RINGS OF ENDOMORPHISMS

by

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SUMMARY

This dissertation reviews some results about rings of endomorphisms of modules, mainly in the form "if a module has the property $\mathcal{P}$ then its ring of endomorphisms has the property Q".

After an introductory Chapter 0, Chapter 1 is devoted to develop some concepts that will be necessary later on; a detailed study of the uniform (Goldie) dimension of a module is carried out and, in this vein, some original results of the author, which will appear elsewhere, are included in Section 4.

In Chapter 2 we present the endomorphism ring of a module as well as a general technique for its study (Sections 5 and 6). The modules whose rings of endomorphisms have been reviewed are detailed next.

In Section 7, injective and quasi-injective modules are considered; it is shown that the factor ring of their endomorphism ring modulo its radical is a regular and (right) self-injective ring.

In Section 8, projective modules are discused; the Morita Theorem is recollected and some properties of a ring which are inherited by the endomorphism rings of its finitely generated projective modules are stated; also, a study of the projective modules with local endomorphism rings is done.

In Section 9, we consider finite dimensional modules. First they are assumed to be also injective and, after dropping this hypothesis, we study the nilpotency of the nil subrings of their rings of endomorphisms; we also answer some questions about the quotient ring of the endomorphism ring of a finite dimensional nonsingular module.

Finally, in Section 10, we look at what happens when the module is assumed to satisfy some chain conditions, in general at a first stage and under the hypothesis of quasi-injectivity or quasi-projectivity in the final paragraph of the dissertation.

CONTENTS

PREFACE AND ACKNOWLEDGEMENTS

This dissertation consists of two quite different chapters. The first of them intends to provide the basic tools of Module Theory that shall be used in the second, and that are usually beyond the scope of a very elementary course in Rings and Modules. There has been assumed some knowledge about direct sums and products, homomorphisms, isomorphism theorems, exact sequences, injectivity, projectivity and chain conditions, while concepts like essential submodules and extensions, complement submodules, singular and nonsingular modules, injective hulls or quasi-injective modules are introduced in some detail.

Particular emphasis is made in the topic of finite (Goldie) dimension, leading to a Section 4 in which the dimension of a sum of finite dimensional modules is studied. This section should be viewed as an Appendix to Chapter 1 and may be omitted without consequences for Chapter 2.

The second chapter deals with the topic announced in the title: the ring of endomorphisms (or endomorphism ring) of a module. It is assumed that the reader is familiar with concepts such as local, simple, semisimple, Artinian or Noetherian rings, Jacobson radical, idempotent elements, nilpotent ideals and subrings, factor rings or lifting of idempotents. Other ideas are introduced here although at times, in order to keep our attention in the main subject of this work, some strong results of Ring Theory are just quoted, e.g. some facts about perfect and semiperfect rings in Section 8, Goldie's theorems in Section 9 or the Hopkins-Levitzki Theorem in Section 10.

The rings of endomorphisms are introduced along with some easy results which show how the structure of a module determines that of its ring of endomorphisms. This is the central idea of this dissertation, namely the search for theorems of the form "if a module has property $\mathcal{P}$ then its endomorphism ring has property Q". In fact, what is done here is a review of the results of that kind which already existed in the literature in case the module is injective, quasi-injective, projective or finite dimensional, or satisfy some chain conditions.

A very general technique to find results of that type (the "correspondence theorems") is then presented; some of the results here will prove very helpful in the following sections.

Next, the endomorphism ring of a quasi-injective module, or rather its factor ring modulo the radical, is studied. This is followed by a quick review of the Morita Theorem, and some classes of projective modules whose rings of endomorphisms have nice properties are briefly introduced.

Injective modules are then revisited, now taking into account their dimension; later, nil subrings and quotient rings of the endomorphism ring of a finite dimensional module are studied. Finally, we state some results about Artinian, Noetherian and finite-length modules, and also about quasi-injective and quasi-projective modules with certain chain conditions.

Throughout the dissertation, an effort to pay tribute to the parents of the ideas which appear there has been made. The references do not necessarily mean that we have followed the proofs given in the quoted paper but that, to the best of the author's knowledge, that is the first time such a result appeared in the literature. Some papers which are not referred to in the text, but contain material which aided in the preparation of this work, have been included after the main text under the common label of List of References. Also, an index with the concepts assumed and defined through the dissertation is provided.

Finally, the author would like to express his gratitude to his supervisor, Professor P.F.Smith, for his guidance and for suggesting the topic of the dissertation, as well as to the British Council and the Caja de Ahorros del Mediterráneo (Spain) for their efficient collaboration in preparing and financially supporting him in this last study's year.

Alberto del Valle Robles. Glasgow, September of 1992.

CHAPTER 0

NOTATION AND CONVENTIONS

Throughout this dissertation, by a ring R we will mean an associative ring with identity 1_R (or 1 if there is no risk of confusion about the ring), and all modules will be unitary (i.e. the product of an element x of the module by the identity of the ring equals x). The following right-sided conventions will also stand in their left-sided form.

The category of all (unitary) right R-modules will be denoted by Mod_R ($_RMod$ for left R-modules). $M=M_R$ will mean that M is an object of Mod_R. Given $M=M_R$ and $N=N_R$, the notation $f:M_R \longrightarrow N_R$ will imply that f is a morphism in Mod_R (i.e., a right R-homomorphism), while $f:M \longrightarrow N$ shall be viewed as a set theoretical map, unless otherwise specified. All morphisms in the categories Mod_R and $_RMod$ will be written in the side opposite to the scalars (i.e., given $f:M_R \longrightarrow N_R$ and $g:{}_RL \longrightarrow {}_RK$, the images of $x \in M$ and $y \in L$ will be $f(x)$ and $(y)g$, or more often fx and yg). In the same way, the image of a submodule $P \subseteq M_R$ will be written f(P) or fP.

Given $M=M_R$ and $N=N_R$ we will denote by $Hom_R(M,N)$, or by $Hom(M_R,N_R)$ if we want to emphasize the side, the set of all right R-homomorphisms from M into N. If $f,g \in Hom_R(M,N)$ then the map $f+g:M \longrightarrow N$ defined via $(f+g)x=fx+gx$ for all $x \in M$, is actually in $Hom_R(M,N)$, and this 'sum of homomorphisms' provides $Hom_R(M,N)$ with the structure of Abelian group (with the zero map as zero element) which shall be assumed in the sequel. In case M=N, we call an element of $Hom_R(M,M)$ an *endomorphism* of M, and write $End_R(M)$ or $End(M_R)$ for $Hom_R(M,M)$.

Given two rings T,R, a (T,R)-*bimodule* is an Abelian group M which is both a left T-module and a right R-module in such a way that, for all $t \in T$, $r \in R$ and $x \in M$, the equality $(tx)r=t(xr)$ is satisfied. We denote this situation by $M={}_TM_R$, and the category of (T,R)-bimodules by $_TMod_R$.

For bimodules $_QM_R$ and $_TN_R$, it is well known that $Hom(M_R,N_R)$ is an object of $_TMod_Q$. Similarly, for $_RA_Q$ and $_RB_T$, $Hom(_RA,{}_RB)$ is in $_QMod_T$.

According to our notation, we will write ${}_T\mathrm{Hom}(M_R,N_R)_Q$ and ${}_Q\mathrm{Hom}({}_RA,{}_RB)_T$. Note that every module M_R (${}_RN$) can be realized as a bimodule ${}_{\mathbb{Z}}M_R$ (${}_RN_{\mathbb{Z}}$), where $\mathbb{Z}$ is the ring of rational integers; therefore, given e.g. ${}_QM_R$ and N_R, we have $\mathrm{Hom}(M_R,N_R)_Q$, and so on.

The symbols $\subseteq$ and $\subset$ will mean inclusion and strict inclusion, respectively. If $M=M_R$, the fact that N is an R-submodule of M will be abbreviated as $N\leq M_R$, while $N\subseteq M$ shall be viewed as a set inclusion. Therefore, $\mathfrak{a}\leq R_R$ will mean that $\mathfrak{a}$ is a right ideal of R. By '$\mathfrak{a}$ is an ideal of R' (without further specification) we will understand '$\mathfrak{a}$ is a two-sided ideal of R'.

For a module M_R, the lattice of submodules of M ordered by inclusion will be denoted by $\mathrm{Lat}(M_R)$. Then $\mathrm{Lat}(R_R)$ ($\mathrm{Lat}({}_RR)$) will stand for the lattice of right (left) ideals of R. Many times, we will speak about chain conditions in a subset Ω of $\mathrm{Lat}(M_R)$ or $\mathrm{Lat}(R_R)$, as for example when we say 'M_R has the descending chain condition (always abbreviated DCC) on complements'; this will mean that the subset Ω of $\mathrm{Lat}(M_R)$ consisting of all complement submodules in M_R satisfies the minimum condition: i.e., every nonempty subset of Ω contains a minimal element or, equivalently, every strictly descending chain of elements of Ω must be finite. Of course, a similar convention stands for the ascending chain condition, or ACC.

Recall that, for a module M_R, the lattice $\mathrm{Lat}(M_R)$ is *modular*, i.e. $N+(L\cap K) = (N+L)\cap K$ whenever N, L, K are submodules of M_R such that $N\leq K$. This *Modular Law* will be used without further reference.

Given $N\leq M_R$ and a nonempty subset X of M, we write $(N:X)$ for the right ideal $\{r\in R:\ xr\in N \text{ for all } x\in X\}$ of R. If X is a singleton $X=\{x\}$, then we write $(N:x)$. If, for example, M is a bimodule ${}_SM_R$, we avoid any confusion by writing $(N:_RX)$ or $(N:_SX)$. If N is the zero submodule then $(0:_RX)$ is usually called the (right) *annihilator* of X in R and written $\mathbf{r}_R(X)$; similarly, $(0:_SX)$ is called the (left) annihilator of X in S, and we write $\mathbf{l}_S(X)$.

If $M=R_R$ then the annihilator ideals of nonempty subsets V of R will be simply called the *right annihilator ideals* of R, and we shall write $\mathcal{R}(V)$ for $\mathbf{r}_R(V)$. Similarly $\mathcal{L}(V)$ will stand for the left annihilator ideal $\mathbf{l}_R(V)$.

Also, for a bimodule ${}_S M_R$, the annihilators in M of nonempty subsets V of R and W of S will be considered, and our notation will be $\mathbf{l}_M(V)=\{m\in M:\ mr=0 \text{ for all } r\in V\}$ and $\mathbf{r}_M(W)=\{m\in M:\ sm=0 \text{ for all } s\in W\}$.

If $N\subseteq M_R$ is a direct summand of M (i.e. if there exists $L\subseteq M_R$ such that $M=N\oplus L$) then we write $N\subseteq_d M$. If M can be decomposed as $M=\bigoplus_{i\in I} M_i$ and $j\in I$ then $f:\bigoplus_{i\in I} M_i \xrightarrow{c} M_j$ will mean that f is the canonical projection of M on M_j for the given decomposition (i.e., if $x=\sum_{i\in F} x_i$, for some finite subset F of I and some $0\neq x_i\in M_i$, is the unique expression of x in $\oplus M_i$, then $f(x)=x_j$ if $j\in F$ and $f(x)=0$ otherwise). Similarly, if $N\subseteq M_R$, then $p:M\xrightarrow{c} M/N$ should be read as 'p is the natural epimorphism of M onto M/N (i.e., $p(x)=x+N$ for all $x\in M$).

Finally, a family $\{M_i : i\in I\}$ of submodules of a module M_R will be said to be *independent* if the sum $\sum_{i\in I} M_i$ is direct, i.e. if, for any two nonempty finite subsets J and K of I, we have $(\sum_{j\in J} M_j)\cap(\sum_{k\in K} M_k)=0$.

CHAPTER 1
GOLDIE DIMENSION

SECTION 1: ESSENTIAL EXTENSIONS AND COMPLEMENT SUBMODULES

In this first section, we introduce several concepts which will be used throughout this dissertation, and establish their first properties. The basic concepts are the essential extensions and complement submodules of a module and the nonsingular modules.

Essential Submodules and Essential Extensions

The concept of essentiality was introduced by R.E.Johnson [26; p.891] in the early fifties, although the terminology is due to B.Eckmann and A.Schopf [12]. Given two right R-modules $N \subseteq M_R$, N is said to be an *essential submodule* of M if N has nonzero intersection with each nonzero submodule of M (A.W.Goldie's terminology [19] was N *meets all submodules of* M). We denote this situation by $N \subseteq_e M$. If $N \subseteq_e M$ and $N \subset M$, then we write $N \subset_e M$ and say that N is a *proper* essential submodule of M. If $M = R_R$ we call an essential submodule of R_R an *essential right ideal* of R.

Clearly, if the zero submodule is an essential submodule of M_R then M itself must be zero. Also, if N is an essential direct summand of M_R then N=M.

If $N \subseteq_e M$ then we also say that M is an *essential extension* of N though, as C.Faith points out in [FA73; p.168], it might well be called an *inessential extension*. In Section 3 we shall slightly generalize our concept of essential extension, but for the other sections the definition given here will suffice.

Next, we give the following useful characterization of essential extensions:

LEMMA 1.1 N *is an essential submodule of* M *if and only if, for all* $x\in M\setminus N$, *there exists* $r\in R$ *such that* $0\neq xr\in N$. *In this case, for all* $x\in M$, $(N:x)\subseteq_e R_R$.

PROOF: If $N\subseteq_e M$ and $x\in M\setminus N$ then, in particular, $x\neq 0$, whence $xR\neq 0$ and thus $xR\cap N\neq 0$. Conversely, if there exists $0\neq L\subseteq M_R$ such that $N\cap L=0$ then, for any $x\in L\setminus N\neq\emptyset$, $xR\cap N=0$.
Assume now that $N\subseteq_e M$, and let $r\in R\setminus(N:x)$; then $xr\notin N$ and hence there exists $s\in R$ such that $0\neq(xr)s\in N$; then $0\neq rs\in(N:x)$ and thus, by the first part, $(N:x)\subseteq_e R_R$. ■

Then, for example, it is clear that $\mathbb{Z}\subseteq_e\mathbb{Q}_\mathbb{Z}$ (or, more generally, every commutative domain is essential in its field of fractions).

The following lemma states, among some other useful properties of essential extensions, that the set of essential submodules of M_R is a filter in the lattice $Lat(M_R)$.

PROPOSITION 1.2 *Let* M_R *be a module with submodules* N,L; $N_1,\dots,N_n$; $L_1,\dots,L_n$; *and let* $f:K_R\longrightarrow M_R$ *be any homomorphism. Then*

a) *if* $N\subseteq L$, *then* $N\subseteq_e M$ *if and only if* $N\subseteq_e L$ *and* $L\subseteq_e M$;

b) *if* $N\subseteq_e N_1$ *and* $L\subseteq_e L_1$, *then* $N\cap L\subseteq_e N_1\cap L_1$; *in particular, if* $N\subseteq_e M$ *and* $L\subseteq_e M$ *then* $N\cap L\subseteq_e M$.

c) *if* $N\subseteq L$ *and* $(L/N)\subseteq_e(M/N)$, *then* $L\subseteq_e M$;

d) *if* $N\subseteq_e M$ *then* $f^{-1}(N)\subseteq_e K$;

e) *if* $N_i\subseteq_e L_i$ *for* $i=1,\dots,n$ *and the sum* ΣN_i *is direct, then* ΣL_i *is also direct and* $(\oplus N_i)\subseteq_e(\oplus L_i)$.

PROOF: a) Assume $N\subseteq_e M$; then for all $0\neq A\subseteq M$, $L\cap A\supseteq N\cap A\neq 0$, whence $L\subseteq_e M$; and for all $0\neq A\subseteq L$, $N\cap A\neq 0$, whence $N\subseteq_e L$. Conversely, if $N\subseteq_e L$, $L\subseteq_e M$ and $A\subseteq M_R$ verifies $A\cap N=0$, then $N\cap(A\cap L)=0$, whence $A\cap L=0$ and thus $A=0$; therefore $N\subseteq_e M$.

b) Let $0\neq A\subseteq N_1\cap L_1$; since $N\subseteq_e N_1$, $0\neq N\cap A$ and then, since $L\subseteq_e L_1$, $L\cap(N\cap A)=(L\cap N)\cap A\neq 0$; therefore $L\cap N\subseteq_e N_1\cap L_1$.

c) Assume $(L/N)\subseteq_e(M/N)$ and let $0\neq A\subseteq M_R$; if $A\subseteq N$ then $A\cap L=A\neq 0$; if $A\nsubseteq N$ then $\frac{A+N}{N}\neq 0$, whence $\frac{A+N}{N}\cap\frac{L}{N}\neq 0$, i.e. $N\subset(A+N)\cap L=(A\cap L)+N$, and thus $A\cap L\neq 0$; therefore $L\subseteq_e M$.

d) This is very easily proved using (1.1); however, in order to obtain a dual proof for the next result, we proceed as follows: Assume first that f is monic; then if $A \subseteq K_R$ is such that $f^{-1}N \cap A=0$, we get

$$0= f(f^{-1}N \cap A)= ff^{-1}N \cap fA= (N \cap fK) \cap fA= N \cap fA,$$

whence fA=0 and thus A=0, proving that $f^{-1}N \subseteq_e K$.

In general, since $g: (K/\mathrm{Ker} f) \longrightarrow M$ given by $g(k+\mathrm{Ker} f)=fk$ is monic, we get $g^{-1}N=\frac{f^{-1}N}{\mathrm{Ker} f} \subseteq_e \frac{K}{\mathrm{Ker} f}$ and hence, by c), $f^{-1}N \subseteq_e K$.

e) Since the case n=2 is easily extended to any finite number of submodules, we prove that $N \subseteq_e L$, $N' \subseteq_e L'$ and $N \cap N'=0$ implies $L \cap L'=0$ and $N \oplus N' \subseteq_e L \oplus L'$; by b), $0=N \cap N' \subseteq_e L \cap L'$, whence $L \cap L'=0$; consider now the projections $\pi: L \oplus L' \longrightarrow L$ and $\rho: L \oplus L' \longrightarrow L'$; by d), $\pi^{-1}N=N \oplus L' \subseteq_e L \oplus L'$ and $\rho^{-1}N'=L \oplus N' \subseteq_e L \oplus L'$, and thus b) gives $N \oplus N' \subseteq_e L \oplus L'$.■

REMARK: (1.2.e) also holds for infinite direct sums [G; Prop.1.4].

There is a dual concept for essentiality which is convenient to introduce here, though it will not be used until the second chapter. A submodule N of M_R is said to be *small* or *superfluous* in M if the only submodule L of M which verifies N+L=M is M itself. Our notation for this situation is $N \ll M$. Dual to (1.2) we have:

PROPOSITION 1.3 *Let* M_R *be a module with submodules* N,L; $N_1,\dots,N_n$; $L_1,\dots,L_n$; *and let* $f: M_R \longrightarrow K_R$ *be any homomorphism. Then*

a) *if* $N \subseteq L$ *then* $L \ll M$ *if and only if* $N \ll M$ *and* $(L/N) \ll (M/N)$;

b) *if* $N \ll M$ *and* $L \ll M$ *then* $N+L \ll M$;

c) *if* $N \subseteq L$ *and* $N \ll L$, *then* $N \ll M$;

d) *if* $N \ll M$ *then* $f(N) \ll K$;

e) *if* $N_i \ll L_i$ *for* i=1,...,n *and the sum* ΣL_i *is direct then* $\oplus N_i \ll \oplus L_i$.■

There are modules which possess submodules which are at the same time essential and superfluous, for example any nontrivial subgroup of the quasi-cyclic group $\mathbb{Z}(p^{\infty})$ (for any prime integer p), or the (two-sided) maximal ideal of any local ring which is not a division ring.

On the other hand, some modules M_R have no essential (resp. superfluous) submodules other than M (resp. 0). These are precisely the semisimple modules (resp. the modules with zero radical). This will follow immediately from Proposition 1.5, but before proving it we need to introduce the concept of relative complement.

Complement Submodules

Given a module M_R and a submodule $N \subseteq M_R$, the set $\Omega=\{L \subseteq M_R : L \cap N=0\}$ is clearly inductive and nonempty ($0 \in \Omega$). Any maximal element of Ω is said to be a *relative complement* for N in M. This concept is reminiscent of the set-theoretical concept of complement subset and has an obvious generalization to arbitrary lattices.

Note that, if $N,K \subseteq M_R$ verify $N \cap K=0$, then $\Omega'=\{L \subseteq M_R: K \subseteq L \text{ and } L \cap N=0\}$ is also inductive and nonempty, so that we can take a relative complement for N which contains K. The following result implies the remarkable fact, essentially proved by R.E.Johnson [26], that every submodule of a module M_R is a direct summand of an essential submodule of M_R.

<u>PROPOSITION 1.4</u> *Let* $N \subseteq M_R$*; if* L *is a relative complement for* N *in* M *then* $N \oplus L$ *is an essential submodule of* M.

PROOF: Since $N \cap L=0$, the sum N+L is direct. Suppose now that $K \subseteq M_R$ is such that $(N \oplus L) \cap K=0$; then $(N \oplus L)+K=N \oplus L \oplus K$, whence $N \cap (L \oplus K)=0$ and then, by maximality of L, we get $L \oplus K=L$, i.e. K=0. ■

<u>PROPOSITION 1.5</u> *For any module* M_R*, the socle of* M *is the intersection of all essential submodules of* M*, and the radical of* M *is the sum of the superfluous submodules of* M.

PROOF: Write SocM and RadM for the socle and the radical of M, (i.e. the sum of all simple submodules of M and the intersection of all maximal submodules of M, respectively); and set

$$A=\cap\{N : N \subseteq_e M\} \qquad B=\Sigma\{N : N \ll M\}.$$

For any simple submodule S of M, and for any $N \subseteq_e M$, we have $0 \neq N \cap S \subseteq S$, whence $N \cap S=S$, i.e. $S \subseteq N$; therefore $\mathrm{Soc}M \subseteq A$. On the other hand, let $L \subseteq A_R$, and let L′ be a relative complement for L in M; then, by (1.4), $L \oplus L' \subseteq_e M$ and hence $A \subseteq L \oplus L'$; by modularity, $L \oplus (A \cap L')=A \cap (L \oplus L')=A$, which proves that every submodule of A is a direct summand of A, i.e. A is

semisimple and hence $A\subseteq SocM$. Therefore $A=SocM$, as desired.
For any maximal submodule L of M, and for any $N\ll M$, we have $N\subseteq N+L\subset M$, whence $N=N+L$, i.e. $L\subseteq N$; therefore $B\subseteq RadM$. To see that $RadM\subseteq B$, we prove that xR is superfluous for all $x\in RadM$; if xR is not superfluous and $N\subset M_R$ is such that $N+xR=M$, then clearly $x\notin N$ and thus, by Zorn's Lemma, $\Delta=\{K\subseteq M_R\colon N\subseteq K \text{ and } x\notin K\}$ has a maximal element K_o, which is in turn a maximal submodule of M, since

$$K_o\subset L\subseteq M \;\Rightarrow\; L\notin\Delta \;\Rightarrow\; x\in L \;\Rightarrow\; M= N+xR\subseteq K_o+xR\subseteq L \;\Rightarrow\; L=M;$$

therefore, since $x\notin K_o$, $x\notin RadM$. This completes the proof.∎

By a *complement* in M we will mean any submodule N of M which is a relative complement in M for some submodule of M. In this case we write $N\subseteq_c M$. If $M=R_R$ then we call a complement submodule of R_R a *right complement* in R. For example, every direct summand N of M is a complement in M (if $M=N\oplus L$, then N is a relative complement for L in M). There exists a close relationship between the concepts of complement and essentiality, as the next result shows (in fact, some authors call complements *closed* or *essentially closed* submodules because of the equivalence a)⇔b)).

PROPOSITION 1.6 *Let* $N\subseteq M_R$. *Then the following are equivalent:*

a) N *is a complement in* M;

b) N *does not admit proper essential extensions within* M;

c) *for any* $L\subseteq M_R$ *such that* $N\subseteq L\subseteq_e M$, $(L/N)\subseteq_e(M/N)$.

PROOF: a)⇒b) Assume that N is a relative complement for some $K\subseteq M_R$, and suppose $N\subseteq_e L\subseteq M$. Since $(L\cap K)\cap N=K\cap N=0$, we get $L\cap K=0$ and then, by maximality of N, it must be N=L; this proves b).

b)⇒c) Assume $N\subseteq L\subseteq_e M$, and suppose that K is such that $N\subseteq K\subseteq M_R$ and $(L/N)\cap(K/N)=0$; then $N=L\cap K\subseteq_e M\cap K=K$ (1.2.b), whence N=K by b), i.e. $K/N=0$; therefore $(L/N)\subseteq_e(M/N)$.

c)⇒a) Let K be a relative complement for N in M; we prove that N is a relative complement for K in M. Since $N\cap K=0$, we can find a complement N′ for K in M with $N\subseteq N'$; then, by modularity, $(N\oplus K)\cap N'=N\oplus(K\cap N')=N$; since $N\subseteq N\oplus K\subseteq_e M$ by (1.4), the hypothesis gives $\frac{N\oplus K}{N}\subseteq_e\frac{M}{N}$, but the previous argument gives $\frac{N\oplus K}{N}\cap\frac{N'}{N}=0$, so that $N=N'$.∎

COROLLARY 1.7 *Let* M_R *be any module, and let* N *be a complement in* M_R. *Then, for any* $K \subseteq M_R$ such that $N \subseteq K$ and $(K/N) \subseteq_c (M/N)$, *we have* $K \subseteq_c M$.

PROOF: If $K \subseteq_e L \subseteq M$ then, by (1.6.c), $\frac{K}{N} \subseteq_e \frac{L}{N}$, whence $\frac{K}{N} = \frac{L}{N}$, i.e. K=L.■

From (1.6.b), it is clear that, if $N \subseteq_c M$ and $L \subseteq M_R$ is such that $N \subseteq L$, then $N \subseteq_c L$. On the other hand, we have the following 'transitive' property of complements.

PROPOSITION 1.8 *Let* $L \subseteq N$ *be submodules of* M_R *such that* $L \subseteq_c N$ *and* $N \subseteq_c M$. *Then* $L \subseteq_c M$.

PROOF [10;Theo.2.2]: By hypothesis, L is a relative complement in N for some $L' \subseteq N$, and N is a relative complement in M for some $N' \subseteq M$. Then $L \cap (L' \oplus N') = 0$ since, for $x \in L$, $y \in L'$ and $z \in N'$, we have

$$x=y+z \Rightarrow z=x-y \in N \cap N'=0 \Rightarrow x=y \in L \cap L'=0.$$

Then we can take a complement K for $L' \oplus N'$ in M such that $L \subseteq K$. Set $P=N \cap (K+N')$; then $P \cap L'=(K+N') \cap L'=0$; for, let $k \in K$, $x \in N'$, $y \in L'$, then

$$k+x=y \Rightarrow k=y-x \in K \cap (L' \oplus N')=0 \Rightarrow x=y \in N' \cap L'=0.$$

Now, since $L \subseteq P \subseteq N$, the maximality of L gives $L=P=N \cap (K+N')$, and from this we get $(K+N) \cap N'=0$; for, let $k \in K$, $x \in N$, $y \in N'$, then

$$y=k+x \Rightarrow x=y-k \in N \cap (K+N')=L \Rightarrow y=k+x \in N' \cap K \subseteq (N'+L') \cap K = 0.$$

Therefore, by maximality of N, we have K+N=N, i.e. $K \subseteq N$ and thus, by modularity, $L= N \cap (K+N')= K+(N \cap N')= K$; hence $L \subseteq_c M$.■

Given $N \subseteq M_R$ we can take first a relative complement K for N in M, and then a relative complement N' for K in M containing N. N' is then a complement and also an essential extension of N. For, if $L \subseteq N'$ verifies $L \cap N=0$ then we have $K \subseteq K \oplus L$ and $(K \oplus L) \cap N=0$, since k+x=n (for $k \in K$, $x \in L$, $n \in N$) implies $k=n-x \in K \cap N'=0$, so that $n=x \in N \cap L=0$. Hence, by maximality of K, $K=K \oplus L$, i.e. L=0, proving the claim. We shall call an *e-closure* (for essential closure) of N in M every complement in M that is an essential extension of N. As we have just seen, e-closures do exist for any submodule of any module M_R, and by (1.6) and (1.2.a) the set of e-closures of N in M coincide with $\mathcal{F}=\{L \subseteq M_R: N \subseteq_e L\}$. The next proposition gives another description or the same set.

PROPOSITION 1.9 *Let* $N \leq M_R$. *Then the e-closures of* N *in* M *are the minimal elements of* $\mathcal{E}=\{K \leq M_R\colon K \leq_c M$ and $N \leq K\}$.

PROOF: If N′ is an e-closure for N in M then clearly $N' \in \mathcal{E}$; if $K \in \mathcal{E}$ is such that $K \leq N'$, then (1.2.a) $K \leq_e N'$ and, since $K \leq_c M$, $K=N'$; therefore N′ is minimal in $\mathcal{E}$.
On the other hand, if K is a minimal element of $\mathcal{E}$, we have to prove that $N \leq_e K$; for, let N′ be an e-closure for N in K; then $N' \leq_c M$ (1.8) and therefore $N' \in \mathcal{E}$, whence $K=N'$ and thus $N \leq_e K$.■

An e-closure for N in M need not be unique: For example, if $R=\mathbb{Z}$ and $M=(\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z})$, then $N=(\bar{0},\bar{2})\mathbb{Z}$ has two e-closures in M, namely $(\bar{0},\bar{1})\mathbb{Z}$ and $(\bar{1},\bar{1})\mathbb{Z}$.

We close this paragraph with an application of the concept of e-closure, which characterizes the essential extensions within a module M_R.

PROPOSITION 1.10 *Let* $N \leq L$ *be submodules of* M_R. *Then the following conditions are equivalent*

a) $N \leq_e L$;

b) N *and* L *have a common relative complement in* M;

c) N *and* L *have a common e-closure in* M.

PROOF: a)⇒b). Let L′ be a relative complement for L in M; then $N \cap L' \leq L \cap L'=0$, and if $K \leq M_R$ is such that $L' \leq K$ and $N \cap K=0$ then (1.2.b) $0=N \cap K \leq_e L \cap K$, i.e. $0=L \cap K$; by maximality of L′, this gives $K=L'$. Therefore L′ is also a relative complement for N in M.

b)⇒c). If K is a common relative complement for N and L, take a relative complement K′ for K containing L (and hence N); K′ is then the desired common e-closure.

c)⇒a). This follows immediately from (1.2.a).■

Nonsingular Modules

The concepts of singular ideal of a ring and nonsingular ring were introduced by R.E.Johnson [26; p.894], and extended some years later to modules by himself [27; p.537]. These concepts are closely related to those of essential and complement submodules and have proved to be very helpful in many different areas of Ring Theory, particularly in the study of quotient rings. They will be used frequently throughout this dissertation, and we shall compile here their definition and first properties.

