SURFACE WAVES IN PRE-STRESSED ELASTIC MATERIALS

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PREFACE

This dissertation is submitted in accordance with the regulations for the degree of Master of Science in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University.

I would like here to express my deepest gratitude to Professor R.W. Ogden, Professor of Mathematics, Glasgow University for his guidance, constant interest and encouragement throughout the period of this research.

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INTRODUCTION

The object of this thesis is to review the existing work on surface waves in pre-stressed elastic materials. In particular we shall be concerned with Rayleigh waves and Love waves.

In Chapter 1 we summarize the main results of non-linear elasticity which will be required in subsequent chapters. In particular, we consider the general forms of strain-energy functions for both incompressible and compressible isotropic elastic materials. In section 5 of this Chapter we establish the equations of motion for both incompressible and compressible materials which are used in our discussion of surface waves. Also, the relevant boundary conditions are noted. In the last part of this Chapter we consider plane waves in an infinite medium. The work in this Chapter is based on, for example, Ogden (1984) and Truesdell and Noll (1965).

Chapter 2 is concerned with Rayleigh waves. We start with the analysis for incompressible materials with particular attention paid to propagation along a principal axis. For a general form of strain-energy function the secular equation for Rayleigh waves in a pre-strained incompressible elastic material is obtained. This generalizes results given by Willson (1973a). Next, we consider some special deformations and obtain some new results along with the well known results for Rayleigh waves in the linear theory. Also, for some particular strain energy functions we obtain explicit solutions of the secular equation.

Rayleigh waves propagating in a general direction in the (x_1,x_2)-plane are also considered. Because, in general, the equation involve complicated algebra, we confine attention to the neo-Hookean material and deduce results equivalent to those given by Flavin (1963).

Corresponding analysis is given for compressible materials.

In Chapter 3, we discuss Love waves in a pre-strained layer on a pre-strained half-space. As for Rayleigh waves we consider both incompressible and compressible materials. We obtain the dispersion equation for waves propagating along a principal axis of the underlying deformation in respect of a general strain-energy function. For illustration we then consider a neo-Hookean material and obtain some numerical results for this case.

CHAPTER 1

Basic equations of non-linear elasticity:

1.1 Kinematics and mass conservation

Here we introduce the notation required for the description of the deformation of an elastic body. We consider a continuous body which occupies the region B_r in some natural (i.e. unstressed) configuration. Let a typical point of B_r, P say, have position vector $\underline{X}$ relative to some (arbitrarily chosen) origin O.

The *motion* of the body, in which the body occupies the region B_t at time t, is described by the one-parameter mapping

$$\chi_t: \quad B_r \to B_t,$$

with $$\underline{x} = \underline{\chi}_t(\underline{X}) \equiv \underline{\chi}(\underline{X},\ t) \tag{1.1.1}$$

being the position occupied by P in B_t. In Cartesian components equation (1.1.1) may be expressed

$$\begin{aligned} x_1 &= \chi_1(X_1,X_2,X_3,t), \\ x_2 &= \chi_2(X_1,X_2,X_3,t), \\ x_3 &= \chi_3(X_1,X_2,X_3,t). \end{aligned} \tag{1.1.2}$$

We require $\underline{\chi}_t$ to have a unique inverse $\underline{\chi}_t^{-1}$ such that

$$\underline{X} = \underline{\chi}_t^{-1}(\underline{x}) \qquad \underline{x} \in B_t \tag{1.1.3}$$

We also assume that $\underline{\chi}$ is twice-continuously differentiable when this degree of regularity is required.

The *velocity* and *acceleration* of the material particle P are given by

$$\underline{v} = \frac{\partial \underline{\chi}}{\partial t}(\underline{X},\ t), \qquad \underline{f} = \frac{\partial^2 \underline{\chi}}{\partial t^2}(\underline{X},\ t) \tag{1.1.4}$$

respectively, where $\partial/\partial t$ here denotes differentiation with respect to t at fixed $\underline{X}$.

The *deformation gradient* tensor, the gradient of (1.1.1), is given by

$$\underset{\sim}{A} = \text{Grad } \underset{\sim}{x}(\underset{\sim}{X}, t), \tag{1.1.5}$$

where Grad denotes the gradient operator with respect to $\underset{\sim}{X}$.

It follows from (1.1.4) and (1.1.5) that

$$\dot{\underset{\sim}{A}} = \underset{\sim}{\Gamma}\,\underset{\sim}{A}, \tag{1.1.6}$$

where the superposed dot denotes $\partial/\partial t$ at fixed $\underset{\sim}{X}$ and

$$\underset{\sim}{\Gamma} = \text{grad } \underset{\sim}{\nu} \tag{1.1.7}$$

is the *velocity gradient* tensor. Note that in (1.1.7) grad denotes the gradient operator with respect to $\underset{\sim}{x}$.

In Cartesian components, we have

$$A_{ij} = \frac{\partial x_i}{\partial X_j}(\underset{\sim}{X}, t) \equiv \frac{\partial x_i}{\partial X_j} \tag{1.1.8}$$

$$\Gamma_{ij} = \frac{\partial \nu_i}{\partial x_j} \tag{1.1.9}$$

We write

$$J = \det \underset{\sim}{A} \tag{1.1.10}$$

and impose the usual constraint

$$J > 0, \tag{1.1.11}$$

which ensures that the deformation is locally invertible, i.e. that $\underset{\sim}{A}^{-1}$ exists. We use the notation

$$\underset{\sim}{B} = (\underset{\sim}{A}^{-1})^T \tag{1.1.12}$$

where ^{T} denotes the transpose a second-order tensor.

In components

$$B_{ij} = \frac{\partial X_j}{\partial x_i} \tag{1.1.13}$$

We shall make use of the following polar decompositions for $\underline{A}$:

$$A = \underline{R}\ \underline{U} = \underline{V}\ \underline{R}, \tag{1.1.14}$$

where $\underline{R}$ is proper orthogonal and $\underline{U}$ and $\underline{V}$ are positive definite and symmetric, the *right* and *left stretch* tensors respectively.

Each of $\underline{U}$ and $\underline{V}$ may be expressed in spectral form

$$\begin{aligned} \underline{U} &= \sum_{i=1}^{3} \lambda_i\ \underline{u}^{(i)} \otimes \underline{u}^{(i)}, \\ \underline{V} &= \sum_{i=1}^{3} \lambda_i\ \underline{v}^{(i)} \otimes \underline{v}^{(i)}, \end{aligned} \tag{1.1.15}$$

where $\lambda_i(>0)$, $i \in \{1,2,3\}$, are the *principal stretches* of the deformation, and $\underline{u}^{(i)}$ and $\underline{v}^{(i)}$ respectively are the unit eigenvectors of $\underline{U}$ and $\underline{V}$. We shall refer to $\underline{u}^{(i)}$ and $\underline{v}^{(i)}$, $i \in \{1,2,3\}$, as the *Lagrangian* and *Eulerian Principal* axes respectively.

The right Cauchy-Green deformation tensor $\underline{C}$ is given by

$$\underline{C} = \underline{A}^T\underline{A} = \underline{U}^2 = \sum_{i=1}^{3} \lambda_i^2\ \underline{u}^{(i)} \otimes \underline{u}^{(i)} \tag{1.1.16}$$

For future reference we note that *principal invariants* of $\underline{C}$, denoted I_1, I_2, I_3, are given by

$$\begin{aligned} I_1 &= \mathrm{tr}(\underline{C}), \\ I_2 &= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\mathrm{tr}(\underline{C}^2), \\ I_3 &= \det \underline{C}. \end{aligned} \tag{1.1.17}$$

Let ρ_r denote the mass density of the material in B_r and ρ the corresponding density in B_t. *Conservation of mass* is expressed by means of the equation

$$\rho_r/\rho = J \equiv \det \underline{A} \tag{1.1.18}$$

For an *isochoric* (volume preserving) deformation $J = 1$ and $\rho=\rho_r$. An *incompressible material* is one for which *every* deformation is necessarily isochoric, i.e.

$$\rho_r/\rho = J = 1 \qquad \underline{X} \in B_r \tag{1.1.19}$$

In view of (1.1.14) and (1.1.15) equation (1.1.18) may also be written

$$\rho_r/\rho = \det\underline{U} = \lambda_1\lambda_2\lambda_3, \tag{1.1.20}$$

with

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{1.1.21}$$

for an isochoric deformation.

We shall also require the rate form of (1.1.18), namely

$$\frac{\partial\rho}{\partial t} + \rho \operatorname{div} \underline{v} = 0, \tag{1.1.22}$$

which yields

$$\operatorname{tr}(\underline{\Gamma}) \equiv \operatorname{div} \underline{v} = 0 \tag{1.1.23}$$

when the motion is isochoric.

Let $\underline{N}$ denote the unit outward normal to the boundary ∂B_r of B_r and $\underline{n}$ the corresponding unit normal to the boundary ∂B_t of B_t. Then, according to Nanson's formula, area elements da_r and da of ∂B_r and ∂B_t are related by

$$\underline{n}da = J\underline{B}\,\underline{N}\,da_r. \tag{1.1.24}$$

1.2 Stress and the equations of motion

The traction (load) on the area element da of the deformed surface ∂B_t is expressible in the form

$$\underline{\sigma}^T\underline{n}\,da = \underline{S}^T\,\underline{N}\,da_r, \tag{1.2.1}$$

where $\underline{\sigma}^T$ is the Cauchy stress tensor (independent of $\underline{n}$) and $\underline{S}$ the nominal stress tensor. In view of (1.1.24) equation (1.2.1) yields

$$\underline{S} = J\, \underline{B}^T \underline{\sigma} \tag{1.2.2}$$

and we shall use this connection later.

In this thesis we shall make use of the equations of motion expressed in terms of nominal stress. Thus

$$\text{Div}\, \underline{S} = \rho_r \underline{f}, \tag{1.2.3}$$

where $\underline{f}$ is the acceleration given by (1.1.4), Div denotes the divergence operator with respect to $\underline{X}$ and body forces are disregarded.

The rate form of (1.2.3) is obtained by differentiating with respect to t at fixed $\underline{X}$ to give, on use of (1.1.4),

$$\text{Div}\, \dot{\underline{S}} = \rho_r \ddot{\underline{\nu}}, \tag{1.2.4}$$

where the dot indicates the differentiation in question.

Furthermore, if the reference configuration is updated from B_r to the current configuration B_t then (1.2.4) is replaced by

$$\text{div}\, \dot{\underline{S}}_0 = \rho\, \ddot{\underline{\nu}}, \tag{1.2.5}$$

where div denotes the divergence operator with respect to $\underline{x}$ and $\dot{\underline{S}}_0$ represents $\dot{\underline{S}}$ evaluated in B_t *after* differentiation with respect to t.

The equations of rotational balance are satisfied when the Cauchy stress tensor $\underline{\sigma}$ is symmetric, or, equivalently,

$$\underline{A}\, \underline{S} = \underline{S}^T \underline{A}^T. \tag{1.2.6}$$

The rate counterpart of (1.2.6) is obtained by differentiating with respect to t and updating the reference configuration to B_t to give

$$\dot{\underline{S}}_0 + \underline{\Gamma}\, \underline{\sigma} = \dot{\underline{S}}_0^T + \underline{\sigma}\, \underline{\Gamma}^T, \tag{1.2.7}$$

where $\underline{\Gamma}$ is defined in (1.1.7).

1.3 Constitutive laws for elastic materials

We consider an elastic material for which there is a strain energy $W(\underline{A})$ per unit reference volume, so that the nominal stress tensor is given by

$$\underline{S} = \frac{\partial W}{\partial \underline{A}} \tag{1.3.1}$$

or, in components,

$$S_{ji} = \frac{\partial W}{\partial A_{ij}} \tag{1.3.2}$$

It is assumed that the material is homogeneous so that W has no explicit dependence on $\underline{X}$ (i.e. it depends on $\underline{X}$ only through $\underline{A}$).

For the function W to be *objective* (i.e. unaffected by a superposed rigid-body rotation *after* deformation), it must depend on $\underline{A}$ only through the right stretch tensor $\underline{U}$ occurring in (1.1.14); thus

$$W(\underline{A}) \equiv W(\underline{U}). \tag{1.3.3}$$

With $\underline{U}$, analogously to (1.3.1), we associate the so-called Biot stress tensor $\underline{T}$ defined by

$$\underline{T} = \frac{\partial W}{\partial \underline{U}}\,. \tag{1.3.4}$$

If the material is *isotropic* relative to B_r then W must also be unaffected by an arbitrary rigid-body rotation *before* deformation. Coupled with the objectivity requirment (1.3.3) this leads to the standard restriction on W, namely

$$W(\underline{Q}\ \underline{U}\ \underline{Q}^T) = W(\underline{U}) \tag{1.3.5}$$

for all orthogonal $\underline{Q}$.

Because of (1.1.15) this ensures that W depends only on the principal stretches λ_1, λ_2, λ_3 and is indifferent to any pairwise interchange of λ_1, λ_2, λ_3. Without changing notation, we express this as

$$W(\lambda_1,\lambda_2,\lambda_3) = W(\lambda_1,\lambda_3,\lambda_2) = W(\lambda_3,\lambda_1,\lambda_2). \qquad (1.3.6)$$

It then follows that $\underline{T}$ is coaxial with $\underline{U}$ and, from (1.3.4), we obtain

$$\underline{T} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \underline{u}^{(i)} \otimes \underline{u}^{(i)}, \qquad (1.3.7)$$

analogously to (1.1.15).

It is convenient to introduce the notation t_i, $i \in \{1,2,3\}$, for the principal values of $\underline{T}$. Then

$$t_i = \frac{\partial W}{\partial \lambda_i} \quad i \in \{1,2,3\} \qquad (1.3.8)$$

for an (unconstrained) isotropic material, and hence

$$\underline{T} = \sum_{i=1}^{3} t_i \, \underline{u}^{(i)} \otimes \underline{u}^{(i)} \qquad (1.3.9)$$

The corresponding expression for the Cauchy stress tensor $\underline{\sigma}$ is

$$\underline{\sigma} = \sum_{i=1}^{3} \sigma_i \, \underline{v}^{(i)} \otimes \underline{v}^{(i)} \qquad (1.3.10)$$

(coaxial with $\underline{V}$), with

$$\sigma_i = J^{-1}\lambda_i t_i = J^{-1} \lambda_i \frac{\partial W}{\partial \lambda_i}. \qquad (1.3.11)$$

For completeness, we note that

$$\underline{S} = \sum_{i=1}^{3} t_i \, \underline{u}^{(i)} \otimes \underline{v}^{(i)}, \qquad (1.3.12)$$

which is analogous to the decomposition

$$\underline{A} = \sum_{i=1}^{3} \lambda_i \, \underline{v}^{(i)} \otimes \underline{u}^{(i)} \qquad (1.3.13)$$

for the deformation gradient (the latter being obtained from (1.1.14) and (1.1.15) on noting $\underline{v}^{(i)} = \underline{R}\,\underline{u}^{(i)}$, $i \in \{1,2,3\}$).

1.3.1 Incompressible elastic materials

For an incompressible material it follows from (1.1.18) - (1.1.21) that the constraint

$$J \equiv \det \underline{A} \equiv \det \underline{U} \equiv \lambda_1\lambda_2\lambda_3 = 1 \tag{1.3.14}$$

must be satisfied at each point $\underline{X} \in B_r$. Then equations (1.3.1) - (1.3.4) are replaced by

$$\underline{S} = \frac{\partial W}{\partial \underline{A}} - p\underline{B}^T \tag{1.3.15}$$

and

$$\underline{T} = \frac{\partial W}{\partial \underline{U}} - p\underline{U}^{-1} \tag{1.3.16}$$

respectively, where p is an arbitrary function of $\underline{X}$ and acts as a Lagrange multiplier in respect of the constraint (1.3.14).

If the material is isotropic then I, $\underline{\sigma}$ and $\underline{S}$ are given by (1.3.9), (1.3.10) and (1.3.12) respectively, but t_i and σ_i become

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1} \qquad i \in \{1,2,3\} \tag{1.3.17}$$

(corresponding to the principal values of (1.3.16)) and

$$\sigma_i = \lambda_i\frac{\partial W}{\partial \lambda_i} - p \qquad i \in \{1,2,3\} \tag{1.3.18}$$

respectively.

1.4 Strain-energy functions for isotropic materials

We noted in (1.3.6) that for an isotropic elastic material the strain energy may be regarded as a symmetric function of λ_1, λ_2, λ_3. Equivalently, it may be considered as a function of the principal invariants I_1, I_2, I_3 defined in (1.1.17); in terms of λ_1, λ_2, λ_3 these are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$I_2 = \lambda_2^2\,\lambda_3^2 + \lambda_3^2\,\lambda_1^2 + \lambda_1^2\,\lambda_2^2, \qquad (1.4.1)$$
$$I_3 = \lambda_1^2\,\lambda_2^2\,\lambda_3^2,$$

and when the material is incompressible $I_3 \equiv 1$, and the remaining independent invariants are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}. \qquad (1.4.2)$$

For future reference we now record some specific forms of W for both incompressible and compressible isotropic elastic materials.

1.4.1 Incompressible materials

With W now regarded as a function of I_1 and I_2, as given in (1.4.2), equations (1.3.17) and (1.3.18) give

$$\lambda_i t_i = \sigma_i = 2\lambda_i^2 \frac{\partial W}{\partial I_1} - 2\lambda_i^{-2} \frac{\partial W}{\partial I_2} - p \quad i \in \{1,2,3\} \qquad (1.4.3)$$

and, on elimination of p, we obtain

$$\lambda_i t_i - \lambda_j t_j = \sigma_i - \sigma_j = 2(\lambda_i^2 - \lambda_j^2) \left[\frac{\partial W}{\partial I_1} + \lambda_i^{-2}\,\lambda_j^{-2}\,\frac{\partial W}{\partial I_2}\right] \qquad (1.4.4)$$

The Mooney (or Mooney-Rivlin) strain-energy function is defined as

$$W = C_1\,(I_1 - 3) + C_2\,(I_2 - 3), \qquad (1.4.5)$$

where C_1 and C_2 are physical constants, and the special case of this corresponding to $C_2 = 0$ yields the neo-Hookean form of strain energy, namely

$$W = C_1\,(I_1 - 3) \qquad (1.4.6)$$

The strain-energy function (1.4.5) is a particular member of the class of strain-energy functions proposed by Ogden (1972a). For these

$$W = \sum_{n=1}^{N} \mu_n \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right)/\alpha_n, \tag{1.4.7}$$

where μ_n and α_n, $n \in \{1,2,\ldots,N\}$, are material constants, and (1.3.17) and (1.3.18) yield

$$\lambda_i t_i = \sigma_i = \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n} - p \quad i \in \{1,2,3\} \tag{1.4.8}$$

Comparison of (1.4.5) and (1.4.7) shows that for the Mooney strain-energy function

$$\left.\begin{array}{l} \alpha_1 = 2, \quad \alpha_2 = -2, \\ \mu_1 = 2C_1, \; \mu_2 = -2C_2, \; \mu_n = 0 \quad n \in \{3,4,\ldots,N\}. \end{array}\right\} \tag{1.4.9}$$

A useful generalization of (1.4.7) is the Valanis Landel strain energy, for which

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3) \tag{1.4.10}$$

and hence

$$\lambda_i t_i = \sigma_i = \lambda_i w'(\lambda_i) - p, \tag{1.4.11}$$

where w is any suitably well-behaved function.

1.4.2 Compressible materials

For a compressible material use of the invariants (1.4.1) in (1.3.11) yields

$$J\sigma_i = \lambda_i t_i = 2\lambda_i^2 \frac{\partial W}{\partial I_1} + 2\lambda_i^2 (I_1 - \lambda_i^2) \frac{\partial W}{\partial I_2} + 2I_3 \frac{\partial W}{\partial I_3}, \tag{1.4.12}$$

and hence

$$J(\sigma_i - \sigma_j) = \lambda_i t_i - \lambda_j t_j = 2(\lambda_i^2 - \lambda_j^2)\left[\frac{\partial W}{\partial I_1} + \lambda_k^2 \frac{\partial W}{\partial I_2}\right], \tag{1.4.13}$$

where (i,j,k) is a permutation of (1,2,3).

For the strain-energy function

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + F(I_3),$$

a modification of (1.4.5), where $F(I_3)$ is a suitably well-behaved function, equation (1.4.13) simplifies to

$$J(\sigma_i - \sigma_j) = \lambda_i t_i - \lambda_j t_j = 2(\lambda_i^2 - \lambda_j^2)(C_1 + C_2\lambda_k^2). \qquad (1.4.14)$$

Finally, we consider a similar modification of (1.4.7), namely

$$W = \sum_{n=1}^{N} \mu_n \, (\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) + g(\lambda_1\lambda_2\lambda_3), \qquad (1.4.15)$$

where g is a function of $J = \lambda_1\lambda_2\lambda_3$. From (1.3.11) we obtain

$$J\sigma_i = \lambda_i t_i = \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n} + Jg'(J). \qquad (1.4.16)$$

1.4.3 Isotropic linear elasticity

For infinitesimal strains we use the variables

$$e_i = \lambda_i - 1 \qquad i \in \{1,2,3\} \qquad (1.4.17)$$

and linearize the stress-strain equations to obtain

$$t_i = \sigma_i = 2\mu e_i + \lambda(e_1 + e_2 + e_3), \; i \in \{1,2,3\} \qquad (1.4.18)$$

where λ and μ are the Lamé elastic moduli, correct to first order in e_1, e_2, e_3. The bulk modulus κ is defined by

$$\kappa = \lambda + \frac{2}{3}\mu \qquad (1.4.19)$$

For incompressible materials (1.4.18) is replaced by

$$t_i = \sigma_i = 2\mu e_i - p \qquad i \in \{1,2,3\} \qquad (1.4.20)$$

subject to

$$e_1 + e_2 + e_3 = 0, \qquad (1.4.21)$$

with p having the same interpretation as in (1.3.16).

