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INJECTIVE MODULES AND REPRESENTATIONAL REPLETENESS

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Abstract

The purpose of this thesis is to introduce a new representation theoretic condition on prime ideals of noncommutative Noetherian rings, to study its properties and to investigate various classes of ring in relation to it, including enveloping algebras of solvable Lie algebras, finite dimensional algebras and skew-polynomial rings.

In Chapter 1, we review the basic theory of injective modules over both commutative and noncommutative rings and the strong connections which these modules have with the ideal structure of the ring. In particular, when R is a Noetherian ring and P a prime ideal of R whose clique satisfies the second layer condition, we recall the description of the fundamental series of a uniform injective module with assassinator prime P and of the associated primes of its non-zero factors, the set of fundamental primes of P.

Our object is to investigate the relationship between this set and the right clique of P and, in Chapter 2, we introduce the concept of *representational repleteness* to refer to such an assassinator where every linked prime is fundamental. Moreover, P satisfies *strong representational repleteness* when, for every natural number n, the n^{th} prime in any chain of links from P is a fundamental prime of the n^{th} layer. Finally, the ring R is said to be *(strongly) representationally replete* provided all its prime ideals are.

In Corollary 2.1.3, we present a classification of the strongly rep. rep. primes of a commutative Noetherian ring as all non-minimal primes together with those minimal primes, the localizations at which are fields; furthermore, we observe that, in a trivial way, the ring itself is rep. rep.. Moreover, in Lemma 2.1.6, we show that any prime of a Noetherian right hereditary ring whose clique satisfies the s.l.c. is strongly rep. rep..

In §2.3, we see that both representational repleteness and strong representational repleteness are preserved under localization at an appropriate Ore set (Theorem 2.3.2) and that they are Morita invariant properties (Theorem 2.3.3). Chapter 2 is concluded by discussing how the repleteness of a ring R relates to the repleteness of a factor R/I where I is polycentral or regularly polynormal (Theorem 2.3.14): in the first case, we find that P/I is rep. rep. in R/I if and only if P is rep. rep. in R ; in the second case, and with the additional assumption that r.cl.(P) is locally finite or that R satisfies the s.l.c., we show the forward implication holds. The first main result is then deduced:

Corollary 2.3.17. *The enveloping algebra of a finite dimensional solvable Lie algebra over $\mathbb{C}$ is representationally replete.* ∎

Finite dimensional algebras over algebraically closed fields are examined in Chapter 3 and we obtain, in Theorem 3.3.6, a necessary and sufficient condition for strong representational repleteness; namely, that the algebra be Morita equivalent to a certain type of factor of a generalized triangular matrix ring over division rings.

In Chapters 4 and 5, our attention is turned to skew-polynomial rings over a commutative Noetherian ring R. Here we provide sufficient conditions for the repleteness of a prime P in an iterated differential operator ring $T = R[\Theta;\Delta]$ of commuting derivations Δ on R (assuming R is a $\mathbb{Q}$-algebra) and in an Ore extension $S = R[\theta;\sigma]$ of a single automorphism σ of R. When $P = (P \cap R)T$ or $(P \cap R)S$ we find that P is rep. rep. and is strongly rep. rep. when $P \cap R$ (in the first case) or a prime minimal over $P \cap R$ (in the second case) is strongly rep. rep. in R (Corollary 4.2.14(i) and Theorem 5.2.3). Primes P of S containing θ are also always rep. rep. and are strongly rep. rep. when $P \cap R$ is strongly rep. rep. in R (Theorem 5.2.1) but this is not a *necessary* condition.

For any other prime P of T or S, we show that P is rep. rep. when $R_{P\cap R}$ is a regular local ring (in the first case) or a regular semilocal ring (in the second case) (Corollary 4.2.14(ii) and Theorem 5.2.4). We also include a sufficient condition for strong representational repleteness for a prime P of T in terms of eigenvectors of the action of Δ on $(P \cap R)/(P \cap R)^2$ (Corollary 4.2.14(iii)). Again, none of these conditions is necessary (see §§4.3 and 5.3).

Our main conclusion for skew-polynomial rings is the following:

Corollaries 5.2.6 and 4.2.15. *Let R be a regular commutative Noetherian ring, let σ be an automorphism of R and let Δ be a finite set of commuting derivations on R. Then, the Ore extension $R[\theta; \sigma]$ is a rep. rep. ring. Furthermore, if R is a $\mathbb{Q}$-algebra, the iterated differential operator ring $R[\Theta; \Delta]$ is a rep. rep. ring.* ∎

Finally, in Chapter 6, we discuss the extent to which repleteness of a Noetherian ring R carries over to the ordinary polynomial ring $R[x]$ which we assume satisfies the second layer condition. The question splits into two cases: when, for the prime P of $R[x]$, $P = (P \cap R)R[x]$ and when $P \supsetneqq (P \cap R)R[x]$. In the first case, we establish a positive result, Theorem 6.2.3, that P is rep. rep. [resp. strongly rep. rep.] in $R[x]$ if and only if $P \cap R$ is rep. rep. [resp. strongly rep. rep.] in R. In the second case, while we show in Theorem 6.2.1 that P is always strongly rep. rep. when R is commutative, we see (in §6.3) that, in general, P need not be rep. rep. in $R[x]$, even when $P \cap R$ is strongly rep. rep. in R.

We do, however, discuss some partial results for the second case, using a method based on [B&G] for constructing $R[x]$-module injective hulls from R-module injective hulls (see (6.2.4) to (6.2.16)). This method is a generalization of the modules of inverse polynomials which arise in [N2] when constructing the injective hull of the trivial module of a commutative polynomial ring over a field (see §2.2).

Throughout the thesis, we include numerous examples and counterexamples illustrating the concepts introduced and the results deduced. These include explicit calculations of injective hulls (§2.2) after the manner of [N2] but mainly rely on results of [B&W] and [L&L] which allow the fundamental primes to be determined from the ideal structure of the ring (§§4.3, and 5.3).

Contents

Introduction

Since 1940, when Baer proved that every module embeds in an injective module, it has been evident that injective modules would play a leading role in representation theory. However, it was not until Eckmann and Schopf developed the notion of the injective hull in 1953 that the study of the structure of injectives began, culminating, for commutative Noetherian rings, in 1958 with Matlis's classical results (§1.1). In particular, if R is a commutative Noetherian ring and E is a uniform right R-module with assassinator prime P, then every element of E can be annihilated by some power of P (see Theorem 1.1.14).

Much work has been done recently in extending this theory to noncommutative rings (see for example [J2], [B&W] and [L&L]) and, in §1.3, we discuss the noncommutative analogue of the above theorem of Matlis, obtained as [B&W, Lemma 5.4]. For a Noetherian ring R, we denote by $A_n(P)$ the associated primes occurring at the n^{th} layer of a uniform injective with assassinator prime P (see 1.3.1) whose first layer we can assume is a torsion-free R/P-module (see 1.3.1 and 1.3.4(ii) for details) and recalling the definition (1.2.1) of second layer links between prime ideals, we denote by $X_n(P)$ the set of primes $(n-1)$ links from P (see 1.2.3). In the commutative case, these sets are either empty or contain only the prime P.

By insisting that P satisfies the second layer condition (1.2.5), Jategaonker's "Main Lemma" (Lemma 1.2.4) requires that $A_2(P) \subseteq X_2(P)$. However, if we assume that the clique of P satisfies the s.l.c., in which situation detailed results are obtained in [B&W], it is known that $A_n(P) \subseteq X_n(P)$ for all $n \in \mathbb{N}$ (Corollary 1.3.5). The question of when equality holds is raised in [B&W, 5.11] where it is noted that $\bigcup_{m=1}^{\infty} X_m(P)$ can be infinite even when $\bigcup_{m=1}^{\infty} A_m(P)$ is finite. (For instance, consider (4.3.4): $R = \mathbb{C}[x, y : yx = x(y+1), x^2 = 0]$.) It is also noted, however, that $X_2(P) = A_2(P)$ always holds and it is the purpose of this thesis to consider the higher layers. We introduce the following definitions.

Let P be a prime ideal of a Noetherian ring R such that cl.(P) satisfies the s.l.c.. Then we say (2.1.1) that P is *(right) representationally replete* in R

provided $\bigcup_{m=1}^{\infty} X_m(P) = \bigcup_{m=1}^{\infty} A_m(P)$ and we say that P is *(right) strongly representationally replete* in R provided $X_m(P) = A_m(P)$ for each $m \in \mathbb{N}$. Thus, representational repleteness asserts that every prime Q in the right clique of P does occur as an associated prime of some layer of a uniform injective module with assassinator prime P, while strong representational repleteness guarantees that, if Q is $(n-1)$ links from P, then Q will occur at the n^{th} layer.

Our main tools in deciding which primes are (strongly) rep. rep., Theorems 1.3.7 and 1.3.11, come from [L&L] and [B&W] and relate to the bimodule structure of the ring. The possession of these properties by a ring depends on whether the bimodules arising from the second layer links can be combined to form an appropriate ideal link between a given prime in r.cl.(P) and P. Repleteness is a kind of smoothness property on a ring.

Trivially, commutative Noetherian rings are rep. rep. (that is, all their primes are) and, in Corollary 2.1.3, we classify their strongly rep. rep. primes as all non-minimal primes together with those minimal primes the localizations at which are fields. We conclude §2.1 by showing:

Lemma 2.1.6. *Any prime of a Noetherian right hereditary ring whose clique satisfies the s.l.c. is strongly rep. rep..* ∎

Since Theorems 1.3.7 and 1.3.11 translate the study of repleteness into a question of the ideal theory of the ring and since injective hulls tend to be inaccessible, most of the examples we consider avoid explicit descriptions of injective hulls. However, in §2.2, we include two examples where we display the repleteness of primes by constructing appropriate hulls. Thus, Example 2.2.5 uses the method of [N2] to express $E_{k[x_1,\ldots,x_n]}(k)$, where k is a field, as a module of inverse polynomials and so illustrates the strong repleteness of $<x_1, \ldots x_n>$ in $k[x_1, \ldots, x_n]$. Using a similar technique, Example 2.2.7 shows that a co-Artinian prime is strongly rep. rep. in the two dimensional solvable non-Abelian Lie algebra over $\mathbb{C}$. In (2.2.6), (2.2.8) and (2.2.9), we discuss generalizations of these constructions.

In §2.3, we consider how these properties behave under various constructions. Thus, Theorem 2.3.2 shows that *(strong) representational repleteness is* *preserved under localization at an appropriate Ore set*, and Theorem 2.3.3 shows that *(strong) representational repleteness is Morita invariant.* The rest of §2.3 is concerned with how the repleteness of a ring R relates to the repleteness of a factor R/I where I is polycentral or regularly polynormal (see Definitions 2.3.4). We prove the following:

Theorem 2.3.14. *Let R be a Noetherian ring with a prime ideal P and let I be an ideal of R with $I \subseteq P$.*

(i) Suppose that cl.(P) *satisfies the second layer condition. If I is polycentral,*

$$P/I \text{ is rep. rep. in } R/I \iff P \text{ is rep. rep. in } R.$$

(ii) Suppose either that cl.(P) *satisfies the second layer condition and* r.cl.(P) *is locally finite (that is, $|X_n(P)|$ is finite for each $n \in \mathbb{N}$) or that R satisfies the second layer condition. If I is regularly polynormal,*

$$P/I \text{ is rep. rep. in } R/I \implies P \text{ is rep. rep. in } R. \qquad \blacksquare$$

By localizing at the appropriate clique and exploiting properties of the localized ring, we use this theorem to obtain:

Corollary 2.3.17. *The enveloping algebra of a finite dimensional solvable Lie algebra over $\mathbb{C}$ is rep. rep..* ■

Chapters 3, 4 and 5 discuss particular classes of rings, in an attempt to decide when their primes are (strongly) rep. rep., drawing heavily on the known structure of these rings. Thus Chapter 3 uses Harada's description [Ha] of hereditary semiprimary rings as certain factors of generalized triangular matrix rings to classify the strongly rep. rep. primes:

Theorem 3.3.6. *Let k be an algebraically closed field and let A be a finite dimensional k-algebra. With the notation of (3.1.3) and (3.1.7), the following conditions are equivalent.*

(i) A is strongly rep. rep..

(ii) There exists T a hereditary g.t.a. matrix ring over division rings with radical J and an ideal $I \subseteq J^2$ such that A is Morita equivalent to T/I and such that $(J^m)_{j,i} \not\subseteq (J^{m+1}+I)_{j,i}$ whenever i, j and $m \in \mathbb{N}$ and $(J^m)_{j,i} \neq (J^{m+1})_{j,i}$. ■

In Chapter 4, we use Goodearl's classification [G2] of the link graph of differential operator rings to provide the following sufficient condition for repleteness in these rings:

Corollary 4.2.14. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \dots, \delta_n\}$ be a set of commuting derivations on R. Suppose P is a prime ideal of $T = R[\Theta; \Delta]$ such that $M = P \cap R$.*

(i) If $P = MT$ then P is rep. rep. in T and is strongly rep. rep. if and only if M is strongly rep. rep. in R;

(i) If R_M is a regular local ring, then P is rep. rep. in T. ∎

We conclude our discussion of differential operator rings with a series of examples (§4.3) which illustrate the concepts introduced and, in particular, this last corollary. Thus, while we see in Example 4.3.2 that regularity is not a necessary condition for the repleteness of P, (4.3.4) and (4.3.5) show that it cannot be dropped altogether. In the situation of Corollary 4.2.14(ii), we also provide a sufficient condition for P to be strongly rep. rep. although Example 4.3.3 shows that this is not a necessary condition either. It may, indeed, be found advantageous to read through these examples before embarking on the details of Chapter 4.

In Chapter 5, we discuss the repleteness of primes in Ore extensions of the form $S = R[\theta; \sigma]$, where σ is an automorphism and R is a commutative Noetherian ring, using Poole's description [Po] of the second layer links. Let P be a prime of S and put $A = P \cap R$. Following Poole, we refer to primes of S containing θ as *type (1) primes*. The other primes are either *upper primes* (when $P \supsetneq AS$) or *lower primes* (when $P = AS$). (See Notation 5.1.1.) In §5.2, we discuss each type of prime in turn. For type (1) primes, A is a prime ideal of R and we prove the following:

Theorem 5.2.1. *With the notation of (5.1.1), let P be a type (1) prime of S. Then P is rep. rep. in S and, furthermore, if A is strongly rep. rep. in R, then P is strongly rep. rep. in S.* ∎

We note, however, that this condition for strong repleteness is not a necessary condition (Remark 5.2.2). On the other hand, *some* condition is required as we see in Example 5.3.2.

For upper and lower primes, we have the additional problem that A may not be a prime ideal of R. However, since A is a σ-*cyclic* semiprime ideal (see (5.1.4)), the repleteness of lower prime ideals is readily understood:

Theorem 5.2.3. *With the notation of (5.1.1), let P be a lower prime ideal of S. Then P is rep. rep. in S. Furthermore, letting J be a prime ideal of R minimal over $P \cap R$, P is strongly rep. rep. in S if and only if J is strongly rep. rep. in R.* ∎

For upper prime ideals, we obtain a result very similar to that for differential operator rings (Corollary 4.2.14(ii)):

Theorem 5.2.4. *With the notation of (5.1.1), let P be an upper prime ideal of S. If R_A is a regular semilocal ring, then P is rep. rep. in S.* ∎

Again, we discuss, in §5.3, some examples (Examples 5.3.3 and 5.3.4) which show, in particular, that this condition is not a necessary one either. Example 5.3.5, however, does show that not every upper prime is rep. rep..

We observe that both T in (4.2.14) and S in (5.2.4) are rep. rep. rings provided their coefficient rings are regular (Corollaries 4.2.15 and 5.2.6).

In Chapter 6, we discuss whether the repleteness of a ring R carries over to the ordinary polynomial ring $R[x]$. We construct various examples in §6.3 (for instance Example 6.3.2) which show that this is not the case. However, we are able to derive some positive results. In particular, for commutative rings we have:

Theorem 6.2.1. *Let R be a commutative Noetherian ring and let P be a prime ideal of the polynomial ring $R[x]$. If $P \supsetneqq (P \cap R)R[x]$ then P is strongly rep. rep. in R while, if $P = (P \cap R)R[x]$, then P is strongly rep. rep. in $R[x]$ if and only if $P \cap R$ is strongly rep. rep. in R.* ∎

For noncommutative rings, we show that a prime $P \supsetneqq (P \cap R)R[x]$ of $R[x]$ need not be rep. rep. even when $P \cap R$ is strongly rep. rep. in R (see Example 6.3.3). However, we can establish:

Theorem 6.2.3. *Let R be a Noetherian ring and assume that the polynomial ring $R[x]$ satisfies the second layer condition. Let J be a prime ideal of R. Then $JR[x]$ is rep. rep. [resp. strongly rep. rep.] in $R[x]$ if and only if J is rep. rep. [resp. strongly rep. rep.] in R.* ∎

Finally, we discuss some conjectures on how we might obtain results for primes $P \supsetneqq (P \cap R)R[x]$ in the noncommutative case. In particular, we show how unpublished work of Brown and Goodearl [B&G] on constructing injective hulls over a q-skew polynomial ring from injectives over the coefficient ring, might apply in this case to ordinary polynomial rings. The method involved is, in fact, a return to the ideas of §2.2 (see (6.2.4) to (6.2.17)).

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A Note on Terminology

The basic terminology and notation will be consistent with those of [G&W], [McC&R], [A&F] and [Rm]. The following conventions, in particular, will be fixed.

Throughout this thesis, all rings contain an identity element and, by a *Noetherian* ring, we mean a *left and right Noetherian* ring. All modules are unital and are right modules unless otherwise indicated.

Given a ring R and a right R-module M, the *annihilator* of a subset N of M, the right ideal

$$\mathrm{ann}_R(N) = \{r \in R : Nr = 0\},$$

is also denoted by $\mathrm{ann}(N)$ if the ring is not in doubt. Moreover, the *annihilator* of N in a subset Y of R is denoted by

$$\mathrm{ann}_Y(N) = \{r \in Y : Nr = 0\}.$$

Similarly, the *annihilator* in N of Y is

$$\mathrm{ann}_N(Y) = \{m \in N : mY = 0\}.$$

If $\mathrm{ann}(M) = 0$, M is said to be *faithful* while M is *fully faithful* if every non-zero submodule of M is faithful. An *associated* prime, P, of M is an annihilator of a non-zero submodule, N, of M such that every non-zero submodule of N has annihilator P, while, if R is right Noetherian and M is uniform, its unique associated prime is termed the *assassinator* prime.

Similar definitions apply for left modules and, if we wish to emphasize (for instance in bi-modules) whether an annihilator is taken with respect to a left or right module, we replace "ann" with "l.ann" or "r.ann" respectively.

By a *regular* element of the ring R, we mean a non-zero-divisor of R, that is, an element $x \in R$ such that $\mathrm{r.ann}(x) = \mathrm{l.ann}(x) = 0$. Given an ideal I of R, we denote by $\mathcal{C}_R(I)$ (or $\mathcal{C}(I)$) the set of elements of R regular modulo I. By a *right*

Ore set in a ring R is meant a multiplicatively closed subset X of R containing the identity which satisfies the *right Ore condition*, namely that

$$rX \cap xR \neq \emptyset$$

for all $r \in R$ and $x \in X$. A *left Ore set* is defined similarly and, of course, a set which is both left and right Ore is called an *Ore set*. It is well known that the regular elements of a semiprime right Noetherian ring form a right Ore set.

When M is a right R-module and X is a right Ore set in R, the *X-torsion submodule* of M is the submodule

$$t_X(M) = \{m \in M : mx = 0 \text{ for some } x \in X\}.$$

If $t_X(M) = M$, M is said to be *X-torsion* while, if $t_X(M) = 0$, M is said to be *X-torsion-free*. In particular, if R is a semiprime right Noetherian ring and X is the right Ore set of regular elements of R, we denote $t_X(M)$ by $Z(M)$ and call it simply the *torsion submodule* of M. Similarly, in this case, M is said to be *torsion* if $Z(M) = M$ and *torsion-free* if $Z(M) = 0$.

If R is a right Noetherian ring and X is a right Ore set, we denote the *(right Ore) localization* with respect to X by RX^{-1} (although R may not embed in RX^{-1} unless X consists of regular elements) and if, moreover, R is commutative, we denote the *localization* at a semiprime ideal A by R_A.

The set of prime ideals of R will be denoted by Spec(R). If $S \subseteq$ Spec(R), then an *S-semiprime ideal* of R will be an intersection of primes of S. Of course, when R is right or left Noetherian, we can insist that this intersection is finite [G&W, Theorem 2.4].

Finally, we use $\mathbb{C}$ for the complex numbers, $\mathbb{Q}$ for the rationals, $\mathbb{Z}$ for the integers and $\mathbb{N}$ for the natural numbers, $\{1, 2, 3, \ldots\}$.

Chapter 1 : Injective Modules

§1.1 Matlis's Theorems

Definition 1.1.1. Let R be any ring and let E be a right R-module. Then, E is *injective*, provided that, whenever A and B are right R-modules with $A \subseteq B$ and f is an R-module homomorphism from A to E, then f extends to an R-module homomorphism from B to E. ∎

What are now known as injective modules, first appeared in 1935, for the ring of integers, in Zippin's work on Abelian groups [Z]. (See Remarks 1.1.5.) However, it was Baer in 1940 who studied the general notion for arbitrary rings [Ba]. His paper is concerned with what were called "complete" modules, that is, modules satisfying condition (c) of the following result which is now known as "Baer's Criterion" (see [G&W, Proposition 4.11). In fact, the modern terminology of "injective" modules, along with that of the dual notion, "projective" (a module is *projective* provided it is a direct summand of a free module), was introduced in 1956 in Cartan and Eilenberg's important work [C&E].

Theorem 1.1.2. *Let R be a ring and A a right R-module. The following statements are equivalent:*

(a) the right R-module A is injective;

(b) for every right ideal I of R and every R-module homomorphism f from I to A, there exists $a \in A$ such that $f(r) = ar$ for all $r \in I$;

(c) for every right ideal I of R and every R-module homomorphism f from I to A, f extends to an R-module homomorphism from R to E. ∎

Remarks 1.1.3. (i) An element a of a $\mathbb{Z}$-module A is said to be *divisible* by a non-zero integer n provided $a \in nA$. The module A is called *divisible* provided every element of A is divisible by every non-zero integer; that is, provided $nA = A$ for all non-zero $n \in \mathbb{Z}$. Since an element $a \in A$ is divisible by a non-zero integer n if and only if the homomorphism $n\mathbb{Z} \longrightarrow A$ sending n to a extends to $\mathbb{Z}$, it follows immediately from Baer's Criterion that a $\mathbb{Z}$-module is injective if and only if it is divisible [G&W, Proposition 4.2].

(ii) It is easily seen that all direct summands and all direct products of injective modules are injective. However, while it follows that all finite direct sums of injectives are injective, this is not true of arbitrary direct sums. In fact, a ring R is right Noetherian if and only if every direct sum of injective right modules is again injective (see [Pa, Theorem 1] and [Bs, Theorem 1.1]). ∎

Our main interest in this section is with Matlis's celebrated paper of 1958; however first we require some basic terminology.

Definitions 1.1.4. Let R be any ring and let M be a right R-module. We say that a submodule N is *essential* in M and write $N \leq_e M$ provided $L \cap N \neq 0$ for all non-zero submodules L of M. In this case, M is said to be an *essential extension* of N. The module M is called *uniform* when M is non-zero and $A \cap B \neq 0$ for all non-zero submodules A and B of M, or, equivalently, when M is non-zero and $N \leq_e M$ for all non-zero submodules N of M. Uniform right ideals were introduced by Goldie [Go1] who extended the concept to modules in [Go2]. (We note that every non-zero Noetherian module has a uniform submodule [G&W, Corollary 4.15].) Then again, we say that M is *indecomposable* provided the only direct summands of M are 0 and M. ∎

Remarks 1.1.5. Clearly uniform modules are non-zero indecomposables. However, $M = \frac{\mathbb{C}[x,y]}{<xy>}$ is not a uniform $\mathbb{C}[x,y]$-module since $xM \cap yM = 0$, while $xM + yM \neq M$ and M is indecomposable. Nonetheless, for injective modules, the terms "uniform" and "non-zero indecomposable" mean the same [Ma, Proposition 2.2]. ∎

In the first work on the subject, it is proved in [Z] that an Abelian group is divisible if and only if it is a direct summand of every Abelian group which contains it. It was soon shown that this is not a special property of $\mathbb{Z}$-modules: the generalization to an arbitrary ring (the equivalence of (a) and (b) in the next theorem) is proved in [Ba, Theorem 1]. The equivalence of (a) and (c) is contained in [E&S, §4].

Theorem 1.1.6. *Let R be a ring and let E be an R-module. Then the following are equivalent:*

(a) E_R is injective;

(b) E_R is a direct summand of every module which contains it;

(c) E_R has no proper essential extensions. ∎

Definitions 1.1.7. In 1940, Baer showed that every module M can be embedded in an injective module [Ba, Theorem 3] and this we call an *injective extension*, while, in 1953, Eckmann and Schopf proved that every module has a maximal essential extension which is also a minimal injective extension [E&S, §4]. We use the term *injective hull of M*, introduced by Rosenberg and Zelinsky in 1959 (see [R&Z]), to refer to such an extension which can be more easily defined as any injective module E such that $M \leq_e E$. (The alternative term *injective envelope* is also frequently encountered.) ∎

A proof of part (i) of the next result can be found in [G&W, Theorem 4.8(a)] while part (ii) is proved in [G&W, Proposition 4.9]. In particular, we see that the injective hull of an R-module M is unique up to isomorphism.

Theorem 1.1.8. *Let R be a ring and let M be an R-module.*

(i) Any injective module containing M contains an injective hull of M_R.

(ii) Let E and F be injective hulls of M_R. Then, the identity map on M extends to an R-module isomorphism between E and F. ∎

In the light of Theorem 1.1.8(ii), we can usually refer to *the* injective hull of M_R and we write it as $E_R(M)$ or $E(M)$ if the ring is not in doubt. We note a standard result which will frequently be useful. It appears as [G&W, Exercise 4E].

Lemma 1.1.9. *Let I be an ideal in a ring R, let A be a right (R/I)-module and let E be an injective hull for A_R. Then, the right (R/I)-module $\mathrm{ann}_E(I)$ is an injective hull for $A_{R/I}$.*

Proof. Since $A \leq_e E$ as R-modules, it is immediate that $A \leq_e \mathrm{ann}_E(I)$ as R-modules and hence as R/I-modules. Now suppose that B and C are R/I-modules with $B \leq C$ and suppose that f is an (R/I)-module homomorphism from B to $\mathrm{ann}_E(I)$. Then, f is an R-module homomorphism from B to E and so extends to an R-module homomorphism, f', from C to E. Since

$$f'(C).I = f'(CI) = f'(0) = 0 ,$$

we see that f' is also an R/I-module homomorphism from C to $\mathrm{ann}_E(I)$ and thus $\mathrm{ann}_E(I)$ is an injective (R/I)-module. This completes the proof of the lemma. ∎

Remark 1.1.10. It is easy to check that any essential extension of a uniform module is uniform (this also follows from [Ma, Proposition 2.2]). However, it is also easy to see that an essential extension of an indecomposable module need not be indecomposable. For if M_R is a non-zero indecomposable which is not uniform then $E_R(M)$ cannot be uniform (non-zero submodules of uniform modules are obviously uniform) and hence is not indecomposable. ∎

We now state a result which was proved independently in [Ma, Theorem 2.5] and [Pa, Theorem 2].

Theorem 1.1.11. *Every injective right module of a right Noetherian ring has a decomposition as a direct sum of indecomposable injective submodules.* ∎

As a consequence of this theorem, we need only consider uniform injective modules if we wish to understand all injectives over a Noetherian ring. Matlis's paper goes on to obtain a description of the uniform injectives over a commutative Noetherian ring and we now state [Ma, Proposition 3.1].

Theorem 1.1.12. *There is a one to one correspondence between the prime ideals of a commutative Noetherian ring R and the uniform injective R-modules given by $P \mapsto E(R/P)$ for each prime ideal P of R. The reverse bijection is given by $E \mapsto$ (assassinator of E) for each uniform injective module E.* ∎

Remarks 1.1.13. Over noncommutative Noetherian rings, this bijection breaks down in general. Of course, $E(R/P)$ need not be uniform but still we can take U to be a uniform right ideal of R/P and then $E_R(U)$ is uniform with assassinator P. However, the mapping $E \mapsto$ (assassinator of E), is a bijection between the uniform injectives and the prime ideals if and only if R is an FBN ring [G&W, Theorem 8.14]. Usually, there are more uniform injectives than there are prime ideals. As an example of this, we can take R to be a simple Noetherian domain which is not a division ring (for instance the Weyl algebra over a division ring of characteristic zero) and A to be a simple right R-module. Then $E(R_R)$ and $E(A)$ are uniform modules which are not isomorphic, although both have assassinator 0, the unique prime of R. ∎

In the commutative case, Matlis then shows how an injective module can be split up into layers which are easier to understand than the whole module.

Theorem 1.1.14. *Let R be a commutative Noetherian ring and E a uniform injective right R-module with assassinator prime P. Set $E_m = \{e \in E : eP^m = 0\}$ for $m \geq 0$. Then, for all $n \in \mathbb{N}$,*

(i) E is isomorphic to the injective hull of $(R/P)_R$;

(ii) $E = \bigcup_{m=1}^{\infty} E_m$;

(iii) $E_n = \left\{e \in E : \text{ann}\left(\frac{eR+E_{n-1}}{E_{n-1}}\right) = P\right\}$;

(iv) $E_n/E_{n-1} = \text{ann}_{E/E_{n-1}}(P)$;

(v) P is the unique associated prime of E/E_{n-1} provided $E \neq E_{n-1}$;

(vi) E_n/E_{n-1} is a finite dimensional vector space over the quotient field, Q, of R/P and this dimension is equal to the dimension over Q of $(PR_P)^{n-1}/(PR_P)^n$. In particular, $E_1 \cong Q$ and $E_n = E_{n-1}$ if and only if $(PR_P)^n = (PR_P)^{n-1}$.

Proof. (i) is [Ma, Prop. 3.1], (ii)–(v) follow from [Ma, Theorem 3.4] and (vi) follows from [Ma, Lemma 3.4 and Theorem 3.9]. ∎

Notwithstanding the failure of Theorem 1.1.12 in the noncommutative case, it still makes sense to ask whether we can extend this description. For instance, if E is a uniform injective with assassinator P, we can let $E_1 = \text{ann}_E(P)$ and form E/E_1. For an analogue of Theorem 1.1.14, we would first need to know the associated primes of this factor. Then again, since every element of E can be annihilated by some power of P when R is commutative, we would ask whether the same happens for noncommutative R.

As we will find in the rest of this chapter, the noncommutative case is very different, although, with suitable assumptions, it can be described in much the same format.

§1.2 Prime Links and the Second Layer Conditions

In order to answer the question introduced at the end of the last section, we require to recall a combinatorial structure of the prime spectrum of a Noetherian ring, the *graph of links*. While further information about links and fundamental primes can be found in [J2], we first recall the basic terminology.

Definitions 1.2.1. For a Noetherian ring R with prime ideals P and Q, we say there is an *ideal link* from Q to P and write $Q \rightsquigarrow\rightsquigarrow P$ if there are ideals $A \subset B$ of R such that B/A is torsion-free (or equivalently, by [G&W, Lemma 7.3 and Proposition 7.4], fully faithful) as a right R/P and as a left R/Q-module. If $QP \subseteq A$ and $B = Q \cap P$, we call such a link a *second layer link* and write $Q \rightsquigarrow P$. A link from a prime ideal to itself will be referred to as a *trivial link*. Unless otherwise specified, links will always be assumed to be second layer links.

Given $P \in \mathrm{Spec}(R)$, the *right clique* of P is defined

$$\begin{aligned}\text{r.cl.}(P) = \big\{Q \in \mathrm{Spec}(R) : Q = P_n \rightsquigarrow P_{n-1} \rightsquigarrow \cdots \rightsquigarrow P_1 = P,\\ \text{for some } P_i \in \mathrm{Spec}(R) \text{ and some } n \in \mathbb{N}\big\},\end{aligned}$$

while the *clique* of P is defined

$$\begin{aligned}\text{cl.}(P) = \big\{Q \in \mathrm{Spec}(R) :\ \text{there exist } P_i \in \mathrm{Spec}(R) \text{ and } n \in \mathbb{N} \text{ such that } P_n = Q,\\ P_1 = P \text{ and for each } 2 \le m \le n,\ \text{either } P_m \rightsquigarrow P_{m-1} \text{ or } P_{m-1} \rightsquigarrow P_m\big\}.\end{aligned}$$

We note that, by definition,

$$P \in \text{r.cl.}(P) \subseteq \text{cl.}(P)$$

even if P is not linked to itself. ∎

The notion of ideal links appeared first in work of Jategaonkar (see [J1]) while second layer links were introduced by Müller for FBN rings as an obstruction to localization (see [Mül]). This latter aspect, together with most of the present section can be found in [J2], however we will not discuss the localization criteria beyond noting the following result which can be found in [G&W, Lemma 12.17] and in the proof of [J2, Theorem 5.4.5].

Theorem 1.2.2. *Let R be a Noetherian ring and let P and Q be prime ideals of R such that $Q \rightsquigarrow P$. Let $\mathcal{C}$ be a right Ore set in R. If the elements of $\mathcal{C}$ are regular modulo P then they are regular modulo Q.* ∎

Of course, in a prime Noetherian ring, a right Ore set which does not contain zero consists of regular elements (see for instance [G&W, Lemma 9.21]) and so the conclusion of Theorem 1.2.2 could be re-expressed to say that if $\mathcal{C}$ is disjoint from P then it is disjoint from Q.

We note that, if P is a prime ideal of a commutative ring, the clique of P is just $\{P\}$ although P is not necessarily linked to itself. We note also that, if P and Q are maximal ideals, $Q \rightsquigarrow P$ if and only if $\frac{Q \cap P}{QP}$ is non-zero. As an example of a second layer link, we can consider the ring $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ which has two prime ideals $\begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ and just one second layer link, namely

$$\begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix} .$$

Other examples can be found in [G&W, Chapter 11].

Before indicating our motivation for considering second layer links, we complete our discussion of the link terminology by introducing some further notation which we will require in the next section and which will, indeed, be fixed throughout this thesis.

Notation 1.2.3. Let R be a Noetherian ring and P a prime ideal of R. We define $X_1(P) = \{P\}$ and for $n \geq 1$,

$$X_{n+1}(P) = \{Q \in \mathrm{Spec}(R) : Q \rightsquigarrow I \text{ for some } I \in X_n(P)\} .$$

Where the prime P is not in doubt, we abbreviate $X_n(P)$ to X_n. Finally, we note that

$$\text{r.cl.}(P) = \bigcup_{n=1}^{\infty} X_n(P) \ . \qquad \blacksquare$$

The fundamental connection between these concepts and the representation theory of Noetherian rings is made clear in the following "Main Lemma" of Jategaonkar [J2, Lemma 6.1.3].

Lemma 1.2.4. *Let R be a Noetherian ring, M_R uniform with assassinator prime P and let $U = \text{ann}_M(P)$. Suppose $0 < U < M$ with $(M/U)_R$ uniform and having assassinator prime Q. Assume $MQ \subseteq U$ and that, for all $M' \leq M$ with $M' \not\leq U$,* $\text{ann}(M') = \text{ann}(M)$.

Then, either (1) $Q \rightsquigarrow P \quad \left(\textit{via } \frac{Q \cap P}{\text{ann}(M)}\right)$,

or else (2) $\text{ann}(M) = Q \subsetneq P$. ■

The occurrence of case (2) in Lemma 1.2.4 is excluded by the following definitions which can be found, for instance, in [G&W, p.183].

Definitions 1.2.5. Given R a Noetherian ring and P a prime ideal, if case (2) cannot occur P is said to satisfy the *right strong second layer condition.* For any $Y \subseteq \text{Spec}(R)$, Y satisfies the right s.s.l.c. if every prime in Y does, and R satisfies the right s.s.l.c. if $\text{Spec}(R)$ does. Analogous definitions apply on the left, and R is said to satisfy the s.s.l.c. if it satisfies the left and right s.s.l.c..

Including the hypothesis "$U_{R/P}$ torsion-free" in the lemma gives rise to similar definitions substituting "*second layer condition*" for "*strong second layer condition*". It is an open question whether a prime can satisfy the s.l.c. without satisfying the s.s.l.c..

Examples of rings which are known to satisfy the s.s.l.c. include commutative rings (trivially), Artinian rings (since they contain no primes P and Q with $Q \subsetneq P$), FBN rings [G&W, p183], group rings of polycyclic-by-finite groups over commutative Noetherian coefficient rings [J2, Theorem A.4.6] and enveloping algebras of solvable Lie algebras [J2, Theorem A.3.9].

On the other hand, [G&W, Exercise 11H] is an example of a Noetherian ring which does not satisfy the second layer condition. ■

In fact, almost all the rings considered in this thesis will satisfy at least the second layer condition and, in such rings, [G1, Lemma 1.3] provides us with simplified characterisations of ideal links and second layer links.

Lemma 1.2.6. *Let R be a Noetherian ring with the s.l.c. and let P and Q be prime ideals. Then,*

(i) $Q \rightsquigarrow P$ *if and only if* ${}_{R/Q}\left|\frac{Q\cap P}{QP}\right|_{R/P}$ *is faithful on both sides, while*

(ii) $Q \rightsquigarrow\rightsquigarrow P$ *if and only if we can find ideals A and B of R such that* $QB + BP \subseteq A$ *and such that* ${}_{R/Q}\left|\frac{B}{A}\right|_{R/P}$ *is faithful on both sides.* ∎

The following lemma will frequently be useful. Part (i) is contained in [J2, Theorem 8.2.4] while part (ii) is [J2, Theorem 8.2.9]. We first recall the definition of *classical Krull dimension*, introduced in this form in [K1, Definition 11].

Definition 1.2.7. Let R be any ring and define the sets $\mathcal{S}_\alpha$ of prime ideals of R, for each ordinal α, by the following transfinite induction. First, we set $\mathcal{S}_{-1}$ equal to the empty set (here, -1 is considered to be an ordinal number). Now let $\alpha \geq 0$ and assume that $\mathcal{S}_\beta$ has been defined for each ordinal $\beta < \alpha$; we let $\mathcal{S}_\alpha$ consist of those prime ideals P of R such that every prime properly containing P is a member of $\bigcup_{\beta<\alpha} \mathcal{S}_\beta$. If there is some ordinal γ such that every prime ideal of R belongs to $\mathcal{S}_\gamma$ then we say that the classical Krull dimension of R exists and is equal to the smallest such γ. We write this as " $\mathrm{Cl.K.dim}(R) = \gamma$ ".

If the classical Krull dimension is finite, then it is simply the supremum of the lengths of chains of prime ideals and, indeed, this was the original definition. If R is a ring which satisfies the ascending chain condition on prime ideals (in particular, if R is Noetherian), then its classical Krull dimension exists (see for instance [G&W, Proposition 12.1]) although it need not be finite. ∎

Lemma 1.2.8. *Suppose R is a Noetherian ring, let P be a prime ideal of R and let $Q_1, Q_2 \in \mathrm{cl.}(P)$.*

(i) If $\mathrm{cl.}(P)$ satisfies the s.l.c. and $Q_1 \subseteq Q_2$, then $Q_1 = Q_2$.

(That is, $\mathrm{cl.}(P)$ satisfies the incomparability condition.)

(ii) If R satisfies the s.l.c., then $\mathrm{Cl.K.dim}(R/Q_1) = \mathrm{Cl.K.dim}(R/Q_2)$. ∎

§1.3 The Fundamental Series

Notation 1.3.1. Let R be a Noetherian ring. Then, for any uniform injective module, E_R, with assassinator prime P, we put $E_0 = 0$ and $E_1 = \operatorname{ann}_E(P)$. Then, for $n \geq 0$, we let

$$A_{n+1}(E) = \{Q \ : \ Q \text{ is an associated prime of } E/E_n\}$$

and

$$E_{n+1} = \left\{e \in E \ : \ \operatorname{ann}\left(\frac{eR + E_n}{E_n}\right) \text{ is a finite intersection of primes in } A_{n+1}(E)\right\}.$$

■

Definitions 1.3.2. If either cl.(P) satisfies the s.l.c. and $E = E(U)$ for some uniform right ideal U of R/P (all such $E(U)$ are isomorphic), or cl.(P) satisfies the s.s.l.c., then $A_n(E)$ is known as the set of n^{th} *layer fundamental primes* of E and $\{E_n\}$ as the *fundamental series* of E. ■

The next result is due to Brown and Warfield [B&W, Lemma 5.4 and Theorem 5.10]. (In fact, (1)–(3) are only stated in [B&W] for the case where the clique of P satisfies the strong second layer condition but the proof works in either case.)

Theorem 1.3.3. *Let R be a Noetherian ring and let P be a prime ideal of R. Suppose that either* cl.(P) *satisfies the s.l.c. and $E = E(U)$ for some uniform right ideal of R/P or* cl.(P) *satisfies the s.s.l.c.. Let $n \in \mathbb{N}$. Then, in the notation of (1.2.3) and (1.3.1),*

(i) $E = \bigcup_{m=1}^{\infty} E_m$;

(ii) $E_n = \left\{e \in E : \operatorname{ann}\left(\frac{eR+E_{n-1}}{E_{n-1}}\right) \text{ is a finite intersection of primes in } X_n(P)\right\}$

$= \left\{e \in E : \operatorname{ann}\left(\frac{eR+E_{n-1}}{E_{n-1}}\right) \text{ is a finite intersection of primes in r.cl.}(P)\right\}$

$= \left\{e \in E : eI^n = 0 \text{ for some r.cl.}(P)\text{-semiprime ideal } I \text{ of } R\right\}$;

(iii) $E_n/E_{n-1} = \Sigma^{\oplus}_{Q \in X_n(P)} \operatorname{ann}_{E/E_{n-1}}(Q)$;

(iv) if cl.(P) *satisfies the s.s.l.c., $A_n(E)$ is determined by the prime P and does not depend on the choice of E.* ■

Remark 1.3.4. (i) We note that the direct sum of (iii) need not be finite, although for many rings it is. (See the discussion on local finiteness in Definition 1.3.9.) Again, some of the summands may be zero and this is the key issue under analysis in this thesis.

We note also that, by Lemma 1.1.9, $(E_1)_{R/P}$ is an injective hull for $U_{R/P}$.

(ii) By Theorem 1.3.3(iv), we can always assume, when considering the fundamental primes, that $E = E_P := E(U)$ for any uniform right ideal U of R/P (or, equivalently, that $(E_1)_{R/P}$ is torsion-free). So, for *any* prime P whose clique satisfies the second layer condition, we use the notation $A_n(P)$ to mean $A_n(E_P)$, which is then known as the set of n^{th} *layer fundamental primes* of P. Where the prime P is not in doubt, we abbreviate $A_n(P)$ to A_n. Furthermore, we put

$$\text{Fund}(P) := \bigcup_{n=1}^{\infty} A_n(P)$$

and call this the set of *fundamental primes* of P. ∎

By Lemma 1.2.4 and the (strong) second layer condition, $A_2(P) \subseteq X_2(P)$. As we now show, Theorem 1.3.3 ensures that a similar result holds for all layers.

Corollary 1.3.5. *Let R be a Noetherian ring and let P be a prime ideal of R whose clique satisfies the second layer condition. Then $A_n(P) \subseteq X_n(P)$ for all $n \in \mathbb{N}$.*

Proof. Let $E = E(U)$ for some uniform right ideal of R/P and let $Q \in A_n(P)$. Then, by definition of an associated prime, Q is the annihilator of some submodule of E/E_{n-1} which we can assume to be cyclic. On the other hand, by Theorem 1.3.3(ii), such a cyclic submodule has annihilator equal to a finite intersection of primes in $X_n(P)$, say $Q_1 \cap \ldots \cap Q_t$. So $Q = Q_1 \cap \ldots \cap Q_t$ and it is easily seen that $Q = Q_i$ for some i (and hence, by Lemma 1.2.8(i), for all i). Thus $Q \in X_n(P)$. ∎

Remark 1.3.6. By definition, $A_1 = X_1 = \{P\}$ and Brown and Jategaonkar have shown that $A_2 = X_2$ ([J2, Lemma 9.1.1(a)] or [G&W, Theorem 11.2]). However, in general $A_n \subsetneq X_n$ for $n \geq 3$. In fact, we will see several examples later where

$\text{Fund}(P) \subsetneqq \text{r.cl.}(P)$. On the other hand, Lenagan and Letzter have obtained the following easily stated description (i) of Fund(P) [L&L, Theorem 2.3]. For (ii), see the proof of [J2, Theorem 8.2.4]. ∎

Theorem 1.3.7. *Let R be a Noetherian ring and P a prime ideal of R. Then,*

(i) if R satisfies the s.l.c.,

$$Q \rightsquigarrow\!\!> \rightsquigarrow\!\!> P \iff Q \in \text{Fund}(P) \ ;$$

(ii) if cl.(P) *or* cl.(Q) *satisfies the s.l.c.,*

$$Q \rightsquigarrow\!\!> \rightsquigarrow\!\!> P \implies Q \in \text{Fund}(P) \ .$$ ∎

Corollary 1.3.8. *Suppose R is Noetherian, Q, $P \in \text{Spec}(R)$ and either* cl.(Q) *or* cl.(P) *satisfies the s.l.c.. If $Q \rightsquigarrow\!\!> \rightsquigarrow\!\!> P$ then $Q \in \text{r.cl.}(P)$.*

Proof. This follows from Theorem 1.3.7(ii) and Corollary 1.3.5. ∎

As suggested by Jategaonkar's "Main Lemma" (Lemma 1.2.4) and by Theorem 1.3.7, bimodules play a crucial role in the study of the fundamental primes. Indeed, before introducing our main definitions, we give a refinement of Theorem 1.3.7 due to Brown and Warfield, describing the n^{th} layer primes, which does, however, require a further condition on r.cl.(P), that of *local finiteness.*

Definition 1.3.9. The right clique of a prime P of a Noetherian ring is said to be *locally finite* provided $|X_n(P)|$ is finite for all $n \in \mathbb{N}$.

This condition holds for many important classes of rings, for instance group rings of polycyclic-by-finite groups over commutative Noetherian coefficient rings [B2, 6.4 and the last paragraph of §1], enveloping algebras of finite dimensional solvable Lie algebras over $\mathbb{C}$ [B3, Theorem 2.9] and Noetherian P.I. rings [Mü2]. This last class was extended to all FBN rings in [St2, Corollary 3.10].

In fact, [St2, Corollary 3.13] shows that in *any* Noetherian ring, every clique is at least countable. On the other hand, [St1, 4.4] shows an example of a Noetherian ring satisfying the strong second layer condition having a prime P such that $|X_2(P)| = |A_2(P)|$ is infinite. ∎

Notation 1.3.10. We use the same notation as Brown and Warfield [B&W, §5] which we list as follows. Let R be a Noetherian ring and let P be a prime ideal of R. For $n \geq 1$, let

$$S_n = \bigcap_{Q \in X_n(P)} Q \ ,$$
$$I_n = S_n S_{n-1} \dots S_1 \cap S_{n+1} S_n \dots S_2 \ ,$$
$$J_n{}' = S_{n+1} S_n \dots S_1 \ ,$$
$$J_n / J_n{}' = \text{the torsion submodule of } (I_n / J_n{}')_{R/P} \ ,$$
$$B_n(P) = I_n / J_n \ , \ B_0(P) = R/P \text{ and}$$
$$B_n(Q, P) = \{b \in B_n(P) : Qb = 0\} \text{ for each } Q \in X_{n+1}(P) \ . \qquad \blacksquare$$

The next result follows from [B&W, Theorem 5.6] although, as for Theorem 1.3.3, it is only stated in the case where the right clique of P satisfies the strong second layer condition.

Theorem 1.3.11. *Let R be a Noetherian ring and P a prime ideal of R. Suppose that* cl.(P) *satisfies the s.l.c. and that* r.cl.(P) *is locally finite. Let $n \in \mathbb{N}$ and $Q \in X_{n+1}(P)$. Then, with the notation of (1.3.1) and (1.3.10),*

$$E_{n+1}/E_n \cong \mathrm{Hom}_{R/P}(B_n(P), E_1)$$

and

$$Q \in A_{n+1}(P) \iff B_n(Q, P) \neq 0 \ . \qquad \blacksquare$$

Remark 1.3.12. The bimodule $B_n(Q, P)$ is, by definition, torsion-free as a right R/P-module while it follows, from [G&W, Proposition 7.5], Corollary 1.3.8 and Lemma 1.2.8(i), that it is torsion-free as a left R/Q-module. Thus, when $B_n(Q, P) \neq 0$, it forms an ideal link between Q and P. So Theorem 1.3.11 provides a converse to Theorem 1.3.7(ii) in the case where r.cl.(P) is locally finite. That is, when cl.(P) satisfies the second layer condition and r.cl.(P) is locally finite,

$$Q \in \mathrm{Fund}(P) \text{ if and only if } Q \rightsquigarrow\!\!\rightsquigarrow P. \qquad \blacksquare$$

§1.4 Notes

The results of this chapter have almost all appeared elsewhere in the literature. Those of §1.1 are well known and, except for Theorem 1.1.14, can be found, for instance, in [G&W, Chapter 4]. We note the following specific references.

