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CONTRIBUTIONS TO COMBINATORIAL SEMIGROUP THEORY

by

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To my father

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SUMMARY

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INTRODUCTION

The main subject of this thesis can be defined as combinatorial semigroup theory; that is, the study of the structure and properties of free objects and presentations in varieties whose elements are semigroups. Most of the results of this type have similar formulations for semigroups and monoids, and for each of them we tried to give the version that involves the higher degree of generality, usually the monoid version.

Our work is mainly about inverse monoids, and so it is not surprising that Chapter I is essentially devoted to the introduction of concepts in this area. Most of the results in this chapter are not original, references being given in the text. Some definitions and results are formulated in terms of varieties of inverse monoids, which play an important role later on.

We give particular importance to the structure of the *free inverse monoid* on a nonempty set X, denoted by $FIM(X)$. Since we introduce languages and automata in Chapter VI, which involve the use of the free monoid, we opted to define $FIM(X)$ to be the quotient $(X\cup X^{-1})^*/\rho$, where X^{-1} denotes a set of formal inverses of X, disjoint from X, and ρ is the *Vagner congruence* on $(X\cup X^{-1})^*$ (the free monoid on $X\cup X^{-1}$). This congruence was introduced by Vagner in 1957 [41]. Since there is no natural canonical form for $(X\cup X^{-1})^*/\rho$, the direct use of this quotient was not very fruitful until 1973, when W.D.Munn solved the corresponding word problem [26]. W.D.Munn also provided a geometrical description of $FIM(X)$, using labelled trees. Previously, in 1972, and independently, H.Scheiblich produced an algebraic description, using left closed subsets of the free group on X [38].

In Chapter II we discuss the intersection of two free inverse submonoids A and B of a free inverse monoid $FIM(X)$. In contrast with analogous results for groups and monoids [40], the intersection $A \cap B$ is not necessarily free. However, we can classify all the possibilities in the case where A and B both have rank 1 [§II.2 and 3], and this discussion is further simplified when $FIM(X)$ has rank 1 itself [§II.4].

A related problem is whether the finitely generated property is preserved by the intersection of free inverse submonoids of a free inverse monoid. A counterexample is provided in Section II.5.

The subject of Chapter III is the semilattice of idempotents E of a free inverse monoid. We introduce some new concepts in semilattice theory, in particular that of a *unique factorization semilattice* (*UFS*) [§III.1]. Some general properties are proved for this class of semilattices, and these results are used to give necessary and sufficient conditions for two principal ideals of E to be isomorphic [§III.2]. This enables us to obtain some properties of T_E, the Munn semigroup of E, such as being E-unitary [§III.3].

In Section III.4 we discuss the embedding of semilattices in a free inverse monoid and some general results are obtained, involving finite semilattices and UFSs. We also provide an example of a countable semilattice S such that the subsemilattices $\{f \in S: f \geq e\}$ are finite for every $e \in S$ and yet S is not embeddable in any free inverse monoid.

In Section III.5 we show that the semilattice of idempotents of a free inverse monoid never is hopfian, in contrast with the situation for $FIM(X)$ itself, which is hopfian if and only if X is finite [26].

One of the key concerns in our work is the *word problem*, which can be defined in its most general form as follows. Let S be a semigroup and let $R \subseteq S \times S$ be a relation on S. Let $R^{\#}$ denote the congruence on S generated by R. Is there an algorithm which determines, for every $u,v \in S$, whether or not $uR^{\#} = vR^{\#}$? If such an algorithm exists, the problem is said to be decidable.

In Chapter IV we define the concept of *normal-convex* embedding for semigroups, which is naturally related to word problems. In fact, let $\varphi : S \longrightarrow T$ be a normal-convex embedding of semigroups and let R be a relation on S. Let $R\varphi$ denote the relation on T induced by R and φ. Then the word problem for R is decidable if the word problem for $R\varphi$ is decidable [§IV.1].

In 1974, McAlister introduced the triples (G,K,L), later known as *McAlister triples*, and the corresponding semigroups $P(G,K,L)$, called *P-semigroups* [20]. When K is a semilattice, we refer to (G,K,L) as a *strong McAlister triple*. In Section IV.2 we show that if (G,K,L) is a strong McAlister triple, then $P(G,K,L)$ admits a normal-convex embedding into a semidirect product of a semilattice by a group.

This result is generalized in Section IV.3, where it is shown that every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group, a stronger version of a result by O'Carroll [30].

McAlister proved that every inverse semigroup is the idempotent-separating image of a E-unitary inverse semigroup [20]. In Section IV.4 we show that every inverse semigroup admits a normal-convex embedding into an idempotent-separating image of a semidirect product of a semilattice by a free group.

Now let V be a variety of inverse monoids. We define a *presentation* in V to be an expression of the form $V\langle X;R\rangle$, where X is a nonempty set and R is a relation on the free monoid $(X \cup X^{-1})^*$. Assuming that the free object of V on X is a quotient of the form $(X \cup X^{-1})^*/\tau$, we define the *word problem* for $V\langle X;R\rangle$ to be the word problem for the relation $\tau\cup R$ on $(X\cup X^{-1})^*$. The *idempotent word problem* for $V\langle X;R\rangle$ is the restriction of the word problem to the words $u \in (X\cup X^{-1})^*$ such that $u\rho = u^2\rho$. The inverse monoid defined by a presentation $V\langle X;R\rangle$ is the quotient $(X\cup X^{-1})^*/(\tau\cup R)^\#$. The presentation $V\langle X;R\rangle$ is said to be finitely generated (respectively finitely related) if X (respectively R) is finite. A presentation is said to be finite if it is both finitely generated and finitely related.

Chapter V is essentially devoted to the study of decidability problems for presentations in the variety of Clifford monoids.

In Section V.1 we show that every finitely presented Clifford monoid can be finitely presented as an inverse monoid, thus establishing a bridge between presentations in the two varieties.

In Section V.2 we solve the word problem for finitely related presentations in the variety of semilattices with unity and we prove that the word problem for a finitely related Clifford monoid presentation is equivalent to the word problems for finitely many group presentations.

In Section V.3 we solve the E-unitary problem for one-relator Clifford presentations, that is, we give an algorithm which determines, for any such presentation, whether or not the corresponding Clifford monoid is E-unitary. A counterexample is given to a conjecture by Margolis and Meakin [19] on the E-unitary problem for one-relator inverse monoid presentations. It is also proved that the E-unitary problem is undecidable for the class of all finite Clifford monoid presentations, and this result is extended to the class of all finite inverse monoid presentations.

Some more decidability results are obtained in Section V.4, concerning triviality, finiteness, freeness and others.

Finally, the results of Section V.2 are applied in Section V.5 to simplify the word problem for finite inverse monoid presentations which define E-reflexive inverse monoids.

Chapter VI is essentially about the idempotent word problem for inverse monoid presentations. Using the techniques of Stephen [39], in the form developed by Margolis and Meakin [18], we obtain a positive decidability result involving any finite presentation and any rational language (§VI.2).

This result can be used to provide an alternative proof to Margolis and Meakin's solution [18] of the word problem for finite idempotent-pure presentations (§VI.3). A generalization of this result is obtained in Section VI.4.

In Section VI.5 we solve the idempotent word problem for the class of finite $\mathcal{R}$-pure presentations, still using the results of Section VI.2.

The bounds of application of rational languages as a technique for solving idempotent word problems are discussed in Section VI.6, where some results are obtained for one-relator presentations.

In Section VI.7 we produce an example of a finite inverse monoid presentation with undecidable idempotent word problem.

Chapter VII presents some results on primeness of semigroup rings. In Section VII.2 we introduce a certain Condition C on semigroups which is proved to be a sufficient condition for primeness of the corresponding semigroup rings.

Condition C is applied in Section VII.3 to prove primeness for semigroup rings of free products of semigroups, and in Section VII.4, for one-relator semigroup presentations (if the generating set has more than two elements).

Finally, Section VII.5 gives a simple generalization of a result by W.D.Munn [27] concerning semigroup rings of inverse semigroups with pseudofinite semilattice of idempotents (which include free inverse semigroup of finite rank). Moreover, Condition C is applied to prove primeness for semigroup rings of free inverse semigroups of infinite rank.

CHAPTER I

GENERAL CONCEPTS

1. Semigroups and monoids

In this section we introduce elementary concepts and terminology of general semigroup theory. The details can be found in Howie [11]. Let S be a nonempty set and let $\cdot$ denote an associative binary operation on S. Then $(S,\cdot)$ is said to be a *semigroup*. In general, we omit the operation symbol. The element $u \in S$ is said to be a *unity* if $us = su = s$ for every $s \in S$. It is immediate that every semigroup has at most one unity. A semigroup with unity is said to be a *monoid*, the unity being usually denoted by 1.

To every semigroup S, we associate a monoid S^1 as follows. If S has a unity, we take $S^1 = S$. If not, we define $S^1 = S\cup\{1\}$ to be the monoid obtained by adjoining the unity 1 to S.

For the remainder of this section, we assume that S and T are semigroups.

For all subsets A,B of S, we write $AB = \{ab\colon a \in A,\ b \in B\}$.

Let A be a nonempty subset of S. We say that A is a *subsemigroup* of S if $A^2 \subseteq A$.

Let $\varphi : S \longrightarrow T$ be a map. We say that φ is a *homomorphism* of semigroups if $(ab)\varphi = (a\varphi)(b\varphi)$ for every $a,b \in S$. We shall also use the following terminology. If φ is injective, φ is said to be an *embedding*. If φ is bijective, φ is said to be an *isomorphism*. If

$S = T$, φ is said to be an *endomorphism*. If $S = T$ and φ is bijective, φ is said to be an *automorphism*.

Let A be a nonempty subset of S. We say that A is an *ideal* of S if $S^1AS^1 \subseteq A$. We denote that fact by $A \trianglelefteq S$. If $A = S^1aS^1$ for some $a \in S$, A is said to be a *principal ideal*.

Let X be a nonempty set. A subset $R \subseteq X \times X$ is said to be a *relation* on X. For all relations R,R' on X, we define the composite relation RoR' on X by

$$(a,b) \in RoR' \iff \exists c \in X\colon (a,c) \in R \text{ and } (c,b) \in R'.$$

Let R be a relation on X. We say that R is an *equivalence* relation on X if the conditions below hold:

$$\forall a \in X,\ (a,a) \in R;$$

$$(a,b) \in R \Rightarrow (b,a) \in R;$$

$$RoR \subseteq R.$$

Let R be an equivalence relation on X. For every $a \in X$, we write $aR = \{b \in X\colon (a,b) \in R\}$, and $X/R = \{aR\colon a \in X\}$.

Now we define the following relations on the semigroup S.

$$(a,b) \in \mathcal{R} \iff aS^1 = bS^1;$$

$$(a,b) \in \mathcal{L} \iff S^1a = S^1b;$$

$$(a,b) \in \mathcal{J} \iff S^1aS^1 = S^1bS^1;$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L};$$

$$\mathcal{D} = \mathcal{R} o \mathcal{L}.$$

These relations are equivalence relations and are called *Green's* relations on S. We note that $\mathcal{R} o \mathcal{L} = \mathcal{L} o \mathcal{R}$. If $\mathcal{D} = S \times S$, we say that S is *bisimple*.

Let τ be an equivalence relation on S. We say that τ is a *congruence* on S if, for every $c \in S$,

$$(a,b) \in \tau \Rightarrow (ac,bc),(ca,cb) \in \tau.$$

If τ is a congruence on S, then $(a\tau)(b\tau) = (ab)\tau$ defines an associative operation on S/τ. Moreover, the map $\tau^{\natural}:S \longrightarrow S/\tau$: $s \mapsto s\tau$ is a surjective homomorphism of semigroups.

Let $\varphi:S \longrightarrow T$ be a homomorphism. The relation $Ker\varphi$ on S defined by

$$(a,b) \in Ker\varphi \iff a\varphi = b\varphi$$

is a congruence. We have

LEMMA 1.1 [11,§I.5]. *Let S and T be semigroups, let $\varphi:S \longrightarrow T$ be a homomorphism and let τ be a congruence on S with $\tau \subseteq Ker\varphi$. Then the map $\overline{\varphi}:S/\tau \longrightarrow T$: $s\tau \mapsto s\varphi$ is a homomorphism and the diagram*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\tau^{\natural}"'] & T \\ S/\tau \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

commutes. Moreover, if $\tau = Ker\varphi$, then $\overline{\varphi}$ is injective and $S\varphi \simeq S/Ker\varphi$.

LEMMA 1.2 [11,§I.5]. *Let S be a semigroup and let τ,ν be congruences on S with $\tau \subseteq \nu$. Let ν/τ denote the relation on S/τ defined by*

$$(a\tau,b\tau) \in \nu/\tau \iff (a,b) \in \nu.$$

Then ν/τ is a congruence on S/τ and the map $(S/\tau)/(\nu/\tau) \longrightarrow S/\nu$: $(s\tau)(\nu/\tau) \mapsto s\nu$ is an isomorphism.

Let R be a relation on S. We define a relation $R^{\#}$ on S as follows. For every $a,b \in S$, $(a,b) \in R^{\#}$ if and only if there exist $w_0,\ldots,w_n \in S$ such that

$w_0 = a;$

$w_n = b;$

$\forall i \in \{1,\dots,n\}\ \exists p_i,q_i \in S\ \exists(u_i,v_i) \in R:$

$(w_{i-1},w_i) = (p_iu_iq_i,p_iv_iq_i).$

It follows easily that $R^{\#}$ is the smallest congruence on S containing the relation R and so it is said to be the congruence *generated* by R. Suppose that $\varphi:S \longrightarrow T$ is a homomorphism. Then the relation $R\varphi = \{(a\varphi,b\varphi): (a,b) \in R\}$ on T is said to be the relation induced by R and φ. It follows easily that

$$R^{\#}\varphi \subseteq (R\varphi)^{\#}. \tag{1.1}$$

If we are working inside the class of monoids, it is useful to strengthen some definitions.

Let M be a monoid and let A be a nonempty subset of M. Then A is a *submonoid* of M if $A^2\cup\{1\} \subseteq A$.

Let M,N be monoids and let $\varphi:M \longrightarrow N$ be a map. Then φ is a *homomorphism* of monoids if $(ab)\varphi = (a\varphi)(b\varphi)$ for every $a,b \in M$ and $1\varphi = 1$. Throughout this work, we shall refer to a homomorphism of monoids simply as a homomorphism.

Everything said before for semigroups holds for monoids with these stronger definitions. In particular, for every congruence τ on a monoid M, the map $\tau^{\natural}:M \longrightarrow M/\tau:\ a \mapsto a\tau$ is a surjective homomorphism.

Now let X denote a nonempty set. We define a *word* on X to be a finite sequence of elements of X; including the empty sequence. Each term of a word is then said to be a *letter*. A nonempty word will be usually written in the form $x_1\dots x_n$, $x_i \in X$, and we identify each $x \in X$ with the word x. The empty word is denoted by 1.

The set of all words on X is denoted by X^*. We define an operation on X as follows. For all nonempty words $x_1\dots x_n,y_1\dots y_m$ on X, we define

$$(x_1 \ldots x_n)(y_1 \ldots y_m) = x_1 \ldots x_n y_1 \ldots y_m.$$

For every $w \in X^*$, we define $1w = w1 = w$. With this operation, X^* is a monoid and the subset $X^+ = X^* \setminus \{1\}$ is a subsemigroup of X^*.

LEMMA 1.3 [34,§I.10]. *Let X be a nonempty set and let i stand for inclusion map.*

(i) Let S be a semigroup and let $\varphi : X \longrightarrow S$ be a map. Then there exists a unique homomorphism of semigroups $\Phi : X^+ \longrightarrow S$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "i"'] & S \\ X^+ \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes.

(ii) Let M be a monoid and let $\varphi : X \longrightarrow M$ be a map. Then there exists a unique homomorphism $\Phi : X^* \longrightarrow M$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "i"'] & M \\ X^* \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes.

With this property, X^+ (respectively X^*) is said to be the *free semigroup* (respectively *free monoid*) on X.

We define the following partial orders on X^*.

$$a \leqslant b \iff b \in X^*aX^*;$$

$$a \leqslant_l b \iff b \in aX^*;$$

$$a \leqslant_r b \iff b \in X^*a.$$

If $a \leqslant b$ (respectively $a \leqslant_l b$, $a \leqslant_r b$) we say that a is a *factor* (respectively *prefix*, *suffix*) of b.

For every $w \in X^*$, we define the *length* $|w|$ of w as follows. If $w = 1$, let $|w| = 0$. If $w = x_1 \ldots x_n$, $x_i \in X$, let $|w| = n$.

2. Inverse monoids

In this section we introduce terminology and notation related to inverse monoids, which constitute the main subject of our work. As in the previous section, every notion has a similar counterpart in the context of inverse semigroups. For further details, see Howie [11] and Petrich [34].

Let M be a monoid. We say that M is *inverse* if

$$\forall a \in M\ \exists! b \in M: aba = a \text{ and } bab = b.$$

The element b is said to be the *inverse* of a and is denoted by a^{-1}. If M is an inverse monoid and N is a submonoid of M, we say that N is an *inverse submonoid* of M if

$$\forall a \in N,\ a^{-1} \in N.$$

LEMMA 2.1 [11,§V.1]. *Let M,N be monoids and let $\varphi: M \longrightarrow N$ be a homomorphism. Suppose that M is inverse. Then $M\varphi$ is inverse.*

Let M be a monoid and let $a \in M$. We say that a is *idempotent* if $a^2 = a$. The subset of all idempotents of M is denoted by $E(M)$. Since $1 \in E(M)$, $E(M)$ is always nonempty. Now we have

LEMMA 2.2 [11,§V.1]. *Let M be a monoid. Then M is inverse if and only if*

$$\forall a \in M\ \exists b \in M\colon aba = a \quad \text{and} \quad \forall e,f \in E(M),\ ef = fe.$$

We define a *semilattice* to be a commutative semigroup whose elements are idempotents. By Lemma 2.2, every semilattice is inverse. Let M be an inverse monoid. It follows easily from Lemma 2.2 that $E(M)$ is an inverse submonoid of M. We refer to $E(M)$ as the *semilattice of idempotents* of M. Now we define a partial order on M as follows. For every $a,b \in M$,

$$a \leqslant b \iff a = eb \text{ for some } e \in E(M).$$

We say that $\leqslant$ is the *natural partial order* of M. We note that, if $a,b \in E(M)$, then $a \leqslant b$ is equivalent to $a = ab$. It follows easily that, for every $a,b \in E(M)$, ab is the greatest lower bound of a and b for the natural partial order of M.

Now suppose that $a,b \in M$ are such that $b < a$ and

$$b \leqslant c < a \Rightarrow b = c$$

for every $c \in M$. Then we say that a *covers* b and we denote this fact by $b \prec a$. We write $Cov(a) = \{b \in M\colon b \prec a\}$.

Let σ be the relation on M defined by

$$(a,b) \in \sigma \iff ea = eb \text{ for some } e \in E(M).$$

Then σ is a congruence and M/σ is a group. Moreover, if M/τ is a group for some congruence τ on M, then $\sigma \subseteq \tau$. Therefore σ is said to be the *least group congruence* on M. It is immediate that $E(M) \subseteq 1\sigma$. We say that M is *E-unitary* if $1\sigma \subseteq E(M)$.

We note that the equivalence

$$(a,b) \in \sigma \iff ae = be \text{ for some } e \in E(M)$$

holds for every $a,b \in M$.

Let *Inv* denote the class of all inverse monoids.

Let X be a nonempty set. We define $X^{-1} = \{x^{-1}: x \in X\}$ to be a set such that

$$X \cap X^{-1} = \emptyset;$$

$$\forall x_1, x_2 \in X,\ x_1^{-1} = x_2^{-1} \Rightarrow x_1 = x_2.$$

Moreover, we define $(x^{-1})^{-1} = x$ for every $x \in X$. Under these conditions, X^{-1} is said to be a set of *formal inverses* of X. For every $w \in (X \cup X^{-1})^*$, we define a formal inverse $w^{-1} \in (X \cup X^{-1})^*$ as follows. If $w = 1$, let $w^{-1} = 1$. If $w = x_1 \dots x_n$, $x_i \in X \cup X^{-1}$, let $w^{-1} = x_n^{-1} \dots x_1^{-1}$.

For every $w \in (X \cup X^{-1})^*$, we define the *content* $\xi(w)$ of w to be $\{x \in X: x < w \text{ or } x^{-1} < w\}$.

Let Y be a nonempty set. A pair $(u,v) \in (Y \cup Y^{-1})^* \times (Y \cup Y^{-1})^*$ is said to be an *identity* of inverse monoids. We often write it in the form $u = v$. Let M be an inverse monoid and let $\varphi: Y \longrightarrow M$ be a map. We define a homomorphism $\tilde{\varphi}: (Y \cup Y^{-1})^* \longrightarrow M$ as follows: for every $y \in Y$, let

$$y\tilde{\varphi} = y\varphi \text{ and } y^{-1}\tilde{\varphi} = (y\varphi)^{-1}. \tag{2.1}$$

We say that M *satisfies* the identity $u = v$ if, for every map $\varphi: Y \longrightarrow M$, we have $u\tilde{\varphi} = v\tilde{\varphi}$. Let Σ be a system of identities. We say that M satisfies Σ if M satisfies every identity in Σ. We denote by $Inv[\Sigma]$ the class of all inverse monoids that satisfy Σ. The class $Inv[\Sigma]$ is said to be a *variety* of inverse monoids. In particular, considering Σ empty, we have that *Inv* is a variety of inverse monoids.

Let X be a nonempty set, and let $u = v$ be an identity, with $u,v \in (Y \cup Y^{-1})^*$. For every map $\varphi: Y \longrightarrow (X \cup X^{-1})^*$, we define a homomorphism $\tilde{\varphi}: (Y \cup Y^{-1})^* \longrightarrow (X \cup X^{-1})^*$ as in (2.1). Let

$$H(u = v) = \{(u\tilde{\varphi}, v\tilde{\varphi}): \varphi: Y \longrightarrow (X \cup X^{-1})^* \text{ is a map}\}.$$

We define the *Vagner* congruence on $(X \cup X^{-1})^*$ to be

$$\rho = [H(xx^{-1}x = x) \cup H(xx^{-1}yy^{-1} = yy^{-1}xx^{-1})]^{\#},$$

that is, $\rho = (\{(ww^{-1}w,w) : w \in (X \cup X^{-1})^*\}$
$\cup\{(uu^{-1}vv^{-1},vv^{-1}uu^{-1}) : u,v \in (X \cup X^{-1})^*\})^\#$.

LEMMA 2.3 [41]. *Let X be a nonempty set. Then $(X \cup X^{-1})^*/\rho$ is inverse. Let $\gamma : X \longrightarrow (X \cup X^{-1})^*/\rho : x \mapsto x\rho$. Let M be an inverse monoid and let $\varphi : X \longrightarrow M$ be a map. Then there exists a unique homomorphism $\Phi : (X \cup X^{-1})^*/\rho \longrightarrow M$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \\ (X \cup X^{-1})^*/\rho \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes.

We define $FIM(X) = (X \cup X^{-1})^*/\rho$ to be the *free inverse monoid* on X. Let Σ be a system of identities. We define

$$\tau(\Sigma) = (\rho \cup [\bigcup_{\sigma \in \Sigma} H(\sigma)])^\#.$$

LEMMA 2.4 [34,§XII.1]. *Let X be a nonempty set and let Σ be a system of identities. Then $(X \cup X^{-1})^*/\tau(\Sigma) \in \mathrm{Inv}[\Sigma]$. Let $\gamma : X \longrightarrow (X \cup X^{-1})^*/\tau(\Sigma) : x \mapsto x[\tau(\Sigma)]$. Let $M \in V = \mathrm{Inv}[\Sigma]$ and let $\varphi : X \longrightarrow M$ be a map. Then there exists a unique homomorphism $\Phi : (X \cup X^{-1})^*/\tau(\Sigma) \longrightarrow M$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \\ (X \cup X^{-1})^*/\tau(\Sigma) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes.

Let $V = Inv[\Sigma]$. We say that $(X \cup X^{-1})^*/\tau(\Sigma)$ is the *free* object of V on X and we denote it by $FV(X)$.

Let A be a nonempty subset of $FV(X)$. Let Y be a set such that there exists a bijection $\varphi: Y \longrightarrow B$. Let $\Phi: FV(Y) \longrightarrow FV(X)$ be the homomorphism induced by φ. Then A is said to be a *basis* if Φ is injective.

The class of all groups, denoted by Gp, is a variety of inverse monoids, since $Gp = Inv[xx^{-1} = 1]$. We write $\pi = \tau(xx^{-1} = 1) = [\rho \cup H(xx^{-1} = 1)]^\# = \{\rho \cup \{(uu^{-1},1): u \in (X \cup X^{-1})^*\}\}^\#$. It is immediate that $\rho \subseteq \{(uu^{-1},1): u \in (X \cup X^{-1})^*\}^\#$ and so $\pi = \{(uu^{-1},1): u \in (X \cup X^{-1})^*\}^\#$. It follows easily that $\pi = \{(xx^{-1},1): x \in X \cup X^{-1}\}^\#$. The quotient $FG(X) = (X \cup X^{-1})^*/\pi$ is the *free group* on X and we define $D_X = \{u \in (X \cup X^{-1})^*: (u,1) \in \pi\}$ to be the set of all *Dyck* words on X.

Let $V = Inv[\Sigma]$ be a variety of inverse monoids. We define a *presentation* in V to be an expression of the form $V\langle X;R\rangle$, where X is a nonempty set and R is a relation on $(X \cup X^{-1})^*$. The inverse monoid defined by this presentation is the quotient $M = (X \cup X^{-1})^*/[\tau(\Sigma) \cup R]^\#$. If X is finite, we say that the presentation is *finitely generated*. If R is finite, we say that the presentation is *finitely related*. If both X and R are finite, we say that the presentation is *finite*. Two presentations $V\langle X;R\rangle$ and $V\langle Y;S\rangle$ are said to be *equivalent* if they define isomorphic inverse monoids.

The next result is easy to obtain.

LEMMA 2.5. *Let $Inv\langle X;R\rangle$ be a presentation and let $M = (X\cup X^{-1})^*/(\rho\cup R)^\#$. Then the map $\varphi:M \longrightarrow (X\cup X^{-1})^*/(\pi\cup R)^\#$: $w(\rho\cup R)^\# \mapsto w(\pi\cup R)^\#$ is a surjective homomorphism and $Ker\varphi = \sigma$, the least group congruence on M.*

Proof. Since $\rho \subseteq \pi$, we have $(\rho\cup R)^\# \subseteq (\pi\cup R)^\#$ and so φ is defined. It is immediate that φ is a surjective homomorphism. Since $\pi \subseteq (\pi\cup R)^\#$, it follows that $(X\cup X^{-1})^*/(\pi\cup R)^\#$ satisfies the identity $xx^{-1} = 1$ and so is a group. Hence $\sigma \subseteq Ker\varphi$. Now we prove that $Ker\varphi \subseteq \sigma$. Suppose that $[a(\rho\cup R)^\#]\varphi = [b(\rho\cup R)^\#]\varphi$ for some $a,b \in (X\cup X^{-1})^*$. Then there exist $w_0,\dots w_n \in (X\cup X^{-1})^*$ such that

$$a = w_0;$$

$$b = w_n;$$

$$\forall i \in \{1,\dots,n\}\ \exists s_i,t_i \in (X\cup X^{-1})^*$$

$$\exists(u_i,v_i) \in \{(xx^{-1},1): x \in X\cup X^{-1}\}\cup R:$$

$$(w_{i-1},w_i) = (s_iu_it_i,s_iv_it_i).$$

Let $Z = \{x \in X: x \in \xi(u_i)\cup\xi(v_i),\ i \in \{1,\dots,n\}\}$ and let $z = \prod_{x\in Z} xx^{-1}.x^{-1}x$. Let $y = \prod_{i=1}^{n} t_i^{-1}zt_i$. We show that $(w_{j-1}y)(\rho\cup R)^\# = (w_jy)(\rho\cup R)^\#$ for every $j \in \{1,\dots,n\}$. Let $j \in \{1,\dots,n\}$.

Suppose first that $(u_j,v_j) \in R$. Then $(s_ju_jt_j)(\rho\cup R)^\# = (s_jv_jt_j)(\rho\cup R)^\#$ and so $w_{j-1}(\rho\cup R)^\# = w_j(\rho\cup R)^\#$. Thus $(w_{j-1}y)(\rho\cup R)^\# = (w_jy)(\rho\cup R)^\#$.

Now suppose that $(u_j,v_j) = (xx^{-1},1)$ for some $x \in X\cup X^{-1}$. Since $t_i^{-1}zt_i \in D_X$ for every $i \in \{1,\dots,n\}$, we have $y\rho = [(t_j^{-1}zt_j)y]\rho$ and so $(s_ju_jt_jy)(\rho\cup R)^\# = [s_jxx^{-1}t_j(t_j^{-1}zt_j)y](\rho\cup R)^\# = (s_jt_jt_j^{-1}xx^{-1}zt_jy)(\rho\cup R)^\# = (s_jt_jt_j^{-1}zt_jy)(\rho\cup R)^\# = (s_jt_jy)(\rho\cup R)^\# = (s_jv_jt_jy)(\rho\cup R)^\#$. Hence

$(w_{j-1}y)(\rho\cup R)^{\#} = (w_j y)(\rho\cup R)^{\#}$.

It follows that $(ay)(\rho\cup R)^{\#} = (by)(\rho\cup R)^{\#}$. Since $y(\rho\cup R)^{\#} \in E(M)$, this yields $[a(\rho\cup R)^{\#}]\sigma = [b(\rho\cup R)^{\#}]\sigma$ and so $Ker\varphi \subseteq \sigma$. Thus $Ker\varphi = \sigma$ and the lemma is proved.

We say that $(X\cup X^{-1})^*/(\pi\cup R)^{\#}$ is the *maximal group homomorphic image* of M. In particular, $FG(X)$ is the maximal group homomorphic image of $FIM(X)$.

We can give a simple description of the free product in a variety using presentations.

LEMMA 2.6. *Let $V = Inv[\Sigma]$ be a variety of inverse monoids and let $V\langle X;R\rangle$, $V\langle Y;S\rangle$ be presentations, with $X\cap Y = \emptyset$. Let $A = (X\cup X^{-1})^*/[\tau(\Sigma)\cup R]^{\#}$, $B = (Y\cup Y^{-1})^*/[\tau(\Sigma)\cup S]^{\#}$ and*

$$A*_V B = [(X\cup Y)\cup(X\cup Y)^{-1}]^*/[\tau(\Sigma)\cup R\cup S]^{\#}.$$

Let $i_A : A \longrightarrow A*_V B$: $w[\tau(\Sigma)\cup R]^{\#} \mapsto w[\tau(\Sigma)\cup R\cup S]^{\#}$ and $i_B : B \longrightarrow A*_V B$: $w[\tau(\Sigma)\cup S]^{\#} \mapsto w[\tau(\Sigma)\cup R\cup S]^{\#}$. Then

(i) for every $C \in V$ and homomorphisms $\varphi_A : A \longrightarrow C$, $\varphi_B : B \longrightarrow C$, there exists a unique homomorphism $\Phi : A*_V B \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[ddr, "\varphi_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, "\varphi_B"] \\
& A*_V B \arrow[d, "\Phi"] & \\
& C &
\end{tikzcd}$$

commutes;

(ii) the homomorphisms i_A and i_B are injective;

Proof. (i) Let $C \in V$ and let $\varphi_A : A \longrightarrow C$, $\varphi_B : B \longrightarrow C$ be homomorphisms. Clearly, the elements $z[\tau(\Sigma)\cup R\cup S]^{\#}$, $z \in X\cup Y$, generate $A*_V B$. Suppose that Φ exists. Let $x \in X$. Then $(x[\tau(\Sigma)\cup R\cup S]^{\#})\Phi = (x[\tau(\Sigma)\cup R]^{\#})i_A\Phi = (x[\tau(\Sigma)\cup R]^{\#})\varphi_A$. Similarly, $(y[\tau(\Sigma)\cup R\cup S]^{\#})\Phi = (y[\tau(\Sigma)\cup S]^{\#})\varphi_B$ for every $y \in Y$. Therefore Φ is uniquely determined on a set of generators of $A*_V B$. Hence, if Φ exists, then Φ is unique.

Let $\theta : (X\cup Y\cup X^{-1}\cup Y^{-1})^* \longrightarrow C$ be the homomorphism defined by

$$x\theta = (x[\tau(\Sigma)\cup R]^{\#})\varphi_A \quad \text{for } x \in X\cup X^{-1};$$

$$y\theta = (y[\tau(\Sigma)\cup S]^{\#})\varphi_B \quad \text{for } y \in Y\cup Y^{-1}.$$

By Lemma 2.4, we have $\tau(\Sigma) \subseteq Ker\theta$. Since $R\cup S \subseteq Ker\theta$ as well, we know, by Lemma 1.2, that there exists a homomorphism $\Phi : A*_V B \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
(X\cup Y\cup X^{-1}\cup Y^{-1})^* \arrow[r, "\theta"] \arrow[d, "\gamma"'] & C \\
A*_V B \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

commutes, where $w\gamma = w[\tau(\Sigma)\cup R\cup S]^{\#}$ for every $w \in (X\cup Y\cup X^{-1}\cup Y^{-1})^*$.

Let $x \in X$. Then $(x[\tau(\Sigma)\cup R]^{\#})(i_A\Phi) = (x[\tau(\Sigma)\cup R\cup S]^{\#})\Phi = (x\gamma)\Phi = x\theta = (x[\tau(\Sigma)\cup R]^{\#})\varphi_A$. Hence $i_A\Phi = \varphi_A$. Similarly, we obtain $i_B\Phi = \varphi_B$ and so Φ satisfies the required conditions.

(ii) Let $C = A$ and let $\varphi_A = 1_A$, φ_B trivial. Then, by (i), there exists $\Phi : A*_V B \longrightarrow A$ such that $i_A\Phi = 1_A$. Hence i_A is injective. Similarly, we prove that i_B is injective and so the lemma is proved.

It follows easily from the lemma that $A*_V B$ is, up to isomorphism, independent of the presentations of A and B. We say that $A*_V B$ is the *free product* in V of A and B.

The concept of algorithm has been used for a long time in mathematics; however there is no agreement yet for a basic, simple definition. For precise discussions on the subject, see [10,§6.4], [17,§V.2] or [23,§I.4]. We mention a rather intuitive approach. An *algorithm* is an explicit effective set of instructions for a computing procedure (not necessarily numerical) which may be used to find the answers to any of a given class of questions [9,§7.1].

A problem that consists of finding an algorithm for a certain class of questions is said to be a *decision* problem. If such an algorithm exists, the problem is said to be *decidable*. Otherwise, we say it is *undecidable*.

If there exists an algorithm for computing a certain structure explicitly, we say that it is *effectively* constructible (or that it can be effectively determined).

We shall be particularly interested in the following problems.

Let $V = Inv(\Sigma)$ be a variety of inverse monoids. Let Γ be a class of presentations in V and let C be a subclass of V. We define the *C-problem* for Γ as follows. Is there an algorithm which determines, for every $P \in \Gamma$, whether or not P defines an element of C?

Now let $V\langle X;R\rangle$ be a presentation. We define the *word problem* for $V\langle X;R\rangle$ as follows. Is there an algorithm which determines, for every $u,v \in (X \cup X^{-1})^*$, whether or not $u[\tau(\Sigma) \cup R]^{\#} = v[\tau(\Sigma) \cup R]^{\#}$?

Finally, the *idempotent word problem* for $V\langle X;R\rangle$. Is there an algorithm which determines, for every $e,f \in D_X$, whether or not $e[\tau(\Sigma) \cup R]^{\#} = f[\tau(\Sigma) \cup R]^{\#}$?

Note that, by Lallement's Lemma [11,§II.4], every idempotent of $(X \cup X^{-1})^*/[\tau(\Sigma) \cup R]^{\#}$ can be written in the form $e[\tau(\Sigma) \cup R]^{\#}$ for some $e \in D_X$.

3. Free inverse monoids

The problem of finding a convenient canonical form for $FIM(X)$ remained unsolved until the early seventies, when Scheiblich [38] and Munn [26] published independent works on the subject.

Let X be a nonempty set. We define

$$R_X = (X\cup X^{-1})^*\setminus[\bigcup_{x\in X\cup X^{-1}}(X\cup X^{-1})^*xx^{-1}(X\cup X^{-1})^*].$$

We say that R_X is the set of all *reduced* words of $(X\cup X^{-1})^*$. We define a map $\iota:(X\cup X^{-1})^* \longrightarrow R_X$ as follows: for every $w \in (X\cup X^{-1})^*$, $w\iota$ is the reduced word obtained from w by successively cancelling all factors of the form xx^{-1}, $x \in X\cup X^{-1}$. This operation is confluent, that is, the final result is independent of the order by which we perform the cancellations [16,§1.4]. Therefore ι is well-defined. Since $\pi = \{(xx^{-1},1)\colon x \in X\cup X^{-1}\}^{\#}$, we have

$$u\iota = v\iota \iff (u,v) \in \pi$$

for every $u,v \in (X\cup X^{-1})^*$.

It follows easily that R_X with the operation $(u,v) \mapsto (uv)\iota$ is isomorphic to $FG(X)$ [16,§1.4]. A reduced word is said to be *cyclically reduced* if its first and last letters are not mutually inverse.

For every $u \in (X\cup X^{-1})^*$, let

$$Q(u) = \{v\iota\colon v \leq_I u\}.$$

It follows easily that $Q(u)$ is *left closed* for every $u \in (X\cup X^{-1})^*$, that is,

$$a \in Q(u) \text{ and } a' \leq_I a \Rightarrow a' \in Q(u).$$

The following result also follows from the definition.

LEMMA 3.1. *For every* $u, v \in (X \cup X^{-1})^*$ *and* $e \in D_X$, *we have*

(*i*) $Q(uv) = Q(u) \cup [u.Q(v)]\iota$;

(*ii*) $Q(u^{-1}) = [u^{-1}Q(u)]\iota$;

(*iii*) $Q(uu^{-1}) = Q(u)$;

(*iv*) $Q(ueu^{-1}) = Q(u) \cup [u.Q(e)]\iota$.

For every $u \in (X \cup X^{-1})^*$, we define a birooted tree

$$MT(u) = (Q(u), \{1\}, \{u\iota\}, E(u) \cup [E(u)]^{-1})$$

as follows. The set $Q(u)$ is the set of vertices of $MT(u)$, 1 and $u\iota$ are the two roots, and $E(u) \cup [E(u)]^{-1}$ is the respective set of edges, with

$$E(u) = \{(w, x, w') \in Q(u) \times (X \cup X^{-1}) \times Q(u) : w' = wx\}.$$

and

$$[E(u)]^{-1} = \{(w', x^{-1}, w) : (w, x, w') \in E(u)\}.$$

By [26], $MT(u)$ is a well-defined birooted tree and is said to be the *Munn tree* of u. Now we have

THEOREM 3.2 [26]. *For every* $u, v \in (X \cup X^{-1})^*$, *the following conditions are equivalent.*

(*i*) $u\rho = v\rho$;

(*ii*) $MT(u) = MT(v)$;

(*iii*) $Q(u) = Q(v)$ *and* $u\iota = v\iota$.

For every $u \in (X \cup X^{-1})^*$, we define $|u\rho|$ to be $|Q(u)|$.

The next results follow easily.

LEMMA 3.3. *For every* $e \in D_X$,

$$e\rho = \prod_{v \in Q(e)} (vv^{-1})\rho.$$

LEMMA 3.4. *Let* $u,v \in (X \cup X^{-1})^*$. *Then*

(i) $u\rho \leq v\rho \iff Q(u) \supseteq Q(v)$ *and* $u\iota = v\iota$;

(ii) $u\rho \prec v\rho \iff Q(u) \supset Q(v)$, $|Q(u)| = |Q(v)|+1$ *and* $u\iota = v\iota$.

LEMMA 3.5. *For every* $u \in (X \cup X^{-1})^*$,

$$u\rho \in E[FIM(X)] \iff u \in D_X.$$

LEMMA 3.6. *Let X be a nonempty set. Then FIM(X) is E-unitary.*

Proof. Let σ denote the least group congruence on $FIM(X)$. Let $u \in (X \cup X^{-1})^*$ and suppose that $(u\rho, 1\rho) \in \sigma$. Then $(u,1) \in \pi$ and so $u \in D_X$. By Lemma 3.5, $u\rho \in E[FIM(X)]$ and so $(1\rho)\sigma \subseteq E[FIM(X)]$. Thus $FIM(X)$ is E-unitary.

The Green's relations on $FIM(X)$ are easy to describe.

LEMMA 3.7 [26],[38]. *Let X be a nonempty set and let* $u,v \in (X \cup X^{-1})^*$. *Then*

(i) $(u\rho, v\rho) \in \mathcal{R} \iff Q(u) = Q(v)$;

(ii) $(u\rho, v\rho) \in \mathcal{L} \iff Q(u^{-1}) = Q(v^{-1})$;

(iii) $(u\rho, v\rho) \in \mathcal{H} \iff u\rho = v\rho$;

(iv) $(u\rho, v\rho) \in \mathcal{D} \iff \exists w \in (X \cup X^{-1})^*: Q(v) = [w.Q(u)]\iota$;

(v) $\mathcal{J} = \mathcal{D}$.

We also have

LEMMA 3.8 [26]. *Let X and Y be nonempty sets. Then*

$$FIM(X) \simeq FIM(Y) \iff |X| = |Y|.$$

Therefore we can define the *rank* of $FIM(X)$ to be $|X|$.

The above results will be used frequently in this work and so we shall omit further reference.

Now let M be an inverse monoid and let τ be a congruence on M. We say that τ is *idempotent-pure* if, for every $a \in M$ and $e \in E(M)$,

$$(a,e) \in \tau \Rightarrow a \in E(M).$$

We say that τ is *idempotent-separating* if, for every $e,f \in E(M)$,

$$(e,f) \in \tau \Rightarrow e = f.$$

We say that a homomorphism φ of inverse monoids is *idempotent-pure* (respectively *idempotent-separating*) if $Ker\varphi$ is idempotent-pure (respectively idempotent-separating).

An inverse monoid M is said to be *quasi-free* if $M \simeq [FIM(X)]/\nu$ for some nonempty set X and some idempotent-pure congruence ν.

Let R be a relation on $(X\cup X^{-1})^*$. We define a relation R_D on $(X\cup X^{-1})^*$ by

$$R_D = \{(aea^{-1},beb^{-1}): (a,b) \in R \text{ and } e \in D_X\}.$$

The following result was proved by Munn and Reilly [28] and we give a new proof using presentations.

LEMMA 3.9. *Let* $Inv\langle X;R\rangle$ *be a presentation. Then* $\rho \subseteq (\rho\cup R_D)^\# \subseteq (\rho\cup R)^\#$ *and the canonical diagram*

$$\begin{tikzcd}
(X\cup X^{-1})^*/\rho \arrow[r, "\alpha"] \arrow[dr] & (X\cup X^{-1})^*/(\rho\cup R_D)^\# \arrow[d, "\beta"] \\
& (X\cup X^{-1})^*/(\rho\cup R)^\#
\end{tikzcd}$$

commutes. Moreover, α *is idempotent-pure,* β *is idempotent-separating and* $(X\cup X^{-1})^*/(\rho\cup R_D)^\#$ *is quasi-free.*

Proof. It is immediate that $\rho \subseteq (\rho\cup R_D)^\# \subseteq (\rho\cup R)^\#$ and that the diagram commutes. Now we prove that α is idempotent-pure.

