the right AR-property. Then each finitely generated projective $R$-module is a generator.*

**PROOF.** Let $P_R$ be a finitely generated projective $R$-module, and put $I = \mathrm{Tr}(P)$, the trace ideal of $P$. By Lemma 1.2.11, $I$ is idempotent. Therefore, by Lemma 1.2.13, $I = R$. So there exist $h_1,\dots,h_t \in \mathrm{Hom}_R(P,R)$ and $x_1,\dots,x_t \in P$ such that

$$\sum\{h_i(x_i) : i = 1,\dots,t\} = 1.$$

Hence we have the $R$-homomorphism

$$h : P^{(t)} \longrightarrow R$$

defined by

$$h((x_1,\dots,x_t)) = \sum\{h_i(x_i) : i = 1,\dots,t\},$$

and $h$ is surjective. So by Lemma 1.2.3(ii), $P$ is a generator, as required.

## §1.3 CROSSED PRODUCTS AND FINITE NORMALISING EXTENSIONS.

**1.3.1 DEFINITION.** Let $R$ be a ring with 1, and $G$ a multiplicative group. A *crossed product* $R*G$ of $G$ over $R$ is a free left $R$-module with $\overline{G}$, a copy of $G$, as a basis. Also, $\overline{g}$ is a unit of $R*G$ for all $g \in G$, and each element of $R*G$ has a unique expression as

$$\sum\{r_g\overline{g} : g \in G\},$$

where each $r_g$ belongs to $R$. Addition is as expected, viz

$$\sum r_g\overline{g} + \sum s_g\overline{g} = \sum(r_g + s_g)\overline{g}$$

for $r_g$, $s_g \in R$. Multiplication is determined by two rules:

(i) (Twisting). For $g$, $h \in G$,

$$\overline{h}\overline{g} = (\overline{gh})\tau(h,g)$$

where $\tau : G \times G \longrightarrow \mathrm{u}(R)$, $\mathrm{u}(R)$ being the set of units of $R$.

(ii) (Action). For $g \in G$ and $r \in R$,

$$\overline{g}r = r^{\varphi(g)}\overline{g}$$

where $\varphi : G \longrightarrow \mathrm{Aut}(R)$ is a group homomorphism from $G$ into the group of $R$-automorphisms. For $g \in G$ and $r \in R$ denote the image of $r$ under $\varphi(g)$ by $r^g$, so we have $\overline{g}r = r^g\overline{g}$.

**1.3.2** Usually, there will be no ambiguity in denoting a crossed product of a group $G$ over a ring $R$ by $R*G$ (without specifying the $(\varphi,\tau)$-structure); if it is necessary to emphasise the twisting and action, then we will do so explicitly.

If the twisting is trivial, so that $\tau(g,h) = 1$ for all $g$, $h \in G$, then $R*G$ is called a *skew group ring*. If in addition the action is trivial, so that $\varphi(g) = 1$ and we have $r^g = r$ for all $g \in G$ and $r \in R$, then $R*G = RG$ the ordinary group ring.

Suppose that the twisting is trivial, so that $R*G$ is a skew group ring. In practice, we identify $G$ with its image in Aut($R$) under $\varphi$ and suppose that $G$ is a subgroup of Aut($R$), and so drop the $\bar{\ }$ notation. Then,

$$gr = r^g g$$

for all $r \in R$ and $g \in G$, so that $r^g = grg^{-1}$ and $g$ acts on $r$ by conjugation.

**1.3.3 LEMMA.** **[P. Lemma 1.3].** *Let $R$ be a ring and $G$ a group. Let $R*G$ denote a crossed product. If $N$ is a normal subgroup of $G$, then*

$$R*G = (R*N) * (G/N)$$

*where the latter is a crossed product of the group $G/N$ over the ring $R*N$.*

**1.3.4 DEFINITIONS.** Let $R$ be a prime Noetherian ring and denote by $E$ the set of all non-zero ideals of $R$. Then $E$ is a right localisation set (see [Mc-R, Example 10.3.4]). The resulting localisation is denoted $R_E$ and is equal to

$$\cup\{\mathrm{Hom}(I, R) : I \in E\}.$$

By [P1, Proposition 10.2], we have the following characterisation of $R_E$:

(i) $R \subseteq R_E$;

(ii) If $q \in R_E$ then there exists a non-zero ideal $A$ of $R$ with $qA \subseteq R$, that is, $q \in A_I{}^*$.

The ring $R_E$ is called the *right Martindale quotient ring* of $R$.

Similarly we define the *left Martindale quotient ring* of $R$, denoted ${}_ER$, which has the corresponding characterisation:

(i) $R \subseteq {}_ER$;

(ii) If $q \in {}_ER$ then there exists a non-zero ideal $B$ of $R$ with $Bq \subseteq R$, that is, $q \in B_r{}^*$.

We now define the *symmetric Martindale quotient ring* of $R$, denoted $R_S$, to be as follows (for further details, see [P1, Proposition 10.6]):

$$R_S = \{q \in R_E : Aq \subseteq R \text{ for some non-zero ideal } A \text{ of } R\}$$
$$= \{q \in {}_ER : qB \subseteq R \text{ for some non-zero ideal } B \text{ of } R\}.$$

Since our ring $R$ is assumed to be prime Noetherian, $R_S$ is contained in $Q(R)$, the quotient ring of $R$ with respect to the set of regular elements of $R$. Moreover, if $R$ is simple, then $R = R_E = R_S$.

Let $\sigma \in \mathrm{Aut}(R)$. We say that $\sigma$ is *X-inner* if there exists a unit $q$ of $R_S$ such that $r^\sigma = q^{-1}rq$ for all $r \in R$. We denote the set of X-inner elements of a subgroup $G$ of $\mathrm{Aut}(R)$ by $G_{\mathrm{inn}}$, and $G_{\mathrm{inn}}$ is a normal subgroup of $G$.

An element $\sigma \in \mathrm{Aut}(R)$ is called *X-outer* if it is not X-inner, and $G$ is *X-outer* on $R$ if and only if $G_{\mathrm{inn}} = \{1\}$.

Furthermore, an $R$-automorphism $\sigma$ is *inner* if there is an element $s \in R$ such that $r^\sigma = s^{-1}rs$ for all $r \in R$. An $R$-automorphism is *outer* if it is not inner. The set of elements of $G$ which are inner is contained in $G_{\mathrm{inn}}$.

**1.3.5** The following result is a special case of more general results, but is stated in as much generality as we will need.

**THEOREM.** [P1, Corollary 12.6] *Let $R$ be a prime Noetherian ring and $G$ a group. Let $R*G$ be a crossed product. If $R*G_{inn}$ is prime, then $R*G$ is prime.*

**1.3.6 DEFINITION.** Let $R$ be a ring and $G$ a group. An ideal $I$ of $R$ is said to be *$G$-invariant* if $I^g = \{a^g : a \in I\} \subseteq I$ for all $g \in G$.

**1.3.7** We denote the set of prime ideals of any ring $R$ by *Spec(R)*. If I is an ideal of $R$ then $C_R(I)$ will be the set of elements of $R$ that are regular modulo $I$. In particular, $C_R(0)$ is the set of regular elements of $R$.

**DEFINITION.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. We say that $I$ is *localisable* if $C_R(I)$ is an Ore set. The resulting localisation is denoted $R_I$.

**1.3.8 LEMMA.** *Let $I$ be a $G$-invariant ideal of a ring $R$ where $G$ is a finite group, and put $T = R*G$, a crossed product. Then $C_R(I)$ is $G$-invariant, and $C_R(I) \subseteq C_T(I*G)$.*

**PROOF.** We show first that $C_R(I)$ is $G$-invariant: Let $x \in C_R(I)$ and suppose that $x^g r \in I$ for some $g \in G$, $r \in R$. Put $h = g^{-1}$. Then

$$xr^h = (x^g r)^h \in I^h = I$$

so that $r^h \in I$. Therefore

$$r = (r^h)^g \in I^g = I,$$

and $x^g \in C_R(I)$.

Now, let $x \in C_R(I)$, $t \in T$ be such that $tx \in I*G$. Write

$$t = \sum\{r_g\bar{g} : g \in G\}$$

with each $r_g$ a member of $R$. Then

$$tx = \sum\{r_g\bar{g}x : g \in G\} = \sum\{r_g y\bar{g} : g \in G\}$$

for some $y \in C_R(I)$, since $C_R(I)$ is $G$-invariant. Therefore $tx \in I*G$ implies that $r_g y \in I$ for each $g \in G$, so that each $r_g \in I$ and so $t \in I*G$. A similar argument works on the left, and so

$$C_R(I) \subseteq C_T(I*G)$$

as required.

**1.3.9** We now quote some results on crossed products which will be of use in the sequel. The proofs can be found in [P1]. Note that some of the following results can be generalised (see §1.4).

**DEFINITION.** Let $R$ be a ring and $G$ a group; suppose that $I$ is a proper $G$-invariant ideal of $R$. We say that $I$ is a *$G$-prime ideal* if for all $G$-invariant ideals $A$, $B$ of $R$, the inclusion $AB \subseteq I$ forces either $A \subseteq I$ or $B \subseteq I$. In particular, $R$ is a *$G$-prime ring* if and only if 0 is a $G$-prime ideal.

**REMARK.** It is readily checked that any $G$-invariant prime ideal of a ring $R$ on which a group $G$ acts is $G$-prime, and that any $G$-prime ideal of a right Noetherian ring is semiprime.

**1.3.10 LEMMA.** [P1, Lemma 14.1]. *Let $R$ be a ring and $G$ a finite group. Let $R*G$ be a crossed product.*

*(i) If $P$ is a prime ideal of $R*G$, then $P \cap R$ is a $G$-prime ideal of $R$;*

*(ii) if $I$ is a $G$-prime ideal of $R$, then there exists a prime ideal $P$ of $R*G$ with $P \cap R = I$.*

1.3.11 **LEMMA.** **[P1, Lemma 14.2]** *Let $R$ be a ring and $G$ a finite group. If an ideal $I$ of $R$ is $G$-prime then there exists a prime ideal $P$ of $R$ such that $I = \cap\{P^g : g \in G\}$.*

1.3.12 The following two results can be generalised, but are stated here in as much generality as suits our present purposes. For the proofs, see [P1, Theorem 14.7, Corollary 14.8] respectively. The following notation is used:

Let $R$ be a ring and $G$ a finite group acting on $R$, and suppose that $R$ is $G$-prime. Let $p$ be a minimal prime ideal of $R$ and put

$$K := \mathrm{Stab}_G(p) = \{g \in G : p^g = p\}.$$

We suppose further that a twisting $\tau$ is given, and let $R*G$ and $R*K$ denote the corresponding crossed products.

**THEOREM.** **(Lorenz, Passman, 1980).** *Let $R$, $G$, $p$ and $K$ be as above. There exists a one-to-one correspondence between the set of prime ideals $P$ of $R*G$ with $P \cap R = 0$ and the set of primes $Q$ of $R*K$ with $Q \cap R = p$.*

1.3.13 We will now discuss how the correspondence in Theorem 1.3.12 actually works. The notation set out in 1.3.12 continues to be in force; in addition, for a minimal prime $p$ of $R$, we let $N = \mathrm{Ann}_R(p)$.

For an ideal $I$ of $R*G$, define

$$I(K) = \{x \in R*K : Nx \subseteq I\},$$

and for an ideal $L$ of $R*K$ define

$$L(G) = \cap\{(L*G)^g : g \in G\}.$$

**REMARK.** It is easy to see that for all ideals $I$ of $R*G$, $I(K)$ is an ideal of $R*K$: since $N$ is an ideal of $R$, $I(K)$ is a left $R$-module, and since $I$ is an ideal of $R*G$, $I(K)$ is a right $(R*K)$-module. Also, $I(K)$ is $K$-invariant. Thus $I(K)$ is an ideal of $R*K$. Similarly if $L$ is an ideal of $R*K$, observe that $L(G)$ is a right $(R*G)$-module and a left $R$-module. It is also clear that $L(G)$ is $G$-invariant. Therefore $L(G)$ is an ideal of $R*G$.

**1.3.14 LEMMA.** [P1, Lemma 14.4]. *Adopt the notation of 1.3.13. Let $I$ be an ideal of $R*G$ and $L$ an ideal of $R*K$.*

*(i) If $I \cap R = 0$, then $I(K) \cap R = p$;*

*(ii) if $L \cap R = p$, then $L(G) \cap R = 0$.*

1.3.15 Retain the notation and hypotheses of 1.3.13 and let $p$ be a minimal prime of $R$. The ideal correspondence of Theorem 1.3.12 is given as follows. The maps

$$P \longrightarrow P(K); \qquad Q \longrightarrow Q(G)$$

as described above yield a one-to-one correspondence between the prime ideals $P$ of $R*G$ with $P \cap R = 0$ and the prime ideals $Q$ of $R*K$ with $Q \cap R = p$.

If $p*K$ is a prime ideal of $R*K$, then it follows in particular that $0 = (p*K)(G)$ is a prime ideal of $R*G$. This yields the following corollary to Theorem 1.3.12.

**COROLLARY TO THEOREM 1.3.12.** *Let $R$, $G$, $p$ and $K$ be as in 1.3.13. Then $R*G$ is a prime ring if and only if*

$$(R*K)/(p*K) \cong (R/p)*K$$

*is prime.*

**1.3.16 THEOREM.** [P1, Theorem 16.2]. *Let $R*G$ be a crossed product with $G$ a finite group and $R$ a $G$-prime ring. Then*

(i) *a prime ideal $P$ of $R*G$ is minimal if and only if $P \cap R = 0$;*

(ii) *there exist finitely many minimal primes $P_1, P_2, \ldots, P_n$ of $R*G$, and $n \leqslant |G|$;*

(iii) *$N := P_1 \cap \ldots \cap P_n$ is the unique maximal nilpotent ideal of $R*G$, and $N^{|G|} = 0$;*

(iv) *if $p$ is a minimal prime ideal of $R$, then $\{p^g : g \in G\}$ is the set of all minimal primes of $R$, and $\cap\{p^g : g \in G\} = 0$.*

**1.3.17 PROPOSITION.** [P1, Proposition 16.4]. *Let $R*G$ be a crossed product with $G$ a finite $\rho$-group for some prime number $\rho$, and $R$ a $G$-prime ring of characteristic $\rho$. Then $R*G$ has a unique minimal prime ideal.*

**1.3.18 DEFINITION.** Let $R \subseteq S$ be rings. If $S$ is finitely generated as an $R$-module by elements $x_1, \ldots, x_n \in S$ such that

$$x_i R = R x_i$$

for each i, then $S$ is called a *finite normalising extension* of $R$. The elements $\{x_1, \ldots, x_n\}$ are called *normalising generators*. As we shall see in Theorem 1.3.20, there exist certain relations between the prime ideals of $S$ and those of $R$. Also, we can link the chain conditions satisfied by the two rings, as shown in Theorem 1.3.19.

**EXAMPLE.** Let $R$ be a ring and $G$ a finite group. Then the crossed product $R*G$ is a finite normalising extension of $R$, with the elements

$$\{\bar{g} \in \mathrm{u}(R) : g \in G\}$$

serving as normalising generators.

**1.3.19 THEOREM.** [Mc-R, Corollary 10.1.11]. *Let $S$ be a finite normalising extension of a ring $R$. Then*

*(i) $S$ is right Artinian if and only if $R$ is right Artinian;*

*(ii) $S$ is right Noetherian if and only if $R$ is right Noetherian.*

**1.3.20 THEOREM.** [P1, Theorem 16.9]. *Let $S$ be a finite normalising extension of the ring $R$, generated by $n$ normalising generators. Then the following relations hold between the prime ideals of $R$ and those of $S$:*

*(i)* Cutting Down. *If $P$ is a prime ideal of $S$ then there exist finitely many primes $p_1,\ldots,p_m$ of $R$ such that each $p_i$ is minimal over $P \cap R$. In fact, $\cap\{p_i : i = 1,\ldots,m\} = P \cap R$, and $1 \leqslant m \leqslant n$. We say that $P$* lies over *$p_i$ for each $i$.*

*(ii)* Lying Over (LO). *If $p \in Spec(R)$ then there exist finitely many primes $P_1,\ldots,P_t$ of $S$ such that $P_i$ lies over $p$ for each $i$, and $1 \leqslant t \leqslant n$.*

*(iii)* Incomparability (INC). *If $P, Q \in Spec(S)$ with $P \subset Q$, then $P \cap R \subset Q \cap R$.*

*(iv)* Going Up (GU). *If $p, q \in Spec(R)$ with $p \subset q$ and if $P \in Spec(S)$ lies over $p$, then there exists $Q \in Spec(S)$ such that $P \subset Q$ and $Q$ lies over $q$.*

*(v)* Going Down (GD). *If $P, Q \in Spec(S)$ with $P \subset Q$ and $Q$ lies over $q \in Spec(R)$, then there exists $p \in Spec(R)$ such that $p \subset q$ and $P$ lies over $p$.*

**1.3.21** The following result is an analogue of Theorem 1.3.20 for the case where $S$ is the crossed product $R*G$, where $G$ is a finite group.

**THEOREM.** **[P1, Theorem 16.6].** *Let $R*G$ be a crossed product over the ring $R$, with $G$ a finite group. The following relations hold between the prime ideals of $R$ and those of $R*G$:*

*(i)* Cutting Down. *If $P \in Spec(R*G)$ then there exists $p \in Spec(R)$, unique up to $G$-conjugation, such that $p$ is minimal over $P \cap R$. In fact, $P \cap R = \cap\{p^g : g \in G\}$. We say that $P$* lies over *$p$.*

*(ii)* Lying Over. *If $p \in Spec(R)$ then there exist primes $P_1,\ldots,P_n$ of $R*G$ with $1 \leqslant n \leqslant |G|$ such that each $P_i$ lies over $p$.*

*(iii)* Incomparability. *If $P, Q \in Spec(R*G)$ with $P \subset Q$, then $P \cap R \subset Q \cap R$.*

*(iv)* Going Up. *(a) If $p, q \in Spec(R)$ with $p \subset q$ and if $P \in Spec(R*G)$ lies over $p$, then there exists $Q \in Spec(R*G)$ such that $Q$ lies over $q$ and $P \subset Q$.*

*(b) Given $P, Q \in Spec(R*G)$ with $P \subset Q$ and if $P$ lies over $p \in Spec(R)$, there exists $q \in Spec(R)$ such that $p \subset q$ and $Q$ lies over $q$.*

*(v)* Going Down. *(a) If $p, q \in Spec(R)$ with $p \subset q$ and if $Q \in Spec(R*G)$ lies over $q$, then there exists $P \in Spec(R*G)$ such that $P$ lies over $p$ and $P \subset Q$.*

*(b) Given $P, Q \in Spec(R*G)$ with $P \subset Q$ and if $Q$ lies over $q \in Spec(R)$, there exists $p \in Spec(R)$ such that $p \subset q$ and $P$ lies over $p$.*

**1.3.22** Again let $S$ be a finite normalising extension of the ring $R$. We have the following result, concerning the Jacobson radicals of $R$ and $S$, denoted $J(R)$, $J(S)$ respectively.

**THEOREM.** **[Mc-R, Corollary 10.4.15].** *Let $S$ be a finite normalising extension of $R$. Then $J(R) = J(S) \cap R$.*

**1.3.23 THEOREM.** **[P2, Theorem 7.2.5].** *Let $S$ be a finite normalising extension of $R$. Then, for some $n \in \mathbb{N}$,*

$$(J(S))^n \subseteq J(R)S \subseteq J(S).$$

**1.3.24** When $S = R*G$, the crossed product, with $G$ finite and $|G|^{-1} \in R$ we can say more.

**THEOREM.** **[P1, Theorem 4.2].** *Let $R*G$ be a crossed product with $G$ finite and $n = |G|$. Then*

$$(J(R*G))^n \subseteq J(R) * G \subseteq J(R*G).$$

*If in addition we suppose that $|G|$ is a unit in $S$, then*

$$J(R*G) = J(R) * G.$$

**REMARK.** Note that the second statement of the above Theorem is a generalisation of Maschke's Theorem.

**1.3.25 DEFINITION.** Let $R$ be any ring and $I$ an ideal of $R$. Then we define

$$\sqrt{I} := \bigcap\{P \in \mathrm{Spec}(R) : I \subseteq P\}.$$

**LEMMA.** *Let $R$ be a Noetherian ring with $G$ a finite group acting on $R$, and let $T$ denote a crossed product $R*G$. Suppose that $I$ is a localisable $G$-invariant semiprime ideal of $R$ such that $C_R(I) \subseteq C_R(0)$. Then $J := \sqrt{(I*G)}$ is a localisable semiprime ideal of $T$, and $T_J \cong R_I*G$.*

**PROOF.** First of all, recall that

$$\sqrt{}(I*G) := \cap\{P \in \mathrm{Spec}(T) : I*G \subseteq P\}.$$

Put $C = C_R(I)$; we know that $C \subseteq C_R(0)$ by hypothesis. By Lemma 1.3.8,

$$C_R(0) \subseteq C_T(0)$$

and so $C \subseteq C_T(0)$.

Now we show that $C$ is an Ore set in $T$. Let $c \in C$,

$$t = \sum\{r_g\bar{g} : g \in G\} \in T.$$

Since $I$ is localisable by hypothesis $C$ is an Ore set in $R$, and so for all $g \in G$ there exists $d_g \in C$, $s_g \in R$ such that

$$cs_g = r_gd_g.$$

Therefore

$$cs_g\bar{g} = r_gd_g\bar{g} = r_g\bar{g}(\bar{g}^{-1}d_g\bar{g}) = r_g\bar{g}(d_g)^h,$$

where $h = g^{-1}$, and $(d_g)^h \in C$. By [Mc-R, Proposition 2.1.16(i)], there exist elements $d \in C$ and $u_g \in T$ such that

$$cu_g = r_g\bar{g}d$$

for all $g \in G$. Adding gives

$$cu = c\sum\{u_g : g \in G\} = \sum\{r_g\bar{g}d : g \in G\} = td$$

where

$$u = \sum\{u_g : g \in G\} \in T.$$

Similarly, there exist $v \in T$, $e \in C$ such that $vc = et$, and so $C$ is a (right and left) Ore set in $T$, as required.

The action of $G$ can be extended to $R_C$, so we can now form

$$T_C = (R*G)_C = R_C * G.$$

Consider the map

$$\psi : R_C * G \longrightarrow (R/I)_C * G$$

defined by

$$\psi\Big(\sum_{g\in G} r_g(c_g)^{-1}\bar{g}\Big) = \Big(\sum_{g\in G} r_g(c_g)^{-1}\bar{g}\Big) + I$$

$$= \sum_{g\in G}(r_g + I)(c_g + I)^{-1}\bar{g}$$

since $I$ is $G$-invariant. It is clear that $\psi$ is a well-defined epimorphism and that $\mathrm{Ker}(\psi) = I_C * G$. Therefore by the Isomorphism Theorem,

$$(R*G)_C/(I*G)_C = (R_C * G)/(I_C * G) \cong (R/I)_C * G.$$

Now, $R/I$ is a semiprime Noetherian ring, so that $(R/I)_C$ is semisimple Artinian. Therefore by Theorem 1.3.19(i) $(R/I)_C * G$ is Artinian, since $G$ is finite. So by the above isomorphism, the factor ring

$$(R*G)_C/(I*G)_C$$

is also Artinian.

Since $J$ is a proper semiprime ideal of $T$, and

$$J \cap C = J \cap R \cap C = I \cap C$$

which is empty, $J_C$ is a proper semiprime ideal of $T_C$; we claim that $J_C$ is the Jacobson radical of $T_C$ (denoted $\mathrm{Jac}(T_C)$). By hypothesis, $I$ is a semiprime localisable ideal of $R$, and $C$ is an Ore set of regular elements. Therefore

$$I_C = \mathrm{Jac}(R_C)$$

by [G-W, Lemma 12.18]. By Theorem 1.3.24,

$$I_C * G \subseteq \mathrm{Jac}(T_C).$$

But

$$J_C = (\surd(I*G))_C = \surd(I_C * G),$$

and $I_C * G \subseteq J_C$. Therefore we must have $J_C = \mathrm{Jac}(T_C)$ as claimed.

Finally, it remains to show that $J$ is localisable. By [G-W, Lemma 10.8], the set of regular elements of $T_C$ is contained in the set of elements of $T_C$ which are regular modulo $J_C$. In other words,

$$C_T(I) \subseteq C_T(J).$$

For the reverse inclusion let $\alpha \in \mathcal{C}_T(J)$. Then $\alpha + J_C$ is a regular element of $T_C/J_C$ which is Artinian, being a factor of $T_C/(I*G)_C$. Hence $\alpha + J_C$ is a unit of $T_C/J_C$, and so there exists an element $\beta$ of $T_C$ such that

$$\alpha\beta - 1 \in J_C.$$

We can choose $\gamma \in J_C$ with $\alpha\beta = 1 + \gamma$, so that $\alpha\beta$ is a unit in $T_C$ since $J_C = \mathrm{Jac}(T_C)$. It follows that $\alpha$ is a unit in $T_C$, and so $\mathcal{C}_T(J)$ consists of units in $T_C$. Therefore

$$\mathcal{C}_T(J) = \mathcal{C}_T(I)$$

and $J$ is indeed localisable in $T$. It is now easy to see that

$$T_J = T_C = R_C * G,$$

as required.

**COROLLARY.** *Let $R$ be a right Noetherian ring with right Artinian quotient ring $Q(R)$, and $G$ a finite group. Put $T = R*G$, a crossed product. Then $T$ has a right Artinian quotient ring $Q(T)$, and $Q(T) = Q(R)*G$.*

**PROOF.** Firstly, note that $Q(R)*G$ is right Artinian since $Q(R)$ is right Artinian and $G$ is finite. Let $N$ denote the prime radical of $T$. Then, by Lemma 1.3.25, $T_N \cong Q(R)*G$. But by [G-W, Lemma 10.8], $\mathcal{C}_T(0) \subseteq \mathcal{C}_T(N)$. Therefore we have

$$Q(T) \subseteq T_N \cong Q(R)*G \subseteq Q(R*G) = Q(T).$$

Hence $Q(T) = Q(R)*G$ and is right Artinian, as required.

## §1.4 INFINITE GROUPS.

**1.4.1 DEFINITION.** A group $G$ is said to be *polycyclic-by-finite* if $G$ has a subseries

$$\{1\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_{n-1} \subseteq G_n = G$$

such that (i) $G_{i-1}$ is a normal subgroup of $G_i$ for all $i = 1,\dots,n$, and

(ii) the factor $G_i/G_{i-1}$ is either finite or infinite cyclic for each i.

The series can be chosen in such a way that $G_n/G_{n-1}$ is finite and all the other factors are infinite cyclic.

**1.4.2 DEFINITION.** A group $G$ is said to be *polycyclic* if there exists a subseries

$$\{1\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_{n-1} \subseteq G_n = G$$

such that (i) $G_{i-1}$ is a normal subgroup of $G_i$ for all $i = 1,\dots,n$, and

(ii) each factor $G_i/G_{i-1}$ is cyclic.

If G is a polycyclic group such that each factor in the series described above is infinite cyclic, then G is called *poly-(infinite cyclic)*. It is clear that poly-(infinite cyclic) groups are polycyclic-by-finite.

**1.4.3** In view of the above definitions and comments, we have the following property of polycyclic-by-finite groups.

**LEMMA.** **[P1, Lemma 21.4(i)].** *If $G$ is a polycyclic-by-finite group, then there exists a normal subgroup $H$ of $G$ such that $H$ is poly-(infinite cyclic) and $G/H$ is finite.*

**1.4.4** The following result is a variant of Hilbert's Basis Theorem, and allows us to extend ascending chain conditions from a ring to a crossed product of that ring by a polycyclic-by-finite group. Compare this with Theorem 1.3.19.

**PROPOSITION.** **[P1, Proposition 1.6].** *Let $R$ be a right Noetherian ring and $G$ a polycyclic-by-finite group. Then any crossed product $R*G$ is right Noetherian.*

**1.4.5** The following should be compared with Lemma 1.3.10.

**LEMMA.** **[P1, Lemma 14.1].** *Let $R$ be a right Noetherian ring and $G$ a polycyclic-by-finite group. Let $R*G$ be a crossed product.*

*(i) If $P$ is a prime ideal of $R*G$, then $P \cap R$ is a $G$-prime ideal of $R$;*

*(ii) if $I$ is a $G$-prime ideal of $R$, then there exists a prime ideal $P$ of $R*G$ with $P \cap R = I$.*

**1.4.6** We can also extend Lemma 1.3.11.

**LEMMA.** **[P1, Lemma 14.2].** *Let $R$ be a right Noetherian ring and $G$ a polycyclic-by-finite group. If $I$ is a $G$-prime ideal of $R$ then there exists a prime ideal $P$ of $R$ such that $I = \cap\{P^g : g \in G\}$, and the number of distinct ideals in this intersection is finite.*

**1.4.7 DEFINITION.** A group $G$ is said to be *nilpotent* if there exists a subseries

$$\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_{n-1} \subseteq G_n = G$$

such that (i) $G_i$ is a normal subgroup of $G$ for all $i = 0,\ldots,n-1$;

(ii) $G_{i+1}/G_i$ lies in the centre of $G/G_i$ for all i.

**1.4.8 LEMMA.** [P2, Lemma 11.1.2]. *Let $G$ be a finitely generated nilpotent group. Then the torsion elements of $G$ form a finite normal subgroup $H$ of $G$, and $G/H$ is torsionfree.*

**1.4.9** The following result is an easy consequence of Definition 1.4.7; see, for example, [Ma, Theorem 9.02].

**THEOREM.** *All subgroups and factor groups of a nilpotent group are themselves nilpotent.*

**1.4.10** We have the following useful fact about finitely generated nilpotent groups. For a proof, see [P2, Lemma 11.4.3].

**LEMMA.** *A finitely generated nilpotent group is polycyclic.*

**REMARK.** Suppose that $G$ is a torsionfree nilpotent group. Recall 1.4.7 and put $G_0 = \{1\}$ and $G_{i+1}/G_i = \mathrm{Centre}(G/G_i)$ for all $i = 1,\ldots,n$. Then $G_{i+1}$ is just the inverse image in $G$ of the centre of $G/G_i$ for each i. By [P2, Lemma 11.1.3], $G/G_i$ is torsionfree for all i. Suppose further that $G$ is finitely generated. Then $G_{i+1}/G_i$ is finitely generated for all i. Therefore each $G_{i+1}/G_i$ is a finitely generated torsionfree

Abelian group, and so is poly-(infinite cyclic), by [Ma, Theorem 5.09]. Hence a finitely generated torsionfree nilpotent group is poly-(infinite cyclic).

**1.4.11 DEFINITION.** A group $G$ is said to be *Abelian-by-finite* if $G$ has an Abelian subgroup $H$ such that $G/H$ is finite.

**REMARK.** Suppose that $G$ is a finitely generated Abelian-by-finite group and $R$ a commutative Noetherian ring. Let $A$ be an Abelian subgroup of $G$ with $G/A$ finite. Then the group ring

$$RG = RA * (G/A),$$

a crossed product of a finite group over the commutative ring $RA$, and $RA$ is Noetherian since $A$ is finitely generated, by [Ro, Lemma 1.43].

**1.4.12 DEFINITION.** Let $N$ and $H$ be groups, and let $\varphi$ be a group homomorphism from $H$ into $\mathrm{Aut}(N)$, defined as follows. For $n \in N$ and $h \in H$, write $n^h$ for the image of $n$ under $\varphi(h)$. Then the *semidirect product* $G$ of $H$ and $N$, denoted $N \rtimes H$, has multiplication

$$(n,f)(m,h) = (nm^f, fh)$$

for $f$, $h \in H$ and $n$, $m \in N$.

**1.4.13 REMARK.** Let $H$, $N$ and $\varphi$ be as in the above definition. If we let $R$ be a ring with $H$ acting as the identity on $R$, then $\varphi$ can be extended to a homomorphism from $H$ into $\mathrm{Aut}(RN)$, where $\mathrm{Aut}(RN)$ is the group of ring automorphisms of $RN$. In this way, we can identify the ordinary group ring $RG$ as the skew group ring $RN*H$. In particular, consider the case where $R$ is a commutative Noetherian ring, $N$ a finitely generated Abelian group and $H$ a finite group. Then, putting

$G = N \rtimes H$, the group ring $RG$ is a skew group ring $RN*H$ of a finite group over a commutative Noetherian ring. This particular construction will be discussed further in the next section.

## §1.5 KNOWN RESULTS ON GROUP RINGS AND MAXIMAL ORDERS.

1.5.1 There has already been much ground covered on the question of when group rings (and skew group rings) are maximal orders. In particular we have the following two results, due to P. F. Smith (in 1984) and K. A. Brown (in the following year) respectively, which deal with the group ring case. The generalisation from the hypotheses of Brown's result to an arbitrary Noetherian coefficient ring (as in Smith's result) is detailed in [B-M-S, Corollary 2.5] and [B-M-S, Proposition 2.7]. This is a stark contrast to the results we prove in the sequel on skew group rings; as we shall see in later Chapters, the generalisation to a non-commutative Noetherian coefficient ring is by no means straightforward, and in fact we have to strengthen our hypotheses somewhat. A known result concerning skew group rings and maximal orders is discussed in 1.5.6.