Theorem 1.1.2 is essentially proved as [G&W, Proposition 4.1], while, that a $\mathbb{Z}$-module is injective if and only if it is divisible, is proved in [G&W, Proposition 4.2].

That a ring is Noetherian if and only if every direct sum of injective modules is injective was proved independently by Bass [Bs, Theorem 1.1] and Papp [Pa, Theorem 1].

That any non-zero indecomposable injective module is uniform, was proved in [Ma, Proposition 2.2]. The characterizations of injective modules in Theorem 1.1.6 are due to Baer [Ba, Theorem 1] and Eckmann and Schopf [E&S, §4]. As noted, the first of these is a generalization of Zippin's result for Abelian groups [Z].

That every module can be embedded in an injective module was shown in [Ba, Theorem 3], while the existence of injective hulls follows from [E&S, §4].

Theorem 1.1.8(i) is proved in [G&W, Theorem 4.8(a)] but is originally due to [E&S]. Part (ii), the uniqueness of injective hulls up to isomorphism, can be found, for instance, in [G&W, Proposition 4.9] but follows from [Ba, Theorem 4].

Lemma 1.1.9 is a well known result and appears as [G&W, Exercise 4E].

Theorem 1.1.11 was proved by Matlis in [Ma, Theorem 2.5] and independently by Papp [Pa, Theorem 2].

Theorem 1.1.12 is taken from [Ma, Proposition 3.1]. A version of this result for noncommutative rings in terms of irreducible ideals instead of primes, can be found in [Ma, §2].

That the mapping $E \mapsto$ (assassinator of E), is a bijection between the uniform injectives and the prime ideals if and only if R is an FBN ring, was proved independently in [G&R, Theorem 8.6 and Corollary 8.11] and [K2, Theorem 3.5]. The reverse implication (that the bijection holds in FBN rings) was also proved in [L&M, Corollary 3.12].

Theorem 1.1.14 is extracted from several results of [Ma, §3].

Most of the material of §1.2 can be found in [J2]. However, a version of Lemma 1.2.4 can also be found in [B1, Lemma 5.3], while Lemma 1.2.6 follows immediately from [G1, Lemma 1.3].

The results of §1.3 are principally due to Brown and Warfield (see [B&W] and [L&W, §7] for some minor corrections to this paper). There is a useful overall review of the representation theory of Noetherian rings in [B5] which also discusses how these results might extend to the case where the clique does not satisfy the second layer condition.

Theorem 1.3.3 was proved in the case where the clique of P satisfies the strong second layer condition in [B&W, Lemma 5.4 and Theorem 5.10]. The slightly generalized form we have quoted is essentially taken from [J2, §§9.1 and 9.2].

Corollary 1.3.5 is noted in [B&W, 5.11] and again in [B5, §3]. A proof is also given in [J2, Theorem 9.1.2].

Part (i) of Theorem 1.3.7 is proved as [L&L, Theorem 2.3]. Part (ii) follows from the proof of [J2, Theorem 8.2.4], although the conclusion as stated there is that Q is in the clique of P.

Corollary 1.3.8 is a well known result but is usually stated (as in [L&L]) for rings satisfying the second layer condition.

The notation of (1.3.10) as well as Theorem 1.3.11 are taken from [B&W, §5].

The observation of Remark 1.3.12 does not appear to have been stated before.

Chapter 2 : Representational Repleteness

§2.1 The Definitions and Elementary Examples

Definitions 2.1.1. For a Noetherian ring R with a prime ideal P where cl.(P) satisfies the s.l.c., we say that P is *right representationally replete (right rep. rep.)* in R provided Fund(P) = r.cl.(P) and we say that P is *right strongly representationally replete (right strongly rep. rep.)* in R provided $A_n(P) = X_n(P)$ for each $n \geq 1$. Of course we say that R is *right (strongly) rep. rep.* provided every prime of R is. Similar definitions can be made by interchanging right and left modules but we consider only the right-handed definitions throughout and, for convenience, abbreviate "right (strongly) rep. rep." to "(strongly) rep. rep.". ∎

Before going on to describe some properties of these definitions, we discuss some examples. In the next section, these will involve detailed constructions of injective hulls but, for the moment, we consider two easy cases. Firstly, when R is commutative, cl.$(P) = \{P\}$ for any prime P of R and, by Corollary 1.3.5 (or by Theorem 1.1.14(v) to which Theorem 1.3.3 reduces), Fund$(P) = \{P\}$. Thus, every prime of a commutative Noetherian ring is rep. rep.. To see that strong representational repleteness can fail, we need only consider:

Example 2.1.2. Let $R = \frac{\mathbb{C}[X]}{<X^2>}$, $P = \frac{<X>}{<X^2>}$ and put

$$E := E_R(R/P) = \mathrm{ann}_{E_{\mathbb{C}[X]}(R/P)}(X^2) \ ,$$

where the equality holds by Lemma 1.1.9. Then, E has only two layers so, while

$$A_1(P) = A_2(P) = X_n(P) = \{P\}$$

for all $n \in \mathbb{N}$,

$$A_m(P) = \emptyset$$

for all $m \geq 3$. (We will discuss this example in more detail in (2.2.9).) ∎

The next result answers the question of which commutative rings are strongly representationally replete.

Corollary 2.1.3. *Let R be a commutative Noetherian ring and P a prime ideal of R. Then P is rep. rep. and, moreover, the following are equivalent:*

(1) P is not strongly rep. rep. in R;

(2) P is a minimal prime of R and $\mathrm{ann}_R(P) \subseteq P$;

(3) P is a minimal prime of R and R_P is not a field;

(4) P is a minimal prime of R and R_P is not semiprime.

Proof. Since R is a commutative ring, either $X_n(P) = \{P\}$ for all $n \geq 1$ (in the case where $P \rightsquigarrow P$) or $X_n(P) = \emptyset$ for all $n \geq 2$ (in the case where $P \not\rightsquigarrow P$). So, if $P \not\rightsquigarrow P$, then P is strongly rep. rep. in R. Now $A_2(P) = X_2(P)$ so, if $A_2(P) = \emptyset$, then P is strongly rep. rep. in R. On the other hand, $A_n(P) \subseteq X_n(P)$ for all $n \geq 1$ and so P will fail to be strongly rep. rep. if and only if there are precisely n non-zero layers in $E(R/P)$ for some finite $n \geq 2$. Thus, by Theorem 1.1.14(vi), P will fail to be strongly rep. rep. precisely when $(PR_P)^n = (PR_P)^{n+1}$ for some $n \geq 2$ but $PR_P \neq (PR_P)^2$, that is, when $(PR_P)^n = 0$ for some $n \geq 2$ but $PR_P \neq 0$.

The equivalence of the statements follows easily. ∎

Examples 2.1.4. For an example where R_P is not semiprime but P is strongly rep. rep, we can take $R = R_P = \frac{\mathbb{C}[x,y]]}{<x^2>}$ and $P = \overline{x}R + \overline{y}R$. On the other hand, the zero ideal in any domain is an example of a minimal prime which is strongly rep. rep.. ∎

Remark 2.1.5. As noted in [B&W, 5.11], noncommutative rings may fail even to be rep. rep.. Indeed, Müller has shown [Mü2] that Fund(P) is finite for any Noetherian P.I. ring even though r.cl.(P) can be infinite. For example,

$$R = \mathbb{C}[x, y : yx = x(y+1), x^2 = 0]$$

is such a ring and the prime $P = xR + yR$ has right clique

$$\{xR + (y-n)R : n \in \mathbb{N} \cup \{0\}\}$$

(see Remarks 2.2.9 and 4.3.4). In fact, his proof extends, by [J2, Theorem 9.3.6(a)], to *centrally separated* rings (that is rings where every non-zero prime ideal of a prime factor ring contains a non-zero central element). By Corollary 2.1.3, commutative domains are always strongly rep. rep.. However, this is not true for noncommutative domains (see Example 4.3.2). Indeed, [Mü1, Counterexample 2] shows that a prime Noetherian P.I. ring may have infinite right cliques, so, by the above, prime rings need not be rep. rep.. On the other hand, we do not know of any domain which is not rep. rep.. ∎

It is, however, easy to see that some noncommutative rings are strongly rep. rep.. We recall that one definition of a *right hereditary ring*, R, is that every factor of an injective right R-module be injective. (*Left hereditary* is defined similarly and R is *hereditary* if it is both left and right hereditary.)

Lemma 2.1.6. *Let P be a prime of a Noetherian right hereditary ring R such that* cl.(P) *satisfies the s.l.c.. Then P is strongly rep. rep. in R.*

Proof. Suppose that $P_{i-1} \in A_{i-1}$ and that $P_i \rightsquigarrow P_{i-1}$. Then since P_{i-1} is an associated prime of E/E_{i-2} and E/E_{i-2} is injective, [J2, Lemma 9,1.1(a)] shows that P_i is an associated prime of E/E_{i-1}. The result follows by an induction argument on i. ∎

On the other hand, as we shall see in due course, there are plenty of noncommutative strongly rep. rep. rings which are not hereditary (for instance, Examples 2.2.7, 3.3.2, and 4.3.3).

§2.2 Calculations on Injective Hulls

Since Theorems 1.3.7 and 1.3.11 transform the study of the fundamental primes into a question in the ideal theory of the ring, explicit descriptions of injective hulls will play a relatively small part in this thesis. Indeed, in most of our examples, the injective hulls are inaccessible and, to determine whether Q is a fundamental prime of P, it is much easier to look for an ideal link between Q and P, and, in particular, to calculate the bimodule $B_n(Q, P)$ of (1.3.10). However, in this section, we discuss two examples where the injective hulls are known and use these decriptions to establish strong representational repleteness directly. We first recall the definition of an *exact* contravariant functor [Rm, pp.34&35].

Definition 2.2.1. Let F be a contravariant additive functor between categories of modules. Then, F is *left exact* if exactness of the sequence

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{0} 0$$

implies exactness of the sequence

$$0 \xrightarrow{0} FZ \xrightarrow{F\beta} FY \xrightarrow{F\alpha} FX \ ;$$

while F is *right exact* if exactness of the sequence

$$0 \xrightarrow{0} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

implies exactness of the sequence

$$FZ \xrightarrow{F\beta} FY \xrightarrow{F\alpha} FX \xrightarrow{0} 0 \ .$$

Furthermore, F is *exact* if and only if it is both left exact and right exact. ■

The first part of the next theorem is taken from [Rm, Theorem 2.9] and the second part from [Rm, Theorem 3.16].

Theorem 2.2.2. *The contravariant functor* $\mathrm{Hom}(\ ,M)$ *is left exact for every module* M *and is exact if and only if* M *is injective.* ■

We now quote the "Adjoint Isomorphism Theorem", which can be found in [Rm, Theorem 2.11].

Theorem 2.2.3. *Consider rings* R *and* S *and let* A *be a right* R*-module,* B *an* (R,S)*-bimodule and* C *a right* S*-module. Then, there is an isomorphism*

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)).$$ ■

Our next result is extracted from [N2, §1] in a slightly more general form.

Corollary 2.2.4. *Consider rings R and S such that S is a subring of R and let E be an injective right S-module. Then, $\mathrm{Hom}_S(R,E)$ is an injective right R-module.*

Proof. First, we note that, for any right R-module A,

$$\mathrm{Hom}_S(A,E) \cong \mathrm{Hom}_S(A \otimes_R R, E) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(R,E))$$

where the second isomorphism holds by Theorem 2.2.3. Thus, $\mathrm{Hom}_S(\ ,E)$ and $\mathrm{Hom}_R(\ ,\mathrm{Hom}_S(R,E))$ are naturally equivalent as functors from the category of right R-modules to the category of Abelian groups. Since E is an injective right S-module, we see from Theorem 2.2.2 that these functors are exact and, applying Theorem 2.2.2 again, it follows that $\mathrm{Hom}_S(R,E)$ is an injective right R-module.∎

We are now in a position to present our examples. In the first, the description of the injective hull of the trivial module over a ring of polynomials over a field, can be found in [N2, Theorem 2] but first appeared as [Hs, Exercise 6.11].

Example 2.2.5. Let k be a field and let $R = k[X] = k[x_1, \dots, x_n]$ be the commutative ring of polynomials in the n indeterminates $X := \{x_1, \dots, x_n\}$. Let $P = x_1R + \cdots + x_nR$, a co-Artinian prime ideal of R, and we note that $R/P \cong k$ as a k-module, the mapping from R to k which sends each polynomial to its constant term, inducing an isomorphism. Indeed, this isomorphism allows us to regard k as a uniform R-module with assassinator P, which we do from now on. We can see from Corollary 2.1.3 that P is a strongly rep. rep. prime of R, however, we now show this directly, by considering $E := E_R(k)$.

We denote by $k[[x_1^{-1}, \dots, x_n^{-1}]]$, the ring of formal power series in the indeterminates $x_1^{-1}, \dots, x_n^{-1}$ and we define a mapping

$$\phi : \mathrm{Hom}_k(k[X], k) \to k[[x_1^{-1}, \dots, x_n^{-1}]]$$

by means of the formula

$$\phi(f) = \sum_{\lambda_i \geq 0} f(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) x_1^{-\lambda_1} x_2^{-\lambda_2} \dots x_n^{-\lambda_n}$$

for each $f \in \mathrm{Hom}_k(k[X], k)$. Clearly, ϕ is an isomorphism of Abelian groups. However, $\mathrm{Hom}_k(R, k)$ is a right R-module with an action of R given by

$$(f.r)(s) = f(rs)$$

for each $f \in \mathrm{Hom}_k(R, k)$ and r and $s \in R$. Thus, we can turn $k[[x_1^{-1}, \ldots, x_n^{-1}]]$ into a right R-module by defining

$$p.r := \phi(\phi^{-1}(p).r)$$

for each $p \in k[[x_1^{-1}, \ldots, x_n^{-1}]]$ and $r \in R$. To see explicitly what this action looks like, we fix $(\mu_1, \ldots, \mu_n)$ and $(\nu_1, \ldots, \nu_n) \in (\mathbb{N} \cup \{0\})^n$, and α and $\beta \in k$. We observe that $\phi^{-1}(\beta x_1^{-\nu_1} \ldots x_n^{-\nu_n})$ is the mapping $f \in \mathrm{Hom}_k(k[X], k)$ given by

$$f(x_1^{\lambda_1} \ldots x_n^{\lambda_n}) = \begin{cases} \beta & \text{if } (\lambda_1, \ldots, \lambda_n) = (\nu_1, \ldots, \nu_n)\ ; \\ 0 & \text{otherwise,} \end{cases}$$

extended linearly to $k[X]$. Thus,

$$\begin{aligned} &\beta x_1^{-\nu_1} \ldots x_n^{-\nu_n}.\alpha x_1^{\mu_1} \ldots x_n^{\mu_n} \\ &\qquad = \sum_{\lambda_i \geq 0} (f.\alpha x_1^{\mu_1} \ldots x_n^{\mu_n})(x_1^{\lambda_1} \ldots x_n^{\lambda_n}) x_1^{-\lambda_1} \ldots x_n^{-\lambda_n} \\ &\qquad = \sum_{\lambda_1 \geq 0} f(\alpha x_1^{\mu_1+\lambda_1} \ldots x_n^{\mu_n+\lambda_n}) x_1^{-\lambda_1} \ldots x_n^{-\lambda_n} \\ &\qquad = \begin{cases} \alpha\beta x_1^{-(\nu_1-\mu_1)} \ldots x_n^{-(\nu_n-\mu_n)} & \text{if } \mu_i \leq \nu_i \text{ for } 1 \leq i \leq n\ ; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Of course, this multiplication is extended by additive and distributive laws to the module action of R on $k[[x_1^{-1}, \ldots, x_n^{-1}]]$.

Since ϕ is now a right R-module isomorphism, we see from Corollary 2.2.4 (with $S = E = k$) that $k[[x_1^{-1}, \ldots x_n^{-1}]]$, under this action, is an injective right R-module and, since it contains k_R, we see that it must also, by Theorem 1.1.8(i), contain an injective hull for k_R. We claim that this hull is simply the set of formal polynomials $k[x_1^{-1}, \ldots, x_n^{-1}]$.

It is obvious that $k[x_1^{-1}, \ldots, x_n^{-1}]$ is an R-submodule of $k[[x_1^{-1}, \ldots, x_n^{-1}]]$ and we show first that it is an essential extension of k. If M is a non-zero R-submodule of $k[x_1^{-1}, \ldots, x_n^{-1}]$, then it contains some non-zero polynomial, p say. Choose a non-zero term, $\beta x_1^{-\nu_1} \dots x_n^{-\nu_n}$, of p such that $\nu_1 + \dots + \nu_n$ is as large as possible. So, for any other non-zero term, $\gamma x_1^{-\zeta_1} \dots x_n^{-\zeta_n}$, for some i, $\zeta_i < \nu_i$ and thus

$$x_1^{-\zeta_1} \dots x_n^{-\zeta_n} . x_1^{\nu_1} \dots x_n^{\nu_n} = 0 \; .$$

We see, therefore, that

$$p . x_1^{\nu_1} \dots x_n^{\nu_n} = \beta x_1^{-\nu_1} \dots x_n^{-\nu_n} . x_1^{\nu_1} \dots x_n^{\nu_n} = \beta.$$

Thus, $\beta \in M \cap k$ and, since $\beta \neq 0$, it follows that k_R is essential in $k[x_1, \dots, x_n]_R$.

We have now established that

$$k \subseteq k[x_1^{-1}, \dots, x_n^{-1}] \subseteq E \subseteq k[[x_1^{-1}, \dots, x_n^{-1}]]$$

where $E = E_R(k)$. To show that $E = k[x_1^{-1}, \dots, x_n^{-1}]$, we let ξ be a non-zero element of E which is therefore a formal power series in $x_1^{-1}, \dots, x_n^{-1}$ over k. By Theorem 1.1.14, $\xi . (x_1 R + \dots + x_n R)^s = 0$ for some non-zero integer, s. Thus, for $1 \leq i \leq n$, $\xi . x_i^s = 0$ and it follows that $\xi \in k[x_1^{-1}, \dots x_n^{-1}]$, which establishes that

$$E = E_R(k) = k[x_1^{-1}, \dots, x_n^{-1}] \; .$$

Now, by Theorem 1.3.3, and in the notation of (1.3.1), we see that

$$\begin{aligned} E_m =& \{e \in E : eP^m = 0\} \\ =& \left\{ p \in k[x_1^{-1}, \dots, x_n^{-1}] : p . (x_1 R + \dots + x_n R)^m = 0 \right\} \\ =& \left\{ \sum_{\substack{\lambda_1, \dots, \lambda_n \geq 0 \\ \lambda_1 + \dots + \lambda_n < m}} \beta_{\lambda_1, \dots, \lambda_n} x_1^{-\lambda_1} \dots x_n^{-\lambda_n} : \beta_{\lambda_1, \dots, \lambda_n} \in k \right\} \; . \end{aligned}$$

Clearly, then, $E \neq E_m$ for any $m \in \mathbb{N}$ and P occurs as the fundamental prime at each layer. (Indeed, for $m \in \mathbb{N}$, E_m / E_{m-1} is an n^{m-1}-dimensional vector space over $k \cong R/P$.)

We have thus shown directly that P is strongly rep. rep. in R. ∎

Remarks 2.2.6. (i) Continuing the notation of (2.2.5), if we write

$$k[[X]] := k[[x_1, \dots, x_n]],$$

for the ring of formal power series over X, then

$$k \cong \frac{k[[X]]}{x_1 k[[X]] + \cdots + x_n k[[X]]}$$

and so we may ask what the injective hull of $k_{k[[X]]}$ would be. In fact, it turns out that the set of *inverse polynomials*, $k[x_1^{-1}, \ldots, x_n^{-1}]$ described above, is also a module over $k[[X]]$ and, as such, is again an injective hull for $k_{k[[X]]}$. (This can be found in [N2, Theorem 3] although it was first proved in [Ga1, p7].

(ii) When k is a field and $R = k[X]$, Northcott's construction provides a description for the injective hull of R/P where P is the maximal ideal $x_1R + \cdots + x_nR$ of R. This result was generalized by Fossum [F] to give a precise (though not explicit) description for $E_R(R/P)$ where P is any prime ideal of a commutative Noetherian ring R. After localization of R at P and completion of the local ring R_P (see Definitions 4.1.9) [F] uses Cohen's Theorem (Theorem 4.1.11) to express R_P as a homomorphic image of a ring of formal power series and then introduces modules of inverse polynomials.

(iii) When k is algebraically closed, any maximal ideal in the ring $R[X]$ can be written in the form $x_1R + \cdots x_nR$, after making appropriate substitutions for $x_1, \ldots, x_n$, if required. So, in this case, Northcott's construction describes $E_R(R/P)$ for *any* maximal ideal P of R. This can be generalized: see [S&S] for the case where k is not algebraically closed but has characteristic zero. For instance,

$$E_{\mathbb{Q}[x,y]}\left(\frac{\mathbb{Q}[x,y]}{(x^2-2, y^2+1)}\right)$$

is described as [S&S, Example 3.12].

(iv) Finally, we mention that Kucera has also repeated Northcott's argument to provide an explicit description of $E_R(R/P)$ where $R = k[x_1, \ldots, x_n]$ and $P = x_1R + \cdots x_tR$ for $t \leq n$. Specifically, [Ku, Corollary 2.10(a)] shows that

$$E_R(R/P) = k(x_{t+1}, \ldots, x_n)[x_1^{-1}, \ldots, x_t^{-1}]$$

with an R-module action similar to that of Example 2.2.5. ■

Given that repleteness for commutative rings is completely solved by Corollary 2.1.3, we are, of course, more interested in descriptions of injective hulls over noncommutative rings. Our next example, however, uses an argument very similar to that of the previous one. While this example is certainly not new either (it appears, for instance, as [Da, §4, Example 1]; see also Remarks 2.2.8, below), similar ideas will recur in Chapter 6. In particular, the method we use to show that $\mathbb{C}[x^{-1}, y^{-1}]$ can be taken as the whole injective hull, will reappear in Theorem 6.2.9, the special positive case of Hypothesis 6.2.8.

Example 2.2.7 will be met again in Example 2.3.16 and in Example 4.3.1 where we will prove strong representational repleteness for a second time, but by consideration of the bimodule $B_n(P)$ of (1.3.10). In fact, it will be convenient for us to borrow results of Chapter 4 for the description of prime links in R.

Example 2.2.7. Let $T = \mathbb{C}[x, y : yx - xy = x]$ (the enveloping algebra of the two-dimensional solvable non-Abelian Lie algebra over $\mathbb{C}$ and which therefore satisfies the s.s.l.c. by, for instance, [J2, Theorem A.3.9]). Let $P = xT + yT$. As in Example 2.2.5, $\mathbb{C} \cong T/P$ and so we can and do regard $\mathbb{C}$ as a uniform right T-module with assassinator prime P. We will show that P is a strongly rep. rep. prime of T, by constructing the injective hull, $E := E_T(T/P)$.

We denote by $\mathbb{C}[[x^{-1}, y^{-1}]]$, the ring of formal power series in x^{-1} and y^{-1} over $\mathbb{C}$ and we define a mapping

$$\phi : \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}) \to \mathbb{C}[[x^{-1}, y^{-1}]]$$

by means of the formula

$$\phi(f) = \sum_{i,j \geq 0} f(x^i y^j) x^{-i} y^{-j}$$

for each $f \in \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$. Clearly, ϕ is an isomorphism of Abelian groups. However, $\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ is a right T-module with an action of T given by

$$(f.t)(s) = f(ts)$$

for each $f \in \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ and t and $s \in T$. Thus, we can turn $\mathbb{C}[[x^{-1}, y^{-1}]]$ into a right T-module by defining

$$p.t := \phi(\phi^{-1}(p).t)$$

for each $p \in \mathbb{C}[[x^{-1}, y^{-1}]]$ and $t \in T$. To see explicitly what this action looks like, we fix (μ, η) and $(\nu, \omega) \in (\mathbb{N} \cup \{0\})^2$, and α and $\beta \in \mathbb{C}$. We observe that $\phi^{-1}(\beta x^{-\nu} y^{-\omega})$ is the mapping $f \in \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ given by

$$f(x^\lambda y^\mu) = \begin{cases} \beta & \text{if } (\lambda, \mu) = (\nu, \omega) \ ; \\ 0 & \text{otherwise,} \end{cases}$$

extended linearly to T. Now, since $yx = x(y+1)$, we see that $y^\lambda x^i = x^i(y+i)^\lambda$ for any i and $\lambda \geq 0$. Thus,

$$\begin{aligned} &\beta x^{-\nu} y^{-\omega} . \alpha x^\mu y^\lambda \\ &\quad = \sum_{i,j \geq 0} (f . \alpha x^\mu y^\lambda)(x^i y^j) x^{-i} y^{-j} \\ &\quad = \sum_{i,j \geq 0} f(\alpha x^\mu y^\lambda x^i y^j) x^{-i} y^{-j} \\ &\quad = \sum_{i,j \geq 0} f(\alpha x^{\mu+i} (y+i)^\lambda y^j) x^{-i} y^{-j} \\ &\quad = \alpha \sum_{i,j \geq 0} \sum_{\rho=0}^{\lambda} \binom{\lambda}{\rho} i^{\lambda-\rho} f(x^{\mu+i} y^{\rho+j}) x^{-i} y^{-j} \\ &\quad = \begin{cases} \alpha\beta x^{-(\nu-\mu)} \sum_{\rho=0}^{\min(\lambda,\omega)} \binom{\lambda}{\rho} (\nu-\mu)^{\lambda-\rho} y^{-(\omega-\rho)} & \text{if } \mu < \nu \ ; \\ \alpha\beta y^{-(\omega-\lambda)} & \text{if } \mu = \nu \text{ and } \lambda \leq \omega \ ; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Of course, this multiplication is extended by additive and distributive laws to the module action of T on $\mathbb{C}[[x^{-1}, y^{-1}]]$.

Since ϕ is now a right T-module isomorphism, we see from Corollary 2.2.4 (with $R = T$ and $S = E = \mathbb{C}$) that $\mathbb{C}[[x^{-1}, y^{-1}]]$, under this action, is an injective right T-module and, since it contains $\mathbb{C}_T$, it must also, by Theorem 1.1.8(i), contain an injective hull for $\mathbb{C}_T$. Of course, it could contain more than one, however, we claim that one such hull is the set of formal polynomials $\mathbb{C}[x^{-1}, y^{-1}]$.

It is obvious that $\mathbb{C}[x^{-1}, y^{-1}]$ is a right T-submodule of $\mathbb{C}[[x^{-1}, y^{-1}]]$ and we show first that it is an essential extension of $\mathbb{C}$. If M is a non-zero T-submodule of $\mathbb{C}[x^{-1}, y^{-1}]$, then it contains some non-zero polynomial, p say. Let $\mathcal{S} = \{\beta_i x^{-\nu} y^{-\omega_i} : i \in I\}$ (for some index set I), the set of non-zero terms of p such that ν is as large as possible. For any other non-zero term, $\gamma x^{-\zeta} y^{-\xi}$, $\zeta < \nu$ and so

$$\gamma x^{-\zeta} y^{-\xi} . x^\nu = 0 \ .$$

Let $\beta x^{-\nu}y^{-\omega}$ be the element of $\mathcal{S}$ such that ω is as large as possible. Then, for every other element of $\mathcal{S}$, $\omega_i < \omega$ so that

$$\beta_i x^{-\nu}y^{-\omega_i}.x^{\nu}y^{\omega} = 0 .$$

We see, therefore, that

$$p.x^{\nu}y^{\omega} = \beta x^{-\nu}y^{-\omega}.x^{\nu}y^{\omega} = \beta.$$

Thus, $\beta \in M \cap \mathbb{C}$ and, since $\beta \neq 0$, it follows that $\mathbb{C}_T$ is essential in $\mathbb{C}[x^{-1}, y^{-1}]_T$.

We have now established that

$$\mathbb{C} \subseteq \mathbb{C}[x^{-1}, y^{-1}] \subseteq E \subseteq \mathbb{C}[[x^{-1}, y^{-1}]]$$

where E is some injective hull of $\mathbb{C}_T$. To show that $E = \mathbb{C}[x^{-1}, y^{-1}]$, we let $q \in E$ so that q is a power series in x^{-1} and y^{-1} over $\mathbb{C}$.

Now, $xT = Tx$ so, as we shall see in Lemma 2.3.5, x is contained in every prime in the right clique of P (alternatively, we can refer to the description of the right clique given in Example 4.3.1 and quoted below) and so, by Theorem 1.3.3, we can find some $s \in \mathbb{N}$ such that $q.x^s = 0$. Since, whenever $0 \leq \omega$ and $s \leq \nu$,

$$x^{-\nu}y^{-\omega}.x^s = x^{-(\nu-s)}y^{-\omega}$$

we see that

$$q \in \mathbb{C}[[y^{-1}]][x^{-1}] ,$$

the algebra of formal polynomials in x^{-1} over $\mathbb{C}[[y^{-1}]]$. Thus, writing the "coefficients" on the right,

$$q = q_0 + x^{-1}q_1 + x^{-2}q_2 + \cdots + x^{-r}q_r$$

for some $q_0, \ldots, q_r \in \mathbb{C}[[y^{-1}]]$ with $r < s$.

We observe that $q_r = q.x^r \in E$. However, if we can establish that $q_r \in \mathbb{C}[y^{-1}]$, then, since

$$x^{-r}q_r \in \mathbb{C}[x^{-1}, y^{-1}] \subseteq E ,$$

we can replace q with

$$q' = q_0 + x^{-1}q_1 + \cdots + x^{-(r-1)}q_{r-1}$$

and an induction argument will show that $q \in \mathbb{C}[x^{-1}, y^{-1}]$.

By Lemma 1.1.9, $\mathrm{ann}_E(xT)$ is an injective hull for $\mathbb{C}_{T/xT}$ while, since $T/xT \cong \mathbb{C}[y]$, Example 2.2.5 shows that

$$E_{T/xT}(\mathbb{C}) = \mathbb{C}[y^{-1}]$$

with the same action as that induced by the T-module action described above. On the other hand, $q_r.x = 0$ in the T-module E so that

$$\{q_r\} \cup \mathbb{C}[y^{-1}] \subseteq \mathrm{ann}_E(xT) .$$

Since there cannot be two copies of the injective hull of $\mathbb{C}_{T/xT}$, one strictly contained inside the other, we see that $q_r \in \mathbb{C}[y^{-1}]$.

As noted, an induction argument now shows that $q \in \mathbb{C}[x^{-1}, y^{-1}]$ and so we have proved that

$$E_T(\mathbb{C}) = \mathbb{C}[x^{-1}, y^{-1}]$$

with the action described above.

To establish the repleteness of the prime P in T, we next require to know the right clique of P. In fact, as we will see in Chapter 4, and in the notation of (1.2.3),

$$X_n(P) = \{xT + (\theta - \alpha)T : \alpha \in \{0, 1, 2, \ldots, (n-1)\}\}$$

for each $n \in \mathbb{N}$ (see Example 4.3.1).

We claim that, in the notation of (1.3.1),

$$E_{n-1} = \left\{ \sum_{\substack{\nu,\omega \geq 0 \\ \nu+\omega < n-1}} \beta_{\nu,\omega} x^{-\nu} y^{-\omega} : \beta_{\nu,\omega} \in \mathbb{C} \right\}$$

and that,

$$A_n(P) = X_n(P)$$

for each $n \in \mathbb{N}$.

The claim is trivially true for $n = 1$ since $E_0 = 0$ and $A_1(P) = X_1(P) = \{P\}$. So we assume it is true for $n \leq k$ for some $k \in \mathbb{N}$ and consider $n = k + 1$.

Now, for $0 \leq \nu < k$ and $\omega = k - 1 - \nu$, certainly

$$x^{-\nu}y^{-\omega}.x = x^{-(\nu-1)}y^{-\omega} = x^{-(\nu-1)}y^{-(k-1-\nu)} \in E_{k-1}$$

while

$$x^{-\nu}y^{-\omega}.y = \begin{cases} \nu x^{-\nu}y^{-\omega} + x^{-\nu}y^{-(\omega-1)} & \\ \qquad = \nu x^{-\nu}y^{-\omega} + x^{-\nu}y^{-(k-2-\nu)} & \text{when } \nu \neq k-1 \;; \\ \nu x^{-\nu} = (k-1)x^{-(k-1)} & \text{when } \nu = k-1 \;. \end{cases}$$

We thus see that, when $0 \leq \nu \leq k-2$ and $\omega = k - 1 - \nu$,

$$x^{-\nu}y^{-\omega}.\,(xT + (y-\nu)T) = x^{-(\nu-1)}y^{-(k-1-\nu)}T + x^{-\nu}y^{-(k-2-\nu)}T \in E_{k-1}$$

while, when $\nu = k - 1$ and $\omega = k - 1 - \nu = 0$,

$$x^{-\nu}y^{-\omega}T.\,(xT + (y-(k-1))\,T = x^{-(\nu-1)}y^{-(k-1-\nu)}T + 0 \in E_{k-1} \;.$$

This establishes two facts: firstly, that each of the primes in $X_k(P)$ is an associated prime of E/E_{k-1} (for, by the above, each prime in $X_k(P)$ annihilates a non-zero element of E/E_{k-1} and so is contained in an associated prime which must equal it by the incomparability property of r.cl.(P)), and so $A_k(P) = X_k(P)$; secondly, that

$$E_k \supseteq \left\{ \sum_{\substack{\nu,\omega\geq 0 \\ \nu+\omega<k}} \beta_{\nu,\omega}x^{-\nu}y^{-\omega} : \beta_{\nu,\omega} \in \mathbb{C} \right\} \;.$$

To complete the induction step, we must show that equality holds. So, consider $p \in E_k(P)$. Since x is contained in every prime in $X_k(P)$, we see that $p.x \in E_{k-1}$. Thus,

$$p.x = \sum_{\substack{\nu,\omega\geq 0 \\ \nu+\omega<k-1}} \beta_{\nu,\omega}x^{-\nu}y^{-\omega}$$

for some $\beta_{\nu,\omega} \in \mathbb{C}$. Since, whenever ν and $\omega \geq 0$,

$$x^{-(\nu+1)}y^{-\omega}.x = x^{-\nu}y^{-\omega},$$

we see that

$$p = q + \sum_{\substack{\nu,\omega \geq 0 \\ \nu+\omega < k-1}} \beta_{\nu,\omega} x^{-(\nu+1)} y^{-\omega}$$

where q is a polynomial in y^{-1} over $\mathbb{C}$ which we observe is also contained in $E_k(P)$. From the description of $X_k(P)$,

$$q.y(y-1)(y-2)\ldots(y-(k-1)) \in E_{(k-1)}(P)$$

and hence has degree (in y^{-1}) less than $k-1$. However, since

$$y(y-1)(y-2)\ldots(y-(k-1)) = (-1)^{k-1}(k-1)!y + y^2.h$$

for some polynomial h in y, multiplying by it reduces the degree of q by only 1. Thus, the degree (in y^{-1}) of q is less than k. It follows that

$$p \in \left\{ \sum_{\substack{\nu,\omega \geq 0 \\ \nu+\omega < n}} \beta_{\nu,\omega} x^{-\nu} y^{-\omega} : \beta_{\nu,\omega} \in \mathbb{C} \right\}$$

and this completes the induction step and hence the proof of the claim.

We have thus shown directly that P is strongly rep. rep. in T. ∎

Remark 2.2.8. The isomorphism between $\mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ and $\mathbb{C}[[x^{-1}, y^{-1}]]$ can be generalized to the enveloping algebra of any Lie algebra. Indeed, by [D2, Proposition 2.7.5], if $\{e_1, \ldots, e_n\}$ is a basis for a Lie algebra L over a field k of characteristic zero and $U(L)$ is its enveloping algebra,

$$f \mapsto \sum_{v_i \geq 0} f\left(\frac{e_1^{v_1} \ldots e_n^{v_n}}{v_1! \ldots v_n!}\right) x_1^{v_1} \ldots x_n^{v_n}$$

provides an isomorphism from the algebra $\mathrm{Hom}_k(U(L), k)$ to the algebra of formal power series $k[[x_1, \ldots, x_n]]$.

The inclusion of $v_1! \ldots v_n!$, means this is a slightly different isomorphism to ours. With this modification, L acts on $k[[x_1, \ldots, x_n]]$ as a *Lie algebra of derivations* [D2, p91]. Specifically, the action of e_j is given by $\sum_{i=1}^n a_{ij}\, \partial/\partial x_i$ where $a_{ij} \in k[x_1, \ldots, x_n]$. Furthermore, when L is a solvable Lie algebra of finite dimension n over $\mathbb{C}$ and P is a co-Artinian prime of $U(L)$, [Lv, Theorem 2.2] uses results of [H1] and [H2] to show that the injective hull of $U(L)/P$ ($\cong \mathbb{C}$) can be identified with $\mathbb{C}[x_1, \ldots, x_n]$, a submodule of $\mathbb{C}[[x_1, \ldots, x_n]]$.

Since [B3, Theorem 2.11] provides a description of the link graph of P (a result based on [Lp]), we might hope to generalize Example 2.2.7 to Lie algebras of arbitrary finite dimension. In fact, we will prove in Corollary 2.3.17 that repleteness holds for *any* prime of a finite dimensional solvable Lie algebra over $\mathbb{C}$. However, it remains open whether *strong* representational repleteness must hold for any such prime. ∎

Remark 2.2.9. Both Examples 2.2.5 and 2.2.7 can easily be extended, by the use of Lemma 1.1.9, to consider injective hulls of $\mathbb{C}$ over factor rings. For instance, in the notation of (2.2.5),

$$E_{R/x^2R}(\mathbb{C}) = \operatorname{ann}_E(x^2)$$

and so can be identified with the first two layers of E_R. In this case, P/x^2R will not be strongly rep. rep., $X_n\left(P/x^2R\right) = \left\{P/x^2R\right\}$ for each $n \in \mathbb{N}$, while there are only two layers in the (R/x^2R)-module injective hull of $\mathbb{C}$. (This was the example which we saw as Example 2.1.2.)

Similarly, in the notation of (2.2.7),

$$\begin{aligned} E_{T/x^2T}(\mathbb{C}) &= \operatorname{ann}_E(x^2) \\ &= \left\{ \sum_{\substack{\nu=0 \text{ or } 1 \\ \omega\geq 0}} \beta_{\nu,\omega} x^{-\nu} y^{-\omega} : \beta_{\nu,\omega} \in \mathbb{C} \right\} \end{aligned}$$

and we see that

$$A_n\left(P/x^2T\right) = \begin{cases} \left\{P/x^2T\right\} & \text{if } n = 1 \; ; \\ \left\{P/x^2T, \left(xT + (y-1)T\right)/x^2T\right\} & \text{if } n \geq 2 \; . \end{cases}$$

Since $x^2 \in QP$ for any linked primes Q and P of T, it is easy to see that Q/x^2T and P/x^2T will still be linked (see Lemma 2.3.7). So, we obtain the link graph of P/x^2T simply by factoring out each prime in the link graph of P by the ideal x^2T. That is,

$$X_n\left(P/x^2T\right) = \left\{\left(xT + (y-\alpha)T\right)/x^2T : \alpha \in \{0, 1, 2, \ldots, (n-1)\}\right\}$$

for each $n \in \mathbb{N}$. Thus, $A_n\left(P/x^2T\right) = X_n\left(P/x^2T\right)$ if and only if $n = 1$ or 2 and so P/x^2T is not strongly rep. rep. in T/x^2T. Indeed, since $|\,\text{r.cl.}\left(P/x^2T\right)|$ is infinite while $\left|A_n\left(P/x^2T\right)\right| = 2$, we see that P/x^2T is not even rep. rep. in T/x^2T. (This observation is made in [B&G, 5.11]. That $\left|A_n\left(P/x^2T\right)\right|$ is finite, follows from [Mü2], since T/x^2T is a P.I. ring, the identity $(ab-ba)^2 = 0$ holding for all $a, b \in T/x^2T$.) (This example, which we saw in (2.1.5), will reappear as Example 4.3.4.) ∎

§2.3 Properties of Representational Repleteness

Remarks 2.3.1. Let $\mathcal{C}$ be a right Ore set in a Noetherian ring R. It is well known (see for instance [G&W, Theorem 9.22]) that contraction and extension are inverse bijections between the set of prime ideals of $R\mathcal{C}^{-1}$ and the set of those prime ideals of R that are disjoint from $\mathcal{C}$. If P and Q are prime ideals disjoint from $\mathcal{C}$, then it is routine to check that $Q\mathcal{C}^{-1} \rightsquigarrow P\mathcal{C}^{-1}$ if and only if $Q \rightsquigarrow P$ (see [G&W, Ex 11S]). This, of course, ensures that, if r.cl.(P) is locally finite, then so too is r.cl.$(P\mathcal{C}^{-1})$. Furthermore, P satisfies the second layer condition if and only if $P\mathcal{C}^{-1}$ does. (For the strong second layer condition, however, only the forward implication is known to be true, the reverse implication being an open question.) ∎

The next result shows that (strong) representational repleteness is also preserved in this situation.

Theorem 2.3.2. *Let R be a Noetherian ring and P a prime ideal of R. Let $\mathcal{C}$ be a right Ore set in R disjoint from P (or, equivalently, $\mathcal{C}$ is a right Ore set in R with all elements of $\mathcal{C}$ regular modulo P). Suppose that* cl.(P) *satisfies the s.l.c.. Then P is rep. rep. [resp. strongly rep. rep.] in R if and only if $P\mathcal{C}^{-1}$ is rep. rep. [resp. strongly rep. rep.] in $R\mathcal{C}^{-1}$.*

Proof. Let $E = E_R(U)$, for some uniform right ideal U of R/P. By [G&W, Corollary 9.16], E is isomorphic as an R-module to $E\mathcal{C}^{-1}$ and we lose nothing by identifying them. Then, again by [G&W, Corollary 9.16], E is an injective RC^{-1}-module. Furthermore, since RC^{-1}-submodules of E are identified with their contractions to E_R, it is easy to see that E is a uniform injective RC^{-1}-module.

We note that, by Theorem 1.2.2, since $\mathcal{C}$ is regular modulo P, $\mathcal{C}$ is regular modulo Q for every $Q \in$ r.cl.(P) So, by induction on n and using Remarks 2.3.1,

$$Q \in X_n(P) \iff Q\mathcal{C}^{-1} \in X_n(P\mathcal{C}^{-1}).$$

We now claim that, for each $n \geq 1$, the n^{th} layers of E as an R-module and as a $R\mathcal{C}^{-1}$-module coincide and that

$$Q \in A_n(P) \iff Q\mathcal{C}^{-1} \in A_n(P\mathcal{C}^{-1}).$$

Obviously this is true for $n = 1$ and the induction step is routine once we recall that the inclusion of $A_n(P)$ in the right clique of P guarantees that all primes in $A_n(P)$ are disjoint from $\mathcal{C}$.

This completes the proof of the theorem. ∎

It is well known that the s.s.l.c. and the s.l.c. are Morita invariant properties and we now show that the same is true of the (strong) rep. rep. properties, the proof being a straightforward application of standard Morita Theory. (The notation of [McC&R, Chapter 3, §5] is used. In particular, $M^* := \text{Hom}_R(M, R)$.)

Theorem 2.3.3. *Suppose that R and S are Morita equivalent Noetherian rings with progenerator ${}_SM_R$ and let P be a prime ideal of R whose clique satisfies the s.l.c.. Then P is rep. rep. [resp. strongly rep. rep.] in R if and only if MPM^* is rep. rep. [resp. strongly rep. rep.] in S.*

Proof. Since

$$\frac{(MQM^*) \cap (MPM^*)}{(MQM^*)(MPM^*)} = \frac{M(Q \cap P)M^*}{M(QP)M^*}$$

and since it is easy to show that ${}_{S/(MQM^*)}\left|\frac{M(Q\cap P)M^*}{M(QP)M^*}\right|_{S/(MPM^*)}$ has a non-zero factor which is torsion-free on both sides if and only if ${}_{R/Q}\left|\frac{Q \cap P}{QP}\right|_{R/P}$ has a non-zero factor which is torsion-free on both sides, it follows that

$$Q \rightsquigarrow P \text{ in } R \iff MQM^* \rightsquigarrow MPM^* \text{ in } S$$

and so, by induction on n, we see that

$$X_n(MQM^*) = \{MQM^* : Q \in X_n(P)\}\,.$$

Let E be a uniform injective R-module with assassinator prime P. Then $E \otimes_R M^*$ is an $MM^* = S$-module which is uniform [McC&R, Lemma 3.5.8(vi)], injective [A&F, Proposition 21.6(2)] and is easily seen to have assassinator prime MPM^*.

Now, $\mathrm{ann}_E(P)$ is a torsion-free $\frac{R}{P}$-module if and only if $\mathrm{ann}_{E\otimes_R M^*}(MPM^*)$ is a torsion-free $\frac{S}{MPM^*}$-module while, if not, then, by Theorem 1.3.3(iv) above, $A_{n+1}(E \otimes_R M^*)$ is the set of assassinator primes of $\frac{E(W)}{E_n(W)}$ for any uniform S-module, W, with assassinator prime MPM^*. In either case, the n^{th} layer of $E \otimes_R M^*$ is $E_n \otimes_R M^*$ and $A_n(E \otimes_R M^*) = \{MQM^* : Q \in A_n(E)\}$. That is,

$$A_n(MPM^*) = \{MQM^* : Q \in A_n(P)\}\,.$$

The result follows. ∎

We now wish to consider extensions of rep. rep. rings and so we must first recall that the second layer conditions are inherited by factor rings. We find, however, that for the repleteness property we need to be more careful about which factors are allowed (see Examples 2.3.15 and 2.3.16). Our starting point is to consider the behaviour of links on factoring. We require the following definitions.

Definitions 2.3.4. An element a of a ring R is *central* provided $ar = ra$ for all $r \in R$ and *normal* provided $aR = Ra$. An ideal I of R is said to be *polycentral* if it is generated by some elements $a_1, \ldots, a_n$ of R where a_1 is central in R and $a_i + \sum_{j=1}^{i-1} a_j R$ is central in $R/\left(\sum_{j=1}^{i-1} a_j R\right)$ for $2 \leq i \leq n$. A *polynormal* ideal is defined similarly and I is said to be *regularly polynormal* if it is generated by some elements $a_1, \ldots, a_n$ of R where a_1 is a normal non-zero-divisor of R and $a_i + \sum_{j=1}^{i-1} a_j R$ is a normal non-zero-divisor of $R/\left(\sum_{j=1}^{i-1} a_j R\right)$ for $2 \leq i \leq n$. ∎

The next result is well known.

Lemma 2.3.5. *Let R be a Noetherian ring and P and Q prime ideals of R with $Q \rightsquigarrow P$. Suppose also that c is a normal element of R. Then,*

$$c \in P \iff c \in Q.$$

Proof. Let $\frac{Q \cap P}{A}$ be a linking bimodule for $Q \rightsquigarrow P$ and assume $c \in P$.

Let $a \in Q \cap P$. Then, by normality, we can find some $b \in R$ such that $ca = bc$. Since $ca \in Q$, $bc \in Q$ and so $bcR = bRc \subseteq Q$. Now, if $c \notin Q$ then $b \in Q$ and so $ca = bc \in QP$. Thus, if $c \notin Q$ then $c(Q \cap P) \subseteq QP \subseteq A$, since a was arbitrary in $Q \cap P$. However, ${}_{R/Q}|\frac{Q \cap P}{A}$ is non-zero and faithful so we conclude that $c \in Q$.

This establishes the "$\Rightarrow$" implication and the "$\Leftarrow$" is similar. ∎

Suppose that $\frac{Q \cap P}{A}$ is a linking bimodule for $Q \rightsquigarrow P$ and that we wish to factor by an ideal I. Lemmas 2.3.6 and 2.3.7 consider the two cases, $I \not\subseteq A$ and $I \subseteq A$. We note, however, that these cases may not be disjoint for primes P and Q since A is not fixed.

Lemma 2.3.6. *Let R be a Noetherian ring with prime ideals P and Q and suppose that $Q \rightsquigarrow P$ via $\frac{Q \cap P}{A}$. Let $c \in P - A$.*

(i) If c is normal in R then $Qc = cP$. If, moreover, c is a non-zero divisor then $Q = \sigma(P)$ where σ is the automorphism of R given by $\sigma(t)c = ct$ for $t \in R$.

(ii) If c is central in R then $Q = P$.

Proof. Firstly, we note that given the hypothesis of (i) or (ii), $c \in Q \cap P$ by Lemma 2.3.5.

(i) Suppose c is normal in R and let $d \in Q$. Then we can find $e \in R$ such that $ce = dc \in QP$ and so $cRe = Rce \subseteq QP \subseteq A$. Now $\frac{cR+A}{A}$ is a non-zero sub-bimodule of $\frac{Q \cap P}{A}$ and, since $\frac{Q \cap P}{A}|_{R/P}$ is fully faithful, it follows that $e \in P$. Thus $Qc \subseteq cP$ and similarly $cP \subseteq Qc$. The first part of (i) follows and the second part is immediate from the existence of the automorphism σ in the case where c is a non-zero-divisor.