Comparison of (1.4.20) with the linearized form of (1.4.8) shows that

$$\sum_{n=1}^{N} \mu_n \alpha_n = 2\mu. \tag{1.4.22}$$

In particular, for the Mooney strain-energy function (1.4.5) we have

$$\mu_1 - \mu_2 = 2\,(C_1 + C_2) = \mu. \tag{1.4.23}$$

The corresponding linearization of (1.4.16) again yields (1.4.22), and, in addition,

$$g'(1) + \sum_{n=1}^{N} \mu_n = 0, \qquad g''(1) = \lambda. \tag{1.4.24}$$

1.5 Elastic moduli

For use in the rate forms of the equations of motion (1.2.4) or (1.2.5) we shall require rate forms of the constitutive laws. First, for *compressible materials*, differentiation of (1.3.1) with respect to t at fixed $\underline{X}$ yields.

$$\dot{\underline{S}} = \underline{\mathcal{A}}\,\dot{\underline{A}}, \tag{1.5.1}$$

where $\underline{\mathcal{A}}$ is the fourth-order tensor given by

$$\underline{\mathcal{A}} = \frac{\partial^2 W}{\partial \underline{A}\,\partial \underline{A}}, \tag{1.5.2}$$

or, in components,

$$\dot{S}_{ji} = \mathcal{A}_{jilk}\,\dot{A}_{kl} \tag{1.5.3}$$

with

$$\mathcal{A}_{jilk} = \frac{\partial^2 W}{\partial A_{ij}\,\partial A_{kl}}. \tag{1.5.4}$$

We refer to $\underline{\mathcal{A}}$ as the tensor of *first-order elastic moduli* associated with the variables ($\underline{S}$, $\underline{A}$) relative to B_r.

If the reference configuration is now updated to coincide with the current configuration B_t, (1.5.1) becomes

$$\dot{\underline{S}}_0 = \mathcal{A}_0 \dot{\underline{A}}_0. \tag{1.5.5}$$

where the subscript zero indicates evaluation in B_t. From (1.1.6) we deduce that $\dot{\underline{A}}_0 = \underline{\Gamma}$. The tensor $\underline{\mathcal{A}}_0$ is called the tensor of first-order *instantaneous* elastic moduli associated with ($\underline{S}$, $\underline{A}$).

For *compressible isotropic materials* the components of $\underline{\mathcal{A}}_0$ referred to the Eulerian principal axes of the underlying deformation are derived in Ogden (1984), and we refer to this book for full details. Here it suffices to state that the only non-zero components of $\underline{\mathcal{A}}_0$ are

$$\begin{aligned}
\mathcal{A}_{0iiii} &= \lambda_i \frac{\partial \sigma_i}{\partial \lambda_i}, \\
\mathcal{A}_{0iijj} &= \lambda_j \frac{\partial \sigma_i}{\partial \lambda_j} + \sigma_i \qquad i \neq j, \\
\mathcal{A}_{0ijij} &= \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 \qquad i \neq j, \\
\mathcal{A}_{0ijji} = \mathcal{A}_{0jiij} &= \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 - \sigma_i \quad i \neq j,
\end{aligned} \tag{1.5.6}$$

where $i, j \in \{1,2,3\}$, and

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}. \tag{1.5.7}$$

In components equation (1.5.5) reads

$$\dot{S}_{0ji} = \mathcal{A}_{0jilk} \Gamma_{kl} \equiv \mathcal{A}_{0jilk} \frac{\partial v_k}{\partial x_l} \tag{1.5.8}$$

on noting (1.1.9).

For *incompressible materials*, differentiation of (1.3.15) with respect to t at fixed $\underline{X}$ and use of (1.1.6) and (1.1.12), followed by an update of the reference configuration to B_t, yields the counterpart of (1.5.5), namely

$$\dot{\underline{S}}_0 = \underline{\mathcal{A}}_0 \underline{\Gamma} + p\underline{\Gamma} - \dot{p}\underline{I}, \tag{1.5.9}$$

where $\underline{I}$ is the identity tensor. This is coupled with the rate form of the incompressibility condition:

$$\mathrm{tr}(\underline{\Gamma}) \equiv \mathrm{div}\, \underline{v} = 0, \tag{1.5.10}$$

as in (1.1.23).

In components

$$\dot{S}_{oji} = \mathcal{A}_{ojilk} \frac{\partial v_k}{\partial x_l} + p \frac{\partial v_j}{\partial x_i} - \dot{p}\delta_{ij} \tag{1.5.11}$$

with

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{1.5.12}$$

For incompressible isotropic materials the components of $\underline{\mathcal{A}}_0$ differ slightly from (1.5.6), and are given by

$$\begin{aligned} \mathcal{A}_{oiijj} &= \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}, \\ \mathcal{A}_{oijij} &= \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 \quad i \neq j, \\ \mathcal{A}_{oijji} &= \mathcal{A}_{ojiij} = \mathcal{A}_{oijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} \quad i \neq j, \end{aligned} \tag{1.5.13}$$

where

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \tag{1.5.14}$$

and $i,\ j \in \{1,2,3\}$.

For the special case in which $\lambda_i = \lambda_j$ for $i \neq j$ the formulae (1.5.6) and (1.5.13) still hold except that in the limit $\lambda_i \to \lambda_j$ A_{oijij} is replaced by

$$\mathcal{A}_{oijij} = \tfrac{1}{2}(\mathcal{A}_{oiiii} - \mathcal{A}_{oiijj} + \sigma_i) \qquad (1.5.15)$$

for *compressible materials*, and

$$\mathcal{A}_{oijij} = \tfrac{1}{2}(\mathcal{A}_{oiiii} - \mathcal{A}_{oiijj} + \lambda_i \frac{\partial W}{\partial \lambda_i}) \qquad (1.5.16)$$

for *incompressible materials*.

The equations of motion (1.2.5) have components form

$$\frac{\partial}{\partial x_j} \dot{S}_{oji} = \rho \ddot{\nu}_i \qquad i \in \{1,2,3\}$$

so, for compressible and incompressible materials respectively, equations (1.5.8), (1.5.11) and (1.5.12) yield

$$\frac{\partial}{\partial x_j} (\mathcal{A}_{ojilk} \frac{\partial \nu_k}{\partial x_l}) = \rho \ddot{\nu}_i \qquad (1.5.17)$$

and

$$\frac{\partial}{\partial x_j} (\mathcal{A}_{ojilk} \frac{\partial \nu_k}{\partial x_l}) + \frac{\partial \nu_j}{\partial x_i} \frac{\partial p}{\partial x_j} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{\nu}_i, \qquad (1.5.18)$$

the latter being coupled with (1.5.12).

When the underlying deformation from $B_r \to B_t$ is homogeneous $\mathcal{A}_o$ and p are independent of $\underline{x}$ and (1.5.17) and (1.5.18) simplify to

$$\mathcal{A}_{ojilk} \frac{\partial^2 \nu_k}{\partial x_j \partial x_l} = \rho \ddot{\nu}_i, \qquad (1.5.19)$$

$$\mathcal{A}_{ojilk} \frac{\partial^2 \nu_k}{\partial x_j \partial x_l} - \frac{\partial \dot{p}}{\partial x_i} = \rho \ddot{\nu}_i \qquad (1.5.20)$$

respectively.

Finally in this section we note that the traction rate $\dot{\underline{S}}_0^T\underline{n}$ on a surface with unit normal $\underline{n}$ in the current configuration B_t has components

$$\dot{S}_{oji}n_j \equiv (\mathcal{A}_{ojilk}\frac{\partial \nu_k}{\partial x_l})\, n_j \tag{1.5.21}$$

and

$$\dot{S}_{oji}n_j \equiv (\mathcal{A}_{ojilk} + p\delta_{jk}\delta_{il})\,\frac{\partial \nu_k}{\partial x_l}\, n_j - \dot{p}n_i \tag{1.5.22}$$

for compressible and incompressible materials respectively.

1.6 Plane waves in an infinite medium

As a prelude to our discussion of surface waves, we consider the propagation of plane waves in an unbounded medium. For a plane wave propagating in the direction of the unit vector $\underline{n}$ with speed c we may write

$$\underline{\nu} = \underline{m}f(t - \frac{\underline{n}.\underline{x}}{c}) \tag{1.6.1}$$

and, additionally, for an incompressible material,

$$\dot{p} = \frac{q}{c}\, f'\, (t - \frac{\underline{n}.\underline{x}}{c}), \tag{1.6.2}$$

where q is a constant and $\underline{m}$ a constant unit vector. We refer to $\underline{m}$ as the unit *amplitude vector.*

For an incompressible material substitution of (1.6.1) into (1.5.10) yields the constraint

$$\underline{m}.\underline{n} = 0. \tag{1.6.3}$$

Substitution of (1.6.1) and (1.6.2) into (1.5.19) and (1.5.20) yields

$$\mathcal{A}_{ojilk}n_jn_l\; f''(t - \frac{\underline{n}.\underline{x}}{c})\; m_k = \rho c^2 m_i\; f''(t - \frac{\underline{n}.\underline{x}}{c})$$

and

$$\mathcal{A}_{ojilk}n_jn_l\, f''(t - \frac{\underline{n} \cdot \underline{x}}{c})\, m_k + qn_i\, f''(t - \frac{\underline{n} \cdot \underline{x}}{c})$$

$$= \rho c^2 m_i\, f''(t - \frac{\underline{n} \cdot \underline{x}}{c})$$

respectively. On the assumption that f is a twice continuously differentiable function we deduce that

$$\mathcal{A}_{ojilk}\, n_jn_lm_k = \rho c^2 m_i \tag{1.6.4}$$

and

$$\mathcal{A}_{ojilk}\, n_jn_lm_k + qn_i = \rho c^2 m_i, \quad m_in_i = 0 \tag{1.6.5}$$

for compressible and incompressible materials respectively.

It is convenient to introduce the notation $\underline{Q}(\underline{n})$ for the second-order tensor (dependent on $\underline{n}$) with components defined by

$$Q_{ik}(\underline{n}) = \mathcal{A}_{ojilk}\, n_jn_l \tag{1.6.6}$$

Then (1.6.4) may be written compactly in the form

$$\underline{Q}(\underline{n})\underline{m} = \rho c^2 \underline{m}, \tag{1.6.7}$$

where, in view of the definitions (1.5.4) and (1.6.6), $\underline{Q}(\underline{n})$ is *symmetric* for each $\underline{n}$.

This guarantees that the secular equation

$$\det[\underline{Q}(\underline{n}) - \rho c^2 \underline{I}] = 0 \tag{1.6.8}$$

yields real eigen values $\rho c^2$for (1.6.7). However, for the existence of plane waves ρc^2must be positive. This follows if the *strong ellipticity condition*

$$\mathrm{tr}\{[\mathcal{A}_o(\underline{m}\otimes\underline{n})]\, (\underline{m}\otimes\underline{n})\} \equiv [\underline{Q}(\underline{n})\underline{m}]\, .\, \underline{m} > 0 \text{ all } \underline{m}\otimes\underline{n} \neq \underline{0} \tag{1.6.9}$$

holds.

From (1.6.7) the wave speed c associated with the direction of propagation $\underline{n}$ and the amplitude $\underline{m}$ is given by

$$\rho c^2 = [\underline{Q}(\underline{n})\underline{m}] \cdot \underline{m} = \mathcal{A}_{0jilk} n_j n_l m_i m_k. \qquad (1.6.10)$$

Equation (1.6.10) applies for compressible materials. For incompressible materials, using the notation (1.6.6), equation (1.6.5) yields

$$\underline{Q}(\underline{n})\underline{m} + q\underline{n} = \rho c^2 \underline{m}, \qquad \underline{m}.\underline{n} = 0. \qquad (1.6.11)$$

Taking the dot product of this with $\underline{n}$ we deduce that

$$q = -[\underline{Q}(\underline{n})\underline{m}].\underline{n},$$

so that (1.6.11) can be rewritten, analogously to (1.6.7), in the form

$$\underline{Q}^* (\underline{n})\, \underline{m} = \rho c^2 \underline{m}, \qquad \underline{m}.\underline{n} = 0, \qquad (1.6.12)$$

where $\underline{Q}^*(\underline{n})$ is defined by

$$\underline{Q}^*(\underline{n}) = \underline{Q}(\underline{n}) - \underline{n} \otimes [\underline{Q}^T(\underline{n})\underline{n}]. \qquad (1.6.13)$$

In this case the wave speed is given by

$$\rho c^2 = [\underline{Q}^*(\underline{n})\underline{m}].\underline{m} = [\underline{Q}(\underline{n})\underline{m}].\underline{m}. \qquad (1.6.14)$$

which is the same expression as (1.6.10) except that the constraint $\underline{m}.\underline{n} = 0$ must be satisfied.

An important distinction between $\underline{Q}(\underline{n})$ and $\underline{Q}^*(\underline{n})$ is that, whereas $\underline{Q}(\underline{n})$ is symmetric, $\underline{Q}^*(\underline{n})$ is not in general symmetric.

Plane waves for which $\underline{m}.\underline{n} = 0$ are said to be *transverse waves*, and the unit amplitude vector is then referred to as the *polarization vector*. Plane waves for which $\underline{m}=\underline{n}$ (in a compressible material) are called *longitudinal waves*. In general, there is no guarantee that either longitudinal or transverse waves will exist for particular choices of the direction of propagation. However, if $\underline{n}$ is along a principal axis of the underlying deformation then some simple results follow if $\underline{m}$ is also along such a principal direction. For future reference we now record these results.

First, for a compressible material, if $\underline{n} = \underline{\nu}^{(i)}$ and $\underline{m} = \underline{\nu}^{(j)}$, where $\underline{\nu}^{(1)}$, $\underline{\nu}^{(2)}$, $\underline{\nu}^{(3)}$ denote the Eulerian principal axes, and c_{ij} denotes the associated wave speed, then from (1.5.6), (1.5.7) and (1.6.10) we obtain

$$\rho c_{ii}^2 = \mathcal{A}_{oiiii} = \lambda_i \frac{\partial \sigma_i}{\partial \lambda_i} \quad i \in \{1,2,3\} \tag{1.6.15}$$

or, equivalently,

$$\rho_r c_{ii}^2 = \lambda_i^2 \frac{\partial^2 W}{\partial \lambda_i^2} \qquad i \in \{1,2,3\} \tag{1.6.16}$$

and also

$$\rho c_{ij}^2 = \mathcal{A}_{oijij} = \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2 \qquad i \neq j. \tag{1.6.17}$$

Equation (1.6.17) is also valid for incompressible materials. We shall make use of the notation defined in (1.6.15) - (1.6.17) in later sections of this thesis.

Finally, for waves propagating in an unstrained material we note that longitudinal and transverse waves exist for *every* direction of propagation. This follows from the fact that the components of $\underline{\mathcal{A}}_0$ reduce to

$$\mathcal{A}_{oijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl} \tag{1.6.18}$$

in B_r, where λ and μ are the Lamé moduli introduced in (1.4.18), and, for an incompressible material, to

$$\mathcal{A}_{oijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.6.19}$$

If C_L and C_T denote the speeds of propagation of longitudinal and transverse waves respectively in this special

$$\rho c_{ii}^2 = \rho c_L^2 = \lambda+2\mu \quad i \in \{1,2,3\}, \tag{1.6.20}$$

$$\rho c_{ij}^2 = \rho c_T^2 = \mu \quad i \neq j \in \{1,2,3\}. \tag{1.6.21}$$

Basic references to work on plane waves in deformed elastic materials are the paper by Hayes and Rivlin (1961a), which is concerned with isotropic materials possessing a strain-energy function, and the monograph by Truesdell and Noll (1965), which generalizes this to the case where the existence of a strain-energy function is not required.

CHAPTER 2

Rayleigh waves on a pre-strained elastic half-space

On the surface of an elastic body it is possible to have waves which are confined to near the surface of the body. Lord Rayleigh (1885) was first to investigate such waves, which now take his name.

Rayleigh's theory related to surface waves on the free surface of a semi-infinite elastic solid; he proved that the motion becomes negligible at a distance of a few wave lengths from the surface.

In this chapter, we shall discuss Rayleigh waves on a pre-strained elastic half-space for both incompressible and compressible materials. Our work is based on the equations derived in Chapter 1, and we recover certain results obtained by Hayes and Rivlin (1961b), who used a different approach, and generalize other results given by Flavin (1963) and Willson (1973a, 1974a,b) for incompressible materials and Willson (1972, 1973b) for compressible materials.

2.1 Analysis for incompressible materials

Consider the large homogeneous pure strain defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \tag{2.1.1}$$

Upon this deformation we superpose a small displacement $\underline{u}$, such that

$$x_1 = \lambda_1 X_1 + u_1, \quad x_2 = \lambda_2 X_2 + u_2, \quad x_3 = \lambda_3 X_3 + u_3 \tag{2.1.2}$$

where u_1, u_2, u_3 (which, in general depend on x_1, x_2, x_3 and t) are the components of $\underline{u}$. The velocity components are given by

$$\nu_i = \left.\frac{\partial u_i}{\partial t}\right|_{\underline{X}}. \tag{2.1.3}$$

From equation (1.3.18), we have the principal components of the Cauchy stress tensor associated with the homogeneous deformation, namely

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \quad i \in \{1,2,3\}.$$

By using the incremental equations (1.5.11) and (1.5.12), we deduce that

$$\begin{aligned}
\dot{s}_{011} &= \mathcal{A}_{01111}\frac{\partial \nu_1}{\partial x_1} + \mathcal{A}_{01122}\frac{\partial \nu_2}{\partial x_2} + \mathcal{A}_{01133}\frac{\partial \nu_3}{\partial x_3} + p\frac{\partial \nu_1}{\partial x_1} - \dot{p},\\
\dot{s}_{022} &= \mathcal{A}_{02211}\frac{\partial \nu_1}{\partial x_1} + \mathcal{A}_{02222}\frac{\partial \nu_2}{\partial x_2} + \mathcal{A}_{02233}\frac{\partial \nu_3}{\partial x_3} + p\frac{\partial \nu_2}{\partial x_2} - \dot{p},\\
\dot{s}_{033} &= \mathcal{A}_{03311}\frac{\partial \nu_1}{\partial x_1} + \mathcal{A}_{03322}\frac{\partial \nu_2}{\partial x_2} + \mathcal{A}_{03333}\frac{\partial \nu_3}{\partial x_3} + p\frac{\partial \nu_3}{\partial x_3} - \dot{p},\\
\dot{s}_{012} &= \mathcal{A}_{01212}\frac{\partial \nu_2}{\partial x_1} + \mathcal{A}_{01221}\frac{\partial \nu_1}{\partial x_2} + p\frac{\partial \nu_1}{\partial x_2},\\
\dot{s}_{021} &= \mathcal{A}_{02112}\frac{\partial \nu_2}{\partial x_1} + \mathcal{A}_{02121}\frac{\partial \nu_1}{\partial x_1} + p\frac{\partial \nu_2}{\partial x_1},\\
\dot{s}_{013} &= \mathcal{A}_{01313}\frac{\partial \nu_3}{\partial x_1} + \mathcal{A}_{01331}\frac{\partial \nu_1}{\partial x_3} + p\frac{\partial \nu_1}{\partial x_3},\\
\dot{s}_{031} &= \mathcal{A}_{03113}\frac{\partial \nu_3}{\partial x_1} + \mathcal{A}_{03131}\frac{\partial \nu_1}{\partial x_3} + p\frac{\partial \nu_3}{\partial x_1},\\
\dot{s}_{023} &= \mathcal{A}_{02323}\frac{\partial \nu_3}{\partial x_2} + \mathcal{A}_{02332}\frac{\partial \nu_2}{\partial x_3} + p\frac{\partial \nu_2}{\partial x_3},\\
\dot{s}_{032} &= \mathcal{A}_{03232}\frac{\partial \nu_2}{\partial x_3} + \mathcal{A}_{03223}\frac{\partial \nu_3}{\partial x_2} + p\frac{\partial \nu_3}{\partial x_2},
\end{aligned} \tag{2.1.4}$$

subject to the incompressibility condition (1.5.12).