LEMMA 1.11 *Let* M_R *be any module. The set* $Z(M_R) = \{x\in M: \mathbf{r}_R(x)\subseteq_e R_R\} = \{x\in M: x\mathfrak{a}=0 \text{ for some } \mathfrak{a}\subseteq_e R_R\}$ *is a submodule of* M_R.

PROOF: First note that, with the help of (1.2.a), it is clear that both sets in the statement of the lemma are actually equal.
Since $\mathbf{r}_R(0)=R\subseteq_e R_R$, we have $0\in Z(M_R)$. Let $x,y\in Z(M_R)$ and write $\mathfrak{a}=\mathbf{r}_R(x)$, $\mathfrak{b}=\mathbf{r}_R(y)$; then $(x-y)(\mathfrak{a}\cap\mathfrak{b})=0$ and $\mathfrak{a}\cap\mathfrak{b}\subseteq_e R_R$, whence $x-y\in Z(M_R)$. Finally, if $r\in R$ then, by (1.1), $(\mathfrak{a}:r)\subseteq_e R_R$ and $xr(\mathfrak{a}:r)\subseteq x\mathfrak{a}=0$, whence $xr\in Z(M_R)$. Therefore $Z(M_R)$ is a submodule of M_R.■

$Z(M_R)$ is called the *singular submodule* of M_R. The module M_R is called nonsingular (resp. singular) if $Z(M_R)=0$ (resp. $Z(M_R)=M$). The *right singular ideal* of a ring R is $Z_r(R)=Z(R_R)$, and R is a *right nonsingular ring* if $Z_r(R)=0$. The left singular ideal and left nonsingular rings are defined similarly. Since all our rings have an identity, it is easy to show that $Z_r(R)\neq R$ for any ring R in which $1\neq 0$, i.e. there do not exist 'singular rings'.

PROPOSITION 1.12 *Let* M_R *be any module and let* $N\subseteq M_R$. *Then*

a) $Z(N_R)=N\cap Z(M_R)$;

b) $Z(M_R)$ *is a singular module;*

c) *if* M_R *is singular (resp. nonsingular), then so is* N_R;

d) *if* N_R *is nonsingular and* $N\subseteq_e M$, *then* M_R *is nonsingular;*

e) *if* $N\subseteq_e M$ *then* M/N *is singular;*

f) *if* M/N *is nonsingular then* $N\subseteq_c M$.

PROOF: a) is clear from the definition.

b) follows by applying a) to the case $N=Z(M_R)$.

c) follows directly from a).

d) Since N_R is nonsingular, $0=Z(N_R)=N\cap Z(M_R)$, and since $N\subseteq_e M$ this implies $Z(M_R)=0$, i.e., M_R is nonsingular.

e) For any $x\in M$, $(N:x)$ is an essential right ideal of R such that $x(N:x)\subseteq N$, i.e. $(x+N)(N:x)=0$ in M/N, and therefore $x+N\in Z(M/N)$, whence $Z(M/N)=M/N$.

f) If M/N is nonsingular and $N\subseteq_e K\subseteq M$, then K/N is singular by e) and nonsingular by c), since $K/N\subseteq M/N$. But obviously the only module which is both singular and nonsingular is the zero module, so that $K/N=0$, i.e. $N=K$, and hence $N\subseteq_c M$. ■

<u>PROPOSITION 1.13</u> *Let* R *be any ring and let* M_R *be a nonsingular module. Then*

a) *for any right ideal* $\mathfrak{a}$ *of* R, $\mathfrak{a}\subseteq_e R_R \Leftrightarrow R/\mathfrak{a}$ *is singular* as a *right* R-*module; in this case* $M\mathfrak{a}\subseteq_e M$;

b) *for any* $N\subseteq M_R$, $N\subseteq_e M \Leftrightarrow M/N$ *is singular;*

c) *for any* $N\subseteq M_R$, $N\subseteq_c M \Leftrightarrow M/N$ *is nonsingular.*

PROOF: a) If $\mathfrak{a}\subseteq R_R$ and $(R/\mathfrak{a})_R$ is singular, then there exists $\mathfrak{b}\subseteq_e R_R$ such that $(1+\mathfrak{a})\mathfrak{b}=0$, i.e. $\mathfrak{b}\subseteq\mathfrak{a}$, whence $\mathfrak{a}\subseteq_e R_R$; the converse follows from (1.12.e); if $\mathfrak{a}\subseteq_e R_R$ then for all $0\neq x\in M$ we have, by nonsingularity of M_R, $0\neq x\mathfrak{a}$; thus there exists $r\in\mathfrak{a}$ such that $0\neq xr\in M\mathfrak{a}$, and therefore $M\mathfrak{a}\subseteq_e M$.

b) Assume that M_R is nonsingular and M/N is singular; then, for all $x\in M\setminus N$, $(N:x)=r_R(x+N)$ is an essential right ideal of R, and hence $x(N:x)\neq 0$; thus there exists $r\in R$ such that $0\neq xr\in N$, and therefore $N\subseteq_e M$. The converse is (1.12.e).

c) Assume that M_R is nonsingular and $N\subseteq_c M$; let K be such that $N\subseteq K\subseteq M$ and $K/N=Z(M/N)$; then, since K is nonsingular and K/N is singular, b) gives $N\subseteq_e K$ and hence $N=K$, i.e. $Z(M/N)=K/N=0$. The converse is (1.12.f). ■

SECTION 2: THE UNIFORM DIMENSION OF A MODULE

A module M_R is called *finite dimensional* (abbreviated *f.d.*) if all direct sums of nonzero submodules of M have a finite number of summands. Thus e.g. all Artinian or Noetherian modules are f.d. We shall show that, if M is f.d., there is a least upper bound for the set $\mathcal{D}(M)=\{n\in\mathbb{Z}$: there is a direct sum of nonzero submodules of M with n summands$\}$. This fact will allow us to define a 'dimension' for finite dimensional modules which generalize the concept of dimension of a vector space.

Both concepts, finite dimensional modules and the dimension of a module, were introduced by A.W.Goldie. In [18] he concerned himself with ideals of a ring, but most of the proofs given there go through with minor changes when extending these concepts to modules, as Goldie did in [19].

Next, we give a first characterization of finite dimensional modules which sharpen our observation that either chain condition implies finite dimensionality.

<u>PROPOSITION 2.1</u> *For any module* M_R*, the following are equivalent:*

a) M_R *is finite dimensional;*

b) M_R *satisfies the* ACC *on complement submodules;*

c) M_R *satisfies the* DCC *on complement submodules.*

PROOF: a)⇒b) If $N_1\subset N_2\subset\cdots$ is an infinite chain of complements in M, then we construct an infinite direct sum of nonzero submodules of M as follows: since each extension $N_i\subset N_{i+1}$ is not essential, choose $0\neq L_i\subseteq N_{i+1}$ such that $L_i\cap N_i=0$; then $N_1\oplus L_1\oplus L_2\oplus\cdots$ is the announced sum.

b)⇒c) If $N_1\supset N_2\supset\cdots$ is an infinite chain of complements in M, then set $L_0=0$ and let L_i $(i=1,2,\ldots)$ be a relative complement for N_i in M containing L_{i-1}; then $L_1\subset L_2\subset\cdots$ is an infinite ascending chain of complements in M in which the inclusions are strict by (1.10).

c)⇒a). If M contains an infinite direct sum $N_1\oplus N_2\oplus\cdots$ of nonzero submodules and K_i is an e-closure for $N_i\oplus N_{i+1}\oplus\cdots$ in K_{i-1} (where

$K_0=M$), then $K_1\supset K_2\supset\cdots$ is an infinite chain of complements in M by (1.8) and (1.10).■

Clearly, every submodule of a f.d. module is f.d. Some other properties of stability for f.d. modules are listed below.

PROPOSITION 2.2 *Let* M_R *be any module and let* $N, N_1, \ldots, N_r$ *be submodules of* M. *Then*

a) *if* N *is f.d. and* $N\subseteq_e M$, *then* M *is f.d.;*

b) *if* M *is f.d. and* $N\subseteq_c M$, *then* M/N *is f.d.;*

c) *if* N *and* M/N *are both f.d., then* M *is f.d.;*

d) *if each* N_i *is f.d. and the sum* ΣN_i *is direct, then* $\oplus N_i$ *is f.d.*

PROOF: a) Clearly, a direct sum $\underset{i\in I}{\oplus}M_i$ of nonzero submodules of M provides a direct sum $\underset{I}{\oplus}N_i$ of nonzero submodules of N, where $N_i=N\cap M_i$; therefore the index set I must be finite and hence M is f.d.

b) By (1.7), an infinite strictly ascending chain of complements in M/N would provide an infinite strictly ascending of complements in M, which is impossible by (2.1); thus, also by (2.1), M/N is f.d.

c) Assume that N and M/N are both f.d., and let $M_1\oplus M_2\oplus\cdots$ be an infinite direct sum of submodules of M; set $T_k=M_k\oplus M_{k+1}\oplus\cdots$; we claim that $N\cap T_k=0$ for some k.
Suppose not; since $N\cap T_1\neq 0$, there exists $r_1\geq 1$ such that $N_1=N\cap(M_1\oplus\cdots\oplus M_{r_1})\neq 0$; but also $N\cap T_{r_1+1}\neq 0$ and hence we have, for some $r_2>r_1$, $N_2=N\cap(M_{r_1+1}\oplus\cdots\oplus M_{r_2})\neq 0$. In this way, we produce an infinite independent set $\{N_1, N_2, \ldots\}$ of nonzero submodules of N, which contradicts the hypothesis and hence proves the claim.
Let now $p:M\xrightarrow{c}M/N$; since $\mathrm{Ker}p\cap T_k=N\cap T_k=0$, we may view T_k as a submodule of the f.d. module M/N, and then all but a finite number of the M_r (for $r\geq k$) must be zero, proving that M is f.d.

d) By induction: If r=1 then there is nothing to prove; if r>1 then N_1 and $(\overset{r}{\underset{1}{\oplus}}N_i)/N_1\cong\overset{r}{\underset{2}{\oplus}}N_i$ are f.d., and hence so is $\overset{r}{\underset{1}{\oplus}}N_i$ by c).■

Note that (2.2.b) fails if N is not a complement in M. For example, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module; since any two nonzero elements of $\mathbb{Q}$ have a common nonzero multiple, any two nonzero submodules of $\mathbb{Q}$ have nonzero intersection and then $\mathbb{Q}$ is finite dimensional; but $\mathbb{Q}/\mathbb{Z}$ may be expressed as the direct sum of all its p-primary components, which are infinitely many and all nonzero, and thus it is not finite dimensional.

The stated property about the submodules of $\mathbb{Q}_{Z}$ is of interest in itself, and will be key in order to define the dimension of a finite dimensional module. A module U_R is *uniform* if $U \neq 0$ and every two nonzero submodules of U have nonzero intersection; or, equivalently, if $U \neq 0$ and every nonzero submodule of U is essential in U. Note that an essential extension of a uniform module is uniform. For a uniform module U it is clear that the least upper bound of $\mathcal{D}(U)$ is 1.

<u>LEMMA 2.3</u> *Every nonzero finite dimensional module* M_R *contains a uniform submodule.*

PROOF [18;Lemma 1.2]: Suppose not. Then M itself is not uniform and so there exist $0 \neq M_1, L_1 \subseteq M_R$ with $M_1 \cap L_1 = 0$. But again L_1 is not uniform, and we can find $0 \neq M_2, L_2 \subseteq L_1$ with $M_2 \cap L_2 = 0$. This process leads to an infinite direct sum $M_1 \oplus M_2 \oplus \cdots$ of nonzero submodules of M, a contradiction. ■

<u>PROPOSITION 2.4</u> *If* $0 \neq M_R$ *is finite dimensional, then there exist uniform submodules* $U_1, \cdots, U_n$ *of* M *such that the sum* $U_1 + \cdots + U_n$ *is direct and* $U_1 \oplus \cdots \oplus U_n \subseteq_e M$.

PROOF: Let U_1 be a uniform submodule of M, and let K_1 be a complement for U_1 in M. If U_1 is not essential in M then $K_1 \neq 0$ and K_1 is f.d., so that K_1 contains a uniform submodule U_2. If $U_1 \oplus U_2$ is not essential in M then it has a nonzero complement K_2 which contains a uniform submodule U_3 with $(U_1 \oplus U_2) \cap U_3 \subseteq (U_1 \oplus U_2) \cap K_2 = 0$.
Since M is f.d., this process must stop at some step n, and then $U_1, \cdots, U_n$ have the desired property. ■

<u>THEOREM 2.5</u> *Let* M_R *be a module, and suppose that* M *contains an essential submodule of the form* $U_1 \oplus \cdots \oplus U_n$ *where the* U_i*'s are uniform. Then any direct sum of nonzero submodules of* M *has at most* n *summands.*

PROOF [18;Theo.6]: Let $V_1, \cdots, V_k$ be an independent family of nonzero submodules of M and suppose $k>n$. Assume also $n\geq 2$ (if $n\leq 1$ then M is uniform or zero, and thus the result is clear).

If $N\leq M_R$ is not essential in M, then N has zero intersection with some U_i. For, suppose $N\cap U_i\neq 0$ $(i=1,\cdots n)$; then $N\cap U_i\leq_e U_i$, whence

$$\bigoplus_1^n (N\cap U_i) \leq_e \bigoplus_1^n U_i \leq_e M$$

by (1.2.e). Then, since $\oplus(N\cap U_i)\leq N$, we have $N\leq_e M$ (1.2.a), a contradiction.

Let $\hat{V}_1=V_2\oplus\cdots\oplus V_k$; $\hat{V}_1$ is not essential in M, and so we can assume e.g. $\hat{V}_1\cap U_1=0$. Therefore the sum $U_1\oplus V_2\oplus\cdots\oplus V_k$ is direct.

Let $\hat{V}_2=U_1\oplus V_3\oplus\cdots\oplus V_k$ which, as above, has zero intersection with some U_i, and not actually with U_1. Assume then $\hat{V}_2\cap U_2=0$; therefore the sum $U_1\oplus U_2\oplus V_3\oplus\cdots\oplus V_k$ is direct.

Since $n<k$ we can, by repeating this argument, give raise to a direct sum $(U_1\oplus\cdots\oplus U_n)\oplus(V_{n+1}\oplus\cdots\oplus V_k)$ with the second parenthesis nonzero, but this contradicts the essentiality of $U_1\oplus\cdots\oplus U_n$. Therefore it must be that $k\leq n$, as desired.■

<u>COROLLARY 2.6</u> *A module* M_R *is finite dimensional if and only if it contains a finite direct sum of uniform submodules which is an essential submodule of* M. *In this case, the number of summands in such a sum is an invariant* n *of* M *which equals the least upper bound of* $\mathcal{D}(M)=\{k\in\mathbb{Z}$: M contains k independent nonzero submodules$\}$.

PROOF: The first statement follows immediately from (2.4) and (2.5). If $U_1\oplus\cdots\oplus U_n$ and $V_1\oplus\cdots\oplus V_k$ are essential submodules of M with each U_i and each V_i uniform, then apply (2.5) twice to obtain $n\leq k$ and $k\leq n$, whence $n=k$. A new application of the previous theorem proves the last statement.■

If M_R is finite dimensional then the integer n of Corollary 2.6 is called the *uniform* or *Goldie dimension* of M. Our notation will be $u(M)=n$. If M is not finite dimensional then we write $u(M)=\infty$. A ring R is said to be right (left) finite dimensional if so is the regular module R_R $({}_RR)$.

For example, a semisimple module M is f.d. if and only if it is finitely generated, if and only if it is of finite length, and in this case u(M)=length(M), the composition length of M. In particular, this shows that our concept of dimension generalizes the usual one for vector spaces.

In (2.1) we showed that the finite dimensionality of M_R depends on the set of complement submodules of M_R; in fact, also the dimension of M_R may be described in terms of its chains of complements. In the next proposition, for an strict chain $K_0 \subset K_1 \subset \cdots \subset K_n$ of complements in M, call n the *length* of the chain.

<u>PROPOSITION 2.7</u> *Let* M_R *be a finite dimensional module, and let* $n=u(M_R)$. *Then* n *is the maximum of the lengths of all chains of complements in* M. *Moreover, a chain* $0=K_0 \subset K_1 \subset \cdots \subset K_r=M$ *of complements in* M *has length* r=n *if and only if* (K_{i+1}/K_i) *is uniform for* i=0,...,n-1.

PROOF [19;Lemma 1.4]: The construction methods used in (2.1) prove the first part: if $K_0 \subset K_1 \subset \cdots \subset K_r$ is a chain of complements, then we get a direct sum of r nonzero submodules of M_R as in 'a)⇒b)'; also, if $N_1 \oplus \cdots \oplus N_r$ is a direct sum of nonzero submodules, then we get a chain of complements with r strict inclusions as in 'c)⇒a)'.
Now, suppose that $0=K_0 \subset K_1 \subset \cdots \subset K_n=M$ is a chain of complements and fix an $i \in \{0,\ldots,n-1\}$. By the first part, we cannot insert any complement between K_i and K_{i+1}; hence, for any $N \leq M_R$ with $K_i \subset N \subset K_{i+1}$, K_{i+1} must be an e-closure of N by (1.9) and thus, by (1.6.c), $(N/K_i) \subseteq_e (K_{i+1}/K_i)$; therefore K_{i+1}/K_i is uniform.
Conversely, let $0=K_0 \subset K_1 \subset \cdots \subset K_r=M$ be a chain of complements with each K_{i+1}/K_i uniform. For i=2,...,r let $L_i (\neq 0)$ be a relative complement of K_{i-1} in K_i; then clearly the sum $K_1 \oplus L_2 \oplus \cdots \oplus L_r$ is direct; moreover, since each L_i embeds in K_i/K_{i-1}, all the summands are uniform; also, we have $K_{i-1} \oplus L_i \subseteq_e K_i$ (1.4); then, repeated applications of (1.2.e) give

$$K_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r \subseteq_e K_2 \oplus L_3 \oplus \cdots \oplus L_r \subseteq_e \cdots \subseteq_e K_{r-1} \oplus L_r \subseteq_e K_r = M,$$

and hence, by (2.6), n=r.■

As we have already remarked, Artinian modules are f.d.; more can be said in this case since, if M_R is Artinian and $U_1,\ldots,U_n$ are uniform submodules of M with $\oplus U_i \subseteq_e M$, then we can find a simple submodule S_i

inside each U_i, and we get $\oplus S_i \subseteq_e M$, whence $Soc(M) \subseteq_e M$ and then $u(M)=u(SocM)=length(SocM)$, as a consequence of part a) of the next result.

PROPOSITION 2.8 *Let* M_R *be any module, and let* $N, N_1, \ldots, N_r$ *be submodules of* M. *Then*

a) *if* $N \subseteq_e M$ *then* $u(N)=u(M)$; *if* M *is f.d. then the converse holds;*

b) *if* $N \subseteq_c M$ *then* $u(M)=u(N)+u(M/N)$;

c) *if* K *is an e-closure for* N *in* M *then* $u(M)+u(K/N) = u(N)+u(M/N)$;

d) *if* $N_1, \ldots, N_r$ *are independent then* $u(\oplus N_i)=\Sigma u(N_i)$.

PROOF: a) If $u(N)=\infty$ then also $u(M)=\infty$; if $u(N)=n$ and $N \subseteq_e M$, then any direct sum $\overset{n}{\underset{1}{\oplus}} U_i$ of uniform submodules of N which is essential in N is also essential in M, whence $u(M)=n$. If $u(M)=u(N)=n$ and $\oplus U_i$ is as before, then $\oplus U_i$ must be essential in M, because otherwise it could be extended to a direct sum with more than n terms, so that $N \subseteq_e M$ (1.2.a).

b) Note first that, if some term is not finite, then the formula holds by (2.2.b) and (2.2.c). Suppose then they are all finite; let $0=N_0 \subset N_1 \subset \cdots \subset N_r=N$ be a chain of complements in N (and hence in M) with each N_{i+1}/N_i uniform; and let $0=(K_0/N) \subset (K_1/N) \subset \cdots \subset (K_s/N)=(M/N)$ be a chain of complements in M/N (whence each $K_i \subseteq_c M$ by (1.7)) with each $(K_{i+1}/N)/(K_i/N)$ uniform (and then so is K_{i+1}/K_i). Thus

$$0=N_0 \subset N_1 \subset \cdots \subset N_r=N=K_0 \subset K_1 \subset \cdots \subset K_s=M$$

is a chain of complements in M with each factor uniform, so that, by (2.7), $u(M)= r+s= u(N)+u(M/N)$.

c) As in b), if some summand is not finite, then the formula holds. Assume then they are all finite; by (1.2.c), $(K/N) \subseteq_c (M/N)$ and thus, applying b) twice, we get

$$u(M) = u(K)+u(M/K) = u(K)+u\left(\frac{M/N}{K/N}\right) = u(K)+u(M/N)-u(K/N),$$

and since $u(N)=u(K)$, the result follows.

d) This follows easily by induction from b) (recall that every direct summand of M is a complement in M).■

SECTION 3: INJECTIVE HULLS; FINITE DIMENSIONAL INJECTIVE MODULES

This section start with a proposition which shows how the injectivity of a module depends on its essential extensions; this is one of the ways in which the concept of injective hull of a module appears naturally (as a maximal essential extension of the module). The fact that every submodule of an injective module E_R admits an injective hull within E will be used to characterize the finite dimensional injective modules. This characterization will be used later when studying the endomorphism ring of E. At the end of the section, we define quasi-injective modules and prove some results which will be used later.

Injective Hulls

In this paragraph we shall change slightly our concept of essential extension. By a (proper) essential extension of M_R we shall henceforth mean a monomorphism f from M_R to any module N_R such that ($f(M)\neq N$ and) $f(M)\subseteq_e N$.

Recall that an injective module is a direct summand of any module containing it (in fact this is also a sufficient condition for the module to be injective). Next, we give two more characterizations of injective modules.

PROPOSITION 3.1 *A module* E_R *is injective if and only if it does not admit proper essential extensions.*

PROOF [12; 4.2]: If E_R is injective and $f:E_R\to N_R$ is an essential extension of E, then $fE(\cong E)$ is injective and hence a direct summand of N, but since $fE\subseteq_e N$ this implies fN and therefore f is not proper.
Conversely, if E_R is not injective, then there exists a module L_R containing E such that E is not a direct summand of L; for a relative complement E′ of E in L, $E\oplus E'$ is a proper essential submodule of L by (1.4), and hence $\frac{E\oplus E'}{E'}\subset_e\frac{L}{E'}$ by (1.6.c); since $E\cap E'=0$, the natural map

$f:E \hookrightarrow L \to L/E'$ is a monomorphism with image $\frac{E\oplus E'}{E'}$, and hence a proper essential extension of E.■

Using (1.6), we get at once:

<u>COROLLARY 3.2</u> *A module is injective if and only if it is a complement submodule in any module containing it.*■

The following lemma states that an injective module E_R containing M_R also contains an isomorphic copy of each essential extension of M_R, so that E_R may be viewed as an 'upper bound' for the essential extensions of M_R.

<u>LEMMA 3.3</u> *Let* E_R *be injective and let* $f:M\to E$ *be a monomorphism. For any essential extension* $g:M_R\to N_R$ *of* M_R *there exists a monomorphism* $h:N\to E$ *such that* $f=hg$.

PROOF: By injectivity of E, there exists $h:N_R\to E_R$ with $f=hg$, and we have $gM\subseteq_e N$ and $gM\cap \mathrm{Ker}h=0$ (since $\mathrm{Ker}f=0$), so that $\mathrm{Ker}h=0$.■

Let $M\subseteq N_R$; if M is injective then $M\subseteq_d N$, and if $M\subseteq_d N$ then $M\subseteq_c N$. When N_R is injective we get both converses.

<u>PROPOSITION 3.4</u> *Let* E_R *be injective. For any* $M\subseteq E_R$ *the following are equivalent:*

a) M *is injective;*

b) M *is a direct summand of* E*;*

c) M *is a complement in* E.

PROOF: We need to prove c)⇒a). Assume c), let $g:M\to N$ be an essential extension of M_R and consider the inclusion $u:M\to E$; by (3.3) there exists a monomorphism $h:N\to E$ with $u=hg$; then $h:N\to hN$ is an isomorphism which carries gM onto $hgM=uM=M$, whence $M\subseteq_e hN\subseteq E$ (1.2.d); by assumption, we get $M=hN$ and hence $gM= h^{-1}hgM= h^{-1}M= N$. Thus M_R does not admit proper essential extensions and then it is injective by (3.1).■

Recall that any module is a submodule of an injective module [A-F; Prop.18.6]. We are now ready to prove the existence of minimal injective extensions and of maximal essential extensions for any module, and also to show that both coincide. The first of these facts was essentially proved by R.Baer [3], and the rest is due to B.Eckmann and A.Schopf [12].

THEOREM 3.5 *Given any module* M_R, *there exists a module* E_R *containing* M *such that*

a) $M \subseteq_e E$ *and, for any essential extension* $g: M \to N$, *there exists a monomorphism* $h: N \to E$ *such that* hg *is the inclusion map;*

b) E *is injective and any monomorphism* $f: M \to E'$ *with* E' *injective extends to a monomorphism* $h: E \to E'$.

PROOF [12; 4.1.4 & 4.3]: Let F_R be an injective module containing M, and let E be an e-closure of M in F. Then E_R is injective by (3.4) and $M \subseteq_e E$, so that we already have the first parts of a) and b).
Since E is injective, the second part of a) follows by taking f in (3.3) as the inclusion map. Also the second part of b) follows from (3.3), taking g as the inclusion $M \hookrightarrow E$. ∎

A module E_R satisfying the conditions of (3.5) is called an *injective hull* for M_R. The injective hull of a module is not unique; in fact, (3.4) and the proof of (3.5) show that, if $M \subseteq F_R$ and F_R is injective, the injective hulls of M inside F coincide with the e-closures of M in F. However, we have the following unicity theorem, which will allow us to speak about 'the' injective hull of M_R when any of the (isomorphic) injective hulls of M_R serves our purposes.

THEOREM 3.6 *If* E *and* E' *are injective hulls of* M_R, *then there exists an isomorphism* $f: E \to E'$ *which is the identity over* M.

PROOF: Since E_R is injective, the inclusion $M \hookrightarrow E$ extends to some $f: E \to E'$ which, by (3.3), is a monomorphism. Moreover, $M = fM \subseteq fE \subseteq E'$ together with $M \subseteq_e E'$ imply $fE \subseteq_e E'$, whence f is an essential extension of E and hence, by (3.1), an isomorphism. ∎

Finite Dimensional Injective Modules

Finite dimensional injective modules admit a well behaved decomposition theory, which in turn serves to characterize all finite dimensional modules, as follows.

PROPOSITION 3.7 *A nonzero module is uniform if and only if its injective hull is indecomposable.*

PROOF: Let $0 \neq M_R$ be any module, E_R an injective hull for M. Since $M \subseteq_e E$, if M is uniform then so is E, and every uniform module is indecomposable. Conversely, if M is not uniform and $0 \neq K, L \subseteq M_R$ are such that $L \cap K = 0$, then, taking e-closures L' and K' for L and K in E, we know that L' and K' are injective (3.4) and that their sum is direct (1.2.e), so that $L' \oplus K'$ is injective and hence a direct summand of E; therefore E is not indecomposable.■

PROPOSITION 3.8 *Let* M_R *be a module which has a finite decomposition* $M = \bigoplus_{i=1}^{n} M_i$ *and let* E *be an injective hull of* M. *Also, for each* $i=1,\dots,n$, *let* E_i *be an injective hull of* M_i *within* E; *then* $E = \oplus E_i$.

PROOF: By (1.2.e), the sum ΣE_i is direct, and hence it is an essential injective submodule of E; then (3.1) gives the result.■

THEOREM 3.9 *A module* M_R *is finite dimensional if and only if its injective hull* E *is a direct sum of finitely many nonzero indecomposable modules* $E_1,\dots,E_n$. *In this case* $u(M)=n$.

PROOF: If $u(M)=n$ then there exist uniform submodules $U_1,\dots,U_n$ of M with $\oplus U_i \subseteq_e M$ (2.6). If E_i is an e-closure of U_i in E (for $i=1,\dots,n$) then $E = \oplus E_i$ by (3.8), and each E_i is indecomposable by (3.7). The converse follows from (3.7), (2.6) and (2.8.a).■

Quasi-Injective Modules

Quasi-injective modules are a generalization of injective modules introduced by R.E.Johnson and E.T.Wong [65; p.260]. Their endomorphism rings have some nice properties that we shall study in Section 7. Here, we introduce them and prove their first properties.

A module M_R is said to be *quasi-injective* if, for every submodule N of M_R and every homomorphism $f:N_R \longrightarrow M_R$, there exists $g:M_R \longrightarrow M_R$ such that $g|_N=f$. Obviously, every injective and every semisimple module is quasi-injective. Also, by Baer's Criterion [A-F; p.205], the regular module R_R (for any ring R) is injective if and only if it is quasi-injective. In this case R is called a right self-injective ring.

The following result characterizes quasi-injective modules in terms of their relationship with their injective hull.

PROPOSITION 3.10 *Let* M_R *be a module and let* E_R *be an injective hull for* M_R. *Then* M_R *is quasi-injective if and only if, for every endomorphism* f *of* E_R, $fM\subseteq M$.

PROOF [65;Theo.1.1]: Assume that M_R is quasi-injective and let $f:E_R \longrightarrow E_R$ be an endomorphism. Then $N=M\cap f^{-1}M$ is a submodule of M_R such that $fN\subseteq M$; thus there exists $g:M_R \longrightarrow M_R$ with $g|_N=f|_N$. Let $u:M_R \longrightarrow E_R$ be the inclusion map; then, by injectivity of E_R, ug extends to some $h:E_R \longrightarrow E_R$, for which we have $hM=huM=ugM=gM\subseteq M$, and hence $M\cap(h-f)^{-1}M\subseteq f^{-1}M$; on the other hand, since h, g and f coincide over N, we have $N\subseteq Ker(h-f)$ and therefore

$$M\cap(h-f)^{-1}M \subseteq M\cap f^{-1}M = N \subseteq Ker(h-f),$$

whence $(h-f)M\cap M=0$. Since $M\subseteq_e E$, this implies $(h-f)M=0$ and hence h and f coincide over M, whence $fM=hM\subseteq M$.
Conversely, if $fM\subseteq M$ for all $f:E_R \longrightarrow E_R$ and we are given a submodule $N\subseteq M_R$ and a homomorphism $g:N_R \longrightarrow M_R$ then, by injectivity of E_R, ug extends to $h:E_R \longrightarrow E_R$ for which $huM= hM\subseteq M$, i.e. hu is an endomorphism of M_R which extends g, and therefore M_R is quasi-injective.∎

COROLLARY 3.11 *If* M_R *is quasi-injective and* E_R *is an injective hull of* M_R, *then any decomposition* $E=\bigoplus_I E_i$ *yields a decomposition* $M=\bigoplus_I (M\cap E_i)$.