**DEFINITION.** The *infinite dihedral group*, usually denoted $D_\infty$, is the group

$$D_\infty = \langle a,\ b\ :\ b^2 = 1,\ a^b = b^{-1}ab = a^{-1}\rangle.$$

**1.5.2 THEOREM.** **[S, Theorem 4.1]** *Let $R$ be a Noetherian maximal order in a simple Artinian ring, and $G$ a nilpotent-by-finite group which is polycyclic-by-finite. Then the group ring $RG$ is a maximal order in a simple Artinian ring if and only if* *(i) $G$ contains no non-trivial finite normal subgroup;*

*(ii) $G$ contains no infinite dihedral normal subgroup.*

**1.5.3 THEOREM.** **[B, Theorem F].** *Let $R$ be a commutative Noetherian domain and $G$ a polycyclic-by-finite group. Then the group ring $RG$ is a maximal order in its (simple Artinian) quotient ring if and only if*

*(i) $R$ is integrally closed;*

*(ii) $G$ contains no non-trivial finite normal subgroup;*

*(iii) $G$ contains no subgroup with finitely many $G$-conjugates which is isomorphic to the infinite dihedral group.*

**1.5.4 REMARK.** Note that condition (i) of Brown's Theorem gives us that $R$ is a maximal order. Condition (ii) is necessary and sufficient for $RG$ to be prime, so that $RG$ is an order in a simple Artinian ring by Goldie's Theorem. So condition (iii) is the key to deciding whether or not $RG$ is a maximal order. Similarly, (ii) of Smith's Theorem is the crucial condition in deciding when the group ring is a maximal order.

**1.5.5** Let $A$ be a finitely generated torsionfree Abelian group and $H$ a finite group. Let $G$ be the semidirect product $A \rtimes H$. If we put $S = RA$, then the group ring $RG$ is equal to the skew group ring $S*G$. Then the question of whether or not this ring is a maximal order can be answered using either Brown's result or Theorem 5.2.7. (See Example 5.2.16).

**1.5.6** As to the problem of when a skew group ring is a maximal order, there is a result credited to Auslander, Goldman and Rim in the literature, which we discuss in the following few paragraphs. For further details, the reader is referred to [A-R, Page 578]. The statement of the Theorem, together with a complete proof can be found in [R, Theorem 40.14]. We will state the result as it stands first of all (as Theorem 1.5.8), and then aim towards restating it (as Theorem 1.5.16) in a form which enables us to make comparisons with results which appear later on in the thesis, in particular, with Theorem 5.2.7. Our treatment of this will be fairly sketchy; further details can be found in [R]. But first some definitions.

**1.5.7 DEFINITIONS** Recall that a *Dedekind domain* is an integrally closed commutative Noetherian domain in which all non-zero prime ideals are maximal. If $A \subseteq B$ are rings, then the *Galois group* of $B/A$ is defined to be the set

$$\mathrm{Gal}(B/A) = \{\sigma \in \mathrm{Aut}(B) : \sigma(a) = a \text{ for all } a \in A\}.$$

It is clear that $\mathrm{Gal}(B/A) \subseteq \mathrm{Aut}(B)$.

**1.5.8** Let $R$ be a Dedekind domain with quotient field $K$, and $L$ an extension field such that $L/K$ is a finite Galois extension with Galois group $G$. Let $S$ be the integral closure of $R$ in $L$, so that $S$ is a Dedekind domain with quotient field $L$ and is finitely generated as a module over $R$. Let $S*G$ denote the skew group ring. Then we have the following result. (See 1.5.11 for a definition of the term *unramified*).

**THEOREM. (Auslander, Goldman, Rim).** *Adopt the notation set out above. Then the skew group ring $S*G$ is a maximal order if and only if $S/R$ is unramified.*

**1.5.9 DEFINITIONS.** Let $A \subseteq B$ be commutative domains with quotient fields $K$ and $L$ respectively. Then each element $b$ of $B$ determines a $K$-linear map ($\sigma_b$, say) from $L$ to $L$ defined by

$$\sigma_b(x) = bx$$

for all $x \in L$. Suppose that $\dim_K(L) = n < \infty$. Then

$$\sigma_b \in \mathrm{End}_K(L) \cong M_n(K).$$

The *characteristic polynomial* of an element $b \in B$ is defined to be the characteristic polynomial of the map $\sigma_b$ as defined above. Denote this polynomial by $f_b$ for each $b \in B$. We then define the *trace* of $b$, $T(b)$, as follows.

$$T(b) = T_{L/K}(b) = -\text{ (coefficient of } X^{n-1} \text{ in } f_b).$$

If we suppose that $B$ is integral over $A$, then $T(b) \in A$ for all $b \in B$, by [R, Theorem 1.14, Theorem 1.7].

Suppose now that $G$ is a finite group acting on $B$ and that $A = B^G$. Then for all $\alpha \in L$, [R, Exercise 4, p.7] implies that

$$T(\alpha) = \sum\{\alpha^g : g \in G\}.$$

Now define

$$\tilde{B} = \{x \in L : \sum_{g \in G}(xb)^g \in A \text{ for all } b \in B\}$$

We call $\tilde{B}$ the *complementary module* of $B$. Consider the ordinary group ring $AG$. The field $L$ is a left $AG$-module by virtue of the fact that $A$ acts on $L$ by multiplication, and where we set

$$gl = l^h \in L$$

for all $l \in L$ and $g \in G$ (with $h = g^{-1}$). Then $\tilde{B}$ is an $(AG - B)$-bisubmodule of $L$, and

$$\tilde{B}/B = \mathrm{Ann}_{L/B}(A\hat{g}A)$$

where $\hat{g} = \sum\{g : g \in G\}$. Also, $A\hat{g}A$ is an ideal of $AG$. We now define the

*Dedekind different*, denoted $D(B/A)$, of $B/A$ to be

$$D(B/A) = \mathrm{Ann}_B(\tilde{B}/B).$$

**1.5.10** We have the following result, a proof of which can be found in [H, Lemma B(ii)].

**PROPOSITION.** *Let $A \subseteq B$ be commutative domains, and suppose that $B$ is a projective $A$-module (for example, if $A$ is a Dedekind domain). Then*

$$\mathrm{Ann}_B(\tilde{B}/B) = (B\hat{g}B) \cap B.$$

**1.5.11 DEFINITION.** Suppose that $A \subseteq B$ are commutative domains; we say that $B$ is an *unramified extension* of $A$ if, for all maximal ideals $P$ of $A$, $PB$ is a semiprime ideal of $B$ (so that $PB = \cap\{Q_i : i = 1,\ldots,n\}$ for some prime ideals $Q_i$ of $B$), and $B/Q_i$ is a separable field extension of $A/P$ for all $i = 1,\ldots,n$.

**1.5.12 THEOREM.** *Let $A \subseteq B$ be Dedekind domains with $B$ integral over $A$. Suppose further that $G$ is a finite group acting on $B$, and that $A = B^G$. Then the following are equivalent:*

*(i)* *$B$ is an unramified extension of $A$;*

*(ii)* $\tilde{B} = B$;

*(iii)* $D(B/A) = B$;

*(iv)* $(B\hat{g}B) \cap B = B$.

**PROOF.** (i) $\Longleftrightarrow$ (ii): [R, Theorem 4.37].

(ii) $\Longleftrightarrow$ (iii): This is clear from the definition of the Dedekind different.

(iii) $\Longleftrightarrow$ (iv): This is immediate from Proposition 1.5.10.

**1.5.13 DEFINITION.** Let $R$ be a commutative ring and $G$ a finite group acting on $R$. For each non-identity element $g$ of $G$ define

$$I(g) = \{s - s^g : s \in S\}S.$$

It is easy to check that $I(g)$ is an ideal of $S$ for each $1 \neq g \in G$. (These ideals will be discussed further in §5.2).

**1.5.14** The following result can be found in [B-L, Lemma 2.2].

**LEMMA.** *Let $S$ be a commutative Noetherian ring and $G$ a finite group acting on $S$. Let $T$ be the skew group ring $S*G$, and denote by $\hat{g}$ the element $\sum\{g : g \in G\}$ of $T$. Then*

$$\prod\{I(g) : 1 \neq g \in G\} \subseteq (T\hat{g}T) \cap S \subseteq \cap\{I(g) : 1 \neq g \in G\}.$$

**1.5.15 REMARK.** In the above situation, it is easy to see that

$$(T\hat{g}T) \cap S = (S\hat{g}S) \cap S.$$

So, regarding Theorem 1.5.12, Theorem 1.5.8 says that $S*G$ is a maximal order if and only if $(S\hat{g}S) \cap S = S$, where $\hat{g} = \sum\{g : g \in G\}$. Then Lemma 1.5.14 gives that $(S\hat{g}S) \cap S = S$ precisely when $I(g) = S$ for all (non-identity) $g \in G$. In view of the above comments, we can now restate Theorem 1.5.8 as follows.

**1.5.16 THEOREM.** *Let $S$ be a Dedekind domain and $G$ a finite group acting on $S$. Then the skew group ring $S*G$ is a (prime Noetherian) maximal order if and only if $I(g) = S$ for all non-identity elements $g$ of $G$.*

**REMARK.** Recall from Example 1.1.2 and Definition 1.5.7 that a Dedekind domain is a maximal order. It is now easy to see that this result is just a special case of Theorem 5.2.7.

## §1.6 ADDITIONAL REMARKS.

**1.6.1** All the results quoted in §1.1 are well-known; [Mc-R] and [P1] were the main sources used. A full account of maximal orders is also given in [M-R].

**1.6.2** (i) Again, there is nothing new in §1.2.

(ii) Lemma 1.2.5, quoted from [Mc-R], is a union of results proved by G.O. Michler and J.C. Robson in [Mi] and [Rob] respectively.

(iii) Proposition 1.2.8 should be compared with [H-L, Proposition 1.3], where it is shown that a prime Noetherian local ring is hereditary if and only if it is a principal right and left ideal ring if and only if its Jacobson radical is invertible. These conditions characterise a *bounded Asano order*, which we do not discuss here.

(iv) As we mentioned in the text, Theorem 1.2.9 is an adaptation of [F-S, Theorem 1].

(v) In Definition 1.2.12, *AR* is an abbreviation for Artin - Rees.

(vi) The proof of Lemma 1.2.13 hinges on the fact that the intersection of the powers of an ideal of a prime Noetherian ring that has the AR-property is zero; this is actually shown in [H-L, Corollary 2.2].

**1.6.3** (i) Most of the material in §1.3 is taken from [P1].

(ii) The (right) Martindale quotient ring defined in 1.3.4 was introduced by W.S. Martindale, III in the paper [Mar]. Our information, however, was taken from [P1] and [Mc-R].

(iii) Theorem 1.3.12 is taken from [P1], but originally appeared in [L-P3] and [P3].

(iv) Theorem 1.3.16 is due to M. Lorenz and D.S. Passman, and first appeared in [L-P2].

(v) Theorem 1.3.19 was proved by E. Formanek and A.V. Jategaonkar, and can be found in [F-J, Theorem 4].

(vi) Theorem 1.3.20 was first proved by A.G. Heinicke and J.C. Robson in [H-R].

(vii) Theorem 1.3.21 is again due to M. Lorenz and D.S. Passman, and can be found in [L-P1] and [L-P2].

**1.6.4** §1.4 is all standard material, mostly coming from [P1], [P2], [Ma], and [Ro].

**1.6.5** (i) Further details of Theorem 1.5.2 and Theorem 1.5.3 can be found in [S] and [B] respectively.

(ii) Full details of Theorem 1.5.8 can be found in [R].

(iii) Proposition 1.5.10 undoubtedly appeared before [H], but this was the only reference we could find.

(iv) Theorem 1.5.12 is not new; (i) $\Longleftrightarrow$ (ii) is in [R], (i) $\Longleftrightarrow$ (iii) was first proved in [A-B], and (i) $\Longleftrightarrow$ (iv) appears in [H].

# CHAPTER 2. $G$-MAXIMAL ORDERS.

## §2.1 DEFINITION AND EXAMPLES.

**2.1.1** In this section we introduce the concept of a *$G$-maximal order* which, as its name suggests, can be thought of as a "$G$-equivariant version" of a maximal order as defined in §1.1 (where $G$ is a group). As we will see in Chapter 3, a skew group ring being a maximal order does not require that the coefficient ring be a maximal order, but we do need it to be a $G$-maximal order (where $G$ is the group acting on the ring in question). For this reason, we devote this chapter to the study of such objects.

The definition of a $G$-maximal order is offered in 2.1.4 and we go on to prove an analogue of Theorem 1.1.2 (Theorem 2.1.8), which will be used often in the sequel. In Lemma 2.1.12 we see that a commutative

Noetherian domain which is a $G$-maximal order is precisely a maximal order; however $G$-maximal orders which are not maximal orders do exist and a *bona fide* example is given in 2.1.13.

As one may expect, there are many results about maximal orders which can be adapted to give results concerning $G$-maximal orders. We give several such results in sections 2.2 and 2.3, with complete proofs and a reference to the original source where appropriate.

**2.1.2 NOTATION.** Let $R$ be a prime Noetherian ring and $G$ a finite group acting on $R$ such that $G \subseteq \mathrm{Aut}(R)$. It follows that $G$ acts on the quotient ring $Q$ of $R$; in particular $G$ permutes the set of right (or left) orders in $Q$.

Of course $R$ is $G$-invariant; we are interested in defining an equivalence relation on the set of $G$-invariant right orders in $Q$.

**2.1.3 DEFINITION.** Let $R$, $Q$ and $G$ be as above, and let $S$ be a right order in $Q$ which is $G$-invariant. Define a relation $\sim$ on $G$-invariant orders in $Q$ to be such that $S \sim R$ if and only if there exist units $a$, $b$, $c$, $d$ in $Q$ with $aRb \subseteq S$ and $cSd \subseteq R$. It is easy to check that $\sim$ is an equivalence relation.

**2.1.4 DEFINITION.** Let $R$, $Q$ and $G$ be as in 2.1.2. Then $R$ is said to be a *$G$-maximal (right) order* precisely when $R$ is maximal within its equivalence class of $G$-invariant (right) orders in $Q$ as defined in 2.1.3.

**2.1.5 LEMMA.** *Let $R$, $Q$ and $G$ be as in 2.1.2. Suppose that $A$ and $B$ are non-zero $G$-invariant subsets of $Q$. Then* *(i)* $(A : B)_l := \{q \in Q : qA \subseteq B\}$ *and* $(A : B)_r := \{q \in Q : Aq \subseteq B\}$ *are G-invariant, and*

*(ii)* $AB$ *is G-invariant.*

**PROOF.** (i) It is enough to show that $(A : B)_l$ is $G$-invariant. Let $q$ belong to $(A : B)_l$ and $g$ be a non-identity element of $G$. Then

$$(q^g)A = (q^g)(A^g) = (qA)^g \subseteq B^g = B$$

so that $q^g \in (A : B)_l$. Therefore

$$((A : B)_l)^g \subseteq (A : B)_l$$

for all $g \in G$.

Now let $q \in (A : B)_l$ and write $q = (q^h)^g$, where $g, h \in G$ and $h = g^{-1}$. Then

$$q = (q^h)^g \subseteq ((A : B)_l{}^h)^g \subseteq ((A : B)_l)^g$$

by the argument above. Therefore

$$(A : B)_l = ((A : B)_l)^g,$$

and $(A : B)_l$ is $G$-invariant. Similarly, $(A : B)_r$ is $G$-invariant.

(ii) Let $g$ be an element of $G$. Then $(AB)^g = (A^g)(B^g) = AB$.

**COROLLARY.** *Let* $R$, $Q$ *and* $G$ *be as in 2.1.2, and* $I$ *a non-zero G-invariant ideal of* $R$. *Then*

*(i)* $I_l{}^*$ *and* $I_r{}^*$ *are G-invariant;*

*(ii)* $(I_l{}^*)_r{}^*$ *and* $(I_r{}^*)_l{}^*$ *are G-invariant;*

*(iii)* $O_l(I)$ *and* $O_r(I)$ *are G-invariant.*

**PROOF.** (i) Take $A = I$ and $B = R$ in part (i) of the lemma.

(ii) Take $A = I_l{}^*$ (resp. $A = I_r{}^*$) and $B = R$ in part (i) of the lemma.

(iii) Take $A = B = I$ in part (i) of the lemma.

**2.1.6 LEMMA.** *Let $R$, $Q$ and $G$ be as in 2.1.2 and let $I$ be a non-zero $G$-invariant ideal of $R$. Then $O_l(I)$ and $O_r(I)$ are $G$-invariant right orders in $Q$ equivalent to $R$.*

**PROOF.** It is clear that $O_l(I)$ and $O_r(I)$ are right orders in $Q$. That they are $G$-invariant follows from Corollary 2.1.5. It remains to show that the equivalence relation defined in 2.1.3 is satisfied. That $R \subseteq O_l(I)$ and $R \subseteq O_r(I)$ is clear. Since $R$ is a prime Noetherian ring, $I$ is essential in $R$. Therefore $I$ contains a regular element, $b$ say, of $R$ which is necessarily a unit of $Q$. Then $bO_r(I) \subseteq R$ and $O_l(I)b \subseteq R$, and the proof is complete.

**2.1.7** The following lemma should be compared with [Mc-R, Lemma 3.1.10].

**LEMMA.** *Let $R$, $Q$ and $G$ be as in 2.1.2. Let $S$ be a $G$-invariant right order in $Q$ such that $S$ contains $R$ and is equivalent to $R$. Then there exist equivalent $G$-invariant right orders $T$ and $U$ in $Q$ with $R \subseteq T \subseteq S$ and $R \subseteq U \subseteq S$ with the following property: there exist units $\hat{a}$, $\hat{b}$ of $Q$ contained in $R$ and such that*

$$\hat{a}S \subseteq T, \quad T\hat{b} \subseteq R, \quad S\hat{b} \subseteq U, \quad \hat{a}U \subseteq R.$$

*In particular, $\hat{a}S\hat{b} \subseteq R$.*

**PROOF.** By Definition 2.1.3 there exist units $x$, $y$ of $Q$ such that $xSy \subseteq R$. Write $x = ac^{-1}$, $y = bd^{-1}$ with $a$, $b$, $c$, $d$ regular elements of $R$. Then $xSy \subseteq R$ implies that

$$ac^{-1}Sbd^{-1} \subseteq R$$

so that

$$ac^{-1}Sb \subseteq Rd \subseteq R.$$

But $cS \subseteq S$, so $S \subseteq c^{-1}S$. Therefore we have

$$aSb \subseteq ac^{-1}Sb \subseteq R.$$

Since $a$ is regular in $R$, $a^g$ is regular in $R$ for all $g \in G$. Therefore each $Ra^g$ is an essential left ideal of $R$, and so we can choose a regular element

$$\hat{a} \in \cap\{Ra^g : g \in G\}.$$

Similarly, there exists a regular element

$$\hat{b} \in \cap\{b^gR : g \in G\}.$$

Now,

$$a^gSb^g = (aSb)^g \subseteq R^g = R$$

for all $g \in G$, and so

$$\hat{a}S\hat{b} \subseteq Ra^gSb^gR \subseteq R$$

for each $g$. Put

$$T := \langle R + Ra^gS\rangle, \quad U := \langle R + Sb^gR\rangle.$$

It is clear that $T$, $U$ are right orders in $Q$ and that they are both $G$-invariant, since $R$ and $S$ have these properties. It is also easily verified that $T$ and $U$ are equivalent, and that $R \subseteq T \subseteq S$, $R \subseteq U \subseteq S$.

Note that

$$a^gS\hat{b} \subseteq a^gSb^gR \subseteq R$$

and

$$\hat{a}Sb^g \subseteq Ra^gSb^g \subseteq R$$

for all $g \in G$. We now have

$$T\hat{b} = \langle R\hat{b} + Ra^gS\hat{b}\rangle \subseteq R$$

and

$$\hat{a}U = \langle \hat{a}R + \hat{a}Sb^gR\rangle \subseteq R.$$

The proof is now complete.

2.1.8 The following result is the analogue for $G$-maximal orders of Theorem 1.2.2. The proof given here is modelled on [Mc-R, Proposition 5.1.4].

**THEOREM.** *Let $R$, $Q$ and $G$ be as in 2.1.2. Then $R$ is a $G$-maximal order if and only if $O_l(I) = O_r(I) = R$ for all non-zero $G$-invariant ideals $I$ of $R$.*

**PROOF.** First suppose that $R$ is a $G$-maximal order. Let $I$ be a non-zero $G$-invariant ideal of $R$. By Lemma 2.1.6 $O_l(I)$ and $O_r(I)$ are $G$-invariant orders in $Q$ equivalent to $R$. Clearly $R \subseteq O_l(I)$ and $R \subseteq O_r(I)$. Therefore by Definition 2.1.4, we must have

$$O_l(I) = O_r(I) = R$$

as required.

Conversely, suppose that $O_l(I) = O_r(I) = R$ for all non-zero $G$-invariant ideals $I$ of $R$. Let $S$ be a $G$-invariant order in $Q$ such that $S$ is equivalent to $R$ and contains $R$. We show that $S = R$. By Lemma 2.1.7 there exists a $G$-invariant order $T$ in $Q$ with $R \subseteq T \subseteq S$, together with units $\hat{a}$, $\hat{b}$ of $Q$ contained in $R$ such that $\hat{a}S \subseteq T$ and $T\hat{b} \subseteq R$.

Consider the set

$$I := \{x \in R : Tx \subseteq R\}.$$

Now $I$ is non-zero since $\hat{b}$ belongs to $I$, and it is easy to see that $I$ is an ideal of $R$. Furthermore, $I$ is $G$-invariant by Lemma 2.1.5 since $I = R \cap (T : R)_r$. Hence

$$O_l(I) = O_r(I) = R$$

by hypothesis.

If $x \in I$, then $T(Tx) = Tx \subseteq R$, so that

$$Tx \subseteq R \cap (T : R)_r = I.$$

Thus $T \subseteq O_l(I) = R$, so that $T = R$. A similar argument shows that

$$S \subseteq O_r(J) = T = R$$

for some $G$-invariant ideal $J$ of $T$. Therefore $S = R$ and $R$ is a $G$-maximal order, as required.

**2.1.9 THEOREM.** *Let R be a ring, and H a subgroup of a finite group G which acts on R. If R is an H-maximal order, then R is a G-maximal order.*

**PROOF.** Suppose that $R$ is an $H$-maximal order. Then $O_l(I) = O_r(I) = R$ for all $H$-invariant ideals $I$ of $R$. Let $A$ be a $G$-invariant ideal of $R$. Then certainly $A$ is $H$-invariant, and so $O_l(A) = O_r(A) = R$. Hence $R$ is a $G$-maximal order, as required.

**2.1.10** The following result should be compared with [Mc-R, Proposition 5.1.5], which is the corresponding result for the maximal order case.

**THEOREM.** *Let R be a semiprime right Goldie ring with right quotient ring Q, and let G be a finite group acting on R. Then R is a G-maximal order if and only if the matrix ring $M_n(R)$ is a G-maximal order for all $n \geqslant 1$.*

**PROOF.** Note first that $G$ acts on $M_n(R)$ element-wise for all $n \in \mathbb{N}$. Now, by [Mc-R, Corollary 3.1.5], $R$ is a semiprime right Goldie ring with right quotient ring $Q$ if and only if $M_n(R)$ is a semiprime right

Goldie ring with right quotient ring $M_n(Q)$ for all $n \geqslant 1$. Moreover the $G$-invariant ideals of $M_n(R)$ are precisely of the form $M_n(I)$ where $I$ is a $G$-invariant ideal of $R$. Suppose that $I$ is a $G$-invariant $R$-ideal, that is, $I$ is a $G$-invariant ideal of $R$ which contains a unit of $Q$. Then $I$ is essential in $R$, and so $M_n(I)$ is essential in $M_n(R)$. Therefore $M_n(I)$ is a $G$-invariant $M_n(R)$-ideal. The above argument is easily reversed to give the converse. So $G$-invariant $M_n(R)$-ideals have precisely the form $M_n(I)$ for some $G$-invariant $R$-ideal $I$. Therefore, using Theorem 2.1.8 we have that

$$\begin{aligned}
& R \text{ is a } G\text{-maximal order in } Q \\
\iff\ & O_l(I) = O_r(I) = R \text{ for all } G\text{-invariant } R\text{-ideals } I \\
\iff\ & O_l(M_n(I)) = O_r(M_n(I)) = M_n(R) \\
\iff\ & M_n(R) \text{ is a } G\text{-maximal order in } M_n(Q).
\end{aligned}$$

**2.1.11** The following lemma is an analogue of Lemma 1.1.3.

**LEMMA.** *Let $R$ be a prime Noetherian ring and $G$ a finite group acting on $R$ with the property that $O_l(P) = O_r(P) = R$ for all non-zero $G$-prime ideals $P$ of $R$. Then $R$ is a $G$-maximal order.*

**PROOF.** Suppose that $R$ satisfies the hypotheses of the lemma, but that $R$ is not a $G$-maximal order. Then by Theorem 2.1.8 there exists a (non-zero) $G$-invariant ideal $I$ of $R$ and an element $x$ such that $x \in O_l(I)\backslash R$ or $x \in O_r(I)\backslash R$. Suppose the former and choose $I$ maximal with respect to this property. We show that $I$ is $G$-prime as follows:

Let $A$ and $B$ be non-zero $G$-invariant ideals of $R$ strictly containing $I$ and such that $AB \subseteq I$. Write

$$A = \{r \in R : rB \subseteq I\}.$$

Now, $x \in O_l(I)$ means that $xI \subseteq I$ so that

$$xAB \subseteq xI \subseteq I.$$

Therefore $xA \subseteq Ol(B) \subseteq R$, so that $xA \subseteq A$. We now have $x \in O_l(A)\backslash R$ with $A \supset I$, contradicting the maximality of $I$. So $I$ is $G$-prime as claimed. The existence of $I$ now contradicts our initial hypotheses, and so $R$ is a $G$-maximal order, as required.

**2.1.12** We now give a result to show that our concept and definition of a $G$-maximal order yields nothing new in the commutative case.

**LEMMA.** *Let $S$ be a commutative Noetherian domain and $G$ a finite group acting as automorphisms on $S$. Then $S$ is a $G$-maximal order if and only if $S$ is a maximal order.*

**PROOF.** It is clear from Theorems 1.1.2 and 2.1.8 that if $S$ is a maximal order then $S$ is a $G$-maximal order.

Conversely, suppose that $S$ is a $G$-maximal order which is not a maximal order. By Lemma 1.1.3 there exists a (non-zero) prime ideal $I$ of $S$ with the property that $O_l(I) \supset S$.

Let $\{I = I_1, I_2, \ldots, I_n\}$ be the $G$-orbit of $I$; it is clear that $I_j$ is a prime ideal of $S$ for each $j = 1,2,\ldots,n$. Choose an element $x \in O_l(I)\backslash S$. For each $j = 1,\ldots,n$ there exists $g(j) \in G$ such that $I_j = I^{g(j)}$. Let $x_j = x^{g(j)}$ so that

$$x_j I_j = (x^{g(j)})(I^{g(j)}) = (xI)^{g(j)} \subseteq I^{g(j)} = I_j.$$

Hence $x_j \in O_l(I_j)\backslash S$ for each $j$. Suppose that $I$ has height $k$ for some $k \in \mathbb{N}$; then $I_j$ has height $k$ for all $j = 1,2,\ldots,n$. Also $x \notin S$ means that $x_j \notin S$ for all such $j$, since $G \subseteq \mathrm{Aut}(S)$.

Now, put

$$I_0 := \prod\{I_j : j = 1,2,\ldots,n\},$$

a non-zero $G$-invariant ideal of $S$ since $S$ is commutative. Let

$$x_0 = \sum\{x_j ; j = 1,2,\ldots,n\}.$$

Then

$$\begin{aligned} x_0 I_0 &= \left(\sum_{j=1}^{n}(x_j)\right)\left(\prod_{j=1}^{n}(I_j)\right) \\ &= \sum_{j=1}^{n}\left((x_j I_j)\left(\prod_{i=1}^{n}\{I_i ; i \neq j\}\right)\right) \\ &\subseteq \sum_{j=1}^{n}\left((I_j)\left(\prod_{i=1}^{n}\{I_i ; i \neq j\}\right)\right) \\ &= I_0. \end{aligned}$$

Therefore $x_0 \in O_1(I_0) = S$ since $S$ is a $G$-maximal order. Consider $(Q/S)_S := \overline{Q}$, where $Q$ is the quotient field of $S$. Put

$$\overline{x}_i = x_i + S$$

for all $i = 1,2,\ldots,n$. We have $\overline{x}_0 = \overline{0}$ since $x_0 \in S$, so that

$$\overline{x}_1 + \ldots + \overline{x}_n = \overline{0}.$$

Hence

$$\overline{x}_1 = -(\overline{x}_2 + \ldots + \overline{x}_n).$$

Now, $\overline{x}_1$ is annihilated in $\overline{Q}$ by $I_1$, and $(\overline{x}_2 + \ldots + \overline{x}_n)$ is annihilated in $\overline{Q}$ by $I_2 \ldots I_n$. Therefore

$$I_2 \ldots I_n \subseteq I_1$$

so that $I_j \subseteq I_1$ for some $j \in \{2,3,\ldots,n\}$, since $I_1$ is a prime ideal of $S$. We must have $I_j \subset I_1$ since equality is impossible; this contradicts the fact that they both have the same height. It follows that

$$\overline{x}_1 = -(\overline{x}_2 + \ldots + \overline{x}_n) = \overline{0}$$

so that $x = x_1 \in S$, again a contradiction. Therefore there is no such ideal $I$ and $S$ is a maximal order, as required.

**2.1.13 EXAMPLE.** We now give an example to show that $G$-maximal orders which are not maximal orders do exist. Let $R = \mathbb{Z}_{2\mathbb{Z}}$ and put

$$S = \begin{bmatrix} R & R \\ 2R & R \end{bmatrix},$$

a prime Noetherian ring. Let

$$P = \begin{bmatrix} 2R & R \\ 2R & R \end{bmatrix}, \quad Q = \begin{bmatrix} R & R \\ 2R & 2R \end{bmatrix};$$

these are the only non-zero prime ideals of $S$, and both have height one. The Jacobson radical of $S$, denoted $J(S)$, is

$$J(S) = P \cap Q = \begin{bmatrix} 2R & R \\ 2R & 2R \end{bmatrix}.$$

Let

$$g = \begin{bmatrix} 0 & \sqrt{\tfrac{1}{2}} \\ \sqrt{2} & 0 \end{bmatrix};$$

then

$$g^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Put $G := \langle g \rangle \cong C_2$, the cyclic group of order 2. We see that $G$ acts on $S$ by conjugation: let $a, b, c, d \in R$. Then

$$\begin{bmatrix} 0 & \sqrt{\tfrac{1}{2}} \\ \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 2c & d \end{bmatrix}\begin{bmatrix} 0 & \sqrt{\tfrac{1}{2}} \\ \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ 2b & a \end{bmatrix} \in S,$$

and $G$ is X-outer on $S$. So we have a prime Noetherian ring $S$ and a finite group $G$ acting on $S$. The action of $G$ permutes $P$ and $Q$:

$$\begin{bmatrix} 0 & \surd\frac{1}{2} \\ \surd 2 & 0 \end{bmatrix}\begin{bmatrix} 2a & b \\ 2c & d \end{bmatrix}\begin{bmatrix} 0 & \surd\frac{1}{2} \\ \surd 2 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ 2b & 2a \end{bmatrix} \in Q,$$

so that $P^g = Q$. Similarly, $Q^g = P$. It is immediate that $J(S)$ is $G$-prime. Now we show that $S$ is not a maximal order. Consider

$$x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin S$$

and

$$p = \begin{bmatrix} 2a & b \\ 2c & c \end{bmatrix} \in P.$$

Then

$$xp = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2a & b \\ 2c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2a & b \end{bmatrix} \in P.$$

Therefore $x$ belongs to $O_l(P)$ but not to $S$, and so $S$ is not a maximal order as claimed. However, $S$ is a $G$-maximal order as follows. Using Lemma 2.1.13 it is enough to show that

$$O_l(J(S)) = O_r(J(S)) = S.$$

To this end, let $\alpha \in O_l(J(S))$ and put

$$\alpha = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

where $x, y, z, w \in Q(S)$, the quotient ring of $S$. Let

$$\begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix}$$

be a non-zero element of $J(S)$. Then

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 2(ax+cy) & bx+2dy \\ 2(az+cw) & bz+2dw \end{bmatrix} \in \begin{bmatrix} 2R & R \\ 2R & 2R \end{bmatrix} = J(S).$$

Since $a$, $b$, $c$, $d$ are arbitrary integers, we must have that $x$, $y$, $w \in R$ and $z \in 2R$ so that $\alpha \in S$. Hence $O_l(J(S)) = S$. Similarly, $O_r(J(S)) = S$ and $S$ is a $G$-maximal order, as claimed.

## §2.2 SOME RESULTS CONCERNING $G$-MAXIMAL ORDERS.

**2.2.1 NOTATION.** Throughout this section, unless stated otherwise, we let $R$ be a prime Noetherian ring with (simple Artinian) quotient ring $Q$, and $G$ a finite group acting on $R$.