(ii) Suppose c is central in R. Then, $cQ = Qc \subseteq A$ and so, again since $\frac{Q \cap P}{A}|_{R/P}$ is fully faithful, we conclude that $Q \subseteq P$ and similarly $P \subseteq Q$ establishing (ii). ∎

The next result is immediate from the definitions.

Lemma 2.3.7. *Let R be a Noetherian ring and P and Q prime ideals of R. Let I be an ideal of R with $I \subset P \cap Q$. Then, for any ideal A with $I \subseteq A \subset Q \cap P$,*

$$Q \rightsquigarrow P \text{ via } \frac{Q \cap P}{A} \iff Q/I \rightsquigarrow P/I \text{ via } \frac{Q/I \cap P/I}{A/I}. \qquad \blacksquare$$

The following result, which is also contained in [J2, Proposition 5.3.12], is an easy consequence of Lemmas 2.3.6 and 2.3.7 and a simple induction argument on the number of generators for I_R. A version for ideal links can be found in [L&L, Corollary 2.5].

Corollary 2.3.8. *Let R be a Noetherian ring with prime ideals P and Q and a polycentral ideal I. Then,*

$$I \subseteq P,\ Q \neq P \text{ and } Q \rightsquigarrow P \iff I \subseteq Q \cap P,\ Q/I \neq P/I \text{ and } Q/I \rightsquigarrow P/I.$$

Proof. The "$\Leftarrow$" implication is obvious, so consider the "$\Rightarrow$" implication.

(a) First suppose $I = cR$ for some central c in R and assume the left-hand side with $Q \rightsquigarrow P$ via $\frac{Q \cap P}{A}$. Then, by Lemma 2.3.6(ii), $c \in A$ and the right-hand side follows by Lemma 2.3.7.

(b) Now suppose the result is known for some polycentral ideal I' and that $c + I'$ is central in R/I' for some $c \in R$. Put $I = I' + cR$ and assume $I \subseteq P$, $Q \neq P$ and $Q \rightsquigarrow P$. Then certainly $c + I' \in P/I'$, $Q/I' \neq P/I'$, and $Q/I' \rightsquigarrow P/I'$ and so, by (a), $c + I' \subseteq Q/I' \cap P/I'$, $\frac{Q/I'}{(cR+I')/I'} \neq \frac{P/I'}{(cR+I')/I'}$ and $\frac{Q/I'}{(cR+I')/I'} \rightsquigarrow \frac{P/I'}{(cR+I')/I'}$. Thus $I \subseteq Q \cap P$, $Q/I \neq P/I$ and $Q/I \rightsquigarrow P/I$.

The result now follows by induction on the number of generators of I_R. $\blacksquare$

This last corollary shows that, for a polycentral ideal I which is contained in P, $Q \in \text{r.cl.}(P)$ if and only if $Q/I \in \text{r.cl.}(P/I)$. Thus, the question of the preservation of representational repleteness under such factors and extensions is equivalent to asking whether $Q \in \text{Fund}(P)$ if and only if $Q/I \in \text{Fund}(P/I)$. In the next two lemmas we turn our attention to the fundamental primes and indeed Lemma 2.3.9 answers this question in the positive.

Lemma 2.3.9. *Let R be a Noetherian ring with prime ideals P and Q and an ideal I such that $I \subseteq P \cap Q$. Suppose also that* cl.(P) *satisfies the second layer condition. Then, for all $n \in \mathbb{N}$,*

$$Q/I \in A_n(P/I) \implies Q \in A_n(P).$$

Further, if I is polycentral we have the partial converse

$$Q \in A_n(P) \implies Q/I \in \bigcup_{k=1}^{n} A_k(P/I).$$

Proof. We choose a uniform submodule, U, of $(R/P)_R$ and, for each $n \in \mathbb{N}$, denote $(E_R(U))_n$ and $\big(E_{R/I}(U)\big)_n$ by $(E_R)_n$ and $(E_{R/I})_n$ respectively.

It is well known that $E_{R/I}(U) = \text{ann}_{E_R(U)}(I)$ and, in fact, we can see from an induction on n using Theorem 1.3.3(ii) that, for all $n \in \mathbb{N}$,

$$(E_{R/I})_n = \text{ann}_{(E_R)_n}(I) \ . \tag{A}$$

Suppose $Q/I \in A_n(P/I)$. Then we can find $a \in (E_{R/I})_n - (E_{R/I})_{n-1}$ such that $aQ \subseteq (E_{R/I})_{n-1}$. By (A), $a \in (E_R)_n$ and $a \notin (E_R)_{n-1}$. However, $aQ = a(Q/I) \subseteq (E_{R/I})_{n-1} \subseteq (E_R)_{n-1}$ and so $Q \in A_n(P)$ establishing the first implication.

To prove the second implication, we first assume that $I = cR$ for some central $c \in R$. Let $n \in \mathbb{N}$, fix $Q \in A_n(P)$ and let $M/(E_R)_{n-1} = \text{ann}_{E_R(U)/(E_R)_{n-1}}(Q)$.

Suppose $Q \notin \bigcup_{k=1}^{n-1} A_k(P)$. Then there are r.cl.$(P)$-semiprimes, $T_1, \dots, T_{n-1}$ with $P = T_1$, over none of which Q is minimal and which satisfy $MQT_{n-1}\dots T_1 = 0$. Now, $MT_{n-1}\dots T_1 \nsubseteq (E_R)_{n-1}$ but $(MT_{n-1}\dots T_1)c = McT_{n-1}\dots T_1 = 0$ since $c \in Q$ and c is central in R. Certainly, then, $MT_{n-1}\dots T_1 \subseteq E_{R/cR}(U)$ and, since

$$(MT_{n-1}\dots T_1)(Q/cR) = MT_{n-1}\dots T_1 Q \subseteq MQ \subseteq (E_R)_{n-1} \ ,$$

it follows from (A) that $(MT_{n-1}\dots T_1)(Q/cR) \subseteq (E_{R/cR})_{n-1}$. Consequently, in this case, $Q/cR \in A_n(P/cR)$.

Since $A_1(P) = \{P\}$ and $A_1(P/I) = \{P/I\}$ the result for $I = cR$ now follows by induction on n. Finally, by a similar argument to the proof (b) of Corollary 2.3.8, the result for any polycentral ideal, I, follows by an induction argument on the number of generators for I_R. ∎

We see in Examples 2.3.15 and 2.3.16 that the second part of this lemma does not extend to a full converse or to the case where I is only polynormal.

We could now state the result for the polycentral case but defer this to allow the remaining technicalities for polynormal extensions to be disposed of. We first require a definition in which it is sufficient for us to restrict our attention to the case where I is cyclic.

Definition 2.3.10. Let R be a Noetherian ring and consider $I = cR$ where c is a normal element. We wish to deduce the repleteness of a prime P of R, containing I, from the repleteness of P/I.

Let $Q' \in \text{r.cl.}(P)$. Lemma 2.3.7 shows that, given a link $Q'' \rightsquigarrow Q'$, on passing to the factor ring we can find a link $Q''/I \rightsquigarrow Q'/I$ provided $I \subseteq A$ for some linking bimodule $\frac{Q'' \cap Q'}{A}$ between Q'' and Q'. Such a link, we call a *lifting link.* (If Q is linked by a chain of lifting links to P and P/I is rep. rep., Lemma 2.3.9 shows that $Q \in A_n(P)$.)

When we can find no such bimodule, we call the link in R a *non-lifting link* and apply Lemma 2.3.6: if c is central, we see from (ii) that the only non-lifting links are self-links; if c is not central, we insist that it is a non-zero-divisor and then, by (i), the link is given by the automorphism σ of R. ∎

We now show that $\text{Fund}(P)$ is closed under σ.

Lemma 2.3.11. *Let R be a Noetherian ring and let P be a prime ideal of R. Suppose either that $\text{cl.}(P)$ satisfies the second layer condition and $\text{r.cl.}(P)$ is locally finite or that R satisfies the second layer condition. Let c be a normal element of R which is a non-zero-divisor and suppose $c \in P$. Define an automorphism, σ, of R by $ct = \sigma(t)c$ for each $t \in R$. Then, for all $Q \in \text{Fund}(P)$ and $n \geq 0$, $\sigma^n(Q) \in \text{Fund}(P)$.*

Proof. By Remark 1.3.12 or by Theorem 1.3.7(i), there are ideals $J \subset I$ of R such that ${}_{R/Q}\left|\frac{I}{J}\right|_{R/P}$ is torsion-free on both sides. Let $n \geq 0$. Since $\sigma^n(Q)c^n = c^nQ$, we can consider $B = {}_{R/\sigma^n(Q)}\left|\frac{c^nI}{c^nJ}\right|_{R/P}$. Now, since c is a non-zero-divisor, $c^nI/c^nJ \cong I/J$ as right R/P-modules and so $B_{R/P}$ is torsion-free.

Now let $u \in R$, regular modulo $\sigma^n(Q)$, let $a \in I$ and suppose $uc^n a \subseteq c^n J$. Since $uc^n = c^n\sigma^{-n}(u)$ and c is a non-zero-divisor, we see that $\sigma^{-n}(u)a \subseteq J$. However, since u is regular modulo $\sigma^n(Q)$, $\sigma^{-n}(u)$ is regular modulo Q. So, by the torsion-freeness of ${}_{R/Q}(I/J)$, it follows that $a \in J$ and $c^n a \in c^n J$. Thus, ${}_{R/\sigma^n(Q)}B$ is torsion-free.

From Theorem 1.3.7(ii), we conclude that $\sigma^n(Q) \in \mathrm{Fund}(P)$. ∎

Where $I = cR$ for a normal, non-zero-divisor c, Lemmas 2.3.6 and 2.3.7 show us that any non-lifting link arises via the automorphism σ, so it follows from Lemma 2.3.11 that a prime Q', linked via n non-lifting links to a fundamental prime Q, is itself fundamental, although it does not say in which layer it lies. For strongly rep. rep. to be preserved, we would, of course, require Q' to be in the n^{th} layer above Q. (However, while we do not know in general that $\sigma(P) \rightsquigarrow P$, Lemma 2.3.11 does show that $\sigma(P) \in \text{r.cl.}(P)$.) In any case, what of a prime Q'' linked to Q' via a lifting link ? Lemma 2.3.9 does not help here since we do not know that $Q'/I \in \mathrm{Fund}(P/I)$. The next lemma uses σ to find a new chain of links from Q'' to P with all of the non-lifting links occurring at the left-hand end where Lemma 2.3.11 can handle them.

Remark 2.3.12. Part (iii) of Lemma 2.3.13 can also be found in [B&duC, Lemma 3.8] where, however, the proof is incomplete, this last question discussed in the previous paragraph being overlooked. ∎

Lemma 2.3.13. *Let R be a Noetherian ring and let P be a prime ideal of R. Also, let c be a normal element of R which is a non-zero-divisor and suppose $c \in P$. Define an automorphism, σ, of R by $ct = \sigma(t)c$ for each $t \in R$.*

(i) If Q is a prime of R with $Q \rightsquigarrow P$ via $\frac{Q \cap P}{A}$ then,

$$(a)\ c \in A \implies \begin{cases} (1) & Q/cR \rightsquigarrow P/cR\,, \\ (2) & \text{for any } a \geq 0,\ c = \sigma^{-a}(c) \in \sigma^{-a}(A) \\ & \text{and } \sigma^{-a}(Q) \rightsquigarrow \sigma^{-a}(P) \text{ via } \frac{\sigma^{-a}(Q)\ \cap\ \sigma^{-a}(P)}{\sigma^{-a}(A)}\,, \\ (3) & \text{for any } a \geq 0,\ \frac{\sigma^{-a}(Q)}{cR} \rightsquigarrow \frac{\sigma^{-a}(P)}{cR}\ ; \end{cases}$$

$$(b)\ c \notin A \implies Q = \sigma(P)\ .$$

(ii) *If $Q' \in X_{n+1}(P)$ for some $n \geq 0$, there exist $Q \in \mathrm{r.cl.}(P)$ and $0 \leq a \leq n$ such that $Q' = \sigma^a(Q)$ and $Q/cR \in X_{n-a+1}(P/cR)$ (where $c \in Q$ by Lemma 2.3.5).*

(iii) *Suppose either that $\mathrm{cl.}(P)$ satisfies the second layer condition and $\mathrm{r.cl.}(P)$ is locally finite or that R satisfies the second layer condition. Then,*

$$\mathrm{r.cl.}(P) = \left\{\sigma^n(Q) : n \geq 0 \text{ and } Q/cR \in \mathrm{r.cl.}(P/cR)\right\} .$$

Proof. (i) Let Q be a prime of R with $Q \rightsquigarrow P$ via $\frac{Q \cap P}{A}$.

(a) If $c \in A$, then (1) is true by Lemma 2.3.7, (2) is easy to see and (3) follows from Lemma 2.3.7 and (2).

(b) This implication is just Lemma 2.3.6(i).

(ii) Since $X_1(P) = \{P\}$, we can take $a = 0$ and $Q = P$ in the case where $n = 0$ and we proceed by induction. So suppose the result is true for $n = m$ for some $m \geq 0$ and consider $Q'' \in X_{m+2}(P)$. Then, by definition, there exists $Q' \in X_{m+1}(P)$ such that $Q'' \rightsquigarrow Q'$, via $\frac{Q'' \cap Q'}{A}$, say, while, by hypothesis, there exists $Q \in \mathrm{r.cl.}(P)$ and $0 \leq a \leq m$ such that $Q' = \sigma^a(Q)$ and $Q/cR \in X_{m-a+1}(P/cR)$. From (i), either

(a) $c \in A$ and, by (3), $\frac{\sigma^{-a}(Q'')}{cR} \rightsquigarrow \frac{\sigma^{-a}(Q')}{cR} = \frac{Q}{cR}$, and so $\frac{\sigma^{-a}(Q'')}{cR} \in X_{m-a+2}(\frac{P}{cR})$ while $Q'' = \sigma^a(\sigma^{-a}(Q''))$;

or else

(b) $c \notin A$ and so $Q'' = \sigma(Q')$.

In either case, the result is true for $n = m + 1$ and (ii) follows by induction on n.

(iii) The left-hand side is contained in the right-hand side as a consequence of (ii), and so we consider the other inclusion. Let $Q/cR \in \mathrm{r.cl.}(P/cR)$ and let $n \geq 0$. Now, by Lemma 2.3.7, $Q \in \mathrm{r.cl.}(P)$ and so $\mathrm{r.cl.}(Q) \subseteq \mathrm{r.cl.}(P)$. In particular, if $\mathrm{r.cl.}(P)$ is locally finite, so is $\mathrm{r.cl.}(Q)$. Certainly $Q \in \mathrm{Fund}(Q)$ and therefore, by Lemma 2.3.11,

$$\sigma^n(Q) \in \mathrm{Fund}(Q) \subseteq \mathrm{r.cl.}(Q) .$$

Thus, $\sigma^n(Q) \in \mathrm{r.cl.}(P)$. ∎

We now give the results for both the polycentral and polynormal cases.

Theorem 2.3.14. *Let R be a Noetherian ring with a prime ideal P and let I be an ideal of R with $I \subseteq P$.*

(i) Suppose that cl.(P) *satisfies the second layer condition. If I is polycentral,*

$$P/I \text{ is rep. rep. in } R/I \iff P \text{ is rep. rep. in } R.$$

(ii) Suppose either that cl.(P) *satisfies the second layer condition and* r.cl.(P) *is locally finite or that R satisfies the second layer condition. If I is regularly polynormal,*

$$P/I \text{ is rep. rep. in } R/I \implies P \text{ is rep. rep. in } R.$$

Proof. (i) Corollary 2.3.8 shows that, for each prime Q of R, $Q \in \text{r.cl.}(P)$ if and only if $Q/I \in \text{r.cl.}(P/I)$ while Lemma 2.3.9 shows that $Q \in \text{Fund}(P)$ if and only if $Q/I \in \text{Fund}(P/I)$ and (i) follows.

(ii) Let c be a regular normal element and, as before, let σ be the automorphism of R defined by $ct = \sigma(t)c$ for each $t \in R$. Let $Q' \in \text{r.cl.}(P)$. Then $Q' \in X_{n+1}$ for some $n \geq 0$ and so, by Lemma 2.3.13(ii), there exist $Q \in \text{r.cl.}(P)$ and $0 \leq a \leq n$ such that $Q' = \sigma^a(Q)$ and $Q/cR \in X_{n-a+1}(P/cR)$. Now, if we assume that P/cR is rep. rep. in R/cR, then $Q/cR \in A_m(P/cR)$ for some $m \in \mathbb{N}$ and, by Lemma 2.3.9, $Q \in A_m(P)$. (Of course, if P/cR is strongly rep. rep. in R/cR, we can take $m = n - a + 1$.) In this case, Lemma 2.3.11 shows that $Q' = \sigma^a(Q) \in \text{Fund}(P)$.

We have thus established (ii) for $I = cR$ and, by a similar argument to the proof (b) of Corollary 2.3.8, the result for any regularly polynormal ideal, I, follows by an induction argument on the number of generators for I_R. ∎

Example 2.3.15. It is easy to see that 2.3.14(i) does not extend to strong representational repleteness in either direction. Consider $R = \mathbb{C}[x]$ and $P = xR$. Then $A_n(P) = X_n(P) = \{P\}$ for each n so that P is strongly rep. rep. in R, yet, as we have seen, $\overline{P} = P/x^2R$ is not strongly rep. rep. in $\overline{R} = R/x^2R$. So "$\Leftarrow$" fails even though x^2 is a regular central element. Then again, $\frac{\overline{P}}{xR/x^2R} \cong 0$ is strongly rep. rep. in $\frac{\overline{R}}{xR/x^2R} \cong \mathbb{C}$, so "$\Rightarrow$" fails.

Since $x + x^2R$ is a zero-divisor in R/x^2R, this example would not prevent (ii) from extending to the analogous statement for strong representational repleteness, a question which remains open. ∎

Example 2.3.16. Certainly, without regularity, (ii) fails and, moreover, the converse to (ii) fails in general, even assuming strong representational repleteness in each case. To see this, consider $T = \mathbb{C}[x, y : yx - xy = x]$ (the enveloping algebra of the two dimensional solvable non-Abelian Lie algebra over $\mathbb{C}$) and $Q = xT + yT$. Then, as we saw in (2.2.5), Q is strongly rep. rep. in T while, as noted in (2.2.9), $\overline{Q} = Q/x^2T$ is not even rep. rep. in $\overline{T} = T/x^2T$, even though x^2 is a regular normal element of T. Then again, $\overline{x}$ is normal but not regular in $\overline{T}$ and, by Corollary 2.1.3, it is easy to see that $\frac{\overline{Q}}{xT/x^2T} \cong \theta\,\mathbb{C}[\theta]$ is strongly rep. rep. in the ring $\frac{\overline{T}}{xT/x^2T} \cong \mathbb{C}[\theta]$.

More will be said about differential operator rings like T in Chapter 4. ∎

We complete this section with an application of the last theorem. Let U be the enveloping algebra of a finite dimensional solvable Lie algebra over $\mathbb{C}$ and P a prime ideal of U such that $U/P \cong \mathbb{C}$. We have seen in (2.2.7) that, when the dimension is two, P is strongly rep. rep. in U and, as commented in (2.2.8), we might hope to extend our argument to Lie algebras of arbitrary finite dimension. However, using Theorem 2.3.14, we can obtain a result for *any* prime of U and, in fact, state it in more generality.

Recall that, for any ring T with an ideal I, we denote by $\mathcal{C}_T(I)$ the set of elements of T regular modulo I.

Corollary 2.3.17. *(i) Let R be a Noetherian ring, let P be a prime ideal of R and let $\mathcal{C}$ be a right Ore set in R such that $\mathcal{C} \subseteq \mathcal{C}_R(P)$. Suppose either that* cl.(P) *satisfies the second layer condition and* r.cl.(P) *is locally finite or that R satisfies the second layer condition. Suppose, further, that $P\mathcal{C}^{-1}$ is regularly polynormal in $R\mathcal{C}^{-1}$. Then P is rep. rep. in R.*

(ii) If U is the enveloping algebra of a finite dimensional solvable Lie algebra over $\mathbb{C}$, then U is rep. rep..

Proof. (i) First, we note that, by Remarks 2.3.1, the above hypotheses tell us either that cl.$(P\mathcal{C}^{-1})$ satisfies the second layer condition and r.cl.$(P\mathcal{C}^{-1})$ is locally finite or that RC^{-1} satisfies the second layer condition. Now, since r.cl.(0) = $\{0\}$, PC^{-1}/PC^{-1} is strongly rep. rep. in RC^{-1}/PC^{-1} and so, by Theorem 2.3.14(ii), PC^{-1} is rep. rep. in RC^{-1}. Theorem 2.3.2 then shows that P is rep. rep. in R, establishing the first part of the corollary.

(ii) Next we recall that the enveloping algebra of any solvable Lie algebra satisfies the s.s.l.c. (for instance see [J2, Theorem A.3.9]). Now, for any prime P of U, let $\mathcal{S}(P) = \bigcap_{Q \in \mathrm{cl.}(P)} \mathcal{C}_U(Q)$. By [B4, Theorem 5.3(ii)], $\mathcal{S}(P)$ is an Ore set and, by [B4, Theorem 6.1(iii)], $P\mathcal{S}(P)^{-1}$ is regularly polynormal in $U\mathcal{S}(P)^{-1}$ so that the first part applies. ∎

As we do not know whether Theorem 2.3.14(ii) extends to strongly rep. rep., it is an open question here whether the enveloping algebras of Corollary 2.3.17(ii) are in fact strongly rep. rep.. However, having seen a class of rings which are at least rep. rep., we consider in the next chapter a class of rings in which we can categorize those which are strongly rep. rep..

§2.4 Notes

Although the question of when the fundamental primes coincide with the right clique of an assassinator prime has been raised before (for instance in [B&W, 5.11]), the definitions of (2.1.1) are here introduced for the first time. Consequently, Corollary 2.1.3 and Lemma 2.1.6 are both new results.

Theorems 2.2.2 and 2.2.3 are both standard results which can be found, for instance, in [Rm]. Specifically, the first is proved in [Rm, Theorems 2.9 and 3.16] and the second in [Rm, Theorem 2.11].

There does not seem to be a specific reference for Corollary 2.2.4 although, as an easy consequence of Theorems 2.2.2 and 2.2.3, it may be well known. It is, however, derived in specific cases in [N2, §1], [Da, Theorem 3] and [G&W, Lemma 4.3].

The derivation of the injective hull in Example 2.2.5 is, as stated, extracted from [N2, §§1&2], although the construction first appeared as [Hs, Exercise 6.11].

Indeed, the use of modules of inverse polynomials goes back to F. S. Macaulay [McA]. As noted in (2.2.6), related constructions and generalizations can be found in [Ga1], [F], [S&S] and [Ku].

Again, the calculation of the injective hull in Example 2.2.7 is a modified version of [Da, §4, Example 1]. This is a special case of [Lv, Theorem 2.2] which is, itself, based on results of [H1] and [H2]. However, that a Lie algebra, L, over a field, k, of characteristic zero acts via derivations on $\mathrm{Hom}_k(U(L), k)$ and that this algebra is isomorphic to an algebra of formal power series over k, can be found in [D2, p91 and Proposition 2.7.5].

The description of the link graph of the prime P in Example 2.2.7, is certainly well known (see, for instance, [G&W, Exercise 11G]). It can also be calculated from Theorem 4.1.7(ii) [G2, Theorem 5.11(a)] (see Example 4.3.1). Alternatively, the links between co-Artinian primes of the enveloping algebra of a finite-dimensional solvable Lie algebra over $\mathbb{C}$ are fully described by [Lp] (quoted in [B3, Theorem 2.11]), a result partially generalized to all primes by [B3, Theorem 2.9].

Theorems 2.3.2 and 2.3.3 are new; however, the preservation of second layer links and of the second layer conditions under localization and under Morita equivalences, are standard results. See [G&W, Ex11S] for the preservation of links under localization and the proof of [J2, Proposition 8.1.4] for the preservation of the second layer conditions under localization. That the converse holds for the second layer condition but is open for the strong second layer condition, is noted in [Be3, p23]; as seen in (1.2.5), there are in any case no known examples of a prime satisfying the second layer condition but not the strong second layer condition. The Morita invariance of the second layer conditions is noted in [J2, Proposition 8.1.5].

Lemma 2.3.5 is a consequence of the facts that any ideal generated by normal elements has the AR property (see, for instance, [J2, Theorem 3.3.16]) and that, if I is an ideal satisfying the AR property and $I \subseteq P$ for some prime ideal P, then $I \subseteq Q$ for each prime Q in the clique of P, and even for each Q ideal linked to P (see [J2, Proposition 5.3.9]).

Lemma 2.3.6(i) is essentially contained in the proof of [B&duC, Lemma 3.8].

Corollary 2.3.8 is contained in [J2, Proposition 5.3.12]. It is also shown in [L&L, Corollary 2.5] that an analogous statement holds for ideal links, namely that, assuming P and Q are prime ideals and I is a polycentral ideal of a Noetherian ring R, if $I \subseteq P$ and $Q \rightsquigarrow\!\!\!> \rightsquigarrow\!\!\!> P$ in R, then $I \subseteq Q$ and $Q/I \rightsquigarrow\!\!\!> \rightsquigarrow\!\!\!> P/I$ in R/I.

Lemmas 2.3.9 and 2.3.11 appear to be new results as does Lemma 2.3.13(ii) however, as noted, Lemma 2.3.13(iii) has appeared as [B&duC, Lemma 3.8] with a gap in the proof.

Theorem 2.3.14 and Corollary 2.3.17 are also new results.

Chapter 3 : Finite Dimensional Algebras

As shown in Lemma 2.1.6, any Noetherian hereditary ring satisfying the second layer condition is strongly rep. rep.. Theorem 2.3.14(i) can therefore be used to provide examples of factors of hereditary rings which will again be rep. rep.. Of course, by Theorem 2.3.3, any ring Morita equivalent to such a factor will also be rep. rep.. Here we consider a finite dimensional k-algebra, A, where k is an algebraically closed field, and show in Theorem 3.3.6, the main result of this chapter, that A is strongly rep. rep. precisely when A is Morita equivalent to a certain kind of factor of a hereditary finite dimensional k-algebra.

§3.1 Harada's Results and Generalized Triangular Matrix Rings

Definitions 3.1.1. We recall that an *idempotent* in a ring R is an element e such that $e = e^2$ and that two idempotents e_1 and e_2 are *orthogonal* provided $e_1e_2 = e_2e_1 = 0$. An idempotent e is called *primitive* when e is non-zero and there do not exist two non-zero orthogonal idempotents e_1 and e_2 such that $e = e_1 + e_2$. Let I be an ideal of the ring R and let $g + I$ be an idempotent element of R/I. Then we say that this idempotent can be *lifted (to e) modulo I* when there is an idempotent e of R such that $e + I = g + I$. When every idempotent of R/I can be lifted to R we say that *idempotents lift modulo I*.

A ring R is called *semiperfect* provided idempotents lift modulo its Jacobson radical $J(R)$ and $R/J(R)$ is Artinian. Let R be a semiperfect ring. Then a module M_R is *primitive* when $M \cong eR$ for some primitive idempotent e of R. A set $\{e_1, \ldots, e_m\}$ of idempotents of R is *basic* when its elements are pairwise orthogonal and the set $\{e_1R, \ldots, e_mR\}$ is a complete irredundant set of representatives of the primitive right R-modules. By [A&F, Proposition 27.10], such a basic set of idempotents always exists for a semiperfect ring.

An idempotent e of a semiperfect ring R is called a *basic* idempotent when it is the sum of the elements of a basic set of primitive idempotents. Indeed, since a basic set of primitive idempotents is pairwise orthogonal, the sum of the elements of any basic set is a basic idempotent. A ring S is a *basic ring for* R provided S is isomorphic to eRe for some basic idempotent e of R. Since, by [A&F, Propsition 27.10], a basic set of idempotents exists for any semiperfect ring, any such ring R has a basic ring. Since all basic rings for R are isomorphic (see [A&F, p308]), we refer to *the* basic ring of R and denote it by $B(R)$. ∎

Remarks 3.1.2. A ring R is called *semiprimary* if its Jacobson radical, $J(R)$ is nilpotent and $R/J(R)$ is Artinian. Obviously, Artinian rings, in particular finite dimensional k-algebras, are semiprimary. On the other hand, it is easy to see that all Noetherian semiprimary rings are Artinian and so the terms "Noetherian semiprimary" and "Artinian" are equivalent. In 1964, Harada [Ha] categorized hereditary semiprimary rings as certain "generalized triangular matrix rings" over semisimple rings.

Now, by [A&F, Proposition 27.1], idempotents lift modulo a nil ideal in any ring and so we see that all semiprimary rings are semiperfect. Let R be a semiperfect ring (so for instance R could be semiprimary). Then R is Morita equivalent to its basic ring, $B(R)$, [A&F, Proposition 27.14]. Furthermore, R is termed a *basic ring* provided 1 is a basic idempotent for R; it is noted in [A&F, p309] that the basic ring of a semiperfect ring is indeed a basic ring. By [A&F, Proposition 27.15], R is basic if and only if $R/J(R)$ is a direct sum of division rings. By the Morita invariance of the strong rep. rep., hereditary and semiprimary properties, we need only consider basic semiprimary rings. ∎

Next, however, we must recall the construction given in [Ha, §2] of a generalized triangular matrix ring.

Notation 3.1.3. Suppose that $R_1, \ldots, R_n$ are rings and let ${}_{R_i}|M_{i,j}|_{R_j}$ be bimodules for $1 \leq j < i \leq n$. Let $T_n(R_i; M_{i,j})$ be the set

$$\left\{ \begin{pmatrix} r_{1,1} & 0 & \cdots & 0 & 0 \\ m_{2,1} & r_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & r_{n-1,n-1} & 0 \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n-1} & r_{n,n} \end{pmatrix} : r_{t,t} \in R_t \text{ and } m_{i,j} \in M_{i,j} \right\}.$$

For convenience, we denote R_t by $M_{t,t}$ for $1 \le t \le n$ and we let $m_{i,j}$ and $m'_{i,j}$ denote elements of $M_{i,j}$ for $1 \le j \le i \le n$. Then, $T_n(R_i; M_{i,j})$ becomes a ring under the definitions

$$(m_{i,j}) \pm (m'_{i,j}) = (m_{i,j} \pm m'_{i,j})$$
$$(m_{i,j}) \,.\, (m'_{i,j}) = \sum_{l=j}^{i} \phi^l_{i,j}(m_{i,l} \otimes m'_{l,j})$$

where the

$$\phi^l_{i,j} : M_{i,l} \otimes_{R_l} M_{l,j} \to M_{i,j}$$

are bilinear (R_i, R_j)-homomorphisms, in particular

and
$$\phi^t_{i,t} : M_{i,t} \otimes_{R_t} R_t \cong M_{i,t}$$
$$\phi^i_{i,t} : R_i \otimes_{R_i} M_{i,t} \cong M_{i,t}$$

being given by the module actions in ${}_{R_i}|M_{i,t}|_{R_t}$, and where the diagrams

$$\begin{array}{ccc} M_{i,j} \otimes_{R_j} M_{j,l} \otimes_{R_l} M_{l,k} & \xrightarrow{\iota_{i,j} \otimes \phi^l_{j,k}} & M_{i,j} \otimes_{R_j} M_{j,k} \\ \big\downarrow{\scriptstyle \phi^j_{i,l} \otimes \iota_{l,k}} & & \big\downarrow{\scriptstyle \phi^j_{i,k}} \\ M_{i,l} \otimes_{R_l} M_{l,k} & \xrightarrow{\phi^l_{i,k}} & M_{i,k} \end{array}$$

with ι denoting the identity map, commute.

Given the above, $T = T_n(R_i; M_{i,j})$ is called a *generalized triangular matrix ring* (or a *g.t.a. matrix ring*) *over the rings* R_i. Let

$$e_i = \mathrm{diag}\,(0_{R_1}, \ldots, 0_{R_{i-1}}, 1_{R_i}, 0_{R_{i+1}}, \ldots, 0_{R_n})\ .$$

Then, for any ideal I of T, $e_i I e_j \subseteq I$ and hence $I = T_n(S_i; N_{i,j})$ where S_i is an ideal of R_i and $N_{i,j}$ is an (R_i, R_j)-sub-bimodule of $M_{i,j}$. In these circumstances, we denote S_i by $I_{i,i}$ and $N_{i,j}$ by $I_{i,j}$. ∎

It is easy to see that the ring $T_n(R_i; M_{i,j})$ is semiprimary if and only if each ring R_i is semiprimary. Thus the hereditary semiprimary generalized triangular matrix rings are characterized in the following result taken from [Ha, Theorem 1].

Theorem 3.1.4. *For $1 \le i \le n$, let R_i be semiprimary rings with radical J_i. A generalized triangular matrix ring $T_n(R_i; M_{i,j})$ over R_i is hereditary if and only if the following conditions are satisfied:*

(a) *all R_i are hereditary;*

(b) *for each triplet (i, j, k), with $1 \le j \le k \le i \le n$, $\phi^j_{i,k}$ is monomorphic;*

(c) *for each pair (i, j), with $1 \le j < i \le n$, $\frac{M_{i,j}}{M_{i,j}J_j + \sum_{t=j+1}^{i-1} M_{i,t}M_{t,j}}$ is a projective left R_i-module; in this case we let $\overline{M_{i,j}}$ be an R_i submodule of $M_{i,j}$ isomorphic to $\frac{M_{i,j}}{M_{i,j}J_j + \sum_{t=j}^{i-1} M_{i,t}M_{t,j}}$;*

(d) *for each pair (i, j) with $1 \le j < i \le n$,*

$$M_{i,j} = \overline{M_{i,j}} \oplus M_{i,j+1}\overline{M_{j+1,j}} \oplus \cdots \oplus M_{i,i-1}\overline{M_{i-1,j}} \oplus M_{i,j}J_j$$

as a left R_i-module. ∎

The next result, taken from [Ha, Theorem 4'] and its proof, shows that every hereditary semiprimary ring is isomorphic to a generalized triangular matrix ring. To state it, however, we will again need to introduce some further notation.

Notation 3.1.5. Let T be a basic semiprimary ring (as is the ring T of Theorem 3.1.6 by Remarks 3.1.2) and let $\{e_i : 1 \le i \le n\}$ be a basic set of primitive idempotents of T such that $1 = \sum_{i=1}^n e_i$ (see Definitions 3.1.1). Since the radical J of T is nilpotent, there is an integer $n(e_i)$ for each e_i such that $J^{n(e_i)}e_i \ne 0$ while $J^{n(e_i)+1}e_i = 0$. We re-order the e_i, if necessary, to ensure that $n(e_i) \ge n(e_{i+1})$ for all i. ∎

Theorem 3.1.6. *Let T be a hereditary semiprimary ring with radical J and suppose that $T/J = \oplus_{i=1}^n R_i$ for some division rings R_i. Let $\{e_i : 1 \le i \le n\}$ be a basic set of primitive idempotents of R such that $1 = \sum_{i=1}^n e_i$ and assume that these have been re-ordered as described in (3.1.5). For $1 \le j < i \le n$, put $M_{i,j} = e_iTe_j$. Then, T is isomorphic to the generalized triangular matrix ring $T_n(R_i; M_{i,j})$ where the $\phi^j_{i,k}$ of (3.1.3) are given by multiplication in T.* ∎

Notation 3.1.7. We assume that the rings R_i are Noetherian and that the bimodules ${}_{R_i}|M_{i,j}|_{R_j}$ are finitely generated on both sides, so that $T = T_n(R_i; M_{i,j})$

is Noetherian. If, moreover, the R_i are semiprimary, we then have that T is a semiprimary Noetherian and hence, as noted in (3.1.2), Artinian ring. In this case, therefore, T satisfies the strong second layer condition and, since T has only finitely many primes, r.cl.(P) is trivially locally finite for each prime P of T. In particular, if the R_i are division rings, it is easily seen that the primes of T are the ideals

$$P_\lambda = T_n(R_1, \ldots, R_{\lambda-1}, 0, R_{\lambda+1} \ldots R_n; M_{i,j})$$

for $1 \leq \lambda \leq n$ and that the Jacobson radical of T is $J = J(T) = T_n(0\,; M_{i,j})$. We retain this notation throughout the present chapter. ∎

§3.2 Quivers

Definitions 3.2.1. Let Δ_0 and Δ_1 be two sets and let s and e be two maps from Δ_1 to Δ_0. Then

$$\Delta := (\Delta_0, \Delta_1, s, e)$$

is called a *quiver*. Here, Δ_0 is known as the set of *vertices* and Δ_1 as the set of *arrows*. Given an arrow α, $s(\alpha)$ is called its *starting point* and $e(\alpha)$ its *end point*. When $a = s(\alpha)$ and $b = e(\alpha)$, it is usual to write the arrow α as

$$a \xrightarrow{\alpha} b$$

and to say that it *points* from a to b. We note that there may be more than one arrow pointing from a to b. The quiver Δ is said to be *finite* provided both Δ_1 and Δ_0 are finite.

Given the quiver Δ, its *opposite quiver* Δ^* is the quiver

$$\Delta^* := (\Delta_0, \Delta_1, e, s)\ .$$

Thus, the opposite quiver is obtained simply by reversing the arrows of Δ.

A *path of length* $l \geq 1$ from vertices x to y in the quiver Δ is a sequence of arrows $\alpha_1, \ldots, \alpha_l$ such that $e(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i \leq l-1$ and where the starting point of α_1 is x and the end point of α_l is y. Such a path can be written

$$(x|\alpha_1, \ldots, \alpha_l|y) \ .$$

Furthermore, for any vertex x in Δ, we define a *path of length* 0 from x to itself and denote it by

$$x|x \ .$$

A path of length greater than or equal to 1 from a vertex to itself is called a *cyclic path.*

Finally, given a field k, we define the *path algebra* on Δ as the k-vector space with basis the set of all paths in Δ and with the product pq of two paths $p = (a|\alpha_1, \ldots, \alpha_l|b)$ and $q = (c|\beta_1, \ldots, \beta_s|d)$ defined as zero when $c \neq b$ and as

$$(a|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_s|d)$$

when $c = b$. The path algebra defined in this way is denoted $k\Delta$.

We note that the path algebra $k\Delta$ is finite dimensional provided Δ is finite and there are no cyclic paths in Δ. We note also that, in this case, the Jacobson radical of $k\Delta$ is the ideal generated by all arrows. We denote this ideal by $k\Delta^+$. Clearly $(k\Delta^+)^n$ is the ideal generated by all paths of length greater than or equal to n. ∎

The following result is proved in [Bm, Corollary 2.6]. A proof for the case where Q is a finite quiver can also be found in [Bn, Theorem 4.1.4].

Lemma 3.2.2. *Suppose that Q is a quiver with finitely many vertices and let k be a field. Then the path algebra kQ is a hereditary algebra.* ∎

Notation 3.2.3. Let k be an algebraically closed field and A a finite dimensional k-algebra. We form the *Ext-quiver* of A, denoted by Δ_A, by taking vertices to be the isomorphism classes of simple right A-modules and by requiring there to be n

arrows from the isomorphism class of the simple module M to that of the simple module N where

$$n = \dim_k \mathrm{Ext}^1_A(M, N) \ .$$

(If A is basic, of course, we can identify the vertices with the right A-modules A/P for each prime P of A and in this case the number of arrows from A/Q to A/P is $\dim_k \mathrm{Ext}^1_A(A/Q, A/P)$), for any two primes Q and P of A.)

Furthermore, we let H_A be the path algebra of the finite quiver, Δ^*_A, the opposite quiver of the Ext-quiver Δ_A of A. ∎

The next result was originally proved in [Ga3, §4.3]. Again, a version of this can be found in [Bn, Proposition 4.1.7]. (In this second reference, however, multiplication of paths is defined in reverse order so that the Ext-quiver is actually the opposite quiver of the above definition.)

Lemma 3.2.4. *Let k be an algebraically closed field and A be a basic finite dimensional k-algebra. Then, adopting the notation of (3.2.3), there is an ideal I of H_A with $I \subseteq (k\Delta^{*+})^2$ such that $A \cong H_A/I$.* ∎

Remark 3.2.5. It follows immediately from these two lemmas that a basic finite dimensional algebra over an algebraically closed field is isomorphic to a factor ring of a hereditary ring. However, although H_A is the path algebra of a finite quiver, this hereditary ring is not in general itself a finite dimensional algebra, since the Ext-quiver may contain cyclic paths. However, as we will see in Lemma 3.3.5, for a strongly rep. rep. algebra A, H_A is in fact finite dimensional and so we will be able to apply the construction of §3.1 to it. ∎

§3.3 The Repleteness of Finite Dimensional Algebras

We return to our consideration of generalized triangular matrix rings and use the notation of §3.1. In the case where the R_i are division rings it is easy to calculate ideal links and second layer links as we now show.

Lemma 3.3.1. *Let $T = T_n(R_i; M_{i,j})$ be a g.t.a. matrix ring where the R_i are division rings and the $M_{i,j}$ are finitely generated on both sides and let J be its Jacobson radical. Then, for any λ and $\mu \in \{1, \ldots, n\}$ and for each $m \in \mathbb{N}$, and with the notation of (3.1.3) and (3.1.7),*

(i) $P_\mu \rightsquigarrow P_\lambda \iff \mu > \lambda$ and $\sum_{i=\lambda+1}^{\mu-1} M_{\mu,i}M_{i,\lambda} \subsetneqq M_{\mu,\lambda}$

$\qquad\qquad \iff (J^2)_{\mu,\lambda} \subsetneqq J_{\mu,\lambda}$;

(ii) $P_\mu \in \mathrm{Fund}(P_\lambda) \iff M_{\mu,\lambda} \neq 0$;

(iii) $P_\mu \in A_{m+1}(P_\lambda) \iff (J^{m+1})_{\mu,\lambda} \subsetneqq (J^m)_{\mu,\lambda}$.

Proof. Let λ and $\mu \in \{1, \ldots, n\}$ and $m \in \mathbb{N}$.

(i) By Lemma 1.2.6, $P_\mu \rightsquigarrow P_\lambda$ if and only if $P_\mu \cap P_\lambda \supsetneqq P_\mu P_\lambda$. Now, if $i \neq \mu$ then $(P_\mu)_{i,i} = R_i$ while if $j \neq \lambda$ then $(P_\lambda)_{j,j} = R_j$ and in either case $(P_\mu P_\lambda)_{i,j} = M_{i,j}$. So, for $\mu > \lambda$, $P_\mu \rightsquigarrow P_\lambda$ if and only if $M_{\mu,\lambda} \supsetneqq (P_\mu P_\lambda)_{\mu,\lambda}$, while $(P_\mu P_\lambda)_{\mu,\lambda} = \sum_{i=\lambda+1}^{\mu-1} M_{\mu,i}M_{i,\lambda}$ since $(P_\mu)_{\mu,\mu} = 0$ and $(P_\lambda)_{\lambda,\lambda} = 0$. Also, for $\mu > \lambda$, $\sum_{i=\lambda+1}^{\mu-1} M_{\mu,i}M_{i,\lambda} = (J^2)_{\mu,\lambda}$ and $M_{\mu,\lambda} = J_{\mu,\lambda}$. Of course, if $\mu \leq \lambda$, $(P_\mu \cap P_\lambda)_{\mu,\lambda} = 0$ so that $P_\mu \not\rightsquigarrow P_\lambda$ and also $(J^2)_{\mu,\lambda} = J_{\mu,\lambda} = 0$. The equivalences of (i) now follow.

(ii) Let A and B be ideals of T with $A \subseteq B$, satisfying $P_\mu B + BP_\lambda \subseteq A$. Then, for any $j \neq \mu$ and for any k, $(P_\mu)_{j,j}B_{j,k} = R_j B_{j,k} = B_{j,k}$. So, $B_{j,k} = A_{j,k}$ for all $j \neq \mu$ and similarly $B_{j,k} = A_{j,k}$ for all $k \neq \lambda$. Thus, if $M_{\mu,\lambda} = 0$, $B/A = 0$ and hence, by Theorem 1.3.7, $P_\mu \notin \mathrm{Fund}(P_\lambda)$. This establishes the "$\Rightarrow$" implication.

On the other hand, if $M_{\mu,\lambda} \neq 0$, then, where "$*$" represents arbitrary entries chosen from the $M_{i,j}$,

$$\left(\begin{array}{cccccccc}
R_1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & R_{\lambda-1} & & & & 0 & \\
 & & & 0 & & & & \\
 & & & \vdots & 0 & & & \\
 & & & \vdots & \vdots & \ddots & & \\
 & & & 0 & 0 & \cdots & 0 & \\
 & & & M_{\mu,\lambda} & 0 & \cdots & \cdots & 0 \\
 & & & & & & & R_{\mu+1} \\
 & * & & & & & & \ddots \\
 & & & & & & & & R_n
\end{array}\right) \Bigg/ \left(\begin{array}{cccccccc}
R_1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & R_{\lambda-1} & & & & 0 & \\
 & & & 0 & & & & \\
 & & & \vdots & 0 & & & \\
 & & & \vdots & \vdots & \ddots & & \\
 & & & 0 & 0 & \cdots & 0 & \\
 & & & 0 & 0 & \cdots & \cdots & 0 \\
 & & & & & & & R_{\mu+1} \\
 & * & & & & & & \ddots \\
 & & & & & & & & R_n
\end{array}\right)$$

is a non-zero $(T/P_\mu, T/P_\lambda)$-bimodule and so, by Lemma 1.2.6 and Theorem 1.3.7, $P_\mu \in \mathrm{Fund}(P_\lambda)$, establishing the "$\Leftarrow$" implication.

(iii) For $t \geq 1$, $1 \leq k \leq n$ and for a fixed prime, P, of T, put

$$ {}^tR_j = \begin{cases} 0 & \text{if } P_j \in X_t(P) \\ R_j & \text{otherwise.} \end{cases} $$

Then, in the notation of (1.3.10), where we consider $P = P_\lambda$ and $Q = P_\mu$, it is easy to see that $S_t = T_n({}^tR_1, \ldots, {}^tR_n; M_{i,j})$. We next show that

$$ S_{t+1}S_t = JS_t + \begin{pmatrix} {}^tR_1 \cap {}^{t+1}R_1 & & 0 \\ & \ddots & \\ 0 & & {}^tR_n \cap {}^{t+1}R_n \end{pmatrix} \tag{A} $$

for $t \geq 1$.

It is clear that the right side is contained in the left side, and so we consider the other inclusion. For this, it is enough to consider the i^{th} row,

$$ (M_{i,1}, \ldots, M_{i,i-1}, {}^{t+1}R_i, 0 \ldots, 0) \ , $$

of S_{t+1} and the j^{th} column,

$$ \text{col}\,(0, \ldots, 0, {}^tR_j, M_{j+1,j}, \ldots, M_{n,j}) \ , $$

of S_t, where $j \leq i$, for those i such that ${}^{t+1}R_i = R_i$, that is, such that $P_i \notin X_{t+1}$. If $j = i$, then

$$ (S_{t+1}S_t)_{i,j} = {}^{t+1}R_i{}^tR_i \subseteq {}^tR_i \cap {}^{t+1}R_i $$

and so we consider $j < i$. If ${}^tR_j = R_j$, then

$$ (JS_t)_{i,j} \supseteq M_{i,j}R_j = M_{i,j} \ . $$

On the other hand, if ${}^tR_j = 0$, then $P_j \in X_t$ and, since $P_i \notin X_{t+1}$, we see that $P_i \not\rightsquigarrow P_j$; so, by (i),

$$ (JS_t)_{i,j} \supseteq \sum_{k=j+1}^{i-1} M_{i,k}M_{k,j} = M_{i,j} \ . $$

In either case,

$$ (JS_t)_{i,j} \supseteq M_{i,j} \supseteq (S_{t+1}S_t)_{i,j} \ , $$

establishing (A).

Now, $S_{t+1}J \subseteq (S_{t+1}S_t) \cap J$, so, by (A), we have

$$S_{t+1}J \subseteq JS_t \qquad (B)$$

for $t \geq 1$.

Obviously, $(S_1)_{k,\lambda} = J_{k,\lambda}$ for all $k \neq \lambda$, while $(S_1)_{\lambda,\lambda} = J_{\lambda,\lambda} = 0$ since $S_1 = P_\lambda$. Suppose $(S_m \dots S_1)_{k,\lambda} = (J^m)_{k,\lambda}$ for all $1 \leq k \leq n$ which we know is true for $m = 1$. Then

$$\begin{aligned}(S_{m+1}S_m \dots S_1)_{k,\lambda} &= (S_{m+1}J^m)_{k,\lambda} \\ &\subseteq (J^m S_1)_{k,\lambda} \quad \text{by } (B) \\ &= (J^{m+1})_{k,\lambda} \quad \text{since } (S_1)_{\mu,\lambda} = J_{\mu,\lambda} \text{ for all } \mu\ .\end{aligned}$$

Thus, by induction, for all $m \geq 1$ and for all $1 \leq k \leq n$,

$$(S_m \dots S_1)_{k,\lambda} = (J^m)_{k,\lambda} \qquad (C)$$

the reverse inclusion being obvious.