PROOF: For each $j\in I$, let $f_j:\bigoplus_I E_i \xrightarrow{c} E_j$; since f_j may be viewed as an endomorphism of E_R, $f_jM\subseteq M$. Thus, if $x\in M$ is expressed in $E=\oplus E_i$ as $x=\sum_K x_k$ for some finite subset K of I and some $x_k\in E_k$, then each $x_k=f_kx\in M\cap E_k$; hence $M\subseteq\sum_I(M\cap E_i)$. That the sum is direct and contained in M is clear.∎

<u>COROLLARY 3.12</u> *Let* M_R *be quasi-injective with finite dimension* n. *Then* M *is the direct sum of* n *uniform submodules.*

PROOF: If an injective hull E_R of M is written as $E=\bigoplus_{i=1}^{n} E_i$ with each E_i uniform (3.9), then $M=\bigoplus_{i=1}^{n}(M\cap E_i)$ with each $M\cap E_i$ uniform.■

Analogously to (3.4), we have:

<u>PROPOSITION 3.13</u> *Let* M_R *be quasi-injective. Then every complement submodule of* M_R *is a direct summand of* M_R, *and every direct summand of* M_R *is quasi-injective.*

PROOF [38;Prop.4.3]: Let $K\subseteq_c M$; let E_R be an injective hull of M_R and let F be an e-closure of K in E (hence an injective hull of K); since $K\subseteq M\cap F\subseteq F$ and $K\subseteq_e F$, we get $K\subseteq_e M\cap F\subseteq M$, i.e. $K=M\cap F$; if G is such that $E=F\oplus G$ then we get (3.11) $M=(M\cap F)\oplus(M\cap G)=K\oplus(M\cap G)$, whence $K\subseteq_d M$.
Now, suppose that $M=N\oplus K$; let E, F be as above, and let G be an e-closure of N, so that $E=F\oplus G$. Let $h:G_R\longrightarrow G_R$ be any endomorphism; if $u:G\longrightarrow E$ and $p:E\longrightarrow G$ are the canonical injection and projection of $E=F\oplus G$, then uhp is an endomorphism of E_R and therefore $uhpM\subset M$ by (3.10); hence $hN=hpN=uhpN\subseteq uhpM\subseteq M$ and $hN\subseteq G$ imply, by the modular law, $hN\subseteq M\cap G=(N\oplus K)\cap G=N+(K\cap G)=N$, i.e. $hN\subseteq N$ for all endomorphism h of the injective hull G of N, and therefore N is quasi-injective.■

SECTION 4: THE DIMENSION FORMULA

Although the uniform dimension of a module is a generalization of the dimension of a vector space, in general the formula

$$u(A+B) = u(A) + u(B) - u(A\cap B),$$

that we shall call the *dimension formula* (for A+B) does not hold for submodules A and B of an arbitrary module M_R. Moreover, taking A, B of dimension 1 (i.e. uniform), u(A+B) may be any positive integer k or even ∞. The following are two easy examples, one of each case:

<u>EXAMPLE 1</u>: Consider the $\mathbb{Z}$-module $M=\mathbb{Z}\oplus(\mathbb{Z}/n\mathbb{Z})$ where n is a product of (powers of) $k-1$ distinct primes (with n=1 if $k=1$), and let $A=(1,0)\mathbb{Z}$, $B=(1,1+n\mathbb{Z})\mathbb{Z}$. Then $A\cong B\cong\mathbb{Z}$, whence u(A)=u(B)=1, but u(A+B)=u(M)=k.

<u>EXAMPLE 2</u>: Consider the $\mathbb{Z}$-module $M=\mathbb{Q}\oplus(\mathbb{Q}/\mathbb{Z})$. Let $A=\{(q,0):q\in\mathbb{Q}\}$ and $B=\{(q,q+\mathbb{Z}):q\in\mathbb{Q}\}$. Now $A\cong B\cong\mathbb{Q}$ and M=A+B, whence u(A)=u(B)=1 and u(M)=∞ ($\mathbb{Q}/\mathbb{Z}\cong\oplus\mathbb{Z}_{p^\infty}$, where the sum runs over all prime integers).

Our purpose in this Section, which contains several results of the author [58], is to impose conditions on a module under which the dimension formula holds for any pair of submodules, as well as to give some alternative general formulae for u(A+B). Another task arises from Example 2: that of characterizing the modules M_R such that the sum of any two finite dimensional submodules of M_R is still finite dimensional; we shall give a partial answer to this in the last part of the section.

A Characterization of Modules which Satisfy the Dimension Formula

We shall say that a module M_R *satisfies the dimension formula* if any sum of two submodules of M_R does. It is obvious that the dimension formula for A+B holds if either u(A)=∞ or u(B)=∞, so that M_R satisfies the dimension formula if any sum of two f.d. submodules does. Here, we shall prove that this is equivalent to the apparently weaker condition that any sum of two (f.d.) complements in M_R satisfies the dimension formula, and this happens when and only when any finite dimensional submodule of M_R has a unique e-closure.

Modules with the property that all their submodules (not only the f.d. ones) have a unique e-closure were studied by G.Rénault in his doctoral thesis [R67; p.42], where they are shown to be exactly those modules such that the intersection of any two complement submodules is again a complement submodule. This will be proved in one of the following preparatory lemmas for our Theorem 4.4.

LEMMA 4.1 *For a module* M_R*, the following statements are equivalent:*
a) M_R *satisfies the dimension formula;*
b) *if* $A\subseteq_e A'$, $B\subseteq_e B'$ *are f.d. submodules of* M_R *then* $A+B\subseteq_e A'+B'$.

PROOF: a)⇒b) By (1.2.b) we get $A\cap B\subseteq_e A'\cap B'$, and hence, using (2.8.a),

$$u(A'+B') = u(A')+u(B')-u(A'\cap B') = u(A)+u(B)-u(A\cap B) = u(A+B)$$

and all terms are finite, whence $A+B\subseteq_e A'+B'$.

b)⇒a) Assume b) and note first that if N and K are complements in M then so is $N\cap K$. For, let $N\cap K\subseteq_e L\subseteq M$; then, by b), $N+(N\cap K)\subseteq_e N+L$, i.e. $N\subseteq_e N+L$, whence $N=N+L$, i.e. $L\subseteq N$; similarly $L\subseteq K$ and thus $N\cap K=L$. Therefore $N\cap K$ is a complement in M.
Now, let A and B be arbitrary finite dimensional submodules of M_R, and take e-closures A' and B' for them in M; then we get $A\cap B\subseteq_e A'\cap B'$, $A'\subseteq_c A'+B'$ and $A'\cap B'\subseteq_c M$, whence $A'\cap B'\subseteq_c B'$; thus, by (2.8.a & b)

$$u(A+B) = u(A'+B') = u(A')+u((A'+B)'/A') = u(A')+u(B'/(A'\cap B'))$$

$$= u(A')+u(B')-u(A'\cap B') = u(A)+u(B)-u(A\cap B).$$

Therefore, by the remark preceding this lemma, M_R satisfies the dimension formula.■

We use now (4.1) to show that the dimension formula holds in all nonsingular modules (more proofs of this fact will come later).

COROLLARY 4.2 *If* M_R *is nonsingular then it satisfies the dimension formula.*

PROOF: Suppose $A\subseteq_e A'\leq M$ and $B\subseteq_e B'\leq M$. Then A'/A and B'/B are singular (1.12.e) and hence so is $(A'+B')/(A+B)$. For, let $a\in A'$ and $b\in B'$; then there exist essential right ideals e and f of R such that $ae\subseteq A$ and $bf\subseteq B$; thus $e\cap f$ is an essential right ideal of R with $(a+b)(e\cap f)\subseteq A+B$. This proves that $(A'+B')/(A+B)$ is singular. Since M is nonsingular, this implies $A+B\subseteq_e A'+B'$ (1.13.c) and therefore Lemma 4.1 applies.■

<u>LEMMA 4.3</u> *For a module* M_R*, the following statements are equivalent:*

a) *the dimension formula holds for all complements* A,B *in* M;

b) *if* A,B *are finite dimensional complements in* M *then so is* $A\cap B$;

c) *each f.d. submodule of* M *has a unique e-closure in* M.

PROOF: a)⇔b): Let A,B be f.d. complements in M; then $A\subseteq_c A+B$, whence

$$u(A+B) = u(A)+u((A+B)/A) = u(A)+u(B/(A\cap B)).$$

Then the dimension formula holds if and only if $u(B/A\cap B)=u(B)-u(A\cap B)$, i.e. if and only if $A\cap B\subseteq_c B$ (2.8.c), but since $B\subseteq_c M$ this is equivalent to saying that $A\cap B\subseteq_c M$.

b)⇒c) Let L be a f.d. submodule of M, and suppose that A and B are e-closures of L in M; then b) implies that $A\cap B$ is a complement in M containing L. By minimality of A and B (1.9) one gets A=B.

c)⇒b) Let A,B be f.d. complements in M. Since $A\cap B$ is f.d. we can take its (unique) e-closure K in M but then, since A and B are complements in M containing $A\cap B$, both must contain K (1.9), and hence $A\cap B=K$, which is a complement in M.■

<u>REMARK:</u> The equivalence of b) and c) without the hypotheses of finite dimension follows exactly in the same way as above.

<u>THEOREM 4.4</u> *The following statements about a module* M_R *are equivalent*

a) M_R *satisfies the dimension formula;*

b) *the dimension formula holds for all complements* A,B *in* M;

c) *if* $A\subseteq_e A'$, $B\subseteq_e B'$ *are f.d. submodules of* M *then* $A+B\subseteq_e A'+B'$;

d) *if* A,B *are f.d. complements in* M *then so is* $A\cap B$;

e) *each f.d. submodule of* M *has a unique e-closure in* M.

PROOF [58;Theo.4]: In view of the previous lemmas, and since a)⇒b) is clear, it suffices to show that b),d) and e) together imply c). Suppose then $A \subseteq_e A' \leq M$, $B \subseteq_e B' \leq M$ with $u(A)<\infty$, $u(B)<\infty$, and let us prove that $A+B \subseteq_e A'+B'$. First, it is clear that we can assume A' and B' to be the e-closures of A and B in M (if not, take e-closures for them and apply (1.2.a)); then b) ensures that A'+B' is f.d. (and thus so is A+B); hence we can take their respective unique e-closures K' and K. Since K' is a complement in M containing A+B, the uniqueness of K as minimal complement over A+B (1.9) implies $K \leq K'$. The same argument applied to the inclusions $A \leq K$ and $B \leq K$ give us $A' \leq K$ and $B' \leq K$, whence $A'+B' \leq K$ and this implies $K' \leq K$. Therefore K=K' and thus, by (1.10), we get $A+B \subseteq_e A'+B'$. This proves c).■

Since, in a nonsingular module M_R, every submodule N has a unique e-closure $\bar{N}=\{x \in M: x\mathfrak{e} \subseteq N \text{ for some } \mathfrak{e} \subseteq_e R_R\}$ [FA67; p.61]; and since every submodule of a semisimple module is a complement, we infer that the dimension formula holds in any nonsingular or semisimple module.

Next, we make use of (4.2) to determine which Abelian groups satisfy the dimension formula: Let M be an abelian group. If M contains an element of infinite order and a nonzero element of order n, then M contains a copy of $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$, and therefore it does not satisfy the formula (see Example 1). If M is torsion and the primary component of M for some prime p is neither semisimple nor uniform, then M contains a copy of $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$. This copy does not satisfy the dimension formula: let $A=(0,1)\mathbb{Z}$, $B=(1,1)\mathbb{Z}$.

Therefore a necessary condition for M to satisfy the dimension formula is: M is either torsion-free, or torsion with each primary component either semisimple or uniform.

This condition is also sufficient. For, if M is torsion-free then (4.2) ensure that the dimension formula holds in M. On the other hand, if M is torsion and M_p denotes the p-primary component of M, then, if A,B are submodules of M, we have clearly $(A \cap B)_p = A_p \cap B_p$, and we claim that $A_p+B_p \subseteq_e (A+B)_p$.

To prove it, let $a \in A$, $b \in B$ be such that $o(a+b)=$ (the order of a+b)$= p^n$ for some $n \geq 1$; we have to prove that some nonzero multiple of a+b lies in A_p+B_p. If $a \in A_p$, say $o(a)=p^r$, then let $m=\max\{n,r\}$ and note that

$0=p^m(a+b)=p^m b$, i.e. $b \in B_p$ and hence we are done. If $a \notin A_p$ then $b \notin B_p$ by the previous argument; in this case let $q,t,r,s \in \mathbb{N}$ be such that $o(a)=qp^r$, $o(b)=tp^s$, $q>1$, $t>1$, $p \nmid q$, $p \nmid t$.

Since p is prime, $p^n \nmid qt$ and hence $qt(a+b)=qta+qtb$ is a nonzero multiple of $a+b$ with $p^r(qta)=0$ and $p^s(qtb)=0$, i.e. $qta \in A_p$ and $qtb \in B_p$, proving our claim.

Hence, if the dimension formula holds in each M_p, then we get

$$u(A+B) = \Sigma u((A+B)_p) = \Sigma u(A_p+B_p) = \Sigma\big(u(A_p)+u(B_p)-u(A_p \cap B_p)\big)$$

$$= \Sigma u(A_p) + \Sigma u(B_p) - \Sigma u(A_p \cap B_p) = u(A)+u(B)-u(A \cap B)$$

(where the sums run over all prime integers p), and this proves that the stated condition is also sufficient.

Some General Formulae for u(A+B)

Suppose we are given submodules A,B of an arbitrary module M. Take e-closures A' for A in A+B, and C' for $C=A \cap B$ in B. By (2.8.c),

$$u(A+B) + u(A'/A) = u(A) + u((A+B)/A) = u(A) + u(B/C)$$

$$= u(A) + u(B) - u(C) + u(C'/C).$$

Note that, by modularity, $A+(A' \cap B) = A'$ and therefore

$$A'/A = [(A+(A' \cap B))/A] \cong (A' \cap B)/(A \cap B).$$

But $A \subseteq_e A'$ implies $A \cap B \subseteq_e A' \cap B$ and therefore, without loss of generality, we could have taken C' such that $A' \cap B \subseteq_e C' \subseteq_c B$, whence $u(A'/A) \leq u(C'/C)$. Thus we can restate our formula as

$$u(A+B) = u(A) + u(B) - u(A \cap B) + [u(C'/C)-u(A'/A)].$$

This shows at once that in general we have

$$u(A+B) \geq u(A)+u(B)-u(A \cap B), \qquad (I)$$

and as a consequence we get the implication b)⇒a) of (4.4). For, given A and B, take e-closures A',B' for them; then b) implies

$$u(A+B) \leq u(A'+B') = u(A')+u(B')-u(A' \cap B') = u(A)+u(B)-u(A \cap B) \leq u(A+B).$$

Therefore all terms are equal, so the dimension formula holds.

Consider now the short exact sequence $0 \to A\cap B \xrightarrow{\alpha} A\oplus B \xrightarrow{\beta} A+B \to 0$, where $\alpha(x)=(x,x)$ and $\beta(a,b)=a-b$, and let $C=\mathrm{Im}(\alpha)$. If C' is an e-closure for C in $A\oplus B$ then by (2.8.c & d) we get

$$u(A+B) = u(A) + u(B) - u(A\cap B) + u(C'/C).$$

In [8] Camillo and Zelmanowitz established, for submodules A,B of a module M, the formula

$$u(A+B) = u(A) + u(B) - u(D) + u(D/C), \qquad \text{(II)}$$

where $C=A\cap B$ and $D\supseteq C$ is a submodule of A maximal with respect to the property of being the domain of a monic extension into B of the identity in C. Note that (II) and (2.8.c) give (I).

We finally use (II) to give another proof of (4.2).

<u>COROLLARY 4.2'</u> *If M_R is nonsingular then it satisfies the dimension formula.*

PROOF: Given $A,B\subseteq M$ let C,D be as in (II) and let $f:D\to B$ be a monic extension of the identity in C. We claim that D/C is nonsingular. For, let $d\in D$ and suppose there exists an essential right ideal ϵ of R such that $d\epsilon\subseteq C$; then for all $e\in\epsilon$, $de=f(de)=f(d)e$ so that $(d-f(d))\epsilon=0$, but since M is nonsingular this implies $d=f(d)$ and hence $d\in C$. That is, $d+C=0$ in D/C, which proves that D/C is nonsingular. Thus (1.12.f) $C\subseteq_c D$ and hence $u(C)=u(D)-u(D/C)$. Therefore (II) takes the form of the dimension formula for A+B, as desired.■

Finiteness of u(A+B).

We give now a partial answer to the following question: If A,B are finite dimensional submodules of a module M_R, when is A+B also finite dimensional?

Following [6], we say that a module M_R is *quotient finite dimensional* (or *q.f.d.* for short) if M/N is f.d. for all submodules N of M_R. The following lemma is straightforward:

LEMMA 4.5 *Let* M_R *be any module and* N *any submodule of* M_R. *Then* M *is q.f.d. if and only if both* N *and* M/N *are q.f.d. In particular, a finite direct sum is q.f.d. if and only if each summand is q.f.d.*■

LEMMA 4.6 *For a ring* R *the following statements are equivalent:*

a) A+B *is f.d. for all f.d. submodules* A,B *of any right* R*-module* M;

b) *every f.d. right* M*-module is q.f.d.;*

c) *every f.d. injective right* R*-module is q.f.d.*

PROOF: a)⇒b) Suppose there exist modules $N \subseteq M_R$ such that $u(M)=n<\infty$ but $u(M/N)=\infty$. Then we can take submodules of $M\oplus(M/N)$ as in Example 2, namely $A=\{(x,0+N):x\in M\}$ and $B=\{(x,x+N):x\in N\}$, such that A and B are f.d. (they are isomorphic to M) but $A+B=M\oplus(M/N)$ is not f.d., contradicting a).

b)⇒a) Note that $u(A)<\infty$ and $u(B)<\infty$ imply $u(A\oplus B)<\infty$ (here $A\oplus B$ is an 'external' direct sum) and that A+B is a quotient of $A\oplus B$. Since $A\oplus B$ is q.f.d. by hypothesis, A+B is finite dimensional.

b)⇔c) This is clear since every finite dimensional module is contained in a finite dimensional injective module (3.9) and every submodule of a q.f.d. is q.f.d.■

Next we make use of (4.6) to study in some detail the commutative case. Recall that a module M_R is *finitely embedded* if its injective hull is a finite direct sum of injective hulls of simple modules (see [59]). The following proposition shows that examples in [7] cannot be extended to the infinite case:

PROPOSITION 4.7 *Let* R *be a commutative Artinian ring. Then every sum of finite dimensional R-modules is finite dimensional.*

PROOF [58;Prop.8]: By (4.6), we just have to prove that every f.d. injective R-module is q.f.d. By [59; Theo.1], every f.d. injective R-module E_R is finitely embedded. Then [34; Prop.3] shows that E_R is Artinian and therefore it is q.f.d.■

For the Noetherian case we have the following:

PROPOSITION 4.8 *Let* R *be a commutative integrally closed Noetherian domain with field of fractions* K. *Then the following statements are equivalent:*

a) *If* A, B *are f.d. submodules of an R-module* M *then* A+B *is f.d.;*

b) *every f.d. (injective)* R*-module is q.f.d.;*

c) K/R *is f.d.;*

d) R *is a semilocal principal ideal domain.*

PROOF [58;Prop.9]: a)⇔b) This is just (4.6).

b)⇒c) Since R is a domain, K_R is uniform and injective. Apply b).

c)⇒d) This is just [52; Prop.4.7]. Note that there the hypothesis 'K has Krull dimension' is only used to get 'K/R is f.d.'

d)⇒b) Since R is commutative Noetherian, every f.d. injective R-module E is a finite direct sum of injective hulls E(R/P) of R/P for some prime ideals P of R [33; Theorem 2.5, Prop.3.1]. But d) implies that each P is either 0 (in which case E(R/P)≅K by [33; Theorem 3.4]) or maximal (and then E(R/P) is Artinian by [34; Prop.3]), and therefore E takes the form $E=K\oplus\cdots\oplus K\oplus A$ where A is Artinian and thus q.f.d. Hence by (4.5) it suffices to show that K is q.f.d. as R-module or, since R is Noetherian, that K/R is q.f.d.. In fact, we prove next that K/R is Artinian.

Let $P_1=Rp_1,\dots,P_n=Rp_n$ be the maximal ideals of R and denote by P_i^* the R-submodule of K generated by $1/p_i$. Since P_i^*/R is annihilated by P_i and nonzero cyclic, it is simple. Moreover, every simple R-submodule of K/R is one of the P_i^*/R. For, let S/R be simple; then it is annihilated by some P_i, i.e. $Sp_i\subseteq R$, whence $R\subset S\subseteq P_i^*$ and thus $S=P_i^*$. Therefore the sum $\sum(P_i^*/R)$ (which is indeed direct, see [52]) is the socle of K/R. But every torsion module over a principal ideal domain has essential socle, whence K/R is finitely embedded and thus Artinian by [34; Prop.3].■

CHAPTER 2

THE RING OF ENDOMORPHISMS OF A MODULE

SECTION 5: ENDOMORPHISM RINGS; FIRST RESULTS.

Given a module $M=M_R$, call $S=End_R(M)=Hom_R(M,M)$. We can view the binary operation 'composition of maps' as a 'product' in S, and easy computations show that the Abelian group S (see Chapter 0) becomes then a ring with identity $1_S=1_M$ (the identity map in M). This ring S is called the *ring of endomorphisms* of M or the *endomorphism ring* of M.

For example, consider the regular module R_R. For each $r\in R$, the map $\lambda_r:R\longrightarrow R$ defined by $\lambda_r(t)=rt$ ('left multiplication by r') is an endomorphism of R_R, and $\lambda_r\neq\lambda_s$ whenever $r\neq s$. On the other hand, if $f\in End(R_R)$ and $f(1)=r$, then $f(t)=f(1t)=f(1)t=rt$ for all $t\in R$, so that $f=\lambda_r$. In fact, the map $r\longmapsto\lambda_r$ defines a ring isomorphism $\lambda:R\longrightarrow End(R_R)$ with inverse given by $f\longmapsto f(1)$.

In the general situation, M becomes a left S-module if we define the product of an element x of M by an element f of S as $fx=f(x)$ (and, fortunately, this agrees with our convention about the notation). Then M is clearly faithful as an S-module and it is an (S,R)-bimodule: $M={}_SM_R$ (see Chapter 0).

Given a submodule N of M and a nonempty subset W of S, we will write WN for the product submodule $\Sigma\{tN:\ t\in W\}$.

An S-submodule N of M need not be an R-submodule. For example consider the regular module R_R; by the previous example, the $End(R_R)$-submodules of R are the left ideals of R, which need not be submodules of R_R.

The (S,R)-submodules of M are usually called *fully invariant* submodules of M, since they are precisely the R-submodules N of M whose image under any endomorphism of M_R remains inside N. Therefore, if $N\leq M_R$ is fully invariant, we can define the "restriction map" $End(M_R)\longrightarrow End(N_R)$ in the obvious way, and it is easy to check that it is a ring homomorphism (and it is a ring epimorphism whenever arbitrary endomorphisms of N can be extended to End(M_R), e.g. if $N\subseteq_d M$ or if M_R is quasi-injective).

For example, we can restate (3.10) as 'A module M_R is quasi-injective if and only if it is a fully invariant submodule of its injective hull'. Other examples of fully invariant submodules are the ideals of R (when $M=R_R$); the annihilators in M_R of left ideals of R; the socle of M (in fact, every sum of homogeneous components of Soc(M)); and the radical of M [FA67; p.179]. Clearly, since Lat($_SM_R$) is a sublattice of Lat(M_R), arbitrary sums and intersections of fully invariant submodules are still fully invariant.

The relationships between properties of M and properties of S have been widely studied, and the next sections are devoted to a review of the main results obtained in this area for certain classes of modules. Sometimes we shall make use of the fact that End(R_R)$\cong$R and rewrite these results in the specific case $M=R_R$, obtaining as corollaries some well-known theorems about rings.

With no further background we can already get some easy properties of the endomorphism rings of well-behaved modules, such as vector spaces, simple, semisimple or free modules. S will always stand for $End_R(M)$.

<u>PROPOSITION 5.1</u> *If* M_R *is a simple module then* S *is a division ring.*

PROOF: If $f\in S$ is not the zero homomorphism then $Kerf\subset M_R$ and $0\neq Imf\subseteq M_R$; since M has no nontrivial submodules we have Ker(f)=0 and Im(f)=M, whence f is invertible in S. Therefore S is a division ring.∎

If M_R admits a finite decomposition as $M=\oplus_1^n M_i$ and we write M_{ij} for $Hom_R(M_j,M_i)$, then S may be identified with the ring of n-square matrices $[f_{ij}]$ with each f_{ij} in M_{ij}. Specifically, if $u_j:M_j\to M$ and $p_i:M\to M_i$ are the injections and projections (respectively) of the coproduct $\oplus M_i$, then the map $f\mapsto[f_{ij}]$, where $f_{ij}=p_ifu_j$, provides the desired ring isomorphism.

In particular, if $M^{(n)}$ represents the direct sum of n copies of M, then $End_R(M^{(n)})$ is isomorphic to the full ring of n by n matrices with entries in $S=End_R(M)$. Using this facts, we can prove the next result.

PROPOSITION 5.2 *If* M_R *is semisimple and finitely generated then* S *is a finite product of matrix rings over division rings (and hence* S *is a semisimple Artinian ring).*

PROOF: We can write $M=\overset{n}{\underset{i=1}{\oplus}}M_i$ where the M_i's are the homogeneous components of M, each of which is a finite direct sum of copies of a simple module, whence each $End_R(M_i)$ is a matrix ring over a division ring (5.1).

On the other hand, it is clear that any homomorphic image of a semisimple homogeneous module is again homogeneous of the same type, and then for the given decomposition of M we have $Hom_R(M_j,M_i)=0$ whenever $i\neq j$. Therefore it is clear that S is the ring product of the $End_R(M_i)$'s, which completes the proof.■

PROPOSITION 5.3 *If* M_R *is free then* S *is a row-finite matrix ring.*

PROOF: Let $\{x_i : i\in I\}$ be a basis for M_R. Denote by $\mathbb{RFM}_I(R)$ the ring of row-finite I-square matrices with entries in R. Then the map $\phi: S\longrightarrow\mathbb{RFM}_I(R)$ given by $\phi(f)=[r_{ij}]_{(i,j)\in I\times I}$ where the r_{ij}'s are such that $f(x_i)=\sum_{j\in I} x_j r_{ij}$ is a ring isomorphism.■

Idempotents

The behavior of the idempotent elements in a ring of endomorphisms is very important, since they are closely related with the direct summands of the module, as the following lemma shows. Before stating it, we recall that a set $\{t_i : i\in I\}$ of idempotents of a ring is said to be *orthogonal* if, for all $i\neq j$ in I, we have $t_it_j=0$. An idempotent t is *primitive* if it cannot be expressed as $t=t_1+\cdots+t_n$, with $\{t_1,\ldots,t_n\}$ a family of orthogonal idempotents and $n>1$. Finally, a finite orthogonal set $\{t_1,\ldots,t_n\}$ of idempotents is said to be *complete* if $t_1+\cdots+t_n=1$.

LEMMA 5.4 *Let* M_R *be any module and let* $S=End(M_R)$. *Then:*

a) *if* $N\subseteq M_R$, *then* $N\subseteq_d M$ *if and only if there exist an idempotent* t *of* S *such that* $N=tM$; *in this case* $M=tM\oplus(1-t)M$, *and* N *is indecomposable if and only if* t *is primitive;*

b) M *is indecomposable if and only if* 0 *and* 1 *are the unique idempotents of* S, *if and only if* 1 *is primitive in* S;

c) *if* $\{t_i : i\in I\}$ *is a family of orthogonal idempotents of* S *then the sum* $\sum_I t_i M$ *is direct; if* I *is finite and* $\sum_I t_i=1$, *then* $M=\oplus_I M_i$;

d) *if* $M=\oplus_{i\in I} M_i$ *then there exists a family* $\{t_i : i\in I\}$ *of orthogonal idempotents in* S *with* $M_i=t_i M$ *and* $Ker(t_i)=\oplus_{j\neq i} M_j$ *for each* $i\in I$. *If* I *is finite then* $\sum_I t_i=1$.

PROOF: a) If $M=N\oplus L$ then the canonical projection $t:N\oplus L\xrightarrow{c} N$ is the desired idempotent. Conversely, it is clear that, for $t^2=t\in S$, we have $M=tM\oplus(1-t)M$. Of course, this t is not unique in general.

If N is the direct sum $N=N_1\oplus N_2$ of two nonzero submodules and we define $t_i:N_1\oplus N_2\oplus L\xrightarrow{c} N_i$ for $i=1,2$, then t_1, t_2 are nonzero orthogonal idempotents with $t=t_1+t_2$, and thus t is not primitive. Conversely, if $t=t_1+t_2$ where t_1 and t_2 are nonzero orthogonal idempotents of S, then it is easy to see that $N=t_1M\oplus t_2M$, and hence N is not indecomposable.

b) is clear from a).

c) To see that the sum $\sum_I t_i M$ is direct, let $\sum_J t_j x_j=0$ for some finite subset J of I and for some $x_j\in M$; then, for all $k\in J$, we have $0 = t_k(\sum_J t_j x_j) = t_k x_k$. Therefore $\Sigma t_i M$ is direct. If I is finite and $\sum_I t_i=1$ then, for all $x\in M$, $x= 1(x)= (\Sigma t_i)x= \Sigma t_i x \in\sum_I t_i M$, so that $M=\oplus M_i$.

d) If $M=\oplus_{i\in I} M_i$ and we define $t_j:M=\oplus M_i\xrightarrow{c} M_j$ then $\{t_j: j\in I\}$ is the desired family of orthogonal idempotents.∎

COROLLARY 5.5 M_R *admits a finite indecomposable decomposition if and only if* S *possesses a complete family of primitive idempotents.*∎

COROLLARY 5.6 *If* $S=End(M_R)$ *is a local ring then* M_R *is an indecomposable module.*

PROOF: Since in any ring the only invertible idempotent is the identity, and since in a local ring the non-invertible elements form an ideal, 1 is a primitive idempotent of S, and therefore M_R is indecomposable by (5.4.b).∎

PROPOSITION 5.7 *Let* M_R *be any module and let* $S=End(M_R)$; *for every idempotent* t *of* S *there is a ring isomorphism* ϕ *between* tSt *and* $End_R(tM)$ *given by* $\phi(tft)(tx)=tftx$ *for all* $f\in S$ *and* $x\in M$.

PROOF: It is clear that the given map is a ring homomorphism and that it is injective. If $g \in End_R(tM)$ then it is easily checked that $g=\phi(tgt)$, and hence ϕ is a ring isomorphism.■

The Dual Module and the Trace Ideal of a Module

Next, we introduce the concepts of the *dual module* of M_R and the *trace ideal* of M in R, whose first properties we state here. These concepts will be used in later sections.

