Wrinkling Problems for Non-Linear Elastic Membranes

by

Barry McKay

A thesis submitted to
the Faculty of Science
at the University of Glasgow
for the degree of
Doctor of Philosophy.

March 1996

Contents

Abstract

In this thesis we study several examples of finite deformations of non-linear, elastic, isotropic membranes consisting of both incompressible and compressible materials which result in the membrane becoming wrinkled. To investigate the nature and occurrence of these wrinkled regions we adapt ordinary membrane theory by using a systematic approach developed by Pipkin (1986) and Steigmann (1990) which accounts for wrinkling automatically. In each problem considered, we employ the relaxed strain-energy function proposed by Pipkin (1986) and assume that the in-plane principal Cauchy stresses are non-negative.

A discussion of the basic equations for a membrane from the three-dimensional theory and the derivation of the relaxed strain-energy function from tension field theory is given. The resulting equations of equilibrium are then used to formulate various problems considered and solutions are obtained by analytical or numerical means for both the tense and wrinkled regions. In particular we consider the deformation of a membrane annulus of uniform thickness which is subjected to either a displacement or a stress on the inner and outer radii. We present the first analytical solution for such a problem, for incompressible and compressible materials, for both the tense and wrinkled regions. This first problem therefore provides a simple example to illustrate the theory of Pipkin (1986).

The second problem studies an elastic, circular, cylindrical membrane which is inflated by an internal pressure and subjected to a flexural deformation. The equations of equilibrium are solved numerically, two different solution methods being described, and results are presented graphically showing the deformed cross-section of the cylinder for incompressible and compressible materials. Particular attention is given to the value of curvature at which wrinkling begins. An incremental deformation is also considered to investigate possible bifurcation solutions which could occur at some finite value of curvature.

The final problem considers two butt jointed, incompressible, elastic, circular, cylindrical membranes of different material and geometric properties. In particular we fix the cylinders to have different initial radii which ensures that wrinkling will occur. The composite cylinder is inflated and subjected to axial loading on either end. This deformation may have useful applications in surgery as it could be considered as a first approximation model for arterial grafts, the wrinkled surface having important implications in the formation of blood clots as blood flows through such a region. Again the equations of equilibrium are solved numerically and graphical results of the deformed, axial length against the deformed radius for a range of values of the parameters are given showing the tense and wrinkled regions.

Preface

This thesis was submitted to the University of Glasgow in accordance with the requirements for the degree of Doctor of Philosophy.

I would sincerely like to thank Dr.D.M. Haughton for his help, guidance and encouragement throughout the period of this research and also for the temporary use of his office in the later stages of this work.

I would like to thank The Engineering and Physical Science Research Council who funded me throughout this research by means of a studentship.

Chapter 1

Introduction

Many authors have studied problems relating to the finite deformation of non-linear, elastic membranes making use of a variety of different methods to investigate the occurrence of wrinkling on the surface of the membrane. In particular, Reissner's tension field theory (Reissner (1938)) has been used in modified form to tackle problems of this nature. One such version of tension field theory was developed by Pipkin (1986), see also Steigmann (1990), who used a systematic approach to account for wrinkling automatically. Within such a wrinkled region it is found that ordinary membrane theory predicts a negative principal stress which is contrary to the basic concepts of membrane theory. To avoid this difficulty Pipkin (1986) modelled the wrinkled region using tension field theory together with a relaxed strain-energy function which replaced the usual strain-energy function. This theory is described in more detail in chapter 3.

Many problems have been considered using this approach. Steigmann and Pipkin (1989) investigated axisymmetric finite deformations of an isotropic, elastic membrane that was initially flat or developable with no inflating pressure. Two specific boundary value problems were detailed, one involving constriction in the middle of the cylinder and the other pure twisting, where the ends of the cylinder were twisted in opposite directions. In each case approximate solutions only were found based on the assumption that the membrane was fully wrinkled. Taut regions were located near the constrained parts of the cylinder but were assumed to be small in comparison to the wrinkled regions. Steigmann (1990) also considered axisymmetric deformations and extended Reissner's theory to develop a general tension field theory for finite deformations of curved membranes consisting of isotropic material. A general solution in implicit form for such a deformation was presented and two specific problems were considered to illustrate the theory. The first problem studied a surface of revolution loaded by axial point forces and the second investigated an asymptotic solution near the tip of a crack in an initially plane sheet of material.

Haseganu and Steigmann (1994) used this theory to investigate the finite bending of an inflated cylinder for an incompressible material. They obtained numerical solutions for the tense and wrinkled regions by reducing the equations of equilibrium to an analysis of quadratures. Li and Steigmann (1993) presented a numerical solution for the finite deformation of an isotropic, incompressible, annular membrane which was fixed at its outer radius while its inner radius was prestretched and attached to a rigid hub. The hub was subsequently rotated through a specified angle. The authors showed that wrinkling occurred for certain combinations of the hub radius and rotation angle. Li and Steigmann (1995) also considered the axisymmetric deformation of an isotropic, incompressible, elastic, hemispherical membrane fixed at its equator and subjected to a force at its pole. Using the theory of Steigmann (1990) the analysis of the problem was again reduced to quadratures and numerical solutions were given for both the tense and wrinkled regions.

The work contained in this thesis considers problems concerned with the wrinkling of non-linear, elastic membranes consisting of both incompressible and compressible, isotropic, hyperelastic materials. The purpose of this thesis is to provide several examples of finite deformations which cause wrinkling to occur in some region of the membrane. We then illustrate how ordinary membrane theory can be adapted to obtain a solution in this wrinkled region by using the systematic approach developed by Pipkin (1986) and Steigmann (1990) which accounts for wrinkling automatically. In each problem considered, we utilise the relaxed strain-energy function proposed by Pipkin (1986). We note from the problems described above that previous work on membranes has been almost exclusively concerned with incompressible materials. A second aim of this thesis is therefore to consider deformations of membranes consisting of compressible materials to extend wrinkling theory to these materials. We also investigate whether the surface area of a membrane composed of compressible material will wrinkle more than one composed of incompressible material for a given deformation.

Having provided the motivation for the work contained in this thesis, we now go on to discuss each chapter in some detail. In chapter 2 we introduce the basic equations of elasticity, following the approach of standard texts given by Truesdell and Noll (1965) and Ogden (1984) while introducing the notation to be adopted in this thesis. This provides the equations of motion and constitutive relations used in finite, three-dimensional, elasticity theory. Membrane theory can be formulated from a two-dimensional or a three-dimensional point of view. Here we adopt the three-dimensional version. From this three-dimensional theory we derive the equations of equilibrium for a membrane. The basic equations of membrane theory can be found in Green and Zerna (1968), Green and Adkins (1970) and Naghdi (1972), however, the formulation given here follows Haughton and Ogden (1978a). Haughton and Ogden (1978a) used the notion of averaging the variables through the thickness of the membrane in the reference configuration. This formulation is different to derivations given by other authors in that Haughton and Ogden (1978a) worked relative to the reference configuration rather than the current configuration. This is of considerable advantage when incremental deformations are being considered as it avoids having to calculate explicitly the changes in geometry from the current to reference configurations.

Chapter 3 deals with the theory required to describe a wrinkled region. As ordinary membrane theory predicts this region to be of negative principal stress which contradicts the basic ideas of membrane theory, we model this wrinkled region through tension field theory. In practice the distribution of wrinkles would be controlled by the small bending stiffness of the material. In membrane theory, however, we assume the bending stiffness of a membrane to be negligible. Consequently, we utilise tension field theory together with the relaxed strain-energy to describe the wrinkled region. To obtain equilibrium solutions we use energy considerations and try to minimize the total strain-energy associated with a deformed membrane. Tension field theory states that this occurs when an infinitesimal distribution of wrinkles have formed. This is because it is assumed that no energy is required to fold the membrane. Details of this energy minimization are given in section 3.2. We then describe the construction of the relaxed strain-energy function using the notion of simple tension which occurs when the membrane has only one principal stress non-zero. Finally we describe wrinkling theory from a physical viewpoint.

In the remaining three chapters we find solutions to problems using the analysis of chapters 2 and 3. Previously, deformations considered in this way have always required extensive numerical work to obtain a solution to the problem. In chapter 4 we present the first analytical solution for such a problem for both incompressible and compressible materials and for both the tense and wrinkled regions. The aim of this first problem is therefore to provide a simple example to illustrate explicitly the theory of Pipkin (1986) and Steigmann (1990).

Chapter 4 considers the deformation of an isotropic, elastic, membrane annulus of uniform thickness which is subjected to either a displacement or a stress on the inner and outer radii. This problem was initially investigated by Rivlin and Thomas (1951) who assumed the membrane to consist of the incompressible Mooney-Rivlin material. The problem was formulated to ensure the membrane was always in tension to avoid the possibility of wrinkling. Although an approximate solution to the problem was obtained for the limiting case of infinitesimally small deformations, the problem had to be solved numerically. This problem has been studied by many other authors, namely, Yang (1967), Wong and Shield (1969), Wu (1978) and Lee and Shield (1980). In each case the authors have had to solve the governing equations by numerical means while assuming the membrane consisted of incompressible material described by the Mooney-Rivlin or neo-Hookean strain-energy functions. More recently, however, Haughton (1991) demonstrated how exact solutions could be obtained for both the incompressible and compressible forms of the Varga material defined below by (4.2.5) and (4.3.1) respectively. Haughton (1991) also found that the principal Cauchy stresses varied monotonically with the undeformed radius of the annulus. These principal Cauchy stresses remained positive throughout the membrane for each displacement imposed on the radii of the annulus except for one case which occurred for the compressible material. This special case was not considered in detail and hence a solution for the region of negative stress was not obtained.

Specifically four different deformations are considered in chapter 4 for both incompressible and compressible Varga materials. We therefore extend the work by Haughton (1991) to obtain exact analytical solutions for incompressible and compressible materials for both the tense and wrinkled regions by considering deformations that will ensure that wrinkling occurs. In each case, it is shown that the principal Cauchy stresses vary monotonically with the undeformed radius simplifying the problem considerably. Subsequently these are studied throughout. An unexpected feature of the deformation is that the deformed thickness of the membrane is uniform for incompressible and compressible materials.

Chapter 5 considers the problem of an isotropic, elastic, circular, cylindrical membrane which is inflated by an internal pressure and subjected to a flexural deformation. This problem was initially formulated by Stein and Hedgepeth (1961) using a modified version of Reissner's tension field theory (Reissner (1938)) to deal with any regions of negative stress, and hence wrinkling of the membrane, which may occur. This solution was based on an approximate theory associated with continuously distributed wrinkles over a smooth surface. Haseganu and Steigmann (1994) also considered this deformation extending the above to finite deformations of an incompressible material described by the Varga strain-energy function. The authors utilised the theory developed by Pipkin (1986) to cope with any wrinkled area. Using the Euler-Lagrange equations which describe the flexural deformation, Haseganu and Steigmann (1994) derived a pair of integrals and used these to reduce the analysis of the problem to quadratures. By numerical evaluation of these integrals the variables required for the solution to the problem were obtained.

Chapter 5 extends this work to consider the same deformation for an incompressible material described by the three-term strain-energy function defined by (5.2.28) with (5.2.29) below and for compressible materials. We begin by formulating the problem and describing the kinematics of flexure for incompressible materials. We obtain the equations of equilibrium for a pressurised membrane under flexural deformations before finding equations of equilibrium for a membrane that has been inflated only. On solving the latter set of equations we obtain expressions for the principal stretches, which provide parameters used in the flexural problem. This follows the work by Haseganu and Steigmann (1994) although we have used a different formulation for the problem. In many ways this is a non-standard problem. In particular, while retaining the spirit of membrane wrinkling theory given by Pipkin (1986), see also Steigmann (1990), it is possible to obtain a solution in at least one other way. We describe the solution procedure for both methods for comparison allowing for the evolution of a wrinkled region. We then present results for the incompressible Varga and three-term materials.

In Haseganu and Steigmann (1994) it was assumed that the inflating pressure remained constant. Clearly as the cylinder is bent further there will be a change in the enclosed volume. This changing volume will lead to a change in inflating pressure and we therefore compare their constant pressure approach with the assumption that the cylinder is inflated by a fixed mass of incompressible gas.

Assuming the deformation progresses smoothly as the curvature of the cylinder increases, there will be some critical value of curvature at which wrinkling becomes possible. This is of considerable interest and results of this critical curvature are given for both methods, the first assuming a constant pressure, and the second, inflation with an ideal gas. At this critical value of curvature at which wrinkling could occur, it may be that the bent configuration found from ordinary membrane theory will bifurcate into some other configuration at some finite value of curvature. This configuration could take the form of a single kink at some point along the axial length of the cylinder as can happen with a cylindrical balloon. The possibilities of bifurcation are therefore considered in this chapter. Finally, we formulate the problem for compressible materials and present numerical results for compressible materials governed by the Varga strain-energy function, choosing the widest possible range of values of bulk modulus.

The final chapter considers two butt jointed, isotropic, elastic, incompressible, right, circular, cylindrical membranes of different geometric and material properties. The composite cylinder is inflated by a hydrostatic pressure and then extended longitudinally with applied axial loads. These axial loads are designed to tether the membrane so that the plane containing the join remains fixed in space. Recently Hart and Shi (1991) considered this problem investigating the behaviour of two joined cylinders which possessed different material properties but the same geometric properties. Hart and Shi (1991) regarded their work as a possible first approximation for surgical implants in an artery. This led the authors to also consider two longer cylinders being joined by a shorter cuff of different material again assuming the same geometric properties for all components. Hart and Shi (1991) obtained exact solutions for this problem for isotropic, incompressible, elastic materials. These solutions were exact only in the sense that the equations of equilibrium were integrated analytically, numerical methods being required to determine a variable used to describe the deformed configurations. Numerical results were presented for isotropic, incompressible, Mooney-Rivlin and neo-Hookean materials. In a subsequent paper by Hart and Shi (1993) a similar analysis was used to extend the solution to orthotropic, incompressible materials, in particular, results being given for the Vaishnav and How-Clarke materials. Again the same geometry was assumed for all components in this second paper.

The deformations considered by Hart and Shi in the above work did not address the possibility of a wrinkled region forming. In our problem the membranes are assumed to be of different materials, shear moduli, undeformed thickness and initial radii. The differing radii will ensure that wrinkling occurs provided the inflating pressures are low. We assume that this is the case to avoid any possible instabilities. If such a theory is used to model arterial grafts (or other anastomoses) the wrinkled region would be very significant in the formation of blood clots as blood flows through such a region. The problem formulation follows Hart and Shi (1991) but again the derivation of the governing equations given here is somewhat different, but equivalent, to their work. To solve the equations of equilibrium we require to use numerical integration. The solution procedure is described. Graphical results showing the tense and wrinkled regions (if one exists) are presented for a range of values of the parameters, predominantly concentrating on the effects of different undeformed radii for the two component cylinders.

The work contained in chapter 4 has been modified and published (jointly with D.M. Haughton) in the International Journal of Engineering Science, Volume 33, (1995) and the work described in each of chapters 5 and 6 has been submitted, in modified form, to appropriate journals.

Chapter 2

Basic Equations for a Membrane

2.1 Introduction

In this chapter we introduce the basic equations of elasticity, following the standard approach of Truesdell and Noll (1965) and Ogden (1984) using the notation to be adopted in this thesis. From this three-dimensional theory we then derive the basic equations required to consider problems in membrane theory. The basic equations of membrane theory can be found in Green and Zerna (1968), Green and Adkins (1970) and Naghdi (1972). Haughton and Ogden (1978a) also formulated the membrane equations in a somewhat different, but equivalent, manner by considering the average of the deformation gradient to be the measure of deformation in the membrane. This average value was taken through the thickness of the shell in the reference configuration and enables an estimate of the order of magnitude of error arising from the approximations associated with the stress and deformation and hence the equations of thin shell theory. Throughout their report Haughton and Ogden (1978a) worked relative to the reference configuration rather than the current configuration. This is particularly useful when considering incremental deformations as the equations are derived with respect to the reference configuration and therefore avoids having to calculate explicitly the changes in geometry for such a deformation.

The notion of averaging the variables will be adopted in this thesis for the purpose of obtaining the equations of equilibrium for a membrane. The errors associated with the approximations made will not be calculated explicitly in this work although an indication of when smaller terms are present will be given. The reader is therefore referred to Haughton and Ogden (1978a) for details about this.

2.2 Kinematics

On choosing an arbitrary origin, let $\mathbf{X}$ be the position vector of a material point in the body B_0 which is in three-dimensional Euclidean space. This region occupied by the body is referred to as the reference configuration. Often it is convenient to take the initial undeformed configuration to be the reference configuration and in this case the body will be unstressed. If the body is deformed into a new configuration B say, henceforth referred to as the current or deformed configuration, then the material point $\mathbf{X}$ moves to the position $\mathbf{x}$ according to

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \tag{2.2.1}$$

where $\boldsymbol{\chi} : B_0 \to B$ is a bijection and is called the *deformation* of B_0 to B. The Cartesian bases $\mathbf{E}_i$ and $\mathbf{e}_i$ $(i = 1, 2, 3)$ are chosen to represent the reference and current configurations respectively. The *deformation gradient tensor,* $\mathbf{F}$, is defined by

$$\mathbf{F} = \operatorname{Grad} \mathbf{x}, \tag{2.2.2}$$

where Grad is the gradient operator in the reference configuration. In general, the deformation gradient $\mathbf{F}$ will depend on $\mathbf{X}$, however, in the special case where $\mathbf{F}$ is independent of $\mathbf{X}$ the deformation is said to be *homogeneous.* If we now consider volume elements dv and dV in B and B_0 at $\mathbf{x}$ and $\mathbf{X}$ respectively we have

$$dv = (\det \mathbf{F}) dV, \tag{2.2.3}$$

where det denotes evaluation of the determinant. By definition, finite volumes are taken to be positive and hence (2.2.3) gives

$$J = \det \mathbf{F} > 0, \tag{2.2.4}$$

where J is known as the *dilatation.* Comparing (2.2.3) and (2.2.4) we note that physically J is defined as the relative change in volume of the volume elements as the body is deformed. If the volume of any region in B_0 is unchanged on deformation we therefore have $dv = dV$ and hence $J = \det \mathbf{F} = 1$. Materials which satisfy this constraint are said to be *incompressible.* From (2.2.4) we conclude that $\mathbf{F}$ is non-singular and apply the Polar Decomposition Theorem to $\mathbf{F}$ which permits the unique decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.2.5}$$

where $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are the symmetric, positive definite, right and left stretch tensors respectively defined by

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T.$$

As $\mathbf{U}$ is symmetric and positive definite, there exists positive values λ_i $(i = 1, 2, 3)$ of $\mathbf{U}$ corresponding to the principal directions $\mathbf{u}^{(i)}$ $(i = 1, 2, 3)$ such that

$$\mathbf{U}\mathbf{u}^{(i)} = \lambda_i \mathbf{u}^{(i)}, \quad i = 1, 2, 3. \tag{2.2.6}$$

Subsequently the subscript or superscript i will denote that $i = 1, 2, 3$ unless stated otherwise. These positive values λ_i are known as the *principal stretches* and $\mathbf{u}^{(i)}$ are the *Lagrangian principal axes* of the deformation. Using (2.2.5) with (2.2.6) we have

$$\mathbf{VR}\mathbf{u}^{(i)} = \mathbf{RU}\mathbf{u}^{(i)} = \lambda_i \mathbf{R}\mathbf{u}^{(i)}, \tag{2.2.7}$$

which shows that λ_i are also the principal stretches of $\mathbf{V}$ corresponding to the *Eulerian principal axes* $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$. We note at this point that since $\mathbf{R}$ is proper orthogonal (2.2.5) gives

$$\det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V},$$

and hence from the above analysis

$$J = \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3. \tag{2.2.8}$$

For incompressible materials, we therefore obtain the constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1, \tag{2.2.9}$$

which implies that the principal stretches are no longer independent. This constraint is known as the *incompressibility condition.*

2.3 Stress and Equilibrium

Let $\mathbf{N}$ and $\mathbf{n}$ be outward unit normals to the surfaces of the bodies B_0 and B in the reference and current configurations respectively. The traction $\mathbf{T}$ per unit area of a surface in the reference configuration can be expressed as

$$\mathbf{T} = \mathbf{s}^T\mathbf{N}, \qquad (2.3.1)$$

where $\mathbf{s}$ is the second order *nominal stress tensor* and the transpose of this quantity is known as the *Piola-Kirchhoff stress.* Similarly the traction $\mathbf{t}$ per unit area of a surface in the current configuration can be written as

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \qquad (2.3.2)$$

where $\boldsymbol{\sigma}$ is the second order *Cauchy stress tensor.* These two second order tensors defined above are related by

$$\mathbf{s} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \qquad (2.3.3)$$

and we note that $\boldsymbol{\sigma}$ is symmetric, that is $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$, but in general, $\mathbf{s}$ is not.

For a self-equilibrated body, in the absence of body forces, the *equation of equilibrium* is

$$\text{div}\ \boldsymbol{\sigma} = \mathbf{0}, \qquad (2.3.4)$$

where div is the divergence operator in the current configuration. The equations of equilibrium (2.3.4) will be employed throughout this work, however, as (2.3.4) uses $\mathbf{x}$ as the independent variable it is not always convenient to formulate the equations of equilibrium relative to the current configuration. For example, when considering an incremental deformation of a body it is simpler to work in the reference configuration so that the geometry of the deformation does not need to be recalculated. We therefore state the corresponding equation to (2.3.4) for the reference configuration which is

$$\text{Div}\ \mathbf{s} = \mathbf{0}, \qquad (2.3.5)$$

where Div is the divergence operator in the reference configuration.

We now define an elastic solid as a material whose components of stress are single-valued functions of the deformation gradient such that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}). \tag{2.3.6}$$

An elastic material is said to be *hyperelastic* if there exists a scalar-valued *strain-energy function* $W(\mathbf{F})$ per unit reference volume which satisfies

$$\frac{d}{dt}W(\mathbf{F}) = Jtr(\boldsymbol{\sigma}\frac{d\mathbf{F}}{dt}),$$

where tr denotes the trace of a second order tensor. In principle the strain-energy W could be a function of the deformation gradient $\mathbf{F}$ and of the position vector $\mathbf{X}$. However, in this thesis we only consider homogeneous materials and in this case the strain-energy W is only a function of $\mathbf{F}$.

The constitutive relation (2.3.6) is required to be *objective* under rigid body motions and this can be expressed as

$$\boldsymbol{\sigma}(\mathbf{QF}) = \mathbf{Q}\boldsymbol{\sigma}(\mathbf{F})\mathbf{Q}^T, \quad \forall \quad \text{proper orthogonal} \quad \mathbf{Q}. \tag{2.3.7}$$

If we now assume the material is *isotropic* , which infers that the material has no preferred directions, we must have

$$\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{FP}), \quad \forall \quad \text{proper orthogonal} \quad \mathbf{P}. \tag{2.3.8}$$

From (2.3.7) and (2.3.8) we conclude that $\boldsymbol{\sigma}$ is an isotropic function of $\mathbf{V}$ and hence the principal components of $\boldsymbol{\sigma}$ are functions of the principal components of $\mathbf{V}$.

For an unconstrained, hyperelastic material we can express the Cauchy stress tensor in terms of the strain-energy function and the deformation gradient by

$$J\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}^T}, \tag{2.3.9}$$

which, on using (2.3.3), allows us to write the nominal stress tensor as

$$\mathrm{s} = \frac{\partial W}{\partial \mathbf{F}^T}. \tag{2.3.10}$$

If we now assume $W(\mathbf{F})$ is isotropic and objective we obtain

$$W(\mathbf{F}) = W(\mathbf{QF}) = W(\mathbf{FP}), \quad \forall \quad \text{proper orthogonal} \quad \mathbf{P} \quad \text{and} \quad \mathbf{Q}. \tag{2.3.11}$$

Hence we conclude that W is a scalar isotropic function of $\mathbf{V}$. W can therefore be expressed in terms of the principal values of $\mathbf{V}$ and hence W is a symmetric function of the principal stretches λ_i only, giving

$$W = W(\lambda_1, \lambda_2, \lambda_3). \tag{2.3.12}$$

Given a homogeneous, isotropic, unconstrained, hyperelastic material we can, after some work, rewrite (2.3.9) when evaluated along a principal axis, as

$$J\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i = 1, 2, 3, \quad \text{no sum.} \tag{2.3.13}$$

For incompressible materials we have the incompressibility condition (2.2.9) which ensures that materials can only be deformed if there is no change in volume. We therefore have to reconsider the constitutive relations. We retain $W = W(\lambda_1, \lambda_2, \lambda_3)$ but to deal with the interdependence of the principal stretches which follows from (2.2.9), we must introduce a Lagrange multiplier, p, and rewrite our constitutive relation (2.3.9) as

$$\boldsymbol{\sigma} = \mathbf{F} \frac{\partial W}{\partial \mathbf{F}^T} - p\mathbf{I}, \tag{2.3.14}$$

where $\mathbf{I}$ is the identity tensor. After further work the principal stresses for an incompressible, homogeneous, isotropic, hyperelastic material can be written in component form as

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i = 1, 2, 3, \quad \text{no sum,} \tag{2.3.15}$$

where we note that p can be regarded as acting as a hydrostatic pressure, evaluation of which can be obtained from (2.3.15) together with the assumption of membrane theory which is explained in the next section.

2.4 Basic Membrane Equations

On choosing an arbitrary origin, let $\mathbf{X}$ be the position vector of a material point in the reference configuration described by θ^i, a triad of orthogonal, curvilinear coordinates, such that

$$\mathbf{X} = \mathbf{A}(\theta^1, \theta^2) + \theta^3 \mathbf{A}_3(\theta^1, \theta^2). \tag{2.4.1}$$

Here $\mathbf{A}$ is the position vector from the origin to the middle surface, $\theta^3 = 0$, and $\mathbf{A}_3$ is the positive outward unit normal to the middle surface with θ^3 the parameter measured in this direction such that $-H/2 \leq \theta^3 \leq H/2$. The membrane is therefore bounded by the surfaces $\theta^3 = \pm H/2$ where H is the thickness of the membrane in the reference configuration. We can introduce a set of orthonormal base vectors for the reference configuration $\mathbf{E}_i$ defined by

$$\mathbf{E}_\mu = \frac{\partial \mathbf{X}}{\partial \theta^\mu} / \left|\frac{\partial \mathbf{X}}{\partial \theta^\mu}\right|, \tag{2.4.2}$$

and

$$\mathbf{E}_3 = \frac{\partial \mathbf{X}}{\partial \theta^3} / \left|\frac{\partial \mathbf{X}}{\partial \theta^3}\right| = \mathbf{A}_3, \tag{2.4.3}$$

where $\mu = 1, 2$ and we have used (2.4.1). Henceforth Greek indices will take the values 1 and 2 only.

If we now let R_{min} be the smallest of the principal radii of curvature of the middle surface of the membrane and H_{max} be the greatest thickness of the membrane in the reference configuration then by an assumption of membrane theory we can write

$$\epsilon = \frac{H_{\text{max}}}{R_{\text{min}}} \ll 1. \tag{2.4.4}$$

We can therefore redefine the base vectors $\mathbf{E}_\mu$ as follows. Using (2.4.1), (2.4.2) and (2.4.4) we have

$$\frac{\partial \mathbf{X}}{\partial \theta^\mu} = \frac{\partial \mathbf{A}}{\partial \theta^\mu} + \theta^3 \frac{\partial \mathbf{A}_3}{\partial \theta^\mu} = \frac{\partial \mathbf{A}}{\partial \theta^\mu} + \mathbf{O}(\epsilon),$$

and so

$$\mathbf{E}_\mu = \frac{\partial \mathbf{X}}{\partial \theta^\mu} / \left|\frac{\partial \mathbf{X}}{\partial \theta^\mu}\right| = \frac{\partial \mathbf{A}}{\partial \theta^\mu} / \left|\frac{\partial \mathbf{A}}{\partial \theta^\mu}\right| + \mathbf{O}(\epsilon),$$

where the $\mathbf{O}$ notation is adopted with the usual meaning. To lowest order we obtain

$$\mathbf{E}_\mu = \mathbf{A}_\mu = \mathbf{A}_{,\mu}, \tag{2.4.5}$$

where we have introduced $()_{,\mu}$ to be the operator $\partial()/\partial\theta^\mu/|\partial()/\partial\theta^\mu|$ and therefore (2.4.3) and (2.4.5) give $\mathbf{E}_i = \mathbf{A}_i$ $(i = 1, 2, 3)$. Membrane theory is concerned with making approximations to the lowest order and the errors involved in making these approximations can be calculated explicitly. However, the errors are not given in this work and henceforth we will work to lowest order terms only, although an indication of when smaller terms are present may be given in certain cases. For details of the calculations concerning these errors the reader is referred to Haughton and Ogden (1978a).

If we now deform the body according to the mapping (2.2.1) we have the material point given by $\mathbf{X}$ defined by the position vector $\mathbf{x}$ in the current configuration. In particular the middle surface $\theta^3 = 0$ deforms into the surface $\lambda_3\theta^3 = 0$ (to lowest order) and we use the same convected, orthogonal, curvilinear coordinates θ^i to describe the membrane by

$$\mathbf{x} = \mathbf{a}(\theta^1, \theta^2) + \lambda_3(\theta^1, \theta^2)\theta^3\mathbf{a}_3(\theta^1, \theta^2). \tag{2.4.6}$$

Here $\mathbf{a}$ is the vector from an arbitrary origin to the deformed middle surface $\lambda_3\theta^3 = 0$ and $\mathbf{a}_3$ is the positive outward unit normal to this surface with θ^3 measured in this direction such that $-h/2 \leq \lambda_3\theta^3 \leq h/2$. Here h is the thickness of the membrane in the current configuration. In general $\lambda_3\theta^3 = 0$ will not represent the middle surface of the deformed membrane as we cannot assume the deformation to be symmetrical about $\theta^3 = 0$. However, to lowest order it can be shown that $\lambda_3\theta^3 = 0$ does represent the middle surface of the deformed shell.

We now define the base vectors $\mathbf{e}_i$ for the current configuration in terms of a set of orthonormal vectors $\mathbf{a}_i$. Using (2.4.6) and neglecting terms of $\mathbf{O}(\epsilon^2)$ we obtain

$$\mathbf{e}_\mu = \frac{\partial\mathbf{x}}{\partial\theta^\mu}/|\frac{\partial\mathbf{x}}{\partial\theta^\mu}| = \mathbf{a}_\mu + \mathbf{O}(\epsilon), \tag{2.4.7}$$

and

$$\mathbf{e}_3 = \frac{\partial \mathbf{x}}{\partial \theta^3} / |\frac{\partial \mathbf{x}}{\partial \theta^3}| = \mathbf{a}_3, \tag{2.4.8}$$

where $\mathbf{a}_\mu = \mathbf{a}_{,\mu} = \partial \mathbf{a}/\partial\theta^\mu / |\partial \mathbf{a}/\partial \theta^\mu|$. To lowest order, neglecting terms of $\mathbf{O}(\epsilon)$, we therefore have $\mathbf{e}_i = \mathbf{a}_i$ $(i = 1, 2, 3)$.

We are now in a position to define the deformation gradient $\mathbf{F}$ for a membrane. From (2.2.2) with (2.4.1) and (2.4.6) we obtain

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{a}}{\partial \mathbf{A}} + \lambda_3 \mathbf{a}_3 \otimes \mathbf{A}_3, \tag{2.4.9}$$

to lowest order. It follows that λ_3 is the principal stretch normal to the middle surface of the membrane and for isotropic materials the shear stress components $\sigma_{\mu 3}$ $(\mu = 1, 2)$ are at most $\mathbf{O}(\epsilon)$.

We now consider (2.3.4) to obtain the *equations of equilibrium* for a membrane referred to the current configuration in terms of the orthonormal vectors $\mathbf{a}_i$. To obtain this system consider

$$\mathbf{a}_i \cdot \text{div}\, \boldsymbol{\sigma} = \sigma_{ji,j} + \sigma_{ji}\mathbf{a}_k \cdot \mathbf{a}_{j,k} + \sigma_{kj}\mathbf{a}_i \cdot \mathbf{a}_{j,k} = 0, \tag{2.4.10}$$

where $()_{,j}$ denotes the operator $\partial()/\partial\theta^j / |\partial()/\partial\theta^j|$ and summation over the indices $j, k = 1, 2, 3$ is implied. In other words we can alternatively write (2.4.10) as

$$\left.\begin{aligned} \sigma_{\mu i,\mu} + \sigma_{3i,3} + \sigma_{\mu i}\mathbf{a}_\nu \cdot \mathbf{a}_{\mu,\nu} + \sigma_{3i}\mathbf{a}_\nu \cdot \mathbf{a}_{3,\nu} + \sigma_{\mu\nu}\mathbf{a}_i \cdot \mathbf{a}_{\nu,\mu} & \\ + \sigma_{\mu 3}\mathbf{a}_i \cdot \mathbf{a}_{3,\mu} + \sigma_{3\nu}\mathbf{a}_i \cdot \mathbf{a}_{\nu,3} &= 0, \end{aligned}\right\} \tag{2.4.11}$$

where $\mu, \nu = 1, 2$, i is taken to be 1, 2 or 3 and we have used the formulae

$$\mathbf{a}_{3,3} = 0, \quad \mathbf{a}_3 \cdot \mathbf{a}_{\mu,3} = 0, \quad \mathbf{a}_3 \cdot \mathbf{a}_{3,\mu} = 0, \tag{2.4.12}$$

which follow from

$$\mathbf{a}_3 \cdot \mathbf{a}_\mu = 0 \quad \text{and} \quad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1. \tag{2.4.13}$$

We want to replace $\sigma_{3i,3}$ in (2.4.11). This is done through the process of *averaging* the variables. The average value of a variable is calculated by considering the average through the current thickness of the membrane. In this work we give a brief insight into this process but the reader is referred to Haughton and Ogden (1978a) for details of this derivation. Consider a deformed membrane of thickness h so that

$$-\frac{h}{2} \leq \lambda_3\theta^3 \leq \frac{h}{2}, \tag{2.4.14}$$

then we simply calculate the average of $\sigma_{3i,3}$ by evaluating

$$\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3i,3}\; d(\lambda_3\theta^3) = \frac{1}{h}[\sigma_{3i}]_{-h/2}^{h/2}. \tag{2.4.15}$$

We need to find $[\sigma_{3i}]_{-h/2}^{h/2}$ $(i = 1, 2, 3)$ before substituting for $\sigma_{3i,3}$ in (2.4.11). We therefore examine the boundary conditions on the surface $\lambda_3\theta^3 = \pm h/2$. Suppose that the surface loading consists only of normal pressures $P^{\pm}$ acting on $\lambda_3\theta^3 = \pm h/2$ respectively in the current configuration. We then obtain

$$\boldsymbol{\sigma}\mathbf{n} = -P^{\pm}\mathbf{n} \quad \text{on} \quad \lambda_3\theta^3 = \pm\frac{h}{2}, \tag{2.4.16}$$

where $\mathbf{n}$ is the positive outward unit normal given by

$$\mathbf{n} = \frac{\mathbf{x}_{,1} \times \mathbf{x}_{,2}}{|\mathbf{x}_{,1} \times \mathbf{x}_{,2}|}, \tag{2.4.17}$$

where $\mathbf{x}_{,\mu}$ $(\mu = 1, 2)$ are tangent vectors to the surfaces $\lambda_3\theta^3 = \pm h/2$ and the $\times$ symbol denotes the vector or cross product of two vectors. If we neglect terms of higher order we can show the normal $\mathbf{n}$ is defined by

$$\mathbf{n} = \pm\mathbf{a}_3 - \frac{1}{2}h_{,\alpha}\mathbf{a}_\alpha \quad \text{on} \quad \lambda_3\theta^3 = \pm\frac{h}{2}, \tag{2.4.18}$$

where we have used (2.4.6), (2.4.14) and, from the geometry of the deformation, taken the normal to be in the inward negative direction on the surface $\lambda_3\theta^3 = -h/2$. Substituting (2.4.18) into (2.4.16) yields

$$\pm\,\sigma_{3i}\mathbf{a}_i - \frac{1}{2}h_{,\alpha}\sigma_{\alpha i}\mathbf{a}_i = -P^{\pm}(\pm\mathbf{a}_3 - \frac{1}{2}h_{,\alpha}\mathbf{a}_\alpha) \quad \text{on} \quad \lambda_3\theta^3 = \pm\frac{h}{2}, \tag{2.4.19}$$

where we have used the symmetry of $\boldsymbol{\sigma}$. Choosing $i = 3$ and neglecting higher order terms gives

$$\sigma_{33} = -P^{\pm} \quad \text{on} \quad \lambda_3\theta^3 = \pm\frac{h}{2}, \tag{2.4.20}$$

since $h_{,\alpha}\sigma_{\alpha i}/2$ is negligible when compared to the lowest order terms. Taking $i = \mu = 1, 2$ yields

$$\sigma_{3\mu} = \pm\frac{1}{2}h_{,\alpha}\sigma_{\alpha\mu} \quad \text{on} \quad \lambda_3\theta^3 = \pm\frac{h}{2}, \tag{2.4.21}$$

again neglecting the higher order term $Ph_{,\alpha}/2$. Finally, for definiteness, we assume $P^- = P$ and $P^+ = 0$ and therefore obtain, from (2.4.20) with (2.4.15),

$$\frac{1}{h}[\sigma_{33}]_{-h/2}^{h/2} = \frac{P}{h}, \tag{2.4.22}$$

and using (2.4.21) with (2.4.15) gives

$$\frac{1}{h}[\sigma_{3\mu}]_{-h/2}^{h/2} = \frac{1}{h}h_{,\alpha}\sigma_{\alpha\mu}. \tag{2.4.23}$$

Taking the average of (2.4.11) and replacing $\sigma_{3i,3}$ with the expressions given in (2.4.23) and (2.4.22) yields the equations of equilibrium for a membrane, for $i = \kappa = 1, 2$ and $i = 3$ respectively, as

$$\left.\begin{aligned} \sigma_{\mu\kappa,\mu} + \sigma_{\mu\kappa}\mathbf{a}_\nu \cdot \mathbf{a}_{\mu,\nu} + \sigma_{\mu\nu}\mathbf{a}_\kappa \cdot \mathbf{a}_{\nu,\mu} + \frac{1}{h}h_{,\mu}\sigma_{\mu\kappa} &= 0, \\ \sigma_{\mu\nu}\mathbf{a}_3 \cdot \mathbf{a}_{\nu,\mu} + \frac{P}{h} &= 0, \end{aligned}\right\} \tag{2.4.24}$$

where $\sigma_{\mu\kappa}$ are now average values.

In membrane theory we assume that

$$\sigma_{3i} = \sigma_{i3} = 0, \quad i = 1, 2, 3, \tag{2.4.25}$$

which, to lowest order, follows from (2.4.21), (2.4.22) and (2.4.19). This condition is known as the *membrane assumption* and is of considerable use in simplifying membrane problems, since taking $i = 3$ leads to the condition

$$\sigma_{33} = 0. \tag{2.4.26}$$

That is the principal Cauchy stress in the normal direction is assumed to be zero. Together with (2.3.15) this condition, given the form of the strain-energy function, provides an equation to calculate the hydrostatic pressure p when dealing with incompressible materials. We can use this fact to show how the strain-energy function, and hence the principal Cauchy stresses, can be expressed in terms of the two principal stretches λ_1 and λ_2 only. If we first consider a strain-energy W which is a function of the principal stretches λ_i such that (2.3.12) holds we can redefine this strain-energy function as

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1}), \tag{2.4.27}$$

for incompressible materials having used the incompressibility condition (2.2.9) and

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_2, \lambda_3(\lambda_1\lambda_2)), \tag{2.4.28}$$

for compressible materials having used (2.4.26) with (2.3.13) to obtain $\partial W/\partial\lambda_3 = 0$. We therefore have reduced the strain-energy to a function of the two principal stretches λ_1 and λ_2 for all materials. We can in turn rewrite the principal Cauchy stresses (2.3.13) and (2.3.15) in terms of this strain-energy function for compressible and incompressible materials respectively. Using the notation $W_i = \partial W/\partial\lambda_i$ we have, for a compressible material,

$$\hat{W}_\mu = W_\mu + W_3\frac{\partial\lambda_3}{\partial\lambda_\mu}, \quad \mu = 1, 2, \quad \text{no sum.}$$

However, $W_3 = 0$ having used (2.4.26) with (2.3.13). Hence $\hat{W}_\mu = W_\mu$ ($\mu = 1, 2$) and clearly $\hat{W}_3 = W_3 = 0$ from the definition given by (2.4.28). We therefore obtain

$$\sigma_{ii} = \frac{\lambda_i}{J}\hat{W}_i, \quad i = 1, 2, 3, \quad \text{no sum,} \tag{2.4.29}$$

from (2.3.13). For an incompressible material we obtain

$$\hat{W}_\mu = W_\mu - \frac{1}{\lambda_\mu^2\lambda_\nu}W_3, \quad \mu, \nu = 1, 2, \quad \mu \neq \nu, \quad \text{no sum,}$$

having used (2.4.27) and (2.2.9). We can write this as

$$\lambda_\mu\hat{W}_\mu = \lambda_\mu W_\mu - \lambda_3 W_3, \quad \mu = 1, 2, \quad \text{no sum,}$$

and consequently

$$\lambda_\mu \hat{W}_\mu = \lambda_\mu W_\mu - p = \sigma_{\mu\mu}, \quad \mu = 1, 2, \quad \text{no sum,} \tag{2.4.30}$$

having used (2.4.26) and (2.3.15). Clearly we also have $\sigma_{33} = \lambda_3 \hat{W}_3 = 0$ from (2.4.26) and (2.4.27) and therefore, for incompressible materials, we obtain

$$\sigma_{ii} = \lambda_i \hat{W}_i, \quad i = 1, 2, 3, \quad \text{no sum.} \tag{2.4.31}$$

The membrane assumption (2.4.25) is also important when considering the *relaxed strain-energy function* which is discussed in the next chapter.

