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GENERALISED TOPOLOGICAL DEGREE AND BIFURCATION THEORY

BY

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To My Mother and Father;

and Nana and Jack

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PREFACE

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. It represents research conducted by me at the University of Glasgow over the period October 1981 to September 1984. With the exception of some results specifically mentioned in the text and attributed there to the authors concerned, and of some work done in collaboration with Dr. J. R. L. Webb, the results in this thesis are the original work of the author alone.

I would like to express my deep gratitude to my supervisor, Dr. J. R. L. Webb, F.R.S.E., for suggesting the problems and for his unswerving assistance and encouragement, without which this thesis would not have been possible; and to Professor I. N. Sneddon, F.R.S., who provided me with the opportunity to study at the University of Glasgow. I should also like to thank the Science and Engineering Research Council for financing this research through an S.E.R.C. studentship.

Stewart C. Welsh

SUMMARY

The objective of this thesis is to give sufficient conditions for global bifurcation of solutions to the nonlinear eigenvalue problem: $F(x,\lambda) = 0$, where $F : X \times \mathbb{R} \to Y$, with $X \times \mathbb{R}$, Y Banach spaces and $(x,\lambda) \in X \times \mathbb{R}$. $F(.,\lambda)$ is assumed to belong to the class of A-proper maps and to be of the non-standard form, an A-proper, linear operator $A - \lambda B : X \to Y$ plus a nonlinear mapping $R(.,\lambda) : X \to Y$. $R(x,\lambda)$ is taken to satisfy a smallness condition in x at the origin in X. Our analysis is based on an extension of known methods, for obtaining global bifurcation results, which have been used successfully when the mappings involved are compact or k-set contractive.

Chapter One is an introduction to the concepts used throughout the thesis, including Fredholm maps of index zero, A-proper maps and generalised topological degree. In Chapter Two we state and prove our main global bifurcation theorem in terms of the generalised degree; this result forms the basis for the proofs of all the main theorems in the thesis. Chapters Three and Four contain various global bifurcation theorems, for different sets of hypotheses imposed on the mapping F and the underlying spaces $X \times \mathbb{R}$ and Y. Finally, in Chapter Five we apply our results to certain classes of ordinary differential equations and obtain existence results, for periodic solutions in one case and not necessarily periodic solutions in another.

The main results are: Theorem 2.10; Theorems 3.3 and 3.13; Theorems 4.7, 4.12, 4.15 and 4.18.

INTRODUCTION

This thesis is concerned with proving existence of nontrivial solutions to a nonlinear operator equation of the form

$$F(x,\lambda) = 0 \tag{0.1}$$

where $F : X \times \mathbb{R} \to Y$ is continuous with X and Y Banach spaces. In particular we wish to study the dependence of the solution set on the parameter λ. Equation (0.1) is often referred to as a nonlinear eigenvalue problem.

Suppose that $F(0,\lambda) = 0$ for all λ in $\mathbb{R}$. Then we call $\{(0,\lambda) : \lambda \in \mathbb{R}\}$ the set of trivial solutions and denote by S the set of non-trivial solutions, so that, $(x,\lambda) \in S$ if and only if $F(x,\lambda) = 0$ with $x \neq 0$.

We say that $\lambda_0 \in \mathbb{R}$ is a bifurcation point of equation (0.1) if there is a sequence of solutions in S converging to the point $(0,\lambda_0)$.

So there is a 'forking' of solutions at the point $(0,\lambda_0)$, where a branch of non-trivial solutions emanates from the set of trivial solutions. By 'branch' we mean a maximal connected subset. The term global bifurcation will be used, which refers to the fact that global properties of this branch of solutions are obtained. Typically we shall see that the branch has at least one of the following properties: it is unbounded in X; it meets the trivial solutions at a point $(0,\lambda_1)$ with $\lambda_1 \neq \lambda_0$; or, it contains elements in S for either all parameters greater than, or all parameters less than, λ_0, for which equation (0.1) is defined. In such a case, we say that λ_0 is a global bifurcation point.

It is our objective to impose conditions on F which are sufficient to ensure global bifurcation occurs for equation (0.1). In order to achieve this goal, we shall assume F has the general form

$$F(x,\lambda) = Ax - T(\lambda)x - R(x,\lambda) = 0 \qquad (0.2)$$

where $A - T(\lambda) : X \to Y$ is the Fréchet derivative of $F(.,\lambda)$ at the fixed point $x = 0$, a bounded, linear operator, and R is the 'higher order' term.

One method of proving bifurcation results for equation (0.2) is to apply the Implicit Function Theorem: if there exist continuous projections $P : X \to X$ and $Q : Y \to Y$, with ranges given, respectively, by $R(P) = N(A - T(\lambda_0))$ and $R(Q) = R(A - T(\lambda_0))$, where λ_0 is the candidate for a bifurcation point, then there exist closed subspaces $X_1 \subset X$ and $Y_2 \subset Y$ with $X = N(A - T(\lambda_0)) \oplus X_1$ and $Y = Y_2 \oplus R(A - T(\lambda_0))$. An application of the Implicit Function Theorem shows that λ_0 must be such that $N(A - T(\lambda_0)) \neq 0$. Then, using the decompositions of X and Y and invoking the Implicit Function Theorem, again, the problem is reduced to an equivalent one on $N(A - T(\lambda_0))$. This reduction argument is known as the Liapunov-Schmidt procedure. In most cases $N(A - T(\lambda_0))$ has a smaller dimension than that of X. In fact, we shall only be dealing with problems where $N(A - T(\lambda_0))$ is finite dimensional, and in this case the classical Brouwer degree theory may be used to obtain the bifurcation result after reduction by the Liapunov-Schmidt method. This approach, however, does not give global results. Moreover, if equation (0.2) involves a class of operators for which there exists a topological degree theory we may use the degree properties directly without performing any reduction. The classes of operators for which there are topological degree theories are quite extensive. In addition to the Brouwer degree for continuous maps acting from oriented finite dimensional spaces onto spaces of equal finite dimension there are: the Leray-Schauder degree, developed in 1934 by Leray and Schauder [15], for maps of the form identity minus compact; the degree of Nussbaum [22], for identity minus k-set contractions; the coincidence degree of Mawhin [18]; the generalised degree of Browder and Petryshyn [5], for so called A-proper maps, which we shall define below; and others.

The method of solving problems by topological degree arguments was one of several used by Krasnosel'skii [13]. He applied the Leray-Schauder degree when the operators involved were compact, and his results were essentially of a local form. Then, in 1971 Rabinowitz [35] proved the first global result when he gave sufficient conditions for global bifurcation of equation (0.2) when $A = I$, $T(\lambda) = \lambda B$ with B and R compact maps. After this important paper, a number of generalisations were made including: C. A. Stuart [36], who allowed $F(.,\lambda)$ to be a k-set contraction with $R(.,\lambda) = \lambda R$; Stuart and Toland [38], they retained the compactness of B and R, but let A have the more general form $I - C$, with I the identity, C compact and $I - C$ not necessarily invertible; Toland [42], let $A = I$, $T(\lambda) = \lambda B$ and obtained global bifuration results when $F(.,\lambda)$ and $I - \lambda B$ are A-proper maps with $X = Y$ a Hilbert space.

Recent global bifurcation results, which use homotopy theory rather than degree theory, seem to be extremely general indeed, see, for example, Alexander and Fitzpatrick [2] and Ize [11]. We have not studied homotopy theory but remark that in order to obtain stronger, more general results, this seems to be the way forward.

In this thesis we shall be concerned with so called A-proper maps. The class of A-proper maps was first studied by Petryshyn, under various guises, and then in 1968 Browder and Petryshyn christened them Approximation-proper, or more concisely, A-proper maps. Their definition requires the idea of an admissible scheme, $\Gamma = \{X_n, Y_n, Q_n\}$, for maps from X into Y : $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences of oriented finite dimensional subspaces, with $\dim X_n = \dim Y_n$ for each $n \in \mathbb{N}$; $\{Q_n\}$ is a sequence of projections of Y onto Y_n for each positive integer n, with $Q_n\, y \to y$ as $n \to \infty$ for each $y \in Y$; and the distance from X_n to x tends to zero as $n \to \infty$, for each $x \in X$. Then, a not necessarily linear mapping, $f : X \to Y$, is said to be A-proper with respect to Γ, if $Q_n\, f$ is continuous, for each n, and whenever $Q_n\, f(x_n) \to y$ as $n \to \infty$, for some $y \in Y$ and some bounded sequence $\{x_n\}$, in X_n, then $\{x_n\}$ has a convergent subsequence converging to x, such that $f(x) = y$. Browder and Petryshyn [5] also developed a degree theory for A-proper maps: they denoted by $\mathrm{Deg}(f,G,0)$, the generalised topological degree of f at 0 relative to the open bounded set G. This degree is well defined, provided that $0 \notin f(\partial G)$, where ∂G is the boundary of G, and although multivalued in general, possesses most of the useful properties of the Brouwer degree. For a comprehensive account of A-proper maps and generalised degree, see the survey article by Petryshyn [31].

In Chapter One of this thesis, we introduce the basic ideas required in the development of our theory, including a reiteration of the definition of A-proper maps and generalised degree, given above. We prove most of the important results except the very well known and excessively long ones and where appropriate we give reference sources.

Also included in this chapter is the definition of Fredholm maps of index zero, which have been successfully employed in bifurcation theory by many authors, including Mawhin [18], Alexander and Fitzpatrick [2] and Ize [11], and shall play an important part in the work contained here. We prove the useful theorem, due to Petryshyn [33], that a linear Fredholm mapping of index zero is necessarily A-proper with respect to a particular admissible scheme.

In Chapter Two, we suppose that $F(.,\lambda)$ and its Fréchet derivative $A - T(\lambda)$ are A-proper with respect to some admissible scheme Γ, for all $\lambda \in (a,b)$, an open interval in $\mathbb{R}$, which may be infinite. Then, we generalise the global bifurcation results - of Rabinowitz [35] for compact maps and the subsequent extension by Stuart [36] to k-set contractions - to the class of A-proper maps. This possibility of global bifurcation for A-proper maps was observed by Toland, whose main theorem, in [42], may be deduced as a special case of the results given in §4.3 of this thesis. Toland stated that the proof follows as a generalisation of Rabinowitz' [35] method for compact maps, but he never gave the details, so we include our own proof for completeness. The global result itself, tells us that if the generalised degree of the linear part of equation (0.2), namely $A - T(\lambda)$, changes as λ moves across an isolated value λ_0 for which $N(A - T(\lambda_0)) \neq \{0\}$ - that is, $\mathrm{Deg}\,(A - T(\underline{\lambda}),G,0) \neq \mathrm{Deg}(A - T(\overline{\lambda}),G,0)$ for $\underline{\lambda} < \lambda_0 < \overline{\lambda}$ sufficiently close to λ_0 with G an open, bounded set in X - then, λ_0 must be a global bifurcation point.

Our main objective, over the subsequent chapters, is to impose conditions on λ_0, A, $T(\lambda)$ and R under which this change in degree takes place.

Chapter Three generalises two different methods of Toland for proving that the degree changes. In §3.1 we take $T(\lambda) = \lambda B : X \to Y$, with B linear and compact and assume, for some $\lambda_0 \in \mathbb{R}$ with $N(A - \lambda_0 B) \neq \{0\}$, that $BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$. The required degree result is shown to hold provided that $\dim N(A - \lambda_0 B)$ is an odd number, and, for all $\lambda \neq \lambda_0$ in an interval $(\lambda_0 - \delta, \lambda_0 + \delta)$, $N(A - \lambda B) = \{0\}$. This generalises Toland's [43] result. The hypothesis $BN(A - \lambda_0 B) \cap R(A - \lambda_0 B)$, known as a transversality condition, was not considered by Toland, but we show that it generalises one of his sets of hypotheses and allows a more general setting. Many authors in bifurcation theory use a transversality condition, see for example Mawhin [18] and Alexander and Fitzpatrick [2]. The oddness requirement on the $\dim N(A - \lambda_0 B)$ is a recurrent condition throughout the thesis and is closely related to the concept of multiplicity of elements λ_0 with $N(A - \lambda_0 B) \neq \{0\}$, which we define in the text. For this reason it is sometimes said that global bifurcation occurs at values λ_0 of odd multiplicity.

The results in §3.1 have been published jointly with J. R. L. Webb, [48].

In §3.2 we generalise a Leray-Schauder degree multiplication formula, due to Krasnosel'skii [13], to a generalised degree version. Our result proves that, when $X = E_1 \oplus E_2$, where E_1 and E_2 are closed subspaces with E_1 finite dimensional; $I - T : X \to X$ is a homeomorphism; $T_i : E_i \to E_i (i = 1,2)$ is the restriction of T to E_i, and $B_i(0,1)$ is the open unit ball in $E_i (i = 1,2)$; then, $\mathrm{Deg}(I - T, B(0,1), 0) = \deg_{LS}(I - T_1, B_1(0,1), 0)\, \mathrm{Deg}(I - T_2, B_2(0,1), 0)$, where $B(0,1)$ is the open unit ball in X. We use this formula to generalise another result of Toland [41]. Our main theorem on global bifurcation applies to the situation: $X = Y$, $A = I$, $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$ for some odd number k, under various hypotheses, such as an odd multiplicity requirement, and a transversality assumption. This generalises Toland [41], who required that $B_j (j = 1,\ldots,k)$ were all compact. A corresponding generalisation is proved when X is a Hilbert space. A more concise version of these results has also been published jointly with J. R. L. Webb, [49].

In Chapter Four, §4.1, we assume that, at some λ_0 with $N(A - T(\lambda_0)) \neq \{0\}$, $A - T(\lambda_0)$ can be decomposed as $H - C$ where H is a homeomorphism and C is a bounded linear mapping. Then, the results of Chapter Two are used to prove that global bifurcation occurs at λ_0 if the degree of $I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}$ changes as λ crosses λ_0. In §4.2 it is assumed that $A - T(\lambda_0)$ is Fredholm of index zero and that the transversality condition $(A - T(\lambda))N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$, for all $\lambda \neq \lambda_0$, holds. Exploiting the properties of Fredholm maps we prove that the decomposition in §4.1, of $A - T(\lambda_0)$, into $H - C$, may be chosen so that C is compact.

It is shown under various hypotheses that global bifurcation occurs at λ_0 if $\dim N(A - T(\lambda_0))$ is an odd number. The hypotheses depend on the form of $T(\lambda)$: when $T(\lambda) = \lambda B$, then λ_0 may be positive, negative or zero; however, for $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, with k finite, we have to let X and Y be Hilbert spaces and either $\lambda_0 = 0$ or λ_0 is a positive value; furthermore, if k is allowed to be infinite, then we further restrict ourselves to $\lambda_0 = 0$, again with X and Y Hilbert spaces.

The results contained in §4.2 generalise some of those which appear in Chapter Three.

In §4.3 it is not assumed that the transversality condition holds. Instead we impose a segment condition which depends on the decomposition H - C of §4.1, where in general C is not compact. It is shown that if the radius of the essential spectrum of CH^{-1} is less than one and the algebraic multiplicity of λ_0, namely $\dim \{ \bigcup_{n=1}^{\infty} N((I - CH^{-1})^n)\}$, is an odd number, then the required degree result holds. Actually, the condition on the essential spectrum of CH^{-1} implies that $A - T(\lambda_0)$ is again Fredholm of index zero and so the decomposition H - C can be chosen such that C is compact. However, the segment and multiplicity conditions depend explicitly on H and C. So, even though we know that such a compact C exists, if we cannot find it explicitly, then we may be unable to verify the other conditions. If, on the other hand, there is a readily available explicit decomposition H - C where C is not compact, but for which the other conditions are satisfied, then this method may be used.

Finally, in Chapter Five we give applications to the problem of the existence of even, T-periodic solutions of the ordinary differential equation

$$x''(t) + b^2 x(t) = g(x,x',x''),$$

where $0 < b \in \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded, and satisfies a smallness condition.

We indicate how this problem may be transformed into an abstract nonlinear eigenvalue problem of the form of equation (0.2). The results of Chapter Four are then used to give a solution to this problem under some additional hypotheses.

We also consider the problem of existence of solutions (x_λ,λ), with x_λ not identically zero, of the ordinary differential equation

$$x''(t) + \lambda x(t) = \lambda g(t,x,x',x''),$$

where $x(0) = x(1) = 0$, $\lambda \in \mathbb{R}$, $x:[0,1] \to \mathbb{R}$ and $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous and satisfies a smallness condition.

Again, we impose additional hypotheses on the equation and invoke the theorems of Chapter Four.

Finally, examples, of the above ordinary differential equations are given, which satisfy the various hypotheses.

CHAPTER ONE

PRELIMINARIES

1.1 Notation and general concepts

We shall write $\mathbb{Z}$ for the set of all integers, $\mathbb{N}$ for the set of all positive integers, 1, 2, ..., and $\mathbb{R}$ for the set of real numbers.

Unless otherwise stated X and Y will denote Banach spaces with norms given, respectively, by $\|x\|$ and $\|y\|$ for all $x \in X$ and $y \in Y$. $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are, then, also Banach spaces with norms given, respectively, by $[\|x\|^2 + |\lambda|^2]^{\frac{1}{2}}$ and $[\|y\|^2 + |\lambda|^2]^{\frac{1}{2}}$ for all $(x,\lambda) \in X \times \mathbb{R}$ and $(y,\lambda) \in Y \times \mathbb{R}$.

For the remainder of this section we take Z and E to be Banach spaces.

If $D \subset Z$ is a linear subspace, then dim D will be written for the dimension of D, which may be infinite. If G is a subset in Z and $z \in Z$ an arbitrary point, then dist(z,G) will denote the distance of z from the set G, that is, $\mathrm{dist}(z,G) = \inf\{\|z-g\| : g \in G\}$. The closure and boundary of a set G will be denoted, respectively, by $\overline{G}$ and ∂G.

$B(z,r)$ will denote the open ball in Z, centre z and radius r with closure $\overline{B}(z,r)$ and boundary $\partial B(z,r)$.

If there exist subspaces Z_1 and Z_2 of Z, such that each z in Z may be written uniquely in the form $z = z_1 + z_2$, with $z_1 \in Z_1$ and $z_2 \in Z_2$, then we write $Z = Z_1 \oplus Z_2$, and call $Z_1 \oplus Z_2$ the direct sum of Z_1 and Z_2.

The next result may be found in Taylor and Lay [39].

Theorem 1.1 If Z has a finite-dimensional, and hence closed, subspace Z_1, then there exists another closed subspace Z_2 such that $Z = Z_1 \oplus Z_2$.

<u>Definition 1.2</u> A continuous mapping $f : Z \to E$, which is one-to-one (injective), onto (surjective) and whose inverse mapping $f^{-1} : E \to Z$ is also continuous, is called a homeomorphism.

<u>Definition 1.3</u> A mapping $f : Z \to E$ is said to be compact if it is continuous and $\overline{f(D)}$ is compact in E whenever D is a bounded subset in Z.

<u>Remark</u> It is well known, see for example Taylor and Lay [39], that if f is linear and $\overline{f(D)}$ is compact in E, whenever D is a bounded subset in Z, then f is continuous.

<u>Definition 1.4</u> Let D be a bounded subset of Z. The set (ball) measure of non-compactness of D, denoted by $\alpha(D)$ $(\beta(D))$ is such that $\alpha(D) = \inf\{d > 0 :$ D can be covered by finitely many sets each of diameter less than or equal to $d\}$ $(\beta(D) = \inf\{r > 0 :$ D can be covered by finitely many balls each of diameter r, with centres in $Z\})$.

<u>Definition 1.5</u> We call a continuous map $f : Z \to E$ a k-set contraction if there is a constant $k \geq 0$ such that for all bounded sets $D \subset Z$. $\alpha(f(D)) \leq k\, \alpha(D)$, and define $\alpha(f) = \inf\{k : f$ is a k-set contraction$\}$. We say that f is set condensing if $\alpha(f(D)) < \alpha(D)$ for all bounded sets $D \subset Z$ such that $\alpha(D) \neq 0$.

Replacing the word "set" by the word "ball" and α by β, we obtain an equivalent definition for the ball measure of noncompactness.

Notice that f is compact if and only if it is a 0-set (0-ball) contraction.

Two important properties of α and β are that, if D, D_1, D_2 are bounded subsets of Z, and $L : Z \to E$ is linear, then

$\alpha(D_1 + D_2) \leq \alpha(D_1) + \alpha(D_2)$ and $\alpha(L(D)) \leq \|L\| \ \alpha(D)$. The same inequalities hold when α is replaced by β.

For a further discussion of the set and ball measures see [14], [16] and [22].

Definition 1.6 A mapping $f : Z \to E$ is said to be Fréchet differentiable at the point $z_0 \in Z$, if there exists a bounded, linear map $f'(z_0) : Z \to E$ such that $f(z_0 + h) - f(z_0) - f'(z_0)h = R(z_0,h)$, where $R : Z \times Z \to E$ is such that $\| R(z_0,h)\| \ / \ \|h\| \to 0$ as $\|h\| \to 0$.

We call $f'(z_0)$ the Fréchet derivative of f at the point z_0.

Remark If f is a k-set contraction, then so is its Fréchet derivative [21]. This is not true, in general, for the A-proper maps we shall define in §1.2.

The next collection of results may be found in the book of Taylor and Lay [39].

Let $L : Z \to Z$ be a bounded, linear operator and denote the null space and range of L by, respectively, $N(L)$ and $R(L)$. Note that $N(L^k) \subset N(L^{k+1})$ and $R(L^{k+1}) \subset R(L^k)$ for each $k \in \mathbb{N}$, so $N(L^k)$ is an increasing family, and $R(L^k)$ is a decreasing family, of subspaces of E. If there exists a smallest positive integer $p(q)$, such that $N(L^p) = N(L^{p+1})$ $(R(L^{q+1}) = R(L^q))$ then $p(q)$ is called the ascent (descent) of L. In general the ascent and descent of L need not be equal or even finite. However, when they are both finite, then they are equal, and $Z = N(L^p) \oplus R(L^p)$.

Two sets which will be frequently encountered are

$\rho(L) = \{\lambda \in \mathbb{C} : (\lambda I - L)^{-1} : Z \to Z$ is a bounded linear operator$\}$,

known as the resolvent set of L, and $\sigma(L) = \{\lambda \in \mathbb{C} : \lambda \notin \rho(L)\}$, called the spectrum of L. An important subset of $\sigma(L)$ is the essential spectrum

$\sigma_e(L)$ of L, which corresponds to all $\lambda \in \sigma(L)$ for which at least one of the following conditions is satisfied:

(1) the range of $\lambda I - L$ is not closed;

(2) λ is a limit point of $\sigma(L)$;

(3) $\dim \bigcup_{n=1}^{\infty} N((\lambda I - L)^n)$ is infinite.

Nussbaum [20] has shown that $\sigma_e(L)$ is a closed, bounded set. Its radius is defined by $r_e(L) = \sup\{|\lambda| : \lambda \in \sigma_e(L)\}$.

Nussbaum [20] related the essential spectrum to the notion of k-set and k-ball contraction by proving that $r_e(L) = \lim_{n\to\infty}\{\alpha(L^n)\}^{1/n}$ $= \lim_{n\to\infty}\{\beta(L^n)\}^{1/n}$. Thus the essential spectrum of a compact mapping is zero.

There are several possible definitions of essential spectrum. The one given here is due to Browder [3] and leads to the largest set. However, Nussbaum [20] has also shown that whichever definition is taken, the radius is the same. Also A. Lebow and M. Schechter [14] prove similar results.

Another important set, which corresponds to the reciprocals of a subset of $\sigma(L)$, is the set of characteristic values, ch(L), of L given by $ch(L) = \{\lambda \in \mathbb{R} : N(I - \lambda L) \neq \{0\}\}$.

For $\lambda \in ch(L)$, we define the algebraic multiplicity $M_a(\lambda)$ and geometric multiplicity $M_g(\lambda)$ of λ by, respectively,
$M_a(\lambda) = \dim\{\bigcup_{n=1}^{\infty} N((I - \lambda L)^n)\}$, $M_g(\lambda) = \dim\{N(I - \lambda L)\}$. In general $M_a(\lambda)$ and $M_g(\lambda)$ need not be equal or even finite. However, when L is compact then $M_a(\lambda)$, and hence $M_g(\lambda)$, is finite; ch(L) is a discrete set with no finite limit points, and is bounded away from zero; the ascent and descent of $I - \lambda L$ are finite, equal to p say; and
$Z = N((I - \lambda L)^p) \oplus R((I - \lambda L)^p)$, with $M_a(\lambda) = \dim\{N(I - \lambda L)^p\}$.

This decomposition is often called the Riesz decomposition of Z.

1.2 A-proper maps

The main results in this thesis will involve, so called, Approximation proper maps, or, more concisely, A-proper maps. This class of maps was first named as such by Browder and Petryshyn [5] in 1968, although Petryshyn had used them earlier in [25], where he referred to them as mappings satisfying condition (H). To define A-proper mappings, we need the following definition.

Definition 1.7 $\Gamma = \{X_n, Y_n, Q_n\}$ is said to be an admissible scheme for maps from X into Y provided that:

(i) $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences or oriented finite dimensional subspaces with $\dim X_n = \dim Y_n$, for each $n \in \mathbb{N}$;

(ii) $\{Q_n\}$ is a sequence of linear, continuous projections, with $Q_n : Y \to Y_n$ for each $n \in \mathbb{N}$, and $Q_n y \to y$ as $n \to \infty$, for each $y \in Y$;

(iii) $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$.

In Definition 1.7 by 'oriented' finite dimensional spaces X_n, Y_n, we mean that bases have been chosen for X_n and Y_n, such that if a bounded, linear operator $L : X_n \to Y_n$ maps the basis in X_n onto the basis in Y_n, then the determinant of the matrix of L is positive.

Remarks (1) By the Uniform Boundedness Theorem, cf. [39], condition (ii) in Definition 1.7, implies that there exists a number $K > 0$ such that $\|Q_n\| \leq K$, for all $n \in \mathbb{N}$.

(2) It is easy to show that if X and Y possess Schauder bases then there exists an admissible scheme [31]. In particular if X and Y are separable Hilbert spaces, then an admissible scheme exists.

Definition 1.8 A, not necessarily, linear map $f : X \to Y$ is said to be A-proper with respect to the admissible scheme $\Gamma = \{X_n, Y_n, Q_n\}$, if $f_n \equiv Q_n f|_{X_n} : X_n \to Y_n$ is continuous for each $n \in \mathbb{N}$, and if whenever $\{x_{n_j} : x_{n_j} \in X_{n_j}\}$ is any bounded sequence with $f_{n_j}(x_{n_j}) \to y$ as $j \to \infty$, for some $y \in Y$, then there exist a subsequence, which we again denote by $\{x_{n_j}\}$, and $x \in X$, such that $x_{n_j} \to x$ as $j \to \infty$ and $f(x) = y$. Sometimes we just speak of an operator being A-proper, without mentioning an admissible scheme; in such cases it is implicit that an admissible scheme exists.

Thus, in the class of A-proper maps the problem of finding solutions to an infinite dimensional problem $f(x) = y$ may be reduced to that of solving the associated finite dimensional problems $Q_m f(x_m) = Q_m y$. The required solution is then the strong limits of some subsequence of $\{x_m\}$, provided the sequence $\{x_m\}$ is bounded.

It follows directly from Definition 1.8 that if $f : X \to Y$ is A-proper with respect to Γ, then $c\, f : X \to Y$ is also A-proper, for any constant $c \in \mathbb{R}$; however, Petryshyn [27] has shown that the sum of two bounded, linear A-proper operators need not be A-proper. Thus, the set of all bounded, linear A-proper operators is not a linear subspace of the space of all bounded, linear operators.

The class of A-proper maps evolved from the concept of a Projectionally-compact mapping, or, more concisely, a P-compact mapping, which was introduced by Petryshyn, [23] in 1966. Petryshyn, [31] has shown that a mapping $f : X \to X$ is P-compact if and only if $T_\lambda = f - \lambda I$ is A-proper for each $\lambda > 0$, where I is the identity mapping. It was shown in [24], that if H is a Hilbert space and $L : H \to H$ is a bounded, linear, monotone decreasing (i.e. $(Lx,x) \leq 0$ for all $x \in H$) operator, then $-L$ is P-compact. Thus, for such an operator L, $L + \lambda I$ is A-proper for each $\lambda > 0$. Other examples of A-proper maps include $I - f : X \to X$, where f is k-ball condensing, provided $\| Q_n \| = 1$ for each $n \in \mathbb{N}$. This fact was proved by Webb, [45] and extended the result that $I - f$ is A-proper when f is compact. In two recent papers, Webb [46,47] has improved a result of Toland [42], which gives another example of an A-proper mapping. In order to cite this example we need some additional information. Recall that if X has a uniformly convex dual space X^*, then it is well known, [31], that the duality map $J : X \to X^*$ is uniquely determined by the requirements $\| Jx \| = \| x \|$ and $(x,Jx) = \| x \|^2$, where (x,f) denotes the value of $f \in X^*$ at $x \in X$. One may then define a mapping $f : X \to X$ to be accretive (or J-monotone) if, for all $x, y \in X$, $(f(x) - f(y), J(x-y)) \geq 0$. If $f - c I$ is accretive for some $c > 0$, then f is said to be strongly accretive with constant c. Webb, [47] has shown that, if X^* is uniformly convex; $\| Q_n \| = 1$ and $X_n \subset X_{n+1}$, for each $n \in \mathbb{N}$; $g : X \to X$ is a k-ball contraction; $f : X \to X$ is strongly accretive with constant c, and demicontinuous - i.e. $x_n \to x$ implies that $f(x_n) \overset{w}{\to} f(x)$ ($\overset{w}{\to}$ denotes weak convergence), - then $f + g$ is A-proper if $k < c$, and $I + g + f$ is A-proper if $k - c < 1$. Notice that f need not be bounded. The class of k-ball contraction plus strongly accretive and demicontinuous mappings is not known to belong to any other class of mappings and, consequently, the A-proper mapping theory is the only one that can handle such equations.

Milojevic, [19], has considered similar problems, and his results imply that if X is a reflexive Banach space, $f : X \to X$ is a linear, continuous, accretive operator and g is a linear, compact operator, then $\alpha I + f + g$ is A-proper for each $\alpha > 0$.

We now look at some properties of A-proper maps. These are all due to Petryshyn and the proofs are included for completeness.

Theorem 1.9 (Petryshyn, [26]). Suppose that $L : X \to Y$ is a bounded, linear, injective, A-proper operator with respect to $\Gamma = \{X_n, Y_n, Q_n\}$. Then L is a homeomorphism.

Proof: We have just to show that L is onto, for then the Open Mapping Theorem, cf. [39], gives the required result.

First, let us prove that there exists a constant $C > 0$ and $N_0 \in \mathbb{N}$, such that $\| Q_n L x_n \| \geq C \| x_n \|$ for all $x_n \in X_n$ with $n \geq N_0$. Suppose the contrary, then there is a sequence $\{x_n\}$, which by linearity of $Q_n L$ we may choose with $\| x_n \| = 1$ for each $n \in \mathbb{N}$, such that $\| Q_n L x_n \| < \frac{1}{n} \| x_n \| = \frac{1}{n} \to 0$ as $n \to \infty$. By A-properness of L this implies the existence of a subsequence, which we again denote by $\{x_n\}$, and an element $x \in X$ with $x_n \to x$ as $n \to \infty$. Clearly, $\| x \| = 1$ and $Lx = 0$. This contradicts the injectiveness of L, so C and N_0 exist. Thus, for $n \geq N_0$, $Q_n L|_{X_n} : X_n \to Y_n$ is injective and therefore onto, since X_n and Y_n are of equal finite dimension n and $Q_n L$ is linear and continuous for each $n \geq N_0$. Thus, for each $y \in Y$ there is a unique $x_n \in X_n$, such that $Q_n L x_n = Q_n y$, for each $n \geq N_0$. Now $C \| x_n \| \leq \| Q_n L x_n \| = \| Q_n y \| \leq K \| y \|$, since the Q_n's are uniformly bounded, cf. Remark (1) preceding Definition 1.8. So, $\| x_n \|$ is a bounded sequence and $Q_n L x_n = Q_n y \to y$, as $n \to \infty$, which implies, again by A-properness of L, that there is an $x \in X$, such that for a subsequence, $x_n \to x$ and $Lx = y$. By the injectiveness property, such an x is unique. Hence L is a bijection and therefore a homeomorphism.

Theorem 1.10 (Petryshyn, [28]). If $f : X \to Y$ (not necessarily linear) is continuous and A-proper with respect to $\Gamma = \{X_n, Y_n, Q_n\}$, then the restriction of f to any closed, bounded subset F of X is proper: where, by proper we mean that for any compact set K in Y, the non-empty set $F \cap f^{-1}(K)$ is also compact in X.

Proof: Let F be a closed, bounded subset of X and $\{x_k\}$ a sequence in $F \cap f^{-1}(K)$, where K is a compact subset in Y. Then $\{f(x_k)\}$ is a sequence in K, which, without loss of generality, we may assume converges. That is, $f(x_k) \to y(\text{say}) \in Y$ as $k \to \infty$.

Now for each $k \in \mathbb{N}$ choose $\varepsilon_k > 0$ with $\varepsilon_k \to 0$ as $k \to \infty$. By continuity of f, there exists $\delta_k > 0$, with $\delta_k \to 0$ as $k \to \infty$, such that, if $\| v - x_k \| < \delta_k$ for $v \in X$, then $\| f(v) - f(x_k) \| < \varepsilon_k$. But, by the properties of an admissible scheme there exists $v_{n(k)} \in X_{n(k)}$ (where we can suppose that $n(k) > k$) with $\| f(v_{n(k)}) - f(x_k) \| < \varepsilon_k$ and $\| v_{n(k)} - x_k \| < \delta_k$. Thus, $\| Q_{n(k)} f(v_{n(k)}) - y \|$

$$\begin{aligned} &\leq \| Q_{n(k)} f(v_{n(k)}) - Q_{n(k)} f(x_k) \| + \| Q_{n(k)} f(x_k) - Q_{n(k)} y \| \\ &\quad + \| Q_{n(k)} y - y \| \\ &\leq K \| f(v_{n(k)}) - f(x_k) \| + K \| f(x_k) - y \| + \| Q_{n(k)} y - y \| , \end{aligned}$$

since the Q_n's are uniformly bounded by the constant K. So $Q_{n(k)} f(v_{n(k)}) \to y$ as $k \to \infty$. Hence, by the A-properness of f, we may assume (passing to a subsequence if necessary) that there exists $x \in X$, such that $v_{n(k)} \to x$ as $k \to \infty$ and $f(x) = y$. This implies that $x_k \to x$ as $k \to \infty$, and, since F is closed, $x \in F \cap f^{-1}(K)$, which is therefore compact, as required.

Theorem 1.11 (Petryshyn, [26]). If $L : X \to Y$ is a bounded, linear, A-proper operator with respect to $\Gamma = \{X_n, Y_n, Q_n\}$, then $N(L)$ is finite dimensional.

Proof: Assume that $N(L)$ is infinite dimensional. Then, since $\partial B(0,1)$ is not compact in the infinite dimensional space $N(L)$, there exists a sequence $\{x_n\}$ in $\partial B(0,1)$ and a constant $C > 0$, such that $\|x_i - x_j\| > C$ for $i \neq j$, and $L(x_i) = 0$ for each $i \in \mathbb{N}$. Now, since $\{x_i\} \subset \partial B(0,1)$, then $\{x_i\}$ is bounded. Also L is a continuous, A-proper operator. Thus, Theorem 1.10 tells us that $\{x_i\}$ is compact and, therefore, has a convergent subsequence $\{x_k\}$, with $x_k \to x$ (say) as $k \to \infty$. Hence, there exists $N_0 \in \mathbb{N}$, such that $\|x_k - x_\ell\| \leq \frac{C}{2}$, for all $k, \ell \geq N_0$ with $k \neq \ell$. This contradiction implies that $N(L)$ is finite dimensional.

Theorem 1.12 (Petryshyn, [26]). If $L : X \to Y$ is a bounded, linear, A-proper operator with respect to $\Gamma = \{X_n, Y_n, Q_n\}$, then $R(L)$ is closed in Y.

Proof: This proof is similar to that for the compact case as in, for example, Yosida's book [51]. Suppose that $R(L)$ is not closed. Then there is a sequence $\{x_n\} \subset X$ such that $Lx_n \to y$ and $y \notin R(L)$. By the linearity of L, $y \neq 0$ and we may assume, without loss of generality, that $x_n \notin N(L)$ for each $n \in \mathbb{N}$. Since $N(L)$ is closed, $d_n = \inf\{\|x_n - x\| : x \in N(L)\} > 0$ for each $n \in \mathbb{N}$. By a property of the infimum, we can choose $s_n \in N(L)$ such that $A_n \equiv \|x_n - s_n\| < 2d_n$ for each $n \in \mathbb{N}$. We shall prove that $A_n \to \infty$ as $n \to \infty$. Suppose not, then $\{x_n - s_n\}$ contains a bounded subsequence $\{x_i - s_i\} \equiv \{k_i\}$ for which $Lk_i = Lx_i - Ls_i = Lx_i \to y$ as $i \to \infty$. Since $\{k_i\}$ is a bounded sequence, then $\{Lk_i\}$ is also bounded and every subsequence converges to y. So, by Theorem 1.10, $\{k_i\}$ is compact and, therefore, has a convergent subsequence $\{k_j\}$ with $k_j \to k$(say) $\in X$ as $j \to \infty$. Thus, $Lk = y \in R(L)$, contrary to our assumption. Hence, $A_n \to \infty$ as $n \to \infty$. Setting

$p_n = A_n^{-1}(x_n - s_n)$ it is easily seen that $\|p_n\| = 1$ for each $n \in \mathbb{N}$, and $Lp_n = A_n^{-1} Lx_n \to 0$ as $n \to \infty$. Again, by Theorem 1.10, $\{p_n\}$ has a convergent subsequence $\{p_j\}$ such that $p_j \to p$(say) $\in X$ as $j \to \infty$. Clearly $\|p\| = 1$ with $Lp = 0$, and so $p \in N(L)$.

Finally, setting $z_n = s_n + A_n p$, we have that $Lz_n = 0$ for each $n \in \mathbb{N}$, implying that $z_n \in N(L)$. Thus, $\|x_n - z_n\| \geq d_n$ for each $n \in \mathbb{N}$. On the other hand $x_n - z_n = A_n p_n + s_n - s_n - A_n p = A_n(p_n - p)$. Now, $A_n < 2d_n$, so $d_n \leq \|x_n - z_n\| \leq A_n\|p_n - p\| \leq 2d_n\|p_n - p\|$. This implies that $1 < 2\|p_n - p\|$ for each $n \in \mathbb{N}$, contradicting the fact that $\{p_n\}$ contains a subsequence $\{p_j\}$ converging to p. Hence $R(L)$ is closed.

1.3 Fredholm maps of index zero

The following class of operators will play an important role in this thesis.

Definition 1.13 A bounded, linear operator $L : X \to Y$ is said to be a Fredholm operator if $\dim N(L) \equiv n(L)$ (say) and $\dim\{Y/R(L)\} \equiv d(L)$ (say) are both finite; where $\dim\{Y/R(L)\} = \operatorname{codim} R(L)$, that is, the dimension of any subspace of Y complementary to $R(L)$. We denote the class of such operators by $\Phi(X,Y)$, or $\Phi(X)$ if $X = Y$. The index of L, denoted by $i(L)$, is given by $i(L) = n(L) - d(L)$. When $i(L) = 0$, L is said to be a Fredholm operator of index zero, the class of which we denote by $\Phi_0(X,Y)$, or $\Phi_0(X)$ if $X = Y$.

Examples of maps belonging to $\Phi_0(X,Y)$ include $B : X \to Y$, where B is a linear, continuous bijection, and $I + C : X \to X$, where C is a compact, linear operator.

Remarks (1) If $L \in \Phi(X,Y)$, then $R(L)$ is closed, cf. Taylor and Lay [39], Theorem IV. 13.2.

(2) It is shown in Theorem 5.26 of Kato [12], that, if $L \in \Phi_0(X,Y)$ and $C : X \to Y$ is a linear, compact operator, then $L + C \in \Phi_0(X,Y)$.

(3) Nussbaum has shown in [20] that if $L : X \to X$ is a bounded, linear operator and $|\lambda| > r_e(L)$, then $\lambda I - L \in \Phi_0(X)$.

(4) If $L \in \Phi_0(X,Y)$ and $T \in \Phi_0(Y,Z)$ for some Banach space Z, then $TL \in \Phi_0(X,Z)$, see Taylor and Lay [39].

(5) Petryshyn [31], Theorem 2.3A, has shown that if L is a bounded, linear, A-proper operator, then either, $N(L) = \{0\}$, in which case L is a homeomorphism, or $N(L) \neq \{0\}$, and in this case $i(L) \geq 0$.

The class $\Phi_0(X,Y)$ has the following useful properties.