**2.2.2 LEMMA.** *Let $R$, $Q$ and $G$ be as in 2.2.1 and suppose that $R$ is a $G$-maximal order. Then $I_l^* = I_r^*$ for all $G$-invariant $R$-ideals $I$.*

**PROOF.** Let $I$ be a $G$-invariant $R$-ideal. Then, by Theorem 2.1.8, $O_l(I) = O_r(I) = R$. Therefore

$$\begin{aligned} I_l^* &= \{q \in Q : qI \subseteq R\} \\ &= \{q \in Q : qI \subseteq O_r(I)\} \\ &= \{q \in Q : IqI \subseteq I\} \\ &= \{q \in Q : Iq \subseteq O_l(I)\} \\ &= \{q \in Q : Iq \subseteq R\} \\ &= I_r^*. \end{aligned}$$

As in Remark 1.1.6, we write $I_l^* = I_r^* = I^*$ for a $G$-invariant $R$-ideal $I$. Note that, by Corollary 2.1.5, $I^*$ is $G$-invariant.

**REMARK.** Recall from 1.1.7 the definition of a (right or left) reflexive ideal of $R$; it follows from the above that if $R$ is a $G$-maximal order and $I$ a (non-zero) $G$-invariant ideal of $R$ then

$$(I_l^*)_r^* = (I_r^*)_l^* := I^{**}.$$

Note that $I \subseteq I^{**}$, and that $I^{**}$ is right and left reflexive.

**2.2.3 DEFINITION.** Let $R$ be a ring and $G$ a finite group. We define the *height* of a $G$-prime ideal $P$ (say) of $R$ to be the length of a longest chain

$$P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_n = P$$

of $G$-prime ideals $P_0, P_1, \ldots, P_n$ of $R$ such that there exists no $G$-prime ideal $Q$ of $R$ with $P_i \subsetneq Q \subsetneq P_{i+1}$ fc. all $i = 0, \ldots, n-1$. (In the case illustrated here, $P$ has height n).

**LEMMA.** *Let $R$ be a ring and $G$ a finite group. Suppose that $I$ is a height-1 $G$-prime ideal of $R$. Then $I = \cap\{P^g : g \in G\}$ for some height-1 prime ideal $P$ of $R$.*

**PROOF.** Let $I$ be as stated. By Lemma 1.3.11, there exists a prime ideal $P$ of $R$ such that

$$I = \cap\{P^g : g \in G\}.$$

Suppose that $P$ has height strictly greater that 1. Then there exist prime ideals $Q_1$, $Q_2$, say, of $R$ with

$$Q_1 \subset Q_2 \subset P.$$

Put $J_1 := \cap\{Q_1{}^g : g \in G\}$ and $J_2 := \cap\{Q_2{}^g : g \in G\}$. Then

$$J_1 \subseteq J_2 \subseteq \cap\{P^g : g \in G\} = I$$

and $J_1$, $J_2$ are both $G$-prime ideals of $R$. But $I$ has height one, and so we must have $J_1 = J_2$ or $J_2 = I$. Without loss of generality, suppose the latter. Then

$$\prod\{P^g : g \in G\} \subseteq \cap\{P^g : g \in G\} = \cap\{Q_2{}^g : g \in G\} \subseteq Q_2$$

which implies that $P^g \subseteq Q_2$ for some $g \in G$, since $Q_2$ is prime. This is a contradiction, and so $P$ has height one, as required.

2.2.4 The following result should be compared with [C-S2, Theorem 1.6]. Observe that by a "maximal reflexive $G$-invariant ideal" of a ring $R$ we mean an ideal of $R$ which is maximal amongst proper reflexive ideals which are $G$-invariant. Notice that it is not obvious from the definition that a ring $R$ will have any proper $G$-invariant reflexive ideals, even if it has a proper reflexive ideal. We will deal with this question in Proposition 2.2.5.

**LEMMA.** *Let $R$ and $G$ be as in 2.2.1 and suppose that $R$ is a $G$-maximal order. Then any non-zero maximal reflexive $G$-invariant ideal of $R$ is $G$-prime and has height 1.*

**PROOF.** Let $P$ be a non-zero maximal reflexive $G$-invariant ideal of $R$. Let $A$, $B$ be (non-zero) $G$-invariant ideals of $R$ properly containing $P$ and such that $AB \subseteq P$. Note that by Lemma 2.2.2,

$$A_r^* = A_l^* = A^*.$$

Suppose that $A^{***} = A^* \supset R$. Then $A^{**}$ is a reflexive ideal of $R$ properly containing $P$, and is $G$-invariant by Corollary 2.1.5(ii); this contradicts the maximality of $P$. Therefore $A^* = R$. Similarly $B^* = R$.

Now, let $q \in P^*$. Then

$$qAB \subseteq qP \subseteq R$$

so that $qA \subseteq B^* = R$. Therefore

$$q \in A^* = R,$$

implying that $P^* \subseteq R$. This is a contradiction to the fact that $P$ is reflexive, by Remark 1.1.9. So there exist no such ideals $A$, $B$ and $P$ is indeed $G$-prime.

Now suppose that $Q$ is a non-zero $G$-prime ideal of $R$ with $Q \subset P$.

Then $P^* \subseteq Q^*$ and $QP^*$ is a $G$-invariant ideal of $R$ by Lemma 2.1.5 and Lemma 2.2.2. We have

$$QP^*P \subseteq QR \subsetneq Q.$$

Since $Q$ is $G$-prime and $P \nsubseteq Q$ it follows that $QP^* \subseteq Q$. Therefore

$$P^* \subseteq O_r(Q) = R,$$

again contradicting the reflexivity of $P$. Hence $P$ has height 1, as required.

**2.2.5 PROPOSITION.** *Let $R$ and $G$ be as in 2.2.1 and suppose that $R$ is a $G$-maximal order. Then the following all hold.*

*(i) Let $I$ be a proper $G$-invariant ideal of $R$. Then the following are equivalent:*

*(a) $I$ is a maximal reflexive $G$-invariant ideal of $R$;*

*(b) $I = \cap\{P^g : g \in G\}$ for some maximal proper right reflexive ideal $P$ of $R$ (so that $P$ is necessarily a height-1 prime of $R$);*

*(c) $I = \cap\{Q^g : g \in G\}$ for some maximal proper left reflexive ideal $Q$ of $R$;*

*(d) $I = \cap\{J^g : g \in G\}$ for some maximal proper (right and left) reflexive ideal $J$ of $R$.*

*(ii) $R$ has a maximal reflexive $G$-invariant ideal if and only if it has a proper right and left reflexive ideal.*

*(iii) The set of maximal right reflexive ideals of $R$ coincides with the set of maximal left reflexive ideals of $R$, and both sets are equal to the set of prime ideals minimal over the maximal $G$-invariant (right and left) reflexive ideals of $R$.*

**PROOF.** (i) (a) $\Longrightarrow$ (b): First suppose that $I$ is a non-zero maximal reflexive $G$-invariant ideal of $R$. Then by Lemma 2.2.4 $I$ is $G$-prime and has height 1. Therefore, by Lemma 2.2.3, there exists a height-1 prime ideal $P$ of $R$ such that

$$I = \cap\{P^g : g \in G\}.$$

It is clear that $P^g$ is also a height-1 prime of $R$ for all $g \in G$. We show next that each $P^g$ is right reflexive. Let $Q(R)$ denote the quotient ring of $R$. Now, $I$ is reflexive by hypothesis so that $R \subset I^*$ by Remark 1.1.9. Hence the $R$-submodule $(I^*/R)_R$ of $(Q(R)/R)_R$ is non-zero. Let $A$ be maximal amongst annihilators of non-zero $R$-submodules of $(I^*/R)_R$. Then $A$ is a prime ideal of $R$ and $I \subseteq A$.

We claim that $A$ is minimal over $I$; suppose that this is not the case. Then there exists a prime ideal $B$ of $R$ such that $I \subseteq B \subset A$. Now $I$ is $G$-invariant, so

$$I = I^g \subseteq B^g \subset A^g$$

for all $g \in G$. Therefore

$$I = \cap\{I^g : g \in G\} \subseteq \cap\{B^g : g \in G\} \subset \cap\{A^g : g \in G\}.$$

Suppose that the order of $G$ is n and that $1 = g_1, g_2, \ldots, g_n$ are the elements of $G$. For any $R$-module $X$, let $X_i$ denote the $R$-module resulting from the action of $g_i$ on $X$. Put $\hat{A} = \cap\{A^g : g \in G\}$ and consider

$$\begin{aligned}
&( \textstyle\sum\{(A^*)_i : i = 1,\ldots,n\} )\hat{A} \\
&= ( A^* + (A^*)_2 + \ldots + (A^*)_n )\hat{A} \\
&= A^*\hat{A} + (A^*)_2\hat{A} + \ldots + (A^*)_n\hat{A} \\
&\subseteq A^*A + (A^*)_2A_2 + \ldots + (A^*)_nA_n \\
&= A^*A + (A^*A)_2 + \ldots + (A^*A)_n \\
&\subseteq R.
\end{aligned}$$

Therefore $\hat{A}$ is contained in the annihilator of a non-zero

$R$-submodule of $(Q(R)/R)_R$, namely

$$(\sum\{(A^*)^g : g \in G\})/R.$$

Hence $\hat{A}$ is contained in a right reflexive ideal, by Proposition 1.1.8. It is clear that $\hat{A}$ is $G$-invariant, so $\hat{A}$ is contained in a maximal right reflexive $G$-invariant ideal of $R$, $J$ say. Let

$$J = \cap\{Q^g : g \in G\}$$

where $Q$ is a height-1 prime of $R$. Then there exists $g \in G$ such that $A^g \subseteq Q$. But $Q$ has height 1 and $A^g$ has the same height as A, which is greater that 1; a contradiction. So $A$ is minimal over $I$, as claimed.

But $I = \cap\{P^g : g \in G\}$, so we must have $A = P^g$ for some $g \in G$. Therefore $P^g$ is right reflexive. Let $k = g^{-1}$. It follows that

$$P^h = ((P^g)^k)^h$$

is right reflexive for all $h \in G$.

Now, suppose that $M$ is a right reflexive ideal of $R$ such that $P \subset M$. Then $P^g \subset M^g$ for all $g \in G$, and so

$$I = \cap\{P^g : g \in G\} \subseteq \cap\{M^g : g \in G\} := \hat{M}.$$

But $\hat{M}$ is a right reflexive $G$-invariant ideal of $R$, and so is also left reflexive since $R$ is a $G$-maximal order. Therefore, due to the maximality of $I$ we must have $I = \hat{M}$. It follows that $P$ is maximal amongst right reflexive ideals of $R$, as required.

(a) $\Longrightarrow$ (c): This follows in the same way as the proof above; we use symmetry together with the fact that $I$ is both right and left reflexive by Lemma 2.2.2 to reach the conclusion that the ideals appearing in the intersection defining $I$ are left reflexive.

(a) $\Longrightarrow$ (d): This is a consequence of the above.

(d) $\Longrightarrow$ (a): Suppose that

$$I = \cap\{J^g : g \in G\}$$

with $J$ a maximal reflexive ideal of $R$. Then each $J^g$ is a maximal reflexive ideal of $R$ for all $g \in G$. It is clear that $I$ is $G$-invariant. Now $I \subseteq J^g$ for all $g \in G$, so it follows that

$$I^{**} \subseteq (J^g)^{**}$$

for all $g$. Therefore

$$I^{**} \subseteq \cap\{(J^g)^{**} : g \in G\} = \cap\{J^g : g \in G\} = I,$$

so that $I$ is reflexive. It remains to show the maximality of $I$. If $I$ is not a maximal reflexive $G$-invariant ideal of $R$, then $I$ is contained in such an ideal, $M$ say. Then, by the proof of (a) $\Longrightarrow$ (d),

$$M = \cap\{A^g : g \in G\}$$

where $A^g$ is a maximal reflexive ideal of $R$ for each $g \in G$. But $I \subseteq M$, so that $I \subseteq A^g$ for all $g \in G$. Hence

$$\prod\{J^h : h \in G\} \subseteq \cap\{J^h : h \in G\} = I \subseteq A,$$

so that $J^h \subseteq A$ for some $h \in G$. It follows that $J^h = A$, since $J^h$ is maximal. Therefore $I = M$ and $I$ is indeed maximal. This proves (d).

(b) $\Longrightarrow$ (a): Let $P$ be a maximal proper right reflexive ideal of $R$ and suppose that

$$I = \cap\{P^g : g \in G\}.$$

Since $P$ is right reflexive, $P^g$ is right reflexive and so

$$P^g = ((P^g)_l^*)_r^*$$

for all $g \in G$. Now,

$$(I_l^*)_r^* \subseteq ((P^g)_l^*)_r^*$$

for each $g$, so that

$$(I_l^*)_r^* \subseteq \cap\{ ((P^g)_l^*)_r^* : g \in G\} = \cap\{P^g : g \in G\} = I.$$

Therefore $I$ is right reflexive. But $I$ is a $G$-invariant ideal of the $G$-maximal order $R$, and so it follows from Lemma 2.2.2 that $I$ is also left reflexive. It is shown in a similar way as in the proof of (d) $\Longrightarrow$ (a) that $I$ is a maximal (reflexive) ideal of $R$, and so (a) holds.

(c) $\Longrightarrow$ (a): This is shown in a similar way to the proof of (b) $\Longrightarrow$ (a) above, using symmetry.

(d) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (c) are both trivial, and so the proof of (i) is complete.

(ii) This is now immediate from (i).

(iii) Let $X = \{$maximal right reflexive ideals of $R\}$;

$Y = \{$maximal left reflexive ideals of $R\}$;

$Z = \{$prime ideals minimal over a maximal reflexive $G$-invariant ideal of $R\}$.

First, let $P \in X$. Then from (i),

$$I := \cap\{P^g : g \in G\} = \cap\{Q^g : g \in G\}$$

for some $Q \in Y$. This implies that

$$\prod\{P^g : g \in G\} \subseteq \cap\{P^g : g \in G\} = \cap\{Q^g : g \in G\} \subseteq Q$$

so that $P^g \subseteq Q$ for some $g \in G$. Similarly there exists an element $h$ of $G$ with $Q^h \subseteq P$. Therefore

$$P^{gh} \subseteq Q^h \subseteq P$$

so that $P^{gh} \subseteq P$. But $P^{gh} \in X$, and so $P^{gh} = P$. Hence $gh = 1$. It follows that

$$P \subseteq Q^h \subseteq P$$

and so $P \in Y$. It is shown in a similar way that members of $Y$ are also members of $X$.

Now we let $P$ belong to $X$ and show that $P \in Z$. Again put

$$I = \cap\{P^g : g \in G\};$$

from (i), $I$ is a maximal reflexive $G$-invariant ideal of $R$, and so is a height-1 $G$-prime ideal of $R$ by Lemma 2.2.4. Then by Lemma 2.2.3 we have that $P$ has height one, and so must be minimal over $I$. Hence $P$ belongs to $Z$, as claimed.

Conversely, let $A \in Z$, that is, $A$ is a prime ideal of $R$ minimal over a maximal reflexive $G$-invariant ideal, $M$ (say), of $R$. We show that $A \in X$. By part (i) we know that

$$M = \cap\{P^g : g \in G\}$$

for some $P \in X$. Since $M$ is $G$-invariant,

$$M = \cap\{P^g : g \in G\} \subseteq \cap\{A^g : g \in G\}.$$

Therefore,

$$\prod\{P^g : g \in G\} \subseteq \cap\{P^g : g \in G\} \subseteq \cap\{A^g : g \in G\} \subseteq A$$

so that $P^g \subseteq A$ for some $g \in G$, since $A$ is prime. We now have

$$M \subseteq P^g \subseteq A$$

and so must have that $A = P^g \in X$ since $A$ is minimal over $M$. This completes the proof.

**2.2.6** The following result should be compared with [C-S2, Proposition 1.7]. Note that there is no ambiguity in the notation $P^*$, in the light of Lemma 2.2.2.

**LEMMA.** *Let $R$ and $G$ be as in 2.2.1 and suppose that $R$ is a $G$-maximal order. Let $P$ be a non-zero $G$-prime ideal of $R$. Then the following are equivalent:*

*(i)* *$P$ is reflexive;*

*(ii)* *$R \subset P^*$ and*

*(iii)* *$P \subset PP^*$.*

**PROOF.** (i) $\Longrightarrow$ (ii): This is immediate from Remark 1.1.9.

(ii) $\Longrightarrow$ (iii): Suppose that $R \subset P^*$. It is clear that $P \subseteq PP^*$; if $P = PP^*$ then $P^* \subseteq O_r(P)$, and $O_r(P) = R$ since $R$ is a $G$-maximal order. Thus

$$P^* \subseteq R \subset P^*,$$

a contradiction. Therefore $P \subset PP^*$, proving (iii).

(iii) $\Longrightarrow$ (i): Suppose that $P \subset PP^*$. Since $P$ is $G$-invariant, $PP^*$ and $P^{**}$ are both $G$-invariant by Corollary 2.1.5. (Note that both are ideals of $R$ by Lemma 2.2.2 and the fact that $1 \in P^*$ respectively.) We have

$$(PP^*)(P^{**}) = P(P^*P^{**}) \subseteq PR \subseteq P.$$

Since $PP^* \nsubseteq P$ and $P$ is $G$-prime it follows that $P^{**} \subseteq P$. Hence $P$ is reflexive, as required.

**2.2.7** The following result is well-known; nevertheless we state it in full as we will generalise it later.

**PROPOSITION.** *Let $R$ be a prime Noetherian semilocal ring. Suppose that the Jacobson radical of $R$, denoted $J$, is right and left reflexive. Then $R$ is hereditary if and only if $J$ is invertible.*

**PROOF.** First suppose that $R$ is hereditary. Since $J \neq 0$, $J$ is invertible by [E-R, Theorem 4.13].

Conversely, suppose that $J$ is invertible. Then by Lemma 1.2.5, ${}_RJ$ and $J_R$ are projective, so that

$$\text{pr.dim.}(R/J)_R = 1.$$

Also, $R$ is semilocal so that $R/J$ is semisimple Artinian and

$$\text{gl.dim.}(R/J) = 0.$$

Therefore by [Mc-R, Theorem 7.3.14],

$$\text{gl.dim.}(R) \leqslant \text{gl.dim.}(R/J) + \text{pr.dim.}(R/J)_R = 0 + 1 = 1.$$

Hence $R$ is hereditary, as required.

**2.2.8** The next proposition is a generalisation of (i) $\Longrightarrow$ (iii) of Proposition 1.2.8.

**PROPOSITION.** *Let $R$ be a prime Noetherian semilocal ring and $G$ a finite group acting on $R$. Suppose that the Jacobson radical of $R$, denoted $J$, is right and left reflexive. If $R$ is a $G$-maximal order, then $J$ is invertible.*

**PROOF.** Let $M$ be a maximal ideal of $R$ and put $M_0 := \cap\{M^g : g \in G\}$. Then

$$J = \cap\{M_0 : M \text{ a maximal ideal of } R\}$$
$$= \prod\{M_0 : M \text{ a maximal ideal of } R\}.$$

We claim that each $M_0$ is reflexive. Since $R$ is a $G$-maximal order, it is enough, by Lemma 2.2.2, to show that $M_0$ is the right annihilator of a non-zero submodule of $Q/R$ (where $Q$ is the quotient ring of $R$). Now, since $J$ is right reflexive by hypothesis there exists a submodule $A$ of $Q$ strictly containing $R$ with $J = \text{r.ann}(A/R)$. Since $R$ is semilocal, $R/J$ is semisimple Artinian. Therefore $R/J$ is isomorphic to a direct sum of matrix rings over division rings, say

$$R/J \cong \bigoplus_{i=1}^{t} \mathrm{M}_{n_i}(D_i),$$

where each $n_i$ is a natural number and each $D_i$ is a division ring. We can also write

$$M_0/J \cong \bigoplus_{i=1}^{r} \mathrm{M}_{n_i}(D_i),$$

with reindexing if necessary, for some $r < t$.

Since $A/R$ is a faithful $R/J$-module, $A/R$ includes at least one copy of each isomorphism class of irreducible $R/J$-modules in its decomposition. For each $i = 1,\ldots,t$ let $V_i$ be an irreducible $M_{n_i}(D_i)$-module. Let $B/R$ denote the sum in $A/R$ of all the irreducible $R/J$-modules isomorphic to $V_i$, where $i = r+1,\ldots,t$. It follows that $M_0$ is the right annihilator (in $R$) of $B/R$. Therefore $M_0$ is right reflexive.

We now show that $M_0$ is invertible. Now, $M_0{}^*M_0$ is an ideal of $R$ containing $M_0$, and is $G$-invariant by Lemma 2.1.5. If $M_0{}^*M_0 \subset R$ then $M_0{}^*M_0 \subseteq M^g$ for some $g \in G$. Then

$$M_0{}^*M_0 = \cap\{(M_0{}^*M_0)^g : g \in G\} \subseteq \cap\{M^g : g \in G\} = M_0,$$

so that

$$M_0{}^* \subseteq O_l(M_0) = R$$

since $R$ is a $G$-maximal order. This is a contradiction to the reflexivity of $M_0$ by Remark 1.1.9, and so $M_0{}^*M_0 = R$. Similarly $M_0M_0{}^* = R$ and $M_0$ is indeed invertible.

Now, $R$ has finitely many maximal ideals; label the $M_0$s as $M_1,\ldots,M_n$. Then, since each $M_i$ is invertible,

$$\begin{aligned}&(M_1M_2\ldots M_n)(M_n{}^*M_{n-1}{}^*\ldots M_1{}^*)\\ &= M_1M_2\ldots M_{n-1}(R)M_{n-1}{}^*\ldots M_1{}^*\\ &= \ldots = R.\end{aligned}$$

A similar argument works on the left, and so by definition of $J$ we have that $J$ is invertible as required.

**2.2.9 COROLLARY TO PROPOSITIONS 2.2.7 AND 2.2.8.** *Let $R$ be a prime Noetherian semilocal ring, and $G$ a finite group acting on $R$. Suppose that the Jacobson radical of $R$, denoted $J$, is (right and left) reflexive. Then the following are equivalent:*

*(i) $R$ is a $G$-maximal order;*

*(ii) $R$ is hereditary;*

*(iii)* $\cap\{M^g : g \in G\}$ *is invertible for each maximal ideal* $M$ *of* $R$.

**PROOF.** (ii) $\Longrightarrow$ (iii). Suppose that $R$ is hereditary; by Proposition 2.2.7, $J$ is invertible. Let $M$ be a maximal ideal of $R$ and put

$$M_0 = \cap\{M^g : g \in G\}.$$

Then

$$J = \prod\{M_0 : M \text{ a maximal ideal of } R\}.$$

Put

$$X = \prod\{N_0 : N \text{ a maximal ideal of } R \text{ and } N \neq M\}.$$

Then $J = M_0X = XM_0$. Since $J$ is invertible,

$$M_0(XJ^*) = (M_0X)J^* = JJ^* = R.$$

Similarly,

$$(J^*X)M_0 = J^*(XM_0) = J^*J = R.$$

Therefore $M_0$ is invertible, proving (iii).

(iii) $\Longrightarrow$ (ii). Suppose that $M_0$ (as defined above) is invertible for all maximal ideals $M$ of $R$. Denote the (distinct) $M_0$s by $M_1, M_2, \ldots, M_n$. Then

$$J = M_1M_2\ldots M_n,$$

a product of invertible ideals which is itself invertible, as in the proof of Proposition 2.2.8. Then $R$ is hereditary by Proposition 2.2.7, and (ii) holds.

(i) $\Longrightarrow$ (iii). This follows from Proposition 2.2.8.

(iii) $\Longrightarrow$ (i). Suppose that

$$M_0 = \cap\{M^g : g \in G\}$$

is invertible for each maximal ideal $M$ of $R$. As in that proof of Proposition 2.2.8 we see that $J$ is also invertible. Suppose that $R$ is not a $G$-maximal order, so that there exists a non-zero $G$-invariant ideal $I$ of $R$ such that $R \subset O_1(I)$. Choose $I$ to be maximal with respect to this property. Now, since $I$ is $G$-invariant, $I$ is contained in $M_0$ for some maximal ideal $M$ of $R$. Since $M_0$ is invertible, $M_0$ is (right and left) projective by Lemma 1.2.5. It thus follows from Lemma 1.2.4 that $M_0$ is reflexive.

Put $K = IM_0{}^*$, an ideal of $R$. By Lemma 2.1.5 $K$ is $G$-invariant. We have

$$I = IM_0{}^*M_0 = KM_0 \subseteq K;$$

suppose that $I = K$. Then $KM_0 = K$ so that

$$K = KM_0 = (KM_0)M_0 = \ldots = K(M_0)^t$$

for all $t \geqslant 0$. Therefore

$$K \subseteq \cap\{(M_0)^i : i = 1,\ldots,\infty\}.$$

But $M_0$ being invertible means that $M_0$ has the $AR$-property, by [Mc-R, Corollary 4.2.5], and so

$$\cap\{(M_0)^i : i = 1,\ldots,\infty\} = 0$$

by [H-L, Corollary 2.2]. Hence $K = 0$, a contradiction and so $I \subset K$.

In the same way as was shown in the proof of Proposition 1.2.8, we have

$$R \subset O_1(I) \subseteq O_1(K).$$

This contradicts our initial choice of $I$, and so $R$ is a $G$-maximal order. Thus (i) holds and the proof is complete.

**2.2.10 DEFINITION.** Let $R$ be a (prime Noetherian) semilocal ring, and $G$ a finite group acting on $R$ such that the Jacobson radical of $R$ is $G$-prime. Then $R$ is said to be *$G$-local*.

**2.2.11** The following result is a special case of Corollary 2.2.9; notice that it is a $G$-equivariant version of Proposition 1.2.8.

**COROLLARY.** *Let $R$ be a prime Noetherian ring and $G$ a finite group acting on $R$. Suppose that $R$ is $G$-local and that the Jacobson radical of $R$, denoted $J$, is (right and left) reflexive. Then the following are equivalent:*

*(i) $R$ is a $G$-maximal order;*

*(ii) $R$ is hereditary;*

*(iii) $J$ is invertible.*

**PROOF.** Note that, since $R$ is $G$-local,

$$J = \cap\{M^g : g \in G\} = M_0$$

for each maximal ideal $M$ of $R$. The result is now immediate from Corollary 2.2.9.

**2.2.12 DEFINITION.** A prime Noetherian ring $R$ is said to have *Krull dimension 1* if $R/E$ is an Artinian module for each essential right ideal $E$ of $R$.

**2.2.13** The following result is a further consequence of Corollary 2.2.9.

**PROPOSITION.** *Let $R$ be a prime Noetherian ring with Krull dimension 1 and $G$ a finite group acting on $R$. Suppose that $R$ is $G$-local. Then the following hold.*

*(i) The Jacobson radical of $R$, denoted $J$, is (right and left) reflexive.*

*(ii) The following are equivalent:*

*(a) $R$ is a $G$-maximal order;*

*(b) $R$ is hereditary;*

*(c) $J$ is invertible.*

**PROOF.** (i) Since $R$ is $G$-local, $J$ is non-zero. Let $Q$ denote the quotient ring of $R$ and let $c$ be a regular element of $R$. Then $c$ is invertible in $Q$ by [G-W, Lemma 5.7], and $c^{-1} + R$ belongs to $Q/R$. Also,

$$(c^{-1} + R)R \cong R/cR.$$

Now, $cR$ is an essential right ideal of $R$ by [G-W, Lemma 5.5], and so $R/cR$ is Artinian by hypothesis. So $(c^{-1} + R)R$ is a simple (Artinian) submodule of $Q/R$, and so $Q/R$ has non-zero socle. Therefore there exists a maximal ideal $M$ of $R$ such that $M_r^*/R$ is non-zero in $Q/R$; in other words, $R \subset M_r^*$. It follows that $R \subset (M_r^*)^g$ for all $g \in G$. To show that $J$ is right reflexive, it is enough to show that $J$ is the annihilator of a non-zero proper right $R$-submodule of $Q/R$. Suppose that

$$\prod\{(M_r^*)^g : g \in G\} = R.$$

Then $R = (M_r^*)^g$ for some $g \in G$, since $R$ is prime. This is clearly a contradiction, and so

$$R \subset \prod\{(M_r^*)^g : g \in G\} := X.$$

Therefore $X/R$ is a non-zero proper right $R$-submodule of $Q/R$. Since $R$ is $G$-local by hypothesis,

$$J = \prod\{M^g : g \in G\} := M_0.$$

Let $1 = g_1, g_2, \ldots, g_n$ be the elements of $G$ and put $M_i$ equal to the ideal formed when $g_i$ acts on $M$. Similarly, let $(M_r^*)_i$ denote the $R$-submodule of $Q$ resulting from the action of $g_i$ on $M_r^*$. Then

$$\begin{aligned} JX &= (M_1M_2\ldots M_n)(\ (M_r^*)_n(M_r^*)_{n-1}\ldots(M_r^*)_1\ ) \\ &\subseteq (M_1M_2\ldots M_{n-1})(R)(\ (M_r^*)_{n-1}\ldots(M_r^*)_1\ ) \\ &\subseteq \ldots \subseteq R. \end{aligned}$$

So $J$ is the annihilator in $Q/R$ of $X/R$, and so is right reflexive. That

$R$ is also left reflexive follows by symmetry, and so (i) holds.

(ii) This is now immediate from Corollary 2.2.11.

**REMARK.** Example 2.1.13 satisfies all the hypotheses of the above result.

## §2.3 LOCALISATION IN $G$-MAXIMAL ORDERS.

**2.3.1** We now aim towards proving the fact that in a $G$-maximal order, all reflexive height-1 $G$-prime ideals are localisable. This result appears as Proposition 2.3.12, and should be compared with [H-W, Lemma 2.1(ii)]. First we require some definitions and preliminary results.

**DEFINITION.** Let $R$ be a prime Noetherian ring and $P$ a height-1 $G$-prime ideal of $R$. Put $P_l^{(0)} = R$ and define

$$P_l^{(n)} = \{s \in R : s(P_r^*) \subseteq P_l^{(n-1)}\} \text{ for all } n \in \mathbb{N}.$$

Similarly, $(P_r)^{(0)} = R$ and

$$P_r^{(n)} = \{s \in R : (P_l^*)s \subseteq P_r^{(n-1)}\} \text{ for all } n \in \mathbb{N}.$$

**REMARK.** Note that

$$P_l^{(1)} = \{s \in R : s(P_r^*) \subseteq R\} = (P_r^*)_l^*$$

$$P_r^{(1)} = \{s \in R : (P_l^*)s \subseteq R\} = (P_l^*)_r^*$$

and both are $G$-invariant. Also, $P_l^{(1)}$ is left reflexive and $P_r^{(1)}$ is right reflexive. Of course, if $P$ is left reflexive then $P_l^{(1)} = P$ and if $P$ is right reflexive then $P_r^{(1)} = P$.

**2.3.2 NOTATION.** As before, we let $R$ be a prime Noetherian ring with (simple Artinian) quotient ring $Q$ and $G$ be a finite group acting on $R$. In addition, $P$ will be a height-1, $G$-prime ideal of $R$.

**2.3.3 LEMMA.** *Let $R$, $P$ be as above and suppose that $P$ is left (resp. right) reflexive. Then $P_l^{(n)}$ (resp. $P_r^{(n)}$) is an ideal of $R$ for all $n \in \mathbb{N}$.*

**PROOF.** It is enough to show that when $P$ is left reflexive $P_l^{(n)}$ is an ideal for each n, the right hand case being shown similarly. We use induction on n. When n = 1, $P_l^{(1)} = P$ is clearly an ideal of $R$. Suppose $P_l^{(n-1)}$ is an ideal of $R$, and let $x, y \in P_l^{(n)}$. Then

$$x(P_r^*) \subseteq P_l^{(n-1)}$$

and

$$y(P_r^*) \subseteq P_l^{(n-1)},$$

an ideal of $R$ by the inductive hypothesis. Therefore

$$(x+y)(P_r^*) \subseteq x(P_r^*) + y(P_r^*) \subseteq P_l^{(n-1)}.$$

Let $r \in R$. Then

$$rx(P_r^*) \subseteq rP_l^{(n-1)} \subseteq P_l^{(n-1)},$$

so that $rx \in P_l^{(n)}$. Also

$$xr(P_r^*) \subseteq x(P_r^*) \subseteq P_l^{(n-1)}$$

since $P_r^*$ is a left $R$-module, and we have $xr \in P_l^{(n)}$. Therefore $P_l^{(n)}$ is an ideal of $R$ for each $n \in \mathbb{N}$.

**2.3.4 LEMMA.** *Let $R$, $P$ and $G$ be as in 2.3.2. Then $P_l^{(n)}$ and $P_r^{(n)}$ are $G$-invariant for all $n \in \mathbb{N}$.*

**PROOF.** To show that $P_l^{(n)}$ is $G$-invariant, take $A = P_r^*$, $B = P_l^{(n-1)}$ in Lemma 2.1.5(i). Similarly for $P_r^{(n)}$.