Chapter 3

Wrinkling Theory

3.1 Introduction

The ordinary membrane theory used in the last chapter to derive the governing equations for a membrane subjected to a finite deformation cannot predict the formation of a wrinkled region when such a region exists. Ordinary membrane theory indicates the wrinkled region to be an area of negative principal stress which contradicts the basic concepts of membrane theory. To avoid the negative stresses predicted by this theory we model the wrinkled region through *tension field theory* which was formulated by Reissner (1938). In practice the bending stiffness of a material would determine the distribution of wrinkles. However, introducing bending stiffness complicates membrane theory and tension field theory to untractable levels and we therefore retain the membrane assumption that the bending stiffness of a membrane is negligible.

Tension field theory assumes that no energy is required to fold the membrane and therefore the total strain-energy of the membrane can be minimized when in such a state. Minimum-energy states can involve a continuous distribution of infinitesimal wrinkles and the stress distribution associated with this wrinkled region is known as the tension field. Consequently tension field theory does not provide details about the local distribution of wrinkles but does give more realistic global results than ordinary membrane theory.

The deformation associated with this continuous distribution of wrinkles forms a sequence of discontinuous deformation gradients. Each sequence has successively more discontinuities and therefore gives a lower total energy as these discontinuities, which represent wrinkles, are formed by folding the membrane which requires no energy. For a general strain-energy function the limit of this sequence may not provide a limiting deformation gradient. To circumvent this difficulty we replace the usual strain-energy function with a *quasiconvexification* of the strain-energy function so that the limiting deformation can be found, see Morrey (1952) and Dacorogna (1982). This quasiconvexification is in fact the *relaxed strain-energy function* proposed by Pipkin (1986), see also Steigmann and Pipkin (1989).

Steigmann (1990) made use of the relaxed strain-energy function to modify Reissner's tension field theory. In this chapter we show how tension field theory can be incorporated into ordinary membrane theory by replacing the usual strain-energy function, $\hat{W}(\lambda_1, \lambda_2)$, by this relaxed strain-energy, $\bar{W}(\lambda_1)$, where $\bar{W}$ is the quasiconvexification of W and λ_1 is the principal stretch in the direction of a tension line. Using this approach, Pipkin (1986) showed that all of the assumptions made by tension field theory could be incorporated as consequences of the relaxed strain-energy function. Pipkin (1986) considered the *Legendre-Hadamard inequality* and showed how this inequality can be expressed in terms of the derivatives of the strain-energy function W, the derivatives being calculated with respect to the principal stretches. These derivatives gave conditions that were shown to be equivalent to the Legendre-Hadamard inequality. Some details regarding this work by Pipkin (1986), together with the ideas behind energy minimization, are included in this chapter.

A distinguishing feature of the relaxed strain-energy is that its derivatives, with respect to the principal stretches, are non-negative. Therefore the advantage of using the relaxed strain-energy is that any compressive stresses that would have occurred in ordinary membrane theory are now assumed to be zero. Although no general algorithm exists to calculate $\bar{W}$ from W we can make use of the notion of *simple tension* to obtain $\bar{W}$. Simple tension occurs when all the principal Cauchy stresses, except one, are zero. In this case the membrane has a *natural width* and Pipkin (1986) showed that the relaxed strain-energy is a function of the larger principal stretch only. This is discussed in more detail in section 3.3.

3.2 Energy Minimization

In this section we consider the minimizing of the total strain-energy associated with a membrane which is deformed according to (2.2.1) with deformation gradient (2.2.2). As no bending stiffness is attributed to the membrane the minimum energy states can be found when an infinitesimal distribution of wrinkles are formed as we assume no energy is required to fold the membrane. Consider a membrane that occupies the region B_0 in the (θ^1, θ^2) plane in two-dimensional space, with the strain-energy a function of the deformation gradient, $W(\mathbf{F})$. Then the total strain-energy of the membrane is

$$E[\mathbf{x}] = \int\int_{B_0} W(\mathbf{F}(\mathbf{X}))d\theta^1 d\theta^2. \tag{3.2.1}$$

In a standard boundary-value problem we want a deformation that will minimize E. If we assume an infinitesimal distribution of wrinkles has formed then the corresponding sequence of deformation gradients will be discontinuous and the limit of this sequence may not approach the limiting deformation. Morrey (1952) and more recently Dacorogna (1982) showed that with such deformations the limiting deformation can be obtained by replacing the strain-energy W in (3.2.1) by the relaxed strain-energy $\bar{W}$. Here $\bar{W}$ represents the smallest function that is consistent with the deformation gradient $\mathbf{F}$ for the limiting deformation and hence ensures the lowest possible energy is attained. Considering a deformation of the form

$$\mathbf{x}(\mathbf{X}) = \mathbf{F}\mathbf{X} + \mathbf{u}(\mathbf{X}),$$

where $\mathbf{F}$ is constant and $\mathbf{u}(\mathbf{X}) = \mathbf{0}$ on the boundary of B_0, C say, we can define the relaxed strain-energy by

$$\bar{W}(\mathbf{F})A(B_0) = \inf \int\int_{B_0} W(\mathbf{F} + \mathrm{Grad}\,\mathbf{u}^T(\mathbf{X}))d\theta^1 d\theta^2, \tag{3.2.2}$$

where $A(B_0)$ is the area of the region B_0. The infimum denoted by inf is taken over all possible values of $\mathbf{u}$ with $\mathbf{u} = \mathbf{0}$ on C. The relaxed strain-energy $\bar{W}$ is therefore the largest quasiconvex function that does not exceed W anywhere. We require $\bar{W}$ to be independent of B_0. For a proof of this the reader is referred to Morrey (1952) who also gives the equivalent definition of $\bar{W}$ as

$$\bar{W}(\mathbf{F}) = \liminf \int_0^1 \int_0^1 W(\mathbf{F} + \mathrm{Grad}\,\mathbf{u}^T(\mathbf{X}))d\theta^1 d\theta^2, \tag{3.2.3}$$

where $\mathbf{u}$ need not be $\mathbf{0}$ on C but the limit is taken over all sequences $\mathbf{u}_n$ that tend to $\mathbf{0}$ uniformly in B_0.

We now define two functions which are bounds for $\bar{W}$, namely the *convexification* W_c and the *rank-one convexification* W_r of W, where these are the largest functions of their type not exceeding W. These are defined by

$$W_c(\mathbf{F} + \theta\mathbf{A}) \leq (1-\theta)W_c(\mathbf{F}) + \theta W_c(\mathbf{F} + \mathbf{A}), \tag{3.2.4}$$

for all $\mathbf{F}$, $\mathbf{A}$ and θ such that $0 \leq \theta \leq 1$ and

$$W_r(\mathbf{F} + \theta\mathbf{u}\mathbf{n}^T) \leq (1-\theta)W_r(\mathbf{F}) + \theta W_r(\mathbf{F} + \mathbf{u}\mathbf{n}^T), \tag{3.2.5}$$

respectively, where we have replaced $\mathbf{A}$ in (3.2.4) by the rank-one matrix $\mathbf{A} = \mathbf{u}\mathbf{n}^T$. Dacorogna (1982) showed that convexity implies quasiconvexity which in turn implies rank-one convexity. Since W_c, $\bar{W}$ and W_r are the largest functions nowhere exceeding W and $W \geq 0$ we obtain

$$0 \leq W_c \leq \bar{W} \leq W_r \leq W, \tag{3.2.6}$$

which gives a bound above and below for the relaxed strain-energy function $\bar{W}$.

Given a deformation $\mathbf{x}$ to be continuous and continuously differentiable in the region B which minimizes the energy $E[\mathbf{x}]$ in (3.2.1), then the strain-energy $W(\mathbf{F})$ is rank-one convex at the deformation gradient $\mathbf{F}$ everywhere in the (θ^1, θ^2) plane. This is known as the *Legendre-Hadamard inequality* which can be written as

$$W_{i\mu j\nu} u_i n_\mu u_j n_\nu \geq 0, \tag{3.2.7}$$

where $W_{i\mu j\nu} = \partial^2 W/\partial F_{i\mu}\partial F_{j\nu}$, is the fourth order elastic moduli, the values $i = 1, 2, 3$, $\mu, \nu = 1, 2$ are summed over and the rank-one convexity of W ensures that the second derivatives of $W(\mathbf{F} + \theta \mathbf{u}\mathbf{n}^T)$ with respect to θ are non-negative for all $\mathbf{F}$, $\mathbf{u}$ and $\mathbf{n}$. The Legendre-Hadamard inequality states that the strain-energy function W is required to be rank-one positive definite for all $\mathbf{u}$ and $\mathbf{n}$.

For isotropic membranes Pipkin (1986) derived necessary and sufficient restrictions on the derivatives of the strain-energy $\hat{W}$, given by (2.4.27) and (2.4.28), that are equivalent to the Legendre-Hadamard inequality. Using the notation $\hat{W}_\mu = \partial \hat{W}/\partial \lambda_\mu$ and $\hat{W}_{\mu\nu} = \partial^2 \hat{W}/\partial \lambda_\mu \partial \lambda_\nu$, these restrictions are given by

$$\hat{W}_1 \geq 0, \quad \hat{W}_2 \geq 0, \quad \hat{W}_{11} \geq 0, \quad \hat{W}_{22} \geq 0, \quad G \geq 0, \tag{3.2.8}$$

$$(\hat{W}_{11}\hat{W}_{22})^{\frac{1}{2}} - \hat{W}_{12} \geq H - G, \quad (\hat{W}_{11}\hat{W}_{22})^{\frac{1}{2}} + \hat{W}_{12} \geq -H - G, \tag{3.2.9}$$

where, for $\lambda_1 \neq \lambda_2$, G and H are defined by

$$G = \frac{\lambda_1 \hat{W}_1 - \lambda_2 \hat{W}_2}{\lambda_1^2 - \lambda_2^2}, \quad H = \frac{\lambda_2 \hat{W}_1 - \lambda_1 \hat{W}_2}{\lambda_1^2 - \lambda_2^2}. \tag{3.2.10}$$

To show that the conditions (3.2.8)-(3.2.9) placed on $\hat{W}$ are equivalent to the Legendre-Hadamard inequality is a straightforward but lengthy process. The proof is therefore omitted here and the reader is referred to Pipkin (1986) where each condition given in (3.2.8)-(3.2.9) is shown to follow as a consequence of (3.2.7). Pipkin (1986) also showed that the strain-energy function $\hat{W}$ is convex with respect to the principal stretches λ_1 and λ_2, in the sense of (3.2.4). By (3.2.6) $\hat{W}$ is quasiconvex which indicates that the lowest possible energy state has been reached. Again no attempt is made to prove this and we simply accept, through physical considerations, that this condition holds so that the resulting derived relaxed strain-energy function will satisfy the convexity conditions.

A particular strain-energy function will generally not satisfy all the restrictions imposed by the conditions (3.2.8)-(3.2.9). To circumvent such difficulties we now construct the relaxed strain-energy function $\bar{W}$ mentioned previously which will automatically satisfy the above inequalities for all $\lambda_1, \lambda_2 \geq 0$.

3.3 The Relaxed Strain-Energy

To allow for the possibility of a wrinkled region forming in the solution process we use tension field theory. A wrinkled region forms when one of the principal stresses becomes negative. Ordinary membrane theory, however, is no longer applicable since membranes cannot withstand compressive stresses. We therefore assume that this compressive stress is in fact zero throughout the wrinkled region. Pipkin (1986) showed how the strain-energy function of the material can be replaced by the relaxed strain-energy function to satisfy all of the required convexity conditions. This ensures that the in-plane principal stresses are non-negative. Using this fact together with the membrane assumption (2.4.25), we have only one in-plane principal stress non-zero. When this occurs the membrane is defined as being in a state of simple tension. The idea is to modify the constitutive equation of the material so that the membrane remains in a state of simple tension throughout the wrinkled region. This is done as follows.

From (2.4.27) and (2.4.28) we have a function $\hat{W}(\lambda_1, \lambda_2)$ which indicates the strain-energy required to deform a unit square of material into a rectangle of dimensions λ_1 and λ_2. Suppose that $\hat{W}(\lambda_1, \lambda_2)$ satisfies (3.2.4) so that it is convex with respect to λ_2 for a given value of λ_1 and that the minimum of $\hat{W}$ is attained at a single point only, namely $\lambda_2 = n(\lambda_1)$. Then the strain-energy $\hat{W}$ is a function of λ_1 only and hence $\hat{W}_2 = 0$. In turn from (2.4.29) and (2.4.31) the principal stress $\sigma_{22} = 0$ and making use of (2.4.26) the membrane is in a state of simple tension. We define $n(\lambda_1)$ to be the natural width of the membrane which occurs when the membrane is in such a state. From the symmetry of $\hat{W}$ and $\boldsymbol{\sigma}$, we could have $\lambda_1 = n(\lambda_2)$ and hence $\sigma_{11} = 0$. In the subsequent chapters choosing $\sigma_{22} = 0$ proves to be a relevant assumption for the problems under consideration in this work.

From the above, the natural width of the membrane in simple tension, $n(\lambda_1)$, is the solution of

$$\sigma_{22} = 0, \tag{3.3.1}$$

which gives

$$\lambda_2 = \lambda_3 = n(\lambda_1), \tag{3.3.2}$$

since $\sigma_{22} = \sigma_{33} = 0$. Using (3.3.2) we can therefore define the energy minimum for a membrane in such a state as the relaxed strain-energy function $\bar{W}$ given by

$$\bar{W}(\lambda_1) = W(\lambda_1, n(\lambda_1), n(\lambda_1)) = \hat{W}(\lambda_1, n(\lambda_1)). \tag{3.3.3}$$

For all isotropic, incompressible materials we can express the natural width $n(\lambda_1)$ in terms of the principal stress λ_1 using (2.2.9) with (3.3.2) to yield

$$n(\lambda_1) = \lambda_1^{-\frac{1}{2}}, \tag{3.3.4}$$

which gives the relaxed strain-energy function (3.3.3), for an incompressible material, as

$$\bar{W}(\lambda_1) = W(\lambda_1, \lambda_1^{-\frac{1}{2}}, \lambda_1^{-\frac{1}{2}}) = \hat{W}(\lambda_1, \lambda_1^{-\frac{1}{2}}). \tag{3.3.5}$$

Although (3.3.2) and hence (3.3.3) hold for all materials there is no condition analogous to the incompressibility constraint (2.2.9) to obtain an expression for $n(\lambda_1)$ for a compressible material. This is generally a far more complex calculation and details of this will be given in subsequent chapters as the evaluation of $n(\lambda_1)$ depends on the form of the strain-energy function.

For completeness, we note from the symmetry of $\hat{W}(\lambda_1, \lambda_2)$ and $\boldsymbol{\sigma}$ that we could also have

$$\sigma_{11} = \sigma_{33} = 0,$$

giving

$$\lambda_1 = \lambda_3 = n(\lambda_2),$$

and hence

$$\bar{W}(\lambda_2) = W(n(\lambda_2), \lambda_2, n(\lambda_2)) = \hat{W}(n(\lambda_2), \lambda_2).$$

We now explain the above ideas of wrinkling theory from a physical viewpoint. We note at this point that if $\lambda_1 > n(\lambda_2)$ and $\lambda_2 > n(\lambda_1)$ the membrane is in a state of tension and no wrinkling will occur, the strain-energy being given by $\hat{W}(\lambda_1, \lambda_2)$. If $\lambda_2 = n(\lambda_1)$ the membrane is in a state of simple tension and the boundary at which wrinkling begins to form has been reached. Reducing the principal stretch further so that $\lambda_2 < n(\lambda_1)$ would result in the strain-energy being lowered to a value less than that associated with the relaxed strain-energy $\bar{W}(\lambda_1)$ and we would expect $\sigma_{22} < 0$. In tension field theory we assume this possibility does not occur. That is, there can be no compressive stresses across any line element of the membrane. In wrinkling theory this assumption is replaced by conditions (3.2.8) which ensure the absence of compressive stresses and allow the membrane to remain in a state of simple tension. In fact (3.2.8) with (2.4.29) and (2.4.31) show that the principal stresses are non-decreasing functions of the corresponding principal stretches. The decrease in width of the membrane, occurring when $\lambda_2 < n(\lambda_1)$, is therefore achieved by wrinkling. This results in the lowest energy level being attained because no energy is required to fold the membrane. For such a deformation the membrane is now wrinkled everywhere satisfying $\lambda_2 < n(\lambda_1)$ and hence $\sigma_{22} = 0$.

Finally if the unit square of material is deformed such that $\lambda_1 \leq 1$ and $\lambda_2 \leq 1$ then the material has been compressed in both the θ^1 and θ^2 directions and we achieve the decrease in width by first folding the material in the θ^1 direction and then in the θ^2 direction. As no stress is exerted across a wrinkled region we therefore require no energy to obtain this deformation and hence $\hat{W}(\lambda_1, \lambda_2) = 0$. This region is described as being *slack*.

To summarise we define the strain-energy function in each of its states, namely tense, wrinkled and slack, by

$$W = \begin{cases} \hat{W}(\lambda_1, \lambda_2) & \text{if} \quad \lambda_1 \geq n(\lambda_2) \text{ and } \lambda_2 \geq n(\lambda_1), \\ \bar{W}(\lambda_1) & \text{if} \quad \lambda_1 \geq 1 \text{ and } \lambda_2 \leq n(\lambda_1), \\ \bar{W}(\lambda_2) & \text{if} \quad \lambda_1 \leq n(\lambda_2) \text{ and } \lambda_2 \geq 1, \\ 0 & \text{if} \quad \lambda_1 \leq 1 \text{ and } \lambda_2 \leq 1, \end{cases} \qquad (3.3.6)$$

respectively. The transition from one strain-energy function to another when a boundary is crossed is continuous although not necessarily smooth.

Chapter 4

Wrinkling of Annular Discs subjected to Radial Displacements

4.1 Introduction

This chapter considers the problem of an isotropic, elastic, annular membrane of uniform thickness which is subjected to either a displacement or a stress on the inner and outer radii. This problem was originally considered by Rivlin and Thomas (1951) and has been studied by many other authors since, for example, Yang (1967), Wong and Shield (1969), Wu (1978) and Lee and Shield (1980). In each case the membrane was assumed to be of incompressible material governed by the neo-Hookean or Mooney-Rivlin strain-energy function. The problem was formulated to ensure that the membrane was always in tension, avoiding the possibility of wrinkling, and the resulting equations of equilibrium had to be solved numerically. Recently, however, Haughton (1991) showed that both the incompressible and compressible forms of the Varga strain-energy function, given below by (4.2.5) and (4.3.1) respectively, allow simple analytical solutions to be obtained.

The possibility of wrinkling, however, was not considered.

The aim of this chapter is therefore to extend the work by Haughton (1991) to obtain the first analytical solution for both incompressible and compressible materials for the tense and wrinkled regions. Hence this problem provides a simple example in which to illustrate the explicit analysis of a wrinkling problem using the theory of Pipkin (1986) and Steigmann (1990).

Four different deformations are considered for both the incompressible and compressible materials, these being a combination of displacement and traction boundary conditions applied to the inner and outer edges of the annulus. We begin by formulating the problem for incompressible materials. We solve the resulting equations of equilibrium analytically and then present various graphical results, concentrating on the displacements which produce wrinkling, indicating the tense and wrinkled areas. In section 4.3 we note the necessary changes to the problem formulation for compressible materials and proceed to solve the governing equations analytically. Again graphical results are given for the different displacements. As previously stated the boundary of the wrinkled region (if one exists) is found when the membrane is in a state of simple tension. Consequently the two principal Cauchy stresses defined by (2.4.29) and (2.4.31) for compressible and incompressible materials respectively are studied throughout to locate this boundary. It is found in every case considered that the principal stresses vary monotonically with the undeformed radius, simplifying each problem considerably. We also note that the deformed thickness of the membrane is constant for incompressible and compressible materials.

4.2 Incompressible Materials

We shall consider an isotropic, incompressible, elastic, uniform, circular, membrane annulus which occupies the region

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad \frac{-H}{2} \leq Z \leq \frac{H}{2},$$

in the undeformed configuration of the body where (R, Θ, Z) are cylindrical coordinates and the thickness H is constant. For membrane theory to be appropriate we require $H/B << 1$. The membrane is then deformed symmetrically to occupy the region

$$0 \leq a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad \frac{-h}{2} \leq z \leq \frac{h}{2},$$

where (r, θ, z) are cylindrical coordinates with $r = r(R)$, $\theta = \Theta$ and $h = h(R)$. The principal stretches for this deformation are

$$\lambda_1 = \frac{dr}{dR}, \quad \lambda_2 = \frac{r}{R}, \quad \lambda_3 = \frac{h}{H}, \tag{4.2.1}$$

where the subscripts (1, 2, 3) correspond to the (r, θ, z) directions respectively. In the absence of body forces the equations of equilibrium (2.4.24) for a membrane reduce to the single equation

$$\frac{d}{dr}(h\sigma_{11}) + \frac{h}{r}(\sigma_{11} - \sigma_{22}) = 0, \tag{4.2.2}$$

the other two equations being trivially satisfied where σ_{ii}, (i = 1,2,3) are the principal Cauchy stresses. The above is valid for both incompressible and compressible materials. However, choosing the material to be one rather than the other will result in different solutions being obtained for (4.2.2) and so we look at each case separately. We consider incompressible materials first.

We shall assume the material is hyperelastic with a strain-energy function of the form $W = W(\lambda_1, \lambda_2, \lambda_3)$. Hence using the analysis of section 2.4 and writing

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3) = W\left(\lambda_1, \lambda_2, \frac{1}{\lambda_1\lambda_2}\right),$$

we obtain from (2.4.31)

$$\sigma_{11} = \lambda_1 \hat{W}_1, \quad \sigma_{22} = \lambda_2 \hat{W}_2, \tag{4.2.3}$$

where $\hat{W}_\mu = \partial \hat{W} / \partial \lambda_i$, $\mu = 1, 2$. The equation of equilibrium (4.2.2) now becomes

$$\hat{W}_{11} r r'' + \lambda_2 \hat{W}_{12}(\lambda_1 - \lambda_2) + (\hat{W}_1 - \hat{W}_2)\lambda_2 = 0, \tag{4.2.4}$$

where $r'' = d^2 r/dR^2$. To obtain a solution to (4.2.4) we consider the incompressible material governed by the Varga strain-energy function which, on use of the incompressibility condition (2.2.9), has strain-energy function

$$\hat{W}(\lambda_1, \lambda_2) = 2\mu(\lambda_1 + \lambda_2 + (\lambda_1 \lambda_2)^{-1} - 3), \tag{4.2.5}$$

where $\mu > 0$ is the ground state shear modulus. Substituting (4.2.5) into (4.2.4) and using (4.2.1) the equation of equilibrium becomes

$$r r'' + r'^2 - \frac{r}{R} r' = 0, \tag{4.2.6}$$

which has the solution

$$r^2 = C_1 R^2 + C_2, \tag{4.2.7}$$

where C_1 and C_2 are constants. This is the general solution for the non-wrinkled region in all cases considered, with C_1, C_2 varying for the different boundary conditions. At this point we note from (4.2.7) that

$$r' = \frac{C_1 R}{r}, \tag{4.2.8}$$

and hence (4.2.1) gives $\lambda_1 \lambda_2 = C_1$ so $\lambda_3 = C_1^{-1}$. This presents an interesting result as it shows the principal stretch λ_3 and hence the deformed thickness of the membrane are constant.

The principal Cauchy stresses are

$$\sigma_{11} = 2\mu\lambda_1 \left(1 - \frac{1}{\lambda_1^2 \lambda_2}\right) = 2\mu \left(\frac{C_1 R}{r} - \frac{1}{C_1}\right), \tag{4.2.9}$$

and

$$\sigma_{22} = 2\mu\lambda_2\left(1 - \frac{1}{\lambda_1\lambda_2^2}\right) = 2\mu\left(\frac{r}{R} - \frac{1}{C_1}\right), \tag{4.2.10}$$

having used (4.2.5) and (4.2.8) in (4.2.3). We also note the derivatives of σ_{11} and σ_{22} with respect to R as these shall be used to determine the monotonicity of the stresses which provides useful information about the wrinkled region (if one exists). Hence (4.2.9) and (4.2.10) yield

$$\frac{d\sigma_{11}}{dR} = \frac{2\mu C_1}{r^3}(r^2 - C_1R^2) = \frac{2\mu C_1C_2}{r^3}, \tag{4.2.11}$$

and

$$\frac{d\sigma_{22}}{dR} = \frac{2\mu}{rR^2}(C_1R^2 - r^2) = \frac{-2\mu C_2}{rR^2}. \tag{4.2.12}$$

To allow for the possibility of wrinkling we now obtain a solution for this wrinkled region by using tension field theory as described in chapter 3. We recall the notion of modifying the constitutive equation of the material so that the membrane remains in a state of simple tension throughout the wrinkled region. A membrane in simple tension is defined as having only one principal stress non-zero and utilising (2.4.26) we only require one other principal stress to be zero. This occurs on the boundary of the wrinkled region. For this problem we expect that a wrinkled region will be formed through σ_{22} becoming zero. To find a solution for the wrinkled region we follow Steigmann and Pipkin (1989) and define a special form of strain-energy function which is called the relaxed strain-energy, a function of λ_1 only. Recall the natural width of the membrane in simple tension, $n(\lambda_1)$, defined to be the solution of $\sigma_{22} = 0$. We can then use the incompressibility condition (2.2.9) to write $\lambda_3 = (\lambda_1\lambda_2)^{-1}$ and hence $\lambda_2 = \lambda_3 = n(\lambda_1)$. We can therefore define the relaxed strain-energy in the wrinkled region by

$$\bar{W}(\lambda_1) = W(\lambda_1, n(\lambda_1), n(\lambda_1)),$$

a function of λ_1 only. For all incompressible materials $n(\lambda_1) = 1/\lambda_1^{\frac{1}{2}}$ having used (2.2.9) with (3.3.2) and so the relaxed strain-energy function corresponding to

(4.2.5) is

$$\bar{W}(\lambda_1) = 2\mu\left(\lambda_1 + 2\lambda_1^{-\frac{1}{2}} - 3\right). \qquad (4.2.13)$$

The equation of equilibrium (4.2.2) becomes

$$\frac{d}{dr}(h\sigma_{11}) + \frac{h}{r}\sigma_{11} = 0, \qquad (4.2.14)$$

and we can rearrange this to obtain

$$rh\sigma_{11} = \text{constant}. \qquad (4.2.15)$$

From (2.3.15) and (3.3.1)-(3.3.4) we have

$$\sigma_{11} = \lambda_1 W_1 - \lambda_1^{-\frac{1}{2}} W_2, \qquad (4.2.16)$$

and

$$\bar{W}_1 = W_1 + W_2\frac{\partial\lambda_2}{\partial\lambda_1} + W_3\frac{\partial\lambda_3}{\partial\lambda_1} = W_1 - \lambda_1^{-\frac{3}{2}} W_2,$$

where $\bar{W}_i = d\bar{W}/d\lambda_i$. On comparison with (4.2.16) we obtain

$$\sigma_{11} = \lambda_1 \bar{W}_1. \qquad (4.2.17)$$

Making use of (4.2.1) and (4.2.17) we can reduce (4.2.15) to

$$R\bar{W}_1 = \text{constant}. \qquad (4.2.18)$$

Using (4.2.13) we can rewrite (4.2.18) as

$$2\mu R(1 - \lambda_1^{-\frac{3}{2}}) = C, \qquad (4.2.19)$$

which gives the integral

$$r(R) = \int \left(\frac{R}{R-\alpha_1}\right)^{\frac{2}{3}} dR + \alpha_2, \qquad (4.2.20)$$

where α_1, α_2 are constants. This integral can be evaluated analytically to give

$$\left.\begin{aligned} r(R) \;=\; & \frac{\alpha_1}{3}\left\{\ln\left|\frac{(x^2+x+1)}{(x-1)^2}\right| + (x-1)^{-1}\right. \\ + \;& \left.\frac{4(2x+1)}{3+(2x+1)^2} - 2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right\} + \alpha_2, \end{aligned}\right\} \qquad (4.2.21)$$

where

$$x = \left(\frac{R}{R - \alpha_1} \right)^{\frac{1}{3}}.$$

Suitable boundary conditions are applied at the inner and outer edges of the wrinkled region to determine the constants α_1 and α_2. We now go on to look at four specific deformations of the annular membrane.

4.2.1 Case(i) : Fixed Outer Boundary

Firstly we consider a fixed outer boundary so that $b = B$ while the inner boundary moves inwards with $0 \leq a < A$. We define the critical radii, R^* before deformation and r^* after deformation, to be the radius at which the wrinkled and non-wrinkled regions meet and hence where $\sigma_{22}(R^*) = 0$. The boundary conditions are therefore

$$\left. \begin{aligned} r(R^*) &= r^*, \\ r(B) &= B, \end{aligned} \right\} \tag{4.2.22}$$

for the non-wrinkled region (if any), and

$$\left. \begin{aligned} r(R^*) &= r^*, \\ r(A) &= a, \end{aligned} \right\} \tag{4.2.23}$$

for the wrinkled region (if any). Unfortunately, this approach leaves R^* undetermined, the four boundary conditions above determining the constants C_1, C_2, α_1 and α_2 in terms of r^* and R^*. Using the equation $\sigma_{22}(R^*) = 0$ we can express r^* in terms of R^*, however, R^* remains essentially arbitrary. However, the deformation given by normal membrane theory is assumed to be correct throughout the region where the membrane is in tension. If a slack region is subsequently discovered wrinkling theory just corrects this region leaving the tense region unchanged. Hence for the problem considered here this implies that the boundary conditions for the unwrinkled region are

$$\left.\begin{aligned} r(A) &= a, \\ r(B) &= B, \\ r(R^*) &= r^*. \end{aligned}\right\} \tag{4.2.24}$$

Using $(4.2.24)_{1,2}$ in (4.2.7) gives

$$C_1 = \frac{B^2 - a^2}{B^2 - A^2}, \quad C_2 = \frac{B^2(a^2 - A^2)}{B^2 - A^2}. \tag{4.2.25}$$

Note that $C_1 > 1$ and $C_2 < 0$ since $a < A$. Hence from (4.2.11) and (4.2.12) σ_{11} is monotonic decreasing and σ_{22} is monotonic increasing with the undeformed radius R. From (4.2.9) and $(4.2.25)_1$ we have

$$\sigma_{11}(B) = 2\mu\left(C_1 - \frac{1}{C_1}\right) > 0,$$

hence σ_{11} is positive for all $R \in [A, B]$, and from (4.2.10) with $(4.2.25)_1$

$$\sigma_{22}(A) = \frac{2\mu(a - A)}{A(B^2 - a^2)}(B^2 - a^2 - aA - A^2). \tag{4.2.26}$$

If we consider the extremes of a given $0 < a < A$ then (4.2.26) shows that

$$\sigma_{22}(A) \to 0 \quad \text{as } a \to A,$$

and

$$\sigma_{22}(A) \to \frac{-2\mu(B^2 - A^2)}{B^2} < 0 \quad \text{as } a \to 0.$$

Hence wrinkling can occur depending on the values of a and A. From (4.2.26) we note that

$$\sigma_{22}(A) < 0 \quad \text{if} \quad (B^2 - a^2 - aA - A^2) > 0,$$

since $a < A$. Hence we can write $B^2 > 3A^2 > a^2 + aA + A^2$ using $a < A$. We can therefore deduce that a wrinkled region <u>will exist for all $a < A$</u> provided $A/B < 1/\sqrt{3}$.

For $A/B > 1/\sqrt{3}$ a wrinkled region will only exist if a/B is made sufficiently small with the above condition $B^2 > a^2 + aA + A^2$ also required. Rearranging this expression gives

$$\frac{4B^2 - 3A^2}{B^2} > \frac{4a^2 + 4aA + A^2}{B^2}.$$

Clearly both sides are positive so taking the square root and rewriting yields

$$a/B \leq (\sqrt{4 - 3(A/B)^2} - A/B)/2,$$

which determines the value of a at which wrinkling is initiated.

We now have to look at the solution for the wrinkled region. Boundary conditions $(4.2.24)_1$ and $(4.2.24)_3$ with (4.2.20) give

$$\begin{aligned} a &= \int_A \left(\frac{R}{R-\alpha_1}\right)^{\frac{2}{3}} dR + \alpha_2, \\ r^* &= \int_{R^*} \left(\frac{R}{R-\alpha_1}\right)^{\frac{2}{3}} dR + \alpha_2. \end{aligned}$$

Hence we require α_1 to satisfy the equation

$$a - r^* + \int_A^{R^*} \left(\frac{R}{R-\alpha_1}\right)^{\frac{2}{3}} dR = 0. \tag{4.2.27}$$

To solve this equation we recall that a, A are given, r^* is obtained from (4.2.7) with (4.2.25) and R^* is determined by setting $\sigma_{22} = 0$ in (4.2.10) with (4.2.25). We find that it is more concise to use the integral representation (4.2.20) rather than the exact solution (4.2.21).

For the purpose of graph plotting we non-dimensionalise the variables, without loss of generality, by dividing by B. Therefore the critical radii will be R^*/B before deformation and r^*/B after deformation but for convenience will still be defined simply by R^* and r^* respectively. The non-dimensionalised inner radius after deformation, a/B, was increased from zero to $0.9A/B$ in steps of $0.1A/B$. Hence ten curves were plotted for each value of A/B considered. Figures 4.1 - 4.3 show the curves for $A/B = 0.45$, 0.65 and 0.85 respectively. The starred symbol on the curves represents the point (R^*, r^*) which denotes the change from the wrinkled to non-wrinkled region. Figure 4.1 shows that for all deformations wrinkling will occur as expected since $A/B < 1/\sqrt{3} \simeq 0.5774$, the stars sloping downwards from left to right indicating a decrease in r^* but an increase in R^* as the deformed inner radius decreases towards zero. The decrease in r^* is unexpected but can be explained as follows. For the uppermost curve in Figure 4.1 the range of values for r/B are 0.405 to 1.0 with $r^* \simeq 0.56$ and for the lowest curve in Figure 4.1 the range is zero to 1.0 with $r^* \simeq 0.52$. Hence in the second case more of the membrane is wrinkled although the deformed critical radius is smaller.

In Figure 4.2 note that the first two curves indicate that no wrinkling has occurred. This is because $A/B > 1/\sqrt{3}$ and therefore a non-trivial deformation is required to produce wrinkling, in fact $a/B < (\sqrt{4-3(A/B)^2} - A/B)/2 \simeq 0.5015$ for this case. Similarly in Figure 4.3 the first seven deformations tried produce no wrinkling effect and we require $a/B \simeq 0.2518$ before a wrinkled region will appear. These results imply that as A/B increases a far greater deformation is required to produce a wrinkled region in the membrane.

Figure 4.4 plots values of R^* against a/B for values of $A/B = 0.2\ (0.1)\ 0.9$. This plot emphasises the results in Figures 4.1 - 4.3 by indicating the amount of region that is wrinkled and also the initiation of wrinkling. Clearly for all values of $A/B < 1/\sqrt{3}$ we obtain a wrinkled surface for any value of $a/B < A/B$ as can be seen for the four lowest curves. We also observe that this wrinkled boundary increases from the minimum value of $1/\sqrt{3}$ as $a/B \to 0$. By contrast, the four uppermost curves require a sufficient deformation before wrinkling occurs on the inner boundary. For example, the uppermost curve corresponding to $A/B = 0.9$ shows that wrinkling does not occur until $a/B \simeq 0.1765$ and reducing this value further only results in a small extension of the wrinkled region.

Lastly in this section, we graph, in Figure 4.5, the difference in solutions described by (4.2.7), the usual membrane solution, and (4.2.20), the wrinkled solution defined in the wrinkled region. In each case we note that the solution has been lowered when using the wrinkling theory and as $a/B \to 0$ the difference becomes more apparent as expected since the amount of wrinkling has increased.

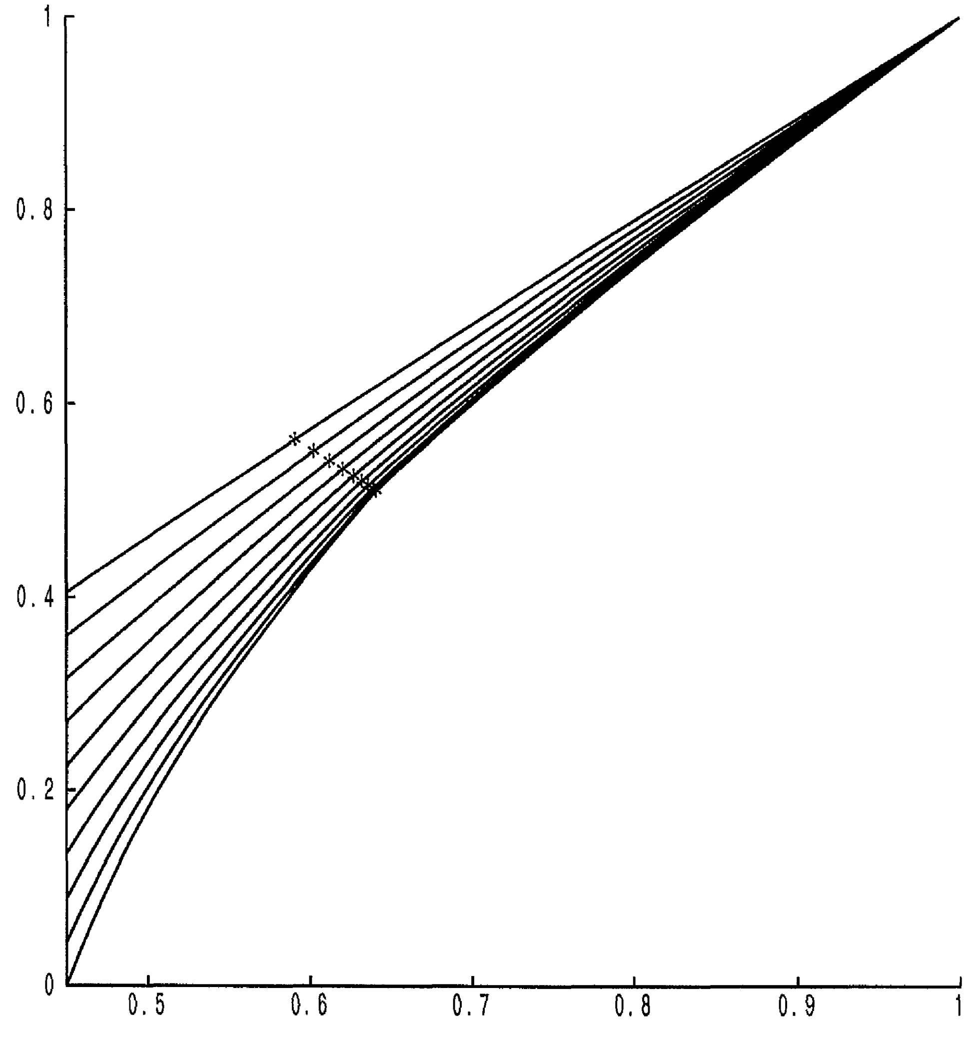

Figure 4.1: Plot of the non-dimensionalised deformed radius r/B against the undeformed radius R/B for an annulus with $A/B = 0.45$ and $a/B = 0.0\ (0.045)\ 0.405$.

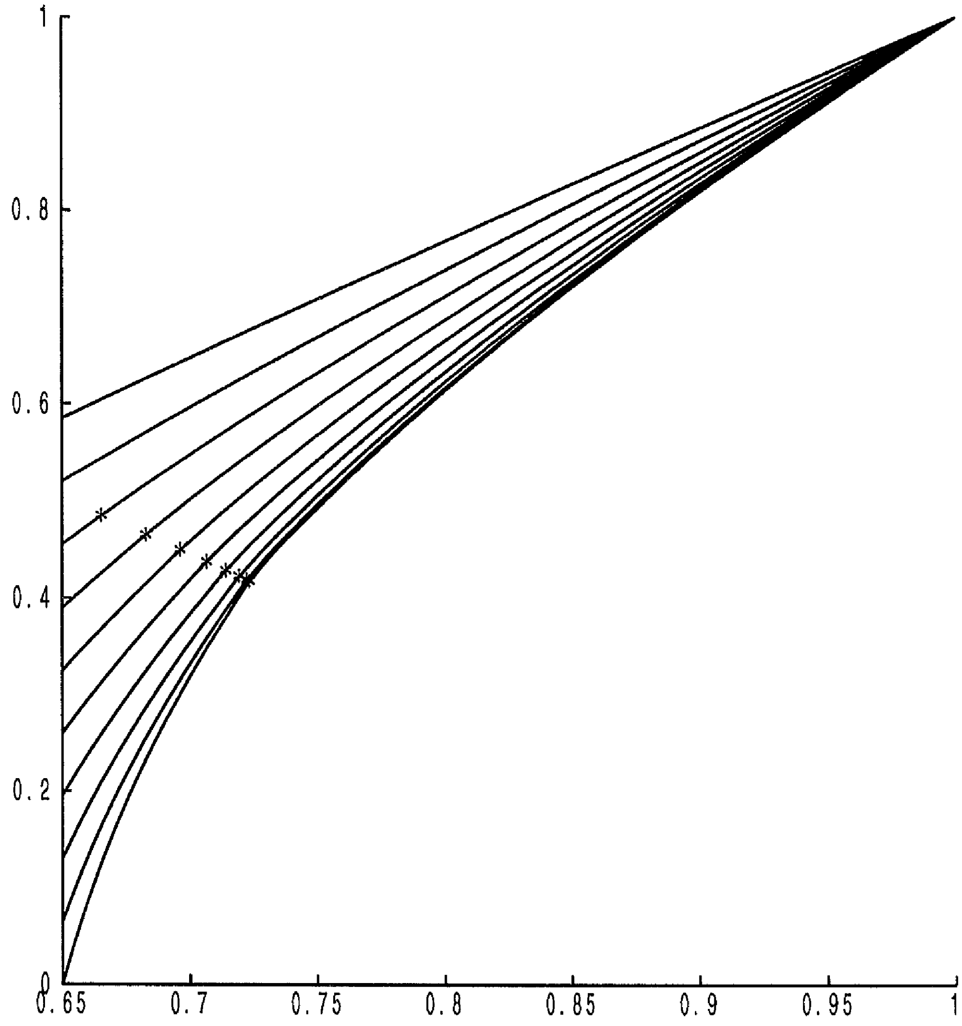

Figure 4.2: Plot of the non-dimensionalised deformed radius r/B against the undeformed radius R/B for an annulus with $A/B = 0.65$ and $a/B = 0.0$ (0.065) 0.585.