Let $a \in (X\cup X^{-1})^*$ and let $e \in D_X$. Suppose that $a(\rho\cup R_D)^\# = e(\rho\cup R_D)^\#$. Then there exist $w_0,\ldots w_n \in (X\cup X^{-1})^*$ such that

$$w_0 = a,$$

$$w_n = e,$$

$$\forall i \in \{1,\ldots,n\}\ \exists c_i,d_i \in (X\cup X^{-1})^*\ \exists (u_i,v_i) \in \rho\cup R_D:$$

$$(w_{i-1},w_i) = (c_iu_id_i,c_iv_id_i).$$

For every $i \in \{1,\ldots,n\}$, we have $u_i\pi = v_i\pi$ and so $w_{i-1}\pi = w_i\pi$. Hence $a\pi = w_0\pi = w_n\pi = e\pi = 1\pi$ and so $a \in D_X$. Thus, by (4.1), $a\rho \in E[FIM(X)]$ and so α is idempotent-pure. By Lemma 1.2, we have $(X\cup X^{-1})^*/(\rho\cup R_D)^\# \simeq [(X\cup X^{-1})^*/\rho]/[(\rho\cup R_D)^\#/\rho] = [FIM(X)]/Ker\alpha$ and so $(X\cup X^{-1})^*/(\rho\cup R_D)^\#$ is quasi-free.

Now we prove that β is idempotent-separating. Suppose that $e,f \in D_X$ and $e(\rho\cup R)^\# = f(\rho\cup R)^\#$. Then there exist $z_0,\ldots,z_m \in (X\cup X^{-1})^*$ such that

$$z_0 = e,$$

$$z_m = f,$$

$$\forall j \in \{1,\ldots,m\}\ \exists g_j,h_j \in (X\cup X^{-1})^*\ \exists (a_j,b_j) \in \rho\cup R:$$

$$(z_{j-1},z_j) = (g_ja_jh_j,g_jb_jh_j).$$

Then we have

$$z_0z_0^{-1} = ee^{-1},$$

$$z_mz_m^{-1} = ff^{-1},$$

$$\forall j \in \{1,\ldots,m\},$$

$$(z_{j-1}z_{j-1}^{-1},z_jz_j^{-1}) = (g_ja_jh_jh_j^{-1}a_j^{-1}g_j^{-1},g_jb_jh_jh_j^{-1}b_j^{-1}g_j^{-1}).$$

Since $(a_jh_jh_j^{-1}a_j^{-1},b_jh_jh_j^{-1}b_j^{-1}) \in \rho\cup R_D$ for every $j \in \{1,\ldots,m\}$, we have $e(\rho\cup R_D)^\# = (ee^{-1})(\rho\cup R_D)^\# = (z_0z_0^{-1})(\rho\cup R_D)^\# = (z_mz_m^{-1})(\rho\cup R_D)^\# = (ff^{-1})(\rho\cup R_D)^\# = f(\rho\cup R_D)^\#$ and so β is idempotent-separating.

CHAPTER II

INTERSECTIONS OF FREE INVERSE SUBMONOIDS OF *FIM(X)*

1. Preliminaries

Let M be a monoid and let A be a nonempty subset of M. Let $A^{(*)}$ be the submonoid of M defined by

$$A^{(*)} = \{1\}\cup\{a_1\ldots a_n\colon a_i \in A\}.$$

We say that $A^{(*)}$ is the submonoid of M *generated* by A.

Let N be an inverse monoid and let B be a nonempty subset of N. We say that $(B\cup B^{-1})^{(*)}$ is the inverse submonoid of N generated by B and we denote it by $\langle B\rangle$. We remark that this notation is not standard.

A monoid (inverse monoid) that is generated by a finite subset is said to be *finitely generated*. A monoid (inverse monoid) that is generated by a single element is said to be *monogenic*.

Let X be a nonempty set and let A be a submonoid of X^*. We say that A is a *free* submonoid of X^* if $A \simeq Y^*$ for some nonempty set Y. Similarly, we define a free inverse submonoid of $FIM(X)$ and a free subgroup of $FG(X)$.

In this chapter, we discuss the following problems. Let X be a nonempty set and let A,B be two free inverse submonoids of $FIM(X)$.

(1) Is $A\cap B$ free?

(2) If A and B are both finitely generated, is $A\cap B$ finitely

generated?

It is interesting to observe how the corresponding problems are answered in the context of groups and monoids. Throughout this chapter, we assume that the trivial group $\{1\}$ is the free group (free monoid) on the empty set. It is well-known that every subgroup of a free group is free [16,§2.3] and so we have

LEMMA 1.1. *Let X be a set and let A,B be free subgroups of $FG(X)$. Then $A \cap B$ is free.*

Tilson proved an identical result for monoids.

LEMMA 1.2 [40]. *Let X be a set and let A,B be free submonoids of X^*. Then $A \cap B$ is free.*

However, the second problem reveals different behaviours.

LEMMA 1.3 [13,§I.3]. *Let X be a set and let A,B be free finitely generated subgroups of $FG(X)$. Then $A \cap B$ is finitely generated.*

LEMMA 1.4. *Let $X = \{x,y\}$ and let $A = \{x,xy\}^{(*)}$, $B = \{x,yx\}^{(*)}$ be submonoids of X^*. Then A and B are free but $A \cap B$ is not finitely generated.*

Proof. Let $Y = \{z,t\}$ and let $\varphi : Y^* \longrightarrow A$ be the homomorphism defined by $z\varphi = x$ and $t\varphi = xy$. Obviously, φ is surjective. Suppose that $u_1,\ldots,u_n,v_1,\ldots,v_m \in Y$ and $(u_1\ldots u_n)\varphi = (v_1\ldots v_m)\varphi$. Then $u_1\varphi\ldots u_n\varphi = v_1\varphi\ldots v_m\varphi$ and it follows easily that $n = m$ and $u_i\varphi = v_i\varphi$ for every $i \in \{1,\ldots,n\}$. Hence $u_i = v_i$ for every i and so φ is injective. Thus A is free. Similarly, we prove that B is free.

Let $C = \{(xy)^n x \colon n \in \mathbb{N}\}$. Obviously, $C \subseteq A \cap B$. Since every element in $(A \cap B) \backslash \{1\}$ must have x as both first and last letter, no element of C can be written as the product of two elements of $(A \cap B) \backslash \{1\}$. Therefore every generating set of $A \cap B$ must contain C and so $A \cap B$ is not finitely generated.

It is well-known [16,§2.2] that, for all nonempty sets X and Y,

$$FG(X) \simeq FG(Y) \iff |X| = |Y|.$$

We define $rank(FG(X))$ to be $|X|$. Now we have

LEMMA 1.5. *Let $u,v \in (X \cup X^{-1})^*$. Then $\{u\pi, v\pi\}$ is a basis in $FG(X)$ if and only if there exist no $w,z \in R_X$ and $r,s \in \mathbb{Z}$ such that $u\iota = wz^r w^{-1}$ and $v\iota = wz^s w^{-1}$.*

Proof. Suppose that there exist $w,z \in R_X$ and $r,s \in \mathbb{Z}$ such that $u\iota = wz^r w^{-1}$ and $v\iota = wz^s w^{-1}$. Then $u\pi = (wz^r w^{-1})\pi$ and $v\pi = (wz^s w^{-1})\pi$.

If $r = 0$ or $s = 0$, then $1 \in \{u\pi, v\pi\}$ and so $\{u\pi, v\pi\}$ is not a basis.

If $r \neq 0$ and $s \neq 0$, then the nontrivial identity $(u\pi)^s = (v\pi)^r$ holds and it follows that $\{u\pi, v\pi\}$ is not a basis either.

Conversely, suppose that $\{u\pi, v\pi\}$ is not a basis. Let $G = \langle u\pi, v\pi \rangle$. Since G is a subgroup of $FG(X)$, G is free. If G is trivial, we take $w = z = 1$, so we assume that G is nontrivial. Since G is generated by a two-element set, we have $rank(G) \leq 2$, and since every two-element set generating a free group of rank 2 is a basis [13,§I.2], we have $rank(G) = 1$. Therefore $G = \langle g \rangle$ for some $g \in FG(X) \backslash \{1\}$. Thus there exist $r,s \in \mathbb{Z}$ such that $u\pi = g^r$ and $v\pi = g^s$. Let $g' \in R_X$ be such that $g'\pi = g$. We can write $g' = wzw^{-1}$ for some $w,z \in R_X$ with z cyclically reduced. Hence $u\iota = (g'^r)\iota = [(wzw^{-1})^r]\iota = wz^r w^{-1}$. Similarly, we obtain $v\iota = wz^s w^{-1}$ and so the lemma is proved.

Now we turn our attention to free inverse monoids, particularly the free inverse monoid of rank 1.

Reilly provided a criterion for determining whether a subset of a free inverse monoid is a basis or not. We shall use it in the following modified version.

LEMMA 1.6 [37]. *Let X be a nonempty set and let K be a nonempty subset of $(X\cup X^{-1})^*$. Then $K\rho$ is a basis in $FIM(X)$ if and only if:*

(i) $(K\rho)\cap(K^{-1}\rho) = \emptyset$;

(ii) for every $u \in K\cup K^{-1}$ there exists $c \in Q(u)$ such that: if $c \in Q(w_1\ldots w_n)$, with $w_i \in K\cup K^{-1}$ for $i \in \{1,\ldots,,n\}$ and $w_{j+1}\rho \neq w_j^{-1}\rho$ for $j \in \{1,\ldots,n-1\}$, then $u\rho = w_1\rho$.

The next result is an easy consequence of the previous lemma.

LEMMA 1.7 [37]. *Let X be a nonempty set and let $u \in (X\cup X^{-1})^*$. Then $\{u\rho\}$ is a basis in $FIM(X)$ if and only if $u \notin D_X$.*

The free inverse monoid of rank 1 was studied by Gluskin in 1957, who produced its first normal form [8]. The normal form that we shall use throughout this chapter follows naturally from the general normal form for a free inverse monoid considered in Theorem I.3.2. By convention, an expression of the form v^0 always denotes the unity 1. We define $X_1 = \{x\}$.

LEMMA 1.8 [34,§IX.1]. *Let $u \in FIM(X_1)$. Then there exist unique $a,b \in \mathbb{N}^0$ and $p \in \mathbb{Z}$ such that $-a \leq p \leq b$ and $u = (x^{-a}x^{a+b}x^{-b}x^p)\rho$. Moreover, $p = 0$ if and only if $u \in E[FIM(X_1)]$.*

For such a,b and p, we denote $x^{-a}x^{a+b}x^{-b}x^p$ by $x(-a,p,b)$. In particular, we have $x(0,0,0) = 1$. It is easy to verify [34,§IX.1] that

$$[x(-a,p,b)]\rho.[x(-c,q,d)]\rho = [x(-\max\{a,c-p\},p+q,\max\{b,d+p\})]\rho,$$
$$([x(-a,p,b)]\rho)^{-1} = [x(-a-p,-p,b-p)]\rho. \tag{1.1}$$

We shall be interested in some particular inverse submonoids of $FIM(X_1)$, which we now define. For every $k \in \mathbb{N}$, let

$$I_k = \{1\} \cup \{[x(-a,p,b)]\rho \in FIM(X_1): a+b \geq k\}.$$

Note, in particular, that $I_1 = FIM(X_1)$.

LEMMA 1.9. *Let $k \in \mathbb{N}$. Then*

(i) I_k is an inverse submonoid of $FIM(X_1)$,

(ii) $I_k = \langle h_1\rho,\ldots,h_k\rho\rangle$, where $h_i = x(0,i,k)$ for all $i \in \{1,\ldots,k\}$.

Proof. (i) Let $[x(-a,p,b)]\rho, [x(-c,q,d)]\rho \in I_k$. We can assume that $[x(-a,p,b)]\rho \neq 1$, so $a+b \geq k$. Then $[x(-a,p,b)]\rho.[x(-c,q,d)]\rho = [x(-\max\{a,c-p\},p+q,\max\{b,d+p\})]\rho$ and $\max\{a,c-p\}+\max\{b,d+p\} \geq a+b \geq k$. Hence I_k is a submonoid of $FIM(X_1)$. Since $([x(-a,p,b)]\rho)^{-1} = [x(-a-p,-p,b-p)]\rho$ and $(a+p)+(b-p) = a+b \geq k$, it follows that I_k is inverse.

(ii) It is trivial that $\langle h_1\rho,\ldots,h_k\rho\rangle \subseteq I_k$. Now let $[x(-a,p,b)]\rho \in I_k\backslash\{1\}$. Then it is easy to see that $[x(-a,p,b)]\rho = [x(-a,-a,b)]\rho.[x(0,a+p,a+b)]\rho = ([x(0,a,a+b)]\rho)^{-1}.[x(0,a+p,a+b)]\rho$. Therefore all we need is to show that for every $n \geq k$ and every $m \in \{0,\ldots,n\}$ we have $[x(0,m,n)]\rho \in \langle h_1\rho,\ldots,h_k\rho\rangle$.

Since $[x(0,0,k)]\rho = (h_1h_1^{-1})\rho$, this is true for $n = k$. Suppose that it is true for some $n \geq k$ and let $m \in \{0,\ldots,n+1\}$.

If $m = 0$, we have $[x(0,m,n+1)]\rho$
$= [x(0,1,n)]\rho.[x(0,0,n)]\rho.[x(-1,-1,n-1)]\rho$
$= [x(0,1,n)]\rho.[x(0,0,n)]\rho.([x(0,1,n)]\rho)^{-1}.$

If $m > 0$, we have $[x(0,m,n+1)]\rho = [x(0,1,n)]\rho.[x(0,m-1,n)]\rho$.

In any case, we obtain $[x(0,m,n+1)]\rho \in \langle h_1\rho,\dots,h_k\rho\rangle$ and so the lemma is proved.

Our next lemma is about a certain type of endomorphism of $FIM(X_1)$.

LEMMA 1.10. *Let* $a_0,b_0 \in \mathbb{N}^0$ *and let* $\lambda \in \mathbb{N}$. *Let* $\Phi:FIM(X_1) \longrightarrow FIM(X_1)$ *be the map defined by* $([x(-a,p,b)]\rho)\Phi = [x(-a_0-\lambda a,\lambda p,b_0+\lambda b)]\rho$. *Then* Φ *is an injective homomorphism.*

Proof. The injectivity is obvious. Now let $[x(-a,p,b)]\rho,[x(-c,q,d)]\rho \in FIM(X_1)$. We have $([x(-a,p,b)]\rho.[x(-c,q,d)]\rho)\Phi = ([x(-max\{a,c-p\},p+q,max\{b,p+d\})]\rho)\Phi$
$= [x(-a_0-\lambda max\{a,c-p\},\lambda(p+q),b_0+\lambda max\{b,p+d\})]\rho$
$= [x(-max\{a_0+\lambda a,a_0+\lambda c-\lambda p\},\lambda p+\lambda q,max\{b_0+\lambda b,b_0+\lambda p+\lambda d\})]\rho$
$= [x(-a_0-\lambda a,\lambda p,b_0+\lambda b)]\rho.[x(-a_0-\lambda c,\lambda q,b_0+\lambda d)]\rho$
$= ([x(-a,p,b)]\rho)\Phi.([x(-c,q,d)]\rho)\Phi$. Hence Φ is a homomorphism and the lemma is proved.

Finally, we need some facts about diophantine equations. For every $a \in \mathbb{Z}\backslash\{0\}$, $b \in \mathbb{Z}$, we denote by $a|b$ the relation "*a divides b*". For every $a,b \in \mathbb{N}$, we denote by (a,b) the *greatest common divisor* of a and b.

LEMMA 1.11 [24,§5.1]. *Let* $a,b \in \mathbb{N}$, *let* $c \in \mathbb{Z}$ *and let* γ *denote the diophantine equation* $ax-by = c$ *in the integer variables* x,y. *Then*

(*i*) γ *has solutions if and only if* $(a,b)|c$;

(*ii*) *if* (x_0,y_0) *is a solution of* γ, *then the set of solutions of* γ *is* $\{(x_0+kb(a,b)^{-1},y_0+ka(a,b)^{-1}): k \in \mathbb{Z}\}$.

2. Classification by isomorphism classes

In this section, we show that intersections of monogenic free inverse submonoids of a free inverse monoid must necessarily be isomorphic to some particular inverse monoids and we produce an algorithm to determine the respective isomorphism class.

We need some preliminary results, the next lemma following from Lemmas 1.7 and 1.8.

LEMMA 2.1. *Let* $u \in (X \cup X^{-1})^* \backslash D_X$ *and let* $v \in (X \cup X^{-1})^*$ *be such that* $v\rho \in \langle u\rho \rangle$. *Then there exist unique* $a,b \in \mathbb{N}^0$ *and* $p \in \mathbb{Z}$ *such that* $-a \leqslant p \leqslant b$ *and* $v\rho = (u^{-a}u^{a+b}u^{-b}u^{p})\rho$. *Moreover,* $p = 0$ *if and only if* $v \in D_X$.

For all such u,a,b and p, we will denote $u^{-a}u^{a+b}u^{-b}u^{p}$ by $u(-a,p,b)$. It follows from (1.1) and Lemma 1.7 that

$$[u(-a,p,b)]\rho.[u(-a',p',b')]\rho$$
$$= [u(-\max\{a,a'-p\},p+p',\max\{b,b'+p\})]\rho,$$
$$([u(-a,p,b)]\rho)^{-1} = [u(-a-p,-p,b-p)]\rho.$$

By convention, we assume that an expression of the form $\bigcup_{i=0}^{-1} A_i$ denotes the empty set.

Now we have

LEMMA 2.2. *Let* $u \in (X \cup X^{-1})^* \backslash D_X$ *and let* $v \in (X \cup X^{-1})^*$. *Then* $v\rho \in \langle u\rho \rangle$ *if and only if there exist* $a,b \in \mathbb{N}^0$ *and* $p \in \mathbb{Z}$ *such that:*

(i) $-a \leqslant p \leqslant b$;

(ii) $Q(v) = \{1\} \cup (\bigcup_{i=-a}^{b-1} [u^{i}.Q(u)]\iota)$;

(iii) $v\iota = u^{p}\iota$.

Moreover, such a,b *and* p *are unique.*

Proof. First we prove that, for every $n \in \mathbb{N}^0$,

$$Q(u^n) = \{1\} \cup \left(\bigcup_{i=0}^{n-1} [u^i Q(u)]\iota\right).$$

We use induction on n. By our previous convention, the case $n = 0$ is trivial. Now suppose that it holds for $n = k$. Then $Q(u^{k+1}) = Q(u^k.u)$ $= Q(u^k) \cup [u^k.Q(u)]\iota = \{1\} \cup \left(\bigcup_{i=0}^{k-1} [u^i.Q(u)]\iota\right) \cup [u^k.Q(u)]\iota$ $= \{1\} \cup \left(\bigcup_{i=0}^{k} [u^i.Q(u)]\iota\right)$, which proves our assertion. Since $Q(u^{-1}) = [u^{-1}.Q(u)]\iota$, we have that, for every $n \in \mathbb{N}$, $Q(u^{-n}) = Q((u^{-1})^n) = \{1\} \cup \left(\bigcup_{i=0}^{n-1} [u^{-i}.Q(u^{-1})]\iota\right) = \{1\} \cup \left(\bigcup_{i=0}^{n-1} [u^{-i-1}.Q(u)]\iota\right)$ $= \{1\} \cup \left(\bigcup_{i=-n}^{-1} [u^i.Q(u)]\iota\right)$.

By Lemma 2.1, we have $v\rho \in \langle u\rho \rangle$ if and only if $v\rho = [u(-a,p,b)]\rho$ for some $a,b \in \mathbb{N}^0$ and $p \in \mathbb{Z}$ such that $-a \leqslant p \leqslant b$. Moreover, such a,b and p are unique. Also, we have $v\rho = [u(-a,p,b)]\rho$ if and only if $Q(v) = Q(u(-a,p,b))$ and $v\iota = [u(-a,p,b)]\iota$. But $Q(u(-a,p,b))$ $= Q(u^{-a}u^a u^b u^{-b} u^p) = Q(u^{-a}u^a) \cup Q(u^b u^{-b}) \cup Q(u^p) = Q(u^{-a}) \cup Q(u^b)$ $= \{1\} \cup \left(\bigcup_{i=-a}^{b-1} [u^i.Q(u)]\iota\right)$ and $[u(-a,p,b)]\iota = u^p\iota$, so the lemma is proved.

Note that, for every $u \in (X \cup X^{-1})^* \backslash D_X$ and $a,b \in \mathbb{N}^0$, we have either $a = b = 0$ or $Q(u) \subseteq \bigcup_{i=-a}^{b-1} [u^i.Q(u)]\iota$ or $Q(u^{-1}) \subseteq \bigcup_{i=-a}^{b-1} [u^i.Q(u)]\iota$. Since $u \neq 1$, we have $|Q(u)| > 1$ and it follows from Lemma 2.2 that

LEMMA 2.3. *Let $u,v \in (X \cup X^{-1})^* \backslash D_X$. Let $a,b,c,d \in \mathbb{N}^0$ and let $p,q \in \mathbb{Z}$ be such that $-a \leqslant p \leqslant b$ and $-c \leqslant q \leqslant d$. Then*

(*i*) $Q(u(-a,p,b)) = Q(v(-c,q,d))$ *if and only if* $\bigcup_{i=-a}^{b-1} [u^i.Q(u)]\iota = \bigcup_{j=-c}^{d-1} [v^j.Q(v)]\iota$.

(*ii*) $[u(-a,p,b)]\iota = [v(-c,q,d)]\iota$ *if and only if* $u^p\iota = v^q\iota$.

The following result is a mere computation.

LEMMA 2.4. *Let* $u \in (X \cup X^{-1})^* \backslash D_X$ *and let* $v = u(-a,p,b)$ *for some* $a,b \in \mathbb{N}^0$ *and* $p \in \mathbb{N}$ *such that* $-a \leqslant p \leqslant b$. *Let* $w = v(-c,q,d)$ *for some* $c,d \in \mathbb{N}^0$ *and* $q \in \mathbb{Z}$ *such that* $-c \leqslant q \leqslant d$ *and* c *or* d *is nonzero. Then*

$$w\rho = [u(-pc-a,pq,pd-p+b)]\rho.$$

Proof. Since $p \in \mathbb{N}$, we have $v \notin D_X$ and so $\langle v\rho\rangle$ is free by Lemma 1.7. Let $c' = pc+a$ and $d' = pd-p+b$. By Lemma 2.3, we only need to prove that

$\bigcup_{i=-c}^{d-1}[v^i.Q(v)]\iota = \bigcup_{j=-c'}^{d'-1}[u^j.Q(u)]\iota$ and $v^q\iota = u^{pq}\iota$. The second equality is obvious. Since $p \in \mathbb{N}$, we have $-a \leqslant b-1$ and so we obtain

$$\bigcup_{i=-c}^{d-1}[v^i.Q(v)]\iota = \bigcup_{i=-c}^{d-1}[u^{pi}.(\bigcup_{k=-a}^{b-1}[u^k.Q(u)])]\iota$$

$$= \bigcup_{i=-c}^{d-1}(\bigcup_{k=-a}^{b-1}[u^{pi+k}.Q(u)])\iota.$$ We must show that

$$\{pi+k\colon -c \leqslant i \leqslant d-1 \text{ and } -a \leqslant k \leqslant b-1\} = \{j\colon -pc-a \leqslant j \leqslant pd-p+b-1\}.$$

Suppose that $-c \leqslant i \leqslant d-1$ and $-a \leqslant k \leqslant b-1$. Since $p > 0$, we have $-pc \leqslant pi \leqslant pd-p$ and so $-pc-a \leqslant pi+k \leqslant pd-p+b-1$.

Conversely, suppose that $-pc-a \leqslant j \leqslant pd-p+b-1$. Suppose first that $j < -pc$. Since $-c \leqslant d-1$, we can take $i = -c$ and $k = j+pc$ to satisfy the required conditions. The case $j > pd-p$ is dealt similarly, with $i = d-1$ and $k = j-pd+p$. Finally, suppose that $-pc \leqslant j \leqslant pd-p$. There exist $i,k \in \mathbb{Z}$ such that $j = pi+k$ and $0 \leqslant k < p$. Since $-pc \leqslant pi+k$, we have $p(c+i) \geqslant -k$. Since $k < p$, this yields $p(c+i) \geqslant 0$. Hence $i \geqslant -c$. Similarly, $pi+k \leqslant pd-p$ yields $i \leqslant d-1$. Since $-a \leqslant 0 \leqslant k \leqslant p-1 \leqslant b-1$, the lemma follows.

Now we fix $u,v \in (X \cup X^{-1})^* \backslash D_X$. The discussion of the intersection $\langle u\rho\rangle \cap \langle v\rho\rangle$ will require a split into two main cases.

Case A. $\{u\pi, v\pi\}$ is not a basis.

By Lemma 1.5, there exist $w, z \in R_X$, with z cyclically reduced, and $r, s \in \mathbb{Z}$ such that $u\iota = wz^r w^{-1}$ and $v\iota = wz^s w^{-1}$. Since $u\iota, v\iota \neq 1$, we have $r, s \neq 0$. Since $\langle y\rho\rangle = \langle y^{-1}\rho\rangle$ for every $y \in (X \cup X^{-1})^*$, we can assume that $r, s \in \mathbb{N}$.

We consider the following as an equation in the nonnegative integer variables (a,b,c,d).

$$\bigcup_{i=-a}^{b-1} [z^{ri} w^{-1}.Q(u)]\iota = \bigcup_{j=-c}^{d-1} [z^{sj} w^{-1}.Q(v)]\iota. \tag{2.1}$$

By Lemma 2.3, (a,b,c,d) is a solution of (2.1) if and only if $[u(-a,0,b)]\rho = [v(-c,0,d)]\rho$. It is clear that the trivial solution $(0,0,0,0)$ always satisfies (2.1), since both sides of the equation become the empty set. Moreover, a and b are both zero if and only if c and d are both zero. If the equation (2.1) has no nontrivial solutions, it follows from Lemma 2.3 that $\langle u\rho\rangle \cap \langle v\rho\rangle = \{1\}$.

Now assume that (2.1) has nontrivial solutions. The following lemma shows how such nontrivial solutions must relate one to each other.

LEMMA 2.5. *Let* (a_1, b_1, c_1, d_1) *and* (a_2, b_2, c_2, d_2) *be nontrivial solutions of* (2.1). *Then there exist* $\lambda, \mu \in \mathbb{Z}$ *such that* $(a_2, b_2, c_2, d_2) = (a_1 + \lambda s(r,s)^{-1}, b_1 + \mu s(r,s)^{-1}, c_1 + \lambda r(r,s)^{-1}, d_1 + \mu r(r,s)^{-1})$.

Proof. For every $t \in (X \cup X^{-1})^*$, let $K_t = \{k \in \mathbb{Z}\colon wz^k \in Q(t)\}$. Whenever $K_t \neq \emptyset$, we define $M_t = max(K_t)$ and $m_t = min(K_t)$.

Suppose that (a,b,c,d) is a nontrivial solution of (2.1). Thus a and b are not both zero. By Lemma 2.3, we have $[u(-a,0,b)]\rho = [v(-c,0,d)]\rho$. Let $e = u(-a,0,b)$. Then $Q(e) = \bigcup_{i=-a}^{b-1} [wz^{ri} w^{-1}.Q(u)]\iota$. Since $w \in Q(u)$, we certainly have $K_e \neq \emptyset$.

It is immediate that $wz^{r(b-1)+M_u} \in Q(e)$ and so

$$r(b-1)+M_u \leq M_e. \tag{2.2}$$

Suppose that $wz^{M_e} = (wz^{ri}w^{-1}t)\iota$ for some $i \in \{-a,\ldots,b-1\}$ and $t \in Q(u)$. We have $[wz^{r(b-1)}w^{-1}t]\iota = [wz^{r(b-1-i)}w^{-1}wz^{ri}w^{-1}t]\iota = wz^{r(b-1-i)+M_e}$. By maximality of M_e, we must have $i = b-1$. Hence $wz^{M_e} = [wz^{r(b-1)}w^{-1}t]\iota$ and so we have $z^{M_e-r(b-1)} = (w^{-1}t)\iota$. By (2.2), and since $M_u \geq r > 0$, we have $M_e > r(b-1)$ and so the first letter of $z^{M_e-r(b-1)}$ is the first letter of z. Since $wz \in R_X$, the first letter of w^{-1} is not the first letter of z and so we have $t = wt'$ for some $t' \in R_X$. Moreover, $t' = z^{M_e-r(b-1)}$ and so $M_e-r(b-1) \leq M_u$. Therefore, by (2.2), $M_e = r(b-1)+M_u$. Similarly, $M_e = s(d-1)+M_v$ and $m_e = -ra+m_u = -sc+m_v$.

Hence (a,c) is a solution of the diophantine equation

$$rx-sy = m_u-m_v \tag{2.3}$$

and (b,d) is a solution of the diophantine equation

$$rx-sy = M_v-M_u+r-s. \tag{2.4}$$

Since (a_1,c_1) and (a_2,c_2) are both solutions of (2.3), we know by Lemma 1.11(ii) that there exists some $\lambda \in Z$ such that $a_2 = a_1+\lambda s(r,s)^{-1}$ and $c_2 = c_1+\lambda r(r,s)^{-1}$.

Similarly, since (b_1,d_1) and (b_2,d_2) are both solutions of (2.4), we have $b_2 = b_1+\mu s(r,s)^{-1}$ and $d_2 = d_1+\mu r(r,s)^{-1}$ for some $\mu \in Z$ and so the lemma is proved.

Now let a_0 be the minimum $a \in N^0$ such that there is a nontrivial solution (a,b,c,d) of (2.1) and let b_0 be the minimum $b \in N^0$ such that there is a nontrivial solution (a_0,b,c,d) of (2.1). Thus a_0 and b_0 are not both zero. By Lemma 2.5, there exist unique $c_0,d_0 \in N^0$ such that (a_0,b_0,c_0,d_0) is a nontrivial solution of (2.1). We define successively

$$\Gamma = \{k \in \mathbb{N}\colon (a_0, b_0+ks(r,s)^{-1}, c_0, d_0+kr(r,s)^{-1}) \textit{ is a solution of } (2.1)\};$$

$$\Lambda = \begin{cases} \Gamma\cup\{0\} & \textit{if } s(r,s)^{-1} \le b_0 \textit{ and } r(r,s)^{-1} \le d_0, \\ \Gamma & \textit{otherwise.} \end{cases}$$

$$\eta = \begin{cases} \min(\Lambda) & \textit{if } \Lambda \neq \emptyset, \\ 0 & \textit{if } \Lambda = \emptyset. \end{cases}$$

$$\omega = \max\{l \in \mathbb{N}^0\colon (l-\eta)s(r,s)^{-1} \le b_0 \textit{ and } (l-\eta)r(r,s)^{-1} \le d_0\}.$$

We show that

$$\omega = 0 \iff \Lambda = \emptyset. \tag{2.5}$$

Suppose that $\omega = 0$. Since $\omega \geq \eta \geq 0$, it follows that $\eta = 0$. Suppose that $0 \in \Lambda$. Then $s(r,s)^{-1} \le b_0$ and $r(r,s)^{-1} \le d_0$. Hence $\omega \geq 1$, a contradiction. Therefore $0 \notin \Lambda$ and so, by definition of η, we must have $\Lambda = \emptyset$.

Conversely, suppose that $\Lambda = \emptyset$. Then $\eta = 0$. Moreover, since $0 \notin \Lambda$, we have $s(r,s)^{-1} > b_0$ or $r(r,s)^{-1} > d_0$. Hence $\omega = 0$ and (2.5) holds.

Finally, we define $g_0 = u(-a_0,0,b_0)$, and if $\omega > 0$, let $g_k = u(-a_0,ks(r,s)^{-1},b_0+\eta s(r,s)^{-1})$ for every $k \in \{1,\ldots,\omega\}$.

LEMMA 2.6. $\langle u\rho\rangle\cap\langle v\rho\rangle = \langle g_0\rho,\ldots,g_\omega\rho\rangle$.

Proof. Since (a_0,b_0,c_0,d_0) is a solution of (2.1), it is clear that $g_0\rho = [v(-c_0,0,d_0)]\rho$ and so $g_0\rho \in \langle u\rho\rangle\cap\langle v\rho\rangle$. Suppose that $\omega > 0$. By (2.5), $\Lambda \neq \emptyset$ and so $(a_0,b_0+\eta s(r,s)^{-1},c_0,d_0+\eta r(r,s)^{-1})$ is a solution of (2.1). Hence $g_k\rho = [v(-c_0,kr(r,s)^{-1},d_0+\eta r(r,s)^{-1})]\rho$ and so $g_k\rho \in \langle u\rho\rangle\cap\langle v\rho\rangle$ for every k. Therefore $\langle g_0\rho,\ldots,g_\omega\rho\rangle \subseteq \langle u\rho\rangle\cap\langle v\rho\rangle$.

Conversely, let $y \in \langle u\rho\rangle\cap\langle v\rho\rangle$. We can assume that $y \neq 1$. By Lemma 2.3, we have $y = [u(-a,p,b)]\rho = [v(-c,q,d)]\rho$ for some nontrivial solution (a,b,c,d) of (2.1) and some $p,q \in \mathbb{Z}$ such that $-a \le p \le b$, $-c \le q \le d$ and $u^{p}\iota = v^{q}\iota$. By Lemma 2.5, there exist $\lambda,\mu \in \mathbb{Z}$ such that

$(a,b,c,d) = (a_0+\lambda s(r,s)^{-1}, b_0+\mu s(r,s)^{-1}, c_0+\lambda r(r,s)^{-1}, d_0+\mu r(r,s)^{-1})$.

Moreover, $u^{p_\iota} = v^{q_\iota}$ is equivalent to $wz^{rp}w^{-1} = wz^{sq}w^{-1}$, that is, to $rp = sq$. By Lemma 1.11, we have $p = ts(r,s)^{-1}$ and $q = tr(r,s)^{-1}$ for some $t \in \mathbb{Z}$. Thus $y = f\rho$, where

$$f = u(-a_0-\lambda s(r,s)^{-1}, ts(r,s)^{-1}, b_0+\mu s(r,s)^{-1}).$$

Since $\langle y\rho\rangle = \langle y^{-1}\rho\rangle$, we can assume that $t \geqslant 0$. By the minimality of a_0, we have $\lambda \geqslant 0$ as well. Let

$$f' = u(-a_0-\lambda s(r,s)^{-1}, -\lambda s(r,s)^{-1}, b_0+\mu s(r,s)^{-1}).$$

It follows easily that $f'\rho$ $= [v(-a_0-\lambda r(r,s)^{-1}, -\lambda r(r,s)^{-1}, b_0+\mu r(r,s)^{-1})]\rho$ and so $f'\rho \in \langle u\rho\rangle \cap \langle v\rho\rangle$. Hence $(f'\rho)^{-1} \in \langle u\rho\rangle \cap \langle v\rho\rangle$. We have

$$(f'\rho)^{-1} = [u(-a_0, \lambda s(r,s)^{-1}, b_0+(\lambda+\mu)s(r,s)^{-1})]\rho. \tag{2.6}$$

By the minimality of b_0, we must have

$$\lambda+\mu \geqslant 0. \tag{2.7}$$

Now we must consider several different cases and subcases, according to the following diagram.

$$\begin{cases} \omega = 0 \\ \omega > 0 \begin{cases} \lambda+\mu < \eta \\ \lambda+\mu \geqslant \eta \begin{cases} t-\mu+\eta \leqslant 1 \\ t-\mu+\eta > 1 \begin{cases} \mu \geqslant \eta \\ \mu < \eta \end{cases} \end{cases} \end{cases} \end{cases}$$

Suppose that $\omega = 0$. By (2.5), we have $\Lambda = \emptyset$ and so $\lambda+\mu \leqslant 0$. Hence $\mu = -\lambda$, by (2.7), and so $(f'\rho)^{-1} = [u(-a_0, \lambda s(r,s)^{-1}, b_0)]\rho$, by (2.6). If $\lambda > 0$, then, since $(f'\rho)^{-1} \in \langle u\rho\rangle \cap \langle v\rho\rangle$, we would have $s(r,s)^{-1} \leqslant b_0$, $r(r,s)^{-1} \leqslant d_0$ and so $0 \in \Lambda$, a contradiction. Hence $\lambda = 0$ and so $f = u(-a_0, ts(r,s)^{-1}, b_0)$. Similarly, we obtain $t = 0$ and so $y = f\rho = g_0\rho$.

Now assume that $\omega > 0$. Suppose first that $\lambda+\mu < \eta$. Then, by (2.7),

$\eta > 0$, and by the minimality of η, we must have $\lambda+\mu \notin \Gamma$. However, by (2.6), $(a_0, b_0+(\lambda+\mu)s(r,s)^{-1}, c_0, d_0+(\lambda+\mu)r(r,s)^{-1})$ is a solution of (2.1). Hence $\lambda+\mu \leqslant 0$. It follows that $\lambda+\mu = 0$, by (2.7). Hence, by (2.6), we have $(f'\rho)^{-1} = [u(-a_0, \lambda s(r,s)^{-1}, b_0)]\rho$. Suppose that $\lambda > 0$. Since $(f'\rho)^{-1} \in \langle u\rho\rangle \cap \langle v\rho\rangle$, we have $0 \in \Lambda$ and so $\eta = 0$, a contradiction. Hence $\lambda = 0$. Similarly, we obtain $t = 0$. Thus $y = g_0\rho$.

Now we assume that

$$\lambda+\mu \geqslant \eta. \tag{2.8}$$

Since $ts(r,s)^{-1} = p \leqslant b = b_0+\mu s(r,s)^{-1}$, then $(t-\mu)s(r,s)^{-1} \leqslant b_0$. Similarly, we obtain $(t-\mu)r(r,s)^{-1} \leqslant d_0$, so

$$t-\mu+\eta \leqslant \omega. \tag{2.9}$$

Suppose that $t-\mu+\eta \leqslant 1$. Then $\mu+1-\eta \geqslant t \geqslant 0$ and by Lemma 2.4, we have $y = f\rho = [u(-a_0-\lambda s(r,s)^{-1}, ts(r,s)^{-1}, b_0+\mu s(r,s)^{-1})]\rho$ $= [g_1(-\lambda, t, \mu+1-\eta)]\rho$.

Now assume that

$$t-\mu+\eta > 1. \tag{2.10}$$

Suppose that

$$\mu \geqslant \eta. \tag{2.11}$$

If $\lambda = \mu-\eta = 0$, then (2.9) and (2.10) yield $1 < t \leqslant \omega$ and so $f = u(-a_0, ts(r,s)^{-1}, b_0+\eta s(r,s)^{-1}) = g_t$. Hence $y = g_t\rho$. Assume that λ or $\mu-\eta$ is nonzero. Then, by (2.11) and Lemma 2.4, we have $[g_1(-\lambda, \mu-\eta, \mu-\eta)]\rho = [u(-\lambda s(r,s)^{-1}-a_0, (\mu-\eta)s(r,s)^{-1}, (\mu-1)s(r,s)^{-1}+b_0)]\rho$.
Hence $(g_1^{-\lambda}g_1^{\lambda+\mu-\eta}g_{t-\mu+\eta})\rho$
$= [u(-\lambda s(r,s)^{-1}-a_0, (\mu-\eta)s(r,s)^{-1}, (\mu-1)s(r,s)^{-1}+b_0)]\rho$
$.[u(-a_0, (t-\mu+\eta)s(r,s)^{-1}, b_0+\eta s(r,s)^{-1})]\rho = f\rho = y$. Thus
$y \in \langle g_0\rho, \ldots, g_\omega\rho\rangle$.

Finally, suppose that

$$\mu < \eta. \tag{2.12}$$

By (2.8) and (2.12), we have $0 < \eta-\mu \leq \lambda$ and so we can write $\lambda = \kappa(\eta-\mu)+\delta$, with $\kappa \in \mathbb{N}$ and $0 \leq \delta < \eta-\mu$. By (2.9), we have $\eta-\mu \leq t-\mu+\eta \leq \omega$, and so we can define $h = g_{\eta-\mu}^{-\kappa} g_\delta^{-1} g_\delta g_{\eta-\mu}^{\kappa-1} g_{t-\mu+\eta}$. We claim that $h\rho = f\rho = y$. In fact,
$h\iota = (g_{\eta-\mu}^{-1} g_{t-\mu+\eta})\iota = [u^{(\mu-\eta)s(r,s)^{-1}} . u^{(t-\mu+\eta)s(r,s)^{-1}}]\iota = u^{ts(r,s)^{-1}}\iota$
$= f\iota$. Since $Q(g_{t-\mu+\eta}) = Q(g_{\eta-\mu})$, we have $Q(h) = Q(g_{\eta-\mu}^{-\kappa} g_\delta^{-1} g_\delta g_{\eta-\mu}^{\kappa})$
$= Q(g_{\eta-\mu}^{-\kappa} g_\delta^{-1})$. By Lemma 2.4, we have $[g_{\eta-\mu}(-\kappa,-\kappa,0)]\rho$
$= [u(-\kappa(\eta-\mu)s(r,s)^{-1}-a_0,-\kappa(\eta-\mu)s(r,s)^{-1},\mu s(r,s)^{-1}+b_0)]\rho$. By (2.8), we obtain $(g_{\eta-\mu}^{-\kappa} g_\delta^{-1})\rho = [g_{\eta-\mu}(-\kappa,-\kappa,0).g_\delta^{-1}]\rho$
$= [u(-\kappa(\eta-\mu)s(r,s)^{-1}-a_0,-\kappa(\eta-\mu)s(r,s)^{-1},\mu s(r,s)^{-1}+b_0)]\rho$
$.[u(-a_0-\delta s(r,s)^{-1},-\delta s(r,s)^{-1},b_0+(\eta-\delta)s(r,s)^{-1})]\rho$
$= [u(-a_0-\lambda s(r,s)^{-1},-\lambda s(r,s)^{-1},b_0+\mu s(r,s)^{-1})]\rho = f'\rho$. Hence $Q(h) = Q(f') = Q(f)$ and so $h\rho = f\rho = y$. Thus $y \in \langle g_0\rho,\ldots,g_\omega\rho\rangle$ and the lemma is proved.

Now we analyse the structure of the inverse monoid $\langle g_0\rho,\ldots,g_\omega\rho\rangle$.

For every inverse monoid M, we define $M^{(1)}$ to be the inverse monoid obtained by adjoining a new unity to M.

LEMMA 2.7.
$$\langle g_0\rho,\ldots,g_\omega\rho\rangle \simeq \begin{cases} I_\omega & \text{if } \omega > 0 \text{ and } \eta = 0, \\ (I_\omega)^{(1)} & \text{if } \omega > 0 \text{ and } \eta > 0, \\ \{1\}^{(1)} & \text{if } \omega = 0. \end{cases}$$

Proof. Suppose that $\omega = 0$. Then $\langle g_0\rho\rangle = \{1,g_0\rho\} \simeq \{1\}^{(1)}$.

Now assume that $\omega > 0$. We prove that $\langle g_1\rho,\ldots,g_\omega\rho\rangle \simeq I_\omega$. Since $p \leq b$ and $(\omega-\eta)s(r,s)^{-1} \leq b_0$, we have $ps(r,s)^{-1} \leq b_0+(b+\eta-\omega)s(r,s)^{-1}$ and so we can define a map $\Phi:\langle u\rho\rangle \longrightarrow \langle u\rho\rangle$ by $([u(-a,p,b)]\rho)\Phi = [u(-a_0-as(r,s)^{-1},ps(r,s)^{-1},b_0+(b+\eta-\omega)s(r,s)^{-1})]\rho$.
By Lemmas 1.7 and 1.10, with b_0 replaced by $b_0+(\eta-\omega)s(r,s)^{-1}$, Φ is an injective homomorphism. For every $k \in \{1,\ldots,\omega\}$, let $h_k = u(0,k,\omega)$.