For any module M_R, we already know that M is an (S,R)-bimodule, where $S=End(M_R)$. Moreover, for each module $N=N_R$, the Abelian group $Hom_R(M,N)$ is, in a natural way (see Chapter 0), a right S-module. In case $N=R_R$, we write $\hat{M}$ for $Hom(M_R,R_R)$, and call it the *dual module* of M_R; since R is an (R,R)-bimodule, $\hat{M}$ has a natural (R,S)-bimodule structure (see again Chapter 0), $\hat{M}={}_R\hat{M}_S$.

For a module $N=N_R$, it may be of interest to know which elements of N appear as images of elements of M under an homomorphism of $Hom(M_R,N_R)$. The *trace* of M_R in N_R, written $t_N(M)$, is the submodule of N generated by these images, i.e.

$$t_N(M) = \Sigma\{\varphi M:\ \varphi\in Hom(M_R,N_R)\}.$$

In case $N=R_R$, the trace of M in R, i.e. $t_R(M)=\Sigma\{\varphi M:\varphi\in\hat{M}\}$, is usually called the *trace ideal* of M_R; it is a two-sided ideal of R since $\hat{M}$ is a left R-module.

Now, consider the bimodules ${}_SM_R$ and ${}_R\hat{M}_S$; the tensor products $\hat{M}\otimes_S M$ and $M\otimes_R\hat{M}$ are, respectively, (R,R)- and (S,S)-bimodules. Given $x\in M$ and $\varphi\in\hat{M}$, let (φ,x) represent the image of x under φ, i.e. $(\varphi,x)=\varphi x$, and let $[x,\varphi]$ be the map $M\longrightarrow M$ defined by $[x,\varphi]y=x(\varphi,y)$ for all $y\in M$. It is easy to check that $[x,\varphi]$ is an endomorphism of M_R and that the maps $(\,,\,):\hat{M}\times M\longrightarrow R$ and $[\,,\,]:M\times\hat{M}\longrightarrow S$ are bilinear, so that they extend to $\mathbb{Z}$-homomorphisms $(\,,\,):\hat{M}\otimes_S M\longrightarrow R$ and $[\,,\,]:M\otimes_R\hat{M}\longrightarrow S$. Easy computations show that, in fact, $(\,,\,)$ is an homomorphism of (R,R)-bimodules and $[\,,\,]$ is an homomorphism of (S,S)-bimodules. We rewrite here their action on generators for easy reference:

$$(\ ,\ ): \hat{M}\otimes_S M \longrightarrow R \qquad\qquad [\ ,\ ]: M\otimes_R \hat{M} \longrightarrow S$$
$$\varphi\otimes x \longmapsto (\varphi, x)=\varphi x \qquad\qquad x\otimes\varphi \longmapsto [x,\varphi]: M \longrightarrow M$$
$$y \longmapsto [x,\varphi]y = x(\varphi, y)$$

Note that the image of $(\ ,\ )$ is precisely $t_R(M)$, the trace ideal of M_R. Note also that, for any $x\in M$, $\varphi\in\hat{M}$, $r\in R$ and $f\in S$, we have

$$[xr,\varphi] = [x,r\varphi] \qquad r(\varphi,x) = (r\varphi,x) \qquad (\varphi,x)r = (\varphi,xr)$$

$$(\varphi f,x) = (\varphi,fx) \qquad f[x,\varphi] = [fx,\varphi] \qquad [x,\varphi]f = [x,\varphi f].$$

All these relations will be assumed in the sequel, and we shall use them without further reference.

SECTION 6: GALOIS CONNECTIONS AND CORRESPONDENCE THEOREMS

A very natural approach to the study of the relationships between properties of a module M_R and properties of the ring $S=End(M_R)$ consists in seeking out bijections between the lattice $\mathcal{L}=Lat(M_R)$ and either of the lattices $\mathcal{L}_l=Lat(_SS)$ or $\mathcal{L}_r=Lat(S_S)$.

The concept of "Galois connection", that we shall introduce shortly, provides a general source to get lattice (anti-) isomorphisms, and some examples of these connections will fit perfectly our purposes. For any module M_R we shall find, in a natural way, two Galois connections: G_1 between $\mathcal{L}$ and $\mathcal{L}_l$ and G_2 between $\mathcal{L}^{op}$ (the opposite lattice of $\mathcal{L}$) and $\mathcal{L}_r$. For each one of these, we shall get the corresponding sets of "closed" elements in $\mathcal{L}$, $\mathcal{L}_l$ or $\mathcal{L}_r$, as well as lattice (anti-) isomorphisms between them.

This general setting seems to have been first introduced by Baer [4], and has been widely employed (see K.Wolfson, G.Tsukerman, and S.Khuri [63, 54, 28, 29, 30, 31]); most of the proofs in this section are to be found in [30] and [31].

If we want to study a property of S which may be stated in terms of a class $\mathcal{C}$ of, say, right ideals, our two tasks will be: First, to check that the right ideals in $\mathcal{C}$ are closed objects of $\mathcal{L}_r$ for G_2 (or, if in general they are not, to find conditions on M_R under which they actually are); and second, to identify the images of the elements of $\mathcal{C}$ via the corresponding isomorphism. In this way, we obtain a bijection involving the ideals we are interested in and a certain class of submodules of M_R (we shall call that a *correspondence theorem*), and from this bijection we can deduce necessary and sufficient conditions on M in order to get the desired property on S.

Fortunately, these bijections will not only preserve (or reverse) the inclusions, but also we will be able to prove that every member of the domain $\mathcal{C}$ is a direct summand of S if and only if every member of the image is a direct summand of M_R (6.2), and this will enlarge the range of the applications of our correspondence theorems.

Let us first introduce the concepts of closure operator in a lattice and Galois connection between two lattices, and state their first properties (for details see e.g. [S; Chap.III, Sec.7 & 8]).

Closure Operators and Galois Connections

Let $(L,\leq)$ be a complete lattice. A *closure operator* in L is a map $c:L\longrightarrow L$ (we shall represent the image of $a\in L$ under c by a^c) which satisfies:

$a\leq a^c$, for all $a\in L$;

$a\leq b \Rightarrow a^c\leq b^c$, for all $a,b\in L$;

$(a^c)^c=a^c$, for all $a\in L$.

For example, if M_R is a module in which every submodule has a unique e-closure, e.g. a nonsingular module or a f.d. module in which the 'dimension formula' holds (see Section 4), then 'taking e-closures' is a closure operator in $\mathrm{Lat}(M_R)$. In a ring R, the most common closure operator acting on $\mathrm{Lat}(R_R)$ is given by $\mathfrak{a}\longmapsto\mathcal{R}\mathcal{L}(\mathfrak{a})$.

An element a of L is said to be *c-closed* if $a^c=a$; the c-closed elements of L are precisely the images under c of elements of L; we denote the set of c-closed elements in L by $L^c=\{a\in L: a^c=a\}=\{a^c: a\in L\}$. In the previous examples, the closed elements were, respectively, the complement submodules of M_R and the right annihilator ideals of R. The set L^c with the order inherited from L forms a complete lattice, (however, it is not in general a sublattice of L).

Let L_1 and L_2 be complete lattices (since the risk of confusion is small, we shall use the same symbol $\leq$ for the partial orders in L_1 and L_2). A *Galois connection* between L_1 and L_2 consists of a pair $G=\{\tau,\sigma\}$ of mappings $\tau:L_1\longrightarrow L_2$, $\sigma:L_2\longrightarrow L_1$ satisfying

$a\leq b \Rightarrow \tau(a)\geq\tau(b)$ for all $a,b\in L_1$;

$x\leq y \Rightarrow \sigma(x)\geq\sigma(y)$ for all $x,y\in L_2$;

$a\leq\sigma\tau(a)$ and $x\leq\tau\sigma(x)$ for all $a\in L_1$, $x\in L_2$.

For example, if R is any ring, then the annihilator operators $\mathcal{L}:\mathrm{Lat}(R_R)\longrightarrow\mathrm{Lat}({}_RR)$ and $\mathcal{R}:\mathrm{Lat}({}_RR)\longrightarrow\mathrm{Lat}(R_R)$ form a Galois connection.

If $G=\{\tau,\sigma\}$ is a Galois connection between L_1 and L_2, then the composition maps $\sigma\tau: L_1 \to L_1$ and $\tau\sigma: L_2 \to L_2$ are closure operators. Let $\hat{L}_1$ (resp. $\hat{L}_2$) represent the lattice of $\sigma\tau$-closed (resp. $\tau\sigma$-closed) elements of L_1 (resp. L_2). It is easy to prove that $\hat{L}_1=\{\sigma(x): x\in L_2\}$ and $\hat{L}_2=\{\tau(a): a\in L_1\}$, and that the restrictions $\tau: \hat{L}_1 \to \hat{L}_2$ and $\sigma: \hat{L}_2 \to \hat{L}_1$ are inverse lattice anti-isomorphisms. The elements of $\hat{L}_i$ are often called the *Galois objects* of L_i (for i=1,2) with respect to G.

For a lattice L, let L^{op} stand for the opposite lattice of L, i.e. the lattice consisting of the same underlying set with the opposite order. A Galois connection $G=\{\tau,\sigma\}$ between L_1^{op} and L_2 must then verify

$a\le b \Rightarrow \tau(a)\le\tau(b)$ for all $a,b\in L_1$;

$x\le y \Rightarrow \sigma(x)\le\sigma(y)$ for all $x,y\in L_2$;

$a\ge\sigma\tau(a)$ and $x\le\tau\sigma(x)$ for all $a\in L_1$, $x\in L_2$;

(where $\le$ always denotes the order in the original lattices L_1 and L_2); in this case the restrictions $\tau: \hat{L}_1 \to \hat{L}_2$ and $\sigma: \hat{L}_2 \to \hat{L}_1$ are inverse lattice isomorphisms.

The Galois Connections G_1 and G_2

Let M_R be any module and let $S=End(M_R)$. Write $\mathfrak{L}=Lat(M_R)$, $\mathfrak{L}_l=Lat(_SS)$ and $\mathfrak{L}_r=Lat(S_S)$. As always, let $\mathcal{L}$ and $\mathcal{R}$ denote the annihilator operators in S, and let $\mathbf{l}_S$ and $\mathbf{r}_M$ denote the annihilators in S of subsets of M and in M of subsets of S, respectively. Specifically, for any nonempty subsets W of S and X of M,

$\mathcal{L}(W)=\{f\in S:\ fg=0\ \forall g\in W\}$ $\qquad$ $\mathbf{l}_S(X)=\{f\in S:\ fx=0\ \forall x\in X\}=\{f\in S:\ X\subseteq Ker f\}$

$\mathcal{R}(W)=\{f\in S:\ gf=0\ \forall g\in W\}$ $\qquad$ $\mathbf{r}_M(W)=\{x\in M:\ gx=0\ \forall g\in W\}=\bigcap_{g\in W} Ker g$

Also let, for any subset W of S and any submodule N of M_R

$\tau_S(N)=\{f\in S:\ fM\subseteq N\}$ $\qquad$ $\sigma_M(W)=\Sigma\{gM:\ g\in W\}$.

With the notation of Chapter 0, we could have written $(N:_S M)$ for $\tau_S(N)$; note also that $\tau_S(N)$ may be identified with Hom(M,N) (since N is a submodule of M), and it is a right ideal of S. On the other hand, $\sigma_M(W)$ is just the 'product' WM, which is an R-submodule of M.

It is easy, although a bit tedious, to check that

1) the mappings $\mathbf{l}_S$ and $\mathbf{r}_M$ form a Galois connection G_1 between $\mathfrak{L}$ and

$\mathfrak{L}_l$; the closed elements of $\mathfrak{L}$ and $\mathfrak{L}_l$ for G_1 will be called, respectively, a-*closed (annihilator-closed) submodules* and a-*closed (left) ideals*, and we shall write $\mathcal{M}_a$ and $\mathcal{I}_a$ for the sets of a-closed elements of $\mathfrak{L}$ and $\mathfrak{L}_l$, i.e. $\mathcal{M}_a=\{\mathbf{r}_M(W):W\subseteq S\}$ and $\mathcal{I}_a=\{\mathbf{l}_S(N):N\subseteq M_R\}$; if N is a submodule of M, we shall sometimes write N^a for $\mathbf{r}_M\mathbf{l}_S(N)$.

2) the mappings τ_S and σ_M form a Galois connection G_2 between $\mathfrak{L}^{op}$ and $\mathfrak{L}_r$; the closed elements of $\mathfrak{L}$ for G_2 will be called, following [30], M-*cotorsionless* submodules of M; the Galois objects of $\mathfrak{L}_r$ for G_2 will be simply called $\tau\sigma$-*closed* (right) ideals. We shall write $\mathcal{M}_{\sigma\tau}=\{\sigma_M(W):W\subseteq S\}$ and $\mathcal{I}_{\tau\sigma}=\{\tau_S(N):N\subseteq M\}$.

3) the mappings $\mathcal{R}$ and $\mathcal{L}$ (as we have already remarked) form a Galois connection between $\mathfrak{L}_l$ and $\mathfrak{L}_r$, whose Galois objects are respectively $\mathcal{A}_l$ and $\mathcal{A}_r$, where we write $\mathcal{A}_l$ (resp. $\mathcal{A}_r$) for the set of left (resp. right) annihilator ideals of S. In particular, from the existence of a lattice anti-isomorphism between $\mathcal{A}_l$ and $\mathcal{A}_r$ we infer the well known fact that, for any ring, ACC (DCC) on right annihilators is equivalent to DCC (ACC) on left annihilators.

We wish to notice here the importance in what follows of the sets $\mathcal{I}_a$ and $\mathcal{I}_{\tau\sigma}$, since for any class $\mathcal{C}$ of ideals of S contained in either of these sets we will get a correspondence theorem involving $\mathcal{C}$.

Apart from the inclusion relations which are inherent to the fact that the above are Galois connections, some other relations always occur, as it is easily verified. They are listed below, and will be used throughout this section without further reference.

LEMMA 6.1 *With the above notation we have, for all nonempty subsets* W *of* S, *for all submodules* N *of* M *and for all* $f,t\in S$ *with* $t^2=t$:

a) $\tau_S\mathbf{r}_M(W)=\mathcal{R}(W)$; $\quad \mathbf{l}_S\sigma_M(W)=\mathcal{L}(W)$;

b) $\mathcal{R}(Sf)=\mathcal{R}(f)$; $\mathcal{L}(fS)=\mathcal{L}(f)$; $\quad \mathcal{R}(t)=(1-t)S$; $\mathcal{L}(t)=S(1-t)$

c) $\mathbf{r}_M(Sf)=\mathbf{r}_M(f)=\mathrm{Ker}f$; $\quad \mathbf{r}_M(St)=(1-t)M$;
   $\mathbf{l}_S(fM)=\mathcal{L}(f)$; $\quad \mathbf{l}_S(tM)=S(1-t)$;

d) $\sigma_M(fS)=fM$; $\quad \tau_S(tM)=tS$;

PROOF: We only prove the second half of d), since the rest of the proofs are mechanical. So let $t^2=t\in S$, then $t\in\tau_S(tM)$ and hence $tS\subseteq\tau_S(tM)$; if $g\in\tau_S(tM)$ then, for all $x\in M$, there exists $y\in M$ with $gx=ty$, whence $tgx=t^2y=ty=gx$, i.e. $g=tg\in tS$, which completes the proof.■

In particular, by (6.1.c,d), for each $f\in S$, Kerf is a-closed and fM is M-cotorsionless. Then, using (5.4.a), we deduce that every direct summand of M_R is a-closed and M-cotorsionless.

We now intend to show some examples of situations in which these Galois connections are particularly useful. For example, as a consequence of (6.1.a), every member of $\mathcal{A}_l$ (resp. $\mathcal{A}_r$) is an a-closed ideal (resp. a $\tau\sigma$-closed ideal), so that the 'first step' outlined at the beginning of the section is already done, and this will be helpful when studying conditions in S which depend on its annihilator ideals, such as being a Baer ring or a ring with chain conditions on annihilator ideals.

In the same way we shall study conditions in M under which the right complements of S will be Galois objects of G_2; we will make further use of these conditions in Section 9.

We close this section with a brief study of the principal and finitely generated left or right ideals of S. This study will be carried on in Section 10, where we shall characterize the quasi- injective and quasi-projective modules whose endomorphism rings are Noetherian, semiprimary or Artinian.

In what follows, a bijection between two partially ordered sets which is order-preserving (resp. order-reversing) will be called a *projectivity* (resp. a *duality*). Note that, if $\varphi:L_1\to L_2$ is a projectivity (e.g. a lattice isomorphism) and K_i is a subset of L_i $(i=1,2)$, then $\varphi:K_1\to K_2$ is a projectivity if and only if $\varphi(K_1)=K_2$, if and only if $\varphi(K_1)\subseteq K_2$ and $\varphi^{-1}(K_2)\subseteq K_1$. Of course, a similar remark holds for dualities.

The following lemma, announced at the beginning of the section, will be used in the proof of most applications of our correspondence theorems:

<u>LEMMA 6.2</u> a) *Assume that* $\mathbf{l}_S$ *and* $\mathbf{r}_M$ *determine a duality between certain subsets* $\mathcal{U}$ *of* $\mathcal{M}_a$ *and* $\mathcal{V}$ *of* $\mathcal{S}_a$. *Then every member of* $\mathcal{U}$ *is a direct summand of* M_R *if and only if every member of* $\mathcal{V}$ *is a direct*

summand of sS.

b) *Assume that* τ_S *and* σ_M *determine a projectivity between the subsets* $\mathcal{U}$ *of* $\mathcal{M}_{\sigma\tau}$ *and* $\mathcal{V}$ *of* $\mathcal{S}_{\tau\sigma}$. *Then every member of* $\mathcal{U}$ *is a direct summand of* M_R *if and only if every member of* $\mathcal{V}$ *is a direct summand of* S_S.

PROOF: a) Let $\mathcal{U},\mathcal{V}$ be as stated, and assume $\mathcal{U}\subseteq\mathcal{D}=\{N\le M_R: N\le_d M\}$; then, for all $\mathfrak{A}\in\mathcal{V}$, there exists $t^2=t\in S$ such that $\mathbf{r}_M(\mathfrak{A})=tM$; also, since $\mathcal{V}\subseteq\mathcal{S}_a$, we have $\mathfrak{A}=\mathbf{l}_S\mathbf{r}_M(\mathfrak{A})$, whence $\mathfrak{A}=\mathbf{l}_S(tM)=S(1-t)$, which is a direct summand of sS. Conversely, if every member of $\mathcal{V}$ is a direct summand of sS and $N\in\mathcal{U}$, then there exists $t^2=t\in S$ with $\mathbf{l}_S(N)=St$ and hence, since $N\in\mathcal{U}\subseteq\mathcal{M}_a$, $N=\mathbf{r}_M\mathbf{l}_S(N)=\mathbf{r}_M(St)=(1-t)M$ is a direct summand of M_R.

b) is proved similarly.■

Correspondence Theorems for Annihilators

As we have already remarked, the class $\mathcal{A}_l$ of left annihilator ideals of S is included in $\mathcal{M}_a$ so that, in order to obtain a correspondence theorem for left annihilators, all we have to do is to identify the a-closed submodules of M_R which correspond to the ideals in $\mathcal{A}_l$.

<u>THEOREM 6.3</u> *For any module* M_R *set* $\mathcal{M}_l=\{N\le M_R: N=[\sigma_M\tau_S(N)]^a\}$. *Then the maps* $\mathbf{l}_S:\mathcal{M}_l\to\mathcal{A}_l$ *and* $\mathbf{r}_M:\mathcal{A}_l\to\mathcal{M}_l$ *are inverse dualities.*

PROOF: Since every element of $\mathcal{M}_l$ is an a-closed submodule and every element of $\mathcal{A}_l$ is an a-closed ideal, the only things we have to check are that $\mathbf{l}_S(\mathcal{M}_l)\subseteq\mathcal{A}_l$ and that $\mathbf{r}_M(\mathcal{A}_l)\subseteq\mathcal{M}_l$.
If $N\in\mathcal{M}_l$ then (6.1.a) gives $\mathbf{l}_S(N)=\mathbf{l}_S\sigma_M\tau_S(N)=\mathcal{L}\tau_S(N)\in\mathcal{A}_l$, i.e. $\mathbf{l}_S(\mathcal{M}_l)\subseteq\mathcal{A}_l$. On the other hand, again using (6.1.a), if $\mathfrak{A}=\mathcal{L}\mathcal{R}(\mathfrak{A})$ then $\mathbf{r}_M\mathbf{l}_S\sigma_M\tau_S\big(\mathbf{r}_M(\mathfrak{A})\big)=\mathbf{r}_M\mathcal{L}\mathcal{R}(\mathfrak{A})=\mathbf{r}_M(\mathfrak{A})$, i.e. $\mathbf{r}_M(\mathfrak{A})\in\mathcal{M}_l$.■

A *Baer ring* is a ring in which every left (or right) annihilator ideal is generated by an idempotent. From (6.2) and (6.3) we get

<u>COROLLARY 6.4</u> a) S *has* ACC *(*DCC*) on left annihilators if and only if* M *has* DCC *(*ACC*) on* $\mathcal{M}_l$.

b) S *is a Baer ring if and only every member of* $\mathcal{M}_l$ *is a direct summand of* M_R.■

A module M_R is a *self-generator* if $t_N(M)=N$ for all $N\leq M_R$ (see Section 5 and compare with the definition of generator in Section 8), i.e. if all its submodules are M-cotorsionless. For a self-generator M_R it is clear that $\mathcal{M}_l=\mathcal{M}_a$; however, there exist modules which are not self-generators but verify $\mathcal{M}_l=\mathcal{M}_a$ [30;p.395]. A module for which $\mathcal{M}_l=\mathcal{M}_a$, i.e. a module M_R such that, for each $N\in\mathcal{M}_a$, we have $N=[\sigma_M\tau_S(N)]^a$, is called an a-*self-generator*. In this case we get not only a duality between $\mathcal{M}_a$ and $\mathcal{A}_l$, but also a projectivity between $\mathcal{M}_a$ and $\mathcal{A}_r$:

<u>THEOREM 6.5</u> *For a module* M_R *the following are equivalent:*

a) M_R *is an a-self-generator;*

b) *the maps* $l_S:\mathcal{M}_a\longrightarrow\mathcal{A}_l$ *and* $r_M:\mathcal{A}_l\longrightarrow\mathcal{M}_a$ *are inverse lattice anti-isomorphisms;*

c) *the maps* $\tau_S:\mathcal{M}_a\longrightarrow\mathcal{A}_r$ *and* $\mathfrak{A}\longmapsto[\sigma_M(\mathfrak{A})]^a$ *from* $\mathcal{A}_r$ *to* $\mathcal{M}_a$ *are inverse lattice isomorphisms.*

PROOF: a)⇒b) Since $\mathcal{M}_a=\mathcal{M}_l$ by hypothesis, (6.3) gives the result.

b)⇒c) Since $\tau_S(\mathcal{M}_a)\subseteq\mathcal{A}_r$ (6.1.a) and $[\sigma_M(\mathfrak{A})]^a\in\mathcal{M}_a$ for all $\mathfrak{A}\in\mathcal{A}_r$, we just have to prove that both mappings in c) are inverse of each other:

If $\mathfrak{A}\in\mathcal{A}_r$ then $\tau_S[\sigma_M(\mathfrak{A})]^a=\tau_S r_M l_S \sigma_M(\mathfrak{A})=\mathcal{R}\mathcal{L}(\mathfrak{A})=\mathfrak{A}$ (6.1.a).

If $N\in\mathcal{M}_a$ then, by b), $l_S(N)\in\mathcal{A}_l$, i.e. $\mathcal{L}\mathcal{R}l_S(N)=l_S(N)$, and hence

$$[\sigma_M\tau_S(N)]^a = r_M l_S \sigma_M \tau_S(r_M l_S(N)) = r_M \mathcal{L}\mathcal{R} l_S(N) = r_M l_S(N) = N.$$

c)⇒a) This is clear from the definition of a-self-generator.■

<u>COROLLARY 6.6</u> *Let* M_R *be an a-self-generator. Then*

a) S *has* ACC *(*DCC*) on left annihilators if and only if* M_R *has* DCC *(*ACC*) on a-closed submodules;*

b) S *is a Baer ring if and only if every a-closed submodule of* M_R *is a direct summand of* M.

PROOF: By Corollary 6.4.■

A module M_R in which every complement submodule is a direct summand is called a CS-*module*. For example, every quasi-injective module is a CS-module (3.13). If M_R is a module for which the a-closed submodules coincide with the complements in M, then we can rewrite (6.6) in terms of the module being a CS-module or finite dimensional.

As we have remarked, for a nonsingular module M_R, 'taking e-closures' is a closure operator in Lat(M_R) and hence, for any $N \subseteq M_R$, we can write N^e for the (unique) e-closure of N in M. For such a module (and by abuse of language for all modules) we shall write $\mathcal{M}_e$ for the set of essentially closed (i.e. complement) submodules of M_R.

PROPOSITION 6.7 a) *If* M_R *is nonsingular then* $\mathcal{M}_a \subseteq \mathcal{M}_e$.
b) *If* M_R *is a* CS-*module then* $\mathcal{M}_e \subseteq \mathcal{M}_a$.

PROOF: a) Assume that M_R is nonsingular, and let $N \in \mathcal{M}_a$. Let $K=N^e$; then $N \subseteq K$ and hence $\mathbf{l}_S(K) \subseteq \mathbf{l}_S(N)$; on the other hand, if $f \in \mathbf{l}_S(N)$ and $x \in K$, then $(N:x) \subseteq_e R_R$ and $fx(N:x) \subseteq fN=0$, whence $fx=0$ by nonsingularity; this means that $f \in \mathbf{l}_S(K)$ and hence $\mathbf{l}_S(K)=\mathbf{l}_S(N)$. Therefore $K \subseteq \mathbf{r}_M\mathbf{l}_S(K)= \mathbf{r}_M\mathbf{l}_S(N)= N$ and thus N=K, so that $N \in \mathcal{M}_e$.
b) If M is a CS-module then every element of $\mathcal{M}_e$ is a direct summand of M_R and hence is a-closed, whence $\mathcal{M}_e \subseteq \mathcal{M}_a$.∎

COROLLARY 6.8 *Let* M_R *be a nonsingular a-self-generator* CS-*module. Then* S *is a Baer ring. If, in addition,* M_R *is finite dimensional, then* S *has* ACC *and* DCC *on left (and right) annihilator ideals.*

PROOF: By (6.7), we have $\mathcal{M}_a = \mathcal{M}_e$. Then (6.6.b) gives the first part, while (6.6.a) and (2.1) yield the second.∎

Correspondence Theorems for Right Complements

In the next lemma, we shall make use of the concepts of the trace ideal T of a module M_R and of the dual module $\hat{M}$ of M_R, as well as of the maps $(\,,\,):\hat{M}\otimes M \longrightarrow R$ and $[\,,\,]:M\otimes\hat{M} \longrightarrow S$, which were defined in Section 5.

LEMMA 6.9 *Let* M_R *be a module with trace ideal* T. *The following statements are equivalent:*

a) $xT \neq 0$ *for all* $0 \neq x \in M$;

b) $[x,\hat{M}] \neq 0$ *for all* $0 \neq x \in M$;

c) $NT \subseteq_e N$ *for all* $N \subseteq M_R$.

PROOF: a)⇒b) If $[x,\hat{M}]=0$ then $0=[x,\hat{M}]M=x(\hat{M},M)=xT$ and thus x=0.
b)⇒c) Let $N \subseteq M_R$; if N=0 there is nothing to prove; otherwise, for each

$0\neq x\in N$, we have $[x,\hat{M}]\neq 0$, i.e. there exists $\varphi\in\hat{M}$ such that $[x,\varphi]\neq 0$; thus there exists $y\in M$ with $0\neq[x,\varphi]y=x(\varphi,y)\in xR\cap NT$, what proves that $NT\subseteq_e N$.
c)⇒a) Let $0\neq x\in M$; thus we get $xR\neq 0$ and $xT=(xR)T\subseteq_e xR$, whence $xT\neq 0$. ∎

A module which satisfies the equivalent conditions of (6.9) is called a *non-degenerate* module. If M_R is a generator of Mod_R (i.e. if $\mathbf{t}_N(M)=N$ for all N in Mod_R, see Section 8) then $T=\mathbf{t}_R(M)=R$ and hence M_R is non-degenerate (and, as we already remarked, self-generator). However, none of these conditions implies that M_R is a generator [30; p.387].

Let $\mathscr{C}_r$ stand for the set of right complement ideals of S. Part d) of the following proposition, namely that every right complement in S is $\tau\sigma$-closed (i.e. $\mathscr{C}_r\subseteq\mathscr{F}_{\tau\sigma}$) whenever M_R is non-degenerate, suggests that non-degeneracy is a suitable condition under which we will be able to obtain correspondence theorems for right complements.