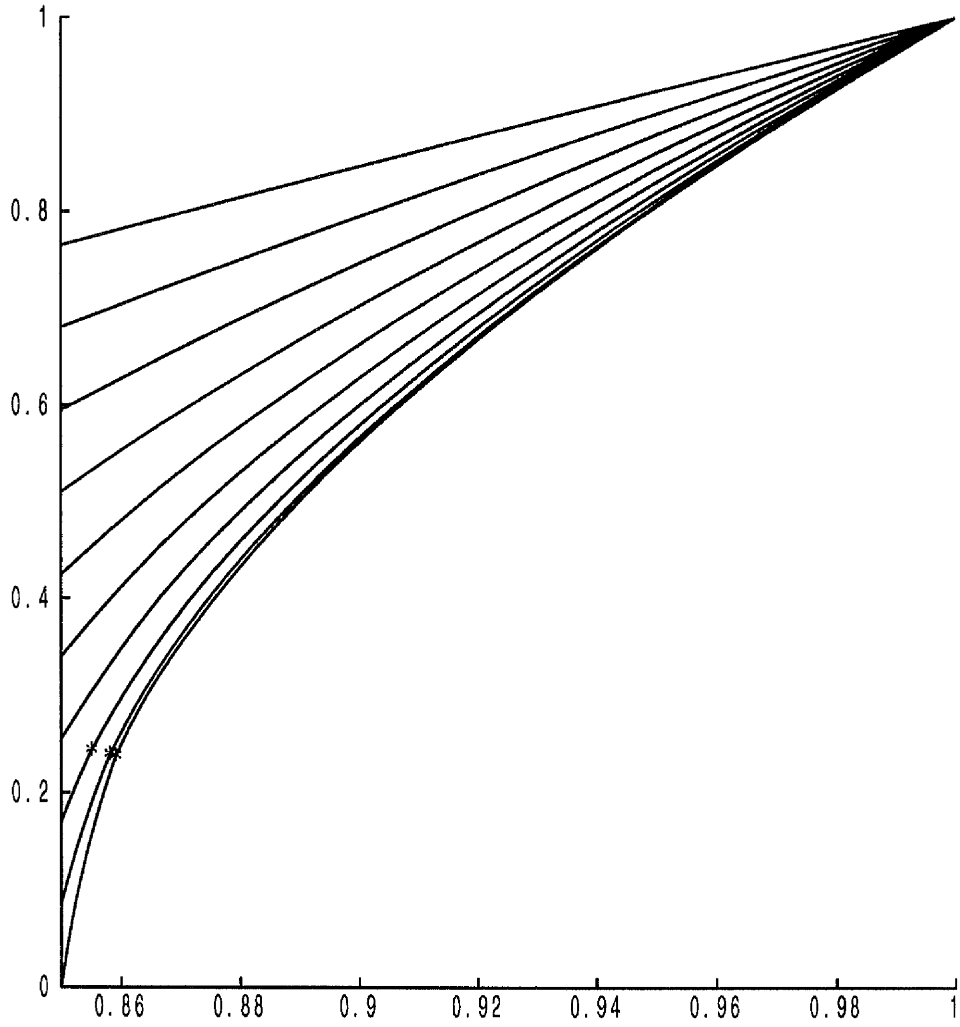

Figure 4.3: Plot of the non-dimensionalised deformed radius r/B against the undeformed radius R/B for an annulus with $A/B = 0.85$ and $a/B = 0.0$ (0.085) 0.775.

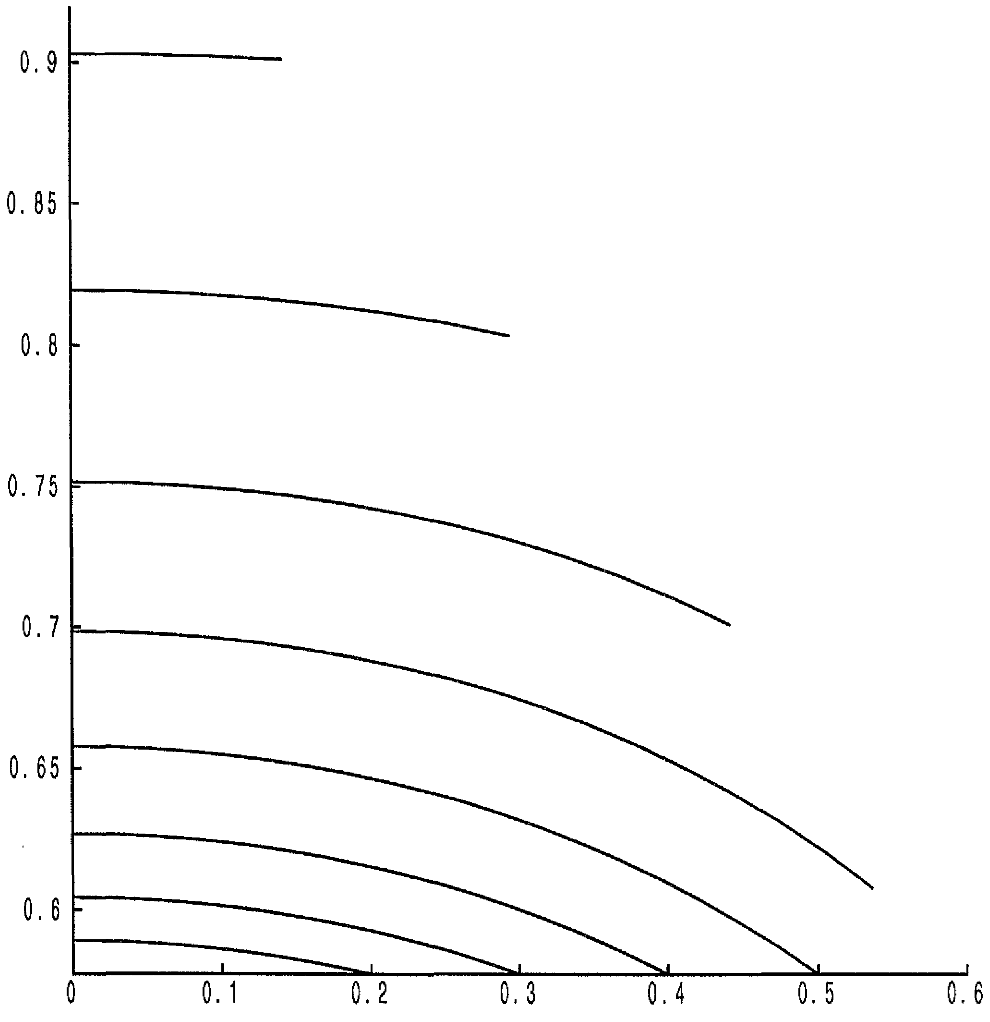

Figure 4.4: Plot of the non-dimensionalised undeformed critical radius R^* against the deformed inner radius a/B for an annulus with $A/B = 0.2\ (0.1)\ 0.9$.

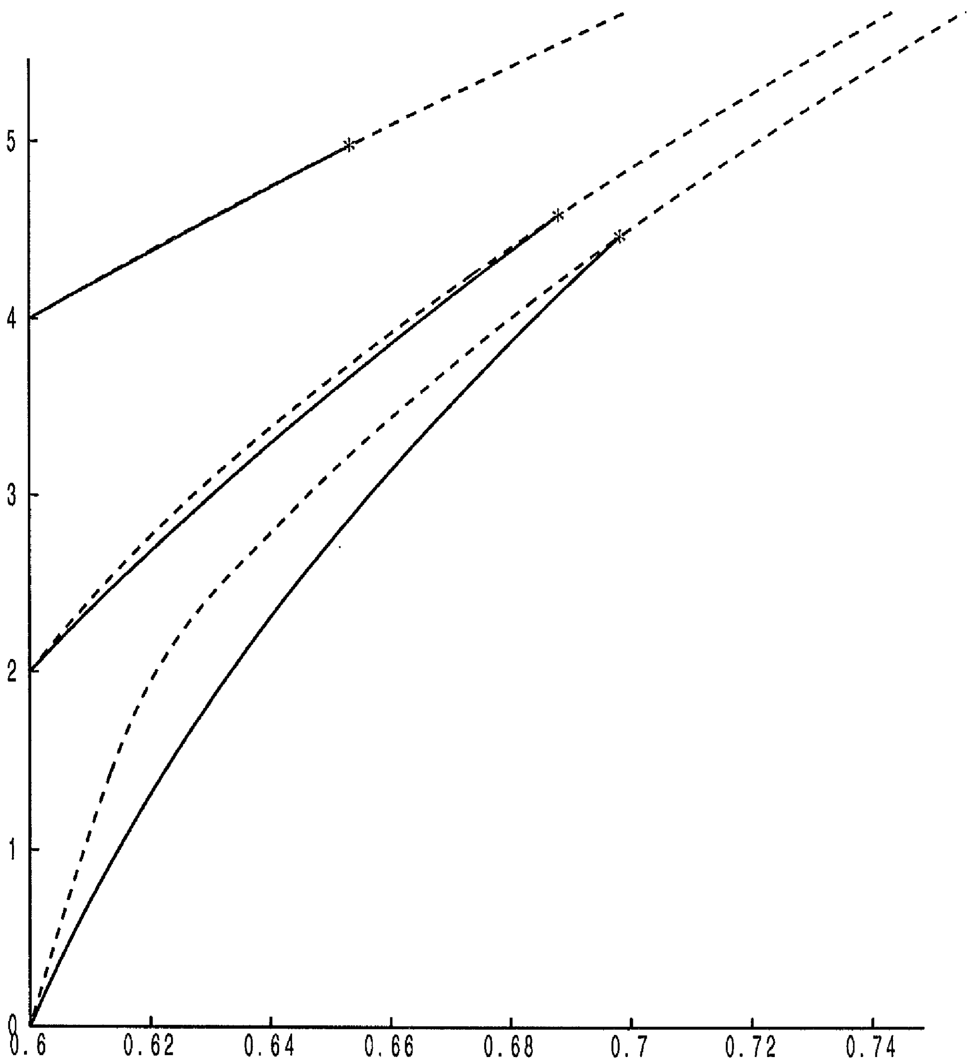

Figure 4.5: Plot of the non-dimensionalised deformed radius r/B against the undeformed radius R/B for an annulus with $A/B = 0.6$ and $a/B = 0.0\ (0.2)\ 0.4$. Both the wrinkled (——) and tense solutions (- - - -) are shown.

4.2.2 Case(ii) : Outer Boundary Traction Free

Consider the problem of forcing the inner boundary inwards such that $0 \leq a < A$ with the outer surface having the zero traction condition

$$\sigma_{11}(B) = 0. \tag{4.2.28}$$

The deformation in the tense region is again given by (4.2.7), $r^2 = C_1 R^2 + C_2$. Using (4.2.7) with $r(A) = a$ and $r(B) = b$, the constants C_1 and C_2, are

$$C_1 = \frac{b^2 - a^2}{B^2 - A^2}, \quad C_2 = \frac{a^2 B^2 - A^2 b^2}{B^2 - A^2}. \tag{4.2.29}$$

We note from (4.2.28) and (4.2.9) that

$$C_1 = \sqrt{\frac{b}{B}}, \tag{4.2.30}$$

and equating $(4.2.29)_1$ and (4.2.30) gives

$$B(b^2 - a^2)^2 = b(B^2 - A^2)^2.$$

If we rearrange the above we obtain $f(b) = 0$ where

$$f(b) = Bb^4 - 2a^2 Bb^2 - (B^2 - A^2)^2 b + a^4 B, \tag{4.2.31}$$

then evaluating this function at a and B gives

$$f(a) = -(B^2 - A^2)^2 a < 0,$$

and

$$f(B) = B(B^2 - a^2)^2 - B(B^2 - A^2)^2 > 0.$$

Hence $f(b)$ has a root such that

$$a < b < B,$$

as we would expect and using (4.2.30) it follows that

$$0 < C_1 < 1.$$

Evaluating (4.2.7) at $r = b$ and using (4.2.30) we note

$$C_2 = C_1^4 B^2 - C_1 B^2 < 0.$$

By (4.2.11) and (4.2.12) σ_{11} is monotonic decreasing and σ_{22} is monotonic increasing with R. With the boundary condition $\sigma_{11}(B) = 0$ we must have $\sigma_{11}(R) \geq 0$ for all $R \in [A, B]$ and from (4.2.10)

$$\sigma_{22}(B) = 2\mu \left(\frac{b}{B} - \frac{1}{C_1} \right) < 0.$$

Consequently $\sigma_{22}(R) < 0$ for all $R \in [A, B]$ and therefore wrinkling must occur throughout the whole membrane. The wrinkled solution will be the valid solution for the problem which is just (4.2.20) with α_1 evaluated using (4.2.27) replacing r^* by b and R^* by B.

4.2.3 Case(iii) : Rigid Inclusion

This problem considers an annulus with its inner radius fixed so that

$$a = A, \tag{4.2.32}$$

and its outer radius subjected to an expansion such that $b > B$. In this problem the constants C_1, C_2 are

$$C_1 = \frac{b^2 - A^2}{B^2 - A^2}, \quad C_2 = \frac{A^2(B^2 - b^2)}{B^2 - A^2}. \tag{4.2.33}$$

We note that $C_1 > 1$ and $C_2 < 0$ so again by (4.2.11) and (4.2.12) σ_{11} is monotonic decreasing and σ_{22} is monotonic increasing with radius R. From (4.2.9)

$$\sigma_{11}(B) = \frac{2\mu}{bC_1}(C_1^2 B - b),$$

the sign of which is undetermined. However, from (4.2.11) and (4.2.7) evaluated at $r = b$ we obtain

$$\frac{d\sigma_{11}}{dR}(B) = \frac{2\mu C_1}{b^3}(b^2 - C_1 B^2) < 0,$$

giving $b^2 < C_1 B^2$. By definition of the problem $b > B$ and $C_1 > 1$ so $b < C_1^2 B$. Consequently $\sigma_{11}(B) > 0$ and hence $\sigma_{11}(R) > 0$ for all $R \in [A, B]$. From (4.2.10)

$$\sigma_{22}(A) = 2\mu\left(1 - \frac{1}{C_1}\right) > 0,$$

and hence

$$\sigma_{22}(R) > 0 \text{ for all } R \in [A, B].$$

Therefore both principal stresses are positive throughout the membrane and hence wrinkling of the membrane does not occur.

4.2.4 Case(iv) : Inner Boundary Traction Free

This deformation again looks at an expansion of the outer boundary so that $b > B$ but this time the boundary condition on the inner surface is

$$\sigma_{11}(A) = 0. \tag{4.2.34}$$

The constants C_1 and C_2 are given by (4.2.29) and similarly as in case(ii) we note from (4.2.34) and (4.2.9) that

$$C_1 = \sqrt{\frac{a}{A}}. \tag{4.2.35}$$

Firstly we show that we must have $A < a < b$. Again equating $(4.2.29)_1$ and (4.2.35) and rearranging the resulting equation we obtain $f(a) = 0$ where

$$f(a) = Aa^4 - 2Ab^2a^2 - (B^2 - A^2)^2 a + Ab^4.$$

Evaluating this function at A and b gives

$$\begin{aligned} f(A) &= A(b^2 - A^2)^2 - A(B^2 - A^2)^2 > 0, \\ f(b) &= -(B^2 - A^2)^2 b < 0. \end{aligned}$$

Therefore a root of $f(a)$ must exist which satisfies $A < a < b$ and hence by (4.2.35) $C_1 > 1$. Using (4.2.7) evaluated at $r = a$ and (4.2.35) we note that

$$C_2 = C_1^4 A^2 - C_1 A^2 > 0.$$

Hence from (4.2.11) and (4.2.12) σ_{11} is monotonic increasing and σ_{22} is monotonic decreasing with R. Noting the above with (4.2.34) ensures

$$\sigma_{11}(R) \geq 0 \text{ for all } R \in [A, B],$$

and from (4.2.10)

$$\sigma_{22}(B) = 2\mu \left(\frac{b}{B} - \frac{1}{C_1} \right) > 0,$$

since $b > B$ and $C_1 > 1$. Hence $\sigma_{22}(R) > 0$ for all $R \in [A, B]$. Therefore both principal stresses are positive throughout the membrane and no wrinkling occurs.

4.3 Compressible Materials

We now consider the same set of problems for compressible materials. Equations (4.2.1) and (4.2.2) will still hold and again we consider the Varga material which has a compressible strain-energy function of the form

$$W(\lambda_1, \lambda_2, \lambda_3) = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - g(J)), \tag{4.3.1}$$

where $\mu > 0$ is the ground state shear modulus of the material and g is an arbitrary function of the dilatation J apart from satisfying the constraints

$$g(1) = 3, \quad g'(1) = 1, \quad g''(1) < -\frac{2}{3}. \tag{4.3.2}$$

These conditions ensure zero energy and zero stress in the undeformed configuration and a positive ground state bulk modulus respectively. From (2.4.29) and (4.3.1) we obtain

$$\sigma_{33} = \frac{2\mu\lambda_3}{J}\left(1 - \frac{Jg'(J)}{\lambda_3}\right),$$

and hence the membrane assumption (2.4.25) gives

$$\lambda_3 = Jg'(J). \tag{4.3.3}$$

Using (4.3.3) with (4.3.1) the equations of equilibrium (4.2.2) become

$$g'(J)\{rr'' + \lambda_1(\lambda_1 - \lambda_2)\}\{g'(J) + 2Jg''(J)\} = 0. \tag{4.3.4}$$

The function $g(J)$ is given and it is not possible to choose a $g(J)$ which satisfies $g'(J) + 2Jg''(J) = 0$ and the conditions (4.3.2). Hence (4.3.4) reduces to

$$rr'' + \lambda_1(\lambda_1 - \lambda_2) = 0.$$

This is the same equation as that for incompressible materials, namely (4.2.6), and therefore the solution is again (4.2.7) for the non-wrinkled region. It is a point of interest that the solution for the non-wrinkled region is the same for all Varga materials, both incompressible and compressible forms. However, the constants will be different if the boundary conditions involve stresses, which in turn requires the strain-energy to be specified. From this solution with (4.2.1) we can again write

$$\lambda_1\lambda_2 = C_1,$$

and hence using (4.3.3)

$$1 = C_1 g'(J). \tag{4.3.5}$$

Since $g''(J) \not\equiv 0$, (4.3.5) shows that J is a constant. Consequently λ_3 and the deformed thickness of the membrane are constant for compressible materials as we found for incompressible materials. Using (2.4.29), (4.3.1) and (4.3.5) the Cauchy stresses can be written as

$$\sigma_{11} = 2\mu\left(\frac{\lambda_1}{J} - \frac{1}{C_1}\right), \tag{4.3.6}$$

and

$$\sigma_{22} = 2\mu\left(\frac{\lambda_2}{J} - \frac{1}{C_1}\right). \tag{4.3.7}$$

At this point we note the derivatives of the Cauchy stresses with respect to R. From (4.3.6) and (4.3.7), using (4.2.1), (4.2.7) and (4.2.8), we obtain

$$\frac{d\sigma_{11}}{dR} = \frac{2\mu C_1 C_2}{Jr^3}, \tag{4.3.8}$$

and

$$\frac{d\sigma_{22}}{dR} = -\frac{2\mu C_2}{JrR^2}, \tag{4.3.9}$$

where C_1 and C_2 are given using appropriate boundary conditions in (4.2.7). Again it is interesting to note the similarity of the incompressible and compressible Varga materials. Comparing the expressions obtained for the principal Cauchy stresses and their derivatives given by (4.3.6) - (4.3.9) for the compressible material we note that substituting $J = 1$ would give the expressions obtained for the incompressible material given by (4.2.9) - (4.2.12).

Before trying to obtain a solution for the wrinkled region we have to find the relaxed strain-energy for compressible Varga materials. Assuming the membrane is in simple tension the stretches λ_2 and λ_3 are equal so with (4.3.3)

$$\lambda_2 = Jg'(J). \tag{4.3.10}$$

Equation (4.3.3) gives an implicit equation for λ_3 as a function of λ_1 and λ_2 and in turn (4.3.10) gives an implicit equation for λ_2 as a function of λ_1 only. Therefore the relaxed strain-energy corresponding to (4.3.1) is

$$\bar{W}(\lambda_1) = 2\mu[\lambda_1 + \lambda_2(\lambda_1) + \lambda_3(\lambda_1, \lambda_2(\lambda_1)) - g(J(\lambda_1))]. \tag{4.3.11}$$

The equation of equilibrium (4.2.18) still holds and to make any progress in solving this we need to specify $g(J)$. For definiteness let us choose the simplest form of $g(J)$ consistent with the constraints (4.3.2). We define $g(J)$ by

$$g(J) = \frac{J^\beta - 1}{\beta} + 3, \tag{4.3.12}$$

which satisfies conditions (4.3.2) provided $\beta < 1/3$, $\beta \neq 0$. The compressible form of the Varga strain-energy function (4.3.1) with (4.3.2) and (4.3.12) was first introduced by Haughton (1987). From (4.3.3), (4.3.10) and (4.3.12) we obtain

$$\lambda_2 = \lambda_3 = J^\beta. \tag{4.3.13}$$

Substituting (4.3.13) into (2.2.8) gives

$$J = \lambda_1\lambda_2\lambda_3 = \lambda_1 J^{2\beta},$$

which can be rearranged to yield

$$J = \lambda_1^{\frac{1}{1-2\beta}}.$$

Finally we can express the interdependence of the principal stretches as

$$\lambda_2 = \lambda_3 = \lambda_1^{\frac{\beta}{1-2\beta}}, \tag{4.3.14}$$

and replace (4.3.11) by the relaxed strain-energy function

$$\bar{W}(\lambda_1) = 2\mu\left[\lambda_1 + (2 - \frac{1}{\beta})\lambda_1^{\frac{\beta}{1-2\beta}} + \frac{1}{\beta} - 3\right]. \tag{4.3.15}$$

The equation of equilibrium (4.2.18) is now

$$2\mu R\left[1-\lambda_1^{\frac{3\beta-1}{1-2\beta}}\right]=\alpha_1,$$

and this reduces to the first order differential equation

$$\frac{dr}{dR}=\left[\frac{R}{(R-\alpha_1)}\right]^{\frac{2\beta-1}{3\beta-1}},$$

which has solution

$$r=\int\left[\frac{R}{R-\alpha_1}\right]^{\frac{2\beta-1}{3\beta-1}}dR+\alpha_2. \tag{4.3.16}$$

This is the solution for the wrinkled part of the membrane where α_1, α_2 are constants. We note the similarity with (4.2.20) particularly as $\beta \to -\infty$ in which case the compressible solution is a limiting case for the incompressible solution. Using suitable boundary conditions in (4.3.16) we can define a function $F(\alpha_1)$ by

$$F(\alpha_1)=a-r^*+\int_A^{R^*}\left[\frac{R}{R-\alpha_1}\right]^{\frac{2\beta-1}{3\beta-1}}dR.$$

The equation $F(\alpha_1)=0$ which defines α_1 will be solved numerically for the purpose of plotting graphs of r against R in the wrinkled region. We now consider the four specific deformations previously described in section 4.2.

4.3.1 Case(i) : Fixed Outer Boundary

As this problem has already been described in section 4.2.1 the boundary conditions, and hence constants C_1 and C_2, will again be given by (4.2.24) and (4.2.25) respectively. Hence from (4.3.8) and (4.3.9) σ_{11} is monotonic decreasing and σ_{22} is monotonic increasing since $C_1 > 1$ and $C_2 < 0$. Note from (4.3.5) and (4.3.12) that

$$J=C_1^{\frac{1}{1-\beta}}. \tag{4.3.17}$$

Then from (4.3.6) using (4.2.1), (4.2.8), $(4.2.24)_2$ and (4.3.17)

$$\sigma_{11}(B)=2\mu\left(C_1^2-C_1^{\frac{1}{1-\beta}}\right)C_1^{\frac{2-\beta}{\beta-1}}>0,$$

since $1/(1-\beta) < 3/2$ for all $\beta < 1/3$ and $C_1 > 1$. Therefore $\sigma_{11}(R) > 0$ for all $R \in [A, B]$.

If we now consider σ_{22}, using (4.3.7) with (4.2.1) and (4.3.17), we obtain

$$\sigma_{22}(B) = 2\mu \left(1 - C_1^{\frac{\beta}{1-\beta}}\right) C_1^{\frac{1}{\beta-1}}. \tag{4.3.18}$$

Noting that $-1 < \beta/(1-\beta) < 1/2$ for all $\beta < 1/3$ we are required to look at two separate cases for (4.3.18):

Case (1) $0 < \beta < 1/3$

Here $0 < \beta/(1-\beta) < 1/2$ and so $\sigma_{22}(B) < 0$.

Hence wrinkling occurs throughout the whole membrane, although we note that as $\beta \to 0^+$, $R^* \to B^+$.

Case (2) $\beta < 0$

In this case $\beta/(1-\beta) < 0$ and so $\sigma_{22}(B) > 0$.

At this point we cannot tell if wrinkling of the membrane occurs. To determine whether or not a wrinkled region appears for $\beta < 0$ we now consider σ_{22} on the inner radius. Using $(4.2.1)_2$, (4.3.7) and (4.3.17) we have

$$\sigma_{22}(A) = \frac{2\mu}{C_1} \left(\frac{a}{A} C_1^{\frac{\beta}{\beta-1}} - 1\right). \tag{4.3.19}$$

From (4.3.19) we see that for wrinkling to occur for all $a < A$ we require

$$C_1 < (\frac{A}{a})^{\frac{\beta-1}{\beta}}.$$

Equating C_1 with $(4.2.25)_1$ we can rearrange the resulting inequality to obtain $f(a) > 0$ where

$$f(a) = a^{\frac{3\beta-1}{\beta}} - a^{\frac{\beta-1}{\beta}} B^2 + A^{\frac{\beta-1}{\beta}} (B^2 - A^2).$$

We remark that $f(0) = A^{\frac{\beta-1}{\beta}} (B^2 - A^2) > 0$ and $f(A) = 0$. We note that

$$f'(a) = \frac{a^{\frac{-1}{\beta}}}{\beta} \left((3\beta - 1)a^2 - (\beta - 1)B^2\right),$$

and it follows that wrinkling will occur for all $0 < a < A$ if and only if $f'(A) < 0$. Evaluating $f'(a)$ at A we can show wrinkling will occur for all a if and only if

$$\left(\frac{A}{B}\right)^2 < \frac{\beta - 1}{3\beta - 1}. \tag{4.3.20}$$

On studying (4.3.20) we note that as $\beta \to -\infty$ we can recover the corresponding result for the incompressible material. This is a beneficial feature of the compressible Varga material that such limiting processes are possible as this is not the case in general for other compressible materials. For $(A/B)^2 > (\beta - 1)/(3\beta - 1)$ a wrinkled region will only be formed if the deformation is made sufficiently large. This can be seen from (4.3.19) as $\sigma_{22}(A) \to -2\mu/C_1$ as $a \to 0$.

To demonstrate this feature we plot R^* against β for an annular disc with initial inner radius $A/B = 0.75$ in Figure 4.6 where again we have non-dimensionalised the variables by dividing by B. We have chosen a/B to vary from zero to 0.75 taking increments of 0.05. In Figure 4.6 having taken $A/B = 0.75$ and hence using (4.3.20) we find that some wrinkling will always occur within the membrane for all values of $\beta > -7/11$. The curve furthest to the right represents $a/B = 0.75$ and shows that as soon as the membrane is deformed wrinkling begins, provided $\beta > -7/11$, with the curve intersecting the horizontal axis at $\beta = -7/11$. Incrementing this value of β even slightly results in a considerable increase in R^*. Considering values of $\beta < -7/11$ we observe that some critical value of a/B has to be reached before wrinkling occurs. For example, taking $\beta = -3$ we note that $a/B \simeq 0.5$ before any wrinkling occurs as can be seen from the curve corresponding to $a/B = 0.5$ crossing the horizontal axis at $\beta \simeq -3$. If we decrease β further to $\beta = -5$ say, then we observe that the wrinkled region increases as $a/B \to 0$ and the value of R^* for the upper curve has increased to somewhere around 0.795.

Figure 4.7 plots values of R^* against a/B for values of β = -0.5 (0.5) -5 and $\beta \to -\infty$. The annulus has again an initial inner radius of $A/B = 0.75$ and for $\beta = -0.5$ we see that wrinkling exists for any deformation satisfying $a < A$ as expected since $\beta > -7/11$. For $\beta \leq -1$ we remark that a finite deformation is required before wrinkling is initiated. This again consolidates the results obtained in Figure 4.6 and in particular if we consider $\beta = -3$ we comment that $a/B \simeq 0.5$ before wrinkling occurs which can be seen from the fact that the curve meets the horizontal axis at this point. Finally we note that the innermost curve corresponding to $\beta \to -\infty$, which is the limiting case for the incompressible material, cuts the x-axis at $a/B \simeq 0.385$. If we again compare this result with Figure 4.6 we observe that the curve corresponding to $a/B = 0.4$ in Figure 4.6 appears to be approaching a limiting value of R^* as the curve is flattening out. Extending a curve with value $a/B \simeq 0.385$ to accommodate $\beta \to -\infty$ would show that a value of R^* does in fact exist and therefore wrinkling has formed for the approximation to the incompressible material. In fact using the analysis given for the incompressible Varga material we can calculate that for $A/B = 0.75$ we require $a/B = (\sqrt{4 - 3(A/B)^2} - A/B)/2 \simeq 0.3853$ for wrinkling to occur.

4.3.2 Case(ii) : Outer Boundary Traction Free

This problem has already been considered for an incompressible material and so the zero traction condition (4.2.28) and constants C_1, C_2 given by (4.2.29) still hold. When this problem was considered for an incompressible material it was shown that $b < B$ given $0 \leq a < A$. As for the incompressible case it is useful to establish the condition $a < b < B$. We now show this condition holds for the compressible Varga material as follows. First note from (4.2.28) and (4.3.6)

$$J = \lambda_1 C_1,$$

and using (4.2.8) with $(4.2.1)_1$ evaluated at $r = b$ gives

$$J = C_1^2 \frac{B}{b}. \tag{4.3.21}$$

From (4.3.17) and (4.3.21) we have

$$C_1 = \left(\frac{b}{B}\right)^{\frac{\beta-1}{2\beta-1}}, \tag{4.3.22}$$

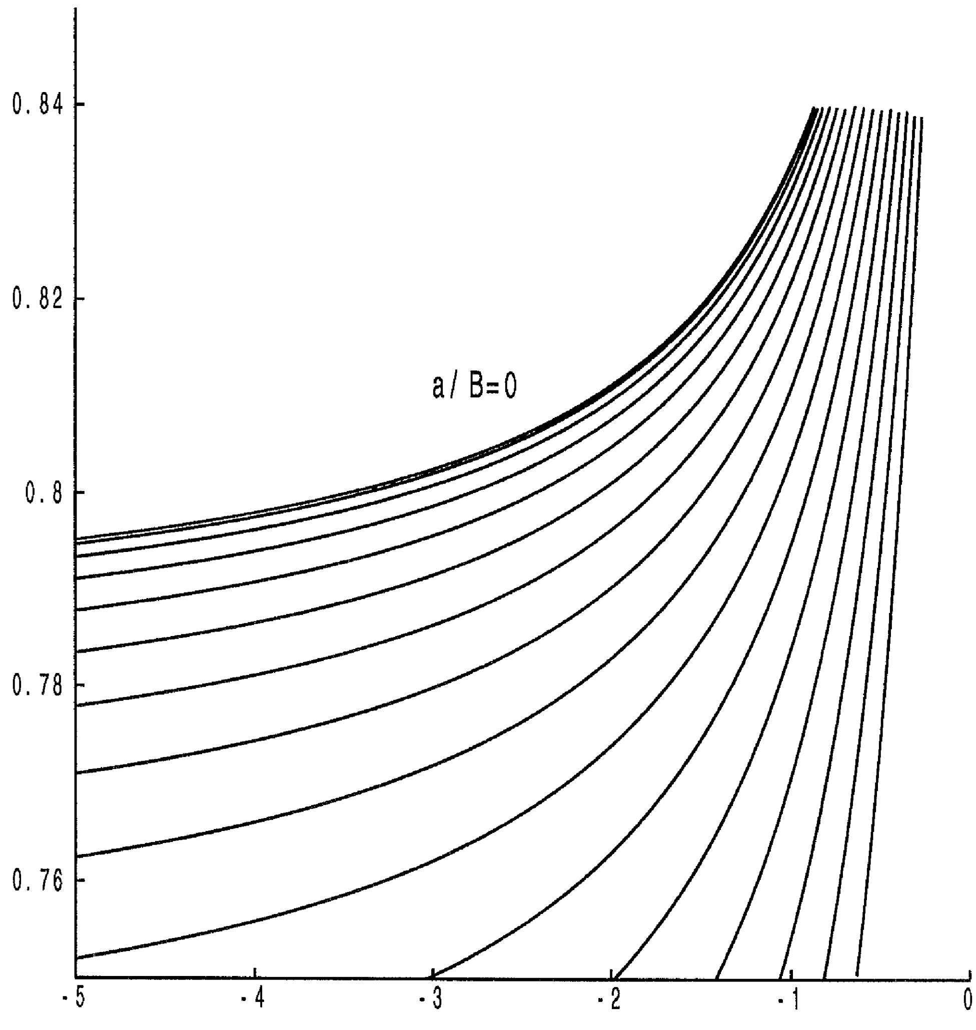

Figure 4.6: Plot of the non-dimensionalised undeformed critical radius R^* against β for an annulus with $A/B = 0.75$ and $a/B = 0.0\ (0.05)\ 0.75$.

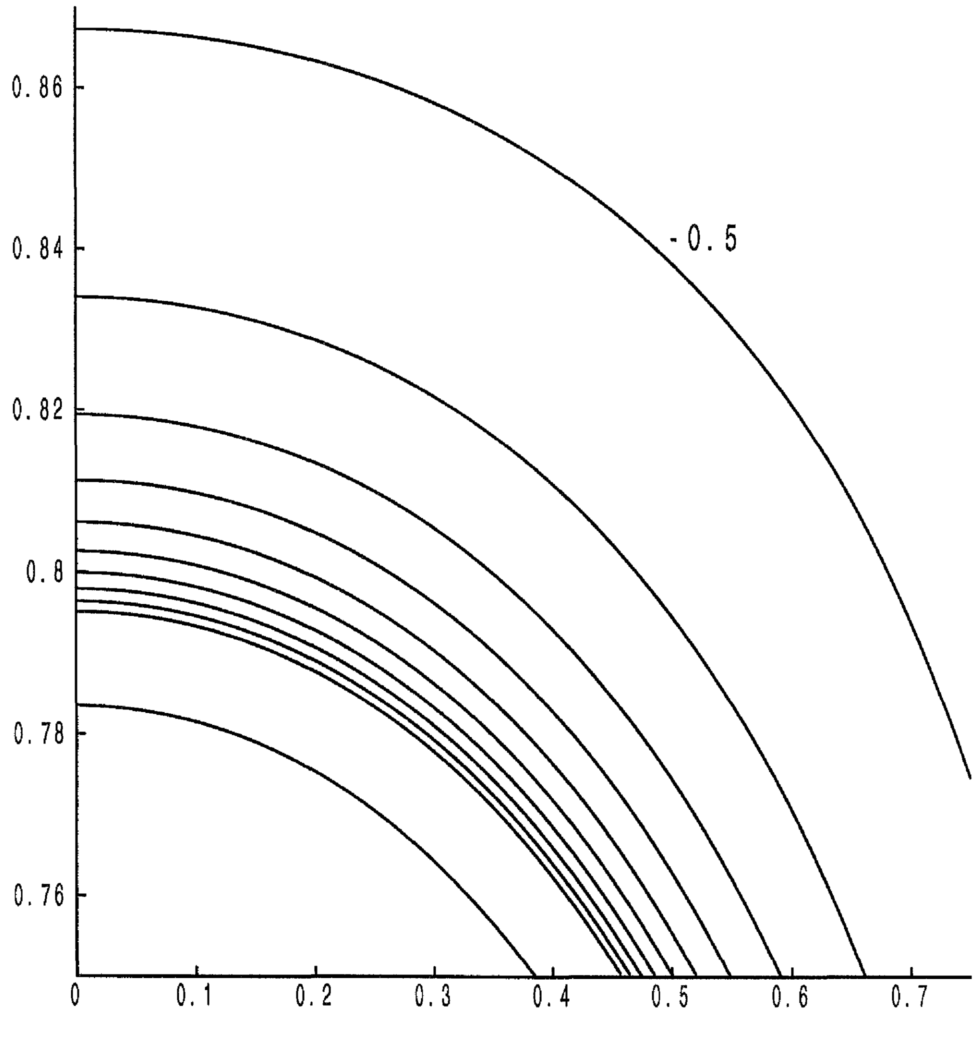

Figure 4.7: Plot of the non-dimensionalised undeformed critical radius R^* against the deformed inner radius a/B for an annulus with $A/B = 0.75$ and values of $\beta =$ -0.5 (0.5) -5 with the innermost curve corresponding to $\beta \to -\infty$.

and note from (4.3.22) that $C_1 > 0$ as b, B are both positive. Equating $(4.2.29)_1$ and (4.3.22) we obtain

$$b^{\frac{\beta-1}{2\beta-1}}(B^2 - A^2) = B^{\frac{\beta-1}{2\beta-1}}(b^2 - a^2). \tag{4.3.23}$$

If we rearrange (4.3.23) we obtain $f(b) = 0$ where

$$f(b) = b^{\frac{\beta-1}{2\beta-1}}(B^2 - A^2) - (b^2 - a^2)B^{\frac{\beta-1}{2\beta-1}}, \tag{4.3.24}$$

and evaluating (4.3.24) at a and B gives

$$f(a) = a^{\frac{\beta-1}{2\beta-1}}(B^2 - A^2) > 0,$$

$$f(B) = B^{\frac{\beta-1}{2\beta-1}}(a^2 - A^2) < 0,$$

since $a < A$. Hence a root of (4.3.24) must exist which satisfies

$$a < b < B.$$

We now determine an expression for C_2 so we can investigate the Cauchy stresses. Rearranging (4.2.7) and using (4.3.22) we find

$$\frac{C_2}{b^2} = 1 - \left(\frac{B}{b}\right)^{\frac{3\beta-1}{2\beta-1}}. \tag{4.3.25}$$

With $b < B$ and $(3\beta - 1)/(2\beta - 1) > 0$ for $\beta < 1/3$ we must have $C_2 < 0$. From (4.3.8) σ_{11} is monotonic decreasing and hence with (4.2.28) $\sigma_{11}(R) \geq 0$ for all $R \in [A, B]$. Similarly from (4.3.9) σ_{22} is monotonic increasing. From (4.3.7) and (4.3.21)

$$\sigma_{22}(B) = \frac{2\mu C_2}{C_1^2 B^2} < 0,$$

and hence $\sigma_{22}(R) < 0$ for all $R \in [A, B]$ and wrinkling must occur throughout the entire membrane. This is in agreement with the incompressible case in which the whole membrane was wrinkled.

4.3.3 Case(iii) : Rigid Inclusion

We considered this problem in section 4.2.3 and therefore the boundary condition (4.2.32) with constants C_1 and C_2 given by (4.2.33) still hold. From (4.3.8) and (4.3.9) with $C_1 > 1$ and $C_2 < 0$, σ_{11} is monotonic decreasing and σ_{22} is monotonic increasing. Substituting (4.2.8) and (4.3.17) into (4.3.6) gives

$$\sigma_{11}(B) = \frac{2\mu}{b}\left(B - bC_1^{\frac{2\beta-1}{1-\beta}}\right)C_1^{\frac{\beta}{\beta-1}}. \tag{4.3.26}$$

We now want to determine the positive or negative nature of (4.3.26). Evaluating (4.2.7) at $r = b$ and recalling that $C_2 < 0$ we find $b^2 - C_1B^2 < 0$. Using this fact and noting that $C_1 > 1$ and $1/2 < (\beta-1)/(2\beta-1) < 2$ for the range of β under consideration, it follows that

$$b < C_1^{\frac{1}{2}}B < C_1^{\frac{\beta-1}{2\beta-1}}B,$$

which can be manipulated and substituted into (4.3.26) to show $\sigma_{11}(B) > 0$. Consequently $\sigma_{11}(R) > 0$ for all $R \in [A, B]$.