Since $(h_k\rho)\Phi = g_k\rho$, we have $\langle h_1\rho,\dots,h_\omega\rho\rangle \simeq \langle g_1\rho,\dots,g_\omega\rho\rangle$. But, by Lemmas 1.7 and 1.9, we have $\langle h_1\rho,\dots,h_\omega\rho\rangle \simeq I_\omega$ and so $\langle g_1\rho,\dots,g_\omega\rho\rangle \simeq I_\omega$.

Suppose that $\eta = 0$. Then $g_0\rho = g_1\rho(g_1\rho)^{-1}$ and so $g_0\rho \in \langle g_1\rho,\dots,g_\omega\rho\rangle$. Therefore $\langle g_0\rho,\dots,g_\omega\rho\rangle \simeq I_\omega$.

Now suppose that $\eta > 0$. Then $|Q(g_0)| < |Q(g_i)|$ for every $i \in \{1,\dots,\omega\}$ and so $g_0\rho \notin \langle g_1\rho,\dots,g_\omega\rho\rangle$. Let $k \in \{1,\dots,\omega\}$. We certainly have $g_0\rho.g_k\rho = g_k\rho$. Supppose that $\omega > \eta$. Then $s(r,s)^{-1} \leq b_0$ and $r(r,s)^{-1} \leq d_0$. Thus $0 \in \Lambda$ and so $\eta = 0$, a contradiction. Hence $\omega \leq \eta$ and so $g_k\rho.g_0\rho = g_k\rho$. Therefore $\langle g_0\rho,\dots,g_\omega\rho\rangle \simeq \langle g_1\rho,\dots,g_\omega\rho\rangle^{(1)} \simeq (I_\omega)^{(1)}$.

Now we produce an algorithm which determines the values of η and ω for every $u,v \in (X\cup X^{-1})^*\backslash D_X$ in case A.

Let $\beta = \max\{|f|: f \in Q(u)\cup Q(v)\}$ and let $\hbar = 4\beta(r,s)^{-1}|z|^{-1}$.

LEMMA 2.8. *If* (2.1) *has nontrivial solutions, then* $a_0+b_0 \leq \hbar$ *or* $c_0+d_0 \leq \hbar$.

Proof. Suppose that $a_0+b_0 > \hbar$ and $c_0+d_0 > \hbar$. Since $wz^r \in Q(u)$, we have $r|z| \leq \beta$. Hence $2r(r,s)^{-1} \leq \hbar$. Similarly, $2s(r,s)^{-1} \leq \hbar$. We prove that either

$$s(r,s)^{-1} \leq a_0 \text{ and } r(r,s)^{-1} \leq c_0$$

or (2.13)

$$s(r,s)^{-1} \leq b_0 \text{ and } r(r,s)^{-1} \leq d_0.$$

Suppose that (2.13) is false. Since $a_0+b_0 > \hbar \geq 2s(r,s)^{-1}$, we can assume, without loss of generality, that $s(r,s)^{-1} > a_0$ and $r(r,s)^{-1} > d_0$. Since (a_0,b_0,c_0,d_0) is a solution of (2.1) and $w \in Q(u)$, we have $z^{r(b_0-1)} = (z^{sj}w^{-1}v')\iota$ for some $j \in \{-c_0,\dots,d_0-1\}$ and $v' \in Q(v)$. Hence $z^{r(b_0-1)-sj} = (w^{-1}v')\iota$ and so $|z|.|r(b_0-1)-sj|$

$< 2\beta$. But $r(b_0-1)-sj \geq r(\hbar-a_0)-sd_0 \geq r\hbar-rs(r,s)^{-1}-rs(r,s)^{-1} = r[\hbar-2s(r,s)^{-1}] = r(r,s)^{-1}[(r,s)\hbar-2s] \geq (r,s)\hbar-2s$ and so $|z|(r,s)\hbar-2|z|s \leq |z|.|r(b_0-1)-sj| < 2\beta$. Therefore $|z|(r,s)\hbar < 2\beta+2|z|s \leq 4\beta$ and so $\hbar < \hbar$, a contradiction. Thus (2.13) holds.

Now we will assume that $s(r,s)^{-1} \leq b_0$ and $r(r,s)^{-1} \leq d_0$, the other case being completely similar. Let $b_1 = b_0-s(r,s)^{-1}$ and $d_1 = d_0-r(r,s)^{-1}$. We shall prove that (a_0,b_1,c_0,d_1) is a solution of (2.1).

Let $i \in \{-a_0,\ldots,b_1-1\}$ and let $u' \in Q(u)$. We want to show that $(z^{ri}w^{-1}u')\iota \in \bigcup_{j=-c_0}^{d_1-1}[z^{sj}w^{-1}Q(v)]\iota$. Since $b_1 < b_0$, $(z^{ri}w^{-1}u')\iota = (z^{sj}w^{-1}v')\iota$ for some $j \in \{-c_0,\ldots,d_0-1\}$ and $v' \in Q(v)$. If $j < d_1$, there is nothing else to prove, so we suppose that $j \geq d_1$. But there exist $j' \in \{-c_0,\ldots,d_0-1\}$ and $v'' \in Q(v)$ such that $(z^{r[i+s(r,s)^{-1}]}w^{-1}u')\iota = (z^{sj'}w^{-1}v'')\iota$.

Suppose that $j' < -c_0+r(r,s)^{-1}$. It follows that $[z^{rs(r,s)^{-1}+sj}w^{-1}v']\iota = [z^{rs(r,s)^{-1}+ri}w^{-1}u']\iota = (z^{sj'}w^{-1}v'')\iota$ and so $z^{rs(r,s)^{-1}+sj-sj'} = (w^{-1}v''v'^{-1}w)\iota$. Therefore $|z|.|rs(r,s)^{-1}+sj-sj'| < |w|+3\beta$. But $rs(r,s)^{-1}+sj-sj' \geq rs(r,s)^{-1}+sd_0-rs(r,s)^{-1}+sc_0-rs(r,s)^{-1} = s(d_0+c_0)-rs(r,s)^{-1} \geq s\hbar-rs(r,s)^{-1} \geq (r,s)\hbar-r$, so we obtain $|z|[(r,s)\hbar-r] \leq |z|.|rs(r,s)^{-1}+sj-sj'| < |w|+3\beta$. Hence $|z|(r,s)\hbar < |z|r+|w|+3\beta \leq 4\beta$ and so $\hbar < \hbar$, a contradiction.

Thus $j' \geq -c_0+r(r,s)^{-1}$. Let $j'' = j'-r(r,s)^{-1}$. Then $j'' \in \{-c_0,\ldots,d_1-1\}$ and $(z^{ri}w^{-1}u')\iota = [z^{sj'-rs(r,s)^{-1}}w^{-1}u']\iota = (z^{sj''}w^{-1}v'')\iota$. Thus $\bigcup_{i=-a_0}^{b_1-1}[z^{ri}w^{-1}Q(u)]\iota \subseteq \bigcup_{j=-c_0}^{d_1-1}[z^{sj}w^{-1}Q(v)]\iota$. The converse inclusion is proved similarly and so (a_0,b_1,c_0,d_1) is a solution of (2.1). Since $a_0+b_0 \geq \hbar \geq 2s(r,s)^{-1}$, we have $a_0+b_1 > 0$ and so (a_0,b_1,c_0,d_1) is nontrivial. This contradicts the minimality of b_0 and so we must have $a_0+b_0 \leq \hbar$ or $c_0+d_0 \leq \hbar$.

We can now establish an algorithm for effectively determining the nature of the intersection $\langle u\rho\rangle\cap\langle v\rho\rangle$. By Lemma 2.8, we only need to test a finite number of values (a,b,c,d) in order to determine whether (2.1) has nontrivial solutions or not and, if the answer is positive, which one is the minimal (a_0,b_0,c_0,d_0). Now we will show how we can compute η.

Suppose that $\eta > 0$. Moreover, suppose that $a_0+b_0+\eta s(r,s)^{-1} > \hbar$ and $c_0+d_0+\eta r(r,s)^{-1} > \hbar$. Since $\eta > 0$, $s(r,s)^{-1} \leqslant b_0+\eta s(r,s)^{-1}$ and $r(r,s)^{-1} \leqslant d_0+\eta r(r,s)^{-1}$. Now we can proceed as in the proof of Lemma 2.8, replacing respectively b_0 by $b_0+\eta s(r,s)^{-1}$ and d_0 by $d_0+\eta r(r,s)^{-1}$. It follows that $(a_0,b_0+(\eta-1)s(r,s)^{-1},c_0,d_0+(\eta-1)r(r,s)^{-1})$ is a solution of (2.1). If $\eta > 1$, this implies $\eta-1 \in \Lambda$, contradiction. Hence $\eta = 1$.

Therefore, for every $u,v \in (X\cup X^{-1})^{*}\backslash D_X$ in Case A, we have that one of the following conditions is satisfied:

$$\eta = 0,$$

$$\eta = 1,$$

$$a_0+b_0+\eta s(r,s)^{-1} \leqslant \hbar,$$

$$c_0+d_0+\eta r(r,s)^{-1} \leqslant \hbar.$$

Whatever the case, this means that we only need to evaluate a finite set of values (a,b,c,d) in order to obtain η. Now ω follows by a simple computation and so we can determine the isomorphism class of $\langle u\rho\rangle\cap\langle v\rho\rangle$.

Case B. $\{u\pi,v\pi\}$ is a basis.

LEMMA 2.9. $|\langle u\rho\rangle\cap\langle v\rho\rangle| \leqslant 2$.

Proof. Consider the following as an equation on the nonnegative integer variables (a,b,c,d).

$$\bigcup_{i=-a}^{b-1}[u^iQ(u)]\iota = \bigcup_{j=-c}^{d-1}[v^jQ(v)]\iota. \tag{2.14}$$

Since $\{u\pi,v\pi\}$ is a basis in $FG(X)$, we have $\langle u\pi\rangle\cap\langle v\pi\rangle = \{1\}$ and so $\langle u\rho\rangle\cap\langle v\rho\rangle$ must be a semilattice. By Lemma 2.2, we only need to show that

$$|\{\bigcup_{i=-a}^{b-1}[u^iQ(u)]\iota : (a,b,c,d) \text{ is a solution of } (2.14)\}| \leq 2. \tag{2.15}$$

Clearly, $(0,0,0,0)$ is always a solution of (2.14). Suppose that (a_0,b_0,c_0,d_0) is a solution of (2.14) with $a_0+b_0 > 1$ and $c_0+d_0 > 1$. We define a sequence $(\varepsilon_k)_k$ on $\{-1,+1\}$ as follows.

Let ε_1 be such that $(u\iota)^{\varepsilon_1} \in \bigcup_{i=-a_0}^{b_0-1}[u^iQ(u)]\iota$. There exist $w \in Q(v)$ and $k \in \{-c_0,\ldots,d_0-1\}$ such that $(u\iota)^{\varepsilon_1} = [v^kw]\iota$. Thus $[v^{-k}(u\iota)^{\varepsilon_1}]\iota \in Q(v)$. Since $c_0+d_0 > 1$, there exists $\varepsilon_2 \in \{-1,+1\}$ such that $(v^{\varepsilon_2}.u^{\varepsilon_1})\iota \in \bigcup_{j=-c_0}^{d_0-1}[v^jv^{-k}(u\iota)^{\varepsilon_1}]\iota \subseteq \bigcup_{j=-c_0}^{d_0-1}[v^jQ(v)]\iota$. Similarly, since $a_0+b_0 > 1$, we can find $\varepsilon_3 \in \{-1,1\}$ with $(u^{\varepsilon_3}.v^{\varepsilon_2}.u^{\varepsilon_1})\iota \in \bigcup_{i=-a_0}^{b_0-1}[u^iQ(u)]\iota$. Continuing this process, since $\bigcup_{i=-a_0}^{b_0-1}[u^iQ(u)]\iota$ is finite we must find some odd $m,n \in \mathbb{N}$ such that $m < n$ and $(u^{\varepsilon_m}.v^{\varepsilon_{m-1}}\ldots u^{\varepsilon_1})\iota = (u^{\varepsilon_n}.v^{\varepsilon_{n-1}}\ldots u^{\varepsilon_1})\iota$. Hence $(u^{\varepsilon_n}\ldots v^{\varepsilon_{m+1}})\iota = 1$ and so $(u\pi)^{\varepsilon_n}\ldots(v\pi)^{\varepsilon_{m+1}} = 1$. Since $\{u\pi,v\pi\}$ is a basis, this is impossible and so any nontrivial solution (a,b,c,d) of (2.14) must satisfy either $a+b = 1$ or $c+d = 1$.

Suppose that (2.14) has two nontrivial solutions (a_0,b_0,c_0,d_0) and (a_1,b_1,c_1,d_1) such that

$$\bigcup_{i=-a_1}^{b_1-1}[u^iQ(u)]\iota \neq \bigcup_{i=-a_0}^{b_0-1}[u^iQ(u)]\iota. \tag{2.16}$$

Let $a' = max\{a_0,a_1\}$, $b' = max\{b_0,b_1\}$, $c' = max\{c_0,c_1\}$ and $d' = max\{d_0,d_1\}$. It follows easily that (a',b',c',d') is a nontrivial solution of (2.14).

Suppose that $a'+b' = 1$. Then $a_0 = a_1$ and $b_0 = b_1$, which contradicts (2.16). Suppose that $a'+b' > 1$. Then $c'+d' = 1$ and so $c_0 = c_1$ and $d_0 = d_1$. Clearly, this also leads to the same contradiction. Hence (2.15) holds and the lemma is proved.

Obviously, we can always compute $\langle u\rho\rangle\cap\langle v\rho\rangle$ for all u,v in this case.

From Lemmas 2.6, 2.7 and 2.9 we now obtain

THEOREM 2.10. *Let* $u,v \in (X\cup X^{-1})^*\backslash D_X$. *Then* $\langle u\rho\rangle\cap\langle v\rho\rangle$ *is isomorphic to one of the following*:

$$\begin{cases} I_k, \ k \in \mathbb{N}, \\ (I_k)^{(1)}, \ k \in \mathbb{N}, \\ \{1\}, \\ \{1\}^{(1)}. \end{cases}$$

3. A few examples

In this section we will prove the existence of intersections $\langle u\rho\rangle\cap\langle v\rho\rangle$ belonging to all isomorphism classes considered in Theorem 2.10.

LEMMA 3.1. *Let* $X = \{x,y\}$. *Let* $k \in \mathbb{N}$ *and let* $u,v \in (X\cup X^{-1})^*$ *be such that* $Q(u) = \{1,x,\dots,x^{k+1},xy,x^{k+1}y\}$, $u\iota = x$, $Q(v) = \{1,x,xy\}$ *and* $v\iota = x$. *Then* $\langle u\rho\rangle\cap\langle v\rho\rangle \simeq I_k$.

Proof. Consider the equation

$$\bigcup_{i=-a}^{b-1}[x^iQ(u)]\iota = \bigcup_{j=-c}^{d-1}[x^jQ(v)]\iota \tag{3.1}$$

on the nonnegative integer variables (a,b,c,d). It is easy to see that $(0,k,0,2k)$ is a solution of (3.1); in fact, $\bigcup_{i=0}^{k-1}[x^iQ(u)]\iota = \{1,x,\dots,x^{2k},xy,x^2y,\dots,x^{2k}y\} = \bigcup_{j=0}^{2k-1}[x^jQ(v)]\iota$. Hence $a_0 = 0$ and $b_0 \leq k$. Suppose that $b_0 < k$. Since $a_0 = 0$ and $u \notin D_X$, we have $b_0 > 0$. We have that $x^ky \notin \bigcup_{i=0}^{b_0-1}[x^iQ(u)]\iota$ and $x^{k+1}y \in \bigcup_{i=0}^{b_0-1}[x^iQ(u)]\iota$. However, $x^{k+1}y \in \bigcup_{j=0}^{d_0-1}[x^jQ(v)]\iota$ implies $d_0 > k$ and so $x^ky \in \bigcup_{j=0}^{d_0-1}[x^jQ(v)]\iota$.

This is a contradiction, so we must have $b_0 = k$. Since $s(r,s)^{-1} - 1 < b_0$ and $r(r,s)^{-1} - 1 < d_0$, we have $0 \in \Lambda$. Hence $\eta = 0$ and $\omega = k$. By Lemmas 2.6 and 2.7, we have $\langle u\rho\rangle \cap \langle v\rho\rangle \simeq I_k$.

LEMMA 3.2. *Let $X = \{x,y\}$. Let $k \in \mathbb{N}$ and let $u,v \in (X \cup X^{-1})^*$ be such that $Q(u) = Q(v)$ $= \{x^i\colon i \in \{0,\dots,6k+4\}\} \cup \{x^2{}^j y\colon j \in \{0,\dots,3k+2\}\} \cup \{xy, x^{6k+3}y\}$, $u\iota = x^2$ and $v\iota = x^3$. Then $\langle u\rho\rangle \cap \langle v\rho\rangle \simeq (I_k)^{(1)}$.*

Proof. Consider the equation

$$\bigcup_{i=-a}^{b-1} [x^{2i}Q(u)]\iota = \bigcup_{j=-c}^{d-1} [x^{3j}Q(v)]\iota \tag{3.2}$$

on the nonnegative integer variables (a,b,c,d). Since $Q(u) = Q(v)$, $(0,1,0,1)$ is a solution, so $a_0 = 0$ and $b_0 = 1$. Since $s(r,s)^{-1} = 3 > b_0$, we have $0 \notin \Lambda$.

We show that $(0,1+3k,0,1+2k)$ is a solution of (3.2). In fact, $\bigcup_{i=0}^{3k} [x^{2i}Q(u)]\iota = \{1,x,\dots,x^{12k+4},y,xy,\dots,x^{12k+4}y\} = \bigcup_{j=0}^{2k} [x^{3j}Q(v)]\iota$. Hence $k \in \Lambda$. Let $k' \in \{1,\dots,k-1\}$ and suppose that $(0,1+3k',0,1+2k')$ is a solution of (3.2). Since $x^3x^{6k-2}y = x^{6k+1}y$, we have $x^{6k+1}y \in \bigcup_{j=0}^{2k'} [x^{3j}Q(v)]\iota$. Hence $x^{6k+1}y \in \bigcup_{i=0}^{3k'} [x^{2i}Q(u)]\iota$ and this clearly implies $k' \geq k$, contradiction. Hence $\eta = k$. Now $\omega = \max\{l \in \mathbb{N}^0\colon 3(l-k) \leq 1 \text{ and } 2(l-k) \leq 1\} = k$. By Lemmas 2.6 and 2.7, we have $\langle u\rho\rangle \cap \langle v\rho\rangle \simeq (I_k)^{(1)}$.

Just for the sake of completeness, we mention the following trivial examples, where $X = \{x,y\}$.

Let $u,v \in (X \cup X^{-1})^*$ be such that $Q(u) = \{1,x\}$, $u\iota = x$, $Q(v) = \{1,y\}$ and $v\iota = y$. Then $\langle u\rho\rangle \cap \langle v\rho\rangle = \{1\}$.

Now let $u,v \in (X \cup X^{-1})^*$ be such that $Q(u) = Q(v) = \{1,x,y\}$, $u\iota = x$ and $v\iota = y$. Then $\langle u\rho\rangle \cap \langle v\rho\rangle = \{1,(uu^{-1})\rho\} \simeq \{1\}^{(1)}$.

Therefore, none of the isomorphism classes mentioned in Theorem 2.10 is superfluous. In contrast with Lemmas 1.1 and 1.2, we can now state the following result.

COROLLARY 3.3. *The intersection of two monogenic free inverse submonoids of a free inverse monoid is not necessarily free.*

4. The rank 1 case

In this section we consider a particular case of the situation discussed in section 2, yielding simple necessary and sufficient conditions for the occurrence of each possible isomorphism class.

We suppose that $u,v \in (X \cup X^{-1})^* \backslash D_X$ are such that $u\rho, v\rho \in \langle w\rho \rangle$ for some $w \in (X \cup X^{-1})^* \backslash D_X$. Consistently with the notation of Lemma 2.5, we assume that $u = w(m_u,r,M_u)$ and $v = w(m_v,s,M_v)$. As in previous cases, we can restrict ourselves to the case $r,s > 0$. Moreover, by Lemma 1.7, we can assume that $w = x \in X$. Since $u\iota = x^r$ and $v\iota = x^s$, we are necessarily in case A. Further, we have $Q(u) = \{x^{m_u},\dots,x^{M_u}\}$ and $Q(v) = \{x^{m_v},\dots,x^{M_v}\}$. Hence (2.1) is equivalent to
$\bigcup_{i=-a}^{b-1} [x^{ri}\{x^{m_u},\dots,x^{M_u}\}]\iota = \bigcup_{j=-c}^{d-1} [x^{sj}\{x^{m_v},\dots,x^{M_v}\}]\iota$. Now it is clear that the nontrivial solutions of (2.1) are exactly the solutions of

$$\begin{cases} -ra+m_u = -sc+m_v \\ r(b-1)+M_u = s(d-1)+M_v, \end{cases} \tag{4.1}$$

where $a,b,c,d \in \mathbb{N}^0$ and $a+b$, $c+d > 0$.

This is a system of diophantine equations and so, by Lemma 1.11(i), (4.1) has solutions if and only if $(r,s) \mid (m_u-m_v)$ and $(r,s) \mid (M_u-M_v+s-r)$.

Now assume that (4.1) has solutions. Let (a_0,b_0,c_0,d_0) be defined as in section 2. By Lemma 1.11(ii), $(a_0,b_0+ks(r,s)^{-1},c_0,d_0+kr(r,s)^{-1})$

is a solution of (4.1) for every $k \in \mathbb{N}$. Hence $\Lambda \neq \emptyset$, and so, by (2.5), $\omega > 0$.

Suppose that $\eta > 1$. By Lemma 1.11(ii), $(a_0, b_0+(\eta-1)s(r,s)^{-1}, c_0, d_0+(\eta-1)r(r,s)^{-1})$ is a solution of (4.1), and since $\eta > 1$, this implies $\eta-1 \in \Lambda$, contradiction. Thus $\eta \leq 1$.

Since $\Lambda \neq \emptyset$, we have $\omega \geq 1$, by (2.5). Suppose that $\omega > 1$. We have $(\omega-\eta)s(r,s)^{-1} \leq b_0$ and $(\omega-\eta)r(r,s)^{-1} \leq d_0$. But $\omega-\eta \geq \omega-1 \geq 1$, so $0 \in \Lambda$ and $\eta = 0$. Since $\omega > 1$, it follows that $(a_0, b_0-s(r,s)^{-1}, c_0, d_0-r(r,s)^{-1})$ is a solution of (4.1), which contradicts the minimality of b_0. Thus $\omega = 1$.

When does $\eta = 0$ occur? Suppose that $\eta = 0$. Then $s(r,s)^{-1} \leq b_0$ and $r(r,s)^{-1} \leq d_0$. Considering (4.1) as an equation on $(\mathbb{N}^0)^4$, $(a_0, b_0-s(r,s)^{-1}, c_0, d_0-r(r,s)^{-1})$ would be a solution, by Lemma 1.11(ii). However, by the minimality of b_0, this is not a solution of (4.1) and so $a_0+b_0-s(r,s)^{-1} \leq 0$ or $c_0+d_0-r(r,s)^{-1} \leq 0$. The other case being dual, we can assume that $a_0+b_0-s(r,s)^{-1} \leq 0$. Since a_0, $b_0-s(r,s)^{-1} \geq 0$, this is equivalent to $a_0 = 0$ and $b_0 = s(r,s)^{-1}$ and therefore to

$$\begin{cases} sc_0 = m_V - m_U \\ s[d_0-r(r,s)^{-1}] = M_U - M_V + s - r. \end{cases}$$

But this is equivalent to $s \mid (m_V - m_U) \geq 0$ and $s \mid (M_U - M_V + s - r) \geq 0$. Considering also the dual case, Lemmas 2.6 and 2.7 yield

THEOREM 4.1. *Let $w \in (X \cup X^{-1})^* \backslash D_X$ and let $m_U, r, M_U, m_V, s, M_V \in \mathbb{Z}$ with $m_U \leq 0 < r \leq M_U$ and $m_V \leq 0 < s \leq M_V$. Let $u = w(m_U, r, M_U)$ and $v = w(m_V, s, M_V)$. Then $\langle u\rho\rangle \cap \langle v\rho\rangle$ is isomorphic to*

$$\begin{cases} \{1\} & \text{if } (r,s) \nmid (m_V - m_U) \text{ or } (r,s) \nmid (M_U - M_V + s - r), \\ I_1 & \text{if } s \mid (m_V - m_U) \geq 0 \text{ and } s \mid (M_U - M_V + s - r) \geq 0, \text{ or dual}, \\ (I_1)^{(1)} & \text{otherwise}. \end{cases}$$

5. The finitely generated problem

In this section we discuss the second of the problems stated in Section 1. The following result follows from Theorem 2.10.

THEOREM 5.1. *Let* $u,v \in (X\cup X^{-1})^*$. *Then* $\langle u\rho\rangle\cap\langle v\rho\rangle$ *is finitely generated.*

This theorem cannot be generalized to the case of non-monogenic free inverse submonoids of a free inverse monoid, as we show next.

Let $X = \{x,y,z\}$ and let $u_1,u_2,v_1,v_2 \in (X\cup X^{-1})^*$ be such that

$$Q(u_1) = \{1,z\},\ u_1\iota = z;$$

$$Q(u_2) = \{1,x,xy,xyx,xyxy\},\ u_2\iota = xyxy;$$

$$Q(v_1) = \{1,x,xy,xyx,z^{-1}\},\ v_1\iota = xyx;$$

$$Q(v_2) = \{1,y,yx,yxy,yxyx,yxyxy\},\ v_2\iota = yxyx.$$

Then we have

$$Q(u_1^{-1}) = \{1,z^{-1}\},\ u_1^{-1}\iota = z^{-1};$$

$$Q(u_2^{-1}) = \{1,y^{-1},y^{-1}x^{-1},y^{-1}x^{-1}y^{-1},y^{-1}x^{-1}y^{-1}x^{-1}\},$$

$$u_2^{-1}\iota = y^{-1}x^{-1}y^{-1}x^{-1},$$

$$Q(v_1^{-1}) = \{1,x^{-1},x^{-1}y^{-1},x^{-1}y^{-1}x^{-1},x^{-1}y^{-1}x^{-1}z^{-1}\},$$

$$v_1^{-1}\iota = x^{-1}y^{-1}x^{-1};$$

$$Q(v_2^{-1}) = \{1,y,x^{-1},x^{-1}y^{-1},x^{-1}y^{-1}x^{-1},x^{-1}y^{-1}x^{-1}y^{-1}\},$$

$$v_2^{-1}\iota = x^{-1}y^{-1}x^{-1}y^{-1}.$$

LEMMA 5.2.

(*i*) $\{u_1\rho,u_2\rho\}$ *is a basis;*

(*ii*) $\{v_1\rho,v_2\rho\}$ *is a basis.*

Proof. (i) We show that $u_1\rho$ and $u_2\rho$ satisfy the conditions of Lemma 1.6. Obviously, $\{u_1\rho,u_2\rho\}\cap\{u_1^{-1}\rho,u_2^{-1}\rho\} = \emptyset$. Let $u \in \{u_1,u_1^{-1},u_2,u_2^{-1}\}$ and suppose that $u\iota \in Q(w_1\ldots w_n)$, with $w_i \in \{u_1,u_1^{-1},u_2,u_2^{-1}\}$ and $w_{i+1}\rho \neq w_i^{-1}\rho$ for every i.

Suppose that $u\iota \notin Q(w_1)$. Then we have $u\iota \in [w_1\ldots w_iQ(w_{i+1})]\iota$ for some $i \geq 1$ and so $(w_i^{-1}\ldots w_1^{-1}u)\iota \in Q(w_{i+1})$. Since $u \neq w_1$ and $w_{i+1} \neq w_i^{-1}$, it follows that $(w_i^{-1}\ldots w_1^{-1}u)\iota = (w_i^{-1}\iota)\ldots(w_1^{-1}\iota)(u\iota)$, and so $(w_i^{-1}\iota)\ldots(w_1^{-1}\iota)(u\iota) \in Q(w_{i+1})$. Since $i \geq 1$, this is clearly impossible. Hence $u\iota \in Q(w_1)$. But this implies $w_1 = u$ and so, by Lemma 1.6, $\{u_1\rho,u_2\rho\}$ is a basis.

(ii) Again, we use Lemma 1.6. We clearly have $\{v_1\rho,v_2\rho\}\cap\{v_1^{-1}\rho,v_2^{-1}\rho\} = \emptyset$. By Lemma 1.5, we have that $\{v_1\pi,v_2\pi\}$ is a basis.

We have $z^{-1} \in Q(v_1)$. Suppose that $z^{-1} \in Q(w_1\ldots w_n)$, with $w_i \in \{v_1,v_1^{-1},v_2,v_2^{-1}\}$ and $w_{i+1}\rho \neq w_i^{-1}\rho$ for every i. We prove that $v_1 = w_1$. Suppose that $z^{-1} \notin Q(w_1)$. Then $z^{-1} \in [w_1\ldots w_iQ(w_{i+1})]\iota$ for some $i \geq 1$. Suppose that $w_{i+1} = v_1$. Since $z \notin \xi((w_1\ldots w_i)\iota)$, then we have $(w_1\ldots w_i)\iota = 1$ and so $(w_1\pi)\ldots(w_i\pi) = 1$. Since $i \geq 1$ and $\{v_1\pi,v_2\pi\}$ is a basis, this is a contradiction. Hence $w_{i+1} \neq v_1$, but then we must have $w_{i+1} = v_1^{-1}$ and $z^{-1} = (w_1\ldots w_ix^{-1}y^{-1}x^{-1}z^{-1})\iota$. Hence $(w_1\ldots w_i)\iota = xyx = v_1\iota$ and so $(w_1\pi)\ldots(w_i\pi) = v_1\pi$. Since $\{v_1\pi,v_2\pi\}$ is a basis, this yields $i = 1$ and $w_1 = v_1$, a contradiction. Hence $z^{-1} \in Q(w_1)$ and so $v_1 = w_1$.

We have $x^{-1}y^{-1}x^{-1}z^{-1} \in Q(v_1^{-1})$. Supposing that $x^{-1}y^{-1}x^{-1}z^{-1} \in Q(w_1\ldots w_n)$, we prove that $v_1^{-1} = w_1$ by a completely similar argument.

Now let $p = xyx$ and $q = yxyx$. For $j \in \{1,2\}$ and $\varepsilon \in \{-1,+1\}$, let $Q'(v_j^\varepsilon) = \{g \in Q(v_j^\varepsilon)\colon g\pi \in \langle p\pi,q\pi\rangle\}$. A simple computation leads to

$$Q'(v_1) = \{1,p\},$$

$$Q'(v_1^{-1}) = \{1,p^{-1}\},$$

$Q'(v_2) = \{1, q, qp^{-1}, q^2p^{-1}\},$

$Q'(v_2^{-1}) = \{1, p^{-1}, q^{-1}, qp^{-1}\}.$

We have $q \in Q(v_2)$. Suppose that $q \in Q(w_1 \ldots w_n)$. We show that $w_1 = v_2$. Suppose that $w_1 \neq v_2$. Then $q \in [w_1 \ldots w_i Q(w_{i+1})]\iota$ for some $i \geq 1$. It follows easily that we must have $q = (w_1 \ldots w_i g)\iota$ for some $g \in Q'(w_{i+1})$. Hence $g = (w_i^{-1} \ldots w_1^{-1} q)\iota$. Since we are assuming that $w_1\iota \neq q$, and $\{p\pi, q\pi\}$ is a basis, we have that g, when expressed as a reduced word on $\{p, p^{-1}, q, q^{-1}\}$, must have length superior to 1 and have q as its final letter. But no such word exists in $Q(w_{i+1})$, whatever w_{i+1} is, so $w_1 = v_2$.

We take $q^{-1} \in Q(v_2^{-1})$ and the case $q^{-1} \in Q(w_1 \ldots w_n)$ is developed in a similar way.

Thus, by Lemma 1.6, $\{v_1\rho, v_2\rho\}$ is a basis.

Now we prove

LEMMA 5.3. *$\langle u_1\rho, u_2\rho\rangle \cap \langle v_1\rho, v_2\rho\rangle$ is not finitely generated.*

Proof. For every $n \in \mathbb{N}$, let $w_n = u_1^{-1}u_1u_2^{n+1}$ and $t_n = v_1v_2^n$. Since $Q(w_n) = Q(u_1^{-1}u_1) \cup Q(u_2^{n+1}) = \{1, z^{-1}\} \cup (\bigcup_{i=0}^{n} [u_2^i . Q(u_2)]\iota)$

$= \{1, z^{-1}\} \cup \{1, x, xy, xyx, \ldots, (xy)^{2n+1}x, (xy)^{2n+2}\}$

$= \{1, z^{-1}, x, xy, xyx, \ldots, (xy)^{2n+1}x, (xy)^{2n+2}\}$

$= \{1, z^{-1}, x, xy, xyx\} \cup \{xyx, xyxy, \ldots, (xy)^{2n+1}x, (xy)^{2n+2}\}$

$= Q(v_1) \cup (\bigcup_{k=0}^{n-1} [v_1v_2^k . Q(v_2)]\iota = Q(t_n)$, we have $(w_nw_n^{-1})\rho = (t_nt_n^{-1})\rho$.

Let $n \in \mathbb{N}$. We show that

$$\nexists g \in (X \cup X^{-1})^* \setminus D_X : g\rho \in \langle u_1\rho, u_2\rho\rangle \cap \langle v_1\rho, v_2\rho\rangle \text{ and } Q(g) \subseteq Q(w_n). \tag{5.1}$$

Suppose that there exists such g. Let $g\rho = (p_1 \ldots p_m)\rho$, with $p_i \in \{v_1, v_1^{-1}, v_2, v_2^{-1}\}$ for all i. Suppose that $p_i \in \{v_2, v_2^{-1}\}$ for all i.

Then $g\pi \in \langle(yxyx)\pi\rangle$. Further, since $g\rho \in \langle u_1\rho, u_2\rho\rangle$, we have $g\pi \in \langle z\pi, (xyxy)\pi\rangle$. It is immediate that $\langle(yxyx)\pi\rangle \cap \langle z\pi, (xyxy)\pi\rangle = \{1\}$, so $g\pi = 1$ and $g \in D_X$, a contradiction. Therefore we can define $k = max\{i \in \{1,\ldots,m\}: p_i = v_1 \text{ or } p_i = v_1^{-1}\}$. Since $Q(g) \subseteq Q(w_n)$, we have $[p_1\ldots p_{k-1}Q(p_k)]\iota \subseteq Q(w_n)$.

Suppose first that $p_k = v_1$. Then $z^{-1} \in Q(p_k)$ and so $(p_1\ldots p_{k-1})\iota = 1$. Now suppose that $p_k = v_1^{-1}$. Then $x^{-1}y^{-1}x^{-1}z^{-1} \in Q(p_k)$ and so $(p_1\ldots p_{k-1})\iota = xyx$.

Whatever case arises, we have $g\pi = (p_1\ldots p_m)\pi = (p_1\ldots p_{k-1})\pi(p_k\ldots p_m)\pi \in \{1,(xyx)\pi\}.\langle(yxyx)\pi\rangle$. Since $g\rho \in \langle u_1\rho, u_2\rho\rangle$, we have $g\pi \in \langle z\pi, (xyxy)\pi\rangle$ as well. But $\langle z\pi, (xyxy)\pi\rangle \cap (\{1,(xyx)\pi\}.\langle(yxyx)\pi\rangle) = \{1\}$ and so we have reached a contradiction. Thus (5.1) holds.

Now suppose that $\langle u_1\rho, u_2\rho\rangle \cap \langle v_1\rho, v_2\rho\rangle = \langle f_1\rho, \ldots, f_s\rho\rangle$ for some $f_1,\ldots f_s \in (X\cup X^{-1})^*$. Let $n \in \mathbb{N}$ and suppose that $(w_n w_n^{-1})\rho = (h_1\ldots h_r)\rho$, where every h_i is either f_{j_i} or $f_{j_i}^{-1}$ for some $j_i \in \{1,\ldots,s\}$. Suppose that $h_i \notin D_X$ for some $i \in \{1,\ldots,r\}$. Let i_0 be the minimum of such i. Then $Q(h_{i_0}) \subseteq Q(w_n)$, which contradicts (5.1). Denoting $\{f_j\rho: j \in \{1,\ldots,s\} \text{ and } f_j \in D_X\}$ by K, we obtain $\{(w_n w_n^{-1})\rho: n \in \mathbb{N}\} \subseteq \langle K\rangle$. Since $\langle K\rangle$ is finite, this is impossible and so $\langle u_1\rho, u_2\rho\rangle \cap \langle v_1\rho, v_2\rho\rangle$ is not finitely generated.

Thus we obtain the following result.

THEOREM 5.4. *There exist finitely generated free inverse submonoids of a free inverse monoid whose intersection is not finitely generated.*

CHAPTER III

THE SEMILATTICE $E[FIM(X)]$

1. Preliminaries

In this section we introduce some concepts in semilattice theory and we relate them to $E[FIM(X)]$.

Let E be a semilattice. Let $e \in E$. We say that e is *irreducible* if, for every $f,g \in E$,

$$e = fg \Rightarrow e = f \text{ or } e = g.$$

The set of all irreducibles of E is denoted by $Irr(E)$.

We say that e is *prime* if, for every $f,g \in E$,

$$e \geq fg \Rightarrow e \geq f \text{ or } e \geq g.$$

LEMMA 1.1. *Let E be a semilattice and let $e \in E$. Then*

$$e \text{ prime} \Rightarrow e \text{ irreducible}.$$

Proof. Suppose that e is prime and suppose that $e = fg$ for some $f,g \in E$. Then $e \leq f$ and $e \leq g$. Further, $e \geq fg$ and so, since e is prime, we have $e \geq f$ or $e \geq g$. Hence $e = f$ or $e = g$. Thus e is irreducible.

The semilattice E is said to be a *unique factorization* semilattice

(*UFS*) if

(i) E is generated by $Irr(E)$;

(ii) every irreducible is prime.

All these concepts are inspired by well-known concepts for integral domains [2,§5.3].

We need some results on UFSs.

LEMMA 1.2. *Let E denote a UFS. Let $e_1,\ldots e_n,f_1,\ldots,f_m \in Irr(E)$ be such that $e_1\ldots e_n = f_1\ldots f_m$. Then, for every $i \in \{1,\ldots,n\}$, there exists $j \in \{1,\ldots,m\}$ such that $e_i \geq f_j$.*

Proof. Let $i \in \{1,\ldots,n\}$. Clearly, $e_i \geq f_1\ldots f_m$. Since E is a *UFS*, e_i is prime and an elementary induction yields $e_i \geq f_j$ for some $j \in \{1,\ldots,m\}$.

LEMMA 1.3. *Let E denote a UFS and let $e \in E$. Then*

(i) *$Irr(Ee) = e.Irr(E)$;*

(ii) *Ee is a UFS.*

Proof. (i) Let $f \in Irr(Ee)$. Since E is a UFS, there exist $g_1,\ldots,g_n \in Irr(E)$ such that $f = g_1\ldots g_n$. Let I be minimal among the nonempty subsets of $\{1,\ldots,n\}$ with respect to $f = e\prod_{i\in I} g_i$. Suppose that $|I| > 1$. Since $eg_i > f$ for every $i \in I$ and $f = \prod_{i\in I} eg_i$, we obtain $f \notin Irr(Ee)$, a contradiction. Hence $|I| = 1$ and so $f \in e.Irr(E)$.

Conversely, let $g \in Irr(E)$ and suppose that $eg = ff'$ for some $f,f' \in Ee$. We have $e \geq f \geq eg$ and $e \geq f' \geq eg$. But $g \geq ff'$ and since E is a UFS, g is prime, so $g \geq f$ or $g \geq f'$. We can assume that $g \geq f$. Hence $eg \geq f$ and so $eg = f$. Thus $eg \in Irr(Ee)$.

(ii) Let $f \in Ee$. Since E is a UFS, there exist $g_1,\ldots,g_n \in Irr(E)$ such that $f = g_1\ldots g_n$. Therefore $f = ef = eg_1\ldots g_n = (eg_1)\ldots(eg_n)$. By (i), $eg_i \in Irr(Ee)$ for every $i \in \{1,\ldots,n\}$. Thus Ee is generated by $Irr(Ee)$.

Now let $h \in Irr(Ee)$ and let $a,b \in Ee$. Suppose that $h > ab$. By (i), we have $h = eg$ for some $g \in Irr(E)$. Hence $g > ab$ and so, since g is prime, $g > a$ or $g > b$. We can assume that $g > a$. Since $e > a$, we have $h = eg > a$. Thus h is prime and the lemma is proved.

We say that a semilattice E is *upper finite* if the sets $\{f \in E: f > e\}$ are finite for all $e \in E$.

The next lemma states some properties of $E[FIM(X)]$.

LEMMA 1.4. *Let X be a nonempty set and let $E = E[FIM(X)]$. Then*

(i) *$Irr(E) = \{(ww^{-1})\rho: w \in R_X\}$;*

(ii) *E is a UFS;*

(iii) *E is upper finite.*

Proof. Let $e\rho \in Irr(E)$. Suppose that $Q(e) = \{u_1,\ldots,u_n\} \subseteq R_X$. We can write $e\rho = (u_1u_1^{-1})\rho\ldots(u_nu_n^{-1})\rho$. Since $e\rho \in Irr(E)$, we have $e\rho = (u_iu_i^{-1})\rho$ for some $i \in \{1,\ldots,n\}$. Therefore $Irr(E) \subseteq \{(ww^{-1})\rho: w \in R_X\}$.

Now suppose that $w \in R_X$. We prove that $(ww^{-1})\rho$ is prime. Suppose that $(ww^{-1})\rho > e\rho.f\rho$ for some $e,f \in D_X$. Then $Q(w) = Q(ww^{-1}) \subseteq Q(ef) = Q(e)\cup Q(f)$. Hence $w \in Q(e)\cup Q(f)$. We can assume that $w \in Q(e)$. But $Q(e)$ is left closed and $Q(w) = \{w' \in R_X: w' <_l w\}$, therefore $Q(w) \subseteq Q(e)$ and $(ww^{-1})\rho > e\rho$. Hence $(ww^{-1})\rho$ is prime.

By Lemma 1.1, this implies $(ww^{-1})\rho$ irreducible and so (i) is proved. Moreover, it follows that every irreducible of E is prime. By (i), $Irr(E)$ generates E and so E is a UFS.

Since $f\rho > e\rho$ implies $Q(f) \subseteq Q(e)$ for every $e,f \in D_X$, it follows easily that E is upper finite.

The next result is immediate.

LEMMA 1.5. *Let X be a nonempty set and let $w \in R_X\backslash\{1\}$. Then* $\{e \in E[FIM(X)]: e \succ (ww^{-1})\rho\} = \{(w_0w_0^{-1})\rho\}$, *where w_0 is the maximal proper prefix of w.*

2. Principal ideals

In this section we shall obtain necessary and sufficient conditions for two principal ideals of $E[FIM(X)]$ to be isomorphic.