2.1.1. Plane incremental motion

Next, we take $\nu_2 \equiv 0$ and assume that ν_1, ν_3 depend only on x_1, x_3. Then equation (1.5.12) reduces to

$$\frac{\partial \nu_1}{\partial x_1} + \frac{\partial \nu_3}{\partial x_3} = 0.$$

Hence there exists a function ψ (x_1, x_3, t) such that

$$\nu_1 = \frac{\partial\psi}{\partial x_3} = \psi_{,3}, \qquad \nu_3 = -\frac{\partial\psi}{\partial x_1} = -\psi_{,1}. \tag{2.1.5}$$

Equations (2.1.4) reduce to

$$\begin{aligned}
\dot{s}_{011} &= \mathcal{A}_{01111}\ \nu_{1,1} + \mathcal{A}_{01133}\ \nu_{3,3} + p\nu_{1,1} - \dot{p},\\
\dot{s}_{022} &= \mathcal{A}_{02211}\ \nu_{1,1} + \mathcal{A}_{02233}\ \nu_{3,3} - \dot{p},\\
\dot{s}_{033} &= \mathcal{A}_{03311}\ \nu_{1,1} + \mathcal{A}_{03333}\ \nu_{3,3} + p\nu_{3,3} - \dot{p},\\
\dot{s}_{013} &= \mathcal{A}_{01313}\ \nu_{3,1} + \mathcal{A}_{01331}\ \nu_{1,3} + p\nu_{1,3},\\
\dot{s}_{031} &= \mathcal{A}_{03131}\ \nu_{1,3} + \mathcal{A}_{03113}\ \nu_{3,1} + p\nu_{3,1}.
\end{aligned} \tag{2.1.6}$$

From (1.2.5), the incremental motion is governed by

$$\dot{s}_{oji,j} = \rho\ddot{\nu}_i. \tag{2.1.7}$$

Thus, from the incremental equations (1.5.11) we obtain

$$\dot{s}_{oji,j} \equiv \mathcal{A}_{ojilk}\ \nu_{k,lj} - p_{,j} = \rho\ddot{\nu}_i. \tag{2.1.8}$$

In equation (2.1.7), if we take i=1 and i=3, we deduce that

$$\begin{aligned}
\dot{s}_{011,1} + \dot{s}_{031,3} &= \rho\ddot{\nu}_1,\\
\dot{s}_{013,1} + \dot{s}_{033,3} &= \rho\ddot{\nu}_3.
\end{aligned} \tag{2.1.9}$$

From (2.1.5) and (2.1.6) we then obtain

$$\begin{aligned}
\dot{s}_{011,1} &= (\mathcal{A}_{01111} - \mathcal{A}_{01133} + p)\ \psi_{,113} - \dot{p}_{,1},\\
\dot{s}_{031,3} &= \mathcal{A}_{03131}\ \psi_{,333} - (\mathcal{A}_{03113} + p)\ \psi_{,113},\\
\dot{s}_{013,1} &= -\mathcal{A}_{01313}\ \psi_{,111} + (\mathcal{A}_{01331} + p)\ \psi_{,331},\\
\dot{s}_{033,3} &= (\mathcal{A}_{03311} - \mathcal{A}_{03333} - p)\ \psi_{,133} - \dot{p}_{,3}.
\end{aligned} \tag{2.1.10}$$

Now substitute (2.1.10) into (2.1.9) to get

$$\begin{aligned}
\rho\ddot{\psi}_{,3} &= -\dot{p}_{,1} + (\mathcal{A}_{01111} - \mathcal{A}_{01133} - \mathcal{A}_{03113})\ \psi_{,113} + \mathcal{A}_{03131}\ \psi_{,333},\\
-\rho\ddot{\psi}_{,1} &= -\dot{p}_{,3} + (\mathcal{A}_{01331} + \mathcal{A}_{03311} - \mathcal{A}_{03333})\ \psi_{,133} - \mathcal{A}_{01313}\ \psi_{,111}.
\end{aligned} \tag{2.1.11}$$

To eliminate $\dot{p}$ we need to differentiate equations (2.1.11) with respect to x_3 and x_1 respectively and obtain the partial differential equation governing ψ , namely

$$-\rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,33}) + \mathcal{A}_{01313}\ \psi_{,1111} +$$
$$(\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01331} - 2\mathcal{A}_{03311})\ \psi_{,1133}$$
$$+ \mathcal{A}_{03131}\ \psi_{,3333} = 0. \qquad (2.1.12)$$

Suppose the elastic medium occupies the half-space defined by $x_3 \geq 0$. In the basic homogeneous configuration the normal stress on the surface $x_3 = 0$ is σ_3. We assume that this is unaffected by the perturbed deformation, so that the incremental boundary tractions vanish. This means that

$$\dot{S}_{031} = 0, \quad \dot{S}_{033} = 0 \qquad \text{on } x_3 = 0,$$

that is

$$\left.\begin{aligned} &\mathcal{A}_{03131}\ v_{1,3} + (\mathcal{A}_{01331} + p)\ v_{3,1} = 0, \\ &(\mathcal{A}_{01133} - \mathcal{A}_{03333} - p)\ v_{1,1} - \dot{p} = 0. \end{aligned}\right\} \text{ on } x_3 = 0 \qquad (2.1.13)$$

or, in terms of ψ ,

$$\left.\begin{aligned} &\mathcal{A}_{03131}\ \psi_{,33} + (\mathcal{A}_{01331} + p)(-\ \psi_{,11}) = 0, \\ &(\mathcal{A}_{01133} - \mathcal{A}_{03333} - p)\ \psi_{,13} - \dot{p} = 0. \end{aligned}\right\} \text{ on } x_3 = 0 \qquad (2.1.14)$$

2.1.2 Propagation along a principal direction

We now assume that $\psi(x_1, x_3, t)$ has the form

$$\psi = f(x_3)e^{i\omega(t - \frac{x_1}{c})}. \qquad (2.1.15)$$

This represents a wave propagating with (constant) wave speed c in the x_1-direction, which is a principal direction of the underlying homogeneous deformation. The frequency of the wave is ω, also a constant.

We also assume that the spatial variation of ψ is of the form $e^{(-ikx_1 - sx_3)}$, where $k = \omega/c$ is the wave number. Then equation (2.1.12) demands that

$$\mathcal{A}_{03131}\, s^4 - (\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01331} - 2\mathcal{A}_{01133})k^2 s^2 + \mathcal{A}_{01313}\, k^4 + \rho\omega^2 (s^2 - k^2) = 0. \tag{2.1.16}$$

This is a quadratic equation for s^2. Suppose it has roots s_1^2 and s_2^2. Then

$$s_1^2 + s_2^2 = \frac{(\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01331} - 2\mathcal{A}_{01133})k^2 - \rho\omega^2}{03131}$$

$$s_1^2 s_2^2 = \frac{(\mathcal{A}_{01313}\, k^2 - \rho\omega^2)\, k^2}{03131}. \tag{2.1.17}$$

Assume $\dot{p}$ has also form of $e^{i\omega(t-\frac{x_1}{c})}\ e^{-sx_3}$.

Then, from (2.1.11), we obtain

$$ik\dot{p} = [(\mathcal{A}_{01111} - \mathcal{A}_{01133} - \mathcal{A}_{03113})\frac{\omega^2}{c^2} - \rho\omega^2]\ \psi_{,3} + \mathcal{A}_{03131}\ \psi_{,333}. \tag{2.1.18}$$

so, from (2.1.14) and (2.1.18), we obtain the boundary conditions in the form

$$\left.\begin{aligned} &\mathcal{A}_{03131}\ \psi_{,33} + (\mathcal{A}_{01331} + p)\, k^2\psi = 0, \\ &\mathcal{A}_{03131}\ \psi_{,333} + [(\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01133} - \mathcal{A}_{03113} + p)k^2 - \rho\omega^2]\ \psi_{,3} = 0. \end{aligned}\right\} \text{on } x_3 = 0 \tag{2.1.19}$$

For surface waves we must have a solution for ψ in equation (2.1.12) which decays when $x_3 \to +\infty$ and which satisfies the boundary conditions in (2.1.19) at the surface $x_3 = 0$. Hence in (2.1.16) and (2.1.17), if a solution of this type is to exist, we should be able to find numbers s_1 and s_2, with positive real parts, and the solution for ψ may then be written

$$\psi = (A_1\, e^{-s_1 x_3} + A_3\, e^{-s_2 x_3})\, e^{i\omega(t-\frac{x_1}{c})}, \tag{2.1.20}$$

where A_1 and A_3 are chosen to satisfy (2.1.19).

Substitution of (2.1.20) into (2.1.19) leads to

$$A_1[\mathcal{A}_{03131}\ s_1^2+k^2(\mathcal{A}_{01331}+p)]+A_3[\mathcal{A}_{03131}\ s_2^2+k^2(\mathcal{A}_{01331}+p)]=0, \tag{2.1.21}$$

$$A_1s_1[-\mathcal{A}_{03131}\ s_1^2+Nk^2-\rho\omega^2]+A_3s_2[-\mathcal{A}_{03131}\ s_2^2+Nk^2-\rho\omega^2] = 0.$$

where

$$N = \mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01133} - \mathcal{A}_{03113} + p.$$

For equations (2.1.21) in A_1 and A_3 to have a non-trivial solution we must have

$$\Delta \equiv \begin{vmatrix} \mathcal{A}_{03131}\ s_1^2+(\mathcal{A}_{01331}+p)k^2 & \mathcal{A}_{03131}\ s_2^2+(\mathcal{A}_{01331}+p)k^2 \\ s_1(-\mathcal{A}_{03131}\ s_1^2+Nk^2-\rho\omega^2) & s_2(-\mathcal{A}_{03131}\ s_2^2+Nk^2-\rho\omega^2) \end{vmatrix} = 0.$$

On use of equations (2.1.17) this becomes

$$\Delta \equiv \begin{vmatrix} \mathcal{A}_{03131}\ s_1^2+(\mathcal{A}_{01331}+p)k^2 & \mathcal{A}_{03131}\ s_2^2+(\mathcal{A}_{01313}+p)k^2 \\ \mathcal{A}_{03131}\ s_1\ (s_2^2+k^2)-\sigma_3k^2s_1 & \mathcal{A}_{03131}\ s_2(s_1^2+k^2)-\sigma_3k^2s_2 \end{vmatrix} = 0.$$

Since, from (1.5.13), $\mathcal{A}_{01331} = \mathcal{A}_{03131} - \lambda_3\dfrac{\partial W}{\partial\lambda_3}$, and hence $\mathcal{A}_{01313} + p = \mathcal{A}_{03131} - \sigma_3$, we may rewrite Δ as

$$\Delta \equiv \begin{vmatrix} \mathcal{A}_{03131}\ s_1^2+(\mathcal{A}_{03131}-\sigma_3)k^2 & \mathcal{A}_{03131}\ s_2^2+(\mathcal{A}_{03131}-\sigma_3)k^2 \\ \mathcal{A}_{03131}\ s_1(s_2^2+k^2)-\sigma_3k^2s_1 & \mathcal{A}_{03131}\ s_2(s_1^2+k^2)-\sigma_3k^2s_2 \end{vmatrix} = 0.$$

$$\begin{aligned} \text{So } \Delta \equiv\ & [\mathcal{A}_{03131}\ s_1^2+(\mathcal{A}_{03131}-\sigma_3)k^2][\mathcal{A}_{03131}\ s_2(s_1^2+k^2)-\sigma_3k^2s_2] \\ & -[\mathcal{A}_{03131}\ s_2^2+(\mathcal{A}_{03131}-\sigma_3)k^2][\mathcal{A}_{03131}\ s_1(s_2^2+k^2)-\sigma_3k^2s_1]=0. \end{aligned}$$

i.e.

$$\Delta \equiv (\mathcal{A}_{03131})^2[s_1s_2(s_1-s_2)(s_1^2-s_1s_2+s_2^2)+(s_1-s_2)(-k^4+2s_1s_2k^2)]$$
$$- 2\sigma_3k^2\mathcal{A}_{03131}[-(s_1-s_2)k^2+s_1s_2(s_1-s_2)]-\sigma_3^2k^4(s_1-s_2) = 0.$$

Therefore, the secular equation is

$$(s_1-s_2)[\mathcal{A}^2_{03131}(s_1^2+s_1s_2+s_2^2)s_1s_2+2s_1s_2k^2-k^4)-2\sigma_3\mathcal{A}_{03131}k^2$$
$$(s_1s_2-k^2)-\sigma_3^2k^4]= 0. \tag{2.1.22}$$

Assuming that $s_1 \neq s_2$, the secular equation becomes

$$(\mathcal{A}_{03131})^2[s_1s_2(s_1^2+s_1s_2+s_2^2)+2s_1s_2k^2-k^4]-2\sigma_3k^2\mathcal{A}_{03131}$$
$$(s_1s_2-k^2) -\sigma_3^2k^4 = 0, \tag{2.1.23}$$

where s_1 and s_2 are given by (2.1.17).

Equation (2.1.23) generalizes the formula given by Willson (1973a) for the special case $\sigma_3=0$. Willson also took $\lambda_1=\lambda_2$ throughout his calculation.

Of particular interest is the case when $\sigma_3=0$. Then equation (2.1.23) reduces to

$$s_1^2s_2^2 - k^4 + s_1s_2(s_1^2 + s_2^2 + 2k^2) = 0. \tag{2.1.24}$$

Next, substitute for s_1^2 and s_2^2 from (2.1.17) into (2.1.24) to obtain

$$\mathcal{A}_{03131} \frac{\omega^2}{c^2} [(\mathcal{A}_{01313} - \mathcal{A}_{03131}) \frac{\omega^2}{c^2} - \rho\omega^2]^2 =$$

$$(\mathcal{A}_{01313}\frac{\omega^2}{c_2} - \rho\omega^2)[(\mathcal{A}_{01111}+\mathcal{A}_{03333}+2\lambda_3\frac{\partial W}{\partial\lambda_3}- 2\mathcal{A}_{01133})\frac{\omega^2}{c^2}- \rho\omega^2]^2.$$

To obtain the corresponding equation for the wave speed c, we must eliminate ω by multiplying this equation by c^4 and dividing it by ω^6. Thus

$$\mathcal{A}_{03131}\,(\mathcal{A}_{01313} - \mathcal{A}_{03131} - \rho c^2)^2 =$$

$$(\mathcal{A}_{01313} - \rho c^2)(\mathcal{A}_{01111} + \mathcal{A}_{03333} + 2\lambda_3 \frac{\partial W}{\partial \lambda_3} - 2\mathcal{A}_{01133} - \rho c^2)^2. \qquad (2.1.25)$$

Equation (2.1.25) is the *secular equation for Rayleigh surface waves in a pre-strained incompressible isotropic elastic medium with a traction-free surface.* In different notation this is equivalent to an equation given by Willson (1973a), although Willson restricted attention to the special case $\lambda_1 = \lambda_2$ from the beginning.

In the above we have assumed that $s_1 \neq s_2$. We now consider the following special case in which $s_1 = s_2 = s$, say, so that the solution (2.1.20) is replaced by

$$\psi = (A + Bx_3)\, e^{-sx_3}\, e^{i\omega(t - \frac{x_1}{c})} .$$

From this equation we have

$$\psi_{,3} = [B - s(A + Bx_3)]\, e^{-sx_3}\, e^{i\omega(t - \frac{x_1}{c})} ,$$

$$\psi_{,33} = [-2sB + s^2 (A + Bx_3)] e^{-sx_3}\, e^{i\omega(t - \frac{x_1}{c})} ,$$

$$\psi_{,333} = [3s^2B - s^3A - s^3Bx_3] e^{-sx_3}\, e^{i\omega(t - \frac{x_1}{c})} .$$

Substitution into the boundary conditions (2.1.19) leads to

$$(k^2 + s^2)\, A - 2SB = 0,$$

$$[k^2N - s^2\mathcal{A}_{03131} - \rho\omega^2]\, sA + [3s^2\mathcal{A}_{03131} + \rho\omega^2 - Nk^2]B = 0,$$

where N is given by

$$N = \mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01133} - \mathcal{A}_{03311} + p,$$

i.e.

$$N = \mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01133} - \mathcal{A}_{03131} + 2p. \qquad (2.1.26)$$

For these equations to have non-trivial solutions for A and B we must have

$$\Delta \equiv \begin{vmatrix} k^2 + s^2 & -2s \\ (k^2N - s^2\mathcal{A}_{03131} - \rho\omega^2)s & 3s^2\mathcal{A}_{03131} + \rho\omega^2 - Nk^2 \end{vmatrix} = 0,$$

which reduces to

$$\mathcal{A}_{03131}\ (3k^2+s^2)s^2 + (s^2-k^2)(Nk^2-\rho\omega^2) = 0. \qquad (2.1.27)$$

Since it is assumed that $s_1=s_2=s$, equation (2.1.16) must give

$$[\rho\omega^2-(\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01331} - 2\mathcal{A}_{01133})k^2]^2 =$$
$$4\mathcal{A}_{03131}\ [\mathcal{A}_{01313}k^4-\rho\omega^2k^2],$$

and equations (2.1.17) become

$$2s^2 = \frac{(\mathcal{A}_{01111}+\mathcal{A}_{03333}-2\mathcal{A}_{01331}-2\mathcal{A}_{01133})k^2-\rho\omega^2}{\mathcal{A}_{03131}},$$

$$s^4 = \frac{(\mathcal{A}_{01313}k^2 - \rho\omega^2)k^2}{\mathcal{A}_{03131}}. \qquad (2.1.28)$$

From (2.1.27) we have

$$\mathcal{A}_{03131}\ s^4 + s^2[3k^2\mathcal{A}_{03131}+Nk^2-\rho\omega^2] - k^2(Nk^2-\rho\omega^2) = 0.$$

Therefore, the sum of roots for this case is

$$2s^2 = \frac{-[3k^2\mathcal{A}_{03131} + Nk_2 - \rho\omega^2]}{\mathcal{A}_{03131}}$$

Using (2.1.26) and (2.1.28) yields

$$s^2 = -k^2.$$

That is, s is purely imaginary, there is no decay when x_3 approaches infinity, so this case cannot arise and we conclude that A = B = 0! This result appears to be new, although a corresponding result for the compressible case has been found by Hayes and Rivlin (1961b).

2.1.3. Results for some special deformations

(a) The case $\lambda_3 = \lambda_1$

Consider the special case of (2.1.25) in which $\lambda_1 = \lambda_3$, so we obtain

$$\mathcal{A}_{01111} = \mathcal{A}_{03333}, \qquad \mathcal{A}_{01313} = \mathcal{A}_{03131},$$
$$\mathcal{A}_{01133} = \mathcal{A}_{03311}, \qquad \mathcal{A}_{01331} = \mathcal{A}_{03113}, \tag{2.1.29}$$

and recall from (1.5.16) that

$$\mathcal{A}_{03131} = \tfrac{1}{2}(\mathcal{A}_{01111} - \mathcal{A}_{01133} + \lambda_1\frac{\partial W}{\partial \lambda_1}). \tag{2.1.30}$$

Using equation (2.1.29) the secular equation (2.1.25) simplifies to

$$\rho^2c^4\mathcal{A}_{03131} = (\mathcal{A}_{01313}-\rho c^2)(2\mathcal{A}_{03333}+2\lambda_1\frac{\partial W}{\partial \lambda_1}-2\mathcal{A}_{01133}-\rho c^2)^2.$$

Also, by using equation (2.1.30), we may rewrite this equation as

$$\mathcal{A}_{03131}\,\rho^2c^4 = (\mathcal{A}_{03131} - \rho c^2)(4\mathcal{A}_{03131} - \rho c^2)^2.$$

On setting $x = \rho c^2/\mathcal{A}_{03131}$ this becomes

$$x^2 = (1-x)(4-x)^2. \tag{2.1.31}$$

The only positive real solution of (2.1.31) is $x=x_0=0.9126$ approximately. Thus there exists a Rayleigh wave with speed c given by

$$\rho c^2 = x_0\mathcal{A}_{03131}$$

provided $\mathcal{A}_{03131} > 0$ (this inequality is often referred to as the Baker-Ericksen inequality). This puts no restriction on the admissible set of values of $\lambda_3 = \lambda_1$ (subject to the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$).

(b) The limiting case $\lambda_1 = \lambda_2 = \lambda_3 = 1$

In the particularly simple case in which the primary stress is zero and $\lambda_1 = \lambda_3 = \lambda_2 = 1$ we have

$$\mathcal{A}_{01313} = \mathcal{A}_{03131} = \mu,$$

where μ is modulus of rigidity in the classical linear theory. Then $\rho c^2 = \mu x_0$. This special case was noted by Willson (1973a).

(c) The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in the presence of hydrostatic pre-stress

If the undeformed configuration is subject to a hydrostatic pre-stress $\sigma_1 = \sigma_2 = \sigma_3$ then we have from (1.6.19)

$$\mathcal{A}_{01111} = \mathcal{A}_{03333} = 2\mu, \mathcal{A}_{01313} = \mathcal{A}_{03131} = \mathcal{A}_{03113} = \mathcal{A}_{01331} = \mu$$
$$\mathcal{A}_{01133} = 0.$$

So, (2.1.17) gives

$$(s_1^2 + s_2^2)/k^2 = 2 - \frac{\rho c^2}{\mu}$$

$$s_1^2 s_2^2/k^4 = 1 - \frac{\rho c^2}{\mu}$$

and we need the secular equation in the form (2.1.23), which yields

$$\mu^2\left[1-\frac{\rho c^2}{\mu} + \sqrt{1-\frac{\rho c^2}{\mu}}\left[2-\frac{\rho c^2}{\mu}\right]-1 + 2\sqrt{1-\frac{\rho c^2}{\mu}}\right]-2\sigma_3\left[\sqrt{1-\frac{\rho c^2}{\mu}}-1\right] - \sigma_3^2 = 0.$$

On rearranging this and setting

$$\sigma = \sigma_3/\mu, \qquad x = \frac{\rho c^2}{\mu}$$

we obtain

$$\sigma^2 + 2\sigma(\sqrt{1-x} - 1) - \sqrt{1-x}\,(4-x) + x = 0.$$

We note that values of x must be restricted to the range $0 \leq x \leq 1$. When $x = 0$, this equation yields $\sigma^2 - 4 = 0$, i.e. $\sigma = \pm 2$, while when $x = 1$ it yields $(\sigma-1)^2 = 0$ i.e. $\sigma = 1$. What we require to find is the range of values of σ for which the above yields solutions for x in the stated interval ($0 \leq x \leq 1$). We therefore re-cast the above equation in form

$$(4-x-2\sigma)\sqrt{1-x} = x+\sigma^2 - 2\sigma$$

and, on squaring and rearranging this is written

$$f(x) \equiv x^3+4(\sigma-2)x^2 + 6(\sigma-2)^2x + (\sigma+2)(\sigma-2)^3 = 0.$$

Then, at the end-points of the interval $0 \leq x \leq 1$, we have

$$f(0) = (\sigma+2)(\sigma-2)^3, \qquad f(1) = (\sigma-1)^4$$

Clearly $f(0) < 0$ for $-2 < \sigma < 2$

while $f(1) > 0$ in the same interval except at $x=1$.

Also, we have

$$f'(x) = 3x^2+8(\sigma-2)x + 6(\sigma-2)^2$$

and this is strictly positive except for $\sigma=2$, $x=0$.

Thus, $f(x)$ is monotonically increasing for $0 \leq x \leq 1$ and changes sign once in this interval provided $-2 < \sigma < 2$ and $\sigma \neq 1$.

Thus, for each σ in the interval $(-2,2)$ there is a unique value of x satisfying the secular equation ($\sigma=1$ corresponding to $x=1$).