Evaluating (4.3.7) at $R = A$ gives

$$\sigma_{22}(A) = 2\mu\left(1 - C_1^{\frac{\beta}{1-\beta}}\right)C_1^{\frac{1}{\beta-1}}, \tag{4.3.27}$$

having used (4.3.17). Clearly (4.3.27) depends on the value of β and as before we consider two different cases.

Case (1) $\beta < 0$

Here $-1 < \beta/(1-\beta) < 0$ and so $\sigma_{22}(A) > 0$ and hence no wrinkling will occur. However we do note that as $\beta \to 0^-$, $\sigma_{22}(A) \to 0^+$. For $\beta < 0$ we therefore have agreement between the incompressible and compressible cases.

Case (2) $0 < \beta < 1/3$

In this case $0 < \beta/(1-\beta) < 1/2$ and so (4.3.27) gives $\sigma_{22}(A) < 0$.

Hence, for a sufficiently compressible material ($\beta > 0$) some wrinkled region will always exist. We note from (4.3.7) with (4.3.17) and (4.2.1) that

$$\sigma_{22}(B) = 2\mu \left(\frac{b}{B} - C_1^{\frac{\beta}{1-\beta}} \right) C_1^{\frac{1}{\beta-1}}. \tag{4.3.28}$$

Using (4.2.33) and $0 < \beta/(1-\beta) < 1/2$ for $0 < \beta < 1/3$ we deduce that for any β in this range we can find a large enough value of $A/B < 1$ to ensure that the whole membrane will initially be wrinkled for $b > B$. Also, we find that for sufficiently large b/B, $\sigma_{22}(B) > 0$ and so wrinkling will be confined to some inner portion of the annulus (since $\sigma_{22}(A) < 0$ there will always be some wrinkled region).

To illustrate this a range of values were considered for each of the three variables $\beta, A/B$ and b/B. In Figure 4.8 we plot the deformed radius against the undeformed radius for the parameter values $\beta = 0.25$ and $A/B = 0.6$ with a number of different curves being indicated as b/B increases from 1.1 to 2.5 in steps of 0.1. The curves to the right of the critical point (R^*, r^*) are given by the non-wrinkled solution (4.2.7) and the inner wrinkled portions are plotted using (4.3.16). We observe for small values of b/B that almost the whole membrane is wrinkled but the outer portion is smoothed out as the deformation is increased. It was also found that for all values of A/B and b/B the wrinkled region increased as β increased. This increase was only slight for small values of β but significant as $\beta \to 0.3$. In fact for $A/B \geq 0.5$ and $\beta = 0.3$ the membrane was completely wrinkled for all values of b/B. For a particular value of A/B and b/B each curve plotted for β from 0.05 to 0.25 was almost identical, however, for $\beta = 0.3$ the wrinkled solution changed slightly. This can be explained by the form of the integral in (4.3.16) in which $(2\beta - 1)/(3\beta - 1)$ will increase rapidly as $\beta \to 1/3$. Another variable which affects the amount of wrinkling is the initial inner radius A/B. For fixed values of β and b/B it was found that the wrinkled region increased as A/B increased. For example, given $b/B = 2.0$ and $\beta = 0.3$ the undeformed critical radius increased from 0.62 to 0.98 as A/B increased from 0.3 to 0.5.

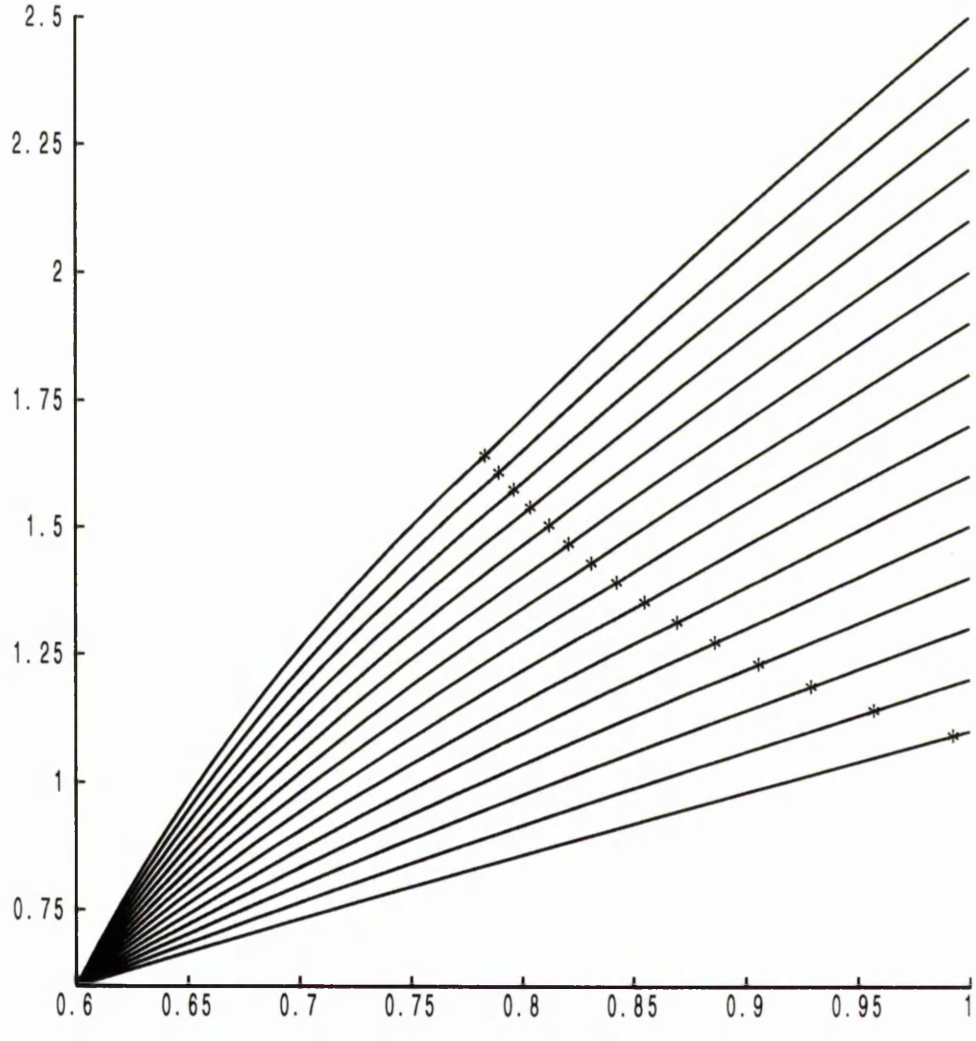

Figure 4.8: Plot of the non-dimensionalised deformed radius r/B against the undeformed radius R/B for an annulus composed of compressible Varga material with $A/B = 0.6$, $\beta = 0.25$ and $b/B = 1.1\ (0.1)\ 2.5$.

4.3.4 Case(iv) : Inner Boundary Traction Free

The problem of displacing the outer boundary so that $b > B$ while the inner surface remained traction free was considered in section 4.2.4. The boundary condition (4.2.34) and constants C_1, C_2 given by (4.2.29) still hold. For the incompressible case it was shown that if $b > B$ then $a > A$. We show this is true for a compressible material also before looking at the Cauchy stresses. From (4.2.34) and (4.3.6) using (4.2.8) evaluated at $r = a$ and (4.2.1) we have

$$J = C_1^2 \frac{A}{a}, \tag{4.3.29}$$

and from (4.3.17) and (4.3.29) we obtain

$$C_1 = \left(\frac{a}{A}\right)^{\frac{\beta-1}{2\beta-1}}. \tag{4.3.30}$$

Equating this expression with $(4.2.29)_1$ we can rearrange the resulting equation as $g(a) = 0$ where

$$g(a) = a^{\frac{\beta-1}{2\beta-1}}(B^2 - A^2) + (a^2 - b^2)A^{\frac{\beta-1}{2\beta-1}},$$

the solution of which determines the deformed inner radius a. We note that

$$\begin{aligned} g(A) &= A^{\frac{\beta-1}{2\beta-1}}(B^2 - b^2) < 0, \\ g(b) &= b^{\frac{\beta-1}{2\beta-1}}(B^2 - A^2) > 0. \end{aligned}$$

Therefore a root must exist which satisfies

$$A < a < b,$$

and since $(\beta - 1)/(2\beta - 1) > 1/2$ then $C_1 > 1$.

We now want to determine the sign of C_2. This can be done in exactly the same way as in case(ii) so that we obtain an equation analogous to (4.3.25) which is

$$\frac{C_2}{a^2} = 1 - \left(\frac{A}{a}\right)^{\frac{3\beta-1}{2\beta-1}}. \tag{4.3.31}$$

For $\beta < 1/3$, we have $(3\beta - 1)/(2\beta - 1) > 0$ and so $C_2 > 0$. Therefore from (4.3.8) and (4.3.9) σ_{11} is monotonic increasing and σ_{22} is monotonic decreasing. Hence from (4.2.34) $\sigma_{11}(R) \geq 0$ for all $R \in [A, B]$. If we now consider σ_{22} using $(4.2.29)_1$, (4.3.7) and (4.3.29) we have

$$\sigma_{22}(B) = \frac{2\mu}{J}\left(\frac{(Aa + bB)(aB - Ab)}{aB(B^2 - A^2)}\right).$$

Since $C_2 > 0$ then from $(4.2.29)_2$ we obtain $aB > Ab$ which yields $\sigma_{22}(B) > 0$ giving $\sigma_{22}(R) > 0$ for all $R \in [A, B]$. Hence no wrinkling occurs giving agreement with the incompressible case.

4.4 Conclusions

To be able to compare the incompressible and compressible materials for each deformation considered we now summarize the results obtained. Firstly though, we again draw attention to an interesting feature exhibited by the Varga strain-energy function in that for incompressible and compressible materials the deformed thickness of the membrane was constant. Also we note that the solutions found for the tense region for both the incompressible and compressible forms of the Varga material were identical while those obtained for the wrinkled region had similar forms with the compressible solution being a limiting case for the incompressible solution as $\beta \to -\infty$. The principal Cauchy stresses always varied monotonically with the undeformed radius R for each deformation considered.

The first problem considered the deformation described by the inner radius of the disc being displaced inwards and the outer radius being held fixed. We recall that a wrinkled region always existed for all $a < A$ for incompressible and compressible materials provided $(A/B)^2 < 1/3$ for the incompressible material and $(A/B)^2 < (\beta - 1)/(3\beta - 1)$ where $\beta < 0$ for the compressible material. If these conditions were not satisfied a wrinkled region only occurred if a/B was made sufficiently small, this condition being $a/B < (\sqrt{4 - 3(A/B)^2} - A/B)/2$ for the incompressible material. Hence we obtained similar results for the materials and we also noted that as $\beta \to -\infty$ the above conditions for wrinkling were equivalent. However, if $0 < \beta < 1/3$ then the entire membrane was wrinkled when considering the compressible material, this possibility never occurring for the incompressible material.

The second problem studied the deformation in which the inner radius was displaced inwards with the outer radius being traction free. The results obtained were the same for the incompressible and compressible materials as the whole membrane wrinkled in each case for all parameter values considered.

The third problem discussed held the inner radius of the disc fixed and displaced the outer radius outwards. It was found for the incompressible material and the compressible material with $\beta < 0$ that no wrinkling occurred. However, the compressible material always produced a wrinkled region on the inner part of the membrane provided $0 < \beta < 1/3$ which increased as β or A/B increased. This increase is particularly significant as β approaches $1/3$ due to the nature of the solution (4.3.16). In contrast as b/B was increased the undeformed critical radius became smaller and the portion of membrane which was wrinkled decreased.

The last deformation left the inner radius stress free and again displaced the outer radius so that $b > B$. Wrinkling did not occur for the incompressible or compressible materials.

Chapter 5

Wrinkling of Inflated Cylindrical Membranes under Flexure

5.1 Introduction

In this chapter we consider the problem of a right, circular, cylindrical, isotropic, elastic membrane which is inflated by a pressure and then subjected to finite bending. We assume the cylinder is closed at both ends and is of finite length. This deformation was previously considered by Stein and Hedgepeth (1961) in which they used a modified version of Reissner's tension field theory (1938) to deal with the wrinkling of the membrane. More recently Haseganu and Steigmann (1994) considered this problem for an incompressible material governed by the Varga strain-energy function. Using the theory of Pipkin (1986), Haseganu and Steigmann (1994) showed that a wrinkled region formed at some critical value of pressure and curvature. This wrinkled region increased as the cylinder was subjected to further bending.

The purpose of this chapter is to extend the work of Haseganu and Steigmann (1994) to other forms of incompressible materials and to the compressible Varga material. We follow the solution approach given by Haseganu and Steigmann

(1994) but offer an alternative method of solution which results in a different cross-section of the cylinder being formed. This allows a comparison of the two methods, the results of which will be discussed in section 5.4. We investigate, for the incompressible Varga material, the effect on the volume caused by this change in cross-sectional shape of the cylinder by assuming the inflating gas is an ideal fluid. This volume change will have an effect on the internal pressure and the initiation of wrinkling.

Finally we consider a 'small' deformation superimposed on the body while in the current configuration, the body being inflated and bent but not wrinkled in this configuration. This new perturbed state becomes the current configuration with the inflated and bent state now being regarded as the reference configuration.

Haughton and Ogden (1979) considered a tube with closed ends which was subjected to an inflating pressure only and found that upon applying an incremental deformation the resulting bifurcation equations could be integrated exactly. Further Haughton and Ogden (1979) concluded that applying fixed end conditions $\dot{\mathbf{x}} = \mathbf{0}$ led to an explicit bifurcation criterion which, for an infinitely long tube, reduced to the equations of equilibrium for an inflated tube with closed ends. This indicated that the infinitely long cylinder was only neutrally stable with respect to a bending mode. As an infinitely long cylinder will undergo a bifurcation immediately upon bending it seems likely that cylinders of finite length will undergo a similar bifurcation at some finite bending. In this work we obtain the system of equations governing the incremental deformation for the inflated and bent cylinder. Unfortunately we cannot solve these equations exactly and instead have to use a standard approach which assumes the equations have a separable solution. This is discussed in more detail in section 5.4.5.

Section 5.2 begins by formulating the problem and describes the kinematics of flexure. We then obtain the equations of equilibrium for a pressurised membrane under flexural deformations before finding expressions for the principal stretches when the membrane has been inflated only. The latter set of equations are considered first as their solution provides parameters required in the flexural deformation. Section 5.3 goes on to solve the equations of equilibrium for the tense and wrinkled regions and explains the solution procedure together with a description of the two solution methods used to plot the deformed cross-section of the membrane. Section 5.4 describes the results obtained for incompressible materials and makes a comparison of the results found by plotting the deformed cross-section for both solution methods. The results obtained for volume changes and initiation of wrinkling considerations are presented and the deformation associated with the incremental displacement is also described for the incompressible Varga material. Finally, section 5.5 derives the equations of equilibrium for a general compressible material which are found to be the same as that for incompressible materials although specifying the strain-energy function will result in different particular governing equations. We again solve the equations of equilibrium for an inflated cylinder first to obtain parameters necessary for the flexural problem. The solution procedure is described and results are presented for the compressible Varga material.

5.2 Kinematics of Flexure

We now describe the kinematics behind the problem and we use these to find a system of equations of equilibrium to be solved. To this end we follow Haseganu and Steigmann (1994) and although many of the equations given in this section were obtained by them, the formulation given here is different, but equivalent, to their derivation.

We consider a right, circular, cylindrical, elastic, isotropic membrane of radius R and length L, the position vector of a point on the undeformed middle surface being

$$\mathbf{X}(\Theta, Z) = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{5.2.1}$$

where Z is the axial length along the fixed unit vector $\mathbf{E}_Z$, $\mathbf{E}_R(\Theta)$ is a unit normal to the cylinder's surface and R is the radius of the middle surface of the cylinder. The ends of the cylinder are closed. The axis of the undeformed cylinder is then deformed into a planar curve by inflating the membrane with a pressure P and subjecting it to a uniform flexural deformation. The position vector of a point on the deformed middle surface is given by

$$\mathbf{x}(\Theta, Z) = (\rho - x(\Theta))\mathbf{e}_r(\theta) + y(\Theta)\mathbf{e}_z, \tag{5.2.2}$$

where ρ is the radius of curvature of the deformed axis of the cylinder. The plane of this curve is described by cylindrical coordinates (ρ, θ) and y is the coordinate perpendicular to this plane along $\mathbf{e}_z$. If we consider a cross-section of the deformed membrane perpendicular to the deformed axis of the cylinder, points in this plane are described by cartesian coordinates x and y where the x-axis is parallel to $\mathbf{e}_r$ and the y-axis is parallel to $\mathbf{e}_z$. We can now write

$$\theta = \frac{\alpha Z}{\rho}, \tag{5.2.3}$$

where α is a positive constant representing the axial stretch after inflation. This will be determined later in this section. The current configuration can now be rewritten as

$$\mathbf{x}(\Theta, Z) = (\rho - x(\Theta))\mathbf{e}_r(\frac{\alpha Z}{\rho}) + y(\Theta)\mathbf{e}_z. \tag{5.2.4}$$

Before going on to obtain the deformation gradient we define the unit base vectors $\mathbf{A}_i$ before deformation by

$$\mathbf{A}_1 = \mathbf{E}_\Theta, \quad \mathbf{A}_2 = \mathbf{E}_Z, \quad \mathbf{A}_3 = \mathbf{E}_R. \tag{5.2.5}$$

From (5.2.4) the unit base vectors $\mathbf{a}_i$ after deformation are given by

$$\mathbf{a}_1 = \frac{-x'\mathbf{e}_r + y'\mathbf{e}_z}{(x'^2 + y'^2)^{\frac{1}{2}}}, \quad \mathbf{a}_2 = \mathbf{e}_\theta, \quad \mathbf{a}_3 = \frac{-x'\mathbf{e}_z - y'\mathbf{e}_r}{(x'^2 + y'^2)^{\frac{1}{2}}}. \tag{5.2.6}$$

We will assume throughout that a prime denotes differentiation with respect to Θ. Therefore the deformation gradient $\mathbf{F}$, given by (2.2.2), can be written as

$$\mathbf{F} = \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{E}_\Theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{E}_Z + \lambda_3 \mathbf{a}_3 \otimes \mathbf{A}_3,$$

where λ_3 is the principal stretch normal to the middle surface of the cylinder. Using (5.2.4) and (5.2.6) gives the deformation gradient as

$$\mathbf{F} = \frac{1}{R}(x'^2 + y'^2)^{\frac{1}{2}}\mathbf{a}_1 \otimes \mathbf{E}_\Theta + \frac{\alpha}{\rho}(\rho - x)\mathbf{a}_2 \otimes \mathbf{E}_Z + \lambda_3 \mathbf{a}_3 \otimes \mathbf{A}_3, \tag{5.2.7}$$

which is clearly diagonal with respect to the unit base vectors $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$. The principal stretches can then be defined by

$$\lambda_1 = \frac{(x'^2 + y'^2)^{\frac{1}{2}}}{R}, \quad \lambda_2 = \alpha(1 - \kappa x), \quad \lambda_3 = \frac{h}{H}, \tag{5.2.8}$$

where $\kappa = 1/\rho$ is the curvature of the axis of the cylinder and H, h are the thicknesses of the membrane before and after deformation respectively, where we assume H to be constant. Substituting (5.2.6) and (5.2.8) into (2.4.24), the general equations of equilibrium for a pressurised membrane, we obtain, in the absence of body forces,

$$\left.\begin{aligned} \frac{1}{h}\frac{\partial(h\sigma_{11})}{\partial\Theta} + \frac{x'}{(\rho - x)}(\sigma_{22} - \sigma_{11}) &= 0, \\ \frac{\sigma_{11}(y'x'' - x'y'')}{(x'^2 + y'^2)^{\frac{3}{2}}} + \frac{\sigma_{22}y'}{(x'^2 + y'^2)^{\frac{1}{2}}(\rho - x)} + \frac{P}{h} &= 0, \end{aligned}\right\} \tag{5.2.9}$$

where σ_{ii}, $i = 1, 2$ are the principal Cauchy stresses and the third equation in the Z direction is trivially satisfied. We shall assume that the material is hyperelastic with strain-energy function $W = W(\lambda_1, \lambda_2, \lambda_3)$. Again using the analysis of section 2.4 for an incompressible material we obtain

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1}),$$

and hence we can show (2.4.31) holds, namely

$$\sigma_{11} = \lambda_1 \hat{W}_1, \quad \sigma_{22} = \lambda_2 \hat{W}_2,$$

where $\hat{W}_i = \partial \hat{W}/\partial \lambda_i$, $i = 1, 2$.

Substituting (5.2.8) and (2.4.31) into the equations of equilibrium (5.2.9) results in

$$\left.\begin{aligned} \lambda_1 \hat{W}_1' + \alpha\kappa x' \hat{W}_2 &= 0, \\ \frac{\lambda_1 \hat{W}_1 (y'x'' - x'y'')}{(x'^2 + y'^2)^{\frac{3}{2}}} + \frac{\alpha\kappa y' \hat{W}_2}{(x'^2 + y'^2)^{\frac{1}{2}}} + \frac{P\lambda_1\lambda_2}{H} &= 0. \end{aligned}\right\} \tag{5.2.10}$$

Following Pipkin (1968) we can integrate $(5.2.10)_1$ with respect to Θ directly to obtain

$$\hat{W} - \lambda_1 \hat{W}_1 = b, \tag{5.2.11}$$

where b is a constant of integration. This equation was obtained by Haseganu and Steigmann (1994) naturally from their variational formulation of the membrane equations. To obtain a second equation of equilibrium from the system (5.2.10) we note that

$$\frac{y'\hat{W}_1(y'x'' - x'y'')}{(x'^2 + y'^2)^{\frac{3}{2}}} = \left(\frac{x'\hat{W}_1}{(x'^2 + y'^2)^{\frac{1}{2}}}\right)' - \frac{x'}{(x'^2 + y'^2)^{\frac{1}{2}}}\hat{W}_1'. \tag{5.2.12}$$

Multiplying $(5.2.10)_2$ by y'/λ_1 allows the first term in $(5.2.10)_2$ to be replaced by the right hand side of (5.2.12) to obtain

$$\left(\frac{x'\hat{W}_1}{(x'^2 + y'^2)^{\frac{1}{2}}}\right)' - \frac{x'}{(x'^2 + y'^2)^{\frac{1}{2}}}\hat{W}_1' + \frac{R\alpha\kappa y'^2 \hat{W}_2}{(x'^2 + y'^2)} + \frac{P\alpha(1 - \kappa x)y'}{H} = 0, \tag{5.2.13}$$

having used (5.2.8). If we now replace $\hat{W}_2$ in (5.2.13) using $(5.2.10)_1$ we obtain

$$x'\left(\frac{x'\hat{W}_1}{(x'^2+y'^2)^{\frac{1}{2}}}\right)' - (x'^2+y'^2)^{\frac{1}{2}}\hat{W}_1' + \frac{P\alpha(1-\kappa x)x'y'}{H} = 0,$$

which can be written as

$$\left(\frac{y'\hat{W}_1}{(x'^2+y'^2)^{\frac{1}{2}}}\right)' = \frac{P\alpha(1-\kappa x)x'}{H}. \tag{5.2.14}$$

We can integrate (5.2.14) with respect to Θ directly to obtain

$$\frac{y'\hat{W}_1}{(x'^2+y'^2)^{\frac{1}{2}}} - \frac{P\alpha x}{2H}(2-\kappa x) = a, \tag{5.2.15}$$

where a is a constant of integration. We now non-dimensionalise the curvature and the coordinates by writing

$$k = \frac{\kappa R}{2}, \quad u(\Theta) = \frac{x(\Theta)}{R}, \quad v(\Theta) = \frac{y(\Theta)}{R}, \tag{5.2.16}$$

which results in the principal stretches being defined by

$$\lambda_1 = (u'^2+v'^2)^{\frac{1}{2}}, \quad \lambda_2 = \alpha(1-2ku), \tag{5.2.17}$$

with λ_3 unchanged. Finally we non-dimensionalise the strain-energy and the pressure by writing

$$\hat{w} = \frac{\hat{W}}{2\mu}, \quad p = \frac{PR}{2\mu H}. \tag{5.2.18}$$

Using (5.2.16) – (5.2.18) the equations of equilibrium, (5.2.15) and (5.2.11), reduce to

$$\left.\begin{aligned} \frac{v'\hat{w}_1}{(u'^2+v'^2)^{\frac{1}{2}}} - p\alpha u(1-ku) &= a, \\ \hat{w} - \lambda_1\hat{w}_1 &= b, \end{aligned}\right\} \tag{5.2.19}$$

respectively, where a and b have been rescaled. Before we proceed in trying to solve (5.2.19) for a and b we want to obtain expressions for the principal stretches λ_1 and λ_2 before any flexural deformation has taken place. This in turn yields a value for α directly from λ_2 by substituting $\kappa = 0$ in $(5.2.8)_2$.

Consider an inflated, cylindrical membrane which has a point on the undeformed middle surface given by position vector (5.2.1) then a point on the deformed middle surface is given by position vector

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{5.2.20}$$

and the unit base vectors associated with this deformation are simply (5.2.5) before deformation and

$$\mathbf{a}_1 = \mathbf{e}_\theta, \quad \mathbf{a}_2 = \mathbf{e}_z, \quad \mathbf{a}_3 = \mathbf{e}_r, \tag{5.2.21}$$

after deformation. The deformation gradient is diagonal with respect to $(\mathbf{A}_i, \mathbf{a}_i)$ and the principal stretches are

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = \alpha, \quad \lambda_3 = \frac{h}{H}. \tag{5.2.22}$$

Using (5.2.21) and (5.2.22) with (2.4.24), the general equations of equilibrium for a pressurised membrane, we find only one resulting equation is non-trivial which reduces to

$$\frac{\sigma_{11}}{r} - \frac{P}{h} = 0. \tag{5.2.23}$$

We can rearrange (5.2.23) to yield

$$\hat{w}_1 = p\lambda_1\lambda_2, \tag{5.2.24}$$

having used (2.4.31), (5.2.18) and (5.2.22). We require a second equation to define the principal stretches λ_1 and λ_2 in terms of the inflating pressure p. This is obtained by considering end conditions. The total load T on one end of the closed cylinder due to the internal pressure will be

$$T = P\pi r^2. \tag{5.2.25}$$

However, this load is supported by the stress in the axial direction acting in the walls of the membrane. If we assume the walls have thickness h such that a point x on the membrane must lie in the region

$$r - \frac{h}{2} \leq x \leq r + \frac{h}{2},$$

then with (5.2.25) we find

$$\sigma_{22}\pi((r+\frac{h}{2})^2-(r-\frac{h}{2})^2)=P\pi r^2,$$

and hence using (2.4.31), (5.2.18) and (5.2.22) as before gives

$$2\hat{w}_2 = p\lambda_1^2. \tag{5.2.26}$$

Given the form of the strain-energy function, equations (5.2.24) and (5.2.26) specify λ_1 and λ_2 in terms of the applied pressure p. Haseganu and Steigmann (1994) were exclusively concerned with the incompressible Varga material which has non-dimensionalised strain-energy function

$$\hat{w}(\lambda_1,\lambda_2)=\lambda_1+\lambda_2+(\lambda_1\lambda_2)^{-1}-3, \tag{5.2.27}$$

having used (4.2.5) and $(5.2.18)_2$. To compare with the results given by Haseganu and Steigmann (1994) we shall consider (5.2.27) but we shall also consider a more realistic form of strain-energy proposed by Ogden (1972) which is given by

$$\hat{w}(\lambda_1,\lambda_2)=\sum_{i=1}^{N}\mu_i\{\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+(\lambda_1\lambda_2)^{-\alpha_i}-3\}/\alpha_i. \tag{5.2.28}$$

This form of strain-energy function was found to give an excellent fit to data obtained for a particular rubber-like material if the (non-dimensionalised) parameters $(N,\mu_i,\alpha_i,i=1...N)$ are chosen to be $N=3$ and

$$\begin{aligned}\mu_1 &= 0.74556, & \alpha_1 &= 1.3,\\ \mu_2 &= 0.00142, & \alpha_2 &= 5.0,\\ \mu_3 &= -0.01189, & \alpha_3 &= -2.0.\end{aligned} \tag{5.2.29}$$

The form of strain-energy function given by (5.2.27) considerably simplifies the analysis of the problem. To illustrate, we can solve (5.2.24) and (5.2.26) with (5.2.27) to yield

$$\left.\begin{aligned}2p\lambda_1^5+p^2\lambda_1^4+4(1-\lambda_1^3+p\lambda_1^2) &= 0,\\ \lambda_2^2-\frac{2}{\lambda_1(2-p\lambda_1^2)} &= 0,\end{aligned}\right\} \tag{5.2.30}$$

which can be solved numerically for λ_1 and λ_2 given a value of the pressure p. Incidentally on solving $(5.2.30)_1$ we find that only two real positive roots exist for λ_1 for a given p between zero and the maximum value. This is shown in Figure 5.1 where we plot the pressure p against the circumferential stretch $\lambda_1 = r/R$ for the two materials given by (5.2.27) and (5.2.28) with (5.2.29). We note that for moderate deformations both materials predict similar behaviour, however, considerable differences are observed as we choose larger values of the stretch. Figure 5.1 also shows that the Varga material reaches a maximum pressure before monotonically decreasing as λ_1 is increased. For a cylinder we would expect the pressure to increase again at some point as the membrane is stretched further as shown for the three-term material. This suggests that for large values of λ_1 the solutions obtained for the Varga material would be questionable. In fact Haughton and Ogden (1979) showed that for an inflated cylinder, bifurcation occurred for values of λ_1 greater than the value of λ_1 associated with the maximum pressure. After this point the cylinder became unstable and a cylindrical bulge formed. This bulge expanded and propagated along the cylinder until it consumed the whole of the membrane and the cylindrical shape became either the stable configuration again for the three-term material or continued to be the unstable configuration as shown for the Varga material. This point occurred before the pressure minimum for the three-term material was reached as shown in Figure 5.1. Haseganu and Steigmann (1994) used these results together with their own stability analysis to show that the smaller value of λ_1 was locally stable but the larger value was unstable. This work concentrates only on the stable value of λ_1 as this produces more interesting results from the point of view of wrinkling theory. For the Varga material we note from Figure 5.1 that the pressure maximum occurs when $\lambda_1 = 1.58740$ with the axial stretch value $\lambda_2 = 1.08422$ hence we shall restrict our attention to the region $1 \leq \lambda_1 \leq 1.5874$. We also note that the values of the two principal stretches at the pressure maximum for the three-term material (5.2.29) are $\lambda_1 = 1.60334$ and $\lambda_2 = 1.11368$. Finally we find that the axial stretch λ_2 increases monotonically with the stretch λ_1 and for large values of λ_1 the relationship is almost linear.

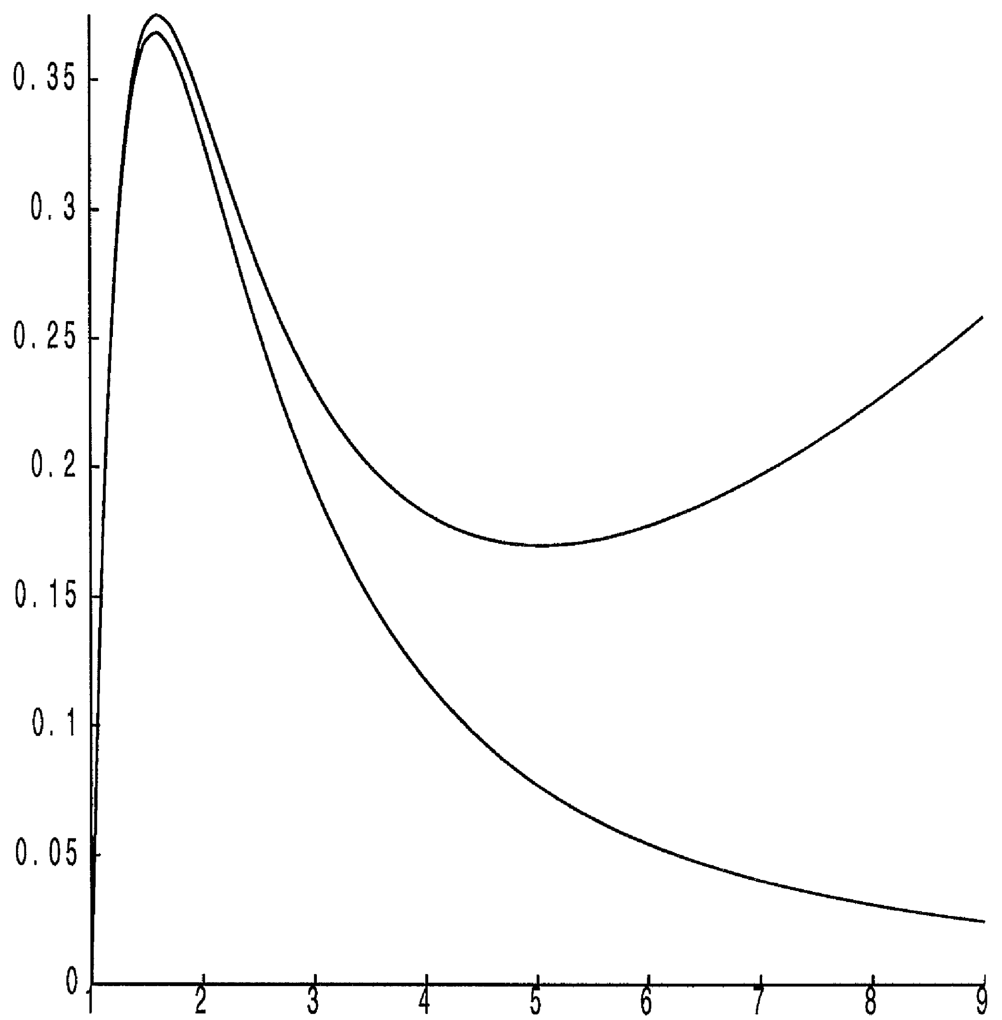

Figure 5.1: Plot of the non-dimensionalised pressure p against λ_1. The upper curve corresponds to the three-term material (5.2.29) and the lower curve to the Varga material (5.2.27).

5.3 Solution Procedure

We now turn our attention to solving (5.2.19) for a and b. For a general strain-energy function these two simultaneous equations have to be solved numerically. For the special case of a Varga material considerable simplification is possible as shown by Haseganu and Steigmann (1994) although numerical techniques still have to be employed to solve the system of equations. We illustrate the method used, for the special case of a Varga material, to construct the cross-sectional shape of the membrane when wrinkling is present. Firstly we recall the definition of the relaxed strain-energy function for the incompressible Varga material which operates in the wrinkled region. Using the theory described in chapter 3 we again obtain the relaxed strain-energy function defined by (4.2.13) which when non-dimensionalised can be written as

$$\bar{w}(\lambda_1) = \lambda_1 + 2\lambda_1^{-\frac{1}{2}} - 3, \tag{5.3.1}$$

having used (5.2.18). Substituting (5.2.27) and (5.3.1) into $(5.2.19)_2$ we can find expressions for λ_1 given by

$$\left.\begin{aligned} \lambda_1 &= \Lambda_t = \frac{2}{\lambda_2(b+3-\lambda_2)}, \\ \lambda_1 &= \Lambda_w = (1+b/3)^{-2}, \end{aligned}\right\} \tag{5.3.2}$$

for the tense and wrinkled regions respectively, where λ_2 is given by $(5.2.17)_2$. We can observe from $(5.3.2)_1$ that Λ_t must be a function of u only and from $(5.3.2)_2$ that Λ_w will be a constant since b is constant. In all succeeding equations λ_1 will be given by the appropriate definition $(5.3.2)_1$ or $(5.3.2)_2$ for the tense or wrinkled region respectively. We now want to define the boundary between the tense and wrinkled regions $u = u^*$ say. This will occur when $\sigma_{22}(u^*) = 0$ and hence (3.3.2) and (3.3.4) will hold. Either Λ_t or Λ_w can be chosen because of continuity across the boundary and so we choose Λ_w as it is a constant. Hence substituting $(5.2.17)_2$,

$(5.3.2)_2$ and (3.3.2) into (3.3.4) yields

$$u^*(b) = \frac{1}{2k}(1 - \alpha^{-1}\Lambda_w^{-1/2}). \tag{5.3.3}$$

If all values of u satisfy $u < u^*$ then the entire membrane is tense and no wrinkling occurs. However, if some u exists such that $u > u^*$ then the membrane must be partly tense and partly wrinkled. We assume that the deformation is symmetric with respect to the u-axis and points $(u_0, 0)$ and $(u_\pi, 0)$ correspond to $\Theta = 0$ and $\Theta = \pi$ respectively. Therefore we need only consider the upper half plane $v \geq 0$. Rearranging $(5.2.19)_1$ gives

$$\frac{dv}{d\Theta} = \frac{\lambda_1}{\hat{w}_1}(a + p\alpha u(1 - ku)) = K(u, a, b), \tag{5.3.4}$$

say. If we now assume that $u'(\Theta) \leq 0$ on the upper half plane then $(5.2.17)_1$ and (5.3.4) yield

$$\frac{du}{d\Theta} = -(\lambda_1^2 - K^2)^{\frac{1}{2}}, \tag{5.3.5}$$

and combining (5.3.4) and (5.3.5) we obtain

$$\frac{dv}{du} = \frac{-K}{(\lambda_1^2 - K^2)^{\frac{1}{2}}}, \tag{5.3.6}$$

which defines the deformed shape of the membrane. From symmetry and continuity dv/du must be infinite at $u = u_0, u_\pi$. Hence from (5.3.6)

$$\lambda_1^2 = K^2, \qquad \text{at } u = u_0, u_\pi. \tag{5.3.7}$$

Note that equations (5.3.4) – (5.3.7) hold for a general material. We can therefore solve (5.3.7) numerically to obtain values for u_0 and u_π for any incompressible material. Returning to consider the material defined by the incompressible Varga strain-energy function we can show how (5.3.7) simplifies by substituting $(5.3.2)_1$, (5.2.27) and (5.3.4) into (5.3.7) to obtain

$$a + p\alpha u(1 - ku) = \pm(1 - \frac{\lambda_2}{4}(b + 3 - \lambda_2)^2), \tag{5.3.8}$$

for the tense region and substituting $(5.3.2)_2$, (5.3.1) and (5.3.4) into (5.3.7) yields

$$a + p\alpha u(1 - ku) = \pm(1 - (1 + \frac{b}{3})^3), \tag{5.3.9}$$

for the wrinkled region. With λ_2 defined by $(5.2.17)_2$ we note that (5.3.8) gives a cubic in u and (5.3.9) a quadratic. Hence if the membrane is tense there exists six choices for u_0 and u_π and if partly tense and partly wrinkled then there are six possibilities for u_π and four for u_0.

Fortunately, for all materials, there are several conditions placed on these parameters which allows us to select the correct values. Clearly $u_0 > u_\pi$ and they both must be real. We also require that $\lambda_2 > 0$ so evaluating $(5.2.17)_2$ at $u = u_0$, since $u \leq u_0$, we have

$$2ku_0 < 1. \tag{5.3.10}$$

Owing to the nature of the deformation we note that on bending the inflated cylinder the outer surface of the cylinder, corresponding to $u = u_\pi$, will be stretched further in the axial direction. This results in the membrane always being tense at $u = u_\pi$ with $\sigma_{22} > 0$ and hence gives the condition

$$\hat{w}_2 > 0 \text{ at } u = u_\pi. \tag{5.3.11}$$

As p, λ_1 and λ_2 are all positive (5.2.24) yields

$$\hat{w}_1 > 0, \text{ at } u_0, u_\pi. \tag{5.3.12}$$

These conditions are sufficient to render the choice of u_0 and u_π unique and we now derive two equations to determine the parameters a and b.

Using symmetry of the deformation with (5.3.6) we obtain

$$v(0) = -\int_{u_\pi}^{u_0} \frac{K du}{(\lambda_1^2 - K^2)^{\frac{1}{2}}} = 0. \tag{5.3.13}$$

If we then define an inverse function of $u(\Theta)$ by $\Theta(u)$ we find that (5.3.5) yields

$$\Theta(u) = \int_{u}^{u_0} \frac{du}{(\lambda_1^2 - K^2)^{\frac{1}{2}}}, \tag{5.3.14}$$

and hence we can write

$$\Theta(u_\pi) = \int_{u_\pi}^{u_0} \frac{du}{(\lambda_1^2 - K^2)^{\frac{1}{2}}} = \pi. \tag{5.3.15}$$

Equations (5.3.13) and (5.3.15) provide two simultaneous equations for the unknowns a and b. Lastly we define

$$v(u) = \int_{u}^{u_\pi} \frac{K du}{(\lambda_1^2 - K^2)^{\frac{1}{2}}}, \tag{5.3.16}$$

which can be used to plot the deformed cross-section of the membrane.

For membrane wrinkling problems we will generally have two solution procedures available. For example, consider a general displacement boundary value problem which requires to be solved numerically using a shooting method. If we encounter a wrinkled region during the iterative process we can continue to use the tense theory to solve over the whole membrane, determine the wrinkled region and correct the solution in the wrinkled region. Alternatively, we can continually test for a negative principal stress and switch between tense and wrinkled theory as appropriate to obtain the valid solution. Generally the boundary between the tense and wrinkled region will be determined using the tense theory and so we will arrive at the same solution using either method described above. For an example of this the reader is referred to the annular problem in chapter 4. However, the problem considered here is rather different because the domain (u_π, u_0) is not fixed and is instead obtained as part of the solution process. This in turn will produce different values for a and b when solving (5.3.13) and (5.3.15). Therefore we investigate both solution methods.