LEMMA 2.1. *Let X be a nonempty set and let $E = E[FIM(X)]$. Let $e \in D_X$. Then*

$$|Cov(e\rho)| = \begin{cases} 2|Q(e)|(|X|-1)+2 & \text{if } X \text{ is finite} \\ |X| & \text{if } X \text{ is infinite.} \end{cases}$$

Proof. We assume that X is finite, the other case being obvious. We use induction on $|Q(e)|$.

Suppose that $|Q(e)| = 1$. Then $e = 1$ and so $Cov(e\rho) = \{(xx^{-1})\rho: x \in X \cup X^{-1}\}$. Hence $|Cov(e)| = 2|X|$ and the lemma holds.

Now suppose that the lemma holds for every $f \in D_X$ such that $|Q(f)| \leq n$, with $n \in \mathbb{N}$. Let $e \in D_X$ be such that $|Q(e)| = n+1$. Since $|Q(e)| > 1$, there exists some $v \in Q(e)\backslash\{1\}$ such that v is an extremal vertex of $MT(e)$. Let $y \in X \cup X^{-1}$ denote the last letter of v. Let $e' \in D_X$ and suppose that $e'\rho \in Cov(e\rho)$. Since $|Q(e')\backslash Q(e)| = 1$, we can define $t_{e'\rho}$ to be the single element of $Q(e')\backslash Q(e)$. Moreover, there exist unique $i_{e'\rho} \in Q(e)$ and $x_{e'\rho} \in X \cup X^{-1}$ such that $(i_{e'\rho}, x_{e'\rho}, t_{e'\rho})$

is an edge of $MT(e')$. We define $A = \{e'\rho \in Cov(e\rho): i_{e'\rho} = v\}$ and $B = [Cov(e\rho)]\backslash A$.

Since v is extremal in $MT(e)$ and y is the last letter of v, we have $A = \{e'\rho: e' \in D_X$ and $Q(e') = Q(e)\cup\{vx\}, x \in (X\cup X^{-1})\backslash\{y^{-1}\}\}$. Hence $|A| = 2|X|-1$.

Since v is extremal in $MT(e)$ and $v \neq 1$, there exists $e_0 \in D_X$ such that $Q(e_0) = Q(e)\backslash\{v\}$. We define a map $\beta: B \longrightarrow Cov(e_0\rho)$ as follows. Suppose that $e'\rho \in B$. Then v is still an extremal vertex of $MT(e')$ and so there exists $e_0' \in D_X$ with $Q(e_0') = Q(e')\backslash\{v\}$. It is clear that $e_0'\rho \in Cov(e_0\rho)$ and so we can define $(e'\rho)\beta = e_0'\rho$. Moreover, β is injective and $[Cov(e_0\rho)]\backslash B\beta = \{e\rho\}$. Hence $|B| = |Cov(e_0\rho)|-1$. Using the induction hypothesis, we obtain $|B| = 2|Q(e_0)|(|X|-1)+2-1 = 2n(|X|-1)+1$. Thus $|Cov(e\rho)| = |A|+|B| = 2|X|-1+2n(|X|-1)+1 = 2(n+1)(|X|-1)+2 = 2|Q(e)|(|X|-1)+2$ and the result follows by induction.

We must introduce some new concepts and notation.

Let $e \in E = E[FIM(X)]$ and let $m = |e|$. For all $k \in \mathbb{N}^0$, we define $Irr_{m+k}(Ee) = \{u \in Irr(Ee): |u| = m+k\}$. Surely, $Irr(Ee) = \dot{\bigcup}_{k\geq 0} Irr_{m+k}(Ee)$. Moreover, $Irr_m(Ee) = \{e\}$ and $Irr_{m+1}(Ee) = Cov(e)$.

For every $k \in \mathbb{N}^0$, we define a map $\delta_{e,k+1}: Irr_{m+k+1}(Ee) \longrightarrow Irr_{m+k}(Ee)$ as follows. Let $g \in Irr_{m+k+1}(Ee)$. By Lemmas 1.3(i) and 1.5, there exists a unique $h \in Irr_{m+k}(Ee)$ such that $g \prec h$. We define $g\delta_{e,k+1} = h$.

Obviously, we have a bijection $\zeta_{e,k}: Irr_{m+k}(Ee) \longrightarrow [Irr_{m+k+1}(Ee)]/Ker(\delta_{e,k+1})$ defined by $g\zeta_{e,k} = g\delta_{e,k+1}^{-1}$. It is easy to see that, for every $g \in Irr_{m+k}(Ee)$, we have

$$|g\zeta_{e,k}| = \begin{cases} 2|X|-1 & \text{if } X \text{ is finite} \\ |X| & \text{if } X \text{ is infinite.} \end{cases} \tag{2.1}$$

Now we obtain a criterion for isomorphism.

LEMMA 2.2. *Let X be a nonempty set and let $E = E[FIM(X)]$. Let $e,f \in E$. Then*

$$Ee \simeq Ef \quad \leftrightarrow \quad |Cov(e)| = |Cov(f)|.$$

Proof. Suppose that $\Phi: Ee \longrightarrow Ef$ is an isomorphism. We certainly have $e\Phi = f$. Let $e' \in Cov(e)$. Since Φ is injective, we have $e'\Phi < f$. Suppose that $e'\Phi < f' < f$ for some $f' \in Ef$. Let $e'' = f'\Phi^{-1}$. It follows easily that $e' < e'' < e$, in contradiction with $e' \in Cov(e)$. Hence no such f' exists and so $e'\Phi \in Cov(f)$. Thus $[Cov(e)]\Phi \subseteq Cov(f)$. Similarly, we obtain $[Cov(f)]\Phi^{-1} \subseteq Cov(e)$. Hence $[Cov(e)]\Phi = Cov(f)$ and so $|Cov(e)| = |Cov(f)|$.

Conversely, suppose that $|Cov(e)| = |Cov(f)|$. Suppose that $m = |e|$ and $n = |f|$. For every $k \in \mathbb{N}^0$, we define a bijection $\varphi_k: Irr_{m+k}(Ee) \longrightarrow Irr_{n+k}(Ef)$ as follows.

Consider $k = 0$. Since $Irr_m(Ee) = \{e\}$ and $Irr_n(Ef) = \{f\}$, we define $e\varphi_0 = f$.

Now suppose that φ_k is defined for some $k \in \mathbb{N}^0$. Let $h \in Irr_{m+k}(Ee)$. Suppose that $k = 0$. We have $|h\zeta_{e,k}| = |Cov(e)| = |Cov(f)| = |h\varphi_k\zeta_{f,k}|$. Suppose now that $k > 0$. Then, by (2.1), we obtain $|h\zeta_{e,k}| = |h\varphi_k\zeta_{f,k}|$ as well. Whatever the case, we can define a bijection $\psi_h: h\zeta_{e,k} \longrightarrow h\varphi_k\zeta_{f,k}$ for every $h \in Irr_{m+k}(Ee)$.

We define $\varphi_{k+1}: Irr_{m+k+1}(Ee) \longrightarrow Irr_{n+k+1}(Ef)$ by $g\varphi_{k+1} = g\psi_h$, where $h = g\delta_{e,k+1}$. Next, we define $\varphi: Irr(Ee) \longrightarrow Irr(Ef)$ by $g\varphi = g\varphi_k$, where $k = |g|-m$. It is immediate that φ is a bijection.

We prove that, for every $g,h \in Irr(Ee)$,

$$g \lessdot h \quad \Leftrightarrow \quad g\varphi \lessdot h\varphi. \tag{2.2}$$

Suppose that $g \lessdot h$. We have $h \in Irr_{m+k}(Ee)$, $g \in Irr_{m+k+1}(Ee)$ and $h = g\delta_{e,k+1}$ for some $k \in \mathbb{N}^0$. Therefore $h\varphi = h\varphi_k$ and $g\varphi = g\varphi_{k+1} = g\psi_h$.

Since $g\psi_h \in h\varphi_k\zeta_{f,k} = (h\varphi_k)\delta^{-1}_{f,k+1}$, we have $(g\varphi)\delta_{f,k+1} = h\varphi_k = h\varphi$ and so $g\varphi \prec h\varphi$.

Conversely, suppose that $g\varphi \prec h\varphi$ and suppose that $h \in Irr_{m+k}(Ee)$ for some $k \in N^0$. Then $h\varphi = h\varphi_k$ and $g\varphi \in Irr_{n+k+1}(Ef)$. Therefore $g \in Irr_{m+k+1}(Ee)$ and $g\varphi = g\varphi_{k+1}$. Hence $(g\varphi_{k+1})\delta_{f,k+1} = h\varphi_k$ and so $g\varphi_{k+1} \in (h\varphi_k)\delta^{-1}_{f,k+1} = (h\varphi_k)\zeta_{f,k} = (h\zeta_{e,k})\psi_h$. But $g\varphi_{k+1} = g\psi_{g\delta_{e,k+1}}$ by definition and so $Im(\psi_h) \cap Im(\psi_{g\delta_{e,k+1}}) \neq \emptyset$. Therefore $\psi_h = \psi_{g\delta_{e,k+1}}$ and so $h = g\delta_{e,k+1}$. Thus $g \prec h$ and (2.2) is proved.

Since E is upper finite, we have that, for every $a,b \in E$ with $a < b$, there exist $c_0,\dots,c_k \in E$ such that $a = c_0 \prec \dots \prec c_k = b$. It follows immediately from (2.2) that, for every $g,h \in Irr(Ee)$,

$$g < h \quad \Leftrightarrow \quad g\varphi < h\varphi. \tag{2.3}$$

Suppose that $g_1\dots g_r = h_1\dots h_s$, with $g_1,\dots,g_r,h_1,\dots,h_s \in Irr(Ee)$. Let $i \in \{1,\dots,r\}$. By Lemmas 1.2 and 1.3(ii), there exists $j \in \{1,\dots,s\}$ such that $g_i \geq h_j$. By (2.3), we have $g_i\varphi \geq h_j\varphi$ and so $g_1\varphi\dots g_r\varphi \geq h_1\varphi\dots h_s\varphi$. Similarly, we obtain $h_1\varphi\dots h_s\varphi \geq g_1\varphi\dots g_r\varphi$ and so $g_1\varphi\dots g_r\varphi = h_1\varphi\dots h_s\varphi$. Also by (2.3), $g_1\varphi\dots g_r\varphi = h_1\varphi\dots h_s\varphi$ implies $g_1\dots g_r = h_1\dots h_s$ and so we can define an injective map $\Phi : Ee \longrightarrow Ef$ as follows. Let $g \in Ee$. By Lemma 1.3, we can write $g = g_1\dots g_r$ for some $g_1,\dots,g_r \in Irr(Ee)$. Then we define $g\Phi = g_1\varphi\dots g_r\varphi$.

We show that Φ is an isomorphism.

Let $g \in Ef$. By Lemma 1.3(ii), there exist $g_1,\dots,g_r \in Irr(Ef)$ such that $g = g_1\dots g_r$. Since φ' is bijective, there exist $h_1,\dots,h_r \in Irr(Ee)$ such that $g_i = h_i\varphi$ for every $i \in \{1,\dots,r\}$. Thus $g = g_1\dots g_r = h_1\varphi\dots h_r\varphi = (h_1\dots h_r)\Phi$ and so Φ is surjective.

Let $g,h \in Ee$. Suppose that $g = g_1\dots g_r$ and $h = h_1\dots h_s$ for some $g_1,\dots,g_r,h_1,\dots,h_s \in Irr(Ee)$. Then $g\Phi.h\Phi = (g_1\dots g_r)\Phi.(h_1\dots h_s)\Phi = g_1\varphi\dots g_r\varphi h_1\varphi\dots h_s\varphi = (g_1\dots g_r h_1\dots h_s)\Phi = (gh)\Phi$. Thus Φ is a homomorphism and the lemma is proved.

We note that every isomorphism $\Phi: Ee \longrightarrow Ef$ must induce bijections between $Irr_{m+k}(Ee)$ and $Irr_{n+k}(Ef)$ and satisfy (2.2).

For every $a \in D_X$, we have $|a\rho| = |Q(a)|$ and so Lemmas 2.1 and 2.2 yield

THEOREM 2.3. *Let X be a nonempty set and let $E = E[FIM(X)]$. Let $e,f \in E$.*

(i) *If X is infinite or $|X| = 1$, then $Ee \simeq Ef$.*

(ii) *If X is finite and $|X| > 1$, then*

$$Ee \simeq Ef \iff |e| = |f|.$$

A semilattice in which all the principal ideals are isomorphic is said to be *uniform*. It follows from Theorem 2.3 that, if X is infinite or $|X| = 1$, then $E[FIM(X)]$ is uniform.

3. The Munn semigroup

We can use the results obtained in Section 2 to get information about the Munn semigroup [25] of the semilattice $E[FIM(X)]$.

Let E be a semilattice and let $U = \{(e,f) \in E \times E: Ee \simeq Ef\}$. For every $(e,f) \in U$, let $T_{e,f}$ denote the set of all isomorphisms from Ee onto Ef. The *Munn semigroup* of E is defined to be $T_E = \bigcup_{(e,f) \in U} T_{e,f}$, with the usual composition of relations. This is an inverse semigroup and $E(T_E) = \{1_{Ee}: e \in E\}$ is isomorphic to E. It follows easily from the definition that, for every $e,f \in E$, $1_{Ee}\mathcal{D} = 1_{Ef}\mathcal{D}$ if and only if $(e,f) \in U$.

THEOREM 3.1. *Let X be a nonempty set and let $E = E[FIM(X)]$. Then T_E is E-unitary.*

Proof. Let $e,f,g \in E$ and let $\Phi: Ee \longrightarrow Ef$ be an isomorphism. Suppose that $1_{Eg}.\Phi \in E(T_E)$. We want to prove that $\Phi \in E(T_E)$. We have that $1_{Eg}.\Phi$ is the restriction of Φ to the semilattice $(Eg)\cap(Ee)$, that is, Ege. Therefore we have $\Phi|_{Ege} = 1_{Ege}$ and we must show that $\Phi = 1_{Ee}$.

Suppose that $\Phi \neq 1_{Ee}$. We show that

$$\exists h \in Irr(E) \text{ such that } h \not\geq e \text{ and } (eh)\Phi \neq fh. \qquad (3.1)$$

Assume first that $e = f$. Since $\Phi \neq 1_{Ee}$, there exists $a \in Ee$ such that $a\Phi \neq a$. Since $e\Phi = f = e$, we have $a \neq e$ and so we can write $a = eh_1 \ldots h_n$ for some $h_i \in Irr(E)$ with $h_i \not\geq e$, $i \in \{1,\ldots,n\}$. It follows that $h_i\varphi \neq h_i$ for some i and so (3.1) holds.

Now assume that $e \neq f$. Since $Cov(e) \subseteq Irr(Ee)$, and by Lemma 1.3(i), there exist $\{h_i : i \in I\} \subseteq Irr(E)$ such that $Cov(e) = \{eh_i : i \in I\}$. Suppose that $(eh_i)\Phi = fh_i$ for every $i \in I$. Since $[Cov(e)]\Phi = Cov(f)$, we have $Cov(f) = \{fh_i : i \in I\}$.

Suppose that $Q(e) \not\subseteq Q(f)$. Let $u \in Q(e)\backslash Q(f)$. Let u' denote the maximum prefix of u contained in $Q(f)$ and suppose that $u = u'xu''$, with $x \in X\cup X^{-1}$ and $u'' \in R_X$. Then $f.(u'xx^{-1}u'^{-1})\rho \in Cov(f)$ and so $f.(u'xx^{-1}u'^{-1})\rho = fh_i$ for some $i \in I$. Since $(u'xx^{-1}u'^{-1})\rho, h_i \in Irr(E)$, we show easily that $(u'xx^{-1}u'^{-1})\rho = h_i$. In fact, $h_i > f.(u'xx^{-1}u'^{-1})\rho$ and $h_i \not> f$ together imply $h_i > (u'xx^{-1}u'^{-1})\rho$. Similarly, $(u'xx^{-1}u'^{-1})\rho > h_i$ and so $(u'xx^{-1}u'^{-1})\rho = h_i$. However, $(u'xx^{-1}u'^{-1})\rho > e$, a contradiction. Thus $Q(e) \subseteq Q(f)$. Similarly, we obtain $Q(f) \subseteq Q(e)$ and so $e = f$, a contradiction. Therefore (3.1) holds.

Now suppose that $h \in Irr(E)$ is such that $h \not\geq e$ and $(eh)\Phi \neq fh$. Let $h' \in Irr(E)$ be such that $h' \prec h$. By Lemma 1.3(i), $eh \in Irr(Ee)$. Hence

$(eh)\Phi \in Irr(Ef)$ and so, by Lemma 1.3(i), $(eh)\Phi = fu$ for some $u \in Irr(E)$. Since $h \not> e$, we have $u \not> f$ and also $h' \not> e$. Hence $eh' \prec eh$ and so $(eh')\Phi \prec (eh)\Phi$. Similarly, $(eh')\Phi = fu'$ for some $u' \in Irr(E)$. Since u is prime, $fu' \prec fu$ and $u \not> f$, we have $u' < u$. If $u' < u'' < u$ for some $u'' \in E$, then $u'' \in Irr(E)$, $u'' \not> f$ and it follows easily that $fu' < fu'' < fu$, a contradiction. Hence $u' \prec u$. Now suppose that $fu' = fh'$. Since $u', h' \in Irr(E)$ and $u' \not> f$, it follows easily that $u' = h'$. But $u' \prec u$ and $h' \prec h$, so, by Lemma 1.5, we have $u = h$, a contradiction. Hence $fu' \neq fh'$, that is, $(eh')\Phi \neq fh'$ and so (3.1) holds for $h \in Irr(E)$ with arbitrary large length. In particular, we can assume that $|h| > |efg|$. Suppose that $(eh)\Phi = fu$, with $u \in Irr(E)$. Then $geh = (geh)\Phi = (ge)\Phi(eh)\Phi = gefu$. Therefore $h > gefu$. Since $|h| > |gef|$, we have $h \not> gef$. Then, since h is prime, we get $h > u$. Hence $|u| > |h| > |efg| > |ge|$ and so $u \not> ge$. But $u > geh$ and so, since u is prime, $u > h$. Therefore $u = h$, a contradiction. Hence $\Phi = 1_{Ee}$ and so T_E is E-unitary.

An inverse monoid M is said to be *completely semisimple* if

$$\forall e,f \in E(M),\quad e\mathcal{D} = f\mathcal{D} \Rightarrow e \not< f.$$

THEOREM 3.2. *Let X be a nonempty set and let $E = E[FIM(X)]$. Then*

(i) T_E is bisimple if and only if X is infinite or $|X| = 1$;

(ii) T_E is completely semisimple if and only if X is finite and $|X| > 1$.

Proof. (i) Since every $\mathcal{D}$-class of an inverse monoid M contains an idempotent, it follows that an inverse monoid M is bisimple if and only if

$$\forall e,f \in E(M),\ e\mathcal{D} = f\mathcal{D}.$$

Let $e,f \in E$. Since $1_{Ee}\mathcal{D} = 1_{Ef}\mathcal{D}$ is equivalent to $Ee \simeq Ef$, we have that T_E is bisimple if and only if E is uniform, and Theorem 2.3 yields the result.

(ii) Suppose that X is infinite or $|X| = 1$. Let $e,f \in E$ be such that $e > f$. We have $1_{Ee}\mathfrak{D} = 1_{Ef}\mathfrak{D}$ and $1_{Ee} > 1_{Ef}$, so T_E is not completely semisimple.

Now suppose that X is finite and $|X| > 1$. Let $e,f \in E$ be such that $1_{Ee}\mathfrak{D} = 1_{Ef}\mathfrak{D}$ and $1_{Ee} < 1_{Ef}$. Since $1_{Ee}\mathfrak{D} = 1_{Ef}\mathfrak{D}$, we have $Ee \simeq Ef$, and by Theorem 2.3, $|e| = |f|$. Since $1_{Ee} < 1_{Ef}$, we have $e < f$. Clearly, $e < f$ and $|e| = |f|$ together imply $e = f$, so T_E is completely semisimple and the lemma is proved.

4. Subsemilattices of $E[FIM(X)]$

The problem of finding necessary and sufficient conditions for a semilattice to be embeddable in $E[FIM(X)]$ is still open. In this section, we obtain some results concerning some particular classes of semilattices.

Since the free inverse monoid of countable rank is itself embeddable in any free inverse monoid of rank greater than 1 [37], we will fix $X = \{x_n : n \in \mathbb{N}\}$ and $E = E[FIM(X)]$ throughout this section.

THEOREM 4.1. *Let L be a finite semilattice. Then L is embeddable in E.*

Proof. We consider E to be the set of all finite nonempty left closed subsets of R_X, with the union operation.

Let $\varphi : L \longrightarrow X$ be an injective map. We define a map $\Phi : L \longrightarrow E$ by $a\Phi = \{1\} \cup (L \backslash L^1 a)\varphi$.

We show that Φ is a homomorphism. Let $a,b \in L$. Since

$L'ab = (L'a) \cap (L'b)$, we have $(ab)\Phi = \{1\} \cup (L \backslash L'ab)\varphi$ $= \{1\} \cup (L \backslash [(L'a) \cap (L'b)])\varphi = \{1\} \cup [(L \backslash L'a) \cup (L \backslash L'b)]\varphi$ $= [\{1\} \cup (L \backslash L'a)\varphi] \cup [\{1\} \cup (L \backslash L'b)\varphi] = (a\Phi) \cup (b\Phi)$. Therefore Φ is a homomorphism.

Now suppose that $a\Phi = b\Phi$. Then $\{1\} \cup (L \backslash L'a)\varphi = \{1\} \cup (L \backslash L'b)\varphi$ and so $L'a = L'b$. Hence $a = cb$ for some $c \in L'$, that is, $a \leq b$. Similarly, $b \leq a$, hence $a = b$. Thus Φ is injective and the theorem is proved.

THEOREM 4.2. *Let L be a countable UFS. Then L is embeddable in E if and only if L is upper finite.*

Proof. Suppose that L is embeddable in E. Clearly, subsemilattices of upper finite semilattices are upper finite. Since E is upper finite, it follows that L is upper finite.

Conversely, suppose that L is upper finite.

We prove that the elements of L can be written as a sequence $(f_n)_{n \in \mathbb{N}}$ such that

$$f_n < f_m \Rightarrow n > m. \tag{4.1}$$

Suppose that $L = \{e_n : n \in \mathbb{N}\}$. We define a sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of L as follows. Assuming that $A_0 = \emptyset$, we define $A_n = \{g \in L : g \geq e_n\} \backslash (A_0 \cup \ldots \cup A_{n-1})$ for every $n \in \mathbb{N}$. Since L is upper finite, every A_n is finite, possibly empty. Moreover, $L = \dot{\bigcup}_{n \in \mathbb{N}} A_n$. Now we define the sequence $(f_n)_{n \in \mathbb{N}}$.

Clearly, $A_1 \neq \emptyset$. Let f_1 be maximal in A_1 for the natural partial order of L.

Suppose that $f_1, \ldots, f_k$ are defined for some $k \in \mathbb{N}$ and suppose that $f_k \in A_n$. If $A_n \backslash \{f_1, \ldots, f_k\} \neq \emptyset$, we choose f_{k+1} to be a maximal element of $A_n \backslash \{f_1, \ldots, f_k\}$. If $A_n \backslash \{f_1, \ldots, f_k\} = \emptyset$, we choose f_{k+1} to be a maximal element of A_{n+m}, where $m = min\{l \in \mathbb{N} : A_{n+l} \neq \emptyset\}$. Note that $\{l \in \mathbb{N} : A_{n+l} \neq \emptyset\}$ is nonempty, since L is countable and $A_1 \cup \ldots \cup A_n$ is finite.

It is immediate that $L = \{f_n: n \in \mathbb{N}\}$ and $(f_n)_{n\in\mathbb{N}}$ satisfies (4.1).

We define a map $\varphi:L \longrightarrow E$ as follows. Since (4.1) holds, we have $f_1 \in Irr(L)$. Let $k \in \mathbb{N}$. The set $B_k = \{i \in \mathbb{N}: f_i \in Irr(L)$ and $f_i > f_k\}$ is clearly finite. Since $Irr(L)$ generates L, there exists some $f_i \in Irr(L)$ such that $f_i > f_k$ and so B_k is nonempty. Since L is a UFS, it is clear that $f_k = \prod_{i\in B_k} f_i$. We define $f_k\varphi = \prod_{i\in B_k} (x_ix_i^{-1})\rho$.

We prove that φ is a homomorphism. Let $m,n \in \mathbb{N}$ and suppose that $f_mf_n = f_k$. We want to show that $f_m\varphi.f_n\varphi = f_k\varphi$, that is, $B_m\cup B_n = B_k$. Since $f_k < f_m$ and $f_k < f_n$, it follows that $B_m\cup B_n \subseteq B_k$. Now suppose that $i \in B_k$. Then $f_i \in Irr(L)$ and $f_i > f_k = f_mf_n$. Since L is a UFS, f_i is prime and so we have $f_i > f_m$ or $f_i > f_n$. Hence $i \in B_m\cup B_n$ and so $B_k \subseteq B_m\cup B_n$. Thus $B_m\cup B_n = B_k$ and φ is a homomorphism.

Now suppose that $f_m\varphi = f_n\varphi$ for some $m,n \in \mathbb{N}$. Then $B_m = B_n$ and so $f_m = \prod_{i\in B_m} f_i = \prod_{i\in B_n} f_i = f_n$. Therefore φ is injective and the theorem is proved.

We note that these results only yield sufficient conditions for a semilattice to be embeddable in E. We can provide a trivial example of a subsemilattice of E which is not a UFS. In fact, let $u,v,w,z \in D_X$ be such that $Q(u) = \{1,x_1,x_2\}$, $Q(v) = \{1,x_1,x_3\}$, $Q(w) = \{1,x_2,x_3\}$ and $Q(z) = \{1,x_1,x_2,x_3\}$. Let $N = \{u\rho,v\rho,w\rho,z\rho\}$. Obviously, N is a subsemilattice of E. However, N is not a UFS, since $u\rho \in Irr(N)$, $u\rho > v\rho.w\rho$, $u\rho \not> v\rho$ and $u\rho \not> w\rho$.

THEOREM 4.3. *There exists a countable upper finite semilattice which is not embeddable in E.*

Proof. Let $M = \{(m,n) \in \mathbb{N}^0 \times \mathbb{N}^0: m > n\}$, with multiplication described by

$$(m,n)(m',n') = \begin{cases} (m,\min\{n,n'\}) & \text{if } m = m' \\ (\max\{m,m'\},0) & \text{if } m \neq m'. \end{cases}$$

It follows from the definition that the groupoid M is commutative and every element of M is idempotent. We note that $M_0 = \{(m,0)\colon m \in \mathbb{N}^0\}$ satisfies $(M_0M)\cup(MM_0) \subseteq M_0$. Let $(m,n),(m',n'),(m'',n'') \in M$. If $m = m' = m''$, then $[(m,n)(m',n')](m'',n'') = (m,\min\{n,n',n''\}) = (m,n)[(m',n')(m'',n'')]$. Otherwise, it follows from the remark on M_0 that $[(m,n)(m',n')](m'',n'') = (\max\{m,m',m''\},0) = (m,n)[(m',n')(m'',n'')]$. Hence M is associative and so a semilattice.

Let $(m,n),(m',n') \in M$. It should be clear that $(m',n') \geq (m,n)$ implies $m' \leq m$. Since $n' \leq m'$, there exist only finitely many $(m',n') \in M$ such that $(m',n') \geq (m,n)$. Hence M is upper finite.

Now suppose that $\varphi\colon M \longrightarrow E$ is an embedding. Let $k = |(0,0)\varphi|$. Since $(k,k) > (k,k-1) > \ldots > (k,0)$, we have $(k,k)\varphi > \ldots > (k,0)\varphi$. Hence $|(k,k)\varphi| < \ldots < |(k,0)\varphi|$ and so $|(k,0)\varphi|-|(k,k)\varphi| \geq k$. Since $|ef| \leq |e|+|f|-1$ for every $e,f \in E$, we have $|(k,0)\varphi| = |(0,0)\varphi.(k,k)\varphi| \leq |(0,0)\varphi|+|(k,k)\varphi|-1$. Hence $|(0,0)\varphi| \geq |(k,0)\varphi|-|(k,k)\varphi|+1 \geq k+1$, a contradiction. Therefore no such embedding exists.

5. The Hopf property

An algebra A is said to be *hopfian* if the only surjective endomorphisms of A are the automorphisms.

It is known that $FIM(X)$ is hopfian if and only if X is finite [26]. However, $E[FIM(X)]$ shows different behaviour.

THEOREM 5.1. *Let X be a nonempty set and let $E = E[FIM(X)]$. Then E is not hopfian.*

Proof. We consider E to be the set of all finite nonempty left closed subsets of R_X, with the union operation.

Let $x \in X$ and let

$$Y = \{u \in R_X: x^2 <_l u\}.$$

Let $A \in E$. We define $A' = (A\backslash Y)\cup[x^{-1}(A\cap Y)]\iota$. Obviously, A' is finite and nonempty. We show that A' is left closed. Let $w \in A'$ and let $w' \in R_X$ with $w' <_l w$.

Suppose first that $w \in A\backslash Y$. Since A is left closed, we have $w' \in A$ and it is clear that $w \notin Y$ implies $w' \notin Y$. Hence $w' \in A'$.

Now suppose that $w \in [x^{-1}(A\cap Y)]\iota$. Since $1 \in A\backslash Y$, we can assume that $w' \neq 1$. Then there exists some $v \in R_X$ such that $x^2v \in A$ and $w = xv$. Since $w' <_l w$ and $w' \neq 1$, there exists $v' \in R_X$ such that $v' <_l v$ and $w' = xv'$. Since A is left closed, $x^2v' \in A$. Hence $w' = xv' = [x^{-1}(x^2v')]\iota \in [x^{-1}(A\cap Y)]\iota \subseteq A'$. Thus A' is left closed.

We define a map $\varphi:E \longrightarrow E$ by $A\varphi = A'$, $A \in E$, and we show that φ is a noninjective surjective homomorphism.

(i) φ is not injective.

It follows from the definition that $\{1,x,x^2\}\varphi = \{1,x\} = \{1,x\}\varphi$, hence φ is not injective.

(ii) φ is surjective.

Let $e \in E$. Suppose that $A\cap Y = \emptyset$. Then it is immediate that $A\varphi = A$.

Now suppose that $A\cap Y \neq \emptyset$. Then $x, x^2 \in A$. Let $B = (A\backslash Y)\cup\{x^2\}\cup[x(A\cap Y)]$. Obviously, B is finite and nonempty. We show that B is left closed. Let $w \in B$ and let $w' \in R_X$ be such that $w' <_l w$. We have seen before that $A\backslash Y$ is left closed, so we can assume that $w \notin A\backslash Y$. Suppose that $w = x^2$. Since $A\cap Y \neq \emptyset$ and A is left closed, we have $x^2 \in A$ and so $w' \in A\backslash Y \subseteq B$. Now suppose that $w = x^3u$ for some $u \in R_X$ such that $x^2u \in A$. We can assume that $w' = x^3u'$ and $u' <_l u$ for some $u' \in R_X$. Since $x^2u' <_l x^2u$ and A is left closed, we have $x^2u' \in A$

and so $w' = x^3u' \in [x(A\cap Y)] \subseteq B$. Thus B is left closed and so $B \in E$. It is immediate that $B\varphi = A$ and so φ is surjective.

(iii) φ is a homomorphism.

Let $A,B \in E$. Then $(A\cup B)\varphi = [(A\cup B)\backslash Y]\cup(x^{-1}[(A\cup B)\cap Y])\iota = (A\backslash Y)\cup(B\backslash Y)\cup[x^{-1}(A\cap Y)]\iota\cup[x^{-1}(B\cap Y)]\iota = (A\varphi)\cup(B\varphi)$ and so φ is a homomorphism and the theorem is proved.

CHAPTER IV

NORMAL-CONVEX EMBEDDINGS

1. Preliminaries

In this chapter we introduce the concept of normal-convex embedding for inverse semigroups and we obtain two new embedding theorems. Normal-convex subgroups of a group were introduced by Papakyriakopoulos [32] and our definition is the natural generalization. All homomorphisms are supposed to be semigroup homomorphisms.

Let $\varphi: S \longrightarrow T$ be an embedding of inverse semigroups. We say that φ is *normal-convex* if and only if, for every relation R on S,

$$(R\varphi)^{\#} \cap (S \times S)\varphi \subseteq R^{\#}\varphi.$$

Note that the inclusion $R^{\#}\varphi \subseteq (R\varphi)^{\#} \cap (S \times S)\varphi$ is always true. By Lemma I.1.1, we know that φ induces, for every relation R on S, a unique homomorphism $\varphi_R: S/R^{\#} \longrightarrow T/(R\varphi)^{\#}$ such that the canonical diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \varphi\ } & T \\ {\scriptstyle (R^{\#})^{\natural}}\downarrow & & \downarrow{\scriptstyle [(R\varphi)^{\#}]^{\natural}} \\ S/R^{\#} & \xrightarrow[\ \varphi_R\ ]{} & T/(R\varphi)^{\#} \end{array} \qquad (1.1)$$

commutes. Now we have

LEMMA 1.1. *Let $\varphi : S \longrightarrow T$ be an embedding of inverse semigroups. Then φ is normal-convex if and only if φ_R is injective for every relation R on S.*

Proof. Suppose that φ is normal-convex and let R be a relation on S. Let $a,b \in S$ be such that $(aR^{\#})\varphi_R = (bR^{\#})\varphi_R$. Since (1.1) commutes, we have $(a\varphi)(R\varphi)^{\#} = (b\varphi)(R\varphi)^{\#}$. Hence $(a\varphi, b\varphi) \in (R\varphi)^{\#} \cap (S \times S)\varphi$. Since φ is normal-convex, this yields $(a\varphi, b\varphi) \in R^{\#}\varphi$. Thus $aR^{\#} = bR^{\#}$ and so φ_R is injective.

Conversely, suppose that φ_R is injective for every relation R on S. Suppose that $(a\varphi, b\varphi) \in (R\varphi)^{\#}$ for some $a,b \in S$. Since (1.1) commutes, we have $(aR^{\#})\varphi_R = (bR^{\#})\varphi_R$, and since φ_R is injective, $aR^{\#} = bR^{\#}$. Therefore $(a\varphi, b\varphi) \in R^{\#}\varphi$ and so φ is normal-convex.

The following result shows that the class of normal-convex embeddings is closed under composition.

LEMMA 1.2. *Let $\varphi : S \longrightarrow T$ and $\psi : T \longrightarrow U$ be normal-convex embeddings of inverse semigroups. Then $\varphi\psi$ is a normal-convex embedding.*

Proof. It is trivial that $\varphi\psi$ is an embedding. Now let R be a relation on S. Since $(\varphi\psi)_R$ is uniquely defined, we certainly have $(\varphi\psi)_R = \varphi_R\psi_{R\varphi}$ and so $(\varphi\psi)_R$ is injective. Thus, by Lemma 1.1, $\varphi\psi$ is normal-convex.

The next result shows an application of the concept of normal-convex embedding.

THEOREM 1.3. *Let $\varphi:S \longrightarrow T$ be a normal-convex embedding of inverse semigroups and let R be a relation on S. Then the word problem for R is decidable if the word problem for $R\varphi$ is decidable.*

Proof. Suppose that the word problem for $R\varphi$ is decidable. Let $a,b \in S$. By Lemma 1.1, φ_R is injective and so $aR^{\#} = bR^{\#} \leftrightarrow (aR^{\#})\varphi_R = (bR^{\#})\varphi_R$. Since (1.1) commutes, we have $(aR^{\#})\varphi_R = (bR^{\#})\varphi_R \leftrightarrow (a\varphi)(R\varphi)^{\#} = (b\varphi)(R\varphi)^{\#}$. But this latter equality is decidable, hence the word problem for R is decidable and the theorem is proved.

Let M denote an inverse monoid with least group congruence σ. Then M is said to be *F-inverse* if every σ-class of M has a maximal element under the natural partial order. It is well-known that every F-inverse monoid is E-unitary [34,§VII.5].

Let G be a group and let K be a semilattice. An *action* of G on K by left automorphisms is a map $G \times K \longrightarrow K$: $(g,A) \mapsto gA$ such that, for every $g,h \in G$ and $A,B \in K$,

$$g(hA) = (gh)A,$$

$$g(AB) = (gA)(gB),$$

$$1A = A.$$

It follows easily that, for every $g \in G$ and $A,B \in K$, we have

$$A \leq B \Rightarrow gA \leq gB.$$

The *semidirect product* of K by G induced by this action is the inverse semigroup $K \times G$ with the operation given by $(A,g)(B,h) = (A(gB),gh)$. When no ambiguity arises about the action, we shall denote this semigroup by $K \rtimes G$.

Now suppose that L is an ideal of K such that $GL = K$. Then we say that (G,K,L) is a *strong McAlister triple* and

$P(G,K,L) = \{(A,g) \in L \times G\colon\ g^{-1}A \in L\}$ is an inverse subsemigroup of $K \mathbin{\bar{\times}} G$ [21].

LEMMA 1.4 [21]. *Let M be an inverse monoid. Then M is F-inverse if and only if $M \cong P(G,K,L)$ for some strong McAlister triple (G,K,L) such that L has a unity.*

LEMMA 1.5 [28]. *Let S be a quasi-free inverse semigroup. Then $S \cong P(G,K,L)$ for some strong McAlister triple (G,K,L) with G free.*

2. Strong McAlister triples

THEOREM 2.1. *Let (G,K,L) be a strong McAlister triple. Then the inclusion map $\varphi\colon P(G,K,L) \longrightarrow K \mathbin{\bar{\times}} G$ is normal-convex.*

Proof. Let $S = P(G,K,L)$ and let $T = K \mathbin{\bar{\times}} G$. Let R be a relation on S, say $R = \{((A_i,g_i),(B_i,h_i))\colon\ i \in I\}$. Without loss of generality, we can assume that R is symmetric. Let $(U,u),(V,v) \in S$ be such that $(U,u)(R\varphi)^{\#} = (V,v)(R\varphi)^{\#}$. We want to prove that $(U,u)R^{\#} = (V,v)R^{\#}$. Since R is symmetric, we know that there exist $(W_0,w_0),\ldots,(W_n,w_n) \in T$ such that

$$(W_0,w_0) = (U,u)$$

$$(W_n,w_n) = (V,v)$$

$$\forall j \in \{1,\ldots,n\}\ \exists (P_j,p_j),(Q_j,q_j) \in T\ \exists i_j \in I\colon$$

$$(W_{j-1},w_{j-1}) = (P_j,p_j)(A_{i_j},g_{i_j})(Q_j,q_j) \text{ and}$$

$$(W_j,w_j) = (P_j,p_j)(B_{i_j},h_{i_j})(Q_j,q_j).$$

Now we show that, for every $m \in \{0,\ldots,n\}$,

$$\exists P'_m, Q'_m, W'_m \in L: \tag{2.1}$$

$$(W'_m, w_m) \in S,$$

$$(W'_m, w_m)R^{\#} = (U,u)R^{\#},$$

$$(W'_m, w_m) = (P'_m, 1)(W_m, w_m)(Q'_m, 1).$$

We use induction on m. Defining $P'_0 = U$, $Q'_0 = u^{-1}U$ and $W'_0 = U$, we see that (2.1) holds for $m = 0$.

Now suppose that (2.1) holds for $m = j-1$, with $j \in \{1,\dots,n\}$. Then $(W'_{j-1}, w_{j-1}) = (W'_{j-1},1)(W'_{j-1},w_{j-1})(w_{j-1}^{-1}W'_{j-1},1)$
$= (W'_{j-1},1)(P'_{j-1},1)(W_{j-1},w_{j-1})(Q'_{j-1},1)(w_{j-1}^{-1}W'_{j-1},1)$
$= (W'_{j-1},1)(P'_{j-1},1)(P_j,p_j)(A_{i_j},g_{i_j})(Q_j,q_j)(Q'_{j-1},1)(w_{j-1}^{-1}W'_{j-1},1)$. It is clear that

$$W'_{j-1} < P'_{j-1}P_j \tag{2.2}$$

and so $(W'_{j-1},1)(P'_{j-1},1)(P_j,p_j) = (W'_{j-1},p_j)$. Similarly, $W'_{j-1} < (p_j g_{i_j} Q_j)(p_j g_{i_j} q_j Q'_{j-1})$ and so

$$g_{i_j}^{-1} p_j^{-1} W'_{j-1} < Q_j(q_j Q'_{j-1}). \tag{2.3}$$

Hence $(Q_j,q_j)(Q'_{j-1},1)(w_{j-1}^{-1}W'_{j-1},1) = (g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j)$. Thus

$$(W'_{j-1},w_{j-1}) = (W'_{j-1},p_j)(A_{i_j},g_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j).$$

Since $W'_{j-1} < p_jA_{i_j}$, we have $p_j^{-1}W'_{j-1} < A_{i_j} \in L$. But $L \trianglelefteq K$ and so $p_j^{-1}W'_{j-1} \in L$. Since $W'_{j-1} \in L$, we obtain $(W'_{j-1},p_j) \in S$. Similarly, we have $g_{i_j}^{-1}p_j^{-1}W'_{j-1} < g_{i_j}^{-1}p_j^{-1}(p_jA_{i_j}) = g_{i_j}^{-1}A_{i_j} \in L$, and $q_j^{-1}g_{i_j}^{-1}p_j^{-1}W'_{j-1} = w_{j-1}^{-1}W'_{j-1} \in L$. Hence $(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j) \in S$.

Let $P'_j = W'_{j-1}$, $Q'_j = w_{j-1}^{-1}W'_{j-1}$ and $W'_j = W'_{j-1}(p_jB_{i_j})(w_jw_{j-1}^{-1}W'_{j-1})$. Obviously, P'_j, $Q'_j \in L$ and since $L \trianglelefteq K$, we have $W'_j \in L$ as well. We have $(W'_j,w_j) = (W'_{j-1},p_j)(B_{i_j},h_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j)$, that is, (W'_j,w_j) is a product of elements of S. Therefore $(W'_j,w_j) \in S$. Moreover, $(W'_j,w_j)R^{\#}$
$= [(W'_{j-1},p_j)(B_{i_j},h_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j)]R^{\#}$
$= [(W'_{j-1},p_j)(A_{i_j},g_{i_j})(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j)]R^{\#} = (W'_{j-1},w_{j-1})R^{\#} = (U,u)R^{\#}$.

It follows from (2.2) that $(W'_{j-1},p_j) = (W'_{j-1},1)(P_j,p_j)$. Similarly, (2.3) yields $(g_{i_j}^{-1}p_j^{-1}W'_{j-1},q_j) = (Q_j,q_j)(w_{j-1}^{-1}W'_{j-1},1)$. Hence (W'_j,w_j)

$= (W'_{j-1}, p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1} p_j^{-1} W'_{j-1}, q_j)$

$= (W'_{j-1}, 1)(P_j, p_j)(B_{i_j}, h_{i_j})(Q_j, q_j)(w_j^{-1} W'_{j-1}, 1) = (P'_j, 1)(W_j, w_j)(Q'_j, 1)$

and so (2.1) holds for $m = j$.

Thus (2.1) holds for every $m \in \{0, \ldots, n\}$. In particular, we have $(W'_n, v)R^{\#} = (W'_n, w_n)R^{\#} = (U, u)R^{\#}$ and $(W'_n, v) = (P'_n, 1)(W_n, w_n)(Q'_n, 1) = (P'_n, 1)(V, v)(Q'_n, 1)$. Therefore $W'_n \leq V$ and so $(W'_n, v) = (W'_n, 1)(V, v)$. It follows that $(U, u)R^{\#} = (W'_n, 1)R^{\#}(V, v)R^{\#}$ and so $(U, u)R^{\#} \leq (V, v)R^{\#}$. Similarly, we obtain $(V, v)R^{\#} \leq (U, u)R^{\#}$ and so $(U, u)R^{\#} = (V, v)R^{\#}$. Thus φ is normal-convex.