<u>Theorem 1.14</u> (Petryshyn, [33]). When $L \in \Phi_0(X,Y)$, there exist closed subspaces X_1 of X and Y_2 of Y such that $X = N(L) \oplus X_1$, $Y = Y_2 \oplus R(L)$; $L_1 = L|_{X_1}$ is injective with $L_1(X_1) = R(L)$; and $\dim Y_2 = \dim N(L)$. Furthermore, L may be decomposed into $H - C : X \to Y$, where $H : X \to Y$ is a linear homeomorphism and $C : X \to Y$ is linear and compact.

<u>Proof</u>: Since $L \in \Phi_0(X,Y)$ there exist Y_2, a complement of $R(L)$ in Y, and $\dim Y_2 = \dim N(L)$ is finite. So, Y_2 is a closed subspace and by Theorem 1.1 there exists a closed subspace X_1 in X such that the decompositions of X and Y hold as required. Let P be the continuous, linear projection of X onto $N(L)$, and M a linear homeomorphism of $N(L)$ onto Y_2. Then, we define $C : X \to Y_2$ by $C = MP$, and since Y_2 is finite dimensional, C is linear and compact. Remark (2) succeeding Definition 1.13 tells us that $L + C \in \Phi_0(X,Y)$. Furthermore $L + C$ is a homeomorphism. To see this we first verify that it is injective. Suppose that $(L + C)(x) = 0$. Then $x = u + v$, with $u \in N(L)$, $v \in X_1$ and $Lv + Mu = 0$.

But $Lv \in R(L)$ and $Mu \in Y_2$, implying that $u = v = 0$. Thus $L + C$ is injective. It is also surjective since it is Fredholm of index zero. Hence by the Open Mapping Theorem $L + C$ is a homeomorphism. Thus, setting $H = L + C$ we have that $L = L + C - C = H - C$, which completes the proof of the theorem.

The next result tells us that, in a space which has an admissible scheme, linear Fredholm maps of index zero are A-proper with respect to a related scheme.

Theorem 1.15 (Petryshyn, [33]). Suppose that $L \in \Phi_0(X,Y)$ and $\Gamma = \{Y_n, Q_n\}$ is an admissible scheme for maps from Y into Y. Then L is A-proper with respect to the admissible scheme $\Gamma_H = \{X_n, Y_n, Q_n\}$, where $X_n = H^{-1}(Y_n)$ for each $n \in \mathbb{N}$ and where $H = L + C$ is the decomposition given in Theorem 1.14.

Proof: First, we show that Γ_H is admissible. Since H is a linear homeomorphism, $\dim H^{-1}(Y_n) = \dim Y_n$, and for each $x \in X$, there exists $y \in Y$ with $x = H^{-1}(y)$, and $\operatorname{dist}(x, H^{-1}(Y_n)) = \operatorname{dist}(H^{-1}(y), H^{-1}(Y_n))$ $\leq \|H^{-1}\| \operatorname{dist}(y, Y_n) \to 0$ as $n \to \infty$.

Finally $Q_n y \to y$ as $n \to \infty$ for each $y \in Y$, since Γ is admissible. Therefore, Γ_H is an admissible scheme.

To prove that L is A-proper with respect to Γ_H, assume $\{x_{n_j} : x_{n_j} \in X_{n_j}\}$ is an arbitrary bounded sequence such that $Q_{n_j} L x_{n_j} \to y$ as $j \to \infty$ for $y \in Y$. Then, $Q_{n_j}(L + C - C)(x_{n_j}) \to y$ and, since C is compact, we may assume that $Q_{n_j} C x_{n_j} \to w$ (say) as $j \to \infty$. Also, there exists $y_{n_j} \in Y_{n_j}$ such that $x_{n_j} = H^{-1}(y_{n_j}) = (L + C)^{-1}(y_{n_j})$, and $Q_{n_j} y_{n_j} = y_{n_j}$ for each $j \in \mathbb{N}$. Thus, $Q_{n_j} y_{n_j} = y_{n_j} \to y + w$ as $j \to \infty$. Therefore, $(L + C) x_{n_j} \to y + w$, which implies that

$x_{n_j} \to (L + C)^{-1}(y + w) = x(\text{say})$ as $j \to \infty$. So $Cx = w$ and $(L + C)(x) = y + w = y + Cx$. Hence, $Lx = y$ and, therefore, L is A-proper with respect to Γ_H.

Remark Examples by Petryshyn, [27] show that:

(i) An A-proper mapping need not be Fredholm of index zero;

(ii) A Fredholm mapping of index zero need not be A-proper with respect to a given scheme; however, if L is a bounded, linear, A-proper operator with $N(L) = \{0\}$, then L is Fredholm of index zero.

If we perturb a mapping in $\Phi_0(X,Y)$, which is also A-proper, by a bounded linear operator of sufficiently small norm then the perturbed map is still A-proper with respect to the same admissible scheme.

Theorem 1.16 (Petryshyn, [30]). If $L \in \Phi_0(X,Y)$ is A-proper with respect to an admissible scheme Γ, then there exists a constant $\gamma > 0$ such that, for each bounded linear operator $T : X \to Y$, with $\|T\| < \gamma$, the map $L + T$ is also A-proper with respect to Γ.

Remark In the book by Taylor and Lay [39], Theorem 13.6 shows that there certainly exists $\gamma > 0$ such that $L + T \in \Phi_0(X,Y)$ for all bounded linear operators $T : X \to Y$ with $\|T\| < \gamma$. So, Theorem 1.15 above implies that $L + T$ is A-proper with respect to $\Gamma_H = \{H^{-1}(X_n), Y_n, Q_n\}$ where $L + T = H - C$. However, Theorem 1.16 says that whatever admissible scheme L is A-proper with respect to, $L + T$ is A-proper with respect to the same scheme, for $\|T\| < \gamma$.

Proof: See Petryshyn, [30].

1.4 Generalised topological degree

One of the main tools available in nonlinear problems is the theory of generalised topological degree. For A-proper maps the theory was developed by Browder and Petryshyn [5]. This degree, although not single valued, possesses most of the useful properties of the classical Brouwer topological degree for maps between oriented normed spaces of equal finite dimension. Throughout the text we shall assume that the reader is familiar with the definition and properties of the classical Brouwer degree, which we denote by deg, and the classical Leray-Schauder topological degree, denoted by $\deg_{LS}$, for infinite dimensional maps of the form identity minus compact. These concepts may be found in the book of N. G. Lloyd [16]. One result on Leray-Schauder degree, which does not appear in Lloyd's book is the following, due to Krasnosel'skii [13], which may be found in Cronin [7], in the form given here.

(The Leray-Schauder Formula). Suppose that $L : X \to X$ is a linear compact operator and $\lambda > 0$ is not a characteristic value of L. Then $\deg_{LS}(I - \lambda L, G, 0) = (-1)^{\nu}$, where $G \subset X$ is an arbitrary open bounded set containing zero, and ν is the sum of the algebraic multiplicities of the characteristic values of L in the interval (o,λ).

We now recall the definition of degree for A-proper mappings.

Definition 1.17 (Browder and Petryshyn [5]) Let $G \subset X$ be an open bounded set and, for each $n \in \mathbb{N}$, define $G_n = G \cap X_n$. Then, $\overline{G}_n = \overline{G} \cap X_n$ and $\partial G_n = \partial G \cap X_n$. If $f : \overline{G} \to Y$ is A-proper with respect to the admissible scheme $\Gamma = \{X_n, Y_n, Q_n\}$ and $0 \notin f(\partial G)$, then we define the generalised topological degree of f at $0 \in Y$ relative to G, denoted by Deg(f,G,0), to be the set $\{m \in \mathbb{Z} \cup \{-\infty,\infty\}$: for a subsequence $\{n_j\}$ of $\mathbb{N}$,

$\deg(Q_{n_j}f, G_{n_j}, 0) \to m$ as $j \to \infty\}$.

<u>Remark</u> The A-properness of f ensures that for sufficiently large j, $0 \notin Q_{n_j} f(\partial G_{n_j})$, and so deg is well defined and Deg(f,G,0) is a non-empty subset of $\mathbb{Z} \cup \{-\infty,\infty\}$.

It is convenient to note that an alternative definition is possible in terms of limits of $\deg_{LS}$, when $f = I - g : \overline{G} \to X$ is A-proper with $0 \notin (I - g)(\partial G)$.

<u>Theorem 1.18</u> Let $G \subset X$ be an open bounded set. If $I - g : X \to X$ is A-proper with respect to the admissible scheme $\Gamma = \{X_n, Y_n, Q_n\}$, and $0 \notin (I - g)(\partial G)$, then $\mathrm{Deg}(I - g, G, 0) = \{m \in \mathbb{Z} \cup \{-\infty,\infty\}$: for a subsequence $\{n_j\}$ of $\mathbb{N}$, $\deg_{LS}(I - Q_{n_j}g, G, 0) \to m$ as $j \to \infty\}$.

<u>Proof</u>: From definition 1.17 we need only show that for j sufficiently large $\deg_{LS}(I - Q_{n_j}g, G, 0) = \deg(Q_{n_j}(I - g), G \cap X_{n_j}, 0)$. Trivially, $Q_{n_j}(I - g)|_{G \cap X_{n_j}} = I - Q_{n_j}g|_{G \cap X_{n_j}}$, and, for all sufficiently large j, $0 \notin (I - Q_{n_j}g)\partial(G \cap X_{n_j})$ by A-properness.

Also, for all $j \in \mathbb{N}$, $0 \in X_{n_j}$ and $Q_{n_j} g(G) \subset X_{n_j}$. Hence, by the definition of Leray-Schauder degree, cf. Lloyd [16], $\deg_{LS}(I - Q_{n_j}g, G, 0) = \deg(I - Q_{n_j}g, G \cap X_{n_j}, 0)$ for all j sufficiently large. The result follows by letting $j \to \infty$.

<u>Remark</u> From Theorem 1.18 it is easily seen that, if $0 \in G$, then $\mathrm{Deg}(I,G,0) = 1$, where I is the identity operator.

Unlike the Brouwer and Leray-Schauder degrees the generalised degree is multi-valued, in general. For example, $\mathrm{Deg}(-I, B(0,1), 0) = \{-1,1\}$. As we shall see, however, many of the useful properties of classical topological degree hold for generalised degree. Results that are well known we shall just cite, the others will be proved. Unless otherwise stated we shall assume the notation of Definition 1.17.

(P1.) (Lloyd, [16]). If $\mathrm{Deg}(f,G,0) \neq \{0\}$ then there exists $x \in G$ such that $f(x) = 0$.

(P2.) (Lloyd, [16]). Let $G = G_1 \cup G_2$ where G_1 and G_2 are open and $0 \notin f(\partial G_1 \cup \partial G_2 \cup (G_1 \cap G_2))$. Then $\mathrm{Deg}(f,G,0) \subseteq \mathrm{Deg}(f,G_1,0) + \mathrm{Deg}(g,G_2,0)$, with equality holding if either $\mathrm{Deg}(f,G_1,0)$ or $\mathrm{Deg}(f,G_2,0)$ is single valued.

(P3.) (Homotopy property). (Toland, [42]). Suppose that $H : X \times [0,1] \to Y$ is such that $H(.,t) : X \to Y$ is A-proper with respect to $\Gamma = \{X_n,Y_n,Q_n\}$ for each $t \in [0,1]$, and $H(x,.) : [0,1] \to Y$ is continuous, uniformly for x in closed, bounded subsets of X. Let $G \subset X \times [0,1]$ be a bounded open set and define $G_t = \{x \in X : (x,t) \in G\}$. Then, $\mathrm{Deg}(H(.,t),G_t,0)$ is independent of $t \in [0,1]$, provided that $0 \notin H(\partial G_t,t)$ for $0 \leq t \leq 1$.

Proof: As Toland does not prove this result we give a proof for completeness.

Since $0 \notin H(\partial G_t,t)$ for each fixed $t \in [0,1]$, then by the remark following Definition 1.17, $\mathrm{Deg}(H(.,t),G_t,0)$ is well defined. The required result holds if we show that, for sufficiently large $j \in \mathbb{N}$, $\deg(Q_{n_j}H(.,t),G_t \cap X_{n_j},0)$ is independent of t in $[0,1]$. Theorem 2.2.4 in Lloyd [16] tells us that this is so, provided there exists $N_0 \geq 1$ such that, for all $n_j \geq N_0$, $0 \notin Q_{n_j} H(\partial(G_t \cap X_{n_j}),t)$ for $0 \leq t \leq 1$. Suppose this is not true, then there exist sequences $\{n_j\} \subset \mathbb{N}$, $\{t_j\} \subset [0,1]$ and $\{x_{n_j}\} \subset \partial(G_{t_j} \cap X_{n_j})$ such that $n_j \to \infty$ and, without loss of generality, $t_j \to t$ as $j \to \infty$ with $Q_{n_j}H(x_{n_j},t_j) = 0$ for each $j \in \mathbb{N}$.

By assumption, $H(x,.) : [0,1] \to Y$ is continuous, uniformly, for x in bounded subsets of X. Now for each $j \in \mathbb{N}$, $\partial(G_{t_j} \cap X_{n_j})$ is contained in the closure of the set $\{G_t : t \in [0,1]\}$, which is closed and bounded. Thus $\| H(x_{n_j},t_j) - H(x_{n_j},t) \| \to 0$ as $j \to \infty$ and so

$\| Q_{n_j} H(x_{n_j},t_j) - Q_{n_j} H(x_{n_j},t) \| = \| Q_{n_j} H(x_{n_j},t) \| \to 0$ as $j \to \infty$.

But $H(.,t) : X \to Y$ is A-proper for each $t \in [0,1]$, therefore, there exists $x \in X$ and a subsequence, which we again denote by $\{x_{n_j}\}$ such that $x_{n_j} \to x$ as $j \to \infty$ and $H(x,t) = 0$. So $(x_{n_j},t_{n_j}) \to (x,t)$ as $j \to \infty$ and since $(x_{n_j},t_{n_j}) \in \partial G$ for each $j \in \mathbb{N}$, it follows that $(x,t) \in \partial G$ and $x \in \partial G_t$. This contradicts the fact that $0 \notin H(\partial G_t,t)$.

Hence the result follows.

(P4.) (Toland, [42]). Let $L : X \to X$ be a bounded linear operator with $r_e(L) < |1/\lambda|$, such that $I - \lambda L$ is A-proper with respect to the admissible scheme $\Gamma = \{X_n,Y_n,Q_n\}$. Then, provided λ is not a characteristic value of L, $\mathrm{Deg}(I - \lambda L,G,0) = \{(-1)^\nu\}$, where ν is the sum of the algebraic multiplicities of the characteristic values of L in the interval $(0,\lambda)$, and G is an arbitrary open, bounded set containing zero.

This result is not given a proof in [42], so we include our own.

<u>Proof</u>: Since $r_e(\lambda L) < 1$, then $M_a(\lambda) = \dim\{ \bigcup_{n=1}^{\infty} N((I - \lambda L)^n)\}$ is finite and so the ascent p (say) of $I - \lambda L$ is finite with $\dim\{N((I - \lambda L)^p)\} = M_a(\lambda)$. Also, $I - \lambda L$ is Fredholm of index zero by Remark (3), following Definition 1.13, and then, by Remark (4), $(I - \lambda L)^n$ is Fredholm of index zero for each $n \in \mathbb{N}$. Hence, since $R((I - \lambda L)^{p+1}) \subseteq R((I - \lambda L)^p)$, then $\dim\{R((I - \lambda L)^p)\} = \dim R((I - \lambda L)^{p+1}$, and we have that the descent of $I - \lambda L$ is also finite. Therefore, by the results of §1.1, $X = N((I - \lambda L)^p) \oplus R((I - \lambda L)^p)$.

Now by a similar method to that used by Nussbaum [21], in his proof of Lemma 8, we may show that $X = E_1 \oplus E_2$, with E_1 finite dimensional, E_2 a closed subspace, $L: E_1 \to E_1$, $L: E_2 \to E_2$, and $I - t\lambda L|_{E_2}$ an A-proper homeomorphism for each t in [0,1]; A-properness of $I - t\lambda L|_{E_2}$ requires an argument using Theorem 1.16. Let P be the projection of X onto E_1, and define $T : X \to X$ by $T = LP$. Then, T has finite dimensional range and is therefore compact. Define the homotopy $H : \overline{B}(0,1) \times [0,1] \to X$ by $H(x,t) = x - t\lambda Tx - (1-t)\lambda Lx$, for $x \in \overline{B}(0,1)$ and $t \in [0,1]$. Since T is compact, $H(.,t) : X \to Y$ is A-proper with respect to Γ for all $t \in [0,1]$. We shall prove that $H(x,t) \neq 0$ for all $x \in \partial B(0,1)$ and $t \in [0,1]$. Suppose the contrary, then $H(x,t) = 0$ for some $x \in \partial B(0,1)$ and $t \in [0,1]$. Then $x = x_1 + x_2$ where $x_1 \in E_1$ and $x_2 \in E_2$. This gives,

$x_1 + x_2 - t\lambda Lx_1 - (1-t)\lambda Lx_1 - (1-t)\lambda Lx_2 = 0$,

since $Tx = LP(x_1 + x_2) = Lx_1$. Thus, $x_1 - \lambda Lx_1 = -(x_2 - (1-t)\lambda Lx_2)$ $= 0$ by the invariance of E_1 and E_2 under L. Hence, $x_2 = 0$ since $I - (1-t)\lambda L|_{x_2}$ is a homeomorphism, and therefore $x_1 = 0$, for λ is not a characteristic value of L by assumption. Therefore, $x = 0$, contradicting the fact that $x \in \partial B(0,1)$. Hence, it follows that

$\mathrm{Deg}(I - \lambda L, B(0,1),0) = \mathrm{Deg}(I - \lambda T, B(0,1),0) = \{\deg_{LS}(I - \lambda T, B(0,1),0)\}$ (by Theorem 1.18) $= \{(-1)^{\nu}\}$ (by the classical Leray-Schauder formula which is stated before Definition 1.17), where ν is the sum of the algebraic multiplicities of the characteristic values of T in the interval $(0,\lambda)$.

To complete the proof we show that ν also equals the sum of the algebraic multiplicities of the characteristic values of L in the interval $(0,\lambda)$.

Suppose that $\mu \varepsilon (0,\lambda)$, $x \varepsilon X$, $x \neq 0$, $n \varepsilon \mathbb{N}$ and $(I - \mu T)^n x = 0$, then, writing $x = x_1 + x_2$, where $x_1 \varepsilon E_1$ and $X_2 \varepsilon E_2$ and $T = LP$, we have that $(I - \mu L)^n x_1 = -\mu^n x_2 = 0$ by invariance of L on E_1 and E_2. So, $x_2 = 0$ and $x = x_1$, implying that $(I - \mu L)^n x = 0$. Conversely, suppose that $(I - \mu L)^n x = 0$ with μ, n and x as before. Then, $(I - \mu L)^n x_1 = -(I - \mu L)^n x_2 = 0$, and, since $I - \mu L|_{E_2}$ is a homeomorphism, then $x_2 = 0$ and $x = x_1 = Px_1$, which implies that $(I - \mu T)^n x = 0$. Hence, $(I - \mu T)^n x = 0$ if and only if $(I - \mu L)^n x = 0$, so $\bigcup_{n=1}^{\infty} \{N((I - \mu T)^n)\} = \bigcup_{n=1}^{\infty} \{N((I - \mu L)^n)\}$, which completes the proof.

(P5.) (Petryshyn, [31]). If $L : X \to Y$ is a linear, continuous, injective, A-proper map and G is an arbitrary open bounded set in X with $0 \varepsilon G$, then for arbitrary $r > 0$, $\text{Deg}(L,B(0,r),0) = \text{Deg}(L,G,0)$.

<u>Proof</u>: For arbitrary ε such that $0 < \varepsilon < r$, it follows easily that $B(0,r) = B(0,r)\setminus\overline{B}(o,\varepsilon/2)) \cup B(0,\varepsilon)$. Now, since L is injective, $Lx \neq 0$ for $x \varepsilon B(0,r)\setminus\overline{B}(0,\varepsilon/2)$ and so (P1) implies that $\text{Deg}(L,B(0,r)\setminus\overline{B}(0,\varepsilon/2),0) = \{0\}$. Thus, by (P2.), we have

$$\text{Deg}(L,B(0,r),0) = \text{Deg}(L,B(0,\varepsilon),0) + \text{Deg}(L,B(0,r)\setminus\overline{B}(0,\varepsilon/2),0)$$
$$= \text{Deg}(L,B(0,\varepsilon),0) + \{0\}.$$

But $0 \varepsilon G$ and G is open, and so there exists ε_0 such that $0 < \varepsilon_0 < r$ with $\overline{B}(0,\varepsilon_0) \subset G$. As above, the injectiveness of L implies that $\text{Deg}(L,G\setminus\overline{B}(0,\varepsilon_0/2),0) = \{0\}$. Thus,

$$\text{Deg}(L,G,0) = \text{Deg}(L,B(0,\varepsilon_0),0) + \text{Deg}(L,G\setminus\overline{B}(0,\varepsilon_0/2),0)$$
$$= \text{Deg}(L,B(0,\varepsilon_0),0) + \{0\}.$$

If we take $\varepsilon = \varepsilon_0$, it is easily seen that $\text{Deg}(L,G,0) = \text{Deg}(L,B(0,r),0)$.

This proves the result.

(P6.) (Fitzpatrick [8]). Let $f : X \to Y$ and $G \subset X$ satisfy the hypotheses of Definition 1.17. Assume that f is continuous and $0 \in G$. Suppose that $g : \overline{G} \to Y$ is also continuous and A-proper with respect to Γ. Then there exists $d > 0$ such that, if $\|g(x) - f(x)\| \leq d$ for all $x \in \partial G$, then $\mathrm{Deg}(f,G,0) = \mathrm{Deg}(g,G,0)$.

Proof: First, we prove that there exists $d > 0$ such that $\mathrm{Deg}(g,G,0)$ is well defined. To do this we need to show that there exists $\delta_1 > 0$ such that $\|f(x)\| > \delta_1$ for all $x \in \partial G$.

Suppose not, then for each $k > 0$ there is a sequence $\{x_k\}$ in ∂G with $\|f(x_k)\| \leq \frac{1}{k} \to 0$ as $k \to \infty$, so $f(x_k) \to 0$ as $k \to \infty$.

However, f is continuous and A-proper, which, by Theorem 1.10, implies the existence of $x \in \partial G$ such that $f(x) = 0$. Thus, δ_1 exists. Hence, for d less than $\frac{\delta_1}{2}$ it follows that $\|g(x)\| \geq \|f(x)\| - d$, $> \frac{\delta_1}{2}$ for all $x \in \partial G$. So $\mathrm{Deg}(g,G,0)$ is well defined for $d < \frac{\delta_1}{2}$.

To complete the proof we show that there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\deg(Q_nf, G \cap X_n, 0) = \deg(Q_ng, G \cap X_n, 0)$. Recall that $\partial(\overline{G} \cap X_n) = \partial G \cap X_n$.

Now there exists $\delta_2 > 0$ and $N_0 \in \mathbb{N}$ with the property that for all $n \geq N_0$, $\|Q_nf(x_n)\| > 2\delta_2$ for all $x_n \in \partial G \cap X_n$. For otherwise A-properness implies $f(x) = 0$ for some $x \in \partial G$, which is a contradiction. Also, for each $n \in \mathbb{N}$,

$$\|Q_n(f(x_n) - g(x_n))\| \leq K\|f(x_n) - g(x_n)\| \leq Kd \text{ for all } x \in \partial G \cap X_n$$

where K is the uniform bound on $\|Q_n\|$. Now choose $d < \min\{\delta_1/2, \delta_2/K\}$ and consider the continuous homotopy $H_n : \overline{G} \cap X_n \times [0,1] \to Y_n$ defined by $H_n(x,t) = t\,Q_n\,g(x) + (1 - t)\,Q_n\,f(x)$ for $x \in \overline{G} \cap X_n$ and $t \in [0,1]$. We shall prove that for each $n \geq N_0$, $H_n(x,t) \neq 0$ for all $x \in \partial G \cap X_n$ and $t \in [0,1]$.

Suppose the contrary, then there exist sequences $\{n_j\} \subset \mathbb{N}$, $\{x_{n_j}\} \subset \partial U \cap X_{n_j}$ and $\{t_{n_j}\} \subset [0,1]$ such that $n_j \geq N_0$ for all $j \in \mathbb{N}$, $n \to \infty$, $t_{n_j} \to t$ as $j \to \infty$ and $H_{n_j}(x_{n_j}, t_{n_j}) = 0$.

This implies that $\| t\, Q_{n_j} g(x_{n_j}) + (1 - t)\, Q_{n_j} f(x_{n_j}) \| \to 0$ as $j \to \infty$. Thus $\| Q_{n_j} f(x_{n_j}) - t\, Q_{n_j}(f(x_{n_j}) - g(x_{n_j})) \| \to 0$ as $j \to \infty$.

We have seen above that for $n_j \geq N_0$

$$\begin{aligned}
&\| Q_{n_j} f(x_{n_j}) - t\, Q_{n_j}(f(x_{n_j}) - g(x_{n_j})) \| \\
&\geq \| Q_{n_j} f(x_{n_j}) \| - \| Q_{n_j}(f(x_{n_j}) - g(x_{n_j})) \| \\
&> 2\delta_2 - \delta_2 = \delta_2 > 0
\end{aligned}$$

This contradiction shows that $H_n(x,t)$ is a valid homotopy and the required result follows by application of the homotopy property for Brouwer degree.

(P7.) (Fitzpatrick, [8]). Assume that $f(0) = 0$ and f is Fréchet differentiable at 0 with the Fréchet derivative $f'(0)$. Suppose $f'(0)$ is injective and A-proper with respect to Γ. Then 0 is an isolated solution of $f(x) = 0$ and there exists $r > 0$ such that $\mathrm{Deg}(f, B(0,r), 0) = \mathrm{Deg}(f'(0), G, 0)$ where G is an arbitrary open bounded set in X containing zero.

Proof: By Definition 1.6, there exist $R : \overline{G} \times X \to Y$ and $r_1 > 0$, with $f(x) = f'(0)(x) + R(0,x)$ for all $x \in X$, such that, whenever $0 < r \leq r_1$, then

$$\|f(x) - f'(0)(x)\| = \begin{cases} \dfrac{\|R(0,x)\|}{\|x\|}\, \|x\|, & \text{for } x \neq 0,\ x \in X \\ 0, & \text{for } x = 0 \end{cases}$$

$\leq d$, for all $x \in \partial B(0,r)$, where $d > 0$ is the constant from (P6.). Now, since $f'(0)$ is injective, then $0 \notin f'(0)(\partial B(0,r))$ for all $r > 0$, and by the proof of (P6.), there exists $r_2 > 0$ such that for $0 < r \leq r_2$, $0 \notin f(\partial B(0,r))$. So, by choosing $r = \min\{r_1, r_2\}$, 0 is the only solution of $f(x) = 0$ in $\overline{B}(0,r)$, and, by (P6.) and (P5.), $\mathrm{Deg}(f,B(0,r),0) = \mathrm{Deg}(f'(0),B(0,r),0) = \mathrm{Deg}(f'(0),G,0)$. The last equality follows by (P5.), since $f'(0)$ is linear, continuous, injective and A-proper with respect to Γ. This completes the proof.

(P8.) <u>The Multiplication Formula</u> (Petryshyn, [32]). Suppose that $L_1 : X \to Y$ and $L_2 : X \to Y$ are bounded, linear operators such that L_1 is injective and A-proper with respect to the admissible scheme $\Gamma = \{X_n,Y_n,Q_n\}$. Assume that L_2 is compact, $L_1 - L_2$ is injective, and let G be an arbitrary open ball in X containing zero. Then,

$$\begin{aligned}\mathrm{Deg}(L_1 - L_2,G,0) &= \mathrm{Deg}((I - L_2L_1^{-1})L_1,G,0) \\ &= \deg_{LS}(I - L_2L_1^{-1},L_1(G),0)\ \mathrm{Deg}(L_1,G,0)\end{aligned}$$

<u>Proof</u>: Since L_2 is compact, $L_1 - tL_2$ is A-proper with respect to Γ for each $t \in [0,1]$. Also, $L_1 - L_2$ and L_1 are linear, continuous, injective and A-proper. So, as in the proof of Theorem 1.9, there exists a constant $C_0 > 0$ and $N_0 \in \mathbb{N}$, such that $\|Q_n(L_1 - L_2)(x_n)\| \geq C_0\|x_n\|$, and $\|Q_n L_1 x_n\| \geq C_0\|x_n\|$ for all $x_n \in X_n$ and all $n \geq N_0$. Also, by Theorem 1.9, L_1 is a homeomorphism, so $I - L_2L_1^{-1} : Y \to Y$ is also linear, continuous and injective; furthermore, since L_2 is compact and $\{Y_n,Q_n\}$ is an admissible scheme for Y, then $I - L_2L_1^{-1}$ is A-proper with respect to $\Gamma_Y = \{Y_n,Q_n\}$ and there exists a constant $C_1 > 0$ and $N_1 \in \mathbb{N}$ for which $\|Q_n(I - L_2L_1^{-1})(y_n)\| \geq C_1\|y_n\|$, for all $y_n \in Y_n$ with $n \geq N_1$. We shall prove that for $n \geq \max\{N_0,N_1\} = N_2$ (say),

$$\begin{aligned}&\deg(Q_n(L_1 - L_2), B(0,1) \cap X_n,0) \\ &= \deg(Q_n(I - L_2L_1^{-1}), L_1(B(0,1)) \cap Y_n,0)\ \deg(Q_nL_1,B(0,1) \cap X_n,0).\end{aligned}$$

From the above argument each Brouwer degree in this equation is well defined, for $n \geq N_2$.

We shall use a homotopy argument to obtain the required result. Define $H_n : (\overline{B(0,1)} \cap X_n) \times [0,1] \to Y_n$ by,

$H_n(x,t) = t\, Q_n(L_1 - L_2)(x_n) + (1 - t)\, Q_n(I - L_2L_1^{-1})\, Q_n\, L_1(x_n)$.

Then, $H_n(x,t) \neq 0$ for $x \in \partial B(0,1) \cap X_n$ and $t \in [0,1]$, with $n \geq N_2$. For, suppose the contrary, then there exist subsequences $\{n_j\} \subset \mathbb{N}$, $\{x_{n_j}\} \subset \partial B(0,1) \cap X_{n_j}$, and $\{t_{n_j}\} \subset [0,1]$ such that $n_j \geq N_2$ for each $j \in \mathbb{N}$, $n_j \to \infty$, and $t_{n_j} \to t \in [0,1]$ as $j \to \infty$ with $H_{n_j}(x_{n_j}, t_{n_j}) = 0$ for each $j \in \mathbb{N}$.

So, $t\, (Q_{n_j}(L_1 - L_2)(x_{n_j}) + (1 - t)Q_{n_j}(I - L_2L_1^{-1})Q_{n_j}L_1(x_{n_j}) \to 0$ as $j \to \infty$,

and, therefore,

$Q_{n_j}L_1(x_{n_j}) - t\, Q_{n_j}\, L_2(x_{n_j}) - (1 - t)\, Q_{n_j}\, L_2L_1^{-1}\, Q_{n_j}L_1(x_{n_j}) \to 0$ as $j \to \infty$.

Now, since $Q_{n_j}\, L_1(x_{n_j})$ is bounded and L_2 is compact, we may assume that $L_2L_1^{-1}\, Q_{n_j}\, L_1(x_{n_j}) \to w$ as $j \to \infty$, therefore

$$\begin{aligned}
&\| (1-t)Q_{n_j}\, L_2L_1^{-1}\, Q_{n_j}\, L_1(x_{n_j}) - (1 - t)\, w\| \\
&\leq \| Q_{n_j}(1 - t)\, L_2L_1^{-1}\, Q_{n_j}\, L_1(x_{n_j}) - Q_{n_j}(1 - t)w\| \\
&+ \| Q_{n_j}(1 - t)\, w - (1 - t)\, w\| \\
&\leq \| Q_{n_j}\| \ (1 - t)\| \ L_2L_1^{-1}\, Q_{n_j}\, L_1(x_{n_j}) - w\| \\
&+ \| Q_{n_j}(1 - t)\, w - (1 - t)\, w\| \to 0 \text{ as } j \to \infty.
\end{aligned}$$

Thus, $(1 - t)\, Q_{n_j}\, L_2L_1^{-1}\, Q_{n_j}\, L_1(x_{n_j}) \to (1 - t)\, w$ as $j \to \infty$. Hence, $Q_{n_j}(L_1 - tL_2)(x_{n_j}) \to (1 - t)\, w$, and, by the A-properness of $L_1 - tL_2$, we may assume that there exists $x \in X$ such that $x_{n_j} \to x$ and $(L_1 - tL_2)\, x = (1 - t)\, w$. Then

$\| Q_{n_j}L_1(x_{n_j}) - L_1(x)\| \leq \| Q_{n_j}\, L_1(x_{n_j}) - Q_{n_j}\, L_1(x)\|$

$+ \| Q_{n_j}\, L_1(x) - L_1(x)\| \to 0$ as $j \to \infty$, which implies that

$L_2L_1^{-1} Q_{n_j} L_1(x_{n_j}) \to L_2L_1^{-1} L_1x = L_2x = w.$

Thus, $(L_1 - tL_2) x - (1 - t) L_2x = 0$, or $(L_1 - L_2)x = 0$ with $\|x\| = 1$, contradicting the injectiveness of $L_1 - L_2$.

Hence, by the homotopy property for Brouwer degree, we have, for each $n \geq N_2$, that

$$\deg(Q_n(L_1 - L_2), B(0,1) \cap X_n, 0)$$
$$= \deg(Q_n(I - L_2L_1^{-1}) Q_nL_1, B(0,1) \cap X_n, 0)$$
$$= \deg(Q_n(I - L_2L_1^{-1}), L_1B(0,1) \cap Y_n,0) \deg(Q_n L_1, B(0,1) \cap X_n, 0),$$

by the multiplication formula for Brouwer degree. Now, by definition of Leray-Schauder degree [16], $\deg(Q_n(I - L_2L_1^{-1}), L_1(B(0,1)) \cap Y_n,0)$ $= \deg_{LS}(I - L_2L_1^{-1}, L_1(B(0,1)),0)$.

Hence, letting $n \to \infty$, we have $\mathrm{Deg}(L_1 - L_2, B(0,1), 0)$ $= \deg_{LS}(I - L_2L_1^{-1}, L_1(B(0,1)),0) \mathrm{Deg}(L_1,B(0,1),0)$, and the result follows by (P5.).

CHAPTER TWO

GLOBAL BIFURCATION

VIA

GENERALISED TOPOLOGICAL DEGREE

Introduction

In this chapter we define, in its most general form, the abstract nonlinear eigenvalue problem to be considered in this thesis. Such a problem has the form $F(x,\lambda) = Ax - T(\lambda)x - R(x,\lambda) = 0$, where $F : X \times \mathbb{R} \to Y$, $A - T(\lambda)$ is linear, R is a higher order term with $F(.,\lambda)$ and $A - T(\lambda)$ both A-proper for λ in some interval in $\mathbb{R}$. We define the concept of bifurcation of solutions to this equation and prove that a sufficient condition for bifurcation is that the generalised degree of the linear part, $A - T(\lambda)$, changes as the real parameter λ moves across a special point λ_0. λ_0 is then called a bifurcation point. Our method provides us with global results, in particular we are able to deduce that from a bifurcation point λ_0 there emanates from $(0,\lambda_0)$ a maximal connected set $C_s \subset X \times \mathbb{R}$, of solutions which satisfies at least one of three properties: namely, it is unbounded; it moves out of the region where our maps are well-defined; or it simply ends at some other point $(0,\hat{\lambda}_0)$ with $\hat{\lambda}_0$ different from λ_0.

The use of degree theory in proving global results, was first made by Rabinowitz [35] when the mappings involved were compact. Generalisations have been given to more general classes of mappings, see, for example, Stuart [36], Toland [42], Stuart and Toland [38] and Mawhin [18]. Stuart and Toland [38] considered problems, where the nonlinear eigenvalue problem has the non-standard form,

$$I - C - \lambda B - R(.,\lambda) = 0$$

with B,C linear compact maps and R a compact continuous map of higher order. They proved a global result when it was not required that I - C be invertible. Stuart [36] also proved a global result for the problem

$$I - \lambda L - \lambda R = 0,$$

where $\lambda(L + R)$ is of the more general class of k-set contractions, with $k < 1$, and R again of higher order.

We shall extend these two methods by replacing I or I - C by a general linear map A, by allowing λL or λB to have the more general form $T(\lambda)$, retaining the linearity and continuity conditions, but assuming that $A - T(\lambda) - R(.,\lambda)$ and $A - T(\lambda)$ are A-proper for certain values of λ.

2.1 The general global bifurcation result

The equation to be studied is as follows:

$$F(x,\lambda) = Ax - T(\lambda)x - R(x,\lambda) = 0 \tag{2.1}$$

with $F : X \times \mathbb{R} \to Y$, where X, Y and $X \times \mathbb{R}$ are Banach spaces.

We impose the following hypotheses:

(H1) $F(.,\lambda) : X \to Y$ is an A-proper mapping with respect to the admissible scheme $\Gamma = \{X_n, Y_n, Q_n\}$ for λ in some real interval (a,b) finite or infinite;

(H2) $A - T(\lambda) : X \to Y$ is a bounded linear, A-proper operator with respect to Γ for all $\lambda \in (a,b)$ (as in (i)) and $T(\lambda)x$ is uniformly continuous in λ for x in bounded subsets of X;

(H3) $R(.,\lambda) : X \to Y$ is a continuous mapping such that $\| R(x,\lambda)\| / \|x\| \to 0$ as $\| x \| \to 0$, uniformly for λ in bounded intervals;

(H4) The mapping $\lambda \to R(x,\lambda)$ is continuous from $\mathbb{R}$ into Y, uniformly for x in bounded subsets of X.

Remark From (H3) it follows that $A - T(\lambda)$ is the Frechét derivative of $F(.,\lambda)$ at the point 0.

We shall refer to equation (2.1) satisfying hypotheses (H1) - (H4) as problem (2.1).

It follows from (H3) that the set $\{(0,\lambda) \in X \times \mathbb{R}\}$ is a solution set for equation (2.1). We call this the set of trivial solutions and make the following definition.

Definition 2.1 S will denote the set of non-trivial solutions of equation (2.1) in $X \times \mathbb{R}$. That is $(x,\lambda) \in S$ if and only if $F(x,\lambda) = 0$ with $\| x\| \neq 0$.

If $(0,\lambda_0) \in X \times \mathbb{R}$ is a point from which emanates a continuous set of non-trivial solutions of equation (2.1), then the value λ_0 is called a bifurcation point. More precisely:

Definition 2.2 A point $\lambda_0 \in \mathbb{R}$ is called a bifurcation point of equation (2.1) if there exists a sequence $\{(x_n,\lambda_n)\}$ in S converging to the point $(0,\lambda_0) \in X \times \mathbb{R}$.

It will be shown that all the bifurcation points of equation (2.1) are "characteristic values" of the linear operators. More precisely we make the following definition.

Definition 2.3 The set of characteristic values of $T(.)$ relative to A, denoted by $C_A(T)$, is given by

$$C_A(T) = \{\lambda \in \mathbb{R} : N(A - T(\lambda)) \neq \{0\}\}$$

This set is a simple generalisation of the set ch(T) of characteristic values of T defined in Chapter One.

Note that for $\lambda \notin C_A(T)$ with $\lambda \in (a,b)$, $A - T(\lambda)$ is a linear, continuous, injective, A-proper map and so is a homeomorphism by Theorem 1.9.

<u>Proposition 2.4</u> All bifurcation points of equation (2.1) in the interval (a,b) are contained in $C_A(T)$.

<u>Proof</u>: Suppose that $\hat{\lambda} \in (a,b)$ with $\hat{\lambda} \notin C_A(T)$. We shall prove that $\hat{\lambda}$ is not a bifurcation point. First we show that there exists a constant $k > 0$ such that $\|(A - T(\hat{\lambda}))x\| \geq k\|x\|$, for all $x \in X$.

For if this is false, then there exists a bounded sequence $\{x_n\}$ in X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, such that $\|(A - T(\hat{\lambda}))(x_n)\| \leq \frac{1}{n} \to 0$ as $n \to \infty$. Thus $\{(A - T(\hat{\lambda}))x_n\}$ is compact in X. Now since $A - T(\hat{\lambda})$ is continuous and A-proper, then by Theorem 1.10 $A - T(\hat{\lambda})$ is proper on closed bounded sets in X. Hence we may assume that there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$ and $(A - T(\hat{\lambda}))\, x = 0$ which contradicts the assumption that $\hat{\lambda} \notin C_A(T)$. So $k > 0$ exists.

Let $\lambda \in \mathbb{R}$ and $x \in X$, $x \neq 0$. Then

$$\begin{aligned}
&\|Ax - T(\lambda)x - R(x,\lambda)\| \\
&= \|Ax - T(\hat{\lambda})x - (T(\lambda) - T(\hat{\lambda}))x - R(x,\lambda)\| \\
&\geq \|Ax - T(\hat{\lambda})x\| - \|T(\lambda) - T(\hat{\lambda})\| \; \|x\| - \|R(x,\lambda)\| \\
&\geq [k - \|T(\lambda) - T(\hat{\lambda})\| - \|R(x,\lambda)\| / \|x\|]\|x\| \text{ , for } \|x\| \neq 0 \\
&> 0,
\end{aligned}$$

when $|\lambda - \hat{\lambda}|$ and $\|x\|$ are sufficiently small. Hence $\hat{\lambda}$ is not a bifurcation point of equation (2.1).