Now consider the bimodule

$$B_m(P_\mu, P_\lambda) = {}_{T/P_\mu}\left|\frac{S_{m+1}\dots S_2 \ \cap \ S_m \dots S_1}{S_{m+1}\dots S_1}\right|_{T/P_\lambda}.$$

By the same reasoning as in part (ii),

$$B_m(P_\mu, P_\lambda) \neq 0 \iff (S_{m+1}\dots S_2)_{\mu,\lambda} \ \cap \ (S_m \dots S_1)_{\mu,\lambda} \supsetneqq (S_{m+1}\dots S_1)_{\mu,\lambda}\ .$$

Now,

$$(J^m)_{\mu,\lambda} \subseteq (S_{m+1}\dots S_2)_{\mu,\lambda} \cap (S_m \dots S_1)_{\mu,\lambda} \subseteq (S_m \dots S_1)_{\mu,\lambda} = (J^m)_{\mu,\lambda}$$

by (C), and so

$$B_m(P_\mu, P_\lambda) \neq 0 \iff (J^m)_{\mu,\lambda} \supsetneqq (J^{m+1})_{\mu,\lambda}.$$

Finally, (iii) follows by Theorem 1.3.11. ∎

The above lemma allows us to test whether a given g.t.a. matrix ring over division rings is strongly rep. rep.. We know, by Lemma 2.1.6, that all hereditary g.t.a. matrix rings, which are characterised by Theorem 3.1.4, must be strongly rep. rep.. Consider, however, the following example:

Example 3.3.2. The ring

$$T = \begin{pmatrix} k & 0 & 0 \\ k \oplus k & k & 0 \\ \frac{k \oplus k \oplus k \oplus k}{k \oplus k \oplus 0 \oplus 0} & k \oplus k & k \end{pmatrix},$$

where the entries multiply via tensoring, has link graph

$$P_3 \rightsquigarrow P_2 \rightsquigarrow P_1$$

and is strongly rep. rep.. To see this, we must show that $P_3 \in A_3(P_1)$. We denote the Jacobson radical of T by J and note that

$$J = \begin{pmatrix} 0 & 0 & 0 \\ k \oplus k & 0 & 0 \\ \frac{k \oplus k \oplus k \oplus k}{k \oplus k \oplus 0 \oplus 0} & k \oplus k & 0 \end{pmatrix} \text{ so } J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{k \oplus k \oplus k \oplus k}{k \oplus k \oplus 0 \oplus 0} & 0 & 0 \end{pmatrix} \text{ and } J^3 = 0.$$

Thus, $(J^3)_{3,1} = 0 \subsetneq \frac{k \oplus k \oplus k \oplus k}{k \oplus k \oplus 0 \oplus 0} = (J^2)_{3,1}$ and it follows from Lemma 3.3.1(iii) that $P_3 \in A_3(P_1)$ whence T is strongly rep. rep.. (That $P_3 \in A_2(P_2)$ and $P_2 \in A_2(P_1)$ hold automatically by Remark 1.3.6.)

On the other hand, T is not hereditary. (Theorem 3.1.4 shows that, if T were hereditary, then all $\phi^j_{i,k}$ would be monomorphisms; here, $\phi^2_{3,1}$ is not.) ∎

Remark 3.3.3. Suppose T is a strongly rep. rep. g.t.a. matrix ring over division rings (for instance, T hereditary) with Jacobson radical J and let $I \subseteq J^2$ for some ideal I of T. Since, by Lemma 2.3.7, T/I has the same link graph as T, and since the linked primes of T coincide with its fundamental primes, we can apply Lemma 3.3.1(iii) to the ring T to determine the link graph of T/I. Of course, T/I is itself a g.t.a. matrix ring over division rings and so we can also apply Lemma 3.3.1(iii) to the ring T/I to determine *its* fundamental primes. By these considerations, we see that T/I is a strongly rep. rep. ring if and only if, whenever $i, j \in \{1, \ldots, n\}$, $m \in \mathbb{N}$ and $(J^m)_{j,i} \neq (J^{m+1})_{j,i}$, then $(J^m)_{j,i} \supsetneq (J^{m+1} + I)_{j,i}$. ∎

Of course, not every factor of a hereditary g.t.a. matrix ring over division rings is strongly rep. rep. or even rep. rep.. For instance, consider:

Example 3.3.4. The ring

$$T = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$$

has link graph

$$P_3 \rightsquigarrow P_2 \rightsquigarrow P_1$$

and $P_3 \in A_3(P_1)$ via the bimodule

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}.$$

Since $I \subseteq J^2$, where J is the radical of T, we see (for the same reason as in Remark 3.3.3) that $P_3/I \in X_3(P_1/I)$. Again, since T/I is itself a g.t.a. matrix ring we can apply Lemma 3.3.1(ii) to show that $P_3 \notin \mathrm{Fund}(P_1)$. Thus, T/I is not even rep. rep.. ∎

While not every factor of a hereditary g.t.a. matrix ring need be rep. rep., Lemma 3.3.5 shows that, up to Morita equivalence, the converse *is* true for finite dimensional algebras over algebraically closed fields. However, to prove this, we require the terminology of §3.2. Furthermore, for the rest of this chapter, we will denote the Jacobson radical of a ring S by $J(S)$.

Lemma 3.3.5. *Let k be an algebraically closed field and A a finite dimensional k-algebra. If A is strongly rep. rep., then A is Morita equivalent to H/I, where H is a basic hereditary finite dimensional k-algebra and $I \subseteq (J(H))^2$. In particular, we can take $H = H_{B(A)}$ with the notation of (3.2.3).*

Proof. First, as noted in (3.1.2), A is Morita equivalent to its basic algebra $B(A)$ so, by Theorem 2.3.3 we can assume that A is basic. Then, by Lemma 3.2.4, and in the notation of (3.2.3) we write $A = H_A/I$ for some ideal I of H_A where $I \subseteq (J(H_A))^2$. Since Δ_A^* (the opposite quiver of the Ext-quiver Δ_A) is finite, H_A

is a hereditary k-algebra by Lemma 3.2.2 and, as noted in Definition 3.2.1, H_A is finite dimensional if and only if there are no cyclic paths in Δ_A^*.

Let P and Q be prime ideals of A. By construction (3.2.3), in the Ext-quiver Δ_A of A there is an arrow

$$A/Q \longrightarrow A/P \iff \operatorname{Ext}^1(A/Q, A/P) \neq 0$$

and so, by [J2, Lemma 6.1.6],

$$A/Q \longrightarrow A/P \iff Q \rightsquigarrow P .$$

Now, for any Artinian ring, the fundamental series of an indecomposable injective is just its socle series and, since modules over Artinian rings have finite Loewy length [G&W, Propositions 3.14 and 3.15], there can be no infinite chain of links in a strongly rep. rep. Artinian ring and therefore no cyclic paths in Δ_A. Clearly, then, there are no cyclic paths in Δ_A^* either and so H_A is finite dimensional.

The result follows. ∎

We can now present the main result of this chapter which characterizes the strongly rep. rep. finite dimensional algebras over algebraically closed fields.

Theorem 3.3.6. *Let k be an algebraically closed field and let A be a finite dimensional k-algebra. With the notation of (3.1.3) and (3.1.7), the following conditions are equivalent.*

(i) A is strongly rep. rep..

(ii) There exists T a hereditary g.t.a. matrix ring over division rings with radical J and an ideal $I \subseteq J^2$ such that A is Morita equivalent to T/I and such that $(J^m)_{j,i} \nsubseteq (J^{m+1} + I)_{j,i}$ whenever i, j and $m \in \mathbb{N}$ and $(J^m)_{j,i} \neq (J^{m+1})_{j,i}$.

Suppose that these equivalent conditions hold. Then, in (ii), and with the notation of (3.2.3), we can take $T \cong H_{B(A)}$ and in this case $T/I \cong B(A)$, the basic algebra of A. Further, if $\widehat{T}$ is any hereditary g.t.a. matrix ring over division rings with radical $\widehat{J}$ and an ideal $\widehat{I} \subseteq \widehat{J}^2$ such that A is Morita equivalent to $\widehat{T}/\widehat{I}$, then $(\widehat{J}^m)_{j,i} \nsubseteq (\widehat{J}^{m+1} + \widehat{I})_{j,i}$ whenever i, j and $m \in \mathbb{N}$ and $(\widehat{J}^m)_{j,i} \neq (\widehat{J}^{m+1})_{j,i}$.

Proof. Assume (i) holds. By Lemma 3.3.5 A is Morita equivalent to $H_{B(A)}/I$ for some ideal $I \subseteq (J(H_{B(A)}))^2$, while $H_{B(A)}$ is itself a basic hereditary finite dimensional k-algebra. In this case, we can apply Theorem 3.1.6 to see that $H_{B(A)}$ is isomorphic to a g.t.a. matrix ring over division rings. Being hereditary, $H_{B(A)}$ is strongly rep. rep. by Lemma 2.1.6 while $H_{B(A)}/I$ is strongly rep. rep. by Theorem 2.3.3. Statement (ii) follows by Remark 3.3.3.

Conversely, assume (ii) holds. Since T is strongly rep. rep. by Lemma 2.1.6, Remark 3.3.3 ensures that T/I is strongly rep. rep. and then (i) follows by Theorem 2.3.3.

In the case where the conditions hold, we have already seen that we can take $T \cong H_{B(A)}$ and that $H_{B(A)}$ and hence $H_{B(A)}/I$ are basic. Finally, assuming A is strongly rep. rep. and is Morita equivalent to $\widehat{T}/\widehat{I}$ as described, then $\widehat{T}$ and $\widehat{T}/\widehat{I}$ are strongly rep. rep. by Lemma 2.1.6 and Theorem 2.3.3 respectively. The conclusion follows by Remark 3.3.3. ∎

Suppose we are given a finite dimensional algebra A over an algebraically closed field. To decide whether A is strongly rep. rep., we would first use the construction in (3.2.1) and (3.2.3) to describe the path algebra $H_{B(A)}$ of the quiver Δ_A^*, the opposite quiver of the Ext-quiver $\Delta_{B(A)}$ of the basic algebra $B(A)$ of A. By the proof of Lemma 3.3.5, it is necessary for $H_{B(A)}$ to be finite dimensional as a k-algebra if A is to be strongly rep. rep.. If this is the case, we would then find I, the kernel of the surjection from $H_{B(A)}$ to $B(A)$ which is defined in [Ga3, §4.3] and also in [Bn, Proposition 4.1.7]. Finally, we would check the condition in Theorem 3.3.6, by using Theorem 3.1.6 which, for a basic Artinian hereditary ring, describes the $M_{i,j}$ in terms of idempotents.

Remark 3.3.7. In particular, returning to Example 3.3.2, for the ring

$$A = \begin{pmatrix} k & 0 & 0 \\ k \oplus k & k & 0 \\ \frac{k\oplus k\oplus k\oplus k}{k\oplus k\oplus 0\oplus 0} & k \oplus k & k \end{pmatrix},$$

which Lemma 3.3.1 showed us was strongly rep. rep., we can take

$$T = \begin{pmatrix} k & 0 & 0 \\ k \oplus k & k & 0 \\ k \oplus k \oplus k \oplus k & k \oplus k & k \end{pmatrix} \text{ and } I = \begin{pmatrix} 0 & 0 & 0 \\ 0 \oplus 0 & 0 & 0 \\ k \oplus k \oplus 0 \oplus 0 & 0 \oplus 0 & 0 \end{pmatrix}$$

in Theorem 3.3.6, to see again that A is strongly rep. rep.. ∎

§3.4 Notes

The theory of idempotents and basic algebras discussed in (3.1.1) and (3.1.2) is well known and can be found for instance in [A&F]. The remainder of the terminology and results of §3.1 are taken from [Ha]. In particular, Theorem 3.1.4 is [Ha, Theorem 1] and Theorem 3.1.6 is [Ha, Theorem 4'].

The description of quivers and the terminology introduced in §3.2 is taken primarily from [Rg, §2]. The term "Ext-quiver", however, is taken from [Bn, Definition 4.1.6]. In fact, [Bn], which deals with left modules, defines multiplication in path algebras to be composition of paths in reverse order and, consequently, our definition of the opposite Ext-quiver is just the Ext-quiver of [Bn]. Our terminology corresponds to that of [Rg].

Lemma 3.2.2 is taken from [Bm, Corollary 2.6] and is well known. It can also be found in [Bn, Theorem 4.1.4] with a proof for the case where Q is a finite quiver and is discussed in [Rg, §2.4].

Lemma 3.2.4 is due to Gabriel [Ga3, §4.3]. Again, this is a well known result which is proved in [Bn, Proposition 4.1.7] and stated in [Rg, Theorem 2.1.2]. Generalizations of this result to fields which are not algebraically closed are discussed in [Bn, Proposition 4.1.10 and Corollary 4.1.11].

All of the results of §3.3 are new, except that Lemma 3.3.1(i) and (ii) are special cases of [J2, Proposition 5.3.13] which describes the second layer links and ideal links for a general Artinian ring.

We saw in the proof of Lemma 3.3.5 that if A is a strongly rep. rep. finite dimensional algebra over an algebraically closed field, then the Ext-quiver of A contains no cyclic paths. This improves [Bn, Lemma 4.2.3] which proves the same result for a hereditary finite dimensional algebra. (Of course, by Lemma 2.1.6, every hereditary Noetherian ring satisfying the second layer condition is strongly rep. rep..)

In fact, suppose R is any ring with an ideal I which is finitely generated as a left and as a right ideal and suppose that $I \subseteq (J(R))^2$. If R/I is hereditary, then $I = 0$ (see [Bn, Lemma 4.2.1]). It follows that if A is a finite dimensional hereditary basic algebra over an algebraically closed field, $A \cong H_A$, the path algebra of its opposite Ext-quiver (see [Bn, Proposition 4.2.4]). It is remarked after [Bn, Proposition 4.2.5], that this is not true for arbitrary finite dimensional algebras and so the assumption of algebraic closure would seem to be necessary in our analysis in §3.2.

Chapter 4 : Differential Operator Rings

Definitions and Notation 4.0.0. Let R be any ring and σ an endomorphism of R. By a *σ-derivation* of R is meant an additive map $\delta : R \to R$ such that

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$$

for all r and $s \in R$. In the case where σ is the identity map, δ is simply called a *derivation* of R.

Given a ring R, an endomorphism σ and a σ-derivation δ, we can form the free left R-module S with basis $\{1, \theta, \theta^2, \ldots\}$. We define a multiplication in S by setting $\theta^i\theta^j = \theta^{i+j}$ and

$$\theta r = \sigma(r)\theta + \delta(r)$$

for each $r \in R$. By [G&W, Proposition 1.10], this multiplication can be extended to the whole of S by imposing associative and distributive laws. Thus, the construction turns S into a ring which is denoted $S = R[\theta; \sigma, \delta]$ and is termed a *skew polynomial ring* over R. If σ is the identity, S is called a *(formal) differential operator ring* over R and is abbreviated to $R[\theta; \delta]$, while if δ is the zero map, S is called an *Ore extension* of R by σ and is written $R[\theta; \sigma]$. (In fact, the term "Ore extension" is often applied to *any* skew polynomial ring, but we will reserve it for the case where $\delta = 0$.)

Provided σ is an automorphism on R, a version of Hilbert's Basis Theorem can be proved for the skew polynomial ring $S = R[\theta; \sigma, \delta]$ [G&W, Theorem 1.12]. Thus, if R is a right [respectively left] Noetherian ring, then S is a right [respectively left] Noetherian ring.

Naturally, the above construction can be used to produce *iterated skew polynomial rings*; that is, rings of the form

$$R[\theta_1; \sigma_1, \delta_1][\theta_2; \sigma_2, \delta_2] \ldots [\theta_n; \sigma_n, \delta_n],$$

where σ_1 is an endomorphism of R and δ_1 is a σ_1-derivation of R, while σ_2 is an endomorphism of $R[\theta; \sigma_1, \delta_1]$ and δ_2 is a σ_2-derivation of $R[\theta_1; \sigma_1, \delta_1]$ and so on.

Suppose we are given sets

$$\Sigma := \{\sigma_1, \ldots, \sigma_n\}$$

and

$$\Delta := \{\delta_1, \ldots, \delta_n\}$$

where the σ_i are commuting endomorphisms on R and the δ_i are commuting σ_i-derivations of R. We can form an iterated skew polynomial ring

$$S = R[\theta_1; \sigma_1, \delta_1][\theta_2; \varsigma_2, \partial_2] \ldots [\theta_n; \varsigma_n, \partial_n]$$

by defining

$$\varsigma_j \left(\sum r_i \theta_1^{i(1)} \ldots \theta_{j-1}^{i(j-1)}\right) = \sum \sigma_j(r_i) \theta_1^{i(1)} \ldots \theta_{j-1}^{i(j-1)}$$

and

$$\partial_j \left(\sum r_i \theta_1^{i(1)} \ldots \theta_{j-1}^{i(j-1)}\right) = \sum \delta_j(r_i) \theta_1^{i(1)} \ldots \theta_{j-1}^{i(j-1)}.$$

We will denote S so formed by

$$S = R[\theta_1, \ldots, \theta_n; \sigma_1, \ldots, \sigma_n, \delta_1, \ldots, \delta_n] = R[\Theta; \Sigma, \Delta]$$

where $\Theta := \{\theta_1, \ldots, \theta_n\}$. By repeated applications of the above Hilbert Basis Theorem, we see that, when the σ_i are automorphisms and R is right [respectively left] Noetherian, then S is right [respectively left] Noetherian. ∎

In Chapter 5, we will turn our attention to Ore extensions by a single automorphism. However, in this chapter, we restrict our attention to an iterated differential operator ring T of commuting derivations on a commutative Noetherian $\mathbb{Q}$-algebra R. Such a ring is, of course, again Noetherian since the σ_i are trivially automorphisms in this case. Furthermore, the ring T satisfies the s.l.c. [Be2, Theorem 7.3] and an explicit description of the link graph of T is provided by [G2]. (In particular, T satisfies the local finiteness condition.)

In §4.2, we use this description to give in Corollary 4.2.14(i), our main result, a sufficient, though not necessary, condition for a prime P of T to be rep. rep.: specifically, P is rep. rep. in T provided $R_{P \cap R}$ is a regular local ring. In Corollary 4.2.14(iii) we also provide a sufficient condition for strong repleteness. In the case when $P = (P \cap R)T$, Corollary 4.2.14(ii) is more precise: namely that P is always rep. rep. and is strongly rep. rep. if and only if $P \cap R$ is strongly rep. rep. in R. Finally, we note that T is rep. rep. when R is regular (Corollary 4.2.15).

§4.1 Prime Links in Differential Operator Rings and Preliminaries

Notation and Remarks 4.1.1. Suppose R is a commutative Noetherian $\mathbb{Q}$-algebra, let $\{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R and put

$$T = R[\theta_1, \ldots, \theta_n; \delta_1, \ldots, \delta_n] = R[\Theta; \Delta].$$

Let P be a prime ideal of T; so P is *completely prime* (that is, T/P is a domain) by [G&W, Theorem 9.24], a special case of [Sg, Corollary 2.6]. Put $M = P \cap R$ and $\mathcal{C} = R - M$. By [G1, Lemma 7.3], M is a prime ideal of R which is Δ-*invariant* (that is to say, $\Delta(M) \subseteq M$ so that $MT = TM$ is an ideal of T) and, if $Q \in \text{cl.}(P)$, then $Q \cap R = M$. Also, as noted in [G2, 1.2], $\mathcal{C}$ is an Ore set in T and, writing T_M for $T\mathcal{C}^{-1}$, we can identify T_M with $R_M[\theta_1, \ldots, \theta_n; \delta_1, \ldots, \delta_n]$, extending the δ_i to R_M via the quotient rule (see, for instance, [McC&R, Lemma 14.2.7]). By Theorem 2.3.2, it is therefore sufficient to consider the case where R is a local ring with maximal ideal M. (See Definitions 4.1.9(i).) ∎

At this point, we note a result from [G&W, Lemma 2.20] which shows, in particular, that any minimal prime of R is Δ-invariant (for recall that we are assuming that R is a $\mathbb{Q}$-algebra).

Lemma 4.1.2. *Let S be any ring, let δ be a derivation on S and let I be a minimal prime ideal of S such that the characteristic of S/I is zero. Then I is a δ-invariant ideal.* ∎

Notation 4.1.3. Adopt the notation of (4.1.1) and we write $K = R/M$. Then, since M is closed under the action of Δ, we can form $U := K[\Theta; \Delta] \cong T/MT$ and we identify these rings where convenient (see [G2, Lemma 3.1]). Of course, since U is a domain, MT is a prime of T. We denote

$$W := \left\{ a_1\theta_1 + \cdots + a_n\theta_n : a_1, \ldots, a_n \in K \text{ and } \sum_{i=1}^{n} a_i\delta_i = 0 \text{ on } K \right\} \subseteq U,$$

so that W is a K-subspace of $\sum_{i=1}^{n} K\theta_i$, and let U_0 be the subring of U generated by K and W. (Indeed, by [P2, Theorem 2.2], U_0 is the centralizer of K in U.) By [G2, Lemma 2.3], U_0 is a commutative domain and an affine K-algebra which is closed under the operation of Δ. After renumbering, if necessary, we may assume that the subset $\{\theta_1, \ldots, \theta_l\}$, of Θ (for some l) is a basis for the K-vector space $\sum_{i=1}^{n} K\theta_i/W$. Now let ϕ be the natural projection from $\sum_{i=1}^{n} K\theta_i = W \oplus \sum_{i=1}^{l} K\theta_i$ onto W and let $w_i = \phi(\theta_i)$ for $l+1 \leq i \leq n$. Then $\{w_{l+1}, \ldots, w_n\}$ forms a basis for W. Furthermore, by [G2, Proposition 2.4], $U = U_0[\theta_1, \ldots, \theta_l; \delta_1, \ldots, \delta_l]$ and $U_0 = K[w_{l+1}, \ldots, w_n]$, the ordinary polynomial ring over K in the indeterminates $w_{l+1}, \ldots, w_n$.

If K' is an isomorphic copy of K, we let W' be the K'-subspace of $\sum_{i=1}^{n} K'\theta_i$ corresponding to W and $w'_{l+1}, \ldots, w'_n$ be the images of $w_{l+1}, \ldots, w_n$.

We note that $\frac{MT}{M^2T}$ is a (U, U)-bimodule and that $\frac{M+M^2T}{M^2T}U_0$ is a right U_0-submodule of $\frac{MT}{M^2T}$. We write $P' = P/MT$ (so P' is a prime ideal of U) and $P_0 = P' \cap U_0$ (so P_0 is a prime Δ-invariant ideal of U_0). ∎

The following result is proved in [G2, Theorem 2.9] and is a special case of the "Passman Correspondence" [P2, Theorem 4.3].

Theorem 4.1.4. *Adopt the notation of (4.1.1) and (4.1.3). Then, contraction ($Q' \mapsto Q' \cap U_0$) and extension ($Q_0 \mapsto Q_0 U$) are inverse bijections between the prime ideals Q' of U and the prime Δ-invariant ideals Q_0 of U_0. In particular, $P_0 U = P'$.* ∎

To state Goodearl's results, we require the following terminology of [G2, §5].

Notation 4.1.5. We maintain the notation of (4.1.1) and (4.1.3).

Let $K^{\#}$ be an algebraic closure of K, let $U_0^{\#} = U_0 \otimes_K K^{\#}$ and identify $U_0^{\#}$ with $K^{\#}[w_{l+1}, \ldots, w_n]$ so that U_0 is a K-subalgebra of $U_0^{\#}$ and $U_0^{\#}$ is integral over U_0. Let $(\frac{M}{M^2})^{\#} = \frac{M}{M^2} \otimes_K K^{\#}$ and identify $U_0^{\#} \otimes_K \frac{M}{M^2}$ and $\frac{M}{M^2} \otimes_K U_0^{\#}$ with $K^{\#} \otimes_K \frac{M+M^2T}{M^2T} U_0$ so that $U_0^{\#} \otimes_K \frac{M}{M^2}$ is a $(U_0^{\#}, U_0^{\#})$-bimodule, containing a natural copy of $(\frac{M}{M^2})^{\#}$. Then a map $\varepsilon \in \mathrm{Hom}_K(W, K^{\#})$ is an *eigenvalue* for W on $(\frac{M}{M^2})^{\#}$ if and only if there exists a non-zero element μ of $(\frac{M}{M^2})^{\#}$ with $w\mu - \mu w = \varepsilon(w)\mu$

in $U_0^{\#} \otimes_K \frac{M}{M^2}$ for all $w \in W$. Any such μ is called an *eigenvector* for W on $(\frac{M}{M^2})^{\#}$ with *eigenvalue* ε. Then, τ_ε is the unique $K^{\#}$-algebra automorphism of $U_0^{\#}$ such that $\tau_\varepsilon(w_i) = w_i + \varepsilon(w_i)$ for $i = l+1, \ldots, n$ and is called a *winding automorphism* of $U_0^{\#}$. It is easy to see that $t\mu = \mu\tau_\varepsilon(t)$ in $U_0^{\#} \otimes_K \frac{M}{M^2}$ for all $t \in U_0^{\#}$ [G2, Lemma 5.5]. ∎

Before stating the main result of [G2], we quote a result extracted from [Z&S, Chapter VII, §11, Corollary p226], which, since every extension field of a field of characteristic zero is separable [McC, Chapter 1, Theorem 7], shows in particular that, with notation as in (4.1.1), (4.1.3) and (4.1.5), the ideal $P_0U_0^{\#}$ of $U_0^{\#}$ is semiprime.

Lemma 4.1.6. *Let F be a field and L a separable extension field of F. Let I be a prime ideal of the polynomial ring $F[X_1, \ldots, X_s]$. Then, the ideal $IL[X_1, \ldots, X_s]$ of the polynomial ring $L[X_1, \ldots, X_s]$ is semiprime.* ∎

Theorem 4.1.7. *Let R be a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M, fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Suppose P and Q are distinct prime ideals of $T = R[\Theta; \Delta]$ such that $M = P \cap R = Q \cap R$. Let U_0, W, K, $K^{\#}$, $U_0^{\#}$ and $(M/M^2)^{\#}$ be defined as in (4.1.3) and (4.1.5) and set $P_0 = (P/MT) \cap U_0$ and $Q_0 = (Q/MT) \cap U_0$.*

(i) Let $P_0^{\#}$ be a prime of $U_0^{\#}$ lying over P_0. Then $Q \rightsquigarrow P$ if and only if there exists an eigenvalue ε for W on $(M/M^2)^{\#}$ such that the prime $\tau_\varepsilon^{-1}(P_0^{\#})$ lies over Q_0.

(ii) Suppose that all eigenvalues for W on $(M/M^2)^{\#}$ map W into K. Then $Q \rightsquigarrow P$ if and only if there exists an eigenvalue ε for W on $(M/M^2)^{\#}$ such that $Q_0 = \tau_\varepsilon^{-1}(P_0)$.

Proof. By [G2, Theorem 1.2], we can assume that $M^2 = 0$ and then (i) follows from [G2, Theorem 5.8] and (ii) from [G2, Theorem 5.11(a)]. ∎

To complete the description of the links of T, we state [G2, Theorem 6.1].

Theorem 4.1.8. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Put $T = R[\Theta; \Delta]$.*

(i) Every non-minimal prime of T is linked to itself.

(ii) The minimal primes of T are the ideals MT where M is a minimal prime of R. A minimal prime MT is linked to itself if and only if $MR_M \neq 0$. ∎

We conclude this section by listing some preliminary results we will require from commutative ring theory. First we recall the concept of complete local rings.

Definitions 4.1.9. (i) Let A be a commutative Noetherian ring. Then A is a *semilocal* ring provided it has only finitely many maximal ideals, while A is a *local* ring provided it has a unique maximal ideal. Assume A is a local ring and let J be the maximal ideal. Let b be an element of A and let $b_1, b_2, b_3, \ldots$ be an infinite sequence of elements of A. We say that *the sequence (b_n) converges to b* and that *b_n tends to b as n tends to infinity* provided, whenever $s \in \mathbb{N}$, we can find some $n_0 = n_0(s) \in \mathbb{N}$ such that, $b - b_n \in J^s$ for all $n > n_0$. In this case, we write "$b_n \to b$ as $n \to \infty$". Furthermore, we say that a sequence (a_n) of elements of A is a *Cauchy sequence* in A if, given any $s \in \mathbb{N}$, we can find some $n_0 = n_0(s) \in \mathbb{N}$ such that $a_n - a_m \in J^s$ whenever $n > m > n_0$. It turns out (see [N1, §5.2 Lemma 1]) that (a_n) is a Cauchy sequence if and only if $a_n - a_{n-1} \to 0$ as $n \to \infty$.

(ii) If $a_n \to a$ in a commutative Noetherian local ring A, then it is easily seen that $a_n - a_{n-1} \to a - a = 0$. Thus, if a sequence has a limit in A, it must be a Cauchy sequence. Following the ideas of ordinary analysis, we define R to be a *complete* local ring if every Cauchy sequence in A has a limit in A.

(iii) Let A and A' be commutative Noetherian local rings with $A \subseteq A'$. If a sequence of elements of A is a Cauchy sequence in A when and only when it is a Cauchy sequence in A', we say that A' is a *concordant extension* of A. If A' is a concordant extension which is complete and whose every element is the limit of a sequence of elements of A, then A' is said to be a *completion of A (at J)*. ∎

The next theorem is well known and can be found, for instance, in [N1, §5.5 Theorem 4].

Theorem 4.1.10. *Let A be commutative Noetherian local ring. Then A has a completion $\overline{A}$. Moreover, if A' is another completion of A, then the identity map on A extends to an isomorphism from $\overline{A}$ to A'.* ∎

In view of this theorem, we can talk of *the* completion of a local ring A which is usually denoted by $\overline{A}$. It will be convenient for us to have an explicit description of the completion, and this is provided by [Na, Corollary 17.6] (a result sometimes known as Cohen's Theorem) as follows.

Theorem 4.1.11. *Let A be a commutative Noetherian local ring with maximal ideal J. Say $J = \sum_{i=1}^{r} j_i A$, for some $j_i \in J$. Denote the completion of A at J by $\overline{A}$. Then*

$$\overline{A} \cong \frac{A[[x_1, \ldots, x_r]]}{\sum_{i=1}^{r}(x_i - j_i)A[[x_1, \ldots, x_r]]}$$

where the x_i are indeterminates. ∎

We now note some important properties of the completion. The following theorem can be found in [N1, §5.5 Proposition 9 and Theorem 6].

Theorem 4.1.12. *Let A be a commutative Noetherian local ring with maximal ideal J and let $\overline{A}$ be its completion.*

(i) The maximal ideal $\overline{J}$ of $\overline{A}$ is given by $\overline{J} = J\overline{A}$ and, for every $s \in \mathbb{N}$, $\overline{J}^s \cap A = J^s$. Furthermore, if I is any ideal of A, then $I\overline{A} \cap A = I$.

(ii) Let I be a proper ideal of A. The natural map from $\overline{A}$ to $\overline{A}/I\overline{A}$ induces a homomorphism from A to $\overline{A}/I\overline{A}$ with kernel $I\overline{A} \cap A (= I$ by part (i)). If we identify A/I with its image in $\overline{A}/I\overline{A}$ then $\overline{A}/I\overline{A}$ is the completion of A/I. ∎

The following corollary is another well known result.

Corollary 4.1.13. *Let A be a commutative Noetherian local ring with maximal ideal J and let $\overline{A}$ be its completion. Then the map*

$$a + J \mapsto a + J\overline{A}$$

is an isomorphism from the ring A/J to the ring $\overline{A}/J\overline{A}$.

Proof. By Theorem 4.1.12(ii), $\overline{A}/J\overline{A}$ is a complete local ring and the map given in the statement of the Corollary is an embedding of A/J into it. On the other hand, since A/J is a field, it is certainly complete and the result follows by Theorem 4.1.10. ∎

The next two results we require are taken from [Z&S, Chapter VIII, §6, Theorem 15(c) and §12, Theorem 27] respectively, although both Theorems 4.1.16 and 4.1.19 are originally due to Cohen [C].

Theorem 4.1.14. *Let A and B be commutative Noetherian local rings and suppose that B is finitely generated as a module over A. If A is a complete local ring then so is B.* ∎

Definition 4.1.15. Let A be a local ring with maximal ideal J. Then A is *equicharacteristic*, provided A and A/J have the same characteristic; in this case, the characteristic is clearly zero or a prime.

If the characteristic of A is a prime p, then A contains the field of p elements. On the other hand, if the characteristics of both A and A/J are zero, then J contains no non-zero integer and so every non-zero integer is a unit in A; in this case, A contains $\mathbb{Q}$.

Since it is obvious that a local ring containing a field is equicharacteristic, we see that a local ring is equicharacteristic if and only if it contains a field. ∎

We now note that an equicharacteristic complete local ring contains a copy of the field A/J. This result, a consequence of Theorem 4.1.11, is also frequently referred to as "Cohen's Theorem".

Theorem 4.1.16. *Let A be a complete commutative Noetherian local ring which is equicharacteristic. Then, A contains a subfield L isomorphic to A/J, with an isomorphism being given by the restriction to L of the canonical epimorphism from A to A/J.* ∎

For our final preliminaries, we require some more terminology, that of *regular* commutative rings.

Definitions 4.1.17. Let A be a commutative Noetherian local ring with maximal ideal J and classical Krull dimension d. (The classical Krull dimension of a commutative Noetherian local ring is finite by [A&McD, Corollary 11.11].) Then A is said to be *regular* provided J can be generated by d elements (or, equivalently, provided the dimension of J/J^2 as a vector space over A/J is d).

Furthermore, if B is an arbitrary commutative Noetherian ring, then B is said to be *regular* provided the localization B_K is a regular local ring for each prime ideal K of B. ■

It is primarily for the following property that regular local rings are of interest to us. This result is taken from [N1, §4.6, Lemma 3].

Lemma 4.1.18. *Let A be a regular commutative Noetherian local ring with maximal ideal J. Suppose that $\alpha \in J^h - J^{h+1}$ and $\beta \in J^k - J^{k+1}$ for some non-negative integers h and k. Then, $\alpha\beta \in J^{h+k} - J^{h+k+1}$.* ■

Since, in the situation of Lemma 4.1.18, the intersection of the powers of J is zero (that is, $\bigcap_{i=1}^{\infty} J^i = 0$, a special case of the Krull Intersection Theorem [Na, Theorem 3.11]), it is an immediate consequence that a regular commutative Noetherian local ring is an integral domain.

We complete our introductory discussion with the characterization of complete regular local rings, known as "Cohen's Structure Theorem" (part (ii) of Theorem 4.1.19). This may be found in [Z&S, Chapter VIII, §12, Corollary to Theorem 27] and its proof. Part (i) may be found in [N1, §5.6, Corollary to Theorem 8].

Theorem 4.1.19. *Let A be a commutative Noetherian local ring with maximal ideal J.*

(i) Let $\overline{A}$ be the completion of A at J. Then $\overline{A}$ is regular if and only if A is regular.

(ii) Suppose that A is equicharacteristic, complete and regular and let d be the classical Krull dimension of A. Then A is isomorphic to $\frac{A}{J}[[X_1, \ldots, X_d]]$, the ring of formal power series in d indeterminates over the field A/J. ■

§4.2 The Repleteness of Differential Operator Rings

To apply Goodearl's results in our context, we need to manoeuvre into a setting where the hypothesis of Theorem 4.1.7(ii) holds. For this, it is sufficient that all the eigenvectors for W on $(M/M^2)^{\#}$ can be chosen from M/M^2 and in fact we will need this stronger condition in Theorem 4.2.12. To this end, we first show that we can assume that the commutative Noetherian $\mathbb{Q}$-algebra R is a complete local ring. By Theorem 4.1.16, R then contains a copy K' of $K = R/M$ (an isomorphism being given by the restriction to K' of the canonical epimorphism from R to R/M) and we show that, for sufficient conditions for repleteness, we can replace T with $T \otimes_{R/M} \widehat{R/M}$ for any finite algebraic extension, $\widehat{R/M}$ of R/M.

Remark 4.2.1. Let R be a commutative Noetherian local ring with maximal ideal M. Say $M = \sum_{i=1}^{r} a_i R$, for some $a_i \in M$. Denote the completion of R at M by $\overline{R}$. Then, by Theorem 4.1.11, there is an isomorphism

$$\psi \;:\; \overline{R} \cong \frac{R[\,[x_1, \ldots, x_r]\,]}{\sum_{i=1}^{r}(x_i - a_i)R[\,[x_1, \ldots, x_r]\,]}$$

where $x_1, \ldots, x_r$ are indeterminates.

If we identify R with the subring $\{s + \sum_{i=1}^{r}(x_i - a_i)R[\,[x]\,] : s \in R\}$, of $\psi(\overline{R})$, we can then identify $\psi(\overline{R})$ with the completion $\overline{R}$ of R. ■

The next result is well known. (See, for instance, [G,L&R, p16].)

Lemma 4.2.2. *Let R be a commutative Noetherian local ring with maximal ideal M and let Δ be a set of commuting derivations on R. Then Δ extends uniquely to a set $\overline{\Delta}$ of commuting derivations on the completion $\overline{R}$ of R at M.*

Proof. We adopt the description of $\overline{R}$ given above. For each $\delta \in \Delta$, define $\delta'(x_i) = \delta(a_i)$ and $\delta'(s) = \delta(s)$ for each $1 \le i \le r$ and $s \in R$. This extends Δ to Δ', commuting derivations on $R[\,[x_1, \ldots, x_r]\,]$. Then, since

$$\sum_{i=1}^{r}(x_i - a_i)R[\,[x_1, \ldots, x_r]\,]$$

is Δ'-invariant, Δ' induces a set $\overline{\Delta}$ of commuting derivations on $\overline{R}$. Now, for each $s \in R$ and $\delta \in \Delta$ we have

$$\overline{\delta}\left(s + \left(\sum_{i=1}^{r}(x_i - a_i)R[[x]]\right)\right) = \delta'(s) = \delta(s)$$

so $\overline{\Delta}$ extends Δ to $\overline{R}$ and this extension is clearly unique. ∎

Notation 4.2.3. In the situation of Lemma 4.2.2 and given the differential operator ring $T = R[\Theta; \Delta]$, we can extend, by Lemma 4.2.2, the set Δ of commuting derivations on R to a set $\overline{\Delta}$ of commuting derivations on the completion $\overline{R}$ of R at M. We form the differential operator ring $\overline{T} = \overline{R}[\Theta; \overline{\Delta}]$ and we consider T to be a subring of $\overline{T}$. Suppose that P is a prime ideal of T such that $P \cap R = M$. By Corollary 4.1.13, $\overline{R}/M\overline{R} \cong R/M$ and hence $\overline{T}/P\overline{T} \cong T/P$ as rings, so that $P\overline{T}$ is a prime ideal of $\overline{T}$. ∎

Lemma 4.2.4. *Let R be a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M, fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Suppose P is a prime ideal of $T = R[\Theta; \Delta]$ such that $M = P \cap R$. With the notation of (4.2.3),*

(i) if I/P is a uniform right ideal of T/P then $E_T(I/P)$ can be made into a $\overline{T}$-module and, as such, is (isomorphic to) $E_{\overline{T}}(I\overline{T}/P\overline{T})$;

(ii) P is rep. rep. [resp. strongly rep. rep.] in T if and only if $P\overline{T}$ is rep. rep. [resp. strongly rep. rep.] in $\overline{T}$.

Proof. (i) Let $c \in \overline{R}$, so that there is a Cauchy sequence $\{{}^n c\}$ in R such that ${}^n c \to c$ (Definition 4.1.9(iii)). Put $E = E_T(I/P)$ and let $x \in E$. By Theorem 1.3.3, $x \in E_i$ for some $i \in \mathbb{N}$, where, in the notation of (1.3.1), E_i is the i^{th} layer of E_T. Then $xM^i = 0$ since every prime in the clique of P contains M by [G1, Lemma 7.3] (see Remark 4.1.1). Now, by the definition of a Cauchy sequence (Definition 4.1.9(i)), we can find $N(c) \in \mathbb{N}$ such that ${}^n c - {}^m c \in M^i$ whenever $n \geq m \geq N(c)$. Thus, $x.{}^n c = x.{}^m c = x.{}^{N(c)} c$. Let $t \in \mathbb{N}$ and, for each $1 \leq i \leq t$, let $(i_1, \ldots, i_n) \in \mathbb{N}^n$, let $c_i \in \overline{R}$ and define:

$$x.\left(\sum_{i=1}^{t} c_i \theta_1^{i_1} \ldots \theta_n^{i_n}\right) := \sum_{i=1}^{t}\left(x.{}^{N(c_i)} c_i\right)\theta_1^{i_1} \ldots \theta_n^{i_n} \;,$$

which extends the action of T on E to make E into a $\overline{T}$-module.

Now let $\overline{E}$ be the $\overline{T}$-module injective hull of E. As noted in (4.2.3), $\overline{T}/P\overline{T} \cong T/P$ as rings and hence as $\overline{T}$-modules and so $\overline{E} = E_{\overline{T}}(I\overline{T}/P\overline{T})$. As a T-module, $\overline{E} = E \oplus L$ for some T-module L. Again by Theorem 1.3.3, if $x \in L$ then $x \in \overline{E}_i$ for some $i \in \mathbb{N}$ and, for the same reason as applied to E_i, $x\overline{M}^i = 0$. Now, for all $\overline{r} \in \overline{R}$, there exists $r \in R$ such that $\overline{r} - r \in \overline{M}^i$. So $x\overline{r} - xr = 0 \in L$ and it follows that L is a $\overline{T}$-module. Since $\overline{E}_{\overline{T}}$ is uniform, we must have $L = 0$ and thus $\overline{E} = E$.

(ii) Now, since $T/MT \cong \overline{T}/M\overline{T}$, contraction ($Q \mapsto Q \cap T$) and extension ($P \mapsto P\overline{T}$) are inverse bijections between the prime ideals of T containing M and of $\overline{T}$ containing $M\overline{T}$. We claim that, for each $i \in \mathbb{N}$,

$$\begin{aligned} A_i(P\overline{T}) &= \{Q\overline{T} : Q \in A_i(P)\} \\ \text{and} \qquad \overline{E}_i &= E_i \ . \end{aligned} \tag{A}$$

Since $A_1(\overline{P}) = \{\overline{P}\}$ and $A_1(P) = \{P\}$, the claim, for $i = 1$, follows from the equivalences

$$e \in E_1 \iff eP = 0 \iff eP\overline{T} = 0 \iff e \in \overline{E}_1 \ .$$

So suppose that, for some $n \in \mathbb{N}$, the claim (A) is true for all $1 \leq i \leq n$.

Now, if $Q \in A_{n+1}(P)$ then Q is an associated prime of $(E/E_n)_T$. So there is a T-submodule, M/E_n say, of E/E_n with M/E_n a fully faithful T/Q-module. Since, by the same argument as before, any T-submodule of E is also a $\overline{T}$-module, it is easily seen that M/E_n is a fully faithful $\overline{T}/Q\overline{T}$-module. So, $Q\overline{T}$ is an associated prime of $(E/E_n)_{\overline{T}} = (\overline{E}/\overline{E}_n)_{\overline{T}}$. That is, $Q\overline{T} \in A_{n+1}(P\overline{T})$.

Conversely, if $\overline{Q} \in A_{n+1}(P\overline{T})$ then, by Corollary 1.3.5, $\overline{Q} \in \text{cl.}(P\overline{T})$ and so, by [G1, Lemma 7.3] (see Remark 4.1.1), $\overline{Q} \cap \overline{R} = P\overline{T} \cap \overline{R} = M\overline{R}$. Put $Q = \overline{Q} \cap T$ and then, by the above bijection, $\overline{Q} = Q\overline{T}$. Now $\overline{Q}$ is an associated prime of $(E/E_n)_{\overline{T}}$ and so there is a $\overline{T}$-submodule, say M, of E with M/E_n a fully faithful $\overline{T}/\overline{Q}$-module and therefore a fully faithful T/Q-module. Thus $Q \in A_{n+1}(P)$.

We have shown that $A_{n+1}(P\overline{T}) = \{Q\overline{T} : Q \in A_{n+1}(P)\}$ and further, that $\overline{E}_{n+1} = E_{n+1}$, follows easily from the definitions of $\overline{E}_{n+1}$ and E_{n+1}. This establishes (A) for $1 \leq i \leq n+1$ and hence, by induction, for all $i \in \mathbb{N}$.

To complete the proof we need to show that, for each $i \in \mathbb{N}$,

$$\text{r.cl.}_{\overline{T}}(P\overline{T}) = \{Q\overline{T} : Q \in \text{r.cl.}_T(P)\}$$

and

$$Q \in X_i(P) \iff Q\overline{T} \in X_i(P\overline{T}) .$$

Clearly, it suffices to show that

$$Q \rightsquigarrow P \text{ in } T \iff Q\overline{T} \rightsquigarrow P\overline{T} \text{ in } \overline{T} . \qquad (B)$$

Now, by Lemma 1.2.6, $Q \rightsquigarrow P$ in T if and only if ${}_{T/Q}\left|\frac{Q \cap P}{QP}\right|_{T/P}$ is faithful on both sides, while $Q\overline{T} \rightsquigarrow P\overline{T}$ in $\overline{T}$ if and only if ${}_{\overline{T}/Q\overline{T}}\left|\frac{Q\overline{T} \cap P\overline{T}}{Q\overline{T}P\overline{T}}\right|_{\overline{T}/P\overline{T}} = {}_{\overline{T}/Q\overline{T}}\left|\frac{(Q \cap P)\overline{T}}{(QP)\overline{T}}\right|_{\overline{T}/P\overline{T}}$ is faithful on both sides.

Suppose $\frac{Q \cap P}{QP}\big|_{T/P}$ is not faithful, and let $t \in T - P$ such that $(Q \cap P)t \subseteq QP$. Then $t \in \overline{T} - P\overline{T}$ while $(Q \cap P)\overline{T}t \subseteq QP\overline{T}$ so that $\frac{(Q \cap P)\overline{T}}{(QP)\overline{T}}\big|_{\overline{T}/P\overline{T}}$ is not faithful.

Conversely, suppose $\frac{(Q \cap P)\overline{T}}{(QP)\overline{T}}\big|_{\overline{T}/P\overline{T}}$ is not faithful and let $t \in \overline{T} - P\overline{T}$ such that $(Q \cap P)\overline{T}t \subseteq QP\overline{T}$. Since $\overline{T} = T + M\overline{T}$, we can find $s \in T$ such that $s + M\overline{T} = t + M\overline{T}$. Then $s \notin P$ while $(Q \cap P)s \subseteq (QP\overline{T}) \cap T = QP$ so that $\frac{Q \cap P}{QP}\big|_{T/P}$ is not faithful.

Since a similar argument holds on the left, we see that (B) holds completing the proof of the lemma. ∎

Notation 4.2.5. For any ring S containing a copy L' of a field L of which $\widehat{L}$ is an extension field, we can form the ring $\widehat{S} = S \otimes_L \widehat{L}$ where the L-module action on S is given by L'. We will usually identify $S \otimes_L L$ with S and regard S as a subring of $\widehat{S}$. ∎

So, for instance, let R be a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M. For the purposes of studying repleteness, we can assume, by Lemma 4.2.4, that R is complete and then, by Cohen's Theorem (Theorem 4.1.16), R must contain a copy K' of $K = R/M$ (an isomorphism being given by the restriction to K' of the canonical epimorphism from R to R/M). Indeed, whenever R contains such a copy and $\widehat{K}$ is any extension field of K, we can form $\widehat{R}$ as above. We are interested in the case where the extension is finite algebraic and, in order to replace R with $\widehat{R}$ it is necessary to extend the derivations Δ to $\widehat{R}$.

Now, it is well known that commuting derivations on a field extend uniquely to commuting derivations on any separable algebraic extension field [Z&S, Chapter II, §17, Corollary 2' to Theorem 39] and so, for instance, the induced derivations on $K = R/M$ extend uniquely to commuting derivations on an algebraic extension $\widehat{K}$ of K. In fact, since the characteristic of K is zero, R being a $\mathbb{Q}$-algebra, and since any algebraic extension of a field of characteristic zero is separable [McC, Chapter 1, Theorem 7], the next lemma shows that the derivations, Δ, of R, extend to derivations on $\widehat{R}$ even though K' is not in general closed under the action of Δ.