Now, Lemma 1.5 and Theorem 2.1 immediately yield

COROLLARY 2.2. *Every quasi-free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

Since every free inverse semigroup is quasi-free, we also obtain

COROLLARY 2.3. *Every free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

3. E-unitary inverse semigroups

In this section we prove that every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.

Let S be a E-unitary inverse semigroup. Let $M(S) = \{\emptyset \neq A \subseteq S\colon E(S).A \subseteq A \subseteq a\sigma \text{ for some } a \in S\}$ with the operation described by $AB = \{ab\colon a \in A \text{ and } b \in B\}$. The following result is due to O'Carroll.

LEMMA 3.1 [30]. *Let S be a E-unitary inverse semigroup. Then $M(S)$ is an F-inverse monoid and the map $\varphi\colon S \longrightarrow M(S)\colon s \mapsto \{t \in S\colon t \leq s\}$ is an embedding. Moreover, if σ_S and $\sigma_{M(S)}$ denote respectively the least group congruences of S and $M(S)$, then $\sigma_{M(S)} \cap (S \times S)\varphi = \sigma_S\varphi$.*

We prove that this embedding is in fact normal-convex.

LEMMA 3.2. *Let S be a E-unitary inverse semigroup. Then the embedding $\varphi\colon S \longrightarrow M(S)\colon s \mapsto \{t \in S\colon t \leq s\}$ is normal-convex.*

Proof. Let R be a relation on S. Without loss of generality, we can assume that R is symmetric. Let $a,b \in S$ be such that $(a\varphi,b\varphi) \in (R\varphi)^{\#}$. We want to prove that $(a,b) \in R^{\#}$.

Since $(a\varphi,b\varphi) \in (R\varphi)^{\#}$, there exist $W_0,\ldots,W_n \in M(S)$ such that

$$W_0 = a\varphi;$$

$$W_n = b\varphi;$$

$$\forall i \in \{1,\ldots,n\}\ \exists P_i,Q_i \in M(S)\ \exists(u_i,v_i) \in R\colon$$

$$W_{i-1} = P_i(u_i\varphi)Q_i \text{ and } W_i = P_i(v_i\varphi)Q_i.$$

We prove the following result. Let $z \in S$ and $C,D \in M(S)$ be such that $C(z\varphi)D \in S\varphi$. Then

$$\exists c,d \in S\colon c\varphi \subseteq C,\ d\varphi \subseteq D \text{ and } (czd)\varphi = C(z\varphi)D. \tag{3.1}$$

Since $C(z\varphi)D \in S\varphi$, there exists some $w \in S$ such that $C(z\varphi)D = w\varphi$. Since $w \in w\varphi$, there exist $c \in C$, $z' \in z\varphi$ and $d \in D$ such that $cz'd = w$. Since $c\varphi \subseteq C$, $z'\varphi \subseteq z\varphi$ and $d\varphi \subseteq D$, we obtain $w\varphi = (cz'd)\varphi = (c\varphi)(z'\varphi)(d\varphi) \subseteq (c\varphi)(z\varphi)(d\varphi) \subseteq C(z\varphi)D = w\varphi$. Therefore $(czd)\varphi = C(z\varphi)D$ and (3.1) holds.

Since S is E-unitary, it is clear that

$$\forall A \in M(S),\ AA^{-1} \subseteq 1\sigma \subseteq E(S). \tag{3.2}$$

Now we show that, for every $j \in \{0,\ldots,n\}$

$$\exists w_j \in S\colon w_j\varphi \subseteq W_j \text{ and } (a,w_j) \in R^{\#}. \tag{3.3}$$

Let $w_0 = a$. It follows that (3.3) holds for $j = 0$.

Now suppose that (3.3) holds for $j = i-1$, with $i > 0$. Then $w_{i-1}\varphi \subseteq W_{i-1}$ and so, since S is inverse, $w_{i-1}\varphi \subseteq W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi)$. By (3.2), we also have $W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) \subseteq w_{i-1}\varphi$. Hence $w_{i-1}\varphi = W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Now we can apply (3.1) with $z = u_i$, $C = P_i$ and $D = Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Hence there exist $p_i,q_i \in S$ such that $p_i\varphi \subseteq P_i$, $q_i\varphi \subseteq Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$ and $(p_iu_iq_i)\varphi = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = w_{i-1}\varphi$. We define $w_i = p_iv_iq_i$. Now $w_i\varphi = (p_i\varphi)(v_i\varphi)(q_i\varphi) \subseteq P_i(v_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = W_iW_{i-1}^{-1}(w_{i-1}\varphi) \subseteq W_iW_{i-1}^{-1}W_{i-1}$ and so, by (3.2), we have $w_i\varphi \subseteq W_i.E(S)$. For every $s \in S$ and $e \in E(S)$, we have $ae = aea^{-1}a$, and hence $W_i.E(S) \subseteq E(S).W_i$. Therefore $w_i\varphi \subseteq W_i.E(S) \subseteq E(S).W_i \subseteq W_i$. Moreover, $w_iR^{\#} = (p_iv_iq_i)R^{\#} = (p_iu_iq_i)R^{\#} = w_{i-1}R^{\#} = aR^{\#}$ and so (3.3) holds for $j = i$. Thus (3.3) holds for every $j \in \{0,\ldots,n\}$.

In particular, $w_n\varphi \subseteq W_n = b\varphi$ and $(a,w_n) \in R^{\#}$. Hence $w_n < b$ and $aR^{\#} = w_nR^{\#} < bR^{\#}$. Similarly, we prove that $bR^{\#} < aR^{\#}$. Thus $(a,b) \in R^{\#}$ and the lemma is proved.

Now we obtain

THEOREM 3.3. *Every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.*

Proof. Let S be a E-unitary inverse semigroup. By Lemma 3.2, the embedding $\varphi: S \longrightarrow M(S): s \mapsto \{t \in S: t \leq s\}$ is normal-convex. By Lemma 3.1, $M(S)$ is F-inverse and so, by Lemma 1.4 and Theorem 2.1, there exists a normal-convex embedding $\psi: M(S) \longrightarrow P$, where P is a semidirect product of a semilattice by a group. By Lemma 1.2, the composition $\varphi\psi: S \longrightarrow P$ is a normal-convex embedding and the theorem is proved.

4. Inverse semigroups

The results of Section 2 can be used to obtain an embedding result concerning general inverse semigroups.

THEOREM 4.1. *Every inverse semigroup admits a normal-convex embedding into an idempotent-separating homomorphic image of a semidirect product of a semilattice by a free group.*

Proof. Let S be an inverse semigroup. By Lemma I.3.9, every inverse semigroup is an idempotent-separating homomorphic image of a quasi-free inverse semigroup, so we can assume that $S = F/\tau$, with F quasi-free and τ idempotent-separating. By Lemma 1.5, we can assume that $F = P(G,K,L)$ for some strong McAlister triple (G,K,L), with G free. By Theorem 2.1, the inclusion $\varphi: F \longrightarrow K \bar{\times} G$ is normal-convex. Therefore, by Lemma 1.1, the induced map $\psi: F/\tau \longrightarrow (K \bar{\times} G)/(\tau\varphi)^{\#}$ defined by $(a\tau)\psi = a(\tau\varphi)^{\#}$ is injective. We must prove that ψ is normal-convex and $(\tau\varphi)^{\#}$ is idempotent-separating.

First we prove that ψ is normal-convex. Let $T = (K \bar{\times} G)/(\tau\varphi)^{\#}$. Let R be a relation on S. We want to show that $(R\psi)^{\#} \cap (S \times S)\psi \subseteq R^{\#}\psi$.

Let μ be the congruence on F such that $\mu/\tau = R^{\#}$. It follows that, for every $a,b \in F$, $(a,b) \in \mu$ if and only if $(a\tau, b\tau) \in R^{\#}$. We prove that

$$(R\psi)^\# \subseteq (\mu\varphi)^\#/(\tau\varphi)^\#. \tag{4.1}$$

Since $\tau \subseteq \mu$, we have $\tau\varphi \subseteq \mu\varphi$ and so $(\tau\varphi)^\# \subseteq (\mu\varphi)^\#$. Hence $(\mu\varphi)^\#/(\tau\varphi)^\#$ is a congruence on T and we only need to show that $R\psi \subseteq (\mu\varphi)^\#/(\tau\varphi)^\#$. Let $a,b \in F$ be such that $(a\tau,b\tau) \in R$. Then $(a\tau,b\tau) \in R^\#$ and so, by definition of μ, we have $(a,b) \in \mu$. Hence $(a\varphi,b\varphi) \in \mu\varphi \subseteq (\mu\varphi)^\#$. Therefore $(a\varphi(\tau\varphi)^\#,b\varphi(\tau\varphi)^\#) \in (\mu\varphi)^\#/(\tau\varphi)^\#$, that is, $((a\tau)\psi,(b\tau)\psi) \in (\mu\varphi)^\#/(\tau\varphi)^\#$. Hence (4.1) holds.

Now suppose that $a,b \in F$ and $((a\tau)\psi,(b\tau)\psi) \in (R\psi)^\#$. Then, by (4.1), we have $((a\tau)\psi,(b\tau)\psi) \in (\mu\varphi)^\#/(\tau\varphi)^\#$. Hence $(a\varphi(\tau\varphi)^\#,b\varphi(\tau\varphi)^\#) \in (\mu\varphi)^\#/(\tau\varphi)^\#$ and so $(a\varphi,b\varphi) \in (\mu\varphi)^\#$. Since φ is normal-convex and μ is a congruence on F, we have $(\mu\varphi)^\# \cap (F \times F)\varphi \subseteq \mu\varphi$. Hence $(a\varphi,b\varphi) \in \mu\varphi$ and so $(a,b) \in \mu$ and $(a\tau,b\tau) \in R^\#$. Therefore $((a\tau)\psi,(b\tau)\psi) \in R^\#\psi$ and so ψ is normal-convex.

Now we prove that $(\tau\varphi)^\#$ is idempotent-separating. Obviously, $E(K \,\bar{\times}\, G) = \{(A,1):\ A \in K\}$. Suppose that $A,B \in K$ are such that $(A,1)(\tau\varphi)^\# = (B,1)(\tau\varphi)^\#$. Since $GL = K$, there exists $g \in G$ and $C \in L$ such that $gC = A$. Hence $g^{-1}A = C \in L$ and we have $(g^{-1}A,1)(\tau\varphi)^\#$ $= [(g^{-1}A,g^{-1})(A,1)(A,g)](\tau\varphi)^\# = [(g^{-1}A,g^{-1})(B,1)(A,g)](\tau\varphi)^\#$ $= ((g^{-1}A)(g^{-1}B),1)(\tau\varphi)^\#$. Since $(g^{-1}A)(g^{-1}B) \leq g^{-1}A \in L$ and $L \trianglelefteq K$, we have $(g^{-1}A)(g^{-1}B) \in L$. Hence $(g^{-1}A,1),((g^{-1}A)(g^{-1}B),1) \in F$. But $[(g^{-1}A,1)\tau]\psi = (g^{-1}A,1)(\tau\varphi)^\# = ((g^{-1}A)(g^{-1}B),1)(\tau\varphi)^\#$ $= [((g^{-1}A)(g^{-1}B),1)\tau]\psi$ and so, since ψ is injective, $(g^{-1}A,1)\tau = ((g^{-1}A)(g^{-1}B),1)\tau$. Since τ is idempotent-separating, we obtain $(g^{-1}A,1) = ((g^{-1}A)(g^{-1}B),1)$, that is, $g^{-1}A = (g^{-1}A)(g^{-1}B)$. Hence $A = AB$ and $A \leq B$. Similarly, we obtain $B \leq A$ and so $A = B$. Thus $(A,1) = (B,1)$ and $(\tau\varphi)^\#$ is idempotent-separating.

CHAPTER V

CLIFFORD MONOID PRESENTATIONS

1. Preliminaries

In this chapter we establish several decidability results for the variety of Clifford monoids.

Let M be an inverse monoid. We say that M is a *Clifford* monoid if

$$\forall a \in M\ \forall e \in E(M),\ ae = ea.$$

Let C be a class of inverse semigroups. The inverse monoid M is said to be a *semilattice of elements of* C if there exists a semilattice E and a homomorphism $\varphi : M \longrightarrow E$ such that $e\varphi^{-1} \in C$ for every $e \in E$.

LEMMA 1.1. *Let M be an inverse monoid. Then the following propositions are equivalent.*

(i) *M is a Clifford monoid*;

(ii) *M is a semilattice of groups*;

(iii) $\forall a \in M,\ aa^{-1} = a^{-1}a$.

Proof. Suppose that (i) holds. Let $\varphi : M \longrightarrow E(M)$ be the map defined by $a\varphi = aa^{-1}$. For every $a,b \in M$, we have $(ab)\varphi = abb^{-1}a^{-1} = aa^{-1}bb^{-1} = (a\varphi)(b\varphi)$. Hence φ is a homomorphism. Let $e \in E(M)$. Since φ is a homomorphism, $e\varphi^{-1}$ is a semigroup. Now suppose that $a \in e\varphi^{-1}$. Then

$a^{-1}\varphi = a^{-1}a = a^{-1}aa^{-1}a = a^{-1}aaa^{-1} = aa^{-1}aa^{-1} = aa^{-1} = a\varphi = e$. Hence $e\varphi^{-1}$ is inverse. Let $f \in E(e\varphi^{-1})$. Then $f = ff^{-1} = f\varphi = e$ and so $e\varphi^{-1}$ has a single idempotent. Thus $e\varphi^{-1}$ is a group and so (ii) holds.

Now suppose that (ii) holds. Let E be a semilattice and let $\varphi : M \longrightarrow E$ be a homomorphism such that $e\varphi^{-1}$ is a group for every $e \in E$. Let $a \in M$. Then $a^{-1} \in a\varphi\varphi^{-1}$ and so $(aa^{-1})\varphi = a\varphi = (a^{-1}a)\varphi$. Hence $aa^{-1}, a^{-1}a \in E(e\varphi^{-1})$. Since $e\varphi^{-1}$ is a group, we have $aa^{-1} = a^{-1}a$ and so (iii) holds.

Finally, suppose that (iii) holds. Let $a \in M$ and let $e \in E(M)$. Then $ae = aea^{-1}a = (ae)(ea^{-1})a = ea^{-1}aea = e(a^{-1}a)a = eaa^{-1}a = ea$. Thus (i) holds and the lemma is proved.

Let *Clf* denote the class of all Clifford monoids. It follows from Lemma 1.1 that $Clf = Inv[xx^{-1} = x^{-1}x]$ and so *Clf* is a variety of inverse monoids. Similarly, $Su = Inv[x^2 = x]$ is the variety of semilattices with unity. Let X be a nonempty set. We define

$$\nu = \tau(xx^{-1} = x^{-1}x) = (\rho\cup\{(uu^{-1},u^{-1}u) : u \in (X\cup X^{-1})^*\})^{\#},$$

$$\eta = \tau(x^2 = x) = (\rho\cup\{(u^2,u) : u \in (X\cup X^{-1})^*\})^{\#}.$$

The quotients $FCM(X) = (X\cup X^{-1})^*/\nu$ and $FSU(X) = (X\cup X^{-1})^*/\eta$ are respectively the *free Clifford monoid* on X and the *free semilattice with unity* on X.

It is not difficult to prove [34,§VIII.2] that, for every $u,v \in (X\cup X^{-1})^*$,

$$(u,v) \in \eta \iff \xi(u) = \xi(v). \tag{1.1}$$

Moreover, we have that, for every $u,v \in (X\cup X^{-1})^*$,

$$(u,v) \in \nu \iff \xi(u) = \xi(v) \text{ and } u\iota = v\iota, \tag{1.2}$$

by [12] (see also [34,§VIII.2]).

It is clear that $\nu \subset \pi$ and $\nu \subset \eta$. The next result is immediate.

LEMMA 1.2. *Let* $u,v \in (X \cup X^{-1})^*$. *Then*

$$u\eta = v\eta \;\Rightarrow\; (uu^{-1})\nu = (vv^{-1})\nu.$$

The following lemma establishes strong connections between **Inv**, **Clf** and **Gp**.

LEMMA 1.3. *(i) Every finitely presented Clifford monoid has a finite inverse monoid presentation.*

(ii) Every finitely presented group has a finite Clifford monoid presentation.

Proof. (i) We define

$$S_1 = \{(xx^{-1}, x^{-1}x) : x \in X\},$$

$$S_2 = \{(xyy^{-1}x^{-1}, xx^{-1}yy^{-1}) : x,y \in X \cup X^{-1}\}.$$

We want to prove that $\nu = (\rho \cup S_1 \cup S_2)^\#$. It is immediate that $\rho \cup S_1 \subseteq \nu$. Let $x,y \in X \cup X^{-1}$. Then $(xyy^{-1}x^{-1})\nu = (xyy^{-1}yy^{-1}x^{-1})\nu = (yy^{-1}x^{-1}xyy^{-1})\nu = (yy^{-1}xx^{-1})\nu = (xx^{-1}yy^{-1})\nu$ and so $S_2 \subseteq \nu$. Hence $(\rho \cup S_1 \cup S_2)^\# \subseteq \nu$.

Every $u \in (X \cup X^{-1})^+$ is of the form $u = x_1 \dots x_k$, with $x_i \in X \cup X^{-1}$ for $i \in \{1,\dots,k\}$. Using induction on k, we will prove that, if $u = x_1 \dots x_k$, then

$$(uu^{-1})(\rho \cup S_1 \cup S_2)^\# = (x_1x_1^{-1} \dots x_kx_k^{-1})(\rho \cup S_1 \cup S_2)^\#.$$

This is clearly true for $k = 1$. Suppose that it is true for $k = n$, and let $u = x_1 \dots x_{n+1}$. Then $(uu^{-1})(\rho \cup S_1 \cup S_2)^\#$

$= [(x_1 \dots x_{n-1})(x_nx_{n+1}x_{n+1}^{-1}x_n^{-1})(x_{n-1}^{-1} \dots x_1^{-1})](\rho \cup S_1 \cup S_2)^\#$

$= [x_1 \dots x_{n-1}(x_nx_n^{-1})(x_{n+1}x_{n+1}^{-1})x_{n-1}^{-1} \dots x_1^{-1}](\rho \cup S_1 \cup S_2)^\#$

$= [x_1 \dots x_{n-1}(x_{n-1}^{-1} \dots x_1^{-1})(x_1 \dots x_{n-1})x_nx_n^{-1}x_{n+1}x_{n+1}^{-1}x_{n-1}^{-1} \dots x_1^{-1}](\rho \cup S_1 \cup S_2)^\#$

$= [x_1 \dots x_{n-1}(x_nx_n^{-1})x_{n-1}^{-1} \dots x_1^{-1}x_1 \dots x_{n-1}x_{n+1}x_{n+1}^{-1}x_{n-1}^{-1} \dots x_1^{-1}](\rho \cup S_1 \cup S_2)^\#$

$= (x_1 \dots x_nx_n^{-1} \dots x_1^{-1})(\rho \cup S_1 \cup S_2)^\#(x_1 \dots x_{n-1}x_{n+1}x_{n+1}^{-1}x_{n-1}^{-1} \dots x_1^{-1})(\rho \cup S_1 \cup S_2)^\#$

$= (x_1x_1^{-1}\ldots x_nx_n^{-1})(\rho\cup S_1\cup S_2)^{\#}(x_1x_1^{-1}\ldots x_{n-1}x_{n-1}^{-1}x_{n+1}x_{n+1}^{-1})(\rho\cup S_1\cup S_2)^{\#}$

$= (x_1x_1^{-1}\ldots x_{n+1}x_{n+1}^{-1})(\rho\cup S_1\cup S_2)^{\#}$. Hence the equality above holds for $k = n+1$ and so for every $k \in \mathbb{N}$.

Let $u = x_1\ldots x_k$. Then $(u^{-1}u)(\rho\cup S_1\cup S_2)^{\#}$

$= (x_k^{-1}x_k\ldots x_1^{-1}x_1)(\rho\cup S_1\cup S_2)^{\#} = (x_1x_1^{-1}\ldots x_kx_k^{-1})(\rho\cup S_1\cup S_2)^{\#}$

$= (uu^{-1})(\rho\cup S_1\cup S_2)^{\#}$. Thus $\nu \subseteq (\rho\cup S_1\cup S_2)^{\#}$ and so $\nu = (\rho\cup S_1\cup S_2)^{\#}$.

Now let M denote the Clifford monoid defined by the finite presentation $Clf\langle X;R\rangle$. Then $M = (X\cup X^{-1})^*/(\nu\cup R)^{\#}$ $= (X\cup X^{-1})^*/(\rho\cup S_1\cup S_2\cup R)^{\#}$ and so M is finitely presented as an inverse monoid by $Inv\langle X;S_1\cup S_2\cup R\rangle$.

(ii) Let G denote the group defined by the finite presentation $Gp\langle X;R\rangle$. Since $\nu \subset \pi$, we have $G = (X\cup X^{-1})^*/(\pi\cup R)^{\#} = (X\cup X^{-1})^*/(\nu\cup\pi\cup R)^{\#}$ and so G is finitely presented as a Clifford monoid by $Clf\langle X;T\cup R\rangle$, with $T = \{(xx^{-1},1) : x \in X\cup X^{-1}\}$.

Let X be a nonempty set and let Y be a subset of X. By Lemma I.1.3(ii), we can define a homomorphism $\theta_Y:(X\cup X^{-1})^* \longrightarrow (Y\cup Y^{-1})^*$ by

$$x\theta_Y = \begin{cases} x & \text{if } x \in Y\cup Y^{-1} \\ 1 & \text{if } x \in (X\cup X^{-1})\backslash(Y\cup Y^{-1}). \end{cases}$$

2. The word problem

In this section we will show how word problems in *Clf* can be related to word problems in *Su* and *Gp*.

LEMMA 2.1. *In Su, every finitely related presentation has decidable word problem.*

Proof. Let R denote a finite relation on $(X\cup X^{-1})^*$. Let $Y = \{x \in X : x \text{ or } x^{-1} \text{ occurs in } R\}$. Since R is finite, Y is finite also. We prove that, for every $u,v \in (X \cup X^{-1})^*$, we have

$$u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#} \tag{2.1}$$

$$\Leftrightarrow \quad u\theta_Y(\eta \cup R)^{\#} = v\theta_Y(\eta \cup R)^{\#} \text{ and } \xi(u)\backslash Y = \xi(v)\backslash Y.$$

Suppose that $u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#}$. Then there exist $w_0,\ldots,w_n \in (X \cup X^{-1})^*$ such that

$$w_0 = u,$$

$$w_n = v,$$

$$\forall i \in \{1,\ldots,n\} \ \exists s_i,t_i \in (X \cup X^{-1})^* \ \exists (a_i,b_i) \in \eta \cup R:$$

$$\{w_{i-1},w_i\} = \{s_i a_i t_i, s_i b_i t_i\}.$$

Since $a_i\theta_Y = a_i$ and $b_i\theta_Y = b_i$ for every i, it follows that

$$w_0\theta_Y = u\theta_Y,$$

$$w_n\theta_Y = v\theta_Y,$$

$$\forall i \in \{1,\ldots,n\}, \ \{w_{i-1}\theta_Y, w_i\theta_Y\} = \{s_i\theta_Y.a_i(t_i\theta_Y), s_i\theta_Y.b_i(t_i\theta_Y)\}.$$

Hence $u\theta_Y(\eta \cup R)^{\#} = v\theta_Y(\eta \cup R)^{\#}$. Similarly, we obtain

$$w_0\theta_{X\backslash Y} = u\theta_{X\backslash Y},$$

$$w_n\theta_{X\backslash Y} = v\theta_{X\backslash Y},$$

$$\forall i \in \{1,\ldots,n\}, \ \{w_{i-1}\theta_{X\backslash Y}, w_i\theta_{X\backslash Y}\} = \{(s_it_i)\theta_{X\backslash Y}\}.$$

Therefore $u\theta_{X\backslash Y} = v\theta_{X\backslash Y}$, in particular $\xi(u)\backslash Y = \xi(v)\backslash Y$.

Conversely, suppose that $u\theta_Y(\eta \cup R)^{\#} = v\theta_Y(\eta \cup R)^{\#}$ and $\xi(u)\backslash Y = \xi(v)\backslash Y$. Let $w \in (X \cup X^{-1})^*$ be such that $\xi(w) = \xi(u)\backslash Y$. Then, by (1.1), we have $u(\eta \cup R)^{\#} = (u\theta_Y.w)(\eta \cup R)^{\#} = u\theta_Y(\eta \cup R)^{\#}.w(\eta \cup R)^{\#}$ $= v\theta_Y(\eta \cup R)^{\#}.w(\eta \cup R)^{\#} = (v\theta_Y.w)(\eta \cup R)^{\#} = v(\eta \cup R)^{\#}$ and so (2.1) holds.

It follows easily that the word problem for $Su\langle X;R\rangle$ is equivalent to the word problem for $Su\langle Y;R\rangle$. By (1.1), we have $|FSU(Y)| = 2^{|Y|} \in \mathbb{N}$. Hence the word problem for $Su\langle X;R\rangle$ is certainly decidable and so the lemma is proved.

Let $Clf\langle X;R\rangle$ be a Clifford monoid presentation. For every

$u \in (X \cup X^{-1})^*$, we define a relation $R(u)$ on $(X \cup X^{-1})^*$ by

$$R(u) = \{(a,b) \in R\colon u(\eta \cup R)^{\#} = (ua)(\eta \cup R)^{\#}\}.$$

It follows from the definition that

$$\xi(u) \subseteq \xi(v) \Rightarrow R(u) \subseteq R(v), \tag{2.2}$$
$$u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#} \Rightarrow R(u) = R(v).$$

Finally, we note that, if R is finite, say $R = \{(a_1,b_1),\ldots,(a_n,b_n)\}$, there exists $w \in (X \cup X^{-1})^*$ such that $R(w) = R$, namely $w = a_1 \ldots a_n$.

LEMMA 2.2. *Let $Clf\langle X;R\rangle$ be a presentation and let $u,v \in (X \cup X^{-1})^*$. Then*

$$u(\nu \cup R)^{\#} = v(\nu \cup R)^{\#}$$
$$\Leftrightarrow \quad u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#} \text{ and } u[\pi \cup R(u)]^{\#} = v[\pi \cup R(u)]^{\#}.$$

Proof. Suppose that $u(\nu \cup R)^{\#} = v(\nu \cup R)^{\#}$. Then there exist $w_0,\ldots,w_n \in (X \cup X^{-1})^*$ such that

$$w_0 = u,$$
$$w_n = v,$$
$$\forall k \in \{1,\ldots,n\}\ \exists s_k,t_k \in (X \cup X^{-1})^*\ \exists (a_k,b_k) \in \nu \cup R\colon$$
$$\{w_{k-1},w_k\} = \{s_k a_k t_k, s_k b_k t_k\}.$$

Since $\nu \subset \eta$, we have $u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#}$. Since $u(\nu \cup R)^{\#} = w_k(\nu \cup R)^{\#}$ and $\xi(a_k) \subseteq \xi(w_k)$, we have $(ua_k)(\eta \cup R)^{\#} = (w_k a_k)(\eta \cup R)^{\#} = w_k(\eta \cup R)^{\#} = u(\eta \cup R)^{\#}$ and so $(a_k,b_k) \in \nu \cup R(u)$ for every $k \in \{0,\ldots,n\}$. Therefore $u[\nu \cup R(u)]^{\#} = v[\nu \cup R(u)]^{\#}$ and since $\nu \subset \pi$, we obtain $u[\pi \cup R(u)]^{\#} = v[\pi \cup R(u)]^{\#}$.

Conversely, suppose that $u(\eta \cup R)^{\#} = v(\eta \cup R)^{\#}$ and $u[\pi \cup R(u)]^{\#} = v[\pi \cup R(u)]^{\#}$. By Lemma 1.2, we have $(uu^{-1})(\nu \cup R)^{\#} = (vv^{-1})(\nu \cup R)^{\#}$. Since $u[\pi \cup R(u)]^{\#} = v[\pi \cup R(u)]^{\#}$, there exist $z_0,\ldots,z_m \in (X \cup X^{-1})^*$ such that

$z_0 = u,$

$z_m = v,$

$\forall l \in \{1,\ldots,m\}\ \exists s_l, t_l \in (X \cup X^{-1})^*$

$\exists (a_l, b_l) \in \{(xx^{-1}, 1) : x \in X \cup X^{-1}\} \cup R(u):$

$(z_{l-1}, z_l) = (s_l a_l t_l, s_l b_l t_l).$

Suppose that $\{(a_1,b_1),\ldots,(a_m,b_m)\} \cap R(u) = \{(a_{l_1},b_{l_1}),\ldots,(a_{l_k},b_{l_k})\}$. Let $p = uva_{l_1} b_{l_1} \ldots a_{l_k} b_{l_k}$. We have $\xi(u),\xi(v),\xi(a_{l_i}),\xi(b_{l_i}) \subseteq \xi(p)$ for every l_i. Let $Z = \xi(p)$ and let $S = \{(a_{l_1},b_{l_1}),\ldots,(a_{l_k},b_{l_k})\}$. By Lemma I.2.6, $(Z \cup Z^{-1})^*/(\pi \cup S)^\#$ embeds in $(X \cup X^{-1})^*/(\pi \cup S)^\#$ and so we can assume that $\xi(a_l),\xi(b_l),\xi(s_l),\xi(t_l) \subseteq Z$ for every l.

Since $u(\eta \cup R)^\# = v(\eta \cup R)^\#$, $u(\eta \cup R)^\# = (ua_{l_i})(\eta \cup R)^\#$ and $a_{l_i}(\eta \cup R)^\# = b_{l_i}(\eta \cup R)^\#$ for every i, we have $u(\eta \cup R)^\# = p(\eta \cup R)^\#$. Hence, by Lemma 1.2, $(uu^{-1})(\nu \cup R)^\# = (pp^{-1})(\nu \cup R)^\#$. It follows that $(vv^{-1})(\nu \cup R)^\# = (pp^{-1})(\nu \cup R)^\#$ as well.

Now we prove that $(pp^{-1}z_{l-1})(\nu \cup R)^\# = (pp^{-1}z_l)(\nu \cup R)^\#$ for every $l \in \{1,\ldots,m\}$. If $(a_l,b_l) \in R$, then $z_{l-1}(\nu \cup R)^\# = z_l(\nu \cup R)^\#$ and so $(pp^{-1}z_{l-1})(\nu \cup R)^\# = (pp^{-1}z_l)(\nu \cup R)^\#$. Suppose now that $(a_l,b_l) = (xx^{-1},1)$, with $x \in Z \cup Z^{-1}$. Since x or x^{-1} occurs in p, we have $p\nu = (xx^{-1}p)\nu$ and so $(pp^{-1}s_l a_l t_l)\nu = (pp^{-1}s_l xx^{-1} t_l)\nu = (xx^{-1}pp^{-1}s_l t_l)\nu = (pp^{-1}s_l t_l)\nu$. Hence $(pp^{-1}z_{l-1})(\nu \cup R)^\# = (pp^{-1}z_l)(\nu \cup R)^\#$.

In particular, we obtain $(pp^{-1}u)(\nu \cup R)^\# = (pp^{-1}v)(\nu \cup R)^\#$. Thus $v(\nu \cup R)^\# = (vv^{-1}v)(\nu \cup R)^\# = (pp^{-1}v)(\nu \cup R)^\# = (pp^{-1}u)(\nu \cup R)^\# = (uu^{-1}u)(\nu \cup R)^\# = u(\nu \cup R)^\#$ and the lemma is proved.

Now suppose that $R = \{(a_i,b_i) : i \in \{1,\ldots,n\}\}$ is a finite relation on $(X \cup X^{-1})^*$. Let Y be defined as in the proof of Lemma 2.1. We define $K(R)$ to be $\{R(u) : u \in (Y \cup Y^{-1})^*\}$.

THEOREM 2.3. *Let $Clf\langle X;R\rangle$ be a finitely related presentation, with $R = \{(a_i,b_i): i \in \{1,\ldots,n\}\}$. Then*

(i) $K(R)$ can be effectively determined;

(ii) the word problem for $Clf\langle X;R\rangle$ is decidable if and only if the word problem for $Gp\langle X;K\rangle$ is decidable for every $K \in K(R)$.

Proof. (i) Let $u \in (Y\cup Y^{-1})^*$. By Lemma 2.1, we can, for every $i \in \{1,\ldots,n\}$, determine whether or not $u(\eta\cup R)^\# = (ua_i)(\eta\cup R)^\#$. Therefore we can effectively compute $R(u)$. By (2.2), and since Y is finite, we only have to compute finitely many $R(u)$ and so $K(R)$ can be effectively determined.

(ii) Suppose that the word problem for $Clf\langle X;R\rangle$ is decidable. Let $K \in K(R)$. Then $K = R(u)$ for some $u \in (Y\cup Y^{-1})^*$. Since Y is finite and we can compute $R(u)$ for every possible $\xi(u)$, we can assume that $\xi(u)$ is maximal with respect to inclusion. Let $Z = \xi(u)$.

Suppose that $(a_i,b_i) \in R(u)$. Then $(ua_i)(\eta\cup R)^\# = u(\eta\cup R)^\#$ and so $R(u) = R(ua_i)$. By the maximality of Z, we have $\xi(a_i) \subseteq Z$. Similarly, we have $(ub_i)(\eta\cup R)^\# = u(\eta\cup R)^\#$ and so $R(u) = R(ub_i)$ and $\xi(b_i) \subseteq Z$.

Thus, by [5,§9.3], the word problem for $Gp\langle X;K\rangle$ is decidable if and only if the word problem for $Gp\langle Z;K\rangle$ is decidable.

Let $w,w' \in (Z\cup Z^{-1})^*$. Since $\xi(uu^{-1}w) = Z = \xi(uu^{-1}w')$, it follows from Lemma 2.2 that

$$w(\pi\cup K)^\# = w'(\pi\cup K)^\# \iff (uu^{-1}w)(\nu\cup R)^\# = (uu^{-1}w')(\nu\cup R)^\#.$$

Hence the word problem for $Gp\langle Z;K\rangle$ is decidable and so the word problem for $Gp\langle X;K\rangle$ is decidable.

Conversely, suppose that the word problem for $Gp\langle X;K\rangle$ is decidable for every $K \in K(R)$. Let $u,v \in (X\cup X^{-1})^*$. By Lemma 2.2, $u(\nu\cup R)^\# = v(\nu\cup R)^\#$ is equivalent to $u(\eta\cup R)^\# = v(\eta\cup R)^\#$ and $u[\pi\cup R(u)]^\# = v[\pi\cup R(u)]^\#$. By Lemma 2.1, we can decide whether or not $u(\eta\cup R)^\# = v(\eta\cup R)^\#$ and so we only need to show that we can decide whether or not $u[\pi \cup R(u)]^\# = v[\pi \cup R(u)]^\#$.

We prove that $R(u) = R(u\theta_Y)$. Since $\xi(u\theta_Y) \subseteq \xi(u)$, we have $R(u\theta_Y) \subseteq R(u)$, by (2.2). Now suppose that $i \in \{1,\ldots,n\}$ and $(ua_i)(\eta \cup R)^\# = u(\eta \cup R)^\#$. Then, by (2.1), we have $(ua_i)\theta_Y(\eta \cup R)^\# = u\theta_Y(\eta \cup R)^\#$. But $\xi(a_i) \subseteq Y$ and so $(u\theta_Y.a_i)(\eta \cup R)^\# = (ua_i)\theta_Y(\eta \cup R)^\# = u\theta_Y(\eta \cup R)^\#$. Thus $R(u) \subseteq R(u\theta_Y)$ and so $R(u) = R(u\theta_Y)$.

Hence $R(u) \in K(R)$ and, by hypothesis, we can decide whether or not $u[\pi \cup R(u)]^\# = v[\pi \cup R(u)]^\#$. Thus the word problem for $Clf\langle X;R\rangle$ is decidable.

Now the case $|R| = 1$ follows easily.

COROLLARY 2.4. *One-relator Clifford presentations have decidable word problem.*

Proof. This is a consequence of Theorem 2.3 and the fact that one-relator group presentations have decidable word problem [15].

3. The E-unitary problem

In this section we study the E-unitary problem for the class of one-relator Clifford monoid presentations and the class of finite Clifford monoid presentations.

We need preliminary results relating the E-unitary property to the concepts with which we have been working.

LEMMA 3.1. *Let M denote the Clifford monoid defined by the presentation $Clf\langle X;R\rangle$. Then M is E-unitary if and only if*

$$\forall u \in (X \cup X^{-1})^+, \qquad (3.1)$$

$$u(\pi \cup R)^\# = 1(\pi \cup R)^\# \Rightarrow u[\pi \cup R(u)]^\# = 1[\pi \cup R(u)]^\#.$$

Proof. Suppose that M is E-unitary. Let $u \in (X \cup X^{-1})^+$ and suppose that $u(\pi \cup R)^\# = 1(\pi \cup R)^\#$. By Lemma I.2.5, we have $[u(\nu \cup R)^\#]\sigma = 1$. Since M is E-unitary, we have $u(\nu \cup R)^\# \in E(M)$ and so $u(\nu \cup R)^\# = (uu^{-1})(\nu \cup R)^\#$. By Lemma 2.2, we must have $u[\pi \cup R(u)]^\# = (uu^{-1})[\pi \cup R(u)]^\#$ and so $u[\pi \cup R(u)]^\# = 1[\pi \cup R(u)]^\#$.

Now suppose conversely that the condition $u(\pi \cup R)^\# = 1(\pi \cup R)^\# \Rightarrow u[\pi \cup R(u)]^\# = 1[\pi \cup R(u)]^\#$ is satisfied for every $u \in (X \cup X^{-1})^+$. Let $u \in (X \cup X^{-1})^*$ be such that $[u(\nu \cup R)^\#]\sigma = 1$. We want to prove that $u(\nu \cup R)^\# \in E(M)$. The case $u = 1$ is trivial, so we assume $u \neq 1$. By Lemma I.2.5, we have $u(\pi \cup R)^\# = 1(\pi \cup R)^\#$ and so, by hypothesis, $u[\pi \cup R(u)]^\# = 1[\pi \cup R(u)]^\#$. Hence, by Lemma 2.2, $u(\nu \cup R)^\# = (uu^{-1})(\nu \cup R)^\#$. Thus $u(\nu \cup R)^\# \in E(M)$ and the lemma is proved.

We can provide a very simple algorithm which solves completely the one-relator case, but first we need some definitions.

For every $u \in R_X$, there exist unique $\alpha(u), \beta(u) \in R_X$ such that $u = \alpha(u).\beta(u).[\alpha(u)]^{-1}$ and $\beta(u)$ is cyclically reduced. We define $\xi_c(u)$ to be $\xi[\beta(u)]$. For every $v \in (X \cup X^{-1})^*$, we define $\beta(v) = \beta(v\iota)$ and $\xi_c(v) = \xi_c(v\iota)$.

THEOREM 3.2. *Let M denote the Clifford monoid defined by the one-relator presentation $Clf\langle X;R\rangle$, with $R = \{(a,b)\}$. Then M is E-unitary if and only if one of the following conditions is satisfied:*

(*i*) $a\iota = b\iota$;

(*ii*) $\xi(a) \subseteq \xi_c(ab^{-1})$;

(*iii*) $\xi(b) \subseteq \xi_c(ab^{-1})$.

Proof. Suppose that $a_\iota = b_\iota$. Then $(\pi \cup R)^\# = \pi$ and (3.1) is trivially satisfied. Hence, by Lemma 3.1, M is E-unitary.

Now assume that $a_\iota \neq b_\iota$. We prove that

$$M \text{ is E-unitary} \Leftrightarrow \beta(ab^{-1})(\eta \cup R)^\# = [\beta(ab^{-1}).a](\eta \cup R)^\#. \quad (3.2)$$

Suppose that M is E-unitary. Since $a(\pi \cup R)^\# = b(\pi \cup R)^\#$, we have $\beta(ab^{-1})(\pi \cup R)^\# = 1(\pi \cup R)^\#$. Since $\beta(ab^{-1}) \neq 1$, by Lemma 3.1, we have $\beta(ab^{-1})[\pi \cup R(\beta(ab^{-1}))]^\# = 1[\pi \cup R(\beta(ab^{-1}))]^\#$. Hence $R(\beta(ab^{-1})) \neq \emptyset$ and since $|R| = 1$, $R(\beta(ab^{-1})) = R$. By definition, we obtain $\beta(ab^{-1})(\eta \cup R)^\# = [\beta(ab^{-1}).a](\eta \cup R)^\#$.

Conversely, suppose that $\beta(ab^{-1})(\eta \cup R)^\# = [\beta(ab^{-1}).a](\eta \cup R)^\#$. Then $R(\beta(ab^{-1})) = R$. Let $u \in (X \cup X^{-1})^+$ be such that $u(\pi \cup R)^\# = 1(\pi \cup R)^\#$. The Freiheitssatz [14] states that $\xi_c(ab^{-1}) \subseteq \xi(u)$. Therefore $R(\beta(ab^{-1})) \subseteq R(u)$ and so $R(u) = R$ and (3.1) holds. By Lemma 3.1, M is E-unitary. Hence (3.2) holds.

Suppose that (ii) or (iii) holds. Then it is clear that $\beta(ab^{-1})(\eta \cup R)^\# = [\beta(ab^{-1}).a](\eta \cup R)^\#$ and so, by (3.2), M is E-unitary.

Now suppose that neither (ii) nor (iii) holds. Then the $(\eta \cup R)^\#$-class and the η-class of $\beta(ab^{-1})$ coincide and so $\beta(ab^{-1})(\eta \cup R)^\# \neq [\beta(ab^{-1}).a](\eta \cup R)^\#$. By (3.2), M is not E-unitary and the theorem is proved.

The next corollary is immediate.

COROLLARY 3.3. *Let M denote the Clifford monoid defined by the one-relator presentation $Clf\langle X;R\rangle$, with $R = \{(a,1)\}$. Then M is E-unitary.*

The E-unitary problem for one-relator inverse monoid presentations is still open. Margolis and Meakin [19] formulated a conjecture on the subject. The conjecture stated that, for every $u \in R_X$ and $R = \{(u,1)\}$,

$(X \cup X^{-1})^*/(\rho \cup R)^\#$ is E-unitary if and only if u is cyclically reduced. We can provide a counterexample.

Let $X = \{x,y\}$, let $u = xyx^{-2}yxyx^{-1}$ and let $R = \{(u,1)\}$. We prove that $(X \cup X^{-1})^*/(\rho \cup R)^\#$ is a group. It is clear that, for every $w,z \in (X \cup X^{-1})^*$,

$$(wzw)(\rho \cup R)^\# = 1(\rho \cup R)^\# \Rightarrow (ww^{-1})(\rho \cup R)^\# = 1(\rho \cup R)^\# = (w^{-1}w)(\rho \cup R)^\#.$$

Let $v = xyx^{-1}$. Since $u = vx^{-1}yv$, we have $(v^{-1}v)(\rho \cup R)^\# = 1(\rho \cup R)^\#$. Therefore $[x^{-1}(yvxy)x^{-1}](\rho \cup R)^\# = (x^{-1}yv^2)(\rho \cup R)^\# = (v^{-1}vx^{-1}yv^2)(\rho \cup R)^\# = (v^{-1}uv)(\rho \cup R)^\# = (v^{-1}v)(\rho \cup R)^\# = 1(\rho \cup R)^\#$ and so $(x^{-1}x)(\rho \cup R)^\# = 1(\rho \cup R)^\# = (xx^{-1})(\rho \cup R)^\#$. Hence $[y(x^{-2}yx)y](\rho \cup R)^\# = (x^{-1}xyx^{-2}yxyx^{-1}x)(\rho \cup R)^\# = (x^{-1}ux)(\rho \cup R)^\# = (x^{-1}x)(\rho \cup R)^\# = 1(\rho \cup R)^\#$ and so $(yy^{-1})(\rho \cup R)^\# = 1(\rho \cup R)^\# = (y^{-1}y)(\rho \cup R)^\#$. Thus $\{(zz^{-1},1): z \in X \cup X^{-1}\} \subseteq (\rho \cup R)^\#$ and so M is certainly a group, in particular E-unitary.