PROPOSITION 6.10 *Let* M_R *be non-degenerate. Then*

a) *for any* $0\neq N\subseteq M_R$ *we have* $\tau_S(N)\neq 0$*;*

b) *if* $\mathfrak{A}\subseteq\mathfrak{B}$ *are right ideals of* S *then* $\mathfrak{A}\subseteq_e\mathfrak{B}\Leftrightarrow\sigma_M(\mathfrak{A})\subseteq_e\sigma_M(\mathfrak{B})$*;*

c) *for any right ideal* $\mathfrak{A}$ *of* S, $\mathfrak{A}\subseteq_e\tau_S\sigma_M(\mathfrak{A})$*;*

d) *every right complement in* S *is* $\tau\sigma$*-closed;*

e) *for all* $N\subseteq M_R$ *we have* $\sigma_M\tau_S(N)\subseteq_e N$*;*

f) *if* $N\subseteq K$ *are submodules of* M_R *then* $N\subseteq_e K\Leftrightarrow\tau_S(N)\subseteq_e\tau_S(K)$*;*

g) M_R *is an a-self-generator.*

PROOF: a) Let $0\neq N\subseteq M_R$; then (6.9) gives $0\neq[N,\hat{M}]$ and hence $0\neq\tau_S(N)$, since clearly $[N,\hat{M}]\subseteq\tau_S(N)$.

b) Assume first that $\mathfrak{A}\subseteq_e\mathfrak{B}$ (as right S-modules); for all $0\neq x\in\sigma_M(\mathfrak{B})$ (which has the form $x=\sum_I f_i x_i$ for some finite set I and some $f_i\in\mathfrak{B}$, $x_i\in M$), we have $0\neq[x,\hat{M}]=\sum_I f_i[x_i,\hat{M}]\subseteq\mathfrak{B}$, and hence $0\neq\mathfrak{A}\cap[x,\hat{M}]$; thus

$$0\neq\big(\mathfrak{A}\cap[x,\hat{M}]\big)M\subseteq\mathfrak{A}M\cap[x,\hat{M}]M=\mathfrak{A}M\cap xT\subseteq\mathfrak{A}M\cap xR,$$

whence $\mathfrak{A}M=\sigma_M(\mathfrak{A})\subseteq_e\sigma_M(\mathfrak{B})$.
Conversely, if $\sigma_M(\mathfrak{A})\subseteq_e\sigma_M(\mathfrak{B})$ then, for all $0\neq f\in\mathfrak{B}$, fM is nonzero, and hence $fM\cap\sigma_M(\mathfrak{A})\neq 0$ whence, by non-degeneracy,

$$0\neq[fM\cap\sigma_M(\mathfrak{A}),\hat{M}]\subseteq[fM,\hat{M}]\cap[\mathfrak{A}M,\hat{M}]\subseteq fS\cap\mathfrak{A}.$$

Therefore $\mathfrak{A}\subseteq_e\mathfrak{B}$, and this finishes the proof of b).

c) Since, for all $\mathfrak{A}\subseteq S_S$, we have $\mathfrak{A}\subseteq\tau_S\sigma_M(\mathfrak{A})$ and $\sigma_M(\mathfrak{A})=\sigma_M\tau_S\sigma_M(\mathfrak{A})$, b) gives $\mathfrak{A}\subseteq_e\tau_S\sigma_M(\mathfrak{A})$.

d) If $\mathfrak{A}$ is a complement then c) implies $\mathfrak{A}=\tau_S\sigma_M(\mathfrak{A})$, as required.

e) Let $0\neq N\subseteq M_R$ and $0\neq x\in N$; then, by a), $\tau_S(xR)\neq 0$, and for any $0\neq f\in\tau_S(xR)$ we have $0 \neq fM \subseteq xR\cap\sigma_M\tau_S(xR) \subseteq xR\cap\sigma_M\tau_S(N)$, whence $\sigma_M\tau_S(N)\subseteq_e N$. If $N=0$ then the result is obvious.

f) The cases $N=0$ or $K=0$ are trivial. Assume then $0\neq N\subseteq_e K$ whence, by e), $\sigma_M\tau_S(N)\subseteq\sigma_M\tau_S(K)\subseteq_e K$ and $\sigma_M\tau_S(N)\subseteq_e N\subseteq_e K$; thus (1.2.a) $\sigma_M\tau_S(N)\subseteq_e\sigma_M\tau_S(K)$ and hence, by b), $\tau_S(N)\subseteq_e\tau_S(K)$. Conversely, if $\tau_S(N)\subseteq_e\tau_S(K)$ then, by b) and e), $\sigma_M\tau_S(N)\subseteq_e\sigma_M\tau_S(K)\subseteq_e K$; but $\sigma_M\tau_S(N)\subseteq N\subseteq K$, and therefore $N\subseteq_e K$ by (1.2.a).

g) Let $N\in\mathcal{M}_a$; we have to prove that $N=[\sigma_M\tau_S(N)]^a$, and for this it will suffice to see that $\mathbf{l}_S(N)=\mathbf{l}_S\sigma_M\tau_S(N)$, since then the action of $\mathbf{r}_M$ on both sides will yield the desired equality. Also, since $\sigma_M\tau_S(N)\subseteq N$, it will suffice to prove that $\mathbf{l}_S\sigma_M\tau_S(N)\subseteq\mathbf{l}_S(N)$.
Let then $f\in\mathbf{l}_S\sigma_M\tau_S(N)$; thus $f(\sigma_M\tau_S(N))=0$, which clearly implies $f\tau_S(N)=0$; now, since $[N,\hat{M}]\subseteq\tau_S(N)$, we get $[fN,\hat{M}]= f[N,\hat{M}]\subseteq f\tau_S(N)= 0$ which, by non-degeneracy, implies $fN=0$, i.e. $f\in\mathbf{l}_S(N)$, as required.■

Now, we are ready to prove the following correspondence theorem:

<u>THEOREM 6.11</u> *Let* M_R *be non-degenerate and let* $\mathcal{M}_2=\{N\subseteq M_R$: N is M-cotorsionless and $\tau_S(N)\in\mathfrak{C}_r\}$. *Then* $\tau_S:\mathcal{M}_2\longrightarrow\mathfrak{C}_r$ *and* $\sigma_M:\mathfrak{C}_r\longrightarrow\mathcal{M}_2$ *are inverse projectivities.*

PROOF: Since $\mathcal{M}_2\subseteq\mathcal{M}_{\sigma\tau}$ by definition and $\mathfrak{C}_r\subseteq\mathcal{S}_{\tau\sigma}$ (6.10), we just have to prove that $\tau_S(\mathcal{M}_2)\subseteq\mathfrak{C}_r$ and that $\sigma_M(\mathfrak{C}_r)\subseteq\mathcal{M}_2$. The first inclusion follows directly from the definition of $\mathcal{M}_2$, and if $\mathfrak{A}\in\mathfrak{C}_r$ then $\mathfrak{A}=\tau_S\sigma_M(\mathfrak{A})$ and hence $\sigma_M(\mathfrak{A})\in\mathcal{M}_2$.■

A ring R is a *right (left)* CS-*ring* if R_R ($_RR$) is a CS-module. And R is said to be a right Goldie ring if it is a *right finite dimensional ring* (i.e. R_R is f.d.) with ACC on right annihilators.

<u>COROLLARY 6.12</u> *Let* M_R *be non-degenerate. Then*

a) S *is a right* CS-*ring if and only if every* $N\in\mathcal{M}_2$ *is a direct* *summand of* M_R;

b) *if* M_R *has* ACC *on* $\mathcal{M}_{\sigma\tau}$ *then* S *is a right Goldie ring.*

PROOF: a) follows from (6.11) and (6.2).

b) Assume that M_R has ACC on M-cotorsionless submodules and let $\mathcal{R}(W_1)\subseteq\mathcal{R}(W_2)\subseteq\cdots$ be an ascending chain in $\mathcal{A}_r$; then, by hypothesis, $\sigma_M\mathcal{R}(W_1)\subseteq\sigma_M\mathcal{R}(W_2)\subseteq\cdots$ gets stationary at some step n and then, for k≥n, we get $\mathcal{L}\mathcal{R}(W_n)= 1_S\sigma_M\mathcal{R}(W_n)= 1_S\sigma_M\mathcal{R}(W_k)= \mathcal{L}\mathcal{R}(W_k)$, and hence $\mathcal{R}(W_n)=\mathcal{R}(W_k)$ for all k≥n. Therefore S has ACC on right annihilator ideals (note that we can prove in the same way that DCC on M-cotorsionless submodules implies DCC on right annihilators).

To see that S is right Goldie, it remains to show that it is right f.d., but ACC on M-cotorsionless submodules implies ACC on $\mathcal{M}_2$, which in turn implies ACC on $\mathcal{C}_r$ (6.11) and hence (2.1) S is right f.d.■

In the previous paragraph, we had to introduce the notion of a-self-generator in order to get $\mathcal{M}_a=\mathcal{M}_1$. Now, it would be of interest to get conditions under which $\mathcal{M}_2$ coincides with the set of complement submodules of M_R. Two of the concepts already introduced will suffice to get $\mathcal{M}_2=\mathcal{M}_e$, though in this case these conditions are not necessary.

<u>THEOREM 6.13</u> *Let* M_R *be non-degenerate. If* M_R *is a self-generator or a* CS-*module, then* $\tau_S:\mathcal{M}_e\longrightarrow\mathcal{C}_r$ *and* $\sigma_M:\mathcal{C}_r\longrightarrow\mathcal{M}_e$ *are inverse projectivities.*

PROOF: By (6.11), it will suffice to prove that, under the stated conditions, $\mathcal{M}_2=\mathcal{M}_e$.

Let $N\in\mathcal{M}_2$ and let K be an e-closure for N in M. Since $N\subseteq_e K$, $\tau_S(N)\subseteq_e\tau_S(K)$ by (6.10.f), but we have $\tau_S(N)\in\mathcal{C}_r$, whence $\tau_S(N)=\tau_S(K)$ and $N=\sigma_M\tau_S(N)=\sigma_M\tau_S(K)$. If M_R is a self-generator then K is M-cotorsionless, and if M_R is a CS-module then $K\subseteq_d M$; in any case $K=\sigma_M\tau_S(K)=N$ and therefore N is a complement in M.

For the converse inclusion, let $K\subseteq_c M$; then, as above, K is M-cotorsionless, and hence it remains to show that $\tau_S(K)\in\mathcal{C}_r$. Let $\mathfrak{A}$ be an e-closure for $\tau_S(K)$ in S_S; then $\tau_S(K)\subseteq_e\mathfrak{A}$ and, by (6.10.b), $K=\sigma_M\tau_S(K)\subseteq_e\sigma_M(\mathfrak{A})$, which implies $K=\sigma_M(\mathfrak{A})$ and therefore $\mathfrak{A}\subseteq\tau_S\sigma_M(\mathfrak{A})=\tau_S(K)$, i.e. $\tau_S(K)=\mathfrak{A}\in\mathcal{C}_r$ and hence $K\in\mathcal{M}_2$.■

COROLLARY 6.14 *Let* M_R *be non-degenerate. Then*

a) *if* M_R *is a self-generator or a* CS-*module, then* $u(M_R)=u(S_S)$*;*

b) *if* M_R *is a self-generator then* M_R *is a* CS-*module if and only if* S *is a right* CS-*ring;*

c) *if* M_R *is a* CS-*module then* S *is a right* CS-*ring.*

PROOF: a) follows from (6.13) and (2.7); b) and c) follow from (6.12.a), using the fact that $\mathcal{M}_2=\mathcal{M}_e$.■

The conditions imposed on the non-degenerate module M_R in (6.13) are not the only ones under which $\mathcal{M}_e$ and $\mathcal{C}_r$ are isomorphic. The next result makes further use of the uniqueness of the e-closures in a nonsingular module to show that also nonsingularity of M_R implies the existence of such an isomorphism, although in this case we have to change slightly the definition of our maps.

THEOREM 6.15 *Let* M_R *be nonsingular and non-degenerate. Then the maps* $\tau_S:\mathcal{M}_e\longrightarrow\mathcal{C}_r$ *and* $\mathfrak{A}\mapsto[\sigma_M(\mathfrak{A})]^e$ *from* $\mathcal{C}_r$ *to* $\mathcal{M}_e$ *are inverse projectivities.*

PROOF: First, we see that $\tau_S(\mathcal{M}_e)\subseteq\mathcal{C}_r$; if $K\leq_c M$ and $\mathfrak{A}$ is an e-closure for $\tau_S(K)$ in S_S, then (6.10.b) $\sigma_M\tau_S(K)\subseteq_e\sigma_M(\mathfrak{A})$; but (6.10.e) K is the e-closure of $\sigma_M\tau_S(K)$ and hence, by nonsingularity, also of $\sigma_M(\mathfrak{A})$, i.e. $K=[\sigma_M(\mathfrak{A})]^e$; in particular, $\sigma_M(\mathfrak{A})\leq K$ and hence $\mathfrak{A}\leq\tau_S\sigma_M(\mathfrak{A})\leq\tau_S(K)$, i.e. $\tau_S(K)=\mathfrak{A}\in\mathcal{C}_r$.

Since the image of $\mathfrak{A}$ under $\mathfrak{A}\mapsto[\sigma_M(\mathfrak{A})]^e$ is in $\mathcal{M}_e$, it only remains to show that both maps are inverse of each other.

As we have mentioned above, $K=[\sigma_M\tau_S(K)]^e$ for all $K\in\mathcal{M}_e$; on the other hand, if $\mathfrak{A}\in\mathcal{C}_r$, then $\mathfrak{A}=\tau_S\sigma_M(\mathfrak{A})$ by (6.10.d); since $\sigma_M(\mathfrak{A})\subseteq_e[\sigma_M(\mathfrak{A})]^e$, (6.10.f) yields $\mathfrak{A}\subseteq_e\tau_S\big([\sigma_M(\mathfrak{A})]^e\big)$ and hence $\mathfrak{A}=\tau_S\big([\sigma_M(\mathfrak{A})]^e\big)$, showing that both maps are inverse of each other.■

COROLLARY 6.16 *Let* M_R *be nonsingular and non-degenerate. Then*

a) $u(M_R)=u(S_S)$*;*

b) S *is a right* CS-*ring if and only if* M_R *is a* CS-*module.*

PROOF: a) follows from (6.15) and (2.7), and b) follows from (6.15) and (6.2).■

Correspondence Theorems for Principal and Finitely Generated Ideals

Write $\mathcal{P}_l$ and $\mathcal{P}_r$ for the sets of left and right principal ideals of S:

$$\mathcal{P}_l=\{Sf: f\in S\} \qquad \mathcal{P}_r=\{fS: f\in S\}.$$

Also, let $\mathcal{F}_l$ and $\mathcal{F}_r$ represent the sets of finitely generated left and right ideals of S, that is

$$\mathcal{F}_l=\{\sum_{i=1}^{n} Sf_i:\ f_1,\ldots,f_n\in S\} \qquad \mathcal{F}_r=\{\sum_{i=1}^{n} f_iS:\ f_1,\ldots,f_n\in S\}.$$

Recall that $r_M(Sf)=\mathrm{Ker}f$ and $\sigma_M(fS)=\mathrm{Im}f$ for all $f\in S$; hence, if we write

$$\mathcal{K}=\{\mathrm{Ker}f; f\in S\} \qquad \mathcal{I}=\{\mathrm{Im}f: f\in S\},$$

then $r_M(\mathcal{P}_l)=\mathcal{K}$ and $\sigma_M(\mathcal{P}_r)=\mathcal{I}$. Further, if we set

$$\mathcal{K}_F=\{\bigcap_{i=1}^{n}\mathrm{Ker}f_i:\ f_1,\ldots,f_n\in S\} \qquad \mathcal{I}_F=\{\sum_{i=1}^{n} f_iM:\ f_1,\ldots,f_n\in S\},$$

then $r_M(\mathcal{F}_l)=\mathcal{K}_F$ and $\sigma_M(\mathcal{F}_r)=\mathcal{I}_F$.

Since, on the other hand, $\mathcal{K}$ and $\mathcal{K}_F$ are always included in $\mathcal{M}_a$, while $\mathcal{I}$ and $\mathcal{I}_F$ always lie in $\mathcal{M}_{\sigma\tau}$, we get the following equivalences:

THEOREM 6.17 *With the above notation,*

a) l_S, r_M *determine a duality between* $\mathcal{K}$ *and* $\mathcal{P}_l$ $\Leftrightarrow$ $\mathcal{P}_l\subseteq\mathcal{S}_a$;

b) τ_S, σ_M *determine a projectivity between* $\mathcal{I}$ *and* $\mathcal{P}_r$ $\Leftrightarrow$ $\mathcal{P}_r\subseteq\mathcal{S}_{\tau\sigma}$;

c) l_S, r_M *determine a duality between* $\mathcal{K}_F$ *and* $\mathcal{F}_l$ $\Leftrightarrow$ $\mathcal{F}_l\subseteq\mathcal{S}_a$;

d) τ_S, σ_M *determine a projectivity between* $\mathcal{I}_F$ *and* $\mathcal{F}_r$ $\Leftrightarrow$ $\mathcal{F}_r\subseteq\mathcal{S}_{\tau\sigma}$. ∎

Recall that a ring is *regular* if every principal (left or right) ideal (equivalently, every f.g. left or right ideal) is generated by an idempotent; and that a ring is *letf (right) perfect* if its principal (equivalently, f.g.) right (left) ideals satisfy the DCC [5]. Therefore we get, from (6.17) and (6.2),

COROLLARY 6.18 *With the above notation, and if* $\mathcal{D}=\{N\subseteq M_R:\ N\subseteq_d M\}$,

a) *if* $\mathcal{P}_l\subseteq\mathcal{S}_a$ *then:* S *is regular* $\Leftrightarrow$ $\mathcal{K}\subseteq\mathcal{D}$;

a') *if* $\mathcal{P}_l\subseteq\mathcal{S}_a$ *then:* S *is right perfect* $\Leftrightarrow$ M *has* ACC *on* $\mathcal{K}$;

b) *if* $\mathcal{P}_r\subseteq\mathcal{S}_{\tau\sigma}$ *then:* S *is regular* $\Leftrightarrow$ $\mathcal{I}\subseteq\mathcal{D}$;

b') *if* $\mathcal{P}_r\subseteq\mathcal{S}_{\tau\sigma}$ *then:* S *is left perfect* $\Leftrightarrow$ M *has* DCC *on* $\mathcal{I}$;

c) *if* $\mathcal{F}_l \subseteq \mathcal{S}_a$ *then:* S *is regular* ⇔ $\mathcal{K}_F \subseteq \mathcal{D}$;

c') *if* $\mathcal{F}_l \subseteq \mathcal{S}_a$ *then:* S *is right perfect* ⇔ M *has* ACC *on* $\mathcal{K}_F$;

d) *if* $\mathcal{F}_r \subseteq \mathcal{S}_{\tau\sigma}$ *then:* S *is regular* ⇔ $\mathcal{I}_F \subseteq \mathcal{D}$;

d') *if* $\mathcal{F}_r \subseteq \mathcal{S}_{\tau\sigma}$ *then:* S *is left perfect* ⇔ M *has* DCC *on* $\mathcal{I}_F$.■

Now, it is of interest to seek for conditions on M_R under which one of the equivalent conditions of (6.17) holds. In fact, conditions a) and c) (resp. b) and d)) hold in any quasi-injective module (resp. quasi-projective module), but we shall postpone the proof of this until Section 10 since then we will have introduced the concept of T-nilpotency, which will be needed when applying these facts.

Nevertheless, we can prove now the following result, which gives as a corollary the fact that S is regular if and only if $\mathcal{K} \subseteq \mathcal{D}$ and $\mathcal{I} \subseteq \mathcal{D}$, with the notation of (6.18).

<u>PROPOSITION 6.19</u> a) *If* $\mathcal{I} \subseteq \mathcal{D}$ *then* $\mathcal{P}_l \subseteq \mathcal{S}_a$; b) *if* $\mathcal{K} \subseteq \mathcal{D}$ *then* $\mathcal{P}_r \subseteq \mathcal{S}_{\tau\sigma}$.

PROOF: a) Assume $\mathcal{I} \subseteq \mathcal{D}$ and let f∈S; we need to prove that $Sf = \mathbf{l}_S\mathbf{r}_M(Sf)$ or, equivalently, that $\mathbf{l}_S\mathbf{r}_M(Sf) \subseteq Sf$. Let $g \in \mathbf{l}_S\mathbf{r}_M(Sf)$; then Kerf⊆Kerg and hence $h_1: fM \to gM$ given by $h_1(fx)=gx$ is a well-defined R-homomorphism. Now, if M=fM⊕N and we define h∈S by $h|_{fM}=h_1$ and $h|_N=0$, then hf=g and therefore g∈Sf, as required.

b) Assume $\mathcal{K} \subseteq \mathcal{D}$ and let f∈S; we have to prove that $\tau_S\sigma_M(fS) \subseteq fS$. Let $g \in \tau_S\sigma_M(fS)$, i.e. gM⊆fM, and let t,q be idempotents of S such that Kerf=Kert and Kerg=Kerq (5.4.a).

For all x∈M, there exists y∈M with gx=fy; we claim that the map $h_1: qM \to tM$ given by $h_1(qx)=ty$ is well-defined (and hence it is clearly an R-homomorphism); for, if qx=qx′ then x-x′∈Kerq=Kerg, so that gx=gx′, and similarly fy=fy′ implies ty=ty′.

Define then h∈S by $h|_{qM}=h_1$ and $h|_{Kerq}=0$; we shall prove that g=hf, which will finish the proof. First note that, for any z∈M, z=tz+(1-t)z with (1-t)z∈Kert=Kerf, so that fz=ftz. Now, for any x∈M, x=qx+(1-q)x and, if y∈M is such that gx=fy, then hx= hqx= $h_1$qx= ty and hence fhx= fty= fy= gx, so that g=fh.■

<u>COROLLARY 6.20</u> *For any module* M_R, S *is a regular ring if and only if the kernel and the image of every endomorphism of* M_R *are direct summands of* M_R.

PROOF: Assume that S is regular; then for every f∈S there exist idempotents t,q of S such that fS=tS and Sf=Sq, and hence we get

$$fM = fSM = tSM = tM \in \mathcal{D} \quad \text{and} \quad \mathrm{Ker} f = \mathbf{r}_M(Sf) = \mathbf{r}_M(Sq) = \mathrm{Ker} q \in \mathcal{D}.$$

Conversely, if $\mathcal{K}\subseteq\mathcal{D}$ and $\mathcal{I}\subseteq\mathcal{D}$ then either (6.19.a) and (6.18.a) or (6.19.b) and (6.18.b) imply that S is regular.■

Finally, we prove a lemma which will allow us to apply our previous results to nonsingular continuous modules. A module M_R is said to be *continuous* if it is a CS-module such that every submodule of M_R isomorphic to a direct summand of M_R is again a direct summand of M_R.

LEMMA 6.21 *Let* M_R *be a module for which* $\mathcal{K}\subseteq\mathcal{D}$. *Then* $\mathcal{I}\subseteq\mathcal{D}$ *if and only if, for all* $N\in\mathcal{D}$ *and for every monomorphism* $h:N\to M$, *we have* $hN\in\mathcal{D}$.

PROOF: The 'only if' part does not need the hypothesis $\mathcal{K}\subseteq\mathcal{D}$: if $\mathcal{I}\subseteq\mathcal{D}$ and N, $h:N\to M$ are as stated, take $K\subseteq M_R$ such that $M=N\oplus K$ and extend h to f∈S by requiring $f|_K=0$; then hN=fN which is in $\mathcal{D}$ by assumption.
Conversely, assume that $\mathcal{K}\subseteq\mathcal{D}$ and that $hN\in\mathcal{D}$ for all $N\in\mathcal{D}$ and every monomorphism $h:N\to M$, and let f∈S; thus $M=\mathrm{Ker}f\oplus K$ for some $K\subseteq M_R$; hence $h=f|_K:K\to M$ is monic and thus $fM=hK\in\mathcal{D}$. Therefore $\mathcal{I}\subseteq\mathcal{D}$.■

COROLLARY 6.22 a) *If* M_R *is continuous then* S *is regular* ⇔ $\mathcal{K}\subseteq\mathcal{D}$;
b) *if* M_R *is a nonsingular* CS-*module then* S *is regular* ⇔ $\mathcal{I}\subseteq\mathcal{D}$;
c) *if* M_R *is nonsingular and continuous then* S *is a regular ring.*

PROOF: a) The 'only if' part follows from (6.20). If M_R is continuous and $\mathcal{K}\subseteq\mathcal{D}$ then (6.21) implies $\mathcal{I}\subseteq\mathcal{D}$ and hence S is regular by (6.20).
b) If M_R is nonsingular and CS then $\mathcal{M}_a\subseteq\mathcal{M}_e$ (6.7.a) and $\mathcal{M}_e=\mathcal{D}$, whence $\mathcal{K}\subseteq\mathcal{M}_a\subseteq\mathcal{M}_e=\mathcal{D}$. Therefore S is regular if and only if $\mathcal{I}\subseteq\mathcal{D}$ by (6.20).
c) If M_R is nonsingular and continuous then, as above, $\mathcal{K}\subseteq\mathcal{D}$ and hence S is regular by a).■

SECTION 7: THE ENDOMORPHISM RING OF A QUASI-INJECTIVE MODULE

Throughout this section we shall study the ring S of endomorphisms of a quasi-injective module M_R. J=J(S) will denote the Jacobson radical of S. The main results in this area concern, rather than S itself, the factor ring S/J, which is sometimes called the *associated ring* of M. They are due to B.Osofsky [41], J.Roos [45] and G.Rénault [R75 & 44], among others, who followed techniques introduced by Y.Utumi [57], E.Wong and R.Johnson [64] to show that S/J is a regular and right self-injective ring (7.11).

Further results on the endomorphism ring of quasi-injective or injective modules with some finiteness conditions (such as chain conditions or finite dimension) will be proved in Sections 9 and 10.

Let us denote the factor ring S/J by $\bar{S}$ and, for any f∈S, let $\bar{f}$ be its image in $\bar{S}$. The ideal Γ=Γ(S), introduced in the following lemma, will be of key importance in what follows.

<u>LEMMA 7.1</u> *Let* M_R *be any module and let* S=End(M_R). *Then the set* Γ(S)={f∈S:Kerf$\subseteq_e$E} *is an ideal of* S.

PROOF: Let f,g∈Γ(S) and h∈S. Then Kerf$\subseteq_e$M and Kerg$\subseteq_e$M, whence Kerf∩Kerg$\subseteq_e$M. Since Kerf∩Kerg⊆Ker(f+g), we have f+g∈Γ(S); and since Kerf⊆Ker(hf), we get hf∈Γ(S). Note that Ker(fh)=h^{-1}(Kerf); then (1.2.d) gives Ker(fh)$\subseteq_e$M, whence fh∈Γ(S). Therefore Γ(S) is a two-sided ideal of S.■

The next proposition was first proved, for M_R injective, by Y.Utumi [55; Lemma 8].

<u>PROPOSITION 7.2</u> *If* M_R *is quasi-injective then* J(S)=Γ(S) *and* $\bar{S}$ *is a (von Neumann) regular ring.*

PROOF [13; Theo.3.1.a]: Write J=J(S) and Γ=Γ(S). First we prove that Γ⊆J: Let f∈Γ; Kerf∩Ker(1-f)=0 implies Ker(1-f)=0, whence (1-f):M⟶(1-f)M is an isomorphism whose inverse g:(1-f)M⟶M extends to some h∈S, for which h(1-f)=1; thus every element of Γ is left quasi-regular and hence $\Gamma \subseteq J$ [A-F; Theo.15.3].

Next we see that S/Γ is a regular ring: Given $f \in S$, set $K = \mathrm{Ker} f$ and take a relative complement N for K in M; then $f|_N$ is monic with inverse $g: fN \to N$. Since M is quasi-injective, g can be lifted to $h \in S$, and then $K \oplus N \subseteq \mathrm{Ker}(f - fhf)$: for, if $k \in K$ and $n \in N$, then

$$fhf(k+n) = fhf(n) = fgf(n) = f(n) = f(k+n).$$

Now, since $K \oplus N \subseteq_e M$, $f - fhf \in \Gamma$, i.e. f and fhf have the same image in S/Γ, and this shows that S/Γ is regular.

Finally, we show that $J \subseteq \Gamma$, which will complete the proof. If $f \in J$, choose $h \in S$ such that $g = f - fhf \in \Gamma$; since $1 - fh$ has an inverse we get $f = (1-fh)^{-1} g \in \Gamma$. Therefore $J \subseteq \Gamma$. ∎

Our purpose now is to show that $\bar{S}$ is a right self-injective ring. At a first stage, we will prove this for the endomorphism ring of an injective module, but at the end of the section we will see that the result also holds for quasi-injectives. We need some technical lemmas about the lifting of idempotents from $\bar{S}$ to S; the proofs given here are due to G.Rénault [44 or R75].

<u>PROPOSITION 7.3</u> *If* E_R *is injective and* $S = \mathrm{End}(E_R)$, *then every idempotent of* $\bar{S}$ *can be lifted to an idempotent of* S.

PROOF [13]: Suppose $\bar{f}^2 = \bar{f} \in \bar{S}$; then $h = f^2 - f \in J$. If $f' = f|_{\mathrm{Ker} h}$ then it is easy to see that $\mathrm{Ker} h = \mathrm{Ker} f \oplus \mathrm{Im} f'$. Now, since $\mathrm{Ker} h \subseteq_e E$ (7.2), if E_1 and E_2 are injective hulls in E of $\mathrm{Ker} f$ and $\mathrm{Im} f'$, respectively, then $E = E_1 \oplus E_2$ (3.8). Let $t: E_1 \oplus E_2 \xrightarrow{c} E_2$; then $\mathrm{Ker} h \subseteq \mathrm{Ker}(t - f)$ for if $x \in \mathrm{Ker} h$ is written as $x = y + fz$ (with $y \in \mathrm{Ker} f$, $z \in \mathrm{Ker} h$) then

$$tx = fz = (f^2 - h)z = f^2 z = f(fz + y) = fx.$$

This shows that $t - f \in J$ (because $h \in J$) and hence $\bar{f} = \bar{t}$ with $t^2 = t \in S$. ∎

<u>REMARK</u> By [S; p.186-7], (7.3) shows that any countable family of orthogonal idempotents of $\bar{S}$ lifts to a family of orthogonal idempotents of S. The same remark applies to the following corollary.

<u>COROLLARY 7.4</u> *Let* R *be a right self-injective ring, and let* $J(R)$ *be*

the Jacobson radical of R. *Then* $J(R)=Z_r(R)$, $R/J(R)$ *is a regular ring and idempotents can be lifted modulo* $J(R)$.

PROOF [57;Theo.4.6 & Cor.4.10]: The last two statements are direct consequences of (7.2) and (7.3), using the ring isomorphism $S=End(R_R)\cong R$ (see Section 5).

To see that $J(R)=Z_r(R)$, note that the ring isomorphism $\varphi:R\longrightarrow S$ carries $r\in R$ to the endomorphism "left multiplication by r", and then $Ker(\varphi(r))=\mathcal{R}(r)$ for all $r\in R$. Therefore

$$r\in J(R) \Leftrightarrow \varphi(r)\in J(S) \Leftrightarrow Ker(\varphi(r))\subseteq_e R_R \Leftrightarrow \mathcal{R}(r)\subseteq_e R \Leftrightarrow r\in Z_r(R).$$

Hence $J(R)=Z_r(R)$.■

<u>LEMMA 7.5</u> *Suppose* E_R *is injective and* $S=End(E_R)$. *Let* $\{t_i:i\in I\}$ *be a family of idempotents of* S *such that the sum* $\sum_I \bar{t}_i\bar{S}$ *is direct. Then the sum* $\sum_I t_iS$ *is direct and, if* I *is finite, then there exists* $t^2=t\in S$ *such that* $\oplus_I t_iS=tS$.

PROOF: Since in general a sum $\sum_I A_i$ is direct if and only if so is $\sum_F A_i$ for every finite subset F of I, we can assume I to be finite, say $I=\{1,...,n\}$.

Set $E_i=t_iE$ for $i=1,...,n$. If we prove that the E_i are independent then so are the t_iS: for, suppose e.g. that $t_1f_1=t_2f_2+\cdots+t_nf_n$ is an element of $(t_1S)\cap(\sum_2^n t_iS)$; then for all $x\in E$ we get

$$t_1f_1(x)=\sum_2^n t_if_i(x) \in E_1\cap(\sum_2^n E_i) = 0;$$

hence $t_1f_1=0$.

Let us now prove that $\sum_1^n E_i=\oplus_1^n E_i$ by induction in n. Suppose the sum $F=\sum_2^n E_i$ is direct and $G=F\cap E_1\neq 0$; let H be an injective hull of G contained in E_1 and let $e^2=e\in S$ be such that $H=eE$. Then clearly $e=t_1e$.

Since F is injective, the inclusion map $G\longrightarrow F$ lifts to some $h:E\longrightarrow F$ for which we have $h=\sum_2^n t_ih$. For, if $x\in E$ then $hx=\sum_2^n t_iy_i$ for some $y_i\in E$, whence

$$\left(\sum_2^n t_ih\right)x = \sum_i\left(\sum_j t_it_jy_j\right) = \sum_2^n t_i^2y_i = \sum_2^n t_iy_i = hx.$$

Now, since $G\subseteq_e H$ and $E=H\oplus Ker(e)$, we get $G\oplus Ker(e)\subseteq_e E$; but e and he coincide in G (in fact they are both the identity in G), and thus so do they in $G\oplus Ker(e)$, whence $\bar{e}=\bar{h}\bar{e}$.