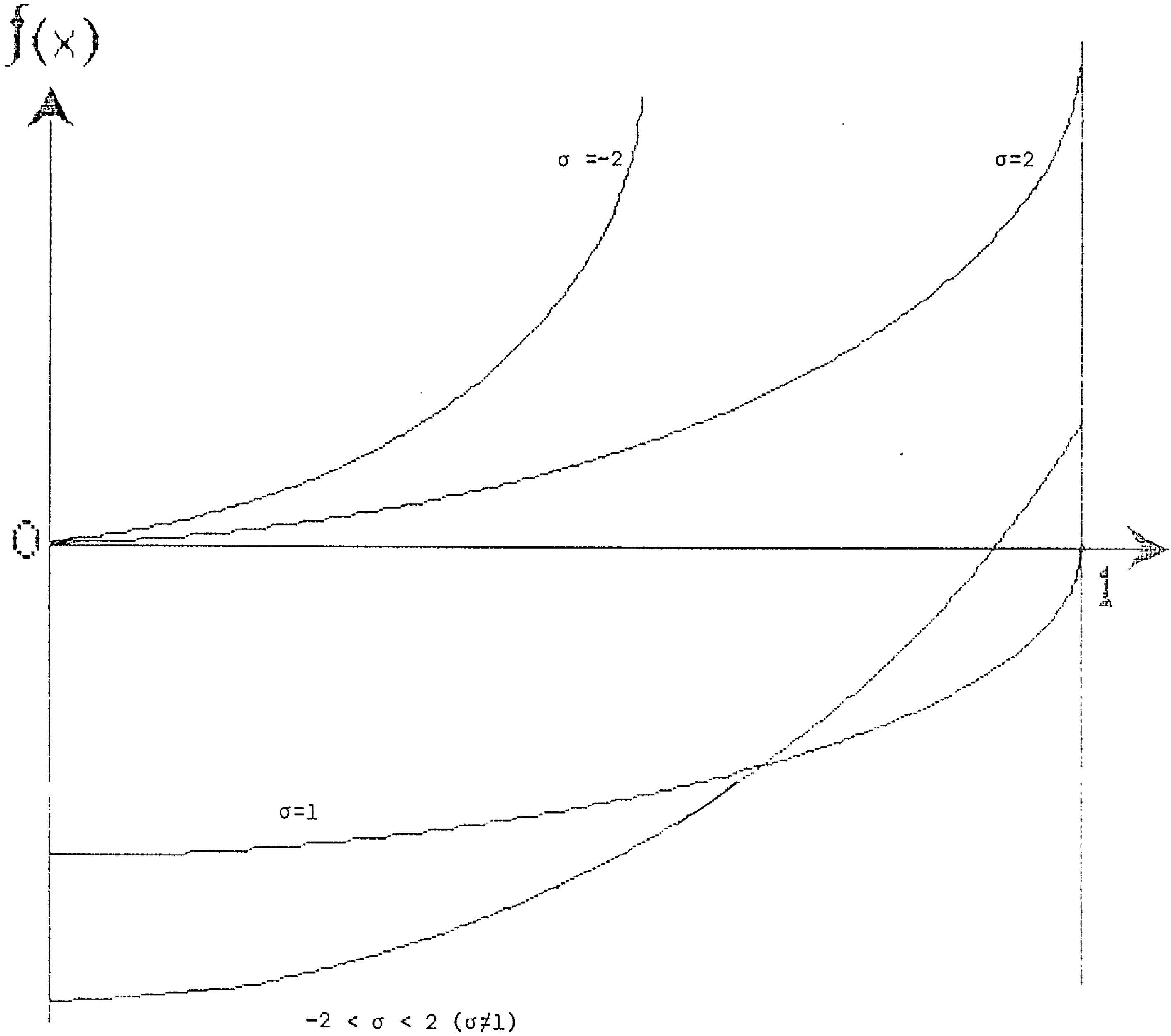


Figure 1

A Rayleigh wave will therefore propagate in a hydrostatically pre-stressed half-space provided the pre-stress satisfies $-2 < (\sigma = \frac{\sigma_3}{\mu}) < 2$. The limiting cases $\sigma = \pm 2$ correspond to situations in which the underlying homogeneous deformation becomes neutrally stable. (The stability of such configurations has been discussed in detail in the book by Ogden, 1984, for example).

The form of the function f(x) is illustrated in Figure 1 for different values of σ.

(d) The case of uniaxial stress: $\sigma_2 = \sigma_3 = 0$, $\lambda_2 = \lambda_3$

Since $\lambda_2 = \lambda_3$ and $\lambda_1\lambda_2\lambda_3 = 1$ we may write

$$\lambda_2 = \lambda_3 = \lambda_1^{-½}$$

and define

$$\hat{W}(\lambda_1) = W(\lambda_1, \lambda_1^{-½}, \lambda_1^{-½}).$$

Thus, we have

$$\frac{d\hat{W}}{d\lambda_1} = \frac{\partial W}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial \lambda_1} + \frac{\partial W}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \lambda_1} + \frac{\partial W}{\partial \lambda_3}\frac{\partial \lambda_3}{\partial \lambda_1}.$$

and hence

$$\lambda_1\frac{d\hat{W}}{d\lambda_1} = \lambda_1\frac{\partial W}{\partial \lambda_1} - \lambda_3\frac{\partial W}{\partial \lambda_3}.$$

Also

$$\lambda_1\frac{d}{d\lambda_1}\left[\lambda_1\frac{d\hat{W}}{d\lambda_1}\right] = \left[\lambda_1\frac{\partial}{\partial \lambda_1} - \lambda_3\frac{\partial}{\partial \lambda_3}\right]\left[\lambda_1\frac{\partial W}{\partial \lambda_1} - \lambda_3\frac{\partial W}{\partial \lambda_3}\right],$$

i.e.

$$\lambda_1^2\frac{d^2\hat{W}}{d\lambda_1^2}+\lambda_1\frac{d\hat{W}}{d\lambda_1}=\lambda_1^2\frac{\partial^2 W}{\partial \lambda_1^2}+\lambda_1\frac{\partial W}{\partial \lambda_1}+\lambda_3^2\frac{\partial^2 W}{\partial \lambda_3^2}+\lambda_3\frac{\partial W}{\partial \lambda_3}-2\lambda_1\lambda_3\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_3}$$

and hence

$$\lambda_1^2 \frac{d^2\hat{W}}{d\lambda_1^2} = \mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01133} + 2\lambda_3 \frac{\partial W}{\partial \lambda_3}.$$

Substituting into the secular equation (2.1.25), we have

$$\frac{\lambda_1\hat{W}'}{\lambda_1^3-1}(\lambda_1\hat{W}' - \rho c^2)^2 = \left[\frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1} - \rho c^2\right](\lambda_1^2\hat{W}'' - \rho c^2)^2.$$

This is a cubic for n, ($n=\rho c^2$), it can be written

$$\frac{\lambda_1\hat{W}'}{\lambda_1^3-1}\left[\lambda_1^2\hat{W}'^2 - 2n\lambda_1\hat{W}' + n^2\right] = \left[\frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1} - n\right]\left[\lambda_1^4\hat{W}''^2 - 2n\lambda_1^2\hat{W}'' + n^2\right]$$

$$= \lambda_1^8\frac{\hat{W}''^2\hat{W}'}{\lambda_1^3-1} - n\left[\lambda_1^4\hat{W}''^2 + 2\frac{\lambda_1^6\hat{W}''\hat{W}'}{\lambda_1^3-1}\right] + n^2\left[\frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1} + 2\lambda_1^2\hat{W}''\right] - n^3,$$

i.e.

$$n^3 - n^2\left[\frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1} + 2\lambda_1^2\hat{W}''\right] + \left[\frac{\lambda_1\hat{W}'}{\lambda_1^3-1}\right]n^2 + n\left[\lambda_1^4\hat{W}''^2 + \frac{2\lambda_1^6\hat{W}''\hat{W}'}{\lambda_1^3-1} - 2\frac{\lambda_1^2\hat{W}'}{\lambda_1^3-1}\right]$$

$$+ \frac{\lambda_1^3\hat{W}'^3}{\lambda_1^3-1} - \lambda_1^8\frac{\hat{W}''^2\hat{W}'}{\lambda_1^3-1} = 0.$$

This can be reduced to

$$n^3 - n^2\left[\lambda_1\hat{W}' + 2\lambda_1^2\hat{W}''\right] + n\left[\lambda_1^2(\lambda_1^3-1)\hat{W}''^2 + 2\lambda_1^4\hat{W}''\hat{W}' - 2\hat{W}'^2\right]\lambda_1^2/(\lambda_1^3-1)$$

$$+ \frac{\lambda_1^3\hat{W}'}{\lambda_1^3-1}\left[\hat{W}'^2 - \lambda_1^5\hat{W}''^2\right] = 0.$$

i.e.

$$f(n) \equiv n^3 - \alpha n^2 + \beta n - \gamma = 0,$$

where α, β, γ are given by

$$\alpha = 2\lambda_1^2\hat{W}'' + \lambda_1\hat{W}', \qquad \beta = \lambda_1^4\hat{W}''^2 + \frac{2\lambda_1^6\hat{W}''\hat{W}'}{\lambda_1^3-1} - \frac{2\lambda_1^2\hat{W}'^2}{\lambda_1^3-1},$$

$$\gamma = \frac{\lambda_1^3\hat{W}'}{\lambda_1^3-1}(\lambda_1^5\hat{W}''^2 - \hat{W}'^2).$$

Recall from (2.1.17) that

$$s_1^2 + s_2^2 = \frac{(\mathcal{A}_{01111} + \mathcal{A}_{03333} - 2\mathcal{A}_{01331} - 2\mathcal{A}_{01133})k^2 - \rho\omega^2}{\mathcal{A}_{03131}},$$

$$s_1^2 s_2^2 = \frac{(\mathcal{A}_{01313}\, k^2 - \rho\omega^2)\, k^2}{\mathcal{A}_{03131}}.$$

Since s_1 and s_2 are to have positive real parts, and s_1^2, s_2^2 are complex conjugates it follows, assuming $\mathcal{A}_{03131} > 0$, that $s_1^2 s_2^2$ must be positive. We therefore require

$$\rho\omega^2 < \mathcal{A}_{01313}\, k^2,$$

i.e.

$$\rho c^2 \equiv n < \mathcal{A}_{01313} \equiv \rho c_{13}^2 = \frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1}.$$

Thus

$$0 < n < n_0 \equiv \frac{\lambda_1^4\hat{W}'}{\lambda_1^3-1} \tag{2.1.32}$$

Then,

$$f(0) = -\gamma, \qquad f(n_0) = \lambda_1\hat{W}'(\lambda_1^2\hat{W}'' - \lambda_1\hat{W}')^2$$

$$f'(n) = 3n^2 - 2\alpha n + \beta.$$

If $\gamma > 0$ (<0) then f changes sign on the interval $0 < \eta < \eta_0$ provided $\lambda_1 \hat{W}'' \neq \hat{W}' > 0$ (<0).

This will ensure the existence of at least one solution of $f(\eta) = 0$, and hence the existence of a Rayleigh wave. We do not investigate here the circumstances in which more than one such wave may exist.

2.1.4 Results for particular strain-energy functions

(a) The neo-Hookean material

We begin by considering the neo-Hookean material, for which

$$W = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3).$$

From equations (1.5.13) we obtain

$$\mathcal{A}_{03131} = \mathcal{A}_{03333} = \mu\lambda_3^2,$$

$$\mathcal{A}_{01111} = \mathcal{A}_{01313} = \mu\lambda_1^2, \qquad (2.1.33)$$

and

$$\mathcal{A}_{01331} = \mathcal{A}_{03113} = \mathcal{A}_{01133} = 0.$$

Next, substitute these into the secular equation (2.1.16); we obtain

$$\mu\lambda_3^2 s^4 - (\mu\lambda_3^2 + \mu\lambda_1^2)\, k^2 s^2 + \mu\lambda_1^2 k^4 + \rho\omega^2(s^2 - k^2) = 0,$$

i.e.

$$\mu\lambda_3^2 s^4 + (\rho\omega^2 - \mu\lambda_1^2 k^2 - \mu\lambda_3^2 k^2) s^2 + \mu\lambda_1^2 k^4 - \rho\omega^2 k^2 = 0.$$

The roots $s^2 = s_1^2$ and $s^2 = s_2^2$ of this quadratic equation are given by

$$s_1^2 = k^2 \text{ and } s_2^2 = \frac{\mu\lambda_1^2 k^2 - \rho\omega^2}{\mu\lambda_3^2}.$$

Alternatively, since $k = \frac{\omega}{c}$ the second of these may be written

$$\frac{s_2^2}{k^2} = \frac{\mu\lambda_1^2-\rho c^2}{\mu\lambda_3^2},$$

Since s_2 must have positive real part and cannot be pure imaginary and ρc^2 must be positive the constraint

$$0 < \rho c^2 < \mu\lambda_1^2$$

must be satisfied. This could be obtained directly by specializing (2.1.32).

Next, using also (2.1.33) in the secular equation (2.1.25), we have

$$\mu\lambda_3^2(\mu\lambda_1^2 - \mu\lambda_3^2 - \rho c^2)^2 = (\mu\lambda_1^2 - \rho c^2)[\mu\lambda_1^2 + 3\mu\lambda_3^2 - \rho c^2]^2.$$

By putting $\bar{n} = \dfrac{\rho c^2}{\mu}$, we reduce this equation to

$$\lambda_3^2(\lambda_1^2 - \lambda_3^2 - \bar{n})^2 = (\lambda_1^2 - \bar{n})(\lambda_1^2 + 3\lambda_3^2 - \bar{n})^2,$$

subject to

$$0 < \bar{n} < \lambda_1^2.$$

This is a cubic equation for $\bar{n}$, which may be rewritten as

$$\bar{n}^3+\bar{n}^2\ (\lambda_3^2-\lambda_1^2-2\lambda_1^2-6\lambda_3^2) + \bar{n}[(\lambda_1^2+3\lambda_3^2)^2 + 2\lambda_1^2(\lambda_1^2+3\lambda_3^2) + 2\lambda_3^2(\lambda_3^2-\lambda_1^2)]$$

$$+ \lambda_3^2(\lambda_1^2-\lambda_3^2)^2 - \lambda_1^2(\lambda_1^2+3\lambda_3^2)^2 = 0$$

and simplified to

$$f(\bar{n})\equiv\bar{n}^3-\bar{n}^2(3\lambda_1^2+5\lambda_3^2)+\bar{n}[11\lambda_3^4+3\lambda_1^4+10\lambda_1^2\lambda_3^2]-\lambda_1^6-5\lambda_1^2\lambda_3^2-11\lambda_1^2\lambda_3^2+\lambda_3^6 = 0.$$

Consider the sign of $f(\bar{n})$ for different values of (λ_1, λ_3) to determine how many roots, subject to $0 < \bar{n} < \lambda_1^2$.

$$f(0) = \lambda_3^6 - 11\lambda_3^4\lambda_1^2 - 5\lambda_3^2\lambda_1^4 - \lambda_1^6.$$

$$f(\lambda_1^2) = \lambda_3^6 > 0.$$

$$f'(\bar{n}) = 3\bar{n}^2 - 2\bar{n}(3\lambda_1^2 + 5\lambda_3^2) + [11\lambda_3^4 + 3\lambda_1^4 + 10\lambda_1^2\lambda_3^2].$$

It is not difficult to show that $f'(\bar{n}) > 0$ for all $\bar{n}$, so in particular $f(\bar{n})$ is monotonic increasing for $0 < \bar{n} < \lambda_1^2$. Therefore, for there to exist a real root $\bar{n}$ we must have

$$f(0) \equiv \lambda_3^6 - 11\lambda_3^4\lambda_1^2 - 5\lambda_3^2\lambda_1^4 - \lambda_1^6 < 0.$$

Let us now consider $x = \lambda_3^2/\lambda_1^2$ and find values of x for which

$$g(x) \equiv x^3 - 11x^2 - 5x - 1 < 0,$$

bearing in mind that $x > 0$.

We have

$$g(0) = -1$$

and

$$g'(x) = 3x^2 - 22x - 5, \quad g'(0) = -5.$$

It follows that $g(x) = 0$ has a unique positive solution, x' say, and hence $g(x) < 0$ for $0 < x < x'$. The approximate value of x' is calculated to be $x' = 11.44$.

Thus, (2.1.33) possesses a positive real root $\bar{n}$ provided

$$0 < \frac{\lambda_3^2}{\lambda_1^2} < x'$$

i.e.

$$\lambda_3 < \lambda_1 \sqrt{x'}$$

The limiting case $\lambda_3 = \lambda_1\sqrt{x'}$ yields the solution $\bar{n} = 0$. This corresponds to the boundary of stability of the underlying homogeneous deformation.

Finally, in the particular case $\lambda_3 = \lambda_1$, equation (2.1.31) gives

$$n = x_o\lambda_3^2, \qquad \text{or } \rho c^2 = \mu x_o\lambda_3^2,$$

and the restriction $\lambda_3 < \lambda_1\sqrt{x'}$ is automatically satisfied.

(b) The Mooney-Rivlin material

For the Mooney-Rivlin material, the strain-energy function is given by

$$W = \tfrac{1}{2}\mu_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \tfrac{1}{2}\mu_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3),$$

where $\mu_1 > o$ and $\mu_2 < o$. The shear modulus μ is given by

$$\mu = \mu_1 - \mu_2.$$

From (1.5.13) we obtain

$$\begin{aligned}
\mathcal{A}_{01111} &= \mu_1\lambda_1^2 - 3\mu_2\lambda_1^{-2},\\
\mathcal{A}_{03333} &= \mu_1\lambda_3^2 - 3\mu_2\lambda_3^{-2},\\
\mathcal{A}_{03131} &= \frac{\mu_1(\lambda_3^4-\lambda_1^2\lambda_3^2) + \mu_2(1-\lambda_1^{-2}\lambda_3^2)}{\lambda_3^2 - \lambda_1^2} = \mu_1\lambda_3^2 - \mu_2\lambda_1^{-2},\\
\mathcal{A}_{01313} &= \frac{\mu_1(\lambda_1^4-\lambda_1^2\lambda_3^2) + \mu_2(1-\lambda_1^2\lambda_3^{-2})}{\lambda_1^2 - \lambda_3^2} = \mu\lambda_1^2-\mu_2\lambda_3^{-2},\\
\mathcal{A}_{01331} &= -\mu_2(\lambda_1^{-2} + \lambda_3^{-2}),\\
\mathcal{A}_{01133} &= 0.
\end{aligned} \tag{2.1.34}$$

Substituting (2.1.34) into the secular equation (2.1.16), we obtain

$$(\mu_1\lambda_3^2-\mu_2\lambda_1^{-2})\ s^4-\{\mu_1\lambda_1^2-3\mu_2\lambda_1^{-2}+\mu_1\lambda_3^2-3\mu_2\lambda_3^{-2}+2\mu_2(\lambda_1^{-2}+\lambda_3^{-2})\}k^2s^2$$

$$+\ (\mu_1\lambda_1^2-\mu_2\lambda_3^2)k^4\ +\ \rho\omega^2(s^2-k^2)\ =\ 0.$$

i.e.

$$(\mu_1\lambda_3^2-\mu_2\lambda_1^{-2})\ s^4-\{\mu_1(\lambda_1^2+\lambda_3^2)k^2-\mu_2(\lambda_1^{-2}+\lambda_3^{-2})k^2+\rho\omega^2\}s^2+(\mu_1\lambda_1^2-\mu_2\lambda_3^{-2})k^2$$

$$-\ \rho\omega^2k^2\ =\ 0.$$

This equation has roots

$$s_1^2 = k^2, \qquad s_2^2 = \frac{(\mu_1\lambda_1^2 - \mu_2\lambda_3^{-2})k^2 - \rho\omega^2}{\mu_1\lambda_3^2 - \mu_2\lambda_1^{-2}}.$$

Alternatively, we may rewrite the second root, since $k = \frac{\omega}{c}$, as

$$\frac{s_2^2}{k^2} = \frac{\mu_1\lambda_1^2 - \mu_2\lambda_3^{-2} - \rho c^2}{\mu_1\lambda_3^2 - \mu_2\lambda_1^{-2}}.$$

Recalling the notation (1.6.17), we may reduce the above root to

$$\frac{s_2^2}{k^2} = \frac{c_{13}^2 - c^2}{c_{31}^2}.$$

Since s_2^2 must have positive real part and cannot be pure imaginary the constraint $\rho c^2 < \mu_1\lambda_1^2 - \mu_2\lambda_3^{-2} \equiv \rho c_{13}^2$ must be satisfied.

Also, substitute (2.1.34) into (2.1.25), to obtain ρc^2, we get

$$(\mu_1\lambda_1^2 - \mu_2\lambda_1^{-2})\ [\mu_1(\lambda_1^2-\lambda_3^2) + \mu_2\ (\lambda_1^{-2} - \lambda_3^{-2}) - \rho c^2]^2 =$$

$$(\mu_1\lambda_1^2-\mu_2\lambda_3^{-2}-\rho c^2)[\mu(\lambda_1^2+3\lambda_3^2)-\mu_2(3\lambda_1^{-2}+\lambda_3^{-2}) - \rho c^2]^2.$$

Since

$$\mu_1\lambda_1^2 - \mu_2\lambda_3^{-2} = \rho c_{13}^2,$$

$$\mu_1\lambda_3^2 - \mu_2\lambda_1^{-2} = \rho c_{31}^2,$$

we may reduce this equation to

$$c_{13}^2[c_{13}^2 - c_{31}^2 - c^2]^2 = (c_{13}^2 - c^2)\ [c_{13}^2 + 3c_{31}^2 - c^2]^2,$$

subject to $0 < \rho c^2 < \rho c_{13}^2$ or $0 < c^2 < c_{13}^2$.

i.e.

$$c^6 + c^4(c_{13}^2 - 2c_{13}^2 - 6c_{31}^2 - c_{13}^2) + c^2(2c_{13}^2c_{31}^2 + 2c_{13}^4 + 6c_{13}^2c_{31}^2 - 2c_{13}^4 + c_{13}^4 + 6c_{13}^2c_{31}^2 + 9c_{31}^2)$$

$$+ c_{13}^6 + c_{13}^2c_{31}^4 - 2c_{13}^4c_{31}^2 - 6c_{13}^4c_{31}^2 - 9c_{13}^2c_{31}^4 - c_{13}^6 = 0.$$

This equation reduces to

$$c^6 - c^4(2c_{13}^2 + 6c_{31}^6) + c^2(c_{13}^4 + 14c_{13}^2c_{31}^2 + 9c_{31}^4) - 8(c_{13}^2c_{31}^4 + c_{13}^4c_{31}^2) = 0.$$

By using the ratio $\varepsilon = {c^2}/{c_{13}^2}$, where $0 < \varepsilon < 1$, and

$a = {c_{31}^2}/{c_{13}^2}$, we get $h(\varepsilon) \equiv \varepsilon^3 - \varepsilon^2(2+6a) + \varepsilon(9a^2+14a+1) - 8(a^2+a) = 0.$

Next, consider the sign of $h(\varepsilon)$ subject to $0 < \varepsilon < 1$:

$$h(0) = -8(a^2+a)$$

$$h(1) = a^2.$$

Thus, h changes sign on $0 < \varepsilon < 1$ so there exists at least one solution. We omit further details here.