Proposition 2.4 tells us that a bifurcation point of equation (2.1) must necessarily be a characteristic value of $T(\lambda)$ relative to A. However, not all characteristic values are bifurcation points. For example, let $X = Y = \mathbb{R}^2$ and A have a matrix representation $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with respect to some basis in $\mathbb{R}^2$. Let $T(\lambda) = \lambda L$ where $L = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and define $R(x,\lambda) = \begin{pmatrix} 0 \\ x_1{}^3 \end{pmatrix}$ where $R : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $x = (x_1, x_2)$ for $x \in \mathbb{R}^2$. Then $\| R(x,\lambda) \| / \| x \| = |x_1{}^3| / (x_1{}^2 + x_2{}^2)^{\frac{1}{2}} \leq |x_1{}^3| / |x_1| = x_1{}^2$ and so $\| R(x,\lambda) \| / \| x \| \to 0$ as $\| x \| \to 0$.

Thus since all maps are compact it is easily seen that this example fits into the framework of problem (2.1)

Now, $C_A(L) = \{\lambda : N(A - \lambda L) \neq \{0\}\}$ is easily seen to be the singleton $\{0\}$. The equation $Ax - \lambda Lx - R(x,\lambda) = 0$ is equivalent to the simultaneous equations

$$\begin{aligned} x_2 + \lambda x_1 &= 0 \\ \lambda x_2 - x_1{}^3 &= 0 \end{aligned}$$

which imply that $x_1(\lambda^2 + x_1{}^2) = 0$.

Hence the only solution to this problem is the trivial one $x_1 = x_2 = 0$. Thus $\lambda_0 = 0$ is not a bifurcation point.

However, as we shall see, isolated elements λ_0 of $C_A(T)$ for which the degree of $A - T(\lambda)$ changes as λ passes through λ_0 are always bifurcation points. Before proving this we require some preliminary results.

Definition 2.5 Denote by S' the set $S \cup \{(0,\lambda) \in X \times \mathbb{R} : \lambda \in C_A(T)\}$.

Lemma 2.6 Let $[c,d]$ be any closed interval contained in (a,b). Define $Z = S' \cap \{X \times [c,d]\}$. Then all closed bounded subsets of Z are compact.

Proof: Let $\{(x_n,\lambda_n)\}$ be a sequence in an arbitrary closed bounded subset of Z. Without loss of generality we may assume that $\lambda_n \to \lambda$ as $n \to \infty$.

For each $n \in \mathbb{N}$ set $\tilde{T}_n = A - T(\lambda_n) - R(.,\lambda_n)$ and $\tilde{T} = A - T(\lambda) - R(.,\lambda)$. Then

$$\|\tilde{T}_n(x) - \tilde{T}(x)\| = \|(T(\lambda) - T(\lambda_n))x + R(x,\lambda) - R(x,\lambda_n)\|$$
$$\leq \|T(\lambda) - T(\lambda_n)\| \; \|x\| + \|R(x,\lambda) - R(x,\lambda_n)\| \to 0 \text{ as } n \to \infty,$$

uniformly for x in bounded subsets of Z. Now since $\{x_n\}$ is a bounded sequence in Z and $\tilde{T}_n x_n = 0$ for all $n \in \mathbb{N}$ we have that $\tilde{T}\, x_n \to 0$ as $n \to \infty$.

But $\tilde{T}$ is continuous and A-proper, therefore it is proper, by Theorem 1.10, on closed, bounded sets. Hence, we may assume that there exists $x \in X$ such that $x_n \to x$ (say) as $n \to \infty$ and $\tilde{T}x = 0$.

Thus, $(x_n,\lambda_n) \to (x,\lambda)$ and it follows that all closed bounded subsets of Z are compact.

Definition 2.7 Let $\lambda_0 \in C_A(T)$ and denote by C_s the component (maximal connected set) of S' containing the point $(0,\lambda_0)$. Then we say that λ_0 is a global bifurcation point of equation (2.1) or that global bifurcation occurs at λ_0, provided C_s satisfies at least one of the following properties:

(i) C_s is an unbounded subset of $X \times \mathbb{R}$;

(ii) $(0,\hat{\lambda}_0) \in C_s$ for some element $\hat{\lambda}_0 \in C_A(T)$ with $\hat{\lambda}_0 \neq \lambda_0$;

(iii) $\inf\{|\lambda - a|: (x,\lambda) \in C_s \text{ for some } x \in X\} = 0$ or
$\inf\{|b - \lambda|: (x,\lambda) \in C_s \text{ for some } x \in X\} = 0$.

Before giving our main bifurcation theorem we state a topological result and prove a Lemma.

Lemma 2.8 (Whyburn, [50]). Let K be a compact metric space and A and B be disjoint closed subsets of K. Then, either there exists a connected set in K meeting both A and B, or $K = K_A \cup K_B$, where K_A, K_B are disjoint compact subsets of K containing A and B respectively.

Lemma 2.9 Suppose that $\lambda_0 \in C_A(T)$ is isolated but λ_0 is not a global bifurcation point of equation (2.1). Then there exist a bounded open set G in $X \times \mathbb{R}$ and positive numbers ε, ρ and η such that:

(a) $\lambda > a + \varepsilon$ and $\lambda < b - \varepsilon$ for all $\lambda \in \mathbb{R}$ such that $(x,\lambda) \in G$ for some $x \in X$;

(b) $(0,\lambda_0) \in G$;

(c) $S \cap \partial G = \phi$;

(d) $\|x\| \geq \eta$ for all $x : (x,\lambda) \in G$ with $|\lambda - \lambda_0| \geq \rho$;

(e) λ_0 is the only element belonging to $C_A(T)$ in the interval $[\lambda_0 - \rho, \lambda_0 + \rho]$.

Remark: Similar to results of Rabinowitz [35] and Stuart [36].

Proof: Let C_s denote the maximal connected subset of S' to which $(0,\lambda_0)$ belongs. Since λ_0 is not a global bifurcation point, then (i) of Definition (2.7) does not hold and C_s is therefore a closed bounded subset of $X \times \mathbb{R}$. Let

$$\varepsilon = \tfrac{1}{4} \inf\{1, \lambda - a, b - \lambda : (x,\lambda) \in C_s \text{ for some } x \in X\}.$$

Then since (iii) of Definition 2.7 fails we must have $\varepsilon > 0$ and therefore $\lambda > a + \varepsilon$, $\lambda < b - \varepsilon$ for all $\lambda \in \mathbb{R}$ such that $(x,\lambda) \in C_s$ for some $x \in X$. Define $Z = S' \cap \{X \times [a + \varepsilon, b - \varepsilon]\}$.

Then, from Lemma 2.6, all closed, bounded subsets of Z are compact, therefore C_s is compact.

Now, since (ii) of Definition 2.7 also fails, if $(x,\hat{\lambda}_0) \in C_s$ with $\hat{\lambda}_0 \in C_A(T)$ and $\hat{\lambda}_0 \neq \lambda_0$, then $\|x\| > 0$. So there exist numbers $\rho_1 > 0$ and $\eta_1 > 0$ such that $\|x\| \geq \eta_1$ for all $(x,\lambda) \in C_s$ with $|\lambda - \hat{\lambda}_0| \leq \rho_1$. Also $\lambda_0 \in C_A(T)$ is isolated so Proposition (2.4) and the previous argument imply the existence of numbers $\rho > 0$ and $\eta > 0$ such that $\|x\| \geq 4\eta$ for all $(x,\lambda) \in C_s$ with $|\lambda - \lambda_0| \geq \frac{1}{4}\rho$ and where λ_0 is the only element of $C_A(T)$ in the interval $[\lambda_0 - \rho, \lambda_0 + \rho]$. Hence (e) is satisfied.

Let $\delta = \min\{\varepsilon, \frac{1}{4}\rho, \eta\}$ and

$V_\delta = \{(x_1,\lambda_1) \in X \times \mathbb{R} : \|x_1 - x\|^2 + |\lambda - \lambda_1|^2 < \delta^2$ for some $(x,\lambda) \in C_s\}$.

Then, by our choice of δ, $\|x\| > 3\eta$ for all $(x,\lambda) \in V_\delta$ with $|\lambda - \lambda_0| \geq \frac{1}{2}\rho$. This tells us that $(0, \lambda_0 \pm \rho) \notin V_\delta$. To see this consider $(0, \lambda_0 + \rho)$ and suppose that the nearest point in V_δ to $(0,\lambda_0 + \rho)$ is (x,λ). If $|\lambda - \lambda_0| \geq \frac{1}{2}\rho$, then $\|x\| > 3\eta$ and so $\text{dist}((0,\lambda_0 + \rho),(x,\lambda)) > 3\eta > 0$. Alternatively if $|\lambda - \lambda_0| < \frac{1}{2}\rho$, then $\text{dist}((0,\lambda_0 + \rho),(x,\lambda)) \geq \frac{1}{2}\rho > 0$. So $(0,\lambda_0 + \rho) \notin V_\delta$ and similarly $(0,\lambda_0 - \rho) \notin V_\delta$.

Now, let $K = Z \cap \overline{V}_\delta = S' \cap \overline{V}_\delta$.

Then K is a closed bounded subset of Z and hence compact by Lemma 2.6. This follows since when $\{(x_n,\lambda_n)\}$ is a sequence in K such that $(x_n,\lambda_n) \to (x,\lambda)$ (say), then by closure and boundedness of $\overline{V}_\delta$, $(x,\lambda) \in \overline{V}_\delta$ and (x,λ) is bounded. Hence, by the continuity of F, $(x,\lambda) \in S'$.

Now, since $\delta > 0$, C_s and ∂V_δ are disjoint closed subsets of K, therefore so are C_s and $S' \cap \partial V_\delta$. Also, by the fact that C_s is a maximal connected subset of S', there is no connected subset of K intersecting both C_s and $S' \cap \partial V_\delta$. Hence, by Lemma 2.8, there exist disjoint compact subsets K_1 and K_2 of K with $K = K_1 \cup K_2$, $C_s \subseteq K_1$ and $S' \cap \partial V_\delta \subseteq K_2$. Let $\text{dist}(K_1,K_2) = m$. Then, by compactness, $m > 0$. Define

$G = \{(x_1,\lambda_1) \in X \times \mathbb{R} : \|x_1 - x\|^2 + |\lambda_1 - \lambda|^2 < \frac{m^2}{16}$ for some $(x,\lambda) \in K_1\}$. Hence, since $(0,\lambda_0) \in C_s \subseteq K_1 \subset G$, (b) holds.

Now $\mathrm{dist}(C_s,\partial G) < \mathrm{dist}(C_s,K_2) \leq \delta$ and so $G \subset V_\delta$ and $\partial G \subset V_\delta$. Also $K \cap \partial G = (K_1 \cup K_2) \cap \partial G = \phi$, therefore, $\phi = K \cap \partial G = (S' \cap \overline{V}_\delta) \cap \partial G = S' \cap \partial G$, and (c) holds.

Furthermore, since $G \subset V_\delta$, then (a) holds and by our observation above that $\|x\| > 3\eta$ for all $(x,\lambda) \in V_\delta$ with $|\lambda - \lambda_0| \geq \frac{1}{2}\rho$ then (d) holds. Hence G satisfies all the conditions (a) - (e).

We can now prove the following global bifurcation result.

<u>Theorem 2.10</u> Let $\lambda_0 \in C_A(T)$ be isolated and suppose that there exists $\delta > 0$ such that $\mathrm{Deg}(A - T(\underline{\lambda}), W, 0) \neq \mathrm{Deg}(A - T(\overline{\lambda}), W, 0)$ for $\lambda_0 - \delta < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \delta$, where W is an arbitrary open bounded set in X containing zero. Then λ_0 is a global bifurcation point of equation (2.1).

<u>Proof:</u> The proof is by contradiction. We assume that λ_0 is not a global bifurcation point and prove, then, that necessarily $\mathrm{Deg}(A - T(\underline{\lambda}), W,0) = \mathrm{Deg}(A - T(\overline{\lambda}),W,0)$ contradicting our assumption.

So suppose λ_0 is not a global bifurcation point. Then by Lemma 2.9 there exist an open subset G of X and positive numbers ε, ρ and η satisfying conditions (a) - (e). For $\lambda \in \mathbb{R}$ we define $G_\lambda = \{x \in X : (x,\lambda) \in G\}$ and $\partial G_\lambda = \{x \in X : (x,\lambda) \in \partial G\}$. Choose $\underline{\lambda},\overline{\lambda}$ with $\lambda_0 - \rho < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \rho$ such that $(0,\lambda) \in G$ for all $\lambda \in [\underline{\lambda},\overline{\lambda}]$, and ρ as defined in Lemma 2.9.

Note that this is possible since $(0,\lambda_0) \in G$, and G is open. So $0 \notin \partial G_\lambda$, and hence $0 \in G_\lambda$ for $\lambda \in [\underline{\lambda},\overline{\lambda}]$. Now, by condition (c) of Lemma 2.9, $S \cap \partial G = \phi$, and, therefore,

$Ax - T(\lambda)x - R(x,\lambda) \neq 0$, for $x \in \partial G_\lambda$ and $\lambda \in [\underline{\lambda},\overline{\lambda}]$.

Also, $\lambda \to Ax - T(\lambda)x - R(x,\lambda)$ is continuous on $[\underline{\lambda},\overline{\lambda}]$, uniformly on $\overline{G}_\lambda$ and $A - T(\lambda) - R(.,\lambda)$ is A-proper for all $\lambda \in [\underline{\lambda},\overline{\lambda}]$. Hence by the homotopy property (P3.), $\mathrm{Deg}(A - T(\lambda) - R(.,\lambda), G_\lambda, 0)$ is defined and independent of λ for $\lambda \in [\underline{\lambda},\overline{\lambda}]$, which implies that

$$\mathrm{Deg}(A - T(\underline{\lambda}) - R(.,\underline{\lambda}), G_{\underline{\lambda}}, 0) = \mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), G_{\overline{\lambda}}, 0) \quad (2.2).$$

We show that

$$\mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), G_{\overline{\lambda}}, 0) = \mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), B(0,r), 0)$$

with $r > 0$ arbitrarily small. It follows, from Proposition 2.4, that there exists a number $r_1(\overline{\lambda})$ such that for every $\lambda \in [\overline{\lambda}, \rho + \lambda_0]$, 0 is the only solution of equation (2.1) in the closed ball $\overline{B}(0,r_1(\overline{\lambda}))$. Let $r_2(\overline{\lambda}) = \min\{\frac{1}{2}r_1(\overline{\lambda}), \frac{1}{4}n\}$. Then, from condition (d) of Lemma 2.9,

$$\overline{B}(0,r_2(\overline{\lambda})) \cap \overline{G}_\lambda = \phi \text{ for } \lambda \geq \rho + \lambda_0.$$

Suppose that $x \in \partial(G_\lambda \setminus \overline{B}(0,r_2(\overline{\lambda})))$ for $\lambda \geq \overline{\lambda}$. Then, either $\|x\| = r_2(\overline{\lambda})$, or else $\|x\| > r_2(\overline{\lambda})$, and $x \in \partial G_\lambda$. By condition (c) of Lemma 2.9 this implies that if $\lambda \geq \overline{\lambda}$ and $x \in \partial(G_\lambda \setminus \overline{B}(0,r_2(\overline{\lambda})))$, then (x,λ) does not satisfy equation (2.1). Also, $A - T(\lambda) - R(.,\lambda)$ is A-proper for $\lambda \in [\overline{\lambda}, b - \varepsilon]$ and $\lambda \to Ax - T(\lambda)x - R(x,\lambda)$ is continuous on $[\overline{\lambda}, b - \varepsilon]$, uniformly for x in $\overline{G_\lambda \setminus \overline{B}(0,r_2(\overline{\lambda}))}$. Hence by the homotopy property (P3.) $\mathrm{Deg}(A - T(\lambda) - R(.,\lambda), G_\lambda \setminus \overline{B}(0,r_2(\overline{\lambda})), 0)$ is defined and independent of $\lambda \in [\overline{\lambda}, b - \varepsilon]$. In particular

$$\mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), G_{\overline{\lambda}} \setminus \overline{B}(0,r_2(\overline{\lambda})), 0)$$

$$= \mathrm{Deg}(A - T(b - \varepsilon) - R(.,b - \varepsilon), G_{b-\varepsilon} \setminus \overline{B}(0,r_2(\overline{\lambda})), 0) = \{0\}.$$

This follows by degree property (P1.) since $G_{b-\varepsilon} \setminus \overline{B}(0,r_2(\overline{\lambda})) = \phi$.

Hence by (P2.),

$$\begin{aligned}\mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}),G_{\overline{\lambda}}, 0) &= \mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}),G_{\overline{\lambda}}\backslash\overline{B}(0,r_2(\overline{\lambda})), 0)\\ &+ \mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), G_{\overline{\lambda}} \cap B(0,2r_2(\overline{\lambda})), 0)\\ &= \mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), B(0,2r_2(\overline{\lambda})), 0) + \{0\} \qquad (2.3)\end{aligned}$$

since $G_{\overline{\lambda}} \cap B(0,2r_2(\overline{\lambda})) = B(0,2r_2(\overline{\lambda}))$. Note that we have equality here since one of the terms in the sum is single valued. It should also be emphasised that we originally chose r_2 half as small as was necessary, so replacing $r_2(\overline{\lambda})$ by $2r_2(\overline{\lambda})$ does not affect any of the important arguments. In particular for $\lambda \in [\overline{\lambda}, \rho + \lambda_0]$, zero is the only solution of equation (2.1) in the closed ball $\overline{B}(0,2r_2(\overline{\lambda}))$.

It may be proved similarly that there exists $r_2(\underline{\lambda})$ such that

$$\mathrm{Deg}(A - T(\underline{\lambda}) - R(.,\underline{\lambda}),G_{\underline{\lambda}},0) = \mathrm{Deg}(A - T(\underline{\lambda}) - R(.,\underline{\lambda}), B(0,2r_2(\underline{\lambda})), 0) + \{0\} \qquad (2.4)$$

Finally by choosing $r_2(\underline{\lambda})$ and $r_2(\overline{\lambda})$ small enough it follows from (P7.) that $\mathrm{Deg}(A - T(\underline{\lambda}) - R(.,\underline{\lambda}), B(0,2r_2(\underline{\lambda})),0) = \mathrm{Deg}(A - T(\underline{\lambda}),B(0,2r_2(\underline{\lambda})),0)$

and

$$\mathrm{Deg}(A - T(\overline{\lambda}) - R(.,\overline{\lambda}), B(0,2r_2(\overline{\lambda}), 0) = \mathrm{Deg}(A - T(\overline{\lambda}), B(0,2r_2(\overline{\lambda})), 0).$$

Then, from equations (2.2), (2.3) and (2.4) and (P5.), we have that $\mathrm{Deg}(A - T(\underline{\lambda}),W,0) = \mathrm{Deg}(A - T(\overline{\lambda}),W,0)$ for $\lambda_0 - \rho < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \rho$ which is a contradiction. Hence λ_0 is a global bifurcation point of equation (2.1) as required.

Remark The remainder of this thesis will be concerned with obtaining sufficient conditions under which Theorem 2.10 may be applied. That is, conditions which imply that the generalised degree of $A - T(\lambda)$ does change as λ moves across $\lambda_0 \in C_A(T)$.

CHAPTER 3

DEGREE MULTIPLICATION FORMULAE LEADING TO GLOBAL BIFURCATION

Introduction

In this chapter we present generalisations of two methods of Toland for obtaining global bifurcation of problem (2.1) via Theorem 2.10. Both of the ideas involve a degree multiplication formula: one for a product of mappings and the other when a direct sum, of the underlying space, exists in a particular form.

3.1 A result using the Leray-Schauder formula

This section extends Toland's work in [43] where he shows that two different sets of hypotheses provide a method for proving global bifurcation of problem (2.1) by a procedure which depends on the multiplication formula for Leray-Schauder degree, cf. Lloyd [16]. The results here were obtained in collaboration with Dr. J. R. L. Webb and a shorter version is to be published [48]. In [48], however, it was assumed for simplicity that $X = Y$. The proofs for the general case are essentially the same.

One extension we make is to allow more general operators. Toland considers problem (2.1) with $X = Y$, $A = I - \tilde{A}$ and $T(\lambda) = \lambda B$ where $\tilde{A}$ and B are linear compact maps and R is continuous and compact. We also consider problem (2.1) with $T(\lambda) = \lambda B : X \to Y$, where B is linear compact but we do not require that $A : X \to Y$ be of the form identity minus compact or that R be compact. Since we replace compact maps by A-proper maps we must also replace the Leray-Schauder degree with the generalised degree. But, as previously noted, the proof adopted by Toland, and suitably modified by us, relies on the multiplication formula for Leray-Schauder degree, which has no direct equivalent in the generalised degree theory. Petryshyn has shown, however, ((P8.) of Chapter 1) that there is a restricted analogue of the Leray-Schauder multiplication formula, in the generalised degree theory, which enables us to obtain a global result in an analogous way to Toland.

Another extension we make is to provide an alternative set of hypotheses, for which the method still works, which involves a condition on the null space $N(A - \lambda_0 B)$ at some characteristic value λ_0. This condition replaces the commutativity demanded by Toland and turns out to be a generalisation of his other set of hypotheses: namely, $Y = X$ is a Hilbert space, A and B are self-adjoint and either A or B is positive semi-definite.

Let us be more precise.

Consider problem (2.1) with the additional hypotheses:

(H5) $T(\lambda) = \lambda B$, where $B : X \to Y$ is a compact linear map;

(H6) For some $\lambda_0 \in (a,b) \cap C_A(T)$, $BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$.

Hypothesis (H6) is known as a transversality condition and is frequently employed in bifurcation theory, as for example in Alexander and Fitzpatrick [2], Mawhin [18], Chow and Hale [6] (Chapter 5), and many others.

Note that A is A-proper since B is compact and so $A - \lambda B$ is A-proper for all $\lambda \in \mathbb{R}$.

The first result is that the compactness of B implies a dichotomy of the set $C_A(T)$.

Proposition 3.1 Either $C_A(T) = \mathbb{R}$, or $C_A(T)$ is a discrete set with no finite limit points.

Proof: Assume that there exists a point μ such that $\mu \notin C_A(T)$. Then $A - \mu B$ is a homeomorphism and we have that

$C_A(T) = \{\lambda + \mu : N[I - \lambda B(A - \mu B)^{-1}]\} \neq \{0\}$.

Since B is compact, $C_A(T)$ is a discrete set with no finite limit points, cf. Chapter One. The other possibility is that $C_A(T) = \mathbb{R}$.

Remark: This proof is exactly the same as the one given by Toland [43], it applies equally well to our situation.

We now give the main result in this section.

Theorem 3.2 Consider problem (2.1) with the additional hypotheses (H5) and (H6). Suppose that $C_A(T) \neq \mathbb{R}$, let $\nu = \dim N(A - \lambda_0 B)$, and suppose $[\underline{\lambda},\overline{\lambda}] \cap C_A(T) = \{\lambda_0\}$ for $\underline{\lambda} < \lambda_0 < \overline{\lambda}$. Then, $\mathrm{Deg}(A - \overline{\lambda}B,G,0) = (-1)^\nu \mathrm{Deg}(A - \underline{\lambda}B,G,0)$ for an arbitrary bounded open set $G \subset X$ containing zero.

Proof: We have that $A - \overline{\lambda}B = A - \underline{\lambda}B - (\overline{\lambda} - \underline{\lambda})B$

$$= [I - (\overline{\lambda} - \underline{\lambda})\, B\, (A - \underline{\lambda}B)^{-1}](A - \underline{\lambda}B)$$

Replacing L_1 by $A - \underline{\lambda}B$ and L_2 by $(\overline{\lambda} - \underline{\lambda})B$ in (P8.) implies that, $\mathrm{Deg}(A - \overline{\lambda}B,G,0) = \deg_{LS}(I - (\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1},D,0)\ \mathrm{Deg}(A - \underline{\lambda}B,G,0)$, where D is the open set $(A - \underline{\lambda}B)(G)$ containing zero.

Now by the Leray-Schauder formula, cf. remarks preceding Definition 1.17, $\deg_{LS}(I - (\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1},D,0) = (-1)^\nu$, where ν is the sum of the algebraic multiplicities of the characteristic values of $(\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1}$ in the interval $(0,1)$. We shall prove that there is only one such value. Suppose $\mu \in (0,1)$ is a characteristic value of

$(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1}$. Then, for some $y \neq 0$, $y - \mu(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1}y = 0$ which implies that $(A - \underline{\lambda}B)x - \mu(\bar{\lambda} - \underline{\lambda})\,Bx = 0$, where $(A - \underline{\lambda}B)^{-1}y = x \neq 0$. So $Ax - (\underline{\lambda} + \mu(\bar{\lambda} - \underline{\lambda}))\,Bx = 0$, where $\mu \in (0,1)$.

Hence $\underline{\lambda} + \mu(\bar{\lambda} - \underline{\lambda}) = \lambda_0$ or $\mu = \dfrac{\lambda_0 - \underline{\lambda}}{\bar{\lambda} - \underline{\lambda}} = \mu_0$ (say).

Next we show that the ascent of $I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1}$ is equal to one, that is $N(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})^2 = N(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})$, which will prove that $\nu = \dim\{N(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})\}$.

So, let $(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})^2 y = 0$, with $y \neq 0$.

Now $I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1}$

$$= I - \frac{(\lambda_0 - \underline{\lambda})(\bar{\lambda} - \underline{\lambda})}{(\bar{\lambda} - \underline{\lambda})}\, B\,(A - \underline{\lambda}B)^{-1}$$

$$= (A - \underline{\lambda}B - (\lambda_0 - \underline{\lambda})B)(A - \lambda B)^{-1}$$

$$= (A - \lambda_0 B)(A - \underline{\lambda}B)^{-1}$$

So $((A - \lambda_0 B)(A - \underline{\lambda}B)^{-1})^2\, y = 0$, with $y \neq 0$

Thus, $(A - \lambda_0 B)(A - \underline{\lambda}B)^{-1}\, w = 0$ where,

$$w = (A - \lambda_0 B)(A - \underline{\lambda}B)^{-1}\, y \in R(A - \lambda_0 B).$$

Therefore, $(A - \underline{\lambda}B)^{-1}\, w \in N(A - \lambda_0 B)$,

$$\begin{aligned} \text{so } w &\in (A - \underline{\lambda}B)N(A - \lambda_0 B) \\ &= (A - \lambda_0 B - (\underline{\lambda} - \lambda_0)B)N(A - \lambda_0 B) \\ &= BN(A - \lambda_0 B). \end{aligned}$$

Hence $w \in BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$, by H6, which implies that $0 = (A - \lambda_0 B)(A - \underline{\lambda}B)^{-1}y = (I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})y$.

Thus $N(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})^2 \subseteq N(I - \mu_0(\bar{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})$.

The reverse inclusion is always valid and so equality holds.

Finally it follows easily that

$$\begin{aligned}&\dim\{N(I - \mu_0(\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})\}\\ &= \dim\{N((A - \lambda_0 B)(A - \underline{\lambda}B)^{-1})\}\\ &= \dim\{N(A - \lambda_0 B)\} = \nu\end{aligned}$$

Hence the theorem is proved.

Remark (1) If A is of the form I - K, where K is a k-set contraction with k < 1, practically the same proof holds using the degree theory of Nussbaum, [22]. Thomas, [40] proves the necessary version of the multiplication formula. The required extension of the Leray-Schauder Formula has been proved by Stuart and Toland [37].

(2) If X is a Hilbert space, a linear operator $T : X \to X$ is said to be positive semi-definite, provided : T is self-adjoint; $(Tx,x) \geq 0$ for all $x \in X$; and $(Tx,x) = 0$ implies that $Tx = 0$.

If $\lambda_0 \neq 0$, condition (H6) generalises one of Toland's [43] set of assumptions; namely, X = Y is a Hilbert and A,B are self-adjoint maps with either A or B positive semi-definite. To see this, assume that $w \in BN(A - \lambda_0 B) \cap R(A - \lambda_0 B)$. Then there exist $x, v \in X$ with $w = Bx$ and $w = (A - \lambda_0 B)v$, where $(A - \lambda_0 B)x = 0$. So $(Bx,x) = (w,x)$ $= ((A - \lambda_0 B)v,x) = (v,(A - \lambda_0 B)x) = 0$ and, if B is positive semi-definite, then $Bx = 0$ and $w = 0$. Also $(Ax,x) = (\lambda_0 Bx,x)$ and the result holds again if A is positive semi-definite. A similar argument may be used to show that (H6) also generalises the assumption : X = Y is a Hilbert space and A,B are self-adjoint maps with either A or B negative semi-definite.

(3) We could prove a result under Toland's other set of hypotheses too, namely that X = Y and A and B commute. These hypotheses would

replace our hypothesis (H6). The conclusions of Theorem 3.2 hold, with this assumption, if we replace $\dim\{N(A - \lambda_0 B)\}$ by $\dim\{N((A - \lambda_0 B)^p\}$, where p is the ascent of $A - \lambda_0 B$. This follows from the proof of Theorem 3.2, since

$$I - \mu(\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1} = (A - \lambda_0 B)(A - \underline{\lambda}B)^{-1}$$
$$= (A - \underline{\lambda}B)^{-1}(A - \lambda_0 B),$$

So, for each $n \in \mathbb{N}$,

$$(I - \mu_0(\overline{\lambda} - \underline{\lambda})B(A - \underline{\lambda}B)^{-1})^n x = 0$$

if and only if $(A - \lambda_0 B)^n x = 0$.

Note, since B is compact this also shows that p is finite and $\dim\{N(A - \lambda_0 B)^p\}$ is finite.

Theorem 3.2 provides us with the following global bifurcation theorem.

<u>Theorem 3.3</u> Consider problem (2.1) with the additional hypothesis (H5). Suppose that $C_A(T) \neq \mathbb{R}$ and, for some $\lambda \in \mathbb{R}$, with $\lambda \notin C_A(T)$, $\mathrm{Deg}(A - \lambda B, G, 0)$ is a singleton, where $G \subset X$ is an arbitrary bounded, open set containing zero. Then, λ_0 is a global bifurcation point if at least one of the following additional hypotheses is satisfied:

(1) hypothesis (H6) holds with $\dim\{N(A - \lambda_0 B)\}$ an odd number;

(2) X = Y, A and B commute and $\dim\{N((A - \lambda_0 B)^p)\}$ is an odd number, where p is the ascent of $A - \lambda_0 B$, which is finite;

(3) X = Y is a Hilbert space, A,B are self-adjoint operators with either A or B positive semi-definite and $\dim\{N(A - \lambda_0 B)\}$ is an odd number, with $\lambda_0 \neq 0$.

<u>Proof</u>: From Theorem 3.2 we have that; if $\underline{\lambda} < \lambda_0 < \overline{\lambda}$, $[\underline{\lambda},\overline{\lambda}] \cap C_A(T) = \{\lambda_0\}$, and $\dim N\{(A - \lambda_0 B)\} = \nu$ is an odd number, then $\mathrm{Deg}(A - \overline{\lambda}B,G,0) = (-1)^\nu \,\mathrm{Deg}(A - \underline{\lambda}B,G,0) = -\mathrm{Deg}(A - \underline{\lambda}B,G,0)$.

Now, since $A - \underline{\lambda}B$ and $A - \overline{\lambda}B$ are homeomorphisms,

$\mathrm{Deg}(A - \underline{\lambda}B,G,0) \subseteq \{-1,1\}$ and

$\mathrm{Deg}(A - \overline{\lambda}B,G,0) \subseteq \{-1,1\}$, cf. [31].

But by assumption, there exists $\lambda \in \mathbb{R}$, with $\lambda \notin C_A(T)$, such that $\mathrm{Deg}(A - \lambda B,G,0)$ is a singleton. Then since $[\underline{\lambda},\overline{\lambda}] \cap C_A(T) = \{\lambda_0\}$, Proposition 3.1 tells us that there is a discrete number of characteristic values, i.e., they are isolated. Thus, by Theorem 3.2, for each $\lambda \in \mathbb{R}$ with $\lambda \notin C_A(T)$, $\mathrm{Deg}(A - \lambda B,G,0)$ is a singleton and alternates between 1 and -1 as λ passes through isolated characteristic values of odd multiplicity. Hence $\mathrm{Deg}(A - \overline{\lambda}B,G,0) \neq \mathrm{Deg}(A - \underline{\lambda}B,G,0)$ and the result of the theorem follows from Theorem 2.10 and the preceding Remarks (2) and (3).

Remark (1) In Theorem 3.3 we have assumed that $\mathrm{Deg}(A - \lambda B,G,0)$ is a singleton for some $\lambda \in \mathbb{R}$ with $\lambda \notin C_A(T)$. In Chapter Four, Theorem 4.12, we prove a global bifurcation result without making this assumption and for not necessarily compact B, which generalises Theorem 3.3 (1). Also in Chapter Four, Theorem 4.18, for the case $\lambda_0 = 0$, we generalise Theorem 3.3 (3), without making the assumption that the degree is a singleton, for the more general $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, where k is finite or infinite. We are able to relax the condition that A and the B_j's are self-adjoint and we require a less stringent condition on the B_j's than positive semi-definite.

(2) There are examples where $\mathrm{Deg}(A - \lambda B,G,0)$ is a singleton. For instance, $A - \lambda B$ is of the form I - compact, I - ball condensing; $A - \lambda B$

is orientation preserving; and others, including the following. Consider problem (2.1) with the additional hypothesis (H5) and where X is reflexive and $A - \lambda B$ is accretive for some $\lambda \in \mathbb{R}$ with $\lambda \notin C_A(T)$. (Accretive maps were given as examples of A-proper maps following Definition 1.8). We define a homotopy $H : \overline{G} \times [0,1] \to X$ by $H(x,t) = (1 - t)x + t(A - \lambda B)x$ for $(x,t) \in \overline{G} \times [0,1]$. $H(x,.) : [0,1] \to X$ is easily seen to be uniformly continuous on $\overline{G}$. Hence, to show that H is a valid homotopy we need only prove that $H(.,t) : X \to X$ is A-proper and $H(\partial G,t) \neq 0$ for each $t \in [0,1]$. First notice that $H(x,1) = A - \lambda B$ is A-proper and for $0 \leq t < 1$, $H(x,t) = (1 - t) I + t(A - \lambda B)$ is of the form αI + accretive and is, therefore, A-proper by [19]. Hence, $H(.,t)$ is A-proper for each $t \in [0,1]$.

Now suppose $H(x_0,t_0) = 0$ for some $x_0 \in \partial G$ and $t_0 \in [0,1]$, that is, $(1 - t_0)x_0 + t_0(A - \lambda B)x_0 = 0$ with $\|x_0\| \neq 0$.

Since $A - \lambda B$ and I are injective maps it follows easily that $t_0 \neq 0$ and $t_0 \neq 1$. Thus $t_0 \in (0,1)$ and, therefore

$$(A - \lambda B)x_0 = - \frac{(1 - t_0)}{t_0} x_0 \text{ and}$$

$$0 \leq ((A - \lambda B)x_0, Jx_0) = - \frac{(1 - t_0)}{t_0} \|x_0\|^2 < 0 ,$$

by the accretiveness of $A - \lambda B$. This contradiction proves that $H(\partial G,t) \neq 0$ for each $t \in [0,1]$. Thus by the homotopy property (P3.) we have that $\mathrm{Deg}(A - \lambda B,G,0) = \mathrm{Deg}(I,G,0) = \{1\}$.

A particular case of the above situation may be seen when $X = Y$ is a Hilbert space. If $0 \in \mathbb{R}$ and there exists $\varepsilon > 0$ such that $(Ax,x) \geq \varepsilon\|x\|^2$ for each $x \in X$, then whenever $0 \leq |\lambda| \leq \varepsilon/\|B\|$, it follows that $((A - \lambda B)x,x) \geq 0$. But X is a Hilbert space which is reflexive and the duality map J equals I. So $A - \lambda B$ is accretive for $0 \leq |\lambda| \leq \varepsilon/\|B\|$ and the above analysis implies that, here, $\mathrm{Deg}(A - \lambda B, G, 0)$ is a singleton for each $\lambda \in \mathbb{R}$ with $\lambda \in C_A(T)$.

3.2 A product formula for generalised degree

The results in this section are again joint work with Dr. J. R. L. Webb and a shorter version is to be published, [49].

We shall extend a Leray-Schauder degree multiplication formula of Krasnosel'skii to a generalised version. Krasnosel'skii [13] showed that if X can be decomposed into the direct sum $E_1 \oplus E_2$ and $T_j : E_j \to E_j$ $(j = 1,2)$ are compact linear operators such that $I - T_j : E_j \to E_j$ are homeomorphisms, then by defining $Tx = T_1x_1 + T_2x_2$, for $x_j \in E_j (j = 1,2)$, the Leray-Schauder degrees are related by

$$\deg_{LS}(I - T, B(0,1), 0) = \deg_{LS}(I - T_1, B_1(0,1), 0)\ \deg_{LS}(I - T_2, B_2(0,1), 0)$$

where $B_1(0,1)$ and $B_2(0,1)$ are the open unit balls in E_1 and E_2 respectively.

We shall assume that $X = E_1 \oplus E_2$, where E_1 is a <u>finite dimensional</u> subspace of X and E_2 is a closed subspace of X. We suppose also, that $I - T : X \to X$ is an A-proper homeomorphism with respect to an admissible scheme $\Gamma = \{X_n, Q_n\}$. Then we prove that the generalised degree multiplication formula

$$\mathrm{Deg}(I - T, B(0,1), 0) = \deg_{LS}(I - T_1, B_1(0,1), 0)\ \mathrm{Deg}(I - T_2, B_2(0,1), 0) \text{ holds.}$$

The proof involves showing that $I - T : X \to X$ is also A-proper with respect to another admissible scheme Γ'' constructed from the original scheme Γ. Relative to Γ'', we are able to prove that the generalised degree multiplication formula does hold and then by a homotopy argument it is shown that the result also holds relative to the original scheme.

By making a transversality assumption similar to that in §3.1 hypothesis (H6) and assuming that $I - T(\lambda_0)$ is Fredholm of index zero we show that a decomposition $X = E_1 \oplus E_2$ exists with dim E_1 finite and such that both E_1 and E_2 are invariant under $T(\lambda)$. Then by hypotheses similar to Toland's [41] we use the derived generalised degree multiplication formula to prove global bifurcation results. We take problem (2.1) with $X = Y$, $A = I$ and $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$. This generalises Toland's work in that he considered the same problem but demanded that B_j be linear and compact for $j = 1,2,\ldots,k$ and R be continuous and compact. The proof we adopt is similar to Toland's, in particular we use the same homotopies. However, we deal with the class of A-proper maps and hence use generalised degree theory.

So, consider problem (2.1) with the additional conditions:

(H7) $X = Y$, $0 \in (a,b)$,

$A = I$, and for $k \in \mathbb{N}$, $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$,

where B_j are bounded linear maps for $j = 1,2,\ldots,k$ and $1 \leq k \in \mathbb{N}$;

(H8) There exists a smallest positive element $\lambda_0 \in C_A(T) \cap (a,b)$ and this is isolated, such that $N(I - T(\lambda_0)) \subset X_n$ for each $n \in \mathbb{N}$;

(H9) $(I - T(\lambda))\, N(I - T(\lambda_0)) \cap R(I - T(\lambda_0)) = \{0\}$ for $\lambda \neq \lambda_0$ with λ_0 as in (H8) and $I - T(\lambda_0)$ is Fredholm of index zero;

(H10) k is an odd integer and for $k \geq 3$, B_k is injective;

(H11) B_i commutes with B_j $(1 \leq i,j \leq k)$;

(H12) If $(I - T(\lambda_0))x = 0$ for $x \neq 0$, then $(I - T(\mu))x \neq 0$ for all $\mu \neq \lambda_0$, $\mu \in \mathbb{R}$, where λ_0 is as in (H8).

We shall prove that λ_0 is a global bifurcation point of problem (2.1), with the above hypotheses, provided that dim $N(I - T(\lambda_0)$ is odd.

The first step is to generalise the following result due to Krasnosel'skii.

<u>Theorem 3.4</u> (Krasnosel'skii, [13], p. 129).

Suppose X, E_1 and E_2 are Banach spaces such that $X = E_1 \oplus E_2$ with compatible norms. Let $T_j : E_j \to E_j$ be a compact linear operator such that $I - T_j : E_j \to E_j$ is a homeomorphism ($j = 1,2$). If $x = x_1 + x_2$ with $x_j \in E_j (j = 1,2)$ define $Tx = T_1x_1 + T_2x_2$. Then the Leray-Schauder degrees are related by $\deg_{LS}(I - T, B(0,1),0) = \deg_{LS}(I - T_1,B_1(0,1),0)$ $\deg_{LS}(I - T_2,B_2(0,1),0)$ where $B_j(0,1)$ is the open unit ball in E_j ($j = 1,2$).

For the remainder of this section we adopt the following notation.