**2.3.5 LEMMA.** *Let $R$, $G$ and $P$ be as in 2.3.2. Suppose further that $R$ is a $G$-maximal order and that $P$ is reflexive. Then*

$$P_l^{(n)} = P_r^{(n)} := P^{(n)}$$

*for all $n \in \mathbb{N}$.*

**PROOF.** We use induction on n. Since $R$ is a $G$-maximal order,

$$P_l^{(1)} = P = P_r^{(1)}$$

by Lemma 2.2.2. Suppose that

$$P_l^{(k)} = P_r^{(k)} := P^{(k)}$$

for all $k < n$. Now, since $R$ is a $G$-maximal order we have that

$$P_r^{*} = P_l^{*} = P^{*},$$

by Lemma 2.2.2. Let $x \in P_l^{(n)}$. Then

$$xP^{*} \subseteq P_l^{(n-1)} = P_r^{(n-1)}$$

by the induction hypothesis. Therefore

$$P^{*}xP^{*} \subseteq P_r^{(n-2)} = P^{(n-2)},$$

so that

$$P^{*}x \subseteq P^{(n-1)}.$$

Hence $x \in P_r^{(n)}$ and so $P_l^{(n)} \subseteq P_r^{(n)}$. The reverse inclusion is shown similarly, and so we have

$$P_l^{(n)} = P_r^{(n)} := P^{(n)}$$

for all $n \in \mathbb{N}$.

**2.3.6 LEMMA.** *Let $R$ and $P$ be as in 2.3.2. Then $P^n \subseteq P_l^{(n)}$ (and $P^n \subseteq P_r^{(n)}$) for all $n \in \mathbb{N}$.*

**PROOF.** Again we use induction on n. When $n = 1$, the claim is clearly true. Suppose that $P^{n-1} \subseteq P_l^{(n-1)}$. Now,

$$P^n(P_r^{*}) = P^{n-1}(P(P_r^{*})) \subseteq P^{n-1}R = P^{n-1} \subseteq P_l^{(n-1)}$$

by the induction hypothesis. Therefore by definition $P^n \subseteq P_l^{(n)}$, and this holds for all $n \in \mathbb{N}$. Similarly we see that $P^n \subseteq P_r^{(n)}$ for all n.

**2.3.7 LEMMA.** *Let R, G and P be as in 2.3.2. Suppose further that R is a G-maximal order and that P is reflexive. Then*

$$P(n) \subset P(n-1)$$

*for all $n \in \mathbb{N}$. In particular, $P^{(n)} \subset P$ for all $n \in \mathbb{N}$.*

**PROOF.** It is clear that $P^{(n)} \subseteq P^{(n-1)}$ for all $n \in \mathbb{N}$, since $1 \in P^*$. Suppose that $P^{(n)} = P^{(n-1)}$ for some n. Then

$$P(n)P^* \subseteq P(n-1) = P(n),$$

and $P^{(n)}$ is $G$-invariant by Lemma 2.3.4. Therefore

$$P^* \subseteq O_r(P^{(n)}) = R$$

since $R$ is a $G$-maximal order. This contradicts the reflexivity of $P$, so we have that $P^{(n)} \subset P^{(n-1)}$ for all $n \in \mathbb{N}$, as required.

**2.3.8 LEMMA.** *Let R, Q and P be as in 2.3.2 and suppose that P is left (resp. right) reflexive. Then $P_l^{(n)}$ (resp. $P_r^{(n)}$) is left (resp. right) reflexive for all $n \in \mathbb{N}$.*

**PROOF.** Again we use induction on n, suppose that $P$ is left reflexive and only show that $P_l^{(n)}$ is left reflexive. It is clear that when $n = 1$, $P_l^{(1)} = P$ is left reflexive by hypothesis. Suppose that $P_l^{(n-1)}$ is left reflexive. Then $P_l^{(n-1)}$ is the annihilator of some non-zero $R$-submodule of $_R(Q/R)$ by Proposition 1.1.8, so that there exists a submodule $X$ of $Q$ strictly containing $R$ and such that

$$P_l^{(n-1)} = \{r \in R : rX \subseteq R\}.$$

Now by definition of $P_l^{(n)}$,

$$(P_l^{(n)})(P_r^*) \subseteq P_l^{(n-1)}.$$

Therefore

$$(P_l^{(n)})(P_r^*)X \subseteq R,$$

so that $P_l^{(n)}$ is contained in the annihilator (in $R$) of the $R$-submodule $((P_r^*)X)/R)$, and $R \subset (P_r^*)X$.

Let $a \in 1.\mathrm{ann}_R((P_r^*)X/R)$. Then

$$a(P_r^*)X \subseteq R,$$

and so

$$aP_r^* \subseteq P_l^{(n-1)}.$$

Hence $a$ belongs to $P_l^{(n)}$ by definition. We have that $P_l^{(n)}$ is the annihilator in $R$ of $((P_r^*)X/R)$, and so by Proposition 1.1.8 $P_l^{(n)}$ is left reflexive. This completes the inductive step, and so $P_l^{(n)}$ is left reflexive for all $n \in \mathbb{N}$. Similarly, $P_r^{(n)}$ is right reflexive for all n.

**2.3.9 LEMMA.** *Let $R$, $Q$, $G$ and $P$ be as in 2.3.2. Suppose that $R$ is a $G$-maximal order and that $P$ is reflexive. Then*

$$\cap\{P^{(n)} : n \in \mathbb{N}\} = 0.$$

**PROOF.** First of all, since $R$ is a $G$-maximal order, $P_l^{(n)} = P_r^{(n)}$ for all $n \in \mathbb{N}$ by Lemma 2.3.5 and $P_l^* = P_r^*$ by Lemma 2.2.2. Let

$$X = \cap\{P^{(n)} : n \in \mathbb{N}\},$$

and suppose that $X \neq 0$. Then, being a non-zero ideal of the prime Noetherian ring $R$, $X$ contains a regular element $d$ say. It follows that $d$ belongs to $P^{(n)}$ for all $n \in \mathbb{N}$. Now,

$$(P^{(n)})^* = \{q \in Q : q(P^{(n)}) \subseteq R\},$$

and so

$$((P^{(n)})^*)d \subseteq R.$$

Therefore

$$(P^{(n)})^* \subseteq Rd^{-1} \cong R$$

for all $n \in \mathbb{N}$. Since $P^{(n)} \subseteq P^{(n-1)}$ for all n, we have that

$$(P^{(n-1)})^* \subseteq (P^{(n)})^*$$

for all n. Suppose for some n that $(P^{(n-1)})^* = (P^{(n)})^*$. Then

$$P^{(n)} = (P^{(n)})^{**} = (P^{(n-1)})^{**} = P^{(n-1)}$$

since $P^{(n)}$, $P^{(n-1)}$ are reflexive by Lemma 2.3.8. This is a contradiction to Lemma 2.3.7, so we have the strictly ascending chain

$$(P^{(1)})^* \subset (P^{(2)})^* \subset \ldots \subset (P^{(n-1)})^* \subset (P^{(n)})^* \subset \ldots \subset Rd^{-1} \cong R.$$

This contradicts the Noetherian property of $R$, so that no such $d$ exists and we have $X = 0$, as required.

**2.3.10 LEMMA.** *Let $R$, $G$ and $P$ be as in 2.3.2. Suppose that $R$ is a $G$-maximal order and that $P$ is reflexive. Then $P/P^{(n)}$ is localisable in $R/P^{(n)}$.*

**PROOF.** We aim to apply Small's Theorem to the Noetherian ring $\bar{R} := R/P^{(n)}$. Since $P$ is $G$-prime, the nilradical of $\bar{R}$,

$$N := \mathrm{N}(\bar{R}) = P/P^{(n)} = \bar{P}.$$

We show that $C_{\bar{R}}(\bar{0}) = C_{\bar{R}}(\bar{P})$. By [G-W, Lemma 10.8],

$$C_{\bar{R}}(\bar{0}) \subseteq C_{\bar{R}}(\bar{P}).$$

For the reverse inclusion it is enough to show that $C_R(P) \subseteq C_R(P^{(n)})$.

Let $c$ be an element of $C_R(P)$ and $x$ an element of $R$ such that $cx \in P^{(n)}$. Note that $x$ belongs to $P$ since $P^{(n)} \subseteq P$ and $c \in C_R(P)$. We use induction on n to show that $c \in C_R(P^{(n)})$. The claim is clear when $n = 1$. Suppose that $c \in C_R(P^{(k)})$ for all $k < n$. We have

$$cx \in P^{(n)} \subseteq P^{(k)}$$

so that $x$ belongs to $P^{(k)}$ for all such k. Also, $cx \in P^{(n)}$ means that

$$cxP^* \subsetneq P^{(n-1)}.$$

Now $xP^* \subseteq R$ since $x \in P$, and therefore

$$xP^* \subsetneq P^{(n-1)}$$

by the induction hypothesis. Hence $x \in P^{(n)}$ by definition, and so $c$ belongs to $C_R(P^{(n)})$. We have shown that $c$ is left regular modulo $P^{(n)}$; the right regular case is shown similarly. This completes the proof.

**2.3.11 LEMMA.** *Let $R$ be a $G$-prime ring (where $G$ is a finite group acting on $R$), and $I$ a non-zero $G$-invariant ideal of $R$. Then $I \cap C_R(0)$ is non-empty.*

**PROOF.** Let $P$ be a minimal prime of $R$. Then

$$\cap\{P^g : g \in G\} = 0.$$

Write

$$0 = P_1 \cap P_2 \cap \ldots \cap P_m$$

where $P_1,\ldots,P_m$ are minimal primes of $R$ with $P_i \neq P_j$ for all $i, j = 1,\ldots,m$. Therefore, for each $i$,

$$\cap\{P_j : j \neq i\} \nsubseteq P_i$$

If $I \subseteq P_i$ for some $i$, then

$$I \subseteq \cap\{(P_i)^g : g \in G\} = 0$$

since $I$ is $G$-invariant. Thus $I \nsubseteq P_i$ for all $i$. Hence, for each $i$,

$$(\ (I \cap (\bigcap_{j\neq i} (P_j))) + P_i\ )/P_i$$

is a non-zero ideal of the prime ring $R/P_i$, and so is essential as a right ideal. Therefore, by [G-W, Proposition 5.9], it contains a regular element, $d_i$ say, of $R/P_i$. Hence

$$d_i \in (I \cap (\bigcap_{j\neq i} (P_j))) \cap C(P_i).$$

Put $d = \sum\{d_i : i = 1,\ldots,m\}$. For each $j$ we have

$$d + P_j = d_j + P_j \in C(P_j).$$

Therefore

$$d \in \cap\{C(P_j) : j = 1,\dots,m\} = C_R(0),$$

by [G-W, Lemma 6.4]. Hence $I \cap C_R(0)$ is non-empty as required.

2.3.12 We are now ready to prove the main result of this section.

**PROPOSITION.** *Let $R$ be a prime Noetherian $G$-maximal order where $G$ is a finite group acting on $R$. If $P$ is a reflexive height-1 $G$-prime ideal of $R$ then $P$ is localisable.*

**PROOF.** We begin by showing that, for all $m \in \mathbb{N}$,

$$P^{(m-1)}P \subsetneq P^{(m)} \qquad \dots (1)$$

by induction on m. When $m = 1$, this is clear. Suppose that

$$P^{(m-2)}P \subseteq P^{(m-1)}.$$

Let $x \in P^{(m-1)}$, $y \in P$. Then $P^*x \subseteq P^{(m-2)}$ so that

$$P^*xy \subseteq P^{(m-2)}P \subseteq P^{(m-1)}$$

by the induction hypothesis. Therefore $xy$ belongs to $P^{(m)}$, by definition of $P^{(m)}$. Hence

$$P^{(m-1)}P \subseteq P^{(m)}$$

for all $m \in \mathbb{N}$, proving (1).

Recall from Lemma 2.3.7 that $P^{(n)} \subset P^{(k)}$ for all $k < n$, and so

$$(P^{(k)})^* \subsetneq (P^{(n)})^*$$

for all such k. Next we claim that

$$(P^{(n)})^*P^{(n)} \nsubseteq P \qquad \dots(2)$$

for all $n \in \mathbb{N}$. Suppose that this is not the case, so that for some $n \in \mathbb{N}$ $(P^{(n)})^*P^{(n)} \subseteq P$. Then

$$P^{(n-1)}(P^{(n)})^*P^{(n)} \subseteq P^{(n-1)}P. \qquad \dots (3)$$

We now have, from (1) and (3), that

$$P^{(n-1)}(P^{(n)})^*P^{(n)} \subseteq P^{(n)}.$$

This implies that

$$P^{(n-1)}(P^{(n)})^* \subseteq O_1(P^{(n)}) = R$$

since $R$ is a $G$-maximal order and $P^{(n)}$ is $G$-invariant by Lemma 2.3.4. It follows that

$$(P^{(n)})^* \subseteq (P^{(n-1)})^*,$$

so we must have

$$(P^{(n)})^* = (P^{(n-1)})^*.$$

Since $P^{(n)}$ and $P^{(n-1)}$ are both reflexive by Lemma 2.3.8, this gives

$$P^{(n)} = (P^{(n)})^{**} = (P^{(n-1)})^{**} = P^{(n-1)},$$

a contradiction to Lemma 2.3.7. Therefore

$$(P^{(n)})^*P^{(n)} \not\subseteq P$$

for all $n \in \mathbb{N}$ and (2) is proved.

We now show that

$$(P^{(k)})^*P^{(n)} \subseteq P \qquad \ldots(4)$$

for all $k < n$. It is enough to show that

$$(P^{(n-1)})^*P^{(n)} \subseteq P,$$

since

$$(P^{(k)})^* \subseteq (P^{(n-1)})^*$$

for all $k < n$. Let $x \in (P^{(n-1)})^*$, $y \in P^{(n)}$. Then

$$yP^* \subseteq P^{(n-1)}.$$

Therefore $xyP^* \subseteq R$ and so $xy$ belongs to $P^{**} = P$, since $P$ is reflexive. So

$$(P^{(k)})^*P^{(n)} \subseteq P$$

for all $k < n$, proving (4).

Now, let $E$ be a right ideal of $R$. Since $R$ is Noetherian, there exists a positive integer n such that

$$\sum_{k=1}^{\infty}(E \cap P^{(k)})(P^{(k)})^* = \sum_{k=1}^{n-1}(E \cap P^{(k)})(P^{(k)})^*.$$

Therefore

$$(E \cap P^{(n)})(P^{(n)})^*P^{(n)} \subseteq \sum_{k=1}^{n-1}(E \cap P^{(k)})(P^{(k)})^*P^{(n)} \subseteq EP$$

by (4). Let $a \in R$, $c \in C(P)$. Then $aR + cR$ is a right ideal of $R$, and by the above we have, for some $n \in \mathbb{N}$,

$$[(aR + cR) \cap P^{(n)}](P^{(n)})^*P^{(n)} \subseteq (aR + cR)P = aP + cP.$$

By Lemma 2.3.10 there exist $b \in R$, $d \in C(P)$ such that

$$ad - cb \in P^{(n)}.$$

Therefore $ad - cb$ belongs to $(aR + cR) \cap P^{(n)}$. Let

$$I := (P^{(n)})^*P^{(n)},$$

an ideal of $R$ not contained in $P$ by (2). Then $(I + P)/P$ is a non-zero $G$-invariant ideal of the $G$-prime ring $R/P$, and so contains a regular element of $R/P$ by Lemma 2.3.11. Hence, without loss of generality, we can choose an element $x \in I\backslash P$ and suppose that $x \in C(P)$. Then

$$(ad - cb)x \subseteq [(aR + cR) \cap P^{(n)}](P^{(n)})^*P^{(n)} \subseteq aP + cP,$$

so that we can write

$$(ad - cb)x = aw_1 + cw_2$$

for some $w_1, w_2 \in P$. Then

$$a(dx - w_1) = c(w_2 + bx),$$

and $dx - w_1 \in C(P)$, $w_2 + bx \in R$. Therefore $R$ satisfies the right Ore condition with respect to $C(P)$. The left hand case follows by symmetry and so $P$ is localisable, as required.

## §2.4 ADDITIONAL REMARKS.

**2.4.1** (i) The notion of a $G$-maximal order is original.

(ii) Much of §2.1 up to (and including) Theorem 2.1.8 is based on the appropriate analogues in [Mc-R].

**2.4.2** (i) Theorem 2.2.4 is a $G$-equivariant version of [C-S2, Theorem 1.6]. See also [H-W, Lemma 2.1(i)].

(ii) Lemma 2.2.6 is a $G$-equivariant analogue of [C-S2, Propsition 1.7].

(iii) The notion of a $G$-local ring (as in Definition 2.2.10) is original.

**2.4.3** (i) The definitions of $P_l{}^{(n)}$ and $P_r{}^{(n)}$ given in 2.3.1 are based on the definition given for the symbolic powers of a (right and left) reflexive ideal of a semiprime Noetherian maximal order in [C-S1]. In this paper it is shown that this definition agrees with that of Goldie's symbolic powers, introduced in [G].

(ii) The proofs of Lemmas 2.3.7, 2.3.8 and 2.3.9 are modelled on [C-S1, Proposition 1.2], parts (3), (2) and (4) respectively.

(iii) Proposition 2.3.12 is a $G$-equivariant version of [H-W, Lemma 2.1(ii)]. Our proof was based on theirs.

# CHAPTER 3. NECESSARY AND SUFFICIENT CONDITIONS FOR A SKEW GROUP RING TO BE A MAXIMAL ORDER.

## §3.1 TWO TEST THEOREMS.

**3.1.1** In this section we give two results which allow us to test when a prime Noetherian ring is a maximal order or a $G$-maximal order. These appear as Theorem 3.1.4 and Theorem 3.1.13 respectively. The first part of this section deals with the maximal order case.

**3.1.2 NOTATION.** Unless stated otherwise, the following will apply. Let $R$ be a prime Noetherian ring with (simple Artinian) quotient ring $Q$. Put

$$\Omega := \{P \in \mathrm{Spec}(R) : P \text{ is reflexive and } \mathrm{ht}(P) = 1\}.$$

Recall from 1.1.4 the definitions of $I_l{}^*$ and $I_r{}^*$ for an ideal $I$ of $R$.

**3.1.3 DEFINITION.** Let $R$ be as in 3.1.2. Then we define

$$R_0 := \{q \in Q : qI \subseteq R \text{ for some non-zero ideal } I \text{ of } R\}, \text{ and}$$

$$R_0' := \{q \in Q : Iq \subseteq R \text{ for some non-zero ideal } I \text{ of } R\}.$$

Note that

$$R_0 = \cup\{I_l{}^* : 0 \neq I \text{ an ideal of } R\}$$

and similarly,

$$R_0' = \cup\{I_r{}^* : 0 \neq I \text{ an ideal of } R\}.$$

**REMARK.** Suppose that $R$ is a maximal order. Then as in Remark 1.1.6, $I_l{}^* = I_r{}^*$ for each ideal $I$ of $R$, and we see that $R_0 = R_0'$.

**3.1.4** We now aim towards proving the following result, the first of our two Test Theorems:

**THEOREM. (Test Theorem)** *Let $R$ be as in 3.1.2. Then $R$ is a maximal order if and only if*

*(i) Each $P \in \Omega$ is localisable;*

*(ii) $R_P$ is a maximal order for all $P \in \Omega$;*

*(iii) $R_0 = R_0'$;*

*(iv) $R = R_0 \cap (\cap\{R_P : P \in \Omega\})$.*

**3.1.5 PROPOSITION. [C, Proposition 1.10(b)].** *Adopt the notation of 3.1.2 and suppose that $R$ is a maximal order. Then each $P \in \Omega$ is localisable, and $R = R_0 \cap (\cap\{R_P : P \in \Omega\})$.*

**PROOF.** By Proposition 2.3.12, $P$ is localisable for all $P \in \Omega$. That

$$R \subseteq R_0 \cap (\cap\{R_P : P \in \Omega\})$$

is clear. Conversely, let

$$x \in R_0 \cap (\cap\{R_P : P \in \Omega\}$$

and consider

$$I := \{r \in R : xr \in R\}.$$

Since $x \in R_0$, there exists a non-zero ideal $J$ of $R$ with $xJ \subseteq R$, so that $J \subseteq I$. Let $A$ be the largest two-sided ideal of $R$ contained in $I$. Suppose that $R \subset A^*$; we show that this is untenable. Consider $(A^*/R)_R$, a non-zero $R$-submodule of $(Q/R)_R$. Let $\hat{P}$ be maximal amongst annihilators of non-zero $R$-submodules of $(A^*/R)_R$. Then $\hat{P}$ is a reflexive prime ideal of $R$ and $A \subseteq \hat{P}$.

Note that since $x \in R_P$ for all $P \in \Omega$, $I \cap C(P)$ is non-empty for all such $P$; choose $c \in I \cap C(\hat{P})$. Because $A \subseteq \hat{P}$ it follows that $\hat{P}^* \subseteq A^*$, so that $\hat{P}^*A \subseteq R$. Now, $R$ is a maximal order so that $A_l^* = A_r^*$ by Remark 1.1.6, and so $Ax \subseteq R$. Therefore

$$\hat{P}\hat{P}^*Ax \subseteq (R \cap \hat{P}x) \subseteq (R \cap \hat{P}c^{-1}) \subseteq \hat{P}.$$

This gives

$$\hat{P}^*Ax \subseteq O_r(\hat{P}) = R,$$

since $R$ is a maximal order, so that

$$x \in (\hat{P}^*A)_r^* = (\hat{P}^*A)_l^*.$$

Hence $x\hat{P}^*A \subseteq R$ and we see that $\hat{P}^*A \subseteq I$.

But $\hat{P}^*A$ is an ideal of $R$, so that $\hat{P}^*A \subseteq A$ due to the maximality of $A$. Then

$$\hat{P}^* \subseteq O_l(A) = R,$$

contradicting the reflexivity of $\hat{P}$. Therefore $A^* = R$. But $A$ is contained in $I$ and so $xA \subseteq R$, that is, $x$ belongs to $A^* = R$. Hence

$$R_0 \cap (\cap\{R_P : P \in \Omega\})$$

is contained in $R$ and we have equality, as required.

**3.1.6 LEMMA.** *Let $R$ and $Q$ be as in 3.1.2. Suppose further that $R$ is a maximal order, and that $C$ is a (non-empty) Ore set of regular elements of $R$. Then $R_C$ is a maximal order.*

**PROOF.** Clearly $R \subseteq R_C \subseteq Q$, so $R_C$ is an order in $Q$. Let $A$ be a non-zero ideal of $R_C$. Then

$$A = (A \cap R)R_C$$

and $A \cap R$ is an ideal of $R$. Let $q \in O_1(A) \subseteq Q$, so that $qA \subseteq A$. Now,

$$q(A \cap R)R_C \subseteq (A \cap R)R_C$$

so in particular

$$q(A \cap R) \subseteq R_C.$$

Since $q(A \cap R)$ is a finitely generated right $R$-module, the common multiple property for $C$ (by [Mc-R, Proposition 2.2.5]) ensures that there exists $x \in C$ such that

$$xq(A \cap R) \subseteq R.$$

Also,

$$q(A \cap R) \subseteq (A \cap R)R_C = A$$

so that

$$xq(A \cap R) \subseteq A.$$

So we have

$$xq(A \cap R) \subseteq A \cap R.$$

Since $R$ is a maximal order,

$$xq \in O_1(A \cap R) = R.$$

But $x \in C$, so $q \in R_C$. Therefore $O_1(A) = R_C$. Symmetry gives that $O_r(A) = R_C$ and hence by Theorem 1.1.2 $R_C$ is a maximal order as required.

**COROLLARY.** *Let $R$ and $Q$ be as in 3.1.2. Suppose that $R$ is a maximal order, and let $P$ be a localisable semiprime ideal of $R$. Then $R_P$ is a maximal order.*

**PROOF.** Since $P$ is localisable, $C(P)$ is an Ore set of regular elements of $R$. The result is now immediate from Lemma 3.1.6.

**3.1.7 PROPOSITION.** *Let $R$ be as in 3.1.2. Suppose that $P$ is localisable for all $P \in \Omega$; that $R_0 = R_0'$, and that $R = R_0 \cap (\cap\{R_P : P \in \Omega\})$ with each $R_P$ being a maximal order. Then $R$ is a maximal order.*

**PROOF.** Let $I$ be a non-zero ideal of $R$ and $x \in O_l(I)$. Then

$$xI \subseteq I \subseteq R,$$

so that $x \in R_0$ by definition of $R_0$. Now, $R_P$ exists for each $P \in \Omega$ since each such $P$ is localisable by hypothesis. Since $I$ is non-zero $IR_P$ is a non-zero ideal of $R_P$ for all $P \in \Omega$. We have

$$xIR_P \subseteq IR_P$$

so that

$$x \in O_l(IR_P) = R_P$$

for all $P \in \Omega$. Therefore

$$x \in R_0 \cap (\cap\{R_P : P \in \Omega\}) = R,$$

and so $O_l(I) = R$.

Now suppose that $x \in O_r(I)$. Since $IR_P$ is a two-sided ideal of $R_P$ for all $P \in \Omega$, it follows as above that

$$x \in \cap\{R_P : P \in \Omega\}.$$

It remains to show that $x \in R_0$. We have $x \in O_r(I)$ implies $Ix \subseteq I \subseteq R$ so that $x \in R_0'$. But $R_0' = R_0$ by hypothesis and so $x \in R_0$. Therefore

$$x \in R_0 \cap (\cap\{R_P : P \in \Omega\}) = R,$$

giving that $O_r(I) = R$. Hence $R$ is a maximal order, as required.

**3.1.8 PROOF OF THEOREM 3.1.4.** First suppose that $R$ is a maximal order. That (i) holds follows from Proposition 3.1.5. That (ii) holds follows from Corollary 3.1.6, and (iii) follows from Remark 3.1.3. Finally, Proposition 3.1.5 gives us (iv).

Conversely, if (i), (ii), (iii) and (iv) all hold than $R$ is a maximal order by Proposition 3.1.7.

**3.1.9 NOTATION.** From now on in this section, $S$ will be a prime Noetherian ring with quotient ring $Q$, and $G$ will be a finite group acting on $S$. Let

$$\Gamma = \{p_0 : p_0 \text{ is a height-1 reflexive } G\text{-prime ideal of } S.\}$$

**3.1.10 REMARK.** Note that, by Lemma 2.2.3 and Proposition 2.2.5, an ideal $p_0$ belongs to $\Gamma$ if and only if there exists a height-1 reflexive prime ideal, $p$ say, of $S$ such that $p_0 = \cap\{p^g : g \in G\}$.

**3.1.11 DEFINITION.** Let $S$, $Q$ and $G$ be as in 3.1.9. Define

$$S_0 = \{q \in Q : qI \subseteq S \text{ for some non-zero } G\text{-invariant ideal } I \text{ of } S\},$$
$$S_0' = \{q \in Q : Iq \subseteq S \text{ for some non-zero } G\text{-invariant ideal } I \text{ of } S\}.$$

**3.1.12 REMARK.** Note that if $S$ is a $G$-maximal order, then $S_0 = S_0'$ by Lemma 2.2.2.

**3.1.13** We now aim to prove a result analogous to Theorem 3.1.4, to test when a prime Noetherian ring $S$ is a $G$-maximal order (where $G$ is a finite group acting on $S$). Many of the results which follow are similar to those given earlier in this section; nevertheless we give complete proofs. This is the second of our Test Theorems, and is as follows:

**THEOREM.** *Adopt the notation of 3.1.9. Then $S$ is a $G$-maximal order if and only if*

*(i)* $p_0$ *is localisable for all* $p_0 \in \Gamma$;

*(ii)* $S_{p_0}$ *is a $G$-maximal order for all* $p_0 \in \Gamma$;

*(iii)* $S_0 = S_0'$;

*(iv)* $S = S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\})$.

**3.1.14** The following lemma is analogous to Proposition 3.1.5, and should be compared with [C, Proposition 1.10(b)].

**LEMMA.** *Adopt the notation of 3.1.9, and suppose that $S$ is a $G$-maximal order. Then $p_0$ is localisable for each $p_0 \in \Gamma$, and*

$$S = S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}).$$

**PROOF.** Let $S$ be a $G$-maximal order. Then each $p_0 \in \Gamma$ is localisable by Proposition 2.3.12. It is clear that

$$S \subseteq S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}.$$

For the reverse inclusion let

$$x \in S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}),$$

and consider the right ideal

$$I := \{s \in S : xs \in S\}$$

of $S$. Since $x \in S_0$, there exists a (non-zero) $G$-invariant ideal $J$ of $S$

with $xJ \subseteq S$, so that $J \subseteq I$.

Let $A$ be the largest $G$-invariant ideal of $S$ contained in $I$ and suppose that $S \subset A^*$; we show that this is untenable. Consider the (non-zero) $S$-submodule $(A^*/S)_S$ of $(Q/S)_S$. Let $M$ be a maximal annihilator ideal of non-zero $G$-invariant submodules of $(A^*/S)_S$, so that there exists a $G$-invariant submodule $X$ of $A^*$ strictly containing $S$ and such that $XM \subseteq S$. Since $S$ is a $G$-maximal order, Lemma 2.2.2 gives that

$$M = X_r{}^* = X_l{}^* = X^*.$$

Now, $A^*A \subseteq S$ and is an ideal of $S$, so by the maximality of $M$, $A \subseteq M$. Since $X$ is $G$-invariant, Lemma 2.1.5 applies to give that $M$ is $G$-invariant. Then by Lemma 2.2.3 and Lemma 2.2.4, $M$ belongs to $\Gamma$. Therefore $x \in S_M$.

Choose an element $c \in I \cap C(M)$. Now, $A \subseteq M$ means that $M^* \subseteq A^*$, and so $M^*A \subseteq S$. Also, $A \subseteq I$ so that $xA \subseteq S$ by definition of $I$. Therefore

$$x \in A_l{}^* = A_r{}^*,$$

and by Lemma 2.2.2 we have that $Ax \subseteq S$. Hence

$$MM^*Ax \subseteq S \cap Mx \subseteq S \cap Mc^{-1} \subseteq M.$$

Therefore

$$M^*Ax \subseteq O_r(M) = S,$$

since $M$ is a $G$-invariant ideal of the $G$-maximal order $S$. Also $M^*A$ is a $G$-invariant ideal of $S$ by Lemma 2.1.5, so that $xM^*A \subseteq S$ by Lemma 2.2.2. Hence $M^*A \subseteq I$ by definition of $I$. Because of the maximality of $A$ we must have $M^*A \subseteq A$, and so

$$M^* \subseteq O_l(A) = S,$$

contradicting the reflexivity of $M$. Therefore $A^* = S$. But $A \subseteq I$ so that $xA \subseteq S$ and we get $x \in A^* = S$. Hence

$$S = S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}),$$

as required.

3.1.15 Our next results are the $G$-equivariant analogues of Lemma 3.1.6 and Corollary 3.1.6 respectively. Recall from Remark 1.3.9 that any $G$-prime ideal of a right Noetherian ring is semiprime.

**LEMMA.** *Let $S$, $Q$ and $G$ be as in 3.1.9. Suppose that $S$ is a $G$-maximal order, and let $C$ be a $G$-invariant Ore set of regular elements of $S$. Then $S_C$ is a $G$-maximal order.*

**PROOF.** It is clear that $S_C$ is an order in $Q$, and that $G$ extends to a group of automorphisms of $S_C$. There is no loss in denoting this group by $G$ also. Let $0 \neq I$ be a $G$-invariant ideal of $S_C$. Then

$$I = (I \cap S)S_C$$

and $I \cap S$ is a $G$-invariant ideal of $S$. Let $x \in O_l(I) \subseteq Q$, so that $xI \subseteq I$. Now,

$$x(I \cap S)S_C \subseteq (I \cap S)S_C$$

so that in particular $x(I \cap S) \subseteq S_C$. Therefore, since $x(I \cap S)$ is a finitely generated right $S$-module, the common multiple property for $C$ ensures that there is an element $c \in C$ such that $cx(I \cap S) \subseteq S$. Also,

$$x(I \cap S) \subseteq (I \cap S)S_C = I$$

so that $cx(I \cap S) \subseteq I$. We now have

$$cx(I \cap S) \subseteq I \cap S.$$

Since $S$ is a $G$-maximal order and $I \cap S$ is $G$-invariant,

$$cx \in O_l(I \cap S) = S.$$

But $c \in C$, so $x \in S_C$. Therefore $O_l(I) = S_C$. Similarly, $O_r(I) = S_C$ and hence by Theorem 2.1.8 $S_C$ is a $G$-maximal order as required.

**COROLLARY.** *Let $S$, $Q$ and $G$ be as in 3.1.9. Suppose that $S$ is a $G$-maximal order, and let $p_0$ be a localisable semiprime $G$-invariant ideal of $S$. Then $S_{p_0}$ is a $G$-maximal order.*

**PROOF.** Since $p_0$ is localisable and $G$-invariant, Lemma 1.3.8 applies to give that $C(p_0)$ is a $G$-invariant Ore set of regular elements of $S$. Lemma 3.1.15 is now easily applied.

**3.1.16 LEMMA.** *Adopt the notation of 3.1.9 and 3.1.11. Suppose that $p_0$ is localisable for all $p_0 \in \Gamma$, and that $S = S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\})$ with each $S_{p_0}$ a $G$-maximal order. Suppose further that $S_0 = S_0'$. Then $S$ is a $G$-maximal order.*

**PROOF.** Let $0 \neq I$ be a $G$-invariant ideal of $S$, and let $x \in O_l(I)$. Then

$$xI \subseteq I \subseteq S,$$

so that $x \in S_0$ by definition of $S_0$. Since $I$ is a non-zero $G$-invariant (2-sided) ideal of $S$, $IS_{p_0}$ is a non-zero $G$-invariant ideal of $S_{p_0}$ for all $p_0 \in \Gamma$. Therefore $xI \subseteq I$ implies that

$$xIS_{p_0} \subseteq IS_{p_0},$$

so that

$$x \in O_l(IS_{p_0}) = S_{p_0}$$

since each $S_{p_0}$ is a $G$-maximal order by hypothesis. We now have

$$x \in S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}) = S,$$

and so $O_l(I) = S$.