The question of whether or not $(X \cup X^{-1})^*/(\rho \cup R)^\#$ is E-unitary when $R = \{(u,1)\}$, for u cyclically reduced, is still open.

Now we consider a more general class of presentations.

THEOREM 3.4. *The E-unitary problem for finite Clifford monoid presentations is undecidable.*

Proof. It is well-known that there exists a finite group presentation with undecidable word problem [3],[29]. Let $Gp\langle Z;T\rangle$ be such a presentation, with $T = \{(h_i,1): i \in \{1,\dots,n\}\}$, and let a be a new element, distinct from the elements of Z. Let $h_{n+1} = a^2$. Let $X = Z \cup \{a\}$ and $I = \{1,\dots,n+1\}$. Let $g \in (Z \cup Z^{-1})^*$ and let $w = ag$. Let y denote a new element, distinct from the elements of X. Let $V = X \cup \{y\}$. Let $p,q \in (V \cup V^{-1})^*$ be such that $\xi(p) = X$ and $\xi(q) = V$.

We define a finite relation R on $(V \cup V^{-1})^*$ by

$$R = \{(pp^{-1}h_i,pp^{-1}): i \in I\} \cup \{(qq^{-1},xx^{-1}): x \in X\}$$
$$\cup \{(qq^{-1}y,qq^{-1}w)\}.$$

There are three $(\eta\cup R)^{\#}$-classes in $(V\cup V^{-1})^{*}$: $\{1\}$, $\{u \in (V\cup V^{-1})^{*}: \xi(u) = y\}$ and $\{u \in (V\cup V^{-1})^{*}: X\cap\xi(u) \neq \emptyset\}$. It is clear from the definitions that $R(1) = R(y) = \emptyset$ and $R(u) = R$ for every $u \in (V\cup V^{-1})^{*}$ such that $X\cap\xi(u) \neq \emptyset$.

Hence (3.1) is trivially satisfied by every $u \in (V\cup V^{-1})^{+}$ except possibly when $\xi(u) = \{y\}$. Since $y^{k_{\pi}} \neq 1_{\pi}$ for every $k \in \mathbb{Z}\setminus\{0\}$, $(V\cup V^{-1})^{*}/(\nu\cup R)^{\#}$ is E-unitary if and only if

$$\forall k \in \mathbb{Z}\setminus\{0\},\ y^{k}(\pi\cup R)^{\#} \neq 1(\pi\cup R)^{\#}. \tag{3.3}$$

Since $(qq^{-1}y, qq^{-1}w) \in R$, the presentation $Gp\langle V;R\rangle$ is clearly equivalent (by a Tietze transformation) to the presentation $Gp\langle X;S\rangle$, where $S = \{(h_i,1): i \in I\}$. Moreover, (3.3) is equivalent to

$$\forall k \in \mathbb{Z}\setminus\{0\},\ w^{k}(\pi\cup S)^{\#} \neq 1(\pi\cup S)^{\#}. \tag{3.4}$$

We prove that (3.4) is equivalent to $g(\pi\cup T)^{\#} \neq 1(\pi\cup T)^{\#}$.

Suppose that $g(\pi\cup T)^{\#} = 1(\pi\cup T)^{\#}$. Then $g(\pi\cup S)^{\#} = 1(\pi\cup S)^{\#}$ and so $w(\pi\cup S)^{\#} = a(\pi\cup S)^{\#}$. But $a(\pi\cup S)^{\#}$ has order 2, so (3.4) does not hold.

Now suppose that $g(\pi\cup T)^{\#} \neq 1(\pi\cup T)^{\#}$. By Lemma I.2.6, $(X\cup X^{-1})^{*}/(\pi\cup S)^{\#}$ is the free product (in Gp) of $(Z\cup Z^{-1})^{*}/(\pi\cup T)^{\#}$ and $\{a,a^{-1}\}^{*}/(\pi\cup\{(a^{2},1)\})^{\#}$. Moreover, $(Z\cup Z^{-1})^{*}/(\pi\cup T)^{\#}$ embeds canonically in $(X\cup X^{-1})^{*}/(\pi\cup S)^{\#}$ and so $g(\pi\cup S)^{\#} \neq 1(\pi\cup S)^{\#}$. Similarly, we have $a(\pi\cup S)^{\#} \neq 1(\pi\cup S)^{\#}$ and so $w(\pi\cup S)^{\#}$ is a nonhomogeneous element of a free product of two groups, that is, $w(\pi\cup S)^{\#}$ is not contained in either of the factor groups. Therefore $w(\pi\cup S)^{\#}$ has infinite order and so (3.4) holds.

Thus, decidability of the E-unitary problem for all finite Clifford monoid presentations would imply decidability of the word problem for $Gp\langle Z;T\rangle$. The result follows.

COROLLARY 3.5. *The E-unitary problem for finite inverse monoid presentations is undecidable.*

Proof. This follows from Theorem 3.4 and Lemma 1.3(i).

4. Other decidability results

The results of Section 2 provide some general positive answers. In contrast, other problems turn out to be undecidable, as a consequence of analogous results on group presentations.

The following result is a corollary of Lemmas 2.1 and 2.2.

THEOREM 4.1. *The idempotent word problem is decidable for every finitely related Clifford monoid presentation.*

This enables us to decide whether or not a finitely presented Clifford monoid is a group.

COROLLARY 4.2. *The group problem for finite Clifford monoid presentations is decidable.*

Proof. Let M be the Clifford monoid defined by the finite presentation $Clf\langle X;R\rangle$. Then M is a group if and only if

$$\forall x \in X\cup X^{-1},\ (xx^{-1})(\nu\cup R)^{\#} = 1(\nu\cup R)^{\#}.$$

Since X is finite, all we need is to apply Theorem 4.1 finitely many times.

It is known to be undecidable whether or not finitely presented groups are trivial (or finite) [1],[35]. This yields some analogous results:

THEOREM 4.3. *It is undecidable whether or not finitely presented Clifford monoids are*

(i) *trivial,*

(ii) *finite,*

(iii) *semilattices,*

(iv) *free.*

Proof. (i) and (ii) follow from Lemma 1.3(ii).

Since a group is a semilattice if and only if it is trivial, the same applies to (iii).

Now we prove (iv). Let G be a group, defined as a Clifford monoid by a finite presentation $Clf\langle X;R\rangle$. Let Y be a finite nonempty set, disjoint from X, and let $Z = X\cup Y$. The group of units of a free Clifford monoid is always trivial: in fact, if $(uu^{-1})\nu = 1\nu$, then, by (1.2), $\xi(uu^{-1}) = \emptyset$, that is, $u = 1$. We prove that $(Z\cup Z^{-1})^*/(\nu\cup R)^\#$ is free if and only if G is trivial.

Suppose that $(Z\cup Z^{-1})^*/(\nu\cup R)^\#$ is free. By Lemma I.2.6, $G = (X\cup X^{-1})^*/(\nu\cup R)^\#$ embeds canonically in $(Z\cup Z^{-1})^*/(\nu\cup R)^\#$. Hence G embeds in the group of units of $(Z\cup Z^{-1})^*/(\nu\cup R)^\#$, which is trivial. Therefore G is trivial.

Conversely, suppose that G is trivial. Then, by Lemma I.2.6, $(Z\cup Z^{-1})^*/(\nu\cup R)^\#$ is isomorphic to $(Y\cup Y^{-1})^*/\nu$ and so it is free.

Since it is not decidable whether or not G is trivial, the theorem follows.

5. E-reflexive inverse monoids

In this section we discuss the word problem for inverse monoid presentations which define E-reflexive inverse monoids.

Let M be an inverse monoid. We say that M is *E-reflexive* if

$$\forall a,b \in M \quad \forall e \in E(M),\ aeb \in E(M) \Rightarrow bea \in E(M).$$

This concept was introduced by O'Carroll [31], who used the expression *strongly E-reflexive*.

Let τ be a congruence on the inverse monoid M. We say that τ is a *Clifford* congruence if M/τ is a Clifford monoid. It is easy to see that the intersection of all Clifford congruences on M is still a Clifford congruence on M. We denote it by ν_M and we say it is the *least Clifford congruence* on M.

LEMMA 5.1 [34,§III.8]. *Let M be an inverse monoid. Then the following propositions are equivalent.*

(i) M is E-reflexive;

(ii) M is a semilattice of E-unitary inverse semigroups;

(iii) ν_M is idempotent-pure.

In order to apply Lemma 5.1, we need a description of ν_M in terms of presentations.

LEMMA 5.2. *Let X be a nonempty set and let R be a relation on $(X\cup X^{-1})^*$. Let $M = (X\cup X^{-1})^*/(\rho\cup R)^\#$ and let $\varphi:M \longrightarrow (X\cup X^{-1})^*/(\nu\cup R)^\#$ be defined by $[w(\rho\cup R)^\#]\varphi = w(\nu\cup R)^\#$. Then $Ker\varphi = \nu_M$.*

Proof. By definition, $Ker\varphi = (\nu\cup R)^\#/(\rho\cup R)^\#$. By Lemma I.1.1, $M/Ker\varphi \simeq (X\cup X^{-1})^*/(\nu\cup R)^\#$ and so $Ker\varphi$ is a Clifford congruence.

Let τ be a Clifford congruence on $(X\cup X^{-1})^*/(\rho\cup R)^\#$. Let $\overline{\tau}$ be the congruence on $(X\cup X^{-1})^*$ defined by

$$(u,v) \in \overline{\tau} \iff (u(\rho\cup R)^{\#}, v(\rho\cup R)^{\#}) \in \tau.$$

It is immediate that $R \subseteq \overline{\tau}$. Since τ is a Clifford congruence, we have $[(uu^{-1})(\rho\cup R)^{\#}]\tau = [(u^{-1}u)(\rho\cup R)^{\#}]\tau$ for every $u \in (X\cup X^{-1})^*$. Hence $(uu^{-1}, u^{-1}u) \in \overline{\tau}$ for every $u \in (X\cup X^{-1})^*$ and so $\nu \subseteq \overline{\tau}$. Thus $(\nu\cup R)^{\#} \subseteq \overline{\tau}$ and so $Ker\varphi = (\nu\cup R)^{\#}/(\rho\cup R)^{\#} \subseteq \overline{\tau}/(\rho\cup R)^{\#} = \tau$. Therefore $Ker\varphi = \nu_M$.

Using the notation of Section 2, we obtain

LEMMA 5.3. *Let $Inv\langle X;R\rangle$ be a presentation such that $(X\cup X^{-1})^*/(\rho\cup R)^{\#}$ is E-reflexive. Let $u,v \in (X\cup X^{-1})^*$. Then $u(\rho\cup R)^{\#} = v(\rho\cup R)^{\#}$ if and only if*

(i) $(u^{-1}u)(\rho\cup R)^{\#} = (v^{-1}v)(\rho\cup R)^{\#}$;

(ii) $(uu^{-1})(\rho\cup R)^{\#} = (vu^{-1}uv^{-1})(\rho\cup R)^{\#}$;

(iii) $(vu^{-1})[\pi\cup R(vu^{-1})]^{\#} = 1[\pi\cup R(vu^{-1})]^{\#}$.

Proof. Consider the condition

(iii)' $(vu^{-1})(\rho\cup R)^{\#} = (vu^{-1}uv^{-1})(\rho\cup R)^{\#}$.

We prove that $u(\rho\cup R)^{\#} = v(\rho\cup R)^{\#}$ holds if and only if (i),(ii) and (iii)' hold. It is immediate that $u(\rho\cup R)^{\#} = v(\rho\cup R)^{\#}$ implies (i),(ii) and (iii)'. Conversely, suppose that (i),(ii) and (iii)' hold. Then $u(\rho\cup R)^{\#} = (uu^{-1}u)(\rho\cup R)^{\#} = (vu^{-1}uv^{-1}u)(\rho\cup R)^{\#} = (vu^{-1}u)(\rho\cup R)^{\#} = (vv^{-1}v)(\rho\cup R)^{\#} = v(\rho\cup R)^{\#}$.

Thus we only need to prove that (iii)' is equivalent to (iii).

Suppose that (iii)' holds. Since $(\rho\cup R)^{\#} \subseteq (\nu\cup R)^{\#}$, it follows that $(vu^{-1})(\nu\cup R)^{\#} = (vu^{-1}uv^{-1})(\nu\cup R)^{\#}$. Now, by Lemma 2.2, we have $(vu^{-1})[\pi\cup R(vu^{-1})]^{\#} = (vu^{-1}uv^{-1})[\pi\cup R(vu^{-1})]^{\#}$. Since $(X\cup X^{-1})^*/[\pi\cup R(vu^{-1})]^{\#}$ is a group, we have $(vu^{-1}uv^{-1})[\pi\cup R(vu^{-1})]^{\#} = 1[\pi\cup R(vu^{-1})]^{\#}$ and so (iii) holds.

Conversely, suppose that (iii) holds. Then $(vu^{-1})[\pi\cup R(vu^{-1})]^{\#} = (vu^{-1}uv^{-1})[\pi\cup R(vu^{-1})]^{\#}$. Since $(X\cup X^{-1})^{*}/(\eta\cup R)^{\#}$ is a semilattice, we have $(vu^{-1})(\eta\cup R)^{\#} = (vu^{-1}uv^{-1})(\eta\cup R)^{\#}$ as well. By Lemma 2.2, it follows that $(vu^{-1})(\nu\cup R)^{\#} = (vu^{-1}uv^{-1})(\nu\cup R)^{\#}$. Hence $[(vu^{-1})(\rho\cup R)^{\#}][(\nu\cup R)^{\#}/(\rho\cup R)^{\#}] = [(vu^{-1}uv^{-1})(\rho\cup R)^{\#}][(\nu\cup R)^{\#}/(\rho\cup R)^{\#}]$. Since $(X\cup X^{-1})^{*}/(\rho\cup R)^{\#}$ is E-reflexive, we have that $(\nu\cup R)^{\#}/(\rho\cup R)^{\#}$ is idempotent-pure, by Lemmas 5.1 and 5.2. Therefore $(vu^{-1})(\rho\cup R)^{\#}$ is idempotent and so (iii)' holds. Thus (iii) is equivalent to (iii)' and the lemma is proved.

THEOREM 5.4. *Let* $Inv\langle X;R\rangle$ *be a finite presentation such that* $(X\cup X^{-1})^{*}/(\rho\cup R)^{\#}$ *is E-reflexive. Then* $K(R)$ *can be effectively determined and the word problem for* $Inv\langle X;R\rangle$ *is decidable if*

(*i*) *the idempotent word problem for* $Inv\langle X;R\rangle$ *is decidable*;

(*ii*) *the word problem for* $Gp\langle X;K\rangle$ *is decidable for every* $K \in K(R)$.

CHAPTER VI

INVERSE MONOID PRESENTATIONS

1. Preliminaries

Let Σ denote a finite nonempty set. A subset $L \subseteq \Sigma^*$ is said to be a Σ-*language*.

The quadruple $A = (Q,I,T,E)$ is said to be a Σ-*automaton* if Q is a nonempty set, I and T are subsets of Q, and $E \subseteq Q \times \Sigma \times Q$. We say that A is *finite* if Q is finite. We say that A is *deterministic* if $|I| = 1$ and

$$(q,\sigma,q'),(q,\sigma,q'') \in E \Rightarrow q' = q''.$$

We can describe A graphically: each element of Q labels a vertex; each $(q,\sigma,q') \in E$ corresponds to an edge oriented from q to q' and labelled by σ; the vertices corresponding to the elements of I (respectively T) are identified by an input sign (respectively output sign). Two Σ-automata are said to be *isomorphic* if their graphical description coincides, up to labelling of vertices.

A *nontrivial path* in A is a finite nonempty sequence on E of the form $(q_0,\sigma_1,q_1),(q_1,\sigma_2,q_2),\ldots,(q_{n-1},\sigma_n,q_n)$. The *label* of such a path is $\sigma_1\ldots\sigma_n$. A *trivial path* in A is a triple $(q,1,q)$, with $q \in Q$. The *label* of such a path is 1. The above nontrivial (respectively trivial) path is said to be *successful* if $q_0 \in I$ and $q_n \in T$ (respectively $q \in I \cap T$).

The *language* accepted by A is

$$L(A) = \{w \in \Sigma^*: w \text{ labels a successful path in } A\}.$$

We say that a Σ-language L is *rational* if $L = L(A)$ for some finite Σ-automaton A. Clearly, all finite Σ-languages are rational, as well as Σ^* or $\emptyset$.

We say that A is *trim* if every $q \in Q$ lies in some successful path of A.

LEMMA 1.1 [7,§III.2]. *Let L be a rational Σ-language. Then there exists a finite trim deterministic Σ-automaton $A = (Q,\{i\},T,E)$ such that $L = L(A)$.*

Now let $A = (Q,\{i\},T,E)$ be a trim deterministic Σ-automaton. For every $q \in Q$, we define $A_q = (Q,\{i\},\{q\},E)$ and $A_{(q)} = (Q,\{q\},T,E)$. Consider the equivalence relation ν on Q given by $(q,q') \in \nu$ if and only if $L[A_{(q)}] = L[A_{(q')}]$. We define $A_{min} = (Q/\nu,\{i\nu\},T\nu,E\nu)$, where $T\nu = \{t\nu: t \in T\}$ and $E\nu = \{(q\nu,\sigma,q'\nu): (q,\sigma,q') \in E\}$.

LEMMA 1.2 [7,§III.5]. *Let $A = (Q,\{i\},T,E)$ be a trim deterministic Σ-automaton. Then*

(i) *A_{min} is a trim deterministic Σ-automaton;*

(ii) *$L(A) = L(A_{min})$;*

(iii) *if B is a trim deterministic Σ-automaton such that $L(B) = L(A)$, then A_{min} and B_{min} are isomorphic.*

The automaton A_{min} is said to be the *minimal automaton* of $L(A)$.

We now state some well-known results on finite Σ-automata.

LEMMA 1.3 [10,§9.2]. *Let A_1 and A_2 be finite Σ-automata. Then we can produce finite Σ-automata accepting the languages $L(A_1) \cap L(A_2)$, $L(A_1).L(A_2)$ and $L(A_1) \setminus L(A_2)$.*

LEMMA 1.4 [10,§14.7]. *Given two finite Σ-automata A_1 and A_2, it is decidable whether or not $L(A_1) \subseteq L(A_2)$.*

Now we introduce a more general class of Σ-languages.

The quintuple $B = (Q,i,\Gamma,s,E)$ is said to be a *pushdown Σ-automaton* if: Q and Γ are finite nonempty sets; $i \in Q$; $s \in \Gamma$; $E \subseteq Q \times (\Sigma \cup \{1\}) \times \Gamma \times Q \times \Gamma^*$ is finite.

Let $q,q' \in Q$. Let $z \in \Gamma^+$ and $z' \in \Gamma^*$. Let y denote the first letter of z and suppose that $z = yc$. Let $\sigma \in \Sigma \cup \{1\}$. If $(q,\sigma,y,q',z') \in E$, we write σ: $(q,z) \vdash (q',z'c)$.

Suppose that $\sigma_1,\ldots,\sigma_n \in \Sigma \cup \{1\}$; $q_0,\ldots,q_n \in Q$; $z_0,\ldots,z_{n-1} \in \Gamma^+$; $z_n \in \Gamma^*$; for every $j \in \{1,\ldots,n\}$, we have $\sigma_j:(q_{j-1},z_{j-1}) \vdash (q_j,z_j)$. Then we say that $\sigma_1\ldots\sigma_n:(q_0,z_0) \overset{*}{\vdash} (q_n,z_n)$. We define the *language accepted by* B as

$$L(B) = \{w \in \Sigma^*: w:(i,s) \overset{*}{\vdash} (q,1) \text{ for some } q \in Q\}.$$

A Σ-language L is said to be *context-free* if $L = L(B)$ for some pushdown Σ-automaton B. It is well-known that rational languages are context-free [10,§2.3].

LEMMA 1.5 [10,§9.2]. *Let A_1 and A_2 be pushdown Σ-automata. Then we can produce a pushdown Σ-automaton accepting $L(A_1) \cup L(A_2)$.*

LEMMA 1.6 [10,§9.2]. *Let A and B be respectively a pushdown Σ-automaton and a finite Σ-automaton. Then we can produce a pushdown Σ-automaton accepting $L(A) \cap L(B)$.*

LEMMA 1.7 [10,§9.3]. *Let A be a pushdown Σ-automaton. Let Δ be a finite nonempty set and let $\varphi:\Sigma^* \longrightarrow \Delta^*$ be a monoid homomorphism. Then we can produce a pushdown Δ-automaton accepting $L\varphi$.*

LEMMA 1.8 [10,§4.1]. *Given a pushdown Σ-automaton B, it is decidable whether or not $L(B)$ is empty.*

Now let X denote a nonempty set. *For the remainder of this chapter, we assume that X is finite.* Let L be an $(X\cup X^{-1})$-language. Then L is said to be *reduced* if and only if $L \subseteq R_X$.

Since $R_X = (X\cup X^{-1})^*/[\bigcup_{x\in X\cup X^{-1}}(X\cup X^{-1})^*xx^{-1}(X\cup X^{-1})^*]$, Lemma 1.3 yields

LEMMA 1.9. *If X is finite, then R_X is a rational $(X\cup X^{-1})$-language.*

For every $n \in \mathbb{N}^0$, we define

$$F_n = \{u \in R_X: |u| = n\},\ F_{(n)} = \{u \in R_X: |u| \leq n\}.$$

Now let $A = (Q,I,T,E)$ be a trim deterministic $(X\cup X^{-1})$-automaton. Let $E^{-1} = \{(q',x^{-1},q): (q,x,q') \in E\}$. Then A is said to be *inverse* if and only if $|I| = |T| = 1$ and $E = E^{-1}$. If A is inverse and the graph of A is a tree, we say that A is an *inverse tree automaton*.

Let A_X denote the class of all inverse tree $(X\cup X^{-1})$-automata of the form $A = (Q,\{i\},\{i\},E)$. Let S_X denote the set of all nonempty left closed subsets of R_X. We define a map $\Upsilon:A_X \longrightarrow S_X$ by

$$\Upsilon(A) = \{u \in R_X: uu^{-1} \in L(A)\}.$$

LEMMA 1.10. *Let $A \in A_X$. Then*

$$L(A) = \{u \in D_X: v\iota \in \Upsilon(A) \text{ for every } v \leq_I u\}.$$

Proof. Let $A = (Q,\{i\},\{i\},E)$. Let $u \in L(A)$. Since A is inverse, we have $u\iota \in L(A)$, and so, since A is a tree automaton, we must have $u\iota = 1$. Hence $u \in D_X$. Suppose that $v \leq_I u$. Then v labels a path in A beginning at i and since A is inverse, so does $v\iota$. Hence $v\iota \in \Upsilon(A)$.

Conversely, suppose that $u \in D_X$ is such that $v\iota \in \Upsilon(A)$ for every $v \leq_I u$. We can assume that $u \neq 1$. Suppose that $u = x_1 \ldots x_n$, $x_j \in X \cup X^{-1}$. We prove that, for every $j \in \{1,\ldots,n\}$, $x_1 \ldots x_j$ labels a path in A, beginning at i.

Since $x_1 \in \Upsilon(A)$, this is certainly true for $j = 1$. Suppose that it is true for j, with $1 \leq j < n$. Then $x_1 \ldots x_j$ labels a path from i to some $q \in Q$, and so does $v = (x_1 \ldots x_j)\iota$. Suppose that $vx_{j+1} \in R_X$. Then $vx_{j+1} = (x_1 \ldots x_{j+1})\iota \in \Upsilon(A)$. Now suppose that $vx_{j+1} \notin R_X$. Then $(vx_{j+1})\iota <_I v \in \Upsilon(A)$. In either case, it follows that vx_{j+1} labels a path in A, beginning at i, and so does $x_1 \ldots x_j x_{j+1}$. By induction, it follows that u labels a path α in A, beginning at i. Since $u\iota$ labels a trivial path at i, it follows that the terminal point of α must be i as well. Hence $u \in L(A)$.

LEMMA 1.11.

(i) Υ is surjective;

(ii) for every $A,B \in A_X$,

$$\Upsilon(A) = \Upsilon(B) \iff A \text{ and } B \text{ are isomorphic.}$$

Proof. (i) Let $W \in S_X$. We define $A = (W,\{1\},\{1\},E \cup E^{-1})$, with $E = \{(w,x,w') \in W \times (X \cup X^{-1}) \times W : w' = wx\}$. It follows easily that $A \in A_X$ and $\Upsilon(A) = W$.

(ii) Let $A,B \in A_X$. Suppose that $\Upsilon(A) = \Upsilon(B)$. Let $u \in L(A)$. By Lemma 1.10, we have $u \in D_X$ and $v\iota \in \Upsilon(A)$ for every $v \leq_I u$. Since $\Upsilon(A) = \Upsilon(B)$, we have $u \in L(B)$ and so $L(A) \subseteq L(B)$. Similarly,

$L(B) \subseteq L(A)$ and so $L(A) = L(B)$.

It is immediate that $A = A_{min}$ and $B = B_{min}$. Therefore, by Lemma 1.2, we have that A and B are isomorphic.

The converse implication is trivial.

Considering the operation $(W,W') \mapsto W \cup W'$ on S_X, we can now define a multiplication on A_X, up to isomorphism. For every $A,B \in A_X$, we define AB by $\Upsilon(AB) = \Upsilon(A) \cup \Upsilon(B)$.

For every $u \in (X \cup X^{-1})^*$, we can consider

$$MT(u) = (Q(u), \{1\}, \{u\iota\}, E(u) \cup [E(u)]^{-1})$$

as an inverse tree automaton. It follows easily that

LEMMA 1.12 [26]. *For every* $u \in (X \cup X^{-1})^*$,

$$L[MT(u)] = \{v \in (X \cup X^{-1})^* : v\rho \geq u\rho\}.$$

Now assume that P is a finite relation on $(X \cup X^{-1})^*$ and let $e \in D_X$. Following the construction of Stephen [39], adapted by Meakin and Margolis [18], we define a sequence of finite inverse tree automata $A_{P,k}(e)$, $k \in \mathbb{N}$. We denote $\Upsilon[A_{P,k}(e)]$ by $W_{P,k}(e)$.

Let $A_{P,1}(e) = MT(e)$. Suppose that $A_{P,k}(e)$ is defined for some $k \in \mathbb{N}$. We can give the following intuitive description of $A_{P,k+1}(e)$, from a geometric point of view.

We consider all the instances of vertices q of $A_{P,k}(e)$, $(a,b) \in P \cup P^{-1}$ and $f \in D_X$ such that the tree $MT(afa^{-1})$ embeds in $A_{P,k}(e)$ at q. Then we define $A_{P,k+1}(e)$ to be the inverse tree automaton obtained by adjoining the tree $MT(bfb^{-1})$ to $A_{P,k}(e)$ at q for all such instances.

In a more algebraic perspective, we define

$$W_{P,k+1}(e) = W_{P,k}(e)\cup\{[q.Q(bf)]\iota : q \in W_{P,k}(e),\ f \in D_X,$$
$$(a,b) \in P\cup P^{-1} \text{ and } [q.Q(af)]\iota \subseteq W_{P,k}(e)\}$$

and we define $A_{P,k+1}(e)$ by $\Upsilon[A_{P,k+1}(e)] = W_{P,k+1}(e)$.

Finally, we define $W_P(e) = \bigcup_{k\geq 1} W_{P,k}(e)$ and we define $A_P(e)$ by $\Upsilon[A_P(e)] = W_P(e)$. It follows from Lemma 1.10 that $L[A_P(e)] = \bigcup_{k\geq 1} L[A_{P,k}(e)]$.

For every $e,f \in D_X$, we have

$$e(\rho\cup P)^{\#} = f(\rho\cup P)^{\#} \iff e(\rho\cup P_D)^{\#} = f(\rho\cup P_D)^{\#}, \qquad (1.1)$$

by Lemma I.3.9. The next result follows easily from [18] and [39], but we give a proof for completeness.

LEMMA 1.13. *Let P be a finite relation on $(X\cup X^{-1})^*$ and let $e \in D_X$. Then*

$$W_P(e) = \{u \in R_X : (uu^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}\}.$$

Proof. We prove that, for every $j \in \mathbb{N}$,

$$\forall u \in W_{P,j}(e),\ (uu^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}. \qquad (1.2)$$

Let $u \in W_{P,1}(e)$. Since $W_{P,1}(e) = Q(e)$, we have $(uu^{-1})\rho \geq e\rho$, by Lemma 1.12. Hence $(uu^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}$ and so (1.2) holds for $j = 1$.

Now suppose that (1.2) holds for $j = k$, with $k \in \mathbb{N}$. Let $u \in W_{P,k+1}(e)$. We can assume that $u \notin W_{P,k}(e)$. Then there exist $q \in W_{P,k}(e)$, $(a,b) \in P\cup P^{-1}$ and $f \in D_X$ such that $[q.Q(af)]\iota \subseteq W_{P,k}(e)$ and $u \in [q.Q(bf)]\iota$. Since $Q(qafa^{-1}q^{-1}) = Q(qaf) = Q(q)\cup[q.Q(af)]\iota$, we have $Q(qafa^{-1}q^{-1}) \subseteq W_{P,k}(e)$.

By Lemma I.3.3, we have $(qafa^{-1}q^{-1})\rho = \prod_{v\in Q(qaf)} (vv^{-1})\rho$ and so $(qafa^{-1}q^{-1})(\rho\cup P_D)^{\#} = \prod_{v\in Q(qaf)} (vv^{-1})(\rho\cup P_D)^{\#}$. By hypothesis, we have $(vv^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}$ for every $v \in Q(qaf)$ and so $(qafa^{-1}q^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}$. Moreover,

$(qafa^{-1}q^{-1})(\rho\cup P_D)^{\#} = (qbfb^{-1}q^{-1})(\rho\cup P_D)^{\#}$ and $u \in [q.Q(bf)]\iota \subseteq Q(qbfb^{-1}q^{-1})$. By Lemma 1.12, we obtain $(uu^{-1})\rho \geq (qbfb^{-1}q^{-1})\rho$ and so $(uu^{-1})(\rho\cup P_D)^{\#} \geq (qbfb^{-1}q^{-1})(\rho\cup P_D)^{\#} = (qafa^{-1}q^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}$. Hence (1.2) holds for $j = k+1$ and so for every $j \in \mathbb{N}$. Thus $W_P(e) \subseteq \{u \in R_X\colon (uu^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}\}$.

Conversely, suppose that $u \in R_X$ and $(uu^{-1})(\rho\cup P_D)^{\#} \geq e(\rho\cup P_D)^{\#}$. Then $uu^{-1}e(\rho\cup P_D)^{\#} = e(\rho\cup P_D)^{\#}$ and so there exist $w_0,\ldots,w_n \in (X\cup X^{-1})^*$ such that

$$w_0 = uu^{-1}e;$$

$$w_n = e;$$

$$\forall j \in \{1,\ldots,n\}\ \exists r_j,s_j \in (X\cup X^{-1})^*\ \exists (a_j,b_j) \in \rho\cup P_D\colon$$
$$\{w_{j-1},w_j\} = \{r_ja_jb_j, r_jb_js_j\}.$$

By Lemma I.3.9, P_D is idempotent-pure and so $w_j \in D_X$ for every $j \in \{0,\ldots,n\}$. We prove that, for every $j \in \{0,\ldots,n\}$,

$$u \in W_{P,j+1}(w_j). \qquad (1.3)$$

Since $W_{P,1}(w_0) = Q(uu^{-1}e)$, (1.3) holds for $j = 0$.

Assume that (1.3) holds for $j = k$, with $0 \leq k < n$. Suppose first that $(a_{k+1},b_{k+1}) \in \rho$. Then $w_{k+1}\rho = w_k\rho$ and so $W_{P,1}(w_{k+1}) = W_{P,1}(w_k)$. Hence $W_{P,k+1}(w_{k+1}) = W_{P,k+1}(w_k)$. Therefore $u \in W_{P,k+1}(w_{k+1}) \subseteq W_{P,k+2}(w_{k+1})$ and (1.3) holds for $j = k+1$.

Now suppose that $(a_{k+1},b_{k+1}) \in P_D$. Then there exist $(a,b) \in P$ and $f \in D_X$ such that $a_{k+1} = afa^{-1}$ and $b_{k+1} = bfb^{-1}$. Without loss of generality, we can assume that $w_k = r_{k+1}afa^{-1}s_{k+1}$ and $w_{k+1} = r_{k+1}bfb^{-1}s_{k+1}$. Since $[r_{k+1}.Q(bf)]\iota \subseteq Q(w_{k+1}) = W_{P,1}(w_{k+1})$, we have $[r_{k+1}.Q(af)]\iota \subseteq W_{P,2}(w_{k+1})$. We have $Q(w_k)$
$= Q(r_{k+1}afa^{-1})\cup[r_{k+1}afa^{-1}.Q(s_{k+1})]\iota$
$= Q(r_{k+1})\cup[r_{k+1}.Q(afa^{-1})]\iota\cup[r_{k+1}.Q(s_{k+1})]\iota$
$= Q(r_{k+1})\cup[r_{k+1}.Q(af)]\iota\cup[r_{k+1}.Q(s_{k+1})]\iota$. Similarly, $Q(w_{k+1}) = Q(r_{k+1})\cup[r_{k+1}.Q(bf)]\iota\cup[r_{k+1}.Q(s_{k+1})]\iota$ and so

$W_{P,1}(w_k) = Q(w_k) \subseteq W_{P,2}(w_{k+1})$. Hence $W_{P,k+1}(w_k) \subseteq W_{P,k+2}(w_{k+1})$ and so, by the induction hypothesis, we have $u \in W_{P,k+2}(w_{k+1})$. Therefore (1.3) holds for $j = k+1$ and so for every $j \in \{0,\ldots,n\}$. In particular, $u \in W_{P,n+1}(w_n) \subseteq W_P(e)$ and so the lemma is proved.

LEMMA 1.14. *Let P be a finite relation on $(X \cup X^{-1})^*$ and let $e,f \in D_X$. Then*

$$e(\rho \cup P_D)^\# = f(\rho \cup P_D)^\# \iff W_P(e) = W_P(f).$$

Proof. The direct implication follows immediately from Lemma 1.13.

Now suppose that $W_P(e) = W_P(f)$. Since $e\rho = \prod_{u \in Q(e)} (uu^{-1})\rho$ and $Q(e) \subseteq W_P(e) = W_P(f)$, we have, by Lemma 1.13,
$e(\rho \cup P_D)^\# = \prod_{u \in Q(e)} (uu^{-1})(\rho \cup P_D)^\# \geqslant f(\rho \cup P_D)^\#$. Similarly, $f(\rho \cup P_D)^\# \geqslant e(\rho \cup P_D)^\#$. Hence $e(\rho \cup P_D)^\# = f(\rho \cup P_D)^\#$ and the lemma is proved.

The next result follows from the definition of $W_P(e)$.

LEMMA 1.15. *Let P be a finite relation on $(X \cup X^{-1})^*$ and let $e \in D_X$. Then $W_P(e)$ is the smallest $(X \cup X^{-1})$-language W such that*

(1) $W \subseteq R_X$;

(2) *W is left closed*;

(3) $Q(e) \subseteq W$;

(4) $\forall w \in W\ \forall (a,b) \in P \cup P^{-1}\ \forall f \in D_X$,

$$[w.Q(af)]\iota \subseteq W \Rightarrow [w.Q(bf)]\iota \subseteq W.$$

We can replace condition (4) by a pair of conditions each with fewer quantifiers. Let $\gamma(X,P,e)$ be the set of all $(X \cup X^{-1})$-languages satisfying

(i) $W \subseteq R_X$;

(ii) W is left closed;

(iii) $Q(e) \subseteq W$;

(iv) $\forall w \in W\ \forall(a,b) \in P\cup P^{-1}$, $[w.Q(a)]\iota \subseteq W \Rightarrow [w.Q(b)]\iota \subseteq W$;

(v) $\forall w \in W\ \forall(a,b) \in P\cup P^{-1}$, $[w.Q(a)]\iota \subseteq W \Rightarrow (wba^{-1}w^{-1}W)\iota \subseteq W$.

LEMMA 1.16. $W_P(e) = \cap\{W\colon W \in \gamma(X,P,e)\}$.

Proof. By Lemma 1.15, it is enough to show that conditions (i)-(v) are equivalent to conditions (1)-(4). Let W be an $(X\cup X^{-1})$-language satisfying (i)-(iii). We must prove that W satisfies (4) if and only if W satisfies (iv) and (v).

Assume that W satisfies (4). Suppose that $[w.Q(a)]\iota \subseteq W$ for some $w \in W$ and $(a,b) \in P\cup P^{-1}$. Let $w' \in W$ and let $f = a^{-1}w^{-1}w'w'^{-1}wa$. Then we have $Q(af) = Q(aa^{-1}w^{-1}w'w'^{-1}wa) = Q(a)\cup[w^{-1}Q(w)]\iota\cup[w^{-1}Q(w')]\iota$ and so $[w.Q(af)]\iota = [w.Q(a)]\iota\cup Q(w)\cup Q(w') \subseteq W$. Thus, by (4), we have $[w.Q(bf)]\iota \subseteq W$. But $Q(bf) \supseteq Q(b)\cup\{(ba^{-1}w^{-1}w')\iota\}$ and so $[w.Q(b)]\iota\cup\{(wba^{-1}w^{-1}w')\iota\} \subseteq W$. Hence W satisfies (iv) and (v).

Now suppose that W satisfies (iv) and (v). Let $w \in W$, let $(a,b) \in P\cup P^{-1}$ and let $f \in D_X$. Suppose that $[w.Q(af)]\iota \subseteq W$. We want to prove that $[w.Q(bf)]\iota \subseteq W$ and we have $Q(bf) = Q(b)\cup[b.Q(f)]\iota$. Since $Q(a) \subseteq Q(af)$, we have $[w.Q(a)]\iota \subseteq W$ and so $[w.Q(b)]\iota \subseteq W$, by (iv). Since $[a.Q(f)]\iota \subseteq Q(af)$, we have $[wa.Q(f)]\iota \subseteq W$. Let $v \in Q(f)$. Then $(wbv)\iota = (wba^{-1}w^{-1}wav)\iota$. Since $(wav)\iota \in W$, we have $(wbv)\iota \in W$, by (v). Hence $[wb.Q(f)]\iota \subseteq W$ and so $[w.Q(bf)]\iota \subseteq W$. Thus W satisfies (4) and the lemma is proved.

2. A result on rational languages

In this section we will prove that it is decidable whether or not a rational $(X \cup X^{-1})$-language W belongs to $\gamma(X,P,e)$.

We need a few more preliminary results.

LEMMA 2.1 [18]. *Let A be a finite $(X \cup X^{-1})$-automaton. Then we can produce a finite $(X \cup X^{-1})$-automaton accepting $[L(A)]\iota$.*

LEMMA 2.2. *Let A be a finite $(X \cup X^{-1})$-automaton. Let $[L(A)]^{-1} = \{w^{-1}: w \in L(A)\}$. Then we can produce a finite $(X \cup X^{-1})$-automaton accepting $[L(A)]^{-1}$.*

Proof. Let $A = (Q,I,T,E)$. Let $A^{-1} = (Q,T,I,E^{-1})$. It follows easily that $L(A^{-1}) = [L(A)]^{-1}$.

LEMMA 2.3. *Let A be a finite $(X \cup X^{-1})$-automaton and let $c \in (X \cup X^{-1})^*$. Let $L = \{wcw^{-1}: w \in L(A)\}$. Then we can produce a pushdown $(X \cup X^{-1})$-automaton accepting L.*

Proof. Let $y \notin X \cup X^{-1}$. Let $\varphi: (X \cup X^{-1} \cup \{y\})^* \longrightarrow (X \cup X^{-1})^*$ be the monoid homomorphism defined by $y\varphi = c$ and $x\varphi = x$, $x \in X \cup X^{-1}$. By Lemma 1.7, we only have to produce a pushdown $(X \cup X^{-1})$-automaton accepting $L' = \{wyw^{-1}: w \in L(A)\}$.

Suppose that $A = (Q,\{i\},T,E)$. Let p be a symbol not in Q and let s be a symbol not in $X \cup X^{-1}$. We define a pushdown $(X \cup X^{-1} \cup \{y\})$-automaton $B = (Q \cup \{p\}, i, X \cup X^{-1} \cup \{s\}, s, E')$, where

$E' = \{(q,x,z,q',zx): (q,x,q') \in E,\ x \in X \cup X^{-1}$ and $z \in X \cup X^{-1} \cup \{s\}\}$
$\cup \{(t,y,z,p,z): t \in T,\ z \in X \cup X^{-1} \cup \{s\}\} \cup \{(p,x^{-1},x,p,1): x \in X \cup X^{-1}\}$
$\cup \{(p,1,s,p,1)\}$.

It is easy to see that $L' \subseteq L(B)$. Conversely, suppose that $u \in L(B)$. Then there exist $x_1,\ldots,x_n \in X \cup X^{-1} \cup \{1,y\}$; $q_0,\ldots,q_n \in Q$; $z_0,\ldots,z_n \in (X \cup X^{-1} \cup \{s\})^*$ such that

$$x_1 \ldots x_n = u;$$

$$q_0 = i;$$

$$z_0 = s;$$

$$z_n = 1;$$

$$\forall j \in \{1,\ldots,n\},\ x_j : (q_{j-1}, z_{j-1}) \vdash (q_j, z_j).$$

It follows easily that we must have $q_n = p$. Let $k = min\{j \in \{1,\ldots,n\}: q_j = p\}$. Then it is not difficult to see successively that

$$q_{k-1} \in T;$$

$$x_1 \ldots x_{k-1} \in W;$$

$$x_k = y;$$

$$z_{k-1} = z_k = sx_1 \ldots x_{k-1};$$

$$x_{k+1} \ldots x_{n-1} = x_{k-1}^{-1} \ldots x_1^{-1};$$

$$x_n = 1.$$

Hence $u \in L'$ and the lemma is proved.

LEMMA 2.4. *Let $A = (Q,\{i\},T,E)$ be a finite $(X \cup X^{-1})$-automaton. Let $c_0 \in (X \cup X^{-1})^*$. Let $L = \{(wc_0w^{-1})\iota : w \in L(A)\}$. Then we can produce a pushdown $(X \cup X^{-1})$-automaton accepting L.*

Proof. Let $W = L(A)$. By Lemma 2.1, we can assume that both $L(A)$ and c_0 are reduced. Moreover, we can assume that c_0 is cyclically reduced. In fact, suppose that $c_0 = uvu^{-1}$, with v cyclically reduced. Then, by Lemmas 1.3 and 2.1, we can produce a finite $(X \cup X^{-1})$-automaton accepting the language $W' = (Wu)\iota$, and $L = \{(w'vw'^{-1})\iota : w' \in W'\}$.