$X = E_1 \oplus E_2$, where E_1 and E_2 are Banach spaces with compatible norms and dim E_1 is finite. $P : X \to E_1$ is the projection of X onto E_1, so that P is compact; $T : X \to X$ is such that $I - T$ is a linear, A-proper homeomorphism with respect to $\Gamma = \{X_n,Q_n\}$, where $E_1 \subset X_n$ for every $n \in \mathbb{N}$. T_1 and T_2 denote the restrictions of T to E_1 and E_2, respectively, and $I - T_j : E_j \to E_j$ are homeomorphisms. Finally, $B_j(0,1)$ denotes the open unit ball in E_j ($j = 1,2$).

First we show that $I - T_2 : E_2 \to E_2$ is A-proper with respect to some admissible scheme.

<u>Lemma 3.5</u> $I - T_2 : E_2 \to E_2$ is A-proper with respect to the admissible scheme $\Gamma' = \{X_n',Q_n'\}$, where we take projections $Q_n' = (I - P)Q_n$ and subspaces $X_n' = Q_n'(X)$.

Proof: First notice that Γ is admissible for maps from E_2 into E_2. For, $\dim X_n' = \dim(I - P)(Q_n(X)) = \dim(I - P)(X_n) \leq \dim X_n < \infty$ for each $n \in \mathbb{N}$. Q_n' is easily seen to be a continuous projection.

Also $Q_n'x = (I - P)\, Q_n x \to (I - P)\, x = x$ as $n \to \infty$ for each $x \in E_2$, and hence $\mathrm{dist}(x, X_n') \to 0$ as $n \to \infty$.

Thus, Γ' is an admissible scheme for maps from E_2 into E_2.

To show that $I - T_2$ is A-proper with respect to Γ', suppose that $\{x_n : x_n \in X_n'\}$ is a bounded sequence with

$x_n - Q_n'\, T_2 x_n \to w$ as $n \to \infty$ for some $w \in X$.

Now, $x_n \in X_n'$ for each $n \in \mathbb{N}$, so there exist $u \in X$ and $u_n = Q_n u \in X_n$ such that $x_n = Q_n' u = (I - P)\, Q_n u = u_n - Pu_n$. Therefore, $u_n - Pu_n - (I - P)Q_n T_2\, (u_n - Pu_n) \to w$ as $n \to \infty$ and, since P is compact, $PQ_n T_2(u_n - Pu_n) \to p$ (say) and $Pu_n \to v$ (say) as $n \to \infty$. So $u_n - Q_n T\, u_n + Q_n\, T\, P\, u_n \to w + v - p$, where we must replace T_2 by T when we split up $(u_n - Pu_n)$.

Hence $Q_n(I - T)u_n \to w + v - p - Tv$ and by A-properness of $I - T$ with respect to Γ, we may assume that there exists $u \in X$ such that $u_n \to u$ as $n \to \infty$ and $u - Tu = w + v - p - Tv$. Therefore, $Pu = v$ and $x_n = u_n - Pu_n \to u - Pu = u - v = x$ (say) as $n \to \infty$.

So $x = (I - P)\, u \in E_2$ and by invariance of E_2 under T, $Tx = T_2\, x \in E_2$.

Hence, since $x_n - Q_n'\, T_2 x_n \to w$, $x_n - (I - P)Q_n T_2 x_n \to w$ and so $x - (I - P)T_2 x = w$, or, equivalently, $x - T_2 x = w$.

Thus, $I - T_2$ is A-proper with respect to Γ', which completes the proof.

Next we show that the fact that $I - T_2 : E_2 \to E_2$ is A-proper with respect to Γ' implies that $I - T : X \to X$ is necessarily A-proper with respect to an admissible scheme Γ'', which we construct from Γ'. In general Γ'' is different from Γ.

Lemma 3.6 Suppose $I - T_2 : E_2 \to E_2$ is A-proper with respect to an admissible scheme $\Gamma' = \{X_n', Q_n'\}$. For $x = x_1 + x_2$ with $x_j \in E_j$ $(j = 1,2)$, let $Tx = T_1x_1 + T_2x_2$. Then, $I - T : X \to X$ is A-proper with respect to the admissible scheme $\Gamma'' = \{X_n'', Q_n''\}$ where, $X_n'' = E_1 \oplus X_n'$ and $Q_n''(x_1 + x_2) = x_1 + Q_n'x_2$, with $x_1 \in E_1$ and $x_2 \in E_2$.

Proof: First we show that Γ'' is an admissible scheme. For, $\dim X_n'' = \dim(E_1 \oplus X_n') = \dim E_1 + \dim X_n' < \infty$ for each $n \in \mathbb{N}$.

Also for each $x \in X$ and $n \in \mathbb{N}$, $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$, and $Q_n''x = Q_n''(x_1 + x_2) = x_1 + Q_n'x_2 \to x_1 + x_2 = x$ as $n \to \infty$, which implies that $\operatorname{dist}(x, X_n'') \to 0$ as $n \to \infty$ for each $x \in X$, and since Q_n'' is a continuous projection, then Γ'' is admissible.

To prove that $I - T : X \to X$ is A-proper with respect to Γ'', suppose that $\{x_n : x_n \in X_n''\}$ is a bounded sequence with $x_n - Q_n'' Tx_n \to w$. We can write $x_n = e_n + x_n'$, where $e_n \in E_1$ and $x_n' \in X_n'$.

Then, since $\{e_n\}$ is a bounded sequence in a finite dimensional space, we may suppose that $e_n \to e \in E_1$ and $Te_n = T_1e_n \to T_1e$ as $n \to \infty$. Also $e_n + x_n' - Q_n''T(e_n + x_n') \to w$ implies that $x_n' - Q_n'T_2x_n' \to w - e + T_1e$. But $(I - Q_n'T_2)x_n' = Q_n'(I - T_2)x_n'$, so $Q_n'(I - T_2)x_n' \to w - e + T_1e$ as $n \to \infty$.

By the A-properness of $I - T_2$, with respect to Γ', we may assume that there exists $x' \in E_2$ such that $x_n' \to x'$ as $n \to \infty$ and so $x_n = e_n + x_n' \to e + x' = x$ (say) as $n \to \infty$.

Hence, $Q_n''Tx_n \to Tx$ as $n \to \infty$ and $(I - T)x = w$, which proves the Lemma. So we have the following result.

Lemma 3.7 $I - T : X \to Y$ is A-proper with respect to $\Gamma'' = \{X_n'', Q_n''\}$, where $X_n'' = E_1 \oplus (I - P)Q_n(X)$ and $Q_n''(x_1 + x_2) = x_1 + (I - P)Q_n x_2$, where $x_j \in E_j$ $(j = 1,2)$.

Proof: Immediate from Lemmas 3.5 and 3.6.

Before proving our generalised degree multiplication formula we need one more preliminary result.

Lemma 3.8 For all sufficiently large $n \in \mathbb{N}$, $\deg_{LS}(I - Q_n''T, B(0,1), 0) = \deg_{LS}(I - Q_nT, B(0,1), 0)$ and $\deg_{LS}(I - Q_n''T_2, B_2(0,1), 0) = \deg_{LS}(I - Q_nT_2, B_2(0,1), 0)$.

Proof: Notice first, from the proof of Theorem 1.18, that for sufficiently large n, $\deg_{LS}(I - Q_n''T, B(0,1), 0) = \deg(I - Q_n''T, B(0,1) \cap X_n'', 0)$ and $\deg_{LS}(I - Q_nT, B(0,1), 0) = \deg(I - Q_nT, B(0,1) \cap X_n, 0)$ and all degrees are well-defined by virtue of the fact that $I - T$ is a homeomorphism.

Also since $X_n = Q_n(X)$, $X_n' = (I - P)(X_n)$ and

$$X_n'' = E_1 \oplus X_n' = E_1 \oplus (I - P)(X_n),$$

then $X_n'' \subset X_n$ and $I - Q_nT : \overline{B}(0,1) \cap X_n \to X_n$. Therefore, by the excision property for Brouwer degree, cf. Lloyd [16],

$$\deg(I - Q_nT, B(0,1) \cap X_n, 0) = \deg(I - Q_nT, B(0,1) \cap X_n'', 0) = \deg_{LS}(I - Q_nT, B(0,1), 0).$$

Now let $H_n : (\overline{B}(0,1) \cap X_n'') \times [0,1] \to X_n''$ be defined by $H_n(x,t) = x - tQ_nTx - (1 - t)Q_n''Tx$, for each $x \in \overline{B}(0,1) \cap X_n''$ and $t \in [0,1]$.

Note that for each $n \in \mathbb{N}$, $H_n(.,t)$ is of the form identity minus linear compact and so is a valid homotopy for both Brouwer and Leray-Schauder degrees. To apply the homotopy property we must show that, for n sufficiently large, $H_n(x,t) \neq 0$ for all $x \in \partial B(0,1) \cap X_n''$ and $t \in [0,1]$.

Suppose the contrary. Then there exist sequences $\{n\} \subset \mathbb{N}$, $\{x_n\} \subset \partial B(0,1) \cap X_n''$ and $\{t_n\} \subset [0,1]$ with $t_n \to t \in [0,1]$ as $n \to \infty$, and such that $x_n - t_nQ_nTx_n - (1 - t_n)Q_n''Tx_n = 0$ for each n. Writing $x_n = e_n + y_n$ with $e_n \in E_1$ and $y_n \in X_n'$, we have

$e_n + y_n - t_nQ_n(Te_n + Ty_n) - (1 - t_n)Q_n''(Te_n + Ty_n) = 0$ for each n.

Now $Q_n''(Te_n + Ty_n) = Te_n + (I - P)\, Q_nTy_n$ and, since $\{e_n\}$ is bounded in E_1, all the terms t_n, e_n, Te_n and PQ_nTy_n may be assumed to converge. Then $e_n + y_n - t_nQ_n(Te_n + Ty_n) - (1 - t_n)(Te_n + (I - P)Q_nTy_n) = 0$ implies that $y_n - (I - P)Q_nTy_n \to w$ (say) as $n \to \infty$. But $y_n \in X_n' = Q_n'(X) = (I - P)Q_n(X) \subset E_2$, so $y_n = Q_n'y_n$ and, therefore, $Q_n'(I - T)y_n = Q_n'(I - T_2)y_n \to w$ as $n \to \infty$.

By the A-properness of $I - T_2$ with respect to Γ' we may assume that there exists $y \in E_2$ such that $y_n \to y$ as $n \to \infty$ and $(I - T_2)y = w$. So $x_n = e_n + y_n \to e + y = x$ (say) as $n \to \infty$, where $e_n \to e$ (say) $\in E_1$ as $n \to \infty$.

Therefore, $0 = x_n - t_nQ_nTx_n - (1 - t_n)Q_n''Tx_n$

$\to x - tTx - (1 - t)Tx$ as $n \to \infty$, and so $(I - T)x = 0$.

Now since $\|x_n\| = 1$ for each n, then $\|x\| = 1$, and this contradicts the injectiveness of $I - T$. Hence, by the homotopy property for Brouwer degree, for n sufficiently large, we have

$\deg(I - Q_n''T, B(0,1) \cap X_n'', 0) = \deg(I - Q_nT, B(0,1) \cap X_n'', 0)$ and so, by the first part of the proof, $\deg_{LS}(I - Q_n''T, B(0,1), 0)$

$= \deg_{LS}(I - Q_nT, B(0,1), 0)$ which proves the first assertion of the lemma.

The result for T_2 follows by the same procedure, but is simpler since all the components of decompositions are zero outside E_2. Hence the lemma is proved.

We may now prove the generalised multiplication formula.

Theorem 3.9 For T, T_1 and T_2 as defined above,

$\mathrm{Deg}(I - T, B(0,1), 0) = \deg(I - T_1, B_1(0,1), 0)\ \mathrm{Deg}(I - T_2, B_2(0,1), 0)$.

Proof: From Theorem 1.18 it follows that $\mathrm{Deg}(I - T, B(0,1), 0)$

$= \{m \in \mathbb{Z} \cup \{-\infty, \infty\}$: there is a sequence $\{n_j\}$ with

$$\deg_{LS}(I - Q_{n_j}T, B(0,1), 0) = m \text{ as } j \to \infty\}$$

Now from Theorem 3.4 and Lemma 3.8, for each sufficiently large $j \in \mathbb{N}$,

$\deg_{LS}(I - Q_{n_j}T, B(0,1), 0) = \deg_{LS}(I - Q_{n_j}{''}T, B(0,1), 0)$

$= \deg_{LS}(I - Q_{n_j}{''}T_1, B_1(0,1), 0)\ \deg_{LS}(I - Q_{n_j}{''}T_2, B_2(0,1), 0)$

$= \deg_{LS}(I - Q_{n_j}{''}T_1, B_1(0,1), 0)\ \deg_{LS}(I - Q_{n_j}T_2, B_2(0,1), 0)$

But $Q_{n_j}{''}T_1x = T_1x$ for all $x \in E_1$ and for all $j \in \mathbb{N}$, so

$\deg_{LS}(I - Q_{n_j}{''}T_1, B_1(0,1), 0) = \deg(I - T_1, B_1(0,1), 0)$.

Therefore,

$\mathrm{Deg}(I - T, B(0,1), 0) = \deg(I - T_1, B_1(0,1), 0)\ \{m \in \mathbb{Z} \cup \{-\infty, \infty\}$:

there is a sequence $\{n_j\}$ with $\deg_{LS}(I - Q_{n_j}T_2, B_2(0,1), 0) \to m$

as $j \to \infty\}$

$= \deg(I - T_1, B_1(0,1), 0)\ \mathrm{Deg}(I - T_2, B_2(0,1), 0)$,

which is the required result.

Theorem 3.9 provides us with a useful multiplication formula whenever there exists a direct decomposition of the space X into $E_1 \oplus E_2$ with E_1 finite dimensional and E_1 and E_2 are both invariant under T. We shall prove that condition (H9) implies such a decomposition.

Proposition 3.10 We may decompose X into

$$X = N(I - T(\lambda_0)) \oplus R(I - T(\lambda_0))$$
$$= E_1 \oplus E_2 \text{ (say)},$$

where λ_0 is as defined in hypothesis (H8), $E_1 = N(I - T(\lambda_0))$ and $E_2 = R(I - T(\lambda_0))$ with dim E_1 finite and E_2 a closed subspace of X.

Proof: Notice first that, by Theorems 1.11 and 1.12, since $I - T(\lambda_0)$ is A-proper, $E_1 = N(I - T(\lambda_0))$ is finite dimensional and $R(I - T(\lambda_0))$ is closed. (Thus, certainly by Theorem 1.1 there exists a decomposition $X = N(I - T(\lambda_0)) \oplus E_2$ with E_2 a closed subspace. However, in order to apply Theorem 3.9 we need to know that E_2 is invariant under $T(\lambda)$. So we must find E_2 explicitly).

From Theorem 1.9, if $\lambda_1 \varepsilon (a,b)$ with $\lambda_1 \not\in C_A(T)$, then $I - T(\lambda_1)$ is a homeomorphism and so $(I - T(\lambda_1))^{-1}$ exists. We show that $N((I - T(\lambda_0))(I - T(\lambda_1))^{-1}) = (I - T(\lambda_1)N(I - T(\lambda_0))$. Let $x \varepsilon N((I - T(\lambda_0))(I - T(\lambda_1))^{-1})$, then $(I - T(\lambda_0))\, w = 0$, where $w = (I - T(\lambda_1))^{-1}\, x$.

Thus $w = (I - T(\lambda_1))^{-1}\, x \varepsilon N(I - T(\lambda_0)$ and $x \varepsilon (I - T(\lambda_1))N(I - T(\lambda_0))$. Hence $N((I - T(\lambda_0))(I - T(\lambda_1))^{-1}) \subseteq (I - T(\lambda_1))N(I - T(\lambda_0))$. The reverse inclusion follows similarly and so equality holds.

Next we show that $N(((I - T(\lambda_0))(I - T(\lambda_1))^{-1})^2)$ $= N((I - T(\lambda_0))(I - T(\lambda_1))^{-1})$.

Suppose that for $x \varepsilon X, ((I - T(\lambda_0))(I - T(\lambda_1))^{-1})^2 x = 0$, then $(I - T(\lambda_0))(I - T(\lambda_1))^{-1} x = w \varepsilon N((I - T(\lambda_0))(I - T(\lambda_1))^{-1})$ $= (I - T(\lambda_1))N(I - T(\lambda_0))$.

Hence $w \varepsilon (I - T(\lambda_1))N(I - T(\lambda_0)) \cap R(I - T(\lambda_0))$ which implies, by assumption (H9) that $w = 0$.

Thus, $(I - T(\lambda_0))(I - T(\lambda_1))^{-1}x = 0$, and so
$N((I - T(\lambda_0))(I - T(\lambda_1))^{-1})^2 \subseteq N((I - T(\lambda_0))(I - T(\lambda_1))^{-1})$.

The reverse inclusion always holds so we have equality.

Now by the commutativity condition (H11), $(I - T(\lambda_1))^{-1}$ commutes with $I - T(\lambda_0)$. Thus $N(((I - T(\lambda_0))(I - T(\lambda_1))^{-1})^j) = N((I - T(\lambda_0))^j)$, $j = 1,2$, and so $N((I - T(\lambda_0))^2) = N(I - T(\lambda_0))$. This proves that the ascent of $(I - T(\lambda_0))$ is equal to one. Also since $I - T(\lambda_0)$ is Fredholm of index zero by (H9), it follows from Remark (4) preceding Theorem 1.14, that $(I - T(\lambda_0))^2$ is also Fredholm of index zero and has the same null space as $I - T(\lambda_0)$. Hence the codimension of $R((I - T(\lambda_0))^2)$ equals the codimension of $R(I - T(\lambda_0))$ and since $R((I - T(\lambda_0))^2)$ $\subseteq R(I - T(\lambda_0))$ we must have $R(I - T(\lambda_0)) = R((I - T(\lambda_0))^2)$. Thus the ascent and descent of $I - T(\lambda_0)$ are both one and, therefore, by the results of Chapter One,

$X = N(I - T(\lambda_0)) \oplus R(I - T(\lambda_0))$ as required.

We may use similar techniques to Toland [41] to prove the following degree result.

<u>Theorem 3.11</u> Consider problem (2.1) with the additional hypotheses (H7)-(H12). If $\dim\{N(I - T(\lambda_0))\}$ is odd, then there exists $\delta > 0$ such that $\mathrm{Deg}(I - T(\underline{\lambda}),G,0) \neq \mathrm{Deg}(I - T(\overline{\lambda}),G,0)$ for $\lambda_0 - \delta < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \delta$, where G is an arbitrary open bounded set in X containing zero.

<u>Proof</u>: First suppose that $\underline{\lambda} \in (0,\lambda_0)$ and consider $H : \overline{G} \times [0,1] \to X$ defined by $H(x,t) = x - T(t\underline{\lambda})x$.

Then $H(\partial G,t) \neq 0$ for $t \in [0,1]$.

For suppose the contrary, then there exist $x \in \partial G$ and $t \in [0,1]$

such that $x - T(t\underline{\lambda})x = 0$, $\|x\| \neq 0$, which implies that $t\underline{\lambda} \in C_A(T)$. But $0 \leq t\underline{\lambda} < \lambda_0$, and if $t\underline{\lambda} = 0$, then $x = 0$, therefore $0 < t\underline{\lambda} < \lambda_0$. Thus by hypothesis (H8) we must have $x = 0$.

This contradiction proves that $H(\partial G,t) \neq 0$ for all $t \in [0,1]$. Moreover, by hypothesis (H7), $(0,\lambda_0) \subseteq (a,b)$ and so $H(.,t)$ is A-proper for all $t \in [0,1]$ and H is continuous in both arguments. Hence, by (P3.), we have that $\text{Deg}(I - T(\underline{\lambda}),G,0) = \text{Deg}(I,G,0) = \{1\}$.

To prove the theorem we will show that for some $\delta > 0$, $\text{Deg}(I - T(\overline{\lambda}),G,0) \neq \{1\}$ for $\lambda_0 < \overline{\lambda} < \lambda_0 + \delta$.

From assumption (H11) it is easily seen that $T(\lambda)$ commutes with $T(\lambda_0)$ for all $\lambda \in \mathbb{R}$ and, therefore, from Proposition 3.10, E_1 and E_2 are invariant under $I - T(\lambda)$.

Now, since $\lambda_0 \in C_A(T)$ is isolated, we can choose $\delta > 0$ such that $\lambda_0 + \delta > \overline{\lambda} > \lambda_0$, $\overline{\lambda} \in (a,b)$ and $\overline{\lambda}$ is less than any other positive element of $C_A(T)$. So $I - T(\overline{\lambda})$ is a homeomorphism.

Next we use the decomposition $X = N(I - T(\lambda_0)) \oplus R(I - T(\lambda_0)) = E_1 \oplus E_2$ and define a homotopy on $E_2 = R(I - T(\lambda_0))$. Let T_j denote the restriction of T to E_j $(j = 1,2)$, then $I - T_j(\overline{\lambda})$ is a homeomorphism on E_j $(j = 1,2)$. For $x \in \overline{B}_2(0,1) \subset E_2$ (the closed unit ball on E_2), let $H(x,t) = x - T_2(t\overline{\lambda})x$ for $t \in [0,1]$. Then $H(\partial B_2(0,1),t) \neq 0$ for $t \in [0,1]$. For if not, there exist $x \in \partial B_2(0,1)$ and $t \in [0,1]$ such that $H(x,t) = 0 = x - T_2(t\overline{\lambda})x = x - T(t\overline{\lambda})x$, since $x \in \overline{B}_2(0,1) \subset E_2$. This implies that $t\overline{\lambda} = \lambda_0$ and so $x \in N(I - T(\lambda_0)) \cap R(I - T(\lambda_0)) = \{0\}$. Therefore, by the homotopy property (P3.), since $I - T(t\overline{\lambda})$ is A-proper for all $t \in [0,1]$, $\text{Deg}(I - T_2(\overline{\lambda}),B_2(0,1),0) = \text{Deg}(I,B_2(0,1),0) = \{1\}$.

In $E_1 = N(I - T(\lambda_0))$ we use the homotopy

$H(x,t) = (2t - 1)x - \sum_{j=1}^{k} \bar{\lambda}^{j}\, t^{j/k}\, (2t - 1)^{(k-j)/k} B_j x$ for $x \in \bar{B}_1(0,1)$ (the closed unit ball in E_1) and $t \in [0,1]$, which is easily seen to be continuous and well defined since $(-1)^{1/k}$ is a real number for k odd.

Since E_1 is finite dimensional we need only use the Brouwer degree.

As before, $H(x,t) \neq 0$ for all $x \in \partial B_1(0,1)$ and $t \in [0,1]$. For, suppose the contrary, then there is $x \in \partial B_1(0,1)$ and $t \in [0,1]$ such that $H(x,t) = 0$. If $t = ½$, then $½\bar{\lambda}^{k} B_k x = 0$ and, by hypothesis (H10), $x = 0$. Note that if $k = 1$, then $B_1 x = 0$ implies that $\lambda_0 B_1 x = 0$ and, since $x \in N(I - \lambda_0 B)$, in this case we must have $x = 0$. So $t \neq ½$ and $x - T_1(\bar{\lambda} t^{1/k}/(2t-1)^{1/k})x = 0$, which implies that $x - T(\bar{\lambda}(t/2t-1)^{1/k})x = 0$, since $x \in E_1$. Thus, by assumption (H12), $\lambda_0 = \bar{\lambda}(\frac{t}{2t-1})^{1/k}$ unless $x = 0$. However, $(\frac{t}{2t-1})^{1/k}$ lies in the range $(-\infty,0] \cup [1,\infty)$, so this is impossible. Therefore $x = 0$, contradicting the fact that $\|x\| = 1$. Hence H is a valid homotopy and by the homotopy property for Brouwer degree

$$\begin{aligned}\deg(I - T_1(\bar{\lambda}), B_1(0,1), 0) &= \deg(-I, B_1(0,1), 0)\\ &= (-1)^{\dim N(I - T(\lambda_0))}\\ &= -1.\end{aligned}$$

Note, the fact that $\deg(-I, B_1(0,1), 0) = (-1)^{\nu}$, where ν is the dimension of the underlying space (in our case $N(I - T(\lambda_0))$ is a well known result and follows easily from the definition of the Brouwer degree, cf. Lloyd [16]. Hence by Theorem 3.9,

$$\begin{aligned}\mathrm{Deg}(I - T(\bar{\lambda}), B(0,1), 0) &= \deg(I - T_1(\bar{\lambda}), B_1(0,1), 0)\, \mathrm{Deg}(I - T_2(\bar{\lambda}), B_2(0,1), 0)\\ &= \{-1\},\end{aligned}$$

and, therefore, $\mathrm{Deg}(I - T(\bar{\lambda}), G, 0) \neq \mathrm{Deg}(I - T(\underline{\lambda}), G, 0)$ where we have applied (P5.). This is the required result.

Another set of hypotheses is possible when X is a Hilbert space.

Theorem 3.12 Consider problem (2.1) when X is a Hilbert space with additional hypotheses (H7), (H8), (H9) and (H11). If $\dim\{N(I - T(\lambda_0))\}$ is odd and $(B_j x,x) \geq 0$ for all $x \in N(I - T(\lambda_0))$, with $\sum_{j=1}^{k} (B_j x,x) > 0$ for all non zero $x \in N(I - T(\lambda_0))$, then the conclusion of Theorem 3.11 holds.

Proof: The proof is almost identical to Theorem 1.25 of Toland [41], but we give it here for completeness.

Exactly as in the proof of Theorem 3.11 we may show that $\mathrm{Deg}(I - T(\underline{\lambda}),G,0) = \{1\}$ and that, on E_2, $\mathrm{Deg}(I - T_2(\overline{\lambda}),G,0) = \{1\}$.

In $E_1 = N(I - T(\lambda_0))$ we use the homotopy $H : \overline{B_1}(0,1) \times [0,1] \to E_1$ defined by

$H(x,t) = (2t - 1)x - t \sum_{j=1}^{k} \overline{\lambda}^j B_j x$, for each $x \in \overline{B_1}(0,1)$ and $t \in [0,1]$.

We shall prove that H is a valid homotopy for Brouwer degree. First notice that the uniform continuity assumptions on $H(.,t)$ hold. Also, suppose there exist $x \in \partial B_1(0,1)$ and $t \in [0,1]$ such that $H(x,t) = 0$. Then it is easily seen that $t \neq 0$ and $t \neq 1$. If $t = \frac{1}{2}$ we have $\frac{1}{2} \sum_{j=1}^{k} \overline{\lambda}^j (B_j x,x) = 0$ which implies, by the monotonicity assumptions that $x = 0$. Thus $t \neq \frac{1}{2}$, and so $x = \frac{t}{2t-1} \sum_{j=1}^{k} \overline{\lambda}^j B_j x$, or

$$\|x\|^2 = \frac{t}{2t-1} \sum_{j=1}^{k} \overline{\lambda}^j (B_j x,x).$$

But $x \in \partial B_1(0,1) \subset E_1 = N(I - T(\lambda_0))$, therefore

$$x = \sum_{j=1}^{k} \lambda_0^{\,j} B_j x \quad \text{and} \quad \|x\|^2 = \sum_{j=1}^{k} \lambda_0^{\,j} (B_j x,x).$$

Hence $\sum_{j=1}^{k} (\lambda_0^{\,j} - \frac{t}{(2t-1)} \overline{\lambda}^j)(B_j x,x) = 0$; however, $t \in (0,1)\setminus \frac{1}{2}$ implies that $\frac{t}{2t-1} \in (-\infty,0) \cup (1,\infty)$, from which we see that $\lambda_0^{\,j} - \frac{t\overline{\lambda}^j}{2t-1}$ is either negative or greater than λ_0, for each $j \in \mathbb{N}$, which again contradicts the monotonicity assumptions. We have, thus, shown that H is a valid homotopy and by the homotopy proper for Brouwer degree, $\deg(I - T_1(\overline{\lambda}), B_1(0,1), 0) = \deg(-I, B_1(0,1), 0) = -1$, since $\dim E_1$ is an odd number. The result follows exactly as in the proof of Theorem 3.11.

Remarks (1) As previously noted, Theorems 3.11 and 3.12 are similar to Theorems 1.24 and 1.25 of Toland, [41] and exactly the same homotopies are used; however, we obtain a different decomposition of the space by assuming condition (H9), which Toland never considered. We have also replaced the compactness condition on the B_j's by the more general A-properness assumption and extended the multiplication result of Krasnosel'skii, to generalised degree.

(2) We could obtain similar results to Toland [41] by replacing hypothesis (H9) by an assumption that $X = N((I - T(\lambda_0))^p) \oplus R((I - T(\lambda_0))^p)$ for some $p \in \mathbb{N}$ and $\dim\{N(I - T(\lambda_0))^p\}$ is finite. In this case we obtain analogues of Theorems 3.11 and 3.12 replacing the condition that $\dim N(I - T(\lambda_0))$ is odd by the condition that $\dim N(I - T(\lambda_0))^p$ is odd. Notice that if we retain the Fredholm of index zero property of (H9), but replace the transversality assumption by the condition that $I - T(\lambda_0)$ has finite ascent p, then this decomposition of X holds due to the Fredholm of index zero property.

(3) By removing the compactness property on the B_j's we lose the result of Friedman and Shinbrot [9] invoked by Toland [41], which guarantees that the set $C_A(T)$ is a discrete set with no finite limit

points and is bounded away from zero. This result ensures that there is a smallest positive element of $C_A(T)$, which is isolated. Since this fact is crucial to the method, we had to assume that such an element exists. It is not obvious that this assumption is valid. The following example, however, indicates that there are linear operators which satisfy our assumptions, but fall outside that covered by Toland.

Let X be a Banach space and $C : X \to X$ be a compact linear map. Define $T(\lambda) = \lambda C + \lambda^2 C + \lambda^3 I$.

Then $I - T(\lambda) = (1 - \lambda^3) I - \lambda(1 + \lambda)C$

$$= \begin{cases} (1 - \lambda^3)\left(I - \dfrac{\lambda(1 + \lambda)}{1 - \lambda^3} C\right) & , \text{ for } \lambda \neq 1 \\ -2C & , \text{ for } \lambda = 1 \end{cases}$$

$$= \begin{cases} (1 - \lambda^3)\ (I - \mu(\lambda)C) & , \text{ for } \lambda \neq 1 \\ -2C & , \text{ for } \lambda = 1, \end{cases}$$

where $\mu(\lambda) = \dfrac{\lambda(1 + \lambda)}{1 - \lambda^3}$.

Thus, $I - T(\lambda)$ is A-proper for all $\lambda \neq 1$. We suppose that the smallest positive characteristic value of C is $\mu_0 = 6/7$. This corresponds to $\lambda_0 = \frac{1}{2}$. By considering the graph of $\mu(\lambda)$ we see that $\mu(\lambda)$ increases for λ between 0 and 1. Also $\mu(\lambda)$ has a positive maximum of approximately 0.23 for λ in the range $(-\infty,0]$ which occurs between -2 and -3. Furthermore, $\mu(\lambda)$ is always negative for $\lambda > 1$. Hence $\lambda_0 = \frac{1}{2}$ is the smallest positive element in $C_A(T)$ and is isolated since C is compact.

Thus, if R is compact or ball-condensing, or -R is accretive and satisfies a smallness condition, then we can satisfy hypotheses (H1) - (H4) of problem (2.1); furthermore, hypotheses (H7), (H8), (H10) and (H11) are easily seen to hold. We shall give conditions under which (H9) and (H12) also hold. First consider (H12). Suppose, for

$\lambda_0 = \frac{1}{2}$, there is $0 \neq x \in X$ with $(I - T(\lambda_0))x = (I - T(\frac{1}{2}))x = 0$ or, equivalently, $(I - \frac{6}{7}C)x = 0$. Then, in order that $(I - T(\lambda))x = 0$, we must have either, $(I - \mu(\lambda)C)x = 0$ if $\lambda \neq 1$, or $-2Cx = 0$ if $\lambda = 1$. But since $(I - \frac{6}{7}C)x = 0$ implies that $x = \frac{6}{7}Cx$, then equality $(I - \mu(\lambda)C)x = 0$ may be rewritten as $(\frac{6}{7} - \mu(\lambda))Cx = 0$, which gives $\mu(\lambda) = \frac{6}{7}$ and $\lambda = \frac{1}{2}$. Also $-2Cx = 0$ implies that $x = \frac{6}{7}Cx = 0$. Thus, we have shown that (H12) holds. Before imposing a further condition on C to make (H9) true, we observe, from the previous remark (2), that Theorems 3.11 and 3.12 give a global bifurcation result when (H9) is replaced by an assumption that $X = N(I - T(\lambda_0))^p \oplus R(I - T(\lambda_0))^p$, for some $p \in \mathbb{N}$ and $\dim\{N(I - T(\lambda_0))^p\}$ is finite and an odd number. Well in this case we have $\lambda_0 = \frac{1}{2}$ with $I - T(\lambda_0) = I - \frac{6}{7}C$, and the compactness of C ensures that such a $p \in \mathbb{N}$ exists, and we may assume that $\dim\{N(I - \frac{6}{7}C)^p\}$ is an odd number. A special case of this situation is when $p = 1$ and $\dim\{N(I - \frac{6}{7}C)\}$ is an odd number. We now show that the transversality condition H9 holds under this assumption. It is required that

$$(I - \mu(\lambda)C)\, N(I - \tfrac{6}{7}C) \cap R(I - \tfrac{6}{7}C) = \{0\} \text{ for } \lambda \neq \tfrac{1}{2} \text{ and } \lambda \neq 1,$$
$$\text{and } C.N(I - \tfrac{6}{7}C) \cap R(I - \tfrac{6}{7}C) = \{0\} \text{ for } \lambda = 1.$$

The first observation is that when $\lambda \neq \frac{1}{2}$ and $\lambda \neq 1$, then

$$\begin{aligned}(I - \mu(\lambda)C)\, N(I - \tfrac{6}{7}C) &= (I - \tfrac{6}{7}C - (\mu(\lambda) - \tfrac{6}{7})C)\, N(I - \tfrac{6}{7}C) \\ &= (\mu(\lambda) - \tfrac{6}{7})C\, N(I - \tfrac{6}{7}C) \\ &= C\, N(I - \tfrac{6}{7}C), \text{ since } \mu(\lambda) \neq \tfrac{6}{7} \text{ for } \lambda \neq \tfrac{1}{2}.\end{aligned}$$

Thus to verify (H9) we need only show that

$\subset N(I - \frac{6}{7}C) \cap R(I - \frac{6}{7}C) = \{0\}$.

Suppose there are $x, y \in X$ such that $(I - \frac{6}{7}C)x = 0$ and $Cx = (I - \frac{6}{7}C)y$.

Then $x = \frac{6}{7}Cx$ and $\frac{7}{6}x = Cx$. So $(I - \frac{6}{7}C)Cx = \frac{7}{6}(I - \frac{6}{7}C)x = 0$, which implies $Cx \in N(I - \frac{6}{7}C) \cap R(I - \frac{6}{7}C) = \{0\}$.

Hence (H9) holds.

Finally when X is a Hilbert space and $x \in N(I - T(\lambda_0))$, then $(I - \frac{6}{7}C)x = 0$ and $x = \frac{6}{7}Cx$. So $(Cx,x) = (\frac{7}{6}x,x) = \frac{7}{6}\|x\|^2 \geq 0$. Thus the positivity conditions of Theorem 3.12 are also satisfied.

We have shown that the conditions (H1) - (H4) and (H7) - (H12) can be satisfied for the above problem which falls outside the class of problem covered by Toland [41] : in infinite dimensional spaces I is not compact.

From the previous theorems and remarks, we may deduce the following global bifurcation result.

<u>Theorem 3.13</u> Consider problem (2.1) with the additional hypotheses (H7), (H8) and (H11). Then λ_0 is a global bifurcation point of problem (2.1) if at least one of the following hypotheses is satisfied:

(1) Assumptions (H9), (H10) and (H12) hold and $\dim\{N(I - T(\lambda_0))\}$ is an odd number;

(2) Assumptions (H10) and (H12) hold and there exists $p \in \mathbb{N}$ with $\dim\{(N(I - T(\lambda_0))^p)\}$ an odd number and $X = N((I - T(\lambda_0))^p) \oplus R((I - T(\lambda_0))^p)$;

(3) X is a Hilbert space with $(B_j x,x) \geq 0$ for each $j = 1, 2, \ldots, k$ and all $x \in N(I - T(\lambda_0))$; $\sum_{j=1}^{k} (B_j x,x) > 0$ for all $x \in N(I - T(\lambda_0))$ with $x \neq 0$, and, either assumption (H9) holds with $\dim\{N(I - T(\lambda_0))\}$ an odd number, or there exists

$p \in \mathbb{N}$ with $\{\dim N((I - T(\lambda_0))^p)\}$ an odd number and
$X = N((I - T(\lambda_0))^p) \oplus R((I - T(\lambda_0))^p)$.

Proof: Follows from Theorems 2.10, 3.11 and 3.12 and Remark (2) above.

Remarks (1) In Chapter Four, Theorem 4.15 we prove a more general result than 3.13(3). In particular, we replace I by the more general bounded, linear operator A and we do not require that the B_j's commute.

(2) There are some problems which satisfy all the hypotheses of both sections 3.1 and 3.2. By comparing Theorems 3.3 and 3.13 it is easily seen that, when $X = Y$, $A = I$, $T(\lambda) = \lambda B$ with $B : X \to X$ linear and compact; λ_0 is the smallest positive element in $C_A(T)$, $(0,\lambda_0) \subset (a,b)$ and $C_A(T) \neq (a,b)$; then, by either theorem, λ_0 is a global bifurcation point of problem (2.1) if:

(1) $BN(I - \lambda_0 B) \cap R(I - \lambda_0 B) = \{0\}$ and $\dim N(I = \lambda_0 B)$ is an odd number; or,

(2) $\dim\{N((I - \lambda_0 B)^p)\}$ is an odd number, where p is the ascent of $I - \lambda_0 B$.

This follows easily since hypotheses (H5) and (H6) of §3.1 and (H7) - (H12) of §3.2 are all satisfied.

CHAPTER FOUR

GLOBAL BIFURCATION OF FREDHOLM MAPS OF INDEX ZERO

Introduction

In this chapter we derive global bifurcation results for problem (2.1) by decomposing $A - T(\lambda_0)$ into $H - C$, where λ_0 is some isolated element in $C_A(T) \cap (a,b)$, H is a linear homeomorphism, and C is a bounded linear operator. In §4.1 we use this decomposition to transform equation (2.1) from $A - T(\lambda) - R(.,\lambda) : X \to Y$ into $I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1} - R(H^{-1}(.),\lambda) : Y \to Y$ for each $\lambda \varepsilon (a,b)$, where the transformed equation has the same continuity conditions and analogous A-properness conditions to the original. In fact we prove, by a suitable definition of global bifurcation, that a global bifurcation point of the new equation is necessarily a global bifurcation point of the original equation. Then, by exploiting the identity operator, which is present in the new equation, we prove a global bifurcation result via the methods of Chapter Two which consequently holds for the original equation.

In §4.2 we first assume that $A - T(\lambda_0)$ is Fredholm of index zero and that the transversality assumption $(A - T(\lambda))\, N\, (A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$ holds for all $\lambda \neq \lambda_0$. Then, from Theorem 1.14, we deduce that a decomposition of $A - T(\lambda_0) = H - C$ exists with the property that C is linear and compact. The methods of §4.1 are then used to prove a global bifurcation result for problem (2.1) when $T(\lambda) = \lambda B$, where, by making a judicious choice for

H and C, we obtain our result when $\dim\{N(A - \lambda_0 B)\}$ is an odd number. We then extend the method to cover the more general case where $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$ with k finite or infinite; however, we must have $X = Y$ a Hilbert space and impose a positivity condition on $N(A - \lambda_0 B)$. We prove two results here: one when $\lambda_0 > 0$ and k is finite, the other when $\lambda_0 = 0$ and k may be infinite.

In the final section we study the same problem but do not assume that the transversality condition holds. We do not suppose that $A - T(\lambda_0)$ is Fredholm of index zero directly, but, as in §4.1, that $A - T(\lambda_0)$ can be decomposed into $H - C$, where C is a general bounded, linear mapping, not necessarily compact. A sufficient condition for global bifurcation, depending upon the mappings C and H, is then proved. Other additional conditions assumed in this proof imply that $A - T(\lambda_0)$ is, in fact, Fredholm of index zero, and so as before $H - C$ certainly exists and C can be chosen to be compact. Since the results here, however, depend explicitly on C and H we must know what these mappings are. In some cases there may be a decomposition $H - C$ readily available, where C is not compact, with no obvious method of obtaining an explicit alternative decomposition in which C is compact. Of course, the method works equally well if we can find explicitly a decomposition with C compact, provided the other hypotheses are satisfied, and the proof in this case is much simpler than the one we give for general C.

4.1 The general operator decomposition

Assume problem (2.1) holds with the additional hypotheses:

(A5.) For some isolated $\lambda_0 \in C_A(T) \cap (a,b)$, $A - T(\lambda_0)$ can be decomposed into $H - C$ where $H : X \to Y$ is a linear homeomorphism and

$C : X \to Y$ is linear and continuous;

(A6.) For a decomposition, as in (A5.) there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that $A - T(\lambda) - \xi C$ is A-proper with respect to Γ, for all λ and ξ with $|\lambda - \lambda_0| < \tau_1$ and $|\xi| < \tau_2$.