To do this we assume that the pressure p, initial axial stretch α and curvature k are given. We can then determine the cross-sectional shape as follows. Initially we require an estimate of the values of a and b. We can take $k = 0$ and evaluate b from $(5.2.19)_2$ when the strain-energy function is known. We also note from $(5.2.19)_1$, taking $u = \lambda_1 \cos\Theta$, $v = \lambda_1 \sin\Theta$ and using (5.2.24) that $a = 0$ for a straight cylinder. We can now determine a and b for any k using an incremental procedure and subsequently we find u_0 and u_π from (5.3.7). We then check for wrinkling by evaluating $\sigma_{22}(u_0)$. If $\sigma_{22}(u_0) \geq 0$ then wrinkling has not occurred and we can evaluate the integrals (5.3.13) and (5.3.15), update a and b and repeat the process until convergence is achieved. We can then plot the deformed cross-section of the membrane using (5.3.16). If $\sigma_{22}(u_0) < 0$, however, then wrinkling has occurred and we go on to use one of the solution procedures described below.

1) Integrated Method

The integrated method described here is that which is adopted by Haseganu and Steigmann (1994) in which we continually check the sign of σ_{22} and switch between the tense and wrinkled solutions, when required, as soon as possible. We begin by estimating the values of a and b, given the strain-energy function of the material, the curvature k, the initial stretch α and the pressure p. We use these values to find u_0 and u_π using (5.3.7) and then calculate the hoop stress $\sigma_{22}(u_0)$. If this is found to be negative then we find u^* by solving $\sigma_{22}(u^*) = 0$ and return to (5.3.7) with $(5.3.2)_2$ to find $\bar{u}_0$, say. If $\sigma_{22}(u_0) \geq 0$ we have no wrinkling present and we set $u^* = u_0 = \bar{u}_0$. We can now evaluate (5.3.13) and (5.3.15) for the incompressible Varga material using (5.2.27) with $(5.3.2)_1$ for the tense region $u_\pi \leq u \leq u^*$ and (5.3.1) with $(5.3.2)_2$ for the wrinkled region $u^* \leq u \leq \bar{u}_0$. As previously described we then adjust a and b if necessary and repeat the process until we achieve convergence. We can therefore write

$$v(u) = \begin{cases} \displaystyle\int_u^{u_\pi} \frac{K\,du}{(\Lambda_t^2 - K^2)^{1/2}}, & u_\pi \leq u \leq u^*, \\ \displaystyle\int_{u^*}^{u_\pi} \frac{K\,du}{(\Lambda_t^2 - K^2)^{1/2}} + \int_u^{u^*} \frac{K\,du}{(\Lambda_w^2 - K^2)^{1/2}}, & u^* \leq u \leq \bar{u}_0, \end{cases} \tag{5.3.17}$$

and use (5.3.17) to plot the deformed cross-section of the membrane.

2) Correcting Method

This method uses the ordinary membrane theory to find a solution for the whole membrane and then uses wrinkling theory to correct any regions where negative principal stretches are subsequently found. Again we begin with estimates for a and b to determine u_0 and u_π from (5.3.7). This time, however, we immediately evaluate (5.3.13) and (5.3.15) and then correct the values of a and b sequentially to achieve convergence of the integrals. Now we compute the solution $v(u)$ predicted by the ordinary membrane theory from $(5.3.17)_1$ with $u^* = u_0$. We then determine the boundary of the wrinkled region (if one exists) $u = u^*$ where again $\sigma_{22}(u^*) = 0$ with $u_\pi < u^* < u_0$. Using wrinkling theory with the relaxed strain-energy function we correct the wrinkled region as follows. Firstly we require a new evaluation of u_0, $\bar{u}_0$, which is obtained by solving (5.3.7) using the relaxed strain-energy function (5.3.1) with $(5.3.2)_2$. We now recalculate the solution $v(u)$, $u^* < u < \bar{u}_0$, from (5.3.16) again using (5.3.1) and $(5.3.2)_2$. To match the two solutions we simply require the wrinkled solution to give the same value of $v(u^*)$ as was found by ordinary membrane theory. Finally we can rewrite (5.3.16) as

$$v(u) = \int_{u^*}^{u_\pi} \frac{K du}{(\Lambda_t^2 - K^2)^{1/2}} + \int_{u}^{u^*} \frac{K du}{(\Lambda_w^2 - K^2)^{1/2}}, \qquad (5.3.18)$$

where $u^* \leq u \leq \bar{u}_0$ which we use to plot the deformed cross-section of the cylinder.

Intuitively, the integrated method should be more accurate as it uses information about the sign of σ_{22} as soon as it is available, although the correcting method appeals to the idea that tense membrane theory is correct except in areas of negative stress. In the absence of any experimental data for comparison, we include the correcting method as an interesting alternative.

5.4 Results for Incompressible Materials

5.4.1 Deformed Cross-Section

This section looks at the results obtained by solving the system of equations numerically as described by the two methods above for both the incompressible Varga material (5.2.27) and the three-term material defined by (5.2.28) with (5.2.29). For comparison with Haseganu and Steigmann (1994) we take the value of the non-dimensionalised, inflating pressure to be $p = 0.3$. This corresponds to an inflated cylinder with $\lambda_1 = 1.26464$, $\lambda_2 = 1.01995$ for the Varga material and $\lambda_1 = 1.25966$, $\lambda_2 = 1.02448$ for the three-term material, all values given correct to five decimal places.

Figure 5.2 shows a plot of the deformed cross-section for the Varga material with curvature $k = 0.05$, 0.1, 0.15 and 0.2 having used the integrated method of solution. The starred symbols signify the boundary between the tense and wrinkled regions. Recalling that $u_0 > u_\pi$ the curves to the right of this symbol will represent the wrinkled region. The dashed curve shows the deformed cross-section when $k = 0.05$ in which no wrinkling occurs. We observe that as the curvature increases the wrinkled region also increases as we would expect. We also note that as the curvature increases from zero, u_π initially moves outwards from its original position, $u_\pi = -1.26$, reaches a minimum sometime shortly after wrinkling occurs and subsequently moves inwards. Similarly the value of u_0 or $\bar{u}_0$ initially decreases to reach a minimum value and then moves outwards but to a far lesser extent.

Figure 5.3 also shows the deformed cross-section of the cylinder for the Varga material again with curvature $k = 0.05$, 0.1, 0.15 and 0.2 but this time uses the correcting method of solution. We note that as no wrinkling occurs for $k = 0.05$, indicated by the dashed curve, the solution is precisely the same as that plotted by the dashed curve in Figure 5.2. Again we observe that as the curvature increases, the wrinkled region increases, with u_π initially moving outwards, reaching

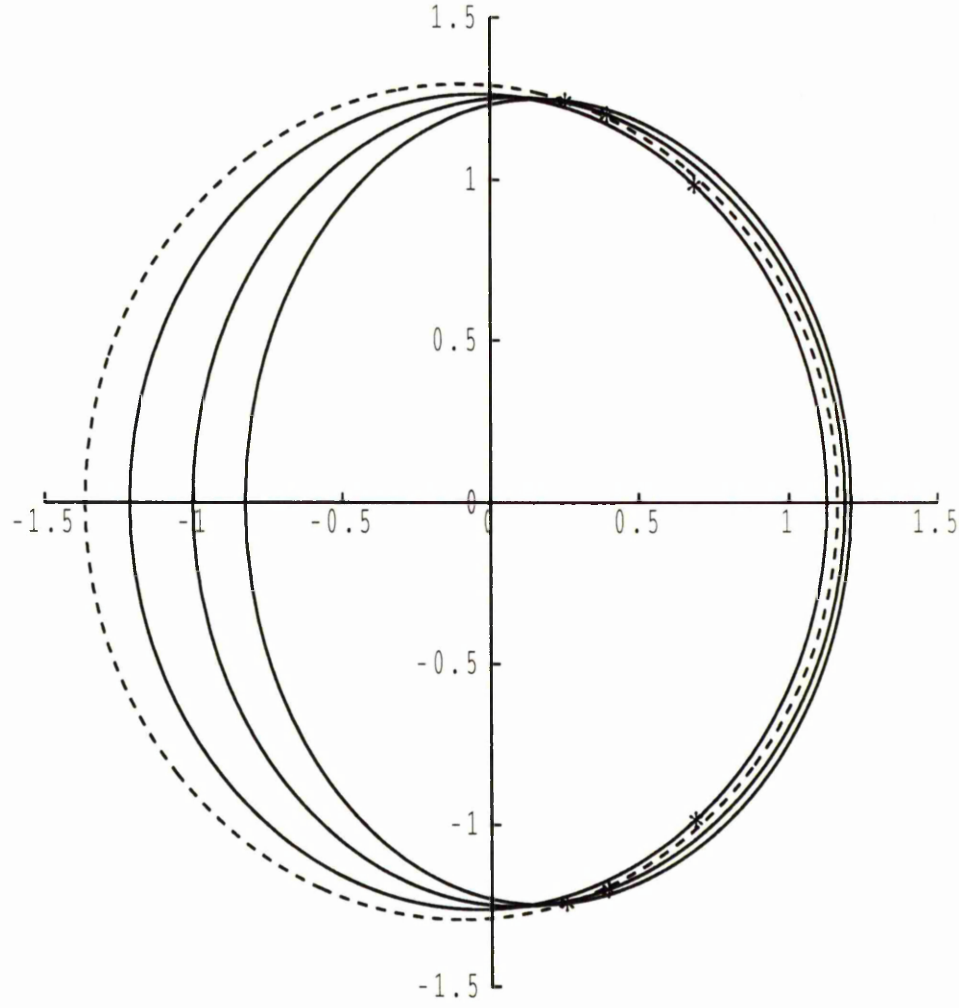

Figure 5.2: Plot of the deformed cross-section for the Varga material using the integrated method with $k = 0.05$ (no wrinkling - - - - -), 0.1, 0.15 and 0.2.

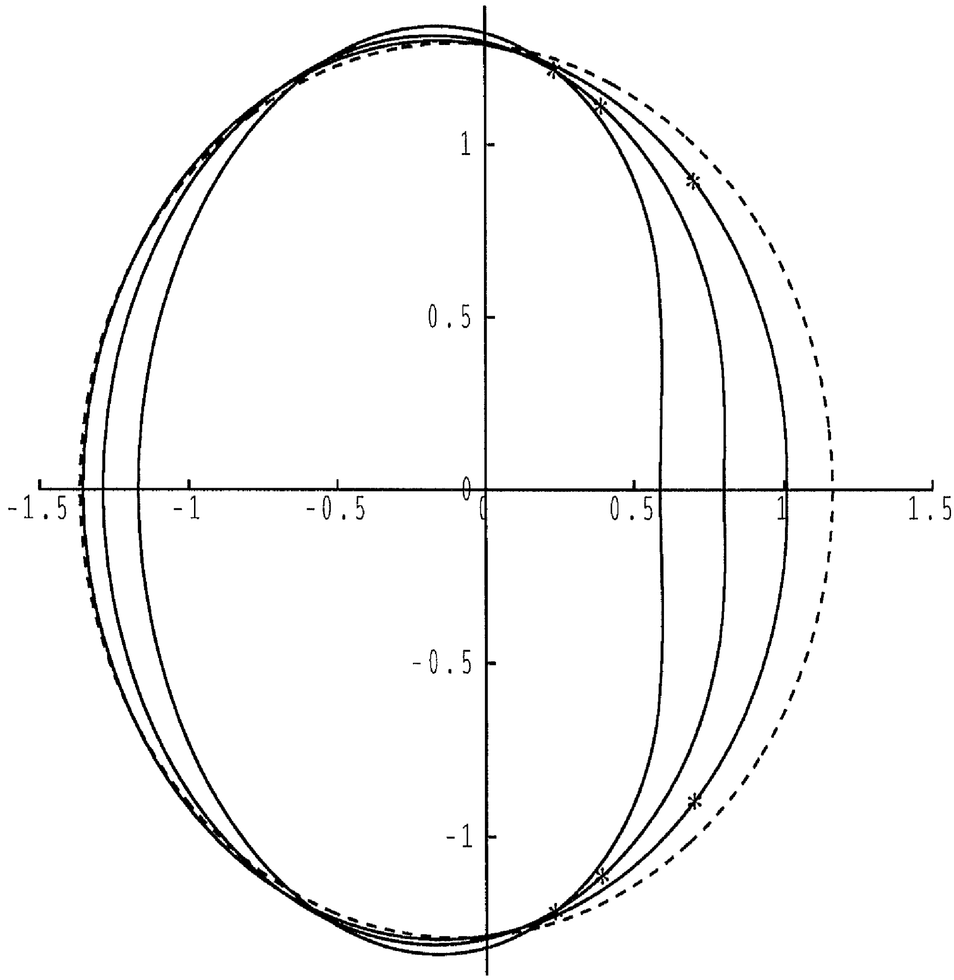

Figure 5.3: Plot of the deformed cross-section for the Varga material using the correcting method with $k = 0.05$ (no wrinkling - - - - -), 0.1, 0.15 and 0.2.

a minimum and then moving inwards as before. However, this time u_0 or $\bar{u}_0$ is a monotonic decreasing function of the curvature and we observe a flattening of the wrinkled part at $\bar{u}_0$ as we increase the curvature.

By comparing graphs of the deformed cross-section for the two methods we find that although the boundary u^* is similar in value for each the proportion of wrinkled membrane is considerably larger for the integrated method since $\bar{u}_0$ is larger. Figures 5.4 – 5.6 show a comparison of methods for $k = 0.1$, 0.15 and 0.2. The solution predicted by tense membrane theory is also plotted, again shown as a dashed curve, to indicate the correcting method simply adjusts the wrinkled region whereas the integrated method produces a completely new solution. For curvature values of 0.1 or less there is little difference between the tense and corrected solutions. However, we start to see the affect of correction in Figures 5.5 and 5.6 as the curvature increases and a more developed wrinkled region forms. We also note the two solution methods become increasingly different as the curvature increases.

This difference is accounted for by changes in a, as shown in Figure 5.7. As the curvature gets larger the value of a for the integrated method is monotonically decreasing and becomes negative when $k \simeq 0.12$. However for the correcting method a increases between $k = 0.06$ and around $k = 0.16$ to reach a peak value of approximately 0.055 before starting to fall slightly. The difference in a for the two solution methods is therefore monotonically increasing as the curvature increases. Note that the plot begins with a curvature value of $k = 0.06$ rather than 0.05 on the x-axis. No wrinkling occurs before this value and so both solution methods will use the tense theory only hence will have equal values of a (and b) for $k \leq 0.06$. The point at which wrinkling is initiated will be discussed in more detail later in this section. Figure 5.8 shows how the values of b vary with curvature and that they are essentially in agreement suggesting they will have little affect on the difference observed in the deformed cross-section of the two methods.

We now turn our attention to the three-term material (5.2.29) and plot the deformed cross-section of the cylinder using both solution methods which have

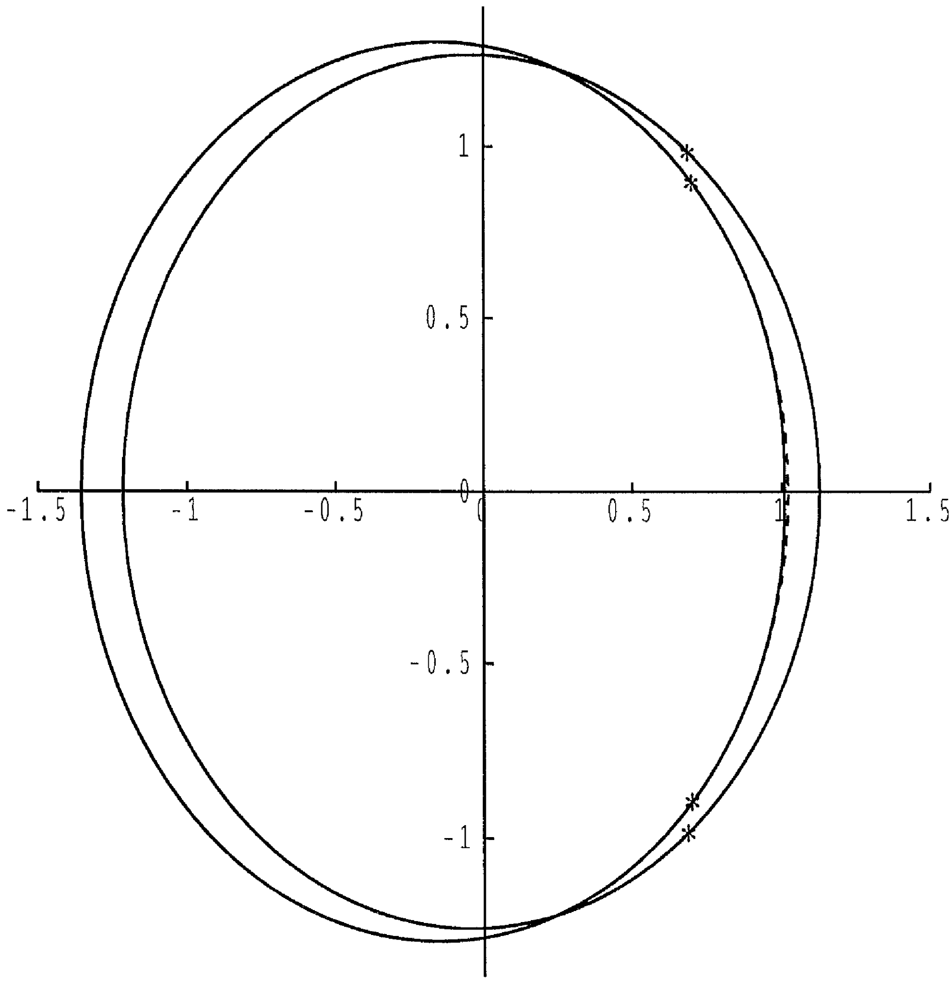

Figure 5.4: Plot of both solution methods and the tense solution (- - - - -) with $k = 0.1$.

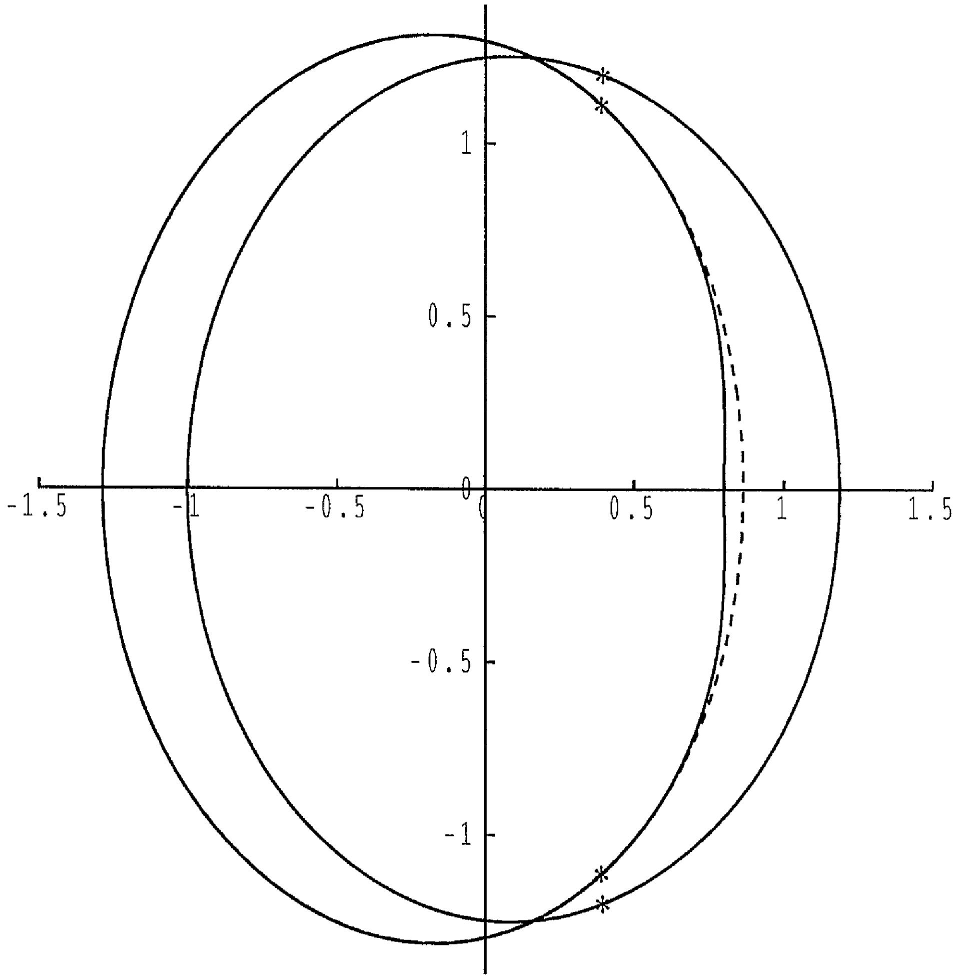

Figure 5.5: Plot of both solution methods and the tense solution (- - - - -) with k = 0.15.

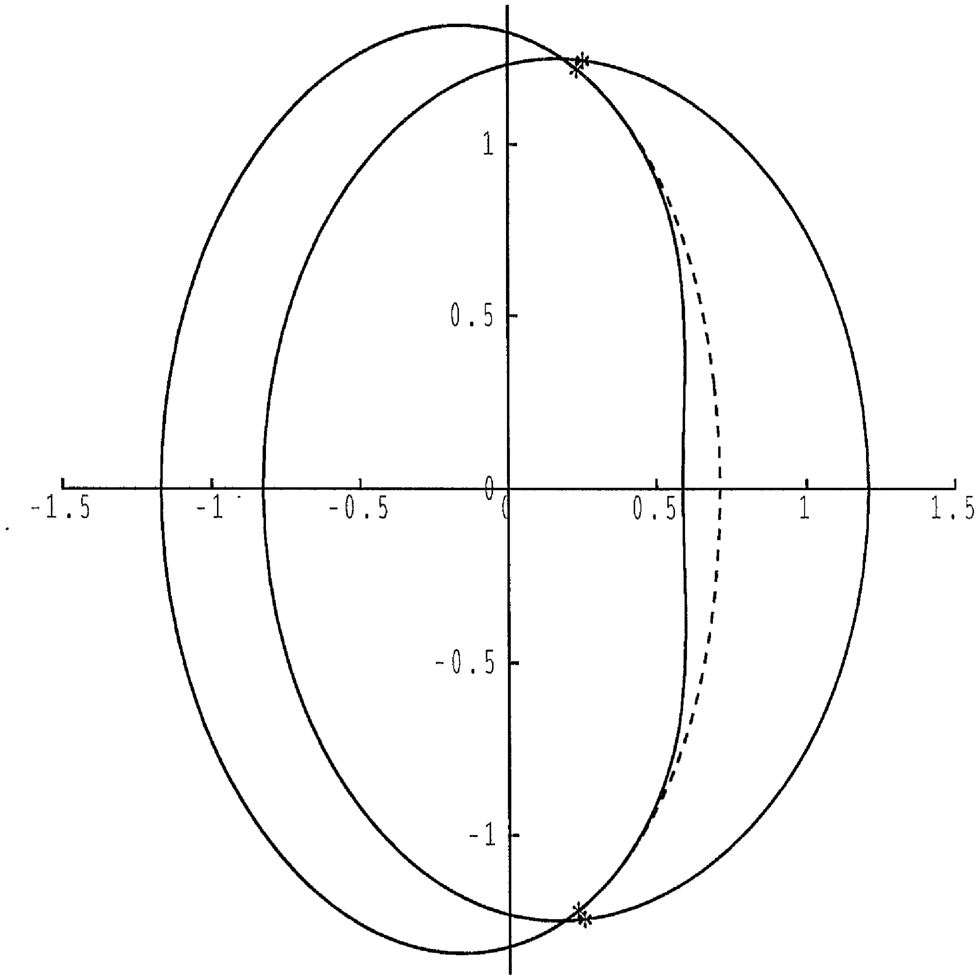

Figure 5.6: Plot of both solution methods and the tense solution (- - - - -) with k = 0.2.

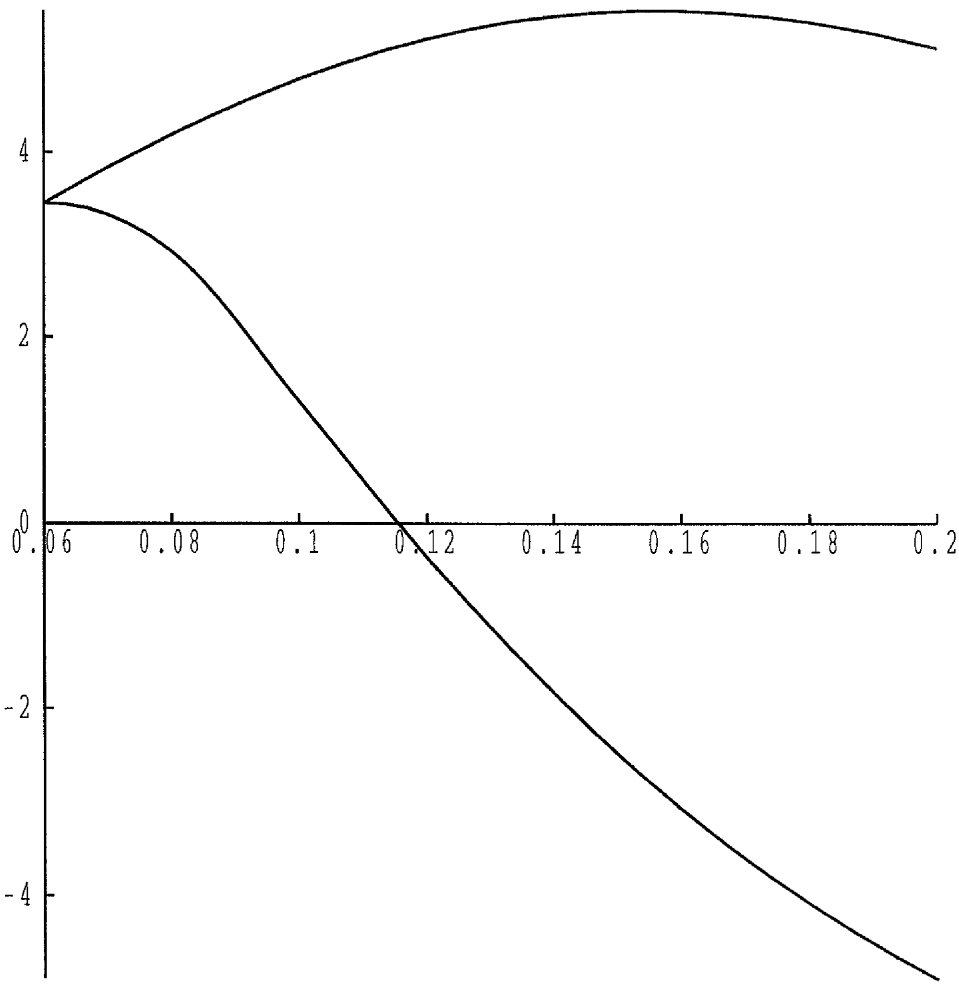

Figure 5.7: A plot of a x 100 against curvature for both solution methods. The upper curve represents the correcting method while the lower curve corresponds to the integrated method.

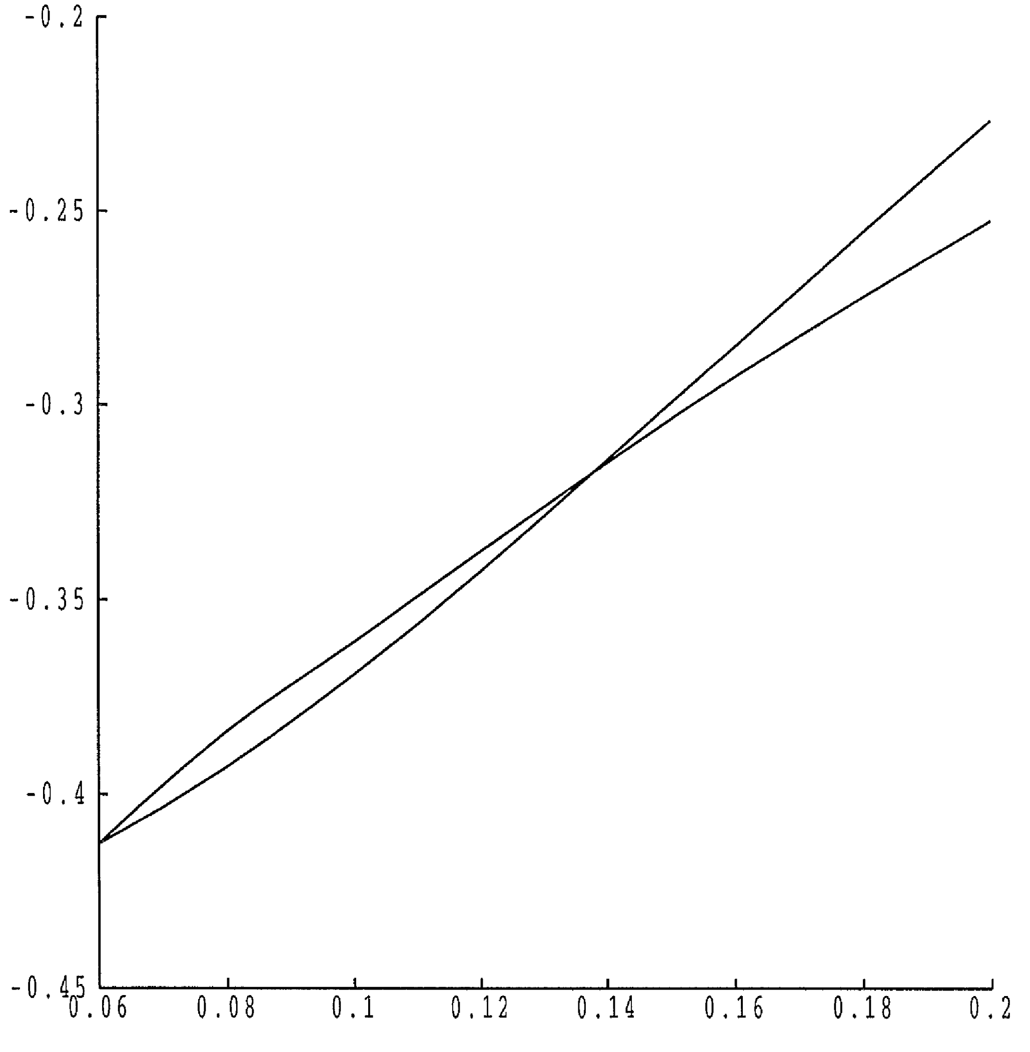

Figure 5.8: A plot of b against curvature for both solution methods.

already been described. As mentioned previously the equations used to find $v(u)$ are simplified considerably for the Varga material and the calculations used for the three-term material are much more complex. The solution methods outlined above were formulated for a general strain-energy function of the form (5.2.28) and taking $N = 1$ and $\alpha = 1$ reproduce the results obtained for the Varga material. Therefore we can employ similar numerical techniques to solve the system of equations for the three-term material (5.2.29). To compare the results obtained with that for the Varga material we again choose the non-dimensionalised, inflating pressure $p = 0.3$ which gives $\lambda_1 = 1.25966$ and $\lambda_2 = 1.02448$. We note the similarity in principal stretch values for the two materials and because the deformation is moderate we do not expect to find large differences in the solutions of the materials.

Figure 5.9 plots the deformed cross-section of the membrane for values of the curvature $k = 0.06$, 0.09, 0.12, 0.15 and 0.18 for the integrated solution method. Wrinkling does not occur for curvature $k = 0.06$, the dashed curve again indicating this. We observe that as the curvature increases the wrinkled region also increases and u_π moves inwards while u_0 or $\bar{u}_0$ moves outwards. This agrees with results found for the Varga material. In Figure 5.10 we plot the curves corresponding to the above in Figure 5.9 this time using the correcting method. Again wrinkling has not occurred for curvature $k = 0.06$ and we note that, as before, the wrinkled region increases as the curvature increases. Again we see a flattening of the wrinkled region at $\bar{u}_0$ as we increase the curvature. Comparing Figures 5.9 and 5.10 with Figures 5.2 and 5.3 we observe the similarities obtained for the two materials. As mentioned above this is not surprising as the pressure is relatively small and so is the resulting deformation. Differences between material models will become more apparent at larger pressure values, and hence deformations, as can be observed from Figure 5.1.

Although the above plots of the deformed cross-section indicate the curvature values that produce wrinkling, the curves do not allow a precise value at which wrinkling first becomes possible. We now consider this aspect in some detail below.

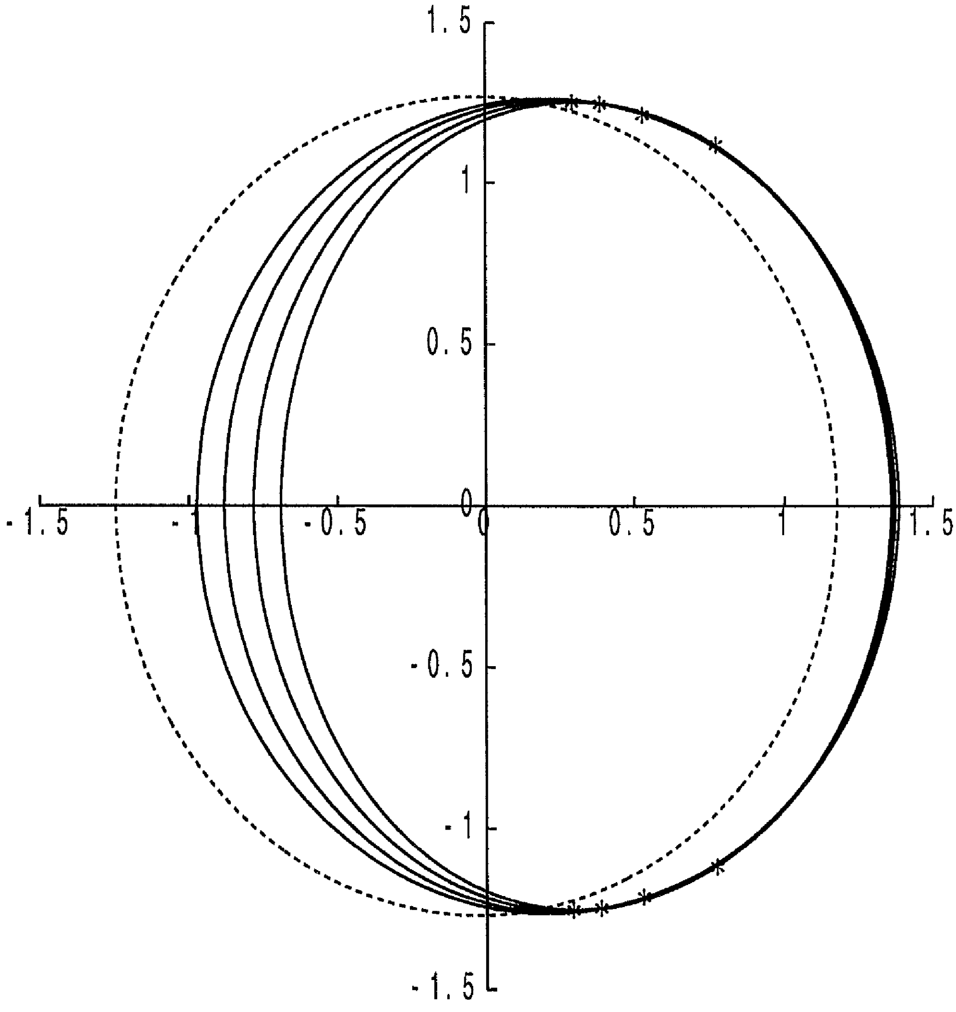

Figure 5.9: Plot of the deformed cross-section for the three-term material using the integrated method with curvature $k = 0.06$ (no wrinkling - - - - -), 0.09, 0.12, 0.15 and 0.18.

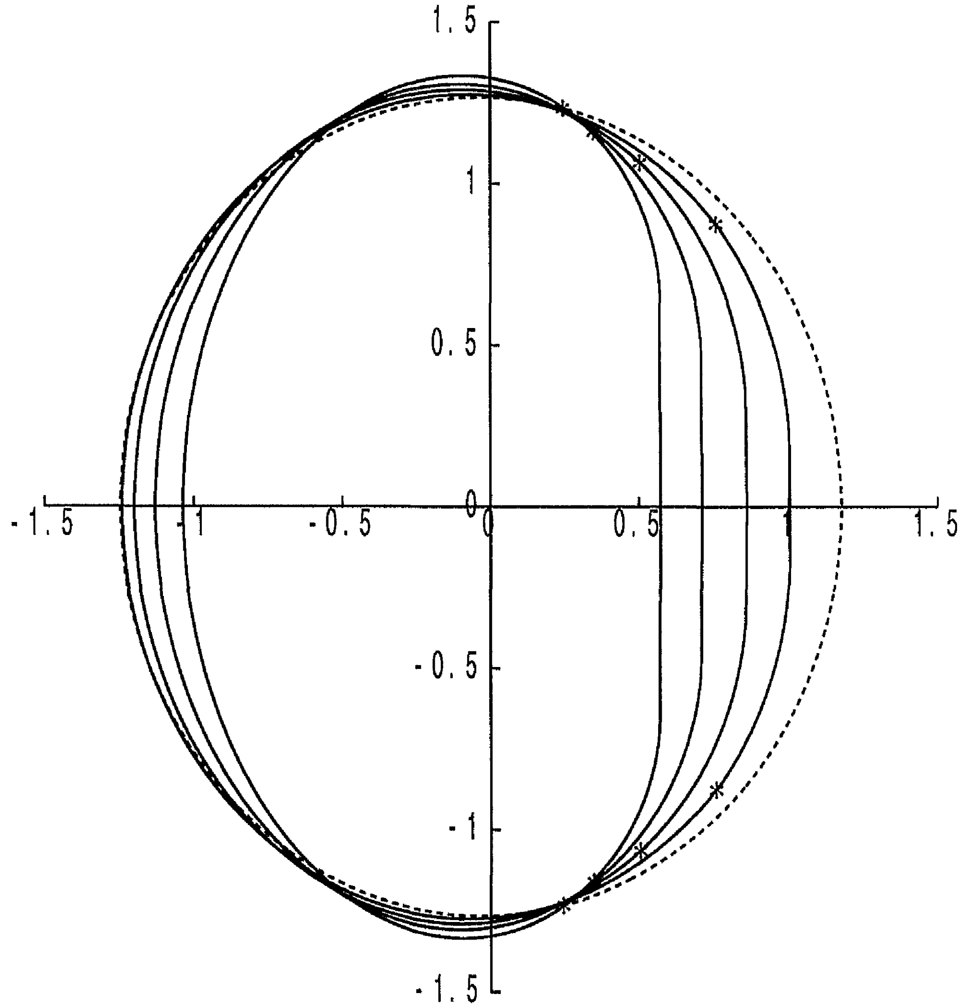

Figure 5.10: Plot of the deformed cross-section for the three-term material using the correcting method with curvature $k = 0.06$ (no wrinkling - - - - -), 0.09, 0.12, 0.15 and 0.18.

5.4.2 Volume Considerations

As we have shown above, the bending of an inflated cylinder produces a change in the cross-sectional shape of the membrane. We have assumed that the cylinder is closed so that the internal pressure acting on the ends of the cylinder produces an axial loading that determines the axial extension of the inflated tube. In the above we have simply taken the pressure to be constant. This was the approach adopted by Haseganu and Steigmann (1994). However, for most practical purposes the cylinder will be inflated by a gas. In this case the change in shape of the cross-section, and the resulting change in enclosed volume will change the inflating pressure of the gas. Hence we have a problem where the loading and the deformed configuration are interdependent.

To investigate how the changing enclosed volume of the cylinder affects the solution we shall assume that the inflating gas is an ideal fluid. In the inflated configuration of the tube the enclosed volume per unit undeformed length of the cylinder is $\pi\lambda^2\lambda_z$ where λ and λ_z are the principal stretches associated with inflation by an ideal fluid. Hence we shall assume that

$$pV = \pi\hat{p}\lambda^2\lambda_z, \tag{5.4.1}$$

where p is the gas pressure, as before, $\hat{p}$ is the pressure when the cylinder is straight and V is the currently enclosed volume per unit undeformed length. This can be written in several different ways and we find the most convenient form to be

$$V = \frac{2\alpha}{\rho}\int_{u_\pi}^{u_0}(\rho - u)v\,du.$$

Using (5.3.6) we may integrate by parts to obtain a single integral expression for V, namely,

$$V = \frac{\alpha}{\rho}\int_{u_\pi}^{u_0} u(2\rho - u)\frac{K}{(\lambda_1^2 - K^2)^{\frac{1}{2}}}du, \tag{5.4.2}$$

where K is given by (5.3.4). To evaluate (5.4.2) we again have to employ numerical techniques. To obtain a solution to the overall problem, allowing the pressure to change during the bending process, we begin by calculating the pressure and enclosed volume (per unit undeformed length) for the inflated cylinder only. Specifying a curvature k we solve the equations of equilibrium (5.2.19) while satisfying (5.3.13) and (5.3.15) as we have previously described using one of the two methods outlined above. For this curvature k we calculate the new volume and then use (5.4.1) to find the associated pressure p. This p is used to adjust α and then we repeat the process of solving the equations of equilibrium with this new pressure value. We continue this iterative scheme until we find values of pressure and volume which satisfy (5.4.1). This process is successful for small curvature, however, as the curvature is made larger divergence occurs. Instead we recast the problem into solving

$$pV - \pi\hat{p}\lambda^2\lambda_z = 0, \tag{5.4.3}$$

where p is now the current pressure and V is the resulting volume. This can then be solved using a standard method and the process outlined above. Clearly the method which uses the uncorrected value of the pressure will be inaccurate and is merely given here to indicate the change in value of the relevant variables.