Lemma 4.2.6. *Let S be a ring and let Δ be a set of commuting derivations on S. Suppose that S contains a copy of a field L of which $\widehat{L}$ is a separable algebraic extension and let $\widehat{S} = S \otimes_L \widehat{L}$. Then Δ extends uniquely to a set $\widehat{\Delta}$ of commuting derivations on the ring $\widehat{S}$.*

Proof. Suppose $x \in \widehat{L}$ and, for $0 \leq i \leq n$, let $a_i \in L$ such that $\sum_{i=0}^n a_i X^i$ is a minimal polynomial for x. We let L' be the copy of L contained in S and denote by λ' the image in L' of each element λ of L. If γ is to be a derivation on $\widehat{T}$, we see by applying it to the identity $\sum_{i=0}^n (a_i' \otimes 1)(1 \otimes x)^i = 0$ that we require

$$\sum_{i=0}^{n} \left(\gamma(a_i' \otimes 1)(1 \otimes x)^i + i(a_i' \otimes 1)(1 \otimes x)^{i-1}\gamma(1 \otimes x)\right) = 0$$

and hence

$$\gamma(1 \otimes x) = -\left[\ \sum_{i=0}^{n} \gamma(a_i' \otimes 1)(1 \otimes x)^i\right] . \left[1 \otimes \sum_{i=0}^{n} i a_i x^{i-1}\right]^{-1}$$

where the denominator is non-zero by the separability of the extension $L \subseteq \widehat{L}$ [McC, Chapter 1, §3, Proposition]. So, for each $\delta \in \Delta$ we require to define a derivation such that

$$\widehat{\delta}(1 \otimes x) = -\left[\ \sum_{i=0}^{n} \delta(a_i') \otimes x^i\right] . \left[\ \sum_{i=0}^{n} i a_i x^{i-1}\right]^{-1} \qquad (A)$$

and to extend this to $\widehat{S} = S \otimes_L \widehat{L}$ by additivity and the product rule for derivations. Assuming that $\widehat{\delta}$ is a well-defined derivation on $\widehat{S}$, we will require that

$$\widehat{\delta}(s \otimes 1) = \delta(s) \otimes 1 \qquad (B)$$

so that, identifying $S \otimes_L L$ with S, $\widehat{\delta}$ extends the action of δ to $\widehat{S}$. Evidently $\widehat{\delta}$ will be the unique such derivation.

Rather than verify directly that the definition (A) gives rise to a set of well defined commuting derivations on $\widehat{S}$, we first restrict our attention to the case where $\widehat{L} = L(x)$, a simple separable algebraic extension. As above, we assume that $\sum_{i=0}^{n} a_i X^i$ is the minimal polynomial for x and that $a_n = 1$. For each $s \in S$, we define $\widehat{\delta}(1 \otimes x)$ and $\widehat{\delta}(s \otimes 1)$ as in (A) and (B) and then set

$$\widehat{\delta}(s \otimes x^m) := \widehat{\delta}(s \otimes 1)(1 \otimes x^m) + (s \otimes m x^{m-1})\widehat{\delta}(1 \otimes x) \tag{C}$$

for each $m \geq 0$. We observe that, since δ is a derivation on S,

$$\widehat{\delta}[(s \otimes x^m)(r \otimes x^l)] = \widehat{\delta}(s \otimes x^m)(r \otimes x^l) + (s \otimes x^m)\widehat{\delta}(r \otimes x^l) \tag{D}$$

and

$$\widehat{\delta}[(s \otimes x^m) + (r \otimes x^m)] = \widehat{\delta}(s \otimes x^m) + \widehat{\delta}(r \otimes x^m) \tag{E}$$

for s, $r \in S$ and l, $m \geq 0$.

Next, since the minimal polynomial for x has degree n, each element t of $\widehat{S}$ can be expressed uniquely in the form

$$t = \sum_{i=0}^{n-1} s_i \otimes x^i$$

for some $s_0, \ldots, s_{n-1} \in S$. Thus, we can extend our definition, (C), to $\widehat{S}$ by setting,

$$\widehat{\delta}\left(\sum_{i=0}^{n-1} s_i \otimes x^i\right) := \sum_{i=0}^{n-1} \widehat{\delta}\left(s_i \otimes x^i\right) \ . \tag{F}$$

To see that $\widehat{\delta}$ is a derivation on $\widehat{S}$, we note that, for t_1 and $t_2 \in \widehat{S}$,

$$\widehat{\delta}(t_1 + t_2) = \widehat{\delta}(t_1) + \widehat{\delta}(t_2)$$

by (E) and (F) while

$$\widehat{\delta}(t_1 t_2) = \widehat{\delta}(t_1)t_2 + t_1\widehat{\delta}(t_2)$$

by a straightforward calculation applying (D) to (F).

If, δ_1 and $\delta_2 \in \Delta$, then we can form the derivations $\widehat{\delta}_1$ and $\widehat{\delta}_2$ on $\widehat{S}$ by the above. Since δ_1 and δ_2 commute, it is easy to see from (A), (B) and (C) that $\widehat{\delta}_1$ and $\widehat{\delta}_2$ commute. Thus, we have extended Δ to a set $\widehat{\Delta}$ of commuting derivations on $\widehat{S}$ in the case where $\widehat{L} = L(x)$.

Now let $\widehat{L} = L(B)$ be an arbitrary separable algebraic extension of L and let I be the set of all subsets, B_α, of B such that Δ can be extended to a set of commuting derivations Δ_α on $S_\alpha = S \otimes_L L(B_\alpha)$. The set I is non-empty since it contains the empty set. If $B_\alpha \subseteq B_\beta$ then Δ_β extends Δ_α from S_α to S_β. It follows that any ascending chain in I has an upper bound in I and so, by Zorn's Lemma, there is a maximal element B' in I. Let $S' = S \otimes_L L(B')$ and let Δ' be the set of commuting derivations extending Δ to S'. If $L(B') \neq L(B)$, then there exists an element $y \in L(B') - L(B)$ and, by the above argument, Δ' can be extended to a set Δ'' of commuting derivations on $L(B' \cup \{y\})$. This contradicts the maximality of B' in I and it follows that $L(B') = L(B)$.

We have thus shown that Δ can be extended to a set of commuting derivations $\widehat{\Delta}$ on $\widehat{S} = S \otimes_L \widehat{L}$ where $\widehat{L}$ is any separable algebraic extension of L. ∎

Remark 4.2.7. In particular, consider the case where R is a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M, R contains a copy K' of $K = R/M$ and $\widehat{K}$ is a finite algebraic extension of K. Assuming this extension is finite, $\widehat{R}$ is again Noetherian. Further, by [P1, Theorem 7.2.5], $M \otimes_K \widehat{K} = J(R) \otimes_K \widehat{K} \subseteq J(\widehat{R})$, while $\frac{R \otimes_K \widehat{K}}{M \otimes_K \widehat{K}} \cong K \otimes_K \widehat{K} \cong \widehat{K}$ so that $J(\widehat{R}) \subseteq J(R) \otimes_K \widehat{K} = M \otimes_K \widehat{K}$. Thus $\widehat{R}$ is local. Furthermore, if R is complete then, by Theorem 4.1.14, $\widehat{R}$ is complete. ∎

Notation 4.2.8. In the situation of Remark 4.2.7 and given the differential operator ring $R[\Theta; \Delta]$, we can extend, by Lemma 4.2.6, the set Δ of commuting derivations on R to a set $\widehat{\Delta}$ of commuting derivations on the ring $\widehat{R}$ and we form the differential operator ring $\widehat{T} = \widehat{R}[\Theta; \widehat{\Delta}]$. Finally we put

$$\widehat{U}_0 = \widehat{K}[w_{l+1}, \ldots, w_n] \cong U_0 \otimes_K \widehat{K} \quad \text{and} \quad \widehat{U} = \widehat{K}[\Theta, \widehat{\Delta}] \cong \widehat{T}/M\widehat{T} \ ,$$

(these isomorphisms hold since the K-module action on R/M arising through multiplication in R by K' coincides with the ring multiplication in R/M) and, where convenient, we will identify these isomorphic rings. We note that, as $\widehat{K}$ vector spaces, $\widehat{T} \cong T \otimes_K \widehat{K}$ and $\widehat{U} \cong U \otimes_K \widehat{K}$. However, since neither T nor U are K-algebras (K not being central in general) the right-hand sides do not automatically have a ring structure. ■

In Lemma 4.2.11, we show that, for the purpose of finding *sufficient* conditions for repleteness in T, we can instead look at the ring $\widehat{T}$, although the lemma does not show that repleteness in $\widehat{T}$ is *necessary* for repleteness in T. First we require two preliminary results. Lemma 4.2.9 is proved in [Lt, Corollary 2.4], while Lemma 4.2.10 is a well known result.

Lemma 4.2.9. *Let S be a right Noetherian subring of a ring V such that V_S is a finitely generated module. Then, the extension $S \subseteq V$ satisfies the incomparability property on prime ideals; that is, for every prime I of S, there do not exist primes $L_1 \supsetneq L_2$ of V such that I is minimal over both $L_1 \cap S$ and $L_2 \cap S$.* ■

Lemma 4.2.10. *Let $S \subseteq V$ be rings with ${}_SV$ a free module. If E_V is an injective module, then E_S is injective.*

Proof. We suppose that I is a right ideal of S and consider $f \in \mathrm{Hom}_S(I, E)$. To define a V-homomorphism, $f' : IV \to E$, fix a free basis $\{v_\lambda : \lambda \in \Lambda\}$ for V over S. Then, for any finite subset J of Λ and for any elements $a_j \in I$, set

$$f'\left(\sum_{j\in J} a_j v_j\right) = \sum_{j\in J} f(a_j)v_j.$$

It is easy to check that $f'(bv) = f'(b)v$ for all $b \in IV$ and $v \in V$ and that $f'(a) = f(a)$ for all $a \in I$. Now, we can extend f' to $f'' \in \mathrm{Hom}_V(V, E)$ by the injectivity of E_V. Then, since the restriction of f'' to S extends the action of f on I, it follows E_S is injective. ■

Lemma 4.2.11. *Let R be a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M, fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Suppose R contains a copy of the field $K = R/M$ of which $\widehat{K}$ is a finite algebraic extension and let*

$\widehat{T}$ *be constructed as in (4.2.8). Suppose* P *is a prime ideal of* $T = R[\Theta;\Delta]$ *with* $M = P \cap R$.

(i) *There is a prime ideal,* $\mathcal{P}$, *of* $\widehat{T}$ *such that* $\mathcal{P} \cap T = P$ *and any such* $\mathcal{P}$ *is minimal over* $P\widehat{T}$.

(ii) *If some such* $\mathcal{P}$ *is rep. rep. [resp. strongly rep. rep.] in* $\widehat{T}$, *then* P *is rep. rep. [resp. strongly rep. rep.] in* T .

Proof. (i) Adopting Notation 4.1.3, we put $P' = P/MT$ and $P_0 = P' \cap U_0$. Then, since R is a $\mathbb{Q}$-algebra, the characteristic of K is zero, $\widehat{K}$ is a separable extension of K [McC, Chapter 1, Theorem 7] and so Lemma 4.1.6 shows that

$$P_0\widehat{U}_0 = P_0\widehat{K} = \mathcal{P}_{10} \cap \ldots \cap \mathcal{P}_{t0}$$

for some primes $\mathcal{P}_{10}, \ldots, \mathcal{P}_{t0}$ of $\widehat{U}_0$, minimal over $P_0\widehat{U}_0$. By Lemma 4.1.2, each $\mathcal{P}_{i0}$ is $\widehat{\Delta}$-invariant. Now,

$$P_0 = P_0\widehat{U}_0 \cap U_0 = (\mathcal{P}_{10} \cap U_0) \cap \ldots \cap (\mathcal{P}_{t0} \cap U_0) = P_{10} \cap \ldots \cap P_{t0}$$

for prime ideals $P_{i0} = (\mathcal{P}_{i0} \cap U_0)$ of U_0. We claim that $P_0 = P_{i0}$ for each i. For, if there is any j for which $P_0 \subsetneq P_{j0}$ then, denoting by $\mathcal{C}_S(I)$ the set of elements of a ring S regular modulo the ideal I, we see by [G&W, Exercise 3L and Proposition 5.9] that $P_{j0} \cap \mathcal{C}_{U_0}(P_0) \neq \emptyset$. In this case, since $\mathcal{C}_{U_0}(P_0) \subseteq \mathcal{C}_{\widehat{U}_0}(P_0\widehat{U}_0)$, we have $\mathcal{P}_{j0} \cap \mathcal{C}_{\widehat{U}_0}(P_0\widehat{U}_0) \neq \emptyset$, contradicting the minimality of $\mathcal{P}_{j0}$ over $P_0\widehat{U}_0$. It follows that $P_0 = P_{i0}$ for every i.

Write $\widehat{U}\mathcal{P}_{i0} = \mathcal{P}'_i$, which is a prime ideal of $\widehat{U}$ by Theorem 4.1.4, and let $\mathcal{P}_i$ be the prime of $\widehat{T}$ such that $\mathcal{P}_i/M\widehat{T} = \mathcal{P}'_i$. Then, for each i,

$$(\mathcal{P}_i \cap T)/MT = \mathcal{P}'_i \cap U = \widehat{U}\mathcal{P}_{i0} \cap U \ .$$

Now, since $\mathcal{P}_i$ is completely prime by [G&W, Theorem 9.24] (see Remark 4.1.1), $\mathcal{P}_i \cap T$ is a prime ideal of T and hence $\widehat{U}\mathcal{P}_{i0} \cap U$ is a prime ideal of U. So, by Theorem 4.1.4, we see that

$$\widehat{U}\mathcal{P}_{i0} \cap U = U\left(\widehat{U}\mathcal{P}_{i0} \cap U \cap U_0\right) = U\left(\widehat{U}\mathcal{P}_{i0} \cap U_0\right) \ .$$

On the other hand,

$$P/MT = UP_0 = UP_{i0} = U(\mathcal{P}_{i0} \cap U_0) = U\left(\widehat{U}\mathcal{P}_{i0} \cap \widehat{U}_0 \cap U_0\right) = U\left(\widehat{U}\mathcal{P}_{i0} \cap U_0\right) ,$$

where we have again applied Theorem 4.1.4 for the first and fourth equalities. We thus obtain, for each i,

$$(\mathcal{P}_i \cap T)/MT = \widehat{U}\mathcal{P}_{i0} \cap U = U\left(\widehat{U}\mathcal{P}_{i0} \cap U_0\right) = P/MT ,$$

so that $\mathcal{P}_i \cap T = P$ and we can take $\mathcal{P} = \mathcal{P}_i$ for any i. Now,

$$\begin{aligned}
P\widehat{T}/M\widehat{T} =& UP_0\widehat{U} = \widehat{U}P_0 = \widehat{U}P_0\widehat{U}_0 \\
=& \widehat{U}(\mathcal{P}_{10} \cap \ldots \cap \mathcal{P}_{t0}) \\
=& \widehat{U}\mathcal{P}_{10} \cap \ldots \cap \widehat{U}\mathcal{P}_{t0} \\
=& \mathcal{P}'_1 \cap \ldots \cap \mathcal{P}'_t \\
=& \mathcal{P}_1/M\widehat{T} \cap \ldots \cap \mathcal{P}_t/M\widehat{T} \\
=& (\mathcal{P}_1 \cap \ldots \cap \mathcal{P}_t)/M\widehat{T}
\end{aligned}$$

so that $P\widehat{T} = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_t$. As above, $\mathcal{P}_i \cap T = P$ for each i and we see that the $\mathcal{P}_i$ are the primes of $\widehat{T}$ minimal over $P\widehat{T}$. Further, since the extension $T \subseteq \widehat{T}$ satisfies the incomparability property for prime ideals (Lemma 4.2.10), no prime of $\widehat{T}$ strictly containing one of the $\mathcal{P}_i$ can contract to P. We will, for convenience, take $\mathcal{P}$ to be $\mathcal{P}_1$.

(ii) As we have seen, every prime of $\widehat{T}$ contracts to a prime of T and, for every prime of T, there is a prime of $\widehat{T}$ which contracts to it. So, by the incomparability property of Lemma 4.2.10, for any prime $\mathcal{Q}$ of $\widehat{T}$,

$$\text{Cl.K.dim}\left(\widehat{T}/\mathcal{Q}\right) = \text{Cl.K.dim}\left(T/(\mathcal{Q} \cap T)\right) . \qquad (A)$$

Now let V be a uniform submodule of $\left(\widehat{T}/\mathcal{P}\right)_{\left(\widehat{T}/\mathcal{P}\right)}$ and put $E = E_{\widehat{T}}(V)$. Since ${}_T\widehat{T}$ is free, it follows, from Lemma 4.2.10, that E is an injective T-module. Let Q be an associated prime of E_T. We claim that

$$Q = P . \qquad (B)$$

By the choice of Q, we can find $e \in E$ such that eT is a fully faithful T/Q-module. Now, $e\widehat{T} \cap V$ is a fully faithful $\widehat{T}/\mathcal{P}$-module and so $Q \subseteq \mathcal{P} \cap T = P$. On the other hand, by Theorem 1.3.3 there exists a r.cl.($\mathcal{P}$)-semiprime ideal, $\mathcal{I}$, of $\widehat{T}$ such that $\mathcal{I}^{n+1} \subseteq \text{ann}(e\widehat{T})$ for some $n \in \mathbb{N}$. So $(\mathcal{I} \cap T)^{n+1} \subseteq \mathcal{I}^{n+1} \cap T \subseteq Q$ and hence $\mathcal{I} \cap T \subseteq Q$. It follows that we can find a prime $\mathcal{A}$ in r.cl.($\mathcal{P}$) such that $\mathcal{A} \cap T \subseteq Q \subseteq P$. However, by Lemma 1.2.8(ii) and (A),

$$\begin{aligned}\text{Cl.K.dim}\left(\frac{T}{\mathcal{A}\cap T}\right) &= \text{Cl.K.dim}\left(\frac{\widehat{T}}{\mathcal{A}}\right) = \text{Cl.K.dim}\left(\frac{\widehat{T}}{\mathcal{P}}\right) \\ &= \text{Cl.K.dim}\left(\frac{T}{\mathcal{P}\cap T}\right) = \text{Cl.K.dim}\left(\frac{T}{P}\right),\end{aligned}$$

and we see that $\mathcal{A} \cap T = Q = P$ establishing (B).

Thus, P is the unique associated prime of $\left(E_{\widehat{T}}(V)\right)_T$ and therefore, as T-modules, $E = E_{\widehat{T}}(V) = \oplus_{\lambda\in\Lambda}{}^{\lambda}E$ for some uniform injectives ${}^{\lambda}E$ each with assassinator prime P and some index set Λ. We observe that, as T-modules,

$$\text{ann}_E(\mathcal{P}) = \oplus_{\lambda\in\Lambda}\text{ann}_{{}^{\lambda}E}(P). \qquad (C)$$

For, if $e \in E$ and $eP = 0$ then $eP\widehat{T} = 0$ so we have $e(\mathcal{P}_1 \cap \ldots \cap \mathcal{P}_t) = 0$. If $e\mathcal{P}_1 \neq 0$ then $(e\mathcal{P}_1) \cap V \neq 0$ and so $\mathcal{P}_2 \cap \ldots \cap \mathcal{P}_t \subseteq \mathcal{P}_1$ which is impossible. Thus $e\mathcal{P} = e\mathcal{P}_1 = 0$. Conversely, if $c \in E$ and $c\mathcal{P} = 0$ then clearly $cP = c(\mathcal{P} \cap T) = 0$.

Now suppose $e \in \text{ann}_{{}^{\lambda}E}(P)$ for some $\lambda \in \Lambda$ and that $et = 0$ for some $t \in T - P$. Clearly, $t \in \widehat{T} - \mathcal{P}$ so that t is regular modulo $\mathcal{P}$ since $\mathcal{P}$ is completely prime. By (C), $e \in \text{ann}_E(\mathcal{P})$ so, by the torsion-freeness of $(\text{ann}_E(P))_{\widehat{T}/\mathcal{P}}$, we see that $e = 0$. Thus, $(\text{ann}_{{}^{\lambda}E}(P))_{T/P}$ is torsion-free for each λ. We can then assume that each ${}^{\lambda}E$ is the injective hull of a uniform right ideal of T/P so that Theorem 1.3.3 applies. In particular, the fundamental primes are identical for each ${}^{\lambda}E$.

As before, we write E_k and ${}^{\lambda}E_k$ for the k^{th} layers of $E_{\widehat{T}}$ and ${}^{\lambda}E_T$ respectively. Then, viewing E_k as a T-module, we claim that, for each $k \in \mathbb{N}$,

$$\begin{array}{lll} & (a) & A_k(P) = \{\mathcal{Q} \cap T : \mathcal{Q} \in A_k(\mathcal{P})\} \\ \text{and} & & \\ & (b) & E_k = \oplus_{\lambda\in\Lambda}{}^{\lambda}E_k\ . \end{array} \qquad (D)$$

By (B) and (C), the claim is true for $k = 1$ and we assume it for all $1 \leq k \leq n$ for some $n \in \mathbb{N}$. We note that, as seen in (i), $\mathcal{Q}$ is minimal over $(\mathcal{Q} \cap T)\widehat{T}$ for any prime $\mathcal{Q}$ of $\widehat{T}$.

(a) Let $\mathcal{Q} \in A_{n+1}(\mathcal{P})$. Then there exists $e \in E - E_n$ such that $e\mathcal{Q} \subseteq E_n$. Writing $e = \oplus_{\lambda\in\Lambda} e_\lambda$, where each $e_\lambda \in {}^\lambda E$ and only finitely many are non-zero, we have, by the induction hypothesis on (Db),

$$\oplus_{\lambda\in\Lambda} e_\lambda(\mathcal{Q} \cap T) \subseteq \oplus_{\lambda\in\Lambda} {}^\lambda E_n \ .$$

So, for some λ, $e_\lambda \in {}^\lambda E - {}^\lambda E_n$ while $e_\lambda(\mathcal{Q} \cap T) \subseteq {}^\lambda E_n$. Consequently, $\mathcal{Q} \cap T \subseteq Q'$ for some $Q' \in A_{n+1}(P) \subseteq X_{n+1}(P)$. Again, by Lemma 1.2.8(ii) and (A),

$$\begin{aligned}\text{Cl.K.dim}\left(\frac{T}{\mathcal{Q}\cap T}\right) = \text{Cl.K.dim}\left(\frac{\widehat{T}}{\mathcal{Q}}\right) &= \text{Cl.K.dim}\left(\frac{\widehat{T}}{\mathcal{P}}\right) \\ &= \text{Cl.K.dim}\left(\frac{T}{P}\right) = \text{Cl.K.dim}\left(\frac{T}{Q'}\right)\end{aligned}$$

so that $\mathcal{Q} \cap T = Q' \in A_{n+1}(P)$.

On the other hand, suppose $Q \in A_{n+1}(P)$. We can write $Q\widehat{T} = \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_l$ for some $l \in \mathbb{N}$ and where each $\mathcal{Q}_i \cap T = Q$. Now, if we fix $\mu \in \Lambda$ then there exists some $e_\mu \in {}^\mu E - {}^\mu E_n$ such that $e_\mu Q \subseteq {}^\mu E_n$. We now put

$$e_\lambda = \begin{cases} e_\mu & \text{if } \mu = \lambda \ , \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e = \oplus_{\lambda\in\Lambda} e_\lambda \ ,$$

and, by the induction hypothesis on (Cb), we see that

$$e\mathcal{Q}_1 \ldots \mathcal{Q}_l \subseteq e(\mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_l) = eQ\widehat{T} \subseteq \oplus_{\lambda\in\Lambda} {}^\lambda E_n = E_n \ .$$

Now, if $e\mathcal{Q}_1 \subseteq E_n$ then $\mathcal{Q}_1 \subseteq \widetilde{\mathcal{Q}_1}$ for some $\widetilde{\mathcal{Q}_1} \in A_{n+1}(\mathcal{P})$; but then

$$\widetilde{\mathcal{Q}_1} \cap T \in A_{n+1}(P) \subseteq \text{cl.}(P)$$

and so by Lemma 1.2.8(i) $Q = \mathcal{Q}_1 \cap T = \widetilde{\mathcal{Q}_1} \cap T$. Thus, since the extension $T \subseteq \widehat{T}$ satisfies the incomparability property for prime ideals (Lemma 4.2.10), we see that $\mathcal{Q}_1 = \widetilde{\mathcal{Q}_1} \in A_{n+1}(\mathcal{P})$. If $e\mathcal{Q}_1 \not\subseteq E_n$, then we can suppose by induction that, for some $1 \leq a \leq l-1$, $e\mathcal{Q}_1 \ldots \mathcal{Q}_a \not\subseteq E_n$ while $a\mathcal{Q}_1 \ldots \mathcal{Q}_{a+1} \subseteq E_n$. In this case, $\mathcal{Q}_{a+1} \in A_{n+1}(\mathcal{P})$. Thus we find $\mathcal{Q} \in A_{n+1}(\mathcal{P})$ such that $\mathcal{Q} \cap T = Q$ completing the induction step for (Da).

(b) Now choose $e \in E - E_n$. Again we write $e = \oplus_{\lambda\in\Lambda} e_\lambda$, where only finitely many e_λ are non-zero, and note that, by the induction hypothesis on (Db), $e_\lambda \in$

${}^{\lambda}E - {}^{\lambda}E_n$, for some λ. Suppose $e \in E_{n+1}$. Then we can find a subset $\{\mathcal{Q}_j : j \in J\}$ of $A_{n+1}(\mathcal{P})$, for some finite index set J, such that $e\left(\bigcap_{j\in J} \mathcal{Q}_j\right) \subseteq E_n$. Then

$$\oplus_{\lambda\in\Lambda} e_\lambda . \bigcap_{j\in J} (\mathcal{Q}_j \cap T) \subseteq {}^{\lambda}E_n \ .$$

Now, for each $j \in J$, we have seen by (Da) that $\mathcal{Q}_j \cap T \in A_{n+1}(P) \subseteq X_{n+1}(P)$ while, for each λ, $e_\lambda . \bigcap_{j\in J}(\mathcal{Q}_j \cap T) \subseteq {}^{\lambda}E_n$ and so, by Theorem 1.3.3(ii), $e_\lambda \in {}^{\lambda}E_{n+1}$. We conclude that

$$E_{n+1} \subseteq \oplus_{\lambda\in\Lambda} {}^{\lambda}E_{n+1} \ .$$

Conversely, suppose that, for each λ, $e_\lambda \in {}^{\lambda}E_{n+1}$ with only finitely many non-zero and put $e = \oplus_{\lambda\in\Lambda} e_\lambda$. For each λ we can find finitely many ${}^{j}f_\lambda \in {}^{\lambda}E$ such that $e_\lambda = \sum_j {}^{j}f_\lambda$ and where ${}^{j}f_\lambda\left({}^{j\lambda}P\right) \subseteq {}^{\lambda}E_n$ for some ${}^{j\lambda}P \in A_{n+1}(P)$. It suffices to show that, for each λ and j, ${}^{j}f_\lambda \in E_{n+1}$. So we fix λ and j and note that, by the induction hypothesis on (Db), ${}^{j}f_\lambda\left({}^{j\lambda}P\right) \subseteq E_n$. Now ${}^{j\lambda}P\widehat{T} = {}^{j\lambda}\mathcal{P}_1 \cap \ldots \cap {}^{j\lambda}\mathcal{P}_s$ where the ${}^{j\lambda}\mathcal{P}_r$ are the primes which contract to ${}^{j\lambda}P$. Clearly, ${}^{j}f_\lambda . \bigcap_r {}^{j\lambda}\mathcal{P}_r \subseteq E_n$. By the same argument as before, one of the ${}^{j\lambda}\mathcal{P}_r$, say ${}^{j\lambda}\mathcal{P}_v$, must be in $A_{n+1}(\mathcal{P})$. In fact, we claim that

$$ {}^{j}f_\lambda . \bigcap_{({}^{j\lambda}\mathcal{P}_r \in A_{n+1}(\mathcal{P}))} {}^{j\lambda}\mathcal{P}_r \subseteq E_n \ . \tag{E}$$

For if this is not the case, then since certainly

$$ {}^{j}f_\lambda . \bigcap_{({}^{j\lambda}\mathcal{P}_r \in A_{n+1}(\mathcal{P}))} {}^{j\lambda}\mathcal{P}_r . \prod_{({}^{j\lambda}\mathcal{P}_r \notin A_{n+1}(\mathcal{P}))} {}^{j\lambda}\mathcal{P}_r \subseteq E_n \ ,$$

we would deduce that $\prod_{({}^{j\lambda}\mathcal{P}_r \notin A_{n+1}(\mathcal{P}))} {}^{j\lambda}\mathcal{P}_r$ is contained in a prime in $A_{n+1}(\mathcal{P})$. So, for instance, we would have ${}^{j\lambda}\mathcal{P}_u \subseteq \mathcal{Q} \in A_{n+1}(\mathcal{P})$ for some ${}^{j\lambda}\mathcal{P}_u \notin A_{n+1}(\mathcal{P})$. Then, since ${}^{j\lambda}\mathcal{P}_v$ and $\mathcal{Q}$ are both in the clique of $\mathcal{P}$ and, by Lemma 1.2.8(ii),

$$\text{Cl.K.dim.}\left(\frac{\widehat{T}}{{}^{j\lambda}\mathcal{P}_v}\right) = \text{Cl.K.dim.}\left(\frac{\widehat{T}}{\mathcal{Q}}\right) \leq \text{Cl.K.dim.}\left(\frac{\widehat{T}}{{}^{j\lambda}\mathcal{P}_u}\right).$$

However, since ${}^{j\lambda}\mathcal{P}_v \cap T = {}^{j\lambda}\mathcal{P}_u \cap T$, we see from (A) that

$$\begin{aligned}\text{Cl.K.dim.}\left(\frac{\widehat{T}}{{}^{j\lambda}\mathcal{P}_v}\right) &= \text{Cl.K.dim.}\left(\frac{T}{{}^{j\lambda}\mathcal{P}_v \cap T}\right) \\ &= \text{Cl.K.dim.}\left(\frac{T}{{}^{j\lambda}\mathcal{P}_u \cap T}\right) = \text{Cl.K.dim.}\left(\frac{\widehat{T}}{{}^{j\lambda}\mathcal{P}_u}\right) \ .\end{aligned}$$

Thus ${}^{j\lambda}\mathcal{P}_u = \mathcal{Q} \in A_{n+1}(\mathcal{P})$ which contradicts our assumption and so (E) is proved. It follows from (E) and Theorem 1.3.3(ii) that ${}^{j}f_\lambda \in E_{n+1}$ completing the induction step of (Db) and therefore the proof of (D) for all $k \in \mathbb{N}$.

Finally, we observe that, given primes ${}^2P \rightsquigarrow\!\!\!> {}^1P$ in T and a prime ${}^1\mathcal{P}$ of $\widehat{T}$ minimal over ${}^1P\widehat{T}$, we can apply (Da) with $P = {}^1P$ and $k = 2$ to produce a prime ${}^2\mathcal{P}$ minimal over ${}^2P\widehat{T}$ which is in $A_2\left({}^1\mathcal{P}\right) \subseteq X_2\left({}^1\mathcal{P}\right)$. Again, given primes ${}^2\mathcal{P} \rightsquigarrow\!\!\!> {}^1\mathcal{P}$ in $\widehat{T}$ we apply (Da) with $\mathcal{P} = {}^1\mathcal{P}$ to see that ${}^2\mathcal{P} \cap T \rightsquigarrow\!\!\!> {}^1\mathcal{P} \cap T$. So, by induction on these arguments, we deduce that, for all $k \in \mathbb{N}$,

$$X_k(P) = \{\mathcal{Q} \cap T : \mathcal{Q} \in X_k(\mathcal{P})\} \tag{F}$$

and any such $\mathcal{Q}$ is minimal over $(\mathcal{Q} \cap T)\widehat{T}$.

Part (ii) of the Lemma follows easily from (Da) and (F). ∎

In the case where R is a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M and R contains a copy of $K = R/M$ (for instance if R is complete), Remark 4.2.7 and Lemma 4.2.11 show that, for finding sufficient conditions for repleteness, we can replace K with any finite algebraic extension, $\widehat{K}$. So taking a suitably large extension, we have reduced to the case where, in the notation of (4.1.5), all the eigenvectors for W on $(M/M^2)^\#$ can be chosen from M/M^2. This last condition is required in Theorem 4.2.12 but also ensures that all eigenvalues for W on $(M/M^2)^\#$ map W into K, as follows from the definition of the action of W on M/M^2.

Thus R satisfies the hypotheses of Theorem 4.1.7(ii), which we wish to use for the description of the clique of a prime of T.

If we make the additional assumption that R is a complete regular local ring (Definition 4.1.17), then, by Cohen's structure theorem (Theorem 4.1.19(ii)), R is isomorphic to a power series ring over K in finitely many indeterminates; however in the following theorem we only require the fact which we noted in Lemma 4.1.18, namely that, in any regular local ring with maximal ideal M, if $m_1 \in M^u - M^{u+1}$ and $m_2 \in M^v - M^{v+1}$ then

$$m_1 m_2 \in M^{u+v} - M^{u+v+1} .$$

Theorem 4.2.12. *Let R be a commutative local Noetherian $\mathbb{Q}$-algebra with maximal ideal M, fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Suppose P is a prime ideal of $T = R[\Theta; \Delta]$ such that $M = P \cap R$.*

(i) If $P = MT$ then P is rep. rep. in T and is strongly rep. rep. if and only if M is strongly rep. rep. in R.

(ii) Suppose that, with the notation of (4.1.5), all the eigenvectors for W on $(M/M^2)^\#$ can be chosen from M/M^2. If R is a regular local ring, then P is rep. rep. in T.

(iii) Suppose that, with the notation of (4.1.5), all the eigenvectors for W on $(M/M^2)^\#$ can be chosen from M/M^2 and that $(M/M^2)_{R/M}$ is generated by the eigenvectors for W. If R is a regular local ring containing a copy K' of $K = R/M$ such that, in the notation of (4.1.3), each element of K' commutes with each element of W' (or, equivalently, with each of $\{w'_{l+1}, \ldots, w'_n\}$), then P is strongly rep. rep. in T.

Proof. We first note that $X_i(P) \subseteq X_{i+1}(P)$ for each $i \in \mathbb{N}$, provided every prime in the right clique of P is linked to itself. In this case, and in the notation of (1.3.10), $S_i \supseteq S_{i+1}$ and

$$\frac{I_m}{J'_m} = \frac{S_m S_{m-1} \ldots S_1 \cap S_{m+1} S_m \ldots S_2}{S_{m+1} S_m \ldots S_1} = \frac{S_{m+1} \ldots S_2}{S_{m+1} \ldots S_1} \qquad (A)$$

for each i and $m \in \mathbb{N}$. By Theorem 4.1.8, (A) will hold except when $P = MT = 0$.

(i) Since $(M^t/M^{t+1})_R$ is isomorphic to a direct sum of copies of R/M for each $t \in \mathbb{N}$, $M^tT/M^{t+1}T$ is isomorphic to a direct sum of copies of T/MT as a right or left T-module. Thus

$$M^tT/M^{t+1}T \text{ is torsion free as a left or right } T/MT \text{ module.} \qquad (B)$$

Since the prime MT of T can be linked to no other prime (any such prime would necessarily contain MT by [G1, Lemma 7.3] (see Remark 4.1.1) and so would equal MT by the incomparability result of Lemma 1.2.8(i)), MT is always rep. rep. in T. Of course, if it is not linked to itself, (that is, by (B), or Theorem 4.1.8(ii), if it is zero) it will be strongly rep. rep.. Since, by (B), $B_m(MT) = I_m/J'_m$, in the case where MT is self-linked (A) becomes $B_m(MT, MT) = B_m(MT) = M^{m-1}T/M^mT$. We thus see from Theorem 1.3.11 that MT fails to be strongly rep. rep. precisely when $M^{m-1}T = M^mT$ for some $m \geq 3$ but $MT \neq 0$, that is, when $M^m = 0$ for some $m \geq 2$ but $M \neq 0$.

Part (i) of the theorem now follows from Corollary 2.1.3.

From (i), the prime MT is certainly strongly rep. rep. when R is regular and we can now restrict our attention to the case where $P \neq MT$. In particular, as noted at the start of the proof, the identity (A) holds. We also assume that the eigenvectors for W on $(M/M^2)^\#$ can be chosen from M/M^2. Suppose that $Q \in X_{m+1}(P)$ and that Q is linked to P by m non-trivial links. We wish to show first that $Q \in \mathrm{Fund}(P)$.

Let $P' = P/MT$, so that P' is a prime ideal of $U = K[\Theta; \Delta]$ and recall that, by Theorem 4.1.4, $P' = (P' \cap U_0)U$. By Theorem 4.1.7(ii),

$$(Q/MT) \cap U_0 = \tau^{-1}_{f_1+\cdots+f_m}(P' \cap U_0) \qquad (C)$$

where $\tau_{f_1+\cdots+f_m}$ is the winding automorphism given by some choice, $f_1, \ldots, f_m$, from the set $\{\varepsilon_1, \ldots, \varepsilon_t\}$, of all eigenvalues for W on M/M^2. Thus, putting

$$C_m = \left(\tau^{-1}_{f_1+\cdots+f_m}(P' \cap U_0)\right) U$$

we have $Q' = Q/MT = C_m$. Now, we can find $g_{f_1}, \ldots, g_{f_m} \in M - M^2$ such that

$$wg_{f_j} - g_{f_j}w + M^2T = f_j(w)g_{f_j} + M^2T$$

for each $w \in W$ and $1 \leq j \leq m$. From this and the definition of the winding automorphism τ_{f_j},

$$wg_{f_j} + M^2T = g_{f_j}\tau_{f_j}(w) + M^2T$$

for all $w \in U_0$. It follows from (C) that, in the (U,U)-bimodule $M^mT/M^{m+1}T$,

$$\begin{aligned}\frac{C_m g_{f_1}\cdots g_{f_m} + M^{m+1}T}{M^{m+1}T} &= \frac{U\left(\tau^{-1}_{f_1+\cdots+f_m}(P' \cap U_0)\right) g_{f_1}\cdots g_{f_m} + M^{m+1}T}{M^{m+1}T} \\ &\subseteq \frac{U\left(g_{f_1}\cdots g_{f_m}(P' \cap U_0) + M^{m+1}T\right)}{M^{m+1}T} \\ &\subseteq \frac{M^mP' + M^{m+1}T}{M^{m+1}T},\end{aligned}$$

from which we conclude that

$$Qg_{f_1}\cdots g_{f_m} \subseteq M^m P \subseteq S_{m+1}\cdots S_1 \ . \tag{D}$$

(ii) Suppose henceforth that R is regular. Now $(M^m P)\cap R = M^{m+1}$ while we see from Lemma 4.1.18 that $g_{f_1}\cdots g_{f_m} \notin M^{m+1}$. Therefore, $g_{f_1}\cdots g_{f_m} \notin M^m P$ and the bimodule

$$G_m = {}_{T/Q}\left|\frac{g_{f_1}\cdots g_{f_m}T + M^m P}{M^m P}\right|_{T/P} \tag{E}$$

is non-zero. Now, we have already noted that $M^m T/M^{m+1}T$ is a free right T/MT-module and so $M^m T/M^m P$ is a free right T/P-module. Since $(G_m)_{T/P}$ is a non-zero submodule, it must be faithful. By [G&W, Lemma 12.3], then, G_m has sub-bimodules $G' > G''$ such that G'/G'' is a torsion-free right T/P and a torsion-free left T/Q'-module for some prime ideal Q' of T with $Q \subseteq Q'$. Of course, by Theorem 1.3.7, $Q' \in \mathrm{Fund}(P) \subseteq \mathrm{cl.}(P)$. However, $Q \in \mathrm{cl.}(P)$ also and so, by Lemma 1.2.8(ii),

$$\mathrm{Cl.K.dim}(T/Q) = \mathrm{Cl.K.dim}(T/P) = \mathrm{Cl.K.dim}(T/Q') \ .$$

Consequently, $Q = Q'$ and we have shown that $Q \in \mathrm{Fund}(P)$.

Part (ii) of the theorem follows.

(iii) Now, we assume that R contains a copy K' of K and we write $K'[\Theta]$ for the set of polynomials in Θ over K' where the coefficients are written on the left. This set is contained in T and, as a left K-module is isomorphic to

$$U = T/MT = K[\Theta;\Delta] \ .$$

Furthermore, as left K-modules, $T = MT + K'[\Theta]$. For each $i \in \mathbb{N}$, we write, in the notation of (1.3.10),

$$D'_i = S_i \cap K'[\Theta] \ .$$

Then, $S_i = MT + D'_i$ and D'_i is a left K-submodule of $K'[\Theta]$ isomorphic to $D_i = S_i/MT$. We assume also the other hypotheses of (iii).

For each $1 \leq j \leq t$, let $\mu_j \in M/M^2$ be the eigenvector corresponding to the eigenvalue ε_j with $\widetilde{\mu_j} \in M$ such that $\widetilde{\mu_j} + M^2 = \mu_j$. Then, for each $i \in \mathbb{N}$,

$$\begin{aligned}
\frac{D_i\widetilde{\mu}_j + M^2T}{M^2T} &= \frac{U(D_i \cap U_0)\widetilde{\mu_j} + M^2T}{M^2T} \\
&\subseteq \frac{U\widetilde{\mu_j}(D_{i-1} \cap U_0) + M^2T}{M^2T} \\
&\subseteq \frac{MU(D_{i-1} \cap U_0) + M^2T}{M^2T} \\
&= \frac{MD_{i-1} + M^2T}{M^2T} .
\end{aligned}$$

Thus, since the μ_j generate M/M^2,

$$S_iM \subseteq MS_{i-1} = M^2T + MD'_{i-1} .$$

An induction argument now shows that, for any s and $t \in \mathbb{N}$ with $1 \leq t \leq s$,

$$\begin{aligned}
S_sS_{s-1}\ldots S_t &= (MT + D'_s)(MT + D'_{s-1})\cdots(MT + D'_t) \\
&= D'_s \ldots D'_t + MTD'_{s-1} \ldots D'_t + \cdots + M^{s+1-t}T . \qquad (F)
\end{aligned}$$

Recalling that identity (A) holds, it follows from (F) that

$$\frac{I_m}{J'_m} = \frac{S_{m+1}\ldots S_2}{S_{m+1}\ldots S_1} = \frac{D'_{m+1}\ldots D'_2 + MTD'_m \ldots D'_2 + \cdots + M^mT}{D'_{m+1}\ldots D'_1 + MTD'_m \ldots D'_1 + \cdots + M^{m+1}T} . \qquad (G)$$

Since Q is m non-trivial links from P but is self-linked, we have to show that $Q \in A_{m+k}(P)$ for each $k \in \mathbb{N}$. Since we are assuming that each element of K' commutes with each of $w'_{l+1}, \ldots, w'_n$, the left K-module isomorphism from $K[\Theta;\Delta]$ to $K'[\Theta]$, restricted to $U_0 = K[w_{l+1}, \ldots, w_n]$, is a ring isomorphism to the commutative subring $U'_0 = K'[w'_{l+1}, \ldots, w'_n]$ of T.

Now, in U'_0,

$$(D'_1 \cap U'_0)(D'_k \cap U'_0) = (D'_k \cap U'_0)(D'_1 \cap U'_0) \qquad (H)$$

for any $k \in \mathbb{N}$. On the other hand, we see from Theorem 4.1.4, that

$$\frac{P}{MT} = \frac{T}{MT}\left(\frac{P}{MT} \cap U_0\right) = \frac{T}{MT}\left(\frac{P}{MT} \cap \frac{U'_0 + MT}{MT}\right)$$

and hence

$$P = T(P \cap U_0') + MT = T(D_1' \cap U_0') + MT \ . \qquad (I)$$

We can also deduce from Theorem 4.1.4 that

$$\frac{S_k}{MT} = \left(\frac{S_k}{MT} \cap U_0\right) \frac{T}{MT} = \left(\frac{S_k}{MT} \cap \frac{U_0' + MT}{MT}\right) \frac{T}{MT}$$

and hence

$$S_k = (S_k \cap U_0')T + MT = (S_k \cap U_0')K'[\Theta] + MT \ .$$

Since $U_0'K' = K'U_0'$, $(S_k \cap U_0')K'[\Theta] \subseteq K'[\Theta]$ and so

$$D_k' = S_k \cap K'[\Theta] = (S_k \cap U_0')K'[\Theta] = (D_k' \cap U_0')K'[\Theta] \ . \qquad (J)$$

Combining (H), (I) and (J), we deduce that

$$\begin{aligned} PD_k' &= T(D_1' \cap U_0')(S_k \cap U_0')K'[\Theta] + MTD_k' \\ &= T(S_k \cap U_0')(D_1' \cap U_0')K'[\Theta] + MTD_k' \\ &\subseteq S_k P \end{aligned}$$

It follows from this and from (D) that

$$Qg_{f_1} \cdots g_{f_m} D_k' \dots D_2' \subseteq M^m PD_k' \dots D_2' \subseteq M^m D_k' \dots D_2' P \subseteq S_{m+k} \dots S_1$$

while $\Lambda := g_{f_1} \cdots g_{f_m} D_k' \cdots D_2' \subseteq S_{m+k} \dots S_2$. By Theorem 1.3.11, it is thus sufficient to show that there is no $u \in T - P$ with $\Lambda u \subseteq S_{m+k} \dots S_1$. We now apply (F) with $s = m + k$ and $t = 1$ and note that, since $\Lambda \subseteq M^m T$, we can assume, for a contradiction, that

$$\Lambda u \subseteq M^m TD_k' \dots D_1' + \cdots + M^{m+k-1}TD_1' + M^{m+k}T \ .$$

Now, since R is regular, Lemma 4.1.18 shows that $\Lambda \notin M^{m+1}T$. Of course, since $T = MT + K'[\Theta; \Delta]$, we can insist that $u \in K'[\Theta; \Delta]$ and then

$$\Lambda u = g_{f_1} \cdots g_{f_m} D_k' \dots D_2' u \in M^m TD_k' \dots D_1' \ .$$

We deduce that $D_k' \dots D_2' u \subseteq D_k' \dots D_1'$ whereby $u \in D_1' \subseteq P$.

This establishes part (iii) of the theorem. ∎

Remark 4.2.13. With the notation of (4.1.3), we note that M/M^2 is a (right) module over $U_0 = K[w_{l+1}, \ldots, w_n]$ where the action of the w_i is given by

$$(m + M^2).w_i := w_1 m - m w_1 + M^2$$

for each $m \in M$. In the notation of (4.1.5) and under the assumption that all eigenvectors for W on $(M/M^2)^\#$ can be chosen from M/M^2, the hypothesis of Theorem 4.2.12(iii), that $(M/M^2)_{R/M}$ is generated by the eigenvectors for W, is equivalent to requiring that the module $(M/M^2)_{U_0}$ be semisimple. ∎

We are now in a position to establish the main result of Chapter 4.

Corollary 4.2.14. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Fix $n \in \mathbb{N}$, let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of indeterminates and let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a set of commuting derivations on R. Suppose P is a prime ideal of $T = R[\Theta; \Delta]$ such that $M = P \cap R$.*

(i) If $P = MT$ then P is rep. rep. in T and is strongly rep. rep. if and only if M is strongly rep. rep. in R; this case is characterised in Corollary 2.1.3.

(ii) If R_M is a regular local ring, then P is rep. rep. in T.

(iii) Let $\overline{R_M}$ denote the completion at MR_M, put $K = R_M/MR_M$ and write $K^\#$ for an algebraic closure of K. Let $\widehat{K}$ be a finite algebraic extension of K such that, with the notation of (4.1.5), all the eigenvectors for W on $\frac{M\overline{R_M}}{M^2\overline{R_M}} \otimes_K K^\#$ can be chosen from $\frac{M\overline{R_M}}{M^2\overline{R_M}} \otimes_K \widehat{K}$. Suppose that $\left(\frac{M\overline{R_M}}{M^2\overline{R_M}} \otimes_K \widehat{K}\right)_{\widehat{K}}$ is generated by the eigenvectors for W. If R_M is a regular local ring, and if there exists an image K' of K in $\overline{R_M}$ such that, in the notation of (4.1.3), each element of K' commutes with each element of W' (or, equivalently, with each of $\{w'_{l+1}, \ldots, w'_n\}$), then P is strongly rep. rep. in T.

Proof. By Remark 4.1.1, it is sufficient to assume that R is a local ring with maximal ideal M and we do so. We note that part (i) is then just part (i) of Theorem 4.2.12. We consider parts (ii) and (iii).

Put $K = R/M$ and suppose R is regular. Then, in the notation of (4.2.3) and by Theorem 4.1.19(i), the completion $\overline{R}$ is regular. By Cohen's structure theorem (Theorem 4.1.19(ii)), $\overline{R} \cong K[[Y_1, \ldots, Y_t]]$ for indeterminates $\{Y_1, \ldots, Y_t\}$ and for some t. Take $\widehat{K}$ to be a suitably large finite algebraic extension of K such that all the eigenvectors on $\left(M\overline{R}/M^2\overline{R}\right)^{\#}$ can be chosen from $\left(M\overline{R}/M^2\overline{R}\right) \otimes_K \widehat{K}$. Now $\overline{R} \otimes_K \widehat{K} \cong \widehat{K}[[Y_1, \ldots, Y_t]]$ is also regular and, thus, in the notation of (4.2.5) and recalling Definition 4.1.15, $\widehat{\overline{R}}$ is an equicharacteristic complete regular local Noetherian $\mathbb{Q}$-algebra to which we can apply Theorem 4.2.12. Finally, Lemma 4.2.4 and Lemma 4.2.11 complete the proof. ∎

The following is an immediate consequence of Corollary 4.2.14(i) and (ii).