Once again we are seeking sufficient conditions for λ_0, satisfying hypothesis (A5.), to be a global bifurcation point of problem (2.1).

From (A5.) we may rewrite equation (2.1) in the form;

$$
\begin{aligned}
F(x,\lambda) &= Ax - T(\lambda_0)x - (T(\lambda) - T(\lambda_0))x - R(x,\lambda)\\
&= Hx - Cx - (T(\lambda) - T(\lambda_0))x - R(x,\lambda) = 0,
\end{aligned}
$$

where $(x,\lambda) \in X \times \mathbb{R}$.

If we set $y = Hx$, then

$$
F(H^{-1}(y),\lambda) = y - CH^{-1}y - (T(\lambda) - T(\lambda_0))H^{-1}y - R(H^{-1}(y),\lambda) = 0 \qquad (4.1)
$$

where $(y,\lambda) \in Y \times \mathbb{R}$ and $F(H^{-1}(.),.) : Y \times \mathbb{R} \to Y$.

We will show that equation (4.1) may be used to obtain global bifurcation results, for problem (2.1), via the methods of Chapter Two.

Our first result is on the smallness of the non-linearity of equation (4.1).

<u>Proposition 4.1</u> For λ in bounded intervals, $I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1} : Y \to Y$ is the Fréchet derivative of $F(H^{-1}(.),\lambda)$ at the point 0.

<u>Proof</u>: First, $I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}$ is clearly seen to be a linear continuous map. For λ in bounded intervals we obtain from conditions (H3) and (H4) of problem (2.1) that $R(H^{-1}(0),\lambda) = 0$ and, if $y \neq 0$,

$$
\frac{\|R(H^{-1}(y),\lambda)\|}{\|y\|} = \frac{\|R(H^{-1}(y),\lambda)\|}{\|H^{-1}(y)\|}\,\frac{\|H^{-1}(y)\|}{\|y\|}
$$

$$\leq \frac{\| R(H^{-1}(y),\lambda)\|}{\| H^{-1}(y)\|} \quad \| H^{-1}\| \to 0 \text{ as } \| y\| \to 0$$, uniformly for λ in bounded intervals.

Hence, by Definition 1.6 of the Fréchet derivative, the result follows.

The next result tells us about the A-properness of equation (4.1).

Proposition 4.2 $F(H^{-1}(.),\lambda)$ and $I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}$ are A-proper with respect to the admissible scheme $\Gamma_H = \{H(X_n),Y_n,Q_n\}$ for all $\lambda \in (a,b)$. Furthermore, provided $|\xi| \leq \tau_2$,
$I -(1 + \xi)CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}$ is A-proper with respect to Γ_H for all λ such that $|\lambda - \lambda_0| \leq \tau_1$.

Proof: First we show that Γ_H is admissible. Since H is a homeomorphism, $\dim H(X_n) = \dim X_n = \dim Y_n$ for each $n \in \mathbb{N}$. Also for each $y \in Y$, $\mathrm{dist}(y,H(X_n)) = \mathrm{dist}(Hx,H(X_n))$, for some $x \in X$, so $\mathrm{dist}(y,H(X_n)) \leq \| H\| \ \mathrm{dist}(x,X_n) \to 0$ as $n \to \infty$, by admissibility of Γ. Since Γ is admissible, then, $Q_n y \to y$ as $n \to \infty$. So Γ_H is admissible.

To see that $F(H^{-1}(.),\lambda)$ is A-proper with respect to Γ_H for $\lambda \in (a,b)$, let $\{x_{n_j} : x_{n_j} \in H(X_{n_j})\}$ be a bounded sequence such that $Q_{n_j} F(H^{-1}(x_{n_j}),\lambda) \to y$ as $j \to \infty$ for $y \in Y$. Then there exists $z_{n_j} = H^{-1}x_{n_j} \in X_{n_j}$ with $Q_{n_j} F(z_{n_j},\lambda) \to y$ as $j \to \infty$. Since $\{z_{n_j}\}$ is bounded we may assume, by the A-properness of F for $\lambda \in (a,b)$, that there exists $z \in X$ such that $z_{n_j} \to z$ as $j \to \infty$ and $F(z,\lambda) = y$. But $z_{n_j} = H^{-1}x_{n_j} \to z$, so $x_{n_j} \to Hz = x$ (say) as $j \to \infty$.

Hence $z = H^{-1}x$ and $F(H^{-1}(x),\lambda) = y$, which proves that $F(H^{-1}(.),\lambda)$ is A-proper with respect to Γ_H.

The above analysis shows that, if $T : X \to Y$ is A-proper with respect to Γ, then $TH^{-1} : Y \to Y$ is necessarily A-proper with respect to Γ_H. The result for $I - CH^{-1} - (T(\lambda) - T(\lambda_0)H^{-1}$ follows similarly.

Also $I - (1 + \xi)CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1} = (A - T(\lambda) - \xi C)H^{-1}$ and so, by hypotheses (A6.), for all λ such that $|\lambda - \lambda_0| \leq \tau_1$ with $|\xi| \leq \tau_2$, $I - (1 + \xi)CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}$ is A-proper with respect to Γ_H. Hence result of Proposition.

Propositions 4.1 and 4.2 tell us that the structure and A-properness properties, which were used to prove the global bifurcation results of Chapter Two, also hold for equation (4.1). Since, in Chapter Two, we only used the fact that our operators were A-proper with respect to some admissible scheme, then the theorems of Chapter Two apply equally well to equation (4.1). Equivalently, we may regard (4.1) as a particular case of problem (2.1); which is not surprising really, in view of its construction. We, therefore, make similar definitions here to those made in Chapter Two.

Notice, by Proposition 4.1, that the set $\{(0,\lambda) : \lambda \in \mathbb{R}\}$ is a solution set of equation (4.1), which we call the set of trivial solutions, and is equal to the corresponding trivial solution set of equation (2.1).

<u>Definition 4.3</u>

$$C_H(T) = \{\lambda \in \mathbb{R} : N(I - CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}) \neq \{0\}\};$$

$$\hat{S} = \{(y,\lambda) \in Y \times \mathbb{R} : F(H^{-1}(y),\lambda) = 0 \text{ with } \|y\| \neq 0\};$$

$$\hat{S}' = \hat{S} \cup \{(0,\lambda) \in Y \times \mathbb{R} : \lambda \in C_H(T)\}$$

The sets $C_H(T)$, $\hat{S}$ and $\hat{S}'$ are analogous to $C_A(T)$, S and S' of Chapter Two and are related as follows.

<u>Proposition 4.4</u> $\qquad C_A(T) = C_H(T)$

and $S' = \{(H^{-1}(y),\lambda) : (y,\lambda) \in \hat{S}'\}$

<u>Proof:</u> By definition $C_A(T) = \{\lambda \in \mathbb{R} : N(A - T(\lambda)) \neq \{0\}\}$

Now $N(A - T(\lambda))$

$$= N(A - T(\lambda_o) - (T(\lambda) - T(\lambda_o)))$$
$$= N(H - C - (T(\lambda) - T(\lambda_o)))$$
$$= N((I - CH^{-1} - (T(\lambda) - T(\lambda_o))H^{-1})H)$$
$$= H^{-1}\, N(I - CH^{-1} - (T(\lambda) - T(\lambda_o))H^{-1});$$

it follows easily that $C_A(T) = C_H(T)$. The result on S' follows directly from the construction of equation (4.1).

Let us now generalise the concepts of algebraic and geometric multiplicity of Chapter One.

Definition 4.5 For $\lambda \varepsilon C_A(T)$ the algebraic multiplicity, denoted by $M_a(\lambda)$, is given by

$$M_a(\lambda) = \dim\{\bigcup_{n=1}^{\infty} N((I - CH^{-1} - (T(\lambda) - T(\lambda_o))H^{-1})^n)\}$$

Similarly the geometric multiplicity of $\lambda \varepsilon C_A(T)$ is given by $M_g(\lambda) = \dim\{N(I - CH^{-1} - (T(\lambda) - T(\lambda_o))H^{-1})\}$.

A global bifurcation point of equation (4.1) is defined exactly as in Definition 2.7, for equation 2.1, replacing X, $C_A(T)$ and S' by respectively Y, $C_H(T)$ (equal to $C_A(T)$ by Proposition 4.4) and $\hat{S}'$. Then the following is true.

Theorem 4.6 λ_o is a global bifurcation point of problem (2.1) if and only if it is a global bifurcation point of equation (4.1).

Proof: Immediate from Definition 2.7 and Proposition 4.4.

The preceding results enable us to prove the next important theorem.

Theorem 4.7 Consider problem (2.1) with the additional hypotheses (A5.) and (A6.). Then λ_o is a global bifurcation point of problem (2.1) if there exists $\delta > 0$ such that $\mathrm{Deg}(I - CH^{-1} - (T(\overline{\lambda}) - T(\lambda_0))H^{-1}, G, 0)$ $\neq \mathrm{Deg}(I - CH^{-1} - (T(\underline{\lambda}) - T(\lambda_0))H^{-1}, G, 0)$, for $\lambda_0 - \delta < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \delta$, where G is an arbitrary open bounded set in Y containing zero.

<u>Proof</u>: By Propositions 4.1, 4.2 and 4.4 we may regard equation (4.1) as a special case of equation (2.1) and, from Theorem 2.10, obtain a global bifurcation result, at $\lambda_0 \in C_H(T) = C_A(T)$, of equation (4.1), when the above degree property holds. Theorem 4.6 then gives us the required result.

<u>Remark</u> For the rest of this chapter we shall consider additional hypotheses which ensure that the degree result in Theorem 4.7 holds.

<u>4.2 The Transversality Condition</u>

Consider problem (2.1) with the additional hypotheses:

(A7.) $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, where k is a positive integer, or is infinite;

(A8.) there exists an isolated element λ_0 of $C_A(T) \cap (a,b)$, and $A - T(\lambda_0)$ is Fredholm of index zero - where, if $k \neq 1$, either;

(A9.) $\lambda_0 \neq 0$ and $k > 1$ is finite, $X = Y$ is a Hilbert space, $0 \in (a,b)$, and λ_0 is a positive element in $C_A(T) \cap (a,b)$; there exists $\eta > 0$ such that $N(A - T(\lambda_0))$ and $R(A - T(\lambda_0))$ are invariant under $A - T(\lambda)$, whenever $\lambda \in (0, \lambda_0 + \eta) \cap (a,b)$; for all $x \in N(A - T(\lambda_0))$, $(B_i x, x) \geq 0$ $(i = 1, \ldots, k)$ and $\sum_{j=1}^{k} (B_j x, x) > 0$ for all $0 \neq x \in N(A - T(\lambda_0))$; or,

(A10.) $\lambda_0 = 0$ and $k > 1$, possibly infinite, $X = Y$ is a Hilbert space and there exists $\eta > 0$ such that $N(A - T(\lambda_0))$ and

$R(A - T(\lambda_0))$ are invariant under $A - T(\lambda)$, whenever $\lambda \in (0,\eta) \cap (a,b)$; $\{B_j\}$ is a uniformly bounded sequence of bounded linear operators; that is, $\sup\{\| B_j \| : j \in \mathbb{N}\}$ is finite; for all $x \in N(A - T(\lambda))$, $(B_i x,x) \geq 0$ for every $i = 1,2,\ldots,k$, and $(B_1 x,x) > 0$ for all $0 \neq x \in N(A - T(\lambda_0))$.

(A11.) The transversality condition,

$$(A - T(\lambda))\, N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\},$$

holds for all $\lambda \in (a,b)$ with $\lambda \neq \lambda_0$.

When $k = 1$ this condition is equivalent to

$$B\, N(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}.$$

From hypothesis (A8.), Theorem 1.14 tells us that $A - T(\lambda_0)$ can be decomposed into H-C with H a linear homeomorphism and C linear and compact. Hence hypotheses (A5.) and (A6.) of §4.1 are satisfied and the methods of that section may be used here.

The transversality condition (A11.) is a generalisation of assumption (H9.) of §3.2 with the identity I replaced by A. We can prove an analogous result to Proposition 3.10.

Proposition 4.8 (i.) If $k = 1$, then

$$X = N(A - \lambda_0 B) \oplus X_2,$$

$$Y = BN(A - \lambda_0 B) \oplus R(A - \lambda_0 B),$$

where $X_2 = (A - \lambda_1 B)^{-1} R(A - \lambda_0 B)$ for a fixed $\lambda_1 \neq \lambda_0$ with $\lambda_1 \notin C_A(T)$ and $\lambda_1 \in (a,b)$; furthermore, $\dim\{BN(A - \lambda_0 B)\} = \dim N(A - \lambda_0 B)$, which is finite by A-properness; $A(X_2) \subset R(A - \lambda_0 B)$, $B(X_2) \subset R(A - \lambda_0 B)$ and $(A - \mu B)X_2 \subset R(A - \lambda_0 B)$ for all $\mu \in (a,b)$.

(ii.) If $k > 1$, then $X = Y = N(A - T(\lambda_0)) \oplus R(A - T(\lambda_0))$,

and $\sum_{j=1}^{k} (\lambda^j - \lambda_0^{\,j}) B_j\, N(A - T(\lambda_0)) \subset N(A - T(\lambda_0))$, for all $\lambda \in (0,\lambda_0 + \eta) \cap (a,b)$.

Proof: (i.) As in the proof of Proposition 3.10, replacing I by A and $T(\lambda)$ by λB, we may show that

$N((A - \lambda_0 B)(A - \lambda_1 B)^{-1})^2 = N((A - \lambda_0 B)(A - \lambda_1 B)^{-1})$. Now since $A - \lambda_1 B$ is a homeomorphism it is Fredholm of index zero and, by Remark (4) following Definition 1.13 and hypothesis (A8.), it follows that $(A - \lambda_0 B)(A - \lambda_1 B)^{-1}$ is Fredholm of index zero. Hence, as in the proof of Proposition 3.10, this implies that the ascent and descent of $(A - \lambda_0 B)(A - \lambda_1 B)^{-1}$ are both equal to one, so

$$\begin{aligned} Y &= N((A - \lambda_0 B)(A - \lambda_1 B)^{-1}) \oplus R((A - \lambda_0 B)(A - \lambda_1 B)^{-1}) \\ &= (A - \lambda_1 B)\, N(A - \lambda_0 B) \oplus R(A - \lambda_0 B) \\ &= (A - \lambda_0 B - (\lambda_1 - \lambda_0)B)\, N(A - \lambda_0 B) \oplus R(A - \lambda_0 B) \\ &= BN(A - \lambda_0 B) \oplus R(A - \lambda_0 B), \end{aligned}$$

and

$$\begin{aligned} X &= N(A - \lambda_0 B) \oplus (A - \lambda_1 B)^{-1} R(A - \lambda_0 B) \\ &= N(A - \lambda_0 B) \oplus X_2, \end{aligned}$$

where $X_2 = (A - \lambda_1 B)^{-1} R(A - \lambda_0 B)$.

Since $A - \lambda_1 B$ is a homeomorphism and $\dim\{N(A - \lambda_0 B)\}$ is finite by A-properness of $A - \lambda_0 B$, then

$$\begin{aligned} \dim\{N(A - \lambda_0 B)\} &= \dim\{(A - \lambda_1 B)\, N(A - \lambda_0 B)\} \\ &= \dim\{B\, N(A - \lambda_0 B)\}. \end{aligned}$$

Next we prove that $(A - \mu B) X_2 \subset R(A - \lambda_0 B)$ for all $\mu \in (a,b)$.

We have that $(A - \lambda_1 B)X_2 = R(A - \lambda_0 B)$. Let x_2 be an arbitrary element of X_2. Then, there exists $x \in X$ such that

$$(A - \lambda_1 B)x_2 = (A - \lambda_0 B)x, \text{ therefore}$$

$$(A - \lambda_0 B - (\lambda_1 - \lambda_0)B)x_2 = (A - \lambda_0 B)x.$$

So, $-(\lambda_1 - \lambda_0)Bx_2 = (A - \lambda_0 B)(x - x_2)$ and $Bx_2 = (A - \lambda_0 B)(x - x_2)/(\lambda_0 - \lambda_1)$. Thus, $BX_2 \subset R(A - \lambda_0 B)$. Also

$Ax_2 = \lambda_1 Bx_2 + (A - \lambda_0 B)x$, so $AX_2 \subset R(A - \lambda_0 B)$, which implies that $(A - \mu B)\, X_2 \subset R(A - \lambda_0 B)$ for all $\mu \in (a,b)$.

(ii.) By the same procedure as in (i.) we may prove that

$X = Y = (A - T(\lambda_1))\, N\, (A - T(\lambda_0)) \oplus R(A - T(\lambda_0))$, for some fixed $\lambda_1 \in (a,b) \cap (0, \lambda_0 + \eta)$ with $\lambda_1 \notin C_A(T)$ (η as defined in (A9.) or (A10.)). But from (A9.) or (A10.), $N\, (A - T(\lambda_0))$ is invariant under $(A - T(\lambda_1))$ and by A-properness, $N(A - T(\lambda_0))$ is finite dimensional. Hence $(A - T(\lambda_1))\, N\, (A - T(\lambda_0)) = N(A - T(\lambda_0))$ and, therefore, $X = Y = N(A - T(\lambda_0)) \oplus R(A - T(\lambda_0))$.

Finally notice that

$$\begin{aligned}
(A - T(\lambda))\, N\, (A - T(\lambda_0)) &= (A - T(\lambda_0) - (T(\lambda) - T(\lambda_0)))\, N\, (A - T(\lambda_0)) \\
&= -\,(T(\lambda) - T(\lambda_0))\, N\, (A - T(\lambda_0)) \\
&= \sum_{j=1}^{k} (\lambda^j - \lambda_0{}^j) B_j\, N\, (A - T(\lambda_0)) \\
&\subset N\, (A - T(\lambda_0)), \text{ for all } \lambda \in (a,b) \cap (0, \lambda_0 + \eta),
\end{aligned}$$

by hypotheses (A9.) or (A10.)

Remark If $A - T(\lambda)$ and $A - T(\lambda_0)$ commute for every $\lambda \in (a,b)$, then it is easily seen that the decomposition in Proposition 4.8 (ii.) holds.

We shall now choose H and C in a particular way, which will reduce the algebraic multiplicity $M_a(\lambda_0)$ to the geometric multiplicity $M_g(\lambda_0)$.

Proposition 4.9 $A - T(\lambda_0)$ may be decomposed into $H - C$, where:

(i.) if $k = 1$, $C : X \to BN(A - \lambda_0 B)$ is defined by $Cx = C(x_1 + x_2) = Bx_1$, with $x_1 \in N(A - \lambda_0 B)$ and $x_2 \in X_2$;

(ii.) if $k > 1$, $C : X \to N(A - T(\lambda_0))$ is defined by $Cx = C(x_1 + x_2) = -(A - T(\lambda))x_1$, with

$x_1 \in N(A - T(\lambda_0))$, $x_2 \in R(A - T(\lambda_0))$, where λ is fixed, $\lambda \in (\lambda_0, \lambda_0 + \eta) \cap (a,b)$ and $0 < \lambda - \lambda_0 < \mathrm{dist}(\lambda_0, C_A(T)\setminus\{\lambda_0\})$. In (i.) and (ii.), H is defined by $Hx = (A - T(\lambda_0) + C)x$, for each $x \in X$. Then, in both cases, C is compact and H is a homeomorphism.

Proof: Note that, since $|\lambda - \lambda_0| < \eta$ and $\lambda \in (a,b)$, then (A9.) or (A10.) imply that C in (ii.) maps X into $N(A - T(\lambda_0))$. Decompositions $x = x_1 + x_2$ in (i.) and (ii.) are guaranteed by Proposition 4.8. Since $BN(A - \lambda_0 B)$ in (i.), and $N(A - T(\lambda_0))$ in (ii.), are both finite dimensional, then C is compact. H is, therefore, A-proper, and by Theorem 1.9 we need only show that H is injective to complete the proof.

Suppose $Hx = 0$, then $(A - T(\lambda_0) + C)x = 0$, which implies that $(A - T(\lambda_0))x = -Cx$. Thus:

(i.) When $k = 1$, $x = x_1 + x_2$, where $x_1 \in N(A - \lambda_0 B)$, $x_2 \in X_2$ and $(A - \lambda_0 B)x_2 = -Bx_1$, so $Bx_1 \in BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$ by (A12.). Since $(A - \lambda_0 B)x_1 = 0$, it follows that $Ax_1 = 0$ and therefore $(A - \lambda B)x_1 = 0$ for any $\lambda \notin C_A(T)$. This implies that $x_1 = 0$. Also $x_2 \in N(A - \lambda_0 B) \cap X_2 = \{0\}$, by Proposition 4.8, therefore $x = x_1 + x_2 = 0$ and H is injective.

(ii.) When $k > 1$, $x = x_1 + x_2$, where $x_1 \in N(A - T(\lambda_0))$, $x_2 \in R(A - T(\lambda_0))$ and $(A - T(\lambda_0))x_2 = (A - T(\lambda))x_1$, so $(A - T(\lambda))x_1 \in (A - T(\lambda))\, N\, (A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$, by (A11.). Hence, $x_1 = 0$, and $x_2 \in N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$ by Proposition 4.8. Thus, $x = x_1 + x_2 = 0$ and H is injective. This completes the proof.

Proposition 4.10 The algebraic multiplicity $M_a(\lambda_0)$ is independent of H and C and is given by the finite number

$$M_a(\lambda_0) = \dim\{N(A - T(\lambda_0))\}$$

(which equals $\dim\{N(I - CH^{-1})\}$), the geometric multiplicity of λ_0.

Proof: Since C maps X into the complement of $R(A - T(\lambda_0))$, then

$$(A - T(\lambda_0) + C)N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}.$$

However, $A - T(\lambda_0) + C = H$, therefore,

$$\begin{aligned} H\, N(A - T(\lambda_0)) &= N((A - T(\lambda_0))H^{-1}) \\ &= N((H - C)H^{-1}) \\ &= N(I - CH^{-1}) \\ \text{and } R(A - T(\lambda_0)) &= R(I - CH^{-1}) \end{aligned}$$

Hence $N(I - CH^{-1}) \cap R(I - CH^{-1}) = \{0\}$, which implies that $N(I - CH^{-1}) = N(I - CH^{-1})^2$. For if $(I - CH^{-1})^2 y = 0$, then $(I - CH^{-1})y \in R(I - CH^{-1}) \cap N(I - CH^{-1}) = \{0\}$.

Hence $N((I - CH^{-1})^2) \subseteq N(I - CH^{-1})$ and, since the reverse inclusion is always valid, equality holds. Thus $M_a(\lambda_0) = \dim\{N(I - CH^{-1})\} = \dim\{N(A - T(\lambda_0))\} = M_g(\lambda_0)$.

We now prove one of the main results in this section.

Theorem 4.11 Consider problem (2.1) with the additional hypotheses (A7.), (A8.) and (A11.) with k=1. Suppose that $\dim\{N(A - \lambda_0 B)\}$ is an odd number. Define

$$\delta = \min\{1, \operatorname{dist}(\lambda_0, C_A(\lambda_0 B)\setminus\{\lambda_0\}), \lambda_0 - a, b - \lambda_0\}$$

Then

$$\operatorname{Deg}(I - CH^{-1} - (\underline{\lambda} - \lambda_0)BH^{-1}, G, 0) \neq \operatorname{Deg}(I - CH^{-1} - (\overline{\lambda} - \lambda_0)BH^{-1}, G, 0)$$

for $\lambda_0 - \delta < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \lambda_0 + \delta$, where G is an arbitrary open, bounded set in Y containing zero, with C and H as defined in Proposition 4.9 (i.).

Proof: First we prove that

$$\deg_{LS}(I - \underline{t}\, CH^{-1}, G, 0) = -\deg_{LS}(I - \overline{t}\, CH^{-1}, G, 0) \tag{4.2}$$

for $0 \leq 1 - \delta < \underline{t} < 1 < \overline{t} < 1 + \delta$.

We may apply the Leray-Schauder Formula, cf. §1.4 preceding Definition 1.17, provided that $\underline{t}$ and $\overline{t}$ are not characteristic values of CH^{-1}. Note that, since $\lambda_0 \in C_A(T)$, then 1 is a characteristic value of CH^{-1}; we shall prove that 1 is the only characteristic value of CH^{-1}. Suppose, for some $t \neq 1$, there exists $y \in Y$ with $\| y \| = 1$ such that $y - tCH^{-1}y = 0$, then, $y - CH^{-1}y - (t - 1)CH^{-1}y = 0$, so $(A - \lambda_0 B - (t - 1)C)H^{-1}y = 0$.

Let $H^{-1}y = w = w_1 + w_2$, where $w_1 \in N(A - \lambda_0 B)$ and $w_2 \in X_2$. Then

$$(A - \lambda_0 B)w_2 = (t - 1)C(w_1 + w_2) = (t - 1)\,Bw_1.$$

However, as we noted in the proof of Proposition 4.9, $(A - \lambda B)N(A - \lambda_0 B) = BN(A - \lambda_0 B)$, so by hypothesis (A11.),

$$(A - \lambda_0 B)w_2 \in BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}.$$

Thus $w_2 \in N(A - \lambda_0 B) \cap X_2 = \{0\}$ by Proposition 4.8 (i.), implying that $(t - 1)\,Bw_1 = 0$. Hence $Bw_1 = 0$, and, since $Aw_1 = \lambda_0 Bw_1 = 0$, then $(A - \lambda B)w_1 = 0$ for $\lambda \neq \lambda_0$ with $\lambda \notin C_A(\lambda B)$ and so $w_1 = 0$. Thus $w = 0$ and, therefore, $y = 0$, which contradicts $\| y \| = 1$. We have shown that 1 is the only element in $ch(CH^{-1})$ and so by the Leray-Schauder Formula

$$\deg_{LS}(I - \underline{t}\,CH^{-1},G,0) = (-1)^0 = 1$$

and $\deg_{LS}(I - \overline{t}\,CH^{-1},G,0) = (-1)^{\dim \{\bigcup_{n=1}^{\infty} N((I - CH^{-1})^n)\}}$

But from the proof of Proposition 4.10, $\dim \{\bigcup_{n=1}^{\infty} N((I - CH^{-1})^n)\}$ $= \dim\{N(A - \lambda_0 B)\}$. Hence

$$\deg_{LS}(I - \overline{t}\,CH^{-1},G,0) = -1^{\dim\{N(A - \lambda_0 B)\}} = -1.$$

So equation (4.2) holds.

To complete the proof we use a homotopy argument. Define the homotopy $H : \overline{G} \times [0,1] \to Y$ by

$$H(y,s) = y - s\underline{t}\, CH^{-1}y - (1 - s)CH^{-1}y - (1 - s)(\underline{t} - 1)BH^{-1}y,$$

for each $(y,s) \in \overline{G} \times [0,1]$.

Let us rewrite this in the form

$$H(y,s) = y - (1 + s(\underline{t} - 1))CH^{-1}y - (1 - s)(\underline{t} - 1)BH^{-1}y.$$

Now, since $|s(\underline{t} - 1)| < \delta \leq \tau_2$ for all $s \in [0,1]$ and $|(1 - s)(\underline{t} - 1)| < \delta \leq \tau_1$, then by Proposition 4.2, $H(.,s)$ is A-proper with respect to Γ_H for all $s \in [0,1]$. Clearly $H(.,s)$ is continuous, uniformly on closed, bounded subsets of Y. We shall prove that $H(\partial G,s) \neq 0$ for each $s \in [0,1]$. Suppose the contrary, then there is $y \in \partial G$ and $s \in [0,1]$ such that
$y - (1 + s(\underline{t} - 1))CH^{-1}y - (1 - s)(\underline{t} - 1)BH^{-1}y = 0$, $y \neq 0$. If $s = 0$, this implies that $\underline{t} - 1 \in C_A(T)$, which is impossible since $|(\underline{t} - 1)| < \delta \leq \mathrm{dist}(\lambda_0, C_A(T)\setminus\{\lambda_0\})$. Also if $s = 1$, then $\underline{t} \in ch(CH^{-1})$ and by the first part of the proof this implies that $\underline{t} = 1$ which is a contradiction. Thus $s \neq 0$ and $s \neq 1$. We may rewrite the above equation in the form

$$[H - C - (1 - s)(\underline{t} - 1)B - s(\underline{t} - 1)C]H^{-1}y = 0.$$

Setting $H - C = A - \lambda_0 B$ and $H^{-1}y = x = x_1 + x_2$, where $x_1 \in N(A - \lambda_0 B)$ and $x_2 \in X_2$, we have

$$[A - \lambda_0 B - (1 - s)(\underline{t} - 1)B - s(\underline{t} - 1)C](x_1 + x_2) = 0.$$

Therefore, replacing C, as in Proposition 4.9 (i.), we find that

$$(A - (\lambda_0 + (1 - s)(\underline{t} - 1))B)x_2 = (1 - s)(\underline{t} - 1)Bx_1 + s(\underline{t} - 1)Bx_1$$
$$= (\underline{t} - 1)Bx_1$$

Now since $0 < |(1 - s)(\underline{t} - 1)| < \delta \leq \min\{\lambda_0 - a, b - \lambda_0\}$, for $s \in (0,1)$, it follows from Proposition 4.8, that $A - (\lambda_0 + (1 - s)(\underline{t} - 1))Bx_2 \in R(A - \lambda_0 B)$. But $(\underline{t} - 1)Bx_1 \in BN(A - \lambda_0 B)$. Hence by assumption (A8.)

$$(A - (\lambda_0 + (1 - s)(\underline{t} - 1)B)x_2 = (\underline{t} - 1)Bx_1 = 0.$$

This implies that $x_2 = 0$ and $Bx_1 = 0$; however, $(A - \lambda_0 B)x_1 = 0$, so $Ax_1 = \lambda_0 Bx_1 = 0$. Hence $(A - \lambda B)x_1 = 0$, for an arbitrary $\lambda \in (a,b)$ with $\lambda \notin C_A(T)$, and so $x_1 = 0$. Thus $x = 0$ and therefore $y = 0$. This contradiction tells us that $H(\partial G,s) \neq 0$ for all $s \in [0,1]$. Hence by the homotopy property (P3.),

$$\mathrm{Deg}(I - \underline{t}\, CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - (\underline{t} - 1)BH^{-1},G,0).$$

Using the homotopy

$$\hat{H}(x,s) = I - s\bar{t}\, CH^{-1} - (1 - s)CH^{-1} - (1 - s)(\bar{t} - 1)BH^{-1}$$

we may prove in the same way that

$$\mathrm{Deg}(I - \bar{t}\, CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - (\bar{t} - 1)BH^{-1},G,0).$$

The result follows easily from equation (4.2) recalling, cf. Theorem 1.18, that

$$\mathrm{Deg}(I - t\, CH^{-1},G,0) = \{\deg_{LS}(I - t\, CH^{-1},G,0)\}$$

and replacing $\underline{t} - 1$ and $\bar{t} - 1$ by, respectively, $\underline{\lambda} - \lambda_0$ and $\bar{\lambda} - \lambda_0$.

We have the corresponding global bifurcation result.

<u>Theorem 4.12</u> Consider problem (2.1) with the additional hypotheses (A7.), (A8.) and (A11.) with $k = 1$. Suppose that $\dim\{N(A - \lambda_0 B)\}$ is an odd number. Then λ_0 is a global bifurcation point of problem (2.1).

<u>Proof</u>: Immediate from Theorems 4.7 and 4.11.

<u>Remark</u> Theorem 4.12 generalises Theorem 3.3 (1) to the case when B is not necessarily compact. Here we do not need to assume that $\mathrm{Deg}(A - \lambda B, G, 0)$ is a singleton for any $\lambda \in (a,b)$.

When $\lambda_0 = 0$ and $X = Y$ is a Hilbert space we can generalise Theorem 4.12 to the more general case, where $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, with k finite or infinite: we prove this result in Theorem 4.18.

In order to extend our results, when $\lambda_0 \neq 0$, to the more general situation $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, where $k > 1$ and finite, we require a Lemma.

<u>Lemma 4.13</u> Let $K_1 > 0$ and $K_2 > 0$ be two constants such that, for λ_0 satisfying hypotheses (A7.), (A8.) and (A9.),

$\|(A - T(\lambda_0))x\| \geq K_1$, for all $x \in R(A - T(\lambda_0))$ with $\|x\| = 1$, and $A - T(\lambda_0) + L$ is A-proper with respect to Γ for all bounded linear operators $L : X \to X$ with

$$\|L\| < K_2.$$

Note that K_2 is guaranteed by Theorem 1.16.

Then there exists $\delta_0 > 0$ such that $\sum_{j=1}^{k} \lambda_0^j |\lambda^j - 1| \, \|B_j\| < \min\{\lambda_0, K_1, K_2\}$, whenever $|\lambda - 1| \leq \delta_0$, with $\lambda > 0$ fixed.

Proof: First, we prove that K_1 exists. Suppose not, then there exists a sequence $\{x_n\}$ in $R(A - T(\lambda_0))$ with $\|x_n\| = 1$ for each $n \in \mathbb{N}$ such that $\|(A - T(\lambda_0))x_n\| < \frac{1}{n}$. So $(A - T(\lambda_0))x_n \to 0$ as $n \to \infty$.

By Theorem 1.10, $\{x_n\}$ is compact, therefore we may assume without loss of generality that there exists $x \in X$ with $\|x\| = 1$ such that $x_n \to x$ and $(A - T(\lambda_0))x = 0$. Hence, $x \in N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$ by Proposition 4.8 (ii.). This contradiction implies that K_1 exists.

Next we prove the inequality. If $\lambda = 1$ the inequality is trivial. When $0 < |\lambda| < 1$ or $|\lambda| > 1$, $\max\{|\lambda^j - 1| : 1 \leq j \leq k\} = |\lambda^k - 1|$ so

$$\sum_{j=1}^{k} \lambda_0^{\,j} |\lambda^j - 1| \, \|B_j\| \leq |\lambda^k - 1| \max\{\|B_j\| : 1 \leq j \leq k\} \sum_{j=1}^{k} \lambda_0^{\,j}$$

$$= \begin{cases} |\lambda^k - 1| \max\{\|B_j\| : 1 \leq j \leq k\} \dfrac{\lambda_0(1 - \lambda_0^{\,k})}{1 - \lambda_0}, & \text{if } \lambda_0 \neq 1 \\ k|\lambda^k - 1| \max\{\|B_j\| : 1 \leq j \leq k\}, & \text{if } \lambda_0 = 1 \end{cases}$$

$$\leq K|\lambda^k - 1|,$$

where K is a finite positive constant. The result follows easily.

We can now prove an important degree result.

<u>Theorem 4.14</u> Consider problem (2.1) with the additional hypotheses (A7.) - (A9.) and (A11.). Suppose that $\dim\{N(A - T(\lambda_0))\}$ is an odd number.

Define

$$\delta = \min\{(\lambda_0 - a), (b - \lambda_0), \delta_0, \lambda_0, \operatorname{dist}(\lambda_0, C_A(T)\backslash\{\lambda_0\}), \eta\}$$

where δ_0 is as defined in Lemma 4.13. Then

$$\operatorname{Deg}(I - CH^{-1} - (T(\underline{\lambda}) - T(\lambda_0))H^{-1}, G, 0)$$

$$\neq \operatorname{Deg}(I - CH^{-1} - (T(\overline{\lambda}) - T(\lambda_0))H^{-1}, G, 0)$$

for $\lambda_0 - \delta < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \delta + \lambda_0$, where G is an arbitrary bounded open set in X containing zero and C is the compact map defined in Proposition 4.9 (ii.) by

$$\begin{aligned} C(x_1 + x_2) &= -(A - T(\lambda))x_1 \\ &= -(A - T(\lambda_0) - (T(\lambda) - T(\lambda_0)))x_1 \\ &= (T(\lambda) - T(\lambda_0))x_1 \\ &= \sum_{j=1}^{k} (\lambda^j - \lambda_0^{\,j})B_j x_1, \end{aligned}$$

for $x_1 \in N(A - T(\lambda_0))$ and $x_2 \in R(A - T(\lambda_0))$, where λ is an arbitrary fixed number such that $0 < \lambda - \lambda_0 < \delta$, and $H = A - T(\lambda_0) + C$.

Proof: In a similar manner to the proof of Theorem 4.11, we first prove that

$$\deg_{LS}(I - \underline{t}\, CH^{-1}, G, 0) = -\deg_{LS}(I - \overline{t}\, CH^{-1}, G, 0)$$

$$\text{for } 0 \leq 1 - \delta/\lambda_0 < \underline{t} < 1 < \overline{t} < 1 + \delta/\lambda_0 \leq 2.$$

Suppose for some $t \neq 1$, there exists $y \in X$ with $\| y \| = 1$ such that $y - t\, CH^{-1}y = 0$.

Then,

$$y - CH^{-1}y - (t - 1)CH^{-1}y = 0,$$

or $\quad [A - T(\lambda_0) - (t - 1)C]H^{-1}y = 0.$

Let $H^{-1}y = w = w_1 + w_2$, where $w_1 \in N(A - T(\lambda_0))$ and $w_2 \in R(A - T(\lambda_0))$. This decomposition is guaranteed by Proposition 4.8 (ii.). Then, from the definition of C,

$$\begin{aligned} (A - T(\lambda_0))w_2 &= (t - 1)\, C\, (w_1 + w_2) \\ &= -(t - 1)(A - T(\lambda))w_1 \\ &= (t - 1)\sum_{j=1}^{k} (\lambda^j - \lambda_0^{\,j})B_j w_1 \end{aligned}$$

But from Proposition 4.8 (ii.) and our choice of λ;

$(t - 1)\sum_{j=1}^{k}(\lambda^j - \lambda_0{}^j)B_j w_1 \subset N(A - T(\lambda_0))$. Thus, again by Proposition 4.8 (ii.),

$(A - T(\lambda_0))w_2 \in N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$, and so,

$w_2 \in N(A - T(\lambda_0)) \cap R(A - T(\lambda_0)) = \{0\}$.

Also, $-(t - 1)(A - T(\lambda))w_1 = 0$, which implies that $(A - T(\lambda))w_1 = 0$, and by our choice of λ we must have $w_1 = 0$.

Hence $w = 0$ and therefore $y = 0$, which contradicts $\|y\| = 1$. We have thus shown that 1 is the only characteristic value of CH^{-1}. So as in the proof of Theorem 4.11 the Leray-Schauder degrees are related as required.

To complete the proof we use the homotopy property (P3.).

Let $H : \overline{G} \times [0,1] \to X$ be defined by

$$H(y,s) = y - s\overline{t}\, CH^{-1}y - (1 - s)CH^{-1}y - \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\overline{t}^j - 1)B_jH^{-1}y,$$

where $\overline{t}$ is arbitrary, but fixed, such that $1 < \overline{t} < \delta/\lambda_0 + 1$.

We may rewrite $H(y,s)$ as

$$H(y,s) = y - CH^{-1}y - \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\overline{t}^j - 1)B_jH^{-1}y - s(\overline{t} - 1)CH^{-1}y$$

and since C is compact we need only prove that

$I - CH^{-1} - \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\overline{t}^j - 1)BH^{-1}$ is A-proper with respect to Γ_H.

But we can write this operator as

$$\begin{aligned}&(H - C - \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\overline{t}^j - 1)B_j)H^{-1}\\ &= (A - \sum_{j=1}^{k}\lambda_0{}^jB_j - \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\overline{t}^j - 1)B_j)H^{-1}.\end{aligned}$$

Now $\|\sum_{j=1}^{k}((1-s)\lambda_0)^j(\bar{t}^j - 1)B_j\|$

$$\leq \sum_{j=1}^{k} \lambda_0{}^j(\bar{t}^j - 1)\|B_j\|$$

$< K_2$, by Lemma 4.13 and our choice of δ.

Hence by Theorem 1.16 and the proof of Proposition 4.2 it follows that $H(.,s)$ is A-proper with respect to Γ_H for all $s \in [0,1]$. We show that $H(\partial G,s) \neq 0$ for $s \in [0,1]$ by a contradiction argument. Indeed, suppose $H(y,s) = 0$ for some $y \in \partial G$ and some $s \in [0,1]$.

First we prove that $s \neq 0$ and $s \neq 1$. If $s = 0$ then $\bar{t}\lambda_0 \in C_A(T)$ which is a contradiction, since $(\bar{t}\lambda_0 - \lambda_0) = \lambda_0(\bar{t} - 1) < \frac{\lambda_0\delta}{\lambda_0} = \delta$ $\leq \text{dist}(\lambda_0, C_A(T)\setminus\{\lambda_0\})$. So $s \neq 0$.