In practice we find that the uncorrected solution produces a decrease in volume with increasing curvature. From (5.4.3) we therefore must have an increase in pressure and subsequently a larger cylinder as $\lambda_1 = r/R$ and $\lambda_2 = \alpha$ both increase. This will cause the corrected volume to increase over its uncorrected value, although it will still have decreased from its original value associated with the inflated configuration only.

These effects are illustrated in Figure 5.11 for the incompressible Varga material using the integrated method where we have chosen an initial inflating pressure to yield $\lambda_1 = 1.45$. Figure 5.11(a) plots both the uncorrected and corrected values of the volume against the curvature k. The lower curve represents the enclosed volume assuming a constant pressure whereas the upper curve shows the corrected volume when we assume the cylinder is inflated with a perfect gas. We observe from Figure 5.11(a) that as the curvature increases the volume decreases for both methods but to a far greater extent if we do not use the correcting method. This can be explained if we consider graphs 5.11(b)-(d) which show plots of the corrected and uncorrected values of the pressure p, the circumferential stretch λ_1 and the axial stretch λ_2 against curvature, respectively. The uncorrected values are indicated by the constant lines and the corrected values by the curves. We note that p, λ_1 and λ_2 all increase as the curvature increases if the correcting method is used. Although the volume must decrease as the pressure increases this causes the principal stretches to increase also. As the principal stretches are directly proportional to the volume this will cause the volume to increase slightly although it will still be less than the value associated with inflation only.

5.4.3 Initiation Of Wrinkling

As mentioned earlier the value of the curvature at which wrinkling first occurs is of considerable interest. Figure 5.12 plots the critical value of the radius of curvature ρ against the initial stretch λ_1 and shows the initiation of wrinkling. The lower curve represents the values for the method which assumes inflation with an ideal gas while the upper curve plots the values for the method which assumes constant pressure. Clearly, the change in volume has only a small effect on the onset of wrinkling which is to delay wrinkling as we correct the volume. This can be explained as follows. As λ_1 increases, a greater deformation, and hence a smaller value of ρ, is required to induce wrinkling. Given an initial stretch λ_1 the corrected value of λ_1 will be larger than the uncorrected value (see Figure 5.11(c)) hence the radius of curvature will need to be decreased further to obtain a wrinkled effect. We note that wrinkling will occur immediately ($\rho \to \infty$) if the membrane is not inflated so that $\lambda_1 = 1$ and then a flexural deformation is applied. We also note that for a cylinder inflated close to its pressure maximum (see Figure 5.1) the volume correction may cause λ_1 to increase to a value associated with an unstable

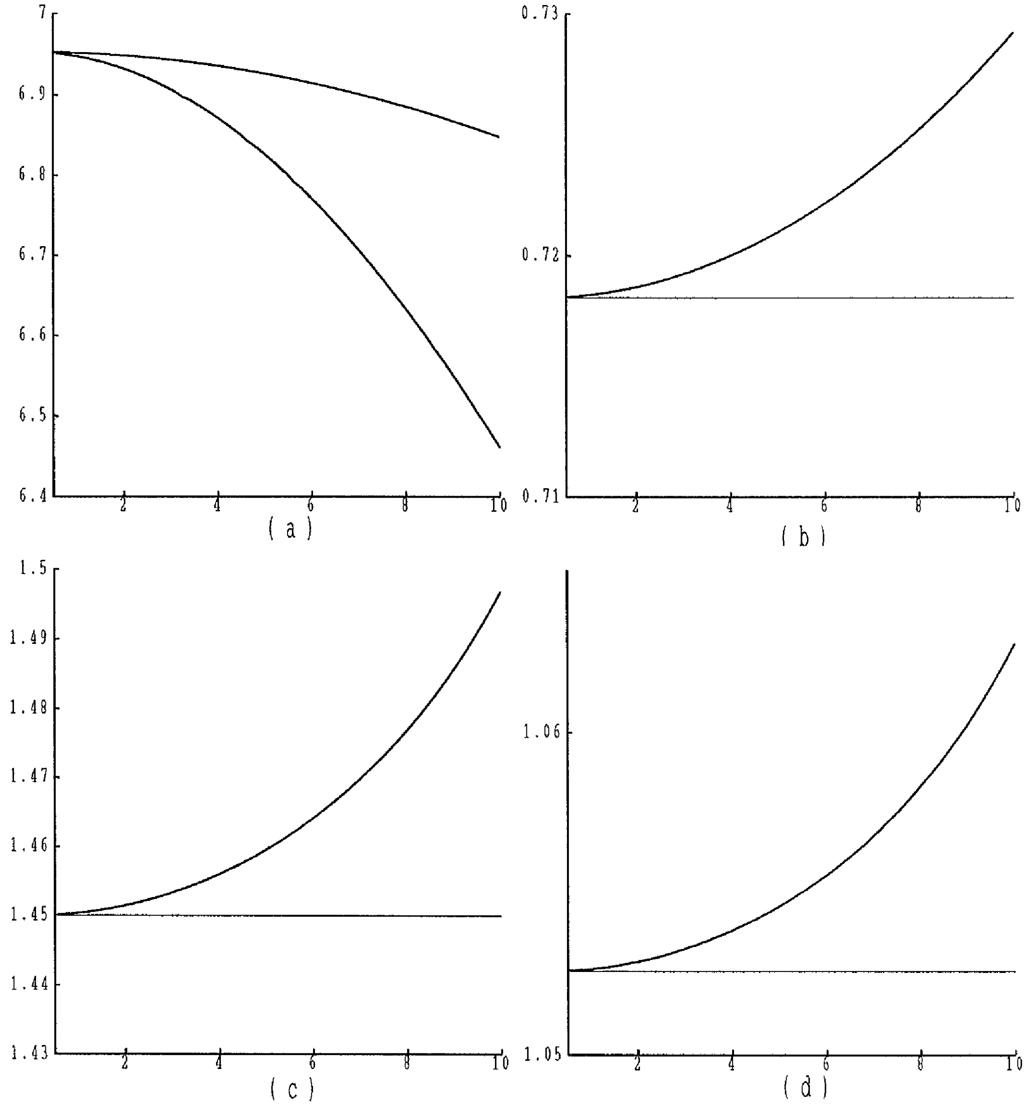

Figure 5.11: Plots of corrected and uncorrected values of (a) volume, (b) pressure, (c) λ_1 and (d) λ_2 against curvature k x 100, respectively, with an initial stretch of $\lambda_1 = 1.45$.

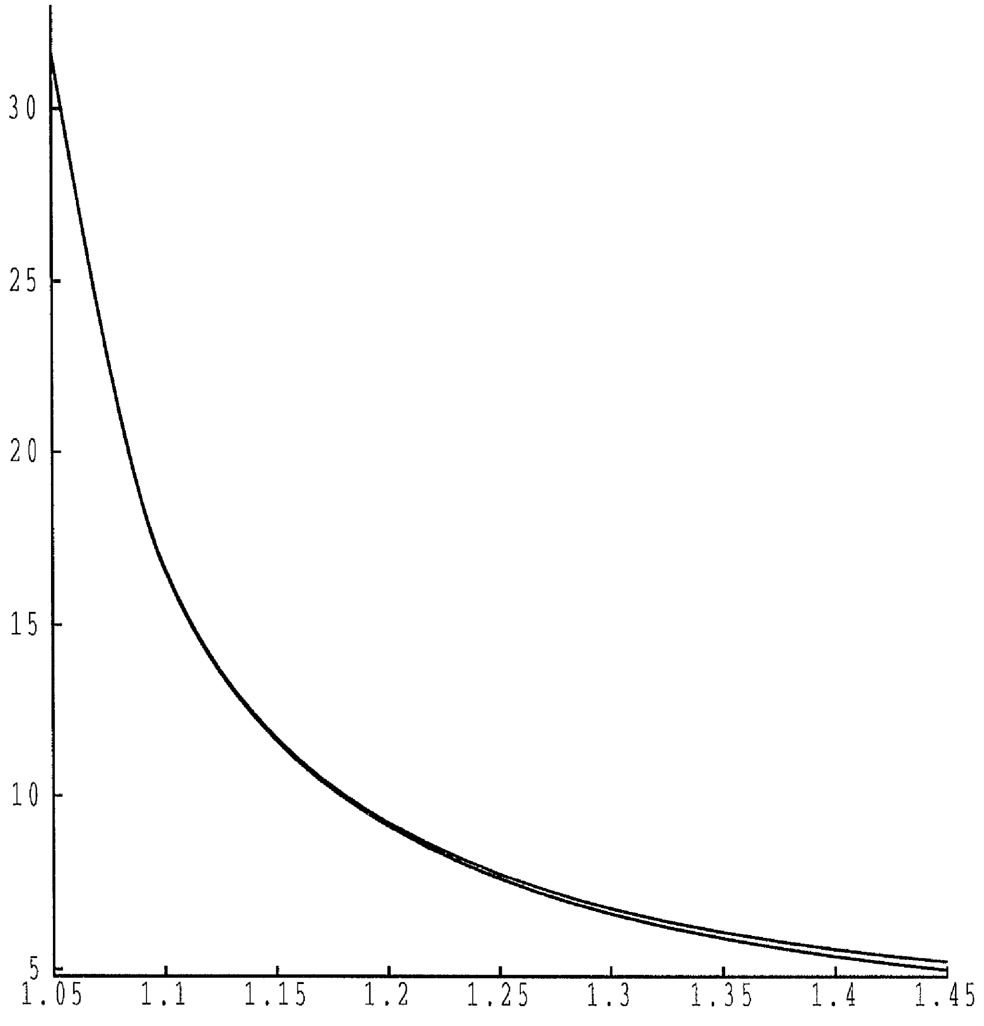

Figure 5.12: Plot of the critical radius of curvature ρ for the Varga material against the principal stretch λ_1. The upper curve assumes a constant pressure, while the lower curve assumes a perfect gas.

pressure value. In this case a bulge will appear before the onset of wrinkling.

5.4.4 Bifurcation Analysis

We now consider a 'small' deformation which is superimposed on the body while in its current configuration. We do not quantify the meaning of 'small' in this work but we assume the deformation is small enough to ignore higher order terms. The equations of equilibrium, in the absence of body forces, can be recast as

$$\text{Div } \mathbf{s} = \mathbf{0}, \tag{5.4.4}$$

where $\mathbf{s}$ is the nominal stress, and Div is the divergence operator in the reference configuration. Applying an increment to $\mathbf{x}$ such that $\mathbf{x} \to \mathbf{x} + \dot{\mathbf{x}}$ results in (5.4.4) becoming

$$\text{Div } \dot{\mathbf{s}} = \mathbf{0}, \tag{5.4.5}$$

where $\dot{\mathbf{s}}$ is the incremental nominal stress. If we now choose the inflated and bent configuration as the reference configuration we can rewrite (5.4.5) as

$$\text{div } \dot{\mathbf{s}}_0 = \mathbf{0}, \tag{5.4.6}$$

where div is the divergence operator in the current configuration and $\dot{\mathbf{s}}_0$ is now the incremental stress referred to the current configuration. Henceforth a superposed dot will represent an increment in the associated quantity and the zero subscript will denote evaluation in the current configuration. The deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{5.4.7}$$

becomes $\mathbf{F} + \dot{\mathbf{F}}$ as $\mathbf{x} \to \mathbf{x} + \dot{\mathbf{x}}$ where

$$\dot{\mathbf{F}} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{X}}. \tag{5.4.8}$$

If we evaluate (5.4.8) in the current configuration we have

$$\dot{\mathbf{F}}_0 = \frac{\partial \dot{\mathbf{x}}_0}{\partial \mathbf{x}} = \boldsymbol{\eta}. \tag{5.4.9}$$

Given the nominal stress, for an unconstrained material, defined by

$$\mathbf{s} = \frac{\partial W}{\partial \mathbf{F}^T}, \tag{5.4.10}$$

where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function of the material, we can write the incremental nominal stress as

$$\dot{\mathbf{s}} = \frac{\partial^2 W}{\partial \mathbf{F}^T \partial \mathbf{F}^T} \dot{\mathbf{F}}^T. \tag{5.4.11}$$

When (5.4.11) is evaluated in the current configuration we obtain

$$\dot{\mathbf{s}}_0 = \left(\frac{\partial^2 W}{\partial \mathbf{F}^T \partial \mathbf{F}^T} \right)_0 \dot{\mathbf{F}}_0^T = \mathbf{B} \boldsymbol{\eta}^T, \tag{5.4.12}$$

where $\mathbf{B}$ is the fourth order tensor of instantaneous moduli and $\boldsymbol{\eta} = \dot{\mathbf{F}}_0$. The incremental equations given above can be used for an unconstrained material. In this section, however, we only consider deformations of incompressible materials and hence we now give the associated equations for such a material. The constitutive law for an incompressible material yields

$$\mathbf{s} = \frac{\partial W}{\partial \mathbf{F}^T} - p\mathbf{F}^{-1}, \tag{5.4.13}$$

where p is a Lagrange multiplier. By taking the increment of (5.4.13) we obtain

$$\dot{\mathbf{s}} = \frac{\partial^2 W}{\partial \mathbf{F}^T \partial \mathbf{F}^T} \dot{\mathbf{F}}^T - \dot{p}\mathbf{F}^{-1} - p\dot{\mathbf{F}}^{-1}, \tag{5.4.14}$$

which in the current configuration can be written

$$\dot{\mathbf{s}}_0 = \mathbf{B} \boldsymbol{\eta}^T - \dot{p}\mathbf{I} + p\boldsymbol{\eta}, \tag{5.4.15}$$

where $\mathbf{I}$ is the identity and $\dot{\mathbf{F}}_0$ has been replaced by $\boldsymbol{\eta}$. In component form (5.4.15) becomes,

$$\dot{s}_{0ij} = B_{ijkl}\eta_{lk} - \dot{p}\delta_{ij} + p\eta_{ij}, \tag{5.4.16}$$

where the only non-zero components of $\mathbf{B}$, given in Haughton and Ogden (1978b), for an incompressible material are

$$\left.\begin{aligned} B_{iijj} &= B_{jjii} = \lambda_i\lambda_j \frac{\partial^2 W}{\partial\lambda_i \partial\lambda_j}, \\ B_{ijij} &= \lambda_i^2 \frac{\sigma_i - \sigma_j}{\lambda_i^2 - \lambda_j^2}, \quad \lambda_i \neq \lambda_j, \\ B_{ijij} - B_{ijji} &= B_{ijij} - B_{jiij} = \sigma_i, \quad i \neq j, \end{aligned}\right\} \tag{5.4.17}$$

where $\sigma_i = \lambda_i \partial W/\partial\lambda_i$. The incremental equations of equilibrium for a membrane of incompressible material when referred to the current configuration can be written as

$$\dot{s}_{0\mu i,\mu} + \dot{s}_{0\mu i}\boldsymbol{a}_\nu \cdot \boldsymbol{a}_{\mu,\nu} + \dot{s}_{0\mu\nu}\boldsymbol{a}_i \cdot \boldsymbol{a}_{\nu,\mu} + \dot{s}_{0\mu 3}\boldsymbol{a}_i \cdot \boldsymbol{a}_{3,\mu} + \frac{1}{h}(h_{,\mu}\dot{s}_{0\mu i} - p\eta_{3i}) = 0, \tag{5.4.18}$$

where $\dot{s}_{0ij}$ are given by (5.4.16) with (5.4.17) and $\mathbf{a}_i$ are unit base vectors (5.2.6), $i = 1,2,3$, $\mu,\nu = 1,2$. The derivation of these equations from (5.4.6) follows in a similar way to the derivation of the equations of equilibrium for a membrane (2.4.24) given in section 2.4. The reader is referred to Haughton and Ogden (1978a) for a full account. Let $\mathbf{x}$ be the position vector of a material point in the body referred to the current configuration. Then

$$\mathbf{x} = \mathbf{a}(\theta^1, \theta^2) + \lambda_3(\theta^1, \theta^2)\theta^3 \mathbf{a}_3(\theta^1, \theta^2), \tag{5.4.19}$$

where θ^i, $i = 1,2,3$, $\lambda_3\theta^3$ and position vectors $\mathbf{a}$ and $\mathbf{a}_3$ have been defined in section 2.4 with regard to the current configuration (2.4.6). Therefore the incremental deformation gradient in the current configuration is given by

$$\boldsymbol{\eta} = \frac{\partial \dot{\mathbf{x}}_0}{\partial \mathbf{x}} = \frac{\partial \dot{\mathbf{a}}}{\partial \mathbf{a}} + \dot{\lambda}_3 {\lambda_3}^{-1} \mathbf{a}_3 \otimes \mathbf{a}_3 + \dot{\mathbf{a}}_3 \otimes \mathbf{a}_3, \tag{5.4.20}$$

where we have used (5.4.9) with (5.4.19).

If we now let the incremental displacement $\dot{\mathbf{a}}$ be defined by

$$\dot{\mathbf{a}}(\Theta, Z) = u(\Theta, Z)\mathbf{a}_1 + v(\Theta, Z)\mathbf{a}_2 + w(\Theta, Z)\mathbf{a}_3, \tag{5.4.21}$$

where $\mathbf{a}_i$ are the unit base vectors given by (5.2.6) then the incremental deformation gradient (5.4.20) gives

$$\begin{aligned}\boldsymbol{\eta} &= \frac{1}{\lambda}\frac{\partial}{\partial\Theta}(u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3)\otimes\mathbf{a}_1 \\ &+ \frac{1}{\lambda_Z}\frac{\partial}{\partial Z}(u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3)\otimes\mathbf{a}_2 + \dot{\lambda}_3\lambda_3^{-1}\mathbf{a}_3\otimes\mathbf{a}_3 + \dot{\mathbf{a}}_3\otimes\mathbf{a}_3,\end{aligned} \tag{5.4.22}$$

where we have used (5.4.21). Following Haughton and Ogden (1978b) we have

$$\eta_{3\mu} = \mathbf{a}_3\cdot\dot{\mathbf{a}}_\mu = -\dot{\mathbf{a}}_3\cdot\mathbf{a}_\mu = -\eta_{\mu 3}, \tag{5.4.23}$$

and

$$\eta_{33} = \dot{\lambda}_3\lambda_3^{-1}. \tag{5.4.24}$$

As we confine attention to incompressible materials in this section, we note the incremental form of the incompressibility condition (2.2.9) is

$$tr(\boldsymbol{\eta}) = 0, \tag{5.4.25}$$

and hence

$$\eta_{33} = -\eta_{11} - \eta_{22}. \tag{5.4.26}$$

Substituting (5.4.23) and (5.4.24) with (5.4.26) into (5.4.22) and using (5.2.6) we have, in component form,

$$\boldsymbol{\eta} = \begin{bmatrix} \dfrac{u'}{\lambda} + wg & \dfrac{1}{\lambda_Z}\dfrac{\partial u}{\partial Z} + vx'f & -\eta_{31} \\ \dfrac{v'}{\lambda} & \dfrac{1}{\lambda_Z}\dfrac{\partial v}{\partial Z} - f(ux' + wy') & -\eta_{32} \\ \dfrac{w'}{\lambda} - ug & \dfrac{1}{\lambda_Z}\dfrac{\partial w}{\partial Z} + fvy' & -\eta_{11} - \eta_{22} \end{bmatrix}, \tag{5.4.27}$$

where $f = \dfrac{\alpha}{\rho\lambda\lambda_Z}$ and $g = \dfrac{x'y'' - x''y'}{\lambda^3}$. The prime again denotes partial differentiation with respect to Θ. In general $\dot{\mathbf{x}}$ will depend on Θ and Z but not on R. It is easily shown that the only non-zero components of $\mathbf{a}_i \cdot \mathbf{a}_{j,k}$ are

$$\left.\begin{aligned} \mathbf{a}_1 \cdot \mathbf{a}_{2,2} &= -\mathbf{a}_2 \cdot \mathbf{a}_{1,2} = fx', \\ \mathbf{a}_1 \cdot \mathbf{a}_{3,1} &= -\mathbf{a}_3 \cdot \mathbf{a}_{1,1} = g, \\ \mathbf{a}_3 \cdot \mathbf{a}_{2,2} &= -\mathbf{a}_2 \cdot \mathbf{a}_{3,2} = fy'. \end{aligned}\right\} \tag{5.4.28}$$

We now define the components of incremental nominal stress $\dot{s}_{0ij}$. First note that the incremental form of the membrane approximation gives

$$\dot{s}_{03i} = 0, \quad (i = 1, 2, 3), \tag{5.4.29}$$

and taking $i = 3$ together with (5.4.16) and (5.4.17) we obtain

$$\dot{p} = B_{1133}\eta_{11} + B_{2233}\eta_{22} + (B_{3333} + p)\eta_{33}. \tag{5.4.30}$$

If we now substitute (5.4.17), (5.4.26) and (5.4.30) into (5.4.16) we obtain the components of stress as

$$\left.\begin{aligned} \dot{s}_{011} &= (B_{1111} + B_{3333} + 2p)\eta_{11} + (B_{3333} + p)\eta_{22}, \\ \dot{s}_{022} &= (B_{3333} + p)\eta_{11} + (B_{2222} + B_{3333} + 2p)\eta_{22}, \\ \dot{s}_{012} &= B_{1212}\eta_{21} + (B_{1221} + p)\eta_{12}, \\ \dot{s}_{021} &= B_{2121}\eta_{12} + (B_{2112} + p)\eta_{21}, \\ \dot{s}_{0\mu 3} &= \sigma_{\mu\mu}\eta_{3\mu}. \end{aligned}\right\} \tag{5.4.31}$$

Substituting (5.4.27), (5.4.28) and (5.4.31) into (5.4.18) gives a system of equations of equilibrium corresponding to $i = 1, 2, 3$ respectively, as

$$\begin{aligned} &\frac{1}{h\lambda}\frac{\partial h}{\partial \Theta}\{(B_{1111} + B_{3333} + 2p)\eta_{11} + (B_{3333} + p)\eta_{22}\} \\ &+\frac{1}{\lambda}\Big\{(B'_{1111} + B'_{3333} + 2p')\eta_{11} + (B'_{3333} + p')\eta_{22} \\ &\qquad +(B_{1111} + B_{3333} + 2p)\eta'_{11} + (B_{3333} + p)\eta'_{22}\Big\} \\ &+\frac{1}{\lambda_Z}\left\{B_{2121}\frac{\partial \eta_{12}}{\partial Z} + (B_{2112} + p)\frac{\partial \eta_{21}}{\partial Z}\right\} + \left\{g(\sigma_1 - \sigma_3) - \frac{P}{h}\right\}\eta_{31} \\ &\qquad +x'f\{(B_{2222} + p)\eta_{22} - (B_{1111} + p)\eta_{11}\} = 0, \end{aligned} \tag{5.4.32}$$

$$\frac{1}{h\lambda}\frac{\partial h}{\partial \Theta}\left\{(B_{1212})\eta_{21}+(B_{1221}+p)\eta_{12}\right\}-fy'(\sigma_2-p-\frac{P}{h})\eta_{32}$$
$$+\frac{1}{\lambda}\left\{B'_{1212}\eta_{21}+(B'_{1221}+p')\eta_{12}+B_{1212}\eta'_{21}+(B_{1221}+p)\eta'_{12}\right\}$$
$$+\frac{1}{\lambda_Z}\left\{(B_{3333}+p)\frac{\partial \eta_{11}}{\partial Z}+(B_{2222}+B_{3333}+2p)\frac{\partial \eta_{22}}{\partial Z}\right\}$$
$$-x'f\left\{(B_{1221}+B_{2121}+p)\eta_{12}+(B_{1212}+B_{2112}+p)\eta_{21}\right\} \quad = \quad 0, \qquad (5.4.33)$$

$$\left\{\frac{1}{h\lambda}\frac{\partial h}{\partial \Theta}-x'f\right\}(\sigma_1-\sigma_3)\eta_{31}+\frac{1}{\lambda}\left\{\sigma'_{11}\eta_{31}+\sigma_{11}\eta'_{31}\right\}+\frac{P}{h}(\eta_{11}+\eta_{22})$$
$$+\frac{1}{\lambda_Z}\left\{\sigma_{22}\frac{\partial \eta_{32}}{\partial Z}\right\}-g\left\{(B_{1111}+B_{3333}+2p)\eta_{11}+(B_{3333}+p)\eta_{22}\right\}$$
$$+fy'\left\{(B_{3333}+p)\eta_{11}+(B_{2222}+B_{3333}+2p)\eta_{22}\right\} \quad = \quad 0. \qquad (5.4.34)$$

We note from the definition of $\mathbf{B}$, (5.4.17), and the nature of the deformation, that the fourth order tensors are functions of Θ only. Before trying to solve the system of equations of equilibrium above, we note some results previously obtained for the bifurcation of an inflated, elastic, cylindrical membrane with closed ends and identify some possible complications associated with solving this system. In particular, Haughton and Ogden (1979) showed that plotting λ against λ_z for such a cylinder of arbitrary length produced single bifurcation curves for given axial loads. Here λ was the stretch in the azimuthal direction and λ_z was the stretch in the axial direction. They also found that the equations of bifurcation obtained for an inflated cylinder could be integrated exactly. Applying fixed end conditions $\dot{\mathbf{x}} = \mathbf{0}$ on $z = 0, l$, Haughton and Ogden (1979) investigated the asymmetric, buckling modes of bifurcation and obtained an explicit bifurcation criterion. For the special case of an infinitely long, cylindrical membrane this criterion reduced to the equations of equilibrium for the inflation of a cylinder with closed ends. The authors concluded that an infinitely long tube must be neutrally stable when inflated only and that any imperfection would result in a bent deformed cylinder rather than a straight one when inflated. For cylinders of finite length the bifurcation curves in the (λ_z, λ) plane are displaced so that a tube with closed ends would need to be inflated followed by an axial compression to obtain such a buckling bifurcation mode.

It seems likely that such buckling modes will have a role to play in the present problem. We therefore investigate these asymmetric, buckling modes for the deformation considered in this chapter. That is, the inflation of an elastic, circular, cylindrical membrane with closed ends which is then subjected to a finite flexure. Unfortunately the equations of bifurcation obtained above are more complicated to solve than those found by Haughton and Ogden (1979). We therefore cannot obtain exact solutions. To attempt to solve the equations (5.4.32) – (5.4.34) we use what has become the standard approach to such problems and assume the equations have a separable solution. However, this method of solution has a disadvantage in that it is not possible to satisfy the fixed end conditions $\dot{\mathbf{x}} = \mathbf{0}$ for all three incremental displacements at once. At most only two of these displacements can be made zero for a given set of boundary conditions.

Assuming the solution to (5.4.32) – (5.4.34) for u, v and w is separable consider

$$\left.\begin{aligned} u &= u(\Theta)\sin(\gamma Z), \\ v &= v(\Theta)\cos(\gamma Z), \\ w &= w(\Theta)\sin(\gamma Z), \end{aligned}\right\} \tag{5.4.35}$$

where $\gamma = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots$. Substituting (5.4.35) into (5.4.32) – (5.4.34) and rearranging gives the following

$$\left.\begin{aligned} u_{\Theta\Theta} &= C_1 u + C_2 u_\Theta + C_3 v + C_4 v_\Theta + C_5 w + C_6 w_\Theta, \\ v_{\Theta\Theta} &= R_1 u + R_2 u_\Theta + R_3 v + R_4 v_\Theta + R_5 w, \\ w_{\Theta\Theta} &= D_1 u + D_2 u_\Theta + D_3 v + D_4 w + D_5 w_\Theta, \end{aligned}\right\} \tag{5.4.36}$$

where lengthy expressions have been replaced by the functions $C_1, \ldots C_6$, $R_1, \ldots R_5$, and $D_1, \ldots D_5$ which will all depend on Θ and can be calculated explicitly using (5.4.32) – (5.4.34). We note that interchanging the sine and cosine functions in

(5.4.35) would produce the same resulting equations (5.4.36). Hence the end conditions are

$$u(0) = u(L) = 0, \quad w(0) = w(L) = 0, \quad v(\Theta, 0) = -v(\Theta, L),$$

since we assume $n = 1$ and take the parameter γ to depend on L only.

We now apply the standard method of solution to such a system of equations known as the compound matrix method, details of which are not given here but can be found in Haughton and Orr (1995). This method avoids the need to calculate large determinants which have to be made zero and therefore eliminates the numerical problems typically encountered. This method has also been shown to give more accurate results than the results associated with the calculation of determinants. If we let

$$\mathbf{y} = (\mathrm{u}, \mathrm{u}', \mathrm{v}, \mathrm{v}', \mathrm{w}, \mathrm{w}')^{\mathrm{T}}, \tag{5.4.37}$$

then equations (5.4.36) can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}, \tag{5.4.38}$$

where the components of A are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ R_1 & R_2 & R_3 & R_4 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ D_1 & D_2 & D_3 & 0 & D_4 & D_5 \end{bmatrix}. \tag{5.4.39}$$

In order to solve the above system of equations (5.4.38) with (5.4.39) we must apply boundary conditions at both ends of the cylinder and at $\Theta = 0$, and $\Theta = 2\pi$. However, we assume that the buckling mode will leave the cylinder with symmetry about the $u = 0$ plane, in the same way that the bent cylinder is symmetric about this plane. We therefore require boundary conditions to be evaluated at $\Theta = 0$ and

$\Theta = \pi$. From (5.4.21) and (5.2.6) we see that the assumed symmetry will require $u(0) = u(\pi) = 0$. For completeness we have tried all combinations of v, v', w and w' being zero or non-zero at $\Theta = 0, \pi$. The boundary conditions complete the system and we can then search through the parameter space of the inflating pressure p, the length of tube L, the constitutive material W and the curvature κ to try to find bifurcation modes.

For example, taking

$$u(\Theta) = 0, \quad v'(\Theta) = 0, \quad w'(\Theta) = 0, \quad \Theta = 0, \pi, \tag{5.4.40}$$

would result in the boundary conditions becoming

$$B\mathbf{y} = \mathbf{0}, \quad \Theta = 0, \pi, \tag{5.4.41}$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.4.42}$$

We now have six equations with three boundary conditions which results in a system of $\binom{6}{3} = 20$ differential equations to be solved. Lindsay and Rooney (1992) have produced a program of standard fortran code to generate these equations given the matrix A. The program was tested as fully as possible and we are sure that there are no significant programming errors. However, it was found that bifurcation did not occur for any set of boundary conditions. Also we have considered the asymmetric case with continuity conditions applied at $\Theta = 0$ and $\Theta = 2\pi$ but again no bifurcation points were found.

The compound matrix method has been shown to be very reliable and discovery of such a bifurcation mode is guaranteed if one exists. We therefore believe that the non-fixed end conditions prevent the bifurcation modes from appearing and conclude that the equations of bifurcation (5.4.32) – (5.4.34) do not have a separable solution. Choosing the form of the solution to be non-separable will allow us to impose fixed end conditions on the cylinder, however, this would force the use of finite differences to model the equations of bifurcation. This would result in the bifurcation criterion being reduced to evaluating a large (typically at least 100 x 100) determinant which invariably will produce numerical problems. Certainly we anticipate such problems for this example. However, this is an area of work that could be considered in the future as we still believe that bifurcation modes do exist for such a deformation.

5.5 Solution and Results for the Compressible Varga Material

We now consider the inflation and bending of a right, circular, cylindrical, isotropic, elastic membrane which consists of the compressible Varga material. Using the notation of section 5.2 we inflate this cylinder with a pressure and then subject it to finite bending. As equations (5.2.1) – (5.2.9) were obtained before the form of the strain-energy function was specified it follows that they will still hold for the compressible material. Using the membrane assumption (2.4.25), with the definition for the principal Cauchy stresses for a compressible material (2.3.13), we have

$$\frac{\partial W}{\partial \lambda_3} = 0, \tag{5.5.1}$$

giving $\lambda_3 = \lambda_3(\lambda_1, \lambda_2)$ which allows us to write

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3(\lambda_1, \lambda_2)). \tag{5.5.2}$$

Using the analysis of section 2.4 we can show (2.4.29) holds, namely

$$\sigma_{11} = \frac{\lambda_1}{J}\hat{W}_1, \quad \sigma_{22} = \frac{\lambda_2}{J}\hat{W}_2.$$

Starting with the equations of equilibrium (5.2.9) and substituting in (5.2.8) and (2.4.29) we can rearrange to obtain

$$\left.\begin{aligned} \lambda_1 \hat{W}_1' + \alpha\kappa x' \hat{W}_2 &= 0, \\ \frac{\lambda_1 \hat{W}_1 (y'x'' - x'y'')}{(x'^2 + y'^2)^{\frac{3}{2}}} + \frac{\alpha\kappa y' \hat{W}_2}{(x'^2 + y'^2)^{\frac{1}{2}}} + \frac{P\lambda_1\lambda_2}{H} &= 0, \end{aligned}\right\} \tag{5.5.3}$$

which are identical to the governing equations derived in section 5.2 for an incompressible material given by (5.2.10). Again $(5.5.3)_1$ can be integrated with respect to Θ directly to yield (5.2.11) and similar analysis can be used to reduce $(5.5.3)_2$ to (5.2.15). Finally we can non-dimensionalise our variables as defined previously by

(5.2.16) – (5.2.18) to obtain the equations of equilibrium $(5.2.19)_1$ and $(5.2.19)_2$. Before solving these equations for a and b we want to obtain expressions for λ_1 and λ_2 prior to any flexural deformations so that we can determine a value for α directly from $(5.2.8)_2$ with curvature $\kappa = 0$. If we consider an inflated, cylindrical membrane with reference configuration (5.2.1), current configuration (5.2.20) and principal stretches (5.2.22) it is clear that we can again obtain equations (5.2.24) and (5.2.26). Specifying the strain-energy function, gives two equations for λ_1, λ_2 and p. We assume that one of the three variables is known and hence can calculate the remaining two. We define the non-dimensionalised strain-energy function for the compressible Varga material, using (4.3.1) with (5.2.18), as

$$w(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3 - g(J). \qquad (5.5.4)$$

Here $g(J)$ is an arbitrary function except that it must satisfy the constraints given by (4.3.2) to ensure that the ground state energy and stress are zero and that the bulk modulus is positive respectively. Hence we again choose the simplest form of $g(J)$ consistent with these constraints, namely, $g(J) = (J^\beta - 1)/\beta + 3$ which satisfies (4.3.2) provided $\beta < 1/3$, $\beta \neq 0$. From (5.5.1) and (5.5.4) with (4.3.12) we can show

$$\lambda_3 = (\lambda_1\lambda_2)^{\frac{\beta}{1-\beta}}, \qquad (5.5.5)$$

and hence using (5.5.5)

$$J^\beta = (\lambda_1\lambda_2)^{\frac{\beta}{1-\beta}}. \qquad (5.5.6)$$

The non-dimensionalised strain-energy function (5.5.4) then becomes

$$\hat{w}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 + (1 - \frac{1}{\beta})(\lambda_1\lambda_2)^{\frac{\beta}{1-\beta}} + \frac{1}{\beta} - 3, \quad \beta \neq 0. \qquad (5.5.7)$$

For a cylinder subjected to inflation only, we substitute (5.5.7) into (5.2.24) and (5.2.26) to obtain

$$\left.\begin{aligned} \lambda_1(2 + p\lambda_1^2)^{\frac{\beta}{1-\beta}} - (2\lambda_1^2)^{\frac{\beta}{1-\beta}} - 2p\lambda_1^3(2 + p\lambda_1^2)^{\frac{2\beta-1}{1-\beta}} &= 0, \\ \lambda_2 - \frac{2\lambda_1}{2 + p\lambda_1^2} &= 0. \end{aligned}\right\} \qquad (5.5.8)$$

Clearly we have to specify β and one of the variables p, λ_1 or λ_2 before solving (5.5.8) numerically. Previously, for the incompressible material, we assigned $p = 0.3$ and solved (5.2.30) to obtain values for λ_1 and λ_2. However as the material becomes more compressible ($\beta \to 1/3$) it is noted that no positive real roots can be found for λ_1 from $(5.5.8)_1$ when $p = 0.3$. To illustrate this effect, Figure 5.13 plots the non-dimensionalised inflating pressure p against the circumferential stretch λ_1 for various values of the parameter β. We observe how the peak pressure decreases as β increases with real solutions only being obtained for a pressure value of $p = 0.3$ for the curves corresponding to the incompressible approximation ($\beta \to -\infty$) and $\beta = -5$. Instead we set $\lambda_1 = 1.26464$ which corresponds to $p = 0.3$ for the incompressible material and solve $(5.5.8)_1$ for p. Figure 5.13 shows that for this value of λ_1 and a given β we can find a unique value of p which occurs before the pressure maximum is reached ensuring stable values of the variables are used. We can then find the associated value of λ_2, and hence α, by solving $(5.5.8)_2$ with these values.

We now return to our original deformation of inflation and flexure of a cylindrical membrane. We will assume, from previous results obtained, that wrinkling will occur for some curvature. We therefore need to find a relaxed strain-energy function for the compressible Varga material using the theory of chapter 3. Recalling the notion of a membrane in simple tension, in which the natural width of the membrane, $n(\lambda_1)$, is defined to be the solution of (3.3.1), we have $n(\lambda_1) = \lambda_2 = \lambda_3$. We can then define the non-dimensionalised, relaxed strain-energy function for the compressible Varga material by

$$\bar{w} = \lambda_1 + (2 - \frac{1}{\beta})\lambda_1^{\frac{\beta}{1-2\beta}} + \frac{1}{\beta} - 3, \quad \beta \neq 0, \tag{5.5.9}$$

where we have used (4.3.14) with (5.5.7). We now have two forms of the strain-energy function for the compressible Varga material, (5.5.7) and (5.5.9), the first being used in the tense region and the second in the wrinkled region.

We are now in a position to solve (5.2.19) for a and b and hence determine the

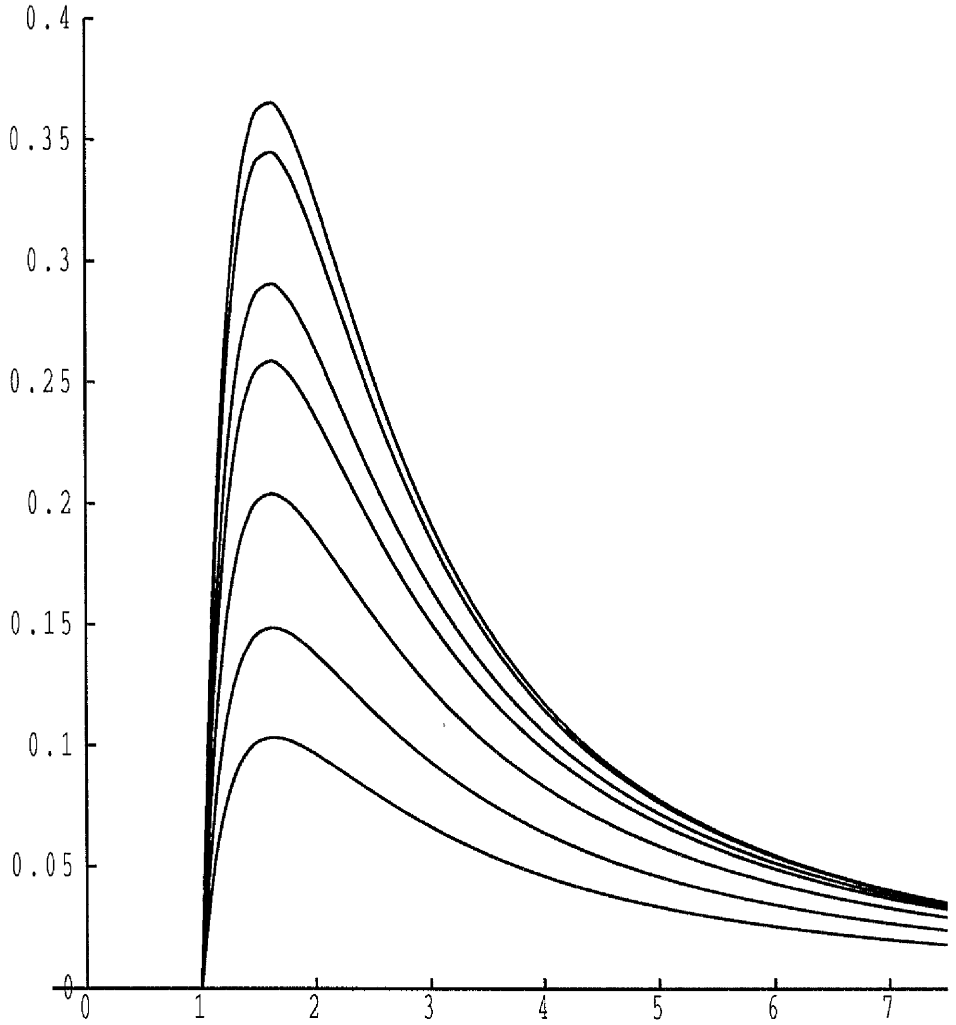

Figure 5.13: Plot of the non-dimensionalised pressure p against $\lambda_1 = r/R$ for the incompressible curve ($\beta \to -\infty$) and for β = -5, -1, -0.5, -0.1, 0.1 and 0.2. The uppermost curve corresponds to the incompressible case and the lowest to $\beta = 0.2$.

deformed cross-sectional shape. Substituting (5.5.7) and (5.5.9) into $(5.2.19)_2$ we obtain expressions for λ_1, in the tense and wrinkled regions respectively, as

$$\left.\begin{aligned} \lambda_1 &= \Lambda_t = \frac{1}{\lambda_2}\left(\frac{\beta}{2\beta-1}(b+3-\frac{1}{\beta}-\lambda_2)\right)^{\frac{1-\beta}{\beta}}, \\ \lambda_1 &= \Lambda_w = \left(1+\frac{b\beta}{3\beta-1}\right)^{\frac{1-2\beta}{\beta}}. \end{aligned}\right\} \tag{5.5.10}$$

As with the incompressible Varga material, Λ_t is a function of u only and Λ_w is constant. We now define the boundary, $u = u^*$, where the tense and wrinkled regions meet which occurs when $\sigma_{22}(u^*) = 0$. To solve this we can choose either definition of λ_1 from (5.5.10) because of continuity across the boundary. As in section 5.3 we choose Λ_w as it is constant. Hence substituting $(5.2.17)_2$, $(5.5.10)_2$ and (3.3.2) into (3.3.4) yields

$$u^*(b) = \frac{1}{2k}(1-\alpha^{-1}\Lambda_w^{\frac{\beta}{1-2\beta}}). \tag{5.5.11}$$

We again assume the deformation is symmetric about the u-axis and so we only need consider the upper half plane $v \geq 0$. Equations (5.3.4) – (5.3.7) will still hold and so we now try to determine u_0 and u_π by solving (5.3.7). Substituting (5.5.7), $(5.5.10)_1$ and (5.3.4) into (5.3.7) gives

$$a + p\alpha u(1-ku) = \pm\left(1-\lambda_2\left(\frac{\beta}{2\beta-1}(b+3-\frac{1}{\beta}-\lambda_2)\right)^{\frac{2\beta-1}{\beta}}\right), \tag{5.5.12}$$

for the tense region and similarly substituting (5.5.9), $(5.5.10)_2$ and (5.3.4) into (5.3.7) yields

$$a + p\alpha u(1-ku) = \pm\left(1-\left(1+\frac{b\beta}{3\beta-1}\right)^{\frac{3\beta-1}{\beta}}\right), \tag{5.5.13}$$

for the wrinkled region. We observe that if the membrane is tense (5.5.12) provides at least four choices for u_0 and u_π depending on the value of β and if the membrane is partly tense and partly wrinkled there are at least four choices for u_π

from (5.5.12) and four choices for $\bar{u}_0$ from (5.5.13) again depending on β. Several conditions that must be satisfied exist for u_0 ($\bar{u}_0$) and u_π which help us to choose the correct values for these parameters. As before $u_0 > u_\pi$ and both must be real. Clearly condition (5.3.10) must still hold also. The analogous result to (5.3.11), namely $\hat{w}_2 > 0$, can be shown to be

$$\alpha(1 - 2ku_\pi) > \Lambda_t^{\frac{\beta}{1-2\beta}}, \tag{5.5.14}$$

having used (5.5.7) and $(5.2.17)_2$. We also require (5.3.12) to be satisfied. For the tense region using (5.5.7) and $(5.2.17)_2$ with (5.3.12) yields

$$\Lambda_t > (\alpha(1 - 2ku))^{\frac{\beta}{1-2\beta}}, \tag{5.5.15}$$

where u is replaced appropriately by u_0 or u_π. There is no equivalent condition for the wrinkled region, namely $\bar{w}_1 > 0$, concerning u_0 or u_π as the relaxed strain-energy is a function of λ_1 only and $\lambda_1 = \Lambda_w$ is constant in this case from $(5.5.10)_2$. However the above conditions are sufficient to determine a unique u_0 and u_π.