Thus, we get $\bar{t}_1\bar{e}=\bar{e}=\bar{h}\bar{e}=\sum_2^n \bar{t}_i\bar{h}\bar{e}$ with $\bar{e}\neq 0$ ($Ker(e)\cap H=0$ with $H\neq 0$), contradicting the hypothesis that the sum $\sum_1^n \bar{t}_i\bar{S}$ is direct. Therefore $\sum_1^n E_i$ is direct, and then so is $\sum_I t_i S$.

Now, each E_i is injective and thus so is $\oplus_1^n E_i$; hence there exists $E_o \subseteq E_R$ such that $E=\oplus_o^n E_i$; let $q_j:\oplus_o^n E_i \xrightarrow{c} E_j$, so that, for $i=1,\dots,n$, we have $q_i=t_iq_i$, and call $t=q_1+\cdots+q_n$. Then, for all $f\in S$, we get

$$tf = \sum_1^n q_i f = \sum_1^n t_i q_i f \in \sum_1^n t_i S,$$

and, for all $f_1,\dots,f_n\in S$, we get $(\sum_1^n t_i f_i)E \subseteq \oplus_1^n E_i$ and thus

$$(\sum_1^n t_i f_i) = t(\sum_1^n t_i f_i) \in tS.$$

Therefore $tS=\sum_1 t_i S$, which proves the last statement of the lemma.∎

<u>LEMMA 7.6</u> *With the above notation, let* $\{t_i : i\in I\}$ *be a family of idempotents of* S *such that the sum* $\sum_I \bar{t}_i\bar{S}$ *is direct, and let* $\mathfrak{B}=\oplus_I t_i S$. *Then every homomorphism* $\phi:\mathfrak{B}_S\to S_S$ *extends to an endomorphism of* S_S.

PROOF: Consider the right R-module $F=\mathfrak{B}E$ and define $g:F_R\to E_R$ as follows: If $z=\sum_J t_j f_j x_j$ for some finite set J (where, for all $j\in J$, $f_j\in S$, $x_j\in E$ and the t_j's are elements of $\{t_i : i\in I\}$, possibly repeated) then set $g(z)=\sum_J \phi(t_j f_j)x_j$.

If g is well-defined, then it is clearly an R-homomorphism. To see that g is actually single-valued, suppose another expression of z is given and let K be the (finite) set consisting of those members of $\{t_i : i\in I\}$ which appear in any of these two expression. Then, by (7.5), there exists an idempotent t of S such that $\oplus_K t_k S=tS$, and then $t_j=tt_j$ for all $j\in J$; hence

$$g(z) = \sum_J \phi(tt_j f_j)x_j = \sum_J \phi(t)t_j f_j x_j = \phi(t)z$$

independently of the representation of z.

Therefore g extends to some $h\in S$; now, if $f\in\mathfrak{B}$ may be written as $f=\sum_K t_k f_k$ for some finite subset K of I and some $f_k\in S$, then, for all $x\in E$,

$$\phi(f)x = \phi(\sum_K t_k f_k)x = \sum_K \phi(t_k f_k)x = g(\sum_K t_k f_k x) = h(\sum_K t_k f_k x) = (hf)x.$$

Therefore $\phi(f)=hf$ for all $f\in\mathfrak{B}$ and thus ϕ can be extended to an endomorphism of S_S.∎

In view of (7.6), the following characterization of regular self-injective rings will clearly help us.

LEMMA 7.7 *Let* R *be a regular ring. Then* R *is right self-injective if and only if for every right ideal* $\mathfrak{b}$ *of* R *of the form* $\mathfrak{b}=\oplus_I t_i R$ *(where* $\{t_i : i\in I\}$ *is a family of idempotents of* R*), every homomorphism* $\mathfrak{b}_R \longrightarrow R_R$ *can be extended to a endomorphism of* R_R.

PROOF: The necessity is clear. For the sufficiency let $\mathfrak{a}$ be any right ideal of R. By Zorn's Lemma there exists a maximal element $\mathfrak{b}$ in the family of all direct sums $\oplus_I t_i R \subseteq \mathfrak{a}$ where the t_i's are idempotents of R, and since R is regular this maximal element is an essential submodule of $\mathfrak{a}$ (recall that every principal right ideal of R is generated by an idempotent).
Then for an arbitrary $f:\mathfrak{a}\longrightarrow R$ let $g=f|_{\mathfrak{b}}$ and let $h\in End(R_R)$ be an extension of g. Then, for all $x\in\mathfrak{a}$, $\delta=(\mathfrak{b}:x)$ is an essential right ideal of R (1.1) and $(h-f)x\delta=0$; since R is right nonsingular [S; p.244], we get hx=fx and therefore h extends f, as desired.∎

THEOREM 7.8 *Let* E_R *be an injective module,* S *its endomorphism ring and* J=J(S) *the Jacobson radical of* S*. Then* S/J *is a regular right self-injective ring.*

PROOF [44;Theo.3.2] or [R75;p.85]: By (7.2), $\bar{S}=S/J$ is regular. Then, by (7.7), it suffices to show that, for each right ideal of $\bar{S}$ of the form $\mathfrak{B}=\oplus_I \bar{t}_i \bar{S}$ (with $\{\bar{t}_i : i\in I\}$ a family of idempotents of $\bar{S}$ which, by (7.5), may be taken in such a way that every $t_i^2=t_i$ in S), and for each homomorphism $\phi:\mathfrak{B}_{\bar{S}}\longrightarrow\bar{S}_{\bar{S}}$, there exists a right $\bar{S}$-endomorphism of $\bar{S}$ which extends ϕ.
Given ϕ, let $\bar{f}_i=\phi(\bar{t}_i)$; then $\bar{f}_i=\phi(\bar{t}_i^2)=\phi(\bar{t}_i)\bar{t}_i=\bar{f}_i\bar{t}_i$. By (7.5), the sum $\mathfrak{A}=\sum_I t_i S$ is direct, so that the correspondence $t_i\longmapsto\varphi(t_i)=f_i t_i$ defines a right S-homomorphism $\varphi:\mathfrak{A}\longrightarrow S$. By (7.6), there exists $f_o\in S$ with $\varphi(t_i)=f_o t_i$ for all $i\in I$. Then the right endomorphism of $\bar{S}$ defined by left multiplication by $\bar{f}_o$ extends ϕ, and this completes the proof of the theorem.∎

Theorem 7.8 yields as a corollary the following result of Y.Utumi [57; Theo.8].

COROLLARY 7.9 *If* R *is a right self-injective ring then* R/J(R) *is also right self-injective.*■

The following proposition allows us to extend our results to quasi-injective modules.

PROPOSITION 7.10 *Let* M_R *be a quasi-injective module,* E_R *its injective hull,* S=End(M_R) *and* H=End(E_R). *Then* H/J(H) *and* S/J(S) *are ring isomorphic.*

PROOF: Since M_R is quasi-injective, (3.10) implies that the map $\phi: H \longrightarrow S$ given by $\phi(f)=f|_M$ is well defined. Clearly, it is a ring homomorphism and, since E_R is injective, ϕ is surjective. Composing with the natural ring epimorphism $\pi: S \longrightarrow S/J(S)$ we get $\frac{S}{J(S)} \cong \frac{H}{Ker(\pi\phi)}$, and then all we have to check is that J(H) coincides with $Ker(\pi\phi)= \phi^{-1}(Ker\pi)= \phi^{-1}J(S)$. For, let $f \in H$; then we have

$$f \in \phi^{-1}J(S) \Leftrightarrow \phi(f) \in J(S) \Leftrightarrow Ker(f|_M) \subseteq_e M \Leftrightarrow M \cap Kerf \subseteq_e M \Leftrightarrow Kerf \subseteq_e E \Leftrightarrow f \in J(H),$$

whence effectively $J(H)=\phi^{-1}J(S)$.■

Thus we get as a corollary the announced result of B.Osofsky [41; Theo.12], G.Rénault [44; Cor.3.5] and J.E.Roos [45; p.176].

THEOREM 7.11 *If* M_R *is a quasi-injective module,* S *its endomorphism ring and* J=J(S) *the Jacobson radical of* S, *Then* S/J *is a regular right self-injective ring.*■

There is an important case, namely when M_R is not only quasi-injective but also nonsingular, in which case J(S)=0 and then we obtain a result which, together with (7.4) and (7.9), was the motivation for the study undertaken in this section. It appeared in the form given here in [64; Theo.5], although the proof of the self-injectivity of S is attributed to Y.Utumi.

THEOREM 7.12 *If* M_R *is a nonsingular quasi-injective module then* S=End(M_R) *is a regular right self-injective ring.*

PROOF: After (7.11) and (7.2) it suffices to see that $\Gamma=\Gamma(S)=0$. For, let $f\in\Gamma$ and $K=\mathrm{Ker} f\subseteq_e M$. For any $x\in M$ we have $\alpha=(K:x)\subseteq_e R_R$ (1.1) and $x\alpha\subseteq K$; then $(fx)\alpha=0$, and the nonsingularity of M gives $fx=0$. Therefore $f=0$ and thus $\Gamma=0$. ∎

<u>REMARKS</u> 1) Nonsingularity is necessary in (7.12). For example, consider the Abelian group $\mathbb{Z}_{p^\infty}$ (the p-primary component of $\mathbb{Q}/\mathbb{Z}$, where p is any prime integer), which is an injective $\mathbb{Z}$-module, but its endomorphism ring is the ring of p-adic integers [F; p.211], which is not self-injective [41; p.897].

2) Any semisimple module is quasi-injective with $\Gamma(S)=0$, so that the endomorphism ring of a semisimple module is right self-injective.

3) B.Osofsky has investigated when S/J is also left self-injective. In [41] she proves, for E_R quasi-injective and using results of Utumi [56] that S/J is left self-injective if, for every orthogonal set $\{t_i : i\in I\}$ of idempotents of S, the map $\phi: E\to\prod_I t_i E$ given by $\phi(m)=\langle t_i m\rangle_{i\in I}$ is onto. In particular, for a right vector space V_D over a division ring D, $S=\mathrm{End}(V_D)$ is always right self-injective by the previous remark, but S is left self-injective if and only if V is finitely generated (i.e. finite dimensional) [G; Prop.2.23]. This may also be proved for free modules over QF-rings [S; p.278].

CHAPTER 8: THE ENDOMORPHISM RING OF A PROJECTIVE MODULE

Although there do not exist results for the endomorphism ring of an arbitrary projective module as strong as those given for (quasi-) injective modules in Section 7 (but see (8.5)), the literature about the subject is fairly wide. In particular, Morita's Theorem characterizes finitely generated projective generators of Mod_R as those modules M_R such that there exists a category equivalence between Mod_R and Mod_S, where $S=End(M_R)$. We begin this section by recalling Morita's Theorem and drawing some consequences.

Next, we study the Jacobson radical of $S=End(M_R)$ when M_R is projective and, as a consequence, we determine when S is a local ring. This prompts us to a brief introductory discussion of the so-called local, regular, perfect and semiperfect (projective) modules, with which we close the section.

The Morita Theorem; Finitely Generated Projective Modules

In this paragraph we shall make use of the language of Category Theory, with which the reader will be assumed to be familiar. For the standard definitions we refer to [A-F].

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F:\mathcal{C}\longrightarrow\mathcal{D}$ is said to be a *category equivalence* if there exists a functor $G:\mathcal{D}\longrightarrow\mathcal{C}$ such that GF (resp. FG) is naturally isomorphic to the identity functor in $\mathcal{C}$ (resp. in $\mathcal{D}$). This occurs if and only if F is full and faithful and, for every object D of $\mathcal{D}$, there exists an object C of $\mathcal{C}$ such that FC and D are isomorphic. Therefore, a category equivalence between $\mathcal{C}$ and $\mathcal{D}$ preserves and reflects most of the categorical properties of the objects and the morphisms of $\mathcal{C}$ and $\mathcal{D}$.

Two rings R and S are *(Morita) equivalent* if there exists a category equivalence between Mod_R and Mod_S (it turns out that it happens if and only if there exists a category equivalence between ${}_RMod$ and ${}_SMod$). Therefore, all the properties of a ring which may be stated in categorical terms are preserved by Morita equivalence. For example, if R and S are equivalent rings, then each one of the properties listed below hold in R if and only if it holds in S (such properties are called *Morita invariants*; see [FA73; p.220] for an extended list):

- (right or left) Noetherian
- (right or left) Artinian
- (von Neumann) regular
- (right or left) self-injective
- (right or left) hereditary
- (right or left) perfect
- (right or left) primitive
- prime
- simple
- semiperfect
- semiprimitive
- semiprime
- semisimple

A *generator* for Mod_R is a right R-module M_R such that, for any module N_R, there exists a set I and an epimorphism $M^{(I)}\longrightarrow N$, where $M^{(I)}$ represent the direct sum of copies of M indexed by I. If we set $H=Hom_R(M,N)$ and define a map $\pi:M^{(H)}\longrightarrow N$ via $\pi(x_h)_{h\in H}=\sum_{h\in H}h(x_h)$, then clearly $Im(\pi)=\mathbf{t}_N(M)$ (see Section 5), and from that it is easy to see that M_R is a generator of Mod_R if and only if $\mathbf{t}_N(M)=N$ for all right R-modules N.

A *progenerator* of Mod_R is just a finitely generated projective generator of Mod_R. A well known theorem of Morita [39] characterizes the rings which are equivalent to a given ring R (see e.g. [A-F;§22]).

<u>THEOREM</u> *For two rings* R,S *the following statements are equivalent:*
a) R *and* S *are Morita equivalent;*
b) *there exists a progenerator* M_R *of* Mod_R *with* $S\cong End(M_R)$*;*
c) *there exists a progenerator* $_RM$ *of* $_RMod$ *with* $S\cong End(_RM)$. ■

Therefore, the endomorphism ring of a progenerator of Mod_R inherits many of the properties of R (but not all, for example being a domain, a field, a commutative ring or an indecomposable ring are not Morita invariants [FA73; p.221]).

Sometimes, however, being a progenerator is too restrictive a condition in M_R for $S=End(M_R)$ to preserve some Morita invariants. In fact, many times 'finitely generated (f.g.) projective' is a sufficient condition for that.

For example, note that a f.g. projective module M_R is isomorphic to a direct summand of some finite direct sum of copies of R and so, in particular, M_R is a direct summand of a progenerator P_R of Mod_R. If we write $S=End(M_R)$, $H=End(P_R)$, and if e is an idempotent of H such that $M=eP$, then it is easily checked that the assignation $f\mapsto efe$ defines a ring isomorphism between S and eHe with inverse $ehe\mapsto eh|_M$. Thus, Morita invariants which do not vanish when one passes from a ring Q to qQq (for some $q^2=q\in Q$) are preserved for the endomorphism ring of f.g. projective modules. Specifically:

THEOREM 8.1 *Let* M_R *be a finitely generated projective module. If* R *has one of the properties below, then so does* $S=End(M_R)$*:*

a) R *is a regular ring;*

b) R *is a semiperfect ring;*

c) R *is a right perfect ring.*

PROOF: All three properties are Morita invariants. We see that eRe is regular whenever R is and $e^2=e\in R$: given $r\in R$, let $x\in R$ be such that $(ere)x(ere)=ere$; then $(ere)(exe)(ere)=ere$, so that eRe is regular. The same property for semiperfect rings follows from [A-F; Cor.27.7], and for perfect rings from [A-F; Theo.28.4.b) and Lemma 28.18].■

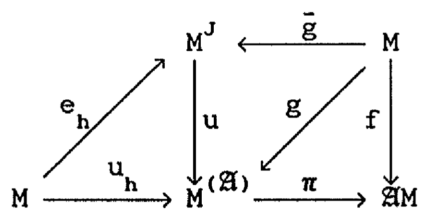
In fact, we can add 'R is right Noetherian' and 'R is right Artinian' to the list in (8.1), and it will follow as a particular case of Theorem 8.3. Before stating it, we need a result which is more easily proved if we introduce the notion of smallness in Mod_R.

A module M_R is said to be *small in* Mod_R [M; p.74] if, for any direct sum $\bigoplus_I M_i$ of right R-modules and for any homomorphism $f:M\longrightarrow\bigoplus_I M_i$, there exist a finite subset J of I and a homomorphism $\bar{f}:M\longrightarrow\bigoplus_J M_j$ such that $f=u_{IJ}\bar{f}$, where u_{IJ} is the canonical inclusion of $\bigoplus_J M_j$ in $\bigoplus_I M_i$.

For example, every f.g. module M_R is small in Mod_R, since the images via $f:M\longrightarrow\bigoplus_I M_i$ of a finite generating set of M_R (and hence all of fM) lie in only finitely many of the M_i's. In fact, for a projective module, finite generation and smallness are equivalent conditions.

<u>PROPOSITION 8.2</u> *If* M_R *is a finitely generated projective module then, for all right ideals* $\mathfrak{A}$ *of* $S=\mathrm{End}(M_R)$, *we have* $\mathfrak{A}=\mathrm{Hom}_R(M,\mathfrak{A}M)$ *(i.e. every right ideal is* $\tau\sigma$*-closed).*

PROOF [23;Lemma 2.6]: Note that, from the definitions, for any right right ideal $\mathfrak{A}$ of S, we get $\tau_S\sigma_M(\mathfrak{A})=\mathrm{Hom}(M,\mathfrak{A}M)$ and $\mathfrak{A}\subseteq\mathrm{Hom}(M,\mathfrak{A}M)$. On the other hand, let $f:M\longrightarrow\mathfrak{A}M$ be any homomorphism and consider the coproduct $M^{(\mathfrak{A})}$ with canonical inclusions $\{u_h:h\in\mathfrak{A}\}$; the maps $\{h:M\longrightarrow\mathfrak{A}M: h\in\mathfrak{A}\}$ induce a homomorphism $\pi:M^{(\mathfrak{A})}\longrightarrow\mathfrak{A}M$ such that $\pi u_h=h$ for all $h\in\mathfrak{A}$, and π is clearly an epimorphism.
Then, by projectivity of M_R, there exists $g:M\longrightarrow M^{(\mathfrak{A})}$ such that $f=\pi g$, and by smallness there exist a finite subset J of $\mathfrak{A}$ and a homomorphism $\bar{g}:M\longrightarrow M^J$ such that (writing u for $u_{\mathfrak{A}J}$) $g=u\bar{g}$.



Then, if $\{e_h:h\in J\}$ and $\{p_h:h\in J\}$ are the injections and projections, respectively, of the coproduct M^J, we get

$$f = \pi g = \pi u\bar{g} = \pi u(\sum_J e_h p_h)\bar{g} = \sum_J \pi(ue_h)p_h\bar{g} = \sum_J \pi u_h p_h\bar{g} = \sum_J h(p_h\bar{g}) \in \mathfrak{A}$$

since J is finite, $h\in\mathfrak{A}$ and, for each $h\in J$, $p_h\bar{g}\in S$.
Therefore $\mathfrak{A}=\mathrm{Hom}_R(M,\mathfrak{A}M)$.∎

<u>REMARK</u> This proof may be slightly modified in order to obtain a similar result in which finite generation is required not in M_R but in $\mathfrak{A}$. This will be done in (10.11) in a more general situation.

<u>THEOREM 8.3</u> a) *If* M_R *is projective and Noetherian then* $S=\mathrm{End}(M_R)$ *is right Noetherian.* b) *If* M_R *is finitely generated, projective and Artinian then* S *is right Artinian*

PROOF: Since in any case M_R is f.g., $\mathfrak{A}=\mathrm{Hom}_R(M,\mathfrak{A}M)$ for every right ideal $\mathfrak{A}$ of S, and therefore for any two right ideals $\mathfrak{A},\mathfrak{B}$ of S we have $\mathfrak{A}M=\mathfrak{B}M$ if and only if $\mathfrak{A}=\mathfrak{B}$. Thus, assuming a) (resp. b)), for a nonempty set $\Sigma=\{\mathfrak{A}_i;i\in I\}$ in $\mathrm{Lat}(S_S)$, the set $\{\mathfrak{A}_iM;i\in I\}$ has a maximal (resp. minimal) element $\mathfrak{A}_iM$, and then $\mathfrak{A}_i$ is maximal (resp. minimal) in Σ.∎

REMARK: In fact, any projective Artinian module is f.g., so that the condition 'f.g.' in (8.3.b) is redundant (see the proof of [22; Theo.2.8]). Moreover, even the endomorphism ring of a Σ-quasi-projective Artinian module is right Artinian [43; Theo.7].

The Jacobson Radical of the Endomorphism Ring of a Projective Module; Local Endomorphism Rings.

The Jacobson radical of the endomorphism ring of a projective module admits a description that is dual to that given for an injective module, and that may be improved if, in addition, RadM (the radical of M, see (1.5)) is a superfluous submodule of M_R.

It is known that, if M_R is projective and $J=J(R)$ is the Jacobson radical of R, then $RadM=MJ\subset M$ [A-F; Prop.17.10 & 17.14]. Also, if M_R is f.g. then every proper submodule L of M_R is included in a maximal submodule K and thus $L+RadM\subseteq K\subset M$; therefore $RadM \ll M$.

The proof of the next lemma is dual to (7.1).

LEMMA 8.4 *Let* M_R *be any module and let* $S=End(M_R)$. *Then the set* $\Delta(S)=\{f\in S:\ fM\ll M\}$ is a two-sided ideal of S.■

PROPOSITION 8.5 *Let* M_R *be a projective module,* $S=End(M_R)$ *and* $N=RadM$. *Then*

a) $J(S)=\Delta(S)\subseteq Hom(M,N)$;

b) *there exists a ring epimorphism* $S\longrightarrow End(M/N)$ *with kernel* $Hom(M,N)$.

c) *if* $N\ll M$ *(e.g. if* M_R *is finitely generated) then* $J(S)=Hom(M,N)$ *and hence* $S/J(S)$ *is ring isomorphic to* $End(M/N)$.

PROOF: a) $\Delta(S)\subseteq J(S)$: Let $f\in\Delta(S)$; since $M=fM+(1-f)M$ and $fM\ll M$, we get $(1-f)M=M$. Then the short exact sequence $0\longrightarrow Ker(1-f)\longrightarrow M\xrightarrow{1-f}M\longrightarrow 0$ splits (by projectivity of M) and hence $Ker(1-f)\subseteq_d M$; but since $Ker(1-f)\subseteq fM$, $Ker(1-f)$ is a superfluous direct summand of M (1.3), i.e. $Ker(1-f)=0$. Hence, $1-f$ is invertible for all $f\in\Delta(S)$, whence $\Delta(S)\subseteq J(S)$.
$J(S)\subseteq\Delta(S)$: Let $f\in J(S)$ and suppose $fM+N=M$ for some $N\subseteq M_R$; let $\pi:M\xrightarrow{c}M/N$, thus for any $x\in M$ we have $x=fy+z$ for some $y\in M$, $z\in N$, whence $x+N=fy+N$; therefore $\pi f:M\longrightarrow M/N$ is epic and thus there exists $g\in S$ such that $\pi fg=\pi$,

i.e. $\pi(1-fg)=0$, which means that $(1-fg)M\subseteq N$; but $(1-fg)M=M$ since $f\in J(S)$, whence $N=M$. Therefore $fM\ll M$, i.e. $f\in\Delta(S)$.
$J(S)\subseteq Hom(M,N)$: if $f\in J(S)$ then $fM\ll M$ and hence $fM\subseteq N$ by (1.5).

b) Define $\phi: S\longrightarrow End_R(M/N)$ as follows: Let $f\in S$; since $N=RadM$ is a fully invariant submodule of M we have $fN\subseteq N$ and then $\bar{f}: x+N\longmapsto fx+N$ defines an endomorphism of M/N. Let then $\phi(f)=\bar{f}$; this clearly defines a ring homomorphism, which is indeed an epimorphism by projectivity of M_R. It is also clear that $\bar{f}=0$ if and only if $fM\subseteq N$, so that $Ker\phi=Hom(M,N)$.

c) If $N\ll M$ and $f\in Hom(M_R,N_R)$ then (1.3.d) $fM\ll M$, whence $f\in\Delta(S)$. Then a) and b) yield c).■

REMARKS 1) Further characterizations of J(S) may be found in [61].

2) A module M_R is said to be *quasi-projective* if for any module $N=N_R$, any epimorphism $f:M\longrightarrow N$ and any homomorphism $g:M\longrightarrow N$, there exists an endomorphism h of M_R such that $g=fh$. A careful look at the proof of (8.5) reveals that it may be proved for M_R quasi-projective.

As a consequence of (8.5) we can characterize those projective modules which have a local endomorphism ring. Recall that a ring R is said to be *local* if its radical J(R) is a maximal right or left ideal or, equivalently, if $J(R)=\{r\in R: r \text{ is not invertible}\}$.

Before stating the next theorem, a dual of which will be proved in Section 9, we need to introduce the dual concept of the injective hull, namely the projective cover: A *projective cover* for M_R is a projective module P_R, together with an epimorphism $\pi:P\longrightarrow M$, such that $Ker\pi$ is a superfluous submodule of P_R.

Unlike injective hulls, projective covers for arbitrary modules seldom exist; e.g. if R is a ring with zero radical then only the projective R-modules possess a projective cover [A-F; Ex.17.14]. In particular, for $R=\mathbb{Z}$, this implies that an Abelian group has a projective cover if and only if it is free. In fact, the only rings R for which every right R-module has a projective cover are the right perfect rings defined in Section 6 (see [5], [A-F; §28] or [FA67; §22]).

THEOREM 8.6 *Let* M_R *be a projective module,* N=RadM *and* S=End(M_R). *The following statements are equivalent:*

a) S *is a local ring;*

b) M_R *is the projective cover of a simple right* R*-module;*

c) M_R *contains a submodule which is both superfluous and maximal;*

d) N *is a superfluous and maximal submodule of* M_R*;*

e) M_R *has a unique maximal submodule (necessarily equal to* N*) which contains every proper submodule of* M_R.

PROOF: a)⇒b) Assume that S is local. In particular, $S\neq 0$ and hence $M\neq 0$; therefore $N\subset M$, i.e. M contains a maximal submodule K. Thus M/K is a simple module, and if we prove $K\ll M$ then obviously M_R will be a projective cover for M/K, proving b).
Suppose then $L\leq M_R$ is such that K+L=M; we have to prove L=M. Since $K\subset M$ and $M/K = (K+L)/K \cong L/(K\cap L)$, there exists a nonzero homomorphism $g:M\longrightarrow L/(K\cap L)$, and by projectivity of M_R there exists $f:M_R\longrightarrow L_R$ such that $g=\pi f$, where $\pi:L\xrightarrow{c}L/(K\cap L)$. Now, since $g\neq 0$, fM is not included in K, and hence K+fM=M by maximality of K; therefore fM is not superfluous in M, i.e. $f\notin J(S)$ (8.5). But, since S is local, this implies that f is invertible, so that $M=fM\leq L$, i.e. L=M, as required.

b)⇒c) Note that b) just means that M_R contains a superfluous submodule K such that M/K is simple, i.e. K is also a maximal submodule of M_R.

c)⇒d) Let $K\leq M_R$ be superfluous and maximal in M_R; then (1.5)

$$K \leq \Sigma\{L\leq M_R: L\ll M\} = N = \cap\{L\leq M_R: \text{ L is maximal in } M_R\} \leq K,$$

i.e. N=K.

d)⇒e) Obviously, if N=RadM is maximal, then it is the only maximal submodule of M_R; now, if $L\subset M_R$ then, since $N\ll M$, $N\leq N+L\subset M$ and thus, by maximality of N, N=N+L, i.e. $L\leq N$.

e)⇒a) Clearly, if N contains every proper submodule of M_R, then $N\ll M$ and hence (8.5) $S/J(S)\cong End_R(M/N)$. Since N is maximal by hypothesis, M/N is simple and hence S/J(S) is a division ring (5.1), i.e. S is a local ring.■

REMARKS: 1) Since every proper submodule of M_R, for M_R satisfying the conditions of (8.6), is superfluous in M_R, we can add to these equivalences 'M$_R$ is a projective cover for all its nonzero quotient modules' (c.f. (9.2.c)).

2) In [60; §4], R.Ware remarks that the equivalence of the conditions which define a local ring depend largely on the fact that any ring is projective as a module over itself, and proves that most of these conditions remain equivalent when translated to an arbitrary projective module (e.g. b), c), d), e) of (8.6)).

He calls a module *local* if it is a projective module which satisfies these conditions, so that with this terminology (8.6) says that a projective module is local if and only if it has a local endomorphism ring [60; Theo.4.2].

3) Also, some of the conditions which define a regular, semiperfect or perfect ring remain equivalent when extended to projective modules. Thus, one can define, always within the classes of projective modules, *regular*, *semiperfect* or *perfect modules*. We state here, without proof, the properties of their endomorphism rings.

Regular modules are defined by R.Ware in [60; §§ 2 and 3], as those projective modules M$_R$ with the property that every cyclic submodule of M$_R$ is a direct summand of M$_R$ (definitions of arbitrary regular modules which agree with this one in the projective case may be found in [14] and [66]). For the given definition, f.g. regular modules have regular endomorphism rings [60; Theo.3.6] and, over a commutative ring, projective modules whose ring of endomorphism is regular are regular [60; Theo.3.9] (finite generation and commutativity are necessary).

Perfect and semiperfect modules were introduced by E.Mares in [32]; a projective module M$_R$ is *semiperfect* if every factor module of M$_R$ has a projective cover or, equivalently, if RadM«M, M/RadM is semisimple and decompositions of M/RadM can be lifted to M [32]. A projective module has a semiperfect endomorphism ring if and only if it is finitely generated and semiperfect ([32; Theo.6.1] and [60; Prop.1.5]).

A projective module M$_R$ is *perfect* if, for every set I and every factor module N of the direct sum $M^{(I)}$, N has a projective cover. A projective module has a perfect endomorphism ring if and only if it is finitely generated and perfect ([32; Theo.2.4 and Cor.7.5] and [60; Prop.5.2]).

SECTION 9: THE ENDOMORPHISM RING OF A FINITE DIMENSIONAL MODULE

We start this section by characterizing finite dimensional injective modules in terms of their endomorphism rings. In fact, an injective module E_R will be f.d. if and only if $S=End(E_R)$ is semiperfect (Theorem 9.5). This will allow us to embed the factor ring $S/\Gamma(S)$ of any f.d. module M_R in a semisimple ring, and as a consequence we will find conditions under which every nil subring of S is nilpotent (Theorems 9.8 and 9.13). These latter results are due to R. Shock [50].

Later on, we shall look for situations in which not only finite dimensionality, but also the dimension of M_R, is inherited by $S=End(M_R)$; this will be used, for example, to characterize some modules which have Goldie rings of endomorphisms (J. Hutchinson and J. Zelmanowitz [25]). In this area we will find some help in the results and techniques of Section 6.