Corollary 4.2.15. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra and let T be an iterated differential operator ring of commuting derivations on R. If R is a regular ring then T is a rep. rep. ring.* ∎

§4.3 Some Examples

Example 4.3.1. As an easy illustration of the above results, we can now present an alternative formulation of Example 2.2.7. Thus, consider $R = \mathbb{C}[x]$, $T = R[\theta; x\frac{d}{dx}]$ and $P = xT + \theta T$. Then, $M = P \cap R = xR$ and clearly R_M is regular (Definition 4.1.17) so that, by Corollary 4.2.14(ii), P is rep. rep. in T. Indeed, since the eigenvector $x + M^2R_M$ generates $\left(MR_M/M^2R_M\right)_{\mathbb{C}}$, we see from Corollary 4.2.14(iii) that P is strongly rep. rep. in T. It may be instructive to show this by calculating, in the notation of (1.3.10), the bimodule $B_m(P)$ directly (since P is co-Artinian, $B_m(P) = I_m/J'_m$) and we do this in the same way as we derived the identity (G) in the proof of Theorem 4.2.12. Now, the only links between primes which contract to M are the trivial links and the links

$$xT + (\theta - \alpha - 1)T \rightsquigarrow xT + (\theta - \alpha)T$$

for all $\alpha \in \mathbb{C}$, corresponding to the eigenvector $x + M^2R_M$. It follows that

$$X_n(P) = \{xT + (\theta - \alpha)T : \alpha \in \{0, 1, 2, \ldots, (n-1)\}\} \ .$$

So, in the notation of (1.3.10), $S_n = xT + \theta \ldots (\theta - n + 1)T$ and therefore, by an induction argument,

$$B_{n-1}(P) = \frac{S_n \ldots S_2}{S_n \ldots S_1} = \frac{\sum_{i=0}^{n-1} x^{n-1-i}\theta^i(\theta-1)^i(\theta-2)^{i-1}\ldots(\theta-i)^1 T}{\sum_{i=0}^{n} x^{n-i}\theta^i(\theta-1)^{i-1}(\theta-2)^{i-2}\ldots(\theta-i+1)^1 T}$$

for each $n \geq 2$. Thus,

$$\gamma_\alpha := x^\alpha \theta^{n-1-\alpha}(\theta-1)^{n-1-\alpha}\ldots(\theta-n+1+\alpha)^1 + S_n \ldots S_1 \in B_{n-1}(P) - \{0\}$$

for each $\alpha \in \{0, 1, 2, \ldots, (n-1)\}$ while

$$(xT + (\theta - \alpha)T) . \gamma_\alpha = 0 \ .$$

Accordingly,

$$B_{n-1}(xT + (\theta-\alpha)T, P) \neq 0$$

and so, by Theorem 1.3.11,

$$xT + (\theta - \alpha)T \in A_n \ .$$

We have shown that $A_n(P) = X_n(P)$ for each $n \in N$. ∎

Example 4.3.2. *An example satisfying all the hypotheses of Corollary 4.2.14(iii), except that R_M is not regular, with P rep. rep. but not strongly rep. rep. in T.*

Thus, in the situation of Corollary 4.2.14(ii), regularity is not a necessary condition. Consider $R = \mathbb{C}[t^2, t^3]$, $T = R[\theta; t\frac{d}{dt}]$ and $M = t^2R + t^3R$. Then the prime $P = MT + \theta T$ is rep. rep. in T although R_M is not regular. To see this, we note that the links are the trivial links together with

$$MT + (\theta - \alpha - 2)T \rightsquigarrow MT + (\theta - \alpha)T \ ,$$

corresponding to the eigenvector $t^2 + M^2R_M \in MR_M/M^2R_M$, and

$$MT + (\theta - \alpha - 3)T \rightsquigarrow MT + (\theta - \alpha)T$$

corresponding to the eigenvector $t^3 + M^2R_M \in MR_M/M^2R_M$. It follows that,

$$X_n(P) = \{MT + (\theta - \alpha)T : \alpha \in \{0, 2, 3, 4, \ldots, 3(n-1)\}\}$$

for each $n \in \mathbb{N}$ and so

$$\text{r.cl.}(P) = \{MT + (\theta - \alpha)T : \alpha \in \{0, 2, 3, 4, \ldots\}\} .$$

On the other hand, it can be seen that $\frac{t^{2n}T + M^{n+1}T}{M^{n+1}T}$ and $\frac{t^{2n+1}T + M^{n+1}T}{M^{n+1}T}$ are ideal links from $MT + (\theta - 2n)T$ to P and from $MT + (\theta - (2n+1))\,T$ to P respectively. (Compare these bimodules with the bimodule G_m defined at (E) in the proof of Theorem 4.2.12.) Theorem 1.3.7 now shows that P is rep. rep..

To see that P is not strongly rep. rep. in T, we note, from the description of $X_n(P)$ above, that

$$S_n = MT + \theta(\theta - 2)(\theta - 3)(\theta - 4) \ldots (\theta - 3(n-1))$$

and then, since the eigenvectors generate MR_M/M^2R_M, a similar argument to that for Example 4.3.1 or for (G) in Theorem 4.2.12 shows that the bimodule $B_{n-1}(P)$ can be written in the form

$$\frac{\sum_{i=0}^{n-1} M^{n-1-i}\theta^i(\theta-2)^i(\theta-3)^i \prod_{j=2}^{i}(\theta-3j+2)^{i+1-j}(\theta-3j+1)^{i+1-j}(\theta-3j)^{i+1-j}T}{\sum_{i=0}^{n} M^{n-i}\theta^i(\theta-2)^{i-1}(\theta-3)^{i-1} \prod_{j=2}^{i-1}(\theta-3j+2)^{i-j}(\theta-3j+1)^{i-j}(\theta-3j)^{i-j}T}$$

for, since P is co-Artinian in T, $B_{n-1}(P) = \frac{S_n \ldots S_2}{S_n \ldots S_1}$. Now, for each $m \in \mathbb{N}$,

$$\frac{M^m}{M^{m+1}} = \frac{t^{2m}R + t^{2m+1}R}{t^{2m+2}R + t^{2m+3}R} .$$

It follows that $S_n \ldots S_2$ is generated by the set

$$\{t^s\theta^i(\theta-2)^i(\theta-3)^i \prod_{j=2}^{i}(\theta-3j+2)^{i+1-j}(\theta-3j+1)^{i+1-j}(\theta-3j)^{i+1-j} :$$
$$0 \leq i \leq n-1 \text{ and } s = 2(n-1-i) \text{ or } 2(n-1-i)+1 \text{ but not } 1\}$$

and that none of these elements is contained in $S_n \ldots S_1$. Finally, we see that

$$(MT+(\theta-s)).t^s\theta^i(\theta-2)^i(\theta-3)^i \prod_{j=2}^{i}(\theta-3j+2)^{i+1-j}(\theta-3j+1)^{i+1-j}(\theta-3j)^{i+1-j} \subseteq S_n \ldots S_1$$

for each $0 \leq i \leq n-1$, whenever $s = 2(n-1-i)$ or $2(n-1-i)+1$ but not 1, and so

$$A_n(P) = \{MT + (\theta - \alpha)T : \alpha \in \{0, 2, 3, 4, \ldots, 2n-1\}\} .$$

Clearly, therefore, $A_n(P) \subsetneq X_n(P)$ except when $n = 1$ or 2. For instance, we can find a chain of links

$$MT + (\theta - 6)T \rightsquigarrow MT + (\theta - 3)T \rightsquigarrow MT + \theta T ,$$

each of which corresponds to the eigenvector $t^3 + M^2R_M$, and a chain of links

$$MT + (\theta - 6)T \rightsquigarrow MT + (\theta - 4)T \rightsquigarrow MT + (\theta - 2)T \rightsquigarrow MT + \theta T ,$$

each of which corresponds to the eigenvector $t^2 + M^2R_M$. Then, $MT + (\theta - 6)T \notin A_3(P)$ although $MT + (\theta - 6)T \in A_4(P)$, basically because $(t^3)^2 \in M^3$, while $(t^2)^3 \notin M^4$. ∎

Example 4.3.3. Furthermore, in Corollary 4.2.14, it is easy to see that the module $\left(MR_M/M^2R_M\right)_{R/M}$ need not be generated by the eigenvectors when P is strongly rep. rep., even for $P \neq MT$. For, we take $R = \mathbb{C}[x, y]$, $\delta(x) = y$, $\delta(y) = 0$, $T = R[\theta]$ and $M = xR + yR$. Then, up to multiplication by scalars, there is only one independent eigenvector, $y + M^2R_M$, which of course does not generate $\left(MR_M/M^2R_M\right)_{\mathbb{C}}$, and the corresponding link is the trivial link. Thus, if $P = xT + yT + \theta T$, then $X_n(P) = \{P\}$ for all $n \in \mathbb{N}$ and, since P is co-Artinian, we see that

$$B_m(P, P) = B_m(P) = I_m/J'_m = P^{m-1}/P^m$$

which is obviously non-zero. So P is strongly rep. rep. by Theorem 1.3.11. For a less trivial example, we could have taken $\delta(y) = y$ and we would again have found P to be strongly rep. rep.. However, we omit this calculation. ∎

Remark 4.3.4. In fact, we do not have any example of a prime satisfying the hypotheses of Corollary 4.2.14(ii) which is not strongly rep. rep. or of a prime which fails to be rep. rep. when R_M is a domain. Certainly, if R_M is not a domain, it is easy to obtain examples which are not rep. rep.. Indeed we have already seen one in (2.1.5), (2.2.9) and (2.3.16) which, in our present notation, we write as $R = \frac{\mathbb{C}[x]}{<x^2>}$, $T = R[\theta; x\frac{d}{dx}]$ and $M = xR$. Then, as well as the trivial links, we have the links

$$xT + (\theta - \alpha - 1)T \rightsquigarrow xT + (\theta - \alpha)T$$

for all $\alpha \in \mathbb{C}$, corresponding to the eigenvector $x+M^2R_M$. However, for any prime P of T, Fund(P) is finite by [Mü2], T being a P.I. ring. (The identity $(ab-ba)^2 = 0$ holds for each a and $b \in T$.) Consequently, no prime of T, other than MT, is rep. rep.. For instance, taking $P = xT + \theta T$, it can be seen that, for each $n \in \mathbb{N}$,

$$X_n(P) = \{MT + (\theta - \alpha)T : \alpha \in \{0, 1, 2, \dots, (n-1)\}\}$$

while

$$A_n(P) = \begin{cases} \{P\} & \text{if } n = 1 \\ \{P, MT + (\theta - 1)T\} & \text{if } n \geq 2 \ . \end{cases}$$

■

Example 4.3.5. As a more interesting example than (4.3.4), we can take $R = \frac{\mathbb{C}[x,y]}{<xy>}$, $\delta(x) = 2x$, $\delta(y) = 3y$, $\delta(\alpha) = 0$ for all $\alpha \in \mathbb{C}$, $T = R[\theta; \delta]$ and $M = xR+yR$. Then $P = xT + yT + \theta T$ is not rep. rep. in T, although in this case, Fund(P) is infinite. The non-trivial links are of the form

$$MT + (\theta - \alpha - 2)T \rightsquigarrow MT + (\theta - \alpha)T$$

$$MT + (\theta - \alpha - 3)T \rightsquigarrow MT + (\theta - \alpha)T$$

corresponding respectively to the eigenvectors $x+M^2R_M$ and $y+M^2R_M$. It is easy to see that the primes $MT + (\theta - 2n)T$ and $MT + (\theta - 3n)T$ are all fundamental, the bimodules $\frac{x^nT+M^{n+1}T}{M^{n+1}T}$ and $\frac{y^nT+M^{n+1}T}{M^{n+1}T}$ respectively being ideal links to P. Indeed, while we omit the details of the calculation, which are similar to those of Example 4.3.2, it can be shown that, for each $n \in \mathbb{N}$,

$$X_n(P) = \{MT + (\theta - \alpha) : \alpha \in \{0, 2, 3, 4, \dots, 3(n-1)\}\}$$

and

$$A_n(P) = \{MT + (\theta - \alpha) : \alpha = 2m \text{ or } 3m, \text{ where } 0 \leq m \leq n-1\} \ .$$

As in Example 4.3.2, $A_n(P) \subsetneq X_n(P)$ except when $n = 1$ or 2. However, unlike (4.3.2), P is not even rep. rep. For example, $MT + (\theta - 5)T$, which arises by connecting together a link corresponding to each eigenvector, is not fundamental, essentially because $xy = 0$ in R.

■

§4.4 Notes

Iterated skew polynomial rings over a field were introduced by Noether and Schiedler in 1920. In particular, in [N&S, Satz III], they proved such rings are Noetherian, generalizing Hilbert's original results for ordinary polynomial rings, [Ht, Theorems I and II]. While Noether and Schiedler were primarily interested in the "unmixed" cases where either all derivations are zero or all automorphisms are the identity, the case of a general skew-polynomial ring in one variable over a division ring was studied in detail by Ore in [O].

That the right Noetherian condition on any ring R is inherited by iterated skew polynomial rings over R, can be found in [G&W, Theorem 1.12].

The material of §4.1 has all appeared elsewhere. We note the following specific references.

The second layer condition for iterated differential operator rings of commuting derivations over a commutative Noetherian ring, was established in [Be, Theorem 7.3] although whether the strong second layer condition holds appears to be an open question. (See also the comments in §5.4.)

The terminology and notation of (4.1.1)–(4.1.8) is taken from [G2] as is the classification of the second layer links in a differential operator ring.

Lemma 4.1.2 was first proved for commutative Noetherian $\mathbb{Q}$-algebras in [Sb, Theorem 1]. For noncommutative $\mathbb{Q}$-algebras, the result for minimal completely prime ideals follows from [D1, Lemma 6.1] and this was generalized to all minimal primes in [Ga2, Lemma 3.4]. The version which we have used is taken from [G&W, Lemma 2.20].

Theorem 4.1.4, quoted from [G2, Theorem 2.9], is, as noted, a special case of [P2, Theorem 4.3].

Lemma 4.1.6 is part of [Z&S, Chapter VII, §11, Corollary p226].

Theorem 4.1.7 combines [G2, Theorem 5.8] and [G2, Theorem 5.11], while Theorem 4.1.8 is taken from [G2, Theorem 6.1]. Corresponding results to these were first proved in [Si] for the case of a single derivation.

The theory of commutative local rings began with the 1938 paper of Krull [Kr], where the question of the structure of complete local rings was raised. This problem was solved in 1946 by Cohen [C]. However, our main sources in (4.1.9)–(4.1.19) are [N1], [Na] and [Z&S].

Theorem 4.1.10 is a standard result. Our specific reference is [N1, §5.5 Theorem 4].

Theorem 4.1.11, can be found in this form in [Na, Corollary 17.6] but is essentially due to [C].

For Theorem 4.1.12, we have combined [N1, §5.5 Proposition 9 and Theorem 6]; Corollary 4.1.13 is a well known consequence.

Theorems 4.1.14 and 4.1.16 are quoted from [Z&S, Chapter VIII, §6, Theorem 15(c) and §12, Theorem 27] respectively. The latter is originally due to Cohen.

Our reference for Lemma 4.1.18 is [N1, §4.6, Lemma 3].

Part (i) of Theorem 4.1.19 is taken from [N1, §5.6, Corollary to Theorem 8] and part (ii) (Cohen's Structure Theorem), from [Z&S, Chapter VIII, §12, Corollary to Theorem 27].

That a derivation on a commutative local ring extends uniquely to the completion (Lemma 4.2.2) is shown on [G,L&R, p16] but without explicitly defining the extension.

Lemma 4.2.4 is a new result, however part (i) is based on the proof of [Ma, Theorem 3.6] which established a similar result for commutative rings.

Lemma 4.2.6 appears to be a new result. That derivations on a field extend uniquely to derivations on a separable algebraic extension field, is proved in [Z&S, Chapter II, §2, Corollary 2' to Theorem 39] and the analogous statement of our result for L-derivations is also well known (see for instance [Mu, §1]).

Lemma 4.2.9 is proved in [Lt, Corollary 2.4].

Lemma 4.2.10 is perhaps a well known result, however we do not have a specific reference for it.

Lemma 4.2.11, Theorem 4.2.12, Corollary 4.2.14 and Corollary 4.1.15 are all new results.

Chapter 5 : Ore Extensions

Having studied repleteness in differential operator rings, it is natural to ask what happens in Ore extensions of the form

$$S = R[\theta_1, \ldots, \theta_n; \sigma_1, \ldots, \sigma_n] = R[\Theta; \Sigma],$$

where R is a commutative Noetherian ring and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a set of commuting automorphisms on R (see Definitions and Notation 4.0.0). (Such rings satisfy the second layer condition by a similar argument to that of [Be2, Theorem 7.3].) We might expect to obtain similar results to those of §4.2. To date, unfortunately, the second layer links in S have only been calculated for the case of a single automorphism (see [Po]) and we now examine repleteness in this situation.

Our main result in the case when $S = R[\theta; \sigma]$ is that a prime P is rep. rep. in S when $R_{P\cap R}$ is a regular semilocal ring (Theorem 5.2.4). Again, for some primes we obtain more precise results (Theorems 5.2.1 and 5.2.3). In particular, P is always rep. rep. if it contains θ or if $P = (P \cap R)S$ and in these cases we give conditions for strong repleteness.

§5.1 Prime Links in Ore Extensions and Preliminaries

Notation 5.1.1. Let R be a commutative Noetherian ring, σ an automorphism on R and form the Ore extension $S = R[\theta; \sigma]$. Let P be a prime ideal of S and put $A = P \cap R$ which, in general, is not a prime ideal of R. Following [Po], we consider the three cases:

(1) $\theta \in P$;

(2) $\theta \notin P$ and $P = AS$;

(3) $\theta \notin P$ and $P \supsetneq AS$.

Type (2) primes are known as *lower prime ideals* and type (3) primes as *upper prime ideals*. ∎

The following classification of type (1) primes is well known.

Lemma 5.1.2. *Adopt the notation of (5.1.1) and assume that P is a type (1) prime. Then A is a prime ideal of R and $P = A + \theta S$. Conversely, if I is a prime ideal of R then $I + \theta S$ is a type (1) prime of S.*

Proof. Clearly, when $\theta \in P$, $P = (P \cap R) + \theta S$. Now suppose that I is an ideal of R. It is easy to check that the map

$$r_0 + r_1\theta + \cdots r_j\theta^j \mapsto r_0 + I,$$

for $r_0, \ldots, r_j \in R$, is a ring epimorphism from S to R/I with kernel $\theta S + I$. Thus $\theta S + I$ is a prime ideal of S if and only if I is a prime ideal of R. ∎

When P is an upper or a lower prime of S, $P \cap R$ need not be prime; however, the next result, taken from [I, Theorems 4.1 and 4.2], shows that it is at least semiprime.

Theorem 5.1.3. *Adopt the notation of (5.1.1) and assume that P is either a lower or an upper prime ideal of S. Then $A = \bigcap_{i=1}^{n} \sigma^i(J)$ for some prime ideal $J = \sigma^n(J)$ of R and some $n \in \mathbb{N}$. Conversely, if I is an ideal of R such that $I = \bigcap_{i=1}^{m} \sigma^i(K)$ for some prime ideal $K = \sigma^m(K)$ of R and some $m \in \mathbb{N}$, then IS is a lower prime ideal of S.* ∎

Definitions 5.1.4. (i) A semiprime ideal of the form of J or I arising in the previous theorem is called *σ-cyclic.*

(ii) By a *σ-invariant* subset of R or S, we mean a subset I such that $\sigma(I) \subseteq I$. (We extend σ to an automorphism on S by defining $\sigma(\theta) = \theta$.) For such a subset, $I \subseteq \sigma^{-1}(I) \subseteq \sigma^{-2}(I) \subseteq \cdots$, and so, if I is a right or left ideal of R or S, the Noetherian condition shows us that $I = \sigma(I)$. Clearly, a σ-cyclic semiprime ideal of R is σ-invariant. We note, moreover that, if P is an upper or lower prime of S, then P is σ-invariant. For in this case, since $P \supseteq \theta P = \sigma(P)\theta$, we deduce that $\sigma(P) \subseteq P$ by the primeness of P.

(iii) A σ-invariant ideal I of R is called *σ-prime* provided, for every two σ-invariant ideals B and C of R such that $BC \subseteq I$, either $B \subseteq I$ or $C \subseteq I$. It is shown in [I, Corollary 3.4] that, by the Noetherian hypothesis on R, σ-cyclic and σ-prime ideals are the same. ∎

The following result will frequently be useful. The proof for the first part is taken from [G&M, Lemma 1.4].

Lemma 5.1.5. *Adopt the notation of (5.1.1) and let X be a σ-invariant Ore set in R. Then X is an Ore set in S. In particular, when P is an upper or lower prime of S, $\mathcal{C}_R(P \cap R)$ is an Ore set in S.*

Proof. Let $c \in X$ and let $f = f_0 + f_1\theta + \cdots f_j\theta^j \in S$. Since X is an Ore set in R, we can find $d_0 \in X$ and $b_0 \in R$ such that $f_0 d_0 = cb_0$. Thus

$$fd_0 = cb_0 + f_1\sigma(d_0)\theta + \cdots + f_j\sigma^j(d_0)\theta^j.$$

Again, we can find $d_1 \in X$ and $b_1 \in R$ such that $f_1\sigma(d_0)d_1 = cb_1$ and so

$$fd_0\sigma^{-1}(d_1) = cb_0\sigma^{-1}(d_1) + cb_1\theta + \cdots + f_1\sigma^j(d_0)\sigma^{j-1}(d_1)\theta^j.$$

Continuing in this way, we can find $d_0, \ldots, d_j \in X$ such that

$$fd_0\sigma^{-1}(d_1)\ldots\sigma^{-j}(d_j) = cg$$

for some $g \in S$. Since X is σ-invariant and multiplicatively closed,

$$d_0\sigma^{-1}(d_1)\ldots\sigma^{-j}(d_j) \in X$$

and this establishes the right Ore condition. The left Ore condition holds similarly.

For an upper or lower prime ideal P of S, we have seen in (5.1.1) that $P \cap R$ is a semiprime ideal of the commutative ring R and thus $\mathcal{C}_R(P \cap R)$ is an Ore set in R. We have also seen that $P \cap R$ is σ-invariant so that

$$\sigma(\mathcal{C}(P \cap R)) \subseteq \mathcal{C}(\sigma(P \cap R)) \subseteq \mathcal{C}(P \cap R).$$

Thus $\mathcal{C}(P \cap R)$ is a σ-invariant Ore set in R to which we can apply the first part of the lemma. ∎

Before discussing the link graph of an Ore extension, we require some further preliminary terminology which will give us a description of the upper prime ideals. The comments of (5.1.6) to (5.1.8) are contained in [I] and [Po].

Remarks 5.1.6. Given a σ-cyclic ideal, A, of R, we have seen by Theorem 5.1.3 that the ideal AS of S is always a lower prime ideal of S. As in this theorem, we can write $A = \bigcap_{i=1}^{n} \sigma^i(J)$ for some prime ideal $J = \sigma^n(J)$ of R and some $n \in \mathbb{N}$. Furthermore, we write $\overline{R}$ for the ring R/A and $\overline{\sigma}$ for the automorphism induced on $\overline{R}$ by σ. By [I, Theorem 4.3], there can be upper prime ideals of S lying over A only in the case when $\overline{\sigma}$ has finite order. When this condition holds, we denote the order by m and put $t = \theta^m$. ∎

As we now see, the finiteness of the order of $\overline{\sigma}$ is, in fact, a sufficient condition for upper primes to exist. The result is taken from [I, Theorem 4.4].

Theorem 5.1.7. *Adopt the notation of (5.1.6) and let $\overline{R}^{\overline{\sigma}}$ be the fixed subring of $\overline{R}$ under the action of $\overline{\sigma}$. Then, $\overline{R}^{\overline{\sigma}}[t]$ is the centre of $\overline{R}[\theta;\overline{\sigma}]$. Furthermore, the upper primes of $\overline{R}[\theta;\overline{\sigma}]$ which intersect $\overline{R}$ in $\{0\}$ are in one to one correspondence with the non-zero primes of $\overline{R}^{\overline{\sigma}}[t]$ which do not contain t and intersect $\overline{R}^{\overline{\sigma}}$ in $\{0\}$.* ∎

Poole's description of the links between upper prime ideals uses this correspondence after first localizing at $\mathcal{C}_R(A)$, which of course, is an Ore set in S by Lemma 5.1.5. We outline the required details, extracted from Irving's proof of Theorem 5.1.7, as follows.

Remarks 5.1.8. (i) Assume we are in the situation of (5.1.6) and (5.1.7). Let $\mathcal{Q}$ be the classical quotient ring of $\overline{R}$ and extend $\overline{\sigma}$ to $\mathcal{Q}$ in the natural way. By the Chinese Remainder Theorem,

$$\mathcal{Q} \cong \oplus_{i=1}^{n} \left(\frac{\mathcal{Q}}{\overline{\sigma}^i(J/A)\mathcal{Q}} \right) \cong \oplus_{i=1}^{n} K_i$$

a direct sum of the quotient fields K_i of $R/\sigma^i(J)$. We identify these isomorphisms where convenient. Furthermore, since each natural epimorphism

$$\begin{aligned} \theta_i \ : \mathcal{Q} &\to \frac{\mathcal{Q}}{\overline{\sigma}^{i+1}(J/A)\mathcal{Q}} \\ q &\mapsto \overline{\sigma}(q) + \overline{\sigma}^{i+1}(J/A)\mathcal{Q} \end{aligned}$$

has kernel $\overline{\sigma}^i(J/A)\mathcal{Q}$, we see that the K_i are cyclically permuted by $\overline{\sigma}$. Indeed,

$$\begin{aligned} &\overline{\sigma}\big(q_1 + \overline{\sigma}(J/A)\mathcal{Q}, \ldots, q_n + \overline{\sigma}^n(J/A)\mathcal{Q}\big) \\ &\qquad = \big(\overline{\sigma}(q_n) + \overline{\sigma}(J/A)\mathcal{Q}, \ldots, \overline{\sigma}(q_{n-1}) + \overline{\sigma}^n(J/A)\mathcal{Q}\big) \qquad (A) \end{aligned}$$

for $q_1, \ldots, q_n \in \mathcal{Q}$.

(ii) Denote by k the fixed subring, $\mathcal{Q}^{\overline{\sigma}}$, of $\mathcal{Q}$ under the action of $\overline{\sigma}$. We observe from (A) that any element of k can be written in the form

$$\left(\overline{\sigma}(q) + \overline{\sigma}(J/A)\mathcal{Q}, \ldots, \overline{\sigma}^n(q) + \overline{\sigma}^n(J/A)\mathcal{Q}\right)$$

for some $q \in \mathcal{Q}$ such that $\overline{\sigma}^n(q) = q$. Thus, regarding $\mathcal{Q}$ as a direct sum of copies of K_n, k is embedded diagonally in $\mathcal{Q}$ and is isomorphic to the fixed subfield $K_n^{\overline{\sigma}^n}$ of K_n under the action of $\overline{\sigma}$. Furthermore, since the order of $\overline{\sigma}$ on $\overline{R}$ is finite (and equals m), the order of $\overline{\sigma}^n$ on K_n is also finite, and so, by [McC, Chapter 2, Theorem 3], the degree of the extension $k \subseteq K_n$ is finite. Thus, K_n and hence $\mathcal{Q}$ are finite dimensional vector spaces over k.

(iii) By Theorem 5.1.7, the upper prime ideals of S lying over A are in one to one correspondence with the non-zero prime ideals of $k[t]$ which do not contain t; that is, with the non-trivial monic irreducible polynomials of $k[t]$ other than t. We note that $k[t]$ is the centre of the ring $\mathcal{Q}[\theta; \overline{\sigma}]$ which can be identified with the right Ore localization of $\overline{R}[\theta; \overline{\sigma}] \cong S/AS$ with respect to the set of regular elements of $\overline{R}$. Each upper prime ideal of this localization is centrally generated by a non-trivial irreducible monic polynomial $\overline{p}[t]$ of $k[t]$ other than t. Following [Po], we denote the corresponding upper prime ideal of S by $[A, \overline{p}(t)]$. ∎

We will now consider the right clique of each type of prime in turn. Our first result, for type (1) primes, can also be found in [Po, Theorem 6] and its proof.

Lemma 5.1.9. *Adopt the notation of (5.1.1), let P and Q be prime ideals of S and suppose that $Q \rightsquigarrow P$. Then, Q is a type (1) prime if and only if P is a type (1) prime; in this case, either $Q = P$ or $Q = \sigma(P)$.*

Proof. That $\theta \in Q$ if and only if $\theta \in P$ is proved in Lemma 2.3.5. The second part of the result follows from Lemmas 2.3.6(i) and 2.3.7. ∎

The proof of the first part of the next Lemma is taken from [Po, Theorem 6].

Lemma 5.1.10. *Adopt the notation of (5.1.1) and suppose that P is a type (1) prime of S. Then $\sigma(P) \rightsquigarrow P$ and, furthermore, $P \rightsquigarrow P$ if and only if* $\mathrm{ann}_{(\sigma(A) \cup \sigma^{-1}(A))}\left(\frac{A}{A^2}\right) \subseteq A$.

Proof. We first recall that, by Lemma 1.2.6, there is a link $Q \rightsquigarrow P$ between prime ideals Q and P of S if and only if $\frac{Q \cap P}{QP}$ is faithful as a left S/Q-module and as a right S/P-module.

We let $\sigma(P) \cap R = B$ and observe that

$$\sigma(P) \cap P = (A \cap B) + \theta S$$

while

$$\begin{aligned}\sigma(P).P = (B + \theta S).(A + \theta S) =& BA + \theta\sigma^{-1}(B)S + \theta AS + \theta^2 S\\ =& BA + \theta AS + \theta^2 S.\end{aligned}$$

Now, let $s = \sum_{i=0}^{j} r_i\theta^i$ and suppose $s.(\sigma(P) \cap P) \subseteq \sigma(P).P$. In particular, $s\theta \in \sigma(P).P$, whence

$$\theta\sigma^{-1}(r_0) = r_0\theta \in \theta A.$$

Thus, $\sigma^{-1}(r_0) \in A$, so $r_0 \in \sigma(A) \subseteq \sigma(P)$ and hence $s \in \sigma(P)$.

We have established that $\frac{\sigma(P) \cap P}{\sigma(P).P}$ is faithful as a left $S/\sigma(P)$-module and similarly it is faithful as a right S/P-module. Consequently, $\sigma(P) \rightsquigarrow P$.

For the second part of the lemma, we note that

$$P^2 = (A + \theta S).(A + \theta S) = A^2 + \theta\left(A + \sigma^{-1}(A)\right)S + \theta^2 S.$$

First suppose that $s \in \sigma(A) - A$ and $sA \subseteq A^2$. Then, certainly $s \in S - P$ while

$$sP = s(A + \theta S) = sA + \theta\sigma^{-1}(s)S \subseteq A^2 + \theta AS \subseteq P^2$$

from which it follows that ${}_{S/P}(P/P^2)$ is unfaithful. Similarly, if we can find some $t \in \sigma^{-1}(A) - A$ such that $At \subseteq A^2$, then $(P/P^2)_{S/P}$ is unfaithful. In either case, $P \not\rightsquigarrow P$. This establishes the forward implication.

Now we suppose that $P \not\rightsquigarrow P$. First assume ${}_{S/P}(P/P^2)$ is unfaithful. Then we can find $s \in S - P$ such that $sP \subseteq P^2$. Since $\theta \in P$, we can assume that $s \in R$. In particular, $s\theta = \theta\sigma^{-1}(s) \in \theta\left(A + \sigma^{-1}(A)\right)$, whence $s \in A + \sigma(A)$. Again since $A \subseteq P$, we can assume that $s \in \sigma(A) - A$. Now, by the choice of s, $sA \subseteq P^2 \cap R$; so we see that $sA \subseteq A^2$. By a similar argument, if $(P/P^2)_{R/P}$ is unfaithful, we can find some $t \in \sigma^{-1}(A) - A$ such that $At \subseteq A^2$. This establishes the reverse implication. ■

Remark 5.1.11. In particular, it follows by this last result that, for type (1) primes, if $A \rightsquigarrow A$ then $P \rightsquigarrow P$. This is of course immediate from the definition of links once we notice that $P/\theta S \cong A$. (See also Lemma 2.3.7.) However, the prime $x\mathbb{C}[x]$ of $\mathbb{C}[x]$ is a type (1) prime which is linked to itself while the contraction 0 to the coefficient ring $\mathbb{C}$ is not. ■

The next result follows immediately from the previous two lemmas. The last part of the Theorem is the conclusion of [Po, Theorem 6].

Theorem 5.1.12. *Adopt the notation of (5.1.1) and suppose that P is a type (1) prime of S. For each $n \geq 0$,*

$$X_n(P) = \left\{\sigma^i(P) : 0 \leq i \leq n-1\right\},$$

provided $\operatorname{ann}_{(\sigma(A)\cup\sigma^{-1}(A))}\left(\frac{A}{A^2}\right) \subseteq A$, *while*

$$X_n(P) = \left\{\sigma^{n-1}(P)\right\}$$

otherwise. In either case,

$$\text{r.cl.}(P) = \left\{\sigma^i(P) : i \geq 0\right\} .$$ ■

As we might expect, the situation for lower prime ideals is essentially the same as for commutative rings. The result is proved in [Po, Theorem 9].

Lemma 5.1.13. *If, in the notation of (5.1.1), P is a lower prime ideal of S, then* r.cl.$(P) = \{P\}$. ■

Remark 5.1.14. As we shall see in the proof of Theorem 5.2.3, if P is a lower prime ideal of S and if J is a prime ideal of R minimal over the σ-cyclic semiprime ideal $P \cap R$ of R, then $P \rightsquigarrow P$ if and only if $J \rightsquigarrow J$. By Lemma 5.1.13, then, $X_n(P) = \{P\}$ when $n = 1$ or $J \rightsquigarrow J$ and is empty otherwise. ■

We now turn our attention to the right clique of an upper prime ideal of S. The first result which we note is taken from [Po, Lemma 10]. We include the proof for the sake of completeness.

Lemma 5.1.15. *Adopt the notation of (5.1.1), let P and Q be prime ideals of S and suppose that $Q \rightsquigarrow P$. Then, Q is an upper prime if and only if P is an upper prime; in this case, $Q \cap R = P \cap R$.*

Proof. That Q is an upper prime if and only if P is, follows from Lemmas 5.1.9 and 5.1.13. We now assume that Q and P are upper prime ideals with $Q \rightsquigarrow P$. By Lemma 5.1.5, $\mathcal{C}_R(P \cap R)$ is an Ore set in S and so, by Theorem 1.2.2, $\mathcal{C}_R(P \cap R) \subseteq \mathcal{C}_R(Q \cap R)$. Similarly, $\mathcal{C}_R(Q \cap R) \subseteq \mathcal{C}_R(P \cap R)$ and thus $\mathcal{C}_R(Q \cap R) = \mathcal{C}_R(P \cap R)$. It follows easily that $Q \cap R = P \cap R$. ∎

We next show that any upper prime is linked to itself. The argument is taken from [Po, p440].

Lemma 5.1.16. *With the notation of (5.1.1), let P be an upper prime ideal of S. Then $P \rightsquigarrow P$.*

Proof. We put $A = P \cap R$ and recall that, by Lemma 5.1.5, $\mathcal{C} = \mathcal{C}_R(A)$ is an Ore set of S. So we can replace S with $S\mathcal{C}^{-1}$ and P with $P\mathcal{C}^{-1}$ (see Remarks 2.3.1); we assume that this has been done. In this case, and putting $\overline{R} = R/A$ in the notation of (5.1.6) and (5.1.8), $\overline{R} = \mathcal{Q}(\overline{R})$ and $k[t]$ is the centre of $\overline{R}[\theta; \overline{\sigma}]$ which we identify with S/AS. We write $\overline{p}(t)$ for the central irreducible polynomial of S/AS such that $P = [A, \overline{p}(t)]$. Thus,

$$P = AS + p(t)S$$

for $p(t) \in S$ such that $p(t) + AS = \overline{p}(t)$. Evidently, $p(t)$ is central in S modulo AS and it is easy to see that

$$P^2 = A^2S + p(t)AS + ASp(t) + p^2(t)S \ .$$

Suppose that $Pd \subseteq P^2$ for some $d \in S$. Then, writing $p = p(t)$,

$$pd = a_1 + pa_2 + a_3p + p^2s$$

for some $a_1 \in A^2S$, a_2, $a_3 \in AS$ and $s \in S$. Thus,

$$pS(d - ps) \subseteq (AS + Sp)(d - ps) = AS(d - ps) + S(a_1 + pa_2 + a_3p) \in AS$$

and, since AS is a prime ideal of S,

$$d - ps \in AS \subseteq P .$$

Thus, $d \in P$ and so we have shown that $(P/P^2)_{S/P}$ is faithful. Similarly, P/P^2 is faithful and it follows from Lemma 1.2.6(i) that $P \rightsquigarrow P$. ∎

Notation 5.1.17. Adopt the notation of (5.1.6) and (5.1.8), let $P = [A, \overline{p}(t)]$ be an upper prime ideal of S. Assume that we have localized S at $\mathcal{C}_R(A)$. Now, if $A = 0$, then P is centrally generated by $p(t)$ and so r.cl.$(P) = \{P\}$. In this case P is trivially rep. rep. and, in fact, since no power of P is zero, it is easy to see from Theorem 1.3.11 that P is strongly rep. rep. in S.

So, assume that $A \neq 0$. Let $\widehat{A} = A/A^2$, which is a finitely generated right R-module and, since $\overline{\sigma}$ has finite order here, a finite dimensional right vector space over k. Let $\{\widehat{a}_1, \ldots, \widehat{a}_n\}$ be a basis for $\widehat{A}$ over k and we note that this induces a basis $\widehat{\mathcal{A}} = \left\{\widehat{a_j x^i} : 1 \leq j \leq n, 0 \leq i \leq m-1\right\}$ of the right $k[t]$-module $\widehat{AS} := AS/A^2S$. Let $\widehat{\sigma}$ be the invertible k-linear operator on $\widehat{A}$ induced by the action of σ on A. For each i and $j \in \{1, \ldots, n\}$, let $u_{i,j} \in k$ such that

$$(\widehat{a}_j)^{\widehat{\sigma}} = \sum_{i=1}^{n} \widehat{a}_i u_{i,j}$$

and let U be the matrix $(u_{i,j})$.

Finally, let $\overline{k}$ be the algebraic closure of k and, for each $\alpha \in \overline{k}$, let $\mu_k(\alpha)$ be the minimal polynomial for α over k. ∎

The next result is taken from [Po, Theorem 13], the main result of that paper. Together with Theorem 5.1.12, Lemma 5.1.13 and Lemma 5.1.16, it completes the description of the link graph of S.

Theorem 5.1.18. *Adopt the notation of (5.1.1), (5.1.6), (5.1.8) and (5.1.17). Assume that A is a non-zero σ-cyclic ideal of R. Let $Q = [A, \overline{q}(t)]$ and $P = [A, \overline{p}(t)]$ be distinct upper prime ideals of S and let $\overline{q}(t) = \mu_k(\alpha)$ for some $\alpha \in \overline{k}$. Then $Q \rightsquigarrow P$ if and only if $\overline{p}(t) = \mu_k(\alpha/\lambda^m)$ for some non-zero eigenvalue λ of U.* ∎

§5.2 The Repleteness of Ore Extensions

We recall from (5.1.1) that, for a type (1) prime P, $A = P \cap R$ is a prime ideal of R. The next result shows that strong repleteness of the contraction carries over to P.

Theorem 5.2.1. *With the notation of (5.1.1), let P be a type (1) prime of S. Then P is rep. rep. in S and, furthermore, if A is strongly rep. rep. in R, then P is strongly rep. rep. in S.*

Proof. Since θ is a regular normal element of S and $S/\theta S$ is commutative, it follows from Theorem 2.3.14(ii) that P is rep. rep. in S. Now, from the proof of Theorem 2.3.14(ii), we see that, for strong representational repleteness to be preserved, it suffices to show that, for each $n \in \mathbb{N}$ and all $Q \in A_n(P)$,

$$\sigma(Q) \in A_{n+1}(P) \ . \tag{A}$$

By Theorem 5.1.12, r.cl.(P) is locally finite, so, by applying Theorem 1.3.11 we assume that $B_{n-1}(Q, P) \neq 0$. So, in the notation of (1.3.10), suppose that $b \in S_{n-1} \dots S_1 \cap S_n \dots S_2$, that there is no element c of S, regular modulo P, such that $bc \in S_n \dots S_1$ but that $Qb \subseteq S_n \dots S_1$. Now, since $\theta \in P$, Lemma 2.3.5 says that $\theta \in I$ for all $I \in$ r.cl.(P). Thus, $\theta \in S_t$ for all $t \in \mathbb{N}$ and so $\theta b \in S_n \dots S_1 \cap S_{n+1} \dots S_2$. Let $c \in S$ and suppose that

$$\theta bc \in S_{n+1} \dots S_1 \ . \tag{B}$$

We claim that

$$bc \in S_n \dots S_1 \tag{C}$$

from which it will follow that c is not regular modulo P.

Since $\theta \in I$ for each $I \in$ r.cl.(P), we can write, for each $i \in \mathbb{N}$,

$$S_i = \theta S + D_i$$

for some ideal D_i of R. By Lemma 5.1.10, there is always a link $\sigma(I) \rightsquigarrow I$ for each $I \in$ r.cl.(P) and so, $\sigma(S_{i-1}) \supseteq S_i$ or, equivalently, $\sigma^{-1}(S_i) \subseteq S_{i-1}$. Then,

$$S_i\theta = \theta\sigma^{-1}(S_i) \subseteq \theta S_{i-1}$$

from which we see that

$$S_{n+1} \dots S_1 = \theta S_n \dots S_1 + D_{n+1} \dots D_1 \ . \qquad (D)$$

It is easy to see that (C) follows from (B) and (D).

Furthermore, $\sigma(Q)\theta = \theta Q$ and it follows that $\sigma(Q)\theta b \in S_{n+1} \dots S_1$. Thus, $B_n(Q, P) \neq 0$ and, by Theorem 1.3.11, this establishes (A). ∎

Remark 5.2.2. It is possible for a type (1) prime P to be strongly rep. rep. in S even when A is *not* strongly rep. rep. in R. (See, for instance, Remark 6.2.2.) In fact, when σ is the identity, we shall see in Theorem 6.2.1 that type (1) primes are *always* strongly rep. rep. and this also follows from the proof of Theorem 5.2.1. However, in Example 5.3.2, we will see that type (1) primes are not always strongly rep. rep. in general. ∎

For an upper or lower prime P, the contraction $A = P \cap R$ need not be prime. However, we recall that $P \cap R = \bigcap_{i=1}^{n} \sigma^i(J)$ for some prime $J = \sigma^n(J)$ of R and some $n \in \mathbb{N}$. We observe that, since σ is an automorphism, the repleteness of $\sigma^i(J)$ and the question of whether it is linked to itself, will be the same for each i. We now show that, when P is a lower prime, the repleteness of P is determined by that of J, as follows.

Theorem 5.2.3. *With the notation of (5.1.1), let P be a lower prime ideal of S. Then P is rep. rep. in S. Furthermore, letting J be a prime ideal of R minimal over $P \cap R$, P is strongly rep. rep. in S if and only if J is strongly rep. rep. in R.*

Proof. By Lemma 5.1.13, the clique of P is a singleton and therefore it is certainly rep. rep. in S.

Recalling that $P \cap R = A$ and $P = AS$, it is easy to check that

$$(A^j/A^{j+1})_{R/A} \text{ is faithful if and only if } (P^j/P^{j+1})_{S/P} \text{ is faithful,} \qquad (A)$$

for each $j \in \mathbb{N}$, and that these statements are equivalent to those for left modules. Let J be as described in the statement of the theorem and assume that n is minimal with the property that $\sigma^n(J) = J$. We now show that

$$(A^j/A^{j+1})_{R/A} \text{ is faithful if and only if } (J^j/J^{j+1})_{R/J} \text{ is faithful.} \qquad (B)$$

For, suppose that $(A^j/A^{j+1})_{R/A}$ is faithful and let $r \in R$ be chosen such that $J^j r \subseteq J^{j+1}$. Clearly, then,

$$A^j r. \left(\prod_{i=1}^{n-1} \sigma^i(J)\right)^{j+1} \subseteq \left(\prod_{i=1}^{n} \sigma^i(J)\right)^{j+1} \subseteq A^{j+1}.$$

By the faithfulness of $(A^j/A^{j+1})_{R/A}$, we see that

$$r. \left(\prod_{i=1}^{n-1} \sigma^i(J)\right)^{j+1} \subseteq A \subseteq J$$

whence $r \in J$.

Conversely, suppose that $(J^j/J^{j+1})_{R/J}$ is faithful and let $r \in R$ such that $A^j r \subseteq A^{j+1}$. We observe that

$$J^j. \left(\prod_{i=1}^{n-1} \sigma^i(J)\right)^{j} .r \subseteq \left(\prod_{i=1}^{n} \sigma^i(J)\right)^{j} .r \subseteq A^j r \subseteq A^{j+1} \subseteq J^{j+1}.$$

By the faithfulness of $(J^j/J^{j+1})_{R/J}$, it follows that

$$\left(\prod_{i=1}^{n-1} \sigma^i(J)\right)^{j} .r \subseteq J$$

whence $r \in J$. Now, we note that $\left((\sigma^i(J))^j \,/\, (\sigma^i(J))^{j+1}\right)_{R/\sigma(J)}$ is faithful for each $1 \leq i \leq n$, since σ is an automorphism. So, by a similar argument, we could deduce that $r \in \sigma^i(J)$ for each $1 \leq i \leq n$. Thus, $r \in A$.

By (A) and (B) we have established that, for each $j \in \mathbb{N}$,

$$(J^j/J^{j+1})_{R/J} \text{ is faithful if and only if } (P^j/P^{j+1})_{S/P} \text{ is faithful,} \qquad (C)$$

and that these statements are equivalent to those for left modules. Taking $j = 1$ in (C), by Lemma 1.2.6, $J \rightsquigarrow J$ if and only if $P \rightsquigarrow P$. In the absence of such links, of course, both J and P are strongly rep. rep.. So assume that the links exist. Now, in the notation of (1.3.10),

$$B_j(P,P) = B_j(P) = \frac{P^j/P^{j+1}}{Z(P^j/P^{j+1})_{S/P}}$$

and

$$B_j(J,J) = B_j(J) = \frac{J^j/J^{j+1}}{Z(J^j/J^{j+1})_{R/J}} \qquad (D)$$

where Z indicates the right torsion submodule. It follows, by [G&W, Lemma 7.3], that $B_j(P,P) \neq 0$ if and only if $(P^j/P^{j+1})_{S/P}$ is faithful and that $B_j(J,J) \neq 0$ if and only if $(J^j/J^{j+1})_{R/J}$ is faithful. By (C) and (D), we see that $B_j(P,P) \neq 0$ if and only if $B_j(J,J) \neq 0$ and therefore, by Theorem 1.3.11, $P \in A_{j+1}$ if and only if $J \in A_{j+1}(J)$.

It follows that P is strongly rep. rep. in S if and only if J is strongly rep. rep. in R. ∎

For upper prime ideals, we obtain a result very similar to Corollary 4.2.14(ii) although, since, as in the previous theorem, $A = P \cap R$ need not be prime, R_A need only be semilocal. (See Definitions 4.1.9 and 4.1.17 for the definitions of semilocal and regular commutative rings.)

Theorem 5.2.4. *With the notation of (5.1.1), let P be an upper prime ideal of S. If R_A is a regular semilocal ring, then P is rep. rep. in S.*

Proof. Let $\mathcal{C} = \mathcal{C}_R(A)$, which, by Lemma 5.2.4, is an Ore set in S. By Theorem 2.3.2, for the purposes of studying repleteness, we can replace S with $S\mathcal{C}^{-1}$ and we do so, as in the proof of Lemma 5.1.16.

Now, suppose $Q \in X_{l+1}(P)$ for some $l \in \mathbb{N}$ and suppose, moreover, that we can find l non-trivial links between Q and P. That is,

$$Q = Q_l \rightsquigarrow Q_{l-1} \rightsquigarrow \cdots \rightsquigarrow Q_1 \rightsquigarrow Q_0 = P$$

where $Q_j \neq Q_{j-1}$ for any $1 \leq j \leq l$. Now, in the notation of (5.1.6) and (5.1.8), we write

$$Q_i = [A, \overline{q_i}(t)]$$

for each $0 \leq i \leq l$ and we suppose that

$$\overline{q_i}(t) = \mu_k(\alpha_i) \ ,$$

the minimal polynomial of some $\alpha_i \in \overline{k}$. Then, by Theorem 5.1.18, and in the notation of (5.1.17), we can find non-zero eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_l$ of U such that, for $1 \leq j \leq l$,

$$\alpha_{j-1} = \alpha_j / \lambda_j^m \ .$$

We see from this that

$$\overline{q_0}(t) = \mu_k(\alpha_l/\lambda_1^m \ldots \lambda_l^m) \; .$$

On the other hand,

$$\overline{q_l}(t) = \mu_k(\alpha_l)$$

and it follows easily that

$$\overline{q_l}\,(\lambda_1^m \ldots \lambda_l^m t) = \gamma \overline{q_0}(t) \qquad (A)$$

for some non-zero $\gamma \in k$.