2.2 Propagation in a general direction in the (x_1, x_2) plane

So far we have considered Rayleigh waves on a pre-strained half-space of incompressible material with propagation along a principal axis. In this section we shall obtain equations for the propagation in a general direction in the (x_1, x_2)-plane. The direction of propagation has direction cosines $(\cos\theta, \sin\theta)$.

From (1.5.20) for an incompressible material the motion is governed by

$$\mathcal{A}_{oji\ell k}\, \nu_{k,j\ell} - \dot{p}_{,i} = \rho \ddot{\nu}_i, \qquad (2.2.1)$$

with $\nu_{i,i} = 0.$

Now let $\underline{\nu}$ and $\dot{p}$ be given by

$$\underline{\nu} = \underline{\Psi}(x_3)\, e^{i\omega\left[t - \frac{x_1\cos\theta + x_2\sin\theta}{c}\right]},$$

$$\dot{p} = \phi(x_3)\, e^{i\omega\left[t - \frac{x_1\cos\theta + x_2\sin\theta}{c}\right]}, \qquad (2.2.2)$$

so that the components of $\underline{\nu}$ are

$$\nu_i = \Psi_i(x_3)\, e^{i\omega\left[t - \frac{x_1\cos\theta + x_2\sin\theta}{c}\right]} \quad (i=1,2,3). \qquad (2.2.3)$$

From (2.2.1) we then obtain

$$\Psi_1(x_3)\left[-\frac{i\omega\cos\theta}{c}\right] + \Psi_2(x_3)\left[-\frac{i\omega\sin\theta}{c}\right] + \Psi'_3(x_3) = 0, \qquad (2.2.4)$$

and

$$\begin{aligned}
&\mathcal{A}_{oj1\ell 1}\, \nu_{1,j\ell} + \mathcal{A}_{oj1\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj1\ell 3}\, \nu_{3,j\ell} - \dot{p}_{,1} = \rho\ddot{\nu}_1,\\
&\mathcal{A}_{oj2\ell 1}\, \nu_{1,j\ell} + \mathcal{A}_{oj2\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj2\ell 3}\, \nu_{3,j\ell} + \dot{p}_{,2} = \rho\ddot{\nu}_2, \qquad (2.2.5)\\
&\mathcal{A}_{oj3\ell 1}\, \nu_{1,j\ell} + \mathcal{A}_{oj3\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj3\ell 3}\, \nu_{3,j\ell} - \dot{p}_{,3} = \rho\ddot{\nu}_3.
\end{aligned}$$

By differentiating (2.2.3) and substituting into (2.2.5) we obtain

$$\begin{aligned}
&\mathcal{A}_{o1111}\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]\Psi_1 + \mathcal{A}_{o2121}\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]\Psi_1 + \mathcal{A}_{o3131}\,\Psi''_1\\
&\quad + (\mathcal{A}_{o1122} + \mathcal{A}_{o2112})\Psi_2\left[-\frac{\omega^2\sin\theta\cos\theta}{c^2}\right]\\
&\quad + (\mathcal{A}_{o1133} + \mathcal{A}_{o3113})\,\Psi'_3\left[-\frac{i\omega\cos\theta}{c}\right] - \left[-\frac{i\omega\cos\theta}{c}\right]\phi = -\rho\Psi_1,
\end{aligned}$$

$$(\mathcal{A}_{02211}+\mathcal{A}_{01221})\,\psi_1\left[-\frac{\omega^2\sin\theta\cos\theta}{c^2}\right]+\mathcal{A}_{01212}\,\psi_2\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]$$

$$+\mathcal{A}_{02222}\,\psi_2\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]+\mathcal{A}_{03232}\psi''_2$$

$$+(\mathcal{A}_{02233}+\mathcal{A}_{03223})\psi'_3\left[-\frac{i\omega\sin\theta}{c}\right]-\left[-\frac{i\omega\sin\theta}{c}\right]\phi=-\rho\psi_2, \quad (2.2.6)$$

$$\mathcal{A}_{03311}\psi'_1\left[-\frac{i\omega\cos\theta}{c}\right]+\mathcal{A}_{01331}\,\psi'_1\left[-\frac{i\omega\cos\theta}{c}\right]+(\mathcal{A}_{03322}+\mathcal{A}_{02332})\,\psi'_2$$

$$\left[-\frac{i\omega\sin\theta}{c}\right]+\mathcal{A}_{01313}\,\psi_3\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]+\mathcal{A}_{02323}\,\psi_3\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]+\mathcal{A}_{03333}$$

$$\psi''_3-\phi'=-\rho\psi_3$$

Suppose now that

$$\psi_1 = A\,\bar{e}^{sx_3}, \quad \psi_2 = B\,\bar{e}^{sx_3}, \quad \psi_3 = C\,\bar{e}^{sx_3}, \quad \phi = D\,\bar{e}^{sx_3}.$$

Equations (2.2.6) and (2.2.4) become

$$
\begin{aligned}
&\Big(\rho + s^2\mathcal{A}_{03131} - \frac{\omega^2\cos^2\theta}{c^2}\mathcal{A}_{01111} - \frac{\omega\sin^2\theta}{c^2}\mathcal{A}_{02121}\Big) A \\
&\qquad - \frac{\omega^2\sin\theta\cos\theta}{c^2}\left(\mathcal{A}_{01122} + \mathcal{A}_{02112}\right) B \\
&\qquad - \frac{is\omega\cos\theta}{c}\left(\mathcal{A}_{01133} + \mathcal{A}_{03113}\right) C + \frac{i\omega\cos\theta}{c} D = 0, \\
&-\frac{\omega^2\sin\theta\cos\theta}{c^2}\left(\mathcal{A}_{02211} + \mathcal{A}_{01221}\right) A + \Big(\rho + s^2\mathcal{A}_{03232} - \frac{\omega^2\cos^2\theta}{c^2}\mathcal{A}_{01212} \\
&\qquad - \frac{\omega^2\sin^2\theta}{c^2}\mathcal{A}_{02222}\Big) B - \frac{i\omega s\sin\theta}{c}\left(\mathcal{A}_{02233} + \mathcal{A}_{03223}\right) C + \frac{i\omega\sin\theta}{c} D = 0, \\
&\frac{is\omega\cos\theta}{c}\left(\mathcal{A}_{03311} + \mathcal{A}_{01331}\right) A + \frac{is\omega\sin\theta}{c}\left(\mathcal{A}_{03322} + \mathcal{A}_{02332}\right) B \\
&\qquad + \Big(\rho + s^2\mathcal{A}_{03333} - \frac{\omega^2\cos^2\theta}{c^2}\mathcal{A}_{01313} - \frac{\omega^2\sin^2\theta}{c^2}\mathcal{A}_{02323}\Big) C + sD = 0, \\
&\qquad isC = \frac{\omega\cos\theta}{c} A + \frac{\omega\sin\theta}{c} B.
\end{aligned}
\tag{2.2.7}
$$

On eliminating C, equations (2.2.7) become

$$
\begin{aligned}
&\Big[\rho + s^2\mathcal{A}_{03131} + \frac{\omega^2\cos^2\theta}{c^2}\left(\mathcal{A}_{01133} + \mathcal{A}_{03113} - \mathcal{A}_{01111}\right) - \frac{\omega^2\sin^2\theta}{c^2}\mathcal{A}_{02121}\Big] A \\
&\qquad + \frac{\omega^2\sin\theta\cos\theta}{c^2}\left(\mathcal{A}_{01133} + \mathcal{A}_{03113} - \mathcal{A}_{01122} - \mathcal{A}_{02112}\right) B + \frac{i\omega\cos\theta}{c} D = 0, \\
&\frac{\omega^2\sin\theta\cos\theta}{c^2}\left(\mathcal{A}_{02233} + \mathcal{A}_{03223} - \mathcal{A}_{02211} - \mathcal{A}_{01221}\right) A + \Big[\rho + s^2\mathcal{A}_{03232} \\
&\qquad - \frac{\omega^2\cos^2\theta}{c^2} + \frac{\omega^2\sin^2\theta}{c^2}\left(\mathcal{A}_{02233} + \mathcal{A}_{03223} - \mathcal{A}_{02222}\right)\Big] B + \frac{i\omega\sin\theta}{c} D = 0,
\end{aligned}
\tag{2.2.8}
$$

$$\left[\rho+s^2(\mathcal{A}_{03333}-\mathcal{A}_{03311}-\mathcal{A}_{01331})-\frac{\omega^2\cos^2\theta}{c^2}\mathcal{A}_{01313}-\frac{\omega^2\sin^2\theta}{c^2}\mathcal{A}_{02323}\right]$$

$$\frac{\omega\cos\theta}{c}A+\left[\rho+s^2(\mathcal{A}_{03333}-\mathcal{A}_{03322}-\mathcal{A}_{02332})-\frac{\omega^2\cos^2\theta}{c^2}\mathcal{A}_{01313}-\frac{\omega^2\sin^2\theta}{c^2}\mathcal{A}_{02323}\right]\frac{\omega\sin\theta}{c}B+is^2D=0.$$

For equations (2.2.8) to yield a non-trivial solution for A, B, D we must have

$$\Delta\equiv\begin{vmatrix}
\begin{array}{l}\rho c^2+s^2c^2\mathcal{A}_{03131}\\ +\omega^2\cos^2\theta(\mathcal{A}_{01133}\\ +\mathcal{A}_{03113}-\mathcal{A}_{01111})\\ -\omega^2\sin^2\theta\mathcal{A}_{02121}\end{array} & \begin{array}{l}\omega^2\sin\theta\cos\theta(\mathcal{A}_{01133}\\ +\mathcal{A}_{03113}-\mathcal{A}_{01122}\\ -\mathcal{A}_{02112})\end{array} & i\omega c\cos\theta \\
\begin{array}{l}\omega^2\sin\theta\cos\theta(\mathcal{A}_{02233}\\ +\mathcal{A}_{03223}-\mathcal{A}_{02211}\\ -\mathcal{A}_{01221})\end{array} & \begin{array}{l}\rho c^2+s^2c^2\mathcal{A}_{03232}\\ -\omega^2\cos^2\theta\mathcal{A}_{01212}\\ +\omega^2\sin^2\theta(\mathcal{A}_{02233}\\ +\mathcal{A}_{03223}-\mathcal{A}_{02222})\end{array} & i\omega c\sin\theta \\
\begin{array}{l}\frac{\omega\cos\theta}{c}[\rho c^2+s^2c^2(\mathcal{A}_{03333}\\ -\mathcal{A}_{03311}-\mathcal{A}_{01331})\\ -\omega^2\sin^2\theta\mathcal{A}_{01313}-\omega^2\sin^2\theta\\ \mathcal{A}_{02323}]\end{array} & \begin{array}{l}\frac{\omega\sin\theta}{c}[\rho c^2+s^2c^2(\mathcal{A}_{03333}\\ -\mathcal{A}_{03322}-\mathcal{A}_{02332})\\ -\omega^2\cos^2\theta\mathcal{A}_{01313}\\ -\omega^2\sin^2\theta\mathcal{A}_{02323}]\end{array} & is^2c^2
\end{vmatrix}=0.$$

$$\therefore \Delta \equiv \omega^4\sin^2\theta\cos^2\theta(\mathcal{A}_{02233}+\mathcal{A}_{03223}-\mathcal{A}_{02211}-\mathcal{A}_{01221})[\rho c^2+s^2c^2$$
$$(\mathcal{A}_{03333}-\mathcal{A}_{03322}-\mathcal{A}_{02332})-\omega^2\cos^2\theta\mathcal{A}_{01313}-\omega^2\sin^2\theta\mathcal{A}_{02323})$$
$$-\omega^2\cos^2\theta[\rho c^2+s^2c^2(\mathcal{A}_{03333}-\mathcal{A}_{03311}-\mathcal{A}_{01331})$$
$$-\omega^2\cos^2\theta\mathcal{A}_{01313}-\omega^2\sin^2\theta\mathcal{A}_{02323}][\rho c^2+s^2c^2\mathcal{A}_{03232}-\omega^2\cos^2\theta$$
$$\mathcal{A}_{01212}+\omega^2\sin^2\theta(\mathcal{A}_{02233}+\mathcal{A}_{03223}-\mathcal{A}_{02222})]-\omega^2\sin^2\theta$$
$$[\rho c^2+s^2c^2\mathcal{A}_{03131}+\omega^2\cos^2\theta(\mathcal{A}_{01133}+\mathcal{A}_{03113}-\mathcal{A}_{01111})-\omega^2\sin^2\theta$$
$$\mathcal{A}_{02121}][\rho c^2+s^2c^2(\mathcal{A}_{03333}-\mathcal{A}_{03322}-\mathcal{A}_{02332})-\omega^2\cos^2\theta\mathcal{A}_{01313}$$
$$-\omega^2\sin^2\theta\mathcal{A}_{02323}]+\omega^4\sin^2\theta\cos^2\theta(\mathcal{A}_{01133}+\mathcal{A}_{03113}-\mathcal{A}_{01122}$$
$$-\mathcal{A}_{02112})[\rho c^2+s^2c^2(\mathcal{A}_{03333}-\mathcal{A}_{03311}-\mathcal{A}_{01331})-\omega^2\cos^2\theta$$
$$\mathcal{A}_{01313}-\omega^2\sin^2\theta\mathcal{A}_{02323}]+s^2c^2[\rho c^2+s^2c^2\mathcal{A}_{03131}+\omega^2\cos^2$$
$$(\mathcal{A}_{01133}+\mathcal{A}_{03113}-\mathcal{A}_{01111})-\omega^2\sin^2\theta\mathcal{A}_{02121}][\rho c^2+s^2c^2$$
$$\mathcal{A}_{03232}-\omega^2\cos^2\theta\mathcal{A}_{01212}+\omega^2\sin^2\theta(\mathcal{A}_{02233}+\mathcal{A}_{03223}-\mathcal{A}_{02222})]$$
$$-\omega^4\sin^2\theta\cos^2\theta(\mathcal{A}_{01133}+\mathcal{A}_{03113}-\mathcal{A}_{01122}-\mathcal{A}_{02112})(\mathcal{A}_{02233}$$
$$+\mathcal{A}_{3223}-\mathcal{A}_{02211}-\mathcal{A}_{01221})s^2c^2=0. \qquad (2.2.9)$$

This is a cubic equation for s^2. Let s_1, s_2, s_3 be the three values of s with positive real part. Then we may write the solution in the form

$$\begin{aligned}
\psi_1 &= A_1e^{-s_1x_3} + A_2e^{-s_2x_3} + A_3e^{-s_3x_3},\\
\psi_2 &= B_1e^{-s_1x_3} + B_2e^{-s_2x_3} + B_3e^{-s_3x_3},\\
\psi_3 &= C_1e^{-s_1x_3} + C_2e^{-s_2x_3} + C_3e^{-s_3x_3},\\
\phi &= D_1e^{-s_1x_3} + D_2e^{-s_2x_3} + D_3e^{-s_3x_3}.
\end{aligned} \qquad (2.2.10)$$

Each ratio $A_i: B_i: C_i: D_i$ $(i=1,2,3)$ is obtained from (2.2.7). For $A_1: A_2: A_3$ to be non-trivial the boundary conditions yield the secular equation. Because of complicated algebra involved we do not give the details of the general case here, but concentrate on the application to the neo-Hookean material.

2.2.1 Propagation in any direction for a neo-Hookean material

For a neo-Hookean material we have

$$W = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),$$

and hence, from (1.5.13), we obtain

$$\begin{aligned}
&\mathcal{A}_{01111}=\mathcal{A}_{01313}=\mathcal{A}_{01212}=\mu\lambda_1^2,\\
&\mathcal{A}_{02222}=\mathcal{A}_{02121}=\mathcal{A}_{02323}=\mu\lambda_2^2,\\
&\mathcal{A}_{03333}=\mathcal{A}_{03131}=\mathcal{A}_{03332}=\mu\lambda_3^2,\\
&\mathcal{A}_{01331}=\mathcal{A}_{03113}=\mathcal{A}_{01133}=\mathcal{A}_{02233}=\mathcal{A}_{03223}=\mathcal{A}_{02211}=\mathcal{A}_{03311}=0.
\end{aligned} \tag{2.2.11}$$

Substituting (2.2.11) into (2.2.9) we obtain

$$\begin{aligned}
&-\omega^2\cos^2\theta(\rho c^2+\mu\lambda_3^2 s^2c^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)^2-\omega^2\sin^2\theta\\
&\quad(\rho c^2+\mu\lambda_3^2 s^2c^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)^2+s^2c^2\\
&\quad(\rho c^2+\mu\lambda_3^2 s^2c^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)^2 = 0,
\end{aligned}$$

i.e.

$$(\rho c^2+\mu\lambda_3^2 s^2c^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)^2(s^2c^2-\omega^2)=0 \tag{2.2.12}$$

Equation (2.2.12) yields two district values of s^2 with positive real part, s_1 and s_2 say, where

$$s_1^2c^2=\omega^2 \quad \text{or} \quad \frac{s_1^2c^2}{\omega^2} = 1$$

and (2.2.13)

$$\frac{s_2^2c^2}{\omega^2} = (\mu\lambda_1^2\cos^2\theta + \mu\lambda_2^2\sin^2\theta - \rho c^2)/\mu\lambda_3^2,$$

this requiring

$$0 < \rho c^2 < \mu(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta).$$

From (2.2.13), we see that s_2^2 is a repeated root, that is $s_2 = s_3$, so equations (2.2.10) become

$$\begin{aligned}
\Psi_1 &= A_1\bar{e}^{s_1x_3} + (A_2+A_3x_3)\bar{e}^{s_2x_3},\\
\Psi_2 &= B_1\bar{e}^{s_1x_3} + (B_2+B_3x_3)\bar{e}^{s_2x_3},\\
\Psi_3 &= C_1\bar{e}^{s_1x_3} + (C_2+C_3x_3)\bar{e}^{s_2x_3},\\
\Phi &= D_1\bar{e}^{s_1x_3} + (D_2+D_3x_3)\bar{e}^{s_2x_3}.
\end{aligned} \qquad (2.2.14)$$

Next, the incremental boundary conditions for propagation in any direction are

$$\dot{S}_{03i} = 0 \qquad \text{on } x_3 = 0$$

i.e.

$$\dot{S}_{031} = 0, \quad \dot{S}_{032} = 0, \quad \dot{S}_{033} = 0 \text{ on } x_3 = 0.$$

On use of equations (2.1.4), (2.2.2) and (2.2.3) with the above boundary conditions, we obtain

$$(\mathcal{A}_{03113} + p)\left[-\frac{i\omega\cos\theta}{c}\right] \psi_3 + \mathcal{A}_{03131}\, \psi'_1 = 0,$$

$$(\mathcal{A}_{03223} + p)\left[-\frac{i\omega\sin\theta}{c}\right] \psi_3 + \mathcal{A}_{03232}\, \psi'_2 = 0, \qquad (2.2.15)$$

$$\mathcal{A}_{03311}\left[-\frac{i\omega\cos\theta}{c}\right] \psi_1 + \mathcal{A}_{03322}\left[-\frac{i\omega\sin\theta}{c}\right] \psi_2 + (\mathcal{A}_{03333}+p)\, \psi'_3 - \Phi = 0$$

on $x_3 = 0$.

Since, from (1.5.13), $\mathcal{A}_{03113} = \mathcal{A}_{03131} - \lambda_3 \frac{\partial W}{\partial \lambda_3}$, $\mathcal{A}_{03113} + p = \mathcal{A}_{03131} - \sigma_3$ and similarly, $\mathcal{A}_{03223} + p = \mathcal{A}_{03232} - \sigma_3$. Also with the case when $\sigma_3 = 0$, equations (2.2.15) become

$$\left[-\frac{i\omega\cos\theta}{c}\right] \psi_3 + \psi'_1 = 0,$$

$$\left[-\frac{i\omega\sin\theta}{c}\right] \psi_3 + \psi'_2 = 0, \qquad (2.2.16)$$

$$\mathcal{A}_{03113}\left[-\frac{i\omega\cos\theta}{c}\right] \psi_1 + \mathcal{A}_{03322}\left[-\frac{i\omega\sin\theta}{c}\right] \psi_2 + (\mathcal{A}_{03333}+p)\, \psi'_3 - \Phi = 0.$$

on $x_3 = 0$.

For the neo-Hookean material the third of (2.2.16) reduces to

$$(2\mu\lambda_3^2)\, \psi'_3 - \Phi = 0 \qquad (2.2.17)$$

since $\sigma_3 = 0$ implies $p = \mu\lambda_3^2$, and the first two of (2.2.16) are unchanged.