Before that, we prove an easy but interesting result, which should be compared with (10.2).

PROPOSITION 9.1 *Let* M_R *be a finite dimensional module and let* $f\in S=End(M_R)$. *Then* f *is invertible if and only if it is left (or right) invertible.*

PROOF: We have to prove that, for all $f,g\in S$, $fg=1$ implies $gf=1$ (in fact, this condition is equivalent to M_R being *directly finite*, i.e. such that M_R is not isomorphic to a proper direct summand of itself, see [G; Lemma 6.9]).

First, note that a f.d. module cannot be isomorphic to a proper direct summand of itself, because if $M=N\oplus L$ with $N\cong M$ and $L\neq 0$ then $M\cong N\oplus L^n$ for all $n\in\mathbb{N}$, which is impossible.

Now suppose that $f,g\in S$ satisfy $fg=1$; then $t=1-gf$ is an idempotent of S such that $tg=0$, and thus $M=gM\oplus tM$: if $gx=gy\in gM\cap tM$ then $ty=t^2y=tgx=0$, whence $gM\cap tM=0$; and for all $x\in M$, $x=gfx+(x-gfx)\in gM+tM$.

Since g is monic ($fg=1$), we get, by the above remark, that $gM=M$, and hence $tM=t(1-gf)M=0$, i.e. $gf=1$.∎

Finite Dimensional Injective Modules

Here, the results of Section 3 will be used to show that an injective module is finite dimensional if and only if its endomorphism ring S is *semiperfect* (i.e. idempotents lift modulo J=J(S) and S/J is semisimple [5]). As a first step towards this result, we characterize those injective modules which are uniform (compare with (8.6)).

PROPOSITION 9.2 *Let* E_R *be a nonzero injective module, and let* $S=End(E_R)$. *The following conditions are equivalent.*

a) E_R *is uniform;*

b) E_R *is indecomposable;*

c) E_R *is the injective hull of all its nonzero submodules;*

d) S *is a local ring.*

PROOF: a)⇔b) follows directly from (3.7).

a)⇔c) since every nonzero submodule of E is essential in E.

c)⇒d). We show that the sum of any two noninvertible elements f,g of S is noninvertible. For, note that $Kerf\neq 0$, because otherwise fE would be a nonzero injective submodule of E and thus, by c), E=fE, whence f would be an isomorphism. Similarly, $Kerg\neq 0$ and hence, by a), $0\neq Kerf\cap Kerg\subseteq Ker(f+g)$, whence f+g is not invertible.

d)⇒b) is (5.6).■

THEOREM 9.3 *Let* E_R *be an injective module,* n *a positive integer and write* $S=End(E_R)$. *The following conditions are equivalent.*

a) E_R *is finite dimensional and* $u(E_R)=n$;

b) E_R *is a finite direct sum of indecomposable injective modules, and any such decomposition of* E *consists of exactly* n *nonzero summands;*

c) S *contains no infinite family of orthogonal idempotents, and* n *is the maximum cardinality of all families of nonzero orthogonal idempotents of* S.

PROOF: a)⇔b) follows directly from (3.9).

b⇒c). By (9.2) and by the Krull-Schmidt-Azumaya Theorem, every direct decomposition of E has at most n summands (and at least one of them has exactly n); then apply (5.4.c).

c)⇒a). Let $\oplus_I M_i$ be any direct sum of nonzero submodules of E_R; for each $i \in I$, let E_i be an e-closure in E for M_i; now, for each $j \in I$, let $p_j : \oplus_I E_i \longrightarrow E_j$ and $e_j : E_j \longrightarrow \oplus_I E_i$ be the canonical projection and injection, and let $u : \oplus_I E_i \longrightarrow E$ be the inclusion map.
By injectivity of E_j (3.4), there exists $f_j : E \longrightarrow E_j$ (which may be viewed as an element of S) such that $f_j u = p_j$; thus, for each $x \in E$, $f_j x \in E_j$ and hence $f_j x = p_j f_j x = u p_j f_j x$. Now let $j, k \in I$ and $x \in E$; we get

$$f_k f_j x = f_k u p_j f_j x = p_k p_j f_j x = \delta_{kj} f_j x$$

(where δ_{jk} is the 'Kroeneker delta'), and therefore $\{f_j : j \in I\}$ is a family of nonzero orthogonal idempotents of S.
Thus, for any direct sum of nonzero submodules of E_R we get a family of nonzero orthogonal idempotents of S with the same cardinality, and vice-versa (5.4.c), whence the implication c)⇒a) follows readily.■

The following lemma is the key to prove our Theorem 9.6.

<u>LEMMA 9.4</u> *Let* R *be a regular ring. Then* R *is semisimple of length* n *(equivalently, of right or left dimension* n*) if and only if* R *contains a family of* n *nonzero orthogonal idempotents, and no set of nonzero orthogonal idempotents of* R *has more than* n *elements.*

PROOF: We prove that a regular ring R such that R does not contain infinite families of orthogonal idempotent elements is semisimple. The lemma follows then easily from the well-behaved decomposition theory of semisimple modules and rings.
From the hypothesis about the idempotents in R, it is easy to see that $1 \in R$ may be written as a finite sum of primitive orthogonal idempotents of R, and therefore R is a finite direct sum of indecomposable right (or left) ideals; but, in a regular ring, an indecomposable right ideal $\mathfrak{a} \neq 0$ must be simple: for any $0 \neq r \in \mathfrak{a}$, there exists $e^2 = e \in R$ with $rR = eR$, whence rR is a direct summand of R_R and hence of $\mathfrak{a}$; since $rR \neq 0$ and $\mathfrak{a}$ is indecomposable, we get $rR = \mathfrak{a}$, and thus $\mathfrak{a}$ is simple. Therefore R is semisimple.■

<u>THEOREM 9.5</u> *Let* E_R *be injective. Then* E *is finite dimensional if and only if* S=End(E_R) *is semiperfect. In this case* u(E_R)=u(S/J(S)).

PROOF: Let J=J(S). Recall that S/J is regular (7.2) and idempotents may be lifted modulo J (7.3). Now (9.3) and (9.4) give:

S is semiperfect with u(S/J)=n ⇔ S/J is semisimple with u(S/J)=n ⇔
the idempotents of S/J verify the conditions of (9.4) ⇔
so do the idempotents of S ⇔ u(E_R)=n.■

<u>COROLLARY 9.6</u> *Let* R *be a right self-injective ring. The following conditions are equivalent.*

a) R *is semiperfect;*

b) R *has no infinite family of orthogonal idempotents;*

c) R *is right finite dimensional;*

d) R *is left finite dimensional.*■

<u>COROLLARY 9.7</u> *The endomorphism ring of a f.d. injective nonsingular module is semisimple.*

PROOF: Such a ring is semiperfect by (9.5) and has zero radical by (7.2) and the proof of (7.12); hence it is semisimple.■

Nil Subrings of the Endomorphism Ring of a Finite Dimensional Module

The fact that the injective hull E_R of a finite dimensional module M_R has the same dimension as M_R, together with the previous study of finite dimensional injective modules and with the fact that S=End(M_R) may be embedded in End(E_R), will allow us to study in some detail the nilpotency of nil subrings of S. A sufficient condition for these to be nilpotent will be found in terms of the rationally closed submodules of M_R, which we shall shortly introduce. The main results in this area are due to R.Shock [50].

<u>THEOREM 9.8</u> *Let* M_R *be any module (resp. a f.d. module), and let* Γ=Γ(S). *Then* S/Γ *is ring isomorphic to a subring of a regular (resp. semisimple) ring.*

PROOF [50;Lemma 2]: Let E_R be an injective hull for M_R, and let H=End(E_R); then J(H)=Γ(H) and H/J(H) is regular (semisimple if M_R is f.d.), and thus it suffices to embed S/Γ in H/Γ(H).

Let $S'=\{h\in H: hM\subseteq M\}$ and $\Gamma'=\{h\in S': \mathrm{Ker}h\subseteq_e E\} = \Gamma(H)\cap S'$. For $f\in S$, let f' be an extension to E of f; then it is easy to check that the correspondence $f+\Gamma\longmapsto f'+\Gamma'$ gives a ring isomorphism between S/Γ and S'/Γ', and S'/Γ' may be embedded in $H/J(H)$ via $h+\Gamma'\longmapsto h+\Gamma(H)$.■

We now turn our attention to the study of the nil subrings of S. The starting point is the next result.

PROPOSITION 9.9 *Let* M_R *be finite dimensional and* $S=\mathrm{End}(M_R)$. *Then*

a) *every nil subring of* $S/\Gamma(S)$ *is nilpotent;*

b) *if* $\Gamma(S)$ *is nilpotent then every nil subring of* S *is nilpotent;*

c) *a nil subring* Q *of* S *is nilpotent if and only if* $Q\cap\Gamma(S)$ *is nilpotent.*

PROOF: a) From a result of I.Hernstein and L.Small [24], in a ring with ACC on right and left annihilators every nil subring is nilpotent (see also [C-H; Theo.1.34]), so that, from (9.8) and by symmetry, it suffices to see that, in general, a subring of a semisimple (Artinian) ring has ACC on right annihilators.

Let B be a subring of the semisimple ring T. Write $\mathcal{R}_B$, $\mathcal{L}_B$ (resp. $\mathcal{R}_T$, $\mathcal{L}_T$) for the annihilator operators in B (resp. T) and note that, for a nonempty subset X of B, $\mathcal{R}_B(X)=\mathcal{R}_T(X)\cap B$. Now suppose $X_1,X_2,\ldots$ are nonempty subsets of B such that $\mathcal{R}_B(X_1)\subseteq\mathcal{R}_B(X_2)\subseteq\cdots$ and set $Y_i=\mathcal{L}_B\mathcal{R}_B(X_i)$ for $i=1,2,\ldots$, so that $\mathcal{R}_B(X_i)=\mathcal{R}_B(Y_i)$ for all i. If $\mathcal{L}_B$ acts in that chain then we get $Y_1\supseteq Y_2\supseteq\cdots$ and hence $\mathcal{R}_T(Y_1)\subseteq\mathcal{R}_T(Y_2)\subseteq\cdots$. By hypothesis there exists $n\in\mathbb{N}$ such that, for all $k\geq n$, $\mathcal{R}_T(Y_k)=\mathcal{R}_T(Y_n)$ and hence

$$\mathcal{R}_B(X_k) = \mathcal{R}_B(Y_k) = \mathcal{R}_T(Y_k)\cap B = \mathcal{R}_T(Y_n)\cap B = \mathcal{R}_B(Y_n) = \mathcal{R}_B(X_n),$$

proving that B has ACC on right annihilators, as desired.

b) This follows then easily from a).

c) If Q is a nil subring of S and we write Γ for $\Gamma(S)$, then $(Q+\Gamma)/\Gamma$ is a nil subring of S/Γ and then $(Q+\Gamma)/\Gamma \cong Q/(Q\cap\Gamma)$ is nilpotent; if $Q\cap\Gamma$ is also nilpotent, then so is Q, and this proves c).■

COROLLARY 9.10 *Let* R *be a right finite dimensional ring; then a nil subring* Q *of* R *is nilpotent if and only if* so is $Q\cap Z_r(R)$. *If* $Z_r(R)$ *is nilpotent then nil subrings of* R *are nilpotent.*

PROOF: As we already remarked in the proof of (7.4), $\Gamma(R)=Z_r(R)$.■

Now, we can look for f.d. modules M_R for which $\Gamma(S)$ is nilpotent, and then use (9.9.b) to deduce that all nil subrings of S are nilpotent. A sufficient condition both for M_R to be f.d. and for $\Gamma(S)$ to be nilpotent will depend on the concept of M-rationally closed submodules of M_R, which we define next.

Let $L\subseteq N$ be submodules of M_R; we say that N is an M-*rational extension* of L provided $\mathrm{Hom}_R(K/L,M)=0$ for all $K\leq M_R$ such that $L\leq K\leq N$. If $L\leq M_R$ has no proper M-rational extensions within M, we say that L is M-*rationally closed*. A concept of rational closure (r-closure), similar to that of e-closure developed in Section 1, may be defined in terms of a class of right ideals of R called M-dense ideals: $\mathfrak{a}\leq R_R$ is M-*dense* if $x\mathfrak{a}\neq 0$ for every nonzero element x in an injective hull of M_R. Now, for $L\leq M_R$, let $L'=\{x\in M: (L:x)$ is M-dense$\}$; L' is called the r-closure of L in M, and it turns out that L is rationally closed in M if and only if $L=L'$ [50].

Before state the next theorem, we need two previous results. We shall again make use of the notation and results of Section 6.

<u>LEMMA 9.11</u> *If* S *has* DCC *on a-closed ideals, then* $\Gamma(S)$ *is nilpotent.*

PROOF [36]: We know from (6.1.a) that every left annihilator ideal of S is a-closed, so that S has DCC on left annihilator ideals and hence ACC on right annihilator ideals, whence the chain $\mathcal{R}(\Gamma)\subseteq\mathcal{R}(\Gamma^2)\subseteq\cdots$ (where $\Gamma=\Gamma(S)$) stops, i.e. $\mathcal{R}(\Gamma^n)=\mathcal{R}(\Gamma^{n+1})$ for some $n\in\mathbb{N}$; now, we shall prove that $\Gamma\subseteq\mathcal{R}(\Gamma^n)$, which will imply $\Gamma^{n+1}=0$, proving the lemma.

Suppose there exists $f\in\Gamma\setminus\mathcal{R}(\Gamma^n)$; since $f\notin\mathcal{R}(\Gamma^{n+1})$, there exists $f_1\in\Gamma$ such that $\Gamma^n f_1 f\neq 0$, i.e. $f_1 f\in\Gamma\setminus\mathcal{R}(\Gamma^n)$; in this way, we can construct an infinite sequence $f,f_1,f_2,\ldots$ of elements of Γ such that $g_r=f_r\cdots f_1 f\neq 0$ for all $r\in\mathbb{N}$. Since $g_r M\neq 0$ and $\mathrm{Ker}(f_{r+1})\subseteq_e M$, $g_r M\cap\mathrm{Ker}(f_{r+1})\neq 0$, whence $\mathrm{Ker}(g_r)\subset\mathrm{Ker}(g_{r+1})$. Since the strictly ascending chain $\mathrm{Ker}(g_1)\subset\mathrm{Ker}(g_2)\subset\cdots$ consists of right annihilators in M of subsets of S, also $l_S(\mathrm{Ker}g_1)\supset l_S(\mathrm{Ker}g_2)\supset\cdots$ is strict, contradicting the hypothesis of the lemma and hence proving our claim.■

<u>LEMMA 9.12</u> *Let* $L\subset N$ *be submodules of* M_R*; then* $L'\subset N'$ *if and only if there exists* $x\in N\setminus L$ *such that* $(L:x)$ *is not* M-*dense.*

PROOF: Clearly, $L'=N'$ if and only if $N\subseteq L'$, and by definition this occurs if and only if $(x:L)$ is M-dense for all $x\in N$. Since $(x:L)=R$ (which is M-dense) for all $x\in L$, the lemma follows.■

<u>THEOREM 9.13</u> *Let* M_R *be a module with* ACC *on rationally closed submodules, and let* $S=\mathrm{End}(M_R)$. *Then*

a) M_R *is finite dimensional;*

b) S *has* DCC *on a-closed left ideals;*

c) *every nil subring of* S *is nilpotent.*

PROOF [50;Theo.3.10]: a) Let L,N be nonzero independent submodules of M_R; for any $0\neq x\in N$ we have $x(L:x)\subseteq N\cap L=0$, so that $(L:x)$ is not M-dense and hence $L'\subset(N\oplus L)'$ by the preceding lemma. Therefore, any infinite direct sum of nonzero submodules of M_R would force a strictly ascending chain of rationally closed submodules of M, a contradiction which proves a).

b) Let $\mathfrak{A},\mathfrak{B}$ be a-closed left ideals of S with $\mathfrak{A}\subset\mathfrak{B}$, and set $E=\mathbf{r}_M(\mathfrak{A})$, $F=\mathbf{r}_M(\mathfrak{B})$; thus $\mathbf{l}_S(E)=\mathfrak{A}\subset\mathfrak{B}=L_S(F)$. Take then $f\in S$ such that $fF=0$ and $fE\neq0$, and pick $z\in E$ with $fz\neq0$; since $fz(F:z)\subseteq fF=0$, $(F:z)$ is not M-dense and thus $F'\subset E'$ by (9.12). Therefore, for each strictly descending chain $\mathbf{l}_S(F)\supset\mathbf{l}_S(E)\supset\cdots$ of left annihilators in S of subsets of M_R, we find a strictly increasing chain $E'\subset F'\subset\cdots$ of rationally closed submodules of M_R, which must be finite by hypothesis.

c) This follows directly from a), b), (9.11) and (9.9.b).■

<u>COROLLARY 9.14</u> *If* M_R *is injective with* ACC *on rationally closed submodules, then* S *is semiprimary.*

PROOF: By (9.13.a) and (9.5), S is semiperfect, and by (7.2), (9.13.b) and (9.11), J(S) is nilpotent. Thus S is semiprimary.■

Quotient Rings of the Endomorphism Ring of a Finite Dimensional Nonsingular Module

In Section 6 we saw that, for a non-degenerate module M_R, we can obtain a satisfactory correspondence theorem for right complements of $S=\mathrm{End}(M_R)$. Next, we shall make use of this and other facts to get some information about the endomorphism ring of a non-degenerate finite dimensional module. The results here are due to J.Hutchinson and J.Zelmanowitz [25].

Our first two result are of key importance in what follows, and they state that, for a non-degenerate module M_R, the dimension of M_R and the right dimension of S coincide, generalizing (6.14.a) and (6.16.a), and that M_R is nonsingular if and only if S is right nonsingular.

<u>THEOREM 9.15</u> *If* M_R *is a non-degenerate module and* $S=\mathrm{End}(M_R)$*, then* $u(M_R)=u(S_S)$.

PROOF [1;Theo.2]: Let $N_1\oplus\cdots\oplus N_r$ be a direct sum of nonzero submodules of M_R; by hypothesis, the right ideals $[N_i,\hat{M}]$ $(i=1,\dots,r)$ of S are nonzero. We claim that they are independent, which will imply $u(M_R)\le u(S_S)$.
Suppose that $f_i\in[N_i,\hat{M}]$ $(i=1,\dots,r)$ are such that $\Sigma f_i=0$; for all $x\in M$ we get $0=(\Sigma f_i)x=\Sigma(f_ix)$ with each $f_ix\in f_iM\subseteq[N_i,\hat{M}]M=N_i t_R(M)\subseteq N_i$, and then, by assumption, $f_ix=0$ for each i; this shows that $f_i=0$ for each i, so that the $[N_i,\hat{M}]$'s are independent.
On the other hand, for each direct sum $\mathfrak{A}_1\oplus\cdots\oplus\mathfrak{A}_t$ of nonzero right ideals of S, we claim that the nonzero submodules $\mathfrak{A}_iM$ $(i=1,\dots,t)$ of M_R are independent, whence $u(S_S)\le u(M_R)$ and hence the theorem is proved.
Suppose then that $x_i\in\mathfrak{A}_iM$ $(i=1,\dots,t)$ are such that $\Sigma x_i=0$; then, for any $\varphi\in\hat{M}$, $y\in M$, we have $(\Sigma[x_i,\varphi])y=(\Sigma x_i)(\varphi,y)=0$, whence $\Sigma[x_i,\varphi]=0$; but, for each $i=1,\dots,t$, we have $[x_i,\varphi]\in[\mathfrak{A}_iM,\hat{M}]=\mathfrak{A}_i[M,\hat{M}]\subseteq\mathfrak{A}_iS=\mathfrak{A}_i$ and thus, by assumption, $[x_i,\varphi]=0$; therefore $[x_i,\hat{M}]=0$ and hence, by hypothesis, $x_i=0$, as required.■

<u>THEOREM 9.16</u> *Let* M_R *be a non-degenerate module. Then* M_R *is nonsingular if and only if* S *is right nonsingular.*

PROOF [25;Prop.2]: Assume first that M_R is nonsingular, and let $f\in Z_r(S)$; then there exists $\mathfrak{A}\subseteq_e S_S$ such that $f\mathfrak{A}=0$. hence $\mathfrak{A}M\subseteq\mathrm{Ker}f$ and thus, since $\mathfrak{A}M\subseteq_e M$ by (6.10.b), $\mathrm{Ker}f\subseteq_e M$. Now, for any $0\ne x\in M$, we have $\mathfrak{e}=(\mathrm{Ker}f:x)\subseteq_e R_R$ (1.1) and $fx\mathfrak{e}=0$; by assumption, this implies $fx=0$, whence $f=0$, i.e. $Z_r(S)=0$.
Conversely, assume that $Z_r(S)=0$ and suppose there exists $0\ne x\in Z(M_R)$. By

hypothesis, we can find $\varphi\in\hat{M}$ such that $f=[x,\varphi]\neq 0$, and we claim that $\mathrm{Ker}f \subseteq_e M$.

To prove so, let $y\in M\setminus \mathrm{Ker}f$; then $0\neq fy=x(\varphi,y)$, whence $(\varphi,y)\neq 0$; now, since $\mathfrak{e}=r_R(x)\subseteq_e R_R$, we can choose $r\in R$ such that $0\neq(\varphi,y)r\in\mathfrak{e}$, i.e. $fyr=x(\varphi,y)r=0$; and then we have got $0\neq yr\in \mathrm{Ker}f$, proving the claim.

Thus, if we prove that $[\mathrm{Ker}f,\hat{M}]\subseteq_e S_S$, since clearly $[\mathrm{Ker}f,\hat{M}]\subseteq\mathcal{R}(f)$, we will get $0\neq f\in Z_r(S)$, a contradiction which will imply $Z(M_R)=0$.

We prove in general that $N\subseteq_e M$ implies $[N,\hat{M}]\subseteq_e S_S$: let $0\neq g\in S$; then $gM\neq 0$ and hence $N\cap gM\neq 0$; thus, by hypothesis,

$$0 \neq [N\cap gM,\hat{M}] \subseteq [N,\hat{M}]\cap[gM,\hat{M}] \subseteq [N,\hat{M}]\cap gS;$$

therefore $[N,\hat{M}]\subseteq_e S_S$.■

With this information in hand, and making use of some results of Section 6, it is straightforward to characterize those non-degenerate modules which have a right Goldie endomorphism ring.

<u>COROLLARY 9.17</u> *Let* M_R *be a non-degenerate module. Then* S *is a right Goldie ring if and only if* M_R *is finite dimensional with* ACC *on a-closed submodules (and both conditions hold if* M_R *has* ACC *on* M-*cotorsionless submodules).*

PROOF: By (9.15), S is right f.d. if and only if M_R is f.d. By (6.10.g), M_R is an a-self-generator and hence (6.6.a) S has ACC on right annihilators if and only if M_R has ACC on a-closed submodules. Therefore the result follows, the statement in parenthesis being a direct consequence of (6.12.b).■

<u>COROLLARY 9.18</u> *Let* M_R *be a non-degenerate nonsingular module. Then* S *is a right Goldie ring if and only if* M_R *is f.d.*

PROOF: Note that every a-closed submodule of M_R is a complement in M (6.7.a), and therefore, if M_R is f.d., then it has ACC on a-closed submodules. Thus (9.17) gives the result.■

Now, we can study when S has a semisimple classical or maximal right quotient ring. Rings with these properties may be characterized as follows:

(1) A ring R has a semisimple (simple) classical right quotient ring if and only if R is a semiprime (prime) right Goldie ring [19] (see e.g. [S; p.54] or [G; Theo.3.35]).

(2) A semiprime (prime) ring R has a semisimple (simple) right quotient ring if and only if R is a right nonsingular and right f.d. ring [G; Cor.3.32].

(3) A ring R has a semisimple maximal right quotient ring if and only if R is a right nonsingular and right f.d. ring [46; Theo.1.6].

Therefore we obtain at once, from (3), (9.15) and (9.16),

COROLLARY 9.19 *Let* M_R *be a non-degenerate module. Then* S *has a semisimple maximal right quotient ring if and only if* M_R *is f.d. and nonsingular.*∎

For the 'classical' case, we need a lemma. Recall that a ring R is prime if aRb is nonzero for any nonzero elements a,b of R; and that R is semiprime if and only if $aRa\neq0$ for any $0\neq a\in R$.

LEMMA 9.20 *Let* M_R *be a nonzero nondegenerate module. Then* S *is a prime (semiprime) ring if and only if* $\bar{R}=R/\mathbf{r}_R(M)$ *is prime (semiprime) and* $[M,\hat{M}]x\neq0$ *for all* $0\neq x\in M$.

PROOF [25;Theo.11]: Assume that S is a prime ring and write $\mathfrak{a}=\mathbf{r}_R(M)$. To see that $\bar{R}$ is prime we have to show that, for $r,s\in R\setminus\mathfrak{a}$, $rRs\nsubseteq\mathfrak{a}$. But in this case we get $Mr\neq0$, $Ms\neq0$, whence $[Mr,\hat{M}]\neq0$, $[Ms,\hat{M}]\neq0$. Thus, since S is prime,

$$0\neq[Mr,\hat{M}][Ms,\hat{M}]=\big[[Mr,\hat{M}]Ms,\hat{M}\big]=[Mr(\hat{M},M)s,\hat{M}],$$

whence $Mr(\hat{M},M)s\neq0$ and thus, in particular, $MrRs\neq0$, i.e. $rRs\nsubseteq\mathfrak{a}$.
Now, if $x\in M$ is such that $[M,\hat{M}]x=0$, then $[M,\hat{M}][x,\hat{M}]=\big[[M,\hat{M}]x,\hat{M}\big]=0$ whence, by primeness of S and since $[M,\hat{M}]\neq0$, $[x,\hat{M}]=0$. Then, by hypothesis, $x=0$.

Conversely, assume that $\bar{R}$ is prime and that $[M,\hat{M}]x=0$ for $x\in M$ implies $x=0$. Let f,g be nonzero elements of S; then $fM\neq0$ and $gM\neq0$ imply $M(\hat{M},fM)=[M,\hat{M}]fM\neq0$ and $M(\hat{M},gM)=[M,\hat{M}]gM\neq0$, i.e $(\hat{M},fM)\nsubseteq\mathfrak{a}$ and $(\hat{M},gM)\nsubseteq\mathfrak{a}$ whence, by primeness of $\bar{R}$, $(\hat{M},fM)(\hat{M},gM)\nsubseteq\mathfrak{a}$. But

$$(\hat{M},fM)(\hat{M},gM) = \big(\hat{M},fM(\hat{M},gM)\big) = (\hat{M},[fM,\hat{M}]gM) = (\hat{M},f[M,\hat{M}]gM),$$

whence $(\hat{M},f[M,\hat{M}]gM)\nsubseteq\mathfrak{a}$ and thus, in particular, $f[M,\hat{M}]g\neq0$ and $fSg\neq0$. Therefore S is a prime ring.

The semiprime case follows by taking r=s and f=g.■

COROLLARY 9.21 *Let* M_R *be a nondegenerate module. Then the following conditions are equivalent:*

a) S *has a simple (semisimple) classical right quotient ring;*

b) M_R *is f.d. and nonsingular,* $\bar{R}=R/\mathbf{r}_R(M)$ *is a prime (semiprime) ring and* $[M,\hat{M}]x\neq0$ *for all* $0\neq x\in M$.

PROOF: This follows from (1), (2), (9.15), (9.16) and (9.20).■

Finally, we go one step further in the study of the maximal quotient ring of S; if S is right nonsingular (i.e. if M_R is nonsingular, always under the hypothesis of non-degeneracy) then it possesses a maximal right quotient ring. We shall describe this maximal quotient ring in the next proposition.

PROPOSITION 9.22 *Let* M_R *be a non-degenerate nonsingular module, let* E_R *be an injective hull for* M_R *and write* $S=End(M_R)$, $H=End(E_R)$. *Then* H *is the maximal right quotient ring of* S.

PROOF [25;Prop.4]: By injectivity of E_R, we may view S as a subring of H; by (1.1.d) and (7.12), H is a regular right self-injective ring, and hence a right nonsingular ring by [G; Prop.1.27] or (9.16).
Suppose we prove that, for any $0\neq h\in H$, $S\cap hS\neq0$; this is clearly equivalent to $S\subseteq_e H_S$, and then H_S is nonsingular (1.12.d), whence it is a rational extension of S_S [G; Lemma 2.24], and thus H is a right quotient ring of S. Now, H_H is injective, and then it has no proper rational extensions [G; Lemma 2.24], which implies that H is a maximal right quotient ring of S [G; Prop.2.28].
Let us then prove that $0\neq h\in H$ implies $0\neq S\cap hS$. Since $M\subseteq_e E$, $N=M\cap h^{-1}M\subseteq_e M$ (1.2.a & d) and hence $hN\neq0$, because $\Gamma(H)=0$ (7.12). Therefore, by non-degeneracy of M_R,

$$0\neq [hN,\hat{M}]= h[N,\hat{M}]= h[M\cap h^{-1}M,\hat{M}]\subseteq h[M,\hat{M}]\cap h[h^{-1}M,\hat{M}]\subseteq hS\cap S.$$ ■

SECTION 10: THE ENDOMORPHISM RING OF MODULES WITH CHAIN CONDITIONS

We start this section by proving the classical Fitting's Lemma, and obtain as a consequence the fact that every indecomposable module of finite length has a local endomorphism ring. Next, we prove a recent result of Camps and Dicks, whose characterization of semilocal rings in [9] gives as a corollary that the endomorphism ring of an Artinian module is semilocal.

Later on we introduce the concept of T-nilpotency and use it to prove that, in the endomorphism ring of a module which is either Noetherian or Artinian, every nil subring is nilpotent; and that every module of finite length has a semiprimary endomorphism ring. These results are due to Fisher and Small [16].

We close Section 10 proving that the correspondence theorems for finitely generated ideals of S studied in Section 6 work for quasi-injective or quasi-projective modules, and under these hypothesis we obtain necessary and sufficient conditions on M_R for S to be Noetherian, semiprimary, or Artinian. The main results in this area are due to M.Harada and T.Ishii ([22] and [23]), though our proofs of them make use of different techniques (those of Section 6).

Fitting's Lemma and Consequences

LEMMA 10.1 (Fitting) *If* M_R *is a module of finite length* n *and* f *is an endomorphism of* M, *then* $M=\mathrm{Im}f^n\oplus\mathrm{Ker}f^n$.

PROOF: Let $K_i=\mathrm{Ker}f^i$; the chain $0\subseteq K_1\subseteq K_2\subseteq\cdots$ becomes stationary at some step j, and for the least j with this property the inclusions $K_{i-1}\subset K_i$ are strict for $1\le i\le j$: Suppose not; then if $x\in K_{i+1}$, $fx\in K_i=K_{i-1}$ and thus $x\in K_i$, i.e. $K_i=K_{i+1}$; by induction $K_{i-1}=K_j$, against the minimality of j. This shows in particular that $j\le n$, whence $K_n=K_{2n}$. Then, if $x\in\mathrm{Ker}f^n\cap\mathrm{Im}f^n$, we get $x=f^ny$ $(y\in M)$ and $0=f^nx=f^{2n}y$, i.e. $y\in K_{2n}=K_n$ and thus $x=0$. Hence $\mathrm{Ker}f^n\cap\mathrm{Im}f^n=0$.