Let $u,v \in R_X$ be such that $c_0 = uv$. Then vu is said to be a *cyclic* *conjugate* of c_0. Let C denote the set of all cyclic conjugates of c_0. We prove that, for every $w \in W$,

$$\forall c \in C\ \exists w' \leq_l w\ \exists c' \in C\colon (wcw^{-1})\iota = w'c'w'^{-1}. \tag{2.1}$$

We use induction on the length of w. The case $|w| = 0$ is trivial. Suppose that (2.1) holds for every $w \in W$ with $|w| = n \in \mathbb{N}^0$. Let $v \in W$ with $|v| = n+1$. If $vcv^{-1} \in R_X$, we take $v' = v$ and $c' = c$. Now suppose that $vcv^{-1} \notin R_X$. Then we have either $vc \notin R_X$ or $cv^{-1} \notin R_X$. Suppose that $vc \notin R_X$. Then there exist $x \in X \cup X^{-1}$ and $u, a \in R_X$ such that $v = ux^{-1}$ and $c = xa$. Obviously, $(vcv^{-1})\iota = (ux^{-1}xaxu^{-1})\iota = (uaxu^{-1})\iota$. Since W is left closed, we have $u \in W$. Moreover, $|u| = n$ and $ax \in C$. By induction hypothesis, we have $(uaxu^{-1})\iota = v'c'v'^{-1}$ for some $v' \leq_l u$ and $c' \in C$. Thus $(vcv^{-1})\iota = v'c'v'^{-1}$ and $v' <_l v$. The case $cv^{-1} \notin R_X$ is similar. Thus (2.1) holds for v and so for every $w \in W$.

In particular, for every $w \in W$, we have $(wc_0w^{-1})\iota = w'c_ww'^{-1}$ for some $w' \leq_l w$ and $c_w \in C$. It is immediate that such w' and c_w are unique.

For every $d \in C$, we define $\Omega_d = \{w'\colon w \in W \text{ and } c_w = d\}$. Our next step is to prove that, for every $d \in C$, Ω_d is a rational $(X \cup X^{-1})$-language. Let

$$V = [\bigcup_{(q,x)\in I} L(A_q).x] \cup (\{1\} \cap \Omega_d),$$

with $I = \{(q,x) \in Q \times (X \cup X^{-1})\colon [L(A_q).x] \cap \Omega_d \neq \emptyset\}$. We claim that $\Omega_d = V$. We certainly have $\Omega_d \subseteq V$. Now suppose that $[L(A_q).x] \cap \Omega_d \neq \emptyset$ for some $q \in Q$ and $x \in X \cup X^{-1}$. Hence there exists some $u \in L(A_q)$ such that $ux \in \Omega_d$. Let $u' \in L(A_q)$. Since $ux \in \Omega_d$, there exists $w \in W$ such that $(wc_0w^{-1})\iota = uxdx^{-1}u^{-1}$. We have $w = uxp$ for some $p \in R_X$ and it follows that $(xpc_0p^{-1}x^{-1})\iota = xdx^{-1}$. Since $u, u' \in L(A_q)$, we have $u'xp \in W$ and $(u'xpc_0p^{-1}x^{-1}u'^{-1})\iota = (u'xdx^{-1}u'^{-1})\iota$. But $u'x$ labels a path in A and so $u'x \in R_X$. Therefore $u'xdx^{-1}u'^{-1} \in R_X$. Hence $L(A_q).x \subseteq \Omega_d$ and so $\Omega_d = V$.

Now we give an algorithm to produce a finite $(X \cup X^{-1})$-automaton accepting Ω_d.

Suppose that $[L(A_q).x] \cap \Omega_d \neq \emptyset$ for some $q \in Q$, $x \in X \cup X^{-1}$ and $d \in C$. Let $w \in W$ be such that $c_w = d$ and $w' \in L(A_q).x$. Then, as shown above, $L(A_q).x \subseteq \Omega_d$. Moreover, there exists $u \in L(A_q)$ with $|u| < |Q|$ and so we can assume that $|w'| \leq |Q|$. Let $Z = \{z \in W\colon z' = w' \text{ and } c_z = d\}$. We can assume too that w has minimal length among all the elements of Z.

Suppose that $w = w'x_1 \ldots x_n$, $x_i \in X \cup X^{-1}$. Suppose that there exist $j,k \in \{1,\ldots,n\}$ and $p \in Q$ such that $j < k$ and

$$x_j = x_k;$$

$$w'x_1 \ldots x_{j-1},\ w'x_1 \ldots x_{k-1} \in L(A_p);$$

$$(x_j \ldots x_n c_0 x_n^{-1} \ldots x_j^{-1})\iota = (x_k \ldots x_n c_0 x_n^{-1} \ldots x_k^{-1})\iota \in C.$$

Note that, by our previous construction of w' and c_w, we can assume that $(x_l \ldots x_n c_0 x_n^{-1} \ldots x_l^{-1})\iota \in C$ for every $l \in \{1,\ldots,n\}$.

Let $v = w'x_1 \ldots x_{j-1}x_k \ldots x_n$. Since $w = w'x_1 \ldots x_n$ and $w'x_1 \ldots x_{j-1}$, $w'x_1 \ldots x_{k-1} \in L(A_p)$, we have $v \in W$. Hence $(vc_0v^{-1})\iota$
$= (w'x_1 \ldots x_{j-1}x_k \ldots x_n c_0 x_n^{-1} \ldots x_k^{-1} x_{j-1}^{-1} \ldots x_1^{-1} w'^{-1})\iota$
$= (w'x_1 \ldots x_n c_0 x_n^{-1} \ldots x_1^{-1} w'^{-1})\iota = (wc_0w^{-1})\iota = w'dw'^{-1}$.

Since $|v| < |w|$, this contradicts the minimality of w and so we must have $n \leq |Q|.|X \cup X^{-1}|.|C|$. Hence $|w| = |w'x_1 \ldots x_n| \leq |Q|+n \leq |Q|+2|Q|.|X|.|c_0|$. Thus, to determine whether or not $[L(A_q).x] \cap \Omega_d$ is empty, we only have to compute w' and c_w for every $w \in W$ with length not exceeding $|Q|+2|Q|.|X|.|c_0|$.

Similarly, if $1 \in \Omega_d$, then we can find some $w \in W$ with length not exceeding $2|Q|.|X|.|c_0|$ such that $w' = 1$ and $c_w = d$.

Let $H = \{w \in W\colon |w| \leq |Q|+2|Q|.|X|.|c_0|\}$. For every $w \in H$, we compute w' and c_w. Suppose that $c_w = d$. If $w' = 1$, then $1 \in \Omega_d$. If $w' \neq 1$, then we can write $w' \in L(A_q).x$ for some $q \in Q$ and $x \in X \cup X^{-1}$, yielding $(q,x) \in I$. Since H is finite, we can perform these computations for all such w. As shown above, this is enough to determine completely I and $\{1\}\cap\Omega_d$. Now we can apply Lemma 1.3 and produce a finite $(X\cup X^{-1})$-automaton accepting Ω_d.

Finally, we have $L = \bigcup_{d\in C}\{wdw^{-1}: w \in \Omega_d\}$ and the lemma follows from Lemmas 1.5 and 2.3.

Let W be a reduced rational $(X\cup X^{-1})$-language and let $a \in W$. We define the language W_a to be $\{w \in W: [w.Q(a)]\iota \subseteq W\}$.

LEMMA 2.5. *Let $A = (Q,\{i\},T,E)$ be a finite $(X\cup X^{-1})$-automaton with $L(A)$ reduced. Let $W = L(A)$ and let $a \in (X\cup X^{-1})^*$. Then we can produce a finite $(X\cup X^{-1})$-automaton accepting W_a.*

Proof. Let $n = |a|+1$. For every $q \in Q$, let

$$V_q = \{v \in (L[A_{(q)}])\cap F_n: [v.Q(a)]\iota \subseteq L[A_{(q)}]\}.$$

Let $V = [W_a\cap F_{(n)}]\cup(\bigcup_{q\in Q}[L(A_q).V_q])$. The languages V_q and $W_a\cap F_{(n)}$ are finite and so, by Lemma 1.3, we can produce a finite $(X\cup X^{-1})$-automaton accepting V. We prove that $W_a = V$.

Let $q \in Q$. Suppose that $u \in L(A_q)$ and $v \in V_q$. Since $L(A_q).L[A_{(q)}] \subseteq W$, we have $uv \in W$. Since $|v| > |a|$, we have $[uv.Q(a)]\iota = (u.[v.Q(a)])\iota = u([v.Q(a)]\iota) \subseteq L(A_q).L[A_{(q)}] \subseteq W$ and so $uv \in W_a$. Hence $V \subseteq W_a$.

Conversely, let $w \in W_a$. If $|w| \leqslant n$, then $w \in W_a\cap F_{(n)} \subseteq V$, so we assume that $|w| > n$. Let $w = uv$, with $|v| = n$. Then u labels a path in A going from the state i to some state $q \in Q$. Hence $u \in L(A_q)$ and $v \in L[A_{(q)}]$. Since $|v| > |a|$, we have $[w.Q(a)]\iota = u([v.Q(a)]\iota)$. Since $w \in W_a$, then $u([v.Q(a)]\iota) \in W$ and so $[v.Q(a)]\iota \in L[A_{(q)}]$. Hence $v \in V_q$ and $W_a \subseteq V$. Thus $W_a = V$ and the lemma is proved.

THEOREM 2.6. *Let X be a finite nonempty set, let P be a finite relation on $(X \cup X^{-1})^*$ and let $e \in D_X$. Let $A = (Q, \{i\}, T, E)$ be a finite $(X \cup X^{-1})$-automaton. Then it is decidable whether or not $L(A) \in \gamma(X,P,e)$.*

Proof. Let $W = L(A)$. Without loss of generality, we can assume that A is minimal. We will consider the five conditions defining $\gamma(X,P,e)$ successively.

(i) W is reduced.

This is equivalent to $W \subseteq R_X$, and this is decidable by Lemmas 1.4 and 1.9.

(ii) W is left closed.

We prove that this is equivalent to having $T = Q$. Suppose that W is left closed and let $q \in Q$. Since A is minimal, A is trim and so q lies in some successful path of A. Since W is left closed, this implies that there exists a path in A, beginning at i and ending at q, labelled by some $w \in W$. Since A is deterministic, this implies $q \in T$. The converse implication is immediate.

Thus it is decidable whether or not W is left closed.

(iii) $Q(e) \subseteq W$.

This is decidable, since $Q(e)$ is finite.

(iv) $\forall (a,b) \in P \cup P^{-1}\ \forall w \in W,\ [w.Q(a)]\iota \subseteq W \Rightarrow [w.Q(b)]\iota \subseteq W$.

This is equivalent to having $W_a = W_b$ for every $(a,b) \in P$. Assuming that W satisfies (i), and since P is finite, this is decidable by Lemmas 1.4 and 2.5.

(v) $\forall (a,b) \in P \cup P^{-1}\ \forall w \in W,\ [w.Q(a)]\iota \subseteq W \Rightarrow (wba^{-1}w^{-1}W)\iota \subseteq W$.

The fifth and final condition is by far the hardest to deal with. We shall assume that W satisfies all four previous conditions.

Moreover, we can assume that $(ba^{-1})\iota \neq 1$, otherwise the condition is trivially satisfied.

Since W is left closed, $1 \in W$. Therefore a necessary condition for W to satisfy (v) is

$$\forall (a,b) \in P \cup P^{-1}\ \forall w \in W,\ [w.Q(a)]\iota \subseteq W \Rightarrow (wba^{-1}w^{-1})\iota \in W.$$

This is the same as

$$\forall (a,b) \in P \cup P^{-1}\ \forall w \in W_a,\ (wba^{-1}w^{-1})\iota \in W. \tag{2.2}$$

Let $N(a,b) = \{(wba^{-1}w^{-1})\iota : w \in W_a\}$. Then (2.2) is clearly equivalent to

$$\forall (a,b) \in P \cup P^{-1},\ N(a,b) \subseteq W. \tag{2.3}$$

Let $(a,b) \in P \cup P^{-1}$. Then, by Lemmas 2.4 and 2.5, we can produce a pushdown $(X \cup X^{-1})$-automaton accepting $N(a,b)$. Moreover, $N(a,b) \subseteq W$ is equivalent to $N(a,b) \cap [(X \cup X^{-1})^* \backslash W] = \emptyset$. By Lemmas 1.3 and 1.6, we can produce a pushdown $(X \cup X^{-1})$-automaton accepting this intersection and so, by Lemma 1.8, we can decide whether or not it is empty. Hence it is decidable whether W satisfies (2.3) or not.

If we verify that W does not satisfy (2.3), then W cannot satisfy (v) either. If W satisfies (2.3), then the algorithm must continue as follows.

As a preliminary step, we must discuss the condition

$$(uW)\iota \subseteq W, \tag{2.4}$$

where $u \in W$. We can assume that $u \neq 1$, say $u = x_1 \ldots x_n$. Let $m = max(\{j \in \{1,\ldots,n\} : x_n^{-1} \ldots x_{n-j+1}^{-1} \in W\} \cup \{0\})$. We partition W into $m+1$ disjoint subsets as follows.

Let $W_0 = W \backslash (x_n^{-1} R_X)$. For every $j \in \{1,\ldots,m-1\}$, let $W_j = [W \cap (x_n^{-1} \ldots x_{n-j+1}^{-1} R_X)] \backslash (x_n^{-1} \ldots x_{n-j}^{-1} R_X)$. Let $W_m = W \cap (x_n^{-1} \ldots x_{n-m+1}^{-1} R_X)$.

Let $J = \{j \in \{0,\ldots,m\} : W_j \neq \emptyset\}$. Of course, (2.4) is equivalent to

$$\forall j \in J,\ (uW_j)\iota \subseteq W. \tag{2.5}$$

Since $u \in W$, there exist unique $q_0,\ldots,q_n \in Q$ such that

$$q_0 = i;$$

$$\forall k \in \{1,\ldots,n\},\ (q_{k-1},x_k,q_k) \in E.$$

Similarly, there exist unique $q'_0,\ldots,q'_m \in Q$ such that

$$q'_0 = i;$$

$$\forall l \in \{1,\ldots,m\},\ (q'_{l-1},x_{n-l+1}^{-1},q'_l) \in E.$$

Now we define $x_0 = 1$. For every $x \in X \cup X^{-1}$, let $P(x) = x^{-1}R_X$. We also define $P(1)$ to be $\emptyset$. We prove that (2.5) is equivalent to

$$\forall j \in J,\ L(A_{q_{n-j}})(L[A_{(q'_j)}]\backslash P(x_{n-j})) \subseteq W. \tag{2.6}$$

For every $k \leq 0$, we assume that a word of the form $x_1x_2\ldots x_k$ is the empty word. This and other similar conventions will be used without further comment.

Suppose that (2.5) holds. Let $j \in J$, let $c \in L(A_{q_{n-j}})$ and let $d \in L[A_{(q'_j)}]\backslash P(x_{n-j})$. We want to prove that $cd \in W$. Since $d \in L[A_{(q'_j)}]$, we have $x_n^{-1}\ldots x_{n-j+1}^{-1}d \in W$. Moreover, since $d \notin P(x_{n-j})$, we have $x_n^{-1}\ldots x_{n-j+1}^{-1}d \in W_j$. Hence, by (2.5), $[u(x_n^{-1}\ldots x_{n-j+1}^{-1}d)]\iota \in W$. Therefore $x_1\ldots x_{n-j}d \in W$. But $x_1\ldots x_{n-j} \in L(A_{q_{n-j}})$ and so $cd \in W$ holds as well. Thus (2.6) holds.

Conversely, suppose that (2.6) holds. Let $j \in J$ and let $c \in W_j$. Then $c = x_n^{-1}\ldots x_{n-j+1}^{-1}c'$ for some $c' \in L[A_{(q'_j)}]\backslash P(x_{n-j})$ and so $(uc)\iota = x_1\ldots x_{n-j}c' \in L(A_{q_{n-j}})(L[A_{(q'_j)}]\backslash P(x_{n-j}))$. Hence, by (2.6), $(uc)\iota \in W$. Therefore (2.5) holds and so (2.6) is equivalent to (2.5).

We define

$$\delta(u) = \{(q_{n-j},q'_j,x_{n-j}):\ j \in J\}.$$

Now we consider the set

$$\Theta = \{(q,q',x) \in Q \times Q \times (X \cup X^{-1} \cup \{1\}):\ (L(A_q)(L[A_{(q')}]\backslash P(x)) \not\subseteq W\}.$$

The set Θ is finite and can be effectively determined, by Lemmas 1.3 and 1.4.

Since (2.4) is equivalent to (2.6), it follows that W satisfies (v) if and only if

$$\forall(a,b) \in P\cup P^{-1}\ \forall w \in W_a,\ (\delta[(wba^{-1}w^{-1})\iota])\cap\Theta = \emptyset, \tag{2.7}$$

or equivalently,

$$\forall(a,b) \in P\cup P^{-1},\ \delta[N(a,b)]\cap\Theta = \emptyset. \tag{2.8}$$

Let $(q,q',x) \in \Theta$. We prove that

$$(q,q',x) \in \delta[N(a,b)] \tag{2.9}$$

$$\Leftrightarrow\quad [N(a,b)]\cap[L(A_q)\cap R_X x][L(A_{q'})]^{-1} \neq \emptyset \text{ and } L[A_{(q')}] \not\subseteq P(x).$$

Suppose that $(q,q',x) \in \delta[N(a,b)]$. Then there exists $u \in N(a,b)$ such that $(q,q',x) \in \delta(u)$. Suppose that $u = x_1\ldots x_n,\ x_i \in X\cup X^{-1}$. With the notation of (2.6), there exists $j \in J$ such that $q_{n-j} = q,\ q'_j = q'$ and $x_{n-j} = x$. Hence $u = x_1\ldots x_n = (x_1\ldots x_{n-j})(x_{n-j+1}\ldots x_n)$ $\in [L(A_{q_{n-j}})\cap R_X x_{n-j}][L(A_{q'_j})]^{-1} = [L(A_q)\cap R_X x][L(A_{q'})]^{-1}$ and so $[N(a,b)]\cap[L(A_q)\cap R_X x][L(A_{q'})]^{-1} \neq \emptyset$. Moreover, since $j \in J$, we have $L[A_{(q'_j)}] \not\subseteq P(x_{n-j})$ and so $L[A_{(q')}] \not\subseteq P(x)$.

Conversely, suppose that there exists some $u \in [N(a,b)]\cap[L(A_q)\cap R_X x][L(A_{q'})]^{-1}$ and $L[A_{(q')}] \not\subseteq P(x)$. Let $u = x_1\ldots x_n,\ x_i \in X\cup X^{-1}$. Then there exists $j \in \{0,\ldots,n\}$ such that $x_1\ldots x_{n-j} \in L(A_q)\cap R_X x$ and $x_{n-j+1}\ldots x_n \in [L(A_{q'})]^{-1}$. Hence $x_n^{-1}\ldots x_{n-j+1}^{-1} \in L(A_{q'})$ and since $L[A_{(q')}] \not\subseteq P(x)$, we must have $j \in J$, with the notation of (2.6). It is immediate that $(q,q',x) = (q_{n-j},q'_j,x_{n-j}) \in \delta(u) \subseteq \delta[N(a,b)]$ and so (2.9) holds.

Now, by Lemmas 1.3 and 1.4, we can decide whether or not $L[A_{(q')}] \subseteq P(x)$. By Lemmas 1.3 and 2.2, we can produce a finite $(X\cup X^{-1})$-automaton accepting the language $[L(A_q)\cap R_X x][L(A_{q'})]^{-1}$. By Lemma 1.6, we can produce a pushdown $(X\cup X^{-1})$-automaton accepting the language $[N(a,b)]\cap[L(A_q)\cap R_X x][L(A_{q'})]^{-1}$ and so, by Lemma 1.8, we can decide whether or not this language is empty. Since Θ is finite, we only need to perform finitely many such computations and so it is decidable whether or not W satisfies (v).

3. Idempotent-pure presentations

In this section we will give an alternative solution to the word problem for finite idempotent-pure presentations, solved by Meakin and Margolis [18].

A presentation $Inv\langle X;P\rangle$ is said to be *idempotent-pure* if $P \subseteq D_X \times D_X$.

LEMMA 3.1 [28]. *The inverse monoid defined by an idempotent-pure presentation is E-unitary.*

Throughout this section, assume that $P \subseteq D_X \times D_X$ is finite.

LEMMA 3.2. *Let $u,v \in (X\cup X^{-1})^*$. Then*

$$u(\rho\cup P)^{\#} = v(\rho\cup P)^{\#} \iff u\iota = v\iota \text{ and } (uu^{-1})(\rho\cup P)^{\#} = (vv^{-1})(\rho\cup P)^{\#}.$$

Proof. Suppose that $u(\rho\cup P)^{\#} = v(\rho\cup P)^{\#}$. We certainly have $(uu^{-1})(\rho\cup P)^{\#} = (vv^{-1})(\rho\cup P)^{\#}$. Moreover, we have $[u(\rho\cup P)^{\#}]\sigma = [v(\rho\cup P)^{\#}]\sigma$, that is, $u(\pi\cup P)^{\#} = v(\pi\cup P)^{\#}$. Since $P \subseteq \pi$, we obtain $u\pi = v\pi$ and so $u\iota = v\iota$.

Now suppose that $u\iota = v\iota$ and $(uu^{-1})(\rho\cup P)^{\#} = (vv^{-1})(\rho\cup P)^{\#}$. Since $u\rho = (uu^{-1})\rho(u\iota\rho)$ and $v\rho = (vv^{-1})\rho(v\iota\rho)$, we have $u(\rho\cup P)^{\#} = (uu^{-1})(\rho\cup P)^{\#}.u\iota(\rho\cup P)^{\#} = (vv^{-1})(\rho\cup P)^{\#}.v\iota(\rho\cup P)^{\#} = v(\rho\cup P)^{\#}$.

We obviously have $(\rho\cup P)^{\#} = (\rho\cup P_D)^{\#}$ and so, by Lemmas 1.14 and 3.2, the word problem for $Inv\langle X;P\rangle$ will be decidable if we can determine whether or not $W_P(e) = W_P(f)$ for every $e,f \in D_X$. The next result is essential in our solution. In [18], it is proved by using second-order monadic logic and Rabin's Tree Theorem [36]. We give a new proof.

LEMMA 3.3 [18]. *Let* $e \in D_X$. *Then we can produce a finite* $(X \cup X^{-1})$*-automaton accepting the language* $W_P(e)$.

Proof. Let $Q = W_P(e)$ and $A_P(e) = (Q,\{1\},\{1\},E)$. Let $E_+ = \{(q,x,q') \in E\colon |q| < |q'|\}$. We define a trim deterministic $(X \cup X^{-1})$-automaton $A = (Q,\{1\},Q,E_+)$. It follows easily that $L(A) = W_P(e)$. By Lemma 1.2, $W_P(e)$ is rational if and only if A_{min} is finite.

Let ν be the equivalence on Q defined by $q\nu = q'\nu$ if and only if $L[A_{(q)}] = L[A_{(q')}]$. Let $N = max(\{2|a|\colon (a,b) \in P \cup P^{-1}\} \cup \{|e|\})$. We prove that

$$\forall q,q' \in Q,\ L[A_{(q)}] \cap F_{(N)} = L[A_{(q')}] \cap F_{(N)} \Rightarrow q\nu = q'\nu. \qquad (3.1)$$

Suppose that $L[A_{(q)}] \cap F_{(N)} = L[A_{(q')}] \cap F_{(N)}$ for some $q,q' \in Q$. We can write Q as a sequence $(q_n)_{n \in \mathbb{N}}$ that successively enumerates all the elements of $W_{P,1}(e)$, then all the elements of $W_{P,2}(e) \backslash W_{P,1}(e)$, then the elements of $W_{P,3}(e) \backslash W_{P,2}(e)$, and so on. We want to prove that $L[A_{(q)}] \subseteq L[A_{(q')}]$. In fact, we will show that for every $j \in \mathbb{N}$,

$$qu = q_j \Rightarrow u \in L[A_{(q')}]. \qquad (3.2)$$

We use induction on j. Suppose that $qu = q_1$. Then $q_1 \in W_{P,1}(e) = Q(e)$ and so $|q_1| \leq |e| \leq N$. Hence $|u| \leq N$ and so $u \in L[A_{(q)}] \cap F_{(N)}$. Therefore $u \in L[A_{(q')}]$ and (3.2) holds for $j = 1$.

Suppose that (3.2) holds for every $j \leq n$. If $q_{n+1} \notin qR_X$, then (3.2) holds for $j = n+1$. Now assume that $q_{n+1} = qu$. If $|u| \leq N$, then (3.2) holds for $j = n+1$. Hence assume that $|u| > N$. Note that $q_{n+1} \notin W_{P,1}(e)$, since otherwise $|q_{n+1}| \leq |e| \leq N$. Thus $q_{n+1} \in W_{P,k+1}(e) \backslash W_{P,k}(e)$ for some $k \in \mathbb{N}$. Hence there exists some

$p \in W_{P,k}(e)$ and some $(a,b) \in P\cup P^{-1}$ such that $[p.Q(a)]\iota \subseteq W_{P,k}(e)$ and $q_{n+1} \in [p.Q(b)]\iota$. Let $v \in Q(b)$ be such that $q_{n+1} = (pv)\iota$. Therefore $qu = (pv)\iota$ and so there exist $p',g,v' \in R_X$ such that $p = p'g$, $v = g^{-1}v'$ and $qu = p'v'$. Since $|v'| \leq |v| \leq N/2$ and $|u| > N$, we must have $p' = qh$ for some $h \in R_X$ with $|h| > N/2$. Hence we have $p = qs_1$ for some $s_1 \in R_X$ such that $|s_1| > N/2$. Since $|a| \leq N/2$, then $[p.Q(a)]\iota \subseteq qR_X$. Assume that $[p.Q(a)]\iota = \{qs_1,\ldots,qs_l\}$. Since $p = qs_1$, we obtain $[s_1.Q(a)]\iota = \{s_1,\ldots,s_l\}$. Moreover, for every $i \in \{1,\ldots,l\}$, we have $qs_i \in W_{P,k}(e)$. Hence $qs_i = q_{j_i}$ for some $j_i \leq n$ and we can apply the induction hypothesis, obtaining $\{s_1,\ldots,s_l\} \subseteq L[A_{(q')}]$. Therefore $\{q's_1,\ldots,q's_l\} \subseteq L(A) = W_P(e)$ and so $[q's_1.Q(a)]\iota \subseteq W_P(e)$. Thus $[q's_1.Q(b)]\iota \subseteq W_P(e)$. In particular, $(q's_1v)\iota \in Q$. But $|s_1| > |v|$ and so $(q's_1v)\iota = q'[(s_1v)\iota]$. Hence $(s_1v)\iota \in L[A_{(q')}]$. But $qu = (pv)\iota = (qs_1v)\iota = q[(s_1v)\iota]$, so $(s_1v)\iota = u$ and $u \in L[A_{(q')}]$. Hence (3.2) holds for $j = n+1$ and so holds for every $j \in \mathbb{N}$. Thus $L[A_{(q)}] \subseteq L[A_{(q')}]$. Similarly, we obtain $L[A_{(q')}] \subseteq L[A_{(q)}]$. Hence $q\nu = q'\nu$ and (3.1) is proved.

For every $k \in \mathbb{N}$, let Λ_k denote the set of all nonempty left closed subsets of $R_X\cap F_{(k)}$ and let $\lambda_k = |\Lambda_k|$. From (3.1) it follows that $|Q/\nu| \leq \lambda_N$. Now we give an algorithm to compute λ_k inductively.

For every $k \in \mathbb{N}$ and every $x \in X\cup X^{-1}$, let

$$\Gamma_k(x) = \{L \subseteq R_X\cap xR_X\cap F_{(k)} \colon L\cup\{1\} \text{ is left closed}\}.$$

Since $|\Gamma_k(x)|$ is independent of x, we denote it by γ_k. It is immediate that $\gamma_1 = 2$. Let $x \in X\cup X^{-1}$ and let $Z = (X\cup X^{-1})\backslash\{x^{-1}\}$. It is not difficult to see that, for every $k \in \mathbb{N}$, the map $\Gamma_{k+1}(x)\backslash\{\emptyset\} \longrightarrow \prod_{y\in Z}\Gamma_k(y)\colon xW \mapsto (W\cap yR_X)_{y\in Z}$ is a bijection. Hence $\gamma_{k+1} = 1+(\gamma_k)^{|Z|} = 1+(\gamma_k)^{2|X|-1}$. It is also easy to verify that the map $\Lambda_k \longrightarrow \prod_{x\in X\cup X^{-1}}\Gamma_k(x)\colon W \mapsto (W\cap xR_X)_{x\in X\cup X^{-1}}$ is a bijection and so $\lambda_k = (\gamma_k)^{2|X|}$ for every $k \in \mathbb{N}$. This completes the algorithm.

Since the number of (minimal) $(X\cup X^{-1})$-automata with at most λ_N

states is bounded, it is decidable, by Theorem 2.6, which of those automata accept languages belonging to $\gamma(X,P,e)$. By Lemma 1.4, we can then determine the smallest of all such languages and so we can produce a finite $(X\cup X^{-1})$-automaton accepting $W_P(e)$.

We remark that, in this particular case, condition (v) in the definition of $\gamma(X,P,e)$ is trivial and so the proof of Theorem 2.6 is very much simplified.

Now, Lemmas 1.14, 3.2 and 3.3 yield

THEOREM 3.4 [18]. *The word problem for a finite idempotent-pure presentation is decidable.*

4. A natural generalization

Let $Y \subseteq X$. We define a homomorphism $\theta_Y:(X\cup X^{-1})^* \longrightarrow (X\cup X^{-1})^*$ as follows. For every $y \in Y$, let $y\theta_Y = y^{-1}\theta_Y = yy^{-1}$; for every $x \in X\backslash Y$, let $x\theta_Y = x$ and let $x^{-1}\theta_Y = x^{-1}$.

It is easy to see that $\rho\theta_Y \subseteq \rho$. This follows from the fact that $[(uu^{-1}u)\theta_Y, u\theta_Y], [(uu^{-1}vv^{-1})\theta_Y, (vv^{-1}uu^{-1})\theta_Y] \in \rho$ for every $u,v \in (X\cup X^{-1})^*$.

LEMMA 4.1. *Let $Y \subseteq X$ and let $T = \{(y,yy^{-1}): y \in Y\}\cup\{(y^{-1},yy^{-1}): y \in Y\}$. Then $Ker(\theta_Y) = T^\#$ and $(\theta_Y)^2 = \theta_Y$.*

Proof. It is trivial that $T \subseteq Ker(\theta_Y)$ and so $T^\# \subseteq Ker(\theta_Y)$. Conversely, suppose that $(u,v) \in Ker(\theta_Y)$. It follows from the definition that $(w,w\theta_Y) \in T^\#$ for every $w \in (X\cup X^{-1})^*$. Hence $uT^\#$

$= (u\theta_Y)T^{\#} = (v\theta_Y)T^{\#} = vT^{\#}$. Thus $Ker(\theta_Y) = T^{\#}$. But then we have $(w, w\theta_Y) \in Ker(\theta_Y)$ for every $w \in (X \cup X^{-1})^*$ and so $(\theta_Y)^2 = \theta_Y$.

We can now generalize Theorem 3.4.

THEOREM 4.2. *Let $P \subseteq (D_X \cup X \cup X^{-1}) \times D_X$ be finite. Then the word problem for $Inv\langle X;P\rangle$ is decidable.*

Proof. Since $(\rho\cup\{(x^{-1},e)\})^{\#} = (\rho\cup\{(x,e)\})^{\#}$ for every $x \in X$ and $e \in D_X$, we can assume that $P \subseteq (D_X \cup X) \times D_X$. Let $Y = \{x \in X: P\cap(\{x\} \times D_X) \neq \emptyset\}$. We prove that, for every $u, v \in (X \cup X^{-1})^*$, we have

$$u(\rho\cup P)^{\#} = v(\rho\cup P)^{\#} \Leftrightarrow (u\theta_Y)[\rho\cup(P\theta_Y)]^{\#} = (v\theta_Y)[\rho\cup(P\theta_Y)]^{\#}. \qquad (4.1)$$

Suppose that $u(\rho\cup P)^{\#} = v(\rho\cup P)^{\#}$. Then $(u\theta_Y)[(\rho\cup P)^{\#}\theta_Y] = (v\theta_Y)[(\rho\cup P)^{\#}\theta_Y]$. By (I.1.1), we have $(\rho\cup P)^{\#}\theta_Y \subseteq [(\rho\cup P)\theta_Y]^{\#}$. Hence $(\rho\cup P)^{\#}\theta_Y \subseteq [(\rho\cup P)\theta_Y]^{\#} = [(\rho\theta_Y)\cup(P\theta_Y)]^{\#} \subseteq [\rho\cup(P\theta_Y)]^{\#}$ and so $(u\theta_Y)[\rho\cup(P\theta_Y)]^{\#} = (v\theta_Y)[\rho\cup(P\theta_Y)]^{\#}$.

To prove the converse implication, we begin by showing that $Ker(\theta_Y) \subseteq (\rho\cup P)^{\#}$. Using the notation of Lemma 4.1, we prove that $T \subseteq (\rho\cup P)^{\#}$. Let $y \in Y$. Then there exists some $e \in D_X$ such that $(y,e) \in P$. Hence $y(\rho\cup P)^{\#} = e(\rho\cup P)^{\#} = (ee^{-1})(\rho\cup P)^{\#} = (yy^{-1})(\rho\cup P)^{\#}$. Moreover, we have $y^{-1}(\rho\cup P)^{\#} = (yy^{-1})^{-1}(\rho\cup P)^{\#} = (yy^{-1})(\rho\cup P)^{\#}$. Thus $T \subseteq (\rho\cup P)^{\#}$ and so, by Lemma 4.1, $Ker(\theta_Y) \subseteq (\rho\cup P)^{\#}$.

Now we prove that $[\rho\cup(P\theta_Y)]^{\#} \subseteq (\rho\cup P)^{\#}$. Let $(a,b) \in P$. Then, by Lemma 4.1, we have $(a, a\theta_Y), (b, b\theta_Y) \in Ker(\theta_Y) \subseteq (\rho\cup P)^{\#}$ and so $(a\theta_Y)(\rho\cup P)^{\#} = a(\rho\cup P)^{\#} = b(\rho\cup P)^{\#} = (b\theta_Y)(\rho\cup P)^{\#}$. Hence
$[\rho\cup(P\theta_Y)]^{\#} \subseteq (\rho\cup P)^{\#}$.

Finally, suppose that $(u\theta_Y)[\rho\cup(P\theta_Y)]^{\#} = (v\theta_Y)[\rho\cup(P\theta_Y)]^{\#}$. Since $[\rho\cup(P\theta_Y)]^{\#} \subseteq (\rho\cup P)^{\#}$, we have $(u\theta_Y)(\rho\cup P)^{\#} = (v\theta_Y)(\rho\cup P)^{\#}$. Since $Ker(\theta_Y) \subseteq (\rho\cup P)^{\#}$, we obtain $u(\rho\cup P)^{\#} = (u\theta_Y)(\rho\cup P)^{\#} = (v\theta_Y)(\rho\cup P)^{\#}$

$= v(\rho\cup P)^{\#}$ and (4.1) is proved.

But $P\theta_Y \subseteq D_X \times D_X$ and so, by Theorem 3.4, it is decidable whether $(u\theta_Y)[\rho\cup(P\theta_Y)]^{\#} = (v\theta_Y)[\rho\cup(P\theta_Y)]^{\#}$ or not. Thus the word problem for $Inv\langle X;P\rangle$ is decidable.

5. $\mathcal{R}$-pure presentations

We say that a relation P on $(X\cup X^{-1})^*$ is *$\mathcal{R}$-pure* if $P\rho^{\natural} \subseteq \mathcal{R}$.

LEMMA 5.1. *Let P be a finite $\mathcal{R}$-pure relation on $(X\cup X^{-1})^*$. Let $e \in D_X$. Then we can produce a finite $(X\cup X^{-1})$-automaton accepting the language $W_P(e)$.*

Proof. As in the proof of Lemma 3.3, let $Q = W_P(e)$, $A_P(e) = (Q,\{1\},\{1\},E)$ and $A = (Q,\{1\},Q,E_+)$. Further, let $A' = (Q,\{1\},Q,E)$. We have $L(A) = W_P(e)$. Let ν and ν' be the equivalences on Q defined by

$$q\nu = q'\nu \iff L[A_{(q)}] = L[A_{(q')}];$$
$$q\nu' = q'\nu' \iff L[A'_{(q)}] = L[A'_{(q')}].$$

For every $q \in Q$, let $\chi(q) = \{x \in X\cup X^{-1}: [q,x,(qx)\iota] \in E\backslash E_+\}$. We have $\chi(1) = \emptyset$ and $|\chi(q)| = 1$ for every $q \in Q\backslash\{1\}$. It is not difficult to see that, for every $q \in Q$, $L[A_{(q)}] = (L[A'_{(q)}]\cap R_X)\backslash[\chi(q)R_X]$. It follows that if $q\nu' = q'\nu'$ and $\chi(q) = \chi(q')$, then $q\nu = q'\nu$. Hence $|Q/\nu| \leqslant |Q/\nu'|(2|X|+1)$. Now we will prove that $|Q/\nu'| \leqslant |Q(e)|$. This will ensure that $|Q/\nu| \leqslant |Q(e)|(2|X|+1)$ and so, by Theorem 2.6 and Lemma 1.4, we can produce a finite $(X\cup X^{-1})$-automaton accepting $W_P(e)$.

Let $Q = (q_n)_{n\in\mathbb{N}}$ be a sequence which enumerates successively the elements of $W_{P,1}(e)$, then those of $W_{P,2}(e)\backslash W_{P,1}(e)$ and so on. It is enough to show that

$$\forall n > |Q(e)|\ \exists m < n: q_m\nu' = q_n\nu'. \tag{5.1}$$

Suppose that $n > |Q(e)|$. Then $q_n \in W_{P,k+1}(e) \backslash W_{P,k}(e)$ for some $k \in \mathbb{N}$. Hence there exists $p \in Q$, $(a,b) \in P \cup P^{-1}$ and $f \in D_X$ such that $[p.Q(af)]\iota \subseteq W_{P,k}(e)$ and $q_n \in [p.Q(bf)]\iota$. Since $Q(bf) = Q(b) \cup [b.Q(f)]\iota$, we have either $q_n \in [p.Q(b)]\iota$ or $q_n \in [pb.Q(f)]\iota$. Suppose that $q_n \in [p.Q(b)]\iota$. Since P is $\mathcal{R}$-pure, we have $Q(b) = Q(a)$ and so $q_n \in [p.Q(a)]\iota$. Since $[p.Q(a)]\iota \subseteq [p.Q(af)]\iota \subseteq W_{P,k}(e)$, we obtain $q_n \in W_{P,k}(e)$, which is a contradiction. Hence $q_n \in [pb.Q(f)]\iota$. Let $v \in Q(f)$ be such that $q_n = (pbv)\iota$. Since $[p.Q(af)]\iota \subseteq W_{P,k}(e)$, we have $(pav)\iota \in W_{P,k}(e)$. Let $q_m = (pav)\iota$. Since $q_m \in W_{P,k}(e)$, we have $m < n$. We now prove that $q_n\nu' = q_m\nu'$.

Suppose that $u \in L[A\wr_{(q_m)}]$. Suppose that $u' \leq_I u$. Then $q_m u' \in L(A')$ and it follows easily that $(q_m u')\iota \in L(A')$ as well. Hence $(q_m u')\iota \in W_P(e)$ and so $[q_m.Q(u)]\iota \subseteq W_P(e)$. Now we show that $[p.Q(avuu^{-1}v^{-1})]\iota \subseteq W_P(e)$. Since $v \in Q(f)$, we have $[p.Q(av)]\iota \subseteq [p.Q(af)]\iota \subseteq W_{P,k}(e)$. Hence $[p.Q(avuu^{-1}v^{-1})]\iota = [p.Q(av)]\iota \cup [pav.Q(u)]\iota = [p.Q(av)]\iota \cup [q_m.Q(u)]\iota \subseteq W_P(e)$. Since $vuu^{-1}v^{-1} \in D_X$, this yields $[p.Q(bvuu^{-1}v^{-1})]\iota \subseteq W_P(e)$. Hence $[q_n.Q(u)]\iota = [pbv.Q(u)]\iota \subseteq [p.Q(bvuu^{-1}v^{-1})]\iota \subseteq W_P(e)$ and so $u \in L[A\wr_{(q_n)}]$. Hence $L[A\wr_{(q_m)}] \subseteq L[A\wr_{(q_n)}]$. Similarly, we obtain $L[A\wr_{(q_n)}] \subseteq L[A\wr_{(q_m)}]$. Hence $q_n\nu' = q_m\nu'$. Thus (5.1) is proved and so is the lemma.

Now, by (1.1) and Lemmas 1.14 and 5.1, we obtain

THEOREM 5.2. *Let P be a finite $\mathcal{R}$-pure relation on $(X \cup X^{-1})^*$. Then the idempotent word problem for Inv<X;P> is decidable.*

6. One-relator presentations

A natural question to ask is which are the instances of one-relator presentations $Inv\langle X;P\rangle$ that yield rational languages $W_P(e)$. Perhaps it is not possible to find clear necessary and sufficient conditions for this to happen in general, but we can do so for $W_P(1)$. This particular language is intimately connected with the *group problem*. By Lemma 1.13, we have

$$W_P(1) = \{u \in R_X : (uu^{-1},1) \in (\rho \cup P_D)^{\#}\}. \qquad (6.1)$$

The next result follows immediately.

LEMMA 6.1. *Let P be a relation on $(X\cup X^{-1})^*$. Then $(X\cup X^{-1})^*/(\rho\cup P)^{\#}$ is a group if and only if $W_P(1) = R_X$.*

Let $P = \{(u,v)\}$ be a relation on $(X\cup X^{-1})^*$. If $u,v \neq 1$, then it is obvious that $W_P(1) = \{1\}$. Hence we will assume that $P = \{(u,1)\}$.

THEOREM 6.2. *Let $P = \{(u,1)\}$ and let $Y = \{x \in X: x$ or x^{-1} occurs in $u\}$. Let ρ denote the Vagner congruence on $(Y\cup Y^{-1})^*$. Then $W_P(1)$ is rational if and only if $u \in D_X$ or $(Y\cup Y^{-1})^*/(\rho\cup P)^{\#}$ is a group.*

Proof. Suppose that $u \in D_X$. Then, by Lemma 3.3, $W_P(1)$ is rational.

Suppose that $(Y\cup Y^{-1})^*/(\rho\cup P)^{\#}$ is a group. Then, by (6.1), we have $W_P(1) = R_Y$, which is rational, by Lemma 1.9.