Next, using equations (2.2.14), (2.2.16) and (2.2.17), we have

$$\left[-\frac{i\omega\cos\theta}{c}\right](C_1+C_2) - s_1A_1 + A_3 - s_2A_2 = 0,$$

$$\left[-\frac{i\omega\sin\theta}{c}\right](C_1+C_2) - s_1B_1 + B_3 - s_2B_2 = 0, \qquad (2.2.18)$$

$$(2\mu\lambda_3^2+p)(-s_1C_1 - s_2C_2 + C_3) - (D_1 + D_2) = 0.$$

Now, we want to determine the ratio A_i: B_i: C_i: D_i from (2.2.8) and (2.2.7). For the neo-Hookean case the first two equations of (2.2.8) reduce to

$$[\rho c^2+\mu\lambda_3^2c^2s^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta]A+i\omega c\cos\theta\ D=0,$$
$$[\rho c^2+\mu\lambda_3^2c^2s^2-\mu\lambda_1^2\omega^2\cos^2-\mu\lambda_2^2\omega^2\sin^2\theta]B+i\omega c\sin\theta\ D=0. \qquad (2.2.19)$$

For $s=s_1$ these give

$$\frac{D_1}{A_1} = \frac{-[\rho c^2+\mu\lambda_3^2c^2s_1^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta]}{i\omega c\cos\theta},$$

$$\frac{D_1}{B_1} = \frac{-[\rho c^2+\mu\lambda_3^2c^2s_1^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta]}{i\omega c\sin\theta},$$

so that

$$\frac{B_1}{A_1} = \tan\theta.$$

For $s = s_2$ we have to consider

$$\Psi_1 = (A_2 + A_3x_3)e^{-s_2x_3},$$
$$\Psi_2 = (B_2 + B_3x_3)e^{-s_2x_3}, \qquad (2.2.20)$$
$$\Psi_3 = (C_2 + C_3x_3)e^{-s_2x_3}.$$
$$\Phi = (D_2 + D_3x_3)e^{-s_2x_3}.$$

Substitution of (2.2.20) into (2.2.6) shows that $A_3=B_3=C_3=D_3=0$ and hence (2.2.19) applies with $s=s_2$, giving

$$\frac{D_2}{A_2} = -\left[\frac{\rho c^2+\mu\lambda_3^2 c^2 s_2^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta}{i\omega c\cos\theta}\right],$$

$$\frac{D_2}{B_2} = -\left[\frac{\rho c^2+\mu\lambda_3^2 c^2 s_2^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta}{i\omega c\sin\theta}\right],$$

and

$$\frac{B_2}{A_2} = \tan\theta.$$

From $(2.2.7)_4$ we also have

$$is_1C_1 = \frac{\omega}{c}\cos\theta\ A_1 + \frac{\omega}{c}\sin\theta\ B_1$$

$$is_2C_2 = \frac{\omega}{c}\cos\theta\ A_2 + \frac{\omega}{c}\sin\theta\ B_2.$$

Thus, the boundary conditions (2.2.18) become

$$\left[\frac{i\omega}{c}\cos\theta\right](C_1+C_2) + s_1A_1 + s_2A_2 = 0$$

$$\left[\frac{i\omega}{c}\sin\theta\right](C_1+C_2) + s_1B_1 + s_2B_2 = 0 \qquad (2.2.21)$$

$$2\mu\lambda_3^2(s_1C_1 + s_2C_2) + D_1+D_2 = 0.$$

Substitution for C_1, C_2, B_1, B_2, D_1, D_2 in terms of A_1, A_2 gives

$$is_1C_1 = \frac{\omega}{c}A_1\frac{1}{\cos\theta}, \qquad is_2C_2 = \frac{\omega}{c}A_2\frac{1}{\cos\theta}.$$

Hence

$$\frac{\omega}{c}\cos\theta\left[\frac{\omega A_1}{s_1c\cos\theta} + \frac{\omega A_2}{s_2c\cos\theta}\right] + s_1A_1 + s_2A_2 = 0,$$

$$2\mu\lambda_3^2\left[\frac{\omega}{c}\frac{A_1}{\cos\theta} + \frac{\omega A_2}{c\cos\theta}\right] + iD_1 + iD_2 = 0,$$

i.e.

$$\left[\frac{\omega^2}{c^2s_1} + s_1\right]A_1 + \left[\frac{\omega^2}{c^2s_2} + s_2\right]A_2 = 0 \qquad (2.2.22)_1$$

and

$$2\mu\lambda_3^2\omega\ (A_1+A_2) + ic\cos\theta D_1 + ic\cos\theta D_2 = 0,$$

and hence

$$2\mu\lambda_3^2\omega^2\ (A_1+A_2)-[\rho c^2+\mu\lambda_3^2c^2s_1^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta]A_1$$

$$-[\rho c^2+\mu\lambda_3^2c^2s_2^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta]A_2 = 0. \qquad (2.2.22)_2$$

For A_1, A_2 in $(2.2.22)_{1,2}$ to be non-trivial solution we require

$$\Delta \equiv \begin{vmatrix} \dfrac{\omega^2}{c^2s_1} + s_1 & \dfrac{\omega^2}{c^2s_2} + s_2 \\ \rho c^2+\mu\lambda_3^2c^2s_1^2-2\mu\lambda_3^2\omega^2 & \rho c^2+\mu\lambda_3^2c^2s_2^2-2\mu\lambda_3^2\omega^2 \\ -\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta & -\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta \end{vmatrix} = 0,$$

Hence

$$\Delta \equiv \left[s_2\frac{\omega^2}{c^2} + s_1^2s_2\right](\rho c^2\omega^2+\mu\lambda_3^2c^2s_2^2-2\mu\lambda_3^2\omega^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)$$

$$-\left[s_1\frac{\omega^2}{c^2} + s_1s_2^2\right](\rho c^2\omega^2+\mu\lambda_3^2c^2s_1^2-2\mu\lambda_3^2\omega^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)=0$$

and, on rearrangement, this becomes

$$(\rho c^2\omega^2-2\mu\lambda_3^2\omega^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)\ \{\frac{\omega^2}{c^2}(s_2-s_1)+s_1s_2(s_1-s_2)\}$$

$$+ \mu\lambda_3^2\omega^2(s_2^3-s_1^3)+\mu\lambda_3^2c^2s_1^2s_2^2(s_2-s_1) = 0. \qquad (2.2.23)$$

Assuming $s_1 \neq s_2$, the secular equation (2.2.23) reduces to

$$(\rho c^2\omega^2-2\mu\lambda_3^2\omega^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)\left[\frac{\omega^2}{c^2} - s_1s_2\right] + \mu\lambda_3^2c^2s_1^2s_2^2$$

$$+ (s_1^2+s_1s_2+s_2^2)\ \mu\lambda_3^2\omega^2 = 0. \qquad (2.2.24)$$

This is the secular equation for the propagation of Rayleigh waves in any direction for a neo-Hookean material. (As in the case of incompressible materials, $s_1=s_2$ gives only the trivial result $A_1=A_2=0$ etc.)

But, from (2.2.13)

$$s_1 = \frac{\omega}{c}$$

$$s_2 = \frac{\omega}{c}\left[\frac{\mu\lambda_1^2\cos^2\theta + \mu\lambda_2^2\sin^2\theta - \rho c^2}{\mu\lambda_3^2}\right]^{½}$$

so,

$$s_1^2 + s_2^2 = \frac{\omega^2}{c^2}\left\{1 + \frac{\mu\lambda_1^2\cos^2\theta + \mu\lambda_2\sin^2\theta - \rho c^2}{\mu\lambda_3^2}\right\}$$

Hence

$$\frac{\omega^2}{c^2}(\rho c^2\omega^2-2\mu\omega^2\lambda_3^2-\mu\lambda_1^2\omega^2\cos^2\theta-\mu\lambda_2^2\omega^2\sin^2\theta)-s_1s_2\omega^2(\rho c^2-3\mu\lambda_3^2$$

$$-\mu\lambda_1^2\cos^2\theta-\mu\lambda_2^2\sin^2\theta)+\mu\lambda_3^2c^2s_1^2s_2^2+\mu\lambda_3^2\omega^2(s_1^2+s_2^2)=0,$$

i.e.

$$(\rho c^2-2\mu\lambda_3^2-\mu\lambda_1^2\cos^2\theta-\mu\lambda_2^2\sin^2\theta)-\left[\frac{\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta-\rho c^2}{\mu\lambda_3^2}\right]^{½}$$

$$(\rho c^2-3\mu\lambda_3^2-\mu\lambda_1^2\cos^2\theta-\mu\lambda_2^2\sin^2\theta)+(\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta-\rho c^2)+\mu\lambda_3^2$$

$$+\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta-\rho c^2 = 0$$

and hence

$$(\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta-\mu\lambda_3^2-\rho c^2) = \left[\frac{\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta-\rho c^2}{\mu\lambda_3^2}\right]^{½}$$

$$(\rho c^2-3\mu\lambda_3^2-\mu\lambda_1^2\cos^2\theta-\mu\lambda_2^2\sin^2\theta).$$

By putting $x = \rho c^2/\mu\lambda_3^2$, $\eta = \dfrac{\mu\lambda_1^2\cos^2\theta+\mu\lambda_2^2\sin^2\theta}{\mu\lambda_3^2}$, the secular equation simplifies to

$$(n - x - 1) = (n - x)^{1/2}(x - n - 3),$$

i.e.

$$(n-x)^2 + 1 - 2(n-x) = (n-x)\,[(x-n)^2 + 6(n-x) + 9]$$

$$= (n-x)^3 + 6(n-x)^2 + 9(n-x).$$

Thus, the secular equation becomes

$$(n-x)^3 + 5(n-x)^2 + 11(n-x) - 1 = 0. \qquad (2.2.25)$$

Equation (2.2.25) gives only one solution for $n-x$, n_0 say, so

$$\mu(\lambda_1^2\cos^2\theta+\lambda_2^2\sin^2\theta) - \rho c^2 = n_0\mu\lambda_3^2.$$

Hence

$$\rho c^2 = \mu(\lambda_1^2\cos^2\theta+\lambda_2^2\sin^2\theta)-n_0\mu\lambda_3^2.$$

This result is equivalent to an equation given by Flavin (1963).

2.3 Rayleigh waves on a pre-strained elastic half-space

2.3.1 Analysis for compressible materials

For compressible materials the components of $\dot{\underline{S}}$ are given by (1.5.8). In full the components of $\dot{\underline{S}}$ are

$$
\begin{aligned}
\dot{s}_{011} &= \mathcal{A}_{01111}\frac{\partial\nu_1}{\partial x_1} + \mathcal{A}_{01122}\frac{\partial\nu_2}{\partial x_2} + \mathcal{A}_{01133}\frac{\partial\nu_3}{\partial x_3},\\
\dot{s}_{022} &= \mathcal{A}_{02211}\frac{\partial\nu_1}{\partial x_1} + \mathcal{A}_{02222}\frac{\partial\nu_2}{\partial x_2} + \mathcal{A}_{02233}\frac{\partial\nu_3}{\partial x_3},\\
\dot{s}_{033} &= \mathcal{A}_{03311}\frac{\partial\nu_1}{\partial x_1} + \mathcal{A}_{03322}\frac{\partial\nu_2}{\partial x_2} + \mathcal{A}_{03333}\frac{\partial\nu_3}{\partial x_3},\\
\dot{s}_{012} &= \mathcal{A}_{01212}\frac{\partial\nu_2}{\partial x_1} + \mathcal{A}_{01221}\frac{\partial\nu_1}{\partial x_2},\\
\dot{s}_{021} &= \mathcal{A}_{02112}\frac{\partial\nu_2}{\partial x_1} + \mathcal{A}_{02121}\frac{\partial\nu_1}{\partial x_2},\\
\dot{s}_{013} &= \mathcal{A}_{01313}\frac{\partial\nu_3}{\partial x_1} + \mathcal{A}_{01331}\frac{\partial\nu_1}{\partial x_3},\\
\dot{s}_{031} &= \mathcal{A}_{03113}\frac{\partial\nu_3}{\partial x_1} + \mathcal{A}_{03131}\frac{\partial\nu_1}{\partial x_3},\\
\dot{s}_{023} &= \mathcal{A}_{02323}\frac{\partial\nu_3}{\partial x_2} + \mathcal{A}_{02332}\frac{\partial\nu_2}{\partial x_3},\\
\dot{s}_{032} &= \mathcal{A}_{03223}\frac{\partial\nu_3}{\partial x_2} + \mathcal{A}_{03232}\frac{\partial\nu_2}{\partial x_3},
\end{aligned}
\tag{2.3.1}
$$

(a) Plane incremental motion

We take $\nu_2 \equiv 0$ and assume that ν_1 and ν_3 depend on x_1 and x_3. Equations (2.3.1) then simplify to

$$\dot{s}_{011} = \mathcal{A}_{01111}\, \nu_{1,1} + \mathcal{A}_{01133}\, \nu_{3,3},$$
$$\dot{s}_{022} = \mathcal{A}_{02211}\, \nu_{1,1} + \mathcal{A}_{02233}\, \nu_{3,3},$$
$$\dot{s}_{033} = \mathcal{A}_{03311}\, \nu_{1,1} + \mathcal{A}_{03333}\, \nu_{3,3}, \qquad (2.3.2)$$
$$\dot{s}_{013} = \mathcal{A}_{01313}\, \nu_{3,1} + \mathcal{A}_{01331}\, \nu_{1,3},$$
$$\dot{s}_{031} = \mathcal{A}_{03113}\, \nu_{3,1} + \mathcal{A}_{03131}\, \nu_{1,3}.$$

By using the incremental equation of motion for a compressible material (1.5.19) with equation (2.1. 7) we get

$$\dot{s}_{oji,j} \equiv \mathcal{A}_{oji\ell k}\, \nu_{k,\ell j} = \rho\ddot{\nu}_i. \qquad (2.3.3)$$

From this equation, when we take i=1 and i=3, we obtain

$$\dot{s}_{011,1} + \dot{s}_{031,3} = \rho\ddot{\nu}_1,$$
$$\qquad (2.3.4)$$
$$\dot{s}_{013,1} + \dot{s}_{033,3} = \rho\ddot{\nu}_3.$$

Next, from equations (2.3.2) we have

$$\dot{s}_{011,1} = \mathcal{A}_{01111}\, \nu_{1,11} + \mathcal{A}_{01133}\, \nu_{3,31},$$
$$\dot{s}_{031,3} = \mathcal{A}_{03131}\, \nu_{1,33} + \mathcal{A}_{03113}\, \nu_{3,13},$$
$$\dot{s}_{013,1} = \mathcal{A}_{01313}\, \nu_{3,11} + \mathcal{A}_{01331}\, \nu_{1,31} \qquad (2.3.5)$$
$$\dot{s}_{033,3} = \mathcal{A}_{03311}\, \nu_{1,13} + \mathcal{A}_{03333}\, \nu_{3,33}.$$

Substituting (2.3.5) into (2.3.4) we have the required equations of plane incremental motion, namely

$$\rho\ddot{\nu}_1 \equiv \mathcal{A}_{01111}\nu_{1,11} + \mathcal{A}_{01133}\nu_{3,31} + \mathcal{A}_{03131}\nu_{1,33} + \mathcal{A}_{03113}\nu_{3,13},$$
$$\rho\ddot{\nu}_3 \equiv \mathcal{A}_{01313}\nu_{3,11} + \mathcal{A}_{01331}\nu_{1,31} + \mathcal{A}_{03311}\nu_{1,13} + \mathcal{A}_{03333}\nu_{3,33}. \qquad (2.3.6)$$

<u>(b) Propagation along a principal axis</u>

We now assume that ν_1 and ν_3 are given by

$$\nu_1 = A_1 e^{-sx_3 + i\omega\left[t - \frac{x_1}{c}\right]},$$
$$\qquad (2.3.7)$$
$$\nu_3 = A_3 e^{-sx_3 + i\omega\left[t - \frac{x_1}{c}\right]},$$

where A_1 and A_3 are constants.

Substitution of (2.3.7) into (2.3.6) yields

$$-\rho\omega^2 A_1 = (\mathcal{A}_{03131}s^2 - \frac{\omega^2}{c^2}\mathcal{A}_{01111})\, A_1 + i\frac{\omega}{c}s(\mathcal{A}_{01133} + \mathcal{A}_{03113})\, A_3,$$

$$-\rho\omega^2 A_3 = i\frac{\omega}{c}s(\mathcal{A}_{01331} + \mathcal{A}_{03311})\, A_1 + (s^2\mathcal{A}_{03333} - \mathcal{A}_{01313}\frac{\omega^2}{c^2})\, A_3. \qquad (2.3.8)$$

For a non-trivial solution for A_1, A_3 we must have

$$\Delta \equiv \begin{vmatrix} \mathcal{A}_{03131}s^2 - \frac{\omega^2}{c^2}\mathcal{A}_{01111} + \rho\omega^2 & i\frac{\omega}{c}s(\mathcal{A}_{01133} + \mathcal{A}_{03113}) \\ i\frac{\omega}{c}s(\mathcal{A}_{01331} + \mathcal{A}_{03311}) & s^2\mathcal{A}_{03333} - \frac{\omega^2}{c^2}\mathcal{A}_{01313} + \rho\omega^2 \end{vmatrix} = 0$$

Therefore,

$$\Delta \equiv (\mathcal{A}_{03131}\, s^2 - \frac{\omega^2}{c^2}\mathcal{A}_{01111} + \rho\omega^2)(s^2\mathcal{A}_{02222} - \frac{\omega^2}{c^2}\mathcal{A}_{01313} + \rho\omega^2) + \frac{\omega^2}{c^2}s^2$$

$$(\mathcal{A}_{01133} + \mathcal{A}_{03113})^2 = 0,$$

i.e.

$$\mathcal{A}_{03333}\mathcal{A}_{03131}\, s^4 - \frac{\omega^2}{c^2}s^2[\mathcal{A}_{01111}\mathcal{A}_{03333} + \mathcal{A}_{03131}\mathcal{A}_{01313} -$$

$$(\mathcal{A}_{01133} + \mathcal{A}_{03113})^2] + \frac{\omega^4}{c^4}\mathcal{A}_{01111}\mathcal{A}_{01313} + \rho\omega^2$$

$$(\rho\omega^2 + \mathcal{A}_{03131}\, s^2 + \mathcal{A}_{03333}\, s^2 - \mathcal{A}_{01111}\,\frac{\omega^2}{c^2} - \mathcal{A}_{01313}\,\frac{\omega^2}{c^2}) = 0 \qquad (2.3.10)$$

This is a quadratic equation for s^2. Suppose it has roots s_1^2 and s_2^2. Then

$s_1^2 + s_2^2 =$

$$\frac{\frac{\omega^2}{c^2}\left[\mathcal{A}_{01111}\mathcal{A}_{03333}+\mathcal{A}_{03131}\mathcal{A}_{01313}-(\mathcal{A}_{01133}+\mathcal{A}_{03113})^2\right]-(\mathcal{A}_{03131}+\mathcal{A}_{03333})\rho\omega^2}{\mathcal{A}_{03333}\mathcal{A}_{03131}},$$

(2.3.11)

$$s_1^2 s_2^2 = \frac{\left[\frac{\omega^4}{c^4}\mathcal{A}_{01111}\mathcal{A}_{01313} - \frac{\rho\omega^4}{c^2}(\mathcal{A}_{01111}+\mathcal{A}_{01313})\right]}{\mathcal{A}_{03333}\mathcal{A}_{03131}}.$$

The incremental boundary conditions are

$$\dot{S}_{031} = 0, \quad \dot{S}_{033} = 0 \qquad \text{on } x_3=0,$$

so from equations (2.3.2) we obtain

$$\begin{aligned} &\mathcal{A}_{03131}\, \nu_{1,3} + \mathcal{A}_{03113}\, \nu_{3,1} = 0, \\ &\mathcal{A}_{03311}\, \nu_{1,1} + \mathcal{A}_{03333}\, \nu_{3,3} = 0. \end{aligned} \qquad \text{on } x_3 = 0 \quad (2.3.12)$$

For Rayleigh surface waves we seek a solution for ν_1 and ν_3 in equation (2.3.7) which vanishes when $x_3 \to +\infty$ and which also satisfies the boundary conditions in equation (2.3.12).

The general solutions for ν_1 and ν_3 are given by

$$\nu_1 = (A_1 e^{-s_1 x_3} + B_1 e^{-s_2 x_3})\, e^{i\omega\left[t-\frac{x_1}{c}\right]},$$

(2.3.13)

$$\nu_3 = (A_3 e^{-s_1 x_3} + B_3 e^{-s_2 x_3})\, e^{i\omega\left[t-\frac{x_1}{c}\right]},$$

where s_1 and s_2 should have positive real part.

Next, substitute equations (2.3.13) into the boundary conditions (2.3.12) to obtain

$$-\mathcal{A}_{03131}\, s_1 A_1 - \frac{i\omega}{c}\mathcal{A}_{03113}\, A_3 - \mathcal{A}_{03131}\, B_1 - \frac{i\omega}{c}\mathcal{A}_{03113}\, B_3 = 0.$$

$$(2.3.14)$$

$$-\, i\frac{\omega}{c}\mathcal{A}_{03311}\, A_1 - \mathcal{A}_{03333}\, s_1 A_3 - \frac{i\omega}{c}\mathcal{A}_{03311}\, B_1 - \mathcal{A}_{03333} s_2 B_3 = 0.$$

From equations (2.3.8) we obtain

$$\frac{iA_3}{A_1} = \frac{(k^2\mathcal{A}_{01111} - s_1^2\mathcal{A}_{03131} - \rho\omega^2)}{ks_1(\mathcal{A}_{01133} + \mathcal{A}_{03113})},$$

and similarly (2.3.15)

$$\frac{iB_3}{B_1} = \frac{(k^2\mathcal{A}_{01111} - s_2^2\mathcal{A}_{03131} - \rho\omega^2)}{ks_2(\mathcal{A}_{01133} + \mathcal{A}_{03113})}.$$

Now, substitute (2.3.15) into (2.3.14). This yields

$$\left\{\mathcal{A}_{03131}\, s_1 + \frac{\mathcal{A}_{03113}(k^2\mathcal{A}_{01111} - s_1^2\mathcal{A}_{03131} - \rho\omega^2)}{s_1(\mathcal{A}_{01133} + \mathcal{A}_{03113})}\right\}^2 A_1 +$$

$$\left\{\mathcal{A}_{03131} s_2 + \frac{\mathcal{A}_{03113}(k^2\mathcal{A}_{01111} - s_2^2\mathcal{A}_{03131} - \rho\omega^2)}{s_2(\mathcal{A}_{01133} + \mathcal{A}_{03113})}\right\} B_1 = 0,$$

$$(2.3.16)$$

$$\left\{i\frac{\omega}{c}\mathcal{A}_{03311} + \frac{\mathcal{A}_{03333}(k^2\mathcal{A}_{01111} - s_1{}^2\mathcal{A}_{03131} - \rho\omega^2)}{ik(\mathcal{A}_{01133} + \mathcal{A}_{03113})}\right\} A_1 +$$

$$\left\{i\frac{\omega}{c}\mathcal{A}_{03311} + \frac{\mathcal{A}_{03333}(k^2\mathcal{A}_{01111} - s_2^2\mathcal{A}_{03131} - \rho\omega^2)}{ik(\mathcal{A}_{01133} + \mathcal{A}_{03113})}\right\} B_1 = 0.$$