On the other hand, let $M_i=\mathrm{Im}f^i$; the chain $M\supseteq M_1\supseteq M_2\supseteq\cdots$ stops at some minimal j and then $M_{i-1} \subset M_i$ for $1 \le i \le j$: If not, for all $x \in M_i$ we have $x = f^i y = f f^{i-1} y$ $(y \in M)$, and $f^{i-1} y \in M_{i-1} = M_i$, whence $f^{i-1} y = f^i z$ $(z \in M)$ and then $x = f^{i+1} z \in M_{i+1}$; thus $M_i = M_{i+1}$ and by induction $M_{i-1} = M_j$, a contradiction. Therefore $j \le n$ and thus $M_n = M_{2n}$.

It remains to see that $M = \mathrm{Ker} f^n + \mathrm{Im} f^n$. For all $x \in M$, $y = f^n x \in M_n = M_{2n}$ and then there exists $z \in M$ such that $y = f^{2n} z$, whence $x - f^n z \in \mathrm{Ker} f^n$ and thus $x = (x - f^n z) + f^n z \in \mathrm{Ker} f^n + \mathrm{Im} f^n$. This completes the proof.■

<u>COROLLARY 10.2</u> *Let* M_R *be a Noetherian (resp. Artinian) module and let* $f \in \mathrm{End}(M_R)$, *then* f *is an epimorphism (resp. a monomorphism) if and only if it is an isomorphism.*

PROOF: Assume that M_R is Noetherian. From the first part of the proof of Fitting's Lemma we know that, for some n, $\mathrm{Ker} f^n \cap \mathrm{Im} f^n = 0$; but if f is epic then so is f^n, whence $\mathrm{Ker} f^n = 0$ and thus $\mathrm{Ker} f = 0$.

The proof when M_R is Artinian follows by duality.■

<u>LEMMA 10.3</u> *Let* R *be a nonzero ring in which every element is either invertible or nilpotent. Then* R *is a local ring.*

PROOF: First note that if $a \in R$ has no right inverse, then it has no left inverse: if $ba = 1$ and n is the least integer such that $a^n = 0$, then $0 = ba^n = (ba)a^{n-1} = a^{n-1}$, a contradiction. This remark and its right-left symmetric show that, for all $a \in R$,

$$a \text{ is invertible} \Leftrightarrow a \text{ is left invertible} \Leftrightarrow a \text{ is right invertible.}$$

Take now two non-invertible elements a,b of R; to see that R is local it suffices to show that a+b is non-invertible. Suppose then that $(a+b)c = 1$ for some $c \in R$; since bc is not invertible (b would then be right invertible), there exists $n \in \mathbb{N}$ with $(bc)^n = 0$; then $(1-bc)(1+bc+\cdots+(bc)^{n-1}) = 1-(bc)^n = 1$, but $1-bc = ac$ cannot have a right inverse (a would also have one). This contradiction implies that a+b is non-invertible, and hence that R is local.■

<u>THEOREM 10.4</u> *If* M_R *is a nonzero indecomposable module of finite length, then* $S = \mathrm{End}(M_R)$ *is a local ring.*

PROOF [17;Satz 3]: Let n be the composition length of M_R; then for all $f \in S$ we get $M = \mathrm{Ker} f^n \oplus \mathrm{Im} f^n$ (10.1). But, since M is indecomposable, it must be that either $Imf^n=0$ (whence $f^n=0$), or $Imf^n=M$, $Kerf^n=0$ (whence Imf=M, Kerf=0 and thus f is invertible). Therefore any element of S is either invertible or nilpotent and then (10.3) applies.■

REMARK H.Fitting [17; Satz 8] also proved that, in the above situation, J(S) is nilpotent of index at most the length of M_R (see [FA73; Theo.17.20]).

The Endomorphism Ring of an Artinian Module is Semilocal

In this paragraph we state one of the implications of a characterization of semilocal rings given by Camps and Dicks in [9], who used it to solve a conjecture made by their teacher P.Menal, namely that the endomorphism ring of an Artinian module is semilocal (i.e. semisimple modulo its radical). Let us first introduce the concept of maximum condition with respect to summands in a set of subgroups of an Abelian group.

Let Ω be a set of subgroups of an Abelian group M. Given X,Y,Z in Ω with X⊕Y=Z, we say that X and Y are *Ω-summands* of Z; if Y≠0 then X is said to be a *proper Ω-summand* of Z. The set Ω *satisfies the maximum condition with respect to summands* if every nonempty subset Δ of Ω contains an element which is a proper Ω-summand of no member of Δ.

For example, given a module M_R, $\Omega=\{\mathbf{1}_M(r):r\in R\}$ is a family of subsets of the Abelian group M_Z. Suppose further that there exists a ring Q such that $M={}_QM_R$ and $_QM$ is finite dimensional; since, for each r∈R, $\mathbf{1}_M(r)$ is a submodule of $_QM$, an easy argument shows that, in this case, Ω satisfies the maximum condition with respect to summands. This fact will be used in the proof of Theorem 10.6.

PROPOSITION 10.5 *Let* R *be a ring such that there exists an* R*-module* M_R *satisfying the following two conditions:*

1) *the set* $\Omega=\{\mathbf{1}_M(r):\ r\in R\}$ *satisfies the maximum condition with respect to summands;*
2) *if* r∈R *is not invertible then* $\mathbf{1}_M(r)\neq 0$.

Then R *is a semilocal ring.*

PROOF [9;Theo.1]: In what follows, let U(R) be the set of all units in R, J=J(R), $\bar{R}$=R/J and, for all r∈R, write $\bar{r}$=r+J. We have to show that $\bar{R}$ is a semisimple ring.

First, we give a partial ordering in R: For a,b∈R, write a>b if $l_M(a)$ is a proper Ω-summand of $l_M(b)$; then the relation ≥ (defined in the obvious way) is a partial ordering in R which, by 1), satisfies the minimum condition. Henceforth, by *minimal* we shall mean minimal with respect to this ordering.

We recall that a∈J if and only if 1-ab (and 1-ba) belongs to U(R) for all b∈R. This will be helpful after proving that

3) a,x∈R, 1-ax∉U(R) ⇒ a>a-axa.

Since, by 2), $l_M(1-ax)\neq 0$, 3) will be proved if we show that $l_M(a-axa)=l_M(a)\oplus l_M(1-ax)$; clearly the sum $l_M(a)+l_M(1-ax)$ is direct and it is contained in $l_M(a-axa)$; on the other hand, if $m\in l_M(a-axa)$, then m=max+m(1-ax) with $max\in l_M(1-ax)$ and $m(1-ax)\in l_M(a)$.

Now let $\mathcal{E}$={a∈R: $\bar{a}^2=\bar{a}$ and $(\bar{1}-\bar{a})\bar{R}$ is semisimple in Mod-$\bar{R}$}; since $\mathcal{E}$ is nonempty (1∈$\mathcal{E}$), it contains a minimal element, say a; clearly, the proposition will be proved if we show that a∈J.

Suppose then that a∉J; thus aR\J is nonempty and hence there exists b∈B such that ab is minimal in aR\J. We claim that

4) x∈R, 1-abx∉U(R) ⇒ $\bar{a}\bar{b}\bar{x}\bar{a}\bar{b}=\bar{a}\bar{b}$.

By 3) we have ab>ab-abxab, and since ab is minimal in aR\J, it follows that (ab-abxab)∈J, proving 4). Now we can prove

5) $\bar{a}\bar{b}\bar{R}$ is a simple right $\bar{R}$-module.

We show that, for any x∈R such that $\bar{a}\bar{b}\bar{x}\neq\bar{0}$, $\bar{a}\bar{b}\bar{x}$ generates $\bar{a}\bar{b}\bar{R}$; for, since abx∉J, there exists y∈R such that 1-abxy∉U(R), whence $\bar{a}\bar{b}\bar{x}\bar{y}\bar{a}\bar{b}=\bar{a}\bar{b}$ and thus $\bar{a}\bar{b}\bar{x}\bar{R}\subseteq\bar{a}\bar{b}\bar{R}=\bar{a}\bar{b}\bar{x}\bar{y}\bar{a}\bar{b}\bar{R}\subseteq\bar{a}\bar{b}\bar{x}\bar{R}$; hence $\bar{a}\bar{b}\bar{x}\bar{R}=\bar{a}\bar{b}\bar{R}$.

Now, since ab∉J, there exists c∈R with 1-abc∉U(R); for this c we claim that a-abca∈$\mathcal{E}$. By 4), $\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}=\bar{a}\bar{b}$; this implies that $\bar{a}\bar{b}\bar{c}\bar{a}\bar{R}=\bar{a}\bar{b}\bar{R}$ (as in the proof of 5)) and that $\bar{a}-\bar{a}\bar{b}\bar{c}\bar{a}$ is idempotent, since

$$(\bar{a}-\bar{a}\bar{b}\bar{c}\bar{a})^2 = \bar{a}^2-\bar{a}^2\bar{b}\bar{c}\bar{a}-\bar{a}\bar{b}\bar{c}\bar{a}^2+\bar{a}\bar{b}\bar{c}\bar{a}^2\bar{b}\bar{c}\bar{a} = \bar{a}-\bar{a}\bar{b}\bar{c}\bar{a}-\bar{a}\bar{b}\bar{c}\bar{a}+\bar{a}\bar{b}\bar{c}\bar{a} = \bar{a}-\bar{a}\bar{b}\bar{c}\bar{a}.$$

It remains to check that $(\bar{1}-\bar{a}+\bar{a}\bar{b}\bar{c}\bar{a})\bar{R}$ is semisimple; now, if $(\bar{1}-\bar{a})\bar{r}=\bar{a}\bar{b}\bar{c}\bar{a}\bar{s}$ for some r,s∈R, then $\bar{r}\in\bar{a}\bar{R}$ and, since $\bar{a}=\bar{a}^2$, this implies $\bar{r}=\bar{a}\bar{r}$, i.e. $(\bar{1}-\bar{a})\bar{r}=\bar{0}$; thus the sum $(\bar{1}-\bar{a})\bar{R}\oplus\bar{a}\bar{b}\bar{c}\bar{a}\bar{R}$ is direct and clearly it contains $(\bar{1}-\bar{a}+\bar{a}\bar{b}\bar{c}\bar{a})\bar{R}$. Since $a\in\mathcal{E}$ and $\bar{a}\bar{b}\bar{c}\bar{a}\bar{R}=\bar{a}\bar{b}\bar{R}$ is simple (5), $(\bar{1}-\bar{a}+\bar{a}\bar{b}\bar{c}\bar{a})\bar{R}$ is a submodule of a semisimple module and hence is semisimple itself, as required.

Finally, since $1-abc\notin U(R)$, we can apply 3) to get $a>a-abca$, but $a-abca\in\mathcal{E}$ and a is minimal in $\mathcal{E}$. This contradiction shows that $a\in J$, as desired.■

<u>THEOREM 10.6</u> *If* M_R *is an Artinian module then* $S=End(M_R)$ *is a semilocal ring.*

PROOF [9;Theo.6]: Consider the left S-module ${}_SM$; for any $f\in S$, $\mathbf{r}_M(f)$ is precisely Kerf, and thus (10.2) shows that $\mathbf{r}_M(f)\neq 0$ for all $f\in S\setminus U(S)$. On the other hand, M_R is finite dimensional and then the remark preceding (10.5) shows that $\mathcal{E}=\{Kerf: f\in S\}$ satisfies the maximum condition with respect to summands. Now the left-right symmetric of (10.5) applies and therefore S is semilocal.■

Modules of Finite Length

We intend to prove that an Artinian or Noetherian module has an endomorphism ring in which every nil subring is nilpotent (here we do not require that subrings contain the identity of the overring), and as a consequence the endomorphism ring of a module of finite length is *semiprimary* (i.e. semilocal with nilpotent radical). The first part was first announced by A.Goldie and L.Small in [20] for Noetherian modules. Later on, J.Fisher [16] gave a proof for the Artinian case which was dualizable. The second part is here easily proved using our Theorem 10.6.

At this point, we need to introduce the concept of T-nilpotency (for transfinite nilpotency). A subset W of a ring R is said to be *left* (resp. *right*) T-*nilpotent* if, for every infinite sequence $w_1,w_2,\dots$ of elements of W, there exists an integer k such that $w_1\cdots w_k=0$ (resp. $w_k\cdots w_1=0$). Every nilpotent subring of R is left (resp. right) T-nilpotent, and every T-nilpotent subset of R is nil. Counter-examples for both converses do exist (see [A-F; Ex.15.8]); however, T-nilpotency does imply nilpotency in the following particular case.

LEMMA 10.7 *Let* R *be a ring with* ACC *on left (right) annihilators, and let* B *be a subring of* R *.Then* B *is nilpotent if and only if* B *is left (right) T-nilpotent.*

PROOF: As we have already remarked, if B is nilpotent then it is (left and right) T-nilpotent. Assume then that B is not nilpotent; by hypothesis, $\mathscr{L}(B)\subseteq\mathscr{L}(B^2)\subseteq\cdots$ gets stationary at some $n\in\mathbb{N}$, and by assumption $B^{n+1}\neq 0$; then there exists $b_1\in B$ with $b_1B^n\neq 0$, i.e. $b_1\in B\setminus\mathscr{L}(B^n)$. Since $\mathscr{L}(B^n)=\mathscr{L}(B^{n+1})$, this implies $b_1B^{n+1}\neq 0$; take then $b_2\in B$ such that $b_1b_2B^n\neq 0$, so that $b_1b_2\notin\mathscr{L}(B^n)$ and in particular $b_1b_2\neq 0$. In this way, we get a sequence $b_1,b_2,..$ of elements of B with $b_1\cdots b_k\neq 0$ for all $k\in\mathbb{N}$, and hence B is not left T-nilpotent.∎

THEOREM 10.8 *If* M_R *is Artinian or Noetherian, then each nil subring of* $S=End(M_R)$ *is nilpotent.*

PROOF [16;Theo.1.5] or [A-F;Theo.29.2]: We assume that M_R is Artinian; the proof if M is Noetherian may follow dually to this one, but in fact is a direct consequence of (9.13).

Let B be a nil subring of S. By (6.4.a) S has ACC on left annihilator ideals, so that it suffices to see that B is left T-nilpotent (10.7).

Let us first introduce the following two concepts: a sequence $\{b_n\}$ in B is an *∞-chain* if $b_1\cdots b_n\neq 0$ for all n (all subscripts will belong to $\mathbb{N}$); clearly every tail $\{b_n,b_{n+1},...\}$ of an ∞-chain is an ∞-chain. An element $b\in B$ *has an ∞-chain* if there exists an ∞-chain $\{b_n\}$ with $b_1=b$. If $\{b_n\}$ is an ∞-chain and $i\leq j$, then the product $b_ib_{i+1}\cdots b_j$ has an ∞-chain.

Suppose that B is not left T-nilpotent; then $\emptyset\neq\Omega_1=\{b\in B$: b has an ∞-chain$\}$ and thus, since M_R is Artinian, there exists $b_1\in\Omega_1$ such that b_1M is minimal in $\{bM: b\in\Omega_1\}$. By induction, and using the previous remarks, we can construct the nonempty set $\Omega_n=\{b\in B$: $b_1b_2\cdots b_{n-1}b$ has an ∞-chain$\}$ and find $b_n\in\Omega_n$ such that b_nM is minimal in $\{bM: b\in\Omega_n\}$. It is clear that $\{b_n\}$ is then an ∞-chain.

Moreover, for each $i\leq j$, we have $(b_i\cdots b_jM)\subseteq b_iM$ and $(b_i\cdots b_j)\in\Omega_i$ whence, by minimality of b_iM, we get

(1) $$(b_i\cdots b_jM)=b_iM.$$

Now, for each n, call $f_n=b_1\cdots b_n(\neq 0)$. By the last remark, $f_nM=f_mM$

$(=b_1M)$ for all m,n. In particular $f_nM=f_{n+1}M$ and thus, for all n,

(2) $$M=(\mathrm{Ker}f_n)+(b_{n+1}M).$$

To see that, let $x\in M$ and take $y\in M$ such that $f_nx=f_{n+1}y$; then $x=(x-b_{n+1}y)+b_{n+1}y$ with $f_n(x-b_{n+1}y)=0$.

Let us now prove that, for $n\geq m$, $f_nb_m=0$. Suppose $n\geq m$ and $f_nb_m\neq 0$; for any $k\geq m$ we have (1) $b_mM=b_m\cdots b_kM$ and thus $f_nb_m\cdots b_kM=f_nb_mM\neq 0$, what means $f_nb_m\cdots b_k\neq 0$ for all $k\geq m$, i.e. f_nb_m has an ∞-chain; but $f_nb_m=(b_1\cdots b_{m-1})(b_m\cdots b_nb_m)$, whence $(b_m\cdots b_nb_m)\in\Omega m$; then, by minimality of b_mM, we get

$$0\neq b_mM = (b_m\cdots b_n)b_mM = (b_m\cdots b_n)^2b_mM = \cdots,$$

contrary to the nilpotency of $b_m\cdots b_n\in B$.

This also proves that $b_mM\subseteq\mathrm{Ker}f_n$ for all $n\geq m$, and thus

(3) $$b_mM\subseteq\bigcap_{n\geq m}\mathrm{Ker}f_n \qquad \text{and} \qquad \sum_{j=1}^{n}b_jM\subseteq\mathrm{Ker}f_n.$$

Next, we show by induction that, for any n,

(4) $$\bigcap_{k=1}^{n}\mathrm{Ker}f_k + \sum_{j=2}^{n+1}b_jM = M.$$

The case n=1 is covered by (2). Assume now that (4) holds for n-1; then using (2), (3) and the modular law we get

$$\Big(\bigcap_{k=1}^{n}\mathrm{Ker}f_k\Big)+\Big(\sum_{j=2}^{n+1}b_jM\Big) = \Big[\Big(\bigcap_{k=1}^{n-1}\mathrm{Ker}f_k\Big)\cap\mathrm{Ker}f_n\Big]+\Big(\sum_{j=2}^{n}b_jM\Big)+b_{n+1}M =$$

$$\Big[\Big[\Big(\bigcap_{k=1}^{n-1}\mathrm{Ker}f_k\Big)+\Big(\sum_{j=2}^{n}b_jM\Big)\Big]\cap\mathrm{Ker}f_n\Big]+b_{n+1}M = \mathrm{Ker}f_n+b_{n+1}M = M.$$

Now, since M is Artinian, the sequence $\Big(\bigcap_{k=1}^{n}\mathrm{Ker}f_k\Big)_{n\in\mathbb{N}}$ stops at some n, for which $\bigcap_{k=1}^{n}\mathrm{Ker}f_k\subseteq\mathrm{Ker}f_{n+1}$. This, together with (3) and (4), shows that $M=\mathrm{Ker}f_{n+1}$, a contradiction since $f_i\neq 0$ for all i. This contradiction proves the theorem.■

<u>COROLLARY 10.9</u> *If* M_R *is Noetherian (resp. Artinian) then* $\Gamma(S)$ *(resp.* $\Delta(S)$*) is nilpotent.*

PROOF: Suppose M_R is Noetherian and recall that $\Gamma=\Gamma(S)=\{f\in S:\mathrm{Ker}f\subseteq_e M\}$. By (10.8), it suffices to show that Γ is nil. Let $f\in\Gamma$ and consider the ascending chain $\mathrm{Ker}f\subseteq\mathrm{Ker}f^2\subseteq\cdots$ of submodules of M_R; by hypothesis there exists $n\in\mathbb{N}$ such that $\mathrm{Ker}f^n=\mathrm{Ker}f^{n+1}=\cdots$. Then it is easy to see that $f^nM\cap\mathrm{Ker}f=0$ and thus, since $\mathrm{Ker}f\subseteq_e M$, we get $f^n=0$. Hence Γ is nil.

The result for $\Delta(S)=\{f\in S:fM\ll M\}$ follows by duality.■

THEOREM 10.10 *If* M_R *has finite length, then* $S=End(M_R)$ *is semiprimary.*

PROOF [A-F;Cor.29.3]: We have to prove that $S/J(S)$ is semisimple and $J(S)$ is nilpotent. Since M has finite length, it is in particular Artinian, and therefore (10.6) S is semilocal.
Then, by (10.8), it suffices to see that $J=J(S)$ is nil. Let $f\in J$; by (10.1) $M=Kerf^m\oplus Imf^m$ for m=composition length of M. Then $Imf^m\cap Kerf=0$ and $Imf^m=Imf^{m+1}$ (see the proof of (10.1)); therefore f is an automorphism of $N=Imf^m$; its inverse $g:N\longrightarrow N$ can be extended to some $h\in S$ since $N\leq_d M$, and then fh is the identity on N, what implies $N\leq Ker(1-fh)$. But $f\in J$ implies that 1-fh is invertible, so that N=0 and thus $f^m=0$, as required.■

REMARK This last theorem may be proved starting with a weaker assumption on M, namely that M_R is Artinian with finite homogeneous length, and the index of nilpotency of J(S) (i.e. the Loewy length of S) may be bounded in terms of all simple submodules of M_R. For details see [48, 49, 51].

Quasi-Injective and Quasi-Projective Modules with Noetherian, Semiprimary and Artinian Endomorphism Rings

We extend now some of the results at the end of Section 6, and characterize the quasi-injective (resp. quasi-projective) modules M_R which have semiprimary or left (resp. right) Noetherian or Artinian endomorphism rings in terms of chain conditions in their lattices of annihilator-closed (resp. M-cotorsionless) submodules. We shall keep the notation introduced in Section 6.

The key steps in what follows are the next two theorems:

THEOREM 10.11 *If* M_R *is quasi-injective then every finitely generated left ideal of* S *is annihilator-closed.*

PROOF [23;Lemma 1]: We prove that, if the left ideal $\mathfrak{A}$ is a-closed and $f\in S$ then $\mathfrak{A}+Sf$ is a-closed; thus, since the zero ideal is a-closed, the result follows by induction.
Assume then $\mathbf{l}_S\mathbf{r}_M(\mathfrak{A})=\mathfrak{A}$ and $f\in S$; our task is reduced to show that

$l_S r_M(\mathfrak{A}+Sf)\subseteq\mathfrak{A}+Sf$. Note that, for arbitrary left ideals $\mathfrak{B},\mathfrak{C}$ of S, we have $r_M(\mathfrak{B}+\mathfrak{C})=r_M(\mathfrak{B})\cap r_M(\mathfrak{C})$. Let then $g\in l_S r_M(\mathfrak{A}+Sf)=l_S(r_M(\mathfrak{A})\cap r_M(f))$; we have at once $r_M(g)\supseteq r_M(\mathfrak{A})\cap r_M(f)$; consider the following diagram in which the rows are exact ($r_M(h)$=Kerh for any $h\in S$), i, j are inclusion maps (so that the square on the left commutes) and $r_M=r$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & r(\mathfrak{A})\cap r(f) & \longrightarrow & r(\mathfrak{A}) & \xrightarrow{f} & fr(\mathfrak{A}) & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow j & & \vdots\, k & & \\ 0 & \longrightarrow & r(g) & \longrightarrow & M & \xrightarrow{g} & gM & \longrightarrow & 0 \end{array}$$

An easy exercise of diagram chasing shows that there exists $k:fr_M(\mathfrak{A})\to gM$ such that the resulting diagram commutes. From the quasi-injectivity of M, k extends to some $h\in S$ and then hf and g coincide in $r_M(\mathfrak{A})$; therefore $g-hf\in l_S r_M(\mathfrak{A})=\mathfrak{A}$, whence $g\in\mathfrak{A}+Sf$, proving the theorem.∎

THEOREM 10.12 *If* M_R *is quasi-projective then every finitely generated right ideal of* S *is* $\tau\sigma$*-closed.*

PROOF [A-N;Prop.4.9]: The proof is dual to that of (10.11), and consists of showing that $\mathfrak{A}+fS$ is $\tau\sigma$-closed whenever the right ideal $\mathfrak{A}$ of S is and $f\in S$.

So, assume $\mathfrak{A}=\tau_S\sigma_M(\mathfrak{A})$, $f\in S$, and let us prove that $\tau_S\sigma_M(\mathfrak{A}+fS)\subseteq\mathfrak{A}+fS$. It is clear that $\sigma_M(\mathfrak{A}+fS)\subseteq\mathfrak{A}M+fM$, so that if $g\in\tau_S\sigma_M(\mathfrak{A}+fS)$ then $gM\subseteq\mathfrak{A}M+fM$; write $N=\mathfrak{A}M=\sigma_M(\mathfrak{A})$, $p:M\xrightarrow{c}M/N$, $\bar{f}=pf$ and $\bar{g}=pg$, and note that

$$\bar{g}M = pgM = \frac{gM+N}{N} \subseteq \frac{fM+N}{N} = pfM = \bar{f}M.$$

Thus, by quasi-projectivity of M, there exists $h\in S$ such that $\bar{g}=\bar{f}h$, i.e. pg=pfh, or p(g-fh)=0, but that just means that $g-fh\in N$ and hence $g\in N+fS=\sigma_M(\mathfrak{A})+fS$, as required.∎

Thus we get the next two corollaries from (6.18):

COROLLARY 10.13 *Let* M_R *be quasi-injective and consider the following statements:*

a) S *is a regular ring;*

b) *the kernel of every element of* S *is a direct summand of* M_R*;*

c) *finite intersections of kernels of elements of* S *are direct summands of* M_R*;*

a′) S *is a right perfect ring;*

b′) M *has* ACC *on* $\mathcal{K}$={Kerf; f∈S};

c′) M *has* ACC *on* $\mathcal{K}_F=\{\bigcap_{i=1}^{n}\mathrm{Ker}f_i:\ f_1,\dots,f_n\in S\}$.

Then we have a)⇔b)⇔c) and a′)⇔b′)⇔c′).■

COROLLARY 10.14 *Let* M_R *be quasi-projective and consider the following statements:*

a) S *is a regular ring;*

b) *the image of every element of* S *is a direct summand of* M_R*;*

c) *finite sums of images of elements of* S *are direct summands of* M_R*;*

a′) S *is a right perfect ring;*

b′) M *has* DDC *on* $\mathcal{I}$={Imf; f∈S};

c′) M *has* DCC *on* $\mathcal{I}_F=\{\sum_{i=1}^{n}\mathrm{Im}f_i:\ f_1,\dots,f_n\in S\}$.

Then we have a)⇔b)⇔c) and a′)⇔b′)⇔c′).■

The following theorems, proved here for quasi-injective and quasi-projective modules, may be proved under the weaker assumptions $\mathcal{F}_l\subseteq\mathcal{M}_a$ and $\mathcal{F}_r\subseteq\mathcal{M}_{\sigma\tau}$, respectively, and in fact the proofs given here use only these hypothesis.

THEOREM 10.15 *If* M_R *is quasi-injective then the following assertions are equivalent:*

a) S *is left Noetherian;*

b) M *has* DCC *on* $\mathcal{K}_F$*;*

c) M *has* DCC on $\mathcal{M}_a$.

PROOF [31;Theo.4.3]: Since S is left Noetherian if and only if it satisfies ACC on finitely generated left ideals, i.e. on $\mathcal{F}_l$, the equivalence a)⇔b) follows from (10.11) and (6.17.c).

The implications a)⇒c)⇒b) follow without requiring M_R to be quasi-injective: by definition, $\mathcal{K}_F\subseteq\mathcal{M}_a$, so that c) implies b). On the other hand, let $N_1\supseteq N_2\supseteq\cdots$ be a chain of a-closed submodules of M and consider $l_S(N_1)\supseteq l_S(N_2)\supseteq\cdots$; if S is left Noetherian then there exits $n\in\mathbb{N}$ such that $l_S(N_n)=l_S(N_k)$ for all k≥n and hence

$$N_n = r_M l_S(N_n) = r_M l_S(N_k) = N_k$$

for all k≥n, and this proves that a) implies c).■

Dualizing the proof of (10.15) we get:

THEOREM 10.16 *If* M_R *is quasi-projective then the following statements are equivalent:*

a) S *is right Noetherian;*

b) M *has* ACC *on* $\mathcal{I}_F$*;*

c) M *has* ACC *on* $\mathcal{M}_{\sigma\tau}$. ■

Now we turn to the question of when is S semiprimary; proofs of Theorems 10.16 and 10.17 which use different techniques may be found in [23; Theorem 1] and [22; Prop.2.4].

THEOREM 10.17 *If* M_R *is a quasi-injective module with* ACC *on a-closed submodules then* S *is a semiprimary ring.*

PROOF [31;Theo.4.5]: Since M_R is quasi-injective, (10.11) and (6.18.c′) imply that S is a right perfect ring, i.e. S is semilocal and its radical J(S) is right T-nilpotent. Now, from (6.4.a) and the hypothesis, we know that S has DCC on left annihilators, i.e. ACC on right annihilators, and hence J(S) is indeed nilpotent by (10.7). Therefore S is semiprimary. ■

Dually, we get

THEOREM 10.18 *If* M_R *is a quasi-projective module with* DCC *on* M-*cotorsionless submodules then* S *is a semiprimary ring.* ■

Next, using the Hopkins-Levitzki Theorem (a ring is left Artinian if and only if it is left Noetherian and semiprimary [S; p.181]), we get:

THEOREM 10.19 *Let* M_R *be quasi-injective. Then* S *is left Artinian if and only if* M *satisfies* ACC *and* DCC *on* $\mathcal{M}_a$.

PROOF [31;Theo.4.6]: If M satisfies both chain conditions on $\mathcal{M}_a$ then it is left Noetherian (10.15) and semiprimary (10.17), i.e. S is left Artinian.

Conversely, if S is left Artinian then it is left Noetherian and hence M has DCC on $\mathcal{M}_a$ (10.15); moreover, if $N_1 \subseteq N_2 \subseteq \cdots$ is a chain in $\mathcal{M}_a$ then the chain $l_S(N_1)\subseteq l_S(N_2)\subseteq\cdots$ stops by hypothesis and thus so does $N_1\subseteq N_2\subseteq\cdots$, whence M has ACC on $\mathcal{M}_a$. Note that this 'only if' part may be proved without requiring M_R to be quasi-injective.■

Dually,

THEOREM 10.20 *Let* M_R *be quasi-projective. Then* S *is right Artinian if and only if* M *satisfies* ACC *and* DCC *on* $\mathcal{M}_{\tau\sigma}$.■

REMARK Theorem 10.19 generalizes the well known fact that the endomorphism ring of a quasi-injective module of finite length is left Artinian (for an easy proof see [21]).

COROLLARY 10.21 *Let* M_R *be quasi-projective and quasi-injective; then*

a) *if* M_R *is Noetherian, then* S *is right Artinian.*

b) *if* M_R *is Artinian, then* S *is left Artinian.*

PROOF: a) S is right Noetherian by (10.16) and semiprimary by (10.17); thus S is right Artinian.

b) This follows similarly from (10.15) and (10.18)■

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