If $s = 1$, then $\bar{t}$ is a characteristic value of CH^{-1}, which is another contradiction, by the argument above. Thus $s \in (0,1)$. Now we have that,

$$H(y,s) = (H - C - s(\bar{t} - 1)C - \sum_{j=1}^{k}((1-s)\lambda_0)^j(\bar{t}^j - 1)B_j)H^{-1}y = 0$$

Setting $H^{-1}y = x = x_1 + x_2$, with $x_1 \in N(A - \sum_{j=1}^{k}\lambda_0{}^jB_j)$ and $x_2 \in R(A - \sum_{j=1}^{k}\lambda_0{}^jB_j)$ we obtain,

$$(A - \sum_{j=1}^{k}\lambda_0{}^jB_j - \sum_{j=1}^{k}((1-s)\lambda_0)^j(\bar{t}^j - 1)B_j - s(\bar{t} - 1)C)(x_1 + x_2) = 0$$

$$\text{So } (A - \sum_{j=1}^{k}\lambda_0{}^jB_j)x_2 + (A - \sum_{j=1}^{k}((1-s)\bar{t}\lambda_0)^jB_j)x_2$$

$$- (A - \sum_{j=1}^{k}((1-s)\lambda_0)^jB_j)x_2$$

$$= \sum_{j=1}^{k}(((1-s)\bar{t}\,\lambda_0)^j - \lambda_0{}^j)B_jx_1$$

$$- \sum_{j=1}^{k}(((1-s)\lambda_0)^j - \lambda_0{}^j)B_jx_1 + s(\bar{t} - 1)\sum_{j=1}^{k}(\lambda^j - \lambda_0{}^j)B_jx_1, \qquad (4.4)$$

using the definition of C.

Since $(1 - s)\,\bar{t}\,\lambda_0$ and $(1 - s)\,\lambda_0$ both belong to the interval $(0, \lambda_0 + n)$, it follows by (A9.) and Proposition 4.8 (ii.) that the left hand side of equation (4.4) belongs to $R(A - T(\lambda_0))$, while the right hand side belongs to $N(A - T(\lambda_0))$. Hence both sides must equal zero by Proposition 4.8. Thus

$$(A - \sum_{j=1}^{k} \lambda_0{}^j B_j)x_2 - \sum_{j=1}^{k} ((1 - s)\lambda_0)^j(\bar{t}^j - 1)B_j x_2$$

$$= \sum_{j=1}^{k} (s(\bar{t} - 1)(\lambda^j - \lambda_0{}^j) + ((1 - s)\lambda_0)^j(\bar{t}^j - 1))B_j x_1$$

$$= 0 \qquad (4.5)$$

But $0 < (1 - s) < 1$, so $0 < (1 - s)^j < 1$, $\lambda_0{}^j > 0$ and $(\lambda^j - \lambda_0{}^j) > 0$ for each $j = 1,\ldots,k$; and, by our choice of $\bar{t}$, $s(\bar{t} - 1) > 0$. Hence taking the inner product of the right hand side of equation (4.5) with x_1, it follows from (A9.) that $(B_j x_1, x_1) = 0$ for each $j = 1,\ldots,k$, which is a contradiction unless $x_1 = 0$. Therefore, $x = x_2 = H^{-1}y \neq 0$ and we may divide the left hand side of equation (4.5) by $\| x_2 \|$ to obtain

$$(A - \sum_{j=1}^{k} \lambda_0{}^j B_j)\left(\frac{x_2}{\| x_2 \|}\right) - \sum_{j=1}^{k} ((1 - s)\lambda_0)^j(\bar{t}^j - 1)B_j\left(\frac{x_2}{\| x_2 \|}\right) = 0$$

which implies that

$$\left\| (A - \sum_{j=1}^{k} \lambda^j B_j)\left(\frac{x_2}{\| x_2 \|}\right)\right\| = \left\| \sum_{j=1}^{k} (1 - s)^j \lambda_0{}^j(\bar{t}^j - 1)B_j\left(\frac{x_2}{\| x_2 \|}\right)\right\|$$

But from Lemma 4.13

$$\left\| (A - \sum_{j=1}^{k} \lambda^j B_j)\left(\frac{x_2}{\| x_2 \|}\right)\right\| \geq K_1$$

and $$\left\| \sum_{j=1}^{k} (1 - s)^j \lambda_0{}^j(\bar{t}^j - 1)B_j\left(\frac{x_2}{\| x_2 \|}\right)\right\| < \sum_{j=1}^{k} \lambda_0{}^j(\bar{t}^j - 1)\| B_j \|$$

$$< K_1.$$

This contradiction proves that $x_2 = 0$, and so $y = 0$, which implies that $H(\partial G,s) \neq 0$ for all $s \in [0,1]$. Hence by the homotopy property (P3.) we have that

$$\mathrm{Deg}(I - \bar{t}CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - \sum_{j=1}^{k}((\lambda_0\bar{t})^j - \lambda_0{}^j)B_jH^{-1},G,0)$$

for $1 < \bar{t} < 1 + \delta/\lambda_0$ or equivalently $\lambda_0 < \lambda_0\bar{t} < \lambda_0 + \delta$.

By using the homotopy $\hat{H} : \bar{G} \times [0,1] \to X$ defined by

$$\hat{H}(y,s) = y - s\underline{t}CH^{-1}y - (1 - s)CH^{-1}y \sum_{j=1}^{k}((1 - s)\lambda_0)^j(\underline{t}^j - 1)B_jHy^{-1},$$

where $\underline{t}$ is arbitrary but fixed, such that $0 \leq 1 - \delta/\lambda_0 < \underline{t} < 1$, we can use a similar procedure to prove that

$$\mathrm{Deg}(I - \underline{t}CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - \sum_{j=1}^{k}((\lambda_0\underline{t})^j - \lambda_0{}^j)B_jH^{-1},G,0)$$

for $0 \leq \lambda_0 - \delta < \underline{t}\,\lambda_0 < \lambda_0$.

Hence, from the fact that $\mathrm{Deg}(I - tCH^{-1},G,0) = \{\deg_{LS}(I - tCH^{-1},G,0)\}$, cf. Theorem 1.18, the result of the Theorem follows by replacing $\underline{t}\lambda_0$ and $\bar{t}\lambda_0$ by, respectively, $\underline{\lambda}$ and $\bar{\lambda}$ and recalling that $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$.

The corresponding global bifurcation result is the following.

<u>Theorem 4.15</u> Consider problem (2.1) with the additional hypotheses (A7.) - (A9.) and (A11.). Suppose that $\dim N(A - T(\lambda_0))$ is an odd number. Then λ_0 is a global bifurcation point of problem (2.1).

<u>Proof:</u> Immediate from Theorems 4.7 and 4.14.

<u>Remark</u> Theorem 4.15 generalises Theorem 3.13 (3) to the case where A replaces I, -- the B_j's, j = 1,2,...,k, do not necessarily commute, and it is not required that $N(A - T(\lambda_0)) \subset X_n$ for every $n \in \mathbb{N}$.

We now seek sufficient conditions for $\lambda_0 = 0$ to be a global bifurcation point of problem (2.1). Again we shall assume that

$T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$; however, this time k may be infinite.

We require a Lemma.

Lemma 4.16 Consider problem (2.1) and assume hypotheses (A7.), (A8.), (A10.) and (A11.) hold. Suppose that $H : X \to X$ is a linear homeomorphism. Let G be an arbitrary open bounded set in X, containing zero. Define

$$M_1 = \inf\{(B_1x,x) : x \in N(A) \cap H^{-1}(\partial G)\}$$

and $M_2 = \sup_{j \in \mathbb{N}}\{\sup(B_jx,x) : x \in N(A) \cap H^{-1}(\partial G)\}$.

Then $M_1 > 0$ and $M_2 < \infty$.

Proof: First consider M_1. Suppose $M_1 = 0$, then there is a sequence $\{x_n\} \in N(A) \cap H^{-1}(\partial G)$ such that $(B_1x_n,x_n) \to 0$ as $n \to \infty$.

But dim N(A) is finite by Theorem 1.11 since A is A-proper, so $N(A) \cap H^{-1}(\partial G)$ is a compact set in N(A) and we may assume without loss of generality that $x_n \to x \in N(A) \cap H^{-1}(\partial G)$. Therefore $(B_1x,x) = 0$ and, since $x \in H^{-1}(\partial G)$, $x \neq 0$. This contradicts hypothesis (A10.) and so $M_1 > 0$.

Next consider M_2. For each $j \in \mathbb{N}$, if $x \in N(A) \cap H^{-1}(\partial G)$, then

$$\begin{aligned}\sup(B_jx,x) &\le \|B_j\| \sup\{\|x\|^2 : x \in N(A) \cap H^{-1}(\partial G)\}\\ &\le \|B_j\| \sup\{\|H^{-1}(y)\|^2 : y \in \partial G\}\\ &\le \|B_j\| \|H^{-1}\|^2 \sup\{\|y\|^2 : y \in \partial G\}\\ &\le \|B_j\| \|H^{-1}\|^2 N,\end{aligned}$$

for some constant $N > 0$, since G is bounded. So

$$M_2 = \sup_{j \in \mathbb{N}}\{\sup(B_jx,x) : x \in N(A) \cap H^{-1}(\partial G)\} \le \|H^{-1}\|^2 N \sup\{\|B_j\| : j \in \mathbb{N}\},$$

which is finite by hypothesis (A10.).

This completes the proof of Lemma 4.16.

We can now prove the following degree result.

Theorem 4.17 Consider problem (2.1) with the additional hypotheses (A7.), (A8.), (A10.) and (A11.).

Suppose that dim N(A) is an odd number and let $\delta_0 > 0$ be such that

$$0 < \frac{M_2|(\lambda - 1)|}{1 - (\lambda - 1)^2} < M_1$$

whenever $0 < |\lambda - 1| < \delta_0$, where M_1 and M_2 are the constants defined in Lemma 4.16.

Define

$\delta = \min\{1, \delta_0, \operatorname{dist}(0, C_A(T) \setminus \{0\}), -a, b, \eta\}$

then,

$$\operatorname{Deg}(I - CH^{-1} - \sum_{j=1}^{k} \underline{\lambda}^j B_j H^{-1}, G, 0) \neq \operatorname{Deg}(I - CH^{-1} - \sum_{j=1}^{k} \overline{\lambda}^j B_j H^{-1}, G, 0)$$

for $-\delta < \underline{\lambda} < 0 < \overline{\lambda} < \delta$, where G is an arbitrary open bounded set in Y containing zero and C and H are as defined in Proposition 4.9 (ii.) with

$C(x_1 + x_2) = \sum_{j=1}^{k} \lambda^j B_j x_1$ for $x_1 \in N(A)$ and $x_2 \in R(A)$, where $0 < \lambda < \delta$.

Proof: As in the proof of Theorem 4.14 we may prove that

$\deg_{LS}(I - \underline{t}CH^{-1}, G, 0) = -\deg_{LS}(I - \overline{t}CH^{-1}, G, 0)$ for

$0 \leq 1 - \delta < \underline{t} < 1 < \overline{t} < 1 + \delta$.

To complete the proof we require a homotopy argument. Define

$H : \overline{G} \times [0,1] \to X$ by

$$H(y,s) = y - s\underline{t}CH^{-1}y - (1 - s)CH^{-1}y - \sum_{j=1}^{k} ((1-s)(\underline{t} - 1))^j B_j H^{-1} y,$$

where $\underline{t}$ is taken arbitrarily, but fixed such that

$0 \leq 1 - \delta < \underline{t} < 1$.

Rewriting H as

$$H(y,s) = y - (1 + s(\underline{t} - 1))CH^{-1}y - \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j H^{-1}$$

and using the fact that $|s(\underline{t} - 1)| < \delta$ and

$0 \leq |(1 - s)(\underline{t} - 1)| \leq \delta < \tau_1$ and C is compact, it follows by Proposition 4.2 that $H(.,s)$ is A-proper with respect to Γ_H for each $s \in [0,1]$. It is easily seen that $H(x,.) : [0,1] \to X$ is uniformly continuous on $\overline{G}$ and $H(.,s)$ is continuous on $[0,1]$. To apply (P3.) we must show that $H(\partial G,s) \neq 0$ for all $s \in [0,1]$. Suppose the contrary, then

$$H(y,s) = 0 \text{ for some } y \in \partial G \text{ and } s \in [0,1].$$

Note that $s \neq 0$ and $s \neq 1$. For then $\underline{t}$ is a characteristic value of CH^{-1} or $\underline{t} - 1 \in C_A(T)$, both contradictions. So $s \in (0,1)$ and

$$(H - C - s(\underline{t} - 1)C - \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j)H^{-1}y = 0.$$

But $H - C = A$, so setting $H^{-1}y = x = x_1 + x_2$ with $x_1 \in N(A)$, $x_2 \in R(A)$ and replacing C, we have that

$$(A - \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j)x_2 = \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j x_1 + s(\underline{t} - 1) \sum_{j=1}^{k} \lambda^j B_j x_1 \tag{4.6}$$

Since $0 < |(1 - s)(\underline{t} - 1)| < \delta$ and $\lambda \in (0,\eta) \cap (a,b)$ then by (A10.) and Proposition 4.8 (ii.), the left hand side of equation (4.6) belongs to $R(A)$, while the right hand side belongs to $N(A)$. Hence by Proposition 4.8 (i.),

$$(A - \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j)x_2 = \sum_{j=1}^{k} (s(\underline{t} - 1)\lambda^j + (1 - s)^j(\underline{t} - 1)^j)B_j x_1 = 0 \tag{4.7}$$

Taking the inner product of the right hand side of this equation with x_1 we obtain

$$\sum_{j=1}^{k} (s(\underline{t} - 1)\lambda^j + (1 - s)^j(\underline{t} - 1)^j)(B_j x_1, x_1) = 0$$

We shall prove that $s(\underline{t} - 1)\lambda^j + (1 - s)^j(\underline{t} - 1)^j$ is negative for each $j = 1,2,\ldots,k$, which, by hypothesis (A10.), implies that $(B_1x_1,x_1) = 0$, and so $x_1 = 0$.

Since the first part of the summation, namely, $\sum_{j=1}^{k} s(\underline{t} - 1)\lambda^j(B_j x_1, x_1)$ is always negative, we shall obtain our result by showing that

$$\Sigma = \sum_{j=1}^{k} (1 - s)^j(\underline{t} - 1)^j(B_j x_1, x_1) < 0.$$

We do this by proving that the sum of all the positive terms in Σ added to the single term $(1 - s)(\underline{t} - 1)(B_1x_1,x_1)$, is negative. So consider the sum of positive terms given by

$$\Sigma_1 = (1 - s)^2(\underline{t} - 1)^2(B_2x_1,x_1) + (1 - s)^4(\underline{t} - 1)^4(B_4x_1,x_1) + \ldots,$$

assuming that k is infinite. If k is finite then there are less positive contributions than we have taken, so an infinite number of terms is the worst case as regards proving negativity. From Lemma 4.16 we have that $(B_jx_1,x_1) \leq M_2$ for each $j \in \mathbb{N}$, so

$$\begin{aligned}\Sigma_1 &\leq M_2[(1 - s)^2(\underline{t} - 1)^2 + (1 - s)^4(\underline{t} - 1)^4 + \ldots] \\ &= \frac{M_2(1 - s)^2(\underline{t} - 1)^2}{1 - (1 - s)^2(\underline{t} - 1)^2},\end{aligned}$$

by summing to infinity and using the fact that $0 < |(1 - s)(\underline{t} - 1)| < 1$. Also $0 < (1 - s) < 1$, so $(1 - s)^2 < (1 - s) < 1$ and

$$\frac{1}{1 - (1 - s)^2(\underline{t} - 1)^2} < \frac{1}{1 - (\underline{t} - 1)^2}, \text{ therefore}$$

$$\Sigma_1 < \frac{M_2(1 - s)(\underline{t} - 1)^2}{1 - (\underline{t} - 1)^2}$$

Now the first negative term is $(1 - s)(\underline{t} - 1)(B_1x_1,x_1)$. But by Lemma 4.16 $(B_1x_1,x_1) \geq M_1 > \dfrac{M_2|\underline{t} - 1|}{1 - (\underline{t} - 1)^2}$

since $|\underline{t} - 1| < \delta \leq \delta_0$.

Thus $(1 - s)|\underline{t} - 1|(B_1x_1,x_1) > \dfrac{M_2(1 - s)(\underline{t} - 1)^2}{1 - (\underline{t} - 1)^2}$

$> \Sigma_1$ by above.

We have therefore shown that even if all the positive terms in Σ are non-zero, their sum is still less than the modulus of the first negative term, $(1 - s)(\underline{t} - 1)(B_1x_1,x_1)$.

Hence, from (A10.) it follows that $x_1 = 0$. So $x = x_2 \neq 0$. But since $0 < |(1 - s)(\underline{t} - 1)| < \delta$, then $A - \sum_{j=1}^{k} ((1 - s)(\underline{t} - 1))^j B_j$ is a homeomorphism which implies from equation (4.7) that $x_2 = 0$. So $y = Hx = 0$ and this contradiction shows that $H(\partial G,s) \neq$ for all $s \in [0,1]$. Hence H is a valid homotopy and (P3.) gives

$$\mathrm{Deg}(I - \underline{t}CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - \sum_{j=1}^{k} (\underline{t} - 1)^j B_j H^{-1},G,0).$$

By use of the homotopy $\hat{H} : \overline{G} \times [0,1] \rightarrow Y$ defined by

$$\hat{H}(y,s) = y - s\overline{t}\, CH^{-1}y - (1 - s)CH^{-1}y - \sum_{j=1}^{k} ((1 - s)(\overline{t} - 1))^j B_j H^{-1}y$$

we may show as above that

$$\mathrm{Deg}(I - \overline{t}CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - \sum_{j=1}^{k} (\overline{t} - 1)^j B_j H^{-1},G,0),$$

for $1 < \overline{t} < 1 + \delta$.

The result then follows easily by noting that

$\mathrm{Deg}(I - tCH^{-1},G,0) = \{\deg_{LS}(I - tCH^{-1},G,0)\}$ and replacing $\underline{t} - 1$ and $\overline{t} - 1$ by, respectively, $\underline{\lambda}$ and $\overline{\lambda}$.

Remark For the proof that $\mathrm{Deg}(I - \overline{t}CH^{-1},G,0)$ $= \mathrm{Deg}(I - CH^{-1} - \sum_{j=1}^{k} (\overline{t} - 1)^j B_j H^{-1},G,0)$ we do not require the sign argument used for $\underline{t}$, since $(\overline{t} - 1)^j$ is positive for all $j \in \mathbb{N}$.

We have the following global bifurcation result.

Theorem 4.18 Consider problem (2.1) with the additional hypotheses (A7.), (A8.), (A10.) and (A11.).

Suppose that dim $N(A)$ is an odd number. Then λ_0 is a global bifurcation point of problem (2.1).

Proof: Immediate from Theorems 4.7 and 4.17.

Remarks (1.) Theorem 4.18 generalises Theorem 3.3(3) for $\lambda_0 = 0$, to the case when $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$ with k finite or infinite; where A and the B_j's ($j = 1,2,\ldots,k$) are not necessarily self-adjoint; a less stringent condition than positive semi-definite is assumed on the B_j's, and we do not demand that $\mathrm{Deg}(A - T(\lambda),G,0)$ is a singleton for any $\lambda \in (a,b)$. Theorem 4.18 also generalises Theorem 4.12, for $\lambda_0 = 0$, when $X = Y$ is a Hilbert space.

(2.) Throughout this section we assume that $\lambda_0 \in C_A(T) \cap (a,b)$ is isolated. When $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, we may ensure that λ_0 is isolated by imposing a more stringent transversality condition: namely, whenever $0 \neq x \in N(A - T(\lambda_0))$, then $\sum_{j=1}^{k} (\lambda^j - \lambda_0{}^j)B_j x \notin R(A - T(\lambda_0))$ for $\lambda_0 \neq \lambda \in (a,b)$. This condition implies hypothesis (A11.) and so the methods outlined above all go through as before. To see that (A11.) holds, suppose the contrary. Then there exist $0 \neq x \in N(A - T(\lambda_0))$, $0 \neq y \in X$, and

$\lambda \neq \lambda_0$ such that $(A - T(\lambda))x = (A - T(\lambda_0))y$. But $(A - T(\lambda))x = (A - T(\lambda_0) - (T(\lambda) - T(\lambda_0)))x = -(T(\lambda) - T(\lambda_0))x = -\sum_{j=1}^{k}(\lambda^j - \lambda_0{}^j)B_jx$ and so $\sum_{j=1}^{k}(\lambda^j - \lambda_0{}^j)B_jx \in R(A - T(\lambda_0))$, where $\lambda \in (a,b)$ with $\lambda \neq \lambda_0$ and $0 \neq x \in N(A - T(\lambda_0))$. We have, thus, shown that (A11.) holds whenever the more stringent transversality condition holds. Now it has been shown by Fitzpatrick, cf. [2], that this stronger condition is equivalent to : There exists $\varepsilon > 0$ such that $\|(A - T(\lambda))\,x\| \geq \varepsilon[\sum_{j=1}^{k}(\lambda^j - \lambda_0{}^j)^2]^{\frac{1}{2}}\|x\|$, whenever λ is sufficiently close to λ_0 and $x \in X$. Hence λ_0 is an isolated element in $C_A(T)$.

(3.) We have considered $T(\lambda)$ to have the form $\sum_{j=1}^{k}\lambda^jB_j$ rather than the, perhaps, more natural form $\sum_{j=1}^{k}\lambda_jB_j$ for some vector parameter $\lambda = (\lambda_1,\ldots,\lambda_k)$ in $\mathbb{R}^k$. This choice has been forced upon us by our use of degree theory to obtain global results. The method requires that an element $\lambda_0 \in C_A(T)$ be isolated and, for $\lambda = (\lambda_1,\ldots,\lambda_k)$, $C_A(T)$ generally corresponds to some hypercurve in $\mathbb{R}^k$ which has no isolated elements in $\mathbb{R}^k$. However, for the summation involving powers of λ, $C_A(T)$ turns out to be the set of roots of a polynomial in λ which are isolated in $\mathbb{R}$. Some authors have obtained global bifurcation results for $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$, for example [2] and [11]; however, these require homotopy theory which we have not considered. It should be noted that our Theorem 4.12 may be deduced as a special case of the results in [2].

4.3 The segment condition

In the previous section we assume that the transversality assumption (A11.) is satisfied and that $A - T(\lambda_0)$ is Fredholm of index zero, which allows us to use the theory of §4.1. In this section we shall not assume the transversality condition, but will again use the results of §4.1. It will be seen that other hypotheses we make, imply, as in the previous section, that $A - T(\lambda_0)$ is Fredholm of index zero. Although this property ensures that hypotheses (A5.) and (A6.) of §4.1 hold, we do not necessarily take a decomposition, $H - C$ of $A - T(\lambda_0)$, where C is compact. This is because one of our conditions will depend explicitly on knowing C and there may be a more accessible C, which is not compact.

We shall take problem (2.1) with hypotheses (A5.) and (A6.) of §4.1.

As in the previous section, we will give sufficient conditions under which Theorem 4.7 applies, where C may not be compact.

We require a definition.

Definition 4.19 $C(T,C) = \{(\mu,\lambda) \in \mathbb{R}^2 : N(A - T(\lambda) - (\mu - 1)C) \neq \{0\}\}$,

$$M^{\pm}(\lambda_0,\varepsilon) = \{(\mu,\lambda) \in \mathbb{R}^2 : 0 < (\mu - 1)^2 + (\lambda - \lambda_0)^2 < \varepsilon^2,$$

$$\text{where } \lambda = \lambda_0 \pm m(\mu - 1), \text{ for some } m \geq 0\}.$$

Lemma 4.20 $C(T,C) = \{(\mu,\lambda) \in \mathbb{R}^2 :$

$$N(I - \mu CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}) \neq \{0\}\}.$$

Proof: Let $(\mu,\lambda) \in C(T,C)$. Suppose $x \in N(A - T(\lambda) - (\mu - 1)C)$, then there is $0 \neq x \in X$ such that $Ax - T(\lambda_0)x - (T(\lambda) - T(\lambda_0))x - (\mu - 1)Cx = 0$, so

$(H - \mu C - (T(\lambda) - T(\lambda_0))x = 0$ and $(I - \mu CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1})Hx = 0$.

Hence $(\mu,\lambda) \in \{(\mu,\lambda) \in \mathbb{R}^2 : N(I - \mu CH^{-1} - (T(\lambda) - T(\lambda_0))H^{-1}) \neq \{0\}\}$.

The converse is proved similarly.

Definition 4.19 may be seen, through Lemma 4.20, to be a generalisation of a couple of sets defined and used by Stuart and Toland [38]. The form of these sets, and the subsequent condition which we will give in the next theorem, were suggested by the homotopy arguments used in the previous section.

We may prove the following degree result.

Theorem 4.21 Consider problem (2.1) with hypotheses (A5.) and (A6.) of §4.1. Suppose that $r_e(CH^{-1}) < 1$, λ_o has odd algebraic multiplicity $M_a(\lambda_o)$ as defined in Definition 4.5, and there exists $\varepsilon > 0$ such that,

$$C(T,C) \cap M^+(\lambda_o,\varepsilon) = \phi, \text{ or}$$
$$C(T,C) \cap M^-(\lambda_o,\varepsilon) = \phi.$$

Let

$$\delta_2 = \mathrm{dist}(1,\sigma(CH^{-1})\setminus\{1\}),$$

$$\delta_1 = \begin{cases} \delta_2/(1+\delta_2), & \text{if } \delta_2 \text{ is finite} \\ 1, & \text{otherwise} \end{cases}$$

$$\delta_3 = \begin{cases} \dfrac{1 - r_e(CH^{-1})}{r_e(CH^{-1})}, & \text{if } 0 < r_e(CH^{-1}) < 1 \\ 1, & \text{if } r_e(CH^{-1}) = 0\,. \end{cases}$$

Define $\delta = \min\{\varepsilon, \mathrm{dist}(\lambda_o, C_A(T)\setminus\{\lambda_o\}), \lambda_o - a, b - \lambda_o, \delta_1, \delta_3, \tau_1, \tau_2.\}$.

Then, $\mathrm{Deg}(I - CH^{-1} - (T(\underline{\lambda}) - T(\lambda_o))H^{-1},G,0)$
$\neq \mathrm{Deg}(I - CH^{-1} - (T(\overline{\lambda}) - T(\lambda_o))H^{-1},G,0)$ for $\lambda_o - \delta < \underline{\lambda} < \lambda_o < \overline{\lambda} < \lambda_o + \delta$,
where G is an arbitrary open bounded set in Y, containing zero.

Remark By Remarks (3.) and (4.) following Definition 1.13, since $r_e(CH^{-1}) < 1$, $I - CH^{-1}$ is Fredholm of index zero and therefore $(I - CH^{-1})H = H - C = A - T(\lambda_0)$ is also Fredholm of index zero. Then by Theorem 1.14 we may decompose $A - T(\lambda_0)$ into $H - C$ with C linear compact. But unless we can find such a map C explicitly, we cannot verify the condition $C(T,C) \cap M^{\pm}(\lambda_0,\varepsilon) = \phi$. For this reason we assume a general decomposition as in hypothesis (A6.) of §4.1.

Proof of Theorem 4.21: As in Theorems 4.11, 4.14 and 4.17 we first show that

$$\mathrm{Deg}(I - \underline{t}CH^{-1},G,0) \neq \mathrm{Deg}(I - \overline{t}CH^{-1},G,0) \tag{4.6}$$

for $0 < 1 - \delta < \underline{t} < 1 < \overline{t} < 1 + \delta$.

We emphasise that now C is not necessarily compact.

We can apply (P4.) to both operators in equation (4.6), provided that $r_e(CH^{-1}) < \frac{1}{\overline{t}} < \frac{1}{\underline{t}}$, if $\overline{t}$, $\underline{t}$ are not characteristic values of CH^{-1} and $I - \underline{t}CH^{-1}$ and $I - \overline{t}CH^{-1}$ are A-proper with respect to Γ_H.

Consider $I - \overline{t}CH^{-1}$

$$= I -(1 +(\overline{t} - 1))CH^{-1}.$$

Since $\overline{t} - 1 < \delta \leq \tau_2$, then from Proposition 4.2, $I - \overline{t}CH^{-1}$ and, similarly, $I - \underline{t}CH^{-1}$ are A-proper with respect to Γ_H.

Now $\delta \leq \dfrac{1 - r_e(CH^{-1})}{r_e(CH^{-1})}$, so

$$r_e(CH^{-1}) = \cfrac{1}{1 + \cfrac{1 - r_e(CH^{-1})}{r_e(CH^{-1})}} \leq \frac{1}{1 + \delta} < \frac{1}{\bar{t}} < \frac{1}{\underline{t}} \ ,$$

as required for an application of (P4.)

Also $1 - \underline{t} < \delta \leq \delta_1 = \dfrac{\delta_2}{1 + \delta_2}$, if δ_2 is finite,

therefore $1 - \dfrac{\delta_2}{1 + \delta_2} < \underline{t}$, which implies that $\dfrac{1}{1 + \delta_2} < \underline{t}$,

or equivalently $\dfrac{1}{\underline{t}} < 1 + \delta_2$.

Hence $0 < \dfrac{1}{\underline{t}} - 1 < \delta_2 = \operatorname{dist}(1, \sigma(CH^{-1}) \setminus \{1\})$.

This inequality is trivially satisfied if δ_2 is infinite.

Similarly $\bar{t} < 1 + \delta \leq 1 + \dfrac{\delta_2}{1 + \delta_2} = \dfrac{1 + 2\delta_2}{1 + \delta_2}$, therefore $\dfrac{1}{\bar{t}} > \dfrac{1 + \delta_2}{1 + 2\delta_2}$,

so $0 < 1 - \dfrac{1}{\bar{t}} < 1 - \dfrac{1 + \delta_2}{1 + 2\delta_2} = \dfrac{\delta_2}{1 + 2\delta_2} < \delta_2$.

Thus we may apply (P4.). If ν is the sum of the algebraic multiplicities of the characteristic values of CH^{-1} in the interval $(0, \underline{t})$, which is finite since $r_e(CH^{-1}) < \dfrac{1}{\bar{t}} < \dfrac{1}{\underline{t}}$, then

$\operatorname{Deg}(I - \underline{t}CH^{-1}, G, 0) = \{(-1)^{\nu}\}$ and

$\operatorname{Deg}(I - \bar{t}CH^{-1}, G, 0) = \{(-1)^{\nu + M_a(\lambda_0)}\} = \{-(-1)^{\nu}\}$.

The second equality holds since by the results in Chapter One, the algebraic multiplicity of the characteristic value $\hat{t}$ of CH^{-1} is given by

$$\dim\{ \bigcup_{n=1}^{\infty} N((I - \hat{t}CH^{-1})^n)\}.$$

However, by the arguments above, 1 is the only such characteristic value in the interval $(\underline{t}, \bar{t})$. So the sum of the algebraic multiplicities of the characteristic values in the interval $(o,\bar{t})$ is given by

$$\alpha + \dim\{\bigcup_{n=1}^{\infty} N((I - CH^{-1})^n)\}$$

which is precisely equal to $\alpha + M_a(\lambda_o)$ by Definition 4.5. Hence equation (4.6) holds as required.

We now use a homotopy argument to obtain the desired degree result.

Define $H : \bar{G} \times [0,1] \to Y$ by

$$H(y,s) = y - s\underline{t}CH^{-1}y - (1 - s)CH^{-1} - (T(\lambda_o + (1 - s)(\underline{t} - 1)) - T(\lambda_o))H^{-1}y,$$

where $\underline{t}$ is arbitrary, but fixed in $(1 - \delta, 1)$.

Now $H(.,s) = I - (s(\underline{t} - 1) + 1)CH^{-1} - (T(\lambda_o + (1 - s)(\underline{t} - 1)) - T(\lambda_o))H^{-1}$, with $|\lambda_o + (1 - s)(\underline{t} - 1) - \lambda_o| = |(1 - s)(\underline{t} - 1)| < \delta \le \tau_1$, and $|s(\underline{t} - 1)| < \delta \le \tau_2$ for each $s \in [0,1]$, and so $H(.,s)$ is A-proper with respect to Γ_H, for all $s \in [0,1]$, by Proposition 4.2. In order to apply (P3.), we must prove that $H(\partial G, s) \neq 0$, for all $s \in [0,1]$. Suppose the contrary, then there is $y \in \partial G$ and s in $[0,1]$ with $H(y,s) = 0$. Notice that $s \neq 0$, for otherwise $\lambda_o + (\underline{t} - 1) \in C_A(T)$, which is a contradiction by the choice of δ. So we have, for some $s \in (0,1]$ and $y \in \partial G$ that

$$y - (s(\underline{t} - 1) + 1)CH^{-1}y - (T(\lambda_o + (1 - s)(\underline{t} - 1) - T(\lambda_o))H^{-1}y = 0,$$

implying that

$$(1 + s(\underline{t} - 1),\ \lambda_o + (1 - s)(\underline{t} - 1)) \in C(T,C).$$

However, the distance, in $\mathbb{R}^2$, from this point to $(1, \lambda_o)$ is given by

$$\begin{aligned} 0 < D^2 &= s^2(\underline{t} - 1)^2 + (1 - s)^2(\underline{t} - 1)^2 \\ &= (\underline{t} - 1)^2(s^2 + (1 - s)^2) \\ &\le (\underline{t} - 1)^2 < \delta^2 \le \varepsilon^2. \end{aligned}$$

So, for $s \in (0,1]$, $(1 + s(\underline{t} - 1),\ \lambda_0 + (1 - s)(\underline{t} - 1)) \in M^+(0,\varepsilon)$ contradicting our assumption that $C(T,C) \cap M^+(\lambda_0,\varepsilon) = \phi$. Hence $H(\partial G,s) \neq 0$, for $s \in [0,1]$ and by (P3.), we have

$$\mathrm{Deg}(I - \underline{t}CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} - (T(\lambda_0 + (\underline{t} - 1))\cdot T(\lambda_0))H^{-1},G,0).$$

Using the homotopy,

$$\hat{H}(y,s) = I - s\bar{t}\,CH^{-1} - (1 - s)CH^{-1} - (T(\lambda_0 + (1 - s)(\bar{t} - 1))-T(\lambda_0))H^{-1},$$

we may prove, in an identical manner, that

$$\mathrm{Deg}(I - \bar{t}\,CH^{-1},G,0) = \mathrm{Deg}(I - CH^{-1} -(T(\lambda_0 + (\bar{t} - 1)-T(\lambda_0))H^{-1},G,0).$$

The result of the theorem follows easily from equation (4.6), by replacing $\lambda_0 + (\underline{t} - 1)$ and $\lambda_0 + (\bar{t} - 1)$ by, respectively, $\underline{\lambda}$ and $\bar{\lambda}$.

<u>Remark</u> In the above proof, we obtained our contradiction by assuming implicitly that $C(T,C) \cap M^+(\lambda_0,\varepsilon) = \phi$, and applying the homotopy arguments using $H(.,s)$ and $\hat{H}(.,s)$. If, however, the alternative hypothesis, namely, $C(T,C) \cap M^-(0,\varepsilon) = \phi$, is assumed to hold, then the same proof applies if we replace the terms $T(\lambda_0 + (1 - s)(\bar{t} - 1))$ and $T(\lambda_0 + (1 - s)(\underline{t} - 1))$ by $T(\lambda_0 + (1 - s)(1 - \bar{t}))$ and $T(\lambda_0 + (1 - s)(1 - \underline{t}))$, respectively, in the above homotopies. In this case we obtain our contradiction via $C(T,C) \cap M^-(\lambda_0,\ ) = \phi$, and here we replace $\lambda_0 + (1 - \bar{t})$ and $\lambda_0 + (1 - \underline{t})$ by $\underline{\lambda}$ and $\bar{\lambda}$, respectively.

The corresponding global bifurcation result is the following.

<u>Theorem 4.22</u> Consider problem (2.1) with the hypotheses (A5.) and (A6.) of §4.1. Suppose that $r_e(CH^{-1}) < 1$, λ_0 has odd algebraic

multiplicity $M_a(\lambda_0)$ and there exists $\varepsilon > 0$ such that,

$$C(T,C) \cap M^+(\lambda_0,\varepsilon) = \phi, \text{ or}$$
$$C(T,C) \cap M^-(\lambda_0,\varepsilon) = \phi.$$

Then λ_0 is a global bifurcation point of problem (2.1).

Proof: Immediate from Theorems 4.7 and 4.21.

Remarks (1.) In the paper by Alexander and Fitzpatrick [2], homotopy theory is used to prove general global bifurcation results for equations similar to equation (2.1), but where λ is allowed to be vector valued; however, Theorem 4.22 cannot be deduced as a special case of their results since they require that the transversality condition, mentioned in Remark (2) at the end of the previous section, should hold.

(2.) In Theorem 4.22, $T(\lambda)$ has a more general form than in the previous sections, where we took $T(\lambda) = \sum_{j=1}^{k} \lambda^j B_j$, for some k, finite or infinite.

We consider an example when Theorem 4.22 is applicable.

Example 1 Consider problem (2.1) with $T(\lambda) = \lambda B$, where B is not necessarily compact and A is an injective, A-proper operator with respect to Γ. Then, by Theorem 1.9, A is a homeomorphism. Suppose $\lambda_0 \in C_A(T) \cap (a,b)$. Let $H = A$, $C = \lambda_0 B$ and assume that $r_e(CH^{-1}) = r_e(\lambda_0 BH^{-1}) < 1$. Then, hypothesis (A5.) holds.

To verify that (A6.) is true we must prove that there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that $A - (\lambda + \xi\lambda_0)B$ is A-proper with respect to Γ, whenever $|\lambda - \lambda_0| < \tau_1$ and $|\xi| < \tau_2$. But $\lambda_0 \varepsilon(a,b)$ and by assumption (H2.) of problem (2.1), $A - \lambda B$ is A-proper for all $\lambda \varepsilon(a,b)$. Thus, if we set $\eta = \min\{\lambda_0 - a, b - \lambda_0\}$ and choose $\tau_1 \leq \eta/2$ and $\tau_2 \leq \eta/2|\lambda_0|$, then

$$
\begin{aligned}
|(\lambda + \xi\lambda_0) - \lambda_0| &= |(\lambda - \lambda_0) + \xi\lambda_0| \\
&\leq |\lambda - \lambda_0| + |\xi|\ |\lambda_0| \\
&< \tau_1 + \tau_2|\lambda_0| \\
&\leq \frac{\eta}{2} + \frac{\eta|\lambda_0|}{2|\lambda_0|} \\
&= \eta
\end{aligned}
$$

So, $(\lambda + \xi\lambda_0) \varepsilon (a,b)$ in this case, therefore (A6.) holds.

The condition $C(T,C) \cap M^{\pm}(\lambda_0,\varepsilon) = \phi$ for some $\varepsilon > 0$ is also satisfied. For, suppose $(\mu,\lambda) \varepsilon C(T,C)$. Then, from Lemma (4.20), there exists $x \neq 0$ such that

$$x - \mu\lambda_0 BA^{-1}x - (\lambda - \lambda_0)BA^{-1}x = 0,$$

which implies that

$$x - (\lambda_0(\mu - 1) + \lambda)BA^{-1}x = 0.$$

But $r_e(\lambda_0 BA^{-1}) < 1$ may be written as $r_e(BA^{-1}) < 1/|\lambda_0|$, so $1/\lambda_0$ is an isolated element in the spectrum of BA^{-1}, or equivalently, λ_0 is an isolated characteristic value of BA^{-1}. Thus, if we choose $(\mu,\lambda) \varepsilon \mathbb{R}^2$ and $\varepsilon > 0$ such that $\lambda = \lambda_0 \pm m(\mu - 1)$ for some $m \geq 0$, and $0 < (\mu - 1)^2 + (\lambda - \lambda_0)^2 < \varepsilon^2$ implies that $0 < |(\lambda_0(\mu - 1) + \lambda) - \lambda_0| < \mathrm{dist}(\lambda_0, ch(BA^{-1})\backslash\{\lambda_0\})$, then $(\mu,\lambda) \varepsilon M^{\pm}(\lambda_0,\varepsilon)$, but $(\mu,\lambda) \not\varepsilon C(T,C)$. Hence, by Theorem 4.22, if $M_a(\lambda_0)$ is an odd number, then λ_0 is a bifurcation point.

<u>Remarks</u> (1) Example 1 was considered by Toland [42] in Hilbert space, with $A = I$.

(2) Example 1 was treated by Petryshyn [32] in the case when B is compact, but he never gave global bifurcation results. The compactness of B ensures that $r_e(\lambda_0 BA^{-1}) = 0 < 1$, and when problem (2.1) holds, then A is automatically A-proper with respect to Γ.

CHAPTER FIVE

APPLICATIONS

5.1 Results on the existence of periodic solutions to a class of ordinary differential equations.

Consider the ordinary differential equation

$$x''(t) + b^2x(t) = g(x(t),x'(t),x''(t)) \tag{5.1}$$

where $0 < b \in \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$ and g satisfies:

(A1.) $g : \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous, $g(x,.,.) : \mathbb{R}^2 \to \mathbb{R}$ is uniformly continuous for x in bounded subsets of $\mathbb{R}$ and $g(x,y,z) = o(\max\{|x|,|y|,|z|\})$ as $x, y, z \to 0$.

From (A1.) it follows that $x = 0$ is a solution of equation (5.1) for each $t \in \mathbb{R}$, called the equilibrium solution. We shall consider the problem of proving the existence of non-trivial, even, T-periodic solutions, that is, solutions such that for some $T > 0$, $x(t + T) = x(t)$ and $x(t) = x(-t)$ for all $t \in \mathbb{R}$.