Now, for $1 \leq j \leq l$, we can find $b_{1,j}, \ldots, b_{n,j} \in k$, not all zero, such that

$$U. \begin{pmatrix} b_{1,j} \\ \vdots \\ b_{n,j} \end{pmatrix} = \lambda_j . \begin{pmatrix} b_{1,j} \\ \vdots \\ b_{n,j} \end{pmatrix} \; .$$

Thus, adopting the notation of (5.1.17),

$$\begin{aligned} \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right)^{\widehat{\sigma}} &= \sum_{i=1}^{n} (\widehat{a_i})^{\widehat{\sigma}} \, b_{i,j} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} \widehat{a_k} u_{k,i} b_{i,j} \\ &= \sum_{k=1}^{n} \widehat{a_k} \sum_{i=1}^{n} u_{k,i} b_{i,j} \\ &= \sum_{k=1}^{n} \widehat{a_k} \lambda_j b_{k,j} \\ &= \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right) \lambda_j \end{aligned}$$

from which it follows that, in $\widehat{AS}$,

$$\overline{\theta} \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right) = \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right) \lambda_j \theta$$

and so

$$\overline{f}(t) \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right) = \left(\sum_{i=1}^{n} \widehat{a_i} b_{i,j} \right) \overline{f}\,(\lambda_j^m t)$$

for each $\overline{f}(t) \in k[t]$. For $1 \leq j \leq l$, let ${}_ja \in A$ such that ${}_ja + A^2 = \sum_{i=1}^n \widehat{a}_i b_{i,j}$. We see that, in $S/A^{l+1}S$,

$$\overline{f}(t).{}_la.{}_{l-1}a \ldots {}_1a + A^{l+1}S = {}_la.{}_{l-1}a \ldots {}_1a\overline{f}\left(\lambda_1^m \ldots \lambda_l^m t\right) + A^{l+1}S \qquad (B)$$

for each $\overline{f}(t) \in k[t]$.

Taking $f = q_l$ in (B), we see from (A) that

$$\begin{aligned} \overline{q_l}(t).{}_la.{}_{l-1}a \ldots {}_1a + A^{l+1}S &= {}_la \ldots {}_1a.\overline{q_l}\left(\lambda_1^m \ldots \lambda_l^m t\right) + A^{l+1}S \\ &= {}_la \ldots {}_1a.\gamma.\overline{q_0}(t) + A^{l+1}S \qquad (C) \end{aligned}$$

for some non-zero $\gamma \in k$. Writing $\gamma = c + A$ for some $c \in R$, invertible modulo A, (C) becomes

$$q_l(t).{}_la.{}_{l-1}a \ldots {}_1a + A^{l+1}S = {}_la \ldots {}_1a.c.q_0(t) + A^{l+1}S \qquad (D)$$

Since $Q = AS + q_l(t)S$ and $P = AS + q_0(t)$, it follows from (D) that

$$Q.{}_la.{}_{l-1}a \ldots {}_1a + A^{l+1}S = {}_la \ldots {}_1a.P + A^{l+1}S\ .$$

Thus,

$$H_l(Q,P) := \frac{{}_la \ldots {}_1aS + A^{l+1}S}{{}_la \ldots {}_1aP + A^{l+1}S}$$

is an $(S/Q, S/P)$-bimodule which we claim is faithful on both sides. Let $s \in S$ such that $H_l(Q,P).(s+P) = 0$ and we require to show that $s \in P$. Then,

$${}_la \ldots {}_1a(s-p) \in A^{l+1}S = \oplus_{i=0}^{\infty} A^{l+1}\theta^i$$

for some $p \in P$. We write

$$s - p = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_w\theta^w$$

for some $r_0, \ldots, r_w \in R$, we fix $z \in \{0, \ldots, w\}$ and we claim that $r_z \in A$. Note that

$${}_la \ldots {}_1ar_z \in A^{l+1} \subseteq J^{l+1}$$

and that, after localizing R at J,

$${}_la \ldots {}_1ar_z1^{-1} \in J^{l+1}(R-J)^{-1} = \left(J(R-J)^{-1}\right)^{l+1}\ .$$

Suppose that $r_z \notin J$. Then, since R_J is a regular local ring, Lemma 4.1.18 shows that, for some $i \in \{1, \dots, l\}$,

$$_ia1^{-1} \in \left(J(R-J)^{-1}\right)^2 = J^2(R-J)^{-1} \ .$$

Thus, we can find some $x \in R - J$ such that $_iax \in J^2$. Now,

$$\begin{aligned} {}_iax\,(\sigma(J))^2\left(\sigma^2(J)\right)^2\dots\left(\sigma^{n-1}(J)\right)^2 &\subseteq J^2.(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2 \\ &\subseteq \left(J\cap\sigma(J)\cap\dots\sigma^{n-1}(J)\right)^2 \\ &= A^2 \end{aligned}$$

and so, in $\widehat{A}_R$,

$$\left(\frac{_ia + A^2}{A^2}\right).x\,(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2 = 0 \ .$$

Since $\sigma(A^2) = A^2$ and $\sigma\left(\frac{_ia+A^2}{A^2}\right) = \left(\frac{_ia+A^2}{A^2}\right).\lambda$ in $\widehat{A}_k$, we see that

$$\left(\frac{_ia + A^2}{A^2}\right).\lambda^j.\sigma^j\left(x\,(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2\right) = 0$$

for any $j \geq 0$. Since λ is a non-zero element of the field k,

$$\left(\frac{_ia + A^2}{A^2}\right).\sigma^j\left(x\,(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2\right) = 0 \ .$$

Of course, since

$$x\,(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2 \nsubseteq J \ ,$$

it follows that

$$\sigma^j\left(x\,(\sigma(J))^2\dots\left(\sigma^{n-1}(J)\right)^2\right) \nsubseteq \sigma^j(J)$$

so we have shown that

$$I := \operatorname{ann}_R\left(\frac{_ia + A^2}{A^2}\right) \nsubseteq \sigma^j(J)$$

for any $j \geq 0$. For each $0 \leq j \leq n-1$, choose

$$y_j \in I.\prod_{\substack{i=0 \\ i\neq j}}^{n-1}\sigma^i(J)$$

such that $y_j \notin \sigma^j(J)$. Since $y_j \in \sigma^i(J)$ for $0 \leq i \leq n-1$, $i \neq j$, $y := \sum_{j=0}^{n-1} y_j \notin \sigma^i(J)$ for any i, while $y \in I$ since each $y_j \in I$. That is, $y \in I \cap \mathcal{C}_R(A)$. However, R is semilocal with semimaximal ideal A and so y has an inverse in R. Thus $1_R \in I$ and it follows that ${}_i a \in A^2$.

Since this is a contradiction of the choice of ${}_i a$, we conclude that $r_z \in J$. Similarly, $r_z \in \sigma^i(J)$ for each $i \geq 0$ so that $r_z \in A$ as claimed. Since z was arbitrary, we see that $s - p \in AS \subseteq P$ and, consequently, $s \in P$. We have thus shown that $H_l(Q, P)$ is faithful as a right S/P-module and, similarly, it is faithful as a left S/Q-module.

By Theorem 1.3.7, this completes the proof. ∎

Remark 5.2.5. As we might expect in view of the differential operator ring examples, regularity is not in fact a necessary condition for repleteness (see Example 5.3.3). (Indeed, we do not know of any prime satisfying the hypothesis of Theorem 5.2.4 which is not *strongly* rep. rep..) Moreover, we do not know of any example where R_A is semiprime and P is not rep. rep. although, even when R_A is prime, P need not be strongly rep. rep. (see Example 5.3.4). Certainly, if R_A is not semiprime, it is easy to find examples where P is not rep. rep. (see Example 5.3.5). ∎

The following is an immediate consequence of Theorems 5.2.1, 5.2.3 and 5.2.4. We observe that this is the same conclusion as that for iterated differential operator rings over a commutative Noetherian $\mathbb{Q}$-algebra (Corollary 4.2.15.).

Corollary 5.2.6. *Adopt the notation of (5.1.1). If R is a regular ring then S is a rep. rep. ring.* ∎

§5.3 Some Examples

Example 5.3.1. Let $R = \mathbb{C}(t)[x]$ and put $S = R[\theta; \sigma]$ where σ is the automorphism of R given by $\sigma(t) = (t+1)$ and $\sigma(x) = x$, so that $\theta t = (t+1)\theta$ and $\theta x = x\theta$. Then, by Theorem 5.2.1, the type (1) prime $P = (x-t)S + \theta S$ is strongly rep. rep.. We now show this by calculating the bimodule $B_m(P)$ directly.

By Lemma 5.1.10, the links of r.cl.(P) are the trivial links together with the links

$$(x-t-\alpha)S+\theta S \rightsquigarrow (x-t-\alpha+1)S+\theta S$$

so that, for $n \in \mathbb{N}$,

$$X_n(P) = \{(x-t-m)S+\theta S : m \in \{0,1,\dots,n-1\}\}$$
$$S_n = (x-t)(x-t-1)\cdots(x-t-n+1)S+\theta S \ .$$

By an induction argument similar to those of Chapter 4,

$$\begin{aligned} B_{n-1}(P) &= \frac{S_nS_{n-1}\dots S_2}{S_nS_{n-1}\dots S_1} \\ &= \frac{\sum_{j=0}^{n-1}\prod_{i=0}^{j}(x-t-n+i)^i(x-t+j-n+1)^j\theta^{n-1-j}S}{\sum_{j=0}^{n}\prod_{i=0}^{j}(x-t-n+i)^i\theta^{n-j}S}. \end{aligned}$$

We note that the elements

$$\gamma_j := \prod_{i=0}^{j}(x-t-n+i)^i(x-t+j-n+1)^j\theta^{n-1-j}+S_n\dots S_1$$

for $0 \le j \le n-1$ are non zero generators of ${}_S(B_{n-1}(P))_S$ and that

$$((x-t-j)S+\theta S)\,.\gamma_{n-1-j}=0$$

in $B_{n-1}(P)$ for each $0 \le j \le n-1$. It follows that $X_n(P) = A_n(P)$ for each $n \in \mathbb{N}$ and so P is strongly rep. rep. in S. ∎

When σ is the identity, we shall see in Theorem 6.2.1 that type (1) primes are always strongly rep. rep.. This is not true in general as we now show.

Example 5.3.2. *An example of a type (1) prime which is not strongly rep. rep..*

Let R be the ring $(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}})$, let σ be the automorphism of R defined by $\sigma(u_1,u_2,u_3) = (u_3,u_1,u_2)$ and put $S = R[\theta;\sigma]$. Then, the primes

$$P_1 = \theta S + \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right) S$$
$$P_2 = \theta S + \left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right) S$$
$$P_3 = \theta S + \left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right) S$$

are co-Artinian, type (1) primes of R. We will show that $P := P_1$ is not strongly rep. rep. (and P_2 and P_3 will, of course, also fail to be strongly rep. rep. by symmetry).

We note that $A = P \cap R = \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right)$, that $A/A^2 = \frac{\left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right)}{\left(0 \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right)}$ and that $\mathrm{ann}_R(A/A^2) = A$. Thus $P \rightsquigarrow P$ by Lemma 5.1.10 (and similarly, $P_2 \rightsquigarrow P_2$ and $P_3 \rightsquigarrow P_3$). By Theorem 5.1.12, then,

$$X_n(P) = \begin{cases} \{P_1\} & \text{when } n = 1 ; \\ \{P_1, P_2\} & \text{when } n = 2 ; \\ \{P_1, P_2, P_3\} & \text{when } n \geq 3 . \end{cases}$$

It follows that

$$S_n = \begin{cases} \theta S + \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right) S & \text{when } n = 1 ; \\ \theta S + \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right) S & \text{when } n = 2 ; \\ \theta S + \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right) S & \text{when } n \geq 3 . \end{cases}$$

Since $\theta r = \sigma(r)\theta$ for each $r \in R$ and since $\sigma\left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right) = \left(\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)$, a simple induction argument shows that, for $n \geq 3$,

$$\begin{aligned} B_{n-1}(P) &= \frac{S_n \dots S_2}{S_n \dots S_1} \\ &= \frac{\theta^{n-1}S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\theta^{n-2}S + \sigma^n\left(0 \oplus 0 \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\theta^{n-3}S}{\theta^n S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\theta^{n-1}S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus 0 \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\theta^{n-2}S + \sigma^n\left(0 \oplus 0 \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\theta^{n-3}S} \end{aligned}$$

and this also holds for $n = 2$ if we replace θ^{-1} with 0. Clearly, $B_{n-1}(P)$ is generated by the non-zero elements

$$\rho := \theta^{n-1} + S_n \dots S_1$$

and

$$\mu := \sigma^n(1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 2 + 4\mathbb{Z})\theta^{n-2} + S_n \dots S_1 .$$

On the other hand, we observe that

$$\left(\theta S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right)\right) \cdot \rho = 0$$

and

$$\left(\theta S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right)\right) \cdot \mu = 0 .$$

Thus, by Theorem 1.3.11,

$$A_n(P) = \left\{\theta S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}}\right), \theta S + \sigma^n\left(\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}\right)\right\}$$

for $n \geq 2$, and we deduce that

$$A_n(P) = \begin{cases} \{P_1\} & \text{if } n = 1 \text{ ;} \\ \{P_1, P_2\} & \text{if } n \equiv 2 \pmod 3 \text{ and } n \in \mathbb{N} \text{ ;} \\ \{P_2, P_3\} & \text{if } n \equiv 0 \pmod 3 \text{ and } n \in \mathbb{N} \text{ ;} \\ \{P_1, P_3\} & \text{if } n \equiv 1 \pmod 3 \text{ and } n \in \mathbb{N} - \{1\} \text{ .} \end{cases}$$

Finally, we note that $A_n(P) = X_n(P)$ if and only if $n = 1$ or 2 and so P is not strongly rep. rep. in S. (Of course, in this case, $P \cap R = (\frac{2\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}})$ cannot be strongly rep. rep. in R.) ∎

Example 5.3.3. *An example of a strongly rep. rep. upper prime which does not satisfy the hypothesis of Theorem 5.2.4.*

Let $R = \frac{\mathbb{C}[x]}{<x^2>}$ and $S = R[\theta]$. The upper prime ideal $P = xS + (\theta - 1)S$ is of course strongly rep. rep. in the commutative ring S since it is not a minimal prime. However, $R_{P \cap R} = \frac{\mathbb{C}[x]}{<x^2>}$ is not even semiprime. ∎

Example 5.3.4. *An example of a rep. rep. upper prime P which is not strongly rep. rep., where $R_{P \cap R}$ is a prime local ring which is not, however, regular.*

Let $R = \mathbb{C}[x^2, x^3]$, let $\sigma(f(x)) = f(2x)$ for each $f(x) \in R$ and put $S = R[\theta; \sigma]$. The ideal $P := (\theta - 1)S + x^2S + x^3S$ of S is a co-Artinian upper prime while $A := P \cap R = x^2R + x^3R$ is a prime ideal of R and R_A is a prime local ring which is not regular. We will show that P is rep. rep. but not strongly rep. rep.. The argument proceeds along the same lines as those of Example 4.3.2.

Since σ is the identity on $\mathbb{C} \cong R/A$, $k \cong \mathbb{C}$ in the notation of (5.1.8) and, since $\{x^2 + A^2, x^3 + A^2\}$ is a basis for A/A^2 over $\mathbb{C}$, it is easy to see that, in the notation of (5.1.17), the only eigenvalues for U are 4 and 8. So, aside from the trivial links of Lemma 5.1.16, there are by Theorem 5.1.18, corresponding respectively to these eigenvalues, the links

$$(\theta - 4\alpha)S + AS \rightsquigarrow (\theta - \alpha)S + AS$$

and

$$(\theta - 8\alpha)S + AS \rightsquigarrow (\theta - \alpha)S + AS$$

between upper prime ideals, for each $\alpha \in \mathbb{C} - \{0\}$. It follows that

$$X_n(P) = \{(\theta - 2^i)S + AS : i \in \{0, 2, 3, 4, \ldots, 3n - 3\}\}$$

and so

$$S_n = (\theta - 1)(\theta - 2^2)(\theta - 2^3)(\theta - 2^4)\dots(\theta - 2^{3n-3})S + AS$$

for each $n \in \mathbb{N}$. Since

$$(\theta - 1)(\theta - 2^2)(\theta - 2^3)\dots(\theta - 2^{3n-3})x^2 = x^2(\theta - \frac{1}{4})(\theta - 2^0)(\theta - 2^1)\dots(\theta - 2^{3n-5})$$

and

$$(\theta - 1)(\theta - 2^2)(\theta - 2^3)\dots(\theta - 2^{3n-3})x^3 = x^3(\theta - \frac{1}{8})(\theta - 2^{-1})(\theta - 2^0)\dots(\theta - 2^{3n-6})$$

we see that

$$S_n A \subseteq AS_{n-1}$$

and, by an induction argument,

$$\begin{aligned} B_{n-1}(P) &= \frac{S_n \dots S_2}{S_n \dots S_1} \\ &= \frac{\sum_{i=0}^{n-1} A^{n-1-i}(\theta-1)^i(\theta-2^2)^i(\theta-2^3)^i \prod_{j=2}^{i}(\theta-2^{3j-2})^{i+1-j}(\theta-2^{3j-1})^{i+1-j}(\theta-2^{3j})^{i+1-j} S}{\sum_{i=0}^{n} A^{n-1}(\theta-1)^i(\theta-2^2)^{i-1}(\theta-2^3)^{i-1} \prod_{j=2}^{i-1}(\theta-2^{3j-2})^{i-j}(\theta-2^{3j-1})^{i-j}(\theta-2^{3j})^{i-j} S} \end{aligned}$$

for each $n \geq 2$. Now, for each $m \in \mathbb{N}$,

$$\frac{A^m}{A^{m+1}} = \frac{x^{2m}R + x^{2m+1}R}{x^{2m+2}R + x^{2m+3}R} .$$

It follows that $S_n \dots S_2$ is generated by the set

$$\Big\{x^s(\theta - 1)^i(\theta - 2^2)^i(\theta - 2^3)^i \prod_{j=2}^{i}(\theta - 2^{3j-2})^{i+1-j}(\theta - 2^{3j-1})^{i+1-j}(\theta - 2^{3j})^{i+1-j} :$$
$$0 \leq i \leq n-1 \text{ and } s = 2(n-1-i) \text{ or } 2(n-i-1)+1 \text{ but not } 1\Big\}$$

and that none of these elements is contained in $S_n \dots S_1$. Finally, we see that

$$((\theta-2^s)S+AS).x^s(\theta-1)^i(\theta-2^2)^i(\theta-2^3)^i \prod_{j=2}^{i}(\theta-2^{3j-2})^{i+j-1}(\theta-2^{3j-1})^{i+1-j}(\theta-2^{3j})^{i+1-j}$$
$$\subseteq S_n \dots S_1$$

for each $0 \leq i \leq n-1$, whenever $s = 2(n-1-i)$ or $2(n-1-i)+1$ but not 1, and so

$$A_n(P) = \{(\theta - 2^\alpha)S + AS : \alpha \in \{0, 2, 3, 4, \dots, 2n-1\}\} .$$

We observe that $A_n(P) = X_n(P)$ if and only if $n = 1$ or 2, so P is not strongly rep. rep. in S. On the other hand,

$$\begin{aligned}\mathrm{Fund}(P) &= \bigcup_{n=1}^{\infty} A_n(P) = \{(\theta - 2^\alpha)S + AS : \alpha \in \{0, 2, 3, 4, \ldots\}\} \\ &= \bigcup_{n=1}^{\infty} X_n(P) = \mathrm{r.cl.}(P)\end{aligned}$$

so that P is rep. rep. in S as claimed. ∎

Example 5.3.5. On the other hand, not every upper prime is rep. rep.. For, let R be the ring $\mathbb{C}[x]$, take σ to be the automorphism such that $\sigma(x) = 2x$ and put $T = R[\theta; \sigma]$. Now, the upper prime $P = xS + (\theta - 1)S$ is certainly rep. rep. by Theorem 5.2.4. We show first that it is strongly rep. rep..

Since σ is the identity on $\mathbb{C} \cong R/xR$, $k \cong \mathbb{C}$ in the notation of (5.1.8) and since $\{x + (P \cap R)^2\}$ is a basis for $\frac{P \cap R}{(P \cap R)^2}$ over $\mathbb{C}$, it is easy to see that, in the notation of (5.1.17), the only eigenvalue for U is 2. It follows then, from Lemma 5.1.16 and Theorem 5.1.18, that

$$X_n(P) = \{xS + (\theta - 2^m)S : 0 \leq m \leq n - 1\}$$

and so

$$S_n = xS + (\theta - 1)(\theta - 2)(\theta - 4) \ldots (\theta - 2^{n-1})S$$

for each $n \in N$. Since

$$(\theta - 1)(\theta - 2)(\theta - 4) \ldots (\theta - 2^{n-1})x = 2^n x(\theta - \frac{1}{2})(\theta - 1)(\theta - 2) \ldots (\theta - 2^{n-2})$$

we see that

$$S_n x \subseteq x S_{n-1}$$

and, by an induction argument, $B_{n-1}(P)$ can be written as

$$\frac{S_n \ldots S_2}{S_n \ldots S_1} = \frac{\sum_{i=0}^{n-1} x^{n-1-i}(\theta - 1)^i(\theta - 2)^i(\theta - 4)^{i-1} \ldots (\theta - 2^i)^1 S}{\sum_{i=0}^{n} x^{n-i}(\theta - 1)^i(\theta - 2)^{i-1}(\theta - 4)^{i-2} \ldots (\theta - 2^{i-1})^1 S}$$

for each $n \geq 2$. Thus,

$$\gamma_i := x^{n-1-i}(\theta - 1)^i(\theta - 2)^i(\theta - 4)^{i-1} \ldots (\theta - 2^i)^1 + S_n \ldots S_1 \in B_{n-1}(P) - \{0\}$$

for each $0 \le i \le n-1$ and these elements generate $B_{n-1}(P)$. Clearly,

$$(xS + (\theta - 2^m)S) . \gamma_{n-1-m} = 0$$

so that

$$B_{n-1}\left(xS + (\theta - 2^m)S, P\right) \neq 0 .$$

Hence, by Theorem 1.3.11,

$$xS + (\theta - 2^m)S \in A_n$$

for each $0 \le m \le n-1$ and, accordingly, P is strongly rep. rep. in S.

However, since $x \in Q$ for each $Q \in$ r.cl.(P), the link graph is unchanged if we replace R by $\overline{R} = \frac{R}{x^2R}$. Furthermore, $B_{n-1}\left(\overline{P}\right)$ is still generated by the $\overline{\gamma_i}$. In this case, however $\overline{\gamma_i} = 0$ for $0 \le i \le n-3$. We see, therefore that

$$A_n\left(\overline{P}\right) = \begin{cases} \{\overline{P}\} & \text{if } n = 1 \\ \{\overline{P}, x\overline{S} + (\theta - 2)\overline{S}\} & \text{if } n \ge 2 \end{cases}$$

and so $A_n\left(\overline{P}\right) = X_n\left(\overline{P}\right)$ if and only if $n = 1$ or 2 and, indeed, $\left|\text{Fund}(\overline{P})\right| = 2$ while $\left|\text{r.cl.}(\overline{P})\right| = \infty$. Thus, $\overline{P}$ is not rep. rep. in $\overline{S}$. ∎

§5.4 Notes

That rings of the form $R[\Theta; \Sigma]$, where R is commutative Noetherian and Σ is a set of commuting automorphisms on R, satisfy the second layer condition, follows by an analogous argument to that for differential operator rings in [Be, Theorem 7.3]. In fact, it is shown in [G3, Theorem 5.1] that the skew polynomial ring $R[\theta; \sigma, \delta]$ (where σ is an automorphism) over a commutative Noetherian ring R (which of course includes the rings considered in this chapter) satisfies the *strong* second layer condition. It appears to be an open question whether rings of the form $R[\theta_1; \delta_1] \ldots [\theta_n; \delta_n]$ and $R[\theta_1; \sigma_1] \ldots [\theta_n; \sigma_n]$ (where R is commutative Noetherian and $\{\delta_1, \ldots, \delta_n\}$ and $\{\sigma_1, \ldots, \sigma_n\}$ are sets of noncommuting derivations and automorphisms respectively) satisfy the second layer condition in general.

The terminology and notation of §5.1 can be found in [Po] and [I]. In particular, the terms "lower" and "upper" primes were introduced in [Po, §1].

Lemma 5.1.2 is a well known result while Theorem 5.1.3 is contained in [I, Theorems 4.1 and 4.2].

Lemma 5.1.5 is essentially [G&M, Lemma 1.4]. In fact, in that reference it is proved that, for *any* right Noetherian semiprime ring R, $S = R[\theta; \sigma]$ satisfies the right Ore condition with respect to $\mathcal{C}_R(0)$.

The comments in (5.1.6) and (5.1.8) are taken from [Po, §1], although the results are mainly proved in [I, §§4&5]. In particular, Theorem 5.1.7 is one of the main results, [I, Theorem 4.4], of Irving's paper.

Lemma 5.1.9 can be found in [Po, Theorem 6] and its proof.

That, for type (1) primes, $\sigma(P) \rightsquigarrow P$, is proved in [Po, Theorem 6], however, the rest of Lemma 5.1.10 appears to be new.

The last part of Lemma 5.1.12 is the conclusion of [Po, Theorem 6] while the other parts follow from the proof of [Po, Theorem 6] in the light of Lemma 5.1.10.

Lemma 5.1.13 follows from [Po, Theorem 9] although this latter result has been independently proved in [Be1, Proposition 7.6].

Lemma 5.1.15 is quoted from [Po, Lemma 10].

Lemma 5.1.16 follows from the discussion of [Po, p440] and Theorem 5.1.18 from [Po, Theorem 13].

Theorems 5.2.1, 5.2.3 , 5.2.4 and 5.2.6 are all new results.

Chapter 6 : Polynomial Rings

In this chapter, we discuss the natural question of whether the repleteness of a ring R carries over to the ordinary polynomial ring $R[x]$. An immediate obstacle is that it is an open question whether the second layer conditions carry over to polynomial rings. For some important classes of rings, however, this question is known to have a positive answer.

Specifically, if R is a fully sub-bounded Noetherian ring, then R satisfies the strong second layer condition [By, proof of Lemma B] and $R[x_1, \ldots, x_n]$ satisfies the second layer condition [By, Corollary D]. (A prime ring R is called *sub-bounded* if every non-zero prime ideal of R contains an element c and a non-zero sub-ideal J such that $J \subseteq cR \cap Rc$; a ring R is *fully sub-bounded* if every prime factor of R is sub-bounded. Fully sub-bounded rings include FBN rings, PI rings, polynormal rings and rings in which all prime ideals are maximal.) Indeed, if R is an FBN ring, then $R[x_1, \ldots, x_n]$ satisfies the strong second layer condition [By, Corollary F].

However, even assuming that $R[x]$ satisfies the second layer condition, we will see that, in general, our question has a negative answer, although we do present some positive results, in particular for commutative rings (Theorem 6.2.1) and for the case where the prime P of $R[x]$ is just $(P \cap R)R[x]$ (Theorem 6.2.3). First, we discuss some preliminaries.

§6.1 Prime Links in Polynomial Rings

For any ring R and a prime ideal P of the polynomial ring $S = R[x]$, it is easily seen that $J = P \cap R$ is a prime ideal of R. In this scenario, and assuming that R (and hence S) is Noetherian, we discuss in this section how the clique of P is related to that of J. The proofs are extracted from those contained in unpublished work of Brown and Goodearl [B&G] which deals with q-skew polynomial rings. These results only require that R satisfies the second layer condition.

Lemma 6.1.1. *Let R be a Noetherian ring which satisfies the second layer condition and P and Q prime ideals of the polynomial ring $S = R[x]$. Suppose that $Q \rightsquigarrow P$. If $Q \cap R \not\rightsquigarrow P \cap R$ then $Q = P \supsetneqq (P \cap R)S$.*

Proof. *(From [B&G, Proposition 1.7 and Corollary 1.8].)*

Put $K = Q \cap R$ and $J = P \cap R$. Suppose that $Q \rightsquigarrow P$ and that $K \not\rightsquigarrow J$. Then, by Lemma 1.2.6, $\frac{K \cap J}{KJ}$ must be unfaithful either as a left R/K-module or as a right R/J-module; we suppose the latter. Let

$$I = \operatorname{ann}_R \left(\frac{K \cap J}{KJ} \right)_R .$$

Since $I \supsetneqq J$, $I \not\subseteq P$ and so, since

$$(K \cap J)SI = S(K \cap J)I \subseteq SKJ \subseteq QP,$$

it follows that $\frac{(K \cap J)S + QP}{QP}$ is unfaithful as a right S/P module and so is torsion. We now suppose that the link $Q \rightsquigarrow P$ arises via the $(S/Q, S/P)$-bimodule $\frac{Q \cap P}{A}$ for some ideal A of S with $A \supseteq QP$ and note that, since this is torsion-free as a right S/P-module,

$$(K \cap J)S \subseteq A.$$

Consequently,

$$Q/(K \cap J)S \rightsquigarrow P/(K \cap J)S \qquad (A)$$

and so, without loss of generality, we may assume that $K \cap J = 0$. By Lemma 1.2.8, the primes K and J of R are minimal and therefore are disjoint from $\mathcal{C}_R(0)$, the regular elements of R. It follows that Q and P are disjoint from $\mathcal{C}_R(0)$ and so, as noted in Remark 2.3.1, we may localize S at $\mathcal{C}_R(0)$.

So, without loss of generality, we assume that R is a semisimple ring and then $J = Re$ for some central idempotent e in R. By Lemma 2.3.5, we see that $e \in K$ and so $J \subseteq K$. Similarly $K \subseteq J$ and hence

$$J = K. \qquad (B)$$

Now, (A) becomes

$$Q/JS \rightsquigarrow P/JS \qquad (C)$$

and so, from the definition of a link, $Q \neq JS$ and $P \neq JS$. Furthermore, we see from (B) that, in fact, R is a simple Artinian ring and, by [L&M, Theorem 2.2 and Proposition 2.9], either S has only one non-zero prime, in which case, clearly, $Q = P$, or else each non-zero prime of S is generated by a central element. In the latter case, we apply Lemma 2.3.5 again to see that $Q = P$. In either case,

$$Q = P \supsetneqq JS$$

and the proof is complete. ∎

Lemma 6.1.2. *Let R be a Noetherian ring which satisfies the second layer condition and P and Q prime ideals of the polynomial ring $S = R[x]$. Suppose that $Q \rightsquigarrow P$. Then $Q = (Q \cap R)S$ if and only if $P = (P \cap R)S$.*

Proof. *(From [B&G, Proposition 1.7 and Corollary 1.9].)*

Put $K = Q \cap R$ and $J = P \cap R$. By Lemma 6.1.1, if $K \not\rightsquigarrow J$, then $Q = P \neq JS$ and, in this case, the result follows trivially. So, we suppose that $K \rightsquigarrow J$ and assume that $\frac{K \cap J}{C}$ is the linking bimodule for some ideal C of R with $KJ \subseteq C$. Without loss of generality, we may assume that C/KJ is a torsion right R/J-module. Furthermore, assume that the linking bimodule for $Q \rightsquigarrow P$ is $\frac{Q \cap P}{A}$ for some ideal A of S with $QP \subseteq A$. We note that $\frac{Q \cap P}{A}$ is a torsion-free right R/J-module. Since $KJ \subseteq QP \subseteq A$, it follows that $C \subseteq A$ and therefore $CS \subseteq A$. Thus, $Q/CS \rightsquigarrow P/CS$ and so we may assume that $C = 0$. Consequently, $KJ = 0$ and so $N = K \cap J$ is the prime radical of R. Furthermore, $\mathcal{C}_R(N) = \mathcal{C}_R(K) \cap \mathcal{C}_R(J)$, while, by the definition of a link, N is torsion-free as a right R/J-module and as a left R/K-module. Hence N is $\mathcal{C}_R(N)$-torsion-free on both sides. Since this is clearly true of R/N as well, we see that $\mathcal{C}_R(N) \subseteq \mathcal{C}_R(0)$. We thus have $\mathcal{C}_R(N) = \mathcal{C}_R(0)$ since the reverse inclusion holds in any Noetherian ring (for instance, by [G&W, Lemma 10.8]) and so, by Small's Theorem [G&W, Corollary 10.10], $\mathcal{C}_R(0)$ is an Ore set in R, the corresponding localization being Artinian. Of course, P and Q are disjoint from $\mathcal{C}_R(N) = \mathcal{C}_R(0)$ and so, by Remark 2.3.1, we can assume that this localization has been made.

Thus, R is an Artinian ring and R/J is a simple Artinian ring. We assume that P/JS is a non-zero prime of S/JS and claim that $Q \neq KS$. The reverse implication will follow by symmetry. To this end, we identify S/JS with $(R/J)[x]$ and note that the leading coefficients of polynomials in P/JS form a non-zero ideal of R/J. Since R/J is simple, there must be a monic polynomial in P/JS and so P contains a monic polynomial. By [R,S&S], the set $\mathcal{M}$ of monic polynomials in S is Ore. Since $\mathcal{M} \not\subseteq \mathcal{C}_S(P)$, and $Q \rightsquigarrow P$, it follows by Lemma 1.2.2 that $\mathcal{M} \not\subseteq \mathcal{C}_S(Q)$. As R is an Artinian ring, this means that we can find some monic polynomial in Q and, consequently, $Q \neq KS$.

This completes the proof of the lemma. ∎

Theorem 6.1.3. *Let R be a Noetherian ring which satisfies the second layer condition and P and Q prime ideals of the polynomial ring $S = R[x]$. Suppose that $P = (P \cap R)S$ and that $Q = (Q \cap R)S$. Then $Q \rightsquigarrow P$ if and only if $Q \cap R \rightsquigarrow P \cap R$.*

Proof. *(From [B&G, Theorem 2.1].)*

Put $K = Q \cap R$ and $J = P \cap R$. We suppose first that $Q \rightsquigarrow P$. If $K \not\rightsquigarrow J$, then, as seen at (C) in the proof of Lemma 6.1.1, $Q/JS \rightsquigarrow P/JS$ which is impossible since, by assumption, $P/JS = 0$. It follows that $K \rightsquigarrow J$.

Conversely, suppose that $K \rightsquigarrow J$ with a linking bimodule $\frac{K \cap J}{A}$ for some ideal A of R with $A \supseteq KJ$. We can assume without loss that $A = 0$ and, in this case, KJ and hence QP are zero. Since $K \cap J$ is a torsion-free right R/J-module, it can be embedded in $(R/J)^n$ for some $n \in \mathbb{N}$. Since ${}_RS$ is a flat module, tensoring by it we obtain a right S-module embedding

$$Q \cap P \cong (K \cap J) \otimes_R S \hookrightarrow (R/J)^n \otimes_R S \cong (S/P)^n.$$

It follows from this that $Q \cap P$ is a torsion-free right S/P-module and, by a similar argument, it is a torsion-free left S/Q-module. Thus, $Q \rightsquigarrow P$ which completes the proof of the lemma. ∎

§6.2 The Repleteness of Polynomial Rings

We first deal with the case where R is a commutative Noetherian ring. Of course, since $R[x]$ is commutative, it is rep. rep. and, indeed, strong representational repleteness is also preserved as follows. (In fact, the second part is also a corollary of Theorem 5.2.3, however we give an alternative proof which relies on our classification of commutative strongly rep. rep. primes.)

Theorem 6.2.1. *Let R be a commutative Noetherian ring and let P be a prime ideal of the polynomial ring $R[x]$. If $P \supsetneq (P \cap R)R[x]$ then P is strongly rep. rep. in R while, if $P = (P \cap R)R[x]$, then P is strongly rep. rep. in $R[x]$ if and only if $P \cap R$ is strongly rep. rep. in R.*

Proof. If $P \supsetneq (P \cap R)R[x]$ then P is not a minimal prime so must be strongly rep. rep. by Corollary 2.1.3. Now suppose $P = (P \cap R)R[x]$. In this case, P is a minimal prime of $R[x]$ if and only if $P \cap R$ is a minimal prime of R. Furthermore,

$$\mathrm{ann}_R(P \cap R) = \mathrm{ann}_{R[x]}((P \cap R)R[x]) \cap R = \mathrm{ann}_{R[x]}(P) \cap R ,$$

from which we see that, if $\mathrm{ann}_{R[x]}P \subseteq P$, then $\mathrm{ann}_R(P \cap R) \subseteq P \cap R$. Conversely, if $f(x) \in \mathrm{ann}_{R[x]}(P) - P$, then, by considering the coefficient of highest degree in $f(x)$ we see that that coefficient annihilates P and, similarly, so does each coefficient. Since not all coefficients are in $P \cap R$, $\mathrm{ann}_{R[x]}(P) \cap R \not\subseteq P \cap R$ and so $\mathrm{ann}_R(P \cap R) \not\subseteq P \cap R$. The result follows from two further applications of Corollary 2.1.3. ∎

Remark 6.2.2. Given a commutative Noetherian ring R and a prime P of $R[x]$, if $P \cap R$ is strongly rep. rep. in R then, by Theorem 6.2.1, P is strongly rep. rep. in $R[x]$. We see, however, that the converse is false: simply take J to be a prime of R which is not strongly rep. rep. and let $P = JR[x] + xR[x]$. For instance, consider $R = \frac{\mathbb{C}[y]}{<y^2>}$. We saw in Example 2.1.2 that yR is not strongly rep. rep. in R while, by the above, $xR[y] + yR[y]$ is strongly rep. rep. in $R[x]$. ∎

As we shall see, for noncommutative rings, a prime $P \supsetneq (P \cap R)R[x]$ need not be rep. rep., even assuming $P \cap R$ is strongly rep. rep.. However, the second part of Theorem 6.2.1 does in fact, carry over to the noncommutative case as we now show. (Although it is much simpler, the proof proceeds along similar lines to that of Lemma 4.2.11 and, furthermore, in this case a converse holds.)

Theorem 6.2.3. *Let R be a Noetherian ring and assume that the polynomial ring $R[x]$ satisfies the second layer condition. Let J be a prime ideal of R. Then $JR[x]$ is rep. rep. [resp. strongly rep. rep.] in $R[x]$ if and only if J is rep. rep. [resp. strongly rep. rep.] in R.*

Proof. By Lemma 6.1.2, if $Q \in \text{cl.}\, JR[x]$, then $Q = (Q \cap R)R[x]$ and, by Theorem 6.1.3, if J_1 and J_2 are primes of R then $J_2 \rightsquigarrow J_1$ if and only if $J_2R[x] \rightsquigarrow J_1R[x]$. It follows that, for all $n \in \mathbb{N}$,

$$\begin{array}{lll} & (1) & X_n(J) = \{Q \cap R : Q \in X_n(JR[x])\} \\ \text{and} & & \\ & (2) & X_n(JR[x]) = \{KR[x] : K \in X_n(J)\}\ . \end{array} \qquad (A)$$

Now let U be a uniform right ideal of $R[x]/JR[x]$ and set $E = E_{R[x]}(U)$. Since E is an injective right $R[x]$-module and ${}_RR[x]$ is free, E is also an injective right R-module by Lemma 4.2.10. Let K be an associated prime of E_R. We claim that

$$K = J\ . \qquad (B)$$

By the choice of K, we can find $e \in E$ such that eR is a fully faithful R/K-module. Now, $eR[x] \cap U$ is a fully faithful $R[x]/JR[x]$-module and so $K \subseteq JR[x] \cap R = J$. On the other hand, by Theorem 1.3.3 there exists a r.cl. $(JR[x])$-semiprime ideal, I, of $R[x]$ such that $I^{n+1} \subseteq \text{ann}\,(eR[x])$ for some $n \in \mathbb{N}$. So $(I \cap R)^{n+1} \subseteq I^{n+1} \cap R \subseteq K$ and hence $I \cap R \subseteq K$. It follows that we can find a prime A in r.cl. $(JR[x])$ such that $A \cap R \subseteq K \subseteq J$. By $(A1)$ above, $A \cap R$ is in r.cl. $(JR[x] \cap R) =$ r.cl.(J) and then, by the incomparability of r.cl.(J) (Lemma 1.2.8(i)), $A \cap R = K = J$ establishing (B).

Thus, J is the unique associated prime of $\left(E_{R[x]}(U)\right)_R$ and therefore, as R-modules, $E = E_{R[x]}(U) = \oplus_{\lambda \in \Lambda} {}^{\lambda}E$ for some uniform injectives ${}^{\lambda}E$ each with assassinator prime J and some index set Λ. Clearly, $\text{ann}_E(J) = \text{ann}_E(JR[x])$ and, since this is a torsion-free $R[x]/JR[x]$-module, it is easy to see that it is also a torsion-free R/J-module. Thus, we can assume that each ${}^{\lambda}E$ is the injective hull of a uniform right ideal of R/J so that Theorem 1.3.3 applies. The fundamental primes are, of course, identical for each ${}^{\lambda}E$.

As before, we write E_k and ${}^\lambda E_k$ for the k^{th} layers of $E_{R[x]}$ and ${}^\lambda E_R$ respectively. Then, viewing E_k as an R-module, we claim that, for each $k \in \mathbb{N}$,

$$\begin{aligned} &(a1) \qquad A_k(J) = \{Q \cap R : Q \in A_k\,(JR[x])\}\,, \\ &(a2) \qquad A_k\,(JR[x]) = \{KR[x] : K \in A_k(J)\}\,, \\ &(b) \qquad\qquad E_k = \oplus_{\lambda\in\Lambda}{}^\lambda E_k\ . \end{aligned} \tag{C}$$

As noted above, the claim is true for $k = 1$ and we assume it for all $1 \le k \le n$ for some $n \in \mathbb{N}$.

(a) Let $Q \in A_{n+1}\,(JR[x])$. Then there exists $e \in E - E_n$ such that $eQ \subseteq E_n$. Writing $e = \oplus_{\lambda\in\Lambda} e_\lambda$, where each $e_\lambda \in {}^\lambda E$ and only finitely many are non-zero, we have, by the induction hypothesis on (Cb)

$$\oplus_{\lambda\in\Lambda} e_\lambda(Q \cap R) \subseteq \oplus_{\lambda\in\Lambda}{}^\lambda E_n\ .$$

So, for some λ, $e_\lambda \in {}^\lambda E - {}^\lambda E_n$ while $e_\lambda(Q \cap R) \subseteq {}^\lambda E_n$. Thus, since, by $(A1)$, $Q \cap R \in X_{n+1}(J)$, and by the incomparability of cl.(J) (Lemma 1.2.8(i)),

$$Q \cap R \in A_{n+1}({}^\lambda E) = A_{n+1}(J)\ .$$

On the other hand, suppose $K \in A_{n+1}(J)$. Now, if we fix $\mu \in \Lambda$ then there exists some $e_\mu \in {}^\mu E - {}^\mu E_n$ such that $e_\mu K \subseteq {}^\mu E_n$. We now put

$$e_\lambda = \begin{cases} e_\mu & \text{if } \mu = \lambda\ , \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e = \oplus_{\lambda\in\Lambda} e_\lambda\ ,$$

and, by the induction hypothesis on (Cb), we see that

$$eKR[x] \subseteq \oplus_{\lambda\in\Lambda}{}^\lambda E_n = E_n\ .$$

It follows that $KR[x] \subseteq \widetilde{Q}$ for some $\widetilde{Q} \in A_{n+1}\,(JR[x]) \subseteq \text{r.cl.}\,(JR[x])$; but since $K \in \text{r.cl.}(J)$, $KR[x] \in \text{r.cl.}\,(JR[x])$ by $(A2)$ and so, by Lemma 1.2.8(i),

$$KR[x] = \widetilde{Q} \in A_{n+1}\,(JR[x])\ ,$$

completing the induction step for $(Ca1)$. That $(Ca2)$ holds, follows from $(Ca1)$ and the fact that, if $Q \in \text{cl.}\ JR[x]$, then $Q = (Q \cap R)R[x]$ by Lemma 6.1.2.

(b) Now choose $e \in E - E_n$. Again we write $e = \oplus_{\lambda\in\Lambda} e_\lambda$, where only finitely many e_λ are non-zero, and note that, by the induction hypothesis on (Cb), $e_\lambda \in {}^\lambda E - {}^\lambda E_n$, for some λ. Suppose $e \in E_{n+1}$. Then we can find a subset $\{Q_\xi : \xi \in \Xi\}$ of $A_{n+1}(JR[x])$, for some finite index set Ξ, such that $e\left(\bigcap_{\xi\in\Xi} Q_\xi\right) \subseteq E_n$. Then

$$\oplus_{\lambda\in\Lambda} e_\lambda . \bigcap_{\xi\in\Xi} (Q_\xi \cap R) \subseteq {}^\lambda E_n \ .$$

Now, for each $\xi \in \Xi$, we have seen by ($Ca1$) that $Q_\xi \cap R \in A_{n+1}(J) \subseteq X_{n+1}(J)$ while, for each λ, $e_\lambda . \bigcap_{\xi\in\Xi}(Q_\xi \cap R) \subseteq {}^\lambda E_n$ and therefore, by Theorem 1.3.3(ii), $e_\lambda \in {}^\lambda E_{n+1}$. We conclude that

$$E_{n+1} \subseteq \oplus_{\lambda\in\Lambda} {}^\lambda E_{n+1} \ .$$

Conversely, suppose that, for each λ, $e_\lambda \in {}^\lambda E_{n+1}$ with only finitely many non-zero and put $e = \oplus_{\lambda\in\Lambda} e_\lambda$. For each λ we can find finitely many ${}^\xi f_\lambda \in {}^\lambda E$ such that $e_\lambda = \sum_\xi {}^\xi f_\lambda$ and where ${}^\xi f_\lambda\left({}^{\xi\lambda}J\right) \subseteq {}^\lambda E_n$ for some ${}^{\xi\lambda}J \in A_{n+1}(J)$. It suffices to show that, for each λ and ξ, ${}^\xi f_\lambda \in E_{n+1}$. So we fix λ and ξ and note that, by the induction hypothesis on (Cb),

$${}^\xi f_\lambda\left({}^{\xi\lambda}J\right) \subseteq E_n \ .$$

Now, as seen by ($Ca2$), $\left({}^{\xi\lambda}J\right) R[x] \in A_{n+1}(JR[x])$ while clearly

$${}^\xi f_\lambda\left({}^{\xi\lambda}J\right) R[x] \subseteq E_n \ . \qquad (D)$$

It follows from (D) and Theorem 1.3.3(ii) that ${}^\xi f_\lambda \in E_{n+1}$ completing the induction step of (Cb) and therefore the proof of (C) for all $k \in \mathbb{N}$.

That J is rep. rep. [resp. strongly rep. rep.] in R if $JR[x]$ is rep. rep. [resp. strongly rep. rep.] in $R[x]$, follows from ($A1$) and ($Ca1$) while the converse follows from ($A2$) and ($Ca2$). ∎

We now consider the case where P is a prime ideal of $R[x]$ and $P \supsetneq (P \cap R)R$. If we let $E = E_{R[x]}(R[x]/P)$ then, as before, E_R is injective and, by a similar argument to that of Theorem 6.2.3, $P \cap R$ is its unique associated prime. If we tried to imitate the rest of the proof of Theorem 6.2.3, two problems would arise.

Firstly, trivial links in cl.(P) need not contract to trivial links in cl.$(P \cap R)$ (for instance, $x\mathbb{C}[x] \rightsquigarrow x\mathbb{C}[x]$ in $\mathbb{C}[x]$ but $0 \not\rightsquigarrow 0$ in $\mathbb{C}$) and so we cannot expect this method to show that *strong* representational repleteness is preserved under polynomial extensions. Secondly, it is not difficult to see that there is an immediate obstacle to the induction argument in the description of E_1. For, if we have an element of E which can be annihilated by the prime $J = P \cap R$ of R, we cannot deduce that it is also annihilated by the prime P of $R[x]$. The identity $P = JR[x]$ which is available in the above proof does not, of course, hold here. This is also a problem in Lemma 4.2.11, but the solution which we used there also fails here, because it would require that P be a prime minimal over $JR[x]$.

It should be noted that, even in Lemma 4.2.11, we were unable to deduce that *all* primes of the extension ring lying over fundamental primes of the base ring were again fundamental, and this prevented us from concluding that the repleteness of the base ring carried over to the extension. Here, the extension fails to satisfy the incomparability property for prime ideals (see Lemma 4.2.9) and, as a result, the method of Theorem 6.2.3 for describing the layers of $E_{R[x]}$ breaks down completely. In an attempt to resolve this situation, we change our approach, and show that our injective $R[x]$-module can be formed starting from an injective R-module.

The first result which we need is taken from [B&G, Lemma 2.4] and, in fact, holds for any *skew* polynomial ring, $S = R[x; \sigma, \delta]$. We include the proof for the ordinary polynomial ring $S = R[x]$ for the sake of completeness.