Now suppose that $x \in O_r(I)$. It is shown in a similar way to the above that $x \in S_{p_0}$ for each $p_0 \in \Gamma$. Now,

$$Ix \subseteq I \subseteq S,$$

so that $x \in S_0' = S_0$. So we have

$$x \in S_0 \cap (\cap\{S_{p_0} : p_0 \in \Gamma\}) = S,$$

and so $O_r(I) = S$. Therefore, by Theorem 2.1.8, $S$ is a $G$-maximal order as required.

**3.1.17 PROOF OF THEOREM 3.1.13.** Suppose first that $S$ is a $G$-maximal order. That (i) holds follows from Proposition 2.3.12, and (ii) holds by Corollary 3.1.15. Condition (iii) is immediate from Remark 3.1.12, and (iv) follows from Lemma 3.1.14.

Conversely, if conditions (i), (ii), (iii) and (iv) all hold, then $S$ is a $G$-maximal order by Lemma 3.1.16.

## §3.2 PROOF OF THE MAIN THEOREM.

**3.2.1 NOTATION AND HYPOTHESES.** Throughout this section, the following will apply. Let $S$ be a prime Noetherian ring and $G$ a finite group acting on $S$. Recall 1.3.4, and suppose that the action of $G$ is X-outer on $S$. Denote the skew group ring $S*G$ by $T$. Let

$$\Omega = \{p \in \mathrm{Spec}(S) : p \text{ is reflexive and has height } 1\}$$

$$\Omega_0 = \{p_0 : p_0 = \cap\{p^g : g \in G\}, \text{ some } p \in \Omega\}$$

We now turn to the proof of the following result, which is the aim of this chapter.

**3.2.2 THEOREM.** (Main Theorem) *Let $S$, $G$, $T = S*G$ be as above. Consider the following conditions:*

*(a) $S$ is a $G$-maximal order;*

*(b) $p_0T$ is a prime ideal of $T$ for all $p_0 \in \Omega_0$.*

*(i) If (a) and (b) both hold, then $T$ is a prime maximal order.*

*(ii) Suppose that the order of $G$ is a unit in $S$. If $T$ is a (prime) maximal order, then (a) and (b) both hold.*

**REMARK.** Note that in general, part (ii) of the above Theorem is not true without imposing the extra hypothesis that the order of $G$ be a unit in $S$, and an example illustrating this is given in 6.2.11. However, when $S$ is commutative, we do have part (ii) holding without insisting on any extra hypothesis on the order of $G$, as we will see in

Theorem 5.2.7.

**3.2.3 REMARK.** Note that, under the hypotheses of 3.2.1, Lemma 1.3.5 guarantees that $T$ is prime.

**3.2.4** Recall from 3.1.3 and 3.1.11 the definitions of $T_0$ and $S_0$.

**LEMMA.** *Let $S$, $G$ and $T$ be as in 3.2.1. Then $T_0 = S_0 * G$.*

**PROOF.** Let $Q(R)$ denote the quotient ring of any ring $R$. By Corollary 1.3.25 $Q(T) = Q(S)*G$. Let

$$x \in T_0 \subseteq Q(T) = Q(S)*G,$$

and write

$$x = \sum\{s_g g : g \in G\}$$

with $s_g \in Q(S)$ for each $g \in G$. By definition of $T_0$, there exists a non-zero ideal $I$ of $T$ with $xI \subseteq T$. Now, $T$ is prime so that by Goldie's Theorem, $Q(T)$ is simple Artinian. Therefore

$$Q(T) = IQ(T) = I(Q(S)*G) = IG(Q(S)) = I(Q(S)).$$

So there exist elements $a \in I$ and $c \in C_S(0)$ with $1 = ac^{-1}$, and so $a = c \in I \cap S$. It follows that $I \cap S$ is a non-zero $G$-invariant ideal of $S$. Therefore $(I \cap S)*G$ is a non-zero ideal of $T$ and

$$(I \cap S)*G \subseteq I.$$

We have

$$x((I \cap S)*G) \subseteq xI \subseteq T,$$

so that

$$(\sum_{g \in G} s_g g)((I \cap S)*G) \subseteq T = S*G.$$

This means that $s_g(I \cap S) \subseteq S$ for all $g \in G$, since $I \cap S$ is $G$-invariant. Therefore $s_g \in S_0$ for all $g \in G$ and so $x \in S_0*G$.

Conversely, let

$$\sum\{r_g g : g \in G\} \in S_0{*}G$$

with $r_g \in S_0$ for all $g \in G$. Then there exist non-zero $G$-invariant ideals $J_g$ of $S$ such that $r_g J_g \subseteq S$ for all $g \in G$. Now let

$$J := \cap\{J_g : g \in G\},$$

also a $G$-invariant ideal of $S$, and of course non-zero due to the primeness of $S$. Then $J \subseteq J_g$ for all $g \in G$, and $JT$ is an ideal of $T$. Since $r_g J \subseteq r_g J_g$ for each $g$, we have $r_g J \subseteq S$. Therefore $r_g(JT) \subseteq T$ for all $g \in G$. Hence

$$(\sum_{g\in G} r_g g\ )JT \subseteq T,$$

and so

$$\sum_{g\in G} r_g g \quad \in T_0.$$

Therefore $T_0 = S_0{*}G$, as claimed.

**3.2.5 LEMMA.** *Adopt the notation and hypotheses of 3.2.1, and suppose that $p_0$ is a localisable ideal of $S$ for all $p_0 \in \Omega_0$. Then*

$$(S_0 * G) \cap (\bigcap_{p_0\in\Omega_0} (S_{p_0} * G)\ ) \;=\; (S_0 \cap (\bigcap_{p_0\in\Omega_0} S_{p_0})) * G.$$

**PROOF.** Let

$$\sum_{g\in G} (s_g g) \in (S_0 * G) \cap (\bigcap_{p_0\in\Omega_0} (S_{p_0} * G)\ ).$$

Since

$$\sum_{g\in G} (s_g g) \in S_0 * G,$$

we have $s_g \in S_0$ for all $g \in G$. Also,

$$\sum_{g\in G} (s_g g) \in (S_{p_0} * G)$$

for all $p_0 \in \Omega_0$, so that $s_g \in S_{p_0}$ for all such $p_0$ and for all $g \in G$.

Therefore

$$s_g \in S_0 \cap \left( \bigcap_{p_0 \in \Omega_0} S_{p_0} \right)$$

for all $g$, and so $\quad p_0 \epsilon \Omega_0 p_0$

$$\sum_{g \in G} (s_g g) \in \left(S_0 \cap \left( \bigcap_{p_0 \in \Omega_0} S_{p_0} \right)\right) * G.$$

Conversely, let

$$\sum_{g \in G} (r_g g) \in \left(S_0 \cap \left( \bigcap_{p_0 \in \Omega_0} S_{p_0} \right)\right) * G.$$

Then

$$r_g \in S_0 \cap \left( \bigcap_{p_0 \in \Omega_0} S_{p_0} \right)$$

for all $g \in G$, and it follows that

$$\sum_{g \in G} (r_g g) \in (S_0 * G) \cap \left( \bigcap_{p_0 \in \Omega_0} (S_{p_0} * G) \right).$$

**3.2.6 REMARK.** Recall from Remark 3.1.10 that an ideal $p_0$ of $S$ belongs to $\Omega_0$ if and only if it is $G$-prime, reflexive and has height 1. In particular, if $S$ is a $G$-maximal order, $p_0$ is localisable by Proposition 2.3.12. Then $p_0T$ is an ideal of $T$, and is localisable by an application of Lemma 1.3.25. We denote the localisation of $T$ at $p_0T$ by $T_{p_0}$; note that Lemma 1.3.25 also gives that $T_{p_0} = S_{p_0} * G$.

**LEMMA.** *Adopt the notation of 3.2.1 and suppose that $S$ is a $G$-maximal order. Then* $T = T_0 \cap \left( \bigcap_{p_0 \in \Omega_0} T_{p_0} \right)$

**PROOF.** If $S$ is a $G$-maximal order, then by Lemma 3.1.14,

$$S = S_0 \cap \left( \bigcap_{p_0 \in \Omega_0} S_{p_0} \right).$$

Therefore

$$T = S*G = ( S_0 \cap ( \bigcap_{p_0 \in \Omega_0} S_{p_0} ) ) * G$$

$$= (S_0 * G) \cap ( \bigcap_{p_0 \in \Omega_0} (S_{p_0} * G) )$$

by Lemma 3.2.5. It now follows that

$$T = T_0 \cap ( \bigcap_{p_0 \in \Omega_0} T_{p_0} )$$

by Lemma 3.2.4, as required.

**3.2.7 LEMMA.** *Adopt the notation of 3.2.1 and suppose that $T = S*G$ is a (prime Noetherian) maximal order. Then $S$ is a $G$-maximal order.*

**PROOF.** Let $I$ be a non-zero $G$-invariant ideal of $S$. Then $\hat{I} := I*G$ is a non-zero ideal of $T$. Now, $O_l(I)I \subseteq I$ implies that

$$O_l(I)\hat{I} = O_l(I)(I*G) \subseteq I*G = \hat{I}.$$

Therefore

$$O_l(I) \subseteq O_l(\hat{I}) = T,$$

since $T$ is a maximal order. Let $Q$ denote the quotient ring of $S$. We have

$$O_l(I) \subseteq T \cap Q = S,$$

so that $O_l(I) = S$. Since $I$ is $G$-invariant

$$\hat{I} = I*G = G*I,$$

so that we can use symmetry along with the above argument to show that $O_r(I) = S$. Hence $S$ is a $G$-maximal order, as required.

**3.2.8 LEMMA.** *Adopt the notation and hypotheses of 3.2.1. Suppose further that $S$ is a semilocal $G$-maximal order with the Jacobson radical $J(S)$ of $S$ reflexive, and that $|G|^{-1} \in S$. Then $T$ is a semilocal hereditary ring.*

**PROOF.** First we show that $T$ is semilocal. Let $J(T)$, $J(S)$ denote the Jacobson radicals of $T$ and $S$ respectively. Since $|G|^{-1} \in S$, Theorem 1.3.24 gives that

$$J(T) = J(S) * G.$$

Now $S$ is semilocal, so $S/J(S)$ is semisimple Artinian. We have

$$T/J(T) = (S*G)/(J(S) * G) \cong (S/J(S)) * G.$$

It follows that $|G|^{-1} \in S/J(S)$ and so by Maschke's Theorem $T/J(T)$ is semisimple Artinian. Hence $T$ is semilocal, as claimed. By Proposition 2.2.8, $J(S)$ is invertible. Therefore

$$\begin{aligned} J(T)(J(T))^* &= (J(S) * G)(J(S) * G)^* \\ &= (J(S) * G)(J(S)^* * G) \\ &= J(S)(J(S))^* * G \\ &= S*G = T \end{aligned}$$

Similarly $(J(T))^*J(T) = T$, and $J(T)$ is also invertible.

Put $J := J(T)$. By [Mc-R, Lemma 5.2.5] ${}_TJ$ and $J_T$ are projective and so

$$\text{pr.dim.}(T/J)_T = 1.$$

Also, $T/J$ is semisimple Artinian so that

$$\text{gl.dim.}(T/J) = 0.$$

Therefore by [Mc-R, Theorem 7.3.14]

$$\begin{aligned} \text{gl.dim.}(T) &\leqslant \text{gl.dim.}(T/J) + \text{pr.dim.}(T/J)_T \\ &= 0 + 1 \\ &= 1. \end{aligned}$$

Hence $T$ is hereditary, as required.

**3.2.9 EXAMPLE.** We now give an example to show that Lemma 3.2.8 does not hold without the requirement that the order of $G$ be a unit of $S$. Consider

$$T = (\mathbb{Z}/2\mathbb{Z})D_\infty,$$

where

$$D_\infty = \langle a,\ b : b^2 = 1,\ bab = a^{-1}\rangle$$

denotes the infinite dihedral group as defined in 1.5.1. Put $S = (\mathbb{Z}/2\mathbb{Z})\langle a\rangle$ and $C_2 = \langle b\rangle$, the cyclic group of order 2. Then

$$T \cong S*C_2,$$

the skew group ring. Clearly $|C_2| = 2$ is not a unit of $\mathbb{Z}/2\mathbb{Z}$. But $\mathbb{Z}/2\mathbb{Z}$ is a field of characteristic 2 and $D_\infty$ contains an element of order 2. Therefore, by [P2, Theorem 10.3.13], $T$ has infinite global dimension and so is certainly not hereditary.

**3.2.10 LEMMA.** *Adopt the notation of 3.2.1 and let $P$ be a height-1 reflexive prime ideal of $T$. Then*

*(i) $P \cap S$ is a $G$-prime ideal of $S$, and*

*(ii) $P \cap S$ belongs to $\Omega_0$.*

**PROOF.** (i). Lemma 1.3.10.

(ii). Put $p_0 = P \cap S$. We need to show that there exists a height-1 reflexive prime ideal $p$ of $S$ such that

$$p_0 = \cap\{p^g : g \in G\}.$$

By (i), $p_0$ is $G$-prime, and it follows from Theorem 1.3.21 that $p_0$ has height one. We can now apply Lemma 2.2.3 and Proposition 2.2.5 to give the desired result.

**3.2.11** Adopt the notation of 3.2.1 and let $P$ be a height-1 reflexive prime ideal of $T$. Put $p_0 = P \cap S$. By Lemma 3.2.10 $p_0$ is $G$-prime and so $p_0T$ is an ideal of $T$, and is contained in $P$. The following results give sufficient conditions for equality.

**LEMMA.** *Let $S$ be a Noetherian ring and $G$ a finite group acting on $S$. Let $T$ denote the skew group ring $S*G$. Let $P$ be a prime ideal of $T$ and put $p_0 = P \cap S$. Suppose that $p_0$ is localisable and that $PT_{p_0} = p_0T_{p_0}$. Then $P = p_0T$.*

**PROOF.** Note first that, as in the proof of Lemma 1.3.25, $C_S(p_0)$ is an Ore set in $T$, and so $T_{p_0}$ exists. It is clear that $p_0T \subseteq P$.

For the reverse inclusion, let $\alpha \in P$ and write $\alpha = ac^{-1}$ with $a \in p_0T$ and $c \in C_S(p_0)$. Then $\alpha c = a \in p_0T$. But

$$c \in C_S(p_0) \subseteq C_T(p_0T)$$

by Lemma 1.3.8, so $\alpha c \in p_0T$ implies that $\alpha \in p_0T$ as required.

**3.2.12 PROPOSITION.** *Adopt the notation and hypotheses of 3.2.1 and suppose that $|G|^{-1} \in S$. Suppose further that $T$ is a prime Noetherian maximal order. Let $P$ be a reflexive prime ideal of $T$ and put $p_0 = P \cap S$. Then $P = p_0T$.*

**PROOF.** By Lemma 3.2.7 $S$ is a $G$-maximal order. Then by Lemma 3.2.10 $p_0 \in \Omega_0$, and so is localisable by Remark 3.2.6. So first we localise $S$ at $p_0$, so that without loss of generality, in view of Lemma 3.2.11, $S$ is a semilocal ring. Of course, $G$ is still outer on $S$. Lemma 3.2.8 now applies to give that $T$ is semilocal and hereditary. Therefore (in particular), all maximal ideals of $T$ are projective and so by Lemma 1.2.4 they are all reflexive. It follows from Lemma 1.1.10 that all maximal ideals of $T$ are invertible, and hence they all have the $AR$-property. We can now use Lemma 1.2.14 to conclude that each finitely generated projective $T$-module is a generator. Then by Theorem

1.2.9 there exists a primitive idempotent $0 \neq e \in T$ such that each finitely generated projective (right) $T$-module is isomorphic to a direct sum of copies of $eT$. In particular we have

$$P \cong (eT)^{(n)}$$

and

$$T \cong (eT)^{(m)}$$

for some $n, m \in \mathbb{N}$.

Now since $T$ is a prime ring and $P$ is a non-zero ideal of $T$, $P$ is essential in $T$. Therefore by [G-W, Corollary 4.17], $P$ and $T$ have the same uniform dimension in $T$. We must have $n = m$ and

$$P_T \cong T_T.$$

Hence $P_T$ is principal. Similarly $_TP$ is principal, and so $P$ is right and left principal. By [Mc-R, Lemma 5.2.8],

$$P = \alpha T = T\alpha$$

for some element $\alpha \in P$.

Let $Q$ denote the quotient ring of $S$. Using [P1, Proposition 12.4(iv)] we can write $\alpha = \alpha_0 g$ , for some $g \in G$, with

$$\alpha_0 \in Q * G_{\text{inn}} = Q,$$

since $G$ is X-outer on $S$ (and hence on $Q$) by hypothesis. But $\alpha \in P \subseteq S{*}G$, so

$$\alpha_0 \in Q \cap S{*}G = S.$$

Therefore without loss of generality we have $\alpha = \alpha_0 g$ with $\alpha_0 \in S$. Hence

$$P = \alpha T = \alpha_0 gT = \alpha_0 T \subseteq (P \cap S)T = p_0 T.$$

That $p_0 T \subseteq P$ is clear, and so $P = p_0 T$.

**3.2.13 LEMMA.** *Let $U \subseteq V$ be rings such that $_UV$ is free. Let $M$ be a projective $U$-module. Then $M \otimes_U V$ is a projective $V$-module.*

**PROOF.** Recall from Definition 1.2.1 that $M$ being projective means that $M$ is a direct summand of a free $U$-module, say there exists a module $X$ such that $M \oplus X = F$ with $F_U$ free. Then

$$(M \otimes_U V) \oplus (X \otimes_U V) = (M \oplus X) \otimes_U V = F \otimes_U V$$

which is a free $V$-module. So $M \otimes_U V$ is a projective $V$-module, as required.

**3.2.14 PROOF OF THE MAIN THEOREM.** We are now in a position to prove the main result of this chapter, which is as follows:

**THEOREM.** *Adopt the notation and hypotheses of 3.2.1 and consider the following hypotheses:*

*(a) $S$ is a $G$-maximal order;*

*(b) $p_0T$ is a prime ideal of $T$ for all $p_0 \in \Omega_0$.*

*(i) If (a) and (b) both hold, then $T$ is a prime maximal order.*

*(ii) Suppose that the order of $G$ is a unit in $S$. If $T$ is a (prime) maximal order, then (a) and (b) both hold.*

**PROOF.** (i) Suppose first that (a) and (b) both hold. Since $S$ is a $G$-maximal order by (a), any ideal $p_0$ in $\Omega_0$ is localisable by Proposition 2.2.5 and Proposition 2.3.12. As in the proof of Lemma 1.3.25 we see that $C_S(p_0)$ is an Ore set in $T$ and that $T_{p_0}$ exists. Recall from Lemma 1.3.5 that $T$ is prime. Now

$$T/p_0T = (S*G)/(p_0*G) \cong (S/p_0)*G$$

for all $p_0 \in \Omega_0$. Therefore $p_0T$ prime implies that $(S/p_0)*G$ is prime for all such $p_0$. Let $Q(R)$ denote the quotient ring of any ring $R$. Then

$$Q(S/p_0)*G = Q((S/p_0)*G)$$

is simple Artinian by Corollary 1.3.25. But

$$Q\left[\frac{S}{p_0}\right] * G \cong \left[\frac{S_{p_0}}{p_0 S_{p_0}}\right] * G = \frac{\left[S_{p_0} * G\right]}{\left[p_0 S_{p_0} * G\right]},$$

so that $p_0 S_{p_0} * G$ is a prime ideal of $S_{p_0} * G$. Let $J(R)$ denote the Jacobson radical of any ring $R$. We have that $J(S_{p_0}) * G$ is prime. Using Theorem 1.3.24 we see that

$$(J(T_{p_0}))^n \subseteq J(S_{p_0}) * G \subseteq J(T_{p_0})$$

for some $n \in \mathbb{N}$. But since $J(S_{p_0}) * G$ is prime, we must have

$$J(S_{p_0}) * G = J(T_{p_0}),$$

so that $J(T_{p_0})$ is prime. Therefore

$$\frac{T_{p_0}}{J(T_{p_0})}$$

is simple Artinian, and so $T_{p_0}$ is local. Now $S$ is a $G$-maximal order, so by Corollary 3.1.15, $S_{p_0}$ is a $G$-maximal order. Therefore $S_{p_0}$ is hereditary by Corollary 2.2.9, and so $J(S_{p_0})$ is projective. Apply Lemma 3.2.13 with $U = S_{p_0}$ and $V = T_{p_0} = S_{p_0} * G$. Then $J(T_{p_0})$ is a projective $T$-module, and hence is reflexive. It follows from [Mc-R, Theorem 7.3.14(ii)] that

$$\begin{aligned} \text{gl.dim}(T_{p_0}) &\leqslant \text{gl.dim.}(T_{p_0}/J(T_{p_0})) + \text{pr.dim.}(T_{p_0}/J(T_{p_0})) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

so that $T_{p_0}$ is hereditary. We can now apply Proposition 1.2.8 to give that $T_{p_0}$ is a maximal order. Observe that, if $P$ is a reflexive height-1 prime of $T$, then $p_0 := P \cap S$ belongs to $\Omega_0$, by Lemma 3.2.10.

Since $P$ is minimal over $p_0T$ by Theorem 1.3.21, hypothesis (b) guarantees that $P = p_0T$. By this fact and Lemma 3.2.6,

$$T = T_0 \cap \left( \bigcap_{p_0 \in \Omega_0} T_{p_0} \right) = T_0 \cap \left( \bigcap_P T_P \right)$$

where $P$ runs over the set of height-1 reflexive primes of $T$. Now Theorem 3.1.4 gives us that $T$ is a maximal order, as required.

(ii) Suppose that the order in $G$ is a unit in $S$, and that $T$ is a prime maximal order. That (a) holds is immediate from Lemma 3.2.7. Let $p_0$ belong to $\Omega_0$. Since $p_0$ is reflexive, it follows that $p_0T$ is the annihilator in $T$ of

$$((p_0{}^*T) + T)/T,$$

and so $p_0T$ is also reflexive. Therefore $p_0T$ is contained in a maximal reflexive prime ideal $P$ of $T$, which is a height-1 prime of $T$ by [H-W, Lemma 2.1] (since $T$ is a maximal order). Then by Theorem 1.3.21, $P \cap S = p_0$. It now follows from Proposition 3.2.12 that $P = p_0T$, so that $p_0T$ is a prime ideal of $T$. This proves (b), and so (ii) holds.

## §3.3 ADDITIONAL REMARKS.

3.3.1 (i) The rings $R_0$ and $R_0'$ defined in 3.1.3 are equivalent to the rings $S(R)$ and $R_0$ introduced in [C-S2] and [C] respectively.

(ii) Theorem 3.1.4 is new in the form it is given here. As noted in the text, it is partially proved in [C, Proposition 1.10].

(iii) Theorem 3.1.13 is a new result (and is, of course, analogous to Theorem 3.1.4).

3.3.2 Theorem 3.2.2 is a new result.

# CHAPTER 4. NECESSARY AND SUFFICIENT CONDITIONS FOR A CROSSED PRODUCT TO BE A LOCAL RING.

## §4.1 LOCAL GROUP RINGS.

**4.1.1** The results of this short section will be used in the proof of the main result of this chapter, which appears as Theorem 4.2.8.

**4.1.2 NOTATION AND DEFINITIONS.** Throughout, $J(X)$ will denote the Jacobson radical of any ring $X$. Let $R$ be a ring and $G$ a finite group, and suppose that $x$ is an element of the group ring $RG$. Then we can write

$$x = \sum\{r_g g : g \in G\}$$

with $r_g$ an element of $R$ for all $g \in G$. We define the *trace* of $x$, denoted $\mathrm{tr}(x)$, to be the coefficient of the identity element of $G$ in $x$. In other words, $\mathrm{tr}(x) = r_1$. The *support* of $x$, denoted $\mathrm{Supp}(x)$, is defined as follows:

$$\mathrm{Supp}(x) = \{g \in G : r_g \neq 0\}.$$

The *augmentation ideal*, denoted $A$, of the ring $RG$ is defined to be

$$A = \sum\{R(g - 1) : g \in G\}.$$

Consider the map $\varphi : RG \longrightarrow R$ defined such that each element of $R$ is mapped to itself, and each group element is mapped to 1. Then a general element $\sum\{r_g g : g \in G\}$ of $RG$ is mapped to the element $\sum\{r_g : g \in G\}$ of $R$. It is easy to check that $\varphi$ is a well-defined surjective ring homomorphism, and that $\mathrm{Ker}(\varphi) = A$. Therefore

$$RG/A \cong R.$$

The map $\varphi$ is called the *augmentation map*.

**4.1.3 LEMMA.** *Let $R$ be a ring and $G$ a finite group. If the group ring $RG$ is local, then $R$ is local.*

**PROOF.** Suppose that $RG$ is local with unique maximal ideal $J = J(RG)$. Then the augmentation ideal $A$ of $RG$ is contained in $J$, and so $RG/A$ is local. That is, $R$ is local, as required.

**4.1.4** The following result is undoubtedly well-known, although a suitable reference could not be found; a complete proof is given below. For the case when $R$ is a field, however, the result is given in [P2, Theorem 10.1.16]].

**THEOREM.** *Let $R$ be a ring and $G$ a finite group. Then the group ring $RG$ is local if and only if*

*(i) $R$ is local;*

*(ii) when $\mathrm{char}(R/J(R)) = 0$, $G = \{1\}$; and when*

*$char(R/J(R)) = \rho > 0$, $G$ is a $\rho$-group.*

**PROOF.** First suppose that $RG$ is local. That $R$ is local is immediate from Lemma 4.1.3. Therefore $R/\mathrm{J}(R)$ is simple Artinian, and so has prime characteristic or zero characteristic. Put

$$\overline{R} = R/\mathrm{J}(R),$$

and let $A$ be the augmentation ideal of the group ring $\overline{R}G$. We show that $A$ is nilpotent. Now, $\mathrm{J}(R)G$ is an ideal of $RG$, and

$$RG/\mathrm{J}(R)G \quad \cong \quad (R/\mathrm{J}(R))G \quad = \quad \overline{R}G.$$

So $\overline{R}G$ is a factor of a local ring, and so is itself local with unique maximal ideal $\mathrm{J}(\overline{R}G)$. Since $\overline{R}$ is Artinian and $G$ is finite, $\overline{R}G$ is Artinian. Now,

$$\overline{R}G/A \cong \overline{R},$$

a simple Artinian ring, and so $A$ is a maximal ideal of $\overline{R}G$. Therefore

$$A = \mathrm{J}(\overline{R}G)$$

and is indeed nilpotent, by [G-W, Theorem 2.11], as claimed.

We now show that $G$ must satisfy condition (ii). Let $g$ be a non-identity element of $G$ and consider $g - 1$, an element of $A$. It is clear that $g - 1$ is a nilpotent element, since $A$ is nilpotent by the above. If $\mathrm{char}(\overline{R}) = 0$ then

$$\mathrm{tr}(g - 1) = 0$$

by [P2, Lemma 2.3.3], so that $g = 1$. So $G = \{1\}$ as required. Now suppose that $\mathrm{char}(\overline{R}) = \rho > 0$ and note that

$$\mathrm{tr}(g - 1) = -1.$$

Consider the set

$$X := \{x \in G : x \neq 1,\ x \in \mathrm{Supp}(g - 1) \text{ and } x \text{ has order a power of } \rho\}.$$

By [P2, Lemma 2.3.3], $X$ is non-empty. Therefore $X = \{g\}$ and so $G$ is a $\rho$-group. This proves (ii).

Conversely, suppose that (i) and (ii) both hold. Since $R$ is a local ring, $\bar{R} = R/J(R)$ is a simple Artinian ring. If $\mathrm{char}(\bar{R}) = 0$ then $G = \{1\}$ and we are done. Assume that $\mathrm{char}(\bar{R}) = p > 0$. Then $G$ is a $p$-group and Proposition 1.3.17 gives that $\bar{R}G$ is local. As before,

$$\bar{R}G \cong RG/J(R)G,$$

so $RG/J(R)G$ is also local. Let $M$ be the unique maximal ideal of $RG/J(R)G$. Then there exists a (necessarily maximal) ideal $P$ (say) of $RG$ containing $J(R)G$ and such that

$$M = P/J(R)G.$$

Let $Q$ be a maximal ideal of $RG$ distinct from $P$; we show that this is untenable. Now, by Theorem 1.3.24,

$$J(R)G \subseteq J(RG) \subseteq Q.$$

Also, $Q/J(R)G$ is an ideal of $RG/J(R)G$. Therefore

$$Q/J(R)G \subseteq M = P/J(R)G$$

which implies that $Q \subset P$, a contradiction to the maximality of $Q$. Hence $P$ is the only maximal ideal of $RG$, and so $RG$ is local, as required.

## §4.2 LOCAL CROSSED PRODUCTS.

**4.2.1** Let $S$ be a ring and $G$ a finite group. Suppose that $G$ acts on $S$, and that a twisting $\tau$ is given. Let $T$ denote the corresponding crossed product $S*G$. In this section, we discuss when the ring $T$ is local; necessary and sufficient conditions for this are given in Theorem 4.2.8. We work towards proving this via the following results.

**4.2.2 NOTATION.** Throughout this section, $S$, $G$ and $T$ will be as above. Denote by $J(X)$ the Jacobson radical of any ring $X$. Let $p$ be a maximal ideal of $S$. We will be considering the cases where either $\mathrm{char}(S/p) = 0$, or $\mathrm{char}(S/p) = \rho$ for some prime number $\rho > 0$. Put

$$H(p) := \{g \in G : p^g = p,\ g \text{ acts as identity on } S/p\}$$

$$K(p) := \mathrm{Stab}_G(p) = \{g \in G : p^g = p\}.$$

**4.2.3 LEMMA.** *Adopt the notation of 4.2.2, and let $p$ be a maximal ideal of $S$. Then $H(p) \subseteq K(p)$ are both subgroups of $G$. Furthermore, $H(p)$ is normal in $K(p)$.*

**PROOF.** It is clear that $H := H(p)$ and $K := K(p)$ are both subgroups of $G$. To show the second claim of the lemma, it is enough to show that

$$x^{-1}Hx = H$$

for all $x \in K$. This is clearly true when $x = 1$. So let $x$ be a non-identity element of $K$, and $h$ a non-identity element of $H$. Then

$$g := x^{-1}hx \in x^{-1}Hx.$$

It is clear that $p^g = p$. It remains to show that $g$ acts as the identity on $S/p$. Let $s \in S$, and put $y = x^{-1}$. Then $s^y \in S$, and

$$(s^y + p)^h = (s^y + p)$$

since $h \in H$. Therefore

$$\begin{aligned}(s + p)^g &= s^g + p = s^{yhx} + p \\ &= (s^y + p)^{hx} = (s^y + p)^x \\ &= s^{yx} + p = s + p,\end{aligned}$$

and so

$$g = x^{-1}hx \in H.$$

The above argument is easily reversed to give $x^{-1}Hx = H$ for all $x \in K$, as required.

**4.2.4** Recall from Definition 2.2.10 what it means for a ring to be $G$-local.

**PROPOSITION.** *Adopt the notation of 4.2.2. If $T$ is local then $S$ is $G$-local, with $J(S) = J(T) \cap S$.*

**PROOF.** Suppose that $T$ is local with unique maximal ideal $J(T)$. It is immediate from Theorem 1.3.22 that

$$J(S) = J(T) \cap S.$$

Put

$$T_1 = T/J(T),$$

and

$$S_1 = S/J(S) = S/(J(T) \cap S).$$

Since $T$ is local, $T_1$ is simple Artinian. Also, $T_1$ is a finite normalising extension of $S_1$, so by Theorem 1.3.19(i) $S_1$ is Artinian.

By Lemma 1.3.10

$$J(S) = J(T) \cap S$$

is a $G$-prime ideal of $S$ and hence is semiprime, so that $S_1$ is semisimple Artinian. Hence $S$ is a semilocal ring. Therefore $S$ is $G$-local, as required.

**4.2.5 LEMMA.** *Adopt the notation of 4.2.2 and suppose that the twisting $\tau$ is trivial (so that $T$ is a skew group ring). Suppose further that $T$ is local. Let $p$ be a maximal ideal of $S$. Then $H(p) = \{1\}$ if $char(S/p) = 0$, and $H(p)$ is a $\rho$-group if $char(S/p) = \rho > 0$.*

**PROOF.** Suppose that $T$ is local, but that $H := H(p)$ does not satisfy the hypotheses of the lemma. Put

$$R := (S/p) * H = (S/p)H,$$

the ordinary group ring (recall that $\tau$ is trivial by hypothesis). We claim that $R$ is not local.

Consider when $\mathrm{char}(S/p) = 0$; then $H \neq \{1\}$. By Maschke's Theorem $R$ is semisimple Artinian, but $R$ is not simple since $H \neq \{1\}$. Therefore $R$ is not local. In the case where $\mathrm{char}(S/p) = \rho > 0$, $H$ is not a $\rho$-group and so $R$ is not local by Theorem 4.1.4. This proves the claim.