Note that T is also an unknown of the problem: we seek $T > 0$ and a solution x of period T.

To obtain our results we shall invoke the global bifurcation analysis of the previous chapters. The first step is, therefore, to transform the problem into an equivalent nonlinear eigenvalue problem. If $T > 0$ is given, then making the change of variable $t \to T\tau$, x is a T-periodic solution of equation (5.1) if and only if $x(t) = z(t/T) = z(\tau)$ is a 1-periodic solution of the equation

$$z''(\tau) + T^2b^2z(\tau) = T^2g(z(\tau),\frac{1}{T}\, z'(\tau),\frac{1}{T^2}\, z''(\tau)) \tag{5.2}$$

This follows since

$$\frac{dx(t)}{dt} = \frac{dz(t/T)}{dt} = \frac{dz(\tau)}{dt} = \frac{d\tau}{dt}\;\frac{dz(\tau)}{d\tau} = \frac{1}{T}\frac{dz}{d\tau}(\tau) \text{ and so}$$

$$\frac{d^2x(t)}{dt^2} = \frac{1}{T^2}\;\frac{d^2z}{d\tau^2}(\tau)$$

Now setting $\lambda = T^2$ and reverting to t for τ and x for z we see that the problem is equivalent to seeking non-trivial, even, 1-periodic solutions of the ordinary differential equation

$$x''(t) + \lambda b^2 x(t) = \lambda g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t)) \tag{5.3}$$

for values of λ in $(0,\infty) \equiv \mathbb{R}_+$.

Since we are looking for even, 1-periodic solutions of equation (5.3) we shall impose the following condition on g.

(A2.) $g(x,y,z) = g(x,-y,z)$ for all $x,y,z \in \mathbb{R}$.

This assumption makes equation (5.3) consistent for all values of $t \in \mathbb{R}$.

We shall convert equation (5.3) into an operator equation of the type studied in the previous chapters. The existence results given there depend on a condition of odd multiplicity at some characteristic value. We shall see that, by restricting ourselves to even solutions, this odd multiplicity property can be satisfied.

We wish to transform equation (5.3) into an abstract, non-linear eigenvalue problem in some Banach space. To this end we make the following definition.

Definition 5.1

$X = \{x \in C^2(\mathbb{R},\mathbb{R})$: x is 1-periodic and is an even function, that is, $x(t + 1) = x(t)$ and $x(t) = x(-t)$ for all $t \in \mathbb{R}\}$;

$Y = \{y \in C(\mathbb{R},\mathbb{R}) : y \text{ is 1-periodic and even}\}$;

$A : X \to Y$ with $Ax(t) = x''(t)$ for each $t \in \mathbb{R}$;

$B : X \to Y$ with $Bx(t) = -b^2x(t)$ for each $t \in \mathbb{R}$;

$R : X \times \mathbb{R}_+ \to Y$ with $R(x(t),\lambda) = \lambda g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))$ for each $(x,\lambda) \in X \times \mathbb{R}_+$.

Let the norms on Y and X be given by $\|y\|_0 = \sup\{|y(t)| : t \in \mathbb{R}\}$ for each $y \in Y$ and $\|x\|_2 = \max\{\|x^{(j)}\|_0 : 0 \leq j \leq 2\}$ for each $x \in X$.

By periodicity we have that

$\|y\|_0 = \sup\{|y(t)| : t \in \mathbb{R}\} = \max\{|y(t)| : t \in [0,1]\}$ for each $y \in Y$, and so X and Y are both Banach spaces and by the well known embedding result, see for example [1], X is compactly embedded into Y.

Note that, since we are using A-proper maps we are able to use two spaces of classical differentiable functions. We could also use Sobolov spaces via a weak formulation but this is not necessary for us. We can rewrite equation (5.3) in the operator form:

$$F(x,\lambda) = Ax - \lambda Bx - R(x,\lambda) = 0, \tag{5.4}$$

where $F : X \times \mathbb{R}_+ \to Y$.

Notice that $X \times \mathbb{R}_+ \subset X \times \mathbb{R}$, the Banach space with norm $\|(x,\lambda)\| = (\|x\|_2^2 + \lambda^2)^{\frac{1}{2}}$ for each $(x,\lambda) \in X \times \mathbb{R}$.

Equation (5.4) is in the standard form of equation (2.1), with $T(\lambda) = \lambda B$. We now verify that the hypotheses (H1) - (H4) of problem (2.1) are all satisfied.

<u>Theorem 5.2</u> $A : X \to Y$ is a Fredholm map of index zero;

$B : X \to Y$ is a compact linear map;

$N(A) = \{x \in X : x(t) = \text{a constant for all } t \in \mathbb{R}\}$;

$R(A) = \{ y \in Y : \int_0^1 y(t)dt = 0\}$;

$X = N(A) \oplus X_1$, for some closed subspace $X_1 \subset X$;

$Y = IN(A) \oplus R(A)$, where I is the inclusion map of X into Y which is compact.

$A_1 \equiv A|_{X_1} : X_1 \to R(A)$ is a homeomorphism; $A - \lambda B$ is Fredholm of index zero for all $\lambda \in \mathbb{R}$, and for each $\lambda \in \mathbb{R}$ there exist a linear homeomorphism $H : X \to Y$ and a linear compact operator $C : X \to Y$ such that $A - \lambda B = H - C$, where in general H and C depend on λ.

Proof: Suppose $Ax = 0$, then from Definition 5.1, $x''(t) = 0$. So $x'(t) = D$ and $x(t) = Dt + E$, where D and E are constants. From the 1-periodicity of x we have, $E = x(0) = x(1) = D + E$, therefore $D = 0$ and $N(A)$ is precisely the set of constant functions in X, which implies that $\dim N(A) = 1$. Hence $X = N(A) \oplus X_1$, where, by Theorem 1.1, X_1 may be chosen to be closed.

Now if $y \in R(A)$, then $y = Ax = x''$ for some $x \in X$ and so $\int_0^1 y(t)dt = \int_0^1 x''(t)dt = x'(1) - x'(0) = 0$ by the 1-periodicity. We shall prove that $R(A)$ is actually equal to $\{y \in Y : \int_0^1 y(t)dt = 0\}$. First notice that if $y \in Y$ with $\int_0^1 y(t)dt = 0$, then $\int_0^1 t\, y(t)dt = 0$. For,

$$\int_0^1 y(t)dt = 0, \; y \in Y$$

if and only if $\int_{-\frac{1}{2}}^{\frac{1}{2}} y(t)dt = 0$ (by 1-periodicity of y)

if and only if $\int_{-\frac{1}{2}}^{0} y(t)dt = 0$ (by evenness of y).

$$\text{Then } \int_0^1 t\, y(t)dt = \int_0^{\frac{1}{2}} t\, y(t)dt + \int_{-\frac{1}{2}}^{0} (s + 1)y(s + 1)ds$$

$$= \int_0^{\frac{1}{2}} t\, y(t)dt + \int_{-\frac{1}{2}}^{0} s\, y(s)ds + \int_{-\frac{1}{2}}^{0} y(s)ds$$

(by 1-periodicity)

$$= \int_0^{\frac{1}{2}} t\, y(t)dt - \int_0^{\frac{1}{2}} s\, y(s)ds$$

(since $s\, y(s)$ is odd)

$$= 0.$$

Now suppose $y \in Y$ with $\int_0^1 y(t)dt = 0$, we must show that there exists $x \in C^2(\mathbb{R},\mathbb{R})$, where x is 1-periodic and even, with $x'' = y$. Setting $x'' = y$ and integrating we have $x'(t) = x'(0) + \int_0^t y(s)ds$ so $x'(0) = x'(1)$. Also

$$\begin{aligned} x'(-t) &= x'(0) + \int_0^{-t} y(s)ds \\ &= x'(0) - \int_0^{t} y(s)ds \text{ (since } y \text{ is even),} \end{aligned}$$

therefore, if we take x with $x'(0) = 0$, then x' is an odd function, such that $x'(t) = \int_0^t y(s)ds = Y(t)$ (say). Again by integration we have $x(t) = x(0) + \int_0^t Y(s)ds$. Since $Y(s)$ is odd, $\int_0^t Y(s)$ is even and so x is even. Finally

$$\begin{aligned} x(1) - x(0) &= \int_0^1 Y(s)ds \\ &= \int_0^1 \left(\int_0^t y(s)ds\right)dt \\ &= \int_0^1 (1 - s)y(s)ds \\ &= \int_0^1 y(s)ds - \int_0^1 sy(s)ds \\ &= 0. \end{aligned}$$

This proves that $R(A) = \{y \in Y : \int_0^1 y(t)dt = 0\}$.

To see that the decomposition $Y = I\, N(A) \oplus R(A)$ holds observe that each $y \in Y$ may be written in the form $y = \int_0^1 y(t)dt + (y - \int_0^1 y(t)dt)$, where $\int_0^1 y(t)dt \in I\, N(A)$ and $\int_0^1 (y(s) - \int_0^1 y(t)dt)ds = 0$, which implies that $y - \int_0^1 y(t)dt \in R(A)$.

Thus $Y = IN(A) + R(A)$. Finally if $\int_0^1 (\int_0^1 y(t)dt)ds = 0$, then $\int_0^1 y(t)dt = 0$.

Hence $Y = I\,N(A) \oplus R(A)$, where I is the inclusion map of X into Y, which is compact. It is easily seen that $A_1 \equiv A|_{X_1} : X_1 \to R(A)$ is a homeomorphism. The fact that $B : X \to Y$ is compact follows trivially, since X is compactly embedded in Y. Finally, we have shown that A is Fredholm of index zero, so by Remark (2.) preceding Theorem 1.14, $A - \lambda B$ is also Fredholm of index zero for all $\lambda \in \mathbb{R}$ and, for each $\lambda \in \mathbb{R}$, the decomposition $H - C$ is guaranteed by Theorem 1.14.

In order to prove that A is an A-proper map we need to define an admissible scheme for maps from X into Y. For each $n \in \mathbb{N}$, define $t_i = \frac{i}{n}$ for $i = 0,1,\ldots,n$ and for each $y \in Y$,

$$Q_n\, y(t) = z(t) \text{ for each } t \in \mathbb{R}, \text{ where}$$

$$z(t) = \begin{cases} y(t), & \text{when } t = t_i \ (i = 0,\ldots,n) \\ y(t_i) + (y(t_{i+1}) - y(t_i))\dfrac{(t - t_i)}{t_{i+1} - t_i}, & \text{when} \\ \quad t \in (t_i, t_{i+1}) \ (i = 0,\ldots,n-1); & \end{cases}$$

and extend $z(t)$ to all of $\mathbb{R}$ by periodicity such that $z(t) = z(t + 1)$ for all $t \in \mathbb{R}$.

Let $Y_n = R(Q_n)$ (the range of Q_n), then the following result holds.

<u>Theorem 5.3</u> $\Gamma = \{Y_n, Q_n\}$ is an admissible scheme for maps from Y into Y, with $\|Q_n\| = 1$ for each $n \in \mathbb{N}$. If H is the homeomorphism from Theorem 5.2 for some fixed value $\lambda_0 \in \mathbb{R}$ then $\Gamma_H = \{H^{-1}(Y_n), Y_n, Q_n\}$ is an admissible scheme for maps from X into Y and $A - \lambda B : X \to Y$ is A-proper with respect to Γ_H for all $\lambda \in \mathbb{R}$.

<u>Proof:</u> First we show that $\{Y_n\} \subset Y$. Clearly if $z \in Y_n$, then z is continuous on $\mathbb{R}$ and 1-periodic. Also, since $t \in (t_i, t_{i+1})$ implies that $-t \in (-t_{i+1}, -t_i)$, it is easily verified that z is an even function by

using the fact that y is an even function. Hence $\{Y_n\} \subset Y$.

Next we prove that, for each $n \in \mathbb{N}$, Q_n is a continuous projection of Y onto Y_n.

Let $y,w \in Y$ and $\alpha,\beta \in \mathbb{R}$. Consider $Q_n(\alpha y + \beta w)$. It follows trivially from the definition that $Q_n(\alpha y + \beta w) = \alpha Q_n y + \beta Q_n w$, and if $\{y_k\}$ is a sequence in Y such that $y_k \to y$ as $k \to \infty$, then $Q_n y_k \to Q_n y$ as $k \to \infty$.

Also $Q_n^{\;2}\, y(t) = Q_n\, y(t) = y(t)$ for each $t = t_i$ $(i=0,\ldots,n)$, and since Q_n joins the points $y(t_i)$ by straight line segments we must have $Q_n^{\;2}\, y(t) = Q_n\, y(t)$ for all $t \in \mathbb{R}$.

Thus, for each $n \in \mathbb{N}$, Q_n is a projection from Y onto Y_n.

Next we prove that Γ is admissible. We have, for each $n \in \mathbb{N}$, that $\dim Y_n = n + 1$. To see this, let $\{e_0,e_1,\ldots,e_n\}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$ and suppose that $z \in Y_n$ as defined above. Then z is uniquely defined by the element $(y(t_0),y(t_1),\ldots,y(t_n))$ in $\mathbb{R}^{n+1}$. Thus every $z \in Y_n$, with $z(t) = Q_n y(t)$ for some $y \in Y$ and all $t \in \mathbb{R}$, is uniquely defined by $\sum_{j=0}^{n} \alpha_j e_j$ for some $(\alpha_0,\ldots,\alpha_n) \in \mathbb{R}^{n+1}$, where $y(t_i) = \alpha_i (i = 0,\ldots,n)$.

We also have that, for each $y \in Y$, $Q_n y \to y$ as $n \to \infty$ in the $\|\cdot\|_0$ norm. For, consider

$$\|Q_n y - y\|_0 = \max\{|Q_n y(t) - y(t)| : t \in [0,1]\}$$

$$= \max\{|y(t_i) + (y(t_{i+1}) - y(t_i))\frac{(t - t_i)}{(t_{i+1} - t_i)} - y(t)| : t \in (t_i,t_{i+1}), \text{ for } i = (0,\ldots,n-1)$$

$$\leq \max\{|y(t_i) - y(t)| : t \in (t_i,t_{i+1}) \text{ for } i = (0,\ldots,n-1\}$$

$$+ \max\{|y(t_{i+1}) - y(t_i)|\, |\frac{t - t_i}{t_{i+1} - t_i}| : t \in (t_i,t_{i+1}), i = (0,\ldots,n-1)\}.$$

$\to 0$ as $n \to \infty$, since $|\frac{t - t_i}{t_{i+1} - t_i}| < 1$ and $|y(t_{i+1}) - y(t_i)| \to 0$ as $n \to \infty$, for each $i = (0,\ldots,n-1)$.

Finally we shall prove that $\| Q_n \| = 1$ for all $n \in \mathbb{N}$.

By definition,

$$\begin{aligned}
\| Q_n \| &= \sup\{ \| Q_n y \|_0 : \| y \|_0 = 1\} \\
&= \sup\{\max(|Q_n y(t)| : t \in [0,1]) : \| y \|_0 = 1\} \\
&\leq \sup\{\max(|y(t)| : t \in [0,1]) : \| y \|_0 = 1\} \\
&= 1.
\end{aligned}$$

But $y(t) = 1$ for all $t \in \mathbb{R}$ is such that $y \in Y$ and $\| Q_n y \|_0 = \| y \|_0 = 1$. Hence $\| Q_n \| = 1$ for each $n \in \mathbb{N}$.

Thus it follows by Theorems 5.2 and 1.15 that $A - \lambda_0 B$ is A-proper with respect to Γ_H and since $B : X \to Y$ is compact, $A - \lambda B$ is A-proper with respect to Γ_H for all $\lambda \in \mathbb{R}$. This completes the proof of Theorem 5.3.

Our next task is to show that $A - \lambda B$ is the Fréchet derivative of $F(.,\lambda)$ at the point 0.

<u>Theorem 5.4</u> $R(x,.) : \mathbb{R}_+ \to Y$ is continuous uniformly for x in bounded subsets of X and $\| R(x,\lambda) \|_0 / \| x \|_2 \to 0$ as $\| x \|_2 \to 0$, uniformly for λ in bounded intervals of R_+, which are bounded away from zero.

<u>Proof:</u> That $R(x,.) : \mathbb{R}_+ \to Y$ is continuous uniformly for x in bounded subsets of X follows easily from (A1.) and Definition (5.1). Now

$$\frac{\| R(x,\lambda) \|_0}{\| x \|_2}$$

$$= \frac{\lambda \max\{|g(x(t),\lambda^{-\frac{1}{2}} x'(t),\lambda^{-1} x''(t))| : t \in [0,1]\} \max\{\| x \|_0, \lambda^{-\frac{1}{2}} \| x' \|_0, \lambda^{-1} \| x'' \|_0\}}{\max\{\| x \|_0, \| x' \|_0, \| x'' \|_0\} \qquad \max\{\| x \|_0, \lambda^{-\frac{1}{2}} \| x' \|_0, \lambda^{-1} \| x'' \|_0\}}$$

But $\max\{\|x\|_0, \lambda^{-\frac{1}{2}}\|x'\|_0, \lambda^{-1}\|x''\|_0\}$

$$\leq \max\{1, \lambda^{-\frac{1}{2}}, \lambda^{-1}\}\max\{\|x\|_0, \|x'\|_0, \|x''\|_0\}$$

$$\to 0 \text{ as } \|x\|_2 \to 0$$

for λ bounded away from zero in R_+.

So $\dfrac{\|R(x,\lambda)\|_0}{\|x\|_2}$

$$\leq \frac{\max\{|g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t)| : t \in [0,1]\}\lambda \max\{1,\lambda^{-\frac{1}{2}},\lambda^{-1}\}\|x\|_2}{\max\{\|x\|_0,\lambda^{-\frac{1}{2}}\|x'\|_0,\lambda^{-1}\|x''\|_0\} \qquad \|x\|_2}$$

$$\to 0 \text{ as } \|x\|_2 \to 0$$

for λ in bounded intervals in R_+, which are bounded away from zero. Hence the result is proved.

In order to prove that $F(.,\lambda) : X \to Y$ is A-proper with respect to Γ_H another assumption on g is required.

(A3.) There exists a constant $q \in (0,1)$ such that

$|g(x,y,z) - g(x,y,w)| \leq q|z - w|$ for $x,y,z,w \in \mathbb{R}$.

Some such restriction is necessary, for we must exclude equations such as $x''(t) = x''(t)$.

We shall also need the following definitions and lemmas.

<u>Definition 5.5</u> (Browder [4]). Let $f : D \to Y$ be continuous and bounded, where D is a closed subset of X. Then f is said to be a k-semicontraction if there exists a continuous and bounded mapping $V : X \times X \to Y$ and a constant k, $0 \leq k < 1$ such that $f(x) = V(x,x)$ for all $x \in D$ and for each fixed x in X, $V(.,x) : X \to Y$ is k-Lipschitzian (that is

$\| V(z,x) - V(w,x)\|_0 \leq k \|z - w\|_2$ for $z,w \in X$) and $V(x,.) : X \to Y$ is compact.

Lemma 5.6 (Webb, [44], Petryshyn, [29]). If $G \subset X$ is open and $f : \overline{G} \to Y$ is a k-semicontraction, then f is a k-ball contraction (cf. Chapter 1).

Definition 5.7 If $L : X \to Y$ is Fredholm of index zero then we define $\ell(L)$ by

$$\ell(L) = \sup\{r > 0 : r\beta(\Omega) \leq \beta(L(\Omega)) \text{ for each bounded } \Omega \subset X\}.$$

Lemma 5.8 (Petryshyn [33]). Suppose that $L : X \to Y$ is Fredholm of index zero and $\Gamma = \{X_n, Y_n, Q_n\}$ is an admissible scheme for maps from X into Y constructed as in Theorem 1.15, with $\| Q_n\| = 1$ for each $n \in \mathbb{N}$. Let $N : X \to Y$ be a bounded k-ball contraction with $0 \leq k < \ell(L)$. Then $L - sN : X \to Y$ is A-proper with respect to Γ for each $s \in (0,1]$.

Before proving that $F(.,\lambda)$ is A-proper we need one more preliminary result about the mapping A.

Lemma 5.9 $\ell(A) \geq 1$, where $\ell(A)$ is defined in Definition 5.7.

Proof: We already know from Theorem 5.2 that $X = N(A) \oplus X_1$, $Y = \mathbb{N}(A) \oplus R(A)$ and $A_1 \equiv A|_{X_1} : X_1 \to R(A)$ is a linear homeomorphism. Thus, for each bounded set $\Omega \subset X$ we have that $\Omega \subset A_1^{-1}A(\Omega) + P(\Omega)$, where P is the projection of X onto N(A), defined by $Px = \int_0^1 x(t)dt$. Now, since $\overline{P(\Omega)}$ is compact, then $\beta(P(\Omega)) = 0$, therefore, by the results of Chapter One, $\beta(\Omega) \leq \beta(A_1^{-1}(A(\Omega))) \leq \| A_1^{-1}\| \beta(A(\Omega))$. We shall complete the proof of the Lemma by showing that $\| A_1^{-1}\| = \sup\{\| A_1^{-1}y\|_2 : \| y \|_0 = 1\} \leq 1$.

For each $y \in R(A)$ with $\|y\|_0 = 1$, there exists $x \in X$ such that $Ax = x''(t) = y$ and, since $x(0) = x(1)$ and $x'(0) = x'(1)$, we may write this x in the form

$$x(t) = \int_0^t (\int_0^s y(\tau)d\tau)ds - t \int_0^1 (\int_0^s y(\tau)d\tau)ds + C.$$

But since y is an even function,

$$\int_0^1 (\int_0^s y(\tau)d\tau)ds = 0. \quad \text{So}$$

$$x(t) = \int_0^t (\int_0^s y(\tau)d\tau)ds + C, \tag{5.5}$$

and if we choose $C = -\int_0^1 [\int_0^t (\int_0^s y(\tau)d\tau)ds]dt$, then $\int_0^1 x(t)dt = 0$, which implies by Theorem 5.2 that $x \in X_1$ is the unique solution of $A_1x = y$.

Now since $\int_0^1 y(t)dt = 0$, then for $0 \leq s,\ t \leq 1$,

$$|\int_s^t y(\tau)d\tau| = |\int_0^1 y(\tau)dt - \int_0^s y(\tau)d\tau \ - \int_t^1 y(\tau)d\tau|$$

$$\leq \int_0^s |y(\tau)|d\tau + \int_t^1 |y(\tau)|d\tau,$$

therefore $2|\int_s^t y(\tau)d\tau| \leq |\int_s^t y(\tau)d\tau| + \int_s^t |y(\tau)|d\tau$

$$\leq \int_0^s |y(\tau)|d\tau + \int_t^1 |y(\tau)|d\tau + \int_s^t |y(\tau)|d\tau$$

$$= \int_0^1 |y(\tau)|d\tau$$

Hence $|\int_s^t y(\tau)d\tau| \leq \frac{1}{2} \int_0^1 |y(\tau)|d\tau$,

for $0 \leq s,t \leq 1$, provided $\int_0^1 y(t)dt = 0$ (5.6)

From equation (5.5) it follows that

$|x(t)| \leq |\int_0^t (\int_0^s y(\tau)d\tau)ds| + |C|$, where $\int_0^1 (\int_0^s y(\tau)d\tau)ds = 0$ since $y \in R(A)$.

Thus by equation (5.6)

$$|x(t)| \leq \tfrac{1}{2} \int_0^1 \left|\int_0^s y(\tau)d\tau\right| ds + |C|$$

$$\leq \tfrac{1}{2}\,\tfrac{1}{2} \int_0^1 \left(\int_0^1 |y(\tau)|d\tau\right) ds + |C|$$

$$\leq \tfrac{1}{4}\, \| y \|_0 + |C|$$

$$= \tfrac{1}{4} + |C|, \text{ since } \| y \|_0 = 1.$$

Similarly $|C| = \left|\int_0^1 \left[\int_0^t \left(\int_0^s y(\tau)\, d\tau\right) ds\right] dt\right|$

$$\leq \int_0^1 \left|\int_0^t \left(\int_0^s y(\tau)d\tau\right) ds\right| dt.$$

But again we have $\int_0^1 \left(\int_0^s y(\tau)d\tau\right) ds = 0$ and so by equation (5.6)

$$|C| \leq \tfrac{1}{2} \int_0^1 \left(\int_0^1 \left|\int_0^s y(\tau)d\tau\right| ds\right) dt$$

$$\leq \tfrac{1}{2}\,\tfrac{1}{2} \int_0^1 \left[\int_0^1 \left(\int_0^1 |y(\tau)|d\tau\right) ds\right] dt$$

$$\leq \tfrac{1}{4}\, \| y \|_0 = \tfrac{1}{4}.$$

Also since $x'(t) = \int_0^t y(\tau)d\tau$, then

$$|x'(t)| = \left|\int_0^t y(\tau)d\tau\right|$$

$$\leq \tfrac{1}{2} \int_0^1 |y(\tau)|d\tau$$

$$\leq \tfrac{1}{2}\, \| y \|_0 = \tfrac{1}{2}.$$

Finally it is trivial that $|x''(t)| = |y(t)|$. Hence

$$\| x \|_2 = \max\{\| x \|_0, \| x' \|_0, \| x'' \|_0\}$$

$$\leq \max\{\tfrac{1}{2}, \tfrac{1}{2}, 1\}$$

$$= 1.$$

Therefore, $\| A_1^{-1} \| = \sup\{\| A_1^{-1} y\|_2 : \| y\|_0 = 1\}$

$= \sup\{\| x\|_2 : x''(t) = y \text{ with } \| y\|_0 = 1\}$

≤ 1 by the above analysis.

Thus $\beta(\Omega) \leq \beta(A(\Omega))$ and therefore $\ell(A) \geq 1$.

Theorem 5.10 If (A1.), (A2.) and (A3.) hold, then $F(.,\lambda) : X \to Y$ is A-proper with respect to Γ_H for λ in bounded subsets of $\mathbb{R}_+$, which are bounded away from zero.

Proof: We prove this assertion by an application of Lemma 5.8, by showing first that $R(.,\lambda) : X \to Y$ is a k-semicontraction for λ in bounded subsets of $\mathbb{R}_+$ which are bounded away from zero and $k \in [0,\ell(A))$. Then Lemma 5.6 tells us that $R(.,\lambda)$ is, therefore, a k-ball contraction, which implies, by Lemma 5.8, that $A - R(.,\lambda)$ is A-proper with respect to Γ_H for λ in bounded subset in $\mathbb{R}_+$, which are bounded away from zero. The required result then follows since B is compact. To prove that $R(.,\lambda)$ is a k-semicontraction, define $V : X \times X \to Y$ by $V(u,x) = \lambda\, g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}u''(t))$, for $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$. Then $R(x,\lambda) = V(x,x)$ for $x \in X$. From (A1.), V is continuous and bounded for λ in bounded subsets of $\mathbb{R}_+$, which are bounded away from zero, and, for each $u \in X$, the mapping $V(u,.) : X \to Y$ is compact and continuous since X is compactly embedded in $\{x \in C'(\mathbb{R},\mathbb{R}) : x \text{ is 1-periodic and even}\}$. Furthermore (A3.) implies that for fixed $x \in X$ and $u,v \in X$,

$$\| V(u,x) - V(v,x)\|_0$$

$$= \lambda \sup\{|g(t,x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}u''(t)) - g(t,x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}v''(t)| : t \in[0,1]\}$$

$$\leq \lambda\, q\, \lambda^{-1} \sup\{|u''(t) - v''(t)| : t \in[0,1]\} = q\| u'' - v''\|_0 \leq q\| u - v\|_2,$$

for $q \in (0,1) \subset [0,\ell(A))$, by Lemma 5.9.

Thus, by Lemma 5.6, $R(.,\lambda) : X \to Y$ is a q-ball contraction for λ in bounded subsets of $\mathbb{R}_+$, which are bounded away from zero. Hence the result follows by Lemma 5.8 as described at the start of the proof.

Remark From Theorems 5.3, 5.4 and 5.10 we see that equation (5.4) satisfies hypotheses (H1) - (H4) of problem (2.1), with (a,b) any bounded interval in $\mathbb{R}_+$, provided $a \neq 0$. From Theorem 5.4, (H3) holds for all $\lambda \in (a,b)$.

We shall obtain our results by invoking Theorem 4.12 of §4.2. First we must verify that all of the conditions of the theorem hold.

Theorem 5.11 $C_A(B) = \{(\frac{2k\pi}{b})^2 : k \in \mathbb{N}\}$ and for each $k \in \mathbb{N}$, $N(A - (\frac{2k\pi}{b})^2 B) = \{D \cos(2k\pi t) : D \in \mathbb{R}\}$, which is one dimensional.

Proof: Suppose that $Ax - \lambda Bx = 0$, with $0 \neq x \in X$ and $\lambda > 0$. Then $x''(t) + \lambda b^2 x(t) = 0$.

Therefore, by the elementary theory of ordinary differential equations,

$$x(t) = D\cos(t\sqrt{\lambda b^2}) + E\sin(t\sqrt{\lambda b^2}), \lambda > 0,$$

where D and E are constants. For x to be an even function we must set $E = 0$. Also from periodicity assumptions, $x(0) = x(1)$, so

$$D = D\cos\sqrt{\lambda b^2},$$

which implies that $\sqrt{\lambda b^2} = 2k\pi$, for some $k \in \mathbb{Z}$.

Therefore, $\lambda = (\frac{2k\pi}{b})^2$ with $k \in \mathbb{N}$ since $\lambda > 0$, and $x(t) = D\cos(2k\pi t)$, $D \in \mathbb{R}$, $t \in \mathbb{R}$, $k \in \mathbb{N}$. Thus for each $k \in \mathbb{N}$, $\lambda_k = (\frac{2k\pi}{b})^2$ is a characteristic value of B relative to A and $N(A - \lambda_k B) = \{D\cos(2k\pi t) : D \in \mathbb{R}\}$, which is one dimensional as required.

Notice that by restricting ourselves to even solutions we have ensured that $\dim N(A - \lambda B)$ is odd for each $\lambda \in C_A(B)$.

Theorem 5.12 For each $\lambda_0 = (\frac{2k_0\pi}{b})^2$, with $k_0 \in \mathbb{N}$, the transversality assumption $BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$ holds; $A - \lambda_0 B$ is Fredholm of index zero; $X = N(A - \lambda_0 B) \oplus X_2$, $Y = \mathbb{N}(A - \lambda_0 B) \oplus R(A - \lambda_0 B)$, where $X_2 \subset X$ is a closed subspace, such that $BX_2 \subset R(A - \lambda_0 B)$.

Proof: First notice that since $B : X \to Y$ is defined by $Bx(t) = -b^2\, x(t)$, for each $t \in \mathbb{R}$, then it follows from the proof of Theorem 5.11 that

$$BN(A - \lambda_0 B) = \mathbb{N}(A - \lambda_0 B) = D \cos 2k_0\pi t$$

Now suppose that

$$Ax - \lambda_0 Bx = D \cos(2k_0\pi t) \tag{5.7}$$

for some $0 \neq x \in X$, $D \in \mathbb{R}$.

Then as in the proof of Theorem 5.11, $D_0 \cos 2k_0\pi t$ is the complementary solution, and so a particular solution must be of the form

$$x_p(t) = E\, t \cos(2k_0\pi t) + F\, t \sin(2k_0\pi t).$$

But, for $x_p(t)$ to be even, we need E to be zero, and for $x_p(t)$ to be 1-periodic we require that F be zero. Thus the only solution of equation 5.7 is $x = 0$, when $D = 0$. Hence

$$BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}.$$

Now we know from Theorem 5.2 that $A - \lambda_0 B$ is Fredholm of index zero. Hence, since equation (5.4) satisfies hypotheses (H1) - (H4) of problem (2.1), it follows from Theorem 5.11 and the above that Proposition 4.8 (i.) of §4.2 may be invoked. This completes the proof.

We can now prove a global bifurcation result for equation (5.4).

Theorem 5.13 Assume that (A1.), (A2.) and (A3.) hold and $\lambda_0 = (\frac{2k_0\pi}{b})^2$ with $k_0 \in \mathbb{N}$. Then λ_0 is a global bifurcation point of equation (5.4).

Proof: Immediate from Remark following Theorem 5.10 and Theorems 5.11, 5.12 and 4.12.

Theorem 5.13 provides us with a result on the existence of T-periodic solutions of equation (5.1). Before proving this, we need the following well known theorem which may be found in the book of Chow and Hale [6].

(Implicit Function Theorem)

Suppose X,Y,Z are Banach spaces, $U \subset X$, $V \subset Y$ are open sets, $F : U \times V \to Z$ is continuously differentiable, $(x_0,y_0) \in U \times V$, $F(x_0,y_0) = 0$ and the Fréchet derivative $F_x'(x_0,y_0)$ of F, with respect to x in U at the point (x_0,y_0), has a bounded inverse. Then there is a neighbourhood $U_1 \times V_1 \subset U \times V$ of (x_0,y_0) and a function $f : V_1 \to U_1$, with $f(y_0) = x_0$ such that $F(x,y) = 0$ for $(x,y) \in U_1 \times V_1$ if and only if $x = f(y)$. If $F \in C^k(U \times V,Z)$, $k \geq 1$, then $f \in C^k(V_1,X)$ in a neighbourhood of y_0.

We shall exclude one of the three possibilities for global bifurcation by assuming that $g(x,y,z)$ is locally Lipschitz in x, y for every z.

Theorem 5.14 Assume that hypotheses (A1.), (A2.), and (A3.) hold and let $T_0 \in (0,\infty)$ with $T_0 = \frac{2k_0\pi}{b}$, for some $k_0 \in \mathbb{N}$. Then at least one of the following properties holds:

(a.) For any number $M > 0$ there exists an even T_M - periodic solution x_M of equation (5.1) such that $\| x_M \|_2 = M$ and if $M \to 0$, $T_M \to T_0$;

(b.) There is an even T-periodic solution x_T of equation (5.1), either for all $T \in (0,T_0)$, or for all $T \in (T_0,\infty)$, such that $\| x_T \|_2 > 0$ for T in the appropriate interval and $\| x_T \|_2 \to 0$, as $T \to T_0$.

Proof: First notice that from Theorem 5.13, λ_0 is a global bifurcation point and so there is a maximal connected set C_s (say) in $X \times \mathbb{R}_+$ which satisfies at least one of the conditions (i.), (ii.) or (iii.) in Definition 2.7. Then, using the fact that a continuous image of a connected set is itself connected, we may take the continuous projection of C_s onto $\mathbb{R}_+$ to obtain an interval on $\mathbb{R}_+$. Transforming these facts for equation (5.4) into the terminology of equation (5.1), it is easily seen that (a.) and (b.) are direct consequences of (i.) and (iii.) of Definition (2.7) for the global bifurcation point, λ_0, of equation (5.4). To prove the theorem we must show that possibility (ii.) in Definition 2.7 is not possible for equation (5.1). We prove this in two steps.

(1.) If $(x,\lambda) \in C_s$ and $0 < |\lambda - \lambda_0|$ is sufficiently small, then $x = ux_0 + o(|u|)$ as $u \to 0$, where $u \in \mathbb{R}$ and x_0 is a non-zero element in $N(A - \lambda_0 B)$, that is, $x_0 = D \cos (2k_0\pi t)$ with $0 \neq D \in \mathbb{R}$.

To see this we apply the Implicit Function Theorem. Suppose $(\lambda,x) \in C_s$ and $0 < |\lambda - \lambda_0|$ is sufficiently small. Then $F(x,\lambda) = Ax - \lambda Bx - R(x,\lambda) = 0$. From Theorem 5.11, $\dim N(A - \lambda_0 B) = 1$. Also from Theorem 5.12 there exists a closed subspace $X_2 \subset X$ with $X = N(A - \lambda_0 B) \oplus X_2$ and $Y = IN(A - \lambda_0 B) \oplus R(A - \lambda_0 B)$.

Writing $x = x_1 + x_2$ with $x_1 \in N(A - \lambda_0 B)$, and $x_2 \in X_2$, we have that

$$F(x_1 + x_2,\lambda) = (A - \lambda_0 B)(x_1 + x_2) - (\lambda - \lambda_0)B(x_1 + x_2) - R(x_1 + x_2,\lambda) = 0$$

If we let $Q_1 : Y \to IN(A - \lambda_0 B)$ and $Q_2 : Y \to R(A - \lambda_0 B)$ be continuous projections, then

$$(A - \lambda_0 B)x_2 - (\lambda - \lambda_0)Bx_2 - Q_2R(X_1 + x_2,\lambda) = (\lambda - \lambda_0)Bx_1 + Q_1R(x_1 + x_2,\lambda) = 0.$$

Notice we have used the fact that $Bx_2 \in R(A - \lambda_0 B)$ which follows by Theorem 5.12.

Let us consider the equation

$$\begin{aligned} 0 &= (A - \lambda_0 B)x_2 - (\lambda - \lambda_0)Bx_2 - Q_2R(x_1 + x_2,\lambda) \\ &= (A - \lambda B)x_2 - Q_2R(x_1 + x_2,\lambda) \\ &= F_2(x_1,x_2,\lambda) \text{ (say).} \end{aligned}$$

Then $F_2(0,0,\lambda) = 0$ for all $\lambda \in \mathbb{R}_+$ and the Fréchet derivative, $F_2'(0,0,\lambda)$, of $F_2(x_1,x_2,\lambda)$ with respect to x_2, at the point $(0,0,\lambda)$, is such that

$$F_2'(0,0,\lambda) = (A - \lambda B),$$

which is a homeomorphism for $|\lambda - \lambda_0| < \operatorname{dist}(\lambda_0, C_A(B) \setminus \{\lambda_0\})$.

Thus by the Implicit Function Theorem, there exists a neighbourhood $N_1 \times N_2$ of $(0,\lambda_0) \in N(A - \lambda_0 B) \times \mathbb{R}$ and a function $f_2 : N_1 \times N_2 \to X_2$, such that $F_2(x_1,x_2,\lambda) = 0$ has the unique solution,

$$x_2 = f_2(x_1,\lambda) \in C^2(N_1 \times N_2,\ X_2) \text{ and } f_2\ (0,\lambda) = 0.$$

Hence $x_2 = f_2(x_1,\lambda) = (A - \lambda B)^{-1}\ Q_2\ R(x_1 + f_2(x_1,\lambda),\lambda)$, and implicit differentiation of this equation with respect to x_1 implies that

$$\frac{\| f_2(x_1,\lambda)\|_2}{\| x_1 \|_2} \to 0 \quad \text{as } \| x_1 \| \to 0$$

Now since $N(A - \lambda_0 B)$ is one-dimensional we may write $x_1 = ux_0$ with $\| x_0 \|_2 = 1$ and $u \in \mathbb{R}$. Hence

$$\begin{aligned} x &= x_1 + x_2 \\ &= ux_0 + f_2(ux_0,\lambda) \\ &= ux_0 + o(|u|) \text{ as } u \to 0, \end{aligned}$$

as required. This follows since

$$\frac{\| f_2(ux_0,\lambda)\|_2}{|u|} \to 0 \text{ as } u \to 0.$$

(2.) Notice that for each $k \in \mathbb{N}$ the element $x_k = D_k \cos(2k\pi t)$, $D_k \neq 0$, of $N(A - \lambda_k B)$, where $\lambda_k = (\frac{2k\pi}{b})^2$, is such that x_k has exactly $2k$ simple zeros in the interval $(0,1)$. Let S^k denote the set of all functions $x(t) \in X$ having exactly $2k$ simple zeros in the interval $(0,1)$ and for which $x(0) = x(1) \neq 0$. Then it is easily seen that, for each $k \in \mathbb{N}$, S^k is open in X and $S^k \cap S^\ell = \phi$ for $k \neq \ell \in \mathbb{N}$. Now, if $(x,\lambda) \in C_s$ and $0 < |\lambda - \lambda_0|$ is sufficiently small, then from step (1.) above, $x = ux_0 + o(|u|)$ as $u \to 0$. Since $x_0(t) = D_0 \cos(2k_0\pi t)$

has exactly $2k_0$ zeros in $(0,1)$, then $\{(x,\lambda) \in C_s : (x,\lambda) \neq (0,\lambda_0)$, $0 < |\lambda_0 - \lambda| + \|x\|_2$ is small$\} \subset S^{k_0} \times \mathbb{R}$. Now if $(x,\lambda) \in C_s \cap (\partial S^{k_0} \times \mathbb{R})$, then x must have a double root in $(0,1)$. To see this notice that since $x \in \partial S^{k_0}$, and ∂S^{k_0} is arbitrarily close to S^{k_0}, certainly x cannot have more than $2k_0$ roots in $(0,1)$. If x is such that $x(0) = x(1) = 0$, then the evenness of $x \in X$ implies that $x'(0) = 0$ and so, by the uniqueness of the initial value problem $x \in C^2[0,1]$, $x(0) = x'(0) = 0$, we must have x identically zero. But $(\lambda,0) = (\lambda,x) \in C_s$ implies that λ is a bifurcation point of equation (5.4), which is a contradiction since $0 < |\lambda_0 - \lambda|$ may be taken less than $\operatorname{dist}(\lambda_0, C_A(B) \setminus \{\lambda_0\})$. Thus x must have a double root in $(0,1)$, but again by the uniqueness of the initial value problem we must have $x = 0$, which is a contradiction by the previous argument.