We now proceed in solving (5.3.13) and (5.3.15) for a and b as described in section 5.3 and we will also use (5.3.16) to determine the deformed cross-section of the cylinder. The results for the incompressible Varga material are used to choose suitable initial intervals for a and b for a specified curvature. Only the integrated method of solution is considered here. In Figure 5.14(a)-(d) we plot the deformed cross-section of the membrane with initial circumferential stretch $\lambda_1 = 1.26464$, for curvature values of $k = 0.05, 0.1, 0.15$ and 0.2 respectively, with the parameter β taking values $\beta = \pm 0.2$. The starred symbols again indicate the point at which wrinkling begins. These two values of β are chosen since it has previously been found for other problems in non-linear elasticity, that for negative values of β the solutions retain the features displayed by the incompressible material ($\beta \to -\infty$) but as β passes through zero there can be a significant difference in solutions. In fact Figure 5.14(a) shows, for small curvature $k = 0.05$, that wrinkling only occurs for $\beta = 0.2$ and the membrane remains tense if $\beta = -0.2$. This indicates that more highly compressible materials are more susceptible to wrinkling. The values of β and k at which wrinkling first occurs will be discussed in more detail later. Figures 5.14(b), (c) and (d), however, show that for curvatures of $k \geq 0.1$ both values of β will cause wrinkling to occur. Comparing Figures 5.14 and 5.2 we observe that the compressible and incompressible materials exhibit the same qualitative behaviour. We note that as the curvature increases u_π moves significantly to the right with u_0 ($\bar{u}_0$) also moving slightly to the right, whereas u^* moves to the left indicating an increase in the wrinkled region. As β increases, each plot shown in Figure 5.14 displays similar effects to the above, namely, u_0 ($\bar{u}_0$) and u_π moving to the right.

Figure 5.15 plots the size of the wrinkled region defined by $\bar{u}_0 - u^*$ against β for an initial stretch of $\lambda_1 = 1.26464$ and curvature values of $k = 0.05$, 0.06, 0.065, 0.075, 0.10, 0.15 and 0.20 with β ranging from -5 to 0.2. The uppermost curve corresponds to $k = 0.2$ and the lowest curve to $k = 0.05$. It is indicated clearly that as the curvature increases the size of the wrinkled region increases. We observe that as $\beta \to -5$ each curve for which $k \geq 0.065$ has reached a limit approximating the incompressible material. This is in agreement with results shown in Figure 5.12 since $k = 0.065$ corresponds to a radius of curvature $\rho = 7.6923$ at which wrinkling first occurs when $\lambda_1 = 1.26464$. For $k = 0.05$ and 0.06 we see that the membrane remains tense ($\bar{u}_0 - u^* = 0$) until β is approximately -1.1 and -0.01 respectively. This shows that for small values of the curvature we require a more highly compressible material before we can initiate wrinkling. We observe that as β increases, causing the material to become more compressible, the wrinkled region again increases. We note in each case the sudden 'jump' in the size of the wrinkled region as β approaches and passes through zero, the significance of which was mentioned earlier, justifying our choice of values for β taken in Figure 5.14. We note, however, a smaller increase in the wrinkled region when k is incremented from 0.15 to 0.2 for all values of β. This suggests that such a large percentage of the membrane is already wrinkled it is becoming more difficult to induce further wrinkling.

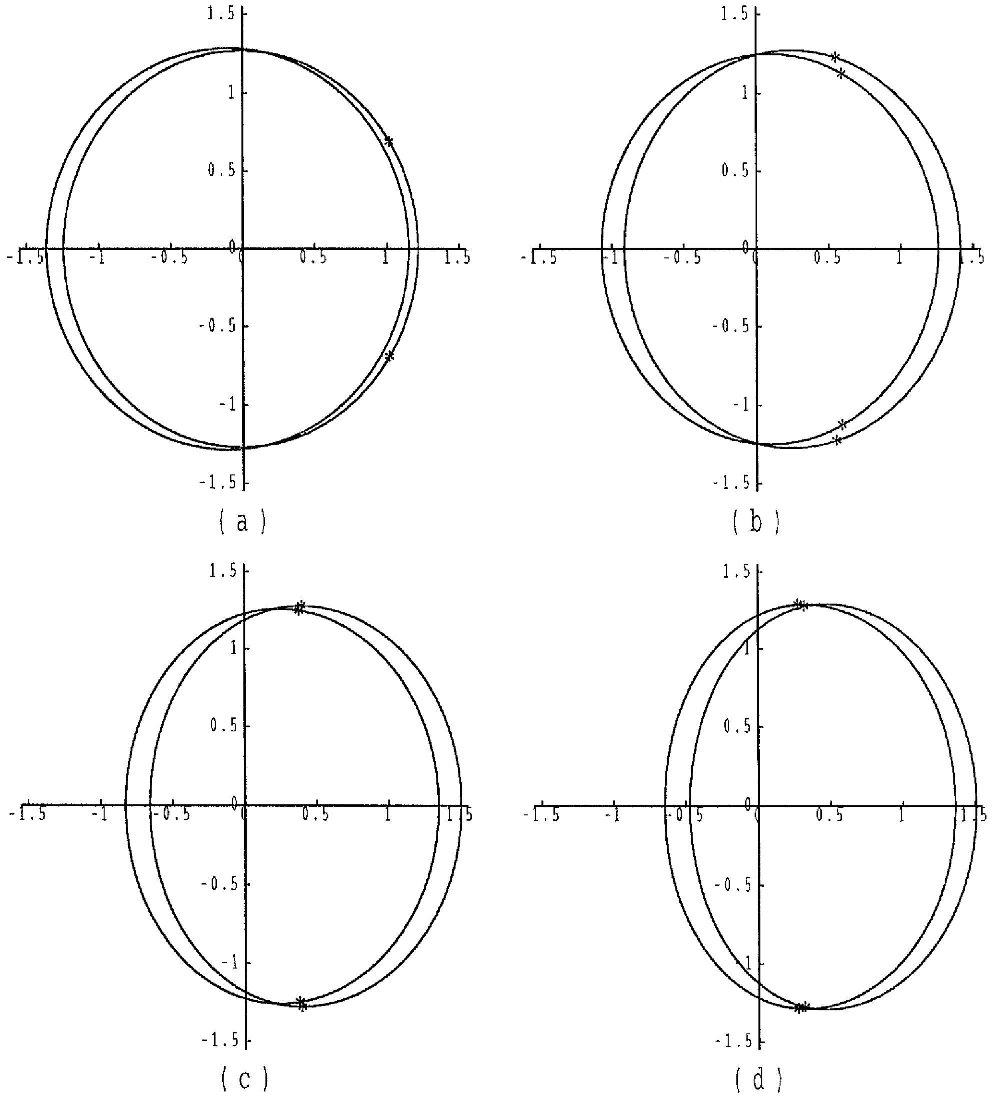

Figure 5.14: Plot of the deformed cross-section with curvature k taking values (a) 0.05, (b) 0.1, (c) 0.15 and (d) 0.2 with $\beta = \pm 0.2$.

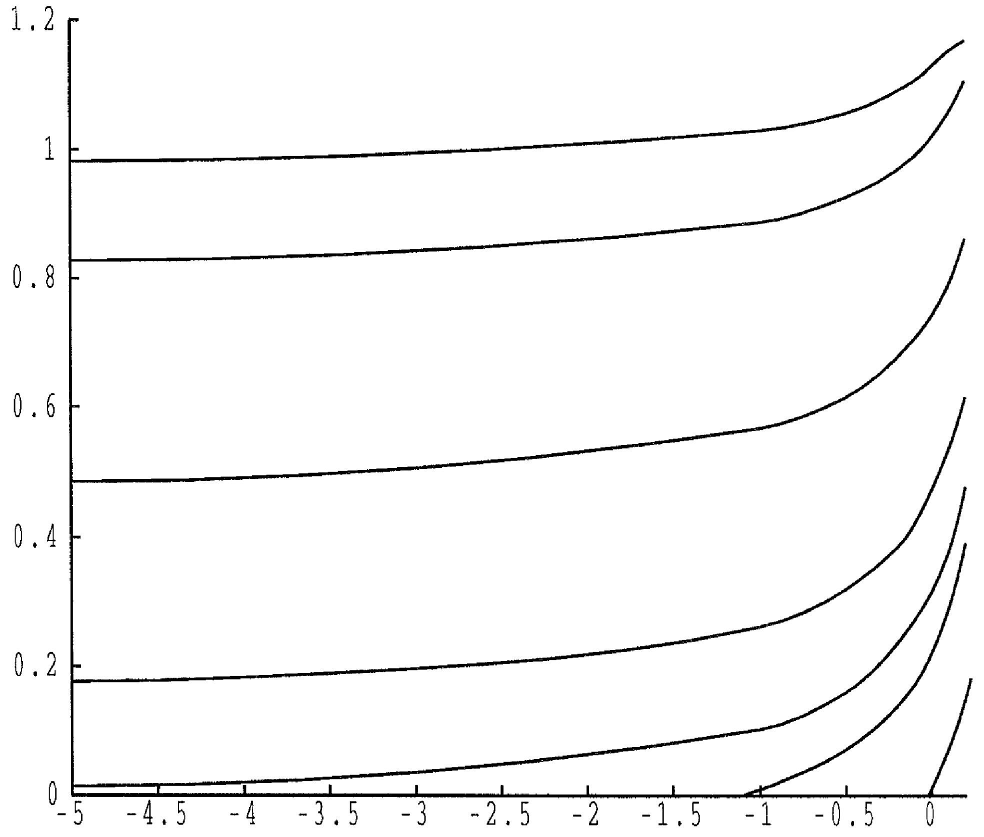

Figure 5.15: Plot of the size of the wrinkled region $\bar{u}_0 - u^*$ against β for $k = 0.05$, 0.06, 0.065, 0.075, 0.1, 0.15, 0.2. The uppermost curve corresponds to $k = 0.2$ and the lowest to $k = 0.05$.

Chapter 6

Wrinkling of Joined Elastic Cylindrical Membranes

6.1 Introduction

In this chapter we consider two isotropic, incompressible, elastic, right, circular, cylindrical membranes which are joined together, inflated with an internal pressure and subjected to axial loading on either end. These tubes are essentially finite in length. However, we assume that they are sufficiently long so as the ends are a suitable distance from the join and we therefore attain the same deformation as would a single, infinite tube subjected to the same axial loads. We find that this will be the case provided the ratio of the undeformed length to undeformed radius $(L/R) > 2.5$ approximately for the loadings considered here. The cylinders are open at the ends and the varying internal cross-sectional area throughout the overall cylinder will result in an imbalance of the components of pressure acting axially in the deformed configuration. We therefore apply axial loads on the ends to keep the system in equilibrium. This ensures the plane containing the join of the membranes remains fixed in space.

Recently Hart and Shi (1991) considered this problem for the isotropic, incompressible, Mooney-Rivlin and neo-Hookean materials and for the orthotropic, incompressible, Vaishnav and How-Clarke materials, Hart and Shi (1993). Although the two cylinders were assumed to have different material properties they possessed the same geometric properties. In other words both tubes had the same undeformed thickness and radius. Hart and Shi (1991) regarded their work as a first approximation model for arterial grafts. We therefore extend this analysis to consider the joining of two cylinders of different geometric properties. In particular, we investigate cylinders of different radii. We believe that this is the most important factor for arterial grafts since the differing radii will produce a wrinkled region adjacent to the join in the cylinder of larger radius. This wrinkled region could have significant consequences, as blood flows through such a graft, in the formation of blood clots which is clearly an important post-operative complication to be avoided. Other factors, however, will affect the resulting deformation and hence wrinkled region. We therefore consider cylinders of different materials, shear moduli and undeformed thicknesses and apply various inflating pressures and axial loads to discover how these parameters affect the wrinkled region of the composite cylinder.

The purpose of this chapter is to investigate the development of a wrinkled region on the surface of the cylinder. Although the other variables will affect this phenomenon it is essentially the differences in initial radii that will contribute to this effect. As the composite cylinder is inflated we expect wrinkling to be confined to the region adjacent to the join. This may disappear completely if the inflating pressure is sufficiently large although not so large as to create any possible instabilities in the form of bulging modes which we do not consider. Section 6.2 derives the governing equations using an alternative, but equivalent, approach to that given by Hart and Shi (1991). As this work extends the problem to consider wrinkling, the equations of equilibrium for this region are also found. In section 6.3 we describe the solution procedure and present graphical results concentrating on the effects of the different undeformed radii for the two component cylinders.

6.2 Problem Formulation

We consider the joining of two isotropic, elastic, incompressible, right, circular, cylindrical membranes of undeformed radius R_T and R_B where the subscripts T and B indicate the top and bottom cylinders respectively. We assume, without loss of generality, that the top cylinder is the one of larger radius. The composite cylinder is then inflated by a hydrostatic pressure P and subjected to axial loading on each end. We assign the undeformed lengths of the individual cylinders to be L_T and L_B which may be finite or (semi) infinite. As stated previously, we assume that the lengths of the finite cylinders, L_T and L_B, are sufficiently long so that at the ends, away from the join, the cylinders behave as if they are not joined but act as an individual cylinder. That is, the deformation of the inflated, composite cylinder away from the join attains the deformation that would occur for two independent cylinders subjected to the same loading.

Although many of the equations described in the following were previously obtained by Hart and Shi (1991), the derivation given here is somewhat different, but equivalent, to their work. Hart and Shi (1991) studied the deformation with particular reference to the membrane generated by rotating a curve about the axis associated with the axial direction. They investigated the angle ω which was made by the tangent to this curve and the axial axis. The authors also made use of the principal curvatures to the surface of the membrane defining these curvatures in terms of ω and then making use of these definitions to obtain the equations of equilibrium. Instead we shall simply apply the equations given in section 2.4 and 3.3 to obtain the equations of equilibrium for the problem.

In the reference configuration a point on the undeformed middle surface of the combined membrane is given by the position vector

$$\mathbf{X}(Z,\Theta) = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z, \quad 0 \leq \Theta \leq 2\pi, \quad -L_B \leq Z \leq L_T, \qquad (6.2.1)$$

where Z is the axial length along the fixed unit vector $\mathbf{E}_Z$, $\mathbf{E}_R(\Theta)$ is a unit outward normal to the cylinder's surface and R is the radius of the middle surface of the bottom cylinder. Inflating the cylinders with a pressure P and applying axial loads to either end causes a point on the deformed middle surface of the membrane to be given by the position vector

$$\mathbf{x}(Z,\Theta) = r\mathbf{e}_r(\theta) + z\mathbf{e}_z, \quad 0 \leq \theta \leq 2\pi, \quad -l_B < z < l_T, \tag{6.2.2}$$

where $r = r(Z)$, $\theta = \Theta$ and $z = z(Z)$. The deformed lengths of the individual cylinders l_B and l_T are unknown but can be obtained, if required, from the final solution. We define the unit base vectors $\mathbf{a}_i$ after deformation to be

$$\mathbf{a}_1 = \frac{r'\mathbf{e}_r + z'\mathbf{e}_z}{(r'^2 + z'^2)^{\frac{1}{2}}}, \quad \mathbf{a}_2 = \mathbf{e}_\theta, \quad \mathbf{a}_3 = \frac{z'\mathbf{e}_r - r'\mathbf{e}_z}{(r'^2 + z'^2)^{\frac{1}{2}}}, \tag{6.2.3}$$

where the prime denotes differentiation with respect to Z and we have arranged $\mathbf{a}_3$ to be a unit outward normal to the middle surface of the membrane. The unit base vectors $\mathbf{A}_i$ before deformation are taken to be

$$\mathbf{A}_1 = \mathbf{E}_Z, \quad \mathbf{A}_2 = \mathbf{E}_\Theta, \quad \mathbf{A}_3 = \mathbf{E}_R. \tag{6.2.4}$$

The deformation gradient $\mathbf{F}$ given by (2.2.2) can therefore be calculated with respect to the base vectors defined in (6.2.3) and (6.2.4) as

$$\mathbf{F} = \begin{bmatrix} (r'^2 + z'^2)^{\frac{1}{2}} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \tag{6.2.5}$$

so that the principal stretches are defined by

$$\lambda_1 = (r'^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_2 = \frac{r}{R}, \quad \lambda_3 = \frac{h}{H}, \tag{6.2.6}$$

where H, h are the thicknesses of the membranes before and after deformation respectively. We assume H to be constant for simplicity, however, the model obtained could still be solved for any $H(Z)$ provided the equations of equilibrium were integrated numerically. We note that although H is (piecewise) constant it may be different in each individual cylinder. From the general equations of equilibrium for a membrane given earlier by (2.4.24) we obtain, in the absence of body forces,

$$\left.\begin{aligned}\frac{1}{h}\frac{d(h\sigma_{11})}{dZ}+\frac{r'}{r}(\sigma_{11}-\sigma_{22}) &= 0,\\ \frac{\sigma_{11}(r''z'-r'z'')}{(r'^2+z'^2)^{\frac{3}{2}}}-\frac{z'\sigma_{22}}{r(r'^2+z'^2)^{\frac{1}{2}}}+\frac{P}{h} &= 0,\end{aligned}\right\} \qquad (6.2.7)$$

the equation in the azimuthal direction being trivially satisfied. The principal Cauchy stresses σ_{ii} are defined, for an incompressible material, by (2.3.15). We assume that the material is hyperelastic with strain-energy $W = W(\lambda_1, \lambda_2, \lambda_3)$. As before, using the incompressibility condition (2.2.9) it is easy to show that (2.4.31) again holds where $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1})$. Using (2.4.31) and the incompressibility condition (2.2.9) with $(6.2.6)_3$ gives the equations of equilibrium (6.2.7) as

$$\left.\begin{aligned}\lambda_1\hat{W}_1'-\frac{r'\hat{W}_2}{R} &= 0,\\ \frac{(r''z'-r'z'')\hat{W}_1}{\lambda_1^2}-\frac{z'\lambda_2\hat{W}_2}{r\lambda_1}+\frac{P\lambda_1\lambda_2}{H} &= 0.\end{aligned}\right\} \qquad (6.2.8)$$

Following Pipkin (1968) and using $(6.2.6)_2$ we can integrate $(6.2.8)_1$ directly with respect to Z to obtain

$$\hat{W}-\lambda_1\hat{W}_1 = C_2, \qquad (6.2.9)$$

where C_2 is a constant of integration. We can also integrate the second equation of equilibrium in the system (6.2.8) as follows. Using $(6.2.8)_1$ to substitute for $\hat{W}_2$ in $(6.2.8)_2$ yields

$$\frac{(r''z'-r'z'')\hat{W}_1}{\lambda_1^2}-\frac{z'\hat{W}_1'}{r'}+\frac{P\lambda_1\lambda_2}{H}=0. \qquad (6.2.10)$$

Using (6.2.6) we also note that

$$\left(\frac{r'\hat{W}_1}{\lambda_1}\right)'=\frac{r'\hat{W}_1'}{\lambda_1}+\frac{(r''z'-r'z'')z'\hat{W}_1}{\lambda_1^3},$$

which, after manipulation allows us to substitute for the first term in (6.2.10) giving

$$\frac{r'}{z'}\left(\frac{r'\hat{W}_1}{\lambda_1}\right)' - \frac{\lambda_1 \hat{W}_1'}{z'} + \frac{P\lambda_2 r'}{H} = 0. \qquad (6.2.11)$$

We can then integrate (6.2.11) directly with respect to Z to find

$$\frac{\hat{W}_1 z'}{\lambda_1} - \frac{P\lambda_2^2 R}{2H} = C_1, \qquad (6.2.12)$$

where C_1 is a constant of integration.

We now consider components of pressure P, stress σ_{ii} and axial force F for the top part of the composite cylinder which has undeformed length L_T. Using $(6.2.6)_2$ we can define the deformed radius at the end of the cylinder by $r(L_T) = R_T\lambda_2(L_T)$ and the deformed radius at the join as $r(0^+) = R_T\lambda_2(0^+)$. Therefore the pressure P exerts an axial force downwards of magnitude

$$\pi P(r^2(L_T) - r^2(0^+)) = \pi P R_T^2(\lambda_2^2(L_T) - \lambda_2^2(0^+)).$$

Considering the axial component of the resultant stress at the origin we note that the tangent to the middle surface of the membrane in the $r - z$ plane will make an angle $\tan^{-1}(z'/r')$ with the r-axis. Therefore the axial component of the resultant stress will have magnitude

$$\sigma_{11}\pi\left((r+\frac{h}{2})^2 - (r-\frac{h}{2})^2\right)\frac{z'}{\lambda_1},$$

where $(r-h/2)$ and $(r+h/2)$ represent the inner and outer radii of the top cylinder after deformation. We assume that the join of the two cylinders remains in the $z = 0$ plane and hence the axial component of pressure, together with the applied axial force F_T, is balanced by the axial component of resultant stress at the join giving

$$\sigma_{11}\pi\left((r+\frac{h}{2})^2 - (r-\frac{h}{2})^2\right)\frac{z'}{\lambda_1} = F_T - P\pi R_T^2(\lambda_2^2(L_T) - \lambda_2^2(0)).$$

Using (6.2.6), (2.2.9) and (2.4.31) this can be rewritten as

$$\left.\begin{aligned} 2\pi R_T H_T \left(\frac{\hat{W}_1 z'}{\lambda_1}\right)(0^+) &= F_T - \pi P R_T^2 (\lambda_2^2(L_T) - \lambda_2^2(0^+)), \\ 2\pi R_B H_B \left(\frac{\hat{W}_1 z'}{\lambda_1}\right)(0^-) &= F_B - \pi P R_B^2 (\lambda_2^2(-L_B) - \lambda_2^2(0^-)), \end{aligned}\right\} \quad (6.2.13)$$

where we have merely stated the equivalent result for the bottom cylinder which can be obtained using similar analysis. If we assume the problem is of a static nature the components of force and pressure must balance for the composite cylinder. Hence considering the components on the ends we have

$$F_B = F_T - \pi P (R_T^2 \lambda_2^2(L_T) - R_B^2 \lambda_2^2(-L_B)), \quad (6.2.14)$$

which defines the required load F_B when the pressure P and top load F_T are given such that the join remains in the plane $z = 0$. In order to use equations (6.2.13) and (6.2.14) we require values of the principal stretch λ_2 at the ends and at the join of the composite cylinder. As stated above, we assume that the individual component cylinders are long enough so that the inflated radius of the overall cylinder at the ends is the same as that which would be obtained by the component cylinders subjected to the same pressure and axial load. This assumption simplifies the calculation of the principal stretches at the open ends of the cylinder as shown below.

If we consider a single component cylinder subjected to a pressure P and axial load F the cylinder deforms into another right, circular cylinder so $r'(Z) = 0$ and the principal stretches (6.2.6) become

$$\lambda_1 = z', \quad \lambda_2 = \frac{r}{R}, \quad (6.2.15)$$

where λ_1 and λ_2 will be constant. The equations of equilibrium (6.2.7) with (6.2.15) become

$$H\hat{W}_2 = RP\lambda_1\lambda_2, \quad (6.2.16)$$

with $(6.2.7)_1$ being trivially satisfied. As the radius remains constant the pressure components at the ends and at the joining section will be equal hence $(6.2.13)_1$ with (6.2.15) gives

$$2\pi RH\hat{W}_1 = F, \tag{6.2.17}$$

for the upper cylinder where $\hat{W}_1 = \hat{W}_1(\lambda_1(L_T), \lambda_2(L_T))$. Given initial values for P, R, H and F and also specifying the strain-energy function we can solve (6.2.16) and (6.2.17) simultaneously for $\lambda_1(L_T)$ and $\lambda_2(L_T)$. For simple strain-energy functions explicit solutions may be found, otherwise numerical methods are required.

If we now return to the combined deformation of the two membranes then from $(6.2.13)_2$ with (6.2.14) we obtain

$$2\pi R_B H_B \hat{W}_1 = F_T - \pi P(R_T^2\lambda_2^2(L_T) - R_B^2\lambda_2^2(-L_B)), \tag{6.2.18}$$

for the lower cylinder where $\hat{W}_1 = \hat{W}_1(\lambda_1(-L_B), \lambda_2(-L_B))$. Similarly given initial values for P, R_B, H_B and F_T, we can find the values of λ_1 and λ_2 at the ends of the bottom membrane, namely $\lambda_1(-L_B)$ and $\lambda_2(-L_B)$, by solving (6.2.16) and (6.2.18).

Having determined the values of the principal stretches at the open ends of the composite cylinder we can then make use of these values to calculate the constant of integration in (6.2.9). We note that as (6.2.9) is applied separately to each cylinder there will be two constants of integration given by

$$C_{2T} = \hat{W}_T(\lambda_1(L_T), \lambda_2(L_T)) - \lambda_1(L_T)\hat{W}_{1T}(\lambda_1(L_T), \lambda_2(L_T)), \tag{6.2.19}$$

where the strain-energy $\hat{W}_T$ is that for the upper cylinder and $\hat{W}_{1T} = \partial\hat{W}_T/\partial\lambda_1$ and

$$C_{2B} = \hat{W}_B(\lambda_1(-L_B), \lambda_2(-L_B)) - \lambda_1(-L_B)\hat{W}_{1B}(\lambda_1(-L_B), \lambda_2(-L_B)), \tag{6.2.20}$$

with the strain-energy $\hat{W}_B$ being that for the lower cylinder and $\hat{W}_{1B} = \partial\hat{W}_B/\partial\lambda_1$. Once the strain-energy function of the material has been specified, (6.2.9) becomes an equation for $\lambda_1(Z)$ in terms of $\lambda_2(Z)$ for some general point Z. For example, if we consider the top part of the composite cylinder to consist of the incompressible Varga material defined by (4.2.5) then (6.2.9) gives

$$C_{2T} = 2\mu_T \left(\lambda_2(L_T) + \frac{2}{\lambda_1(L_T)\lambda_2(L_T)} - 3 \right), \tag{6.2.21}$$

which can be rearranged and evaluated at a general point Z by

$$\lambda_1(Z) = \frac{2}{\lambda_2(Z)(\hat{C}_{2T} - \lambda_2(Z))}, \tag{6.2.22}$$

where $\hat{C}_{2T} = C_{2T}/2\mu_T + 3$. Clearly a similar result will hold for the bottom cylinder depending on the strain-energy function $\hat{W}_B$.

We now turn our attention to finding the deformed radius at the join $r(0) = r_0$ say, where we have assumed the join occurs at $z = 0$. If we consider the continuity of the stress resultants across the join we obtain the condition

$$r_T h_T \sigma_{11}(0^+) = r_B h_B \sigma_{11}(0^-). \tag{6.2.23}$$

Since the deformed radius of the top and bottom membranes must be equal at the join we have $r_T(0) = r_B(0) = r_0$. We can then recast (6.2.23), using $(6.2.6)_3$ with the incompressibility condition (2.2.9), as

$$R_T H_T \hat{W}_{1T}(\lambda_1(0^+), \lambda_2(0^+)) = R_B H_B \hat{W}_{1B}(\lambda_1(0^-), \lambda_2(0^-)), \tag{6.2.24}$$

where $\lambda_2(0^+) = r_0/R_T$, $\lambda_2(0^-) = r_0/R_B$ and $\lambda_1(0^\pm)$ is calculated from (6.2.9) using the appropriate values of λ_2 and C_2. Hence r_0 can be found from (6.2.24) as it is the only unknown. However, assuming that a wrinkled region exists on the surface of the upper cylinder, (6.2.24) will use the incorrect form of the strain-energy function $\hat{W}$. We require to use the relaxed strain-energy function $\bar{W}$ in the wrinkled region as described in chapter 3. We therefore have to evaluate $\sigma_{22}(0^+)$ to examine whether the stress in the azimuthal direction is positive or negative at the join in the cylinder of larger radius. As before if the stress is negative wrinkling will occur. We therefore assume the membrane to be in a state of simple tension, that is having only one principal stress non-zero, to obtain the relaxed strain-energy function $\bar{W}_1(\lambda_1)$. This function replaces $\hat{W}_{1T}(\lambda_1, \lambda_2)$ in (6.2.24) and we can find the new deformed radius at the join when wrinkling occurs. Incidentally when considering a wrinkled region, with the upper cylinder consisting of the incompressible Varga material, we obtain

$$\lambda_{1T} = \frac{9}{\hat{C}_{2T}^2}, \tag{6.2.25}$$

having used (6.2.9) and (4.2.13). We therefore note that λ_1 is constant in the wrinkled region. Using (6.2.25) with the relaxed strain-energy for an incompressible Varga material (4.2.13), (6.2.24) becomes

$$2\mu_T R_T H_T(1 - \frac{\hat{C}_{2T}^3}{27}) = R_B H_B \hat{W}_{1B}(\lambda_1(0^-), \lambda_2(0^-)). \tag{6.2.26}$$

It only remains to find $r(Z)$ and $z(Z)$ so that we can plot the deformed shape of the membrane along its axial length. If we consider

$$z' = \frac{dz}{dZ} = \frac{dz}{d\lambda_2}\frac{r'}{R},$$

then we obtain

$$z(Z) = R\int_{\lambda_2(0)}^{\lambda_2(Z)} \frac{z'}{r'} d\lambda_2, \tag{6.2.27}$$

and $z(Z)$ can be calculated numerically for a given $\lambda_2(Z)$ provided we know $z'(Z)$ and $r'(Z)$, where (6.2.27) is applied to the upper and lower cylinders separately. To find an expression for z' we utilise (6.2.12) evaluated at the join to obtain an expression for C_1 which can then be used to solve (6.2.12) for a general point Z. Again as (6.2.12) is solved separately for the upper and lower cylinders we will have two constants of integration. To illustrate we consider the upper tube only but clearly an analogous result can be found for the lower one. Hence (6.2.12) yields

$$C_{1T} = \frac{z'\hat{W}_1}{\lambda_1}(0^+) - \frac{PR_T\lambda_2^2(0^+)}{2H_T}. \tag{6.2.28}$$

Substituting C_{1T} from (6.2.28) back into (6.2.12) and evaluating for a general point Z on the top membrane we find

$$\frac{z'\hat{W}_1}{\lambda_1}(Z) + \frac{PR_T}{2H_T}(\lambda_2^2(0^+) - \lambda_2^2(Z)) - \frac{z'\hat{W}_1}{\lambda_1}(0^+) = 0, \tag{6.2.29}$$

which is an implicit equation for $\lambda_1(Z)$ given $\lambda_2(Z)$. Rearranging $(6.2.13)_1$ gives an expression for $z'\hat{W}_1/\lambda_1$ evaluated at the join and we can substitute this into (6.2.29) which yields

$$z' = \frac{\lambda_1(Z)}{\hat{W}_1}\left[\frac{F_T}{2\pi R_T H_T} + \frac{PR_T}{2H_T}(\lambda_2^2(Z) - \lambda_2^2(L_T))\right], \tag{6.2.30}$$

where $\lambda_1(Z)$ is evaluated from (6.2.9) with C_2 given by (6.2.19) (or (6.2.20) for the lower cylinder). Finally recasting $(6.2.6)_1$ gives

$$r'(Z) = (\lambda_1^2(Z) - z'^2(Z))^{\frac{1}{2}}. \tag{6.2.31}$$

Using (6.2.30) and (6.2.31) in (6.2.27) yields $z(Z)$ for a given value of $\lambda_2(Z)$. We can then rearrange $(6.2.6)_2$ to obtain $r(Z)$ and plot the deformed axial length against the deformed radius of the membrane.

Note however, if wrinkling does occur (6.2.30) must use two different forms of the strain-energy function, namely $\hat{W}$ and $\bar{W}$, to calculate z' for the different solutions associated with the tense and wrinkled regions respectively. Therefore we need to find the value of λ_2 (and hence λ_1) at which $\sigma_{22} = 0$, λ_1^* and λ_2^* say. This is a simple procedure which will be described in the next section along with the results obtained by solving the system of equations derived in this section, attention being given to the existence and initiation of wrinkling.

6.3 Numerical Results

This section presents graphical results for the problem, considering different material and geometric properties but restricting attention to incompressible materials governed by the Varga and three-term strain-energy functions. Firstly, we non-dimensionalise all of the variables as follows. We divide the radii R, r and axial lengths Z, z by the radius of the smaller tube R_B. The thicknesses of both tubes H and h are then divided by the thickness of the smaller tube H_B and similarly the strain-energy functions are divided by the shear modulus of the smaller tube μ_B. We also non-dimensionalise the pressure P by dividing by $\mu_B H_B/R_B$ and the axial load F is divided by $\mu_B R_B H_B$. Once the material parameters are specified for both cylinders and the pressure P, which is assumed to be constant throughout the deformation, and the axial load on the top membrane F_T are fixed, the axial load on the bottom membrane F_B, can be calculated from (6.2.14). Numerical techniques can then be employed to solve the above system of equations as follows.

Using a simple root finding method, such as the secant method, we obtain the principal stretch values on each end of the composite cylinder by solving equations (6.2.16) and (6.2.17) simultaneously for $\lambda_1(L_T)$ and $\lambda_2(L_T)$ and (6.2.16) and (6.2.18) for $\lambda_1(-L_B)$ and $\lambda_2(-L_B)$ choosing appropriate initial values. This allows us to calculate C_{2T} and C_{2B} using (6.2.19) and (6.2.20) respectively. Knowing the values of these constants we can obtain $\lambda_1(Z)$, given a value for $\lambda_2(Z)$, from (6.2.9) for any Z along the axial length of the composite cylinder. We then find the deformed radius at the join, r_0, using (6.2.24), where r_0 is known to be in the range $(R_B\lambda_2(-L_B), R_T\lambda_2(L_T))$. We then calculate $\lambda_2(0) = r_0/R$, with λ_1 being found directly from (6.2.22) for the Varga material or from (6.2.9) for a general strain-energy function. With these values for the principal stretches we evaluate σ_{22} to discover whether its sign is negative, therefore producing a wrinkled region, or positive. If σ_{22} is positive no wrinkling has occurred and we can proceed in calculating the coordinates to be plotted $r(Z)$ and $z(Z)$. To do this we increment

$\lambda_2(Z)$ from $\lambda_2(0)$ to $\lambda_2(L_T)$ for the top cylinder or from $\lambda_2(0)$ to $\lambda_2(-L_B)$ for the bottom one. Substituting (6.2.30) and (6.2.31) into the integral (6.2.27) we can calculate $z(Z)$ for a specified value of $\lambda_2(Z)$, where $\lambda_1(Z)$ is obtained from (6.2.9). Lastly, we evaluate $r(Z)$ from $(6.2.6)_2$ using the given value of $\lambda_2(Z)$.

If σ_{22} is negative, however, we have to recalculate r_0 using (6.2.26) if the top membrane is composed of the incompressible Varga material or (6.2.24) replacing $\hat{W}_{1T}$ by $\bar{W}_{1T}$ if not. If a wrinkled region exists we must solve $\sigma_{22} = 0$ to obtain the principal stretches λ_1^* and λ_2^* corresponding to the point at which the wrinkled region ends and the tense solution takes over. This is done by choosing a relevant value for λ_2, such that $\lambda_2(0^+) < \lambda_2 < \lambda_2(L_T)$, calculating the corresponding value for λ_1 from (6.2.9) and then evaluating σ_{22}. We increment λ_2 until the correct values of λ_1^* and λ_2^*, which satisfy $\sigma_{22} = 0$, are found. The integral defined in (6.2.27) will now have to be split into two intervals, namely, $(\lambda_2(0^+), \lambda_2^*)$ for the wrinkled region and $(\lambda_2^*, \lambda_2(L_T))$ for the tense region and can be written

$$z(Z) = \begin{cases} R\displaystyle\int_{\lambda_2(0)}^{\lambda_2(Z)} \frac{z' d\lambda_2}{r'}, & \lambda_2(0) < \lambda_2 < \lambda_2^*, \\ R\displaystyle\int_{\lambda_2(0)}^{\lambda_2^*} \frac{z' d\lambda_2}{r'} + R\int_{\lambda_2^*}^{\lambda_2(Z)} \frac{z' d\lambda_2}{r'}, & \lambda_2^* < \lambda_2 < \lambda_2(L_T). \end{cases} \tag{6.3.1}$$

The wrinkled region will still use (6.2.30) to calculate z' with $\hat{W}_1$ being replaced by $\bar{W}_1$ and λ_1 being evaluated using (6.2.25) for the incompressible Varga material and (6.2.9) for the three-term material. Note that these intervals are required for the top tube only, the bottom tube using only one interval, namely $(\lambda_2(0), \lambda_2(-L_B))$ as the stresses are always positive at any point Z satisfying $-L_B \leq Z < 0$.

Given the above we are now in a position to plot the deformed, axial length $z(Z)$ against the deformed radius r(Z) for the composite cylinder. We must recall that the lengths of the component cylinders, L_T and L_B, are large enough so as the deformation is equivalent to that of a single, infinitely long cylinder that has been subjected to the same loads. Hence we increment λ_2 from $\lambda_2(0)$ to $\lambda_2(L_T)$ or $\lambda_2(-L_B)$ depending on which part of the cylinder is being plotted. We continue to plot the deformed, axial length until $\lambda_2(Z)/\lambda_2(L_T) > 0.99$ and $\lambda_2(Z)/\lambda_2(-L_B) > 0.99$. This allows the height of the deformed profiles to indicate how quickly the shape of the cylinder attains the values it would have reached if a single cylinder had simply been inflated and axial loads applied. As the deformation is axisymmetrical we only plot one half of the cylinder.

Figure 6.1 plots the deformed, axial length z against the deformed radius r with an initial, non-dimensionalised, inflating pressure $P = 0.15$, a non-dimensionalised, axial load $F_T = 2.0$ and initial radii $R_T/R_B = 1.1$ to 1.5. The curve furthest to the left represents the deformation with initial radius $R_T/R_B = 1.1$. Both membranes consist of the incompressible Varga material of equal shear moduli and undeformed thickness. The starred symbols, as before, indicate the point at which wrinkling ends as we trace the curve upwards from the r-axis, (r^*, z^*) say, which is found using the principal stretches λ_1^* and λ_2^*. We observe that this wrinkled region increases as the initial radius is increased with no wrinkling occurring when $R_T/R_B = 1.1$. This is because the inflating pressure P is sufficiently large to remove the wrinkling that will exist in the upper cylinder prior to applying the pressure. We also note as R_T/R_B increases, the height of the curves increase from 1.5341 to 2.2151 for the top cylinder, indicating for a larger undeformed radius it takes longer for the cylinder to reach its asymptotic shape. However for the lower cylinder the opposite effect can be observed as the height of the curves decrease from 1.4258 to 1.1947.