Lemma 6.2.4. *Let R be any ring and let S be the polynomial ring, $R[x]$. Let A be a right S-module and suppose that A is an R-submodule of an injective right R-module E. Then there is an S-module structure on E compatible with its right R-module structure and also with the right S-module structure on A.*

Proof. We set

$$B = (E \otimes 1) \oplus (E \otimes x) \subseteq E \otimes_R S$$

and

$$C = (E \otimes 1) \oplus (A \otimes x) \subseteq B$$

which are right R-submodules of $E \otimes_R S$ and define an additive map $f : C \to E$ by

$$f(e \otimes 1 + a \otimes x) = e + ax$$

for all $e \in E$ and $a \in A$. Now, f is a right R-module homomorphism, since

$$
\begin{aligned}
f((e \otimes 1 + a \otimes x)r) &= f(e \otimes r + a \otimes rx) = f(er \otimes 1 + ar \otimes x) \\
&= er + arx = (e + ax)r = (f(e \otimes 1 + a \otimes x))\, r
\end{aligned}
$$

for all $r \in R$, and so it extends to a right R-module homomorphism $g : B \to E$. Furthermore, we define an additive map $h : E \to E$ by

$$h(e) = g(e \otimes x)$$

for each $e \in E$. Again, we note that

$$
\begin{aligned}
(h(e))\, r &= (g(e \otimes x))\, r = g(e \otimes rx) \\
&= g(er \otimes x) = h(er)
\end{aligned}
$$

for each $e \in E$ and $r \in R$ so that we can extend the right R-module structure on E to a right S-module structure by defining

$$ex := h(e)$$

for each $e \in E$. Since, for each $a \in A$,

$$h(a) = g(a \otimes x) = f(a \otimes x) = ax \ ,$$

this S-module structure extends that on A. ■

Thus, if P is a prime ideal of the ring $S = R[x]$, if U is a uniform right ideal of S/P and if we let $E = E_R(U)$, then we can regard E as an S-module. The next lemma allows us to construct an injective S-module using E. The proof is extracted from [B&G, Lemma 3.8] which again deals with skew polynomial extensions.

Lemma 6.2.5. *Let R be any ring and let S be the polynomial ring $R[x]$. Let E be a right S-module and let $\widetilde{E}$ be the Abelian group $\prod_{i=0}^{\infty} E$.*

(a) $\widetilde{E}$ can be made into a right S-module using a multiplication $*$ such that

$$
\begin{aligned}
(e_0, e_1, e_2, \ldots) * r &= (e_0 r, e_1 r, e_2 r, \ldots) \\
(e_0, e_1, e_2, \ldots) * x &= (e_0 x + e_1, e_1 x + e_2, e_2 x + e_3, \ldots)
\end{aligned}
$$

for each $(e_0, e_1, e_2, \ldots) \in \widetilde{E}$ *and* $r \in R$.

(b) Each of the sets $W_n = \{(e_0, e_1, e_2, \ldots) \in \widetilde{E} : e_i = 0$ *for all* $i > n\}$ *is an* S*-submodule of* $\widetilde{E}$.

(c) If E *is an injective* R*-module then* $\widetilde{E}$ *is an injective* S*-module.*

Proof. (a) It is clear that $\widetilde{E}$ is a right R-module under the action $*$ and so, to see that it is an S-module we note that, for each $e \in \widetilde{E}$, $r \in R$ and $i \geq 0$,

$$\begin{aligned}[(e * x) * r]_i &= (e_i x + e_{i+1}) * r = e_i x r + e_{i+1} r \\ &= e_i r x + e_{i+1} r = [(e * r) * x]_i\end{aligned}$$

and therefore $(e * x) * r = (e * r) * x$.

(b) If $e_i = 0$ for all $i \geq n$ then clearly $[e * x]_i = [e * r]_i = 0$ for all $r \in R$ and all $i \geq n$.

(c) For each $i \geq 0$, let $\pi_i : \widetilde{E} \to E$ be the right R-module homomorphism defined by

$$\pi_i(e_0, e_1, e_2, \ldots) = e_i$$

where $(e_0, e_1, e_2, \ldots) \in \widetilde{E}$. We note that,

$$\pi_i(ex) = (\pi_i(e))\, x + \pi_{i+1}(e)$$

for each $i \geq 0$ and $e \in \widetilde{E}$. Now, fix a right S-module A and let $\pi_0^* : \mathrm{Hom}_S(A, \widetilde{E}) \to \mathrm{Hom}_R(A, E)$ be the map defined by

$$(\pi_0^*(f))\,(a) = \pi_0\,(f(a))$$

for each $f \in \mathrm{Hom}_S(A, \widetilde{E})$ and $a \in A$. It is sufficient to show that the map π_0^* is a bijection.

So, suppose $\pi_0^*(f) = \pi_0^*(g)$ for some f and $g \in \mathrm{Hom}_S(A, \widetilde{E})$. We claim that $\pi_i\,((f - g)(a)) = 0$ for each $i \geq 0$ and all $a \in A$. Certainly, this is true for $i = 0$ since

$$\begin{aligned}\pi_0\,((f - g)(a)) = \pi_0\,(f(a) - g(a)) &= \pi_0\,(f(a)) - \pi_0\,(g(a)) \\ &= (\pi_0^*(f))\,(a) - (\pi_0^*(g))\,(a) = 0\end{aligned}$$

for each $a \in A$ where the second equality holds since π_0 is an R-module homomorphism. Suppose it is true for $i = k$ for some $k \geq 0$. Then, since

$$\pi_k\left((f-g)(ax)\right) = \left(\pi_k\left((f-g)(a)\right)\right)x + \pi_{k+1}\left((f-g)(a)\right)$$

and since A is an S-module, we see that $\pi_{k+1}\left((f-g)(a)\right) = 0$ for all $a \in A$. This establishes the claim and it follows that $f = g$ so that π_0^* is injective.

Now let $f_0 \in \mathrm{Hom}_R(A, E)$. We inductively define maps $f_i : A \to E$ for $i > 0$ by

$$f_i(a) = f_{i-1}(ax) - \left(f_{i-1}(a)\right)x$$

for each $a \in A$. We claim that the f_i are R-module homomorphisms for each $i \geq 0$. This is true for $i = 0$ and we assume it is true for $i = k$ for some $k \geq 0$. Then, for all a_1 and $a_2 \in A$,

$$\begin{aligned} f_{k+1}(a_1 + a_2) &= f_k\left((a_1 + a_2)x\right) - \left(f_k(a_1 + a_2)\right)x \\ &= f_k(a_1 x) + f_k(a_2 x) - \left(f_k(a_1)\right)x - \left(f_k(a_2)\right)x \\ &= f_{k+1}(a_1) + f_{k+1}(a_2) \end{aligned}$$

while, for all $a \in A$ and for all $r \in R$,

$$\begin{aligned} f_{k+1}(ar) &= f_k(arx) - \left(f_k(ar)\right)x \\ &= f_k(ax)r - \left(f_k(a)\right)xr \\ &= \left(f_k(ax) - \left(f_k(a)\right)x\right)r \ . \end{aligned}$$

Thus, f_{k+1} is an R-module homomorphism and it follows that the same is true for f_i, for all $i \geq 0$. It follows that the map

$$f = (f_0, f_1, f_2, \ldots)$$

is an R-module homomorphism from A to $\widetilde{E}$. Furthermore, for each $i \geq 0$ and for all $a \in A$,

$$\begin{aligned} \pi_i\left(f(ax)\right) &= f_i(ax) = f_{i+1}(a) + \left(f_i(a)\right)x \\ &= \pi_{i+1}\left(f(a)\right) + \left(\pi_i\left(f(a)\right)\right)x \\ &= \pi_i\left(\left(f(a)\right)x\right) \end{aligned}$$

so that $f(ax) = (f(a))\, x$ from which it follows that f is a right S-module homomorphism. Finally, for each $a \in A$,

$$(\pi_0^*(f))\,(a) = \pi_0\,(f(a)) = f_0(a)$$

so

$$\pi_0^*(f) = f_0 \ .$$

Thus, π_0^* is surjective.

Since π_0^* is a bijection, it follows, as noted above, that $\widetilde{E}$ is an injective right S-module. ∎

Lemma 6.2.6. *Construct the $R[x]$-module $\widetilde{E}$ as in Lemma 6.2.5 and, for each element $e \in \widetilde{E}$, write e_i for its i^{th} component. Then, for each $f \in R[x]$ and $i \geq 0$,*

$$(e * f)_i = \sum_{j=0}^{\infty} e_{i+j} . \frac{1}{j!} \frac{d^j f}{dx^j} \ .$$

Proof. It is sufficient to establish the identity for $f = x^t$ for each $t \geq 0$ and we note that it is clearly true for $t = 0$. Now suppose that the Lemma holds for $f = x^k$ for some $k \geq 0$. Then,

$$\begin{aligned}
(e * x^{k+1})_i &= ((e * x^k) * x)_i = (e * x^k)_i . x + (e * x^k)_{i+1} \\
&= \left(\sum_{j=0}^{\infty} e_{i+j} . \frac{1}{j!} \frac{d^j (x^k)}{dx^j} \right) . x + \sum_{j=0}^{\infty} e_{i+j+1} . \frac{1}{j!} \frac{d^j (x^k)}{dx^j} \\
&= \left(\sum_{j=0}^{k} e_{i+j} . \frac{1}{j!} \frac{d^j (x^k)}{dx^j} \right) . x + \sum_{j=1}^{k+1} e_{i+j} . \frac{1}{(j-1)!} \frac{d^{j-1} (x^k)}{dx^{j-1}} \\
&= e_i x^{k+1} + \sum_{j=1}^{k} e_{i+j} . \frac{1}{j!} . \left(x \frac{d^j (x^k)}{dx^j} + j \frac{d^{j-1} (x^k)}{dx^{j-1}} \right) + e_{i+k+1} \\
&= e_i x^{k+1} + \sum_{j=1}^{k} e_{i+j} . \frac{1}{j!} . \left(\frac{k!}{(k-j)!} + \frac{j\, k!}{(k-j+1)!} \right) . x^{k-j+1} + e_{i+k+1} \\
&= e_i x^{k+1} + \sum_{j=1}^{k} e_{i+j} . \frac{1}{j!} . \frac{(k+1)!}{(k-j+1)!} . x^{k-j+1} + e_{i+k+1} \\
&= \sum_{j=0}^{\infty} e_{i+j} . \frac{1}{j!} \frac{d^j (x^{k+1})}{dx^j}
\end{aligned}$$

so that the Lemma holds for $f = x^{k+1}$, by induction for $f = x^t$, for all $t \geq 0$, and hence for any $f \in R[x]$. ■

When E_R is injective, the injective module $\widetilde{E}_S$ must, of course, contain an injective hull of E_S for which the next result provides us with a candidate in the case when S satisfies the second layer condition. A related result is proved as [B&G, Proposition 3.9] for S being the skew polynomial ring $R[x;\sigma,\delta]$ but in the case where $\sigma(P \cap R) \neq P \cap R$. We do, however, require the additional condition that R be a $\mathbb{Q}$-algebra.

From now on, we will supress the notation "$*$" for the action of S on $\widetilde{E}$.

Lemma 6.2.7. *Let R be a $\mathbb{Q}$-algebra, S be the polynomial ring $R[x]$ and P be a prime ideal of S such that $P \supsetneqq (P \cap R)S$. Let I be a right ideal of S/P, set $E = E_R(I)$ and endow E with the S-module structure of Lemma 6.2.4. Construct $\widetilde{E}$, $W_0, W_1, \ldots$ as in Lemma 6.2.5 and set $W = \bigcup_{n=0}^{\infty} W_n$. Then W_0 is an essential S-submodule of W and so $I_S \leq W \leq E_S(I) \leq \widetilde{E}$.*

Proof. It suffices to show that $W_{n-1} \leq_e W_n$ for each $n \in \mathbb{N}$ and so we consider $w \in W_n - W_{n-1}$ and show that we can find some $s \in S$ such that $0 \neq ws \in W_{n-1}$.

Since $I_R \leq_e E$, there exists some $r \in R$ with $0 \neq w_n r \in I$ and so we can assume that $w_n \in I$. Similarly, if $w_{n-1} \neq 0$, then we can find $r' \in R$ such that $0 \neq (wr')_{n-1} = w_{n-1}r' \in I$. In this case, if $(wr')_n = 0$, then $0 \neq wr' \in W_{n-1}$ and the proof is complete, so we assume instead that $0 \neq (wr')_n \in I$. So, without loss of generality, we will assume from now on that both w_n and w_{n-1} are in I, with $w_n \neq 0$.

Suppose $w_{n-1} = 0$ so that $(wx)_{n-1} = w_n \neq 0$. In this case, if $w_n x = 0$, then wx is a non-zero member of W_{n-1} and the proof is complete, while otherwise, both $(wx)_{n-1}$ and $(wx)_n$ $(= w_n x)$ are non-zero members of I. Thus, replacing w with wx if necessary, we can now assume that both w_n and w_{n-1} are non-zero members of I.

Now, if $f \in P$, then we see from Lemma 6.2.6 that

$$(wf)_n = w_n f = 0$$

and that

$$(wf)_{n-1} = w_{n-1}f + w_n\frac{df}{dx} = w_n\frac{df}{dx} \ .$$

So, we require to show that we can find some $f \in P$ for which $w_n\frac{df}{dx} \neq 0$. Writing $w_n = s + P$ for some $s \in S - P$, we can assume, for a contradiction that, for all $f \in P$, $s\frac{df}{dx} \in P$. Thus, for each $f \in P$ and $t \in S$,

$$st\frac{df}{dx} = s\frac{d(tf)}{dx} - s\frac{dt}{dx}f \in P$$

and, since P is a prime ideal of S, it follows that $\frac{df}{dx} \in P$ for all $f \in P$. Now, since $P \neq (P \cap R)S$, we can find a polynomial of minimal degree

$$f = \sum_{i=0}^{m} a_i x^i \in P - (P \cap R)S \ ,$$

with the $a_i \in R$ and $a_m \notin P \cap R$. Since

$$\sum_{i=1}^{m} i a_i x^{i-1} = \frac{df}{dx} \in P$$

we see from the minimality of m that $ma_m \in P \cap R$ and hence $a_m \in P \cap R$ since R is a $\mathbb{Q}$-algebra. This contradicts the choice of a_m and so we have established that $(W_0)_S \leq_e W_S$.

Finally, by Theorem 1.1.8(i),

$$I_S \leq W \leq E_S(I) \leq \widetilde{E}$$

since $I_S \leq_e E_S \cong W_0$ and $\widetilde{E}$ is an injective S-module. ∎

Hypothesis 6.2.8. Let I be a uniform right ideal of S/P and set $E = E_R(I)$. In the situation of Lemma 6.2.7 above, we see that

$$I_S \leq W \leq E_S(I) \leq \widetilde{E}$$

and we assume that, in the case where R is a Noetherian ring and $R[x]$ satisfies the second layer condition, $E_S(I) = W$. This is, of course, equivalent to the requirement that W be essentially closed in $\widetilde{E}$.

It is, of course, possible to consider Example 2.2.5 (with $n = 1$) as a special case of this construction: if we take $E = E_K(K) = K$, then $\widetilde{E}$ corresponds to the injective $K[x]$-module $K[[x^{-1}]]$, and W corresponds to the submodule $K[x^{-1}]$, an essential extension of $K_{K[x]}$. So, our proof that $E_{K[x]}(K) = K[x^{-1}]$, shows that the hypothesis is true in this particular instance.

In the context of a q-skew polynomial ring $S = R[x; \sigma, \delta]$, where W was first constructed, this hypothesis is certainly true for the case where $\sigma(P \cap R) \neq P \cap R$ (see [B&G, Remark 3.10]). We do not know whether it is always true in our present setting, although we note, in the next result, one further case where it does hold. Indeed, all the examples of §6.3 are of this type. ∎

Theorem 6.2.9. *In the situation of (6.2.7), suppose that S satisfies the second later condition and that the natural embedding of $R/(P \cap R)$ into S/P is an isomorphism. Then $E_S(I) = W$.*

Proof. Suppose, as an induction hypothesis, that we know

$$E_S(I)_n \subseteq W_{n-1}$$

for some $n \geq 0$ (with the convention that $W_{-1} = 0$) and let

$$(e_0, e_1, e_2 \ldots) \in (E_S(I))_{n+1}$$

for some $e_i \in E_R(I)$. Now, there is a r.cl.(P)-semiprime ideal N of S such that

$$(e_0, e_1, e_2, \ldots).N \subseteq (E_S(I))_n \subseteq W_{n-1} \ .$$

In particular, we observe that

$$(e_n, e_{n+1}, e_{n+2}, \ldots).N = 0$$

and it follows, by Theorem 1.3.3, that

$$\widehat{e} := (e_n, e_{n+1}, e_{n+2}, \ldots) \in (E_S(I))_1 \ .$$

We claim that $\widehat{e} \in W_0$ and observe that this will complete the induction step. Since, by assumption,

$$E_{S/P}(I) = E_{R/(P\cap R)}(I) \ , \qquad (A)$$

we see that

$$\left(\widehat{e} + \operatorname{ann}_{E_R(I)}(P \cap R) \oplus 0 \oplus 0 \oplus \cdots\right).P = 0 \tag{B}$$

On the other hand, applying Lemma 1.1.9 twice to (A),

$$\operatorname{ann}_{E_S(I)}(P) \cong \operatorname{ann}_{E_R(I)}(P \cap R)\ . \tag{C}$$

Thus, since there cannot be two copies of $\operatorname{ann}_{E_R(I)}(P \cap R)$ ($\cong E_{R/P\cap R}(I)$), one strictly contained in the other, we conclude from (B) and (C) that $\widehat{e} \in W_0$ which establishes Hypothesis 6.2.8 in this case. ∎

Remarks 6.2.10. (i) In particular, the condition of Theorem 6.2.9 is satisfied in all of the examples of §6.3. Also, if P contains $x - \alpha$, for some $\alpha \in R$, and $x - \alpha$ is normal in S, then every element of S can be written as a polynomial in $x - \alpha$, with the coefficients on the left, and it is then easy to see that the natural embedding of $R/(P \cap R)$ into S/P is an isomorphism. So Hypothesis 6.2.8 holds by the last result.

(ii) Let I be a uniform right ideal of S/P and set $E = E_R(I)$. Suppose that K is an assassinator prime of E_R. Then $\operatorname{ann}_E(K)$ is a fully faithful R/K-module. However, $(\operatorname{ann}_E(K))\,S \cap \operatorname{ann}_E(P)$ is a fully faithful S/P-module and thus, $K = P \cap R$. So we can write

$$E_R = \oplus_{\lambda \in \Lambda} {}^{\lambda}E$$

for some uniform injective R-modules, ${}^{\lambda}E$, each with assassinator $P \cap R$, and where Λ is some index set. Furthermore, we note that $\mathcal{C}_R(P \cap R) \subseteq \mathcal{C}_S(P)$. (To see this, we can assume without loss that $P \cap R = 0$ so that $\mathcal{C} := \mathcal{C}_R(P \cap R) = \mathcal{C}_R(0)$ is an Ore set in R and in S. Then, since $\mathcal{C}$ is disjoint from P, $\mathcal{C} \subseteq \mathcal{C}_S(P)$, by [G&W, 9.21] for instance.) It follows that S/P is a torsion-free right $R/(P \cap R)$-module. Thus, the right submodule $I_{R/(P\cap R)}$ is also torsion-free and, for each $\lambda \in \Lambda$, so must be $(\operatorname{ann}_{{}^{\lambda}E}(P \cap R))_{R/(P\cap R)}$ which is an essential extension of $(I \cap \operatorname{ann}_{{}^{\lambda}E}(P \cap R))_{R/(P\cap R)}$. Thus, Theorem 1.3.3 applies to the uniform injectives $({}^{\lambda}E)_R$. The fundamental primes are, of course, identical for each $({}^{\lambda}E)_R$. ∎

Notation 6.2.11. Assume we are in the situation of (6.2.7) and (6.2.10(ii)) and suppose S satisfies the second layer condition. Then, for each $i \geq 0$, we put

$$E_i = \oplus_{\lambda \in \Lambda} ({}^{\lambda}E)_i$$

where $({}^{\lambda}E)_i$ is the i^{th} layer of ${}^{\lambda}E$, as in the notation of (1.3.1). ∎

Theorem 6.2.12. *Let R be a Noetherian $\mathbb{Q}$-algebra, let S be the polynomial ring $R[x]$ which we assume satisfies the second layer condition and let P be a prime ideal of S such that $P \supsetneqq (P \cap R)S$. Let U be uniform right ideal of S/P, set $E = E_R(U)$ and endow E with the S-module structure of Lemma 6.2.4. Construct $\widetilde{E}$, $W_0, W_1, \ldots$ as in Lemma 6.2.5, set $W = \bigcup_{n=0}^{\infty} W_n$ and assume that Hypothesis 6.2.8 holds so that we can (and do) identify W with $E_S(U)$. With the notation of (1.3.1) and (6.2.11),*

$$(E_S(U))_i \subseteq E_i \oplus E_{i-1} \oplus \cdots \oplus E_2 \oplus E_1 \oplus 0 \oplus 0 \oplus \cdots \qquad (\#)$$

for each $i \geq 0$. In particular, equality holds when $i = 0$ or 1. Furthermore:

(i) if $Q \in A_k(P)$ and equality holds in $(\#)$ for $i = k-1$ and for $i = k$, then $Q \in A_{k+1}(P)$;

(ii) if $K \in A_k(P \cap R)$, then there exists a $Q \in A_j(P)$ for some $j \geq k$ such that $Q \cap R = K$; when equality holds in $(\#)$ for $i = k$, we can take $j = k$;

(iii) if $Q \in A_j(P)$, then $Q \cap R \in A_k(P \cap R)$ for some $k \leq j$; when equality holds in $(\#)$ for $i = j-1$, we can take k to be the minimal value of j such that $Q \in A_j(P)$.

Proof. We note first that $(E_S(U))_1$ is isomorphic to a simple right module over $Q(S/P)$, the Goldie quotient ring of S/P, while $E_1 = \oplus_{\lambda \in \Lambda} \left({}^{\lambda}E\right)_1$ where each $\left({}^{\lambda}E\right)_1$ is isomorphic to a simple right module over $Q\left(R/(P \cap R)\right)$, the Goldie quotient ring of $R/(P \cap R)$. Since $R/(P \cap R)$ embeds naturally in S/P, the same can be said of their Goldie quotient rings and it follows that

$$E_1 \oplus 0 \oplus 0 \oplus \cdots \subseteq (E_S(U))_1 \ . \qquad (A)$$

We note that $(\#)$ certainly holds for $i = 0$ and we assume it is true for all $i \leq n-1$ for some $n \in \mathbb{N}$. So let

$$e = (e_0, e_1, e_2, \ldots) \in (E_S(U))_n \ .$$

Then, we can find a r.cl.(P)-semiprime ideal I of S such that

$$eI \subseteq (E_S(U))_{n-1} \subseteq E_{n-1} \oplus E_{n-2} \oplus \cdots \oplus E_1 \oplus 0 \oplus 0 \oplus \cdots \tag{B}$$

where the second inequality holds by the induction hypothesis. Now, $I \cap R$ is a r.cl.$(P \cap R)$-semiprime ideal by Theorem 6.1.3 and we observe that

$$\{(e_0a, e_1a, e_2a, \ldots) : a \in I \cap R\} = e(I \cap R) \subseteq eI$$

so that $e_j(I \cap R) \subseteq E_{n-1-j}$ for each $0 \leq j \leq n-1$ and $e_j(I \cap R) = 0$ for each $j \geq n-1$. It follows that $e_j \in E_{n-j}$ for $0 \leq j \leq n-1$ while $e_j \in E_1$ for $j \geq n-1$. We require to show that, in fact $e_j = 0$ for $j \geq n$.

By Hypothesis 6.2.8, we can find some $m \geq 0$ such that $e_j = 0$ for $j > m$. So, we can assume that $m \geq n$ and it is sufficient to show that $e_m = 0$. Now, letting f be an element of P and denoting its first derivative by f', we see from Lemma 6.2.6 that

$$(ef)_{m-1} = e_{m-1}f + e_mf' = e_mf' \tag{C}$$

where the second equality holds by (A) since $e_{m-1} \in E_1$. By (B), $(eI)_{m-1} \subseteq E_0 = 0$ since $m \geq n$. For the same reason, $(e_m)I = (eI)_m = 0$, so if we assume that $e_m \neq 0$, $I \subseteq P$. If we denote by M' the set of first derivatives of the elements of an ideal M of S, it follows from (C) that

$$0 = (eI)_{m-1} = e_m(I)'$$

whence

$$I' \subseteq P \ .$$

Suppose that $I = P_1 \cap \ldots \cap P_t$ for some incomparable prime ideals $P_k \in$ r.cl.(P) and assume that $P_1 = P$. Then

$$P \supseteq I' + P \supseteq (P_1 \ldots P_t)' + P = P_1'P_2 \ldots P_t + P \supseteq P_1'P_2 \ldots P_t.$$

and, since P is prime and the P_k are incomparable,

$$P \supseteq P' \ .$$

As we saw in the proof of Lemma 6.2.7, this conclusion is impossible for a prime $P \neq (P \cap R)S$ and so we conclude that $e_m = 0$ as required. This completes the proof of (#).

Of course, it is trivial that equality holds in (#) for $i = 0$ and, furthermore, equality holds for $i = 1$ by (A) above.

(i) Let $k \in \mathbb{N}$, let $Q \in A_k(P)$ and assume that equality holds in (#) for $i = k - 1$ and for $i = k$.

Now, we can find $e \in (E_S(U))_k - (E_S(U))_{k-1}$ such that $eQ \in (E_S(U))_{k-1}$. If we write

$$e = (e_0, e_1, \ldots, e_{k-1}, 0, 0, \ldots)$$

then $e_m \in E_{k-m}$ for $0 \leq m \leq k-1$ and, by the equality of (#) at $i = k-1$, $e_m \notin E_{k-m-1}$ for some m. Take m maximal with respect to this property. Without loss of generality, we can assume that all other $e_t = 0$; for clearly, by the equality of (#) at $i = k$, and with the assumption that $e_t = 0$ for all $t \neq m$, $e \in (E_S(U))_k - (E_S(U))_{k-1}$ while

$$(eQ)_t = e_m \frac{1}{(m-t)!} \frac{d^{m-t}Q}{dx^{m-t}} \in E_{k-t-1}$$

for each $t \leq m$. It then follows by (#) that

$$\widehat{e} := (\overbrace{0, \ldots, 0}^{m+2 \text{ zeros}}, e_m, 0, 0, \ldots) \notin (E_S(U))_k \ .$$

On the other hand,

$$(\widehat{e}Q)_t = e_m \frac{1}{(m-t+1)!} \frac{d^{m-t+1}Q}{dx^{m-t+1}} \in E_{k-t}$$

for each $t \leq m+1$. It then follows by the equality of (#) at $i = k$ that $\widehat{e}Q \subseteq (E_S(U))_k$ and thus $Q \in A_{k+1}(P)$.

(ii) Let $K \in A_k(P \cap R)$. Then, we can find $e \in E_k - E_{k-1}$ such that $\mathrm{ann}_R\left(\frac{eR+E_{k-1}}{E_{k-1}}\right) = K$. We put

$$\widetilde{e} = (e, 0, 0, \ldots)$$

so that, by (#), $\widetilde{e} \notin (E_S(U))_{k-1}$. We can, however, choose $j \geq k$ such that $\widetilde{e} \in (E_S(U))_j$. Then there exist primes $Q_1, \ldots, Q_s \in \mathrm{Fund}(P)$, where each $Q_\lambda \in A_{\zeta(\lambda)}(P)$ for some $j \geq \zeta(\lambda) \geq k$, such that

$$\widetilde{e}.\left(\bigcap_\lambda Q_\lambda\right)^{j-k+1} \subseteq (E_S(U))_{k-1} \ .$$

We note therefore that, by (#) again,

$$e.\left(\bigcap_\lambda (Q_\lambda \cap R)\right)^{j-k+1} \subseteq (E)_{k-1}$$

and by the faithfulness of $\frac{eR+E_{k-1}}{E_{k-1}}\big|_{R/K}$ it follows that $\bigcap_\lambda (Q_\lambda \cap R) \subseteq K$. It is then easy to see that $(Q_\lambda \cap R) \subseteq K$ for some $1 \leq \lambda \leq s$. However, by Lemma 6.1.1, $(Q_\lambda \cap R) \in \mathrm{cl.}(P \cap R) = \mathrm{cl.}(K)$ so by the incomparability property of $\mathrm{cl.}(P \cap R)$, we in fact have $(Q_\lambda \cap R) = K$. Since $Q_\lambda \in A_{\zeta(\lambda)}(P)$ and $j \geq \zeta(\lambda) \geq k$, the first part of (i) has been proved. Of course, if equality holds in (#) at $i = k$, then $\widetilde{e} \in (E_S(U))_k$ so we can take $j = k$ above. In this case, $\zeta(\lambda) = k$ and the second part of (ii) follows.

(iii) Let $Q \in A_j(P)$. Then, we can find $e \in (E_S(U))_j - (E_S(U))_{j-1}$ such that $\mathrm{ann}_S\left(\frac{eS+(E_S(U))_{j-1}}{(E_S(U))_{j-1}}\right) = Q$. If we write

$$e = (e_0, e_1, \ldots, e_{j-1}, 0, 0, \ldots)$$

then $e_m \in E_{j-m}$ for $0 \leq m \leq j-1$. Now, we can find primes $K_1, \ldots, K_s \in \mathrm{r.cl.}(P \cap R)$, where $K_\lambda \in X_{\zeta(\lambda)}(P \cap R)$ for some $1 \leq \zeta(\lambda) \leq j$ and such that

$$e.\left(\bigcap_\lambda K_\lambda\right)^j = 0 \ .$$

By the faithfulness of $\frac{eS+(E_S(U))_{j-1}}{(E_S(U))_{j-1}}\big|_{S/Q}$, we see that $(\bigcap_\lambda K_\lambda)^j \subseteq Q$ whence $\bigcap_\lambda K_\lambda \subseteq Q$. It follows easily that $K_\lambda \subseteq Q$ for some $1 \leq \lambda \leq s$. By the same argument as that in (ii), $K_\lambda = Q \cap R$ which proves the first part of (iii) since $K_\lambda \in X_{\zeta(\lambda)}$ and $\zeta(\lambda) \leq j$.

Now let j be minimal such that $Q \in A_j(P)$ and assume that equality holds in $(\#)$ at $i = j-1$. We claim that

$$e_0 \notin E_{j-1}. \tag{D}$$

Otherwise, we observe that

$$\overline{e} := (e_1, e_2 \ldots, e_{j-1}, 0, 0, \ldots) \notin (E_S(U))_{j-2}\ . \tag{E}$$

We can find r.cl.(P)-semiprime ideals $I_1, \ldots, I_{j-2}$ such that

$$e.Q.I_1 \ldots I_{j-2} \subseteq (E_S(U))_1 = E_1 \oplus 0 \oplus 0 \oplus \cdots$$

from which we see that

$$\overline{e}.Q.I_1 \ldots I_{j-2} = 0\ .$$

It follows from (E) and Theorem 1.3.3 that $Q \in A_{j-1}(P)$ a contradiction of the choice of j. This establishes (D).

Since $e.Q \subseteq E_S(U)_{j-1}$, we see from $(\#)$ that

$$e.(Q \cap R) \subseteq E_{j-1} \oplus E_{j-2} \oplus \cdots \oplus E_1 \oplus 0 \oplus 0 \oplus \cdots$$

whence

$$e_0.(Q \cap R) \subseteq E_{j-1}.$$

By this observation and (D) above, $Q \cap R \in A_j(P \cap R)$. This completes the proof of (iii). ∎

Remark 6.2.13. (i) Unfortunately, equality need not hold at $(\#)$, even for $i = 2$ as we will see in the next section (Example 6.3.4). Of course, by Remark 1.3.6, we cannot find $Q \in X_2(P) - A_2(P)$ but, in fact, in this example, $X_i(P) = A_i(P) = \{P\}$ for *each* i so that strong repleteness is preserved.

(ii) Since $P \supsetneq (P \cap R)S$, the same can be said for all primes in the clique of P by Lemma 6.1.2. So, applying Theorem 6.2.12(ii) with $k = 2$ together with

Remark 1.3.6, to each prime in the clique of P, we see that, whenever Q' is a prime in the clique of P and $K \rightsquigarrow Q' \cap R$, we can find a prime Q of S such that $Q \cap R = K$ and $Q \in \text{r.cl.}(Q')$. Of course, if equality holds at (#) for $i = 2$ with Q' in place of P in Theorem 6.2.12, we can insist that $Q \rightsquigarrow Q'$. Thus, whenever $K \in X_n(P \cap R)$, we can find a prime $Q \in X_m(P)$, for some $m \geq n$, with $Q \cap R = K$ and when equality holds at (#) for $i = 2$ (replacing P with Q' in Theorem 6.2.12), we can take $m = n$. ■

Corollary 6.2.14. *In the situation of Theorem 6.2.12 (but without assuming equality at (#)), if P is rep. rep. in S, then $P \cap R$ is rep. rep. in R.*

Proof. Let $K \in \text{r.cl.}(P \cap R)$. By Remark 6.2.13(ii), we can find a prime $Q \in \text{r.cl.}(P)$ such that $Q \cap R = K$. Assuming that P is rep. rep., $Q \in \text{Fund}(P)$ and, by Theorem 6.2.12(iii), $K = Q \cap R \in \text{Fund}(P)$. Since this holds for all $K \in \text{r.cl.}(P)$, the result follows. ■

Remark 6.2.15. As seen in (6.2.2), the above Corollary does not extend to the analogous statement substituting strongly rep. rep. for rep. rep.. In that example, both S and R are rep. rep. rings (being commutative) which are not strongly rep. rep.. In the next section, we will see a noncommutative counter-example (6.3.2) (where, furthermore, S is not rep. rep. and R is rep. rep.). We do not know of any example where S is strongly rep. rep. and R is not strongly rep. rep.. ■

Corollary 6.2.16. *Adopt the notation and hypotheses of Theorem 6.2.12.*

(i) Suppose that, whenever Q and $Q' \in \text{r.cl.}(P)$ and $Q \cap R = Q' \cap R$, then $Q = Q'$. If $P \cap R$ is rep. rep. in R, then P is rep. rep. in S.

(ii) Suppose that, whenever Q and $Q' \in X_i(P)$, for some $i \geq 0$, and $Q \cap R = Q' \cap R$, then $Q = Q'$. Furthermore, assume that equality holds at (#) in Theorem 6.2.12 for each $i \geq 0$. If $P \cap R$ is strongly rep. rep. in R, then P is strongly rep. rep. in S.

Proof. Suppose $Q \in X_i(P)$ for some $i \geq 0$ and that $Q \notin X_j(P)$ for any $j < i$. By Lemma 6.1.1, $Q \cap R \in X_i(P \cap R)$.

(i) If $P \cap R$ is rep. rep. in R, then $Q \cap R \in \text{Fund}(P \cap R)$ and, by Theorem 6.2.12(ii), there exists some $Q' \in \text{Fund}(P)$ such that $Q' \cap R = Q \cap R$. Assuming the hypothesis of (i), $Q = Q'$. Since this holds for each $Q \in \text{r.cl.}(P)$, P is rep. rep. in T.

(ii) If $P \cap R$ is strongly rep. rep. in R, then $Q \cap R \in A_i(P \cap R)$ and, by Theorem 6.2.12(ii), there exists some $Q' \in A_i(P)$ such that $Q' \cap R = Q \cap R$. Assuming the hypothesis of (ii), $Q = Q'$ so that $Q \in A_i(P)$ and then, by Theorem 6.2.12(i), $Q \in A_k(P)$ for each $k \geq i$. Since this holds for each $Q \in X_i(P)$, P is strongly rep. rep. in S. ∎

Remark 6.2.17. As noted at (6.2.15), we will see in the next section that, while the additional hypotheses of Corollary 6.2.16 are certainly not necessary conditions, without them we can find a ring R which is rep. rep, but for which $R[x]$ is not rep. rep.. In fact, even assuming that $P \cap R$ is strongly rep. rep., P need not be rep. rep., although in our example of this, R is not a rep. rep. ring. (In the examples which we consider, it is, however, the first hypothesis of part (ii) which fails and this is weaker than that of part (i).) We do not know of any strongly rep. rep. ring R for which $R[x]$ is not strongly rep. rep. or even of any rep. rep. ring R where $P \cap R$ is strongly rep. rep. and P is not strongly rep. rep.. In the next section, we do present an example where equality does not hold at (#) in Theorem 6.2.12, but even here strong repleteness carries over to the polynomial ring. ∎

§6.3 Some Examples

Example 6.3.1. We first show that strong representational repleteness may carry over to a polynomial ring even where the first hypothesis of Corollary 6.2.16(ii) does not hold.

Let $R = \mathbb{C}(t)[\theta; \sigma]$ where σ is the automorphism of $\mathbb{C}(t)$ given by $\sigma(t) = (t+1)$, so that $\theta t = (t+1)\theta$. The co-Artinian prime $J = \theta R$ is strongly rep. rep. by Theorem 5.2.1. (In fact, R is a strongly rep. rep. ring.) Put $S = R[x]$ and

$P = (x-t)S + \theta S$ so that $P \cap R = J$. By Theorem 5.2.1 (or as shown in Example 5.3.1), P is a strongly rep. rep. prime of S.

Furthermore, we note that

$$R/J \cong S/P \cong \mathbb{C}(t) \ ,$$

with the natural embedding providing an isomorphism, so Hypothesis 6.2.8 holds by Theorem 6.2.9.

On the other hand, by Lemma 5.1.10, the links of r.cl.(P) are the trivial links together with the links

$$(x-t-\alpha-1)S + \theta S \rightsquigarrow (x-t-\alpha)S + \theta S$$

and consequently,

$$X_n(P) = \{(x-t-m)S + \theta S : m \in \{0, 1, \ldots, n-1\}\}$$

for each $n \in N$. Now clearly, $\big((x-t-m)S + \theta S\big) \cap R = \theta S$ for $0 \leq m \leq n-1$ so that $Q \cap R = \theta S = J$ for every $Q \in$ r.cl.(P). (Of course, since r.cl.$(J) = \{J\}$, this is inevitable by Lemma 6.1.1.) It follows that the hypothesis of Corollary 6.2.16(ii) fails for every pair of primes Q and $Q' \in X_i(P)$ and for every i except $i = 0$. ∎

Example 6.3.2. *An example where the coefficient ring is rep. rep. but not strongly rep. rep. and the polynomial ring is not rep. rep.; where, furthermore, the polynomial ring contains a strongly rep. rep. prime whose contraction to the coefficient ring is not strongly rep. rep..*

We let R be the same ring as in the previous example and put $\overline{R} = \frac{R}{<\theta^2>}$. Then, putting $\overline{J} = \frac{\theta R}{<\theta^2>}$, we see that $\overline{J}$ is the unique prime ideal of $\overline{R}$ and so R is rep. rep.. However, since $\frac{\overline{J}}{\overline{J}^2}$ is non-zero, $\overline{J} \rightsquigarrow \overline{J}$. So, since $\overline{J}^2 = 0$, $\overline{J}$ cannot be strongly rep. rep. in $\overline{R}$.

Now let $\overline{S} = \overline{R}[x]$ and consider the prime $\overline{P} = (x-t)\overline{S} + \theta\overline{S}$, which satisfies Hypothesis 6.2.8 for the same reason as in Example 6.3.1. Since, in the previous example, every prime in the right clique of P contained θ, we see that the link graph of $\overline{P}$ is obtained from that of P by factoring out by the ideal $\theta^2 S$. That is,

$$X_n(P) = \{(x-t-m)S + \theta S : m \in \{0, 1, \ldots, n-1\}\}$$

for each $n \in \mathbb{N}$. Furthermore, an identical argument to that of Example 5.3.1 produces again the elements $\overline{\gamma_j}$ as the generators of $B_{n-1}(\overline{P})$. However, since $\theta^2 = 0$ in $\overline{S}$, there are only two non-zero $\overline{\gamma_j}$ which generate the bimodule in this case, namely

$$\overline{\gamma_{n-1}} := \prod_{i=0}^{n-1}(x-t-n+i)^i(x-t)^{n-1} + S_n \ldots S_1$$

and

$$\overline{\gamma_{n-2}} := \prod_{i=0}^{n-2}(x-t-n+i)^i(x-t-1)^{n-2}\theta + S_n \ldots S_1 \ .$$

Of course,

$$\left((x-t)\overline{S} + \theta\overline{S}\right).\overline{\gamma_{n-1}} = 0$$

and

$$\left((x-t-1)\overline{S} + \theta\overline{S}\right).\overline{\gamma_{n-2}} = 0$$

in $B_{n-1}\left(\overline{P}\right)$ and so

$$A_n\left(\overline{P}\right) = \begin{cases} \{\overline{P} = (x-t)\overline{S} + \theta\overline{S}\} & \text{if } n = 1, \\ \{\overline{P}, (x-t-1)\overline{S} + \theta\overline{S}\} & \text{if } n \geq 2. \end{cases}$$

Thus, $A_n\left(\overline{P}\right) = X_n\left(\overline{P}\right)$ if and only if $n = 1$ or 2 and, indeed, $\left|\text{Fund}\left(\overline{P}\right)\right| = 2$ while $\left|\text{r.cl.}\left(\overline{P}\right)\right| = \infty$.

While we have seen that $\overline{S}$ is not a rep. rep. ring and $\overline{R}$ is, we note that the prime $\overline{Q} = x\overline{S} + \theta\overline{S}$ (which also satisfies Hypothesis 6.2.8) is strongly rep. rep. in $\overline{S}$ although $\overline{Q} \cap \overline{R} = \overline{J}$ is not strongly rep. rep. in $\overline{R}$. To see that $\overline{Q}$ is strongly rep. rep., we note that, by Theorem 5.1.12, or since $\overline{Q}$ is generated by normal elements, r.cl. $\left(\overline{Q}\right) = \overline{Q}$ while $\overline{Q}$ is a co-Artinian prime with $\overline{Q}^n \neq 0$ for any $n \in \mathbb{N}$ and so $B_{n-1}\left(\overline{Q}, \overline{Q}\right) \neq 0$. ∎

Example 6.3.3. *An example where a prime of the polynomial ring is not rep. rep. but its contraction to the coefficient ring is strongly rep. rep., but where the coefficient ring is not, however, rep. rep..*

We take, for our coefficient ring $\widehat{R}$, the ring $\overline{S} = \frac{R}{<\theta^2>}[x]$ of the previous example which, as we saw, is not rep. rep.. Furthermore, we take $\widehat{J}$ to be the prime $\overline{Q}$ $\left(= x\widehat{R} + \theta\widehat{R}\right)$ which, as noted, is strongly rep. rep. in $\widehat{R}$. Now, although the prime $\widehat{P} = y\widehat{S} + x\widehat{S} + \theta\widehat{S}$ of $\widehat{S} = \widehat{R}[y]$ satisfies $\widehat{P} \cap \widehat{R} = \widehat{J}$, we will show that $\widehat{P}$ is not rep. rep. in $\widehat{S}$.

First, however, we note once again that $\widehat{P}$ satisfies Theorem 6.2.9 and hence Hypothesis 6.2.8.

By Lemma 5.1.10, the links of r.cl.$(\widehat{P})$ are the trivial links together with the links

$$(y - t - \alpha)\widehat{S} + x\widehat{S} + \theta\widehat{S} \rightsquigarrow (y - t - \alpha + 1)\widehat{S} + x\widehat{S} + \theta\widehat{S}$$

so that, for $n \in \mathbb{N}$,

$$\begin{aligned} X_n(\widehat{P}) &= \left\{(y - t - m)\widehat{S} + x\widehat{S} + \theta\widehat{S} : m \in \{0, 1, \ldots, n-1\}\right\} \\ S_n &= (y-t)(y-t-1)\cdots(y-t-n+1)\widehat{S} + x\widehat{S} + \theta\widehat{S}\ . \end{aligned}$$

By an induction argument similar to those of Chapter 4,

$$\begin{aligned} B_{n-1}(\widehat{P}) &= \frac{S_n S_{n-1} \ldots S_2}{S_n S_{n-1} \ldots S_1} \\ &= \frac{\sum_{j=0}^{n-1} \sum_{k=0}^{j} \prod_{i=k}^{j} (y-t-n+i)^{i-k} (y-t+j-n+1)^{j-k} x^k \theta^{n-j-1} \widehat{S}}{\sum_{j=0}^{n} \sum_{k=0}^{j} \prod_{i=k}^{j} (y-t-n+i)^{i-k} x^k \theta^{n-j} \widehat{S}}. \end{aligned}$$

Recalling that $\theta^2 = 0$ in $\widehat{S}$, we note that the non-zero elements

$$\widehat{\gamma_{n-1,k}} := \prod_{i=k}^{n-1} (y-t-n+i)^{i-k} (y-t)^{n-k-1} x^k + S_n \ldots S_1,$$

for $0 \leq k \leq n-1$, and

$$\widehat{\gamma_{n-2,k}} := \prod_{i=k}^{n-2} (y-t-n+i)^{i-k} (y-t-1)^{n-k-2} x^k \theta + S_n \ldots S_1,$$

for $0 \leq k \leq n-2$, generate ${}_{\widehat{S}}\left(B_{n-1}(\widehat{P})\right)_{\widehat{S}}$. Of course,

$$\left((y-t)\widehat{S} + x\widehat{S} + \theta\widehat{S}\right) .\widehat{\gamma_{n-1,k}} = 0,$$

for all $0 \leq k \leq n-1$, and

$$\left((y-t-1)\widehat{S} + x\widehat{S} + \theta\widehat{S}\right) .\widehat{\gamma_{n-2,j}} = 0,$$

for all $0 \leq k \leq n-2$, in $B_{n-1}(\widehat{P})$ and so

$$A_n(\widehat{P}) = \begin{cases} \left\{\widehat{P} = (y-t)\widehat{S} + x\widehat{S} + \theta\widehat{S}\right\} & \text{if } n = 1, \\ \left\{\widehat{P}, (y-t-1)\widehat{S} + x\widehat{S} + \theta\widehat{S}\right\} & \text{if } n \geq 2. \end{cases}$$

Thus, $A_n(\widehat{P}) = X_n(\widehat{P})$ if and only if $n = 1$ or 2 and, indeed, $\left|\mathrm{Fund}(\widehat{P})\right| = 2$ while $\left|\mathrm{r.cl.}(\widehat{P})\right| = \infty$. ∎

Example 6.3.4. *An example where equality fails at (#) in Theorem 6.2.12 for $i = 2$.*

Form the differential operator ring

$$T = \mathbb{C}[w, x][y; \delta]$$

where $\delta(w) = 0$ and $\delta(x) = w$. We note that w is central in T and that, since $yx - xy = w$, x is central modulo wT. Thus, the prime ideal $J = wT + xT$ is polycentral. In particular, by Corollary 2.3.8, the clique of J is a singleton. It follows by [G&W, Theorem 12.20] (or by [J2, Corollary 7.3.10]), that J is classically localizable in T and we let R be the corresponding localization. Furthermore, we let S be the ordinary polynomial ring $R[z]$ and we let P be the prime ideal $wS + xS + (z-y)S$. Since P is polycentral in T, we see that its clique is also a singleton.

We note that $S/JS \cong \mathbb{C}(y)[z]$ and that the dimension of the vector space $(J/J^2)_{\mathbb{C}(y)}$ is two. Moreover, since

$$w = yx - xy = (z-y)x - x(z-y) \in P^2 ,$$

we see that the dimension of the vector space $(P/P^2)_{\mathbb{C}(y)}$ is also two. Now, in the notation of (1.3.10),

$$B_1(P) = B_1(P, P) = P/P^2$$

and

$$B_1(J) = B_1(J, J) = J/J^2$$

while, by Theorem 1.3.11, and in the notation of (6.2.10(ii)) to (6.2.12),

$$(E_S(U))_2 \,/\, (E_S(U))_1 \cong \mathrm{Hom}_{S/P}\left(B_1(P), (E_S(U))_1\right)$$

and

$$({}^{\lambda}E)_2/({}^{\lambda}E)_1 \cong \mathrm{Hom}_{R/J}\left(B_1(J), ({}^{\lambda}E)_1\right)$$

for each $\lambda \in \Lambda$. It follows that the dimension of $\frac{(E_S(U))_2}{(E_S(U))_1}|_{\mathbb{C}(y)}$ is two while the dimension of $\frac{E_2 \oplus E_1 \oplus 0 \oplus \cdots}{E_1 \oplus 0 \oplus \cdots}|_{\mathbb{C}(y)}$ is three. Thus, although P satisfies Hypothesis 6.2.8 by Theorem 6.2.9, we see that equality fails at (#) in Theorem 6.2.12 for $i = 2$.

However, in this case, since the cliques of both P and J are singletons and both $B_n(P)$ and $B_n(J)$ are non-zero for each $n \in \mathbb{N}$, P is strongly rep. rep. in S and J is strongly rep. rep. in R. ∎

§6.4 Notes

As noted above, the results of §6.1 are extracted from unpublished work of Brown and Goodearl [B&G]. In particular, Lemma 6.1.1 is taken from [B&G, Theorem 1.7 and Corollary 1.8], Lemma 6.1.2 from [B&G, Corollary 1.9] and Theorem 6.1.3 from [B&G, Theorem 2.1].

Theorems 6.2.1 and 6.2.3 are both new results.

Lemma 6.2.4 is a special case of [B&G Lemma 2.4] and Lemma 6.2.5 a special case of [B&G, Lemma 3.8].

Lemmas 6.2.6 and 6.2.7 are both new: the proof of the analogous result to Lemma 6.2.7 for the q-skew polynomial case is proved as [B&G, Proposition 3.9] but requires a condition which cannot apply to ordinary polynomial rings.

Theorems 6.2.10 and 6.2.12, and Corollaries 6.2.14 and 6.2.16 are all new results.

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