We have, therefore, proved that if $(x,\lambda) \in C_S$, then $(x,\lambda) \in S^{k_0} \times \mathbb{R}$ and in particular $(x,\lambda) \neq (0,\lambda_1)$ for any $\lambda_1 = \left(\frac{2k_1\pi}{b}\right)^2$ with $k_0 \neq k_1 \in \mathbb{N}$. This completes the proof of Theorem 5.14.

<u>Remark</u> If the function g in equation (5.1) is independent of x'', then the map $R(.,\lambda) : X \to Y$ defined in Definition 5.1 is compact for λ in compact intervals in $\mathbb{R}_+$, since X is compactly embedded in $\{x \in C'(\mathbb{R},\mathbb{R}) : x$ is an even, 1-periodic function$\}$. The conclusions of Theorem 5.14 then hold without requiring condition (A3.). This case has been studied by many authors, including [17], where the main tool used is the Leray-Schauder degree theory. Note that this method cannot be used when g also depends on x'', since then g is not compact.

Conclusion (i.) of Theorem 5.14 says that the periodic solutions x_M are unbounded. If we know, a priori, that for certain periods, even periodic solutionsare bounded, then for these periods conclusion (i.) is redundant. We shall now impose further conditions on equation (5.1), which ensure that, for certain periods, such a priori bounds exist.

In addition to the hypotheses (A1.) - (A3.) assume that the following two conditions are satisfied by equation (5.1).

(Q1.) There exist non-negative constants D, E, F and λ_2 in $\mathbb{R}$ with $E > 0$ and $\lambda_2 > 0$ such that

$$|g(x,y,0) - b^2x| \leq D + E|x| + F|y|,$$

for x and y in $\mathbb{R}$ with

$$(\frac{\pi}{2E})^2 \; [-F + \sqrt{F^2 + 8E(1-q)}\;]^2 \geq \lambda_2$$

where $q \in (0,1)$ is the Lipschitz constant from (A3.)

(Q2.) There exist $\lambda_1 \geq 0$ and $M > 0$ such that for each λ with $0 \leq \lambda_1 < \lambda < \lambda_2$,

$$\int_0^1 \lambda\{g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t)) - b^2x(t)\}dt \neq 0,$$

for each $x \in X$ with $|x(t)| \geq M$ for all $t \in \mathbb{R}$.

We have the following result on constants E and λ_2 appearing in (Q1.)

<u>Theorem 5.15</u> If there exist $x,y \in \mathbb{R}$ such that $g(x,y,0) = 0$, then the constants E and λ_2 in hypotheses (Q1.) are such that $E \geq b^2$ and $\lambda_2 < \frac{2\pi^2}{b^2}$.

<u>Proof</u>: Suppose that $g(x,y,0) = 0$, then trivially

$$|g(x,y,0) - b^2x| = b^2|x|.$$

Hence $E \geq b^2$.

Now it is easily verified that

$(\frac{\pi}{2E})^2\ [-F + \sqrt{F^2 + 8E(1-q)}]^2$ decreases as E increases from b^2 and $-F + \sqrt{F^2 + 8E(1-q)}$ decreases as F increases from 0. Thus

$$(\frac{\pi}{2E})^2\ [-F + \sqrt{F^2 + 8E(1-q)}]^2 \le \frac{\pi^2 8b^2(1-q)}{4b^4}$$

$$= \frac{2\pi^2(1-q)}{b^2}$$

$$< \frac{2\pi^2}{b^2}\ ,\ \text{since } q \in (0,1)$$

Hence $\lambda_2 < \frac{2\pi^2}{b^2}$.

We now prove the following result on a priori bounds.

<u>Theorem 5.16</u> If (A1.), (A2.), (A3.), (Q1.) and (Q2.) hold and $Ax - \lambda Bx - R(x,\lambda) = 0$ for $x \in X$ with $\lambda \in (\lambda_1,\lambda_2)$, then there exists a constant $M_1 > 0$, independent of x and λ, such that $\| x \|_2 \le M_1$.

<u>Proof</u>: Let $x \in X$ and $\lambda \in (\lambda_1,\lambda_2)$ with $Ax - \lambda Bx - R(x,\lambda) = 0$, then

$$-x''(t) = \lambda b^2 x(t) - \lambda g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t)) \tag{5.8}$$

On integration from 0 to 1 equation (5.8) becomes

$$\lambda \int_0^1 \{b^2 x(t) - g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))\}dt = 0 \tag{5.9}$$

which implies by assumption (Q2.) and the 1-periodicity of x that there exists $t_0 \in [0,1]$ such that $|x(t_0)| < M$. Writing $x(t) = a_0 + u(t)$ with $a_0 = \int_0^1 x(t)dt$ it follows that

$$\int_0^1 u(t)dt = 0, \ x'(t) = u'(t).$$

Since for $t_o \varepsilon [0,1]$ we may write $x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds$, we have that

$$|x(t)| \leq M + \|x'\| = M + \|u'\| \text{ for all } t \ \varepsilon \ \mathbb{R} \tag{5.10}$$

where $\|\nu\| = [\int_0^1 |\nu(t)|^2 dt]^{\frac{1}{2}}$.

Notice that the norm $\|.\|$ is different from both $\|.\|_0$ and $\|.\|_2$.

Next we prove that $\|x\| \leq M + \frac{1}{\pi}\|u'\|$.

To see this consider

$$w(t) = \begin{cases} x(t + t_0 - 1) - x(t_0), \text{ if } 1 - t_0 \leq t \leq 1 \\ x(t + t_0) - x(t_0), \text{ if } 0 \leq t < 1 - t_0 \end{cases}$$

Since $w(0) = w(1) = 0$ and $w \ \varepsilon \ C'[0,1]$, then by Theorem 257 in [10],

$$\|w\| \leq \frac{1}{\pi}\|w'\|.$$

Now since $\|w + x(t_0)\|^2 = \int_0^{1-t_0} |x(t + t_0)|^2 dt + \int_{1-t_0}^{1} |x(t + t_0 - 1)|^2 dt$

$$= \int_0^1 |x(t)|^2 dt,$$

then
$$\begin{aligned} \|x\| &= \|w + x(t_0)\| \\ &\leq \|w\| + \|x(t_0)\| \\ &\leq \|w\| + M, \end{aligned}$$

and $\|w'\|^2 = \int_0^{1-t_0} |x'(t + t_0)|^2 dt + \int_{1-t_0}^{1} |x'(t + t_0 - 1)|^2 dt$

$$= \int_0^1 |x'(t)|^2 dt,$$

therefore, $\| w' \| = \| x' \| = \| u' \|$.

Thus $\| x \| \leq M + \| w \| \leq M + \frac{1}{\pi} \| w' \|$

$$= M + \frac{1}{\pi} \| u' \| . \tag{5.11}$$

Now from the equality

$$x''(t)^2 = \lambda\, g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x'')x''(t) - \lambda b^2\, x(t)x''(t)$$

it follows that

$$\int_0^1 x''(t)^2 dt = \int_0^1 |x''(t)|^2 dt$$

$$\leq \lambda \int_0^1 |g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))-b^2\, x(t)|\; |x''(t)|dt$$

Thus $\| x'' \|^2$

$$\leq \lambda \int_0^1 [|g(x(t),\lambda^{-\frac{1}{2}}x'(t),0)-b^2x(t)| + |g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))$$

$$- g(x(t),\lambda^{-\frac{1}{2}}x'(t),0)|]\; |x''(t)|dt$$

implying by (A3.) and (Q1.) that,

$$\| x'' \|^2 \leq \lambda \int_0^1 [D + E|x(t)| + F\lambda^{-\frac{1}{2}}|x'(t)| + q\lambda^{-1}|x''(t)|]\; x''(t)\; dt,$$

and by Hölder's inequality,

$$\| x'' \|^2 \leq \lambda[D\| x'' \| + E\| x \|\; \| x'' \| + F\lambda^{-\frac{1}{2}}\| x' \|\; \| x'' \| + q\lambda^{-1}\| x'' \|^2],$$

$$\text{so } \| x'' \| \leq \frac{\lambda}{1-q} [D + E\| x \| + F\, \lambda^{-\frac{1}{2}}\| x' \|]. \tag{5.12}$$

Moreover, from the equality,

$$-x''(t)x(t) = \lambda b^2x^2(t) - \lambda g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))x(t)$$

we have that

$$\int_0^1 |x'(t)|^2 dt = \lambda\int_0^1 [b^2x(t) - g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))]x(t)dt$$

$$= \lambda\int_0^1 [b^2x(t) - g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))](a_0 + u(t))dt$$

implying by (A3.), (Q1.) and equation (5.9) that

$$\|x'\|^2 \le \lambda\int_0^1 |b^2x(t) - g(x(t),\lambda^{-\frac{1}{2}}x'(t),\lambda^{-1}x''(t))||u(t)|dt$$

$$\le \lambda\int_0^1 [D + E|x(t)| + F\lambda^{-\frac{1}{2}}|x'(t)| + q\lambda^{-1}|x''(t)|]|u(t)|dt$$

and so by Holder's inquality

$$\|x'\|^2 \le \lambda[D + E\|x\| + \lambda^{-\frac{1}{2}}F\|x'\| + q\lambda^{-1}\|x''\|]\|u\| .$$

But, equation (5.12) implies that

$$\|x'\|^2 \le \lambda[D + E\|x\| + \lambda^{-\frac{1}{2}}F\|x'\| + \frac{q}{1-q}(D + E\|x\| + F\lambda^{-\frac{1}{2}}\|x'\|)]\|u\| .$$

Thus $\|x'\|^2 \le \frac{\lambda}{1-q}[D + E\|x\| + F\lambda^{-\frac{1}{2}}\|u'\|]\,\|u\|$

Also by the definition of u(t) we have $\int_0^1 u(t)dt = 0$, therefore, by Wirtinger's inequality [10], it follows that,

$$\|u\| \le \frac{1}{2\pi}\|u'\| .$$

Hence from equation (5.11)

$$\|x'\|^2 = \|u'\|^2 \le \frac{\lambda}{1-q}[D + E(M + \frac{1}{\pi}\|u'\|) + F\lambda^{-\frac{1}{2}}\|u'\|]\ \frac{1}{2\pi}\|u'\|$$

$$= (D + EM)\frac{\lambda\|u'\|}{2\pi(1-q)} + (E + \pi F\lambda^{-\frac{1}{2}})\frac{\lambda\|u'\|^2}{2\pi^2(1-q)}$$

so $\frac{2\pi^2(1-q)}{\lambda}\|u'\| \le \pi(D + EM) + (E + \pi F\lambda^{-\frac{1}{2}})\|u'\|$.

Thus $\|x'\| = \|u'\| \le \dfrac{\pi\lambda(D + EM)}{2\pi^2(1-q) - (E\lambda + \pi F\lambda^{\frac{1}{2}})} \le A_1$ (say) (5.13)

where $A_1 > 0$ is a constant independent of x and $\lambda\ \varepsilon(\lambda_1,\lambda_2)$.

Notice that $2\pi^2(1-q) - (E\lambda + \pi F\lambda^{\frac{1}{2}}) > 0$, since (Q1.) implies the following

$$2\pi^2(1-q) - (E\lambda + \pi F\lambda^{\frac{1}{2}})$$

$$> 2\pi^2(1-q) - \frac{\pi^2}{4E}[-F + \sqrt{F^2 + 8E(1-q)}]^2 - \frac{\pi^2 F}{2E}[-F + \sqrt{F^2 + 8E(1-q)}]$$

$$= 2\pi^2(1-q) - \frac{\pi^2}{4E}[-F + \sqrt{F^2 + 8E(1-q)}][-F + \sqrt{F^2 + 8E(1-q)} + 2F]$$

$$= 2\pi^2(1-q) - \frac{\pi^2}{4E}[-F^2 + F^2 + 8E(1-q)]$$

$= 2\pi^2(1-q) - 2\pi^2(1-q) = 0$ as asserted.

Thus from equations (5.10) and (5.13),

$$|x(t)| \leq M + A_1 = A_2 \text{ (say) for all } t \in \mathbb{R} \tag{5.14}$$

and from equations (5.11) and (5.13)

$$\|x\| \leq M + \frac{A_1}{\pi} \tag{5.15}$$

Therefore, from equations (5.13) and (5.15), it follows by equation (5.12) that

$$\|x''\| \leq \frac{\lambda}{1-q}[D + E(M + \frac{A_1}{\pi}) + F\lambda^{-\frac{1}{2}}A_1]$$

$$= \frac{1}{1-q}[\lambda D + \lambda E(M + \frac{A_1}{\pi}) + F\lambda^{\frac{1}{2}}A_1]$$

$$\leq A_3 \text{ (say)},$$

where $A_3 > 0$ is a constant independent of x and λ in (λ_1, λ_2).

Now, since $x(0) = x(1)$, there must exist $t_1 \in (0,1)$ such that $x'(t_1) = 0$, so

$$x'(t) = \int_{t_1}^{t} x''(s)ds \text{ and}$$

$|x'(t)| \leq \int_0^1 |x''(s)|ds \leq \|x''\|$ by Holder's inequality.

Thus $|x'(t)| \leq A_3$ for all $t \in \mathbb{R}$, by periodicity. (5.16)

Also from equation (5.8), (A3.) and (Q1.)

$$|x''(t)| \le \lambda[D + E|x(t)| + \lambda^{-\frac{1}{2}}F|x'(t)| + \lambda^{-1}q\;|x''(t)|]$$

and so $|x''(t)| \le \frac{\lambda}{1-q}\,[D + E|x(t)| + \lambda^{-\frac{1}{2}}|x'(t)|]$ therefore equations (5.14), (5.16) imply that

$$|x''(t)| \le \frac{\lambda}{1-q}\,[D + E\,A_2 + \lambda^{-\frac{1}{2}}A_3] \le A_4 \text{ (say) for all } t \in \mathbb{R} \qquad (5.17)$$

where A_4 is a positive constant independent of x and $\lambda\;\varepsilon(\lambda_1,\lambda_2)$.

Finally by equations (5.14), (5.16) and (5.17)

$$\|x\|_2 = \max\{\|x\|_0, \|x'\|_0, \|x''\|_0\} \le \max\{A_2, A_3, A_4\} = M_1 \text{ (say)}$$

which is a finite number as required. This completes the proof of Theorem 5.16.

We can now prove the following improved version of Theorem 5.14.

Theorem 5.17 Assume that hypotheses (A1.), (A2.), (A3.), (Q1.) and (Q2.) hold and let $T_0\;\varepsilon(0,\infty)$, with $T_0 = \frac{2k_0\pi}{b}$, for some $k_0 \in \mathbb{N}$. Let M_1 be the constant defined in Theorem 5.16. Then at least one of the following properties holds:

(i.) For any number $M > 0$ there exists an even T_M-periodic solution x_M of equation (5.1) such that $\|x_M\|_2 = M$ and:

(a.) if $M \to 0$, then $T_M \to T_0$;
(b.) if $M > M_1$, then $T_M \not\in (\lambda_1^{1/2}, \lambda_2^{1/2})$;

(ii.) There is an even T-periodic solution x_T of equation (5.1), either for all $T\;\varepsilon(0,T_0)$, or for all $T\;\varepsilon(T_0,\infty)$ such that

$\| x_T \|_2 > 0$, and if $T \in (\lambda_1^{1/2}, \lambda_2^{1/2})$ then $0 < \| x_T \|_2 \leq M_1$. Furthermore, if $\| x_T \|_2 \to 0$, then $T \to T_0$.

Proof: As in the proof of Theorem 5.14 using the additional results supplied by Theorems 5.15 and 5.16.

Remarks (1.) As far as the author is aware the above application is a new result.

(2.) Equation (5.1) may be regarded as a special case of an equation considered by Petryshyn and Yu [34]. They prove existence results for an equation of the form

$$(p(t)x'(t))' + f(t,x(t),x'(t),x''(t)) = y(t);$$

$x(0) = x(1)$, $x'(0) = x'(1)$ under various conditions on the functions p, y and f. However, their results cannot pick out even periodic solutions and their method does not determine any properties of the solution.

(3.) Mahwin [18] gives results on periodic solutions to systems of ordinary differential equations using a bifurcation argument akin to ours. However, the non-linear term considered there cannot depend on the highest derivative and they employ coincidence degree theory.

(4) By a similar procedure we can find odd, T-periodic solutions of equation (5.1). We make a hypothesis akin to (A2.) and a definition analogous to Definition (5.1). In this case $N(A) = \{0\}$ and $R(A)$ is the whole space. This case is, therefore, somewhat simpler.

5.2 Existence results for a class of ordinary differential equations.

Consider the ordinary differential equation

$$x''(t) + \lambda x(t) = \lambda g(t,x(t),x'(t),x''(t)), \tag{5.18}$$

$\lambda \in \mathbb{R}$, $x : [0,1] \to \mathbb{R}$,
where $x(0) = x(1) = 0$ and g satisfies:

(C1.) $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous and

$$g(t,x,y,z) = o(\max\{|x|,|y|,|z|\})$$

as $x,y,z \to 0$, uniformly for $t \in [0,1]$.

From (C1.) it follows that $x = 0$ is a solution of equation 5.18 for each $t \in [0,1]$ and for all $\lambda \in \mathbb{R}$. We shall consider the problem of proving the existence of solutions (x,λ) with x not identically zero. We shall again employ the global bifurcation results of the previous chapters and the analysis will be similar to that in the previous section. First we must transform equation (5.18) into an abstract, nonlinear eigenvalue problem.

Definition 5.18

$X = \{x \in C^2[0,1] : x(0) = x(1) = 0\}$;

$Y = \{y \in C[0,1]\}$;

$A : X \to Y$, where $Ax(t) = x''(t)$ for each $t \in [0,1]$;

$B : X \to Y$, where $Bx(t) = -x(t)$ for each $t \in [0,1]$;

$R : X \times \mathbb{R}$, where $R(x,\lambda) = \lambda\, g(t,x,x',x'')$ for each $(x,\lambda) \in X \times \mathbb{R}$.

If we denote the norms on Y and X by

$\|y\|_0 = \max\{|y(t)| : t \in [0,1]\}$ for each $y \in Y$ and
$\|x\|_2 = \max\{\|x^{(j)}\|_0 : 0 \le j \le 2\}$ for each $x \in X$,
then X and Y are Banach spaces.

Thus we can rewrite equation (5.18) as

$$F(x,\lambda) = Ax - \lambda Bx - R(x,\lambda) = 0, \tag{5.19}$$

where $(x,\lambda) \in X \times \mathbb{R}$ and $F : X \times \mathbb{R} \to Y$.

We have the following analogue to Theorem 5.2.

Theorem 5.19 $A : X \to Y$ is a bijection, that is, $N(A) = 0$ and $R(A) = Y$; A is a Fredholm operator of index zero; $B : X \to Y$ is a compact linear operator; $A - \lambda B$ is Fredholm of index zero for all $\lambda \in \mathbb{R}$; for each $\lambda \in \mathbb{R}$ there exist a linear homeomorphism $H : X \to Y$ and a linear compact operator $C : X \to Y$ such that $A - \lambda B = H - C$, where in general C and H depend on λ.

Proof: Suppose $Ax = 0$, then $x''(t) = 0$, so $x(t) = Ct + D$. The boundary conditions $x(0) = x(1) = 0$ imply that $x(0) = D = 0 = C$ and, therefore, $N(A) = \{0\}$.

To prove that $R(A) = Y$, we must show that for each $y \in Y$, there exists $x \in X$ such that $x''(t) = y(t)$ for all $t \in [0,1]$. Integrating we have that

$$x'(t) = x'(0) + \int_0^t y(s)ds$$
$$\text{and } x(t) = tx'(0) + \int_0^t\left(\int_0^s y(u)du\right)ds$$

So $x(0) = 0$. We must prove that $x(1) = 0$.

$$\begin{aligned} x(1) &= x'(0) + \int_0^1\left(\int_0^s y(u)du\right)ds \\ &= x'(0) + \int_0^1\left(\int_u^1 y(u)ds\right)du \\ &= x'(0) + \int_0^1(1 - u)y(u)du \\ &= x'(0) + \int_0^1 y(u)du - \int_0^1 u\, y(u)du. \end{aligned}$$

But $\int_0^1 y(u)du = \int_0^1 x''(t)dt = x'(1) - x'(0)$, and $\int_0^1 u\, y(u)du = \int_0^1 t\, x''(t)dt$

$$= [t\, x'(t)]_0^1 - \int_0^1 x'(t)dt$$

$$= x'(1) - (x(1) - x(0))$$

$$= x'(1).$$

Hence $x(1) = x'(0) + x'(1) - x'(0) - x'(1)$

$$= 0.$$

Thus $R(A) = Y$.

As we noted in Chapter One, a bounded, linear bijection is Fredholm of index zero. Hence A is Fredholm of index zero. B is easily seen to be compact and the remainder of proof follows exactly as in Theorem 5.2.

The next result is similar to Theorems 5.3 and 5.11.

Theorem 5.20 Let Q_n and Y_n be as defined in the last section preceding Theorem 5.3. Suppose H is the homeomorphism from Theorem 5.19 for some fixed $\lambda_0 \in \mathbb{R}$. Then, $\Gamma_H = \{H^{-1}(Y_n), Y_n, Q_n\}$ with $\|Q_n\| = 1$ for each $n \in \mathbb{N}$ is an admissible scheme for maps from X into Y; $A - \lambda B: X \to Y$ is A-proper with respect to Γ_H for all $\lambda \in \mathbb{R}$; $C_A(B) = \{\lambda_k = (k\pi)^2 : k \in \mathbb{N}\}$, and $N(A - \lambda_k B) = \{D \sin k\pi t : D \in \mathbb{R}\}$, which is one dimensional.

Proof: That Γ_H is admissible and $A - \lambda B$ is A-proper with respect to Γ_H follows in a similar way to Theorem 5.3 : at the point where we prove that $\|Q_n\| = 1$, we show that $\|Q_n\| \leq 1$, as before and then use $y \in Y$ such that $y(t) = -2\,|t - \tfrac{1}{2}| + 1$ to deduce that $\|Q_n\| = 1$ for each $n \in \mathbb{N}$.

Now suppose that $(A - \lambda B)x = 0$, $0 \neq x \in X$. Then $x''(t) + \lambda x(t) = 0$, therefore $x(t) = D \sin \sqrt{\lambda}\, t + E \cos \sqrt{\lambda}\, t$, if $\lambda > 0$ and $x(t) = Fe^{\sqrt{\lambda t}} + Ge^{-\sqrt{\lambda t}}$, if $\lambda < 0$. Notice that if $\lambda = 0$, then $x = 0$

which is a contradiction. Using the boundary conditions $x(0) = x(1) = 0$ we have that $E = 0$ and $\sqrt{\lambda} = k\pi$ for $k \in \mathbb{N}$. So $\lambda = (k\pi)^2$ and $x(t) = D \sin k\pi t$, $k \in \mathbb{N}$. Also $0 = F + G$ and $0 = Fe^{\sqrt{\lambda}} + Ge^{-\sqrt{\lambda}}$. Therefore,

$$F = -G \text{ and}$$

$$0 = 2F\frac{(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})}{2}$$

$$= 2F \sinh \sqrt{\lambda}$$

Thus $F = G = 0$. So $C_A(B) = \{\lambda_k = (k\pi)^2 : k \in \mathbb{N}\}$ and $N(A - \lambda_k B) = \{D \sin k\pi t : D \in \mathbb{R}\}$, for each $k \in \mathbb{N}$ where $\lambda_k = (k\pi)^2$ which is 1 dimensional. This completes the proof of Theorem 5.20.

It is a trivial consequence of hypothesis (C1.) and Definition 5.18 that R, in equation (5.19), satisfies hypotheses (H3) and (H4) of problem (2.1); furthermore, from Theorem 5.20, (H2) is satisfied for all $\lambda \in \mathbb{R}$, so $(a,b) = \mathbb{R}$. In Theorem 5.23 we shall see that (H1) also holds. Before verifying (H1) we prove that equation (5.19) satisfies a transversality condition.

<u>Theorem 5.21</u> $BN(A - \lambda_k B) \cap R(A - \lambda_k B) = \{0\}$ for each $k \in \mathbb{N}$ such that $\lambda_k = (k\pi)^2$.

<u>Proof</u>: We have seen in Theorem 5.20, that $N(A - \lambda_k B) = \{D \sin k\pi t: D \in \mathbb{R}\}$. It follows easily that $BN(A - \lambda_k B) = IN(A - \lambda_k B)$, where I is the inclusion map of X into Y, which is compact. Then, if $D \sin k\pi t \in BN(A - \lambda_k B) \cap R(A - \lambda_k B)$, we must have $D = 0$. For, suppose $(A - \lambda_k B)x = D \sin k\pi t$, $0 \neq x \in X$, then $x''(t) + \lambda_k x(t) = D \sin k\pi t$. The complementary function is given by $x_c(t) = F \sin k\pi t + G \cos k\pi t$, for some constants F and G, so the particular integral must be of the form

$X_p(t) = Pt \sin k\pi t + Qt \cos k\pi t.$

Thus we must have

$x(t) = F \sin k\pi t + G \cos k\pi t + Pt \sin k\pi t + Qt \cos k\pi t.$

Since we require that $x \varepsilon X$, then $x(0) = x(1) = 0$, which implies that $G = Q = 0$ and

$$x(t) = F \sin k\pi t + Pt \sin k\pi t, \text{ with}$$

$$x''(t) = -F(k\pi)^2 \sin k\pi t + 2Pk\pi \cos k\pi t$$
$$-Pt(k\pi)^2 \sin k\pi t.$$

Hence $x''(t) + (k\pi)^2 x(t) = D \sin k\pi t$, which implies that $2Pk\pi \cos k\pi t = D \sin k\pi t$, which can only be true when $D = P = 0$.

Thus the transversality condition holds.

It follows from Theorem 5.21 that we can use similar results for solving equation 5.19 as we used for equation 5.4; in particular the theorems from §4.2 apply since a transversality condition holds. First we need to prove that $F(.,\lambda)$ is A-proper for λ in some open interval of the real line. A further assumption on g is needed.

(C2.) There exists a constant $q \varepsilon (0,1)$ such that $|g(t,x,y,z) - g(t,x,y,w)| \leq q|z - w|$, for $x,y,z,w \varepsilon \mathbb{R}$ and $t \varepsilon [0,1]$.

We again assume that g is Locally Lipschitz with respect to x and y.

The statement of the next theorem is exactly the same as Lemma 5.9, but the proof is different.

Lemma 5.22 $\ell(A) \geq 1$.

Proof: Since A is a bijection, it is a homeomorphism, so for each bounded set $\Omega \subset X$, $\beta(\Omega) \leq \|A^{-1}\| \; \beta(A(\Omega))$.

We shall prove that $\|A^{-1}\| = \sup\{\|A^{-1}y\|_2 : \|y\|_0 = 1\} \leq 1$.

For each $y \in Y$ with $\|y\|_0 = 1$, there exists $x \in X$ such that $Ax = x'' = y$.

Integrating we obtain that

$$x'(t) = x'(0) + \int_0^t y(s)ds, \text{ and}$$
$$x(t) = tx'(0) + \int_0^t (\int_0^s y(u)du)ds + C$$

But $x(0) = 0$ implies that $C = 0$ and $x(1) = 0$ gives
$x'(0) = -\int_0^1 (\int_0^s y(u)du)ds$

So $x(t) = \int_0^t (\int_0^s y(u)du)ds) - t\int_0^1 (\int_0^s y(u)du)ds$. Or, equivalently,

$$x(t) = \int_0^t (\int_0^s y(u)du)ds - t\int_0^1 (\int_0^\nu y(u)du)d\nu$$

So
$$\begin{aligned}|x(t)| &= |\int_0^t (\int_0^s y(u)du)ds - t\int_0^1 (\int_0^\nu y(u)du)d\nu| \\
&= |\int_0^t [\int_0^s y(u)du - \int_0^1 (\int_0^\nu y(u)du)d\nu]ds| \\
&\le \int_0^t |\int_0^s y(u)du - \int_0^1 (\int_0^\nu y(u)du)d\nu|ds \\
&= \int_0^t |\int_0^1 (\int_0^s y(u)du)d\nu - \int_0^1 (\int_0^\nu y(u)du)d\nu|ds \\
&= \int_0^t |\int_0^1 (\int_\nu^s y(u)du)d\nu|ds \\
&\le \int_0^1 \int_0^1 |\int_\nu^s y(u)du|d\nu\, ds \\
&\le \int_0^1 \int_0^1 \int_0^1 |y(u)|du\, d\nu\, ds \\
&\le \|y\|_0 = 1.\end{aligned}$$

Also, since $x'(t) = \int_0^t y(s)ds - \int_0^1 (\int_0^s y(u)du)ds$, then

$$\begin{aligned}x'(t) &= \int_0^1 (\int_0^t y(s)ds)d\nu - \int_0^1 (\int_0^\nu y(u)du)d\nu \\
&= \int_0^1 (\int_\nu^t y(s)ds)d\nu.\end{aligned}$$

So as above $|x'(t)| \leq \|y\|_0 = 1$. Hence, since $\|x''\|_0 = \|y\|_0 = 1$, we have

$$\|x\|_2 = \max\{\|x\|_0, \|x'\|_0, \|x''\|_0\} \leq 1, \text{ and so } \|A^{-1}\| \leq 1$$

which implies that $\beta(\Omega) \leq \beta(A(\Omega))$ and $\ell(A) \geq 1$.

Theorem 5.23 If (C1.) and (C2.) hold, then $F(.,\lambda) : X \to Y$ is A-proper with respect to Γ_H for all $\lambda \in \mathbb{R}$.

Proof: Exactly as in the proof of Theorem 5.10, using Theorem 5.22

The preceding results tell us that equation (5.19) satisfies the hypotheses (H1) - (H4) of problem (2.1) with $(a,b) = \mathbb{R}$ and that $BN(A - \lambda_0 B) \cap R(A - \lambda_0 B) = \{0\}$, for each $\lambda_0 = (k_0\pi)^2$, $k \in \mathbb{N}$. We can use Theorem 4.12 to prove that such a λ_0 is necessarily a global bifurcation point of equation (5.19).

Theorem 5.24 Assume that (C1.) and (C2.) are satisfied. Then, $\lambda_0 = (k_0\pi)^2$ is a global bifurcation point of equation (5.19) for each $k_0 \in \mathbb{N}$.

Proof: Immediate from Theorem 4.12 and the preceding results.

Transforming the conclusions of Theorem 5.24 into an existence theorem for equation 5.18 we have the following analogue to Theorem 5.14.

Theorem 5.25 Assume that equation 5.18 satisfies (C1.) and (C2.) and $\lambda_0 = (k_0\pi)^2$ for $k_0 \in \mathbb{N}$. Then at least one of the following must hold:

(a.) For any $M > 0$ there exist $\lambda_M > 0$ and $x_M \varepsilon X$ such that $\|x_M\| = M$ and (x_M,λ_M) satisfies equation (5.18); furthermore, if $M \to 0$, then $\lambda_M \to \lambda_o$.

(b.) There is $x_\lambda \varepsilon X$ such that for all $\lambda \varepsilon(\lambda_o,\infty)$, $\|x_\lambda\|_2 > 0$ and (x_λ,λ) satisfies equation (5.18). If $\|x_\lambda\|_2 \to 0$, then $\lambda \to \lambda_o$.

Proof: The proof is similar to that of Theorem 5.14 with $\mathbb{R}_+$ replaced by $\mathbb{R}$: as in step (1.) we may take $X = N(A - \lambda_o B) \oplus X_2$,

$$Y = I\, N(A - \lambda_o B) \oplus R(A - \lambda_o B),$$

and, using the Liapunov-Schmidt procedure, show that if $(x,\lambda) \varepsilon C_s$ (the maximal connected subset of $X \times \mathbb{R}$ guaranteed by Theorem 5.24), and $0 < |\lambda - \lambda_o|$ is sufficiently small, then $x = ux_o + o(|u|)$ as $u \to 0$, where $u \varepsilon \mathbb{R}$ and $x_o = D_o \sin k_o \pi t$ with $0 \neq D_o \varepsilon \mathbb{R}$. We can then denote by Z^k the set of all functions $x \varepsilon X$ having exactly $k-1$ simple zeros in the open interval $(0,1)$ and for which $x(0) = x(1) = 0$, $x'(0) \neq 0$ and $x'(1) \neq 0$. Then for each $k \varepsilon N$, Z^k is open in X and $Z^k \cap Z^\ell = \phi$ for $k \neq \ell \varepsilon \mathbb{N}$. Proceeding again as in the proof of Theorem 5.14 we may show that possibility (ii.) of Definition 2.7 is impossible. Finally, observe that if $(x,\lambda) \varepsilon C_s$ with $\lambda = 0$, then $x'' = 0$ and so x is identically zero and therefore $\lambda = 0$ is a bifurcation point. But $0 \notin C_A(\lambda B)$ which implies that $\lambda = 0$ is not a bifurcation point. This contradiction tells us that equation 5.18 cannot have solutions (x,λ) with $\|x\|_2 \neq 0$, if $\lambda = 0$. Hence by Theorem 5.24, Definition 2.7 (iii.) and Definition 5.18, the result follows.

In almost exactly the same way as in the previous section, we may find a priori bounds for x, whenever (x,λ) is a solution of equation (5.18), provided λ lies in some specified interval. We make the following assumptions, which correspond to (Q1.) and (Q2.) of section 5.1.

(C3.) There exist non-negative constants, D, E, F and λ_2 in $\mathbb{R}$ with $E > 0$ and $\lambda_2 > 0$ such that $|g(t,x,y,0) - x| \leq D + E|x| + F|y|$, for x and y in $\mathbb{R}$ and $t \in [0,1]$, with $(\frac{\pi}{2E})^2 [-F + \sqrt{F^2 + 8E(1-q)}]^2 \geq \lambda_2$, where $q \in (0,1)$ is the constant from C_2.

(C4.) There exist $\lambda_1 \geq 0$ and $M > 0$ such that for each λ with $0 \leq \lambda_1 < \lambda < \lambda_2$, $\int_0^1 \lambda\{g(t,x,x',x'') - x(t)\}dt \neq 0$, for every $x \in X$ with $|x(t)| \geq M$ for all $t \in \mathbb{R}$.

Proceeding exactly as in section 5.1 we have the following theorem which is a consequence of Theorems 5.15 and 5.16.

<u>Theorem 5.26</u> If there exist t, x, $y \in \mathbb{R}$ such that $g(t,x,y,0) = 0$, then the constants E and λ_2 in (C3.) are such that $E \geq 1$ and $\lambda_2 < 2\pi^2$. If (C1.) - (C4.) are satisfied and (x,λ) is a solution of equation (5.18) with $\lambda \in (\lambda_1,\lambda_2)$, where λ_1 is as defined in (C4.) then $\|x\|_2 \leq M_1$ for some finite number $M_1 > 0$ which is independent of x and λ.

<u>Proof</u>: Immediate from Theorems 5.15 and 5.16.

Theorem 5.26 provides us with an improved version of Theorem 5.25.

Theorem 5.27 Assume that hypotheses (C1.) - (C4.) are satisfied and $\lambda_0 = (k_0\pi)^2$, for $k_0 \in \mathbb{N}$. Let M_1 be the finite number defined in Theorem 5.26. Then at least one of the following properties holds:

(a.) For any $M > 0$ there exists $\lambda_M > 0$ and $x_M \in X$ such that $\| x_M \|_2 = M$ and (x_M, λ_M) satisfies equation (5.18). If $M \to 0$, then $\lambda_M \to \lambda_0$ and, furthermore, if $M > M_1$ then $\lambda \notin (\lambda_1, \lambda_2)$;

(b.) There is $x_\lambda \in X$ such that for all $\lambda \in (\lambda_0, \infty)$, $\| x_\lambda \|_2 > 0$ and (x_λ, λ) satisfies equation 5.18. Furthermore, if $\|x_\lambda\|_2 > M_1$, then $\lambda \notin (\lambda_1, \lambda_2)$ and if $\| x_\lambda \|_2 \to 0$, then $\lambda \to \lambda_0$.

Proof: Follows from Theorem 5.26.

Corollary 5.28 Suppose there exist t, x, y $\in \mathbb{R}$ such that $g(t,x,y,0) = 0$ and hypotheses (C1.) - (C4.) are satisfied such that $\lambda_1 \leq \pi^2$ and $\lambda_2 \in (\pi^2, 2\pi^2)$. Then there is a solution (x_λ, λ) of equation 5.18 for every $\lambda \in (\pi^2, \lambda_2]$ such that $0 < \| x_\lambda \|_2 \leq M_1$, where M_1 is the finite number defined in Theorem 5.26.

Proof: In Theorem 5.27 set $\lambda_0 = \pi^2$ and the result is immediate. Notice that $\lambda_2 < 2\pi^2$ follows by Theorem 5.26.

Remark (1.) An equation similar to equation (5.18) is considered by Chow and Hale [6], Chapter 5, §5.8. They obtain a global bifurcation result when the nonlinear term g has the form $g(t,x,x')$. Since g does not depend on x'' it is compact and they use the Leray-Schauder degree to obtain their result.

(2.) The application given in this section is a new result.

5.3 Examples

In this final section, we give an example of an equation which satisfies the hypotheses (C1.) - (C4.) of the previous section and an example which satisfies (A1.) - (A3.), (Q1.) and (Q2.) of Section 5.1. Assume notation as before. Consider,

$x''(t) + \lambda x(t) = \lambda g(t,x,x',x'') = \lambda q \sin x(t) \sin(x''(t))$, where $\lambda \in \mathbb{R}$, $q \in (0,1)$ and $x : [0,1] \to \mathbb{R}$.

Then $q \dfrac{\| \sin x(t) \sin(x''(t)) \|_0}{\| x \|_2}$

$$= q \frac{\max\{|\sin x(t) \sin x''(t)| : t \in [0,1]\}}{\max\{\| x \|_0, \| x' \|_0, \| x'' \|_0}$$

$$\leq q \frac{\max\{|\sin x(t)|\ |\sin x''(t)| : t \in [0,1]\}}{\| x'' \|_0}$$

$$\leq q \max\{|\sin x(t)|\ \left|\frac{\sin x''(t)}{x''(t)}\right| : t \in [0,1]\}$$

$\to 0$ as $\| x \|_2 \to 0$. So (C1.) is satisfied.

Now $\| q \sin x(t) \sin x''(t) - q \sin x(t) \sin \hat{x}''(t) \|_0$

$$\leq q \| \sin x(t) \|_0 \| \sin x''(t) - \sin \hat{x}''(t) \|_0$$

$$\leq q\, 1 \| 2 \cos\left(\frac{x''(t) + \hat{x}''(t)}{2}\right) \sin\left(\frac{x''(t) - \hat{x}''(t)}{2}\right) \|_0$$

$$\leq q\, 2 \| \cos\left(\frac{x''(t) + \hat{x}''(t)}{2}\right) \|_0 \| \sin\left(\frac{x''(t) - \hat{x}''(t)}{2}\right) \|_0$$

$$\leq 2q \| \sin\left(\frac{x''(t) - \hat{x}''(t)}{2}\right) \|_0$$

$$= 2q \max\{|\sin\left(\frac{x''(t) - \hat{x}''(t)}{2}\right)| : t \in [0,1]\}$$

$$\leq 2q \max\{\left|\frac{x''(t) - \hat{x}''(t)}{2}\right| : t \in [0,1]\}$$

$$= \frac{2q}{2} \| x'' - \hat{x}'' \|_0 \leq q \| x - \hat{x} \|_2$$

implying that (C2.) holds since $q \in (0,1)$. Now $|g(t,x,y,0) - x| = |x|$, so (C3.) holds with $E = 1$, $D = F = 0$ and

$$\left(\frac{\pi}{2E}\right)^2 \left[-F + \sqrt{F^2 + 8E(1-q)}\right]^2 = \left(\frac{\pi}{2}\right)^2 8\ (1-q) = 2\pi^2(1-q) \leqslant \lambda_2.$$

Finally consider $\int_0^1 \lambda\{q \sin x(t) \sin(x''(t)) - x(t)\}dt$. Since $|q \sin x(t) \sin(x''(t))| \leq q$ for all $t \in [0,1]$ and for all $\lambda > 0$, then provided that $|x(t)| > q$ for all $t \in [0,1]$,

$\int_0^1 \lambda\{q \sin x(t) \sin x''(t) - x(t)\}dt \neq 0$ for all $\lambda \in (0,\lambda_2)$.

Hence (C4.) holds with $\lambda_1 = 0$ and M any number greater than q.

Thus (C1.) - (C4.) are all satisfied. Notice that if $0 < q < \frac{1}{2}$, then Corollary 5.29 applies, since $g(t,x,y,0) = 0$ for all $x, y \in \mathbb{R}$.

By considering the equation

$$x''(t) + b^2x(t) = g(x,x',x'') = q \sin x(t) \sin x''(t),$$

where $0 < b \in \mathbb{R}$, $q \in (0,1)$ and $x : \mathbb{R} \to \mathbb{R}$, a similar procedure shows that (A1.) - (A3.), (Q1.) and (Q2.) are satisfied.

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