In the following, $\mathrm{char}(S/p)$ may be either 0 or $\rho$. Since $H$ is normal in $K := K(p)$,

$$(S/p) * K = (S/p)H * (K/H) = R * (K/H).$$

Consider the augmentation ideal $A$ of $R$. Since $R$ is not local, there exists a prime ideal $B$ of $R$ different from $A$. If a non-identity element $k$ of $K$ is such that $B^k = A$, then

$$B = (B^k)^l = A^l = A$$

(where $l = k^{-1}$) since A is $K$-invariant, a contradiction. Hence no $K$-conjugate of $B$ can equal $A$, and so there are distinct $K$-orbits $O_1$, $O_2$ of primes of $R$. Put

$$N_1 = \cap\{P : P \in O_1\}$$

and

$$N_2 = \cap\{Q : Q \in O_2\}.$$

By LO (Theorem 1.3.21) there exist maximal ideals $M_1$, $M_2$ of $R * (K/H)$ with

$$M_1 \cap R = N_1$$

and

$$M_2 \cap R = N_2.$$

If $M_1 = M_2$ then $N_1 = N_2$, a contradiction to the fact that $O_1 \neq O_2$. So $M_1 \neq M_2$ and we have distinct maximal ideals of $(S/p) * K$. Put

$$p_0 = \cap\{p^g : g \in G\}$$

and let $\overline{M}_1$, $\overline{M}_2$ be the inverse images in $(S/p_0) * K$ of $M_1$, $M_2$ respectively. Note that $\overline{M}_1 \neq \overline{M}_2$, but

$$p/p_0 = \overline{M}_1 \cap (S/p_0) = \overline{M}_2 \cap (S/p_0).$$

Using Theorem 1.3.12 we see that there exist distinct maximal ideals of $T$, a contradiction to the fact that $T$ is local. Hence the result holds.

**REMARK.** Note that Lemma 4.2.5 above is false for general crossed products. For example, let $k$ be any field, n $\in \mathbb{N}$, and $<x>$ an infinite cyclic group. Put $U := k<x^n>$ and $W := k<x>$. Therefore $W$ is a crossed product of $U$ by the cyclic group of order n, that is,

$$W = U \cdot G$$

where $G = <x>/<x^n>$. Since $W$ is a commutative domain, it is clear that the multiplicatively closed set $C := U - \{0\}$ is an Ore set in $W$. It follows that

$$W_C \cong U_C * G,$$

a crossed product of $G$ over the field $U_C$. The ring $W_C$ is a subring of the quotient field of $W$; also it is finitely generated as a vector space over $U_C$, and so is Artinian. Therefore all its regular elements are units, and so $W_C$ is in fact the full quotient field of $W$. In particular, it is a local crossed product of a finite group $G$. In this case, $p = 0$ and $H = K = G$, but clearly, for suitable choices of n, Lemma 4.2.5 does not hold.

**4.2.6 LEMMA.** *Let $S$ and $G$ be as in 4.2.2 and suppose that the twisting $\tau$ is trivial on $S$. Suppose further that $T$ is local. Let $p$ be a maximal ideal of $S$ and put $H(p) = H$. Then $(S/p)H$ is local.*

**PROOF.** We will use Theorem 4.1.4. If $\mathrm{char}(S/p) = 0$, then $H = \{1\}$ by Lemma 4.2.5 and so we just have the simple ring $S/p$ which is clearly local. If $\mathrm{char}(S/p) = \rho$ for some prime number $\rho$, then Lemma 4.2.5 gives that $H$ is a $\rho$-group and then $(S/p)H$ is local by Theorem 4.1.4. So in either case we have the desired result.

**4.2.7 LEMMA.** *Let $S$, $G$ and $T = S*G$ be as in 4.2.2. Let $p$ be a maximal ideal of $S$, and put $K := K(p)$. Then if $T$ is local, $(S/p) * K$ is local.*

**PROOF.** Suppose that $T$ is local, but that there exist distinct maximal ideals $M_1$ and $M_2$ (say) of $(S/p) * K$. Put $\bar{S} = S/p$. Now, $M_i \cap \bar{S}$ is a proper ideal of the simple ring $\bar{S}$ for i = 1,2, and so $M_i \cap \bar{S} = 0$. But by definition of $K$, $\bar{S} * K$ is a factor of $S*K$. So there exist distinct primes $P_1$ and $P_2$ of $S*K$ such that

$$P_i \cap S = p.$$

Then by Theorem 1.3.12, we can find distinct primes $Q_1$ and $Q_2$ of $S*G = T$ such that

$$Q_i \cap S = 0$$

for i = 1,2, contradicting the fact that $T$ is local. So there exist no such ideals $M_1$, $M_2$ and $\overline{S} * K$ is a local ring, as required.

**4.2.8 THEOREM.** *Let $S$, $G$, $\tau$ and $T$ be as in 4.2.2. The crossed product $T = S*G$ is a local ring if and only if*

*(a) $S$ is $G$-local;*

*(b) $(S/p) * K$ is local, where $p$ is a maximal ideal of $S$ and $K := K(p)$.*

**REMARK.** Condition (a) of Theorem 4.2.8 ensures that all the maximal ideals of $S$ are $G$-conjugate, so that given (a), statement (b) is true if and only if it is true for any one maximal ideal of $S$.

**PROOF OF THEOREM 4.2.8.** Suppose first that conditions (a) and (b) both hold. Put $S_1 := S/J(S)$ and $p_1 := p/J(S)$, a minimal prime of $S_1$. Since $J(S) * G$ is contained in $J(T)$ by Theorem 1.3.24, in order to show that $T$ is local it is enough to show that

$$(S/J(S)) * G$$

is local. Now, $S_1$ is semisimple Artinian since $S$ is $G$-local, and is $G$-prime since $J(S)$ is $G$-prime. Also $S_1/p_1 \cong S/p$, so that

$$(S_1/p_1) * K \cong (S/p) * K$$

is local, since $(S/p) * K$ is local by (b). But $(S_1/p_1) * K$ is a factor of $S_1*K$ so that there exists a unique maximal ideal of $S_1*K$ containing $p_1*K$. Therefore, by Theorem 1.3.12, there exists a unique prime ideal

$P$ (say) of $S_1{*}G$ with $P \cap S_1 = 0$. By Theorem 1.3.16(i), $P$ is a minimal prime of $S_1{*}G$ and so is the only one. But $S_1{*}G$ is Artinian; hence

$$S_1{*}G = (S/J(S)) * G$$

is local. Therefore $T$ is local, as required.

Conversely, suppose that the crossed product $T$ is local. Then condition (a) is seen to hold by Proposition 4.2.4, and (b) holds by Lemma 4.2.7.

**COROLLARY.** *Let $S$, $G$, $\tau$ and $T$ be as in 4.2.2. Suppose further that the twisting $\tau$ is trivial. If the skew group ring $T = S{*}G$ is local, then*

*(a) $S$ is $G$-local;*

*(b) for each maximal ideal $p$ of $S$, $(S/p){*}K$ is local where $K := K(p)$;*

*(c) $H := H(p) = \{1\}$ if $\mathrm{char}(S/p) = 0$, and $H$ is a $\rho$-group if $\mathrm{char}(S/p) = \rho > 0$.*

**PROOF.** Use Theorem 4.2.8 and Lemma 4.2.5.

**4.2.9 REMARK.** If $S$ is a commutative ring satisfying hypothesis (a) of Theorem 4.2.8, then condition (b) is automatically satisfied. To see this, consider

$$W := (S/p) * K = (S/p)H * (K/H) = R * (K/H)$$

where $R$ denotes the ordinary group ring $(S/p)H$. Using Lemma 4.2.6, we see that $R$ is local, and so $R/J(R)$ is a simple Artinian ring.

Now in this situation, $K/H$ is a finite group of *outer* automorphisms of $R$. Therefore Lemma 1.3.5 implies that

$$(R/J(R)) * (K/H) = W/J(R)W$$

is prime, and hence simple Artinian. So $J(R)W$ is a maximal ideal of $W$.

But $J(R)W \subseteq J(W)$ by Theorem 1.3.24, so we must have equality. In other words, $W/J(W)$ is a simple Artinian ring and so $W = (S/p) * K$ is a local ring as claimed. We therefore have the following special case of Theorem 4.2.8.

**COROLLARY.** *Let $S$, $G$, $\tau$ and $T$ be as in 4.2.2 and suppose further that $S$ is commutative. Then the crossed product $T = S*G$ is local if and only if $S$ is $G$-local.*

**4.2.10 EXAMPLE.** We now give an example to illustrate that condition (b) of Theorem 4.2.8 is not vacuous in the general case, that is when we have a non-commutative coefficient ring.

Let $S = M_2(\mathbb{Q})$, the 2x2 matrix ring with entries in $\mathbb{Q}$. Then $S$ is a simple ring and so is trivially local. Put $G = \langle g \rangle$, the group acting on $S$ by conjugation by

$$u = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

an element of $S$. Note that $u^2$ is the identity element so that $G \cong C_2$. Put $T = S*G$, the skew group ring. Then $T \cong S\langle gu \rangle$, the ordinary group ring, which is not local by Theorem 4.1.4.

## §4.3 ADDITIONAL REMARKS.

**4.3.1** As mentioned in the text, Theorem 4.1.4 is a known result. For the case of a group ring of a finite group over a field, the result is given in [P2].

**4.3.2** Theorem 4.2.8 is a new result.

# CHAPTER 5. COMMUTATIVE COEFFICIENTS.

## §5.1 PRELIMINARY RESULTS.

5.1.1 DEFINITIONS. Let $R$ be a Noetherian ring and $P, Q \in \text{Spec}(R)$. There is said to be a *link* from $Q$ to $P$ if there is an ideal $A$ of $R$ such that $Q \cap P > A \geqslant QP$ and $(Q \cap P)/A$ is (non-zero and) torsion-free both as a right $(R/P)$-module and as a left $(R/Q)$-module. The bimodule $(Q \cap P)/A$ is called a *linking bimodule* between $Q$ and $P$.

We denote by $\text{Cl}(P)$ the *clique* containing $P$. The prime ideal $Q$ of $R$ belongs to $\text{Cl}(P)$ if and only if there is a chain of links between $P$ and $Q$.

A subset $X$ of $\text{Spec}(R)$ is said to be *right link closed* if whenever $P \in X$ and there is a link from $Q$ to $P$ for some $Q \in \text{Spec}(R)$, then we must have $Q \in X$. Moreover, the set $X$ is *link closed* if $X$ is a union of cliques.

5.1.2 The following result is known as Jategaonkar's Main Lemma. A proof can be found in [G-W, Theorem 11.1].

**THEOREM. (Jategaonkar).** *Let $R$ be a Noetherian ring and $M$ a finitely generated right $R$-module. Suppose that $0 < U < M$ is an affiliated series for $M$ with corresponding affiliated primes $P$, $Q$ such that $U$ is essential in $M$. Let $N$ be a submodule of $M$ with $U \subset N$ and such that $A := \mathrm{Ann}_R(N)$ is maximal amongst annihilators of submodules of $M$ properly containing $U$. Then precisely one of the following occurs:*

*(i) $Q < P$ and $NQ = 0$. In this case, $N$ and $N/U$ are faithful torsion $(R/Q)$-modules;*

*(ii) there is a link from $P$ to $Q$ and $(Q \cap P)/A$ is a linking bimodule. In this case, if $U$ is torsionfree as a right $(R/P)$-module, then $N/U$ is torsionfree as a right $(R/Q)$-module.*

**5.1.3 DEFINITIONS.** Let $R$ be a Noetherian ring. A prime ideal $P$ of $R$ is said to satisfy the *right strong second layer condition* (rsslc) if, given the hypotheses of Theorem 5.1.2, situation (i) never occurs.

Moreover, $P$ is said to satisfy the *right second layer condition* (rslc) if, given the hypotheses of Theorem 5.1.2 together with the additional hypothesis that $U$ is torsionfree as a right $(R/P)$-module, situation (i) never occurs.

We say that the ring $R$ has rsslc (resp. rslc) if each $P \in \mathrm{Spec}(R)$ has rsslc (resp. rslc). The *left strong second layer condition* (lsslc) and *left second layer condition* (lslc) are defined similarly, and $R$

has sslc (or slc) if the conditions hold on both the left and the right.

**5.1.4 EXAMPLES.** (i) A commutative Noetherian ring has sslc, since a commutative Noetherian domain has no faithful finitely generated torsion modules.

(ii) Any right Artinian ring has sslc, since such a ring has no prime ideals $P$, $Q$ (say) with $Q \subset P$.

(iii) Let $R$ be a commutative Noetherian ring and $G$ a polycyclic-by-finite group. Then, by [Mc-R, §4.3.14], the group ring $RG$ has slc.

(iv) Let $R$ be a FBN (fully bounded Noetherian) ring. Then $R$ satisfies slc by [Mc-R, §4.1.14]. In particular, any Noetherian ring finitely generated as a module over its centre has slc.

**5.1.5** Our next result, due to Müller and Jategaonkar, helps with the problem of localisation; for a proof, see [G-W, Theorem 12.21].

**THEOREM.** *Let $R$ be a Noetherian ring satisfying slc, and let $N$ be a semiprime ideal of $R$. If the set of prime ideals of $R$ minimal over $N$ is link closed, then $N$ is localisable.*

**5.1.6** For proofs of the following lemma and Theorem 5.1.7, see [G-W, Lemma 12.17] and [G-W, Lemma 11.20] respectively. For further details of Theorem 5.1.7, the reader is referred to [M].

**LEMMA.** *Suppose that $R$ is a Noetherian ring and that $P, Q \in Spec(R)$ are such that there is a link from $Q$ to $P$. Let $C$ be a right Ore set in $R$*

*and suppose that* $C \subseteq C(P)$. *Then* $C \subseteq C(Q)$.

**5.1.7 THEOREM. (Müller, 1976.)** *Let* $R$ *be a ring finitely generated as a module over its centre* $Z$ *and suppose that* $Z$ *(and therefore* $R$*) is Noetherian. Then, for* $P \in Spec(R)$, $P \cap Z \in Spec(Z)$ *and*

$$Cl(P) = \{Q \in Spec(R) : Q \cap Z = P \cap Z\}.$$

**5.1.8 REMARK.** Observe that one inclusion of the Theorem 5.1.7 is clear, as can be seen by the following result. The main point of Theorem 5.1.7, therefore, lies in the (not so obvious) converse.

**LEMMA.** *Let* $R$ *be a Noetherian ring with centre* $Z$, *and let* $P$, $Q$ *be prime ideals of* $R$. *Suppose that there exists either a link from* $P$ *to* $Q$ *or a link from* $Q$ *to* $P$. *Then* $P \cap Z = Q \cap Z$.

**PROOF.** Using symmetry, it is enough to assume that there is a link from $Q$ to $P$, and show that this forces

$$P \cap Z = Q \cap Z.$$

So, we make these assumptions and let $x \in P \cap Z$. By definition, there exists a linking bimodule $(Q \cap P)/A$ where $QP \subseteq A$. Certainly

$$(Q \cap P)P \subseteq A;$$

therefore

$$(Q \cap P)x \subseteq A.$$

Since $x$ belongs to the centre of $R$, we also have that

$$x(Q \cap P) \subseteq A.$$

Hence

$$x \in \text{l.ann}_R((Q \cap P)/A) = Q.$$

Therefore $P \cap Z \subseteq Q \cap Z$. The argument is similar for the reverse inclusion, and so the lemma is proved.

**5.1.9** We will use the results in 5.1.6 and 5.1.7 to prove the following

**PROPOSITION.** *Let $R$ be a prime Noetherian ring finitely generated as a module over its centre $Z$ and $P \in Spec(R)$. Then $P$ is localisable if and only if $P$ is the unique prime lying over $P \cap Z$.*

**PROOF.** Suppose first that $P$ is localisable. By Theorem 5.1.7 it is enough to show that

$$\mathrm{Cl}(P) = \{P\}.$$

If not, then there exists $Q \in \mathrm{Spec}(R)$, different from $P$, with $Q \in \mathrm{Cl}(P)$. Then there is either a link from $Q$ to $P$, or a link from $P$ to $Q$. Consider the former case. Lemma 5.1.6 gives

$$C(P) \subseteq C(Q).$$

(Note that $C(P)$ is an Ore set since $P$ is localisable by hypothesis). We claim that $Q \subseteq P$. Suppose not. Then $P \subset Q + P$. Now, $R/P$ is a prime Noetherian ring, so $(Q + P)/P$ is an essential ideal of $R/P$ and so contains a regular element of $R/P$, $c + P$ say, with

$$c \in C(P) \cap (Q + P).$$

Write $c = q + p$ for some $q \in Q$, $p \in P$. Let $x \in R$; then $px \in P$. Now $qx \in P$ implies that

$$qx + px = (q + p)x \in P,$$

which means that $x \in P$, since $q + p \in C(P)$. Therefore

$$q \in C(P) \cap Q,$$

which contradicts the fact that $C(P) \subseteq C(Q)$. So $Q \subseteq P$ as claimed. But, using Theorem 1.3.20, INC implies that $Q = P$, since $P \cap Z = Q \cap Z$ by

Theorem 5.1.7.

Now, if there were a link from $P$ to $Q$, then since $C(P)$ is a left and right Ore set we can use the left analogue of Lemma 5.1.6 to show that $C(P) \subseteq C(Q)$, and the argument then follows as above. So in either case we have that

$$\mathrm{Cl}(P) = \{P\},$$

and so $P$ is the unique prime lying over $P \cap Z$.

On the other hand, suppose that $P$ is the one and only prime lying over $P \cap Z$. Then it follows from Lemma 5.1.8 and Theorem 5.1.5 (together with the fact that, by Example 5.1.4(iv), a ring finitely generated as a module over its centre satisfies the slc), that $P$ is localisable. For the sake of completeness, however, we give a direct elementary proof here.

Put $p = P \cap Z$, a prime ideal of $Z$ by Theorem 5.1.7. Consider the multiplicatively closed set

$$C := Z \backslash (P \cap Z).$$

As in the proof of Lemma 1.3.8, it is easy to show that $C \subseteq C(P)$, so that we can localise and form the partial quotient ring of $R$ with respect to $C$. Put $S = R_C$. Since $C$ consists of regular elements, $S \subseteq Q(R)$, the quotient ring of $R$ with respect to $C_R(0)$. We show that $S = R_P$, and so to form $R_P$ it is enough to invert the elements of $C$. Firstly, $PS$ is contained in the Jacobson radical of $S$. This is because $Z_C$ is local with unique maximal ideal $(P \cap Z)Z_C$, and for some $t \geqslant 1$,

$$(J(S))^t \subseteq (P \cap Z)S = ( (P \cap Z)Z_C )S \subseteq J(S)$$

by Theorem 1.3.23. By hypothesis, $P$ is the one and only prime ideal of $R$ minimal over $(P \cap Z)R$, so that we have

$$(PS)^m \subseteq (P \cap Z)S \subseteq J(S)$$

for some natural number m. It follows that $PS \subseteq J(S)$, as claimed. We now show that

$$S/PS \cong Q(R/P)$$

by considering the well-defined ring homomorphism

$$\varphi : S \longrightarrow Q(R/P)$$

defined by

$$\varphi(rc^{-1}) = (r + P)(c + P)^{-1}$$

for some $r \in R$ and $c \in C$. It is clear that $\ker(\varphi) = PS$. Therefore

$$S/PS \cong \mathrm{Im}(\varphi)$$

and it remains to show that $\varphi$ is surjective. Now, $S$ is finitely generated as a module over $Z$, so

$$S/PS = (RC^{-1})/(PRC^{-1})$$

is finitely generated as a module over $(Z\backslash p)C^{-1}$, which is a field. Hence $S/PS$ is (in particular) Artinian, and therefore so is $\mathrm{Im}(\varphi)$, by the above isomorphism. Thus all the regular elements of $\mathrm{Im}(\varphi)$ are units. But

$$R/P \subseteq \mathrm{Im}(\varphi),$$

so regular elements of $R/P$ are units of $\mathrm{Im}(\varphi)$, that is for each $c \in C(P)$, $c + P$ is a unit of $\mathrm{Im}(\varphi)$. Then a general element of $Q(R/P)$, $z$ say, has the form $z = xy^{-1}$ with $x$, $y \in R/P$ and $y$ regular. We can write $x = r + P$ for some $r \in R$, and $y = c + P$ for some $c \in C(P)$. Then $x$ and $y$ both belong to $\mathrm{Im}(\varphi)$ with $y$ a unit. Therefore $z \in \mathrm{Im}(\varphi)$ and $\varphi$ is indeed surjective. So we now have

$$S/PS \cong \mathrm{Im}(\varphi) = Q(R/P) \cong R_P/PR_P.$$

In particular, $PS$ is a maximal ideal of $S$ and so $PS \subseteq J(S)$ forces $PS = J(S)$. It is routine to show that $S$ satisfies the definition of the quotient ring of $R$ with respect to $C(P)$, so by the uniqueness of $R_P$ we must have $S = R_P$. So $P$ is localisable, and the proof is complete.

5.1.10 The following two results are taken from [Y], and will be used in the next section.

**THEOREM.** [Y, Theorem 5.2]. *Let $R$ be a commutative Noetherian ring, let $G$ be a finite group acting on $R$, and denote by $R*G$ the skew group ring. The following are equivalent:*

*(i)* $gl.dim.(R*G) < \infty$;

*(ii)* *(a)* $gl.dim.(R) < \infty$;

*(b) for all maximal ideals $M$ of $R$ with $char(R/M) = \rho > 0$, $(R/M) * G_M$ is semisimple Artinian, where $G_M := Stab_G(M)$;*

*(iii)* *(a)* $gl.dim.(R) < \infty$;

*(b) for all maximal ideals $M$ of $R$ with $char(R/M) = \rho > 0$, $G(M)$ contains no element of order $\rho$, where*

$$G(M) = \{g \in G : r - r^g \in M \text{ for all } r \in R\}$$

5.1.11 Recall from Definition 1.2.12 what it means for an ideal of a ring to have the right AR-property.

**LEMMA.** [Y, Lemma 4.4] *Let $R$ be a right Noetherian ring and $G$ a finite group; let $R*G$ be a crossed product. Suppose that $I$ is a $G$-invariant ideal of $R$ satisfying the right AR-property. Then $I*G$ (is an ideal of $R*G$ and) has the right AR-property.*

## §5.2 THE COMMUTATIVE CASE.

**5.2.1** In this section we consider when a skew group ring $T = S*G$ is a prime maximal order in the case where the coefficient ring $S$ is a commutative Noetherian domain and $G$ is a finite group acting on $S$. In this setting, we no longer require the hypothesis that the order of $G$ is a unit in $S$ (as we did in Theorem 3.2.2). Moreover, as we shall see, there is a nice test for when the ideal $(P \cap S)T$ of $T$ is prime, where $P$ is a prime ideal of $T$. This means that condition (ii) of Theorem 3.2.2 can be replaced by a condition which is much easier to check.

**5.2.2 NOTATION.** Throughout, let $S$ be a commutative Noetherian domain and $G$ a finite group acting on $S$. As usual, $T$ will denote the skew group ring $S*G$.

**5.2.3** If we impose the condition that $G$ acts non-trivially on $S$, then this ensures that $T$ is prime:

**LEMMA.** *Let $S$, $G$ and $T$ be as in 5.2.2. Then $T$ is a prime ring if and only if there does not exist a non-identity element of $G$ which acts trivially on $S$.*

**PROOF.** First suppose that $T$ is prime. Let

$$H = \{g \in G : g \text{ acts trivially on } S\}$$

and suppose that $H \neq \{1\}$. Note first that $H$ is a normal subgroup of $G$, being the kernel of the map from $G$ into $\mathrm{Aut}(S)$. Put $\hat{H} = \sum\{h : h \in H\}$; it is clear that $\hat{H} \neq 0$. Note that $\hat{H}$ commutes with all $g \in G$. Also, $\hat{H}$ acts trivially on elements of $S$ by definition, so that $\hat{H}$ is central in $T$. Choose $1 \neq h \in H$. Then, for all $t \in T$,

$$0 = t\hat{H}(h - 1) = \hat{H}t(h - 1)$$

so that

$$T\hat{H}T(h - 1)T = 0;$$

a contradiction to the primeness of $T$. Therefore $H = \{1\}$, and no non-identity element of $G$ acts trivially on $S$.

On the other hand, assume that no non-identity element of $G$ acts trivially on $S$. Write $Q(S)$ for the quotient field of $S$. By Corollary 1.3.25, $T$ has quotient ring

$$Q(T) = Q(S)*G.$$

By hypothesis, $G$ is a group of X-outer automorphisms of $S$, and hence of $Q(S)$. Therefore the set of X-inner automorphisms, $G_{inn}$, of $Q(S)$ is just equal to $\{1\}$. Now $Q(S)$ is a field and so is certainly prime. Using Lemma 1.3.5 $Q(S)*G$ is also prime. Therefore

$$Q(T) = Q(S)*G$$

is simple Artinian and so $T$ is prime, as required.

**5.2.4** Adopt the notation of 5.2.2. Recall from 3.1.3 the definitions of $S_0$ and $S_0'$, and let $Q$ be the quotient field of $S$. It is clear, since $S$ is commutative, that

$$\begin{aligned} S_0 &= \{q \in Q : qI \subseteq S \text{ for some non-zero ideal } I \text{ of } S\} \\ &= \{q \in Q : Iq \subseteq S \text{ for some non-zero ideal } I \text{ of } S\} = S_0'. \end{aligned}$$

We can adapt Theorem 3.1.4 to our current situation by making the following observation:

**LEMMA.** *Let $S$ be a commutative Noetherian domain with quotient field $Q$. Then $S_0 = Q$.*

**PROOF.** It is clear that $S_0 \subseteq Q$. Conversely, let $q \in Q$ and write $q = ac^{-1}$ for some non-zero elements $a$, $c$ of $S$. Consider $cS$, an ideal of $S$. Then

$$q(cS) = ac^{-1}(cS) = aS \subseteq S,$$

so that $q \in S_0$.

**5.2.5** Theorem 3.1.4 can thus be modified to give the following well-known result. A specific reference for the theorem could not be found, but the necessity of the given conditions appears in [R, Theorem 4.25]. Note that condition (i) of Theorem 3.1.4 is automatically satisfied when $S$ is commutative.

**THEOREM.** *Let $S$ be a commutative Noetherian domain. Then $S$ is a maximal order (that is, $S$ is integrally closed) if and only if*

*(i) $S_p$ is a maximal order for all height-1 primes $p$ of $S$;*

*(ii) $S = \cap\{S_p : p$ a height-1 prime of $S\}$.*

**REMARK.** Condition (i) of the above Theorem is equivalent to the requirement that each $S_p$ is integrally closed, and since each $S_p$ is also local, this is the same as saying that $S_p$ is a discrete valuation ring for all height-1 primes $p$ of $S$.

**5.2.6 DEFINITION.** Let $S$ and $G$ be as in 5.2.2. For each $1 \neq g \in G$, define

$$\mathrm{I}(g) = \{s - s^g : s \in S\}S.$$

(Recall that $\mathrm{I}(g)$ was briefly introduced in 1.5.14). It is clear that each $\mathrm{I}(g)$ is an ideal of $S$. The size of the $\mathrm{I}(g)$s is crucial; the smaller they are, the larger the part of $S$ which is fixed by the action of $G$. In particular, note that $\mathrm{I}(g) = 0$ if and only if $g$ acts trivially on $S$. This is directly related to the question of whether or not certain prime ideals of $T$ are localisable, as we see in Proposition 5.2.8 and Proposition 5.2.9.

**REMARK.** Note that for a maximal ideal $M$ of a commutative Noetherian ring $R$, the set $G(M)$ defined in condition (iii)(b) of Theorem 5.1.10 is just the set

$$\{g \in G : \mathrm{I}(g) \subseteq M\},$$

where $G$ is a finite group acting on $R$. It is easy to check that, for any ideal $J$ of $R$, the subset

$$\{g \in G : \mathrm{I}(g) \subseteq J\}$$

is a subgroup of $G$; in fact it is the largest subgroup of $G$ which fixes $J$ as a set and acts trivially on $R/J$.

**5.2.7** We now turn to the main result of this chapter, the proof of which occupies the remainder of this section, and appears as 5.2.15.

**THEOREM.** *Let $S$, $G$ and $T$ be as in 5.2.2. Then $T$ is a prime maximal order if and only if*

*(a) $S$ is integrally closed;*

*(b) there exists no non-identity element $g$ of $G$ such that*

*$I(g) \subseteq p$ for some height-1 prime $p$ of $S$.*

**REMARK.** Recall from Example 1.1.2 that a commutative Noetherian domain is a maximal order precisely when it is integrally closed; Lemma 2.1.12 then implies the equivalence of condition (a) above and condition (a) of Theorem 3.2.2. The following result together with Lemma 3.2.10 shows how condition (b) above implies condition (b) of Theorem 3.2.2.

**5.2.8 PROPOSITION.** *Adopt the notation of 5.2.2 and suppose there exist no non-identity elements of $G$ acting trivially on $S$. Let $P \in Spec(T)$. Put*

$$\Psi := \{p \in Spec(S) : p \text{ is minimal over } P \cap S\}.$$

*Suppose further that $I(g) \nsubseteq p$ for all $1 \neq g \in G$, $p \in \Psi$. Then $P = (P \cap S)T$.*

**PROOF.** We know that $(P \cap S)T \subseteq P$ and that by Lemma 1.3.10 $P \cap S$ is a $G$-prime ideal of $S$. Now, $P$ lies over $P \cap S$ and using Theorem 1.3.21, INC implies that there does not exist $P_1 \in \mathrm{Spec}(T)$ with $P_1 \subset P$ and $P_1 \cap S = P \cap S$. Therefore $P$ is minimal over $(P \cap S)T$. To show that $(P \cap S)T = P$ it is enough to show that $(P \cap S)T \in \mathrm{Spec}(T)$. Consider

$$T/((P \cap S)T) \cong (S/(P \cap S)) * G;$$

we show that the latter ring is prime. It is clear that $S/(P \cap S)$ is $G$-prime. Let $p \in \Psi$. Then $p/(P \cap S)$ is a minimal prime of $S/(P \cap S)$. Put

$$G_p := \{g \in G : p^g = p\},$$

the stabiliser of $p$ in $G$. Since $I(g) \nsubseteq p$ for all $1 \neq g \in G$, no element of $G_p$ acts trivially on $S/p$. Lemma 5.2.3 gives that $(S/p) * G_p$ is prime. Now,

$$\{(S/(P \cap S)) \; / \; (p/(P \cap S))\} * G_p \cong (S/p) * G_p$$

and so by the corollary to Theorem 1.3.12, $(S/(P \cap S)) * G$ is prime. Therefore $T/((P \cap S)T)$ is prime, $(P \cap S)T \in \mathrm{Spec}(T)$ and $P = (P \cap S)T$, as required.

**5.2.9 PROPOSITION.** *Let $S$, $G$, and $T$ be as in 5.2.2, and let $P$ be a prime ideal of $T$. If $P = (P \cap S)T$, then $P$ is localisable.*

**PROOF.** Recall from Definition 1.2.12 what it means for an ideal of $S$ to have the right $AR$-property. By [Mc-R, Theorem 4.2.7], all ideals of $S$ have the right $AR$-property since $S$ is a commutative Noetherian domain. In particular, $P \cap S$ does. Then, by Lemma 5.1.11, $P = (P \cap S)T$ has the right and left $AR$-property. Using [Mc-R, Proposition 6.8.21(ii)], we see that $P$ is localisable.

**5.2.10** In view of Proposition 5.2.8, it follows immediately from Theorem 3.2.2 that conditions (a) and (b) of Theorem 5.2.7 are sufficient for $T$ to be a maximal order. We now concentrate on the necessity of these conditions.

**NOTATION.** Let $S$ be a commutative Noetherian domain and $G$ a finite group acting on $S$. As before, $T$ denotes the skew group ring $S*G$. For any height-1 prime ideal $p$ of $S$, we put

$$p_0 = \cap\{p^g : g \in G\}.$$

**Unless stated otherwise we suppose that $T$ is a prime maximal order,** so by Lemma 5.2.3 all non-identity elements of $G$ act non-trivially on $S$.

In Lemma 5.2.12 we prove that under the above hypotheses

$$T_{p_0} = S_{p_0} * G$$

is local for each $p_0$ as defined above; we use this to show that conditions (a) and (b) of Theorem 5.2.7 are both satisfied. A key tool in proving this is one of the results of Chapter 4, Corollary 4.2.8.