To explain this we note that as R_T/R_B (or P) increases, the deformed radius at the joining section will also increase and so the bottom cylinder is being stretched further in the azimuthal direction as it is joined to successively larger upper cylinders. This increasing radial displacement will be accompanied by some positive, axial displacement of the bottom tube which has the affect of moving the asymptotic shape upwards. We also note that F_B given by (6.2.14) decreases from 1.8910 to 1.3189 so the bottom cylinder has a smaller, axial load placed on it as the undeformed radius increases.

This effect is more pronounced in Figure 6.2 which again plots the deformed,

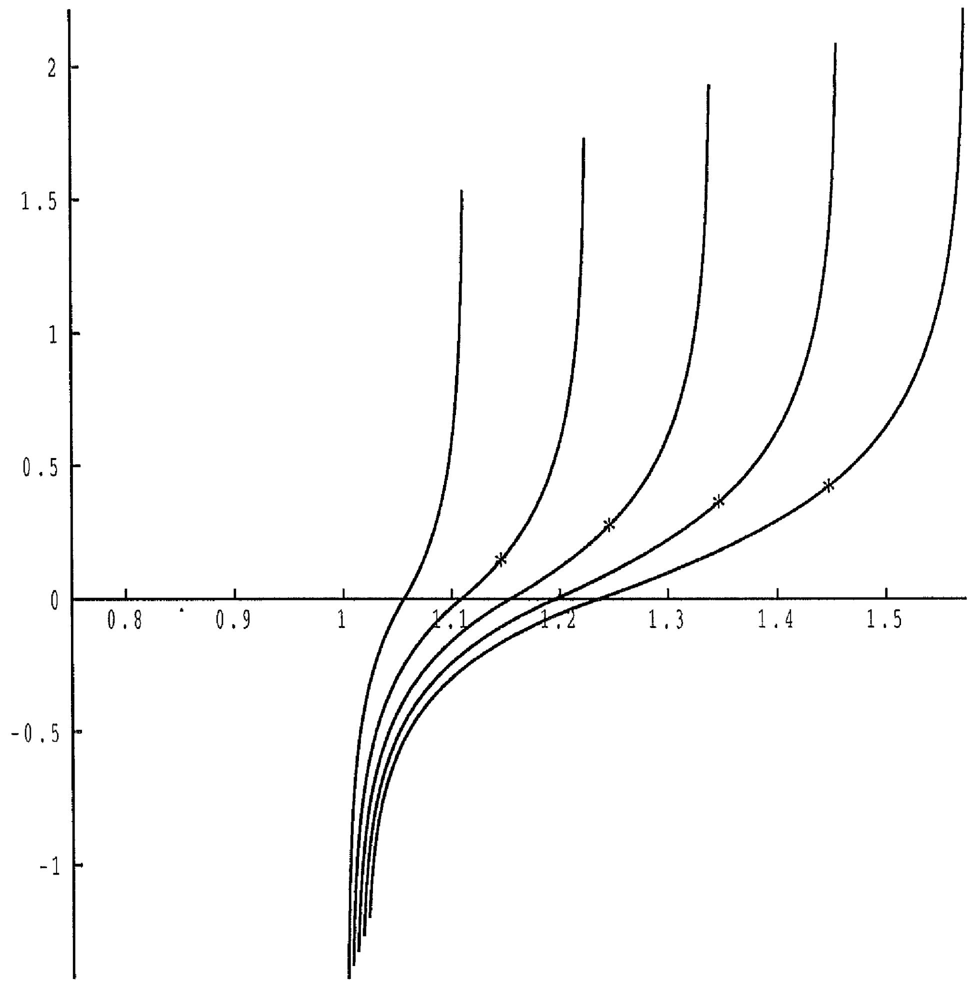

Figure 6.1: Plot of the deformed, axial length against the deformed radius with $P = 0.15$, $F_T = 2.0$ and $R_T/R_B = 1.1\ (0.1)\ 1.5$ for the incompressible Varga material.

axial length against the deformed radius using the same material and parameter values as in Figure 6.1, with the exception of pressure which is taken to be $P = 0.25$. To put this in context we note that a membrane cylinder of incompressible Varga material will, when inflated and subjected to an axial loading of $F = 2.0$, obtain a pressure maximum of $P = 0.8919$ attained at a stretch of $\lambda_2 = 2.378$. We note that the higher pressure has reduced the amount of wrinkling in each of the upper cylinders as expected and in particular no wrinkling has occurred for $R_T/R_B = 1.1$ or 1.2. As R_T/R_B increases the height of the curves again increase for the top cylinder but by a smaller amount than before, the minimum and maximum values, corresponding to $R_T/R_B = 1.1$ and $R_T/R_B = 1.5$ respectively, being 1.6062 and 2.0022. Again this shows how the asymptotic shape is reached more quickly for smaller undeformed radii. For the bottom cylinder we observe a significant decrease in height profiles from 1.4524 to 0.9272 as R_T/R_B increases with F_B decreasing from 1.7955 to 0.6690. This can again be explained by the increasingly larger, azimuthal stretch $\lambda_2(0) = r_0/R_B$ as the cylinder is joined to successively larger, upper cylinders causing a corresponding increase in radial displacement and a decrease in axial displacement.

Figure 6.3 assigns parameter values of $P = 0.35$, $F_T = 5.0$ and $R_T/R_B = 1.2$, 1.4, 1.6 and 1.8. The larger value chosen for F_T is to ensure that F_B remains positive throughout the deformations. Wrinkling has again been postponed, this time until $R_T/R_B = 1.6$, and even at this value only a small amount develops. The above effect of the upper membrane reaching its asymptotic shape more slowly while the lower membrane undergoes a positive, axial displacement can again be seen as R_T/R_B is made larger.

In Figure 6.4 we let the radii be constant ascribing the value $R_T/R_B = 1.5$ but this time vary P from 0.15 to 0.35 in steps of 0.05. The lefthand most curve corresponds to $P = 0.15$ and the righthand most curve to $P = 0.35$. We choose $F_T = 4.0$, again to avoid F_B becoming negative as the pressure is increased, but also to display the result that $r_B(-L_B)/R_B < 1.0$, taking the values 0.9611 and 0.9921

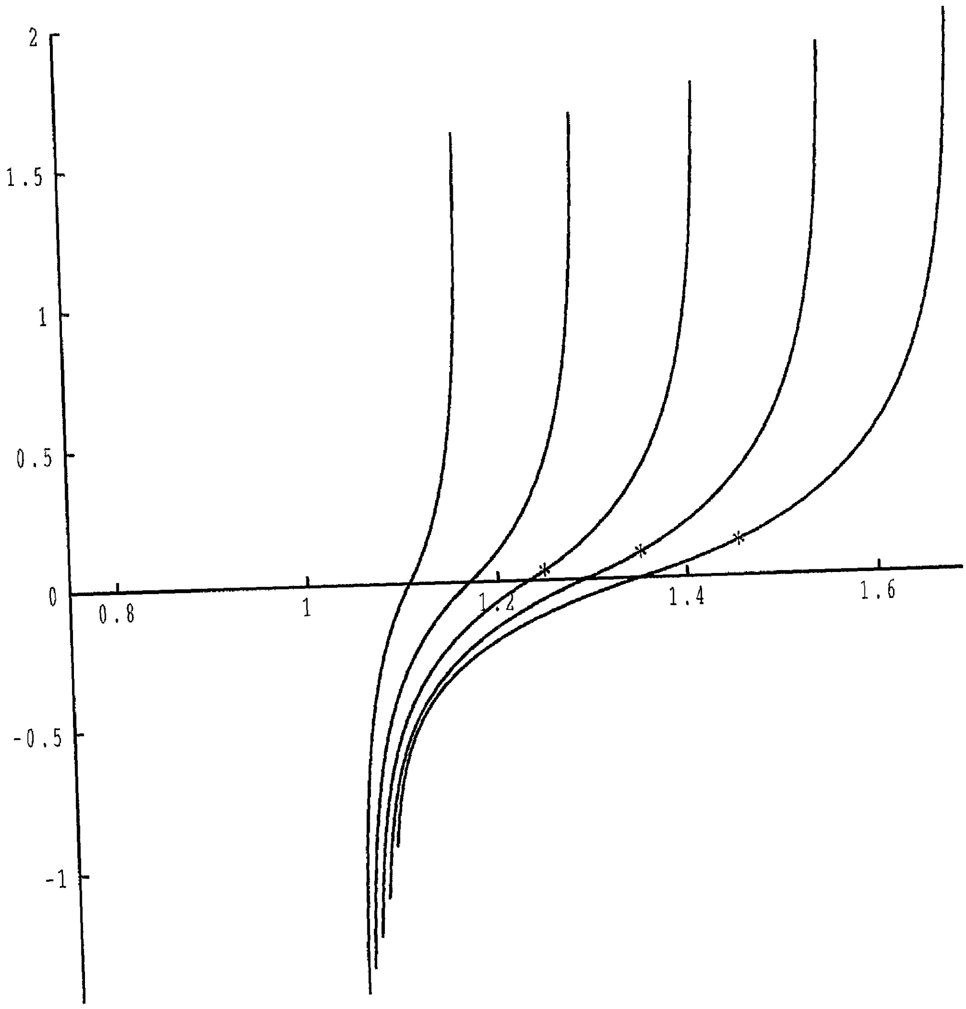

Figure 6.2: Plot of the deformed, axial length against the deformed radius with $P = 0.25$, $F_T = 2.0$ and $R_T/R_B = 1.1\ (0.1)\ 1.5$ for the incompressible Varga material.

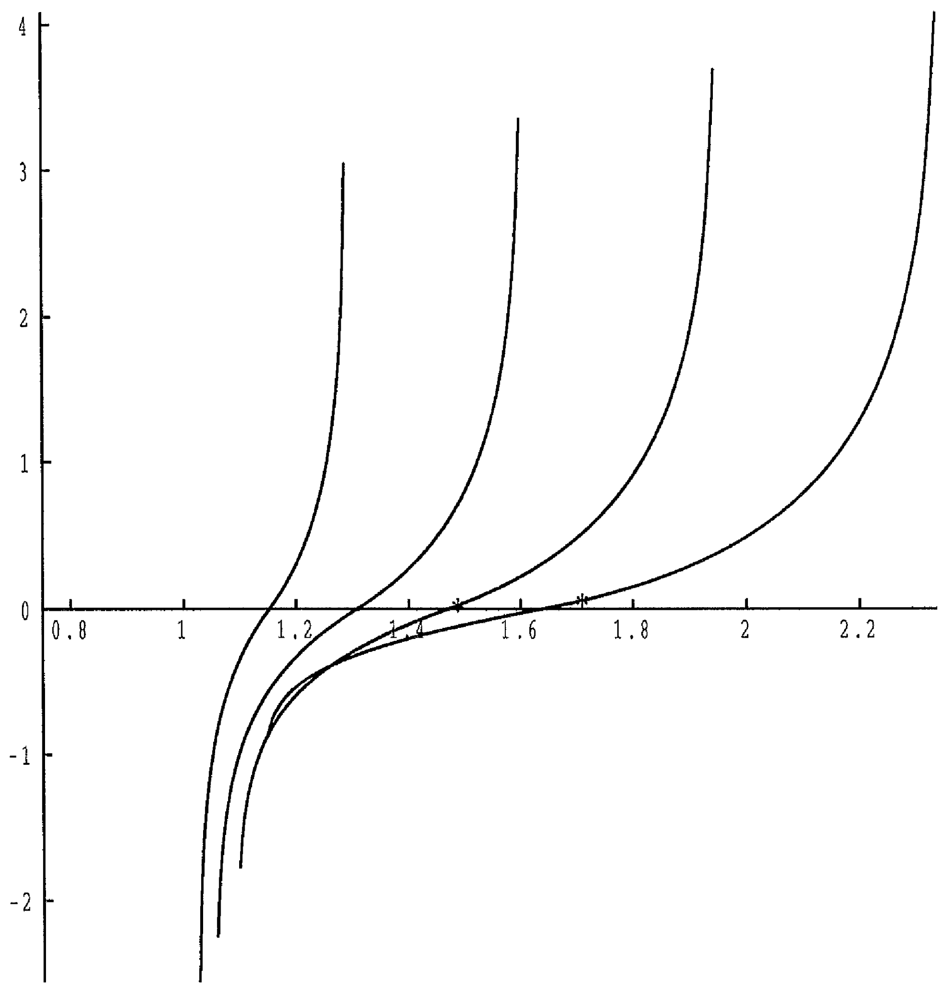

Figure 6.3: Plot of the deformed, axial length against the deformed radius with $P = 0.35$, $F_T = 5.0$ and $R_T/R_B = 1.2\ (0.2)\ 1.8$ for the incompressible Varga material.

when $P = 0.15$ and 0.20 respectively. This shows that the bottom tube, although initially increasing in radius due to the inflation, has been elongated to such an extent that its radius has been reduced to less than R_B. This can be justified by the larger value of F_T, which is required to keep the membrane axially tense at higher pressures. In turn F_B is required to be larger to balance F_T, stretching the membrane by a greater amount in the axial direction, the values being $F_B = 3.3472$ and 3.0598 for $P = 0.15$ and 0.20 respectively. Wrinkling occurs for each pressure, except when $P = 0.35$, with the amount of surface crumpled decreasing with increasing pressure. This is expected since a larger pressure value 'irons out' the otherwise affected region. We also note the plotted length shortens with increasing pressure until the curve associated with $P = 0.35$ in which the tense solution only is required. This can again be interpreted by the infinite solution being reached more quickly as the wrinkled region decreases. The last curve, corresponding to $P = 0.35$, in which wrinkling does not occur, shows an increase in the height of the profile. This is simply associated with the fact that this curve is completely tense and will therefore be plotted using (6.2.27) rather than (6.3.1).

In Figure 6.5 we have chosen parameter values $P = 0.25$, $F_T = 2.0$ and $R_T/R_B = 1.3$ with values of $H_T/H_B = 0.5$, 0.75, 1.0, 1.5 and 2.0 to investigate the effect of varying the thickness of the top membrane. The right-hand most curve corresponds to $H_T/H_B = 0.5$ and the left-hand most one to $H_T/H_B = 2.0$. It can be seen that as we increment H_T/H_B from 0.5 to 2.0, the radius and the plotted length of the top cylinder decrease considerably showing that the constant, inflating pressure P is having less affect as we increase the thickness of the top membrane. We require $H_T/H_B \geq 1$ for wrinkling to form and a larger region is creased as the top membrane is made thicker. We obtain identical results if instead we vary the shear modulus μ_T/μ_B, rather than the thickness, of the composite cylinder. This is accounted for on examination of the equations given in the previous section as μH can always be forced to occur as a product. Hence changing μ_T/μ_B has the equivalent affect as varying H_T/H_B.

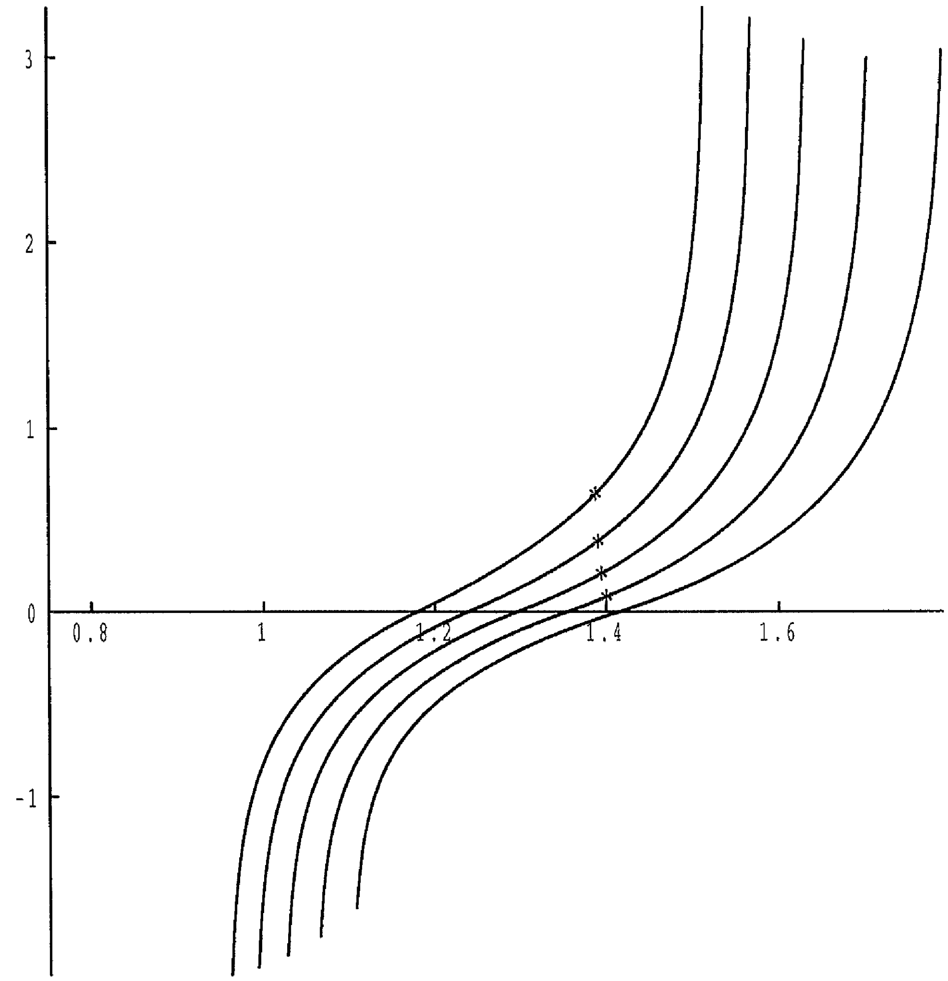

Figure 6.4: Plot of the deformed, axial length against the deformed radius with $R_T/R_B = 1.5$, $F_T = 4.0$ and $P = 0.15$ (0.05) 0.35 for the incompressible Varga material.

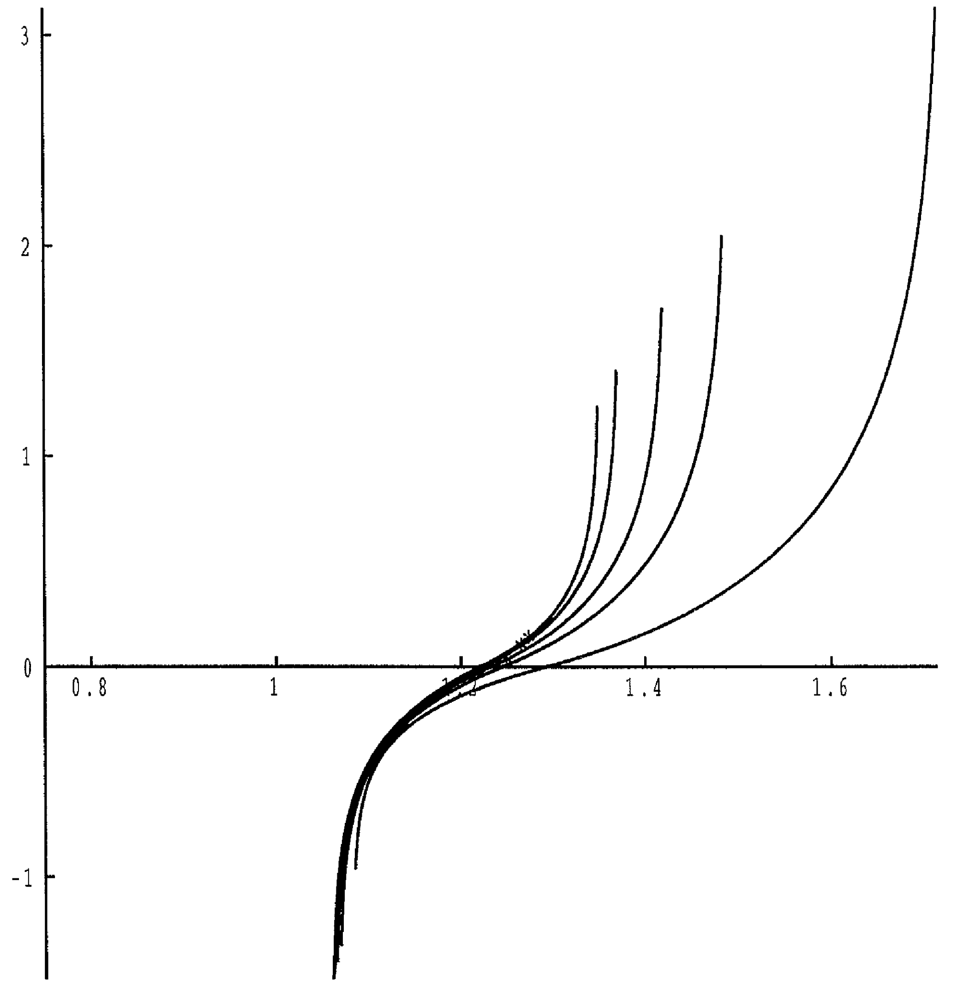

Figure 6.5: Plot of the deformed, axial length against the deformed radius with $P = 0.25$, $F_T = 2.0$, $R_T/R_B = 1.3$ and H_T/H_B (or μ_T/μ_B) $= 0.5, 0.75, 1.0, 1.5$ and 2.0 for the incompressible Varga material.

It is of interest to ask what effect the inclusion of wrinkling theory has on the solution obtained using ordinary membrane theory. To illustrate this we plot in Figure 6.6 the curves obtained from the ordinary membrane theory alongside the solution which makes use of the relaxed strain-energy. The variables have the values associated with the deformation given in Figure 6.1 and the upper curve represents the solution which utilises $\bar{W}$ in each case. We observe the solution obtained by ordinary membrane theory is the lower curve for a given pair. There is a considerable increase in the height of the upper curves of larger radius R_T/R_B when wrinkling theory is used. This is expected and can be explained as the curves with a larger wrinkled region will take longer to reach the asymptotic shape of the cylinder. We observe a slight increase in the wrinkled region when the relaxed strain-energy function is employed although the overall solution is substantially different when wrinkling occurs. We also note that the curve plotted with $R_T/R_B = 1.1$ is identical for both solution methods, as expected, as no wrinkling develops.

If we now consider an example analogous to that described in Figure 6.1 but with the bottom tube consisting of the three-term material defined by (5.2.29) which has the same shear modulus and undeformed thickness as the top tube we can plot the deformed, axial length against the deformed radius as shown in Figure 6.7. Again wrinkling is delayed until $R_T/R_B \geq 1.2$ and there is in fact a slight increase in the amount of wrinkling obtained when compared to that observed in Figure 6.1. Figure 6.8 shows a plot of Figures 6.1 and 6.7 together to compare the differences obtained when the Varga and the three-term materials are chosen for the bottom cylinder. The upper curves indicate the deformation achieved when we select the three-term material for the bottom tube. We also note the significant decrease in height for the lower membrane when defined by the three-term material again indicating how quickly the infinite solution is achieved.

This effect becomes more apparent if we take a pressure $P = 0.25$ as shown in Figure 6.9. Only a small creased region is formed when $R_T/R_B \geq 1.3$ which is in agreement with results obtained for the Varga material. Again the amount of wrinkling is slightly greater than that produced when the lower cylinder is defined by the Varga material (see Figure 6.2). We also observe, for the bottom tube, that the height of the curves initially increase before the onset of wrinkling and hence a reduction in this height. The numerical method used can explain this phenomenon. Essentially if we considered the inflation of two membranes of equal, initial radii they would deform into another right, circular cylinder with $\lambda_2(0) = \lambda_2(L_T)$ and so the plotting routine would define a single point only on the r-axis. Therefore as we increase R_T/R_B, the height of the lower membrane increases before reaching a maximum and then decreases as the lower membrane is joined to successively, larger, upper cylinders which results in an increasingly larger, azimuthal stretch $\lambda_2(0) = r_0/R_B$ and a positive, axial displacement shifting the asymptotic shape upwards.

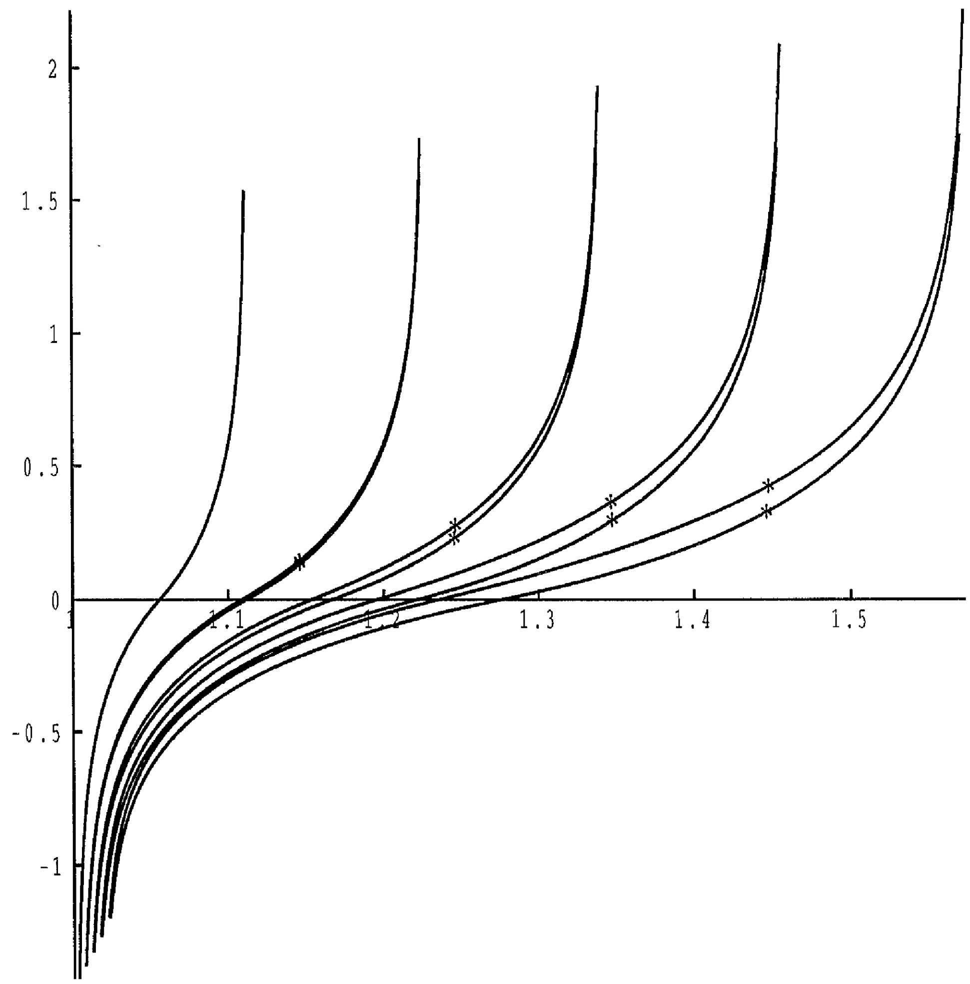

Figure 6.6: Plot of the deformed, axial length against the deformed radius with $P = 0.15$, $F_T = 2.0$ and $R_T/R_B = 1.1$ (0.1) 1.5 for the solutions predicted by ordinary membrane theory (lower curves) and wrinkling theory for the incompressible Varga material.

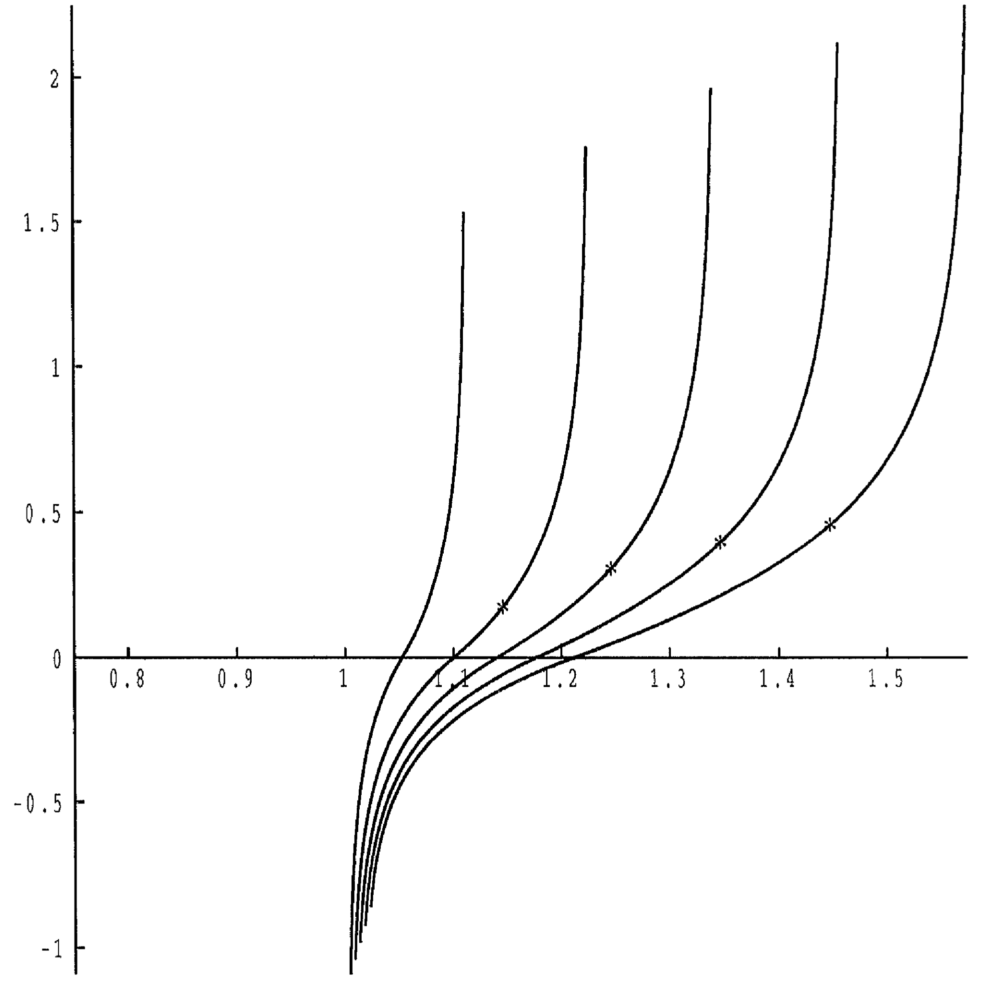

Figure 6.7: Plot of the deformed, axial length against the deformed radius with $P = 0.15$, $F_T = 2.0$ and $R_T/R_B = 1.1\ (0.1)\ 1.5$ with the upper cylinder consisting of the incompressible Varga material and the lower cylinder consisting of the three-term material.

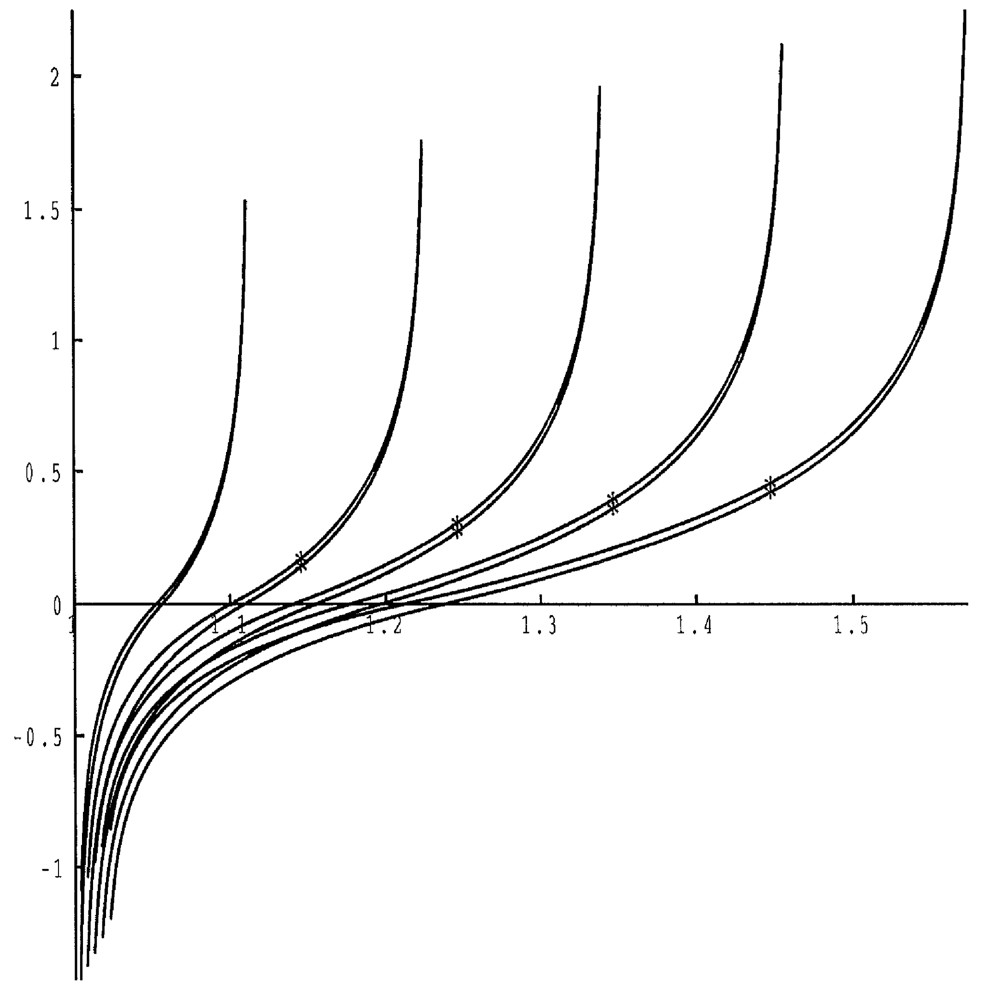

Figure 6.8: Combination of Figures 6.1 and 6.7 to compare the incompressible Varga (lower curves) against the three-term material for the lower cylinders.

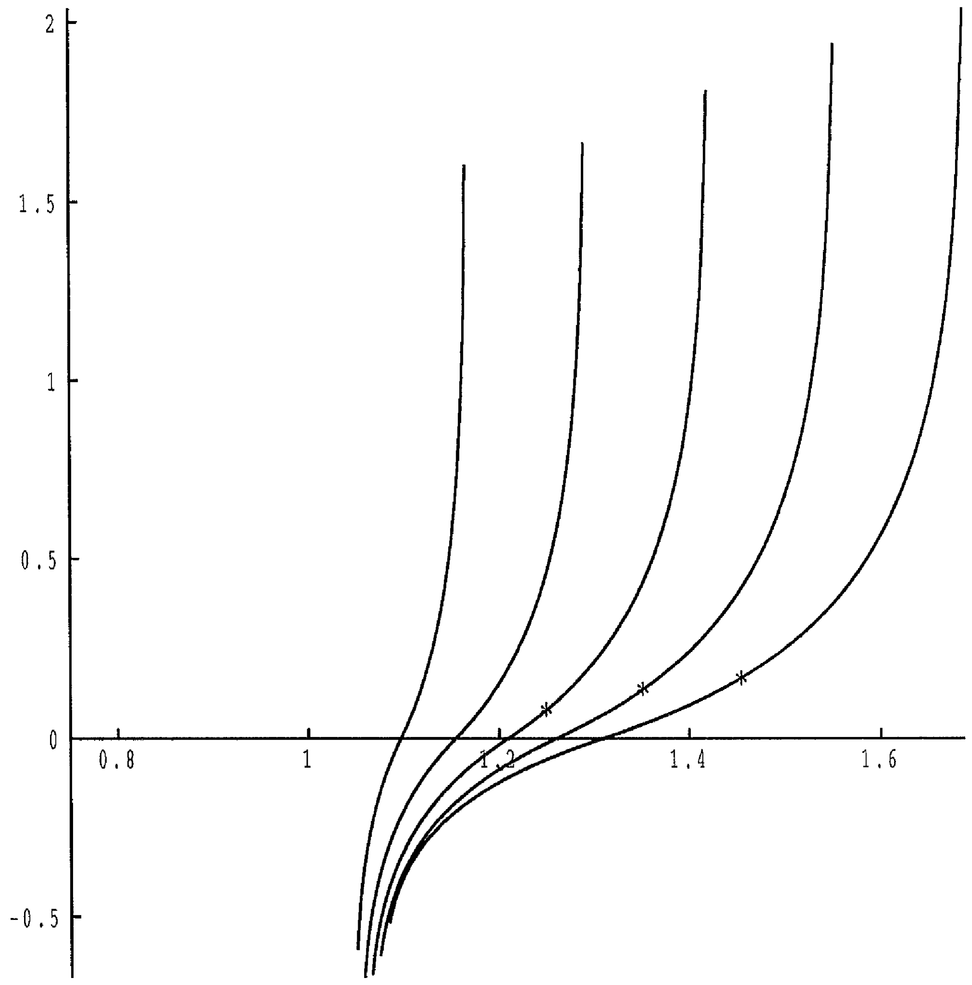

Figure 6.9: Plot of the deformed, axial length against the deformed radius with $P = 0.25$, $F_T = 2.0$ and $R_T/R_B = 1.1\ (0.1)\ 1.5$ using the incompressible Varga material for the upper cylinder and the three-term material for the lower cylinder.

References

B. Dacorogna (1982), Quasiconvexity and relaxation of nonconvex problems in the calculus of variations, J. Func. Anal., **46**, 102-118.

A.E. Green and J.E. Adkins (1970), Large Elastic Deformations, 2nd Edition, Oxford University Press.

A.E. Green and W. Zerna (1968), Theoretical Elasticity, 2nd Edition, Oxford University Press.

V.G. Hart and Jingyu Shi (1991), Joined dissimilar isotropic elastic cylindrical membranes under internal pressure and longitudinal tension, Q. J. Mech. Appl Math., **44 Pt 4**, 581-600.

V.G. Hart and Jingyu Shi (1993), Joined dissimilar orthotropic elastic cylindrical membranes under internal pressure and longitudinal tension, J. Austal. Math. Soc. Ser B, **34**, 296-317.

E.M. Haseganu and D.J. Steigmann (1994), Theoretical flexure of a pressurized cylindrical membrane, Int. J. Solids Struct., **31**, 27-50.

D.M. Haughton (1987), Inflation and bifurcation of thick-walled compressible elastic spherical shells, **39**, 259-272.

D.M. Haughton (1991), An exact solution for the stretching of elastic membranes containing a hole or inclusion, Mech. Res. Comm., **18**, 29-39.

D.M. Haughton and B.A. McKay (1995), Wrinkling of annular discs subjected to radial displacements, Int. J. Engng. Sci., **33**, 335-350.

D.M. Haughton and R.W. Ogden (1978a), On the incremental equations in non-linear elasticity- I. Membrane theory, J. Mech. Phys. Solids, **26**, 93-110.

D.M. Haughton and R.W. Ogden (1978b), On the incremental equations in non-linear elasticity- II. Bifurcation of pressurised spherical shells, J. Mech. Phys Solids, **26**, 111-138.

D.M. Haughton and R.W. Ogden (1979), Bifurcation of inflated circular cylinders of elastic material under axial loading- I. Membrane theory for thick walled tubes, J. Mech. Phys. Solids, **27**, 179-212.

D.M. Haughton and A. Orr (1995), On the eversion of incompressible elastic cylinders, Int. J. Non-Linear Mech., **30**, 81-95.

S.J. Lee and R.T. Shield (1980), Applications of variational principles in finite elasticity, Z.A.M.P., **31**, 454-472.

X. Li and D.J. Steigmann (1993), Finite plane twist of an annular membrane, Q.J. Mech. Appl. Math., **46**, 601-626.

X. Li and D.J. Steigmann (1995), Point loads on a hemispherical elastic membrane, Int. J. Non-linear Mech., **30 Pt. 4**, 569-581.

K.A. Lindsay and C.E. Rooney (1992), A note on compound matrices, J. of Computational Phys., **133**, 472-477.

C.B. Morrey (1952), Quasiconvexity and the lower semicontinuity of multiple integrals, Pacific J. Math., **2**, 25-53.

P.M. Naghdi (1972), Theory of Shells and Plates, Handbuch der physik, **VI a/2**, Springer-Verlag, Berlin and Heidelberg.

R.W. Ogden (1972), Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubberlike solids, Proc. Roy. Soc. Lond., **A 328**, 567-583.

R.W. Ogden (1984), Non-Linear Elastic Deformations, Ellis-Horwood, Chichester.

A.C. Pipkin (1968), Integration of an equation in membrane theory, Z.A.M.P., **19**, 818-819.

A.C. Pipkin (1986), The relaxed energy density for isotropic elastic membranes, IMA J. Appl. Math., **36**, 85-99.

E. Reissner (1938), On tension field theory, Proc. Fifth Int. Congress on Appl. Mechanics, 88-92.

R.S. Rivlin and A.G. Thomas (1951), Large elastic deformations of isotropic materials viii. Strain distribution around a hole in a sheet, Phil. Trans. Roy. Soc. Lond. A., **243**, 289-298.

D.J. Steigmann (1990), Tension field theory, Proc. Roy. Soc. Lond., **A 429**, 141-173.

D.J. Steigmann and A.C. Pipkin (1989), Finite deformations of wrinkled membranes, Q. J. Mech. Appl. Math., **42, Pt 3**, 427-440.

M. Stein and J.M. Hedgepeth (1961), Analysis of partly wrinkled membranes, NASA TN D-813.

C. Truesdell and W. Noll (1965), The Non-Linear Field Theories of Mechanics, Handbuch der physik, **III/3**, Springer-Verlag, Berlin and Heidelberg.

F.S. Wong and R.T. Shield (1969), Large plane deformations of thin elastic sheets of neo-Hookean material, Z.A.M.P., **20**, 176-199.

C.H. Wu (1978), Nonlinear wrinkling of nonlinear membranes of revolution, J. Appl. Mech., **45**, 533-538.

W.H. Yang (1967), Stress concentration in a rubber sheet under axially symmetric stretching, J. Appl. Mech., **34**, 942-946.