Now suppose that $u \notin D_X$ and $(Y\cup Y^{-1})^*/(\rho\cup P)^{\#}$ is not a group. Then there exists some $u \in R_Y$ such that $(uu^{-1},1) \notin (\rho\cup P)^{\#}$ and so

$$\exists y \in Y\cup Y^{-1}: (y^{-1}y,1) \notin (\rho\cup P)^{\#}. \qquad (6.2)$$

We claim that

$$\exists y \in Y \cup Y^{-1}: (y^{-1}y, 1) \notin (\rho \cup P)^{\#} \text{ and } (yy^{-1}, 1) \in (\rho \cup P)^{\#}. \tag{6.3}$$

Suppose that $u = y_1 \ldots y_m$. Then $(y_1 y_1^{-1})(\rho \cup P)^{\#} = (y_1 y_1^{-1} y_1 \ldots y_m)(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$. If $(y_1^{-1} y_1)(\rho \cup P)^{\#} \neq 1(\rho \cup P)^{\#}$, then (6.3) holds. If not, then $(y_2 \ldots y_m y_1)(\rho \cup P)^{\#} = (y_1^{-1} y_1 y_2 \ldots y_m y_1)(\rho \cup P)^{\#} = (y_1^{-1} y_1)(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$. Now we continue the procedure and, by (6.2), we know that (6.3) must hold for $y = y_j$ for some $j \in \{1, \ldots, m\}$. Let y be such an element. By (6.1), we have $y^n \in W_P(1)$ for every $n \in \mathbb{N}$. Since $u \notin D_X$, we have $u\iota \neq 1$. Since $u(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$ and $[uu^{-1}(u\iota)]\rho = u\rho$, we have $(u\iota)(\rho \cup P)^{\#} = (uu^{-1})(\rho \cup P)^{\#}(u\iota)(\rho \cup P)^{\#} = u(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$. Suppose that the first letter of $u\iota$ is y^{-1}. Then $(y^{-1}y)(\rho \cup P)^{\#} = [y^{-1}y(u\iota)](\rho \cup P)^{\#} = (u\iota)(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$, in contradiction with (6.3). Hence the first letter of $u\iota$ is not y^{-1}. Similarly, the last letter of $u\iota$ is not y and so $y^n(u\iota)y^{-n} \in R_Y$ for every $n \in \mathbb{N}$. Since $(u\iota)(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$, we have $[(u\iota)y^{-n}y^n(u\iota)^{-1}](\rho \cup P_D)^{\#} = (y^{-n}y^n)(\rho \cup P_D)^{\#}$. Hence $[y^n(u\iota)y^{-n}y^n(u\iota)^{-1}y^{-n}](\rho \cup P_D)^{\#} = (y^n y^{-n})(\rho \cup P_D)^{\#} = 1(\rho \cup P_D)^{\#}$ and so, by (6.1), $y^n(u\iota)y^{-n} \in W_P(1)$. Suppose that $A_P(1) = (Q, \{1\}, \{1\}, E)$ and $A = (Q, \{1\}, Q, E_+)$. As in the previous sections, we have $L(A) = W_P(1) = Q$. Let ν be defined for A as in Lemma 1.2. We will prove that for every $m, n \in \mathbb{N}$, $m \neq n$, we have $y^m\nu \neq y^n\nu$.

Let $m, n \in \mathbb{N}$, $m < n$. Since $y^n(u\iota)y^{-n} \in W_P(1)$, we have $(u\iota)y^{-n} \in L[A_{(y^n)}]$. Suppose that $(u\iota)y^{-n} \in L[A_{(y^m)}]$. Then $y^m(u\iota)y^{-n} \in W_P(1)$ and so, by (6.1), we have $(y^m(u\iota)y^{-n}y^n(u\iota)^{-1}y^{-m}, 1) \in (\rho \cup P)^{\#}$. But then, since $(u\iota)(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$, we must have $(y^m y^{-n} y^n y^{-m}, 1) \in (\rho \cup P)^{\#}$. Since $m < n$, and by (6.3), we have $[y^{-(n-m)}y^{n-m}](\rho \cup P)^{\#} = [y^m y^{-m} y^{-(n-m)} y^{n-m} y^m y^{-m}](\rho \cup P)^{\#} = (y^m y^{-n} y^n y^{-m})(\rho \cup P)^{\#} = 1(\rho \cup P)^{\#}$. Therefore $(y^{-1}y, 1) \in (\rho \cup P)^{\#}$, in contradiction with (6.3). Hence $(y^m)\nu \neq (y^n)\nu$, which shows that A_{min} is not finite. Thus $W_P(1)$ is not rational and the theorem is proved.

7. Undecidability of the idempotent word problem

In this section we will prove the existence of finitely presented inverse monoids with undecidable idempotent word problem.

LEMMA 7.1. *Let P be a finite relation on $(X\cup X^{-1})^*$ such that $(X\cup X^{-1})^*/(\rho\cup P)^\#$ is a group. Let $z \notin X\cup X^{-1}$ and let $X' = X\cup\{z\}$. Consider P as a relation on X' and let $j \in \mathbb{N}$. Then, for every $u_1,\dots,u_n \in R_X\backslash\{1\}$, $z_0 \in R_{\{z\}}$ and $z_1,\dots,z_n \in R_{\{z\}}\backslash\{1\}$, we have*

$$z_0u_1z_1\dots u_nz_n \in W_{P,j}(zz^{-1}) \Rightarrow u_i(\rho\cup P)^\# = 1(\rho\cup P)^\# \text{ for every } i.$$

Proof. We prove the lemma by induction on j. Since $W_{P,1}(zz^{-1}) = Q(zz^{-1}) = \{1,z\}$, the result holds trivially for $j = 1$. Now suppose that it holds for $j = k$. We will use a secondary induction on n, proving that

$$z_0u_1z_1\dots u_nz_n \in W_{P,k+1}(zz^{-1}) \quad (7.1)$$
$$\Rightarrow u_i(\rho\cup P)^\# = 1(\rho\cup P)^\# \text{ for every } i$$

holds for every $n \in \mathbb{N}^0$. Obviously, (7.1) holds trivially for $n = 0$.

Suppose that (7.1) holds for $n = t-1$, with $t \in \mathbb{N}$. Suppose that $w = z_0u_1z_1\dots u_tz_t \in W_{P,k+1}(zz^{-1})$ for some $u_1,\dots,u_t \in R_X\backslash\{1\}$, $z_0 \in R_{\{z\}}$ and $z_1,\dots,z_t \in R_{\{z\}}\backslash\{1\}$. We can assume that $w \notin W_{P,k}(zz^{-1})$. Hence there exists some $q \in W_{P,k}(zz^{-1})$, $(a,b) \in P\cup P^{-1}$ and $f \in D_{X'}$ such that $[q.Q(af)]\iota \subseteq W_{P,k}(zz^{-1})$ and $w \in [q.Q(bf)]\iota$. Since $Q(bf) = Q(b)\cup[bQ(f)]\iota$, we have either $w \in [q.Q(b)]\iota$ or $w \in [qb.Q(f)]\iota$.

Suppose that $w \in [q.Q(b)]\iota$. Since $Q(b) \subseteq R_X$ and the last letter of w is either z or z^{-1}, w must then be a prefix of q. But $W_{P,k}(zz^{-1})$ is left closed, hence $w \in W_{P,k}(zz^{-1})$ and so we reach a contradiction. Therefore $w \in [qb.Q(f)]\iota$. Let $v \in Q(f)$ be such that $w = (qbv)\iota$. Let

$w' = (qav)\iota$. Since $W_{P,k+1}(zz^{-1})$ is left closed, we have $z_0u_1z_1\ldots u_{t-1}z_{t-1} \in W_{P,k+1}(zz^{-1})$. Therefore, by induction hypothesis, we must have $u_i(\rho\cup P)^{\#} = 1(\rho\cup P)^{\#}$ for every $i \in \{1,\ldots,t-1\}$.

Suppose that the last letter of v is neither z nor z^{-1}. Since $Q(b) \subseteq R_X$, the same applies to $(bv)\iota$. Since $w = [q.(bv)\iota]\iota$ and the last letter of w is either z or z^{-1}, w must be a prefix of q, which we know to be impossible. Therefore the last letter of v is either z or z^{-1} and so the last letter of $(av)\iota$ must be either z or z^{-1}. Now two cases may occur.

Suppose that the last letter of w' is either z or z^{-1}. Then we can write $w' = z_0'u_1'z_1'\ldots u_m'z_m'$. Applying the induction hypothesis on k, we get $(u_1'\ldots u_m')(\rho\cup P)^{\#} = u_1'(\rho\cup P)^{\#}\ldots u_m'(\rho\cup P)^{\#} = 1(\rho\cup P)^{\#}$.

Suppose that the last letter of w' is neither z nor z^{-1}. Since the last letter of $(av)\iota$ is either z or z^{-1}, we have that either $w'z$ or $w'z^{-1}$ must be a prefix of q. Without loss of generality, we can assume that $w'z$ is a prefix of q. We can apply induction again to $w'z = z_0'u_1'z_1'\ldots u_m'z \in W_{P,k}(zz^{-1})$ and get $(u_1'\ldots u_m')(\rho\cup P)^{\#} = 1(\rho\cup P)^{\#}$.

Let $\psi:(X'\cup X'^{-1})^* \longrightarrow (X\cup X^{-1})^*$ be the homomorphism defined by

$$y\psi = \begin{cases} y & \text{if } y \in X\cup X^{-1} \\ 1 & \text{if } y \in \{z,z^{-1}\}. \end{cases}$$

Let ρ' denote the Vagner congruence on $(X'\cup X'^{-1})^*$. It is easy to see that $(\rho'\cup P)\psi \subseteq (\rho\cup P)^{\#}$. Therefore, for every $g,h \in (X'\cup X'^{-1})^*$,

$$g(\rho'\cup P)^{\#} = h(\rho'\cup P)^{\#} \Rightarrow g\psi(\rho\cup P)^{\#} = h\psi(\rho\cup P)^{\#}. \tag{7.2}$$

It follows easily that

$$\forall g \in (X'\cup X'^{-1})^*,\ g\psi\iota = g\iota\psi\iota.$$

Since $(X\cup X^{-1})^*/(\rho\cup P)^{\#}$ is a group, we have $\pi \subseteq (\rho\cup P)^{\#}$ and so

$$\forall g \in (X\cup X^{-1})^*,\ (g\iota,g) \in \pi \subseteq (\rho\cup P)^{\#}.$$

Since $(qbv)(\rho'\cup P)^{\#} = (qav)(\rho'\cup P)^{\#}$, we have $(qbv)\psi(\rho\cup P)^{\#}$

$= (qav)\psi(\rho\cup P)^{\#}$, by (7.2). By induction hypothesis, we have $(u_1\ldots u_{t-1})(\rho\cup P)^{\#} = 1(\rho\cup P)^{\#}$ and so $u_t(\rho\cup P)^{\#} = (u_1\ldots u_{t-1})(\rho\cup P)^{\#}u_t(\rho\cup P)^{\#} = (u_1\ldots u_t)(\rho\cup P)^{\#} = w\psi(\rho\cup P)^{\#} = (qbv)\iota\psi(\rho\cup P)^{\#} = (qbv)\iota\psi\iota(\rho\cup P)^{\#} = (qbv)\psi\iota(\rho\cup P)^{\#} = (qbv)\psi(\rho\cup P)^{\#} = (qav)\psi(\rho\cup P)^{\#} = (qav)\psi\iota(\rho\cup P)^{\#} = (qav)\iota\psi\iota(\rho\cup P)^{\#} = (qav)\iota\psi(\rho\cup P)^{\#} = w'\psi(\rho\cup P)^{\#} = (u_1'\ldots u_m')(\rho\cup P)^{\#} = 1(\rho\cup P)^{\#}$. Therefore (7.1) holds for $n = t$ and so for every $n \in \mathbb{N}^0$. Thus the lemma holds for $j = k+1$ and so for every $j \in \mathbb{N}$.

THEOREM 7.2. *There exists a finitely presented inverse monoid with an undecidable idempotent word problem.*

Proof. Let $Gp\langle X_1;T\rangle$ denote a finite group presentation with undecidable word problem [3],[29]. Let $y \notin X_1\cup X_1^{-1}$ and let $X_2 = X_1\cup\{y\}$. Let $z \notin X_2\cup X_2^{-1}$ and let $X_3 = X_2\cup\{z\}$. For every $i \in \{1,2,3\}$, we denote by ρ_i the Vagner congruence on $(X_i\cup X_i^{-1})^*$ and we write $\pi_i = \{(xx^{-1},1)\colon x \in X_i\cup X_i^{-1}\}^{\#}$.

Now we define $P = T\cup\{(y^2,1)\}\cup\{(xx^{-1},1)\colon x \in X_1\cup X_1^{-1}\}$. Let G be the group defined by the presentation $Gp\langle X_2;P\rangle$. We prove that

$$\forall u \in (X_1\cup X_1^{-1})^*,\ (uy)^2(\rho_2\cup P)^{\#} = 1(\rho_2\cup P)^{\#} \tag{7.3}$$
$$\Leftrightarrow\ u(\pi_1\cup T)^{\#} = 1(\pi_1\cup T)^{\#}.$$

Let $u \in (X_1\cup X_1^{-1})^*$. Suppose that $u(\pi_1\cup T)^{\#} = 1(\pi_1\cup T)^{\#}$. Since $(\pi_1\cup T)^{\#} \subseteq (\rho_2\cup P)^{\#}$, then $u(\rho_2\cup P)^{\#} = 1(\rho_2\cup P)^{\#}$ and so $(uy)^2(\rho_2\cup P)^{\#} = y^2(\rho_2\cup P)^{\#} = 1(\rho_2\cup P)^{\#}$.

Now suppose that $u(\pi_1\cup T)^{\#} \neq 1(\pi_1\cup T)^{\#}$. As we see in [16,§4.1], G is the free product in the category of groups of $(X_1\cup X_1^{-1})^*/(\pi_1\cup T)^{\#}$ and $\{y,y^{-1}\}^*/\{(yy^{-1},1),(y^{-1}y,1),(y^2,1)\}^{\#}$. Hence $(X_1\cup X_1^{-1})^*/(\pi_1\cup T)^{\#}$ embeds canonically in G and so $u(\pi_2\cup P)^{\#} \neq 1(\pi_2\cup P)^{\#}$. Similarly, we have $y(\pi_2\cup P)^{\#} \neq 1(\pi_2\cup P)^{\#}$ and so $(uy)(\pi_2\cup P)^{\#}$ is a nonhomogeneous element of a free product of two groups, that is, $(uy)(\pi_2\cup P)^{\#}$ is not contained in either of the factor groups. It follows that $(uy)(\pi_2 \cup P)^{\#}$ has infinite order. Since $(\pi_2 \cup P)^{\#} = (\rho_2 \cup P)^{\#}$, it follows that $(uy)^2(\rho_2 \cup P)^{\#} \neq 1(\rho_2 \cup P)^{\#}$. Thus (7.3) holds.

Now let M be the inverse monoid defined by $Inv\langle X_3;P\rangle$.

Let $u \in (X_1 \cup X_1^{-1})^*$ and consider $uyzz^{-1}y^{-1}u^{-1}$, $y^{-1}u^{-1}zz^{-1}uy \in D_X$. We prove that

$$(uyzz^{-1}y^{-1}u^{-1})(\rho_3 \cup P)^{\#} = (y^{-1}u^{-1}zz^{-1}uy)(\rho_3 \cup P)^{\#} \qquad (7.4)$$
$$\Leftrightarrow \quad u(\pi_1 \cup T)^{\#} = 1(\pi_1 \cup T)^{\#},$$

which implies that the idempotent word problem for $Inv\langle X_3;P\rangle$ is undecidable.

Suppose that $(uyzz^{-1}y^{-1}u^{-1})(\rho_3 \cup P)^{\#} = (y^{-1}u^{-1}zz^{-1}uy)(\rho_3 \cup P)^{\#}$. Then we have $[(uy)^2(zz^{-1})(y^{-1}u^{-1})^2](\rho_3 \cup P)^{\#} = (zz^{-1})(\rho_3 \cup P)^{\#}$. By (1.1) and Lemma 1.14, we obtain $W_P[(uy)^2(zz^{-1})(y^{-1}u^{-1})^2] = W_P(zz^{-1})$. Hence $(u\iota .y)^2 z = [(uy)^2 z]\iota \in Q[(uy)^2(zz^{-1})(y^{-1}u^{-1})^2] = W_{P,1}[(uy)^2(zz^{-1})(y^{-1}u^{-1})^2] \subseteq W_P[(uy)^2(zz^{-1})(y^{-1}u^{-1})^2] = W_P(zz^{-1})$ and so, by Lemma 7.1, we must have $(u\iota .y)^2(\rho_2 \cup P)^{\#} = 1(\rho_2 \cup P)^{\#}$. But now, by (7.3), we have $u\iota(\pi_1 \cup T)^{\#} = 1(\pi_1 \cup T)^{\#}$. Thus $u(\pi_1 \cup T)^{\#} = 1(\pi_1 \cup T)^{\#}$.

Conversely, suppose that $u(\pi_1 \cup T)^{\#} = 1(\pi_1 \cup T)^{\#}$. It follows easily that $u(\rho_3 \cup P)^{\#} = 1(\rho_3 \cup P)^{\#}$. Moreover, $y^2(\rho_3 \cup P)^{\#} = 1(\rho_3 \cup P)^{\#}$ yields $y(\rho_3 \cup P)^{\#} = y^{-1}(\rho_3 \cup P)^{\#}$ and so we have $(uyzz^{-1}y^{-1}u^{-1})(\rho_3 \cup P)^{\#} = (y^{-1}u^{-1}zz^{-1}uy)(\rho_3 \cup P)^{\#}$. Thus (7.4) holds and the theorem is proved.

CHAPTER VII

SEMIGROUP RINGS

1. Preliminaries

Let R be a nonempty set and let $+$ and $\cdot$ denote two binary operations on R. We say that $(R,+,\cdot)$ is a *ring* if:

(*i*) $(R,+)$ is an abelian group;

(*ii*) $(R,\cdot)$ is a semigroup;

(*iii*) $\forall a,b,c \in R$, $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$ and $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$.

We will refer to $(R,+,\cdot)$ simply as R. The unity of $(R,+)$ is said to be the *zero* of R and is denoted by 0. Whenever $(R,\cdot)$ has a unity, we denote it by 1 and we say it is the *unity* of R.

The ring R is said to be *prime* if $aRb \neq 0$ for every $a,b \in R\backslash\{0\}$.

Let R be a ring and let S be a semigroup. The *semigroup ring* $R[S]$ is the ring consisting of all finite formal sums $\sum_i \lambda_i s_i$ ($\lambda_i \in R$, $s_i \in S$) with the natural addition and multiplication defined by

$$\sum_i \lambda_i s_i \cdot \sum_j \mu_j s_j' = \sum_{i,j} (\lambda_i \mu_j)(s_i s_j').$$

Let $w \in R[S]$. If $w \neq 0$, say $w = \sum_{i=1}^{n} \lambda_i s_i$, where $s_1,\ldots,s_n$ are distinct elements of S and $\lambda_1,\ldots,\lambda_n$ are nonzero elements of R, we define $supp(w)$ to be $\{s_1,\ldots,s_n\}$. If $w = 0$, we define $supp(w)$ to be $\emptyset$.

Suppose that R is a ring with a unity. Then, for every $s \in S$, we denote $1s \in R[S]$ simply by s.

The problem of finding necessary and sufficient conditions on S for $R[S]$ to be prime is not solved yet. However, a famous result was obtained by Connell in the case of groups.

THEOREM 1.1 [6]. *Let R be a nontrivial ring with unity and let G be a group. Then $R[G]$ is prime if and only if R is prime and G has no nonidentity finite normal subgroups.*

2. The Condition C

Consider the following condition on a semigroup S.

CONDITION C. *For any pair A,B of finite nonempty subsets of S there exist $c,d \in S$ such that $|d\varphi^{-1}| = 1$, where $\varphi:A \times B \longrightarrow S$ is the map defined by $(a,b)\varphi = acb$.*

We note that if, for any pair A,B of finite nonempty subsets of S, there exists $c \in S$ such that φ is injective, then the condition is satisfied.

Condition C may be regarded as a weakening of the *u.p. property* [33,§13.1] for semigroups. A semigroup S is said to have the u.p. property if and only if, for any pair A,B of finite nonempty subsets of S, there exists $d \in S$ such that $|d\varphi^{-1}| = 1$, where $\varphi:A \times B \longrightarrow S$ is the map defined by $(a,b)\varphi = ab$. We prove that a semigroup with the u.p. property satisfies Condition C.

Suppose that S has the u.p. property. Let A and B be finite nonempty subsets of S. Let $c \in S$. Since S has the u.p. property, there exists $d \in S$ such that $|d\psi^{-1}| = 1$, where $\psi: A \times (cB) \longrightarrow S$ is the map defined by $(a,cb)\psi = acb$. Consider the map $\varphi: A \times B \longrightarrow S$ defined by $(a,b)\varphi = acb$. We show that $|d\varphi^{-1}| = 1$. Since $|d\psi^{-1}| = 1$, we certainly have $|d\varphi^{-1}| \geq 1$.

Suppose that $(a,b)\varphi = (a',b')\varphi$. If $a \neq a'$ or $cb \neq cb'$, then $|d\psi^{-1}| > 1$, which is impossible. Hence $a = a'$ and $cb = cb'$. But, by the u.p. property, we must have $|(cb)\eta^{-1}| = 1$, where $\eta:\{c\} \times \{b,b'\} \longrightarrow S$ is the map defined by $(c,x) = cx$. Hence $b = b'$. Thus $|d\varphi^{-1}| = 1$ and S satisfies Condition C.

Now we have

THEOREM 2.1. *Let R be a ring and let S be a semigroup satisfying Condition C. Then*

$$R[S] \text{ is prime} \Longleftrightarrow R \text{ is prime.}$$

Proof. Assume that R is not prime. Then there exist $\alpha,\beta \in R\backslash\{0\}$ such that $\alpha R\beta = 0$. Let $s \in S$. Then if $w \in R[S]\backslash\{0\}$, say $w = \sum_{i=1}^{n} \lambda_i s_i$, we have that $\alpha s.w.\beta s = \sum_{i=1}^{n} (\alpha\lambda_i\beta)(ss_i s) = 0$. Thus $\alpha s.R[S].\beta s = 0$ and $R[S]$ is not prime.

Now assume that $R[S]$ is not prime. Then there exist $u,v \in R[S]\backslash\{0\}$ such that $u.R[S].v = 0$. By Condition C, there exist $c,d \in S$ such that $|d\varphi^{-1}| = 1$, where $\varphi: supp(u) \times supp(v) \longrightarrow S$ is defined by $(a,b)\varphi = acb$. Let $\gamma \in R$. Then we have $u.\gamma c.v = 0$. Writing $u = \sum_{i=1}^{n} \lambda_i a_i$ and $v = \sum_{j=1}^{m} \mu_j b_j$, where $a_1,\ldots,a_n$ are the elements of $supp(u)$ and $b_1,\ldots,b_m$ are the elements of $supp(v)$, we obtain $\sum_{i=1}^{n}\sum_{j=1}^{m} (\lambda_i\gamma\mu_j)(a_i c b_j) = 0$. There exist unique i and j such that $a_i c b_j = d$ and so $\lambda_i\gamma\mu_j = 0$. Hence $\lambda_i R\mu_j = 0$. Thus, since λ_i and μ_j are nonzero, R is not prime.

3. Free products of semigroups

Let S and T be semigroups. Assume that $S \cap T = \emptyset$. We define an equivalence χ on $S \cup T$ by

$$a\chi = b\chi \iff a,b \in S \text{ or } a,b \in T.$$

The *free product* of S and T in the category of semigroups [4,§9.4], denoted by $S *_{sgp} T$, is defined as the set of all nonempty sequences $(w_1,\ldots,w_n)$ on $S \cup T$ such that $w_i\chi \neq w_{i+1}\chi$ for every i, with the following operation:

$$(u_1,\ldots,u_n)(v_1,\ldots,v_m) = \begin{cases} (u_1,\ldots,u_n,v_1,\ldots,v_m) & \text{if } u_n\chi \neq v_1\chi \\ (u_1,\ldots,u_nv_1,\ldots,v_m) & \text{if } u_n\chi = v_1\chi. \end{cases}$$

For every $w = (w_1,\ldots,w_n) \in S *_{sgp} T$, we define the length $|w|$ of w to be n. It is a simple fact that $|uv| \leq |u|+|v|$ for every $u,v \in S *_{sgp} T$.

As a very important example, we note that every free semigroup of rank $n > 1$ is the free product of free semigroups of rank $n-1$ and 1.

LEMMA 3.1. *Let S and T be semigroups. Then $S *_{sgp} T$ satisfies Condition C.*

Proof. Assume that $S \cap T = \emptyset$.

Let A and B be finite nonempty subsets of $S *_{sgp} T$. Let a_1 and b_1 be elements with maximal length in A and B, respectively. Fix $s \in S$, $t \in T$. Let u and v denote respectively the last component of a_1 and the first component of b_1. We define

$$c = \begin{cases} (t,s) & \text{if } (u,v) \in S \times T \\ (s,t) & \text{if } (u,v) \in T \times S \\ (s) & \text{if } (u,v) \in T \times T \\ (t) & \text{if } (u,v) \in S \times S. \end{cases}$$

Let $\varphi : A \times B \longrightarrow S*_{sgp}T$ be defined by $(a,b)\varphi = acb$. Suppose that $(a_2,b_2)\varphi = (a_1,b_1)\varphi$ for some $(a_2,b_2) \in A \times B$. Since $|a_1|$ and $|b_1|$ are maximal, we have $|a_2cb_2| \leqslant |a_2|+|c|+|b_2| \leqslant |a_1|+|c|+|b_1|$. By choice of c, we have $|a_1|+|c|+|b_1| = |a_1cb_1|$. Since $a_1cb_1 = a_2cb_2$, this yields $|a_2|+|c|+|b_2| = |a_1|+|c|+|b_1|$. Hence $a_1 = a_2$ and $b_1 = b_2$. Therefore $|(a_1cb_1)\varphi^{-1}| = 1$ and $S*_{sgp}T$ satisfies Condition C.

Lemma 3.1 and Theorem 2.1 immediately yield

THEOREM 3.2. *Let R be a ring and let S,T be semigroups. Then*

$$R[S*_{sgp}T] \text{ is prime } \Leftrightarrow R \text{ is prime.}$$

4. One-relator semigroup presentations

Let X denote a nonempty set. Let $F = X^*$. For every $k \in \mathbb{Z}$, we define $F_k = \{w \in F\colon |w| = k\}$. Note, in particular, that $F_k = \emptyset$ if $k < 0$.

Let $a,b \in X^+$. Let $n = |a|-1$. We define

$$O(a,b) = \{k \in \mathbb{N}^0\colon b \in F_k aF\},$$

$$I(a,b) = \{k \in \mathbb{Z}\colon (F_n bF_n)\cap(F_{k+n}aF) \neq \emptyset\}.$$

Note that $O(a,b) \subseteq I(a,b) \subseteq \{-|a|+1,\ldots,|b|-1\}$.

When $O(a,b) \neq \emptyset$, we say that *a occurs* in b. Each element of $O(a,b)$ is then said to be an *occurrence* of a in b.

When $I(a,b) \neq \emptyset$, we say that *a intersects* b. Each element of $I(a,b)$ is then said to be an *intersection* of b by a.

We apply these concepts to the study of one-relator semigroup presentations. A *semigroup presentation* is an expression of the form $Sgp\langle X;T\rangle$, where X is a nonempty set and T is a relation on X^+. The semigroup defined by this presentation is the quotient $X^+/T^\#$.

LEMMA 4.1. *Let S be the semigroup defined by the presentation $Sgp\langle X;T\rangle$, where $T = \{(u,v)\}$ is a relation on X^+. Suppose that, for every $m \in \mathbb{N}$ such that $m > max\{|u|,|v|\}$, there exists $c_m \in X^+$ such that*

(i) $I(u,c_m) = I(v,c_m) = \emptyset$;

(ii) $|c_m| > m$;

(iii) $I(c_m,c_m)\cap\{-m,\ldots,m\} = \{0\}$.

Then S satisfies Condition C.

Proof. Let A,B be finite nonempty subsets of X^+ and let $m \in \mathbb{N}$ be such that $m > max(\{|p|: p \in A\cup B\}\cup\{|u|,|v|\})$. Let $a,a' \in A$, let $b,b' \in B$ and suppose that $(ac_mb)T^\# = (a'c_mb')T^\#$. Then there exist $w_0,\ldots,w_n \in X^+$ such that

$$w_0 = ac_mb,$$

$$w_n = a'c_mb',$$

$$\forall k \in \{1,\ldots,n\}\ \exists r_k,s_k \in X^+\colon \{w_{k-1},w_k\} = \{r_kus_k,r_kvs_k\}.$$

For $w \in X^+$, denote $|O(c_m,w)|$ by $\gamma(w)$ and define $\gamma(1) = 0$. Let $w = puq$, $w' = pvq$ for some $p,q \in F$. By (i), $\gamma(w) = \gamma(p)+\gamma(q) = \gamma(w')$ and so γ is invariant on each $T^\#$-class of X^+.

Now we want to show that $\gamma(w_0) = 1$. Suppose that c_m has a second occurrence in ac_m. By (ii), we know that c_m does not occur in a, hence $I(c_m,c_m)$ has a nonzero element. By (iii), this would imply that $|a| > m$, which is impossible. Hence c_m has a unique occurrence in ac_m. Similarly, we prove that c_m has a unique occurrence in c_mb, so $\gamma(w_0) = 1$. Therefore, for every k, $\gamma(w_k) = 1$. Thus, for every k, we can write $w_k = a_kc_mb_k$ unambiguously and it is clear that $a_{k-1}T^\# = a_kT^\#$ and $b_{k-1}T^\# = b_kT^\#$. Hence $aT^\# = a'T^\#$, $bT^\# = b'T^\#$, and so the map $\varphi\colon(AT^\#) \times (BT^\#) \longrightarrow S$ defined by $(aT^\#,bT^\#)\varphi = (ac_mb)T^\#$ is injective. Therefore S satisfies Condition C.

LEMMA 4.2. *Let S be the semigroup defined by the presentation* $Sgp\langle X;T\rangle$, *where* $T = \{(u,v)\}$ *is a relation on* X^+. *Suppose that, for every* $m \in \mathbb{N}$ *such that* $m > max\{|u|,|v|\}$, *there exist* $c_m, d_m \in X^+$ *such that*

(i) $|I(u,c_mud_m)| = |I(v,c_mvd_m)| = 1$, $I(u,c_mvd_m) = I(v,c_mud_m) = \emptyset$;

(ii) $|c_m|, |d_m| > m$.

Then S satisfies Condition C.

Proof. Let A,B be finite nonempty subsets of X^+ and let $m \in \mathbb{N}$ be such that $m > max(\{|p|: p \in A\cup B\}\cup\{|u|,|v|\})$. Let $a,a' \in A$, let $b,b' \in B$ and suppose that $(ac_mud_mb)T^{\#} = (a'c_mud_mb')T^{\#}$. Then there exist $w_0,\ldots,w_n \in X^+$ such that

$$w_0 = ac_mud_mb,$$

$$w_n = a'c_mud_mb',$$

$$\forall k \in \{1,\ldots,n\}\ \exists r_k,s_k \in X^+:\ \{w_{k-1},w_k\} = \{r_kus_k, r_kvs_k\}.$$

Consider an occurrence of c_mud_m (respectively c_mvd_m) in ac_mud_m. By (ii), the corresponding factor u (respectively v) must occur in c_mud_m. But, by (i), there is exactly one (respectively none) such occurrence in c_mud_m. The case c_mud_mb runs similarly, so $|O(c_mud_m,w_0)|+|O(c_mvd_m,w_0)| = 1$. For $w \in X^+$, denote $|O(c_mud_m,w)|+|O(c_mvd_m,w)|$ by $\delta(w)$ and define $\delta(1) = 0$.

Let $w = puq$, $w' = pvq$, for some $p,q \in F$. If $c_m <_r p$ and $d_m <_l q$, let $\epsilon = 1$. Otherwise, let $\epsilon = 0$. Then, by (i), $\delta(w) = \delta(p)+\epsilon+\delta(q) = \delta(w')$ and so δ must be invariant on each $T^{\#}$-class of X^+. Hence $\delta(w_k) = 1$ for every k. Therefore we can write $w_k = a_kc_mq_kd_mb_k$ unambiguously for every k, with $q_k \in \{u,v\}$. As in the proof of Lemma 5.1, we can now obtain $aT^{\#} = a'T^{\#}$ and $bT^{\#} = b'T^{\#}$, so the map $\varphi:(AT^{\#}) \times (BT^{\#}) \longrightarrow S$ defined by $(aT^{\#},bT^{\#})\varphi = (ac_mud_mb)T^{\#}$ is injective and S satisfies Condition C.

LEMMA 4.3. *Let S be the semigroup defined by the presentation $Sgp\langle X;T\rangle$, where $|X| \geq 3$ and $T = \{(u,v)\}$ is a relation on X^+. Then S satisfies Condition C.*

Proof. Let u_1 and u_2 denote respectively the first and last letters of u and let v_1 and v_2 denote respectively the first and last letters of v.

First, we will consider three particular cases.

(i) $|u| = 1$, $O(u_1,v) = \emptyset$.

In this case we have $S \simeq (X\backslash\{u_1\})^+$ and so, by Lemma 3.1, S satisfies Condition C.

(ii) $|\{u_1,u_2,v_1,v_2\}| \leq 2$, $|u| > 1$.

Let $x \in X\backslash\{u_1,u_2,v_1,v_2\}$ and let $y \in X\backslash\{x,v_1\}$. For every $m \in \mathbb{N}$ such that $m > max\{|u|,|v|\}$, let $c_m = x^m y x^m$. It is easy to see that the conditions of Lemma 4.1 are satisfied, so S satisfies Condition C.

(iii) $u = x^j y^k$, $v = x^l z$ (or $v = zy^n$), where x,y,z are distinct letters of X and $j,k,l,n \in \mathbb{N}$.

For every $m \in \mathbb{N}$ such that $m > max\{|u|,|v|\}$, let $c_m = x^{m+1}$, $d_m = y^{m+1}$. It is not difficult to see that the conditions of Lemma 4.2 are fulfilled, so S satisfies Condition C.

Now consider the general case. Let $x \in X\backslash\{u_2,v_2\}$.

Suppose that $X = \{u_1,v_1,x\}$. Then $|\{u_1,u_2,v_1,v_2\}| = 2$. If $|u| = |v| = 1$, then we are in case (i). If not, then we can assume, without loss of generality, that $|u| > 1$ and so we are in case (ii).

Hence we can assume that $X\backslash\{u_1,v_1,x\} \neq \emptyset$. Let $y \in X\backslash\{u_1,v_1,x\}$.

Suppose first that neither u nor v is of the form $x^j y^k$, $j,k \in \mathbb{N}$. Then, for every $m \in \mathbb{N}$ such that $m > max\{|u|,|v|\}$, we define $c_m = x^m y^m$ and we apply Lemma 4.1.

Finally, suppose that $u = x^j y^k$ (the case $v = x^j y^k$ being identical). Let $z \in X\backslash\{x,y\}$ and for every $m \in \mathbb{N}$ such that $m > max\{|u|,|v|\}$, define $c_m = x^m z y^m$. Clearly, $I(u,c_m) = \emptyset$.

Suppose that $I(v,c_m) \neq \emptyset$. Since $|v| < m$, $v_2 \neq x$ and $v_1 \neq y$, we have $I(v,c_m) = O(v,c_m)$. This yields the following possibilities:

$v = z$ (dual of case (i));

$v = x^l z$ (case (iii));

$v = zy^n$ (case (iii));

$v = x^l zy^n$ (case (ii)).

Now suppose that $I(v,c_m) = \emptyset$. Then the conditions of Lemma 4.1 are clearly fulfilled, so S satisfies Condition C and the lemma is proved.

From Lemma 4.3 and Theorem 2.1, we obtain

THEOREM 4.4. *Let R be a ring and let S be the semigroup defined by the presentation $Sgp\langle X;T\rangle$, where $|X| \geq 3$ and $T = \{(u,v)\}$ is a relation on X^+. Then*

$$R[S] \text{ is prime } \iff R \text{ is prime.}$$

This result cannot be extended to the case $|X| = 2$, as the following example shows.

Let $X = \{x,y\}$ and let S be the semigroup defined by the presentation $Sgp\langle X;T\rangle$, where $T = \{(xy,x)\}$. It is easy to see that $xT^\# = \{x\}\cup\{xy^n: n \in \mathbb{N}\}$. Let R be a nontrivial ring. Let $r \in R\backslash\{0\}$. Then $r(xT^\#)-r[(yx)T^\#]$ is a nonzero element of $R[S]$ and it is not difficult to verify that $r(xT^\#).R[S].[r(xT^\#)-r[(yx)T^\#]] = 0$. Hence $R[S]$ is not prime.

The case $|X| = 1$ is much simpler. Suppose that $X = \{x\}$ and $T = \{(u,v)\}$ is a relation on X^+. Let S be the semigroup defined by the presentation $Sgp\langle X;T\rangle$. In the trivial cases $u = v$ and $\{u,v\} = \{x,x^2\}$, Condition C is satisfied; but the other cases yield finite nontrivial monogenic semigroups and produce semigroup rings which are not prime, as we show.

Suppose that $u = x^n$ and $v = x^{n+k}$, with $n,k \in \mathbb{N}$ and $n+k > 2$. Let R be a nontrivial ring with unity. If $n = 1$, then S is the cyclic group of order $k > 1$ and so $R[S]$ is not prime, by Theorem 1.1. Now suppose that $n > 1$. Let $r \in R\backslash\{0\}$ and let $w = rx^{n-1} - rx^{n+k-1}$. Then $w \neq 0$ and $w.R[S] = 0$, hence $R[S]$ is not prime.

5. Free inverse monoids

Let E be a semilattice. We say that E is *pseudofinite* if E satisfies the following conditions:

(i) for every $e,f \in E$ with $e > f$, there exists $g \in Cov(e)$ such that $g \geqslant f$;

(ii) for every $e \in E$, $Cov(e)$ is finite.

Let S be an inverse semigroup with $E(S)$ pseudofinite. Let F be a ring with unity. For every $e \in E(S)$, let

$$\sigma(e) = \begin{cases} \prod_{g \in Cov(e)} (e-g) & \text{if } Cov(e) \neq \emptyset, \\ e & \text{if } Cov(e) = \emptyset. \end{cases} \tag{5.1}$$

For every $e \in E(S)$, it is clear that $e \in supp[\sigma(e)]$, hence $\sigma(e) \neq 0$. Now we have

LEMMA 5.1 [27]. *Let S be a non-bisimple inverse semigroup with $E(S)$ pseudofinite. Let F be a ring with unity. Let $e,f \in E(S)$ with $e\mathcal{D} \neq f\mathcal{D}$. Then $\sigma(e).F[S].\sigma(f) = 0$.*

Our purpose is to generalize this result to the case of arbitrary nontrivial rings. Of course, in the definition of $\sigma(e)$ we are using the fact that F has a unity, but we can write

$$\sigma(e) = e + \sum_{\emptyset \neq I \subseteq Cov(e)} (-1)^{|I|}\left(\prod_{g \in I} g\right)$$

for every $e \in E(S)$.

Let R be a nontrivial ring and let $r \in R\setminus\{0\}$. For all $e \in E(S)$, we define

$$\sigma_r(e) = re + \sum_{\emptyset \neq I \subseteq Cov(e)} (-1)^{|I|} r\left(\prod_{g \in I} g\right).$$

Clearly, $\sigma_r(e) \neq 0$. Now we have

LEMMA 5.2. *Let S be a non-bisimple inverse semigroup with $E(S)$ pseudofinite. Let R be a nontrivial ring and let $r \in R\setminus\{0\}$. Let $e,f \in E(S)$ with $e\mathcal{D} \neq f\mathcal{D}$. Then $\sigma_r(e).R[S].\sigma_r(f) = 0$.*

Proof. Let $F = R \times \mathbb{Z}$ and consider the operations on $\mathbb{Z}$ defined by $(r_1,p_1)+(r_2,p_2) = (r_1+r_2,p_1+p_2)$ and $(r_1,p_1)(r_2,p_2) = (r_1r_2+p_1r_2+p_2r_1,p_1p_2)$. It is well-known [22,§1.2] that F is a ring with unity $(0,1)$ and that the map $\theta:R \longrightarrow F$ defined by $x\theta = (x,0)$ is a ring embedding.

We can extend θ to a ring embedding Θ: $R[S] \longrightarrow F[S]$, defined by $(\sum_i r_is_i)\Theta = \sum_i (r_i\theta)s_i$. For every $h \in E(S)$, we define $\sigma(h) \in F[S]$ as in (5.1). Now, we have $[\sigma_r(h)]\Theta = (r,0)h + \sum_{\emptyset \neq I \subseteq Cov(h)} ((-1)^{|I|}r,0)\left(\prod_{g \in I} g\right)$ $= (r,0)h.[h + \sum_{\emptyset \neq I \subseteq Cov(h)} (-1)^{|I|}\left(\prod_{g \in I} g\right)] = (r,0)h.\sigma(h)$. By Lemma 5.1, we have $\sigma(e).F[S].\sigma(f) = 0$. Therefore $[\sigma_r(e).R[S].\sigma_r(f)]\Theta$ $= [\sigma_r(e)]\Theta.(R[S])\Theta.[\sigma_r(f)]\Theta = (r,0)e.\sigma(e).(R[S])\Theta.(r,0)f.\sigma(f)$ $\subseteq (r,0)e.\sigma(e).F[S].\sigma(f) = 0$. Thus $\sigma_r(e).R[S].\sigma_r(f) = 0$.

Since free inverse semigroup (monoid) S of finite rank satisfies the conditions of Lemma 5.2, we obtain the next result, proved by Munn [27] for rings with unity.

THEOREM 5.3. *Let X be a finite nonempty set and let R be a nontrivial ring. Then $R[FIS(X)]$ and $R[FIM(X)]$ are not prime.*

The infinite rank case provides us with opposite results.

LEMMA 5.4. *A free inverse monoid (semigroup) of infinite rank satisfies Condition C.*

Proof. Let X be an infinite set and let A,B be finite nonempty subsets of $(X\cup X^{-1})^*$. Since X is infinite, there exists $x \in X\setminus[\bigcup_{c\in A\cup B} \xi(c)]$. We claim that the mapping $\varphi:(A\rho) \times (B\rho) \longrightarrow FIM(X)$ defined by $(a\rho,b\rho)\varphi = (axb)\rho$ is injective. Suppose that $(axb)\rho = (a'xb')\rho$ for some $a,a' \in A$ and $b,b' \in B$. Since $Q(axb) = Q(a)\cup[ax.Q(b)]\iota$, we have that

$$\{u \in Q(axb):\ x \notin \xi(u)\} = Q(a).$$

Similarly,

$$\{u \in Q(a'xb'):\ x \notin \xi(u)\} = Q(a').$$

But $Q(axb) = Q(a'xb')$ and so $Q(a) = Q(a')$. Moreover, we have

$$\{u \in Q(axb):\ x \leq_r u\} = \{a\iota.x\},\ \{u \in Q(a'xb'):\ x \leq_r u\} = \{a'\iota.x\}.$$

Hence $a\iota = a'\iota$ and so $a\rho = a'\rho$. We also have

$$\{u \in Q(axb):\ x \in \xi(u)\} = a\iota.x.Q(b),$$
$$\{u \in Q(a'xb'):\ x \in \xi(u)\} = a'\iota.x.Q(b');$$

therefore $Q(b) = Q(b')$. Further, since $(axb)\rho = (a'xb')\rho$, we have

$$a\iota.x.b\iota = a'\iota.x.b'\iota,$$

and so $b\iota = b'\iota$. Hence $b\rho = b'\rho$. Thus φ is injective and Condition C is satisfied.

The proof for the free inverse semigroup is identical.

From Lemma 5.4 and Theorem 2.1, we now have

THEOREM 5.5. *Let R be a ring and let S be a free inverse monoid [semigroup] of infinite rank. Then*

$$R[S] \text{ is prime} \Leftrightarrow R \text{ is prime.}$$

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