We may rewrite equation (2.3.16) as

$$s_2\{\, s_1^2\mathcal{A}_{03131}(\mathcal{A}_{01133} + \mathcal{A}_{03113}) + \mathcal{A}_{03113}(k^2\mathcal{A}_{01111} - s_1^2\mathcal{A}_{03131} - \rho\omega^2)\} A_1$$

$$+ s_1\{\, s_2^2\mathcal{A}_{03131}(\mathcal{A}_{01133} + \mathcal{A}_{03113}) + \mathcal{A}_{03113}(k^2\mathcal{A}_{01111} - s_2^2\mathcal{A}_{03131} - \rho\omega^2)\} B_1 = 0.$$

$$\{k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)\}A_1$$

$$+\{k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)\}B_1=0.$$

For these equations to have non-trivial solutions for A_1 and B_1, we must have

$$\Delta \equiv \begin{vmatrix} s_1^2 s_2\mathcal{A}_{03131}(\mathcal{A}_{01133}+\mathcal{A}_{03113})+s_2\mathcal{A}_{03113}(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2) & s_2^2 s_1\mathcal{A}_{03131}(\mathcal{A}_{01133}+\mathcal{A}_{03113})+s_1\mathcal{A}_{03113}(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2) \\ k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2) & k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2) \end{vmatrix} = 0$$

and hence

$$\Delta\equiv[s_1^2 s_2\mathcal{A}_{03131}(\mathcal{A}_{01133}+\mathcal{A}_{03113})+\mathcal{A}_{03113}s_2(k_2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)]$$

$$[k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)]$$

$$-[s_2^2 s_1\mathcal{A}_{03131}(\mathcal{A}_{01133}+\mathcal{A}_{03113})+s_1\mathcal{A}_{03113}(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)]$$

$$[k^2\mathcal{A}_{03311}(\mathcal{A}_{01133}+\mathcal{A}_{03113})-\mathcal{A}_{03333}(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)]=0.$$

Thus

$$s_2[s_1^2k^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})^2\mathcal{A}_{03131}\mathcal{A}_{03311}+k^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})$$

$$(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)\mathcal{A}_{03113}\mathcal{A}_{03311}-s_1^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})$$

$$(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)\mathcal{A}_{03131}\mathcal{A}_{03333}-\mathcal{A}_{03113}\mathcal{A}_{03333}$$

$$(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)]-s_1[s_2^2k^2$$

$$(\mathcal{A}_{01133}+\mathcal{A}_{03113})^2\mathcal{A}_{03131}\mathcal{A}_{03311}-(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)$$

$$(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)\mathcal{A}_{03113}\mathcal{A}_{03333}+k^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})$$

$$(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)\mathcal{A}_{03311}\mathcal{A}_{03113}-s_2^2(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)$$

$$(\mathcal{A}_{01133}+\mathcal{A}_{03113})\mathcal{A}_{03131}\mathcal{A}_{03333}] = 0.$$

Gathering together like terms, we obtain

$$k^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})^2\mathcal{A}_{03131}\mathcal{A}_{01133}\ s_1s_2(s_1-s_2)+(s_1-s_2)$$

$$(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)\mathcal{A}_{1133}\mathcal{A}_{3113}$$

$$+k^2(\mathcal{A}_{01133}+\mathcal{A}_{03113})\mathcal{A}_{03113}\mathcal{A}_{01133}[s_2(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)$$

$$-s_1(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)]+[(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)s_1s_2^2$$

$$-(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)s_1^2s_2](\mathcal{A}_{01133}+\mathcal{A}_{03113})\mathcal{A}_{03131}\mathcal{A}_{03333}=0$$

i.e.

$$(s_1-s_2)]k^2s_1s_2\mathcal{A}_{03131}\mathcal{A}_{01133}(\mathcal{A}_{01133}+\mathcal{A}_{03113})^2+\mathcal{A}_{01133}\mathcal{A}_{03113}$$

$$(k^2\mathcal{A}_{01111}-s_1^2\mathcal{A}_{03131}-\rho\omega^2)(k^2\mathcal{A}_{01111}-s_2^2\mathcal{A}_{03131}-\rho\omega^2)$$

$$+k^2\mathcal{A}_{03113}\mathcal{A}_{01133}(\mathcal{A}_{01133}+\mathcal{A}_{03113})\{\rho\omega^2-k^2\mathcal{A}_{01111}-s_1s_2\mathcal{A}_{03131}\}$$

$$+\mathcal{A}_{03131}\mathcal{A}_{03333}(\mathcal{A}_{01133}+\mathcal{A}_{03113})\{\rho\omega^2s_1s_2-s_1^2s_2^2\mathcal{A}_{03131}-k^2s_1s_2$$

$$\mathcal{A}_{01111}\}] = 0.$$

As for incompressible materials the case $s_1 = s_2$ does not lead to the existence of Rayleigh waves. We therefore assume that $s_1 \neq s_2$, and hence

$$
\begin{aligned}
& k^2 s_1 s_2 \mathcal{A}_{03131} \mathcal{A}_{01133} (\mathcal{A}_{01133} + \mathcal{A}_{03113})^2 + \mathcal{A}_{01133} \mathcal{A}_{03113} (k^2 \mathcal{A}_{01111} - \rho\omega^2)^2 \\
& + \mathcal{A}_{01133} \mathcal{A}_{03113} \mathcal{A}_{03131}^2 s_1^2 s_2^2 - (s_1^2 + s_2^2) \mathcal{A}_{03131} (k^2 \mathcal{A}_{01111} - \rho\omega^2) \\
& \mathcal{A}_{01133} \mathcal{A}_{03113} + k^2 \mathcal{A}_{03113} \mathcal{A}_{01133} (\mathcal{A}_{01133} + \mathcal{A}_{03113}) (\rho\omega^2 - k^2 \mathcal{A}_{01111}) \\
& - k^2 s_1 s_2 \mathcal{A}_{03113} \mathcal{A}_{01133} \mathcal{A}_{03131} (\mathcal{A}_{01133} + \mathcal{A}_{03113}) + s_1 s_2 \mathcal{A}_{03131} \mathcal{A}_{03333} \\
& (\mathcal{A}_{01133} + \mathcal{A}_{03113}) (\rho\omega^2 - k^2 \mathcal{A}_{01111}) - s_1^2 s_2^2 \mathcal{A}_{03131}^2 \mathcal{A}_{03333} \\
& (\mathcal{A}_{01133} + \mathcal{A}_{03113}) = 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
& s_1^2 s_2^2 \mathcal{A}_{03131}^2 \{\mathcal{A}_{01133} \mathcal{A}_{03113} - \mathcal{A}_{03333} (\mathcal{A}_{01133} + \mathcal{A}_{03113})\} \\
& + k^2 s_1 s_2 \mathcal{A}_{03131} \mathcal{A}_{01133} \{(\mathcal{A}_{01133} + \mathcal{A}_{03113})^2 - \mathcal{A}_{03113} (\mathcal{A}_{01133} + \mathcal{A}_{03113})\} \\
& + s_1 s_2 \mathcal{A}_{03131} \mathcal{A}_{03333} (\mathcal{A}_{01133} + \mathcal{A}_{03113}) (\rho\omega^2 - k^2 \mathcal{A}_{01111}) - (s_1^2 + s_2^2) \\
& \mathcal{A}_{03131} \mathcal{A}_{01133} \mathcal{A}_{03113} (k^2 \mathcal{A}_{01111} - \rho\omega^2) + \mathcal{A}_{01133} \mathcal{A}_{03113} \\
& (k^2 \mathcal{A}_{01111} - \rho\omega^2) \{(k^2 \mathcal{A}_{01111} - \rho\omega^2) + k^2 (\mathcal{A}_{01133} + \mathcal{A}_{03113})\} = 0, \qquad (2.3.17)
\end{aligned}
$$

where, from (2.3.11),

$$
\frac{c^2}{\omega^2} (s_1^2 + s_2^2) =
$$

$$
\frac{\mathcal{A}_{01111} \mathcal{A}_{03333} + \mathcal{A}_{03131} \mathcal{A}_{01313} - (\mathcal{A}_{01133} + \mathcal{A}_{03113})^2 - (\mathcal{A}_{03131} + \mathcal{A}_{03333}) \rho c^2}{\mathcal{A}_{03131} \mathcal{A}_{03333}}, \qquad (2.3.18)
$$

$$
\frac{c^4}{\omega^4} s_1^2 s_2^2 = \frac{\mathcal{A}_{01111} \mathcal{A}_{01313} - \rho c^2 (\mathcal{A}_{01111} + \mathcal{A}_{01313})}{\mathcal{A}_{03131} \mathcal{A}_{03333}}.
$$

Equation (2.3.17) is the secular equation for Rayleigh surface waves propagating along a principal axis in a pre-strained compressible elastic medium.

2.3.2 Results for some particular deformations

(a) The case $\lambda_1=\lambda_3$

Consider the special case of (2.3.17) in which $\lambda_3=\lambda_1$, so we have

$$\mathcal{A}_{01111}=\mathcal{A}_{03333},\qquad \mathcal{A}_{01313}=\mathcal{A}_{03131},$$
$$\mathcal{A}_{01133}=\mathcal{A}_{03311},\qquad \mathcal{A}_{01331}=\mathcal{A}_{03113}. \tag{2.3.19}$$

Recall from (1.5.6) that

$$\mathcal{A}_{03113}=\mathcal{A}_{01313}-\sigma_3, \tag{2.3.20}$$

and from (1.5.15) that

$$\mathcal{A}_{01313}=\tfrac{1}{2}(\mathcal{A}_{01111}-\mathcal{A}_{01133}+\sigma_3).$$

Thus,

$$\mathcal{A}_{01133}=\mathcal{A}_{01111}-2\mathcal{A}_{01313}+\sigma_3 \tag{2.3.21}$$

For convenience we write $\alpha=\mathcal{A}_{01111}$, $\beta=\mathcal{A}_{01313}$. Then use of (2.3.19) in equations (2.3.11) leads to

$$\frac{s_1^2+s_2^2}{k^2}=2-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2,$$
$$\frac{s_1^2 s_2^2}{k^4}=1-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2 \tag{2.3.22}$$

Also, by using (2.3.19), (2.3.20), (2.3.21) in the secular equation (2.3.17), we have

$$s_1^2 s_2^2 \beta^2 \{(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)-\alpha(\alpha-\beta)+k^2 s_1 s_2 \beta(\alpha-2\beta+\sigma_3)\{(\alpha-\beta)^2-(\beta-\sigma_3)$$

$$(\alpha-\beta)\}+s_1 s_2 \beta\alpha(\alpha-\beta)(\rho\omega^2-k^2\alpha)-(s_1^2+s_2^2)\beta(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(k^2\alpha-\rho\omega^2)$$

$$+(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(k^2\alpha-\rho\omega^2)\{(k^2\alpha-\rho\omega^2)+k^2(\alpha-\beta)\}=0.$$

Thus,

$$k^4\left[1-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2\right]\beta^2 \ \{(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)-\alpha(\alpha-\beta)\}$$

$$-\left[2-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2\right]\beta k^2(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(k^2\alpha-\rho\omega^2)$$

$$+(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(k^2\alpha-\rho\omega^2)\{k^2(2\alpha-\beta)-\rho\omega^2\}$$

$$+\beta s_1 s_2\{\alpha(\alpha-\beta)(\rho\omega^2-k^2\alpha)+k^2(\alpha-2\beta+\sigma_3)(\alpha-\beta)(\alpha-2\beta+\sigma_3)\}=0.$$

i.e.

$$\beta k^2\left[1-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2\right]\{(\beta k^2-\alpha k^2+\rho\omega^2)(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)-\alpha\beta k^2(\alpha-\beta)\}$$

$$+(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(k^2\alpha-\rho\omega^2)\{k^2(2\alpha-2\beta)-\rho\omega^2\}$$

$$+\beta s_1 s_2(\alpha-\beta)\{\alpha(\rho\omega^2-k^2\alpha)+k^2(\alpha-2\beta+\sigma_3)^2\}=0,$$

and after further rearrangement this can be expressed as

$$(\rho c^2)^2\beta(\beta-\sigma_3)(\alpha-2\beta+\sigma_3)$$

$$-\rho c^2(\alpha-\beta)\{(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)(\beta-2\alpha)+\alpha\beta(\alpha+\beta)\}$$

$$+(\alpha-\beta)\{(\beta-\alpha)(\alpha-2\beta+\sigma_3)(\beta-\sigma_3)\alpha+\alpha^2\beta^2\}$$

$$-\alpha\beta(\alpha-\beta)\left\{1-\left[\frac{\alpha+\beta}{\alpha\beta}\right]\rho c^2\right\}^{1/2}\{\alpha(\rho c^2-\alpha)+\alpha-2\beta+\sigma_3)^2\}=0. \tag{2.3.23}$$

Equation (2.3.23) is the secular equation for Rayleigh waves in pre-strained compressible isotropic elastic medium, for the case $\lambda_3=\lambda_1$. For the case $\sigma_3=0$ the existence of surface waves has been discussed from a general standpoint by Chadwick and Jarvis (1979) using a different approach; also, for the case $\sigma_3=0$ an equivalent result, but in different notation, was given by Hayes and Rivlin (1961b). For the special case $\lambda_1=\lambda_2$ and $\sigma_3=0$ corresponding results can be found in Willson (1972, 1973b).

(b) The case $\lambda_1=\lambda_2=\lambda_3$

Here $\alpha=\lambda+2\mu$ and $\beta=\mu$, as from equation (1.6.18) where λ and μ are the classical Lamé moduli.

If, in particular, $\sigma_3=0$ then (2.3.23) reduces to the corresponding result of the linear theory, namely

$$(\rho c^2)^2\mu^2\lambda-\rho c^2\mu(\lambda+\mu)\{(\lambda+2\mu)(\lambda+3\mu)-\lambda(2\lambda+3\mu)\}$$

$$+\mu(\lambda+\mu)(\lambda+2\mu)\{\mu(\lambda+2\mu)-\lambda(\lambda+\mu)\}$$

$$-\mu(\lambda+2\mu)(\lambda+\mu)\left\{1-\left[\frac{\lambda+3\mu}{\mu(\lambda+2\mu)}\right]\rho c^2\right\}^{\frac{1}{2}}\{(\lambda+2\mu)\rho c^2+4\mu^2-4\mu(\lambda+2\mu)\}=0.$$

This is equivalent to the results derived using the linear theory from beginning, namely

$$\left[\frac{\rho c^2}{\mu}\right]\left[\left[\frac{\rho c^2}{\mu}\right]^3-8\left[\frac{\rho c^2}{\mu}\right]+\left[24-16\,\frac{\mu}{\lambda+2\mu}\right]\left[\frac{\rho c^2}{\mu}\right]+16\left[1-\frac{\mu}{\lambda+2\mu}\right]\right]=0.$$

(see, for example, Eringen and Suhubi (1975)).

(c) The case of uniaxial stress: $\sigma_2=\sigma_3=0$,

This special case has been considered by Willson (1972) using a special form of strain-energy function. Because of the cumbersome algebra involved we omit details here.

2.4 Propagation in a general direction in the (x_1, x_2) plane

In this final section of chapter two, we shall deduce the equation for propagation in a general direction in the (x_1, x_2) plane.

From (1.5.19), the motion for compressible material is governed by

$$\mathcal{A}_{oji\ell k}\, \nu_{k,j\ell} = \rho \ddot{\nu}_i . \tag{2.4.1}$$

Now suppose $\underline{\nu}$ is given by

$$\underline{\nu} = \underline{\Psi}(x_3)\, e^{i\omega\left[t - \frac{x_1\cos\theta + x_2\sin\theta}{c}\right]}, \tag{2.4.2}$$

i.e. the components of $\underline{\nu}$ are

$$\nu_i = \Psi_i(x_3)\, e^{i\omega\left[t - \frac{x_1\cos\theta + x_2\sin\theta}{c}\right]}. \tag{2.4.3}$$

From (2.4.1) we deduce

$$\begin{aligned}
\mathcal{A}_{oj1\ell 1}\, \nu_{1,j\ell} + \mathcal{A}_{oj1\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj1\ell 3}\, \nu_{3,j\ell} &= \rho \ddot{\nu}_1 ,\\
\mathcal{A}_{oj2\ell 2}\, \nu_{1,j\ell} + \mathcal{A}_{oj2\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj2\ell 3}\, \nu_{3,j\ell} &= \rho \ddot{\nu}_2 ,\\
\mathcal{A}_{oj3\ell 3}\, \nu_{1,j\ell} + \mathcal{A}_{oj3\ell 2}\, \nu_{2,j\ell} + \mathcal{A}_{oj3\ell 3}\, \nu_{3,j\ell} &= \rho \ddot{\nu}_3 .
\end{aligned} \tag{2.4.4}$$

Differentiating equation (2.4.3) and substituting into equations (2.4.4) we get

$$\mathcal{A}_{01111}\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]\psi_1+\mathcal{A}_{02121}\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]\psi_1+\mathcal{A}_{03131}\psi''_1$$

$$+(\mathcal{A}_{01122}+\mathcal{A}_{02112})\psi_2\left[-\frac{\omega^2\sin\theta\cos\theta}{c^2}\right]+(\mathcal{A}_{01133}+\mathcal{A}_{03113})\psi'_3$$

$$\left[-\frac{i\omega\cos\theta}{c}\right]=-\rho\psi_1,$$

$$(\mathcal{A}_{02211}+\mathcal{A}_{01221})\psi_1\left[-\frac{\omega^2\sin\theta\cos\theta}{c^2}\right]+\mathcal{A}_{01212}\psi_2\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]+\mathcal{A}_{02222}$$

$$\psi_2\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]+\mathcal{A}_{03232}\psi''_2+(\mathcal{A}_{02233}+\mathcal{A}_{03223})\psi'_3\left[-\frac{i\omega\sin\theta}{c}\right]=-\rho\psi_2,$$

$$(2.4.5)$$

$$\mathcal{A}_{03311}\psi'_1\left[-\frac{i\omega\cos\theta}{c}\right]+\mathcal{A}_{01331}\psi'_1\left[-\frac{i\omega\cos\theta}{c}\right]+(\mathcal{A}_{03322}+\mathcal{A}_{02332})\psi'_2$$

$$\left[-\frac{i\omega\sin\theta}{c}\right]+\mathcal{A}_{01313}\psi_3\left[-\frac{\omega^2\cos^2\theta}{c^2}\right]+\mathcal{A}_{02323}\psi_3\left[-\frac{\omega^2\sin^2\theta}{c^2}\right]+\mathcal{A}_{03333}\psi''_3$$

$$=-\rho\psi_3.$$

Suppose ψ_1, ψ_2 and ψ_3 have the same form as in the incompressible case, namely

$$\psi_1=Ae^{-sx_3},\quad \psi_2=Be^{-sx_3},\quad \psi_3=Ce^{-sx_3}$$

Equations (2.4.5) then become

$$(\rho c^2+s^2c^2\mathcal{A}_{03131}-\omega^2\cos^2\theta\mathcal{A}_{01111}-\omega^2\sin^2\theta\mathcal{A}_{02121})A-\omega^2\sin\theta\cos\theta$$

$$(\mathcal{A}_{01122}+\mathcal{A}_{02112})B+ics\cos\theta(\mathcal{A}_{01133}+\mathcal{A}_{03113})C=0,$$

$$-\omega^2\sin\theta\cos\theta(\mathcal{A}_{02211}+\mathcal{A}_{01221})A+(\rho c^2+s^2c^2\mathcal{A}_{03232}-\omega^2\cos^2\theta\mathcal{A}_{01212}$$

$$-\omega^2\sin^2\theta\mathcal{A}_{02222})B+ics\sin\theta(\mathcal{A}_{02233}+\mathcal{A}_{03223})C=0, \qquad (2.4.6)$$

$$ics\cos\theta(\mathcal{A}_{03311}+\mathcal{A}_{01331})A+ics\sin\theta(\mathcal{A}_{03322}+\mathcal{A}_{02332})B$$

$$+(\rho c^2+s^2c^2\mathcal{A}_{03333}-\omega^2\cos^2\theta\mathcal{A}_{01313}-\omega^2\sin^2\theta\mathcal{A}_{02323})C=0.$$

For A, B and C are a non-trivial solution, we must have

$$\Delta \equiv \begin{vmatrix} \rho c^2+s^2c^2\mathcal{A}_{01313}-\omega^2\cos^2\theta\mathcal{A}_{01111}-\omega^2\sin^2\theta\mathcal{A}_{02121} & -\omega^2\sin\theta\cos\theta(\mathcal{A}_{01122}+\mathcal{A}_{02112}) & ics\cos\theta(\mathcal{A}_{01133}+\mathcal{A}_{03113}) \\ -\omega^2\sin\theta\cos\theta(\mathcal{A}_{02211}+\mathcal{A}_{01221}) & \rho c^2+s^2c^2\mathcal{A}_{03232}-\omega^2\cos^2\theta\mathcal{A}_{01212}-\omega^2\sin^2\theta\mathcal{A}_{02222} & ics\sin\theta(\mathcal{A}_{02233}+\mathcal{A}_{03223}) \\ ics\cos\theta(\mathcal{A}_{03311}+\mathcal{A}_{01331}) & ics\sin\theta(\mathcal{A}_{03322}+\mathcal{A}_{02332}) & \rho c^2+s^2c^2\mathcal{A}_{03333}-\omega^2\cos^2\mathcal{A}_{01313}-\omega^2\sin^2\theta\mathcal{A}_{02323} \end{vmatrix} = 0,$$

So, Δ gives a cubic equation for s^2, which is similar to equation (2.2.9).

We also must have three values of s, s_1, s_2, s_3, say, with positive real parts. We then write the solution in the form,

$$\Psi_1 = A_1 e^{-s_1x_3} + A_2 e^{-s_2x_3} + A_3 e^{-s_3x_3},$$

$$\Psi_2 = B_1 e^{-s_1x_3} + B_2 e^{-s_2x_3} + B_3 e^{-s_3x_3},$$

$$\Psi_3 = C_1 e^{-s_1x_3} + C_2 e^{-s_2x_3} + C_3 e^{-s_3x_3}.$$

The ratio A_i: B_i: C_i (i=1,2,3) is deduced from (2.4.6), for A_1: A_2: A_3 to be non-trivial solution gives determinat as zero to provide equations involves s_1, s_2, s_3 for c^2. Because of complicated algebra involved we do not give any details of this general case. We refer to Chadwick and Jarvis (1979) for an alternative approach to this problem.

CHAPTER 3

Love waves

Surface waves with horizontally polarized displacement do not exist on a half-space, but if there is a layer they do. The explanation of this phenomenon was first given by Love (1911), who showed that such waves are essentially horizontally polarized shear waves trapped in a superficial layer and propagated with multiple total reflections between the boundaries of the layer.

In this chapter we shall discuss Love waves in a pre-strained layer and half-space of different material. In particular, we consider pre-strained half-space with a pre-strained layer of different material. We assume that the pre-strains are coaxial and, in particular, we consider propagation along a principal axis for both incompressible and compressible elastic materials.