Let $p$ be a height-1 prime ideal of $S$. Note that $pS_{p_0}$ is a maximal ideal of $S_{p_0}$. Now, $G$ acts on $S_{p_0}$, and for all $g \in G$,

$$(pS_{p_0})^g = pS_{p_0}$$

if and only if $p^g = p$. Also, if $g \in G$ fixes $p$ and acts trivially on

$$S_{p_0}/pS_{p_0} \cong Q(S/p),$$

then $g$ acts trivially on $S/p$. As in 4.1.2 we put $H(p)$ equal to the set

$$\begin{aligned} H(pS_{p_0}) &= \{g \in G : (pS_{p_0})^g = pS_p, \ g \text{ acts as identity on } S_{p_0}/pS_{p_0}\} \\ &= \{g \in G : p^g = p, \ g \text{ acts as identity on } S/p\}. \end{aligned}$$

**5.2.11** The proof of the following lemma is straightforward, and is therefore left to the reader.

**LEMMA.** *Let $S$ be a ring and $G$ be a finite group acting on $S$. Let $I$ be a finite index set. Suppose that $\{S_i : i \in I\}$ is a collection of $G$-invariant partial quotient rings of $S$. Then*

$$\cap\{S_i * G : i \in I\} = (\bigcap_{i \in I} S_i) * G.$$

**5.2.12 LEMMA.** *Assume the notation and hypotheses of 5.2.10. Let $P$ be a height-1 prime of $T$ with $p_0 = P \cap S$, so that $p_0 = \cap\{p^g : g \in G\}$ for some height-1 prime $p$ of $S$. Put $q := P \cap Z = p_0 \cap Z$, a height-1 prime of $Z$, where $Z = \mathrm{Centre}(T)$. Then*

$$T_P = T_{p_0} = T_q.$$

*In particular, $T_{p_0}$ is a local ring.*

**PROOF.** Since $T$ is a maximal order by hypothesis, $P$ is localisable by Theorem 3.1.4. Then by Proposition 5.1.9, $P$ is the one and only prime ideal of $T$ lying over $P \cap Z = q$. It was shown in the proof of Proposition 5.1.9 that $T_P = T_q$. Therefore since $T_P$ is a local ring, to show that $T_{p_0}$ is local, it is enough to show that $T_{p_0} = T_q$.

First of all, it is shown in a similar way to the proof of Lemma 1.3.8 that

$$C_Z(q) = Z\backslash q \subseteq C_S(p_0)$$

so that, since $C_Z(q)$ is clearly an Ore set in $T$, it is possible to localise at $q$ and form the partial quotient ring $T_q$. Now consider the well-defined ring homomorphism

$$\varphi : T_q \longrightarrow Q(T/p_0T)$$

defined by

$$\varphi(tc^{-1}) = (t + p_0T)(c + p_0T)^{-1}$$

for some $t \in T$ and $c \in Z\backslash q \subseteq C_S(p_0)$. Clearly

$$\mathrm{Ker}(\varphi) = \{tc^{-1} \in T_q : t \in p_0T\} = p_0T_q,$$

an ideal of $T_q$. Therefore

$$T_q/p_0T_q \cong \mathrm{Im}(\varphi).$$

We see, in a similar way as in the proof of Proposition 5.1.9, that $\varphi$ is surjective. So, by the Isomorphism Theorem we have

$$T_q/p_0T_q \cong Q(T/p_0T) \cong T_P/p_0T_P,$$

the second isomorphism being the canonical one. Therefore

$$T_{p_0} = T_q = T_P$$

and $T_{p_0}$ is a local ring, as required.

**5.2.13 PROPOSITION.** *Assume the notation and hypotheses of 5.2.10. Let $p$ be a height-1 prime of $S$, and suppose $char(S/p) = 0$. Then condition (b) of Theorem 5.2.7 is satisfied for $p$, that is there exists no non-identity element $g$ of $G$ such that $I(g) \subseteq p$.*

**PROOF.** Since $T_{p_0}$ is local by Lemma 5.2.12, Corollary 4.2.8 implies that $H(p) = \{1\}$. It is now immediate that there exists no non-identity element $g \in G$ such that $I(g) \subseteq p$, as required.

**5.2.14** We now prove an analogous result to Proposition 5.2.13 for the case where $S/p$ has positive characteristic.

**PROPOSITION.** *Assume the hypotheses of 5.2.10. Let $p$ be a height-1 prime of $S$, and suppose that $char(S/p) = \rho > 0$. Then condition (b) of Theorem 5.2.7 is satisfied for $p$, that is no non-identity element $g$ of $G$ is such that $I(g) \subseteq p$.*

**PROOF.** Firstly, $T_{p_0}$ is a local ring by Lemma 5.2.12. Let $H = H(p)$. By Corollary 4.2.8, $H$ is a $\rho$-group. Let $1 \neq x \in H$ be such that $x$ has order $\rho$. First we claim that gl.dim.$(T_{p_0}) = \infty$. For, put

$$N := pS_{p_0},$$

a maximal ideal of $S_{p_0}$. Note that

$$S_{p_0}/N \cong Q(S/p),$$

the quotient ring of $S/p$, and that $char(S_{p_0}/N) = \rho$. Define

$$G(N) := \{g \in G : s - s^g \in N \text{ for all } s \in S_{p_0}\}.$$

We have $x \in G(N)$ and $x$ has order $\rho$. Therefore, by Theorem 5.1.10

$$\text{gl.dim.}(T_{p_0}) = \infty,$$

as claimed.

Now, $S_{p_0}$ is a semisimple Artinian ring, and so (in particular), $p_0S_{p_0}$ is a reflexive ideal. Consider

$$(p_0T_{p_0})^{**} = (p_0S_{p_0} * G)^{**} = (p_0S_{p_0})^{**} * G = p_0S_{p_0} * G = p_0T_{p_0}.$$

So $p_0T_{p_0}$ is a reflexive ideal of $T_{p_0}$, and so is contained in a maximal reflexive ideal ($M$ say) of $T_{p_0}$, which is necessarily prime. But $T_{p_0}$ is local as we pointed out at the start of the proof, so that $M$ must be $J(T_{p_0})$, the Jacobson radical of $T_{p_0}$. We are now in a position to apply Proposition 1.2.8 to give that $T_{p_0}$ is not a maximal order.

Therefore, by Lemma 5.2.12, $T_P$ is not a maximal order for some height-1 prime ideal $P$ of $T$. Then Lemma 3.1.4 gives that $T$ is not a maximal order, a contradiction to the hypotheses laid out in 5.2.10. Therefore there exists no such element $x$, and so $H = \{1\}$. Hence condition (b) of Theorem 5.2.7 is satisfied for $p$, as required.

**5.2.15 PROOF OF THEOREM 5.2.7.** As was noted in 5.2.10, the sufficiency of conditions (a) and (b) of Theorem 5.2.7 is a direct consequence of Theorem 3.2.2. For the converse, suppose that $T$ is a prime maximal order. That (a) holds follows from Lemma 3.2.7 together with Lemma 2.1.12 and Example 1.1.2. Condition (b) is then proved necessary by Proposition 5.2.13 and Proposition 5.2.14.

**5.2.16 EXAMPLE** The following example illustrates how we can compare Theorem 5.2.7 with Theorem 1.5.3 (see [B]).

Let $k$ be a field with characteristic not equal to 2, and consider

$$S := k[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

Let $G$ be the cyclic group of order 2, and let $G$ be generated by an element $g$. Suppose that $G = \langle g \rangle$ acts on $S$ as follows:

$$g : X_i \longrightarrow X_i^{-1},$$

for all $i = 1, \ldots, j$ where $1 \leqslant j \leqslant n$, and

$$g : X_i \longrightarrow X_i,$$

for $i = j + 1, \ldots, n$. We will allow $j$ to vary, and investigate for which values of $j$ $S*G$ is a prime maximal order, first of all using methods discussed in this section, and then using Brown's Theorem.

If $j = 0$, then $G$ fixes $X_i$ and $X_i^{-1}$ for all $i = 1, \ldots, n$, so that $G$ fixes the whole of $S$. By Lemma 5.2.3, this means that $S*G$ is not prime. Conversely, if $j \neq 0$ then $S*G$ is prime.

If $j = 1$, then $G$ fixes $X_2, X_3, \ldots, X_n$. Consider

$$I(g) \quad = \quad \{s - s^g : s \in S\}S \quad = \quad \langle X_1 - X_1^{-1} \rangle,$$

so that

$$I(g) \quad = \quad (X_1 - 1)(X_1 + 1)S$$

which is the intersection of two height-1 prime ideals of $S$. So by Theorem 5.2.7, $S*G$ is not a maximal order.

If $j > 1$, then

$$I(g) \quad = \quad \langle X_1 - X_1^{-1}, \ldots, X_j - X_j^{-1} \rangle,$$

which is not contained in any height-1 prime ideal of $S$. Therefore in this case, Theorem 5.2.7 gives that $S*G$ is a maximal order.

Now, let

$$A = \langle X_1, X_2, \ldots, X_n \rangle$$

and consider the semidirect product $\Gamma := A \rtimes G$ of $A$ by $G$. (Recall the definition of a semidirect product of two groups from 1.4.12). Then the skew group ring $S*G$ is equal to the ordinary group ring $k\Gamma$. Consider again when $j = 1$. Then

$$k\Gamma \supseteq \langle X_1, g\rangle \cong D_\infty,$$

the infinite dihedral group. For $i = 2,\ldots,n$,

$$X_i^{-1}X_1X_i = X_1$$

and

$$X_i^{-1}gX_i = g(gX_i^{-1}g)X_i = gX_i^{-1}X_i = g \in \langle X_1, g\rangle.$$

It follows that $\langle X_1, g\rangle$ is normal in $\Gamma$, so by Theorem 1.5.3, $S*G = k\Gamma$ is not a maximal order.

However, if $j > 1$, then $\Gamma$ contains no subgroup isomorphic to $D_\infty$ which has finite index in $\Gamma$. Therefore (as expected!), by Theorem 1.5.3, $S*G = k\Gamma$ is a maximal order.

**5.2.17 REMARK.** In 6.2.11 we give an example which shows (amongst other things) that condition (b) of Theorem 5.2.7 is not a reliable test for an ideal $p_0T$ of a skew group ring $T$ to be prime (where $p_0$ is a $G$-prime ideal of the coefficient ring $S$), when the coefficient ring is non-commutative. In other words, we show that condition (b) of Theorem 5.2.7 does not imply condition (b) of Theorem 3.2.2 when $S$ is non-commutative.

## §5.3 ADDITIONAL REMARKS.

**5.3.1** (i) Jategaonkar's Main Lemma (Theorem 5.1.2) was first proved in full in [Ja1, Lemma 2.2]. A version for one-sided Noetherian rings can be found in [Ja2, Lemma 6.1.3].

(ii) Theorem 5.1.5 was proved by B.J. Müller for semiprime ideals of FBN rings in [M, Theorem 5]. It was generalised by A.V. Jategaonkar in [Ja2, Theorem 7.3.1].

(iii) Theorem 5.1.7 is due to Müller, and appears in [M].

(iv) Theorem 5.1.10 and Lemma 5.1.11 were proved by Yi Zhong in [Y], to appear in Journal of Algebra.

**5.3.2** (i) As noted in the text, Theorem 5.2.5 is undoubtedly well-known. One direction of the proof is given in [R].

(ii) Theorem 5.2.7 is a new result.

# CHAPTER 6. NECESSARY AND SUFFICIENT CONDITIONS FOR A SKEW GROUP RING TO BE A TAME ORDER.

## §6.1 PRELIMINARY RESULTS.

**6.1.1** In this section, we give some definitions and results which will be of use in §6.2.

**6.1.2 DEFINITIONS.** Let $A$ and $B$ be rings with $A$ contained in the centre of $B$. Recall that an element $b$ of $B$ is said to be *integral* over $A$ if $b$ is the root of a monic polynomial with coefficients in $A$. The ring $A$ is then *integrally closed* in $B$ if all the elements of $B$ which are integral over $A$ belong to $A$. If every element of $B$ is integral over $A$ we say that $B$ is integral over $A$, or that $B$ is an *integral extension* of $A$.

If $A$ is a commutative domain with quotient field $B$ then $A$ is *completely integrally closed* in $B$ if, whenever we have $bq^n \in A$ for all $n \in \mathbb{N}$ where $0 \neq b$, $q \in B$, then this forces $q$ to be in $A$. Note that any completely integrally closed commutative domain is integrally closed. When $A$ is Noetherian, then we can say more.

**LEMMA.** [Mc-R, Lemma 5.3.2]. *A commutative Noetherian domain is completely integrally closed precisely when it is integrally closed.*

**6.1.3 LEMMA.** [Mc-R, Lemma 5.3.2] *Let $A$ and $B$ be rings with $A$ contained in the centre of $B$. An element $b \in B$ is integral over $A$ if and only if there exists a ring $C$ with $A \subseteq C \subseteq B$ and $b \in C$ such that $C$ is finitely generated as a module over $A$.*

**6.1.4 DEFINITION.** A *polynomial identity ring* (PI ring) is a ring $R$ for which there exists a non-zero monic (non-commutative) polynomial in a finite number of indeterminates with coefficients in $\mathbb{Z}$ which vanishes identically when computed in $R$.

**6.1.5 EXAMPLE.** Let $R$ be a commutative ring and consider the identity $f(X_1,X_2) = X_1X_2 - X_2X_1$. Then for all $r$, $s \in R$, $f(r,s) = 0$. Therefore $R$ is a PI ring.

**6.1.6 REMARK.** Any Noetherian ring finitely generated as a module over its centre is a PI ring, by [Mc-R, Corollary 13.1.13]. The converse is false, however, as the following example shows.

**6.1.7 EXAMPLE.** Let $k$ be an algebraically closed field of characteristic zero and let $S = k[Y][X;\delta]$ where $\delta = Y(d/dY)$. Put $R = S/Y^2S$. Let $\mathrm{N}(R)$ be the nilradical of $R$. Then

$$\mathrm{N}(R)^2 = (YS/Y^2S)^2 = 0$$

and

$$R/\mathrm{N}(R) \cong k[X],$$

a commutative ring. So $R$ satisfies the identity

$$(X_1X_2 - X_2X_1)^2 = 0,$$

and so is a Noetherian PI ring. The centre of $R$ is

$$(k + Y^2S)/Y^2S,$$

but $R$ is not finitely generated as a module over $(k + Y^2S)/Y^2S$.

**6.1.8** The following result is a consequence of Kaplansky's Theorem, which can be found in [Mc-R, Theorem 13.3.8], and states that a primitive PI ring of minimal degree d is a central simple algebra of dimension $(d/2)^2$ over its centre.

**THEOREM.** [Mc-R, Theorem 13.3.8]. *Any primitive Noetherian PI ring is finitely generated as a module over its centre.*

**6.1.9** Recall from §1.3 our discussion on finite normalising extensions and the relationships between certain prime ideals of a ring and a finite normalising extension of that ring. We have similar correspondences in integral extensions, as can be seen in the following result.

**THEOREM.** [Mc-R, Theorem 13.8.14]. *Let $R$ be a PI ring integral over $C$, a subring of its centre. Then the following all hold.*

*(i)* Lying Over (LO). *If $p \in Spec(C)$ then there exists $P \in Spec(R)$ with $P \cap C = p$;*

*(ii)* Going Up (GU). *If $p, q \in Spec(C)$ with $p \subset q$ and if $P \in Spec(R)$ is such that $P \cap C = p$, then there exists $Q \in Spec(R)$ with $P \subset Q$ and $Q \cap C = q$.*

*(iii)* Incomparability (INC). *If $P, Q \in Spec(R)$ with $P \subset Q$, then $P \cap C \subset Q \cap C$.*

*Suppose further that $R$ is prime and $C$ is integrally closed. Then we have:*

*(iv)* Going Down (GD). *If $p, q \in Spec(C)$ with $p \subset q$ and if $Q \in Spec(R)$ is such that $Q \cap C = q$, then there exists $P \in Spec(R)$ with $P \subset Q$ and $P \cap C = p$.*

**6.1.10 DEFINITION.** A *Krull domain* $R$ is a commutative domain which satisfies all of the following conditions:

(i) For all height-1 primes $P$ of $R$, $R_P$ is a discrete valuation ring (that is, a principal ideal domain with unique maximal ideal $PR_P$);

(ii) $R = \cap\{R_P : P \text{ a height-1 prime of } R\}$;

(iii) If $x$ is a non-zero element of $R$, then $x$ lies in only a finite number of height-1 primes of $R$.

**REMARKS.** (i) By [Mc-R, Proposition 5.1.10], a characterisation of a Krull domain is that it is a completely integrally closed commutative domain which satisfies the ascending chain condition on reflexive ideals, and so the centre of a prime Noetherian maximal order is a Krull domain.

(ii) Note that any integrally closed commutative Noetherian domain is a Krull domain.

(iii) The integral closure of a Noetherian domain is a Krull domain, by [F2, Proposition 1.3], but need not be Noetherian, as the following example shows.

**EXAMPLE.** Not all Krull domains are Noetherian; $\mathbb{Z}[X_1, X_2, \ldots]$ is an example of a non-Noetherian Krull domain.

**6.1.11 LEMMA.** *Let $S$ be a commutative Noetherian domain. Suppose that $G$ is a finite group acting non-trivially on $S$ and put $T = S*G$, the skew group ring. Then the centre of $T$ is equal to the fixed ring $S^G$.*

**PROOF.** It is clear that $S^G \subseteq \mathrm{Centre}(T)$. For the converse, let $x$ belong to the centre of $T$. Let $h$ be a non-identity element of $G$ and write

$$x = rh + \sum\{r_g g : h \neq g \in G\}$$

where $r$, $r_g \in S$ for all $g \in G$. Let $s \in S$. Then

$$sx = srh + \sum\{sr_g g : h \neq g \in G\}$$

and

$$\begin{aligned} xs &= rhs + \sum\{r_g gs : h \neq g \in G\} \\ &= rs^h h + \sum\{r_g s^g g : h \neq g \in G\}. \end{aligned}$$

Since $x$ is in the centre of $T$, $sx = xs$ and in particular the coefficient of $h$ is equal in each expression. That is, $sr = rs^h$; since $S$ is commutative we have $r(s - s^h) = 0$. Suppose $r \neq 0$. Then $s = s^h$ since $S$ is a domain. But $h$ is a non-identity element of $G$, and so must act non-trivially on $S$ by hypothesis. Therefore $s \neq s^h$. So we must have that $r = 0$ and so $x \in S$. But $x$ is in the centre of $T$, so $xg = gx$

for all $g \in G$. That is, $x = x^g$ for all $g \in G$ and so $x \in S^G$ as required.

**6.1.12 LEMMA.** *Let $S$ be a prime Noetherian ring integral over its centre $C$ and $G$ a finite group acting on $S$. Let $p$ be a height-1 prime of $C^G$ and $P_1,\dots,P_n$ all the height-1 primes of $C$ with $P_i \cap C^G = p$. Put $Q = \cap\{P_i : i = 1,\dots,n\}$. Then $S_Q = S_p$.*

**PROOF.** Note first that

$$C^G \backslash p \subseteq \mathcal{C}_C(P_i)$$

for all $i = 1,\dots,n$, and that

$$\cap\{\mathcal{C}_C(P_i) : i = 1,\dots,n\} = \mathcal{C}_C(Q)$$

by [G-W, Proposition 6.5]. Firstly, since

$$C^G \backslash p \subseteq \mathcal{C}_C(Q),$$

we have that $S_p \subseteq S_Q$. For the reverse inclusion, it is enough to show that every element of $\mathcal{C}_C(Q)$ is a unit of $S_p$. To this end, let

$$x \in \mathcal{C}_C(Q).$$

Since $Q$ is $G$-invariant,

$$x^g \in \mathcal{C}_C(Q)$$

for all $g \in G$. Consider

$$\hat{x} := \prod\{x^g : g \in G\}.$$

It is easy to see that $\hat{x} \in \mathcal{C}_C(Q)$. But $C$ is a commutative ring, so $\hat{x} \in C^G$. Observe that $Q \cap C^G = p$. Therefore

$$\hat{x} \in C^G \backslash p,$$

and so $\hat{x}$ is a unit in $S_p$. It follows that $x$ is a unit in $S_p$, and the proof is complete.

**6.1.13 LEMMA.** *Let $R$ be a commutative domain, and $\{p_1,\ldots,p_n\}$ a collection of (non-zero) prime ideals of $R$ such that $p_i \nsubseteq p_j$ for all $i, j = 1,\ldots,n$. Put $q = \cap\{p_i : i = 1,\ldots,n\}$. Then $R_q = \cap\{R_{p_i} : i = 1,\ldots,n\}$.*

**PROOF.** It is easy to see that

$$C(q) = \cap\{C(p_i) : \mathrm{i} = 1,\ldots,\mathrm{n}\}.$$

Let $x \in R_q$. Then $x = sc^{-1}$ with $s \in R$ and $c \in C(q)$. But then $c \in C(p_i)$ for all i = 1,...,n and so $x \in R_{p_i}$ for all i. Therefore

$$R_q \subseteq \cap\{R_{p_i} : \mathrm{i} = 1,\ldots,\mathrm{n}\}.$$

Conversely, suppose that $x \in R_{p_i}$ for each i. Then $x = s_i c_i^{-1}$ with $s_i \in R$ and $c_i \in C(p_i)$. Define

$$I := \{r \in R : xr \in R\}.$$

It is clear that $I$ is an ideal of $R$. Now, $c_i$ belongs to $I$ for all i = 1,...,n, so $I$ is not contained in any of the ideals $p_i$. Suppose that

$$I \subseteq \cup\{p_i : \mathrm{i} = 1,\ldots,\mathrm{n}\}.$$

Then by the Prime Avoidance Theorem (for example, [G-W, Exercise 2ZI]), $I \subseteq p_j$ for some j; a contradiction. Therefore $I$ is not contained in the union of the $p_i$. Hence there exists an element $c$ such that

$$c \in I \cap (R\backslash \cup p_i) = I \cap (\cap\{R\backslash p_i : \mathrm{i} = 1,\ldots,\mathrm{n}\})$$
$$= I \cap (\cap\{C(p_i) : \mathrm{i} = 1,\ldots,\mathrm{n}\}).$$

So $c \in I \cap C(q)$. We now have $xc \in R$ and so $x = rc^{-1}$ for some $r \in R$ and with $c \in C(q)$. Therefore we have equality and the proof is complete.

**COROLLARY.** *Let $S$ be a prime Noetherian ring integral over its centre $C$, and let $\{P_1,\ldots,P_n\}$ be a collection of (non-zero) prime ideals of $C$ such that $P_i \not\subseteq P_j$ for all $i$, $j = 1,\ldots,n$. Put $Q = \cap\{P_i : i = 1,\ldots,n\}$. Then*

$$S_Q = \cap\{S_{P_i} : i = 1,\ldots,n\}.$$

**PROOF.** Note that $C$ is a commutative domain. Let $Q$, $P_1,\ldots,P_n$ be as in the statement of the Corollary. It is easy to see that $Q$ and each $P_i$ satisfy the hypotheses of Lemma 6.1.13. Therefore

$$C_Q = \cap\{C_{P_i} : i = 1,\ldots,n\}. \quad \ldots \; (*)$$

Now, since

$$C_C(Q) = \cap\{C_C(P_i) : i = 1,\ldots,n\},$$

we have that

$$S_Q \subseteq \cap\{S_{P_i} : i = 1,\ldots,n\}.$$

But, because of (*), every unit of

$$\cap\{S_{P_i} : i = 1,\ldots,n\}$$

is also a unit of $S_Q$. Therefore we have equality, as required.

**6.1.14** In the next section, we will make use of the following results; the proofs can all be found in [Y]. The first of these is a particular case of [Y, Lemma 2.2], and we state it here in only as much generality as we need. Similarly, Theorem 6.1.16 appears in [Y] in a more general setting than we require here.

**LEMMA. [Y, Lemma 2.2]** *Let $R$ be a Noetherian ring and $G$ a finite group acting on $R$. Let $R*G$ be the skew group ring. If gl.dim.$(R*G) < \infty$, then*

$$gl.dim.(R) = gl.dim.(R*G).$$

**6.1.15 PROPOSITION.** [Y, Proposition 2.7] *Let $R$ be a ring, $G$ a finite group and $R*G$ a crossed product. Let $M$ be a maximal ideal of $R$, and put $M_0 = \cap\{M^g : g \in G\}$. Then $(R/M_0) * G$ is semisimple Artinian if and only if $(R/M) * G_M$ is semisimple Artinian, where $G_M = Stab_G(M)$.*

**6.1.16 THEOREM.** [Y, Theorem 3.2] *Let $R$ be a Noetherian PI ring with finite global dimension. Let $G$ be a finite group and $R*G$ a crossed product. Suppose that for all maximal ideals $M$ of $R$ with $char(R/M) = \rho > 0$, $(R/M) * G_M$ is semisimple Artinian, where $G_M = Stab_G(M)$. Then $R*G$ has finite global dimension.*

**6.1.17 PROPOSITION.** [Y, Proposition 2.5]. *Let $S$ be a simple ring with $char(S) = \rho > 0$. Let $G = \langle g \rangle$ be a cyclic group of order $\rho$ with $G$ inner on $S$, and $S*G$ be the skew group ring. Then the following are equivalent:*

*(i)* *$S*G$ is a simple ring;*

*(ii)* *if $v \in S$ is such that $s^g = vsv^{-1}$ for all $s \in S$, then $v^\rho \neq 1$;*

*(iii)* *$S*G$ is not isomorphic to an ordinary group ring of a cyclic group of order $\rho$ over $S$.*

## §6.2 TAME ORDERS.

6.2.1 In this section, we define tame orders and give sufficient conditions for a skew group ring of a finite group over a non-commutative ring to be a tame order. These conditions also turn out to be necessary when the coefficient ring is commutative, and this result appears as Theorem 6.2.9. But first some definitions.

**6.2.2 DEFINITION.** Let $R$ be a prime Noetherian ring with centre $C$ and quotient ring $Q$. Suppose that $C$ is a Krull domain. We say that $R$ is a *tame order over* $C$ in $Q$ precisely when

(i) $R$ is integral over $C$;

(ii) $R_p$ is hereditary for all height-1 primes $p$ of $C$;

(iii) $R = \cap\{R_p : p \text{ is a height-1 prime of } C\}$.

**REMARKS.** (i) This definition was introduced by R.M. Fossum in [F1, Page 325].

(ii) By Remark 6.1.10(i), the centre of a prime Noetherian maximal order is a Krull domain; consequently it is easy to see that any prime Noetherian maximal order which is integral over its centre is a tame order. For skew group rings, this will be clear once we have proved Theorem 6.2.9, and compared it with Theorem 3.2.2. The converse for skew group rings is not true however, and an example is given in 6.2.12.

**6.2.3 LEMMA.** *Let $R$ be a commutative Noetherian domain. Then $R$ is a tame order if and only if $R$ is a maximal order.*

**PROOF.** Suppose first that $R$ is a tame order. Then by definition $R$ is integrally closed, and so $R$ is a maximal order by Example 1.1.2.

Conversely, assume that $R$ is a maximal order. Then by Theorem 5.2.5,

$$R = \cap\{R_P : P \text{ is a height-1 prime of } R\}$$

and each $R_P$ is a maximal order. Again using Example 1.1.2 we see that $R$ is integrally closed; it follows that each $R_P$ is also integrally closed. Therefore by [R, Theorem 18.4], $R_P$ is hereditary for all height-1 primes $P$ of $R$. Hence $R$ is a tame order, as required.

**REMARK.** It is implicit in the above proof that a commutative Noetherian domain is a tame order precisely when it is integrally closed.

**6.2.4 LEMMA.** *Let $S$ be a commutative Noetherian domain and $G$ a finite group acting on $S$. Let $p$ be a height-1 prime of $S^G$ and suppose that $S$ is integrally closed. Then $S_p$ is hereditary.*

**PROOF.** Let $\Omega$ denote the set of height-1 primes of $S^G$ and consider $p \in \Omega$. There exist height-1 primes $P_1, \ldots, P_n$ (say) of $S$ with

$$P_i \cap S^G = p$$

for all i; put

$$Q = \cap\{P_i : i = 1, \ldots, n\}.$$

Then $S_Q = S_p$ by Lemma 6.1.12. Now, since $S$ is an integrally closed

Noetherian domain, it follows that $S_Q$ is an integrally closed Noetherian domain. Also, $S_Q$ is semilocal, since $Q$ is semiprime.

Let $I$ be a non-zero prime ideal of $S_Q$. Then $I = JS_Q$ for some non-zero prime ideal $J$ of $S$, and $J \subseteq P_i$ for some $i = 1,\ldots,n$ (since $P_1S_Q,\ldots,P_nS_Q$ are all the maximal ideals of $S_Q$). But $P_i$ has height one, so that $J = P_i$. Therefore $I$ is maximal, and so all prime ideals of $S_Q$ are maximal ideals. Hence by [J, Theorem 10.3], $S_Q$ is a Dedekind domain. It then follows from [Mc-R, Example 5.2.7] that $S_p = S_Q$ is hereditary.

**6.2.5 LEMMA.** *Let $S$ be a commutative Noetherian domain and $G$ a finite group acting on $S$; put $T = S*G$. Let $Q$ be a height-1 $G$-prime ideal of $S$ and suppose that $S$ is integrally closed. Then $QT$ is a semiprime ideal of $T$ if and only if $S_p*G$ is hereditary, where $p = Q \cap S^G$.*

**PROOF.** Suppose first that $QT$ is semiprime. Then

$$QT_p = Q(S_p*G)$$

is a semiprime ideal of $S_p*G$. In other words,

$$(S_p/QS_p) * G = (S_p*G)/Q(S_p*G)$$

is semisimple Artinian. Note that $QS_p$ is a $G$-invariant ideal of $S_p$. Now, Proposition 6.1.15 and Theorem 5.1.10 apply to give that $S_p*G$ has finite global dimension. Then by Lemma 6.1.14,

$$\text{gl.dim.}(S_p*G) = \text{gl.dim.}(S_p),$$

and this is less than or equal to 1 by Lemma 6.2.4, since $S$ is integrally closed. So $S_p*G$ is hereditary, as required.

Conversely, suppose that $S_p*G$ is hereditary. Then

$$\text{gl.dim.}(S_p*G) \leqslant 1$$

and so is certainly finite. By Lemma 6.1.12, $S_p = S_Q$ so that

$$J(S_p) = J(S_Q) = QS_Q = QS_p.$$

Consider

$$T_p/QT_p \cong (S_p/QS_p) * G;$$

by Theorem 5.1.10 and Proposition 6.1.15, the ring on the right hand side (and therefore also the one on the left) is semisimple Artinian. Let $\mathrm{Qn}(T/QT)$ denote the quotient ring of $T/QT$. We have

$$T_p/QT_p \cong \mathrm{Qn}(T/QT),$$

and so $T/QT$ is a semiprime ring. Therefore $QT$ is a semiprime ideal of $T$, and the proof is complete.

**6.2.6 NOTATION.** In what follows, $S$ will be a prime Noetherian PI ring integral over its centre $C$, and we assume that $C$ is integrally closed. Let $G$ be a finite group acting on $S$. As usual, $T$ denotes the skew group ring $S*G$. We also suppose that $T$ is a prime ring. Note that a sufficient condition for this is that the action of $G$ is X-outer on $S$ (see Definition 1.3.4 and Theorem 1.3.5). Recall that for any subring $R$ of $S$ the fixed ring

$$R^G = \{r \in R : r = r^g \text{ for all } g \in G\}$$

is a subring of $R$. Consider the following sets.

$$X = \{p \in \mathrm{Spec}(C^G) : \mathrm{ht}(p) = 1\}$$

$$Y = \{q \in \mathrm{Spec}(C) : \mathrm{ht}(q) = 1\}$$

$$Y_0 = \{q_0 : q_0 \text{ is a } G\text{-prime ideal of } C \text{ and } \mathrm{ht}(q_0) = 1\}$$

$$Z = \{P \in \mathrm{Spec}(S) : \mathrm{ht}(P) = 1\}$$

$$Z_0 = \{Q : Q \text{ is a } G\text{-prime ideal of } S \text{ and } \mathrm{ht}(Q) = 1\}.$$

**REMARK.** Note that $C$ is an integral extension of $C^G$, and both are commutative domains.

**6.2.7 LEMMA.** *Adopt the notation and hypotheses of 6.2.6. Then we have the following:*

*(i) For all $q \in Y$, $q \cap C^G \in X$;*

*(ii) for all $P \in Z$, $P \cap C \in Y$;*

*(iii) for all $P \in Z$, $P \cap C^G \in X$;*

*(iv) for all $q_0 \in Y_0$, $q_0 \cap C^G \in X$;*

*(v) for all $Q \in Z_0$, $Q \cap C^G \in X$.*

**PROOF.** (i) Put $p = q \cap C^G$. It is clear that $p$ is a prime ideal of $C^G$. Recall from Remark 6.2.6 that $C$ is integral over $C^G$; we can now use Theorem 6.1.9 to give that $p \in X$.

(ii) This is immediate from Theorem 6.1.9(iv).

(iii) Let $P \in Z$. By (ii), $P \cap C \in Y$. Then from (i),

$$P \cap C^G = (P \cap C) \cap C^G \in X.$$

(iv) Let $q_0$ belong to $Y_0$. Then by Lemma 2.2.3,

$$q_0 = \cap\{q^g : g \in G\}$$

for some $q \in Y$. Then by (i),

$$q^g \cap C^G \in X$$

for all $g \in G$. But $q^g \cap C^G$ is clearly $G$-invariant and so, letting $h = g^{-1}$, we have

$$q^g \cap C^G = (q^g \cap C^G)^h = q \cap C^G$$

for all $g \in G$. Therefore

$$\begin{aligned} q_0 \cap C^G &= (\cap\{q^g : g \in G\}) \cap C^G \\ &= \cap\{q^g \cap C^G : g \in G\} \\ &= q \cap C^G \\ &\in X. \end{aligned}$$

(v) Let $Q \in Z_0$. Then by Lemma 2.2.3,

$$Q = \cap\{P^g : g \in G\}$$

for some $P \in Z$. By (iii) we have that

$$P \cap C^G \in X$$

and so in a similar way to the proof of the above,

$$P^g \cap C^G = P \cap C^G$$

for all $g \in G$. Therefore

$$Q \cap C^G = P \cap C^G \in X,$$

as required.