3.1. Results for incompressible materials

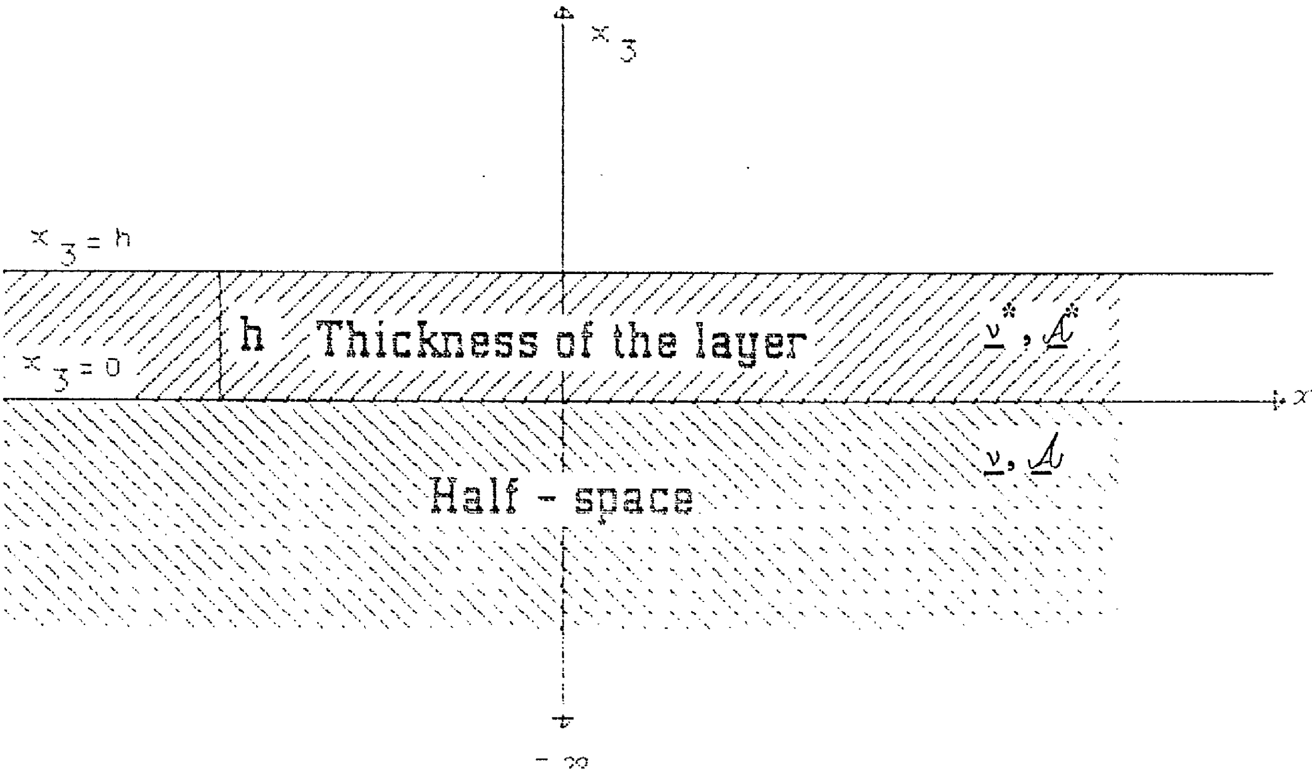
3.1.1 Propagation along a principal axis

We consider a pre-strained half-space defined by $x_3 \leq 0$ on which there is a layer of different pre-strained material of uniform thickness h with boundaries $x_3=0$ and $x_3=h$. The axes of Cartesian coordinates correspond to the principal axes of homogeneous pure strain in both half-space and layer.

We are seeking to find waves such that the traction and the displacement are continuous across $x_3=0$ and the traction is zero on $x_3=h$.

Let $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*)$ be the stretches of the deformation in the half-space and layer respectively and let W and W^* be the corresponding strain-energy functions.

We wish to solve the equations of motion for incompressible material, (1.5.20) with the following boundary conditions.



x_3
$x_3 = h$
$x_3 = 0$
h
Thickness of the layer
$\underline{\nu}^*, \mathcal{A}^*$
x
$\underline{\nu}, \mathcal{A}$
Half - space
$-\infty$

Figure 2

$$\overset{*}{\dot{S}}_{o3i} = 0 \quad \text{on } x_3=h,$$

$$\underline{\nu} = \underline{\nu}^*, \qquad \dot{S}_{o3i} = \overset{*}{\dot{S}}_{o3i} \quad \text{on } x_3 = 0. \tag{3.1.1}$$

The asterisk refers to quantities in the layer. Thus, $\mathcal{A}$ and $\mathcal{A}^*$ are the elastic modulus tensors in the half-space and layer respectively. We assume that

$$\underline{\nu} = (0,\ \nu_2, 0) = (0,\ Ae^{sx_3}e^{i\omega\left[t-\frac{x_1}{c}\right]},\ 0). \tag{3.1.2}$$

$$\underline{\nu}^* = (0,\ \overset{*}{\nu}_2, 0) = (0,\ f(x_3)\ e^{i\omega\left[t-\frac{x_1}{c}\right]},\ 0), \tag{3.1.3}$$

where $f(x_3)$ is given by

$$f(x_3) = (A'\cos s^*x_3 + A''\sin s^*x_3). \tag{3.1.4}$$

Substitution of (3.1.2) into equations of motion (1.5.20) gives

$$\dot{p}_{,1} = 0,$$

$$\mathcal{A}_{01212}\ \nu_{2,11} + \mathcal{A}_{03232}\ \nu_{2,33} = \rho\ddot{\nu}_2, \tag{3.1.5}$$

$$\dot{p}_{,3} = 0.$$

and substituting (3.1.3) into (1.5.20), we deduce

$$\overset{*}{\dot{p}}_{,1} = 0,$$

$$\overset{*}{\mathcal{A}}_{01212}\ \overset{*}{\nu}_{2,11} + \overset{*}{\mathcal{A}}_{03232}\ \overset{*}{\nu}_{2,33} = \rho\overset{*}{\ddot{\nu}}_2, \tag{3.1.6}$$

$$\overset{*}{\dot{p}}_{,3} = 0.$$

On use of (3.1.2) in $(3.1.5)_2$ we obtain

$$\mathcal{A}_{01212}\left[-\frac{\omega^2}{c^2}\right] + \mathcal{A}_{03232}\ s^2 = \rho\omega^2 \tag{3.1.7}$$

and hence

$$s^2 = \frac{\omega^2(\mathcal{A}_{01212} - \rho c^2)}{c^2\mathcal{A}_{03232}} \tag{3.1.8}$$

similarly

$$s^{*2} = \frac{\omega^2}{c^2}\left[\frac{\rho c^2 - \overset{*}{\mathcal{A}}_{01212}}{\overset{*}{\mathcal{A}}_{03232}}\right]. \tag{3.1.9}$$

Next, on introducing the notation (1.6.17), we have

$$\mathcal{A}_{01212} = \rho c_{12}^2, \qquad \mathcal{A}_{03232} = \rho c_{32}^2,$$

$$\overset{*}{\mathcal{A}}_{01212} = \rho^* c^{*2}_{12}, \qquad \overset{*}{\mathcal{A}}_{03232} = \overset{*}{\rho} c^{*2}_{32}.$$

So, we may write equations (3.1.8) and (3.1.10) as

$$s^2 = \frac{\omega^2(c_{12}^2 - c^2)}{c^2 c_{32}^2},$$

$$s^{*2} = \frac{\omega^2(c^2 - c^{*2}_{12})}{c^2 c^{*2}_{32}}. \tag{3.1.10}$$

Let us now substitute these solutions into the boundary conditions (3.1.1). We have

$$\dot{S}_{032} = (\mathcal{A}_{03223} + p)\, v_{3,2} + \mathcal{A}_{03232}\, v_{2,3} = \mathcal{A}_{03232}\, v_{2,3}$$

The boundary condition $(3.1.1)_1$ then gives $\overset{*}{v}_{2,3}=0$ on $x_3=h$ and hence from (3.1.3)

$$A'\sin s^*h - A''\cos s^*h = 0. \tag{3.1.11}$$

From $(3.1.1)_2$ we obtain

$$A = A' \tag{3.1.12}$$

and

$$sA\mathcal{A}_{03232} = s^{*}\,A''\overset{*}{\mathcal{A}}_{03232} \qquad (3.1.13)$$

Next, using (3.1.11), (3.1.12) and (3.1.13), we obtain the secular equation

$$\frac{s^{*}\overset{*}{\mathcal{A}}_{03232}}{s\mathcal{A}_{03232}} = \cot s^{*}h. \qquad (3.1.14)$$

On use of equation (3.1.10) in equation (3.1.14), the secular equation becomes

$$\tan s^{*}h = \frac{s\rho c_{32}^{2}}{s^{*}\rho^{*}\overset{*}{c}{}_{32}^{2}}\,, \qquad (3.1.15)$$

where

$$\overset{*}{c}{}_{12}^{2} < c^{2} < c_{12}^{2} \qquad (3.1.16)$$

Also, by using (3.1.10), the secular equation can be written as

$$\tan\left\{\frac{\omega h}{c}\sqrt{\frac{c^{2}-\overset{*}{c}{}_{12}^{2}}{\overset{*}{c}_{32}}}\right\} = \frac{\rho}{\rho^{*}}\,\frac{c_{32}\sqrt{c_{12}^{2}-c^{2}}}{\overset{*}{c}_{32}\sqrt{c^{2}-\overset{*}{c}{}_{12}^{2}}}\,. \qquad (3.1.17)$$

We want to solve equation (3.1.17) for the wave speed c when ρ/ρ^{*}, ωh, c_{12}, c_{32}, $\overset{*}{c}_{12}$ and $\overset{*}{c}_{32}$ are specified. That is we want to solve equation (3.1.17) for c as a function of ωh for fixed ρ/ρ^{*}, c_{12}, c_{32}, $\overset{*}{c}_{12}$ and $\overset{*}{c}_{32}$. Note that, unlike Rayleigh waves, Love waves with speed c given by (3.1.17) are dispersive.

For illustration we consider neo-Hookean material so that

$$\rho c_{ij}^{2} = \mu\lambda_{i}^{2}, \qquad \rho^{*}\overset{*}{c}{}_{ij}^{2} = \mu^{*}\overset{*}{\lambda}{}_{i}^{2}.$$

and hence (3.1.16) becomes

$$\frac{\mu^{*}}{\rho^{*}}\,\overset{*}{\lambda}{}_{1}^{2} < c^{2} < \frac{\mu}{\rho}\,\lambda_{1}^{2}.$$

Equation (3.1.17) then becomes

$$\tan\left[kh\,\frac{\sqrt{\frac{\rho^*}{\mu^*}c^2-\lambda_1^{*2}}}{\lambda_3^*}\right]=\frac{\lambda_3\mu}{\lambda_3^*\mu^*}\,\frac{\sqrt{\lambda_1^2-\frac{\rho}{\mu}c^2}}{\sqrt{\frac{\rho^*}{\mu^*}c^2-\lambda_1^{*2}}},\tag{3.1.18}$$

where $k=\frac{\omega}{c}$.

Next, specify ρ/μ, ρ^*/μ^*, λ_1, λ_3, λ_1^*, λ_3^* and μ/μ^*, subject to

$$\frac{\mu^*}{\rho^*}\lambda_1^{*2}<\frac{\mu}{\rho}\lambda_1^2.\tag{3.1.19}$$

Now, when we take the following value for the physical constants $\lambda_3^*=1$, $\lambda_3=0.75$, $\mu^*=1$, $\mu=2$, $\rho=\rho^*=3$ then we may write equations (3.1.18) as follows:

$$\tan\left\{kh\sqrt{3c^2-\lambda_1^{*2}}\right\}=\frac{3}{2}\,\frac{\sqrt{\lambda_1^2-\frac{3}{2}c^2}}{\sqrt{3c^2-\lambda_1^{*2}}}\tag{3.1.20}$$

and (3.1.19) reduces to

$$\lambda_1^{*2}<2\lambda_1^2.\tag{3.1.21}$$

By choosing $\lambda_1=\lambda_2$ and using the incompressibility conditions we deduce that

$$\lambda_1^{*2}<\frac{8}{3}\simeq 2.66.\tag{3.1.22}$$

We choose $\lambda_1^* = 1.2$ so that (3.1.20) becomes

$$\tan\left\{kh\sqrt{3c^2-1.44}\right\} = \frac{3}{2}\,\frac{\sqrt{4/3 - 3/2\,c^2}}{\sqrt{3c^2 - 1.44}} \tag{3.1.23}$$

Next, we want to obtain some numerical results, that is, we wish to solve equation (3.1.23) to find kh as a function of c or conversely. Subject to the constraints $0.48 < c^2 < 8/9$ we choose the following values for c to obtain the corresponding values of kh, as shown in the table below

c	kh
0.70	5.145
0.71	3.834
0.74	2.313
0.77	1.656
0.81	1.149
0.84	0.884
0.89	0.531
0.93	0.232

Figure 3 shows kh plotted as a function of c. There is a vertical asymptote at $c^2 = 0.48$.

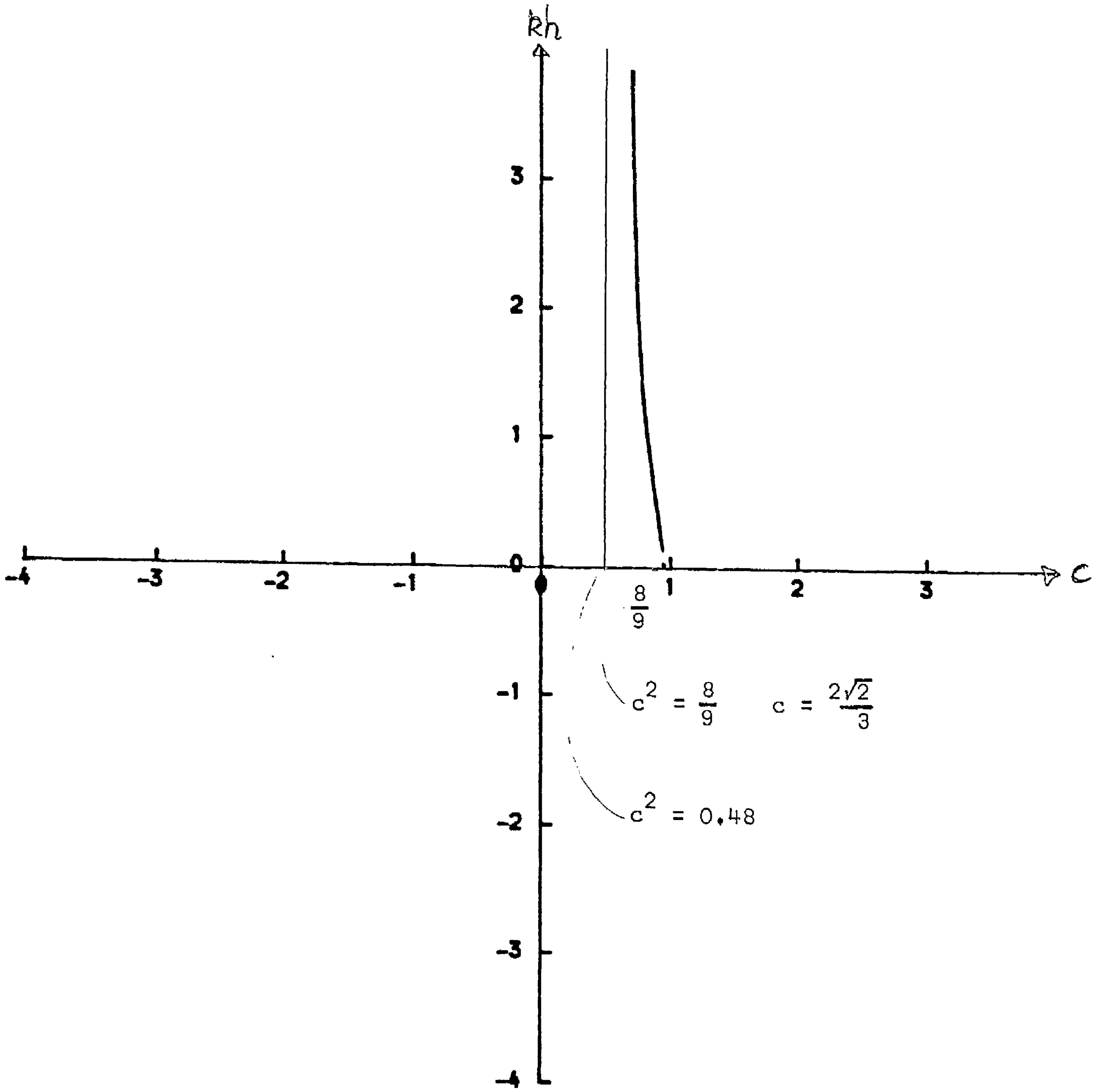


Figure 3 Solution of equation (3.1.23)

3.1.2 Results for the linear theory

Consider the special case of (3.1.18) in which $\lambda_3=\lambda_1=\lambda_2=1$ and $\overset{*}{\lambda}_3=\overset{*}{\lambda}_1=\overset{*}{\lambda}_2=1$. Equation (3.1.18) gives

$$\tan\left\{\left[\left[\frac{c}{\overset{*}{c}_T}\right]^2-1\right]^{½} kh\right\} - \frac{\mu}{\overset{*}{\mu}}\,\frac{[1-(c/c_T)^2]^{½}}{[(c/\overset{*}{c}_T)^2-1]^{½}} = 0, \qquad (3.1.24)$$

where

$$c_T^2 = \frac{\mu}{\rho}, \qquad \overset{*}{c}{}_T^2 = \frac{\overset{*}{\mu}}{\overset{*}{\rho}}.$$

Equation (3.1.24) is the well known *dispersion relation* for Love waves in the linear theory; see, for example, Achenbach (1984). The inequality (3.1.16) requires that

$$\overset{*}{c}_T < c < c_T.$$

3.2 Results for compressible materials

Let us assume, as in the incompressible case, that λ_1, λ_2, λ_3 and $\overset{*}{\lambda}_1$, $\overset{*}{\lambda}_2$, $\overset{*}{\lambda}_3$ be the stretches of the deformation in the half-space and layer respectively and let W and W^* be the corresponding strain-energy functions. Also, we assume that the elastic modulus tensor in the layer is $\underset{\sim}{\mathcal{A}}^*$ and in the half-space $\underset{\sim}{\mathcal{A}}$.

For a compressible elastic material, we want to solve the equations of motion (1.5.19) with the boundary condition

$$\begin{aligned} &\dot{S}_{03i} = 0 && \text{on } x_3=h \\ &\underset{\sim}{\nu} = \underset{\sim}{\nu}^*, \quad \dot{S}_{03i} = \dot{\overset{*}{S}}_{03i} && \text{on } x_3=0, \end{aligned} \tag{3.2.1}$$

where $\underset{\sim}{\nu}$ and $\underset{\sim}{\nu}^*$ are given by (3.1.2) and (3.1.3).

The equations of motion are

$$\begin{aligned} &\mathcal{A}_{01212}\, \nu_{2,11} + \mathcal{A}_{03232}\, \nu_{2,33} = \rho\ddot{\nu}_2, \\ &\overset{*}{\mathcal{A}}_{01212}\, \overset{*}{\nu}_{2,11} + \overset{*}{\mathcal{A}}_{03232}\, \overset{*}{\nu}_{2,33} = \rho\ddot{\overset{*}{\nu}}_2, \end{aligned} \tag{3.2.2}$$

in the half-space and layer respectively.

Also, using (3.1.2) in $(3.2.2)_1$, we deduce

$$s^2\mathcal{A}_{03232} - \frac{\omega^2}{c^2}\mathcal{A}_{01212} = -\rho\omega^2 \tag{3.2.3}$$

i.e.

$$s^2=k^2\left[\frac{\mathcal{A}_{01212} - \rho c^2}{\mathcal{A}_{03232}}\right], \tag{3.2.4}$$

where

$$k^2 = \frac{\omega^2}{c^2} .$$

Similarly

$$s^{*2} = k^2\left[\frac{\rho^* c^2 - \mathcal{A}^*_{01212}}{\mathcal{A}^*_{03232}}\right]. \qquad (3.2.5)$$

Next, on introducing the notation (1.6.17), we have

$$\mathcal{A}_{01212} = \rho c_{12}^2, \qquad \mathcal{A}_{03232} = \rho c_{32}^2,$$

$$\mathcal{A}^*_{01212} = \rho^* c^{*2}_{12}, \qquad \mathcal{A}^*_{03232} = \rho c^{*2}_{32}.$$

So, we write (3.2.4) and (3.2.5), in terms of c_{12}, c_{32}, $\overset{*}{c}_{12}$, $\overset{*}{c}_{32}$ as

$$s^2 = \frac{k^2(c_{12}^2 - c^2)}{c_{32}^2},$$

$$(3.2.6)$$

$$s^{*2} = \frac{k^2(c^2 - \overset{*2}{c_{12}})}{\overset{*2}{c_{32}}}$$

Let us now substitute these solutions into the boundary conditions (3.2.1). We have

$$\dot{s}_{032} = (\mathcal{A}_{03223} + p)\, v_{3,2} + \mathcal{A}_{03232}\, v_{2,3} = \mathcal{A}_{03232}\, v_{2,3}.$$

The boundary condition (3.2.1) then gives $\overset{*}{v}_{2,3}=0$ on $x_3=h$ and hence from (3.1.3), we get the same results as in the incompressible case.

That is, the secular equation (3.1.17) is the same as in the incompressible case namely,

$$\tan\left\{kh\sqrt{\frac{c^2 - \overset{*2}{c_{12}}}{\overset{*}{c}_{32}}}\right\} = \frac{\rho}{\rho^*}\,\frac{c_{32}\sqrt{c_{12}^2 - c^2}}{\overset{*}{c}_{32}\sqrt{c^2 - \overset{*2}{c_{12}}}}.$$

This result, but not the corresponding result for incompressible materials, was given by Hayes and Rivlin (1961b), but expressed in different notation.

3.3 Further Problems

The case of propagation in a general direction is worth considering, but this is left for future work.

Also, Rayleigh-type waves propagating in a half-space with a superficial layer have not been considered here. Those will also be examined in future work.

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