**6.2.8 LEMMA.** *Adopt the notation and hypotheses of 6.2.6. Then $T$ is integral over $C^G$.*

**PROOF.** Since $S$ is integral over $C$ by hypothesis, and $C$ is integral over $C^G$, it follows from [P-S, Corollary 3] that $S$ is integral over $C^G$. Now, we can write

$$T = S*G = \oplus\{Sg : g \in G\}.$$

Let $sg \in Sg$ for some $g \in G$ and $s \in S$, and put $n = |G|$. Then

$$\begin{aligned}(sg)^n &= (sg)(sg)\ldots(sg) \\ &= s(s^g g)gsg\ldots sg \\ &= \ldots = ss^g(s^{gg})\ldots g^n.\end{aligned}$$

But by Lagrange's Theorem $g^n = 1$ and so $(sg)^n := r \in S$. Therefore $sg$ satisfies $X^n - r = 0$, a monic polynomial with coefficients in $S$. So $Sg$ is integral over $S$, and so is integral over $C^G$.

Let $t \in T$ and write

$$t = s_1 g_1 + \ldots + s_n g_n$$

where $s_1, \ldots, s_n \in S$ and $1 = g_1, g_2, \ldots, g_n$ are the elements of $G$. Put

$$\Gamma = C^G\langle s_1 g_1, \ldots, s_n g_n\rangle,$$

a $C^G$-subalgebra of $T$ generated by the $s_i g_i$ for $i = 1,\ldots,n$. Each monomial in $s_i g_i$ is integral over $C^G$, since each $s_i g_i$ is integral over

$C^G$ as was shown above. Therefore, by [Pr, Theorem VI.3], $\Gamma$ is finitely generated as a $C^G$-module. So $\Gamma$ is integral over $C^G$, by Lemma 6.1.3. In particular, $t$ is integral over $C^G$ and so $T$ is integral over $C^G$, as required.

6.2.9 We are now in a position to prove the main result of this chapter, which is as follows.

**THEOREM.** *Adopt the notation and hypotheses of 6.2.6 and recall that the skew group ring $T$ is prime. Consider the following conditions:*

*(a) $S$ is a tame order;*

*(b) for all $Q \in Z_0$, $QT$ is a semiprime ideal of $T$.*

*(i) If (a) and (b) both hold, then $T$ is a tame order.*

*(ii) Suppose that $S$ is commutative. If $T$ is a tame order, then (a) and (b) both hold.*

*(iii) Suppose that the order of $G$ is a unit of $S$. If (a) holds, then $T$ is a tame order.*

**PROOF.** (i) Suppose that conditions (a) and (b) both hold. Since $C^G$ is contained in the centre of $T$, it is immediate from Lemma 6.2.8 that $T$ is integral over its centre. Let $p \in X$. First we show that $S_p$ is hereditary. Since $S$ is tame by hypothesis, $S_q$ is hereditary for all $q \in Y$. Let $q_1,\ldots,q_n$ be the primes of $Y$ for which

$$q_i \cap C^G = p$$

for all i = 1,...,n, and put

$$q_0 = \cap\{q_i : i = 1,\ldots,n\}.$$

Now, note that each $S_{q_i}$ is a localisation of $S_{q_0}$. Let $M$ be a maximal ideal of $S_{q_0}$. Then by [Ba, Corollary 6.6],

$$\mathrm{pr.dim.}(M) = \sup\{\mathrm{pr.dim.}M_N : N \text{ is a maximal ideal of Centre}(S_{q_0})\}.$$

But $C_{q_0}$ is the centre of $S_{q_0}$, and the maximal ideals of this ring are $q_{1_{q_0}},\ldots,q_{n_{q_0}}$. Also, the projective dimension of any ideal of $S_{q_i}$ is zero, since each $S_{q_i}$ is hereditary for all $i = 1,\ldots,n$. Therefore by [Ba, Proposition 6.7],

$$\begin{aligned}\mathrm{gl.dim.}(S_{q_0}) &= \sup\{\mathrm{pr.dim.}(A) : A \text{ is a simple right } S_{q_0}\text{-module}\}\\ &= 1.\end{aligned}$$

So $S_{q_0}$ is hereditary. But Lemma 6.1.12 gives that $S_{q_0} = S_p$, so that $S_p$ is indeed hereditary.

Now, let $Q$ be a height-1 $G$-prime ideal of $S$; condition (b) then gives us that $QT$ is a semiprime ideal of $T$. Set

$$p = Q \cap C^G,$$

so that $p \in X$ by Lemma 6.2.7(v). It follows that

$$QT_p = Q(S_p * G)$$

is a semiprime ideal of $T_p$. Hence

$$(S_p/QS_p) * G = (S_p * G)/Q(S_p * G) = T_p/QT_p$$

is semisimple Artinian. Then Proposition 6.1.15 implies that

$$(S_p/PS_p) * G(P)$$

is semisimple Artinian, where $P$ is a height-1 prime ideal of $S$ for which

$$Q = \cap\{P^g : g \in G\}$$

and

$$G(P) = \{g \in G : P^g = P\}.$$

Now, let $P_1,\ldots,P_n$ be the height-1 primes of $S$ for which

$$P_i \cap C^G = p$$

for all $i = 1,\ldots,n$, and let

$$Q_i = \cap\{(P_i)^g : g \in G\}$$

for each i. Then each $Q_i T$ is a semiprime ideal of $T$, and by the above argument we have that

$$(S_p / P_i S_p) * G(P_i)$$

is semisimple Artinian for all $i = 1, \ldots, n$. By Theorem 6.1.16 it follows that $S_p * G$ has finite global dimension, and we are now in a position to apply Lemma 6.1.14 to give that

$$\text{gl.dim.}(S_p * G) = \text{gl.dim.}(S_p) \leqslant 1.$$

Therefore

$$T_p = S_p * G$$

is hereditary for all $p \in X$.

Now we show that $T = \bigcap\{T_p : p \in X\}$. Since $S$ is tame, we know that

$$S = \bigcap\{S_q : q \in Y\}$$

and so by Corollary 6.1.13,

$$S = \bigcap\{S_{q_0} : q_0 \in Y_0\}.$$

Then Lemma 6.1.12 gives us that

$$S = \bigcap\{S_p : p \in X\}.$$

Therefore, using Lemma 5.2.11,

$$\begin{aligned} T = S*G &= (\bigcap\{S_p : p \in X\}) * G \\ &= \bigcap\{S_p * G : p \in X\} \\ &= \bigcap\{T_p : p \in X\}. \end{aligned}$$

It remains to show that $T$ is a tame order over its centre. Let $D$ denote the centre of $T$, and $W$ the set of height-1 primes of $D$. Since $T$ is integral over $C^G$ by Lemma 6.2.8, it follows that $D$ is integral over $C^G$. So we can use Theorem 6.1.9, and in a similar way to the proof of Lemma 6.2.7 we see that for all $P \in W$, $P \cap C^G$ belongs to $X$, and for all $p \in X$ there exists an ideal $P$ in $W$ such that $P \cap C^G = p$.

Now, fix an element $P$ in $W$, and put $P \cap C^G = p$. Then $T_p$ is hereditary, as we have shown above, and $T_p \subseteq T_P$. Therefore by

[K, Proposition 1.6], $T_p$ is also hereditary. We have that $T_p$ is a hereditary prime PI ring with centre $D_p$, and so by [R-S, Theorem 3], $D_p$ is a Dedekind domain and $T_p$ is finitely generated as a module over $D_p$. Hence, by [R, Theorem 4.21]

$$T_p = \cap\{(T_p)_M : \text{M is a maximal ideal of } D_p\}$$

Let $P_1, P_2, \ldots, P_n \in W$ be all the ideals of $D$ for which $P_i \cap C^G = p$. Then each $(P_i)_p$ is a maximal ideal of $D_p$. It follows that

$$T_p = \cap\{T_{P_i} : P_i \in W \text{ and } P_i \cap C^G = p\}$$

But we already have that $T = \cap\{T_p : p \in X\}$, and so

$$T = \cap\{T_P : P \in W\}.$$

Finally, we confirm that $D$ is a Krull domain. Since $T = \cap\{T_p : p \in X\}$, it follows that

$$D = \cap\{D_p : p \in X\}.$$

Each $D_p$ is a Dedekind domain, and so is integrally closed. Therefore $D$ is integrally closed. Let $K$ be the quotient field of $D$. Since $D$ is integral over $C^G$, $D$ is the integral closure of $C^G$ in $K$. Now,

$$C^G = C \cap K^G$$

and so by [F2, Proposition 1.2], $C^G$ is a Krull domain. Then Remark 6.1.10(iii) gives us that $D$ is a Krull domain. This completes the proof of (i).

(ii) Suppose that $S$ is commutative and that $T$ is a tame order. Then $S = C$ is a commutative Noetherian domain, and by Lemma 5.2.3 and Lemma 6.1.11 the centre of $T$ is $S^G$. So we now have that $X$ is the set of height-1 primes of $S^G$, and $Y = Z$. We show first that $S$ is tame. Since $T$ is tame, we have (in particular) that $T_p = S_p{*}G$ is hereditary for all $p \in X$. This implies that $S_p$ is hereditary for all such $p$, by Lemma 6.1.14. Fix a member $p$ of $X$. Let $P_1, \ldots, P_n$ be all the height-1 primes of $S$ with

$$P_i \cap S^G = p$$

and put

$$Q = \cap\{P_i : i = 1,\ldots,n\}.$$

Then by Lemma 6.1.12, $S_Q = S_p$ and so $S_Q$ is also hereditary. Let $P \in \{P_1,\ldots,P_n\}$; then $S_P$ is a localisation of $S_Q$ and so is itself hereditary by [Mc-R, Corollary 7.4.3]. Therefore $S_P$ is hereditary for all $P \in Y$. Now we also have that

$$\begin{aligned} T = S*G &= \cap\{T_p : p \in X\} \\ &= \cap\{S_p*G : p \in X\} \\ &= (\cap\{S_p : p \in X\}) * G \end{aligned}$$

by Lemma 5.2.11. Therefore

$$S = \cap\{S_p : p \in X\}.$$

But then by Lemma 6.1.12,

$$S = \cap\{S_Q : Q \in Y_0\}.$$

Finally, Lemma 6.1.13 implies that

$$S = \cap\{S_P : P \in Y\}$$

so that $S$ is tame, as claimed.

It remains to show that condition (b) holds; let $Q$ be a height-1 $G$-prime ideal of $S$ and put $p = Q \cap S^G$. Then $S_p$ is hereditary as we saw above. Note that since $S$ is tame, $S$ is integrally closed by Remark 6.2.3. We can now apply Lemma 6.2.5 to give immediately that $QT$ is a semiprime ideal of $T$, proving (b). This completes the proof of part (ii).

(iii) Suppose now that $S$ is tame, and that the order of $G$ is a unit in $S$. We will show that hypothesis (b) of the theorem is satisfied, and then apply part (i). So, let $Q$ belong to $Z_0$; then $Q$ is a semiprime ideal of $S$ and $QT$ is an ideal of $T$. It follows that $S/Q$ is a semiprime Noetherian ring, and so [P1, Theorem 4.4], the version of

Maschke's Theorem for crossed products, gives us that

$$T/QT = (S*G)/Q*G) \cong (S/Q)*G$$

is a semiprime ring. Therefore $QT$ is a semiprime ideal of $T$, and so by part (i) $T$ is a tame order, as required.

**COROLLARY** *Let $S$ be a commutative Noetherian domain and $G$ a finite group acting non-trivially on $S$. Suppose that the order of $G$ is a unit in $S$, and let $T$ denote the skew group ring $S*G$. Then $T$ is a tame order over $S^G$ if and only if $S$ is a maximal order.*

**PROOF.** By Lemma 6.1.11, the centre of $T$ is $S^G$. Recall from Lemma 6.2.3 that $S$ is a maximal order precisely when it is a tame order. The result now follows easily from Theorem 6.2.9, parts (ii) and (iii).

**6.2.10 REMARKS.** (i) Part (iii) of Theorem 6.2.9 is a just a special case of [N-O, Theorem 3.1], which proves a similar result for a ring $R$ strongly graded by a finite group $G$, with the order of $G$ being a unit in $R$.

(ii) Comparing Theorem 6.2.9 with Theorem 3.2.2 and Theorem 5.2.7 shows that, restricting ourselves to PI rings integral over their centres, the key property which distinguishes skew group rings which are maximal orders from those which are tame orders is the *primeness* (as opposed to the *semiprimeness*) of the ideals $QT$ of $T$, where $Q$ is a height-1 $G$-prime ideal of the coefficient ring. The example given in 6.2.12 below shows that this is a genuine distinction.

(iii) In view of Theorem 3.2.2, one may be inclined to define the notion of a "$G$-tame order" ($G$ being a group, of course), whereby one would expect the analogue of Theorem 3.2.2 for tame orders to require that the coefficient ring $S$ be a "$G$-tame order". However, it is easy to check that, if we replace conditions (a) and (b) of Theorem 6.2.9 with appropriate hypotheses involving height-1 $G$-prime ideals of the centre of $S$, then this yields nothing new; what we are dealing with is merely the same situation as before. So (unfortunately) it seems unlikely that any refinement in this direction is possible.

**6.2.11 EXAMPLE.** *There is a skew group ring $T = S*G$ which satisfies the following properties:*

*(i)* *$T$ is a prime Noetherian maximal order;*

*(ii)* *$S$ and $T$ are finitely generated as modules over their centres;*

*(iii)* *condition (b) of Theorem 6.2.9 is not satisfied by $T$;*

*(iv)* *condition (b) of Theorem 3.2.2 is not satisfied by $T$;*

*(v)* *condition (b) of Theorem 5.2.7 is satisfied by $T$.*

**PROOF.** Let $k = \mathbb{Z}/2\mathbb{Z}$ and put

$$S = k[x, x^{-1}][y; d/dx],$$

the differential operator ring. Let $G = \langle g\rangle$ be a cyclic group of order 2 acting on $S$ via $s^g = xsx^{-1}$ for all $s \in S$. Now, $S$ is finitely generated as a module over its centre $k[x^2, x^{-2}, y^2]$, and so is certainly a prime Noetherian PI ring. Let

$$M = (x^2 - 1)S,$$

a $G$-invariant height-1 prime ideal of $S$. Put $\bar{S} = S/M$ and consider the skew group ring $\bar{S}*G$. Then

$$(\bar{x}g - \bar{1})^2 = (\bar{x}^2g^2 - 2\bar{x}g + \bar{1}) = \bar{x}^2g^2 - \bar{1} = \bar{x}^2 - \bar{1} = 0$$

since $k$ has characteristic 2 and $G$ has order 2. Hence $(\bar{x}g - \bar{1})$ is a non-zero central nilpotent element of $\bar{S}*G$, and so

$$(S*G)/(M*G) \cong (S/M) * G = \bar{S}*G$$

is not semisimple Artinian. Therefore $M*G$ is not a semiprime ideal of $T = S*G$, and so condition (b) of Theorem 6.2.9 does not hold. But $T$ is a maximal order, as we will now show by using Theorem 3.1.4. First we need to show that all height-1 primes of $T$ are localisable. There are two cases to consider; firstly, let $P$ be a height-1 prime ideal of $T$ such that $(x^2 - 1) \notin P$. Put $p = P \cap S$, a height-1 $G$-prime ideal of $S$ with $(x^2 - 1) \notin p$. Since $G$ is inner on $S$, $p$ is actually prime. Consider the ring

$$T/pT = (S*G)/(p*G) \cong (S/p) * G.$$

This has Artinian quotient ring $Q(S/p) * G$, by Corollary 1.3.5. Since $(x^2 - 1) \notin p$, we see that condition (ii) of Proposition 6.1.17 is satisfied. Therefore $Q(S/p) * G$ is a simple ring, so that $pT$ is a prime ideal of $T$. It follows that $pT = P$. But $p$ is generated by a central element of $S$, and so $p$ has the $AR$-property by [G-W, Theorem 11.13]. Then Lemma 5.1.11 implies that $P$ has the $AR$-property. Hence by [Mc-R, Proposition 6.8.21], $P$ is localisable.

Now let $P$ be a height-1 prime of $T$ containing $(x^2 - 1)$. Note that

$$(xg - 1)^2 = x^2g^2 - 1 = x^2 - 1 \in P,$$

so that $(xg - 1) \in P$. It is clear that $(x^2 - 1)S$ is a $G$-prime ideal of $S$, so that $S/(x^2 - 1)S$ is a $G$-prime ring. By Proposition 1.3.17,

$$((S/(x^2 - 1)S) * G$$

has a unique minimal prime, and this is

$$[(x^2 - 1)T + (xg - 1)T] / (x^2 - 1)T.$$

Hence

$$P = (x^2 - 1)T + (xg - 1)T$$

is a height-1 prime of $T$, and is the only one containing $(x^2 - 1)$. We have that $C_S((x^2 - 1)S) \subseteq C_S(0)$ and $P$ is the unique prime lying over $(x^2 - 1)S$, so that by Lemma 1.3.25, $P$ is localisable in $T$.

Now, $S$ is a maximal order, so that

$$S = \cap\{S_p : p \text{ a height-1 prime of } S\}.$$

Therefore

$$\begin{aligned} T &= S*G \\ &= \cap\{S_p : p \text{ a height-1 prime of } S\} * G \\ &= \cap\{S_{P\cap S} : P \text{ a height-1 prime of } T\} * G \\ &= \cap\{(S*G)_P : P \text{ a height-1 prime of } T\} \\ &= \cap\{T_P : P \text{ a height-1 prime of } T\}. \end{aligned}$$

It remains to show that each $T_P$ is hereditary, where $P$ is a height-1 prime of $T$. By [Y, Example 6.2], $\text{gl.dim}(T) = 2 < \infty$. Then [Mc-R, Corollary 7.4.3] gives that

$$\text{gl.dim}(T_P) \leqslant \text{gl.dim}(T) = 2.$$

Suppose that $T_P$ has global dimension equal to 2. Now, $T_P$ has Krull dimension 1, being a finite module over $D_q$ (where $D$ is the centre of $T$ and $q = P \cap D$). So it follows from [B-H, Lemma 3.2] that the projective dimension of $T_P$ as a $T_P$-module is equal to 1. But this contradicts the fact that $T_P$ is free (and so has projective dimension equal to 0). So $T_P$ cannot have global dimension 2 and hence $T_P$ is hereditary, as required. Also each $T_P$ is local with reflexive Jacobson radical (by Remark 1.2.6 and Lemma 1.2.4), and so by Proposition 1.2.8, each $T_P$ is a maximal order. Therefore $T$ is a maximal order as claimed. So this example also shows that $T$ being a prime maximal order does not imply conditions (a) and (b) of Theorem 3.2.2 when the order of $G$ is not a unit in $S$ (see Remark 3.2.2).

Recall from 5.2.6 the definition of the ideal $I(g)$ of $S$, and consider the element

$$\begin{aligned} y - y^g &= y - xyx^{-1} \\ &= y - (1 + yx)x^{-1} \\ &= y - x^{-1} + y \\ &= x^{-1} \\ &\in \mathrm{I}(g). \end{aligned}$$

So $\mathrm{I}(g)$ contains a unit of $S$, and so must be the whole of $S$. Hence condition (b) of Theorem 5.2.7 is satisfied, but as we have seen above, the ideal $M*G$ of $T$ is not even semiprime. So this example shows that condition (b) of Theorem 5.2.7 does not imply condition (b) of Theorem 3.2.2 when the coefficient ring is non-commutative (see Remark 5.2.17).

**6.2.12 EXAMPLE.** This example illustrates the fact that not all skew group rings which are tame orders are maximal orders.

Let $k$ be an algebraically closed field of characteristic not equal to 2. As in Definition 1.5.1, $\mathrm{D}_\infty$ will denote the infinite dihedral group generated by elements $a$ and $b$, with $b$ having order 2. Put $S = k\langle a\rangle$ and $G = \langle b\rangle$, the cyclic group of order 2. Then

$$T := k\mathrm{D}_\infty = S*G,$$

the skew group ring. It is clear that $S$ is tame. We use Theorem 3.2.2 together with Theorem 6.2.9 to show that $T$ is tame, but is not a maximal order. Recall from 4.1.2 the augmentation ideal $A$ of $S$; in particular, $A$ is $G$-invariant and $S/A \cong k$. Then $AT$ is an ideal of $T$, and $G$ acts as the identity on $S/A$.

Consider the ring

$$T/AT \cong (S/A)*G = (S/A)G = kG,$$

the ordinary group ring. Since $G$ is cyclic of order 2 and the characteristic of $k$ is different from 2,

$$kG \cong k \oplus k$$

as $k$-modules. Therefore $T/AT$ is semiprime, but not prime. So $AT$ is a semiprime ideal of $T$ but not a prime ideal of $T$, satisfying condition (b) of Theorem 6.2.9, but not condition (b) of Theorem 3.2.2. Hence $T$ is a tame order but is not a maximal order.

## §6.3 ADDITIONAL REMARKS.

**6.3.1** (i) The main source of information on PI rings was [Mc-R].

(ii) The result due to I. Kaplansky mentioned in 6.1.8 was first proved in 1948 in [Ka].

(iii) Theorem 6.1.9 was proved by W.D. Blair in 1973 in [Bl].

(iv) Lemma 6.1.14, Proposition 6.1.15, Theorem 6.1.16 and Proposition 6.1.17 are all taken from [Y], to appear in Journal of Algebra.

**6.3.2** (i) As mentioned in the text, tame orders were introduced by R.M. Fossum in [F1].

(ii) Lemma 6.2.8 appears as part of the proof of [N-O, Theorem 3.1].

(iii) Theorem 6.2.9 was motivated by [N-O], and is a new result.

(iv) Example 6.2.11 was based on [Y, Example 6.2].

# CHAPTER 7. CROSSED PRODUCTS OF A FINITELY GENERATED NILPOTENT GROUP AND MAXIMAL ORDERS.

## §7.1 THE MAIN RESULT.

7.1.1 In this short chapter, we consider a crossed product of a finitely generated nilpotent group over a prime Noetherian ring and discuss when such a ring is a maximal order.

7.1.2 PROPOSITION. [Mc-R, Proposition 1.5.11]. *Let $R$ be a ring and $G$ a group with normal subgroup $N$ such that $G/N$ is infinite cyclic. Then the crossed product*

$$R*G \cong (R*N)[X, X^{-1}; \sigma]$$

*for some $\sigma \in Aut(R*N)$, where $R*N$ is the crossed product of $N$ over $R$.*

**7.1.3 THEOREM.** *Let $R$ be a prime Noetherian ring and $G$ an infinite cyclic group. Suppose that $R$ is a maximal order. Then the crossed product $R*G$ is a maximal order.*

**PROOF.** Let $R$ and $G$ be as stated in the hypotheses of the Theorem. By Proposition 7.1.2, $R*G \cong R[X,X^{-1};\sigma]$ for some automorphism $\sigma$ of $R$. Now, since $R$ is a maximal order, $R[X;\sigma]$ is a maximal order by [M-R, Corollary V.2.6]. But $R[X,X^{-1};\sigma]$ is just the localisation of $R[X;\sigma]$ at the set $\{X^i : i = 1,2,\ldots\}$, and so by Lemma 3.1.6 it is also a maximal order. Therefore $R*G$ is a maximal order, as required.

**7.1.4 NOTATION AND HYPOTHESES.** Let $R$ be a prime Noetherian ring and $G$ a finitely generated nilpotent group. As usual, $T$ denotes the crossed product $R*G$. Denote by $H$ the torsion subgroup of $G$; then $G/H$ is a finitely generated torsionfree nilpotent group. Let $S$ be the crossed product $R*H$, and put $\bar{G} = G/H$. In addition, we assume that the elements of $H$ are X-outer on $R$.

**7.1.5 COROLLARY TO THEOREM 7.1.3.** *Adopt the notation and hypotheses of 7.1.4. Suppose that $S = R*H$ is a maximal order. Then $T = R*G$ is a maximal order.*

**PROOF.** Firstly, note that

$$T = R*G = (R*H) * G/H = S*\bar{G}$$

by Lemma 1.3.3. Since $H$ is X-outer on $R$, Lemma 1.3.5 implies that $S$ is a prime ring. Also, it follows from Remark 1.4.10 that $\bar{G}$ is poly-(infinite cyclic). However, by induction there is no loss in assuming $\bar{G}$ to be infinite cyclic. So we have that $T$ is a crossed product of an infinite cyclic group over a prime Noetherian maximal order, and so is itself a maximal order by Theorem 7.1.3.

**7.1.6 REMARK.** The converse to Theorem 7.1.3 is false, as the following example shows. Compare this to the fact that in a skew group ring $S*G$ with $S$ a prime Noetherian ring and $G$ a finite group acting on $S$, $S*G$ a maximal order does not imply that $S$ is a maximal order (see Theorem 3.2.2).

**EXAMPLE.** Recall Example 2.1.13. Let $R = \mathbb{Z}_{2\mathbb{Z}}$ and put

$$S = \begin{bmatrix} R & R \\ 2R & R \end{bmatrix}.$$

Now, however, we take $G = \langle g \rangle$, an infinite cyclic group acting on $S$ by conjugation by the element

$$u = \begin{bmatrix} 0 & \surd\frac{1}{2} \\ \surd 2 & 0 \end{bmatrix}$$

of $M_2(R)$. Note in particular that the element $g^2$ centralises $S$. Then $S$ is not a maximal order, as in 2.1.13. We have that the Jacobson radical of $S$, $J(S)$ is

$$J(S) = \begin{bmatrix} 2R & R \\ 2R & 2R \end{bmatrix}$$

and this is the only non-zero $G$-prime ideal of $S$. Also note that

$$S/J(S) = \begin{bmatrix} \frac{R}{2R} & 0 \\ 0 & \frac{R}{2R} \end{bmatrix} \cong \mathbb{Z}_{2\mathbb{Z}} \oplus \mathbb{Z}_{2\mathbb{Z}}.$$

It can be shown that $J(S) * G$ is prime, and so it follows that

$$(S/J(S)) * G \cong (S*G) / (J(S) * G)$$

is prime. Therefore by Theorem 3.2.2, $S*G$ is a maximal order, as required.

## §7.2 ADDITIONAL REMARKS.

**7.2.1** (i) Theorem 7.1.3 is undoubtedly well-known, although a precise reference could not be found.

(ii) Corollary 7.1.5 seems to be a new result.

## REFERENCES.


[A-B] Auslander, M. and Buchsbaum, D.A., On ramification theory in Noetherian rings, *Amer. J. Math.* 81 (1959), 749-765.

[A-R] Auslander, M. and Rim, D.S., Ramification index and multiplicity, *Illinois J. Math.* 57 (1962), 566-581.

[Ba] Bass, H., *Algebraic K-Theory*, W.A. Bergman, Inc. (New York), (1968).

[Bl] Blair, W.D., Right Noetherian rings integral over their centres, *J. Algebra* 27 (1973), 187-198.

[B] Brown, K.A., Height one primes of polycyclic group rings, *J. London Math. Soc.* (2) 32 (1985), 426-438.

[B-H] Brown, K.A. and Hajarnavis, C.R., Homologically homogeneous rings, *Trans. Amer. Math. Soc.* (281) 1 (1984), 197-208.

[B-L] Brown, K.A. and Lorenz, M., Grothendiek groups of invariant rings and of group rings, *University of Glasgow Preprint* (1991), to appear in *Proc. London Math. Soc.*.

[B-M-S] Brown, K.A., Marubayashi, M. and Smith, P.F., Group rings which are *v–HC* orders and Krull orders, *Proc. Edinburgh Math. Soc.* 34 (1991), 217-228.

[C] Chamarie, M., Anneaux de Krull non commutatifs, *J. Algebra* 72 (1981), 210-222.

[C-S1] Cozzens, J.H. and Sandomierski, F.L., Reflexive primes, localisation and primary decomposition in maximal orders, *Proc. Amer. Math. Soc.* 58 (1976), 44-50.

[C-S2] Cozzens, J.H. and Sandomierski, F.L., Maximal orders and localisation I, *J. Algebra* 44 (1987), 319-338.

[E-R] Eisenbud, D. and Robson, J.C., Hereditary Noetherian prime rings, *J. Algebra* 16 (1970), 86-104.

[F-J] Formanek, E. and Jategaonkar, A.V., Subrings of Noetherian rings, *Proc. Amer. Math. Soc.* 46 (1974), 181-186.

[F1] Fossum, R.M., Maximal orders over Krull domains, *J. Algebra* 10 (1968), 321-332.

[F2] Fossum, R.M., *The Divisor Class Group Of A Krull Domain*, Springer-Verlag (Berlin), (1973).

[F-S] Fuller, K.R. and Shutters, W.A., Projective modules over non-commutative semilocal rings, *Tôhoku Math. Journal* (2) 27 (1975), 303-311.

[G] Goldie, A.W., Localisation in non-commutative Noetherian rings, *J. Algebra* 5 (1967), 89-105.

[G-W] Goodearl, K.R. and Warfield, R.B.,Jr., *An Introduction To Noncommutative Noetherian Rings*, L.M.S. Student Texts 16, C.U.P. (Cambridge), (1989).

[H-L] Hajarnavis, C.R. and Lenagan, T.H., Localisation in Asano orders, *J. Algebra* 21 (1972), 441-449.

[H-W] Hajarnavis, C.R. and Williams, S., Maximal orders in Artinian rings, *J. Algebra* 90 (1984), 375-384.

[H-R] Heinicke, A.G. and Robson, J.C., Normalising extensions: Prime ideals and incomparability, *J. Algebra* 72 (1981), 237-268.

[H] Hodges, T.J., Equivariant *K*-theory for Noetherian rings, *J. London Math. Soc.* (2) 39 (1989), 414-426.

[J] Jacobson, N., *Basic Algebra II*, Freeman (New York), (1980).

[Ja1] Jategaonkar, A.V., Solvable Lie algebras, polycyclic-by-finite groups and bimodule Krull dimensions, *Comm. In Algebra* 10 (1982), 19-69.

[Ja2] Jategaonkar, A.V., *Localisation In Noetherian Rings*, L.M.S. Lecture Note Series 98, C.U.P. (Cambridge), (1986).

[Ka] Kaplansky, I., Rings with a polynomial identity, *Bull. Amer. Math. Soc.* 54 (1948), 575-580.

[K] Kuzmanovich, J., Localisations of Dedekind prime rings, *J. Algebra* 21 (1972), 378-393.

[L-P1] Lorenz, M. and Passman, D.S., Integrality and normalising extensions of rings, *J. Algebra* 61 (1979), 289-297.

[L-P2] Lorenz, M. and Passman, D.S., Prime ideals in crossed products of finite groups, *Israel J. Math.* 33 (1979), 89-132.

[L-P3] Lorenz, M. and Passman, D.S., Addendum - Prime ideals in crossed products of finite groups, *Israel J. Math.* 35 (1980), 311-322.

[Mc-R] McConnell, J.C. and Robson, J.C., *Noncommutative Noetherian Rings*, Wiley-Interscience (Chichester), (1987).

[Ma] MacDonald, I.D., *The Theory Of Groups*, Oxford University Press (Oxford), (1968).

[Mar] Martindale, W.S.,III, Prime rings satisfying a generalised polynomial identity, *J. Algebra* 12 (1969), 576-584.

[M-R] Maury, G. and Raynaud, J., *Ordres Maximaux Au Sens De K. Asano*, Springer-Verlag 808 (Berlin), (1980).

[Mi] Michler, G.O., Asano orders, *Proc. London Math. Soc.* 19 (1969), 421-443.

[M] Müller, B.J., Localisation in fully bounded Noetherian rings, *Pacific J. Math.* 67 (1976), 233-245.

[N-O] Nauwelaerts, E. and van Oystaeyen, F., Finite generalised crossed products over tame and maximal orders, *J. Algebra* 101 (1986), 61-68.

[P-S] Paré, R. and Schelter, W., Finite extensions are integral, *J. Algebra* 53 (1978), 477-479.

[P1] Passman, D.S., *Infinite Crossed Products*, Academic Press (London), (1989).

[P2] Passman, D.S., *The Algebraic Structure Of Group Rings*, Wiley-Interscience (New York), (1977).

[P3] Passman, D.S., Prime ideals in enveloping rings, *Trans. Amer. Math. Soc.* 302 (1987), 535-560.

[Pr] Procesi, C., *Rings With Polynomial Identities*, Marcel Dekker, Inc. (New York), (1973).

[R] Reiner, I., *Maximal Orders*, Academic Press (London), (1975).

[Ro] Robinson, D.J.S., *Finiteness Conditions And Generalised Soluble Groups (Part 1)*, Spinger-Verlag (Berlin), (1972).

[Rob] Robson, J.C., Non-commutative Dedekind rings, *J. Algebra* 9 (1968), 249-265.

[R-S] Robson, J.C. and Small, L.W., Hereditary prime PI rings are classical hereditary orders, *J. London Math. Soc.* (2) 8 (1974), 499-503.

[S] Smith, P.F., Some examples of maximal orders, *Math. Proc. Camb. Phil. Soc.* 98 (1985), 19-32.

[Y] Yi, Z., Homological dimension of skew group rings and crossed products, *University of Glasgow Preprint* (1992), to appear in *J. Algebra*.